PERTURBATIONS OF SEMI-FREDHOLM OPERATORS IN L.C.T.V.S.

PERTURBATIONS OF SEMI-FREDHOLM OPERATORS

IN

LOCALLY CONVEX TOPOLOGICAL VECTOR SPACES

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LE QUANG CHU, LIC., M.SC.

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AUTHOR: Le Quang Chu, Lic. (Université de Liège, Belgium)
M.Sc. (University of Saskatchewan)

SUPERVISOR: Professor T. Husain

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ABSTRACT

It is a well-known result of I.C. Gohberg, M.G. Krein and T. Kato that if T is a semi-Fredholm operator between Banach spaces and P a bounded operator of norm small enough, or a compact operator, then T+P is a semi-Fredholm operator with the same index as T.

This thesis is concerned with extensions of this result to more general locally convex spaces. A systematic study is made of suitably defined small bounded or precompact perturbations of Φ_+ and Φ_--operators. The results obtained apply in particular to Fréchet spaces and effectively extend the theorems of I.C. Gohberg, M.G. Krein and T. Kato as well as several of Ju.N. Vladimirski.

Duality is shown to be a convenient tool to prove many of these results. Some applications are also given.

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CONTENTS

INTRODUCTION

An important theorem concerning non-analytic perturbations of linear operators is the following result of Gohberg, Krein (5) and Kato (10) :

If T is a semi-Fredholm operator between Banach spaces and P a bounded operator of norm small enough, or a compact operator, then T+P is a semi-Fredholm operator with the same index as T (cf. definitions in Chapter 0).

This index theorem has numerous applications. It can be used in particular to prove the existence of solutions of certain functional equations (cf. (11)).

There are similar situations with linear partial differential operators, where the spaces involved are Fréchet spaces rather than Banach spaces. The question then arises whether a generalized version of the index theorem holds.

This thesis deals with extensions of the theorem of Gohberg, Krein and Kato to more general locally convex spaces, in particular to Fréchet spaces.

In the setting of general locally convex spaces, it is possible to define several concepts of "small" perturbations which reduce in case of normed spaces to operators with small norms. In Chapter I we first consider small, compact or precompact perturbations in the sense that, when restricted to certain suitable subspaces equipped with convenient norms, the perturbing operators are small in norm, compact or precompact. It is assumed that these normed "subspaces" are Banach spaces and that the restricted operators satisfy the hypotheses of the index theorem of Kato. We establish several stability theorems, including the stability of the index.

In Chapter II we investigate small or precompact perturbations in the sense that the perturbing operator maps a neighborhood of the origin into a small bounded disk or a precompact disk. We make extensive use of duality. It turns out that in the duals, the adjoint operators constitute perturbations of the type considered in Chapter I. The suitable Banach "subspaces" in this case are those generated by closed equicontinuous disks. We apply the results of Chapter I, and obtain by duality stability theorems for small bounded and precompact perturbations of Φ_+ and Φ_--operators. (The case of compact perturbations is settled by some theorems of Schwartz (23), Köthe (12) and Schaefer (22); it is also covered by our approach.)

Our results in Chapters I and II yield several generalizations of the theorems of Gohberg, Krein and Kato, especially for Φ_--operators. They also strengthen many results of Vladimirski, announced without proof in (26;27).

Chapter III deals with some variations of the previously considered perturbations, and contains some applications. In particular, the stability results are extended to the case where the perturbing operators are bounded "relative to the unperturbed operators". This follows a concept introduced by

Sz-Nagy (24) for Banach spaces. Next, perturbations of semi-Fredholm operators with complemented ranges and kernels are studied. Pietsch (20) considered this problem for operators acting from a space into itself. We treat it for different domain and range spaces. We also study the behaviour under perturbations of operators which "lift" certain families of weakly compact disks. Finally, the thesis ends with some spectral properties of bounded operators in sequentially complete locally convex spaces, and an example of bounded perturbations of (Φ_-) partial diferential operators between Fréchet spaces.

The main text is preceded by a preliminary Chapter 0, where notations and some frequently used lemmas are gathered.

The cross-references are self-explanatory, and of the forms Section III.4.1, Theorem II.12, etc... . The first (Roman) numeral refers to the chapter. The theorems, propositions, lemmas and corollaries are numbered consecutively within each chapter.

Part of our results appeared or will appear in (3;15; 16;17;18).

CHAPTER 0

NOTATIONS AND SOME AUXILIARY RESULTS

In this chapter, we define some of the less common terminologies and notations that are used in the text. For those terminologies and concepts that are not defined here, we refer the reader to (13, Köthe).

We also prove succinctly some of the auxiliary results to which we shall often implicitly or explicitly refer in the main text. These are mostly well-known.

0.1. Notations relating to the spaces

Throughout the text, we will be concerned mainly with vector spaces, which we assume to be defined on a same real or complex scalar field.

Let E be a vector space. A subset $A \subset E$ is an *absolutely convex* set, or a *disk*, if A contains all linear combinations of the form $\sum_{i=1}^{n} c_i x_i$ such that $x_i \in A$, $i=1,\ldots,n$, and $\sum_{i=1}^{n} |c_i| \leq 1$.

If A contains all linear combinations of the form $\sum_{i=1}^{n} c_i x_i$, $x_i \in A$, $i=1,\ldots,n$, for all scalars c_i, then A is a *linear subspace*. For convenience, we also call a linear subspace simply a *subspace* (as opposed to subset, when the subset is not linear).

If $A \subset E$ then $\langle A \rangle$ (resp. $\rangle A \langle$) denotes the *absolutely convex* hull (resp. linear hull) of A, that is the intersection of all disks (resp. subspaces) containing A.

If L is a subspace of E, then dim L denotes the (algebraic) dimension of L and codim L (or more precisely $\text{codim}_E L$) the codimension of L in E.

Let A, B be two subsets of E. We say that A absorbs B if $B \subset \lambda A$ for some scalar $\lambda > 0$. If L is a subspace in E, we say that A is absorbent in L if $L \subset \rangle A \langle$.

By locally convex space we mean a locally convex topological vector space, which is assumed Hausdorff unless otherwise stated.

Let E be a locally convex space. By neighborhood in E we mean a (not necessarily open) neighborhood of the origin in E, unless otherwise specified. The closure of a subset A is denoted by $(A)^{-}$.

Let $A \subset E$ be a disk. We denote by E_A the space generated by A, that is $\rangle A \langle$ topologized by the Minkowski gauge of A (equivalently, E_A has a base of neighborhoods composed of the sets λA, $\lambda > 0$).

We say that the disk A is norming if E_A is a normed space, that is $\bigcap_{\varepsilon > 0} \varepsilon A = \{0\}$. If moreover E_A is a Banach space, then A is said to be completing or a Banach disk. We say that A is a finite disk if E_A is a finite dimensional (euclidean) space.

Let L, M be two subspaces in E. We say that L is a complement, or algebraic complement, of M in E, and write $E = L \oplus M$, if $L + M = E$ and $M \cap L = \{0\}$. We say that L is an algebraic and topological complement of M if moreover the projection of E onto M along L is continuous.

We denote by E^+ the dual of E, that is the space of all continuous (linear) functionals defined on E. We refer to the pointwise convergence topology on E or E^+ in the duality (E, E^+) as the weak topology, characterized by the term "weak" (e.g. weak closure, weakly compact etc...). Unless otherwise specified, E^+ is always equipped with the weak topology.

In the duality (E, E^+), the polar of a subset A is denoted by A° : if $A \subset E$ then $A^\circ = \{f \in E^+ : |f(x)| \leq 1, \ \forall x \in A\}$; if $A \subset E^+$ then $A^\circ = \{x \in E : |f(x)| \leq 1, \ \forall f \in A\}$.

0.2. Notations relating to the operators

Let E, F be two vector spaces. By operator from E into F we mean a linear operator defined on a (linear) subspace of E, with values in F.

Let T be an operator from E into F. Then D(T) and R(T) denote the domain (of definition) and the range (of values) of T respectively. The graph of T is $G(T)=\{(x,y) \in E\times F : x \in D(T), y=Tx\}$. We denote by N(T) the kernel (or null-space) of T : $N(T) = \{x \in D(T) : Tx = 0\}$.

We use the notations nul(T) = dim N(T) for the nullity of T, and def(T) = codim R(T) for the deficiency of T. Notice that nul(T) and def(T) may be infinite. If at least one of them is finite, then we define the index of T to be ind(T) = nul(T) - def(T), and we say that T has an index.

In this connection, we should stress that we will not distinguish between different cardinalities of infinity, that is we also write $a = b$ if both a and b are $+\infty$ or both are $-\infty$ (but $+\infty \neq -\infty$).

For $A \subset E$ and $B \subset F$, we write $TA = \{Tx : x \in A \cap D(T)\}$ and $T^{-1}B = \{x \in D(T) : Tx \in B\}$.

Let E, F be locally convex spaces and T an operator from E into F.

We say that T <u>has a closed graph</u> if G(T) is closed in ExF (and not only in D(T)xF).

We say that T is <u>open</u> (resp. <u>almost open</u>) if TU (resp. $(TU)^{-} \cap R(T)$) is a neighborhood in R(T), for any neighborhood U in E. (It could be proved easily that it is equivalent to say that T is almost open if $(TU)^{-}$ is a neighborhood in $(R(T))^{-}$ for any neighborhood U in E. See (*) in the proof of Theorem 0.15.) We say that T is <u>weakly open</u> if it is open when E and F are equipped with the weak topologies.

The operator T is called a ϕ_+ (resp. ϕ_-) <u>-operator</u> if (a) T has a closed graph, (b) T is open, (c) R(T) is closed and (d) nul(T) (resp. def(T)) is finite, which we denote by nul(T) $< \infty$ (resp. def(T) $< \infty$).

We say that T is a <u>semi-Fredholm operator</u> if it is either a ϕ_+ or a ϕ_--operator, and a <u>Fredholm operator</u> if it is both a ϕ_+ and a ϕ_--operator (after the familiar terminology in Banach spaces).

We say that T is a <u>bounded</u> (resp. <u>compact</u>, <u>precompact</u>, or <u>weakly compact</u>) operator if $(TU)^{-}$ is bounded (resp. compact, precompact i.e. totally bounded, or weakly compact) in F for some neighborhood U in E.

Finally, if $D(T)$ is dense in E, then T^+ denotes the adjoint operator of T, from F^+ into E^+.

0.3. Some lemmas

We gather here some of the auxiliary (known) results which we shall often use in the main text. We use the symbol ./. to mark the end of a proof.

We begin with some lemmas on norming and completing disks.

LEMMA 0.1. *Let* L *be a linear subspace and* A *a disk in a vector space* E. *The following are equivalent* :

(a) $L \cap E_A$ *is closed in* E_A,

(b) $\bigcap_{\varepsilon>0} (L+\varepsilon A) = L$.

Proof. (a) => (b) : Let $x \in \bigcap_{\varepsilon>0}(L+\varepsilon A)$. Let $\varepsilon_n > 0$ be any sequence converging to 0. For each ε_n there is $x_n \in L$ such that $x - x_n \in \varepsilon_n A$. We have $x_n - x_1 \in L \cap E_A$, and $x_n - x_1 \to x - x_1$ in E_A. By assumption (a), $x - x_1 \in L \cap E_A$; hence $x \in L$.

(b) => (a) : Let $x_n \in L \cap E_A$ and $x_n \to x$ in E_A. For any $\varepsilon > 0$, there is x_m such that $x - x_m \in \varepsilon A$. Thus $x \in L+\varepsilon A$. Since $\varepsilon > 0$ is arbitrary, we have $x \in \bigcap_{\varepsilon>0}(L+\varepsilon A)$; hence $x \in L \cap E_A$ ./.

LEMMA 0.2. *If* A, B *are two Banach disks in a vector space* E *such that* $A+B$ *is norming, then* $A+B$ *is a Banach disk*.

Proof. We consider the operator $T : E_A \times E_B \to E_{A+B}$ defined by $(x,y) \to x+y$. Obviously T is continuous, open and onto. Moreover $N(T)$ is closed. Indeed, E_{A+B} is Hausdorff, and T is defined everywhere on $G = E_A \times E_B$ and continuous, thus T has a closed graph. Since $G(T) \cap (G \times \{0\}) = N(T) \times \{0\}$ is closed in $G \times E_{A+B}$, it follows that $N(T)$ is closed. Now $G = E_A \times E_B$ topologized by $A \times B$ is a Banach space (as a product of two Banach spaces), therefore E_{A+B}, being isomorphic to $G/N(T)$, is also a Banach space ./.

LEMMA 0.3. Let E, F be two vector spaces, $A \subset E$, $B \subset F$ be two disks, and T an operator from E into F.

(a) If $N(T) \cap E_A$ is closed in E_A then TA is norming.

(b) If A, B are Banach disks and $G(T) \cap (E_A \times F_B)$ is closed in $E_A \times F_B$, then $A \cap T^{-1}B$ is a Banach disk.

Proof. (a) : Let $N(T) \cap E_A$ be closed in E_A. In view of Lemma 0.1, $\bigcap_{\varepsilon>0} (N(T)+\varepsilon A) = N(T)$. On the other hand,

$$\bigcap_{\varepsilon>0} \varepsilon TA = T(\bigcap_{\varepsilon>0} (\varepsilon A+N(T))) = T(N(T)) = \{0\}.$$

(b) : Let x_n be a Cauchy sequence in E_C, where $C = A \cap T^{-1}B$. Then x_n is a Cauchy sequence in E_A and Tx_n a Cauchy sequence in F_B (as $x_n - x_m \in \varepsilon T^{-1}B$ implies $Tx_n - Tx_m \in \varepsilon B$). Since A and B are completing, there exist x and y such that $x_n \to x$ in E_A and $Tx_n \to y$ in F_B. Since $G(T) \cap (E_A \times F_B)$ is closed in $E_A \times F_B$, it follows that $(x,y) \in G(T) \cap (E_A \times F_B)$, that is $x \in D(T)$ and $Tx = y$. For any $\varepsilon > 0$, there is an integer m such that $x_n - x \in \varepsilon A$ and $Tx_n - Tx \in \varepsilon B$, thus $(x_n - x) \in \varepsilon A \cap T^{-1}B$, for all $n \geq m$. This shows that $x_n \to x$ in E_C ./.

LEMMA 0.4. Let E be a locally convex space and K a bounded disk which is sequentially complete in E. Then K is a Banach disk..

<u>In particular compact disks are Banach disks.</u>

<u>Proof</u>. Since K is bounded, E_K has a topology finer than that induced by E. Thus K is norming. Let x_n be a Cauchy sequence in E_K. Then x_n is a Cauchy sequence in E, and $x_n \in \lambda K$, $\forall n$, for some λ large enough. Thus x_n converges to some $x \in \lambda K$ in E. We now prove that $x_n \to x$ in E_K. Let $\varepsilon > 0$ be given. Then $x_n - x_m \in \varepsilon K$ for n, m large enough. Now $x_n - x_m \to x - x_m$ in E, and $x - x_m \in \varepsilon K$ since K is sequentially complete ./.

We now turn to some results involving an operator. In the remainder of this section, we shall always assume E, F to be locally convex spaces, and T an operator from E into F such that $D(T)$ is dense in E (so that the adjoint operator T^+ is defined).

LEMMA 0.5. $\quad G(T)^\circ = \{(-T^+g, g) \in E^+ \times F^+ : g \in D(T^+)\}$.

<u>In particular</u>, $G(T^+)$ <u>is weakly closed in</u> $F^+ \times E^+$.

<u>Proof</u>. It is well-known that $u \in (E\times F)^+$ is of the form $u = (f,g)$, where $f \in E^+$ and $g \in F^+$; if $h = (x,y) \in E\times F$, then $u(h) = f(x) + g(y)$.

It follows that $u = (f,g) \in G(T)^\circ$ if and only if $u(h) = 0$, $\forall h \in G(T)$, that is $f(x) + g(y) = 0$ or equivalently $f(x) = -g(Tx)$, for all $x \in D(T)$. This is equivalent to : $g \in D(T^+)$ and $f = -T^+g$. Hence $G(T)^\circ = \{(-T^+g, g) : g \in D(T^+)\}$.

Since $G(T)^\circ$ is weakly closed in $(E\times F)^+$, and the weak topology on $(E\times F)^+$ is the product of the weak topologies on E^+ and F^+, it follows that $G(T^+)$ is also weakly closed in $F^+ \times E^+$ ./.

LEMMA 0.6. *If* T *has a closed graph, then* $D(T^+)$ *is weakly dense in* F^+.

Proof. It is equivalent to prove that $D(T^+)^\circ = \{0\}$ in F. Let $y \in D(T^+)^\circ$. Then $g(y) = 0$, $\forall g \in D(T^+)$. In view of Lemma 0.5, $(0,y) \in G(T)^{\circ\circ}$. But $G(T)^{\circ\circ} = (G(T))^- = G(T)$. Therefore $y=T0=0$ ./.

LEMMA 0.7. *Let* L *be a closed subspace of* E. *Then* $\dim L < \infty$ *if and only if* $\operatorname{codim} L^\circ < \infty$ *in* E^+. *Furthermore* $\dim L = \operatorname{codim} L^\circ$. *By duality*, $\operatorname{codim} L < \infty$ *if and only if* $\dim L^\circ < \infty$, *and*

$$\operatorname{codim} L = \dim L^\circ .$$

Proof. If E^+ is equipped with the weak topology, then E^{++} may be canonically identified (via the evaluation operator) with E. On the other hand, L° is weakly closed and $L = (L)^- = L^{\circ\circ}$. Thus we need only prove the second (dual) part.

Assume that $\operatorname{codim} L \geq n$, n an integer. We prove that $\dim L^\circ \geq n$. Let $x_1,\ldots,x_n$ be linearly independent and such that $>\{x_1,\ldots,x_n\}< \cap L = \{0\}$. By the Hahn-Banach theorem, there are $f_1,\ldots,f_n \in E^+$ such that $f_i \in L^\circ$ and $f_i(x_j) = \delta_{ij}$ (Kronecker symbol), for $i,j = 1,\ldots,n$.

These functionals are linearly independent, for if c_i, $i = 1,\ldots,n$ are scalars such that $\sum_{i=1}^{n} c_i f_i = 0$, then $\sum_{i=1}^{n} c_i f_i(x_j) = c_j = 0$, $j = 1,\ldots,n$. Thus $\dim L^\circ \geq n$.

We now prove that if $\operatorname{codim} L = n$, then $\dim L^\circ = n$. Let x_i, f_i be as above. Let $N(f_i)$ be the null-space of f_i. We prove first that $L = \bigcap_{i=1}^{n} N(f_i)$. Let $x \notin L$ and $x = x' + x''$ with $x' = \sum_{i=1}^{n} a_i x_i$ and $x'' \in L$. Since $x' \neq 0$ by assumption, at least one scalar a_i should be non zero. For that a_i, we have $f_i(x') = a_i \neq 0$, whereas $f_i(x'') = 0$. Thus $f_i(x) \neq 0$, and $x \notin \bigcap_{i=1}^{n} N(f_i)$. This shows that $\bigcap_{i=1}^{n} N(f_i) \subset L$ (the converse inclusion is trivial).

Now let $f \in L^\circ$. By the above, $N(f) \supset \bigcap_{i=1}^{n} N(f_i)$. By a well-known property of linear functionals, f is a linear combination of the f_i's. Hence $\dim L^\circ = n$ ./.

LEMMA 0.8. <u>Assume that</u> $D(T) = E$, <u>and</u> T <u>has a closed graph. Then</u> $N(T)^\circ = (R(T^+))^-$ <u>and</u> $R(T)^\circ = N(T^+)$, <u>the closure being taken with respect to the weak topology</u>.

<u>Proof</u>. The second equality is a direct consequence of the definition of T^+ (and is true even without the assumptions $D(T) = E$ and $G(T)$ closed).

For the first, we prove that $N(T) = R(T^+)^\circ$. It is obvious that $N(T) \subset R(T^+)^\circ$. Conversely, let $x \in R(T^+)^\circ$. Then $Tx \in D(T^+)^\circ$. In view of Lemma 0.6, $Tx = 0$, that is $x \in N(T)$ ./.

For the next few results, we shall need the following (cf. (8, Grothendieck)) :

LEMMA 0.9. <u>Let</u> N <u>be a closed subspace of</u> E. <u>Let</u> E_w <u>denote</u> E <u>equipped with the weak topology. Then</u> $E_w/N = (E/N)_w$, <u>and the common topology is the weak topology in the duality</u> $(E/N, N^\circ)$, N° <u>being the polar of</u> N <u>in</u> E^+.

<u>Proof</u>. We first prove that $(E/N)^+ = N^\circ$. Let $f \in (E/N)^+$; then f defines in a canonical way an $\tilde{f} \in E^+$ such that $\tilde{f} \in N^\circ$. This association is clearly linear, one-to-one and onto (by direct examination of the topology on E/N and the definition of $\tilde{f} \in N^\circ$). Since E and E_w have the same dual, we have proved in fact that

$(E/N)^+ = (E_w/N)^+ = N^\circ$. It remains only to prove that the weak topology on E/N in the duality $(E/N, N^\circ)$ is finer than that of E_w/N. Let A be a (weak) neighborhood in E_w. We have to prove that there is a finite disk $D \subset N^\circ$ such that $D^\circ \subset A+N$ (polar taken in the duality (E, E^+)). Now trivially $(A+N)^\circ = A^\circ \cap N^\circ$, N being linear. Since A° is a finite disk in E^+, $A^\circ \cap N^\circ$ is contained in a finite disk $D' \subset N^\circ$. We have then $D'^\circ \subset (A+N)^- \subset 2A+N$. Take $D = 2D'$ ./.

As a direct consequence of Lemma 0.9, we have :

LEMMA 0.10. <u>If</u> T <u>is open then</u> T <u>is weakly open</u>.

<u>Proof</u>. It suffices to remark that if $f \in N(T)^\circ$ then g defined by $g(Tx) = f(x)$ is a continuous functional on $R(T)$, which extends to a continuous functional on F, by the Hahn-Banach theorem ./.

The argument in the preceding proof also yields immediately the following

LEMMA 0.11. <u>Assume that</u> $D(T) = E$ <u>and</u> T <u>has a closed graph. If</u> T <u>is weakly open then</u> $R(T^+)$ <u>is weakly closed</u>.

<u>Proof</u>. We already know that $(R(T^+))^- = N(T)^\circ$ (Lemma 0.8). We now prove that $R(T^+) = N(T)^\circ$. But this is immediate, for if $f \in N(T)^\circ$, then g defined by $g(Tx) = f(x)$ extends to a continuous functional $\bar{g} \in E^+$. Thus $\bar{g} \in D(T^+)$ and $T^+\bar{g} = f \in R(T^+)$ ./.

If we return to the assumption $D(T)$ dense in E, then $R(T^+)$ is a fortiori weakly closed in the duality (E, E^+) if T is weakly open. We now prove that the converse holds true.

LEMMA 0.12. <u>Assume that</u> T <u>has a closed graph. Then</u> $R(T^+)$ <u>is weakly closed if and only if</u> T <u>is weakly open</u>.

<u>Proof</u>. It remains only to prove the "only if" part. Equip E^+ and F^+ with the weak topologies. Since T has a closed graph, $D(T^+)$ is dense in F^+, therefore T^{++} is defined from $E^{++} = E$ into $F^{++} = F$. In view of Lemma 0.5,

$$\begin{aligned} G(T)^{\circ\circ} &= \{(x,y) \in E \times F : (-T^+g)(x) + g(y) = 0, \ \forall g \in D(T^+)\} \\ &= \{(x,y) \in E \times F : g(y) = (T^+g)(x), \ \forall g \in D(T^+)\} \\ &= \{(x,y) \in E \times F : x \in D(T^{++}), \ y = T^{++}x\} \\ &= G(T^{++}). \end{aligned}$$

Since $G(T) = (G(T))^- = G(T)^{\circ\circ}$, we infer that $T = T^{++}$. The Lemma is now a consequence of the following :

LEMMA 0.13. <u>If</u> $R(T)$ <u>is closed, then</u> T^+ <u>is open from</u> F^+ <u>into</u> E^+ <u>equipped with the weak topologies</u>.

<u>Proof</u>. We should prove that T^+ induces (in a canonical way) a one-to-one operator from $F^+/N(T^+)$ into E^+, which is open when F^+, E^+ are equipped with the weak topologies. In view of Lemma 0.9, we should prove that if $y \in N(T^+)^\circ$ and $A = \{g \in F^+ : |g(y)| < \varepsilon\}$ is a (weak) neighborhood in F^+, then T^+A is a (weak) neighborhood in $R(T^+)$. But this is immediate because $N(T^+)^\circ = (R(T))^- = R(T)$ by assumption, therefore $y = Tx$ for some $x \in D(T)$. Now $T^+A = \{f \in E^+ : |f(x)| < \varepsilon\} \cap R(T^+)$ ./.

Finally we prove some sufficient conditions from which to infer that T is open when it is almost open.

LEMMA 0.14. T <u>is open if and only if it is weakly open and almost open</u>.

Proof. The "only if" part is obvious. Conversely, let T be weakly open and almost open. We have to prove that the one-to-one operator from $E/N(T)$ into F induced by T is open. In view of Lemma 0.9, we may assume without loss of generality that T is one-to-one.

Let U be any closed neighborhood in E. Then $(TU)^- \cap R(T)$ is a neighborhood in $R(T)$. We now prove that "weak openness" of T implies $TU = (TU)^- \cap R(T)$. Indeed, since T^{-1} is weakly continuous from $R(T)$ into E, and the closure of a disk is the same whether it be in the original or in the weak topology, we have

$$T^{-1}(TU)^- \subset (T^{-1}(TU))^- \cap D(T)$$
$$\subset U \cap D(T).$$

Thus $(TU)^- \cap R(T) = TT^{-1}(TU)^- \subset TU$, that is TU is a neighborhood in $R(T)$.

If now U is an arbitrary neighborhood in E, and $U' = (1/2\ U)^- \subset U$, then by the preceding paragraph TU' (and a fortiori TU) is a neighborhood in $R(T)$. Hence the conclusion ./.

DEFINITION. A locally convex space E is said to be fully complete if it has the following property :

(P) : If H is a linear subspace in E^+ such that $H \cap U^\circ$ is weakly closed, for all neighborhoods U in E, then H is weakly closed.

We refer the reader to (21, Robertson and Robertson) for the various properties (and their proofs) of fully complete spaces. We recall only that

(a) Fully complete spaces are indeed complete,

(b) A quotient of a fully complete space by a closed subspace is fully complete,

(c) Frechet spaces are fully complete,

(d) The following theorem holds, to which we shall often refer in the main text :

THEOREM 0.15. <u>Assume that</u> E <u>is a fully complete locally convex space and</u> T <u>an operator from a subspace</u> $D(T) \subset E$ <u>into a locally convex space</u> F, <u>such that the graph</u> G(T) <u>is closed in</u> ExF.

<u>If</u> T <u>is almost open, then</u> T <u>is open and</u> R(T) <u>is closed</u>.

<u>Proof</u>. A proof of this theorem may be inferred directly from (21, Robertson and Robertson, Proposition 10). We simply notice that (a) the one-to-one operator from E/N(T) into F induced by T has a closed graph in (E/N(T))xF, because $N(T) \times \{0\} \subset G(T)$, thus the projection of G(T) into (E/N(T))xF along $N(T) \times \{0\}$ is still closed (consider the set theoretic complement of G(T), and the fact that the projection is an open operator), and (b) E/N(T) is also fully complete. Therefore we may assume without loss of generality that T is one-to-one (T^{-1} is "nearly continuous" (21, Robertson and Robertson) from R(T) into E and has a closed graph). The operator T is open by (21, Robertson and Robertson, Proposition 10). Now let $\mathfrak{F}$ be a filter in R(T) converging to y in F. The filter in E generated by $T^{-1}\mathfrak{F}$ is Cauchy, thus converges to x in E (E being in particular complete). Since G(T) is closed in ExF, we infer that $x \in D(T)$ and $Tx = y \in R(T)$.

We may also prove directly that T is open as follows.

In view of Lemma 0.14 it suffices to prove that T is weakly open.

Let $G = (D(T))^- \subset E$. Then $G^+ = E^+/G^\circ$. In view of Lemma 0.12, we should prove that $R(T^+)$ is weakly closed in the duality $(G, E^+/G^\circ)$, if we consider T as an operator from G into F.

Let H be the inverse image of $R(T^+)$ by the canonical quotient operator $Q : E^+ \to E^+/G^\circ$. The problem reduces to showing that H is weakly closed in E^+ (cf. Lemma 0.9) because $G^\circ \subset H$ (so that if H is closed, the image of H by Q, which is $R(T^+)$, will be closed : consider the set-theoretic complement of H, and the fact that Q is an open operator). Since E is fully complete, it is sufficient to prove that $H \cap U^\circ$ is weakly closed for any neighborhood U in E (property (P)). We may assume without loss of generality that U is closed. Then $(U \cap G)^\circ = (U^\circ + G^\circ)^- = U^\circ + G^\circ$, since U° is (weakly) compact and G° (weakly) closed. This shows that if $\bar{U} = U \cap G$ then, in the dual G^+, $\bar{U}^\circ = QU^\circ$. Therefore

$$\begin{aligned} U^\circ \cap H &= U^\circ \cap Q^{-1}(R(T^+)) \\ &= U^\circ \cap (U^\circ + G^\circ) \cap Q^{-1}(R(T^+)) \\ &= U^\circ \cap Q^{-1}((QU^\circ) \cap R(T^+)) \\ &= U^\circ \cap Q^{-1}(\bar{U}^\circ \cap R(T^+)). \end{aligned}$$

We now show that $\bar{U}^\circ \cap R(T^+)$ is weakly closed in G^+, from which it will follow that $U^\circ \cap H$ is weakly closed (Q being continuous), and the proof will be finished.

Now there is a neighborhood V in F such that $V \cap R(T) \subset (T\bar{U})^-$. We may assume V to be open; then $V \cap R(T)$ is dense in $V \cap (R(T))^-$. Thus $V \cap (R(T))^- \subset (V \cap R(T))^- \subset (T\bar{U})^-$. Moreover, if

$x \in (V)^- \cap (R(T))^-$, then $\lambda x \in V \cap (R(T))^-$, $\forall \lambda : 0 \leq \lambda < 1$. Thus

$$(V)^- \cap (R(T))^- \subset (V \cap (R(T))^-)^- \subset (T\bar{U})^-.$$

((*) Incidentally, this proves that T is almost open if $(TU)^-$ is a neighborhood in $(R(T))^-$ (or $R(T)$) for any neighborhood U in E.)

Taking the polars, we obtain

$$T^{+-1}\bar{U}^\circ \subset (V^\circ + R(T)^\circ)^-$$
$$\subset V^\circ + N(T^+),$$

V° being compact and $N(T^+) = R(T)^\circ$ closed.

Now let $\mathcal{F}$ be a filter in $\bar{U}^\circ \cap R(T^+)$, weakly convergent to $f \in \bar{U}^\circ$ in G^+. If $A \in \mathcal{F}$ then $\{(T^{+-1}A) \cap V^\circ\}$ define a filter $\mathcal{G}$ in V°. There is an ultrafilter $\mathcal{G}'$ finer than $\mathcal{G}$ which converges to an element $g \in V^\circ$, since V° is compact. The filter generated by $T^+\mathcal{G}'$ is finer than $\mathcal{F}$ and converges to f in F^+. Since $G(T^+)$ is closed in $F^+ x G^+$ (Lemma 0.5), we have $g \in D(T^+)$ and $T^+g = f \in \bar{U}^\circ \cap R(T^+)$. Thus $\bar{U}^\circ \cap R(T^+)$ is closed ./.

CHAPTER I

KATO'S THEOREM AND SOME EXTENSIONS

In this chapter, we present the statement of a classical result of Gohberg, Krein (5) and Kato (10) concerning small and compact perturbations of semi-Fredholm operators between Banach spaces and some extensions to more general locally convex spaces.

