FOURIER TRANSFORMS OF LIPSCHITZ FUNCTIONS ON COMPACT GROUPS

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SCOPE AND CONTENTS: If a function f is in $L^p(G)$, where $1 < p \leq 2$ and G is a locally compact abelian group, it is well-known that the Fourier transform $\hat{f}$ of f lies in $L^q(\Gamma)$, where $1/p + 1/q = 1$ and Γ is the dual group of G. This thesis is concerned with how this fact can be strengthened if it is known that f satisfies a Lipschitz condition. For certain kinds of compact groups (the circle and 0-dimensional groups) we prove that if f is in Lip(α;p) then $\hat{f}$ lies in $L^\beta(\Gamma)$ for $\beta > p/(p+\alpha p-1)$, and a similar result holds for the n-dimensional torus. These results are generalizations and analogues of classical theorems of Bernstein and Titchmarsh about Fourier series and integrals. Furthermore we obtain more precise information for the case $p = 2$.

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CONTENTS

INTRODUCTION

The purpose of the present work is to study the Fourier transforms of functions that satisfy Lipschitz conditions of certain orders. Thus we study the Fourier transforms of Lipschitz function in the functions space $L^p(G)$, $1 < p \leq 2$, where G is a compact group which will be specified in due course.

Our investigation into the problem was motivated by two theorems proved by Titchmarsh [25, Theorem 84, 85] for Lipschitz functions on the real line. The principal result of those two theorems is that if f belongs to $L^p(R)$, $1 < p \leq 2$, and also belongs to $\mathrm{Lip}(\alpha;p)$, i.e.,

$$||f(x+h) - f(x)||_p = O(h^\alpha)$$

as h approaches zero, $0 < \alpha \leq 1$, then the Fourier transform $\hat{f}$ of f belongs to $L^\beta(R)$ for

$$p/(p + \alpha p - 1) < \beta \leq q = p/p-1 .$$

In [29], this theorem was studied for higher differences and for several variables as well. Ultimately we would like to prove results similar to those of Titchmarsh on any locally compact abelian group for which the concept of Lipschitz conditions makes sense (in particular on metric groups). However, in this work Titchmarsh's results are examined for the circle group T, for the n dimensional torus, T^n and for compact metrizable 0-dimensional groups.

There are three chapters in the thesis. Chapter I starts with a brief historical introduction in which we trace Lipschitz conditions back to their origin. The chapter proceeds with some basic facts about the modulus of continuity and Lipschitz conditions. Although this material is, for the most part, elementary, we have included some proofs for the sake of completeness of exposition. There does not seem to be any book or monograph devoted to a systematic exposition of Lipschitz functions. Some of the theorems are probably well known to workers in the field, but we have not seen them in print. The chapter closes with some applications of Lipschitz functions in several areas of analysis.

In chapter two we prove the analogue of Titchmarsh's Theorems 84,85 for the circle group T, and deduce the classical Bernstein theorem [2] as well as the more general theorem of Szász [23] from the main result of chapter which asserts that if f belongs to $L^p(T) \cap \mathrm{Lip}(\alpha;p)$, then the sequence of its Fourier coefficients $\{\hat{f}(n)\}$ belongs to ℓ^β, where α, β and p are as above.

There are several new results in this chapter including theorems for higher differences and for functions of several variables. For more than one variable the situation depends heavily on just which definition of Lipschitz functions is employed. With the usual definition of $\mathrm{Lip}(\alpha;p)$ for T^n the conclusion is that

$$\beta > \frac{p}{p + \frac{\alpha}{n}\ p - 1}.$$

However if the multiplicative definition of Lipschitz functions is used, then the conclusion can be strengthened to read

$$\beta > \frac{p}{p+\alpha p-1} .$$

The last section of this chapter deals with the special case where f is in $L^2(T)$ and $0 < \alpha < 1$. It turns out in this case that the results can be put in a reversible form (which is the analogue of Titchmarsh's theorem 85).

Chapter three is devoted to the study of Fourier transforms of Lipschitz functions on compact metrizable 0- dimensional groups. In 3.1 we collect some basic material on harmonic analysis on groups for further reference. Section 3.2 deals mainly with the structure of compact metrizable 0-dimensional groups; the latter part of it is an elaboration of the relevant parts of Walker's paper [27a] on Lipschitz classes on 0-dimensional groups.

In 3.3 we combine the techniques of Walker with those of Titchmarsh to prove for compact metrizable 0-dimensional groups the analogues of the above mentioned theorems of Titchmarsh. Namely we prove that if f belongs to $L^p(G)$ and is in Lip(α;p), then the Fourier transform $\hat{f}$ belongs to $L^\beta(\Gamma)$, where Γ is the dual group of G. As in chapter two, this chapter concludes by examining the special case of Lipschitz functions in $L^2(G)$.

It should be mentioned that although the condition on the exponent in those theorems is formulated as

$$\frac{p}{p+\alpha p-1} < \beta \leq q$$

in order to make the analogy with Titchmarsh's theorems clear, we could just as well have written

$$\beta > \frac{p}{p+\alpha p-1}$$

since ℓ^{β} is contained in ℓ^{γ} for $\beta < \gamma$.

CHAPTER I

LIPSCHITZ SPACES

1.1 Historical Remarks

Since the main purpose of the present work is to study the Fourier transforms of functions which satisfy Lipschitz conditions, it would be convenient to give in this chapter a general survey of Lipschitz classes available in the literature.

Lipschitz classes have been constantly employed in Fourier analysis, although they appear in the realm of trigonometric series more than they occur in Fourier transforms.

On historical grounds it is significant to observe that Lipschitz conditions have their genesis in connection with the idea of representing a function by its Fourier series.

In 1864, R. Lipschitz proved the following result. (We quote from Lipschitz [14] in the translation of Manheim [16, p.62-63]).

"We designate by g and h quantities satisfying the inequalities

$$0 \leq g < h \leq \frac{\pi}{2},$$

and [we] let $f(\beta)$ be a function which, in the interval (g,h) remains bounded between the positive and negative values of a constant; [we require] that the difference

$$f(g+\delta) - f(g)$$

approaches zero with δ [and] that the difference

$$f(\beta+\delta) - f(\beta), \text{ for } g < \beta < h$$

have an absolute value less than the product of a constant by an arbitrary positive power of δ; then the integral

$$\int_g^h \frac{f(\beta)\sin k\beta}{\sin\beta}\, d\beta$$

has a limit when k is increased indefinitely. The limit is zero if g is positive and $\frac{\pi}{2}\,\phi(0)$ if g is zero".

The last portion of the hypothesis before the conclusion was meant to say that

$$|f(\beta+\delta) - f(\beta)| \leq A\,\delta^{\alpha}\,, \qquad (1)$$

where A is constant and $\alpha > 0$, which is what we know now as a Lipschitz condition.

Dirichlet, in Crelle's Journal for 1828, proved the following theorem (Boyer [3, p.600]).

If f(x) is 2π-periodic, if for $-\pi < x < \pi$ f(x) has a finite number of maximum and minimum values and a finite number of discontinuities, and if

$$\int_{-\pi}^{\pi} f(x)dx$$

is finite, then the Fourier series converges to f(x) at all points where f(x) is continuous, and at jump-points it converges to the arithmetic mean of the right-hand and left-hand limits of the function.

Lipschitz - using his theorem which we have already mentioned - was able to improve Dirichlet's theorem by proving the following theorem concerning the representation by Fourier series of a function with an infinite number of maxima and minima (Lipschitz [14, p.307-308]).

"[The Fourier series for a function $\phi(x)$] is convergent and always has

$$\tfrac{1}{2}[\phi(x-\varepsilon) + \phi(x+\varepsilon)]$$

for its sum, ε being an arbitrary small quantity, for all x in the interval $(-\pi,\pi)$ when the function presents its oscillations for certain particular values of the variable and even in every case, with one exception, where the function presents its oscillations in certain finite segments of the total interval. This exception occurs when in a finite segment, although the difference

$\phi(x+\delta) - \phi(x)$ approaches zero with δ, ...

the difference diminishes in such a way that one can never find a positive power of δ which remains superior to [the difference] for all values of the variable contained in the segment. This evidently occurs if $\phi(x+\delta) - \phi(x)$ decreases in the same manner or more slowly than the function $1/(\log 1/\delta)$."

The other main historical instance is in connection with the existence theorems in the theory of differential equations, and again it was Lipschitz himself who saw the utility of condition (1) (see Kline [13, p.717-718]). Thus if we have a certain differential equation, then does it have a solution for given initial and boundary conditions? This was first considered by Cauchy who showed that the equation

$$\frac{dy}{dx} = f(x,y)$$

has a unique solution $y = f(x)$ with the initial condition $y_0 = f(x_0)$, assuming that $f(x,y)$ and f_y are continuous for all real values of x and y in the rectangle determined by $[x_0,x]$ and $[y_0,y]$.

In 1876, Lipschitz [15] weakened the hypothesis of the theorem, his essential condition being that for all (x,y_1) and (x,y_2) in the rectangle

$$|x-x_0| \leq a, \quad |y-y_0| \leq b$$

there is a constant K such that

$$|f(x,y_2) - f(x,y_1)| \leq K\, |y_2-y_1|.$$

It is because of this last condition that the existence theorem is called the Cauchy-Lipschitz theorem.

Lipschitz conditions are known sometimes as Lipschitz-Hölder conditions. In Mirkil's paper [17] on Lipschitz-Hölder functions, and also in other papers and books, one might get the impression that Lipschitz conditions are just the special case when $\alpha=1$, whereas Hölder conditions include the general case $0 < \alpha \leq 1$. But the previous paragraphs indicate that Lipschitz was well aware of them both as early as 1864 when Hölder was five years old [Hölder 1859-1936]. We do feel however that the outstanding celebrity of the existence theorems made the special case $\alpha=1$, rather than the general one, to come to be associated so dominantly with the name of Lipschitz.

1.2 Modulus of continuity

Lipschitz conditions are related to the concept of the modulus of continuity of a function, a brief account of which is given here.

Notations Let $C(I)$ denote the space of all bounded continuous complex-valued functions on a closed interval (finite or infinite) of R, with the norm

$$||f||_\infty = \sup_{x \in I}|f(x)| .$$

For $1 \leq p < \infty$, Let $L^p(I)$ denote the space of all functions whose pth powers are Lebesgue integrable over I, with the norm

$$||f||_p = \left[\int_I |f(x)|^p dx\right]^{1/p} < \infty .$$

For $p = \infty$, we denote by $L^\infty(I)$ the space of essentially bounded [i.e., bounded almost everywhere] functions on I, with the norm

$$||f||_\infty = \operatorname*{ess\,sup}_{x\in I}|f(x)| = \inf\{M, |f(x)| \leq M \text{ a.e.}\} .$$

<u>Definition 1.1</u> Let $f \in C(I)$ or $L^\infty(I)$ and let $\omega(\delta) = \omega(\delta;f)$ $= \sup\limits_{|h| \leq \delta}||f(x+h) - f(x)||_\infty$. Then $\omega(\delta)$ is called the <u>modulus of continuity</u> of f.

<u>Definition 1.2</u> For $f \in L^p$, $1 \leq p < \infty$, the quantity $\omega_p(\delta)$ $= \omega_p(\delta;f) = \sup\limits_{|h| \leq \delta}||f(x+h) - f(x)||_p$ is called the <u>integral modulus of continuity</u> of f.

The modulus of continuity $\omega(\delta)$ has the following properties.

<u>THEOREM 1.3</u> Let f(x) belong to C(I). Then (i) $\omega(\delta)$ is a monotone increasing function of $\delta, \delta \geq 0$.

(ii) $\lim\limits_{\delta \to 0} \omega(\delta) = 0$.

(iii) If $\frac{\omega(\delta)}{\delta} \longrightarrow 0$ as $\delta \longrightarrow 0+$, then f is a constant.

If $f \in L^p(I)$, then the same properties hold for $\omega_p(\delta)$.

<u>PROOF</u> See Natanson [18, p.76] for (i),(ii). For (iii), see

Achieser [1, p.162].

<u>Definition 1.4</u> If $f \in C(I)$, the quantity

$$\omega^*(\delta) = \sup_{|h| \leq \delta} ||f(x+2h) - 2f(x+h) + f(x)||_\infty$$

is called the <u>generalized modulus of continuity of f</u>.

