ON LOCALLY M-CONVEX FUNCTION ALGEBRAS

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SCOPE AND CONTENTS: In this thesis an analogue of B-completeness is defined for locally m-convex algebras. Namely, a commutative locally m-convex algebra A is said to be a $B(\mathcal{L})$ algebra if every continuous and almost open homomorphism from A onto any locally m-convex algebra is open. A characterization is obtained of those completely regular spaces X for which C(X) with the compact-open topology is a $B(\mathcal{L})$ algebra. Permanence properties of $B(\mathcal{L})$ algebras are investigated, and extensions of the closed graph theorem are obtained. In addition, a categorical treatment of commutative locally m-convex algebras and their relationship with completely regular spaces is given.

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On Locally m-convex Function Algebras

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INTRODUCTION

Locally m-convex algebras were introduced by Arens and Michael in order to extend the theory of Banach algebras to more general topological algebras. The fundamental theory may be found in [5] and [39] as well as in [57]. The prototype of these algebras is C(X), the algebra of all continuous complex-valued functions on a completely regular space X endowed with the topology of uniform convergence on the compact subsets of X (the compact-open topology). The compact open topology is a natural generalization of the uniform topology on C(X) for compact X. This naturalness is reflected in the intimate relationship between the topological properties of X and the algebraic and topological properties of C(X).

Indeed, many authors have tried to obtain characterizations of C(X) for a given topological property of X. Characterization theorems of this type have been obtained for hemicompact spaces [2], locally compact and paracompact spaces [5], realcompact spaces [43] and [51]; μ-spaces [43] and [51], locally compact spaces [56] and [1], and hemicompact k-spaces [55]. Additional theorems of this type may be found in [55].

One of the main results of this thesis is a characterization of completely regular k-spaces in terms of a certain open-mapping property of C(X).

V. Ptak [46] introduced the notion of B-complete and B_r-complete spaces in order to extend the classical open-mapping and closed graph theorems to locally convex spaces. It is implicit in his paper that if C(X) is

B-complete then X is a normal k-space. Necessary and sufficient conditions have not as yet been obtained. W.H. Summers [53] has obtained a partial answer. He has considered $C^*_\beta(X)$, the bounded continuous functions on a locally compact space X endowed with the "strict" topology, and has obtained necessary and sufficient conditions on X for $C^*_\beta(X)$ to be B_r-complete.

This approach is not followed in this thesis. Rather, C(X) is studied as a locally m-convex algebra and a locally m-convex analogue of B-completeness is defined. This approach is motivated by T. Husain's extensions of the notion of B-completeteness, primarily by his definition of $B_r(\mathcal{A})$ and $B(\mathcal{A})$ topological abelian groups [32]. The study of these groups has been pursued by Baker [7], Sulley [52] and Grant [25].

Chapter 0 contains the basic definitions and the main known results which will be used in subsequent chapters. In particular, it covers topological vector spaces, B-complete spaces, locally m-convex algebras, completely regular spaces, and basic notions of category theory. It should be pointed out that some of the theorems are not stated in their greatest generality.

In Chapter I the notions of $B_r(\mathcal{L})$ and $B(\mathcal{L})$ algebras are introduced. These are analogues of B_r-completeness and B-completeness for locally m-convex algebras. Section one consists of definitions and elementary properties. In section 2, a very useful criterion of L.J. Sulley [52], which was proved for $B_r(\mathcal{A})$ and $B(\mathcal{A})$ groups, is adapted for $B_r(\mathcal{L})$ and $B(\mathcal{L})$ algebras. In section three, $B_r(\mathcal{L})$ and $B(\mathcal{L})$ algebras of the type C(X) are investigated. Necessary and sufficient conditions on X are obtained for

$C(X)$ to be a $B_r(\mathcal{L})$ and a $B(\mathcal{L})$ algebra, respectively. In section 4 several important counterexamples are constructed. In section 5, weaker notions of the $B_r(\mathcal{L})$ and $B(\mathcal{L})$ properties are considered by placing restrictions on the codomain. In particular $B_r(\mathcal{T})$ and $B(\mathcal{T})$ algebras are investigated, $\mathcal{T}$ being the class of all m-barrelled algebras. As a consequence of a characterization theorem of this section it follows that $C(X)$ is a $B(\mathcal{T})$ algebra whenever X is pseudocompact.

In chapter II, section 1, two classes of continuous maps are introduced, full maps and CR-quotient maps. Properties of these maps are used in subsequent sections to obtain certain counterexamples. In section 2 permanence properties of $B_r(\mathcal{L})$ and $B(\mathcal{L})$ algebras are investigated. Also, it is shown that local compactness and paracompactness of X are not sufficient for $C(X)$ to be B_r-complete as a locally convex space. In section 3, closed graph theorems are obtained for $B_r(\mathcal{L})$ algebras.

Chapter III is devoted to categorical investigations of the category $\underline{L}$ of commutative locally m-convex algebras with identity and all continuous unitary homomorphisms, and its relationship to the category $\underline{CR}$ of all completely regular spaces and all continuous maps. The motivation comes from the known adjoint situation between compact Hausdorff spaces and commutative Banach algebras with identity.

In section 1, the functors M, C and C_p are discussed. In section 2, it is shown that $\underline{L}$ is cocomplete, and a representation of coproducts is obtained. In section 3, the subcategory $\underline{CL}$ of $\underline{L}$ is investigated. The main result being that $\underline{CL}$ is a coreflective and productive subcategory of $\underline{L}$.

It is also shown that $M(\Pi A_i) \simeq \coprod_C M(A_i)$. In section four it is shown that the functors $\underline{CR} \underset{M}{\overset{C}{\rightleftarrows}} \underline{CL}$ and $\underline{CR} \underset{M}{\overset{P}{\rightleftarrows}} \underline{L}$ are adjoint on the right. When restricted to suitable subcategories, the former yields a duality which extends the known duality between compact Hausdorff spaces and B* algebras. In section 5 $\mathfrak{C}$-injectivity and $\mathfrak{F}$-projectivity are discussed.

CHAPTER 0

Most of the notation which is used is standard. The symbol $\subset$ is used in the broad sense, strict inclusion being written as $\subsetneq$. All vector spaces and algebras are over the field of complex numbers $\mathbb{C}$. Algebra homomorphism is written simply as homomorphism and isomorphism means algebraic isomorphism as well as topological homeomorphism. The letters f, g, h; ϕ, Ψ; α, β ; are continuous maps, homomorphism or linear maps, and complex-valued homomorphisms or linear functionals, respectively. For technical reasons, in Chapters II and III complex-valued continuous functions are denoted by the letters r or s.

1. Topological vector spaces.

A topological vector space (TVS) is a pair (E, u) consisting of a vector space E and a Hausdorff topology u on E such that addition is a continuous map from E × E to E, and scalar multiplication is a continuous map from $\mathbb{C}$ × E to E. Henceforth, (E, u) will be written as E_u or simply as E if no confusion is likely to arise.

For an element $x \in E$, $\mathcal{N}(E, x)$ is the neighborhood filter of x in E. When x is the zero element of E, $\mathcal{N}(E,0)$ will be written as $\mathcal{N}(E)$.

1.1. Definition. A subset U of E is said to be absorbing if for each $x \in E$ there exists $\lambda_o \in \mathbb{C}$ such that $x \in \lambda U$ whenever $|\lambda| \geq |\lambda_o|$. U is said to be circled if $\lambda U \subset U$ whenever $|\lambda| \leq 1$. U is said to be bounded if it is absorbed by every V in $\mathcal{N}(E)$.

The neighborhood filter $\mathcal{N}(E)$ has certain properties which will be used frequently. In particular, a filter $\mathcal{F}$ on a vector space E is the zero neighborhood filter for some Hausdorff topology which is compatible with the linear structure of E iff $\mathcal{F}$ satisfies the following axioms:

(TVS 1) For each $U \in \mathcal{F}$ there exists a $V \in \mathcal{F}$ such that $V + V \subset U$.

(TVS 2) For each $U \in \mathcal{F}$ there exists a circled and absorbing $V \in \mathcal{F}$ such that $V \subset U$.

(TVS 3) $\bigcap_{U \in \mathcal{F}} U = \{0\}$.

For a subset U of a TVS E, $cl_E(U)$ is the topological closure of U in E. It is well known that $cl_E(U) = \bigcap\{U + V : V \in \mathcal{N}(E)\}$. In particular, if $U \in \mathcal{N}(E)$ then $cl_E(U) \subset U + U$. Hence, in view of the above axioms, it follows that for every TVS E, $\mathcal{N}(E)$ has a basis consisting of closed and circled sets.

1.2. Definition. A subset U of E is said to be convex if $(1 - \lambda)x + \lambda y$ is in U whenever x and y are in U and $0 \leq \lambda \leq 1$. A TVS E is said to be locally convex if $\mathcal{N}(E)$ has a basis consisting of convex sets.

For a locally convex space E, E' is the set of all continuous linear functionals on E. E' is a vector space and is called the dual of E. It may be endowed with several locally convex topologies. Of these the only one used in this dissertation is the weak* (w*) topology which is the relative product topology from $\mathbf{C}^E$.

1.3. Definition. (a) Let U be a subset of a locally convex space E, and let H be a subset of E'. Then, $U^o = \{\alpha \in E' : |\alpha(x)| \leq 1 \ \forall\, x \in U\}$ and $H^o = \{x \in E : |\alpha(x)| \leq 1 \ \forall\, \alpha \in H\}$ are called the polars of U and H.

(b) A subset H of E' is said to be equicontinuous if for each $V \in \mathcal{N}(\mathbb{C})$ there exists $U \in \mathcal{N}(E)$ such that $\alpha(U) \subset V$ for each $\alpha \in H$.

1.4. Theorem. Let E be a locally convex space.

(a) $H \subset E'$ is equicontinuous iff $H^o \in \mathcal{N}(E)$

(b) Every w*-closed and equicontinuous subset of E' is w*-compact.

Proof: (a) follows directly from the definition. A proof of (b) may be found in [49 ; Corollary 4.3].

1.5. Definition. (a) A subset U of a TVS E is said to be a barrel if U is closed, circled, convex and absorbing. A locally convex space E is said to be barrelled if each barrel is in $\mathcal{N}(E)$.

(b) A locally convex space E is said to be bornological if every circled and convex subset of E which absorbs every bounded set in E is in $\mathcal{N}(E)$.

1.6. Theorem. Let E be a locally convex space. Then the following statements are equivalent:

(a) E is barrelled.

(b) Every w*-bounded subset of E' is equicontinuous.

Proof: [31; Theorem 7, p. 30].

Since each w*-compact subset of E' is w*-bounded, the following is immediate.

1.7. <u>Corollary</u>. If E is barrelled then every w*-compact subset of E' is equicontinuous.

2. Completeness, B-completeness, and B_r-completeness.

A uniform space is a pair $(X, \mathcal{U})$ where X is a set and $\mathcal{U}$ is a filter on $X \times X$ which satisfies the following conditions:

(1) Each $W \in \mathcal{U}$ contains $\{(x,x) : x \in X\}$.

(2) If $W \in \mathcal{U}$ then $W^{-1} \in \mathcal{U}$, where $W^{-1} = \{(x,y) : (y,x) \in W\}$.

(3) For each $W \in \mathcal{U}$, there exists $W' \in \mathcal{U}$ such that $W' \circ W' \subset W$

where $W' \circ W' = \{(x,y) : (x,z) \in W'$ and $(z,y) \in W'$ for some $z \in X.\}$
The filter $\mathcal{U}$ is known as a uniformity on X.

A filter $\mathcal{F}$ on a uniform space $(X, \mathcal{U})$ is called a Cauchy filter if for each $W \in \mathcal{U}$ there exists $F \in \mathcal{F}$ such that $F \times F \subset W$; $(X, \mathcal{U})$ is said to be complete if every Cauchy filter converges.

With each TVS E there is associated a natural uniformity which arises from $\mathcal{N}(E)$. For each $V \in \mathcal{N}(E)$ let $W_V = \{(x,y) : (x,y) \in E \times E$ and $x - y \in V\}$. Then the collection $\{W_V : V \in \mathcal{N}(E)\}$ forms a base for a uniformity $\mathcal{U}$ on E. E is said to be complete if it is complete with respect to this uniformity.

2.1. Theorem. Let E be a TVS. Then E can be embedded as a dense subspace of a complete TVS $\tilde{E}$. $\tilde{E}$ is unique up to isomorphism. Moreover, $\{cl_{\tilde{E}}(V) : V \in \mathcal{N}(E)\}$ is a basis for $\mathcal{N}(\tilde{E})$.

Proof: [49; I, 1.5].

2.2. Definition. Let E be a locally convex space and let S be a linear subspace of E'. Then S is said to be almost closed if $V^o \cap S$ is w*-closed for each $V \in \mathcal{N}(E)$.

The following characterization of completeness is due to Grothendieck.

2.3. Theorem. Let E be a locally convex space. Then the following statements are equivalent.

(1) E is complete

(2) Every almost closed maximal subspace of E' is w*-closed.

Proof: [49; IV, 6.2 Corollary 2].

2.4. Theorem. Let E_u be a complete locally convex space. Let v be a second locally convex topology on E such that $u \subset v$, i.e. v is finer than u, and $\mathcal{N}(E_v)$ has a basis consisting of u-closed sets. Then E_v is complete.

Proof: [49; I, 1.6].

The following definitions and theorems are due to Ptak [46].

2.5. Definition. A linear map $f: E \to F$ is said to be almost open if $cl_F(f(V)) \in \mathcal{N}(F)$ whenever $V \in \mathcal{N}(E)$.

Every linear map from a locally convex space onto a barrelled space is almost open.

2.6. Definition. A locally convex space E is said to be B-complete if every continuous and almost open linear map onto any locally convex space F is open; B_r-complete if every one-to-one linear map with the above properties is open.

It is obvious that every B-complete space is B_r-complete. Every complete metrizable locally convex space is B-complete. For examples of non-metrizable B-complete spaces the reader is referred to [49; p. 162] or [31; chapter 4].

The following theorem is due to Ptak.

2.7. Theorem. Let E be a locally convex space, then:

(a) E is a B-complete iff every almost closed linear subspace of E' is w*-closed.

(b) E is B_r-complete iff every dense and almost closed linear subspace of E' is w*-closed, and hence coincides with E'.

Proof: [49; IV, Theorem 8.3].

In view of Theorem 2.3 the following is immediate.

2.8. Corollary. Every B_r-complete space is complete.

The converse is not true. Ptak's original counterexample appears in section I.4. A simpler example appears in [31; p. 45, Prop. 1].

2.9. Proposition. A closed subspace of a B-complete (B_r-complete) space is B-complete (B_r-complete).

Proof: [31; Prop. 4, p. 41].

2.10. Proposition. Let E be a B-complete space and let M be a closed linear subspace. Then E/M is B-complete.

Proof: [31; Corollary 2, p. 48].

2.11. Definition. Let $\phi : E \to F$ be a linear map. ϕ is said to be almost continuous if $cl_E\, \phi^{-1}(V) \in \mathcal{N}(E)$ for each $V \in \mathcal{N}(F)$. ϕ is said to have a closed graph if $\{(x, \phi(x)) : x \in E\}$ is a closed subspace of $E \times F$.

Every linear map from a barrelled space into any locally convex space is almost continuous.

2.12. Theorem. Let E be a B_r-complete and let $\phi : F \to E$ be an almost continuous linear map with closed graph where F is locally convex. Then ϕ is continuous.

Proof: [31; Theorem 6, p. 57].

In particular every linear map from a barrelled space to a B_r-complete space which has a closed graph is continuous.

3. Locally m-convex algebras.

A topological algebra is an associative algebra A which is also a topological vector space and such that multiplication is a continuous function from A × A to A. (Some authors require that multiplication be separately continuous only.)

3.1. Definition. Let A be a topological algebra.

(a) A subset V of A is said to be idempotent if $V\,V \subset V$; it is said to be m-convex if it is convex and idempotent.

(b) A is said to be locally m-convex (LMC) if $\eta(A)$ has a basis consisting of m-convex and circled sets.

Equivalently, a topological algebra A is locally m-convex iff its topology is generated by a set $\{p_i\}_{i \in I}$ of seminorms, each satisfying $p_i(xy) \leq p_i(x)\, p_i(y)$. Such seminorms are called submultiplicative. A general treatment of locally m-convex algebras may be found in [39] or [57].

3.1. Proposition. (a) Every normed algebra is an LMC algebra.

(b) A subalgebra of an LMC algebra is an LMC algebra.

(c) The cartesian product of LMC algebras is an LMC algebra.

(d) If A is an LMC algebra and I is a closed ideal then A/I is an LMC algebra.

Proof: [39; Proposition 2.4].

3.2. Theorem. Let A be an LMC algebra,

(a) If A is a division algebra then A is isomorphic with $\mathbb{C}$.

(b) If A is commutative and M is a closed regular maximal ideal, then A/M is isomorphic to $\mathbb{C}$.

Proof: (a) is a special case of [4; Theorem 1].

(b) follows immediately from (a).

For a locally m-convex algebra A, M(A) is the subset of A' consisting of all nonzero continuous homomorphisms with the relative w*topology. By Theorem 3.2(b) there exists a one-to-one correspondence between M(A) and the maximal regular closed ideals of A.

The following proposition is well known for commutative Banach algebras with identity.

3.3. Proposition. Let A be a commutative locally m-convex algebra with identity. Then M(A) is not empty.

Proof: Let p be a continuous submultiplicative seminorm on A. Then $A_p = A/\ker(p)$ is a commutative normed algebra with identity under the norm induced by p, and so its completion $\tilde{A}_p$ is a Banach algebra. Let $\alpha \in M(\tilde{A}_p) \neq \emptyset$ and let $\phi : A \to A_p$ be the canonical homomorphism. Since ϕ is continuous $\alpha \circ \phi \in M(A) \neq \emptyset$.

In general, Proposition 3.3 fails for topological algebras. In fact, in [3] there is an example of such a topological algebra A which is also complete and metrizable.

3.4. Theorem. Let A be a complete locally m-convex algebra. Then A is isomorphic to a projective limit of Banach algebras.

Proof: [39; Theorem 5.1].

3.5. Definition. Let A be an LMC algebra.

(a) $R(A) = \bigcap_{\alpha \in M(A)} \ker(\alpha)$ is called the strong radical of A. A is said to be semisimple if $R(A) = \{0\}$.

(b) A is said to be functionally continuous if every homomorphism $\alpha: A \to \mathbb{C}$ is continuous.

(c) A is said to be a $*$-algebra if there exists a unary operation $a \to a^*$ satisfying the following conditions:

(1) $(a^*)^* = a$

(2) $(\lambda a + b)^* = \bar{\lambda} a^* + b^*$ for each $\lambda \in \mathbb{C}$.

(3) $(ab)^* = b^* a^*$

(d) Let A and B be $*$-algebras. A homomorphism $\phi: A \to B$ is said to be a $*$-homomorphism if $\phi(a^*) = \phi(a)^*$.

(e) A $*$-algebra A with identiy is said to be symmetric if $1 + xx^*$ has an inverse in A for each $x \in A$.

4. Completely regular spaces and C(X)

A Hausdorff topological space X is said to be completely regular if whenever F is a closed subset of X and x is a point in its complement there exists a continuous real-valued function f such that $0 \leq f \leq 1$, $f(x) = 1$ and $f(F) = \{0\}$. Henceforth all topological spaces are assumed to be completely regular.

For a topological space X, C(X) is the algebra of all continuous complex-valued functions on X. C(X) may be endowed with several topologies. Unless otherwise specified it is assumed that C(X) has the compact-open topology which is also known as the topology of uniform convergence on the compact subsets of X. A basis for the neighborhoods of zero given by sets of the form $N(K, \xi) = \{f \in C(X) : |f(x)| \leq \xi$ for all $x \in K\}$ where K varies over all the compact subsets of X and $\xi > 0$. Equivalently the compact open topology is generated by the submultiplicative seminorms $\{p_K\}_{K \subset X}$ where $p_K(f) = \sup_{x \in K} |f(x)|$, K varying over all compact subsets of X. It is clear that C(X) is a locally m-convex algebra.

4.1. Definition. (a) X is <u>hemicompact</u> if there exists a countable set $\{K_n\}_{n \in \mathbb{N}}$ of compact subsets of X such that every compact subset of X is contained in some K_n.

(b) A complex homomorphism $\alpha : C(X) \to \mathbb{C}$ is said to be <u>fixed</u> if $\alpha(f) = f(x)$ for some $x \in X$. X is said to be <u>realcompact</u> if every complex homomorphism of C(X) is fixed.

(c) A subset Y of X is said to be bounded if $f|Y$ is bounded for each $f \in C(X)$. X is said to be a <u>μ-space</u> if each closed and bounded subset of X is compact.

(d) A subset Y of X is said to be <u>k-closed</u> if $Y \cap K$ is closed in X for each compact subset K of X. X is said to be a <u>k-space</u> if each k-closed subset of X is closed.

(e) A function $f : X \to \mathbb{C}$ is said to be <u>k-continuous</u> if $f|K$ is continuous for each compact subset K of X. X is said to be a k_R-space if every k-continuous function $f : X \to \mathbb{C}$ is continuous.

Since each homomorphism from C(X) to $\mathbb{C}$ maps real-valued functions to real numbers, definition (b) is equivalent to the usual definition of a realcompact space. The terminology of (c) is due to Buchwalter [16]. The term k_R-space appears in [40] and [45]. These same spaces are called k'-spaces in [34].

The following are some of the existing theorems which illustrate the intricate relationship between X and C(X) with the compact open topology. The first theorem is obvious in view of the fact that a TVS is normable iff it has a bounded convex neighborhood of zero.

4.2. <u>Theorem</u>. C(X) is normable iff X is compact.

4.3. <u>Theorem</u>. C(X) is metrizable iff X is hemicompact.

<u>Proof</u>: [2; Theorems 7 and 8].

4.4. <u>Theorem</u>. C(X) is complete and metrizable iff X is a hemicompact k-space.

<u>Proof</u>: [55; Theorem 2].

4.5. <u>Theorem</u>. C(X) is complete iff X is a k_R-space.

<u>Proof</u>: [55; Theorem 1].

The following two theorems were proved independently by Nachbin and Shirota in [43] and [51].

4.6. <u>Theorem</u>. $C(X)$ is bornological iff X is realcompact

4.7. <u>Theorem</u>. $C(X)$ is barrelled iff X is a μ-space.

Additional theorems of this type may be found in [55].

It should be pointed out that theorems 4.6 and 4.7 remain true when the terms bornological and barrelled are replaced by their locally m-convex analogues. The following definitions are due to Warner [54].

4.8. <u>Definition</u>. Let A be an LMC algebra.

(a) A barrel V in A is an <u>m-barrel</u> if $VV \subset V$. A is said to be m-barrelled if every m-barrel is in $\mathcal{N}(A)$.

(b) A subset V of A is said to be <u>i-bounded</u> if λV is contained in a bounded idempotent subset of A for some $\lambda \in \mathbb{C}$. A is said to be <u>i-bornological</u> if every circled, m-convex and absorbing subset of A which absorbs every i-bounded subset of A is in $\mathcal{N}(A)$.

(4.6)' <u>Theorem</u>. $C(X)$ is i-bornological iff X is realcompact

<u>Proof</u>: [54; Theorem 5].

Since $C(X)$ is m-barrelled iff it is barrelled the following is immediate.

(4.7)' <u>Theorem</u>. $C(X)$ is m-barrelled iff X is a μ-space

The following has been proved by Morris and Wulbert [42; Theorem 2.1].

4.9. <u>Theorem</u>. Every closed ideal of $C(X)$ is of the form $I_F = \{f : f \in C(X) \text{ and } f(F) = \{0\}\}$ for some closed $F \subset X$.

4.10. Corollary. Every maximal closed ideal of $C(X)$ is of the form $I_{\{x\}}$ for some $x \in X$.

It is well known [35; Lemma 5, p. 116] that the evaluation map $E : X \to M[C(X)]$, defined by $E(x) = \alpha_x$, is a homeomorphism into. By Corollary 4.10 it follows that this map is onto, hence $X \cong M[C(X)]$ for each completely regular space X.

4.11. Proposition. Let K be a compact subset of X. Then every continuous function $f : K \to \mathbb{C}$ can be extended to a continuous bounded function $f' : X \to \mathbb{C}$. In particular the restriction map $\phi : C(X) \to C(K)$ is onto.

Proof: Since K is a closed subset of βX, the Stone-Čech compactification of X, f can be extended to a continuous function on βX by the Tietze extension theorem. By restricting to X the desired function is obtained.

4.12. Proposition. Let A be a $*$-subalgebra of $C(X)$ which separates the points of X and contains the constant functions. Then A is dense in $C(X)$ (with the compact open topology).

Proof: [39; Proposition 6.8].

