POSITRONIUM DECAY IN A STRONG MAGNETIC FIELD

ONE AND TWO PHOTON ANNIHILATION PROCESSES IN A STRONG MAGNETIC FIELD

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A Thesis

Submitted to the School of Graduate Studies

in Partial Fulfilment of the Requirements

for the Degree

Master of Science

McMaster University

December 1977

MASTER OF SCIENCE (1977) McMASTER UNIVERSITY
(Physics) Hamilton, Ontario

TITLE: One and Two Photon Annihilation Processes in a Strong Magnetic Field

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NUMBER OF PAGES: vi,74

ABSTRACT

We present theoretical calculations of the annihilation rates for one and two photon decay of ground state positronium in a strong magnetic field, of order (10^{12}-10^{13}) Gauss such as assumed for neutron star surfaces. The calculations indicate that the one photon process at a field of 10^{13} Gauss is of order (10^{-4}) compared to the two photon process. Consequently, a one photon annihilation line at 1.022 MeV is not expected in the spectra of neutron stars. The two photon rate is of order (5×10^{13} sec^{-1}) at 10^{13} G; the half width is 43 keV. Thus, at such fields, the two photon line is fairly narrow and not smeared out through the magnetic field effects.

ACKNOWLEDGEMENTS

My thanks to my supervisor, Dr. P. G. Sutherland, for his patient encouragement and valuable discussions throughout the course of this research.

I would like to acknowledge the financial support provided by the Commonwealth Scholarship and Fellowship Committee in making my studies in Canada possible.

I particularly thank Mrs. Helen Kennelly for her eleventh hour typing of this thesis.

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CHAPTER 1

INTRODUCTION

Neutron stars and similar astrophysical objects provide a challenge to both the theoretician and the observational astronomer, because the extreme nature of the physical conditions found in such stars can produce effects with no laboratory equivalent. In particular, neutron stars are believed to have a surface magnetic field of 10^{12}-10^{13} G. (Ruderman 1972); in such fields the magnetic forces on electrons become much stronger than the Coulomb forces in atoms and thus matter under these conditions is qualitatively different from that of our normal experience (Ruderman 1974).

Magnetic fields of this order produce an extremely complex magnetosphere outside the neutron star, and so will affect the various possible radiation emission processes such as synchrotron radiation by electrons, curvature radiation and so on. Typical electron energies are in the range 10^{10}-10^{14} eV and produce γ-rays with energies of order 10^{9} eV. These are sufficiently energetic to create electron-positron pairs within a few stellar radii of the surface (Ruderman 1972). The models proposed by Sturrock (1971), Soldreich and Julian (1969) and others suggest that at least part of the magnetosphere consists of relativistic electron-positron pairs.

Positrons can also be produced in nuclear reactions in accreted material on the surfaces of neutron stars in close binary systems, as suggested by Ramaty, Borner and Cohen (1973). The calculations in the above paper indicate that photons produced through pair annihilation on the surface of the star will escape from the star without significant scattering. This second mechanism for pair production and thus annihilation appears to be more important: Brecker (1977) concludes that the spectral lines corresponding to pair annihilation processes are more likely to arise near the surface of the neutron star than in the magnetosphere. Such lines will be redshifted to lower energies.

The fundamental difference between pair annihilation processes in a magnetic field instead of in field-free space is that the magnetic field can take up momentum from the reaction. In particular, one photon annihilation is possible in a magnetic field, unlike field-free space. The emitted photon has an energy of 1.022 MeV. Conventional two photon annihilation also occurs but with the transition rate and momenta changed because of the magnetic field effect cited above. There will be a spread of photons about the free space annihilation line at 511 keV (rest mass of the electron).

In this thesis, we calculate the transition rates for one and two photon annihilation in fields of order 10^{12}-10^{13} G. The motivation behind the calculation is that there have been various reports in the literature over the last few years

of observations of a 511 keV line from neutron star sources. These include the observations of Johnson, Harnden, Haymes (1972) and Levental, MacCallum, Watts (1977). From a series of observations of the galactic centre, Johnson et al reported a statistically significant flux of $1.8 \cdot 10^{-3}$ photons $cm^{-2} s^{-1}$ at an energy of (473±30) keV. This radiation can be interpreted as a redshifted 511 keV line (Ramaty et al 1973) if it is assumed the average neutron star mass is about 0.6 $M_\odot$ with a radius of 10 km. This is consistent with the usual estimates of possible mass and radius. Leventhal et al, from observations of the Crab pulsar, report the existence of a possible γ-ray line at (400±1) keV with a flux of $(2.24 \pm 0.65) \cdot 10^{-3}$ photons cm^{-2} s^{-1}. This corresponds to $z_{Crab} \sim 0.28$, with $M_{Crab} \sim 1.4$ $M_\odot$. There are at present no reported observations either for or against a spectral line at 1.022 MeV, indicating a one photon annihilation process.

Because the one photon and two photon annihilation processes will compete with each other, it is of interest to know the relative rate, which is a function of the field strength. We may estimate the magnetic field at which one photon annihilation becomes important as follows. The photon has a momentum $q = 2 mC$ which must be provided by the magnetic field. Electrons and positrons in magnetic fields are confined by the field in Landau orbitals of radius $\sim \hat{\rho} \equiv (\hbar C/eB)^{1/2}$ and momenta are available of order $\hbar/\hat{\rho}$. Setting this equal to q

we find $B \sim 4\ B_q$ where

$$B_q = m^2c^3/e\hbar = 4.4 \times 10^{13}\ G.$$

The present calculation indicates that for fields of 10^{12} G the one photon rate is of order (10^{-5}) in comparison to the two photon annihilation rate. The rate vanishes exponentially for fields below B_q as $\exp(-\frac{4B_q}{B})$. At fields comparable to B_q the one photon process begins to dominate the positronium decay. At 10^{12} G, the two photon rate is $\tau^{-1} \gtrsim 10^{12}\ s^{-1}$ and increases with increasing field; the rate of increase of τ^{-1} is much slower than for the one photon problem. As expected, the photons show a spread in energy and momentum centred on the free space line at 511 keV. The calculation suggests that with the currently accepted range of neutron star fields (10^{12}-10^{13} G) one photon annihilation is too slow to produce a significant, observable spectral line at 1.022 MeV.

Positronium in a Strong Magnetic Field

The Quantum-Mechanical description of the bound state of an electron positron pair (henceforth, positronium) in a magnetic field poses some interesting problems because the presence of such a field means that there can be no transformation to a relative coordinate system. To see this more explicitly, consider a region of space with a uniform magnetic field in the z direction. The electron and positron will spiral around the field lines, but in opposite directions since they have

opposite charges. It is precisely this opposite rotation that means no relative angular coordinate can be introduced. The absence of a relative angular coordinate has important applications because processes are allowed in which the field can absorb or provide momentum. Synchrotron radiation is one such process in which momentum balance is provided through the field. In contrast, positronium in free space can be described using a relative coordinate system so that it is mathematically equivalent to a hydrogen atom with an appropriate reduced mass particle ($m_e/2$) corresponding to the electron. The equivalent Bohr radius for positronium is twice that of hydrogen and with this modification, the wavefunctions for the system (in the relative coordinates) are identical to those for hydrogen.

Returning to the description of positronium in a strong magnetic field, the field has no effect on motion in the z direction and so the Coulomb force provides the necessary binding force. Magnetic forces constrain the electron and positron to spiral about the field lines and so the motion in the radial direction must resemble that in the absence of the other particle. The solutions for a non-relativistic particle in a magnetic field are known as Landau orbitals. (Landau and Lifshitz, 1965). Landau orbitals however neglect the spin of the electron or positron and so are not a complete description. Thus, in a strong field such as mentioned in the introduction, the radial wavefunction for positronium will be (to

first order) a product of Landau orbitals, one for each particle. The Coulomb force is too weak to significantly perturb the radial wavefunctions and determines the form of the z wavefunction as that appropriate to a bound state.

The parameter that characterises the range of the radial wavefunction is the radial length $\hat{\rho}$, where $\hat{\rho}$ is given by

$$\hat{\rho}^2 = \frac{\hbar c}{eB} . \qquad (1.1)$$

For fields of 10^9 G, $\hat{\rho}$ is of the same order as the Bohr radius (a_0) and the Coulomb attraction and the magnetic forces are of comparable strength. For weaker fields, the Coulomb force will dominate and the bound state will tend to resemble hydrogenic positronium; for stronger fields the bound state will be formed from the product of two Landau orbitals with some appropriate z wavefunction reflecting Coulomb binding. With this quantitative variation as a guide, we now form a variational bound state wavefunction (Rau and Spruch 1976) that is exact in both the low and high field limits. The general form will be

$$\psi_b(\underset{\sim}{r}_+, \underset{\sim}{r}_-) = N(B) L_m(\rho_+, \theta_+) L_{m'}(\rho_-, \theta_-) Z(z_+, z_-) \psi_H(\underset{\sim}{r}_+ - \underset{\sim}{r}_-) . \qquad (1.2)$$

In (1.2) $L_m(\rho_\pm, \theta_\pm)$ are Landau orbitals with the form

$$L_m(\rho, \theta) = \frac{1}{(2\pi)^{1/2}} \exp(-\frac{\xi}{2}) \, \xi^{|m|/2} \, W(\xi) \exp im\theta \qquad (1.3)$$

where $\xi = \frac{1}{2\hat{\rho}^2} \rho^2$

$$W(\xi) = F(-(\beta - \tfrac{1}{2}|m| - \tfrac{1}{2}), |m| + 1, \xi) \qquad (1.4)$$

is a confluent hypogeometric function: $\beta - \frac{1}{2}|m| - \frac{1}{2} \geq 0$

$\psi_H(\underset{\sim}{r}_+ - \underset{\sim}{r}_-)$ is the hydrogenic wavefunction in the relative coordinates $\underset{\sim}{r} = (\underset{\sim}{r}_+ - \underset{\sim}{r}_-)$

$$\psi_H(\underset{\sim}{r}) = R_{n\ell}(r) Y_{\ell m}(\theta,\phi) \qquad (1.5)$$

where $R_{n\ell}(r)$, $Y_{\ell m}(\theta,\phi)$ are the usual radial and spherical harmonic functions respectively. In (1.2), N(B) reflects an overall normalization, a function of the field B, which is necessary because of the mixed cylindrical and spherical coordinate systems. In this problem, we are concerned with the ground state of positronium and hence we take the lowest energy forms of (1.3), (1.4), (1.5)

$$L_0(\rho,\theta) = \frac{1}{(2\pi)^{1/2}} \exp - \frac{\rho^2}{4\hat{\rho}^2} \qquad (1.6)$$

$$\psi_0(\underset{\sim}{r}) = \frac{1}{(4\pi)^{1/2}} \frac{2}{(2a_0)^{3/2}} \exp - \frac{r}{2a_0} . \qquad (1.7)$$

$2a_0$ appears in (1.7) instead of a_0 because of the reduced mass mentioned earlier. As a final note, it is possible to introduce a screened charge effect into (1.7) to improve the accuracy of calculations.

The z wavefunction $Z(z_+,z_-)$ is effectively a function in the relative coordinate $z = (z_+ - z_-)$ because the presence of the field does not affect motion in this direction and so

the wavefunction has a similar dependence on the relative z coordinate as for the free space description. In addition, this means that the total z momentum of the system will vanish, with equal and opposite z momentum for the electron and positron respectively. We choose $Z(z_+,z_-)$ to be

$$Z(z_+,z_-) = L^{-1/2}(\lambda\sqrt{\frac{2}{\pi}})^{1/2} \exp-\lambda^2 z^2 \quad . \qquad (1.8)$$

For future use, we define $F(z)$ to be

$$F(z) = (\lambda\sqrt{\frac{2}{\pi}})^{1/2} \exp-\lambda^2 z^2$$

so that $Z(z_+,z_-) = L^{-1/2}\ F(z)$. L is the periodic length. A gaussian is chosen as the first order approximation (one parameter λ) to the z wavefunction corresponding to a bound state. The parameter λ is chosen via a variational calculation so as to minimize the total energy. The calculations (Table I, later in this chapter) show that $\lambda^{-1} \sim a_0$ for fields of 10^{12} G. Putting (1.8), (1.7), (1.6) together, the appropriate ground state variational wavefunction has the form

$$\psi(\underset{\sim}{r}_+,\underset{\sim}{r}_-) = N^*(B)\ \exp(-\frac{\rho_+^2}{4\hat{\rho}^2})\exp(-\frac{\rho_-^2}{4\hat{\rho}^2})\exp(-\lambda^2 z^2)\exp(-\frac{r}{2a_0}) \qquad (1.9)$$

All the appropriate constants have been grouped into $N^*(B)$.

To see that (1.9) is a suitable wavefunction, consider the behaviour as $B \to 0$ and $B \to \infty$. In the low field limit, as $B \to 0$, $\hat{\rho} \to \infty$ and $\lambda \to 0$ so that

$$\psi(\underset{\sim}{r}_+,\underset{\sim}{r}_-) \sim \exp - \frac{r}{2a_0} \qquad (1.10)$$

corresponding to a hydrogenic wavefunction. In the high field limit, as $B \to \infty$, $\hat{\rho} \to 0$ so that $\hat{\rho} << a_0$. This means that the term $\exp(-\frac{r}{2a_0})$ is effectively constant over the region of interest - the cylinder of radius $\hat{\rho}$, length $\lambda^{-1/2}$ - which contains the volume of the bound state. Thus, as $B \to \infty$

$$\psi_b(\underset{\sim}{r}_+, \underset{\sim}{r}_-) \sim \exp(-\frac{\rho_+^2}{4\hat{\rho}^2}) \exp(-\frac{\rho_-^2}{4\hat{\rho}^2}) \exp -\lambda^2 z^2 \qquad (1.11)$$

which corresponds to the expected high field form. Nonetheless, in spite of the favourable limits exhibited by (1.9), in this calculation for reasons of simplicity we will ignore the low field limit and use (1.11) as the ground state wavefunction. This is strictly true only for high fields, but at $B \sim 10^{12}$ Gauss, the relevant parameters are

$$\begin{aligned} \hat{\rho} &= 2.56\times10^{-10} \text{ cm} \\ a_0 &= 52.9\times10^{-10} \text{ cm} \\ \lambda^{-1} &= 65.1\times10^{-10} \text{ cm} . \end{aligned} \qquad (1.12)$$

Thus $\hat{\rho} << 2a_0$ and so the approximation inherent in (1.11) is by no means inaccurate for the range of field strengths under consideration: $10^{12} \lesssim B \lesssim 10^{13}$ Gauss.

The problem with (1.11) is that the formalism makes no allowance for the inclusion of spin effects. For fields of 10^{12} Gauss, the energy ε_m required to achieve spin flip is

$$\varepsilon_m \sim 2\, e\hbar B/2mc = 11.1 \text{ keV} . \qquad (1.13)$$

At the surface of a neutron star, the relevant thermal energy is $kT \lesssim 1$ keV and this ensures that the electron and positron will have their spins aligned with the field, but in opposite directions because of the opposite charge. The ground state spin configuration will consist of a spin down electron (anti-parallel to the field) and a spin up positron (parallel to the field). In principle, the modification to (1.11) necessary to incorporate spin is to replace the Landau orbitals with the appropriate relativistic solutions: the Landau orbitals are in fact the non-relativistic limit of these relativistic solutions, outlined in Appendix A. In this problem, while the radial velocities may be relativistic, the z momentum of the particles is small (as shown later) and so for the ground state we now use the usual non-relativistic two component spinors, $\chi(\pm \frac{1}{2})$. Such an approximation is really only possible for the ground state because at these fields of 10^{12} Gauss, the next lowest relativistic solution includes a 4-spinor in which the so called "large" and "small" components are in fact comparable and the separation into two component spinors is no longer valid. Thus, including spin, the appropriately normalized ground state wavefunction is

$$\psi_b(\underset{\sim}{r}_+, \underset{\sim}{r}_-) = R_{e+}(w_+)\chi_{e+}(\tfrac{1}{2}) R_{e-}(w_-)\chi_{e-}(-\tfrac{1}{2}) Z(z_+, z_-) \qquad (1.14)$$

with

$$w_\pm = \gamma \rho_\pm^2 \quad : \quad \gamma = \frac{1}{2\hat{\rho}^2} \qquad (1.15)$$

$$R_{e^{\pm}}(w_{\pm}) = \left(\frac{\gamma}{\pi}\right)^{1/2} \exp\left(-\frac{\gamma}{2}\rho_{\pm}^2\right) \qquad (1.16)$$

and $Z(z_+,z_-)$ is as defined in (1.8). In (1.14), the energy of both the electron and positron is $E_+ = E_- = m$ in spite of the circular motion of the particles; this is because the kinetic energy due to the radial motion is exactly cancelled by the spin energy. The wavefunction defined in (1.14) satisfies the usual normalization condition. If we work in cylindrical coordinates (ρ,θ,z) then

$$\begin{aligned}\langle\psi_b|\psi_b\rangle &= \int_0^{\infty} R^*_{e^+}(w_+)R_{e^+}(w_+)\rho_+ d\rho_+ \int_0^{\infty} R^*_{e^-}(w_-)R_{e^-}(w_-)\rho_- d\rho_- \\ &\times \int_0^{2\pi} d\theta_+ \int_0^{2\pi} d\theta_- \int_{-L/2}^{L/2} dZ \int_{-L/2}^{L/2} Z(z_+,z_-)dz \qquad (1.17)\\ &= 1 \,.\end{aligned}$$

In (1.17), $Z = \frac{1}{2}(z_+ + z_-)$ is the centre of mass z coordinate. In particular, $\lambda^{-1} << L$ and so the limits on the integration over the relative z coordinate $(z_+ - z_-)$ are effectively $(\pm\infty)$. Implicit in (1.17) is the normalization of the spinors:

$$\chi^t_{e^+}(\tfrac{1}{2})\chi_{e^+}(\tfrac{1}{2}) = \chi^t_{e^-}(-\tfrac{1}{2})\chi_{e^-}(-\tfrac{1}{2}) = 1 \qquad (1.18)$$

Representation of $\psi_b(r_+,r_-)$ as a Sum Over Orthonormal States

The ground state wavefunction can also be represented as a sum over the orthonormal solutions for electrons as positrons as defined in Appendix A. The only part affected is the

wavefunction $Z(z_+,z_-)$ because the radial component is already in a form appropriate to solutions in a magnetic field. The appropriate states are plane waves ($\exp ik_3 z_\pm$) as these correspond to the z component wavefunctions defined in (A.7) and (A.21) as the z component of the solutions to the Dirac equation in a magnetic field. To see this in more detail, if we define the state vector for ground state positronium to be $|e^+e^-\rangle$ then

$$|e^+e^-\rangle = \sum_{k_3} \phi(k_3) |0\ 0\ -k_3\rangle_{e+}\ |0\ 0\ k_3\rangle_{e-} \quad . \qquad (1.19)$$

In (1.19), $|0\ 0\ \mp k_3\rangle_{e^\pm}$ represents the state vectors for a positron and electron with quantum numbers $n = \ell = s = 0$. (n,ℓ,s identify a solution for a particle in a field, as summarized in Appendix A.) $\phi(k_3)$ is the expansion coefficient for a particular value of z momentum k_3: as mentioned previously, the positron and electron have equal and opposite momenta, and hence have z momenta $(-k_3)$ and k_3 respectively.

