FRÉCHET ALGEBRAS WITH SCHAUDER BASES

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SCOPE AND CONTENTS:

Let A be a Fréchet algebra with the defining sequence $\{q_n\}_{n\geq 1}$ of seminorms and identity e. Let $\{x_i\}$ be a Schauder basis in A. Then each $x \varepsilon A$ can be written as: $x = \sum_{i=1}^{\infty} \alpha_i x_i$, where $\{\alpha_i\}$ is a unique sequence of complex numbers depending upon x. α_i's are called coordinate functionals or coefficients. This thesis is concerned with some relations among coefficients, seminorms and the identity e of A. Further, it is shown that each multiplicative linear functional on A is continuous provided a certain condition is satisfied. Some of the results needed to prove the above results are shown to be true for Fréchet spaces. Finally, a representation theorem for Fréchet * - algebra is given.

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INTRODUCTION

In 1946, Arens [1] first introduced the concept of a locally m-convex topological algebra. A locally m-convex algebra is a topological algebra over the complex field with a base $\{U_i\}_{i\varepsilon T}$ for the neighborhoods of the origin such that each U_i is convex, symmetric and $U_i U_i \subseteq U_i$. Equivalently, E is locally m-convex if and only if, there is a set $\{q_i : i\varepsilon T\}$ of semi-norms on E such that (i) the sets of form $\{x\varepsilon E : q_i(x) \leq b\}$, for $i\varepsilon T$, $b>0$, form a base of neighborhoods of 0, and (ii) $q_i(xy) \leq q_i(x) q_i(y)$ for $i\varepsilon T$, $x,y\varepsilon E$. A Fréchet algebra is a complete, metrizable, locally m-convex algebra. For example: (i) A Banach algebra is a Fréchet algebra, but the converse is not true. (ii) The albegra C(R) of continuous real valued functions on the real line is a Fréchet algebra with semi-norms $q_n(f) = \sup_{|x| \leq n} |f(x)|$, $n \geq 1$ and pointwise multiplication. The standard work on locally m-convex algebra is the memoirs of Michael [12].

In this thesis, we mostly deal with Fréchet algebras with a Schauder base and establish some propositions which establish some relations with semi-norms, coordinate functionals defined by the base and identity, with a view to answering partly the following open question for all commutative Fréchet algebras: Is every multiplicative linear functional on a commutative Fréchet algebra continuous? Unfortunately, we do not as yet have a solution to this problem.

In Chapter 1, we state some important propositions, taken from [5] and [16], which are needed to prove our results in Chapters 2 to 5.

Definitions introduced here can be found in [12], [14], and [16].

Chapter 2 deals with the following problem: If a Fréchet algebra A possess a sequence $H = \{x_i\}_{i\varepsilon N}$ such that $x = \sum_{i=1}^{\infty}\alpha_i x_i$, does there exist a relation between the coefficients $\{\alpha_i\}$ and the semi-norms $\{q_n\}_{n\varepsilon N}$ where the latter form an increasing sequence of semi-norms and define the topology of A? Moreover, we also investigate the relations among $\{\alpha_i\}$, $\{q_n\}$ and the identity e, if $H = \{x_i\}_{i\varepsilon N}$ is a Schauder basis of A and A has an identity e. We have found some useful relations among them e.g. see theorem 2.1, theorem 2.2, theorem 2.3, and theorem 2.4. These results will be used to prove some major results in Chapter 3.

Chapter 3 is motivated by Michael's problem mentioned above: In 1952, Michael asked whether every multiplicative linear functional on a commutative Fréchet algebra is continuous. This problem is still open. However, we prove that the answer for certain classes of Fréchet algebras is affirmative.

In Chapter 4, we prove some results on Fréchet spaces and Fréchet algebras. In particular, considerations are given to the case where the Schauder basis consists of $H = \{e, z, z^2, \ldots\}$ or $H = \{z, z^2, z^3, \ldots\}$.

In Chapter 5 we generalize some representation theorems on Banach * - algebras to Fréchet * - algebras.

Finally, the results of Chapters 2-Chapter 4 constitute the main part of the thesis, while the results of Chapter 5 are generalized cases of two well-known representation theorems. They are given here to complete a reasonably comprehensive picture of our study.

Chapter 1

PRELIMINARIES

(1) Fundamental properties of topological vector spaces.

We say that a family $\mathcal{T}$ of semi - norms on a vector space is saturated if for any finite subfamily (q_i) of $\mathcal{T}$ the semi - norm $\mathrm{Max}_i q_i$ also belongs to $\mathcal{T}$. A locally convex topology can always be defined by a saturated family of semi-norms.

Proposition 1.1. Let E be a locally convex space whose topology is defined by a saturated family $\{q_i\}_{i\varepsilon I}$ of semi-norms. A linear form f on E is continuous if and only if there exists a semi-norm q_r and a positive number M such that $|f(x)|\leq Mq_r(x)$ for every $x\varepsilon E$.

Proof. See page 97 of [5]. It is a particular case of proposition 2.

Proposition 1.2. Let the locally convex topology $\mathcal{T}$ on the vector space E be defined by the family $\{q_i\}_{i\varepsilon I}$ of semi-norms. Then $\mathcal{T}$ is Hausdorff if and only if for every $x\neq 0$ in E there exists an index $i\varepsilon I$ such that $q_i(x)\neq 0$.

Proof. See page 96 of [5].

Proposition 1.3. Let E be a locally convex Hausdorff space whose topology $\mathcal{T}$ is defined by an increasing sequence $\{q_n\}_{n\varepsilon N}$ of semi-norms. The map $x \to |x|$ from E into R_+, where

$$|x| = \sum_{n=0}^{\infty} \frac{1}{2^n} \frac{q_n(x)}{1+q_n(x)}$$

has the following properties:

(a) $|x|=0$ if and only if $x=0$,

(b) $|x|=|-x|$,

(c) $|x+y|\leq|x|+|y|$,

(d) $|\lambda|\leq 1$ implies $|\lambda x|\leq|x|$,

(e) $\lambda \to 0$ implies $|\lambda x| \to 0$ for every $x\varepsilon E$,

(f) $x \to 0$ implies $|\lambda x| \to 0$ for every $\lambda\varepsilon C$.

Furthermore, the metric $d(x,y)=|x-y|$ defines the topology $\mathcal{T}$ and is translation invariant.

<u>Proof.</u> See page 114 of [5].

N.B. A metric δ on a vector space E is said to be translation-invariant if $\delta(x+a,y+a)=\delta(x,y)$.

<u>Proposition 1.4.</u> A locally convex space is metrizable if and only if it is Hausdorff and its topology can be defined by a countable family of semi-norms.

<u>Proof.</u> See page 113 of [5].

<u>Proposition 1.5.</u> If $L\neq\{0\}$ is a locally compact Hausdorff topological vector space over the complex field, then L is of finite dimension.

<u>Proof.</u> See page 23 of [6].

<u>Proposition 1.6.</u> A Hausdorff topological vector space L is normable if and only if L possesses a bounded, convex neighborhood of 0.

<u>Proof.</u> See page 41 of [16].

<u>Proposition 1.7.</u> The product of a family of normable spaces is normable if and only if the number of factors $\neq\{0\}$ is finite.

<u>Proof.</u> See page 41 of [16].

<u>Proposition 1.8.</u> Let A be a locally convex Hausdorff space whose

topology τ is defined by an increasing sequence $\{q_n\}_{n\in N}$ of semi-norms.

Then $|x_i-x_j| = \sum_{n=0}^{\infty} \frac{1}{2^n} \frac{q_n(x_i-x_j)}{1+q_n(x_i-x_j)} \to 0$ as $i,j \to \infty$ if and only if for each

q_n, $q_n(x_i-x_j) \to 0$ as $i,j \to \infty$.

Proof. Necessity: For each q_n, put $U_{n,\varepsilon}=\{x\varepsilon A|q_n(x)<\varepsilon\}$.

By proposition 1.3, since the topology defined by the metric $d(x,y)=|x-y|$ coincides with τ, there exists a $V_\delta=\{x\varepsilon A||x|<\delta\}$ such that $V_\delta \subseteq U_{n,\varepsilon}$. $|x_i-x_j| \to 0$ as $i,j \to \infty$ implies that there exists a positive integer $N_\delta>0$ such that $|x_i-x_j|<\delta$ whenever $i,j>N_\delta$. Therefore, $x_i-x_j\varepsilon V_\delta \subseteq U_{n,\varepsilon}$ whenever $i,j>N_\delta$; i.e. for each $\varepsilon>0$, there exists an N_δ dependent on ε & n, such that $q_n(x_i-x_j)<\varepsilon$ whenever $i,j>N_\delta$. This implies that, for each q_n, $q_n(x_i-x_j) \to 0$ as $i,j \to \infty$.

Sufficiency: For each $\varepsilon>0$, let $V_\varepsilon=\{x\varepsilon A||x|<\varepsilon\}$. Then there exists a $U_{n,\delta}=\{x\varepsilon A|q_n(x)<\delta\}$ such that $U_{n,\delta}\subseteq V_\varepsilon$. $q_n(x_i-x_j) \to 0$ as $i,j \to \infty$ implies that for each $\delta>0$, there exists a positive integer $N_\delta>0$ such that $q_n(x_i-x_j)<\delta$ whenever $i,j>N_\delta$, i.e. $x_i-x_j\varepsilon U_{n,\delta} \subseteq V_\varepsilon$ whenever $i,j>N_\delta$. Therefor, for each $\varepsilon>0$, there exists a N_δ depending on ε such that $|x_i-x_j|<\varepsilon$ whenever $i,j>N_\delta$. This implies that $|x_i-x_j| \to 0$ as $i,j \to \infty$.

Proposition 1.8'. Let A be a locally convex Hausdorff space whose topology is defined by an increasing sequence $\{q_n\}_{n\varepsilon N}$ of semi-norms. Then

$|x_i-x| = \sum_{n=0}^{\infty} \frac{1}{2^n} \frac{q_n(x_i-x)}{1+q_n(x_i-x)} \to 0$ as $i \to \infty$ if and only if for each q_n,

$q_n(x_i-x) \to 0$ as $i \to \infty$.

Proof. Using a proof similar to that of Proposition 1.8, the result follows.

(2) Fundamental Concepts of topological algebras

Definition 1.1. A topological algebra is an algebra over the complex field which is a topological linear space such that the ring multiplication is separately continuous, i.e. the map $\phi: A\times A \to A$, $\phi(x,y)=xy$, is a continuous function of each argument separately.

Definition 1.2. Let A be an algebra. A subset U in A is said to be symmetric if $x\in U$ and $|\alpha|=1$ (α is any complex number) imply $\alpha x\in U$.

Definition 1.3. A subset U in an algebra A is called m-convex if U is convex and $UU\subseteq U$.

Definition 1.4. A topological algebra is said to be locally m-convex if there exists a basis for the neighborhoods of the origin consisting of m-convex symmetric sets.

Definition 1.5. A Fréchet algebra is a complete, metrizable locally m-convex algebra.

By Arens theorem ([21], page 23) in each Fréchet algebra, the map ϕ of definition 1.1 is continuous in both variables together.

Definition 1.6. A Fréchet space is a complete and metrizable locally convex space.

Example 1.1. The algebra $C(-\infty,\infty)$ of all complex valued continuous functions on the real line with the topology defined by the semi-norms $q_n(x)=\max_{|\lambda|\le n}|x(\lambda)|$ where $x\in C(-\infty,\infty)$, $n=1,2,3\ldots$ is a Fréchet algebra. It is known that ([12], page 56) every multiplicative linear functional on $C(-\infty,\infty)$ is continuous.

Example 1.2. The topological algebra $S=\{(\alpha_1,\alpha_2,\ldots)\,|\,\alpha_i\in C\}$, where C is the complex field, whose ring multiplication is defined by $(\alpha_1,\alpha_2,\ldots)\times(\beta_1,\beta_2,\ldots)=(\alpha_1\beta_1,\alpha_2\beta_2,\ldots)$ and whose topology is defined by the semi-norms $q_n[(\alpha_1,\alpha_2,\ldots)]=\max_{1\le i\le n}|\alpha_i|$ is a Fréchet algebra. The algebra S contains an identity $=(1,1,1,\ldots)$ while the Banach algebra C_0 of sequences converging to zero does not have an identity.

Example 1.3. Every Banach algebra is a Fréchet algebra.

Definition 1.7. If A is a (topological) algebra, then A^+ will denote the algebra which we get by adding a unit to A. A^+ is merely the vector space, direct sum of A and the complex numbers (in the Cartesian product topology), with multiplication defined by "$(x,\lambda)(y,\mu)=(xy+\lambda y+\mu x,\lambda\mu)$" and (0,1) is the identity of A^+. If A is a Fréchet (or Banach) algebra, so is A^+.

Definition 1.8. A topological algebra A is called functionally continuous if every multiplicative linear functional on A is continuous.

Lemma 1.1. If A is a locally m-convex algebra, then so is A^+.

Proof. See pp. 10 of [12].

Lemma 1.2. Let A be a locally m-convex algebra.

(a) If A is an ideal in a locally m-convex algebra B, and if B is functionally continuous, then so is A.

(b) If A^+ is functionally continuous, then so is A.

Proof. See pp. 53 of [12].

Definition 1.9. Let A be a topological vector space, and $\{x_i\}_{i\geq 1}$ is a sequence of A. We write $x=\sum_{i=1}^{\infty}\alpha_i x_i \in A$ if the partial sums $\sum_{i=1}^{n}\alpha_i x_i$ converge to $x(\in A)$, as $n\to\infty$.

Lemma 1.3. Let A be a Fréchet algebra. Then, for arbitrary elements $\sum_{i=1}^{\infty}\alpha_i x_i$, $\sum_{j=1}^{\infty}\beta_j x_j'$, y of A, where $\{x_i\}$, $\{x_j'\}\in A$, we have

(i) $(\sum_{i=1}^{\infty}\alpha_i x_i)y=\sum_{i=1}^{\infty}\alpha_i(x_i y)$ and $y(\sum_{i=1}^{\infty}\alpha_i x_i)=\sum_{i=1}^{\infty}\alpha_i(yx_i)$.

