BIFRAMES

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Abstract

Biframes and their homomorphisms are the subject of this thesis. Various traditional notions from topology and frame theory are extended to this setting: compactness, regularity, normality, zero-dimensionality, Booleanness, coherence, supercoherence, continuity, supercontinuity and connectedness. The category of biframes is shown to be complete and cocomplete, and its monomorphisms, epimorphisms and projectives characterized. Of particular importance are the compactifications — their correspondence with strong inclusions, the existence of zero-dimensional ones and of least ones, and their relations to rim-compactness and normality.

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Contents

Introduction

Biframes, which were defined by Banaschewski, Brümmer and Hardie in 1983 [10], are simultaneously generalizations of frames (locales) and of bitopological spaces.

Bispaces were introduced by Kelly [23, 1963], and have been studied by many authors since then. We do present results on spatiality where appropriate, but it is on the relationship with frames that we concentrate.

Frame theory, in one respect, may be regarded as doing topology by making the open-set lattice the basic notion. (For some of its history, see [22]). This algebraic approach had its origins in the 1930's with the work of Stone (for example, [27, 28]), but frames were only defined in 1957 by Ehresmann [18] (who called them "local lattices"; C.H.Dowker coined the term "frame"). Frame theory has the advantage that many theorems which require some axiom of choice in the topological setting, can be proved constructively in the frame setting. (One might mention the Tychonoff Product Theorem [20, 29] and the construction of the Stone-Čech compactification [11].) Often, when the frame and topological situations differ, the former is better — for example, coproducts of regular frames preserve the Lindelöf property [17]; products of regular spaces do not.

Before proceeding to an outline of this thesis, we give brief references, for the interested reader, of work on biframes by other authors which has not been discussed in this thesis. In [19], Frith defines the concept of a quasi-uniform frame, and shows that, given the axiom of countable dependent choice, the completely regular biframes are exactly those with a compatible quasi-uniform structure. Choe and Chae [15] present adjunctions between categories of convex ordered topological spaces and biframes and relate them to notions of separation and compactness. Banaschewski and Brümmer [9] examine the concept of strong zero-dimensionality for bispaces, and extend it to frames and biframes. In the process they introduce a

De Morgan property for biframes and prove that any zero-dimensional De Morgan biframe is strongly zero-dimensional.

Outline

The Preliminaries give basic definitions, mainly about frames, that we assume in the ensuing chapters.

Chapter 1 defines the categories of biframes and bispaces (**BiFrm** and **BiTop**) and describes the dual adjunction which relates them. Various subcategories — those of the compact, the regular, the Stone and the zero-dimensional biframes — are introduced, and related to the whole category. Congruences are discussed, and the congruence biframes of bilattices characterized.

In Chapter 2 we give further categorical information about **BiFrm** — its completeness and cocompleteness, its monomorphisms and epimorphisms. A forgetful functor from **BiFrm** to the square of the category of sets provides a partial substitute for the non-existent free biframes.

Compactifications are of central interest in this thesis. In Chapter 3 we describe the isomorphism between the compactifications and the strong inclusions of a biframe; we discuss the zero-dimensional compactifications and how one can recognize them from their strong inclusions; and give partial results on the tricky question of least biframe compactifications.

Chapters 4 and 5 proceed more or less in parallel — coherence and supercoherence, relations to lattices and to semilattices, continuity and supercontinuity. However, we emphasize spatiality in Chapter 4 and discuss projectivity only in Chapter 5.

Chapter 6 describes the compact, regular coreflection of a normal, regular biframe, provides a Urysohn Lemma in this context and shows that any regular, Lindelöf biframe must be normal.

In Chapter 7 the notions of rim-compactness and of perfectness of compactifications are extended to the biframe setting. A Π-compact basis for a rim-compact biframe always produces a strong inclusion, and hence a compactification, of the biframe in question.

The Booleanization of a biframe is given in Chapter 8. We discuss several properties of biframe homomorphisms, particularly their behaviour with regard to the pseudocomplement, in an (as yet incomplete) search for appropriate morphisms in the study of Boolean biframes.

Chapter 0

Preliminaries

Our main reference for information on frames is [21], and on category theory, [24]. For topology, any introductory text will do — [14] is an example.

The Boolean Ultrafilter Theorem, which states that any non-trivial Boolean algebra has an ultrafilter, is a choice principle strictly weaker than the Axiom of Choice. Results which require its use are indicated by the initials (BUT).

We draw the attention of the reader to the list of category names and page references in the appendix (on page 86).

0.1 Frames

A **frame** is a complete lattice L in which the distributive law $x \wedge \bigvee Y = \bigvee_{y \in Y} x \wedge y$ holds for all $x \in L$, $Y \subseteq L$. A **frame homomorphism**, or frame map, is a set function between frames which preserves finite meets and arbitrary joins (and thus also the top (e) and the bottom (0) of the frame). The category of frames and frame homomorphisms will be written **Frm**. It is complete and cocomplete. (Johnstone [21] works to a large extent with the dual category of **Frm** — the category of locales

— but we prefer the somewhat more algebraic approach given by **Frm**.)

A frame map $h : L \to M$ is called **dense** iff $h(x) = 0$ implies $x = 0$ for all $x \in L$. It is called **codense** iff $h(x) = e$ implies $x = e$ for all $x \in L$.

0.2 Topological Spaces

The category of topological spaces and continuous maps will be denoted **Top**. There is a dual adjunction between it and the category of frames, given by the following contravariant functors.

- $\mathcal{O}$: **Top**$\to$**Frm** assigns to each topological space its frame of open sets and to each continuous function the frame map taking pre-images with respect to that function.
- Σ : **Frm**$\to$**Top** can be expressed in a variety of ways, of which we shall use these two:
  For $h : L \to M$ in **Frm**,
  1. ΣL = the space of all frame homomorphisms $\xi : L \to \mathbf{2}$ with open sets $\Sigma_a = \{\xi |\ \xi(a) = 1\}$ (**2** denotes the two-point frame),
     Σh = composition on the right with h, or
  2. ΣL = the space of all completely prime filters P of L with open sets $\Sigma_a = \{P|\ a \in P\}$,
     Σh = taking pre-images with respect to h.

The fixed elements with respect to this adjunction are called **sober** spaces and **spatial** frames, respectively.

0.3 Subframes, congruences and nuclei

A **subframe** of a frame L is a subset that is closed under the finite meet and arbitrary join of L.

A **congruence** on a frame L is an equivalence relation on L that is a subframe of the product $L \times L$. We usually denote congruences by Greek letters. The set CL of all congruences of a frame L is closed under arbitrary intersections, thus forms a complete lattice; in fact, a frame, with $\bigwedge = \bigcap$, $e = L \times L$ and $0 = \{(x,x) |\ x \in L\}$. The correspondence is functorial, with $Ch(\theta)$ being the congruence generated by $(h \times h)[\theta]$, for any $h : L \to M$ and θ a congruence on L. Of special interest are congruences of the form $\nabla_a = \{(x,y) \in L \times L |\ x \vee a = y \vee a\}$ (the congruence generated by the pair $(0,a)$) and $\Delta_a = \{(x,y) \in L \times L |\ x \wedge a = y \wedge a\}$ (the congruence generated by the pair (a,e)), for $a \in L$. They are complements in CL, and $\{\nabla_a |\ a \in L\} \cup \{\Delta_a |\ a \in L\}$ generates CL. Furthermore, the map $\nabla : L \to CL$ by $\nabla(a) = \nabla_a$ is the universal frame homomorphism among those with image contained within the set of complemented elements of the range frame.

Congruences may equally well be defined for (bounded) distributive lattices, as equivalence relations that are sublattices of the product. The set of congruences again forms a frame; in fact, a Stone frame (for the definition of this, see Section 0.5).

A **nucleus** on L is a closure operator that preserves binary meet. There is a one-one correspondence between the set of nuclei and the set of congruences of a frame (via the fact that the kernel of a nucleus is a congruence).

0.4 Downsets, upsets and ideals

For any element x in a partially ordered set X, we write $\downarrow x = \{y \in X | \ y \leq x\}$ and $\uparrow x = \{y \in X | \ y \geq x\}$. A subset Y of X will be called a **downset** iff $z \leq y$, $y \in Y$ implies $z \in Y$, and an **upset** iff $z \geq y$, $y \in Y$ implies $z \in Y$.

One may define a functor $\mathcal{D}$ from the category of meet-semilattices to that of frames, by letting $\mathcal{D}A$ be the frame of downsets of A, and $\mathcal{D}h$ give the downset generated by the image of h.

An **ideal** of a frame (or a distributive lattice) is a downset that is closed under finite joins; a **filter** is an upset that is closed under finite meets.

The functor $\mathcal{J}$ from the category of (bounded) distributive lattices to frames is given by letting $\mathcal{J}A$ be the frame of ideals of A and $\mathcal{D}h$ giving the ideal generated by the image of h.

Analogous definitions may be used to define functors $\mathcal{D}$: **Frm**→**Frm** and $\mathcal{J}$: **Frm**→**Frm**.

0.5 Separation and covering properties

The following terminology has been used to describe frames. In this section, L will denote a frame, and $a, b, c \ldots$ elements of L.

- **Regular frames.** We write $a \prec b$ (and read 'a is rather below b') iff there exists $c \in L$ such that $a \wedge c = 0$ and $b \vee c = e$. Then L is **regular** iff $a = \bigvee\{b | \ b \prec a\}$ for all $a \in L$.
- **Compact frames.** An element $c \in L$ is called **compact** iff, whenever $c \leq \bigvee X$ for some $X \subseteq L$, it follows that $c \leq \bigvee F$ for some finite subset $F \subseteq X$. Then L is called **compact** iff $e \in L$ is compact.

- **Zero-dimensional frames, Stone frames and Boolean frames.**
  Every element $a \in L$ has a **pseudocomplement** given by $a^* = \bigvee\{b \mid b \wedge a = 0\}$. L is called **Boolean** iff $a \vee a^* = e$ for *all* $a \in L$, and **zero-dimensional** iff it is generated by such a. A **Stone** frame is a compact, zero-dimensional frame.

- **Coherent frames.**
  L is called **coherent** iff the set of all its compact elements forms a sublattice generating it. The morphisms used in the category of coherent frames are those that preserve compact elements.

- **Continuous and stably continuous frames.**
  The relation $\ll$ on a frame is given by: $a \ll b$ ('a is way below b') iff, whenever $b \leq \bigvee X$ for some $X \subseteq L$ it follows that $a \leq \bigvee F$ for some finite $F \subseteq X$. Then L is called **continuous** iff $a = \bigvee\{b \mid b \ll a\}$ for each $a \in L$. A compact element is then one which satisfies $c \ll c$. Furthermore, a frame is **stably continuous** iff it is continuous and the relation $\ll$ is closed under finite meets (including the top) of L. The morphisms used in the category of stably continuous frames are those which preserve $\ll$.

- **Supercoherent and stably supercontinuous frames.**
  The relation $\lll$ is given by: $a \lll b$ iff $b \leq \bigvee X$ for some $X \subseteq L$ implies that $a \leq x$ for some $x \in X$. An element $c \in L$ is called **supercompact** iff $c \lll c$. L is **supercoherent** iff every element is a join of supercompact elements, and finite meets preserve supercompactness. L is **stably supercontinuous** if $a = \bigvee\{b \mid b \lll a\}$ for each $a \in L$, and $\lll$ is a meet-subsemilattice of $L \times L$.

- **Normal frames.**
  L is **normal** iff, whenever $a \vee b = e$ in L, there exist $u, v \in L$ such that $u \wedge v = 0$ and $a \vee u = e = b \vee v$.

Chapter 1

Basic definitions and constructions

1.1 Biframes and bispaces

The definition of a biframe that we present here, was first given in the 1983 paper [10]. The results of this section are also taken from there.

Definition 1.1 *1. A* **biframe** $L = (L_0, L_1, L_2)$ *is a triple in which L_0 is a frame and L_1 and L_2 are subframes of L_0 which together generate it. This means that any element a of L_0 can be expressed as an arbitrary join of finite meets of elements of $L_1 \cup L_2$, or equivalently, $a = \bigvee_\alpha x_\alpha \wedge y_\alpha$ for some $x_\alpha \in L_1$, $y_\alpha \in L_2$.*

2. A **biframe homomorphism** *(or, simply a biframe map) $h : L \to M$ between biframes is a frame homomorphism from L_0 to M_0 for which the restrictions $h|L_i : L_i \to M_i$ $(i = 1, 2)$ are also frame homomorphisms.*

3. The category of biframes and their homomorphisms we denote by **BiFrm**.

Terminology

- We refer to L_0 as the **total part** of L, and L_1 and L_2 as its **first** and **second** parts.
- The notation L_i, L_k will be reserved for referring to the first or second parts of L, so $i, k = 1, 2$, $i \neq k$.
- The restrictions of a biframe map $h : L \to M$ to its various parts will be written $h_0 : L_0 \to M_0$, $h_i : L_i \to M_i$, or equivalently, $h|L_0$, $h|L_i$. If no confusion can result, we may also simply write $h : L_0 \to M_0$, $h : L_i \to M_i$.

The following notions about biframe maps will come in useful:

Definition 1.2 *The biframe map $h : L \to M$ will be called*

- **dense** *iff h_0 is dense, that is $a = 0$ whenever $h(a) = 0$, for any $a \in L_0$*
- **codense** *iff h_0 is codense, that is $a = e$ whenever $h(a) = e$, for any $a \in L_0$*
- **onto** *iff h_1 and h_2 are both onto.*

It is clear that a dense (respectively, codense) biframe map has first and second parts dense (respectively, codense), and that an onto biframe map has its total part onto.

The functor $D : \mathbf{Frm} \to \mathbf{BiFrm}$ given by $D(L) = (L, L, L)$ (and the obvious action on maps) is an embedding of **Frm** into **BiFrm**, and allows us to regard biframe notions as extensions of frame notions. The left adjoint of D is given by $T : \mathbf{BiFrm} \to \mathbf{Frm}$, $T(L_0, L_1, L_2) = L_0$ and $Th = h|L_0$.

Definition 1.3 *[23]*

1. *A* **bispace** *(or bitopological space)* $X = (|X|, \mathcal{U}_1, \mathcal{U}_2)$ *is a triple consisting of a set* $|X|$ *and two topologies* $\mathcal{U}_1$ *and* $\mathcal{U}_2$ *on* $|X|$.

2. *A* **bicontinuous map** $f : X \to Y$ *between bispaces* $X = (|X|, \mathcal{U}_1, \mathcal{U}_2)$ *and* $Y = (|Y|, \mathcal{V}_1, \mathcal{V}_2)$ *is a function between their underlying sets for which* $f : (|X|, \mathcal{U}_i) \to (|Y|, \mathcal{V}_i)$ *is continuous for* $i = 1, 2$.

3. *The category of bispaces and bicontinuous maps will be called* **BiTop**.

The well-known dual adjunction between topological spaces and frames may be extended to one between bispaces and biframes. (We use the same notation for both.)

- The contravariant functor $\mathcal{O}$: **BiTop**$\to$**BiFrm** is defined as follows:
  For $X = (|X|, \mathcal{U}_1, \mathcal{U}_2)$, $\mathcal{O}X = (\mathcal{O}_0 X, \mathcal{O}_1 X, \mathcal{O}_2 X) = (\mathcal{U}_1 \vee \mathcal{U}_2, \mathcal{U}_1, \mathcal{U}_2)$ where $\mathcal{U}_1 \vee \mathcal{U}_2$ is the coarsest topology finer than $\mathcal{U}_1$ and $\mathcal{U}_2$.
  For $f : X \to Y$, $\mathcal{O}f : \mathcal{O}Y \to \mathcal{O}X$ is given by $\mathcal{O}f(U) = f^{-1}(U)$.

- The contravariant functor Σ : **BiFrm**$\to$**BiTop** may be described in various ways, of which we use the following two. For $L = (L_0, L_1, L_2)$,
  $\Sigma L = (|\Sigma L_0|, \{\Sigma_x : x \in L_1\}, \{\Sigma_y : y \in L_2\})$ and for $h : L \to M$, $\Sigma h : \Sigma M \to \Sigma L$ are given by

  1. $|\Sigma L_0|$ is the set of all completely prime filters in the frame L_0 and
     $\Sigma_a = \{P \in |\Sigma L_0| : a \in P\}$ for $a \in L_0$.
     $\Sigma h(P) = h^{-1}(P)$.

  2. $|\Sigma L_0|$ is the set of all frame homomorphisms $\xi : L \to 2$ and
     $\Sigma_a = \{\xi |\ \xi(a) = 1\}$.
     $\Sigma h(\xi) = \xi.h$.

- The natural transformations are:
  For a bispace X, $\sigma_X : X \to \Sigma\mathcal{O}X$ sends $x \in |X|$ to the filter of open neighbourhoods of x in the topology $\mathcal{O}_0 X$.
  For a biframe L, $o_L : L \to \mathcal{O}\Sigma L$ sends $a \in L_0$ to Σ_a.

Proposition 1.1 *The contravariant functors $\mathcal{O}$:* **BiTop**$\to$**BiFrm** *and Σ :* **BiFrm**$\to$**BiTop** *are adjoint on the right with respect to the adjunctions $\sigma_X : X \to \Sigma\mathcal{O}X$ and $o_L : L \to \mathcal{O}\Sigma L$.*

Definition 1.4 1. *A bispace X is* **sober** *iff the map $\sigma_X : X \to \Sigma\mathcal{O}X$ is a* **BiTop**-*isomorphism.*

2. *A biframe L is* **spatial** *iff the map $o_L : L \to \mathcal{O}\Sigma L$ is a* **BiFrm**-*isomorphism.*

The dual adjunction between $\mathcal{O}$ and Σ restricts to a duality between the sober bispaces and the spatial biframes, and this duality is the largest one contained in the given dual adjunction.

Proposition 1.2 1. *The bispace $X = (|X|, \mathcal{U}_1, \mathcal{U}_2)$ is sober iff the topological space $(|X|, \mathcal{U}_1 \vee \mathcal{U}_2)$ is sober.*

2. *The biframe $L = (L_0, L_1, L_2)$ is spatial iff the frame L_0 is spatial.*

1.2 Compactness and regularity

The following definitions and results are also taken from [10]. The notions of compactness and regularity for frames were defined in Section 0.5.

Definition 1.5 1. *Let $L = (L_0, L_1, L_2)$ be a biframe. For $x, y \in L_i$, we write $x \prec_i y$ iff there exists $c \in L_k$ $(i \neq k)$ such that $x \wedge c = 0$ and $y \vee c = e$.*

2. *L is called* **regular** *iff* $x = \bigvee z(z \in L_i,\ z \prec_i x)$ *for all* $x \in L_i$ $(i = 1, 2)$.

3. *A bispace X is* **regular** *iff the biframe* $\mathcal{O}X$ *is regular.*

Remarks

- If (L_0, L_1, L_2) is regular, then L_0 is a regular frame.
- The image of a regular biframe under a biframe map is again regular.
- A bispace X is regular iff, whenever $x \in U$ for some $U \in \mathcal{O}_i X$, there exists $V \in \mathcal{O}_i X$ such that $x \in V$ and the $\mathcal{O}_k X$-closure of V is contained in U $(i \neq k)$.

Definition 1.6 1. *A biframe* $L = (L_0, L_1, L_2)$ *is* **compact** *iff* L_0 *is a compact frame.*

2. *A bispace X will be called* **compact** *iff* $\mathcal{O}X$ *is a compact biframe.*

3. *A bispace* $X = (|X|, \mathcal{U}_1, \mathcal{U}_2)$ *will be called* T_0 *iff* $(|X|, \mathcal{U}_1 \vee \mathcal{U}_2)$ *is a* T_0 *space.*

Definition . *The full subcategory of* **BiFrm** *consisting of the compact, regular biframes will be denoted* **KRBiFrm**.

Proposition 1.3 *(BUT) Under the duality induced by* $\mathcal{O}$ *and* Σ*, the compact, regular biframes correspond to the compact, regular,* T_0 *bispaces.*

The result that the compact, regular biframes are coreflective in **BiFrm** is also given in [10]. We present it in detail here because we shall need the functors constructed later, and because it is an important example of biframe compactifications, which we study in more detail in a later chapter.

Lemma 1.1 *Any biframe has a largest regular subbiframe $\mathcal{R}L$. This defines a coreflection functor $\mathcal{R}$ from* **BiFrm** *to its full subcategory of regular biframes.*

PROOF. Suppose $M \subseteq L$ is generated by the regular subbiframes $R_\alpha \subseteq L$. Then each $a \in M_i$ is a join of elements $b_1 \wedge \ldots \wedge b_n$, $b_t \in (R_\alpha)_i$ (some α). Suppose $b_t = \bigvee_\gamma x_{t\gamma}$, where $x_{t\gamma} \in (R_\alpha)_i$ and $x_{t\gamma} \prec_i b_t$, witnessed by $x_{t\gamma} \wedge z_{t\gamma} = 0$ and $b_t \vee z_{t\gamma} = e$, $z_{t\gamma} \in (R_\alpha)_k$ $(k \neq i)$. Then $b_1 \wedge \ldots \wedge b_n = \bigvee x_{1\gamma_1} \wedge \ldots \wedge x_{n\gamma_n}$, where $(x_{1\gamma_1} \wedge \ldots \wedge x_{n\gamma_n}) \wedge (z_{1\gamma_1} \vee \ldots \vee z_{n\gamma_n}) = 0$ and $(b_1 \wedge \ldots \wedge b_n) \vee (z_{1\gamma_1} \vee \ldots \vee z_{n\gamma_n}) = e$. Hence M is a regular biframe. The coreflection property then follows from the fact that the image of a regular biframe is regular. ∎

In the next definition we use the same notation, namely $\mathcal{J}$, for the functor taking ideals, whether of frames or biframes. The context should make clear which is meant.

Definition 1.8 *The functor $\mathcal{J}$:* **BiFrm**$\rightarrow$**BiFrm** *is given by the following:*
$\mathcal{J}L_0$ denotes the frame of ideals of L_0. Then
$(\mathcal{J}L)_i = \{J \in \mathcal{J}L_0 |\ J$ *is generated by* $J \cap L_i\}$
$(\mathcal{J}L)_0 =$ *the subframe of* $\mathcal{J}L_0$ *generated by* $(\mathcal{J}L)_1 \cup (\mathcal{J}L)_2$.
For maps, $\mathcal{J}h : \mathcal{J}L \rightarrow \mathcal{J}M$ *is* $\mathcal{J}h(J) =$ *the ideal generated in* $(\mathcal{J}M)_0$ *by the image* $h[J]$.