The main problem shall be to define a suitable concept of "small" perturbations when the spaces are no longer normed. Several such definitions may be possible. In this chapter we study a type of perturbations based on the following idea : a perturbation preserves "nice" properties if it does when the operators are restricted to suitable subspaces. We consider "small" perturbations accordingly and obtain some extensions of the result of Gohberg, Krein and Kato. Some other possible extensions are derived in the next chapter where some principles and results developed in this chapter are used in the duals. Several results of Vladimirski (26), Goldman and Krackowski (7) are also obtained, and in many cases strengthened by this approach.

I.1. The theorem of Gohberg, Krein and Kato

THEOREM I.1 (Gohberg, Krein (5), Kato (10))*. _Let_ E, F _be Banach_

<u>spaces,</u> T <u>a semi-Fredholm operator and</u> P <u>a continuous operator from</u> E <u>into</u> F <u>such that</u> $D(T) \subset D(P)$.

(A) <u>Let</u> $\gamma(T) = \inf\{\|Tx\|/d(x, N(T)) : x \in D(T), x \notin N(T)\}$ <u>if</u> $T \neq 0$ <u>and</u> $\gamma(T) = \infty$ <u>if</u> $T = 0$ (<u>in which case</u> E <u>or</u> F <u>should be finite dimensional). If</u> $\|P\| < \gamma(T)$, <u>then</u> T+P <u>is a semi-Fredholm operator</u>, $\mathrm{nul}(T+P) \leq \mathrm{nul}(T)$, $\mathrm{def}(T+P) \leq \mathrm{def}(T)$ <u>and</u> $\mathrm{ind}(T+P) = \mathrm{ind}(T)$.

(B) <u>If</u> P <u>is a compact operator, then</u> T+P <u>is a semi-Fredholm operator and</u> $\mathrm{ind}(T+P) = \mathrm{ind}(T)$.

(C) <u>There exists</u> $\rho > 0$ <u>such that, for all</u> λ <u>in the annulus</u> $0 < |\lambda| < \rho$, $\mathrm{nul}(T+\lambda P)$ <u>and</u> $\mathrm{def}(T+\lambda P)$ <u>are constant</u>.

We refer the reader to (6, Goldberg) for a proof of this theorem.

Part (A) of this theorem was essentially established by Gohberg and Krein in (5), but only for a less precise upper bound of $\|P\|$. The formulation $\|P\| < \gamma(T)$ is due to Kato in (10). This bound is the best possible if one considers the following simple counter-example : take $T = I$, $P = -I$, I being the identity operator; then $\gamma(T) = 1$, $\|P\| = 1$ and $T+P = 0$.

Part (B) in fact holds for more general locally convex spaces, as was proved earlier by Schwartz (23), Köthe (12) and Schaefer (22) (cf. Theorems I.7 & I.8). Kato (10) proved it for P belonging to a larger class of operators (in Banach spaces) which he called "strictly singular".

A generalized version of Part (C) shall be proved in Chapter II, Theorem II.5.

For convenience, we will refer henceforth to this theorem as Kato's theorem.

If B, B' denote the unit balls of E, F respectively, then $\|P\| = \inf\{\lambda > 0 : PB \subset \lambda B'\}$ and $\gamma(T) = \sup\{\mu \geq 0 : \mu B' \cap R(T) \subset TB\}$.

The first equality follows immediately from the definition of $\|P\|$. For the second, let θ denote the right-hand side. If $T = 0$ then obviously $\gamma(T) = \theta = \infty$. Suppose $T \neq 0$. Then

$$\gamma(T) = \inf\{\|Tx\| : x \in D(T),\ d(x,N(T)) = 1\}.$$

We prove first that $\theta \leq \gamma(T)$. Let $\mu \geq 0$ be such that $\mu B' \cap R(T) \subset TB$, and consider any $x \in D(T)$ such that $d(x,N(T)) = 1$ ($d(x,L)$ denotes the distance of x to L in terms of the norm of the space). If $\|Tx\| < \mu$ then $\|Tx\| < (1-\varepsilon)\mu$ for some $0 < \varepsilon < 1$. Thus $Tx \in (1-\varepsilon)TB$, which would mean that $x \in (1-\varepsilon)B + N(T)$. This would show that $d(x,N(T)) \leq 1-\varepsilon$, contrary to our assumption on x. Thus $\|Tx\| \geq \mu$, therefore $\theta \leq \gamma(T)$. We now prove that for any $0 < \varepsilon < 1$, we have $(1-\varepsilon)\gamma(T)B' \cap R(T) \subset TB$, and this would show that $\theta = \gamma(T)$. Let $y = Tx \in (1-\varepsilon)\gamma(T)B' \cap R(T)$. We may assume that $Tx \neq 0$. Then $\|Tx\| \leq (1-\varepsilon)\gamma(T) \leq (1-\varepsilon)\|Tx\|/d(x,N(T))$. This shows that $d(x,N(T)) \leq 1-\varepsilon$. There is thus an $x_0 \in B$ such that $x - x_0 \in N(T)$, that is $Tx = Tx_0 \in TB$. The proof of the equality under consideration is now complete.

Part (A) of Kato's theorem now reads :

THEOREM I.2. <u>Let</u> E, F <u>be Banach spaces with unit balls</u> B, B' <u>respectively. Let</u> T <u>be a semi-Fredholm operator and</u> P <u>a continuous operator from</u> E <u>into</u> F <u>such that</u> $D(T) \subset D(P)$.

<u>Assume that there exist positive constants</u> μ <u>and</u> λ <u>such that</u>

$$\mu B' \cap R(T) \subset TB, \ PB \subset \lambda B' \ \underline{\text{and}} \ \mu > \lambda .$$

Then $T+P$ is a semi-Fredholm operator, $\mathrm{nul}(T+P) \leq \mathrm{nul}(T)$, $\mathrm{def}(T+P) \leq \mathrm{def}(T)$ and $\mathrm{ind}(T+P) = \mathrm{ind}(T)$.

REMARK. The condition $\mu B' \cap R(T) \subset TB$ implies that T is open. Now as $E/N(T)$ is complete (which is a consequence of the fact that T has a closed graph, hence $N(T)$ is closed, and that E is a Banach space), it also implies that $R(T)$ is closed. Thus the above theorem holds with T (a priori) an operator with a closed graph (in $E \times F$) and which has an index.

There are several proofs of Kato's theorem (cf. (10; 11, Kato), (6, Goldberg), (14, Le Quang Chu)), all of which depend on a crucial theorem of topology due to Borsuk, concerning odd mappings of a sphere into another of smaller dimension (cf. (5, Gohberg, Krein)). The result of Borsuk is used via the following

LEMMA I.3. Let M, N be subspaces of a normed space E. If $\dim N < \infty$ and $\dim M > \dim N$, then there exists an $x \in M$, $x \neq 0$ such that $d(x,N) = \|x\|$.

We refer the reader to (6, Goldberg) for more details about this lemma.

Kato's theorem has numerous applications to ordinary differential equations. The stability of the index is a useful tool to prove several existence theorems. We refer the reader to (11, Kato) and (6, Goldberg) for the many examples of these applications.

We reproduce here one brief example from (6, Goldberg) to illustrate some general ideas about such applications.

EXAMPLE. Let τ be a differential expression of the form

$$\tau = \sum_{k=0}^{n} a_k D^k$$

where each a_k is a complex-valued function on a compact interval I. For each positive integer n, we define $A_n(I)$ to be the set of complex-valued functions f on I for which $D^{n-1}f = f^{(n-1)}$ (the $(n-1)^{th}$ derivative of f) exists and is absolutely continuous on I. Let $A_0(I) = C(I)$ (continuous functions on I), and $L_1(I)$ be the set of Lebesgue-integrable functions on I.

We say that T is the <u>maximal</u> operator associated with τ if T is defined as follows :

$$D(T) = \{f : f \in A_n(I) \cap L_1(I),\ \tau f \in L_1(I)\},$$
$$Tf = \tau f = \sum_{k=0}^{n} a_k D^k f .$$

(Since absolutely continuous functions are differentiable a.e., τf is defined a.e. on I for $f \in A_n(I)$.)

We have the following

THEOREM. <u>Let</u> k <u>be a bounded measurable function on</u> $I \times I$, $a_i \in L_1(I)$, $0 \le i \le n-1$ <u>and</u> $\frac{1}{a_n} \in L_\infty(I)$. <u>Then there exists</u> $\rho > 0$ <u>such that for</u> $|\lambda| < \rho$, <u>the equation</u>

$$\sum_{i=0}^{n} a_i(t) f^{(i)}(t) + \lambda \int_I k(s,t) f(s)\,ds = y(t) \text{ a.e.}$$
$$y \in L_1(I)$$

<u>has precisely</u> n <u>linearly independent solution in</u> $A_n(I)$.

<u>Proof</u>. Let T be the maximal operator associated with $\tau = \sum_{i=0}^{n} a_i D^i$. Define P on $L_1(I)$ by

$$(Pf)(t) = \int_I k(s,t) f(s)\,ds.$$

It is proved in (6, Goldberg, Theorems VI.3.1 and VI.3.2) that T is a Fredholm operator, nul(T) = n and def(T) = 0.

Obviously, P is a bounded operator; hence for $\|\lambda P\| < \gamma(T)$ we have $\mathrm{nul}(T+\lambda P) \leq \mathrm{nul}(T)$, $\mathrm{def}(T+\lambda P) \leq \mathrm{def}(T)$ and $\mathrm{ind}(T+\lambda P) = \mathrm{ind}(T)$. Consequently $\mathrm{def}(T+\lambda P) = 0$ and $\mathrm{nul}(T+\lambda P) = n$ ./.

REMARK. In fact in the preceding proof P is a strictly singular operator and, as a consequence, the equation has precisely n linearly independent solutions in $A_n(I)$ for all λ except for at most a countable set of exceptional isolated points $\{\lambda_i\}$ with no accumulation point at finite distance (cf. (6, Goldberg, Theorem VI.8.7), from which our example is taken).

There are similar situations for linear partial differential equations (see Example III.4.3). But there, the spaces are frequently Frechet spaces or more general locally convex spaces (of distributions). Therefore it seems interesting to determine whether or not generalized versions of Kato's theorem, for locally convex spaces, exist. It is this question that we treat in this thesis.

I.2. First extensions of Kato's theorem

The question we are concerned with is how Kato's theorem can be extended to more general locally convex spaces. The most natural answer is to look first at normed spaces, where, if nothing else, at least the concept of small perturbations is still clear.

We notice that in Kato's theorem, the completeness of F is not really essential.

I.2.1. Extension to the case of normed range spaces

If F is complete we may extend P by continuity, and therefore $D(T) \subset D(P)$ may be replaced by $(D(T))^{-} \subset D(P)$. With this remark, Kato's theorem holds true in the following slightly more general case :

THEOREM I.4. Let E be a Banach space, F a normed space, T a semi-Fredholm operator and P a continuous operator from E into F such that $(D(T))^{-} \subset D(P)$.

(A) Assume that there exist a neighborhood U in E, a neighborhood V in F and $0 < \varepsilon < 1$ such that $V \cap R(T) \subset TU$ and $PU \subset \varepsilon V$. Then T+P is a semi-Fredholm operator, $\mathrm{nul}(T+P) \leqslant \mathrm{nul}(T)$, $\mathrm{def}(T+P) \leqslant \mathrm{def}(T)$ and $\mathrm{ind}(T+P) = \mathrm{ind}(T)$.

(B) Assume that P is a precompact operator, then T+P is a semi-Fredholm operator and $\mathrm{ind}(T+P) = \mathrm{ind}(T)$.

(C) There exists $\rho > 0$ such that, for all λ in the annulus $0 < |\lambda| < \rho$, $\mathrm{nul}(T+\lambda P)$ and $\mathrm{def}(T+\lambda P)$ are constant.

Proof. The proof will be quite characteristic of those in Chapter II. It makes use of Kato's theorem and duality.

That T+P has a closed graph in ExF is straightforward, because T has a closed graph, P is continuous and $(D(T))^{-} \subset D(P)$.

We now prove that T+P is almost open. We may assume without loss of generality that $D(T) = D(T+P) = E$.

Let U, V be as in the statement of the theorem. For (B) we may assume that PU is precompact in F.

We may assume without loss of generality that V is closed. Then from

$$V \cap R(T) \subset TU$$

(R(T) is closed), by taking the polars, and using the fact that $(TU)^\circ = T^{+-1}U^\circ$, $R(T)^\circ = N(T^+)$, we obtain

$$T^{+-1}U^\circ \subset (V^\circ + N(T^+))^- \subset V^\circ + N(T^+)$$

since V° is weakly compact, $N(T^+)$ weakly closed, and the closure is taken with respect to the weak topology in the duality (F,F^+). Thus $U^\circ \cap R(T^+) \subset T^+V^\circ$.

This means that T^+ is open from the Banach space F^+ into the Banach space E^+ generated by V° and U° respectively (this is equivalent to using the strong topologies on F^+ and E^+). In view of Lemma 0.5, T^+ has a closed graph.

On the other hand, $N(T^+) = R(T)^\circ$, $R(T^+) = N(T)^\circ$ (Lemmas 0.8 and 0.11), so that $\mathrm{nul}(T)^+ = \mathrm{def}(T^+)$, $\mathrm{def}(T^+) = \mathrm{nul}(T)$ and $\mathrm{ind}(T^+) = -\mathrm{ind}(T)$ (Lemma 0.7). Thus T^+ is a semi-Fredholm operator.

In case (A), from $PU \subset \varepsilon V$ we get $P^{+-1}U^\circ \supset \varepsilon^{-1}V^\circ$, thus $P^+V^\circ \subset \varepsilon U^\circ$. We apply Kato's theorem, part (A), and obtain that $T^+ + P^+$ is a semi-Fredholm operator, $\mathrm{nul}(T^+ + P^+) \leq \mathrm{nul}(T^+)$, $\mathrm{def}(T^+ + P^+) \leq \mathrm{def}(T^+)$ and $\mathrm{ind}(T^+ + P^+) = \mathrm{ind}(T^+)$.

In case (B), from the precompactness of PU, it follows that V° is precompact with respect to the semi-norm generated by $P^{+-1}U^\circ$, by a well-known property of precompactness in duality (8, Grothendieck). For any $\lambda > 0$, there is a finite set

$F \subset \rangle P^{+-1}U^\circ \langle\ (=F^+)$ such that $V^\circ \subset F + \lambda P^{+-1}U^\circ$; therefore $P^+V^\circ \subset P^+F + \lambda U^\circ$ and this shows that P^+ is a compact operator. Now Kato's theorem, part (B), applies. As a result $T^+ + P^+$ is a semi-Fredholm operator and $\operatorname{ind}(T^+ + P^+) = \operatorname{ind}(T^+)$.

In both cases, $T^+ + P^+$ is open : there is $\mu > 0$ such that

$$\mu U^\circ \cap R(T^+ + P^+) \subset (T^+ + P^+)V^\circ,$$

that is

$$\mu (T^+ + P^+)^{-1}U^\circ \subset V^\circ + N(T^+ + P^+).$$

As $T^+ + P^+ = (T+P)^+$, by duality we obtain

$$\mu V \cap R(T+P) \subset ((T+P)U)^-,$$

thus T+P is almost open.

Since E is a Banach space, hence fully complete, we deduce further that T+P is open and R(T+P) is closed (Theorem 0.15).

Now from the fact that $R(T+P)^\circ = N(T^+ + P^+)$, $N(T+P)^\circ = R(T^+ + P^+)$, we infer $\operatorname{ind}(T+P) = \operatorname{ind}(T)$, and furthermore in case (A), $\operatorname{nul}(T+P) \leq \operatorname{nul}(T)$, $\operatorname{def}(T+P) \leq \operatorname{def}(T)$.

There is $\rho > 0$ such that, for all λ in the annulus $0 < |\lambda| < \rho$, $\operatorname{nul}(T^+ + \lambda P^+)$ and $\operatorname{def}(T^+ + \lambda P^+)$ are constant. By duality part (C) is proved ./.

We now give some counter-examples to show that Kato's theorem fails when E is not complete. More precisely, what we shall fail to obtain is that the range of the perturbed operator T+P be closed in the case of small perturbations, or in the case of compact perturbations of Φ_--operators.

I.2.2. Some counter-examples

We present first a counter-example concerning small perturbations of Φ_+-operators and Φ_--operators.

COUNTER-EXAMPLE (i). Let E be the space of all polynomials defined on the interval $(0,1/2)$, normed by the sup norm : $f \in E$ if and only if f is of the form $f = f(x) = \sum_{i=0}^{n} a_i x^i$, $n=1,2,\ldots$, and $\|f\| = \sup\{|f(x)| \ : x \in (0,1/2)\}$. Obviously E is a normed space but not a Banach space (E is a dense subspace of $C(0,1/2)$, by Weierstrass' theorem). Let $F = E$, $T = I$ the identity operator, and P be the operator "multiplication by x" defined by $Pf(x)=xf(x)$. Then

$$\begin{aligned}\|Pf\| &\leq \sup\{|x| \ |f(x)| \ : x \in (0,1/2)\} \\ &\leq 1/2 \sup\{|f(x)| \ : x \in (0,1/2)\} \\ &\leq 1/2 \ \|f\| \quad .\end{aligned}$$

Therefore $\|P\| \leq 1/2$, whereas $\gamma(T) = \gamma(I) = 1$. The operator T is of course an onto isomorphism (a very particular Fredholm operator) but $R(T+P)$ fails to be closed, and since $\operatorname{def}(T) = 0$, we have $\operatorname{def}(T+P) > \operatorname{def}(T)$ as a consequence. Indeed, $(T+P)f(x) = (1+x)f(x)$, therefore $R(T+P)$ is the subspace of all polynomials divisible by $(1+x)$. It is of course a proper subspace of E (consider $g(x) = 1$ for instance), and in fact it is a dense subspace of E (which is consistent with the fact that $N(T^+ + P^+) = 0$, from which we get $(R(T+P))^- = F$, if we pass to the duals as in the proof of Theorem I.4).

In this counter-example we even know that T+P is one-to-one and open (with a closed graph), as shown by the following

LEMMA I.5. *Let* E, F *be normed spaces,* T *an injective open operator and* P *a bounded operator from* E *into* F *such that* $D(T) \subset D(P)$ *and* $\|P\| < \gamma(T)$. *Then* $T+P$ *is injective and open*.

REMARK. This result in fact extends to the case of general locally convex spaces, as will be proved in Chapter II (Theorem II.2). For convenience we present a direct proof here.

Proof. We prove first that $T+P$ is one-to-one. Assume $x \neq 0$ be such that $(T+P)x = 0$. We may choose $\|x\| = 1$. Then $\|Tx\| \geq \gamma(T)$ whereas $\|Px\| < \gamma(T)$, in contradiction with the assumption $Tx = -Px$.

We now prove that $T+P$ is open. Let $y_n = (T+P)x_n$ be a sequence in $R(T+P)$ converging to $y = (T+P)x$. We have to prove that $x_n \to x$. By considering $y_n - y$ on the one hand and $x_n - x$ on the other, we may assume that $y_n \to 0$. If $x_n \not\to 0$ then there exists a subsequence, rebaptized x_n, such that $\|x_n\| \geq \varepsilon$ for some $\varepsilon > 0$. Consider $h_n = x_n/\|x_n\|$; then $\|h_n\| = 1$ and $(T+P)h_n \leq \|(T+P)x_n\|/\varepsilon$, thus $(T+P)h_n \to 0$. But this leads to a contradiction because $\|(T+P)h_n\| \geq \|Th_n\| - \|Ph_n\| \geq \gamma(T) - \|P\| > 0$ ./.

Before passing on to the next counter-example, we need a lemma.

LEMMA I.6. *Let* E, F *be locally convex spaces and* T *an operator from* E *into* F, *with* $D(T) = E$.

i. *Let* E_1 *be a subspace of* E *and* T_1 *denote the restriction of* T *to* E_1 *with range space* F.

If $N(T) \subset (N(T_1))^-$ *and* T *is open, then* T_1 *is open*.

<u>Proof</u>. Let $U_1 = U \cap E_1$ be a neighborhood in E_1, U being a neighborhood in E. Then

$$\begin{aligned}(U+N(T)) \cap E_1 &\subset (U+(N(T_1))^-) \cap E_1 \\ &\subset (2U+N(T_1)) \cap E_1 \\ &\subset 2U_1+N(T_1).\end{aligned}$$

The second inclusion is due to the fact that $(A)^- \subset A+U$ for any set A and any neighborhood U. The last inclusion is proved as follows. Let $x = x_1+x_2$ with $x_1 \in 2U$, $x_2 \in N(T_1)$ and $x \in E_1$. Then $x_1 = x-x_2 \in E_1$ because $N(T_1) \subset E_1$. Thus $x_1 \in 2U \cap E_1$ and $x \in (2U \cap E_1) + N(T_1)$.

We now prove that $(TU) \cap R(T_1) \subset 2TU_1$. Let $y = T_1x_1$, $x_1 \in E_1$, be such that $y \in TU$, that is $y = Tx$, with $x \in U$. Then $x_1-x \in N(T)$, thus $x_1 \in (U+N(T)) \cap E_1 \subset 2U_1+N(T_1)$ from what is proved in the preceding paragraph. Therefore $y \in 2TU_1$.

Since TU is a neighborhood in $R(T)$, and $R(T_1) \subset R(T)$, it follows that T_1U_1 is also a neighborhood in $R(T_1)$ ./.

COUNTER-EXAMPLE (ii) (Vladimirski). In (25), Vladimirski gave the following counter-example to show that a compact perturbation of a Φ_--operator from a normed space into a Banach space need not give rise to a Φ_--operator.

Let X be an infinite dimensional Banach space and K a compact, injective operator from X into itself. We may take for instance $X = l_2$ and K the operator defined by $K(e_i) = e_i/i$, $i=1,2,\ldots$, where e_i is a base element $e_i = (0,0,\ldots,1,0,\ldots)$ with 1 at the i^{th} position.

Let $F = X$ and $Z = X \oplus F$, the Banach space constructed as the sum of two copies of X. Consider K now as an operator from X into F. Let Q_X (resp. Q_F) be the canonical projection from Z onto X (resp. F). Let $P = KQ_X$. It is clear that P is a compact operator and Q_F a ϕ_--operator (both from Z into F).

We have $N(Q_F) = X$ and $N(P) = F$, thus $N(Q_F+P) \cap X = \{0\}$ because $N(Q_F+P) \cap X = N(Q_F+P) \cap N(Q_F) = N(P) \cap N(Q_F) = X \cap F = \{0\}$.

Let L be an algebraic complement of $N(Q_F+P) + X$ in Z, that is $N(Q_F+P)+X+L = Z$ and $(N(Q_F+P)+X) \cap L = \{0\}$ (the three subspaces $N(Q_F+P)$, X and L form an algebraic decomposition of Z).

Let H be a dense subspace of codimension 1 in X (H is the null-space of a non continuous functional on X).

Let $E = H + N(Q_F+P) + L$. Then E is a subspace of codimension 1 in Z. Moreover E is not closed in Z, because $E \cap X = H$ and H is not closed in X.

Let T (resp. P') be the restriction of Q_F (resp. P) to E, with range space F. Obviously P' is a compact operator. We prove now that T is a ϕ_--operator. It is clear that G(T) is closed in ExF, since it is the restriction of $G(Q_F)$. Moreover, $R(T) = F$ because $TE = Q_F(E+N(Q_F)) = Q_F(E+X) = Q_F(Z) = F$.

Finally, T is open in view of Lemma I.6, and of the fact that $N(T) = H$ is dense in $N(Q_F) = X$.

Now $R(T+P')$ is not closed in F, because otherwise $(Q_F+P)^{-1}R(T+P')$ would be closed in Z (Q_F+P being continuous). But $(Q_F+P)^{-1}R(T+P') = E+N(Q_F+P) = E$ since $N(Q_F+P) \subset E$ by construction. The counter-example is established.

COUNTER-EXAMPLE (iii). We notice that the preceding counter-example is still valid for λK in place of K, for any $\lambda > 0$. The operator P' is bounded (being compact), thus $\|\lambda P'\| < \gamma(T)$ for λ small enough. This proves that a small perturbation of a ϕ_--operator from a normed space into a Banach space need not yield a ϕ_--operator.

The only case we have left out up to now is compact perturbations of ϕ_+-operators. It turns out that these perturbations are "good", without assumption on E, as shown by a theorem of Schwartz, Köthe and Schaefer below.

I.2.3. The theorems of Schwartz, Köthe and Schaefer

When E, F are locally convex spaces, the case of compact perturbations is settled by the following theorems due to Schwartz (23), Köthe (12) and Schaefer (22).

THEOREM I.7 (Schwartz, Köthe, Schaefer). Let E, F be two locally convex spaces, T a ϕ_+-operator and P a compact operator from E into F such that $D(T) \subset D(P)$. Then $T+P$ is a ϕ_+-operator and $\mathrm{ind}(T+P) = \mathrm{ind}(T)$.

REMARK. We may no longer have $\mathrm{nul}(T+P) \leq \mathrm{nul}(T)$. Let T be a one-to-one ϕ_+-operator. Let $x_0 \neq 0$ be a point in E, and L a closed subspace, topological and algebraic complement of $>\{x_0\}<$, i.e. such that $L + >\{x_0\}< = E$ and $x_0 \notin L$. Let P be defined by $Px_0 = -Tx_0$ and $Px = 0$ if $x \in L$ (P is a continuous operator of rank 1 (i.e. $R(P) = >\{Px_0\}<$). Then $\mathrm{nul}(T+P) = 1$, whereas $\mathrm{nul}(T) = 0$.

As a consequence of the equality $\mathrm{ind}(T+P) = \mathrm{ind}(T)$, we may also have $\mathrm{def}(T+P) > \mathrm{def}(T)$, if we choose T with $\mathrm{def}(T) < \infty$.

THEOREM I.8 (Schwartz, Kőthe, Schaefer). Let E, F be two Frechet spaces, T a ϕ_--operator and P a compact operator from E into F such that $D(T) \subset D(P)$. Then $T+P$ is a ϕ_--operator and $\mathrm{ind}(T+P)=\mathrm{ind}(T)$.

Schwartz and Kőthe proved Theorem I.8 by using Theorem I.7 in the duals F^+ and E^+ equipped with the compact convergence topology (i.e. topology generated by the polars of the compact disks). Vladimirski (25) used a technique of "extraction and completion" to construct Frechet spaces out of general locally convex spaces and then applied Theorem I.8 to obtain a generalization of it. He proved that Theorem I.8 holds for E a fully complete space (in particular E a Frechet space), F any locally convex space and P a precompact operator. Counter-example (ii) shows that the completeness of E is essential.

We will prove in Chapter II the theorems of Schwartz, Kőthe and Schaefer and the result of Vladimirski by a different approach, which gives relatively short and convenient proofs (Theorems II.22 and II.24). We use duality, but unlike Schwartz and Kőthe, we do not seek locally convex topologies on the whole of the spaces F^+ and E^+ for which P^+ may be a convenient perturbation of T^+. Instead, we consider convenient subspaces of F^+ and E^+ (which are essentially Banach spaces generated by closed equicontinuous sets), such that restricted to them, P^+ is a "good" perturbation of T^+. This approach is partly motivated by the following (cf. Theorem I.9) : a perturbation leaves the index invariant if it does when the operators are restricted to suitable subspaces.

The preceding remark gives us a means of using results proved for Banach spaces in a more general setting of locally convex spaces. In particular, we will be able to consider "small" perturbations when the spaces are not normed, in the sense that they be small when restricted to suitably normed subspaces. Another type of "small" perturbations (small bounded operators) will be studied in Chapter II.

I.3. Some basic results

I.3.1. Stability of the index

The following algebraic result forms the basis of our approach.

THEOREM I.9. *Let* E, F *be vector spaces and* T, P *operators from* E *into* F *such that* $D(T) \subset D(P)$.

A. (H_1) *Assume that*

(a) T *has an index,*

(b) *Given any finite dimensional subspace* $N \subset N(T)$, *there are subspaces* $E' \subset E$, $F' \subset F$ *such that*

$$TE' = F' \cap R(T), \quad N \subset E', \quad R(P) \subset F'$$

and $\mathrm{nul}(T'+P') \leq \mathrm{nul}(T')$, $\mathrm{def}(T'+P') \leq \mathrm{def}(T')$, $\mathrm{ind}(T'+P') = \mathrm{ind}(T')$, T', P' *being the restrictions of* T, P *to* E' *with range space* F'.

Then $\mathrm{nul}(T+P) \leq \mathrm{nul}(T)$, $\mathrm{def}(T+P) \leq \mathrm{def}(T)$ *and* $\mathrm{ind}(T+P) = \mathrm{ind}(T)$.

B. (H_2) *Assume that*

(a) T *has an index*,

(b) *There exist subspaces* $E' \subset E$, $F' \subset F$ *such that*

$$TE' = F' \cap R(T), \quad R(P) \subset F'$$

and $\mathrm{ind}(T'+P') = \mathrm{ind}(T')$, T', P' *being the restrictions of* T, P *to* E' *with range space* F'.

Then $\mathrm{ind}(T+P) = \mathrm{ind}(T)$.

We need a lemma :

LEMMA I.10. *Let* E, F *be vector spaces*, T, T' *two operators from* E *into* F *such that* $D(T') \subset D(T)$, $\dim (D(T)/D(T')) = n < \infty$ *and* $T = T'$ *on* $D(T')$ (*we henceforth refer to* T *as a finite dimensional extension of* T'). *Assume that* T' *has an index*.

Then $\mathrm{ind}(T) = \mathrm{ind}(T')+n$.

Proof. It suffices clearly to prove this for $n = 1$. If there is $x \neq 0$, $x \in D(T)$ and $x \notin D(T')$ such that $Tx = 0$, then $\mathrm{nul}(T) = \mathrm{nul}(T')+1$ and $R(T) = R(T')$, thus $\mathrm{def}(T) = \mathrm{def}(T')$. If there is no such x, then $\mathrm{nul}(T) = \mathrm{nul}(T')$ and $R(T')$ is a proper subspace of $R(T)$. Indeed, fix $x \neq 0$, $x \in D(T)$, $x \notin D(T')$; if $R(T) = R(T')$ then $Tx = T'x'$ for some $x' \in D(T')$. Then $T(x-x') = 0$, and $x-x' \notin D(T')$, contrary to our assumption. Thus $\mathrm{def}(T) = \mathrm{def}(T')-1$. In both cases, $\mathrm{ind}(T) = \mathrm{ind}(T')+1$ ./.

For later references, we also prove the following.

LEMMA I.11. *In the setting of Lemma* I.10, *if we furthermore assume that* E, F *are locally convex spaces, then the following holds* :

(a) *If* T' *has a closed graph, then* T *has a closed graph*.

(b) *If* $R(T')$ *is closed then* $R(T)$ *is closed*.

(c) *If* T' *is open and* $R(T')$ *is closed, then* T *is open* (*and* $R(T)$ *is closed*).

(d) *If* T' *is almost open then* T *is almost open*.

(e) *If* L *is a subspace in* E *such that* $D(T) \subset L$ *and* $(D(T'))^- \subset L$, *then* $(D(T))^- \subset L$.

Proof. (a) & (b) follow from the fact that the sum of a closed subspace and a finite dimensional subspace is closed : we have $\dim (G(T)/G(T')) < \infty$, $\dim (R(T)/R(T')) < \infty$. Similarly, $(D(T'))^-$ has a finite codimension in $(D(T'))^- + D(T)$, and the latter is consequently closed. For (e), remark that thus $(D(T))^- \subset (D(T'))^- + D(T) \subset L$. For (c), we may assume without loss of generality that $D(T) = E$. There exist finite dimensional subspaces $M \subset N(T)$ and $N \subset E$ such that, algebraically, $D(T') \oplus M = D(T') + N(T)$ and $D(T') \oplus M \oplus N = E$. Then $R(T) = R(T') \oplus TN$. Since $R(T')$ is closed and $\dim TN < \infty$, the decomposition $R(T) = R(T') \oplus TN$ is also topological.