(ii) $(\sum_{i=1}^{\infty}\alpha_i x_i)(\sum_{j=1}^{\infty}\beta_j x_j')=\sum_{i=1}^{\infty}[\sum_{j=1}^{\infty}\alpha_i\beta_j(x_i x_j')]=\sum_{j=1}^{\infty}[\sum_{i=1}^{\infty}\alpha_i\beta_j(x_i x_j')]$.

Proof. (i) Consider the map $\phi: A\times A \to A$ where $\phi(<x,y>)=xy$. We observe that $\phi(<\sum_{i=1}^{n}\alpha_i x_i, y>)=(\sum_{i=1}^{n}\alpha_i x_i)y=\sum_{i=1}^{n}\alpha_i(x_i y)$ and $\phi(<\sum_{i=1}^{\infty}\alpha_i x_i, y>)=(\sum_{i=1}^{\infty}\alpha_i x_i)y$. Since $<\sum_{i=1}^{n}\alpha_i x_i, y>$ converges to $<\sum_{i=1}^{\infty}\alpha_i x_i, y>$, and ϕ is continuous, it follows that $\sum_{i=1}^{n}\alpha_i(x_i y)$ converges to $(\sum_{i=1}^{\infty}\alpha_i x_i)y$. Hence $(\sum_{i=1}^{\infty}\alpha_i x_i)y=\sum_{i=1}^{\infty}\alpha_i(x_i y)$. Similarly, we can prove that $y(\sum_{i=1}^{\infty}\alpha_i x_i)=\sum_{i=1}^{\infty}\alpha_i(yx_i)$.

(ii) $(\sum_{i=1}^{\infty}\alpha_i x_i)(\sum_{j=1}^{\infty}\beta_j x_j')=\sum_{i=1}^{\infty}[\alpha_i x_i(\sum_{j=1}^{\infty}\beta_j x_j')]=\sum_{i=1}^{\infty}[\sum_{j=1}^{\infty}\alpha_i\beta_j(x_i x_j')]$

$(\sum_{i=1}^{\infty}\alpha_i x_i)(\sum_{j=1}^{\infty}\beta_j x_j')=\sum_{j=1}^{\infty}[(\sum_{i=1}^{\infty}\alpha_i x_i)\beta_j x_j']=\sum_{j=1}^{\infty}[(\sum_{i=1}^{\infty}\alpha_i\beta_j(x_i x_j')]$

Definition 1.10. A sequence $\{x_n\}$ in a Fréchet space is called a Schauder basis if each $x\in E$ has a unique representation $x=\sum_{n=1}^{\infty}\alpha_n x_n$, i.e. there exists a unique sequence $\{\alpha_n\}$ of scalars such that the partial sums $\sum_{n=1}^{k}\alpha_n x_n$ converge to x as $k\to\infty$.

Remark. Some other books ([11] and [17]) call such a sequence $\{x_n\}$ a basis of A, and the coefficients $\{\alpha_n\}$ which depend linearly on x the coefficient functionals of the basis $\{x_n\}$. Further, if each element of $\{\alpha_n\}$ is continuous on A, they call such a basis a Schauder basis. By Newn's theorem (page 126 of [11]), every basis for a complete metrizable linear space is a Schauder basis. Therefore, the coefficient functionals $\{\alpha_n\}$ of the basis $\{x_n\}$ for a Fréchet space are all continuous.

Example 1.4. The sequence $\{e_n\}_{n\geq 1}$ ($e_n=(x_1,x_2,\ldots x_n,\ldots)$ for which $x_n=1$ and $x_m=0$ when $m\neq n$) constitute a Schauder basis in each of the spaces l^p $(1\leq p<\infty)$, C_o and S (see example 1.2). If we make the space l^p or C_o into an algebra, the multiplication is the same as that in S.

Definition 1.11. A semi-norm q on a topological algebra A is called submultiplicative if $q(xy)\leq q(x)q(y)$ for all $x,y\in A$.

(3) Some known results on the continuity of multiplicative linear functionals

Definition 1.12. A linear functional on an algebra is said to be multiplicative if $f(xy)=f(x)f(y)$, for $x,y\epsilon A$.

Theorem 1.1. Let A be a complex (or real) Banach algebra. Then every multiplicative linear functional f on A is continuous.

Proof. Let x be any element of A. We claim that $|f(x)|\leq||x||$. If f is zero-functional, then this inequality is obvious. Assume $f(x)\neq 0$, and $|f(x)|>||x||$ or $1>||\frac{x}{f(x)}||$.

Put $Z = \frac{x}{|f(x)|}$, $Y_n = \sum_{i=1}^{n} Z^i$.

First observe that $||Z||<1$ implies that $\sum_{i=1}^{\infty}||Z^i||\leq\sum_{i=1}^{\infty}||Z||^i<\infty$. Therefore, for each positive number $\varepsilon>0$, there exists a positive integer $N\varepsilon>0$ such that $\sum_{i=t+1}^{\infty}||Z||^i<\varepsilon$ whenever $t>N\varepsilon$. This implies that, for each $\varepsilon>0$, $||Y_n-Y_m||=||\sum_{i=m+1}^{n}Z^i||\leq\sum_{i=m+1}^{n}||Z||^i\leq\sum_{i=m+1}^{\infty}||Z||^i<\varepsilon$, whenever $n\geq m>N$.

This implies that $||Y_n-Y_m|| \to 0$ as $n,m \to \infty$. Since A is complete, $Y_n \to Y=\sum_{i=1}^{\infty}Z^i\epsilon A$, $ZY=\sum_{i=2}^{\infty}Z^i=Y-Z$ implies that $f(ZY)=f(Y)-f(Z)$ or $f(Y)=f(Y)-1$, which is impossible. So $|f(x)|\leq||x||$. By proposition 1.1, f is continuous.

In particular, C_o, l^p are functionally continuous. It is also known that the Fréchet algebra S is functionally continuous (see Example 1.2). However, the question, whether every multiplicative linear functional on a commutative F-algebra is continuous, is still unsolved in spite of considerable interest and effort. The following are a number of results due to Michael [12], Husain and Ng [8], [9], in which an affirmative answer is given for commutative Fréchet algebras satisfying certain conditions.

Theorem 1.2. (Michael). Let T be a connected, open subset of the complex plane. Let A be the algebra of functions which are analytic on T, in the compact open topology. Then A is a functionally continuous Fréchet algebra.

Theorem 1.3. (Michael). Let A be a commutative Fréchet algebra which is also a symmetric *-algebra (a symmetric *-algebra A is a *-algebra with the property that xx* is quasi-regular for every $x \in A$). Then A is functionally continuous.

Theorem 1.4. (Husain and Ng [9]). If A is a commutative sequentially complete locally m-convex topological algebra over the complex field that is symmetric with a continuous involution, then every multiplicative linear functional on A is bounded.

Remark. Theorem 1.4 generalizes theorem 1.3. Since every commutative symmetric Fréchet algebra A is a commutative sequentially complete locally m-convex topological algebra, by theorem 1.4, every multiplicative linear functional on A is bounded. But every bounded linear functional on a metrizable, locally convex space is continuous, and hence A is functionally continuous.

Theorem 1.5. (Husain and Ng [8]). Let A be a real complete metrizable topological algebra and B, a real commutative Banach algebra, such that B satisfies condition (C): For any sequence $\{y_n\} \subset B$, $||y_n|| \geq 1$, there exists a sequence $\{f_n\}$ of real multiplicative linear functionals on B such that $\inf_n |f_n(y_n)| = \varepsilon > 0$. Then each linear mapping T of A into B such that $T(x^2) = T^2(x)$, $x \in A$, is continuous.

Corollary 1.5.1. (Husain and Ng [8]). Let A, B be as in theorem 1.5. Then each algebra homomorphism of A into B is continuous.

Corollary 1.5.2. (Husain and Ng [8]). If A is real commutative sequentially complete, bornological locally m-convex algebra and B a real commutative Banach algebra satisfying (C), then each algebra homomorphism T of A into B is continuous.

Corollary 1.5.3. (Husain and Ng [8]). Each real algebra homomorphism on a real commutative Fréchet algebra is continuous.

(4) Nuclear mappings and spaces.

If E is a vector space over K and V is a convex, circled, and radial subset of E, then $\{n^{-1}V : n \in N\}$ is a 0-neighborhood base of a locally convex topology Γ_V on E. The Hausdorff topological vector space associated with (E, Γ_V) is the quotient space $(E, \Gamma_V)/p^{-1}(0)$, where p is the gauge of V; this quotient space is normable by the norm $\hat{x} \to ||\hat{x}|| = p(x)$, where $x \in \hat{x}$. We shall denote by E_V the normed space $(E/p^{-1}(0), || \ ||)$ just introduced, and by $\tilde{E}_V$ its completion, which is a Banach space.

Dually, if E is a locally convex space and $B \neq \emptyset$ a convex, circled, and bounded subset of E, then $E_1 = \bigcup_{n=1}^{\infty} nB$ is a (not necessarily closed) subspace of E. The guage function p_B of B in E_1 is quickly seen to be a norm on E_1; the normed space (E_1, p_B) will henceforth be denoted by E_B.

Definition 1.13. A linear map $u \in L(E,F)$ is nuclear if it is of the form $x \to u(x) = \sum_{n=1}^{\infty} \lambda_n f_n(x) y_n$, where $\sum_{n=1}^{\infty} |\lambda_n| < \infty$, $\{f_n\}$ is an equicontinuous sequence in E', and $\{y_n\}$ is a sequence contained in a convex, circled, and bounded subset B of F for which F_B is complete.

<u>Definition 1.14.</u> A locally convex space E is nuclear if there exists a base H of convex, circled, 0-neighborhoods in E such that for each $V \epsilon H$ the canonical mapping $E \to \tilde{E}_V$ is nuclear.

<u>Definition 1.15.</u> A family $\{x_\alpha : \alpha \epsilon A\}$ is summable to x if for each 0-neighborhood U in E, there exists a finite subset $\phi_U \subset A$ such that for each finite set ϕ, $\phi_U \subset \phi \subset A$ implies that $\sum_{\alpha \epsilon \phi} x_\alpha \epsilon x+U$. This is expressed by writing $x=\Sigma_\alpha x_\alpha$. If A=N and $\{x_n\}$ is summable (a summable sequence), the series $\sum_1^\infty x_n$ is called unconditionally convergent.

<u>Definition 1.16.</u> Suppose E to be locally convex. A family $\{x_\alpha : \alpha \epsilon A\} \subset E$ is absolutely summable if it is summable in E and if for each continuous semi-norm p on E, the family $\{p(x_\alpha) : \alpha \epsilon A\}$ is summable (in R). If A=N and $\{x_n\}_{n \epsilon N}$ is absolutely summable, the series $\sum_1^\infty x_n$ is called absolutely convergent.

The following theorem is very useful to prove theorem 3.4.

<u>Theorem 1.6.</u> A Fréchet space E is nuclear if and only if every summable sequence in E is absolutely summable.

<u>Proof.</u> See corollary 2 of page 184 in [16].

<u>Definition 1.17.</u> An unconditional basis is a Schauder basis in a Fréchet space A such that for each x, $\{f_n(x)x_n : n \epsilon N\}$ is summable to x, where $\{f_n\}_{n \geq 1}$ are linear forms on A such that $x=\sum_{n=1}^\infty f_n(x)x_n$.

(5) Fundamental concepts and propositions in a *-algebra.

<u>Definition 1.18.</u> R is called a *-algebra if

1) R is an algebra.
2) an operation * is defined in R which assigns to each element x in R the element x* in R in such a way that the following conditions are

satisfied:

a) $(\lambda x+\mu y)^*=\bar{\lambda}x^*+\bar{\mu}y^*$

b) $x^{**}=x$

c) $(xy)^*=y^*x^*$.

The operation $x \to x^*$ will be called involution and the element x^* and x will be said to be adjoint to each other.

<u>Definition 1.19.</u> An element x is said to be Hermitian if $x^*=x$.

<u>Proposition 1.8.</u> Every element of the form x^*x is Hermitian. In fact, by virtue of Definition 1.18 (c) and (b), $(x^*x)^*=x^*x^{**}=x^*x$.

<u>Proposition 1.9.</u> The identity e is a Hermitian element. In fact, $e^*=e^*e$ is a Hermitian element: consequently $e^*=e$.

<u>Definition 1.20.</u> The mapping $x \to x^*$ of a *-algebra R into the *-algebra R' is called a *-homomorphism if

α) $x \to x'$ is a homomorphism

β) $x \to x'$ implies that $x^* \to (x')^*$.

<u>Definition 1.21.</u> A linear function f is said to be positive if $f(x^*x)\geq 0$ for an arbitrary element x of the *-algebra R.

<u>Proposition 1.10.</u> Let R be a *-algebra, and f a positive functional, then

$$|f(x^*y)| \leq \sqrt{f(xx^*)}\ \sqrt{f(yy^*)} \quad \text{for } x, y\in A. \tag{0}$$

<u>Proof.</u> Since for any scalar λ, $f((x+\lambda y)^*(x+\lambda y))\geq 0$, we obtain

$$0\leq f((x+\lambda y)^*(x+\lambda y))=f(x^*x)+\lambda f(x^*y)+\bar{\lambda}f(y^*x)+\lambda\bar{\lambda}f(y^*y) \tag{1}$$

Also since f is positive, $f(x^*x)\geq 0$, $f(yy^*)\geq 0$, and therefore $\lambda f(x^*y)+\bar{\lambda}f(y^*y)$ is real, in particular for $\lambda=1$, $f(x^*y)+f(xy^*)$ is real, and then

$$\operatorname{im}f(x^*y)+\operatorname{im}f(y^*x)=0 \quad \text{or} \quad \operatorname{im}f(x^*y)=-\operatorname{im}f(y^*x) \tag{2}$$

Further, in particular for $\lambda=i$, we get

$$i\mathrm{Re}f(x^*y)-i\mathrm{Re}f(xy^*)=0 \quad \text{or} \quad \mathrm{Re}f(x^*y)=\mathrm{Re}f(xy^*) \tag{3}$$

Combining (2) and (3) we have

$$f(x^*y)=\overline{f(y^*x)} \tag{4}$$

From (1) and (4) we obtain that

$$f((x+\lambda y)^*(x+\lambda y))=f(x^*x)+\lambda f(x^*y)+\overline{\lambda f(x^*y)}+\lambda\bar{\lambda}f(y^*y)=f(x^*x)+$$
$$2\mathrm{Re}\lambda f(x^*y)+\lambda\bar{\lambda}f(yy^*) \tag{5}$$

Suppose that $f(yy^*)=0$. We claim $f(x^*y)=0$. If not, choose $\lambda=-\frac{\frac{1}{2}f(x^*x)+1}{f(x^*y)}$.

