QUASIGRAPHS IN N-DIMENSIONAL MANIFOLDS

By

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A Thesis
Submitted to the Faculty of Graduate Studies
in Partial Fulfillment of the Requirements
for the Degree
Master of Science

McMaster University
October 1976

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QUASIGRAPHS IN N-DIMENSIONAL MANIFOLDS
TITLE: Quasigraphs in n-Dimensional Manifolds
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NUMBER OF PAGES: (vii), 61

SCOPE AND CONTENTS: A definition of quasigraphs in n-dimensional manifolds is given. A continuous family of quasigraphs is then introduced and the intersection and support properties of these quasigraphs are studied.
ACKNOWLEDGEMENT

I would like to express my sincerest appreciation to my supervisor, Dr. N.D. Lane, for the invaluable guidance and encouragement he has given me during the preparation of this thesis.

Thanks are also due to McMaster University for their generous financial support that I have received as a graduate student.

Finally, I am indebted to Mrs. Phyllis Matsos who devoted her time and effort in the typing of this thesis.
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INTRODUCTION

In the study of direct differential geometry, families of oriented arcs, curves, conic sections and graphs of polynomials of degree at most \( n \) have been employed to define the differentiability of an arc at a point in various kinds of planes; cf. [1], [4], [5], [7] and [8].

These studies followed similar patterns and led to the search of a general theory of differentiability. In [6], N.D. Lane, P. Scherk and J.M. Turgeon introduced the notion of a quasigraph in the unit disk as a basis for such a general theory. Dr. Ralph Park first suggested that the idea of a quasigraph could be extended to higher dimensional spaces and this led to the study of quasigraphs in \( n \)-dimensional manifolds.

In Chapter I quasigraphs in an \( n \)-dimensional connected manifold \( G \) are introduced and some basic properties looked at. A quasigraph \( K \) is a subset \([K]\) in \( G \) of inductive dimension \( \leq n-1 \), together with a decomposition of \( G \ \setminus [K] \) into two disjoint open sets \( K^1 \) and \( K^{-1} \).

In Chapter II, convergent sequences of quasigraphs are studied and the definition of a continuous family of quasigraphs is introduced.
In Chapter III we look at the properties of a continuous family \( \mathcal{A} = \{ K_\sigma \mid \sigma \in I = (0,1) \} \) of quasigraphs \( K_\sigma = ([K_\sigma], K_{\sigma}^1, K_{\sigma}^{-1}) \), obtained by means of an isotopy of \( G \).

It is assumed that if \( Q_s \in [K_s] \cap [K_t] \), \( s, t \in I \), \( s \neq t \), then \( Q_s \in \bigcap_{s \in I} [K_s] \).

In Chapter IV the global decompositions of the continuous family \( \mathcal{A} \) of quasigraphs are studied. Let \( 0 < t_1 < \ldots < t_h < 1 \). It is shown that if \( K_{t_i} = ([K_{t_i}], K_{t_i}^1, K_{t_i}^{-1}) \), \( i = 1, \ldots, h \), are quasigraphs of \( \mathcal{A} \), then not more than \( 2^h \) of the \( 2^h \) sets \( \bigcap_{i=1}^h K_{t_i}^{a_i} \), \( a_i = \pm 1 \), are non-void.

Chapter V deals with the support and intersection properties of a finite number of quasigraphs of \( \mathcal{A} \). It is shown that if any two quasigraphs of \( \mathcal{A} \) support each other (intersect each other, both decompose \( G \) in the same way) at a point, then so do any two quasigraphs of that family. We also obtain the result that if any two quasigraphs of \( \mathcal{A} \) support (intersect) each other at \( Q \) and \( 0 < t_1 < \ldots < t_h < 1 \), then for every small neighbourhood \( N \) of \( Q \) exactly \( h+1 \) \([2h]\) of the \( 2^h \) open sets \( K_{t_1}^{\pm 1} \cap \ldots \cap K_{t_h}^{\pm 1} \cap N \), \( h \geq 2 \) are non-void, where \( K_{t_i} \in \mathcal{A} \) for \( i = 1, \ldots, h \).
1. Quasigraphs in n-Dimensional Manifolds

1.1. An n-dimensional topological manifold is a Hausdorff space each of whose points has a neighbourhood homeomorphic to $E^n$. Let $G$ be a connected n-dimensional topological manifold, with or without boundary, satisfying the second axiom of countability. Thus $G$ is a separable metric space.

Define a point $P$ of a set $S$ in $G$ to be of dimension $k$, $0 \leq k \leq n$, if some neighbourhood $B$ of $P$ in $G$ is such that $B \cap S$ is homeomorphic to $E^k$. $S$ has dimension $k$ if each point of $S$ has dimension $k$.

Define the inductive dimension of the empty set and only the empty set to be $-1$.

A space $S$ has inductive dimension $\leq k$ ($k \geq 0$) at a point $P$ if $P$ has arbitrarily small neighbourhoods whose boundaries have inductive dimension $\leq k-1$.

$S$ has inductive dimension $\leq k$, if $S$ has inductive dimension $\leq k$ at each of its points. $S$ has inductive
dimension k at point if it is true that S has inductive
dimension \( \leq k \) at P and it is false that S has inductive
dimension \( \leq k-1 \) at P. S has inductive dimension k if
the inductive dimension of S is \( \leq k \), is true and the
inductive dimension of S is \( \leq k-1 \), is false.

Define a k-hyperface F in G to be either a connected
subset of the interior of G such that for each point P in
F, there is a small neighbourhood B of P in G and a homeo-
morphism \( h_P : B \to E^n \) such that \( h_P(B) = E^n \) and \( h_P(B \cap F) = E^k \), \( k \leq n-1 \) or F is a connected subset of the boundary
of G such that for each point P in F there is a small
neighbourhood B of P in the boundary of G homeomorphic
to \( E^k \), \( k \leq n-2 \).

**Remark 1.** A subspace S of a space G has inductive
dimension \( \leq k \) if and only if every point of S has
arbitrarily small neighbourhoods, in G, whose boundaries
have intersections with S of inductive dimension \( \leq k-1 \).

**Remark 2.** The inductive dimension of a set is a
topological invariant and \( E^k \) has inductive dimension k.
Thus if P ∈ S, S ⊂ G has dimension k, then the inductive
dimension of P is k. However, there do exist points
which have inductive dimension but no ordinary dimension.

**Remark 3.** If $F$ is an $(n-1)$-hyperface in $G$ and $P \in F$, then $F$ decomposes the interior of a small neighbourhood $B$ of $P$ in $G$ into exactly two disjoint open subsets.

1.2. Let $[K]$ be the union of a finite number of hyperfaces in $G$. Let $F_i ; i = 1, 2, \ldots, m$, be the $(n-1)$-hyperfaces of $[K]$ and assume that

\[
[K] \setminus \bigcup_{i=1}^{m} F_i
\]

has no points of inductive dimension $n-1$.

Define a point of $[K]$ to be **non-singular** if it belongs to an $(n-1)$-hyperface of $[K]$, otherwise it is **singualr**. Thus if $Q \in [K]$ has dimension $n-1$ it is non-singular.

We call an $(n-1)$-hyperface of $[K]$, a **face** of $[K]$.

1.2.1. **THEOREM:** (Hurewicz and Wallman) A connected $n$-dimensional manifold cannot be disconnected by a subset of inductive dimension $n-2$.

Hence $G$ cannot be disconnected by

\[
[K] \setminus \bigcup_{i=1}^{m} F_i
\]
Remark: (Hurewicz and Wallman) Let $U$ be an open set in an $k$-dimensional manifold which is neither empty nor dense, and let $B$ be the boundary of $U$. Then the inductive dimension of $B$ is $k-1$.

1.2.2. Let $K^1$ and $K^{-1}$ be any open sets which partition $G \setminus [K]$. Thus every connected component of $G \setminus [K]$ lies entirely in $K^1$ or $K^{-1}$. Then call the ordered triple $([K], K^1, K^{-1})$ a quasigraph and denote it by $K$. In particular call (1.2.2-1) $(\emptyset, G, \emptyset)$ and $(\emptyset, \emptyset, G)$ the void quasigraphs. We have (1.2.2-2) $G = [K] \cup K^1 \cup K^{-1}$ and $[K] = \mathcal{F}^{K^1} \cap \mathcal{F}^{K^{-1}}$.

We say $K$ decomposes $G$ if both $K^1$ and $K^{-1}$ are non-void. $K$ decomposes $G$ at a point $Q$ if $K^1 \cap N \neq \emptyset$ and $K^{-1} \cap N \neq \emptyset$ for every neighbourhood $N$ of $Q$.

† I.e., both $U$ and its complement contain a non-empty set.
If \( K = ( [K], K^1, K^{-1} ) \) is a quasigraph, then so is \( L = ([K], K^{-1}, K^1) \) and we call \( K \) and \( L \) opposite quasigraphs. If \( K \) is a quasigraph which does not decompose \( G \) at any point of \( [K] \) then call \( K \) a degenerate quasigraph. Note that if \( [K] \) has no faces then \( K \) is degenerate.

If \( F \) is a face of \( [K] \), then we say \( F \) is a face of \( K \).

1.3. If \( K \) decomposes \( G \) at one point of a face \( F \), then it will do so at every point of \( F \).

Proof. Let \( K \) decompose \( G \) at a point \( P \) of a face \( F \). Let \( N \) be a connected neighbourhood of \( P \) such that \( N \cap [K] \setminus F = \emptyset \). Thus \( F \) separates \( N \) into two open connected sets \( N_1 \) and \( N_2 \) and \( N \cap F \) is part of the boundary of both \( N_1 \) and \( N_2 \). Thus every point \( Q \) of \( N \cap F \) is a point of \( N_1 \) and \( N_2 \). Since \( N_1 \) and \( N_2 \) are connected open sets of \( G \setminus [K] \), we have that \( N_1 \) is completely contained in \( K^1 \) or \( K^{-1} \) and \( N_2 \) is completely contained in \( K^1 \) or \( K^{-1} \). But \( N \setminus F = N_1 \cup N_2 \) meets both \( K^1 \) and \( K^{-1} \). Thus, \( N \subseteq K^1 \) and \( N_2 \subseteq K^{-1} \) or \( N_2 \subseteq K^1 \) and \( N_1 \subseteq K^{-1} \). Hence every neighbourhood \( M \) of a point \( Q \in N \cap F \) meets both \( N_1 \) and \( N_2 \); hence \( M \) meets both \( K^1 \) and \( K^{-1} \).

Thus every point of \( N \cap F \) decomposes \( G \). Thus we have that for any decomposing point of \( F \) there is an open neighbourhood in \( F \) which consists of decomposing points of \( G \). Since \( F \) is connected any two points in \( F \) can be joined by a simple chain of open sets in \( F \). Using remark 3 in 1.1, we have that every point in \( F \) decomposes \( G \).
If K decomposes G at one point of a face F, then it will do so at every point of F. We then call F odd. Any non-odd face is called even.

1.4: K decomposes G at a singular point Q of K if and only if Q is in the closure of at least one odd face.

Proof. Let Q be in the closure of an odd face F of K. Then every neighbourhood N of Q meets F. Hence N is a neighbourhood of some point of F. Hence N meets both \(K^1\) and \(K^{-1}\). Thus K decomposes G at Q.

Let K decompose G at a singular point Q of K. Assume that \(\bar{Q}\) is not in the closure of an odd face of K. Then either there is a neighbourhood N of Q such that N meets no faces of K or Q is in the closure of even faces of K.

If there is a neighbourhood N of Q such that N meets no face of K, then \(N \cap [K]\) has no points of dimension n-1. Thus by 1.2.1, N cannot be separated by [K], that is K does not decompose G at Q. Contradiction.

Let Q be in the closure of even faces of K. Let N be a connected neighbourhood of Q such that N meets no odd faces of K. Let S be the set of singular points of [K] in N. By 1.2.1, \(N \setminus S\) is connected. Thus any two points P, R in \(N \setminus S\) can be joined by an arc. Let \(P \in K^\alpha\). Let P be joined to R by an arc A which does not pass through any singular points of N. Since A does not pass through any odd face, \(R \in K^\alpha\). Hence \(N \setminus S \subset K^\alpha \cup [K]\) and \(N \subset K^\alpha \cup [K]\). Thus, \(N \cap K^{-\alpha} = \emptyset\); a contradiction since N is a neighbourhood of Q.
1.5. The open sets \( K^1 \) and \( K^{-1} \) being disjoint we have

\[
(1.5-1) \quad \overline{K^\alpha} \cap K^{-\alpha} = \emptyset
\]

and

\[
(1.5-2) \quad \overline{K^\alpha} \cap \text{int } K^{-\alpha} = \emptyset; \quad \alpha = \pm 1.
\]

Proof. Let \( P \in \overline{K^\alpha} \cap K^{-\alpha} \). Thus \( P \in \overline{K^\alpha} \) and \( P \in K^{-\alpha} \). Let \( N \) be a neighbourhood of \( P \). Then \( N \cap K^\alpha \neq \emptyset \).

As \( N \) is a neighbourhood of \( P \), then \( N \cap K^{-\alpha} \) is a neighbourhood of \( P \). Thus, we have that \( N \cap K^{-\alpha} \cap K^\alpha \neq \emptyset \); a contradiction since \( K^{-\alpha} \) and \( K^\alpha \) are disjoint.

By \((1.5-1)\), \( \overline{K^\alpha} \cap K^{-\alpha} = \emptyset \). Hence \( K \cap \text{int } K^{-\alpha} = \emptyset \).

By the argument in the proof of \((1.5-1)\), \( \overline{K^\alpha} \cap \text{int } K^{-\alpha} = \emptyset \).

By \((1.5-1)\) and \((1.2.2-2)\), i.e., \( \overline{K^\alpha} \subset \overline{K^{-\alpha}} \subset K^\alpha \cup [K] \), we have

\[
(1.5-3) \quad \overline{K^\alpha} \subset K^\alpha \cup [K].
\]

1.5.1. We have

\[
G = K^1 \cup K^{-1}
\]

Proof. Every neighbourhood of a point in \([K]\) contains points of \( K^1 \) or \( K^{-1} \). Hence \([K] \subset \overline{K^1} \cup \overline{K^{-1}} \). Also,

\[
G = [K] \cup K^1 \cup K^{-1}
\]

\[
\subset \overline{K^1} \cup \overline{K^{-1}} \cup \overline{K^1} \cup \overline{K^{-1}}
\]

\[
\subset \overline{K^1} \cup \overline{K^{-1}}.
\]
By (1.5-3) \( \overline{K^a} \subset K^a \cup \overline{[K]} \).

Thus \( \overline{K^1} \cup \overline{K^{-1}} \subset K^1 \cup K^{-1} \cup [K] = G \).

Hence \( G = \overline{K^1} \cup \overline{K^{-1}} \).

Since \( G = \overline{K^1} \cup \overline{K^{-1}} \),
i.e. \( \mathcal{F} \overline{K^a} \subset \overline{K^{-a}} \), \( a = \pm 1 \),
we have i.e. \( \mathcal{F} \overline{K^a} \subset \text{int} \overline{K^{-a}} \).

Hence i.e. \( \mathcal{F} \overline{K^a} \subset \text{int} \overline{K^{-a}} \).

