QUASIGRAPHS IN N-DIMENSIONAL MANIFOLDS

By

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SCOPE AND CONTENTS: A definition of quasigraphs in n-dimensional manifolds is given. A continuous family of quasigraphs is then introduced and the intersection and support properties of these quasigraphs are studied.

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INTRODUCTION

In the study of direct differential geometry, families of oriented arcs, curves, conic sections and graphs of polynomials of degree at most n have been employed to define the differentiability of an arc at a point in various kinds of planes; cf. [1], [4], [5], [7] and [8].

These studies followed similar patterns and led to the search of a general theory of differentiability. In [6], N.D. Lane, P. Scherk and J.M. Turgeon introduced the notion of a quasigraph in the unit disk as a basis for such a general theory. Dr. Ralph Park first suggested that the idea of a quasigraph could be extended to higher dimensional spaces and this led to the study of quasigraphs in n-dimensional manifolds.

In Chapter I quasigraphs in an n-dimensional connected manifold G are introduced and some basic properties looked at. A quasigraph K is a subset [K] in G of inductive dimension $\leq n-1$, together with a decomposition of $G \setminus [K]$ into two disjoint open sets K^1 and K^{-1}.

In Chapter II, convergent sequences of quasigraphs are studied and the definition of a continuous family of quasigraphs is introduced.

In Chapter III we look at the properties of a continuous family $\mathcal{A} = \{ K_s \mid s \in I = (0,1) \}$ of quasigraphs $K_s = ([K_s], K_s^1, K_s^{-1})$, obtained by means of an isotopy of G. It is assumed that if $Q_s \in [K_s] \cap [K_t]$, $s,t \in I$, $s \neq t$, then $Q_s \in \bigcap_{s \in I} [K_s]$.

In Chapter IV the global decompositions of the continuous family $\mathcal{A}$ of quasigraphs are studied. Let $0 < t_1 < \ldots < t_h < 1$. It is shown that if $K_{t_i} = ([K_{t_i}], K_{t_i}^1, K_{t_i}^{-1})$, $i = 1,\ldots,h$, are quasigraphs of $\mathcal{A}$, then not more than 2h of the 2^h sets $\bigcap_{i=1}^{h} K_{t_i}^{\alpha_i}$, $\alpha_i = \pm 1$, are non-void.

Chapter V deals with the support and intersection properties of a finite number of quasigraphs of $\mathcal{A}$. It is shown that if any two quasigraphs of $\mathcal{A}$ support each other [intersect each other, both decompose G in the same way] at a point, then so do any two quasigraphs of that family. We also obtain the result that if any two quasigraphs of $\mathcal{A}$ support [intersect] each other at Q and $0 < t_1 < \ldots < t_h < 1$, then for every small neighbourhood N of Q exactly h+1 [2h] of the 2^h open sets $K_{t_1}^{\pm 1} \cap \ldots \cap K_{t_h}^{\pm 1} \cap N$, $h \geq 2$ are non-void, where $K_{t_i} \in \mathcal{A}$ for $i = 1,\ldots,h$.

CHAPTER I

1. Quasigraphs in n-Dimensional Manifolds

1.1. An n-dimensional topological manifold is a Hausdorff space each of whose points has a neighbourhood homeomorphic to E^n. Let G be a connected n-dimensional topological manifold, with or without boundary, satisfying the second axiom of countability. Thus G is a separable metric space.

Define a point P of a set S in G to be of dimension k, $0 \leq k \leq n$, if some neighbourhood B of P in G is such that $B \cap S$ is homeomorphic to E^k. S has dimension k if each point of S has dimension k.

Define the inductive dimension of the empty set and only the empty set to be -1.

A space S has inductive dimension $\leq k$ ($k \geq 0$) at a point P if P has arbitrarily small neighbourhoods whose boundaries have inductive dimension $\leq k-1$.

S has inductive dimension $\leq k$, if S has inductive dimension $\leq k$ at each of its points. S has inductive dimension k at point if it is true that S has inductive dimension $\leq$ k at P and it is false that S has inductive dimension $\leq$ k-1 at P. S has inductive dimension k if the inductive dimension of S is $\leq k$, is true and the inductive dimension of S is $\leq k-1$, is false.

Define a <u>k-hyperface</u> F in G to be either a connected subset of the interior of G such that for each point P in F there is a small neighbourhood B of P in G and a homeomorphism $h_P : B \longrightarrow E^n$ such that $h_P(B) = E^n$ and $h_P (B \cap F) = E^k$, $k \leq n-1$ or F is a connected subset of the boundary of G such that for each point P in F there is a small neighbourhood B of P in the boundary of G homeomorphic to E^k, $k \leq n-2$.

<u>Remark 1</u>. A subspace S of a space G has inductive dimension $\leq$ k if and only if every point of S has arbitrarily small neighbourhoods, in G, whose boundaries have intersections with S of inductive dimension $\leq$ k-1.

<u>Remark 2</u>. The inductive dimension of a set is a topological invariant and E^k has inductive dimension k. Thus if $P \in S$, $S \subset G$ has dimension k, then the inductive dimension of P is k. However, there do exist points

which have inductive dimension but no ordinary dimension.

Remark 3. If F is an (n-1) - hyperface in G and $P \varepsilon F$, then F decomposes the interior of a small neighbourhood B of P in G into exactly two disjoint open subsets.

1.2. Let [K] be the union of a finite number of hyperfaces in G. Let F_i ; i = 1,2,...,m, be the (n-1) - hyperfaces of [K] and assume that

$$[K] \setminus \bigcup_{i=1}^{m} F_i$$

has no points of inductive dimension n-1.

Define a point of [K] to be non-singular if it belongs to an (n-1)- hyperface of [K], otherwise it is singular. Thus if Q ε [K] has dimension n-1 it is non-singular.

We call an (n-1)- hyperface of [K], a face of [K].

1.2.1. THEOREM: (Hurewicz and Wallman) A connected n-dimensional manifold cannot be disconnected by a subset of inductive dimension n-2.

Hence G cannot be disconnected by

$$[K] \setminus \left(\bigcup_{i=1}^{m} F_i \right) .$$

Remark: (Hurewicz and Wallman) Let U be an open set in an k-dimensional manifold which is neither empty nor dense†, and let B be the boundary of U. Then the inductive dimension of B is k-1.

1.2.2. Let K^1 and K^{-1} be any open sets which partition $G \setminus [K]$. Thus every connected component of $G \setminus [K]$ lies entirely in K^1 or K^{-1}. Then call the ordered triple

$$([K], K^1, K^{-1})$$

a quasigraph and denote it by K. In particular call

(1.2.2-1) $$(\emptyset, G, \emptyset) \text{ and } (\emptyset, \emptyset, G)$$

the void quasigraphs. We have

(1.2.2-2) $$G = [K] \cup K^1 \cup K^{-1}$$

and

$$[K] = \not{\ell} K^1 \cap \not{\ell} K^{-1}.$$

We say K decomposes G if both K^1 and K^{-1} are non-void. K decomposes G at a point Q if

$$K^1 \cap N \neq \emptyset \text{ and } K^{-1} \cap N \neq \emptyset$$

for every neighbourhood N of Q.

† I.e., both U and its complement contain a non-empty set.

If $K = ([K], K^1, K^{-1})$ is a quasigraph, then so is $L = ([K], K^{-1}, K^1)$ and we call K and L opposite quasigraphs. If K is a quasigraph which does not decompose G at any point of [K] then call K a degenerate quasigraph. Note that if [K] has no faces then K is degenerate.

If F is a face of [K], then we say F is a face of K.

1.3. If K decomposes G at one point of a face F, then it will do so at every point of F.

Proof. Let K decompose G at a point P of a face F. Let N be a connected neighbourhood of P such that $N \cap [K] \setminus F = \emptyset$. Thus F separates N into two open connected sets N_1 and N_2 and $N \cap F$ is part of the boundary of both N_1 and N_2. Thus every point Q of $N \cap F$ is a point of $\overline{N}_1$ and $\overline{N}_2$. Since N_1 and N_2 are connected open sets of $G \setminus [K]$, we have that N_1 is completely contained in K^1 or K^{-1} and N_2 is completely contained in K^1 or K^{-1}. But $N \setminus F = N_1 \cup N_2$ meets both K^1 and K^{-1}. Thus, $N \subset K^1$ and $N_2 \subset K^{-1}$ or $N_2 \subset K^1$ and $N_1 \subset K^{-1}$. Hence every neighbourhood M of a point $Q \in N \cap F$ meets both N_1 and N_2 ; hence M meets both K^1 and K^{-1}. Thus every point of $N \cap F$ decomposes G. Thus we have that for any decomposing point of F there is an open neighbourhood in F which consists of decomposing points of G. Since F is connected any two points in F can be joined by a simple chain of open sets in F. Using remark 3 in 1.1, we have that every point in F decomposes G.

If K decomposes G at one point of a face F, then it will do so at every point of F. We then call F <u>odd</u>. Any non-odd face is called <u>even</u>.

<u>1.4.</u> <u>K decomposes G at a singular point Q of K if and only if Q is in the closure of at least one odd face.</u>

<u>Proof</u>. Let Q be in the closure of an odd face F of K. Then every neighbourhood N of Q meets F. Hence N is a neighbourhood of some point of F. Hence N meets both K^1 and K^{-1}. Thus K decomposes G at Q.

Let K decompose G at a singular point Q of K. Assume that Q is not in the closure of an odd face of K. Then either there is a neighbourhood N of Q such that N meets no faces of K or Q is in the closure of even faces of K.

If there is a neighbourhood N of Q such that N meets no face of K, then $N \cap [K]$ has no points of dimension n-1. Thus by <u>1.2.1</u>, N cannot be separated by [K], that is K does not decompose G at Q. Contradiction.

Let Q be in the closure of even faces of K. Let N be a connected neighbourhood of Q such that N meets no odd faces of K. Let S be the set of singular points of [K] in N. By <u>1.2.1</u>, $N \setminus S$ is connected. Thus any two points P, R in $N \setminus S$ can be joined by an arc. Let $P \in K^\alpha$. Let P be joined to R by an arc A which does not pass through any singular points of N. Since A does not pass through any odd face, $R \in K^\alpha$. Hence $N \setminus S \subset K^\alpha \cup [K]$ and $N \subset K^\alpha \cup [K]$. Thus, $N \cap K^{-\alpha} = \emptyset$; a contradiction since N is a neighbourhood of Q.

1.5. The open sets K^1 and K^{-1} being disjoint we have

(1.5-1) $$\overline{K^\alpha} \cap K^{-\alpha} = \emptyset$$

and

(1.5-2) $$\overline{K^\alpha} \cap \operatorname{int} \overline{K^{-\alpha}} = \emptyset ; \quad \alpha = \pm 1.$$

Proof. Let $P \in \overline{K^\alpha} \cap K^{-\alpha}$. Thus $P \in \overline{K^\alpha}$ and $P \in K^{-\alpha}$. Let N be a neighbourhood of P. Then $N \cap K^\alpha \neq \emptyset$. As N is a neighbourhood of P, then $N \cap K^{-\alpha}$ is a neighbourhood of P. Thus, we have that $N \cap K^{-\alpha} \cap K^\alpha \neq \emptyset$; a contradiction since $K^{-\alpha}$ and K^α are disjoint.

By (1.5-1), $K^\alpha \cap \overline{K^{-\alpha}} = \emptyset$. Hence $K \cap \operatorname{int} \overline{K^{-\alpha}} = \emptyset$. By the argument in the proof of (1.5-1), $\overline{K^\alpha} \cap \operatorname{int} \overline{K^{-\alpha}} = \emptyset$.

By (1.5-1) and (1.2.2-2), i.e., $\overline{K^\alpha} \subset \not{K}^{-\alpha} \subset K^\alpha \cup [K]$, we have

(1.5-3) $$\overline{K^\alpha} \subset K^\alpha \cup [K] .$$

1.5.1. We have

$$G = \overline{K^1} \cup \overline{K^{-1}} .$$

Proof. Every neighbourhood of a point in [K] contains points of K^1 or K^{-1}. Hence $[K] \subset \overline{K^1} \cup \overline{K^{-1}}$. Also,

$$\begin{aligned} G &= [K] \cup K^1 \cup K^{-1} \\ &\subset \overline{K^1} \cup \overline{K^{-1}} \cup K^1 \cup K^{-1} \\ &\subset \overline{K^1} \cup \overline{K^{-1}} . \end{aligned}$$

By (1.5-3) $\overline{K^\alpha} \subset K^\alpha \cup [K]$.

Thus $\overline{K^1} \cup \overline{K^{-1}} \subset K^1 \cup K^{-1} \cup [K] = G$.

Hence $G = \overline{K^1} \cup \overline{K^{-1}}$.

Since $G = \overline{K^1} \cup \overline{K^{-1}}$,

i.e. $\complement \overline{K^\alpha} \subset \overline{K^{-\alpha}}$, $\alpha = \pm 1$,

we have $\operatorname{int} \complement \overline{K^\alpha} \subset \operatorname{int} \overline{K^{-\alpha}}$,

i.e. $\complement \overline{K^\alpha} \subset \operatorname{int} \overline{K^{-\alpha}}$.

Hence

(1.5-4) $G = \overline{K^\alpha} \cup \operatorname{int} \overline{K^{-\alpha}}$, $\alpha = \pm 1$.

__1.6__. Define

(1.6-1) $[\tilde{K}] = \overline{K^1} \cap \overline{K^{-1}}$

and

(1.6-2) $\tilde{K}^\alpha = \operatorname{int} \overline{K^\alpha}$.

__1.6.1__. We have $[\tilde{K}] \subset [K]$.

__Proof__. $\overline{K^1} \cap \overline{K^{-1}} \subset (K^1 \cup [K]) \cap (K^{-1} \cup [K])$

$\subset (K^1 \cap K^{-1}) \cup [K]$

$= [K]$.

__1.6.2__. We have $K^\alpha \subset \tilde{K}^\alpha \subset (K^\alpha \cup [K])$.

__Proof__. $K^\alpha \subset \overline{K^\alpha}$. Thus $\operatorname{int} K^\alpha \subset \operatorname{int} \overline{K^\alpha} = \tilde{K}^\alpha$.

Hence $K^\alpha \subset \tilde{K}^\alpha$. $\tilde{K}^\alpha = \operatorname{int} \overline{K^\alpha} \subset \operatorname{int} (K^\alpha \cup [K]) \subset K^\alpha \cup [K]$.

__1.6.3__. We have $[K] = \operatorname{bd} K^1 \cap \operatorname{bd} K^{-1}$.

Proof. Using (1.5-1),

$$\begin{aligned}[\tilde{K}] &= \overline{K^1} \cap \overline{K^{-1}} \\ &= \overline{K^1} \cap \overline{K^{-1}} \cap \complement K^1 \cap \complement K^{-1} \\ &= \overline{K^1} \cap \overline{K^{-1}} \cap \overline{\complement K^1} \cap \overline{\complement K^{-1}} \\ &= \text{bd } K^1 \cap \text{bd } K^{-1} .\end{aligned}$$

1.6.4. $[\tilde{K}]$ *is the union of the closures of the odd faces of* K.