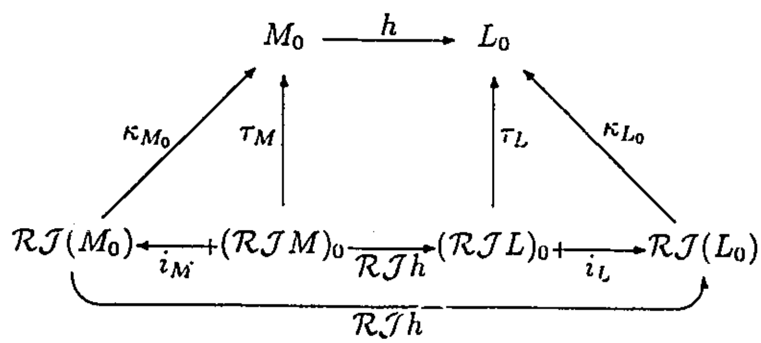
Taking joins of ideals gives a biframe map $\mathcal{J}L \rightarrow L$ (since $J \in (\mathcal{J}L)_i$ implies that $\bigvee J \in L_i$), which we restrict to $\tau_L : \mathcal{R}\mathcal{J}L \rightarrow L$.

Lemma 1.2 *For a compact, regular biframe L, $\tau_L : \mathcal{R}\mathcal{J}L \rightarrow L$ is an isomorphism.*

PROOF. Consider the function $j : L \rightarrow \mathcal{J}L$ given by $j(a) = \{x \in L_0 |\ x \prec a\}$. Now $j(a)$ is an ideal, and we check that $a \in L_i$ implies that $j(a) \in (\mathcal{J}L)_i$: Take $x \in j(a)$, $a = \bigvee y(y \prec_i a)$ so $x \wedge c = 0$, $\bigvee y(y \prec_i a) \vee c = e$ for some $c \in L_k$. Compactness gives a $\hat{y}$ for which $\hat{y} \vee c = e$, $\hat{y} \prec_i a$ so that $x \prec \hat{y} \in j(a) \cap L_i$. Thus $j(a)$ is generated by $j(a) \cap L_i$. Compactness and regularity of L_0 show that $j : L_0 \to (\mathcal{J}L)_0$ is a frame map. Since the image of a regular biframe is regular, we can regard j as a biframe map $j : L \to \mathcal{RJ}L$. Now $\tau_L.j$ is the identity map on L by the regularity of L_0. Further, $\tau_L : (\mathcal{RJ}L)_0 \to L_0$ is codense (by compactness of L_0), hence one-one, and onto, by the above. Hence τ_L is a biframe isomorphism. ∎

Proposition 1.4 *The compact, regular biframes are coreflective in* **BiFrm**, *with coreflection maps* $\tau_L : \mathcal{RJ}L \to L$.

PROOF. Let $h : M \to L$ be a biframe map, and M be compact, regular. By the previous lemma and the naturality of τ, we may write $h = \tau_L.\mathcal{RJ}h.\tau_M^{-1}$. Uniqueness of this factorization follows from the corresponding frame fact: Let $h = \tau_L.g$ for some $g : M \to \mathcal{RJ}L$ and consider the following *frame* diagram.



Here κ_{L_0}, κ_{M_0} are join maps, and i_L, i_M are inclusions. We obtain $i_L.g = \mathcal{RJ}h.\kappa_{M_0}^{-1}$, and hence $g = \mathcal{RJ}h.\tau_M^{-1}$. ∎

Remark

The definition of $\mathcal{J}$ used in [10] was slightly different to the one above. It used

$(\tilde{\mathcal{J}}L)_i = \{J \in \mathcal{J}L_0 | \bigvee J \in L_i\}$

$(\tilde{\mathcal{J}}L)_0 =$ the subframe of $\mathcal{J}L_0$ generated by $(\tilde{\mathcal{J}}L)_1 \cup (\tilde{\mathcal{J}}L)_2$.

Certainly $(\mathcal{J}L)_i \subseteq (\widetilde{\mathcal{J}}L)_i$.

$\mathcal{J}$ and $\widetilde{\mathcal{J}}$ need not coincide, even for regular biframes, as the following example shows:

$\mathcal{L}_0$ = all open subsets of the real line, **R**

$\mathcal{L}_1$ = all open downsets

$\mathcal{L}_2$ = all open upsets

For $U, V \in \mathcal{L}_i$, $U \prec_i V$ iff $U \subset V$, so $\mathcal{L}$ is a regular biframe. Let J be the ideal generated by all open intervals of finite length. Then $\bigvee J = \mathbf{R}$, but J contains no downsets nor upsets. Thus $\bigvee J \in \mathcal{L}_i$ ($i = 1$ or 2) but J is not generated by $J \cap \mathcal{L}_i$, because $J \cap \mathcal{L}_i = \{\emptyset\}$.

1.2.1 Connectedness and the compact, regular coreflection

For a discussion of the frame notions on which this section is based, see [1].

Definition 1.9 *For a biframe L,*

- *we call $c \in L_0$ a* **connected** *element iff whenever $c = x \vee y$, $x \wedge y = 0$ for some $x \in L_i$, $y \in L_k$, it follows that $c = x$ or $c = y$,*
- *and we call L a* **connected** *biframe iff $e \in L_0$ is connected.*

In the following, **2** as usual denotes the two-element biframe; **4** denotes the biframe with total part the four-element Boolean algebra $\{0, a, b, e\}$, first part $\{0, a, e\}$ and second part $\{0, b, e\}$.

Lemma 1.3 *A biframe L is connected iff each homomorphism $\mathbf{4} \to L$ factors through the unique map $\mathbf{2} \to L$.*

PROOF. ($\Longrightarrow$) For any biframe map $h : \mathbf{4} \to L$, $h(a) \vee h(b) = e$, $h(a) \wedge h(b) = 0$ and $h(a) \in L_1$, $h(b) \in L_2$. Since $e \in L_0$ is connected, $h(a) = e$ or $h(b) = e$ and we obtain the desired factorization by defining $\bar{h} : \mathbf{4} \rightarrow \mathbf{2}$ by $\bar{h}(a) = e$ or $\bar{h}(a) = 0$, respectively. ($\Longleftarrow$) Suppose $x \vee y = e$, $x \wedge y = 0$ and $x \in L_1$, $y \in L_2$. The map $h : \mathbf{4} \rightarrow L$ defined by $h(a) = x$ and $h(b) = y$ factors through $\mathbf{2} \rightarrow L$, so $x = e$ or $y = e$. ∎

Lemma 1.4 *Any dense biframe homomorphism $h : M \rightarrow L$ reflects connected elements, that is, $h(c) \in L$ connected implies $c \in M$ connected. In fact, it suffices for $h|M_1$ and $h|M_2$ to be dense.*

PROOF. Let $x \vee y = c$ and $x \wedge y = 0$ for $x \in M_i$, $y \in M_k$. Then $h(x) \vee h(y) = h(c)$, $h(x) \wedge h(y) = 0$ and $h(x) \in L_i$, $h(y) \in L_k$. Since $h(c)$ is connected, $h(x) = h(c)$ or $h(y) = h(c)$, so $h(y) = 0$ or $h(x) = 0$, and by denseness $y = 0$ or $x = 0$; so $x = c$ or $y = c$. ∎

Proposition 1.5 *A biframe is connected iff its compact, regular coreflection is connected.*

PROOF. Let $\tau : M \rightarrow L$ be the compact, regular coreflection of L. If L is connected, so is M, since τ is dense. Conversely, assume that M is connected and consider any $h : \mathbf{4} \rightarrow L$. The biframe $\mathbf{4}$ is compact, regular, so the universal property of τ gives $f : \mathbf{4} \rightarrow M$ with $\tau.f = h$. Since M is connected, $f : \mathbf{4} \rightarrow M$ factors through $\mathbf{2} \rightarrow M$, so $h : \mathbf{4} \rightarrow L$ factors through $\mathbf{2} \rightarrow L$. Thus L is connected. ∎

1.3 Zero-dimensional and Stone biframes

The following notions and results can be found in [4] and [9]. We repeat the construction of the functors we shall use later, but omit proofs of the theorems.

Definition 1.10 *1. For $x \in L_i$, we denote by $x^\bullet$ the largest $y \in L_k$ for which $x \wedge y = 0$; that is, $x^\bullet = \bigvee z(z \wedge x = 0,\ z \in L_k)$.*

2. *A biframe L will be called* **Boolean** *iff $x \vee x^\bullet = e$ for all $x \in L_1 \cup L_2$*
We note that whenever $x \vee x^\bullet = e$ for $x \in L_i$, $x^\bullet$ is the complement of x in L_0, and $x^\bullet \in L_k$.

3. *A biframe L is* **zero-dimensional** *if and only if each L_i is generated by those $x \in L_i$ satisfying $x \vee x^\bullet = e$.*

4. (a) *A compact, zero-dimensional biframe will be called a* **Stone** *biframe.*
   (b) *The full subcategory of* **BiFrm** *consisting of the Stone biframes will be written* **StBiFrm**.

5. (a) *A* **Boolean bilattice** *$B = (B_0, B_1, B_2)$ is a triple in which B_0 is a Boolean algebra, B_1 and B_2 are sublattices of B_0 such that B_0 is generated by $B_1 \cup B_2$ and an element of B_0 is in B_i if and only if its complement is in B_k $(i \neq k)$.*
   (b) *The homomorphisms of Boolean bilattices are the Boolean homomorphisms between their total parts that preserve the two specified sublattices.*
   (c) *The resulting category will be written* **BooBiLatt**.

For a correspondence between Stone biframes and Boolean bilattices we consider the following functors:

- The **Boolean part** BL of a biframe L is the bilattice whose i-th part is $(BL)_i = \{x \in L_i | \, x \vee x^\bullet = e\}$ and whose total part $(BL)_0$ is the sublattice of L_0 generated by $(BL)_1 \cup (BL)_2$.
- The **ideal biframe** $\mathcal{J}A$ of a Boolean bilattice A has total part $(\mathcal{J}A)_0$ the ideal frame of A_0 and i-th part $(\mathcal{J}A)_i = \{J \in (\mathcal{J}A)_0 | \, J \text{ is generated by } J \cap A_i\}$.

(This is the same notation as that used for the ideal biframe functor from **BiFrm** to **BiFrm** in Definition 1.8: the context should suffice to prevent confusion.)

Proposition 1.6 *The categories of Stone biframes and Boolean bilattices are equivalent, by the functors B and $\mathcal{J}$.*

The natural transformations are $\alpha_A : A \to B\mathcal{J}A$ by $\alpha_A(a) = \downarrow a$ (for bilattices A) and $\sigma_L : \mathcal{J}BL \to L$ by the map taking joins of ideals (for biframes L).

Proposition 1.7 *The Stone biframes are coreflective in* **BiFrm**, *with coreflection maps $\sigma_L : \mathcal{J}BL \to L$ by taking joins.*

Definition 1.11 *Let $X = (|X|, \mathcal{U}_1, \mathcal{U}_2)$ be a bispace.*

1. *X is* **pairwise Hausdorff** *iff any two unequal points can be separated by disjoint open sets, one of which is in $\mathcal{U}_1$ and the other in $\mathcal{U}_2$.*

2. *X is* **pairwise zero-dimensional** *iff each $\mathcal{U}_i$ is generated by those $U \in \mathcal{U}_i$ for which there exists $V \in \mathcal{U}_k$ disjoint from U, satisfying $U \cup V = X$.*

3. *X is* **Boolean** *iff it is compact, pairwise Hausdorff and pairwise zero-dimensional.*

Proposition 1.8 *(BUT) The category of Boolean bispaces is dually equivalent to the category of Boolean bilattices.*

Proposition 1.9 *(BUT) The Boolean bispaces are reflective in* **BiTop**.

1.3.1 Compact, regular biframes versus stably continuous frames

An interesting connection between the compact, regular biframes and the stably continuous frames (for unfamiliar definitions, see Section 0.5) was given in the paper [8], which presents the following category equivalence.

- F : **KRBiFrm**$\rightarrow$**StContFrm** is given by taking first parts, that is, $FL = L_1$ and $Fh = h_1$.

- G : **StContFrm**$\rightarrow$**KRBiFrm** needs these definitions.
  A filter F in a frame L is called **Scott open** iff, whenever $\bigvee D \in F$ it follows that $D \cap F \neq \emptyset$, for any updirected $D \subseteq F$. (D is updirected iff whenever d_1, $d_2 \in D$ there exists $d_3 \in D$ such that $d_1 \leq d_3$ and $d_2 \leq d_3$.) For stably continuous frames, the set of all Scott open filters forms a (stably continuous) frame. Let $\Delta_F = \bigvee \Delta_a (a \in F)$ for any Scott open filter F. Then define $GL = (\mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2)$, where
  $\mathcal{L}_1 = \{\nabla_a |\ a \in L\}$
  $\mathcal{L}_2 = \{\Delta_F |\ F \text{ is a Scott open filter on } L\}$
  $\mathcal{L}_0 =$ the subframe of CL_0 generated by $\mathcal{L}_1 \cup \mathcal{L}_2$.
  For any $h : L \rightarrow M$ between stably continuous frames, $Gh : GL \rightarrow GM$ is given by the restriction of $Ch : CL \rightarrow CM$. (See Section 0.3.)

- The functor F induces a right adjoint equivalence between **KRBiFrm** and **StContFrm**. It has inverse G.

We do not pursue this equivalence any further, other than to note that the following equivalences are contained in it. (For the equivalence concerning the Stone biframes, see [4].)

$$\begin{array}{ccccc}
\mathbf{KRBiFrm} & \supseteq & \mathbf{StBiFrm} & \supseteq & \mathbf{KBooBiFrm} \\
F \downarrow \uparrow G & & \downarrow \uparrow & & \downarrow \uparrow \\
\mathbf{StContFrm} & \supseteq & \mathbf{CohFrm} & \supseteq & \mathbf{CFrm}
\end{array}$$

where **CohFrm** denotes the coherent frames (see Section 0.5), **KBooBiFrm** denotes the compact, regular, Boolean biframes and **CFrm** stands for those frames in which *every* element is compact.

There certainly are non-finite frames in which every element is compact: for example, the cofinite topology on an infinite set, or a dually well-ordered set. Thus the compact, Boolean biframes need not be finite. This contrasts with the fact that, under the assumption of the Boolean Ultrafilter Theorem, every compact, Boolean frame is finite.

1.4 Congruences

1.4.1 Congruence biframes of biframes

The congruences of a frame were defined and discussed in Section 0.3.

Definition 1.12 *The* **congruence biframe** $CL = (C_0L_0, C_1L_1, C_2L_2)$ *of a biframe* L *is given as follows:* C_iL_i *is the subframe of* CL_0 *(the frame of congruences of* L_0*) generated by* $\{\nabla_x | \ x \in L_i\} \cup \{\Delta_y | \ y \in L_k\}$ $(i \neq k)$ *and* C_0L_0 *is the subframe of* CL_0 *generated by* $C_1L_1 \cup C_2L_2$.

The function $\nabla = \nabla^L : L \rightarrow CL$ given by $\nabla(a) = \nabla_a$ for all $a \in L_0$, is a biframe embedding. It is also an epimorphism in **BiFrm**, since $\nabla : L_0 \rightarrow C_0L_0$ is an epimorphism in **Frm** (see Proposition 2.2): if $f.\nabla_x = g.\nabla_x$ for all $x \in L_i$, then $f.\Delta_x = g.\Delta_x$ for all $x \in L_i$, since frame homomorphisms preserve complements, and C_0L_0 is generated by congruences of this form, so $f = g$.

The correspondence $L \rightarrow CL$ yields a functor C : **BiFrm**$\rightarrow$**BiFrm**. For $h : L \rightarrow M$, the map $Ch : CL \rightarrow CM$ is given by $Ch(\theta)$ = congruence on M_0 generated by $(h \times h)[\theta]$. Since $Ch(\nabla_x) = \nabla_{h(x)}$ and $Ch(\Delta_x) = \Delta_{h(x)}$ for $x \in L_i$, Ch is a biframe homomorphism.

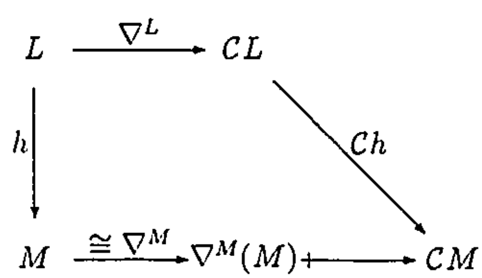
We recall from Definition 1.10 that a biframe L is called **Boolean** if every $x \in L_i$ has a complement in L_0, and that complement is in L_k $(i \neq k)$. The **Boolean part** $BL = (B_0L_0, B_1L_1, B_2L_2)$ of any biframe L has
$B_iL_i = \{x \in L_i |\ x$ has a complement in L_0, and that complement is in $L_k\}$ and B_0L_0 the sublattice of L_0 generated by $B_1L_1 \cup B_2L_2$.

Lemma 1.5 $\nabla : L \to CL$ *is an isomorphism if and only if L is Boolean.*

PROOF. ∇ is an isomorphism
iff the first and second parts of ∇ are both onto
iff for any $x \in L_i$ there is a $y \in L_k$ with $\nabla_y = \Delta_x$
iff for any $x \in L_i$ there is a $y \in L_k$ with y the complement of x in L_0. ∎

Proposition 1.10 $\nabla : L \to CL$ *is the universal biframe homomorphism among those $h : L \to M$ with $Image(h) \subseteq BM$.*

PROOF. Suppose $h : L \to M$ is given with $Image(h) \subseteq BM$. Consider the diagram



We check that $Ch(\theta) \in \nabla^M(M_0)$ for any $\theta \in C_0L_0$: For $x \in L_i$, $Ch(\nabla_x) = \nabla_{h(x)} \in \nabla^M(M_i)$ and $Ch(\Delta_x) = \Delta_{h(x)} = \nabla_{h(x)^*} \in \nabla^M(M_k)$ since $Image(h) \subseteq BM$

(here $h(x)^\bullet$ indicates the complement of $h(x)$). Since $\nabla^M : M \rightarrow \nabla^M(M)$ is an isomorphism, we may define $\tilde{h} : CL \rightarrow M$ by $\tilde{h} = (\nabla^M)^{-1}.Ch$. Uniqueness of $\tilde{h}$ follows from ∇^L being epic. ∎

1.4.2 Congruence biframes of bilattices

For a definition of a bounded distributive bilattice, we refer the reader to Definition 4.2. The **congruence biframe** $CA = (CA_0, C_1A_1, C_2A_2)$ of a bilattice (A_0, A_1, A_2) is given as follows:
C_iA_i is the subframe of CA_0 (the frame of congruences of the bounded distributive lattice A_0) generated by $\{\nabla_x | \ x \in A_i\} \cup \{\Delta_y | \ y \in A_k\}$ $(i \neq k)$. Since CA_0 is generated by $C_1A_1 \cup C_2A_2$, CA is a biframe. It is compact (since ∇ is finitely generated) and zero-dimensional (since, for any $x \in A_i$, ∇_x and Δ_x are complements in CA_0). We recall that a bilattice (A_0, A_1, A_2) is called **Boolean** if each $x \in A_i$ has a complement in A_0, and that complement is in A_k $(i \neq k)$.

Lemma 1.6 *For any Boolean bilattice B, CB is isomorphic to $\mathcal{J}B$.*

PROOF. Define $f : CB \rightarrow \mathcal{J}B$ by $f(\theta) = \theta[0] = \{a \in B | \ (0, a) \in \theta\}$. For any $\theta \in CB$, $\theta[0]$ is certainly an ideal of B_0; if $\theta \in C_iB_i$, then $J = \theta[0]$ is generated by $J \cap B_i$: Since B is Boolean, $\theta = \bigvee \nabla_x(\ (0, x) \in \theta, \ x \in B_i) = \bigcup \nabla_x(\ (0, x) \in \theta, \ x \in B_i)$. If $y \in \theta[0]$, $(0, y) \in \nabla_x$ for some $x \in B_i$ such that $(0, x) \in \theta$. So $y \leq x \in J \cap B_i$, as required.
Now f preserves and reflects inclusion, and is one-one. We show it is onto: For $J \in \mathcal{J}_iB_i$, let $\theta = \bigcup \nabla_x(\ x \in J \cap B_i)$. Then $\theta \in C_iB_i$ and $\theta[0] = J$. Hence f is a biframe isomorphism. ∎

Proposition 1.11 *The CA, $A \in$* **BiLatt**, *are exactly the Stone biframes.*

PROOF. It was already noted above that each CA is compact and zero-dimensional. Conversely, if L is a Stone biframe, $L \cong \mathcal{J}BL \cong CBL$. ∎

1.4.3 Congruence biframes of frames

There is a natural way of regarding the congruences of a frame as a biframe (see [9]).

For a *frame* L, the congruence biframe CL has

$(CL)_0 = CL$, the frame of all congruences of L

$(CL)_1 = \{\nabla_a |\ a \in L\}$

$(CL)_2 =$ the subframe of CL generated by $\{\Delta_a |\ a \in L\}$.

This congruence biframe is zero-dimensional, in fact strongly zero-dimensional, in the sense of [9].

Chapter 2

Some categorical aspects of biframes

The category **BiFrm** is complete and cocomplete. We describe its products, coproducts, equalizers and coequalizers.

Products The product of the biframes $L_\alpha = (L_0{}^\alpha, L_1{}^\alpha, L_2{}^\alpha)$, $\alpha \in I$, is given by $\prod_{\alpha\in I} L_\alpha = (\prod_\alpha L_0{}^\alpha, \prod_\alpha L_1{}^\alpha, \prod_\alpha L_2{}^\alpha)$, with projection maps determined by the frame projections $p_\beta : \prod_\alpha L_0{}^\alpha \to L_0{}^\beta$. (We note that $\prod_\alpha L_0{}^\alpha$ is generated by $\prod_\alpha L_1{}^\alpha \cup \prod_\alpha L_2{}^\alpha$.)
This has the universal property of a categorical product:
Given biframe maps $h_\alpha : M \to L_\alpha$ we obtain a unique frame homomorphism $e : M_0 \to \prod_\alpha L_0{}^\alpha$ for which $p_\alpha.e = h_\alpha$ for each $\alpha \in I$. For $x \in M_i$, $p_\alpha.e(x) \in L_i{}^\alpha$ for all α, so $e(x) \in \prod_\alpha L_i{}^\alpha$; hence $e : M \to \prod_\alpha L^\alpha$ is the unique biframe map required.

Equalizers Given biframe maps $f, g : L \to M$, let $K_i = \{x \in L_i |\ f(x) = g(x)\}$ (this is certainly a subframe of L_0) and let K_0 be the subframe of L_0 generated by

$K_1 \cup K_2$. Then $h : K \rightarrow L$ is the equalizer of f and g, where h is the natural embedding map. Certainly $f.h = g.h$. Suppose $l : N \rightarrow L$ satisfies $f.l = g.l$. Then $l(N_i) \subseteq K_i$, so $l(N_0) \subseteq K_0$ and l factors via h. The factorization is unique because h is an embedding.

