RANDOM SECANTS AND RAYS OF

SOME CONVEX GEOMETRICAL SHAPES

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By

MD. KHADEMUL ISLAM, B.A., M.A.

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AUTHOR: Md. Khademul Islam, B.A. (Hons.) (University of Dacca)

M.A. (University of Dacca)

M.A. (University of Western Ontario)

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SCOPE AND CONTENTS:

This thesis is devoted to investigating certain regular features of random secants and rays of some convex configurations. The randomness of a secant of a convex configuration does not arise uniquely. Several mechanisms under which the randomness of straight line paths through convex bodies commonly arise are discussed and the problem in geometrical probability of the distributions of lengths of random secants has been solved by a general and direct geometrical method based on geometrical arguments for the following configurations and types of randomness: (i) the general triangle, the rectangle, regular polygons, the circle and the sphere under S_1-randomness; (ii) regular polygons under S_2-randomness, (iii) the circle under I_1-randomness. In both (i) and (ii), it is shown that if the polygon P_N of N sides is inscribed in a circle of constant radius r for any N, then the corresponding sequences of distribution functions $F_N^{S_1}(l)$, and $F_N^{S_2}(l)$, N = 3,4,..., of the random secant length l of the polygons P_N, N = 3,4,..., under S_1- and S_2-randomness, respectively, both converge

to the same distribution function $F(\ell)$ of the random secant length L of the circle under S_1- and S_2-randomness. Since the distributions of the lengths of random secants of a circle under S_1-randomness and S_2-randomness are the same, we have here a remarkable instance of two different sequences of distribution functions $F_N^{S_1}(\ell)$ and $F_N^{S_2}(\ell)$, $N = 3,4,\ldots$, (which are 'paradoxical' according to Bertrand) of the random secant length L of a polygon under S_1-randomness and S_2-randomness, converging to the same (hence non paradoxical) distribution function $F(\ell)$ of the random secant length L of a circle. The probability laws of the random lengths of rays emanating in a random direction from random sources within the rectangle, circle and sphere have been formulated.

To My Parents

And

To The Memory Of My Beloved Younger Brother,

MD. EKRAMUL ISLAM, Ph.D.,

Whose Dedication And Sacrifice

Have Made This Work Possible.

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CHAPTER ZERO

0. INTRODUCTION

Geometrical probability deals with probabilities concerning geometrical elements such as points, lines, planes, triangles, polygons, etc. One of the first problems occurring in the literature is Buffon's needle problem [31, p. 70]. This problem is concerned with the determination of the probability that a needle of length ℓ, when dropped randomly on a plane covered by parallel lines at unit distances, intersects at least one of these lines. There are many variations of this problem (cf. [31, p. 70-73], [12], [69]) of which we give only one to illustrate the flavor of geometrical probability. Let the needle be of length ℓ and surrounded by a circle of unit diameter with the midpoint of the needle at the centre. The circle intersects only one of these parallel lines giving a random chord of the circle. The probability that this chord intersects the needle is found to be $\frac{2\ell}{\pi}$.

Perhaps the most famous of problems in geometrical probability is the problem considered by Bertrand. His problem was to find the probability that a random chord in a circle of unit radius has length greater than or equal to $\sqrt{3}$ (cf. [31, p. 9]). There are three different solutions $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{2}$ depending on the reference set chosen to specify the chords. This problem illustrates the fact that 'paradoxes' can arise because of the vague specification of the word "random".

Further problems of this type were considered by Crofton (cf. [31, p. 24]) and Sylvester (cf. [31, p. 42]). Their problems were concerned with the distribution of quantities dependent on random points inside geometrical figures or with the relationships between random lines, planes and geometrical figures. The problems in this thesis are of this type, i.e., they are concerned with the distribution of quantities dependent on random points chosen in some geometrical configurations. Crofton (cf. [31, p. 24]) derived differential equations for solving certain problems of geometrical probability concerning a fixed number of points. By use of Crofton's formula, the probability distribution of the distance D between two random points in the interior of a circle of radius r has been obtained ([31], p. 41). The probability density of D is given by

$$P(d,r) = \frac{1}{\pi r^2} \{\pi d^2 + (r^2 - d^2)(\pi - 2\alpha) - \frac{1}{2}(2r^2 + d^2)\sin 2\alpha\} ,$$

where $d = 2r\cos\alpha$ and $0 \le d \le 2r$.

The problem of Sylvester [31, p. 42] is concerned with the determination of the probability that four randomly selected points inside a convex domain form a convex quadrilateral. The probability is

$$0.667,\ 0.7045,\ 0.6944,\ \text{and}\ 0.7029 ,$$

where the convex domain is a triangle, circle, parallelogram and regular hexagon, respectively.

Garwood and Tanner [19] considered the problem of the distribution of the distance between two random points in a circle, using direct integration. Extending this problem, Fairthorne [17] considered the distribution and the mean of the distances between a random point P in a circle of radius a and an independent random point Q in a concentric circle of radius b, where $b > a$. The problem arose from a theoretical study of distances travelled in a town. The density he derived is given by

$$f(s) = \begin{cases} \dfrac{2s}{b^2} & , \text{ if } 0 \le s \le b-a , \\ \dfrac{s}{\pi}\left(\dfrac{2\alpha - \sin 2\alpha}{b^2} + \dfrac{2\beta - \sin 2\beta}{a^2}\right) & , \text{ if } b-a \le s \le b+a , \end{cases}$$

where $\alpha = \cos^{-1}(\frac{s^2-b^2+a^2}{2as})$, $\beta = \cos^{-1}(\frac{s^2+b^2-a^2}{2bs})$.

Garwood and Holroyed [18] considered the problem of the probability distribution of the distance s of a random chord from the centre of a circle, the random chord being determined by two points uniformly distributed on the perimeter of the circle. They showed that the density of s is given by

$$f(s) = \frac{16(1-s^2)^{3/2}}{3\pi} , \quad \text{for } 0 \le s \le 1 .$$

Barton, David and Fix [4] have used the distribution of the distance between two points independently and uniformly distributed in the same circle for the analysis of chromosome patterns. David and Fix [10] considered the number of intersections of random chords of a circle. They found the mean and variance of the number of intersections.

Borel (cf. [31, p. 42]) considered the distribution of distance between two random points in a triangle, circle, rectangle and polygon in general, by direct integration. The distribution of the distance between two random points in a sphere has been considered by Delthiel by use of Crofton's theorem, by direct integration by Lord [39], and by use of characteristic functions by Hammersley [24]. Horowitz [27] obtained the probability density functions and average path lengths of the random secants under S_1-randomness of the rectangle, the circle, the sphere and the cube. Coleman [8] obtained probability density functions of the random secant length of the circle, the rectangle and the cube under four different specifications of randomness. Kingman [35] has discovered some relationships between two specifications of randomness.

Of many interesting papers in geometric probability concerning random lines, points and simple configurations, e.g., circles, squares, rectangles, triangles, etc., we mention the work of Mack [40], [41], Robbins [61], Kendall [32], [33], Takacs [70], Votaw [71], Morgenthaler [50], Mauldon [53], Renyi [59], Soloman [66A], [66B], Bronowski [7], Krengel [37], Dvoretzky [13], Gilbert [21], [22], Efron [14], DeBruin [11], Miles [44], [45], [46], Ailam [1], [2], Matern [41], Morton [51], Wolfowitz [73], Santalo [62], [63], [64]. For a detailed summary of these and other works, see Kendall and Moran [31], Moran [47], [48], [49].

Geometrical probability is rich in many other types of problems involving probabilities of geometric configurations. There are problems concerned with random division of space by various methods, the measurement of lengths of curves in two and three dimensions, the packing of objects of fixed size into a confined region and the covering of geometrical bodies with other geometrical bodies. Also out of problems in geometrical probability arose the study of integral geometry as introduced by Blaschke [6] and later by Santalo [65], and Stoka [67].

In the study of problems in geometrical probability concerning random points, lines, planes, circles, triangles, rectangles, for example, Kendall and Moran [31, Chapters 1, 2, 3, and 4], Santalo [65], and [19], [17], [24], [27], [28], [35], [43], etc. the following questions present themselves:

1. How are the lengths of random secants of a convex body distributed?

2. How are the lengths of random rays within a convex body distributed?

(By a random ray we mean a line in a random direction emanating from a random point called 'source' in the interior of a geometrical configuration.)

Referring to the first question one finds that the randomness of a secant of a convex body does not arise uniquely. As a result, 'paradoxical' results (as in the case of Bertrand) are obtained for the probability distributions of the lengths of the secants of a convex body, (cf. [31, p. 9]). Random secants may arise in many ways, of which some are as follows, (cf. Coleman [8]).

(A) S_1-randomness. A chord $K \cap \alpha$ ($\neq \phi$), where K is a convex body and α is a random secant, is defined by a point P on the surface of K and the angle θ which it makes with a suitable given direction. The point P is uniformly distributed over the surface of K and θ has a uniform distribution in its domain and P and θ are independent.

(B) S_2-randomness. A chord $K \cap \alpha$ ($\neq \phi$) is defined by two points P and Q having uniform and independent distributions on the surface of K subject to the condition that the join of PQ is a secant of K.

(C) I_1-randomness. A chord $K \cap \alpha$ ($\neq \phi$) is defined by the point P in the interior of K and a direction θ, where P and θ are distributed independently and uniformly.

(D) I_2-randomness. A chord $K \cap \alpha$ ($\neq \phi$) is defined by two points P and Q having uniform and independent distributions in the interior of K.

In this thesis we take up these two questions and study (1) random secants of geometrical configurations extensively covering some general configurations and (2) random rays of some common geometrical configurations, and obtain some 'paradoxical results' (according to Bertrand) for the distribution of the lengths of random secants of a configuration, e.g., polygon, under different randomness. A direct geometrical method for dealing with the problems is provided.