Proof. Let F be an odd face of [K]. Let P be any point in $\overline{F}$. For any neighbourhood N of P, $N \cap K^{\alpha} \neq \emptyset$, $\alpha = \pm 1$. Hence P is in the closure of K^{α}; i.e. in $\overline{K^{\alpha}}$. Thus, $P \in \overline{K^1} \cap \overline{K^{-1}}$ for all P in $\overline{F}$; i.e. $\overline{F} \subset \overline{K^1} \cap \overline{K^{-1}}$.

Conversely, let $P \in \overline{K^1} \cap \overline{K^{-1}}$. Hence for any neighbourhood N of P, $N \cap K^{\alpha} \neq \emptyset$, $\alpha = \pm 1$. As P is also in [K], it is decomposing. Hence P is in the closure of an odd face of [K].

1.7. Now [K] is a closed subset of G and $[\tilde{K}] \subset [K]$, while $\tilde{K}^1$ and $\tilde{K}^{-1}$ are open. *Every point of G belongs to one and only one of the three sets*

$$[\tilde{K}],\ \tilde{K}^1,\ \tilde{K}^{-1}$$

i.e. $G = [\tilde{K}] \cup \tilde{K}^1 \cup \tilde{K}^{-1}$ *with* $[\tilde{K}], \tilde{K}^1, \tilde{K}^{-1}$ *disjoint*.

Proof. $[\tilde{K}] \cap \tilde{K}^{\alpha} = \overline{K^1} \cap \overline{K^{-1}} \cap \text{int } \overline{K^{\alpha}} = \emptyset$, since by (1.5-2), $\overline{K^{-\alpha}} \cap \text{int } \overline{K^{\alpha}} = \emptyset$. Also

$$\tilde{K}^{\alpha} \cap \tilde{K}^{-\alpha} = \text{int } \overline{K^{\alpha}} \cap \text{int } \overline{K^{-\alpha}} = \emptyset .$$

Finally using (1.5-4), we have

$$[\tilde{K}] \cup \tilde{K}^1 \cup \tilde{K}^{-1}$$
$$= (\overline{K^1} \cap \overline{K^{-1}}) \cup \text{int } \overline{K^{-1}} \cup \text{int } \overline{K^1}$$
$$= (\overline{K^1} \cup \text{int } \overline{K^{-1}} \cup \text{int } \overline{K^1}) \cap (\overline{K^{-1}} \cup \text{int } \overline{K^1} \cup \text{int } K^{-1})$$
$$= (G \cup \text{int } \overline{K^1}) \cap (G \cup \text{int } \overline{K^{-1}})$$
$$= G.$$

Hence the triple $\tilde{K} = ([\tilde{K}], \tilde{K}^1, \tilde{K}^{-1})$ is a quasigraph. We call $\tilde{K}$ the <u>reduced quasigraph</u> of K.

<u>Remark</u>. $\tilde{K}$ decomposes G at Q if and only if K does.

<u>Remark</u>. If K is a degenerate quasigraph, then $\tilde{K}$ is a void quasigraph.

<u>1.8</u>. <u>Let $\tilde{K}$ and $\tilde{L}$ be reduced quasigraphs and let $[\tilde{K}] = [\tilde{L}]$. Then $\tilde{K}$ and $\tilde{L}$ are equal or opposite</u>.

<u>Proof</u>. Let P lie on a face F of $[\tilde{K}] = [\tilde{L}]$. For every neighbourhood N of P, $N \cap \tilde{K}^\alpha \neq \emptyset$ and $N \cap \tilde{L}^\alpha \neq \emptyset$, $\alpha = \pm 1$. Choose N sufficiently small that N meets no other face and $N \cap \tilde{K}^\alpha$ and $N \cap \tilde{L}^\alpha$ are connected. Then either the connected component of $\tilde{K}^\alpha$ which contains $N \cap \tilde{K}^\alpha$ is equal to the connected component of $\tilde{L}^\alpha$ which contains $N \cap \tilde{L}^\alpha$ or it is equal to the connected component of $\tilde{L}^{-\alpha}$ which contains $N \cap \tilde{L}^{-\alpha}$.

If $N \cap \tilde{K}^\alpha = N \cap \tilde{L}^\alpha$, then as one travels in G along any arc not passing through any singular points of $[\tilde{K}]$, the moving point remains both in $\tilde{K}^\alpha$ and $\tilde{L}^\alpha$ unless the arc crosses a face of $[\tilde{K}]$, in which case it moves into $\tilde{K}^{-\alpha}$ and $\tilde{L}^{-\alpha}$ simultaneously. The case $N \cap \tilde{K}^\alpha = N \cap \tilde{L}^{-\alpha}$ is similar. Hence $\tilde{K}$ and $\tilde{L}$ are

Remark. Every face of $\tilde{K}$ is odd and is the union of singular points and odd faces of K. Conversely, every odd face of K is contained in some face of $\tilde{K}$.

Remark. The set of singular points of $\tilde{K}$ is a (possibly improper) subset of the set of singular points of K.

1.9. $\tilde{K}^{-1} = \complement\overline{K^1}$ or equivalently $\overline{K^1} = \complement\tilde{K}^{-1} = [\tilde{K}] \cup \tilde{K}^1$.

Proof. By (1.6-1), (1.6-2) and (1.5-4),

$$
\begin{aligned}
[\tilde{K}] \cup \tilde{K}^1 &= (\overline{K^1} \cap \overline{K^{-1}}) \cup (\text{int } \overline{K^1}) \\
&= (\overline{K^1} \cup \text{int } \overline{K^1}) \cap (\overline{K^{-1}} \cup \text{int } \overline{K^1}) \\
&= \overline{K^1} \cap G \\
&= \overline{K^1}.
\end{aligned}
$$

Taking compliments, $\complement\overline{K^1} = \complement([\tilde{K}] \cup \tilde{K}^1) = \tilde{K}^{-1}$.

1.10. Let K be any quasigraph and S be any open set in G. If $[\tilde{K}] \cap S \neq \emptyset$, then $S \cap K^\alpha \neq \emptyset$, $\alpha = \pm 1$.

Proof. Let $P \in [\tilde{K}] \cap S$. Then P is in the closure of an odd face of K and hence K decomposes G at P. Since S is open, there is a neighbourhood N of P contained in S. Hence $N \cap K^\alpha \neq \emptyset$ for $\alpha = \pm 1$. Hence $S \cap K^\alpha \neq \emptyset$ for $\alpha = \pm 1$.

1.11. Let $\tilde{K}$ and $\tilde{L}$ be reduced quasigraphs such that $[\tilde{K}]$ and $[\tilde{L}]$ are homeomorphic and $[\tilde{K}] \subset [\tilde{L}]$. Then $[\tilde{K}] = [\tilde{L}]$.

Proof. Since $[\tilde{K}]$ and $[\tilde{L}]$ are homeomorphic, $\tilde{K}$ and $\tilde{L}$ have the same finite number of k-hyperfaces, $0 \leq k \leq n-1$. If F is a k-hyperface of $\tilde{K}$, then F is contained in a k-hyperface F' of $\tilde{L}$.

Assume F is a proper subset of F'. Then since F' is connected, F' contains a boundary point of F. But if P is a boundary point of F in F' then the inductive dimension of P is k-1. Hence F' contains a point of inductive dimension k-1. Contradiction. Hence $F = F'$.

Hence every k-hyperface of $\tilde{L}$ is a k-hyperface of $\tilde{K}$. Thus $[\tilde{K}] = [\tilde{L}]$. Since $\tilde{K}$ and $\tilde{L}$ are reduced, then $\tilde{K}$ and $\tilde{L}$ are equal or opposite.

1.12. Let $\tilde{\tilde{K}}$ be the reduced quasigraph of $\tilde{K}$. Then $\tilde{\tilde{K}} = \tilde{K}$.

Proof. By 1.6.1, $[\tilde{\tilde{K}}] \subset [\tilde{K}]$ and by (1.6-1) and 1.6.2, $[\tilde{K}] = \overline{K^1} \cap \overline{K^{-1}} \subset \overline{\tilde{K}^1} \cap \overline{\tilde{K}^{-1}} = [\tilde{\tilde{K}}]$. Hence $[\tilde{\tilde{K}}] = [\tilde{K}]$. By 1.8, $\tilde{\tilde{K}}$ and $\tilde{K}$ are equal or opposite.

By 1.6.2, $\tilde{K}^\alpha \subset \tilde{\tilde{K}}^\alpha$ and by (1.6-2) and (1.5-3), $\tilde{\tilde{K}}^\alpha = \operatorname{int} \overline{\tilde{K}^\alpha} \subset \overline{\tilde{K}^\alpha} \subset \tilde{K}^\alpha \cup [\tilde{K}]$. But $\tilde{\tilde{K}}^\alpha \cap [\tilde{\tilde{K}}] = \emptyset$. Thus by the above, i.e., $[\tilde{\tilde{K}}] = [\tilde{K}]$ and $\tilde{\tilde{K}}^\alpha \cap [\tilde{K}] = \emptyset$. Hence $\tilde{\tilde{K}}^\alpha \subset \tilde{K}^\alpha$. Therefore $\tilde{\tilde{K}}^\alpha = \tilde{K}^\alpha$.

This gives the desired results.

CHAPTER II

2. Convergence of sequences of quasigraphs.

2.1. Let $\{S_i\}$ be a sequence of non-empty subsets of G, $i = 1,2,3,\ldots$.

Define a point of P to be an accumulation point of $\{S_i\}$ if every neighbourhood of P contains points of S_i for infinitely many i.

Define a point P to be a limit point of $\{S_i\}$ if every neighbourhood of P contains points of S_i for all but a finite number of i.

Define lim sup S_i to be the set of all accumulation points of $\{S_i\}$.

Define lim inf S_i to be the set of all limit points of $\{S_i\}$.

Note that $\lim\inf S_i \subset \lim\sup S_i$.

We say that $\{S_i\}$ converges if $\lim\inf S_i = \lim\sup S_i$ and write $\lim S_i = \lim\inf S_i = \lim\sup S_i$.

We say that $\{S_i\}$ converges to S if $\{S_i\}$ converges and $\lim S_i = S$.

<u>2.1.1. Lim sup S_i and lim inf S_i are closed sets.</u>

<u>Proof</u>. Let P be a point in $\overline{\text{lim sup } S_i}$. Then every neighbourhood N of P contains points of lim sup S_i. Hence N is a neighbourhood of some point in lim sup S_i. Thus N contains points of S_i for infinitely many i. Thus P ε lim sup S_i. Hence lim sup S_i is closed.

A similar proof holds for lim inf S_i.

<u>2.1.2.</u> Let K be a quasigraph, K = ([K], K^1, K^{-1}).

We say a sequence of quasigraphs $\{K_i\}$ <u>converges</u> to K if $[K_i]$ converges to [K] and ℓK_i^{α} converges to ℓK^{α}, $\alpha = \pm 1$.

<u>2.2.</u> Let $\{ K_s \mid s \varepsilon I = (0,1) \}$ be a family of quasigraphs. We define K_t to be <u>continuous at</u> s in I if and only if for any sequence s_i ε I which converges to s we have that K_{s_i} converges to K.

<u>Write</u> $\lim_{t \to s} K_t = K_s$ if and only if K_t is continuous at s.

<u>We study a family of quasigraphs</u>

$$\{ K_s \mid s \varepsilon I = (0,1) \}$$

<u>where</u> K_s <u>depends continuously on</u> s.

2.3. *If* $P \in K_s^\alpha$ *then* $P \in K_t^\alpha$ *for all* t *near* s. *Thus if* $P \in K_s^\alpha$ *then there exits an open interval* J *in* I *about* s *such that* $P \in K_t^\alpha$ *for all* $t \in J$.

Proof. Let $P \in K_s^\alpha$. Suppose it is false that $P \in K_t^\alpha$ for all t near s. Then there is a sequence t_i converging to s, such that $P \in \complement K_{t_i}^\alpha$ for all i sufficiently large. Then P is an accumulation point of the sequence $\complement K_{t_i}^\alpha$. Thus $P \in \complement K_s^\alpha$. Contradiction.

2.4. *Let* J *be an open subinterval of* $I = (0,1)$. *If* $P \notin \bigcup_{s \in J} [K_s]$ *then there is an* $\alpha \in \{1,-1\}$ *such that* $P \in K_s^\alpha$ *for all* $s \in J$.

Proof. Let $J_\alpha = \{s \in J \mid P \in K_s^\alpha\}, \alpha = \pm 1$. Then J_1 and J_{-1} are open. Since J is connected, J_1 or J_{-1} is void. Hence $P \in K_s^1$ for all $s \in J$ or $P \in K_s^{-1}$ for all $s \in J$.

2.4.1. *Corollary*. *Let* $J = (s_1, s_2)$, $\bar{J} \subset I$. *Then*

$$(K_{s_1}^1 \cap K_{s_2}^{-1}) \cup (K_{s_1}^{-1} \cap K_{s_2}^1) \subset \bigcup_{s \in J} [K_s] .$$

Proof. Let $P \in K_{s_1}^\alpha \cap K_{s_2}^{-\alpha}$, $\alpha \in \{1,-1\}$. Suppose $P \notin \bigcup_{s \in J} [K_s]$. Then by 2.3 and 2.4, $P \in K_s^\alpha$ for all's in J. Hence $P \in \complement K_s^{-\alpha}$ for all s in J. Since K_s is continuous at s_2 , i.e. $\lim_{s \to s_2} \complement K_s^{-\alpha} = \complement K_{s_2}^{-\alpha}$, we have $P \in \complement K_{s_2}^{-\alpha}$. But $P \in K_{s_2}^{-\alpha}$. Hence we obtain a contradiction.

<u>2.4.2.</u> <u>Corollary</u>. <u>Let</u> $J = (s_1, s_2), \bar{J} \subset I$. <u>Then</u>

$$(K^1_{s_1} \cap K^{-1}_{s_2}) \cup (K^{-1}_{s_1} \cap K^1_{s_2}) \subset \bigcup_{s \, \varepsilon J} [K_s] \setminus \bigcap_{s \varepsilon J} [K_s].$$

<u>Proof</u>. Let $P \,\varepsilon\, [K_s]$ for all $s \,\varepsilon\, J$. Then $P \notin K^\alpha_s$, $\alpha = \pm 1$, for all $s \,\varepsilon\, J$. Hence $P \,\varepsilon\, \complement K^\alpha_s$ for all $s \,\varepsilon J$ and we obtain $P \,\varepsilon\, \complement K^\alpha_{s_1}$. Thus $P \notin K^\alpha_{s_1} \cap K^{-\alpha}_{s_2}$.

CHAPTER III

3. Families of Quasigraphs.

3.1. Let $I = (0,1)$. In the following we study families

$$\mathfrak{a} = \{ K_s \mid s \in I \}$$

of quasigraphs with the following properties.

There exists a quasigraph K and a continuous map

$$H : G \times I \longrightarrow G$$

such that, for each s, $H|_{G\times s}$ is a homeomorphism satisfying $H([K] \times s) = [K_s]$, $H(K^1 \times s) = K_s^1$; hence $H(K^{-1} \times s) = K_s^{-1}$.