Example Clearly $K_0 \subseteq \{x \in L_0 |\ f(x) = g(x)\}$. We give an example to show that these two sets are not necessarily equal. Let
$\mathcal{L}_0$ = all open subsets of the closed unit interval E
$\mathcal{L}_1$ = all open downsets
$\mathcal{L}_2$ = all open upsets
Let U be an open interval with end-points different from 0 and 1. Let $f : \mathcal{L}_0 \rightarrow \uparrow U$ be given by $f(W) = W \cup U$, and finally let $(\mathcal{M}_0, \mathcal{M}_1, \mathcal{M}_2) = (\uparrow U, f(\mathcal{L}_1), f(\mathcal{L}_2))$. Then we have a biframe map $f : \mathcal{L} \rightarrow \mathcal{M}$ by $f(W) = W \cup U$. Now $\mathcal{L}_0$ is spatial, so there is a frame map $l : \mathcal{L}_0 \rightarrow \mathbf{2}$ such that $l(U) = 0$. Composing this with the unique map $\mathbf{2} \rightarrow M_0$ we obtain a biframe map $g : \mathcal{L} \rightarrow \mathcal{M}$.
If $g(W) = f(W) = U$ for some $W \in \mathcal{L}_0$, then $W \cup U = U$, that is $W \subseteq U$. If $g(W) = f(W) = E$, then $W \cup U = E$. So $\{W \in \mathcal{L}_0 |\ f(W) = g(W)\} =$
$\{W \in \mathcal{L}_0 |\ W \subseteq U \text{ or } W \cup U = E\}$. However $K_i = \{W \in \mathcal{L}_i |\ f(W) = g(W)\} = \{\emptyset, E\}$.

Coproducts For the biframes $L_\alpha = (L_0{}^\alpha, L_1{}^\alpha, L_2{}^\alpha)$, $\alpha \in I$ let
$h_\alpha : L_0{}^\alpha \rightarrow \bigoplus_{\alpha \in I} L_0{}^\alpha$ be the (frame) coproduct injections. $\bigoplus_\alpha L_\alpha$ has $\bigoplus_\alpha L_0{}^\alpha$ as total part, and the subframe of $\bigoplus_\alpha L_0{}^\alpha$ generated by $\bigcup_\alpha h_\alpha(L_i{}^\alpha)$ as i-th part. $\bigoplus_\alpha L_\alpha$ is a biframe, because $\bigoplus_\alpha L_0{}^\alpha$ is generated by $\bigcup_\alpha h_\alpha(L_0{}^\alpha)$, and the h_α clearly become biframe maps.
This construction has the universal property of the categorical coproduct:
Given $f_\alpha : L_\alpha \rightarrow M$, we obtain a unique $g : \bigoplus_\alpha L_0{}^\alpha \rightarrow M_0$ for which $g.h_\alpha = f_\alpha$, $\alpha \in I$. By the definition of the first and second parts of $\bigoplus_\alpha L_\alpha$, g is a biframe homomorphism.

Coequalizers For $f, g : L \to M$, let $h : M_0 \to K_0$ be the coequalizer, in **Frm**, of $f, g : L_0 \to M_0$. Then $h : M \to K$ is the coequalizer of f and g in **BiFrm**, where $K = (K_0, h(M_1), h(M_2))$. If $l : M \to N$ satisfies $l.f = l.g$, there exists a unique $k : K_0 \to N_0$ for which $k.h = l$. Further, k is a biframe map because $k(h(M_i)) = l(M_i) \subseteq N_i$.

The **initial object** and **terminal object in BiFrm** are $(\mathbf{2}, \mathbf{2}, \mathbf{2})$ and $(\mathbf{1}, \mathbf{1}, \mathbf{1})$ respectively (where **2** and **1** are the two- and one-point frames).

Pullbacks and pushouts Given $h : L \to N$ and $k : M \to N$, let $K_i = \{(x, y) \in L_i \times M_i |\ h(x) = k(y)\}$ and K_0 the subframe of $L_0 \times M_0$ generated by $K_1 \cup K_2$. Then the pullback of h and k is given by $K = (K_0, K_1, K_2)$ with the obvious restrictions of the projection maps.
The example in the section on equalizers may be adapted to show that in general $K_0 \neq \{(x, y) \in L_0 \times M_0 |\ h(x) = k(y)\}$.
To obtain the pushout of $u : L \to M$ and $v : L \to N$, let $s : M \to M \oplus N$ and $t : N \to M \oplus N$ be the coproduct injections and let $h : M \oplus N \to P$ be the coequalizer of $s.u$ and $t.v$. Then $h.s : M \to P$ and $h.t : N \to P$ give the required pushout.

The next propositions describe the monomorphisms and the epimorphisms in **BiFrm**.

Proposition 2.1 *a) h is monic in* **BiFrm** *iff h_1 and h_2 are both one-one.*

b) h_0 one-one implies that h is monic, but not conversely.

PROOF. a) ($\Longrightarrow$) Suppose $h : L \to M$ is monic in **BiFrm** and $k, l : N \to L_1$ are frame maps for which $h_1 k = h_1 l$. Then the biframe maps $k, l : (N, N, \mathbf{2}) \to (L_0, L_1, L_2)$ satisfy $hk = hl$, and so $k = l$. Hence h_1 is monic in **Frm**, and so one-one. A similar argument shows h_2 one-one.

($\Longleftarrow$) If $k, l : N \to L$ are maps in **BiFrm** for which $hk = hl$, then $k_1 = l_1$ and $k_2 = l_2$, and, since N_0 is generated by $N_1 \cup N_2$, $k = l$.

b) The first claim is clear, from (a). For the second, consider the following example: let $\mathcal{L}_0$ = all open subsets of the real unit interval, $\mathcal{L}_1$ = all its open downsets, $\mathcal{L}_2$ = all its open upsets and $\mathcal{M}_0$ = all open subsets of the rational unit interval, $\mathcal{M}_1$ = all its open downsets, $\mathcal{M}_2$ = all its open upsets. Then the biframe map $h : \mathcal{L} \to \mathcal{M}$ given by taking intersections with the rationals has h_1 and h_2 one-one, in fact isomorphisms, but h_0 is not one-one. ∎

Proposition 2.2 *a) h is epic in* **BiFrm** *iff h_0 is epic in* **Frm**.

b) For any h in **BiFrm**, *having h_1 and h_2 onto implies that h_0 is onto, which implies that h is epic, but neither of these implications can be reversed.*

PROOF. a) This holds since the functor T : **BiFrm** $\to$ **Frm** which takes total parts, is faithful and thus preserves and reflects epimorphisms.

b) These two implications are easily checked. The identity function $h : (N, N, \mathbf{2}) \to (N, N, N)$, $N \neq \mathbf{1}$, $\mathbf{2}$, has h_0 onto but h_2 not; the embedding $L \to CL$ of a biframe into its congruence biframe is epic, but need not have total part onto. ∎

In the next result we shall use the characterization of epimorphisms of frames described in [25] to give an analogous characterization for biframes. On page 21 we defined the epic embedding $\nabla : L \to CL$ of a biframe into its congruence biframe. For ordinals α, we define $\mathcal{C}_\alpha(L)$ by the following (cf. [25]):

$$\mathcal{C}_0(L) = L$$

$$\mathcal{C}_{\alpha+1}(L) = \mathcal{C}(\mathcal{C}_\alpha(L))$$

$$\mathcal{C}_\lambda(L) = \varinjlim\{\mathcal{C}_\alpha(L) : \alpha < \lambda\} \text{ for limit ordinals } \lambda$$

The map $\nabla_\alpha : L \to \mathcal{C}_\alpha(L)$ is then defined by the obvious composition; $\mathcal{C}_\alpha$ becomes a functor and ∇_α has total part one-one.

Proposition 2.3 *A biframe map $h : L \to M$ is an epimorphism iff there is an ordinal α for which $\nabla_\alpha(M_0) \subseteq \mathcal{C}_\alpha h((\mathcal{C}_\alpha L)_0)$.*

PROOF. This is easily verified using the fact that maps in **BiFrm** are epimorphisms iff their total parts are epimorphisms in **Frm**, and the characterization of [25] mentioned above. ∎

In the important subcategory consisting of the compact, regular biframes we know a little more about the monics. We recall (see [3]) that in the category of compact, regular *frames*, a map is monic iff it is one-one iff it is dense iff it is codense.

Proposition 2.4 *For $h : L \to M$ in* **KRBiFrm**, *we have that (1) implies (2) and (1) implies (3), where*
(1) h_0 is one-one iff h_1 and h_2 are both one-one.
(2) h_1 and h_2 are both dense iff h_1 and h_2 are both codense.
(3) h is monic in **KRBiFrm**.

PROOF. That (1) implies (2) and (1) implies (3) is clear.
Re (1): ($\Longleftarrow$) This uses the fact (see [8, Lemma 7]) that, in **KRBiFrm**, if h_1 is an isomorphism , h is also an isomorphism. If each h_i is one-one, then $h : L \to Image(h)$ has each h_i one-one, onto, hence an isomorphism, so h_0 is also an isomorphism, hence one-one.
Re (2): ($\Longrightarrow$) Suppose h_1 and h_2 are dense. Take $x \in L_i$ with $h(x) = e$. Then $x = \bigvee z\,(z \in L_i,\ z \prec_i x)$, and compactness gives $z \in L_i$, $z \prec_i x$ with $h(z) = e$. Thus $z \wedge c = 0$, $c \vee x = e$ for some $c \in L_k$; so $h(z) \wedge h(c) = 0$, which gives $h(c) = 0$ and thus $c = 0$ (by the density of h on L_k). So $x = e$, as required.

($\Longleftarrow$) Similar to ($\Longrightarrow$). ∎

We note that h_1 dense does not imply h dense, for h in **KRBiFrm**: Let $L_0 = \{0, a, b, e\}$, the four-element Boolean algebra, $L_1 = \{0, a, e\}$, $L_2 = \{0, b, e\}$ and $h : L \to \mathbf{2}$ the map satisfying $h(a) = e$, $h(b) = 0$. Then h_1 is dense but h_0 is not.

The above characterizations of monomorphisms and epimorphisms show that **BiFrm** is **wellpowered**, but **not co-wellpowered** (since the category **Frm** is not).

BiFrm has a **generating object** (or separator), namely, $(\mathbf{3} \bigoplus \mathbf{3}, \mathbf{3}, \mathbf{3})$: for any $f, g : L \to M$ with $f \neq g$, there exists $x \in L_i$ with $f(x) \neq g(x)$, and we may define a map $(\mathbf{3} \bigoplus \mathbf{3}, \mathbf{3}, \mathbf{3}) \to L$ by sending the middle element of the corresponding $\mathbf{3}$ to x.

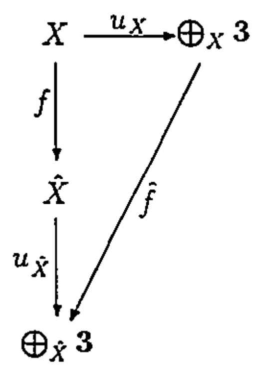
BiFrm has **no cogenerating set** (since this, taken with the properties of completeness and wellpoweredness, would make it cowellpowered).

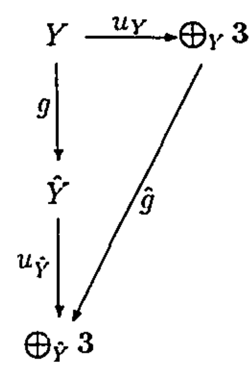
There is no free biframe on one generator: if there were, its total part would have to be $\mathbf{3}$ (the free frame on one generator), so the only possibilities would be $(\mathbf{3}, \mathbf{3}, \mathbf{3})$, $(\mathbf{3}, \mathbf{3}, \mathbf{2})$ or $(\mathbf{3}, \mathbf{2}, \mathbf{3})$. These are not free, since a set map which specifies an element of the total frame not in the first or second parts, cannot be extended to a biframe map.

We do, however, have the following (**Ens** is the category of sets and set functions):

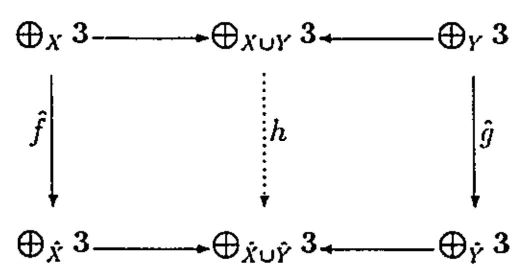
Proposition 2.5 *The functor* U : **BiFrm**$\to$**Ens** $\times$ **Ens** *has a left adjoint, where* $U(L) = (|L_1|, |L_2|)$, $U(h) = (|h_1|, |h_2|)$ *and* $|\ldots|$ *is used to denote underlying sets and set functions, respectively.*

PROOF. We recall that the free frame on a set X may be given by $\bigoplus_X \mathbf{3}$, the X-fold copower of $\mathbf{3}$, with the set map $X \to \bigoplus_X \mathbf{3}$ denoted by u_X. Since taking free frames is left adjoint to the forgetful functor from **Frm** to **Ens**, it must preserve colimits. So $\bigoplus_X \mathbf{3} \oplus \bigoplus_Y \mathbf{3} = \bigoplus_{X \cup Y} \mathbf{3}$. To define a functor $G : \mathbf{Ens} \times \mathbf{Ens} \to \mathbf{BiFrm}$, we let $G(X, Y) = (\bigoplus_{X \cup Y} \mathbf{3}, \bigoplus_X \mathbf{3}, \bigoplus_Y \mathbf{3})$ for any $(X, Y) \in \mathbf{Ens} \times \mathbf{Ens}$. The effect of G on morphisms is given as follows. For $(f, g) : (X, Y) \to (\hat{X}, \hat{Y})$ in $\mathbf{Ens} \times \mathbf{Ens}$, we obtain unique frame homomorphisms $\hat{f}$ and $\hat{g}$ making these diagrams commute:





Since $\bigoplus_{X \cup Y} \mathbf{3} = \bigoplus_X \mathbf{3} \oplus \bigoplus_Y \mathbf{3}$, there is a unique $h : \bigoplus_{X \cup Y} \mathbf{3} \to \bigoplus_{\hat{X} \cup \hat{Y}} \mathbf{3}$ for which the next diagram commutes.



Now define $G(f, g)$ as this $h : (\bigoplus_{X \cup Y} \mathbf{3}, \bigoplus_X \mathbf{3}, \bigoplus_Y \mathbf{3}) \to (\bigoplus_{\hat{X} \cup \hat{Y}} \mathbf{3}, \bigoplus_{\hat{X}} \mathbf{3}, \bigoplus_{\hat{Y}} \mathbf{3})$. An argument similar to the one above may be used to check the universal property for the adjunction. ∎

Chapter 3

Compactifications of biframes

3.1 Compactifications and strong inclusions

In frame theory, the compactifiable biframes are exactly those which admit strong inclusions (see [5]). We present definitions of compactifications and strong inclusions for biframes which allow us to obtain a similar relationship here: we acknowledge our debt to [7].

Definition 3.1 *A* **compactification** *of a biframe L is a dense, onto biframe homomorphism $h : M \to L$ from a compact, regular biframe M to L.*

Definition 3.2 *A* **strong inclusion** *on a biframe L is a pair $\lhd = (\lhd_1, \lhd_2)$ of relations on L_1 and L_2 respectively such that (for $i, k = 1, 2$, $i \neq k$)*

(SI1) $y \leq x \lhd_i a \leq b$ *implies that* $y \lhd_i b$

(SI2) $\lhd_i$ *is a sublattice of* $L_i \times L_i$

(SI3) $x \lhd_i a$ *implies that* $x \prec_i a$

(SI4) $x \lhd_i a$ *implies that there exists* $y \in L_i$ *with* $x \lhd_i y \lhd_i a$

(SI5) *if* $x \lhd_i a$ *then there exist* $u, v \in L_k$ *such that* $u \lhd_k v$, $x \wedge v = 0$ *and* $a \vee u = e$

(SI6) $a = \bigvee x (x \lhd_i a)$

Remark

The condition (SI5) above (in the presence of the other properties of strong inclusions) is the same as: $x \lhd_i a$ implies that $a^\bullet \lhd_k x^\bullet$.

($\Longrightarrow$) If $u \lhd_k v$, $x \wedge v = 0$, $a \vee u = e$ then $a^\bullet \leq u \lhd_k v \leq x^\bullet$.

($\Longleftarrow$) Take $x \lhd_i y \lhd_i a$. Then $a^\bullet \lhd_k y^\bullet \lhd_k x^\bullet$, so that $y^\bullet \lhd_k x^\bullet$, $x \wedge x^\bullet = 0$, $a \vee y^\bullet = e$ (since $y \lhd_i a$ implies that $y \prec_i a$, which is the same as $y^\bullet \vee a = e$).

Lemma 3.1 *On a compact, regular biframe, $(\prec_1, \prec_2)$ is a strong inclusion.*

PROOF. We check only (SI5):
If $x \prec_i a$ then $x \wedge v = 0$, $a \vee v = e$ for some $v \in L_k$. Compactness and regularity give $u \prec_k v$ with $a \vee u = e$. ∎

Lemma 3.2 *For any onto $h : N \to L$, if $\lhd = (\lhd_1, \lhd_2)$ is a strong inclusion on N, then $\widehat{\lhd} = (h \times h[\lhd_1], h \times h[\lhd_2])$ is a strong inclusion on L.*

PROOF. (SI1) Take $x \leq h(a) \widehat{\lhd}_i h(b) \leq y$ with $a \lhd_i b$ in N_i and $x, y \in L_i$. Now $x = h(s)$, $y = h(t)$ so $h(s \wedge a) \leq h(a) \widehat{\lhd}_i h(b) \leq h(b \vee t)$ and $s \wedge a \lhd_i b \vee t$.

(SI2) $h \times h$ preserves sublattices.

(SI3) If $x \lhd_i a$ in N_i (that is, $h(x) \widehat{\lhd}_i h(a)$), then $x \prec_i a$, so $h(x) \prec_i h(a)$.

(SI4) If $h(x) \widehat{\lhd}_i h(a)$ then $x \lhd_i y \lhd_i a$ so that $h(x) \widehat{\lhd}_i h(y) \widehat{\lhd}_i h(a)$.

(SI5) If $x \lhd_i a$ in N_i, there exist $u \lhd_k v$ in N_k with $x \wedge v = 0$, $a \vee u = e$. Then $h(x) \widehat{\lhd}_k h(v)$, $h(x) \wedge h(v) = 0$, $h(a) \vee h(u) = e$.

(SI6) For $a \in N_i$, $a = \bigvee x (x \lhd_i a)$ so $h(a) = \bigvee h(x)(x \lhd_i a)$ hence also $h(a) = \bigvee z (z \widehat{\lhd}_i h(a))$. ∎

Corollary 3.1 *If L has a compactification, it has a strong inclusion.*

We now consider the converse of this corollary; that is, we construct a compactification of L from a given strong inclusion.

Let $(\triangleleft_1, \triangleleft_2)$ be a strong inclusion on L. For an ideal $J \in (\mathcal{J}L)_i$ (that is, an ideal J generated by $J \cap L_i$), J will be called **strongly regular** iff $x \in J \cap L_i$ implies that there exists a $y \in J \cap L_i$ with $x \triangleleft_i y$. Let $\mathcal{R}_i$ consist of these J and $\mathcal{R}_0 \subseteq \mathcal{J}(L_0)$ be the subframe generated by $\mathcal{R}_1 \cup \mathcal{R}_2$. Then $\mathcal{R}_0 \subseteq (\mathcal{J}L)_0$. Now $\mathcal{R}_i$ is a subframe of $\mathcal{J}(L_0)$: it is closed under binary meets and binary joins by (SI2); 0 and $\downarrow e$ are in it, and it is trivially closed under updirected joins (unions). Thus $\mathcal{R}$ is a compact biframe; we verify that it is regular.

Define $r_i : L_i \rightarrow \mathcal{R}_i$ by $r_i(a) = [x \mid x \triangleleft_i a]$, where $[\ldots]$ denotes the ideal generated in L_0. Then $r_i(a) \in \mathcal{R}_i$ by (SI2), (SI4).

Claim: $a \triangleleft_i b$ implies that $r_i(a) \prec_i r_i(b)$.

Proof: Take $a \triangleleft_i c \triangleleft_i b$ and $w \triangleleft_k u$ with $a \wedge u = 0$, $c \vee w = e$. Then $r_i(a) \cap r_k(u) = 0$ and $c \vee w \in r_i(b) \vee r_k(u)$. Hence $r_i(b) \vee r_k(u) = \downarrow e$ and so $r_i(a) \prec_i r_i(b)$.

Finally, for any $J \in \mathcal{R}_i$, $J = \bigvee r_i(a) (a \in J \cap L_i)$ and $a \triangleleft_i b \in J \cap L_i$ implies that $r_i(a) \prec_i r_i(b) \subseteq J$, which gives the regularity of $\mathcal{R}$.

The join map $r_L : \mathcal{R} \rightarrow L$ provides the required compactification of L. It maps $\mathcal{R}_i$ onto L_i since $\bigvee r_i(a) = a$, $a \in L_i$, by (SI6). That it is dense is clear. We have thus shown the first part of the next proposition. For its second part, we need the following terminology:

The compactifications (up to isomorphism) of a biframe L form a partially ordered set under the partial order given by $h : M \rightarrow L \leq \tilde{h} : \widetilde{M} \rightarrow L$ iff there exists a biframe map $f : M \rightarrow \widetilde{M}$ satisfying $\tilde{h}.f = h$.

The strong inclusions of L form a partially ordered set under set inclusion.

We denote these two sets by $\mathbf{K}L$ and $\mathbf{S}L$, respectively.

Proposition 3.1 *A biframe L has a compactification if and only if it has a strong inclusion. Moreover, the above constructions provide isomorphisms between* $\mathbf{K}L$ *and* $\mathbf{S}L$ *inverse to each other.*

PROOF. We first check that both constructions are order-preserving:
Given $\lhd_i \subseteq \widehat{\lhd}_i$, one gets $\mathcal{R}_i \subseteq \widehat{\mathcal{R}}_i$, hence $\mathcal{R} \subseteq \widehat{\mathcal{R}}$ so that $\mathcal{R}\to L \leq \widehat{\mathcal{R}}\to L$ by the inclusion map $\mathcal{R}\to\widehat{\mathcal{R}}$.
Given $h : M\to L \leq \hat{h} : \widehat{M}\to L$ with $f : M\to\widehat{M}$ satisfying $\hat{h}.f = h$, we obtain $h \times h[\prec] = \hat{h}.f \times \hat{h}.f[\prec] = (\hat{h}\times\hat{h})(f\times f)[\prec] \subseteq \hat{h}\times\hat{h}[\hat{\prec}]$, where $\hat{\prec}$ denotes the relation $\prec$ on $\widehat{M}$.

Next we verify that they are inverse to each other.