The generalized integral modulus of continuity $\omega_p^*(\delta)$ is defined for functions in L^p, $1 \leq p < \infty$, as

$$\omega_p^*(\delta) = \sup_{|h| \leq \delta} ||f(x+2h) - 2f(x+h) + f(x)||_p \ .$$

<u>Definition 1.5</u> The rth difference of f with step h is defined by

$$\Delta_h^r f(x) = \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} f(x+jh), \text{ r being an integer.}$$

<u>Definition 1.6</u> The rth <u>modulus of continuity</u> $\omega^r(\delta)$ is defined as

$$\omega^r(\delta) = \sup_{|h| \leq \delta} ||\Delta_h^r f||_\infty \quad f \text{ in } C(I) \ .$$

For f in L^p, $1 \leq p < \infty$, one can also define the <u>rth integral modulus of continuity</u> $\omega_p^r(\delta)$ to be

$$\omega_p^r(\delta) = \sup_{|h| \leq \delta} ||\Delta_h^r f||_p \ .$$

It turns out that $\omega^*(\delta)$ and $\omega_p^*(\delta)$ are just the special cases (r=2)

1.3 Lipschitz conditions

Definition 1.7 Let $f(x)$ be defined over the closed interval I, and let

$$|f(x+h) - f(x)| \leq M|h|^{\alpha}$$

for all x in I and for all sufficiently small h, M being a constant which may depend on f. Then we say that f satisfies a Lipschitz condition of order α, or f belongs to Lip(α).

Definition 1.8 If however

$$\frac{|f(x+h) - f(x)|}{h^{\alpha}} \longrightarrow 0 \text{ as } h \longrightarrow 0 \text{ ,}$$

then f is said to belong to the little Lipschitz class lip(α).

Remark 1.9 It follows immediately from these definitionsthat lip(α) $\subset$Lip(α) and that any function in Lip(α), $\alpha > 0$, is continuous.

O - o Notations The two symbols O, o (read "of the order of", and "of the little order of", respectively) will be widely used in this work, and so they will be defined here.

Definition 1.10 Let f and g be two functions of x such that

$$\leq A \text{ for all } x > x \quad [\text{or} |x|<\delta] .$$

Then we write

$$f(x) = O(g(x)) \text{ as } x \longrightarrow \infty \ [\text{or } x \longrightarrow 0] .$$

If, on the other hand,

$$\lim_{x \longrightarrow a} \left|\frac{f(x)}{g(x)}\right| = 0 ,$$

we write $f(x) = o(g(x))$ as $x \longrightarrow a$.

With these notations in hand, if $\omega(h) = O(h^{\alpha})$ as $h \longrightarrow 0$ then $f \in \mathrm{Lip}(\alpha)$, whereas if $\omega(h) = o(h^{\alpha})$ as $h \longrightarrow 0$, then $f \in \mathrm{lip}(\alpha)$.

Definition 1.11 In addition, if $\omega^*(h) = O(h^{\alpha})$, or $\omega^*(h) = o(h^{\alpha})$, then f is said to belong to the generalized Lipschitz class, or to the generalized little Lipschitz class $\mathrm{Lip}^*(\alpha)$, $\mathrm{lip}^*(\alpha)$ respectively.

THEOREM 1.12 If $f \in \mathrm{Lip}(\alpha)$, then $f \in \mathrm{Lip}^*(\alpha)$, $0 < \alpha \leq 1$. The converse is true only for $0 < \alpha < 1$ (Butzer and Nessel [5, Theorem 2,4,2]).

The function $f(x) = \sin x \log|\sin x|$ belongs to $\mathrm{Lip}^*(1)$ but not to $\mathrm{Lip}(1)$ [5, p.76].

THEOREM 1.13 If $f \in \mathrm{Lip}(\alpha)$, $\alpha > 1$, then f is constant.

PROOF For $x \in I$ and h small we have

$$|f(x+h) - f(x)| \leq Mh^{\alpha}, \text{ i.e, } \left|\frac{f(x+h)-f(x)}{h}\right| \leq Mh^{\alpha-1} ,$$

which tends to zero with h. Thus $f'(x)$ exists and is equal to zero everywhere, and so $f(x)$ is constant.

Q. E. D.

Example 1 Let $0 < \alpha < 1$, $x > 0$, $h \geq 0$. Then

$$\frac{d}{dx}\,[(x+h)^{\alpha} - x^{\alpha}] = \alpha[(x+h)^{\alpha-1} - x^{\alpha-1}] < 0 \ .$$

Therefore, $(x+h)^{\alpha} - x^{\alpha}$ is decreasing for all $x \geq 0$. Hence, $(x+h)^{\alpha} - x^{\alpha} \leq h^{\alpha}$, which shows that x^{α} is in $\mathrm{Lip}(\alpha)$ on any positive interval.

Example 2 A special class of Lip(1) is that of functions defined and having a continuous derivative on a closed finite interval I. If f is such a function, then f' is bounded. By the mean value theorem, for each x_1, x_2 in I, there is an $\bar{x}$ between x_1, x_2 such that

$$|f(x_2) - f(x_1)| = |x_2 - x_1|\ |f'(\bar{x})| \leq M|x_2 - x_1| \ ,$$

since $f'(x)$ is bounded, i.e., f belongs to Lip(1)

In general, we prove the following theorem for Lip(1).

THEOREM 1.14 A function f belongs to Lip(1) if and only if it is the integral of a bounded function, i.e.,

$$f(x) = f(a) + \int_a^x g(t)\,dt \ , \qquad (2)$$

where $a \in I$, $g \in L^{\infty}(I)$.

PROOF Let $f \in Lip(1)$. Then for any set of intervals (α_1, β_1), (α_2, β_2), ... in the closed interval I

$$\sum_{i=1}^{n} |f(\beta_i) - f(\alpha_i)| \leq \sum_{i=1}^{n} M|\beta_i - \alpha_i|$$

(where M is the Lipschitz constant)

which shows that the sum on the left side of the last expression is less than ε for $\sum|\beta_i - \alpha_i| \leq \delta = \frac{\varepsilon}{M}$, so that f is absolutely continuous, and hence it is the integral of its derivative $f'(x)$ (which exists almost everywhere).

Thus $f(x) = f(a) + \int_a^x f'(t)dt$.

But

$$\frac{|f(x+h) - f(x)|}{h} \leq M \text{ for sufficiently small } h$$

and so $|f'(x)| \leq M$ when it exists, which indicates that $f'(x)$ is in $L^{\infty}(I)$.

On the other hand, if (2) holds with g in L^{∞}, then $|g(x)| \leq M$, a.e. say, and so

$$|f(x+h) - f(x)| \leq \left|\int_x^{x+h} g(t)\, dt\right| \leq M|h|, \text{ i.e., } f \in Lip(1).$$

Q. E. D.

<u>Relations between various Lipschitz classes</u>

We include here some of the relations that exist between various Lipschitz classes.

<u>THEOREM 1.15</u> If $\alpha < \beta$, then $Lip(\alpha) \supset Lip(\beta)$ and $lip(\alpha) \supset lip(\beta)$.

<u>PROOF</u> This simply follows from the fact that for $0 \leq h \leq 1$ we have $h^{\beta} \leq h^{\alpha}$.

Q. E. D.

<u>Definition 1.16</u> The function is said to belong to $Lip(\alpha;p)$ if

$$||f(x+h) - f(x)||_p = O(h^{\alpha}),$$

i.e., $\omega_p(h) = O(h^{\alpha})$ as $h \to o$. Similarly, $lip(\alpha;p) = \{f;\ \omega_p(h,f) = o(h^{\alpha})\}$.

<u>THEOREM 1.17</u> Suppose that I is a finite interval. (i) If $\alpha_1, \leq \alpha_2$ and $p_1 \leq p_2$, then $Lip(\alpha_1;p_1) \supset Lip(\alpha_2;p_2)$.
(ii) $Lip\ (\alpha;p) \supset Lip\ (\alpha)$ for any $p > 1$.

<u>PROOF</u> (i) If $p > 1$, let $\frac{1}{p} + \frac{1}{q} = 1$, and $M(I)$ = Lebesque measure of I. Hölder's inequality gives

$$\left\{ \frac{1}{M(I)} \int_I |f(x)|dx \right\} \leq \left[\frac{1}{M(I)} \int_I |f(x)|^p dx \right]^{1/p} \left[\frac{1}{M(I)} \int_I dx \right]^{1/q}$$

$$= \left[\frac{1}{M(I)} \int_I |f(x)|^p dx \right]^{1/p}.$$

Using this with $p = p_2/p_1$ we have

$$\frac{1}{M(I)} \int_I |f(x)|^{p_1} dx \leq \left[\frac{1}{M(I)} \int_I |f(x)|^{p_2} dx \right]^{p_1/p_2} .$$

or

$$\left[\frac{1}{M(I)} \int_I |f(x)|^{p_1} dx \right]^{1/p_1} \leq \left[\frac{1}{M(I)} \int |f(x)|^{p_2} dx \right]^{1/p_2} .$$

This inequality shows that

$$\omega_{p_1}(h) \leq K\, \omega_{p_2}(h) ,$$

and so $\mathrm{Lip}(\alpha_2;p_2) \subset \mathrm{Lip}(\alpha_2;p_1)$.

But as in theorem 1.15, we have $\mathrm{Lip}(\alpha_2;p_1) \subset \mathrm{Lip}(\alpha_1;p_1)$, and so $\mathrm{Lip}(\alpha_1;p_1) \supset \mathrm{Lip}(\alpha_2;p_2)$.

(ii) If $f \in \mathrm{Lip}(\alpha)$, then $|f(x+h) - f(x)| \leq M|h|^\alpha$ for all $x \in I$. Thus

$$\int_I |f(x+h) - f(x)|^p dx \leq \int_I M^p |h|^{\alpha p} dx = M^p |h|^{\alpha p} M(I)$$

$$\Rightarrow \quad \omega_p(h) \leq M\, [M(I)]^{1/p} |h|^\alpha$$

$$\Rightarrow \quad f \in \mathrm{Lip}(\alpha;p) .$$

Q. E. D.

<u>Notice</u> Theorem 1.17 also holds with Lip replaced by lip or $\overset{*}{\mathrm{Lip}}$.

THEOREM 1.18 Let $f, g \in \mathrm{Lip}(\alpha)$ on the closed finite interval I. Then $fg \in \mathrm{Lip}(\alpha)$.

PROOF Since $f, g \in \mathrm{Lip}(\alpha)$, we have, for all x in I,

$$|f(x+h) - f(x)| \leq M_f h^{\alpha}, \quad |g(x+h) - g(x)| \leq M_g h^{\alpha}.$$

It is clear that $|f(x+h)g(x+h) - f(x)g(x)|$

$$= |f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)|$$

$$\leq |g(x+h)||f(x+h) - f(x)| + |f(x)||g(x+h) - g(x)|.$$

But f, g are bounded on I since both belong to $\mathrm{Lip}(\alpha)$, and therefore are continuous. Thus $|f(x+h)g(x+h) - f(x)g(x)| \leq AM_f h^{\alpha} + BM_g h^{\alpha}$, where $|f| \leq B$, $|g| \leq A$ over I. Hence $|f(x+h)g(x+h) - f(x)g(x)| \leq kh^{\alpha}$, and k is independent of h, which shows that $fg \in \mathrm{Lip}(\alpha)$.

Q. E. D.

1.4 Lipschitz Conditions on Metric Spaces

Lipschitz conditions can be defined for functions on general metric spaces.

Definition 1.19 Let $[X, d_1]$, $[Y, d_2]$ be metric spaces, g being a real-valued continuous function such that $g(0) = 0$, $g(t) > 0$ for $t > 0$. Let f be a function from a subset S of X into Y such that

$$d_2(f(x_2), f(x_1)) \leq g(d_1(x_1,x_2))$$

for all x_1, x_2 in S. In case $g = Mt^\alpha$, the continuity condition becomes

$$d_2(f(x_1), f(x_2)) \leq M(d_1(x_1,x_2))^\alpha$$

for all x_1, $x_2 \in S$.

If the last expression holds, we say that f satisfies a Lipschitz condtion of order α.

We specialize definition 1.19 to the case where $X = R^n$ and $Y = C$ (the complex numbers).

There

$|f(x) - f(y)| \leq M[d(x,y)]^\alpha$, where

$x = (x_1,x_2,\ldots, x_n)$, $y = (y_1,y_2,\ldots,y_n)$. For d we could take any of the following equivalent metrics:

$$d_1(x,y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \ldots + (x_n - y_n)^2},$$

$$d_2(x,y) = |x_1 - y_1| + |x_2 - y_2| + \ldots + |x_n - y_n|,$$

$$d_3(x,y) = \max[|x_i - y_i|, i = 1, 2, \ldots, n].$$

They give the same class of Lipschitz functions. In particular, $d_2(x,y)$ gives

$$|f(x_1+h_1,\ x_2+h_2,\ \ldots,\ x_n+h_n) - f(x_1,\ x_2,\ldots,x_n)| \leq M[|h_1| + |h_2| + \ldots + |h_n|]^{\alpha}.$$

This is equivalent to

$$|f(x_1+h_1,\ x_2+h_2,\ \ldots,\ x_n+h_n) - f(x_1,\ x_2,\ldots,x_n)| \leq M[|h_1|^{\alpha} + |h_2|^{\alpha} + \ldots + |h_n|^{\alpha}].$$

Remark 1.20 Some authors use the following definition for Lipschitz functions of several variables.