More specifically, in Chapter One, we find by direct geometrical method how the lengths of random secants of the following configurations under S_1-randomness are distributed: the regular polygon of N sides, (arbitrary) triangles, rectangles, circles and spheres. The problem relating to the regular polygon considered here when stated precisely is as follows: we consider a regular polygon of N sides. A random secant of the polygon is defined by a point P on the perimeter of the regular polygon and an angle θ which it makes with a suitable given direction. The point P is uniformly distributed over the perimeter of the polygon and θ is uniformly distributed over its specified domain. Further we assume that P and θ are independent. We have, following Coleman, called this type of random specification, S_1-randomness. The problem is to find the distribution of the length L of this random secant.

We find the distribution of L for the general regular polygon of N sides. It is further shown that if the polygon of N sides is always inscribed within a circle of constant radius r for any N, then the corresponding sequence of distribution functions $F_N(\ell)$, $N = 3, 4, \ldots,$ of the random secant lengths of the polygons under S_1-randomness converges to the distribution function $F(\ell)$ of the random secant length L of the circle under S_1-randomness. The distribution of L for the special cases of the rectangle, circle and sphere were found by Horowitz [27] and Coleman [8]. The corresponding problem relating to the general triangle is solved here. The procedure based on direct geometrical arguments for the solution of problems is illustrated in a case of the general triangle. This is then used to find the probability distribution of the lengths of random secants in the cases of rectangles and circles. In Chapter One, we have included the case of the distribution of the length of a random secant of a circle under I_1-randomness, where a secant is specified by a random interior point and a random direction.

In Chapter Two, we consider the problem of finding the distribution of the length L of a random secant for a regular polygon of N sides under S_2-randomness. A random secant here is defined by two random points P and Q on the perimeter of the polygon subject to the condition that with probability zero the two random points are on the same side of the polygon. The particular cases of rectangles and circles were solved by Matern [43]. It is further shown that if the polygon of N sides is always inscribed within a circle of constant radius r for any N, then the corresponding sequence of distribution functions $F_N(l)$, N = 3, 4, ..., of the random secant length L of the polygons under S_2-randomness converges to the distribution function F(l) of the random secant length L of the circle under S_2-randomness.

Since the distributions of the lengths of 'random' secants of a circle under S_1-randomness and S_2-randomness are the same, we are providing here a remarkable instance of two different sequences of distribution functions $F_N(l)$ and $F_N(l)$, N = 3, 4, ..., (which are 'paradoxical' according to Bertrand) of the 'random' secant length L of a polygon under S_1-randomness and S_2-randomness, converging to the same (hence nonparadoxical) distribution function F(l) of the random secant length L of a circle under S_1- or S_2-randomness.

In Chapter Three, we consider the problem of determining the distributions of the length L of a random ray in a random direction emanating from a 'random source' within the following configurations: rectangles, circles and spheres. In each of the cases of the circles and the spheres, the 'random source' is confined to a diametral line. The problems in geometrical probability of the distributions of the lengths of 'random rays' from 'random sources' do not appear to have been considered before.

Finally, in Applications, we refer to some practical situations where the results of the thesis can be fruitfully applied.

In order to provide the reader with a graphical view of the regions that contribute to the distribution function $F(\ell)$ of the random secant length L, we have given, in the form of appendices, descriptions of the parameter spaces for certain sample values ℓ of the random secant length L of the polygonal cases in Chapters One and Two. The discussion on the parameter spaces for different sample values ℓ of L leads to an alternative method of the solution of the problem. It may be noted that it is this method that Horowitz [27] followed to solve the corresponding problem in the case of the rectangle.

Method of Solution. The general procedure of solving problems of distributions of quantities (real) related to or functions defined on random geometrical objects (e.g., lengths of secants of a convex configuration) in geometrical probability may be described as usually involving the following steps:

1. Parametrization: First, we provide an explicit definition of "randomness" of the geometric object and then determine "judiciously" a system of coordinates which define the geometric object uniquely. The coordinates are referred to as 'parameters'. Thus we obtain a parameter space corresponding to the set of random secants of a configuration, for example. The randomness is defined by the appropriate

probability measure on the parameter space. In all of our cases in this thesis, this probability measure is the uniform measure (i.e., normalized Lebesgue measure). Parametrization of the geometric object is, in most cases, essential, since we cannot deal directly with geometrical objects. The particular way in which a 'random secant' of a regular polygon has been parametrized in this thesis may be noted.

2. Use of a direct geometrical argument for deriving the distribution of L: In the problems considered in this thesis we do not require the setting up of a functional equation of the parameters and L (cf., e.g., Horowitz [27], Coleman [8]). The problem of the determination of the distribution of the random length L of a secant or a ray of a convex configuration reduces to the determination of the set of parameters (x,θ), for example, in the case of S_1-random secant of a polygon, such that $L(x,\theta) \le \ell$ in the parameter space. In this thesis we use an argument based on geometrical intuition and derive the set $\{(x,\theta): L(x,\theta) \le \ell\}$ in the parameter space directly from the corresponding geometrical figure. We do not require a graphical description of the parameter space. However, if we discuss the sets thus obtained in the parameter space contributing to $F(\ell)$, we find an alternative way of the solution of this problem, the argument adopted by Horowitz to solve the case of the rectangle.

3. Reduction of the parameter space: In many situations there are certain symmetries of the geometrical configurations and the function defined on a relevant set of geometrical objects. In such cases, instead of considering the set of all geometric objects, we can limit ourselves to a proper subset of the geometric objects, and thus reduce the parameter space considerably (cf. Coleman [8]).

<u>An alternative view of the problem</u>. Essentially the problem of finding the probability distribution of the random length L of a secant/ray of a convex configuration can be looked upon as the problem of construction of a measure on the Borel subsets of the range I of L, where $I = [0,d]$, d is the diameter of the convex configuration.

First, to give an idea of what <u>probability</u> is here, let E be an experiment of drawing a random secant or ray from the perimeter or interior of a convex configuration C in E^n, the Euclidean n-space, where the random secant or ray is defined in any one of the above ways. Let A be an event that a random secant has a length ℓ where $0 \leq \ell \leq d$, where d is the diameter of C. The frequency F(A) of the event A in a series of observations of repetitions of the same experiment E is the number of those observations at which the event A has happened. The relative frequency f(A) of the event A in such a sequence of observations is the ratio of the frequency of A and of the total number of observations, and the <u>probability</u> is the theoretical value of long-range relative frequency.

Let Ω be the set of all random secants of a convex configuration where a random secant is defined in any one of the processes, say process (A), i.e., by a surface point and a direction. Let w denote a random secant. We parametrize w by means of the parameters (x,θ) with reference to a fixed point and a fixed direction. Thus we get a mapping $T(w) = (x,\theta)$ of the space Ω onto the parameter space. The dimension P of the parameter space S is the number of the parameters necessary to specify a secant. Let $\mathcal{A}$ be the set of inverse images under the mapping T of Borel subsets of the parameter space S, and let μ_p be defined on $\mathcal{A}$ by putting

$$\mu_p(A) = \frac{1}{V_p}\lambda_p(T(A)) \quad \text{for} \quad A \in \mathcal{A} \ ,$$

where λ_p is the p-dimensional Lebesgue measure in the Euclidean space R^p and V_p is the p-dimensional Lebesgue measure of the parameter space S. Then $(\Omega, \mathcal{A}, \mu_p)$ is a probability space. The probability space $(\Omega, \mathcal{A}, \mu_p)$ which is constructed this way may be interpreted as describing an experiment in which a secant is chosen at random by a perimeter point and a direction each having uniform distributions. Now with each secant ω we associate a value ℓ of random secant length L. L is then a function defined on the space Ω with values ℓ in the interval $I = [0,d]$, where d is the diameter of the configuration. If B is any Borel subset of I, then $L^{-1}(B) \in \mathcal{A}$. Thus L is a random variable defined on the experiment $(\Omega, \mathcal{A})$. X, Θ, L are therefore functions defined on the probability space S, equipped with a normalised Lebesgue measure; X, Θ being the projections. The function L carries a measure induced by X and Θ from the parameter space S to I. The distribution function $F(\ell)$ of L uniquely defines this measure $\mathcal{L}$ on the Borel subsets of the interval I.

<u>Future of the Work</u>. In this thesis, we investigate and discover the regular features of certain random phenomena, viz. the lengths of secants or rays, related to certain geometrical configurations and see how the regularity changes from one geometrical configuration to another. Hopefully, this investigation will raise more questions and inspire further work in the area of geometrical probability, especially concerning random secants or rays in the cases of higher dimensional convex bodies. From a knowledge of the regular features of certain random phenomena, e.g., the lengths of secants or rays, one could possibly try

to characterize the corresponding convex domains. The consideration of the problems on random secants and rays and their solutions presents an interesting direction of work in the area of geometric probability.

CHAPTER 1

1. DISTRIBUTION OF LENGTHS OF S_1-RANDOM SECANTS.

1.0. Introduction.

This chapter is devoted to finding the probability distributions of lengths of random secants of some compact convex geometrical configurations. The general problem when stated precisely is as follows:

Consider a compact convex body K in an n-dimensional Euclidean space with a non-void interior. Let S denote the surface of K. Choose a point P at random on the surface of the body and consider a straight line G in a random direction through P. The random line G will intersect the body at another point Q. Consider the length $L = |PQ|$. Let P be determined by $\underline{X}$ and the direction of G by $\underline{\theta}$ with respect to a set of suitably chosen axes of reference (i) $\underline{X}$ is uniformly distributed on S, (ii) $\underline{\theta}$ is uniformly distributed on its domain, (iii) the distributions of $\underline{X}$ and $\underline{\theta}$ are independent. The set of all possible values $(\underline{x}, \underline{\theta})$ of $(\underline{X}, \underline{\theta})$ determining G such that $G \cap K \neq \phi$ is the parameter space Ω. The parameter space Ω, equipped with a normalized Lebesque measure, is a probability space. L is a function on Ω. In fact, L is a random variable. Our object is to find the probability distribution function $F(\ell)$ of L induced by the probability distributions of $\underline{X}$ and $\underline{\theta}$.