Let $\lhd$ be a strong inclusion on L, $\tau_L : \mathcal{R}\to L$ the compactification associated with it, and $\widehat{\lhd}$ the strong inclusion associated with that. Then $a \lhd_i b$ implies that $r_i(a) \prec_i r_i(b)$, which, by the definition of $\widehat{\lhd}$, gives $\bigvee r_i(a) \widehat{\lhd}_i \bigvee r_i(b)$ and hence $a \widehat{\lhd}_i b$. Conversely, $a \widehat{\lhd}_i b$ means that $a = \bigvee J$, $b = \bigvee I$ where $I, J \in \mathcal{R}_i$ and $J \prec_i I$. Then there exists $H \in \mathcal{R}_k$ with $J \cap H = 0$ and $I \vee H = \downarrow e$. Thus $x \vee y = e$ for some $x \in I$, $y \in H$. Also, $x \lhd_i b$ since I is strongly regular. Now $J \cap H = 0$ implies that $a \wedge y = 0$, so $a = a \wedge x \lhd_i b$. Hence $\lhd_i = \widehat{\lhd}_i$.

Beginning with a compactification $h : M\to L$ of L, let $\lhd$ be its associated strong inclusion, and $\tau_L : \mathcal{R}\to L$ the compactification associated with that. Consider the diagram

$$\begin{array}{ccc} \mathcal{R} & \xrightarrow{\tau_L} & L \\ {\scriptstyle \hat{h}}\uparrow & & \uparrow{\scriptstyle h} \\ \widehat{\mathcal{R}} & \xrightarrow[\tau_M]{} & M \end{array}$$

where $\widehat{\mathcal{R}}$ is given by the strongly regular ideals with respect to $\prec_i$ on M_i, and $\hat{h}$ is the restriction of $\mathcal{J}h$. Since $\prec_i$ is a strong inclusion on the compact, regular biframe M, $\tau_M : \widehat{\mathcal{R}} \to M$ is a compactification. It is codense (that is, its total part is) by compactness of M, hence an isomorphism. We show that $\hat{h}$ is also an isomorphism. Now $\hat{h}$ is dense because h is, and so one-one. (See [3].) It remains to show $\hat{h}$ onto, and for this it suffices to prove $r_i(a) \in Image(\hat{h})$, $a \in L_i$. Let $J = [z \in M_i | \ h(z) \in r_i(a)]$. Then $(\mathcal{J}h)(J) = r_i(a)$ because h is onto. To check that $J \in \widehat{\mathcal{R}}$, take $z \in J \cap M_i$, so $h(z) \triangleleft_i a$. Then $h(z) = h(u)$, $a = h(v)$ for some $u \prec_i v$. Take $u \prec_i w \prec_i v$, hence $u \wedge s = 0$, $w \vee s = e$ for some $s \in L_k$. Now $h(z \wedge s) = h(u \wedge s) = 0$, so $z \wedge s = 0$, by the density of h. So $z \prec_i w \in J \cap M_i$, as required. ∎

Remark

It is clear that if (L_0, L_1, L_2) has a compactification, so does L_0. The converse is not true, as may be seen by considering $(L, L, \mathbf{2})$ for any non-trivial, compactifiable frame L.

3.2 Zero-dimensional compactifications

Definition 3.3 *$h : M \to L$ is a* **zero-dimensional compactification** *of L iff M is compact, zero-dimensional and h is dense, onto.*

Remark

L has a zero-dimensional compactification iff the join map $\mathcal{J}BL \to L$ is onto iff L is zero-dimensional.

Hence we consider only zero-dimensional biframes in this section.

Let $\mathrm{K}_0 L$ be the partially ordered set of zero-dimensional compactifications of L, modulo isomorphism.

Definition 3.4 *A* **basic Boolean bilattice** *of a (zero-dimensional) biframe L is any Boolean subbilattice A of BL which generates L (that is, A_i generates L_i). Ordering these basic Boolean bilattices of L by inclusion gives a partially ordered set, which we denote by* $\mathrm{BBB}(L)$.

We now describe two correspondences between zero-dimensional compactifications and basic Boolean bilattices of a biframe.

- For any such bilattice A, the join map $\sigma_L : \mathcal{J}A \to L$ is a zero-dimensional compactification of L.
- For any zero-dimensional compactification $h : M \to L$, $h[BM] \subseteq BL$ is a basic Boolean bilattice of L.

Proposition 3.2 *The correspondences above are mutually inverse isomorphisms between* $\mathrm{BBB}(L)$ *and* $\mathrm{K}_0(L)$.

PROOF. Let A be a basic Boolean bilattice of L, $\sigma : \mathcal{J}A \to L$ the compactification associated with it, and $\sigma[B\mathcal{J}A] \subseteq BL$ the bilattice associated with that. Since $(B\mathcal{J}A)_i = \{\downarrow a |\ a \in A_i\}$, we obtain $\sigma[B\mathcal{J}A] = A$. (See Proposition 1.6.)
Let $h : M \to L$ be a zero-dimensional compactification of L, $A = h[BM]$ and $\sigma : \mathcal{J}A \to L$ the compactification associated with A. Consider

$$\begin{array}{ccc} \mathcal{J}A & \xrightarrow{\ \sigma\ } & L \\ \hat{h}\uparrow & & \uparrow h \\ \mathcal{J}BM & \xrightarrow[\ \bigvee\ ]{} & M \end{array}$$

where $\hat{h}$ is given by $\hat{h}(J) = \bigcup\{\downarrow h(a) |\ a \in J\}$ for $J \in (\mathcal{J}BM)_0$. The bottom join map is an isomorphism because M is compact, regular. $\hat{h}$ is onto since $h : (BM)_i \to A_i$ is onto, and $\hat{h}$ is dense because h is (and dense maps between compact, regular biframes are one-one). Thus $\hat{h}$ is an isomorphism too, and σ is isomorphic to h. ∎

The next result shows how zero-dimensional compactifications may be identified by looking at their strong inclusions.

Proposition 3.3 *The compactification associated with $\lhd$ is zero-dimensional if and only if, for any $a \lhd_i b$, there exists $c \in L_i$ with $a \leq c \lhd_i c \leq b$.*

PROOF. ($\Longrightarrow$) Suppose we are given a zero-dimensional compactification $h : M \rightarrow L$ with associated $\lhd_i = h \times h[\prec_i]$. Let $a \lhd_i b$ and $u \prec_i v$ with $a = h(u)$, $b = h(v)$. Now $u \prec_i v$ implies $u \ll v$, by compactness, and since $v = \bigvee z\,(z \in M_i \mid z$ complemented, with complement in $M_k)$, there exists a complemented $w \in M_i$ with $u \leq w \leq v$. Then $w \prec_i w$ so $c = h(w)$ satisfies $c \lhd_i c$ and $a \leq c \leq b$.

($\Longleftarrow$) Claim: If $A_i = \{c \in L_i \mid c \lhd_i c\}$ and A_0 is generated by $A_1 \cup A_2$, then (A_0, A_1, A_2) is a basic Boolean bilattice of L.
Proof: A_i is a lattice by (SI2). A is Boolean since $c \lhd_i c$ implies $c \prec_i c$, that is, c is complemented with complement in L_k. By (SI6) and our assumption, A_i generates L_i, hence is basic.
Further, any strongly regular ideal $J \in \mathcal{R}_i$ is generated by $J \cap A_i$: $x \in J$ implies $x \lhd_i y \in J$, which gives $x \leq c \lhd_i c \leq y \in J$, and thus $x \leq c \in J \cap A_i$. Conversely, any ideal J generated by $J \cap A_i$ is in $\mathcal{R}_i$: $x \in J$ implies $x \leq y \in J \cap A_i$, so that $x \leq y \lhd_i y \in J$.
Hence $\mathcal{J}A \cong \mathcal{R}$ and $\mathcal{J}A$ is a zero-dimensional compactification. ∎

3.3 Least biframe compactifications

It is known (see [5]) that a frame has a least compactification iff it is regular and continuous, and that any such compactification may be written in the form $\downarrow a$ for some element a maximal in a compact, regular frame. Further, a zero-dimensional frame has a smallest zero-dimensional compactification iff it is continuous, and that compactification is its smallest. (Any unfamiliar definitions may be found in Section 0.5.)

We do not have analogous characterizations in the case of biframes. We present two partial results, with examples.

Lemma 3.3 *Let L be a regular biframe such that each L_i is stably continuous and condition (SI5) holds for $\ll_i$ (the relation $\ll$ with respect to the frame L_i). This is a necessary and sufficient condition for $\ll_i$ to be a strong inclusion on L, and it is then necessarily the least.*

PROOF. (SI1) and (SI2) follow from the properties of $\ll$, with stable continuity.
(SI3) comes from the regularity of L.
(SI4) holds since $\ll$ interpolates in a continuous lattice.
(SI5) is postulated explicitly.
(SI6) holds because each L_i is continuous.
Furthermore, if $\lhd_i$ is any strong inclusion on L, then $x \ll_i y = \bigvee z(z \lhd_i y)$, so that $x \lhd_i y$. ∎

Example 1
Let $\mathcal{L}_0$ = all open subsets of the **rational** unit interval E
$\mathcal{L}_1$ = all open downsets
$\mathcal{L}_2$ = all open upsets
In $\mathcal{L}_i$, $U \ll_i V$ iff $U \subset V$ or $U = V = E$ or $U = V = \emptyset$. Hence $\mathcal{L}_i$ is stably continuous (note that $\mathcal{L}_i$ is compact). $\mathcal{L}$ is regular because $U \subset V$ implies $U \prec_i V$; and (SI5) is equally easy to check. So $\mathcal{L}$ has a smallest compactification, by the lemma above. It is given by taking $(\mathcal{M}_0, \mathcal{M}_1, \mathcal{M}_2)$ analogous to $(\mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2)$ for the **real** unit interval. $\mathcal{M}$ is compact, regular, and the restriction map $\mathcal{M} \to \mathcal{L}$ is dense, onto. In $\mathcal{M}_i$, $U \prec_i V$ iff $U \subset V$ iff $U \cap E \subset V \cap E$. Hence the strong inclusion induced by this compactification is the smallest, and $\mathcal{M}$ is the least compactification of L.

Remarks

- The smallest compactification of $\mathcal{L}$ has many new points.
- $\mathcal{L}$ is zero-dimensional, but its smallest compactification is not.

- $\mathcal{L}_0$ is not continuous, and thus does not have a smallest frame compactification.

Lemma 3.4 *Let L be a regular biframe in which each L_i is continuous, and $a \prec_i b$ implies $a \ll_i b$ whenever $a < e$. Then L has a unique compactification.*

PROOF. We verify that $\prec_i$ is a strong inclusion on L.
(SI1)–(SI3) always hold.
(SI4) For $a < e$, $a \prec_i b$ iff $a \ll_i b$ (by regularity) and $\ll$ interpolates on a continuous frame.
(SI5) If $a \prec_i b$ there exists $c \in L_i$ with $a \prec_i c \prec_i b$, witnessed by $s, t \in L_k$, $a \wedge s = 0$, $c \vee s = e$, $c \wedge t = 0$, $b \vee t = e$. Then $t \prec_k s$, as required.
(SI6) holds since L_i is continuous.
Now $\prec_i$ is certainly the largest strong inclusion on L. It is also the smallest: if $\lhd_i$ is another, $a < e$, $a \prec_i b$ implies $a \ll_i b$, so that $a \lhd_i b$. ∎

Example 2
$\mathcal{L}_0$ = all open subsets of the open unit interval E
$\mathcal{L}_1$ = all open downsets
$\mathcal{L}_2$ = all open upsets
$\mathcal{L}$ is certainly regular. Also, $U \prec_i V$, $U \neq E$ holds iff $U \subset V$, which implies that $U \ll_i V$, thus $\mathcal{L}_i$ is continuous. So $\mathcal{L}$ has a unique compactification.

Remarks

- The unique compactification is again the biframe given by the closed unit interval (mentioned in the previous example), with the relevant restriction maps.
- The least compactification of $\mathcal{L}$ is not obtained from the least compactification of $\mathcal{L}_0$, since the open unit interval is locally compact, Hausdorff and hence has a one-point compactification.

Example 3

$\mathcal{L}_0$ = all subsets of the natural numbers, N= $\{0, 1, 2, 3, \ldots\}$
$\mathcal{L}_1$ = all downsets
$\mathcal{L}_2$ = all upsets
$\mathcal{L}$ is a biframe, using $\wedge = \cap$ and $\bigvee = \bigcup$. It is clearly Boolean, and hence regular. $\mathcal{L}_1$ and $\mathcal{L}_2$ are compact. For $U, V \in \mathcal{L}_i$, $U \subseteq V$ iff $U \ll_i V$, so $\mathcal{L}_i$ is stably continuous. Also if $U \neq \mathrm{N}$, $U \prec_i V$ implies $U \subseteq V$, which implies $U \ll_i V$. Thus $\mathcal{L}$ has a unique compactification, by the lemma above.
This compactification is given as follows. Let the set $M = \mathrm{N} \cup \{\star\}$, where $\star \notin \mathrm{N}$ and $n \leq \star$ for all $n \in \mathrm{N}$. Let
$\mathcal{M}_1$ = all downsets of M
$\mathcal{M}_2 = \{U \cup \{\star\} \mid U$ is an upset of $\mathrm{N}\}$
$\mathcal{M}_0$ be generated by $\mathcal{M}_1 \cup \mathcal{M}_2$.
Then $\mathcal{M}$ is compact, zero-dimensional and the restriction map $\mathcal{M} \rightarrow \mathcal{L}$ is clearly dense, onto.

The analogous example with N replaced by **Z**, the integers, also has a compactification, using two points at infinity instead of one, as above.

Chapter 4

Coherent and continuous biframes

Compact elements and coherent frames were defined in Section 0.5. We denote the set of compact elements of a frame L by KL.

Definition 4.1 1. *A biframe L is called* **coherent** *if and only if L_0 is coherent and $KL_0 \cap L_i$ generates L_i $(i = 1, 2)$.*

2. *A map $h : L \to M$ between coherent biframes is called* **coherent** *iff $h|L_0$ is a coherent frame homomorphism, that is, if $h(a)$ is compact for every compact element a in L_0.*

3. **CohBiFrm** *is the category of coherent biframes and coherent maps.*

Remark

L coherent implies L_1 and L_2 coherent, since $KL_0 \cap L_i \subseteq KL_i$. The converse need not hold, however: Let L be any coherent frame with a non-compact element, a. The biframe $(L, \{0, a, e\}, L)$ provides a counter-example.

Lemma 4.1 1. *The Stone biframes form a full subcategory of the coherent biframes.*

2. *A biframe is Stone if and only if it is regular and coherent.*

PROOF.

1. If L is a Stone biframe, L_0 is compact, zero-dimensional and hence coherent (since one may show that an element of L_0 is compact iff it is complemented). Since $KL_0 \cap L_i \supseteq \{x \in L_i | \ x \vee x^\bullet = e\}$, and the latter set generates L_i, L is coherent. A map between Stone biframes is clearly coherent.

2. ($\Longrightarrow$) If L is Stone, then for $x \in L_i$,
$x = \bigvee z(z \in L_i,\ z \leq x,\ z \vee z^\bullet = e) = \bigvee z(z \in L_i,\ z \leq x,\ z \prec_i z)$, hence L is regular.
($\Longleftarrow$) If L is regular, $x = \bigvee z(z \in L_i,\ z \prec_i x)$ for each $x \in L_i$. So if $x \in KL_0$ then $x \prec_i x$. Coherence of L gives $y = \bigvee x(x \in L_i,\ x \leq y,\ x \in KL_0)$ and hence L Stone.

∎

In order to obtain an analogue of the category equivalence between the coherent frames and bounded distributive lattices (see [21]), we make the following definitions.

Definition 4.2 *1. (a) A* **(bounded) distributive bilattice** *is a triple $A = (A_0, A_1, A_2)$ where A_0 is a bounded distributive lattice and A_1 and A_2 are sublattices whose union generates A_0.*

(b) A map $h : A \to A'$ between distributive bilattices is a homomorphism $h : A_0 \to A'_0$ of bounded distributive bilattices which preserves the two specified sublattices.

(c) The category of bounded distributive bilattices and their homomorphisms will be written BiLatt.

2. We define a functor K : **CohBiFrm→BiLatt** *as follows:*
$K(L_0, L_1, L_2) = (KL_0, KL_0 \cap L_1, KL_0 \cap L_2)$

$Kh = h|KL_0$
Note that $(KL_0, KL_0 \cap L_1, KL_0 \cap L_2) \in$**BiLatt** *because* $(KL_0 \cap L_1) \cup (KL_0 \cap L_2)$ *generates* L_0 *(as a frame), and hence* KL_0 *(as a lattice).* Kh *is a morphism of bilattices since* h *preserves compact elements.*

3. *The functor* $\mathcal{J}$: **BiLatt**$\to$**CohBiFrm** *is given by:*
$\mathcal{J}(A_0, A_1, A_2) = (\mathcal{J}A_0, \mathcal{J}_1A_1, \mathcal{J}_2A_2)$
where $\mathcal{J}A_0$ *is the frame of ideals of* A_0 *and* $\mathcal{J}_iA_i$ *consists of those ideals* J *generated by* $J \cap A_i$.
For $h : A \to A'$, $\mathcal{J}h$ *is given by* $\mathcal{J}h(J)$ *being the ideal generated by* $h[J]$.
Certainly $\mathcal{J}A_0$ *is a coherent frame, and since its compact elements are exactly the principal ideals,* $\mathcal{J}_iA_i$ *is generated by* $\mathcal{J}_iA_i \cap K\mathcal{J}A_0$*; so* $\mathcal{J}(A_0, A_1, A_2) \in$ **CohBiFrm**. $\mathcal{J}h$ *preserves principal ideals, hence compact elements.*

Proposition 4.1 K : **CohBiFrm**$\to$**BiLatt** *is an equivalence, with inverse* $\mathcal{J}$.

PROOF. For any bilattice A, let $\alpha_A : A \to K\mathcal{J}A$ be given by $\alpha_A(a) = \downarrow a$. Now $K\mathcal{J}A_0 \cap \mathcal{J}_iA_i = \{\downarrow x \mid x \in A_i\}$, so α_A is easily seen to be a lattice homomorphism, in fact, an isomorphism. For $h : A \to A'$ in **BiLatt**, $\alpha_{A'}.h(a) = K\mathcal{J}h.\alpha_A(a) = \downarrow a$ for all $a \in A_0$, so α_A depends naturally on A.
For a coherent biframe L, let $\sigma_L : \mathcal{J}KL \to L$ be given by the join map. It is clear that σ preserves first and second parts. We check that $\sigma_L|(\mathcal{J}KL)_0$ is a frame isomorphism. Order is obviously preserved; it is also reflected: Suppose that $\bigvee I \leq \bigvee J$ for I and J ideals of compact elements. Then $c \in I$ implies $c \leq \bigvee J$ and hence $c \in J$. Also $\sigma_L|(\mathcal{J}KL)_i$ is onto, because any $a \in L_i$ may be expressed as $a = \bigvee c(c \leq a,\ c \in KL_0 \cap L_i)$, by the coherence of L. Further, σ_L depends naturally on L, since for any $h : L \to M$ in **CohBiFrm**, $\sigma_M.\mathcal{J}Kh(I) = h.\sigma_L(I) = h(\bigvee I)$ for all $I \in (\mathcal{J}KL)_0$. ∎

Corollary 4.1 *K induces an equivalence* **StBiFrm→BooBiLatt**.

PROOF. This equivalence was already mentioned in [4]. We simply note that, for Stone biframes, the functors K and B coincide, and for Boolean bilattices the two ideal biframe functors (both denoted $\mathcal{J}$) also coincide. ∎

In Definition 1.8 we introduced an ideal biframe functor between biframes. Its range, however, is contained in the coherent biframes: The compact elements of $\mathcal{J}_0 L_0$ are precisely the principal ideals, and these form a sublattice generating $\mathcal{J}_0 L_0$. Also, any $J \in \mathcal{J}_i L_i$ can be expressed as $J = \bigvee \downarrow x$ $(x \in J \cap L_i)$, so $K\mathcal{J}_0 L_0 \cap \mathcal{J}_i L_i$ generates $\mathcal{J}_i L_i$. Further, the map $\mathcal{J}h$ certainly preserves compact elements and first and second parts, so it is coherent. We thus have a functor $\mathcal{J}$: **BiFrm→CohBiFrm**.

Note also that the map $\tau_L : \mathcal{J}L \to L$ given by taking joins of the ideals in question, is a biframe map.

Proposition 4.2 **CohBiFrm** *is coreflective in* **BiFrm** *with coreflection maps $\tau_L : \mathcal{J}L \to L$ given by taking joins.*

PROOF. Let $h : M \to L$ be given with M coherent, and consider the following diagram:

$$\begin{array}{ccc} \mathcal{J}L & \xrightarrow{\tau_L} & L \\ {\scriptstyle \mathcal{J}h}\uparrow & & \uparrow{\scriptstyle h} \\ \mathcal{J}M & \underset{k}{\overset{\tau_M}{\rightleftarrows}} & M \end{array}$$

$\mathcal{J}h$ is a coherent biframe map. Define k by sending any $x \in M_0$ to the ideal generated (in $\mathcal{J}_0 M_0$) by $KM_0 \cap \downarrow x$.

Then $k : M_0 \to \mathcal{J}_0 M_0$ is a coherent frame homomorphism: It is easily seen that k preserves the bottom, the top and updirected joins. Preservation of binary meets uses the compact elements being closed under these meets, and preservation of binary joins uses the fact that any element can be expressed as a join of the compact elements below it. For compact c, $k(c) = \downarrow c$, so k is coherent.
We check that, for $x \in M_i$, $k(x) \in \mathcal{J}_i M_i$:
If $y \in k(x)$, then $y \leq c \leq x$ for some $c \in KM_0$. But $x = \bigvee z(z \in KM_0 \cap M_i,\ z \leq x)$, so compactness of c gives $y \leq z$ for some $z \in k(x) \cap M_i$. So $k(x)$ is indeed generated by $k(x) \cap M_i$.
Furthermore, since $\tau_L.\mathcal{J}h.k(c) = \bigvee(\mathcal{J}h) \downarrow c = h(c)$ for compact c, and M_0 is generated by KM_0 we obtain $\tau_L.\mathcal{J}h.k = h$, and this factorization is easily seen to be unique. ∎

We now turn again to the relationship between biframes and bispaces, and for that recall the dual adjunction given by the functors $\mathcal{O}$ and Σ on page 11. Since a biframe is spatial iff its total part is spatial (see Proposition 1.2), we obtain the following lemma:

Lemma 4.2 *The following are equivalent:*

1. *Every coherent biframe is spatial.*
2. *Every coherent frame is spatial.*
3. *The Boolean Ultrafilter Theorem holds.*

In this section we shall refer to the **(bilattice) spectrum functor** Π : **BiLatt**$\to$**BiTop**. For $A \in$ **BiLatt**, let ΠA_0 be the lattice spectrum of A_0. It has as points all bounded lattice homomorphisms $\xi : A_0 \to \mathbf{2}$ and as basic open sets $\Pi_a = \{\xi |\ \xi(a) = 1\}$, $a \in A_0$. We then let $\Pi(A_0, A_1, A_2) = (|\Pi A_0|, \mathcal{O}_1, \mathcal{O}_2)$

where $\mathcal{O}_i$ is the topology with base $\{\Pi_a | \ a \in A_i\}$. For $h : A \rightarrow A'$ in **BiLatt**, define $\Pi h : \Pi A \rightarrow \Pi A'$ by $\Pi h(\xi) = \xi.h$. Since $(\Pi h)^{-1}(\Pi_a) = \Pi_{h(a)}$ for any $a \in A_0$, Πh is bicontinuous.