Then from (5), $f(x^*x)+2(-\frac{1}{2}f(x^*x)-1)\geq 0$, and therefore $-2\geq 0$, which is absurd. Thus, the inequality (0) follows. Suppose that $f(yy^*)\neq 0$. Put $\lambda=-\frac{\overline{f(x^*y)}}{f(y^*y)}$, by using (5) again, we obtain

$$f(x^*x) - 2\frac{|f(x^*y)|^2}{f\ (y^*y)} + \frac{|f(x^*y)|^2}{f^2(y^*y)}\ f(y^*y)\geq 0$$

and hence

$$|f(x^*y)| \leq \sqrt{f(x^*x)}\ \ \sqrt{f(y^*y)}$$

<u>Proposition 1.11.</u> Every positive functional f in a *-algebra with identity is real and

$$|f(x)|^2 \leq f(e)f(x^*x).$$

<u>Proof.</u> Put x=e in (0) of proposition 1.10, since $x^*=e^*=e$, we have

$$f(e)=f(e^*e)\geq 0 \text{ and } |f(y)| \leq \sqrt{f(y^*y)}\ \ \sqrt{f(e)}$$

or

$$|f(y)|^2 \leq f(y^*y)f(e) \quad \text{or} \quad |f(x)|^2 \leq f(x^*x)f(e).$$

Definition 1.22. R is called a normed *-algebra if

a) R is a normed algebra

b) R is a *-algebra

c) $||x^*||=||x||$.

It follows from condition c) that the involution operator is an isometry and hence continuous.

Chapter 2

RELATIONS AMONG COEFFICIENTS, SEMI-NORMS, AND IDENTITIES OF FRÉCHET ALGEBRAS WITH SCHAUDER BASES

Let A be a Fréchet space whose topology is defined by an increasing sequence $\{q_n\}_{n\varepsilon N}$ of semi-norms. Let $H=\{x_i\}_{i\varepsilon N}$ be any sequence of A. Suppose $\{\alpha_i\}$ is a complex sequence such that $\sum_{i=1}^{\infty}\alpha_i x_i \varepsilon A$, the following question naturally arises: Does there exist any relation between $\{\alpha_i\}_{i\varepsilon N}$ and semi-norms $\{q_n\}_{n\varepsilon N}$? We establish a number of results giving some relations. In particular, if A is a Fréchet algebra with a Schauder basis and if A contains an identity e, then we establish a relation among coefficients $\{\alpha_i\}_{i\varepsilon N}$, semi-norms $\{q_n\}$, and the identity e. These results are very useful in proving the main results of Chapter 3.

In what follows, we assume that $H=\{x_i\}_{i\varepsilon N}$ consists of non zero distinct elements, i.e. H is a countable set of non zero elements. We also note that the topology of any Fréchet algebra can be defined by an increasing sequence of submultiplicative semi-norms.

Theorem 2.1. Let A be a Fréchet space whose topology is defined by a sequence $\{q_n\}_{n\varepsilon N}$ of semi-norms. Let $H = \{x_i\}_{i\varepsilon N}$ be any sequence of A. Then, for each complex sequence $\{\alpha_i\}_{i\varepsilon N}$, $\sum_{i=1}^{\infty}\alpha_i x_i \varepsilon A$ if and only if, for each q_n, there is some positive integer N_n depending upon n such that $q_n(x_i)=0$ whenever $i>N_n$.

Proof. ("Only if" part). Suppose for each complex sequence $\{\alpha_i\}$,

$\sum_{i=1}^{\infty}\alpha_i x_i \varepsilon A$. Assume there exists a q_n and a subsequence $\{x_{n_k}\}_{k\geq 1}$ of H which has infinitely many distinct elements such that $q_n(x_{n_k})\neq 0$ for all $k\geq 1$. Put $\alpha_{n_k}=[q_n(x_{n_k})]^{-1}$ then $q_n(\alpha_{n_k}x_{n_k})=1$ for all $k\geq 1$. Setting $\alpha_m=\alpha_{n_k}$ if $m=n_k$ and 0 otherwise, we have $\sum_{i=1}^{\infty}\alpha_i x_i \varepsilon A$ by assumption. Hence $|\sum_{i=1}^{m}\alpha_i x_i-\sum_{i=1}^{\ell}\alpha_i x_i| \to 0$ as $m,\ell \to \infty$. By proposition 1.8, it follows that $q_n(\sum_{i=1}^{m}\alpha_i x_i-\sum_{i=1}^{\ell}\alpha_i x_i) \to 0$ as $\ell,m \to \infty$. Hence there exists N such that $q_n(\sum_{j=1}^{m}\alpha_j x_j-\sum_{j=1}^{k}\alpha_j x_j)<1$ for all $m,k\geq N$. In particular, for any $n_k>N+1$, we have $1=q_n(\alpha_{n_k}x_{n_k})=q_n(\sum_{j=1}^{n_k}\alpha_j x_j-\sum_{j=1}^{n_k-1}\alpha_j x_j)<1$, which is absurd. Hence for each q_n there exists N_n such that $q_n(x_i)=0$ for all $i>N_n$.

("if" part). Suppose that for each semi-norm q_n there is a positive integer N_n such that $q_n(x_i)=0$ for all $i>N_n$. Let $\{\alpha_i\}$ be any complex sequence and put $y_m=\sum_{i=1}^{m}\alpha_i x_i$. We show that $\{y_m\}$ is a Cauchy sequence. Let $\varepsilon>0$ be given. For any semi-norm q_n by assumption there is N_n such that $q_n(x_i)=0$ for all $i>N_n$. But then

$$q_n(y_k-y_j) \leq \sum_{i\geq j+1}^{k}|\alpha_i|q_n(x_i) = 0<\varepsilon,$$

whenever $k>j>N_n$. Since q_n is arbitrary, by proposition 1.8, $|y_j-y_k| \to 0$ as $j,k \to \infty$, and hence $\{y_m\}$ is a Cauchy sequence. Thus, completeness of A implies that $\sum_{i=1}^{\infty}\alpha_i x_i \varepsilon A$.

<u>Theorem 2.2.</u> Let A be a Fréchet algebra whose topology is defined by an increasing sequence $\{q_n\}_{n\varepsilon N}$ of submultiplicative semi-norms. If $H=\{x_i\}_{i\varepsilon N}$ is a Schauder basis of A and, for any $x=\sum_{i=1}^{\infty}\alpha_i x_i \varepsilon A$, $\alpha_i \not\to 0$, then there exists a q_n such that $q_n(x_i)\neq 0$ for all $x_i \varepsilon H$. Conversely, if

$H=\{x_i\}_{i\varepsilon H}$ is any sequence of A such that

(i) for each $x_i \varepsilon H$, $x_i x_i = c_i x_i$ where $\inf|c_i|>0$,

(ii) there exists a q_n such that $q_n(x_i) \neq 0$ for all $x_i \varepsilon H$,

then, for any $x=\sum_{i=1}^{\infty}\alpha_i x_i \varepsilon A$, we have $\alpha_i \to 0$.

Proof. Suppose that, for each $x=\sum_{i=1}^{\infty}\alpha_i x_i \varepsilon A$, we have $\alpha_i \to 0$. We claim that there exists a q_n such that $q_n(x_i) \neq 0$ for all $x_i \varepsilon H$. If not, then, for each q_n, there is some $x_{1n} \varepsilon H$, such that

$$q_n(x_{1n}) = 0 \tag{2.2.1}$$

and hence, for each q_m, $n \geq m$ implies that

$$q_m(x_{1n}) \leq q_n(x_{1n}) = 0. \tag{2.2.2}$$

The subsequence $H_1=\{x_{1n}\}_{n\geq 1}$ of H has infinitely many distinct elements, because if there are only finitely many distinct elements in the subsequence $H_1=\{x_{1n}\}$ of H, then, since A is Hausdorff, there exists a semi-norm q_m, such that

$$q_m(x_{1n}) \neq 0 \text{ for all } x_{1n} \varepsilon H_1.$$

In particular, $q_m(x_{1m}) \neq 0$ which is contrary to our assumption that

$$q_m(x_{1m}) = 0.$$

Since (2.2.2) satisfies the "only if" part of theorem 2.1, we obtain $\sum_{n=1}^{\infty}\beta_n x_{1n} \varepsilon A$ for any complex sequence $\{\beta_n\}$. In particular, we have $\sum_{n=1}^{\infty} x_{1n} \varepsilon A$. Since $\{x_i\}$ is a Schauder basis of A, there is a unique sequence $\{\alpha_i\}$ of complex numbers such that $\sum_{i=1}^{\infty}\alpha_i x_i = \sum_{n=1}^{\infty} x_{1n}$. Clearly $\alpha_i = 1$ if $x_i \in H_1$ and $= 0$ if $x_i \in H \sim H_1$. Since H_1 has infinitely many distinct elements, this shows that $\alpha_i \not\to 0$, a contradiction. Conversely, suppose

(i) for each x_i, $x_i x_i = c_i x_i$ where $\inf|c_i| > 0$,

(ii) there exists a semi-norm q_n such that $q_n(x_i) \neq 0$ for all $x_i \in H$.

Then we claim that $x = \sum_{i=1}^{\infty}\alpha_i x_i \in A$ implies that $\alpha_i \to 0$.

Since $\sum_{i=1}^{n}\alpha_i x_i \to \sum_{i=1}^{\infty}\alpha_i x_i \in A$ implies that $|\sum_{i=1}^{j}\alpha_i x_i - \sum_{i=1}^{k}\alpha_i x_i| \to 0$ as $j,k \to \infty$, this also implies that $q_n(\sum_{i=1}^{j}\alpha_i x_i - \sum_{i=1}^{k}\alpha_i x_i) \to 0$ as $j,k \to \infty$. It means that, for each $\varepsilon > 0$, there is some positive integer N_ε such that $q_n(\alpha_j x_j) = q_n(\sum_{i=1}^{j}\alpha_i x_i - \sum_{i=1}^{j-1}\alpha_i x_i) < \varepsilon$ whenever $j-1 > N_\varepsilon$, or $|\alpha_j| q_n(x_j) < \varepsilon$ for $j-1 > N_\varepsilon$

or $$|\alpha_j| < \frac{\varepsilon}{q_n(x_j)} \quad \text{for } j-1 > N_\varepsilon. \tag{2.2.3}$$

Since $x_j^2 = c_j x_j$ for all $j \geq 1$, $|c_j| q_n(x_j) = q_n(x_j^2) \leq q_n^2(x_j)$ for all $j \geq 1$.

It follows that $q_n(x_j)[q_n(x_j) - |c_j|] \geq 0$ for all $j \geq 1$. But $q_n(x_j) \neq 0$ (i.e. $q_n(x_j) > 0$) for all $j \geq 1$ implies $q_n(x_j) \geq |c_j|$ for all $j \geq 1$. Combining this with (2.2.3), we have

$$|\alpha_j| < \frac{\varepsilon}{q_n(x_j)} \leq \frac{\varepsilon}{|c_j|} \leq \frac{\varepsilon}{\inf|c_j|} \quad \text{for } j-1 > N_\varepsilon$$

This proves that $\alpha_i \to 0$.

<u>Corollary 2.2.1.</u> Let A be a Fréchet algebra whose topology is defined by an increasing sequence $\{q_n\}_{n \in N}$ of submultiplicative semi-norms. If A possesses a Schauder basis $H = \{x_i\}_{i \geq 1}$ such that, for each $x_i \in H$, $x_i x_i = c_i x_i$ where $\inf|c_i| > 0$, then, for any $x = \sum_{i=1}^{\infty}\alpha_i x_i \in A$, $\alpha_i \to 0$ if and only if there

exists a q_n such that $q_n(x_i)\neq 0$ for all $x_i \varepsilon H$.

Proof. This is a particular case of Theorem 2.2.

In the case of Fréchet space, we have the following:

Corollary 2.2.2. Let A be a Fréchet space whose topology is defined by an increasing sequence $\{q_n\}_{n\varepsilon N}$ of semi-norms. Let $H=\{x_i\}_{i\varepsilon N}$ be a Schauder basis of A. If, for any $x=\sum_{i=1}^{\infty}\alpha_i x_i \varepsilon A$, $\alpha_i \to 0$, then there exists a q_n such that $q_n(x_i)\neq 0$ for all $x_i \varepsilon H$.

Conversely, if there exists a q_n such that $\inf_{i\varepsilon N} q_n(x_i)>0$ then, for any $x=\sum_{i=1}^{\infty}\alpha_i x_i \varepsilon A$, $\alpha_i \to 0$.

Proof. This follows from the proof of Theorem 2.2.

Theorem 2.3. Let A be a Fréchet algebra whose topology is defined by an increasing sequence $\{q_n\}_{n\varepsilon N}$ of submultiplicative semi-norms. Let $H=\{x_i\}$ be a Schauder basis of A such that

$$x_i x_j = 0 \text{ whenever } i\neq j.$$

Then A contains an identity if and only if $x_i x_i \neq 0$ for each $x_i \varepsilon H$ and for any complex sequence $\{\beta_i\}$, $\sum_{i=1}^{\infty}\beta_i x_i \varepsilon A$.