(1.5-4) \( G = \overline{K^a} \cup \text{int} \overline{K^{-a}} \), \( a = \pm 1 \).

1.6. Define

(1.6-1) \( [\tilde{K}] = \overline{K^1} \cap \overline{K^{-1}} \)

and

(1.6-2) \( K^a = \text{int} \overline{K^a} \).

1.6.1. We have \( [\tilde{K}] \subset [K] \).

Proof. \( \overline{K^1} \cap \overline{K^{-1}} \subset (K^1 \cup [K]) \cap (K^{-1} \cup [K]) \)
\( \subset (K^1 \cap K^{-1}) \cup [K] \)
\( = [K] \).

1.6.2. We have \( K^a \subset \tilde{K}^a \subset (K^a \cup [K]) \).

Proof. \( K^a \subset \tilde{K}^a \). Thus \( \text{int} K^a \subset \text{int} \tilde{K}^a = \tilde{K}^a \).

Hence \( K^a \subset \tilde{K}^a \). \( \tilde{K}^a = \text{int} \tilde{K}^a \subset \text{int} (K^a \cup [K]) \subset K^a \cup [K] \).

1.6.3. We have \( [K] = \text{bd} \overline{K^1} \cap \text{bd} \overline{K^{-1}} \).
Proof. Using (1.5-1),
\[ [\tilde{K}] = \tilde{K}^1 \cap \tilde{K}^{-1} \]
\[ = \tilde{K}^1 \cap \tilde{K}^{-1} \cap \tilde{F}_K \cap \tilde{F}_{K^{-1}} \]
\[ = \tilde{K}^1 \cap \tilde{K}^{-1} \cap \tilde{F}_K \cap \tilde{F}_{K^{-1}} \]
\[ = \text{bd } K^1 \cap \text{bd } K^{-1} . \]

1.6.4. \([\tilde{K}]\) is the union of the closures of the odd faces of \(K\).

Proof. Let \(F\) be an odd face of \([K]\). Let \(P\) be any point in \(\bar{F}\). For any neighbourhood \(N\) of \(P\), \(N \cap K^\alpha \neq \emptyset, \alpha = \pm 1\). Hence \(P\) is in the closure of \(K^\alpha\); i.e. in \(\bar{K^\alpha}\). Thus, \(P \in \bar{K}^1 \cap \bar{K}^{-1}\) for all \(P\) in \(\bar{F}\); i.e. \(\bar{F} \subset \bar{K}^1 \cap \bar{K}^{-1}\).

Conversely, let \(P \in \bar{K}^1 \cap \bar{K}^{-1}\). Hence for any neighbourhood \(N\) of \(P\), \(N \cap K^\alpha \neq \emptyset, \alpha = \pm 1\). As \(P\) is also in \([K]\), it is decomposing. Hence \(P\) is in the closure of an odd face of \([K]\).

1.7. Now \([K]\) is a closed subset of \(G\) and \([\tilde{K}] \subset [K]\), while \(\tilde{K}^1\) and \(\tilde{K}^{-1}\) are open. Every point of \(G\) belongs to one and only one of the three sets
\([\tilde{K}], \tilde{K}^1, \tilde{K}^{-1}\)
i.e. \(G = [\tilde{K}] \cup \tilde{K}^1 \cup \tilde{K}^{-1}\) with \([\tilde{K}], \tilde{K}^1, \tilde{K}^{-1}\) disjoint.

Proof. \([\tilde{K}] \cap \tilde{K}^\alpha = \tilde{K}^1 \cap \tilde{K}^{-1} \cap \text{int } \tilde{K}^\alpha = \emptyset\), since by (1.5-2), \(\tilde{K}^{-\alpha} \cap \text{int } \tilde{K}^\alpha = \emptyset\). Also
\(\tilde{K}^\alpha \cap \tilde{K}^{-\alpha} = \text{int } \tilde{K}^\alpha \cap \text{int } \tilde{K}^{-\alpha} = \emptyset\).

Finally using (1.5-4), we have
\[ \tilde{K} \cup \tilde{K}^1 \cup \tilde{K}^{-1} \]
\[ = (\tilde{K}^1 \cap \tilde{K}^{-1}) \cup \text{int} \tilde{K}^{-1} \cup \text{int} \tilde{K}^1 \]
\[ = (\tilde{K}^1 \cup \text{int} \tilde{K}^{-1} \cup \text{int} \tilde{K}^1) \cap (\tilde{K}^{-1} \cup \text{int} \tilde{K}^1 \cup \text{int} \tilde{K}^{-1}) \]
\[ = (G \cup \text{int} \tilde{K}^1) \cap (G \cup \text{int} \tilde{K}^{-1}) \]
\[ = G. \]

Hence the triple \( \tilde{K} = ([\tilde{K}], \tilde{K}^1, \tilde{K}^{-1}) \) is a quasigraph. We call \( \tilde{K} \) the reduced quasigraph of \( K \).

**Remark.** \( \tilde{K} \) decomposes \( G \) at \( Q \) if and only if \( K \) does.

**Remark.** If \( K \) is a degenerate quasigraph, then \( \tilde{K} \) is a void quasigraph.

1.8. Let \( \tilde{K} \) and \( \tilde{L} \) be reduced quasigraphs and let \([\tilde{K}] = [\tilde{L}]\). Then \( \tilde{K} \) and \( \tilde{L} \) are equal or opposite.

**Proof.** Let \( P \) lie on a face \( F \) of \([\tilde{K}] = [\tilde{L}]\). For every neighbourhood \( N \) of \( P \), \( N \cap \tilde{K}^\alpha \neq \emptyset \) and \( N \cap \tilde{L}^\alpha \neq \emptyset \), \( \alpha = 1 \). Choose \( N \) sufficiently small that \( N \) meets no other face and \( N \cap \tilde{K}^\alpha \) and \( N \cap \tilde{L}^\alpha \) are connected. Then either the connected component of \( \tilde{K}^\alpha \) which contains \( N \cap \tilde{K}^\alpha \) is equal to the connected component of \( \tilde{L}^\alpha \) which contains \( N \cap \tilde{L}^\alpha \) or it is equal to the connected component of \( \tilde{L}^{-\alpha} \) which contains \( N \cap \tilde{L}^{-\alpha} \).

If \( N \cap \tilde{K}^\alpha = N \cap \tilde{L}^\alpha \), then as one travels in \( G \) along any arc not passing through any singular points of \([\tilde{K}]\), the moving point remains both in \( \tilde{K}^\alpha \) and \( \tilde{L}^\alpha \) unless the arc crosses a face of \([\tilde{K}]\), in which case it moves into \( \tilde{K}^{-\alpha} \) and \( \tilde{L}^{-\alpha} \) simultaneously. The case \( N \cap \tilde{K}^\alpha = N \cap \tilde{L}^{-\alpha} \) is similar. Hence \( \tilde{K} \) and \( \tilde{L} \) are
Remark. Every face of $\tilde{K}$ is odd and is the union of singular points and odd faces of $K$. Conversely, every odd face of $K$ is contained in some face of $\tilde{K}$.

Remark. The set of singular points of $\tilde{K}$ is a (possibly improper) subset of the set of singular points of $K$.

1.9. $\tilde{K}^{-1} = \Phi_{\tilde{K}^{-1}}$ or equivalently $\tilde{K}^{-1} = \Phi_{\tilde{K}^{-1}} = [\tilde{K}] \cup \tilde{K}^{-1}$.

Proof. By (1.6-1), (1.6-2) and (1.5-4),

$$[\tilde{K}] \cup \tilde{K}^{-1} = (\tilde{K}^{-1} \cap \tilde{K}^{-1}) \cup (\text{int } \tilde{K}^{-1})$$

$$= (\tilde{K}^{-1} \cup \text{int } \tilde{K}^{-1}) \cap (\tilde{K}^{-1} \cup \text{int } \tilde{K}^{-1})$$

$$= \tilde{K}^{-1} \cap G$$

$$= \tilde{K}^{-1}.$$

Taking compliments, $\Phi_{\tilde{K}^{-1}} = \Phi([\tilde{K}] \cup \tilde{K}^{-1}) = \tilde{K}^{-1}$.

1.10. Let $K$ be any quasigraph and $S$ be any open set in $G$. If $[\tilde{K}] \cap S \neq \emptyset$, then $S \cap K^\alpha \neq \emptyset$, $\alpha = \pm 1$.

Proof. Let $P \in [\tilde{K}] \cap S$. Then $P$ is in the closure of an odd face of $K$ and hence $K$ decomposes $G$ at $P$. Since $S$ is open, there is a neighbourhood $N$ of $P$ contained in $S$. Hence $N \cap K^\alpha \neq \emptyset$ for $\alpha = \pm 1$. Hence $S \cap K^\alpha \neq \emptyset$ for $\alpha = \pm 1$. 
1.11. Let $\tilde{K}$ and $\tilde{L}$ be reduced-quasigraphs such that $[\tilde{K}]$ and $[\tilde{L}]$ are homeomorphic and $[\tilde{K}] \subset [\tilde{L}]$. Then $[\tilde{K}] = [\tilde{L}]$.

Proof. Since $[\tilde{K}]$ and $[\tilde{L}]$ are homeomorphic, $\tilde{K}$ and $\tilde{L}$ have the same finite number of $k$-hyperfaces, $0 \leq k \leq n-1$. If $F$ is a $k$-hyperface of $\tilde{K}$, then $F$ is contained in a $k$-hyperface $F'$ of $\tilde{L}$.

Assume $F$ is a proper subset of $F'$. Then since $F'$ is connected, $F'$ contains a boundary point of $F$. But if $P$ is a boundary point of $F$ in $F'$ then the inductive dimension of $P$ is $k-1$. Hence $F'$ contains a point of inductive dimension $k-1$. Contradiction. Hence $F = F'$.

Hence every $k$-hyperface of $\tilde{L}$ is a $k$-hyperface of $\tilde{K}$. Thus $[\tilde{K}] = [\tilde{L}]$. Since $\tilde{K}$ and $\tilde{L}$ are reduced, then $\tilde{K}$ and $\tilde{L}$ are equal or opposite.

1.12. Let $\tilde{K}$ be the reduced quasigraph of $\tilde{K}$. Then $\tilde{K} = \tilde{K}$.

Proof. By 1.6.1, $[\tilde{K}] \subset [\tilde{K}]$, and by (1.6-1) and 1.6.2, $[\tilde{K}] = K^{1} \cap K^{-1} \subset K^{1} \cap K^{-1} = [\tilde{K}]$. Hence $[\tilde{K}] = [\tilde{K}]$.

By 1.8, $\tilde{K}$ and $\tilde{K}$ are equal or opposite.

By 1.5.2, $\tilde{K}^{\alpha} \subset \tilde{K}^{\alpha}$ and by (1.6-2) and (1.5-3), $\tilde{K}^{\alpha} = \text{int} \tilde{K}^{\alpha} \subset \tilde{K}^{\alpha} \subset \tilde{K}^{\alpha} \cup [\tilde{K}]$. But $\tilde{K}^{\alpha} \cap [\tilde{K}] = \emptyset$. Thus by the above, i.e., $[\tilde{K}] = [\tilde{K}]$ and $\tilde{K}^{\alpha} \cap [\tilde{K}] = \emptyset$. Hence $\tilde{K}^{\alpha} \subset \tilde{K}^{\alpha}$. Therefore $\tilde{K}^{\alpha} = \tilde{K}^{\alpha}$.

This gives the desired results.
CHAPTER II

2. Convergence of sequences of quasigraphs.

2.1. Let \( \{S_i\} \) be a sequence of non-empty subsets of \( G, \ i = 1,2,3,\ldots \).

Define a point of \( P \) to be an accumulation point of \( \{S_i\} \) if every neighbourhood of \( P \) contains points of \( S_i \) for infinitely many \( i \).

Define a point \( P \) to be a limit point of \( \{S_i\} \) if every neighbourhood of \( P \) contains points of \( S_i \) for all but a finite number of \( i \).

Define \( \lim \sup S_i \) to be the set of all accumulation points of \( \{S_i\} \).

Define \( \lim \inf S_i \) to be the set of all limit points of \( \{S_i\} \).

Note that \( \lim \inf S_i \subset \lim \sup S_i \).

We say that \( \{S_i\} \) converges if \( \lim \inf S_i = \lim \sup S_i \) and write \( \lim S_i = \lim \inf S_i = \lim \sup S_i \).

We say that \( \{S_i\} \) converges to \( S \) if \( \{S_i\} \) converges and \( \lim S_i = S \).
2.1.1. \( \limsup S_i \) and \( \liminf S_i \) are closed sets.

Proof. Let \( P \) be a point in \( \limsup S_i \). Then every
neighbourhood \( N \) of \( P \) contains points of \( \limsup S_i \).
Hence \( N \) is a neighbourhood of some point in \( \limsup S_i \).
Thus \( N \) contains points of \( S_i \) for infinitely many \( i \).
Thus \( P \in \limsup S_i \). Hence \( \limsup S_i \) is closed.

A similar proof holds for \( \liminf S_i \).

2.1.2. Let \( K \) be a quasigraph, \( K = ([K], K^1, K^{-1}) \).

We say a sequence of quasigraphs \( \{K_i\} \) converges to \( K \)
if \( [K_i] \) converges to \( [K] \) and \( \|K_i^\alpha\| \) converges to \( \|K^\alpha\|, \alpha = \pm 1 \).

2.2. Let \( \{ K_s \mid s \in I = (0,1) \} \) be a family of
quasigraphs. We define \( K_t \) to be continuous at \( s \) in \( I \) if
and only if for any sequence \( s_i \in I \) which converges to \( s \)
we have that \( K_{s_i} \) converges to \( K_s \).

Write \( \lim_{t \to s} K_t = K_s \) if and only if \( K_t \) is continuous at \( s \).

We study a family of quasigraphs
\( \{ K_s \mid s \in I = (0,1) \} \)

where \( K_s \) depends continuously on \( s \).
2.3. If \( P \in K_s^\alpha \) then \( P \in K_t^\alpha \) for all \( t \) near \( s \). Thus if \( P \in K_s^\alpha \) then there exists an open interval \( J \) in \( I \) about \( s \) such that \( P \in K_t^\alpha \) for all \( t \in J \).

Proof. Let \( P \in K_s^\alpha \). Suppose it is false that \( P \in K_t^\alpha \) for all \( t \) near \( s \). Then there is a sequence \( t_i \) converging to \( s \), such that \( P \in K_{t_i}^\alpha \) for all \( i \) sufficiently large. Then \( P \) is an accumulation point of the sequence \( K_{t_i}^\alpha \). Thus \( P \in K_s^\alpha \). Contradiction.

2.4. Let \( J \) be an open subinterval of \( I = (0,1) \).

If \( P \notin \bigcup_{s \in J} [K_s^\alpha] \) then there is an \( \alpha \in \{1, -1\} \) such that \( P \in K_s^\alpha \) for all \( s \in J \).

Proof. Let \( J_\alpha = \{ s \in J \mid P \in K_s^\alpha \}, \alpha = \pm 1 \). Then \( J_1 \) and \( J_{-1} \) are open. Since \( J \) is connected, \( J_1 \) or \( J_{-1} \) is void. Hence \( P \in K_s^1 \) for all \( s \in J \) or \( P \in K_s^{-1} \) for all \( s \in J \).