Proposition 4.3 Π *is naturally isomorphic to* $\Sigma\mathcal{J}$.

PROOF. For $A \in$ **BiLatt**, let $\alpha_A : \Pi A \rightarrow \Sigma\mathcal{J}A$ be given by $\alpha_A(\xi) = \hat{\xi}$, where $\hat{\xi}(J) = 1$ if $1 \in \xi[J]$, and 0 otherwise. This correspondence is one-one and onto; it is bicontinuous because $\alpha_A^{-1}(\Sigma_J) = \bigcup \Pi_a (a \in J)$ for $J \in \mathcal{J}A_0$, and it is open on each part because $\alpha_A(\Pi_a) = \Sigma_{\downarrow a}$ for $a \in A_0$. Thus α_A is a homeomorphism of bispaces.

∎

Definition 4.3 *1. Call a bispace X* **spectral** *iff X is sober and $\mathcal{O}X$ is a coherent biframe.*

2. A bicontinuous function $f : (|X|, \mathcal{U}_1, \mathcal{U}_2) \rightarrow (|X'|, \mathcal{U}'_1, \mathcal{U}'_2)$ between spectral bispaces is called **spectral** *iff, for any $U \in \mathcal{U}'_1 \vee \mathcal{U}'_2$ which is compact in $\mathcal{U}'_1 \vee \mathcal{U}'_2$, $f^{-1}(U)$ is compact in $\mathcal{U}_1 \vee \mathcal{U}_2$.*

3. The category of spectral bispaces and spectral maps will be denoted **SpecBiTop**.

The definition of a Boolean bispace was given on page 19.

Lemma 4.3 **BooBiTop** *is a full subcategory of* **SpecBiTop**.

PROOF. This follows from the fact that, in Boolean bispaces, the compact open sets are exactly the closed open sets. Alternatively, see Lemma 4.1 . ∎

The following dual equivalences are then easy to verify.

Proposition 4.4 *(BUT)*

1. **SpecBiTop** $\cong^*$ **CohBiFrm** *by the functors* $\mathcal{O}$ *and* Σ.

2. **BiLatt** $\cong^*$ **SpecBiTop** *by the functors* Π *and* $K\mathcal{O}$.

3. **BooBiLatt** $\cong^*$ **BooBiTop** *by the functors* Π *and* $K\mathcal{O}$.

Using Propositions 4.2 and 4.3 we obtain:

Proposition 4.5 *(BUT)* **SpecBiTop** *is reflective in* **BiTop**, *with reflection maps* $\tau_X : X \to \Pi\mathcal{O}X$ *given by* $\tau_X(x)(U) = 1$ *if* $x \in U$ *and 0 otherwise.*

Lemma 4.4 *For any biframe* L, $\mathcal{J}L$ *is regular if and only if* L *is Boolean.*

PROOF. ($\Longrightarrow$) If $a \in L_i$, then $\downarrow a \in \mathcal{J}_i L_i$. Regularity of $\mathcal{J}L$ and compactness of $\downarrow a$ gives $\downarrow a \prec_i \downarrow a$. This means that there exists $J \in \mathcal{J}_k L_k$ which is the complement of $\downarrow a$; but J must then have the form $\downarrow b$, $b \in L_k$ and b is the required complement of a.

($\Longleftarrow$) For each $a \in L_i$, $\downarrow a \prec_i \downarrow a$ and these principal ideals generate $\mathcal{J}_i L_i$. ∎

4.1 Continuity and stable continuity

We recall (see Section 0.5) that the relation $\ll$ on a frame is defined by $a \ll b$ iff, whenever $b \leq \bigvee X$ for some $X \subseteq L$, it follows that $a \leq \bigvee F$ for some finite subset $F \subseteq X$. In a regular, continuous *frame*, it is known ([5]) that $a \ll b$ if and only if $a \prec b$ and $\uparrow a^*$ is compact.

Definition 4.4 *L will be called a* **continuous** *biframe iff* L_0 *is a continuous frame and, for each* $x \in L_i$, $x = \bigvee z(z \in L_i,\ z \ll x$ *in* $L_0)$.

Lemma 4.5 *Let L be a regular, continuous biframe. For $x, y \in L_i$, the following are equivalent:*
(1) $x \ll_i y$ (that is, $x \ll y$ in L_i)
(2) $x \ll y$ (that is, $x \ll y$ in L_0)
(3) $x \prec_i y$ and $\uparrow x^\bullet = \{z \in L_0 | \ z \geq x^\bullet\}$ is compact
(4) $x \prec y$ and $\uparrow x^* = \{z \in L_0 | \ z \geq x^*\}$ is compact.

PROOF. (1) $\Longleftrightarrow$ (2): That (2) implies (1) is trivial. Conversely, if $x \ll_i y$ and $y = \bigvee t(t \in L_i, \ t \ll y$ in $L_0)$ then $x \leq t_1 \vee \ldots \vee t_n \ll y$ for some finite subset $t_1, \ldots, t_n$ of the t's.
(1) $\Longrightarrow$ (3): Take $x \ll_i y$, or equivalently, by the above, $x \ll y$. Since $y = \bigvee s(s \in L_i, \ s \prec_i y)$ by regularity, $x \prec_i y$. Now let J be an ideal in $\uparrow x^\bullet$ with $\bigvee J = e$. Since $\ll_i$ interpolates on L_i, we obtain $u \in L_i$ with $x \ll_i u \ll_i y$. Then $x^\bullet \vee u = e$ and $u \ll y \leq e = \bigvee J$ implies $u \leq j_1 \vee \ldots \vee j_n$ for some finite subset $\{j_1, \ldots, j_n\} \subseteq J$. So $x^\bullet \vee (j_1 \vee \ldots \vee j_n) = e$, and we get $e \in J$, as required.
(3) $\Longrightarrow$ (2): Take $x \prec_i y$ and $\uparrow x^\bullet$ compact. Let $y \leq \bigvee J$ for some ideal J of L_0. We have $x^\bullet \vee y = e$; thus $x^\bullet \vee \bigvee J = e$ and so $\bigvee_{j \in J}(x^\bullet \vee j) = e$. But $\uparrow x^\bullet$ is compact, so $e = x^\bullet \vee j_0$, for some $j_0 \in J$. Thus $x = x \wedge j_0$, that is, $x \leq j_0 \in J$ and $x \in J$, as required.
(2) $\Longleftrightarrow$ (4): Similar to (1) $\Longleftrightarrow$ (3). ∎

Remark

- By the lemma above, L continuous implies L_1 and L_2 continuous. The converse does not hold (see the remark after Definition 4.1).
- The conditions for continuity are fairly restrictive, as is illustrated by the following biframe *not* being continuous:
  $\mathcal{L}_0$ = all open subsets of the real line, **R**
  $\mathcal{L}_1$ = all open downsets
  $\mathcal{L}_2$ = all open upsets

For any $U \in \mathcal{L}_0$, $\uparrow U$ is isomorphic to $\mathcal{O}(\mathbf{C}U)$, where $\mathbf{C}$ denotes set-theoretic complement. For $U = (-\infty, 0)$, $U^\bullet = (0, \infty)$, $\mathbf{C}U^\bullet = (-\infty, 0]$ and $\mathcal{O}((-\infty, 0])$ is not compact. So $\uparrow U^\bullet$ is not compact. Thus $U \ll V$ never occurs, for U, V downsets, unless $U = \emptyset$, and so $\mathcal{L}$ is not continuous.

Definition 4.5 *1. L is said to be a* **stably continuous** *biframe iff L is a continuous biframe and L_0 is a stably continuous frame.*

2. A biframe map $h : L \to M$ will be called **proper** *if $h : L_0 \to M_0$ preserves the relation $\ll$.*

3. The category of stably continuous biframes and proper maps will be called **StContBiFrm**.

Remark
Every compact, regular biframe is stably continuous, and every coherent biframe is stably continuous. (The former because $a \ll_i b$ iff $a \prec_i b$, the latter because $a \ll_i b$ iff there exists $c \in KL_0 \cap L_i$ with $a \leq c \leq b$.)

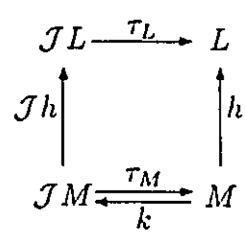
Proposition 4.6 *A biframe is stably continuous if and only if it is the retract of a coherent biframe.*

PROOF. ($\Longrightarrow$) For stably continuous L, define a function $k : L \to \mathcal{J}L$ by $k(a) = \{x \in L_0 | \; x \ll a\}$. Since L_0 is a stably continuous frame, $k : L_0 \to \mathcal{J}_0 L_0$ is a frame map: it preserves the bottom and the top (use $e \ll e$), updirected joins, binary meets (because $\ll$ is closed under them) and binary joins (because $\ll$ is closed under them, and each element can be expressed as the join of elements way below itself). If $a \in L_i$, $k(a)$ is generated by $k(a) \cap L_i$, so k is a biframe map. Now the join map $\tau_L : \mathcal{J}L \to L$ is a biframe homomorphism, and since $\bigvee k(a) = a$ for each $a \in L_0$ (by the continuity of L_0), k is one-one. Hence L is a retract of $\mathcal{J}L$.
($\Longleftarrow$) We show that the retract of any stably continuous frame is stably continuous.

Suppose $j : L \to M$ and $r : M \to L$ are biframe maps satisfying $r.j = 1$, and M is stably continuous. For any $x \in L_i$, $j(x) = \bigvee\{t \mid t \in M_i \text{ and } t \ll j(x) \text{ in } M_0\}$. Now $x = r.j(x) = \bigvee\{r(t) \mid t \in M_i \text{ and } t \ll j(x) \text{ in } M_0\}$. But $t \ll j(x)$ in M_0 implies that $r(t) \ll x$ in L_0, yielding the required expression of x. An analogous argument shows that L_0 is continuous, so it remains to prove stability. That $e \ll e$ is clear. We have seen that if $x \ll y$ and $x \ll z$ in L_0, then $x \leq r(t)$ and $x \leq r(s)$ for some $t \ll j(y)$, $s \ll j(z)$, respectively, in M_0. Then $x \leq r(t \wedge s)$ and $t \wedge s \ll j(y \wedge z)$ (by the stable continuity of M_0), giving $x \ll y \wedge z$ in L_0. ∎

Proposition 4.7 StContBiFrm *is coreflective in* **BiFrm**, *with coreflection maps* $\tau_L : \mathcal{J}L \to L$ *given by taking joins.*

PROOF. Let $h : M \to L$ be given with M stably continuous, and consider the following diagram:



$\mathcal{J}h$ is a proper biframe map. k is the biframe homomorphism (considered in the previous proposition) given by $k(a) = \{x \in M_0 \mid x \ll a\}$, $a \in M_0$. We check that it is proper: Suppose $a \ll b$ in M_0. Express b as $b = \bigvee\{t \mid t \ll b \text{ and } t \in M_0\}$ and each such t as $t = \bigvee\{x \wedge y \mid x \in M_1, y \in M_2, x \wedge y \leq t\}$. Since $a \ll b$, we obtain $a \leq (x_1 \wedge y_1) \vee \ldots \vee (x_n \wedge y_n) \leq b$ for some $x_1, \ldots, x_n \in M_1$ and $y_1, \ldots, y_n \in M_2$. Let the middle element of this inequality be called x. Then $\downarrow x$ is a compact element of $\mathcal{J}_0 M_0$ (being an ideal generated by $\mathcal{J}_1 M_1 \cup \mathcal{J}_2 M_2$), and $k(a) \subseteq \downarrow x \subseteq k(b)$. Thus $k(a) \ll k(b)$.

Since $\tau_L.\mathcal{J}h = h.\tau_M$, we see that $\tau_L.(\mathcal{J}h.k) = h$. To check uniqueness of $\mathcal{J}h.k$, we suppose there were another map g preserving $\ll$ and satisfying $\tau_L.g = h$.

Fix $a \in M_0$.
If $t \ll a$ in M_0, $g(t) \ll g(a)$ in $\mathcal{J}_0 L_0$, which means that $g(t) \subseteq \downarrow s \subseteq g(a)$ for some $s \in L_0$. But then applying τ_L gives $h(t) \leq s \in g(a)$, so that $h(t) \in g(a)$, and $(\mathcal{J}h.k)(a) \subseteq g(a)$. Conversely, if $t \in g(a)$, we have $\downarrow t \ll g(a) = \bigcup\{g(s) |\ s \ll a\}$, so $\downarrow t \subseteq g(s)$ for some $s \ll a$. Acting τ_L on this inclusion gives $t \leq h(s)$, $s \ll a$ so that $t \in (\mathcal{J}h.k)(a)$ and $g(a) \subseteq (\mathcal{J}h.k)(a)$. Thus $g = \mathcal{J}h.k$. ∎

For a discussion of projectivity versus stable continuity in biframes, we refer the reader to Proposition 5.5.

Chapter 5

Supercoherence and projectivity

This section largely parallels the previous one—the concepts of coherence, compactness, and so on, can be obtained from those of supercoherence, supercompactness, etc. by replacing the conditions involving arbitrary joins with updirected ones. We have, however, discussed projectivity in connection with stably continuous and stably supercontinuous biframes together, at the end of this section (see page 58).

Supercoherence and stable supercontinuity of frames were defined in Section 0.5. In this section, we denote the set of all supercompact elements of a frame L by SL. A supercoherent frame is necessarily stably supercontinuous, since in this case $x \lll y$ if and only if $x \leq c \leq y$ for some supercompact c.

Definition 5.1 *1. (a) A biframe L is* **supercoherent** *iff L_0 is a supercoherent frame, and each element of L_i is a join of members of $SL_0 \cap L_i$.*

(b) A map $h : L \to M$ between supercoherent biframes is called supercoherent iff its total part $h|L_0$ preserves supercompact elements.

(c) The resulting category will be called **SCohBiFrm**.

2. (a) A biframe L is **stably supercontinuous** *if L_0 is stably supercontinuous, and $x = \bigvee z\,(z \in L_i,\ z \lll x \text{ in } L_0)$ for each $x \in L_i$.*

(b) The category consisting of the stably supercontinuous biframes and the biframe maps whose total parts preserve the relation $\lll$, will be denoted **StSContBiFrm**.

Remarks

1. Any supercoherent biframe is stably supercontinuous.

2. Any supercoherent biframe is spatial, since $\uparrow c$ is a completely prime filter exactly when c is supercompact.

3. **1** and **2** are the only supercoherent, regular (or stably supercontinuous, regular) biframes. This follows because, for each $x \in L_i$, $x = \bigvee z(z \prec_i x)$ and $z \prec_i x$ means that there exists $c \in L_k$ with $z \wedge c = 0$, $c \vee x = e$. Supercompactness of e gives $x = e$ or $c = e$, and in the latter case $z = 0$.

The following definitions pave the way for an analogue of the category equivalence between semilattices and supercoherent frames (see [12]).

Definition 5.2 *1. (a) A* **bisemilattice** *is a triple $A = (A_0, A_1, A_2)$ in which A_0 is a meet-semilattice with unit and A_1 and A_2 are subsemilattices whose union generates A_0.*

(b) A map $h : A \to A'$ between bisemilattices is a homomorphism $h : A_0 \to A'_0$ of meet-semilattices which preserves the specified subsemilattices.

(c) The resulting category will be denoted **BiSLatt**.

2. We define a functor S : **SCohBiFrm** $\to$ **BiSLatt** *by the following:*

$$S(L_0, L_1, L_2) = (SL_0, SL_0 \cap L_1, SL_0 \cap L_2)$$

$$Sh = h|SL_0$$

where SL_0 denotes, as before, the set of supercompact elements of the frame L_0. Note that, since every element of L_i is a join of elements of SL_0, $(SL_0 \cap L_1) \cup (SL_0 \cap L_2)$ generates SL_0, as a semilattice. So $S(L_0, L_1, L_2)$ is a bisemilattice and Sh is a morphism in **BiSLatt** *because h preserves supercompact elements.*

3. *We define the functor $\mathcal{D}$:* **BiSLatt** $\to$ **SCohFrm** *as follows:*

$$\mathcal{D}(A_0, A_1, A_2) = (\mathcal{D}A_0, \mathcal{D}_1 A_1, \mathcal{D}_2 A_2)$$

where $\mathcal{D}A_0$ is the frame of down-sets of A_0 and $\mathcal{D}_i A_i$ consists of those down-sets X generated by $X \cap A_i$.
For $h : A \to A'$, the morphism $\mathcal{D}h : \mathcal{D}A \to \mathcal{D}A'$ is given by

$$\mathcal{D}h(X) = \bigcup\{\downarrow h(x) : x \in X\}$$

for any $X \in \mathcal{D}A_0$. In $\mathcal{D}A_0$, a down-set X is supercompact iff $X = \downarrow a$ for some $a \in A$, so $\mathcal{D}A$ is easily seen to be a supercoherent biframe. Since $\mathcal{D}h(\downarrow a) = \downarrow h(a)$, $\mathcal{D}h$ preserves supercompact elements.

Proposition 5.1 *S :* **SCohBiFrm** $\to$ **BiSLatt** *is an equivalence, with inverse $\mathcal{D}$.*

PROOF. For any bisemilattice A, let $\eta_A : A \to S\mathcal{D}A$ be given by $\eta_A(a) = \downarrow a$. Now $S\mathcal{D}A_0 \cap \mathcal{D}_i A_i = \{\downarrow a | a \in A_i\}$, and η_A is easily seen to be a bisemilattice isomorphism. η_A depends naturally on A, since for $h : A \to A'$ in **BiSLatt**, $\eta_{A'}.h(a) = S\mathcal{D}h.\eta_A(a) = \downarrow h(a)$ for all $a \in A_0$.
For any supercoherent L, $\varepsilon_L : \mathcal{D}SL \to L$ is given by the join map.
ε certainly preserves first and second parts; we show that $\varepsilon_L|(\mathcal{D}SL)_0$ is a frame isomorphism. Order is clearly preserved; it is also reflected: Suppose $\bigvee X \leq \bigvee Y$ for X and Y down-sets of supercompact elements. Then $c \in X$ implies $c \leq \bigvee Y$, so that $c \leq t$ for some $t \in Y$ and so $c \in Y$. Also $\varepsilon_L|(\mathcal{D}SL)_i$ is onto, because any $a \in L_i$ may be expressed as $a = \bigvee\{s : s \leq a, s \in SL_0 \cap L_i\}$. Further, ε_L depends naturally on L, since for any $h : L \to M$, $\varepsilon_M.\mathcal{D}Sh(X) = h.\varepsilon_L(X) = h(\bigvee X)$ for all

$X \in (\mathcal{D}SL)_0$. ∎

We now define a down-set functor from biframes to supercoherent biframes, which we also denote by $\mathcal{D}$.

Definition 5.3 *For any biframe L, let*

$$\mathcal{D}L = (\mathcal{D}_0 L_0, \mathcal{D}_1 L_1, \mathcal{D}_2 L_2)$$

where $\mathcal{D}_i L_i$ consists of those down-sets X in $\mathcal{D}L_0$ (the frame of down-sets of L_0) which are generated by $X \cap L_i$, and $\mathcal{D}_0 L_0$ is the subframe of $\mathcal{D}L_0$ generated by $\mathcal{D}_1 L_1 \cup \mathcal{D}_2 L_2$.
$\mathcal{D}L$ is a supercoherent biframe: The supercompact elements of $\mathcal{D}_0 L_0$ are exactly its principal down-sets; they are closed under finite meets and generate $\mathcal{D}_0 L_0$. Also, any $X \in \mathcal{D}_i L_i$ may be written $X = \bigcup\{\downarrow x : x \in X \cap L_i\}$, so $S\mathcal{D}_0 L_0 \cap \mathcal{D}_i L_i$ generates $\mathcal{D}_i L_i$.
For $h : L \to M$ in **BiFrm**, *$\mathcal{D}h : \mathcal{D}L \to \mathcal{D}M$ is given by*

$$\mathcal{D}h(X) = \bigcup\{\downarrow h(x) : x \in X\}.$$

$\mathcal{D}h|\mathcal{D}_0 L_0$ is a frame homomorphism which preserves supercompact elements and first and second parts.

We also note that $\tau_L : \mathcal{D}L \to L$ given by taking joins of down-sets, is a biframe map.

Proposition 5.2 *A biframe is stably supercontinuous if and only if it is the retract of a supercoherent biframe.*

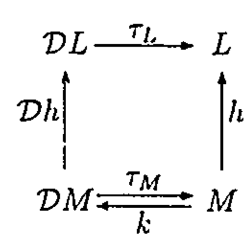
PROOF. ($\Longrightarrow$) For stably supercontinuous L, define $k : L \to \mathcal{D}L$ by $k(a) = \{x \in L_0 : x \lll a\}$ for any $a \in L_0$. Then $k|L_0$ is a frame homomorphism: it preserves binary meets because $\lll$ is closed under them, and arbitrary joins because any element a can be expressed as $a = \bigvee\{t : t \lll a\}$. If $a \in L_i$, $k(a)$ is generated by $k(a) \cap L_i$, so k is a biframe map. The join map $\tau_L : \mathcal{D}L \to L$ is a biframe homomorphism, and k is one-one, because $\bigvee k(a) = a$ for each $a \in L_0$ (by the supercontinuity of L_0). Hence L is a retract of $\mathcal{D}L$.

($\Longleftarrow$) We show that the retract of any stably supercontinuous biframe is stably supercontinuous.