Remark. H is an open mapping. Thus the map $\bar{H} : G \times I \longrightarrow G \times I$, defined by $\bar{H}(x,t) = (H(x,t),t)$ is an open mapping. Since $\bar{H}$ is also bijective and continuous, we have that $\bar{H}$ is a homeomorphism.

Remark. Since $H|_{G\times s}$ is a homeomorphism, it maps each k-hyperface of K onto a k-hyperface of K_s. If F is a k-hyperface of K, and $Q \in G$, write $F_s = H(F,s)$, $Q_s = H(Q,s)$.

Remark. If $J = [s,t]$ is a closed subinterval of I, and R is an interior point of G, then $(H \mid_{G \times J})^{-1}(R)$ is a Jordan arc in $G \times J$ whose endpoints lie in $\operatorname{int}(G \times \{s\})$ and $\operatorname{int}(G \times \{t\})$ and which does not meet the boundary of $G \times J$ elsewhere.

3.2. Given $\mathcal{A}$ satisfying 3.1, the reduced family

$$\tilde{\mathcal{A}} = \{ \tilde{K}_s \mid s \in I,\ K_s \in \mathcal{A} \}$$

also satisfies 3.1.

Proof. Let $\tilde{K}$ be the reduced quasigraph of K. Since $H \mid_{G \times s}$ is a homeomorphism we have

$$\begin{aligned}
[\tilde{K}_s] &= \overline{K_s^1} \cap \overline{K_s^{-1}} \\
&= \overline{H(K^1 \times s)} \cap \overline{H(K^{-1} \times s)} \\
&= H(\overline{K^1 \times s}) \cap H(\overline{K^{-1} \times s}) \\
&= H(\overline{K^1} \times s) \cap H(\overline{K^{-1}} \times s) \\
&= H((\overline{K^1} \times s) \cap (\overline{K^{-1}} \times s)) \\
&= H((\overline{K^1} \cap \overline{K^{-1}}) \times s)) \\
&= H([\tilde{K}] \times s) .
\end{aligned}$$

We also have

$$\begin{aligned}
\tilde{K}_s^\alpha &= \operatorname{int} \overline{K_s^\alpha} \\
&= \operatorname{int} \overline{H(K^\alpha \times s)} \\
&= \operatorname{int} H(\overline{K^\alpha \times s}) \\
&= \operatorname{int} H(\overline{K^\alpha} \times s) \\
&= H((\operatorname{int} \overline{K^\alpha}) \times s) \\
&= H(\tilde{K}^\alpha \times s) .
\end{aligned}$$

<u>3.3</u>. $\mathcal{Q} = \{K_s \mid s \in I\}$ <u>is a continuous family of quasigraphs, i.e.</u> K_t <u>is continuous at</u> s <u>for each</u> $s \in I = (0,1)$.

<u>Proof</u>.

<u>Claim 1</u>: $[K_s] \subset \limsup \{[K_{s_i}]\}$.

Let P be any point in $[K_s]$. Then $P = H(Q,s)$ where $Q \in [K]$. Let $\{s_i\}$, $i = 1,2,\ldots$, be a sequence in I converging to s. Then (Q,s_i) is a sequence of points in $G \times I$ converging to (Q,s). Since H is continuous, $H(Q,s_i)$ is a sequence of points in G converging to $H(Q,s)$. But $Q_{s_i} = H(Q,s_i) \in H([K] \times s_i) = [K_{s_i}]$. Hence every neighbourhood of P contains Q_{s_i} for all but a finite number of s_i. Hence every neighbourhood of P meets $[K_{s_i}]$ for all but a finite number of s_i. Thus P is a limit point of $\{[K_{s_i}]\}$. Hence $[K_s] \subset \liminf \{[K_{s_i}]\} \subset \limsup \{[K_{s_i}]\}$.

<u>Claim 2</u>: $\limsup \{[K_{s_i}]\} \subset [K_s]$. Let $P \in \limsup \{[K_{s_i}]\}$. By choosing a suitable subsequence of $\{s_i\}$, we may assume that there is a sequence $\{P_i\}$ of points $P_i \in [K_{s_i}]$ such that $\{P_i\}$ converges to P. For each i there exists a unique point $Q_i \in [K] \subset G$ such that $H(Q_i,s_i) = P_i$ and there exists a unique point $Q \in G$ such that $H(Q,s) = P$. Let J be a closed subinterval of I; $s \in \text{int } J$. The map $\bar{H} : G \times J \to G \times J$, defined by $\bar{H}(x,t) = (H(x,t),t)$, is a homeomorphism. Let $s_i \in J$. Then $(P_i,s_i) = \bar{H}(Q_i,s_i)$ and $(P,s) = \bar{H}(Q,s)$. Since (P_i,s_i) converges to (P,s), we have that (Q_i,s_i) converges to (Q,s). Since $[K] \times J$ is closed in $G \times J$ and

$(Q_i, s_i) \in [K] \times J$, we obtain $(Q,s) \in [K] \times J$ and thus $(Q,s) \in [K] \times s$. Finally, $P = H(Q,s) \in H([K] \times s) = K_s$.

Claim 3: $\not{\!\partial} K_s^\alpha \subset \limsup \{\not{\!\partial} K_{s_i}^\alpha\}$.

Claim 4: $\limsup \{\not{\!\partial} K_{s_i}^\alpha\} \subset \not{\!\partial} K_s^\alpha$.

The proofs of claims 3 and 4 are similar to those of claims 1 and 2, respectively. From claims 1 to 4, K_t is continuous at s; thus $\mathcal{Q}$ is a continuous family of quasigraphs.

3.3.2. Corollary. By 3.2, $\{ \tilde{K}_s \mid s \in I \}$ is a continuous family of quasigraphs.

Remark. Let F be a k-hyperface of K and let F_s denote the corresponding k-hyperface of K_s. Then by 2.1.1, and the continuity of H restricted to F x I, we have that the closure $\overline{F_s}$ of F_s depends continuously on s.

3.4. Let $M = \bigcap_{s \in I} [K_s]$ and $\tilde{M} = \bigcap_{s \in I} [\tilde{K}_s]$.

We assume:

3.4.1. If $s \neq t$, then $K_s \neq K_t$ and $[K_s] \cap [K_t] = M$.

3.4.2. Either $[\tilde{K}_s] = \tilde{M}$ for all $s \in I$ or if $s \neq t$ then $\tilde{K}_s \neq \tilde{K}_t$ and $[\tilde{K}_s] \cap [\tilde{K}_t] = \tilde{M}$.

The following example shows that 3.4.1 does not imply 3.4.2.

Let G be open unit ball in R^3.

Let $F_1 = \{(x,y,o) \mid -1 < x < 0\}$,

$F_2 = \{(x,y,0) \mid 0 < x < 1\}$,

$F_3 = \{(0,y,z) \mid 0 < z < 1\}$,

$E = \{(0,y,0) \mid -1 < y < 1\}$,

where F_1, F_2, F_3 are 2-hyperfaces (faces). E is a 1-hyperface. Let $[K]$ be the union of F_1, F_2, F_3 and E.

Let $K^1 = \{(x,y\ z) \in G \mid x < 0 \text{ or } z < 0\}$

$K^{-1} = \{(x,y,z) \in G \mid x > 0,\ z > 0\}$.

Then $K = ([K], K^1, K^{-1})$ is a quasigraph in G. Define K_s by sliding E on the xy plane from $(-\frac{1}{2},y,0)$ to $(\frac{1}{2},y,o)$ moving F_3 parallel to itself, expanding $F_{1,s}$ and shrinking $F_{2,s}$. Then

$[K_s] \cap [K_t] = M = \{(x,y,0) \mid -1 \leq x \leq 1,\ -1 \leq y \leq 1\}$. But

$[\tilde{K}_s] \cap [\tilde{K}_t] \neq \tilde{M} = \{(x,y,0) \mid \frac{1}{2} \leq x,\ -1 \leq y \leq 1\}$.

3.4.3. $[\tilde{K}_s] = \tilde{M}$ *for all* $s \in I$ or $[\tilde{K}_s] \neq \tilde{M}$ *for all* $s \in I$.

Proof. Let $[\tilde{K}_s] = \tilde{M}$ for some $s \in I$. Then $[\tilde{K}_s] = \tilde{M}$ for all $s \in I$ or if $s \neq t$ then $\tilde{K}_s \neq \tilde{K}_t$ and $[\tilde{K}_s] \cap [\tilde{K}_t] = \tilde{M}$. Thus $[\tilde{K}_s] \subset [\tilde{K}_t]$. By 1.11, $[\tilde{K}_s]^{\circ} = [\tilde{K}_t]$. Hence $[\tilde{K}_s] = \tilde{M}$ for all $s \in I$. Hence assertion is proven.

3.5. Suppose a singular point of K_s is such that any small neighbourhood of it is disconnected by $[K_s]$ into at least three open sets. Then it lies in M. In particular every singular point of $\tilde{K}_s$ in the interior of G belongs to $\tilde{M}$.

Proof. Let $Q_s = H(Q,s)$ be a singular point of K_s such that any small neighbourhood of it is disconnected by $[K_s]$ into at least three open sets. Assume that $Q_s \notin M$. Then there is a neighbourhood N of Q_s such that $N \cap M = \emptyset$. Choose N so small that its closure does not meet M or any face of K_s which does not have Q_s in its closure.

Let $N \setminus [K_s]$ be the disjoint union of the open connected subsets of N, $\{ B_{i,s} \mid 1 \leq i \leq j;\ j \geq 3 \}$. Then $\bigcup_{i=1}^{j} (\text{bd } B_{i,s}) = N \cap [K_s]$ and $Q_s \in \bigcap_{i=1}^{j} (\text{bd } B_{i,s})$.

Let $B_{i,u} = H(B_i,u) \cap N$. By 2.3, $B_{i,u} \cap B_{i,s} \neq \emptyset$ for u close to s. Since the $B_{i,u}$'s and $B_{i,s}$'s are connected and the boundaries of $B_{i,s}$ and $B_{i,u}$ cannot meet we have that $B_{i,s} \subset B_{i,u}$ and bd $B_{i,s} \subset B_{i,u}$ or $B_{i,u} \subset B_{i,s}$ and bd $B_{i,u} \subset B_{i,s}$.

Let $t > s$. Then $Q_t \neq Q_s$. Since H is continuous there exists a $t > s$, close to s, such that $Q_u \in N$ for all u with $s < u \leq t$. Now $Q_u \notin M$. Thus $Q_u \notin N \cap [K_s]$. Hence Q_s lies in one and only one of the sets $B_{i,s}$. Let $Q_u \in B_{m,s}$; $m \in \{1,\ldots,j\}$. Then bd $B_{i,u} \subset B_{m,s}$ and $B_{i,u} \subset B_{m,s}$ for all $i \neq m$. $Q_s \notin M$. Thus Q_s lies in one and only one of the sets $B_{i,u}$.

Let $Q_s \varepsilon B_{n,u}$; $n \varepsilon \{1,\ldots,j\}$ .. Then bd $B_{i,s} \subset B_{n,u}$ for $i \neq n$, and $B_{i,s} \subset B_{n,u}$ for all $i \neq n$. Hence $B_{m,s} \subset B_{n,u} \subset B_{m,s}$. Hence $B_{m,s} = B_{n,u}$ and thus bd $B_{m,s} \cap$ bd $B_{n,u} \neq \emptyset$. Contradiction.

<u>3.6.</u> <u>Let</u> F_s <u>be</u> <u>a</u> <u>face</u> <u>of</u> K_s <u>and</u> $Q_s \varepsilon F_s \setminus M$. <u>Then there exists a neighbourhood</u> N' <u>of</u> Q_s <u>in</u> G <u>and an interval</u> $[s_1, s_2]$ <u>containing</u> s <u>such that</u>

$$N' \subset \bigcup_{t\varepsilon[s_1,s_2]} F_t .$$

<u>Proof</u>. Since M is closed, G regular, there is a neighbourhood N of Q_s such that $N \cap M = \emptyset$. Thus each point P of N lies on not more than one $[K_t]$. In particular every $P \varepsilon N$ lies on not more than one F_t.

Now since H is continuous $B = H^{-1}(N)$ is open in G x I. Let $F_s = H(F,s)$. Let $Q_s = H(Q,s)$. Then $(Q,s) \varepsilon B = H^{-1}(N)$. Since G is locally compact there is a compact subset A of F, $Q \varepsilon A$, such that $A \times s \subset B$. Hence there are s_1, s_2 such that $s_1 < s < s_2$ and $A \times [s_1,s_2] \subset B$. Let $S = A \times [s_1,s_2] \subset B$. Then $H(S) \subset N$ and $A_t = H(A \times t) \subset F_t \cap N$ for $s_1 \leq t \leq s_2$.

Since H is an open mapping, H(int S)

is a non-void open set containing Q_s. Every point of this set lies on some F_t; $s_1 \leq t \leq s_2$. Thus let N' be a neighbourhood of Q_s contained in H(int S).

$$\text{Thus } N' \subset H(\text{int } S) \subset \bigcup_{t\varepsilon[s_1,s_2]} F_t .$$

3.7. Let F be a face of $\tilde{K}$. We study the restriction of H to F x I.

3.7.1. Let B be a connected component of $\tilde{M}$ which contains a singular point Q_s of $\tilde{K}_s$ in int G. Then $Q_t \varepsilon$ B for all t ε I.

Proof. As Q_s is a singular point of $\tilde{K}_s$, then Q is a singular point of $\tilde{K}$ and Q_t is a singular point of $\tilde{K}_t$, for all t ε I. Now Q_t ε $\tilde{M}$ for all t ε I. Since H is continuous the set given by $\{ Q_u \mid Q_u = H(Q,u) \; ; \; u \varepsilon I \}$ is connected and lies in $\tilde{M}$. Since Q_s ε B, B connected we have that Q_t ε B for all t ε I.

3.7.2. Let B be a connected component of $\tilde{M}$ such that B lies on a face F_s of $\tilde{K}_s$. Then B $\subset$ F_t for all t ε I.

Proof. Let B lie on a face $F_s = H(F,s)$ of $\tilde{K}_s$. Then B contains no singular points of $\tilde{K}_t$ for all t ε I. Hence B lies on a face D(t) of $\tilde{K}_t$ for all t ε I and $D(t) = H(F^{\lambda(t)},t)$.

where F^i, $i = 1,\ldots,m$ is a face of $\tilde{K}$ and $\lambda(t) \in \{1,2,\ldots,m\}$. Let $A_i = \{ t \in I \mid D(t) = H(F^i,t) \}$; $i = 1,\ldots,m$. Let $F_s = F_s^j = H(F^j,s)$. Let $P \in B \subset F_s$. Then there exists a neighbourhood N of P such that no F_s^i meets N for $i \neq j$. If t is sufficiently close to s no F_t^i, $i \neq j$ will meet N. Hence $P \notin F_t^i$ for t close to s. But $P \in \tilde{M}$. Since P is a non-singular point of $\tilde{K}_t$, P lies on some face of $\tilde{K}_t$. But P is not an accumulation point of F_t^i, $i \neq j$. Hence P lies on F_t^j. P is an arbitrary point of B. Hence $B \subset F_t^j$ for all t close to s. Hence each set A_i is open and every $t \in I$ belongs to only one A_i, $i = 1,\ldots,m$. Since I is connected, $I \setminus A_j$ is open, and the open se $A_j = \{ t \in I \mid D(t) = H(F^j,t) = H(F,t) \}$ is not empty, we have that $A_j = I$.