4.13. Remark. In view of Theorem 4.5, k_R-spaces form a very important subclass of completely regular spaces. As is pointed out by Hušek [34], to each completely regular space there is associated a k_R-space in a certain unique way. Given a completely regular space X, let kX be the k-space associated with X [35; p. 241 (K)] and let $k_R X$ be the complete regularization of kX [27; Satz 1.3.3]. Then $k_R X$ is a k_R-space with the same underlying set as X and whose topology is the weak topology generated by all the k-continuous complex valued functions on X. Clearly, X and $k_R X$ have the same compact sets. Thus C(X) can be embedded (topologically and algebraically) into $C(k_R X)$, the embedding being dense by Proposition 4.12. Since $C(k_R X)$ is complete it follows that $C^{\gamma}(X) = C(k_R X)$.

5. Categories.

The following two definitions are essentially those of [41; pages 1 and 49].

5.1. Definition. A category $\underline{C}$ is a class $\mathcal{O}$, together with a class $\mathcal{M}$ which is a disjoint union of the form

$$\mathcal{M} = \bigcup_{(A,B) \in \mathcal{O} \times \mathcal{O}} [A,B]$$

where each [A,B] is a set. Furthermore, for each triple (A,B,C) of members of $\mathcal{O}$ there is a function from $[A,B] \times [B,C]$ to [A,C]. The image of the pair (m,n) under the function is called the composition of m and n, and is denoted by $n \circ m$. The composition functions are subject to two axioms:

(1) $(m \circ n) \circ \ell = m \circ (n \circ \ell)$, whenever the compositions make sense.

(2) For each A in $\mathcal{O}$ there exists an element $e_A \in [A,A]$ such that $e_A \circ m = m$ and $n \circ e_A = n$ whenever the compositions make sense.

The members of $\mathcal{O}$ are called objects and the members of $\mathcal{M}$ are called morphisms. The statement "$m \in [A,B]$" is represented symbolically as $m : A \to B$.

5.2. Definition. Let $\underline{C}$ and $\underline{K}$ be categories. A functor $F : \underline{C} \to \underline{K}$ is an assignment of an object $F(A)$ in $\underline{K}$ to each object A in $\underline{C}$ and a morphism $F(m) : F(A) \to F(B)$ in $\underline{K}$ to each morphism $m : A \to B$ in $\underline{C}$, subject to the following conditions:

(1) If $m \circ n$ is defined in $\underline{C}$, then $F(m \circ n) = F(m) \circ F(n)$

(2) For each A in $\underline{C}$, $F(e_A) = e_{F(A)}$.

If $F(m) : F(B) \to F(A)$ whenever $m:A \to B$ and $F(m \circ n) = F(n) \circ F(m)$, then F is called a contravariant functor.

A category $\underline{C}$ is said to be small if its class of objects is actually a set. Small categories will be denoted by $\underline{I}$ and its objects by i,j.

5.3. Definition. Let $\underline{C}$ be a category and let $\underline{I}$ be a small category. An I-diagram over $\underline{C}$ is a functor $D : \underline{I} \to \underline{C}$. A lower bound for the diagram D is a pair $(L, \{f_i\}_{i \in I})$ where L is an object of $\underline{C}$ and $\{f_i : L \to D(i)\}_{i \in \underline{I}}$ is a set of morphisms in $\underline{C}$ such that for any morphism $m : i \to j$ in $\underline{I}$, $D(m) \circ f_i = f_j$.

A lower bound $(L, \{f_i\})_{i \in \underline{I}}$ said to be a limit of D, written as $\lim D$, if for any lower bound $(L', \{g_i\})_{i \in \underline{I}}$ of D there exists a unique morphism $f : L' \to L$ in $\underline{C}$ such that $f_i \circ f = g_i$ for each i in $\underline{I}$.

5.4. Definition. A category $\underline{C}$ is said to be I-complete if every $\underline{I}$-diagram over $\underline{C}$ has a limit. $\underline{C}$ is said to be complete if it is $\underline{I}$-complete for every small category $\underline{I}$.

Dually one defines an upper bound of a diagram, colimit and cocompleteness. A colimit of a diagram D is denoted by colim D.

When $\underline{I}$ is actually a down directed (up directed) partially ordered set, then an $\underline{I}$ diagram D is called an inductive (projective) system and colim D (lim D) is called an inductive (projective) limit, and is denoted by $\underrightarrow{\text{colim}}$ D ($\underleftarrow{\text{lim}}$ D), respectively.

When $\underline{I}$ is discrete, i.e. when the only morphisms in $\underline{I}$ are the identities then lim D (colim D) is called a product (coproduct) and is denoted by $\prod_{i \in I} D(i)$ ($\coprod_{i \in I} D(i)$), respectively.

When $\underline{I}$ has exactly two objects i and j and exactly two morphisms m_1 and m_2 in [i,j] then lim D (colim D) is called an equalizer (coequalizer), respectively.

5.5. Theorem. Let $\underline{C}$ be a category. The following statements are equivalent.

(1) $\underline{C}$ is complete

(2) $\underline{C}$ has products and equalizers.

Proof: [28; Theorem 23.8].

Dually, $\underline{C}$ is cocomplete iff $\underline{C}$ has coproducts and coequalizers.

5.6. Definition. Let $\underline{C} \underset{F}{\overset{G}{\rightleftarrows}} \underline{K}$ be two functors. F is said to be a left adjoint of G, and G a right adjoint of F, if there exists a natural set isomorphism

$$\eta_{A,B} : [FA, B] \longrightarrow [A, GB] \text{ for each } A \text{ in } \underline{C} \text{ and } B \text{ in } \underline{K}.$$

If F and G are contravariant, then F and G are said to be adjoint on the right if there exists a natural set isomorphism

$$\eta_{A,B} : [A, GB] \longrightarrow [B, FA] \quad \text{for each } A \text{ in } \underline{C} \text{ and } B \text{ in } \underline{K}.$$

5.7. Theorem. Let F be a left adjoint of G.

Then: (1) F preserves limits.

(2) G preserves colimits.

Proof: [28; Theorem 27.7]

For contravariant functors the above theorem can be formulated as follows:

5.8. Theorem. Let F and G be adjoint on the right. Then F and G transform colimits into limits.

5.9. Definition. The categories $\underline{C}$ and $\underline{K}$ are said to be dually equivalent if there exist contravariant functors $\underline{C} \underset{G}{\overset{F}{\rightleftarrows}} \underline{K}$ and natural isomorphisms $\eta_A : A \to GF(A)$, $\varepsilon_B : B \to FG(B)$, for each A in $\underline{C}$ and B in $\underline{K}$.

For example, the category of compact Hausdorff spaces and continuous maps is dually equivalent to the category of commutative B* algebras with identity and unitary algebra homomorphisms.

5.10. Definition. Let $\underline{C}$ be a subcategory of $\underline{K}$. Then $\underline{C}$ is said to be a coreflective (reflective) subcategory of $\underline{K}$ if the inclusion functor $Inc : \underline{C} \to \underline{K}$ has a right adjoint (left adjoint), respectively.

Equivalently, $\underline{C}$ is a coreflective subcategory of $\underline{K}$ if to each $A \in \underline{K}$ there exist $A' \in \underline{C}$ and $f : A' \to A$ in $\underline{K}$ having the following property: given any B in $\underline{C}$ and $g : B \to A$ in $\underline{K}$ there exists a unique $h : B \to A'$ in $\underline{C}$ such that $g = f \circ h$. Dually for a reflective subcategory.

For example the category of compact Hausdorff spaces and continuous maps is a reflective subcategory of the category of completely regular spaces and continuous maps. Whereas, k-spaces form a coreflective subcategory of Hausdorff spaces. From Remark 4.13, it follows directly that k_R-spaces form a coreflective subcategory of completely regular spaces.

Further examples of the concepts mentioned in this section may be found in [28].

CHAPTER I

$B(\mathcal{L})$ and $B_r(\mathcal{L})$ Algebras

In this chapter an analogue of B-completeness is defined for LMC algebras. The central result is a characterization of those completely regular spaces X for which C(X) is a $B_r(\mathcal{L})$ and a $B(\mathcal{L})$ algebra, respectively.

1. Definitions and Elementary Properties.

Let $\mathcal{L}$ be the class of all commutative LMC algebras. The following definition is analogous to the definitions of $B(\mathcal{C})$ spaces [31] and $B(\mathcal{A})$ groups [32].

1.1. Definition. An algebra A in $\mathcal{L}$ is said to be a $B(\mathcal{L})$ algebra if every continuous and almost open homomorphism from A onto any algebra B in $\mathcal{L}$ is open; a $B_r(\mathcal{L})$ algebra if every one-to-one homomorphism with the above properties is open.

From the definition it follows that every $B(\mathcal{L})$ algebra is a $B_r(\mathcal{L})$ algebra. Also, every algebra A in $\mathcal{L}$ which is B-complete (B_r-complete) as a locally convex space is a $B(\mathcal{L})$ algebra ($B_r(\mathcal{L})$ algebra). Thus the following is immediate.

1.2. Theorem. Every algebra in $\mathcal{L}$ which is complete and metrizable is a $B(\mathcal{L})$ algebra. In particular every commutative Banach algebra is a $B(\mathcal{L})$ algebra.

1.3. Examples. (a) $C[0,1]$ is a commutative Banach algebra and therefore a $B(\mathcal{L})$ algebra.

(b) $\mathcal{C}(\mathbb{R})$ is complete and metrizable by theorem 0; 4.4 and therefore a $B(\mathcal{L})$ algebra.

(c) For any set I, $\mathbb{C}^I$ is B-complete as a locally convex space [49; p.162, Ex. 3] and therefore a $B(\mathcal{L})$ algebra. Whenever I is uncountable, $\mathbb{C}^I$ is not metrizable.

1.4. Proposition. (a) Let A be a $B(\mathcal{L})$ algebra and let $\phi: A \to B$, $B \in \mathcal{L}$, be a continuous, onto and almost open homomorphism. Then B is also a $B(\mathcal{L})$ algebra.

(b) Let A be a $B_r(\mathcal{L})$ algebra and let ϕ be as above and one-to-one. Then B is a $B_r(\mathcal{L})$ algebra.

Proof: (a) Let $\Psi : B \to C$, $C \in \mathcal{L}$ be a continuous, onto, and almost open homomorphism. Since A is a $B(\mathcal{L})$ algebra it follows from the definition that ϕ is open, and hence $\Psi \circ \phi$ is continuous and almost open. Again, since A is a $B(\mathcal{L})$ algebra, $\Psi \circ \phi$ is open and since ϕ is open it follows that Ψ is also open. Thus B is a $B(\mathcal{L})$ algebra. The proof of (b) is similar.

1.5. Corollary. Let A be a $B(\mathcal{L})$ algebra and let I be a closed ideal. Then A/I is a $B(\mathcal{L})$ algebra.

Proof: This is a particular case of (a) above.

In section 3 a characterization is obtained of those completely regular spaces X for which C(X) is a $B_r(\mathcal{L})$ algebra and a $B(\mathcal{L})$ algebra, respectively. This characterization produces a large class of $B(\mathcal{L})$ algebras which are not B-complete as locally convex spaces.

2. An adaptation of criterion of L.J. Sulley.

In [52] Sulley found criteria for dense subgroups of Abelian $B(\mathcal{A})$ and $B_r(\mathcal{A})$ groups to inherit the respective properties. 2.1 - 2.6 are adaptations of Sulley's criteria for $B(\mathcal{L})$ and $B_r(\mathcal{L})$ algebras. Some of these results will be used in section 3 for the proof of the main theorem.

2.1. Lemma. Let A be in $\mathcal{L}$ and let B be a dense subalgebra. Let I be a closed ideal of A and let $\phi: A \to A/I$ be the quotient map. Then the restriction of ϕ, $\phi_B: B \to \phi(B)$, is continuous and almost open. Furthermore, ϕ_B is open iff $I \cap B$ is dense in I.

Proof: The continuity of ϕ_B follows from the continuity of ϕ. To see that ϕ_B is almost open let $V \in \mathcal{N}(B)$. Since B is dense in A it follows that $cl_A(V) \in \mathcal{N}(A)$. Thus $cl_{A/I}(\phi_B(V)) = cl_{A/I}(\phi(V)) \supset \phi(cl_A(V))$, the latter being in $\mathcal{N}(A/I)$ since ϕ is open. Since $\phi(cl_A(V)) \cap \phi(B) \subset cl_{A/I}(\phi_B(V)) \cap \phi(B) = cl_{\phi(B)}(\phi_B(V))$ it follows that $cl_{\phi(B)}(\phi_B(V))$ is in $\mathcal{N}(\phi(B))$ and hence ϕ_B is almost open.

Next suppose that ϕ_B is open. To show that $I = cl_A(I \cap B)$ it suffices to show that $I \subset (I \cap B) + U$ for each U in $\mathcal{N}(A)$. Let U be in $\mathcal{N}(A)$.

Then there exists $V \in \eta(A)$ such that $V = -V$ and $V + V \subset U$. Since ϕ_B is open $\phi_B(V \cap B) = \phi(V \cap B)$ is in $\eta(\phi(B))$. This implies that there exists $W \in \eta(A/I)$ such that $W \cap \phi(B) \subset \phi(V \cap B)$. By applying ϕ^{-1} to both sides the following is obtained:

(i) $\quad \phi^{-1}(W) \cap (B + I) \subset (V \cap B) + I.$

Let $x \in I$. Since B is dense in A, there exists $b \in B$ such that $b \in x + (V \cap \phi^{-1}(W))$. This implies that

(ii) $\quad b - x \in V \cap \phi^{-1}(W).$

Also, $b - x \in \phi^{-1}(W) \cap (B + I) \subset (V \cap B) + I$ by (i). Thus there exist $c \in V \cap B$ and $y \in I$ such that $b - x = c + y$. Now $x = (b - c) - y$ where $b - c \in I \cap B$ and $-y = c - (b - x) \in V - V = V + V \subset U$. Hence $x \in (I \cap B) + U$. Since x and U were arbitrarity chosen it follows that $I \subset cl_A(I \cap B)$. Thus $I \cap B$ is dense in I.

Conversely, suppose that $I \cap B$ is dense in I. Let U be in $\eta(A)$. Then there exists $V \in \eta(A)$ such that $V = -V$ and $V + V \subset U$. Now $I = cl_A(I \cap B) \subset (I \cap B) + V$. It will be shown that $V \cap (B + I) \subset (U \cap B) + I$. Let $v \in V \cap (B + I)$. Then $v = b + x$ for some $b \in B$ and $x \in I$. Since $x \in (I \cap B) + V$ there exist $y \in I \cap B$ and $w \in V$ such that $x = y + w$. Thus $v = b + x = b + (y + w) = (b + y) + w$. Now $w = x - y \in I$, $b + y \in B$ and $b + y = v - w \in V - V = V + V \subset U$. Thus $v = (b + y) + w \in (B \cap U) + I$ and since V was arbitrarily chosen it follows that $V \cap (B + I) \subset (U \cap B) + I$. Hence $\phi(V) \cap \phi(B) \subset \phi(U \cap B) = \phi_B(U \cap B)$. Since every member of $\eta(B)$ contains $U \cap B$ for some U in $\eta(A)$ it follows that ϕ_B is open.

2.2. <u>Theorem</u>. Let A be a complete algebra in $\mathcal{L}$ and let B be a dense subalgebra.

(a) If A is a $B(\mathcal{L})$ algebra and $I \cap B$ is dense in I for each closed ideal I of A, then B is a $B(\mathcal{L})$ algebra.

(b) If A is a $B_r(\mathcal{L})$ algebra and B has nonzero intersection with every nonzero closed ideal of A, then B is a $B_r(\mathcal{L})$ algebra.

<u>Proof</u>: (a) Let $\phi : B \to C$, $C \in \mathcal{L}$, be a continuous, onto and almost open homomorphism. Since ϕ is uniformly continuous, it extends to a continuous homomorphism $\tilde{\phi} : A \to \tilde{C}$, $\tilde{C}$ being the completion of C.

Let $U \in \mathcal{N}(A)$. Then $cl_{\tilde{C}}\, \tilde{\phi}(U) \supset cl_{\tilde{C}} \phi(U \cap B) = cl_{\tilde{C}}[cl_C \phi(U \cap B)]$. Since ϕ is almost open, $cl_C\ \phi(U \cap B) \in \mathcal{N}(C)$, hence its closure in $\tilde{C}$ is in $\mathcal{N}(\tilde{C})$. Thus $\tilde{\phi}$ is almost open and since A is a $B(\mathcal{L})$ algebra it follows that $\tilde{\phi} : A \to \tilde{\phi}(A)$ is open.

Let $I = \ker(\tilde{\phi})$ and let $\Psi : A \to A/I$ be the quotient map. By hypothesis, $I \cap B$ is dense in I and so by Lemma 2.1, $\Psi_B : B \to \Psi(B)$ is open. Let $\Psi' : A/I \to \tilde{\phi}(A)$ be the isomorphism such that $\Psi' \circ \Psi = \tilde{\phi}$. Then $\phi = \Psi' \circ \Psi_B$ is open onto $\Psi' \circ \Psi_B(B) = \tilde{\phi}(B) = C$. Thus B is a $B(\mathcal{L})$ algebra.

(b) Let ϕ, C, $\tilde{\phi}$ and $\tilde{C}$ be as in (a) with the additional assumption that ϕ is one-to-one. It follows as in (a) that $\tilde{\phi} : A \to \tilde{\phi}(A)$ is almost open. Now $\ker(\tilde{\phi})$ is a closed ideal and $B \cap \ker(\tilde{\phi}) = \ker(\phi) = \{0\}$. By hypothesis $\ker(\tilde{\phi}) = \{0\}$. Since A is a $B_r(\mathcal{L})$ algebra, $\tilde{\phi}$ is open; hence ϕ is open onto $\phi(B) = C$. Thus B is a $B_r(\mathcal{L})$ algebra.

2.3. <u>Theorem</u>. Let B be a dense subalgebra of A ε $\mathcal{L}$.

(a) If B is a B($\mathcal{L}$) algebra, then I $\cap$ B is dense in I for each closed ideal I of A.

(b) If B is a $B_r(\mathcal{L})$ algebra, then B has nonzero intersection with every closed nonzero ideal of A.

<u>Proof</u>: Let I be a closed ideal of A and let ϕ and ϕ_B be defined as in Lemma 2.1. By this Lemma ϕ_B is continuous and almost open.

(a) If B is a B($\mathcal{L}$) algebra then ϕ_B is open and so by Lemma 2.1, I $\cap$ B is dense in I.

(b) If I $\cap$ B = {0} then ϕ_B is one-to-one and hence open since B is a $B_r(\mathcal{L})$ algebra. Thus again by Lemma 2.1, {0} = I $\cap$ B is dense in I. But this implies that I = {0} since A is Hausdorff. Therefore, B has nonzero intersection with every closed nonzero ideal of A.

2.4. <u>Theorem</u>. Let B be a dense subalgebra of A ε $\mathcal{L}$.

(a) If B is a B($\mathcal{L}$) algebra, then so is A.

(b) If B is a $B_r(\mathcal{L})$ algebra, then so is A.

<u>Proof</u>: (a) Let $\phi : A \to C$ be a continuous, onto and almost open homomorphism. Let $\phi_B : B \to \phi(B)$ be the restriction of ϕ . It will first be shown that ϕ_B is almost open.

Let V be in $\mathcal{N}(B)$. Then there exists U ε $\mathcal{N}(A)$ such that U is open and $U \cap B \subset V$. Since U is open and B is dense, $cl_A(U \cap B) \supset U$. Thus $cl_C\, \phi_B(V) = cl_C\, \phi(V) \supset cl_C\, \phi(U \cap B) \supset \phi(cl_A(U \cap B)) \supset \phi(U)$. So $cl_C\, \phi(U) \subset cl_C\, \phi_B(V)$, the former being in $\mathcal{N}(C)$ since ϕ is almost open.

Thus $cl_C\ \phi_B(V) \in \mathcal{N}(C)$ and consequently $cl_{\phi(B)}\ \phi_B(V) = cl_C\ \phi_B(V) \cap \phi(B) \in \mathcal{N}(\phi(B))$. Thus ϕ_B is almost open; and since B is a $B_r(\mathcal{L})$ algebra, ϕ_B is open onto $\phi(B)$.

To show that ϕ is open, let $U \in \mathcal{N}(A)$. Then there exists $V \in \mathcal{N}(A)$ such that $V + V \subset U$. Since ϕ_B is open there exists $W \in \mathcal{N}(C)$ such that W is open and $W \cap \phi(B) \subset \phi_B(V \cap B) \subset \phi(V)$. This implies that $\phi^{-1}(W) \cap B \subset \phi^{-1}(\phi(V))$ and so $cl_A[\phi^{-1}(W) \cap B] \subset \phi^{-1}(\phi(V)) + V$. Since $\phi^{-1}(W)$ is open in A and B is dense in A, it follows that $\phi^{-1}(W) \subseteq cl_A[\phi^{-1}(W) \cap B] \subset \phi^{-1}(\phi(V)) + V$. Since ϕ is onto $W \subset \phi(V) + \phi(V) \subset \phi(U)$; hence ϕ is open. Thus, A is a $B(\mathcal{L})$ algebra.

(b) The proof is identical with (a), with certain simplifications since ϕ is one-to-one.

2.5. Corollary: Let B be a dense subalgebra of $A \in \mathcal{L}$.

(a) If A is a $B(\mathcal{L})$ algebra and $I \cap B$ is dense in I for each closed ideal I of A, then B is a $B(\mathcal{L})$ algebra.

(b) If A is a $B_r(\mathcal{L})$ algebra and B has nonzero intersection with every closed nonzero ideal of A, then B is a $B_r(\mathcal{L})$ algebra.

Proof: (a) Let A be a $B(\mathcal{L})$ algebra. By Theorem 2.4 (a) it follows that $\tilde{A}$ is also a $B(\mathcal{L})$ algebra. Let I be a closed ideal of $\tilde{A}$. Then, by Theorem 2.3 (a) $I \cap A$ is dense in I. By our hypothesis $(I \cap A) \cap B$ is dense in $I \cap A$. Consequently $I \cap B = (I \cap A) \cap B$ is dense in I, and so by Theorem 2.2 (a) B is a $B(\mathcal{L})$ algebra.

(b) The proof is analogous with (a).

The following theorem summarizes the preceding results.

2.6. Theorem. Let B be a dense subalgebra of $A \in \mathcal{L}$.

(a) B is a $B(\mathcal{L})$ algebra iff A is a $B(\mathcal{L})$ algebra and $I \cap B$ is dense in I for each closed ideal I of A.

(b) B is a $B_r(\mathcal{L})$ algebra iff A is a $B_r(\mathcal{L})$ algebra and B has nonzero intersection with every nonzero closed ideal of A.

This section is concluded with some applications of the above criterion.

2.7. Definition. Let X be a completely regular space. Then $C^*(X)$ is the algebra of all bounded continuous complex-valued functions on X with the relative compact-open topology from $C(X)$; $C_p(X)$ is $C(X)$ with the topology of pointwise convergence.

2.8. Proposition (a) If $C(X)$ is a $B(\mathcal{L})$ algebra, then so is $C^*(X)$.

(b) If $C(X)$ is a $B_r(\mathcal{L})$ algebra, then so is $C^*(X)$.

Proof: (a) In view of Prop. 0; 4.12, it follows that $C^*(X)$ is dense in $C(X)$. Let I be a closed nonzero ideal of $C(X)$. By Theorem 0; 4.9, $I = I_F$ for some proper closed subset F of X. Let $f \in I_F$ and let $N(f, K, \xi)$ be a neighborhood of f. Let $\lambda = \sup_{x \in K} |f(x)|$ and let $W = \{x \in X : |f(x)| \geq \lambda + 1\}$. Then W is closed and $W \cap K = \emptyset$. Since X is completely regular there exists $g \in C(X)$ such that $0 \leq g \leq 1$, $g(K) = \{1\}$ and $g(W) = \{0\}$. Then $g \cdot f \in C^*(X) \cap I_F$ and $g\, f \in N(f; K, \xi)$. Thus $C^*(X) \cap I_F$ is dense in I_F, and so by Corollary 2.5 (a) $C^*(X)$ is a $B(\mathcal{L})$ algebra.

(b) follows from Corollary 2.5 (b).

The converse of the above proposition follows from Theorem 2.4.

2.9. Proposition $C_p(X)$ is a $B_r(\mathcal{L})$ algebra iff X is discrete.

Proof: If X is discrete then $C_p(X) = \mathbb{C}^X$ which is B-complete by [49; p. 162, Ex. 3]. In particular $C_p(X)$ is a $B_r(\mathcal{L})$ algebra.

Conversely let $C_p(X)$ be a $B_r(\mathcal{L})$ algebra. Suppose X is not discrete. Then there exists an element $x \in X$ such that $S = X \setminus \{x\}$ is dense in X. It is well known that the completion of $C_p(X)$ is $\mathbb{C}^X$. Let I be the set of all functions in $\mathbb{C}^X$ which are zero on S. Then I is a nonzero closed ideal of $\mathbb{C}^X$ and $C_p(X) \cap I = \{0\}$. This is a contradiction, in view of Theorem 2.3. Therefore X is discrete.

