GAME THEORY MODELS AND THEIR APPLICATIONS IN SOME INVENTORY CONTROL AND NEW PRODUCT MANAGEMENT PROBLEMS

BY

QINAN WANG

A DISSERTATION

Submitted to the School of Graduate Studies in

Partial Fulfillment of the Requirements

for the Degree

Doctor of Philosophy

McMaster University

1991

GAME THEORY MODELS AND THEIR APPLICATIONS

IN

SOME INVENTORY CONTROL AND NEW PRODUCT MANAGEMENT PROBLEMS

DOCTOR OF PHILOSOPHY (1991)
(Management Science/Systems)

McMASTER UNIVERSITY
Hamilton, Ontario
Canada

TITLE: **GAME THEORY MODELS AND THEIR APPLICATIONS IN SOME INVENTORY CONTROL AND NEW PRODUCT MANAGEMENT PROBLEMS**

AUTHOR: QINAN WANG

B. Sc. Hunan Normal University
M.B.A. NanKai University

SUPERVISORY COMMITTEE: Professors Mahmut Parlar (Chairman)
Sherman Cheung
Yufei Yuan

NUMBER OF PAGES: xi, 200

ABSTRACT

This thesis deals with game theory and its applications in management science and focuses upon some management science areas such as inventory control and new product development. It consists of six chapters, each of which is written as a separate paper except the concluding chapter. Some interesting theoretical findings and new policies are obtained by using the game theoretical approach to analyze certain management science problems.

The discussion starts with a review of static game theory models and their applications in management science. Of particular interest here is the state of the art of game theory as an analytical technique in management science. Its strengths and weaknesses are summarized. The review reveals some new research problems and future research directions in this field. A few of these problems are addressed in this dissertation.

Chapters Two and Three discuss the discount problem. Particular attention is paid to the gaming nature and the buyer's demand aspect of the problem. It is shown that, if they work independently and rationally, the seller and the buyer can gain from price discount only if it can attract more demand from the buyer. Nevertheless, they can gain from quantity discount even if demand is constant. Quantity discount is always better than a price discount

for the seller and, in certain situations, can be very efficient in obtaining the maximum profit. Optimal decisions are obtained for both the seller and the buyer under various conditions.

Chapter Four studies the order quantities of substitutable products with stochastic demands. This analysis extends the newsboy problem analysis into situations with three or more players. It is shown that there is one Nash equilibrium for the problem. If any player(s) acts irrationally, the other players' decision problem reduces to the one without the irrational player(s). If cooperation is possible, their decisions depend on whether side payments are allowed. If side payments are allowed, they will determine their order quantities together. If side payments are not allowed, secure strategies exist for each player. It is also shown that all players' cooperation is often worthwhile and feasible.

Chapter Five analyzes the growth of new repeat purchasing products. The purpose of this analysis is to extend the current research on the diffusion of new consumer durable products to repeat purchasing products in competitive markets. It is shown that markets of repeat purchasing products will never saturate like that of consumer durable products unless customers are extremely loyal to at least one product. For new repeat purchasing products, the optimal advertising strategy is increasing at the introductory stage and then decreasing or possibly terminating after some time and, the optimal service strategy is monotonically increasing at the introductory stage and then possibly maintained constant at a certain level. Especially, more advertising should be done at early stages against competitors' advertising campaign. The game is solved analytically for optimal strategies in the case where all the control functions representing the effects of advertising and service are linear in the control variables.

Finally, the main findings and possible extensions to this research are briefly summarized in Chapter Six.

Acknowledgements

I wish to express my sincerest gratitude to my supervisor, Professor Mahmut Parlar, for his consistent support, advice and constant inspiration during the period in which this thesis was prepared. Without his guidance and support, this thesis would have never been finished. I would also like to express my greatest appreciation to Professor C. S. Sherman Cheung and Professor Yufei Yuan for their kind support and invaluable comments which motivated much of the refinement in the writing of this thesis.

My thanks are also due to other professors in the Faculty of Business, McMaster University. In Particular, I would like to thank Professor M. W. Luke Chan who was a constant source of encouragement and help during my studies at McMaster. I am also thankful to Professor Norman P. Archer and Professor William G. Truscott for their support at the earlier years of my studies at McMaster.

I gratefully acknowledge the financial support provided by the McMaster Graduate Scholarship during my four year studies at McMaster.

Last, but certainly not the least, my thanks go to my parents, my wife and my daughter who provided all the moral support I needed for the completion of this study.

All errors and omissions are my own responsibility.

Table of Contents

Page

List of Figures and Tables

Chapter One

Static Game Theory Models and Their Applications in Management Science: A Survey*

In this chapter, we give a brief review of static game theory models and their applications in management science. Our intention here is twofold: On the one hand, we would like to provide the reader with an overview of game theory and its applications in management science. On the other hand, we intend to explore the mathematical tractability of management science problems when formulated as game theory models. We restrict our attention to static game theory models and five management science related areas. Our discussion starts with a general description of static game theory models and their solution schemes and follows by an investigation of game theory applications. At the end of the chapter, we provide a brief description of some other approaches of game theory.

* This paper has been published in **European Journal of Operational Research**, Vol. 42, No. 1, pp. 1-21, 1989.

1.1. Introduction

The outstanding feature of game theory is modeling conflict and cooperation in explicit mathematical forms. Thus it appears to be an applicable method in management science. Strategy, cooperation, offers-counteroffers, etc., are all factors that should be considered by management scientists. In this chapter, we provide a survey of game theory models and their applications in management science.

Because of the huge literature on this topic, the survey may not be exhaustive and complete. The primary intention of it is to gain some insights into game theory models and their applications to management science problems on one hand, and to provide an overview of the mathematical tractability of real games on the other. Furthermore, we restrict our attention to static game theory models and several areas in management science because a good survey on dynamic (differential) game theory models in management science has already been published by Feichtinger and Jorgensen (1983).

The chapter is organized as follows. In Section 1.2, we provide an overview of static game theory models and their solution schemes. Then, in Section 1.3, we investigate the applications of these models in the management science area. Some comments and views about game theory and its applications are provided in Section 1.4.

1.2. Game Theory Models

Game theory is a mathematical theory of decision-making by participants in conflicting situations. It is generally attributed to John von Neumann in his papers of 1928 and 1937 (von Neumann, 1928; 1937), although its origins can be dated back to the eighteenth century (Rives, 1975). Nevertheless, the establishment point of game theory is generally accepted to be 1928 when von Neumann proved the minmax theorem. But only when John von Neumann and Oskar Morgenstern published their impressive work: Theory of Games and Economic Behavior (von Neumann and Morgenstern, 1944) in 1944, did game theory receive much attention (Dresher, 1961). Since then, an enormous body of work has been done in this area. A few references on these works are Kuhn and Tucker (1950, 1953), Luce and Raiffa (1957), Tucker and Luce (1959), Basar and Olsder (1982), and Owen (1982).

Game theory attempts to model conflict and cooperation and study them by means of quantitative methods. A typical game has three major components: a group of players, a set of rules of play, and a payoff system. The players are the participants of the competitive situation. Each of them makes a choice or choices from a set of alternatives according to the rules of play which specify clearly what each player is allowed or required to do under every possible circumstance. The payoff system assigns an amount of payoff to each player under each possible outcome. A game theory model is simply a mathematical presentation of a game. It abstracts the three components into formal mathematical expressions and is concerned with finding the optimal decision(s) for each player and describes how each player should behave in order to obtain the best possible outcome, considering that the payoff to any player depends not only on his own choices but also on the choices of the other players. In the present literature, various game theory models have been developed to describe different game situations. In this section, we first present several classification methods of game theory models and then discuss their mathematical formulations and solucion schemes.

1.2.1. Classifications

Game theory has been developed into a sound mathematical theory and applied widely in social science and some other areas such as the military. Its scope is so broad that one has to find oneself the appropriate model for his problem to obtain usable results. Therefore classification of game theory models might be helpful in modeling conflicting situations. In the following we give several classification schemes which we found useful in our investigation. However the actual scheme used for a particular game depends on the situation it is designed to describe. The analyst should decide which factor(s) should be used.

(1) Number of players. Game theory models can be classified according to the number of players involved, i.e., two-person games, three-person games, and n-person games, where $n > 2$ and is a positive integer. For example, in games such as chess, there are two players,

hence they are two-person games. In games such as horse racing, there are usually a large number, say n, of participants, hence they are n-person games. Clearly, two players is the minimum number for conflict and cooperation to be present and three or more players can lead to coalition formation, which brings about a major complexity in analyzing n-person ($n>2$) games.

(2) Nature of payoff functions. Payoff function is another classifying factor of game theory models. When the sum of the players' payoffs under every situation is zero, the game is called a zero-sum game. Otherwise it is called a nonzero-sum game. In the case of two-person games, players will work strictly competitively in zero-sum games and somewhat in cooperation in nonzero-sum games because, in zero-sum games, one player's gain is always the other's loss. This distinction, however, is not so obvious in the case of n-person games. Some players may still work cooperatively in zero-sum games and strictly competitively in nonzero-sum games. In fact, we can always add a dummy player to make n-person nonzero-sum games become zero-sum and by doing so the nature of the problem will not be affected.

(3) Nature of preplay negotiation. Games can also be classified into non-cooperative games and cooperative games. A non-cooperative game is one in which the players do not communicate with each other and work independently to achieve one's objective, and a cooperative game is one in which the players or a subgroup of them can discuss their strategies and make binding agreements. For example, in

a 3-person game where each of the three players shows either 1, 2, or 3 fingers simultaneously and player i wins $1 from each of the other two if none of them shows i fingers. Otherwise it is a draw. If each of the three players makes his choice independently, the game is non-cooperative; if any two of them agree to act together (obviously it is profitable for them to do so), the game is cooperative. It will be clear later that this classifying factor is crucial because it determines the formulation and solution scheme to the problem under consideration. As in the above example, the normal form might be used for the non-cooperative case whereas the characteristic form should be used for the cooperative case. The mathematical formulation of games will be discussed later in this section.

(4) Number of strategies. If the number of alternatives available to each player is finite, the game is called a finite game; It is called an infinite game if at least one player has an infinite number of alternatives. For example, in a two-person game, suppose player 1 has m strategies and player 2 has n strategies, the game is finite only if both m and n are finite.

(5) State of information available to each player. This classification is usually used in the extensive formulation to be discussed shortly. If every player's information sets are singletons, that is, any information set of any player includes only one node of the topological tree and each player knows exactly where he is in at each move, a game is said to have complete information. For example, in games such as chess, each player knows exactly what the other and

himself did in the past, the game is one with perfect information. Otherwise a game has incomplete information.

(6) Involvement of time. If time is a factor considered in any player's decision-making, the game is dynamic in the sense that the players' optimal decisions are changing through the decision process. In particular, when time is changing continuously and the state of the system can be described by a set of differential equations, the game is a differential game. On the other hand, when time does not affect decisions at all, the game is static. In our discussion we concentrate on the second type and all the models to be discussed, with a few exceptions, are static game theory models.

There are still other classifying factors such as restrictions on side-payments, deterministic or probabilistic payoffs, etc. However the above factors represent most of the generally used classifying scenarios.

1.2.2. Mathematical Formulations

Most of the static game theory models can be described by one of the three mathematical formulations: the normal form, the extensive form, and the characteristic form. In this section, we discuss briefly each of the three formulations as well as their applicability. An excellent discussion on their definitions, development, and applications before early 1970's has been provided by Lucas (1972).

1.2.2.1. The Normal Form

The normal form is the most intuitive mathematical presentation of a game. In brief, a game model in the normal form consists of a set of players $N = \{P_1, P_2, \ldots, P_n\}$, a strategy space M_i for each player P_i, and a payoff system which assigns a certain amount of payoff to each player for each possible combination of their strategy choices. Let $M_i = \{a_i |$ where a_i is a strategy of player $i\}$ and let $P^i(a_1, a_2, \ldots, a_n)$ denote player i's payoff under the strategy combination $(a_1, a_2, \ldots, a_n)$. The game is played when each player selects an element from his strategy space. At the end of play, each player gets $P^i(a_1, a_2, \ldots, a_n)$ when player i chooses strategy a_i. An element of M_i, a_i, is called a pure strategy of player i. When a player uses a random device to choose a strategy from his strategy space, he is using a mixed strategy. In other words, a mixed strategy for player i is a randomization over his pure strategies or a probability distribution over his strategy space M_i. It is easy to note that a pure strategy a_i of player i is a special case of a mixed strategy with probability of 1 for a_i and probability of zero for any of the other pure strategies. Furthermore, it is usually assumed that each player knows all the strategy spaces and the payoffs under each possible outcome. Therefore a game is played in a way that each of the players selects a strategy from his strategy space, without knowing the others' decisions, to maximize his own (expected) payoff.

The presentation of games in the normal form is usually of a function form in which a payoff function is defined for each player on the product space of the players' strategy spaces. However, two-person finite games can be conveniently described by matrices with one player, usually player 1, as the "row" player and the other, player 2, as the "column" player. They are usually called "matrix games". In a matrix game, each row of the matrix corresponds to a pure strategy of player 1, each column to a pure strategy of player 2 and an element, usually being a two-dimensional vector, denotes the payoffs to the players under the corresponding strategy combination. For example, in the game shown in Figure 1.1, each of the players has three pure strategies and their payoffs under each possible combination of their choices are shown by an element of the matrix. For instance, player 1 gets -1 and player 2 gets 0 when player 1 chooses strategy 1 and player 2 chooses strategy 2. A player's mixed strategy is a three-dimensional vector $s = (x_1, x_2, x_3)$ satisfying $x_j \geq 0$ $(j = 1, 2, 3)$

	1	2	3
1	(1,-2)	(-1,0)	(-2,1)
2	(0,3)	(2,2)	(-2,-1)
3	(-2,1)	(1,1)	(3,2)

Figure 1.1. Matrix Game

and $x_1 + x_2 + x_3 = 1$. To explore matrix games in more depth, the reader might refer to many game theory texts such as Luce and Raiffa (1957), Dresher (1961), Owen (1982) and Basar and Olsder (1982).

The normal form appears to be well suited to conflicting situations where each of the decision makers has a partial control over the decision process through his decision variable and the payoff to each player is determined when each of the players makes a choice from a set of alternatives. Especially, when the decision variables are continuous, a game can be well described by a set of continuous payoff functions. The analysis of the game, in this case, is often fairly "well-behaved" because of the convenience of analyzing continuous functions. However, as the normal form treats players' strategies as primitive elements and suppresses the decision process into each player's selecting a strategy from his strategy space, it is inappropriate to use it to describe games where the decision making process and information rather than the strategy spaces and payoffs are important to players' decision making. Besides, a game theory model in the normal form is concerned with each player's optimal decision, considering the impact of the other players' decisions to his own payoff, but does not specify what will happen if the players cooperate. It is non-cooperative in nature although it may also be used to analyze cooperation.

1.2.2.2. The Extensive Form

The extensive form is another mathematical presentation of a game. It was first presented by von Neumann and Morgenstern (1944) and then modified by Kuhn (1950). Kuhn's definition is usually used in the present literature (Lucas, 1972).

The extensive form provides an explicit description of the order of play and the information structure of each player at each move of a game by using a specially designed topological tree structure. The structure starts with a single node denoting the beginning of play and evolves in such a way that several branches emanate from each node representing the alternatives available to the player at this node. A node presents a choice point for one of the n players. Again at the end of play there is assigned an n-dimensional payoff vector with the ith element being the payoff to player i for each possible outcome. Furthermore, each player's choice points are partitioned into information sets, specifying what he knows at each move. At the extreme case, every information set of any player contains only one node and each player knows exactly where he is in at each move. This is the case of perfect information.

A typical but simple example of an extensive game is depicted in Figure 1.2.

In this game there are two players P_1 and P_2. P_1 has three alternatives L, M, N and acts first in the play and P_2 has two alternatives R and B and acts after P_1's decision. At the terminal of each ending branch there is a two-dimensional vector with the first element being the payoff to P_1 and the second to P_2. For instance, P_1 and P_2 both can get 2 if P_1 chooses N and P_2 chooses R. The broken-lined circles represent the information sets of the players, where these sets have the following interpretation: P_1 has only one information set and knows his position when making his decision and P_2 has two and knows exactly P_1's decision if P_1 chooses L but cannot tell which decision, M or N, that P_1 made when P_1 chooses either M or N.

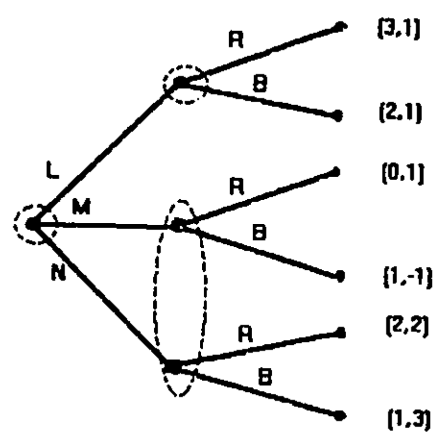

Figure 1.2. Extensive Game

Unlike the normal form case where a strategy for one player is simply a scheme for him to select an element from his strategy space, a player's strategy in the extensive form case is a complete plan telling the player what to do at each of his information sets. A pure strategy is then a complete plan in which the player picks a particular alternative at each of his information sets and a mixed strategy, as in the normal form case, is a probability distribution over the set of his pure strategies. In this case, the player might also use a chance device to determine what to do at each of his information sets. Such a complete plan is called a behavioral strategy. For a more detailed discussion, the reader might refer to Basar and Olsder (1982, pp. 39-65, pp. 92-149) and Luce and Raiffa (1957, Chapter 3).

The extensive form improves the normal form in giving a whole picture of the decision process. In fact, the normal form is only a special case of the extensive form. It is then no doubt that the extensive form is a more powerful presentation of a game, hence it is natural to expect more applications of it to real problems. Unfortunately, few applications of game models in the present literature have been found to be of the extensive form. The explanation of this fact lies mainly in two aspects. First, it is often difficult, if not impossible, to depict explicitly the whole decision process of a game. Especially, it is usually strenuous to specify the information sets for the players. Secondly, it is often not easy to solve a game in the extensive form, especially, when it has a large number of choice points. Although it was pointed out almost two decades ago that it was time to search for solution algorithms for games in the extensive form (Lucas, 1972), the problem seems still being bypassed, possibly because of the difficulty of doing so. Except for a few cases, for example, the case of perfect information (von Neumann, 1928; Kuhn, 1950; Nash, 1950a, 1951) and the case of perfect recall (Kuhn, 1952), there is no general method to find the equilibria. The usual method to solve an extensive game is to partition it (if possible) into simpler games and solve them by means of techniques used in the normal form case. These facts substantially restrict its applicability.

1.2.2.3. The Characteristic Form

The characteristic form is a special presentation for n-person ($n > 2$) cooperative games. It concerns the possible coalitions among the players and attempts to find out what will happen if the players cooperate. This form was originally suggested by von Neumann (Tucker and Luce, 1959) and developed in detail by von Neumann and Morgenstern (1944).

An n-person game in the characteristic form consists of a set $N = \{1, 2, \ldots, n\}$ or the set of players 1, 2, ..., n and a real-valued function v(S) defined on the set of N's subset, including N itself. The value of v(S) measures the total payoff or worth of S when its members act together. More generally, v(S) might be defined as the payoff to S when S is taken as one player in a game played between S and its complementary set N-S. However, in this case, more assumptions concerning the action of N-S are often needed to define a well-behaved function v(S). Such an example is provided by Sherali and Rajan (1987), which will be discussed in detail in Section 1.3. Usually it is assumed that cooperation is always worthwhile, that is, any two groups act together will get no less than that when they act independently, or, mathematically,

$$v(S \cup T) \geq v(S) + v(T) \quad \text{where } S \cap T = \phi.$$

This property is called superadditivity. Obviously, v(N) is then the largest amount of payoff that the players can possibly obtain.

The primal concern of the characteristic form is the distribution of payoff among the players. So a game theory model in the characteristic form is designed to describe how the players should or will distribute a certain amount of payoff. An imputation for an n-person game is defined as a distribution vector $X = (x_1, x_2, \ldots, x_n)$ satisfying

(1) $\sum_{i=1}^{n} x_i = v(N)$,

(2) $x_i \geq v(\{i\})$ for all $i \in N$,

where X is simply a way to distribute the joint maximum payoff v(N) among all the n players, with x_i being the payoff to player i, requiring that each player gets no less than the amount he can get by acting independently. The first condition is often referred to as group rationality and the second condition as individual rationality.

A three-person game in the characteristic form is given by the following example.

Example: $N = \{1, 2, 3\}$,

$v(N) = 1$,

$v(\{1,2\}) = v(\{2,3\}) = v(\{1,3\}) = 0.5$,

$v(\{1\}) = v(\{2\}) = v(\{3\}) = 0.1$.

In words, each player gets 0.1 should all the three players work independently; any two of the three players get 0.5 jointly should they act together against the other who, in this case, gets 0.1; and the three players get 1 jointly should they work together. An imputation is then a vector $X = (x_1, x_2, x_3)$ satisfying

$x_1 + x_2 + x_3 = 1,$

$x_i \geq 0.1$ for $i = 1, 2, 3.$

All imputations for the above example are shown by the shadowed area in the equilateral triangle in Figure 1.3.

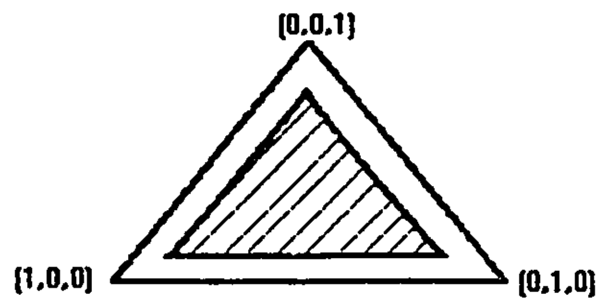

Figure 1.3. Imputation Set

The set of imputations represents all the possible distribution schemes that all the players might agree on and it is usually non-empty. Clearly, to solve a game in the characteristic form is equivalent to finding out some imputation(s) as equilibrium to the problem. However, there is no generally accepted solution scheme for games in the characteristic form, although quite a few solution schemes have been developed based on different arguments. More discussion on these schemes will be given in the following section.

Works including Lucas (1981), Owen (1982), Dresher (1961), etc., provide extensive discussions on games in the characteristic form.

1.2.3. Solution Schemes

There is a basic dichotomy of solution schemes of game theory models, namely that between non-cooperative and cooperative solutions (Shubik, 1981). Non-cooperative solutions characterize how each player should act when the players work independently and cooperative solutions specify what might be the outcome of a game if the players can communicate and make binding agreements.

The solution scheme(s) used for a particular game depends on its formulation (Shubik, 1981). As a matter of fact, the formulation of a model is a pre-solution to the problem under consideration in the sense that a good model should capture the nature of the problem, e.g., how many players are in the game, whether the players make preplay communication, whether the players' decisions depend on time, etc. Among the three basic formulations, since the characteristic form is designed particularly for n-person ($n > 2$) games with coalitions, its solution schemes are cooperative. Indeed most of the cooperative solution schemes are designed for games in the characteristic form. On the other hand, the normal form treats strategies as primitive elements and the extensive form gives a full description of the play of a game and the information structure of each player. But both of them fail to specify what will happen if the players or any subgroup of them form a coalition. Although these two forms can be converted into the characteristic form under certain conditions, non-cooperative solution schemes are usually used for them.

1.2.3.1. Non-cooperative Solutions

There are two major types of non-cooperative solution, namely, the Nash equilibrium and the Stackelberg equilibrium.

(a) Nash equilibrium. The Nash equilibrium is the main solution concept for all kinds of non-cooperative games. It was developed by Nash (Nash, 1950a) and then extended to all kinds of games by many authors.

A Nash equilibrium is an n-tuple of "optimal" strategies, one for each player, such that anyone who deviates from it unilaterally cannot possibly improve his (expected) payoff. Letting s_i denote Player i's strategy, then a Nash equilibrium point is an n-tuple $(s_1^*, s_2^*, \ldots, s_n^*)$ such that

$$P^i(s_1^*, s_2^*, \ldots, s_n^*) \geq P^i(s_1^*, \ldots, s_{i-1}^*, s_i, s_{i+1}^*, \ldots, s_n^*),$$

for $i = 1, 2, \ldots, n$, where $P^i(s_1, s_2, \ldots, s_n)$ is the payoff to Player i under the strategy combination $(s_1, s_2, \ldots, s_n)$. In particular, when s_i^* is a pure strategy for Player i, say a_i^*, for every i, the solution is called a saddle point. In the case of two-person zero-sum games, the saddle point, if it exists, corresponds to one player's maxmin strategy and the other's minmax strategy. In addition, if a saddle point does not exist and the game is finite, we can always find a mixed strategy Nash equilibrium point (von Neumann, 1928). More generally, as Nash has proved, mixed strategy Nash equilibria always exist for finite n-person games (Nash, 1951).

However, there is no general method of finding Nash equilibria. Even for finite games, although the graphical method and linear programming method are usually used in the case of two-person games, they are not generally applicable. To find out if there is any saddle point for an n-person game, one basically has to check exhaustively all possible combinations of pure strategies (Basar and Olsder, 1982). This is often strenuous because of the large number of strategy combinations. For example, a five-person game in the normal form with five strategies for each player would have $5^5 = 3125$ possibilities! In the mixed strategy case, if a given game admits an inner solution, that is, it assigns positive probability to each pure strategy of each player, the solution can be found by solving a set of algebraic equations (Basar and Olsder, 1982). Another method is proposed by Scarf using an approximation algorithm (Scarf, 1967). Nevertheless, for continuous games, the search of Nash equilibria is often simplified by using differential calculus. As proved by Nikaido and Isora (1955), pure strategy Nash equilibrium always exists for convex games which are defined in such a way that each player's payoff function is concave with respect to his own strategy variable and continuous in others' strategy variables. Moreover each player's strategy space is compact and convex (Nikaido and Isora, 1955). When the payoff functions are differentiable, Nash equilibrium can be found by letting the first partial derivative of each player's payoff function with respect to his own strategy variable be zero. A two-person continuous game is provided in the following.

Example: $P^1(x_1,x_2) = -\frac{1}{2}x_1^2 + (1 - \frac{1}{1+x_2})x_1, \quad x_1 \epsilon M_1 = \{x_1 | x_1 > 0\}$, (1)

$P^2(x_1,x_2) = \ln(1 + x_2) - x_1x_2, \qquad x_2 \epsilon M_2 = \{x_2 | x_2 > 0\}$, (2)

where x_i is player i's strategy variable, $P^i(x_1, x_2)$ is Player i's payoff function and M_i is Player i's strategy space (i = 1, 2). It is easy to verify that the game is a convex one and therefore has at least one Nash equilibrium, which satisfies

$$\partial P^1(x_1, x_2)/\partial x_1 = -x_1 + (1 - \frac{1}{1+x_2}) = 0, \qquad (3)$$

$$\partial P^2(x_1, x_2)/\partial x_2 = \frac{1}{1+x_2} - x_1 = 0. \qquad (4)$$

Solving the set of equations gives a unique solution point ($\frac{1}{2}$, 1). Then the game has a unique Nash equilibrium ($\frac{1}{2}$, 1) with $P^1 = \frac{1}{8}$ and $P^2 = \ln 2 - \frac{1}{2}$.

The analysis is comparatively easy to handle in this case. We note in the survey that it is this relative facility of analysis that often motivates analysts to formulate real game problems as continuous games. For more discussion on continuous games, the reader might refer to Nikaido and Isora (1955), Owen (1982, Chapter IV), and Basar and Olsder (1982, pp. 165-194).

(b) Stackelberg equilibrium. Another important solution concept of non-cooperative games is the Stackelberg equilibrium. As the Nash equilibrium provides only the solution to a given game if no one of the players dominates the decision process, Stackelberg equilibrium specifies how one should behave if one has the potential to enforce his strategy on the others. Following the original work of economist von Stackelberg (1934), the one who has the powerful position in such a decision process is called the leader, and the other players who react to the leader's decision are called the followers. The relevant concepts are illustrated in the following by a two-person game in the normal form.

Suppose Player 1 is the leader and Player 2 is the follower who reacts to Player 1's decision. If Player 1 announces his strategy in advance, Player 2 will response in such a way as to maximize his own payoff. Let $R(a_1)$ denote Player 2's optimal strategy(ies) to Player 1's strategy a_1, that is,

$$R(a_1) = \{ a_2^* \mid P^2(a_1, a_2^*) \geq P^2(a_1, a_2), a_2 \epsilon M_2 \}.$$

$R(a_1)$ is referred to as Player 2's response set. A Stackelberg equilibrium with Player 1 as leader and Player 2 as follower is a strategy combination (a_1^*, a_2^*) such that

$$P^1(a_1^*, a_2^*) \geq P^1(a_1^*, R(a_1^*)).$$

Thus a Stackelberg strategy with Player 1 as leader is the optimal strategy for him if he announces his decision first and both players' goals are to maximize their payoffs. If Player 1 chooses any other strategy a_1', Player 2 will choose a strategy a_2' to maximize $P^2(a_1', a_2')$ and the resulting payoff to Player 1 will be less than or at most equal to that when the Stackelberg strategy with Player 1 as the leader is used.

Consider again the example provided in the Nash equilibrium discussion. Because $P^1(x_1, x_2)$ is continuous and concave in x_1, Player 1's response set to Player 2's strategy x_2 is uniquely determined by (3), or,

$$x_1 = 1 - \frac{1}{1+x_2}. \tag{5}$$

Similarly Player 2's response set to Player 1's strategy x_1 is uniquely determined by (4), or,

$$x_2 = \frac{1}{x_1} - 1. \tag{6}$$

If Player 1 acts as the leader, Player 2 will respond according to (6) and Player 1's payoff, in this case, becomes

$$P^1(x_1, x_2) = -\frac{1}{2}x_1^2 + (1 - x_1)x_1 = -\frac{3}{2}x_1^2 + x_1, \tag{7}$$

which is obviously concave in x_1. Hence the optimal strategy for Player 1 can be obtained by letting

$$dP^1/dx_1 = -3x_1 + 1 = 0 \quad \text{or} \quad x_1 = \frac{1}{3}. \tag{8}$$

Substituting x_1 into (6), we obtain $x_2 = 2$ and accordingly, $P^1(1/3, 2) = \frac{1}{6}$ and $P^2(1/3, 2) = \ln 3 - \frac{2}{3}$. Therefore the Stackelberg equilibrium when Player 1 acts as the leader and Player 2 acts as the follower is (1/3, 2) with $P^1 = 1/6$ and $P^2 = \ln 3 - 2/3$. Both players are better-off in this case. Similarly we can obtain the Stackelberg equilibrium when Player 2 acts as the leader and Player 1 acts as the follower as $(\frac{3-\sqrt{5}}{2}, \frac{\sqrt{5}-1}{2})$ with $P^1 = \frac{7-3\sqrt{5}}{4}$ and $P^2 = \ln(\frac{\sqrt{5}+1}{2})-(\sqrt{5}-2)$. In this case, Player 2 is better-off but Player 1 is worse-off than using their Nash equilibrium strategies.

It is shown in Basar and Olsder (1982) that if the follower's response to every strategy of the leader is unique, the leader's Stackelberg strategy will not be worse than his Nash strategy.

However, the follower might be worse off because of his lower position in the decision-making process. If the follower's response is not unique, we might modify the leader's optimal decision by his maxmin strategy to the follower's responses. In this case the above statement may not hold and the leader might gain less by using the Stackelberg strategy than that of using his Nash strategy.

The Stackelberg equilibrium can also be extended to mixed strategies and/or n-person games (n>2) in any form. For such extensions, the reader may consult Chen and Cruz (1972), Simaan and Cruz (1973a; 1973b) and Basar and Olsder (1982).

1.2.3.2. Cooperative Solutions

Most of cooperative solution schemes fall into the category of solution schemes to games in the characteristic form. Although some of them can also be applied to games in other forms, generally accepted cooperative solution schemes do not exist.

(a) The negotiation set. The negotiation set in some sense is a general solution scheme to cooperative games as it characterizes the basic properties a cooperative solution should have. Let P_0^i denote the payoff to Player i under the Nash strategy combination. Then this is the least amount Player i can get by acting independently and is called his security level. When the players decide to cooperate, the basic properties a cooperative solution should have include (1) the players should not be able to improve their payoffs jointly from any such solution, that is, any such solution should be Pareto optimal, and (2) any such solution must not represent payoffs less than their security levels. The set of such solution points is called the negotiation set for the game (Luce and Raiffa, 1957, p. 118).

(b) Cooperative solutions to games in the normal form. Cooperative solution to a given game in the normal form is often specified as the negotiation set. But the negotiation set itself usually cannot be taken directly as the solution to a game as it often contains infinite number of points. It is more like a pre-solution. To further single out a point, Nash developed a bargaining problem solution under a set of assumptions (Nash, 1950a; 1953). Although the scheme received much criticism, it is a point in the negotiation set and is fair in some sense. Other similar schemes include the Shapley value (Shapley, 1953) and the works of Raiffa (1951; 1953).

(c) Solution schemes for games in the characteristic form. The discussion of games in the characteristic form is very extensive. This can be judged from the number of solution schemes that have been developed for this form. These schemes include

(1) Core

(2) Shapley value

(3) von Neumann-Morgenstern stable set

(4) Bargaining set

(5) Kernel

(6) Nucleolus

(7) ϵ-core

(8) Inner core.

As an excellent discussion and review of these solution schemes has been provided by Shubik, we do not repeat the work here and refer the reader to Shubik (1981).

1.3. Applications

In the previous section we investigated the basic mathematical formulations and different solution schemes of game theory. In this section we will present a survey of the applications of these models and solution schemes in management science.

1.3.1. Production and Inventory Management

Production and inventory management is a broad area of research and decision models in this area are usually mixed with more than one issue of production, inventory, investment, and pricing. Therefore models reviewed here might be relevant to marketing and other areas and models reviewed in other areas might be relevant to production and inventory decisions. According to our investigation, the game theory models of production and inventory decisions in the present literature can be roughly classified into two categories: those that assume the market is determined entirely by the players and help to find the market equilibrium conditions and those that assume the players under consideration are only individuals in a competitive market and help the decision makers find the optimal decisions. We investigate both types of models here, but with the emphasis on the second which we believe is more relevant to our subject.

1.3.1.1. Models from the Market Perspective.

The point of view of this category of models is broad and the main results obtained so far with such models are usually of a theoretical and conceptual nature. They aim at obtaining the conditions of market equilibria under certain market conditions and investigate the existence, the uniqueness and possibly the stability of such equilibria, with little attention paid to the optimal decisions for individual players. A casual survey would reveal that there are many such game theory models, most of them constructed by economists under market conditions of duopoly, oligopoly, etc., in the present literature. In the following, we discuss only a few of these models and refer the reader to Shubik (1981, 1984) for an accurate review. Especially, a fairly complete bibliography is provided by Shubik (1981, 1984).

The so called oligopolist theory includes many of these models. By assuming that the duopolists or oligopolists produce homogeneous or differentiated but substitutable products, the models analyze market equilibria given that the competitors have the common objective of seeking profit maximization. For example, Levitan and Shubik (1971) considered a non-cooperative duopolistic market where the duopolists produce two substitutable products. The players' objective functions were defined as

$$\pi_i = p_i \min (q_i, x_i) - \rho_i \max (0, x_i - q_i), \quad \text{for } i = 1, 2,$$

where p_i denotes the price, q_i the demand, x_i Player i's decision variable or quantity to be produced, and ρ_i holding cost per unit of Product i on inventory at the end of the period. The demand was assumed to consist of two components: a random variable capturing the uncertainty and a linear function of p_1 and p_2. They analyzed the conditions for symmetric equilibria both generally and specifically for the case of uniform demand, although some open questions were left. The main results of works discussing non-cooperative oligopolistic markets are contained in Shubik (1984, Chapters 3 and 4).

A second group of game theory models in this category discuss cooperative oligopolistic markets. A typical and interesting example was provided by Sherali and Rajan (1987) recently, studying the situation where n oligopolistic producers are producing a homogeneous product. They investigated the cooperative equilibria of the market using models stemming from the potential threat or bargaining powers of the players, assuming each player's profit to be the difference between his sales revenue and production cost, that is,

$$\pi_i(q_1, \ldots, q_n) = q_i P(Q) - C_i(q_i), \quad \text{for } i = 1, 2, \ldots, n,$$

where $Q = \sum_{i=1}^{n} q_i$ and q_i and $C_i(q_i)$ are the quantity to be produced by Player i and his average unit production cost. The worth of a coalition S under a given combination of production decisions was simply given by the sum of its members' profits, or,

$$\pi_S(q^S, q^{N-S}) = \sum_{i \in S} \pi_i(q_1, \ldots, q_n), \quad \text{for all } S,$$

where $q^T = \{q_i: i \in T\}$ and T is an arbitrary subset of N. Then, based on different assumptions concerning the actions of N-S, they built three cooperative games.

Game 1: N-S plays against S and acts before S's decision,

$$v(S) = \max_{q^S \in Q^S} \min_{q^{N-S} \in Q^{N-S}} \{\pi_S(q^S, q^{N-S})\}.$$

Game 2: N-S plays against S and acts after S's decision,

$$v(S) = \min_{q^{N-S} \in Q^{N-S}} \max_{q^S \in Q^S} \{\pi_S(q^S, q^{N-S})\}.$$

Game 3: The members of N-S act independently,

$$v(S) = \max_{q^S \geq 0} \{\pi_S(q^S, q_0^{N-S})\}.$$

Q^S denotes S's strategy space above. They showed that Game 1 and Game 2 are identical and by taking the Shapley value as the equilibrium of the problem and considering the possible emerging coalition structure of the players all the three games have equilibria under certain conditions.