The procedure for solving this problem can be stated, in general. $Pr(L<\ell)$ depends on the Lebesque measure of the set $\{(\underline{x},\underline{\theta}):L(\underline{x},\underline{\theta}) < \ell\}$, where $L(\underline{x},\underline{\theta})$ is the length of the secant determined by $(\underline{x},\underline{\theta})$. Although this can be stated in general, there is no known method that will provide the answer in general for all convex bodies. The problem of finding the appropriate set $\{(\underline{x},\underline{\theta}): L(\underline{x},\underline{\theta}) < \ell\}$ in the parameter space is particularly difficult even for simple configurations in R^2 and demands a considerable amount of intuition. For example, in the case of a regular polygon, where conditions on its sides are as ideal as possible, the distribution function of L is given by (1.1.1) of this chapter. This probability law is rather complicated. In the case of an irregular polygon, it has not been possible to find the distribution function of L.

SECTION ONE

1.1. DISTRIBUTIONS OF LENGTHS OF S_1-RANDOM SECANTS OF A REGULAR POLYGON.

1.1.0. Introduction.

In this section, the probability distribution of the length of a random secant of a regular polygon of N sides, where a side is of length a, under S_1-randomness is obtained. Here, a random secant is defined by a random point on the perimeter and a random direction. A precise statement of the problem is the following.

Let a point P be chosen at random on the perimeter of a regular polygon of N sides. A ray emanates in a random direction from P and intersects a side of the polygon at another point Q. The length $L = |PQ|$ is a random variable. The problem is to determine the probability distribution of the random variable L.

1.1.1. The Probability Distribution of L.

Theorem 1. The probability distribution function $F_N(\ell)$ of the random secant length L of a regular polygon of N sides under S_1-randomness is given by

$$
(1.1.1)\quad F_N(\ell) \begin{cases} = \phi_1(\ell) & \text{for } \ell \in [0, \ell_1], & \text{(A)} \\ = \phi_2(\ell) & \text{for } \ell \in [\ell_{k-1}, \ell_k],\ k = 2,3,\dots,n-1 & \text{(B)} \\ = \phi_3(\ell) & \text{for } \ell \in [\ell_{n-1}, \ell_n],\ N \text{ even} & \text{(C)} \\ = \phi_4(\ell) & \text{for } \ell \in [\ell_{n-1}, \ell_{n-1,1}],\ N \text{ odd} & \text{(D)} \\ = \phi_5(\ell) & \text{for } \ell \in [\ell_{n-1,1}, \ell_{n-1,2}],\ N \text{ odd} & \text{(E)} \\ = \phi_6(\ell) & \text{for } \ell \in [\ell_{n-1,2}, \ell_n],\ N \text{ odd}, & \text{(F)} \end{cases}
$$

where

$$
(1.1.1a)\quad \phi_1(\ell) = \frac{2\ell}{a\tau}\tan \xi ,
$$

(1.1.1b) $$\psi_2(\ell) = \frac{2}{a\pi} [2ka\delta - 2\delta(a - x_k(\ell)) + W(k) + d_k \phi(0,k) - (x_k(\ell) + d_k)\phi(x_k(\ell),k) + \ell\{U(0,k) - U(x_k(\ell),k)\} / S(k)],$$

(1.1.1c) $$\psi_3(\ell) = \frac{2}{a\pi} [a(\pi/2 - \beta_n) + x_n(\ell)\beta_n + 2(n+1)\delta(a - x_n(\ell)) + W(n)],$$

(1.1.1d) $$\psi_4(\ell) = \frac{2}{a\pi} [x_n(\ell)\beta_n + 2(n-1)\delta(a - x_n(\ell)) + 2d_n(\beta_n - n\delta) + \ell\{\cos\beta_n - \cos(2n\delta - \beta_n)\} / S(n) + W(n)],$$

(1.1.1e) $$\psi_5(\ell) = \frac{2}{a\pi} [\beta_n x_n(\ell) + 2(n-1)\delta(a - x_n(\ell)) + 2(\beta_n - n\delta)d_n + \ell(\cos\beta_n - \cos(2n\delta - \beta_n))/S(n) + W(n)],$$

(1.1.1f) $$\psi_6(\ell) = \frac{2}{a\pi} [a(\theta_1 - \theta_2 + \theta_3 - \theta_5 + \pi/2) + d_{n-1}(\theta_1 - \beta_n) + d_n(\beta_n - \theta_2 + \theta_3 + \theta_4 - 2\theta_5) + \frac{\ell}{S(n-1)}\{\cos(2(n-1)\delta - \beta_n) - \cos(2(n-1)\delta - \theta_1)\} + \frac{\ell}{S(n)}\{\cos(2n\delta - \theta_2) - \cos(2n\delta - \beta_n) + \cos(2n\delta - \theta_4)$$

$$- \cos (2n\delta - \theta_3) + \cos (2n\delta - \theta_5) - \cos (2n\delta - \theta_4)$$

$$+ \cos (2n\delta + \theta_5) - \cos (2n\delta + \theta_4)\}] ,$$

where:

$\delta = \frac{\pi}{N}$, $n = \frac{N}{2}$, or $\frac{N-1}{2}$ for N even or odd respectively,

a = length of a side of the polygon,

$\ell_k = \frac{a \sin k\delta}{\sin \delta}$ for $k = 1, 2, \ldots, n-1$; $\ell_{n-1,1} = - \ell_{n-1} / \cos 2n\delta$,

$\ell_{n-1,2} = a \tan (n\delta) / 2$, $d_k = \ell_{k-1} \frac{\sin k\delta}{\sin 2k\delta}$ for $k = 1, 2, \ldots, n$;

$\beta_k = \sin^{-1} (\frac{\ell_{k-1} \sin k\delta}{\ell})$ for $k = 1, 2, \ldots, n$;

$x_k(\ell) = \frac{\ell \sin (k\delta - \beta_k)}{\sin k\delta}$ for $k = 1, 2, \ldots, n$;

$\phi(x_k(\ell), i) = \sin^{-1} \{(x_k(\ell) + d_i) \sin (2i\delta)/\ell\}$ for $i, k = 1, 2, \ldots, n$,

$S(i) = \sin 2i\delta$, $i = 1, 2, \ldots, n$,

$U(x_k(\ell), i) = [1 - (x_k(\ell) + d_i)^2 \sin^2 (2i\delta)/\ell^2]^{\frac{1}{2}}$ $i, k = 1, 2, \ldots, n$,

$$\theta_1 = 2(n-1) - \phi(a,(n-1)), \quad \theta_2 = (2n-1)\delta/2 - \cos^{-1}\left(\frac{\ell_{n-1,2}}{\ell}\right),$$

$$\theta_3 = \frac{\pi}{2} - \delta + \cos^{-1}(\ell_{n-1,2}/\ell), \quad \theta_4 = \delta + \phi(a, n-1),$$

$$\theta_5 = \sin^{-1}(\ell_{n-1,2}/\ell), \text{ and}$$

$$W(k) = (x_k(\ell) + d_{k-1})\,\phi\,(x_k(\ell), k-1) - (a + d_{k-1})\,\phi\,(a, k-1)$$

$$+ \frac{\ell}{S(k-1)}\{U(x_k(\ell), k-1) - U(a, k-1)\}.$$

The proof of this theorem will need the following: parameterization of the secants, reduction of the parameter space, certain properties of a regular polygon, decomposition of the range of L, determination of the set of parameters for which $L < \ell$ for different intervals of ℓ.

1.1.2. Parametrization.

The secants are parametrized conveniently as follows. Let $A_0A_1 \ldots A_{N-1}$ be the regular polygon (cf. Figure 1). Let a be the length of each side of the polygon. Let X be the distance of the point P measured along the perimeter from the reference point A_1 in the direction A_1A_0 of the polygon. From the point P a ray is drawn making the angle θ with the side of the polygon on which P lies as measured in the counter-clockwise direction.