The following examples show that a $B(\mathcal{L})$ algebra need not be complete and that a normed algebra need not be a $B_r(\mathcal{L})$ algebra.

2.10. Examples. (a) As was pointed out in Example 1.3 (b), $C(\mathbb{R})$ is a $B(\mathcal{L})$ algebra. By Proposition 2.8, $C^*(\mathbb{R})$ with the compact-open topology is a $B(\mathcal{L})$ algebra which is not complete.

(b) The subalgebra P of all polynomials in C[0,1] is a normed algebra under the supremum norm which is not a $B(\mathcal{L})$ algebra. To see this, let I be the closed ideal of C[0,1] consisting of all functions which are zero on the closed interval $[\frac{1}{2}, \frac{3}{4}]$. Then $I \cap P = \{0\}$. Since P is dense in C[0,1] it follows by Theorem 2.3 that P is not a $B_r(\mathcal{L})$ algebra.

Some further applications of Sulley's criterion occur in the next section.

3. Necessary and Sufficient conditions for $C(X)$ to be a $B(\mathcal{L})$ algebra.

Theorem 3.2 and the first part of Theorem 3.5 were proved by Ptak [46; Theorems 6.7 and 6.4]. The proofs are included for completeness and also because the original paper is in Russian. Thanks are due to Dr. M. Novotny for his indispensable help with the translation.

3.1. Definition. Let X be a completely regular space and let F be a closed subset. $C_F(X)$ is the subalgebra of $C(F)$ consisting of all functions which are restrictions of members of $C(X)$. Unless otherwise specified, it is assumed that $C_F(X)$ has the relative compact-open topology from $C(F)$.

3.2. Theorem. Let F be a closed subspace of X. Then the restriction map $\phi : C(X) \to C_F(X)$ is continuous and open.

Proof: The continuity of ϕ is clear. To show that ϕ is open, let K be a compact subset of X and let $\xi > 0$. Then $N(K, \xi) = \{f \in C(X): |f(X)| \leq \xi \ \forall x \in K\}$ is a basic closed neighborhood of zero in $C(X)$. Choose $0 < \delta < \xi$ and let $V = \{f \in C(F) : |f(x)| \leq \delta \ \forall x \in K \cap F\} \cap C_F(X)$. Then V is a neighborhood of zero in $C_F(X)$. It will be shown that $V \subset \phi(N(K, \xi))$.

Let f be in V. Then $f = g|F$ for some $g \in C(X)$. Let $H = \{x \in K : |g(x)| \geq \xi\}$. Then H is a compact subset of X and $H \cap F = \emptyset$. Since X is completely regular there exists $h \in C(X)$ such that $0 \leq h \leq 1$, $h(F) = \{1\}$ and $h(H) = \{0\}$. Let $g' = gh \in C(X)$. Then, for each $x \in F$, $g'(x) = g(x)\ h(x) = g(x)$; and so $g'|F = g|F = f$. To show that $g' \in N(K, \xi)$, let $x \in K$. If $x \notin H$, then $|g(x)| < \xi$ and so $|g'(x)| \leq \xi$; if $x \in H$, then $h(x) = 0$ and $g'(x) = 0$. Hence, $g' \in N(K, \xi)$ and $\phi(g') = g'|F = f$. Consequently, $V \subset \phi(N(K, \xi))$, thus showing that ϕ is open.

It should be pointed out that $C_F(X)$ is isomorphic to $C(X)/I_F$; and for compact $K \subset X$, $C_K(X) = C(K)$ by Theorem 0; 4.11.

3.3. <u>Lemma</u>. Let A_u be in $\mathcal{L}$. Then the following are equivalent:

(a) A_u is a $B_r(\mathcal{L})$ algebra.

(b) For any Hausdorff locally m-convex topology v on A, if the identity map $i : A_u \to A_v$ is continuous and almost open, then $u = v$.

<u>Proof</u>: (a) $\Rightarrow$ (b) follows from the definition of a $B_r(\mathcal{L})$ algebra.

(b) $\Rightarrow$ (a) Let $\phi : A_u \to B$, $B \in \mathcal{L}$ be a continuous, one-to-one, onto and almost open homomorphism. Let v be the topology generated on A by $\{\phi^{-1}(V) : V \in \mathcal{N}(B)\}$. Then v is a Hausdorff locally m-convex topology such that $v \subset u$. Let $i : A_u \to A_v$ be the identity map, and $\phi' : A_v \to B$ the isomorphism which coincides with ϕ pointwise; it follows that $\phi' \circ i = \phi$. Since ϕ is almost open and ϕ' is open, i is almost open. By the hypothesis, i is open and hence ϕ is open. Thus, A_u is a $B_r(\mathcal{L})$ algebra.

3.4. <u>Lemma</u>. Let the identity map $i : C(X) \to C(X)_u$ be almost open. If α_x (evaluation at x) is u-continuous for each $x \in X$, then i is open.

<u>Proof</u>: Since α_x is u-continuous for each $x \in X$, $N(\{x\}, \xi)$ is u-closed for each $x \in X$ and each $\xi > 0$. Consequently $N(K, \xi) = \bigcap_{x \in K} N(\{x\}, \xi)$

is u-closed for each compact $K \subset X$ and each $\xi > 0$. Since i is almost open, $N(K, \xi) = cl_u\, i(N(K, \xi))$ is a u-neighborhood of zero for each compact $K \subset X$ and each $\xi > 0$; hence i is open.

3.5. Theorem. $C(X)$ is a $B_r(\mathcal{L})$ algebra iff every dense and k-closed subset of X coincides with X.

Proof: Suppose that $C(X)$ is a $B_r(\mathcal{L})$ algebra. Let S be a dense and k-closed subset of X. It will be shown that $S = X$.

Define a topology u on $C(X)$ by taking sets of the form $N(K \cap S, \xi)$, where K is a compact subset of X and $\xi > 0$, as a basis for the u-neighborhoods of zero. Since S is dense u is Hausdorff; since each $N(K \cap S, \xi)$ is m-convex, u is m-convex. Consequently $C(X)_u$ is in $\mathcal{L}$. Since $N(K, \xi) \subset N(K \cap S, \xi)$, the identity map $i : C(X) \to C(X)_u$ is continuous. It will be shown that i is almost open by showing that $N(K \cap S, \frac{1}{2}\xi) \subset cl_u(N(K, \xi))$.

To prove the above inclusion it suffices to show that for each $f \in N(K \cap S, \frac{1}{2}\xi)$ and each u-neighborhood V of f, $V \cap N(K, \xi) \neq \emptyset$. So, let $f \in N(K \cap S, \frac{1}{2}\xi)$ and let $N(f; H \cap S, \delta)$ be a basic u-neighborhood of f, H being a compact subset of X and $\delta > 0$. Let $L = \{x: x \in K \text{ and } |f(x)| \geq \xi\}$. Then, L is a compact subset of K and $L \cap S = \emptyset$. Since S is k-closed, $(H \cap K) \cap S$ is compact; and since $L \cap [(H \cup K) \cap S] = \emptyset$, there exists, by the complete regularity of X, a function $g \in C(X)$ such that $0 \leq g \leq 1$, $g = 0$ on L, and $g = 1$ on $(H \cup K) \cap S$. Let $h = f\, g \in C(X)$. If $x \in H \cap S$, then $g(x) = 1$ and so $h(x) = f(x)$. Thus $h \in N(f; H \cap S, \delta)$.

It remains to show that $h \in N(K, \xi)$. Let $x \in K$; if $x \in L$, then $g(x) = 0$ and hence $h(x) = 0$; if $x \notin L$ then $|f(x)| < \xi$ and hence $|h(x)| < \xi$.

Thus, $h \in N(K, \xi) \cap N(f, H \cap S, \delta)$, and since f, H and δ were arbitrarily chosen it follows that $N(K \cap S, \frac{1}{2}\xi) \subset cl_u\, N(K, \xi)$. This says that $i : C(X) \to C(X)_u$ is almost open. Since $C(X)$ is a $B_r(\mathcal{L})$ algebra, i is open.

Now it readily follows that $S = X$. For, let $x \in X$. Since i is open $N(\{x\}, 1)$ is a u-neighborhood of 0. By the definition of u, there exist a compact $K \subset X$ and $\xi > 0$ such that $N(K \cap S, \xi) \subset N(\{x\}, 1)$. By the complete regularity of X it immediately follows that $x \in K \cap S \subset S$. Hence, $S = X$.

Conversely, suppose that every dense and k-closed subset of X coincides with X. Let the identity map $i : C(X) \to C(X)_u$, $C(X)_u \in \mathcal{L}$, be continuous and almost open. In view of Lemma 3.4, to show that i is open it suffices to show that α_x is u-continuous for each $x \in X$. It will be shown that the set $\{x : x \in X \text{ and } \alpha_x \text{ is u-continuous}\}$ is a dense and k-closed subset of X.

Since u is a locally m-convex topology, it is generated by a set $\{p_j\}_{j \in J}$ of submultiplicative seminorms. Using the fact that i is almost open we will first show that actually u is generated by a subset of $\{p_K\}_{K \subset X}$. To see this, let $p \in \{p_j\}_{j \in J}$. Since p is continuous with respect to the compact-open topology there exist a compact $K \subset X$ and $\xi > 0$ such that $N(K,\xi) \subset N(p, 1) = \{f \in C(X) : p(f) \leq 1\}$. [Note: $N(K, \xi)$ need not be a u-neighborhood of zero.] This implies that $I_K \subset \ker(p) = I$, say. Since I is a closed ideal in $C(X)_u$, it is also closed in $C(X)$, hence by Theorem 0; 4.9 $I = I_H$ for some closed $H \subset X$.

Since $I_K \subset I = I_H$, it follows that $H \subset K$ and so H is compact. To see that p_H is u-continuous, consider the following diagram:

$$\begin{tikzcd} C(X) \arrow[r, "i"] \arrow[d, "\phi"'] & C(X)_u \arrow[r, "\Psi"] & C(H)_u \\ C(H) \arrow[urr, "i^*"'] & & \end{tikzcd}$$

where ϕ and Ψ are the quotient maps and i^* is the unique continuous homomorphism such that $i^* \circ \phi = \Psi \circ i$. Since I_H is u-closed, $C(H)_u = C(X)_u / I_H$ is Hausdorff. By proposition 0; 4.11 ϕ and Ψ are onto. It follows that i^* is continuous, one-to-one, onto and almost open since i is such and since ϕ and Ψ are open. Since $C(H)$ is a Banach algebra it is a $B(\mathcal{L})$ algebra, hence i^* is open. Consequently:

(a) p_H is u- continuous.

Also, since i^* is continuous, it follows that $i^* \circ \phi(N(p_H, \xi)) \subset \Psi(N(p, 1))$ for some $\xi > 0$. Since $I_H = \ker(p_H) = \ker(p) = \ker(\phi) = \ker(\Psi)$ it follows that

(b) $N(p_H, \xi) \subset N(p, 1)$.

Combining (a) and (b) and recalling that p was arbitrarily chosen, we can conclude that the topology u is generated by a <u>subset</u> $\{p_H\}$ of $\{p_K\}_{K \subset X}$.

Let $S = \bigcup \{H : H \subset X,\ H \text{ compact, and } p_H \text{ is u-continuous}\}$

$= \bigcup \{H : H \subset X \text{ and } N(H, 1) \text{ is a u-neighborhood of zero}\}$.

To see that $S = \{x : x \varepsilon X$ and α_x is u-continuous$\}$, let $x \varepsilon S$. By the definition of S there exists a compact subset H of X such that $x \varepsilon H$ and $N(H, 1)$ is a u-neighborhood of zero. Now $N(H, 1) \subset N(\{x\}, 1)$ is u-neighborhood of zero, hence α_x is u-continuous. If α_x is u-u-continuous then clearly $x \varepsilon S$. Hence equality holds.

Let $\overline{S} = cl_X(S)$ and suppose there exists $x \varepsilon X \setminus \overline{S}$. By the complete regularity of X there exists $f \varepsilon C(X)$ such that $f(x) = 1$ and $f(\overline{S}) = \{0\}$. Since u is a Hausdorff topology which is generated by a subset of $\{p_K\}_{K \subset X}$, there exists a compact $H \subset X$ such that p_H is u-continuous and $p_H(f) \neq 0$. But since p_H is u-continuous $H \subset S$ and so $p_H(f) = 0$. This contradiction shows that $\overline{S} = X$.

To see that S is k-closed, let K be a compact subset of X. Since i is almost open, $cl_u\, N(K, 1)$ is a u-neighborhood of 0. Since α_x is u-continuous for each $x \varepsilon S$ $N(K \cap S, 1)$ is u-closed. Thus $cl_u\, N(K, 1) \subset N(K \cap S, 1) = N(\overline{K \cap S}, 1)$, and so the latter is also a u-neighborhood of 0. Hence, by the definition of S, $\overline{K \cap S} \subset S$. Since $\overline{K \cap S} \subset K$, it follows that $K \cap S = \overline{K \cap S}$ is closed. Thus, S is a dense and k-closed subset of X, and by the hypothesis $S = X$. So α_x is u-continuous for each $x \varepsilon X$, hence by Lemma 3.4 i is open.

In view of Lemma 3.3. it follows that C(X) is a $B_r(\mathcal{L})$ algebra.

From the above theorem it follows that if X is a k-space then C(X) is a $B_r(\mathcal{L})$ algebra. An example in section 4 shows that the converse is false.

In order to simplify the proof of the next theorem, the following two lemmas are proved first.

3.6. <u>Lemma</u>. Let $A \in \mathcal{L}$. Then A is a $B(\mathcal{L})$ algebra iff A/I is a $B_r(\mathcal{L})$ algebra for each closed ideal I of A.

<u>Proof</u>: If A is a $B(\mathcal{L})$ algebra, then A/I is a $B(\mathcal{L})$ algebra, in particular a $B_r(\mathcal{L})$ algebra, for each closed ideal I of A, by Corollary 1.5.

Conversely, suppose that A/I is a $B_r(\mathcal{L})$ algebra for each closed ideal I of A. Let $\phi : A \to B$, $B \in \mathcal{L}$, be a continuous, onto, and almost open homomorphism. Then $I = \ker(\phi)$ is a closed ideal of A. Let $\Psi : A \to A/I$ be the quotient map, and let $\phi' : A/I \to B$ be the unique continuous, one-to-one and onto homomorphism such that $\phi = \phi' \circ \Psi$. Since Ψ is open and ϕ is almost open, it follows that ϕ' is almost open. By the hypothesis, ϕ' is open. Hence, $\phi = \phi' \circ \Psi$ is open, and A is a $B(\mathcal{L})$ algebra.

3.7. <u>Lemma</u>. Let F be a closed subset of X. Then C(F) is a $B_r(\mathcal{L})$ algebra iff $C_F(X)$ is a $B_r(\mathcal{L})$ algebra.

<u>Proof</u>: Suppose $C_F(X)$ is a $B_r(\mathcal{L})$ algebra. In view of Theorem 0; 4.12 $C_F(X)$ is dense in C(F), thus by Theorem 2.4 (b) it follows that C(F) is also a $B_r(\mathcal{L})$ algebra.

Conversely, suppose C(F) is a $B_r(\mathcal{L})$ algebra. Let I be a closed nonzero ideal of C(F). By Theorem 0; 4.9, $I = I_G$ for some closed subset $G \subset F$. Let $x \in F \setminus G$. Since X is completely regular there exists $f \in C(X)$ such that $f(x) = 1$ and $f(G) = \{0\}$. Then $0 \neq f|F \in C_F(X) \cap I$.

By Corollary 2.5 (b) it follows that $C_F(X)$ is a $B_r(\mathcal{L})$ algebra.

3.8. Theorem. $C(X)$ is a $B(\mathcal{L})$ algebra iff X is a k-space.

Proof: Suppose $C(X)$ is a $B(\mathcal{L})$ algebra. Let S be a k-closed subset of X. Then S is dense and k-closed in $\overline{S} = cl_X(S)$. By Theorem 3.2, the restriction map $\phi : C(X) \to C_{\overline{S}}(X)$ is continuous, open and onto. Hence, by Proposition 1.4 (a), $C_{\overline{S}}(X)$ is also a $B(\mathcal{L})$ algebra, in particular a $B_r(\mathcal{L})$ algebra. By Lemma 3.7, it follows that $C(\overline{S})$ is also a $B_r(\mathcal{L})$ algebra. Since S is a dense and k-closed subspace of $\overline{S}$, by Theorem 3.5 $S = \overline{S}$. Thus X is a k-space.

Conversely, suppose that X is a k-space. Let I be a closed ideal of $C(X)$. By Theorem 0; 4.9, $I = I_F$ for some closed $F \subset X$. Since k-spaces are closed hereditary, F is also a k-space; hence, by Theorem 3.5, $C(F)$ is a $B_r(\mathcal{L})$ algebra. By Lemma 3.7, $C_F(X)$ is also a $B_r(\mathcal{L})$ algebra. Thus, $C(X)/I = C(X)/I_F \cong C_F(X)$ is a $B_r(\mathcal{L})$ algebra. Since I was arbitrarily chosen, it follows by Lemma 3.6 that $C(X)$ is a $B(\mathcal{L})$ algebra.

Theorems 3.5 and 3.8 give rise to large classes of $B_r(\mathcal{L})$ and $B(\mathcal{L})$ algebras, respectively. Moreover, from the following proposition which is implicit in Ptak's paper [46], it follows that there is an abundance of complete $B(\mathcal{L})$ algebras which are not B-complete as locally convex spaces.

3.9. Proposition. If $C(X)$ is B-complete, then X is a normal k-space.

Proof: That X is a k-space follows directly from Theorem 3.8. Let F be a closed subspace of X. Then, by Theorem 3.2, the restriction map $\phi : C(X) \to C_F(X)$ is continuous, open and onto. By Proposition 0; 2.10

$C_F(X)$ is also B-complete, in particular complete. Since $C_F(X)$ is dense in $C(F)$, $C_F(X) = C(F)$. Hence, by Urysohn's lemma X is normal.

As an example of a complete $B(\mathcal{L})$ algebra which is not B-complete as a locally convex space, let X be a k-space which is not normal, in particular let X be the Tychonoff plank. Then by Theorems 0; 4.5 and 3.8, C(X) is a complete $B(\mathcal{L})$ algebra; but, by Proposition 3.9, C(X) is not B-complete.

4. Counterexamples.

In [46; Theorem 6.17] Ptak gave an example, credited to M. Katětov, of a topological space X such that C(X) is complete but not B_r-complete. This also provides an example of a complete LMC algebra which is not a $B_r(\mathcal{L})$ algebra. Since this space is used very frequently for counterexamples, a complete proof of its construction is given.

The following two lemmas are stated but not proved in [46; Theorem 6.17]. The proof of Lemma 4.2 is due to E. Michael [40; Lemma 2.1].

4.1. Definition. A function $f : \mathbb{R}^2 \to \mathbb{R}$ (usual topology) is said to be separately continuous if $f|L$ is continuous for each vertical or horizontal line $L \subset \mathbb{R}^2$.

4.2. Lemma. Let $\{(a_n, b_n)\}_{n \in \mathbb{N}}$ be a strictly monotone sequence in $\mathbb{R}^2$ which converges to (a,b). Then there exists a separately continuous function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f(a,b) = 0$ and $f(a_n, b_n) = 1$ for each $n \in \mathbb{N}$.

Proof. Let $Y = \mathbb{R}^2 \setminus \{(a,b)\}$, and let $A = \{(a_n, b_n)\}_{n \in \mathbb{N}}$,

$B = \{(x,y) \in Y : x = a \text{ or } y = b\}$. Since $(a_n, b_n)_{n \in \mathbb{N}}$ is strictly monotone and $(a_n, b_n) \to (a,b)$, A and B are disjoint closed subsets of Y. Since Y is normal, there is a continuous function $g : Y \to R$ such that $g(A) = \{1\}$ and $g(B) = \{0\}$. Extend g to a function $f : \mathbb{R}^2 \to \mathbb{R}$ by letting $f(a,b) = 0$. Then f is separately continuous at (a,b) and continuous elsewhere; and $f(a_n, b_n) = 1$ for all n, while $f(a,b) = 0$.

In the following lemma $[m,n]$ denotes the greatest common divisor of m and n.

4.3. Lemma. There exists a countable dense subset S of $\mathbb{R}^2$ having the property that $L \cap S$ is finite (possibly empty) for each vertical or horizontal line $L \subset \mathbb{R}^2$.

Proof: For each $m \in \mathbb{N}$, let

$$S_m = \{(\tfrac{i}{m}, \tfrac{j}{m^2}) \in \mathbb{R}^2 : i,j \in \mathbb{Z},\ [i,m] = [j,m] = 1,\ |i| \leq m^2 \text{ and } |j| \leq m^4\}.$$

Let $S = \bigcup_{m \in \mathbb{N}} S_m$. It is clear that $S \cap L$ is finite for each vertical and horizontal line $L \subset \mathbb{R}^2$.

To see that S is dense, let V be any open subset of $\mathbb{R}^2$. Since S is symmetrical, it may be assumed that V has nonempty intersection with the interior of the first quadrant. Choose $p, q \in \mathbb{Q}$ and $\xi > 0$ such that $p > 0$, $q > 0$ and $N((p,q), \xi) \subset V$. Let $p = \frac{a}{b}$ in reduced form. Choose $0 \neq k \in \mathbb{N}$ such that

(1) $b|k$

(2) $\frac{1}{k} < \min\{\frac{\xi}{4}, \frac{1}{a}, \frac{1}{q}\}$.

By the division algorithm, there exists $n \in \mathbb{N}$ such that $q = n(\frac{1}{b^2k}) + r$,

$0 \leq r < \frac{1}{b^2k}$.

It will be shown that $\left(\frac{ak+1}{bk}, \frac{nk+1}{b^2k^2}\right) \in S \cap N((p,q), \xi) \subset V$.

First: since $b|k$, $[ak + 1, bk] = 1$; and since $a < k$, $(bk)^2 = b^2k^2 > a\, b^2k \geq ak + 1$. Also, $\left|p - \frac{ak+1}{bk}\right| = \left|\frac{a}{b} - \frac{a}{b} - \frac{1}{bk}\right| < \frac{\xi}{4}$.

Second: again, since $b|k$, $[nk + 1, b^2k^2] = 1$; and $\left|q - \frac{nk+1}{b^2k^2}\right| = \left|\frac{n}{b^2k} + r - \frac{n}{b^2k} - \frac{1}{b^2k^2}\right| \leq r + \frac{1}{b^2k^2} \leq \frac{1}{b^2k} + \frac{1}{b^2k^2} < 2(\frac{\xi}{4}) = \frac{\xi}{2}$. Moreover, since $q < k$ and $\frac{nk}{b^2k^2} < q < k$, it follows that $nk < k(bk)^2 < ak(bk)^2$. This implies that $nk + 1 \leq ak(bk)^2 + (bk)^2 = (ak + 1)(bk)^2 \leq (bk)^4$, since $ak + 1 \leq (bk)^2$.

By combining the last two paragraphs, it follows that

$$\left(\frac{ak+1}{bk}, \frac{nk+1}{(bk)^2}\right) \in S \cap N((p,q), \xi) \subset V;$$

consequently S is dense in $\mathbb{R}^2$.

4.4. <u>Theorem</u>. There exists a completely regular space X such that:

(a) Every k-continuous $f: X \to \mathbb{R}$ is continuous (i.e., X is a k_R-space).

(b) X contains a proper dense and k-closed subset.

<u>Proof</u>: Let $\mathbb{R}^2_u$ be $\mathbb{R}^2$ with the usual Euclidean topology and let $\mathcal{F}$ be the set of all separately continuous functions $f : \mathbb{R}^2_u \to \mathbb{R}$. Let v be the weakest topology which makes every member of $\mathcal{F}$ continuous. Then $\mathbb{R}^2_v$ is homeomorphic to a subspace of $\prod_{f\in\mathcal{F}} \mathbb{R}_f$, and hence $\mathbb{R}^2_v$ is completely

regular. Also,

(1) v is finer than u.

(2) v and u coincide on each vertical and horizontal line.

To show that $\mathbb{R}_v^2$ is the required space, the first step is to prove the following statement:

(I) A set $K \subset \mathbb{R}_v^2$ is compact iff K is contained in the union of finitely many vertical and horizontal closed line segments.

The "only if" part follows from the fact that each closed vertical and horizontal line segment is compact in $\mathbb{R}_v$, hence also finite unions thereof.

Conversely, suppose that K is v-compact. Then K is also u-compact, hence closed and bounded in $\mathbb{R}_u^2$. Suppose that K is not contained in the union of finitely many horizontal and vertical closed line segments. Then there exists $(a,b) \in K$ and a strictly monotone sequence $\{(a_n, b_n)\}_{n \in \mathbb{N}}$ in K such that $(a_n, b_n) \to (a,b)$ in $\mathbb{R}_u$.