Hence there is a face F of $\tilde{K}$ such that $D(t) = F_t$ for all $t \in I$. Thus $B \subset F_t$ for all $t \in I$.

<u>3.7.3.</u> <u>Let</u> B <u>be any connected component of</u> $\check{M}$ <u>such that</u> $B \cap \overline{F_s} \cap \operatorname{int} G \neq \emptyset$. <u>Then</u> $B \cap \overline{F_t} \neq \emptyset$ <u>for all</u> $t \in I$.

<u>Proof</u>. Assume $B \cap \overline{F_t} \cap \operatorname{int} G \neq \emptyset$. Then by <u>3.7.1</u> and <u>3.7.2</u>, $B \cap \overline{F_t} \neq \emptyset$ for all $t \in I$.

3.8. Let F denote a face of $\tilde{K}$.

Let $s \in I$. Then $F_s \setminus \tilde{M}$ is an open set in F_s. Hence it is the union of a most countably many disjoint open subfaces of F_s. $F_s \setminus \tilde{M} = \bigcup_{i=1}^{\infty} R_i(s)$ where $R_i(s)$ is an open, connected subset of F_s and $R_i(s) \cap R_j(s) = \emptyset$ for $i \neq j$.

Let R(s) be one of these sets. Then there is an open connected subset R of F such that $H(Q,t) \notin \tilde{M}$ for all $Q \in R$ and all $t \in I$.

Let $P_s \in R(s)$. Then the curve given by

$$P(t) = \{ H(P,t) \mid t \in I \}$$

passes through P_s and $P(t) \in F_t \setminus \tilde{M}$.

Define $R(t) = \{ H(P,t) \mid P \in R \}$. Then R(t) is an open connected subset of $F_t \setminus \tilde{M}$. Hence $R(t) \subset R^1(t)$ where $R^1(t)$ is a connected component of $F_t \setminus \tilde{M}$.

If $R(t) \subset R^1(t)$ and $R(t) \neq R^1(t)$, then there is an open connected subset R^1 of F such that $H(R^1, t) = R^1(t)$ and $R \subset R^1$ with $R \neq R^1$. Hence $\overline{R} \subset \overline{R^1}$ with $\overline{R} \neq \overline{R^1}$. Thus there is an accumulation point P of R contained in R^1. But H(P,t) is a point on the bd G or $H(P,t) \in \tilde{M}$. Hence $R^1(t) \cap \text{bd } G \neq \emptyset$ or $R^1(t) \cap \tilde{M} \neq \emptyset$. Contradiction since $R^1(t) \subset F_t \setminus \tilde{M}$. Hence $R(t) = R^1(t)$.

3.8.1. Let $\{s_i\}_{i\varepsilon N}$ be a sequence in I converging to s ε I. Then $R(s_i)$ converges to $\overline{R(s)}$.

Proof. The proof of this assertion is similar to that in 3.3.1.

Remark. $\overline{R(t)}$ depends continuously on t; i.e., for every sequence $\{s_i\}_{i\varepsilon N}$ converging to s in I, the sequence $\{\overline{R(s_i)}\}$ converges to $\overline{R(s)}$.

3.8.2. Let J be an open interval in I = (0,1). Then $\bigcup_{t\varepsilon J} R(t)$ is open in G.

Proof. $\bigcup_{t\varepsilon J} R(t) = \bigcup_{t\varepsilon J} H(R \times t) = H(\bigcup_{t\varepsilon J} (R \times t)) = H(R \times J)$ is open, since R x J is open and H is an open mapping.

Remark. Each point of $\bigcup_{t\varepsilon J} R(t)$ lies on exactly one of the curves,

$$C_Q = \{ P(t) = H(Q,t) \mid t \varepsilon I \} , \quad Q \varepsilon R .$$

CHAPTER IV

4. Global Decomposition

4.1. Let $P_s \in F_s \setminus \tilde{M}$. Construct the arc $\{ P(u) \mid u \in I \}$ with $P(s) = P_s$ as in 3.8. Then $P(u) \in F_u \setminus \tilde{M}$ for all $u \in I$.

Let $s < s'$. Let $P(s^1) \in \tilde{K}_s^{\alpha}$.

Then

$$P(u) \in \tilde{K}_s^{\alpha} \quad \text{for all } u > s$$

and

$$P(u) \in \tilde{K}_s^{-\alpha} \quad \text{for all } u < s .$$

Proof. The arc $\{ P(t) \mid t > s \}$ does not meet $[\tilde{K}_s]$. Hence it lies entirely in $\tilde{K}_s^{\alpha}$ since $P(s') \in K_s^{\alpha}$.

4.2. Let $R(s)$ denote the connected component of $F_s \setminus \tilde{M}$ containing $P(s)$. Then $R(s)$ decomposes $\bigcup_{t \in I} R(t)$ into two subsets, one in $\tilde{K}_s^1$, the other in $\tilde{K}_s^{-1}$.

Proof. $\bigcup_{t \in I} R(t)$ is an open subset of G. Hence $(\bigcup_{t \in I} R(t)) \cap \tilde{K}_s^{\alpha} \neq \emptyset$ by 1.10. Now $R(t) \cap R(s) = \emptyset$ for $t \neq s$. In fact $R(t) \cap [\tilde{K}_s] = \emptyset$, since $R(t) \cap \tilde{M} = \emptyset$. Hence

$R(t) \subset \tilde{K}_s^1$ or $R(t) \subset \tilde{K}_s^{-1}$. By 4.1, there is an $\alpha \in \{1,-1\}$ such that $P(t) \in \tilde{K}_s^{\alpha}$ for all $t > s$. However, $P(t) \in R(t)$ for all $t \in I$. Hence $R(t) \in \tilde{K}_s^{\alpha}$ for all $t > s$ and $R(t) \in \tilde{K}_s^{-\alpha}$ for all $t < s$. Hence $R(s)$ decomposes $\bigcup_{t \in I} R(t)$ into two subsets; one in K_s^1, the other in K_s^{-1}.

4.3. Let $P_s \in F_s \setminus \tilde{M}$. Construct the arc $\{P(u) \mid u \in I\}$ as in 4.1. Let $t \neq s$. Then $P(s) = P_s \in \tilde{K}_t^{\alpha}$ if and only if $P(t) \in \tilde{K}_s^{-\alpha}$.

Proof. Assume $s < t$. Choose u, v such that $u < s < t < v$. Let $P(s) \in \tilde{K}_t^{\alpha}$. Then $P(u) \in \tilde{K}_t^{\alpha}$. Now $P(u) \notin \bigcup_{x \in (u,v)} [\tilde{K}_x]$.

Hence by 2.4, $P(u) \in \tilde{K}_x^{\beta}$ for all $x \in (u,v)$, $\beta = \pm 1$. In particular $P(u) \in \tilde{K}_s^{\beta}$ and $P(u) \in \tilde{K}_t^{\beta}$; but $P(u) \in \tilde{K}_t^{\alpha}$. Hence $\alpha = \beta$. Thus $P(u) \in \tilde{K}_s^{\alpha}$ and since $u < s < t$ we have $P(t) \in \tilde{K}_s^{-\alpha}$.

4.4. If t and u lie on the same side of s in $I = (0,1)$ then $[K_s] \cap K_t^{\alpha} = [K_s] \cap K_u^{\alpha}$; $\alpha = \pm 1$.

Proof. We may assume that $0 < t < u < s < 1$ and that $[K_s] \cap K_t^{\alpha} \neq \emptyset$. Let $P \in [K_s] \cap K_t^{\alpha}$. Then $P \notin [K_t]$ and hence $P \notin M$ and $P \notin [K_u]$, for all $u \neq s$; cf. 3.4.1. Hence $P \notin [K_u]$ for all $u \in J = (0,s)$. By 2.4, $P \in K_u^{\beta}$ for all $u \in J = (0,s)$. Hence $P \in K_t^{\alpha}$ if and only if $P \in K_u^{\alpha}$ for all

$u \in J$. Since P is arbitrary in $[K_s] \cap K_t^{\alpha}$, this proves the result.

Corollary. If t and u lie on the same side of s in $I = (0,1)$, then $[\tilde{K}_s] \cap \tilde{K}_t^{\alpha} = [\tilde{K}_s] \cap \tilde{K}_u^{\alpha}$; $\alpha = \pm 1$.

4.5. We note:

4.5.1.
$$[\tilde{K}_s] \cap K_t^{\alpha} = [\tilde{K}_s] \cap [K_s] \cap K_t^{\alpha}$$
$$= [\tilde{K}_s] \cap [K_s] \cap K_u^{\alpha}$$
$$= [\tilde{K}_s] \cap K_u^{\alpha}$$

is independent of t; $t,u \in (0,s)$ or $t,u \in (s,1)$.

4.5.2.
$$[\tilde{K}_s] \cap [K_t] = [\tilde{K}_s] \cap [K_s] \cap [K_t]$$
$$= [\tilde{K}_s] \cap M$$

is independent of t; $t \neq s$.

4.5.3. By 4.5.2 and 3.4.2,

$$([\tilde{K}_s] \cap [K_t]) \setminus [\tilde{K}_t] = ([\tilde{K}_s] \cap [K_t]) \setminus ([\tilde{K}_s] \cap [\tilde{K}_t])$$
$$= ([\tilde{K}_s] \cap M) \setminus \tilde{M}$$

is independent of t, $t \neq s$.

4.5.4. Let $0 < u < s < v < 1$. Then by 4.4,

$$[K_u] = [K_u] \cap G = [K_u] \cap (K_s^{1} \cup K_s^{-1} \cup [K_s])$$

$$= ([K_u] \cap K_s^1) \cup ([K_u] \cap K_s^{-1}) \cup ([K_u] \cap [K_s])$$

$$= ([K_u] \cap K_s^1 \cap K_v^1) \cup ([K_u] \cap K_s^{-1} \cap K_v^{-1}) \cup M$$

$$\subset (K_s^1 \cap K_v^1) \cup (K_s^{-1} \cap K_v^{-1}) \cup M \ .$$

More generally, <u>if</u> $0 < t_0 < t_1 < \dots < t_h < 1$, <u>then</u>

$$[K_{t_0}] \subset (K_{t_1}^1 \cap \dots \cap K_{t_h}^1) \cup (K_{t_1}^{-1} \cap \dots \cap K_{t_h}^{-1}) \cup M \ .$$

In particular,

$$[\tilde{K}_{t_0}] \subset (\tilde{K}_{t_1}^1 \cap \dots \cap \tilde{K}_{t_h}^1) \cup (\tilde{K}_{t_1}^{-1} \cap \dots \cap \tilde{K}_{t_h}^{-1}) \cup M \ .$$

<u>4.6</u>. <u>Let</u> $0 < u < s < v < 1$. <u>Then</u>

$$[\tilde{K}_s] \subset (\tilde{K}_u^1 \cap \tilde{K}_v^{-1}) \cup (\tilde{K}_u^{-1} \cap \tilde{K}_v^1) \cup \tilde{M} \ .$$

<u>Proof</u>. Let $P_s \in [\tilde{K}_s] \setminus \tilde{M}$. If P_s is a non singular point of $[\tilde{K}_s]$, then P_s lies on some face F_s of $[\tilde{K}_s]$ and on an open connected component of $F_s \setminus \tilde{M}$. Construct the arc $\{P(t) \mid t \in I\}$.

Suppose $P(s) \in \tilde{K}_u^{\alpha}$. Then by <u>4.1</u>, $P(v) \in \tilde{K}_u^{\alpha}$ and by <u>4.3</u>, $P(u) \in \tilde{K}_v^{-\alpha}$. Hence $P(s) \in \tilde{K}_v^{-\alpha}$. Thus $P_s = P(s) \in \tilde{K}_u^{\alpha} \cap \tilde{K}_v^{-\alpha}$. Hence $P_s \in (\tilde{K}_u^1 \cap \tilde{K}_v^{-1}) \cup (\tilde{K}_u^{-1} \cap \tilde{K}_v^1)$.

If P_s' is a singular point of $[\tilde{K}_s] \setminus \tilde{M}$, then P_s' is in the closure of a face F_s of $[\tilde{K}_s]$. But $F_s \subset (\tilde{K}_u^{\alpha} \cap \tilde{K}_v^{-\alpha})$.

Hence $P'_s \in \overline{(\tilde{K}^{\alpha}_u \cap \tilde{K}^{-\alpha}_v)}$. Thus

$$P'_s \in \overline{\tilde{K}^{\alpha}_u \cap \tilde{K}^{-\alpha}_v} \cap [\tilde{K}_s] \setminus \tilde{M}$$

$$\subset \overline{K^{\alpha}_u} \cap \overline{K^{-\alpha}_v} \cap [\tilde{K}_s] \setminus \tilde{M}$$

$$= (\tilde{K}^{\alpha}_u \cup [\tilde{K}_u]) \cap (\tilde{K}^{-\alpha}_v \cup [K_v]) \cap [\tilde{K}_s] \setminus \tilde{M}$$

$$\subset \tilde{K}^{\alpha}_u \cap \tilde{K}^{-\alpha}_v \quad .$$

Hence $[\tilde{K}_s] \subset (\tilde{K}^1_u \cap \tilde{K}^{-1}_v) \cup (\tilde{K}^{-1}_u \cap \tilde{K}^1_v) \cup \tilde{M}$.

<u>4.6.1</u>. <u>Let</u> u < v. <u>Then</u>

$$\bigcup_{u<s<v} [\tilde{K}_s] = (\tilde{K}^1_u \cap \tilde{K}^{-1}_v) \cup (\tilde{K}^{-1}_u \cap \tilde{K}^1_v) \cup \tilde{M} .$$

<u>Proof</u>. By <u>4.6</u>,

$$\bigcup_{u<s<v} [\tilde{K}_s] \subset (\tilde{K}^1_u \cap \tilde{K}^{-1}_v) \cup (\tilde{K}^{-1}_u \cap \tilde{K}^1_v) \cup \tilde{M} .$$

By <u>2.4.2</u>, $(\tilde{K}^1_u \cap \tilde{K}^{-1}_v) \cup (\tilde{K}^{-1}_u \cap \tilde{K}^1_v) \subset \bigcup_{u<s<v} [\tilde{K}_s] \setminus \bigcap_{s\in I} [\tilde{K}_s]$.

Hence $(\tilde{K}^1_u \cap \tilde{K}^{-1}_v) \cup (\tilde{K}^{-1}_u \cap \tilde{K}^1_v) \cup \tilde{M} \subset \bigcup_{u<s<v} [\tilde{K}_s]$

and the result follows.

<u>4.7</u>. <u>If</u> $0 < t_1 < \dots < t_h < 1$. <u>Then</u>

$$[K_{t_i}] \subset \left(\bigcap_{j=1}^{i-1} K^1_{t_j} \cap \bigcap_{j=i+1}^{h} K^{-1}_{t_j}\right) \cup \left(\bigcap_{j=1}^{i-1} \tilde{K}^{-1}_{t_j} \cap \bigcap_{j=i+1}^{h} \tilde{K}^1_{t_j}\right) \cup \tilde{M} ;$$

$i = 2, \dots , h-1$.