Suppose $j : L \to M$ and $r : M \to L$ are biframe maps satisfying $r.j = 1$, and M is stably supercontinuous. For any $x \in L_i$, $j(x) = \bigvee\{t | t \in M_i$ and $t \lll j(x)$ in $M_0\}$. Now $x = r.j(x) = \bigvee\{r(t) | t \in M_i$ and $t \lll j(x)$ in $M_0\}$. But $t \lll j(x)$ in M_0 implies that $r(t) \lll x$ in L_0, yielding the required expression of x. An analogous argument shows that L_0 is supercontinuous, so it remains to prove stability. That $e \lll e$ is clear. We have seen that if $x \lll y$ and $x \lll z$ in L_0, then $x \leq r(t)$ and $x \leq r(s)$ for some $t \lll j(y)$, $s \lll j(z)$, respectively, in M_0. Then $x \leq r(t \wedge s)$ and $t \wedge s \lll j(y \wedge z)$ (by the stable supercontinuity of M_0), giving $x \lll y \wedge z$ in L_0. ∎

Proposition 5.3 **SCohBiFrm** *and* **StSContBiFrm** *are coreflective in* **BiFrm**, *with coreflection maps $\tau_L : \mathcal{D}L \to L$ given by taking joins.*

PROOF. Since **SCohBiFrm** is a full subcategory of **StSContBiFrm**, it suffices to prove this result for the latter category. Let $h : L \to M$ be given, with M stably supercontinuous, and consider the following diagram:



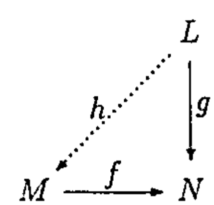
$\mathcal{D}h$ is a biframe map that preserves the relation $\lll$. k is the biframe map (considered in the previous proposition) given by $k(a) = \{x \in M_0 | x \lll a\}$ for any $a \in M_0$. We verify that it preserves $\lll$: Suppose $a \lll b$ in M_0. Express b as $b =$

$\bigvee\{t : t \lll b \text{ and } t \in M_0\}$ and each such t as $t = \bigvee\{x \wedge y : x \in M_1, y \in M_2, x \wedge y \leq t\}$. Since $a \lll b$, we obtain $a \leq x \wedge y \leq b$ for some $x \in M_1$ and $y \in M_2$ with $x \wedge y \lll b$. Now $\downarrow (x \wedge y)$ is a supercompact element of $\mathcal{D}_0 M_0$ (being a principal down-set generated by $\mathcal{D}_1 M_1 \cup \mathcal{D}_2 M_2$), and $k(a) \subseteq \downarrow (x \wedge y) \subseteq k(b)$. Thus $k(a) \lll k(b)$. Since $\tau_L . \mathcal{D}h = h.\tau_M$, we see that $\tau_L.(\mathcal{D}h.k) = h$. To check the uniqueness of $\mathcal{D}h.k$, we suppose there were another map g preserving $\lll$ and satisfying $\tau_L.g = h$. Fix $a \in M_0$.

If $t \lll a$ in M_0, $g(t) \lll g(a)$ in $\mathcal{D}_0 L_0$, which means that $g(t) \subseteq \downarrow s \subseteq g(a)$ for some $s \in L_0$. But then applying τ_L gives $h(t) \leq s \in g(a)$, so that $h(t) \in g(a)$, and $(\mathcal{D}h.k)(a) \subseteq g(a)$. Conversely, if $t \in g(a)$, we have $\downarrow t \lll g(a) = \bigcup\{g(s) | s \lll a\}$, so $\downarrow t \subseteq g(s)$ for some $s \lll a$. Acting τ_L on this inclusion gives $t \leq h(s)$, $s \lll a$ so that $t \in (\mathcal{D}h.k)(a)$ and $g(a) \subseteq (\mathcal{D}h.k)(a)$. Thus $g = \mathcal{D}h.k$. ∎

5.1 Projectivity

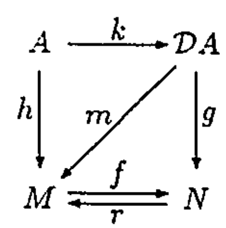
Definition 5.4 *We call a biframe L* **semi-projective** *if the following condition holds: for any onto biframe map $f : M \to N$ for which the right adjoint of the total part preserves first and second parts, and for an arbitrary biframe map $g : L \to N$, there exists a biframe map $h : L \to M$ satisfying $f.h = g$, that is, making the following diagram commute:*



Lemma 5.1 *For any bisemilattice A, $\mathcal{D}A$ is semi-projective in* **BiFrm**.

PROOF. Let $f : M \to N$ be onto, and let $r : N_0 \to M_0$ be the right adjoint of

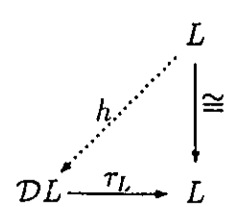
$f|M_0$. We assume that $r[N_i] \subseteq M_i$. Let $g : \mathcal{D}A \to N$ be any biframe homomorphism. Consider the diagram:



The function $k : A \to \mathcal{D}A$ given by $k(a) = \downarrow a$ is a morphism in **BiSLatt**, as is $h = r.g.k$, since r preserves meets and first and second parts. The map $m : \mathcal{D}A \to M$ defined by $m(X) = \bigvee h[X]$ is a biframe homomorphism (it preserves finitary meets and first and second parts because h does), and it satisfies $m.k = h$. Then $f.m.k = f.h = f.r.g.k = g.k$, since $f.r = 1$. This shows that $f.m = g$, since the elements $k(a)$, $a \in A$ generate $\mathcal{D}A$. ∎

Proposition 5.4 *The semi-projectives in* **BiFrm** *are exactly the stably supercontinuous biframes.*

PROOF. ($\Longrightarrow$) For any biframe L, the join map $\tau_L : \mathcal{D}L \to L$ is a biframe homomorphism for which the right adjoint $r : L_0 \to \mathcal{D}_0 L_0$ is given by
$r(a) = \bigcup X (X \in \mathcal{D}_0 L_0$ and $\bigvee X = a)$. For $a \in L_i$, $r(a) = \downarrow a$, so r does preserve first and second parts. Using the projectivity of L with respect to τ_L we obtain $h : L \to \mathcal{D}L$ making the following diagram commute:



So $\tau_L.h = 1$, that is, L is a retract of $\mathcal{D}L$, which is supercoherent. This makes L stably supercontinuous.
($\Longleftarrow$) This follows from Lemma 5.1, and the fact that retracts of semi-projectives are again semi-projective. ∎

We remark that since the category **Frm** has no non-trivial (ordinary) projectives (see [12]), neither does **BiFrm**. (Replacing the condition "onto" by "epi" in the definition of semi-projectives, gives the definition of ordinary projectives.) An argument similar to the one above gives the following result:

Proposition 5.5 *The stably continuous biframes are the projectives relative to those onto biframe homomorphisms for which the right adjoint of the total part preserves finite joins as well as first and second parts.*

Chapter 6

Normality for biframes

Frith [19] gave the following definition for normality of biframes:

Definition 6.1 *A biframe L is called* **normal** *if, whenever $x \vee y = e$ for some $x \in L_i$ and $y \in L_k$, there exist $u \in L_k$ and $v \in L_i$ with $u \wedge v = 0$ and $x \vee u = e = y \vee v$.*

We remark that normality can equivalently be expressed by the condition: whenever $x \vee y = e$ for $x \in L_i$ and $y \in L_k$, there exists $u \in L_k$ such that $x \vee u = e = y \vee u^{\bullet}$.

Lemma 6.1 *Any compact regular biframe is normal.*

PROOF. Suppose $x \vee y = e$ for some $x \in L_i$, $y \in L_k$. By regularity, $x = \bigvee\{s \mid s \prec_i x,\ s \in L_i\}$ and $y = \bigvee\{t \mid t \prec_k y,\ t \in L_k\}$, so by compactness there exist $s \prec_i x$ and $t \prec_k y$ with $s \vee t = e$. Then $s \wedge c = 0$, $x \vee c = e$ and $t \wedge d = 0$, $y \vee d = e$ for some $c \in L_k$, $d \in L_i$. Also, $c \wedge d = c \wedge d \wedge (s \vee t) = (c \wedge d \wedge s) \vee (c \wedge d \wedge t) = 0$, as required. ∎

We recall (see [9]) that a biframe L is called **strictly zero-dimensional** if it satisfies the following condition (or the corresponding one with L_1 and L_2

reversed): for each $x \in L_1$, $x^\bullet$ is a complement of x in L_0 and L_2 is generated by these complements.

Lemma 6.2 *Any strictly zero-dimensional biframe is normal.*

PROOF. Suppose that L satisfies the condition as stated above, and that $x \vee y = e$ for some $x \in L_1$, $y \in L_2$. Let $u = y \wedge x^\bullet$ and $v = x$. Then $u \wedge v = y \wedge x^\bullet \wedge x = 0$. Also $x \vee u = x \vee (y \wedge x^\bullet) = (x \vee y) \wedge (x \vee x^\bullet) = e$ (since x and $x^\bullet$ are complements, by assumption), and $y \vee v = y \vee x = e$; and $u \in L_2$, $v \in L_1$ as required. ∎

Corollary 6.1 1. *Any Boolean biframe is normal.*

2. *([19]) The congruence biframe of a frame is normal.*

3. *The Skula biframe of a topological space is normal.*
(SkX is the biframe for which $(SkX)_1$ is the topology $\mathcal{O}X$ of X, $(SkX)_2$ is the topology on the underlying set of X generated by the closed sets of X, and $(SkX)_0$ is the topology generated by all the open and all the closed subsets of X.)

6.1 The compact regular coreflection of a normal regular biframe.

The frame notions on which this section is based may be found in [1]. To study this coreflection we look at the pair $(\prec_1, \prec_2)$ of relations, and begin with the following lemma.

Lemma 6.3 *If L is normal and regular, $(\prec_1, \prec_2)$ is a strong inclusion on L.*

PROOF. We check that $\prec_i$ interpolates (the other conditions clearly hold). If $z \prec_i x$ for z, $x \in L_i$, there exists $c \in L_k$ such that $z \wedge c = 0$ and $x \vee c = e$. By normality there exist $u \in L_k$, $v \in L_i$ with $u \wedge v = 0$, $x \vee u = e = c \vee v$. So $z \prec_i v \prec_i x$. ∎

The **compact regular coreflection** $\mathcal{R}L$ of a normal, regular L may thus be given as follows:

An ideal J of L_0 generated by $J \cap L_i$ is called **regular** if $x \in J \cap L_i$ implies that there exists an element $y \in J \cap L_i$ for which $x \prec_i y$. Now $(\mathcal{R}L)_i$ consists of these regular ideals and $(\mathcal{R}L)_0$ is the subframe of $\mathcal{J}L_0$ (the frame of ideals of L_0) generated by $(\mathcal{R}L)_1 \cup (\mathcal{R}L)_2$. The compactification is given by the join map $\bigvee : \mathcal{R}L \to L$.

The first and second parts of this join map have right adjoints given by $r_i : L_i \to (\mathcal{R}L)_i$, where $r_i(x) = [z \in L_i | z \prec_i x]$, the square brackets denoting the ideal generated in L_0 by the z's in question.

With the above terminology, we obtain the following lemma.

Lemma 6.4 *If $x \vee y = e$ for some $x \in L_i$, $y \in L_k$ then $r_i(x) \vee r_k(y) = L_0$.*

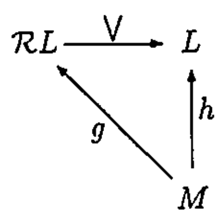
PROOF. Suppose $x \vee y = e$ for $x \in L_i$, $y \in L_k$. By normality, there exist $u \in L_k$, $v \in L_i$ with $u \wedge v = 0$, $x \vee u = e = y \vee v$. In particular, $u \prec_k y$. We now apply normality to $x \vee u = e$ to get $s \in L_k$, $t \in L_i$ such that $s \wedge t = 0$, $x \vee s = e = u \vee t$. Now $t \prec_i x$, so we have $u \vee t = e$ for $u \in r_k(y)$ and $t \in r_i(x)$ and thus $r_i(x) \vee r_k(y) = L_0$. ∎

The above lemma characterizes this compactification, in the following sense:

Proposition 6.1 *Let $h : M \to L$ be a biframe compactification and q_i the right adjoint of $h|M_i$. Suppose that $x \vee y = e$ for $x \in L_i$, $y \in L_k$ implies that $q_i(x) \vee q_k(y) = e$. Then L is a normal regular biframe and the compactification $h : M \to L$ is isomorphic to $\bigvee : \mathcal{R}L \to L$.*

PROOF. To see that L is normal, take $x \vee y = e$, $x \in L_i$, $y \in L_k$. By assumption, $q_i(x) \vee q_k(y) = e$, and $q_i(x) \in M_i$, $q_k(y) \in M_k$. Since M is compact regular, and hence normal, there exist $u \in M_k$, $v \in M_i$ such that $u \wedge v = 0$, $q_i(x) \vee u = e = q_k(y) \vee v$. Then $h(u) \wedge h(v) = 0$ and $x \vee h(u) = e = y \vee h(v)$, as required.

Let r_1 and r_2 denote the right adjoints of the first and second parts of the compactification $\bigvee : \mathcal{R}L \to L$. By the universal property of this compactification there exists a biframe homomorphism $g : M \to \mathcal{R}L$ making the following diagram commute:



We show that g is an isomorphism. Since h is dense, g is also dense and so, being a map between compact regular biframes, is one-one. To see that g is onto, it is sufficient to show that each $r_i(x)$, $x \in L_i$ is in the image of $g|M_i$, since these ideals generate $(\mathcal{R}L)_i$. We claim that $g.q_i(x) = r_i(x)$, $x \in L_i$.

($\subseteq$) Take $z \in g.q_i(x) \cap L_i$. Since $g.q_i(x)$ is a regular ideal, there exists $y \in g.q_i(x) \cap L_i$ such that $z \prec_i y$. Then $z \prec_i y \leq \bigvee g.q_i(x) = h.q_i(x) = x$ and so $z \in r_i(x)$.

($\supseteq$) Take $z \in L_i$, $z \prec_i x$. There exists $c \in L_k$ with $z \wedge c = 0$, $x \vee c = e$. By assumption $q_i(x) \vee q_k(c) = e$, so $g.q_i(x) \vee g.q_k(c) = L_0$. This means that $s \vee t = e$ for some $s \in g.q_i(x) \cap L_i$, $t \in g.q_k(c) \cap L_k$. Now $z = z \wedge (s \vee t) = (z \wedge s) \vee (z \wedge t) = z \wedge s$, since $z \wedge c = 0$ and $t \leq c$. So, finally, $z \leq s \in g.q_i(x)$. ∎

Remark

If $h : M \rightarrow L$ is a biframe compactification for which q (the right adjoint of $h|M_0$) preserves first and second parts, and preserves all binary joins, then L is certainly normal and a similar argument to the one above shows that L_0 is normal and $(\mathcal{R}L)_0$ is isomorphic to $\mathcal{R}(L_0)$. (See [1].)

6.2 Urysohn's Lemma for normal biframes.

This section is based on Dowker and Papert's 1966 paper, "On Urysohn's Lemma" [16], which extends Urysohn's classical topological theorem to frames. We shall need the following lemma from that source:

Lemma 6.5 *Let L and M be frames, let B be a base of L and let $\varphi : B \rightarrow M$ be a function such that if $\{b_i\}$ is finite and $\bigwedge b_i \leq \bigvee c_\alpha$, then $\bigwedge \varphi(b_i) \leq \bigvee \varphi(c_\alpha)$. Then φ extends to a frame homomorphism $\mu : L \rightarrow M$.*
(μ is given by $\mu(a) = \bigvee\{\varphi(b)|b \in B,\ b \leq a\}$.)

Proposition 6.2 *A biframe L is normal if and only if it satisfies the following condition: whenever $x \vee y = e$ for $x \in L_1$, $y \in L_2$, there exists a biframe homomorphism $\mu : (\mathcal{O}\mathbf{R}, \text{open downsets}, \text{open upsets}) \longrightarrow (L_0, L_1, L_2)$ such that $\mu((-\infty,1)) \leq x$ and $\mu((0,\infty)) \leq y$.*

PROOF. ($\Longleftarrow$) To show L normal, consider $x \vee y = e$ for $x \in L_1$, $y \in L_2$, and μ as given. Since $(-\infty,1/2) \cap (1/2,\infty) = \emptyset$ and $(-\infty,1) \cup (1/2,\infty) = \mathbf{R} = (0,\infty) \cup (-\infty,1/2)$, we obtain $\mu((-\infty,1/2)) \wedge \mu((1/2,\infty)) = 0$, $x \vee \mu((1/2,\infty)) \geq \mu((-\infty,1)) \vee \mu((1/2,\infty)) = e$, $\mu((-\infty,1/2)) \vee y \geq \mu((-\infty,1/2)) \vee \mu((0,\infty)) = e$, and $\mu((-\infty,1/2)) \in L_1$, $\mu((1/2,\infty)) \in L_2$, as required.
($\Longrightarrow$) Let $\mathbf{Q}$ denote the set of rational numbers. For each $p \in \mathbf{Q}$ we construct $g_p \in L_2$, $h_p \in L_1$ so that $g_p \wedge h_p = 0$ and, if $p < q$ then $g_p \vee h_q = e$. Then, for $p < q$,

$h_p = h_p \wedge e = h_p \wedge (g_p \vee h_q) = h_p \wedge h_q$,
$g_q = g_q \wedge e = g_q \wedge (g_p \vee h_q) = g_q \wedge g_p$,
so that $h_p \leq h_q$ and $g_p \geq g_q$. Let $r_1, r_2, r_3, \ldots$ be a list of the rationals between 0 and 1. Then define
$\mathbf{Q}_n = \{r_1, r_2, \ldots, r_n\} \cup \{p \in \mathbf{Q} | p \leq 0\} \cup \{p \in \mathbf{Q} | p \geq 1\}$. For $p \in \mathbf{Q}_0$ define $g_p \in L_2$, $h_p \in L_1$ by:

$$\begin{aligned} \text{for } p < 0, \quad & g_p = e, \quad h_p = 0 \\ & g_0 = y, \quad h_0 = 0 \\ & g_1 = 0, \quad h_1 = x \\ \text{for } p > 1, \quad & g_p = 0, \quad h_p = e. \end{aligned}$$

Suppose g_p, h_p have been defined for $p \in \mathbf{Q}_n$. We now define g_r, h_r for $r = r_{n+1}$. Take the largest $p \in \mathbf{Q}_n$ with $p < r$ and the least $q \in \mathbf{Q}_n$ with $r < q$. Then $p < q$ and $g_p \vee h_q = e$, $g_p \in L_2$, $h_q \in L_1$. Apply normality to obtain $g_r \in L_2$, $h_r \in L_1$ with $g_r \wedge h_r = 0$ and $g_p \vee h_r = e = g_r \vee h_q$. If $s \in \mathbf{Q}_{n+1}$ and $s < r$ then $s \leq p$, so $g_s \geq g_p$ and $g_s \vee h_r = e$. If $s \in \mathbf{Q}_{n+1}$ and $s > r$ then $s \geq q$, so $h_s \geq h_q$ and $g_r \vee h_s = e$. We have thus defined g_s, h_s with the required properties, for all $s \in \mathbf{Q}_{n+1}$, and by induction they may be defined for all $s \in \mathbf{Q}$.

Let B be the base of $\mathcal{O}\mathbf{R}$ consisting of the open intervals (a, b) with $a < b$. Define the function $\varphi : B \rightarrow L_0$ by $\varphi((a,b)) = \bigvee\{g_p \wedge h_q | \ a < p < q < b\}$. The proof that φ extends to a frame homomorphism $\mu : \mathcal{O}\mathbf{R} \rightarrow L_0$ is the same as in the Dowker/Papert case. We repeat it here for completeness' sake.

Let (a_j, b_j), $j = 1, \ldots, n$ be a non-empty finite family of intervals, and let (a_α, b_α) be a family of intervals such that $\bigcap(a_j, b_j) \subseteq \bigcup(a_\alpha, b_\alpha)$. Then

$$\begin{aligned} \wedge\varphi((a_j, b_j)) &= \Big(\bigvee_{a_1 < p_1 < q_1 < b_1} g_{p_1} \wedge h_{q_1}\Big) \wedge \ldots \wedge \Big(\bigvee_{a_n < p_n < q_n < b_n} g_{p_n} \wedge h_{q_n}\Big) \\ &= \bigvee \cdots \bigvee g_{p_1} \wedge h_{q_1} \wedge \ldots \wedge g_{p_n} \wedge h_{q_n} \\ &= \bigvee \cdots \bigvee \{g_{\max \mathrm{p}} \wedge h_{\min \mathrm{q}}\} \\ &= \bigvee \{g_p \wedge h_q | \ \max a_j < p < q < \ \min b_j\} \\ &= \varphi(\bigcap(a_j, b_j)). \end{aligned}$$

For any rational numbers p and q such that $\max a_j < p < q < \min b_j$, the compact interval $[p, q]$ is contained in
$\bigcup\{(a_\alpha, b_\alpha)|\text{ all }\alpha\} = \bigcup \dots \bigcup\{(r,s)|a_\alpha < r < s < b_\alpha, \text{ all } \alpha\}$, for r, s rational. Hence $[p,q]$ is contained in a finite number of such intervals (r,s), and so the open interval (p,q) is some finite union $\bigcup\{(r_l, s_l)|\text{ all } l\}$ of these. We may assume that no (r_l, s_l) can be omitted from the union and that (r_l, s_l) overlaps (r_{l+1}, s_{l+1}). If $r < t < s < u$ we have $(g_r \wedge h_s) \vee (g_t \wedge h_u) = (g_r \vee g_t) \wedge (g_r \vee h_u) \wedge (h_s \vee g_t) \wedge (h_s \vee h_u) = g_r \wedge h_u$. Hence $g_p \wedge h_q = \bigvee g_{r_l} \wedge h_{s_l} \leq \bigvee \varphi((a_\alpha, b_\alpha))$, and so $\wedge\varphi((a_j, b_j)) \leq \bigvee \varphi((a_\alpha, b_\alpha))$. If $\bigcup(a_\alpha, b_\alpha) = \mathbf{R}$ then $(-2,3) \subseteq \bigcup(a_\alpha, b_\alpha)$ and hence $e = g_{-1} \wedge h_2 \leq \varphi((-2,3)) \leq \bigvee\varphi((a_\alpha, b_\alpha))$. Thus $\wedge\varphi((a_j, b_j)) \leq \bigvee\varphi((a_\alpha, b_\alpha))$ even when the family (a_j, b_j) is empty. Thus, by the lemma quoted above, φ extends to a frame homomorphism $\mu : \mathcal{O}\mathbf{R} \rightarrow L_0$ given by $\mu(U) = \bigvee\{\varphi((a,b))|(a,b) \subseteq U\}$ for all $U \in \mathcal{O}\mathbf{R}$.
We now check that μ preserves the first and second parts of
$(\mathcal{O}\mathbf{R}, \textit{open downsets}, \textit{open upsets})$, and thus extends to a biframe homomorphism, as required. For $c \in \mathbf{R}$,

$$\begin{aligned}
\mu((-\infty, c)) &= \bigvee\{\varphi((a,b))|(a,b) \subseteq (-\infty, c)\} \\
&= \bigvee\bigvee\{g_p \wedge h_q | a < p < q < b,\ (a,b) \subseteq (-\infty, c)\} \\
&= \bigvee\{h_q | q < c\} \text{ since } g_p = e \text{ for } p < 0. \\
&\in L_1 \text{ and, similarly,} \\
\mu((c, \infty)) &= \bigvee\bigvee\{g_p \wedge h_q | a < p < q < b,\ (a,b) \subseteq (c, \infty)\} \\
&= \bigvee\{g_p | c < p\} \text{ since } h_p = e \text{ for } p > 1 \\
&\in L_2.
\end{aligned}$$