By Lemma 4.2 there exists a separately continuous function $f \in \mathcal{F}$ such that $f(a,b) = 0$ and $f(a_n, b_n) = 1$ for each $n \in \mathbb{N}$. Hence, $\{(a_n, b_n)\}_{n \in \mathbb{N}}$ is a v-closed subset of K and therefore v-compact. It follows that $\{(a_n, b_n)\}_{n \in \mathbb{N}}$ is also u-compact; but this contradicts the fact that $(a_n, b_n) \to (a,b)$ in $\mathbb{R}_u^2$. Thus K is contained in the union of finitely many vertical and horizontal closed line segments.

In view of the form of the compact subsets of $\mathbb{R}_v^2$, it follows that each k-continuous function $f : \mathbb{R}_v^2 \to \mathbb{R}$ is separately continuous with respect to R_u^2; hence v-continuous by the very definition of v. Also, the subset $S \subset \mathbb{R}^2$ given by Lemma 4.3 is clearly k-closed in $\mathbb{R}_v^2$. To complete the proof of the theorem it will be shown that S is dense in $\mathbb{R}_v^2$.

By Lemma 4.3, S is dense in $\mathbb{R}_u^2$. It will be shown that S is also dense in $\mathbb{R}_v^2$ by showing that each v-open set contains a u-open set. Let V be a v-open set and let $(a,b) \in V$. By the definition of v, there exists $f \in \mathcal{F}$ and $\xi > 0$ such that $W = \{(x,y) : |f(x,y) - f(a,b)| < \xi\} \subset V$. Since $f(a,y)$ is a continuous function of y, there exists $\delta > 0$ such that $|y - b| \leq \delta$ implies $|f(a,y) - f(a,b)| < \frac{1}{2}\xi$. Let

$T_n = \{y \in R : |y - b| \leq \delta$ and $|x - a| \leq \frac{1}{n}$ imply $|f(x,y) - f(a,b)| < \frac{1}{2}\xi\}$

<u>Claim</u>: $T = \bigcup_{n \in \mathbb{N}} T_n = [b - \delta, b + \delta]$.

Let $y_o \in [b - \delta, b + \delta]$. Then $|f(a, y_o) - f(a,b)| = \gamma < \frac{1}{2}\xi$. Since f is separately continuous, there exists $n_o \in \mathbb{N}$ such that $|x - a| \leq 1/n_o$ implies

$$|f(x,y_o) - f(a,y_o)| < (\tfrac{1}{2}\xi - \gamma) .$$

Thus, $|f(x,y_o) - f(a,b)| \leq |f(x,y_o) - f(a,y_o)| + |f(a,y_o) - f(a,b)| < \gamma + (\frac{1}{2}\xi - \gamma) = \frac{1}{2}\xi$; hence $y_o \in T_{n_o}$ and so $T = \bigcup_{n \in \mathbb{N}} T_n = [b - \delta, b + \delta]$.

By the Baire category theorem, $cl_R (T_m)$ has nonempty interior for some $m \in \mathbb{N}$. Thus $[y_1, y_2] \subset cl_{\mathbb{R}}(T_m) \subset [b - \delta, b + \delta]$.

It remains to show that

$$U = [a - \frac{1}{m}, a + \frac{1}{m}] \times [y_1,y_2] \subset W.$$

Indeed, fix x_1 such that $|x_1 - a| \leq \frac{1}{m}$. Consider the vertical segment $L = \{(x_1,y) : y_1 \leq y \leq y_2\} \subset \mathbb{R}^2$. The points (x_1, y) with $y \in T_m$ are u-dense in L and satisfy the inequality

$$|f(x_1, y) - f(a,b)| < \tfrac{1}{2}\xi .$$

Since f is separately continuous,

$$|f(x_1,y) - f(a,b)| < \tfrac{1}{2}\xi < \xi \text{ for all } (x_1,y) \in L.$$

Thus $L = \{(x_1,y) : y_1 \leq y \leq y_2\} \subset W$. Since x_1 was arbitrarily chosen in $[a - \frac{1}{m}, a + \frac{1}{m}]$, it follows that $U = [a - \frac{1}{m}, a + \frac{1}{m}] \times [y_1,y_2] \subset W \subset V$. Since U contains a u-open set, it follows that every v-open set contains a u-open set. Consequently every u-dense subset of $\mathbb{R}^2$ is v-dense. Thus the set S of Lemma 4.3. is a countable dense and k-closed subset of $\mathbb{R}^2_v$.

In conclusion, $\mathbb{R}^2_v$ is a k_R-space which has a countable dense and k-closed subset.

In view of Theorems 0; 4.5 and 3.5, it follows that $C(\mathbb{R}^2_v)$ is a complete locally m-convex algebra, but not a $B_r(\mathcal{L})$ algebra.

4.5. <u>Example</u>. There exists a completely regular space X such that C(X) is a $B_r(\mathcal{L})$ algebra but not a $B(\mathcal{L})$ algebra.

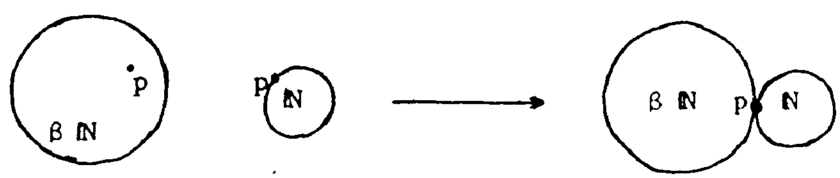
Let $p \in \beta\mathbb{N}\setminus\mathbb{N}$. Then $\mathbb{N} \cup \{p\}$ is not discrete but every compact subset is finite. To see this, let K be an infinite compact subset of $\mathbb{N} \cup \{p\}$; and let $\{x_n\}_{n \in \mathbb{N}}$ be an infinite subset of $K \cap \mathbb{N}$. Since K is compact, there exists a subsequence $\{x_{n_i}\}$ such that $x_{n_i} \to p$ (necessarily).

Define $f : \{x_{n_i}\} \to [0,1]$ by

$$f(x_{n_i}) = \begin{cases} 0 & \text{if } i \text{ is odd} \\ 1 & \text{if } i \text{ is even.} \end{cases}$$

Then there exists a continuous extension $f : \beta\mathbb{N} \to [0,1]$. But clearly f cannot be continuously extended to p. This contradiction shows that every compact subset of $\mathbb{N} \cup \{p\}$ is finite.

Let $Z = \beta\mathbb{N} \amalg (\mathbb{N} \cup \{p\})$, the disjoint topological union. Let X be the quotient of Z obtained by identifying the element $p \in \beta\mathbb{N}$ with the element p in $\mathbb{N} \cup \{p\}$. Graphically, one has the following:



It is clear that X, with the quotient topology, is completely regular. Since p is a limit point of $\mathbb{N}$ in X, $\mathbb{N}$ is not a closed subset of X. However, $\mathbb{N}$ is a k-closed subset of X since $K \cap \mathbb{N}$ is clearly finite for each compact subset K of X (in view of the above paragraph). Thus, X is not a k-space.

Let D be a dense and k-closed subspace of X. Since D is dense it must contain all the isolated points of X. In particular,

(1) $\mathbb{N} \subset D$

(2) $\mathbb{N} \subset D \cap \beta\mathbb{N}$ (since $\mathbb{N}$ is a discrete subset of $\beta\mathbb{N}$).

Since D is k-closed it follows that $\beta\mathbb{N} = cl_X\,\mathbb{N} = D \cap \beta\mathbb{N}$. Hence $D = X$, and so every dense and k-closed subset of X coincides with X.

By Theorem 3.5 it follows that $C(X)$ is a $B_r(\mathcal{L})$ algebra. Since X is not a k-space, $C(X)$ is not a $B(\mathcal{L})$ algebra by Theorem 3.8.

It should be noted that even though $C(X)$ is not a $B(\mathcal{L})$ algebra, its completion $C^{\gamma}(X)$ is a $B(\mathcal{L})$ algebra. By Remark 0; 4.13, $C^{\gamma}(X) = C(k_R X)$. Since X is a hemicompact space, $k_R X = kX$ by [55; Lemma 1]. So by Theorem 3.8, $C^{\gamma}(X) = C(kX)$ is a $B(\mathcal{L})$ algebra. (Note: for this particular X, $kX = \beta \mathbb{N} \amalg \mathbb{N}$.)

4.6. Example: There exists a completely regular space X such that $C(X)$ is a complete $B_r(\mathcal{L})$ algebra but not a $B(\mathcal{L})$ algebra.

The following modification of the space $\mathbb{R}^2_v$ of Theorem 4.4 has been constructed by E. Michael [40; section 3].

Let $\mathbb{R}^2_u$ be the plane with the usual topology and let $A \subset \mathbb{R}^2_u$ be the x-axis. Let $\mathcal{F}$ be the set of all $f : \mathbb{R}^2_u \to \mathbb{R}$ which are continuous on $\mathbb{R}^2_u \setminus A$ and separately continuous on A, and let w be the weakest topology making every $f \in \mathcal{F}$ continuous. Then u and w agree on both A and $\mathbb{R}^2 \setminus A$ and on each vertical or horizontal line; and $\mathbb{R}^2_w$ is a σ-compact k_R-space which is not a k-space [40; section 3].

Let D be a dense and k-closed subset of $\mathbb{R}^2_w$. Since $\mathbb{R}^2_w \setminus A$ is open, $D \cap (\mathbb{R}^2_w \setminus A)$ is dense and k-closed in $\mathbb{R}^2_w \setminus A$. But since $u = w$ on $\mathbb{R}^2_w \setminus A$, $\mathbb{R}^2_w \setminus A$ is a k-space and so $D \cap (\mathbb{R}^2_w \setminus A) = \mathbb{R}^2_w \setminus A$. Let $(x,0) \in A$. Then the vertical segment $B_x = \{(x,y) : -1 \leq y \leq 1\}$ is a compact subset of $\mathbb{R}^2_w$ because $u = w$ on each vertical or horizontal line.
Now $B_x \setminus \{(x,0)\} \subset D \cap B_x$, the latter being closed since D is k-closed.

Hence $(x,0) \in cl_w(B_x \setminus \{(x,0)\}) = D \cap B_x$. Since $(x,0)$ was arbitrarily chosen, $A \subset D$, and so $D = \mathbb{R}^2_w$. By Theorems 0; 4.5 and 3.5, $C(\mathbb{R}^2_w)$ is a complete $B_r(\mathcal{L})$ algebra. Since $\mathbb{R}^2_w$ is not a k-space, $C(\mathbb{R}^2_w)$ is not a $B(\mathcal{L})$ algebra.

5. $B_r(\mathcal{T})$ and $B(\mathcal{T})$ algebras.

In his paper [30], T. Husain introduced the notion of a $B(\mathcal{T})$ locally convex space . These are spaces which satisfy a weakened form of the B-completeteness condition; namely : a locally convex space E is said to be a $B(\mathcal{T})$ space if every continuous linear map from E onto any barrelled space F is open. Husain obtained characterizations of $B(\mathcal{T})$ spaces and showed that they need not be B-complete, in fact not even complete. In subsequent papers, Husain introduced the notion of a $B(\mathcal{C})$ space, $\mathcal{C}$ being a suitable subclass of locally convex spaces. A systematic treatment of $B(\mathcal{C})$ spaces may be found in [31].

The purpose of this section is to investigate analogous weaker forms of the $B(\mathcal{L})$ condition for LMC algebras. These weaker notions are obtained by imposing restrictions on the codomain.

Let $\mathcal{T}$ be the subclass of $\mathcal{L}$ consisting of all m-barrelled algebras [Definition 0; 4.8]. Clearly every barrelled LMC algebra is in $\mathcal{T}$. An example of an m-barrelled algebra which is not barrelled was constructed by Warner [54, Example 5].

Lemma 5.1. Let $A \in \mathcal{L}$, $B \in \mathcal{T}$ and let $\phi : A \to B$ be a continuous, onto homomorphism. Then ϕ is almost open.

Proof: Let $U \in \mathcal{N}(A)$. Then there exists $V \in \mathcal{N}(A)$ which is circled and m-convex and such that $V \subset U$. Since $\phi(U)$ is circled and m-convex, it follows from [39; Lemma 1.4 (b)] that $cl_B \phi(V)$ is circled and m-convex, hence an m-barrel. Since B is m-barrelled $cl_B \phi(V) \in \mathcal{N}(B)$. Thus $cl_B \phi(V) \subset cl_B \phi(U) \in \mathcal{N}(B)$ and so ϕ is almost open.

5.2. Definition. An algebra $A \in \mathcal{L}$ is said to be a $B(\mathcal{T})$ algebra if every continuous homomorphism onto any algebra $B \in \mathcal{T}$ is open; a $B_r(\mathcal{T})$ algebra if every continuous and one-to-one homomorphism onto any algebra $B \in \mathcal{T}$ is open.

In view of lemma 5.1 it follows that every $B(\mathcal{L})$ algebra is a $B(\mathcal{T})$ algebra. Subsequent results will show that the converse is not true.

5.3. Lemma. Let U be a barrel in C(X). Then there exists a closed and bounded [Def. 0; 4.1 (c)] subset T of X such that $N(T, \xi) \subset U$ for some $\xi > 0$.

Proof: This follows from [43, Theorem 1].

5.4. Lemma. Let S be a dense and k-closed subset of X and let u be the topology on C(X) as defined in the first part of theorem 3.5. Let T be a closed subset of X such that $T \cap S$ is compact.

Then $cl_u N(T, \xi) = N(T \cap S, \xi)$.

Proof: Clearly $N(T, \xi) \subset N(T \cap S, \xi)$, the latter being u-closed. In the first part of theorem 3.5 it was shown that for each compact $K \subset X$ $N(K \cap S, \frac{1}{2}\xi) \subset cl_u N(K, \xi)$. An investigation of the proof shows that this holds whenever T is a closed subset of X and $T \cap S$ is compact.

Also, $\frac{1}{2}\xi$ may be replaced by any $0 < \delta < \xi$; hence $N(T \cap S, \delta) \subset cl_u N(T, \xi)$ for each $0 < \delta < \xi$. Thus $N(T \cap S, \xi) = cl_u \bigcup_{0<\delta<\xi} N(K \cap S, \delta) \subset cl_u N(T, \xi)$, and so equality holds.

5.5. Definition. A subset S of X is said to be free if $T \cap S$ is compact whenever T is a closed and bounded subset of X.

5.6. Proposition. C(X) is a $B_r(\mathcal{T})$ algebra iff every dense, k-closed and free subset of X coincides with X.

Proof: Suppose C(X) is a $B_r(\mathcal{T})$ algebra. Let S be a dense, k-closed and free subset of X, and let u be the topology on C(X) whose basic neighborhoods of zero are of the form $N(K \cap S, \xi)$. As in Theorem 3.5, the identity map $i : C(X) \to C(X)_u$ is continuous and almost open. Let U be a barrel in $C(X)_u$. Then U is a barrel in C(X), and so by Lemma 5.3 there exists a closed and bounded subset T of X and $\xi > 0$ such that $N(T,\xi) \subset U$. By Lemma 5.4, $N(T \cap S, \xi) = cl_u N(T, \xi) \subset U$. Since S is a free subset of X $T \cap S$ is compact, hence U is a neighborhood of 0 in $C(X)_u$. Thus $C(X)_u$ is m-barrelled. Since C(X) is a $B_r(\mathcal{T})$ algebra, it follows that i is open; consequently S = X as in Theorem 3.5.

Conversely, suppose that every dense, k-closed and free subset of X coincides with X. Let $i : C(X) \to C(X)_u$, $C(X)_u \in \mathcal{T}$, be continuous. Let $S = \{x : x \in X \text{ and } \alpha_x \text{ is u-continuous}\}$.

As in Theorem 3.5, S is a dense and k-closed subset of X and a basis for the neighborhoods of zero in $C(X)_u$ is given by all sets of the form $N(K \cap S, \xi)$. Let T be a closed and bounded subset of X. Then N(T, 1) is an m-barrel in C(X) and by [39; Lemma 1.4 (b)] $cl_u N(T,1)$ is an m-barrel in $C(X)_u$. Since $C(X)_u \in \mathcal{T}$, $cl_u N(T,1)$ is a u-neighborhood of zero. Thus there exists a compact $K \subset X$ and $\xi > 0$ such that $N(K \cap S, \xi) \subset cl_u N(T,1) \subset N(T \cap S,1)$. Since X is completely regular, $T \cap S \subset K \cap S$; hence $T \cap S = (T \cap K) \cap S$ is compact since S is k-closed. Thus S is a dense k-closed and free subset of X. By hypothesis S = X and by Lemma 3.4 i is open. In view of lemma 3.3, C(X) is a $B_r(\mathcal{T})$ algebra.

It should be pointed out that not every completely regular space satisfies the hypothesis of Proposition 5.5. For example let Y be the space $\mathbb{N} \cup \{p\}$ as in example 4.5. Since Y is countable, it is realcompact by [23; Theorem 8.2]. From [43] and [51] it follows that every closed and bounded subset of a realcompact space is compact. Thus every closed and bounded subset of Y is compact, hence finite by Example 4.5. So $\mathbb{N}$ is a proper dense, k-closed and free subset of Y, consequently C(Y) is not a $B_r(\mathcal{T})$ algebra.

5.7. Proposition. C(X) is a $B(\mathcal{T})$ algebra iff every k-closed and free subset of X is closed.

Proof: Suppose C(X) is a $B(\mathcal{T})$ algebra. Let S be a k-closed and free subset of X. By Theorem 3.2, the restriction map $\phi : C(X) \to C_{\overline{S}}(X) \subset C(\overline{S})$ is continuous and open. Let u be the topology on $C_{\overline{S}}(X)$ whose basic neighborhoods of zero are of the form $\{f \in C_{\overline{S}}(X) : |f(x)| \le \xi \ \forall x \in K \cap S\}$, K being a compact subset of $\overline{S}$ and $\xi > 0$. As in Theorem 3.5, the identity map $i : C_{\overline{S}}(X) \to C_{\overline{S}}(X)_u$ is continuous and almost open. Let U be a m-barrel in $C(X)_u$. Then $\phi^{-1}(U)$ is an m-barrel in C(X). By Lemma 5.3 there exists a closed and bounded subset T of X such that $N(T, \xi) \subset \phi^{-1}(U)$ for some $\xi > 0$.

From the proof of Theorem 3.2, it follows that $cl_{C_{\bar{S}}(X)}\ \phi(N(T,\ \xi)) = \{f \in C_{\bar{S}}(X) : |f(x)| \leq \xi\ \forall\ x \in \bar{S} \cap T\} = W$. By Lemma 5.4, $cl_u(W) = \{f \in C_{\bar{S}}(X) : |f(X)| \leq \xi\ \forall\ x \in (\bar{S} \cap T) \cap S = T \cap S\} = V$. Since S is a free subset of X, $S \cap T$ is compact and consequently V is a neighborhood of zero in $C_{\bar{S}}(X)_u$. Thus $V = cl_u(W) = cl_u\ \phi(N(T,\xi) \subset U$ which implies that U is a neighborhood of zero in $C_{\bar{S}}(X)_u$. Thus $C_{\bar{S}}(X)_u$ is m-barrelled. Since $C(X)$ is a $B(\mathcal{T})$ algebra it follows that $\phi \circ i$ is open. Since ϕ is open, i is also open, hence $S = \bar{S}$ as in Theorem 3.5.

Conversely, suppose every k-closed and free subset of X is closed. Let $\Psi : C(X) \to B$, $B \in \mathcal{T}$, be a continuous and onto homomorphism. By Theorem 0; 4.9 $\ker(\phi) = I_F$ for some closed $F \subset X$, hence $B \simeq C_F(X)_u$ for some barrelled locally m-convex topology u. Consider the following diagram.

$$\begin{tikzcd}
C(X) \arrow[r, "\Psi"] \arrow[d, "\phi"'] & C_F(X)_u \\
C_F(X) \arrow[ur, "i"'] &
\end{tikzcd}$$

By Lemma 5.1, i is almost open. Let $S = \{x \in F : \alpha_x$ is u-continuous$\}$. As in theorem 3.5, S is a dense and k-closed subset of F. Also, a basis for the u-neighborhoods of zero is given by sets of the form $\{f \in C_F(X) : |f(x)| \leq \xi\ \forall\ x \in K \cap S\}$, K being a compact subset of F and $\xi > 0$. Let T be a closed and bounded subset of X. Then $V = \{f \in C_F(X) : |f(x)| \leq 1\ \forall\ x \in T \cap S\}$ is a barrel in $C_F(X)_u$, hence a u-neighborhood of zero. It follows that there exists a compact $K \subset F$ and $\xi > 0$ such that $\{f \in C_F(X) : |f(X)| \leq \xi\ \forall\ x \in K \cap S\} \subset V$. Since X is completely regular,

$T \cap S \subset K \cap S$; hence $T \cap S = (T \cap K) \cap S$ is compact since S is a k-closed subset of F. Thus S is a k-closed and free subset of X. By hypothesis $S = \overline{S} = F$, consequently i is open. By Theorem 3.2 ϕ is open, hence $\Psi = i \circ \phi$ is open. Thus $C(X)$ is a $B(\mathcal{T})$ algebra.

5.8. <u>Corollary</u>: If X is pseudocompact, then $C(X)$ is a $B(\mathcal{T})$ algebra.

<u>Proof</u>: Every free subset of a pseudocompact space is compact.

From [45, Construction 2.3] it follows that every completely regular space can be embedded as a closed subspace of a pseudocompact k_R-space. Hence a pseudocompact space need not be a k-space. From Corollary 5.8 and Theorem 3.8, it follows that a $B(\mathcal{T})$ algebra need not be a $B(\mathcal{L})$ algebra. However, the following holds.

5.9. <u>Proposition</u>. Let A be a $B(\mathcal{T})$ algebra which is in $\mathcal{T}$. Then A is a $B(\mathcal{L})$ algebra.

<u>Proof</u>: Let $\phi : A \to B$, $B \in \mathcal{L}$, be a continuous, almost open, onto homomorphism. Let U be an m-barrel in B. Then $\phi^{-1}(U)$ is an m-barrel in A, hence $\phi^{-1}(U) \in \mathcal{M}(A)$ since $A \in \mathcal{T}$. Since ϕ is almost open, $U = \phi(\phi^{-1}(U)) = cl_u\, \phi[\phi^{-1}(U)] \in \mathcal{M}(B)$. Thus B is m-barrelled and so ϕ is open since A is a $B(\mathcal{T})$ algebra.

5.10. <u>Proposition</u>. Let A be a complete metrizable algebra in $\mathcal{L}$ and let $\phi : C(X) \to A$ be a continuous onto homomorphism. Then ϕ is open.

<u>Proof</u>: By Theorem 0; 4.9 $\ker(\phi) = I_F$ for some closed $F \subset X$ and consequently $A \simeq C_F(X)_u$. Since $C(X)$ is a symmetric algebra $C_F(X)$ is also symmetric. By [39; Theorem 12.6] it follows that $C_F(X)_u$ is

functionally continuous, hence α_x is u-continuous for each $x \in F$. Since ϕ is almost open, it follows as in Lemma 3.4 that ϕ is open.

Proposition 5.10 need not hold if A is not complete. Let $Y = \mathbb{N} \cup \{p\}$ be as in Example 4.5 and let u be the topology on C(Y) of pointwise convergence on $\mathbb{N}$. Since $\mathbb{N}$ is a dense and k-closed subset of Y, $i : C(Y) \to C(Y)_u$ is continuous and almost open. Moreover, u is a metrizable topology, but i is not open.

The following proposition will not be proved.

Let $\mathcal{M}$ be the class of all metrizable algebras in $\mathcal{L}$.

5.11. Proposition. (a) C(X) is a $B_r(\mathcal{M})$ algebra iff every dense, k-closed and hemicompact subset of X coincides with X.

(b) C(X) is a $B(\mathcal{M})$ algebra iff every k-closed and hemicompact subset of X is closed.

5.12. Remark. It has been pointed out by B. Banaschewski that $B_r(\mathcal{L})$ algebras may be viewed as minimal [10] objects in a suitable category. Let $\mathcal{L}^*$ be the category of all commutative LMC algebras and all continuous and relatively almost open homomorphisms ($\phi : A \to B$ is relatively almost open if $\phi: A \to \phi(A)$ is almost open). An algebra $A \in \mathcal{L}^*$ is minimal if whenever $\phi : A \to B$ in $\mathcal{L}^*$ is one-to-one, then $A \simeq \phi(A)$. It follows immediately that the minimal objects in $\mathcal{L}^*$ are exactly the $B_r(\mathcal{L})$ algebras.

CHAPTER II

Permanence Properties and Closed Graph Theorems

In this chapter certain permanence properties of $B_r(\mathcal{L})$ and $B(\mathcal{L})$ algebras are investigated and several closed graph theorems are obtained for $B_r(\mathcal{L})$ algebras. In the first section two special classes of continuous maps are introduced. Properties of these maps are used in subsequent sections.

1. Full maps and CR - quotient maps.

1.1. Definition. Let $f : X \to Y$ be a continuous map.

(a) f is said to be full if for each compact $K \subset Y$ there exists a compact $H \subset X$ such that $f(H) = K$. Clearly every full map is onto.