Let U be a neighborhood in E. There are neighborhoods U', D_1, D_2 in $D(T')$, M, N respectively such that $U' + D_1 + D_2 \subset U$. Now $T(U' + D_1 + D_2) = T'U' + TD_2$ where $T'U'$ is a neighborhood in $R(T')$ (T' is open) and TD_2 is a finite disk generating TN. Thus $T'U' + TD_2$, and a fortiori TU, is a neighborhood in $R(T)$. For (d), keeping the notations of the proof of (c), we remark that $(R(T))^- = (R(T'))^- + TN$ as $(R(T'))^- + TN \subset (R(T))^-$ and $(R(T'))^- + TN$ is closed. On the other hand,

$$(T(U'+D_1+D_2))^- = (T'U'+TD_2)^- = (T'U')^- + (TD_2)^-$$

as $(TD_2)^-$ is compact. Since $(T'U')^-$ is a neighborhood in $(R(T'))^-$, it is easily seen that $(T'U')^- + (TD_2)^-$, and consequently $(TU)^-$ are neighborhoods in $(R(T))^-$ ./.

<u>Proof of Theorem</u> I.9. The proof is elementary but somewhat long. We break it down to several smaller steps.

(a) We first prove that under the assumption (H_1) or (H_2) we have $R(T) + F' = R(T+P) + F'$. Indeed,

$$\begin{aligned} R(T) + F' &= R(T+P-P) + F' \\ &\subset R(T+P) + R(P) + F' \\ &\subset R(T+P) + F' \end{aligned},$$

and
$$R(T+P) + F' \subset R(T) + R(P) + F' \subset R(T) + F',$$

because $R(P) \subset F'$.

Let $F'' = F' + R(T) = F' + R(T+P)$. It follows that

$$\operatorname{codim}_{F''} R(T) = \operatorname{codim}_{F'} R(T) \cap F' ,$$

$$\operatorname{codim}_{F''} R(T+P) = \operatorname{codim}_{F'} R(T+P) \cap F' .$$

For if $N \subset F'$ is such that $N \oplus (R(T) \cap F') = F'$, then $N \oplus R(T) = F''$, and the first relation follows. The second is obtained similarly.

We shall make use of the above in the following form, which gives the connection between T, P and T', P' :

(1) $$\operatorname{def}(T) = \operatorname{def}(T') + \operatorname{codim}_F F'' ,$$

(2) $$\operatorname{def}(T+P) = \operatorname{def}(T'+P') + \operatorname{codim}_F F'',$$

if $R(T) \cap F' = R(T')$ and $R(T+P) \cap F' = R(T'+P')$ in F'. The former is satisfied by assumption; the latter shall be proved to hold for suitable E', F' (cf. relation (4) in (c)).

(b) Let N be any finite dimensional subspace of $N(T)$. We may assume that $N \subset E'$, where E' satisfies (H_1) or (H_2) : in case (H_1), this is included in the assumptions; in case (H_2), let $E'' = E'+N$, where E' is given in (H_2); we still have $F' \cap R(T) = TE''$; let T'', P'' be the restrictions of T, P of E'' with range space F', then in view of Lemma I.10 and the assumption (H_2), we have $\mathrm{ind}(T''+P'') = \mathrm{ind}(T'')$; it suffices now to rebaptize E'' as E'.

(c) We now prove A and B in the case $\mathrm{nul}(T) < \infty$. In view of (b), we may choose E', F' satisfying (H_1) or (H_2) and such that $N(T) \subset E'$.

The following relations hold, and are central to our proof :

(3) $$N(T+P) = N(T'+P') \quad ,$$

(4) $$R(T+P) \cap F' = R(T'+P') \quad .$$

For the relation (3), let $x \in N(T+P)$. Then $Tx = -Px$ and consequently $Tx \in F' \cap R(T)$. Since $F' \cap R(T) = R(T')$ by assumption, there is $x' \in E'$ such that $x-x' \in N(T)$. As $E'+N(T) = E'$, we infer that $x \in E'$, that is $N(T+P) \subset E'$. Thus $N(T'+P') = N(T+P)$.

For the relation (4), let $y \in R(T+P) \cap F'$. Then $y = (T+P)x$ and $Tx = y-Px$. It follows that $Tx \in R(T) \cap F'$ and, as above, $x \in E'$, that is $y \in R(T'+P')$.

We now have, in addition to the relations (1) and (2), the following :

(5) $$\mathrm{nul}(T) = \mathrm{nul}(T') \quad ,$$

(6) $$\mathrm{nul}(T+P) = \mathrm{nul}(T'+P') \quad ,$$

and

(7) $$\operatorname{ind}(T'+P') = \operatorname{ind}(T') \quad .$$

In case A, we have furthermore

(8) $$\operatorname{nul}(T'+P') \leq \operatorname{nul}(T'), \ \operatorname{def}(T'+P') \leq \operatorname{def}(T').$$

Whether $\operatorname{def}(T')$ and $\operatorname{codim}_F F''$ are finite or infinite (we do not distinguish between different cardinalities of infinity), the relations (1), (2), (5), (6), (7), (8) together show that $\operatorname{ind}(T+P) = \operatorname{ind}(T)$ and, in case A,

$$\operatorname{nul}(T+P) \leq \operatorname{nul}(T), \ \operatorname{def}(T+P) \leq \operatorname{def}(T).$$

(d) We now prove A and B in the case $\operatorname{nul}(T) = \infty$ and $\operatorname{def}(T) < \infty$.

Let N again be any finite dimensional subspace of N(T), E', F' satisfy (H_1) or (H_2) with $N \subset E'$ (cf. (b)) and T', P' be the restrictions of T, P to E' with range space F'.

We have immediately

$$\dim N \leq \operatorname{nul}(T'), \ \operatorname{nul}(T'+P') \leq \operatorname{nul}(T+P).$$

Now the relation (1) still holds, with $\operatorname{codim}_F F'' < \infty$, but the relation (2) may fail to be true. In its place, we have

(2') $$\operatorname{def}(T+P) \leq \operatorname{def}(T'+P') + \operatorname{codim}_F F''$$

because $R(T'+P') \subset R(T+P) \cap F'$ trivially.

In case A, we have from the relations (2') and (1)

$$\begin{aligned}\operatorname{def}(T+P) &\leq \operatorname{def}(T') + \operatorname{codim}_F F'' \\ &\leq \operatorname{def}(T).\end{aligned}$$

In case B, we simply notice that $\operatorname{def}(T'+P') < \infty$ because $\operatorname{ind}(T'+P') = \operatorname{ind}(T')$ and $\operatorname{def}(T') < \infty$. In view of the relation (2') it follows that $\operatorname{def}(T+P) < \infty$.

We now prove that in both cases A and B, we have $\mathrm{nul}(T+P) = \infty$, and the proof will be complete.

This is indeed the case because

$$\begin{aligned}\mathrm{nul}(T+P) &\geqslant \mathrm{nul}(T'+P') \\ &\geqslant \mathrm{ind}(T'+P') \\ &\geqslant \mathrm{ind}(T') \\ &\geqslant \mathrm{nul}(T') - \mathrm{def}(T') \\ &\geqslant \dim N - \mathrm{def}(T) + \mathrm{codim}_F F'' \\ &\geqslant \dim N - \mathrm{def}(T).\end{aligned}$$

Since $\dim N$ may be arbitrarily large, we infer that $\mathrm{nul}(T+P) = \infty$ ./.

IMPORTANT REMARK. A closer examination of the proof shows that in Theorem I.9, the assumption $R(P) \subset F'$ can be replaced by $P(D(T)) \subset F'$. Indeed, since $D(T+P) = D(T)$, all that we use in the proof, as far as P is concerned, is the restriction of P to $D(T)$.

This remark applies implicitly in all that follows henceforth.

Stronger conclusions may be drawn from Theorem I.9 if furthermore $R(P) \subset R(T)$, by taking $F = R(T)$ (cf. Theorems I.17, I.18, I.20).

In this line of thought, we have the following.

THEOREM I.12. *Let* E, F *be vector spaces and* T, P *operators from* E *into* F *such that* $D(T) \subset D(P)$.

(H_3) *Assume that*

(a) $R(P) \subset R(T)$,

(b) <u>Given any finite dimensional subspace</u> $N \subset D(T)$, <u>there is a subspace</u> $E' \subset E$ <u>such that</u> $N \subset E'$, $PE' \subset TE'$ <u>and</u>

$$\mathrm{nul}(T'+P') = \mathrm{nul}(T'),\ R(T'+P') = R(T')$$

<u>if</u> T', P' <u>denote the restrictions of</u> T, P <u>to</u> E'.

<u>Then</u> $\mathrm{nul}(T+P) = \mathrm{nul}(T)$ <u>and</u> $R(T+P) = R(T)$.

<u>Proof</u>. It follows from the assumption $R(P) \subset R(T)$ that $R(T+P) \subset R(T)+R(P) \subset R(T)$. We may thus assume that $F = R(T)$, that is $\mathrm{def}(T) = 0$.

We now prove that $R(T+P) = R(T)$ $(= F)$, by showing that if L is any finite dimensional subspace of F such that $L \cap R(T+P) = \{0\}$, then $L = \{0\}$.

There exists a finite dimensional subspace $N \subset D(T)$ such that $TN = L$. Let E' satisfy (H_3) with $N \subset E'$. Since $L \subset TE'$ and, by assumption, $(T+P)E' = TE'$, it follows that $L = \{0\}$.

We now prove that $\mathrm{nul}(T+P) = \mathrm{nul}(T)$.

Assume that $\mathrm{nul}(T) < \infty$. Consider any finite dimensional subspace $N \subset N(T+P)$. Let E' satisfy (H_3) with $N+N(T) \subset E'$, and T', P' be the restrictions of T, P to E'. We have $N \subset N(T'+P')$ thus $\dim N \leq \mathrm{nul}(T'+P')$. But $\mathrm{nul}(T'+P') = \mathrm{nul}(T')$ and $\mathrm{nul}(T') = \mathrm{nul}(T)$ because $N(T) \subset E'$ (whence $N(T) = N(T')$). Thus $\dim N \leq \mathrm{nul}(T)$. Since $N \subset N(T+P)$ is arbitrary, this shows that $\mathrm{nul}(T+P) \leq \mathrm{nul}(T) < \infty$. By replacing N by $N(T+P)$ in the above, we obtain $\mathrm{nul}(T+P) = \mathrm{nul}(T)$ (because now $N(T'+P') = N(T+P)$).

Assume now that $\mathrm{nul}(T) = \infty$. Take any finite dimensional subspace $N \subset N(T)$ and let E' satisfy (H_3) with $N \subset E'$. Then

$$\begin{aligned}\mathrm{nul}(T+P) &\geq \dim N(T+P)\cap E' \\ &\geq \dim N(T)\cap E' \\ &\geq \dim N.\end{aligned}$$

Since dim N is arbitrary, it follows that $\mathrm{nul}(T+P) = \infty$ ./.

I.3.2. Stability of the topological characteristics

Theorems I.9 and I.12 are purely algebraic in nature. We now prove some sufficient conditions for the perturbed operator to be a ϕ_+ or ϕ_--operator when its restriction to some suitable subspaces is one.

PROPOSITION I.13. *Let* E, F *be locally convex spaces and* T, P *operators from* E *into* F *such that* $D(T)\subset D(P)$.

Assume that

(a) T *is open,* R(T) *is closed and* $\mathrm{def}(T) < \infty$,

(b) *There exist two subspaces* $E'\subset E$ *equipped with a locally convex topology* t_1 *finer than that induced by* E, *and* $F'\subset F$ *equipped with a (not necessarily Hausdorff a priori) locally convex topology* t_2 *with the following conditions* :

- P *maps some neighborhood* U_0 *in* E *into a bounded disk in* (F', t_2),

- $TE'\subset F'$,

- *The restriction* $T'+P'$ *of* T+P *is open from* (E', t_1) *into* (F', t_2), $R(T'+P')$ *is closed in* (F', t_2) *and* $\mathrm{codim}_{F'} R(T'+P') < \infty$.

Then T+P *is open.*

Proof. We may assume without loss of generality that $F = R(T)+R(T+P)$.

Let $U \subset U_0$ be any neighborhood in E; we will prove that $(T+P)U$ is a neighborhood in $R(T+P)$.

Since T is open, there is a neighborhood $V \subset F$ such that $V \cap R(T) \subset TU$.

Let $L \subset R(T+P)$ be an algebraic and topological complement of $R(T)$: $R(T) \oplus L = F$. There are a finite disk D generating L and a neighborhood V' in F such that

$$V' \subset V \cap R(T+P) + D \subset TU + D$$
$$\subset (T+P)U + PU + D.$$

Let D' be a finite disk in E such that $D = (T+P)D'$. We have $V' \cap R(T+P) \subset (T+P)U + (T+P)D' + (PU) \cap R(T+P)$.

Now $PU \cap R(T+P)$ is a bounded disk in (F', t_2). Moreover $R(T'+P')$ is closed and has a finite codimension in (F', t_2), and $R(T'+P') \subset R(T+P) \cap F'$. By considering a finite dimensional subspace, complement of $R(T'+P')$ in $R(T+P) \cap F'$, we see that there exist a finite disk $K \subset R(T+P)$ and a bounded disk B in $(R(T',P'), t_2)$ such that $(PU) \cap R(T+P) \subset K+B$. Since $K = (T+P)K'$, K' a finite disk in E, and $(T'+P')U$ is a neighborhood in $(R(T',P'), t_2)$ ($T'+P'$ being open from (E', t_1) into (F', t_2)), there exists $\lambda > 0$ such that

$$V' \cap R(T+P) \subset (T+P)U + (T+P)D' + (T+P)K' + \lambda(T'+P')U.$$

As U absorbs D' and K', there is $\mu > 0$ such that

$$\mu V' \cap R(T+P) \subset (T+P)U \text{ ./.}$$

We will use this proposition in the following form :

COROLLARY I.14. _Let_ E, F _be locally convex spaces, and_ T, P _operators from_ E _into_ F _such that_ $D(T) \subset D(P)$.

Assume that

(a) T is open, $R(T)$ is closed and $\mathrm{def}(T) < \infty$,

(b) There are a bounded disk $B \subset E$, a disk $B' \subset F$ and a neighborhood U_0 in E such that

- $PU_0 \subset B'$, $TE_B \subset F_{B'}$,

- The restriction $T'+P'$ of $T+P$ is open from E_B into $F_{B'}$, $R(T'+P')$ is closed in $F_{B'}$, and $\mathrm{codim}_{F'}\ R(T'+P') < \infty$.

Then $T+P$ is open.

If furthermore T has a closed graph (i.e. T is now a ϕ_--operator), P is continuous, $(D(T))^- \subset D(P)$ and $E/N(T+P)$ is complete (which is the case if E is fully complete) then $R(T+P)$ is closed and $T+P$ is a ϕ_--operator.

Proof. The first part follows directly from Proposition I.13. The last part follows from Theorem 0.15. The assumption $(D(T))^- \subset D(P)$ ensures that $G(T+P)$ is closed in $E \times F$ ./.

Another simple sufficient condition for $R(T+P)$ to be closed is given by the following

PROPOSITION I.15. Let E, F be locally convex spaces, and T, P operators from E into F such that $D(T) \subset D(P)$.

Assume that T is a ϕ_+-operator, $T+P$ is open, $\mathrm{nul}(T+P) < \infty$ and there exist a neighborhood U in E and a bounded Banach disk $B \subset F$ such that $PU \subset B$.

Then $R(T+P)$ is closed and $T+P$ is a ϕ_+-operator.

Proof. Since B is a Banach space, we remark that P has a continuous extension from $(D(P))^-$ into F_B. Thus we may assume that $(D(T))^{-} \subset D(P)$, and consequently $G(T+P)$ is closed.

As both nul(T) and nul(T+P) are finite, there exists a closed subspace L such that (algebraically and topologically) $E = L \oplus (N(T)+N(T+P))$. It suffices now to prove that $(T+P)L$ is closed because $R(T+P) = (T+P)L + (T+P)N(T)$ and $(T+P)N(T)$ is finite dimensional. For simplicity, we will keep the notations T, P to denote the one-to-one restrictions to L.

Let $y \in ((T+P)L)^{-}$ and $\mathcal{F}$ be a filter in $(T+P)L$ converging to y. Then $(T+P)^{-1}\mathcal{F} = \mathcal{G}$ is a Cauchy filter base in L. Now P is continuous from L into F_B, $P\mathcal{G}$ is a Cauchy filter base in F_B, thus converging to an element $z \in F$. This implies that the filter base $T\mathcal{G} = (T+P-P)\mathcal{G}$ converges to $y-z \in R(T)$ (R(T) is closed). The operator T^{-1} being continuous from R(T) into L, it follows that $\mathcal{G} = T^{-1}T\mathcal{G}$ converges to $x = T^{-1}(y-z) \in L$. Due to the closed graph of T+P, we have $(T+P)x = y$ and $y \in (T+P)L$ ./.

Proposition I.15 shall be used in Chapter II (Section II.1.2) in connection with perturbations by small bounded operators. Now we are considering small perturbations in the sense that, restricted to suitable subspaces equipped with a norm, the perturbations are small.

I.4. Small perturbations

Kato's theorem on small perturbations makes use of Lemma I.3, which requires the setting of normed spaces. It seems that to obtain a version of Kato's theorem in more general locally convex spaces, it may be appropriate to assume the existence of some suitably large norming disks in the spaces, which satisfy Kato's conditions.

In this section, we will examine a series of situations centering upon the above idea and to which the general results of the preceding section apply.

The stability of the index is carried through nicely. However, when we turn to the stability of the topological characteristics ("semi-Fredholmness"), difficulties arise. Generalizations of Kato's theorem for Φ_--operators are obtained here, along with some related results. We will deal with the topological aspect of small perturbations of Φ_+-operators in Chapter II.

I.4.1. Stability of the index

THEOREM I.16. *Let* E, F *be vector spaces, and* T, P *operators from* E *into* F *such that* $D(T) \subset D(P)$.

Assume that

(a) T *has an index*,

(b) *There exist a Banach disk* $B \subset E$ *and a norming disk* $B' \subset F$ *such that*

- $R(P) \subset F_{B'}$, $TE_B \subset F_{B'}$,

- *if* T', P' *denote the restrictions of* T, P *to* E_B *with range space* $F_{B'}$, *then* $(D(T'))^{-} \subset D(P')$, *the closure being taken in* E_B,

- T' *has a closed graph in* $E_B \times F_{B'}$,

(c) $B' \cap R(T) \subset TB$ *and* $PB \subset \varepsilon B'$ *for some* $0 < \varepsilon < 1$.

Then $\operatorname{nul}(T+P) \leq \operatorname{nul}(T)$, $\operatorname{def}(T+P) \leq \operatorname{def}(T)$ *and*

$$\operatorname{ind}(T+P) = \operatorname{ind}(T).$$

If $\operatorname{nul}(T) < \infty$, *then there exists* $\rho > 0$ *such that for all* λ *in the annulus* $0 < |\lambda| < \rho$, $\operatorname{nul}(T+\lambda P)$ *is a constant (not exceeding* $\operatorname{nul}(T)$).

Assumption (b) *may be replaced by*

(b') *There exist Banach disks* $B \subset E$ *and* $B' \subset F$ *such that* $R(P) \subset F_{B'}$ *and* $G(T) \cap (E_B \times F_{B'})$ *is closed in* $E_B \times F_{B'}$.

Proof. Let N be a finite dimensional subspace of $N(T)$, generated by a finite disk D.

Fix any ε' such that $\varepsilon < \varepsilon' < 1$. For $\lambda > 0$ small enough, $\lambda PD \subset (\varepsilon' - \varepsilon)B'$ because $R(P) \subset F_{B'}$. Write $D' = \lambda D$; then

$$PB + PD' = P(B+D') \subset \varepsilon' B'.$$

On the other hand, $TE_{B+D'} \subset F_{B'}$ because $D' \subset N \subset N(T)$, and

$$B' \cap R(T) \subset TB \subset T(B+D').$$

Moreover $B+D'$ is a Banach disk. Indeed, let $N = N_1 \oplus N_2$, where $N_1 \subset E_B$ and $N_2 \oplus E_B = E_{B+D'}$ ($\dim N_1 < \infty$, $\dim N_2 < \infty$). We may choose $D' = D_1 + D_2$, with D_1 and D_2 generating N_1 and N_2 respectively; then $B+D' = (B+D_1)+D_2$. In E_B, $B+D_1$ generates the same topology as B because D_1 is absorbed by B. It is now clear that $E_{B+D'} = E_{B+D_1} \oplus E_{D_2}$ is a Banach space (cf. also Lemma 0.2).

Also, by virtue of Lemma I.11, the restriction of T to $E_{B+D'}$ has a closed graph in $E_{B+D'} \times F_{B'}$.

Theorems I.4 (A) and I.9 A give the conclusions. If B' is completing, we may apply directly Kato's theorem (Theorem I.2) instead of Theorem I.4.

It may happen that we have Banach disks B, B' such that $R(P) \subset F_{B'}$, $B' \cap R(T) \subset TB$ and $PB \subset \varepsilon B'$, $0 < \varepsilon < 1$, but that a priori $TE_B \not\subset F_{B'}$. If $G(T) \cap (E_B \times F_{B'})$ is closed in $E_B \times F_{B'}$, then $B \cap T^{-1}B'$ is a Banach disk by virtue of Lemma 0.3. We may then apply Theorem I.16 with $B \cap T^{-1}B'$ in place of B, dropping the assumption $(D(T'))^{-} \subset D(P')$.

That $\mathrm{nul}(T+\lambda P)$ is constant for $|\lambda| \neq 0$ and small enough, if $\mathrm{nul}(T) < \infty$, is proved as follows. With reference to the first part of the proof, let $N = N(T)$ and T', P' denote the restrictions of T, P to $E_{B+D'}$, with range space $F_{B'}$. Then by Theorem I.4 (C), $\mathrm{nul}(T'+\lambda P')$ is constant for $|\lambda| \neq 0$ and small enough. But $\mathrm{nul}(T'+\lambda P') = \mathrm{nul}(T+\lambda P)$. Indeed, if $x \in N(T+\lambda P)$, then $Tx = -\lambda Px$ belongs to $R(T) \cap F_{B'}$, as $R(P) \subset F_{B'}$. From $B' \cap R(T) \subset T(B+D')$ it follows that $R(T) \cap F_{B'} = R(T')$. Therefore $x \in E_{B+D'} + N(T)$. Since $N(T) \subset E_{B+D'}$, it follows that $x \in E_{B+D'}$, thus $x \in N(T'+\lambda P')$; hence $N(T'+\lambda P') = N(T+\lambda P)$ ./.

REMARK. In view of the important remark following Theorem I.9, we may replace in Theorem I.16 the assumption $R(P) \subset F_{B'}$ by $P(D(T)) \subset F_{B'}$.

If furthermore $R(P) \subset R(T)$, we have the following :

THEOREM I.17. <u>Let</u> E, F <u>be vector spaces, and</u> T, P <u>operators from</u> E <u>into</u> F <u>such that</u> $D(T) \subset D(P)$.

<u>Assume that there exist a Banach disk</u> $B \subset E$ <u>and a norming disk</u> $B' \subset F$ <u>such that</u>

(a) $R(P) \subset R(T) \cap F_{B'}$,

(b) $TE_B \subset F_{B'}$,

(c) *if* T', P' *denote the restrictions of* T, P *to* E_B *with range space* $F_{B'}$, *then* $(D(T'))^{-} \subset D(P')$, *the closure being taken in* E_B,

(d) T' *has a closed graph in* $E_B \times F_{B'}$

(e) $B' \cap R(T) \subset TB$ *and* $PB \subset \varepsilon B'$ *for some* $0 < \varepsilon < 1$.

Then $\mathrm{nul}(T+P) = \mathrm{nul}(T)$ *and* $R(T+P) = R(T)$.

If B' *is completing, we may replace assumptions* (b), (c) *and* (d) *by* "$G(T) \cap (E_B \times F_{B'})$ *is closed in* $E_B \times F_{B'}$".

Proof. Since $R(T+P) \subset R(T)+R(P) \subset R(T)$, we may assume without loss of generality that $F = R(T)$, thus $\mathrm{def}(T) = 0$. Apply now Theorem I.16; $\mathrm{def}(T+P) \leq \mathrm{def}(T)$ implies that $\mathrm{def}(T+P) = 0$ that is $R(T+P) = R(T)$. Now $\mathrm{ind}(T+P) = \mathrm{ind}(T)$ implies that $\mathrm{nul}(T+P) = \mathrm{nul}(T)$. We need only justify that we may take $B' \cap R(T)$ instead of B'. In case B' is a norming disk, no difficulty arises. In case B' is completing, we may assume that $TE_B \subset F_{B'}$ by taking $B \cap T^{-1}B'$ and we remark that $R(T')$ is closed in $F_{B'}$ because T' has a closed graph and is open (and E_B is a Banach space); therefore $B' \cap R(T) = B' \cap R(T')$ is also completing ./.

REMARK. The assumption $(D(T'))^{-} \subset D(P')$ is realized for instance if P is everywhere defined, i.e. $D(P) = E$, or more generally if $B \subset D(P)$.

As a corollary of Theorem I.17, we have the following

THEOREM I.18. *Let* E, F *be vector spaces, and* T, P *operators from* E *into* F *such that* $D(T) \subset D(P)$.

<u>Assume that there exists a Banach disk</u> $B \subset D(P)$ <u>and</u> $0 < \varepsilon < 1$ <u>such that</u>

- $N(T) \cap E_B$ <u>is closed in</u> E_B,
- $R(P) \subset F_{TB}$,
- $PB \subset \varepsilon TB$.

<u>Then</u> $nul(T+P) = nul(T)$, $R(T+P) = R(T)$.

<u>Proof</u>. We prove that the assumptions of Theorem I.17 are satisfied for $B' = TB$. It suffices to show that TB is a norming disk and the restriction of G(T) is closed in $E_B x F_{TB}$.

That TB is a norming disk follows from Lemma 0.3. Now let x, $x_m \in D(T) \cap E_B$, $x_m \to x$ in E_B and $Tx_m \to y$ in F_{TB}. Since $y \in {>}TB{<}$, there is $x' \in D(T) \cap E_B$ such that $Tx' = y$. As $Tx_m - y \to 0$ in F_{TB}, there is $u_m \in D(T) \cap E_B$ such that $Tu_m = Tx_m - y$ and $u_m \to 0$ in E_B. We have then $x_m - u_m - x' \in N(T) \cap E_B$ and $x_m - u_m - x' \to x - x'$ in E_B. By assumption $N(T) \cap E_B$ is closed in E_B, thus $x - x' \in N(T)$. Hence $x \in D(T)$ and $Tx = y$ ./.

I.4.2. <u>Stability of the topological characteristics</u>

The following generalizes Kato's theorem on small perturbations of ϕ_--operators.

THEOREM I.19. <u>Let</u> E, F <u>be locally convex spaces, and</u> T, P <u>operators from</u> E <u>into</u> F <u>such that</u> $D(T) \subset D(P)$.

<u>Assume that</u>

(a) T <u>is a</u> ϕ_-<u>-operator</u>,

(b) <u>There exist a neighborhood</u> U <u>in</u> E, <u>bounded Banach disks</u> $B \subset E$, $B' \subset F$ <u>and</u> $0 < \varepsilon < 1$ <u>such that</u>

$$B' \cap R(T) \subset TB, \quad PB \subset \varepsilon B', \quad PU \subset B'.$$

Then $T+P$ is open and has a closed graph, $\mathrm{nul}(T+P) \leq \mathrm{nul}(T)$, $\mathrm{def}(T+P) \leq \mathrm{def}(T)$ and $\mathrm{ind}(T+P) = \mathrm{ind}(T)$.

The range $R(T+P)$ is closed (and thus $T+P$ is a ϕ_--operator) under either of the following additional assumptions :

(c) $E/N(T+P)$ is complete, which is realized if E is a Frechet space, or more generally, fully complete,

(c') $\mathrm{nul}(T) < \infty$, in which case both T and $T+P$ are Fredholm-operators.

Proof. Since P is continuous from E into $F_{B'}$, and $F_{B'}$ is a Banach space, we may extend P by continuity and therefore assume that $(D(T))^- \subset D(P)$. It follows that $G(T+P)$ is closed in $E\times F$. On the other hand, $G(T) \cap (E_B \times F_{B'})$ is closed in $E_B \times F_{B'}$, because the topology of E_B (resp. $F_{B'}$) is finer than that induced by E (resp. F). All the assertions follow now directly from Theorem I.16, Corollary I.14 and Proposition I.15. To apply Corollary I.14, we need only notice that if we replace B by $B \cap T^{-1}B'$, then T', P' satisfy Kato's theorem (Theorem I.2) ./.

REMARK. In the preceding theorem, assumption (b), we may replace "B' a bounded Banach disk" by "B' a bounded disk, $TE_B \subset F_{B'}$ and $(D(T'))^- \subset D(P')$, where T', P' denote the restrictions of T, P to E_B with range space $F_{B'}$ and the closure is taken in E_B". The condition $(D(T'))^- \subset D(P')$ in turn is satisfied in particular if $(D(T))^- \subset D(P)$ (closure in E) or if $D(P) = E$. This simply reflects the possibility of using Theorem I.4 instead of Theorem I.2.

Another situation where we may infer that $R(T+P)$ is closed occurs when $\mathrm{def}(T) = 0$, or $R(P) \subset R(T)$, in such a way that $R(T+P) = R(T)$. In this line of thought, we have the following

THEOREM I.20. _Let_ E _be a locally convex space_, F _a vector space and_ T, P _operators from_ E _into_ F _such that_ $(D(T))^{-} \subset D(P)$.

Assume that there exist a neighborhood U _in_ E, _a bounded Banach disk_ $B \subset E$ _and_ $0 < \varepsilon < 1$ _such that_

- $N(T) \cap E_B$ _is closed in_ E_B,
- $PU \subset TB$,
- $PB \subset \varepsilon TB$.

Then $\mathrm{nul}(T+P) = \mathrm{nul}(T)$, $R(T+P) = R(T)$ _and the images of the topology of_ E _by_ T _and_ $T+P$ _are the same_.

The assumption "$N(T) \cap E_B$ _is closed in_ E_B" _is satisfied in particular if_ $\mathrm{nul}(T) < \infty$ _or if_ F _is also a locally convex space and_ T _has a closed graph in_ $E \times F$.

Proof. By considering the Banach disk $B \cap (D(T))^{-}$, we may assume that $B \subset D(P)$. That $\mathrm{nul}(T+P) = \mathrm{nul}(T)$ and $R(T+P) = R(T)$ is a consequence of Theorem I.18.

To prove the second part, we may assume without loss of generality that $R(T) = F$. Let $U' \subset U$ be any neighborhood in E. We have

$$(T+P)U' \subset TU' + PU' \subset TU' + TB \subset \lambda TU'$$

for $\lambda > 0$ large enough, since B, being bounded, is absorbed by U'.

If T', P' denote the restrictions of T, P to E_B with range space F_{TB}, then T', P' satisfy Theorem I.4. Thus $T'+P'$ is onto and open : there is $\mu > 0$ such that $TB \subset \mu(T'+P')B \subset \mu(T+P)B$.

We then have

$$\begin{aligned} TU' &\subset (T+P)U' + PU' \\ &\subset (T+P)U' + TB \\ &\subset (T+P)U' + \mu(T+P)B \\ &\subset \xi(T+P)U', \end{aligned}$$

for some $\xi > 0$ large enough.

Thus the topologies defined by $\{TU'\}$ and $\{(T+P)U'\}$ when U' runs through the neighborhoods in E are the same ./.

REMARKS. The preceding proof shows a little more : for each neighborhood $U' \subset U$ in E, the semi-norms generated by TU' and $(T+P)U'$ are equivalent. It follows in particular that T is open (resp. weakly open, or almost open) if and only if T+P is open (resp. weakly open,or almost open).

We also notice that the last part of the proof is essentially that of Proposition I.13.