2.4.1. Corollary. Let \( J = (s_1, s_2), \overline{J} \subset I \). Then

\[
(K_{s_1}^1 \cap K_{s_2}^{-1}) \cup (K_{s_1}^{-1} \cap K_{s_2}^1) \subset \bigcup_{s \in J} [K_s^\alpha].
\]

Proof. Let \( P \in K_{s_1}^\alpha \cap K_{s_2}^{-\alpha}, \alpha \in \{1, -1\} \). Suppose \( P \notin \bigcup_{s \in J} [K_s^\alpha] \). Then by 2.3 and 2.4, \( P \in K_s^\alpha \) for all \( s \) in \( J \).

Hence \( P \notin K_{s_2}^{-\alpha} \) for all \( s \) in \( J \). Since \( K_s \) is continuous at \( s_2 \), i.e. \( \lim_{s \to s_2} K_s^{-\alpha} = K_{s_2}^{-\alpha} \), we have \( P \notin K_{s_2}^{-\alpha} \). But \( P \in K_{s_2}^{-\alpha} \). Hence we obtain a contradiction.
2.4.2. Corollary. Let $J = (s_1, s_2), \overline{J} \subset I$. Then

$$(K_{s_1}^{-1} \cap K_{s_2}^{-1}) \cup (K_{s_1}^{-1} \cap K_{s_2}^{1}) \subset \bigcup_{s \in J} [K_s] \setminus \bigcap_{s \in J} [K_s].$$

Proof. Let $P \in [K_s]$ for all $s \in J$. Then $P \notin K_s^{\alpha}$, $\alpha = \pm 1$, for all $s \in J$. Hence $P \notin \mathcal{K}_s^{\alpha}$ for all $s \in J$ and we obtain $P \in \mathcal{K}_{s_1}^{\alpha}$. Thus $P \notin K_{s_1}^{-\alpha} \cap K_{s_2}^{-\alpha}$. 
CHAPTER III

3. Families of Quasigraphs.

3.1. Let $I = (0,1)$. In the following we study families

$$\alpha = \{ K_s \mid s \in I \}$$

of quasigraphs with the following properties.

There exists a quasigraph $K$ and a continuous map

$$H : G \times I \rightarrow G$$

such that, for each $s, H \big|_{G \times s}^{G\times s}$ is a homeomorphism satisfying

$$H([K] \times s) = [K_s], \quad H(K^{-1} \times s) = K^{-1}_s;$$

hence $H(K^{-1} \times s) = K^{-1}_s$.

Remark. $H$ is an open mapping. Thus the map

$$\tilde{H} : G \times I \rightarrow G \times I,$$

defined by $\tilde{H}(x,t) = (H(x,t),t)$ is an open mapping. Since $\tilde{H}$ is also bijective and continuous, we have that $\tilde{H}$ is a homeomorphism.

Remark. Since $H \big|_{G \times s}^{G \times s}$ is a homeomorphism, it maps each $k$-hyperface of $K$ onto a $k$-hyperface of $K_s$. If $F$ is a $k$-hyperface of $K$, and $Q \in G$, write $F_s = H(F,s)$, $Q_s = H(Q,s)$.
Remark. If \( J = [s, t] \) is a closed subinterval of \( I \), and \( R \) is an interior point of \( G \), then \((H \mid_{G \times J})^{-1}(R)\) is a Jordan arc in \( G \times J \) whose endpoints lie in \( \text{int} \sigma(G \times \{s\}) \) and \( \text{int} (G \times \{t\}) \) and which does not meet the boundary of \( G \times J \) elsewhere.

3.2. Given \( \alpha \) satisfying 3.1, the reduced family
\[
\tilde{\alpha} = \{ \tilde{K}_s \mid s \in I, K_s \in \alpha \}
\]
also satisfies 3.1.

Proof. Let \( \tilde{K} \) be the reduced quasigraph of \( K \).

Since \( H \mid_{G \times s} \) is a homeomorphism we have

\[
[\tilde{K}_s] = \overline{K_s} \cap \overline{K_s}^{-1} = H(K_s^1 \times s) \cap H(K_s^{-1} \times s) = H((K_s^1 \times s) \cap (K_s^{-1} \times s)) = H((\tilde{K} \cap \tilde{K}^{-1}) \times s).
\]

We also have
\[
\tilde{K}_s^\alpha = \text{int} \overline{K_s^\alpha} = \text{int} H(K_s^\alpha \times s) = \text{int} H(K_s^\alpha \times s) = \text{int} H(K_s^\alpha \times s) = H((\text{int} \tilde{K}^\alpha) \times s) = H(\tilde{K}^\alpha \times s).
\]
3.3. \( \mathcal{A} = \{ K_s \mid s \in I \} \) is a continuous family of quasigraphs, i.e. \( K_t \) is continuous at \( s \) for each \( s \in I = (0,1) \).

Proof.

Claim 1: \( [K_s] \subseteq \limsup \{ [K_{s_i}] \} \).

Let \( P \) be any point in \( [K_s] \). Then \( P = H(Q,s) \) where \( Q \in [K] \). Let \( \{s_i\} \), \( i = 1,2,\ldots, \) be a sequence in \( I \) converging to \( s \). Then \( (Q,s_i) \) is a sequence of points in \( G \times I \) converging to \( (Q,s) \). Since \( H \) is continuous, \( H(Q,s_i) \) is a sequence of points in \( G \) converging to \( H(Q,s) \). But \( Q_{s_i} = H(Q,s_i) \in H([K] \times s_i) = [K_{s_i}] \). Hence every neighbourhood of \( P \) contains \( Q_{s_i} \) for all but a finite number of \( s_i \). Hence every neighbourhood of \( P \) meets \( [K_{s_i}] \) for all but a finite number of \( s_i \). Thus \( P \) is a limit point of \( \{ [K_{s_i}] \} \). Hence \( [K_s] \subseteq \liminf \{ [K_{s_i}] \} \subseteq \limsup \{ [K_{s_i}] \} \).

Claim 2: \( \limsup \{ [K_{s_i}] \} \subseteq [K_s] \). Let \( P \in \limsup \{ [K_{s_i}] \} \).

By choosing a suitable subsequence of \( \{s_i\} \), we may assume that there is a sequence \( \{P_i\} \) of points \( P_i \in [K_{s_i}] \) such that \( \{P_i\} \) converges to \( P \). For each \( i \) there exists a unique point \( Q_i \in [K] \subseteq G \) such that \( H(Q_i,s_i) = P_i \) and there exists a unique point \( Q \in G \) such that \( H(Q,s) = P \). Let \( J \) be a closed subinterval of \( I, s \in \text{int } J \). The map \( \tilde{H} : G \times J \to G \times J \), defined by \( \tilde{H}(x,t) = (H(x,t),t) \), is a homeomorphism. Let \( s_i \in J \). Then \( (P_i,s_i) = \tilde{H}(Q_i,s_i) \) and \( (P,s) = \tilde{H}(Q,s) \). Since \( (P_i,s_i) \) converges to \( (P,s) \), we have that \( (Q_i,s_i) \) converges to \( (Q,s) \). Since \( [K] \times J \) is closed in \( G \times J \) and
$(Q_i, s_i) \in [K] \times J$, we obtain $(Q, s) \in [K] \times J$ and thus $(Q, s) \in [K] \times s$. Finally, $P = H(Q, s) \in H([K] \times s) = K_s$.

**Claim 3:** $\mathcal{K}_s \subseteq \limsup \{ K_{s_i} \}$.

**Claim 4:** $\limsup \{ K_{s_i}' \} \subseteq \mathcal{K}_s$.

The proofs of claims 3 and 4 are similar to those of claims 1 and 2, respectively. From claims 1 to 4, $K_t$ is continuous at $s$; thus $\mathcal{A}$ is a continuous family of quasigraphs.

3.3.2. **Corollary.** By 3.2, $\{ \mathcal{K}_s | s \in I \}$ is a continuous family of quasigraphs.

**Remark.** Let $F$ be a $k$-hyperface of $K$ and let $F_s$ denote the corresponding $k$-hyperface of $K_s$. Then by 2.1.1, and the continuity of $H$ restricted to $F \times I$, we have that the closure $\overline{F}_s$ of $F_s$ depends continuously on $s$.

3.4. Let $M = \bigcap_{s \in I} [K_s]$ and $\bar{M} = \bigcap_{s \in I} [\mathcal{K}_s]$.

We assume:

3.4.1. If $s \neq t$, then $K_s \neq K_t$ and $[K_s] \cap [K_t] = M$.

3.4.2. Either $[\mathcal{K}_s] = \bar{M}$ for all $s \in I$

or if $s \neq t$ then $\mathcal{K}_s \neq \mathcal{K}_t$ and $[\mathcal{K}_s] \cap [\mathcal{K}_t] = \bar{M}$. 
The following example shows that 3.4.1 does not imply 3.4.2.

Let \( G \) be open unit ball in \( \mathbb{R}^3 \).

Let \( F_1 = \{(x,y,0) \mid -1 < x < 0\} \),
\( F_2 = \{(x,y,0) \mid 0 < x < 1\} \),
\( F_3 = \{(0,y,z) \mid 0 < z < 1\} \),
\( E = \{(0,y,0) \mid -1 < y < 1\} \),

where \( F_1, F_2, F_3 \) are 2-hyperfaces (faces). \( E \) is a 1-hyperface.

Let \([K]\) be the union of \( F_1, F_2, F_3 \) and \( E \).

Let \( K^1 = \{(x,y,z) \in G \mid x < 0 \text{ or } z < 0\} \)
\( K^{-1} = \{(x,y,z) \in G \mid x > 0, z > 0\} \).

Then \( K = ([K], K^1, K^{-1}) \) is a quasigraph in \( G \). Define \( K_s \) by sliding \( E \) on the xy plane from \((-\frac{1}{2},y,0)\) to \((\frac{1}{2},y,0)\) moving \( F_3 \) parallel to itself, expanding \( F_1, s \) and shrinking \( F_2, s \). Then

\([K_s] \cap [K_t] = M = \{(x,y,0) \mid -1 < x < 1, -1 < y < 1\} \). But
\([\tilde{K}_s] \cap [\tilde{K}_t] \neq \tilde{M} = \{(x,y,0) \mid \frac{1}{2} < x, -1 < y < 1\} \).

3.4.3. \( \tilde{K}_s = \tilde{M} \) for all \( s \in I \) or \( \tilde{K}_s \neq \tilde{M} \) for all \( s \in I \).

**Proof.** Let \( \tilde{K}_s = \tilde{M} \) for some \( s \in I \). Then \( \tilde{K}_s = \tilde{M} \)
for all \( s \in I \) or if \( s \neq t \) then \( \tilde{K}_s \neq \tilde{K}_t \) and \( [\tilde{K}_s] \cap [\tilde{K}_t] = \tilde{M} \).
Thus \( [\tilde{K}_s] \subset [\tilde{K}_t] \). By 1.11, \( [\tilde{K}_s] = [\tilde{K}_t] \). Hence \( [\tilde{K}_s] = \tilde{M} \)
for all \( s \in I \). Hence assertion is proven.
3.5. Suppose a singular point of $K_s$ is such that any small neighbourhood of it is disconnected by $[K_s]$ into at least three open sets. Then it lies in $M$. In particular every singular point of $\tilde{K}_s$ in the interior of $G$ belongs to $\tilde{M}$.

Proof. Let $Q_s = H(Q,s)$ be a singular point of $K_s$ such that any small neighbourhood of it is disconnected by $[K_s]$ into at least three open sets. Assume that $Q_s \notin M$. Then there is a neighbourhood $N$ of $Q_s$ such that $N \cap M = \emptyset$. Choose $N$ so small that its closure does not meet $M$ or any face of $K_s$ which does not have $Q_s$ in its closure.

Let $\bigcup_{i=1}^{j} (\text{bd } B_{i,s}) = N \cap [K_s]$ and $Q_s \in \bigcap_{i=1}^{j} (\text{bd } B_{i,s})$.

Let $B_{i,u} = H(B_{i,u}) \cap N$. By 2.3, $B_{i,u} \cap B_{i,s} \neq \emptyset$ for $u$ close to $s$. Since the $B_{i,u}$'s and $B_{i,s}$'s are connected and their boundaries of $B_{i,s}$ and $B_{i,u}$ cannot meet we have that $B_{i,s} \subset B_{i,u}$ and $\text{bd } B_{i,s} \subset B_{i,u}$ or $B_{i,u} \subset B_{i,s}$ and $\text{bd } B_{i,u} \subset B_{i,s}$.

Let $t > s$. Then $Q_t \neq Q_s$. Since $H$ is continuous there exists a $t > s$, close to $s$, such that $Q_u \in N$ for all $u$ with $s < u \leq t$. Now $Q_u \notin M$. Thus $Q_u \notin N \cap [K_s]$. Hence $Q_s$ lies in one and only one of the sets $B_{i,s}$. Let $Q_u \in B_{m,s}$, $m \in \{1, \ldots, j\}$. Then $\text{bd } B_{i,u} \subset B_{m,s}$ and $B_{i,u} \subset B_{m,s}$ for all $i \neq m$. $Q_s \notin M$. Thus $Q_s$ lies in one and only one of the sets $B_{i,u}$. 
Let $Q_s \in B_{n,u} ; n \in \{1, \ldots, j\}$. Then $bd B_{i,s} \subseteq B_{n,u}$ for $i \neq n$, and $B_{i,s} \subseteq B_{n,u}$ for all $i \neq n$. Hence $B_{m,s} \subseteq B_{n,u} \subseteq B_{m,s}$. Hence $B_{m,s} = B_{n,u}$ and thus $bd B_{m,s} \cap bd B_{n,u} \neq \emptyset$. Contradiction.

3.6. Let $F_s$ be a face of $K_s$ and $Q_s \in F_s \setminus M$. Then there exists a neighbourhood $N'$ of $Q_s$ in $G$ and an interval $[s_1, s_2]$ containing $s$ such that $N' \subseteq \bigcup_{t \in [s_1, s_2]} F_t$.

Proof. Since $M$ is closed, $G$ regular, there is a neighbourhood $N$ of $Q_s$ such that $N \cap M = \emptyset$. Thus each point $P$ of $N$ lies on not more than one $[K_t]$. In particular every $P \in N$ lies on not more than one $F_t$.

Now since $H$ is continuous $B = H^{-1}(N)$ is open in $G \times I$. Let $F_s = H(F_s)$. Let $Q_s = H(Q,s)$. Then $(Q,s) \in B = H^{-1}(N)$. Since $G$ is locally compact there is a compact subset $A$ of $F_s \times A$, such that $A \times s \subseteq B$. Hence there are $s_1$, $s_2$ such that $s_1 < s < s_2$ and $A \times [s_1, s_2] \subseteq B$.

Let $S = A \times [s_1, s_2] \subseteq B$. Then $H(S) \subseteq N$ and $A_t = H(A \times t) \subseteq F_t \cap N$ for $s_1 < t < s_2$.

Since $H$ is an open mapping, $H(\text{int } S)$
is a non-void open set containing $Q_s$. Every point of this set lies on some $F_t$; $s_1 \leq t \leq s_2$. Thus let $N^i$ be a neighbourhood of $Q_s$ contained in $H(\text{int } S)$.

Thus $N^i \subset H(\text{int } S) \subset \bigcup_{t \in [s_1, s_2]} F_t$.