(b) f is said to be perfect if f is closed and $f^{-1}(y)$ is compact for each $y \in Y$.

(c) f is said to be a quotient map if V is open in Y whenever $f^{-1}(V)$ is open in X.

Since perfect maps preserve compactness under inverse images [11; I, 10, Prop. 6], it follows that every perfect onto map is full. A full map need not be perfect, a simple example being given by $f : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$. The importance of full maps is illustrated in Theorem 1.4.

1.2. Example. A quotient map need not be full and a full map need not be a quotient map.

(a) Let $\mathcal{H}$ be the set of all compact and countable subsets of $[0,1]$, and let $f : \coprod_{H \in \mathcal{H}} H \to [0,1]$ be the map which maps each H identically into $[0,1]$. By [39; Example D.2], a subset of $[0,1]$ is open iff its intersection with every compact and countable subset is relatively open. Hence f is a quotient map. Since $[0,1]$ is not the image of any compact subset of $\coprod_{H \in \mathcal{H}} H$, f is not a full map.

(b) Let X be a completely regular space which is not a k_R-space; in particular let $X = \mathbb{N} \cup \{p\}$, where $p \in \beta\mathbb{N} \setminus \mathbb{N}$. Then the identity map $i : k_R X \to X$ is full but not a quotient map.

Henceforth, unless otherwise specified, all topological spaces are assumed to be completely regular. Given a continuous map $f : X \to Y$, define $C(f) : C(Y) \to C(X)$ by $C(f)[r] = r \circ f$. Note that whenever $f(X)$ is dense in Y, C(f) is one-to-one.

1.3. Lemma. If $f : X \to Y$ is continuous, then $C(f) : C(Y) \to C(X)$ is a continuous homomorphism.

Proof: That C(f) is a homomorphism is clear. Let $U = N(K, \xi)$ be a neighborhood of zero in C(X). Then $V = N(f(K), \xi)$ is a neighborhood of zero in C(Y), and since $C(f)[V] \subset U$ it follows that C(f) is continuous.

1.4. Theorem. Let $f : X \to Y$ be continuous. Then $C(f) : C(Y) \to C(X)$ is an embedding iff f is full.

Proof. Suppose $f : X \to Y$ is a full map. Since f is continuous and onto, C(f) is a continuous, one-to-one homomorphism. Let $V = N(K, \xi)$ be a neighborhood of zero in C(Y). Since f is full there exists a compact $H \subset X$ such that $f(H) = K$. Now $U = N(H, \xi)$ is a neighborhood of zero in C(X) and

$$U \cap C(f)\ [C(Y)] \subset C(f)\ [V].$$

Hence $C(f)$ is an embedding.

Conversely, suppose $C(f) : C(Y) \to C(X)$ is an embedding. Let K be a compact subset of Y. Then there exists a compact $H \subset X$ and $0 < \xi < 1$ such that

$$N(H, \xi) \cap C(f)\ [C(Y)] \subset C(f)\ [N(K,1)].$$

Since Y is completely regular, it follows that $K \subset f(H)$. Let $H' = H \cap f^{-1}(K)$. Then H', being a closed subset of H, is compact and $f(H') = K$. Thus f is full.

Theorem 1.4 will not be used in full until the next chapter. It should be pointed out that the first part of the proof appears as a footnote in [15; p. 247].

1.5. Definition. A continuous, onto map $f : X \to Y$ is said to be a CR-quotient map if whenever $g \circ f$ is continuous it follows that g is continous for any $g : Y \to Z$. (Note: X,Y and Z are completely regular.)

1.6. Theorem. A map $f : X \to Y$ is continuous iff $r \circ f \in C(X)$ for each $r \in C(Y)$.

Proof: A proof mav be found in [23; Theorem 3.8].

1.7. Proposition. Let $f : X \to Y$ be a continuous onto map. Then the following statements are equivalent.

(a) f is a CR-quotient map.

(b) $r : Y \to \mathbb{C}$ is continuous whenever $r \circ f$ is continuous.

Proof: (a) $\Rightarrow$ (b) follows from the definition.

(b) $\Rightarrow$ (a) Let the composition $X \xrightarrow{f} Y \xrightarrow{g} Z$ be continuous. Then for each $r \in C(Z)$, $r \circ g \circ f$ is continuous; and so by (b) it follows that $r \circ g$ is continuous for each $r \in C(Z)$. By Theorem 1.6, g is continuous, hence f is a CR-quotient map.

Every quotient map between completely regular spaces is a CR-quotient map. However, the converse is not true, as the example following the proposition indicates.

1.8. Proposition. Every full map onto a k_R-space is a CR-quotient map.

Proof: Let $f : X \to Y$ be a full map, where Y is a k_R-space. Suppose $X \xrightarrow{f} Y \xrightarrow{r} \mathbb{C}$ is continuous. Let K be a compact subset of Y. Since f is a full map, there exists a compact $H \subset X$ such that $f(H) = K$. Now, $H \xrightarrow{f|H} K \xrightarrow{r|K} r(K)$ is continuous. Since H and K are compact, $f|H$ is a quotient map, hence $r|K$ is continuous. Thus, r is k-continuous and hence continuous since Y is a k_R-space. By Proposition 1.7, f is a CR-quotient map.

1.9. Example. A CR-quotient map need not be a quotient map.

Let X be a k_R-space which is not a k-space, in particular let $X = R^2_v$ of Theorem I; 4.4. Let $\mathcal{K}$ be the set of all compact subsets of X and let $f : \coprod_{K \in \mathcal{K}} K \to X$ be the map which maps each K identically into X. By Proposition 1.8, f is a CR-quotient map. Since X is not a k-space, the identity map $i : X \to kX$ is not continuous; however $i \circ f$ is continuous. Hence f is not a quotient map.

1.10. Proposition. Let $f : X \to Y$ be a CR-quotient map and let X be a k_R-space. Then Y is also a k_R-space.

Proof: Let $r : Y \to \mathbb{C}$ be k-continuous. Then $r \circ f$ is also k-continuous and hence continuous, since X is a k_R-space. Since f is a CR-quotient map it follows that r is continuous. Thus, Y is a k_R-space.

The properties of full maps and CR-quotient maps will be used in the following two sections for constructing counterexamples.

2. Permanence properties of $B(\mathcal{L})$ and $B_r(\mathcal{L})$ algebras.

By Corollary I, 1.5, it follows that the quotient of a $B(\mathcal{L})$ algebra modulo a closed ideal is again a $B(\mathcal{L})$ algebra. In other respects, however, the permanence properties of $B_r(\mathcal{L})$ and $B(\mathcal{L})$ algebras are very poor, as the following examples illustrate.

2.1. Example. $B(\mathcal{L})$ algebras are not closed hereditary. In fact a closed subalgebra of a $B(\mathcal{L})$ algebra need not be a $B_r(\mathcal{L})$ algebra.

Let $X = \mathbb{R}^2_v$ be the space as in Theorem I;4.4. Then $C(X)$ is complete but not a $B_r(\mathcal{L})$ algebra. Let $\mathcal{K}$ be the set of all compact subsets of X and let $f : \coprod_{K \in \mathcal{K}} K \to X$ be the natural map. Clearly f is a full map. Thus by Theorem 1.4, $C(f) : C(X) \to C(\coprod_{K \in \mathcal{K}} K)$ is an embedding. Since $C(X)$ is complete it follows that $C(f)[C(X)]$ is a closed subalgebra of $C(\coprod_{K \in \mathcal{K}} K)$. Since $\coprod_{K \in \mathcal{K}} K$ is locally compact it is a k-space, hence $C(\coprod_{K \in \mathcal{K}} K)$ is a $B(\mathcal{L})$ algebra by Theorem I; 3.8. However, $C(f)\,[C(X)] \cong C(X)$ is not a $B_r(\mathcal{L})$ algebra.

2.2. Example. The quotient of a $B_r(\mathcal{L})$ algebra need not be a $B_r(\mathcal{L})$ algebra.

As in Example I; 4.5, let X be the quotient of $\beta\mathbb{N} \amalg (\mathbb{N} \cup \{p\})$ obtained by identifying the element $p \in \beta\mathbb{N}$ with the element $p \in (\mathbb{N} \cup \{p\})$. By Example I; 4.5, C(X) is a $B_r(\mathcal{L})$ algebra. Now $\mathbb{N} \cup \{p\}$ is a closed subset of X and the restriction map $\phi : C(X) \to C(\mathbb{N} \cup \{p\})$ is onto since X is hemicompact and therefore normal. By Theorem I; 3.2, ϕ is also continuous and open. Since $\mathbb{N}$ is a proper dense and k-closed subset of $\mathbb{N} \cup \{p\}$, $C(\mathbb{N} \cup \{p\})$ is not a $B_r(\mathcal{L})$ algebra by Theorem I; 3.5. Thus a quotient of a $B_r(\mathcal{L})$ algebra need not be a $B_r(\mathcal{L})$ algebra.

A result of a positive nature will now be proved.

2.3. Definition. Let A be in $\mathcal{L}$.

(a) An ideal I of A is said to be a retract of A if there exists a continuous homomorphism $\phi : A \to I$ such that $\phi|I$ is the identity. ϕ is then said to be a retraction.

(b) An ideal I of A is said to be a topological direct summand of A if there exists an ideal J such that $I \oplus J$ is isomorphic to A via the map $(a,b) \to a + b$.

The following were proved in [25] for topological groups. The lemma is a known standard result about topological direct sums.

2.4. Lemma: Let I be an ideal of A. Then the following are equivalent.

(a) I is a topological direct summand of A.

(b) I is a retract of A.

Proof: (a) $\Rightarrow$ (b) Let $\phi : A \to I \oplus J$ be the inverse of the addition map, and let $p : I \oplus J \to I$ be the projection. Then $p \circ \phi : A \to I$ is a continuous homomorphism and $(p \circ \phi)|I$ is the identity. Hence I is a retract of A.

(b) $\Rightarrow$ (a) Let $\phi : A \to I$ be a retraction. Then $I \oplus \ker(\phi)$ is algebraically isomorphic with A. Define $\tilde{\phi} : A \to I \oplus \ker(\phi)$ by $\tilde{\phi}(a) = (\phi(a), a - \phi(a))$. Since $\tilde{\phi}$ composed with each projection is continuous, it follows that $\tilde{\phi}$ is continuous. Since the inclusions $I \to A$ and $\ker(\phi) \to A$ are continuous, it follows that the addition map $\tilde{\phi}^{-1} : I \oplus \ker(\phi) \to A$ is also continuous. Thus $I \oplus \ker(\phi)$ is isomorphic with A. Hence I is a topological direct summand of A.

2.5. Theorem. Let A be a $B_r(\mathcal{L})$ algebra. Then any retract of A is also a $B_r(\mathcal{L})$ algebra.

Proof: Let $\phi : A \to I$ be a retraction and let $J = \ker(\phi)$. As in Lemma 2.4, $\tilde{\phi} : I \oplus J \to A$ is an isomorphism. Let $\Psi : I \to B$ be a continuous, one-to-one, onto and almost open homomorphism. Define $\tilde{\Psi} : A \to B \oplus J$ by $\tilde{\Psi}(a) = (\Psi \times id_J)(\tilde{\phi}^{-1}(a))$, so that the following diagram commutes:

$$\begin{tikzcd}
I \oplus J \arrow[r, shift left, "\tilde{\phi}"] \arrow[d, "\Psi \times id_J"'] & A \arrow[l, shift left, "\tilde{\phi}^{-1}"] \arrow[dl, "\tilde{\Psi}"] \\
B \oplus J &
\end{tikzcd}$$

Clearly $\tilde{\Psi}$ is continuous, one-to-one and onto. Since Ψ is almost open $(\Psi \times id_J)$ is almost open, because $cl_{B \oplus J}(\Psi \times id_J)(V_1 \oplus V_2) = cl_{B \oplus J}(\Psi(V_1) \oplus V_2) \supseteq [cl_B \Psi(V_1)] \oplus V_2$. Since $\tilde{\phi}^{-1}$ is open, $\tilde{\Psi} = (\Psi \times id_J) \circ \tilde{\phi}^{-1}$ is almost open. Since A is a $B_r(\mathcal{L})$ algebra, $\tilde{\Psi}$ is open, and since $\tilde{\phi}^{-1}$ is open it follows that $\Psi \times id_J$ is open. Hence Ψ is open, and so I is a $B_r(\mathcal{L})$ algebra.

2.6. Theorem. Let A be a $B(\mathcal{L})$ algebra. Then every retract of A is a $B(\mathcal{L})$ algebra.

Proof: A retract is a special quotient, and so the Theorem follows from Corollary I; 1.5.

Those completely regular spaces for which C(X) is a $B_r(\mathcal{L})$ and $B(\mathcal{L})$ algebra, have been characterized in Theorems [illegible] 3.5 and I; 3.8 respectively. However the analogous problem for B_r- and B-complete locally convex spaces still remains open.

This section ends with an application of Theorem 1.4 and Proposition 1.10 to obtain a certain property of the class $\mathcal{B}$ consisting of all completely regular spaces X for which C(X) is B-complete.

2.7. Definition. Let $\mathcal{C}$ be a class of completely regular spaces and let $\mathcal{M}$ be a class of continuous maps. $\mathcal{C}$ is said to be right fitting

<u>with respect to $\mathcal{M}$</u> if whenever $X \in \mathcal{C}$ and $f : X \to Y$ is in $\mathcal{M}$, Y completely regular, then Y is also in $\mathcal{C}$.

Let $\mathcal{B}$ be the class of all completely regular spaces X such that $C(X)$ is B-complete. By Proposition I; 3.9, every member of $\mathcal{B}$ is a normal k-space. By Proposition 0; 2.10, the quotient of a B-complete space modulo a closed subspace is again B-complete, hence $\mathcal{B}$ is closed hereditary (if $X \in \mathcal{B}$ and $F \subset X$ is closed then $F \in \mathcal{B}$). Let $\mathcal{Q}$ be the class of all full CR-quotient maps. Then the following is obtained.

2.8. <u>Proposition</u>. The class $\mathcal{B}$ is right fitting with respect to $\mathcal{Q}$.

<u>Proof</u>: Let $X \in \mathcal{B}$ and let $f : X \to Y$ be in $\mathcal{Q}$. Since f is a CR-quotient map and X is a k-space, Y is a k_R-space by Proposition 1.10. So $C(Y)$ is complete. Since f is a full map, $C(f) : C(Y) \to C(X)$ is an embedding by Theorem 1.4. Thus $C(Y)$, being complete, is isomorphic to a closed subalgebra of $C(X)$. By Proposition 0; 2.9, B-complete spaces are closed hereditary; so $C(Y)$ is B-complete, hence $Y \in \mathcal{B}$.

By Proposition 0; 2.8, B_r-complete spaces are complete, so if $C(X)$ is B_r-complete then X is a k_R-space. Also, by Proposition 0; 2.9, B_r-complete spaces are closed hereditary. Hence, Proposition 2.8 also holds for the class $\mathcal{B}_r$ of all completely regular spaces X such that $C(X)$ is B_r-complete.

Proposition 2.8 may be used to find conditions on X which are not sufficient for $C(X)$ to be B-complete or B_r-complete.

2.9. Example. There exists a normal k-space Y such that C(Y) is not B_r-complete.

Let X be the space $\mathbb{R}^2_v$ constructed in Theorem I; 4.4. Then C(X) is not B_r-complete. By Example 2.1, C(X) is isomorphic to a closed subalgebra of $C(\coprod_{K \in \mathcal{K}} K)$, where $\mathcal{K}$ is the set of all compact subsets of X. Since B_r-complete spaces are closed hereditary, it follows that $C(\coprod_{K \in \mathcal{K}} K)$ is not B_r-complete. So $Y = \coprod_{K \in \mathcal{K}} K$ is the required space.

Observe that the space Y in the above example, being the coproduct of compact spaces, is both locally compact and paracompact. So not even these stronger properties are sufficient for C(X) to be B_r-complete.

Since every complete metrizable locally convex space is B-complete, it follows from Theorem 0; 4.4 that the class $\mathcal{B}$ contains all hemicompact k-spaces. By [49; p. 162, Ex. 3] $\mathcal{B}$ contains all the discrete spaces. The following question is raised. If X is compact and D is discrete, is $X \amalg D$ in $\mathcal{B}$ or in $\mathcal{B}_r$? The author does not know the answer as yet.

3. Closed Graph Theorems.

In order to prove the main closed graph theorem for $B_r(\mathcal{L})$ algebras, the following lemma is needed. A proof may be found in [39; Lemma 1.4 (b)].

3.1. Lemma. Let A be in $\mathcal{L}$. If U is an m-convex subset of A, then so are

(a) the image and inverse image of U under a homomorphism.

(b) $cl_A(U)$.

The proof of the main closed graph theorem requires a technique originally used by Husain [32; Theorem 5, p. 94] for $B(\mathfrak{A})$ groups; namely the construction of a new topology on the codomain.

Let A_u and B_v be in $\mathcal{L}$, and let $\phi : B_v \to A_u$ be a homomorphism such that $\phi(B_v)$ is dense in A_u. For $U \in \eta(A_u)$, let

$$U^* = cl_u\phi(cl_v\phi^{-1}(U))$$

and let w be the topology on A which has $\{U^* : U \in \eta(A_u)\}$ as a basis for the neighborhoods of zero.

3.2. Lemma. w is a (not necessarily Hausdorff) locally m-convex topology on A which is coarser than u.

Proof: Since $\phi(B_v)$ is dense in A_u it follows that $Int_u(U) \subset Int_u(U^*)$ for each $U \in \eta(A_u)$, and so w is coarser than u. That w is locally m-convex follows from the fact that $V \subset U$ implies $V^* \subset U^*$ and that U^* is m-convex whenever U is, by Lemma 3.1.

3.3. Lemma. If the graph of ϕ is closed, then w is a Hausdorff topology.

Proof: Suppose $y \in \bigcap_{U \in \eta(A_u)} U^*$ and let $G \subset B_v \times A_u$ be the graph of ϕ. It will be shown that $(0,y) \in G$.

Let $\mathcal{O}$ be a neighborhood of $(0,y)$ in $B_v \times A_u$. Then there exist $V \in \eta(B_v)$ and $U \in \eta(A_u)$ such that $V \times (y + U) \subset \mathcal{O}$. Choose $W \in \eta(A_u)$ such that W is circled and $W + W \subset U$. By our assumption, $y \in W^* = cl_u\phi(cl_v\phi^{-1}(W)) \subset \phi(cl_v\phi^{-1}(W)) + W$. Thus there exists an element

$x \in cl_v \phi^{-1}(W)$ such that $y \in \phi(x) + W$, which implies that

$$\text{(i)} \quad \phi(x) \in y - W = y + W$$

Since $x \in cl_v\, \phi^{-1}(W) \subset \phi^{-1}(W) + V$, there exists $z \in V$ such that $x \in \phi^{-1}(W) + z$. Thus, $x - z \in \phi^{-1}(W)$, which implies that $\phi(x) - \phi(z) \in W$. Since W is circled, $\phi(z) \in \phi(x) + W$. Hence by (i), $\phi(z) \in (y + W) + W \subset y + U$. Thus, $(z, \phi(z)) \in V \times (y + U) \cap G \subset \mathcal{O} \cap G$. Since $\mathcal{O}$ was an arbitrarily chosen neighborhood of $(0,y)$, it follows that $(0,y) \in \overline{G} = G$ by our hypothesis. Therefore $y = \phi(0) = 0$. Thus $\bigcap_{U \in \mathcal{N}(A_u)} U^* = \{0\}$, and so w is Hausdorff.

3.4. <u>Theorem</u>. Let A_u be a $B_r(\mathcal{L})$ algebra. Let $\phi : B_v \to A_u$, $B_v \in \mathcal{L}_p$ be an almost continuous homomorphism having a closed graph and such that $\phi(B_v)$ is dense in A_u. Then ϕ is continuous.

Proof: Define the new topology w on A as before. By Lemmas 3.2 and 3.3 it follows that w is a Hausdorff locally m-convex topology and the identity map $i : A_u \to A_w$ is continuous. Since ϕ is almost continuous, $cl_v\phi^{-1}(U) \in \mathcal{N}(B_v)$ for each $U \in \mathcal{N}(A_u)$. Thus, since $cl_v\phi^{-1}(U) \subset \phi^{-1}(i^{-1}(U^*))$ for each $U^* \in \mathcal{N}(A_w)$ and since $\{U^* : U \in \mathcal{N}(A_u)\}$ forms a basis of $\mathcal{N}(A_w)$, it follows that $i \circ \phi: B_v \to A_w$ is continuous.

Let $U \in \mathcal{N}(A_u)$. Then $(i \circ \phi)^{-1}\,[cl_w\, i(U)]$ is closed in B_v because $i \circ \phi$ is continuous. Since $\phi^{-1}(U) \subset (i \circ \phi)^{-1}\,[cl_w\, i(U)]$, it follows that $cl_v\phi^{-1}(U) \subset (i \circ \phi)^{-1}\,[cl_w\, i(U)]$. Thus $i \circ \phi[cl_v\phi^{-1}(U)] \subset cl_w i(U)$, and so $U^* = i[cl_u\phi(cl_v\phi^{-1}(U))] \subset cl_w\, i(U)$, because $w \subset u$. But $U^* \in \mathcal{N}(A_w)$, and so $i : A_u \to A_w$ is almost open. Since A_u is a $B_r(\mathcal{L})$ algebra, i is open and so $u = w$; consequently ϕ is continuous since $i \circ \phi$ is continuous.

3.5. Corollary. Let A be a $B_r(\mathcal{L})$ algebra and let B be an m-barrelled algebra in $\mathcal{L}$. Let $\phi : B \to A$ be a homomorphism having a closed graph and such that $\phi(B)$ is dense in A. Then ϕ is continuous.

Proof: It suffices to show that ϕ is almost continuous. Let $U \in \mathcal{N}(A)$. Since $A \in \mathcal{L}$, there exists $V \in \mathcal{N}(A)$ such that V is m-convex, circled and $V \subset U$. By Lemma 3.1, $cl_B\phi^{-1}(V)$ is also m-convex. Thus $cl_B\phi^{-1}(V)$ is closed, circled, m-convex and absorbing, hence an m-barrel. Since B is m-barrelled, $cl_B\phi^{-1}(V) \in \mathcal{N}(B)$, and consequently ϕ is almost continuous.

3.6. Remarks. (a) In general the assumption that ϕ is almost continuous cannot be dropped.

Let $C[0, \Omega)$ and $C[0,\Omega]$ be topologized with the compact open topology. Then $C[0, \Omega]$ is a Banach algebra, hence a $B_r(\mathcal{L})$ algebra, and the identity map $i : C[0, \Omega] \to C[0, \Omega)$ is continuous onto but not open. Hence $i^{-1} : C[0, \Omega) \to C[0, \Omega]$ is onto and has a closed graph, but it is not almost continuous.

(b) In general the assumption that $\phi(B)$ is dense in A cannot be dropped.

Let $X = \mathbb{R}^2_v$ be the space constructed in Theorem I; 4.4, and let S be its proper dense and k-closed subset. Let $C(f) : C(X) \to C(\coprod_{K \in \mathcal{K}} K)$ be the embedding as in Example 2.1. Let u be the topology on C(X) generated by sets of the form $N(K \cap S, \xi)$, where K varies over all compact subsets of X and $\xi > 0$. As in the proof of the first part of Theorem I; 3.5, $C(X)_u \in \mathcal{L}$ and the identity map $i : C(X) \to C(X)_u$ is continuous and almost open.

Thus $i^{-1} : C(X)_u \to C(X)$ is almost continuous, has closed graph, and is not continuous. Since $C(f)$ is an embedding, $C(f) \circ i^{-1}$ is almost continuous, has a closed graph, and is not continuous. Thus, $C(f) \circ i^{-1} : C(X)_u \to C(\coprod_{K \in \mathcal{K}} K)$ is a homomorphism into a $B_r(\mathcal{L})$ algebra which is almost continuous and has a closed graph, but which is not continuous.

(c) In general, the assumption that A is a $B_r(\mathcal{L})$ algebra cannot be dropped.

Let $C(X)_u$ and $C(X)$ be as in (b). Then the identity map $i : C(X)_u \to C(X)$ is almost continuous, onto, has closed graph, but is not continuous.

Under certain conditions, the graph of a homomorphism $\phi: B \to A$ is automatically closed.

3.7. Lemma. Let B be functionally continuous and let A be semisimple . Then every homomorphism $\phi: B \to A$ has closed graph.

Proof: Suppose $b_i \to b$ and $\phi(b_i) \to a$. Then for each $\alpha \in M(A)$, $\alpha \circ \phi(b_i) \to \alpha(a)$. Since B is functionally continuous, $\alpha \circ \phi \in M(B)$ for each $\alpha \in M(A)$; hence $\alpha \circ \phi(b_i) \to \alpha \circ \phi(b)$ for each $\alpha \in M(A)$. It follows that $\alpha(\phi(b) - a) = 0$ for each $\alpha \in M(A)$, thus $\phi(b) = a$ since A is semisimple. Therefore the graph of ϕ is closed.