In (1.19) there is the implicit assumption that the range of k_3 contributing to (1.19) is $k_3 \sim 0$, otherwise the non-relativistic form of the electron and positron spinors is no longer valid and (1.19) is not true. This approximation can be put on a rather more rigorous footing if we use a result to be derived in (1.28), namely

$$\phi(k_3) \sim \exp(-\frac{k_3^2}{4\lambda^2}) \quad . \tag{1.20}$$

From this result, it is apparent that any important physical effects involve k_3: $0 \leq k_3 \leq 2\lambda$. The calculation of λ as a function of the field shows that for 10^{12} Gauss $\lambda \sim a_0^{-1}$ (quoted earlier) and thus

$$k_3 \sim \frac{2}{a_0} \sim 2\alpha m_0 \tag{1.21}$$

where α is the fine structure constant and m_0 is the electron mass. The parameter that appears in the relativistic spinors is $(k_3/E+m_0)$ and from (1.21)

$$\frac{k_3}{E+m_0} \sim \alpha = \frac{1}{137} \tag{1.22}$$

and so the assumption $k_3 \sim 0$ is sufficiently accurate that the non-relativistic spinors can be used without seriously altering the physical aspects of the problem.

Returning to (1.19), the bound state wavefunction is given by

$$\psi_b(\underset{\sim}{r}_+,\underset{\sim}{r}_-) = \langle \underset{\sim}{r}_+\underset{\sim}{r}_- | e^+e^- \rangle = \sum_{k_3} \phi(k_3) \langle \underset{\sim}{r}_+ | 00\text{-}k_3 \rangle_{e^+} \langle \underset{\sim}{r}_- | 00\text{+}k_3 \rangle_{e^-} \tag{1.23}$$

with

$$\langle \underset{\sim}{r}_+ | 00\text{-}k_3 \rangle_{e^+} = \frac{1}{\sqrt{L}}(\frac{\gamma}{\pi})^{1/2} \exp(-ik_3 z_+)\exp(-\frac{\gamma}{2}\rho_+^2) \tag{1.24}$$

$$\langle r_- | 00\ k_3 \rangle_{e^-} = \frac{1}{\sqrt{L}}(\frac{\gamma}{\pi}) \exp(ik_3 z_-)\ \exp(-\frac{\gamma}{2}\rho_-^2) \quad . \tag{1.25}$$

The variational form of $\psi_b(\underset{\sim}{r}_+,\underset{\sim}{r}_-)$ is, from (1.14),

$$\psi_b(\underset{\sim}{r}_+,\underset{\sim}{r}_-) = (\tfrac{\gamma}{\pi})\exp(-\tfrac{\gamma}{2}\rho_+^2)\exp(-\tfrac{\gamma}{2}\rho_-^2)\ \frac{1}{\sqrt{L}}\ (\lambda\sqrt{\tfrac{2}{\pi}})^{1/2}\exp-\lambda^2 z^2 \quad (1.26)$$

If we equate (1.23) and (1.26), using (1.24) and (1.25), then

$$\sum_{k_3}\ \phi(k_3)\ \frac{\exp(-ik_3 z)}{L} = \frac{1}{\sqrt{L}}\ (\lambda\sqrt{\tfrac{2}{\pi}})^{1/2}\ \exp-\lambda^2 z^2\ . \quad (1.27)$$

This is merely a representation of $Z(z_+,z_-)$ in terms of plane waves, as stated earlier in the chapter. Multiplying (1.27) by $\exp(ik_3' z)$ and integrating over z leads to, after some simplification,

$$\phi(k_3) = \frac{1}{\sqrt{L}}\ (\lambda\sqrt{\tfrac{2}{\pi}})^{1/2}\ \sqrt{\frac{\pi}{\lambda^2}}\ \exp(-\frac{k_3^2}{4\lambda^2}) \quad (1.28)$$

the result used in (1.20) to justify the approximation $k_3 \sim 0$.

The variational bound state can also be represented as a sum of free particle plane wave solutions in place of the magnetic field solutions in (1.19). Such a representation is useful in the calculation of the two photon annihilation process. The appropriate expansion coefficient $\phi(\underset{\sim}{p}_+,\underset{\sim}{p}_-)$ is given by

$$|e^+e^-> = \sum_{\underset{\sim}{p}_+}\ \sum_{\underset{\sim}{p}_-}\ \phi(\underset{\sim}{p}_+,\underset{\sim}{p}_-)\,|\underset{\sim}{p}_+,\underset{\sim}{p}_-> \quad (1.29)$$

where $\underset{\sim}{p}_+,\underset{\sim}{p}_-$ are the positron and electron plane wave momenta. The bound state wavefunction is given by

$$\psi_b(\underset{\sim}{r}_+,\underset{\sim}{r}_-) = <\underset{\sim}{r}_+\underset{\sim}{r}_-|e^+e^-> = \sum_{\underset{\sim}{p}_+}\ \sum_{\underset{\sim}{p}_-}\phi(\underset{\sim}{p}_+,\underset{\sim}{p}_-)<\underset{\sim}{r}_+|\underset{\sim}{p}_+><\underset{\sim}{r}_-|\underset{\sim}{p}_-> \quad (1.30)$$

with

$$\langle \underset{\sim}{r}_+ | \underset{\sim}{p}_+ \rangle = \frac{1}{\sqrt{V}} \exp i\underset{\sim}{p}_+ \cdot \underset{\sim}{r}_+$$
$$\langle \underset{\sim}{r}_- | \underset{\sim}{p}_- \rangle = \frac{1}{\sqrt{V}} \exp i\underset{\sim}{p}_- \cdot \underset{\sim}{r}_- \qquad (1.31)$$

where $V = L^3$ is the volume of the box used for normalization. Using the orthogonality properties of plane waves, (1.30) can be rearranged to yield $\phi(\underset{\sim}{p}_+, \underset{\sim}{p}_-)$:

$$\phi(\underset{\sim}{p}_+, \underset{\sim}{p}_-) = \int \psi_b(\underset{\sim}{r}_+, \underset{\sim}{r}_-) \frac{\exp -i\underset{\sim}{p}_+ \cdot \underset{\sim}{r}_+}{\sqrt{V}} \frac{\exp -i\underset{\sim}{p}_- \cdot \underset{\sim}{r}_-}{\sqrt{V}} d^3\underset{\sim}{r}_+ d^3\underset{\sim}{r}_- \qquad (1.32)$$

Substitution of $\psi_b(\underset{\sim}{r}_+, \underset{\sim}{r}_-)$ from (1.14) gives an explicit form for $\phi(\underset{\sim}{p}_+, \underset{\sim}{p}_-)$:

$$\phi(\underset{\sim}{p}_+, \underset{\sim}{p}_-) = \frac{1}{L^2\sqrt{L}} \frac{(2\pi)^2}{\gamma\pi} (\lambda \sqrt{\frac{\pi}{2}})^{1/2} \sqrt{\frac{\pi}{\lambda^2}} \exp -\frac{1}{2\gamma} (p_{+\perp}^2 + p_{-\perp}^2)$$
$$\times \exp(-\frac{p_{+z}^2}{4\lambda^2}) \delta_{p_{+z}+p_{-z},0} \qquad (1.33)$$

In (1.33) $p_{\pm\perp}^2 = p_{\pm x}^2 + p_{\pm y}^2$.

The Kronecker delta function $\delta_{p_{+z}+p_{-z},0}$ contains the information that there is no total z momentum, as mentioned earlier.

Variational Calculation of λ as a Function of the Field

A quantitative analysis of the effect of the magnetic field on the binding energy suggests that as the field increases, the electron and positron move closer together, and so the binding energy increases. This is reflected in a larger λ as this indicates a more peaked Gaussian and hence a higher probability of being found near the origin, thus minimizing the (negative) electrostatic energy. To investigate this in more detail, consider a variational calculation to minimize the Coulomb energy as a function of λ. Strictly,

$$E = \langle \psi_b | H | \psi_b \rangle \tag{1.34}$$

where $H = K(\underset{\sim}{p}_+, \underset{\sim}{p}_-) + V(\underset{\sim}{r}_+, \underset{\sim}{r}_-)$. However, because we are interested in the variation of λ, it is only necessary to evaluate the terms appropriate to motion in the z direction and so we now remove the radial energy operators from $H(\underset{\sim}{p}_+, \underset{\sim}{p}_-)$. Thus, the kinetic energy contribution to (1.34) is

$$\begin{aligned} \langle \psi_b | K_z | \psi_b \rangle &= \int_{-L/2}^{L/2} F(z)^* K_z F(z) dz \\ &= -\frac{1}{m_e} \int_{-\infty}^{+\infty} \lambda \sqrt{\frac{2}{\pi}} \exp -\lambda^2 z^2 \{ \frac{\partial^2}{\partial z^2} \exp -\lambda^2 z^2 \} dz \end{aligned} \tag{1.35}$$

with $\lambda^{-1} << L$ as before.

Evaluation of (1.35) gives

$$\langle \psi_b | K_z | \psi_b \rangle = \frac{1}{m_e} \quad . \tag{1.36}$$

The potential energy term has the form

$$\langle \psi_b | V | \psi_b \rangle = \langle \psi_b | \frac{-e^2}{|\underset{\sim}{r}_+ - \underset{\sim}{r}_-|} | \psi_b \rangle \tag{1.37}$$

$$= -\frac{e^2}{L} \int R^*(w_+) R^*(w_-) F^*(z) \frac{1}{|\underset{\sim}{r}_+ - \underset{\sim}{r}_-|} R(w_+) R(w_-) F(z)$$

$$\times \rho_+ d\rho_+ \rho_- d\rho_- d\theta_+ d\theta_- \, dz dZ \quad . \tag{1.38}$$

The easiest way to evaluate (1.38) is to use an expansion for $\frac{1}{|\underset{\sim}{r}_+ - \underset{\sim}{r}_-|}$ in Bessel functions (Jackson, 1962), as in

$$\frac{1}{|\underset{\sim}{r}_+ - \underset{\sim}{r}_-|} = \sum_{m=-\infty}^{m=+\infty} \exp im(\theta_+ - \theta_-) \int_0^\infty \exp(-k|z|) J_m(k\rho_+) J_m(k\rho_-) dk. \tag{1.39}$$

Substitution of (1.39) into (1.38), together with the definitions for $R(w_\pm)$, $F(z)$ given earlier yields

$$\langle \psi_b | V | \psi_b \rangle = \sum_{m=-\infty}^{m=+\infty} \left(\frac{\gamma}{\pi}\right)^2 \lambda \sqrt{\frac{2}{\pi}} \int_0^\infty dk \int_0^\infty e^{-\gamma \rho_+^2} J_m(k\rho_+) \rho_+ d\rho_+$$

$$\times \int_0^\infty e^{-\gamma \rho_-^2} J_m(k\rho_-) \rho_- d\rho_- \int_0^{2\pi} d\theta_+ \int_0^{2\pi} \exp im(\theta_+ - \theta_-) d\theta_-$$

$$\times \int_{-\infty}^{+\infty} e^{-2\lambda^2 z^2 - k|z|} dz. \tag{1.40}$$

In (1.40) the centre of mass integration has already been completed:

$$\frac{1}{L}\int_{-L/2}^{L/2} dZ = 1.$$

Equation (1.40) consists of separable integrals and evaluation of the angular integral gives:

$$\int_0^{2\pi} d\theta_+ \int_0^{2\pi} d\theta_- \exp\, im(\theta_+ - \theta_-) = (2\pi)^2 \delta_{m,0} \qquad (1.41)$$

and so only the term with m = 0 contributes. Hence the radial integrals are

$$\int_0^{\infty} \exp(-\gamma\rho_\pm^2) J_0(k\rho_\pm)\rho_\pm d\rho_\pm = \frac{1}{2\gamma} \exp - \frac{k^2}{4\gamma} . \qquad (1.42)$$

The z integral is

$$\int_{-\infty}^{\infty} \exp(-2\lambda^2 z^2 - k|z|)dz = 2\int_0^{\infty} \exp(-2\lambda^2 z^2 - kz)dz$$

$$= \frac{1}{2\lambda} \exp\left(\frac{k^2}{8\lambda^2}\right) \sqrt{\frac{\pi}{2}} \left\{1-\phi\left(\frac{k}{2\sqrt{2}\lambda}\right)\right\} \qquad (1.43)$$

where $\phi\left(\frac{k}{2\sqrt{2}\lambda}\right)$ is the Probability Integral, defined by

$$\phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt.$$

Thus the potential energy becomes

$$\langle \psi_b | V | \psi_b \rangle = \int_0^\infty \exp\left(-\frac{k^2}{2\gamma} + \frac{k^2}{8\lambda^2}\right)\left\{1 - \left(\frac{k}{2\sqrt{2}\lambda}\right)\right\} dk \quad \ldots \quad (1.44)$$

After some manipulation, this integral can be evaluated so that

$$\langle \psi_b | V | \psi_b \rangle = \sqrt{\frac{2}{\pi}} \quad a \ln\left(\frac{a+1}{a-1}\right) \qquad (1.45)$$

where $a = \left(1 - \frac{4\lambda^2}{\gamma}\right)^{-1/2}$.

Combining the kinetic (1.36) and potential (1.45) energy terms, the total energy is

$$E = \frac{1}{m_e} \lambda^2 - e^2 \sqrt{\frac{2}{\pi}} \lambda a \ln\left(\frac{a+1}{a-1}\right)$$

or

$$E = \frac{e^2}{2a_0} \left(2\lambda^2 - a_0 \sqrt{\frac{8}{\pi}} \lambda a \ln\left(\frac{a+1}{a-1}\right)\right) \qquad (1.46)$$

where in (1.46) λ is in units of a_0^{-1} and the energy E is in Rydbergs ($e^2/2a_0$). To find the value of λ for some magnetic field, it is necessary to minimize (1.46) with respect to λ. This can be done either of two ways.

1. Equating $\frac{\partial}{\partial \lambda}$ (E) to zero and solving the resulting equation - this is fairly complicated.
2. Numerically minimizing (1.46) through a suitable iteration procedure. The results of such a calculation are shown below and demonstrate the expected behaviour - as the field increases the binding energy increases and the

Gaussian defined by λ becomes more peaked.

Table I: λ as a Function of the Magnetic Field

B_{12} (units of 10^{12} Gauss)	λ (a_0^{-1})	E(Rydbergs)
1	1.23	-6.79
2	1.39	-8.18
3	1.49	-9.09
4	1.56	-9.77
5	1.62	-10.33
10	1.80	-12.19

1 Rydberg = 13.6 eV

As a comparison, for no magnetic field, the ground state energy for hydrogenic positronium is 0.5 Rydberg, and so the field considerably increases the binding.

Table I indicates that $\lambda^{-1} \sim a_0$ and so, as argued previously, the magnetic field has not significantly changed the physical characteristics of the z wavefunction. Thus, while the radial motion is relativistic, accurate (non-relativistic) approximations can be made for k_3 and similarly for p_+, p_- defined in the expansions for the two photon process.

CHAPTER 2

PAIR ANNIHILATION IN FREE SPACE

As mentioned in the Introduction, the characteristics of positronium decay in a magnetic field differ from the corresponding results in free space because the presence of the field means that it is no longer necessary for the annihilation process to conserve energy and momentum simultaneously - the field can provide the required momentum components. This means that there is a possible spread in the energy and momentum of the emergent photons, centred about the free space values i.e. the total energy of the photons remains $2\ m_e c^2$. To provide a basis for comparison, consider the calculations for pair annihilation in free space.

For a first order interaction - corresponding to one photon decay - the scattering matrix element $S_{fi}^{(1)}$ vanishes because in free space it is impossible to simultaneously conserve both energy and momentum. Thus, the transition rate for a one photon annihilation in a magnetic field should vanish as $B \to 0$.