Proof. "Only if" part: Suppose A has an identity; since $\{x_i\}_{i\varepsilon N}$ is a Schauder basis of A, let $e=\sum_{i=1}^{\infty}\alpha_i x_i$ be the identity of A. Then, for each $x_i \varepsilon H$, $ex_i=x_i$ implies $\alpha_i x_i x_i = x_i$. Also, $x_i\neq 0$ implies $x_i x_i \neq 0$ and $\alpha_i \neq 0$. Therefore

$$x_i x_i = \frac{1}{\alpha_i} x_i \text{ for each } i\geq 1. \tag{2.3.1}$$

Since $\sum_{i=1}^{n}\alpha_i x_i \to \sum_{i=1}^{\infty}\alpha_i x_i$ as $n\to\infty$, it follows that, for each q_m, there is a positive integer N_m such that

$$q_m(\alpha_n x_n)<1 \qquad \text{whenever } n>N_m,$$

or

$$q_m(x_n)<\frac{1}{|\alpha_n|} \qquad \text{whenever } n>N_m,$$

or

$$q_m(x_n)-\frac{1}{|\alpha_n|} < 0 \qquad \text{whenever } n>N_m. \tag{2.3.2}$$

On the other hand, from (2.3.1), we have

$$q_m(x_n x_n) = \frac{1}{|\alpha_n|}\, q_m(x_n),$$

or

$$q_m^2(x_n) \geq \frac{1}{|\alpha_n|}\, q_m(x_n),$$

or

$$q_m(x_n)[q_m(x_n) - \frac{1}{|\alpha_n|}] \geq 0 \qquad \text{for all } n\geq 1.$$

Since, by (2.3.2), for each $n>N_m$, $q_m(x_n) - \frac{1}{|\alpha_n|} < 0$, it follows that, for each $n>N_m$, $q_m(x_n)=0$. By Theorem 2.1, for any complex sequence $\{\beta_i\}_{i\varepsilon N}$, we have $\sum_{i=1}^{\infty}\beta_i x_i \varepsilon A$.

"If" part: Suppose for each complex sequence $\{\alpha_i\}$, $\sum_{i=1}^{\infty}\alpha_i x_i \varepsilon A$. Let $x_i x_i = \sum_{j=1}^{\infty}\alpha_j x_j$. Then, for $k\neq i$, we have

$$0 = x_k x_i x_i = x_k(\sum_{j=1}^{\infty}\alpha_j x_j) = x_k\alpha_k x_k = \alpha_k x_k x_k.$$

But then $x_k x_k\neq 0$ implies $\alpha_k=0$ for all $k\neq i$. Thus $\alpha_k=0$ for all $k\neq i$. So $x_i x_i=\alpha_i x_i$ and $x_i x_i\neq 0$ implies $\alpha_i\neq 0$. Therefore, by assumption, we have $\sum_{i=1}^{\infty} \frac{1}{\alpha_i}\, x_i \varepsilon A$.

Put $\qquad e = \sum_{i=1}^{\infty} \frac{1}{\alpha_i}\, x_i.$

Now, for each $y=\sum_{j=1}^{\infty}\beta_j x_j \varepsilon A$, using the fact that $x_i x_j=0$ for $i\neq j$, and $\frac{1}{\alpha_i} x_i x_i = x_i$, we have

$$ye = (\sum_{j=1}^{\infty}\beta_j x_j)(\sum_{j=1}^{\infty} \frac{1}{\alpha_j} x_j) = \sum_{j=1}^{\infty} \beta_j \frac{1}{\alpha_j} x_j x_j = \sum_{j=1}^{\infty}\beta_j x_j = y,$$

$$ey = (\sum_{j=1}^{\infty} \frac{1}{\alpha_j} x_j)(\sum_{j=1}^{\infty}\beta_j x_j) = \sum_{j=1}^{\infty} \beta_j \frac{1}{\alpha_j} x_j x_j = \sum_{j=1}^{\infty}\beta_j x_j = y.$$

Hence e is an identity of A.

Theorem 2.4. Let A be a Fréchet algebra whose topology is defined by an increasing sequence $\{q_n\}_{n\varepsilon N}$ of submultiplicative semi-norms. Let $H=\{x_i\}$ be a Schauder basis of A such that

$x_i x_j=0$ whenever $i\neq j$,

$x_i x_i\neq 0$ for all $x_i \varepsilon H$.

Then the following conditions are equivalent:

(i) For each complex sequence $\{\alpha_i\}$, $\sum_{i=1}^{\infty}\alpha_i x_i \varepsilon A$.

(ii) For each q_n, there is a positive integer N_n such that $q_n(x_i)=0$ whenever $i>N_n$.

(iii) A contains an identity.

Proof. (i) <=> (ii) is clear from Theorem 2.1.

(i) <=> (iii) is clear from Theorem 2.3.

and hence (ii) <=> (iii) follows.

Chapter 3

THE CONTINUITY OF MULTIPLICATIVE LINEAR FUNCTIONALS

In 1952, E. A. Michael [12] asked whether every multiplicative linear functional on a commutative Fréchet algebra is continuous, or equivalently, whether every multiplicative linear functional on a commutative complete locally m-convex algebra is bounded, [4]. So far this problem is still open. There are a number of results due to Michael [12], Husain and Ng [8] and [9], in which an affirmative answer is given for commutative Fréchet algebras satisfying certain conditions. In this chapter, we show that the answer to Michael's question is also affirmative for some other classes of Fréchet algebras.

Theorem 3.1. Let A be a Fréchet algebra whose topology is defined by an increasing sequence $\{q_n\}_{n\in N}$ of submultiplicative semi-norms. If there exists a Schauder basis $H=\{x_i\}_{i\in N}$ such that $x_i x_j=0$ whenever $i\neq j$, then every non-zero multiplicative linear functional f on A is continuous if and only if there exists $x_i \in H$ such that $f(x_i)\neq 0$.

Proof "If" part. Let $x=\sum_{j=1}^{\infty}\alpha_j x_j$ be an arbitrary element of A. Clearly, by hypothesis, $x x_i=\alpha_i x_i^2$.

Since f is multiplicative and $f(x_i)\neq 0$, we have

$$f(xx_i)=f(x)f(x_i)=\alpha_i f^2(x_i) \quad\text{or}\quad f(x)=\alpha_i f(x_i). \tag{3.1.1}$$

Also $f(x_i)\neq 0$ implies $f(x_i x_i)=f(x_i)f(x_i)\neq 0$. Hence $x_i x_i\neq 0$. Since $x_i\neq 0$, $x_i x_i\neq 0$ and A is Hausdorff, there exists a semi-norm q_n such that $q_n(x_i)\neq 0$ and $q_n(x_i x_i)\neq 0$.

Put $\quad M = |f(x_i)| \dfrac{q_n(x_i)}{q_n(x_i x_i)}$ (3.1.2)

On the other hand,

$$xx_i=\alpha_i x_i x_i \text{ implies } |\alpha_i| q_n(x_i x_i) \le q_n(x) q_n(x_i).$$

But $q_n(x_i) \neq 0$ implies

$$|\alpha_i| \frac{q_n(x_i x_i)}{q_n(x_i)} \le q_n(x). \tag{3.1.3}$$

Thus from (3.1.1), (3.1.2), and (3.1.3), we have:

$$Mq_n(x) \ge |f(x_i)| \frac{q_n(x_i)}{q_n(x_i x_i)} |\alpha_i| \frac{q_n(x_i x_i)}{q_n(x_i)} = |f(x_i)||\alpha_i| = |f(x)|$$

for all $x \varepsilon A$. By proposition 1.1, we observe that f is continuous.

"Only if" part.

Let f be a non-zero continuous multiplicative linear functional on A. If $f(x_i)=0$ for all $x_i \varepsilon H$, since f is continuous, $f(x)=0$ for all $x=\sum_{i=1}^{\infty} \alpha_i x_i \varepsilon A$ and hence f is identically zero which is a contradiction. Thus, there exists $x_i \varepsilon H$ such that $f(x_i) \neq 0$.

Theorem 3.2. Let A be a Fréchet algebra whose topology is defined by an increasing sequence $\{q_n\}_{n \varepsilon N}$ of submultiplicative semi-norms. If there exists a Schauder basis $H=\{x_i\}_{i \varepsilon N}$ such that $x_i x_j = 0$ whenever $i \neq j$, and, for any complex sequence $\{\alpha_i\}$, $\sum_{i=1}^{\infty} \alpha_i x_i \varepsilon A$, then every multiplicative linear functional f on A is continuous.

We need the following:

Lemma. Let A be a Fréchet algebra whose topology is defined by an increasing sequence $\{q_n\}_{n \varepsilon N}$ of submultiplicative semi-norms. If f is a multiplicative linear functional on A and if there exists a countable set $H=\{x_i\}_{i \varepsilon N}$ such that

(i) $\quad f(x_i)=0$ for all $x_i \varepsilon H$.

(ii) $\quad$ For each complex sequence $\{\alpha_i\}_{i \varepsilon N}$, $\sum_{i=1}^{\infty} \alpha_i x_i \varepsilon A$.

(iii) $x_i x_j = 0$ whenever $i \neq j$,

then, for any complex sequence $\{\alpha_i\}_{i\in N}$, $f(\sum_{i=1}^{\infty}\alpha_i x_i)=0$.

<u>Proof.</u> First we claim that $f(\sum_{i=1}^{\infty}x_i)=0$. Suppose $f(\sum_{i=1}^{\infty}x_i)\neq 0$. Then by hypothesis (ii), we have

$$f(\sum_{i=1}^{\infty} i\, x_i)f(\sum_{i=1}^{\infty} \frac{1}{i}\, x_i) = f(\sum_{i=1}^{\infty}x_i x_i) = [f(\sum_{i=1}^{\infty}x_i)]^2 \neq 0$$

This implies that $f(\sum_{i=1}^{\infty} i x_i)\neq 0$.

Put $f(\sum_{i=1}^{\infty}x_i)= \alpha$ and $f(\sum_{i=1}^{\infty} i\, x_i)= \beta\alpha$

Then $f(\sum_{i=1}^{\infty}(\beta-i)x_i) = f(\beta\sum_{i=1}^{\infty}x_i - \sum_{i=1}^{\infty} i\, x_i) = \beta f(\sum_{i=1}^{\infty}x_i)-f(\sum_{i=1}^{\infty} i\, x_i) = \beta\alpha-\beta\alpha=0.$

There are two cases.

Case (i), β is not a postive integer. Then

$$0 = f(\sum_{i=1}^{\infty}(\beta-i)x_i)f(\sum_{i=1}^{\infty} \frac{1}{\beta-i}\, x_i) = f(\sum_{i=1}^{\infty}x_i x_i) = [f(\sum_{i=1}^{\infty}x_i)]^2 \neq 0$$

which is a contradiction.

Case (ii), β is a positive integer, say $\beta=n$.

Since $f(x_i)=0$ for all i,

$$0 = f(\sum_{i=1}^{\infty}(\beta-i)x_i)f(\sum_{\substack{i=1\\ i\neq n}}^{\infty} \frac{1}{\beta-i}\, x_i) = f(\sum_{\substack{i=1\\ i\neq n}}^{\infty}x_i x_i) = [f(\sum_{\substack{i=1\\ i\neq n}}^{\infty}x_i)]^2 =$$

$$[f(\sum_{i=1}^{\infty}x_i)]^2 \neq 0$$

which is impossible.

Thus, $f(\sum_{i=1}^{\infty}x_i)=0$.

Let $\{\alpha_i\}_{i\in N}$ be any complex sequence. We observe that

$$f(\sum_{i=1}^{\infty}\alpha_i x_i)f(\sum_{i=1}^{\infty}\alpha_i x_i) = f(\sum_{i=1}^{\infty}\alpha_i{}^2 x_i x_i) = f(\sum_{i=1}^{\infty}\alpha_i{}^2 x_i)f(\sum_{i=1}^{\infty}x_i).$$

Since $f(\sum_{i=1}^{\infty} x_i)=0$, it follows that $f(\sum_{i=1}^{\infty}\alpha_i x_i)=0$. Thus $f(\sum_{i=1}^{\infty}\alpha_i x_i)=0$ for any complex sequence $\{\alpha_i\}_{i\epsilon N}$.

Proof of Theorem 3.2. (a) If there exists a $x_i \epsilon H$ such that $f(x_i)\neq 0$, then, by Theorem 3.1, f is continuous.

(b) If $f(x_i)=0$ for all $x_i \epsilon H$, then, by the above Lemma, $f(\sum_{i=1}^{\infty}\alpha_i x_i)=0$ for any complex sequence $\{\alpha_i\}_{i\epsilon N}$. Since H is a Schauder basis, for each element $x \epsilon A$, there exists a complex sequence $\{\alpha_i\}_{i\epsilon N}$ such that $x=\sum_{i=1}^{\infty}\alpha_i x_i$. Hence $f(x)=0$ for all $x \epsilon A$. This implies that f is identically zero, and hence continuous.

Theorem 3.3. Let A be a Fréchet algebra with identity whose topology is defined by an increasing sequence $\{q_n\}_{n\epsilon N}$ of submultiplicative semi-norms. If there exists a Schauder basis $H=\{x_i\}_{i\epsilon N}$ such that $x_i x_j=0$ whenever $i\neq j$, then each multiplicative linear functional on A is continuous.

Proof. By Theorem 2.3, the fact that A contains an identity implies that, for any complex sequence $\{\alpha_i\}$, we have $\sum_{i=1}^{\infty}\alpha_i x_i \epsilon A$. By Theorem 3.2, the result follows.

Corollary 3.3.1. Let A be a Fréchet algebra. If A^+ possesses a Schauder basis $H=\{x_i\}_{i\geq 1}$ such that $x_i x_j=0$ whenever $i\neq j$, then A is functionally continuous.

Proof. It is clear that A^+ is complete and metrizable. By Lemma 1.1 (see page 7), A^+ is also a locally m-convex algebra. Thus, A^+ is a Fréchet algebra. Using Lemma 1.2 and Theorem 3.3, A is functionally continuous.