Finally, $\mu((-\infty, 1)) = \bigvee\{h_q | q < 1\} \leq h_1 = x$ and
$\mu((0,\infty)) = \bigvee\{g_p | 0 < p\} \leq g_0 = y$. ∎

6.3 Lindelöf and normal biframes

Definition 6.2 *A biframe L will be called* **Lindelöf** *if and only if its total part is Lindelöf, that is, whenever $\bigvee X = e$ there exists a countable subset Y of X with $\bigvee Y = e$.*

Proposition 6.3 *Any regular, Lindelöf biframe is normal.*

PROOF. Suppose $x \vee y = e$ for some $x \in L_i$, $y \in L_k$. Since L is regular, we may write $x = \bigvee\{s \in L_i | \, s \prec_i x\} = \bigvee\{s \in L_i | \, s^\bullet \vee x = e\}$ and $y = \bigvee\{t \in L_k | \, t^\bullet \vee y = e\}$. Let $\mathbb{N}$ denote the set of natural numbers. Since L is Lindelöf, there exist countable subsets $S = \{s_\alpha | \, \alpha \in \mathbb{N}\} \subseteq \{s \in L_i | \, s^\bullet \vee x = e\}$ and $T = \{t_\beta | \, \beta \in \mathbb{N}\}$ $\subseteq \{t \in L_k | \, t^\bullet \vee y = e\}$ such that $\bigvee S \vee y = e$ and $\bigvee T \vee x = e$. For α, $\beta \in \mathbb{N}$, we define $m_\alpha = s_\alpha \wedge \bigwedge\{t^\bullet_\gamma | \, \gamma \leq \alpha\}$ and $n_\beta = t_\beta \wedge \bigwedge\{s^\bullet_\delta | \, \delta \leq \beta\}$. Then $m_\alpha \wedge n_\beta = 0$ for all α, $\beta \in \mathbb{N}$, so $m \wedge n = 0$ for $m = \bigvee\{m_\alpha | \, \alpha \in \mathbb{N}\}$ and $n = \bigvee\{n_\beta | \, \beta \in \mathbb{N}\}$; also $m \in L_i$ and $n \in L_k$. Now

$$
\begin{aligned}
m \vee y &= \bigvee_\alpha m_\alpha \vee y \\
&= \bigvee_\alpha (s_\alpha \wedge \bigwedge\{t^\bullet_\gamma : \gamma \leq \alpha\}) \vee y \\
&= \bigvee_\alpha (s_\alpha \vee y) \wedge (\bigwedge\{t^\bullet_\gamma : \ \gamma \leq \alpha\} \vee y) \\
&= \bigvee_\alpha (s_\alpha \vee y) \wedge (\bigwedge\{t^\bullet_\gamma \vee y : \ \gamma \leq \alpha\}) \\
&= \bigvee_\alpha s_\alpha \vee y \\
&= \bigvee S \vee y \\
&= e
\end{aligned}
$$

and, similarly,

$$n \vee x = \bigvee_\beta (t_\beta \vee x) \wedge (\bigwedge\{s^\bullet_\delta \vee x : \ \delta \leq \beta\}) = \bigvee T \vee x = e.$$

∎

Chapter 7

Perfect compactifications

Skljarenko [26] introduced perfect compactifications of topological spaces — they are those compactifications Y of a space X with the property that $fr_Y(Y - cl_Y(X - U)) = cl_Y fr_X U$ for all open sets U in X (where fr stands for the frontier operator and cl the closure operator). This idea was extended to frames in [2], on which we base this section.

Definition 7.1 *1. For any $x \in L_1 \cup L_2$, a compactification $h : M \rightarrow L$ will be called perfect with respect to x if and only if r (the right adjoint of $h|M_0$) satisfies $r(x \vee x^\bullet) = r(x) \vee r(x^\bullet)$.*

2. The compactification will be called **perfect** *if and only if r preserves first and second parts, and h is perfect with respect to every $x \in L_1 \cup L_2$.*

3. For $u, v, w \in L_0$, (u, v) **disconnects** *w if $w = u \vee v$, $u \wedge v = 0$ and $u, v \neq 0$.*

Lemma 7.1 *For $h : M \rightarrow L$ dense, onto and $r_i : L_i \rightarrow M_i$ the right adjoint of $h|M_i$, we have:*

1. $r_i(x)^\bullet = r_k(x^\bullet)$, $x \in L_i$

2. $h(x^\bullet) = h(x)^\bullet$, $x \in M_i$

PROOF.

1. For $x \in L_i$, $h(r_i(x) \wedge r_k(x^\bullet)) = x \wedge x^\bullet = 0$. Density of h gives $r_i(x) \wedge r_k(x^\bullet) = 0$ and so $r_k(x^\bullet) \leq r_i(x)^\bullet$. Conversely, $r_i(x) \wedge r_i(x)^\bullet = 0$ gives $x \wedge h(r_i(x)^\bullet) = 0$ and so $h(r_i(x)^\bullet) \leq x^\bullet$ and $r_i(x)^\bullet \leq r_k(x^\bullet)$.

2. For $x \in M_i$, $x \wedge r_k(h(x)^\bullet) = x \wedge \bigvee z(z \in M_k,\ h(z) = h(x)^\bullet)$. Now $h(z) = h(x)^\bullet$ implies that $h(z) \wedge h(x) = 0$, so by the density of h, $z \wedge x = 0$. Thus $x \wedge r_k(h(x)^\bullet) = 0$, and so $r_k(h(x)^\bullet) \leq x^\bullet$. Applying h gives $h(x)^\bullet \leq h(x^\bullet)$.

∎

We note that if r (as mentioned above) preserves first and second parts, then $r|L_i = r_i$ (the right adjoint of $h|M_i$). Further, having a biframe map onto (both parts) does not guarantee that the right adjoint preserves first and second parts: Take L_0 to be the four-element Boolean algebra $\{0, a, b, e\}$, with $L_1 = \{0, a, e\}$ and $L_2 = \{0, b, e\}$. Let $h : L \rightarrow \mathbf{2}$ be given by $h(a) = 0$, $h(b) = e$. The corresponding $r : \mathbf{2} \rightarrow L_0$ has $r(0) = a$, $r(e) = e$ and so does not preserve the second part.

Proposition 7.1 *Let $h : M \rightarrow L$ be a biframe compactification for which r preserves both parts. Then the following are equivalent:*

1. *$h : M \rightarrow L$ is perfect.*

2. *If (u, v) disconnects w (for some $u \in L_i$, $v \in L_k$ and $w \in L_0$), then $(r(u), r(v))$ disconnects $r(w)$.*

3. *$r(x \vee y) = r(x) \vee r(y)$ for $x \in L_i$, $y \in L_k$ and $x \wedge y = 0$.*

PROOF. (1) $\Longrightarrow$ (3): Suppose $x \in L_i$, $y \in L_k$ and $x \wedge y = 0$. Since r preserves order, $r(x) \vee r(y) \leq r(x \vee y)$. For the reverse inequality, we have $y \leq x^{\bullet}$, and so $r(x \vee y) \leq r(x \vee x^{\bullet}) = r(x) \vee r(x^{\bullet})$, by assumption. Similarly, $r(x \vee y) \leq r(y) \vee r(y^{\bullet})$. Thus

$$\begin{aligned} r(x \vee y) &\leq [r(x) \vee r(x^{\bullet})] \wedge [r(y) \vee r(y^{\bullet})] \\ &= r(x \wedge y) \vee r(x^{\bullet} \wedge y) \vee r(x \wedge y^{\bullet}) \vee r(x^{\bullet} \wedge y^{\bullet}) \\ &= r(y) \vee r(x) \vee r(x^{\bullet} \wedge y^{\bullet}), \end{aligned}$$

since $y \leq x^{\bullet}$, $x \leq y^{\bullet}$, $x \wedge y = 0$ and $r(0) = 0$ by the density of h. Now

$$\begin{aligned} r(x \vee y) &= r(x \vee y) \wedge [r(y) \vee r(x) \vee r(x^{\bullet} \wedge y^{\bullet})] \\ &= [r(x \vee y) \wedge (r(y) \vee r(x))] \vee [r(x \vee y) \wedge r(x^{\bullet} \wedge y^{\bullet})] \\ &= [r(x \vee y) \wedge (r(y) \vee r(x))] \vee [r((x \vee y) \wedge x^{\bullet} \wedge y^{\bullet})] \\ &= [r(x \vee y) \wedge (r(y) \vee r(x))] \vee r(0) \\ &= r(x \vee y) \wedge (r(y) \vee r(x)). \end{aligned}$$

Thus $r(x \vee y) \leq r(x) \vee r(y)$, as required.
(3) $\Longrightarrow$ (2): Suppose $w = u \vee v$ for some $u \in L_i$, $v \in L_k$, $u, v \neq 0$, $u \wedge v = 0$. (3) gives $r(w) = r(u) \vee r(v)$ and $r(u), r(v) \neq 0$ (since $r(u) = 0$ implies that $u = 0$). Also $r(u) \in M_i$, $r(v) \in M_k$ since r preserves both parts.
(2) $\Longrightarrow$ (1): For $x \in L_1 \cup L_2$, with $x, x^{\bullet} \neq 0$, (2) gives $r(x \vee x^{\bullet}) = r(x) \vee r(x^{\bullet})$ since $x \wedge x^{\bullet} = 0$. If $x = 0$, then $x^{\bullet} = e$, so $r(x \vee x^{\bullet}) = r(x) \vee r(x^{\bullet})$. ∎

For the following proposition we briefly review the correspondence between compactifications and strong inclusions for frames (see [5]).

For a compactification $h : M \to L$ with right adjoint r, the strong inclusion $\lhd$ is defined by $a \lhd b$ iff $r(a) \prec r(b)$. For a strong inclusion $\lhd$ on a frame L, call an ideal J of L strongly regular iff, for any $a \in J$, there exists $b \in J$ with $a \lhd b$. Then the join map $\bigvee : \mathcal{R}L \to L$ from the frame of strongly regular ideals of L to L is the associated compactification.

Proposition 7.2 *Let $h : M \to L$ be a biframe compactification for which r preserves both parts, and let $\lhd$ denote the strong inclusion on L_0 corresponding to the frame compactification $h : M_0 \to L_0$. Then h is perfect if and only if $a \leq x$, $a \lhd x \vee x^\bullet$ implies that $a \lhd x$ for all $a \in L_0$ and $x \in L_i$.*

PROOF. ($\Longrightarrow$) Take $a \in L_0$, $x \in L_i$ such that $a \leq x$ and $a \lhd x \vee x^\bullet$. By the definition of $\lhd$ and the assumption that h is perfect, we get
$r(a) \prec r(x \vee x^\bullet) = r(x) \vee r(x^\bullet)$. Then there exists $t \in L_0$ such that $r(a) \wedge t = 0$ and $t \vee r(x) \vee r(x^\bullet) = e$. Now

$$\begin{aligned} r(a) \wedge (t \vee r(x^\bullet)) &= (r(a) \wedge t) \vee (r(a) \wedge r(x^\bullet)) \\ &= 0 \vee r(a \wedge x^\bullet) \\ &= r(0) \text{ since } x \wedge x^\bullet = 0 \text{ and } a \leq x \\ &= 0 \text{ since } h \text{ is dense.} \end{aligned}$$

So $r(a) \prec r(x)$ with separating element $t \vee r(x^\bullet)$, and we have shown that $a \lhd x$.
($\Longleftarrow$) We recall that r may be given by $r(a) = \{z \in L_0 | \; z \lhd a\}$, all $a \in L_0$. We must show that, for $x \in L_i$, $r(x \vee x^\bullet) \subseteq r(x) \vee r(x^\bullet)$. So take $a \in r(x \vee x^\bullet)$, that is, $a \in L_0$ with $a \lhd x \vee x^\bullet$. Then $a = (a \wedge x) \vee (a \wedge x^\bullet)$.
Now $a \wedge x \leq x$, $a \wedge x \lhd x \vee x^\bullet$ implies that $a \wedge x \lhd x$ by the assumption.
Also, $a \lhd x \vee x^\bullet$ implies that $a \lhd x^\bullet \vee x^{\bullet\bullet}$, since $x \leq x^{\bullet\bullet}$.
So $a \wedge x^\bullet \leq x^\bullet$, $a \wedge x^\bullet \lhd x^\bullet \vee x^{\bullet\bullet}$ implies that $a \wedge x^\bullet \lhd x^\bullet$, again by the assumption.
Thus $a \in r(x) \vee r(x^\bullet)$, as required. ▮

Proposition 7.3 *Let $h : M \to L$ be a compactification for which r preserves both parts. If $x \lhd_i y^\bullet$ for some $x \in L_i$, $y \in L_k$, then $r(x \vee y) = r(x) \vee r(y)$.*

PROOF. Take $x \in L_i$, $y \in L_k$ with $x \lhd_i y^\bullet$. Then $r(x) \prec_i r(y^\bullet)$, so that $r(x)^\bullet \vee r(y^\bullet) = e$ and thus $r(x^\bullet) \vee r(y^\bullet) = e$, by Lemma 7.1 . Now

$$\begin{aligned} r(x \vee y) &= r(x \vee y) \wedge [r(x^\bullet) \vee r(y^\bullet)] \\ &= r((x \vee y) \wedge x^\bullet) \vee r((x \vee y) \wedge y^\bullet) \end{aligned}$$

$$= r(y \wedge x^\bullet) \vee r(x \wedge y^\bullet)$$
$$\leq r(y) \vee r(x).$$

We note that this result could also be obtained by observing that $x \lhd_i y^\bullet$ implies that $x \lhd y^\bullet$, which implies that $x \lhd y^*$; and applying the corresponding result for frames. ∎

Remark

For a biframe compactification $h : M \to L$, r preserves first and second parts if and only if the corresponding strong inclusions satisfy the condition:
For any $a \in L_0$ and $x \in L_i$ with $a \lhd x$, there exists $z \in L_i$ with $a \leq z \lhd_i x$.
Thus the assumption "$x \lhd_i y^\bullet$" in the above proposition could have been replaced by "$x \lhd y^\bullet$."

7.1 Rim-compact biframes

Definition 7.2 *1. A regular biframe L is* **rim-compact** *if and only if each $x \in L_i$ is a join of elements $z \in L_i$ such that $\uparrow(z \vee z^\bullet)$ is compact.*

2. Let L be a rim-compact biframe. A **Π-compact basis** *$B = (B_1, B_2)$ for L is a pair of bases, B_i for L_i $(i = 1, 2)$, such that*

(a) $x \in B_i$ implies that $\uparrow(x \vee x^\bullet)$ is compact

(b) $x \in B_i$ implies that $x^\bullet \in B_k$

(c) $x, y \in B_i$ imply that $x \wedge y$, $x \vee y \in B_i$.

Lemma 7.2 *Let L be a rim-compact biframe.*
Taking $B_i = \{x \in L_i \mid \uparrow(x \vee x^\bullet) \text{ compact}\}$ gives a Π-compact basis for L.

PROOF. We verify the last two conditions in the above definition.
For the second, $\uparrow(x \vee x^\bullet)$ compact implies $\uparrow(x^\bullet \vee x^{\bullet\bullet})$ compact, since

$x \leq x^{\bullet\bullet}$.

For the third, let $x, y \in L_i$ such that $\uparrow(x \vee x^{\bullet})$ and $\uparrow(y \vee y^{\bullet})$ are compact. Since $\uparrow a$ and $\uparrow b$ are both compact iff $\uparrow(a \wedge b)$ is compact, we get $\uparrow(x \vee x^{\bullet}) \wedge (y \vee y^{\bullet})$ compact. Now $(x \wedge y) \vee (x \wedge y)^{\bullet} = (x \vee (x \wedge y)^{\bullet}) \wedge (y \vee (x \wedge y)^{\bullet}) \geq (x \vee x^{\bullet}) \wedge (y \wedge y^{\bullet})$, so $\uparrow(x \wedge y) \vee (x \wedge y)^{\bullet}$ is compact.

Also $(x \vee y) \vee (x \vee y)^{\bullet} = x \vee y \vee (x^{\bullet} \wedge y^{\bullet}) = (x \vee y \vee x^{\bullet}) \wedge (x \vee y \vee y^{\bullet}) \geq (x \vee x^{\bullet}) \wedge (y \vee y^{\bullet})$, so $\uparrow(x \vee y) \vee (x \vee y)^{\bullet}$ is compact. ∎

Lemma 7.3 *Let L be a rim-compact biframe and B be a Π-compact basis for L. If $w \in L_i$ and $u \in B_k$ with $w \vee u = e$, then there exists $v \in B_k$ such that $v \prec_k u$ and $w \vee v = e$.*

PROOF. Since L is regular and B_i is a basis for L_i, we have that $w = \bigvee x (x \prec_i w,\ x \in B_i)$. Then $u \vee \bigvee x (x \prec_i w,\ x \in B_i) = e$ and so $u \vee u^{\bullet} \vee \bigvee x (x \prec_i w,\ x \in B_i) = e$. Now $\uparrow(u \vee u^{\bullet})$ is compact, so there exist $x_j \in B_i$, $x_j \prec_i w$ for $j = 1, \ldots, n$ such that $u \vee u^{\bullet} \vee (x_1 \vee \ldots \vee x_n) = e$. Put $x = x_1 \vee \ldots \vee x_n$. Then $x \in B_i$, $x \prec_i w$ and $u \vee u^{\bullet} \vee x = e$. Let $v = u \wedge x^{\bullet}$. Then $v \in B_k$ (by the second and third properties of Π-compact bases), and $w \vee v = w \vee (u \wedge x^{\bullet}) = (w \vee u) \wedge (w \vee x^{\bullet}) = e \wedge e = e$. Further, $v \prec_k u$ since $v \wedge (u^{\bullet} \vee x) = (v \wedge u^{\bullet}) \vee (v \wedge x) = (u \wedge x^{\bullet} \wedge u^{\bullet}) \vee (u \wedge x^{\bullet} \wedge x) = 0$ and $u \vee (u^{\bullet} \vee x) = e$, and $u^{\bullet} \vee x \in L_i$. ∎

Proposition 7.4 *Let $B = (B_1, B_2)$ be a Π-compact basis for a rim-compact biframe L. Define $\lhd_i$ on L_i by: $x \lhd_i y$ iff there exists $u \in B_i$ such that $x \prec_i u \prec_i y$. Then $\lhd_B = (\lhd_1, \lhd_2)$ is a strong inclusion on L.*

PROOF. (SI1) If $s \leq x \lhd_i y \leq t$, there exists $u \in B_i$ with $x \prec_i u \prec_i y$, so $s \prec_i u \prec_i t$ also.

(SI2) Since $0 \prec_i 0$, $e \prec_i e$ and $0, e \in B_i$, we get $0 \lhd_i 0$ and $e \lhd_i e$.

That $x \lhd_i y, z$ implies $x \lhd_i y \wedge z$ and $x, y \lhd_i z$ implies $x \vee y \lhd_i z$ follows from the properties of the relation $\prec_i$ and the fact that B_i is closed under finite meets and finite joins.

(SI3) That $x \lhd_i y$ implies $x \prec_i y$ is clear.
(SI4) Take $x \lhd_i y$, and $u \in B_i$ such that $x \prec_i u \prec_i y$. Then $x^\bullet \vee u = e$, so, using the above lemma, we obtain $v \in B_i$ satisfying $v \prec_i u$ and $x^\bullet \vee v = e$. Hence $x \prec_i v \prec_i u \prec_i y$. Similarly we can get $w \in B_i$ such that $x \prec_i v \prec_i w \prec_i u \prec_i y$, and then $x \lhd_i w \lhd_i y$.
(SI5) Take $x \lhd_i y$ and $u \in B_i$ with $x \prec_i u \prec_i y$. Then $y^\bullet \prec_k u^\bullet \prec_k x^\bullet$ and $u^\bullet \in B_k$, so $y^\bullet \lhd_k x^\bullet$.
(SI6) For any $x \in L_i$, $x = \bigvee z(z \prec_i x,\ z \in B_i)$. We now show that $z \prec_i x$, $z \in B_i$ implies $z \lhd_i x$ to obtain $x = \bigvee z(z \lhd_i x)$. Since $z^\bullet \vee x = e$ and $z^\bullet \in B_k$ we obtain, by the above lemma, $v \in B_k$, $v \prec_k z^\bullet$ and $v \vee x = e$. Now $v \prec_k z^\bullet$ iff $v^\bullet \vee z^\bullet = e$ iff $z \prec_i v^\bullet$, and $v \vee x = e$ implies that $v^{\bullet\bullet} \vee x = e$, so $v^\bullet \prec_i x$. Hence $z \prec_i v^\bullet \prec_i x$ and $v^\bullet \in B_i$, showing that $z \lhd_i x$. ∎

Lemma 7.2 and Proposition 7.4 together give a description of a compactification for any rim-compact L — in agreement with the terminology for spaces [26] and frames [2], we call this the **Freudenthal** compactification of L.

Lemma 7.4 *Any zero-dimensional biframe L is rim-compact.*
In fact, taking $B_i = \{x \in L_i \mid x \prec_i x\}$ provides a Π-compact basis for L, and the corresponding compactification is the largest zero-dimensional one.

PROOF. Since L is zero-dimensional, $x = \bigvee z(z \leq x,\ z \prec_i z)$ for each $x \in L_i$. But $z \prec_i z$ implies that $z \vee z^\bullet = e$, so that $\uparrow(z \vee z^\bullet)$ is certainly compact. The remaining conditions for B to be a Π-compact basis are clearly satisfied. The final claim follows from this characterization: the compactification corresponding to the strong inclusion $\lhd_i$ is zero-dimensional iff, for any $x \lhd_i y$, there exists $z \in L_i$ with $x \leq z \lhd_i z \leq y$. ∎

Lemma 7.5 *If L is rim-compact, and if $B_i = L_i$ forms a Π-compact basis for L, then $\lhd_B$ corresponds to the largest compactification of L.*

PROOF. If $(\lhd_1{}', \lhd_2{}')$ is another strong inclusion on L, $x \lhd_i{}' y$ implies that $x \lhd_i{}' z \lhd_i{}' y$ for some $z \in L_i$ (by the interpolation property of strong inclusions).

Then $x \prec_i z \prec_i y$, $z \in B_i$, as required. ∎

In the theory of frames (and of spaces), the Stone-Čech and the Freudenthal compactifications of a rim-compact frame (or space) are both perfect. We do not know whether this is also the case for biframes or not.

To conclude, we take another look at two previous examples of compactifications, in connection with the concepts of this section.