If we define $\bigcap_{j=1}^{0} \tilde{K}^{\alpha}_{t_j} = \bigcap_{j=h+1}^{h} \tilde{K}^{\alpha}_{t_j} = G$, then <u>4.7</u> holds for i = 1 and i = h.

Proof. By 4.6.1 and $j < i < k$,

$$[\tilde{K}_{t_i}] \subset (\tilde{K}^{1}_{t_j} \cap \tilde{K}^{-1}_{t_k}) \cup (\tilde{K}^{-1}_{t_j} \cap \tilde{K}^{1}_{t_k}) \cup \tilde{M} .$$

Hence $[\tilde{K}_{t_i}] \subset \bigcap_{j=1}^{i-1} [(\tilde{K}^{1}_{t_j} \cap \tilde{K}^{-1}_{t_k}) \cup (\tilde{K}^{-1}_{t_j} \cap \tilde{K}^{1}_{t_k}) \cup \tilde{M}]$

and $[\tilde{K}_{t_i}] \subset \bigcap_{k=i+1}^{h} [(\tilde{K}^{1}_{t_j} \cap \tilde{K}^{-1}_{t_k}) \cup (\tilde{K}^{-1}_{t_j} \cap \tilde{K}^{1}_{t_k}) \cup \tilde{M}]$.

Thus

$$[\tilde{K}_{t_i}] \subset (\bigcap_{j=1}^{i-1} \tilde{K}^{1}_{t_j} \cap \tilde{K}^{-1}_{t_k}) \cup (\bigcap_{j=1}^{i-1} \tilde{K}^{-1}_{t_j} \cap \tilde{K}^{1}_{t_k}) \cup \tilde{M}$$

$$\cap (\tilde{K}^{1}_{t_j} \cap \bigcap_{k=i+1}^{h} \tilde{K}^{-1}_{t_k}) \cup (\tilde{K}^{-1}_{t_j} \cap \bigcap_{k=i+1}^{h} \tilde{K}^{1}_{t_k}) \cup \tilde{M}$$

$$\subset (\bigcap_{j=1}^{i-1} \tilde{K}^{1}_{t_j} \cap \bigcap_{k=i+1}^{h} \tilde{K}^{-1}_{t_k}) \cup (\bigcap_{j=1}^{i-1} \tilde{K}^{-1}_{t_j} \cap \bigcap_{k=i+1}^{h} \tilde{K}^{1}_{t_k}) \cup \tilde{M} .$$

If $i = 1$,

$$[\tilde{K}_{t_1}] = (G \cap \bigcap_{j=2}^{h} \tilde{K}^{-1}_{t_j}) \cup (G \cap \bigcap_{j=2}^{h} \tilde{K}^{1}_{t_j}) \cup \tilde{M}$$

$$= \bigcap_{j=2}^{h} \tilde{K}^{-1}_{t_j} \cup \bigcap_{j=2}^{h} \tilde{K}^{-1}_{t_j} \cup \tilde{M} .$$

If $i = h$,

$$[\tilde{K}_{t_h}] = (\bigcap_{j=1}^{i-1} \tilde{K}^{1}_{t_j} \cap G) \cup (\bigcap_{j=1}^{i-1} \tilde{K}^{-1}_{t_j} \cap G) \cup \tilde{M}$$

$$= \bigcap_{j=1}^{h-1} \tilde{K}^{1}_{t_j} \cup \bigcap_{j=1}^{h-1} \tilde{K}^{-1}_{t_j} \cup \tilde{M} .$$

4.7.1. Corollary. If $K_0, K_1, \ldots, K_h$ are distinct quasigraphs of $\mathfrak{a}$ then there exists $\alpha_i = \pm 1$, $i = 1, \ldots, h$, such that

$$[\tilde{K}_o] \subset (\bigcap_{i=1}^{h} \tilde{K}_i^{\alpha_i}) \cup (\bigcap_{i=1}^{h} \tilde{K}_i^{-\alpha_i}) \cup \tilde{M} .$$

Proof. Since $K_0, \ldots, K_h \in \mathcal{O}\!\mathcal{l}$ there are $0 < t_0 < t_1 < t_2 < \ldots, < t_h < 1$ such that for each $i \in (0,1, \ldots, h)$, $\tilde{K}_i = \tilde{K}_{t_j}$ for some $j \in (0,\ldots,h)$.

Let $\tilde{K}_0 = \tilde{K}_{t_i}$. Then

$$[\tilde{K}_0] = (\bigcap_{j=0}^{i-1} \tilde{K}_{t_j}^{1} \cap \bigcap_{j=i+1}^{h} \tilde{K}_j^{-1}) \cup (\bigcap_{j=0}^{i-1} \tilde{K}_{t_j}^{-1} \cap \bigcap_{j=i+1}^{h} \tilde{K}_{t_j}^{1}) \cup \tilde{M}$$

$$= (\bigcap_{j=1}^{h} \tilde{K}_i^{\alpha_i}) \cup (\bigcap_{j=1}^{h} \tilde{K}_i^{-\alpha_i}) \cup \tilde{M}$$

where $\alpha_i = \pm 1$, $i = 1, 2, \ldots, h$.

4.7.2. Corollary. If $K_0, K_1, \ldots, K_h$ are distinct quasigraphs of $\mathcal{O}\!\mathcal{l}$ and $[K_0] \cap K_1^{\alpha_1} \cap \ldots \cap K_h^{\alpha_h} \neq \emptyset$, then $[\tilde{K}_0] \subset (\bigcap_{i=1}^{h} \tilde{K}_i^{\alpha_i}) \cup (\bigcap_{i=1}^{h} \tilde{K}_i^{-\alpha_i}) \cup \tilde{M}$:

Proof. By 4.7.1, there exist $\beta_1, \ldots, \beta_h$ such that

$$[\tilde{K}_0] \subset (\bigcap_{i=1}^{h} \tilde{K}_i^{\beta_i}) \cup (\bigcap_{i=1}^{h} \tilde{K}_i^{-\beta_i}) \cup \tilde{M}.$$

Thus any point $P \in [\tilde{K}_0] \setminus \tilde{M}$ lies either in $\bigcap_{i=1}^{h} \tilde{K}_i^{\beta_i}$ or in $\bigcap_{i=1}^{h} \tilde{K}_i^{-\beta_i}$. Let $P \in [\tilde{K}_0] \cap \bigcap_{i=1}^{h} \tilde{K}_i^{\alpha_i}$. Suppose $P \in \bigcap_{i=1}^{h} \tilde{K}_i^{\beta_i}$. Then $P \in \bigcap_{i=1}^{h} (\tilde{K}_i^{\alpha_i} \cap \tilde{K}_i^{\beta_i})$ and $P \in \tilde{K}_i^{\alpha_i} \cap \tilde{K}_i^{\beta_i} \neq \emptyset$.

Hence $\alpha_i = \beta_i$, $i = 1, \ldots, h$.

<u>4.7.3.</u> <u>If</u> $[\tilde{K}_0] \cap (\bigcap_{i=1}^{h} \tilde{K}_i^{\alpha_i}) \neq \emptyset$, <u>then</u>

$$\bigcap_{i=1}^{h} \tilde{K}_i^{\alpha_i} \cap \tilde{K}_0^{\alpha} \neq \emptyset \ ; \quad \alpha = \pm 1 \ .$$

<u>Proof.</u> $\bigcap_{i=1}^{h} \tilde{K}_i^{\alpha_i}$ is open. Hence by <u>1.10</u>, the result follows.

<u>4.8.</u> <u>If</u> $0 < s \leq t < u \leq v < 1$, <u>then</u>

$$K_t^{\alpha} \cap K_u^{-\alpha} \subset K_s^{\alpha} \cap K_v^{-\alpha} \ ; \quad \alpha = \pm 1 \ .$$

<u>Proof.</u> If w lies between u and 1, then by <u>4.4</u> ,

$$[K_w] \subset (K_t^1 \cap K_u^1) \cup (K_t^{-1} \cap K_u^{-1}) \cup M \ .$$

Hence $[K_w]$ has no points in $K_t^{\alpha} \cap K_u^{-\alpha}$. Let $P \in K_t^{\alpha} \cap K_u^{-\alpha}$. Then $P \notin [K_w]$. Since $P \in K_u^{-\alpha}$, then by <u>2.4</u>, $P \in K_w^{-\alpha}$ for all w with $u \leq w < 1$. Hence $P \in K_t^{\alpha} \cap K_v^{-\alpha}$ and $K_t^{\alpha} \cap K_u^{-\alpha} \subset K_t^{\alpha} \cap K_v^{-\alpha}$.

Similarily $K_t^{\alpha} \cap K_v^{-\alpha} \subset K_s^{\alpha} \cap K_v^{-\alpha}$ and we have $K_t^{\alpha} \cap K_u^{-\alpha} \subset K_s^{\alpha} \cap K_v^{-\alpha}$.

4.8.1. Let $s < u < t$. Then $\tilde{K}_s^1 \cap \tilde{K}_t^1 \subset \tilde{K}_u^1$.

Proof. By 4.8, $\tilde{K}_s^1 \cap \tilde{K}_u^{-1} \subset \tilde{K}_s^1 \cap \tilde{K}_t^{-1}$. Now

$$\tilde{K}_s^1 \cap \tilde{K}_t^1 = (\tilde{K}_s^1 \cap \tilde{K}_t^1) \cap G$$

$$= (\tilde{K}_s^1 \cap \tilde{K}_t^1 \cap \tilde{K}_u^1) \cup (\tilde{K}_s^1 \cap \tilde{K}_t^1 \cap \tilde{K}_u^{-1}) \cup (\tilde{K}_s^1 \cap \tilde{K}_t^1 \cap [\tilde{K}_u])$$

But by 4.6.1, $[\tilde{K}_u] \subset (\tilde{K}_s^1 \cap \tilde{K}_t^{-1}) \cup (\tilde{K}_s^{-1} \cap \tilde{K}_t^1)$.

Hence $\tilde{K}_s^1 \cap \tilde{K}_t^1 \cap [\tilde{K}_u] = \emptyset$ and we have

$$\tilde{K}_s^1 \cap \tilde{K}_t^1 = (\tilde{K}_s^1 \cap \tilde{K}_t^1 \cap \tilde{K}_u^1) \cup (\tilde{K}_s^1 \cap \tilde{K}_t^1 \cap \tilde{K}_u^{-1}).$$

But $\tilde{K}_s^1 \cap \tilde{K}_u^{-1} \cap \tilde{K}_t^1 \subset \tilde{K}_s^1 \cap \tilde{K}_t^{-1} \cap \tilde{K}_t^1 = \emptyset$, by 4.8.

Hence $\tilde{K}_s^1 \cap \tilde{K}_t^1 = \tilde{K}_s^1 \cap \tilde{K}_t^1 \cap \tilde{K}_u^1$.

Hence $\tilde{K}_s^1 \cap \tilde{K}_t^1 \subset \tilde{K}_u^1$.

4.8.2. Let $s < u < t$. Then $\tilde{K}_s^{-1} \cap \tilde{K}_t^{-1} \subset \tilde{K}_u^{-1}$.

Proof. The proof of this assertion is identical to 4.8.1.

4.9. Let $0 < t_1 < \dots < t_h < 1$. Then not more than $2h$ of the 2^h sets

(4.9-1) $$\bigcap_{i=1}^{h} \tilde{K}_{t_i}^{\alpha_i} \; ; \; \alpha_i = \pm 1$$

are non-void.

Proof. If $[\tilde{K}_{t_i}] = \tilde{M}$, then by 1.11, $[\check{K}_s] = \tilde{M}$ for all $s \in I$ and only two of the sets (4.9-1) are non-void; i.e., only the sets $\bigcap_{i=1}^{h} \check{K}_{t_i}^{\alpha}$, $\alpha = \pm 1$ are non-void. We may therefore assume that $[\tilde{K}_{t_i}] \setminus \tilde{M} \neq \emptyset$.

We wish to show that only the 2h sets

(4.9-2) $$\bigcap_{i=1}^{j} \tilde{K}_{t_i}^{\alpha} \cap \bigcap_{i=j+1}^{h} \tilde{K}_{t_i}^{-\alpha} \; ; \; i = 0,1,\dots,h \; ; \; \alpha = 1,-1$$

can be non-void, where $\bigcap_{i=1}^{0} \tilde{K}_{t_i}^{\alpha} = \bigcap_{i=h+1}^{h} \tilde{K}_{t_i}^{\alpha} = G$.

The cases $h < 3$ are trivial. Let $h \geq 3$. Let S be one of the sets (4.9-1) which does not belong to the sets (4.9-2). Then there are three indices r_1, r_2, r_3 such that

$$1 \leq r_1 \leq r_2 \leq r_3 \leq h \quad \text{and} \quad \alpha_{r_1} = -\alpha_{r_2} = \alpha_{r_3} .$$

But then $S \subset \tilde{K}_{t_{r_1}}^{\alpha_{r_1}} \cap \tilde{K}_{t_{r_2}}^{-\alpha_{r_2}} \cap \tilde{K}_{t_{r_3}}^{\alpha_{r_3}}$.

By 4.8.1 and 4.8.2, $\tilde{K}_{t_r}^{\alpha_{r_1}} \cap \tilde{K}_{t_{r_3}}^{\alpha_{r_3}} \subset \tilde{K}_{t_{r_2}}^{\alpha_{r_2}}$.

Hence $\tilde{K}_{t_{r_1}}^{\alpha_{r_1}} \cap \tilde{K}_{t_{r_2}}^{-\alpha_{r_2}} \cap \tilde{K}_{t_{r_3}}^{\alpha_{r_3}} = \emptyset$.

Thus S must be void. Hence only the 2h sets (4.9-2) may be non-void.

CHAPTER V

5. Local Decompositions

5.1. Two quasigraphs K_1 and K_2 support [intersect] each other at Q if exactly one [none] of the four open sets

$$K_1^{\pm 1} \cap K_2^{\pm 1} \cap N$$

is void for sufficiently small neighbourhoods N of Q.

Claim. $Q \in [\tilde{K}_1] \cap [\tilde{K}_2]$ in either case and $[\tilde{K}_1] \cap N \neq [\tilde{K}_2] \cap N$.

Proof. If at least three of the sets $K_1^{\pm 1} \cap K_2^{\pm 1} \cap N$ are non-void then $K_1^{\pm 1} \cap N \neq \emptyset \neq K_2^{\pm 1} \cap N$ for all neighbourhoods N of Q. Hence $Q \in \overline{K_1^1} \cap \overline{K_1^{-1}}$ and $Q \in \overline{K_2^1} \cap \overline{K_2^{-1}}$. Thus by (1.6-1), $Q \in [\tilde{K}_1] \cap [\tilde{K}_2]$.