The two photon calculation is somewhat more complicated and the theory outlined here basically follows that in Bjorken and Drell (Volume I, 1964). The appropriate second order Feynman diagrams are

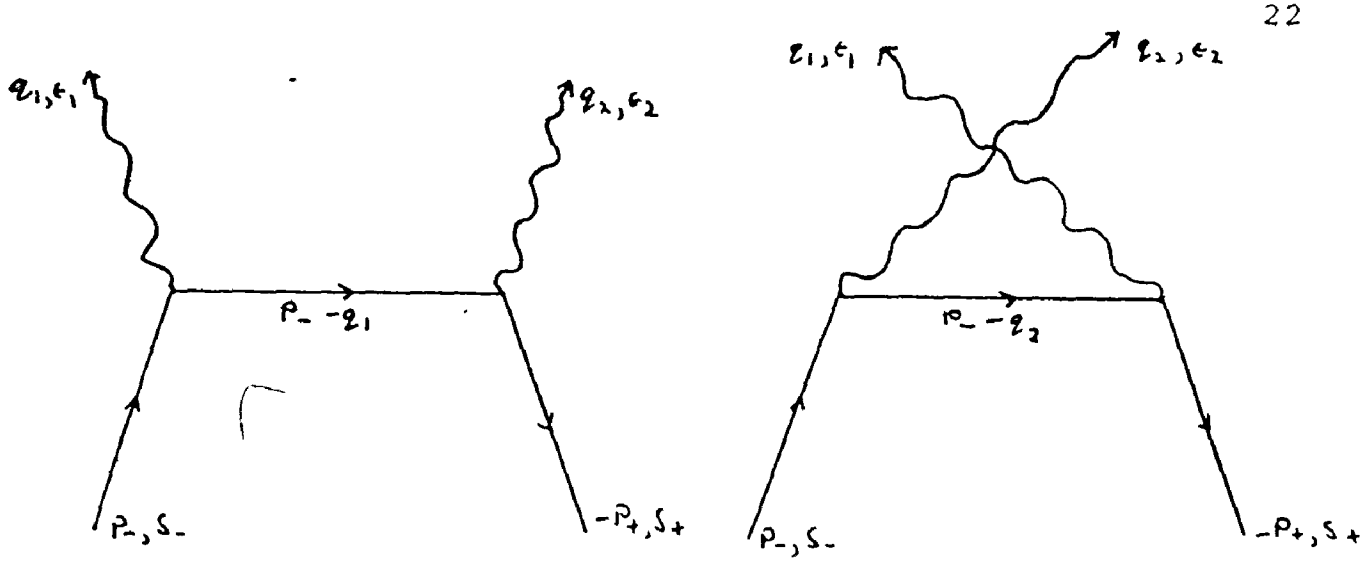


Fig. 2:1 Two Photon Annihilation Diagrams

where ε_1, ε_2 are the polarization vectors for the photons (four momenta q_1, q_2) and the four momenta ($p_\pm$) and spins ($s_\pm$) for the positron and electron are as indicated. Using the rules for calculating Feynman diagrams, the scattering matrix element $S_{fi}^{(2)}$ is

$$S_{fi}^{(2)} = \frac{e^2}{V^2}\left(\frac{m^2}{E_+E_-2q_1 2q_2}\right)^{1/2}(2\pi)^4\delta^4(q_1+q_2-p_+-p_-)M_{fi} \quad (2.1)$$

$$M_{fi} = \bar{v}(p_+,s_+)\{(-i\not{\varepsilon}_2)\frac{i}{\not{p}_- - \not{q}_1 - m}(-i\not{\varepsilon}_1)+(-i\not{\varepsilon}_1)\frac{i}{\not{p}_- - \not{q}_2 - m}(-i\not{\varepsilon}_2)\}$$

$$\times u(p_-,s_-) \quad (2.2)$$

In (2.2) the notation $\not{A}$ means $\gamma_\mu A^\mu$. V is the volume of the box used for normalization and $u(p_-,s_-)$, $\bar{v}(p_+,s_+)$ are the plane wave spinors for the electron and positron. We form

the differential cross section for an annihilation process by squaring (2.1), dividing by VT, dividing by an incident flux $|\underset{\sim}{v}|/V$ ($\underset{\sim}{v}$ is the relative velocity) and dividing by the number of target particles per unit volume (1/V). After some manipulation, the spin averaged differential cross section is given by

$$d\bar{\sigma} = \frac{e^4}{(2\pi)^2} \int \frac{m^2}{E_+E_-} (-\frac{1}{4}) \mathrm{Tr}\{\frac{m-\not{p}_+}{2m} (\frac{\not{\varepsilon}_2\not{q}_1\not{\varepsilon}_1}{2p_-\cdot q_1} + \frac{\not{\varepsilon}_1\not{q}_2\not{\varepsilon}_2}{2p_-\cdot\varepsilon_2})$$

$$\times \frac{\not{p}_-+m}{2m} (\frac{\not{\varepsilon}_1\not{q}_1\not{\varepsilon}_2}{2p_-\cdot q_1} + \frac{\not{\varepsilon}_2\not{q}_2\not{\varepsilon}_1}{2p_-\cdot q_2})\} \frac{d^3q_1}{2q_1} \frac{d^3q_2}{2q_2} \delta^4(q_1+q_2-p_+-p_-). \quad (2.3)$$

In (2.3) we have integrated over the outgoing photons' phase space - $\frac{V^2}{(2\pi)^6} d^3q_1 d^3q_2$. The spinor product $|M_{fi}|^2$ has been expressed as a spin averaged trace, corresponding to averaging over the initial spin states of both the electron and positron. This is possible because the ground state energy does not depend on the spin orientation of the interacting particles and is consequently degenerate. It is important to note that this is not the case for positronium in a magnetic field, in which the spin orientation is fixed by the $-\underset{\sim}{\mu}\cdot\underset{\sim}{B}$ magnetic dipole energy, and hence the technique of using a spin averaged trace is not applicable. The correct procedure, using the explicit spinors, will be discussed in detail later in this chapter. In passing, the basic approach is to consider the low energy limit of the annihilation process in terms of singlet and triplet spin state matrix elements. This is in keeping with the

previous arguments that k_3, or equivalently $\underset{\sim}{p}_+$, are essentially non-relativistic.

The next step is to perform the phase space integrations in (2.3) and after some simplifying algebra

$$\frac{d\bar{\sigma}}{d\Omega_{q_1}} = \frac{\alpha^2(m+E_+)}{8p_+(m+E_+-p_+\cos\theta)^2}\left\{\frac{q_2}{q_1} + \frac{q_1}{q_2} - 4(\varepsilon_1\cdot\varepsilon_2)^2 + 2\right\} \quad (2.4)$$

where a transformation to the laboratory frame (electron at rest) has been made. In this frame, β_+ is the positron velocity ($= p_+/E_+$), θ is the angle between p_+ and q_1, $\alpha=(e^2/4\pi)$ is the fine structure constant and Ω_{q_1} is the solid angle for photon (q_1). Finally, to get a total cross section it is necessary to sum (2.5) over the final photon polarizations $(\varepsilon_1,\varepsilon_2)$ and to integrate (2.4) over the solid angle $d\Omega_{q_1}$. To take care of the indistinguishability of the two outgoing photons the cross section in (2.4) is divided by two, which is equivalent to performing the angular integration over half the total solid angle. As mentioned earlier, we are most interested in the low energy approximation to the cross section, and so considering (2.4) as $\underset{\sim}{p}_+ \to 0$, $\underset{\sim}{q}_1 \to -\underset{\sim}{q}_2$, the differential cross section becomes

$$\frac{d\bar{\sigma}}{d\Omega_{q_1}} = \frac{\alpha^2}{4\beta_+ m^2}\left(1 - |\underset{\sim}{\varepsilon}_1\cdot\underset{\sim}{\varepsilon}_2|^2\right) . \quad (2.5)$$

Performing the above mentioned integration, and summing over the photon polarization states, the total cross section cor-

responding to (2.5) is

$$\bar{\sigma} = \frac{\alpha^2 \pi}{\beta_+ m^2} \quad . \tag{2.6}$$

The physical significance of (2.6) can be better understood if the appropriate low energy substitutions are made in M_{fi} (2.2) before the integration over free space is carried out. (This section follows the discussion in Sakurai (1967).) With this in mind the propagators can be rewritten:

$$\frac{1}{(\not{p}_- - \not{q}_1) - m} = \frac{\not{p}_- - \not{q}_1 + m}{(p_- - q_1)^2 - m^2} \tag{2.7}$$

and similarly for $\frac{1}{\not{p}_- - \not{q}_2 - m}$. $(p_- - q_1)$ is a four-vector and $(p_- - q_1)^2$ the appropriate scalar product, so that

$$(p_- - q_1)^2 - m^2 = p_-^2 - 2p_- \cdot q_1 + q_1^2 - m^2 \quad . \tag{2.8}$$

In the laboratory frame, $p_- = (m,0,0,0)$ and so (2.8) becomes $(-2m\omega_1)$ with $q_1 = (\omega_1, \underset{\sim}{q}_1)$. Thus, in the laboratory frame, M_{fi} is simplified:

$$M_{fi} = i\bar{v}(p_+,s_+)\{\frac{\not{\varepsilon}_2 \not{q}_1 \not{\varepsilon}_1}{2m\omega_1} + \frac{\not{\varepsilon}_1 \not{q}_2 \not{\varepsilon}_2}{2m\omega_2}\}u(p_-,s_-) \quad . \tag{2.9}$$

In accord with the low energy approximations, we can define $\omega = \omega_1 \doteq \omega_2$, $\underset{\sim}{q} = \underset{\sim}{q}_1 \doteq -\underset{\sim}{q}_2$ with $\omega = |\underset{\sim}{q}|$. In this limit, (2.9) becomes

$$M_{fi} = \frac{i}{2m\omega}\,\bar{v}(p_+,s_+)\{\not\varepsilon_2\not k\not\varepsilon_1 - \not\varepsilon_1\not k\not\varepsilon_2\}u(p_-,s_-) \qquad (2.10)$$

and after some further algebra on the matrices in (2.10), it can be shown that

$$\not\varepsilon_2\not k\not\varepsilon_1 - \not\varepsilon_1\not k\not\varepsilon_2 = -2iq\cdot(\underset{\sim}{\varepsilon}_1\times\underset{\sim}{\varepsilon}_2)\begin{pmatrix} 0 & I \\ -I & 0\end{pmatrix} \qquad (2.11)$$

where I is the 2×2 identity matrix. Thus, (2.10) is

$$M_{fi} = \bar{v}(p_+,s_+)\,\frac{2q\cdot(\underset{\sim}{\varepsilon}_1\times\underset{\sim}{\varepsilon}_2)}{2mq}\begin{pmatrix} 0 & I \\ -I & 0\end{pmatrix}u(p_-,s_-). \qquad (2.12)$$

To use (2.12), we must identify the spinors $\bar{v}(p_+,s_+)$ and $u(p_-,s_-)$ for the electron-positron pair.

The ground state of a pair can be either a singlet ($J=0$, $J_z=0$) or triplet ($J=1$, $J_z=\pm1,0$) spin angular momentum state. The three members of the triplet state ($J_z=\pm1,0$) are rotationally equivalent and so in a comparison of the singlet and triplet cross sections, it is necessary to consider only the $|00>$, $|10>$ spin states. The two states may be schematically represented as

$$\text{Singlet: } |00> = \frac{1}{\sqrt{2}}\{|\uparrow_{e^-}\downarrow_{e^+}> - |\downarrow_{e^-}\uparrow_{e^+}>\} \qquad (2.13)$$

$$\text{Triplet: } |10> = \frac{1}{\sqrt{2}}\{|\uparrow_{e^-}\downarrow_{e^+}> + |\downarrow_{e^-}\uparrow_{e^+}>\}\,. \qquad (2.14)$$

In the representation defined in Appendix A, the spinors have the low energy form

Spinor: $u(0,\frac{1}{2})$ $u(0,-\frac{1}{2})$ $v(0,\frac{1}{2})$ $v(0,-\frac{1}{2})$

Representation:

$$\begin{pmatrix}1\\0\\0\\0\end{pmatrix} \quad \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \quad \begin{pmatrix}0\\0\\0\\-i\end{pmatrix} \quad \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \tag{2.15}$$

Schematically: $\uparrow_{e^-}$ $\downarrow_{e^-}$ $\uparrow_{e^+}$ $\downarrow_{e^+}$

Substitution of these spinors into the singlet and triplet states (2.13), (2.14) and then into M_{fi} (equation (2.12)) yields

$$\begin{aligned} |M_{fi}|^2_{\text{triplet}} &= 0 \\ |M_{fi}|^2_{\text{singlet}} &= \frac{2}{m^2}\,(1-|\underset{\sim}{\varepsilon}_1\cdot\underset{\sim}{\varepsilon}_2|^2) \end{aligned} \tag{2.16}$$

These results are in agreement with the low energy limit of (2.2) since

$$\frac{1}{4}\sum_{s^+}\sum_{s^-}|M_{fi}|^2 = \frac{3}{4}\,|M_{fi}|^2_{\text{triplet}} + \frac{1}{4}|M_{fi}|^2_{\text{singlet}} \tag{2.17}$$

where $\frac{1}{4}\sum_{s^+}\sum_{s^-}|M_{fi}|^2 = \frac{1}{2m^2}\,(1-|\underset{\sim}{\varepsilon}_1\cdot\underset{\sim}{\varepsilon}_2|^2)$ is the low energy limit of the spin averaged trace.

The significance of the vanishing matrix element for a triplet state is twofold. Firstly, for positronium in free space the ground state is a singlet state because of the dipole-dipole interaction and so the cross section will be four times the spin averaged cross section described by (2.6), i.e.

$$\bar{\sigma}(\text{positronium}) = 4\bar{\sigma}_0 \quad \text{where} \quad \bar{\sigma}_0 = \frac{\alpha^2 \pi}{\beta_+ m^2} . \tag{2.18}$$

Secondly, for positronium in a magnetic field in which the spins are aligned with the field, the ground state spin orientation - as shown in Chapter I - must be

$$|e^+e^-\rangle_{spin} = |\uparrow_{e^-}\uparrow_{e^+}\rangle = \frac{1}{\sqrt{2}} \{|10\rangle - |00\rangle\} . \tag{2.19}$$

Thus the spinor matrix element (2.12) in a magnetic field is

$$|M_{fi}|^2_{\text{magnetic field}} = \frac{1}{2} |M_{fi}|^2_{\text{singlet}} \tag{2.20}$$

which can be shown using either the explicit form of the spinors (2.15), or alternatively (2.19), remembering that singlet and triplet states are orthogonal. Consequently, the cross section for pair annihilation in a magnetic field where the particles are described by low energy plane waves will be $2\bar{\sigma}_0$. This result will be used in Chapter 4, where the bound state is expressed as a sum over low energy plane wave states.

As a comparison with the two photon annihilation rate to be calculated in Chapter 4, the analogous rate for positronium in free space is given by

$$\tau^{-1} = \bar{\sigma}\beta_+\rho . \tag{2.21}$$

ρ = number of electrons/unit vol (i.e., the electron density at the point considered)

$\bar{\sigma} = 4\bar{\sigma}_0$ and v_+ is the incident positron velocity.

For a bound state ρ is given simply by the value of the square of the bound state wavefunction, evaluated at the origin; for positronium the appropriate relative wavefunction is the hydrogenic 1s orbital (1.7) so that

$$\rho = |\psi_0(0)|^2 = \frac{1}{\pi}(2a_0)^{-3}$$

and so

$$\tau^{-1} = \frac{4\alpha^2}{m^2}\frac{1}{(2a_0)^3} \qquad (2.22)$$

$$= \frac{1}{2}\alpha^5 m$$

in this system of units. Hence $\tau^{-1} = 8\times10^9$ sec^{-1} .

In fact, equation (2.22) can be derived in a more rigorous manner if we express the bound state as a sum over complete states (plane waves). This technique is used in Chapter 4 and will be mentioned then.

CHAPTER 3

THE ONE PHOTON PAIR ANNIHILATION PROCESS IN A MAGNETIC FIELD

Calculation of the transition rate τ^{-1} for one photon pair annihilation involves use of the usual Fermi Golden Rule for first order processes. The perturbation Hamiltonian H' is given by

$$H' = \int J^{\mu}(\underset{\sim}{x}) A_{\mu}(\underset{\sim}{x}) d^3\underset{\sim}{x} \qquad (3.1)$$

where, according to the usual field theory, $J^{\mu}(\underset{\sim}{x})$ (the particle current) is

$$J^{\mu}(\underset{\sim}{x}) = e\bar{\psi}(\underset{\sim}{x})\gamma^{\mu}\psi(\underset{\sim}{x}) \quad . \qquad (3.2)$$

$\psi(\underset{\sim}{x})$, $\psi^{\dagger}(\underset{\sim}{x})$ are the field operators equation (A-32) and $A_{\mu}(\underset{\sim}{x})$ is the vector potential representing the photon field. The golden rule is

$$\tau^{-1} = 2\pi \sum_{\substack{\text{phase} \\ \text{space}}} |\langle f|H'|i\rangle|^2 \delta(\varepsilon_f - \varepsilon_i) \qquad (3.3)$$

with $|f\rangle$, ε_f and $|i\rangle$, ε_i representing the final, initial states and energies of the system. In order to calculate the matrix element $\langle f|H'|i\rangle$ it is useful to represent the vector potential A_{μ} and the field operators ψ, $\psi^{\dagger}$ in second quantized

form appropriate to the problem: In the radiation gauge

$$A_{\underset{\sim}{}}(\underset{\sim}{x}) = \sum_{\underset{\sim}{k}} \sqrt{\frac{1}{2\omega_k V}} \sum_{\lambda=1}^{2} \{a(\underset{\sim}{k},\lambda) e^{i\underset{\sim}{k}\cdot\underset{\sim}{x}} \underset{\sim}{\varepsilon}(\underset{\sim}{k},\lambda) + a^{\dagger}(\underset{\sim}{k},\lambda) e^{-i\underset{\sim}{k}\cdot\underset{\sim}{x}} \underset{\sim}{\varepsilon}^{*}(\underset{\sim}{k},\lambda)\} \quad (3.$$

V = volume of the system.