3.7. Let $F$ be a face of $\tilde{K}$. We study the restriction of $H$ to $F \times I$.

3.7.1. Let $B$ be a connected component of $\tilde{M}$ which contains a singular point $Q_s$ of $\tilde{K}_s$ in $\text{int } G$. Then $Q_t \in B$ for all $t \in I$.

Proof. As $Q_s$ is a singular point of $\tilde{K}_s$, then $Q$ is a singular point of $\tilde{K}$ and $Q_t$ is a singular point of $\tilde{K}_t$, for all $t \in I$. Now $Q_t \in \tilde{M}$ for all $t \in I$. Since $H$ is continuous the set given by \{ $Q_u$ | $Q_u = H(Q,u)$ ; $u \in I$ \} is connected and lies in $\tilde{M}$. Since $Q_s \in B$, $B$ connected we have that $Q_t \in B$ for all $t \in I$.

3.7.2. Let $B$ be a connected component of $\tilde{M}$ such that $B$ lies on a face $F_s$ of $\tilde{K}_s$. Then $B \subset F_t$ for all $t \in I$.

Proof. Let $B$ lie on a face $F_s = H(F, s)$ of $\tilde{K}_s$. Then $B$ contains no singular points of $\tilde{K}_t$ for all $t \in I$. Hence $B$ lies on a face $D(t)$ of $\tilde{K}_t$ for all $t \in I$ and $D(t) = H(F^\lambda(t), t)$.
where \( \tilde{F}^i, i = 1, \ldots, m \) is a face of \( \tilde{K} \) and \( \lambda(t) \in \{1, 2, \ldots, m\} \). Let 
\[ A_i = \{ t \in I \mid \hat{D}(t) = H(F^i_t, t) \} ; i = 1, \ldots, m. \]

Let \( F^s = F^j_s = H(F^j, s) \). Let \( P \subset B \subset F^s \). Then there exists a neighbourhood \( N \) of \( P \) such that no \( F^i_s \) meets \( N \) for \( i \neq j \). If \( t \) is sufficiently close to \( s \) no \( F^i_t, i \neq j \) will meet \( N \). Hence \( P \notin F^i_t \) for \( t \) close to \( s \). But \( P \in \tilde{M} \). Since \( P \) is a non-singular point of \( \tilde{K}_t \), \( P \) lies on some face of \( \tilde{K}_t \). But \( P \) is not an accumulation point of \( F^i_t, i \neq j \). Hence \( P \) lies on \( F^j_t \) \( \forall t \). \( P \) is an arbitrary point of \( B \). Hence \( B \subset F^j_t \) for all \( t \) close to \( s \). Hence each set 
\[ A_i \text{ is open and every } t \in I \text{ belongs to only one } A_i, \]
\( i = 1, \ldots, m \). Since \( I \) is connected, \( I \setminus A_j \) is open, and the set 
\[ A_j = \{ t \in I \mid \hat{D}(t) = H(F^j_t, t) = H(F, t) \} \] is not empty, we have that \( A_j = I \).

Hence there is a face \( F \) of \( \tilde{K} \) such that \( \hat{D}(t) = F_t \) for all \( t \in I \). Thus \( B \subset F_t \) for all \( t \in I \).

\[ \text{3.7.3. Let } B \text{ be any connected component of } \tilde{M} \text{ such that } B \cap F_s \cap \text{int } G \neq \emptyset. \text{ Then } B \cap F_t \neq \emptyset \text{ for all } t \in I. \]

**Proof.** Assume \( B \cap F_t \cap \text{int } G \neq \emptyset \). Then by 3.7.1 and 3.7.2, \( B \cap F_t \neq \emptyset \) for all \( t \in I \).
3.8. Let $F$ denote a face of $\tilde{K}$.

Let $s \in I$. Then $F_s \setminus \tilde{M}$ is an open set in $F_s$. Hence it is the union of a most countably many disjoint open subfaces of $F_s$. $F_s \setminus \tilde{M} = \bigcup_{i=1}^{\infty} R_i(s)$ where $R_i(s)$ is an open, connected subset of $F_s$ and $R_i(s) \cap R_j(s) = \emptyset$ for $i \neq j$.

Let $R(s)$ be one of these sets. Then there is an open connected subset $R$ of $F$ such that $H(Q,t) \not\subset \tilde{M}$ for all $Q \in R$ and all $t \in I$.

Let $P_s \in R(s)$. Then the curve given by

$$P(t) = \{ H(P,t) \mid t \in I \}$$

passes through $P_s$ and $P(t) \in F_t \setminus \tilde{M}$.

Define $R(t) = \{ H(P,t) \mid P \in R \}$. Then $R(t)$ is an open connected subset of $F_t \setminus \tilde{M}$. Hence $R(t) \subset R^1(t)$ where $R^1(t)$ is a connected component of $F_t \setminus \tilde{M}$.

If $R(t) \subset R^1(t)$ and $R(t) \neq R^1(t)$, then there is an open connected subset $R^1$ of $F$ such that

$H(R^1, t) = R^1(t)$ and $R \subset R^1$ with $R \neq R^1$. Hence $\overline{R} \subset \overline{R^1}$ with $\overline{R} \neq \overline{R^1}$. Thus there is an accumulation point $P$ of $R$ contained in $R^1$. But $H(P,t)$ is a point on the bd $G$ or $H(P,t) \subset \tilde{M}$. Hence $R^1(t) \cap \text{bd } G \neq \emptyset$ or $R^1(t) \cap \tilde{M} \neq \emptyset$.

Contradiction since $R^1(t) \subset F_t \setminus \tilde{M}$. Hence $R(t) = R^1(t)$. 
3.8.1. Let \( \{ s_i \}_{i \in \mathbb{N}} \) be a sequence in \( I \) converging to \( s \in I \). Then \( R(s_i) \) converges to \( R(s) \).

**Proof.** The proof of this assertion is similar to that in 3.3.1.

**Remark.** \( R(t) \) depends continuously on \( t \); i.e., for every sequence \( \{ s_i \}_{i \in \mathbb{N}} \) converging to \( s \) in \( I \), the sequence \( \{ R(s_i) \} \) converges to \( R(s) \).

3.8.2. Let \( J \) be an open interval in \( I = (0,1) \).

Then \( \bigcup_{t \in J} R(t) \) is open in \( G \).

**Proof.** \( \bigcup_{t \in J} R(t) = \bigcup_{t \in J} H(R \times t) = H(\bigcup_{t \in J} (R \times t)) = H(R \times J) \)

is open, since \( R \times J \) is open and \( H \) is an open mapping.

**Remark.** Each point of \( \bigcup_{t \in J} R(t) \) lies on exactly one of the curves,

\[
C_Q = \{ P(t) = H(Q,t) \mid t \in I \}, \quad Q \in \mathbb{R}.
\]
4. Global Decomposition

4.1. Let \( P_s \in F_s \setminus \hat{\mathfrak{M}} \). Construct the arc \( \{ P(u) \mid u \in I \} \) with \( P(s) = P_s \) as in 3.8. Then \( P(u) \in F_u \setminus \hat{\mathfrak{M}} \) for all \( u \in I \).

Let \( s < s' \). Let \( P(s') \in K_s^\alpha \).

Then

\[ P(u) \in K_s^\alpha \quad \text{for all} \quad u > s \]

and

\[ P(u) \in K_s^{-\alpha} \quad \text{for all} \quad u < s \]

Proof. The arc \( \{ P(t) \mid t > s \} \) does not meet \([\hat{K}_s]\). Hence it lies entirely in \( K_s^\alpha \) since \( P(s') \in K_s^\alpha \).

4.2. Let \( R(s) \) denote the connected component of \( F_s \setminus \hat{\mathfrak{M}} \) containing \( P(s) \). Then \( R(s) \) decomposes \( \bigcup_{t \in I} R(t) \) into two subsets, one in \( K_s^1 \), the other in \( K_s^{-1} \).

Proof. \( \bigcup_{t \in I} R(t) \) is an open subset of \( G \). Hence

\[ ( \bigcup_{t \in I} R(t)) \cap K_s^\alpha \neq \emptyset \] by 1.10. Now \( R(t) \cap R(s) = \emptyset \) for \( t \neq s \).

In fact \( R(t) \cap [\hat{K}_s] = \emptyset \), since \( R(t) \cap \hat{\mathfrak{M}} = \emptyset \). Hence

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$R(t) \subseteq \tilde{K}_s^1$ or $R(t) \subseteq \tilde{K}_s^{-1}$. By 4.1, there is an $\alpha \in \{1,-1\}$ such that $P(t) \in \tilde{K}_s^\alpha$ for all $t > s$. However, $P(t) \in R(t)$ for all $t \in I$. Hence $R(t) \in \tilde{K}_s^\alpha$ for all $t > s$ and $R(t) \in \tilde{K}_s^{-\alpha}$ for all $t < s$. Hence $R(s)$ decomposes $\bigcup_{t \in I} R(t)$ into two subsets; one in $K_s^1$, the other in $K_s^{-1}$.

4.3. Let $P_s \in F_s \setminus M$. Construct the arc $\{P(u) \mid u \in I\}$ as in 4.1. Let $t \uparrow s$. Then $P(s) = P_s \in \tilde{K}_t^\alpha$ if and only if $P(t) \in \tilde{K}_s^{-\alpha}$.

Proof. Assume $s < t$. Choose $u, v$ such that $u < s < t < v$. Let $P(s) \in \tilde{K}_t^\alpha$. Then $P(u) \in \tilde{K}_t^\alpha$. Now $P(u) \in \bigcup_{x \in (u,v)} [\tilde{K}_x]$. Hence by 2.4, $P(u) \in \tilde{K}_x^\beta$ for all $x \in (u,v)$, $\beta = \pm 1$. In particular $P(u) \in \tilde{K}_s^\beta$ and $P(u) \in \tilde{K}_t^\beta$, but $P(u) \in \tilde{K}_t^\alpha$. Hence $\alpha = \beta$. Thus $P(u) \in \tilde{K}_s^\alpha$ and since $u < s < t$ we have $P(t) \in \tilde{K}_s^{-\alpha}$.

4.4. If $t$ and $u$ lie on the same side of $s$ in $I = (0,1)$ then $[K_s] \cap K_t^\alpha = [K_s] \cap K_u^\alpha$; $\alpha = \pm 1$.

Proof. We may assume that $0 < t < u < s < 1$ and that $[K_s] \cap K_t^\alpha \neq \emptyset$. Let $P \in [K_s] \cap K_t^\alpha$. Then $P \notin [K_t]$ and hence $P \notin M$ and $P \notin [K_u]$, for all $u \neq s$; cf. 3.4.1.

Hence $P \notin [K_u]$ for all $u \in J = (0,s)$. By 2.4, $P \in K_u^\beta$ for all $u \in J = (0,s)$. Hence $P \in K_t^\alpha$ if and only if $P \in K_u^\alpha$ for all
\( u \in J. \) Since \( P \) is arbitrary in \([K_s] \cap K_t^\alpha\), this proves the result.

**Corollary.** If \( t \) and \( u \) lie on the same side of \( s \) in \( I = (0,1) \), then \( [\tilde{K}_s] \cap \tilde{K}_t^\alpha = [\tilde{K}_s] \cap [K_u] \cap K_u^\alpha \); \( \alpha = \pm 1 \).

4.5. We note:

4.5.1. \( [\tilde{K}_s] \cap K_t^\alpha = [\tilde{K}_s] \cap [K_s] \cap K_t^\alpha \)

\[ = [\tilde{K}_s] \cap [K_s] \cap K_u^\alpha \]

\[ = [\tilde{K}_s] \cap K_u^\alpha \]

is independent of \( t; t,u \in (0,s) \) or \( t,u \in (s,1) \).

4.5.2. \( [\tilde{K}_s] \cap [K_t] = [\tilde{K}_s] \cap [K_s] \cap [K_t] \)

\[ = [\tilde{K}_s] \cap M \]

is independent of \( t; t \neq s \).

4.5.3. By 4.5.2 and 3.4.2,

\[ ([\tilde{K}_s] \cap [K_t]) \setminus \tilde{K}_t = ([\tilde{K}_s] \cap [K_t]) \setminus ([\tilde{K}_s] \cap [K_t]) \]

\[ = ([\tilde{K}_s] \cap M) \setminus \tilde{M} \]

is independent of \( t, t \neq s \).

4.5.4. Let \( 0 < u < s < v < 1 \). Then by 4.4,

\( [K_u] = [K_u] \cap G = [K_u] \cap (K_s^1 \cup K_s^{-1} \cup [K_s]) \).
More generally, if \(0 < t_0 < t_1 < \ldots < t_h < 1\), then

\[
[K_{t_0}] \subset (K_{t_1}^1 \cap \ldots \cap K_{t_h}^1) \cup (K_{t_1}^{-1} \cap \ldots \cap K_{t_h}^{-1}) \cup M.
\]

In particular,

\[
[K_{t_0}] \subset (K_{t_1}^1 \cap \ldots \cap K_{t_h}^1) \cup (K_{t_1}^{-1} \cap \ldots \cap K_{t_h}^{-1}) \cup M.
\]

4.6. Let \(0 < u < s < v < 1\). Then

\[
[K_s] \subset (K_u^1 \cap K_v^{-1}) \cup (K_u^{-1} \cap K_v^1) \cup \tilde{M}.
\]

**Proof.** Let \(P_s \in [\tilde{K}_s] \setminus \tilde{M}\). If \(P_s\) is a non singular point of \([\tilde{K}_s]\), then \(P_s\) lies on some face \(F_s\) of \([\tilde{K}_s]\) and on an open connected component of \(F_s \setminus \tilde{M}\). Construct the arc \(\{P(t) \mid t \in I\}\).

Suppose \(P(s) \in K_u^\alpha\). Then by 4.1, \(P(v) \in K_u^\alpha\) and by 4.3, \(P(u) \in K_v^{-\alpha}\). Hence \(P(s) \in K_v^{-\alpha}\). Thus \(P_s = P(s) \in K_u^\alpha \cap K_v^{-\alpha}\).

Hence \(P_s \in (K_u^1 \cap K_v^{-1}) \cup (K_u^{-1} \cap K_v^1)\).

If \(P_s'\) is a singular point of \([\tilde{K}_s] \setminus \tilde{M}\), then \(P_s'\) is in the closure of a face \(F_s\) of \([\tilde{K}_s]\). But \(F_s \subset (K_u^\alpha \cap K_v^{-\alpha})\).
Hence \( P'_s \in (\tilde{K}_u^a \cap \tilde{K}_v^{-a}) \). Thus
\[
P'_s \in \tilde{K}_u^a \cap \tilde{K}_v^{-a} \subseteq [\tilde{K}_s] \setminus \tilde{M}
= (\tilde{K}_u^a \cup [\tilde{K}_u]) \cap (\tilde{K}_v^{-a} \cup [\tilde{K}_v]) \subseteq [\tilde{K}_s] \setminus \tilde{M}
\subseteq \tilde{K}_u^a \cap \tilde{K}_v^{-a}.
\]

Hence \([\tilde{K}_s] \subseteq (\tilde{K}_u^a \cap \tilde{K}_v^{-a}) \cup (\tilde{K}_u^{-1} \cap \tilde{K}_v) \cup \tilde{M}\).