Definition 1.21 $f(x) = f(x_1, x_2, \ldots, x_n)$ is said to belong to $\mathrm{Lip}(\alpha_1, \alpha_2, \ldots, \alpha_n)$ if

$$|f(x_1+h_1,\ x_2+h_2,\ \ldots,\ x_n+h_n) - f(x_1,\ x_2,\ \ldots,\ x_n)| \leq M[|h_1|^{\alpha_1} + |h_2|^{\alpha_2} + \ldots + |h_n|^{\alpha_n}].$$

This reduces to the above special case if

$$\alpha_1 = \alpha_2 = \ldots = \alpha_n = \alpha .$$

THEOREM 1.22 A function is in $\mathrm{Lip}(\alpha_1, \alpha_2, \ldots, \alpha_n)$ $\Longleftrightarrow$ it is in $\mathrm{Lip}(\alpha_i)$ as a function of x_i.

PROOF Let $f \in \mathrm{Lip}(\alpha_i)$ with respect to each x_i; $i=1,2,\ldots,n$, i.e.,

$$|f(x_1, x_2, \ldots, x_i+h_i, \ldots, x_n) - f(x_1, x_2, \ldots, x_n)| \leq M_i|h_i|^{\alpha_i} .$$

Then

$$\begin{aligned} |f(x_1+h_1, x_2+h_2, \ldots, x_n+h_n) - f(x_1, x_2, \ldots, x_n)| &\leq M_1|h_1|^{\alpha_1} + M_2|h_2|^{\alpha_2} + \ldots + M_n|h_n|^{\alpha_n} \\ &\leq M[|h_1|^{\alpha_1} + |h_2|^{\alpha_2} + \ldots + |h_n|^{\alpha_n}] . \end{aligned}$$

On the other hand, let the last expression hold. Then it holds for the particular choice $h_j = 0$ $(j \neq i)$, i.e.,

$$|f(x_1, x_2, \ldots, x_i+h_i, \ldots x_n) - f(x_1, x_2, \ldots, x_n)| \leq M|h_i|^{\alpha_i} \text{ for } i = 1, 2, \ldots, n.$$

Q. E. D.

A still further definition of Lipschitz functions of several variables is available in the literature (Bugaec [4]).

<u>Definition 1.23</u> $f(x,y) \in \mathrm{Lip}^M(\alpha, \beta)$ if

$$|f(x+h, y+k) - f(x, y+k) - f(x+h, y) + f(x,y)| \leq C\, h^{\alpha} k^{\beta} ,$$

$$0 < \alpha,\ \beta \leq 1.$$

This definition does not seem to be equivalent to definition 1.21.

Similarly, one can define Lip($\alpha,\beta;p$) for functions of two variables to be all $f = f(x,y)$ such that

$$||f(x+h,y+k) - f(x,y)||_p = O(h^\alpha + k^\beta) ,$$

and $\text{Lip}^M(\alpha,\beta;p)$ to be all $f = f(x,y)$ such that

$$||f(x+h,y+k) - f(x,y+k) - f(x+h,y) + f(x,y)||_p = O(h^\alpha k^\beta) .$$

For these classes, the analogues of theorems 1.15 and 1.17 hold, viz.,

(i) $\text{Lip}(\alpha_1,\beta_1) \supset \text{Lip}(\alpha_2,\beta_2)$ for $\alpha_1 \leq \alpha_2, \beta_1 \leq \beta_2$.

(ii) $\text{Lip}(\alpha, \beta; p) \supset \text{Lip}(\alpha, \beta)$.

(iii) $\text{Lip}(\alpha_1,\beta_1;p_1) \supset \text{Lip}(\alpha_2,\beta_2;p_2)$ for $\alpha_1 \leq \alpha_2$, $\beta_1 \leq \beta_2$, $p_1 \leq p_2$.

Where (ii), (iii) hold for bounded domains.

1.5 Lipschitz Spaces

Although Lipschitz functions were known for a long time, their systematic study started rather late. Previously, they were encountered through their wide applications in the field of analysis. It is quite recently (in the last decade or so) that mathematicians have started to study spaces of Lipschitz functions for their own sake. Several authors have contributed to the field of Lipschitz spaces. Their contributions cover various aspects of the theory. However, a few papers deserve special consideration; they will be mentioned in due course.

In theorem 1.18, we proved that the product of two functions of a certain Lipschitz class ~~is~~ again in that class. This indicates that Lip(α) is in fact an algebra.

In the previous section, Lipschitz functions were defined on a general metric space [X,d], with a special orientation to the Euclidean space R^n. A norm can be introduced on this class of functions to make it a Banach space.

Let Lip(X,d,α) denote the collection of all bounded complex-valued functions defined on the metric space [X,d] that satisfy the Lipschitz condition of order α in the metric d. Thus Lip(X,d,α) consists of all f in X such that both

$$||f||_\infty = \sup[|f(x)|; \ x \in X] \text{ and}$$

$$||f||_d = \sup[|f(x) - f(y)|/[d(x,y)]^\alpha; x,y \in X, \ x \neq y]$$

are finite. We define the norm $|| \ ||$ to be

$$||f|| = ||f||_\infty + ||f||_d.$$

With this norm, Lip(α) is a Banach space.

Taibleson [24] in his study of the theory of Lipschitz spaces of distributions, introduces several norms on various Lipschitz classes and discusses the equivalence of those norms. It turns out that lip(α) is a closed linear subspace of Lip(α). De Leeuw [7] proved that the space Lip(α) is isometrically isomorphic to the second dual of lip(α), i.e.,

$$\mathrm{Lip}(\alpha) \cong \mathrm{lip}^{**}(\alpha).$$

The simple fact that Lip(α) is an algebra on the one hand, and a Banach space (with the appropriate norm) on the other, motivated some authors to study it as a Banach algebra (Sherbert [22]).

In particular, we shall be concerned with the Banach algebra Lip(T,α) (where T is the circle group) with norm

$$||f|| = ||f||_\infty + \sup_{\substack{t\in T \\ h\neq 0}} \left|\frac{f(t+h) - f(t)}{h^\alpha}\right| .$$

1.6 Applications of Lipschitz conditions

We have already mentioned in the historical remarks the application of Lipschitz conditions to existence theorems in differential equations, and to the representation of functions by Fourier series. Another important application in the field of Fourier analysis is the following theorem of S. Bernstein [2].

THEOREM 1.24 If $f(x) \in \mathrm{Lip}(\alpha)$, $1/2 < \alpha \leq 1$, then the Fourier series of f is absolutely convergent.

The two theorems proved by Titchmarsh[25, Theorems 84, 85], which are related to Bernstein's theorem will be discussed thoroughly in Chapters two and three.

For the Hilbert transforms of Lipschitz functions, Titchmarsh proved the following theorem.

THEOREM 1.25 [25, THEOREM 106] Let $f \in L^p(R)$, $1 < p$, and let $f \in \mathrm{Lip}(\alpha)$. Then the Hilbert reciprocal formulae

$$g(x) = \frac{1}{\pi} \int_0^\infty \frac{f(x+t) - f(x-t)}{t}\, dt$$

and

$$f(x) = \frac{-1}{\pi} \int_0^\infty \frac{g(x+t) - g(x-t)}{t}\, dt$$

hold for all x, and $g(x) \in L^p$ and belongs to $\mathrm{Lip}(\alpha)$ too.

It is well known that if $f \in \mathrm{Lip}(\alpha)$, then the nth Fourier coefficient is $O(n^{-\alpha})$ as $n \longrightarrow \infty$.

In one of his notes on Fourier analysis, Izumi [12] remarked that if $f \in \mathrm{Lip}(\alpha)$, $0 < \alpha \leq 1$ then it is not generally true that

$$f(x) - S_n(x) = O(n^{-\alpha})$$

where $S_n(x)$ is the nth partial sum of the Fourier series of f. However, he proves the following.

THEOREM 1.26 Let $f(x) \in \mathrm{Lip}(\alpha;p)$, $0 < \alpha \leq 1$, $p > 1$, $\alpha p > 1$. Then

$$f(x) - S_n(x) = O\left(\frac{1}{n^{\alpha - 1/p}}\right)$$

uniformly almost everywhere.

We conclude this section by giving some applications of Lipschitz functions to the theory of approximation. In conformity with the present work, we confine ourselves to approximation of functions by trigonometric polynomials.

Let $f(x)$ be a 2π-periodic continuous function. $T(x)$ denotes the trigonometric polynomial of nth degree

$$T(x) = a_o + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx) ,$$

a_o, a_k, b_k being real. H_n^T denotes the set of all trigonometric polynomials not exceeding nth order.

$\Delta(T) = \text{Max}|T(x) - f(x)|$ is called the deviation of $T(x)$ from $f(x)$.

Now let $T(x)$run through the entire set H_n^T. Then we obtain a whole set of non-negative deviations. Let the exact lower bound of this set be

$$E_n = E_n(f) = \inf \{\Delta(T)\} .$$

Then E_n is called the least deviation from or the best approximation to f by polynomials belonging to H_n^T.

We now mention some applications of Lipschitz functions to approximation theory.

THEOREM 1.27 [Jackson's theorem] [18, p.84]

For each 2π-periodic continuous function

$$E_n \leq 12\omega(1/n) ,$$

where ω is the modulus of continuity.

One corollary of this theorem is the following

THEOREM 1.28 If f is in Lip(α), $0 < \alpha \leq 1$.

$$E_n \leq \frac{12M}{n^\alpha} .$$

The following theorem of S. Bernstein is the converse of Jackson's theorem.

THEOREM 1.29 If f is a 2π-periodic continuous function, and if for each n

$$E_n \leq \frac{A}{n^\alpha} , \quad 0 < \alpha \leq 1 .$$

Then $f(x) \in \mathrm{Lip}(\alpha)$.

Combining this theorem with theorem 1.28 we get

THEOREM 1.30 If f(x) is a 2π-periodic continuous function, then f is in Lip(α), $0 < \alpha \leq 1$ if and only if

$$E_n \leq \frac{A}{n^\alpha} .$$

This gives a good characterization of Lipschitz functions in terms of the best approximation by trigonometric polynomials.

CHAPTER TWO

Lipschitz Functions on the Circle Group

2.1 Introduction

In this chapter we investigate the Fourier transforms of Lipschitz functions in $L^p(T)$, where T is the circle group of complex numbers with absolute value 1 (which can be identified with the interval $[-\pi,\pi]$). Unless the contrary is stated p will denote a number such that $1 < p \leq 2$, and q is related to p by the relation $1/p + 1/q = 1$. All integrations are taken over T, and all summations are taken over the range $[-\infty,\infty]$ unless otherwise stated. Thus we write

$$\int f(x)dx = \int_T f(x)dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx$$

in order to be consistent with the convention for Haar measure on compact groups (§ 3.1). The norm on $L^p(T)$ is given by

$$||f||_p = \left[\int |f(x)|^p dx\right]^{1/p} = \left[\frac{1}{2\pi}\int_{\pi}^{\pi} |f(x)|^p dx\right]^{1/p}.$$

<u>Definition 2.1</u> Let f belong to L'(T). The nth Fourier coefficient of f is defined by

$$\hat{f}(n) = \int_T f(x)e^{-inx}dx \;;$$

the Fourier series of f is the trigonometric series

$$S[f] = \sum \hat{f}(n)e^{inx} .$$

The function $\hat{f}$ defined on Z (the integers) is referred to as the Fourier Transform of f. For Z the L^p spaces are denoted by $\ell^p = L^p(Z) = \{g; \sum |g(n)|^p < \infty\}$.

If f belongs to $L'(R)$, where R is the real line, the Fourier Transform of f is the function $\hat{f}$ defined by

$$\hat{f}(u) = \int_{-\infty}^{\infty} f(x)e^{-iux}dx .$$

If f belongs to $L^p(R)$, $1 < p \leq 2$, then $\hat{f}$ can still be defined as in Titchmarsh [25, p.96].

Titchmarsh proved the following two theorems.

<u>THEOREM 2.2</u> If f belongs to $L^p(R)$, $1 < p \leq 2$, and if

$$||f(x+h) - f(x)||_p = O(h^\alpha) \text{ as } h \longrightarrow 0,\ 0 < \alpha \leq 1,$$

then the Fourier transform $\hat{f}$ of f belongs to $L^\beta(R)$ for $p/(p + \alpha p - 1) < \beta \leq q$. [25, Theorem 84].

<u>THEOREM 2.3</u> If f belongs to $L^2(R)$, then the conditions

$$||f(x+h) - f(x)||_2 = O(h^\alpha) \text{ as } h \longrightarrow 0, 0 < \alpha < 1,$$

and $\left[\int_{-\infty}^{-X} + \int_{X}^{\infty}\right] |f(u)|^2 du = O[X^{-2\alpha}]$ as $X \longrightarrow \infty$ are equivalent. [25, theorem 85].