Example 1

$\mathcal{L}_0$ = all open subsets of the (real) **open** unit interval E
$\mathcal{L}_1$ = all open downsets
$\mathcal{L}_2$ = all open upsets
We recall that $\mathcal{L}$ has a unique compactification, given by $\mathcal{M}_0$ = the closed unit interval, $\mathcal{M}_1$ = all open downsets, $\mathcal{M}_2$ = all open upsets, and the obvious restriction maps. It may be viewed as a compactification derived from a Π-compact basis by taking $B_i = \{U \in \mathcal{L}_i\}$, since, for any $U \in \mathcal{L}_i$, $\uparrow(U \vee U^\bullet)$ is isomorphic to $\mathcal{O}(\mathrm{C}(U \vee U^\bullet))$ (where C denotes set-theoretic complement), and the latter is equal to $\mathcal{O}\{x\}$ for some $x \in E$, and this is certainly compact.
Let r denote the right adjoint, as usual. Then $r((0,a)) = [0,a)$ and $r((a,1)) = (a,1]$ for $a \in E$, so r preserves first and second parts. Also $r(U \vee U^\bullet) = r(U) \vee r(U^\bullet)$ for $U \in \mathcal{L}_i$, so this compactification is perfect.

Example 2

$\mathcal{L}_0$ = all open subsets of the (closed) **rational** unit interval E
$\mathcal{L}_1$ = all open downsets
$\mathcal{L}_2$ = all open upsets

1. On the smallest compactification of $\mathcal{L}$:
   We recall that this is given by the restriction map $\mathcal{M} \rightarrow \mathcal{L}$, where $\mathcal{M}_0$ = the real (closed) unit interval, $\mathcal{M}_1$ = all open downsets, $\mathcal{M}_2$ = all open upsets. The corresponding strong inclusion is given by $U \ll_i V$ iff $U \subset V$ or $U = V = E$ or $U = V = \emptyset$.
   This compactification may be obtained from the Π-compact basis $B_i = \{U \in \mathcal{L}_i | \ U \text{ has a rational end-point}\}$ (let its strong inclusion be denoted by $\lhd_i$): If $U \ll_i V$, then $U \subset V$ so there exists $W \in B_i$ with $U \subset W \subset V$ and thus $U \prec_i W \prec_i V$. Conversely, if $U \lhd_i V$, then $U \prec_i W \prec_i V$ for some $W \in B_i$, hence $U \neq V$, that is $U \ll_i V$ (since the $U \in \mathcal{L}_i$ for which $U \prec_i U$ are exactly those with irrational end-points). We also note that, for $U \in B_i$, $r(U \vee U^\bullet) = r(U) \vee r(U^\bullet)$.

2. On the largest compactification of $\mathcal{L}$:
   For $U \in \mathcal{L}_i$, $\uparrow(U \vee U^\bullet) \cong \mathcal{O}(\mathrm{C}(U \vee U^\bullet)) = \mathcal{O}\{x\}$ for some $x \in E$, or $\mathcal{O}\emptyset$. In either case, $\uparrow(U \vee U^\bullet)$ is compact, so $B_i = \mathcal{L}_i$ provides a Π-compact basis for $\mathcal{L}$, which necessarily corresponds to the largest compactification of $\mathcal{L}$ (by the previous lemma). It could also be obtained by taking
   $B_i = \{U \in \mathcal{L}_i | \ U \text{ has an irrational end-point}\} = \{U \in \mathcal{L}_i | \ U \prec_i U\}$. This description makes it immediately clear that this compactification is zero-dimensional.

3. On other compactifications of $\mathcal{L}$:
   Fix some $V \in \{U \in \mathcal{L}_i | \ U \text{ has an irrational end-point}\}$.
   Let $B_i = \{U \in \mathcal{L}_i | \ U \text{ has a rational end-point}\} \cup \{V\}$
   and $B_k = \{U \in \mathcal{L}_k | \ U \text{ has a rational end-point}\} \cup \{V^\bullet\}$.
   Then (B_1, B_2) is a Π-compact basis for $\mathcal{L}$, and the resulting compactification differs from the largest and the smallest. In this way, we can generate uncountably many compactifications of $\mathcal{L}$.

Chapter 8

Boolean biframes

In the setting of frames, the notions of a pseudocomplement and of Booleanness were defined in Section 0.5. The set $\mathcal{B}L = \{x \in L | \ x = x^{**}\}$ is a complete Boolean algebra (called the Booleanization of L) and the function $\beta : L \rightarrow \mathcal{B}L$ given by $\beta(x) = x^{**}$ is a frame homomorphism. (A reference for these results is [13].) The biframe pseudocomplement, $x^{\bullet}$, and the Boolean biframes were introduced in Definition 1.10.

8.1 The Booleanization of a biframe

Consider an arbitrary biframe $L = (L_0, L_1, L_2)$. For any congruence θ on L_0, let θ_i denote the congruence $\theta|L_i \times L_i$ on L_i, and let k_0 and k_i denote the respective nuclei. The function $h : L_i/\theta_i \rightarrow L_0/\theta$ defined by $h(k_i(x)) = k_0(k_i(x)) = k_0(x)$ is a one-one frame homomorphism, so L_i/θ_i may be regarded as a subframe of L_0/θ. Further, L_0/θ is generated by $L_1/\theta_1 \cup L_2/\theta_2$, since, if $a = \bigvee_\alpha x_\alpha \wedge y_\alpha$ for some $x_\alpha \in L_1$, $y_\alpha \in L_2$ and $k_0(a) = a$, then $a = \bigvee_\alpha k_1(x_\alpha) \wedge k_2(y_\alpha)$. Thus $(L_0/\theta, L_1/\theta_1, L_2/\theta_2)$ is a biframe and the map $(L_0, L_1, L_2) \rightarrow (L_0/\theta, L_1/\theta_1, L_2/\theta_2)$ is a biframe homomorphism.

The Booleanization BL of a biframe L is constructed as follows:

Let $\mu = \mu_L$ be the congruence on L_0 generated by $\{(x \vee x^\bullet, e) | \ x \in L_1 \cup L_2\}$. Let $\mu_i = \mu | L_i \times L_i$ and $\beta_L : (L_0, L_1, L_2) \rightarrow (L_0/\mu, L_1/\mu_1, L_2/\mu_2)$ be the biframe map described above. We denote this map by $\beta_L : L \rightarrow BL$. Certainly BL is a Boolean biframe, because, for $x \in L_i$, $k_0(x) \wedge k_0(x^\bullet) = k_0(0)$ and $k_0(x) \vee k_0(x^\bullet) = k_0(x \vee x^\bullet) = k_0(e)$.

The Booleanization of a *frame* is dense: we do not know whether this is always the case for biframes.

The following is clear:

Lemma 8.1 *A biframe L is Boolean if and only if $\beta_L : L \rightarrow BL$ is an isomorphism.*

8.2 Weakly open homomorphisms

We return to the frame context for a moment, to state the following result ([13]):

For any frame homomorphism $h : L \rightarrow M$, these are equivalent—

1. For any $a \in L$, $h(a^{**}) \leq h(a)^{**}$.
2. For any $a \in L$, $h(a)^* = 0$ whenever $a^* = 0$.
3. There exists a map $\tilde{h} : BL \rightarrow BM$ satisfying $\beta_M.h = \tilde{h}.\beta_L$; that is, making the following diagram commute

$$\begin{array}{ccc} L & \xrightarrow{h} & M \\ {\scriptstyle \beta_L}\downarrow & & \downarrow{\scriptstyle \beta_M} \\ BL & \xrightarrow[\tilde{h}]{} & BM \end{array}$$

Such frame homomorphisms have been called "weakly open". (It is known that, in general, the inequality in the first property mentioned above cannot be replaced by equality — [13].)

The equivalence mentioned above makes these maps particularly useful in the study of Boolean frames. For biframes the situation is not so clear. We discuss the following candidates for weakly open biframe maps:

Definition 8.1 *A biframe map* $h : L \to M$ *will be called*

1. *an A-map iff* $h(x^\bullet) = h(x)^\bullet$ *for all* $x \in L_1 \cup L_2$,
2. *a B-map iff* $h(x^{\bullet\bullet}) = h(x)^{\bullet\bullet}$ *for all* $x \in L_1 \cup L_2$,
3. *a C-map iff* $h(x^{\bullet\bullet}) \leq h(x)^{\bullet\bullet}$ *for all* $x \in L_1 \cup L_2$,
4. *a D-map iff* $h(x)^\bullet = 0$ *whenever* $x^\bullet = 0$, *for all* $x \in L_1 \cup L_2$,
5. *an E-map iff there is a map* $\tilde{h} : BL \to BM$ *satisfying* $\beta_M.h = \tilde{h}.\beta_L$; *that is, making the following diagram commute:*

$$\begin{array}{ccc} L & \xrightarrow{h} & M \\ {\scriptstyle \beta_L}\downarrow & & \downarrow{\scriptstyle \beta_M} \\ BL & \xrightarrow[\tilde{h}]{} & BM \end{array}$$

Lemma 8.2 *An A-map is a B-map is a C-map is a D-map.*

Remark

- Any dense, onto biframe map is an A-map.
  This was proved in Lemma 7.1.

- A C-map need not necessarily be a B-map, since these two concepts are distinct even in the analogous frame case. ([13])

Lemma 8.3 *h is a C-map iff $h(x)^\bullet = h(x^{\bullet\bullet})^\bullet$ for all $x \in L_1 \cup L_2$.*

PROOF. ($\Longrightarrow$) $h(x^{\bullet\bullet}) \leq h(x)^{\bullet\bullet}$ implies that $h(x^{\bullet\bullet})^\bullet \geq h(x)^{\bullet\bullet\bullet} = h(x)^\bullet$.
($\Longleftarrow$) $h(x^{\bullet\bullet}) \leq h(x^{\bullet\bullet})^{\bullet\bullet} = h(x)^{\bullet\bullet}$. ∎

Lemma 8.4 *If L is a De Morgan biframe, that is, it satisfies $x^\bullet \vee x^{\bullet\bullet} = e$ for all $x \in L_1 \cup L_2$, then $h : L \rightarrow M$ is an A-map iff it is a C-map.*

PROOF. ($\Longleftarrow$) For $x \in L_1 \cup L_2$, $x^\bullet \vee x^{\bullet\bullet} = e$, hence $h(x^\bullet) \vee h(x^{\bullet\bullet}) = e$ and also $h(x^\bullet) \vee h(x)^{\bullet\bullet} = e$. Then $h(x^\bullet) \wedge h(x)^\bullet = h(x)^\bullet$ so that $h(x)^\bullet \leq h(x^\bullet)$. ∎

Lemma 8.5 *Any composite $f.\beta_L : L \rightarrow BL \rightarrow M$ is an A-map.*

PROOF. For $x \in L_1 \cup L_2$, $f.\beta_L(x \vee x^\bullet) = e$. Taking meets with $f.\beta_L(x)^\bullet$ gives $f.\beta_L(x)^\bullet \leq f.\beta_L(x^\bullet)$. ∎

Lemma 8.6 *Any A-map is an E-map.*

PROOF. Let $h : L \rightarrow M$ be an A-map. For $x \in L_1 \cup L_2$, $\beta_M.h(x \vee x^\bullet) =$
$\beta_M(h(x) \vee h(x)^\bullet) = e$, since $h(x^\bullet) = h(x)^\bullet$. Thus μ_L (the congruence used in the

formation of $\mathcal{B}L$) is contained in the kernel of $\beta_M.h$, so there exists a function $\tilde{h} : L_0/\mu_L \to M_0/\mu_M$ for which $\tilde{h}.\beta_L|L_0 = \beta_M.h|L_0$. Further, $\tilde{h}$ is a biframe map because β_L is onto. ∎

We note that an E-map need not, in general, be an A-map, because these properties differ in the analogous frame setting. ([13])

Corollary 8.1 *$h : L \to M$ is an E-map iff $\beta_M.h : L \to \mathcal{B}M$ is an A-map.*

PROOF. ($\Longrightarrow$) Follows from Lemma 8.5.
($\Longleftarrow$) Follows from Lemma 8.6. ∎

Corollary 8.2 *Boolean biframes are reflective in the category of biframes and A-maps (or biframes and E-maps).*

Proposition 8.1 *1. L is Boolean iff each $h : L \to M$ is an A-map, iff each $h : L \to M$ satisfies $h(x \vee x^\bullet) = e$, $x \in L_1 \cup L_2$.*

2. L satisfies $x = x^{\bullet\bullet}$, $x \in L_1 \cup L_2$ iff each $h : L \to M$ is a C-map.

3. L satisfies the condition: $x = e$ whenever $x^{\bullet\bullet} = e$ iff each $h : L \to M$ is a D-map.

PROOF.

1. If L is Boolean, $x \vee x^\bullet = e$ for all $x \in L_1 \cup L_2$ and so $h(x \vee x^\bullet) = e$. If this latter condition is satisfied, $h(x)$ and $h(x^\bullet)$ are complements in M_0, so certainly $h(x^\bullet) = h(x)^\bullet$. Conversely, suppose that each $h : L \to M$ is an A-map, but L is not Boolean. Then there is some $x \in L_1 \cup L_2$ for which $a = x \vee x^\circ < e$. The function $h : (L_0, L_1, L_2) \to (\uparrow a, \uparrow a, \uparrow a)$ given by $h(z) = z \vee a$, is a biframe map. Now $h(x) = h(x^\bullet) = a$, so if h is an A-map, $a = a^\bullet$, which is a contradiction, since a is the zero but not the top in $\uparrow a$.

2. ($\Longrightarrow$) For any $x \in L_1 \cup L_2$, $h(x^{\bullet\bullet}) = h(x) \leq h(x)^{\bullet\bullet}$.
   ($\Longleftarrow$) Suppose there is an $x \in L_1 \cup L_2$ with $x \neq x^{\bullet\bullet}$. Then $h : (L_0, L_1, L_2) \to (\uparrow x, \uparrow x, \uparrow x)$ given by $h(z) = z \vee x$, is a biframe map. Now $h(x) = x$, $h(x^{\bullet\bullet}) = x^{\bullet\bullet}$ and $x^{\bullet\bullet} = x$ in $(\uparrow x, \uparrow x, \uparrow x)$, so if h is a C-map, $x^{\bullet\bullet} \leq x$, a contradiction.

3. ($\Longrightarrow$) For $x \in L_1 \cup L_2$, if $x^{\bullet} = 0$ then $x^{\bullet\bullet} = e$, so $x = e$, so $h(x) = e$, and so $h(x)^{\bullet} = 0$.
   ($\Longleftarrow$) Suppose there exists an $x \in L_1 \cup L_2$ such that $x < e$ and $x^{\bullet\bullet} = e$. The map $h : (L_0, L_1, L_2) \to (\uparrow x, \uparrow x, \uparrow x)$ by taking joins with x satisfies $h(x) = x$, thus $h(x)^{\bullet} = e$, although $x^{\bullet} = 0$.

∎

For the next lemma we recall that, for any biframe L, the ideal biframe $\mathcal{J}L$ is given as follows:

$(\mathcal{J}L)_i$ consists of those ideals J in L_0 that are generated by $J \cap L_i$, and $(\mathcal{J}L)_0$ is the subframe of $\mathcal{J}L_0$ (the frame of ideals of L_0) generated by $(\mathcal{J}L)_1 \cup (\mathcal{J}L)_2$.

For $I \in (\mathcal{J}L)_i$, $J \in (\mathcal{J}L)_k$ $(i \neq k)$, we have

$$\begin{aligned} I \cap J = 0 \quad & \text{iff} \quad \downarrow x \cap J = 0, \ \text{all } x \in I \cap L_i \\ & \text{iff} \quad x \wedge y = 0, \ \text{all } x \in I \cap L_i, \ y \in J \\ & \text{iff} \quad x \wedge \bigvee J = 0, \ \text{all } x \in I \cap L_i \\ & \text{iff} \quad x \leq (\bigvee J)^{\bullet}, \ \text{all } x \in I \cap L_i \ (\text{note that } \bigvee J \in L_k) \end{aligned}$$

So $J^{\bullet} = \downarrow (\bigvee J)^{\bullet}$ and $J^{\bullet\bullet} = \downarrow (\bigvee J)^{\bullet\bullet}$.

Proposition 8.2 *h is an A-map (respectively a B-map, C-map, D-map) if and only if $\mathcal{J}h$ is an A-map (respectively a B-map, C-map, D-map).*

PROOF. For A-maps:
($\Longrightarrow$) For $J \in (\mathcal{J}L)_i$, $\mathcal{J}h(J^\bullet) = \mathcal{J}h(\downarrow (\bigvee J)^\bullet) = \downarrow h((\bigvee J)^\bullet)$ and $\mathcal{J}h(J)^\bullet = \downarrow (\bigvee \mathcal{J}h(J))^\bullet = \downarrow (\bigvee h[J])^\bullet$. Now $h((\bigvee J)^\bullet) = h(\bigvee J)^\bullet = (\bigvee h[J])^\bullet$, since h is an A-map; so $\mathcal{J}h(J^\bullet) = \mathcal{J}h(J)^\bullet$.
($\Longleftarrow$) For $x \in L_i$, $\mathcal{J}h((\downarrow x)^\bullet) = \mathcal{J}h(\downarrow x)^\bullet$ by assumption. Now $\mathcal{J}h((\downarrow x)^\bullet) = \mathcal{J}h(\downarrow (x^\bullet)) = \downarrow h(x^\bullet)$ and $\mathcal{J}h(\downarrow x)^\bullet = \downarrow h(x)^\bullet$ hence $h(x^\bullet) = h(x)^\bullet$.

For C-maps (the proof for B-maps is similar):
($\Longrightarrow$) For $J \in (\mathcal{J}L)_i$, $\mathcal{J}h(J^{\bullet\bullet}) = \mathcal{J}h(\downarrow (\bigvee J)^{\bullet\bullet}) = \downarrow h((\bigvee J)^{\bullet\bullet})$ and $\mathcal{J}h(J)^{\bullet\bullet} = \downarrow (\bigvee \mathcal{J}h(J))^{\bullet\bullet} = \downarrow (h(\bigvee J))^{\bullet\bullet}$.
So if $h((\bigvee J)^{\bullet\bullet}) \leq h(\bigvee J)^{\bullet\bullet}$ then $\mathcal{J}h(J^{\bullet\bullet}) \subseteq \mathcal{J}h(J)^{\bullet\bullet}$.
($\Longleftarrow$) For $x \in L_i$, $\mathcal{J}h((\downarrow x)^{\bullet\bullet}) \subseteq \mathcal{J}h(\downarrow x)^{\bullet\bullet}$ implies that $\mathcal{J}h(\downarrow (x^{\bullet\bullet})) \subseteq \downarrow (\bigvee \mathcal{J}h(\downarrow x))^{\bullet\bullet}$, so that $\downarrow h(x^{\bullet\bullet}) \subseteq \downarrow h(x)^{\bullet\bullet}$ and thus $h(x^{\bullet\bullet}) \leq h(x)^{\bullet\bullet}$.

For D-maps:
($\Longrightarrow$) For $J \in (\mathcal{J}L)_i$, $J^\bullet = 0$ implies $\downarrow (\bigvee J)^\bullet = 0$, so that $(\bigvee J)^\bullet = 0$ and $h(\bigvee J)^\bullet = 0$ (since h is a D-map). Thus $\mathcal{J}h(J)^\bullet = \downarrow (h(\bigvee J)^\bullet) = 0$.
($\Longleftarrow$) For $x \in L_i$, $x^\bullet = 0$ implies that $(\downarrow x)^\bullet = 0$ so that $\mathcal{J}h(\downarrow x)^\bullet = 0$ (since $\mathcal{J}h$ is a D-map), and, finally, $\downarrow h(x)^\bullet = 0$ and $h(x)^\bullet = 0$. ∎

Proposition 8.3 *If $\mathcal{J}h$ is an E-map, then h is an E-map.*

PROOF. We first note that, for any biframe L, the join map $\tau_L : \mathcal{J}L \rightarrow L$ is dense and onto, and so an A-map, by the remark after Lemma 8.2. Consider the following commuting diagram:

$$\begin{array}{ccccc}
\mathcal{J}L & \xrightarrow{\mathcal{J}h} & \mathcal{J}M & \xrightarrow{\beta_{\mathcal{J}M}} & B\mathcal{J}M \\
\downarrow{\scriptstyle \tau_L} & & \downarrow{\scriptstyle \tau_M} & & \downarrow{\scriptstyle \widetilde{\tau_M}} \\
L & \xrightarrow[h]{} & M & \xrightarrow[\beta_M]{} & BM
\end{array}$$

$\widetilde{\tau_M}$ is the biframe map making the right-hand square commute (it exists by Lemma 8.6). By assumption, $\mathcal{J}h$ is an E-map, that is, $\beta_{\mathcal{J}M}.\mathcal{J}h$ is an A-map (see corollary to Lemma 8.6). $\widetilde{\tau_M}$, being a map between Boolean biframes, is also an A-map. Thus the commutativity of the diagram shows that $\beta_M.h.\tau_L$ is an A-map. We now obtain the result that $\beta_M.h$ is an A-map from the claim that follows.
Claim: If $f : N \rightarrow P$ is onto and $g.f : N \rightarrow P \rightarrow Q$ is an A-map, then g is an A-map.
Proof: Let $x \in P_i$. Then $x = f(z)$ for some $z \in N_i$. Then $g(x)^\bullet = g(f(z))^\bullet = gf(z^\bullet)$ and hence $gf(z)^\bullet = g(f(z)^\bullet)$. ∎

Lemma 8.7 *Biframe products are the products in the category of biframes and A-maps (or B-, C- or D-maps).*

PROOF. Let $L = \prod_\alpha L^\alpha$ and $p_\alpha : L \rightarrow L^\alpha$ denote the projection maps.
For $x = (x_\alpha) \in L_i$, $x^\bullet = (x_\alpha^\bullet)$, so $p_\alpha(x^\bullet) = p_\alpha(x)^\bullet$. Given A-maps $f_\alpha : M \rightarrow L^\alpha$, the map $h : M \rightarrow L$ given by $h(a) = (f_\alpha(a))$ is an A-map, since, for $z \in M_i$, $h(z^\bullet) = (f_\alpha(z^\bullet)) = (f_\alpha(z)^\bullet) = (f_\alpha(z))^\bullet = h(z)^\bullet$.
The proofs for B-, C- and D-maps are similar. ∎

Appendix: List of categories

The following is a list of the categories for which we use abbreviated names, their objects (their morphisms are given in the text) and the page on which they are defined.

Category	Objects	Page
Ens	sets	30
Frm	frames	4
Top	topological spaces	5
BiFrm	biframes	9
BiTop	bitopological spaces	11
KRBiFrm	compact regular biframes	13
StBiFrm	Stone biframes	18
KBooBiFrm	compact Boolean biframes	20
CohBiFrm	coherent biframes	42
StContBiFrm	stably continuous biframes	50
SCohBiFrm	supercoherent biframes	53
StSContBiFrm	stably supercontinuous biframes	54
BooBiTop	Boolean bispaces	47
SpecBiTop	spectral bispaces	47
BiLatt	bilattices	43
BiSLatt	bisemilattices	54
BooBiLatt	Boolean bilattices	18
CFrm	frames with all elements compact	20
CohFrm	coherent frames	20
StContFrm	stably continuous frames	50

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