4.6.1. Let \( u < v \). Then
\[
\bigcup_{u < s < v} [\tilde{K}_s] = (\tilde{K}_u^a \cap \tilde{K}_v^{-a}) \cup (\tilde{K}_u^{-1} \cap \tilde{K}_v) \cup \tilde{M}.
\]

Proof. By 4.6,
\[
\bigcup_{u < s < v} [\tilde{K}_s] \subseteq (\tilde{K}_u^a \cap \tilde{K}_v^{-a}) \cup (\tilde{K}_u^{-1} \cap \tilde{K}_v) \cup \tilde{M}.
\]

By 2.4.2, \((\tilde{K}_u^a \cap \tilde{K}_v^{-a}) \cup (\tilde{K}_u^{-1} \cap \tilde{K}_v) \subseteq \bigcup_{u < s < v} [\tilde{K}_s] \setminus \cap_{s \in I} [\tilde{K}_s].
\]

Hence \((\tilde{K}_u^a \cap \tilde{K}_v^{-a}) \cup (\tilde{K}_u^{-1} \cap \tilde{K}_v) \cup \tilde{M} \subseteq \bigcup_{u < s < v} [\tilde{K}_s]
\]

and the result follows.

4.7. If \( 0 < t_1 < \ldots < t_h < 1 \). Then
\[
[K_{t_i}] \subseteq (\bigcap_{j=1}^{i-1} K_{t_j}^1 \cap \bigcap_{j=i+1}^h K_{t_j}^{-1}) \cup (\bigcap_{j=1}^{i-1} K_{t_j}^{-1} \cap \bigcap_{j=i+1}^h K_{t_j}^1) \cup \tilde{M};
\]

\( i = 2, \ldots , h-1 \).

If we define \( \cap_{j=1}^h K_{t_j}^a = \bigcap_{j=1}^h K_{t_j}^a = G \), then 4.7
holds for \( i = 1 \) and \( i = h \).
Proof. By 4.6.1 and \( j < i < k \),

\[
\tilde{K}_{t_i} \subset (\bigcap_{j=1}^{i-1} \tilde{K}_{t_j}^{-1}) \cup (\bigcap_{t_j}^{i-1} \tilde{K}_{t_k}^{-1}) \cup \tilde{M}.
\]

Hence

\[
\tilde{K}_{t_i} \subset \bigcap_{j=1}^{i-1} [(\tilde{K}_{t_j}^{-1} \cap \tilde{K}_{t_k}^{-1}) \cup (\tilde{K}_{t_j}^{-1} \cap \tilde{K}_{t_k}^{1}) \cup \tilde{M}]
\]

and

\[
\tilde{K}_{t_i} \subset \bigcap_{k=i+1}^{h} [(\tilde{K}_{t_j}^{-1} \cap \tilde{K}_{t_k}^{-1}) \cup (\tilde{K}_{t_j}^{-1} \cap \tilde{K}_{t_k}^{1}) \cup \tilde{M}].
\]

Thus

\[
\tilde{K}_{t_i} \subset (\bigcap_{j=1}^{i-1} \tilde{K}_{t_j}^{-1}) \cup (\bigcap_{j=1}^{i-1} \tilde{K}_{t_k}^{1}) \cup \tilde{M}
\]

\[
\bigcap_{k=i+1}^{h} \tilde{K}_{t_j}^{-1} \cup \bigcap_{k=i+1}^{h} \tilde{K}_{t_k}^{1} \cup \tilde{M}
\]

\[
\subset (\bigcap_{j=1}^{i-1} \tilde{K}_{t_j}^{-1} \cap \bigcap_{k=i+1}^{h} \tilde{K}_{t_j}^{-1}) \cup (\bigcap_{j=1}^{i-1} \tilde{K}_{t_k}^{1} \cap \bigcap_{k=i+1}^{h} \tilde{K}_{t_k}^{1}) \cup \tilde{M}.
\]

If \( i = 1 \), \( \tilde{K}_{t_1} = (G \cap \bigcap_{j=2}^{h} \tilde{K}_{t_j}^{-1}) \cup (G \cap \bigcap_{j=2}^{h} \tilde{K}_{t_j}^{1}) \cup \tilde{M} \)

\[
= \bigcap_{j=2}^{h} \tilde{K}_{t_j}^{-1} \cup \bigcap_{j=2}^{h} \tilde{K}_{t_j}^{1} \cup \tilde{M}.
\]

If \( i = h \), \( \tilde{K}_{t_h} = (\bigcap_{j=1}^{i-1} \tilde{K}_{t_j}^{-1} \cap G) \cup (\bigcap_{j=1}^{i-1} \tilde{K}_{t_j}^{1} \cap G) \cup \tilde{M} \)

\[
= \bigcap_{j=1}^{h-1} \tilde{K}_{t_j}^{-1} \cup \bigcap_{j=1}^{h-1} \tilde{K}_{t_j}^{1} \cup \tilde{M}.
\]

4.7.1. Corollary. If \( K_0, K_1, \ldots, K_h \) are distinct quasigraphs of \( \mathcal{G} \) then there exists \( \alpha_i = \pm 1 \), \( i = 1, \ldots, h \), such that

\[
\tilde{K}_0 \subset (\bigcap_{i=1}^{h} \tilde{K}_{t_i}^{\alpha_i}) \cup (\bigcap_{i=1}^{h} \tilde{K}_{t_i}^{-\alpha_i}) \cup \tilde{M}.
\]
Proof. Since $K_0, \ldots, K_h \in \mathcal{A}$ there are

$0 < t_0 < t_1 < t_2 < \ldots, < t_h < 1$ such that for each

$i \in (0, 1, \ldots, h)$, $\tilde{K}_i = \tilde{K}_{t_j}$ for some $j \in (0, \ldots, h)$.

Let $\tilde{K}_0 = \tilde{K}_{t_i}$. Then

$$[\tilde{K}_0] = (\bigcap_{j=0}^{i-1} \tilde{K}_{t_j} \cap \bigcap_{j=i+1}^{h} \tilde{K}_{t_j}^{-1}) \cup (\bigcap_{j=0}^{i-1} \tilde{K}_{t_j}^{-1} \cap \bigcap_{j=i+1}^{h} \tilde{K}_{t_j}) \cup \tilde{M}$$

$$= (\bigcap_{j=1}^{h} \tilde{K}_{i}^{\alpha_i}) \cup (\bigcap_{j=1}^{h} \tilde{K}_{i}^{-\alpha_i}) \cup \tilde{M}$$

where $\alpha_i = \pm 1$, $i = 1, 2, \ldots, h$.

4.7.2. Corollary. If $K_0, K_1, \ldots, K_h$ are distinct quasigraphs of $\mathcal{A}$ and $[K_0] \cap K_1^{\alpha_1} \cap \ldots \cap K_h^{\alpha_h} \neq \emptyset$,

then $[\tilde{K}_0] \subset (\bigcap_{i=1}^{h} \tilde{K}_{i}^{\alpha_i}) \cup (\bigcap_{i=1}^{h} \tilde{K}_{i}^{-\alpha_i}) \cup \tilde{M}$.

Proof. By 4.7.1, there exist $\beta_1, \ldots, \beta_h$ such that

$$[\tilde{K}_0] \subset (\bigcap_{i=1}^{h} \tilde{K}_{i}^{\beta_i}) \cup (\bigcap_{i=1}^{h} \tilde{K}_{i}^{-\beta_i}) \cup \tilde{M}.$$ 

Thus any point $P \in [\tilde{K}_0] \setminus \tilde{M}$ lies either in $\cap_{i=1}^{h} \tilde{K}_{i}^{\beta_i}$ or in $\cap_{i=1}^{h} \tilde{K}_{i}^{-\beta_i}$. Let $P \in [\tilde{K}_0] \cap \cap_{i=1}^{h} \tilde{K}_{i}^{\alpha_i}$. Suppose $P \in \cap_{i=1}^{h} \tilde{K}_{i}^{\beta_i}$.

Then $P \in \cap_{i=1}^{h} (\tilde{K}_{i} \cap \tilde{K}_{i}^{\beta_i})$ and $P \in \tilde{K}_{i}^{\alpha_i} \cap \tilde{K}_{i}^{\beta_i} \neq \emptyset$.

Hence $\alpha_i = \beta_i$, $i = 1, \ldots, h$.  

4.7.3. If \( [\tilde{K}_0] \cap (\bigcap_{i=1}^{h} \tilde{K}_i^i) \neq \emptyset \), then

\[ \bigcap_{i=1}^{h} \tilde{K}_i^i \cap \tilde{K}_0 \neq \emptyset; \quad \alpha = \pm 1. \]

Proof. \( \bigcap_{i=1}^{h} \tilde{K}_i^i \) is open. Hence by 1.10, the result follows.

4.8. If \( 0 < s < t < u < v < 1 \), then

\[ K^\alpha_t \cap K^{-\alpha}_u \subseteq K^\alpha_s \cap K^{-\alpha}_v; \quad \alpha = \pm 1. \]

Proof. If \( w \) lies between \( u \) and \( 1 \), then by 4.4,

\[ [K_w] \subseteq (K^1_t \cap K^1_u) \cup (K^{-1}_t \cap K^{-1}_u) \cup M. \]

Hence \([K_w]\) has no points in \( K^\alpha_t \cap K^{-\alpha}_u \). Let \( P \in K^\alpha_t \cap K^{-\alpha}_u \). Then \( P \notin [K_w] \). Since \( P \in K^{-\alpha}_u \), then by 2.4, \( P \in K^{-\alpha}_w \) for all \( w \) with \( u \leq w < 1 \). Hence \( P \in K^\alpha_t \cap K^{-\alpha}_v \) and \( K^\alpha_t \cap K^{-\alpha}_u \subseteq K^\alpha_t \cap K^{-\alpha}_v \).

Similarly \( K^\alpha_t \cap K^{-\alpha}_v \subseteq K^\alpha_s \cap K^{-\alpha}_v \) and we have

\[ K^\alpha_t \cap K^{-\alpha}_u \subseteq K^\alpha_s \cap K^{-\alpha}_v. \]
4.8.1. Let \( s < u < t \). Then \( \tilde{K}_s^1 \cap \tilde{K}_t^1 \subset \tilde{K}_u^1 \).

Proof. By 4.8, \( \tilde{K}_s^1 \cap \tilde{K}_u^{-1} \subset \tilde{K}_s^1 \cap \tilde{K}_t^{-1} \). Now

\[
\tilde{K}_s^1 \cap \tilde{K}_t^1 = (\tilde{K}_s^1 \cap \tilde{K}_t^1) \cap \mathcal{G}
\]

\[
= (\tilde{K}_s^1 \cap \tilde{K}_t^1 \cap \tilde{K}_u^1) \cup (\tilde{K}_s^1 \cap \tilde{K}_t^1 \cap \tilde{K}_u^{-1}) \cup (\tilde{K}_s^1 \cap \tilde{K}_t^{-1} \cap [\tilde{K}_u])
\]

But by 4.6.1, \([\tilde{K}_u] \subset (\tilde{K}_s^1 \cap \tilde{K}_t^{-1}) \cup (\tilde{K}_s^{-1} \cap \tilde{K}_t^1)\).

Hence \( \tilde{K}_s^1 \cap \tilde{K}_t^1 \cap [\tilde{K}_u] = \emptyset \) and we have

\[
\tilde{K}_s^1 \cap \tilde{K}_t^1 = (\tilde{K}_s^1 \cap \tilde{K}_t^1 \cap \tilde{K}_u^1) \cup (\tilde{K}_s^1 \cap \tilde{K}_t^1 \cap \tilde{K}_u^{-1})
\]

But \( \tilde{K}_s^1 \cap \tilde{K}_u^{-1} \cap \tilde{K}_t^{-1} \subset \tilde{K}_s^1 \cap \tilde{K}_t^{-1} \cap \tilde{K}_t^1 = \emptyset \), by 4.8.

Hence \( \tilde{K}_s^1 \cap \tilde{K}_t^1 = \tilde{K}_s^1 \cap \tilde{K}_t^1 \cap \tilde{K}_u^1 \).

Hence \( \tilde{K}_s^1 \cap \tilde{K}_t^1 \subset \tilde{K}_u^1 \).

4.8.2. Let \( s < u < t \). Then \( \tilde{K}_s^{-1} \cap \tilde{K}_t^{-1} \subset \tilde{K}_u^{-1} \).

Proof. The proof of this assertion is identical to 4.8.1.
4.9. Let \( 0 < t_1 < \ldots < t_h < 1 \). Then not more than 
\( 2h \) of the \( 2^h \) sets 
\[
\bigcap_{i=1}^{h} K_{t_i}^{\alpha_i} ; \ \alpha_i = \pm 1
\]
are non-void.

**Proof.** If \( [\tilde{K}_{t_i}] = \tilde{M} \), then by 1.11, \( [\tilde{K}_s] = \tilde{M} \) for all 
\( s \in \mathcal{I} \) and only two of the sets (4.9-1) are non-void; i.e., 
only the sets 
\[
\bigcap_{i=1}^{h} K_{t_i}^{\alpha} ; \ \alpha = \pm 1
\]
are non-void. We may therefore 
assume that \( [\tilde{K}_{t_i}] \setminus \tilde{M} \neq \emptyset \).

\[
\bigcap_{i=1}^{j} \tilde{K}_{t_i}^{\alpha_i} \cap \bigcup_{i=j+1}^{h} \tilde{K}_{t_i}^{\alpha} ; \ i = 0, 1, \ldots, h ; \ \alpha = 1, -1
\]

can be non-void, where 
\[
\bigcap_{i=1}^{0} \tilde{K}_{t_i}^{\alpha} = \bigcap_{i=h+1}^{h} \tilde{K}_{t_i}^{\alpha} = G.
\]

The cases \( h < 3 \) are trivial. Let \( h \geq 3 \).

Let \( S \) be one of the sets (4.9-1) which does not belong to the 
sets (4.9-2). Then there are three indices \( r_1, r_2, r_3 \) such 
that 
\[
1 \leq r_1 \leq r_2 \leq r_3 \leq h \text{ and } \alpha_{r_1} = -\alpha_{r_2} = \alpha_{r_3}.
\]

But then \( S \subset \tilde{K}_{t_{r_1}}^{\alpha_{r_1}} \cap \tilde{K}_{t_{r_2}}^{\alpha_{r_2}} \cap \tilde{K}_{t_{r_3}}^{\alpha_{r_3}} \).

By 4.8.1 and 4.8.2, 
\[
\tilde{K}_{t_{r_1}}^{\alpha_{r_1}} \cap \tilde{K}_{t_{r_2}}^{\alpha_{r_2}} \subset \tilde{K}_{t_{r_3}}^{\alpha_{r_3}}.
\]

Hence 
\[
\tilde{K}_{t_{r_1}}^{\alpha_{r_1}} \cap \tilde{K}_{t_{r_2}}^{\alpha_{r_2}} \cap \tilde{K}_{t_{r_3}}^{\alpha_{r_3}} = \emptyset.
\]
Thus $S$ must be void. Hence only the $2h$ sets (4.9-2) may be non-void.
CHAPTER V

5. Local Decompositions

5.1. Two quasigraphs $K_1$ and $K_2$ support each other at $Q$ if exactly one of the four open sets

$$K_1^+ \cap K_2^+ \cap N$$

is void for sufficiently small neighbourhoods $N$ of $Q$.