The generalization of Titchmarsh's theorems to higher differences and to functions on R^n was the subject matter of an M. Phil. thesis by the present author [29]. In this chapter we develop the subject for the circle group T. The following lemma will be needed in due course.

<u>Lemma 2.4</u> (Duren [8, p.101]) Suppose $b_n \geq 0$, and $0 < c < d$. Then

$$\sum_{n=1}^{N} n^d b_n = O(N^c) \text{ iff } \sum_{n=N}^{\infty} b_n = O(N^{c-d}) .$$

<u>PROOF</u>. Let the first condition hold, and let

$S_n = \sum_{k=1}^{n} k^d b_k$. Then using the partial summation formula

(Rudin [21, Theorem 3.41]) we have

$$\begin{aligned} \sum_{n=N}^{M} b_n &= \sum_{n=N}^{M-1} S_n[n^{-d} - (n+1)^{-d}] + S_M M^{-d} - S_{N-1} N^{-d} , \\ &\leq C \sum_{n=N}^{M-1} n^{c-d-1} + CM^{c-d} . \end{aligned}$$

Letting $M \longrightarrow \infty$, we obtain the second assertion. Conversely, if the second assertion holds, let

$$R_n = \sum_{k=n}^{\infty} b_k \text{ , then}$$

$$\sum_{n=1}^{N} n^d b_n = \sum_{n=1}^{N}[n^d - (n-1)^d]R_n - N^d R_{N+1}$$

$$\leq CN^c .$$

Q. E. D.

In due course, lemma 2.4 will be needed in dealing with functions of several variables. Its generalization for double sequences is the following.

Lemma 2.5 Let $B_{mn} \geq 0$, and $0 < a < b$, $0 < c < d$. Then the conditions

$$\sum_{m=1}^{M} \sum_{n=1}^{N} m^b n^d B_{mn} = O[M^a N^c] \tag{1},$$

$$\sum_{m \geq M} \sum_{n=1}^{N} n^d B_{mn} = O[M^{a-b} N^c] \tag{2},$$

$$\sum_{m=1}^{M} \sum_{n \geq N} m^b B_{mn} = O[M^a N^{c-d}] \tag{3},$$

and

$$\sum_{m \geq M} \sum_{n \geq N} B_{mn} = O[M^{a-b} N^{c-d}] \tag{4}.$$

are equivalent.

PROOF Let $C_{mN} = \sum_{n=1}^{N} n^d B_{mn}$. Then $C_{mN} \geq 0$. By applying Lemma 2.4,

$$\sum_{m=1}^{M} m^b\, C_{mN} = O[M^a N^c] \iff \sum_{m\geq M} C_{mN} = O[M^{a-b} N^c] .$$

Thus (1) and (2) are equivalent. Similarly, (1) and (3) are equivalent.

Let $D_{mN} = \sum_{n\geq N} B_{mn}$ (which converges if (3) or (4) holds). Applying Lemma 2.4 again, one has

$$\sum_{m=1}^{M} m^b D_{mn} = O[M^a N^{c-d}] \iff \sum_{m\geq M} D_{mn} = O[M^{a-b} N^{c-d}] ,$$

which shows that (3) implies (4) and conversely, and hence the four conditions are equivalent.

Q. E. D.

2.2 Fourier Transforms of Lipschitz Functions in $L^p(T)$, $1 < p \leq 2$

We now prove the following theorem for the circle group T, which is the analogue of theorem 2.2.

THEOREM 2.6 Let f belong to $L^p(T)$, $1 < p \leq 2$, and let f also belong to $\mathrm{Lip}(\alpha;p)$, i.e.,

$$||f(x+h) - f(x)||_p = O(h^\alpha) \text{ as } h \longrightarrow 0,$$

$0 < \alpha \leq 1$. Then $\hat{f}(n)$ belongs to ℓ^β for

$$p/(p + \alpha p - 1) < \beta \leq q = p/p - 1 .$$

PROOF First we note that the conclusion of the theorem is true for $\beta = q$ by the Hausdorff-Young theorem (3.8) and so we may assume that $\beta < q$.

For a fixed h, the Fourier transform of $f(x+h)$ is $e^{inh}\hat{f}(n)$, and so the Fourier transform of $f(x+h) - f(x)$ is $[e^{inh}-1]\hat{f}(n) = 2ie^{inh/2} \sin\frac{nh}{2}\hat{f}(n)$. The Hausdorff-Young theorem would give

$$\sum|2\sin\frac{nh}{2}\hat{f}(n)|^q \leq [||f(x+h) - f(x)||_p]^q = O[h^{\alpha q}] .$$

Since $|\sin\frac{nh}{2}| > Anh$ for $n \leq \frac{1}{h}$, (A constant), we have

$$\sum_{n=0}^{\infty}|\sin\frac{nh}{2}\hat{f}(n)|^q > A' \sum_{n=1}^{[1/h]}|nh\hat{f}(n)|^q ,$$

from which we obtain

$$\sum_{n=1}^{[1/h]}|n\hat{f}(n)|^q = O[h^{(\alpha-1)q}] .$$

Here $[1/h]$ is the integral part of $1/h$

Let $\phi(N) = \sum_{n=1}^{N}|n\hat{f}(n)|^\beta$

Then for $\beta < q$, by taking $\gamma = \beta/q$, $\delta = 1 - \beta/q$, $a_n = |n\hat{f}(n)|^q$, $b_n = 1$ for all n, and applying Hölder's inequality in the form

$\sum a_n^{\gamma} b_n^{\delta} \leq (\sum a_n)^{\gamma} (\sum b_n)^{\delta}$ where $\gamma + \delta = 1$, we get

$$\phi(N) \leq \left[\sum_{n=1}^{N} |n\hat{f}(n)|^q\right]^{\beta/q} \left[\sum_{n=1}^{N} 1\right]^{1-\beta/q}$$

$$= 0 \left[N^{(1-\alpha)q}\right]^{\beta/q} \left[N^{1-\beta/q}\right] = 0 \left[N^{1-\alpha\beta+\beta/p}\right] .$$

Applying lemma 2.4 with $b_n = |\hat{f}(n)|^{\beta}$, $d = \beta$, $c = 1 - \alpha\beta + \beta/p$ we get

$$\sum_{n=1}^{N} n^{\beta} |\hat{f}(n)|^{\beta} = 0 \left[N^{1-\alpha\beta+\beta/p}\right] \quad \text{if and only if}$$

$$\sum_{n=N}^{\infty} |\hat{f}(n)|^{\beta} = 0 \left[N^{1-\beta-\alpha\beta+\beta/p}\right] .$$

But the validity of the Lemma in this case depends on the condition $0 < 1 - \alpha\beta + \beta/p < \beta$. The condition $0 < 1 - \alpha\beta + \beta/p$, or equivalently, $\alpha < 1/\beta + 1/p$ holds because for $\beta < q$ we have

$$\alpha \leq 1 = 1/q + 1/p < 1/\beta + 1/p .$$

The other requirement is

$$1 - \alpha\beta + \beta/p - \beta < 0, \text{ i.e., } p/(p + \alpha p - 1) < \beta, \quad \text{which}$$

we have assumed. Thus

$$\sum_{n=N}^{\infty} |\hat{f}(n)|^{\beta} = 0[N^{1-\beta-\alpha\beta+\beta/p}] .$$

This will approach 0 as $N \longrightarrow \infty$ if

$1 - \beta - \alpha\beta + \beta/p < 0$ i.e., $p/(p+\alpha p-1) < \beta$. A similar result holds for the range $[-\infty, -N]$.

Q. E. D.

Remark 2.7 The theorem of Bernstein (namely that any function in $\mathrm{Lip}(\alpha)$, $\alpha > 1/2$ has an absolutely convergent Fourier series) is a consequence of theorem 2.6. In fact a more general theorem of Szász [23] follows also from theorem 2.6. Szász proved that if f belongs to $\mathrm{Lip}(\alpha)$, then $\hat{f}$ belongs to ℓ^{β} for $\beta > 2/(2\alpha + 1)$. To see this, we note that if f is in $\mathrm{Lip}(\alpha)$, then it is in $\mathrm{Lip}(\alpha;2)$, and so applying theorem 2.6 with $p = 2$ we see that f belongs to ℓ^{β}, where $\beta > 2/(2 + 2\alpha - 1) = 2/(2\alpha + 1)$. If $\alpha > 1/2$, then $2/(2\alpha + 1) < 1$, and so $\hat{f}$ is in ℓ^{1}, i.e., it has an absolutely convergent Fourier series.

Remark 2.8 The function $f(x) = \sum_{n=1}^{\infty} \frac{e^{in\log n}}{n^{1/2+\alpha}} e^{inx}$, $0 < \alpha < 1$, (first considered by Hardy and Littlewood) is in $\mathrm{Lip}(\alpha)$ but $\hat{f} \notin \ell^{2/2\alpha+1}$ (see Zygmund [30] Vol.1, p.243), and so the range for β cannot be strengthened in theorem 2.6.

Remark 2.9 It goes without mention that the previous results hold true for functions in $\mathrm{lip}(\alpha)$. Since functions in $\mathrm{lip}(\alpha)$ satisfy a stronger condition than those in $\mathrm{Lip}(\alpha)$, one wonders whether this would improve upon the main conclusion, the range of β for example, or the inclusion of $\{\hat{f}; f \in \mathrm{lip}(\alpha)\}$ in a specific portion of ℓ^{β}?

The same remark can be made about the original result of Bernstein, and how it is affected if we confine ourselves to $lip(\alpha)$.

2.3 Differences of Higher Orders

We now examine the validity of the preceding theorem when replacing the first difference $f(x+h) - f(x)$ with a difference of higher order. Recall that the difference of order m with step h is

$$\Delta_h^m f(x) = \sum_{r=0}^{m} (-1)^{m-r} \binom{m}{r} f(x+rh).$$

THEOREM 2.10 Let $f \in L^p(T)$, $1 < p \leq 2$, and let $||\Delta_h^m f||_p = O(h^\alpha)$ as $h \longrightarrow 0$, $0 < \alpha \leq 1$. Then $\hat{f} \in \ell^\beta$ for $p/(p + \alpha p - 1) < \beta \leq q$.

PROOF Since for fixed h the Fourier transform of $f(x+rh)$ is $e^{irhn}\hat{f}(n)$, it can be seen that $\widehat{\Delta_h^m f}(n) = \sum_{r=0}^{m} (-1)^{m-r} \binom{m}{r} e^{irhn} \hat{f}(n)$

$$= (e^{inh} - 1)^m \hat{f}(n) = (2i)^m e^{i\frac{mnh}{2}} [\sin\tfrac{nh}{2}]^m \hat{f}(n) .$$

Thus the theorem of Hausdorff-Young gives in this case

$$\sum |\sin^m \tfrac{nh}{2} \hat{f}(n)|^q \leq Mh^{\alpha q} .$$

Using $|\sin \frac{nh}{2}| > Anh$ for $0 < n \leq 1/h$, we have

$$\sum_{n=0}^{[1/h]} |n^m h^m \hat{f}(n)|^q = O(h^{\alpha q}) , \quad \text{i.e.,}$$

$$\sum_{n=0}^{[1/h]}|n^m\hat{f}(n)|^q = O[h^{(\alpha-m)q}] .$$

As in the previous case, let

$$\phi(N) = \sum_{n=1}^{N} n^{m\beta}|f(n)|^{\beta}.$$

Then Hölder's inequality yields

$$\phi(N) = O[N^{(m-\alpha)\beta + 1 - \beta + \beta/p}] .$$

Applying Lemma (2.4) with $d = m\beta$,

$c = (m-\alpha)\beta + 1 - \beta + \beta/p$, $b_n = |\hat{f}(n)|^{\beta}$, we get

$$\sum_{n=N}^{\infty}|\hat{f}(n)|^{\beta} = O[N^{(m-\alpha)\beta + 1 - \beta + \beta/p - m\beta}] ,$$

which tends to 0 as $N \longrightarrow \infty$ if $p/(p + \alpha p-1) < \beta \leq q$ as before. A similar result holds for $[-\infty,-N]$ and the proof is complete.

2.4 Functions of Several Variables

For functions of several variables we introduce the following notation.

Definition 2.11 Suppose that $f = f(x_1,x_2,\ldots,x_m)$ is integrable over the m-dimensional torus T^m. The Fourier coefficient of f is defined to be

$$\hat{f}(n_1,n_2,\ldots,n_m) = \int_{T^m} f(x_1,x_2,\ldots,x_m)e^{-i(n_1x_1+n_2x_2+\ \ldots\ +n_mx_m)}dx_1dx_2\ldots dx_m.$$

For later purposes, the following lemma will be needed.