For further exploration in this field the interested reader might consult the references provided above.

1.3.1.2. Models from Decision Makers' Perspective.

The second category of game theory models of production and inventory decisions focus on the optimal decisions for individual decision makers. Although some market conditions still have to be assumed here, the primal concern of such models is the players' optimal decisions, which are rightly what management scientists are working on. Unfortunately, we have found in our survey that this field has not been adequately studied by using game theory methods. In the following, we survey this field in some detail, as we hope to shed some light on the strategic nature of production and inventory decisions as well as problems which are possibly worth more research efforts.

In a recent paper Parlar (1988) developed a game theory model addressing an inventory control problem where two individual retailers are selling two substitutable products. Each retailer faces a random demand. The retailers' concern is to order optimal quantities hence to maximize their expected profit from selling the products, which they order from other suppliers. Considering the substitution effect and uncertainty of demand, the author first found the profit for one player, say Player 1, under each possible situation. For example, when both retailers order more than the actual demands, Player 1's profit becomes

$$\pi_1 = s_1 x + q_1(u - x) - c_1 u, \qquad x \le u \text{ and } y \le v.$$

Then Player 1's expected profit was found as

$$\begin{aligned} J_1(u, v) = {} & (s_1+p_1)[\int_0^u xf(x)dx + u\int_u^\infty f(x)dx] - p_1 E(X) \\ & + (s_1 - q_1)\int_0^u \int_v^B b(y-v)g(y)f(x)dydx \\ & + (s_1 - q_1)\int_0^u \int_B^\infty (u-x)g(y)f(x)dydx + q_1\int_0^u (u-x)f(x)dx - c_1 u, \end{aligned}$$

where u(v) denotes the order quantity chosen by Player 1(2); X(Y) the random demand for Player 1's(2's) product with density f(x)(g(y)); a(b) the substitution rate of Player 1's(2's) product with the other when Player 1(2) is sold out, $0 \le a \le 1$ ($0 \le b \le 1$); s_i unit selling price; c_i unit ordering cost; q_i unit salvage value; p_i unit lost sales penalty for Player i's product; and $B = [(u-x)/b]+v$ and $A = [(v-y)/a]+u$. Player 2's expected profit was found analogously.

He demonstrated the existence and uniqueness of the Nash equilibrium for each decision maker with the above payoff functions and also showed that if anyone acts irrationally and tries to inflict maximum damage to the other he will incur a loss and the other's decision problem becomes a single player problem. Moreover the players will gain if they cooperate.

Parlar's work provides some insights into the strategic nature of inventory decisions which may have been overlooked in the present literature of inventory management. Traditionally, inventory problems are studied in a framework in which a decision maker (a firm), facing a certain pattern of demand, makes a decision of his ordering or producing quantity under certain market and production conditions, without considering other competitors' decisions. Although such models capture some elementary aspects of inventory problems, they might also result in undesirable outcomes as they totally ignore competitors. Especially, in inventory problems where there exist substitution, discount, etc., one's optimal decision may heavily depend on other competitors' decisions. Models developed from exclusively one decision maker's perspective cannot adequately describe such situations and game theory method should be used. In Chapters two and three of this dissertation, we will discuss the quantity discount problem in a game theory framework and show that game theoretical method would be a better way to study such decision-making processes.

In the present literature, there seems to be only one type of inventory problem that has attracted much attention from game theorists, namely, that of stockpiling of commodities such as petroleum that are subject to periodic supply and/or demand fluctuations. In fact, this problem has become an important subject of policy since 1979 when Iran slashed its oil exports by 5.5 billion barrels per day, which drove the United States, Japan, and many other countries into very difficult situations with oil supply.

An early work on this topic was done by Nichols and Zeckhauser (1977), addressing the problem from a perspective in which governments (representing consuming nations) stockpile a commodity in order to suppress future prices set by a cartel of producers in future periods. With the cartel's objective being to maximize its present value of profits in n periods and the governments' objective being to maximize the consumers' surplus less costs associated with the stockpiling, the authors investigated the problem in several cases under varying conditions of resource constraints, storage and production costs, and time horizon. For example, in the simplest one-cartel one-nation two-period case, a government, starting with no stockpiles and required to end with no inventory, tries to decide its ordering quantity for each of the two periods under the prices set by a cartel. The demand for the product within the nation in period t, C_t, is determined by

$C_t = K - \alpha P_t$,

where P_t is the price set by the cartel in period t and K and α are constants. Suppose the government decides to stockpile S units in Period 1, the government's ordering quantities, D_t (t = 1, 2), are

$D_1 = C_1 + S$ and $D_2 = C_2 - S$.

The cartel's discounted revenue, Y, can be found as

$Y = P_1 D_1 + \beta_p P_2 D_2$,

where β_p is the cartel's one-period discount factor. The gain of the nation, V, has two components: the consumers' surplus and the government's net gain. With some manipulation, it turns out to be

$V = V_1 + \beta_c V_2$,

where $V_1 = [(K/\alpha) - P_1]C_1/2 - P_1 S$, $V_2 = [(K/\alpha) - P_2]C_2/2 - P_2 S$, and β_c is the nation's one-period discount factor. With this basic model, the authors found that stockpiling aids both the consumers and the producers, and one possible strategy for the producers is to charge a unique monopoly price in every period regardless of the government's decision. In sequel, they generalized the basic model to several more realistic cases and found, in most cases, both the producers and the consumers benefit from the government's stockpiling, although the net benefits in the case where the quantity of the commodity is limited are unclear.

The problem was discussed by Nti (1987) from another perspective in which n countries compete in procuring a homogeneous commodity from a single supplier. At the beginning of the period, each country j, facing a random domestic demand $D_j = g_j(P) + \xi_j$ over the commodity, where $g_j(P)$ is the deterministic component of domestic demand at price P and ξ_j is the random component of its domestic demand, must order a certain amount I_j and sell it at price P_j. It is assumed that the procuring cost, denoted by C(I), depends on the total ordering size $I = \Sigma I_j$ of the n countries and it is increasing and convex in I. Then each country's social profit is expressed as

$$\pi_j(D_j, P_j, I_j, I) = P_jD_j - I_jC(I) - h_j(I_j - D_j) \quad \text{if } D_j < I_j,$$
$$= P_jD_j - I_jC(I) - r_j(D_j - I_j) \quad \text{if } D_j \geq I_j,$$

where h_j and r_j are Country j's inventory carrying and shortage penalty costs, respectively. The model was analyzed both non-cooperatively and cooperatively. The Nash equilibrium for each country was investigated for existence and uniqueness. In addition, he also showed that the incentive for the countries to cooperate is very significant. The policy implications of the solutions were then discussed.

Other game theory models on this topic include Balas (1981), Hogan (1983), etc., and follow a similar way.

1.3.2. Bidding and Auctions

Game theory models on bidding and auctions represent a major portion of the present literature of game theory applications in management science. This is basically because the specific strategic nature of bidding and auction problems makes game theoretical approach well suited to the study of such decision-making situations. In a typical bidding and auction problem, each of the bidders, aiming at winning the contract, selects a strategy to maximize his potential profit, with an assessment of the others' decisions. This is rightly the phenomenon studied in game theory. On the other hand, as Rothkopf (1969) pointed out, the traditional probabilistic approach is limited as it ignores all the other bidders and suppresses their conflict of interests. It is then natural to expect that game theory would attract much attention in this area. Indeed bidding and auction problems have been discussed extensively in game theory approach. This fact can be judged from the number of works done in this area. Especially several reviews have been published (King and Mercer, 1988; Engelbrecht-Wiggans, 1980), with a major portion devoted to game theory models.

The pioneering work using game theoretical approach to bidding and auction problems was done by Griesmer and Shubik in a series of articles (Griesmer and Shubik, 1963a, 1963b, 1963c). Although, as reviewed by King and Mercer (1988), these works do not really adequately address the problem, they did introduce a powerful method into this area. Later, in 1967, Griesmer, Levitan, and Shubik (1967) built a more realistic model in which two bidders, each facing a random cost, compete in winning a contract. The payoff for each player was specified as the difference between the bid and his cost. They worked with expected values.

Since their works, a wide variety of game theory models have been built to study bidding and auction problems. A typical model has the following features. (1) It is probabilistic. Either the bid or the bidders' costs for the contract or both are uncertain, hence each player's payoff, usually expressed as the surplus of the bid over his cost, is probabilistic. The players work with expected values. This is realistic as each player can only make an assessment of his own cost and/or the bid at the time of bidding. The distributions used for such random variables include rectangular (Griesmer, Levitan and Shubik, 1967), normal (Wilson, 1969), Weibull (Rothkopf, 1969), loglogistic (Smith and Case, 1975), etc. The solutions to many models depend heavily on the assumption of distribution. (2) The Nash equilibrium has been the major solution scheme. As rules in bidding and auction often restrict communications among bidders, bidding and auction problems are usually non-cooperative and the Nash equilibrium is then the optimal strategy of each player, although cooperation has been also explored in some cases (Smith and Case, 1975). In addition, the normal form is usually used for bidding and auction models.

The basic questions asked in bidding and auction problems are usually how much for a bidder to bid and/or at what level the seller should set his reservation bid. These questions have been successfully answered by many authors under certain conditions. The reader might refer to King and Mercer (1988) for an accurate description of this point. In the following, we would like to point out two problems that deserve more attention. First, many solutions are dependent on the distributions of the random variables under consideration, which are subjective in the sense that they are assumed by the analyst. As noted by Rothkopf (1980a), only a few models in the present literature succeed in attaining analytically unrestricted equilibrium. Secondly, the Nash equilibrium is usually taken as solution to various bidding and auction problems without attempting to question whether it is proper to do so. Recently, Palfrey (1980) investigated the use of the Nash equilibrium as a solution concept for a bidding and auction game in which exposure constraints are present, that is, the sum of a player's bids in a multiple-object auction cannot exceed a certain amount. Let M^i denote this amount for bidder i. Suppose there are I (≥ 2) bidders competing in obtaining J (≥ 2) items and V_j^i is the value of item j to bidder i, which is constant and known with certainty. A feasible pure strategy for bidder i was characterized by a non-negative J-vector

$$\sigma^i = (b_1^i, \ldots, b_J^i), \qquad i = 1, 2, \ldots, I,$$

where b_j^i denotes the bid of Bidder i for Item j satisfying $b_j^i \geq b_j^*$, the seller's reservation bid and $\sigma^i x\ 1 \leq M^i$, 1 is a unit column vector here. The payoff function, π^i for Player i, is assumed to be

$$\pi^i = M^i + \sum_{j=1}^{J} \delta_j^i (V_j^i - b_j^i),$$

where $\delta_j^i = 1$ if $b_j^i > b_j^k$ for all $k \neq i$ or Player i wins Item j; $\delta_j^i = 0$ otherwise. He investigated the Nash equilibrium under several conditions regarding the numbers of bidders and items and the values of the items and found that the Nash equilibrium does not always exist and if it does exist, it is not necessarily unique. In addition, the Nash equilibrium typically yields zero "surplus". Based on these results, we might challenge the Nash equilibrium as solution to bidding and auction problems because solution might exist for various real problems and usually does not generate zero "surplus" for the players. More empirical evidence is needed on the validity of game theory models as well as their solution schemes on bidding and auction problems.

There are many other bidding and auction models using game theory framework that are not reviewed here. A complete bibliography on bidding up to early 1977 with nearly 500 papers, which includes many game theory models, has been provided by Stark and Rothkopf (1979) and some more recent references are provided at the end of this dissertation.

1.3.3. Marketing

Judging from the number of papers published on game theoretical approach in marketing, it has been another attractive area of game theory applications. Especially, there are a number of differential game theory models, part of which are reviewed in Feichtinger and Jorgensen (1983), dealing with marketing problems. This observation seems to fit well to the nature of marketing problems as decision makers in such situations are often facing predictable but unstable pattern of seasonal demand, which makes time become an important factor under decision makers' consideration. Another common feature of these models is that they are often mixed with issues of production, pricing, inventory, etc.

An early work using game theory to marketing problems was done by Friedman (1958). He offered a game theoretic analysis of duopolistic or more generally oligopolistic competition using the market share attraction model, assuming the firms (players) compete only on the advertising dimension and try to maximize their total sales. Suppose there are n areas, each of which generates an amount of sales, s_i, that can be influenced by advertising. By assuming that the sales obtained by each player in each area is proportional to the share of his advertising expenditure of the total advertising expenditure in this area, the objective functions were defined as

$$P_1 = \sum_{i=1}^{n} [x_i/(x_i + y_i)]s_i \text{ and } P_2 = \sum_{i=1}^{n} [y_i/(x_i + y_i)]s_i,$$

where $x_i(y_i)$ denotes Player 1's (2's) advertising expenditure in Area i. Then the Nash equilibrium strategies were found and the implications of such strategies were discussed. Following this work, many other models have been built by employing the market share attraction model in a similar way that each player has only one control variable (Mills, 1961; Shakun, 1965, 1966; Baligh and Richartz, 1967a; Balch, 1971; and Schmalensee, 1976). Baligh and Richartz (1967a) discussed the television programming problem in which n players, each creating a program to be shown at a certain time of a day using a constrained or unconstrained budget, compete in attracting viewers. The problem was formulated as an n-person nonzero-sum game and analyzed both cooperatively and non-cooperatively. Especially, some light was shed on the problem of optimum budget. On the other hand, Shakun discussed the problem in coupled markets where each of two players (Shakun, 1965) or many players (Shakun, 1966) sells a different class of products whose potential customers can be influenced by the other players' advertising. In particular, Karnani (1985) used the game theory framework to examine the practical implications of market share attraction models recently and showed that they are consistent with previous empirical research in marketing and business policy. Generally, these models are of the normal form and Nash equilibrium strategies are used as optimal strategies for the players. Furthermore as each player has only one control variable they are comparatively easy to handle. However they over-simplify the reality where often many factors rather than only one are present.

Another group of game theory models in the present marketing literature consider the marketing channel problem. As a matter of fact, the coordination function of market distribution channels has been attracting increasing attention in recent years. The interested reader might refer to Eliashberg and Steinberg (1987) and McGuire and Staelin (1983a, 1983b) for some references of such works. For game theory applications, Baligh and Richartz are among the first to study market channel problems in game theoretic approach (Baligh and Richartz, 1967b; Richartz, 1970). They analyzed vertical market systems with several levels and investigated the optimal strategies of the players (firms) regarding the number of levels to be used. However, their analysis was built on a general basis. Later Zusman and Etgar (1981) discussed market channels dominated by contractual exchange by using Nash bargaining theory and economic contract theory. McGuire and Staelin (1983a, 1983b) addressed the problem from a point of view of investigating the conditions desirable for a manufacturer to use intermediaries between itself and the ultimate consumers of his product. By assuming a market structure where each of two manufacturers sells his product through a single retailer, who in return orders exclusively from the manufacturer, they built three game theory models regarding whether both, one or none of the two pairs of manufacturer and retailer is integrated. In other words, the situation was described by a two-person game when both groups are integrated; a three-person game when one group is integrated; and a four-person game when none of them is integrated. Unlike most works done on this topic, the manufacturers' decision variables were defined as their wholesale prices rather than quantities and the retailers' decision variables as their retailing prices. Under certain assumptions, for instance, each retailer's demand is linear in the retailing prices and each manufacturer has a constant variable production and selling cost, they formulated each player's profit (objective) function. Then the Nash equilibria were found and the results were compared to see which structure and under what conditions provides the best results. They found that the substitutability of the two products is crucial in the determination of the preferred structure and even if manufacturers can do the selling tasks as well as intermediaries they have a reason to place a "middleman" between them and the market. Thereafter their works were extended by Coughlan

(1985) by generalizing the original linear demand assumption into several linear demand cases as well as into cases of demand concavity or convexity with respect to own price. Especially some empirical evidence on the validity of their models was obtained in the semi-conductor industry. Some differential game theory models have been also built recently to discuss this problem (Feichtinger and Hartl, 1985; Jorgensen, 1986; Eliashberg and Steinberg, 1987).

It is also worth mentioning that a few stochastic game theory models have been built recently studying marketing, especially advertising, problems. A stochastic game is a discrete time Markov process in which conflicts of interests exist among the participants (players), each of whom has a partial influence over the process through his control variable. It is well suited to the studying of decision process over several periods. However it is usually not easy to find out the equilibrium point for a stochastic game. One way used to resolve this problem is to analyze subclasses of such models (Dirven and Verize, 1986; Sobel, 1981).

Based on our investigation, there seems to be a tendency that recent research in marketing area in game theoretical approach is becoming more involving and more dependent on time effect. Early works usually deal with cases of two players, each of whom has one decision variable, in a single or a few periods. However, more of recent models are dynamic and involve more players and with more factors under consideration.

1.3.4. Queueing

Queueing theory is one of the traditional areas of management science. However there have not been many game theory models built in this area yet. Holt and Sherman studied a particular type of queueing problem in a series of papers (Holt and Sherman, 1980; 1982; 1984), from a point of view that a queue is like an auction. They considered the situation where there are n people competing in obtaining m ($m < n$) prizes that are awarded at a certain time to the first m people in line. Each participant, aiming at winning the prize, makes a decision of arrival time, with an assessment of other participants' arrival times. This is like a sealed bid at an auction. Each participant i's payoff function was defined as

Winner: $V(w_i) - t_i w_i - kw_i$,

Loser: $- kw_i$,

where $V(w_i)$ is Player i's valuation function of the prize, which depends only on his opportunity cost of time w_i, t_i is the time he has to wait in line before awarding the prize and k is a constant amount of time that everyone has to spend to reach the place. It was also assumed that one can know whether he can win the prize or not when he arrives and, if not, he does not have to wait. Clearly, one can win the prize only if there are fewer than m people in line when he arrives. Holt and Sherman investigated the optimal strategies, $t_i = \sigma(w_i)$, for the participants and discussed the factors affecting transaction costs associated with waiting in line.

Clearly, the situation discussed by Holt and Sherman represents only a peculiar case of queueing problems. In fact, as noted by Holt and Sherman (1984), a typical queue has individuals joining and leaving all day along and is not like an auction at all. Therefore their works, though coping with conflict among participants, are not widely applicable. Nevertheless, they did disclose an important fact, which seems to be overlooked in the present context of queueing, that entities in queue as customers may select their own arrival times and it is always desirable for them to do so.

1.3.5. Finance

Another active area of game theory application is the area of finance. Milnor and Shapley (1961) studied a corporation with two major stockholders and many minor stockholders using oceanic game theory models. It was assumed that the major stockholders compete only in obtaining the control of the corporation, which hinges entirely on a simple majority vote of the stock, and the minor stockholders, collectively referred to as a "ocean", do not have significant voting power individually. The problem was formulated as a simple oceanic game:

$$[\ \tfrac{1}{2};\ w_1,\ w_2;\ \alpha],$$

where w_i denotes Player i's power ($i = 1, 2$), $\alpha = 1 - w_1 - w_2$, and 1/2 is the majority quota. They investigated the distribution as well as migration of shares and, thus, of voting power among the stockholders. The major results were demonstrated graphically. Recently Powers

(1987) extended their model to include exchange between cash and shares and studied the feasibility and profitability of corporate takeovers. They also showed the existence of the Nash equilibrium strategies that result in one major stockholder's taking control of the company under certain conditions.

Aivazian and Callen (1980) discussed the leverage problem that is faced by firms facing growth opportunities, recognizing that shareholders and bondholders in such situations are actually involved in a cooperative game. Consider an all-equity firm in a frictionless financial market, its value at $t = 0$ is

$$V_A = \int_{S_a}^{\infty} q(S)[V(S) - I]dS,$$

where $q(S)$ is the firm's equilibrium price at $t = 0$ of a dollar delivered at $t = 1$ under state S, $V(S)$ is the value of the investment I contingent on S, and S_a is the breakeven point corresponding to $V(S) = I$. The firm will invest if and only if $V(S) \geq I$. In the case of leverage, the firm will invest only if $V(S) \geq I+P$ where P is the additional amount the firm promises to pay to the bondholders after state S is revealed. The firm's value, in this case, is

$$V_L = \int_{S_b}^{\infty} q(S)[V(S) - I]dS,$$

where S_b is the breakeven point corresponding to $V(S) = I+P$. Then the returns to shareholders and bondholders under state S, using the notations in the original paper, are

$$X_E(S) = 0 \quad \text{and} \quad X_B(S) = 0 \quad \text{when } S < S_a;$$

$$X_E(S) = V(S) - I \quad \text{and} \quad X_B(S) = P \quad \text{when } S > S_b;$$

$$X_E(S) + X_B(S) = V(S) - I \qquad \text{when } S_a < S < S_b.$$

This is a game in the characteristic form. Aivazian and Callen showed, using this model and Shapley value, that the value of the firm is equal in both the leverage and all-equity cases. Then, firm's investment decisions are independent of its financial structure. However, they also showed by an example that this conclusion may not be true in cases that involve more than two parties. This is because the core of games with more than two players may not exist, which, in a financial institutional arrangement with no transaction cost, may bring about endless negotiation and a firm's financial and investment decisions may be dependent on each other in this case. The discussion was further extended to other financial instruments such as bankruptcy, mergers, etc., using the game-theoretic insights.

Aside from the game theory models in the areas surveyed above, there are also many applications of game theory in other areas. For example, Tijs and Driessen (1986) provided an excellent review of game theory applications in cost allocation with about 50 papers; Orgler and Tauman (1986) built a model for cash management; Mamer and McCardle (1987) discussed the adoption of new technology; and McKelvey (1981) addressed the problem of agenda design. We do not discuss these works in detail here. However some references on these topics are provided in the reference.

1.4. Concluding Remarks and Discussions

It is not surprising, with the plethora of game theory models surveyed in this paper, we can conclude now that game theory has been widely applied in management science. This is particularly true when we consider the fact that the survey is restricted to static game theory models and only some areas of management science. As we mentioned at the beginning of the survey, game theory models conflict and cooperation, which the classical single decision maker models cannot represent adequately. It is then natural to expect game theory to attract much attention. Indeed it has become one of the major analytical methods of management science.

Nevertheless, we have also noted that some traditional management science areas such as inventory and queueing have not attracted the same amount of attention as others such as marketing and bidding. This might be due to not only the less explicit strategic nature of inventory and queueing decisions, but also the lack of interest of game theorists. This, of course, calls for more attention into these areas as well as the development of more realistic and then more complicated game theory models for such decision making problems.

As shown in the survey, most of the game theory models have been built in the normal form, with a few in the characteristic form and almost none in the extensive form. This is primarily because, on the one hand, the normal form is the most convenient form to formulate and analyze and can also be used to analyze both non-cooperative and cooperative decisions; on the other hand, authors often used two-person game theory models, which are presented basically in the normal form, as their basic formulation to avoid complicated discussion and left the n-person case as a further research direction, which, however, has been rarely explored.

Despite the extensive discussion on and wide application of game theory, none of its solution schemes has been accepted to be absolutely applicable. For non-cooperative games, the Nash equilibrium is usually used as solution to various problems. However, as shown by Palfrey (1980) in the bidding situation, it may not provide practical solution to certain real problems, although it does provide good solutions in many other cases (Karnani, 1985; Coughlan, 1985). On the other hand, as we discussed in Section 1.2, Stackelberg strategy is not worse than Nash strategy when the follower's response to the leader's decision is unique, but it seems to be ignored in many cases. For cooperative games, the core is generally accepted both theoretically and empirically. But it is more a pre-solution than a solution scheme per se. Various other solution schemes, especially Shapley value and nucleolus, have been usually taken as cooperative equilibria, which unfortunately have been tested negatively by some empirical tests (Williams, 1988). Therefore care should be taken in selecting game theory solution schemes for real problems.

Clearly, game theory is widely applied but is far from perfect. There have been many studies on game theory in the past, attempting to avoid its weaknesses. However none of these works seems to have been widely accepted. To gain a deeper understanding of game theory we also provide a brief survey of some other approaches of game theory.

Game theory is normative rather than predictive. Some previous experimental evidence showed that minmax players (in two- or three-player cases) are only the exception (Rapoport and Orwant, 1962; Lieberman, 1962). Therefore the question concerning about the validity and applicability of game theory has been long raised.

(1) The players' risk attitude. Eliashberg and Winkler (1978) investigated what will happen if the assumption of linear utility function made by von Neumann and Morgenstern (1944) does not hold. By assuming exponential utility functions, they showed that players' risk attitudes do play a role in the determination of the solution to a game and players' decisions usually do not coincide with that under the original assumption. Unfortunately, this work has not yet attracted much attention.

(2) The hypergame approach. The theory of hypergames was developed in later 1970's (Bennett, 1977) and seems to have been gaining increasing attention in both its development per se and its applications (Fraser and Hipel, 1984). It objects to the regular game theory by arguing that the players usually do not have a common perception of the problem in hand. Instead, they are actually trying to play different games for the same problem in hand (Bennett, 1980; Bryant, 1983). Clearly this argument is often correct. But in this case the formulation of a game theory model would be very strenuous.

(3) The Bayesian approach. As stated by Kadane and Larkey (1982), "We do not understand the search for solution concepts that do not depend on the beliefs of each player about the other's likely actions...". The Bayesian approach is one way to resolve this problem (Harsanyi, 1967, 1968a, 1968b, 1975; Kadane and Larkey, 1982), which makes solution concepts to games depend on individuals' subjective assessment over the strategies the other players are going to use. Recently there were discussions on this issue by Kadane and Larkey (1982), Kahan (1983), Rothkopf (1983), etc.

(4) The metagame approach. The idea of metagames was originated by von Neumann and Morgenstern (1944) and developed formally by Howard (1966a; 1966b). The basic argument here is that the assumption of rationality is inconsistent in game theory and some other approaches should be developed to resolve these problems. The theory of metagames is one of these theories that attempt to do so. It is non-quantitative and built on the metagame tree structure. As tested by Howard (1966a), it is behaviorally accurate. However it is often very strenuous to analyze a metagame. Therefore its application is yet very restricted. For more discussion, the reader might refer to Howard (1966a; 1966b; 1971) and Fraser and Hipel (1984).

Clearly all the above approaches improve the regular game theory in some aspects. However the improvement is often offset by the increased difficulty in some other aspects such as formulation and analysis. Because of this reason, their applications are very limited in certain fields. We do not review these works here but some references are provided.

In the rest of this dissertation, we will investigate some management science problems using game theory.

Chapter Two

A Game Theoretical Analysis of the Discount Problem Under Constant Buyer's Demand

In this and the following chapter, we investigate the discount problem. We treat the seller and the buyer as different players in a competitive situation and analyze their discounting decisions in a game theory framework. We consider the seller's optimal discount schedule when discount does not affect the buyer's demand in this chapter and prove that it is always possible for the seller and the buyer to benefit from discounting. While a simple price discount will always make the seller lose, quantity discount can make both the seller and the buyer gain and significantly improve their positions if carefully selected. By using a general all-unit quantity discount schedule, we give both the seller's and the buyer's optimal decisions. It is shown numerically that our model gives significantly better results than that of some models in the literature. Quantity discount provides an efficient and easy to be implemented solution to the discount problem.

2.1. Introduction

Traditional discount models analyze primarily the buyer's best reactions to various price and quantity discount schedules provided by the seller. They minimize the buyer's total inventory related cost, assuming that the seller offers a discount and then accepts the orders, usually of larger sizes, that the buyer places (Sethi, 1984; Hadley and Whitin, 1963; Peterson and Silver, 1979).

However, many authors feel that traditional discount models are biased on the side of the buyer. They point out that the seller uses the discount structure to lure the buyer to order larger sizes, hence to maximize his profit. Decision models have been developed solely from the perspective of the seller by Monahan (1984), Rosenblatt and Lee (1985), Lee and Rosenblatt (1986), Dada and Srikanth (1987), and others, assuming implicitly that the buyer will cooperate as long as the seller's decision will not make him worse-off. A more generalized solution procedure to the model developed by Lee and Rosenblatt (1986) has also been provided by Goyal (1987).

Unfortunately, there is still one important property overlooked in these two types of models. Consider a situation where a buyer orders periodically from a seller, aside from the problem of ordering size, another critical issue here is that of pricing. The ordering size issue is controlled primarily by the buyer, whereas the pricing issue is basically determined by the seller. A settlement occurs only if one agrees with the other's decision. Traditionally these issues are usually settled through negotiations between the two parties. Models developed from exclusively one party's point of view suppress the conflict of the seller and the buyer and fail to incorporate the negotiation nature of the problem.

Game theoretic method seems to be a better way to deal with this situation because it treats the seller and the buyer as parties in a competitive situation and models their conflict and cooperation. We present an economic analysis of the discount problem in a game theoretical approach in this dissertation, treating each of the seller and the buyer as one player in a two-person game. In this chapter, we study the case where discount does not affect the buyer's demand, while in the following chapter we analyze the discounting decisions of the seller and the buyer when discount does affect the buyer's demand.

We have shown in Chapter One that game theory has become an important analytical method in management science. But few game theory models have been developed to deal with inventory control problems. One of these models was built by Parlar (1988) who considered the problem of two retailers who sell substitutable products and attempt to determine their optimal order quantities when demand is stochastic. Our work here represents another inventory control problem that should be considered in the game theory framework.

This chapter is organized as follows. In Section 2.2, we formulate the discount problem as a two-person nonzero-sum game and characterize the feasible solution area for it. In Section 2.3, we solve the problem with the model built in Section 2.2, considering both price discount and quantity discount. The main findings and possible extensions to this research are summarized in Section 2.4.

2.2. The Model

In this section we formulate our model with the following assumptions.

(1) The seller's unit selling price is the market price and independent of the buyer's ordering size and the buyer uses the EOQ policy to determine his ordering size initially.

(2) There is only one single seller, one single buyer, and one single product under consideration.

(3) The buyer's demand is deterministic and constant, lead times are known with certainty, and no backlogging and lost-sales are allowed.

(4) The seller and the buyer are motivated only by profit maximization or cost savings.

(5) The seller buys the product from yet another external supplier and the lead time is known with certainty.

(6) The players are rational and use only pure strategies.

We define the following parameters and variables:

A_b - the buyer's fixed ordering cost per order;

A_s - the seller's fixed cost of processing one order placed by the buyer;

A_e - the seller's fixed ordering cost per order when ordering from the external supplier;

D - the buyer's deterministic annual demand rate for the product;

P - the seller's unit selling price or the buyer's unit buying cost in the absence of discount;

C - the seller's unit acquisition cost from the external supplier;

H_b - the buyer's inventory carrying cost per dollar per year;

H_s - the seller's inventory carrying cost per dollar per year;

Q - the ordering size of the buyer under the competitive unit selling price P, $Q = \sqrt{(2DA_b/PH_b)}$;

x - the factor by which the buyer will increase his ordering size, i.e., he will order (1+x)Q units each time;

y - the factor by which the seller will decrease his unit selling price, i.e., he will offer a unit selling price of (1-y)P.

A discount scheme is defined as an agreement between the seller and the buyer in which the seller agrees to offer a lower unit selling price and the buyer agrees to order larger sizes. As the seller offers the market unit selling price P and the buyer orders Q units each time initially, a discount scheme is then a pair of ordering size (1+x)Q and unit selling price (1-y)P, which is determined entirely by (x, y) and can be simply referred to as (x, y) in the following discussion. x and y are such that

$$0 \le x, \tag{1}$$

$$0 \le y \le y_0, \tag{2}$$

where $y_0 > 0$ is a point that the seller will not let his unit selling price be less than $(1-y_0)P$. Since it is normally unrealistic for the seller to sell the product at a price lower than his acquisition price C, we let $y_0 = 1-C/P$. More generally, we might have $y_0 < 1-C/P$ as the seller may want a positive profit margin. Here, x and y are the buyer's and the seller's decision variables, respectively.

When a discount scheme is formulated, the following factors might be modified. (1) The buyer incurs extra holding costs associated with the extra quantity ordered each time. This factor is also alleviated by his decreased unit buying cost. (2) The seller reduces his inventory related cost as the buyer orders larger sizes. This point will be clarified later. (3) The seller reduces order processing cost and the buyer reduces ordering cost because the buyer orders larger sizes hence fewer number of orders are placed each year. (4) The seller decreases his sales revenue and the buyer reduces his buying cost because of the lowered unit selling price.

The buyer's concern in this problem is his total inventory related and buying cost of the product. We express the buyer's total annual relevant cost, TC, as

$$\begin{aligned} TC &= \text{(Purchase cost)} + \text{(Ordering cost)} + \text{(Carrying cost)} \\ &= DP + A_bD/Q + QPH_b/2. \end{aligned} \quad (3)$$

If a discount scheme is formulated, the new total cost, TC_b, is

$$TC_b = DP(1-y) + DA_b/[(1+x)Q] + (1+x)(1-y)QPH_b/2. \quad (4)$$

Therefore, what the buyer can gain under (x, y) is $\pi_b(x, y) = TC - TC_b$, or

$$\pi_b(x, y) = yDP + [1-1/(1+x)]DA_b/Q + [1-(1+x)(1-y)]Qh_b/2, \qquad (5)$$

where $h_b = PH_b$ is the buyer's inventory holding cost per unit per year in the absence of discount.

On the other hand, the seller's concern is also what he can gain from discounting. The relevant factors for him include the sales revenue, the acquisition cost, the order processing cost and the inventory related cost, which includes his inventory carrying cost and ordering cost from the external supplier. His annual profit, denoted by TP, can be expressed as

TP = (Sales revenue) - (Acquisition cost) - (Order processing cost) - (Inventory related cost).

The first three terms are, clearly, DP, DC, and A_sD/Q. However, to obtain the inventory related cost, we have to consider his optimal replenishment policy or his optimal ordering size from the external supplier.

Suppose the seller buys a quantity at a time enough to meet N orders of size Q. Because the first three terms in TP are independent of N, the optimal N can be determined by solely examining his inventory related cost. Consider the situation where the buyer's ordering size is given, say Q. The seller's demand pattern is determined in which he provides Q units to the buyer every Q/D time units (Figure 2.1). As his replenishment should only take place when the inventory level is zero (Peterson and Silver, 1979, p. 309), the optimal N is an integer and his inventory carrying cost per cycle, I_c, becomes

$$I_c = [(N-1)Q + (N-2)Q + \dots + Q](Q/D)h_s$$
$$= [N(N-1)Q/2](Q/D)h_s, \qquad (6)$$

where $h_s = CH_s$ is the seller's inventory holding cost per unit per year.

His average annual inventory carrying cost, I_y, can then be obtained by dividing I_c by the length of one cycle, NQ/D, which turns out to be

$$I_y = [(N-1)/2]Qh_s. \qquad (7)$$

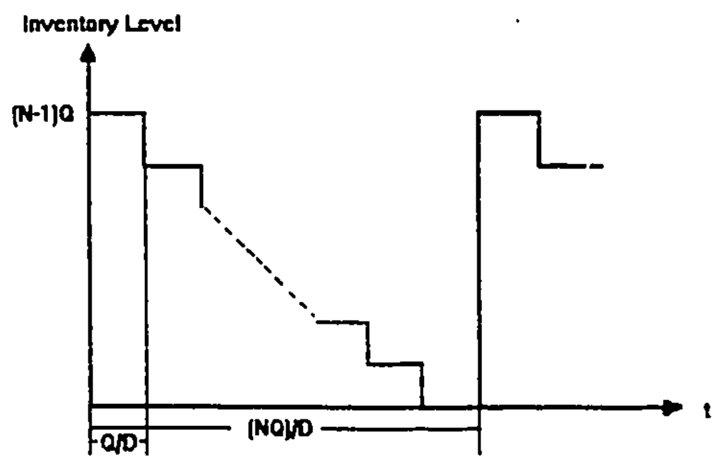

Figure 2.1. The Seller's Inventory Carrying Pattern

Since his average annual ordering cost is $(DA_e)/(NQ)$, his total annual inventory related cost, denoted by TI, can be expressed as

$$TI = [(N-1)/2]Qh_s + (DA_e)/(NQ). \qquad (8)$$

By differentiating TI with respect to N, assuming N to be a continuous variable, we obtain

$$dTI^2/dN^2 = (2DA_e)/(N^3Q) > 0. \quad (9)$$

TI is a convex function of N. The seller should order NQ units each time if TI(N) ≤ TI(N+1) and TI(N) < TI(N-1) which, after some modification, turn out to be

$$Q_0/\surd[N(N+1)] \le Q < Q_0/\surd[N(N-1)], \quad (10)$$

where $Q_0 = \surd(2A_eD/h_s)$ is the seller's EOQ when he faces a uniform demand of D units per year. Thus, the seller's optimal replenishing policy can be determined by comparing the buyer's EOQ with his own EOQ such that

$$N^* = 1 \text{ if } Q_0/\surd 2 \le Q,$$
$$= 2 \text{ if } Q_0/\surd 6 \le Q < Q_0/\surd 2,$$
$$= 3 \text{ if } Q_0/\surd 12 \le Q < Q_0/\surd 6, \ \ldots .$$

Since the seller has to fulfill the buyer's demand, we have $N \ge 1$. If $Q \ge Q_0/\surd 2$, the optimal N is 1 and the seller should order each time what the buyer orders and carry no inventory (lot-for-lot policy). If $Q < Q_0/\surd 2$, the lot-for-lot policy is not optimal. (10) should be used to determine the seller's optimal replenishing policy. In both cases, the seller's total inventory related cost is given by (8).