Claim. $Q \in [\tilde{K}_1] \cap [\tilde{K}_2]$ in either case and

$$[\tilde{K}_1] \cap N \neq [\tilde{K}_2] \cap N.$$

Proof. If at least three of the sets $K_1^+ \cap K_2^+ \cap N$ are non-void then $K_1^+ \cap N \neq \emptyset \neq K_2^+ \cap N$ for all neighbourhoods $N$ of $Q$. Hence $Q \in \overline{K_1^+} \cap \overline{K_2^+}$ and $Q \in \overline{K_2^+} \cap \overline{K_2^+}$. Thus by (1.6-1), $Q \in [\tilde{K}_1] \cap [\tilde{K}_2]$.

If $[\tilde{K}_1] \cap N = [\tilde{K}_2] \cap N$, then $\tilde{K}_1 \cap N$ and $\tilde{K}_2 \cap N$ are equal or opposite. Thus $\tilde{K}_1^\alpha \cap N = \tilde{K}_2^{-\alpha} \cap N$ or $\tilde{K}_1^\alpha \cap N = \tilde{K}_2^\alpha \cap N$; $\alpha = \pm 1$. Hence at least two of the $\tilde{K}_1^+ \cap \tilde{K}_2^+ \cap N$ are void.

But since by 1.6, $K_1^{\alpha} \cap \tilde{K}_2^\alpha$, at least two of the sets $K_1^+ \cap K_2^+ \cap N$ are void.
5.1.1. Remark. If $[\tilde{K}_1] \cap N = [\tilde{K}_2] \cap N$, then $K_1$ and $K_2$ neither support nor intersect each other at $Q$.

5.1.2. Remark. $K_1^{a_1} \cap K_2^{a_2} \cap N \neq \emptyset$ if and only if $\tilde{K}_1^{a_1} \cap \tilde{K}_2^{a_2} \cap N \neq \emptyset$. In fact if $B$ is an open set in $G$.

$k^{a} \cap B \neq \emptyset$ if and only if $\tilde{k}^{a} \cap B \neq \emptyset$; $a, a_i = \pm 1$, $i \in \{1, 2\}$.

Proof. Let $K \cap B \neq \emptyset$. By 1.6, $k^{a} \subset \tilde{k}^{a}$. Thus $k^{a} \cap B \subset \tilde{k}^{a} \cap B$ and hence $\tilde{k}^{a} \cap B \neq \emptyset$.

Let $\tilde{k}^{a} \cap B \neq \emptyset$. Then int $k^{a} \cap B \neq \emptyset$. Thus $\overline{k}^{a} \cap B \neq \emptyset$ and $k^{a} \cap B \neq \emptyset$.

More generally $\bigcap_{j=1}^{h} k_{1}^{a_j} \cap B \neq \emptyset$ if and only if $\bigcap_{j=1}^{h} \tilde{k}_{1}^{a_j} \cap B \neq \emptyset$.

5.2. Suppose $Q \in [K_1] \cap [K_2]$ and $K_1$ and $K_2$ neither support nor intersect each other at $Q$. Then either

$$k_i^{a} \cap N = \emptyset; \text{i.e. } N \subset \tilde{k}_i^{-a}$$

for some $i \in \{1, 2\}$, $a \in \{1, -1\}$

or

$$[\tilde{K}_1] \cap N = [\tilde{K}_2] \cap N$$

for every small neighbourhood $N$ of $Q$. 
In the first case, at least one of the quasigraphs does not decompose $G$ at $Q$. In the second, $\tilde{K}_1$ and $\tilde{K}_2$ may both decompose $G$ at $Q$, but they do so in the same way or in opposite ways.

**Proof.** Since $K_1$ and $K_2$ neither support nor intersect each other at $Q$, then at least two of the four open sets $\alpha_i \cap K_2 \cap N$ are void. Suppose $\gamma_1 \cap K_2 \cap N = \emptyset$ and $\beta_i \cap \gamma_i \cap N = \emptyset$ ; $\beta_i, \gamma_i \in \{1,-1\}$, $i \in \{1,2\}$.

Then only two cases are essentially different

$$\gamma_1 = -\beta_1$$

or

$$\gamma_1 = -\beta_1$$

If $\gamma_1 = -\beta_1$ and $\gamma_2 = -\beta_2$ then assume by way of example that $K_1^{-1} \cap K_2^{-1} \cap N = \emptyset$ and $K_1^{-1} \cap K_2^{-1} \cap N = \emptyset$. Then $K_1^{-1} \cap N \subseteq [K_2^1]$. Since $K_1^{-1} \cap N$ is open, $K_1^{-1} \cap N \subseteq \text{int } [K_2] = \emptyset$. Thus $K_1^{-1} \cap N = \emptyset$ and $K_1^{-1} \cap N = \emptyset$ or $N \subseteq N_1^{-1} = K_1^{1}$.

If $\gamma_1 = -\beta_1$ and $\gamma_2 = -\beta_2$ assume that for some $\alpha \in \{1,-1\}$, $K_1^\alpha \cap K_2^1 \cap N = \emptyset$ and $K_1^{-\alpha} \cap K_2^{-1} \cap N = \emptyset$.

Now $K_1^\alpha \cap K_2 \cap N = \emptyset$ and $K_1^{-\alpha} \cap K_2^{-1} \cap N = \emptyset$. Hence $K_1^\alpha \cap N \subseteq N_1^1$ and by 1.9, $K_1^\alpha \cap N \subseteq N_1^{-1} \cap N$. 

Similarly \[ K_2^1 \cap N \subset K_1^\alpha \cap N. \]

Taking the relative closure on each side we obtain

(5.2-1) \[ \overline{K_1^\alpha} \cap N \subset \overline{K_2^{-1}} \cap N \]

and

(5.2-2) \[ K_2^{-1} \cap N \subset \overline{K_1^\alpha} \cap N. \]

Similarly, from \( K_1^{-\alpha} \cap K_2^{-1} \cap N = \emptyset \) we obtain

(5.2-3) \[ \overline{K_1^{-\alpha}} \cap N \subset \overline{K_2^1} \cap N, \]

and

(5.2-4) \[ K_2^{-1} \cap N \subset \overline{K_1^\alpha} \cap N. \]

Hence by (5.2-1) and (5.2-3),

\[ [\overline{K_1^\alpha}] \cap N = (\overline{K_1^\alpha} \cap N) \cap (\overline{K_1^{-\alpha}} \cap N) \]

\[ \subset (\overline{K_2^{-1}} \cap N) \cap (\overline{K_2^1} \cap N) \]

\[ = [\overline{K_2^1}] \cap N. \]

Similarly, by (5.2-2) and (5.2-4),

\[ [\overline{K_2^{-1}}] \cap N \subset [\overline{K_1^\alpha}] \cap N. \]

Hence \[ [\overline{K_2^1}] \cap N = [\overline{K_1^\alpha}] \cap N. \]
5.3. Consider the following example. Let \( G = \mathcal{T}^3 \).

\[ \mathcal{T}^3 = \{ (x,y,z) \mid 0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1 \} \]

Let \( \mathcal{M} = \{ (x,y,\frac{1}{2}) \mid 0 \leq x \leq 1, 0 \leq y \leq 1 \} \)

\[ \cup \{ (x,\frac{1}{2},z) \mid 0 \leq x \leq 1, 0 \leq z \leq 1 \} \]

Let \( F_s = \{ (s,y,z) \mid 0 \leq y \leq 1, 0 \leq z \leq 1 \} \).

Let \( \tilde{K}_s = \mathcal{M} \cup F_s \).

Let \( \tilde{K}_s^{-1} = \{ (x,y,z) \in \mathcal{T}^3 \mid x > s, y > \frac{1}{2}, z > \frac{1}{2} \text{ or } \]

\[ x > s, y < \frac{1}{2}, z < \frac{1}{2} \text{ or } x < s, y < \frac{1}{2}, z > \frac{1}{2} \]

\[ \text{or } x < s, y > \frac{1}{2}, z < \frac{1}{2} \} \].

Let \( \tilde{K}_s^{-1} = \{ (x,y,z) \in \mathcal{T}^3 \mid x < s, y > \frac{1}{2}, z > \frac{1}{2} \text{ or } \]

\[ x < s, y < \frac{1}{2}, z < \frac{1}{2} \text{ or } x > s, y < \frac{1}{2}, z > \frac{1}{2} \]

\[ \text{or } x > s, y > \frac{1}{2}, z < \frac{1}{2} \} \].

Put \( \tilde{K} = \tilde{K}_s^{-1} \). Then \( \mathcal{A} = \{ \tilde{K}_s \mid s \in I \} \) satisfies requirements of 3.1.

Note:

(1) The 0-hyperface \( Q_s = (s,\frac{1}{2},\frac{1}{2}) \) of \( \tilde{K}_s \) is not fixed.

Nor are the 1-hyperfaces fixed; e.g., \( \mathcal{F}_s = \{ (s,\frac{1}{2},z) \in \mathcal{T}^3 \mid z > \frac{1}{2} \} \).
(2) The quasigraphs $\tilde{K}_1$ and $\tilde{K}_2$ [or $\tilde{K}_1$ and $\tilde{K}_3$] decompose $\tilde{I}^3$ at $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ in the same way [in opposite ways].

(3) $\tilde{K}_1$ and $\tilde{K}_2$ intersect each other at $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$.

We make the assumption:

The point $Q_s \in \tilde{M}$ is a non-singular point of every or of no $\tilde{K}_t$, $t \in I$.

5.4. Let $0 < s < t < 1$; $Q_s \in \tilde{M}$; $\alpha \in \{1, -1\}$.

Suppose

$$\tilde{K}_s^\alpha \cap \tilde{K}_t^{-\alpha} \cap N \neq \emptyset$$

for every neighbourhood $N$ of $Q_s$. Then there exists a face $F$ of $\tilde{K}$ such that $Q_s \in \tilde{F}_u$ and $F_u \cap \tilde{K}_s^\alpha \cap \tilde{K}_t^{-\alpha} \cap N \neq \emptyset$ for all $u \in (s, t)$.

Proof. Since $s < t$, we have by 2.4.2,

$$(\tilde{K}_s^1 \cap \tilde{K}_t^{-1}) \cup (\tilde{K}_s^{-1} \cap \tilde{K}_t^1) \subset \bigcup_{s < v < t} [\tilde{K}_v] \setminus \tilde{M}$$

(thus $\tilde{K}_s^\alpha \cap \tilde{K}_t^{-\alpha} \subset \bigcup_{s < v < t} [\tilde{K}_v] \setminus \tilde{M}$)

and there is a $v = v_N \in (s, t)$ such that

$$[\tilde{K}_v] \cap \tilde{K}_s^\alpha \cap \tilde{K}_t^{-\alpha} \cap N \neq \emptyset;$$

in fact

$$([\tilde{K}_v] \setminus \tilde{M}) \cap \tilde{K}_s^\alpha \cap \tilde{K}_t^{-\alpha} \cap N \neq \emptyset.$$
Thus there is a face $F = F(N)$ of $\tilde{K}$ such that
\[ (F_v \setminus \tilde{M}) \cap \tilde{K}^a_s \cap \tilde{K}^{-a}_t \cap N \neq \emptyset. \]

This holds true for every choice of $N$. As $\tilde{K}$ has only a finite number of faces, there is a face $F$ of $\tilde{K}$ such that $F_v \cap \tilde{K}^a_s \cap \tilde{K}^{-a}_t \cap N \neq \emptyset$ for all neighbourhoods $N$ and a suitable $v = v_N \in (s, t)$. Hence $Q_s \in \overline{F_v}$.

Let $Q_s \in F_v$. Then $Q_s$ is a point of a face of $[\tilde{K}_u]$ for all $u \in I$. Let $N$ be a sufficiently small neighbourhood of $Q_s$. If $u$ is close to $s$, there is only one face $F_u$ which meets $N$. But $Q_s \in \tilde{M}$ so $Q_s$ meets $[\tilde{K}_u]$. Hence $Q_s$ must lie on $F_u$. Hence for $u$ close to $s$, $F_u$ passes through $Q_s$. Thus the set of $u$ such that $F_u$ passes through $Q_s$ is open.

The set of $u$ such that $F_u$ does not pass through $Q_s$ is open.

Since $I$ is connected, the set of $u$ such that $F_u$ passes through $Q_s$ is equal to $I$. Hence $Q_s \in \overline{F_u}$ for all $u \in I$.

Let $Q_s$ be a singular point in $\overline{F_v}$. Then $Q_s$ is a singular point of $[\tilde{K}_u]$ for all $u \in I$ and $Q_s$ lies on the closure of a face $D(u)$ of $[\tilde{K}_u]$ for each $u \in I$. If $Q_s$ lies on $\overline{F_v}$, then the set of all $u \in I$ such that $Q_s$ lies on $\overline{F_u}$ is open. If $Q_s$ does not lie on $\overline{F_u}$, then $Q_s$ lies on $\overline{F_r}$ and $\overline{F_r} \cap F_r$ is empty. Hence $Q_s$ lies on $\overline{F_u}$ for all $u$ close to $r$ and $\overline{F_u} \cap F_u = \emptyset$.

Hence the set of all $u$ such that $Q_s$ does not lie on $F_v$ is open.

Since $I$ is connected and $Q_s \in \overline{F_v}$, we obtain that $Q_s \in \overline{F_u}$ for all $u \in I$.

Hence $Q_s \in \overline{F_u}$ for all $u \in I$. 
As \((F_v \setminus \tilde{M}) \cap \tilde{K}_s \cap \tilde{K}_s^{-\alpha} \cap N \neq \emptyset\) for all neighbourhoods of \(Q_s\), there is a connected component \(R(v)\) of \(F_v \setminus \tilde{M}\) such that \(R(v) \cap N \neq \emptyset\) for all neighbourhoods \(N\) of \(Q_s\).

**Claim:** \(R(v) \subset \tilde{K}_s^\alpha \cap \tilde{K}_t^{-\alpha}\).

**Proof.** \([\tilde{K}_v] \subset (\tilde{K}_s^\alpha \cap \tilde{K}_t^{-\alpha}) \cup (\tilde{K}_s^{-\alpha} \cap \tilde{K}_t^\alpha) \cup \tilde{M}\).
Thus \(R(v) \subset (\tilde{K}_s^\alpha \cap \tilde{K}_t^{-\alpha}) \cup (\tilde{K}_s^{-\alpha} \cap \tilde{K}_t^\alpha)\).

Assume \(R(v) \cap (\tilde{K}_s^{-\alpha} \cap \tilde{K}_t^\alpha) \neq \emptyset\).

Let \(P_1 \in R(v) \cap \tilde{K}_s^\alpha \cap \tilde{K}_t^{-\alpha}\) and \(P_2 \in R(v) \cap \tilde{K}_s^{-\alpha} \cap \tilde{K}_t^\alpha\).
Thus \(P_1 \in \tilde{K}_s^\alpha\) and \(P_2 \in \tilde{K}_s^{-\alpha}\). Since \(R(v)\) is connected, there exists a \(P \in [\tilde{K}_s] \cap R(v)\). Hence \(P \in \tilde{M}\); a contradiction.