3.8. Proposition. Let $B \in \mathcal{L}$ be functionally continuous and let A be a semisimple $B_r(\mathcal{L})$ algebra. Then every almost continuous homomorphism $\phi: B \to A$ such that $\phi(B)$ is dense in A is continuous.

Proof: By Lemma 3.7 the graph of ϕ is closed, hence by Theorem 3.4, ϕ is continuous.

In proposition 3.8, if B is m-barrelled then ϕ is automatically almost continuous.

For the special case when the codomain is $B_r(\mathcal{L})$ algebra of the form C(X), there is another type of closed graph theorem. The proof requires the following proposition which may be found in [31; p. 57, Lemma 3].

3.9. Proposition. Let E and F be locally convex spaces and let $\phi : E \to F$ be an almost continuous linear map. Then $\{\alpha \in F' : \alpha \circ \phi \in E'\}$ is almost closed in F'.

3.10. Theorem. Let C(X) be a $B_r(\mathcal{L})$ algebra and let $B \in \mathcal{L}$. Let $\phi : B \to C(X)$ be an almost continuous homomorphism such that $\{x \in X : \alpha_x \circ \phi \in B'\}$ is dense in X. Then ϕ is continuous.

Proof: Let $S = \{x \in X : \alpha_x \circ \phi \in B'\}$ and let $D = \{\alpha \in C(X)' : \alpha \circ \phi \in B'\}$. Since ϕ is almost continuous, D is almost closed by Proposition 3.9. Thus $N(K,1)^{\circ} \cap D$ is w*-closed in C(X)' for each compact subset K of X. By Corollary 0; 4.10, X can be w*-embedded into C(X)' via the map $x \to \alpha_x$. Let $\hat{X}$ denote the image of X under this embedding. Clearly $\hat{S} \subset D$. For each compact $K \subset X$, $K \cap S \simeq \hat{K} \cap \hat{S} = \hat{K} \cap D = \hat{K} \cap [N(K,1)^{\circ} \cap D]$ which is w*-closed since K is w*-compact and $N(K,1)^{\circ} \cap D$ is w*-closed. Hence S is a dense and k-closed subset of X and since C(X) is a $B_r(\mathcal{L})$ algebra, S = X by Theorem I; 4.5. Thus, $\alpha_x \circ \phi \in B'$ for each $x \in X$, which implies that $\phi^{-1}[N(\{x\}, \xi)] = \phi^{-1}(\alpha_x^{-1}[-\xi,\xi])$ is closed in

B for each $x \in X$. Hence, $\phi^{-1}[N(K, \xi)] = \bigcap_{x \in K} \phi^{-1}[N(\{x\}, \xi)]$ is closed in B for each compact $K \subset X$ and each $\xi > 0$. Since ϕ is almost continuous, $\phi^{-1}[N(K, \xi)] = cl_B \phi^{-1}[N(K, \xi)] \in \mathcal{N}(B)$. Thus ϕ is continuous.

If B is m-barrelled, then the assumption that ϕ is almost continuous is redundant. In general however, the hypotheses cannot be weakened as Remarks 3.6, (a), (b), (c) illustrate.

Barrelled locally convex spaces have been characterized by the following result due to M. Mahowald. A proof may be found in [49; Theorem 8.6, p. 166].

3.11. <u>Theorem</u>. A locally convex space E is barrelled iff for each Banach space F, a linear map $\phi : E \to F$ with closed graph is continuous.

The following analogous result characterizes m-barrelled LMC algebras.

3.12. <u>Theorem</u>. An algebra $A \in \mathcal{L}$ is m-barrelled iff for each Banach algebra B, a homomorphism $\phi : A \to B$ with closed graph is continuous.

<u>Proof</u>: Suppose A is m-barrelled. Let B be a Banach algebra and let $\phi : A \to B$ be a homomorphism with closed graph. Since $cl_B\phi(A)$ is a Banach algebra, $\phi : A \to cl_B\phi(A)$ is continuous by Corollary 3.5. Thus $\phi: A \to B$ is continuous.

For the converse, let D be an m-barrel in A and let p_D be the Minkowski functional associated with D. Since D is an m-barrel, p_D is a submultiplicative seminorm, hence $\ker(p_D)$ is an ideal of A.

Thus $A_D = A/\ker(p_D)$ becomes a normed algebra under the norm induced by p_D; hence its completion $\tilde{A}_D$ is a Banach algebra.

Let $\phi : A \to \tilde{A}_D$ be the canonical homomorphism. Since D is closed, $D = \{x \in A : p_D(x) \leq 1\} = \{x \in A : \|\phi(x)\| \leq 1\} = \phi^{-1}(U)$, where U is the closed unit ball of $\tilde{A}_D$. In view of the hypothesis, to show that $D \in \mathfrak{N}(A)$ it suffices to show that the graph of ϕ is closed in $A \times \tilde{A}_D$.

Let G be the graph of ϕ and let $(a,b) \notin G$. Then $\phi(a) \neq b$ and so there exists $\xi > 0$ such that $\|\phi(a) - b\| > 2\xi$. Since $\phi(A)$ is dense in $\tilde{A}_D$, there exists $c \in \phi(A)$ such that $\|b - c\| < \xi$, and it follows that $\|\phi(a) - c\| > \xi$. The set $H = \{x \in A : \|\phi(x) - c\| \leq \xi\}$ is closed in A since it is a translate of ξD; hence $W = A \backslash H$ is open. Letting $V = \{y \in \tilde{A}_D : \|y - c\| < \xi\}$, it follows that $W \times V \cap G = \emptyset$; for $(x,y) \in W \times V$ implies that $\|\phi(x) - c\| > \xi$ and $\|y - c\| < \xi$ and consequently $\phi(x) \neq y$. Since $(a,b) \in W \times V$ it follows that G is closed. By hypothesis ϕ is continuous, hence $D = \phi^{-1}(U) \in \mathfrak{N}(A)$.

CHAPTER III

CATEGORICAL CONSIDERATIONS

This chapter is devoted to a categorical treatment of LMC algebras and their relationship to completely regular spaces.

1. The contravariant functors C, C_p and M.

Throughout this chapter $\underline{L}$ is the category of all commutative LMC algebras with identity and all continuous unitary homomorphisms; $\underline{CR}$ is the category of all completely regular spaces and all continuous maps. It should be noted that the algebra $\{0\}$, consisting of the zero element only, is in $\underline{L}$.

The contravariant functors to be considered are $C : \underline{CR} \to \underline{L}$, $C_p : \underline{CR} \to \underline{L}$ and $M : \underline{L} \to \underline{CR}$ which are defined as follows:

(a) For $X \in \underline{CR}$, $C(X)$ is the algebra of all continuous complex valued functions on X endowed with the compact open topology. For $f : X \to Y$ in $\underline{CR}$, $C(f) : C(Y) \to C(X)$ is defined by $C(f)[r] = r \circ f$. $C(f)$ is a unitary homomorphism which is continuous by Lemma II, 1.3.

(b) For $X \in \underline{CR}$, $C_p(X)$ is $C(X)$ endowed with the topology of pointwise convergence. For $f : X \to Y$, $C_p(f) : C_p(Y) \to C_p(X)$ is defined in the same manner as $C(f)$. The continuity of $C_p(f)$ is clear.

(c) For $A \in \underline{L}$, $M(A)$ is the set of all nonzero continuous complex-valued homomorphisms endowed with the relative w*-topology from A'. Since (A', w*) is a completely regular space, so is (M(A), w*). For $\phi: A \to B$

in $\underline{L}$, $M(\phi) : M(B) \to M(A)$ is defined by $M(\phi)[\alpha] = \alpha \circ \phi$, $\alpha \circ \phi$ being a continuous unitary homomorphism since α and ϕ are. Since $M(\phi) = \phi'|M(B)$, where $\phi' : B' \to A'$ is the adjoint of ϕ, and since the adjoint of a continuous linear map is always w*-continuous, it follows that $M(\phi)$ is continuous.

Since the conditions of Definition 0; 5.2 are trivial to verify, it follows that C, C_p and M are contravariant functors. From now on they will be called simply functors.

1.1. Remark. If $A \in \underline{L}$ is not equal to $\{0\}$, then $M(A) \neq \emptyset$ by Proposition 0; 3.3.

Every $A \in \underline{L}$ can be algebraically mapped into CM(A) as follows: define $\phi_A : A \to CM(A)$ by $\phi_A(a) = \hat{a}$, where $\hat{a}(\alpha) = \alpha(a)$ for $\alpha \in M(A)$. Recalling that $M(A) \subset A' \subset \mathbb{C}^A$ and that the w* topology is the relative product topology from $\mathbb{C}^A$, it follows that $\hat{a}$, being the restriction of the projection at a, is continuous on M(A). It can be easily verified that ϕ_A is a unitary homomorphism. In general it is neither continuous with respect to the compact open topology on CM(A), nor one-to-one, nor onto. It is known that ϕ_A is one-to-one iff A is semisimple.

1.2. Definition. The homomorphism $\phi_A : A \to CM(A)$ (compact open topology) is called the Gelfand map of A, and $\hat{a}$ is called the Gelfand transform of a .

The subcategory of $\underline{L}$ consisting of all algebras whose Gelfand map is continuous turns out to be very important, and it is studied in section 3.

It should be pointed out that the map $\phi_A : A \to C_pM(A)$ is always continuous. Also, the evaluation map $E_X : X \to MC(X)$ is always a homeomorphism by Corollary 0; 4.10 and [35; Lemma 5, p. 116].

The relationship between M and C, and between M and C_p is investigated in section 4.

2. Categorical properties of L.

From Proposition 0; 3.1, it follows that whenever $A \in \underline{L}$, then any (unitary) subalgebra and any quotient modulo a closed ideal are again in $\underline{L}$. Also, given any subset $\{A_i\}_{i \in I}$ of algebras in $\underline{L}$, their cartesian product with the product topology is again in $\underline{L}$.

2.1. Lemma. Let $\{A_i\}_{i \in I}$ be a subset of $\underline{L}$. Then the categorical product $(\Pi A_i, \{\phi_i\})$ is the cartesian product with the product topology together with the projections. Hence $\underline{L}$ has products.

Proof: The proof is obvious and will be omitted.

2.2. Lemma. $\underline{L}$ has equalizers.

Proof: Given $A \underset{\phi_2}{\overset{\phi_1}{\rightrightarrows}} B$ in $\underline{L}$, let $C = \{a \in A : \phi_1(a) = \phi_2(a)\}$ with the relative topology. Clearly the natural embedding $i : C \to A$ is in $\underline{L}$ and $\phi_1 \circ i = \phi_2 \circ i$. Suppose there exists $\Psi: D \to A$ in $\underline{L}$ such that $\phi_1 \circ \Psi = \phi_2 \circ \Psi$. By the definition of C, it follows that $\Psi(D) \subset C$. So there exists a unique $\Psi' : D \to C$ in $\underline{L}$, namely the corestriction of Ψ, such that $\Psi = i \circ \Psi'$. Thus i is the equalizer of the pair (ϕ_1, ϕ_2) and so $\underline{L}$ has equalizers.

2.3. Proposition. $\underline{L}$ is complete.

Proof: $\underline{L}$ has products and equalizers, hence $\underline{L}$ is complete by Theorem 0; 5.5.

2.4. Lemma. $\underline{L}$ has coequalizers.

Proof: Given $A \underset{\phi_2}{\overset{\phi_1}{\rightrightarrows}} B$, let I be the closure of the ideal generated by $\{\phi_1(a) - \phi_2(a) : a \in A\}$. Then the canonical quotient map $Q : B \to B/I$ is in $\underline{L}$ and $Q \circ \phi_1 = Q \circ \phi_2$. Suppose there exists $\Psi : B \to D$ in $\underline{L}$ such that $\Psi \circ \phi_1 = \Psi \circ \phi_2$. Since Ψ is continuous it follows that $I \subset \ker(\Psi)$. Hence there exists a unique $\Psi' : B/I \to D$ in $\underline{L}$ such that $\Psi' \circ Q = \Psi$. Thus Q is the coequalizer of the pair (ϕ_1, ϕ_2) and so $\underline{L}$ has coequalizers.

Next, it will be shown that $\underline{L}$ is coproductive. In view of Lemma 2.4. and the dual of Theorem 0; 5.5, it will then follow that $\underline{L}$ is cocomplete. It turns out that finite coproducts in $\underline{L}$ coincide with the tensor products endowed with the projective tensor product topology as defined by Grothendieck [26].

Let E and F be locally convex spaces and let $E \otimes F$ be their tensor product. Given continuous seminorms p and q on E and F respectively, define $p \otimes q$ on $E \otimes F$ as follows: given $u = \sum_{i=1}^{n} x_i \otimes y_i \in E \otimes F$,

$$p \otimes q\ (u) = \inf \sum_{i=1}^{n} p(x_i)\, q(y_i),$$

the inf taken over all representations of u. Grothendieck [26, Proposition 1, p. 28] proved that $p \otimes q$ is a seminorm on $E \otimes F$, and that $p \otimes q\ (a \otimes b) = p(a)\, q(b)$.

The following theorem is a reformulation of [26; Proposition 2, p.30].

2.5. Theorem. Let E and F be Hausdorff locally convex spaces. Then there exists a unique Hausdorff locally convex topology τ on $E \otimes F$ having the following universal property: given a continuous bilinear map $\phi : E \times F \to G$, G locally convex, there exists a unique continuous linear map $\phi' : E \otimes F \to G$ such that $\phi' \circ \Psi = \phi$, where $\Psi : E \times F \to E \otimes F$ is the canonical map. Furthermore, if the seminorms $\{p_i\}_{i \in I}$ and $\{q_j\}_{j \in J}$ generate the topology of E and F respectively, then the seminorms $\{p_i \otimes q_j\}_{i,j}$ generate τ .

2.6. Proposition. Let A and B be unitary algebras over $\mathbb{C}$. Then $A \otimes B$ can be given a well-defined algebra structure by defining $(a_1 \otimes b_1)(a_2 \otimes b_2) = a_1a_2 \otimes b_1b_2$. Also, the maps $i_A : A \to A \otimes B$ and $i_B : B \to A \otimes B$ defined by $i_A(a) = a \otimes 1$ and $i_B(b) = 1 \otimes b$ are algebraic embeddings.

Proof: A proof may be found in [12; Chapter III pp. 5 and 40].

2.7. Proposition. If A and B are in $\underline{L}$, $A \underset{\tau}{\otimes} B$ is also in $\underline{L}$.

Proof: In view of Theorem 2.5 and Propositon 2.6, it suffices to prove that if p and q are submultiplicative seminorms on A and B respectively, then $p \otimes q$ is submultiplicative on $A \otimes B$.

Let x and y be in $A \otimes B$, and let $c_1 = p \otimes q(x)$, $c_2 = p \otimes q(y)$. Given $0 < \xi < 1$, there exist representations $x = \sum_{i=1}^{n} a_i \otimes b_i$ and $y = \sum_{j=1}^{m} a_j \otimes b_j$ such that:

$$\sum_{i=1}^{n} p(a_i)\ q(b_i) < p \otimes q(x) + \frac{\xi}{4(1+c_2)}$$

and

$$\sum_{j=1}^{m} p(a_j)\ q(b_j) < p \otimes q(y) + \frac{\xi}{4(1+c_1)} .$$

Now $xy = (\sum_{i=1}^{n} a_i \otimes b_i)(\sum_{j=1}^{m} a_j \otimes b_j) = \sum_{i,j} a_i a_j \otimes b_i b_j$.

Thus, since p and q are submultiplicative, $p \otimes q(xy) \leq \sum_{i,j} p(a_i a_j)\ q(b_i b_j)$
$\leq \sum_{i,j} p(a_i)\ p(a_j)\ q(b_i)\ q(b_j) = (\sum_{i=1}^{n} p(a_i)\ q(b_i))(\sum_{j=1}^{m} p(a_j)\ q(b_j))$

$< (p \otimes q(x))(p \otimes q(y)) + \frac{\xi}{4} + \frac{\xi}{4} + \frac{\xi}{16}$.

This holds for every $0 < \xi < 1$, hence $p \otimes q(xy) \leq (p \otimes q(x))(p \otimes q(y))$. Thus $p \otimes q$ is submultiplicative.

So if the topology of A and B is generated by the submultiplicative seminorms $\{p_i\}_{i \in I}$ and $\{q_j\}_{j \in J}$ respectively, then by Theorem 2.5 the topology τ is generated by the submultiplicative seminorms $\{p_i \otimes q_j\}_{i,j}$ Consequently $A \underset{\tau}{\otimes} B$ is in <u>L</u>.

2.8. <u>Corollary</u>. The embeddings $i_A : A \to A \underset{\tau}{\otimes} B$ and $i_B : B \to A \underset{\tau}{\otimes} B$ are continuous.

<u>Proof</u>: Let $\{p_i\}_{i \in I}$, $\{p_j\}_{j \in J}$ and $\{p_i \otimes q_j\}_{i,j}$ be as in the theorem. The corollary follows directly from the fact that $p_i \otimes q_j(a \otimes 1) = p_i(a)\ q_j(1)$ and $p_i \otimes q_j(1 \otimes b) = p_i(1)\ q_j(b)$. Thus $(p_i \otimes q_j) \circ i_A$ and $(p_i \otimes q_j) \circ i_B$ are continuous for each $i \in I$, $j \in J$. Since $\{p_i \otimes q_j\}_{i,j}$ generate τ , it follows that i_A and i_B are continuous.

2.9. Proposition. Let A and B be in $\underline{L}$. Then $(A \underset{\tau}{\otimes} B, i_A, i_B)$ is the coproduct in $\underline{L}$ of A and B.

Proof. Suppose there exist $\phi_1 : A \to C$ and $\phi_2 : B \to C$ in $\underline{L}$. Define $\phi : A \times B \to C$ by $\phi(a,b) = \phi_1(a)\,\phi_2(b)$. Then ϕ is continuous, multiplicative and bilinear. Hence, by Theorem 2.5 there exists a unique continuous linear map $\phi' : A \otimes B \to C$ such that $\phi'(a \otimes b) = \phi[(a,b)]$. Clearly ϕ' is also multiplicative and unitary, and $\phi' \circ i_A = \phi_1$, $\phi' \circ i_B = \phi_2$. Thus $(A \underset{\tau}{\otimes} B, i_A, i_B)$ is the coproduct in $\underline{L}$ of A and B.

2.10. Corollary. $\underline{L}$ has finite coproducts.

Proof. The proof follows immediately from the fact that $(A \otimes B) \otimes C \simeq A \otimes (B \otimes C)$.

The proof of the existence of arbitrary set indexed coproducts in $\underline{L}$ is a particular case of a theorem which is valid for certain categories of universal algebras. To the best of the author's knowledge, the proof is due to K. Golema.

2.11. Theorem. $\underline{L}$ is coproductive.

Proof: Let $\{A_i\}_{i \in I}$ be a set of algebras in $\underline{L}$. Let $\underline{\mathfrak{F}}$ be the class of all algebras B_ϕ in $\underline{L}$ satisfying the following two properties:

(a) for each $i \in I$ there exists $\phi_i : A_i \to B_\phi$ in $\underline{L}$.

(b) B_ϕ is generated (as an algebra) by $\bigcup_{i \in I} \phi_i(A_i)$.

By condition (b), the cardinality of each $B_\phi \in \mathfrak{F}$ depends on the cardinality of the disjoint union, $\dot{\bigcup}_{i \in I} A_i$, and on the cardinality of of the operations which is three. Since A_i is a set for each $i \in I$, and since I is a set, it follows that Card $(\dot{\bigcup} A_i) = \lambda$ for some cardinal λ . Thus for each $B_\phi \in \mathfrak{F}$, Card $(B_\phi) \leq \aleph_0 \cdot \lambda$.

Since the members of $\mathfrak{F}$ are of bounded cardinality, by the axiom of choice, there exists a representative set Φ having the following property: given $B_\phi \in \mathfrak{F}$, there exist $B'_{\phi'} \in \Phi$ and an isomorphism $\Psi : B_\phi \to B'_{\phi'}$, which are unique with respect to the property that the diagram

$$\begin{tikzcd}
A_i \arrow[r, "\phi_i"] \arrow[dr, "\phi'_i"'] & B_\phi \arrow[d, "\Psi"] \\
& B'_{\phi'}
\end{tikzcd}$$

commutes for each $i \in I$.

By Proposition 0; 3.1, $\prod_{B_\phi \in \Phi} B_\phi$ is in $\underline{L}$. For each $i \in I$, there is a continuous unitary homomorphism $\sqcap\phi_i : A_i \to \prod_{B_\phi \in \Phi} B_\phi$ defined by $\sqcap\phi_i(x) = (\phi_i(x))_{B_\phi \in \Phi}$. Let A be the subalgebra of $\prod_{B_\phi \in \Phi} B_\phi$ generated by $\bigcup_{i \in I} (\sqcap\phi_i(A_i))$.

It is now easy to check that $(A, \{\sqcap\phi_i\}_{i \in I})$ is the coproduct of $\{A_i\}_{i \in I}$. For suppose there exist $\Psi_i : A \to C$ in $\underline{L}$ for each $i \in I$. Let D be subalgebra of C generated by $\bigcup_{i \in I} \Psi_i(A_i)$. Then there exist $B_\phi \in \Phi$ and an isomorphism $\Psi : D \to B_\phi$, which are unique with respect to the property that $\Psi \circ \phi_i = \Psi_i$ for each $i \in I$. Let $\pi : A \to B_\phi$ be the projection.

Then $\Psi^{-1} \circ \pi : A \to D$ in $\underline{L}$ is unique with respect to the property that $\Psi_i = (\Psi^{-1} \circ \pi) \circ \prod \phi_i$ for each $i \in I$. Thus $(A, \{\prod \phi_i\}_{i \in I})$ is the coproduct of $\{A_i\}_{i \in I}$.

2.12. Theorem. $\underline{L}$ is cocomplete.

Proof: The theorem follows from Theorem 2.11, Lemma 2.4, and the dual of Theorem 0; 5.5.

Theorem 2.11 shows that $\underline{L}$ has coproducts, however, in general they cannot be realized in a concrete way. It will be shown now that if the algebras in question are semisimple, then their coproduct can be represented as a subalgebra of a certain function algebra.

Let $\{A_i\}_{i \in I}$ be a set of semisimple algebras in $\underline{L}$. Then for each $i \in I$ the Gelfand map $\phi_{A_i} : A_i \to CM(A_i)$ is one-to-one. Let $\pi_i : \Pi M(A_i) \to M(A_i)$ be the projection, and let $C(\pi_i) : CM(A_i) \to C(\Pi M(A_i))$ be the induced homomorphism. Since π_i is onto, $C(\pi_i)$ is one-to-one for each $i \in I$, hence each A_i can be algebraically embedded into $C(\Pi M(A_i))$ via the map $\phi_i = C(\pi_i) \circ \phi_{A_i}$, which maps a_i to $\hat{a}_i \circ \pi_i$. Let A be the subalgebra of $C(\prod_{i \in I} M(A_i))$ generated by $\bigcup_{i \in I} \phi_i(A_i)$.

Define a topology τ on A as follows: a submultiplicative seminorm p on A is continuous iff $p \circ \phi_i$ is continuous for each $i \in I$. Since $\phi_i : A_i \to C_p(\prod_{i \in I} M(A_i))$ (pointwise topology) is continuous for each $i \in I$, it follows that the topology τ is finer than the relative pointwise topology from $C_p(\prod_{i \in I} M(A_i))$. Hence τ is a Hausdorff locally m-convex topology on A making each ϕ_i continuous. It will be shown that $(A_\tau, \{\phi_i\}_{i \in I})$ is the coproduct in $\underline{L}$ of the algebras $\{A_i\}_{i \in I}$.

From now on, $\phi_i(a_i) = \hat{a}_i \circ \pi_i$ will be written simply as $\hat{a}_i$. From the definition of A, it follows that an arbitrary element of A is a finite sum of elements of the form $\hat{a}_{i_1} \hat{a}_{i_2} \dots \hat{a}_{i_k}$. It is clear that any such finite sum may be equivalently rewritten as a finite sum of the form $\sum_{j=1}^{m} (\hat{a}_{i_{1_j}} \hat{a}_{i_{2_j}} \dots \hat{a}_{i_{n_j}})$ where exactly the same indices $i_1, \dots, i_n$ appear in each term. This may be achieved by multiplying a given term by suitable $\hat{e}_{i_k}$, where e_{i_k} is the identity element of A_{i_k}. In particular, by fixing a Hamel basis $\mathcal{B}_i$ for each A_i, every element of A may be written as :

$$\sum_{j=1}^{m} \lambda_j (\hat{b}_{i_{1_j}} \hat{b}_{i_{2_j}} \dots \hat{b}_{i_{n_j}}),\ b_{i_{k_j}} \in \mathcal{B}_{i_k}$$

2.13. <u>Lemma</u>. For each $i \in I$ let $\mathcal{B}_i$ be a Hamel basis of A_i. Then for $b_{i_{k_j}} \in \mathcal{B}_{i_k}$, $f = \sum_{j=1}^{m} \lambda_j (\hat{b}_{i_{1_j}} \hat{b}_{i_{2_j}} \dots \hat{b}_{i_{n_j}}) = 0$ iff $\lambda_1 = \lambda_2 = \dots = \lambda_m = 0$.