Let N_0 denote the seller's initial optimal replenishing policy. That is, the seller orders N_0Q units each time when the buyer orders Q units each time. The seller's annual profit TP in the absence of discount is

$$TP = DP - DC - A_sD/Q - [(N_0-1)/2]Qh_s - (DA_e)/(N_0Q). \tag{11}$$

When a discount scheme is formulated, the buyer's ordering size is (1+x)Q and the above analysis still holds. The seller's optimal ordering policy is determined by

$$Q_0/\sqrt{[N_x(N_x+1)]} \le (1+x)Q < Q_0/\sqrt{[N_x(N_x-1)]}. \tag{12}$$

N_x is the seller's optimal N for (1+x)Q and its value is determined by

$$N_x = N_0-i \qquad \text{when } x_i \le x < x_{i+1}, \tag{13}$$

where $i = 0, 1, \ldots, N_0-1$, $x_0 = 0$, $x_i = a/\sqrt{[(N_0-i)(N_0-i+1)]}-1$ for $0 < i < N_0$, $a = Q_0/Q$, and $x_{N_0} = \infty$.

In this case, the seller's total inventory related cost is

$$TI_x = [(N_x-1)/2](1+x)Qh_s + (DA_e)/[N_x(1+x)Q]. \tag{14}$$

It is important to note that, at each x_i for $0 < i < N_0$, $TI_x(N_0-i+1) = TI_x(N_0-i)$ and TI_x is continuous. By differentiating TI_x in (x_i, x_{i+1}), we have

$$dTI_x/dx = (N_0-i)(Qh_s/2)[1-(1+x_{i+1})^2/(1+x)^2] < 0. \tag{15}$$

TI_x decreases as x increases. Therefore, the seller reduces his inventory related cost when the buyer increases his ordering size.

The seller's annual profit is

$$TP_s = (1-y)DP - DC - A_sD/[Q(1+x)] - (N_x-1)(1+x)Qh_s/2$$
$$- DA_e/[N_x(1+x)Q]. \tag{16}$$

What the seller can gain under (x, y) is $\pi_s(x, y) = TP_s - TP$, or,

$$\pi_s(x, y) = -yDP + [1-1/(1+x)]A_sD/Q + (Qh_s/2)[(N_0-1)-(N_x-1)(1+x)] - (DA_e/Q)[1/(1+x)N_x-1/N_0]. \quad (17)$$

$\pi_s(x, y)$ is continuous in both x and y.

We have thus formulated the discount problem as follows. The buyer and the seller formulate a discount scheme (x, y), over which the buyer exerts his control through x and the seller exerts his control through y. By doing so the buyer gains $\pi_b(x, y)$ and the seller gains $\pi_s(x, y)$. We assume that the seller and the buyer know all the parameters in the two functions. This is a two-person nonzero-sum ($\pi_s + \pi_b \neq 0$) game.

Consider the impact of an increase in the buyer's order quantity on the seller. If the buyer's ordering size is not less than $Q_0/\sqrt{2}$ initially, he should use the lot-for-lot policy and any further increase in the buyer's ordering size will not alter this policy. His gain by inducing the buyer to order large quantities in this case comes from only the decrease in his order processing cost and ordering cost associated with the buyer's reduced frequency of ordering. If the buyer's ordering size is less than $Q_0/\sqrt{2}$ initially, he should not use the lot-for-lot policy and order in batches including fewer and fewer number of orders of the buyer when the buyer increases his ordering size. His gain by inducing the buyer to order large quantities comes from not only the decrease in his ordering and order processing cost but also the reduction in his inventory carrying cost.

It is worth noting that, if the seller is using the lot-for-lot policy initially, $N_0 = 1$ and $N_x = 1$ for any $x > 0$. (17) becomes

$$\pi_s(x, y) = - yDP + [1-1/(1+x)](A_e+A_s)D/Q. \quad (18)$$

This can be considered to be the case discussed by Monahan (1984). He assumed implicitly that the seller is always using the lot-for-lot policy and carries no inventory. Our discussion may be then helpful in clarifying two points. First, as also pointed out by others (Lee and Rosenblatt, 1986; Joglekar, 1988), Monahan's assumption of the seller's lot-for-lot policy is usually unrealistic. As shown above, the seller's initial optimal replenishing policy depends on his own EOQ and the buyer's EOQ. The seller, as a major intermediary, usually has better inventory facilities and thus his EOQ is often much larger than that of the buyer. Secondly, the buyer's inventory carrying cost formula has been used to obtain the seller's saving in inventory carrying cost (Lal and Staelin, 1984; Dada and Srikanth, 1987). This extension should be used only in situations where the seller always carries a large inventory and determines his replenishing policy independently of the buyer's order quantity. It can be seen in Dada and Srikanth (1987) that this translation requires an additional condition $H_b > H_s$ for the existence of an optimal solution. By including the seller's replenishment policy, this condition is unnecessary.

By the nature of the problem, no one will play the game if the resulting discount scheme gives him negative payoff. Hence $\pi_b(x, y) \geq 0$ and $\pi_s(x, y) \geq 0$. After some algebra, these conditions become

$$y \geq [(C_b/2)x^2/(1+x)]/[DP+(C_b/2)(1+x)], \quad (19)$$

$$y \leq [1-1/(1+x)]A_s/(PQ) + [Qh_s/(2DP)][(N_0-1)-(N_x-1)(1+x)] - [A_e/(PQ)][1/(1+x)N_x-1/N_0], \quad (20)$$

where $C_b = \sqrt{(2DA_bh_b)}$. Both (19) and (20) are continuous in x.

Therefore, a possible discount scheme is determined by (19) and (20) with $x \geq 0$ and $0 \leq y \leq y_0$. The set of such points is shown by the shadowed area in Figure 2.2.

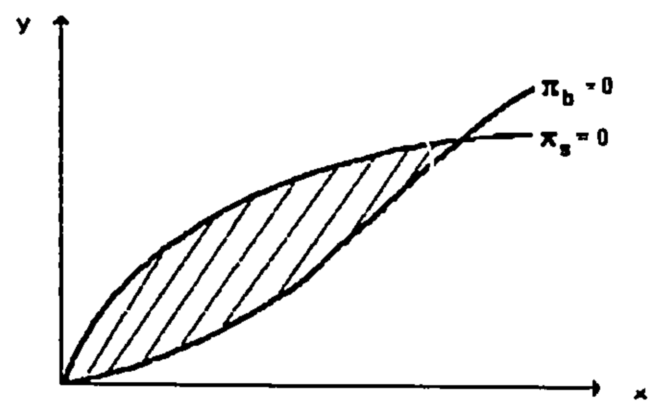

Figure 2.2. The Feasible Solution Area

Note that we assume that the condition $y \leq y_0$ is not restrictive in determining the shadowed area. As the shadowed area is bound by $\pi_s \geq 0$, the value of y_0 is not crucial in this case because the seller can increase it if this can make $\pi_s \geq 0$. We maintain this condition in our analysis. The shadowed area characterizes the feasible solution area for the problem. Any point on the boundary of it represents a discount scheme which makes one gain and the other neither lose nor gain, except the two end points at which no one gains or loses; any point inside it represents a discount scheme which makes both players gain; and any point outside of it makes at least one lose.

Proposition 1. It is always possible for both the seller and the buyer to benefit from discounting.

Proof. We consider only $H_s > 0$ and $A_e > 0$. What we need to prove is that the feasible solution area contains at least one point at which $\pi_b > 0$ and $\pi_s > 0$.

We define a function Y(x) by subtracting (19) from (20). Y(x) is continuous in x as both (19) and (20) are continuous in x and the feasible solution area contains at least one point at which both the seller and the buyer gain if Y(x) > 0 for some x > 0.

By differentiating Y(x) with respect to x at x = 0, we obtain

$$dY/dx|_{x=0} = A_s/(PQ) + (1/DP)[DA_e/(N_0Q)-(N_0-1)Qh_s/2]$$
$$= A_s/(PQ) + [Q_0^2-N_0(N_0-1)Q^2]h_s/(2DPN_0Q)$$
$$> 0 \text{ since } A_s/(PQ) \geq 0 \text{ and } Q < Q_0/\sqrt{[N_0(N_0-1)]}. \qquad (21)$$

Y(x) is strictly increasing at x = 0. Since Y(0) = 0, there is an $x_0 > 0$ such that Y(x) > 0 in $(0, x_0)$. Q.E.D.

Thus, for any product that the seller and the buyer are exchanging, it is always possible and worthwhile for both the seller and the buyer to exploit the benefit of discounting. In the following we will discuss how to formulate a discount scheme for the seller and the buyer.

2.3. Solutions to the Problem

We consider independent suppliers and buyers. In this case, each one in a supplier-buyer relationship pursues his own interest, which is partly conflicting with the other's. It is then, we feel, usually inappropriate to assume, as done by some writers (Banerjee, 1986a), that the seller and the buyer will work in full cooperation in the discount problem.

A careful examination of the discounting process reveals that discount schedules are usually given by suppliers. Buyers react to discount schedules by ordering each time the quantity which minimizes their total relevant cost. Then the seller usually acts as the leader in the process. We focus on this case in our analysis.

2.3.1. The Players' Reaction Curves

If one player's decision is announced to the other in advance, the other will surely respond in such a way so as to maximize his own payoff. For our problem, if the seller's decision y is given, the buyer will choose an x to maximize $\pi_b(x, y)$. When y goes through its domain $[0, y_0]$, the buyer's decision will form a curve over $0 \le y \le y_0$. This curve is called the buyer's reaction curve, which gives the buyer's best responses to the seller's possible decisions (Basar and Olsder, 1982). Similarly, we can define the seller's reaction curve as the seller's best responses to the buyer's possible decisions.

For a given y, we can observe that

$$d\pi_b/dx = [1/(1+x)^2]A_bD/Q - Qh_b(1-y)/2, \quad (22)$$

$$d^2\pi_b/dx^2 = -[2/(1+x)^3]A_bD/Q < 0, \quad (23)$$

hence π_b is a concave function in x, $x \geq 0$. The optimal value of x, for a given y, is obtained by setting (22) equal to zero, which gives

$$x = [1/\surd(1-y)] - 1. \quad (24)$$

We note that (24) is obtained by using $Q = \surd(2DA_b/h_b)$. The reader might verify that this is the same result as that obtained by using the EOQ model when y is given.

On the other hand, for a given x, $d\pi_s/dy = -DP < 0$. π_s is strictly decreasing in y and attains its maximum at $y = 0$ for any x. The reaction curve of the seller is $y = 0$. He will never lower his selling price if the buyer announces his decision in advance. Indeed the buyer does not have the potential to act as the leader in the discounting process.

2.3.2. Price Discount

We classify discount scenarios into two categories. A price discount is one in which the seller gives a discount or simply a lower unit selling price to the buyer, without setting a minimum for an order to be eligible for the discount. In a quantity discount, an explicit restriction is imposed on the order quantity for each discount. Either of the these two discounts is widely observed in actual business practice. We study price discount in this section and quantity discount in the next section.

Proposition 2. If the buyer's demand is constant, the seller will lose if he gives a price discount to the buyer.

Proof. When the seller gives a price discount to the seller, he acts as the leader in the discounting process. The buyer will follow his decision by ordering each time his economic order quantity. His reaction to any y given by the seller is uniquely determined by (24).

By substituting (24) into (17), we get the seller's payoff function as

$$\pi_s = -yDP + [1-\surd(1-y)]A_sD/Q + (Qh_s/2)[(N_0-1)-(N_x-1)/\surd(1-y)]$$
$$-(DA_e/Q)[\surd(1-y)/N_x-1/N_0], \qquad (25)$$

where N_x is given by

$$N_x(N_x-1)/a^2 < (1-y) \leq N_x(N_x+1)/a^2. \qquad (26)$$

Since $N_x = N_0$ when $y = 0$, from (26) we get

$$N_x = N_0-i \text{ when } y_i \leq y < y_{i+1}, \qquad (27)$$

where $i = 0, 1, \ldots, N_0-1$, $y_0 = 0$, $y_{N_0} = y_0$ and $y_i = 1-(N_0-i+1)(N_0-i)/a^2$ for $0 < i < N_0$.

In each interval (y_i, y_{i+1}), N_x is constant and π_s is continuous and differentiable. We can also observe that N_0-i and N_0-i-1 give the same total inventory related cost to the seller when $y = y_i$. Thus π_s is continuous at y_i for $0 < i < N_0$. π_s is continuous for $y \geq 0$.

Differentiating π_s with respect to y in each interval, we get

$$d\pi_s/dy = -DP + (A_sD/Q)(1-y)^{-1/2}/2 - (Qh_s/2)(N_x-1)(1-y)^{-3/2}/2$$
$$+ (DA_e/Q)(1-y)^{-1/2}/(2N_x)$$

$$\le - DP + (A_sD/Q)(1-y)^{-1/2}/2 + (DA_e/Q)(1-y)^{-1/2}/(2N_x)$$
$$\le - \{(1-y)DP - A_sD/[(1+x)Q]/2 - DA_e/[2N_x(1+x)Q)]\}/(1-y)$$
$$\le - \{(1-y)DP - A_sD/[(1+x)Q] - DA_e/[N_x(1+x)Q)]\}/(1-y)$$
$$\le - (TP_s+DC)/(1-y). \quad (28)$$

We use (24) or $1+x = 1/\surd(1-y)$ to obtain (28).

It is very reasonable to assume that the seller gets a positive profit for trading the product initially, i.e., $TP > 0$. Then, for any $y > 0$ and $TP_s \le 0$, $\pi_s = TP_s - TP < 0$. On the other hand, for any $y > 0$ and $TP_s > 0$, $d\pi_s/dy < 0$ and π_s is strictly decreasing. Since π_s is continuous for $y \ge 0$ and $\pi_s(0) = 0$, $\pi_s < 0$.

The seller will lose if he gives a price discount to the buyer. Q.E.D.

Proposition 2 suggests that suppliers should not provide any price discount to their customers when the customers' demand is constant. If a seller gives a price discount to a buyer in this case, the buyer will increase his ordering size according to his EOQ formula. But this increase is always not sufficient to cover the seller's reduction in sales revenue. This finding seems to be contradictory to the reality where, as we observe, suppliers often give price discount to their customers. An explanation to this observation is that price discount is a competitive strategy for suppliers to attract more demand from their customers or to accelerate the selling process of a product rather than a tool for channel efficiency. In the next chapter, we will study this problem in the situation where the seller's discount will increase the buyer's demand. Our conclusions are consistent with these observations.

2.3.3. Quantity Discount

A quantity discount schedule is characterized by a direct association between the discount and the order quantity. Then different restrictions on the discount and order quantity will result in different quantity discount schedules. Indeed, different quantity discount schedules are used by traders in the industries. A few of such schedules as all-unit quantity discount, incremental quantity discount, and carload lot discount, are discussed in Jucker and Rosenblatt (1985). They showed that, for these schedules, an all-unit quantity discount schedule is general enough to admit others as special cases. Thus we consider only all unit quantity discount schedules in the following analysis.

An all-unit quantity discount schedule can be defined as follows. The seller offers a unit selling price P_i for any order between Q_i and Q_{i+1} where $i = 1, 2, \ldots, n$, $P_1 > P_2 > \ldots > P_n$, $Q_1 < Q_2 < \ldots < Q_n$ and Q_{n+1} is infinity. All units in an order are eligible for the appropriate discount.

When a single buyer is considered, only one order quantity will be selected. Thus a quantity discount with only one break is adequate for this situation. In the framework under discussion, an all-unit quantity discount schedule with only one break can be defined in terms of x and y as follows. The seller offers no discount for any order such that $0 \le x < x_1$ and a discount y_1 for any order such that $x_1 \le x$, where $x_1 > 0$ and $y_1 > 0$. A quantity discount schedule can be represented by the break point or (x_1, y_1). Obviously the seller is to find the optimal discount y_1 as well as the optimal break point x_1.

Because the seller gains nothing if the buyer stays at $x = 0$ or the initial order quantity, he has to provide some incentive for the buyer to move away from $x = 0$. Let $w \ge 0$ be the least amount that the buyer will be interested in quantity discount. Then under a quantity discount given by the seller, the buyer should get at least w or $\pi_b \ge w$.

Now consider the buyer's reaction to a quantity discount schedule defined above given by the seller. The buyer's optimal order quantity is Q^* if $Q^* \ge (1+x_1)Q$ or $(1+x_1)Q$ if $Q^* < (1+x_1)Q$, where $Q^* = Q/\sqrt{(1-y_1)}$. In both cases we should have $\pi_b \ge w$ or the buyer's optimal order quantity is Q.

Lemma 1. If the seller gives a quantity discount to the buyer, the break point has to satisfy the condition

$$(1+x_1)\sqrt{(1-y_1)} > 1. \tag{29}$$

Proof. $(1+x_1)\sqrt{(1-y_1)} > 1$ is equivalent to $(1+x_1)Q > Q/\sqrt{(1-y_1)} = Q_1^*$.

Suppose $(1+x_1)\sqrt{(1-y_1)} \le 1$. We have that $Q_1^* \ge (1+x_1)Q$ or the break point is below the buyer's EOQ when a discount y_1 is provided. The buyer will order according to his EOQ formula or (24). As shown in Proposition 2, the seller will lose if $y_1 > 0$ in this case. Hence, if the seller is going to offer a quantity discount to the buyer, it must be such that $Q^* < (1+x_1)Q$ or $(1+x_1)\sqrt{(1-y_1)} > 1$. Q.E.D.

Lemma 1 suggests that if the seller is going to give a quantity discount to the buyer, he should give his quantity discount schedule in such a way that the buyer has to order more than his EOQ.

Lemma 2. It is always possible for the seller and the buyer to gain from quantity discount.

Proof. From Proposition 1, we have at least one point inside the feasible solution area at which $\pi_b > 0$ and $\pi_s > 0$. From the proof of Proposition 2, we can see that the condition for Lemma 1 is satisfied for any point at which $\pi_s > 0$. Then, if we assign a point inside the feasible solution area as the break point for a quantity discount schedule, the buyer's optimal order quantity is the break point. With this schedule, both the seller and the buyer can gain. Q.E.D.

We now prove the following proposition.

Proposition 3. The seller's optimal quantity discount schedule (x_1, y_1) is given by

$$y_1 = [w+(C_b/2)x_1^2/(1+x_1)]/[DP+(C_b/2)(1+x_1)], \tag{30}$$

where x_1 maximizes π_s.

Proof. The seller is to find the optimal break point (x_1, y_1) which maximizes π_s. By Lemma 1 and Lemma 2, such a point exists for an appropriate w. As the buyer has to gain w to order more than Q, we have $\pi_b \geq w$. This condition has to be tight in maximizing π_s because, as we can see from (17) and (5), the seller will set $x = \infty$ and make the buyer lose otherwise.

Solving $\pi_b = w$ for y, we obtain (30). Q.E.D.

Corollary 1. The buyer will get w, the least amount that keeps him interested in ordering more than Q under the quantity discount schedule.

Proof. This is obvious from Proposition 3. **Q.E.D.**

The seller has the full control of the situation as he maximizes his own payoff by letting the buyer gain w. When $w = 0$, the buyer gains nothing and the seller obtains all the benefit from discounting. This specific situation is the case discussed by many writers (Monahan, 1984; Lee and Rosenblatt, 1986). We are cautious about it because the buyer may always need some positive incentive to respond positively to a quantity discount schedule. Generally a quantity discount schedule with a positive w seems to be more appropriate.

Since $\pi_b \geq w$ is always tight when maximizing π_s, decreasing w usually means increasing the seller's gain. Then the seller gets his maximum gain when $w = 0$ and will set w as small as possible. However, as the buyer may not change his ordering policy without an attractive incentive, in which case the seller gains nothing, the seller should use a carefully selected $w > 0$ in a quantity discount schedule.

From Proposition 3, we obtain a solution procedure for (x_1, y_1) as follows.

(a) Substitute y_1 given by (30) in π_s, which then gives the seller's payoff function as a function of x_1 alone.

(b) Find x_1 which maximizes π_s given in (a).

(c) Substitute x_1 into (30) to obtain y_1.

As N_x is a discrete function of x, finding x_1 by maximizing π_s given in (a) analytically is usually cumbersome. To avoid this difficulty, we can also use the following non-linear programming model to obtain the optimal solution.

$$\max_{x,y,N} \; -yDP + [1-1/(1+x)]A_sD/Q + (Qh_s/2)[(N_0-1)-(N-1)(1+x)] - (DA_e/Q)[1/(1+x)N-1/N_0],$$

$$\text{S.T.: } yDP + [1-1/(1+x)]DA_b/Q + [1-(1+x)(1-y)]Qh_b/2 - w \geq 0,$$

$$x < a/\sqrt{[N(N-1)]},$$

$$x \geq a/\sqrt{[N(N+1)]},$$

$$N \geq 1 \text{ and integer},$$

$$y > 0 \text{ and } y \leq y_0. \qquad (31)$$

Using a non-linear programming package such as GINO (Liebman, et al, 1984), this problem can be easily solved.

2.3.4. Numerical Examples and Discussions

The following numerical examples will demonstrate our results and illustrate some of the difference between our model and that of Monahan (1984) or Lee and Rosenblatt (1986). The latter is a generalized model of the former by relaxing the lot-for-lot assumption. We will also provide a brief discussion on the model of Rosenblatt and Lee (1985) as they considered explicitly the buyer's reaction in determining the seller's optimal quantity discount schedule. They used a linear quantity discount schedule in their analysis.

Example One. $D = 14400$, $A_b = \$200$, $A_s = \$100$, $A_e = \$400$, $P = \$10$, $H_b = 0.4$, $h_s = \$2$.

If no discount is considered, we have $Q = 1200$, $Q_0 = 2400$, $a = Q_0/Q = 2$ and $N_0 = 2$.

If a quantity discount is to be given and the seller is willing to let the buyer gain $w = \$200$, we obtain the seller's optimal quantity discount schedule as: no discount for any order between 1200 units and 2279 units and a 0.82% discount for any order greater than or equal to 2280 units. Under this schedule, the buyer will order 2280 units each time and the seller will use the lot-for-lot policy. The buyer gains \$200 and the seller gains \$456.49. □

In order to compare our model with Monahan's or Lee and Rosenblatt's model, we use the data given in Example 1 of Rosenblatt and Lee's paper in the following example.

Example Two. $D = 100$, $P = \$10$, $A_s = \$0$, $A_b = \$1200$, $A_e = \$1200$, $H_b = 0.5$, $h_s = \$2.5$.

When no discount is considered, $Q = 219$, $Q_0 = 310$, $a = \sqrt{2}$ and $N_0 = 1$. Then the lot-for-lot policy is optimal for any ordering size larger than Q.

When a quantity discount is considered, by using Monahan's model or Lee and Rosenblatt's model (they are the same in this particular case), the improved ordering size is 310 units and the discount term is \$0.6645. Under this arrangement, the seller gains \$93.97 and the buyer gains \$50.85.

If our model is used, letting w = \$50.85 we obtain the seller's optimal quantity discount schedule as: no discount for any order between 219 units and 409 units and a 13.45% discount for any order larger than or equal to 410 units. The buyer will order 410 units each time, gaining \$50.85, and the seller gains \$120.18. On the other hand, if the seller is satisfied with the gain by using Monahan's model, he can provide the following quantity discount schedule to the buyer: no discount for any order between 219 units and 419 units and a 16.75% discount for any order larger than or equal to 420 units. Under this schedule, the seller gains \$93.97 but the buyer gains \$104.21.

Evidently, our model gives significantly better results than theirs in the example. This is generally true because, by using the game theoretical approach, our model adopts a more general framework to form a discount scheme for the seller and the buyer. In their model they assume that a "break even" discount is given to the buyer. By doing so, they restrict the feasible solution area shown in Figure 2.2 onto a curve determined by their Equation (2). But our model imposes no restrictions on the discount and order quantity. Note that the buyer's gain w can be changed according to their agreement, our model provides a more efficient, more flexible and thus perhaps more equitable way for the seller and the buyer to exploit the profit from discounting.

Now let us consider Rosenblatt and Lee's model. They give a linear quantity discount schedule for the seller as p = 10 - 0.0059x, where p is the unit selling price and x is the ordering size. If this schedule is given to the buyer, by substituting it into the buyer's total cost function, we obtain

$$C(x) = 120000/x + 0.25(10-0.0059x)x + 100(10-0.0059x). \qquad (32)$$

It can be verified that C(x) is monotonically decreasing in x if x > 0. Therefore, the buyer's optimal order quantity under this schedule is not 438 units, as given in their paper, but infinity. This schedule is clearly not optimal for the seller if such a schedule exists at all. It can be shown that similar situations exist for their Example 3.

It appears that some aspects may have been overlooked in their analysis. First, quantity discount should be available to the buyer only if he orders more than the order quantity when no discount is provided. Therefore, if a linear quantity discount schedule is to be given, it should be defined as $p = a - b(x-x^*)$ with the notations in their paper. Secondly, should such a quantity discount schedule be given, the seller gives the discount rate b and it is the buyer who will choose an order quantity under this schedule. If the game theoretical approach is used, the optimal b should be obtained by finding the buyer's optimal order quantity x for b (x as a function of b), substituting x into the seller's payoff function, and then maximizing it with respect to b. Clearly this is not what was obtained in their analysis. The authors found b as a function of x by minimizing the buyer's cost function with respect to x and then substituted it into the seller's profit function to get the optimal x.

By using some numerical examples, we also found that, should the two factors above had been taken into consideration, the seller's gain would usually be marginal, if not zero, by using a linear quantity discount schedule. Using a continuous approximation for a quantity discount schedule is usually not efficient.

2.3.5. Further Considerations

In this analysis, we concentrate on the case where the seller and the buyer work independently. We give the seller's optimal quantity discount schedule by considering explicitly the buyer's reaction to it. In the literature, several models have been developed to address the joint solution of the problem (Goyal, 1976; Banerjee, 1986a; Joglekar and Tharthare, 1990). Under our model, by letting π be the joint gain of the seller and the buyer, or $\pi = \pi_b + \pi_s$, we have

$$\pi = [1-1/(1+x)](A_b+A_s)D/Q + (DA_e/Q)[1/N_0-1/(1+x)N_x] + (Qh_s/2)[(N_0-1)-(N_x-1)(1+x)] + (Qh_b/2)[1-(1+x)(1-y)]. \quad (33)$$

Differentiating π with respect to the discount term y, we get

$$\partial\pi/\partial y = (Qh_b/2)(1+x) > 0. \quad (34)$$

Clearly, π increases as y increases. Then the seller should set y as high as possible or his unit selling price as low as possible. At the extreme, he should sell the product to the buyer at his unit acquisition cost. In this case, however, his profit is negative. Such a joint solution is highly artificial. Even if there is some collaborative agreement for them to cooperate, letting one's profit be negative will certainly put oneself in a position of great disadvantage in bargaining. It is more reasonable to assume that the seller and the buyer will consider a discount scheme only if it will not make him worse-off than his initial position.

Proposition 4. Quantity discount can be used to obtain the joint maximum gain of the seller and the buyer when no one is willing to accept a negative gain.

Proof. The joint maximum gain of the seller and the buyer when no one is willing to accept a negative gain can be obtained by maximizing (34) under the condition $\pi_s \geq 0$ and $\pi_b \geq 0$. It can be seen from Figure 2.2 that $\pi_s \geq 0$ and $\pi_b \geq 0$ form a closed and non-empty area. Then the problem has a feasible solution, denoted by (x_0, y_0). Letting $x_1 = x_0$ and $y_1 = y_0$, the seller and the buyer can obtain the joint maximum gain at (x_1, y_1). Q.E.D.

As the joint gain π increases when the discount term y increases, the joint maximum occurs at the upper bound of the feasible solution area or $\pi_s = 0$. The seller gains nothing at the joint optimal point. We found that even this joint solution would be very difficult to be implemented as the seller has to get compensation and the problem becomes a bargaining problem here. Offering a carefully selected quantity discount schedule to the buyer is not only more convenient but also gives the seller more control of the process than in a bargaining problem. It seems to us that joint solutions are usually difficult to be implemented, although they are attractive in term of joint gain. An appropriate quantity discount schedule is more reasonable and feasible for the discount problem. As some numerical examples show, quantity discount schedules are very efficient in obtaining the maximum profit increase in certain situations. For instance, in Example One above, the joint maximum gain is $670.96 but the worst the seller and the buyer can gain together by using a quantity discount schedule is $650.36 or 97% of the maximum.

2.4. Conclusions and Possible Extensions

In this chapter, we have discussed the discount problem under the situation of a single seller and a single buyer with a single product. Game theoretical approach is used and the main conclusions are as follows.

(1) It is always possible for the seller and the buyer to benefit from discounting as long as the seller's inventory holding cost and set-up cost are not zero.

(2) Price discount will make the seller lose if the buyer's demand is constant. Thus suppliers should not offer any price discount to their customers when it can not attract more demand from them.

(3) Quantity discount can always make both the seller and the buyer gain. By forming an appropriate all-unit quantity discount schedule, they can significantly improve their positions.

(4) Our model generally gives better or at least no worse results than that of Monahan (1984) or Lee and Rosenblatt (1986).

(5) Quantity discount provides an efficient, flexible and easy to be implemented solution to the discount problem.

In this research, we have restricted our analysis to the case of a single seller with a single buyer and a single product. We realize that this is a major limitation of our study as well as many others in the literature. In reality, a supplier usually has many buyers. An interesting and challenging extension to our research would be the analysis of a supplier's optimal quantity discount schedule when many buyers or many products are involved. In this case, the seller' optimal replenishing policy should include at least one order of each buyer and a quantity discount schedule with many breaks should be used. The problem would be not only a tough task but also has a high potential of real business implementation.

Chapter Three

A Game Theoretical Analysis of the Discount Problem Under Linear Demand

In this chapter, we analyze the discount problem for a supplier and a buyer where the buyer, as a retailer, faces a linear demand. Both price discount and quantity discount scenarios are considered and optimal decision rules are obtained for both the seller and the buyer in various situations. It is shown that, when the seller and the buyer work independently, price discount can improve their positions only if the market demand is sensitive to price changes. Therefore, suppliers provide price discounts primarily to attract more demand and secondarily to reduce inventory related cost. Nevertheless, quantity discount can be of benefit to both parties even if demand is constant and it will always bring a higher profit to the seller than a price discount. In certain situations, it can be very efficient in obtaining the maximum profit and then can be used as a tool for both the seller and the buyer to improve their profitability.

3.1. Introduction

The pricing policy of offering discounts to customers has become a well-known practice in today's industries and an active research area in marketing and inventory management. Recently Dolan (1987) provided a review of models on quantity discounts. He reviewed three types of models according to the three principle motivations for quantity discounts suggested by Buchanan (1953): (1) perfect price discrimination against a single or a group of homogeneous customers, (2) partial price discrimination against a group of heterogeneous customers, and (3) improving channel efficiency. The first two types of models focus primarily on economic issues such as consumers' surplus and the third is mainly concerned with pricing and ordering decisions of suppliers and buyers.

Our main concern in the present analysis is the pricing and ordering decisions of suppliers and buyers. There are a number of articles studying the discount problem for this purpose (Goyal, 1976; Jucker and Rosenblatt, 1985; Lal and Staelin, 1984; Lee and Rosenblatt, 1986; Sethi, 1984). Classical EOQ models discuss exclusively the buyer's best reaction to discount schedules provided by the seller (Snyder, 1973). These models capture the basic issue of the buyer's ordering decisions, but they ignore the seller's principal role in determining such discount schedules. Noting this shortcoming, Monahan (1984) initiated and many others (Banerjee, 1986; Lee and Rosenblatt, 1986) followed the work on finding the seller's best discount schedules. They give the seller's optimal discount schedules under various conditions, assuming implicitly or explicitly that the buyer will cooperate as long as the seller's decision will not make him worse-off. In addition to these, a few models have been built discussing the joint optimal decisions of the seller and the buyer in the selling and buying process (Banerjee, 1986; Chakravarty, 1988).

The problem has also been studied from a game theoretical approach. As we argued in Chapter Two, looking at the problem from solely the perspective of either the buyer or the seller suppresses the conflict as well as the cooperative property of the problem. Only the game theoretical approach takes account of the actions of both the seller and the buyer in a proper manner. Kohli and Park (1989) analyzed the cooperative decisions of the seller and the buyer and we discussed in Chapter Two the non-cooperative aspect of the problem under constant buyer's demand.

In this chapter, unlike most of the studies aimed to improve channel efficiency, which assume that the seller's discount will not affect his demand, we investigate the case where the seller's discount can attract more demand from the buyer. We analyze both price discounts and quantity discounts. The efficiency of each schedule is also discussed.

The chapter is organized as follows. We first formulate a game theory model for a supplier and a buyer and then discuss their price and order quantity when no discounting is considered. In the subsequent two sections, we analyze the optimal decisions of the seller and the buyer for price discount and quantity discount. A brief analysis is then done for their joint decision and the efficiency of the discount schemes. Finally the main findings and possible extensions to this study are summarized.

3.2. The Model

In this section we build a model for a market where a single supplier (seller) sells a product to a group of customers (buyers) who, as retailers, sell the product to ultimate consumers. The customers are homogeneous with respect to their demand pattern and cost behavior. Therefore, we can analyze the problem in terms of the supplier and a single buyer. We maintain the usual assumptions of a basic EOQ model in this study. That is, all lead times are known with certainty, no lost sales and backlogging are allowed, etc.

Secondly, we assume that the buyer faces a linear demand as a function of his unit retailing price, which increases as the price decreases.

Thirdly, we assume that the buyer uses a constant profit margin pricing policy, i.e., he charges a constant profit margin over his unit acquisition cost on each unit he sells to ultimate consumers. For instance, if his unit buying cost from the supplier is $1.00 and he wants to earn a 20% profit margin, he sets his unit retailing price as $1.20. This pricing policy, as we observe, is often used by retailers, especially small retailers and franchise outlets of manufacturers. Nevertheless, if this is not the case, the profit margin may be treated as a decision variable of the buyer. We conjecture that this will not change the main conclusions of our analysis.

We use the following notations.

A_s - the seller's fixed set-up cost of handling each order from the buyer;

A_b - the buyer's fixed ordering cost per order;

H_s - the seller's inventory carrying cost, expressed as the cost of carrying one unit of the product for one year;

H_b - the buyer's inventory carrying cost, expressed as the cost of carrying one unit of the product for one year, $H_b > H_s$;

P - the seller's unit selling price;

C - the seller's unit acquisition cost, $C < P$;

Q - the buyer's ordering size;

α - the buyer's constant profit margin on each unit he sells, that is, he sets his unit retailing price as $(1+\alpha)P$.

D - the buyer's annual demand. Since it is a linear function of his unit retailing price, it can be expressed as $D = -a'(1+\alpha)P + b$, or, simply $D = -aP + b$ where $a = a'(1+\alpha)$ and $C < P < b/a$.

The supplier-buyer relationship under discussion is depicted in Figure 3.1.

The buyer's total profit or payoff function, denoted by π_b, can be expressed as

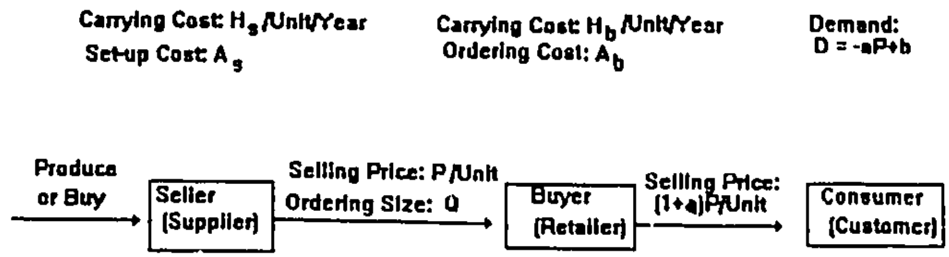

Figure 3.1. Supplier-Buyer Relationship

π_b = (Sales revenue) - (Purchase cost) - (Ordering cost) - (Carrying cost), or

$$\pi_b(Q, P) = (1+\alpha)P(-aP + b) - P(-aP + b) - A_b(-aP + b)/Q - QH_b/2$$
$$= \alpha(-aP + b)P - A_b(-aP + b)/Q - QH_b/2. \quad (1)$$

The seller's annual profit or payoff function, denoted by π_s, can be expressed as

π_s = (Sales revenue) - (Acquisition cost) - (Set-up Cost) + (Saving in Inventory Holding Cost due to the Buyer's Ordering).

Or

$$\pi_s(Q, P) = (-aP+b)(P-C) - A_s(-aP+b)/Q + QH_s/2. \quad (2)$$

Note that, in obtaining the seller's annual profit, we assume that the buyer's ordering size will not alter the seller's ordering or producing policy. Therefore, the seller's saving in inventory holding cost due to the buyer's ordering is $QH_s/2$. This will occur if the seller's customers are many and small and decide their ordering quantities independently and the supplier, on the other hand, determines his ordering or production policy on the basis of an overall estimation of customer demand pattern. In situations where the supplier decides his ordering or production policy on an individual buyer basis, the analysis should be extended to include the determination of the supplier's ordering or production policy (Goyal, 1976; Lee and Rosenblatt, 1986; Rosenblatt and Lee, 1985).