Corollary 3.3.2. Let A be a Fréchet algebra. If A^+ possesses a Schauder basis $H=\{x_i\}_{i\geq 1}$ and A^+ does not contain all $\sum_{i=1}^{\infty}\alpha_i x_i$ where $\{\alpha_i\}_{i\epsilon N}$ is any complex sequence, then there exists x_i, x_j, $i \neq j$, such that $x_i x_j \neq 0$.

Proof. If $x_i x_j = 0$ whenever $i \neq j$, by Theorem 2.3, A^+ contains all $\sum_{i=1}^{\infty} \alpha_i x_i$ where $\{\alpha_i\}_{i \in N}$ is any complex sequence, which is a contradiction. Hence there exist x_i, x_j, $i \neq j$, such that $x_i x_j \neq 0$.

Remark. Note that each Fréchet algebra mentioned in Theorem 3.2 or Theorem 3.3 consists of all $\sum_{i=1}^{\infty} \alpha_i x_i$ where $\{\alpha_i\}_{i \in N}$ is any complex sequence. (We call it a whole-sequence algebra.) However, $x_i x_i$ can be defined as any element of A in different ways and therefore, two whole-sequence algebras need not be isomorphic.

On the other hand, there are Fréchet algebras which need not be the whole-sequence algebra, yet they are functionally continuous as follows from the previous theorems in this chapter and chapter 2.

Theorem 3.4. Let A be a Fréchet algebra whose topology is defined by an increasing sequence $\{q_n\}_{n \in N}$ of submultiplicative semi-norms. If there exists a Schauder basis $H = \{x_i\}$ such that

(i) $x_i x_j = 0$ whenever $j \neq i$,

and $x_i x_i = c_i x_i$ where $0 < a \leq |c_i| \leq b$, $i = 1, 2, 3, \ldots$,

(ii) $\sum_{i=1}^{\infty} \alpha_i a_i x_i \in A$ whenever $\sum_{i=1}^{\infty} \alpha_i x_i \in A$ and $|a_i| \leq 1$,

then every multiplicative linear functional on A is continuous.

Proof. Let f be a multiplicative linear functional on A. We consider two cases:

Case (A). If there exists a $x_i \in H$ such that $f(x_i) \neq 0$, then by Theorem 3.1, the result follows.

Case (B). If $f(x_i) = 0$ for all $x_i \in H$, we claim that f is identically zero. Let $B_n = \{x \in H \mid q_n(x) = 0\}$, $n = 1, 2, 3, \cdots$.

Since $\{q_n\}_{n\in N}$ is an increasing sequence of semi-norms, we have

$$B_n \supseteq B_{n+1} \qquad \text{for all } n\geq 1.$$

Let $H_1=\{x\in H|q_1(x)\neq 0\}$ and $H_n=B_{n-1}\sim B_n$, $n\geq 2$.

Clearly, $H_n\cap B_n=\emptyset$ and hence we have

$$q_n(x)\neq 0 \qquad \text{for all } x\in H_n. \tag{3.4.1}$$

Clearly, for $m>n$, $H_m \subseteq B_{m-1} \subseteq B_n$,

and so, we have

$$q_n(x)=0 \qquad \text{for all } x\in H_m,\ m>n. \tag{3.4.2}$$

We show that $\bigcap_{i=1}^{\infty} B_i=\emptyset$.

If $x\in \bigcap_{i=1}^{\infty} B_i$, then $q_n(x)=0$ for all n. Since A is a Hausdorff topological vector space, this implies that $x=0$. Since $0\notin H$, it proves that $\bigcap_{i=1}^{\infty} B_i=\emptyset$.

It is clear by definition that $\{H_i\}_{i\geq 1}$ are pairwise disjoint and

$$\bigcup_{i=1}^{\infty} H_i = H_1\cup \bigcup_{i=2}^{\infty} H_i = H_1\cup[\bigcup_{i=2}^{\infty}(B_{i-1}\sim B_i)] = H_1 \cup (B_1\sim \bigcap_{i=1}^{\infty} B_i) = H_1\cup B_1 = H.$$

Observe that $H_j\subset H$ for $j\geq 1$. Set $H_j=\{x_{jm}\}_{m\geq 1}$. For each j, we can arrange $\{x_{jm}\}_{m\geq 1}$ such that $jm\geq jn$ if and only if $m>n$. We claim that, for each $x=\sum_{i=1}^{\infty}\alpha_i x_i\in A$, we can write:

$$x = \sum_{i=1}^{\infty}\alpha_i x_i = \sum_{j\geq 1} \sum_{m\geq 1}\alpha_{jm}x_{jm}.$$

Indeed, for each positive integer k, we put $a_i = \begin{cases} 1 & \text{if } x_i\in H_k, \\ 0 & \text{if } x_i\notin H_k. \end{cases}$

Clearly, $|a_i|\leq 1$. Hence by hypothesis (ii), we have

$$\sum_{i=1}^{\infty} a_i\alpha_i x_i = \sum_{x_i \varepsilon H_k} \alpha_i x_i \varepsilon A.$$

Since $k_m > k_n$ if and only if $m > n$, $\sum_{x_i \varepsilon H_k} \alpha_i x_i = \sum_{m\geq 1} \alpha_{km} x_{km} \varepsilon A$.

Set $y_k = \sum_{m\geq 1}^{\infty} \alpha_{km} x_{km}$ $\quad k=1,2,3,\ldots,$

From (3.4.2), we observe that, for $k>j$, $q_j(x_{km})=0$ for all $x_{km} \varepsilon H_k$.

Therefore, $\sum_{m\geq 1}^{p_k} \alpha_k x_{km} \to \sum_{m=1}^{\infty} \alpha_{km} x_{km} = y_k$ as $p_k \to \infty$ implies that $0=q_j(\sum_{m=1}^{p_k} \alpha_{km} x_{km}) \to$

$q_j(y_k)=0$ as $p_k \to \infty$ whenever $k>j$. Thus by Theorem 2.1, it follows that, for any complex sequence $\{\beta_k\}$,

$$\sum_{k=1}^{\infty} \beta_k y_k \varepsilon A. \tag{3.4.3}$$

In particular, $\sum_{k=1}^{\infty} y_k \varepsilon A$, or $\sum_{k\geq 1} \sum_{m\geq 1} \alpha_{km} x_{km} \varepsilon A$.

To show that $\sum_{i=1}^{\infty} \alpha_i x_i = \sum_{k\geq 1} \sum_{m\geq 1} \alpha_{km} x_{km}$, we observe that

$\sum_{m=1}^{p_k} \alpha_{km} x_{km} \to \sum_{m=1}^{\infty} \alpha_{km} x_{km} = y_k$ implies that, for each q_n,

$$q_n\left(\sum_{m=1}^{\infty} \alpha_{km} x_{km} - \sum_{m=1}^{p_k} \alpha_{km} x_{km}\right) \to 0 \quad \text{as } p_k \to \infty.$$

By virtue of (3.4.2), for each q_n, we have

$$q_n\left(\sum_{k\geq 1} \sum_{m\geq 1} \alpha_{km} x_{km} - \sum_{i=1}^{p} \alpha_i x_i\right) = q_n\left(\sum_{k\geq 1}\left(\sum_{m\geq 1} \alpha_{km} x_{km} - \sum_{m=1}^{p_k} \alpha_{km} x_{km}\right)\right)$$

$$\leq q_n\left[\sum_{k=1}^{n}\left(\sum_{m\geq 1} \alpha_{km} x_{km} - \sum_{m=1}^{p_k} \alpha_{km} x_{km}\right)\right] + q_n\left(\sum_{k\geq n+1}\left(\sum_{m\geq 1} \alpha_{km} x_{km} - \sum_{m=1}^{p_k} \alpha_{km} x_{km}\right)\right)$$

$$\leq \sum_{k=1}^{n} q_n\left(\sum_{m\geq 1} \alpha_{km} x_{km} - \sum_{m=1}^{p_k} \alpha_{km} x_{km}\right) \quad (\text{because } q_n(x_{km})=0 \text{ for } k>n)$$

where p is split up into p_k's in the obvious way.

Since $p \to \infty$ implies that $p_k \to \infty$ for each k, and it also implies that

$$q_n\left(\sum_{m\geq 1}\alpha_{km}x_{km}-\sum_{m=1}^{p_k}\alpha_{km}x_{km}\right)\to 0,$$

we have

$$q_n\left(\sum_{k\geq 1}\sum_{m\geq 1}\alpha_{km}x_{km}-\sum_{i=1}^{p}\alpha_i x_i\right)\leq\sum_{k=1}^{n}q_n\left(\sum_{m\geq 1}\alpha_{km}x_{km}-\sum_{m=1}^{p_k}\alpha_{km}x_{km}\right)\to 0 \text{ as } p\to\infty.$$

Since q_n is arbitrary, it follows that

$$\left|\sum_{k\geq 1}\sum_{m\geq 1}\alpha_{km}x_{km}-\sum_{i=1}^{p}\alpha_i x_i\right|\to 0 \quad\text{as } p\to\infty.$$

This implies that $\sum_{i=1}^{p}\alpha_i x_i \to \sum_{k\geq 1}\sum_{m\geq 1}\alpha_{km}x_{km}$.

But on the other hand, $\sum_{i=1}^{p}\alpha_i x_i \to \sum_{i=1}^{\infty}\alpha_i x_i$, and hence, for each $x\varepsilon A$, we can write $x=\sum_{i=1}^{\infty}\alpha_i x_i=\sum_{k\geq 1}\sum_{m\geq 1}\alpha_{km}x_{km}=\sum_{k=1}^{\infty}y_k$.

We wish to show that $f(y_i)=0$ for all $i\geq 1$.

Observe that, for each $y_i=\sum_{m\geq 1}\alpha_{im}x_{im}\varepsilon A$, since $q_i(x_{im})\neq 0$ for all $x_{im}\varepsilon H_i$, and $\inf_{m\geq 1}|c_{im}|\geq a>0$, by Theorem 2.2, $\alpha_{im}\to 0$ as $m\to\infty$.

Now suppose $f(y_i)=\alpha\neq 0$ for some i.

Put $Z_i=\frac{y_i}{\alpha}=\sum_{m\geq 1}\beta_{im}x_{im}$. Then $f(Z_i)=\frac{f(y_i)}{\alpha}=1$ and $f(Z_i^2)=1=f(Z_i)$.

Hence $f(Z_i-Z_i^2)=f[\sum_{m=1}^{\infty}(\beta_{im}-c_{im}\beta_{im}^2)x_{im}]=0$.

Since $|c_{im}|\leq b$ and $\beta_{im}\to 0$ as $m\to\infty$, it follows that

$|\beta_{im}|<\frac{1}{2b}<\frac{1}{2|c_{im}|}$ for sufficiently large m, and hence, if $\beta_{im}\neq 0$,

$$\left|\frac{c_{im}\beta_{im}^2}{\beta_{im}-c_{im}\beta_{im}^2}\right|=\left|\frac{c_{im}\beta_{im}}{1-c_{im}\beta_{im}}\right|\leq\frac{|c_{im}|\frac{1}{2|c_{im}|}}{1-|c_{im}|\frac{1}{2|c_{im}|}}=1$$

for sufficiently large m.

(Note that there exists a $\beta_{im}\neq 0$ because of our assumption $f(y_i)\neq 0$.)

It follows by hypothesis (ii) that

$$\sum_{\substack{m=1\\ \beta_{im}\neq 0}}^{\infty} \frac{c_{im}\beta_{im}^3}{\beta_{im}-c_{im}\beta_{im}^2}\, x_{im}\in A.$$

Thus we have

$$0 = f\Big(\sum_{m=1}^{\infty}(\beta_{im}-c_{im}\beta_{im}^2)x_{im}\Big)\, f\Big(\sum_{\substack{m=1\\ \beta_{im}\neq 0}}^{\infty}\frac{c_{im}\beta_{im}^3}{\beta_{im}-c_{im}\beta_{im}^2}\, x_{im}\Big)$$

$$= f\Big(\sum_{\substack{m=1\\ \beta_{im}\neq 0}}^{\infty} c_{im}\beta_{im}^3 x_{im}x_{im}\Big) = f\Big(\sum_{\substack{m=1\\ \beta_{im}\neq 0}}^{\infty} c_{im}^2\beta_{im}^3 x_{im}\Big)$$

$$= f\Big[\Big(\sum_{\substack{m=1\\ \beta_{im}\neq 0}}^{\infty} \beta_{im}x_{im}\Big)^3\Big] = \Big[f\Big(\sum_{m=1}^{\infty}\beta_{im}x_{im}\Big)\Big]^3 = 1$$

which is impossible. Therefore, $f(y_i)=0$ for all y_i, $i=1,2,3,\ldots$.

Now we observe that $y_e y_n=0$ for $e\neq n$. Without any loss of generality, we assume $n>e$. By definition, $H_n\subseteq B_{n-1}\subseteq B_e$ implies $H_e\cap H_n\subseteq H_e\cap B_e=\emptyset$. Since $x_i x_j=0$ for $i\neq j$, we have $y_e y_n=(\sum_{m\geq 1}\alpha_{em}x_{em})(\sum_{m\geq 1}\ {}_{nm}x_{nm})=0$ whenever $e>n$.

Thus, we have shown that the sequence $\{y_k\}_{k\in N}$ satisfies the hypothesis of lemma of Theorem 3.2 and so $f(\sum_{k=1}^{\infty}y_k)=0$. Since $x=\sum_{k=1}^{\infty}y_k$ is any element of A, f is identically zero. This completes the proof.

Remark. If for each Fréchet algebra A in Theorem 3.4, A^+ possesses a Schauder basis $H=\{x_i'\}_{i\geq 1}$ such that $x_i'x_j'=0$ whenever $i\neq j$, then by Lemma 1.2, the truth of Theorem 3.3 implies the truth of Theorem 3.4. However, the problem is whether A^+ always possesses such a Schauder basis. We note that if A possesses a Schauder basis $H=\{x_i\}_{i\geq 1}$ such that $x_i x_j=0$ whenever $i\neq j$, the Schauder basis $H'=\{x_i'\}_{i\geq 0} = (0,1)\cup\{(x_i,0)\}_{i\geq 1}$ of A^+ does not satisfy the condition: $x_i' x_j'=0$ whenever $i\neq j$, because $(x_i,0)(0,1)=(x_i,0)$. Therefore, Theorem 3.3 and Theorem 3.4 must be proved separately.