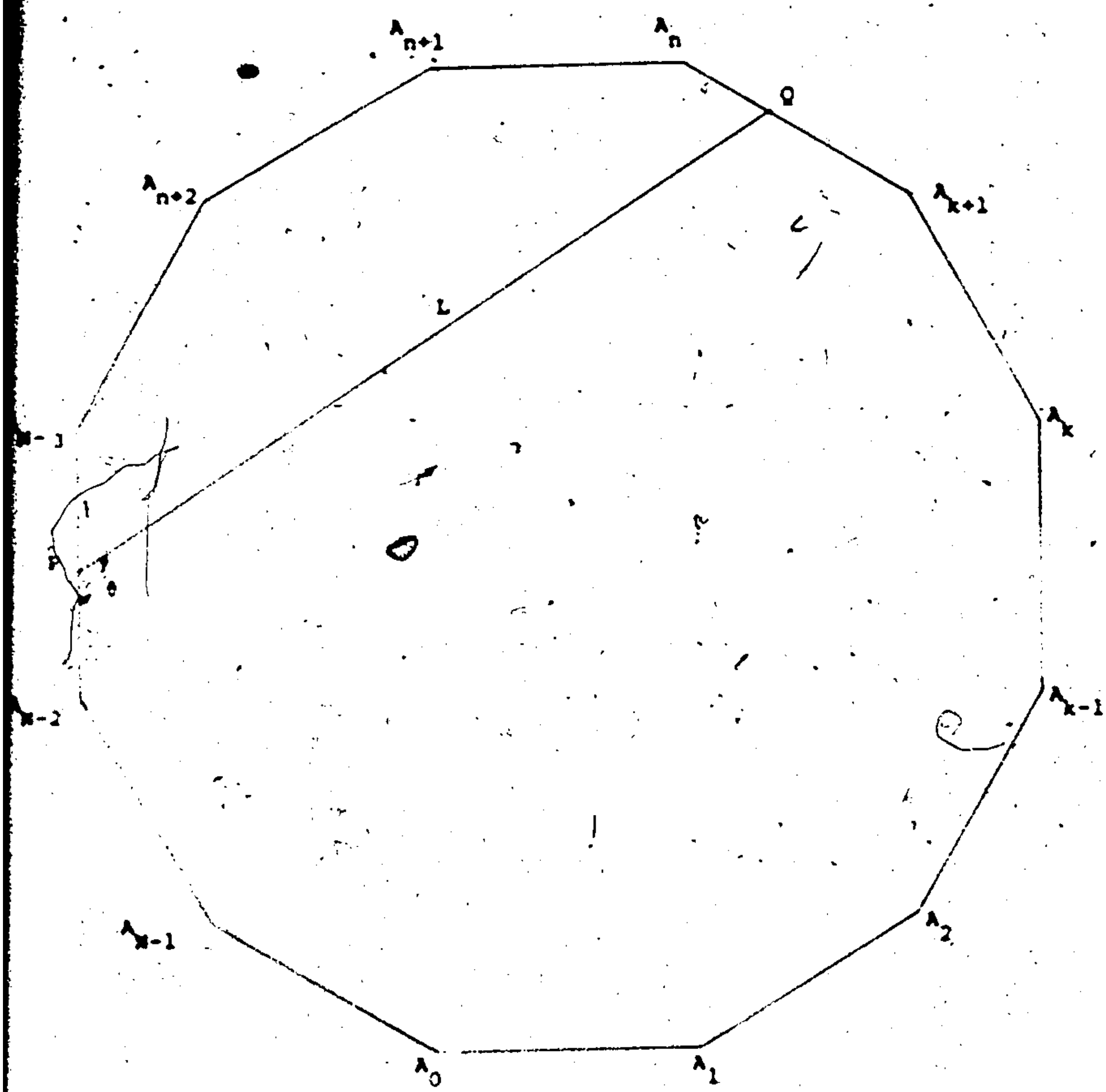

Fig. 1

We assume that

(1) X is uniformly distributed on $[0, Ma]$,

(2) θ is uniformly distributed on $[0, \pi]$, and

(3) the distributions of X and θ are independent.

The joint density of X and θ is given by

(1.1.2)
$$P(x,\theta) = \begin{cases} \dfrac{1}{Ma\pi} \text{ for } \{(x,\theta): 0 < x < Ma,\ 0 < \theta < \pi\}, \\ 0 \text{ elsewhere in the } (x,\theta)\text{-plane.} \end{cases}$$

Clearly, L is a function of X and θ, and to emphasize this fact we denote L by $L(x,\theta)$.

Our object is to find

(1.1.3)
$$F_M(\ell) = P_r(L < \ell) = \frac{1}{Ma\pi}\int_{D(\ell)} dx\, d\theta \ ,$$

where

(1.1.4)
$$D(\ell) = \{(x,\theta): L(x,\theta) < \ell,\ 0 < x < Ma,\ 0 < \theta < \pi\}.$$

The set

(1.1.5)
$$S' = \{(x,\theta): 0 < x < Ma,\ 0 < \theta < \pi\}$$

will be called the <u>parameter space</u>.

1.1.3. Reduction of the Parameter Space.

Using symmetry we can reduce the parameter space. We write $D(\ell)$ defined in (1.1.4) as the union of 2N "equal sets", disjoint except possibly for boundary points, as

$$D(\ell) = \bigcup_{\substack{i=1,2,\ldots,N \\ j=1,2}} D_{ij}(\ell) , \tag{1.1.6}$$

where

$$D_{ij}(\ell) = \{(x,\theta) : L(x,\theta) < \ell, \ (i-1)\,a < x < ia, \ (j-1)\frac{\pi}{2} < \theta < j\frac{\pi}{2}\}, \tag{1.1.7}$$

$$i = 1, 2, \ldots, N, \quad j = 1, 2.$$

We now prove the following:

<u>Lemma 1</u>. Let $F_N(\ell)$ be the distribution function of L. Then

$$F_N(\ell) = \frac{1}{Na\pi}\int_{D(\ell)} dx\, d\theta = \frac{2}{a\pi}\int_{D_{11}(\ell)} dx\, d\theta \tag{1.1.8}$$

<u>Proof</u>. We first show that

$$(1.1.9)\qquad \int_{D_{11}(\ell)} dx\, d\theta = \int_{D_{ij}(\ell)} dx\, d\theta\,, \quad i = 1, 2, 3, \ldots, N\ ;\ j = 1, 2 .$$

Since $L(x,\theta) = L((i-1)a + x,\theta)$, $i = 1, 2, \ldots, N$ for $0 < x < a$, and θ fixed in $[0,\pi]$ (cf. Figure 2), we have

$$(1.1.10)\qquad \int_{D_{1j}(\ell)} dx\, d\theta = \int_{D_{ij}(\ell)} dx\, d\theta\,, \quad i = 1, 2, \ldots, N .$$

Also, since $L(x,\theta) = L(a - x, \pi - \theta)$, for $0 < x < a$, $0 < \theta < \pi/2$ we have

$$(1.1.11)\qquad \int_{D_{11}(\ell)} dx\, d\theta = \int_{D_{12}(\ell)} dx\, d\theta .$$

Combining (1.1.10) and (1.1.11), we have (1.1.9). Therefore

$$F_N(\ell) = \frac{1}{Na\pi} \int_{D(\ell)} dx\, d\theta = \frac{1}{Na\pi} \int_{\substack{\bigcup D_{ij}(\ell) \\ i = 1,2,\ldots,N \\ j = 1,2.}} dx\, d\theta\,, \quad i = 1, 2, \ldots, N$$

$$= \frac{2}{a\pi} \int_{D_{11}(\ell)} dx\, d\theta .$$

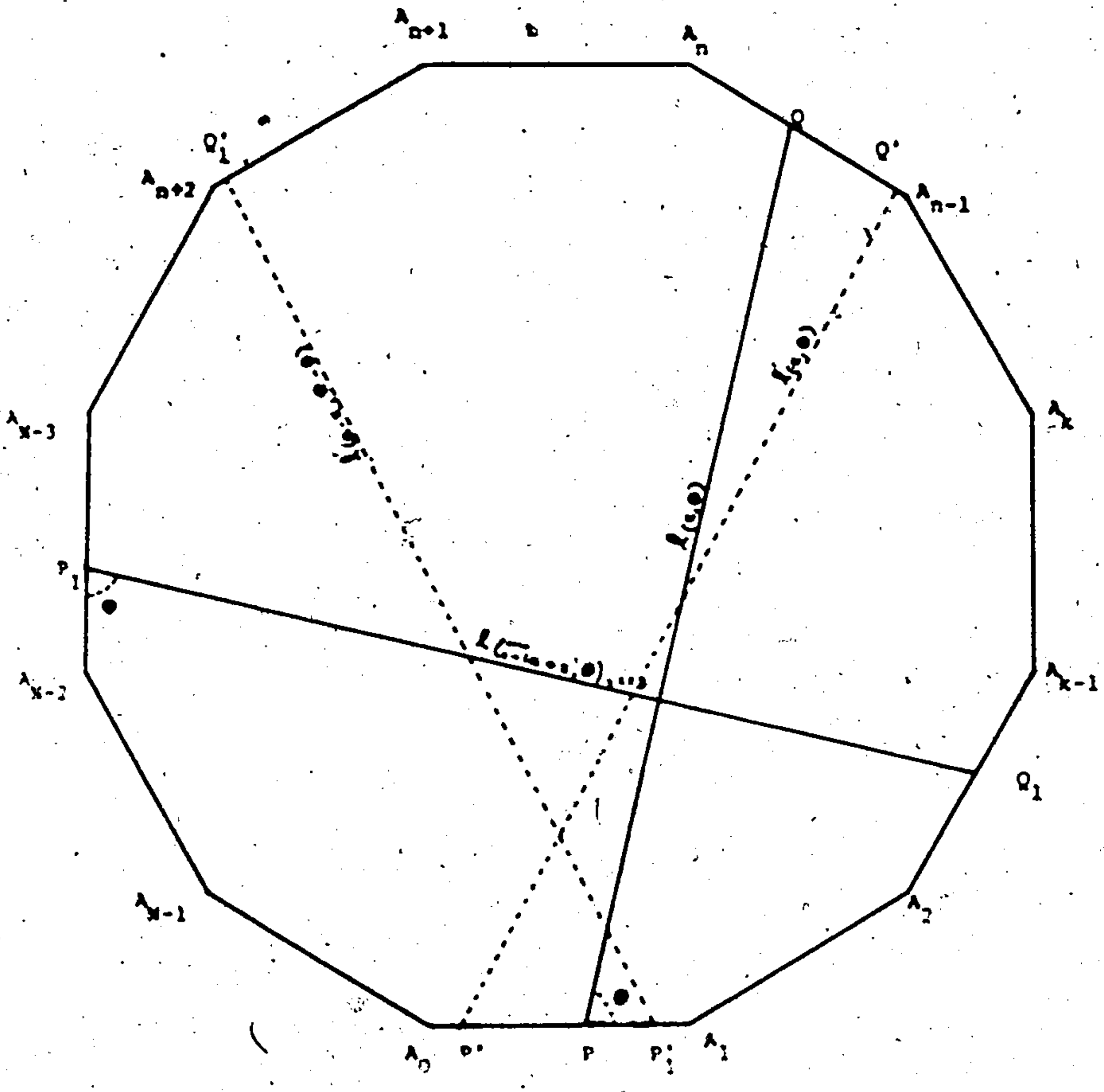

Fig. 2

<u>Definition</u>. The set

$$S = \{(x,\theta) : 0 < x < a,\ 0 < \theta < \pi/2\} \tag{1.1.12}$$

is called the reduced parameter space. The reduced parameter space will, in short, be referred to as the parameter space. For a diagrammatic exposition of random secants of equal length emanating from different points on a side of a regular polygon, refer to Figures 2A, 2B, and 2C.