Theorem I.20 strengthens a result of Vladimirski announced without proof in (26). He stated the theorem under the assumptions "F is a locally convex space, T has a closed graph in ExF, P is continuous, $PU \subset TB$ and $B \subset \varepsilon U$" instead of our weaker assumptions "$N(T) \cap E_B$ is closed in E_B, $PU \subset TB$ and $PB \subset \varepsilon TB$".

By taking the quotient of the range space by a finite dimensional subspace, we obtain a relaxation of the assumption $R(P) \subset R(T)$ as follows.

PROPOSITION I.21. <u>Let</u> E <u>be a locally convex space</u>, F <u>a vector space and</u> T, P <u>operators from</u> E <u>into</u> F <u>such that</u> $(D(T))^{-} \subset D(P)$.

Assume that there exist a neighborhood U in E, a bounded Banach disk $B \subset E$, a finite dimensional subspace $N \subset F$ and $0 < \varepsilon < 1$ such that

- $N(T) \cap E_B$ is closed in E_B,
- $PU \subset TB+N$,
- $PB \subset \varepsilon TB+N$.

Then $R(T+P)+N = R(T)+N$ and $\operatorname{ind}(T+P) = \operatorname{ind}(T)$.

If furthermore F is a locally convex space, T is open, has a closed graph (which implies that $N(T) \cap E_B$ is closed in E_B), $R(T)$ is closed and P is continuous, then $T+P$ is open, has a closed graph and $R(T+P)$ is closed.

The proof of Proposition I.21 is a combination of Theorem I.20 and the following

LEMMA I.22. Let E, F, G be locally convex spaces, T an operator from E into F which has a closed graph, and S an operator from F into G such that $D(S) = F$ and $\operatorname{nul}(S) < \infty$.

(a) If $\operatorname{def}(S) < \infty$, then T and ST have simultaneously an index and $\operatorname{ind}(ST) = \operatorname{ind}(S)+\operatorname{ind}(T)$.

(b) If S is continuous and ST is open then T is open.

(c) If S is continuous, ST open and $R(ST)$ is closed, then $R(T)$ is closed (and T is open).

Proof. (a) : this part is purely algebraic. Let

$$N(S) = (N(S) \cap R(T)) \oplus L ,$$

$$F = R(T) \oplus L \oplus M ,$$

$$R(S) = R(ST) \oplus SM .$$

Since S is one-to-one on M, we have

$$\begin{aligned} \mathrm{nul}(ST) &= \mathrm{nul}(T) + \dim (N(S)\cap R(T)), \\ \mathrm{def}(ST) &= \mathrm{def}(S) + \dim SM \\ &= \mathrm{def}(S) + \dim M \\ &= \mathrm{def}(S) + \mathrm{def}(T) - \dim L \\ &= \mathrm{def}(S) + \mathrm{def}(T) - \mathrm{nul}(S) + \dim (N(S)\cap R(T)). \end{aligned}$$

Because $\mathrm{nul}(S) < \infty$ and $\mathrm{def}(S) < \infty$, the preceding equalities show that $\mathrm{nul}(ST) < \infty$ if and only if $\mathrm{nul}(T) < \infty$ and $\mathrm{def}(ST) < \infty$ if and only if $\mathrm{def}(T) < \infty$. Moreover $\mathrm{ind}(ST) = \mathrm{ind}(S) + \mathrm{ind}(T)$.

(b) and (c) : by considering the quotient $E/N(T)$ we may assume without loss of generality that T is one-to-one. Write $N = T^{-1}N(S) = N(ST)$; $\dim N < \infty$. Let E' be a closed subspace such that (algebraically and topologically) $E = N \oplus E'$.

Let $\mathcal{F}$ be an ultrafilter converging to y in R(T) (resp. in F, for (c)). Let $\mathcal{G}$ be the ultrafilter $T^{-1}\mathcal{F}$ (we make no difference between a filter defined on a subset of a set and the filter it generates on the whole set). Since T has a closed graph, we need only show that $\mathcal{G}$ converges to some $\hat{x}$ in E.

Let $\mathcal{G}'$ denote the projection (ultrafilter) of $\mathcal{G}$ on E' along N.

Since S is continuous, $S\mathcal{F}$ converges to $Sy \in R(ST)$ (for (c), it is because R(ST) is assumed closed). Furthermore the restriction $(ST)'$ to E' is injective and open. Thus $\mathcal{G}' = (ST)'^{-1}S\mathcal{F}$ converges to $x' = (ST)'^{-1}Sy \in E'$.

Let Q denote the projection of E onto N along E', and $\|.\|$

a euclidean norm on N. Let $\mathcal{H}$ be the ultrafilter on the real line defined by $\{\|Qx\| : x \in A, A \in \mathcal{G}\}$, which is the image of the ultrafilter $\mathcal{G}$ by the mapping $x \to \|Qx\|$. If we consider the compact extended real line $(-\infty, +\infty)$, then $\mathcal{H}$ converges to a point h, $0 \leq h \leq +\infty$.

If $h < \infty$ then the set $A = \{x \in E : \|Qx\| \leq h+1\}$ belongs to $\mathcal{G}$. Hence its compact projection $QA = \{x'' \in N : \|x''\| \leq h+1\}$ belongs to the ultrafilter $Q\mathcal{G}$. This shows that the ultrafilter $Q\mathcal{G}$ converges to some $x'' \in QA$. Thus $\mathcal{G} \to x' + x''$.

If $h = \infty$, then the set $B = \{x \in E : \|Qx\| > 0\}$ belongs to $\mathcal{G}$. The ultrafilter $\mathcal{U}$ on E defined by $\{x/\|Qx\| : x \in A, A \in \mathcal{G}, A \subset B\}$ has the following characteristics : $(1-Q)\mathcal{U} \to 0$ in E', $Q\mathcal{U} \to x''$ in N such that $\|x''\| = 1$ (compactness of $\{x'' \in N : \|x''\| = 1\}$) and $T\mathcal{U} \to 0$. Thus $\mathcal{U} \to x'' \neq 0$ and $T\mathcal{U} \to 0$ in contradiction with the assumption that T be injective ./.

REMARK. We could prove (b) and (c) by using filters instead of ultrafilters, that is without the axiom of choice (cf. (3, M. De Wilde, Le Quang Chu)).

<u>Proof of Proposition I.21</u>. We may assume without loss of generality that $F = R(T)+N$. Let $G = F/N$ and S be the canonical quotient operator from F onto G. We have $SPU \subset STB$, $SPB \subset \varepsilon STB$.

Since $N(ST) = T^{-1}N$, with $\dim N < \infty$, it follows that $N(T)$ has a finite codimension in $N(ST)$. Thus we may write $N(ST) \cap E_B = (N(T) \cap E_B) \oplus L$ with $\dim L < \infty$ ($\dim L$ cannot be infinite as $L \subset N(ST)$ and $L \cap N(T) = \{0\}$). As $N(T) \cap E_B$ is closed in

E_B, we infer that $N(ST) \cap E_B$ is closed in E_B.

By Theorem I.20, we have

$$nul(S(T+P)) = nul(ST+SP) = nul(ST),$$

$$R(S(T+P)) = R(ST+SP) = R(ST) \; (= G).$$

The last equality shows that $R(T+P)+N = R(T)+N$. Since $ind(S(T+P)) = ind(ST)$, and $ind(S)$ is finite, it follows from Lemma I.22 that $ind(T+P) = ind(T)$.

Assume now that T is open, has a closed graph and $R(T)$ is closed. We first show that ST is also open. Let W be an open neighborhood in E, then TW is open in $R(T)$. Let $N' \subset N$ be such that (algebraically and topologically) $R(T) \oplus N' = F$. Now $TW+(R(T) \cap N)$ is open in $R(T)$, thus $TW+N = TW+(R(T) \cap N)+N'$ is open in F. This shows that STW is an open neighborhood in $G = R(ST)$. Since $T+P$ has a closed graph (because P is continuous and $G(T)$ is closed), we may apply Theorem I.20 and Lemma I.22, and infer that $T+P$ is open and $R(T+P)$ is closed in $F = R(T)+N$ (recall that $R(S(T+P)) = G$). If we return to the initial F in the statement, then $R(T+P)$ is still closed in F because $R(T)+N$ is ./.

REMARK. The last remark following Theorem I.20 applies here as well : Proposition I.21 was announced without proof in (26, Vladimirski) under similar but more restrictive assumptions.

I.5. Compact perturbations

We now apply Theorem I.9 B. Generally, we may do so when, restricted to suitable subspaces, the perturbations preserve only the index as opposed to the case of "small" perturbations, where a kind of semi-continuity of the nullity and the deficiency takes place as well. This often happens when we deal with compact perturbations.

I.5.1. Stability of the index

THEOREM I.23. *Let* E, F *be vector spaces, and* T, P *operators from* E *into* F *such that* $D(T) \subset D(P)$.

Assume that

(a) T *has an index,*

(b) *There exist Banach disks* $B \subset E$, $B' \subset F$ *such that*

- $G(T) \cap (E_B \times F_{B'})$ *is closed in* $E_B \times F_{B'}$,

- $R(P) \subset F_{B'}$ *and* PB *is relatively compact in* $F_{B'}$ (*i.e.* $(PB)^-$ *is compact in* $F_{B'}$),

- $B' \cap R(T) \subset TB$.

Then $\operatorname{ind}(T+P) = \operatorname{ind}(T)$.

Proof. We may replace B by the Banach disk $B \cap T^{-1}B'$ which we rebaptize as B. We have then the additional relation : $TE_B = F_B \cap R(T)$. If T', P' denote the restrictions of T, P to E_B with range space $F_{B'}$, then T' has a closed graph and is open from the Banach space E_B into the Banach space $F_{B'}$. Hence R(T) is closed, that is T' is a semi-Fredholm operator, and P' is a compact perturbation of P'. That $\operatorname{ind}(T+P) = \operatorname{ind}(T)$ follows now from Kato's theorem and Theorem I.9 B ./.

If in the preceding proof, we use Theorem I.4 instead of

Kato's theorem, we obtain the following

PROPOSITION I.24. Let E, F be vector spaces, and T, P operators from E into F such that $D(T) \subset D(P)$.

Assume that

(a) T has an index,

(b) There exist a Banach disk $B \subset E$, a norming disk $B' \subset F$ such that

- $R(P) \subset F_{B'}$, $TE_B \subset F_{B'}$,

- If T', P' denote the restrictions of T, P to E_B with range space $F_{B'}$, then $(D(T'))^- \subset D(P')$, the closure being taken in E_B,

- T' has a closed graph in $E_B \times F_{B'}$,

- $B' \cap R(T) \subset TB$,

- PB is precompact in $F_{B'}$.

Then $\operatorname{ind}(T+P) = \operatorname{ind}(T)$.

One may relax somewhat the assumption $R(P) \subset F_{B'}$ as follows.

PROPOSITION I.25. Let E, F be vector spaces, and T, P operators from E into F such that $D(T) \subset D(P)$.

Assume that

(a) T has an index

(b) There exist a Banach disk $B \subset E$, a norming disk $B' \subset F$ and a finite dimensional subspace $N \subset F$ such that

- $R(P) \subset F_{B'} + N$, $TE_B \subset F_{B'}$,

- If T' denotes the restriction of T to E_B with range space $F_{B'}$, then $(D(T'))^- \subset D(P)$, the closure being taken in F_B',

- T' *has a closed graph in* $E_B x F_{B'}$,

- $B' \cap R(T) \subset TB$,

- $PB \subset \varepsilon B'$ *for some* $0 < \varepsilon < 1$, *or* PB *is precompact in* $F_{B'}+N$, *topologized by* $B'+D$, D *being a finite disk generating* N.

Then $\mathrm{ind}(T+P) = \mathrm{ind}(T)$.

If B' *is completing, we may replace the assumptions* $(D(T'))^- \subset D(P)$, $TE_B \subset F_{B'}$, *and* "T' *has a closed graph in* $E_B x F_{B'}$" *by* "$G(T) \cap E_B x F_{B'}$ *is closed in* $E_B x F_{B'}$".

Proof. From the relations $TE_B \subset F_{B'}$ and $B' \cap R(T) \subset TB$ we get $TE_B = R(T) \cap F_{B'}$. Moreover $R(T) \cap F_{B'}$ has a finite codimension in $R(T) \cap (F_{B'}+N)$. Consequently, there exists a finite dimensional subspace L generated by a finite disk D', such that $L \subset D(T)$ and $T(E_B+L) = R(T) \cap (F_{B'}+N)$. Now $B+D'$ is a Banach disk generating E_B+L (cf. Proof of Theorem I.16), and it induces on E_B the same topology as B. Similarly $B'+D$ defines a norm on $F_{B'+D}$, which is equivalent to that by B' on $F_{B'}$.

If T'', P'' denote the restrictions of T, P to $E_{B+D'}$ with range space $F_{B'+D}$, then T'' is a semi-Fredholm operator, finite dimensional extension of T', in view of Lemmas I.10 and I.11.

If $PB \subset \varepsilon B'$, and P' denotes the restriction of P to E_B with range space $F_{B'}$, then $\mathrm{ind}(T'+P') = \mathrm{ind}(T')$ by Theorem I.4. Thus $\mathrm{ind}(T''+P'') = \mathrm{ind}(T'')$ by Lemma I.10.

If PB is precompact in $F_{B'+D}$, then $P(B+D')$ is also precompact in $F_{B'+D}$ because PD' is a finite disk in $F_{B'+D}$. Thus $\mathrm{ind}(T''+P'') = \mathrm{ind}(T'')$ by Theorem I.4. Theorem I.9 B shows that $\mathrm{ind}(T+P) = \mathrm{ind}(T)$ ./.

I.5.2. Stability of the topological characteristics

We now consider the question whether T+P is open and R(T+P) is closed, when T has these properties and P is a "compact" perturbation. If we consider compact perturbations in the sense that P is a compact operator then the answers are given by the theorems of Schwartz, Köthe and Schaefer (Theorems I.7 and I.8) and Vladimirski (Theorem II.22). We will prove these in Chapter II (see remark following Corollary I.28). Here we consider compact perturbations in the sense of Theorem I.23, that is the restriction of P to certain subspaces is a compact operator.

As in the case of small perturbations, we obtain positive answers for Φ_--operators, essentially via the techniques of Proposition I.13. It turns out in fact that compact perturbations of Φ_--operators essentially reduce to small perturbations modulo a finite dimensional subspace.

THEOREM I.26. Let E, F be locally convex spaces, and T, P operators from E into F such that $(D(T))^- \subset D(P)$.

Assume that

(a) T is a Φ_--operator,

(b) There exist a neighborhood U in E, a bounded Banach disk $B \subset E$, and a bounded disk $B' \subset F$ such that

- $TE_B \subset F_{B'}$, $B' \cap R(T) \subset TB$,
- $PU \subset B'$ and PB is precompact in $F_{B'}$.

Then T+P is a Φ_--operator and $\mathrm{ind}(T+P) = \mathrm{ind}(T)$.

If B' is a bounded Banach disk, then we may drop the assumption $TE_B \subset F_{B'}$ and we may assume $D(T) \subset D(P)$ a priori.

<u>Proof</u>. Since $R(T)$ is closed in F, $R(T)\cap F_{B'}$ is closed in $F_{B'}$ (this in fact also follows from the fact that T is open from the Banach space E_B into $F_{B'}$ and has a closed graph). There exists a finite dimensional subspace $N\subset F_{B'}$ such that (algebraically and topologically in $F_{B'}$) $F_{B'} = N \oplus (R(T)\cap F_{B'})$. There is thus $\lambda > 0$ such that $B' \subset \lambda B'\cap R(T)+N$.

Let $U' = \lambda^{-1}U$. Then

$$PU'\subset B'\cap R(T)+N$$
$$\subset TB+N \ .$$

On the other hand, PB is precompact in $F_{B'}$. Consequently, for a fixed $0 < \varepsilon < 1$, there is a finite set $\underset{\sim}{F}$ such that

$$PB\subset \underset{\sim}{F}+ \varepsilon \lambda^{-1}B'\subset \underset{\sim}{F}+ \varepsilon TB+N \ .$$

If $N' = N+ \rangle \underset{\sim}{F}\langle$, then $\dim N' < \infty$ and

$$PU'\subset TB + N' \ ,$$
$$PB \subset \varepsilon TB + N' \ .$$

Proposition I.21 applies to give the conclusions.

If B' is a Banach disk, we replace B by $B\cap T^{-1}B'$. We may also extend P by continuity and obtain $(D(T))^{-}\subset D(P)$ ./.

The following results of Goldman, Krackowski (7) and Vladimirski (26) are direct consequences of Theorem I.26 :

COROLLARY I.27. Finite dimensional perturbations (Goldman and Krackowski). <u>Let</u> E, F <u>be locally convex spaces and</u> T, P <u>operators from</u> E <u>into</u> F <u>such that</u> $D(T)\subset D(P)$.

<u>Assume that</u> P <u>is a continuous operator of finite rank, i.e.</u> $R(P)$ <u>is finite dimensional. If</u> T <u>is an open operator, with</u>

a closed graph and a closed range, then $T+P$ *is an open operator, with a closed graph and a closed range.*

If T *is a* ϕ_-*-operator then* $T+P$ *is a* ϕ_-*-operator and* $\mathrm{ind}(T+P) = \mathrm{ind}(T)$.

Proof. By assumption, there is a finite disk D such that $PU \subset D$. We may assume that $F = R(T) + >D<$. Now there is a finite disk D' in E such that $D \cap R(T) \subset TD'$. It is immediate that PD' is precompact in F_D (euclidean space). It suffices now to apply Theorem I.26 ./.

REMARK. It can be seen from the proof that if T is a ϕ_+-operator, then $T+P$ is also a ϕ_+-operator and $\mathrm{ind}(T+P) = \mathrm{ind}(T)$. This is also an immediate consequence of the theorem of Schwartz, Köthe and Schaefer (Theorem I.7), since P is a fortiori a compact operator.

COROLLARY I.28. *Let* E, F *be locally convex spaces, and* T, P *operators from* E *into* F *such that* $D(T) \subset D(P)$.

Assume that T *is a* ϕ_-*-operator, and* P *a compact operator. Let* K *be a compact disk in* F *such that* $PU \subset K$ *for some neighborhood* U *in* E.

If there exists a compact disk K' *in* E *such that* $K \cap R(T) \subset TK'$, *then* $T+P$ *is a* ϕ_-*-operator and* $\mathrm{ind}(T+P) = \mathrm{ind}(T)$.

Proof. It suffices to prove that PK' is precompact in F_K. But this is trivial since P is continuous from E into F_K, and K' is compact in E ./.

REMARK. Corollary I.28 covers the theorem of Schwartz, Köthe and Schaefer on compact perturbations of Φ_--operators (Theorem I.8). Indeed, when E, F are Frechet spaces, then for any compact disk $K \subset F$, there exists a compact disk $K' \subset E$ such that $K \cap R(T) \subset TK'$. (A proof of this is based on the fact that any precompact disk in a metrizable locally convex space is contained in the closed absolutely convex hull of a sequence converging to 0.) We say that such an operator T <u>lifts compact disks</u>.

Vladimirski (26) in fact announced, without proof, a result more general than Corollary I.28. He considered perturbations of the type $PU \subset TB+K$, where B is a bounded Banach disk such that $B \subset \varepsilon U$ for some $0<\varepsilon<1$, and K a compact disk. His result may also be strengthened by our approach :

PROPOSITION I.29. <u>Let</u> E, F <u>be locally convex spaces and</u> T, P <u>operators from</u> E <u>into</u> F <u>such that</u> $(D(T))^- \subset D(P)$.

<u>Assume that</u>

(a) T <u>is a</u> Φ_-<u>-operator</u>,

(b) <u>There exist a neighborhood</u> U <u>in</u> E, <u>a bounded Banach disk</u> $B \subset E$, <u>a compact disk</u> $K \subset R(T)$, <u>a finite dimensional subspace</u> $N \subset F$ <u>and</u> $0<\varepsilon<1$ <u>such that</u>

$$PU \subset TB + K + N \quad \text{and} \quad PB \subset \varepsilon TB + K + N,$$

(c) <u>There exists a compact disk</u> $K' \subset E$ <u>such that</u> $K \subset TK'$.

<u>Then</u> T+P <u>is a</u> Φ_-<u>-operator and</u> $\operatorname{ind}(T+P) = \operatorname{ind}(T)$.

<u>If</u> T <u>lifts compact disks, then we may simply assume in</u> (b) <u>that</u> K <u>be a compact disk in</u> F (<u>instead of</u> $K \subset R(T)$).

Proof. We may assume without loss of generality that $D(P) = E$. Replacing K by $\varepsilon^{-1}K$, we may also assume that

$$PU \subset TB + K + N \, , \; PB \subset \varepsilon(TB+K) + N \, .$$

Fix any $\varepsilon' > 0$ such that $\varepsilon'' = \varepsilon + \varepsilon' < 1$. There is a finite disk $D \subset E$ such that $K' \subset D + \varepsilon' U$. We have then

$$\begin{aligned} P(B+K') &\subset P(B+D+\varepsilon' U) \\ &\subset \varepsilon(TB+K) + N + PD + \varepsilon' PU \\ &\subset \varepsilon(TB+K) + \varepsilon'(TB+K) + PD + N \\ &\subset \varepsilon'' T(B+K') + N' \, , \end{aligned}$$

where $N' = N + {>}PD{<}$ is finite dimensional.

On the other hand, ?

$$PU \subset T(B+K') + N' \, .$$

Since $B+K'$ is a Banach disk (Lemma 0.2), the conclusions follow from Proposition I.21.

Assume now that T lifts compact disks, and that $K \subset F$ instead of $K \subset R(T)$. Let L be such that (algebraically and topologically) $F = R(T) \oplus L$, $\dim L < \infty$. Let K' be the projection of K onto $R(T)$, along L. Then there exists a compact disk $K'' \subset E$ such that $K' \subset TK''$. We have then

$$PU \subset TB + K' + L + N \, ,$$

$$PB \subset \varepsilon TB + K' + L + N \, ,$$

and the first part applies ./.

CHAPTER II

DUALITY AND BOUNDED OR PRECOMPACT PERTURBATIONS OF SEMI-FREDHOLM OPERATORS

In this chapter, we study perturbations of semi-Fredholm operators by addition of small bounded operators or precompact operators (definitions in Chapter 0). The main tool used is duality.

In Chapter I, we considered "small" or precompact perturbations in the sense that the perturbing operators, when restricted to suitable subspaces equipped with convenient norms, are small in norm or precompact. Now, when we deal with "globally" bounded or precompact perturbations, in the sense that the perturbing operator maps some neighborhood into a bounded or precompact disk, it turns out that the approach developed in Chapter I applies handily in the duals, where the suitable subspaces are Banach spaces generated by closed equicontinuous disks. One obtains easily the stability of "almost openness". The stability of the index is readily secured when suitable assumptions of completeness are placed on the spaces in such a way that the perturbed operators become semi-Fredholm operators. These assumptions are satisfied in Frechet spaces.

II.1. Bounded perturbations of Φ_+-operators

II.1.1. General results

The following simple lemma shall be important for our proofs :

LEMMA II.1. *Let* E, F *be locally convex spaces,* T *an injective open operator and* P *a bounded operator from* E *into* F *such that* $D(T) \subset D(P)$.

If there exist a neighborhood U *in* E, *a neighborhood* V *in* F *and* $0 < \varepsilon < 1$ *such that* PU *is bounded,* $V \cap R(T) \subset TU$ *and* $PU \subset \varepsilon V$, *then there exist a base of neighborhoods* U *in* E *for which these relations hold.*

Proof. Let $U' \subset U$ be any neighborhood in E. We need only prove that there exists a neighborhood $U'' \subset U'$ which satisfies relations similar to those in the statement.

There is a neighborhood V' in F such that $V' \cap R(T) \subset TU'$, T being open. Since PU is bounded, there is $\lambda > 0$ small enough such that $\lambda PU \subset \varepsilon V'$. If we take $U'' = U' \cap \lambda U$ and $V'' = V' \cap \lambda V$, then $V'' \cap R(T) \subset TU''$ because T is injective. On the other hand

$$PU'' \subset \lambda PU \subset (\varepsilon V') \cap (\varepsilon \lambda V)$$

$$\subset \varepsilon V'' \quad ./.$$

Using this lemma, we can prove immediately the following (cf. Lemma I.5 and (16, Le Quang Chu, Proposition 3)) :

THEOREM II.2. *Let* E, F *be locally convex spaces,* T *an injective open operator and* P *a bounded operator from* E *into* F *such that* $D(T) \subset D(P)$.

<u>If, for a neighborhood</u> U <u>in</u> E, <u>a neighborhood</u> V <u>in</u> F <u>and</u> $0 < \varepsilon < 1$, PU <u>is bounded</u>,

$$V \cap R(T) \subset TU \quad \underline{\text{and}} \quad PU \subset \varepsilon V ,$$

<u>then</u> T+P <u>is an injective open operator</u>.

<u>If furthermore</u> $(D(T))^{-} \subset D(P)$, T <u>has a closed graph and either</u> E <u>or</u> R(T) <u>is complete, then both</u> T <u>and</u> T+P <u>are injective</u> ϕ_+<u>-operators</u>.

<u>Proof</u>. By Lemma II.1, the relations

$$(*) \qquad V \cap R(T) \subset TU \quad , \quad PU \subsetneq \varepsilon V$$

hold in fact for a base of neighborhoods U in E (to each of which corresponds a neighborhood V in F).

If A is an absorbent disk, let p_A denote the semi-norm associated with A (the Minkowski gauge of A). Then it follows from the relations (*) that (cf. Theorem I.3)

$$p_V(Tx) \geqslant p_U(x) \ , \ p_V(Px) \leqslant \varepsilon p_U(x) \ , \ x \in D(T) \ .$$

Consequently,

$$(**) \qquad \begin{aligned} p_V(Tx+Px) &\geqslant p_V(Tx) - p_V(Px) \\ &\geqslant p_U(x) - \varepsilon p_U(x) \\ &\geqslant (1-\varepsilon)p_U(x) \ , \quad \forall x \in D(T+P) \ . \end{aligned}$$

This shows that T+P is injective and open. Indeed, if $(T+P)x = 0$, then (**) implies that $p_U(x) = 0$ for U belonging to a base of neighborhoods in E. Since E is Hausdorff, $x = 0$. Moreover, we may infer immediately from (**) that

$$(1-\varepsilon)V \cap R(T+P) \subset (T+P)U$$

if we assume U to be closed (which is no loss of generality).

If $(D(T))^{-} \subset D(P)$ and T has a closed graph, then $T+P$ has a closed graph, P being continuous. Let $\mathcal{G}$ be a filter in $R(T+P)$ which converges to $y \in F$. Let $\mathcal{F} = (T+P)^{-1}\mathcal{G}$. Then $\mathcal{F}$ is a Cauchy filter base, as $T+P$ is open. If E is complete, then $\mathcal{F}$ converges to $x \in E$ and the closed graph of $(T+P)$ implies that $x \in D(T+P)$ and $y = (T+P)x \in R(T+P)$. If $R(T)$ is complete, we proceed as follows : we remark that $P\mathcal{F}$ is a Cauchy filter base, therefore $T\mathcal{F} = (T+P-P)\mathcal{F}$ also defines a Cauchy filter in $R(T)$, which converges to some $y' \in R(T)$ ($R(T)$ is complete, hence closed). As T is injective and open, this shows that $\mathcal{F}$ converges to $x = T^{-1}y'$ ./.

REMARK. We note that, for a base of closed neighborhoods U in E, we have $V \cap R(T) \subset TU$, $PU \subset \varepsilon V$, $0 < \varepsilon < 1$, implying $(1-|\lambda|\varepsilon)V \cap R(T+\lambda P) \subset (T+\lambda P)U$, $\forall\, |\lambda| \leq 1$ (T injective). In normed spaces, this means that the minimum modulus $\gamma(T+\lambda P)$ is not less than $(1-|\lambda|\varepsilon)\gamma(T)$ (T being injective).

Theorem II.2 deals with the topological characteristics of $T+P$. The stability of the index is asserted in the following

THEOREM II.3. *Let* E, F *be locally convex spaces*, T *an injective open operator and* P *a bounded operator from* E *into* F *such that* $D(T) \subset D(P)$.

Assume that there exist a neighborhood U *in* E, *a neighborhood* V *in* F *and* $0 < \varepsilon < 1$ *such that* PU *is bounded,*

$$V \cap R(T) \subset TU \quad \text{and} \quad PU \subset \varepsilon V .$$

Then $T+P$ *is injective, open and*

$$\operatorname{codim}\,(R+P))^{-} = \operatorname{codim}\,(R(T))^{-} .$$

<u>In particular, if</u> $R(T)$ <u>and</u> $R(T+P)$ <u>are closed then</u>

$$\mathrm{ind}(T+P) = \mathrm{ind}(T).$$

<u>Proof</u>. That $T+P$ is injective and open is proved in Theorem II.2. For the last part, we may assume that $E=D(T)=D(T+P)$.

From $V \cap R(T) \subset TU$ we infer quite easily (cf. proof of Theorem 0.15) that we may write $(V)^- \cap (R(T))^- \subset (TU)^-$. By duality we get $T^{+-1}U^\circ \subset V^\circ + N(T^+)$, that is

$$U^\circ \cap R(T^+) \subset T^+V^\circ.$$

In fact $R(T^+) = E^+$ because T is injective and open (cf. Lemmas 0.8 and 0.12).

On the other hand, from $PU \subset \varepsilon V$ we infer that $P^{+-1}U^\circ \supset \varepsilon^{-1}V^\circ$, that is $P^+V^\circ \subset \varepsilon U^\circ$.

Since PU is bounded, we have $D(P^+) = F^+$, and $R(P^+) \subset \rangle U^\circ \langle$.

Now U° and V° are weakly bounded Banach disks, being weakly compact. Moreover $G(T^+)$ is weakly closed (Lemma 0.5). All hypotheses of Theorem I.17 are satisfied. We infer in particular that $R(T^++P^+) = R(T^+) = E^+$, which yields again by duality $\mathrm{nul}(T+P) = \mathrm{nul}(T) = 0$, and $\mathrm{nul}(T^++P^+) = \mathrm{nul}(T^+)$, which yields $\mathrm{codim}\ (R(T+P))^- = \mathrm{codim}\ (R(T))^-$. We need only ascertain that $(T+P)^+ = T^++P^+$, but this follows from the fact that P is continuous. Indeed, it is trivial that $D(T^++P^+) = D(T^+) \subset D((T+P)^+)$. Conversely, if $g \in D((T+P)^+)$ and $f = (T+P)^+g$, then for any $x \in E$, we have $f(x) = g((T+P)x) = g(Tx) + g(Px)$. Consequently, $g(Tx) = f(x) - g(Px)$ is a continuous functional on E (by the composite continuity of f, g, P). Therefore $g \in D(T^+)$, which implies that $D((T+P)^+) \subset D(T^++P^+)$ ./.

As a consequence of Theorems II.2 and II.3, we have

THEOREM II.4. Let E, F be locally convex spaces and T, P operators from E into F such that $(D(T))^{-} \subset D(P)$.

Assume that (a) T is a ϕ_+-operator, (b) P is a bounded operator and (c) either E or R(T) is complete.

Then there is $\rho > 0$ such that for all $|\lambda| < \rho$, $T+\lambda P$ is a ϕ_+-operator, $\mathrm{nul}(T+\lambda P) \leq \mathrm{nul}(T)$, $\mathrm{def}(T+\lambda P) \leq \mathrm{def}(T)$ and

$$\mathrm{ind}(T+\lambda P) = \mathrm{ind}(T).$$

Proof. Let L be a closed subspace such that $E = L \oplus N(T)$ (algebraically and topologically). Let T', P' denote the restrictions of T, P to L with range space F, then T' is an injective ϕ_+-operator and P' a bounded operator. Let U', V' be neighborhoods in L, F respectively such that PU' is bounded and $V' \cap R(T') \subset T'U'$. Let $\rho = \sup\{ |\lambda| : \lambda P'U' \subset V' \}$. Then $\rho > 0$, and for all $|\lambda| < \rho$, there is $0 < \varepsilon < 1$ such that $V' \cap R(T') \subset T'U'$ and $\lambda PU' \subset \varepsilon V'$. By Theorem II.2 and Theorem II.3 $T' + \lambda P'$ is an injective ϕ_+-operator and $\mathrm{ind}(T'+\lambda P') = \mathrm{ind}(T')$ (remark that either L or $R(T') = R(T)$ is complete). By Lemma I.11, $T+\lambda P$ is a ϕ_+-operator and $\mathrm{ind}(T+\lambda P) = \mathrm{ind}(T)$. Moreover, it follows from $N(T+\lambda P) \cap L = \{0\}$ that $\mathrm{nul}(T+\lambda P) \leq \mathrm{nul}(T)$, and as a result $\mathrm{def}(T+\lambda P) \leq \mathrm{def}(T)$ ./.