If $[\tilde{K}_1] \cap N = [\tilde{K}_2] \cap N$, then $\tilde{K}_1 \cap N$ and $\tilde{K}_2 \cap N$ are equal or opposite. Thus $\tilde{K}_1^{\alpha} \cap N = \tilde{K}_2^{-\alpha} \cap N$ or $\tilde{K}_1^{\alpha} \cap N = \tilde{K}_2^{\alpha} \cap N$; $\alpha = \pm 1$. Hence at least two of the $\tilde{K}_1^{\pm 1} \cap \tilde{K}_2^{\pm 1} \cap N$ are void. But since by 1.6, $K^{\alpha} \subset \tilde{K}^{\alpha}$, at least two of the sets $K_1^{\pm 1} \cap K_2^{\pm 1} \cap N$ are void.

5.1.1. Remark. If $[\tilde{K}_1] \cap N = [\tilde{K}_2] \cap N$, then K_1 and K_2 neither support nor intersect each other at Q.

5.1.2. Remark. $K_1^{\alpha_1} \cap K_2^{\alpha_2} \cap N \neq \emptyset$ if and only if $\tilde{K}_1^{\alpha_1} \cap \tilde{K}_2^{\alpha_2} \cap N \neq \emptyset$. In fact if B is an open set in G. $K^{\alpha} \cap B \neq \emptyset$ if and only if $\tilde{K}^{\alpha} \cap B \neq \emptyset$; α, $\alpha_i = \pm 1$, $i \varepsilon \{1,2\}$.

Proof. Let $K^{\alpha} \cap B \neq \emptyset$. By 1.6, $K^{\alpha} \subset \tilde{K}^{\alpha}$. Thus $K^{\alpha} \cap B \subset \tilde{K}^{\alpha} \cap B$ and hence $\tilde{K}^{\alpha} \cap B \neq \emptyset$.

Let $\tilde{K}^{\alpha} \cap B \neq \emptyset$. Then $\text{int}\, \overline{K^{\alpha}} \cap B \neq \emptyset$. Thus $\overline{K^{\alpha}} \cap B \neq \emptyset$ and $K^{\alpha} \cap B \neq \emptyset$.

More generally $\bigcap_{j=1}^{h} K_j^{\alpha_j} \cap B \neq \emptyset$ if and only if $\bigcap_{j=1}^{h} \tilde{K}_j^{\alpha_j} \cap B \neq \emptyset$.

5.2. Suppose $Q \varepsilon [K_1] \cap [K_2]$ and K_1 and K_2 neither support nor intersect each other at Q. Then either

$$K_i^{\alpha} \cap N = \emptyset \text{ ; i.e. } N \subset \tilde{K}_i^{-\alpha}$$

for some $i \varepsilon \{1,2\}$, $\alpha \varepsilon \{1,-1\}$

or $$[\tilde{K}_1] \cap N = [\tilde{K}_2] \cap N$$

for every small neighbourhood N of Q.

In the first case, at least one of the quasigraphs does not decompose G at Q. In the second, $\tilde{K}_1$ and $\tilde{K}_2$ may both decompose G at Q, but they do so in the same way or in opposite ways .

Proof. Since K_1 and K_2 neither support nor intersect each other at Q, then at least two of the four open sets $K_1^{\alpha_1} \cap K_2^{\alpha_2} \cap N$ are void. Suppose $K_1^{\gamma_1} \cap K_2^{\gamma_2} \cap N = \emptyset$ and $K_1^{\beta_1} \cap K_2^{\beta_2} \cap N = \emptyset$; $\beta_i, \gamma_i \in \{ 1,-1 \}$, $i \in \{ 1,2 \}$.

Then only two cases are essentially different

$$\gamma_1 = \beta_1 \text{ and } \gamma_2 = -\beta_2$$

or

$$\gamma_1 = -\beta_1 \text{ and } \gamma_2 = -\beta_2 .$$

If $\gamma_1 = \beta_1$ and $\gamma_2 = -\beta_2$ then assume by way of example that $K_1^{-1} \cap K_2^{1} \cap N = \emptyset$ and $K_1^{-1} \cap K_2^{-1} \cap N = \emptyset$. Then $K_1^{-1} \cap N \subset [K_2^1]$. Since $K_1^{-1} \cap N$ is open, $K_1^{-1} \cap N \subset \operatorname{int} [K_2] = \emptyset$. Thus $K_1^{-1} \cap N = \emptyset$ and $\overline{K_1^{-1}} \cap N = \emptyset$ or $N \subset \complement \overline{K_1^{-1}} = \tilde{K}_1^1$.

If $\gamma_1 = -\beta_1$ and $\gamma_2 = -\beta_2$ assume for some $\alpha \in \{ 1,-1 \}$, $K_1^{\alpha} \cap K_2^{1} \cap N = \emptyset$ and $K_1^{-\alpha} \cap K_2^{-1} \cap N = \emptyset$.

Now $K_1^{\alpha} \cap \overline{K_2^{1}} \cap N = \emptyset$ and $\overline{K_1^{\alpha}} \cap K_2^{1} \cap N = \emptyset$. Hence $K_1^{\alpha} \cap N \subset \complement \overline{K_2^{1}}$ and by 1.9, $K_1^{\alpha} \cap N \subset \tilde{K}_2^{-1} \cap N$.

Similarly $K_2^1 \cap N \subset \tilde{K}_1^{-\alpha} \cap N$.

Taking the relative closure on each side we obtain

(5.2-1) $$\overline{K_1^{\alpha}} \cap N \subset \overline{\tilde{K}_2^{-1}} \cap N$$

and

(5.2-2) $$\overline{K_2^1} \cap N \subset \overline{\tilde{K}_1^{-\alpha}} \cap N .$$

Similarly from $K_1^{-\alpha} \cap K_2^{-1} \cap N = \emptyset$ we obtain

(5.2-3) $$\overline{K_1^{-\alpha}} \cap N \subset \overline{\tilde{K}_2^1} \cap N,$$

and

(5.2-4) $$\overline{K_2^{-1}} \cap N \subset \overline{\tilde{K}_1^{\alpha}} \cap N .$$

Hence by (5.2-1) and (5.2-3),

$$[\tilde{K}_1] \cap N = (\overline{K_1^{\alpha}} \cap N) \cap (\overline{K_1^{-\alpha}} \cap N)$$

$$\subset (\overline{\tilde{K}_2^{-1}} \cap N) \cap (\overline{\tilde{K}_2^1} \cap N)$$

$$= [\tilde{K}_2] \cap N .$$

Similarly by (5.2-2) and (5.2-4),

$$[\tilde{K}_2] \cap N \subset [\tilde{K}_1] \cap N .$$

Hence $[\tilde{K}_2] \cap N = [\tilde{K}_1] \cap N$.

5.3. Consider the following example. Let $G = \bar{I}^3$.

$\bar{I}^3 = \{ (x,y,z) \mid 0 \leq x \leq 1 , 0 \leq y \leq 1, 0 \leq y \leq 1 \}$.

Let $\tilde{M} = \{ (x,y,\frac{1}{2}) \mid 0 \leq x \leq 1 , 0 \leq y \leq 1 \}$
$\cup \{ (x,\frac{1}{2},z) \mid 0 \leq x \leq 1 , 0 \leq z \leq 1 \}$.

Let $F_s = \{ (s,y,z) \mid 0 \leq y \leq 1 , 0 \leq z \leq 1 \}$.

Let $[\tilde{K}_s] = \tilde{M} \cup F_s$.

Let $\tilde{K}_s^1 = \{ (x,y,z) \in I^3 \mid x > s , y > \frac{1}{2} , z > \frac{1}{2}$ or
$x > s , y < \frac{1}{2}, z < \frac{1}{2}$ or $x < s , y < \frac{1}{2} , z > \frac{1}{2}$
or $x < s , y > \frac{1}{2} , z < \frac{1}{2} \}$.

Let $\tilde{K}_s^{-1} = \{ (x,y,z) \in I^3 \mid x < s , y > \frac{1}{2} , z > \frac{1}{2}$ or
$x < s , y < \frac{1}{2} , z < \frac{1}{2}$ or $x > s, y < \frac{1}{2} , z > \frac{1}{2}$
or $x > s , y > \frac{1}{2} , z < \frac{1}{2} \}$.

Put $\tilde{K} = \tilde{K}_{\frac{1}{2}}$. Then $\mathfrak{A} = \{ \tilde{K}_s \mid s \in I \}$ satisfies requirements of 3.1.

Note:

(1) The 0-hyperface $Q_s = (s,\frac{1}{2}, \frac{1}{2})$ of $\tilde{K}_s$ is not fixed. Nor are the 1-hyperfaces fixed; e.g. $E_s = \{(s,\frac{1}{2},z) \in I^3 \mid z > \frac{1}{2}\}$.

(2) The quasigraphs $\tilde{K}_{\frac{1}{4}}$ and $\tilde{K}_{\frac{1}{8}}$ [$\tilde{K}_{\frac{1}{4}}$ and $\tilde{K}_{\frac{3}{4}}$] decompose $\bar{I}^3$ at $(\frac{1}{2},\frac{1}{2},\frac{1}{2})$ in the same way [in opposite ways].

(3) $\tilde{K}_{\frac{1}{2}}$ and $\tilde{K}_{\frac{1}{4}}$ intersect each other at $(\frac{1}{2},\frac{1}{2},\frac{1}{2})$.

We make the assumption:

The point $Q_s \in \tilde{M}$ is a non-singular point of every or of no $\tilde{K}_t$, $t \in I$.

5.4. Let $0 < s < t < 1$; $Q_s \in \tilde{M}$; $\alpha \in \{1,-1\}$. Suppose

$$\tilde{K}_s^{\alpha} \cap \tilde{K}_t^{-\alpha} \cap N \neq \emptyset$$

for every neighbourhood N of Q_s. Then there exists a face F of $\tilde{K}$ such that $Q_s \in \bar{F}_u$ and $F_u \cap \tilde{K}_s^{\alpha} \cap \tilde{K}_t^{-\alpha} \cap N \neq \emptyset$ for all $u \in (s,t)$.

Proof. Since $s < t$, we have by 2.4.2,

$$(\tilde{K}_s^{1} \cap \tilde{K}_t^{-1}) \cup (\tilde{K}_s^{-1} \cap \tilde{K}_t^{1}) \subset \bigcup_{s<v<t} [\tilde{K}_v] \setminus \tilde{M}$$

$$(\text{thus } \tilde{K}_s^{\alpha} \cap \tilde{K}_t^{-\alpha} \subset \bigcup_{s<v<t} [\tilde{K}_v] \setminus \tilde{M})$$

and there is a $v = v_N \in (s,t)$ such that

$$[\tilde{K}_v] \cap \tilde{K}_s^{\alpha} \cap \tilde{K}_t^{-\alpha} \cap N \neq \emptyset ;$$

in fact $\quad \left([\tilde{K}_v] \setminus \tilde{M}\right) \cap \tilde{K}_s^{\alpha} \cap \tilde{K}_t^{-\alpha} \cap N \neq \emptyset .$

Thus there is a face $F = F(N)$ of $\tilde{K}$ such that

$$(F_v \setminus \tilde{M}) \cap \tilde{K}_s^{\alpha} \cap \tilde{K}_t^{-\alpha} \cap N \neq \emptyset .$$

This holds true for every choice of N. As $\tilde{K}$ has only a finite number of faces, there is a face F of $\tilde{K}$ such that $F_v \cap \tilde{K}_s^{\alpha} \cap \tilde{K}_t^{-\alpha} \cap N \neq \emptyset$ for all neighbourhoods N and a suitable $v = v_N \varepsilon (s,t)$. Hence $Q_s \varepsilon \overline{F_v}$.

Let $Q_s \varepsilon F_v$. Then Q_s is a point of a face of $[\tilde{K}_u]$ for all $u \varepsilon I$. Let N be a sufficiently small neighbourhood of Q_s. If u is close to s, there is only one face F_u which meets N. But $Q_s \varepsilon \tilde{M}$ so Q_s meets $[\tilde{K}_u]$. Hence Q_s must lie on F_u. Hence for u close to s, F_u passes through Q_s. Thus the set of u such that F_u passes through Q_s is open. The set of u such that F_u does not pass through Q_s is open. Since I is connected, the set of u such that F_u passes through Q_s is equal to I. Hence $Q_s \varepsilon F_u$ for all $u \varepsilon I$.

Let Q_s be a singular point in $\overline{F_v}$. Then Q_s is a singular point of $[\tilde{K}_u]$ for all $u \varepsilon I$ and Q_s lies on the closure of a face D(u) of $[\tilde{K}_u]$ for each $u \varepsilon I$. If Q_s lies on $\overline{F_v}$, then the set of all $u \varepsilon I$ such that Q_s lies on $\overline{F_u}$ is open. If Q_s does not lie on $\overline{F_r}$, then Q_s lies on $\overline{F_r'}$ and $\overline{F_r'} \cap \overline{F_r}$ is empty. Hence Q_s lies on $\overline{F_u'}$ for all u close to r and $\overline{F_u'} \cap \overline{F_u} = \emptyset$. Hence the set of all u such that Q_s does not lie on F_v is open.

Since I is connected and $Q_s \varepsilon \overline{F_v}$, we obtain that $Q_s \varepsilon \overline{F_u}$ for all $u \varepsilon I$.

Hence $Q_s \varepsilon \overline{F_u}$ for all $u \in I$.

As $(F_v \setminus \tilde{M}) \cap \tilde{K}_s^{\alpha} \cap \tilde{K}_s^{-\alpha} \cap N \neq \emptyset$ for all neighbourhoods of Q_s, there is a connected component $R(v)$ of $F_v \setminus \tilde{M}$ such that $R(v) \cap N \neq \emptyset$ for all neighbourhoods N of Q_s.

<u>Claim</u>: $R(v) \subset \tilde{K}_s^{\alpha} \cap \tilde{K}_t^{-\alpha}$.

<u>Proof</u>. $[\tilde{K}_v] \subset (\tilde{K}_s^{\alpha} \cap \tilde{K}_t^{-\alpha}) \cup (\tilde{K}_s^{-\alpha} \cap \tilde{K}_t^{\alpha}) \cup \tilde{M}$.

Thus $R(v) \subset (\tilde{K}_s^{\alpha} \cap \tilde{K}_t^{-\alpha}) \cup (\tilde{K}_s^{-\alpha} \cap \tilde{K}_t^{\alpha})$.

Assume $R(v) \cap (\tilde{K}_s^{-\alpha} \cap \tilde{K}_t^{\alpha}) \neq \emptyset$.

Let $P_1 \in R(v) \cap \tilde{K}_s^{\alpha} \cap \tilde{K}_t^{-\alpha}$ and $P_2 \in R(v) \cap \tilde{K}_s^{-\alpha} \cap \tilde{K}_t^{\alpha}$. Thus $P_1 \in \tilde{K}_s^{\alpha}$ and $P_2 \in \tilde{K}_s^{-\alpha}$. Since $R(v)$ is connected, there exists a $P \in [\tilde{K}_s] \cap R(v)$. Hence $P \in \tilde{M}$; a contradiction Hence $R(v) \subset \tilde{K}_s^{\alpha} \cap \tilde{K}_t^{-\alpha}$.

Consider the arc $\{ P(u) \mid u \in I \}$; cf. <u>3.8</u>. Now let $P(v) = P_v \in R(v) \cap \tilde{K}_s^{\alpha} \cap \tilde{K}_t^{-\alpha} \cap N$. Thus $P_v \in \tilde{K}_s^{\alpha} \cap \tilde{K}_t^{-\alpha}$, where $s < v < t$.