<u>1.1.4</u>. <u>Certain Properties of a Regular Polygon</u>.

<u>Lemma 2.</u> Let $A_0 A_1 \ldots A_{N-1}$ be the regular polygon (cf. Figure 3). Let A_0 be joined to $A_2, A_3 \ldots A_{N-1}$. Let $\ell_k = |A_0 A_k|$, $k = 1, 2, \ldots N-1$. Extend $A_{k+1} A_k$ to meet $A_0 A_1$ extended at D_k, $k = 1, 2, \ldots N-1$, where $N = 2n$ or $2n + 1$ for N even or odd, respectively. Geometrical considerations for the regular polygon yield the following:

(1.1.13) $\measuredangle A_1 A_0 A_2 = \angle A_k A_0 A_{k+1} = \delta = \frac{\pi}{N}$, $k = 1,2,\ldots N-2$.

(1.1.14) $\measuredangle A_1 A_0 A_k = (k-1)\delta$, $k = 1,2,\ldots N-1$.

(1.1.15) $\measuredangle A_0 A_k A_{k+1} = \pi - (k+1)\delta$, $k = 1,2,\ldots N-1$.

(1.1.16) $\measuredangle A_0 A_1 A_k = \pi - k\delta$, $k = 2,\ldots N-1$.

(1.1.17) $\measuredangle A_1 D_k A_k = \pi - 2k\delta$, $k = 2,3,\ldots n$.