Lemma 2.12 Suppose $b_{mn} \geq 0$, $0 < a < b$, $0 < c < d$, and suppose

$$\sum_{m=1}^{M}\sum_{n=1}^{N} m^b n^d b_{mn} = O(M^a N^c) .$$

Then the series $\sum\sum b_{mn}$ is convergent.

PROOF Define the partial sums

$$S_{mn} = \sum_{k=1}^{n} k^d b_{mk},\quad t_{mk} = \sum_{\ell=1}^{m} \ell^b b_{\ell k} .$$

Using the partial summation formula, one has

$$\sum_{m=1}^{M}\sum_{n=1}^{N} b_{mn} = \sum_{m=1}^{M}\sum_{n=1}^{N}(n^d b_{mn})n^{-d}$$

$$= \sum_{m=1}^{M}[\sum_{n=1}^{N-1}[S_{mn}[n^{-d} - (n+1)^{-d}] + S_{mN}N^{-d}]$$

$$\leq \sum_{m=1}^{M}\sum_{n=1}^{N-1}\sum_{k=1}^{n} k^d b_{mk}\ .dn^{-d-1} + \sum_{m=1}^{M}\sum_{k=1}^{N} k^d b_{mk}N^{-d}$$

$$= A_{MN} + B_{MN}\ \text{, say.}$$

By the partial summation formula,

$$
\begin{aligned}
A_{MN} &= d \sum_{n=1}^{N-1} n^{-d-1} \sum_{k=1}^{n} k^{d} \sum_{m=1}^{M} (m^{b} b_{mk}) m^{-b} \\
&= d \cdot \sum_{n=1}^{N-1} n^{-d-1} \sum_{k=1}^{n} k^{d} \sum_{m=1}^{M-1} t_{Mk}[m^{-b} - (m+1)^{-b}] + t_{Mk} M^{-b} \\
&\leq bd \sum_{n=1}^{N-1} n^{-d-1} \sum_{m=1}^{M-1} m^{-b-1} \sum_{\ell=1}^{m} \sum_{k=1}^{n} \ell^{b} k^{d} b_{\ell k} \\
&\qquad + d\, M^{-b} \sum_{n=1}^{N-1} n^{-d-1} \sum_{k=1}^{n} k^{d} \sum_{\ell=1}^{M} \ell^{b} b_{\ell k} \\
&= A_{MN}^{(1)} + A_{MN}^{(2)} \text{, say.}
\end{aligned}
$$

By the hypothesis of the lemma,

$$
\begin{aligned}
A_{MN}^{(1)} &\leq K \sum_{m=1}^{M-1} m^{-b-1} \sum_{n=1}^{N-1} n^{-d-1} . m^{a} n^{c} \\
&= K \sum_{m=1}^{M-1} m^{a-b-1} \sum_{n=1}^{N-1} n^{c-d-1} ,
\end{aligned}
$$

and this sum is bounded as $M, N \longrightarrow \infty$ since $a < b$, $c < d$.

Similarly,

$$
\begin{aligned}
A_{MN}^{(2)} &\leq C\, M^{-b} \sum_{n=1}^{N-1} n^{-d-1} . M^{a} n^{c} \\
&= C\, M^{a-b} \sum_{n=1}^{N-1} n^{c-d-1} ,
\end{aligned}
$$

which approaches zero as $M, N \longrightarrow \infty$ since $a < b$, $c < d$.

In the same manner

$$B_{MN} = N^{-d} \sum_{m=1}^{M} \sum_{k=1}^{N} k^d b_{mk}$$

$$= N^{-d} \sum_{k=1}^{N} k^d \sum_{m=1}^{M} (m^b b_{mk}) m^{-b}$$

Again, by the partial summation formula,

$$B_{MN} = N^{-d} \sum_{k=1}^{N} k^d \sum_{m=1}^{M-1} t_{mk}[m^{-b}-(m+1)^{-b}] + t_{Mk}M^{-b}$$

$$\leq N^{-d} \sum_{k=1}^{N} k^d \sum_{m=1}^{M-1} bm^{-b-1} t_{mk} + t_{Mk}M^{-b}$$

$$\leq bN^{-d} \sum_{m=1}^{M-1} m^{-b-1} \sum_{\ell=1}^{m} \ell^b \sum_{k=1}^{N} k^d b_{\ell k}$$

$$+ bM^{-b}N^{-d} \sum_{\ell=1}^{M} \sum_{k=1}^{N} \ell^b k^d b_{\ell k}$$

$$= B_{MN}^{(1)} + B_{MN}^{(2)} , \quad \text{say,}$$

$$B_{MN}^{(1)} \leq CN^{-d} \sum_{m=1}^{M-1} m^{-b-1} . m^a N^c \leq CN^{c-d} \sum_{m=1}^{M-1} m^{a-b-1} ,$$

which goes to 0 as $M, N \longrightarrow \infty$ since $a < b$, $c < d$.

$$B_{MN}^{(2)} = bM^{-b}N^{-d} \sum_{\ell=1}^{M} \sum_{k=1}^{N} \ell^b k^d b_{\ell k}$$

$\leq CM^{a-b}N^{c-d}$, by the hypothesis of the lemma, which approaches 0 as $M, N \longrightarrow \infty$ for the same reason. Hence

$\sum_{m=1}^{M} \sum_{n=1}^{N} b_{mn}$ is bounded as $M, N \longrightarrow \infty$, and $b_{mn} \geq 0$,

and so the series is convergent.

Q. E. D.

For functions of two variables, one has the following theorem.

THEOREM 2.13 If $f=f(x,y)\in L^p(T^2)$, $1 < p \leq 2$ and if $||\Delta f||_p = O\,[h^{\alpha_1}k^{\alpha_2}]$ as $h,k \longrightarrow 0$, $0 < \alpha_1, \alpha_2 \leq 1$, where

$$\Delta f(x) = f(x+h,y+k) - f(x,y+k) - f(x+h,y) + f(x,y) ,$$

Then $\sum_{m\neq o,n\neq o}|\hat{f}(m,n)|^\beta<\infty$ for $p/(p + \alpha p - 1) < \beta \leq q = p/p-1$, $\alpha = \min(\alpha_1,\alpha_2)$.

PROOF As usual, a consequence of the Hausdorff-Young theorem in this case is

$$\sum_{m=1}^{[1/h]}\sum_{n=1}^{[1/k]}|mn\hat{f}(m,n)|^q = O[h^{(\alpha_1-1)q}k^{(\alpha_2-1)q}] .$$

$$\text{Let } \phi(M,N) = \sum_{m=1}^{M}\sum_{n=1}^{N}|mn|^\beta|\hat{f}(m,n)|^\beta .$$

For $\beta < q$,

$$\begin{aligned}\phi &\leq \left[\sum_{m=1}^{M}\sum_{n=1}^{N}|mn\hat{f}(m,n)|^q\right]^{\beta/q}\left[\sum_{1}^{M}\sum_{1}^{N}1\right]^{1-\beta/q}\\ &= O[M^{(1-\alpha_1)q}N^{(1-\alpha_2)q}]^{\beta/q}[MN]^{1-\beta/q}\\ &= O[M^{\beta-\alpha_1\beta+1-\beta/q}\quad N^{\beta-\alpha_2\beta+1-\beta/q}]\\ &= O[M^{1-\alpha_1\beta+\beta/p}][N^{1-\alpha_2\beta+\beta/p}] .\end{aligned}$$

Applying lemma 2.12, with $b_{mn} = |\hat{f}(m,n)|^\beta$,

$$a = 1 - \alpha_1\beta + \beta/p,\ c = 1 - \alpha_2\beta + \beta/p,\ b = d = \beta\ ,$$

one gets that

$\sum\sum|\hat{f}(m,n)|^{\beta}$ is convergent provided that

$p/(p+\alpha p-1) < \beta \leq q\ ,\ \alpha = \min(\alpha_1,\alpha_2)$.

Q. E. D.

For higher differences of $f(x,y)$, one has the following theorem.

THEOREM 2.14 Let the hypothesis of theorem 2.13 hold with $\Delta f = \Delta^J\Delta^L f$. Then the same conclusion holds, where $\Delta^J\Delta^L f$ means the Jth difference in x of the Lth difference in y of $f(x,y)$.

PROOF In this case, one arrives at

$$\sum_{m=1}^{[1/h]}\sum_{n=1}^{[1/k]}|m^J n^L\hat{f}(m,n)|^q = O[h^{(\alpha_1-J)}k^{(\alpha_2-L)}]^q\ .$$

And as in the previous theorem, let

$$\phi(M,N) = \sum_{m=1}^{M}\sum_{n=1}^{N}|m^J n^L\hat{f}(m,n)|^{\beta}\ .$$

Then for $\beta < q$,

$$\phi(M,N) = O[M^{(J-\alpha_1)\beta + 1 - \beta/q}N^{(L-\alpha_2)\beta + 1 - \beta/q}]\ .$$

Again, by applying lemma 2.12, with $a = (J-1-\alpha_1)\beta + \beta/p + 1$ $c = (L-1-\alpha_2)\beta + \beta/p + 1$, $b = J\beta$, $d = L\beta$, one arrives at the same conclusion.

Q. E. D.

The next step in this section is to examine the validity of theorem 2.13 by employing the additive form of Lipschitz condition for several variables (definition 1.21). Since this has not been dealt with in (Younis [29]), we would like to treat it for R.

THEOREM 2.15 Let $f = f(x_1, x_2, \ldots, x_n) \in L^p(R^n)$, $1 < p \leq 2$, and let

$$||f(x_1+h_1, x_2+h_2, \ldots, x_n+h_n) - f(x_1, x_2, \ldots, x_n)||_p$$

$$= O[h_1^{\alpha_1} + h_2^{\alpha_2} + \ldots + h_n^{\alpha_n}] \text{ as } h_i \longrightarrow 0$$

$$0 < \alpha_i \leq 1, \ (i = 1,2,\ldots,n) \ .$$

Then $\hat{f}(u_1, u_2, \ldots, u_n) \in L^\beta$ for

$$p/(p + \frac{\alpha}{n} p - 1) < \beta \leq q, \ \alpha = \min(\alpha_1, \alpha_2, \ldots, \alpha_n) \ .$$

PROOF In this case,

$$\int_0^{1/h_1} \cdots \int_0^{1/h_n} |u_1h_1 + u_2h_2 + \ldots + u_nh_n|^q |\hat{f}|^q du_1 du_2 \ldots du_n$$

$$= O[h_1^{\alpha_1} + h_2^{\alpha_2} + \ldots + h_n^{\alpha_n}]^q \ .$$

Let

$$G(X_1,X_2,\ldots,X_n)=\int_1^{X_1}\cdots\int_1^{X_n}|u_1+u_2+\ldots+u_n|^\beta|\hat{f}|^\beta du_1\ldots du_n ,$$

and $G(X) = G(X,X,\ldots,X)$. Then it is clear from the previous analysis that

$$\int_1^X\cdots\int_1^X \left|\frac{u_1+u_2+\ldots+u_n}{X}\right||\hat{f}|^q du_1\ldots du_n = O[X^{-\alpha q}]$$

or equivalently

$$\int_1^X\cdots\int_1^X |u_1+u_2+\ldots+u_n|^q|\hat{f}|^q du_1 du_2\ldots du_n = O[X^{(1-\alpha)q}]$$

(by taking $\alpha = \min[\alpha_1,\alpha_2,\ldots,\alpha_n]$). By Hölder's inequality

$$G(X) \leqslant \left[\int_1^X\cdots\int_1^X|u_1+\ldots+u_n|^q|\hat{f}|^q du_1\ldots du_n\right]^{\beta/q}$$

$$\times\left[\int_1^X\cdots\int_1^X du_1\ldots du_n\right]^{1-\beta/q}$$

$$= O[X^{(1-\alpha)q}]^{\beta/q}[X^{n-n\beta/q}]$$

$$= O[X^{\beta-\alpha\beta+n-n\beta+n\beta/p}] .$$

It is clear that

$$|\hat{f}|^{\beta} = |u_1 + u_2 + \ldots + u_n|^{-\beta} \frac{\partial^n G}{\partial u_1 \partial u_2 \ldots \partial u_n}$$

and so

$$\int_1^X \cdots \int_1^X |\hat{f}|^{\beta} du_1 \ldots du_n = \int_1^X \cdots \int_1^X |u_1 + \ldots + u_n|^{-\beta} \frac{\partial^n G}{\partial u_1 \ldots \partial u_n} du_1 \ldots du_n$$

$$= [|u_1 \cdots + u_n|^{-\beta} G(u_1, \ldots, u_n)]_{u_i=1}^{u_i=X}$$

$$+ \text{ other terms of the same order } + O(1)$$

$$= O[X^{-\beta}][X^{\beta - \alpha\beta + n - n\beta + n\beta/p}] + O(1)$$

$$= O[X^{-\alpha\beta + n - n\beta + n\beta/p}] + O(1) .$$

The last quantity is bounded as $X \longrightarrow \infty$ if $-\alpha\beta + n - n\beta + n\beta/p < 0$, which gives $p/(p + \frac{\alpha}{n} p - 1) < \beta \leq \alpha$, with similar results for other parts of of R^n .