In (3.4) the $\underset{\sim}{\varepsilon}(\underset{\sim}{k},\lambda)$ are the polarization vectors for the two polarizations defined by $\lambda = 1,2$, $\omega_k = |\underset{\sim}{k}|$ is the frequency of the emitted photon. The $a(\underset{\sim}{k},\lambda)$, $a^{\dagger}(\underset{\sim}{k},\lambda)$ are the photon annihilation and creation operators; they satisfy the Bose-Einstein commutation relations. The field operators $\psi(\underset{\sim}{x})$, $\psi^{\dagger}(\underset{\sim}{x})$ in (3.2) are taken from Appendix A, equation (A-32). Explicitly,

$$\psi(\underset{\sim}{x}) = \sum_{n'} \sum_{s'} \sum_{k_3'} \sum_{\xi'} \{a(\ell',s',k_3',\xi') \eta_{k_3'\ell'}(\underset{\sim}{x}) f_{+}(n',s',k_3',\xi',\underset{\sim}{x})$$

$$+ b^{\dagger}(\ell',s',k_3',\xi') \eta_{-k_3'\ell'}(\underset{\sim}{x}) f_{-}^{\dagger}(n',s',-k_3',-\xi',\underset{\sim}{x})\} \quad (3.5)$$

$$\psi^{\dagger}(\underset{\sim}{x}) = \sum_{n''} \sum_{s''} \sum_{k_3''} \sum_{\xi''} \{a^{\dagger}(\ell'',s'',k_3'',\xi'') \eta^{\dagger}_{k_3''\ell''}(\underset{\sim}{x}) f_{+}^{\dagger}(n'',s'',k_3'',\xi'',\underset{\sim}{x})$$

$$+ b(\ell'',s'',k_3'',\xi'') \eta^{\dagger}_{-k_3''\ell''}(\underset{\sim}{x}) f_{-}^{\dagger}(n'',s'',-k_3'',-\xi'',\underset{\sim}{x})\} \quad (3.6$$

Returning to (3.3), state $|i\rangle$ is the ground state of positronium (in a strong magnetic field) and the final state $|f\rangle$ contains a single photon labelled by $\underset{\sim}{q},\lambda$. The bound state is (1.19) with the spinors included:

$$|e^{+}e^{-}\rangle = \sum_{k_3} \phi(k_3) \quad |0\ 0\ k_3\ -\tfrac{1}{2}\ ;\quad 0\ 0\ -k_3\ +\tfrac{1}{2}\rangle \quad (3.7)$$

where the first set of labels refer to the electron and the second set to the positron. Hence

$$\langle f|H'|i\rangle = \int \langle f|\underset{\sim}{J}\cdot\underset{\sim}{A}|i\rangle d^3\underset{\sim}{x}$$

$$= e \int \langle \underset{\sim}{q}\lambda|\langle 00|\bar{\psi}\gamma^{\mu}\psi A_{\mu}|e^{+}e^{-}\rangle d^3\underset{\sim}{x} \quad \mu = 1....3 \qquad (3.8)$$

since in the radiation gauge the $\varepsilon^{\mu}(\underset{\sim}{k},\lambda)$ have no time component: $\varepsilon^{o}(\underset{\sim}{k},\lambda) = 0$. Equation (3.8) can be separated into a product of photon and particle matrix elements:

$$\langle f|H'|i\rangle = e \int \langle \underset{\sim}{q}\lambda|A_{\mu}|0\rangle\langle 00|\bar{\psi}\gamma^{\mu}\psi|e^{+}e^{-}\rangle d^3\underset{\sim}{x} \; . \qquad (3.9)$$

The photon matrix element is trivial

$$\langle \underset{\sim}{q}\lambda|A_{\mu}|0\rangle = \sqrt{\frac{1}{2\omega_q V}}\, e^{-i\underset{\sim}{q}\cdot\underset{\sim}{x}}\, \varepsilon^{*}_{\mu}(\underset{\sim}{q},\lambda) \qquad (3.10)$$

and

$$\langle 00|\bar{\psi}\gamma^{\mu}\psi|e^{+}e^{-}\rangle = \sum_{k_3} \phi(k_3)\langle 00|\bar{\psi}\gamma^{\mu}\psi|00\; k_3\; -\tfrac{1}{2}\;;\quad 00\; -k_3\; \tfrac{1}{2}\rangle. \qquad (3.11)$$

The ψ annihilates the electron, and $\bar{\psi}$ the positron so that with the appropriate coefficients from (3.5), (3.6), the electronic matrix element (3.11) is

$$\sum_{n'n''}\sum_{s's''}\sum_{\xi'\xi''}\sum_{k_3'k_3''}\sum_{k_3} \phi(k_3)\langle 00|\{\eta^{\dagger}_{-k_3''\ell''}(x) f^{\dagger}_{-}(n'',s'',-k_3'',-\xi'',\underset{\sim}{x})\}$$

$$\times \gamma^{o}\gamma^{\mu}\{\eta_{k_3'\ell'}(\underset{\sim}{x}) f_{+}(n',s',k_3',\xi',\underset{\sim}{x})\}$$

$$\times b(\ell'',s'',k_3'',\xi'')a(\ell',s',k_3',\xi')|00\; k_3\; -\tfrac{1}{2}\;;\quad 00\; -k_3\; \tfrac{1}{2}\rangle \qquad (3.12)$$

The appropriate matrix element for the electron, positron second quantization operators is

$$\langle 00|b(\ell'',s'',k_3'',\xi'')a(\ell',s',k_3',\xi')|00\ k_3\ -\tfrac{1}{2}\ ;\ \ 00\ -k_3\ \tfrac{1}{2}\rangle$$

$$= \delta_{k_3'',-k_3}\,\delta_{\ell'',0}\delta_{s'',0}\delta_{\xi'',1/2}\,\delta_{k_3',k_3}\delta_{\ell',0}\delta_{s',0}\delta_{\xi',-1/2} \tag{3.13}$$

Using (3.13), the double sum in (3.12) becomes contracted to one term

$$\langle 00|\bar{\psi}\gamma^\mu\psi|e^+e^-\rangle = \sum_{k_3} \phi(k_3)\eta^\dagger_{k_3,0}(x)f^\dagger_-(0,0,k_3,-\tfrac{1}{2},\underset{\sim}{x})\gamma^0\gamma^\mu\eta_{k_3,0}(\underset{\sim}{x})$$

$$\times f_+(0,0,k_3,-\tfrac{1}{2},\underset{\sim}{x}) \tag{3.14}$$

The spatial functions $\eta^\dagger_{k_3,0}(\underset{\sim}{x}),\eta_{k_3,0}(\underset{\sim}{x})$ have the form

$$\eta^\dagger_{k_3,0}(\underset{\sim}{x}) = \frac{1}{\sqrt{L}}\ e^{-ik_3z}\ \frac{1}{\sqrt{2\pi}}\ e^{i\theta/2}$$

$$\eta_{k_3,0}(\underset{\sim}{x}) = \frac{1}{\sqrt{L}}\ e^{ik_3z}\ \frac{1}{\sqrt{2\pi}}\ e^{-i\theta/2} \tag{3.15}$$

which follows from (A.26). With (3.15), equation (2.14) is simplified:

$$\langle 00|\bar{\psi}\gamma^\mu\ \psi|e^+e^-\rangle = \sum_{k_3} \phi(k_3)\frac{1}{L}\ \frac{1}{2\pi}\ f^\dagger_-(0,0,k_3,-\tfrac{1}{2},\underset{\sim}{x})\gamma^0\gamma^\mu f_+(0,0,k_3,-\tfrac{1}{2},\underset{\sim}{x}) \tag{3.16}$$

In the limit $k_3 \sim 0$, the spinors $f^\dagger_-(0,0,k_3,-\tfrac{1}{2},\underset{\sim}{x})$, $f_+(0,0,k_3,-\tfrac{1}{2},\underset{\sim}{x})$ are

$$f_-^\dagger(0,0,0,-\tfrac{1}{2},\underset{\sim}{x}) = \sqrt{2\gamma}\quad I_{00}(w)e^{-i\theta/2}(0\ 0\ 0\ i) \tag{3.17}$$

$$f_+(0,0,0,-\tfrac{1}{2},\underset{\sim}{x}) = \sqrt{2\gamma}\quad I_{00}(w)e^{i\theta/2}\begin{pmatrix}0\\ i\\ 0\\ 0\end{pmatrix} \tag{3.18}$$

which follows from equation (A-27): $w = \gamma\rho^2$ for $x = (\rho,\theta,z)$ with $I_{00}(w) = \exp(-w/2)$.

Explicit calculation, using the standard set of γ-matrices shows that

$$f_-^\dagger(0,0,0,-\tfrac{1}{2},\underset{\sim}{x})\gamma^0\gamma^\mu f_+(0,0,0,-\tfrac{1}{2},\underset{\sim}{x}) = 2\gamma\quad I_{00}^2(w)\delta_{\mu 3} \tag{3.19}$$

so that only $\mu=3$ contributes.

Equations (3.10), (3.19) contain the two component parts of the total matrix element $\langle f|H'|i\rangle$. Remembering that only $\mu=3$ is non-zero because of (3.19),

$$\langle f|H'|i\rangle = e\int \sum_{k_3}\phi(k_3)\sqrt{\frac{1}{2\omega_q V}}\,e^{-i\underset{\sim}{q}\cdot\underset{\sim}{x}}\,\frac{1}{L}\,\frac{\gamma}{\pi}\,I_{00}^2(w)\varepsilon_3^*(\underset{\sim}{q},\lambda)\,d^3\underset{\sim}{x}$$

$$= \sum_{k_3}\phi(k_3)\,e\sqrt{\frac{1}{2\omega_q V}}\,\frac{1}{L}\,\frac{\gamma}{\pi}\,\varepsilon_3^*(\underset{\sim}{q},\lambda)\int e^{-i\underset{\sim}{q}\cdot\underset{\sim}{x}}\,I_{00}^2(w)\,d^3(\underset{\sim}{x}). \tag{3.20}$$

Changing to cylindrical coordinates

$$\int e^{-i\underset{\sim}{q}\cdot\underset{\sim}{x}}\,I_{00}^2(w)\,d^3x = \int_0^\infty\int_0^{2\pi}\int_{-L/2}^{L/2} e^{-\gamma\rho^2}\rho e^{-i\underset{\sim}{q}_\perp\cdot\underset{\sim}{x}_\perp}e^{-iq_z z}\,d\rho d\theta dz \tag{3.21}$$

where the photon momentum vector $\underset{\sim}{q}$ has been represented as

$$\left.\begin{aligned}\underset{\sim}{q} &= (\underset{\sim}{q}_\perp, q_2)\\ \text{and } \underset{\sim}{x} &= (\underset{\sim}{x}_\perp, z)\end{aligned}\right\}\quad \text{so that } \underset{\sim}{q}\cdot\underset{\sim}{x} = \underset{\sim}{q}_\perp\cdot\underset{\sim}{x}_\perp + q_z z\ .$$

Because of the azimuthal symmetry in the problem, we can choose $q_x = 0$ and $q_y = q_\perp$ so that (3.21) is

$$\int_0^\infty \int_0^{2\pi} \int_{-L/2}^{L/2} e^{-\gamma\rho^2} \rho \, e^{-iq_\perp \rho \sin\theta} \, e^{-iq_z z} \, d\rho d\theta dz. \quad (3.22)$$

Now

$$\int_{-L/2}^{L/2} e^{-iq_z z} dz = L \, \delta_{q_z,0} . \quad (3.23)$$

Thus momentum in the z direction is conserved - the photon must come out at right angles to the direction of the field. If we substitute $\theta' = (\theta - \pi)$, then the angular integration in (3.22) can be performed:

$$\int_{-\pi}^{\pi} e^{iq_\perp \rho \sin\theta'} d\theta' = 2\pi \, J_0(q_\perp \rho) . \quad (3.24)$$

Substituting for the z,θ integrations from (3.23), (3.24), equation (3.22) is

$$2\pi \, L \, \delta_{q_z,0} \int_0^\infty \rho e^{-\gamma\rho^2} J_0(q_\perp \rho) d\rho = L \, \frac{\pi}{\gamma} \exp\left(-\frac{q_\perp^2}{4\gamma}\right) \delta_{q_z,0} \quad (3.25)$$

With (3.25), the matrix element in (3.20) is

$$\langle f|H'|i\rangle = \sum_{k_3} \phi(k_3) e \sqrt{\frac{1}{2\omega_q V}} \exp\left(-\frac{q_\perp^2}{4\gamma}\right) \delta_{q_z,0} \, \varepsilon_3^*(\underset{\sim}{q},\lambda) . \quad (3.26)$$

The final step in calculating the transition rate is the sum over final states:

$$\tau^{-1} = 2\pi \sum_{\substack{\text{phase}\\ \text{space}}} |\langle f|H'|i\rangle|^2 \delta(\varepsilon_f - \varepsilon_i)$$

$$= 2\pi \sum_{\substack{\text{phase}\\ \text{space}}} \frac{e^2}{2\omega_q V} \exp\left(-\frac{q_\perp^2}{2\gamma}\right) \varepsilon_3^*(\underset{\sim}{q},\lambda)\,\varepsilon_3(\underset{\sim}{q},\lambda)$$

$$\times \Big|\sum_{k_3} \phi(k_3)\Big|^2 \delta_{q_{z,0}}\, \delta(\varepsilon_f - \varepsilon_i) \quad . \tag{3.27}$$

For photons at right angles to the field, the sum over the polarization states of the photon is

$$\sum_{\lambda=1}^{2} |\varepsilon_3(\underset{\sim}{q},\lambda)|^2 = 1 \tag{3.28}$$

which follows immediately because we take one photon to have polarization vector along $B\hat{\underline{k}}$ and the other at right angles to $B\hat{\underline{k}}$.

Now, consider $\Big|\sum_{k_3} \phi(k_3)\Big|^2$. If we recall from equation (1.27) that

$$\sum_{k_3} \phi(k_3)\, \frac{\exp -ik_3 z}{L} = Z(z_+, z_-) = L^{-1/2} F(z)$$

then

$$\sum_{k_3} \phi(k_3) = \sqrt{L}\, F(0). \tag{3.29}$$

Hence

$$\Big|\sum_{k_3} \phi(k_3)\Big|^2 = L\,|F(0)|^2 \quad . \tag{3.30}$$

Substituting (3.30) and (3.28) into (3.27),

$$\tau^{-1} = 2\pi \sum_{\substack{\text{phase}\\ \text{space}}} \frac{e^2}{2\omega_q V} L\delta_{q_{z,0}} |F(0)|^2 \exp(-\frac{q_\perp^2}{2\gamma}) \delta(\varepsilon_f - \varepsilon_i) \quad . \qquad (3.31)$$

At this point we can dispense with the box normalization and let $L \to \infty$. In this limit,

$$\sum_{\substack{\text{phase}\\ \text{space}}} \to \frac{V}{(2\pi)^3} \int d^3\underset{\sim}{q}$$

$$L\delta_{q_{z,0}} \to 2\pi\delta(q_z) \quad .$$

With this, τ^{-1} in (3.31) becomes

$$\frac{1}{2\pi} \frac{e^2}{2} \int \frac{d^3\underset{\sim}{q}}{\omega_q} \exp(-\frac{q_\perp^2}{2\gamma}) \; |F(0)|^2 \delta(q_z) \delta(\varepsilon_f - \varepsilon_i) .$$

Now in cylindrical coordinates $(q_\perp, \theta_q, q_z)$

$$\omega_q = (q_\perp^2 + q_z^2)^{1/2} \qquad (3.33)$$

$$\delta(\varepsilon_f - \varepsilon_i) = \delta(\omega_q - 2m) \quad .$$

Performing the z integration with the results in (3.33)

$$\tau^{-1} = \frac{e^2}{(2\pi)} \frac{|F(0)|^2}{2} \int \exp(-\frac{q_\perp^2}{2\gamma}) \delta(q_\perp - 2m) d_{q_\perp} d\theta_q$$

$$= \frac{e^2}{2} |F(0)|^2 \exp(-\frac{2m^2}{\gamma}) \qquad (3.34)$$

From (1.8), $|F(0)|^2 = \lambda\sqrt{\frac{2}{\pi}}$ and with $\alpha = e^2/4\pi$ the final form of τ^{-1} is

$$\tau^{-1} = 2\pi\sqrt{\frac{2}{\pi}}\ \lambda\ \alpha\ \exp(-\frac{2m^2}{\gamma})$$

$$= 2\pi\sqrt{\frac{2}{\pi}}\ \lambda\ \alpha\ \exp(-\frac{4B_q}{B}) \qquad (3.35)$$

In these units ($\hbar = c = 1$) $B_q = \frac{m^2}{e}$.

Note that as $B \to 0$, τ^{-1} in (3.35) decreases exponentially, which is in agreement with the discussion mentioned in Chapter I that the transition rate should vanish as $B \to 0$.

CHAPTER 4

TWO PHOTON ANNIHILATION OF POSITRONIUM IN A STRONG MAGNETIC FIELD

The calculation of the transition rate for two photon annihilation is again rather more complicated that the free space analogue (discussed in Chapter 2) because of the form of the bound state wavefunction. In principle, the calculation can be done in a manner similar to the one photon process, using $S^{(2)}$, the second order scattering matrix. The field operators can again be expanded in the orthonormal set based upon relativistic wavefunctions as shown in Chapter 3. Unfortunately, this approach leads to some complex mathematical problems because the electron propagator in a magnetic field is complicated and awkward to use: the interaction does not conserve momentum and consequently there are no corresponding δ functions to simplify the phase space integrals. However, as any set of complete orthonormal functions provides a basis set, we can expand the bound state (and the field operators) in terms of the field free plane wave solutions. This technique, which necessitates an approximation to the physical problem, will be used to calculate the two photon rate.