<u>Proof</u>: If $\lambda_1 = \lambda_2 = \dots = \lambda_m = 0$ then clearly $f = 0$.

For the converse choose $\alpha_{i_{k_1}} \in M(A_{i_k})$ such that $\alpha_{i_{k_1}}(b_{i_{k_1}}) \neq 0$ for $k = 1,2,\dots, n-1$. This can be done since each A_{i_k} is semisimple and each $b_{i_{k_1}}$ is nonzero. Now for each $(\alpha_i)_{i \in I}$ in $\prod_{i \in I} M(A_i)$,

$f((\alpha_i)_{i \in I}) = \sum_{j=1}^{m} \lambda_j [\alpha_{i_1}(b_{i_{1_j}}) \alpha_{i_2}(b_{i_{2_j}}) \dots \alpha_{i_n}(b_{i_{n_j}})] = 0$. In particular,

$\sum_{j=1}^{m} \lambda_j [\alpha_{i_{1_1}}(b_{i_{1_j}}) \alpha_{i_{2_1}}(b_{i_{2_j}}) \dots \alpha_{i_n}(b_{i_{n_j}})] = 0$ for each $\alpha_{i_n} \in M(A_{i_n})$.

Let $\mu_j = \prod_{k=1}^{n-1} \alpha_{i_{k_1}}(b_{i_{k_j}})$. Then $\sum_{j=1}^{m} \lambda_j \mu_j [\alpha_{i_n}(b_{i_{n_j}})] = 0$ for each $\alpha_{i_n} \in M(A_{i_n})$.

Since A_{i_n} is semisimple, it follows that $\sum_{j=1}^{m} \lambda_j \mu_j b_{i_{n_j}} = 0$; and since each $b_{i_{n_j}} \in \mathcal{B}_{i_n}$ it follows that $\lambda_j \mu_j = 0$ for $j = 1,2,\ldots, m$. By the choice of $\alpha_{i_{k_1}}$, $k = 1,\ldots, n-1$, it follows that $\mu_1 \neq 0$, hence $\lambda_1 = 0$. Analogously, it can be shown that $\lambda_j = 0$ for $j = 2,\ldots, m$.

2.14. Proposition. $(A_\tau , \{\phi_i\}_{i \in I})$ is the coproduct in $\underline{L}$ of the algebras $\{A_i\}_{i \in I}$.

Proof: Suppose there exist $\Psi_i : A_i \to B$ in $\underline{L}$ for each $i \in I$. Define $\Psi : A_\tau \to B$ by

$$\Psi \left(\sum_{j=1}^{m} (\hat{a}_{i_{1_j}} \hat{a}_{i_{2_j}} \cdots \hat{a}_{i_{n_j}}) \right)$$

$$= \sum_{j=1}^{m} [\Psi_{i_1}(a_{i_{1_j}})\Psi_{i_2}(a_{i_{2_j}}) \cdots \Psi_{i_n}(a_{i_{n_j}})].$$

From the Lemma 2.13 it follows that Ψ is well defined. It can easily be verified that Ψ is a unitary homomorphism. The continuity of Ψ follows directly from the definition of the topology τ. Finally $\Psi \circ \phi_i = \Psi_i$ for each $i \in I$, and Ψ is unique with respect to this property since A_τ is generated algebraically by $\bigcup_{i \in I} \phi_i(A_i)$. Thus $(A_\tau,\{\phi_i\}_{i \in I})$ is the coproduct in L of the algebras A_i.

So it has been shown that the coproduct of semisimple algebras $\{A_i\}_{i \in I}$ can be identified algebraically with a subalgebra of $C(\prod_{i\in I} M(A_i))$.

3. The subcategory CL.

In section 1 it was noted that, in general, the Gelfand map $\phi_A : A \to CM(A)$ is not continuous. The purpose of this section is to investigate the subcategory CL consisting of all algebras in L whose Gelfand map is continuous. It turns out that CL is a coreflective subcategory of L and, as will be seen in the next section, is closely related to CR.

3.1. Proposition. Let A be in L. Then A is in CL iff every compact subset of M(A) is equicontinuous.

Proof: A proof may be found in [21; Prop. 1.4] or [38; Theorem 3.1].

3.2. Corollary. CL contains all m-barrelled algebras; in particular all Banach algebras and all complete metrizable LMC algebras.

Clearly, C(X) is in CL for every completely regular space X. The following are examples of algebras which are in L but not in CL.

3.3. Examples (a) $C_p(\mathbb{R})$ is in L but not in CL. In general, $C_p(X)$ is in CL iff every compact subset of X is finite.

(b) Let A = C[0,1] with the topology of uniform convergence on the countable and compact subsets of [0,1]. Then A is in L and is complete, but A is not in CL. The completeness follows from [39; Lemma D. 5].

The following are two algebras which are in CL, but which are not m-barrelled.

3.4. Examples (a) $C^*(\mathbb{R})$ is in CL but is not m-barrelled. $B = \{r \in C^*(R) : |r(x)| \leq 1 \text{ for all } x \in R\}$ is an m-barrel which is not a neighborhood of zero. In general, $C^*(X)$ is in CL but is not m-barrelled unless X is compact.

(b) $C[0, \Omega)$ is in CL and is complete, but it is not m-barrelled.

3.5. Lemma: Let $\Psi : A \to B$ be in L and let ϕ_A and ϕ_B be the respective Gelfand maps. Then $CM(\Psi) \circ \phi_A = \phi_B \circ \Psi$.

Proof: Consider the following diagram:

$$\begin{tikzcd} A \arrow[r, "\Psi"] \arrow[d, "\phi_A"'] & B \arrow[d, "\phi_B"] \\ CM(A) \arrow[r, "CM(\Psi)"] & CM(B) \end{tikzcd}$$

Let $a \in A$ and let $\alpha \in M(B)$. Then

$$\{CM(\Psi) \circ \phi_A(a)\}[\alpha] = \{CM(\Psi)\ [\hat{a}]\}[\alpha] = \{\hat{a} \circ M(\Psi)\}[\alpha] = \hat{a}(M(\Psi)[\alpha]) = \hat{a}(\alpha \circ \Psi)$$

$= \alpha \circ \Psi(a) = \alpha(\Psi(a)) = \widehat{\Psi(a)}[\alpha] = \{\phi_B(\Psi(a))\}[\alpha] = \{\phi_B \circ \Psi(a)\}[\alpha]$. Since this holds for every $a \in A$ and every $\alpha \in M(B)$, $CM(\Psi) \circ \phi_A = \phi_B \circ \Psi$.

3.6. Proposition. Let A be in CL and let I be a closed ideal of A. Then A/I is in CL.

Proof: Consider the following diagram:

$$\begin{tikzcd} A \arrow[r, "\Psi"] \arrow[d, "\phi_A"'] & A/I \arrow[d, "\phi_{A/I}"] \\ CM(A) \arrow[r, "CM(\Psi)"] & CM(A/I) \end{tikzcd}$$

where Ψ is the canonical quotient map. By Lemma 3.5, $CM(\Psi) \circ \phi_A = \phi_{A/I} \circ \Psi$, since $CM(\Psi) \circ \phi_A$ is continuous, $\phi_{A/I} \circ \Psi$ is also continuous. Since Ψ is a quotient map it follows that $\phi_{A/I}$ is also continuous, hence $A/I \in$ CL.

The following rather obvious lemma is needed to prove the next proposition.

3.7. Lemma. Let $\{A_i\}_{i \in I}$ be a set of algebras in L and let $(A, \{\Psi_i\}_{i \in I})$ be their coproduct. Let $\phi : A \to B$ be a unitary homomorphism. If $\phi \circ \Psi_i$ is continuous for each $i \in I$, then ϕ is continuous.

Proof: Since $\phi \circ \Psi_i : A_i \to B$ is in L for each $i \in I$, there exists a unique $\phi' : A \to B$ in L such that $\phi' \circ \Psi_i = \phi \circ \Psi_i$. In view of Theorem 2.11, A is generated algebraically by $\bigcup_{i \in I} \Psi_i (A_i)$; it follows that $\phi = \phi'$, and so ϕ is continuous since ϕ' is.

3.8. Proposition. Let $\{A_i\}_{i \in I}$ be a set of algebras in CL and let $(A, \{\Psi_i\}_{i \in I})$ be their coproduct in L. Then A is in CL.

Proof: Since each A_i is in CL, $\phi_{A_i} : A_i \to CM(A_i)$ is continuous for each $i \in I$. Consider the following diagram:

$$\begin{tikzcd} A_i \arrow[r, "\Psi_i"] \arrow[d, "\phi_{A_i}"'] & A \arrow[d, "\phi_A"] \\ CM(A_i) \arrow[r, "CM(\Psi_i)"'] & CM(A) \end{tikzcd}$$

By Lemma 3.5, $\phi_A \circ \Psi_i = CM(\Psi_i) \circ \phi_{A_i}$, hence $\phi_A \circ \Psi_i$ is continuous for each $i \in I$. By Lemma 3.7, ϕ_A is continuous; consequently A is in CL.

Propositions 3.6 and 3.8 can be combined to obtain the following theorem.

3.9. Theorem. CL is cocomplete and the inclusion functor $\mathrm{Inc} : \underline{CL} \to \underline{L}$ preserves colimits.

Next, it will be shown that CL is closed under cartesian products. First, the following lemma is proved.

3.10. Lemma. Let $\{A_i\}_{i \in I}$ be a set of algebras in $\underline{L}$ and let $(\prod_{i \in I} A_i, \{\pi_i\}_{i \in I})$ be their product. Then every $\alpha \in M(\prod_{i \in I} A_i)$ is of the form $\alpha_i \circ \pi_i$ for some (necessarily unique) $\alpha_i \in M(A_i)$.

Proof: Let $\alpha \in M(\prod_{i \in I} A_i)$. Define $\phi_i : A_i \to \prod_{i \in I} A_i$ by:

$$\pi_j \circ \phi_i(x) = \begin{cases} x & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

(The fact that ϕ_i is not unitary is of no consequence in this case.) Since $\pi_j \circ \phi_i$ is continuous for each projection π_j, it follows that ϕ_i is continuous for each $i \in I$.

Suppose $\alpha \circ \phi_i = 0$ for each $i \in I$. Then ker (α) contains the weak product which is dense in $\prod_{i \in I} A_i$. Since α is continuous this would imply that $\ker(\alpha) = \prod_{i \in I} A_i$. This contradicts the fact that $\alpha \neq 0$. So $\alpha \circ \phi_{i_0} \neq 0$ for some $i_0 \in I$. Since α is an algebra homomorphism i_0 is necessarily unique and $\alpha = (\alpha \circ \phi_{i_0}) \circ \pi_{i_0}$ where $\alpha \circ \phi_{i_0} \in M(A_{i_0})$.

3.11. Corollary. Every closed maximal ideal of $\prod_{i\in I} A_i$ is of the form $M_i \times (\prod_{i\neq j} A_j)$ for some unique closed maximal ideal M_i of A_i.

Proof: By Theorem 0; 3.2 (b) for every $A \in \underline{L}$ there exists a one-to-one correspondence between closed maximal ideals of A and the elements of M(A). Hence the corollary follows immediately from Lemma 3.10.

3.12. Proposition. Let $\{A_i\}_{i \in I}$ be in $\underline{L}$. Then $M(\prod_{i\in I} A_i) \sim \coprod_{i\in I} M(A_i)$ (disjoint topological union).

Proof: For each $i \in I$ consider the map $A_i \xrightarrow{\phi_i} \prod_{i\in I} A_i \xrightarrow{\pi_i} A_i$, where ϕ_i is defined as in Lemma 3.10. As in section 1, definition (c), the map $M(A_i) \xrightarrow{M(\pi_i)} M(\prod_{i\in I} A_i) \xrightarrow{M(\phi_i)} M(A_i) \cup \{0\}$ is continuous and $M(\phi_i) \circ M(\pi_i) = id_{M(A_i)}$. [Note: Since ϕ_i is not unitary the image of $M(\phi_i)$ is $M(A_i) \cup \{0\}$.] Since A_i has an identity, 0 is an isolated point of $M(A_i) \cup \{0\}$, hence $M(\pi_i)\ [M(A_i)] = M(\phi_i)^{-1}\ [M(A_i)]$ is an open-closed subset of $M(\prod_{i\in I} A_i)$.

Define $f : \coprod_{i\in I} M(A_i) \to M(\prod_{i\in I} A_i)$ by

$$f(\alpha_i) = \alpha_i \circ \pi_i = M(\pi_i)[\alpha_i].$$

By Lemma 3.10 f is onto. Since $M(\pi_i)$ is one-to-one for each $i \in I$, it follows that f is one-to-one. Since $M(\pi_i)[M(A_i)]$ is open-closed in $M(\prod_{i\in I} A_i)$ for each $i \in I$, it follows that f is a homeomorphism.

3.13. Corollary. Let (P) be a topological property which is preserved under coproducts. Then $M(\Pi A_i)$ has (P) whenever each $M(A_i)$ has (P).

In particular if each $M(A_i)$ is locally compact, paracompact, normal, k-space, or locally connected, then so is $M(\Pi A_i)$, respectively.

3.14. Theorem. Let $\{A_i\}_{i \in I}$ be a subset of CL. Then $\prod_{i\in I} A_i$ is also in CL.

Proof: Let K be a compact subset of $M(\prod_{i\in I} A_i)$. Since $M(\prod_{i\in I} A_i) \simeq \coprod_{i\in I} M(A_i)$, K is the disjoint topological union of $H_{i_1}, \ldots, H_{i_n}$ where H_{i_k} is a compact subset of $M(A_{i_k})$ for $k = 1,\ldots, n$. Since each A_{i_k} is in CL, it follows by Proposition 3.1 that each H_{i_k} is equicontinuous. Hence $H^o_{i_k}$ (the polar taken in A_{i_k}) is a neighborhood of zero in A_{i_k}. Thus

$$U = H^o_{i_1} \times H^o_{i_2} \times \ldots \times H^o_{i_n} \times (\prod_{\substack{i \neq i_k \\ k=1,\ldots,n}} A_i)$$

is a neighborhood of 0 in $\prod_{i\in I} A_i$. Since K^o, the polar of K taken in $\prod_{i\in I} A_i$, is exactly equal to U, it follows that K is equicontinuous. Thus every compact subset of $M(\Pi A_i)$ is equicontinuous. By Proposition 3.1 it follows that $\prod_{i\in I} A_i$ is in CL.

It should be pointed out that CL is not closed hereditary. Otherwise, every complete algebra in L, being a closed subalgebra of a product of Banach algebras by Theorem 0; 3.4, would be in CL. But the algebra in Example 3.3 (b) is complete and is not in CL.

For an algebra $A \in \underline{L}$, let $R(A)$ be the radical of A [Def. 0; 3.5 (a)]. It is known that $A_R = A/R(A)$ is semisimple.

3.15. Lemma. Let A be in $\underline{L}$. Then $M(A) \approx M(A_R)$.

Proof: Let $\phi: A \to A_R$ be the quotient map. Then $M(A_R) \xrightarrow{M(\phi)} M(A)$ is continuous and one-to-one. Since every α in $M(A)$ can be factored through A_R, $M(\phi)$ is also onto. It remains to show that $M(\phi)$ is open. Let $V = N(\alpha', \phi(a), \xi)$ be a neighborhood in $M(A_R)$. Then $U = N(\alpha' \circ \phi, a, \xi)$ is a neighborhood in $M(A)$. For $\beta \in U$, $\beta = \beta' \circ \phi$ for some β' in $M(A_R)$ and

$$|\alpha' \circ \phi(a) - \beta(a)| \leq \xi$$

$$\Longrightarrow \quad |\alpha' \circ \phi(a) - \beta' \circ \phi(a)| \leq \xi$$

$$\Longrightarrow \quad \beta' \text{ is in } V.$$

Thus $\beta = \beta' \circ \phi = M(\phi)(\beta') \in M(\phi)(V)$. Hence $U \subset M(\phi)[V]$, implying that $M(\phi)$ is open. Therefore $M(A) \approx M(A_R)$.

3.16. Proposition. Let A be in $\underline{L}$. Then A is in $\underline{CL}$ iff A_R is in $\underline{CL}$.

Proof: If A is in $\underline{CL}$, then A_R is in $\underline{CL}$ by Proposition 3.6.

Conversely, suppose A_R is in $\underline{CL}$. Then by Lemma 3.15 $M(\phi) : M(A_R) \to M(A)$ is a homeomorphism. Thus $CM(\phi) : CM(A) \to CM(A_R)$ is an isomorphism. By Lemma 3.5, $\phi_{A_R} \circ \phi = CM(\phi) \circ \phi_A$, hence $\phi_A = CM(\phi)^{-1} \circ \phi_{A_R} \circ \phi$ is continuous. Thus A is in $\underline{CL}$.

3.17. Proposition. Let A be in $\underline{L}$ and let $\tilde{A}$ be its completion. If A is in $\underline{CL}$ then so is $\tilde{A}$.

Proof: Let $\phi : A \to \tilde{A}$ be the canonical embedding. Then $M(\phi) : M(\tilde{A}) \to M(A)$ is continuous, one-to-one and onto. Let K be a compact subset of $M(\tilde{A})$. Then $H = M(\phi)[K]$ is a compact subset of M(A), and since A is in CL it follows that H^o is in $\mathcal{N}(A)$. Let K^o be the polar of K in $\tilde{A}$. Then $\phi(H^o) \subset K^o$ and so $cl_{\tilde{A}}\, \phi(H^o) \subset K^o$, since K^o is closed. By Theorem 0; 2.1, it follows that $cl_{\tilde{A}}\, \phi(H^o) \in \mathcal{N}(\tilde{A})$, hence K is equicontinuous. Thus every compact subset of $M(\tilde{A})$ is equicontinuous. By Proposition 3.1, $\tilde{A}$ is in CL.

The following counterexample shows that the converse of Proposition 3.16 does not hold.

3.18. Example. The algebra $A = C_p(\mathbb{R})$ of Example 3.3 (a) is not in CL. It is known that $\tilde{A} = \mathbb{C}^{\mathbb{R}}$, the algebra of all complex-valued functions on $\mathbb{R}$ with the topology of pointwise convergence. Now $\mathbb{C}^{\mathbb{R}} = C(\mathbb{R}_d)$, where $\mathbb{R}_d$ is $\mathbb{R}$ with the discrete topology. Since $C(\mathbb{R}_d)$ is in CL it follows that $\tilde{A}$ is in CL, but A is not in CL.

3.19. Definition. For an algebra $A \in L$, $M_1(A)$ is the set of all nonzero, complex-valued homomorphisms on A, with the relative w* topology from the algebraic dual A*.

M(A) is a topological subspace of $M_1(A)$, in general, the containment being proper. For example, if X is a completely regular space, then $M[C(X)] = X$ and $M_1[C(X)] = \upsilon X$, the realcompactification of X.

The following theorem shows that there is a close relationship between L and CL.

3.20. Theorem. CL is a coreflective sub-category of L.

Proof: Let $(A, \{p_i\}_{i \in I})$ be in L. Proceed by transfinite induction on the class of ordinals as follows:

$$A_0 = (A, \{p_i\}_{i \in I})$$

$$A_1 = (A, \{p_i\}_{i \in I} \cup \{p_K\}_{K \subset M(A_0)})$$

where $\{p_K\}_{K \subset M(A_0)}$ are the submultiplicative seminorms determined by the compact subsets of $M(A_0)$ and defined by $p_K(a) = \sup_{\alpha \in K} |\hat{a}(\alpha)|$, $\hat{a}$ being the Gelfand transform of a. Given A_λ for all $\lambda < \mu$, define:

$$A_\mu = \begin{cases} (A, \{p_i\}_{i \in I} \cup \{p_K\}_{K \subset M(A_\lambda)}) & \text{if } \mu = \lambda + 1 \\ (A, \{p_i\}_{i \in I} \cup \{p_K\}_{K \subset \bigcup_{\lambda<\mu} M(A_\lambda)}) & \end{cases}$$

if μ is a limit ordinal.

Since each $M(A_\lambda)$ is a topological subspace of $M_1(A)$, $\bigcup_{\lambda<\mu} M(A_\lambda)$ is understood to be the union taken in $M_1(A)$.

Clearly, A_λ is in L and the identity map $i : A_\lambda \to A$ is continuous for each λ. Since $M(A_\lambda)$ is a subset of $M_1(A)$ for each λ, by a simple cardinality argument it follows $M(A_\lambda) = M(A_{\lambda+1})$ whenever Card (λ) > Card $(M_1(A$ Let λ_0 be the first ordinal for which $M(A_{\lambda_0}) = M(A_{\lambda_0+1})$. Then the Gelfand map $\phi_0 : A_{\lambda_0+1} \to CM(A_{\lambda_0+1}) = CM(A_{\lambda_0})$ is continuous by the very definition of the topology on A_{λ_o+1}. Hence A_{λ_o+1} is in CL and the identity map $i : A_{\lambda_o+1} \to A$ is continuous. The following lemma shows that the map $i : A_{\lambda_o+1} \to A$ is actually a coreflection.

3.21. Lemma. Let B be in CL and let $\phi : B \to A$ be a continuous unitary homomorphism. Then $\phi : B \to A_{\lambda_o+1}$ is continuous.

Proof: Clearly $\phi : B \to A_1$ is continuous since for any compact subset K of M(A) and $b \in B$, $(p_K \circ \phi)(b) = \sup_{\alpha \in K} |\widehat{\phi(b)}(\alpha)| = \sup_{\alpha \in K} |\alpha \circ \phi(b)|$ $= \sup_{\alpha \in K} |M(\phi)\ [\alpha]\ (b)| = \sup_{\beta \in M(\phi)[K]} |\beta(b)| = p_{M(\phi)[K]}(b)$; hence $p_K \circ \phi = p_{M(\phi)[K]}$ is continuous since $B \in$ CL.

Suppose $\phi : B \to A_\lambda$ is continuous for all $\lambda < \mu$. The nontrivial case is when μ is a limit ordinal. Since $\phi: B \to A_\lambda$ is continuous for all $\lambda < \mu$, $M(\phi)$ maps $M(A_\lambda)$ into $M(B)$. Thus $M(\phi)[\bigcup_{\lambda<\mu} M(A_\lambda)] = \bigcup_{\lambda<\mu} M(\phi)[M(A_\lambda)]$ $\subset M(B)$. So for any compact subset K of $\bigcup_{\lambda<\mu} M(A_\lambda)$, $p_K \circ \phi = p_{M(\phi)[K]}$ is continuous on B since $B \in$ CL. Thus, by the definition of the topology on A_μ, it follows that $\phi : B \to A_\mu$ is continuous. Hence $\phi : B \to A_\lambda$ is continuous for each λ, in particular for $\lambda = \lambda_o + 1$. Thus $i : A_{\lambda_o+1} \to A$ is a coreflection of A in CL. Therefore CL is a coreflective subcategory of L.

It should be pointed out that Propositions 3.6 and 3.8 can be obtained as corollaries of Theorem 3.20, since coreflections preserve coequalizers and coproducts.

This section ends with a description of the not-necessarily continuou complex-valued homomorphism on a product of LMC algebras.

3.22. Definition. A set S is said to be measurable if there exists a {0,1} -valued measure μ on the subsets of S such that $\mu(S) = 1$, and $\mu(\{x\}) = 0$ for each $x \in S$. S is called nonmeasurable otherwise.

3.23. Theorem. Let S be a discrete set. Then S is nonmeasurable iff S is real compact.

Proof: [23; Theorem 12.3].

3.24. Proposition. Let $\{A_i\}_{i \in I}$ be a subset of L. Suppose that I is a nonmeasurable set. Then every $\alpha \in M_1(\prod_{i\in I} A_i)$ is of the form $\alpha_i \circ \pi_i$ for some (necessarily unique) $\alpha_i \in M_1(A_i)$.

Proof: Let α be in $M_1(\prod_{i\in I} A_i)$ and let $\phi : \prod_{i\in I} \mathbb{C} \to \prod_{i\in I} A_i$ be the canonical embedding defined by $\phi[(\lambda_i)_{i \in I}] = (\lambda_i e_i)_{i \in I}$ where e_i is the identity of A_i. Now $\prod_{i\in I} \mathbb{C} \simeq C(I)$ and since I is nonmeasurable, I is realcompa hence $\alpha \circ \phi$ is fixed. Thus there exists $i_o \in I$ such that $\alpha \circ \phi[(\lambda_i)_{i \in I}] = \lambda_{i_o}$. As in Lemma 3.10 it follows that $\alpha = \alpha_{i_o} \circ \pi_{i_o}$ for some unique $\alpha_{i_o} \in M(A_{i_o})$.