Hence \(R(v) \subset \tilde{K}_s^\alpha \cap \tilde{K}_t^{-\alpha}\).

Consider the arc \(\{ P(u) \mid u \in I \}\); cf. 3.8. Now let \(P(v) = P_v \in R(v) \cap \tilde{K}_s^\alpha \cap \tilde{K}_t^{-\alpha} \cap N\). Thus \(P_v \in \tilde{K}_s^\alpha \cap \tilde{K}_t^{-\alpha}\), where \(s < v < t\).

By 4.1, \(P_u \in \tilde{K}_s^\alpha\) for all \(u > s\) and \(P_u \in \tilde{K}_t^{-\alpha}\) for all \(u < t\). Hence \(P_u \in \tilde{K}_s^\alpha \cap \tilde{K}_t^{-\alpha}\) for all \(u \in (s,t)\). Thus \(R(u) \cap \tilde{K}_s^\alpha \cap \tilde{K}_t^{-\alpha} \neq \emptyset\) for all \(u \in (s,t)\). By above argument \(R(u) \subset \tilde{K}_s^\alpha \cap \tilde{K}_t^{-\alpha}\) for all \(u \in (s,t)\).

To complete the proof of 5.4, it is sufficient to show that if \(N\) is a neighbourhood of \(Q_s\) such that \(N \cap R(v) \neq \emptyset\), then \(R(u) \cap N \neq \emptyset\) for \(s < u < t\).
Let \( Q_s \in F_v \). Then \( Q_s \in F_u \) for all \( u \in I \).

Now \( (F_v \setminus \tilde{M}) \cap K_s^a \cap K_t^{-a} \cap N \neq \emptyset \) for every neighbourhood \( N \) of \( Q_s \) implies that every neighbourhood \( N \) of \( Q_s \) meets \( F_u \setminus \tilde{M} \) for all \( u \in I \), since if \( N \cap F_u \setminus \tilde{M} = \emptyset \) for some \( N \), then \( N \cap F_u \subseteq \tilde{M} \) and hence \( N \cap F_v \setminus \tilde{M} \), i.e. \( N \cap F_v \setminus \tilde{M} = \emptyset \); a contradiction.

If every neighbourhood \( N \) of \( Q_s \) meets \( R(v) \), then \( Q_s \in \overline{R(v)} \). Let \( N \) be a neighbourhood of \( Q_s \). Then by 3.8.1, \( N \) meets \( R(u) \) for all \( u \) close to \( s \). Hence the set of all \( u \) such that \( Q_s \in \overline{R(u)} \) is open.

Assume \( Q_s \notin \overline{R(u)} \). Then \( Q_s \in R'(u) \) and \( R'(u) \cap R(u) = \emptyset \).

The set of all \( r \) such that \( Q_s \in \overline{R'(r)} \) is open. If \( Q_s \in R'(r) \), then \( Q_s \notin \overline{R(r)} \). Hence the set of all \( u \) such that \( Q_s \notin \overline{R(u)} \) is open. Since \( I \) is connected and \( Q_s \in \overline{R(v)} \), we have that \( Q_s \in \overline{R(u)} \) for all \( u \in I \).

Hence \( R(u) \cap N \neq \emptyset \) for all neighbourhoods \( N \) of \( Q_s \) and for all \( u \in I \).

This completes the proof of 5.4.
5.5. Suppose \( \tilde{K}_s \) and \( \tilde{K}_t \) support each other at \( Q_s \). Then there exist a neighbourhood \( N \) of \( Q_s \) and an \( \alpha \in \{1, -1\} \) such that
\[
\tilde{K}_s^\alpha \cap \tilde{K}_t^{-\alpha} \cap N = \emptyset.
\]

Proof. Let \( s < t \). Since \( \tilde{K}_s \) and \( \tilde{K}_t \) support each other at \( Q_s \), every neighbourhood \( N \) of \( Q_s \) contains points \( P_s \in [\tilde{K}_s] \setminus \tilde{M} \). By the proof of 5.4, there is an arc \( \{ P(u) \mid u \in I \} \) such that \( \{ P(u) \mid s < u < t \} \) lies entirely in \( N \).

Suppose \( P(s) \in \tilde{K}_s^\alpha \). Then by 5.1, \( P(t) \in \tilde{K}_t^{-\alpha} \).

As \( [\tilde{K}_s] \cap \tilde{K}_t^\alpha \cap N \) is not void, 1.10 implies \( \tilde{K}_s^{\pm1} \cap \tilde{K}_t^\alpha \cap N \neq \emptyset \).

Similarly, \( [\tilde{K}_t] \cap \tilde{K}_s^{-\alpha} \cap N \neq \emptyset \) implies \( \tilde{K}_s^{-\alpha} \cap \tilde{K}_t^{\pm1} \cap N \neq \emptyset \).

This yields \( \tilde{K}_s^\beta \cap \tilde{K}_t^{-\beta} \cap N \neq \emptyset \) for \( \beta = \pm1 \).

Hence \( \tilde{K}_s^{\pm1} \cap \tilde{K}_t^{-\pm1} \cap N = \emptyset \) or \( \tilde{K}_s^{-\pm1} \cap \tilde{K}_t^{\pm1} \cap N = \emptyset \).

5.6. Let \( 0 < s < v < 1 \), \( 0 < t < u < 1 \). Let \( Q_s \in \tilde{M} \); \( \alpha \in \{1, -1\} \). Let \( N \) be a small neighbourhood of \( Q_s \). Then
\[
K_s^\alpha \cap K_v^{-\alpha} \cap N \neq \emptyset \iff K_t^\alpha \cap K_u^{-\alpha} \cap N \neq \emptyset.
\]

Proof. We need consider only the special case \( 0 < s \leq t < u \leq v < 1 \).

By 5.1.2, it suffices to consider the quasigraphs of the reduced family \( \tilde{\mathcal{A}} \).
Suppose \( \tilde{K}_{t}^{\alpha} \cap \tilde{K}_{u}^{-\alpha} \cap N \neq \emptyset \). By 4.9,

\[
\tilde{K}_{t}^{\alpha} \cap \tilde{K}_{u}^{-\alpha} \cap N \subseteq \tilde{K}_{s}^{\alpha} \cap \tilde{K}_{v}^{-\alpha} \cap N.
\]

Hence \( \tilde{K}_{s}^{\alpha} \cap \tilde{K}_{v}^{-\alpha} \cap N \neq \emptyset \).

Conversely, suppose \( \tilde{K}_{s}^{\alpha} \cap \tilde{K}_{v}^{-\alpha} \cap N \neq \emptyset \). Choose \( w \in (t,u) \cap (s,v) \). Then, by 5.4, there is a point \( P_{w} \in [\tilde{K}_{r}] \) such that \( P_{w} \in \tilde{K}_{s}^{\alpha} \cap \tilde{K}_{v}^{-\alpha} \cap N \). Since \( P_{w} \notin \tilde{M} \), we have \( P_{w} \in [\tilde{K}_{r}] \) for \( r \in [s,t] \cup [u,v] \subseteq (0,w) \cup (w,1) \) and thus \( P_{w} \in \tilde{K}_{t}^{\alpha} \cap \tilde{K}_{u}^{-\alpha} \cap N \). Hence \( \tilde{K}_{t}^{\alpha} \cap \tilde{K}_{u}^{-\alpha} \cap N \neq \emptyset \).

5.7. Let \( \tilde{K}_{s} \) and \( \tilde{K}_{t} \) be any two quasigraphs of \( \tilde{G} \).

Let \( Q_{s} \in \tilde{M} \) then exactly one of the following occurs:

i) \( \tilde{K}_{s} \) and \( \tilde{K}_{t} \) support each other at \( Q_{s} \).

ii) \( \tilde{K}_{s} \) and \( \tilde{K}_{t} \) intersect each other at \( Q_{s} \).

iii) \( \tilde{K}_{s} \) and \( \tilde{K}_{t} \) decompose \( G \) at \( Q_{s} \) in the same way.

Proof. Now \( \tilde{K}_{s} \) and \( \tilde{K}_{t} \) support, intersect or neither support nor intersect each other at \( Q_{s} \). If \( \tilde{K}_{s} \) and \( \tilde{K}_{t} \) neither support nor intersect, then by 5.2, and the fact that both \( \tilde{K}_{s} \) and \( \tilde{K}_{t} \) decompose \( G \) at \( Q_{s} \in \tilde{M} \), there exists a neighbourhood \( N' \) of \( Q_{s} \) such that \( [\tilde{K}_{s}] \cap N' = [\tilde{K}_{t}] \cap N' \).

Thus \( \tilde{K}_{s} \) and \( \tilde{K}_{t} \) decompose \( G \) at \( Q_{s} \) in the same way or in opposite ways.
Suppose $\tilde{K}_s$ and $\tilde{K}_t$ decompose $G$ at $Q_s$ in opposite ways. Then $N \cap \tilde{K}_s = N \cap \tilde{K}_t^{-\alpha}$, i.e., $N \cap \tilde{K}_s \cap \tilde{K}_t^{-\alpha} \neq \emptyset$ for all neighbourhoods $N$ of $Q_s$. By 5.4,

$[\tilde{K}_u] \cap \tilde{K}_s \cap \tilde{K}_t^{-\alpha} \cap N \neq \emptyset$. But $[\tilde{K}_u] \cap N = [\tilde{K}_s] \cap N$. Hence $[\tilde{K}_s] \cap \tilde{K}_s \cap \tilde{K}_t^{-\alpha} \cap N \neq \emptyset$; a contradiction. Hence $\tilde{K}_s$ and $\tilde{K}_t$ decompose $G$ at $Q_s$ in the same way.

5.8. If two quasigraphs of an $\tilde{K}$-family support each other [intersect each other, both decompose $G$ in the same way] at $Q_s \in \tilde{M}$, then so do any two quasigraphs of that family.

Proof.

Case 1. Let $\tilde{K}_v$ and $\tilde{K}_u$ decompose $G$ at $Q_s$ in the same way. Then $\tilde{K}_v^1 \cap \tilde{K}_v^{-1} \cap N = \tilde{K}_v^{-1} \cap \tilde{K}_u \cap N = \emptyset$ for any small neighbourhood $N$ of $Q_s$. Thus by 5.6, $\tilde{K}_s^{-1} \cap \tilde{K}_v^1 \cap N = \emptyset$ and $\tilde{K}_s^{-1} \cap \tilde{K}_t^1 \cap N = \emptyset$ for any $s, t \in I$.

If $P \in N \cap [\tilde{K}_s] \setminus \tilde{M}$, then $P \in N \cap \tilde{K}_t^\alpha$ or $P \in N \cap \tilde{K}_t^{-\alpha}$.

Let $P \in N \cap \tilde{K}_t^\alpha$. Then by 1.10, $N \cap \tilde{K}_s^{-\alpha} \cap \tilde{K}_t^\alpha \neq \emptyset$; a contradiction. Thus $N \cap [\tilde{K}_s] \setminus \tilde{M} = \emptyset$. Similarly $N \cap [\tilde{K}_t] \setminus \tilde{M} = \emptyset$. 
Now \( N \subseteq (\bar{K}_s^1 \cap \bar{K}_t^1) \cup (\bar{K}_s^{-1} \cap \bar{K}_t^{-1}) \cup (\bar{K}_s^1 \cap \bar{K}_t^{-1}) \cup (\bar{K}_s^{-1} \cap \bar{K}_t^1) \cup \bar{M} \).

Hence \( N \subseteq (\bar{K}_s^1 \cap \bar{K}_t^1) \cup (\bar{K}_s^{-1} \cap \bar{K}_t^{-1}) \cup \bar{M} \).

Thus \( N \cap \bar{K}_s^1 \subseteq ((\bar{K}_s^1 \cap \bar{K}_t^1) \cup (\bar{K}_s^{-1} \cap \bar{K}_t^{-1}) \cup \bar{M}) \cap \bar{K}_s^1 \)
\[= (\bar{K}_s^1 \cap \bar{K}_t^1 \cap \bar{K}_s^1) \cup (\bar{M} \cap \bar{K}_s^1) \]
\[= (\bar{K}_s^1 \cap \bar{K}_t^1) \cup (\bar{M} \cap \bar{K}_s^1) \]
\[= \bar{K}_s^1 \cap (\bar{K}_t^1 \cup \bar{M}) \]
\[\subseteq \bar{K}_t^1 \cup \bar{M} \]

Thus \( N \cap \bar{K}_s^1 \subseteq (\bar{K}_t^1 \cup \bar{M}) \cap N = (\bar{K}_t^1 \cap N) \cup (\bar{M} \cap N) \).

But \( N \cap \bar{K}_s^1 \cap \bar{M} \cap N = \emptyset \). Therefore \( N \cap \bar{K}_s^1 \subseteq N \cap \bar{K}_t^1 \).

Symmetrically \( N \cap \bar{K}_t^1 \subseteq N \cap \bar{K}_s^1 \). Hence \( N \cap \bar{K}_s^1 = N \cap \bar{K}_t^1 \) and \( N \cap \bar{K}_s^{-1} = N \cap \bar{K}_t^{-1} \). and \( \bar{K}_s \) and \( \bar{K}_t \) decompose \( N \) in the same way.

**Case 2.** Let two quasigraphs of \( \tilde{\mathcal{G}} \) intersect each other at \( Q_s \). Then by 5.6, \( \bar{K}_s^1 \cap \bar{K}_t^{-1} \cap N \neq \emptyset \) and
\( \bar{K}_s^{-1} \cap \bar{K}_t^1 \cap N \neq \emptyset \) for all \( s, t \in I \) and for all neighbourhoods \( N \) of \( Q_s \). By 5.7, \( \bar{K}_s \) and \( \bar{K}_t \) intersect, support or decompose \( G \) at \( Q_s \) in the same way. But by case 1, any two quasigraphs of \( \tilde{\mathcal{G}} \) decompose \( G \) at \( Q_s \) in the same way, a contradiction.

If \( \bar{K}_s \) and \( \bar{K}_t \) support each other at \( Q_s \), then by 5.5 and 5.6 there exist a neighbourhood \( N \) of \( Q_s \) and an \( \alpha \in \{1, -1\} \) such that \( \bar{K}_s^\alpha \cap \bar{K}_t^{-\alpha} \cap N = \emptyset \) for all \( s, t \in I \); a contradiction.
Hence \( \tilde{K}_s \) and \( \tilde{K}_t \) intersect each other at \( Q_s \) for all \( s, t \in I \).

**Case 3.** Let two quasigraphs of \( \tilde{\alpha} \) support each other at \( Q_s \). Then by 5.7, they do not intersect each other at \( Q_s \), or decompose each other at \( Q_s \) in the same way.

If \( \tilde{K}_s \) and \( \tilde{K}_t \) do not support each other at \( Q_s \), then by cases 1 and 2, any two quasigraphs of \( \tilde{\alpha} \) do not support each other at \( Q_s \); a contradiction.

Hence \( \tilde{K}_s \) and \( \tilde{K}_t \) support each other at \( Q_s \) for all \( s, t \in I \).

**5.9.** Let \( s \neq t; s, t \in I \) and \( Q_s \in \tilde{M} \). Then \( K_s \) and \( K_t \) intersect each other at \( Q_s \) if and only if

\[
[\tilde{K}_s] \cap \tilde{K}_t^\alpha \cap N \neq \emptyset, \alpha = \pm 1
\]

for every neighbourhood \( N \) of \( Q_s \).