By <u>4.1</u>, $P_u \in \tilde{K}_s^{\alpha}$ for all $u > s$ and $P_u \in \tilde{K}_t^{-\alpha}$ for all $u < t$. Hence $P_u \in \tilde{K}_s^{\alpha} \cap \tilde{K}_t^{-\alpha}$ for all $u \in (s,t)$. Thus $R(u) \cap \tilde{K}_s^{\alpha} \cap \tilde{K}_t^{-\alpha} \neq \emptyset$ for all $u \in (s,t)$. By above argument $R(u) \subset \tilde{K}_s^{\alpha} \cap \tilde{K}_t^{-\alpha}$ for all $u \in (s,t)$.

To complete the proof of <u>5.4</u>, it is sufficient to show that if N is a neighbourhood of Q_s such that $N \cap R(v) \neq \emptyset$, then $R(u) \cap N \neq \emptyset$ for $s < u < t$.

Let $Q_s \in \overline{F_v}$. Then $Q_s \in \overline{F_u}$ for all $u \in I$.
Now $(F_v \setminus \tilde{M}) \cap \tilde{K}_s^{\alpha} \cap \tilde{K}_t^{-\alpha} \cap N \neq \emptyset$ for every neighbourhood N of Q_s implies that every neighbourhood N of Q_s meets $F_u \setminus \tilde{M}$ for all $u \in I$, since if $N \cap F_u \setminus \tilde{M} = \emptyset$ for some N, then $N \cap F_u \subset \tilde{M}$ and hence $N \cap F_v \setminus \tilde{M}$, i.e. $N \cap F_v \setminus \tilde{M} = \emptyset$; a contradiction.

If every neighbourhood N of Q_s meets R(v), then $Q_s \in \overline{R(v)}$. Let N be a neighbourhood of Q_s. Then by <u>3.8.1</u>, N meets R(u) for all u close to s. Hence the set of all u such that $Q_s \in \overline{R(u)}$ is open.

Assume $Q_s \notin \overline{R(u)}$. Then $Q_s' \in \overline{R'(u)}$ and $\overline{R'(u)} \cap \overline{R(u)} = \emptyset$. The set of all r such that $Q_s \in \overline{R'(r)}$ is open. If $Q_s \in \overline{R'(r)}$, then $Q_s \notin \overline{R(r)}$. Hence the set of all u such that $Q_s \notin \overline{R(u)}$ is open. Since I is connected and $Q_s \in \overline{R(v)}$, we have that $Q_s \in \overline{R(u)}$ for all $u \in I$.

Hence $R(u) \cap N \neq \emptyset$ for all neighbourhoods N of Q_s and for all $u \in I$.

This completes the proof of <u>5.4</u>.

5.5. *Suppose $\tilde{K}_s$ and $\tilde{K}_t$ support each other at Q_s. Then there exist a neighbourhood N of Q_s and an $\alpha \in \{1,-1\}$ such that*

$$\tilde{K}_s^{\alpha} \cap \tilde{K}_t^{-\alpha} \cap N = \emptyset \quad .$$

Proof. Let $s < t$. Since $\tilde{K}_s$ and $\tilde{K}_t$ support each other at Q_s, every neighbourhood N of Q_s contains points $P_s \in [\tilde{K}_s] \setminus \tilde{M}$. By the proof of 5.4, there is an arc $\{P(u) \mid u \in I\}$ such that $\{P(u) \mid s < u < t\}$ lies entirely in N.

Suppose $P(s) \in \tilde{K}_t^{\alpha}$. Then by 5.1, $P(t) \in \tilde{K}_s^{-\alpha}$.

As $[\tilde{K}_s] \cap K_t^{\alpha} \cap N$ is not void, 1.10 implies $\tilde{K}_s^{\pm 1} \cap \tilde{K}_t^{\alpha} \cap N \neq \emptyset$.

Similarly $[\tilde{K}_t] \cap \tilde{K}_s^{-\alpha} \cap N \neq \emptyset$ implies $\tilde{K}_s^{-\alpha} \cap \tilde{K}_t^{\pm 1} \cap N \neq \emptyset$.

This yields $\tilde{K}_s^{\beta} \cap \tilde{K}_t^{\beta} \cap N \neq \emptyset$ for $\beta = \pm 1$.

Hence $\tilde{K}_s^{1} \cap \tilde{K}_t^{-1} \cap N = \emptyset$ or $\tilde{K}_s^{-1} \cap \tilde{K}_t^{1} \cap N = \emptyset$.

5.6. *Let $0 < s < v < 1$, $0 < t < u < 1$. Let $Q_s \in \tilde{M}$; $\alpha \in \{1,-1\}$. Let N be a small neighbourhood of Q_s . Then*

$$K_s^{\alpha} \cap K_v^{-\alpha} \cap N \neq \emptyset \Leftrightarrow K_t^{\alpha} \cap K_u^{-\alpha} \cap N \neq \emptyset \quad .$$

Proof. We need consider only the special case

$$0 < s \leq t < u \leq v < 1 .$$

By 5.1.2, it suffices to consider the quasigraphs of the reduced family $\tilde{\mathfrak{A}}$.

Suppose $\tilde{K}_t^{\alpha} \cap \tilde{K}_u^{-\alpha} \cap N \neq \emptyset$. By 4.9,

$$\tilde{K}_t^{\alpha} \cap \tilde{K}_u^{-\alpha} \cap N \subset \tilde{K}_s^{\alpha} \cap \tilde{K}_v^{-\alpha} \cap N .$$

Hence $\tilde{K}_s^{\alpha} \cap \tilde{K}_v^{-\alpha} \cap N \neq \emptyset$.

Conversely, suppose $\tilde{K}_s^{\alpha} \cap \tilde{K}_v^{-\alpha} \cap N \neq \emptyset$. Choose $w \in (t,u) \subset (s,v)$. Then, by 5.4, there is a point $P_w \in [\tilde{K}_w]$ such that $P_w \in \tilde{K}_s^{\alpha} \cap \tilde{K}_v^{-\alpha} \cap N$. Since $P_w \notin \tilde{M}$, we have $P_w \notin [\tilde{K}_r]$ for $r \in [s,t] \cup [u,v] \subset (0,w) \cup (w,1)$ and thus $P_w \in \tilde{K}_t^{\alpha} \cap \tilde{K}_u^{-\alpha} \cap N$. Hence $\tilde{K}_t^{\alpha} \cap \tilde{K}_u^{-\alpha} \cap N \neq \emptyset$.

5.7. Let $\tilde{K}_s$ and $\tilde{K}_t$ be any two quasigraphs of $\tilde{\mathcal{U}}$. Let $Q_s \in \tilde{M}$ then exactly one of the following occurs:

i) $\tilde{K}_s$ and $\tilde{K}_t$ support each other at Q_s.

ii) $\tilde{K}_s$ and $\tilde{K}_t$ intersect each other at Q_s.

iii) $\tilde{K}_s$ and $\tilde{K}_t$ decompose G at Q_s in the same way.

Proof. Now $\tilde{K}_s$ and $\tilde{K}_t$ support, intersect or neither support nor intersect each other at Q_s. If $\tilde{K}_s$ and $\tilde{K}_t$ neither support nor intersect, then by 5.2, and the fact that both $\tilde{K}_s$ and $\tilde{K}_t$ decompose G at $Q_s \in \tilde{M}$, there exists a neighbourhood N' of Q_s such that $[\tilde{K}_s] \cap N' = [\tilde{K}_t] \cap N'$. Thus $\tilde{K}_s$ and $\tilde{K}_t$ decompose G at Q_s in the same way or in opposite ways.

Suppose $\tilde{K}_s$ and $\tilde{K}_t$ decompose G at Q_s in opposite ways. Then $N \cap \tilde{K}_s^{\alpha} = N \cap \tilde{K}_t^{-\alpha}$, i.e., $N \cap \tilde{K}_s^{\alpha} \cap \tilde{K}_t^{-\alpha} \neq \emptyset$ for all neighbourhoods N of Q_s. By 5.4, $[\tilde{K}_u] \cap \tilde{K}_s^{\alpha} \cap \tilde{K}_t^{-\alpha} \cap N \neq \emptyset$. But $[\tilde{K}_u] \cap N = [\tilde{K}_s] \cap N$. Hence $[\tilde{K}_s] \cap \tilde{K}_s^{\alpha} \cap \tilde{K}_t^{-\alpha} \cap N \neq \emptyset$; a contradiction. Hence $\tilde{K}_s$ and $\tilde{K}_t$ decompose G at Q_s in the same way.

/

5.8. *If two quasigraphs of an $\tilde{\mathcal{R}}$-family support each other [intersect each other, both decompose G in the same way] at $Q_s \in \tilde{M}$, then so do any two quasigraphs of that family.*

Proof.

Case 1. Let $\tilde{K}_v$ and $\tilde{K}_u$ decompose G at Q_s in the same way. Then $\tilde{K}_v^{1} \cap \tilde{K}_v^{-1} \cap N = \tilde{K}_v^{-1} \cap \tilde{K}_u^{1} \cap N = \emptyset$ for any small neighbourhood N of Q_s. Thus by 5.6, $\tilde{K}_s^{-1} \cap \tilde{K}_t^{1} \cap N = \emptyset$ and $\tilde{K}_s^{-1} \cap \tilde{K}_t^{1} \cap N = \emptyset$ for any $s,t \in I$. If $P \in N \cap [\tilde{K}_s] \setminus \tilde{M}$, then $P \in N \cap \tilde{K}_t^{\alpha}$ or $P \in N \cap \tilde{K}_t^{-\alpha}$. Let $P \in N \cap \tilde{K}_t^{\alpha}$. Then by 1.10, $N \cap \tilde{K}_s^{-\alpha} \cap \tilde{K}_t^{\alpha} \neq \emptyset$; a contradiction. Thus $N \cap [\tilde{K}_s] \setminus \tilde{M} = \emptyset$. Similarly $N \cap [\tilde{K}_t] \setminus \tilde{M} = \emptyset$.

Now $N \subset (\tilde{K}_s^1 \cap \tilde{K}_t^1) \cup (\tilde{K}_s^{-1} \cap \tilde{K}_t^{-1}) \cup (\tilde{K}_s^1 \cap \tilde{K}_t^{-1}) \cup (\tilde{K}_s^{-1} \cap \tilde{K}_t^1) \cup \tilde{M}$.

Hence $N \subset (\tilde{K}_s^1 \cap \tilde{K}_t^1) \cup (\tilde{K}_s^{-1} \cap \tilde{K}_t^{-1}) \cup \tilde{M}$.

Thus
$$
\begin{aligned}
N \cap \tilde{K}_s^1 &\subset ((\tilde{K}_s^1 \cap \tilde{K}_t^1) \cup (\tilde{K}_s^{-1} \cap \tilde{K}_t^{-1}) \cup \tilde{M}) \cap \tilde{K}_s^1 \\
&= (\tilde{K}_s^1 \cap \tilde{K}_t^1 \cap \tilde{K}_s^1) \cup (\tilde{M} \cap \tilde{K}_s^1) \\
&= (\tilde{K}_s^1 \cap \tilde{K}_t^1) \cup (\tilde{M} \cap \tilde{K}_s^1) \\
&= \tilde{K}_s^1 \cap (\tilde{K}_t^1 \cup \tilde{M}) \\
&\subset \tilde{K}_t^1 \cup \tilde{M} .
\end{aligned}
$$

Thus $N \cap \tilde{K}_s^1 \subset (\tilde{K}_t^1 \cup \tilde{M}) \cap N = (\tilde{K}_t^1 \cap N) \cup (\tilde{M} \cap N)$.

But $N \cap \tilde{K}_s^1 \cap \tilde{M} \cap N = \emptyset$. Therefore $N \cap \tilde{K}_s^1 \subset N \cap \tilde{K}_t^1$.

Symmetrically $N \cap \tilde{K}_t^1 \subset N \cap \tilde{K}_s^1$. Hence $N \cap \tilde{K}_s^1 = N \cap \tilde{K}_t^1$ and $N \cap \tilde{K}_s^{-1} = N \cap \tilde{K}_t^{-1}$ and $\tilde{K}_s$ and $\tilde{K}_t$ decompose N in the same way.

<u>Case 2</u>. Let two quasigraphs of $\tilde{\mathcal{A}}$ intersect each other at Q_s. Then by <u>5.6</u>, $\tilde{K}_s^1 \cap \tilde{K}_t^{-1} \cap N \neq \emptyset$ and $\tilde{K}_s^{-1} \cap \tilde{K}_t^1 \cap N \neq \emptyset$ for all $s,t \in I$ and for all neighbourhoods N of Q_s. By <u>5.7</u>, $\tilde{K}_s$ and $\tilde{K}_t$ intersect, support or decompose G at Q_s in the same way. But by case 1, any two quasigraphs of $\tilde{\mathcal{A}}$ decompose G at Q_s in the same way; a contradiction. If $\tilde{K}_s$ and $\tilde{K}_t$ support each other at Q_s, then by <u>5.5</u> and <u>5.6</u> there exist a neighbourhood N of Q_s and an $\alpha \in \{1,-1\}$ such that $\tilde{K}_s^{\alpha} \cap \tilde{K}_t^{-\alpha} \cap N = \emptyset$ for all $s,t \in I$; a contradiction.

Hence $\tilde{K}_s$ and $\tilde{K}_t$ intersect each other at Q_s for all $s,t \in I$.

Case 3. Let two quasigraphs of $\tilde{\mathcal{A}}$ support each other at Q_s. Then by 5.7, they do not intersect each other at Q_s, or decompose each other at Q_s in the same way. If $\tilde{K}_s$ and $\tilde{K}_t$ do not support each other at Q_s, then by cases 1 and 2, any two quasigraphs of $\tilde{\mathcal{A}}$ do not support each other at Q_s; a contradiction.

Hence $\tilde{K}_s$ and $\tilde{K}_t$ support each other at Q_s for all $s,t \in I$.

5.9. Let $s \neq t$; $s,t \in I$ and $Q_s \in \tilde{M}$. Then K_s and K_t intersect each other at Q_s if and only if

$$[\tilde{K}_s] \cap \tilde{K}_t^{\alpha} \cap N \neq \emptyset \ , \ \alpha = \pm 1$$

for every neighbourhood N of Q_s.

Proof. Let $[\tilde{K}_s] \cap \tilde{K}_t^{\alpha} \cap N \neq \emptyset$, $\alpha = \pm 1$, for every neighbourhood of Q_s. Now $\tilde{K}_t^{\alpha} \cap N$ is an open and a non-empty subset of G. Hence by 1.10, $\tilde{K}_s^{1} \cap \tilde{K}_t^{\alpha} \cap N \neq \emptyset$ and $\tilde{K}_s^{-1} \cap \tilde{K}_t^{\alpha} \cap N \neq \emptyset$ for all neighbourhoods N of Q_s and $\alpha = \pm 1$. By 5.1.2, $K_s^{1} \cap K_t^{\pm 1} \cap N \neq \emptyset \neq K_s^{-1} \cap K_t^{\pm 1} \cap N$. Thus by definition K_s and K_t intersect at Q_s.