Proposition 3.24 may also be obtained as a corollary of [58; Theorem

3.25. Corollary. Let I and $\{A_i\}_{i \in I}$ be as above. Then every maximal ideal of $\prod_{i\in I} A_i$ which is the kernel of a complex homomorphism is of the form $M_j \times (\prod_{i\neq j} A_i)$ where M_j is the kernel of some $\alpha \in M_1(A_j)$.

3.26. Corollary. Let I be a nonmeasurable set and let $\{A_i\}_{i \in I}$ be a subset of L consisting of functionally continuous algebras. Then $\prod_{i\in I} A_i$ is also functionally continuous.

3.27. Proposition. Let I and $\{A_i\}$ be as in Proposition 3.24. Then $M_1(\prod_{i\in I} A_i) \approx \coprod_{i\in I} M_1(A_i)$.

Proof: In view of Proposition 3.24, the proof is exactly the same as in Proposition 3.12, replacing M by M_1.

In general, in Proposition 3.24, the assumption that I is nonmeasurable cannot be dropped. By Theorem 3.23, if I is measurable, then I is not realcompact, hence $\prod_{i\in I} \mathbb{C} = C(I)$ has a nonfixed complex homomorphism. It should be noted that in Lemma 3.10 the assumption that I is nonmeasurable is not required.

4. The adjoint situation between CR and CL, CR and L.

It is known that the functors $\text{Comp Haus} \underset{M}{\overset{C}{\rightleftarrows}} \text{Ban A}$ are adjoint on the right; Comp Haus being the category of compact Hausdorff spaces and continuous maps and Ban A being the category of commutative Banach algebras with identity and continuous unitary homomorphisms. This adjoint situation extends to CR and CL. Herein lies one of the main reasons for studying CL.

For X, Y ε CR and A, B ε CL, [X,Y] and [A,B] are the set of all continuous functions $f : X \to Y$ and the set of all continuous unitary homomorphisms $\phi: A \to B$, respectively. Also, $E_X : X \to MC(X)$ is the evaluation map and $\phi_A : A \to CM(A)$ is the Gelfand map.

4.1. Theorem. The functors $\underline{CR} \underset{M}{\overset{C}{\rightleftarrows}} \underline{CL}$ are adjoint on the right.

Proof: Given $X \in CR$ and $A \in CL$, define

$$[X, M(A)] \underset{\mu}{\overset{\eta}{\rightleftarrows}} [A, C(X)] \quad \text{by}$$

$$(1) \quad \eta(f) = C(f) \circ \phi_A : A \to CM(A) \to C(X)$$

$$(2) \quad \mu(\phi) = M(\phi) \circ E_X : X \to MC(X) \to M(A).$$

Since A is in $\underline{CL}$, ϕ_A is continuous, hence $\eta(f) \in [A, C(X)]$; since E_X is a homeomorphism $\mu(\phi) \in [X, M(A)]$.

To prove that C and M are adjoint on the right, it will be shown that η is a natural set isomorphism.

(a) $\mu \circ \eta(f) = f$

Let $x \in X$ and let $a \in A$. Then:

$\{[\mu \circ \eta(f)]\ (x)\}(a) = \{\mu[C(f) \circ \phi_A](x)\}\ (a) = \{(M[C(f) \circ \phi_A] \circ E_X)(x)\}(a)$

$= \{M[C(f) \circ \phi_A](\alpha_x)\}(a) = [\alpha_x \circ C(f) \circ \phi_A](a) = [\alpha_x \circ C(f)](\hat{a}) = \alpha_x(\hat{a} \circ f)$

$= (\hat{a} \circ f)\ (x) = \hat{a}[f(x)] = [f(x)](a).$

Since this holds for each $x \in X$ and each $a \in A$, it follows that $\mu \circ \eta(f) = f$.

(b) $\eta \circ \mu(\phi) = \phi$

Let $a \in A$ and let $x \in X$. Then:

$\{[\eta \circ \mu\ (\phi)](a)\}(x) = \{[\eta(M(\phi) \circ E_X)](a)\}(x) = \{[C(M(\phi) \circ E_X) \circ \phi_A](a)\}(x)$

$= \{C[M(\phi) \circ E_X](\hat{a})\}(x) = [\hat{a} \circ M(\phi) \circ E_X](x) = [\hat{a} \circ M(\phi)](\alpha_x) = \hat{a}(\alpha_x \circ \phi)$

$= \alpha_x \circ \phi(a) = \alpha_x[\phi(a)] = [\phi(a)](x)$. Since this holds for each $a \in A$ and each $x \in X$, it follows that $\eta \circ \mu(\phi) = \phi$.

Thus η is a set isomorphism. It remains to show that η is natural.

(c) η is natural.

Let $f : X \to Y$ be in <u>CR</u>. It must be shown that the following diagram commutes:

$$\begin{tikzcd}
{[X, M(A)]} \arrow[r, "\eta_{X,A}"] & {[A, C(X)]} \\
{[Y, M(A)]} \arrow[u, "(-) \circ f"'] \arrow[r, "\eta_{Y,A}"] & {[A, C(Y)]} \arrow[u, "C(f) \circ (-)"']
\end{tikzcd}$$

Given $g \in [Y, M(A)]$, $\eta_{X,A}(g \circ f) = [C(g \circ f)] \circ \phi_A$.
On the other hand, $C(f) \circ [\eta_{Y,A}(g)] = C(f) \circ [C(g) \circ \phi_A] = [C(f) \circ C(g)] \circ \phi_A$ $= [C(g \circ f)] \circ \phi_A$. Hence the diagram commuted.

Let $\phi : A \to B$ be in <u>CL</u>. It must be shown that the following diagram commutes:

$$\begin{tikzcd}
{[X, M(A)]} \arrow[r, "\eta_{X,A}"] & {[A, C(X)]} \\
{[X, M(B)]} \arrow[u, "M(\phi) \circ (-)"'] \arrow[r, "\eta_{X,B}"] & {[B, C(X)]} \arrow[u, "(-) \circ \phi"']
\end{tikzcd}$$

Given $g \in [X, M(B)]$ it must be shown that $\eta_{X,A}[M(\phi) \circ g] = \eta_{X,B}(g) \circ \phi$, or equivalently that $C[M(\phi) \circ g] \circ \phi_A = [C(g) \circ \phi_B] \circ \phi$.

Let $a \in A$ and let $x \in X$. Then:

$\{(C[M(\phi) \circ g] \circ \phi_A)(a)\}(x) = \{C[M(\phi) \circ g](\hat{a})\}(x) = [\hat{a} \circ M(\phi) \circ g](x)$

$= [\hat{a} \circ M(\phi)](g(x)) = \hat{a}[g(x) \circ \phi] = [g(x) \circ \phi](a) = g(x)\ [\phi(a)]$. (i)

On the other hand, $\{[C(g) \circ \phi_B \circ \phi](a)\}(x) = \{C(g) \circ \phi_B(\phi(a))\}(x)$

$= \{C(g)[\widehat{\phi(a)}]\}(x) = \widehat{\phi(a)} \circ g(x) = \widehat{\phi(a)}[g(x)] = g(x)[\phi(a)]$. (ii)

Since (i) = (ii) for each $a \in A$ and each $x \in X$, it follows that the diagram commutes. Hence η is a natural set isomorphism for each $X \in$ CR and each $A \in$ CL. By Definition 0; 5.6 it follows that the functors $\underline{CR} \underset{M}{\overset{C}{\rightleftarrows}} \underline{CL}$ are adjoint on the right.

In view of Theorem 0; 5.8 the following is immediate.

4.2. Corollary (a) Let $(A, \{\phi_i\}_{i \in \underline{I}})$ be the colimit of the $\underline{I}$-diagram D over CL. Then $(M(A), \{M(\phi_i)\}_{i \in \underline{I}})$ is the limit of the $\underline{I}$-diagram M o D over CR.

(b) Let $(X, \{f_i\}_{i \in \underline{I}})$ be the colimit of the $\underline{I}$-diagram D over CR. Then $(C(X), \{C(f_i)\}_{i \in \underline{I}})$ is the limit of the $\underline{I}$ diagram C o D over CL. In particular, $C(\coprod_{i \in I} X_i) \simeq \prod_{i \in I} C(X_i)$ and $M(\coprod_{i \in I} A_i) \simeq \prod_{i \in I} M(A_i)$.

4.3. Corollary. Let $(A, \{\phi_i\}_{i \in \underline{I}})$ be the colimit of the $\underline{I}$-diagram D over CL. Suppose that D(i) is a normed algebra for each $i \in \underline{I}$. Then M(A) is compact.

Proof: By Corollary 4.2, $(M(A), \{M(\phi_i)\}_{i \in \underline{I}})$ is the limit of the $\underline{I}$-diagram M o D. Since D(i) is a normed algebra, M o D(i) is a compact space for each $i \in \underline{I}$. Since the limit of M o D is obtained as a suitable closed subspace of $\prod_{i\epsilon I}$ M o D(i), it follows that M(A) is compact.

4.4. Theorem. The functors $\underline{CR} \underset{M}{\overset{C_p}{\rightleftarrows}} \underline{L}$ are adjoint on the right.

Proof: As was noted in Section 1, the map $\phi_A : A \to C_p M(A)$ is continuous for each $A \in \underline{L}$. The diagrams and the proof are exactly the same as in Theorem 4.1.

4.5. Corollary. Let $(A, \{\phi_i\}_{i \in \underline{I}})$ be the colimit of the $\underline{I}$-diagram D over $\underline{L}$. Then $(M(A), \{M(\phi_i)\}_{i \in \underline{I}})$ is the limit of the $\underline{I}$-diagram M o D over $\underline{CR}$.

The adjoint situation between $\underline{CR}$ and $\underline{CL}$ becomes a duality when restricted to suitable subcategories. This duality provides an extension of the known Gelfand duality between compact Hausdorff spaces and commutative B* algebras with identity.

4.6. Definition. An algebra A in $\underline{L}$ is said to be a b*-algebra if A is a complete *-algebra whose topology is generated by a set $\{p_i\}_{i \in I}$ of submultiplicative seminorms, each satisfying: $p_i(xx^*) = p_i(x)^2$. If, in addition, A is in $\underline{CL}$, then A is called a Cb* algebra.

The term b* algebra comes from [1]. Clearly, C(X) is a Cb* algebra for every k_R-space X, the completeness following from Theorem 0; 4.5.

However, not every b* algebra is Cb* algebra, as Example 3.3 (b) shows. In fact, if X is any first countable space in CR which has an uncountable compact subset, then C(X) with the topology of uniform convergence on the countable compact subsets of X is a b* algebra but not a Cb* algebra. The completeness follows from [39; Lemma D. 5].

Let $\underline{K}_R$ and Cb* be the full subcategories of CR and CL consisting of k_R-spaces and Cb* algebras, respectively. Using the following important theorem, due to Morris and Wulbert, we will show that the categories $\underline{K}_R$ and Cb* are dually equivalent.

4.7. Theorem. Let A be a b* algebra. Then $\phi_A : A \to CM(A)\tau_o$ is a topological *-isomorphism, where τ_o is the topology of uniform convergence on the closed equicontinuous subsets of M(A).

Proof: [42; Theorem 5.2 and Corollary 5.3].

4.8. Corollary. Let A be a Cb* algebra. Then $A \simeq CM(A)$ (compact-open topology).

Proof: Since $A \in$ CL, every compact subset of M(A) is equicontinuous, hence τ_o is equal to the compact-open topology.

4.9. Corollary. If A is a Cb* algebra then M(A) is a k_R-space.

Proof: By the above corollary, $A \simeq CM(A)$ is complete. By Theorem 0; 4.5, M(A) is a k_R-space.

By combining the above results with Theorem 0;4.5, we obtain the following theorem.

4.10. <u>Theorem</u>. The functors $\underline{K}_R \underset{M}{\overset{C}{\rightleftarrows}} \underline{Cb^*}$ give rise to a dual equivalence.

<u>Proof</u>: By Corollary 0; 4.10, the evaluation map $E_X : X \to MC(X)$ is a homeomorphism for each $X \in \underline{K}_R$. By Theorem 4.7 and its corollaries, the Gelfand map $\phi_A : A \to CM(A)$ is a topological isomorphism for each $A \in \underline{Cb^*}$. The naturalness conditions follow from Theorem 4.1. So by Definition 0; 5.9, $\underline{K}_R$ and $\underline{Cb^*}$ are dually equivalent.

4.11. <u>Corollary</u>. The category of commutative B* algebras with identity is dually equivalent to the category of compact Hausdorff spaces.

4.12. <u>Corollary</u>. The category of metrizable Cb* algebras is dually equivalent to the category of hemicompact k-spaces.

Proof: By Theorem 0; 4.4.

4.13. <u>Corollary</u>. The category of bornological Cb* algebras is dually equivalent to the category of realcompact k_R-spaces.

<u>Proof</u>: By Theorem 0; 4.6.

4.14. <u>Corollary</u>. The category of $B(\mathcal{L})$ Cb* algebras is dually equivalent to the category of completely regular k-spaces.

<u>Proof</u>: By Theorem I; 3.8.

The following proposition is a direct consequence of Theorem 4.10.

4.15. <u>Proposition</u>. Let X and Y be k_R-spaces. Then the function $\eta: [X,Y] \to [C(Y), C(X)]$ defined by $\eta(f) = C(f)$ is one-to-one and onto.

Proof: A proof may be found in [28; Theorem 14.11].

4.16. Corollary. Every continuous unitary homomorphism between Cb* algebras is a *-homomorphism.

Proof: Let $\phi : A \to B$ be in Cb*. Since A and B are topologically *-isomorphic to CM(A) and CM(B), respectively, without loss of generality we may suppose that $\phi : C(Y) \to C(X)$ where Y and X are k_R-spaces. By Propostion 4.15, $\phi = C(f)$ for some continuous $f : X \to Y$. Since C(f) is a *-homomorphism so is ϕ.

Actually, Proposition 4.15 is valid whenever X and Y are in CR. The proof is analogous to the proof of [23; Theorem 10.6].

4.17. Proposition: Let X and Y be in CR. Then every $\phi: C(Y) \to C(X)$ in L is of the form C(f) for some $f : X \to Y$ in CR.

Proof: Let x be in X. Since ϕ is unitary, $\alpha_x \circ \phi$ is nonzero and hence in M C(Y). By Corollary 0; 4.10, $\alpha_x \circ \phi = \alpha_y$ for some (necessarily unique) $y \in Y$. (i)

Define $f : X \to Y$ by $f(x) = y$, where y is uniquely determined by (i). For $r \in C(Y)$, $\phi(r)[x] = \alpha_x[\phi(r)] = \alpha_x \circ \phi(r) = \alpha_y(r) = r(y) = r[f(x)] = r \circ f(x) = (C(f)[r])(x)$. Thus $\phi(r) = C(f)[r]$ for each $r \in C(Y)$, and consequently $\phi = C(f)$. Now, for each $r \in C(Y)$, $r \circ f = C(f)[r] = \phi(r) \in C(X)$. Thus by Theorem II; 1.6, f is continuous.

4.18 Corollary. Every $\phi : C(Y) \to C(X)$ in L in a *-homomorphism.

This section ends with the following:

4.19. Proposition. If A is a b* algebra, then M(A) is a k_R-space.

<u>Proof</u>: By Theorem 4.7, $A \simeq CM(A)_{\tau_o}$. Now for every compact subset K of M(A), $N(K, \xi) = \{\hat{a} : |\hat{a}(\alpha)| \leq \xi \ \forall \ \alpha \in K\}$ is closed in the topology of pointwise convergence and therefore closed in the τ_o topology. Since these sets form a basis for the compact-open topology on CM(A), it follows by Theorem 0; 2.4 that CM(A) is complete. Thus, by Theorem 0; 4.5, M(A) is a k_R-space.

It can easily be seen that Cb* algebras form a coreflective subcategory of b* algebras. With each b* algebra A associate the algebra A_C consisting of A with the induced compact-open topology from CM(A). Then A_C is a Cb* algebra and the identity map $i : A_C \to A$ is a coreflection.

5. <u>Projectivity in K_R and Injectivity in Cb*</u>.

5.1. <u>Definition</u>. Let <u>C</u> be a category and let $\mathcal{P}$ be a class of morphisms in <u>C</u>. An object A of <u>C</u> is said to be $\mathcal{P}$-projective if for any morphism $f : A \to B$ in <u>C</u> and for any $g : D \to B$ in $\mathcal{P}$ there exists a morphism $h : A \to D$ in <u>C</u> such that $g \circ h = f$.

Dually, if $\mathcal{E}$ is a class of morphisms in <u>C</u>, an object A of <u>C</u> is said to be $\mathcal{E}$ - injective if for any morphism $f : B \to A$ in <u>C</u> and for any $g : B \to D$ in $\mathcal{E}$ there exists a morphism $h : D \to A$ in <u>C</u> such that $f = h \circ g$.

The purpose of this section is to study $\mathcal{F}$-projectivity in <u>CR</u>, $\mathcal{F}$ being the class of all full maps. In view of Theorem II; 1.4, full maps arise in a very natural way. So it is natural to ask what the $\mathcal{F}$-projective spaces are in <u>CR</u>.

Projectivity with respect to onto maps was first studied by Gleason [24] in <u>Comp Haus</u> the category of compact Hausdorff spaces and continuous maps. Subsequently, projectivity with respect to perfect onto maps ($\mathcal{P}$- projectivity) was studied for larger categories of Hausdorff spaces. B. Banaschewski [9] has developed a systematic theory of $\mathcal{P}$-projectivity for the category <u>Haus</u> and subcategories thereof.

5.2. <u>Definition</u>. A topological space X is said to be <u>extremally disconnected</u> if every open set has open closure.

The following has been proved by B. Banaschewski [9].

5.3. <u>Theorem</u>. In <u>Haus</u>, the $\mathcal{P}$- projective spaces are exactly the extremally disconnected spaces. The same holds for any full subcategory of <u>Haus</u> which is productive and closed hereditary.

5.4. <u>Corollary</u>. In <u>CR</u> and in <u>Comp Haus</u>, the $\mathcal{P}$- projective spaces are exactly the extremally disconnected spaces.

<u>Proof</u>: <u>CR</u> and <u>Comp Haus</u> are closed hereditary and productive subcategories of <u>Haus</u>.

5.5. <u>Corollary.</u> In <u>CR</u>, every $\mathfrak{F}$ - projective space is extremally disconnected.

<u>Proof</u>: Since $\mathfrak{F} \supset \mathcal{P}$, every $\mathfrak{F}$ - projective is $\mathcal{P}$ - projective and hence extremally disconnected.

5.6. <u>Lemma</u>. Every compact extremally disconnected space is $\mathfrak{F}$ -projective.

Proof: Let X be a compact extremally disconnected space. Let $f : X \to Y$ be in CR and $g : Z \to Y$ in $\mathfrak{F}$.

Since f is continuous f(X) is compact; thus since $g \in \mathfrak{F}$ there exists a compact subset H of Z such that $g(H) = f(X)$. Consider the following diagram:

$$\begin{tikzcd} & X \arrow[d, "f"] \arrow[dl, dashed, "h"'] \\ H \arrow[r, "g|H"] \arrow[d, phantom, "\cap"] & f(X) \arrow[d, phantom, "\cap"] \\ Z \arrow[r, "g"] & Y \end{tikzcd}$$

Since $g|H$ is onto and H and f(X) are compact, it follows that $g|H$ is in $\mathcal{P}$. X, being extremally disconnected, is $\mathcal{P}$-projective; hence there exists $h : X \to H$ is CR such that $g \circ h = f$. Thus X is $\mathfrak{F}$-projective.

5.7. Corollary: Let X be the coproduct (disjoint topological union) of $\{K_i\}_{i \in I}$, where each K_i is compact and extremally disconnected. Then X is $\mathfrak{F}$-projective.

Proof: Let $f : X \to Y$ be in CR and let $g : Z \to Y$ be in $\mathfrak{F}$. For each $i \in I$ let $f_i : K_i \to X$ be the natural embedding into the coproduct. By Lemma 5.6 each K_i is $\mathfrak{F}$-projective, thus for each $i \in I$ there exists $h_i : K_i \to Z$ such that $g \circ h_i = f \circ f_i$. By the coproduct property, there exists a unique $h : X \to Z$ such that $h_i = h \circ f_i$. Thus $g \circ h \circ f_i = g \circ h_i = f \circ f_i$ for each $i \in I$. Hence $g \circ h = f$ and so X is $\mathfrak{F}$-projective.

5.8. Lemma. Extremally disconnected spaces are open hereditary..

Proof: Let X be an extremally disconnected space and let Y be an open subset. Let V be an open subset of Y. Then V is open in X, hence $cl_X V$ is open in X. Thus $cl_Y V = cl_X V \cap Y$ is open in Y. Therefore Y is extremally disconnected.

5.9. Proposition: X is $\mathfrak{F}$-projective in CR iff X is the coproduct of compact extremally disconnected spaces.

Proof: The "only if" part follows from Corollary 5.7.

Conversely, suppose that X is $\mathfrak{F}$-projective. Then by Corollary 5.5 X is extremally disconnected. Consider the following diagram:

$$\begin{tikzcd} & X \arrow[dl, dashed, "g"'] \arrow[d, equal, "id_X"] \\ \coprod\limits_{K_i \subset X} K_i \arrow[r, "f"] & X \end{tikzcd}$$

where $\coprod\limits_{K_i \subset X} K_i$ is the coproduct of all the compact subsets of X and f is the canonical map. Trivially $f \in \mathfrak{F}$, hence there exists $g : X \to \coprod\limits_{K_i \subset X} K_i$ such that $f \circ g = id_X$. Now for each K_i, $g^{-1}(K_i)$ is open-closed in X and so by Lemma 5.8, $g^{-1}(K_i)$ is extremally disconnected. Also, $g^{-1}(K_i) = f \circ g(g^{-1}(K_i)) \subset f(K_i) = K_i$, hence $g^{-1}(K_i)$ is compact (possibly empty). So $\{g^{-1}(K_i)\}_{K_i \subset X}$ is a decomposition of X into disjoint, compact, open-closed subsets of X. Thus $X \approx \coprod\limits_{K_i \subset X} g^{-1}(K_i)$ where $g^{-1}(K_i)$ is compact and extremally disconnected.

So in CR the $\mathfrak{F}$-projectives are exactly those spaces which are coproducts of compact extremally disconnected spaces. It will be shown that the same holds in the subcategory $\underline{K}_R$.

5.10. Lemma. Let X be an extremally disconnected space and let Y be a dense subspace. Then Y is also extremally disconnected.

Proof: Let V be an open subset of Y. Then $V = U \cap Y$ where U is an open subset of X. Since Y is dense and U is open in X, $cl_X(U \cap Y) = cl_X(U)$ is open in X. Thus $cl_Y(V) = cl_X(V) \cap Y = cl_X(U \cap Y) \cap Y$ is open in Y. Therefore Y is extremally disconnected.

5.11. Lemma. If X is $\mathcal{P}$-projective in $\underset{\sim}{K}_R$ then X is extremally disconnected.

Proof: Suppose X is $\mathcal{P}$-projective in $\underset{\sim}{K}_R$. It will be shown that βX, the Stone-Čech compactification of X, is $\mathcal{P}$-projective in Comp Haus. Suppose $f : \beta X \to Y$ is in Comp Haus and let $g : Z \to Y$ be in $\mathcal{P}$ (Z compact). Since X is $\mathcal{P}$-projective in $\underset{\sim}{K}_R$, there exists a continuous map $h : X \to Z$ such that $g \circ h = f \circ i$ where $i : X \to \beta X$ is the natural embedding. By the extension property of the Stone-Čech compactification there exists a unique continuous map $\tilde{h} : \beta X \to Z$ such that $h = \tilde{h} \circ i$. Thus $g \circ \tilde{h} \circ i = g \circ h = f \circ i$, and since $i(X)$ is dense, $g \circ \tilde{h} = f$. Therefore βX is $\mathcal{P}$-projective in Comp Haus, hence extremally disconnected by Corollary 5.4. By Lemma 5.10 it follows that X is also extremally disconnected.

5.12. Proposition. X is $\mathcal{F}$-projective in $\underset{\sim}{K}_R$ iff X is the coproduct of compact extremally disconnected spaces.

Proof: In view of Lemma 5.11, if X is $\mathcal{F}$-projective in $\underset{\sim}{K}_R$ then X is extremally disconnected. The rest of the proof is exactly the same as in 5.9.

Using Theorem II; 1.4 and the dual equivalence $\underline{K}_R \underset{M}{\overset{C}{\rightleftarrows}} \underline{Cb}^*$, we can dualize Proposition 5.12 in terms of $\mathcal{E}$-injectivity in $\underline{Cb}^*$, $\mathcal{E}$ being the class of all embeddings in $\underline{Cb}^*$.

5.13. Proposition. A is $\mathcal{E}$-injective in Cb* iff M(A) is the coproduct of compact extremally disconnected spaces.

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