In the supplier-buyer relationship under discussion, the supplier sells the product at a unit price P to the buyer who orders Q units each time. Their annual profits are determined by $\pi_s(Q, P)$ and $\pi_b(Q, P)$, respectively. Clearly each one's profit (payoff) depends on not only his own decision but also the other's decision. Thus their decisions should be analyzed in a game theoretical framework.

3.3. Initial Price and Order Quantity

We first consider the seller's price and the buyer's order quantity in the absence of discount. In the case under discussion, the supplier and the buyer consist of a market. Therefore, the market equilibrium, if any, will determine the unit selling price and the order quantity when no discount is considered. As each player may wish to use the strategy which gives him the maximum payoff regardless of what the other will do, the Nash Equilibrium seems to be appropriate.

Proposition 1. The game admits a Nash equilibrium which is given by

$$Q_0 = (2/\sqrt{3})\sqrt{[(-aC+b)A_b/H_b]}\cos(\beta+\pi/6), \tag{3}$$

$$P_0 = (aC+b)/(2a)+A_s/(2Q_0), \tag{4}$$

where $\beta = (1/3)\mathrm{ATAN}(3\sqrt{3}aA_s/\sqrt{[4(-aC+b)^3A_b/H_b - 27A_s^2a^2]}$.

Proof. We can observe

$$\partial\pi_b(Q, P)/\partial Q = A_b(-aP+b)/Q^2 - H_b/2, \tag{5}$$

$$\partial\pi_s(Q, P)/\partial P = -2aP+aC+b+aA_s/Q, \tag{6}$$

$$\partial^2\pi_b(Q, P)/\partial Q^2 = -2A_b(-aP+b)/Q^3 < 0, \tag{7}$$

$$\partial^2\pi_s(Q, P)/\partial P^2 = -2a < 0. \tag{8}$$

Hence $\pi_b(Q, P)$ and $\pi_s(Q, P)$ are concave in Q and P, respectively. The game admits at least one Nash equilibrium which is determined by $\partial\pi_b(Q, P)/\partial Q = 0$ and $\partial\pi_s(Q, P)/\partial P = 0$ (Nikaido and Isora, 1955). Solving the system of equations (see the appendix), we can get (3) and (4). Q.E.D.

Consider the situation where the seller, if in a position of selling the product to ultimate consumers, faces a demand of $-aP + b$. To maximize his profit $(-aP+b)(P-C)$, his optimal unit selling price is $(aC+b)/(2a)$. When he is a supplier of a retailer, his optimal unit selling price consists of $(aC+b)/(2a)$ and half of A_s/Q, which is his average annual unit ordering cost when the buyer orders Q units each time. The seller passes on half of this cost to the buyer. On the other hand, if the seller offers the buyer the unit selling price $(aC+b)/(2a)$, the buyer's annual demand is $(-aC+b)/2$ and hence his optimal ordering size is $\sqrt{[A_b(-aC+b)/H_b]}$. When the seller passes on half of his ordering cost to him he orders less each time by a factor of $1-(2/\sqrt{3})\cos(\beta+\pi/6)$.

When aA_s is small compared to $-aC+b$, β is small and we can approximate Q_0 by

$$Q_0 \approx (2/\sqrt{3})\sqrt{[(-aC+b)A_b/H_b]}\cos(\pi/6) = \sqrt{[(-aC+b)A_b/H_b]}. \quad (9)$$

Note that the necessary condition for β to be real is $4(-aC+b)^3A_b/H_b - 27A_s^2a^2 \geq 0$. As $-aC+b$ is the demand when the seller's unit selling price is C and is usually very large, we maintain this condition for our analysis.

Therefore, when no discount is considered, the seller will set his unit selling price at P_0 and the buyer will order Q_0 units each time. We use P_0 and Q_0 given by (3) and (4) as the seller's initial unit selling price and the buyer's initial ordering size. However, the reader might note, this is not necessary for the analysis and our model is applicable for any given initial unit selling price P_0 and ordering size Q_0. If discount is considered, $P \leq P_0$ and $Q \geq Q_0$.

3.4. Price Discount

We have all seen ads like "20% off!", "save up to 50%", etc. In these cases the seller simply offers his customers a lower unit selling price by a price discount. To our knowledge, though there are many studies on this common phenomenon, there have not been many decision rules for the seller to follow in such situations.

A price discount is a discount schedule where the seller offers a certain amount of discount or simply a lower unit selling price to his customers, no matter how many units a customer will order from him. Unlike quantity discount schedules, in which price is explicitly related to quantities, a price discount schedule does not necessarily associate with quantities. In today's industries, as we observe, suppliers often simply give price discounts, especially in cases where the seller's customers are many and small or the seller does not know, as it usually happens in practice, his potential customers at the time he makes his discounting decision.

In a selling and ordering situation, when the buyer's ordering size is given, the seller will not offer any discount as it will simply decrease his profit. Then the buyer does not have any potential to act as the leader and the seller always has the leading position in the process when they work independently. Although we do observe that many buyers may propose a larger ordering size to suppliers in exchange of a discount, this is more likely to be a situation of quantity discounting because of the direct association of price and quantity in the relation. We will discuss it in the next section. In this section, we analyze the seller's price discount decision.

When the seller gives a price discount to the buyer, the buyer will react by ordering the quantity which maximizes his annual profit under the discounted price. The determination of the seller's price discount should then take the buyer's reaction into consideration. We analyze the seller's price discount in terms of Stackelberg equilibrium. (Basar and Olsder, 1984).

Let x be the seller's price discount in absolute dollar value. Then the seller's unit selling price is $P = P_0 - x$, $0 \le x < P_0 - C$ since $C < P \le P_0$. Substituting $P = P_0 - x$ into (1) and (2), we obtain the seller's and the buyer's payoff functions as

$$\pi_b(Q, x) = \alpha(D_0+ax)(P_0-x) - A_b(D_0+ax)/Q - QH_b/2, \qquad (10)$$

$$\pi_s(Q, x) = (D_0+ax)(P_0-C-x) - A_s(D_0+ax)/Q + QH_s/2, \qquad (11)$$

where $D_0 = -aP_0+b$. Since $\pi_b(Q, x)$ is concave in Q as shown by (7), the buyer's reaction to x is uniquely determined by $\partial\pi_b/\partial Q = 0$ which, after some simple modification, turns out to be his EOQ formula, or

$$Q = \surd[2(D_0+ax)A_b/H_b]. \qquad (12)$$

Substituting (12) into (11), we obtain the seller's payoff function as

$$\pi_s(Q, x) = (D_0+ax)(P_0-C-x) - A_s(D_0+ax)/\surd[(2(D_0+ax)A_b/H_b] + (H_s/2)\surd[(2(D_0+ax)A_b/H_b]. \qquad (13)$$

Differentiating π_s with respect to x, we get

$$d\pi_s/dx = - 2ax - D_0 + a(P_0-C) - (a/2)[A_s\surd(H_b/(2A_b)) - H_s\surd(A_b/(2H_b))]/\surd(D_0+ax), \qquad (14)$$

$$\begin{aligned} d^2\pi_s/dx^2 &= -2a + (a^2/4)[A_s\surd(H_b/2A_b) - H_s\surd A_b/(2H_b)]/\surd(D_0+ax)^3 \\ &\le -2a + (a^2/4)A_s\surd(H_b/2A_b)/\surd(D_0+ax)^3 \\ &< -2a + (a^2/4)A_s\surd(H_b/2A_b)/\surd(-aC+b)^3 \text{ since } x < P_0-C. \end{aligned} \qquad (15)$$

Proposition 2. The seller and the buyer have unique Stackelberg strategies which are given by

$$Q^* = (2/\surd 3)\surd[(-aC+b)A_b/H_b]\text{COS}(\gamma+\pi/6) \text{ if } A_s-(H_s/H_b)A_b > 0, \qquad (16a)$$

$$= (2/\sqrt{3})\sqrt{[(-aC+b)A_b/H_b]}\cos(\gamma-\pi/6) \text{ if } A_s-(H_s/H_b)A_b < 0, \quad (16b)$$

$$x^* = P_0 - (aC+b)/(2a) - (1/4)(A_s-H_sA_b/H_b)/Q^*$$

$$= A_s/(2Q_0)-(1/4)(A_s-H_sA_b/H_b)/Q^*, \quad (17)$$

where $\gamma = (1/3)\mathrm{ATAN}\{3\sqrt{3}/\sqrt{[16(-aC+b)^3A_b/(H_ba^2(A_s-H_sA_b/H_b)^2) - 27]}\}$.

Proof. Using the condition that $4(-aC+b)^3A_b/H_b - 27A_s^2a^2 \geq 0$ in the third expression of (15), we obtain $d^2\pi_s/dx^2 < -2a + a/\sqrt{216} < 0$. Hence π_s is concave in x. The seller has a unique Stackelberg strategy which is determined by $d\pi_s/dx = 0$. On the other hand, the buyer's response curve is uniquely determined by (12). Solving $d\pi_s/dx = 0$ and $\partial\pi_b/\partial Q = 0$, we can get (16) and (17). Q.E.D.

We need the condition that $A_s-(H_s/H_b)A_b \neq 0$ for (16) and (17) to be real. This condition is violated when the inventory related costs are identical for the seller and the buyer, i.e., $H_s = H_b$ and $A_s = A_b$. For this special case, we can obtain from (14) that $x = A_s/(2Q_0)$ or $P = (aC+b)/(2a)$. The seller should not pass on any inventory related cost to the buyer. More generally, as the seller's inventory holding cost is usually sufficiently smaller than that of the buyer, we have $A_s-(H_s/H_b)A_b > 0$.

From (17), we get

$$x^* \geq [(1/2)A_s/Q^*-(1/4)A_s/Q^*] + (1/4)(H_s/H_b)A_b/Q^* \text{ since } Q_0 \leq Q^*$$

$$= (1/4)[A_s+ (H_s/H_b)A_b]/Q^*. \quad (18)$$

The seller should offer a price discount at least equivalent to one-fourth of $[A_s+(H_s/H_b)A_b]/Q^*$.

Two important observations can be made from the results of Proposition 2. First, the seller's price discount increases as his inventory related cost increases. This can be seen by simply looking at (17). When A_s and H_s increase, x^* increases. The seller is willing to offer more discount to the buyer in exchange for a larger ordering size. This finding is consistent with that of many authors (Lee and Rosenblatt, 1986; Monahan, 1984). They suggested that the primary incentive for the seller to provide discount is to induce the buyer to order more each time and hence to reduce his inventory related cost.

This is not the only reason for the seller to do so, however. We can also observe from (16) and (17) that the seller's price discount also increases as a, which can be considered to be the sensitivity of demand to price changes, increases. This is less obvious than the first observation. But if we consider γ above, we can see that, when a increases, γ increases, Q^* decreases and hence x^* increases. Therefore, when the buyer's demand is more sensitive to price changes, the seller is willing to provide a larger price discount to attract more demand from the buyer. Especially, if the buyer's demand is totally insensitive to price changes, say, it is constant at D_0, the buyer's and the seller's profit functions are

$$\pi_b(Q, x) = \alpha D_0(P_0-x) - A_bD_0/Q - QH_b/2, \tag{19}$$

$$\pi_s(Q, x) = D_0(P_0-x) - A_sD_0/Q + QH_s/2. \tag{20}$$

The buyer's optimal ordering size is given by $\partial\pi_b/\partial Q = 0$ or his EOQ formula

$$Q = \sqrt{(2A_bD_0/H_b)}, \tag{21}$$

which is independent of x. The buyer will not change his ordering size even if the seller offers a discount and the seller will incur a loss if he does so. We have shown in Chapter Two that, even the buyer increases his ordering size in this case, he will always under-react to the seller's discount by ordering less than the seller expects and makes the seller lose.

Therefore the objective or rationale for suppliers to provide price discounts to their customers is twofold: to attract more demand from their customers and to reduce their inventory related cost by inducing the buyer to reduce his frequency of ordering. The former is more important because the seller should not provide price discount at all when it can not attract more demand. Unfortunately this intention has often been overlooked as many writers over-emphasized the inventory cost saving intention. Our analysis shows that price discount is more a mechanism of promotion or competitive strategy to increase demand rather than a tool to improve channel efficiency and thus should be studied including the demand relationship.

3.5. Quantity Discount

In this section we study quantity discount. It has been shown in Chapter Two in the case of constant demand that quantity discount schedules can be of benefit to both the seller and the buyer. In the following we provide an analysis of quantity discount considering that discount can attract more demand from the buyer. Instead of using a continuous approximation of a quantity discount schedule (Lal and Staelin, 1984, Lee and Rosenblatt, 1986), we study quantity discount in a more general framework. We focus on all-unit quantity discount schedule as all other quantity discount schedules can be viewed as special cases of it (Jucker and Rosenblatt, 1985). In addition, some light will be shed on the buyer's optimal decision when he tries to take the initiative in quantity discounting.

When the seller's initial unit selling price is P_0 and the buyer orders Q_0 units each time under P_0, a quantity discount schedule can be defined as follows: the seller gives no discount for any order $Q_0 \le Q < Q_1$, a discount x_1 for any order $Q_1 \le Q < Q_2$, and so on to a discount x_n for $Q_n \le Q$, where x_i is in absolute dollar value, $0 < x_1 < \ldots < x_n$ and $Q_0 < Q_1 < \ldots < Q_n$. In practice, a discounted price or a percentage discount might be used for each step. But they are simply different expressions of discount.

3.5.1. The Seller Acts as the Leader

As we observe in reality, quantity discount schedules are usually given by suppliers. Customers react in their best way to suppliers' quantity discount schedules. In the designing of a quantity discount schedule, the seller should choose the best discounts x_i as well as the break points Q_i, $i = 1, 2, \ldots, n$.

Note that, if the buyer stays at Q_0, the seller gains nothing. Then the seller has to offer an incentive or a discount sufficiently large to induce the buyer to move away from Q_0. Let w be the least amount of gain that will make the buyer change his ordering policy, where $w \geq 0$. The necessary condition for the buyer to order $Q > Q_0$ units each time when a discount x is given is

$$\pi_b(Q, x) \geq \pi_b(Q_0, 0) + w. \tag{23}$$

Consider the buyer's reaction to a quantity discount schedule given by the seller. He will choose the ordering quantity which maximizes his payoff function. Let $Q_i^* = \sqrt{[2(ax_i+D_0)A_b/H_b]}$ or the buyer's EOQ when a discount x_i is given. For $Q_i \leq Q < Q_{i+1}$ with a discount x_i, his best ordering quantity is Q_i^* if $Q_i \leq Q_i^* \leq Q_{i+1}$; Q_i if $Q_i > Q_i^*$; and he will not consider it at all if $Q_{i+1} < Q_i^*$. We then need only to consider the first two types of steps for the seller.

When a single customer is considered, only one ordering quantity will be selected. Therefore, the seller's optimal quantity discount schedule can be analyzed by using a quantity discount schedule with only one break such that no discount for $Q_0 \leq Q < Q_1$ and a discount x for $Q \geq Q_1$, where $x > 0$.

As the buyer's response depends on how the seller assigns the value of Q_1 and x, we consider two policies for the seller in the following.

Policy I. Assign Q_1 and x such that $Q_1 \leq \sqrt{[2(ax+D_0)A_b/H_b]}$.

The buyer will order according to his EOQ or (12). By substituting (12) into (23), we obtain

$$\alpha(2aP_0-b-ax)x - \sqrt{[2(ax+D_0)A_bH_b]} + \sqrt{(2D_0A_bH_b)} - w \geq 0. \tag{24}$$

Solving (24) as an equation for x, we can get a positive x, denoted by x_0, which is independent of Q_1. The necessary condition for the seller to induce the buyer to order at least Q_1 units each time becomes $x \geq x_0$.

In this case, the seller's payoff function is determined by (13). The seller is to find the optimal x and Q_1 which maximize (13) subject to $x \geq x_0$ and $Q_1 \leq \sqrt{[2(ax+D_0)A_b/H_b]}$. Note that the second constraint can be written as $x \geq x_1 = H_b Q_1^2/(2aA_b) - D_0/a$.

We have shown in Section 3.4 that π_s is concave in x and $\partial\pi_s/\partial x = 0$ if $x = x^*$ where x^* is given by (17). Then π_s is increasing when $x < x^*$ and decreasing when $x > x^*$.

If $Q_1 \leq Q^*$, $x_1 \leq H_b(Q_1^*)^2/(2aA_b) - D_0/a = x^*$ since Q^* and x^* satisfy (12). Hence the optimal x is x^* or x_0 whichever is larger. The value of Q_1 will not affect the seller's profit in this case. We can arbitrarily assign any value between Q_0 and Q^* to Q_1.

If $Q_1 > Q^*$, on the other hand, $x_1 > x^*$. To maximize π_s subject to $x \geq x_1$ and $x \geq x_0$, the optimal x is x_1 or x_0 whichever is larger. The seller's gain, π_s, will be less than (if $x_0 < x_1$) or at most equal to (if $x_0 = x_1$) that of assigning $Q_1 \leq Q^*$. We can eliminate $Q_1 > Q^*$.

Then the seller's optimal x is x^* or x_0 whichever is larger. In particular, if (24) is satisfied with x^*, x^* is optimal. Q_1 can be any value between Q_0 and Q^* and the buyer's optimal order quantity is given by (12) with the optimal x.

Policy II. Assign Q_1 and x such that $Q_1 > \sqrt{[2(ax+D_0)A_b/H_b]}$.

The buyer will order Q_1 units each time and the necessary condition (23) becomes

$$\alpha(2aP_0-b-ax)x - A_b(ax+D_0)/Q_1 - Q_1H_b/2 + \sqrt{(2D_0A_bH_b)} - w \geq 0. \quad (25)$$

The seller's payoff function is

$$\pi_s = (D_0+ax)(P_0-C-x) - A_s(D_0+ax)/Q_1 + H_sQ_1/2. \quad (26)$$

The seller's decision problem is to maximize π_s subject to (25) and $x < x_1$. Note that (25) gives a relationship between x and Q_1, which can not be conveniently simplified to express x as a function of Q_1 in some simple form. Therefore, solving the problem analytically would be cumbersome in this case. However, consider the following non-linear programming model

$$\max_{Q_1,x} \quad (D_0+ax)(P_0-C-x) - A_s(D_0+ax)/Q_1 + H_sQ_1/2 \quad (27)$$

$$\text{S.T.} \quad \alpha(2aP_0-b-ax)x - A_b(ax+D_0)/Q_1 - Q_1H_b/2 + \sqrt{(2D_0A_bH_b)} - w \geq 0,$$

$$x \leq Q_1^2H_b/(2aA_b) - D_0/a - \delta,$$

$$Q_1 \geq Q_0 + \delta, \; x > \delta,$$

where δ is a very small positive number to ensure that $0 < x < x_1$ and $Q_1 > Q_0$. The model gives the optimal x and Q_1 which maximize π_s under the condition that the buyer gets at least w and he will order Q_1 units each time. Using a non-linear programming package such as GINO (Liebman, et al, 1986), it can be easily solved.

For the non-linear programming model (27), by differentiating the objective function or π_s with respect to Q_1, we get

$$\partial\pi_s/\partial Q_1 = A_s(D_0+ax)/Q_1^2 + H_s/2 > 0. \quad (28)$$

Then π_s is increasing with Q_1 and the first constraint must be tight because the optimal Q_1 would be infinity otherwise. In this case, the second constraint is automatically satisfied for any limited x but the buyer incurs a loss of infinity. Therefore, if Policy II is used by the seller, the seller gains full control of the situation in the sense that he maximizes his profit by letting the buyer gain w, the least amount that will make him interested in ordering more than Q_0 units each time. As w is determined by the seller, the seller might make the buyer gain very little by letting w be very small.

We can also observe from the above analysis that the seller should not use price discount whenever a quantity discount schedule is possible. Because of the well-known insensitivity of the EOQ formula to parameter changes, the buyer's response to a price discount is low. Then the seller's gain from price discount is usually very small. If a quantity discount is possible, by letting w be the buyer's gain from price discount, the seller can develop a quantity discount schedule to gain at least what he can gain from price discount.

Let us consider the following numerical example.

Example One. $D = -200P + 2000$, $A_b = \$100$, $A_s = \$80$, $H_b = \$3$, $H_s = \$2$, $C = \$4$, $\alpha = 0.5$, and $4 < P < 10$.

By using the decision procedures obtained above, we get the following results.

(a) No discount is used.

$P_0 = \$7.2073$, $Q_0 = 193$, $\pi_b = \$1433.89$, $\pi_s = \$1752.81$.

(b) Price discount is used.

$x^* = \$0.1906$, $Q^* = 200$, $\pi_b = \$1494.96$, $\pi_s = \$1761.28$.

(c) Quantity discount is used.

We consider three different w's: \$0.00, \$14.34 or 1% of $\pi_b(Q_0, P_0)$, and \$143.39 or 10% of $\pi_b(Q_0, P_0)$.

Policy I. At $x^* = \$0.1906$, the buyer will order 200 units each time with a gain of \$61.07. Then (24) is satisfied at $x = 0.1906$ when $w \leq 61.07$. For $w > 61.07$, we have to solve (24) as an equation of x to obtain x_0. The results are summarized in Table 3.1.

Table 3.1. Optimal Solutions by Using Policy I

w	x	Q_1	Q	π_b	Profit Increase	π_s	Profit Increase
\$0.00	\$0.1906	199	200	\$1494.96	\$61.07	\$1761.28	\$8.47
14.34	0.1906	199	200	1494.96	61.07	1761.28	8.47
143.39	0.4893	199	209	1577.28	143.39	1742.25	-10.56

Q_1 is assigned to be 199 above although it could be any value between 193 and 200.

Policy II. Solving the non-linear programming model with $\delta = 0.001$ for each w given above, we obtain the optimal discount decisions as shown in table 3.2.

Table 3.2. Optimal Solutions by Using Policy II

w	x	Q_1	Q	π_b	Profit Increase	π_s	Profit Increase
\$0.00	\$0.7727	473	473	\$1433.89	\$0.00	\$2088.51	\$335.70
14.34	0.7796	462	462	1448.23	14.34	2073.09	320.28
143.39	0.8912	365	365	1577.28	143.39	1910.20	157.39

For each w, Policy II gives the seller a higher profit and then it gives the seller's optimal quantity discount schedule. For instance, if w = \$143.49, the seller's optimal quantity discount schedule is that no discount for $193 \leq Q \leq 364$ and a discount of \$0.8912 for $Q \geq 365$.

The profit increase from quantity discount for the seller, compared with his profit when no discount is considered, is very significant in the example. The best the seller can get is a \$335.70 or a 19% profit increase when the buyer gets nothing. He can get a \$157.39 or a near 9% profit increase even if he allows a 10% profit increase for the buyer!

The seller's gain from quantity discount is much more than that from price discount. He can gain only \$8.47 from price discount while the buyer gains \$61.07. If he allows the buyer to gain the same amount, but he offers a quantity discount schedule instead of a price discount to the buyer, his optimal quantity discount schedule can be found to be that no discount for $193 \leq Q \leq 427$ and a discount of \$0.8080 for $Q \geq 428$. The buyer has to order 428 units in order to benefit from quantity discount. The buyer gains \$61.07 and the seller gains \$270.40!

In the above analysis, we use a quantity discount schedule with only one break. There is no doubt that the seller can offer a quantity discount schedule with more than one break to the buyer. However, it is usually to the seller's benefit to limit the number of breaks in a quantity discount schedule. As shown by the example above, Policy II will usually give $Q_1 > Q^*$ and a discount higher than that from Policy I. Then the seller can offer a quantity discount schedule with two breaks using the two policies. But, in this case, the buyer's response may not be what the seller wants. For instance, if $w = \$14.34$ in the above example, the seller can offer the buyer a quantity discount schedule such that no discount for $193 \leq Q \leq 199$, a discount of \$0.1906 for $200 \leq Q \leq 461$, and a discount of \$0.7796 for $462 \leq Q$. The buyer's response to this schedule would be to order 200 units each time. The buyer gains \$61.07 and the seller gains \$8.47. If the seller provides a quantity discount schedule with only the second break, the buyer will order 462 units each time, gaining \$14.34. The seller gains \$320.28!

3.5.2. The Buyer Acts as the Leader

Although the seller often leads in quantity discounting, the buyer may also act as the leader by proposing a larger ordering quantity to the seller in exchange of a discount. This is especially true when the buyer is a major customer of the seller.

In this case, the seller's reaction to any Q is uniquely determined by $\partial\pi_s/\partial Q = 0$ or $P = (aC+b)/(2a)+A_s/(2Q)$, the buyer's profit function is

$$\pi_b = (\alpha/4)(-aC+b-aA_s/Q)(C+A_s/Q+b/a) - A_b(-aC+b-aA_s/Q)/(2Q) - QH_b/2. \qquad (29)$$

Differentiating π_b with respect to Q, we have

$$d\pi_b/dQ = aA_s(\alpha A_s - 2A_b)/(2Q^3)+[\alpha aCA_s+(-aC+b)A_b]/(2Q^2)-H_b/2, \qquad (30)$$

$$d^2\pi_b/dQ^2 = -3aA_s(\alpha A_s-2A_b)/(2Q^4) - [\alpha aCA_s+(-aC+b)A_b]/Q^3. \qquad (31)$$

Let $q = 3aA_s(A_b-\alpha A_s/2)/[\alpha aCA_s+(-aC+b)A_b]$ if $2A_b-\alpha A_s \geq 0$ and $q = 0$ if $2A_b-\alpha A_s < 0$. Then π_s is concave if $Q > q$ and convex if $0 \leq Q < q$. By solving $d\pi_b/dQ = 0$, we get

$$Q = (2/\sqrt{3})\sqrt{[(\alpha aCA_s+A_bD)/H_b]}\,COS(\theta+\pi/6) \text{ if } 2A_b-\alpha A_s \geq 0, \qquad (32a)$$

$$= (2/\sqrt{3})\sqrt{[(\alpha aCA_s+A_bD)/H_b]}\,COS(\theta-\pi/6) \text{ if } 2A_b-\alpha A_s < 0, \qquad (32b)$$

where $\theta = (1/3)ATAN(3\sqrt{3}/\sqrt{[4(\alpha aC+A_bD/A_s)^3/(a^2(2A_b/A_s-\alpha)^2H_bA_s)-27]})$ and $D = -aC+b$. This is then the optimal quantity the buyer may propose to the seller unless it is less than q, in which case the buyer's optimal decision is q.

Look at the seller's reaction curve $P = (aC+b)/(2a)+A_s/(2Q)$ in this case. When Q_0 is fairly large, an increase in Q from Q_0 will not result in a significant decrease in price. Then the buyer's gain will not be dramatic. Using the data in the above example, we obtain Q = 222 and P = \$7.1805. The buyer gains only \$3.71 while the seller gains \$59.47.

The buyer's potential to get quantity discount from the seller also depends on his inventory related costs and the sensitivity of demand to price changes. This can be seen by looking at Q given by (32) which increases as a and A_b increase and/or H_b decreases. As shown previously, the buyer will not move away from his EOQ when the demand is constant. Therefore the buyer and the seller have similar interests in leading in discounting in the sense that they might gain much only if their inventory related costs are large and the demand is sensitive to price changes.

3.6. The Joint Solution

We have outlined the discounting decisions for the seller and the buyer when they work independently. However, the seller and the buyer might also be interested in finding out what they can possibly obtain if they work together.

By letting π be the joint profit of the seller and the buyer, $\pi = \pi_b + \pi_s$ or

$$\pi = (1+\alpha)(-aP+b)P - (-aP+b)C - (A_b+A_s)(-aP+b)/Q - Q(H_b-H_s)/2, \quad (33)$$

we obtain

$$\partial\pi/\partial Q = (A_b+A_s)(-aP+b)/Q^2 - (H_b-H_s)/2, \quad (34)$$

$$\partial\pi/\partial P = (1+\alpha)(-2aP+b) + aC + a(A_b+A_s)/Q, \quad (35)$$

$$\partial^2\pi/\partial Q^2 = - 2(A_b+A_s)(-aP+b)/Q^3 < 0, \quad (36)$$

$$\partial^2\pi/\partial P^2 = -2a(1+\alpha) < 0. \quad (37)$$

Then Q_J and P_J which maximize π can be obtained by solving $\partial\pi/\partial Q = 0$ and $\partial\pi/\partial P = 0$, or,

$$Q_J = (2/\sqrt{3})\sqrt{[(A_b+A_s)(-aC/(1+\alpha)+b)/(H_b-H_s)]}\cos(\psi+\pi/6), \tag{38}$$

$$P_J = b/(2a) + C/[2(1+\alpha)] + (A_b+A_s)/[2(1+\alpha)Q_J], \tag{39}$$

where $\psi = (1/3)\mathrm{ATAN}\{3\sqrt{3}/\sqrt{[4[-aC+(1+\alpha)b]^3/((1+\alpha)a^2(A_b+A_s)(H_b-H_s))-27)]}\}$.

Their joint gain JG, compared with that at (Q_0, P_0), is

$$JG = \pi(Q_J, P_J) - \pi(Q_0, P_0). \tag{40}$$

By comparing the optimal Q for each case discussed previously with Q_J, we can see that using discount schemes can rarely attain the same level of ordering quantity as the joint solution. Then the seller and the buyer can hardly obtain the maximum profit they can get when working together.

However, quantity discount can be very efficient. Using the data given in the numerical example above, we get P_J = \$6.45, Q_J = 513, JG = \$336.93. If quantity discount is used, their joint gain is \$335.70, \$334.62, and \$300.78 for w = \$0.0, \$14.34, and \$143.39, respectively. They represent 99.6%, 99.3%, and 89.3%, respectively, of \$336.93 or the maximum profit increase they can possibly obtain together! In such cases, if the seller and the buyer decide to cooperate, they can simply use a quantity discount schedule. The distribution of their joint gain can be determined by negotiating a suitable w. Quantity discount can be used not only for the seller to increase profit but also as a way of cooperation for the seller and the buyer.

3.7. Conclusions and Possible Extensions

In this chapter, we have discussed the discounting decisions for both the seller and the buyer when they work independently and discount will increase the demand. As a result, both of them can improve their profitability by using proper discounting policies. Decision rules are derived for both the seller and the buyer in various situations. The other main conclusions include:

First, the objective for suppliers to provide price discounts to their customers is always twofold: reducing their inventory-related costs on one hand and increasing the market demand on the other. Particularly, as shown in the study, attracting more demand is crucial as none of them can benefit from price discount when the demand is constant, even though there is much room to improve over their inventory related costs. Price discount is then more likely to be a competitive marketing strategy for the seller rather than a way of cooperation for the seller and the buyer.

Secondly, the seller should use quantity discount whenever it is possible. Although, the buyer might prefer a simple price discount in some situations, quantity discount will always bring a higher profit to the seller.

Thirdly, quantity discount schedules can be very efficient in obtaining the maximum profit increase that the seller and the buyer can possibly obtain together. Therefore, they can be used not only for the seller to increase his own profit but also for both the seller and the buyer to improve their profitability.

In the present study, we set out to analyze the optimal discounting decisions of the seller and the buyer by including the demand relationship. In our view, our analysis reveals some very important properties of discounting decisions of the seller and the buyer. However, we also realize, the analysis has some limitations. First of all, it is done in a much simplified setting by considering a supplier with a single customer. For the discussion of quantity discount including many customers might give more representative results. Secondly, we assume that the buyer uses a constant profit margin pricing policy, which may not always be the reality. Considering a profit maximizing customer regarding his retailing price would be more appropriate but at the same time will tremendously increase the difficulty of analysis. These limitations represent some of the possible extensions to our study.

Appendix

Solution for $\partial\pi_b(Q, P)/\partial Q = 0$ and $\partial\pi_s(Q, P)/\partial P = 0$.

By setting $\partial\pi_b(Q, P)/\partial Q = 0$ and $\partial\pi_s(Q, P)/\partial P = 0$, we have

$$A_b(-aP+b)/Q^2 - H_b/2 = 0, \tag{A1}$$

$$-2aP + aC + b + aA_s/Q = 0. \tag{A2}$$

(A1) and (A2) give the reaction curves of the buyer and the seller, respectively, as shown in Figure 3.2. It can be seen that the two reaction curves intersect at two points when $Q > 0$ and $P > 0$.

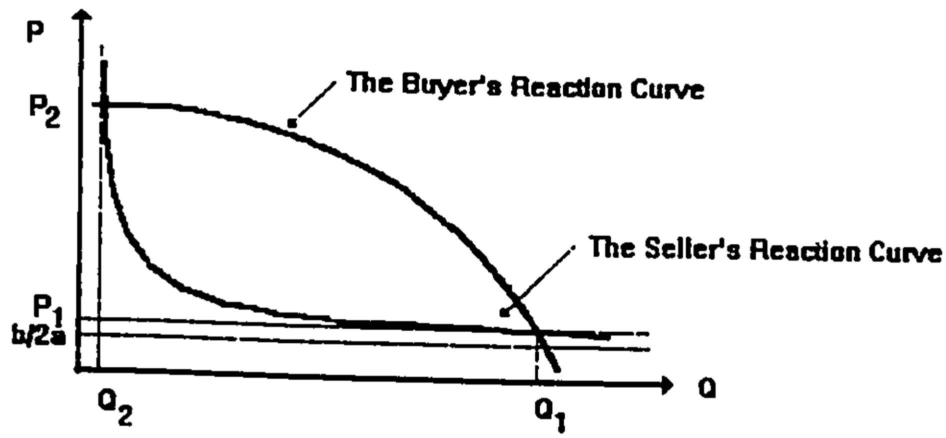

Figure 3.2. The Reaction Curve

Solving (A2) for P, we get

$$P = (aC+b)/(2a) + A_s/(2Q). \quad \text{(A3)}$$

Substituting (A3) into (A1), after some modification, we obtain an equation of Q as

$$Q^3 - [(-aC+b)A_b/H_b]Q + aA_sA_b/H_b = 0. \quad \text{(A4)}$$

The necessary condition for (A4) to have three real unequal roots is $4(-aC+b)^3A_b/H_b - 27A_s^2a^2 \geq 0$. Since $-aC+b$, which is the demand when the seller's unit selling price is C, is usually very large compared to the inventory related cost, we assume this condition in our analysis.

Solving (A4) for Q, we obtain

$$Q_1 = (2/\sqrt{3})\sqrt{[(-aC+b)A_b/H_b]}\cos(\beta+\pi/6), \quad \text{(A5)}$$

$$Q_2 = (2/\sqrt{3})\sqrt{[(-aC+b)A_b/H_b]}\,\mathrm{COS}(\beta-\pi/2), \qquad \text{(A6)}$$

$$Q_3 = (2/\sqrt{3})\sqrt{[(-aC+b)A_b/H_b]}\,\mathrm{COS}(\beta+5\pi/6), \qquad \text{(A7)}$$

where $\beta = (1/3)\mathrm{ATAN}\ (3\sqrt{3}aA_s/\sqrt{[4(-aC+b)^3A_b/H_b - 27A_s^2a^2]}$. Note that $0 \le \beta \le \pi/6$. Then $Q_3 < 0$ and $Q_2 < Q_1$.

Q_3 can be eliminated as the buyer's ordering size cannot be negative.

Q_2 can not be optimal. This can be seen by looking at the case where $A_s = 0$. In this case, $P = (aC+b)/(2a)$ from (A3) and then the buyer's demand is $(-aC+b)/2$ and his optimal ordering quantity is $\sqrt{[(-aC+b)A_b/H_b]}$. However, using Q_2, we get that the buyer's ordering quantity is zero since $\beta = 0$.

Q_1 gives the optimal decision in any case. By substituting Q_1 into (A3), we get the optimal price P_1. (Q_1, P_1) is taken as the solution, denoted by (Q_0, P_0), namely

$$Q_0 = (2/\sqrt{3})\sqrt{[(-aC+b)A_b/H_b]}\,\mathrm{COS}(\beta+\pi/6), \qquad \text{(A8)}$$

$$P_0 = (aC+b)/(2a)+A_s/(2Q_0), \qquad \text{(A9)}$$

where $\beta = (1/3)\mathrm{ATAN}\ (3\sqrt{3}aA_s/\sqrt{[4(-aC+b)^3A_b/H_b - 27A_s^2a^2]}$. This point is indicated by (Q_1, P_1) in Figure 3.2.

The other solutions as (16), (32) and (38) are obtained similarly. We will not repeat the solution procedures.

Chapter Four

A Three-Person Game Theory Model of The Problem of Substitutable Products with Stochastic Demands

In this chapter, we build a game theory model for the single-period inventory problem where each of three (or more) retailers in a common market tries to determine his optimal order quantity. Their products are substitutable and have random demands. Therefore, multiple-direction demand transfers occur when one or more retailers are sold out. It is shown that there is at least one Nash equilibrium for the problem if the players act independently and rationally. If one or more players act irrationally to damage the others, the decision problem for the latter reduces to that without the irrational player(s). We also study the cooperation of the players. Cooperative players will switch excess inventory to those who have excess demand and determine their order quantities collectively depending on whether side payments are allowed. We show that, if side payments are not allowed, secure (Nash) strategies always exist for each player in any case of cooperation. We also give conditions for cooperation in both cases where side payments are and are not allowed and demonstrate that all players' cooperation is often worthwhile and feasible, especially when side payments are allowed.

4.1. Introduction

In many situations, different products sold by different retailers may be substitutable. Decision making issues related to substitutability was first studied by McGillivray and Silver (1978) for inventory control in the EOQ context. Since then several other papers considering substitutability in inventory control have been published (Parlar and Goyal, 1984; Parlar, 1985).