The bound state of positronium can be represented as a sum over plane waves with the appropriate expansion coefficient $\phi(\underset{\sim}{p}_+,\underset{\sim}{p}_-)$. From equations (1.29) and (1.33)

$$|i\rangle = |e^+e^-\rangle = \sum_{\underset{\sim}{p}_+,\underset{\sim}{p}_-} \phi(\underset{\sim}{p}_+,\underset{\sim}{p}_-)\ |\underset{\sim}{p}_+,\underset{\sim}{p}_-\rangle \qquad (4.1)$$

where

$$\phi(\underset{\sim}{p}_+,\underset{\sim}{p}_-) = \frac{(2\pi)^2}{\gamma\pi}\frac{1}{L^2\sqrt{L}}\left(\lambda\sqrt{\frac{2}{\pi}}\right)^{1/2}\sqrt{\frac{\pi}{\lambda^2}}\exp -\frac{1}{2\gamma}\left(\underset{\sim}{p}_{+\perp}^{\ 2}+\underset{\sim}{p}_{-\perp}^{\ 2}\right)$$

$$\times \exp\frac{p_{+z}^{\ 2}}{4\lambda^2}\ \delta_{p_{+z}+p_{-z},0} \qquad (4.2)$$

As there are now two emergent photons, the final state $|f\rangle$ is

$$|f\rangle = |\underset{\sim}{q}_1\lambda_1,\underset{\sim}{q}_2\lambda_2\rangle \qquad (4.3)$$

With $|i\rangle$, $|f\rangle$ as above, we can calculate the second order scattering matrix element $S_{fi}^{(2)}$

$$S_{fi}^{(2)} = \langle f|S^{(2)}|i\rangle = \sum_{\underset{\sim}{p}_+,\underset{\sim}{p}_-} \phi(\underset{\sim}{p}_+,\underset{\sim}{p}_-)\langle\underset{\sim}{q}_1\lambda_1,\underset{\sim}{q}_2\lambda_2|S^{(2)}|\underset{\sim}{p}_+,\underset{\sim}{p}_-\rangle \qquad (4.4)$$

Consider

$$\langle\underset{\sim}{q}_1\lambda_1,\underset{\sim}{q}_2\lambda_2|S^{(2)}|\underset{\sim}{p}_+,\underset{\sim}{p}_-\rangle\ . \qquad (4.5)$$

This is simply the matrix element (2.1) for a field free two photon annihilation process , in which an electron and positron of momenta $\underset{\sim}{p}_-$ and $\underset{\sim}{p}_+$ respectively annihilate into two photons. It is apparent from (4.2) that the momenta of interest are $p_{\pm\perp} \sim \hat{\rho}^{-1} \lesssim a_0^{-1}$ and so the arguments given in Chapter 1 regarding $k_3 \lesssim a_0^{-1}$ can also be applied here. Thus, the form of the

transition rate of interest here is the low energy limit and so the appropriate approximation is to take the non-relativistic limit for the spinor product in $S_{fi}^{(2)}$. Thus, as shown in Chapter 2, the spinor product M_{fi} (2.2) is

$$M_{fi(\text{mag field})} = \frac{1}{\sqrt{2}} M_{fi(\text{singlet})} = \sqrt{2}\left\{\frac{1}{2m^2}\left(1-|\underset{\sim}{\varepsilon}_1\cdot\underset{\sim}{\varepsilon}_2|^2\right)\right\}^{1/2} . \quad (4.6)$$

Thus, with this result from Chapter 2, (4.5) is

$$\langle \underset{\sim}{q}_1\lambda_1, \underset{\sim}{q}_2\lambda_2 | S^{(2)} | \underset{\sim}{p}_+, \underset{\sim}{p}_- \rangle = \frac{1}{V}\frac{e^2}{(2\omega_1\ 2\omega_2)^{1/2}}\sqrt{2}\left\{\frac{1}{2m^2}(1-|\underset{\sim}{\varepsilon}_1\cdot\underset{\sim}{\varepsilon}_2|^2)\right\}^{1/2}\delta_{\underset{\sim}{q}_1+\underset{\sim}{q}_2,\underset{\sim}{p}_++\underset{\sim}{p}_-}$$

where $V = L^3$, the volume of the box used for normalization. The total matrix element, equation (4.4), becomes

$$S_{fi}^{(2)} = \sum_{\underset{\sim}{p}_+,\underset{\sim}{p}_-} \phi(\underset{\sim}{p}_+,\underset{\sim}{p}_-)\frac{e^2}{V(2\omega_1\ 2\omega_2)^{1/2}}\sqrt{2}\left\{\frac{1}{2m^2}\left(1-|\underset{\sim}{\varepsilon}_1\cdot\underset{\sim}{\varepsilon}_2|^2\right)\right\}^{1/2} \times \delta_{\underset{\sim}{q}_1+\underset{\sim}{q}_2,\underset{\sim}{p}_++\underset{\sim}{p}_-} \quad (4.7)$$

The transition rate τ^{-1} is given, as before, by

$$\tau^{-1} = 2\pi \sum_{\substack{\text{phase}\\ \text{space}}} |S_{fi}^{(2)}|^2 \delta(\varepsilon_f-\varepsilon_i) . \quad (4.8)$$

Neglecting the Coulomb binding energy ($\sim$ 10 Rydbergs) $\varepsilon_i = 2m$ and $\delta(\varepsilon_f-\varepsilon_i) = \delta(\omega_1+\omega_2-2m)$. If we substitute (4.7) for $S_{fi}^{(2)}$ in (4.8), then

$$\tau^{-1} = 2\pi \cdot \sum_{\substack{\text{phase}\\ \text{space}}} \frac{e^4}{V^2(2\omega_1 2\omega_2)} 2\{\frac{1}{2m^2}(1-|\underset{\sim}{\varepsilon}_1\cdot\underset{\sim}{\varepsilon}_2|^2)\}\delta(\omega_1+\omega_2-2m)$$

$$\times \Big|\sum_{\underset{\sim}{p}_+,\underset{\sim}{p}_-} \phi(\underset{\sim}{p}_+,\underset{\sim}{p}_-)\delta_{\underset{\sim}{q}_1+\underset{\sim}{q}_2,\underset{\sim}{p}_++\underset{\sim}{p}_-}\Big|^2 \qquad (4.9)$$

Now consider $\sum_{\underset{\sim}{p}_+,\underset{\sim}{p}_-} \phi(\underset{\sim}{p}_+,\underset{\sim}{p}_-)\delta_{\underset{\sim}{q}_1+\underset{\sim}{q}_2,\underset{\sim}{p}_++\underset{\sim}{p}_-}$. (4.10)

Using the definition of $\phi(\underset{\sim}{p}_+,\underset{\sim}{p}_-)$ as in (4.2), then explicitly performing the $\sum_{\underset{\sim}{p}_+}$ sum over the Kronecker δ's yields

$$\sum_{\underset{\sim}{p}_+,\underset{\sim}{p}_-} \phi(\underset{\sim}{p}_+,\underset{\sim}{p}_-)\delta_{\underset{\sim}{q}_1+\underset{\sim}{q}_2,\underset{\sim}{p}_++\underset{\sim}{p}_-}$$

$$= \sum_{\underset{\sim}{p}_-} \frac{(2\pi)^2}{\gamma\pi}\frac{1}{L^2\sqrt{L}}\left(\lambda\sqrt{\frac{2}{\pi}}\right)^{1/2}\sqrt{\frac{\pi}{\lambda^2}}\exp-\frac{1}{2\gamma}|\underset{\sim}{q}_{1\perp}+\underset{\sim}{q}_{2\perp}|^2$$

$$\times \exp-\frac{1}{2\gamma}\{2p_{-x}^2+2p_{-y}^2-2(q_{1x}+q_{2x})p_{-x}-2(q_{1y}+q_{2y})p_{-y}\}$$

$$\times \exp\left(-\frac{p_{-z}^2}{4\lambda^2}\right)\delta_{q_{1z}+q_{2z},0} \qquad (4.11)$$

To evaluate the sum over $\underset{\sim}{p}_-$ in (4.11) we make the change to an integral:

$$\sum_{\underset{\sim}{p}_-} \rightarrow \frac{V}{(2\pi)^3}\int d^3\underset{\sim}{p}_- \ .$$

Integration of (4.11) over $d^3\underset{\sim}{p}_-$ is not difficult, and so finally

$$\sum_{\underset{\sim}{p}_+,\underset{\sim}{p}_-} \phi(\underset{\sim}{p}_+,\underset{\sim}{p}_-)\delta_{\underset{\sim}{q}_1+\underset{\sim}{q}_2,\underset{\sim}{p}_++\underset{\sim}{p}_-} = \sqrt{L}\left(\lambda\sqrt{\frac{2}{\pi}}\right)^{1/2}\exp-\frac{1}{4\gamma}|\underset{\sim}{q}_{1\perp}+\underset{\sim}{q}_{2\perp}|^2$$

$$\times \delta_{q_{1z}+q_{2z},0} \ . \qquad (4.12)$$

Substitution of (4.12) into (4.9) gives

$$\tau^{-1} = 2\pi \sum_{\substack{\text{phase}\\ \text{space}}} \frac{e^4}{V^2(2\omega_1 2\omega_2)} 2\{\frac{1}{2m^2}(1-\underset{\sim}{\varepsilon}_1\cdot\underset{\sim}{\varepsilon}_2|^2)\}L\,\delta_{q_{1z}+q_{2z},0}(\lambda\sqrt{\tfrac{2}{\pi}})$$

$$\times \exp -\frac{1}{2\gamma}|\underset{\sim}{q}_{1\perp}+\underset{\sim}{q}_{2\perp}|^2\,\delta(\omega_1+\omega_2-2m) \qquad (4.13)$$

At this point we can dispense with the box normalization, so that

$$\sum_{\substack{\text{phase}\\ \text{space}}} \to \frac{V}{(2\pi)^3}\frac{V}{(2\pi)^3}\int d^3\underset{\sim}{q}_1\, d^3\underset{\sim}{q}_2 \qquad (4.14)$$

$$L\,\delta_{q_{1z}+q_{2z},0} \to 2\pi\,\delta(q_{1z}+q_{2z})\ . \qquad (4.15)$$

With (4.14) and (4.15), (4.13) can be rewritten and after cancelling the common factors, is

$$\tau^{-1} = \frac{e^4}{(2\pi)^4}\frac{1}{4m^2}(1-|\underset{\sim}{\varepsilon}_1\cdot\underset{\sim}{\varepsilon}_2|^2)\lambda\sqrt{\tfrac{2}{\pi}}\int \exp -\frac{1}{2\gamma}|\underset{\sim}{q}_{1\perp}+\underset{\sim}{q}_{2\perp}|^2\delta(q_{1z}+q_{2z})$$

$$\times\ \frac{\delta(\omega_1+\omega_2-2m)}{\omega_1\omega_2}\,d^3\underset{\sim}{q}_1 d^3\underset{\sim}{q}_2 \qquad (4.16)$$

It is interesting to note that the zero field limit ($B \to 0$) can be obtained from (4.21). If we let $\gamma \to 0$, then $\exp -\frac{1}{2\gamma}|\underset{\sim}{q}_{1\perp}+\underset{\sim}{q}_{2\perp}|^2 \to 2\pi\gamma\,\delta^2(\underset{\sim}{q}_{1\perp}+\underset{\sim}{q}_{2\perp})$ and performing the integrals in (4.16) leads to the result described in Chapter 2, i.e. the field free transition rate for positronium is

$$\tau^{-1}(B=0) \sim |\psi_b(00)|^2 \, 4\tau_0$$

with $\tau_0 = \left(\frac{e^2}{4\pi}\right)^2 \frac{\pi}{m^2}$ (free space annihilation rate).

We return to (4.16) to calculate the rate for a non zero field. If we define

$$I = \int \exp - \frac{1}{2\gamma} |\underset{\sim}{q}_{1\perp}+\underset{\sim}{q}_{2\perp}|^2 \frac{\delta(q_{1z}+q_{2z})}{\omega_1\omega_2} \delta(\omega_1+\omega_2-2m) d^3\underset{\sim}{q}_1 d^3\underset{\sim}{q}_2 \quad (4.17)$$

and change to cylindrical coordinates $(q_{1\rho},\theta_1,q_{1z})$ and $(q_{2\rho},\theta_2,q_{2z})$ then (4.17) can be written as

$$I = \int \exp - \frac{1}{2\gamma}\{q_{1\rho}^2+q_{2\rho}^2 + 2q_{1\rho}q_{2\rho}\cos(\theta_1-\theta_2)\} \frac{\delta(q_{1z}+q_{2z})}{\omega_1\omega_2}$$

$$\times \delta(\omega_1+\omega_2-2m) q_{1\rho}q_{2\rho} dq_{1\rho}dq_{2\rho}d\theta_1 d\theta_2 dq_{1z}dq_{2z} \; . \quad (4.18)$$

Now $\delta(\omega_1+\omega_2-2m) = \delta\left((q_{1\rho}^2+q_{1z}^2)^{1/2} + (q_{2\rho}^2+q_{2z}^2)^{1/2}-2m\right)$ and using the result $\delta(f(x)) = \frac{\delta(x-x_0)}{|f'(x)|_{x=x_0}}$ where x_0 is the root of $f(x)$: $f(x_0) = 0$, it follows that the delta function $\delta(\omega_1+\omega_2-2m)$ is equivalent to

$$\delta(q_2-q_0) \frac{\omega_2}{q_2} \theta(q_0) \quad (4.19)$$

where

$$q_0 = [\{2m-(q_{1\rho}^2+q_{1z}^2)^{1/2}\}^2 - q_{1z}^2]^{1/2} \; . \quad (4.20)$$

$\theta(q_0)$ expresses the requirement that the range of $q_{1\rho},q_{1z}$ is

such that $q_0 \neq 0$. With (4.14), (4.20), equation (4.18) becomes more tractable:

$$I = \int \frac{q_{1\rho}}{(q_{1\rho}^2+q_{1z}^2)^{1/2}} \exp - \frac{1}{2\gamma} \{q_{1\rho}^2+q_{2\rho}^2+2q_{1\rho}q_{2\rho}\cos(\theta_1-\theta_2)\}$$

$$\times \delta(q_{1z}+q_{2z})\delta(q_2-q_0)\theta(q_0)dq_{1\rho}dq_{2\rho}d\theta_1 d\theta_2 dq_{1z}dq_{2z} . \quad (4.21)$$

Integration over the $q_{2z}, q_{2\rho}$ coordinates gives

$$I = \int \frac{q_{1\rho}}{(q_{1\rho}^2+q_{1z}^2)^{1/2}} \exp - \frac{1}{2\gamma}\{q_{1\rho}^2+q_0^2+2q_{1\rho}q_0\cos(\theta_1-\theta_2)\}$$

$$\times \theta(q_0)dq_{1\rho}d\theta_1 d\theta_2 dq_{1z} \quad (4.22)$$

The angular integrals can be evaluated using the result

$$\int_0^{2\pi} d\theta_1 \int_0^{2\pi} d\theta_2 \, f(\theta_1-\theta_2) = 2\pi \int_0^{2\pi} f(\theta)d\theta$$

if f is periodic. $\theta = (\theta_1-\theta_2)$ is the relative angular coordinate. The angular integral in (4.22) is of the form

$$2\pi \int_0^{2\pi} \exp(-\frac{1}{\gamma} q_{1\rho}q_0\cos\theta)d\theta = (2\pi)^2 I_0(\frac{1}{\gamma} q_{1\rho}q_0) \quad (4.23)$$

where I_0 is a Bessel function of imaginary argument. Substituting (4.23) into (4.22) leaves a two dimensional integral:

$$I = (2\pi)^2 \int \frac{q_{1\rho}}{(q_{1\rho}^2+q_{1z}^2)^{1/2}} \exp - \frac{1}{2\gamma}(q_{1\rho}^2+q_0^2) I_0(\frac{1}{\gamma} q_{1\rho}q_0) \times \vartheta(q_0)dq_{1\rho}dq_{1z} \tag{4.24}$$

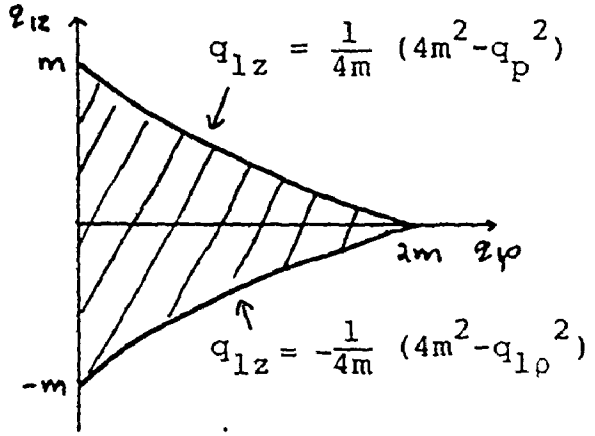
with $q_0 = [\{2m-(q_{1\rho}^2+q_{1z}^2)^{1/2}\}^2 - q_{1z}^2]^{1/2}$.

Range of Integrations

The theta function $\vartheta(q_0)$ limits the range of $q_{1\rho}, q_{1z}$ to regions in the two dimensional $q_{1\rho}, q_{1z}$ plane where $q_0 \geq 0$ i.e.

$$[\{2m-(q_{1\rho}^2+q_{1z}^2)^{1/2}\}^2-q_{1z}^2]^{1/2} \geq 0 . \tag{4.25}$$

The area for which (4.25) is satisfied is depicted in Figure 4.1.