**Proof.** Let \([\tilde{K}_s] \cap \tilde{K}_t^\alpha \cap N \neq \emptyset, \alpha = \pm 1\), for every neighbourhood of \( Q_s \). Now \( \tilde{K}_t^\alpha \cap N \) is an open and a non-empty subset of \( G \). Hence by 1.10, \( \tilde{K}_s^{\pm 1} \cap \tilde{K}_t^\alpha \cap N \neq \emptyset \) and \( \tilde{K}_s^{-1} \cap \tilde{K}_t^\alpha \cap N \neq \emptyset \) for all neighbourhoods \( N \) of \( Q_s \) and \( \alpha = \pm 1 \). By 5.1.2,

\[
K_s^{\pm 1} \cap K_t^{\pm 1} \cap N \neq \emptyset \neq K_s^{-1} \cap K_t^{\pm 1} \cap N.
\]

Thus by definition \( K_s \) and \( K_t \) intersect at \( Q_s \).
Conversely, suppose $K_s$ and $K_t$ intersect at $Q_s$. Then $\tilde{K}_s$ and $\tilde{K}_t$ intersect at $Q_s$. Assume that $s < t$.

Let $u < s$. Then by 5.8, $\tilde{K}_u$ and $\tilde{K}_t$ intersect at $Q_s$ and by 5.4, there is for each $\alpha \in \{1, -1\}$ a face $F_s$ of $K_s$ such that $F_s \cap \tilde{K}_t^\alpha \cap \tilde{K}_u^\alpha \cap N \neq \emptyset$ and $Q_s \in \overline{F_s}$. Thus $[\tilde{K}_s] \cap \tilde{K}_t^\alpha \cap N \neq \emptyset$ for all neighbourhoods $N$ of $Q_s$.

5.10. Let $s \neq t$; $Q_s \in \tilde{M}$. Then $K_s$ and $K_t$ support each other at $Q_s$ if and only if the following are satisfied:

$(5.10-1)$ \quad $[\tilde{K}_s] \cap N \neq [\tilde{K}_t] \cap N$;

$(5.10-2)$ \quad $[\tilde{K}_s] \cap N \subset \tilde{K}_t^\alpha \cup \tilde{M}$ and $[\tilde{K}_t] \cap N \subset \tilde{K}_s^{-\alpha} \cup \tilde{M}$ for some $\alpha \in \{1, -1\}$.

Proof. Suppose $K_s$ and $K_t$ support each other at $Q_s$. Then $[\tilde{K}_s] \cap N \neq [\tilde{K}_t] \cap N$ by 5.1. Now for $s$ between $u$ and $t$,

$[\tilde{K}_s] \cap N \subset (\tilde{K}_u^\alpha \cap \tilde{K}_t^\alpha) \cup (\tilde{K}_u^\alpha \cap \tilde{K}_t^{-\alpha}) \cup \tilde{M} \subset \tilde{K}_t^\alpha \cup \tilde{K}_t^{-\alpha} \cup \tilde{M}$.

By 5.9 and the fact that $K_s$ and $K_t$ do not intersect, we have that $[\tilde{K}_s] \cap N \cap \tilde{K}_t$ is void.

Thus there is an $\alpha \in \{1, -1\}$ such that $[\tilde{K}_s] \cap N \cap \tilde{K}_t^\alpha$ or $[\tilde{K}_s] \cap N \cap \tilde{K}_t^{-\alpha}$ is void.

Similarly, there is an $\beta \in \{1, -1\}$ such that $[\tilde{K}_t] \cap N \cap \tilde{K}_s^\beta$ or $[\tilde{K}_t] \cap N \cap \tilde{K}_s^{-\beta}$ is void.

Hence by 1.10, $K_s^\pm \cap K_t^\pm \cap N \neq \emptyset$ and $K_t^\pm \cap K_s^\pm \cap N \neq \emptyset$. 

By 5.5 and since $K_s$ and $K_t$ support each other at $Q_s$, one of the two sets $K_s^{-1} \cap K_t^{-1} \cap N$ and $K_s^{-1} \cap K_t \cap N$ must be void. Thus $\alpha = -\beta$.

Conversely, assume (5.10-1) and (5.10-2). Now $K_s$ and $K_t$ either support or intersect at $Q_s$ or neither support nor intersect. Since $[K_s] \cap N \neq [K_t] \cap N$ then either $K_s$ and $K_t$ support each other at $Q_s$ or intersect each other at $Q_s$. Since $[\tilde{K}_s] \cap N \subseteq K_t^\alpha \cap \tilde{M}$ and $[\tilde{K}_t] \cap N \subseteq K_s^{-\alpha} \cup \tilde{M}$ for some $\alpha \in \{1, -1\}$, we obtain $[\tilde{K}_s] \cap K_t^{-\alpha} \cap N = \emptyset$ for some $\alpha \in \{1, -1\}$. Thus by 5.9, $K_s$ and $K_t$ do not intersect and hence $K_s$ and $K_t$ support each other at $Q_s$.

5.11. Suppose any two quasigraphs of $\mathcal{A}$ support each other at $Q_s$. Let $0 < t_1 < t_2 < \ldots < t_h < 1$. Then for every small neighbourhood $N$ of $Q_s$ exactly $h+1$ of the $2^h$ open sets

$$K_{t_1}^{\pm 1} \cap \ldots \cap K_{t_h}^{\pm 1} \cap N$$

are non-void; $h \geq 2$.

Proof. By 5.1.2, we may replace $\alpha$ by $\tilde{\alpha}$; i.e.,

$$\bigcap_{j=1}^{h} K_j^\alpha \cap N \neq \emptyset \iff \bigcap_{j=1}^{h} K_j^{-\alpha} \cap N \neq \emptyset , \quad h \geq 2 .$$

By definition the assertion is true for $h = 2$. 
Suppose that \( h > 2 \) and our statement has been proved up to \( h-1 \). Then exactly \( h \) of the \( 2^{h-1} \) open sets
\[
\tilde{K}_1^a \cap \ldots \cap \tilde{K}_{h-1}^a \cap N \text{ are non-void. Now } N = N \cap (\tilde{K}_h^a \cup \tilde{K}_h^{-1} \cup \{\tilde{K}_h^a\}).
\]
Thus if \( \tilde{K}_1^a \cap \ldots \cap \tilde{K}_{h-1}^a \cap N \neq \emptyset \) then either
\[
\tilde{K}_1^a \cap \ldots \cap \tilde{K}_1^a \cap N \neq \emptyset \text{ or } \tilde{K}_1^a \cap \ldots \cap \tilde{K}_{h-1}^a \cap N \neq \emptyset \text{ or by 1.10 both are non-void.}
\]

Since any two quasigraphs of \( \mathcal{A} \) support at \( Q_s \), then
\[
[\tilde{K}_h^a] \cap N \setminus \tilde{M} \text{ is non-void. Let } P \in [\tilde{K}_h^a] \cap N \setminus \tilde{M}. \text{ Then }
\]
P \notin [K_r] \text{ for } r \in \{t_1, t_{h-1}\}. \text{ By 2.4, there is an } \alpha \in \{1, -1\}
for which \( P \in \tilde{K}_1^a \cap \ldots \cap \tilde{K}_{h-1}^a \cap N. \text{ Thus by 5.10, }
\]
\[
[\tilde{K}_h^a] \cap N \subset \tilde{M} \cup \tilde{K}_t^a, i = 1, \ldots, h-1; \text{ i.e., }
\]
\[
[\tilde{K}_h^a] \cap N \setminus \tilde{M} \subset \tilde{K}_t^a. \text{ Thus } [\tilde{K}_h^a] \cap N \setminus \tilde{M} \subset \tilde{K}_t^a \cap \ldots \cap \tilde{K}_{h-1}^a
\]
and \( [\tilde{K}_h^a] \cap \tilde{K}_1^a \cap \ldots \cap \tilde{K}_{h-1}^a \cap N \neq \emptyset \). \text{ By 1.10, }
\]
\[
\tilde{K}_1^a \cap \ldots \cap \tilde{K}_{h-1}^a \cap \tilde{K}_h \cap N \neq \emptyset. \text{ Thus at least } h+1 \text{ of the sets }
\]
\[
\tilde{K}_1^a \cap \ldots \cap \tilde{K}_{h-1}^a \cap N \text{ are non-void.}
\]

Now suppose \( [\tilde{K}_h^a] \cap \tilde{K}_1^b \cap \ldots \cap \tilde{K}_{h-1}^b \cap N \neq \emptyset \).

Then \( \emptyset \neq \tilde{K}_1^b \cap \ldots \cap \tilde{K}_{h-1}^b \cap \tilde{K}_h^b \cap N \subset \tilde{K}_t^b \cap \tilde{K}_h \cap N \),
\[
i = 1, \ldots, h-1. \text{ However, } \tilde{K}_t^a \cap \tilde{K}_h \cap N \neq \emptyset. \text{ Hence,}
\]
since $\tilde{K}_t$ and $\tilde{K}_h$ support each other at $Q_s$, $\beta_i = \alpha$, $i = 1, 2, \ldots, h-1$. Thus $\tilde{K}^{\beta_1}_{t_1} \cap \ldots \cap \tilde{K}^{\beta_h}_{t_h}$ meets both $\tilde{K}^1_{t_1}$ and $\tilde{K}^{-1}_{t_h}$ only if $\beta_i = \alpha$. Thus exactly $h+1$ of the sets $\tilde{K}^\pm_{t_i} \cap \ldots \cap \tilde{K}^\pm_{t_h} \cap N$ are non-void.

5.11. Suppose $K^{-\alpha}_{s} \cap K^\alpha_{t} \cap N = \emptyset$ for $0 < s < t < 1$ and $\alpha \in \{1, -1\}$. Then the $h+1$ non-void sets obtained in 5.11 are

$$\bigcap_{j=1}^{i} K^\alpha_{t_j} \cap \bigcap_{j=i+1}^{h} K^{-\alpha}_{t_j} \cap N; \ i = 0, 1, \ldots, h,$$

where,

$$\bigcap_{j=1}^{0} K^\alpha_{t_j} = \bigcap_{j=h+1}^{h} K^{-\alpha}_{t_j} = G.$$

Proof. By 4.9, not more than $2^h$ of the $2^h$ sets

$$\bigcap_{i=1}^{h} K^\alpha_{t_i}; \alpha_i = \pm 1$$

are non-void. In fact if $\bigcap_{i=1}^{h} K^\alpha_{t_i} \neq \emptyset$, then $\bigcap_{i=1}^{h} K^\alpha_{t_i} \in \{ \bigcap_{j=1}^{i} K^\alpha_{t_j} \cap \bigcap_{j=i+1}^{h} K^{-\alpha}_{t_j} \mid \alpha = 1, -1; \ i = 0, 1, \ldots, h \}.$

If $K^{-\alpha}_{s} \cap K^\alpha_{t} \cap N = \emptyset$, then $\bigcap_{j=1}^{i} K^{-\alpha}_{t_j} \cap \bigcap_{j=i+1}^{h} K^\alpha_{t_j} = \emptyset$ by 5.6.

$i = 1, \ldots, h-1$. Thus the $h+1$ sets

$$\bigcap_{j=1}^{i} K^\alpha_{t_j} \cap \bigcap_{j=i+1}^{h} K^{-\alpha}_{t_j} \cap N; \ i = 0, 1, \ldots, h$$

are non-void.
5.12. Suppose any two quasigraphs of $\alpha$ intersect each other at $Q_s$. Let $0 < t_1 < t_2 < \ldots < t_h < 1$; $h \geq 2$. Then for every small neighbourhood $N$ of $Q_s$ exactly $2h$ of the $2^h$ open sets

$$\bigcap_{1 \leq i \leq h} K_{t_i}^{a_i}; a_i = \pm 1,$$

are non-void.

Proof. By 4.9, not more than $2h$ of the open sets

$$\bigcap_{i} K_{t_i}^{a_i}, a_i = \pm 1, \text{are non-void.}$$

The case $h = 2$ is the definition of intersection for a pair of quasigraphs.

Suppose $h \geq 2$ and suppose our statement has been proved up to $h-1$. Then exactly $2(h-1)$ of the $2^{h-1}$ open sets

$$K_{t_1}^{a_1} \cap \ldots \cap K_{t_{h-1}}^{a_{h-1}} \cap N \text{ are non-void. Now } N = N \cap (K_{t_h}^{1} \cup K_{t_h}^{-1} \cup [K_{t_h}]).$$

If $K_{t_1}^{a_1} \cap \ldots \cap K_{t_{h-1}}^{a_{h-1}} \cap N \neq \emptyset$; $a_i = \pm 1$, $i = 1, \ldots, h-1$,

then either $K_{t_1}^{a_1} \cap \ldots \cap K_{t_{h-1}}^{a_{h-1}} \cap K_{t_h}^{1} \cap N \neq \emptyset$ or

$K_{t_1}^{a_1} \cap \ldots \cap K_{t_{h-1}}^{a_{h-1}} \cap K_{t_h}^{-1} \cap N \neq \emptyset$ or by 1.10, both are non-void. Thus at least $2(h-1)$ of the sets $K_{t_1}^{a_1} \cap \ldots \cap K_{t_{h-1}}^{a_{h-1}} \cap N \neq \emptyset$.

By 5.9 and the fact that $K_{t_h}^{1}$ and $K_{t_h}^{-1}$ intersect at $Q_s \in \mathcal{N}$, we have $[K_{t_h}^{1}] \cap K_{t_{h-1}}^{1} \cap N \neq \emptyset$. Let

$p^1 \in [K_{t_h}^{1}] \cap K_{t_{h-1}}^{1} \cap N$ and let $p^{-1} \in [K_{t_h}^{1}] \cap K_{t_{h-1}}^{-1} \cap N$.

Then $p^1, p^{-1} \in \mathcal{N}$ and $p^1, p^{-1} \in [K_r]$ for $r \in [t_1, t_{h-1}]$. 
Hence $P^1 \in \mathcal{K}_{r}^1$ and $P^{-1} \in \mathcal{K}_{r}^{-1}$ for all $r \in [t_1, t_{h-1}]$.

Thus $[\mathcal{K}_{t}^{-1}] \cap \mathcal{K}_{t_1}^1 \cap \mathcal{K}_{t_2}^1 \cap \ldots \cap \mathcal{K}_{t_{h-1}}^1 \cap N \neq \emptyset$ for $\alpha = 1$ and $\alpha = -1$. Hence by 1.10,

$$\mathcal{K}_{t_1}^1 \cap \ldots \cap \mathcal{K}_{t_{h-1}}^1 \cap \mathcal{K}_{t_h}^{\pm 1} \cap N \neq \emptyset, \quad \alpha = \pm 1.$$  

Hence $2h$ of the sets $\mathcal{K}_{t_1}^{\pm 1} \cap \ldots \cap \mathcal{K}_{t_h}^{\pm 1} \cap N$ are non-void.

Remark. The $2h$ non-void sets obtained in 5.12 are, by 4.9, the sets

$$\bigcap_{j=1}^{i} \mathcal{K}_{t_j}^{\alpha} \cap \bigcap_{j=i+1}^{h} \mathcal{K}_{t_j}^{-\alpha}; \quad i = 0, 1, \ldots, h-1.$$