More recently Parlar introduced the game theoretical approach to study this problem in the newsboy problem context (Parlar, 1988). He observed that substitution often takes place between different products sold by different retailers when the products have stochastic demands. In such situations, each retailer's profit function is determined not only by his own order decision but also by his competitors' order decisions. Thus, the game theoretical approach should be used to analyze each retailer's order decision. He analyzed the problem when two retailers are present by formulating it as a two-person nonzero sum game. It is shown in his study that there exists a unique Nash equilibrium for the problem and, if one of the two players acts irrationally to damage the other, the optimal (defensive) strategy for the latter reduces to the optimal order size in the classical single-period newsboy problem. His model extends the classical newsboy problem into situations with two retailers.

In this research, we study the substitutable product inventory problem when three or more players are present. The presence of additional players brings about multiple-direction two-way demand transfers and coalitions between any two or among all of the three players, which can not be dealt with by either one- or two-decision maker models. For simplicity of presentation, we present a three-person game theory model in this chapter and analyze the problem using both non-cooperative and cooperative solution concepts. The reader might note in the following analysis that the major results of this study can be generalized to situations with more than three players.

The chapter is organized as follows. In Section 4.2, we develop our model. Then we analyze the model in the subsequent two sections for non-cooperative and cooperative solutions, respectively. Finally, in Section 4.5, we summarize our findings and discuss possible extensions to our study.

4.2. The Model

We study the situation where each of three retailers tries to choose his best order quantity when substitution exists among their products. Each retailer faces an independent stochastic demand. If one is short of supply its excess demand will be fulfilled partly or fully by those who have excess supply. The actual substitution between any two products takes place according to a substitution rate which depends on the products and other factors such as retailers' geographical locations. We consider only a single period with no inventory at the beginning and no inventory carrying cost. The inventory control problem in multiple periods represents a direction of future research for this problem.

We use the following notations ($i, j = 1, 2, 3$).

Pi := Player i (or Retailer i).

u, v, w := Order quantity chosen by P1, P2, P3, respectively;

X, Y, Z := Random demand for P1's, P2's, P3's product with p.d.f. $f(x)$, $g(y)$, $h(z)$ and c.d.f. $F(x) = \int_0^x f(t)dt$, $G(y) = \int_0^y g(t)dt$, $H(z) = \int_0^z h(t)dt$, respectively;

s_i := Sales price/unit for Pi's product;

c_i := Purchase cost/unit for Pi's product, $c_i < s_i$;

p_i := Lost sales penalty/unit for Pi's product;

q_i := Salvage value/unit for Pi's product, $q_i < s_i$;

a_{ij} := The fraction of Pi's demand which will switch to Pj's product when Pi is sold out, $0 \le a_{ij} \le 1$ and $\sum_{j=1}^{3} a_{ij} \le 1$ $(i \ne j)$;

Π_i := Random profit for Pi with $J_i = E(\Pi_i)$.

If no substitution exists, each decision maker's problem is the classical single-period newsboy problem. If substitution exists, however, each decision maker's problem is much more complicated as substitution may take place across different products. His profit function will depend on not only his own order quantity but also the others' order quantities. Thus their optimal order decisions should be analyzed in the context of game theory. In the sequel, we will use the term "player" interchangeably with "decision maker".

Let us consider Player 1 first. For any given order sizes u, v and w, there are five possible exclusive cases for demand transfers between P1's product and the others' products to take place, depending on the realized values of the random variables X, Y and Z. His profit function in each case is shown in the following.

(1) $x \geq u$,

$$\pi_1^1 = s_1 u - p_1(x - u) - c_1 u. \tag{1}$$

P1 has a shortage. Substitution does not affect his profit function as he cannot satisfy any of his competitors' unsatisfied demand.

(2) $x \leq u$, $y \leq v$, $z \leq w$,

$$\pi_1^2 = s_1 x + q_1(u-x) - c_1 u. \tag{2}$$

All players have excess supply and hence no substitution takes place.

(3) $x \leq u$, $y \leq v$, $z \geq w$,

$$\pi_1^3 = s_1 x + s_1 \min[u-x,\ a_{31}(z-w)] + q_1 \max[0,\ (u-x) - a_{31}(z-w)] - c_1 u. \tag{3}$$

P1 and P2 have excess supply and P3 has excess demand. The excess demand of P3 is then substituted by the excess supply of P1 and P2. The amount of P3's excess demand to be substituted by P1's excess supply is $a_{31}(z-w)$ or u-x, whichever is less, and P1's excess inventory after substitution is 0 or $(u-x)-a_{31}(z-w)$, whichever is greater.

(4) $x \le u,\ y \ge v,\ z \le w,$

$$\pi_1^4 = s_1x+s_1\min[u-x,\ a_{21}(y-v)]+q_1\max[0,\ (u-x)-a_{21}(y-v)]-c_1u. \quad (4)$$

P1 and P3 have excess supply and P2 has excess demand. Similar to (3), Player 1's profit is given in (4).

(5) $x \le u,\ y \ge v,\ z \ge w,$

$$\pi_1^5 = s_1x + s_1\min[u-x,\ a_{21}(y-v)+a_{31}(z-w)] + q_1\max[0,\ (u-x)-a_{21}(y-v)-a_{31}(z-w)] - c_1u. \quad (5)$$

P1 has excess supply and P2 and P3 have excess demand. The excess demands of P2 and P3 are then substituted by the excess supply of P1. The amount of P2's and P3's excess demand to be substituted by P1's excess supply is $a_{21}(y-v)+a_{31}(z-w)$ or u-x, whichever is less.

The expected profit of Player 1 is obtained by integrating π_1^k (k = 1, 2, 3, 4, 5) over the respective regions as

$$J_1 = \int_u^{\infty}[s_1u-p_1(x-u)-c_1u]fdx + \int_0^u\int_0^v\int_0^w[s_1x+q_1(u-x)-c_1u]fghdxdydz$$
$$+ \int_0^u\int_0^v\int_w^{\infty}\{s_1x+s_1\min[u-x,a_{31}(z-w)]+q_1\max[0,(u-x)-a_{31}(z-w)]-c_1u\}fghdxdydz$$
$$+ \int_0^u\int_v^{\infty}\int_0^w\{s_1x+s_1\min[u-x,a_{21}(y-v)]+q_1\max[0,(u-x)-a_{21}(y-v)]-c_1u\}fghdxdydz$$
$$+ \int_0^u\int_v^{\infty}\int_w^{\infty}\{s_1x+s_1\min[u-x,a_{21}(y-v)+a_{31}(z-w)]+q_1\max[0,(u-x)-a_{21}(y-v)$$

$$-a_{31}(z-w)]-c_1u\}fghdxdydz, \qquad (6)$$

where f, g and h are the demand densities of Players 1, 2 and 3, respectively.

If there is no substitution for Player 1's product, i.e., $a_{21} = a_{31} = 0$, then from (6), Player 1's expected profit function turns out to be

$$J_{10} = \int_u^\infty [s_1u-p_1(x-u)-c_1u]fdx+\int_0^u [s_1x+q_1(u-x)-c_1u]fdx$$

$$= (s_1+p_1)\int_0^u xfdx+(s_1+p_1)u\int_u^\infty fdx+q_1\int_0^u (u-x)fdx-p_1E(X)-c_1u, \qquad (7)$$

which is the objective function of the classical newsboy problem. The optimal order quantity for this problem is given in Parlar (1988) and elsewhere (Hillier and Lieberman, 1990, p. 709). In the following, we assume that $a_{ij} > 0$ ($i < j$, $i, j = 1, 2, 3$).

After some simplifications, J_1 becomes

$$J_1 = \int_u^\infty [s_1u-p_1(x-u)]fdx + s_1\int_0^u xfdx + q_1G(v)H(w)\int_0^u (u-x)fdx$$

$$+ s_1G(v)\int_0^u [1-H(A)](u-x)fdx + s_1H(w)\int_0^u [1-G(B)](u-x)fdx$$

$$+ G(v)\int_0^u\int_w^A [(s_1-q_1)a_{31}(z-w)+q_1(u-x)]hfdzdx$$

$$+ H(w)\int_0^u\int_v^B [(s_1-q_1)a_{21}(y-v)+q_1(u-x)]gfdydx + s_1\int_v^\infty\int_w^\infty\int_C^u (u-x)fghdxdydz$$

$$+ \int_v^\infty\int_w^\infty\int_0^C [s_1(u-C)+q_1(C-x)]fghdxdydz - c_1u, \qquad (8)$$

where $A = w+(u-x)/a_{31}$, $B = v+(u-x)/a_{21}$ and $C = u-a_{21}(y-v)-a_{31}(z-w)$.

Similarly we obtain the expected profit of Player 2 and Player 3 as

$$J_2 = \int_v^\infty [s_2v-p_2(y-v)]g(y)dy + s_2\int_0^v ygdy + q_2F(u)H(w)\int_0^v (v-y)gdy$$
$$+ s_2F(u)\int_0^v [1-H(D)](v-y)gdy + s_2H(w)\int_0^v [1-F(E)](v-y)gdy$$
$$+ F(u)\int_0^v\int_w^D [(s_2-q_2)a_{32}(z-w) + q_2(v-y)]hgdzdy$$
$$+ s_2\int_u^\infty\int_w^\infty\int_F^v (v-y)gfhdydxdz + H(w)\int_0^v\int_u^E [(s_2-q_2)a_{12}(x-u)+q_2(v-y)]fgdxdy$$
$$+ \int_u^\infty\int_w^\infty\int_0^F [s_2(v-F)+q_2(F-y)]gfhdydxdz - c_2v, \qquad (9)$$

where $D = w+(v-y)/a_{32}$, $E = u+(v-y)/a_{12}$ and $F = v-a_{12}(x-u)-a_{32}(z-w)$,

and

$$J_3 = \int_w^\infty [s_3w-p_3(z-w)]hdz + s_3\int_0^w zhdz + q_3F(u)G(v)\int_0^w (w-z)hdz$$
$$+ s_3G(v)\int_0^w [1-F(G)](w-z)hdz + s_3F(u)\int_0^w [1-G(H)](w-z)hdz$$
$$+ G(v)\int_0^w\int_u^G [(s_3-q_3)a_{13}(x-u)+q_3(w-z)]fhdxdz$$
$$+ s_3\int_v^\infty\int_u^\infty\int_I^w (w-z)hfgdzdxdy + F(u)\int_0^w\int_v^H [(s_3-q_3)a_{23}(y-v)+q_3(w-z)]ghdydz$$
$$+ \int_v^\infty\int_u^\infty\int_0^I [s_3(w-I)+q_3(I-z)]hgfdzdydx - c_3w, \qquad (10)$$

where $G = u+(w-z)/a_{13}$, $H = v+(w-z)/a_{23}$ and $I = w-a_{13}(x-u)-a_{23}(y-v)$.

If there are more than three players, each player's expected profit function can be obtained similarly. However the expressions would be much more lengthy and the analysis would then be very cumbersome. This is the main reason why we analyze a three-person game instead of an n-person game in this study. Nevertheless, it can be seen in the following analysis, all results in our study can be generalized into the case of n ($n > 3$) players.

4.3. Non-cooperative Solutions

We first consider the problem when the players make decisions independently. There is no communication among the players in this case. It is then important for each of them to use a secure strategy, if possible, to guarantee himself a certain amount of payoff regardless of what the others do. Such a strategy was proposed by Nash (Nash, 1950). Mathematically it is the strategy (u^*, v^*, w^*) such that

$$J_1(u^*, v^*, w^*) \geq J_1(u, v^*, w^*), \tag{11}$$

$$J_2(u^*, v^*, w^*) \geq J_2(u^*, v, w^*), \tag{12}$$

$$J_3(u^*, v^*, w^*) \geq J_3(u^*, v^*, w). \tag{13}$$

This strategy, called Nash Equilibrium or Nash strategy, ensures that Player i gets at least $J_i(u^*, v^*, w^*)$ if he stays on it and he can not get more than this amount if he deviates from it unilaterally.

Lemma 1. If each player's payoff function is continuous in all decision variables and concave in its decision variable, the game has at least one Nash equilibrium which is determined by letting the first partial derivative of each player's payoff function with respect to its own decision variable be zero.

Proof. Player i's strategy space can be stated as $[0, M_i]$ which is compact and convex, where M_i is Player i's inventory capacity. If each player's payoff function is continuous in all decision variables and concave in its own decision variable, the game is convex. Lemma 1 holds (Nikaido and Isora, 1955). **Q.E.D.**

We now prove the following theorem.

Theorem 1. The game admits a Nash equilibrium which is given by

$$\partial J_1/\partial u = 0, \tag{14}$$

$$\partial J_2/\partial v = 0, \tag{15}$$

$$\partial J_3/\partial w = 0. \tag{16}$$

Proof. By differentiating J_1 with respect to u, after some simplifications, we obtain

$$\begin{aligned}\partial J_1/\partial u = {} & (s_1+p_1)[1-F(u)] + q_1F(u)G(v)H(w) + s_1G(v)\int_0^u[1-H(A)]fdx \\ & + q_1G(v)\int_0^u\int_w^A fhdzdx + s_1H(w)\int_0^u[1-G(B)]fdx + q_1H(w)\int_0^u\int_v^B gfdydx \\ & + s_1\int_v^\infty\int_w^\infty\int_C^u fghdxdydz + q_1\int_v^\infty\int_w^\infty\int_0^C fghdxdydz - c_1,\end{aligned} \tag{17}$$

$$\begin{aligned}\partial^2 J_1/\partial u^2 = {} & -[p_1f(u)+(s_1-q_1)G(v)H(w)f(u)+(s_1-q_1)G(v)\int_0^u h(A)fdx/a_{31} \\ & +(s_1-q_1)H(w)\int_0^u g(B)fdx/a_{21}+(s_1-q_1)\int_u^\infty\int_w^\infty f(C)ghdydz] \\ & < 0 \text{ since } s_1 > q_1.\end{aligned} \tag{18}$$

Similarly we have

$$\begin{aligned}\partial J_2/\partial v = {} & (s_2+p_2)[1-G(v)]+q_2G(v)F(u)H(w)+s_2F(u)\int_0^v[1-H(D)]gdy \\ & + q_2F(u)\int_0^v\int_w^D hgdzdy+s_2H(w)\int_0^v[1-F(E)]gdy+q_2H(w)\int_0^v\int_u^E fgdxdy \\ & + s_2\int_u^\infty\int_w^\infty\int_F^v gfhdxdydz+q_2\int_u^\infty\int_w^\infty\int_0^F gfhdydxdz-c_2,\end{aligned} \tag{19}$$

$$\partial^2 J_2/\partial v^2 = -[p_2g(v)+(s_2-q_2)F(u)H(w)g(v)+(s_2-q_2)F(u)\int_0^v h(D)gdy/a_{32}$$

$$+(s_2-q_2)H(w)\int_0^v f(E)gdy/a_{12}+(s_2-q_2)\int_v^\infty\int_w^\infty g(F)fhdxdz]$$

$$< 0 \text{ since } s_2 > q_2, \tag{20}$$

$$\partial J_3/\partial w = (s_3+p_3)[1-H(w)]+q_3H(w)G(v)F(u)+s_3G(v)\int_0^w [1-F(G)]hdz$$
$$+ q_3G(v)\int_0^w\int_u^G hfdxdz+s_3F(u)\int_0^w [1-G(H)]hdz+q_3F(u)\int_0^w\int_v^H ghdydz$$
$$+ s_3\int_u^\infty\int_v^\infty\int_I^w hgfdzdxdy+q_3\int_u^\infty\int_v^\infty\int_0^I hfgdzdxdy - c_2, \tag{21}$$

$$\partial^2J_3/\partial w^2 = -[p_3h(w)+(s_3-q_3)h(w)F(u)G(v)+(s_3-q_3)G(v)\int_0^w f(G)hdz/a_{13}$$
$$+(s_3-q_3)F(u)\int_0^w g(H)hdz/a_{23}+(s_3-q_3)\int_u^\infty\int_v^\infty h(I)fgdxdy]$$

$$< 0 \text{ since } s_3 > q_3. \tag{22}$$

Then J_i ($i = 1, 2, 3$) is concave in Player i's decision variable. Note that J_i also is continuous in u, v and w. Therefore, the game admits at least one Nash equilibrium which is determined by $\partial J_1/\partial u = 0$, $\partial J_2/\partial v = 0$ and $\partial J_3/\partial w = 0$ (Lemma 1) **Q.E.D.**

In the case of two players, Parlar (1988) proved that there exists a unique Nash equilibrium for the game. In the case of three players, although we conjecture that there would be still a unique Nash equilibrium, it is very difficult to prove the uniqueness. Nonetheless it can be seen from the formulation of the game and the proof of Theorem 1 that the existence theorem of Nash equilibrium can be easily generalized to the case of n ($n > 3$) players.

The Nash equilibrium is obtained under the assumption that all the players are rational, i.e., no one will risk lowering his own payoff for the purpose of damaging the others. This is usually true in an actual market. However it would also be interesting to look at the case where some players act irrationally. In this situation what an irrational player can do is only to order infinitely many units (Parlar, 1988). For instance, if Player 3 acts irrationally to damage Player 1 and Player 2, letting $w = \infty$ and hence obtaining $A = \infty$ and $C = \infty$, we have

$$J_1 = \int_u^\infty [s_1 u - p_1(x-u)] f dx + s_1 \int_0^u x f dx + q_1 G(v) \int_0^u (u-x) f dx$$
$$+ s_1 \int_0^u [1-G(B)](u-x) f dx$$
$$+ \int_0^u \int_v^B [(s_1-q_1)a_{21}(y-v)+q_1(u-x)] g f dy dx - c_1 u, \tag{23}$$

$$J_2 = \int_v^\infty [s_2 v - p_2(y-v)] g dy + s_2 \int_0^v y g dy + q_2 F(u) \int_0^v (v-y) g dy$$
$$+ s_2 \int_0^v [1-F(E)](v-y) g dy$$
$$+ \int_0^v \int_u^E [(s_2-q_2)a_{12}(x-u)+q_2(v-y)] f g dx dy - c_2 v. \tag{24}$$

It is easy to verify that (23) and (24) are the objective functions of the two-person game discussed by Parlar (1988). Therefore the optimal (defensive) strategy of Player 1 and Player 2 reduces to that when Player 1 and Player 2 are the only players. In other words, if Player 3 acts irrationally to inflict damage on Player 1 and Player 2, Player 1 and Player 2 can simply ignore Player 3. As shown by Parlar (1988), if Player 2 also acts irrationally to damage Player 1, Player 1's optimal decision is equivalent to that when he is the sole decision-maker in the classical newsboy problem. We may further generalize the result as follows. In an n-person game, if m

($m < n$) players act irrationally to inflict damage on the others, the optimal decisions of the latter reduce to that in the (n-m)-person game without the irrational players.

4.4. Cooperative Solutions

In this section, we discuss cooperation of the players. Cooperative players might (1) switch their excess inventory, if any, to anyone who has excess demand so that the latter can save in lost sales penalty cost, and (2) determine their order quantities together to maximize their joint profit. The latter, however, depends very much on how they can divide their joint profit. If they can compensate each other in any way possible, they will determine their order quantities collectively to obtain maximum joint profit. Otherwise they may determine their order quantities independently. The condition that compensation can be made among cooperative players in any way possible is referred to as "side payment" condition (Owen, 1982). In the situation where side payments are not allowed, each player keeps his own payoff under any decision combination. We consider both cases where side payments are and are not allowed in our analysis.

We will use the following notation in this section.

J_{ij} := the joint expected profit of Player i and Player j when they cooperate;

$J_{i(ij)}$:- Player i's expected profit when Player i and Player j cooperate;

J_{123} :- the joint expected profit of Players 1, 2 and 3 when they cooperate;

$J_{i(123)}$:- Player i's expected profit when all players cooperate;

Consider the case where Player 2 and Player 3 cooperate. For given u, v and w, their joint profit equals to the sum of their profits when they work independently if there are no demand transfers between them. This way they can save lost sales penalty costs if there are demand transfers between them.

There are four possible exclusive cases where demand transfers between Player 2 and Player 3 can take place. The lost sales penalty cost saving in each case is as follows.

(i) $y \le v$, $z \ge w$, $x \le u$.

Player 1 and Player 2 have excess supply and Player 3 has excess demand. The amount of Player 3's excess demand substituted by Player 2's excess supply is either v-y or $a_{32}(z-w)$, whichever is less. Hence the lost sales penalty cost saving is $p_3\min[v-y,\ a_{32}(z-w)]$.

(ii) $y \ge v$, $z \le w$, $x \le u$.

Player 1 and Player 3 have excess supply and Player 2 has excess demand. Similar to (i), the lost sales penalty cost saving is $p_2\min[w-z,\ a_{23}(y-v)]$.

(iii) $y \le v$, $z \ge w$, $x \ge u$.

Player 2 has excess supply and Player 1 and Player 3 have excess demand. As Player 2 will switch his excess inventory to Player

3, substitution between the products of Player 2 and Player 3 will occur before that between the products of Player 2 and Player 1. Then the amount of Player 2's excess supply going to Player 3's product is either v-y or $a_{32}(z-w)$, whichever is less. The lost sales penalty cost saving is $p_3\min[v-y,\ a_{32}(z-w)]$.

(iv) $y \geq v,\ z \leq w,\ x \geq u$.

Player 3 has excess supply and Player 1 and Player 2 have excess demand. Similar to (iii), the lost sales penalty cost saving is $p_2\min[w-z,\ a_{23}(y-v)]$.

Taking the expectations of the lost sales penalty cost savings over the respective regions and noting that $\pi_{23} = \pi_2 + \pi_3$ if lost sales penalty cost is not considered, we obtain

$$\begin{aligned} J_{23} &= J_2 + J_3 + p_2\int_v^\infty\int_0^w \min[w-z,\ a_{23}(y-v)]ghdydz \\ &\quad + p_3\int_0^v\int_w^\infty \min[v-y,\ a_{32}(z-w)]ghdydz \\ &= J_2 + J_3 + p_2[\int_0^w\int_H^\infty (w-z)ghdydz + \int_0^w\int_v^H a_{23}(y-v)ghdydz] \\ &\quad + p_3[\int_0^v\int_D^\infty (v-y)ghdydz + \int_0^v\int_w^D a_{32}(z-w)ghdydz]. \end{aligned} \tag{25}$$

Note that the cooperation of Player 2 and Player 3 has no effect on Player 1's expected profit for given u, v and w. Player 1's expected profit function in this case is J_1.

When no side payments are allowed, the expected profits of Player 2 and Player 3 are

$$J_{2(23)} = J_2 + p_2[\int_0^w\int_H^\infty (w-z)ghdydz + \int_0^w\int_v^H a_{23}(y-v)ghdydz], \tag{26}$$

$$J_{3(23)} = J_3 + p_3[\int_0^v\int_D^\infty (v-y)ghdydz + \int_0^v\int_w^D a_{32}(z-w)ghdydz]. \qquad (27)$$

Similarly we can obtain the expected profit functions for the players when Player 1 and Player 2 cooperate and when Player 1 and Player 3 cooperate.

When all the three players cooperate, they will switch excess inventory to those who have excess demand. Then each player can reduce his lost sales penalty cost. However, a conflict may exist in this case. Consider the situation where Player 1 has 2 units excess supply but each of Player 2 and Player 3 has 10 units excess demand. If $a_{21} = 0.3$ and $a_{31} = 0.4$, Player 2 has 3 units and Player 3 has 4 units of excess demand that can be satisfied by Player 1's excess inventory if such excess inventory is available. Clearly, in this case, such excess inventory is not available. Then the players have to decide how much of Player 1's excess inventory goes to each of Player 2 and Player 3. We assume, if such a conflict exists, the product with a higher lost sales penalty cost has the priority to receive the excess inventory. Without loss of generality, we let $p_1 \geq p_2 \geq p_3$ as we can always arrange the players in such a way. By following similar procedures as above, each player's expected profit function is obtained as follows.

$$J_{1(123)} = J_1 + p_1\int_u^\infty\int_0^v \min[a_{12}(x-u),\ v-y]fgdxdy$$

$$+ p_1\int_u^\infty\int_0^w \min[a_{13}(x-u),\ w-z]fhdxdz$$

$$= J_1 + p_1[a_{12}\int_u^\infty G(S)(x-u)fdx + \int_u^\infty\int_S^v (v-y)fgdxdy]$$

$$+ p_1[a_{13}\int_u^\infty H(K)(x-u)f dx + \int_u^\infty \int_K^w (w-z) fh dx dz], \quad (28)$$

where $S = v - a_{12}(x-u)$ and $K = w - a_{13}(x-u)$.

$$J_{2(123)} = J_2 + p_2\int_0^u \int_v^\infty \min[a_{21}(y-v),\ u-x] fg dx dy$$

$$+ p_2F(u)\int_v^\infty \int_0^w \min[a_{23}(y-v),\ w-z] gh dy dz$$

$$+ p_2\int_u^\infty \int_v^\infty \int_0^w \max(0,\ \min[a_{23}(y-v),\ w-z-a_{13}(x-u)]) fgh dx dy dz$$

$$= J_2+p_2[a_{21}\int_v^\infty F(L)(y-v) g dy+\int_v^\infty \int_L^u (u-x) fg dx dy]$$

$$+p_2F(u)[a_{23}\int_v^\infty H(M)(y-v) g dy+\int_v^\infty \int_M^w (w-z) gh dy dz]$$

$$+p_2[\int_u^\infty \int_v^\infty \int_N^K (K-z) fgh dx dy dz+a_{23}\int_u^\infty \int_v^\infty \int_0^N (y-v) fgh dx dy dz], \quad (29)$$

where $L = u-a_{21}(y-v)$, $M = w-a_{23}(y-v)$ and $N = w-a_{13}(x-u)-a_{23}(y-v)$.

$$J_{3(123)} = J_3+p_3\int_0^u \int_v^\infty \int_w^\infty \max(0,\ \min[a_{31}(z-w),\ u-x-a_{21}(y-v)]) fgh dx dy dz$$

$$+p_3\int_0^u \int_0^v \int_w^\infty (\min[a_{31}(z-w), u-x]+\min[a_{32}(z-w), v-y]) fgh dx dy dz$$

$$+p_3\int_u^\infty \int_0^v \int_w^\infty \max(0,\ \min[a_{32}(z-w),\ v-y-a_{12}(x-u)]) fgh dx dy dz$$

$$= J_3+p_3[\int_v^\infty \int_w^\infty \int_T^L (L-x) fgh dx dy dz+a_{31}\int_v^\infty \int_w^\infty \int_0^T (z-w) fgh dx dy dz]$$

$$+p_3[a_{31}\int_0^v \int_w^\infty \int_0^P (z-w) fgh dx dy dz+\int_0^v \int_w^\infty \int_P^u (u-x) fgh dx dy dz]$$

$$+p_3[a_{32}\int_0^u \int_w^\infty \int_0^Q (z-w) fgh dx dy dz+\int_0^u \int_w^\infty \int_Q^v (v-y) fgh dx dy dz]$$

$$+p_3[\int_u^\infty \int_w^\infty \int_R^S (S-y) fgh dx dy dz+a_{32}\int_u^\infty \int_w^\infty \int_0^R (z-w) fgh dx dy dz], \quad (30)$$

where $T = u-a_{21}(y-v)-a_{31}(z-w)$, $P = u-a_{31}(z-w)$, $Q = v-a_{32}(z-w)$ and $R = v-a_{12}(x-u)-a_{32}(z-w)$.

The joint expected profit function of the players is $J_{123} = J_{1(123)} + J_{2(123)} + J_{3(123)}$.

We first consider the case where side payments are not allowed. In this case, cooperative players are bound only by an agreement that any one who has excess inventory will switch the inventory to others in cooperation whenever they need it. Since no compensation can be made, conflict of interest still exists among the players and each player will determine its order quantity independently. Therefore, in either case where any two players cooperate or all players cooperate, they are still forming a non-cooperative game.

Lemma 2. If two players act against the third one and no side payments are allowed between them, the game admits a Nash equilibrium.

Proof. In the following we prove the lemma in the case where Players 2 and 3 act against Player 1. The other cases are similar.

As shown in Section 4.3, J_1 is concave in u when v and w are given. By differentiating $J_{2(23)}$ with respect to v, given u and w, we get

$$\partial J_{2(23)}/\partial v = \partial J_2/\partial v - p_2 a_{23}\int_0^w\int_v^H ghdydz, \qquad (31)$$

$$\begin{aligned}\partial^2 J_{2(23)}/\partial v^2 &= \partial^2 J_2/\partial v^2 - p_2 a_{23}\int_0^w g(H)hdz + p_2 a_{23} H(w)g(v) \\ &= [\partial^2 J_2/\partial v^2 + p_2 g(v)] - p_2 a_{23}\int_0^w g(H)hdz - p_2[1-a_{23}H)]g(v)] \\ &< 0, \end{aligned} \qquad (32)$$

since $1-a_{23}H(w) \geq 0$ and $\partial^2 J_2/\partial v^2 + p_2 g(v) \leq 0$.

Similarly, by differentiating $J_{3(23)}$ with respect to w, given u and v, we get

$$\partial J_{3(23)}/\partial w = \partial J_3/\partial w - p_3 a_{32}\int_0^v\int_w^D ghdydz, \qquad (33)$$

$$\partial^2 J_{3(23)}/\partial w^2 = \partial^2 J_w/\partial w^2 - p_3 a_{32}\int_0^v h(D)gdy + p_3 a_{32} G(v)h(w) < 0. \qquad (34)$$

Therefore, the game is convex and admits at least one Nash equilibrium (Lemma 1). **Q.E.D.**

When all the players decide to cooperate, if side payments are not allowed, cooperation means only that they will exchange excess inventory. They still have competing objectives and will determine their order quantities independently. Therefore, they are forming a non-cooperative game and Nash strategy should still be used.

Lemma 3. If all the players cooperate and no side payments are allowed, each player has a Nash equilibrium strategy.

Proof. When no side payments are allowed among the players, the expected profit functions (payoff functions) are given by $J_{i(123)}$ (i = 1, 2, 3). By differentiating each player's payoff function with respect to his decision variable, we obtain

$$\partial J_{1(123)}/\partial u = \partial J_1/\partial u - p_1[a_{12}\int_u^\infty G(S)fdx + a_{13}\int_u^\infty H(K)fdx], \qquad (35)$$

$$\partial^2 J_{1(123)}/\partial u^2 = \partial^2 J_1/\partial u^2 - p_1[a_{12}^2\int_u^\infty g(S)fdx + a_{13}^2\int_u^\infty h(K)fdx]$$
$$+ [a_{12}G(v) + a_{13}H(w)]p_1 f(u) < 0, \qquad (36)$$

since $a_{12}G(v) + a_{13}H(w) \le 1$,

$$\partial J_{2(123)}/\partial v = \partial J_2/\partial v - p_2[a_{21}\int_v^\infty F(L)gdy + a_{23}F(u)\int_v^\infty H(M)gdy], \qquad (37)$$

$$\partial^2 J_{2(123)}/\partial v^2 = \partial^2 J_2/\partial v^2 - p_2[a_{21}^2\int_v^\infty f(L)gdy + a_{23}^2\int_u^\infty h(M)gdy]$$
$$+ [a_{21}F(u) + a_{23}F(u)H(w)]p_2 g(v) < 0, \qquad (38)$$

since $a_{21}F(u) + a_{23}F(u)H(w) \le 1$,

$$\partial J_{3(123)}/\partial w = \partial J_3/\partial w - p_3 a_{31}[\int_v^\infty\int_w^\infty F(T)ghdydz + \int_0^v\int_w^\infty F(P)ghdydz]$$
$$- p_3 a_{32}[\int_u^\infty\int_w^\infty G(R)fhdxdz + \int_0^u\int_w^\infty G(Q)fhdydz], \quad (39)$$

$$\partial^2 J_{3(123)}/\partial w^2 = \partial^2 J_3/\partial w^2 - p_3[a_{31}^2\int_v^\infty\int_w^\infty f(T)ghdydz + a_{31}^2\int_0^v\int_w^\infty f(P)ghdydz$$
$$+ a_{32}^2\int_u^\infty\int_w^\infty g(R)fhdxdz + a_{32}^2\int_0^u\int_w^\infty g(Q)fhdxdz]$$
$$- [a_{31}(\int_v^\infty F(L)gdy + \int_0^v F(u)gdy) + a_{32}\int_u^\infty G(J)fdx + \int_0^u G(v)fdx)]p_3 h(w)$$
$$< 0, \qquad (40)$$

since $a_{31}[\int_v^\infty F(L)gdy + \int_0^v F(u)gdy) + a_{32}\int_u^\infty G(S)fdx + \int_0^u G(v)fdx] \le 1$.

Therefore the game is convex and admits at least one Nash equilibrium (Lemma 1). **Q.E.D.**

From Lemma 2 and Lemma 3, optimal strategies always exist for each player in any case of cooperation when side payments are not allowed. We use a superscript "*" to represent "optimality" when side payments are not allowed. For instance, $J^*_{i(ij)}$ is Player i's optimal expected payoff when Player i and Player j work together. The reader should note that the optimal values are obtained under different order quantities u, v and w. We can define the following cooperative game.

Cooperative Game 1:

$V(i) = J^*_i, \qquad i = 1, 2, 3,$

$V(i, j) = J^{*}_{i(ij)} + J^{*}_{j(ij)}, \ i \neq j, \ i, j = 1, 2, 3,$

$V(1, 2, 3) = J^{*}_{1(123)} + J^{*}_{2(123)} + J^{*}_{3(123)}.$

In the case where side payments are allowed, cooperative players will act as one player and order quantities that maximize their joint expected profit. It is usually true that any two players who decide to cooperate will not inform others of their cooperation. The third player will, then, keep his Nash strategy when all the players work independently and the cooperative players will choose the best order sizes to maximize their joint expected profit. We use a superscript "**" to represent "optimality" when side payments are allowed. For instance, $J^{**}_{i(ij)}$ represents the value of $J_{i(ij)}$ under the optimal order quantities when Player i and Player j work together. Note again that optimal values are obtained under different order quantities. We can also define the following cooperative game.

Cooperative Game 2:

$V(i) = J^{*}_{i}, \ i = 1, 2, 3,$

$V(i, j) = J^{**}_{ij}, \ i \neq j, \ i, j = 1, 2, 3,$

$V(1, 2, 3) = J^{**}_{123}.$

If the players are going to cooperate, it is necessary that they can find a solution such that no subset of the players can jointly get more by forming a coalition against the rest. The set of such solutions is called the core, denoted by C(V). Mathematically C(V) is the set of (m_1, m_2, m_3) such that (Owen, 1982)

(a) $\sum_{i \in S} m_i \geq V(S)$ where S is a subset of $N = \{1, 2, 3\}$, (41)

(b) $\sum_{i \in N} m_i = V(N)$. (42)

Let us consider Game 1 first. Since it is defined in the case where side payments are not allowed, Player i gets $J^*_{i(123)}$ if all players cooperate. The following theorem gives the necessary and sufficient condition for the core of the game to be non-empty.

Theorem 2. The core of Game 1 is non-empty if and only if

$$J^*_{i(123)} + J^*_{j(123)} \geq J^*_{i(ij)} + J^*_{j(ij)}, \text{ for all } i \neq j. \tag{43}$$

Proof. Let $m_i = J^*_{i(123)}$. Since, by cooperating, each player will do at least as well as when all the players work independently, hence, $J^*_{i(123)} \geq J^*_i$ or $m_i \geq V(i)$. If (43) holds, $m_i + m_j = J^*_{i(123)} + J^*_{j(123)} \geq J^*_{i(ij)} + J^*_{j(ij)} = V(i, j)$. As $\sum_{i=1}^{3} m_i = J^*_{123} = V(1,2,3)$, (m_1, m_2, m_3) is in the core.

On the other hand, because (m_1, m_2, m_3) is the only possible way to distribute V(1,2,3) among the players, it is the only imputation (Owen, 1982) for the game. Condition (43) is also necessary. **Q.E.D.**

Theorem 2 provides the necessary and sufficient condition for all the players to cooperate in the case where side payments are not allowed.

Theorem 3. The core of Game 3 is non-empty if

$$J^{**}_{123} \geq \sum_{i=1}^{3} \max_j \{J^{**}_{i(ij)}\}. \tag{44}$$

Proof. Letting $m_i = \max_j \{J^{**}_{i(ij)}\} + \delta_i$ where $\sum_{i=1}^{3} \delta_i = J^{**}_{123} - \sum_{i=1}^{3} \max_j \{J^{**}_{i(ij)}\} \geq 0$ and $\delta_i \geq 0$, we have

$$m_i \geq J^{**}_{i(ij)} \geq J^{*}_{i} - V(i), \tag{45}$$

$$m_i + m_j \geq J^{**}_{i(ij)} + J^{**}_{j(ij)} - V(i,\ j), \tag{46}$$

$$m_1 + m_2 + m_3 - J^{**}_{123} - V(1,2,3). \tag{47}$$

Then $(m_1,\ m_2,\ m_3)$ is in the core, which is non-empty. **Q.E.D.**

The reader might compare the full expression of J_{123} with J_{ij}. It is not too difficult to observe that the necessary condition in Theorem 3 is much less restrictive than it appears to be. Then the core of Game 2 is often non-empty. As a matter of fact, even the condition in Theorem 2 is not very restrictive. If it is, we feel that it is mainly due to the assumption that the product with higher lost sales penalty cost has the priority to receive excess inventory. This assumption makes an uneven distribution of the lost sales cost saving among the players. If excess inventory is shared evenly when a conflict exists, the core of Game 1 would be more likely to be non-empty.