Note that
$-m \leq q_{1z} \leq m$
$0 \leq q_{1\rho} \leq 2m$

Figure 4:1: Allowed region of integration: $0 \leq \omega_1 \leq 2m$

These limits have an obvious physical significance: since the maximum energy is 2m, and $\omega_1+\omega_2 = 2m$, the limit $|q_{1z}| = m$ corresponds to $q_{1\rho} = q_{2\rho} = 0$ with the second photon momentum $|q_{2z}| = m$ because of the conservation of z-momentum. The

limit $q_{1\rho} = 2m$ corresponds to $q_{1z} = q_{2z} = 0$, with $q_2 = 0$.

Thus, with the appropriate limits included, (4.24) is

$$I = (2\pi)^2 \int_{-m}^{m} dq_{1z} \int_{0}^{2(m^2-mq_{1z})^{1/2}} \frac{q_{1\rho}}{(q_{1\rho}^2+q_{1z}^2)^{1/2}} \exp - \frac{1}{2\gamma} (q_{1\rho}^2+q_0^2) \times I_0(\tfrac{1}{\gamma} q_{1\rho}q_0)dq_{1\rho} \qquad (4.26)$$

The form of I in (4.26) is too difficult to integrate analytically (because of the form of q_0) and so it must be evaluated numerically. To aid in this, it is useful to change to dimensionless variables x, z, x_0 where

$$\begin{aligned} z &= q_{1z}/m \\ x &= q_{1\rho}/m \\ x_0 &= q_0/m \end{aligned} \qquad (4.27)$$

and to define $\gamma' = \gamma/m^2$ so that

$$\begin{aligned} -\frac{1}{2\gamma} (q_{1\rho}^2+q_0^2) &= -\frac{1}{2\gamma'} (x^2+x_0^2) \\ \frac{1}{\gamma} (q_{1\rho}q_0) &= \frac{1}{\gamma'} (xx_0) \end{aligned} \qquad (4.28)$$

Now, if we note that

$$x^2+x_0^2 = (x-x_0)^2 + 2xx_0$$

then

$$-\frac{1}{2\gamma'} (x^2+x_0^2) = -\frac{1}{2\gamma'} (x-x_0)^2 - \frac{1}{\gamma'} xx_0 \qquad (4.29)$$

and using (4.27)-(4.29), Equation (4.26) becomes

$$I = (2\pi)^2 m^2 \int_{-1}^{1} dz \int_{0}^{2(1-z)^{1/2}} \frac{x}{(x^2+z^2)^{1/2}} \exp - \frac{1}{2\gamma'} (x-x_0)^2$$

$$\times \exp(-\frac{1}{\gamma'} xx_0) I_0(\frac{1}{\gamma'} xx_0) dx. \qquad (4.30)$$

The motivation in using (4.29) is that to integrate (4.30) numerically, it is necessary to use an approximation for $I_0(\frac{1}{\gamma'} xx_0)$. Because $I_0(\frac{1}{\gamma} xx_0)$ diverges as $(\frac{1}{\gamma} xx_0)$ increases, the numerical approximation must be made in the form $\exp(-x) I_0(x)$ which can be expressed as a convergent series and hence the above manipulations. Finally, the integration over z in (4.30) is over an even function and consequently can be written as twice the integral from 0 to 1, i.e.

$$I = 2(2\pi)^2 m^2 \int_{0}^{1} dz \int_{0}^{2(1-z)^{1/2}} \frac{x}{(x^2+z^2)^{1/2}} \exp - \frac{1}{2\gamma'} (x-x_0)^2$$

$$\times \exp(-\frac{1}{\gamma'} xx_0) I_0(\frac{1}{\gamma'} xx_0) dx \qquad (4.31)$$

Analytic integration yields the high and low field limits:

$$\begin{aligned} I &= 2(2\pi)^2 m^2 \qquad \text{as } B \to \infty \\ I &= (2\pi)^2 \gamma \qquad \text{as } B \to 0 \,. \end{aligned} \qquad (4.32)$$

The high field limit follows because the integral in (4.31) tends to 1 as $\gamma \to 0$; the low field limit was mentioned earlier in relation to the field-free annihilation process.

At higher and higher fields there is expected to be an increasing number of photons which emerge with a spread in energy and momentum about the peak at $\omega = m$. To calculate a measure of the spreading, we require the energy of one of

the photons (say for photon 1) to be larger than some value fm where $1 \leq f \leq 2$. This is equivalent to selecting the region of integration for which $\omega_1^2 = (q_{1\rho}^2 + q_{1z}^2) \geq f^2 m^2$ or, in dimensionless variables

$$x^2 + z^2 \geq f^2 \ . \tag{4.33}$$

The solution of this inequality is shown in Figure (4.2).

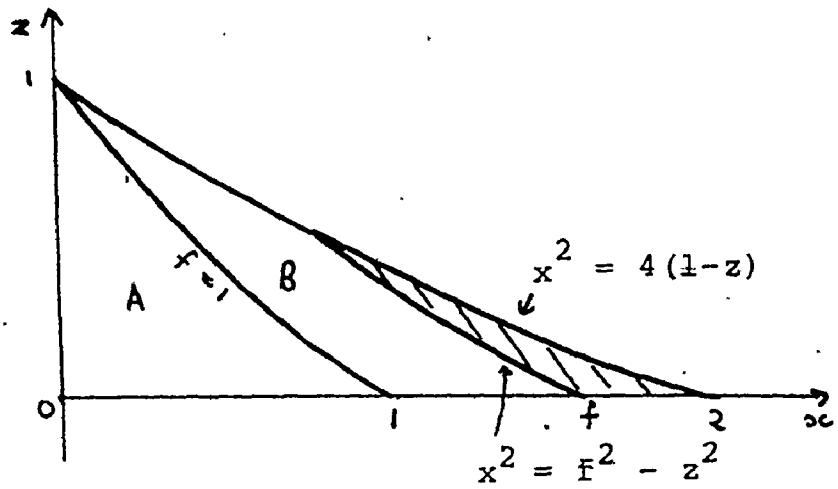


Figure 4.2: Allowed region of integration: $\omega_1 \geq$ fm

The shaded area represents the allowed region for some value f. We can define $\tilde{I}(f)$, the integral analogous to I for the contribution to the transition rate for two photon production processes with $\omega_1 \geq$ fm:

$$\tilde{I}(f) = 2(2\pi)^2 m^2 \int_0^{2-f} dz \int_{(f^2-z^2)^{1/2}}^{2(1-z)^{1/2}} \frac{x}{(x^2+z^2)^{1/2}} \exp - \frac{1}{2\gamma'} (x-x_0)^2$$

$$\times \exp(-\frac{1}{\gamma'} xx_0) I_0(\frac{1}{\gamma'} xx_0) dx \ . \tag{4.34}$$

Analogously, define I' to be the integral over the area marked A in Figure 4.2: this corresponds to $0 \leq \omega_1 \leq m$

$$I' = 2(2\pi)^2 m^2 \int_0^{(1-z^2)^{1/2}} \frac{x}{(x^2+z^2)^{1/2}} \exp - \frac{1}{2\gamma'} (x-x_0)^2 \exp(- \frac{1}{\gamma} xx_0)$$

$$\times I_0(\frac{1}{\gamma'} xx_0)dx \qquad (4.35)$$

Note that $\tilde{I}(1)$, which corresponds to integration over area B, is in fact $\tilde{I}(1) = \frac{1}{2} I$ with I as in (4.31). This is because $\tilde{I}(f)$ represents the value of f for which $\omega_1 \geq fm$, and so $\omega_2 \leq (2-f)m$ of necessity, given energy conservation. Now, for f=1, we could equally ask for $\omega_2 \geq m$ (and therefore $0 \leq \omega_1 \leq m$) which corresponds to region A. Thus, we expect $I' = \tilde{I}(1)$ and since $I' + \tilde{I}(1) = I$, it follows that $I' = \tilde{I}(1) = \frac{1}{2} I$. In fact, with an appropriate change of variables, the two integrals I' and $\tilde{I}(1)$ can be shown to be mathematically equivalent, as expected from the physical argument outlined above.

We now return to calculating the transition rate τ^{-1}. With I as in equation (4.31), τ^{-1} in equation (4.16) is

$$\tau^{-1} = \frac{e^4}{(2\pi)^4} \frac{1}{4m^2} (1-|\underset{\sim}{\varepsilon}_1 \cdot \underset{\sim}{\varepsilon}_2|^2) \lambda \sqrt{\frac{2}{\pi}}\, 2(2\pi)^2 m^2$$

$$\times \int_0^1 dz \int_0^{2(1-z)^{1/2}} \frac{x}{(x^2+z^2)^{1/2}} \exp - \frac{1}{2\gamma'} (x-x_0)^2$$

$$\times \exp(- \frac{1}{\gamma'} xx_0) I_0(\frac{1}{\gamma'} xx_0)dx \; . \qquad (4.36)$$

As before, the sum over the polarization vectors yields 2; this is cancelled by the $\frac{1}{2}$ arising from the indistinguishability of the outgoing photons. Thus

$$\tau^{-1} = \frac{e^4}{(2\pi)^2} \frac{1}{2} \lambda \sqrt{\frac{2}{\pi}} \int_0^1 dz \int_0^{2(1-z)^{1/2}} \frac{x}{(x^2+z^2)^{1/2}} \exp - \frac{1}{2\gamma'} (x-x_0)^2$$

$$\times \exp(- \frac{1}{\gamma'} xx_0) I_0(\frac{1}{\gamma'} xx_0) dx \qquad (4.37)$$

With $\alpha = \frac{e^2}{4\pi}$, this is equivalent to

$$\tau^{-1} = 2\alpha^2 \lambda \sqrt{\frac{2}{\pi}} \int_0^{2(1-z)^{1/2}} \frac{x}{(x^2+z^2)^{1/2}} \exp - \frac{1}{2\gamma'} (x-x_0)^2$$

$$\times \exp(- \frac{1}{\gamma'} xx_0) I_0(\frac{1}{\gamma'} xx_0) dx \; . \qquad (4.38)$$

In Appendix B we list a computer code for the evaluation of the integrals I, $\tilde{I}(f)$, I' and the calculated transition rates (from (4.38)) are discussed in Chapter 5.

CHAPTER 5

CONCLUSION AND ANALYSIS OF RESULTS

The annihilation rates for one and two photon processes in various magnetic fields are compared in Table II:

Table II. Calculated Rates for One and Two Photon Annihilation in a Strong Magnetic Field

B_{12}	1 Photon Rate (s^{-1})	2 Photon Rate (s^{-1})	Ratio	Half width for 2 photon process (keV)
1	4.74×10^{-60}	3.35×10^{12}	1.41×10^{-72}	13.8
3	8.17×10^{-9}	1.23×10^{13}	6.04×10^{-22}	23.5
10	7.91×10^{9}	5.09×10^{13}	1.56×10^{-4}	43.4
44	8.42×10^{15}	1.18×10^{14}	7.16×10^{1}	86.4

The numerical values in columns 2 and 3 of Table II were calculated using the results derived in Chapters 3 and 4, i.e.

$$\tau^{-1}(1 \text{ photon}) = 2\pi\sqrt{\frac{2}{\pi}}\,\lambda\alpha\,\exp\left(-\frac{4B_q}{B}\right) \qquad (3.35)$$

$$\tau^{-1}(2 \text{ photon}) = 2\sqrt{\frac{2}{\pi}}\,\lambda\alpha^2 \int_0^1 dz \int_0^{2(1-z)^{1/2}} \frac{x}{(x^2+z^2)^{1/2}} \times \exp - \frac{1}{2\gamma'}(x-x_0)^2 \exp\left(-\frac{1}{\gamma'}xx_0\right) I_0\left(\frac{1}{\gamma'}xx_0\right) dx \qquad (4.38)$$

with λ (in units of a_0^{-1}) as evaluated in Table I. The integral

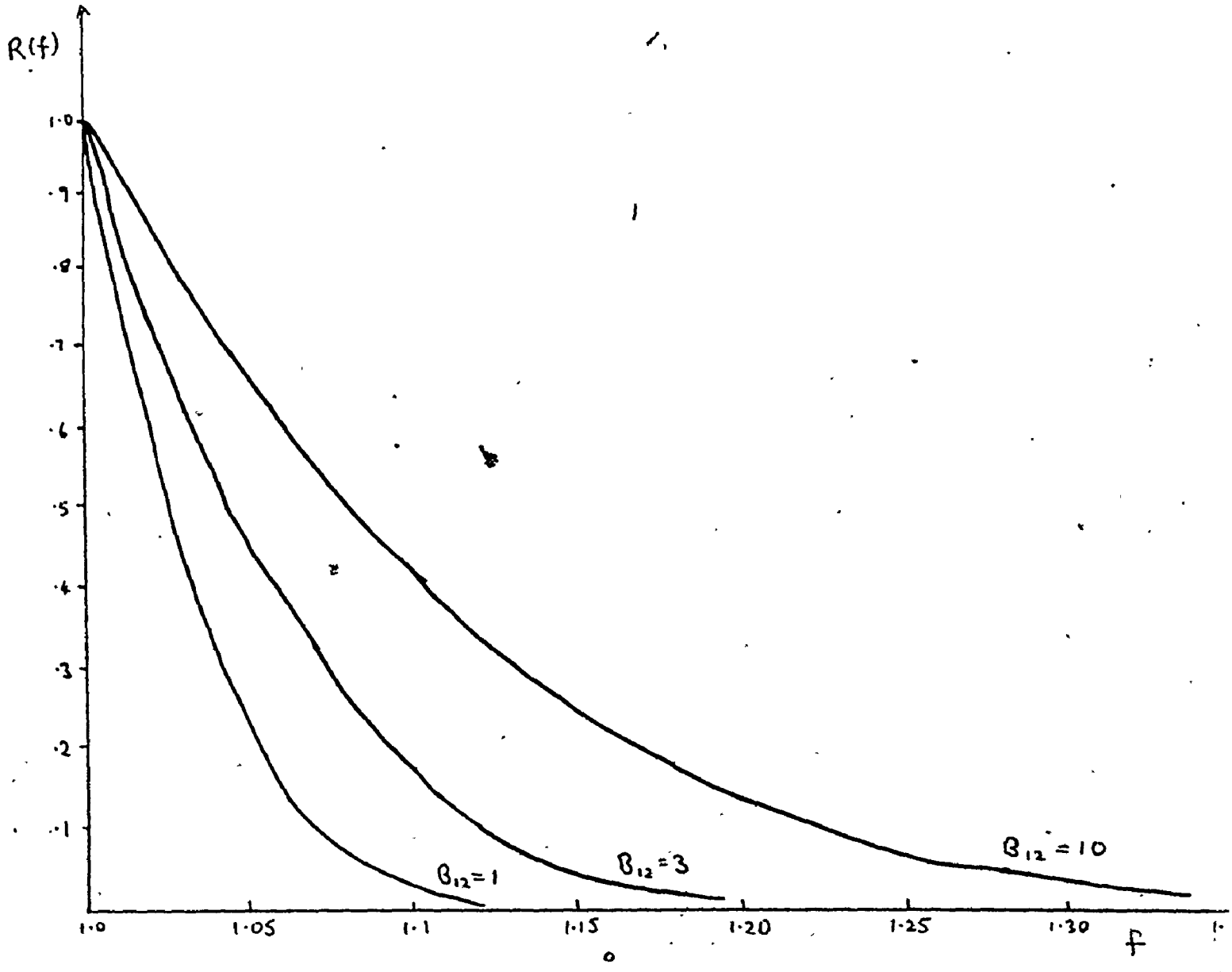


Figure 5.1: Distribution of photon frequency as a function of the magnetic field

in (4.38) was done numerically using the computer code listed in Appendix B.

The half widths for the two photon process, listed in column 5, were obtained from Figure 5.1 which demonstrates the expected spread of the emergent photons about the free space line at 511 keV. As the field increases, the half width increases producing a broader line. In Figure 5.1, the vertical axis is defined in terms of a function R(f), where

$$R(f) = \frac{2\tilde{I}(f)}{I} \qquad (5.1)$$

for each appropriate field with I, $\tilde{I}(f)$ as in equations (4.31), (4.34) respectively. R(f) is essentially a measure of the number of photons that are produced with energies $fm \leq \omega \leq 2m$ normalized to 1 so that the half width can be easily measured. The half width corresponds to f: R(f) = 0.5.

The two photon process is faster than the free space annihilation, for which $\tau^{-1} \sim 8\times10^{9}$ sec^{-1} (Chapter 2). The one photon rate vanishes exponentially as $B \to 0$; at $B_q = 4.4\times10^{13}$ G it is of order (7×10^{1}) faster than the two photon process. Nevertheless, it is clear from Table II that for the fields of current interest in neutron stars (10^{12}-10^{13} G) the one photon annihilation rate is too slow compared to the usual two photon decay. It seems reasonably certain from the results of this calculation that the 1.022 MeV line is not

sufficiently intense to be observed in neutron stars. For these same fields, the half width for the two photon decay is sufficiently small that the 511 keV line will remain fairly narrow.