The Fréchet algebra A in Theorem 3.6 is a whole-sequence algebra. However, the condition "$x_i x_j=0$ whenever $i\neq j$" is replaced by another condition "$x_i x_j=x_j$ whenever $j\geq i$". We will show that such a Fréchet algebra is also functionally continuous.

Theorem 3.5. Let A be a nuclear Fréchet algebra whose topology is defined by an increasing sequence $\{q_n\}$ of submultiplicative semi-norms. If there exists an unconditional basis $H=\{x_i\}$ (see Definition 1.17) such that

$x_i x_j=0$ whenever $j\neq i$,

$x_i x_i=c_i x_i$ where $0<a\leq|c_i|\leq b$, $i=1,2,3,\cdots$,

then each multiplicative linear functional on A is continuous.

Proof. Assuming that $\sum_{i=1}^{\infty}\alpha_i x_i\in A$ and $|a_i|\leq 1$, we claim that $\sum_{i=1}^{\infty}\alpha_i a_i x_i\in A$. Since A is nuclear, by Theorem 1.6, every summable sequence in A is absolutely summable. Therefore, by the definition 1.16 of absolute summability, $\sum_{i=1}^{m}\alpha_i x_i \to \sum_{i=1}^{\infty}\alpha_i x_i$ implies that, for each q_n, $\sum_{i=1}^{m} q_n(\alpha_i x_i)\to$ $\sum_{i=1}^{\infty} q_n(\alpha_i x_i)\in R$ (real line) as $m\to\infty$;

i.e. $\sum_{i=1}^{\infty} q_n(\alpha_i x_i)<\infty$. This implies that

$$\sum_{i=1}^{\infty} q_n(\alpha_i a_i x_i) = \sum_{i=1}^{\infty}|a_i|q_n(\alpha_i x_i) \leq \sum_{i=1}^{\infty} q_n(\alpha_i x_i)<\infty.$$

Therefore, for each q_n, $q_n(\sum_{i=1}^{m}\alpha_i a_i x_i-\sum_{i=1}^{k}\alpha_i a_i x_i)\to 0$ as $m,k\to\infty$.

Hence it follows that $|\sum_{i=1}^{m}\alpha_i a_i x_i-\sum_{i=1}^{k}\alpha_i a_i x_i|\to 0$ as $m,k\to\infty$.

Since A is complete, $\sum_{i=1}^{\infty}\alpha_i a_i x_i \varepsilon A$. By Theorem 3.4, every multiplicative linear functional on A is continuous.

Theorem 3.6. Let A be a Fréchet algebra whose topology is defined by an increasing sequence $\{q_n\}$ of submultiplicative semi-norms. If there exists a Schauder basis $H=\{x_i\}_{i\varepsilon N}$ such that

(i) $q_i(x_i)\neq 0$, $q_i(x_{i+1})=0$ for $i=1,2,3,4,\ldots$

(ii) $x_j x_i = x_i x_j = x_j$ whenever $i\leq j$,

then every multiplicative linear functional on A is continuous.

Proof. $q_i(x_{i+1})=0$ implies that, for each $j\geq i+1$, we have

$$q_i(x_j)=q_i(x_{i+1}x_j) \leq q_i(x_{i+1})q_i(x_j)=0$$

or $\quad q_i(x_j)=0$ for all $i<j$. (3.6.1)

By Theorem 2.1, it follows that, for any sequence $\{\alpha_i\}$, $\sum_{i=1}^{\infty}\alpha_i x_i \varepsilon A,\ldots(*)$.

Now we consider any $x=\sum_{i=1}^{\infty}\alpha_i x_i \varepsilon A$. By virtue of (3.6.1), we have

$$|\alpha_i x_i|=\sum_{n=1}^{\infty}\frac{1}{2^n}\frac{q_n(\alpha_i x_i)}{1+q_n(\alpha_i x_i)}=\sum_{n=i}^{\infty}\frac{1}{2^n}\frac{q_n(\alpha_i x_i)}{1+q_n(\alpha_i x_i)}<\sum_{n=i}^{\infty}\frac{1}{2^n}=\frac{1}{2^{i-1}},$$

and hence $|x|\leq\sum_{i=1}^{\infty}|\alpha_i x_i|<2$. This means that the series $x=\sum_{i=1}^{\infty}\alpha_i x_i$ is absolutely convergent. Hence the terms after multiplication of any two vectors, $x=\sum_{i=1}^{\infty}\alpha_i x_i$ and $y=\sum_{i=1}^{\infty}\beta_i x_i$, can be grouped in any way whatever without changing the sum.

Now we want to prove that if f is a multiplicative linear functional on A, then there exists $x_i \varepsilon H$ such that $f(x_i)=0$. If not, then $f(x_i)\neq 0$

for all $i=1,2,3,4,\ldots$. Since $x_ix_i=x_i$, we have $f(x_i)=f^2(x_i)$, or $[f(x_i)-1]f(x_i)=0$. Since $f(x_i)\neq 0$ by assumption, we have

$$f(x_i)=1 \quad \text{for all } i=1,2,3,4,\cdots . \tag{3.6.2}$$

By (*), $x=\sum_{i=1}^{\infty} i\,x_i\in A$. Set $f(x)=\alpha$.

(i) Suppose α is not a positive integer.

Let $x'=x-\alpha x_1=\sum_{i=1}^{\infty} i\,x_i-\alpha x_1=(1-\alpha)x_1+\sum_{i=2}^{\infty} i\,x_i$. Since $f(x_i)=1$ for $i\geq 1$ and $f(x)=\alpha$, we have

$$f(x')=0 . \tag{3.6.3}$$

Put $y=\sum_{i=1}^{\infty}\beta_i x_i$, where $\beta_1=\frac{-1}{1-\alpha}$, and for $n\geq 2$, $\beta_n=\dfrac{-n\sum_{i=1}^{n-1}\beta_i}{(\sum_{i=1}^{n} i)-\alpha}$. Since the series $\sum_{i=1}^{\infty}\alpha_i x_i$ converges absolutely for any complex sequence $\{\alpha_i\}_{i\in N}$, clearly, $x'y=-x_1+\sum_{n=2}^{\infty}\{n\sum_{i=1}^{n-1}\beta_i+\beta_n[(\sum_{i=1}^{n} i)-\alpha]\}x_n=-x_1$. Applying (3.6.3), it follows that

$$0=f(x')f(y)=f(x'y)=f(-x_1)=-1$$

which is impossible.

(ii) Suppose α is an integer, say $\alpha=n$, but $\alpha\neq 1$.

Let $x''=x-\alpha x_n=x-nx_n=\sum_{\substack{i=1\\ i\neq n}}^{\infty} i\,x_i$. Clearly, $f(x'')=0$.

Put $y=\sum_{\substack{i=1\\ i\neq n}}\beta_i x_i$, where $\beta_1=-1$, and for $m\geq 2$, $m\neq n$, $\beta_m=-\dfrac{m\sum_{\substack{i=1\\ i\neq n}}^{m-1}\beta_i}{\sum_{\substack{i=1\\ i\neq n}}^{m} i}$. Then

$$x''y=-x_1+\sum_{\substack{m=2\\m\neq n}}^{\infty}[m\sum_{\substack{i=1\\i\neq n}}^{m-1}\beta_i+\beta_m\sum_{\substack{i=1\\i\neq n}}^{m}i]x_m=-x_1,$$ so $0=f(x'')f(y)=f(x''y)=f(-x_1)=-1$,

which is impossible.

(iii) Suppose $\alpha=1$. Let $x'''=x-x_1=\sum_{i=2}^{\infty}ix_i$. Clearly, $f(x''')=0$.

Put $y=\sum_{i=2}^{\infty}\beta_i x_i$ where $\beta_2=-1$ and for $m\geq 3$, $\beta_m=-\dfrac{m\sum_{i=2}^{m-1}\beta_i}{\sum_{i=2}^{m} i}$. Then $x'''y=$

$-x_2+\sum_{m=3}^{\infty}[m\sum_{i=2}^{m-1}\beta_i+\beta_m\sum_{i=2}^{m}i]x_m=-x_2$. So $f(x'''y)=f(-x_2)=-1$ which is impossible.

Thus, there must exist $x_i\varepsilon H$ such that $f(x_i)=0$ in both cases. Let us take $f(x_i)=0$. Then for each $k\geq i$,

$$f(x_k)=f(x_kx_i)=f(x_k)f(x_i)=0 \tag{3.6.4}$$

(i) If $f(x_j)=0$ for all j, $1\leq j\leq i-1$, then, for each $x\varepsilon A$, $x=\sum_{i=1}^{\infty}\alpha_i x_i$, $xx_1=(\sum_{i=1}^{\infty}\alpha_i x_i)x_1=\sum_{i=1}^{\infty}\alpha_i x_i x_1=\sum_{i=1}^{\infty}\alpha_i x_i=x$ implies $f(x)=f(xx_1)=f(x)f(x_1)=0$, i.e. f is identically zero.

(ii) If there exists a $x_e\varepsilon H$, $1\leq e\leq i-1$, such that $f(x_e)\neq 0$.

Put $j=\text{Max}\{e:f(x_e)\neq 0\}$.

By virtue of (3.6.4), we have $j\leq i-1$. Hence

$$f(x_{j+1})=0 \text{ and } f(x_j)\neq 0. \tag{3.6.5}$$

Since $0\neq f(x_j)=f(x_j^2)=f(x_j)f(x_j)$, it follows that $f(x_j)=1$. For $k\leq j$, $f(x_j)=1$ and $x_j=x_kx_j$ imply that

$$f(x_k)=f(x_k)f(x_j)=f(x_kx_j)=f(x_j)=1. \tag{3.6.6}$$

For each $x=\sum_{k=1}^{\infty}\alpha_k x_k\varepsilon A$, applying (3.6.5), (3.6.6) and hypothesis (ii), we have

$$f(x) = f(\sum_{k=1}^{\infty}\alpha_k x_k)=f(\sum_{k=1}^{j}\alpha_k x_k)+f(\sum_{k=j+1}^{\infty}\alpha_k x_k)$$

$$= f(\sum_{k=1}^{j}\alpha_k x_k)+f(x_{j+1}\sum_{k=j+1}^{\infty}\alpha_k x_k) \text{ (since } x_{j+1}x_k=x_k \text{ for } k\geq j+1)$$

$$= \sum_{k=1}^{j}\alpha_k+f(x_{j+1})f(\sum_{k=j+1}^{\infty}\alpha_k x_k)=\sum_{k=1}^{j}\alpha_k. \tag{3.6.7}$$

Since $x_k x_j=x_j$ for $k\leq j$, we have

$$|\sum_{k=1}^{j}\alpha_k|q_j(x_j)=q_j(\sum_{k=1}^{j}\alpha_k x_j)=q_j(\sum_{k=1}^{j}\alpha_k x_k x_j)=q_j[(\sum_{k=1}^{j}\alpha_k x_k)x_j]$$

$$\leq q_j(\sum_{k=1}^{j}\alpha_k x_k)q_j(x_j)\leq q_j(\sum_{k=1}^{\infty}\alpha_k x_k-\sum_{k=j+1}^{\infty}\alpha_k x_k)q_j(x_j)$$

$$\leq[q_j(\sum_{k=1}^{\infty}\alpha_k x_k)+q_j(\sum_{k=j+1}^{\infty}\alpha_k x_k)]q_j(x_j)\leq[q_j(x)+q_j(x_{j+1}\sum_{k=j+1}^{\infty}\alpha_k x_k)]q_j(x_j)$$

$$\leq[q_j(x)+q_j(x_{j+1})q_j(\sum_{k=j+1}^{\infty}\alpha_k x_k)]q_j(x_j)\leq q_j(x_j)q_j(x).$$

Since by hypothesis $q_j(x_j)\neq 0$, we have $|\sum_{k=1}^{j}\alpha_k|\leq q_j(x)$, and from (3.6.7), we have $|f(x)|=|\sum_{k=1}^{j}\alpha_k|\leq q_j(x)$. Since x is any element of A, it follows that f is continuous.

Theorem 3.7. Let A be a locally m-convex algebra whose topology is defined by a family $\{q_t\}_{t\in T}$ of submultiplicative semi-norms. If there exists a Hamel basis H such that, for $x,y\in H$, $xy=0$ whenever $x\neq y$, then every multiplicative linear functional on A is continuous.

Proof. (a) Suppose there exists $x\in H$ such that $f(x)\neq 0$.

Let $Z=\sum_{x\in H}\alpha_x X$ be an arbitrary element of A, where $\alpha_x=0$ for all $x\in H$ except a finite number of α_x's. Clearly, by hypothesis $Zx=\alpha_x x^2$, or

$$f(Zx)=f(Z)f(x)=\alpha_x f^2(x). \text{ Since } f(x)\neq 0,\ f(Z)=\alpha_x f(x). \tag{3.7.1}$$

But $f(x)\neq 0$ implies $f(xx)=f(x)f(x)\neq 0$, and hence $xx\neq 0$.

Since $x\neq 0$, $xx\neq 0$, and A is Hausdorff, there exists a semi-norm q_t such that $q_t(x)\neq 0$ and $q_t(xx)\neq 0$.

Put $$M=|f(x)|\frac{q_t(x)}{q_t(xx)}. \tag{3.7.2}$$

Since $Zx=\alpha_x x^2$ implies $|\alpha_x|q_t(x^2)\leq q_t(Z)q_t(x)$, and $q_t(x)\neq 0$ implies $|\alpha_x|\frac{q_t(x^2)}{q_t(x)}\leq q_t(Z)$, from (3.7.1), (3.7.2) and the above result, we have

$$Mq_t(Z)\geq|f(x)|\frac{q_t(x)}{q_t(x^2)}\,|\alpha_x|\,\frac{q_t(x^2)}{q_t(x)}=|f(x)||\alpha_x|=|f(Z)|,\text{ or }|f(Z)|\leq Mq_t(Z).$$

Since Z is any element of A, this shows that f is continuous.