REMARKS. Theorem II.4 does not fully render Kato's theorem in case of Banach spaces. Our estimate ρ makes use of a complement L of N(T). Assume that E, F are Banach spaces with unit balls B, B', and $B' \cap R(T) \subset TB$, $PB \subset \varepsilon B'$, $0 < \varepsilon < 1$. Let $B_0 = (B+N(T)) \cap L$

and $B_1 = B \cap L$. We have then $B' \cap R(T) \subset TB_0$ but a priori $PB_0 \not\subset \varepsilon B'$. However, by virtue of a lemma attributed to Auerbach (cf. (19, Piestch)), we may choose L such that the projection from E onto L along N(T) has a norm not exceding (n+1), where $n = \dim N(T)$. Thus $B_0 \subset (n+1)B_1$. As a result, $(n+1)^{-1}PB_0 \subset \varepsilon B'$. We may say therefore that in Banach spaces, our proof could only yield a ρ about (nul(T)+1) times less precise than Kato's estimate $\gamma(T)$. It should be noted that to prove that T+P is open, Kato uses the powerful Lemma I.2. In fact, our estimate is roughly that given by Gohberg and Krein in (5).

The proof of Theorem II.4 also allows the following formulation : there exist a neighborhood U in E, a neighborhood V in F such that if PU is bounded and $PU \subset V$, then T+P is a ϕ_+-operator, $\mathrm{nul}(T+P) \leq \mathrm{nul}(T)$, $\mathrm{def}(T+P) \leq \mathrm{def}(T)$ and ind(T+P)=ind(T). Indeed, it suffices to take $U = U'+U''$, where U'' is a neighborhood in N(T), and $V = \varepsilon V'$ with any $0 < \varepsilon < 1$.

This formulation, for T a continuous ϕ_+-operator and E a complete locally convex space, is given without proof in (27, Vladimirski).

We now prove an extension of Part (C) of Theorem I.1. The proof partially follows (6, Goldberg, Cor. V.1.7), which in turn is an adaptation of (5, Gohberg and Krein, Lemma 8.1).

THEOREM II.5. <u>Under the assumptions of Theorem</u> II.4, <u>there exists</u> $\rho > 0$ <u>such that, for all</u> λ <u>in the annulus</u> $0 < |\lambda| < \rho$, $\mathrm{nul}(T+\lambda P)$ <u>is a constant (not exceeding</u> nul(T)).

<u>Proof</u>. We now construct two closed subspaces $E_1 \subset E$ and $F_1 \subset F$ such that the restriction T_1 of T to E_1 is a surjective Φ_+-operator onto F_1, the restriction P_1 of P is a bounded perturbation of T_1 and that $N(T+\lambda P) = N(T_1+\lambda P_1)$, $\forall\ \lambda \neq 0$. Theorem II.4 will then show that, for $|\lambda| \neq 0$ small enough $T_1+\lambda P_1$ is surjective and $\operatorname{ind}(T_1+\lambda P_1) = \operatorname{ind}(T_1)$. From that it will follow that $\operatorname{nul}(T+\lambda P) = \operatorname{nul}(T_1+\lambda P_1) = \operatorname{nul}(T_1)$ which is a constant.

We may assume without loss of generality that $E = (D(T))^- = D(P)$. Let $x \in N(T+\lambda P)$, $\lambda \neq 0$. Then $Tx = -\lambda Px$, and $Px \in R(T)$, which means that $x \in P^{-1}(R(T))$. Let $R_1 = R(T)$ and $D_1 = P^{-1}R_1$. Now since $x \in D_1$, $-\lambda Px \in TD_1$. Let $TD_1 = R_2$, then $x \in P^{-1}R_2 = D_2$. We construct by induction $D_n \subset E$, $R_n \subset F$, $n=1,2,\ldots$, such that $TD_n = R_{n+1}$ and $D_{n+1} = P^{-1}R_{n+1}$. We have immediately $D_1 \supset D_2 \supset \ldots$, $R_1 \supset R_2 \supset \ldots$. Let $E_1 = \bigcap_{n=1}^{\infty} D_n$ and $F_1 = \bigcap_{n=1}^{\infty} R_n$.

By construction, if $x \in N(T+\lambda P)$, $\lambda \neq 0$, then $x \in D_n$, $n=1,2,\ldots$, thus $N(T+\lambda P) \subset E_1$, $\forall\ \lambda \neq 0$.

Moreover $TE_1 \subset F_1$, $PE_1 \subset F_1$ by definition of E_1, F_1.

We now prove that E_1 and F_1 are closed, by showing inductively that D_n, R_n are closed, $n=1,2,\ldots$. First, R_1 is closed by assumption, and D_1 is closed because P is continuous. If D_n is closed, then $D_n+N(T)$ is closed as $\operatorname{nul}(T) < \infty$. Since T is open, T maps the set-theoretic complement (in E) of $D_n+N(T)$ into an open set A of $R(T)$. But $TD_n = T(D_n+N(T))$ is obviously the set-theoretic complement of A in $R(T)$. Thus TD_n is closed in $R(T)$, hence in F. As a result, $D_{n+1} = P^{-1}R_{n+1}$ is also closed in E.

It follows in particular that if E or $R(T)$ is complete, then E_1 or F_1 is complete.

Obviously $G(T_1) = G(T) \cap (E_1 x F_1)$ is closed in $E_1 x F_1$. We now show that T_1 is open. Let U_1 be a neighborhood in E_1. Let $E_1 + N(T) = E_1 \oplus L$, with $L \subset N(T)$ and $\dim L < \infty$. Since E_1 is closed, the decomposition $E_1 \oplus L$ is also topological; it follows that $U_1 + N(T) = U_1 + (N(T) \cap E_1) + L$ is a neighborhood in $E_1 + N(T)$, the restriction of a neighborhood U in E. Now TU is a neighborhood in $R(T)$. We show that $T_1 U_1 = TU \cap R(T_1)$. Indeed, let $y = T_1 x_1 = Tx$ with $x_1 \in E_1$ and $x \in U$. Then $x = x_1 + (x - x_1)$ belongs to $U \cap (E_1 + N(T)) = U_1 + N(T)$. Thus $y = Tx \in TU_1 = T_1 U_1$. Consequently $TU \cap R(T_1) \subset T_1 U_1$. (The converse inclusion is trivial and not needed.) As a result, $T_1 U_1$ is a neighborhood in $R(T_1)$ (cf. also Lemma I.6).

We now prove that T_1 is surjective. Let $y \in F_1$. For each integer $n \geq 1$, there is $x_n \in D_n$ such that $Tx_n = y$. Since $\mathrm{nul}(T) < \infty$ and $D_n \supset D_{n+1}$, it follows that, starting from some rank k,

$$N(T) \cap D_n = N(T) \cap D_k, \qquad \forall n \geq k.$$

By definition of the x_n's, we have $x_n - x_k \in N(T) \cap D_k = N(T) \cap D_n$, $\forall n \geq k$. Thus $x_k \in D_n$, $\forall n \geq k$; consequently $x_k \in E_1$ and $Tx_k = y \in R(T_1)$. To sum up, T_1 is a surjective Φ_+-operator.

We now remark that if PU is bounded in F for some neighborhood U in E, then $P(U \cap E_1)$ is bounded in F_1, therefore P_1 is a bounded operator. Our proof is complete, in view of the opening observation ./.

REMARK. If E is a Banach space, F a normed space, T a Φ_--operator and P a bounded operator, then we may apply the preceding result to T^+, P^+ in the duals. We then infer by duality that def$(T+\lambda P)$ is a constant for $|\lambda| \neq 0$ small enough. This completes the proof of Theorem I.1 (C).

We have been dealing with $T+\lambda P$ for $|\lambda|$ small enough. With a (more general) condition "à la Kato", we may get some information on nul$(T+P)$ and codim $(R(T+P))^-$ as follows.

PROPOSITION II.6. Let E, F ~~be~~ locally convex spaces and T, P operators from E into F such that $D(T) \subset D(P)$.

Assume that there exist a neighborhood U in E, a neighborhood V in F and $0 < \varepsilon < 1$ such that PU is bounded,

$$V \cap R(T) \subset (TU)^- \quad \text{and} \quad PU \subset \varepsilon V.$$

(a) If codim $(R(T))^- < \infty$, then

$$\operatorname{codim}\,(R(T+P))^- \leq \operatorname{codim}\,(R(T))^-,$$

and there exists $\varrho > 0$ such that codim $(R(T+\lambda P))^-$ is a constant for all λ in the annulus $0 < |\lambda| < \varrho$.

(b) If T is weakly open, has a closed graph and nul$(T) < \infty$, then nul$(T+P) \leq$ nul(T). In this case, if codim $(R(T))^- = \infty$ then codim $(R(T+P))^- = \infty$; if codim $(R(T))^- < \infty$ then, in addition to (a),

$$\operatorname{nul}(T+P) - \operatorname{codim}\,(R(T+P))^- \leq \operatorname{nul}(T) - \operatorname{codim}\,(R(T))^-,$$

and the equality holds if and only if T+P is weakly open.

Proof. As usual, we may assume that $E = D(T) = D(T+P)$. From $V \cap R(T) \subset (TU)^-$, where V may be assumed open, we infer that

$(V)^- \cap (R(T))^- \subset (TU)^-$. By duality we get $T^{+-1}U^\circ \subset V^\circ + N(T^+)$, that is $U^\circ \cap R(T^+) \subset T^+V^\circ$.

From $PU \subset \varepsilon V$ we obtain by duality $P^+V^\circ \subset \varepsilon U^\circ$. Since PU is bounded (thus P is continuous), we have $D(P^+) = F^+$ and $R(P^+) \subset \rangle U^\circ \langle$.

Moreover, $G(T^+)$ is weakly closed in $F^+ x E^+$, and U°, V° are weakly bounded Banach disks, thus $G(T^+) \cap (F^+_{V^\circ} \times E^+_{U^\circ})$ is closed in $F^+_{V^\circ} \times E^+_{U^\circ}$, if $E^+_{U^\circ}$ (resp. $F^+_{V^\circ}$) denotes the Banach space generated by U° (resp. V°).

In case (a), it follows from codim $(R(T))^- < \infty$ that $\mathrm{nul}(T^+) < \infty$. Theorem I.16 shows that $\mathrm{nul}(T^+ + P^+) \leq \mathrm{nul}(T^+)$ and that there exists $\rho > 0$ such that $N(T^+ + \lambda P^+)$ is constant for $0 < |\lambda| < \rho$. Since $(T+P)^+ = T^+ + P^+$, by duality we get codim $(R(T+P))^- \leq$ codim $(R(T))^-$, and codim $(R(T+\lambda P))^-$ is constant for $0 < |\lambda| < \rho$.

In case (b), since T has a closed graph and is weakly open, $R(T^+)$ is weakly closed ($R(T^+) = N(T)^\circ$, by Lemmas 0.8 and 0.12). Moreover, $\mathrm{def}(T^+) = \mathrm{nul}(T) < \infty$. Theorem I.16 shows that $\mathrm{def}(T^+ + P^+) \leq \mathrm{def}(T^+)$ and $\mathrm{ind}(T^+ + P^+) = \mathrm{ind}(T^+)$.

Consequently, with respect to the weak topology,

$$\mathrm{codim}\,(R(T^+ + P^+))^- \leq \mathrm{codim}\,R(T^+ + P^+) \leq \mathrm{codim}\,R(T^+)$$
$$\leq \mathrm{codim}\,(R(T^+))^-,$$

as $R(T^+) = (R(T^+))^-$. By duality, $\mathrm{nul}(T+P) \leq \mathrm{nul}(T)$.

If codim $(R(T))^- = \infty$, then $\mathrm{nul}(T^+) = \infty$. It follows from $\mathrm{ind}(T^+ + P^+) = \mathrm{ind}(T^+)$ that $\mathrm{nul}(T^+ + P^+) = \infty$, hence

codim $\quad ^- = \infty$

If $\operatorname{codim}\,(R(T))^{-} < \infty$, then $\operatorname{nul}(T^{+}) < \infty$ and

$$\begin{aligned}\operatorname{codim}\,(R(T^{+}+P^{+}))^{-} - \operatorname{nul}(T^{+}+P^{+}) &\overset{(*)}{\leq} -\operatorname{ind}(T^{+}+P^{+})\\ &\leq -\operatorname{ind}(T^{+})\\ &\leq \operatorname{codim}\,(R(T^{+}))^{-} - \operatorname{nul}(T^{+}).\end{aligned}$$

By duality, $\operatorname{nul}(T+P) - \operatorname{codim}\,(R(T+P))^{-} \leq \operatorname{nul}(T) - \operatorname{codim}\,(R(T))^{-}$. We can see that the equality holds if and only if we have equality at $(*)$, that is if and only if $\operatorname{codim}\,(R(T^{+}+P^{+}))^{-} = \operatorname{codim} R(T^{+}+P^{+})$. Since $\operatorname{codim} R(T^{+}+P^{+}) < \infty$, this is equivalent to saying that $(R(T^{+}+P^{+}))^{-} = R(T^{+}+P^{+})$ or $R(T^{+}+P^{+})$ is weakly closed. By Lemma 0.12, $R(T^{+}+P^{+})$ is weakly closed if and only if T+P is weakly open (T+P has a closed graph) ./.

II.1.2. <u>Banach-bounded perturbations. Weakly compact perturbations</u>

In Theorem II.4, assumption (c) that either E or R(T) is complete, is designed to ensure that R(T+P) is closed. There is an alternative assumption to the same effect, which is an immediate application of Proposition I.15.

DEFINITION. Let P be an operator between two locally convex spaces E, F. We say that P is <u>Banach-bounded</u> if $PU \subset B$ for some neighborhood U in E and a bounded Banach disk B in F.

In particular, if P is a bounded operator and F is sequentially complete, then P is a Banach-bounded operator. Indeed, if $PU \subset B$ where B is a bounded disk, then $(B)^{-}$ is sequentially complete, thus a Banach disk by Lemma 0.4.

Weakly compact operators are also Banach-bounded operators.

Theorems II.3, II.4 and Proposition I.15 yield immediately the following

THEOREM II.7. Let E, F be locally convex spaces, T an injective ϕ_+-operator and P a Banach-bounded operator from E into F such that $D(T) \subset D(P)$.

Assume that there exist a neighborhood U in E, a neighborhood V in F and $0 < \varepsilon < 1$ such that PU is bounded,

$$V \cap R(T) \subset TU \quad \text{and} \quad PU \subset \varepsilon V .$$

Then T+P is an injective ϕ_+-operator and $\text{ind}(T+P) = \text{ind}(T)$. In particular $\text{def}(T+P) = \text{def}(T)$.

REMARK. Under the assumptions of Theorem II.7, there exist in fact neighborhoods U in E, V in F, a bounded Banach disk $B \subset F$ and $0 < \varepsilon < 1$ such that $V \cap R(T) \subset TU$, $PU \subset B \subset \varepsilon V$.

Indeed assume that $PU_0 \subset K$, $V \cap R(T) \subset TU$, $PU \subset \varepsilon V$, PU is bounded and $0 < \varepsilon < 1$, for neighborhoods U, U_0 in E, V in F, and a bounded Banach disk $K \subset F$. Then, in view of Lemma II.1, we may assume $U \subset U_0$ and V to be closed. Let $B = \bigcap_{\varepsilon > 0} (1+\varepsilon)K \cap (PU)^-$. Obviously B is a bounded disk. It is a Banach disk because B is closed in the Banach space F_K ($(PU)^-$ is also closed in F_K, and $\bigcap_{\varepsilon > 0} (1+\varepsilon)K$ is the closure of K in F_K) and B generates a topology finer than that of F_K.

Since $(PU)^- \subset \varepsilon V$ and $PU \subset PU_0 \subset K$, we have then

$$V \cap R(T) \subset TU \quad , \quad PU \subset B \subset \varepsilon V \quad .$$

THEOREM II.8. Let E, F be locally convex spaces, T a ϕ_+-operator and P a Banach-bounded operator from E into F such that $D(T) \subset D(P)$.

<u>Then there exists</u> $\rho > 0$ <u>such that for</u> $|\lambda| < \rho$, $T+\lambda P$ <u>is</u> <u>a</u> ϕ_+<u>-operator</u>, $\mathrm{nul}(T+\lambda P) \leq \mathrm{nul}(T)$, $\mathrm{def}(T+\lambda P) \leq \mathrm{def}(T)$ <u>and</u>

$$\mathrm{ind}(T+\lambda P) = \mathrm{ind}(T).$$

<u>We may choose</u> $\rho > 0$ <u>such that</u> $\mathrm{nul}(T+\lambda P)$ <u>is constant for</u> $0 < |\lambda| < \rho$.

<u>The theorem holds in particular if</u> P <u>is a weakly compact operator</u>.

<u>Proof</u>. It remains to prove only that $\mathrm{nul}(T+\lambda P)$ is constant for $0 < |\lambda| < \rho$. But a close examination of the proof shows that Theorem II.5 also holds true under the assumptions of Theorem II.8. Indeed, let $PU \subset B$, where U is a neighborhood in E and B a bounded Banach disk. We may assume B to be closed in the Banach space F_B, otherwise we take $\bigcap\limits_{\varepsilon>0} (1+\varepsilon)B$. Then, with the notations of the proof of Theorem II.5, $B \cap F_1$ is a bounded Banach disk (F_1 is closed), and $P_1(U \cap E_1) \subset B \cap F_1$. Thus P_1 is a Banach-bounded operator, and the proof of Theorem II.5 is carried through ./.

II.1.3. <u>Isomorphism of the ranges</u>

If T is an injective open operator, and P a bounded operator small enough, it is easy to infer from Theorem II.2 that R(T) is isomorphic to R(T+P) :

PROPOSITION II.9. <u>Under the assumptions of Theorem</u> II.2 (<u>the first part), there is an isomorphism from</u> R(T) <u>onto</u> R(T+P).

<u>Proof</u>. Let I denote the canonical injection of R(T) into E,

There is a neighborhood U in E, a neighborhood V in F and $0 < \varepsilon < 1$ such that PU is bounded, $V \cap R(T) \subset TU$ and $PU \subset \varepsilon V$.

We remark that $R(T) = R(I)$ and $T^{-1}V \subset U$. Therefore $V \cap R(I) = IV$ and $PT^{-1}V \subset \varepsilon V$, where $PT^{-1}V \subset PU$ is bounded. Theorem II.2 shows that $I+PT^{-1}$ is injective and open; moreover it is continuous, therefore it is an isomorphism from $R(T)$ onto $R(I+PT^{-1})$. But it is immediate from $T+P = (I+PT^{-1})T$ that $R(I+PT^{-1}) = R(T+P)$ ./.

For a non injective ϕ_+-operator T, we have :

PROPOSITION II.10. <u>Let</u> E, F <u>be locally convex spaces</u>, T <u>a</u> ϕ_+-<u>operator and</u> P <u>a bounded operator from</u> E <u>into</u> F <u>such that</u> $(D(T))^- \subset D(P)$.

<u>Assume that either</u> E <u>or</u> $R(T)$ <u>is complete, or</u> P <u>is a Banach-bounded operator. Then there exists</u> $\rho > 0$ <u>such that for any</u> $0 < |\lambda_1| \leq |\lambda_2| < \rho$, <u>there is an isomorphism from</u> $R(T+\lambda_1 P)$ <u>onto</u> $R(T+\lambda_2 P)$ (<u>and</u> $T+\lambda_1 P$, $T+\lambda_2 P$ <u>are</u> ϕ_+-<u>operators</u>).

<u>Proof</u>. Let L be a closed subspace such that $E = L \oplus N(T)$ algebraically and topologically. Let T', P' denote the restrictions of T, P to L. In view of Theorems II.4, II.5, II.8 and Proposition II.9, there exists $\rho > 0$ such that, for $0 < |\lambda| < \rho$, $T'+\lambda P'$ is an injective ϕ_+-operator, $R(T'+\lambda P')$ is isomorphic to $R(T') = R(T)$ and $\mathrm{nul}(T+\lambda P)$ is constant. In particular, if λ_1 and λ_2 are such that $0 < |\lambda_1| \leq |\lambda_2| < \rho$, then $R(T'+\lambda_1 P')$, $R(T'+\lambda_2 P')$ are closed and there is an isomorphism from $R(T'+\lambda_1 P')$ onto $R(T'+\lambda_2 P')$, since both are isomorphic to $R(T)$. It suffices now

at

<u>Then there exist a base $\mathcal{U}$ of neighborhoods in</u> E <u>such that to any</u> $U \in \mathcal{U}$, <u>there corresponds a bounded disk</u> $B \subset E$ <u>such that</u> $B \subset \varepsilon U$ <u>and</u> $PU \cap (R(T))^- \subset (TB)^- + A$.

<u>Proof</u>. It suffices to prove that for any neighborhood U' in E, there is a neighborhood $U \subset U'$ which has the property in the statement.

Since B_0 is bounded, there is $\lambda > 0$ such that $\lambda B_0 \subset \varepsilon U'$. Let $U = U' \cap \lambda U_0$ and $B = \lambda B_0$. Then $B \subset (\varepsilon U') \cap (\lambda \varepsilon U_0) \subset \varepsilon(U' \cap \lambda U_0) \subset \varepsilon U$. Moreover,

$$\begin{aligned} PU \cap (R(T))^- &\subset \lambda PU_0 \cap (R(T))^- \\ &\subset \lambda(TB_0)^- + \lambda A \\ &\subset (T(\lambda B_0))^- + \lambda A \\ &\subset (TB)^- + A \quad , \end{aligned}$$

if we choose $\lambda \leqslant 1$ ./.

II.2.1. <u>An extension of Kato's theorem</u>

In Chapter I, Theorem I.19, we already obtained an extension of Kato's theorem on small perturbations of ϕ_--operators. The perturbations considered there are of the type "small when restricted to suitably normed subspaces". We now prove another possible extension, where the perturbations are of the type $PU \cap R(T) \subset TB$ and $B \subset \varepsilon U$, $0 < \varepsilon < 1$.

The main tool is duality. We apply the results developed in Chapter I, notably Theorem I.16, to obtain the stability of the index. The suitable Banach spaces are readily provided by [illegible] in the duals. An [illegible] of

$R(T+\lambda_2 P) = R(T'+\lambda_2 P') \oplus M_2$ (algebraically and topologically) with $\dim M_1 = \dim M_2 < \infty$. We know that $\dim N(T+\lambda_1 P) = \dim N(T+\lambda_2 P)$ and $N(T+\lambda_1 P) \cap L = N(T+\lambda_2 P) \cap L = \{0\}$. As a result, there exist finite dimensional subspaces L_1, L_2 such that $L \oplus N(T+\lambda_1 P) \oplus L_1 = L \oplus N(T+\lambda_2 P) \oplus L_2 = E$. It is obvious from $\dim N(T+\lambda_1 P) = \dim N(T+\lambda_2 P)$ that $\dim L_1 = \dim L_2$. Take $M_1 = (T+\lambda_1 P)L_1$ and $M_2 = (T+\lambda_2 P)L_2$. Then $T+\lambda_1 P$ (resp. $T+\lambda_2 P$) is one-to-one on $L \oplus L_1$ (resp. $L \oplus L_2$), thus $M_1 \cap R(T'+\lambda_1 P') = M_2 \cap R(T'+\lambda_2 P') = \{0\}$ and $\dim M_2 = \dim M_1$ $(= \dim L_1 = \dim L_2)$ ./.

II.2. Bounded perturbations of Φ_--operators

Part (a) of Proposition II.6 already gives an indication pertaining to def(T+P) if def(T) $< \infty$. Because we do not have Lemma II.1 for Φ_--operators, we cannot prove the stability of the topological characteristics with the mere assumption that P be a bounded operator. We need to assume further the existence of a suitable bounded disk in the domain space. This additional hypothesis intervenes via the following lemma, which plays a role similar to that of Lemma II.1.

LEMMA II.11. Let E, F be locally convex spaces and T, P operators from E into F such that $D(T) \subset D(P)$.

Assume that there exist a neighborhood U_0 in E, a bounded disk $B_0 \subset E$, a disk $A \subset F$ and $0 < \varepsilon < 1$ such that $B_0 \subset \varepsilon U_0$ and

$$PU_0 \cap (R(T))^- \subset (TB_0)^- + A \quad .$$

Kato's theorem in these Banach spaces, combined with duality, yields the stability of "almost openness". If the domain space is fully complete, we infer that the perturbed operator remains a ϕ_--operator with preservation of the index.

The result which we obtain fully renders Kato's theorem (for ϕ_--operators) in the case of Banach spaces.

THEOREM II.12. *Let* E, F *be locally convex spaces and* T, P *operators from* E *into* F *such that* $D(T) \subset D(P)$.

Assume that

(a) T *is almost open with* codim $(R(T))^- < \infty$ *and* P *is continuous*,

(b) *There exist a base* $\mathcal{U}$ *of neighborhoods in* E *such that for each* $U \in \mathcal{U}$, *there is* $0 < \varepsilon < 1$ *for which*

$$PU \cap (R(T))^- \subset \varepsilon (TU)^- ,$$

(c) *There exist a neighborhood* U_0 *in* E *and bounded disks* $B \subset E$, $B' \subset F$ *such that*

$$PU_0 \subset (TB)^- + B' .$$

Then T+P *is almost open and* codim $(R(T+P))^- \leq$ codim $(R(T))^-$. *There is* $\rho > 0$ *such that* codim $(R(T+\lambda P))^-$ *is constant for* $0 < |\lambda| < \rho$.

If T *and* T+P *are* ϕ_-*-operators, which is the case in particular if* T *has a closed graph*, $(D(T))^- \subset D(P)$ *and* E *is fully complete, then* ind(T+P) = ind(T).

Proof. We may assume without loss of generality that $E = D(T) = D(T+P)$, $F = (R(T))^- + R(P)$ and $U \subset U_0$ if $U \in \mathcal{U}$.

We prove first a lemma.

LEMMA II.13. <u>Let</u> $\mathcal{W}$ (<u>resp.</u> $\mathcal{U}$) <u>denote a base of neighborhoods in</u> $(R(T))^-$ (<u>resp.</u> E). <u>Then</u> $\{\langle W \cup PU \rangle : W \in \mathcal{W}, U \in \mathcal{U}\}$ <u>form a base of neighborhoods in</u> F.

Indeed, $\langle W \cup PU \rangle$ is absorbent in F because of our assumption $(R(T))^- + R(P) = F$. Moreover, given $\langle W_1 \cup PU_1 \rangle$ and $\langle W_2 \cup PU_2 \rangle$ we can find $U_3 \in \mathcal{U}$, $W_3 \in \mathcal{W}$ such that $U_3 \subset U_1 \cap U_2$ and $W_3 \subset W_1 \cap W_2$. Thus $W_3 \cup PU_3 \subset \langle W_1 \cup PU_1 \rangle \cap \langle W_2 \cup PU_2 \rangle$, from which it follows $\langle W_3 \cup PU_3 \rangle \subset \langle W_1 \cup PU_1 \rangle \cap \langle W_2 \cup PU_2 \rangle$. Since all neighborhoods involved are absolutely convex, it is clear that $\{\langle W \cup PU \rangle : W \in \mathcal{W}, U \in \mathcal{U}\}$ define a locally convex topology t on F.

Let V be an arbitrary neighborhood in F, and $W = V \cap (R(T))^-$. Since P is continuous, there is a neighborhood U in E such that $PU \subset V$. Thus $\langle W \cup PU \rangle \subseteq V$, which shows that t is finer than the topology of F. The construction of t indicates that t coincides with the topology induced by F on $(R(T))^-$. Also $(R(T))^-$ is closed for the (a priori finer) topology t.

Let N be a finite dimensional subspace such that $N \oplus (R(T))^- = F$ algebraically and topologically. The topology of F and t also coincide on N, being both Hausdorff. As a result, t is equal to the topology of F.

We now return to the proof of Theorem II.12. Since $(TU)^-$ is a neighborhood in $(R(T))^-$, $V = \langle (TU)^- \cup (\varepsilon^{-1} PU) \rangle$ is a neighborhood in F, by virtue of Lemma II.13.

$\varepsilon D^{\circ} \cap R(T)^{\circ} = \varepsilon D^{\circ} \cap N(T^{+}) \subset V^{\circ}$, and D° is absorbent in F^{+}).

Let $L = F^{+}_{V^{\circ}}$ (resp. $M = E^{+}_{U^{\circ}}$) be the Banach space generated by V° (resp. U°). Let T', P' denote the restrictions of T^{+}, P^{+} to L with range space M. In view of the relations (1), (2) we have $U^{\circ} \cap R(T^{+}) = U^{\circ} \cap R(T') = T'V^{\circ}$ and $P'V^{\circ} \subset \varepsilon U^{\circ}$.

The operator T' has a closed graph in $L \times M$ ($G(T^{+})$ is weakly closed in $F^{+} \times E^{+}$), and $\mathrm{nul}(T') = \mathrm{nul}(T^{+}) = \mathrm{codim}\ (R(T))^{-} < \infty$. Thus T' is a Φ_{+}-operator. We remark that $D(P^{+}) = F^{+}$, P being continuous; in particular $D(P') = L$. Kato's theorem applies to show that $T'+P'$ is a Φ_{+}-operator, $\mathrm{nul}(T'+P') \leq \mathrm{nul}(T')$, $\mathrm{def}(T'+P') \leq \mathrm{def}(T')$ and $\mathrm{ind}(T'+P') = \mathrm{ind}(T')$.

Assumption (c) implies $PU \subset (TB)^{-} + B'$, which in turn implies by duality $B'^{\circ} \cap T^{+-1}B^{\circ} \subset B'^{\circ} \cap (TB)^{\circ} \subset 2P^{+-1}U^{\circ}$. Since B'° is absorbent in F^{+} and B° in E^{+}, this means that $P^{+}(D(T^{+})) \subset M$.

Theorem I.16 (cf. also the subsequent remark) applies to show that $\mathrm{nul}(T^{+}+P^{+}) \leq \mathrm{nul}(T^{+})$, $\mathrm{def}(T^{+}+P^{+}) \leq \mathrm{def}(T^{+})$ and $\mathrm{ind}(T^{+}+P^{+}) = \mathrm{ind}(T^{+})$. Furthermore, there is $\rho > 0$ such that $\mathrm{nul}(T^{+} + \lambda P^{+})$ is constant for $0 < |\lambda| < \rho$. As $T^{+}+P^{+} = (T+P)^{+}$, we obtain by duality that $\mathrm{codim}\ (R(T+P))^{-} \leq \mathrm{codim}\ (R(T))^{-}$ and $\mathrm{codim}\ (R(T+\lambda P))^{-}$ is constant for $0 < |\lambda| < \rho$ (this is in fact already proved in Proposition II.6 (a)). If T and T+P are Φ_{-}-operators, then $R(T^{+}) = N(T)^{\circ}$, $R(T) = N(T^{+})^{\circ}$, $R(T^{+}+P^{+}) = N(T+P)^{\circ}$ and $R(T+P) = N(T^{+}+P^{+})^{\circ}$, thus $\mathrm{ind}(T^{+}+P^{+}) = \mathrm{ind}(T^{+})$ yields $\mathrm{ind}(T+P) = \mathrm{ind}(T)$.

It remains only to prove that T+P is almost open.