APPENDIX A

SOLUTION OF DIRAC EQUATION IN A MAGNETIC FIELD

This appendix closely follows the discussion in Chapter 2 of Sokolov and Ternov (1968). Consider the Dirac equation for a relativistic electron moving in a magnetic field (with $\hbar = c = 1$)

$$i \frac{\partial}{\partial t} \psi(\underset{\sim}{r},t) = H\psi(\underset{\sim}{r},t) \qquad \text{(A-1)}$$

where $H = \underset{\sim}{\alpha}\cdot\underset{\sim}{P} + \beta m_0$ and α, β are the usual matrices. In a magnetic field

$$\underset{\sim}{P} = -i\nabla + e\underset{\sim}{A} \qquad \text{(A-2)}$$

where $(-e)$ is the charge on the electron. If we consider a time independent, homogeneous magnetic field $\underset{\sim}{B} = B\hat{\underset{\sim}{k}}$ and assume that the four spinor $\psi(\underset{\sim}{r},t)$ is given by

$$\psi(\underset{\sim}{r},t) = \exp(-i\varepsilon t)\psi(\underset{\sim}{r}) \qquad \text{(A-3)}$$

where ε is the energy, then using the explicit representation of the matrices $\underset{\sim}{\alpha},\beta$ equation (6.1) becomes a set of four coupled equations:

$$\begin{aligned}
(\varepsilon - m_0)\psi_1 - (P_x - iP_y)\psi_4 - P_z\psi_3 &= 0 \\
(\varepsilon - m_0)\psi_2 - (P_x + iP_y)\psi_3 + P_z\psi_4 &= 0 \\
(\varepsilon + m_0)\psi_3 - (P_x - iP_y)\psi_2 - P_z\psi_1 &= 0 \\
(\varepsilon + m_0)\psi_4 - (P_x + iP_y)\psi_1 + P_z\psi_2 &= 0
\end{aligned} \qquad \text{(A-4)}$$

ψ_μ are the components of the four spinor $\psi(\underset{\sim}{r})$. The magnetic field is described by the vector potential

$$A_x = -\frac{1}{2} yB$$
$$A_y = \frac{1}{2} xB \qquad \text{where } \underset{\sim}{B} = \nabla \times \underset{\sim}{A} \tag{A-5}$$
$$A_z = 0 \ .$$

In a cylindrical coordinate system ρ,θ,z ($x = \rho\cos\theta, y = \rho\sin\theta$) equations (A-4) can be separated and so we assume

$$\psi(\underset{\sim}{r}) = \psi_{k_3 \ell}(z,\theta) f(\rho) \tag{A-6}$$

where

$$\psi_{k_3 \ell}(z,\theta) = \frac{\exp ik_3 z}{\sqrt{L}} \frac{\exp i(\ell - \frac{1}{2})\theta}{\sqrt{2\pi}} \tag{A-7}$$

$$f(\rho) = \begin{pmatrix} f_1(\rho) & \exp(-i\theta/2) \\ f_2(\rho) & \exp(i\theta/2) \\ f_3(\rho) & \exp(-i\theta/2) \\ f_4(\rho) & \exp(i\theta/2) \end{pmatrix} \tag{A-8}$$

ℓ is the azimuthal quantum number ($\ell = 0, \pm 1, \ldots$) and k_3 ($\frac{2\pi n_3}{L}$, $n_3 = 0, \pm 1 \ldots$) is the z component of the electron momentum. $\psi_{k_3 \ell}(z,\theta)$ satisfies the orthonormality condition

$$\int_0^{2\pi} d\theta \int_{-L/2}^{L/2} \psi^*_{k_3' \ell'}(z,\theta) \psi_{k_3 \ell}(z,\theta) dz = \delta_{\ell\ell'} \delta_{k_3 k_3'} \ . \tag{A-9}$$

In this coordinate system the operators $(P_x \pm iP_y)$, P_z have the form:

$$(P_x \pm iP_y) = -i \exp(\pm i\theta)\{\frac{\partial}{\partial\rho} + \frac{i}{\rho}\frac{\partial}{\partial\theta} \mp \gamma\rho\}$$
$$P_z = -i\frac{\partial}{\partial z} \qquad \text{where } \gamma = \frac{eB}{2} \tag{A-10}$$

Substitution of (A-5) to (A-10) into the Dirac equation (A-4) yields the differential equations satisfied by the functions $f_\mu(\rho)$

$$(\varepsilon \mp m_0)f_{1,3}(\rho) + iR_2 f_{4,2}(\rho) - k_3 f_{3,1}(\rho) = 0$$
$$(\varepsilon \mp m_0)f_{2,4}(\rho) + iR_1 f_{3,1}(\rho) + k_3 f_{4,2}(\rho) = 0 \tag{A-11}$$

in which the operators $R_{1,2}$ are

$$R_1 = \frac{\partial}{\partial\rho} - \frac{(\ell-1)}{\rho} - \gamma\rho$$
$$R_2 = \frac{\partial}{\partial\rho} + \frac{\ell}{\rho} + \gamma\rho \,. \tag{A-12}$$

Introducing a change of variables: $\omega = \gamma\rho^2$ and using (A-11) to obtain the second order differential equations for the components f_μ:

$$\{w\frac{d^2}{dw^2} + \frac{d}{dw} + \lambda - \frac{\ell}{2} - \frac{w}{4} - \frac{(\ell-1)^2}{4w}\} f_{1,3}(w) = 0$$
$$\{w\frac{d^2}{dw^2} + \frac{d}{dw} + \lambda - \frac{(\ell-1)}{2} - \frac{w}{4} - \frac{\ell^2}{4w}\} f_{2,4}(w) = 0 \tag{A-13}$$

where $\lambda = \frac{1}{4\gamma}\{\varepsilon^2 - m_0{}^2 - k_3{}^2\}$.

So that the solution vanishes as $\rho \to \infty$, the eigenvalues of λ are n: $n = 0,1,2 \ldots$.

Hence

$$\varepsilon = \pm(m_0^2+k_3^2+4\gamma n)^{1/2} = \pm E \qquad \text{(A-14)}$$

where $E = (m_0^2+k_3^2 + 4\gamma n)^{1/2}$. The two possible signs of the energy are interpreted as corresponding to electrons and positrons, as in the field free case in which the negative energy electrons are understood to be positive energy positrons. The solutions to the differential equations (A-13) are known as Laguerre Functions $I_{n,s}(w)$, where s the radial quantum number, is given by $s = n-\ell = 0,1,2\ldots$.

The Laguerre Functions are defined by

$$I_{n,s}(w) = \frac{1}{\sqrt{n!s!}} e^{-w/2} w^{(n-s)/2} \phi_s^{n-s}(w) \qquad \text{(A-15)}$$

where $\phi_s^{\ell}(w) = w^{-\ell} e^{w} \frac{d^s}{dw^s} (w^{\ell+s} e^{-w})$. (A-16)

Incorporating all these results, the final form of $f_\mu(w)$ is

$$f = \sqrt{2\gamma} \begin{pmatrix} C_1 I_{n-1,s}(w) & \exp(-i\theta/2) \\ C_2 I_{n,s}(w) & \exp(i\theta/2) \\ C_3 I_{n-1,s}(w) & \exp(-i\theta/2) \\ C_4 I_{n,s}(w) & \exp(i\theta/2) \end{pmatrix} \qquad \text{(A-17)}$$

[Note that C_2, C_4 differ from those of Sokolov and Ternov: the factor i has been included in (A-17)].

The coefficients C_μ reflect the type of spin polarization chosen as basis and satisfy

$$(\varepsilon \mp m_0)C_{1,3} + i\sqrt{4\gamma n}\, C_{4,2} - k_3 C_{3,1} = 0$$
$$(\varepsilon \mp m_0)C_{2,4} + i\sqrt{4\gamma n}\, C_{3,1} + k_3 C_{4,2} = 0 \,. \qquad \text{(A-18)}$$

The radial wavefunction is normalized:

$$\int_0^\infty f^\dagger f\, \rho d\rho = 1$$

and this requires that the C_μ be normalized:

$$\sum_{\mu=1}^{4} |C_\mu|^2 = 1. \qquad \text{(A-19)}$$

In strong magnetic fields the spin of the electron will be antiparallel to the field and so the most useful spinor functions will be in a basis corresponding to spin up, spin down states. Imposing this condition on equations (A-18), (A-19), the four independent spinors - two for each sign of the energy - are

$$\{C\} = \left(\frac{E+m_0}{2E}\right)^{1/2} \begin{bmatrix} 1 & 0 & \frac{-k_3}{E+m_0} & \frac{\sqrt{4\gamma n}}{E+m_0} \\ 0 & 1 & \frac{-i\sqrt{4\gamma n}}{E+m_0} & \frac{-ik_3}{E+m_0} \\ \frac{k_3}{E+m_0} & \frac{\sqrt{4\gamma n}}{E+m_0} & 1 & 0 \\ \frac{i\sqrt{4\gamma n}}{E+m_0} & \frac{-ik_3}{E+m_0} & 0 & -1 \end{bmatrix} \qquad \text{(A-20)}$$

Spin Index: ξ =	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$
Energy: ε =	E	E	E	-E
	positive energy		negative energy	

ξ is the z component of the spin, with quantum number $\pm \frac{1}{2}$.

If we return to the Dirac equation (A-1) and instead of the electron consider a particle with charge (+e), then the solutions obtained represent the wavefunctions for a positron. Similar algebra to that already described for the electron solutions leads to a set of wavefunctions $\phi(\underset{\sim}{r},t)$, which have the form

$$\phi(\underset{\sim}{r},t) = \exp(-i\varepsilon t)\ \frac{\exp ik_3 z}{\sqrt{L}}\ \frac{\exp -i(\ell-\frac{1}{2})\theta}{\sqrt{2\pi}}\ f'(\rho) \qquad \text{(A-21)}$$

where

$$f'(\rho) = \sqrt{2\gamma}\begin{pmatrix} C_1' & I_{n,s}(w) & \exp(-i\theta/2) \\ C_2' & I_{n-1,s}(w) & \exp(i\theta/2) \\ C_3' & I_{n,s}(w) & \exp(-i\theta/2) \\ C_4' & I_{n-1,s}(w) & \exp(i\theta/2) \end{pmatrix} \qquad \text{(A-22)}$$

The coefficients C' are

$$\{C'\} = \sqrt{\frac{E+m_0}{2E}}\left[\begin{pmatrix} i \\ 0 \\ \frac{ik_3}{E+m_0} \\ \frac{\sqrt{4\gamma n}}{E+m_0} \end{pmatrix}\begin{pmatrix} 0 \\ -1 \\ -\frac{i\sqrt{4\gamma n}}{E+m_0} \\ \frac{k_3}{E+m_0} \end{pmatrix}\begin{pmatrix} \frac{-ik_3}{E+m_0} \\ -\frac{\sqrt{4\gamma n}}{E+m_0} \\ i \\ 0 \end{pmatrix}\begin{pmatrix} \frac{-i\sqrt{4\gamma n}}{E+m_0} \\ \frac{k_3}{E+m_0} \\ 0 \\ 1 \end{pmatrix}\right] \qquad \text{(A-23)}$$

Spin Index:	$\xi =$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$ $\quad$ $-\frac{1}{2}$
Energy:	$\varepsilon =$	E	E	$-E$ $\quad$ $-E$
		positive energy		negative energy

The electron and positron solutions are transformed into each other through use of the charge conjugation operator C which

can be represented as

$$C\psi = i\gamma_2\psi^* \quad . \qquad \text{(A-24)}$$

In the limit $n = 0$ and $k_3 \to 0$ which corresponds to the electron and positron ground states, these spinors (A-20), (A-23) can be separated into "large" and "small" components, allowing the appropriate non-relativistic representation through two component spinors. It is important to note that in the basis chosen, the transverse and longitudinal motion - effectively represented by n and k_3 - are decoupled and the electron or positron may be extremely relativistic but the wavefunction still has the same form as in the non-relativistic limit - viz a Landau orbital (equivalent to $I_{ns}(\rho)$) and a plane wave z component.

It is convenient to define a notation for the electron solutions which is useful for the calculations in Chapter 3. We define $\psi(\underset{\sim}{x},t)$ to be the electron solutions to the Dirac equation where

$$\psi(\underset{\sim}{x},t) = \exp(-i\varepsilon t)\eta_{k_3\ell}(\underset{\sim}{x})\, f_{\pm}(n,s,k_3,\xi,\underset{\sim}{x}) \qquad \text{(A-25)}$$

with

$$\eta_{k_3\ell} \doteq \frac{\exp ik_3z}{\sqrt{L}} \; \frac{\exp\,(\ell-\frac{1}{2})\theta}{\sqrt{2\pi}} \quad . \qquad \text{(A-26)}$$

The (±) refers to positive and negative energy solutions and the $f_{\pm}(n,s,k_3,\xi,\underset{\sim}{x})$ are given by

$$f_{\pm}(n,s,k_3,\xi,\underset{\sim}{x}) = \sqrt{2\gamma}\begin{pmatrix} C_1^{\pm}(\xi)\ I_{n-1,s}(w)\ \exp(-i\theta/2) \\ C_2^{\pm}(\xi)\ I_{n,s}(w)\ \exp(i\theta/2) \\ C_3^{\pm}(\xi)\ I_{n-1,s}(w)\ \exp(-i\theta/2) \\ C_4^{\pm}(\xi)\ I_{n,s}(w)\ \exp(i\theta/2) \end{pmatrix} \qquad \text{(A-27)}$$

where the $C_\mu^{\pm}(\xi)$ are defined by the appropriate coefficients in (A-20).

Quantization of the Dirac Field

The procedure in quantizing the Dirac field using the electron and positron solutions developed earlier is similar to the technique described in Bjorken and Drell (1964) using plane wave solutions. Basically, the field operators ψ, $\psi^\dagger$ are expanded in terms of the basis set represented by the electron solutions of equation (A-25) and made to satisfy the requisite anti-commutation relations, namely

$$\begin{aligned} \{\psi_\alpha(\underset{\sim}{x},t)\ ,\ \psi_\beta(\underset{\sim}{x}',t)\} &= 0 \\ \{\psi_\alpha^\dagger(\underset{\sim}{x},t)\ ,\ \psi_\beta^\dagger(\underset{\sim}{x}',t)\} &= 0 \\ \{\psi_\alpha(\underset{\sim}{x},t)\ ,\ \psi_\beta^\dagger(\underset{\sim}{x}',t)\} &= \delta_{\alpha\beta}\delta^3(\underset{\sim}{x}-\underset{\sim}{x}'). \end{aligned} \qquad \text{(A-28)}$$

With this in mind, we define the field operators

$$\psi(\underset{\sim}{x},t) = \sum_{n}\sum_{s}\sum_{k_3}\sum_{\xi}\{a_+(\ell,s,k_3,\xi)\exp(-iEt)\eta_{k_3\ell}(\underset{\sim}{x})f_+(n,s,k_3,\xi,\underset{\sim}{x})$$

$$+ a_-(\ell,s,k_3,\xi)\exp(iEt)\eta_{k_3\ell}(\underset{\sim}{x})f_-(n,s,k_3,\xi,\underset{\sim}{x})\}$$

$$\psi^\dagger(\underset{\sim}{x}',t) = \sum_{n'}\sum_{s'}\sum_{k_3'}\sum_{\xi'}\{a_+^\dagger(\ell',s',k_3',\xi')\exp(iE't)\eta^\dagger_{k_3'\ell'}(\underset{\sim}{x}')f_+^\dagger(n',s',k_3',\xi',\underset{\sim}{x}'$$

$$+ a_-^\dagger(\ell',s',k_3',\xi')\exp(-iE't)\eta^\dagger_{k_3'\ell'}(\underset{\sim}{x}')f_-^\dagger(n',s',k_3',\xi',\underset{\sim}{x}')\} \qquad \text{(A-29)}$$

in which $a_\pm(\ell,s,k_3,\xi)$ annihilate positive and negative energy electrons respectively, while $a_\pm^\dagger(\ell,s,k_3,\xi)$ create positive and negative energy electrons. In keeping with the electron-positron (hole) formalism, the negative energy operators can be replaced by positive energy positron operators:

$$a_-(\ell,s,k_3,\xi) = b_+^\dagger(\ell,s,-k_3,-\xi)$$
$$a_-^t(\ell,s,k_3,\xi) = b_+(\ell,s,-k_3,-\xi). \qquad \text{(A-30)}$$

These operators (a_+, b_+) satisfy the anti-commutation relations:

$$\{a,b'\} = \{a^\dagger,b^{\dagger'}\} = \{a,b^{\dagger'}\} = \{a^\dagger,b'\} = 0$$
$$\{a,a^{\dagger'}\} = \{b,b^{\dagger'}\} = \delta_{\ell\ell'}\delta_{ss'}\delta_{k_3k_3'}\delta_{\xi\xi'}\,. \qquad \text{(A-31)}$$

With the positron operators, we can write (A-24) in a more familiar form

$$\psi(\underset{\sim}{x},t) = \sum_{n}\sum_{s}\sum_{k_3}\sum_{\xi}\{a(\ell,s,k_3,\xi)\exp(-iEt)\eta_{k_3\ell}(\underset{\sim}{x})f_{+}(n,s,k_3,\xi,\underset{\sim}{x})$$

$$+ b^{\dagger}(\ell,s,k_3,\xi)\exp(iEt)\eta_{-k_3\ell}(\underset{\sim}{x})f_{-}(n,s,-k_3,-\xi,\underset{\sim}{x})\}$$

$$\psi^{\dagger}(\underset{\sim}{x}',t)=\sum_{n's'k_3'\xi'}\{a^{\dagger}(\ell',s',k_3',\xi')\exp(iE't)\eta^{\dagger}_{k_3'\ell'}(\underset{\sim}{x})f^{\dagger}_{+}(n',s',k_3',\xi',\underset{\sim}{x}')$$

$$+ b(\ell',s',k_3',\xi')\exp(-iE't)\eta^{\dagger}_{-k_3'\ell'}(\underset{\sim}{x}')f^{\dagger}_{-}(n',s',-k_3',-\xi',\underset{\sim}{x}')\} \qquad \text{(A-32)}$$