(b) If $f(x)=0$ for all $x\in H$, then it is clear that f is identically zero.

Chapter 4

ON FRÉCHET SPACES AND FRÉCHET ALGEBRAS WITH CYCLIC SCHAUDER BASES

In theorem 4.1, we wish to prove the existence of a subset $\{x_n\}_{n\varepsilon T}$ in some Fréchet spaces such that $q_n(x_n)=1$ and $q_m(x_n)=0$ whenever $m<n$. Furthermore, in theorem 4.2, we prove that the elements of this subset are linearly independent. Finally, we show that the topology of every locally compact Fréchet space can only be defined by a family $\{q_n\}_{n\varepsilon T}$ of finitely many non-equivalent increasing semi-norms. This result follows if we use first the well known fact that each locally compact locally convex space is finite - dimensional. But the interest of our result lies in the fact that it is derived from the theory of Fréchet spaces.

Theorems 4.6 and 4.7 give some interesting results namely, (i) If a Fréchet algebra A possesses a cyclic Schauder basis $H=\{Z,Z^2,Z^3,\cdots\}$ and $F=\{\sum_{i=1}^{\infty}\alpha^i Z^i \mid \alpha\varepsilon C\}$ A, then every multiplicative linear functional on A is identically zero. (ii) If a Fréchet algebra A possesses a cyclic Schauder basis $H=\{e,Z,Z^2,Z^3,\cdots\}$ and $F=\{\sum_{i=1}^{\infty}\alpha^i Z^i \mid \alpha\varepsilon C\}$ A, then there exists precisely one non-zero multiplicative linear functional on A.

<u>Theorem 4.1.</u> Let A be a Fréchet space whose topology is defined by a family $\{q_n\}_{n\varepsilon T}$ of increasing non-equivalent semi-norms. If A is locally compact, then there is some positive integer N such that for each semi-norms q_n, $n\geq N$, there exists an $x_n\varepsilon A$ such that $q_n(x_n)=1$ and $q_m(x_n)=0$ whenever $m<n$.

Proof. Since A is locally compact, there exists a compact neighborhood W of 0. For each q_n, set $V_n=\{x\varepsilon A|q_n(x)\leq 1\}$, and $\hat{V}_n=\{x\varepsilon A|q_n(x)=1\}$. Since the family $\{\frac{1}{m}V_n\}_{m,n=1,2,3,\ldots}$ forms a fundamental system of neighborhoods of 0, there exist two positive integers m,N such that $\frac{1}{m}V_n\subset W$ whenever $n\geq N$. Since V_n is closed, so is $\frac{1}{m}V_n$. The fact that W is compact implies $\frac{1}{m}V_n$ is compact for all $n\geq N$. This also implies that V_n is compact for all $n\geq N$. Since $\hat{V}_n = \{x\varepsilon A|q_n(x)=1\}$ is closed, the compactness of $\hat{V}_n$, $n\geq N$, follows.

Since q_{n-1}, q_n are non-equivalent, for each $m>0$, there exists

$$y_m'\varepsilon A \text{ such that } mq_{n-1}(y_m')<q_n(y_m') \text{ or } q_{n-1}\left(\frac{y_m'}{q_n(y_m')}\right)<\frac{1}{m}\,, \tag{4.1.1}$$

Put $y_m = \dfrac{y_m'}{q_n(y_m')}$, then $q_n(y_m)=1$ for all $m=1,2,3,\ldots$. And hence the sequence $\{y_m\}_{m\geq 1}\subset\hat{V}_n$. Since $\hat{V}_n$ is compact for $n\geq N$, there exists a subsequence $\{y_{n_k}\}_{k\geq 1}$ of $\{y_m\}_{m\geq 1}$ such that $\{y_{n_k}\}\to x_n\varepsilon\hat{V}_n$. So

$q_n(y_{n_k})=1\to q_n(x_n)=1$ where $n\geq N$. On the other hand, by virtue of (4.1.1), we have

$$q_{n-1}(y_{n_k})=q_{n-1}\left(\frac{y_{n_k}'}{q_n(y_{n_k}')}\right)<\frac{1}{n_k}\,.$$

This implies that $q_{n-1}(y_{n_k})\to 0$ as $n_k\to\infty$ and hence

$$q_{n-1}(y_{n_k})\to q_{n-1}(x_n)=0 \text{ as } y_{n_k}\to x_n.$$

For $m<n$, $q_m(x_n)\leq q_{n-1}(x_n)=0$. Thus, $q_m(x_n)=0$ wherever $m<n$ and $n\geq N$. This completes the proof.

Theorem 4.2. Let A be a locally compact Fréchet space. Then its topology can only be defined by finitely many non-equivalent increasing semi-norms.

Proof. By theorem (4.1), there is some positive integer N such that, for each semi-norm q_n with $n \geq N$, there exists $x_n \in A$ satisfying

$$q_n(x_n)=1 \text{ and } q_m(x_n)=0 \text{ whenever } m<n . \qquad (4.2.1)$$

We claim that $\{x_n\}_{n \geq N}$ is linearly independent.

Indeed, suppose $\alpha_n x_n + \alpha_{n+1} x_{n+1} + \cdots + \alpha_{n+k} x_{n+k} = 0$, then

$\alpha_n x_n = -\alpha_{n+1} x_{n+1} - \alpha_{n+2} x_{n+2} - \cdots - \alpha_{n+k} x_{n+k}$.

From (4.2.1), we have

$$|\alpha_n| q_n(x_n) = q_n(\alpha_n x_n) = q_n(-\alpha_{n+1} x_{n+1} - \alpha_{n+2} x_{n+2} - \cdots - {}_{n+k} x_{n+k})$$

$$\leq \sum_{r=1}^{k} |\alpha_{n+r}| q_n(x_{n+r}) = 0.$$

Thus $|\alpha_n| q_n(x_n) = 0$.

But $q_n(x_n)=1$ implies $|\alpha_n|=0$ or $\alpha_n=0$.

Similarly, we can prove that $\alpha_{n+1} = \alpha_{n+2} = \ldots = \alpha_{n+k} = 0$.

Thus $\{x_n\}_{n \geq N}$ is linearly independent. By proposition 1.5, since every locally compact Hausdorff topological vector space is of finite dimension, the number of $\{x_n\}_{n \geq N}$ must be finite.

This implies that the number of $\{q_n\}_{n \geq N}$ is finite.

Using the above theorem, the following well-known result follows:

Corollary 4.2.1: Every locally compact Fréchet space (Fréchet algebra) is a Banach space (Banach algebra).

Proof. By theorem 4.2, since the number of $\{q_n\}$ is finite, put $q(x) = \max_n q_n(x)$. $x \neq 0$ implies that there exists q_n such that $q_n(x) \neq 0$ and hence $q(x) \neq 0$. So $q(x)=0$ implies $x=0$.

Thus, $q(x)$ is a norm which defines the topology of the Fréchet space and every Frechet space is complete. So it is a Banach space.

Theorem 4.3. Let A be a Fréchet space whose topology is defined by an increasing sequence $\{q_n\}_{n>1}$ of semi-norms. Then, for each sequence $\{x_i\}_{i\varepsilon N}$ of A, there exists $\{\alpha_i\}_{i\varepsilon N}\subset C$ such that $\sum_{i=1}^{\infty}\alpha_i x_i \varepsilon A$ where $\alpha_i\neq 0$ for all i.

Proof. For each $x_i\neq 0$, there exists some $q_{f(i)}$ such that $q_{f(i)}(x_i)\neq 0$ and $f(i)\geq i$.

Put $\alpha_i = \dfrac{1}{i^2 q_{f(i)}(x_i)}$. We claim that $\sum_{i=1}^{\infty}\alpha_i x_i \varepsilon A$.

Indeed, for each $\varepsilon>0$, there exists a positive integer $N\varepsilon$ such that $\sum_{i=N\varepsilon}^{\infty}\frac{1}{i^2}<\varepsilon$. For each $\varepsilon>0$ and q_n, taking $N=\max\{n,N\varepsilon\}$ then

$m>k>N$ implies $q_n(\sum_{i=1}^{m}\alpha_i x_i - \sum_{i=1}^{k}\alpha_i x_i) = q_n(\sum_{i=k+1}^{m}\alpha_i x_i)\leq\sum_{i=k+1}^{m}q_n(\alpha_i x_i)\leq\sum_{i=k+1}^{\infty}q_n(\alpha_i x_i)$

$$<\sum_{i=N}^{\infty}|\alpha_i|q_n(x_i)\leq\sum_{i=N}^{\infty}\frac{q_n(x_i)}{i^2 q_{f(i)}(x_i)}\leq\sum_{i=N}^{\infty}\frac{1}{i^2}\leq\sum_{i=N_\varepsilon}^{\infty}\frac{1}{i^2}<\varepsilon,$$

because $f(i)\geq i\geq N\geq n$ implies $q_{f(i)}(x_i)\geq q_n(x_i)$.

This means that, for each q_n, $q_n(\sum_{i=1}^{m}\alpha_i x_i - \sum_{i=1}^{k}\alpha_i x_i)\to 0$ as $m,k\to\infty$, or $|\sum_{i=1}^{m}\alpha_i x_i - \sum_{i=1}^{k}\alpha_i x_i|\to\infty$ as $m,k\to\infty$.

Since A is complete, $\sum_{i=1}^{n}\alpha_i x_i \to \sum_{i=1}^{\infty}\alpha_i x_i \ \varepsilon A$.

Theorem 4.4. Let A be a Fréchet algebra whose topology is defined by an increasing sequence $\{q_n\}_{n\varepsilon N}$ of submultiplicative semi-norms. Let f be a multiplicative linear functional on A. Then, for each $x\varepsilon A$, there exists a submultiplicative semi-norm q_n depending on x such that $|f(x)|\leq q_n(x)$.

Proof. If not, then for some $x\varepsilon A$, $|f(x)|>q_n(x)\geq 0$ for all $n\geq 1$, or

$1>q_n(\frac{x}{f(x)})$ for all $n\geq 1$, Put $y=\frac{x}{f(x)}$ and $z_m=y+y^2+y^3+\ldots+y^m$.

For each n, $q_n(y)<1$ implies $q_n(z_m-z_k)\leq\sum_{i=k+1}^{m} q_n(y^i)\leq\sum_{i=k+1}^{m} q_n^i(y)\to 0$ as $m,k\to\infty$.

This implies that $|z_m-z_k|=\sum_{i=1}^{\infty}\frac{1}{2^i}\frac{q_i(z_m-z_k)}{1+q_i(z_m-z_k)}\to 0$ as $m\ k\to\infty$.

But A is complete, and hence $z_m\to z=\sum_{i=1}^{\infty} y^i\varepsilon A$.

Since $yz+y=z$ and $f(y)=f(\frac{x}{f(x)})=1$, it follows that

$f(yz)+f(y)=f(z)$ or $f(y)f(z)+f(y)=f(z)$ or $f(z)+1=f(z)$, which is impossible, and so $|f(x)|\leq q_n(x)$ for some n.

Theorem 4.5. Let A be a Fréchet algebra, f a multiplicative linear functional on A, and $Z\varepsilon A$.

(a) If $\sum_{i=1}^{\infty}\alpha^i Z^i\varepsilon A$, then $f(Z)\neq\frac{1}{\alpha}$, where $\alpha\neq 0$.

(b) If $F=\{\sum_{i=1}^{\infty}\alpha^i Z^i|\alpha\varepsilon C\}\subset A$, then $f(Z)=0$.

(c) If $\sum_{i=1}^{\infty}|\alpha|^i|Z^i|<\infty$ then $|f(Z)|<\frac{1}{|\alpha|}$.

Proof. (a) Assume $f(Z)=\frac{1}{\alpha}\neq 0$. Then $f^2(Z)=\frac{1}{\alpha^2}$ $\frac{1}{\alpha}f(Z)-f^2(Z)=0$ or $f(\frac{1}{\alpha}Z-Z^2)=0$, or $f(Z-\alpha Z^2)=0$. Since $0=f(Z-\alpha Z^2)f(\sum_{i=1}^{\infty}\alpha^i Z^i)=f[(Z-\alpha Z^2)(\sum_{i=1}^{\infty}\alpha^i Z^i)]$ $=f(\alpha Z^2)=\alpha f(Z^2)=\frac{1}{\alpha}\neq 0$, which is impossible. Therefore, $f(Z)\neq\frac{1}{\alpha}$.

(b) Assume $f(Z)\neq 0$. Put $f(Z)=\frac{1}{\alpha}$, $\alpha\varepsilon C$. By (a), $\sum_{i=1}^{\infty}\alpha^i Z^i\varepsilon A$ implies

$f(Z)\neq\frac{1}{\alpha}$, which is a contradiction. Thus $f(Z)=0$.

(c) If $|f(Z)|\geq\frac{1}{|\alpha|}$, denote $f(Z)=\frac{1}{\beta}$. Then $\frac{1}{|\beta|}\geq\frac{1}{|\alpha|}$ implies $|\alpha|\geq|\beta|$, and hence $\sum_{i=1}^{\infty}|\beta|^i|Z^i|<\sum_{i=1}^{\infty}|\alpha|^i|Z^i|<\infty$. This means that $\sum_{i=1}^{\infty}\beta^i Z^i\epsilon A$. By (a), $f(Z)\neq\frac{1}{\beta}$ which is a contradiction. Thus $|f(Z)|<\frac{1}{|\alpha|}$.

Theorem 4.6. Let A be a Fréchet algebra with a Schauder basis $\{Z,Z^2,Z^3,\cdots\}$ such that $F=\{\sum_{i=1}^{\infty}\alpha^i Z^i|\alpha\epsilon C\}\subset A$. Then each multiplicative linear functional on A is identically zero.

Proof. Let $X=\sum_{i=1}^{\infty}\alpha_i Z^i\epsilon A$ be an arbitrary element of A, then

$f(x)=f(\alpha_1 Z)+f(\sum_{i=2}^{\infty}\alpha_i Z^i)=f(\alpha_1 Z)+f(Z\sum_{i=2}^{\infty}\alpha_i Z^{i-1})=\alpha_1 f(Z)+f(Z)f(\sum_{i=2}^{\infty}\alpha_i Z^{i-1})$.