We remark, as in the proof of Theorem I.9, that

$R(T^{+}+P^{+}) \cap M = R(T'+P')$. Indeed, if $f \in R(T^{+}+P^{+}) \cap M$ and $f=(T^{+}+P^{+})g$, then $T^{+}g = f-P^{+}g \in M \cap R(T^{+})$ (recall that $P^{+}(D(T^{+})) \subset M$). But $M \cap R(T^{+}) = R(T')$ because $U^{\circ} \cap R(T^{+}) = T'V^{\circ}$. Thus $g \in L+N(T^{+}) = L$. As a result $f \in R(T'+P')$ (similarly, we also have $N(T^{+}+\lambda P^{+}) = N(T'+\lambda P')$, $\forall \lambda$).

From the fact that $T'+P'$ is open, there is $\mu > 0$ such that

$$\mu U^{\circ} \cap R(T^{+}+P^{+}) = \mu U^{\circ} \cap R(T'+P') \subset (T'+P')V^{\circ}.$$

As a result, $\mu(T^{+}+P^{+})^{-1}U^{\circ} \subset V^{\circ} + N(T^{+}+P^{+})$.

As $(T+P)^{+} = T^{+}+P^{+}$, we have by duality

$$((T+P)U)^{-} \supset \mu V \cap R(T+P) .$$

Since U is arbitrary in the base $\mathcal{U}$ of neighborhoods in E, this shows that $T+P$ is almost open ./.

As a consequence of Theorem II.12, we have the following extension of Kato's theorem for Φ_{-}-operators

THEOREM II.14. Let E, F be locally convex spaces and T, P operators from E into F such that $D(T) \subset D(P)$.

Assume that

(a) T is almost open with codim $(R(T))^{-} < \infty$, and P is a bounded operator,

(b) There exist a neighborhood U_{0} in E, a bounded disk $B_{0} \subset E$ and $0 < \varepsilon < 1$ such that

$$B_{0} \subset \varepsilon U_{0} \quad \text{and} \quad PU_{0} \cap (R(T))^{-} \subset (TB_{0})^{-} .$$

Then $T+P$ is almost open and

$$\text{codim } (R(T+P))^{-} \leq \text{codim } (R(T))^{-} .$$

There is $\rho > 0$ *such that* codim $(R(T+\lambda P))^-$ *is a constant for* $0 < |\lambda| < \rho$.

If T *and* $T+P$ *are* ϕ_-*-operators, which is the case if* T *has a closed graph,* $(D(T))^- \subset D(P)$ *and* E *is a Frechet space, then*

$$\text{ind}(T+P) = \text{ind}(T).$$

Proof. In view of Lemma II.11, there exist a base $\mathcal{U}$ of neighborhoods in E such that to $U \in \mathcal{U}$, there corresponds a bounded disk $B \subset E$ with $B \subset \varepsilon U$ and $PU \cap (R(T))^- \subset (TB)^- \subset \varepsilon(TU)^-$. Moreover, $PU \subset B'$ for some neighborhood U in E and a bounded disk $B' \subset F$. We may therefore apply Theorem II.12 ./.

If $(R(T))^- = F$, we have a more precise relationship between $((T+P)U)^-$ and $(TU)^-$:

PROPOSITION II.15. *Let* E, F *be locally convex spaces and* T, P *operators from* E *into* F *such that* $D(T) \subset D(P)$.

Let U *be a neighborhood in* E. *Assume that*

(a) $(TU)^-$ *is a neighborhood in* F,

(b) $PU \subset \varepsilon(TU)^-$, $\quad 0 < \varepsilon < 1$,

(c) P *is continuous and* $PU \subset (TB)^- + B'$, *for some bounded disks* $B \subset E$, $B' \subset F$.

Then $(1-\varepsilon)(TU)^- \subset ((T+P)U)^-$.

Proof. We may assume that $E = D(T) = D(T+P)$. Let $V=(TU)^-$, then as in the proof of Theorem II.12, by duality,

$$U^\circ \cap R(T^+) = T^+V^\circ \quad \text{and} \quad P^+V^\circ \subset \varepsilon U^\circ.$$

Let L (resp. M) be the Banach space generated by V° (resp. U°) and T', P' the restrictions of T^+, P^+ to L with range M. Th T'

<u>If</u> T <u>and</u> T+P <u>are</u> ϕ_-<u>-operators, which is the case in particular if</u> T <u>has a closed graph</u>, $(D(T))^- \subset D(P)$ <u>and</u> E <u>is fully complete, then</u> $\operatorname{ind}(T+P) = \operatorname{ind}(T)$.

<u>Proof</u>. We may assume that $E = D(T) = D(T+P)$, $F = (R(T))^- + R(P)$ and $U \subset U_0$.

In view of Lemma II.13, $V = \langle (TU)^- \cup \varepsilon^{-1} PU \rangle + N$ is a neighborhood in F. Since $PU \cap (R(T))^- \subset \varepsilon(TU)^- + N$, we have

$$\begin{aligned} V \cap (R(T))^- &\subset \langle (TU)^- \cup \varepsilon^{-1} PU \rangle \cap (R(T))^- + N \\ &\subset \langle (TU)^- \cup (\varepsilon^{-1} PU \cap (R(T))^-) \rangle + N \\ &\subset (TU)^- + N. \end{aligned}$$

On the other hand, $(TU)^- + N \subset V \cap (R(T))^-$, by definition of V (and the assumption $N \subset (R(T))^-$). Consequently,

$$V \cap (R(T))^- = (TU)^- + N \text{ and } (V)^- \cap (R(T))^- = ((TU)^- + N)^-.$$

By duality we get

(1) $$(T^{+-1} U^\circ) \cap N^\circ = V^\circ + N(T^+).$$

On the other hand, $PU \subset \varepsilon V$ implies

(2) $$P^+ V^\circ \subset \varepsilon U^\circ.$$

We remark that N° is weakly closed in F^+, $\operatorname{codim} N^\circ = \dim N < \infty$ and $N(T^+) \subset N^\circ$ (as $N \subset (R(T))^-$). We may also assume, as in the proof of Theorem II.12, that $N(T^+) \subset \rangle V^\circ \langle$, P being continuous.

Let L (resp. M) be the Banach space generated by V° (resp. U°). We notice that $L \subset N^\circ$. Let T', P' be the restrictions of T^+; P^+ to L with range space M.

We prove first that

(3) $$U^\circ \cap R(T') = T' V^\circ.$$

The inclusion $T'V^\circ \subset U^\circ \cap R(T')$ is trivial, from the relation (1). Conversely, let $f \in U^\circ \cap R(T')$. Then $f = T'g$ with $g \in (T^{+-1}U^\circ) \cap L \subset (T^{+-1}U^\circ) \cap N^\circ$. Consequently, $g \in V^\circ + N(T^+)$, in view of the relation (1); hence $f \in T'V^\circ$.

On the other hand, $P'V^\circ \subset \varepsilon U^\circ$.

As $\dim N(T') = \dim N(T^+) = \operatorname{codim} (R(T))^- < \infty$, and T' has a closed graph, Kato's theorem applies and shows that $T'+P'$ is a ϕ_+-operator, $\operatorname{nul}(T'+P') \leq \operatorname{nul}(T')$, and $\operatorname{ind}(T'+P') = \operatorname{ind}(T')$.

We cannot yet conclude as we did in the proof of Theorem II.12, because a priori $M \cap R(T^+) \neq R(T')$ and $P^+(D(T^+)) \not\subset M$.

We remark however that $PU \subset (TB)^- + B' + \hat{N}_0$ implies $B'^\circ \cap T^{+-1}B^\circ \cap N_0^\circ \subset 2P^{+-1}U^\circ$, where B'° is absorbent in F^+, B° in E^+ and $\operatorname{codim} N_0^\circ < \infty$. This means that $P^+(D(T^+) \cap N_0^\circ) \subset M$. There is as a result a finite dimensional subspace $M_1 \subset E^+$ such that $P^+(D(T^+)) \subset M+M_1$. We may assume that $M_1 \cap M = \{0\}$, and M_1 is generated by a finite disk D.

The relation (1) shows that L has a finite codimension in $T^{+-1}M$, since $\operatorname{codim} N^\circ < \infty$. On the other hand, M has a finite codimension in $M+M_1$. Thus L has a finite codimension in $T^{+-1}(M+M_1)$. As a result, there is a finite dimensional subspace $L_1 \subset F^+$, generated by a finite disk D', such that $L \cap L_1 = \{0\}$ and $L+L_1 = T^{+-1}(M+M_1)$.

Let $L_2 = L+L_1$ (resp. $M_2 = M+M_1$) be the Banach space generated by $V^\circ+D'$ (resp. $U^\circ+D$). Let T'', P'' denote the restrictions of T^+, P^+ to L_2 with range space M_2. Then T'' (resp. P'') is a finite dimensional extension of T' (resp. P'), if we regard

T', P' as operators from L_2 into M_2. With domain space L_2 and range space M_2, it is clear that T', $T'+P'$ are Φ_+-operators (M is closed in the Banach space M_2). By virtue of Lemmas I.10 and I.11, T'', $T''+P''$ are Φ_+-operators, $\mathrm{ind}(T''+P'') = \mathrm{ind}(T'')$, and

$$\mathrm{nul}(T''+P'') \leq \mathrm{nul}(T'+P') + \dim L_1 < \infty .$$

Furthermore, as P'' is obviously a bounded operator, there is $\rho > 0$ such that $\mathrm{nul}(T''+ \lambda P'')$ is constant for $0 < |\lambda| < \rho$.

From $L_2 = T^{+-1}M_2$, we deduce that

$$M_2 \cap R(T^+) = R(T'').$$

Moreover $P^+(D(T^+)) \subset M_2$. As in the proof of Theorem II.12, because $N(T^+) \subset L_2$, we have $R(T^+ + P^+) \cap M_2 = R(T''+P'')$, and $N(T^+ + \lambda P^+) = N(T''+ \lambda P'')$, $\forall \lambda$. By duality, $\mathrm{codim}\ (R(T+P))^- < \infty$ and $\mathrm{codim}\ (R(T+ \lambda P))^-$ is constant for $0 < |\lambda| < \rho$. Moreover, Theorem I.16 (or Theorem I.9) shows that $\mathrm{ind}(T^+ + P^+) = \mathrm{ind}(T^+)$. Consequently, if T and $T+P$ are Φ_--operators, then $\mathrm{ind}(T+P)=\mathrm{ind}(T)$.

It remains to prove that $T+P$ is almost open. As $T''+P''$ is open, there is $\mu > 0$ such that

$$\mu (U^\circ + D) \cap R(T^+ + P^+) = \mu (U^\circ + D) \cap R(T''+P'') \subset (T''+P'')(V^\circ + D').$$

A fortiori

$$\mu U^\circ \cap R(T^+ + P^+) \subset (T^+ + P^+)(V^\circ + D'),$$

hence

$$\mu (T^+ + P^+)^{-1} U^\circ \subset V^\circ + D' + N(T^+ + P^+).$$

Since $T^+ + P^+ = (T+P)^+$, by duality we have

$$2((T+P)U)^- \supset \mu(V \cap D'^\circ) \cap R(T+P).$$

As D'° is a (weak) neighborhood in F, and $U \in \mathcal{U}$ is arbitrary, this shows that T+P is almost open.

To obtain an upper bound of codim $(R(T+P))^-$, we may proceed as follows. First, in the assumption (c) we may assume that $N_0 \subset (R(T))^-$, since P is continuous and we may choose U_0 such that the projection of PU_0 on a complement of $(R(T))^-$ along $(R(T))^-$ is bounded.

We may replace N by $N+N_0$ in the main proof, that is we may assume that $N_0 \subset N$, thus $N^\circ \subset N_0^\circ$. This means that $P^+(D(T^+)\cap N^\circ) \subset P^+(D(T^+)\cap N_0^\circ) \subset M$. If $g \in N(T^+ + P^+)\cap N^\circ$, then $T^+ g = -P^+ g \in M \cap R(T^+)$. Therefore $g \in (T^{+-1}M)\cap N^\circ$. But the relation (1) shows that $(T^{+-1}M)\cap N^\circ = L$; hence $g \in L$ and $g \in N(T'+P')$. As a result, $N(T^+ + P^+)\cap N^\circ = N(T'+P')$. From this it follows that

$$\begin{aligned} \mathrm{nul}(T^+ + P^+) &\leq \mathrm{nul}(T'+P') + \mathrm{codim}\, N^\circ \\ &\leq \mathrm{nul}(T') + \mathrm{codim}\, N^\circ \\ &\leq \mathrm{nul}(T^+) + \mathrm{codim}\, N^\circ . \end{aligned}$$

By duality, and returning to the initial notations of the statement, we obtain

$$\mathrm{codim}\, (R(T+P))^- \leq \mathrm{codim}\, (R(T))^- + \dim\, (N+N_0) \ ./.$$

As a consequence of Theorem II.16, we have the following

THEOREM II.17. *Theorem* II.16 *holds if the assumption* (b) *is replaced by*

(b') *There exist a neighborhood* U *in* E, *a bounded disk* $B_0 \subset E$, *a precompact disk* $K \subset F$, *a finite dimensional subspace* $N \subset F$ *and* $0<\varepsilon<1$ *such that*

$$B_0 \subset \varepsilon U \ \text{and}\ PU \cap (R(T))^- \subset (TB_0)^- + K + N.$$

In particular, Theorem II.16 holds if both assumptions (b) and (c) are replaced by.

(b") There exist a neighborhood U in E, a bounded disk $B \subset E$, a precompact disk $K \subset F$, a finite dimensional subspace $N \subset F$ and $0 < \varepsilon < 1$ such that

$$B \subset \varepsilon U \text{ and } PU \subset (TB)^- + K + N.$$

If $PU \subset (TB)^- + N$, then $(R(T+P))^- + N = (R(T))^- + N$.

Proof. Let M be a finite dimensional subspace in F such that $F = (R(T))^- \oplus M$. If K' is the projection of K on $(R(T))^-$ along M, then K' is precompact and $K \subset K'+M$. Thus

$$PU \cap (R(T))^- \subset (TB_0)^- + K' + M + N .$$

We may decompose $M+N$ into N_1+N_2 with $N_1 \subset (R(T))^-$ and $N_2 \cap (R(T))^- = \{0\}$. Then

$$(*) \qquad PU \cap (R(T))^- \subset (TB_0)^- + K' + N_1 .$$

By virtue of Lemma II.11, we may assume that (*) holds for a base of neighborhoods U in E.

Fix any $\varepsilon' > 0$ such that $\varepsilon'' = \varepsilon + \varepsilon' < 1$. As $(TU)^-$ is a neighborhood in $(R(T))^-$, there is a finite dimensional subspace $N_3 \subset (R(T))^-$ such that $K' \subset \varepsilon'(TU)^- + N_3$. Consequently,

$$PU \cap (R(T))^- \subset \varepsilon(TU)^- + \varepsilon'(TU)^- + N_3 + N_1$$
$$\subset \varepsilon''(TU)^- + N_3 + N_1 ,$$

hence the assumption (b) in Theorem II.16 is satisfied.

If $PU \subset (TB)^- + N$, to prove that $(R(T+P))^- + N = (R(T))^- + N$, we may use duality, or proceed as follows. We may assume that $F = (R(T))^- + N$. Let $S : F \to F/N$ be the canonical quotient operator. If U is a neighborhood in E then $(TU)^-$ is a neighborhood in

$(R(T))^-$. Let $N = (N \cap (R(T))^-) \oplus N'$. Then $(TU)^- + (N \cap (R(T))^-)$ is a neighborhood in $(R(T))^-$, thus $(TU)^- + N$ is a neighborhood in F, and $S((TU)^- + N) = S((TU)^-)$ is a neighborhood in F/N. Since S is continuous, $S((TU)^-) \subset (STU)^-$, therefore $(STU)^-$ is a neighborhood in F/N. We now prove that $(R(ST))^- = F/N$. We know that R(T) is dense in $(R(T))^-$, therefore $R(ST) = S(R(T))$ is dense in $F/N = S((R(T))^-)$. This shows that $(STU)^-$ is a neighborhood in $(R(ST))^-$, hence ST is almost open. Moreover, codim $(R(ST))^- = 0$, SP is continuous and $SPU \subset S((TB)^-) \subset (STB)^- \subset (STU)^-$. Lemma II.11 and Theorem II.12 (cf. also Theorem II.14) show that

$$\operatorname{codim} (R(ST+SP))^- = 0.$$

We want to infer that $S((R(T+P))^-) = F/N$.

We remark that $(R(T+P))^- + N$ is closed as $\dim N < \infty$. Since S is open and $N(S) \subset (R(T+P))^- + N$, the familiar argument using set-theoretic complements shows that $S((R(T+P)^- + N) = S((R(T+P)^-)$ is closed in F/N. Consequently, $S((R(T+P))^-)$ is exactly the closure in F/N of $S(R(T+P)) = R(ST+SP)$. But we have proved that $(R(ST+SP))^- = F/N$, therefore $S((R(T+P))^-) = F/N$. As a result, $(R(T+P))^- + N = (R(T))^- + N$ ./.

REMARKS. Theorem II.17 generalizes at the same time several results announced without proof by Vladimirski : Theorem 4.b and the remarks following Theorems 2 and 4 in (26). There, Vladimirski considered perturbations of the type $B \subset \varepsilon U$, $PU \subset (TB)^- + N$ and $PU \subset (TB)^- + K$, with K compact. We note that Vladimirski's results do not fully reduce to Kato's theorem for ϕ_--operators in Banach spaces.

Theorem II.17 also provides a short proof of the theorem of Vladimirski on precompact perturbations of Φ_--operators (25, Theorem 2). It shows that precompact perturbations of Φ_--operators can be essentially reduced to "small" perturbations.

Theorem II.17 should be compared with Proposition I.29.

II.2.2. Bounded perturbations

We now examine a few cases where a bounded perturbation also entails the existence of a suitable bounded disk in the domain space, in such a way that the general results of the preceding section may apply.

PROPOSITION II.18. Let E, F be locally convex spaces and T, P operators from E into F such that $D(T) \subset D(P)$.

Assume that

(a) T is a Φ_--operator,

(b) P is a bounded operator,

(c) E is such that any bounded disk of $E/N(T)$ is contained in the closure of the image of a bounded disk in E, by the canonical quotient operator.

Then there is $\rho > 0$ such that $T+\lambda P$ is almost open and codim $(R(T+\lambda P))^- \leq$ codim $(R(T))^-$ for $|\lambda| < \rho$. We may choose ρ such that codim $(R(T+\lambda P))^-$ is constant for $0 < |\lambda| < \rho$.

If $T+\lambda P$, $|\lambda| < \rho$, is a Φ_--operator, which is the case if $(D(T))^- \subset D(P)$ and E is fully complete, then $\operatorname{ind}(T+\lambda P) = \operatorname{ind}(T)$.

Proof. Let U_0 be a neighborhood in E and B' a bounded disk in F such that $PU_0 \subset B'$. Let $\bar{T}$ denote the injective

ϕ_--operator from $E/N(T)$ into F induced by T. Since $\bar{T}^{-1}$ is continuous, it follows that $\bar{T}^{-1}B'$ is a bounded disk in $E/N(T)$.

Let $\bar{A}$ denote the image by the quotient operator from E onto $E/N(T)$ of a set A. By the assumption (c), there is a bounded disk $B_o \subset E$ such that $\bar{T}^{-1}B' \subset (\bar{B}_o)^-$.

Let $\rho = \sup\{|\lambda| : \lambda B_o \subset U_o\}$. Fix any λ such that $|\lambda| < \rho$. Then $\lambda B_o \subset \varepsilon U_o$ for some $0 < \varepsilon < 1$, $\lambda PU_o \subset \lambda B'$ and

$$\lambda PU_o \cap R(T) \subset \lambda B' \cap R(T) \subset \lambda \bar{T}((\bar{B}_o)^-)$$
$$\subset \bar{T}((\lambda \bar{B}_o)^-).$$

We prove, as in Lemma II.11, that relations similar to the above hold for a base of neighborhoods U in E. Indeed, let U' be any neighborhood in E and ξ be such that $\xi(\lambda B_o) \subset \varepsilon U'$. Let $U = U' \cap \xi U_o$, and $B = \xi \lambda B_o$. Then $B \subset \varepsilon U' \cap \varepsilon \xi U_o \subset \varepsilon U$ and

$$\lambda PU \cap R(T) \subset \lambda P(\xi U_o) \cap R(T) \subset \bar{T}((\xi \lambda \bar{B}_o)^-)$$
$$\subset \bar{T}((\bar{B})^-).$$

Fix any $\varepsilon' > 0$ such that $\varepsilon'' = \varepsilon + \varepsilon' < 1$. Then

$$(\bar{B})^- \subset (\varepsilon \bar{U})^- \subset \varepsilon \bar{U} + \varepsilon' \bar{U} \subset \varepsilon'' \bar{U}.$$

Consequently,

$$\lambda PU \cap R(T) \subset \bar{T}((\bar{B})^-) \subset \bar{T}(\varepsilon'' \bar{U}) \subset \varepsilon'' TU.$$

Since this holds for a base of neighborhoods U in E and P is bounded, we may apply Theorem II.12 ./.

Since (DF)-spaces (8, Grothendieck) satisfy the assumption (c), we obtain as a corollary of Proposition II.18 the following result of Vladimirski announced without proof in (26) :

COROLLARY II.19 (Vladimirski). Let E be a fully complete (DF)-space, F a locally convex space, T a Φ_--operator and P a bounded operator from E into F such that P is everywhere defined.

Then there exists $\rho > 0$ such that $T+\lambda P$ is a Φ_--operator, $\mathrm{def}(T+\lambda P) \leq \mathrm{def}(T)$ and $\mathrm{ind}(T+\lambda P) = \mathrm{ind}(T)$, for $|\lambda| < \rho$.

REMARK. The assumption that P be everywhere defined may be replaced by $(D(T))^- \subset D(P)$.

The assumption (c) of Proposition II.18 is satisfied in particular if any bounded disk in E/N(T) is contained in the image of a bounded disk in E by the quotient operator.

A Schwartz space is a locally convex space E such that, for any neighborhood U in E, there is a neighborhood V in E which is precompact with respect to the semi-norm associated with U.

A Montel space is a locally convex space with the property that any closed bounded disk is compact.

If E is a Frechet-Schwartz space (i.e. a Frechet space which is also a Schwartz space) and N(T) is closed, then E/N(T) is also a Frechet-Schwartz space, thus a Frechet-Montel space. (A Frechet-Schwartz space is a Montel space because every bounded set is precompact, thus relatively compact. On the other hand, the quotient of a Schwartz space by a closed subspace is a Schwartz space : cf. for instance (4, Garnir, De Wilde and Schmets, p. 146).) As a result, any bounded disk in E/N(T) is contained in a compact disk in E/N(T). Since E is a Frechet space, any compact disk in E/N(T) is the image of a compact disk in E by the quotient operator. (The proof goes as follows. If K is a compact disk in E/N(T), then K is contained in the closed absolutely convex hull K' of a sequence converging to 0 in E/N(T). This sequence is in turn the image of a sequence converging to 0 in E, the closed absolutely convex hull of which is a compact disk K". Now the image of K" is a compact disk and it coincides with K'.) All this proves the following

THEOREM II.20. Let E be a Frechet-Schwartz space, F a locally convex space, T a ϕ_--operator and P a bounded operator from E into F such that P is everywhere defined.

There exists $\rho > 0$ such that $T+\lambda P$ is a ϕ_--operator, $\operatorname{def}(T+\lambda P) \leq \operatorname{def}(T)$ and $\operatorname{ind}(T+\lambda P) = \operatorname{ind}(T)$ for $|\lambda|<\rho$. We may choose ρ such that $\operatorname{def}(T+\lambda P)$ is constant for $0<|\lambda|<\rho$.

REMARK. A locally convex space E is said to be quasi-normable if for any neighborhood U in E, there is a neighborhood V in E such that, for any $\varepsilon > 0$, there exists a bounded disk B satisfying $V \subset B + \varepsilon U$. Schwartz spaces are obviously quasi-normable. By virtue of a result of Palamqdov (see (2, De Wilde)), if N(T) is a Frechet, quasi-normable space, then any bounded disk in E/N(T) is the image of a bounded disk in E by the quotient operator. Therefore Theorem II.20 holds also under the (more general) assumptions that E be a Frechet space and N(T) quasi-normable.

Finally, it is immediate that if N(T) has an algebraic and topological complement L in E, then any bounded disk in E/N(T) is the image of a bounded disk in L by the quotient operator. In fact, in this case the bounded perturl [illegible] of T may be to a bounded perturbation of the injective Fredholm operator T', the restriction of T to L, as is shown in the following

THEOREM II.21. *Let* E, F *be locally convex spaces and* T, P *operators from* E *into* F *such that* P *is everywhere defined.*

Assume that

(a) T *is a* Φ_-*-operator such that* $N(T) \oplus L = E$ *algebraically and topologically, for some closed subspace* $L \subset E$,

(b) P *is a bounded operator and either* E *or* $R(T)$ *is complete, or* P *is a Banach-bounded operator*.

Then there exists $\rho > 0$ *such that* $T+\lambda P$ *is a* Φ_-*-operator*, $\mathrm{def}(T+\lambda P) \leq \mathrm{def}(T)$ *and* $\mathrm{ind}(T+\lambda P) = \mathrm{ind}(T)$ *for* $|\lambda| < \rho$. *We may choose* ρ *such that* $\mathrm{def}(T+\lambda P)$ *is constant for* $0 < |\lambda| < \rho$.

Proof. Let T', P' denote the restrictions of T, P to L with range space F. Then T' is an injective Fredholm operator, P' is a bounded operator with either L or $R(T') = R(T)$ being complete, or P' is a Banach-bounded operator. By Theorems II.4, II.5, II.8, there exists $\rho > 0$ such that $T'+\lambda P'$ is an injective Fredholm operator and $\mathrm{ind}(T'+\lambda P') = \mathrm{ind}(T')$ (thus $\mathrm{def}(T'+\lambda P') = \mathrm{def}(T')$) for $|\lambda| < \rho$.

We now prove that, for $|\lambda| < \rho$, $T+\lambda P$ is a Φ_--operator. That $T+\lambda P$ has a closed graph is immediate because P is continuous and everywhere defined.

The range $R(T+\lambda P)$ is closed because $R(T'+\lambda P')$ is closed and has a finite codimension in $R(T+\lambda P)$, as $\mathrm{def}(T'+\lambda P') < \infty$.

It remains to prove that $T+\lambda P$ is open. Any neighborhood U in E contains a neighborhood of the form $U'+U''$, where U'

(resp. U'') is a neighborhood in L (resp. $N(T)$). We remark that $(T+\lambda P)U' = (T'+\lambda P')U'$ is a neighborhood in $R(T'+\lambda P')$. Let $M \subset (T+\lambda P)(N(T)) \subset P(N(T))$ be such that $M \oplus R(T'+\lambda P') = R(T+\lambda P)$. Then $\dim M < \infty$, and clearly there is a finite disk D generating M such that $D \subset (T+\lambda P)U''$. Therefore $(T+\lambda P)U' + D \subset (T+\lambda P)U$, where $(T+\lambda P)U' + D$ and a fortiori $(T+\lambda P)U$ are neighborhoods in $R(T+\lambda P)$.

For the last part, we remark that if $PU_0 \subset B'$ for some neighborhood U_0 in E and a bounded disk $B' \subset F$, and $B = T'^{-1}B'$, then for $|\lambda|$ small enough, we have $\lambda B \subset \varepsilon U_0$, $0 < \varepsilon < 1$, and $\lambda PU_0 \cap R(T) \subset T(\lambda B)$. We may therefore apply Theorem II.14 ./.

In Chapter III, Section III.2, we will study perturbations of semi-Fredholm operators with complemented ranges and kernels, in particular the question under what conditions in Theorem II.21 $N(T+\lambda P)$ is also complemented (algebraically and topologically).

II.3. Precompact perturbations of Φ_--operators

Theorem II.17 contains the following result of Vladimirski: if T is almost open with codim $(R(T))^- < \infty$ and P is precompact, then T+P is almost open with codim $(R(T+P))^- < \infty$.

The proof of Vladimirski (25) is completely different from ours, and rather lengthy. It involves a technique of extraction, separation and completion of countable subsystems of continuous semi-norms to derive Frechet spaces from the initial spaces and to apply the theorem of Schwartz, Köthe and Schaefer (Theorem I.8).

We now present another short proof which uses the same principles as for Theorem II.17. The only difference is that we will use the theorem of Schwartz, Köthe and Schaefer on precompact perturbations of Φ_+-operators (Theorem I.7 or Theorem I.1 (B)) rather than Kato's theorem on small perturbations (Theorem I.1 (A)).

THEOREM II.22 (Vladimirski). *Let* E, F *be locally convex spaces and* T, P *operators from* E *into* F *such that* $D(T) \subset D(P)$.

Assume that T *is almost open with* $\operatorname{codim} (R(T))^- < \infty$ *and* P *is precompact. Then* $T+P$ *is almost open with* $\operatorname{codim} (R(T+P))^- < \infty$.

Moreover, $\operatorname{codim} (R(T+\lambda P))^- = n$ *is a constant for all scalars* λ, *except for at most a countable set of exceptional points* $\{\lambda_i\}$ *with no accumulation point at finite distance. At these exceptional points,* $\operatorname{codim} (R(T+\lambda_i P))^- > n$.

If T *and* $T+P$ *are* Φ_-*-operators then* $\operatorname{ind}(T+P) = \operatorname{ind}(T)$.

Proof. We may assume that $E = D(T) = D(T+P)$. Let $PU_0 \subset K$ for some neighborhood U_0 in E and a precompact disk K in F.

Let $U \subset U_0$ be any neighborhood in E. There is a neighborhood V in F such that $(V)^- \cap (R(T))^- \subset (TU)^-$. By duality, we obtain $U° \cap R(T^+) \subset T^+ V°$. On the other hand, since PU is precompact, it is a fortiori precompact with respect to the semi-norm associated with V. Therefore, by duality, $V°$ is precompact with respect to the semi-norm associated with $(PU)° = P^{+-1}U°$. Hence $P^+V°$ is precompact with respect to the semi-norm associated with $U°$.

Let $A = V° \cap T^{+-1}U°$, then A is a (weakly bounded) Banach disk, because $T^{+-1}U° = (TU)°$ is weakly closed and $V°$ weakly compact (see also Lemma 0.3). We have immediately $U^\circ \cap R(T^+) = T^+A$ and P^+A is precompact in the Banach space generated by U°.

We also remark that $\mathrm{nul}(T^+) = \mathrm{codim}\ (R(T))^- < \infty$. Let D be a finite disk generating $N(T^+)$. Let L (resp. M) be the Banach space generated by A+D (resp. U°), and T', P' denote the restrictions of T^+, P^+ to L with range space M.

It is clear that T' is a Φ_+-operator and P' a compact operator from L into M. Theorem I.7 (or Theorem I.1 (B)) shows that $T'+P'$ is a Φ_+-operator and $\mathrm{ind}(T'+P') = \mathrm{ind}(T')$. Moreover, a result of Gohberg, Krein (5) and Kato (10) also indicates that $\dim N(T'+\lambda P') = n$ is a constant for all scalars λ, except for at most a countable set of exceptional points $\{\lambda_i\}$ with no accumulation point at finite distance, where $\dim N(T'+\lambda_i P') > n$. (Briefly, the proof runs as follows. We know that $T'+\lambda P'$ is a Φ_+-operator for any scalar λ. By virtue of Theorem I.1 (C), $\mathrm{nul}(T'+\lambda' P')$ is a constant $n(\lambda)$ (not exceeding $\mathrm{nul}(T+\lambda P)$) for $0 < |\lambda' - \lambda| < \varrho(\lambda)$ and $\varrho(\lambda) > 0$ small enough. Any compact disk in the scalar field is covered by a finite number of such annuli (together with their centers). Since there must be overlapping of these open annuli, it is easily seen that the constants $n(\lambda)$ are the same from one annulus to another. The exceptional points are among the centers λ, which are isolated.)