4.5. Concluding Remarks

In this chapter, we analyze retailers' best order decisions in a market where each retailer's product has a random demand and can be substituted by others' products. We extend Parlar's two-person game theory model as well as his results for this inventory control problem to cases where more than two players are present. Because the presence of additional players brings about many directions of two-way demand transfers among different products and coalition and cooperation among different players, our analysis also features a study of the players's cooperation decisions.

It is shown in our study that there always exists a Nash equilibrium for the game when the players work independently and act rationally. If anyone works irrationally to damage the others, the decision problem for the rational players reduces to that without the irrational player. On the other hand, we have analyzed the cooperation of the players in both cases where side payments are or are not allowed. If side payments are not allowed, conflict of interest still exists among all players, who will then determine their order quantities independently. Our analysis has shown that secure (Nash) strategies always exist for each player in any case of cooperation. We also consider two cooperative games when side payments are or are not allowed and give conditions for the core of each cooperative game to be non-empty. Our results can be generalized to situations with more than three players.

Chapter Five

Strategic Planning for The Growth of New Repeat Purchasing Products

In the last two decades, the growth of non-repeat purchasing products has been studied extensively, but there has been very little work on the growth of repeat purchasing products. In this chapter, we build a diffusion (differential game) model, considering the customers' replacement purchases in the diffusion process of new products, to study the growth of repeat purchasing products and firms' optimal marketing strategies in the growth process. We show that, for repeat purchasing products, the market will never saturate unless customers are extremely loyal to at least one of the products in the market. Thus advertising and promotional activities are always desirable. The optimal advertising strategies are increasing at the introductory stage and then decreasing or possibly terminating after some time. Especially more advertising should be done at early stages against competitors' advertising campaign. In addition to advertising, we also introduce a firm's effort to maintain high customer satisfaction as a control variable, which we call service. The optimal service strategy is found to be monotonically increasing at the introductory stage and then possibly maintained constant at a certain level. The game is solved analytically for optimal strategies in the case where all the control functions representing the effects of advertising and service are linear in their control variables.

5.1. Introduction

In the last two decades, a number of diffusion models have been developed to study the acceptance level of new products. Using the analogy between new product growth and an epidemic, these models attempt to represent the acceptance level of a new product, in a given potential market, as a mathematical function of the time elapsed since its introduction and other marketing mix variables such as advertising and price. Their purpose has been (a) to depict the successive growth in the number of buyers of a new product and forecast its future demand in terms of the diffusion process in progress, and (b) to depict the relationship of various marketing variables to the diffusion process and analyze firms' optimal decisions with respect to advertising, pricing, etc.

The literature on the diffusion of new products is substantial. However we have found in our investigation that two important factors have not received much attention. First, most diffusion models deal with consumer durable products that each potential consumer buys at most once within the planning horizon (Bass, 1969; Mahajan and Muller, 1979; Kalish, 1983; etc.). Few authors have considered repeat purchasing products, which represent a majority of the products sold in every day life even if the planning horizon is intermediate. Second, the aspect of competition has been long-avoided in diffusion models. Although work on competitive problems has recently begun, the progress on analytical solutions has been slow (Clarke and Dolan, 1984; Dockner and Jorgensen, 1988).

In this chapter we develop a diffusion model to study the growth of repeat purchasing products in a duopoly market. Our concern is the firms' optimal marketing decisions in the diffusion process. In the analysis, we observe that consumers' buying behavior of repeat purchasing products is different from that of infrequently purchased products. This makes our model different from those found in the literature. As competition is introduced into the diffusion process, the model gets complex. This complexity enables us to make many useful observations on the diffusion process but at the same time makes explicit solutions difficult to obtain.

The chapter is structured as follows. In Section 5.2 we provide a brief review of some of the models that are relevant to our study. In Section 5.3 we formulate a game theoretical model for repeat purchasing products in a duopoly. Then in the subsequent two sections, we analyze the model and solve it in the case of linear control functions. Finally we discuss the conclusions and possible extensions to our research.

5.2. The Literature Review

Researchers such as Bass (1969) have found that two groups of new product adopters exist: innovators who adopt a new product independently of others' decisions and imitators who are influenced by others' decisions. It has also been found that the attributes of a product can be differentiated by search attributes and experience attributes (Nelson, 1970; 1974). Search attributes, e.g., style of dresses, can be easily observed and verified and thus effectively transmitted by producer originated advertising activities. However, experience attributes, e.g., taste of food, can only be revealed by using the product and spread by word-of-mouth communications. Thus the conditional probability that an individual who has not adopted a new product by time t, $P(t)$, is mainly influenced by advertising and word-of-mouth communication. $P(t)$ is hypothesized to be linear in the market penetration level as $P(t) = a + bX(t)$, where $X(t)$ is the total number of adopters up to time t and a and b are parameters representing the effects of advertising and word-of-mouth communication, respectively. If each individual makes at most one purchase and the potential market size is N, the total number of adopters up to time t is determined by the differential equation (Bass, 1969)

$$\dot{X}(t) = [N-X(t)][a+bX(t)], \qquad X(0) = X_0.$$

Note that only those who have not adopted the product, namely, N-X(t), can be influenced by additional information. This equation is central to the development of diffusion models in the new product context.

Ozga (1960) is among the earliest to build mathematical diffusion mod 's in the economics and management context. He assumes that information can only be effectively spread by word-of-mouth communication. That is, people are informed of the existence and quality of a new product only through social contacts with those who have already had the information. Therefore the number of adopters up to a specific time t is determined by $\dot{X}(t) = c[1-X(t)/N]X(t)$, where c is called the contact coefficient. When advertising is considered, c is set to be d + e where d is the natural contact coefficient and e represents the effect of advertising. His analysis reveals many important properties of the innovation process, but it ignores the primary effect of advertising to disseminate information about new products (Horsky and Simon, 1983). Meanwhile Stigler (1961) studied firms' advertising decisions by ignoring the word-of-mouth effect (i.e., $b = 0$) and assuming $a = \beta A(t)$ where A(t) is the advertising expenditure at t.

Starting with Bass (1969), diffusion models have included both a and b. Bass used regression analysis to estimate a and b and applied the model in forecasting the future demand of new consumer durable products. The behavioral rationale in his model is found to be consistent with other relevant studies in the social science literature (Norton and Bass, 1987). Since then a number of applications of the model for forecasting purposes have been reported (Nevers, 1972; Dodds, 1973; Bass, 1980). By incorporating other control variables into it, the model has also been widely extended to analyze the relationship between the diffusion of new products and various marketing and economic forces such as advertising, price, learning effect, etc.

Advertising and price have usually been considered to be the major marketing forces firms can use to control the diffusion process. The coefficient of innovation, a, is generally accepted to be a function of advertising. However there is no consensus among researchers about where to place the price parameter. In fact, authors have different views on the impact of price changes. Horsky and Simon (1983) argue that price affects essentially the potential market size which can be determined outside of the scope of a diffusion model. They assume N to be a constant and analyze firms' advertising decisions using a specific function for the coefficient of innovation: $a = \alpha + \beta \ln A(t)$. Their model was extended into the stochastic framework by Monahan (1984). Other models studying advertising alone include Sethi (1973), Kotowitz and Mathewson (1979) and Teng and Thompson (1983).

In contrast to Horsky and Simon (1983), many authors claim that price is essential to the diffusion process. In the last decade, more than a dozen papers have been published on pricing in diffusion processes, considering the diffusion demand and/or the experience curve costs. Robinson and Lakhani (1975) did the initial work in this field. They introduced a price multiplicative into the basic diffusion model such that $\dot{X} = (N-X)(a+bX)e^{-BP}$, where B is a constant and P is the price at time t. They analyzed, considering the learning effect, optimal price paths for rapidly growing products. They seem to have opened the field for further research and their work has been a key stimulus for a number of other papers (Bass, 1980; Dolan and Jeuland, 1981; Spence, 1981; Bass and Bultz, 1982; Jeuland and Dolan, 1982; Kalish, 1983; Clarke and Dolan, 1984; Eliashberg and Jeuland, 1986; Dockner and Jorgensen, 1988).

Instead of treating either advertising or price alone, a few authors have considered them simultaneously in a diffusion model. Thompson and Teng (1984) developed a new product growth model, combining the advertising models developed by Vidale and Wolfe (1957) and Ozga (1960) and incorporating price by using the price multiplicative introduced by Robinson and Lakhani (1975). By assuming linear advertising cost, they derived a set of optimal advertising and pricing rules for a monopoly in an experience curve cost situation. Meanwhile they provided many insightful conjectures based on numerical results for an oligopoly where price is determined by the market leader. Unlike Thompson and Teng (1984), Kalish (1985) ignored the dynamics of cost. He separated the adoption process of a new product into two steps: Awareness and Adoption. Awareness is generated through advertising and word-of-mouth communication. State equation of awareness growth is similar to Bass's model (Bass, 1969), except that Kalish differentiated the effectiveness of word-of-mouth communication by those who are merely aware of the product and those who have already adopted it. Adoption occurs only if one is aware and finds the adjusted price (to risk) acceptable. Thus the total number of potential adopters at time t equals the portion that are aware multiplied by the number of individuals to whom the price is acceptable. The number of actual adopters at t, X, is determined by $\dot{X} = [N(P)I-X]k$, where P is the price, N(P) the total number of potential adopters, I the portion that are aware, and k a constant. By using optimal control techniques, Kalish obtained encouraging results. Particularly, some lights were shed on repeat purchasing products.

In the new product diffusion literature, there are also several game theory models dealing with advertising and price competition. Considering the fact that there are usually several firms competing in the same market, the game theory approach is appropriate for new product growth. However the research in this field is just at its beginning (Dockner and Jorgensen, 1988), and in many cases it is merely an extension of a monopoly model. Teng and

Thompson (1983) developed an advertising model for a monopoly and then extended it into the case of oligopoly such that

$$\dot{x}_i = (\gamma_{i1}+\gamma_{i2}u_i)(1-x)+(\gamma_{i3}+\gamma_{i4}u_i)(1-x)x_i, \quad x_i(0) = x_{i0}, \quad i = 1,\ldots,n,$$

where x_i is the market share of Firm i, u_i the advertising rate, γ_{ij} parameters and $x = \sum_{i=1}^{n} x_i$. Each firm is to maximize its total profit with a linear advertising cost. Due to the difficulties involved in obtaining analytical solutions, they generated a series of computational results under different parameter settings to demonstrate useful properties of optimal control trajectories. In a later paper, Thompson and Teng (1984) extended the model to include price by using the price multiplicative in Robinson and Lakhani (1975) and assuming price is determined by the largest firm. See Dockner and Jorgensen (1988) for a discussion of some other game theory models in this field.

In summary:

(a) Research on new product diffusion has made progress in recent years. However works treating advertising or pricing individually are more common than that treating them simultaneously;

(b) The (differential) game theory approach seems to be an appropriate technique to study new product growth. But it has not been developed adequately yet, mainly due to the difficulties in obtaining analytical solutions; and

(c) Most of the diffusion models deal with merely the introductory phase of new products or consumer durable products. Few models have been developed to study repeat purchasing products.

5.3. The Model

In this chapter we study the growth of repeat purchasing products in a duopoly market. We start with a general model with a few assumptions. This allows us to observe some general competitive properties of a duopoly market. In the subsequent developments, we narrow down our analysis to more specific cases in order to obtain stronger results.

We consider a duopoly where each of two firms, Firm 1 and Firm 2, produces a new product (brand) using similar technologies. Their products, which are sold to a common market, are perfectly substitutable with each other and frequently purchased by consumers. The average consumption rate of each consumer is one unit for each time unit. In the literature, to our knowledge, the only model explicitly studying repeat purchasing products is Kalish (1985), in which it is assumed that customers are extremely loyal to the brand.

In the marketing situations of an infrequently purchased product, it is typically assumed that each potential customer buys the product at most once. Therefore total sales equals the cumulative number of buyers. The concern of a firm at any time, if its objective is to maximize total sales, is only those who have not purchased its product up to that time. Advertising and price are often considered to be the major marketing forces that a firm can use to influence consumers' purchasing decisions.

In the marketing situations of a frequently purchased product, however, sales is composed of both first-time purchases and replacement purchases. While new buyers are attracted, its current customers may leave the market. Thus a firm's concern at any time should be not only those who are not buying its product but also those who are currently buying its product. In addition to advertising and price, its effort to maintain high customer satisfaction, which may include sales related services, improving quality, etc., would be its another key strategic consideration. We call this service for simplicity of exposition and will treat it as a control variable in our analysis.

5.3.1. Assumptions

The following assumptions characterize our model.

(1) Price as well as average production cost is constant over time and is determined exogenously. As shown by Thompson and Teng (1984), a monopolist can adopt an optimal constant pricing rule without loss of its profit if the price elasticity of demand is large. The result might also hold for duopolists or oligopolists. Meanwhile, when the technology is not new, cost saving by learning is usually not significant.

(2) The conditional probability that a customer who is not buying any of the products will buy a firm's product at time t equals to f(u) + bx, where u is the firm's advertising spending at t, f(u) a function of u, x the firm's market share at t, b a constant, and $0 \leq f(u) + bx \leq 1$. This formula was proposed by Bass (1969) for consumer durable products and used by many other authors (Teng and Thompson, 1984; Wilson and Norton, 1989). A behavioral rationale can be found for each component. Namely, f(u) represents the effect of advertising, usually with $df/du \geq 0$ and $d^2f/du^2 \leq 0$ (diminishing return), bx represents the effect of word-of-mouth communication, and b can be called either the coefficient of imitation (Bass, 1969) or the contact coefficient (Ozga, 1960). This assumption has been tested theoretically and empirically to be consistent with consumers' actual purchasing behavior. The reader should note, however, that in Bass's model, the target market is the market portion that has not adopted the new product. We will use it for the market portion that are currently not buying, which may include both those who have never adopted the products and those who have adopted (either one or both) but not currently buying.

(3) The conditional probability that a customer who is currently buying a product will repurchase it (repurchasing rate), r, is a function of the firm's current service expenditure and the quality level of its product. As pointed out by Parfitt and Collins (1968), it is comparatively easy to influence first-time buyers, but it is extremely difficult to influence repeat purchasing. However we

observe that service does make a difference, within limits, with respect to repurchasing rate, especially when it means quality improvements. Therefore we take the repeat purchasing rate as a function of service and quality. As quality is usually determined at the product development stage, we treat it as an important parameter for firms to refer to when making their marketing strategies rather than a decision variable in our model.

5.3.2. Notations

To define our model we use the following notations. As we will use a continuous approximation in our analysis, all the variables and functions are defined to be continuous and differentiable, except possibly at a finite number of points. For $i = 1, 2$,

$u_i(t)$ = Firm i's advertising expenditure at time t, $0 \le u_i(t) \le U_i$;

$v_i(t)$ = Firm i's service expenditure at time t, $0 \le v_i(t) \le V_i$;

$x_i(t)$ = Firm i's market share at time t and $x(t) = \Sigma x_i(t)$;

$f_i(u_i)+bx_i$ = the conditional probability that a customer who is buying neither firm's product will buy Firm i's product at time t, $df_i/du_i \ge 0$ and $d^2f_i/du_i^2 \le 0$;

$r_i(v_i)$ = the repurchasing rate of Firm i's product or the probability that a customer who is buying Firm i's product at t will buy the product at $t+\Delta t$, $dr_i/dv_i \ge 0$ and $d^2r_i/dv_i^2 \le 0$;

m_i = the profit margin of Firm i's product;

g_i = Firm i's valuation of its market share at the end of the planning horizon;

N - the average size of the potential market;

T - the terminating time of the planning horizon, i.e., the planning horizon is [0, T];

α - the rate of customers who stop buying Firm 2's product and switch to Firm 1's product;

β - the rate of customers who stop buying Firm 1's product and switch to Firm 2's product;

To improve clarity of expression, we drop the time argument as well as other functional arguments whenever there is no confusion. In the following discussion we will also use $\dot{X}$ to denote the derivative of X with respect to t, Y' the first derivative and Y'' the second derivative of Y with respect to its control variable. Note that all the control functions defined above have only one control variable. This notation will not cause any confusion.

5.3.3. The Market Demand

In the case of a consumer durable, the market operates in a simple way that its market share increases as new buyers are attracted. For a repeat purchasing product, however, its market share increases as new buyers are attracted and decreases as current buyers leave the market. When competitors are present, a product's market share also increases as buyers switch from competitors' products. Consider Firm 1 in [t, t+Δt]. During the period, $(1-x)(f_1+bx_1)\Delta t$ can be attracted from the market portion, 1-x, that is buying neither firm's product at t, $(1-r_1)x_1\Delta t$ will stop buying its product, and $\alpha(1-r_2)x_2\Delta t$ will switch to its product from the rival's (Firm 2's) product. Therefore Firm 1's market share at t+Δt is

$$x_1(t+\Delta t) = x_1 + (1-\dot{x})(f_1+bx_1)\Delta t - (1-r_1)x_1\Delta t + \alpha(1-r_2)x_2\Delta t. \quad (1)$$

Its market share variation in [t, t+Δt], Δx_1, can be expressed as $\Delta x_1 = x_1(t+\Delta t) - x_1(t)$, or

$$\underbrace{\Delta x_1 = (1-x)(f_1+bx_1)\Delta t}_{\text{Market Share Gains by attracting new Buyers}} - \underbrace{(1-r_1)x_1\Delta t}_{\text{Market Share Loss due to non-repurchase}} + \underbrace{\alpha(1-r_2)x_2\Delta t}_{\text{Market Share Gains from the Competitor}}. (2)$$

Dividing both sides of (2) by Δt and letting Δt → 0, Firm 1's market share is determined by

$$\dot{x}_1 = (1-x)(f_1+bx_1) - (1-r_1)x_1 + \alpha(1-r_2)x_2, \quad (3)$$

$$x_1(0) = x_{10}, \quad (4)$$

where x_{10} is Firm 1's market share at t = 0.

Similarly Firm 2's market share is determined by

$$\dot{x}_2 = (1-x)(f_2+bx_2) + \beta(1-r_1)x_1 - (1-r_2)x_2, \quad (5)$$

$$x_2(0) = x_{20}, \quad (6)$$

where x_{20} is Firm 2's market share at t = 0.

When all the parameters are given and all the decision variables are determined, the firms' market shares are fully determined by (3) and (5) with the initial conditions (4) and (6).

5.3.4. Profit Maximization

Each firm is to maximize its accumulated profit across the planning horizon. Meanwhile, the ending market share is also important, e.g., to maintain market leader. Since, for Firm 1, its demand at time t is Nx_1, its instantaneous profit is $Nm_1x_1-u_1-v_1$. Therefore its total gain from the market over [0, T] can be expressed as

$$\pi_1 = \text{(Cumulative Profit)} + \text{(Valuation of Ending Market Share)},$$

$$= \int_0^T (Nm_1x_1-u_1-v_1)dt + g_1x_1(T)$$

$$= \int_0^T (Nm_1x_1-u_1-v_1)dt + g_1\int_0^T \dot{x}_1(t)dt + g_1x_{10}$$

$$= \int_0^T \{Nm_1x_1-u_1-v_1+g_1[(1-x)(f_1+bx_1)-(1-r_1)x_1+\alpha(1-r_2)x_2]\}dt + g_1x_{10}. \quad (7)$$

We use (3) to obtain (7).

Firm 2's total gain from the market over [0, T] can be similarly obtained as

$$\pi_2 = \int_0^T \{Nm_2x_2-u_2-v_2+g_2[(1-x)(f_2+bx_2)+\beta(1-r_1)x_1-(1-r_2)x_2]\}dt + g_2x_{20}. \quad (8)$$

Our model is generally consistent with many previous modeling efforts in the diffusion process. When the market is near saturation, no advertising should be done since its contribution to market share increase is negligibly small. This can be seen by taking $(1-x) \to 0$ in (3) and (5). Furthermore, diminishing return of advertising in time is incorporated into the model in a way that its contribution to market share increase shrinks as market penetration level increases.

It is also possible, with little modification, to extend our model to include price as a control variable. When price is considered, it affects essentially both the conditional probability that a customer will buy a product given that he is not currently buying and the conditional probability that a customer will repurchase a product given that he is currently buying. By letting the control functions (f_i and r_i, $i = 1, 2$) be also functions of price, we can incorporate price into our model. But this will dramatically increase the difficulties of analysis. We leave it as a possible future research topic.

As the reader might have noted, we implicitly assume in the formulation that customers who leave the market simply join those who are not currently buying the products and can be influenced by advertising and word-of-mouth communication in the same way as those who have never bought the products. Clearly our model simplifies the reality since customers who leave the market, especially because of unsatisfaction, are usually more difficult to be influenced by these factors. However, when we use the model for the introductory phase of new products, especially when α and β are close to 1, the market portion that leave the market is small and our model can still well represent the reality.

5.4. Policy Implications

For non-repeat purchasing products, researchers have suggested various optimal advertising and pricing policies for the introduction of new products, using different diffusion models (Horsky and Simon, 1983; Monahan, 1984; Robinson and Lakhani, 1975; Dolan and Jeuland, 1981; Thompson and Teng, 1984; Kalish, 1985; etc.). In particular, it has been demonstrated that the optimal advertising policy is to advertise heavily at early stages of the introduction and to reduce the level of advertising as sales increases (Horsky and Simon, 1983). Optimal pricing policies are more dependent on the products and other market conditions (Kalish, 1985).

For repeat purchasing products, there has not been much research done. In the following, we will derive a series of optimal marketing policies, using the model developed above, for the introduction of new repeat purchasing products. We consider only open-loop strategies (strategies that are functions of time alone) in our analysis for mathematical tractability.

Using the payoff functions (7) and (8), we obtain the Hamiltonians for the game as (Starr and Ho, 1969)

$$H_1 = Nm_1x_1 - u_1 - v_1 + (g_1+\lambda_1)[(1-x)(f_1+bx_1)-(1-r_1)x_1+\alpha(1-r_2)x_2] + \lambda_2[(1-x)(f_2+bx_2)+\beta(1-r_1)x_1-(1-r_2)x_2], \quad (9)$$

$$H_2 = Nm_2x_2 - u_2 - v_2 + \psi_1[(1-x)(f_1+bx_1)-(1-r_1)x_1+\alpha(1-r_2)x_2] + (g_2+\psi_2)[(1-x)(f_2+bx_2)+\beta(1-r_1)x_1-(1-r_2)x_2], \quad (10)$$

where λ_i and ψ_i are auxiliary variables.

The optimal (Nash Equilibrium) open-loop trajectories $u_i(t)$ and $v_i(t)$ maximize H_i ($i = 1, 2$) and satisfy Conditions (3) - (6) and the following conditions (Basar and Olsder, 1982, Chapter 6)

$$\dot{\lambda}_i = - \partial H_1/\partial x_i \ (i = 1, 2), \tag{11}$$

$$\dot{\psi}_i = - \partial H_2/\partial x_i \ (i = 1, 2), \tag{12}$$

$$\lambda_i(T) = \psi_i(T) = 0 \ (i = 1, 2). \tag{13}$$

λ_i and ψ_i are the net benefit of having the constraints, (3) and (5), respectively, relaxed by one unit and can be called as shadow prices. An economic interpretation of them in the context of strategic pricing can be found in Simon (1982).

We now prove the following theorem.

Theorem 1. If $f_i'' \neq 0$ and $r_i'' \neq 0$, $u_i(t)$ and $v_i(t)$ are optimal only if

$$\text{(a)}\ \dot{u}_1 f_1''/(z f_1'^2) = Nm_1 + b/f_1' + [f_2 + bx_2 - \beta(1-r_1)]/(\beta x_1 r_1') - (1-\beta)A_1 - (1-\alpha)B_1, \tag{14a}$$

$$\dot{u}_2 f_2''/(z f_2'^2) = Nm_2 + b/f_2' + [f_1 + bx_1 - \alpha(1-r_2)]/(\alpha x_2 r_2') - (1-\alpha)A_2 - (1-\beta)B_2, \tag{14b}$$

$$\text{(b)}\ \dot{v}_1 r_1''/r_1'^2 = Nm_1 x_1 - [z f_1 + (1-r_2)(x_1 + \alpha x_2)]/(x_1 r_1') - (1-\beta)C_1 + (1-\alpha\beta)D_1, \tag{15a}$$

$$\dot{v}_2 r_2''/r_2'^2 = Nm_2 x_2 - [z f_2 + (1-r_1)(\beta x_1 + x_2)]/(x_2 r_2') - (1-\alpha)C_2 + (1-\alpha\beta)D_2, \tag{15b}$$

where $z = 1-x$, $A_i = [(f_j + bx_j)z + (1-r_i)x_i]/(z^2 f_i')$, $B_i = (1-r_j)x_j/(z^2 f_i')$, $D_i = (1-r_j)x_i/(z f_i')$, $C_1 = (f_2 + bx_2)/(\beta r_1') + [f_1 + bx_1 + (f_2 + bx_2)/\beta]x_1/(z f_1')$, and $C_2 = (f_1+bx_1)/(\alpha r_2')+[f_2+bx_2+(f_1+bx_1)/\alpha]x_2/(zf_2')$, $i, j = 1, 2$ and $i \neq j$.

Proof. See Appendix I. Q.E.D.

Note that if $\alpha = \beta = 1$, (14) and (15) become

(a) $\dot{u}_1 f_1''/(zf_1'^2) = Nm_1+b/f_1'+[f_2+bx_2-(1-r_1)]/(x_1r_1')$, (14a*)

$\dot{u}_2 f_2''/(zf_2'^2) = Nm_2+b/f_2'+[f_1+bx_1-(1-r_2)]/(x_2r_2')$, (14b*)

(b) $\dot{v}_1 r_1''/r_1'^2 = Nm_1x_1-[zf_1+(1-r_2)x]/(x_1r_1')$, (15a*)

$\dot{v}_2 r_2''/r_2'^2 = Nm_2x_2-[zf_2+(1-r_1)x]/(x_2r_2')$. (15b*)

Theorem 1 gives necessary conditions for optimal advertising and service strategies. When all the functions are known and all the parameters are given, optimal trajectories can be obtained by solving these differential equations and Equations (3) - (6) simultaneously. However, it is usually not easy to do so. In the following we will use these conditions to derive optimal advertising and service policies.

5.4.1. Market Perspective

When two products are launched into the same market, they compete for market share. In the competition, optimal marketing strategies usually determine a product's temporary success or failure. It is a product's quality level that determines its long-term performance. We assume that advertising and service will be maintained at a certain level, i.e., f_i and r_i are constant, after some time. Then we have the following proposition.

Proposition 1. (a) if α, $\beta \neq 1$, $x = \sum_{i=1}^{2} x_i$ will never approach 1 unless $r_1 = 1$ or/and $r_2 = 1$;

(b) if $r_i = 1$ and $r_j < 1$, $x_i \to 1$ and $x_j \to 0$ ($i,j = 1, 2$; $i \neq j$);

(c) if $\alpha = 1$ and $\beta = 1$, $x_1 \to (1-r_2)/(2-r_1-r_2)$ and $x_2 \to (1-r_1)/(2-r_1-r_2)$;

Proof. Note that if the limits of x_1 and x_2 ($t \to \infty$) exist, $\dot{x}_1 \to 0$ and $\dot{x}_2 \to 0$.

(a) Assume that the limit of x_i, denoted by x_i^0 ($i = 1, 2$), exists and $x_1^0+x_2^0 = x^0 = 1$. Taking the limit on both sides of (3) and (5), we obtain

$$-(1-r_1)x_1^0 + \alpha(1-r_2)x_2^0 = 0, \tag{16}$$

$$\beta(1-r_1)x_1^0 - (1-r_2)x_2^0 = 0. \tag{17}$$

Multiplying (16) by β and adding it to (17), we get

$$-(1-\alpha\beta)(1-r_2)x_2^0 = 0. \tag{18}$$

Similarly we have

$$-(1-\alpha\beta)(1-r_1)x_1^0 = 0. \tag{19}$$

Since α, $\beta \neq 1$, if $r_i < 1$ ($i = 1, 2$), $x_1^0 = x_2^0 = 0$. This contradicts the assumption. Hence we have either the limit of x_i does not exist or $x_1^0 + x_2^0 = x^0 \neq 1$.

(b) For $i = 1$ and $j = 2$, substituting $r_1 = 1$ into (3) and (5) and taking the limit on both sides of each equation, we have

$$(1-x^0)(f_1+bx_1^0)+\alpha(1-r_2)x_2^0 = 0, \tag{20}$$

$$(1-x^0)(f_2+bx_2^0) - (1-r_2)x_2^0 = 0. \tag{21}$$

Multiplying (21) by α and adding it to (20), we get

$$(1-x^0)[f_1+bx_1^0+\alpha(f_2+bx_2^0)] = 0. \tag{22}$$

Hence $1-x^0 = 0$ since $f_1+bx_1^0+\alpha(f_2+bx_2^0) > 0$. Substituting $1-x^0 = 0$ into (21), we have $-(1-r_2)x_2^0 = 0$ or $x_2^0 = 0$ since $1-r_2 > 0$. Then $x_1^0 = 1$ or $x_1 \to 1$ and $x_2 \to 0$.

Analogously we have $x_2 \to 1$ and $x_1 \to 0$ if $r_1 < 1$ and $r_2 = 1$.

(c) Adding (3) to (5) and taking the limit on the equation, we have

$$(1-x^0)(f_1+bx_1^0+f_2+bx_2^0) = 0. \tag{23}$$

Then $1-x^0 = 0$ since $f_1+bx_1^0+f_2+bx_2^0 > 0$ and (16) and (17) hold with $\alpha = \beta = 1$. Solving the equations we obtain $x_1^0 = (1-r_2)/(2-r_1-r_2)$ and $x_2^0 = (1-r_1)/(2-r_1-r_2)$. Q.E.D.

In a regular market, α, $\beta \neq 1$ and the quality level of a product is usually not so high as to keep all its customers (to repurchase it). Our proposition suggests that such a market will never saturate. Therefore, there is always a portion of the potential market not buying any product. Advertising and promotional activities are always desirable. This provides an explanation of the common phenomenon that advertising and promotional activities are always done, no matter how long a product has been in the market.

If the quality level of a firm's product is exceptionally high and the competitor's is not, the firm will eventually dominate the market. In more general situations, each firm shares part of the market and the actual market share achieved by each firm is determined by the quality level of its product as well as its advertising and service strategies.

In the rest of this section we assume that $\alpha = \beta = 1$ and $x_{10} = x_{20} = 0$.

5.4.2. Advertising

Using the necessary conditions for optimal advertising and service strategies given in Theorem 1, we obtain the following propositions. Because the firms are symmetric with respect to policy implications, we state the results in terms of Firm 1. All findings are equally applicable to Firm 2.

Proposition 2. If the quality level of its product is extremely high, i.e., $r_1 = 1$, Firm 1's optimal advertising strategy is monotonically decreasing.

Proof. If $r_1 = 1$, the RHS of $(14a^*) = Nm_1 + b/f_1' + (f_2+bx_2)/(x_1r_1') > 0$. Then $\dot{u}_1 < 0$ since $f_1'' < 0$. Q.E.D.

Proposition 2 suggests that optimal advertising strategies in the situation assumed should be strictly decreasing over time, regardless of what the competitor does. If customers are extremely loyal to the brand, what is important to a firm at any time is only those who have not adopted its product up to that time, as in the situation of non-repeat purchasing products. It is important to realize, however, that there is a time preference of the adoption of a repeat purchasing product. In addition to the word-of-mouth effect which exists for both repeat and non-repeat purchasing products, each adoption of a repeat purchasing product may generate a stream of sales starting from the time of adoption. If the competitor's product is not as good as its product, as stated in Proposition 1, the firm will eventually dominate the market. Advertising in this case is only to speed up the adoption process. If the competitor's product is as good as its product, advertising is both to speed up the adoption process and to attract customers from adopting the other product. Since advertising is more efficient when the market penetration level is low, more advertising should be done at early stages. The proposition manifests itself by relating advertising with its efficiency.

The results in Proposition 2 replicate what has been found by Kalish (1985) in a monopoly. However the reader might note that we are studying a duopoly market instead of a monopoly. This makes the result more representative of an actual market.

Proposition 3. If $1-r_1 > f_2$, Firm 1's optimal advertising strategy is increasing at the introductory stage (x_1 is very small) and decreasing afterwards (possibly terminating after a certain period.).

Proof. If $1-r_1 > f_2$, $[f_2+bx_2-(1-r_1)]|_{t=0} = [f_2-(1-r_1)]|_{t=0} < 0$. Since $f_2+bx_2-(1-r_1)$ is continuous in t, when t is near zero, it is negative. If x_1 is very small, $Nm_1+b/f_1'+[f_2+bx_2-(1-r_1)]/(x_1r_1') < 0$. Then, by ($14a^*$), $\dot{u}_1 > 0$ since $f_1'' < 0$.

On the other hand, as x_1 increases (x_2 also increases), $Nm_1+b/f_1' +[f_2+bx_2-(1-r_1)]/(x_1r_1')$ increases and could be positive after a certain time. Then $\dot{u}_1 < 0$ since $f_1'' < 0$. Q.E.D.

Bass's studies indicate that the parameter a for consumer durables is typically a few hundredths (Bass, 1969). Although no similar estimates have been found for repeat purchasing products, the magnitude of the parameter should be also around that area. Meanwhile, Parfitt and Collins's study shows a less than fifty percent repeat purchasing rate (Parfitt and Collins, 1968). Therefore $1-r_1$ is usually greater than f_2, regardless of what advertising and service strategies a firm uses. The condition of Proposition 3 usually holds. A firm's optimal advertising strategy is to use increasing advertising at the beginning of the introduction of a new product tc convince its potential customers to buy its product and decrease its advertising expenditure gradually when its market share reaches a certain level.

The situation in the above proposition represents a regular product. It also includes the extreme that r_1 is very small, e.g., because the quality level of the product is low. In this case, customers may either leave the market or switch to the competitor's product after the first purchase. The best a firm can do is to use increasing advertising to convince those who have not adopted its product to buy it. After this it may have to either maintain a low market share or quit from the market.

Proposition 4. If a firm anticipates an advertising campaign from its competitor, its best counter-strategy is to launch the campaign before the competitor starts it (to spend more before the competitor begins the campaign).

Proof. When the competitor launches an advertising campaign, f_2 is higher (than no campaign). The RHS of ($14a^*$) is higher if it is positive and lower if it is negative. If the other conditions (functions and quantities) are unchanged, this implies that $|\dot{u}_1|$ is higher if $\dot{u}_1 < 0$ and lower if $\dot{u}_1 > 0$. Therefore the advertising spending curve becomes steeper when it is decreasing and flatter when it is increasing. If the budget of advertising is constant, both the cases imply more spending at early stages. Q.E.D.

5.4.3. Service

Proposition 5. A firm's optimal service strategy is monotonically increasing at the initial stage (then possibly maintaining at a certain level).

Proof. It can be seen from ($15a^*$) that its RHS < 0 when x_1 is sufficiently small, which implies $\dot{v}_1 > 0$ since $r_1'' < 0$. Q.E.D.

The proposition suggests that the optimal service strategy is to increase its service expenditure at early stages as its market share increases.

5.5. Optimal Strategies

As pointed out by many authors (Kalish, 1985), word-of-mouth communication is not important for repeat purchasing products, especially those that are relatively inexpensive. This is because customers can afford to sample themselves. Therefore we can drop b in our model and the state equations (3) and (5) become

$$\begin{aligned}\dot{x}_1 &= (1-x)f_1 - (1-r_1)x_1 + \alpha(1-r_2)x_2, \\ &= f_1 - (f_1+1-r_1)x_1 - [f_1-\alpha(1-r_2)]x_2, \qquad (24)\end{aligned}$$

$$\begin{aligned}\dot{x}_2 &= (1-x)f_2 + \beta(1-r_1)x_1 - (1-r_2)x_2, \\ &= f_2 - [f_2-\beta(1-r_1)]x_1 - (f_2+1-r_2)x_2. \qquad (25)\end{aligned}$$

The payoff functions are still given in (7) and (8) and the initial conditions are (4) and (6).

Note that, after dropping b, the state equation system becomes a linear differential equation system of x_1 and x_2. If it can be solved for x_1 and x_2, the firms' market shares can be expressed explicitly in terms of their advertising and service strategies. Because it is difficult to obtain general solutions of the system, we consider a special case in the following by assuming that $\alpha = \beta = 1$.

Solving the system (see Appendix II), we obtain

$$x_1(t) = L(t)\{\int_0^t [cf_1W(\tau)+(1-r_2)(1-cW(\tau))]L^{-1}(\tau)d\tau+x_{10}\}, \qquad (26)$$

$$x_2(t) = L(t)\{\int_0^t [cf_2W(\tau)+(1-r_1)(1-cW(\tau))]L^{-1}(\tau)d\tau+x_{20}\}, \qquad (27)$$

where $c = 1-x_{10}-x_{20}$, $W(\tau) = \mathrm{EXP}[-\int_0^{\tau}(f_1+f_2)d\eta]$, and $L(\tau) = \mathrm{EXP}[-\int_0^{\tau}(2-r_1-r_2)d\eta]$. When the starting time is t_0, $x_i(t)$ can be simply obtained by changing the lower limit for all integrations from zero to t_0. If f_i and r_i $(i = 1,2)$ are constant and $(x_{10}, x_{20}) = (0, 0)$,

$$x_1(t) = -\mu_1(e^{-\theta_1 t}-e^{-\theta_2 t})/\delta + (1-r_2)(1-e^{-\theta_2 t})/\theta_2, \tag{28}$$

$$x_2(t) = -\mu_2(e^{-\theta_1 t}-e^{-\theta_2 t})/\delta + (1-r_1)(1-e^{-\theta_2 t})/\theta_2, \tag{29}$$

where $\mu_1 = 1-r_2-f_1$, $\mu_2 = 1-r_1-f_2$, $\theta_1 = f_1+f_2$, $\theta_2 = 2-r_1-r_2$ and $\delta = 2-r_1-r_2-f_1-f_2$.