Q. E. D.

<u>Remark 2.16</u> Although the conclusion of the last result is weaker than that of theorem 2.13, it has the merit of indicating how the number of variables affects the main result. The analogous result for functions in $L^p(T^n)$ can be proved using the partial summation formula instead of integration by parts.

2.5 Fourier Transforms of Lipschitz Functions in $L^2(T)$

We begin by proving the analogue in $L^2(T)$ of theorem 2.3.

THEOREM 2.17 Let f belong to $L^2(T)$. Then the conditions

$$||f(x+h) - f(x)||_2 = O(h^\alpha),\ 0 < \alpha < 1, \text{ and}$$

$$\sum_{|n| \geq N} |\hat{f}(n)|^2 = O(N^{-2\alpha}) \text{ as } N \longrightarrow \infty \text{ are equivalent.}$$

PROOF Let the first condition hold. Then by Parseval's theorem

$$\sum 4\left|\sin\frac{nh}{2}\,\hat{f}(n)\right|^2 = \int_T |f(x+h) - f(x)|^2 dx = O(h^{2\alpha}) .$$

Since $\left|\sin\frac{nh}{2}\right| > A|nh|$ for $0 \leq n \leq \frac{1}{h}$, we have

$$\sum_{n=1}^{[1/h]} |nh\hat{f}(n)|^2 = O(h^{2\alpha}) .$$

and so

$$\sum_{n=1}^{[1/h]} |n\hat{f}(n)|^2 = O[h^{2\alpha-2}] .$$

Thus $$\sum_{n=1}^{N} |n\hat{f}(n)|^2 = O(N^{2-2\alpha}) .$$

Hence lemma 2.4 gives

$$\sum_{n=N}^{\infty} |\hat{f}(n)|^2 = O(N^{2-2\alpha-2}) = O(N^{-2\alpha}),$$

since $0 < \alpha < 1$.

On the other hand, let the second condition hold. Then again by lemma 2.4

$$\sum_{1}^{N} |n\hat{f}(n)|^2 = O(N^{2-2\alpha}) ,$$

and so, taking $N = [1/h]$

$$\sum |\sin \frac{nh}{2} \hat{f}(n)|^2 \leq h^2 \sum_{n=-N}^{N} |n\hat{f}(n)|^2 + \sum_{|n| \geq N} |\hat{f}(n)|^2$$

$$= O(h^{2+2\alpha-2}) + O(h^{2\alpha}) = O(h^{2\alpha}) .$$

Hence

$$||f(x+h) - f(x)||_2 = O(h^{\alpha}) .$$

Q. E. D.

For higher differences in one variable, we have the following theorem.

THEOREM 2.18 Let the hypothesis of the previous theorem hold, with Δf being replaced by $\Delta^m f$. Then the conclusions are the same.

PROOF . A consequence of Parseval's theorem in this case is

$$\sum |\sin^m \frac{nh}{2} \hat{f}(n)|^2 = O(h^{2\alpha}) ,$$

and the first part of the proof follows the same line as in the previous one.

If the second condition holds, then by lemma 2.4

$$\sum_{n=1}^{N} |n^m \hat{f}(n)|^2 = O(N^{2m-2\alpha})$$

so that

$$\sum |\sin^m \frac{nh}{2} \hat{f}(n)|^2 \leq \sum |(\frac{nh}{2})^m \hat{f}(n)|^2 + \sum_{|n| \geq N} |\hat{f}(n)|^2$$

$$= O(h^{2m})(h^{2\alpha-2m}) + O(h^{2\alpha}) ,$$

and the rest of the proof follows.

Q. E. D.

We examine the validity of the previous two theorems for several variables. For convenience, we take up the problem for two variables.

<u>THEOREM 2.19</u> If $f = f(x,y)$ belongs to $L^2(T^2)$, then the conditions

$$||\Delta f||_2 = O(h^\alpha K^\beta) \tag{5}$$

$$\sum_{|m| > M} \sum_{|n| \leq N} |n\hat{f}(m,n)|^2 = O\left[M^{-2\alpha} N^{-2\beta+2}\right] \tag{6}$$

$$\sum_{|m| > M} \sum_{|n| \leq N} |m\hat{f}(m,n)|^2 = O\left[M^{-2\alpha+2} N^{-2\beta}\right] \tag{7}$$

and

$$\sum_{|m|>M} \sum_{|n|\leq N} |\hat{f}(m,n)|^2 = O\,[M^{-2\alpha}N^{-2\beta}] \tag{8}$$

as $M, N \longrightarrow \infty$ are equivalent. Here

$$\Delta f = f(x+h,y+k) - f(x,y+k) - f(x+h,y) + f(x,y) ,$$

and $\quad 0 < \alpha,\ \beta < 1.$

PROOF If (5) holds, then again by Parseval's theorem,

$$\sum \sum |\sin \frac{nh}{2} \sin \frac{mk}{2} \hat{f}(m,n)|^2 = O[h^{2\alpha}k^{2\beta}] ,$$

which leads to

$$\sum_{|m|=1}^{M} \sum_{|n|=1}^{N} |mn\hat{f}(m,n)|^2 = O[M^{-2\alpha+2}N^{-2\beta+2}] . \tag{9}$$

But lemma 2.5 shows that (9), (6), (7) and (8) are equivalent. Hence (5) implies each of (6), (7) and (8).

On the other hand, if any one of (6), (7) and (8) is valid, then again lemma 2.5 asserts that (9), (6), (7) and (8) are all valid. As in the single variable case, by taking $M = [1/h]$, $N = [1/k]$, we have

$$\sum \sum |\sin \frac{mh}{2} \sin \frac{nk}{2} \hat{f}(m,n)|^2$$

$$\leq h^2k^2 \sum_{|m|=1}^{M} \sum_{|n|=1}^{N} |mn\hat{f}(m,n)|^2 + h^2 \sum_{|m|=1}^{M} \sum_{|n|>N} |m\hat{f}(m,n)|^2$$

$$+ k^2 \sum_{|m|>M} \sum_{|n|=1}^{N} |n\hat{f}(m,n)|^2 + \sum_{|m|>M} \sum_{|n|>N} |\hat{f}(m,n)|^2$$

$$= O(h^{2+2\alpha-2}k^{2+2\beta-2}) + O(h^{2+2\alpha-2}k^{2\beta})$$

$$+ O(h^{2\alpha}k^{2+2\beta-2}) + O(h^{2\alpha}k^{2\beta})$$

$$= O(h^{2\alpha}k^{2\beta}).$$

The rest of the proof follows by Parseval's theorem.

Q. E. D.

CHAPTER THREE

LIPSCHITZ FUNCTIONS ON COMPACT ZERO-DIMENSIONAL GROUPS

3.1 Fourier Transforms on Groups

The idea of a topological group arises from the notion of two structures imposed on a set at the same time; one is algebraic and the other is topological.

Definition 3.1 A topological group is a Hausdorff space G which is also a group, provided the map

$$(x,y) \longrightarrow x-y$$

is a continuous map of the product space GxG onto G. If in addition G is a locally compact space (i.e, every point has a compact neighbourhood) and at the same time an abelian group, then it is called a locally compact abelian (LCA) group.

Characters on Groups

Definition 3.2 A complex-valued function γ on a LCA group G is called a character of G if $|\gamma(x)| = 1$ for all $x \in G$, and if the functional equation

$$\gamma(x+y) = \gamma(x)\gamma(y) \quad (x,y \in G)$$

is satisfied.

The Dual Group

Definition 3.3 The set of all continuous characters of G forms a group Γ - called the dual group of G - if the binary operation - addition in this case - is defined by

$$(\gamma_1+\gamma_2)(x) = \gamma_1(x)\gamma_2(x), \ (x \in G, \ \gamma_1, \gamma_2 \ \in \Gamma).$$

From now on, $\gamma(x)$ will be denoted by $[x,\gamma]$. Some of the elementary properties of $[x,\gamma]$ are:

$$[0,\gamma] = [x,0] = 1, \ (x \in G, \ \gamma \in \Gamma)$$

$$[-x,\gamma] = [x,-\gamma] = [x,\gamma]^{-1} = \overline{[x,\gamma]} .$$

It is essential to note that all integrations which will be used subsequently in this chapter are taken with respect to Haar measure. The basic result in this connection is the following theorem.

THEOREM 3.4 On every LCA group G there exists a non-negative regular measure m - called Haar measure of G - which is not identically zero, and which is translation-invariant, that is to say

$$m(E+x) = m(E)$$

for every $x \in G$ and for every Borel set E in G.

For the construction of such a measure on G, as well as for its basic properties, one may refer to Hewitt and Ross [Vol. 1, chapter 4]. One important point to be mentioned here is that the Haar measure m is unique up to a multiplicative positive constant.

If G is compact, it is customary to normalize m so that $m(G) = 1$. If G is discrete, any set consisting of a single point is assigned the measure 1.

Definition 3.5 Let $1 \leq p < \infty$. Then $L^p(G)$ is the space of all measurable functions for which the pth norm

$$||f||_p = \left[\int_G |f(x)|^p dx\right]^{1/p}$$

is finite, where dx stands for the Haar measure dm(x) on the LCA group G.

The Fourier Transforms

Definition 3.6 For all $f \in L^1(G)$, the function f defined on the dual group Γ by

$$\hat{f}(\gamma) = \int_G f(x)[-x,\gamma]dx \quad (\gamma \in \Gamma)$$

is called the Fourier transform of f.

This general definition embraces the three famous and classical definitions of the Fourier transform. Thus if $G = R$ (the real line), then it is known that the dual of R is R itself, and that the characters on the dual take the form

$$\gamma(x) = e^{iyx} \qquad (y \in R),$$

and so for G = R we have

$$\hat{f}(y) = \int_{-\infty}^{\infty} f(x)e^{-iyx}dx \qquad (y \in R) .$$

If G is the circle group T, then the dual group in this case is Z, the additive group of integers, and we have

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta})e^{-in\theta} d\theta \qquad (n \in Z) ,$$

whereas if G = Z, then

$$\hat{f}(e^{i\alpha}) = \sum_{n=-\infty}^{\infty} f(n)e^{-in\alpha} \qquad (e^{i\alpha} \in T) .$$

A few facts about the Fourier transforms which will be needed later are mentioned here.

It is well known that if f is in $L^1(G)$ then $\hat{f}$ is continuous on Γ and that $||\hat{f}||_\infty \leq ||f||_1$. We shall be concerned only with compact groups in this chapter, and so $L^p(G) \subset L^1(G)$ for $p > 1$. Thus in the following two theorems it should be borne in mind that, although they are valid for any LCA group, the Fourier transforms of functions in

$L^p(G)$ need no special definition in our case.

<u>THEOREM 3.7</u> Let $f, g \in L^2(G)$. Then the formula

$$\int_G f(x)\overline{g(x)}dx = \int_G \hat{f}(\gamma)\overline{\hat{g}(\gamma)}d\gamma \quad \text{holds.}$$

This is known as Parseval's identity. See Rudin [21,p27].

<u>THEOREM 3.8</u> Let $f \in L^p(G)$, where G is compact, $1 < p < 2$. Then $\hat{f} \in L^q(G)$ for $\frac{1}{p} + \frac{1}{q} = 1$, and

$$||\hat{f}||_q \leq ||f||_p \quad \text{holds.}$$

This is the Hausdorff-Young inequality (Hewitt and Ross [11, vol 2, p227]).

For the sake of completeness, we prove the following two theorems about compact groups.

<u>THEOREM 3.9</u> The orthogonality relations

$$\int_G [x,\gamma]dx = \begin{cases} 1 \text{ if } \gamma = 0 \\ 0 \text{ if } \gamma \neq 0 \end{cases}$$

hold if G is compact.

PROOF If $\gamma = 0$, then $[x,0] = 1$ and so

$$\int_G [x,\gamma]dx = \int_G dx = m(G) = 1 .$$

If $\gamma \neq 0$, then $[x,0] \neq 1$ for some x_0 in G, and hence

$$\int_G [x,\gamma]dx = [x_0,\gamma] \int_G [x-x_0,\gamma]dx = [x_0,\gamma] \int_G [x,\gamma]dx .$$

But the last equation cannot hold for $[x_0,\gamma] \neq 1$ unless

$$\int_G [x,\gamma]dx = 0 .$$

Q. E. D.

THEOREM 3.10 If G is compact, then Γ is discrete.

PROOF If $f(x) = 1$ for all $x \in G$, then $f \in L^1(G)$, and $\hat{f}(0) = 1$, $\hat{f}(\gamma) = 0$, if $\gamma \neq 0$, by the previous theorem. Since f is continuous, the set consisting of 0 alone is open in G and so G is discrete.