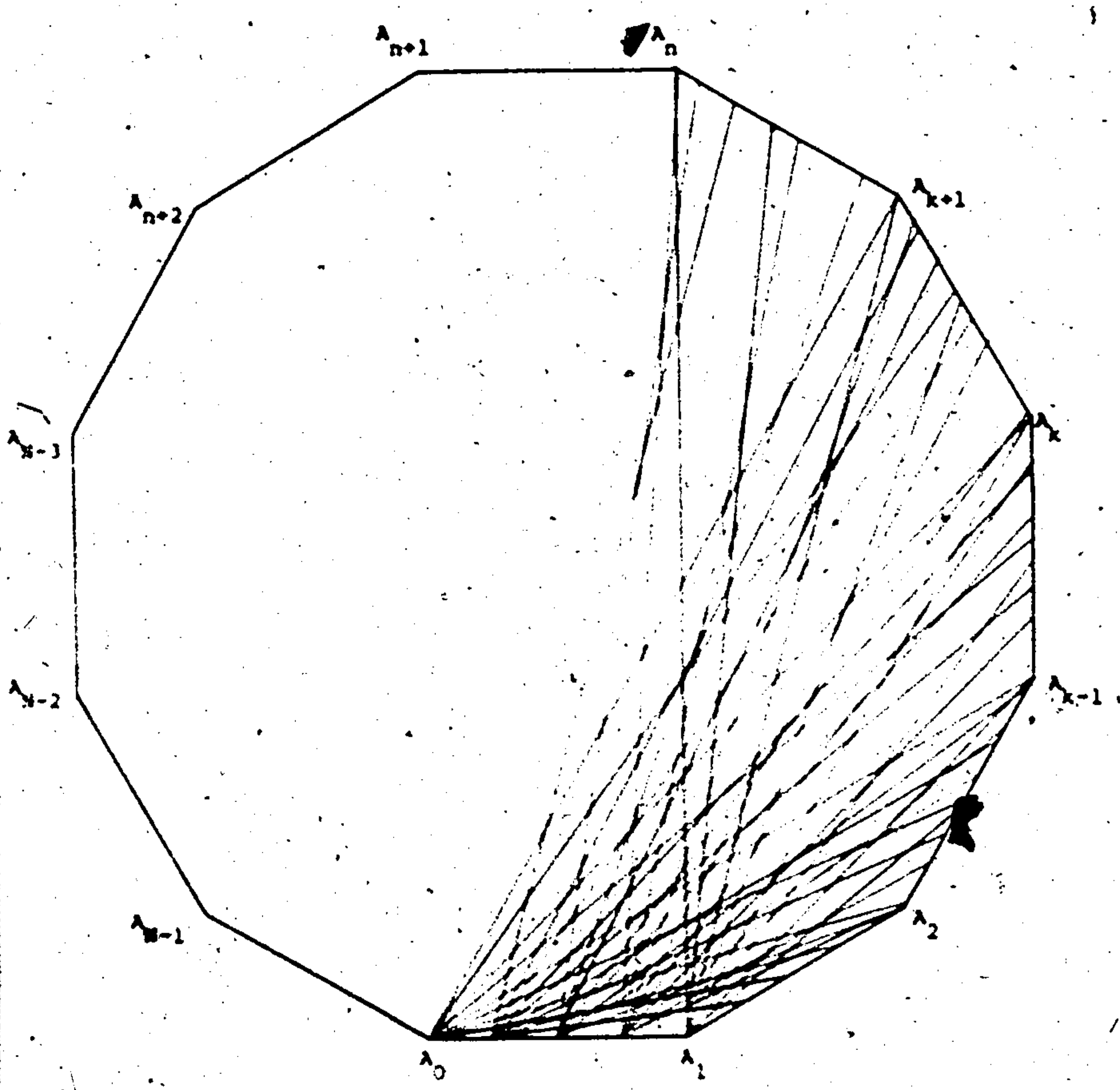

Fig. 2A

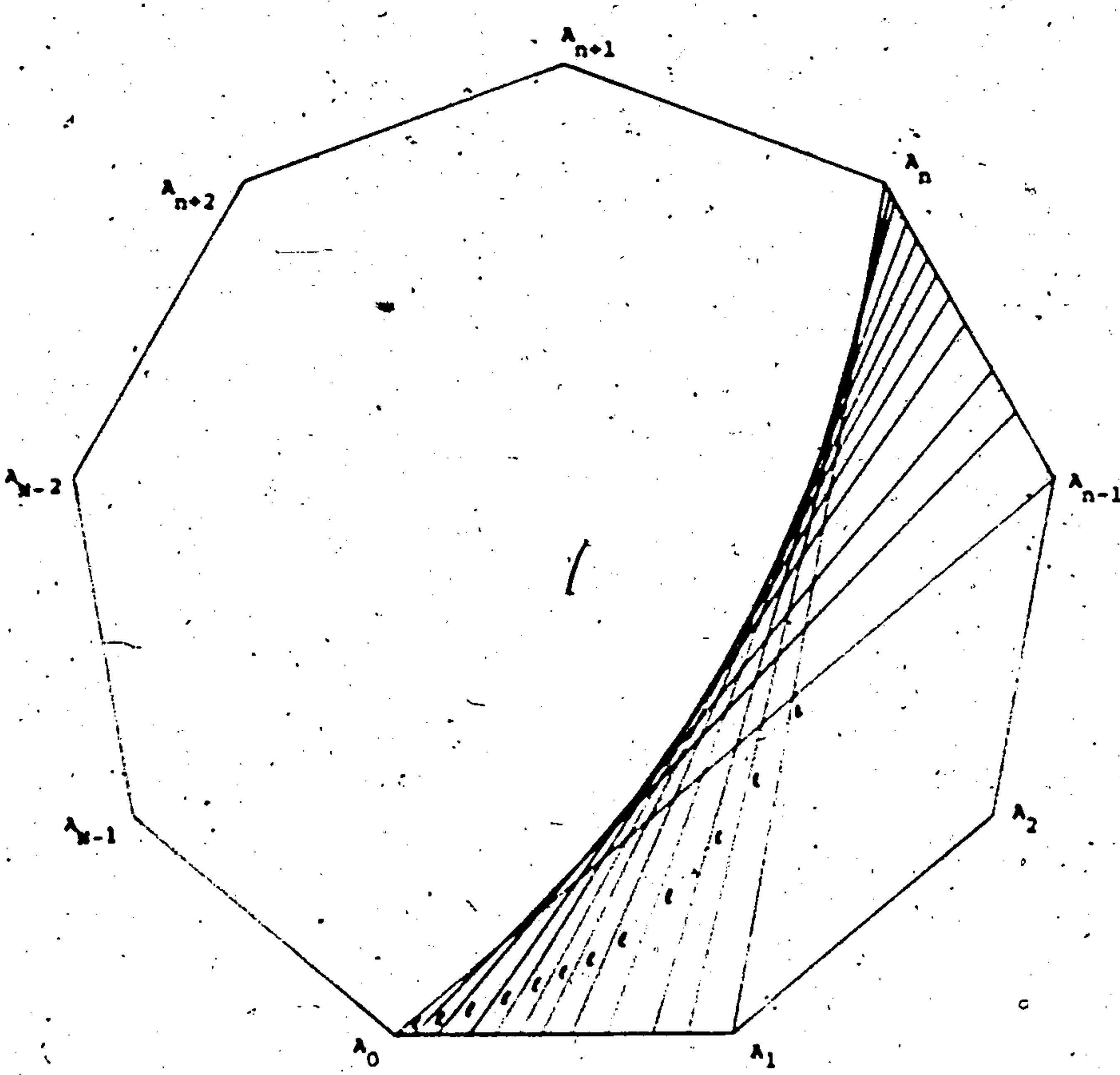

Fig. 2B

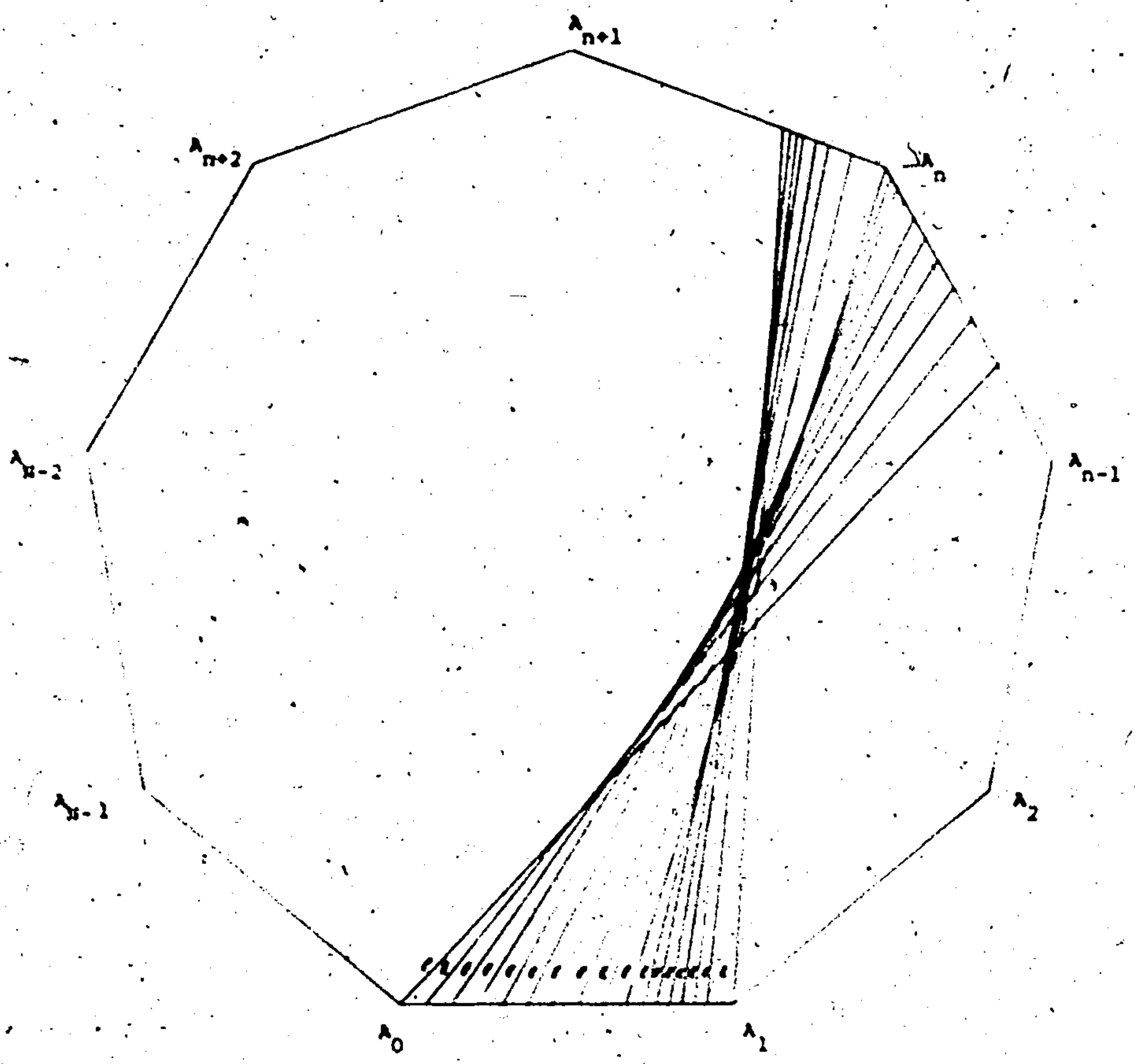

Fig. 2C

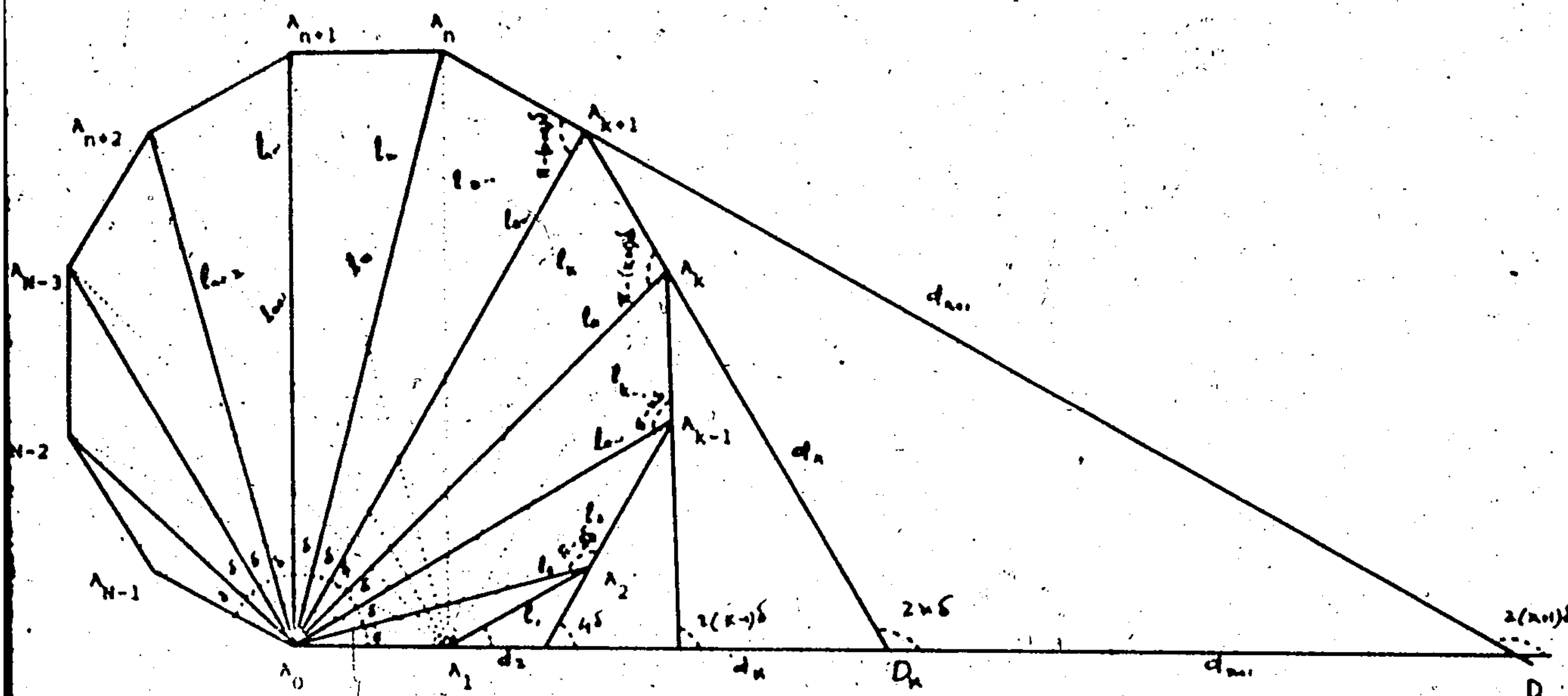
A_{n+1}
A_n
A_{n+2}
A_{k+1}
A_{N-3}
A_k
A_{k-1}
$N-2$
A_{N-1}
A_2
A_0
A_1
d_2
d_k
D_k
D_{n+1}
d_{n+1}
d_n
4δ
$2(k-1)\delta$
$2n\delta$
$2(n+1)\delta$

Fig. 3

(1.1.18) $$\ell_k = \frac{a \sin k\delta}{\sin \delta}, \qquad k = 1,2,\ldots N-1 ,$$

(1.1.19) $$\ell_{n-k} = r(x^k + \frac{1}{x^k}), \qquad k = 1,2,\ldots n-1$$ where $N = 2n$, r is the radius of the circumscribing circle and $x = e^{i\pi/N}$,

(1.1.20) $$d_k = \frac{\sin k\delta}{\sin 2k\delta} \cdot \ell_{k-1} , \qquad k = 1,2,\ldots n.$$

1.1.5. Decomposition of the Range of L.

In order to find the distribution function $F_N(\ell)$, we need to find the set $D_{11}(\ell)$, defined in (1.1.7), in the reduced parameter space S, which is defined in (1.1.12). To find $D_{11}(\ell)$ in S we decompose the interval $[0,\ell_n]$, the range of L, as follows:

(1.1.21) $$[\ell_0,\ell_n] = \bigcup_{k=1}^{n} [\ell_{k-1},\ell_k] .$$

where $N = 2n$ or $2n + 1$ for N even or odd respectively and ℓ_k is given by (1.1.18).

We now prove a few lemmas for determining the set $D_{11}(\ell)$. A graphical description of the set $D_{11}(\ell)$ is provided in each case for ℓ lying in different intervals of the range of L.

1.1.6. Determination of the Set $D_{11}(\ell)$ for $\ell \in [0,\ell_1]$.

In order to find the distribution function $F_N(\ell)$ of L for $\ell \in [0,\ell_1]$ we require the set $\{(x,\theta): L(x,\theta) < \ell\}$. In the following lemma we obtain this set.