As usual, we remark that $R(P^+) \subset M$, $N(T^+) \subset L$ and $R(T^+) \cap M = R(T')$. As a consequence, $R(T^+ + P^+) \cap M = R(T'+P')$ and $N(T^+ + \lambda P^+) = N(T'+\lambda P')$, $\forall \lambda$. By duality, we obtain $\mathrm{codim}\ (R(T+P))^- < \infty$ and $\mathrm{codim}\ (R(T+\lambda P))^-$ has the property enunciated in the theorem.

We prove that T+P is almost open as in the proof of Theorem II.16 : there is $\mu > 0$ such that

$$\mu\, U^\circ \cap R(T^+ + P^+) = \mu U^\circ \cap R(T' + P') \subset (T' + P')(A + D).$$

Hence $\mu(T^+ + P^+)^{-1} U^\circ \subset A + D + N(T^+ + P^+)$, and by duality,

$$2\mu^{-1}((T+P)U)^- \supset A^\circ \cap D^\circ \cap R(T+P) \supset V \cap D^\circ \cap R(T+P),$$

where D° is a (weak) neighborhood in F.

Theorem I.23 shows that $\operatorname{ind}(T^+ + P^+) = \operatorname{ind}(T^+)$. If both T and T+P are Φ_--operators, then $\operatorname{ind}(T+P) = \operatorname{ind}(T)$ ./.

An immediate consequence of Theorem II.22 is the following generalization of the theorem of Schwartz, Köthe and Schaefer (Theorem I.8) :

THEOREM II.23 (Vladimirski). Let E be a Frechet space (or more generally a fully complete space), F a locally convex space and T, P operators from E into F such that $(D(T))^- \subset D(P)$.

If T is a Φ_--operator and P a precompact operator, then T+P is a Φ_--operator and $\operatorname{ind}(T+P) = \operatorname{ind}(T)$.

Moreover, $\operatorname{def}(T+\lambda P) = n$ is a constant except for at most a countable of exceptional points $\{\lambda_i\}$ with no accumulation point at finite distance, where $\operatorname{def}(T+\lambda_i P) > n$.

II.4. Precompact perturbations of Φ_+-operators

We now prove a stability result pertaining to precompact perturbations of Φ_+-operators similar to the theorem of Schwartz, Köthe and Schaefer (Theorem I.7). The stability of the index is proved by duality, the stability of the topological characteristics by adapting the argument of Schwartz (23).

THEOREM II.24. _Let_ E, F _be locally convex spaces and_ T, P _operators from_ E _into_ F _such that_ $D(T) \subset D(P)$.

Assume that T _is a_ Φ_+-_operator and_ P _a precompact operator_.

Then T+P _is almost open and_ $nul(T+P) < \infty$. _If_ T+P _is a_ Φ_+-_operator, then_ $ind(T+P) = ind(T)$.

The operator T+P _is a_ Φ_+-_operator (thus_ $ind(T+P)=ind(T)$) _if_ P _is a compact operator, or if_ $(D(T))^- \subset D(P)$ _and either_ E _or_ R(T) _is complete. In this case_ $nul(T+\lambda P) = n$ _is a constant for all scalars_ λ, _except for at most a countable set of exceptional points_ $\{\lambda_i\}$ _with no accumulation at finite distance, where_ $nul(T+\lambda_i P) > n$.

Proof. We may assume without loss of generality that $E = D(T) = D(T+P)$. Let $PU_o \subset K$ for some neighborhood U_o in E and a precompact disk K in F.

Let $U \subset U_o$ be any neighborhood in E, and V a neighborhood in F such that $(V)^- \cap R(T) \subset (TU)^-$. By duality we arrive at $U° \cap R(T^+) \subset T^+V°$. Let $A = V° \cap T^{+-1}U°$, then A is weakly compact, thus a Banach disk. We have

$$U° \cap R(T^+) = T^+A \quad ,$$

and P^+A is precompact in the Banach space generated by U° (cf. proof of Theorem II.22). Since T is open, it follows that $R(T^+) = N(T)°$, therefore $def(T^+) = nul(T) < \infty$. Theorem I.23 shows that $ind(T^++P^+) = ind(T^+)$, thus $ind(T+P) = ind(T)$ if both T and T+P are Φ_+-operators. Also $def(T^++P^+) < \infty$ implies codim $N(T+P)° < \infty$, hence $nul(T+P) < \infty$ (this may also be proved directly by the argument of Schwartz (23) : considering the restrictions of the operators to a complement of $N(T)$ in E, we may assume T injective; restricted to $N(T+P)$, $T = -P$ where T is open and P precompact; it follows easily that there is a neighborhood U' in $N(T+P)$ which is precompact because $U' = -T^{-1}PU'$; as a result $\dim N(T+P) < \infty$).

We now prove that T+P is almost open. Let L (resp. M) be the Banach space generated by A (resp. U°), and T', P' the restrictions of T^+, P^+ to L with range space M. We have proved that T' is a Φ_--operator and P' a compact operator, thus by part (B) of Kato's theorem (cf. also Theorem II.23), $T'+P'$ is a Φ_--operator.

In particular $\operatorname{def}(T'+P') < \infty$. Consequently, $\operatorname{codim}_M R(T^++P^+)\cap M \leq \operatorname{def}(T'+P') < \infty$. Let $M_1 \subset R(T^++P^+)\cap M$ be a finite dimensional subspace such that

$$M_1 \oplus R(T'+P') = R(T^++P^+)\cap M \quad .$$

Let $f \in M_1$, then $f = (T^++P^+)g$. As $R(P^+) \subset M$ (PU being bounded), $T^+g = f-P^+g$ is an element of $R(T^+)\cap M = R(T')$. Thus $g \in L+N(T^+)$. As a result, there is a finite dimensional subspace $L_1 \subset N(T^+)$, generated by a finite disk D such that $(T^++P^+)L_1 = M_1$ $(= P^+L_1)$. Let $L_2 = L+L_1$ be the Banach space topologized by A+D, and T'', P'' the restrictions of T^+, P^+ to L_2 with range space M. Then $T''+P''$ is a Φ_--operator (cf. Lemma I.11). Now we have

$$R(T^++P^+)\cap M = R(T''+P'') \quad .$$

There is $\mu > 0$ such that

$$\mu U^\circ \cap R(T^++P^+) = \mu U^\circ \cap R(T''+P'') \subset (T''+P'')(A+D) \ ,$$

hence $\mu(T^++P^+)^{-1}U^\circ \subset A+D+N(T^++P^+)$, and by duality,

$$2\mu^{-1}((T+P)U)^{-} \supset A^{\circ} \cap D^{\circ} \cap R(T+P)$$
$$\supset V \cap D^{\circ} \cap R(T+P) \ ,$$

which shows that T+P is almost open.

That T+P is a ϕ_+-operator if P is a compact operator is part of the theorem of Schwartz, Köthe and Schaefer. To prove that it is so if P is a precompact operator, $(D(T))^{-} \subset D(P)$ and either E or R(T) is complete, we adapt the argument of Schwartz (23).

By considering a closed subspace L such that $L \oplus (N(T) + N(T+P)) = E$ (algebraically and topologically), and the restrictions of T, P to L, we may assume that both T and T+P are injective. We notice in particular that $TL = T(L + N(T))$ is closed in R(T) since L+N(T) is closed, T is open and $N(T) \subset L+N(T)$; thus if E or R(T) is complete, then L or TL is complete.

Let $\mathcal{G}$ be an ultrafilter in R(T+P) converging to y in F. We should prove that the ultrafilter defined by $\mathcal{F} = (T+P)^{-1}\mathcal{G}$ converges in E. Let p be the continuous semi-norm associated with U_0 (such that PU_0 is precompact). Consider the ultrafilter defined by $\{p(x) : x \in F, F \in \mathcal{F}\}$ on the (compact) extended real line $(-\infty, +\infty)$. Let a be its limit. If $a < +\infty$, there exists $F \in \mathcal{F}$ such that $p(x) < a+1$, $\forall x \in F$. A fortiori $(a+1)U_0 \in \mathcal{F}$, as $F \subset (a+1)U_0$. Consequently $P\mathcal{F}$, defining an ultrafilter in the precompact disk $(a+1)PU_0$, is itself Cauchy (cf. (8, Grothendieck) or (13, Köthe)). As a result, $T\mathcal{F} = (T+P-P)\mathcal{F}$ defines a Cauchy ultrafilter (since $(T+P)\mathcal{F} = \mathcal{G}$ does). If R(T) is complete then

$T\mathcal{F}$ converges to a certain $Tx \in R(T)$, hence $\mathcal{F}$ converges to x in E (T is open). If E is complete we remark that $\mathcal{F} = T^{-1}T\mathcal{F}$ is Cauchy, thus converges to $x \in E$. This proves at the same time that $T+P$ is open and $R(T+P)$ is closed, since it follows from $\mathcal{F} \to x$, $(T+P)\mathcal{F} = \mathcal{G} \to y$ and the closed graph of $T+P$ (assumption $(D(T))^{-} \subset D(P)$), that $y = (T+P)x$. (*)

If P is a compact operator, or if $(D(T))^{-} \subset D(P)$ and either E or $R(T)$ is complete, then $T+\lambda_0 P$ is a Φ_+-operator for all scalars λ_0. By virtue of Theorem II.5 (cf. Theorem II.8 for the case P a compact operator), $\mathrm{nul}(T+\lambda P)$ is a constant $n(\lambda_0)$ for $0<|\lambda-\lambda_0| < \rho(\lambda_0)$ and $\rho(\lambda_0) > 0$ small enough. The argument of Gohberg, Krein and Kato (cf. proof of Theorem II.22) shows that $\mathrm{nul}(T+\lambda P)$ has the property enunciated in the statement of the theorem ./.

REMARKS. In the preceding proof, if P is a compact operator then $P\mathcal{F}$ is a convergent filter (with the assumption that $PU_0 \subset K$, K compact). Thus $T\mathcal{F} = (T+P-P)\mathcal{F}$ converges to a certain $Tx \in R(T)$, hence $\mathcal{F} \to x$ without any assumption of completeness of E or $R(T)$. This is the original argument of Schwartz (23).

Part of Theorem II.24 (that $T+P$ is Φ_+-operator and $\mathrm{ind}(T+P) = \mathrm{ind}(T)$), if T is continuous, may be deduced from some results announced without proof by Vladimirski (27). Judging by his statements, we think that his methods may be different, using completions of the spaces as in (25) rather than duality.

(*) The case $a = +\infty$ is impossible, because if one considers the ultrafilter defined by $\mathcal{F}' = \{x/p(x) : x \in F, F \in \mathcal{F}\}$ then the preceding argument shows that $\mathcal{F}' \to x' \neq 0$ whereas $(T+P)\mathcal{F}' \to 0$.

CHAPTER III

SOME FURTHER RESULTS AND APPLICATIONS

In this chapter, we first study some extensions of the results in Chapter II to the case where P may not necessarily be bounded, but "T-bounded". This corresponds to the concept of "relative boundedness" in (11, Kato). Then we study perturbations of semi-Fredholm operators with complemented kernels and ranges. We are mainly concerned with the stability of the latter property. This extends a result of Pietsch (20). Next we apply the results of Chapter II in the duals, using suitable locally convex topologies on the duals for which P^+ is a convenient perturbation of T^+. This gives some stability results involving the property that an operator lifts certain families of weakly compact disks (cf. Corollary I.28 and subsequent remark). Finally we briefly derive some spectral properties of bounded operators in a sequentially complete locally convex space, and present an example of small perturbations of linear partial differential operators.

III.1. T-boundedness

In (24), Sz-Nagy remarked that stability results may be extended to the case where P need not be bounded, but is "T-bounded", that is $\|Px\| \leq a\|x\| + b\|Tx\|$, $\forall x \in D(T)$, for some $a, b \geq 0$.

This amounts to saying that P is bounded from D(T) into F if D(T) is equipped with the (stronger) norm $\|x\| + \|Tx\|$. We refer the reader to (6, Goldberg) and (11, Kato) for examples and applications.

We have a similar situation in locally convex spaces. We may assume that P is a bounded operator from D(T) into F if D(T) is equipped with the (finer) topology generated by $\{U \cap T^{-1}V\}$ where U (resp. V) runs through a base of neighborhoods in E (resp. F). (In case of normed spaces we would get this way the norm $\sup\{\|x\|, \|Tx\|\}$ rather than $\|x\|+\|Tx\|$, but the two norms are equivalent.) We call this topology on D(T) the T-<u>topology</u>. The prefix T shall serve to indicate that D(T) is equipped with the T-topology (e.g. P is T-<u>bounded</u> or T-<u>precompact</u>). The T-topology coincides with the topology induced by E if T is continuous. On the other hand, T is always T-continuous.

Most of the stability results could be generalized to this case, because of the following :

LEMMA III.1. <u>Let</u> E, F <u>be locally convex spaces and</u> T <u>an operator from</u> E <u>into</u> F. <u>Then</u>

(a) T <u>is open if and only if</u> T <u>is</u> T-<u>open</u>,

(b) T <u>is almost open if and only if</u> T <u>is</u> T-<u>almost-open</u>.

<u>Proof</u>. The part "if" in both cases is trivial since the T-topology is finer than the topology on D(T) induced by E. We now prove the "only if" part.

(a) Let $U \cap T^{-1}V$ be a T-neighborhood in D(T). Then

$$T(U \cap T^{-1}V) = (TU) \cap V .$$

For if $y \in (TU) \cap V$. Then $y = Tx$ with $x \in U$, thus $x \in U \cap T^{-1}V$. Hence $(TU) \cap V \subset T(U \cap T^{-1}V)$. (The converse inclusion is trivial and not needed.)

As TU is a neighborhood in R(T) and V a neighborhood in F, $T(U \cap T^{-1}V)$ is a neighborhood in R(T).

(b) Let $U \cap T^{-1}V$ be a T-neighborhood in D(T). We may assume that V is open (otherwise we take its interior). Since TU is dense in $(TU)^-$, we have $V \cap (TU)^- \subset (V \cap TU)^- \subset (T(U \cap T^{-1}V))^-$. Indeed let $y \in V \cap (TU)^-$ and Ω be a neighborhood of y in F, then $\Omega \cap V$ is a neighborhood of y. Thus $V \cap \Omega \cap TU$ is not empty.

Now $V \cap (TU)^-$ is a neighborhood in R(T) and so is $(T(U \cap T^{-1}V))^-$ ./.

The general approach in dealing with T-bounded perturbations is the following. Let T be a semi-Fredholm operator and P a T-bounded, T-compact or T-precompact operator. We consider T, P as operators from D(T), equipped with the T-topology, into F. By Lemma III.1, T remains a semi-Fredholm operator, and P becomes a bounded, compact or precompact perturbation of T. Under suitable assumptions as given in Chapter II, T+P is a semi-Fredholm operator. If we return to the initial topology on E, then T+P is obviously open. The main stumbling block remaining is whether or not T+P has a closed graph in ExF. This is answered in the following

THEOREM III.2. _Let_ E, F _be locally convex spaces and_ T, P _operators from_ E _into_ F _such that_ $D(T) \subset D(P)$.

Assume that

(a) T *has a closed graph*,

(b) P *is a* T-*bounded operator*.

Assume further that either

(c) $P(U_o \cap T^{-1}V_o) \subset B$ *for some* T-*neighborhood* $U_o \cap T^{-1}V_o$ *and a bounded Banach disk* $B \subset F$ (*i.e.* P *is a* T-*Banach-bounded operator, which is the case in particular if* F *is sequentially complete*),

or

(c') G(T) *is complete*.

Then there exists $\rho > 0$ *such that* $T + \lambda P$ *has a closed graph for* $|\lambda| < \rho$.

Proof. Let X = G(T) be equipped with the topology induced by Z = ExF.

Let $W = (U \times V) \cap G(T)$ be a neighborhood in X, where U (resp. V) is a neighborhood in E (resp. F). If Q denotes the projection of G(T) on E, along F, then $QW = U \cap T^{-1}V$. Indeed $x \in QW$ if and only if $(x, Tx) \in U \times V$, that is if and only if $x \in U \cap T^{-1}V$.

Let I be the canonical injection of X into Z and S the operator from X into Z defined by $S(x, Tx) = (0, Px)$, $x \in D(T)$.

Obviously I is an injective, continuous Φ_+-operator (because $R(I) = G(T)$ is closed in Z). On the other hand S is a bounded operator, for if $P(U_o \cap T^{-1}V_o) \subset B$ for a T-neighborhood $U_o \cap T^{-1}V_o$ and a bounded disk B, and $W_o = (U_o \times V_o) \cap G(T)$, then $SW_o = \{(0, Px) : x \in U_o \cap T^{-1}V_o\} \subset \{0\} \times B$, where $\{0\} \times B$ clearly is a bounded disk in Z. We also remark that

$$R(I+\lambda S) = \{(x, Tx+\lambda Px) : x \in D(T)\}$$
$$= G(T+\lambda P) .$$

Let $\varrho = \sup\{|\lambda| : \lambda B \subset V_0\}$. If $|\lambda|<\varrho$ then we have

$$IW_0 = (U_0 x V_0) \cap R(I) ,$$
$$\lambda SW_0 \subset \{0\} x \lambda B \subset \varepsilon(U_0 x V_0)$$

for some $0 < \varepsilon < 1$.

Consequently $I+\lambda S$ is an injective (continuous) Φ_+-operator by virtue of Theorems II.2 and II.7 (if B is a bounded Banach disk then $\{0\}$ x B is also a bounded Banach disk). In particular $R(I+\lambda S) = G(T+\lambda P)$ is closed ./.

THEOREM III.3. *Let* E, F *be locally convex spaces and* T, P *operators from* E *into* F *such that* $D(T) \subset D(P)$.

Assume that

(a) T *has a closed graph*

and either

(b) P *is a* T-*compact operator*

or

(b') P *is a* T-*precompact operator and* G(T) *is complete*.

Then T+P *has a closed graph*.

Proof. With the same setting as in the proof of Theorem II.2, S is now either a compact operator, or a precompact operator with a complete domain space. The theorem of Schwartz, Köthe and Schaefer or Theorem II.24 gives the conclusion ./.

REMARK. Theorem III.3 is also given in (25, Vladimirski).

Lemma III.1, Theorems III.2, III.3 and the results of Chapter II immediately yield the following

THEOREM III.4. Let E, F be locally convex spaces and T, P operators from E into F such that $D(T) \subset D(P)$.

Assume that

(a) T is a Φ_+-operator,

(b) P is a T-Banach-bounded operator, which is the case in particular if P is T-bounded and F is sequentially complete.

Then there exists $\rho > 0$ such that $T+\lambda P$ is a Φ_+-operator, $\mathrm{nul}(T+\lambda P) \leq \mathrm{nul}(T)$, $\mathrm{def}(T+\lambda P) \leq \mathrm{def}(T)$ and $\mathrm{ind}(T+\lambda P) = \mathrm{ind}(T)$ for $|\lambda| < \rho$. We may choose $\rho > 0$ such that $\mathrm{nul}(T+\lambda P)$ is constant for $0 < |\lambda| < \rho$.

Proof. For $|\lambda|$ small enough, $T+\lambda P$ is a Φ_+-operator from $D(T)$ equipped with the T-topology into F, $\mathrm{nul}(T+\lambda P) \leq \mathrm{nul}(T)$, $\mathrm{def}(T+\lambda P) \leq \mathrm{def}(T)$ and $\mathrm{ind}(T+\lambda P) = \mathrm{ind}(T)$, by virtue of Lemma III.1 and Theorem II.8. Also $\mathrm{nul}(T+\lambda P)$ is constant for $0 < |\lambda| < \rho$ and ρ small enough.

If we return to the topology of E, then $T+\lambda P$ is obviously open from E into F, for the T-topology is finer than that induced by E. Theorem III.2 shows that $T+\lambda P$ has a closed graph. Moreover $R(T+\lambda P)$ is closed; hence $T+\lambda P$ is a Φ_+-operator from E into F ./.

REMARK. The proof shows that Theorem III.4 also holds for P a T-bounded operator if E and $G(T)$ are complete. The completion of $G(T)$ ensures that $G(T+\lambda P)$ is closed, by Theorem III.2, and the completion of E ensures that $R(T+\lambda P)$ is closed, $T+\lambda P$ being open.

Similarly, for T-compact or T-precompact perturbations of Φ_+-operators, we have, by using Theorems II.24 and III.3 :

THEOREM III.5. *Let* E, F *be locally convex spaces and* T, P *operators from* E *into* F *such that* $D(T) \subset D(P)$.

Assume that

(a) T *is a* ϕ_+*-operator*

and either

(b) P *is* T-*compact*

or

(b') P *is* T-*precompact*, G(T) *and* R(T) *are complete*.

Then T+P *is a* ϕ_+*-operator and* $\operatorname{ind}(T+P) = \operatorname{ind}(T)$. *Furthermore* $\operatorname{nul}(T+\lambda P) = n$ *is constant for all scalars* λ, *except for at most a countable set of exceptional points* $\{\lambda_i\}$ *with no accumulation point at finite distance, where* $\operatorname{nul}(T+\lambda_i P) > n$.

REMARK. Let L be a closed subspace such that $N(T) \oplus L = D(T)$ (algebraically and topologically). If D(T) is equipped with the T-topology, then the restriction of T to L is T-continuous and T-open (Lemma III.1); hence it is a T-isomorphism from L onto R(T). Therefore R(T) is complete if and only if D(T) is complete when equipped with the T-topology.

For small perturbations of ϕ_--operators we have:

PROPOSITION III.6. *Let* E, F *be locally convex spaces and* T, P *operators from* E *into* F *such that* $D(T) \subset D(P)$.

Assume that

(a) T *is almost open and* $\operatorname{codim} (R(T))^- < \infty$,

(b) P *is* T-*bounded, and there exists a bounded disk* $B \subset E$ *such that* TB *is bounded and*

$$B \subset \varepsilon(U_0 \cap T^{-1}V_0),\ P(U_0 \cap T^{-1}V_0) \cap (R(T))^- \subset (TB)^-$$

for some T-neighborhood $U_0 \cap T^{-1}V_0$ and $0 < \varepsilon < 1$.

Then T+P is almost open, codim $(R(T+P))^- \leq$ codim $(R(T))^-$ and codim $(R(T+\lambda P))^-$ is constant for $0 < |\lambda| < \rho$ and $\rho > 0$ small enough.

Proof. If $B \subset E$ is bounded and TB is bounded then obviously $B \cap D(T)$ is T-bounded. Moreover, P is T-bounded. It suffices now to apply Theorem II.14 and Lemma III.1 ./.

For precompact perturbations of Φ_--operators, we have the following result of Vladimirski (25, Theorem 2), which can be inferred immediately from Lemma III.1 and Theorem II.22 :

PROPOSITION III.7. Let E, F be locally convex spaces and T, P operators from E into F such that $D(T) \subset D(P)$.

Assume that T is almost open with codim $(R(T))^- < \infty$ and P is T-precompact.

Then T+P is almost open with codim $(R(T+P))^- < \infty$. Moreover, codim $(R(T+\lambda P))^- = n$ is constant for all scalars λ except for at most a countable set of exceptional points $\{\lambda_i\}$ with no accumulation point at finite distance, where codim $(R(T+\lambda_i P))^- > n$.

III.2. Perturbations of semi-Fredholm operators with complemented ranges and kernels

In (20), Pietsch studied small bounded perturbations of semi-Fredholm endomorphisms (operators from a locally convex space into itself) with complemented ranges and kernels. Here we extend some of his results to the case of different domain and range spaces. In this section we consider only everywhere defined continuous operators.

DEFINITIONS. Let E, F be locally convex spaces and T an operator from E into F (with D(T) = E).

We say that T is a Ψ_+ (resp. Ψ_-) -operator if T is a continuous Φ_+ (resp. Φ_-) -operator such that R(T) (resp. N(T)) has an algebraic and topological complement in F (resp. E).

In particular if T is a continuous Fredholm operator, then T is both Ψ_+ and Ψ_-.

The following two propositions characterize Ψ_+ and Ψ_--operators (in the remainder of this section, we shall always assume E, F to be locally convex spaces and T, P continuous operators from the whole of E into F) :

PROPOSITION III.8. Let T be an operator from E into F. If T is a Ψ_+(resp. Ψ_-) -operator, then there exists a continuous operator U (resp. V) such that S = UT (resp. Z = TV) is a continuous Fredholm operator of index zero.

Proof. Assume that T is a Ψ_+-operator. Let L, M be closed subspaces such that $E = L \oplus N(T)$ and $F = M \oplus R(T)$ algebraically and topologically. Let $\hat{T}$ denote the isomorphism from L onto R(T) induced by T.

Define U by letting $Uy = \hat{T}^{-1}y$ if $y \in R(T)$ and $Uy = 0$ if $y \in M$. Obviously U is continuous, as $\hat{T}^{-1}$ is. Let S = UT. Then S reduces to the identity operator on L and S = 0 on N(T). Therefore S is a continuous Fredholm operator with

$$\text{nul}(S) = \text{def}(S) = \dim N(T).$$

We proceed similarly for a ψ_--operator T. With the same notations as above and V = U, then Z = TV is the identity operator on R(T) and Z = 0 on M with dim $M < \infty$ ./.

Conversely, we have

PROPOSITION III.9. <u>If there exists a continuous operator</u> U (<u>resp</u>. V) <u>such that</u> S = UT (<u>resp</u>. Z = TV) <u>is a</u> ψ_+(<u>resp</u>. ψ_-) -<u>operator</u>, <u>then</u> T <u>is a</u> ψ_+ (<u>resp</u>. ψ_-) -<u>operator</u>.

<u>Proof</u>. Assume first that S = UT is a ψ_+-operator. We have:

(a) $N(T) \subset N(S)$, therefore $\mathrm{nul}(T) \leq \mathrm{nul}(S) < \infty$.

(b) Let L_1 be a finite dimensional subspace such that $N(S) = N(T) \oplus L_1$. Let L_2 be a closed subspace such that $E = N(S) \oplus L_2 = N(T) \oplus L_1 \oplus L_2$. Then $TL_1 \subset N(U)$, $(TL_2) \cap N(U) = \{0\}$ and $TL_1 + TL_2 = R(T)$.

On the other hand, $R(S) = U(TL_2)$. Let T' (resp. T") denote the injective operator from $L_1 \oplus L_2$ (resp. L_2) into R(T) induced by T, and U' the restriction of U to TL_2, into R(S). Let S' be the isomorphism from L_2 onto R(S) induced by S. We recall that U' is injective on TL_2. We have the following diagram :

$$\begin{array}{ccc} L_2 & \xrightarrow{T''} & TL_2 \\ {\scriptstyle S'}\downarrow & \swarrow {\scriptstyle U'} & \\ R(S) & & \end{array}$$

All three operators are continuous and bijective (i.e. injective and surjective), and $S' = U'T''$. Now S' is an isomorphism. It follows that $T''^{-1} = S'^{-1}U'$ is continuous.

Let $\mathcal{G} = T''\mathcal{F}$ be a filter in TL_2 which converges in F to y. Then $U'\mathcal{G}$ converges to $Uy \in R(S)$ because U is continuous and $R(S)$ is closed by assumption. Therefore $S'^{-1}U'\mathcal{G} = T''^{-1}T''\mathcal{F} = \mathcal{F}$ converges to $x = S'^{-1}U'y \in L_2$. As T'' is continuous $\mathcal{G} = T''\mathcal{F}$ converges to $T''x = y \in TL_2$.

This shows that TL_2 is closed in F; hence $R(T)$ is closed in F, as TL_2 has a finite codimension in $R(T)$. It also follows from Lemma I.11 that T is open.

(c) Let Q_1 be the continuous projection from E onto $R(S)$. Define an operator Q_2 from F into TL_2 by

$$Q_2 y = TS'^{-1}Q_1 Uy \quad , \quad y \in F \; .$$

Obviously Q_2 is continuous. Now if $y \in TL_2$, then $y = T''x$, $x \in L_2$ and $Uy = U'y = S'x$. Therefore, as $Q_1 S'x = S'x$,

$$\begin{aligned} Q_2 y &= TS'^{-1}Q_1 Uy \\ &= TS'^{-1}Q_1 S'x \\ &= TS'^{-1}S'x \\ &= Tx \\ &= y \quad . \end{aligned}$$

This shows that $R(Q_2) = TL_2$ and Q_2 is a continuous projection from F onto TL_2. As a result, $N(Q_2) \oplus TL_2 = F$ algebraically and topologically. From this and the fact that TL_2 is closed and has a finite codimension in $R(T)$, it can be deduced that there is a continuous projection from F onto $R(T)$. We remark that $TL_1 \subset N(U) \subset N(Q_2)$. As $\dim(TL_1) < \infty$, there is a closed subspace $M \subset N(Q_2)$ such that $(TL_1) \oplus M = N(Q_2)$ algebraically and topologically. As a result, $M \oplus (TL_1) \oplus (TL_2) = F$, hence $M \oplus R(T) = F$ algebraically and topologically.

This proves that T is a Ψ_+-operator.

Assume now that $Z = TV$ is a Ψ_--operator. We have :

(a') $R(Z) \subset R(T) \Rightarrow \operatorname{def}(T) \leq \operatorname{def}(Z) < \infty$.

(b') Let M be a closed subspace such that $F = M \oplus N(Z)$ algebraically and topologically. Then, as $N(Z) = V^{-1}N(T)$, we have $(VM) \cap N(T) = \{0\}$. On the other hand $M \cap N(V) = \{0\}$ and $TVM = R(Z)$. We have the following diagram

$$\begin{tikzcd} VM \arrow[dr, "T'"'] & M \arrow[l, "V'"'] \arrow[d, "Z'"] \\ & R(Z) \end{tikzcd}$$

where T', V', Z' are the appropriate restrictions of T, V, Z. All three operators T', V', Z' are continuous and bijective, and $Z' = T'V'$. Since Z' is an isomorphism, $T'^{-1} = V'Z'^{-1}$ is continuous. Similarly, $V'^{-1} = Z'^{-1}T'$ is continuous. If $\mathfrak{F}$ is a filter in VM converging to x in E, then $T\mathfrak{F} = T'\mathfrak{F}$ converges to $Tx \in R(Z)$, as $R(Z)$ is closed. Thus $Z'T\mathfrak{F}$ converges to $Z'^{-1}Tx \in M$. As a result, $\mathfrak{F} = V'Z'^{-1}T\mathfrak{F}$ converges to $V'Z'^{-1}Tx = x \in VM$, that is VM is closed.

Let $E' = VM + N(T) = T^{-1}(R(Z))$. Since T is continuous and $R(Z)$ is closed, it follows that E' is closed in E.

Let Q be an operator from E' into VM defined by

$$Qx = V'Z'^{-1}Tx \quad , \quad x \in E' .$$

Obviously Q is continuous. If $x \in VM$ then $Qx = V'Z'^{-1}T'x = x$. Therefore Q is a continuous projection from E' onto VM. Moreover $N(Q) = N(T)$ by the definition of Q.

We remark that E' has a finite codimension in E because $TE' = R(Z)$ has a finite codimension in $R(T)$. Moreover E' is closed, therefore if L is a subspace such that $\dim L < \infty$, $L \oplus E' = E$, we have $N(T) \oplus VM \oplus L = E$, which shows that there is a continuous projection from E onto N(T).

(c') As $R(T) \supset R(Z)$, $R(Z)$ is closed and $\operatorname{def}(Z) < \infty$, it follows that $R(T)$ is closed. Moreover, since T' is open from VM onto $R(Z)$, it is easily seen that T is open from E into F ./.

As a consequence of these characterizations and the stability results in Chapter II, we have the following

THEOREM III.10. _If_ T _is a_ ψ_+ (_resp._ ψ_-) -_operator and_ P _is a compact operator, then_ T+P _is a_ ψ_+ (_resp._ ψ_-) -_operator_.

Proof. There exists a continuous operator U (resp. V) such that S = UT (resp. Z = TV) is a continuous Fredholm operator of index zero, by Proposition III.8. Consequently U(T+P) = S+UP (resp. (T+P)V = Z+PV) is a continuous Fredholm operator of index zero by virtue of Theorem II.24 (or the theorem of Schwartz, Köthe and Schaefer), as UP (resp. PU) is a compact operator. Proposition III.9 shows that T+P is a ψ_+ (resp. ψ_-) -operator ./.