Conversely, suppose K_s and K_t intersect at Q_s. Then $\tilde{K}_s$ and $\tilde{K}_t$ intersect at Q_s. Assume that $s < t$. Let $u < s$. Then by 5.8, $\tilde{K}_u$ and $\tilde{K}_t$ intersect at Q_s and by 5.4, there is for each $\alpha \in \{1,-1\}$ a face F_s of K_s such that $F_s \cap \tilde{K}_t^{\alpha} \cap \tilde{K}_u^{-\alpha} \cap N \neq \emptyset$ and $Q_s \in \overline{F_s}$. Thus $[\tilde{K}_s] \cap \tilde{K}_t^{\alpha} \cap N \neq \emptyset$ for all neighbourhoods N of Q_s.

5.10. *Let $s \neq t$; $Q_s \in \tilde{M}$. Then K_s and K_t support each other at Q_s if and only if the following are satisfied:*

(5.10-1) $[\tilde{K}_s] \cap N \neq [\tilde{K}_t] \cap N$;

(5.10-2) $[\tilde{K}_s] \cap N \subset \tilde{K}_t^{\alpha} \cup \tilde{M}$ *and* $[\tilde{K}_t] \cap N \subset \tilde{K}_s^{-\alpha} \cup \tilde{M}$ *for some* $\alpha \in \{1,-1\}$.

Proof. Suppose K_s and K_t support each other at Q_s. Then $[\tilde{K}_s] \cap N \neq [\tilde{K}_t] \cap N$ by 5.1. Now for s between u and t,

$$[\tilde{K}_s] \cap N \subset (\tilde{K}_u^{-1} \cap \tilde{K}_t^{1}) \cup (\tilde{K}_u^{1} \cap \tilde{K}_t^{-1}) \cup \tilde{M} \subset \tilde{K}_t^{1} \cup \tilde{K}_t^{-1} \cup \tilde{M} .$$

By 5.9 and the fact that K_s and K_t do not intersect, we have that $[\tilde{K}_s] \cap N \cap \tilde{K}_t^{1}$ or $[\tilde{K}_s] \cap N \cap \tilde{K}_t^{-1}$ is void. Thus there is an $\alpha \in \{1,-1\}$ such that $[\tilde{K}_s] \cap N \subset \tilde{K}_t^{\alpha} \cup \tilde{M}$. Similarly there is an $\beta \in \{1,-1\}$ such that $[\tilde{K}_t] \cap N \subset \tilde{K}_s^{\beta} \cup \tilde{M}$. Hence by 1.10, $\tilde{K}_s^{\pm 1} \cap \tilde{K}_t^{\alpha} \cap N \neq \emptyset$ and $\tilde{K}_t^{\pm 1} \cap \tilde{K}_s^{\beta} \cap N \neq \emptyset$.

By 5.5 and since K_s and K_t support each other at Q_s, one of the two sets $\tilde{K}_s^1 \cap \tilde{K}_t^{-1} \cap N$ and $\tilde{K}_s^{-1} \cap \tilde{K}_t^1 \cap N$ must be void. Thus $\alpha = -\beta$.

Conversely, assume (5.10-1) and (5.10-2). Now K_s and K_t either support or intersect at Q_s or neither support nor intersect. Since $[K_s] \cap N \neq [K_t] \cap N$ then either K_s and K_t support each other at Q_s or intersect each other at Q_s. Since $[\tilde{K}_s] \cap N \subset \tilde{K}_t^{\alpha} \cap \tilde{M}$ and $[\tilde{K}_t] \cap N \subset \tilde{K}_s^{-\alpha} \cup \tilde{M}$ for some $\alpha \in \{1,-1\}$, we obtain $[\tilde{K}_s] \cap \tilde{K}_t^{-\alpha} \cap N = \emptyset$ for some $\alpha \in \{1,-1\}$. Thus by 5.9, K_s and K_t do not intersect and hence K_s and K_t support each other at Q_s.

5.11. Suppose any two quasigraphs of $\mathfrak{A}$ support each other at Q_s. Let $0 < t_1 < t_2 < \dots < t_h < 1$. Then for every small neighbourhood N of Q_s exactly $h+1$ of the 2^h open sets

$$K_{t_1}^{\pm 1} \cap \dots \cap K_{t_h}^{\pm 1} \cap N$$

are non-void; $h \geq 2$.

Proof. By 5.1.2, we may replace $\mathfrak{A}$ by $\tilde{\mathfrak{A}}$; i.e.,

$$\bigcap_1^h K_j^{\alpha_j} \cap N \neq \emptyset \longleftrightarrow \bigcap_1^h \tilde{K}_j^{\alpha_j} \cap N \neq \emptyset \text{ , } h \geq 2 \text{ .}$$

By definition the assertion is true for $h = 2$.

Suppose that h > 2 and our statement has been proved up to h-1. Then exactly h of the 2^{h-1} open sets $\tilde{K}_{t_1}^{\pm 1} \cap \ldots \cap \tilde{K}_{t_{h-1}}^{\pm 1} \cap N$ are non-void. Now $N = N \cap (\tilde{K}_{t_h}^{1} \cup \tilde{K}_{t_h}^{-1} \cup [\tilde{K}_{t_h}])$. Thus if $\tilde{K}_{t_1}^{\alpha_1} \cap \ldots \cap \tilde{K}_{t_{h-1}}^{\alpha_{h-1}} \cap N \neq \emptyset$ then either $\tilde{K}_{t_1}^{\alpha_1} \cap \ldots \cap \tilde{K}_{t_h}^{1} \cap N \neq \emptyset$ or $\tilde{K}_{t_1}^{\alpha_1} \cap \ldots \cap \tilde{K}_{t_h}^{-1} \cap N \neq \emptyset$ or by 1.10 both are non-void.

Since any two quasigraphs of $\mathcal{A}$ support at Q_s, then $[\tilde{K}_{t_h}] \cap N \setminus \tilde{M}$ is non-void. Let $P \in [\tilde{K}_{t_h}] \cap N \setminus \tilde{M}$. Then $P \notin [K_r]$ for $r \in [t_1, t_{h-1}]$. By 2.4, there is an $\alpha \in \{1,-1\}$ for which $P \in \tilde{K}_{t_1}^{\alpha} \cap \ldots \cap \tilde{K}_{t_{h-1}}^{\alpha} \cap N$. Thus by 5.10, $[\tilde{K}_{t_h}] \cap N \subset \tilde{M} \cup \tilde{K}_{t_i}^{\alpha}$, $i = 1, \ldots, h-1$; i.e., $[\tilde{K}_{t_h}] \cap N \setminus \tilde{M} \subset \tilde{K}_{t_i}^{\alpha}$. Thus $[\tilde{K}_{t_h}] \cap N \setminus \tilde{M} \subset \tilde{K}_{t_1}^{\alpha} \cap \ldots \cap \tilde{K}_{t_{h-1}}^{\alpha}$ and $[\tilde{K}_{t_h}] \cap \tilde{K}_{t_1}^{\alpha} \cap \ldots \cap \tilde{K}_{t_{h-1}}^{\alpha} \cap N \neq \emptyset$. By 1.10, $\tilde{K}_{t_1}^{\alpha} \cap \ldots \cap \tilde{K}_{t_{h-1}}^{\alpha} \cap K_{t_h}^{\pm 1} \cap N \neq \emptyset$. Thus at least h+1 of the sets $\tilde{K}_{t_h}^{\pm 1} \cap \ldots \cap \tilde{K}_{t_h}^{\pm 1} \cap N$ are non-void.

Now suppose $[\tilde{K}_{t_h}] \cap \tilde{K}_{t_1}^{\beta_1} \cap \ldots \cap \tilde{K}_{t_{h-1}}^{\beta_{h-1}} \cap N \neq \emptyset$. Then $\emptyset \neq \tilde{K}_{t_1}^{\beta_1} \cap \ldots \cap \tilde{K}_{t_{h-1}}^{\beta_{h-1}} \cap \tilde{K}_{t_h}^{\pm 1} \cap N \subset \tilde{K}_{t_i}^{\beta_i} \cap K_{t_h}^{\pm 1} \cap N$, $i = 1, \ldots, h-1$. However, $\tilde{K}_{t_i}^{\alpha} \cap \tilde{K}_{t_h}^{\pm 1} \cap N \neq \emptyset$. Hence, since $\tilde{K}_{t_i}$ and $\tilde{K}_{t_h}$ support each other at Q_s, $\beta_i = \alpha$, $i = 1,2,\ldots,h-1$. Thus $\tilde{K}_{t_1}^{\beta_1} \cap \ldots \cap \tilde{K}_{t_{h-1}}^{\beta_{h-1}}$ meets both $\tilde{K}_{t_h}^{1}$ and $\tilde{K}_{t_h}^{-1}$ only if $\beta_i = \alpha$. Thus exactly h+1 of the sets $\tilde{K}_{t_1}^{\pm 1} \cap \ldots \cap \tilde{K}_{t_h}^{\pm 1} \cap N$ are non-void.

__5.11.1.__ *Suppose* $K_s^{-\alpha} \cap K_t^{\alpha} \cap N = \emptyset$ *for* $0 < s < t < 1$ *and* $\alpha \in \{1,-1\}$. *Then the* h+1 *non-void sets obtained in* __5.11__ *are*

$$\bigcap_{j=1}^{i} K_{t_j}^{\alpha} \cap \bigcap_{j=i+1}^{h} K_{t_j}^{-\alpha} \cap N \; ; \; i = 0, 1, \ldots, h ,$$

where,

$$\bigcap_{j=1}^{0} K_{t_j}^{\alpha} = \bigcap_{j=h+1}^{h} K_{t_j}^{-\alpha} = G .$$

__Proof.__ By __4.9__, not more than 2h of the 2^h sets $\bigcap_{i=1}^{h} \tilde{K}_{t_i}^{\alpha_i}$; $\alpha_i = \pm 1$ are non-void. In fact if $\bigcap_{i=1}^{h} \tilde{K}_{t_i}^{\alpha_i} \neq \emptyset$, then $\bigcap_{i=1}^{h} \tilde{K}_{t_i}^{\alpha_i} \in \{ \bigcap_{j=1}^{i} \tilde{K}_{t_j}^{\alpha} \cap \bigcap_{j=i+1}^{h} \tilde{K}_{t_j}^{-\alpha} \mid \alpha = 1,-1; \; i = 0,1,\ldots h \}$.

If $K_s^{-\alpha} \cap K_t^{\alpha} \cap N = \emptyset$, then $\bigcap_{j=1}^{i} \tilde{K}_{t_j}^{-\alpha} \cap \bigcap_{j=i+1}^{h} \tilde{K}_{t_j}^{\alpha} = \emptyset$ by __5.6__, $i = 1,\ldots,h-1$. Thus the h+1 sets

$$\bigcap_{j=1}^{i} K_{t_j}^{\alpha} \cap \bigcap_{j=i+1}^{h} K_{t_j}^{-\alpha} \cap N \; ; \; i = 0,1,\ldots,h$$

are non-void.

5.12. Suppose any two quasigraphs of $\mathfrak{A}$ intersect each other at Q_s. Let $0 < t_1 < t_2 < \ldots < t_h < 1$; $h \geq 2$. Then for every small neighbourhood N of Q_s exactly $2h$ of the 2^h open sets

$$\bigcap_1^h K_{t_i}^{\alpha_i} \quad ; \alpha_i = \pm 1 ,$$

are non-void.

Proof. By 4.9, not more than $2h$ of the open sets $\bigcap_i^h K_{t_i}^{\alpha_i}$, $\alpha_i = \pm 1$, are non-void.

The case $h = 2$ is the the definition of intersection for a pair of quasigraphs.

Suppose $h \geq 2$ and suppose our statement has been proved up to $h-1$. Then exactly $2(h-1)$ of the 2^{h-1} open sets $\tilde{K}_{t_1}^{\pm 1} \cap \ldots \cap \tilde{K}_{t_{h-1}}^{\pm 1} \cap N$ are non-void. Now $N = N \cap (\tilde{K}_{t_h}^{1} \cup \tilde{K}_{t_h}^{-1} \cup [\tilde{K}_{t_h}])$. If $\tilde{K}_{t_1}^{\alpha_1} \cap \ldots \cap \tilde{K}_{t_{h-1}}^{\alpha_{h-1}} \cap N \neq \emptyset$; $\alpha_i = \pm 1$, $i = 1, \ldots, h-1$, then either $\tilde{K}_{t_1}^{\alpha_1} \cap \ldots \cap \tilde{K}_{t_{h-1}}^{\alpha_{h-1}} \cap \tilde{K}_{t_h}^{1} \cap N \neq \emptyset$ or $\tilde{K}_{t_1}^{\alpha_1} \cap \ldots \cap \tilde{K}_{t_{h-1}}^{\alpha_{h-1}} \cap \tilde{K}_{t_h}^{-1} \cap N \neq \emptyset$ or by 1.10, both are non-void. Thus at least $2(h-1)$ of the sets $\tilde{K}_{t_1}^{\pm 1} \cap \ldots \cap \tilde{K}_{t_h}^{\pm 1} \cap N \neq \emptyset$.

By 5.9 and the fact that K_{t_h} and $K_{t_{h-1}}$ intersect at $Q_s \in \tilde{M}$, we have $[\tilde{K}_{t_h}] \cap \tilde{K}_{t_{h-1}}^{\pm 1} \cap N \neq \emptyset$. Let $P^1 \in [\tilde{K}_{t_h}] \cap \tilde{K}_{t_{h-1}}^{1} \cap N$ and let $P^{-1} \in [\tilde{K}_{t_h}] \cap \tilde{K}_{t_{h-1}}^{-1} \cap N$. Then $P^1, P^{-1} \notin \tilde{M}$ and $P^1, P^{-1} \notin [K_r]$ for $r \in [t_1, t_{h-1}]$.

Hence $P^1 \in \tilde{K}_r^1$ and $P^{-1} \in \tilde{K}_r^{-1}$ for all $r \in [t_1, t_{h-1}]$.

Thus $[\tilde{K}_{t_h}] \cap \tilde{K}_{t_1}^{\alpha} \cap \tilde{K}_{t_2}^{\alpha} \cap \ldots \cap \tilde{K}_{t_{h-1}}^{\alpha} \cap N \neq \emptyset$ for $\alpha = 1$ and for $\alpha = -1$. Hence by 1.10,

$$\tilde{K}_{t_1}^{\alpha} \cap \ldots \cap \tilde{K}_{t_{h-1}}^{\alpha} \cap \tilde{K}_{t_h}^{\pm 1} \cap N \neq \emptyset \ , \ \alpha = \pm 1 \ .$$

Hence 2h of the sets $\tilde{K}_{t_1}^{\pm 1} \cap \ldots \cap \tilde{K}_{t_h}^{\pm 1} \cap N$ are non-void.

Remark. The 2h non-void sets obtained in 5.12 are, by 4.9, the sets

$$\bigcap_{j=1}^{i} K_{t_j}^{\alpha} \cap \bigcap_{j=i+1}^{h} K_{t_j}^{-\alpha} \quad ; \quad i = 0,1,\ldots,h-1.$$

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