Q. E. D.

3.2 The Structure of Compact 0-Dimensional Groups

One of the reasons that we have chosen compact zero-dimensional groups for our investigations is that there have been a good number of successful studies of some parts of harmonic analysis on those groups. Vilenkin [26] defined the Dirichlet kernel and the Lebesgue functions for such groups and discussed the convergence of Fourier series, summability questions analogous to the Fejér kernel method, as well as absolute convergence and uniqueness theorems. Much related to our purpose is the paper of Walker [27a] on Lipschitz classes on 0-dimensional groups, where he obtains an analogue of the classical Bernstein Theorem [Theorem 1.24]. (Vilenkin had done this for the case where G is primary). An important special case occurs when G is the direct product of cyclic groups of order 2, for then Γ can be identified with the system of Walsh functions.

The study of harmonic analysis-in general- on zero-dimensional groups is also important because it is a natural step towards the study on the more general class of finite - dimensional groups (Walker [27b]).

We turn now to study the structural properties of compact, metrizable, 0-dimensional, abelian groups.

Definition 3.11 A metrizable topological group is a group whose topology can be completely described by a suitable metric employed on it.

One of the main facts about metrizable groups is the following.

THEOREM 3.12. Let G be a topological group. Then G is metrizable if and only if there is a countable open basis at e, where e is the identity elements of G [11, vol 1, theorem 8.3 p. 70].

<u>Definition 3.13</u> A topological space X is called <u>0(zero) dimensional</u> if the family of all sets that are both open and closed is an open basis for the topology of X.

<u>Definition 3.14</u> A cover A of a set X is said to be a <u>refinement</u> of a cover B if each member of A is a subset of a member of B.

<u>Definition 3.15</u> Let X be a set, and $\mathcal{A}$ a finite family of subsets of X. For $x \in X$, let $m(x)$ be the cardinal number of the subfamily $\{A \in \mathcal{A};\ x \in A\}$. The multiplicity $m(\mathcal{A})$ is defined as $\max\{m(x) : x \in X\}$.

Now let X be a compact Hausdorff space, and let n be a nonnegative integer. Then X is said to have <u>dimension n</u> if the following two conditions are satisfied:

(i) every finite open covering of X admits a finite closed refinement $\mathcal{C}$ for which $m(\mathcal{C}) \leq n+1$,

(ii) there is some finite open covering $\mathcal{U}$ of X such that if $\mathcal{C}$ is a finite closed refinement of $\mathcal{U}$, then $m(\mathcal{C}) \geq n+1$.

For the special case of 0-dimensional spaces we have the following theorem.

<u>THEOREM 3.16</u> For a compact Hausdorff space X, the definitions 3.13 and 3.15 of zero-dimensionality are equivalent.

<u>PROOF</u> [11, vol.1, p.15].

3.17 Examples of 0-Dimensional Groups

As an example of those groups we take the product group

$$G = \prod_{i=1}^{\infty} Z(S_i)$$

where $Z(S_i)$ is a cyclic group of arbitrary order S_i with the discrete topology.

G, with the product topology, is compact by Tychonoff's Theorem. A sequence x^n converges to $x=(x_i) \in G \Leftrightarrow$ for every positive integer k there exists an integer N such that

$$x_i^n = x_i \text{ for } n \geq N, \ i \leq k.$$

Thus the metric

$$d(x,x') = \frac{1}{S_1 S_2 \ldots S_n}$$

if $x_i = x_i'$ $(i = 1,2,\ldots, n-1)$ and $x_n \neq x_n'$ is equivalent to the product topology on G, and so G is metrizable. The subsets $\Delta_n=\{x \in G; x_i = 0$ for $1 \leq i \leq n\}$ are open and closed and form a basis for the topology of G, which shows that G is 0-dimensional.

In particular, if $S_i = 2$ for all i, this gives us a group connected with the Walsh functions (see Walsh [28]). Fine [9] shows that the system of Walsh functions can be identified with the dual of the group $\prod_{i=1}^{\infty} Z(2)$. Chrestenson [6] studied the generalized Walsh functions which are identified with Γ when $S_i = r$ for all i, where r is a positive integer.

Another example of a compact metric 0-dimensional group is the group of p-adic integers [11, Vol. 1, p.109-110, 408].

<u>THEOREM 3.18</u> On any locally compact 0-dimensional group G, there is a neighbourhood basis at e consisting of compact open subgroups of G.

<u>PROOF</u> (adapted from [21,p.41]). Since G is 0-dimensional there is a neighbourhood base at 0 consisting of sets which are both open and closed. Since G is locally compact the neighbourhood base can be chosen to consist of compact open sets. Let U be a compact open neighbourhood of 0. Since U is open for each $x \in U$ there is a symmetric neighbourhood V_x of 0 such that $x + V_x + V_x \subset U$. Since U is compact there is a finite collection of points $\{x_1, x_2, \ldots, x_n\} \subset U$ such that

$$U = \bigcup_{i=1}^{n} (x_i + V_{x_i})$$

Let $W = \bigcap_{i=1}^{n} V_{x_i}$ and let H be the subgroup of G generated by W. Then $H \subset U$. [For if $y \in W$, then $x \in U \Rightarrow x \in x_i + V_{x_i}$ for some $i \Rightarrow x-y \in x_i + V_{x_i} + W \subset x_i + V_{x_i} + V_{x_i} \subset U$. Thus $U \pm W \subset U$ and so $H \subset U$]. Also H is open because W is open. Any open subgroup is closed and so H is closed and therefore compact (as U is compact). Hence H is a compact open subgroup of G which is contained in U.

Q. E. D.

The remainder of this section consists of an elaboration of §2 of Walkers paper [27a]. We assume henceforth that G is a compact, metrizable, 0-dimensional, abelian group. By theorem 3.18 and theorem 3.12 there is a countable set of neighbourhoods of 0 $\{\Delta_n\}$ which are open compact subgroups of G, and are ordered by inclusion

$$G = \Delta_o \supset \Delta_1 \supset \Delta_2 \supset \cdots \supset \Delta_n \supset \Delta_{n+1} \supset \cdots \supset \{0\} ,$$

and $$\bigcap_{n=o}^{\infty} \Delta_n = \{0\} .$$

If G is infinite, then one can assume that the inclusions are proper.

<u>Definition 3.19</u> Let $p_n = \dfrac{1}{m(\Delta_n)}$, where m is the Haar measure on G. Then $p_o=1$ and $p_n \longrightarrow \infty$ as $n \longrightarrow \infty$.

Define

$$f_n(x) = \begin{cases} p_n & \text{if } x \in \Delta_n \\ 0 & \text{if } x \notin \Delta_n \end{cases} .$$

Then

$$\int_G f_n(x)\,dx = \int_{\Delta_n} p_n\,dx = 1 .$$

THEOREM 3.20 The order of the quotient group G/Δ_n is p_n.

PROOF Let the order of G/Δ_n be N. There are N disjoint cosets of Δ_n and since m is invariant under translation each of them has measure

$$m(\Delta_n+x) = m(\Delta_n) .$$

Thus

$$1 = m(G) = Nm(\Delta_n) ,$$

which implies that

$$N = \frac{1}{m(\Delta_n)} = p_n .$$

Q. E. D.

Definition 3.21 The metric on G.

Let $\{\beta_n\}$ be a sequence which is monotonically decreasing and such that $\beta_n \longrightarrow 0$ as $n \longrightarrow \infty$. For x,y in G, define $d(x,y) = |x-y|$, where $|x| = \beta_n$ if x is in $\Delta_n \setminus \Delta_{n+1}$ and $|0| = 0$. (Here $\Delta_n \setminus \Delta_{n+1} = \{x \in \Delta_n, x \notin \Delta_{n+1}\}$). Since each Δ_n is a subgroup of G, we have

$$|x+y| \leq \max(|x|,|y|)$$

and this that d is a metric on G. The topology generated by d is clearly equivalent to the original topology on G.

In particular we choose $\beta_n = 1/p_{n+1}$. This choice is motivated by Example 3.17 where $p_n = S_1 S_2 \ldots S_n$.

<u>Definition 3.22</u> Let H be subgroup of G. Then the <u>annihilator</u> of H in Γ is

$$A(H) = \{\gamma \in \Gamma; [x,\gamma] = 1 \text{ for all } x \text{ in } H\} .$$

In particular we define $V_n = A(\Delta_n)$.

It is clear that

$$\{0\} = V_0 \subset V_1 \subset \ldots \subset V_n \subset V_{n+1} \subset \ldots \subset \Gamma \text{ , and}$$

again all inclusions are proper if G is infinite.

<u>THEOREM 3.23</u> V_n is a finite group with order p_n.

<u>PROOF</u> If $\gamma \in V_n$, then

$$\hat{f}(\gamma) = \int_G fn(x)[-x,\gamma]dx = \int_{\Delta_n} fn(x)[-x,\gamma]dx,$$

since $f_n = 0$ outside Δ_n. But $[-x,\gamma] = 1$ for all x in Δ_n and γ in V_n, so that

$$\hat{f}_n(\gamma) = \int_{\Delta_n} f_n(x)dx = \int_{\Delta_n} p_n dx$$

$$= [m(\Delta_n)]^{-1} \int_{\Delta_n} dx = 1 .$$

If γ is not in V_n, then it is a non-constant character on Δ_n and hence $\hat{f}_n(\gamma) = 0$ (by the same argument as in Theorem 3.9).

We deduce from the Riemann-Lebesgue Lemma that V_n is finite. The inversion theorem then gives

$$f_n(x) = \int_{\Gamma} \hat{f}_n(\gamma)\ [x,\gamma]d\gamma = \sum_{V_n} [x,\gamma].$$

Since $p_n = f_n(0) = \sum_{V_n} 1$, it follows that V_n has order p_n.

Q. E. D.

<u>THEOREM 3.24</u> The dual group $\Gamma = \bigcup_{n=0}^{\infty} V_n$.

<u>PROOF</u> If γ is in Γ, there is a neighbourhood Δ_k of 0 such that $\mathrm{Re}[x,\gamma] > \frac{1}{2}$ for x in Δ_k. Then

$$\mathrm{Re}\int_G f_k(x)[-x,\gamma]dx = \mathrm{Re}\int_{\Delta_k} p_k[-x,\gamma]dx > \frac{1}{2} .$$

This means that $\mathrm{Re}\widehat{f}_k(\gamma) > \frac{1}{2}$. But from the proof of Theorem 3.22 we know that $\widehat{f}_k(\gamma) = 0$ whenever $\gamma \notin V_k$, and so $\gamma \in V_k$.

Q. E. D.

Definition 3.25 If every element of G has finite order, then G is called a torsion group.

THEOREM 3.26 If G is compact, metrizable, and 0-dimensional, then Γ is a discrete countable torsion group.

PROOF

(i) If G is compact, then Γ is discrete (by Theorem 3.10).

(ii) If G is a compact metric space, then $C(G)$ (the space of continuous functions over G with the sup-norm) is separable [31, p.160].

Now for $\gamma_1 \neq \gamma_2$, Theorem 3.9 gives

$$||\gamma_1-\gamma_2||^2 = \int_G |[x,\gamma_1]-[x,\gamma_2]|^2 dx$$

$$= \int_G |[x,\gamma]|^2 dx + \int_G |[x,\gamma_2]|^2 dx - 2\mathrm{Re}\int_G [x,\gamma_1]\overline{[x,\gamma_2]}dx$$

$$= 2 .$$

Hence if Γ were uncountable, then the relation $||\gamma_1-\gamma_2||_\infty = \sqrt{2}$ for $\gamma_1 \neq \gamma_2$ in Γ would imply that $C(G)$ is not separable. This shows that Γ is countable.

(iii) The assertion that if G is 0-dimensional than Γ is a torsion group follows immediately from Theorem 3.24 since each V_n is finite.

Q. E. D.

<u>Definition 3.27</u> Let $S_n = p_n/p_{n-1}$.

<u>Remark 3.28</u> The order of V_{n+1}/V_n is $p_{n+1}/p_n = S_{n+1}$. If G is infinite (as we shall henceforth assume) then V_{n+1}/V_n is a non-trivial group and so each S_n is an integer which is ≥ 2, which implies that $p_n \geq 2^n$. We may assume that each S_n is prime because we can first achieve V_{n+1}/V_n cyclic by interpolating further groups between V_n and V_{n+1} (each generated by the preceding group and an additional element), and then we can repeat this process to achieve V_{n+1}/V_n of prime order. Since

$$(G/\Delta_{n+1})/(\Delta_n/\Delta_{n+1}) \simeq G/\Delta_n$$

and G/Δ_n has order p_n it follows that the order of Δ_n/Δ_{n+1} is $p_{n+1}/p_n = S_{n+1}$. Thus we have

3.3 Lipschitz Functions in $L^p(G)$

In this section we prove an analogue for compact metrizable 0-dimensional groups of Theorem 2.2 (Titchmarsh's Theorem 84) and Theorem 2.6.