Lemma 3. Let $\ell \in [0,\ell_1]$. Then

(1.1.22) $$D_{11}(\ell) = S_{11}(\ell) ,$$

where

(1.1.23) $$S_{11}(\ell) = \{(x,\theta): 0 < x < \ell,\ 0 < \theta < \frac{2\pi}{N} - \sin^{-1}(\frac{x \sin \pi/N}{\ell})\}.$$

Proof. (cf. Figure 4). Let B be the point on $A_1 A_0$ such that $|A_1 B| = \ell$. Let P be a point on segment $A_1 B$. The point P determines a distance $X = x$ from A_1. To emphasize this we denote P by P_x. Obviously, $0 < x < \ell$. With P_x as the centre we draw a circle of radius ℓ. The circle intersects $A_1 A_2$ at a point Q. (We need only be concerned with the point Q on $A_1 A_2$ since our parameter space involves θ only for $0 < \theta < \pi/2$.) Then from the triangle $QP_x A_1$ denoting $\angle QP_x A_1$ by $\theta_1(x,\ell)$, we have by the sine law

(1.1.24) $$\ell = \frac{x \sin 2\delta}{\sin (2\delta - \theta_1(x,\ell))}$$

so that

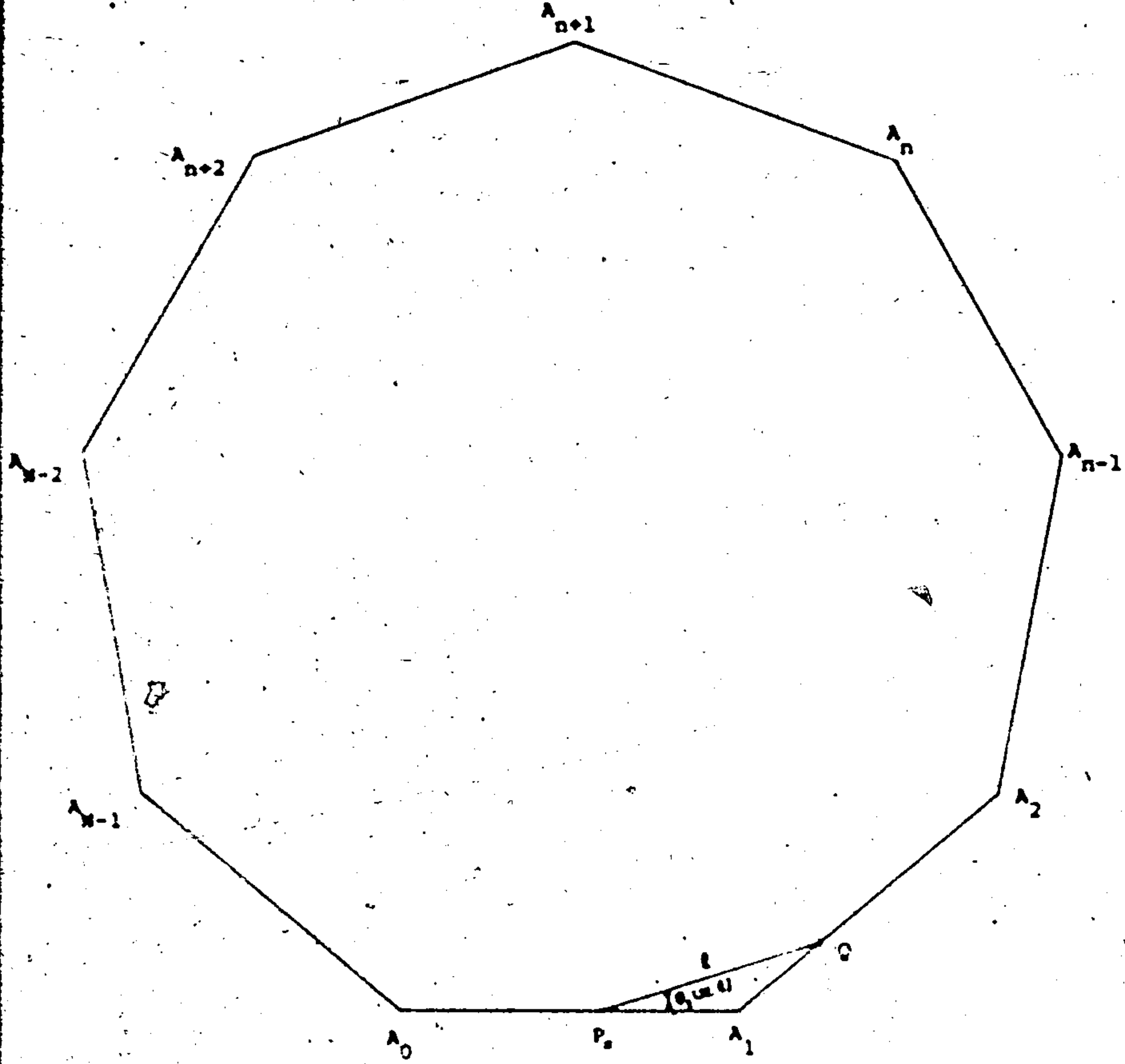

FIG. 4

(1.1.25) $$\theta_1(x,\ell) = 2\delta - \sin^{-1}\left(\frac{x \sin 2\delta}{\ell}\right) .$$

Now in (1.1.24)

$$0 < \theta < \theta_1(x,\ell) \Rightarrow \sin(2\delta - \theta) > \sin(2\delta - \theta_1(x,\ell)) .$$

Consequently,

$$L(x,\theta) < \ell \quad \text{for} \quad 0 < \theta < \theta_1(x,\ell), \quad 0 < x < \ell .$$

Therefore,

(1.1.26) $$(x,\theta) \in S_{11}(\ell) \Rightarrow L(x,\theta) < \ell .$$

Thus,

(1.1.27) $$S_{11}(\ell) \subset D_{11}(\ell) .$$

From the relation (1.1.24), it follows that:

$$2\delta > \theta > \theta_1(x,\ell) \Rightarrow \sin(2\delta - \theta)) < \sin(2\delta - \theta_1(x,\ell)) .$$

and consequently:

$$L(x,\theta) > \ell \quad \text{for} \quad \theta_1(x,\ell) < \theta < 2\delta, 0 < x < \ell .$$

Also, a ray from a point P_x, where $\ell < x < a$, has the corresponding length $L(x,\theta) > \ell$, since $|A_1P_x| = x > \ell$. Thus:

(1.1.28) $$(x,\theta) \notin S_{11}(\ell) \Rightarrow L(x,\theta) > \ell .$$

Therefore,

(1.1.29) $$S_{11}(\ell) \supset D_{11}(\ell) .$$

Combining (1.1.27) and (1.1.29), we obtain (1.1.22) and (1.1.23). For a graphical description of the set $D_{11}(\ell)$ for $\ell \in [0,\ell_1]$ see Appendix 1.1.6A.

1.1.6A. Graphical Description of the Set $D_{11}(\ell)$ for $\ell \in [0,\ell_1]$.

We provide a graphical description of the set $D_{11}(\ell)$ (contributing to Pr $(L < \ell)$) in the parameter space S for certain values of $\ell \in [\ell_{k-1},\ell_k]$, k = 1, 2,...,n.

For $\ell \in [0,\ell_1]$, we have in Lemma 3

$$D_{11}(\ell) = S_{11}(\ell) \quad (1.1.22)$$

where:

$$S_{11}(\ell) = \{(x,\theta): 0 < x < \ell,\ 0 < \theta < 2\delta - \sin^{-1}(\frac{x \sin 2\delta}{\ell})\}. \quad (1.1.23)$$

From (1.1.23) it follows that the set of points in the parameter space S satisfying the equation:

$$x = \frac{\ell \sin (2\delta - \theta)}{\sin 2\delta} \quad (1.1.6A.1)$$

which is obtained by equating θ with $2\delta - \sin^{-1}(\frac{x \sin 2\delta}{\ell})$, map onto ℓ.

Let OABC be the parameter space S (cf. Fig. 5). The curve

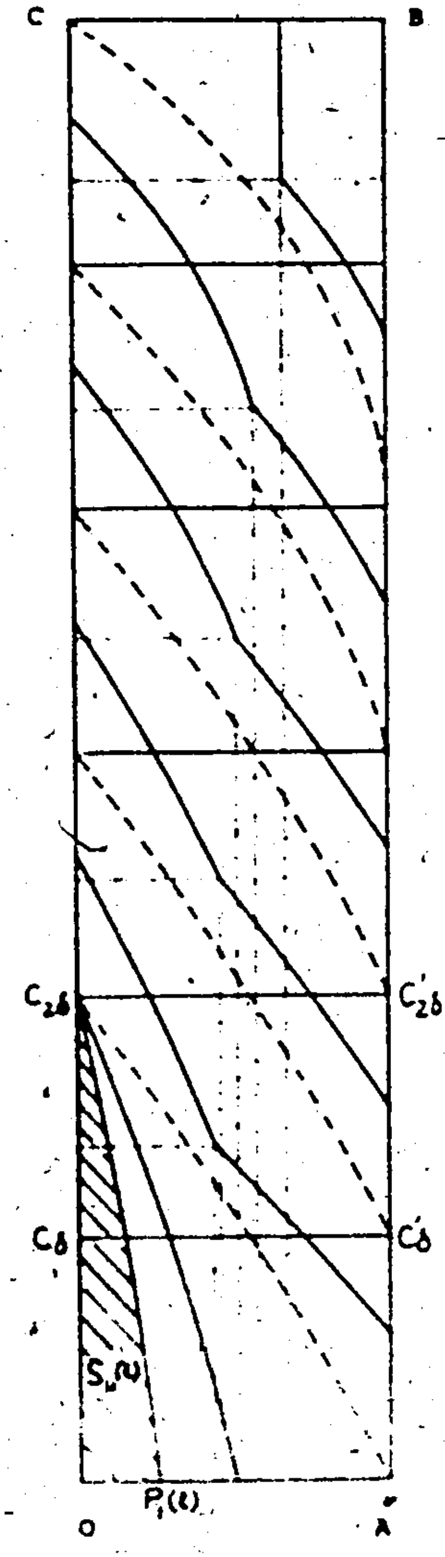

Fig. 5

(1.1.6A.1) passes through $(0,2\delta)$ and $(\ell,0)$ which we denote by $C_{2\delta}$ and $P_1(\ell)$ respectively. It may be noted that as ℓ increases from 0 to ℓ_1, $P_1(\ell)$ moves along OA from 0 to A while $C_{2\delta}$ remains fixed.