THEOREM III.11. _If_ T _is a_ ψ_+-_operator_, P _a precompact operator and_ E _is complete, then_ T+P _is a_ ψ_+-_operator_.

Proof. As above, we have U(T+P) = S+UP where UP is now a precompact thus compact operator ./.

THEOREM III.12. _If_ T _is a_ ψ_+ (_resp._ ψ_-) -_operator_, P _a bounded operator and_ E (_resp_. F) _is sequentially complete, then_ T+ λ P _is a_ ψ_+ (_resp._ ψ_-) -_operator for_ $|\lambda|$ _small enough_.

Proof. As in the proof of Theorem III.10, we have $U(T+\lambda P) = S+\lambda UP$ (resp. $(T+\lambda P)V = Z+\lambda PV$), where UP (resp. PV) is a bounded, thus Banach-bounded operator. It suffices to apply Theorem II.8 ./.

REMARK. If A is a Banach disk in E, $D(T) = E$ and TA is norming, then TA is a Banach disk in F. It suffices indeed to notice that T is a continuous, open operator from E_A onto F_{TA}. With this observation, if $B \subset F$ is a bounded Banach disk, then $\overline{UB}$ is a bounded Banach disk in E. Therefore we have the following

THEOREM III.13. *If T is a ψ_+ (resp. ψ_-) -operator and P is a Banach-bounded operator, then $T+\lambda P$ is a ψ_+(resp. ψ_-) -operator for $|\lambda|$ small enough.*

REMARK. In Theorems III.10 to III.13, the stability of the index is already established in Chapter II.

III.3. Lifting of weakly compact disks

In Chapter II we used duality without explicitly considering perturbations of the adjoint operators with respect to some locally convex topologies on the duals. We only used the Banach spaces generated by closed equicontinuous disks. This is in contrast with Schwartz's proof of the theorem on compact perturbations of Φ_--operators between Frechet spaces (23), where he used the compact convergence topology (polars of compact disks) on the duals.

In this section, we use Schwartz's approach. We apply the stability results established in Chapters I and II to T^+, P^+, when E^+, F^+ are equipped with suitable locally convex topologies arising from duality. This leads naturally to the concept of lifting of weakly compact disks in E, F.

DEFINITIONS. Let $\mathcal{F}$ be a family of weakly compact disks of a locally convex space E. We say that $\mathcal{F}$ is <u>saturated</u> if

(a) $A \in \mathcal{F} \Rightarrow \lambda A \in \mathcal{F}$ for all scalars λ ,

(b) $A, B \in \mathcal{F} \Rightarrow \exists C \in \mathcal{F}: \ A \cup B \subset C$,

(c) $E \subset \bigcup \{A : A \in \mathcal{F}\}$.

In this section, unless there is explicit mention to the contrary, a family of weakly compact disks is always assumed to be saturated.

We use the affix $\mathcal{F}$ (e.g. $\mathcal{F}$-compact, $E^+_{\mathcal{F}}$, $(A)^{-\mathcal{F}}$) to indicate that E^+ is equipped with the locally convex topology generated by the polars A°, $A \in \mathcal{F}$.

We remark that the $\mathcal{F}$-topology is always coarser than the Mackey topology and finer than the weak topology. If A is a disk, then $(A)^{-\mathcal{F}}$ is the same as the weak closure of A, therefore we will simply write $(A)^-$.

Let E, F be locally convex spaces and T an operator from E into F. Let $\mathcal{F}$ (resp. $\mathcal{G}$) be a saturated family of weakly compact disks on E (resp. F). We say that T <u>lifts</u> $\mathcal{G}$ <u>by</u> $\mathcal{F}$ if for any $B \in \mathcal{G}$, there is $A \in \mathcal{F}$ such that $B \cap R(T) \subset TA$. If $\mathcal{F}$ and $\mathcal{G}$ are the families of all weakly compact disks (compact disks, or finite disks), we simply say that T <u>lifts weakly compact disks</u> (<u>compact disks</u>, or <u>finite disks</u>). Remark that T always lifts finite disks.

There is a simple relationship between the lifting property of T and the "openness" of T^+ :

LEMMA III.14. Let E, F be locally convex spaces and $\mathcal{F}$, $\mathcal{G}$ saturated families of weakly compact disks on E, F respectively. Let T be an operator from E into F such that $D(T) = E$, $R(T)$ is closed and T is weakly continuous.

Then T lifts $\mathcal{G}$ by $\mathcal{F}$ if and only if T^+ is open from $F^+_{\mathcal{G}}$ into $E^+_{\mathcal{F}}$.

Proof. Since T is weakly continuous and $D(T) = E$, TA is weakly compact, thus closed, for any $A \in \mathcal{F}$.

If for any $B \in \mathcal{G}$, there is $A \in \mathcal{F}$ such that $B \cap R(T) \subset TA$, then $T^{+-1}A^\circ \subset (TA)^\circ \subset (B^\circ + N(T^+))^-$.

We prove that $(TA)^\circ = T^{+-1}A^\circ$. The inclusion $T^{+-1}A^\circ \subset (TA)^\circ$ is trivial. Conversely, if $g \in (TA)^\circ$, then $g \in D(T^+)$ as $D(T^+) = F^+$, T being weakly continuous. Let $f = T^+g$, then obviously $f \in A^\circ$, which shows that $g \in T^{+-1}A^\circ$.

On the other hand B° is a Mackey neighborhood in F^+, B being weakly compact. Thus $(B^\circ + N(T^+))^- \subset (1+\varepsilon)B^\circ + N(T^+)$ for any $\varepsilon > 0$.

As a result, $B \cap R(T) \subset TA$ entails $A^\circ \cap R(T^+) \subset (1+\varepsilon)T^+B^\circ$, $\varepsilon > 0$, hence T^+ is open from $F^+_{\mathcal{G}}$ into $E^+_{\mathcal{F}}$.

Conversely, if T^+ is open from F^+ into E^+, then for any $B \in \mathcal{G}$, there is $A \in \mathcal{F}$ such that $A^\circ \cap R(T^+) \subset T^+B^\circ$, which implies $T^{+-1}A^\circ = (TA)^\circ \subset B^\circ + N(T^+)$. As a result, $B \cap R(T) \subset (TA)^- \subset TA$ ./.

REMARK. A closer examination of the preceding proof shows that the assumption "R(T) is closed" is not necessary for the "if" part and the assumptions "D(T) = E" and "T is weakly continuous" are not necessary for the "only if" part.

With this lemma, we can now apply the stability results in Chapters I and II to T^+, P^+ from $F^+_{\mathcal{G}}$ into $E^+_{\mathcal{F}}$.

DEFINITION. We say that T is a <u>weak homomorphism</u> if T is weakly continuous and weakly open.

PROPOSITION III.15. <u>Let</u> E, F <u>be locally convex spaces and</u> $\mathcal{F}$, $\mathcal{G}$ <u>saturated families of weakly compact disks in</u> E, F <u>respectively</u>.

<u>Let</u> T, P <u>be operators defined everywhere from</u> E <u>into</u> F. <u>Assume that</u>

(a) T <u>is a weak homomorphism</u>, R(T) <u>is closed and</u> $\mathrm{def}(T) < \infty$,

(b) T <u>lifts</u> $\mathcal{G}$ <u>by</u> $\mathcal{F}$,

(c) <u>There exists a</u> $\mathcal{F}$-<u>compact disk</u> $K \subset E^+_{\mathcal{F}}$ <u>and</u> $B \in \mathcal{G}$ <u>such that</u> $P(K^\circ) \subset B$.

<u>Then</u> T+P <u>is a weak homomorphism</u>, R(T+P) <u>is closed</u>, $\mathrm{def}(T+P) < \infty$, $\mathrm{ind}(T+P) = \mathrm{ind}(T)$ <u>and</u> T+P <u>lifts</u> $\mathcal{G}$ <u>by</u> $\mathcal{F}$.

<u>Proof</u>. By duality T^+ is a Φ_+-operator from $F^+_{\mathcal{G}}$ into $E^+_{\mathcal{F}}$ (Lemma III.14).

On the other hand, K° is a Mackey neighborhood in E, thus P is continuous from E equipped with the Mackey topology into F. In particular P is weakly continuous; hence T+P is weakly continuous and $D(P^+) = F^+$. From $P(K^\circ) \subset B$ we infer that $P^+B^\circ \subset K$, where B° is a $\mathcal{G}$-neighborhood. This shows that P^+ is a compact operator from F^+ into E^+. By the theorem of Schwartz, Köthe and Schaefer, $T^+ + P^+$ is a Φ_+-operator and $\mathrm{ind}(T^+ + P^+) = \mathrm{ind}(T^+)$. In $T^+ + P^+$ is weakly open and $R(T^+ + P^+)$ is weakly closed. By Lemma 0.12, T+P is weakly open and R(T+P) is closed. Also $\mathrm{ind}(T+P) = \mathrm{ind}(T)$ by duality. Lemma III.14 shows that T+P lifts $\mathcal{G}$ by $\mathcal{F}$ ./.

By taking for $\mathcal{F}$ and $\mathcal{G}$ various families of weakly compact disks, we obtain various corollaries of the preceding proposition. The more notable families are of course those of all finite disks, weakly compact disks or compact disks.

THEOREM III.16 (Finite dimensional perturbations). *Let* E, F *be locally convex spaces and* T, P *operators from* E *into* F *such that* $D(T) \subset D(P)$. *Assume that*

(a) T *is a weak homomorphism*, R(T) *is closed and* $\mathrm{def}(T) < \infty$,

(b) P *is a continuous operator of finite rank* (i.e. R(P) *is finite dimensional*).

Then T+P *is a weak homomorphism*, R(T+P) *is closed*, $\mathrm{def}(T+P) < \infty$ *and* $\mathrm{ind}(T+P) = \mathrm{ind}(T)$.

Proof. It suffices to assume $E = D(T) = D(P)$ and take for $\mathcal{F}$ and $\mathcal{G}$ the families of all finite disks. Obviously T and T+P lift finite disks. On the other hand, U° is weakly compact, thus we may apply Proposition III.15 ./.

REMARK. It is obvious that it suffices a priori to assume that $PU \subset D$ for a Mackey neighborhood U in E and a finite disk $D \subset F$, but this is equivalent to the fact that P is a continuous operator of finite rank, as N(P) is closed and has a finite codimension. Theorem III.16 is also a consequence of Corollary I.27.

THEOREM III.17 (Weakly compact perturbations). *Let* E, F *be locally convex spaces and* T, P *operators everywhere defined from* E *into* F.

Assume that

(a) T *is a weak homomorphism*, R(T) *is closed and* $\mathrm{def}(T) < \infty$,

(b) P *is a weakly compact operator*,

(c) E <u>is a Frechet-Schwartz space</u>.

<u>Then</u> T+P <u>is a weak homomorphism</u>, R(T+P) <u>is closed</u>, def(T+P) $< \infty$ <u>and</u> ind(T+P) = ind(T).

<u>If furthermore</u> F <u>is a countable inductive limit of metrizable spaces and</u> T <u>is continuous, then</u> T <u>and</u> T+P <u>are</u> Φ_-<u>-operators</u>.

<u>Proof</u>. Take for $\mathfrak{F}$ the family of all compact disks and $\mathcal{G}$ the family of all weakly compact disks. If $B \in \mathcal{G}$, then $B \cap R(T)$ is a weakly compact disk in R(T), thus $\tilde{T}^{-1}B$ is a weakly compact disk in E/N(T), where $\tilde{T}$ denotes the injective weak isomorphism from E/N(T) into F induced by T. Since E/N(T) is itself a Frechet-Schwartz space, thus a Montel space (cf. remarks preceding Theorem II.20), it follows that $\tilde{T}^{-1}B$ is in fact a compact disk, the image of a compact disk $A \in \mathfrak{F}$ in E by the quotient operator. Therefore $TA = B \cap R(T)$, which means that T lifts $\mathcal{G}$ by $\mathfrak{F}$. On the other hand, U° is $\mathfrak{F}$-compact. We may therefore apply Proposition III.15.

The last part is a consequence of (1, De Wilde, Theorem 4, p. 87) which asserts that under the assumptions on E, F, a (continuous) weak homomorphism is also open ./.

REMARKS. If in Theorem III.17 F is a countable inductive limit of Frechet spaces, then T is automatically continuous, and even continuous from E into one of the constituent Frechet spaces, by a closed graph theorem of De Wilde (1, p. 54).

The proof of Theorem III.17 shows that we may assume $PU \subset K$ for a Mackey neighborhood U in E and a weakly compact disk $K \subset F$. Also, both T and T+P lift $\mathcal{G}$ by $\mathfrak{F}$.

THEOREM III.18 (Lifting of compact disks). *Let* E, F *be locally convex spaces and* T, P *operators everywhere defined from* E *into* F.

Assume that

(a) T *is a weak homomorphism*, R(T) *is closed and* def(T) $< \infty$,

(b) T *lifts compact disks*,

(c) P *is a compact operator (or even a compact operator when* E *is equipped with the Mackey topology*).

Then T+P *is a weak homomorphism*, R(T+P) *is closed*, def(T+P) $< \infty$ *and* ind(T+P) = ind(T). *Moreover*, T+P *lifts compact disks*.

Proof. It suffices to apply Proposition III.15 with $\mathcal{F}$, $\mathcal{G}$ the families of all compact disks ./.

For small perturbations, we have

PROPOSITION III.19. *Let* E, F *be locally convex spaces*, $\mathcal{F}$, $\mathcal{G}$ *saturated families of weakly compact disks in* E, F *respectively and* T, P *operators everywhere defined from* E *into* F.

Assume that

(a) T *is a weak homomorphism*, R(T) *is closed and* def(T) $< \infty$,

(b) T *lifts* $\mathcal{G}$ *by* $\mathcal{F}$,

(c) *There exists a Mackey neighborhood* W *in* E (*i.e.* $W = K^{\circ}$ *for a weakly compact disk* K *in* E^{+}) *and* $B \in \mathcal{G}$ *such that* $PW \subset B$.

Then there is $\rho > 0$ *such that for* $|\lambda| < \rho$, $T + \lambda P$ *is a weak homomorphism*, $R(T+\lambda P)$ *is closed*, $\mathrm{def}(T+\lambda P) < \infty$ *and* $\mathrm{ind}(T+\lambda P) = \mathrm{ind}(T)$. *Moreover* $T+\lambda P$ *lifts* $\mathcal{G}$ *by* $\mathcal{F}$.

Proof. The proof runs exactly as for Proposition III.15, except that W° is now a bounded Banach disk in $E^+_{\mathcal{F}}$, and we apply Theorem II.8 ./.

REMARK. We may assume that W is simply an absorbent disk in E, and that P be weakly continuous, if $F^+_{\mathcal{G}}$ or $E^+_{\mathcal{F}}$ is complete, in particular if E^+ or F^+ is complete : we then apply Theorem II.4 instead of II.8.

For operators T with $\mathrm{nul}(T) < \infty$, we have

PROPOSITION III.20. Let E, F be locally convex spaces and $\mathcal{F}$, $\mathcal{G}$ saturated families of weakly compact disks in E, F respectively.

Let T, P be operators everywhere defined from E into F. Assume that

(a) T is weakly continuous and a Φ_+-operator,

(b) T lifts $\mathcal{G}$ by $\mathcal{F}$,

(c) There exist a neighborhood U_0 in E and $B \in \mathcal{G}$ such that $PU_0 \subset B$.

Then there exists $\rho > 0$ such that for $|\lambda| < \rho$, $T+\lambda P$ is a (weakly continuous) Φ_+-operator, $\mathrm{nul}(T+\lambda P) \leq \mathrm{nul}(T)$, $\mathrm{def}(T+\lambda P) \leq \mathrm{def}(T)$, $\mathrm{ind}(T+\lambda P) = \mathrm{ind}(T)$ and $T+\lambda P$ lifts $\mathcal{G}$ by $\mathcal{F}$.

If furthermore P is a ~~precompact operator~~, then $T+\lambda P$ lifts $\mathcal{G}$ by $\mathcal{F}$ for all scalars λ.

Proof. By duality T^+ is a Φ_--operator from $F^+_{\mathcal{G}}$ into $E^+_{\mathcal{F}}$. Let $U \subset U_0$ be any neighborhood in E. There is a neighborhood V in F such that $(V)^- \cap (R(T))^- \subset (TU)^-$. By duality, $U^\circ \cap R(T^+) \subset T^+V^\circ$.

From $PU \subset B$ we obtain $P^+B^\circ \subset U^\circ$. But V° is absorbed by B°. Therefore there exists $\rho > 0$ such that for $|\lambda| < \rho$,

$\lambda P^+V^\circ \subset \varepsilon P^+B^\circ \subset \varepsilon U^\circ$ for some $0 < \varepsilon < 1$. Since $P^+B^\circ \subset U^\circ$ and B° is a $\mathcal{G}$-neighborhood, we may apply Theorem I.19 to infer that $T^+ + \lambda P^+$ is open from $F^+_{\mathcal{G}}$ into $E^+_{\mathcal{T}}$, that is $T+\lambda P$ lifts $\mathcal{G}$ by $\mathcal{T}$. If P is precompact then we may assume that PU is precompact; hence P^+V° is precompact in the Banach space generated by U°. We may then apply Theorem I.26 to infer that $T^+ + \lambda P^+$ is open for all λ.

The other assertions are proved in Theorem II.7, P being a Banach-bounded operator ./.

III.4. Some further applications

III.4.1. Extension of a theorem of Sz-Nagy on isomorphisms of subspaces

The following is a version in locally convex spaces of a result of Sz-Nagy for Banach spaces (cf. (5, Gohberg and Krein, Theorem I.2)) :

PROPOSITION III.21. Let X be a locally convex space and P, Q two continuous projections, projecting X onto the subspaces $L = PX$, $M = QX$ respectively. If there exist a neighborhood U, a bounded Banach disk B and $0 < \varepsilon < 1$ such that $(P-Q)U \subset B < \varepsilon U$, then the subspaces L, M are isomorphic. (The restriction of P to M is an isomorphism of M onto L.)

In particular, $\dim L = \dim M$, and the theorem holds for B a priori a bounded disk if X is sequentially complete.

Proof. The identity operator I is an isomorphism of X onto itself, and P-Q is a small Banach-bounded perturbation of I. By virtue of Theorem II.7, I-(P-Q) is an isomorphism of X onto itself.

We have then

$$(I - P + Q)X = X \ ,$$

$$P(I - P + Q)X = PX = L \ .$$

Since $P^2 = P$, it follows that $PQX = PM = L$, which shows that the restriction of P to M maps M onto L. We now prove that it is in fact an isomorphism from M onto L.

We remark first of all that L, M are closed, being the ranges of continuous projections (if $\mathcal{F}$ is a filter in L converging to $x \in X$, then $P\mathcal{F} = \mathcal{F}$ converges to $Px = x \in L$). Let P_M, Q_M denote the restrictions of P, Q to M and I_M the canonical injection of M into X. Then $I_M = Q_M$ and $P_M = I_M + (P_M - Q_M)$.

Consider $U_M = U \cap M$ as a neighborhood in M. Then

$$I_M U_M = U \cap R(I_M) \ ,$$

$$(P_M - Q_M)U_M \subset B \subset \varepsilon U \ .$$

Theorem II.7 shows that $P_M = I_M + (P_M - Q_M)$ is an isomorphism from M onto $R(P_M) = L$ ./.

III.4.2. Some spectral properties of bounded operators in sequentially complete locally convex spaces

In this section we shall always assume X to be a sequentially complete locally convex space defined on the complex field, P a bounded everywhere defined operator and I the identity operator in X.

The <u>resolvent set</u> of P is the set of all scalars λ such that $P+\lambda I$ is an isomorphism from X onto itself. The set-theoretic complement of the resolvent set is the <u>spectrum</u>. The ϕ-<u>set</u> of P is the set of all scalars λ such that $P+\lambda I$ is a Fredholm operator. An element of the ϕ-set is called a ϕ-<u>point</u>.

Following are some known results concerning the spectrum of a bounded operator (cf. (4, Garnir, De Wilde, Schmets)), which may be derived immediately from Theorems II.7, II.8 :

PROPOSITION III.22. <u>If there is a neighborhood</u> U <u>in</u> X, <u>a bounded disk</u> $B\subset X$ <u>and</u> $0<\varepsilon<1$ <u>such that</u> $PU\subset B\subset\varepsilon U$, <u>then</u> I-P <u>is an isomorphism from</u> X <u>onto itself</u>.

REMARK. Obviously, Proposition III.22 holds also for an arbitrary locally convex space X if B is a bounded Banach disk (Theorem II.7). This is also proved in (Martens, E.: Invertibility of an operator. J. London Math. Soc. (2), 12 (1976), 467-468).

PROPOSITION III.23. <u>The resolvent set of</u> P <u>is not empty</u>.

<u>Proof</u>. Indeed, $P+\lambda I = \lambda(\lambda^{-1}P+I)$, $\lambda\neq 0$, is an isomorphism from X onto itself if $|\lambda|$ is large enough ./.

PROPOSITION III.24. <u>The resolvent set and the</u> ϕ-<u>set of</u> P <u>are open</u>.

<u>Proof</u>. If $\lambda_0\neq 0$ belongs to the resolvent set of P then for $\lambda\neq 0$ we have

$$P+\lambda I = \lambda(\lambda^{-1}P+I) = \lambda((\lambda_0^{-1}P+I) + (\lambda^{-1}-\lambda_0^{-1})P) ,$$

where the operator in the brackets is an isomorphism from X onto itself together with $\lambda_o^{-1}P+I$, for $|\lambda^{-1}-\lambda_o^{-1}|$ small enough, or equivalently, for $|\lambda-\lambda_o|$ small enough.

If $\lambda_o = 0$ belongs to the resolvent set of P, then X is in fact a Banach space and P an onto isomorphism. Therefore $P+\lambda I$ is also an onto isomorphism for $|\lambda|$ small enough.

We proceed similarly for the ϕ-set (cf. (5, Gohberg and Krein)) ./.

PROPOSITION III.25. $\mathrm{ind}(P+\lambda I)$ <u>is constant in each (connected) component of the</u> ϕ <u>-set of</u> P.

<u>Proof</u>. If λ, $\lambda_o \neq 0$ are ϕ-points, let

$$P+\lambda I = \lambda((\lambda_o^{-1}P+I) + (\lambda^{-1}-\lambda_o^{-1})P) .$$

By Theorem II.8, $\mathrm{ind}(P+\lambda I) = \mathrm{ind}(\lambda_o^{-1}P+I) = \mathrm{ind}(P+\lambda_o I)$ for $|\lambda^{-1}-\lambda_o^{-1}|$ small enough, or equivalently for $|\lambda-\lambda_o|$ small enough. If 0 is a ϕ-point, then P is a Fredholm operator and in fact X is a Banach space (if $L \oplus N(P) = X$, $\dim N(P)<\infty$, then the restriction of P is an isomorphism from L onto R(P) (codim $R(P)<\infty$); as P is bounded, this shows that N(P) is a normed, thus Banach space). Therefore $P+\lambda I$ is a Fredholm operator and $\mathrm{ind}(P+\lambda I) = \mathrm{ind}(P)$ for $|\lambda|$ small enough.

As a consequent $\mathrm{ind}(P+\lambda I)$ is a continuous function of λ on the ϕ-set of P, therefore it is constant on each component of the ϕ-set of P ./.

REMARK. More generally, if X is a Banach space and $P(\lambda)$ is a continuous function of $\lambda \in \Omega$ with values $P(\lambda)$ Fredholm operators in X, then $\mathrm{ind}(P(\lambda))$ is similarly proved to be constant on each component of Ω. This fact is a useful tool to prove several existence theorems in applications of Perturbation Theory to Differential Equations for instance (cf. (6, Goldberg) and (11, Kato)). We may have similar results for suitably defined "continuous" functions T+P(λ) with T+P(λ) Fredholm operators and P(λ·) Banach-bounded, in a locally convex space.

It is proved in the preceding proof that if 0 is a Φ -point then X is in fact a Banach space. The following is then an immediate consequence of (5, Gohberg and Krein, Theorem 3.2) :

PROPOSITION III.26. <u>If every point in the complex plane is a</u> Φ -<u>point of</u> P, <u>then</u> X <u>is finite dimensional. Equivalently, if</u> X <u>is infinite dimensional, then there should be at least a point</u> λ <u>such that</u> P+ λ I <u>is not a Fredholm operator</u>.

III.4.3. <u>An application in Linear Partial Differential Equations</u>

We now present an example of bounded perturbations of Φ_- -operators in Linear Partial Differential Equations, in which Theorem II.14 can be used. Our standard reference for this example is Hörmander's book (9), to which we refer the reader for more details concerning the various definitions and theorems used in this section.

First we need some definitions.

DEFINITIONS. (a) <u>Temperate weight function</u>. A positive function k defined in the n-dimensional euclidean space R^n is called a <u>temperate weight function</u> if there exist positive constants C and N such that

$$k(\xi+\eta) \leq (1+C|\xi|)^N k(\eta) \quad , \quad \forall\ \xi,\eta \in R^n.$$

A continuous functional on $\mathcal{S}$ is called a <u>temperate distribution</u>. The set of all temperate distribution is the (continuous) dual $\mathcal{S}^+$ of $\mathcal{S}$.

The Fourier transform $f \to \hat{f}$ in $\mathcal{S}^+$ is defined as the adjoint operator of the (classical) Fourier transform in $\mathcal{S}$, i.e. $\hat{f}(\varphi) = f(\hat{\varphi})$, $\varphi \in \mathcal{S}$. The Fourier transform in $\mathcal{S}^+$ extends that in $\mathcal{S}$ if $\mathcal{S}$ is considered as a subspace of $\mathcal{S}^+$ (i.e. $\varphi \in \mathcal{S}$ is considered as a distribution), and it is a (weak) isomorphism of $\mathcal{S}^+$ onto itself.

Let $k \in \mathcal{K}$ and $1 \leq p \leq \infty$. The space $B_{p,k}$ is the set of all temperate distributions $f \in \mathcal{S}^+$ such that

- The Fourier transform $\hat{f}$ is a function,
- $k.\hat{f} \in L^p(R^n)$.

$B_{p,k}$ is a Banach space when equipped with the norm

$$\|f\|_{p,k} = ((2\pi)^{-n}\int|k(\xi)\hat{f}(\xi)|^p \, d\xi)^{1/p}$$

if $p < \infty$, or

$$\|f\|_{\infty,k} = \operatorname*{essential\ sup}_{\xi \in R^n} |k(\xi)\hat{f}(\xi)|$$

if $p = \infty$.

Let Ω be an open set in R^n and $C_0^\infty(\Omega)$ denote the space of C^∞ functions with compact support in Ω. Let $D'(\Omega)$ denote the space of distributions in Ω ($D'(\Omega)$ is the dual of $C_0^\infty(\Omega)$ if $C_0^\infty(\Omega)$ is equipped with the familiar locally convex inductive limit topology).

The space $B_{p,k}^{loc}(\Omega)$ is defined to be the space of $f \in D'(\Omega)$ such that $\varphi.f \in B_{p,k}$ for all $\varphi \in C_0^\infty(\Omega)$. Equipped with the topology defined by the semi-norms

We now come to our example.

EXAMPLE. Let Ω be a convex open set in R^n, $k \in \mathcal{K}$, $1 \leq p < \infty$ and $P(D)$ a differential operator with constant coefficients.

By virtue of (9, Hörmander, Theorem 3.5.5), $P(D)$ is a surjective operator from $B^{loc}_{p,k\tilde{P}}(\Omega)$ onto $B^{loc}_{p,k}(\Omega)$. Since the two spaces are Frechet spaces, it follows that $P(D)$ is a surjective (continuous) Φ_--operator.

Let $Q(D)$ be a differential operator with constant coefficients, weaker than $P(D)$. Let $a = a(x) \in C_0^\infty(\Omega)$. We now consider the differential operator

$$D(\lambda) = P(D) + \lambda\, a(x)Q(D)\ ,\ \lambda \text{ scalar},$$

and show that $D(\lambda)$ is a surjective Φ_--operator and $\mathrm{ind}(D(\lambda)) = \mathrm{ind}(P(D))$ for $|\lambda|$ small enough.

Let $K \subset \Omega$ be the compact support of $a(x)$.

Let V be a neighborhood in $B^{loc}_{p,k}(\Omega)$ defined by

$$V = \left\{ g \in B^{loc}_{p,k}(\Omega) \ :\ \|a.g\|_{p,k} \leq 1 \right\} .$$

We now prove that $a.V = \left\{ a(x).g \ :\ g \in V \right\} = P(D)B$ for some bounded disk B in $B^{loc}_{p,k\tilde{P}}(\Omega)$.

Let $g \in V$. We may consider $a.g$ as a distribution with compact support in R^n. By virtue of (9, Hörmander, Theorem 3.2.1), there exists a fundamental solution E of $P(D)$ (i.e. $P(D)E = \delta$, δ being the Dirac measure), such that

- $E \in B^{loc}_{\infty,\tilde{P}}(R^n)$
- If $f = E * (a.g)$, $*$ being the convolution, then

$$P(D)f = a.g \ .$$

We now show that, when g runs through V, the set B of all such f, considered as distributions in Ω, is bounded in $B^{loc}_{p,k\tilde{P}}(\Omega)$.

Fix any $\varphi \in C_0^{\infty}(\Omega)$. Let K_0 denote the compact support of φ, $K_1 = \{x \in R^n : (\{x\}+K) \cap K_0 \neq \emptyset\}$ (recall that K is the support of a(x)). Then K_1 is a compact set (independent of $g \in V$).

Fix a $\psi \in C_0^{\infty}(R^n)$ such that $\psi = 1$ on a neighborhood of K_1. Then the support of a.g is contained in K $(\forall g \in V)$ and the support of

$$E*(a.g) - (\psi.E)*(a.g) = ((1-\psi).E)*(a.g)$$

does not meet the support K_0 of φ (as supp $(u_1 * u_2) \subset$ supp u_1 + supp u_2).

Therefore $\varphi f = \varphi(E*(a.g)) = \varphi((\psi.E)*(a.g))$, with $\psi.E \in B_{\infty,\tilde{P}}$. By virtue of (9, Hörmander, Theorems 2.2.5, 2.2.6), there exists a constant $C>0$ (depending on φ) such that

$$\|\varphi f\|_{p,k\tilde{P}} \leq C\|a.g\|_{p,k} \cdot \|\psi.E\|_{\infty,\tilde{P}}$$
$$\leq C\|\psi.E\|_{\infty,P} .$$

The latter constant being independent of $g \in V$, this shows that B is bounded in $B^{loc}_{p,k\tilde{P}}(\Omega)$.

The preceding argument is essentially an adaptation of (9, Hörmander, Theorem 2.3.6).

Now Q(D) is a continuous operator. Consequently, there is a neighborhood U in $B^{loc}_{p,k\tilde{P}}(\Omega)$ such that $Q(D)U \subset V$. As a result,

$$a(x)Q(D)U \subset a(x)V \subset P(D)B .$$

If $|\lambda|$ is small enough such that $\lambda B \subset \varepsilon U$ for some $0<\varepsilon<1$, then $D(\lambda) = P(D) + \lambda a(x)Q(D)$ is a surjective ϕ_--operator with $\mathrm{ind}(D(\lambda)) = \mathrm{ind}(P(D))$, by virtue of Theorem II.14.

If Q(D) is such that $\tilde{Q}(\xi)/\tilde{P}(\xi) \to 0$, when $\xi \to \infty$, then

$a(x).Q(D)$ is a compact operator, therefore $D(\lambda) = P(D) + a(x)Q(D)$ is a Φ_--operator with $\mathrm{ind}(D(\lambda)) = \mathrm{ind}(P(D))$ for all λ. Further, $D(\lambda)$ is surjective except for at most a countable set of exceptional points $\{\lambda_i\}$, with no accumulation point at finite distance.

The preceding also holds for a P-convex open set Ω (cf. definition in (9, Hörmander)).

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