In (A-32), the substitutions $k_3 \to -k_3$ and $\xi \to -\xi$ have been made - this is possible because the sums over k_3, ξ extend over the entire range of allowed values. To show that this is a correct quantization of the Dirac field, consider for example

$$\{\psi_{\alpha}(\underset{\sim}{x},t)\ ,\ \psi^{\dagger}_{\beta}(\underset{\sim}{x}',t)\}\ .$$

Using equation (A-32) for the field operators ψ, $\psi^{\dagger}$, this can be written as

$$\{\psi_{\alpha}(\underset{\sim}{x},t),\psi^{\dagger}_{\beta}(\underset{\sim}{x}',t)\} = \sum_{nn'}\sum_{ss'}\sum_{k_3k_3'}\sum_{\xi\xi'} X_{\alpha\beta} \qquad \text{(A-33)}$$

where

$$X_{\alpha\beta} = \begin{aligned} &\{(aa^{\dagger\prime}+a^{\dagger\prime}a)\eta(\underset{\sim}{x})\eta^{\dagger}(\underset{\sim}{x}')f_{+\alpha}f^{\dagger\prime}_{+\beta}\ \exp -i(E-E')t \\ &+ (b'b^{\dagger}+b\,b^{\dagger\prime})\eta(\underset{\sim}{x})\eta^{\dagger}(\underset{\sim}{x}')f_{-\alpha}f^{\dagger\prime}_{-\beta}\ \exp\ i(E-E')t \\ &+ (ab'+b'a)\eta(\underset{\sim}{x})\eta^{\dagger}(\underset{\sim}{x}')f_{+\alpha}f^{\dagger\prime}_{+\beta}\ \exp -i(E+E')t \\ &+ (b^{\dagger}a^{\dagger\prime}+a^{\dagger\prime}b^{\dagger})\eta(\underset{\sim}{x})\eta^{\dagger}(\underset{\sim}{x}')f_{-\alpha}f^{\dagger\prime}_{-\beta}\ \exp\ i(E+E')t \end{aligned} \qquad \text{(A-34)}$$

where the condensed notation is used to avoid explicitly writing out the indices, variables etc. in (A-32). Equation (A-34) can

can be considerably simplified using the commutation relations (A-31): the double sum collapses to [with E = E']

$$\{\psi_\alpha, \psi_\beta^\dagger\} = \sum_n \sum_s \int dk_3 \sum_\xi Y_{\alpha\beta} \qquad \text{(A-35)}$$

where

$$Y_{\alpha\beta} = \eta_{k_3 \ell}(\underset{\sim}{x})\eta^\dagger_{k_3 \ell}(\underset{\sim}{x}')f_{+\alpha}(n,s,k_3,\xi,\underset{\sim}{x})f^\dagger_{+\beta}(n,s,k_3,\xi,\underset{\sim}{x}') + \eta_{-k_3 \ell}(\underset{\sim}{x})\eta^\dagger_{-k_3 \ell}(\underset{\sim}{x}')f_{-\alpha}(n,s,-k_3,-\xi,\underset{\sim}{x})f^\dagger_{-\beta}(n,s,-k_3,-\xi,\underset{\sim}{x}') \qquad \text{(A-36)}$$

Using the explicit form of $\eta_{k_3 \ell}(\underset{\sim}{x})$ as defined in (A-26), $Y_{\alpha\beta}$ becomes:

$$Y_{\alpha\beta} = \frac{1}{L}\frac{1}{2\pi}\{f_{+\alpha}(n,s,k_3,\xi,\underset{\sim}{x})f^\dagger_{+\beta}(n,s,k_3,\xi,\underset{\sim}{x}') + f_{-\alpha}(n,s,k_3,\xi,\underset{\sim}{x})f^\dagger_{-\beta}(n,s,k_3,\xi,\underset{\sim}{x}')\} \times \exp ik_3(z-z')\exp\{i(\ell - \tfrac{1}{2})(\theta-\theta')\} \qquad \text{(A-37)}$$

In the second term of (A-37), the transformation $k_3 \to -k_3$, $\xi \to -\xi$ has been made since the summation in (A-35) is over the complete range - this is essentially reversing the transformation made in deriving the field operators in terms of positron operators (A-32).

The next step is to evaluate the spinor contribution in equation (A-37). Using the appropriate components from the electron solutions (equation (A-27), which leads to (A-20)), much tedious algebra shows that $\sum_\xi Y_{\alpha\beta}$ can be written as

$$\sum_{\xi} Y_{\alpha\beta} = \frac{1}{2\pi}\frac{1}{L}\exp ik_3(z-z')\exp\{i(\ell - \tfrac{1}{2})(\theta-\theta')\}F_{\alpha\beta} \qquad (A-38)$$

where

$$F_{\alpha\beta} = \delta_{\alpha\beta}\, 2\gamma \begin{cases} I_{n-1,s}(w)I_{n-1,s}(w')\exp\{-i(\theta-\theta')/2\} & \alpha = 1,3 \\ I_{n,s}(w)I_{n,s,}(w')\ \exp\{i(\theta-\theta')/2\} & \alpha = 2,4 \end{cases} \qquad (A-39)$$

Thus, finally

$$\{\psi_\alpha,\psi_\beta\} = \sum_n \sum_s \sum_{k_3} \frac{\exp ik_3(z-z')}{L} \frac{\exp\{i(\ell - \tfrac{1}{2})(\theta-\theta')\}}{2\pi} F_{\alpha\beta} \qquad (A-40)$$

Now

$$\sum_{k_3} \frac{\exp ik_3(z-z')}{L} = \delta(z-z') \qquad (A-41)$$

and so equation (A-40) can be expressed as

$$\{\psi_\alpha,\psi_\beta\} = \sum_n \sum_s \frac{\exp\{i(\ell-1)(\theta-\theta')\}}{2\pi} I_{n-1,s}(w)I_{n-1,s}(w')2\gamma\delta_{\alpha\beta}\delta(z-z') \qquad (A-42)$$

for $\alpha = 1,3$

$$\{\psi_\alpha,\psi_\beta\} = \sum_n \sum_s \frac{\exp\{i\ell(\theta-\theta')\}}{2\pi} I_{n,s}(w')I_{n,s}(w)2\gamma\delta_{\alpha\beta}\delta(z-z'). \qquad (A-43)$$

for $\alpha = 2,4$

In (A-40) the sum is from n=1 (as $I_{-1,0}$ is not defined) and so if we replace (n-1) with n, both expressions can be written as

$$\{\psi_\alpha,\psi_\beta\} = \sum_n \sum_s \frac{\exp i\ell(\theta-\theta')}{2\pi} I_{n,s}(w)I_{n,s}(w')2\gamma\delta_{\alpha\beta}\delta(z-z')$$

$$\equiv \frac{1}{\rho'}\delta(\rho-\rho')\delta(\theta-\theta')\delta(z-z')\delta_{\alpha\beta}\,. \qquad (A-44)$$

The identification of $\{\psi_\alpha,\psi_\beta\}$ as a product of Dirac delta

functions follows because the sum $\sum_n \sum_s$ is over a complete set of states - viz equation (A-41), a more obvious application of this result. The delta equivalence can be seen more explicitly by multiplying the two forms of $\{\psi_\alpha, \psi_\beta\}$ in (A-44) by $\exp(i\ell'\theta') I_{n',s'}(w')$ and integrating over θ', w':

$$\int_0^{2\pi} \int_0^{\infty} \exp(i\ell'\theta') I_{n',s'}(w') \frac{\delta(\rho-\rho')}{\rho'} \delta(\theta-\theta') \delta(z-z') \delta_{\alpha\beta} \, d\theta' dw'$$

$$= \int_0^{2\pi} \exp(i\ell'\theta') \delta(\theta-\theta') d\theta' \int_0^{\infty} I_{n',s'}(\gamma\rho'^2) \delta(\rho-\rho')$$

$$\times \delta(z-z') \delta_{\alpha\beta} \frac{1}{\rho'} \underbrace{2\gamma\rho' d\rho'}_{dw'}$$

$$= 2\gamma \, I_{n',s'}(w) \exp(i\ell'\theta) \delta(z-z') \delta_{\alpha\beta} . \qquad \text{(A-45)}$$

Using the alternative form of $\{\psi_\alpha, \psi_\beta\}$, we have

$$\int_0^{\infty} \int_0^{2\pi} \exp(i\ell'\theta') d\theta' \{\psi_\alpha, \psi_\beta\} I_{n',s'}(w') dw'$$

$$= 2\gamma \delta_{\alpha\beta} \delta(z-z') \sum_{n,s} \sum \int_0^{\infty} I_{n,s}(w) I_{n,s}(w') I_{n',s'}(w') dw'$$

$$\times \int_0^{2\pi} \frac{\exp(i\ell\theta) \exp(i(\ell'-\ell)\theta') d\theta'}{2\pi}$$

$$= 2\gamma \delta_{\alpha\beta} \delta(z-z') \sum_n \sum_s \{ I_{n,s}(w) \exp i\ell\theta \int_0^{\infty} I_{n,s}(w') I_{n',s'}(w') dw'$$

(continued next page)

$$\times \int_0^{2\pi} \frac{\exp i(\ell'-\ell)\theta' d\theta'}{2\pi} \}$$

$$= 2\gamma\delta_{\alpha\beta}\delta(z-z')I_{n',s'}(w)\exp i\ell'\theta \qquad \text{(A-46)}$$

using the orthogonality properties of $I_{ns}(w')$ and $\exp i\ell'\theta'$. Thus the two forms of $\{\psi_\alpha,\psi_\beta\}$ defined in (A-44) are equivalent. Finally, recognizing that $\frac{1}{\rho'}\delta(\rho-\rho')\delta(\theta-\theta')\delta(z-z')$ is $\delta^3(\underset{\sim}{x}-\underset{\sim}{x}')$ in cylindrical coordinates

$$\{\psi_\alpha(\underset{\sim}{x},t),\psi_\beta(\underset{\sim}{x}',t)\} = \delta(\underset{\sim}{x}-\underset{\sim}{x}')\delta_{\alpha\beta} \quad . \qquad \text{(A-47)}$$

Similarly, it can be shown that

$$\{\psi_\alpha(\underset{\sim}{x},t),\psi_\beta(\underset{\sim}{x}',t)\} = \{\psi_\alpha^\dagger(\underset{\sim}{x},t),\psi_\beta^\dagger(\underset{\sim}{x}',t) = 0$$

and so the field operators defined in (A-32) do satisfy the required commutation relations.

APPENDIX B

NUMERICAL EVALUATION OF THE INTEGRALS

This program is designed to simultaneously evaluate the two dimensional integrals described in Chapter 4. With the constants removed, these are:

$$I = \int_0^1 dz \int_0^{2(1-z)^{1/2}} \frac{x}{(x^2+z^2)^{1/2}} \exp - \frac{1}{2\gamma'} (x-x_0)^2 \times \exp(-\frac{1}{\gamma'} xx_0) I_0(\frac{1}{\gamma'} xx_0)\,dx \quad K=1 \tag{4.31}$$

$$\tilde{I}(f) = \int_0^{2-f} dz \int_{(f^2-z^2)^{1/2}}^{2(1-z)^{1/2}} \frac{x}{(x^2+z^2)^{1/2}} \exp - \frac{1}{2\gamma'} (x-x_0)^2 \times \exp(-\frac{1}{\gamma'} xx_0) I_0(\frac{1}{\gamma'} xx_0)\,dx \quad K=2 \tag{4.34}$$

$$I' = \int_0^1 dz \int_0^{(1-z^2)^{1/2}} \frac{x}{(x^2+z^2)^{1/2}} \exp - \frac{1}{2\gamma'} (x-x_0)^2 \times \exp(-\frac{1}{\gamma'} xx_0) I_0(\frac{1}{\gamma'} xx_0)\,dx \quad K=3 \tag{4.35}$$

The K label appears in the program and allows selection of the appropriate x,z limits for each of the three integrals defined above. I' corresponds to the integral for which $0 \leq \omega_1 \leq m$ and is used as a check on the accuracy of the integration: $I' = \tilde{I}(1) = \frac{1}{2} I$, as discussed previously and

this comparison ensures the accuracy of the results. The integration scheme was also tested on several similar known two dimensional integrals.

The method of integration used consisted of applying Simpson's Rule twice, firstly for the x integration and then for the z integration. The particular form of Simpson's Rule employed was that used in the Scientific subroutine QSF. The remaining subroutine, Bessel, calculates a numerical approximation to $e^{-x}I_0(x)$, as defined on page 378 of Abramowitz and Stegun (1965). The input requirements for this program consist of a magnetic field (ARRAY), the number of steps (NDIM) and various DO loop parameters (K,L,N) which select the appropriate integrals, magnetic field, f values, etc.

The program is as follows:

```
DIMENSION FX(800),FZ(800),RES(800),BESS(800)
DIMENSION ARRAY(10)

C  ARRAY IS A LIST OF THE POSSIBLE MAGNETIC FIELDS OF
C  INTEREST IN UNITS OF 10^12 GAUSS. THE APPROPRIATE FIELD
C  (BM = ARRAY(L)) FOR A PARTICULAR CALCULATION IS SELECTED
C  BY THE INDICE L, WITH 1 < L < 10.  ARRAY(L) IS INPUTTED
C  THROUGH THE FOLLOWING DATA CARD

DATA ARRAY/1.0,3.0,10.0/
DO 300 K = 1,3
DO 300 L = 1,3
DO 300 N = 3,20

C  N IS THE NUMBER OF POINTS AT WHICH f VALUES ARE EVALUATED

BM = ARRAY(L)
QQ = 0.01132,
GAMMA = QQ*BM
TEST = GAMMA/2
```

```
C  GAMMA = γ/m^2 = γ'; BETA = ρ̂m; TEST (γ'/2) IS THE THEORETICAL
C  VALUE OF THE INTEGRAL AT B = 0.

IF(L.EQ.1)  ST = 0.005
IF(L.EQ.2)  ST = 0.01
IF(L.EQ.3)  ST = 0.02
F = 1.0 + (N-1)*ST
TAU = 1.0 + 2.0*GAMMA

C  ST CHOOSES AN APPROPRIATE STEP FOR CALCULATING VALUES OF
C  F: TAU LIMITS THE CALCULATION TO APPROXIMATELY THE VALUE
C  OF F AT THE HALFWIDTH: TAU = 1+2γ'

NDIM = 100
NDIMZ = NDIM
NDIMX = 2* NDIM

C  NDIMX, NDIMZ ARE NUMBER OF STEPS IN THE X,Z INTEGRATIONS,

WRITE (6,115)
115 FORMAT (1X,'NDIMX', 1X, 'NDIMZ', 4X, 'BM',11X, 'BETA',11X,
1 'GAMMA', 11X, 'TEST')
WRITE (6,275) NDIMX, NDIMZ, BM, BETA, GAMMA, TEST
275 FORMAT (1X, 2I4, 1P4E15.6//)

IF (K.EQ.1)  ZLIM = 1.0
IF (K.EQ.2)  ZLIM = 2.0-F
IF (K.EQ.3)  ZLIM = 1.0
HZ = ZLIM/(NDIMZ-1)

C  ABOVE SELECTS APPROPRIATE Z LIMIT FOR PARTICULAR INTEGRATION,
C  AND CALCULATES HZ = STEP SIZE IN Z.

DO 200 J=1, NDIMZ
IF(K.EQ.1) XLIML = 0.0
IF(K.EQ.2) XLIML = SQRT(F*F-Z*Z)
IF(K.EQ.3) XLIML = 0.0
IF(K.EQ.1) XLIMU = Z*SQRT(1-Z)
IF(K.EQ.2) XLIMU = Z*SQRT(1-Z)
IF(K.EQ.3) XLIMU = SQRT(1-Z*Z)
HX = (XLIMU-XLIML)/(NDIMX-1)

C  ABOVE SELECTS UPPER (XLIMU) AND LOWER (XLIML)
C  LINES FOR X INTEGRATION, AND CALCULATES  HX = STEP
C  SIZE FOR X .
```

```
DO 100 I =  HX,NDIMX
X = (I-1)*HX+XLIML
A = SQRT(X*X + Z*Z)
IF(A·EQ·0·0)  A = 1.0 E-100
B = 4 - 4*A + X*X
IF(B·LT·0·0)  B = 0.0
XO = SQRT(B)
ARG = (X*X0)/GAMMA
CALL BESSEL (I,ARG,BESS(I))
C = ((X-X0)**2)/(2*GAMMA)
IF(C·GT.320)  C = 320
FX(I) = (X/EXP(C))*(BESS(I)/A)

C  ABOVE CALCULATES VALUE OF INTEGRAND FX(I) AT X(I), Z(J)
C  BESS(I) IS THE NUMERICAL APPROXIMATION MENTIONED ABOVE
C  THE VARIOUS IF STATEMENTS ARE TO PREVENT UNDEFINED
C  OPERATIONS (O/O) OR VERY LARGE NUMBERS C> e320)

100 CONTINUE
CALL QSF(HX,FX,RES,NDIMX)
FZ(J) = RES(NDIMX)

C  FZ(J) IS X INTEGRAL AT Z(J)

200 CONTINUE

CALL QSF(HZ,FZ,RES,NDIMZ)
ANS = RES(NDIMZ)
RATIO = ANS/TEST

C  ANS IS THE VALUE OF THE 2-DIMENSIONAL INTEGRAL
C  RATIO IS A COMPARISON WITH THE LOW FIELD LIMIT

WRITE(6,259)
259 FORMAT(1X,2X,'K',6X,'XLIMU',10X,'XLIML',10X,'ZLIM',11X,'RATIO'
1  12Z,'ANS',13X,'F',13X,'N'
WRITE(6,375) K, XLIMU, XLIML, ZLIM, RATIO,ANS,F,N
375 FORMAT (1X,I4,1P6E15.6,I4//)
300 CONTINUE
STOP
END
```

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