THEOREM 3.29 If $f(x)$ belongs to $L^p(G)$, $1 < p \leq 2$, and if f belongs to $\mathrm{Lip}(\alpha;p)$ for $\alpha > 0$, i.e.,

$$\int_G |f(x+h) - f(x)|^p dx = O(|h|^{\alpha p}) , \tag{1}$$

where $|h|$ denotes the metric on G given by Definition 3.21, then $\hat{f}$ belongs to $L^\beta(\Gamma)$, where

$$p/(p+\alpha p-1) < \beta \leq q = p/(p-1) .$$

PROOF Since the Fourier transform of

$$g(x) = f(x+h) \text{ is } \hat{g}(\gamma) = [h,\gamma]\hat{f}(\gamma) ,$$

the Hausdorff-Young Theorem (3.8) yields

$$\sum_\Gamma |[h,\gamma] - 1|^q |\hat{f}(\gamma)|^q \leq M |h|^{\alpha q} \tag{2}$$

where h is any element in G.

We shall employ the notation introduced in the previous section. Take h to be in $\Delta_{n-1} \setminus \Delta_n$. Then according to Definition 3.21,

$|h| = p_n^{-1}$.

Since S_n is prime, $h+\Delta_n$ generates the cyclic group Δ_{n-1}/Δ_n (See Remark 3.28).

Define $T_n = V_n \setminus V_{n-1}$. We claim that

$$\gamma \in T_n \Rightarrow [h,\gamma] \neq 1.$$

Suppose $\gamma \in V_n$, and $[h,\gamma] = 1$. Then for all x in Δ_{n-1}, we have $x+\Delta_n = kh+\Delta_n$ for some integer k and so

$$x = kh+x', \text{ where } x' \text{ is in } \Delta_n .$$

Therefore

$$[x,\gamma] = [h,\gamma]^k[x',\gamma] = 1 .$$

This shows that γ belongs to V_{n-1} and so the claim is proved.

But $S_n h$ is in Δ_n, and so $[h,\gamma]^{S_n} = 1$, which gives $[h,\gamma] = \exp(\frac{2\pi ik}{S_n})$ and $|[h,\gamma]-1|^q = 2^q \sin^q(\frac{\pi k}{S_n})$, where $1 \leq k \leq S_n-1$, and k depends on γ.

Thus we have

$$\sum_{T_n} |\hat{f}(\gamma)|^q \sin^q(\frac{\pi k}{S_n}) = O(p_n^{-\alpha q}) .$$

Now h was chosen to be any element in $\Delta_{n-1} \setminus \Delta_n$, and so one can replace h by 2h, 3h, ..., $(S_{n-1})h$. Therefore k can be replaced by

$2k, 3k, \ldots, (S_{n-1})k$, i.e., for all t, $1 \leq t \leq S_{n-1}$, we have

$$\sum_{T_n} |\hat{f}(\gamma)|^q \sin^q\left(\frac{\pi t k}{S_n}\right) = O(p_n^{-\alpha q}).$$

Define

$$T_{n1} = \{\gamma; \gamma \in T_n, \frac{1}{4} < \frac{k}{S_n} < \frac{3}{4}\}$$

$$T_{n2} = \{\gamma; \gamma \in T_n, \frac{1}{8} < \frac{k}{S_n} \leq \frac{1}{4} \text{ or } \frac{3}{4} \leq \frac{k}{S_n} < \frac{7}{8}\}$$

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$$T_{nm} = \{\gamma; \gamma \in T_n, \frac{1}{2^{m+1}} < \frac{k}{S_n} \leq \frac{1}{2^m} \text{ or } 1 - \frac{1}{2^m} \leq \frac{k}{S_n} < 1 - \frac{1}{2^{m+1}}\}$$

where $m = \left[\frac{\log S_n}{\log 2}\right]$, i.e.,

$$\frac{1}{2^{m+1}} < 1/S_n \leq \frac{1}{2^m}.$$

In particular, for $1 \leq j \leq m$, take $t_j = 2^{j-1}$. Then on T_{nj} we have

$$\sin^q\left(\frac{\pi t_j k}{S_n}\right) \geq \left(\sin\frac{\pi}{4}\right)^q = \left(\frac{1}{\sqrt{2}}\right)^q$$

i.e.,

$$\sum_{T_{nj}} |\hat{f}(\gamma)|^q \leqslant (\sqrt{2})^q \sum_{T_{nj}} |\hat{f}(\gamma)|^q \mathrm{Sin}^q\left(\frac{\pi t_j k}{S_n}\right) = O(p_n{}^{-\alpha q}) ,$$

and so

$$\sum_{T_n} |\hat{f}(\gamma)|^q = \sum_{j=1}^{m} \sum_{T_{nj}} |\hat{f}(\gamma)|^q = O(mp_n{}^{-\alpha q}) = O(\log S_n p_n{}^{-\alpha q}) .$$

Let

$$\phi(n) = \sum_{T_n} |\hat{f}(\gamma)|^\beta .$$

For $\beta < q$, Hölder's inequality yields

$$\phi(n) \leqslant [\sum_{T_n} |\hat{f}(\gamma)|^q]^{\beta/q} [\sum_{T_n} 1]^{1-\beta/q}$$

$$= O\,[\log S_n p_n{}^{-\alpha q}]^{\beta/q} [p_n - p_{n-1}]^{1-\beta/q}$$

$$= O\,[(\log S_n)^{\beta/q} p_n{}^{-\alpha\beta}]\; O\,[p_n]^{1-\beta/q}$$

$$= O\,[(\log S_n)^{\beta/q} p_n{}^{1-\alpha\beta-\beta/q}]$$

$$= O\,[(\log S_n)^{\beta/q} p_n{}^{1-\alpha\beta-\beta+\beta/p}] .$$

Hence

$$\sum_{\Gamma}|\hat{f}(\gamma)|^{\beta} = |\hat{f}(0)|^{\beta} + \sum_{n=1}^{\infty}\sum_{T_n}|\hat{f}(\gamma)|^{\beta}$$

$$\leq |\hat{f}(0)|^{\beta} + K\sum_{n=1}^{\infty}[(\log S_n)^{\beta/q} p_n^{1-\alpha\beta-\beta+\beta/p}]$$

$$\leq |\hat{f}(0)|^{\beta} + K\sum_{n=1}^{\infty}(\log S_n)^{\beta/q}(S_n)^{1-\alpha\beta-\beta+\beta/p}[p_{n-1}^{1-\alpha\beta-\beta+\beta/p}]$$

$$\leq |\hat{f}(0)|^{\beta} + K\sum^{\infty}\left[\frac{(\log S_n)^{\beta/q}}{S_n^{\beta/q+\alpha\beta-1}}\right] p_{n-1}^{1-\alpha\beta-\beta+\beta/p} \quad . \tag{3}$$

But by the hypothesis

$$p/(p+\alpha p-1) < \beta$$

which gives

$$1-\alpha\beta-\beta/q = 1-\alpha\beta-\beta+\beta/p < 0 .$$

It follows that

$$\frac{(\log S_n)}{S_n^{\beta/q+\alpha\beta-1}} = O(1) .$$

Thus using the fact that $p_{n-1} \geq 2^{n-1}$, and the hypothesis that

$1-\alpha\beta-\beta+\beta/p < 0$, we see that the series in (3) is convergent, from which we deduce that $\hat{f}$ is in $L^{\beta}(\Gamma)$.

Q. E. D.

Corollary 3.30 If f belongs to Lip(α), $\alpha > 0$, on a compact metrizable 0-dimensional group G, then

$$\hat{f} \in L^{\beta}(\Gamma) \text{ for } \beta > 2/(2\alpha+1).$$

PROOF Any f which belongs to Lip(α) is continuous and therefore is in $L^2(G)$. The corollary then follows from Theorem 3.29 by taking p=2.

Remark 3.31 The analogues of Bernstein's theorem proved by Vile [26, Theorem 5] and Walker [27a, Theorem 1] are immediate consequences of corollary 3.30.

Remark 3.32 Theorem 3.29 is also valid if equation (1) is replaced by

$$\int_G |\Delta_h^r f(x)|^p dx = 0\,[\cdot|h|^{\alpha p}]$$

where $\Delta_h^r f$ is the difference of order r and step h. (See Theorem 2.10). The only changes in the proof stem from the fact that inequality (2) is replaced by

$$\sum_{\Gamma} |[h,\gamma]-1|^{rq} |\hat{f}(\gamma)|^q \leq M|h|^{\alpha q} .$$

3.4 Lipschitz Functions in $L^2(G)$

In this section we explore the validity of Theorem 2.3 (Titchmarsh's Theorem 85) for compact metrizable 0-dimensional groups.

THEOREM 3.33 Suppose that f belongs to $L^2(G)$ and $\alpha > 0$. If f is in Lip(α;2), i.e.,

$$\int_G |f(x+h) - f(x)|^2 dx = O\,(|h|^{2\alpha}) \text{ as } h \longrightarrow 0 \tag{4}$$

then

$$\sum_{\Gamma \setminus V_n} |\hat{f}(\gamma)|^2 = O(p_n^{-2\alpha}) \text{ as } n \longrightarrow \infty \tag{5}$$

where $|h|$, V_n, and p_n are as defined in §3.2.

PROOF By Parseval's theorem,

$$\sum_{\Gamma} |[x,\gamma]- \;|^2 |\hat{f}(\gamma)\;^2 = \int |f(x+h)-f(x)|^2 dx = O(|h|^{2\alpha}),$$

and as in the proof of Theorem 3.29 we obtain

$$\sum_{T_n} |\hat{f}(\gamma)|^2 = O\,(\log S_n\, p_n^{-2\alpha}) \;,$$

where $T_n = V_n \setminus V_{n-1}$.

Hence

$$\sum_{\Gamma\setminus V_n} |\hat{f}(\gamma)|^2 = \sum_{T_{n+1}} + \sum_{T_{n+2}} + \ldots$$

$$= O(\log S_{n+1}\, p_{n+1}{}^{-2\alpha}) + O(\log S_{n+2}\, p_{n+2}{}^{-2\alpha}) + \ldots$$

Now

$$\sum_{k=n+1}^{\infty} \log S_k (p_k)^{-2\alpha} = \sum_{k=n+1}^{\infty} \log S_k (S_k S_{k-1} \ldots S_{n+1} p_n)^{-2\alpha}$$

$$= \frac{p_n{}^{-2\alpha}}{S_{n+1}{}^{2\alpha}} \left[\log S_{n+1} + \frac{\log S_{n+2}}{S_{n+2}{}^{2\alpha}} + \sum_{k=n+3} \frac{\log S_k}{S_k{}^{2\alpha}} (S_{k-1} \ldots S_{n+2})^{-2\alpha}\right].$$

Using the fact that $S_n \geq 2$ for all n, we find that this expression is less than

$$p_n{}^{-2\alpha}\left[\frac{\log S_{n+1}}{S_{n+1}{}^{2\alpha}} + \frac{\log S_{n+2}}{S_{n+2}{}^{2\alpha}} + \sum_{k=n+3}^{\infty} \frac{\log S_k}{S_k{}^{2\alpha}} (2^{k-n-2})^{-2\alpha}\right]$$

$$= O(p_n{}^{-2\alpha})$$

because $\dfrac{\log S_k}{S_k{}^{2\alpha}} = O(1)$ and the series is convergent for $\alpha > 0$. Thus

$$\sum_{\Gamma \setminus V_n} |\hat{f}(\gamma)|^2 = O(p_n^{-2\alpha}) \text{ as } n \longrightarrow \infty .$$

Q. E. D.

Remark 3.34 We conjecture that the converse of Theorem 3.33 is also true, i.e., we conjecture that conditions (4) and (5) are actually equivalent, but we have been unable to prove this.

Remark 3.35 Since $p_n \geq 2^n$, then Theorem 3.33 gives the following. If f belongs to Lip$(\alpha;2)$, then

$$\sum_{\Gamma \setminus V_n} |\hat{f}(\gamma)|^2 = O(2^{-2\alpha n}) \text{ as } n \longrightarrow \infty$$

In fact if we are not concerned with formulating condition (5) so as to be equivalent to (4), but merely a consequence of (4), then the proof of Theorem 3.33 extends to cover the case where $f \in L^p(G)$, $1 < p \leq 2$, and so we have the following result:

If $f \in$ Lip$(\alpha;p)$, $1 < p \leq 2$, then

$$\sum_{\Gamma \setminus V_n} |\hat{f}(\gamma)|^q = O(p_n^{-q\alpha}) = O(2^{-q\alpha n}) \text{ as } n \longrightarrow \infty$$

where $q = p/p-1$.

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