In the following we solve the game for optimal strategies in the case where all the functions f_i and r_i $(i = 1, 2)$ are linear in their control variables and each firm has a budgetary constraint for both advertising and service spending. We assume the existence of solutions. Then there exist four continuous auxiliary variables and the optimal (Nash) strategies maximize the corresponding Hamiltonian at every instant (Basar and Olsder, 1982, Chapter 6).

The Hamiltonians in this case are

$$\begin{aligned} H_1 = Nm_1x_1-u_1-v_1 &+ (g_1+\lambda_1)[(1-x)f_1-(1-r_1)x_1+(1-r_2)x_2] \\ &+ \lambda_2[(1-x)f_2+(1-r_1)x_1-(1-r_2)x_2], \end{aligned} \tag{30}$$

$$\begin{aligned} H_2 = Nm_2x_2-u_2-v_2 &+ \psi_1[(1-x)f_1-(1-r_1)x_1+(1-r_2)x_2] \\ &+ (g_2+\psi_2)[(1-x)f_2+(1-r_1)x_1-(1-r_2)x_2], \end{aligned} \tag{31}$$

where λ_i and ψ_i $(i = 1, 2)$ are auxiliary variables.

The necessary conditions for the optimal (Nash Equilibrium) open-loop trajectories $u_i(t)$ and $v_i(t)$ $(i = 1, 2)$, after some simplifications, become

$$\dot{\lambda}_1 = -Nm_1+(g_1+\lambda_1)(f_1+1-r_1)+\lambda_2[f_2-(1-r_1)], \tag{32}$$

$$\dot{\lambda}_2 = (g_1+\lambda_1)[f_1-(1-r_2)]+\lambda_2(f_2+1-r_2), \tag{33}$$

$$\dot{\psi}_1 = \psi_1(f_1+1-r_1)+(g_2+\psi_2)[f_2-(1-r_1)], \tag{34}$$

$$\dot{\psi}_2 = -Nm_2+\psi_1[f_1-(1-r_2)]+(g_2+\psi_2)(f_2+1-r_2), \tag{35}$$

$$\lambda_i(T) = 0 \ (i = 1, 2), \tag{36}$$

$$\psi_i(T) = 0 \ (i = 1, 2). \tag{37}$$

The partial derivatives of the Hamiltonians are

$$\partial H_1/\partial u_1 = -1+(1-x)(g_1+\lambda_1)f_1', \tag{38}$$

$$\partial H_1/\partial v_1 = -1+r_1'x_1(g_1+\lambda_1-\lambda_2), \tag{39}$$

$$\partial H_2/\partial u_2 = -1+(1-x)(g_2+\psi_2)f_2', \tag{40}$$

$$\partial H_2/\partial v_2 = -1+r_2'x_2(g_2+\psi_2-\psi_1). \tag{41}$$

Lemma 1. If the control function is linear in the control variable, the optimal strategy with respect to the corresponding variable is a piecewise function admitting either the budgetary constraint or zero with at most a finite number of steps.

Proof. We prove the lemma in the case where f_1 is a linear function of u_1. The other cases are similar.

If f_1 is linear in u_1, $\partial H_1/\partial u_1 = -1+(1-x)(g_1+\lambda_1)f_1'$ is independent of u_1. To maximize H_1, the optimal u_1 is determined by

$$u_1 = \begin{cases} U_1 & \text{if } \partial H_1/\partial u_1 > 0, \\ 0 & \text{if } \partial H_1/\partial u_i \le 0, \end{cases}$$

where U_1 is Firm 1's budgetary constraint for advertising. Since λ_1 and (1-x) are both continuous and non-constant, $\partial H_1/\partial u_1$ is continuous and non-constant. It can change signs for at most a finite number of times in [0, T]. Therefore the optimal u_1 has at most a finite number of steps. Q.E.D.

Lemma 2. If f_i and r_i (i = 1, 2) are constant,

(a) $\partial H_i/\partial u_i$ is strictly decreasing over time if $g_i \le Nm_i[(\mu_i/\mu_j+\theta_1/\theta_2)/\delta+1/\theta_2]$ where i, j = 1, 2 and i ≠ j; and

(b) $\partial H_i/\partial v_i$ is strictly increasing over time if $g_i > Nm_i/\theta_2$ and $\dot{x}_i \ge 0$.

Proof. We prove the lemma for i = 1.

When f_i and r_i (i = 1, 2) are constant, λ_1 and λ_2 are given by (See Appendix III)

$$\lambda_1 = -g_1+Nm_1\mu_1/(\theta_1\delta)+Nm_1\mu_2/(\theta_2\delta)+c_1e^{\theta_1 t}-c_2\mu_2e^{\theta_2 t}, \quad (42)$$

$$\lambda_2 = Nm_1\mu_1/(\theta_1\delta)-Nm_1\mu_1/(\theta_2\delta)+c_1e^{\theta_1 t}+c_2\mu_1e^{\theta_2 t}, \quad (43)$$

where $c_1 = [(g_1-Nm_1/\theta_1)\mu_1/\delta]e^{-\theta_1 T}$ and $c_2 = [(Nm_1/\theta_2-g_1)/\delta]e^{-\theta_2 T}$.

(a) Differentiating $\partial H_1/\partial u_1$ with respect to t, we obtain

$$\begin{aligned} d(\partial H_1/\partial u_1)/dt &= d[-1+(1-x)(g_1+\psi_1)f_1']/dt \\ &= -\dot{x}(g_1+\lambda_1)f_1'+(1-x)f_1'\dot{\lambda}_1 \\ &= -(1-x)f_1'[Nm_1(\mu_1+\mu_2\theta_1/\theta_2)/\delta + c_2\mu_2\delta e^{\theta_2 t}] \\ &< 0 \quad \text{if } Nm_1(\mu_1+\mu_2\theta_1/\theta_2)/\delta + c_2\mu_2\delta e^{\theta_2 t} > 0. \end{aligned} \quad (44)$$

Note that we obtain (44) by using $\dot{x} = (1-x)\theta_1$ and (42). Since

$Nm_1(\mu_1+\mu_2\theta_1/\theta_2)/\delta + c_2\mu_2\delta e^{\theta_2 t} > 0$ if $g_1 \le Nm_1[(\mu_1/\mu_2+\theta_1/\theta_2)/\delta+1/\theta_2]$, $\partial H_1/\partial u_1$ is strictly decreasing if $g_1 \le Nm_1[(\mu_1/\mu_2+\theta_1/\theta_2)/\delta+1/\theta_2]$.

(b) Differentiating $\partial H_1/\partial v_1$ with respect to t and using (42) and (43), we obtain

$$d(\partial H_1/\partial v_1)/dt = d[-1+r_1'x_1(g_1+\lambda_1-\lambda_2)]/dt$$
$$= r_1'[\dot{x}_1(Nm_1/\theta_2-c_2\delta e^{\theta_2 t}) - x_1c_2\delta\theta_2 e^{\theta_2 t}]$$
$$> 0 \text{ if } \dot{x}_1 \geq 0 \text{ and } c_2 < 0. \quad (45)$$

Since $c_2 < 0$ if $g_1 > Nm_1/\theta_2$, $\partial H_1/\partial v_1$ is strictly increasing if $g_1 > Nm_1/\theta_2$ and $\dot{x}_1 \geq 0$. Q.E.D.

If u_i and v_i ($i = 1, 2$) are piecewise functions admitting either the budgetary limits or zero with at most a finite number of steps, f_i and r_i are also piecewise functions with at most a finite number of steps. Then we can divide [0, T] into a finite number of subintervals such that, in each such subinterval, f_i and r_i ($i = 1, 2$) are constant and hence λ_1 and λ_2 are given by (42) and (43). Consider any two such consecutive subintervals $[t_1, t_2]$ and $[t_2, t_3]$. Since λ_1 and λ_2 are continuous, $\lambda_1^1(t_2) = \lambda_1^2(t_2)$ and $\lambda_2^1(t_2) = \lambda_2^2(t_2)$. Solving these equations, we get

$$c_2^1\delta^1 e^{\theta_2^1 t_2} = c_2^2\delta^2 e^{\theta_2^2 t_2} + Nm_1/\theta_2^1 - Nm_1/\theta_2^2. \quad (46)$$

We use a superscript j ($j = 1, 2$) to distinguish the functions and parameters for the two subintervals.

In the following, we let $\theta_{i0} = \text{Min}(\theta_i)$, $\theta_{i1} = \text{Max}(\theta_i)$, $\mu_{i0} = \text{Min}(\mu_i)$, $\mu_{i1} = \text{Max}(\mu_i)$, $\delta_0 = \text{Min}(\delta)$ and $\delta_1 = \text{Max}(\delta)$. Note that $\theta_{i0} \leq \theta_i \leq \theta_{i1}$, $\mu_{i0} \leq \mu_i \leq \mu_{i1}$, and $\delta_0 \leq \delta \leq \delta_1$.

We now prove the following theorems.

Theorem 2. If f_i and r_i ($i = 1, 2$) are linear, r_i ($i = 1, 2$) is not decreasing over time and $g_i \le G_{i0} - Nm_i[(\mu_{i0}/\mu_{j1}+\theta_{10}/\theta_{21})/\delta_1 + 1/\theta_{21}]$ ($i, j = 1, 2$; $i \neq j$), the optimal advertising strategies are given by

$$u_i^* = \begin{cases} U_i & \text{when } 0 \le t < \tau_i, \\ 0 & \text{when } \tau_i \le t \le T, \end{cases} \quad i = 1, 2, \tag{47}$$

where U_i ($i = 1, 2$) is Firm i's budgetary constraint and τ_i is T if $\partial H_i/\partial u_i \ge 0$ in [0, T] or otherwise determined by

$$[g_1+\lambda_1(\tau_1)][1-x(\tau_1)]f_1' - 1 = 0, \tag{48}$$

$$[g_2+\psi_2(\tau_2)][1-x(\tau_2)]f_2' - 1 = 0. \tag{49}$$

Proof. From Lemma 1, the optimal trajectories u_i^* and v_i^* ($i = 1, 2$) are piecewise functions admitting either the budgetary limit or zero with at most a finite number of steps. Thus we can divide [0, T] into N (N is a finite integer) subintervals such that, in each subinterval, f_i and r_i ($i = 1, 2$) are constant and, for any two such consecutive subintervals, (44) and (46) hold.

Consider $\partial H_1/\partial u_1$ in [0, T]. In the last such subinterval $[t_{N-1}, T]$, we have $d(\partial H_1/\partial u_1)/dt < 0$ if $g_1 \ge G_{10}$ from Lemma 2(a).

In the second last such subinterval $[t_{N-2}, t_{N-1}]$, since $\theta_2^{N-1} \le \theta_2^N$, using (46) we obtain

$$c_2^{N-1}\delta^{N-1}e^{\theta_2^{N-1}t_{N-1}} \ge (Nm_1/\theta_2^N - g_1)e^{-\theta_2^N(T-t_{N-1})}. \tag{50}$$

Substituting (50) into (44), we also get $d(\partial H_1/\partial u_1)/dt < 0$ if $g_1 \ge G_{10}$.

Following the same procedures as above and working backwards, we obtain $d(\partial H_1/\partial u_1)/dt < 0$ in each subinterval. As $\partial H_1/\partial u_1$ is continuous, it is monotonically decreasing in $[0, T]$.

Similarly, $\partial H_2/\partial u_2$ is monotonically decreasing in $[0, T]$.

Then the optimal advertising strategies are given by (47) with τ_i being T if $\partial H_i/\partial u_i \geq 0$ in $[0, T]$ or determined by $\partial H_i/\partial u_i = 0$ or (48) and (49) otherwise. Q.E.D.

Theorem 2 states that, provided that the firms' service strategies are not decreasing over time, Firm i's optimal advertising strategy is to advertise at its budgetary level from the beginning and possibly stop completely some time before T if $g_i \leq G_{i0}$. If $g_i > G_{i0}$, we do not give the optimal advertising strategies here. However, by differentiating (44) with respect to t, we get

$$d^2(\partial H_1/\partial u_1)/dt^2 = (1-x)f_1'[Nm_1\theta_1(\mu_1+\mu_2\theta_1/\theta_2)/\delta - c_2\mu_2\delta^2 e^{\theta_2 t}]$$
$$> 0 \quad \text{if } c_2 \leq 0 \text{ or } g_1 \geq Nm_1/\theta_2. \tag{51}$$

Hence $\partial H_1/\partial u_1$ is convex if $g_i > Nm_1/\theta_2$. It is then our conjecture that the optimal advertising strategy in this case starts with the budgetary limit, drops to zero after some time, and jumps to the budgetary limit again near the end of the planning horizon. Especially, if g_1 is very large, we can observe from (44) that $d(\partial H_1/\partial u_1)/dt > 0$. Then $\partial H_1/\partial u_1$ is increasing in $[0, T]$ and, if $\partial H_1/\partial u_1 > 0$ at $t = 0$, $\partial H_1/\partial u_1 > 0$ in $[0, T]$. The optimal advertising strategy is constant at the budgetary level over $[0, T]$.

The optimal advertising strategies above are obtained under the condition that the service strategies are not decreasing over time. In the following we will prove that the firms' optimal service strategies are indeed not decreasing over time if $g_i > Nm_i/\theta_2$. From (24) and (25), we have $\dot{x}_i = f_i+\mu_i x-\theta_2 x_i$. We assume that $\dot{x}_i \geq 0$ for any service strategies, i.e., $\dot{x}_i = f_i+\mu_i x-\theta_2 x_i \geq f_i+\mu_i x-\theta_{21} x_i \geq 0$.

Theorem 3. If f_i and r_i are linear, $g_i \geq Nm_i/\theta_{20}$ and $\dot{x}_i \geq 0$ for any service strategies (i=1,2), the optimal service strategies are given by

$$v_i^* = \begin{cases} 0 & \text{when } 0 \leq t \leq \sigma_i, \\ V_i & \text{when } \sigma_i < t \leq T, \end{cases} \quad i = 1,\ 2, \tag{52}$$

where V_i is Firm i's budgetary constraint and σ_i is determined by

$$[g_1+\lambda_1(\sigma_1)-\lambda_2(\sigma_1)]x_1(\sigma_1)r_1' - 1 = 0, \tag{53}$$

$$[g_2+\psi_2(\sigma_2)-\psi_1(\sigma_2)]x_2(\sigma_2)r_2' - 1 = 0. \tag{54}$$

Proof. By Lemma 1, the optimal trajectories u_i^* and v_i^* (i = 1, 2) are piecewise functions with at most a finite number of steps.

Consider the subinterval $[t_0,\ T]$ of $[0,\ T]$. We assume that v_2 has three steps in $[t_0,\ T]$. Then v_2 has either the form

$$\text{(a)} \quad v_2 = \begin{cases} V_2 & \text{if } t_0 \leq t < t_1 \\ 0 & \text{if } t_1 \leq t < t_2, \\ V_2 & \text{if } t_2 \leq t \leq T \end{cases} \quad \text{or (b)} \quad v_2 = \begin{cases} 0 & \text{if } t_0 \leq t < t_1 \\ V_2 & \text{if } t_1 \leq t < t_2. \\ 0 & \text{if } t_2 \leq t \leq T \end{cases}$$

If v_2 has Form (a), for any r_1 which is constant over $[t_0,\ T]$, we have $\theta_2^1 = \theta_2^3 > \theta_2^2$. c_2 in the subintervals is determined by (46) as

$$c_2^1 \delta^1 e^{\theta_2^1 t_1} = (Nm_1/\theta_2^3 - g_1)e^{-\theta_2^3(T-t_2)-\theta_2^2(t_2-t_1)}$$

$$+ (Nm_1/\theta_2^2 - Nm_1/\theta_2^3)e^{-\theta_2^3(T-t_2)} + Nm_1/\theta_2^1 - Nm_1/\theta_2^2$$

$$< 0 \text{ if } g_1 > Nm_1/\theta_{20} \text{ since } \theta_2^1 - \theta_2^3 > \theta_2^2, \text{ hence } c_2^1 < 0, \quad (55)$$

$$c_2^2\delta^2 e^{\theta_2^2 t_2} = (Nm_1/\theta_2^3 - g_1)e^{-\theta_2^3(T-t_2)} + Nm_1/\theta_2^2 - Nm_1/\theta_2^3$$

$$< Nm_1/\theta_2^2 - Nm_1/\theta_2^3, \text{ if } g_1 > Nm_1/\theta_{20}, \quad (56)$$

$$c_2^3\delta^3 e^{\theta_2^3 T} = Nm_1/\theta_2^3 - g_1 < 0 \text{ if } g_1 > Nm_1/\theta_{20}, \text{ and then } c_2^3 < 0. \quad (57)$$

If $\dot{x}_1 \geq 0$, from (45), we have $d(\partial H_1/\partial v_1)/dt > 0$ when $t_0 \leq t < t_1$ and $t_2 \leq t \leq T$ since $c_2 < 0$. When $t_1 \leq t < t_2$,

$$d(\partial H_1/\partial v_1)/dt = r_1'[(f_1^2+\mu_1^2 x)(Nm_1/\theta_2^2 - c_2^2\delta^2 e^{\theta_2^2 t}) - Nm_1 x_1]$$

$$= r_1'[(f_1^2+\mu_1^2 x)(Nm_1/\theta_2^2 - Nm_1/\theta_2^3 + c_2^2\delta^2 e^{\theta_2^2 t}) + (f_1^2+\mu_1^2 x - \theta_2^3 x_1)Nm_1/\theta_2^3]$$

$$\geq r_1'(f_1^2+\mu_1^2 x)[(Nm_1/\theta_2^2 - Nm_1/\theta_2^3) + (Nm_1/\theta_2^3 - Nm_1/\theta_2^2)e^{-\theta_2^2(t_2-t)}]$$

$$> 0, \text{ since } \theta_2^2 < \theta_2^3. \quad (58)$$

Then, $d(\partial H_1/\partial v_1)/dt > 0$ in each subinterval and $\partial H_1/\partial v_1$ is monotonically increasing in $[t_0, T]$ since it is continuous.

The same is true when v_2 has Form (b).

Then v_1 can have at most two steps in $[t_0, T]$.

On the other hand, if v_1 has only two steps in $[t_0, T]$, following exactly the same procedures above, v_2 can have at most two steps in $[t_0, T]$.

Therefore, both v_1 and v_2 can have at most two steps in $[t_0, T]$. As t_0 is arbitrary, v_1 and v_2 can have at most two steps in $[0, T]$.

Since $\partial H_i/\partial v_i$ is monotonically increasing in [0, T], the optimal service strategies are given by (52) with σ_i determined by $\partial H_i/\partial v_i = 0$ (i = 1, 2) or (53) and (54). Q.E.D.

Theorem 3 states that, when the market is rising, a firm's optimal service strategy is to start service at its budgetary level some time after the beginning if $g_i > Nm_i/\theta_{20}$. If $g_i \leq Nm_i/\theta_{20}$, the optimal service strategies are not given here. However, we conjecture that the optimal service strategies in this case are to start service at the budgetary level some time after the introduction and drop to zero some time before T.

Theorem 2 and Theorem 3 give the firms' optimal advertising and service strategies. The optimal strategies depend on the firms' valuation of their ending market share. In general, a firm's optimal advertising strategy is to start advertising at its budgetary level from the very beginning and its optimal service strategy is to start service at its budgetary level when its market share reaches a certain level. If the firm's valuation of ending market share is low, advertising and service are not justified and should be completely stopped near the end of the planning horizon.

Note that, although each of Theorem 2 and Theorem 3 gives only one type of the optimal advertising and service strategies, the results can be combined. In the following we use a numerical example to demonstrate the procedures of finding the optimal strategies.

Example: $r_1 = 0.4 + 0.001v_1, \quad 0 \leq v_1 \leq 4,$

$r_2 = 0.45 + 0.002v_2, \quad 0 \leq v_2 \leq 3,$

$f_1 = 0.02 + 0.005u_1, \quad 0 \le u_1 \le 4,$

$f_2 = 0.02 + 0.010u_2, \quad 0 \le u_2 \le 2,$

$N = 1{,}000,\ T = 12,\ m_1 = 8,\ m_2 = 10,\ g_1 = 10{,}000,\ g_2 = 14{,}000.$

It is easy to verify that the conditions for both Theorem 2 and Theorem 3 are satisfied. Then the optimal advertising and service strategies are determined by (47) and (52), respectively. At $t = 0$, we have $v_i = 0$ and $u_i = U_i$ if $\partial H_i/\partial u_i > 0$. Letting $u_1 = 4$, $u_2 = 2$ and $v_i = 0$, we obtain $\partial H_i/\partial u_i > 0$ when $t = 0$. Therefore the optimal strategies start with $v_1 = v_2 = 0$, $u_1 = 4$ and $u_2 = 3$. These strategies should be changed only if one of $\partial H_i/\partial u_i$ and $\partial H_i/\partial v_i$ ($i = 1, 2$) changes sign. By using (42) and (43) for λ_i (ψ_i) and (26) and (27) for x_i with the strategies above, we obtain that $\partial H_i/\partial u_i \ge \partial H_i/\partial u_i|_{t=T} > 0$ since $\partial H_i/\partial u_i$ is strictly decreasing over time and $\partial H_i/\partial v_i = 0$ has a unique solution in (0, 12). Solving $\partial H_i/\partial v_i = 0$, we get $\sigma_1 = 4.43$ and $\sigma_2 = 1.49$. Then, at $t = 1.49$, Firm 2 should start its service at its budgetary level, i.e., $v_2 = 3$ and this strategy should not be changed since $\partial H_1/\partial v_1$ is monotonically increasing. However the other strategies may change if the corresponding partial derivative of the Hamiltonian changes sign.

With $u_1 = 4$, $u_2 = 2$, $v_1 = 0$ and $v_2 = 3$, we obtain similarly that $\partial H_i/\partial u_i > 0$ ($i = 1, 2$) in [1.49, 12] and $\partial H_1/\partial v_1 = 0$ has a unique solution in (1.49, 12). Solving $\partial H_1/\partial v_1 = 0$, we get $\sigma_1 = 4.42$. Then Firm 1 should start its service at its budgetary level at $t = 4.42$ or $v_1 = 4$ and this strategy should not be changed according to Theorem 3.

But the advertising strategies may still change depending on if $\partial H_i/\partial u_i$ ($i = 1, 2$) will change sign.

With $u_1 = 4$, $u_2 = 2$, $v_1 = 4$ and $v_2 = 3$ and following similar procedures above, we obtain $\partial H_i/\partial u_i > 0$ ($i = 1, 2$) in [4.42, 12].

Then the optimal strategies are given by

$$u_1^* = 4, \quad 0 \le t \le 12; \qquad u_2^* = 2, \quad 0 \le t \le 12;$$

$$v_1^* = \begin{cases} 0, & 0 \le t \le 4.42, \\ 4, & 4.42 < t \le 12; \end{cases} \qquad v^* = \begin{cases} 0, & 0 \le t \le 1.49, \\ 3, & 1.49 < t \le 12. \end{cases}$$

In the above analysis, the budgetary limits of each firm's advertising and service spending are not treated as control variables. In many situations, firms' game may also be very much in determining their budget in advertising and service. However, the inclusion of such variables will not change the structure of optimal strategies given above but creates the interdependency between the budgetary limits and the time to terminate or start advertising and/or service. With the results obtained above the problem becomes a static game in this case.

5.6. Conclusions and Possible Extensions

The model presented in this paper studies the growth of new repeat purchasing products, which has not been studied adequately in the literature. Its contribution includes the development of a diffusion model for repeat purchasing products, taking account of both first-time purchases and replacement purchases, and the derivation of a set of optimal advertising and service policies and strategies for such products in a duopoly market. The main findings can be briefly summarized as follows.

(1) For repeat purchasing products, the market will never saturate unless customers are extremely loyal to the products, which could happen only in some special cases, for instance, if the quality level of one producer's product is extremely high. In more regular situations, there is always a portion of the potential market that is not buying any product. Thus advertising and promotional activities are always desirable. This provides an explanation of the common phenomenon that advertising is always done by producers, no matter how long they have been in the market.

(2) It is usually demonstrated for non-repeat purchasing products that the optimal advertising strategy is monotonically decreasing over time. This strategy, as shown by our study, is optimal for repeat purchasing products only if the quality level of the products is extremely high. For a regular repeat purchasing product, the optimal advertising strategy is increasing at the introductory stage (some period starting from the introduction) and then decreasing. If competition exists, more advertising should be done at early stages to protect its market share increase from competitors' advertising campaign.

(3) Optimal service strategies are increasing at the initial stage and then decreasing or possibly maintaining constant at a certain level afterwards.

(4) When the control functions are linear in their control variables, it is possible to solve the game analytically for optimal strategies. We have solved the problem for optimal advertising and service strategies when the control functions representing the effects of advertising and service are linear in their control variables and a budgetary constraint exists for both advertising and service.

As an early attempt to combine repeat purchasing and the growth of new products, our model leaves many problems unsolved, which are possible directions of further research. We assume in the formulation of our model that customers who leave the market can be influenced by advertising and word-of-mouth communication as easily as those who have never bought the product. In fact customers who stop buying a product are usually more difficult to be influenced by these factors. Future research might incorporate this fact by differentiating the susceptibility to advertising and word-of-mouth communication of different customers. In addition, price is not considered in this study, although it can be well incorporated into our model. It is not only a challenge but also a tough task to study pricing of repeat purchasing products, especially when competition is considered.

APPENDICES

Appendix I: Proof of Theorem 1.

$$H_1 = Nm_1x_1-u_1-v_1 + (g_1+\lambda_1)[(1-x)(f_1+bx_1)-(1-r_1)x_1+\alpha(1-r_2)x_2] + \lambda_2[(1-x)(f_2+bx_2)+\beta(1-r_1)x_1-(1-r_2)x_2], \quad \text{(A1-1)}$$

$$\dot{\lambda}_1 = - Nm_1+(g_1+\lambda_1)[f_1+bx_1-b(1-x)+1-r_1]+\lambda_2[f_2+bx_2-\beta(1-r_1)], \quad \text{(A1-2)}$$

$$\dot{\lambda}_2 = (g_1+\lambda_1)[f_1+bx_1-\alpha(1-r_2)]+\lambda_2[f_2+bx_2-b(1-x)+1-r_2]. \quad \text{(A1-3)}$$

Since optimal strategies maximize H_1 at every instant, the first-degree partial derivatives of H_1 with respect to u_1 and v_1, respectively, vanish, that is,

$$\partial H_1/\partial u_1 = - 1 + (g_1+\lambda_1)(1-x)f_1' = 0, \quad \text{(A1-4)}$$

$$\partial H_1/\partial v_1 = - 1 + (g_1+\lambda_1)x_1r_1' - \lambda_2\beta x_1r_1' = 0, \quad \text{(A1-5)}$$

or

$$\lambda_1 = - g_1 + 1/[(1-x)f_1'], \quad \text{(A1-6)}$$

$$\lambda_2 = - 1/(\beta x_1r_1') + 1/[\beta(1-x)f_1']. \quad \text{(A1-7)}$$

Differentiating (A1-6) and (A1-7), we obtain

$$\dot{\lambda}_1 = [\dot{x}f_1'-(1-x)f_1''\dot{u}_1/[(1-x)^2f_1''], \quad \text{(A1-8)}$$

$$\dot{\lambda}_2 = (\dot{x}_1r_1'+x_1r_1''\dot{v}_1)/(\beta x_1^2r_1'^2) + \dot{\lambda}_1/\beta. \quad \text{(A1-9)}$$

Substituting (A1-6) and (A1-7) into (A1-2) and (A1-3), we also get

$$\dot{\lambda}_1 = - Nm_1 + [f_1+bx_1-b(1-x)+1-r_1]/[(1-x)f_1'] + [f_1+bx_2-\beta(1-r_1)]\{- 1/(\beta x_1r_1')+1/[\beta(1-x)f_1']\}, \quad \text{(A1-10)}$$

$$\dot{\lambda}_2 = [f_1+bx_1-\alpha(1-r_1)]/[(1-x)f_1']$$

$$+ [f_2+bx_2-b(1-x)+1-r_1]\{-1/(\beta x_1 r_1')+1/[\beta(1-x)f_1'']\}. \qquad \text{(A1-11)}$$

Equating (A1-8) with (A1-10) and (A1-9) with (A1-11), with certain simplification, we obtain (14a) and (15a).

Similarly we obtain (14b) and (15b).

Appendix II: Solution of the Market Share Model

The differential equation system becomes

$$\dot{x}_1 = (1-x)f_1 - (1-r_1)x_1 + (1-r_2)x_2, \qquad \text{(A2-1)}$$

$$\dot{x}_2 = (1-x)f_2 + (1-r_1)x_1 - (1-r_2)x_2, \qquad \text{(A2-2)}$$

$$x_1(0) = x_{10}, \qquad \text{(A2-3)}$$

$$x_2(0) = x_{20}. \qquad \text{(A2-4)}$$

By adding (A2-1) to (A2-2), we have

$$\dot{x} = (1-x)(f_1+f_2), \qquad \text{(A2-5)}$$

$$dx/(1-x) = (f_1+f_2)dt. \qquad \text{(A2-6)}$$

Integrating both sides of (A2-6), with the initial condition $x(0) = x_{10}+x_{20}$, we get

$$1-x = cW(t), \qquad \text{(A2-7)}$$

where $c = (1-x_{10}-x_{20})$ and $W(t) = \text{EXP}[-\int_0^t (f_1+f_2)d\tau]$.

By substituting (A2-7) into (A2-1), the state equation becomes

$$\dot{x}_1 = cf_1W(t) - (2-r_1-r_2)x_1 + (1-r_2)[1-cW(t)]. \qquad \text{(A2-8)}$$

Let $x_1 = D(t)L(t)$ in (A2-8) where $L(t) = \text{EXP}[-\int_0^t (2-r_1-r_2)d\tau]$. Differentiating it with respect to t, we obtain

$$\dot{D}(t) = [cf_1W(t)+(1-r_2)(1-cW(t))]L^{-1}(t), \quad (A2\text{-}9)$$

or

$$D(t) = \int_0^t [cf_1W(\tau)+(1-r_2)(1-cW(\tau))]L^{-1}(\tau)d\tau + d, \quad (A2\text{-}10)$$

where d is an arbitrarily selected constant.

Substituting D(t) into $x_1 = D(t)L(t)$ with the initial condition $x_1(0) = x_{10}$ we obtain

$$x_1(t) = L(t)\{\int_0^t [cf_1W(\tau)+(1-r_2)(1-cW(\tau))]L^{-1}(\tau)d\tau+x_{10}\}. \quad (A2\text{-}11)$$

Similarly we get

$$x_2(t) = L(t)\{\int_0^t [cf_2W(\tau)+(1-r_1)(1-cW(\tau))]L^{-1}(\tau)d\tau+x_{20}\}. \quad (A2\text{-}12)$$

Note that, if the starting time is not zero but t_0, we only need to change the lower limit for all integrations from zero to t_0.

Appendix III: Solution of the Auxiliary Variable System

When f_i and r_i $(i = 1, 2)$ are constant, the auxiliary system is

$$\dot{\lambda}_1 = -Nm_1 + (g_1+\lambda_1)(1-r_1+f_1) - \lambda_2(1-r_1-f_2), \quad (A3\text{-}1)$$

$$\dot{\lambda}_2 = \quad - (g_1+\lambda_1)(1-r_2-f_1) + \lambda_2(1-r_2+f_2), \quad (A3\text{-}2)$$

with ending condition $\lambda_1(T) = \lambda_2(T) = 0$.

The engenvalues for the corresponding homogeneous system are found by letting

$$\begin{vmatrix} 1-r_1+f_1-\theta & -(1-r_1-f_2) \\ -(1-r_2-f_1) & 1-r_2+f_2-\theta \end{vmatrix} = 0$$

to be $\theta_1 = f_1+f_2$ and $\theta_2 = 2-r_1-r_2$. The two engenvectors are found by letting

$$\begin{vmatrix} 1-r_1+f_1-\theta & -(1-r_1-f_2) \\ -(1-r_2-f_1) & 1-r_2+f_2-\theta \end{vmatrix} Z = 0$$

to be $Z_1^T = (1, 1)$ and $Z_2^T = (-\mu_2, \mu_1)$, where $\mu_1 = 1-r_2-f_1$, $\mu_2 = 1-r_1-f_2$ and T denotes 'transpose'.

Therefore the general solution to the homogeneous system is

$$(g_1+\lambda_1, \lambda_2)^T = c_1Z_1e^{\theta_1 t} + c_2Z_2e^{\theta_2 t}, \qquad \text{(A3-3)}$$

where c_1 and c_2 are arbitrarily selected constants.

For the non-homogeneous system, we let

$$(g_1+\lambda_1, \lambda_2)^T = c_1(t)Z_1e^{\theta_1 t} + c_2(t)Z_2e^{\theta_2 t}, \qquad \text{(A3-4)}$$

where $c_1(t)$ and $c_2(t)$ are to be selected.

Differentiating (A3-4), we obtain

$$d[(g_1+\lambda_1, \lambda_2)]^T/dt = \dot{c}_1(t)Z_1e^{\theta_1 t} + \dot{c}_2(t)Z_2e^{\theta_2 t} + c_1(t)\theta_1Z_1e^{\theta_1 t} + c_2(t)\theta_2Z_2e^{\theta_2 t}. \qquad \text{(A3-5)}$$

Substi.uting (A3-4) into (A3-1) and (A3-2) and simplifying, we also obtain

$$d[(g_1+\lambda_1, \lambda_2)]^T/dt = (-Nm_1, 0)^T + c_1(t)\theta_1Z_1e^{\theta_1 t} + c_2(t)\theta_2Z_2e^{\theta_2 t}. \text{(A3-6)}$$

Equating the RHS's of (A3-5) and (A3-6), we get a system of $c_1(t)$ and $c_2(t)$ as

$$\dot{c}_1(t)e^{\theta_1 t} - \dot{c}_2(t)\mu_2e^{\theta_2 t} = -Nm_1, \qquad \text{(A3-7)}$$

$$\dot{c}_1(t)e^{\theta_1 t} - \dot{c}_2(t)\mu_1e^{\theta_2 t} = 0, \qquad \text{(A3-8)}$$

or

$$\dot{c}_1(t) = -(\mu_1 Nm_1/\delta)\, e^{-\theta_1 t}, \tag{A3-9}$$

$$\dot{c}_2(t) = (Nm_1/\delta) e^{-\theta_2 t}, \tag{A3-10}$$

where $\delta = 2-r_1-r_2-(f_1+f_2)$. By integrating both sides of (A3-9) and (A3-10), $c_1(t)$ and $c_2(t)$ are found to be

$$c_1(t) = [Nm_1\mu_1/(\theta_1\delta)]e^{-\theta_1 t} + c_1, \tag{A3-11}$$

$$c_2(t) = -[Nm_1/(\theta_2\delta)]e^{-\theta_2 t} + c_2, \tag{A3-12}$$

where c_1 and c_2 are constants to be determined by $\lambda_1(T) = \lambda_2(T) = 0$.

The solution to the non-homogeneous system is then found by substituting $c_1(t)$ and $c_2(t)$ back into (A3-4) as

$$g_1+\lambda_1 = Nm_1\mu_1/(\theta_1\delta)+Nm_1\mu_2/(\theta_2\delta)+c_1e^{\theta_1 t}-c_2\mu_2e^{\theta_2 t}, \tag{A3-13}$$

$$\lambda_2 = Nm_1\mu_1/(\theta_1\delta)-Nm_1\mu_1/(\theta_2\delta)+c_1e^{\theta_1 t}+c_2\mu_1e^{\theta_2 t}, \tag{A3-14}$$

where c_1 and c_2 are determined by

$$Nm_1\mu_1/(\theta_1\delta)+Nm_1\mu_2/(\theta_2\delta)+c_1e^{\theta_1 T}-c_2\mu_2e^{\theta_2 T} = g_1, \tag{A3-15}$$

$$Nm_1\mu_1/(\theta_1\delta)-Nm_1\mu_1/(\theta_2\delta)+c_1e^{\theta_1 T}+c_2\mu_1e^{\theta_2 T} = 0, \tag{A3-16}$$

or

$$c_1 = [(g_1-Nm_1/\theta_1)\mu_1/\delta]e^{-\theta_1 T}, \tag{A3-17}$$

$$c_2 = [(Nm_1/\theta_2-g_1)/\delta]e^{-\theta_2 T}. \tag{A3-18}$$

Chapter Six

Conclusions and Discussions

Game theory models conflict and cooperation in explicit mathematical forms. These factors usually cannot be modeled by single decision maker models. Because of the general gaming nature of real business problems, it is natural to expect game theory to be an applicable method in Management Science. Indeed, as our survey shows, it provides better solutions to real business problems in many situations and has become one of the major analytical techniques in Management Science.