By theorem 4.5 (b), $f(Z)=0$ implies $f(x)=0$.

Thus f is identically zero.

Theorem 4.7. Let A be a Fréchet algebra with a Schauder basis $\{e,Z,Z^2,\cdots\}$ where e is the identity of A. Suppose A contains $F=\{\sum_{i=1}^{\infty}\alpha^i Z^i|\alpha\epsilon C\}$. Then there exists only one non-zero multiplicative linear functional on A. i.e. for each vector $x=\sum_{i=0}^{\infty}\alpha_i Z^i\epsilon A$ (where $Z^0=e$), $f(x)=f(\sum_{i=0}^{\infty}\alpha_i Z^i)=\alpha_0$.

Proof: $f(e)=f(e^2)=f(e)f(e)$ implies $f(e)[f(e)-1]=0$.

f being non-zero implies $f(e)\neq 0$ and hence $f(e)=1$.

Consider $f(x)=f(\sum_{i=0}^{\infty}\alpha_i Z^i)=f(\alpha_0 e+Z\sum_{i=1}^{\infty}\alpha_i Z^{i-1})=\alpha_0 f(e)+f(Z)f(\sum_{i=1}^{\infty}\alpha_i Z^{i-1})$

$=\alpha_0+f(Z)f(\sum_{i=1}^{\infty}\alpha_i Z^{i-1})$.

By theorem 4.5(b), $f(Z)=0$, implies $f(x)=\alpha_0$.

Remark. If in theorem 4.7, A does not contain $F=\{\sum_{i=1}^{\infty}\alpha^i Z^i|\alpha\epsilon C\}$, then the non-zero multiplicative linear functional on A is not necessarily unique. For example, let

$$A=\{\sum_{i=0}^{\infty}\alpha_i Z^i\Big|\sum_{i=1}^{\infty}|\alpha_i|<\infty\}$$

Define $f(Z)=\alpha$, where $|\alpha|<1$, and for each $x=\sum_{i=0}^{\infty}\alpha_i Z^i \varepsilon A$, $f(x)=\sum_{i=0}^{\infty}\alpha_i \alpha^i$. Then f is a multiplicative linear functional on A, but $f(\sum_{i=0}^{\infty}\alpha_i Z^i)\neq\alpha_0$.

Theorem 4.8. Let A be a Frechet algebra with a Schauder basis $\{e,Z,Z^2,Z^3\cdots\}$. Then each multiplicative linear functional f on A can be extended to $\overline{f}$ on a smallest Fréchet algebra E which contains A and $F=\{\sum_{i=1}^{\infty}\alpha^i Z^i|\alpha\varepsilon C\}$ if and only if $f(Z)=0$.

Proof. If f on A can be extended to $\overline{f}$ on E, by theorem 4.5 (b), $\overline{f}(z)=0$ and hence $f(Z)=0$. Conversely, if $f(Z)=0$, then, for each $x=\sum_{i=1}^{\infty}\alpha^i Z^i \varepsilon F$, since $\overline{f}(x)=\overline{f}(\sum_{i=1}^{\infty}\alpha^i Z^i)=\overline{f}(Z(\sum_{i=1}^{\infty}\alpha^i Z^{i-1}))=\overline{f}(Z)\overline{f}(\sum_{i=1}^{\infty}\alpha^i Z^{i-1})$

$=f(Z)\overline{f}(\sum_{i=1}^{\infty}\alpha^i Z^{i-1})=0$, so we must define $\overline{f}(x)=\begin{cases} f(x), & x\varepsilon A \\ 0, & x\varepsilon F \end{cases}$. Then $\overline{f}$ extends f and $\overline{f}$ is a multiplicative linear functional on E.

As a consequence of our results, we derive the following:

Theorem 4.9. Let A be a division algebra. If there exists a non-zero multiplicative linear functional on A, then A is isomorphic to the algebra of complex numbers.

Proof. Let $x\varepsilon A$. If $x\neq 0$, then there exists $x^{-1}\varepsilon A$ such that $xx^{-1}=e$. Hence $f(x)f(x^{-1})=f(e)$. f is non-zero implies $f(e)\neq 0$ and hence $f(x)\neq 0$. This implies that there exists $\alpha_x \varepsilon C$ (α_x depending upon x only) such that $f(\alpha_x e)=f(x)$ or $f(\alpha_x e-x)=0$.

We claim that $\alpha_x e-x=0$.

If not, then there exists $(\alpha_x e-x)^{-1}\varepsilon A$ such that

$$0=f(\alpha_x e-x)f[(\alpha_x e-x)^{-1}]=f(e)\neq 0,$$

which is impossible.

Therefore, $\alpha_x e-x=0$ or $x=\alpha_x e$. Clearly the mapping: $x\to\alpha_x$ is one-to-one and onto. Hence A is isomorphic to $\mathbb{C}$.

Example. Let A be a Fréchet algebra with a Schauder basis $\{e,Z,Z^2,\ldots\}$ whose topology is defined by an increasing sequence $\{q_n\}_{n\in N}$ of submultiplicative semi-norms, where $q_n(\sum_{i=0}^{\infty}\alpha_i Z^i) = \sum_{i=0}^{n}|\alpha_i|$. Since, for each $i\geq 1$, $q_n(Z^i)=0$ whenever $i>n$, by theorem 2.1, A contains $F=\{\sum_{i=0}^{\infty}\alpha^i Z^i \mid \alpha\in C\}$. Hence by theorem 4.7, there is only one non-zero multiplicative linear functional on A, i.e. $f(\sum_{i=0}^{\infty}\alpha_i Z^i)=\alpha_0$.

Chapter 5

POSITIVE FUNCTIONALS AND *-REPRESENTATIONS OF FRÉCHET *-ALGEBRAS

A representation of an algebra R is a homomorphism of R into the algebra of linear operators on some vector space. This vector space is called a representation space.

In the sequel, we shall restrict ourselves to the study of *-representations of *-algebras. An example of a *-algebra is the set B(H) of all bounded linear operators on a given Hilbert space H. The algebra B(H) plays an essential role in the sequel. A *-representation of a *-algebra R is a *-homomorphism $x \to A_x$ of R into the algebra B(H), where B(H) is a *-algebra, with A^* being the adjoint operator of $A \in B(H)$.

There are two well-known theorems:

Theorem (i). Every positive functional f in a Banach *-algebra R with identity satisfies the inequality:

$$|f(x)| \le f(e)||x||_R \quad \text{where } x \in R \text{ and } || \ ||_R \text{ is the norm of R.}$$

Theorem (ii). Every *-representation A of a Banach *-algebra R into B(H) satisfies $||A_x||_B \le ||x||_R$ where $|| \ ||_B$ is a norm of B(H).

In this chapter, we wish to generalize these results to Fréchet *-algebra. First of all, we define the Fréchet *-algebra. Let E be a Fréchet algebra whose topology is defined by an increasing sequence $\{q_n\}_{n \in N}$ of submultiplicative semi-norms. E is called a Fréchet *-algebra if:

a) E is a *-algebra which is also a Fréchet algebra.

b) $q_n(x)=q_n(x^*)$ for all $n=1,2,3,\cdots$.

It follows from b) and Proposition 1.3 that $|x|=|x^*|$ and hence the involution operation is continuous. Also $q_n(xx^*)\leq q_n^2(x)$.

Now, we start to prove a generalization of Theorem (i) mentioned above.

Theorem 5.1. Let R be a Fréchet *-algebra with identity whose topology is defined by an increasing sequence $\{q_n\}_{n\varepsilon N}$ of submultiplicative seminorms. Then every positive functional f in R satisfies the inequality

$$|f(x)|\leq f(e)\lim_{n\to\infty} q_n(x).$$

Proof. (i) Suppose $\lim_{n\to\infty} q_n(x)=\infty$. Since $e^*=e$ (by Proposition 1.9), we have $f(e)=f(ee^*)\geq 0$. If $f(e)=0$, by Proposition 1.11, $0=|f(x)|^2\leq f(e)f(x^*x)=0$, and hence $0=|f(x)|\leq f(e)\lim_{n\to\infty} q_n(x)$. If $f(e)>0$, $|f(x)|<f(e)\lim_{n\to\infty} q_n(x)=\infty$.

(ii) If $\lim_{n\to\infty} q_n(x)\leq 1$ and x is a Hermitian element of R, consider the binomial series

$$(1-\lambda)^{\frac{1}{2}}= 1-\frac{1}{2}\lambda- \frac{1}{2!}\,\frac{1}{2}\,\frac{1}{2}\,\lambda^2-\cdots.$$

It converges absolutely for $|\lambda|\leq 1$. Replace λ by x and 1 by e in this series: Since $\lim_{n\to\infty} q_n(x)\leq 1$ implies $q_n(x)\leq 1$ for all $n\varepsilon N$, we obtain

$$q_n(e-\frac{1}{2}x-\frac{1}{2!}\,\frac{1}{2}\,\frac{1}{2}\,x^2-\cdots) \leq q_n(e)+\frac{1}{2}q_n(x)+\frac{1}{2!}\,\frac{1}{2}\,\frac{1}{2}\,q_n^2(x)+\cdots<\infty, \text{ for all } n\varepsilon N.$$

This implies that $e-\frac{1}{2}x-\frac{1}{2!}\,\frac{1}{2}\,\frac{1}{2}\,x^2-\cdots$ converges absolutely in A.

In virtue of the continuity of involution, its sum, which we shall denote by y, is a Hermitian element of R. Furthermore

$$y^*y=y^2=e-x.$$

Hence it follows that

$$f(e-x)=f(yy^*)\geq 0.$$

Conversely, $f(x)\leq f(e)$.

Replacing x everywhere by $-x$, we obtain that $-f(x)\leq f(e)$.

Consequently,

$$|f(x)|\leq f(e).$$

(iii) If $\lim_{n\to\infty} q_n(x)<\infty$ and x is a Hermitian element of R, suppose $x=0$, then $f(x)=0$ and $\lim_{n\to\infty} q_n(x)=0$ and the inequality follows. Suppose $x\neq 0$, then $\lim_{n\to\infty} q_n(x)>0$, we set $x_1 = \dfrac{x}{\lim_{n\to\infty} q_n(x)}$, then x_1 is also Hermitian, and $\lim_{n\to\infty} q_n(x_1)=1$. Consequently, by what we have already proved,

$$|f(x_1)|\leq f(e), \text{ i.e. } \frac{|f(x)|}{\lim_{n\to\infty} q_n(x)} \leq f(e), \text{ or } |f(x)|\leq f(e)\lim_{n\to\infty} q_n(x),$$

and thus inequality (1) has been proved for an arbitrary Hermitian element of R.

(iv) If $\lim_{n\to\infty} q_n(x)<\infty$,

Then x^*x is a Hermitian element of R and

$$\lim_{n\to\infty} q_n(x^*x)\leq\lim_{n\to\infty} q_n(x^*)q_n(x)=\lim_{n\to\infty} q_n^2(x)<\infty.$$

By (iii), $f(x^*x) \le f(e)\lim_{n\to\infty} q_n(x^*x) \le f(e)\lim_{n\to\infty} q_n^2(x)$. ....(*)

On the other hand, by Proposition 1.11, $|f(x)|^2 \le f(e)f(x^*x)$, it follows from this and the inequality (*) that

$$|f(x)|^2 \le f^2(e)\lim_{n\to\infty} q_n^2(x)$$

or $$|f(x)| \le f(e)\lim_{n\to\infty} q_n(x).$$

Theorem 5.2. Let R be a Fréchet *-algebra whose topology is defined by an increasing sequence $\{q_n\}_{n\in N}$ of submultiplicative semi-norms. Then every *-representation A of R into B(H) satisfies $||A_x||_B \leq \lim_{n\to\infty} q_n(x)$.

We need the following:

Lemma 5.3. Suppose $x \to A_x$ is a representation of a *-algebra in the Hilbert space H and $y \neq 0$ is a vector of H. We set

$$f(x) = (A_x y, y)$$

then f is a positive functional.

Proof of Lemma 5.3. Since $A: x \to A_x$ is a *-homomorphism $x \to A_x$ of R into the algebra B(H), by Definition 1.20, $x \to A_x$ and $x^* \to A_{x^*}$ imply $x^*x \to A_{x^*x} = A_x{}^*A_x$ and $A_x^* = A_{x^*}$.

Thus,

$$f(x^*x) = (A_{x^*x}y, y) = (A_{x^*}A_x y, y) = (A_x^*A_x y, y) = (A_x y, A_x y) \ge 0 .$$

Proof of Theorem 5.2. Without loss of generality, we may assume that R contains an identity. In fact, in the contrary case, by setting $A_{\lambda e+x} = \lambda I + A_x$, we could extend the representation $x \to A_x$ of the *-algebra

R to a representation of the Fréchet *-algebra obtained from R by the adjunction of an identity. But if R contains an identity, then, applying Theorem 5.1 to the positive functional $f(x)=(A_x y,y)$, we obtain

$$|(A_x y,y)|\le f(e)\lim_{n\to\infty} q_n(x)=(A_e y,y)\lim_{n\to\infty} q_n(x)=(y,y)\lim_{n\to\infty} q_n(x).$$

If we replace x by x*x everywhere in the last inequality, we conclude that

$$|(A_{x^*x}y,y)|\le(y,y)\lim_{n\to\infty} q_n(x^*x)\le(y,y)\lim_{n\to\infty} q_n^2(x).$$

By the proof of Lemma 5.3, we have

$$|f(x^*x)|=|(A_{x^*x}y,y)|=||A_x y||^2_H$$

Therefore

$$||A_x y||^2_H \le ||y||^2_H \left(\lim_{n\to\infty} q_n(x)\right)^2$$

or $$||A_x y||_H \le ||y||_H \lim_{n\to\infty} q_n(x).$$

Since y is an arbitrary non-zero vector in H, it follows from the last inequality that $||A_x||_B \le \lim_{n\to\infty} q_n(x)$.

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