However, the present form of game theory may be inadequate for certain problems. One of its shortcomings is the lack of applicable solution techniques. Despite the extensive discussion on and wide application of game theory, none of its solution schemes has been accepted to be absolutely applicable (Palfrey, 1980; Williams, 1988). Therefore, care should be taken in selecting solution concepts.

There is no doubt that game theory has been widely applied in management science. Nevertheless some areas such as inventory control and queueing have not attracted the same amount of attention as others such as marketing and bidding. This might be due to not only the less explicit strategic nature of such decision problems, but also the lack of interest of game theorists. This calls for more attention into these areas as well as the development of more realistic and then more complicated game theory models for such decision making problems. In this dissertation, we have studied a few of such problems. The main findings and possible extensions are briefly summarized for each problem as follows.

6.1. The Discount Problem

The discussion on the discount problem in the literature focuses upon two perspectives. Classical discount models analyze the buyer's best reactions to various price and quantity discount schedules provided by the seller. They give the buyer's best order quantity (Hadley and Whitin, 1963; Peterson and Silver, 1979; Sethi, 1984). Recently, decision models have also been developed for the seller. These models give the seller's best discount schedules (Monahan 1984; Rosenblatt and Lee 1985; Lee and Rosenblatt 1986; Dada and Srikanth 1987).

There are two important factors overlooked in these two types of models. First, a discount scheme consists of an order quantity and a discount term. The order quantity issue is controlled primarily by the buyer, whereas the discount issue is basically determined by the seller. A settlement occurs only if one agrees with the other's decision and can only be achieved through negotiations between the two parties. Secondly, in many situations, suppliers provide discount to attract more demand from their customers. Our research in this thesis has taken these two aspects of the problem, which have not been studied adequately in the literature, into consideration. The main findings are summarized as follows.

(1) If demand is constant, the seller should not provide any price discount to the buyer and then neither the buyer nor the seller can gain from discounting. In this case they should resort to either quantity discount or cooperation which may make both of them gain.

(2) If demand increases with discount, price discount may make both the seller and the buyer gain. However, their gain in this case, especially that of the seller, is usually not very significant compared with the maximum benefit they can possibly obtain. Therefore, price discount is more likely to be a competitive marketing strategy of suppliers rather than a tool to improve channel efficiency.

(3) Quantity discount can be of benefit to both the seller and the buyer. Especially, it always brings a higher profit to the seller than a price discount. Thus, the seller should use a quantity discount schedule instead of a simple price discount whenever it is possible. In many situations, quantity discount schedules can be very

efficient in obtaining the maximum profit the seller and the buyer can possibly obtain. They can be used not only as a tool for the seller to increase profit but also as a way of cooperation for both the seller and the buyer.

This analysis of the discount problem may be extended in the following directions. First, demand could be stochastic. In this case, expected values should be used in the computation of payoffs. Secondly, there might be multiple sellers and/or multiple buyers and/or multiple products. The research in the second case would be not only very interesting but also a tough task.

6.2. The Order Quantity Problem Of Substitutable Products with Stochastic Demands

The decision model for the classical newsboy problem deals with a single decision maker who faces a stochastic demand and gives the optimal order quantity for the decision maker (Hillier and Lieberman, 1990). Recently this model was extended by Parlar (1988) into situations of two retailers whose products are substitutable and having random demands. He introduced the game theoretical approach to analyze the retailers' order decisions. It is shown in his study that there exists a unique Nash equilibrium for the problem and, if one of the two players acts irrationally to damage the other, the optimal (defensive) strategy for the latter reduces to the optimal order size in the classical single-period newsboy problem.

In more general situations, there may be more than two retailers in a common market. We have extended Parlar's two-person game theory model into situations where three or more retailers are present. The presence of additional players brings about multiple-direction two-way demand transfers and coalitions between any two or more players, which can not be dealt with by either one- or two-decision maker models.

It is shown in our study that there always exists a Nash equilibrium for the game when the players work independently and act rationally. If anyone works irrationally to damage the others, the decision problem for the rational players reduces to that without the irrational player(s). If a group or all of the players decide to cooperate, their decisions depend on if side payments are allowed. If side payments are not allowed, conflict of interest still exists and the players will determine their order quantities independently. In this case, secure (Nash) strategies always exist for each player. If side payments are allowed, the players will determine their order quantities collectively. In either case, all players' cooperation is often desirable and feasible.

6.3. The New Product Growth Problem

In the last two decades, a number of diffusion models have been developed to study the acceptance level of new products. In these models, the acceptance level of a new product in a given potential market is represented as a mathematical function of the time elapsed since its introduction and other marketing mix variables such as advertising and price. They have been used to forecast future demand of new products and to derive optimal marketing policies and strategies concerning advertising, pricing, etc. The analysis has been restricted to consumer durable products (Bass, 1969; Manajan and Mullier, 1979; Kalish, 1983; etc.) and the aspect of competition has been long-avoided (Clarke and Dolan, 1984; Dockner and Jorgensen, 1988).

Our investigation has extended the analysis to repeat purchasing products in competitive markets. In the analysis, we observe that consumers' buying behavior of repeat purchasing products is different from that of consumer durable products. Then, a firm's effort to maintain high customer satisfaction, which we call service, is another key consideration for a firm in our study. The main findings of our study are as follows.

(1) For repeat purchasing products, the market will never saturate unless customers are extremely loyal to at least one product. This could happen only in some special cases, for instance, the quality level of one product is extremely high. In more regular

situations, there is always a portion of the potential market not buying any of the products. Thus advertising and promotional activities are always desirable. This provides an explanation of the common phenomenon that advertising is always done by producers, no matter how long they have been in the market.

(2) For a non-repeat purchasing product, the optimal advertising strategy is usually monotonically decreasing over time (Horsky and Simon, 1983; Kalish, 1985). This strategy, however, is optimal for a repeat purchasing product only if its quality level is extremely high. For a normal repeat purchasing product, the optimal advertising strategy is increasing from the introduction for a certain period and decreasing afterwards. Especially, more advertising should be done at early stages to protect its market share increase from competitors' advertising campaign.

(3) Optimal service strategies are increasing at the initial stage and then decreasing or possibly maintaining constant at a certain level afterwards.

(4) We have solved the problem for optimal advertising and service strategies when the control functions representing the effects of advertising and service are linear in their control variables and a budgetary constraint exists for both advertising and service.

The present research represents only an early attempt to combine repeat purchasing and the growth of new products. We assume in the formulation of our model that customers who leave the market can be influenced by advertising and word-of-mouth communication as easily as those who have never bought the product. This limits our model to the introductory phase of new repeat purchasing products or situations where the portion of customers leaving the market is small. Future research might improve our model by differentiating the susceptibility to advertising and word-of-mouth communication of different customers. In addition, price might be considered in the analysis.

In conclusion, our analysis in this dissertation has shown that game theory does provide a better understanding for some Management Science problems. It is not perfect. However, with new developments in it, more game theory models will certainly be developed to provide a better representation and generate a better understanding of real business problems.

REFERENCES

Aivazian, V. A. and J. L. Callen (1980), " Corporate Leverage and Growth--The Game-Theoretic Issues", **Journal of Financial Economics**, Vol. 8, 379-399.

Aumann, R. J. and M. Maschler (1964), "The Bargaining Set for Games", **Annals of Mathematics Studies**, Vol. 52, 443-475.

Banerjee, A. (1986), "A Joint Economic-Lot-Size Model for Purchaser and Vendor", **Decision Science**, Vol. 17, 292-311.

Banerjee, A. (1986), "Notes on 'A Quantity Discount Pricing Model to Increase Vendor's Profit'", **Management Science**, Vol. 32, No. 11, 1513-1517.

Bass, F. M. (1969), "A New Product Growth for Model Consumer Durables", **Management Science**, Vol. 15, No. 5, 215-227.

Bass, F. M. (1980), "The Relationship between Diffusion Rates, Experience Curves, and Demand Elasticities for Consumer Durable Technological Innovations", **Journal of Business**, Vol. 53, No. 3, S52-67.

Bass, F. and A. Bultez (1982), "A Note on Optimal Strategic Pricing of Technological Innovations", **Marketing Science**, Vol. 1, 371-378.

Balas, E. (1981), "The Strategic Petroleum Reserve: How Large Should It Be?", B. A. Bayraktar, et al (eds), Energy Policy Planning, Plenum Press, New York, 335-386.

Balch, M. (1971), "Oligopolies, Advertising, and Non-cooperative Games", H. W. Kuhn and G. P. Szego (eds), Differential Games and Related Topics, North-Holland, New York.

Baligh, H. H. and L. E. Richartz (1967a), "Variable-Sum Game Models of Marketing Problems", **Journal of Marketing Research**, Vol. 4, 173-185.

Baligh, H. H. and L. E. Richartz (1967b), Vertical Market Structures, Allyn and Bacon, Inc., Boston, Mass.

Basar, T. and G. J. Olsder (1982), Dynamic Noncooperative Game Theory, Academic Press, New York.

Bennett, P. G. (1977), "Toward a Theory of Hypergames", **Omega**, Vol. 6, No. 6, 749-751.

Bennett, P. G. (1980), "Hypergames: Developing a Model of Conflict", **Futures**, 489-507.

Bryant, J. (1983), "Hypergames: A Presentation of Perceptions of Conflicts", **Omega**, Vol. 11, No. 6, 575-586.

Buchanan, J. (1953), "The Theory of Monopolistic Quantity Discounts", **Review of Economic Studies**, Vol. 20, 199-208.

Chakravarty, A. K. and G. E. Martin(1988), "An Optimal Joint Buyer-Seller Discount Pricing Model", **Computers and Operations Research**, Vol. 15, No. 3, 271-281.

Chen, C. I. and J. B. Cruz, Jr. (1972), "Stackelberg Solution for Two-person Games with Biased Information Patterns", **IEEE Transactions on Automatic Control**, Vol. AC-17, No. 6, 791-798.

Clarke, F., M. Darrough, and J. Heineke (1982), "Optimal Pricing Policy in the Presence of Experience Effects", **Journal of Business**, Vol. 55, 517-530.

Clarke, D. G. and R. J. Dolan (1984), "A Simulation Analysis of Alternative Pricing Strategies for Dynamic Environments", **Journal of Business**, Vol. 57, No. 1, S179-S200.

Coughlan, A. T. (1985), "Competition and Cooperation in Marketing Channel Choice: Theory and Application", **Marketing Science**, Vol. 4, No. 2, 110-129.

Dada, M. and K. N. Srikanth (1987), "Pricing Policies for Quantity Discounts", **Management Science**, Vol. 33, No. 10, 1247-1252.

Dirven, C.A.J.M. and O. J. Verize (1986), "Advertising Models, Stochastic Games and Myopic Strategies", **Operations Research**, Vol. 34, No. 4, 645-649.

Dockner, E. and S. Jorgensen (1988), "Optimal Pricing Strategies for New Products in Dynamic Oligopolies", **Marketing Science**, Vol. 7, No. 4, 315-334.

Dodds, W. (1973), "An Application of the Bass Model in Long-term New Product Forecasting, **Journal of Marketing Research**, Vol. 10, 308-311.

Dodson, J. A., Jr., and E. Muller (1978), "A Model of New Product Diffusion through Advertising and Word-of-Mouth", **Management Science**, Vol. 24, 1589-1597.

Dolan, R. and A. Jeuland (1981), "Experience Curves and Dynamic Demand Models: Implications for Optimal Pricing Strategies", **Journal of Marketing**, Vol. 45, 52-62.

Dolan, R. J.(1987), "Quantity discounts: Managerial Issues and Research Opportunities", **Marketing Science**, Vol. 6, 1-22.

Dresher, M. (1961), Games of Strategy--Theory and Applications, Prentice-Hall, Inc., Englewood Cliffs, New Jersy.

Driessen, T.S.H. (1986), "Solution Concepts of k-Convex n-Person Games", **International Journal of Game Theory**, Vol. 15, 201-231.

Eliashberg, J. and A. Jeuland (1986), "The Impact of Competitive Entry in a Developing Market upon Dynamic Pricing Strategies", **Marketing Science**, Vol. 5, No. 1, 20-36.

Eliashberg, J. and R. L. Winkler (1978), "The Role of Attitude toward Risk in Strictly Competitive Decision-Making Situations", **Management Science**, Vol. 24, No. 12, 1231-1241.

Eliashberg, J. and R. Steinberg (1987), "Marketing-Production Decisions in an Industrial Channel of Distribution", **Management Science**, Vol. 33, No. 8, 981-1000.

Engelbreght-Wiggans, R. (1980), "Auctions and Bidding Models: A Survey", **Management Science**, Vol. 26, No. 2, 119-142.

Engelbrecht-Wiggans, R., M. Shubik, and R. M. Stark (1983), Auctions, Bidding, and Contracting: Uses and Theory, New York University Press, New York.

Feichtinger, G. and R. Hartl (1985), "Optimal Pricing and Production in an Inventory Model", **European Journal of Operational Research**, Vol. 19, No. 1, 45-56.

Feichtinger, G. and S. Jorgensen (1983), "Differential Game Models in Management Science", **European Journal of Operational Research**, Vol. 14, No. 2, 137-155.

Fink, A. M. (1964), "Equilibrium in a Stochastic n-Person Game", **Journal of Science**, Hiroshima Univ. (Ser. A-I), 28, 89-93.

Fraser, N. M. and K. M. Hipel (1984), Conflict Analysis: Models and Resolutions, North-Holland, New York.

Friedman, L. (1958), "Game Theory Models in the Allocation of Advertising Expenditure", **Operations Research**, Vol. 6, 699-708.

Friedman, J. W. (1977), Oligopoly and the Theory of Games, North-Holland, Amsterdam.

Goyal, S. K. (1976), "An Integrated Inventory Model for a Single supplier-Single Customer Problem", **International Journal of Production Research**, Vol. 15, No. 1, 107-111.

Goyal, S. K. (1987), "Comment on: A Generalized Quantity Discount Pricing Model to Increase Supplier's Profits", **Management Science**, Vol. 33, No. 12, 1635-1636.

Goyal, S. K. (1988), "'A Joint Economic-Lot-Size Model for Purchaser and Vendor': A Comment", **Decision Sciences**, Vol. 19, No. 1, 236-241.

Griesmer, J.H. and M. Shubik (1963a), "Toward a Study of Bidding Processes, Part 1: Some Constant Sum Games", **Naval Research Logistics Quarterly**, Vol. 10, 11-12.

Griesmer, J.H. and M. Shubik (1963b), "Toward a Study of Bidding Processes, Part 2: Games with Capacity Limitations", **Naval Research Logistics Quarterly**, Vol. 10, 151-173.

Griesmer, J.H. and M. Shubik (1963c), "Toward a Study of Bidding Processes, Part 3: Some Special Models", **Naval Research Logistics Quarterly**, Vol. 10, 191-217.

Griesmer, J.H., R. E. Levitan, and M. Shubik (1967), "Toward a Study of Bidding Processes, Part 4: Games with Unknown Costs", **Naval Research Logistics Quarterly**, Vol. 14, 415-433.

Hadley, G. and T. M. Whitin (1963), Analysis of Inventory Systems, Prentice-Hall, Englewood Cliffs, New York.

Harsanyi, J. C. (1962), "Bargaining in Ignorance of the Opponent's Utility Function", **Journal of Conflict Resolution**, Vol. 6, 29-38.

Harsanyi, J. C. (1967, 1968a, 1968b), "Games with Incomplete Information Played by 'Bayesian' Players" (I-III), **Management Science**, Vol. 14, No. 3, 1967, 159-182; Vol. 14, No. 5, 1968, 320-334; Vol. 14, No. 7, 1968, 486-502.

Harsanyi, J. C. (1975), "The Tracing Procedure: A Bayesian Approach to Defining a Solution for n-Person Non-cooperative Games", **International Journal of Game Theory**, Vol. 4, No. 2, 61-94.

Harsanyi, J. C. (1982), "Subjective Probability and the Theory of Games: Comments on Kadane and Larkey's Paper", **Management Science**, Vol. 28, No. 2, 120-125.

Hillier, F. S. and G. J. Lieberman (1990), Introduction to Operations Research, Mcgraw-Hill, Inc., New York.

Hogan, W. W. (1981), "Important Management and Oil Emergencies", Deese, D. and J. Nye (eds), Energy and Security, Ballinger Publishing, Cambridge, Mass., 261-301.

Holt, C. A., Jr. and R. Sherman (1980), "Waiting-Line Auctions", unpublished working paper, University of Virginia, July.

Holt, C. A., Jr. and R. Sherman (1982), "Waiting-Line Auctions", Journal of Political Economy, Vol. 90, 280-294.

Holt, C. A., Jr. and R. Sherman (1984), "When A Queue Is Like An Auction", Engelbrecht-Wiggans, R., M. Shubik, and R. M. Stark (eds), Auctions, Bidding, and Contracting: Uses and Theory, New York University Press, New York.

Horsky, D. and L. S. Simon (1983), "Advertising and the Diffusion of New Products", Marketing Science, Vol. 2, 1-17.

Howard, N. (1966a), "The Theory of Metagames", General Systems, Yearbook of the Society for General Systems Research, 11, 169-186.

Howard, N. (1966b), "The Mathematics of Metagames", General Systems, Yearbook of the Society for General Systems Research, 11, 187-200.

Howard, N. (1971), Paradoxes of Rationality: Theory of Metagames and Political Behavior, MIT Press, Cambridge, Mass..

Hurwicz, L. (1945), "The Theory of Economic Behavior", The American Economic Review, December, 909-925.

Huxham, C. and P. M. Bennett (1985), "Floating Ideas -- An Experiment in Enhancing Hypergames with Maps", Omega, Vol. 13, No. 4, 331-347.

Jeuland, A. and R. Dolan (1982), "An Aspect of New Product Planning: Dynamic Pricing", In A. Zoltners (ed.), Marketing Planning Models, TIMS studies in the Management Sciences, Vol. 18, New York, Elsevier.

Joglekar, P. N. (1988), 'Comments on "A Quantity Discount Pricing Model to Increase Vendor Profits"', Management Science, Vol. 34, No. 11, 1391-1398.

Joglekar, P. and S. Tharthare (1990), "The Individually Responsible and Rational Decision Approach to Economic Lot Sizes for One vendor and Many Purchasers", Decision Sciences, Vol. 21, 492-506.

Jorgensen, S. (1986), "Optimal Production, Purchasing and Pricing: A Differential Games Approach", **European Journal of Operational Research**, Vol. 24, 64-76.

Jucker, J. V. and M. J. Rosenblatt (1985), "Single-Period Inventory Models: Behavioral Implications and A New Solution Procedure", **Naval Research Logistics Quarterly**, Vol. 32, No. 4, 537-550.

Kadane, J. B. and P. D. Larkey (1982), "Subjective Probability and the Theory of Games", **Management Science**, Vol. 28, No. 2, 113-120.

Kahan, J. P. (1983), "On Choosing between Rev. Bayes and Prof. von Neumann", **Management Science**, Vol. 29, No. 11, 1334-1336.

Kalish, S. (1983), "Monopolist Pricing with Dynamic Demand and Production Cost", **Marketing Science**, Vol. 2 135-159.

Kalish, S. (1985), "A New Product Adoption Model with Price, Advertising, and Uncertainty", **Management Science**, Vol. 31, No. 12, 1569-1585.

Karnani, A. (1985), "Strategic Implications of Market Share Attraction Models", **Management Science**, Vol. 31, No. 5, 536-547.

King, M. and A. Mercer (1988), "Recurrent Competitive Bidding", **European Journal of Operational Research**, Vol. 33, 2-16.

Kirman, A. P. and M. J. Sobel (1974), "Dynamic Oligopoly with Inventories", **Econometrica**, 42, 279-287.

Kohli, R. and H. Park(1989), "A Cooperative Game Theory Model of Quantity Discounts", **Management Science**, Vol. 35, No. 6, 693-707.

Kotowitz, Y. and F. Mathewson (1979), "Informative Advertising and Welfare", **American Economic Review**, Vol. 69, 284-294.

Kuhn, H. W. (1950), "Extensive Games", **Proceedings of the Natural Academy of Science**, Vol. 36, 570-576.

Kuhn, H. W. (1952), "Lectures on the Theory of Games", issued as a report of the Logistics Research Project, Office of Naval Research, Princeton University, Priceton.

Kuhn, H. W. and A. W. Tucker (eds) (1950), Contributions to the Theory of Games, Vol. I, **Annals of Mathematics Studies**, No. 24, Princeton University Press, Princeton, New Jersey.

Kuhn, H. W. and A. W. Tucker (eds) (1953), Contributions to the Theory of Games, Vol. II, **Annals of Mathematics Studies**, No. 28, Princeton University Press, Princeton, New Jersey.

Ladany, S. and A. Sternlieb (1974), "The Interaction of Economic Ordering Quantities and Marketing Policies", **AIIE Transactions**, Vol. 6, No. 1, March, 35-40.

Lal, R. and R. Staelin (1984), "An Approach for Developing An Optimal Discount Pricing Policy", **Management Science**, Vol. 30, No. 12, 1524-1539.

Lee, H. L., and M. J. Rosenblatt (1986), "A Generalized Quantity Discount Pricing Model to Increase Vendor's Profit", **Management Science**, Vol. 32, No. 9, 1177-1185.

Levitan, R. and M. Shubik (1971), "Price Variation Duopoly with Differentiated Products and Random Demands", **Journal of Economic Theory**, Vol. 3, 23-39.

Lieberman, B. (1962), "Experimental Studies of Conflict in Some Two-Person and Three-Person Games", J. H. Crswell, H. Soloman, and P. Suppes (eds), Mathematical Methods in Small Group Processes, Stanford University Press, Stanford, California.

Liebman, J., L. Lasdon, L. Schrage, and A. Waren (1986), Modeling and Optimization with GINO, The Scientific Press.

Lucas, W. F. (1967), "A Counterexample in Game Theory", **Management Science**, Vol. 13, 766-767.

Lucas, W. F. (1972), "An Overview of the Mathematical Theory of Games", **Management Science**, Vol. 18, No. 5, January, Part 2, 3-19.

Lucas, W. F. (1981), Game Theory and Its Applications, **Proceedings of Symposia in Applied Mathematics**, Vol. 24, The American Mathematical Society.

Luce, R. and H. Raiffa (1957), Games and Decisions, Wiley, New York.

Mamer J. W. and K. F. McCardle (1987), "Uncertainty, Competition, and the Adoption of New Technology", **Management Science**, Vol. 33, No. 2, 161-177.

Mahajan, V. and E. Muller (1979), "Innovation Diffusion and New Product Growth Models in Marketing", **Journal of Marketing**, Vol. 43, No. 4, 55-68.

Mahajan, V. and R. A. Peterson (1978), "Innovation Diffusion in a Dynamic Potential Adopter Population", **Management Science**, Vol. 24, 1589-1597.

Mahajan, V. and R. A. Peterson (1982), "Erratum to: Innovation Diffusion in a Dynamic Potential Adopter Population", **Management Science**, Vol. 28, 1087.

McGillivray, A. R. and E. A. Silver (1978), "Some Concepts for Inventory Control under Sunstitutable Demand", **INFOR**, Vol. 16, 47-63.

McGuire, T. M. and R. Staelin (1983a), "The Effects of Channel Member Efficiency on Channel Structure", D. Gautschi (eds), Productivity and Efficiency in Distribution Systems, Elsevier Science Publishing Co., New York.

McGuire, T. M. and R. Staelin (1983b), "An Industry Equilibrium Analysis of Downstream Vertical Integration", **Marketing Science**, Vol. 2, No. 2, 161-191.

McKelvey, R. D. (1981), "A Theory of Optimal Agenda Design", **Management Science**, Vol. 27, No. 3, 303-321.

Midgley, D. F. (1976), "A Simple Mathematical Theory of Innovative Behavior", **Journal of Consumer Research**, Vol. 3, 31-41.

Miller, R. L. (1978), Intermediate Microeconomics, McGraw-Hill, New York.

Mills, H. D. (1961), "A Study in Promotional Competition", F. M. Bass, et al (eds), Mathematical Models and Methods in Marketing, Richard D. Irwin, Homewood, Ill.

Milnor, J. W. and L. S. Shapley (1978), "Values of Large Games II: Oceanic Games", **Mathematics of Operations Research**, Vol. 3, No. 4, 290-307.

Monahan, G. E. (1984), "A Pure Birth Model of Optimal Advertising with Word-of Mouth", **Marketing Science**, Vol. 3, No. 2, 168-178.

Monahan, J. P. (1984), "A Quantity Discount Pricing Model to Increase Vendor's Profit", **Management Science**, Vol. 30, No 6, 720-726.

Monahan, J. P. (1988), 'On "Comments on A Quantity Discount Pricing Model to Increase Vendor Profits"', **Management Science**, Vol. 34, No. 11, 1398-1340.

Mossin, J. (1968), "Merger Agreements: Some Game Theoretic Considerations", **Journal of Business**, Vol. 41, 460-471.

Nash, J. F. (1950a) "Equilibrium Points in n-Person Games", **Proceedings of the National Academy of Science**, U.S.A., 36, 48-49.

Nash, J. F. (1950b) "The Bargaining Problem", **Econometrica**, Vol. 18, 155-162.

Nash, J. F. (1951), "Non-cooperative Games," **Annals of Mathematics**, Vol. 54, 286-295.

Nash, J. F. (1953), "Two-person Cooperative Games," **Econometrica**, Vol. 21, 128-140.

Nelson, P. (1970), "Information and Consumer Behavior", **Journal of Political Economy**, Vol. 78, No. 2, 311-329.

Nelson, P. (1974), "Advertising as Information", **Journal of Political Economy**, Vol. 82, 729-754.

von Neumann, J. (1928), "Zur theorie der Gesellschaftspiele", **Mathematische Annalen**, Vol. 100, 295-320.

von Neumann, J. (1937), "Über ein okonomisches Gleichungssytem und eine Verallgemeinerung des Brouwerschen Fixpunktsatzes", **Ergebnisse eines Mathematik Kolloquiums**, Vol. 8, 73-83.

Nevers, J. V. (1972), "Extensions of a New Product Growth Model", **Sloan Management Review**, Vol. 13, 77-91.

von Neumann, J. and O. Morgenstern (1944), Theory of Games and Economic Behavior, Princeton University Press, Princeton, New Jersey.

Nichols, A. L. and R. J. Zeckhauser (1977), "Stockpiling Strategies and Cartel Prices", **Bell Journal of Economics**, Vol. 8, 66-96.

Nikaido, H. and K. Isora (1955), "Notes on Noncooperative Convex Games", **Pacific Journal of Mathematics**, Vol. 5, 807-815.

Norton, J. A. and F. M. Bass (1987), "A Diffusion Theory Model of Adoption and Substitution for Successive Generations of High-Technology Products", **Management Science**, Vol. 33, No. 9, 1069-1086.

Nti, K. O. (1987), "Competitive Procurement under Demand Uncertainty", **Management Science**, Vol. 33, No. 11, 1489-1500.

Oren, S. S. and M. H. Rothkopf (1975), "Optimal Bidding in Sequential Auctions", **Operations Research**, Vol. 23, 1080-1090.

Orgler, Y. E. and Y. Tauman (1986), "A Game Theoretic Approach to Collections and Disbursements", **Management Science**, Vol. 32, No. 8, 1029-1039.

Owen, G. (1982), Game Theory, Academic Press, New York.

Ozga, S. (1960), "Imperfect Markets Through Lack of Knowledge", **Quarterly Journal of Economics**, Vol. 59, 29-52.

Palfery, T. R. (1980), "Multiple-Object, Discriminatory Auctions with Bidding Constraints: A Game-Theoretic Analysis", **Management Science**, Vol. 26, No. 9, 935-946.

Parfitt, J. H. and B. J. K. Collins (1968), "Use of Consumer Panels for Brand Share Predictions", **Journal of Marketing Research**, Vol. 5, 131-145.

Parlar, M. (1985), "Optimal Ordering Policies for a Perishable and Substitutable Product: A Markov Decision Model", **INFOR**, Vol. 23, 182-195.

Parlar, M. (1986), "A Heuristic Method for the Multi-item Inventory Problem with Stochastic and Substitutable Demands", **Opsearch**, Vol. 23, No. 4, 250-257.

Parlar, M. (1988), "Game Theoretic Analysis of the Substitutable Product Inventory Problem with Random Demands", **Naval Research Logistics**, Vol. 35, 397-409.

Parlar, M. and S. K. Goyal (1984), "Optimal Ordering Decisions for Two Substitutable Products with Stochastic Demands", **Opsearch**, Vol. 21, 1-15.

Parthasarathy, T., S. H. Tijs and O. J. Vrieze (1984), "Stochastic Games with State Independent Transitions and Separable Rewards", G. Hammer and D. Pallaschke (eds), Selected Topics in Operations Research and Mathematical Economics, 226-236, Springer Verlag, New York.

Peterson, R. and E. A. Silver (1979), Decision Systems for Inventory Management and Production Planning, Wiley, New York.

Pontryagin, L. D. et al. (1962), The Mathematical Theory of Optimal Processes, Interscience Publishers Inc., New York.

Powers, I. Y. (1987), "A Game-Theoretic Model of Corporate Takeovers by Major Stockholders", **Management Science**, Vol. 33, No. 4, 467-483.

Raghavan, T. E. S., S. H. Tijs and O. J. Vrieze (1985), "On Stochastic Games with Additive Reward and Transition Structure", **Journal of Optimization Theory and Applications**, 47, 451-464.

Raiffa, H. (1951), "Arbitration Schemes for Generalized Two-Person Games", Report M720-1, R30, Engineering Research Institute, University of Michigan, Ann Arbor.

Raiffa, H. (1953), "Arbitration Schemes for Generalized Two-Person Games", in Kuhn and Tucker (1953) pp. 361-387.

Rao, A. G. and M. F. Shakun (1972), "A Quasi-Game Theory Approach to Pricing", **Management Science**, Vol. 18, No. 5, 110-123.

Rapoport, A. and C. Orwant (1962), "Experimental Games: A Review", **Behavioral Science**, Vol. 7, 1-37.

Richartz, L. E. (1970), "A Game Theoretic Formulation of Vertical Market Structure", L. P. Buklin (ed.), Vertical Market Systems, Scott, Foresman and Company, Glenview, Ill.

Rives Jr., N. W. (1975), "On the History of the Mathematical Theory of Games", **History of Political Economy**, Vol. 7, No. 4, 549-565.

Robinson, B. and C. Lakhani (1975), "Dynamic Price Models for New Product Planning", **Management Science**, Vol. 21, 1113-1122.

Rosenblatt, M. J. and H. L. Lee(1985), "Improving Profitability with Quantity Discounts under Fixed Demand", **IIE Transactions**, Vol. 17, No. 4, 388-395.

Roth, A. E. and F. Schoumarker (1983), "Subjective Probability and the Theory of Games: Some Further Comments", **Management Science**, Vol. 29, No. 11, 1337-1340.

Rothkopf, M. H. (1969), "A Model of Rational Competitive Bidding", **Management Science**, Vol. 15, 362-373.

Rothkopf, M. H. (1980a), "Equilibrium Linear Bidding Strategies", **Operations Research**, Vol. 28, 576-583.

Rothkopf, M. H. (1980b), "On Multiplicative Bidding Strategies", **Operations Research**, Vol. 28,570-575.

Rothkopf, M. H. (1983), "modelling Semirational Competitive Behavior", **Management Science**, Vol. 29, No. 11, 1341-1345.

Scarf, H. (1967), "The Approximation of Fixed Points of a Continuous Mapping", **SIAM Journal of Applied Mathematics**, Vol. 15, No. 5, 1328-1343.

Schmalensee, R. (1976), "A Model of Promotional Competition in Oligopoly", **Review of Economic Studies**, Vol. 43, 493-507.

Schemeidler, D. (1969), "The Nucleolus of a Characteristic Function Game", **SIAM Journal of Applied Mathematics.**, 17, 1163-1170.

Sethi, S. P. (1973), "Optimal Control of Vidale-Wolfe Advertising Model", **Operations Research**, Vol. 21, 998-1013.

Sethi, S. P. (1984), "A Quantity Discount Lot Size Model with Disposals", **International Journal of Production Research**, Vol. 22, No 1, 31-39.

Shakun, M. F. (1965), "Advertising Expenditures in Coupled Markets - A Game-Theoretic Approach", **Management Science**, Vol. 11, B42-47.

Shakun, M. F. (1966), "A Dynamic Model of Competitive Marketing in Coupled Markets", **Management Science**, Vol. 12, B525-530.

Shapley, L. S. (1953), "A Value for n-Person Games", **Annals of Mathematics Studies**, No. 28 (in Kuhn and Tucker, 1953), 307-317.

Shapley, L. S. (1967), "On Balanced Sets and Cores", **Naval Research Logistics Quarterly**, Vol. 14, 453-460.

Shapley, L. S. and M. Shubik (1969), "Price Strategy Oligopoly with Product Variation", Kyklos 1.

Sherali, H. and R. Rajan (1986), "A Game Theoretic-Mathematical Programming Analysis of Cooperative Phenomena in Oligopolistic Markets", **Operations Research**, Vol. 34, No. 5, 683-697.

Shubik, M. (1981), Market Structure and Behavior, Harvard University Press, Cambridge, Massachusetts and London, England.

Shubik, M. (1981), "Game Theory Models and Methods in Political Economy", K. J. Arrow, at al (eds), Handbook of Mathematical Economics, Vol. 1, Chapter 7, North-Holland, Amsterdan, New York.

Shubik, M. (1982), Game Theory in the Social Sciences--Concepts and Solutions, The MIT Press, Cambridge, Massachusetts and London, England.

Shubik, M. (1984), A Game-Theoretic Approach to Political Economy, The MIT Press, Cambridge and London, England.

Simaan, M. and J. B. Cruz, Jr (1973a) "On the Stackelberg Strategy in Nonzero-Sum Games", **Journal of Optimization Theory and Applications**, Vol. 11, No. 5, 533-555.

Simaan, M. and J. B. Cruz, Jr (1973b) "Additional Aspects of the Stackelberg Strategy in Nonzero-Sum Games", **Journal of Optimization Theory and Applications**, Vol. 11, No. 6, 613-626.

Simon, H. (1982), "PRICESTRAT - An Applied Strategic Pricing Model for Non-Durables', TIMS Studies in the Management Sciences, Special Issue on Marketing Planning Models, A. Zoltners, ed.

Smith, B. T. and J. H. Case (1975), "Nash Equilibria in a Sealed Bid Auction", **Management Science**, Vol. 22, No. 4, 487-497.

Snyder, R. D. (1973), "The Classical Economic Order Quantity Formula", **Operational Research Quarterly**, Vol. 24, No 1, 125-127.

Sobel, M. J. (1981), "Myopic Solutions of Markov Decision Processes and Stochastic Games", **Operations Research**, Vol. 29, 995-1009.

Spence. M. (1981), "The Learning Curve and Competition", **Bell Journal of Economics**, Vol. 12, 49-70.

von Stackelberg, H. (1934), Marketform and Gleichgewicht, Springer, Berlin.

Stark, R. M. and M. H. Rothkopf (1979), "Competitive Bidding: A Comprehensive Bibliography", **Operations Research**, Vol. 27, No. 2, 364-390.

Starr, A. W. and Y. C. Ho (1969), "Nonzero-Sum Differential Games", **Journal of Optimization Theory and Applications**, Vol. 3, No. 3, 184-206.

Tijs, S. H. and T. S. Driessen (1986), "Game Theory and Cost Allocation Problems", **Management Science**, Vol. 32, No. 8, 1015-1028.

Teng, J. and G. L. Thompson (1983), "Oligopoly Models for Optimal Advertising when Production Costs Obey a Learning Curve", **Management Science**, Vol. 29, No. 9, 1087-1101.

Thompson, G. L. and J. Teng, "Optimal Pricing and Advertising Policies for New Product Oligopoly Models", **Marketing Science**, Vol. 3, No. 2, 148-168.

Tucker, A. W. and R. D. Luce (eds) (1959), Contributions to the Theory of Games, Vol. 4, **Annals of Mathematics Studies**, No. 40, Princeton University Press, Princeton.

Vidale, M. L. and H. B. Wolfe (1957), "An Operations Research Study of Sales Response to Advertising", **Operations Research**, Vol. 5, 370-381.

Wang, M., K. W. Hipel, and N. M. Fraser (1988), "Modelling Misperceptions in Games", **Behavioral Science**, Vol. 33, 207-223.

Wang, Q. and M. Parlar (1989), "Static Game Theory Models and Their Applications in Management Science", **European Journal of Operational Research**, Vol. 42, 1-21.

Williams, M. A. (1988), "An Empirical Test of Cooperative Game Solution Concepts", **Behavioral Science**, Vol. 33, 224-237.

Wilson, R. B. (1969), "Competitive Bidding with Disparate Information", **Management Science**, Vol. 15, 446-448.

Wilson. L. O. and J. A. Norton (1989), "Optimal Entry Timing for A Product Line Extension", **Marketing Science**, Vol. 8, No. 1, 1-17.

Wittman, D. (1986), "Final-Offer Arbitration", **Management Science**, Vol. 32, No. 12, 1551-1561.

Zermelo, E. (1912), "Uber eine Anwendung der Mengenlehre auf die Theorie des Schachspiels", Proc. Fifth International Congress of Mathematics (Cambridge, England, 1912), Vol. II, E. W. Hobson and A. E. H. Love, eds., Cambridge University Press, Cambridge, 1913, 501-504.

Zusman, P. and M. Etgar (1981), "The Marketing Channel as An Equilibrium Set of Contracts", **Management Science**, Vol. 27, No. 3, 284-302.