The set $S_{11}(\ell)$ is bounded by $x = 0$, $\theta = 0$ and the sine curve given by (1.1.6A.1). In what follows we describe the set $S_{11}(\ell)$ for extreme values of ℓ satisfying $0 < \ell < \ell_1$.

We note that $\ell = 0$ implies $x = 0$ and θ varies from 0 to 2δ. Consequently

(1.1.6A.2) $$S_{11}(0) = \{(0,\theta);\ 0 < \theta < 2\delta\}$$

which is the line segment $OC_{2\delta}$ of the θ-axis.

$\ell = \ell_1$ implies $0 < x < \ell_1$ and the curve (1.1.6A.1) passes through $(a,0)$. Therefore

(1.1.6A.3) $$S_{11}(\ell_1) = \{(x,\theta);\ 0 < x < a,\ 0 < \theta < 2\delta - \sin^{-1}\left(\frac{x \sin 2\delta}{\ell_1}\right)\}.$$

The set $S_{11}(\ell_1)$ is represented by $OAC_{2\delta}$ in Figure 5.

1.1.7. <u>Determination of the Set $D_{11}(\ell)$ for $\ell \in [\ell_{k-1},\ell_k]$, $k = 2,3,\ldots,n-1$.</u>

In order to find the distribution function $F_N(\ell)$ of L for $\ell \in [\ell_{k-1},\ell_k]$, we need the set $D_{11}(\ell) = \{(x,\theta);\ L(x,\theta) < \ell\}$. In the following lemma we obtain this set.

<u>Lemma 4.</u> Let $\ell \in [\ell_{k-1}, \ell_k]$, $k = 2,3,\ldots, n-1$, where $n = N/2$ if N is even and $n = (N-1)/2$ if N is odd. Then

$$D_{11}(\ell) = S_{k1}(\ell) \cup S_{k2}(\ell) \cup S_{k3}(\ell) \; . \tag{1.1.30}$$

where

$$S_{k1}(\ell) = \{(x,\theta) : 0 < x < a,\; 0 < \theta < \alpha_k(\ell)\} \; . \tag{1.1.31}$$

$$S_{k2}(\ell) = \{(x,\theta) : x_k(\ell) < x < a,\; \alpha_k(\ell) < \theta < \theta_k(x,\ell)\}, \tag{1.1.32}$$

$$S_{k3}(\ell) = \{(x,\theta) : 0 < x < x_k(\ell),\; \alpha_k(\ell) < \theta < \theta'_k(x,\ell)\} \; , \tag{1.1.33}$$

where

$$\alpha_k(\ell) = 2(k-1)\delta - \sin^{-1}\left(\frac{\ell_{k-1} \sin k\delta}{\ell}\right) \; ,$$

$$\theta'_k(x,\ell) = 2k\delta - \sin^{-1}\left\{\left(\frac{x + d_k}{\ell}\right) \sin 2k\delta\right\} \; .$$

$$\theta_k(x,\ell) = 2(k-1)\delta - \sin^{-1}\left\{\frac{(x+d_{k-1})}{\ell} \sin 2(k-1)\delta\right\}$$

and

$$x_k(\ell) = \frac{\ell}{\sin k\delta} \sin\left\{k\delta - \sin^{-1}\left(\frac{\ell_{k-1} \sin k\delta}{\ell}\right)\right\} \; .$$

<u>Proof.</u> First we show that $S_{ki} \subset D_{11}(\ell)$, $i = 1,2,3$. With A_0 as the centre we draw a circle of radius ℓ (cf. Figure 6.) The circle intersects the side $A_{k-1} A_k$ of the polygon at a point $Q_k(\ell)$. Let $\angle Q_k(\ell) A_0 A_1 = \alpha_k(\ell)$.

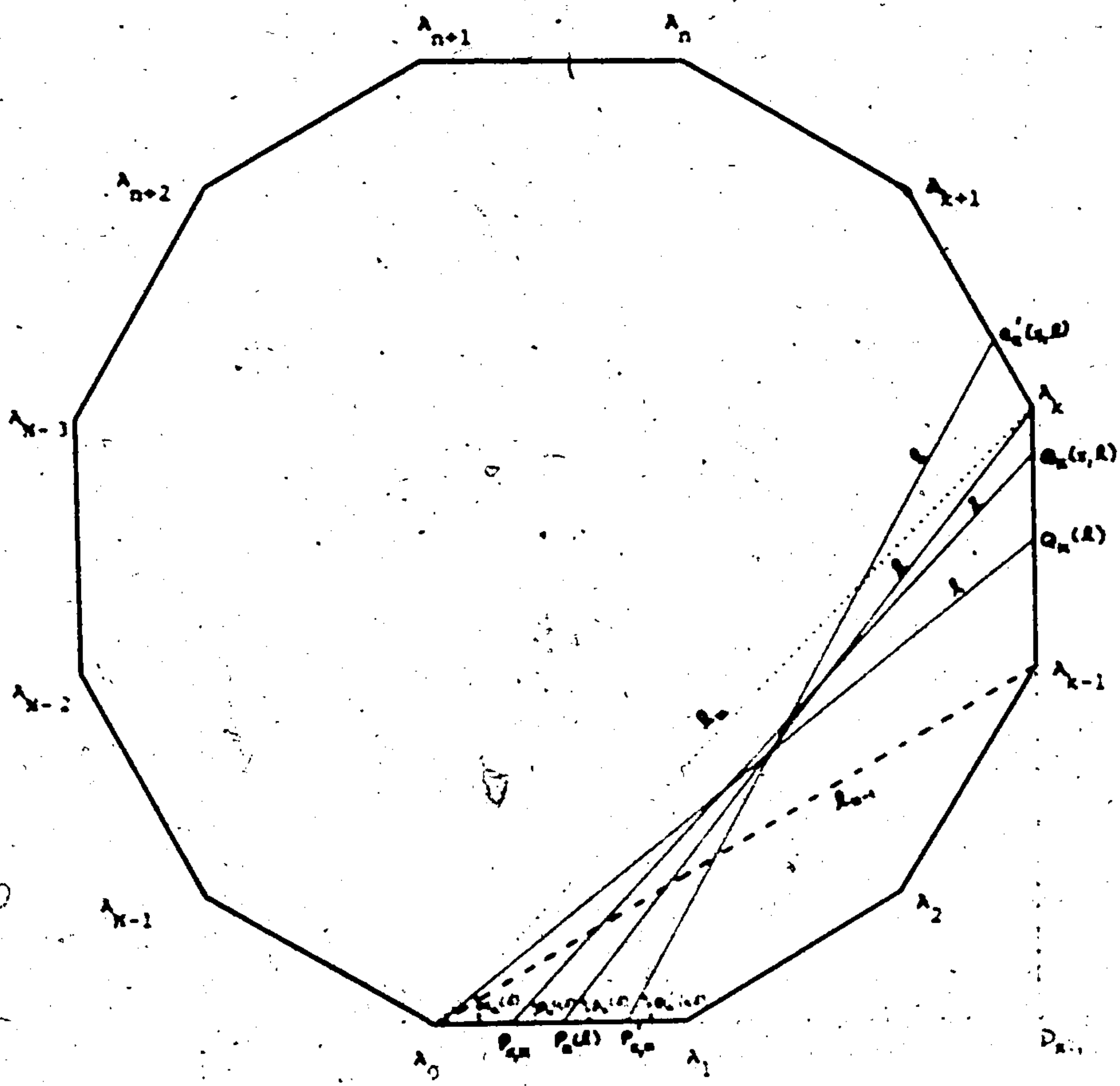

Fig. 6

Then, using trigonometry, from the triangle $A_0 A_{k-1} Q_k(\ell)$, we have

$$\ell = \frac{\ell_{k-1} \sin k\delta}{\sin \{2(k-1)\delta - \alpha_k(\ell)\}} \tag{1.1.34}$$

From (1.1.34), we obtain

$$\alpha_k(\ell) = 2(k-1)\delta - \sin^{-1}\left(\frac{\ell_{k-1} \sin k\delta}{\ell}\right) . \tag{1.1.35}$$

From (1.1.34), it follows that

$$0 < \theta < \alpha_k(\ell) \Rightarrow \sin\{2(k-1)\delta - \theta\} > \sin\{2(k-1)\delta - \alpha_k(\ell)\} .$$

Hence

$$L(x,\theta) < \ell, \text{ for } 0 < \theta < \alpha_k(\ell), \ 0 < x < a .$$

In other words

$$(x,\theta) \in S_{k1}(\ell) \Rightarrow L(x,\theta) < \ell . \tag{1.1.36}$$

Hence

$$S_{k1}(\ell) \subset D_{11}(\ell) . \tag{1.1.37}$$

Next, we show that $S_{k2}(\ell) \subset D_{11}(\ell)$.

With A_k as the centre, we draw a circle of radius ℓ (cf. Fig. 6). Since $|A_1A_k| = \ell_{k-1}$, $|A_0A_k| = \ell_k$ and $\ell \in [\ell_{k-1}, \ell_k]$, the circle intersects A_0A_1 at a point $P_k(\ell)$. Let $|A_1P_k(\ell)| = x_k(\ell)$ and $\measuredangle A_kP_k(\ell)A_1 = \beta_k(\ell)$. We find $x_k(\ell)$ and $\beta_k(\ell)$. Then $\measuredangle A_kA_1P_k(\ell) = \pi - k\delta$, $k = 1,2,\ldots,N-1$, and $\ell_{k-1} = |A_1A_k|$. Therefore, from the triangle $P_k(\ell)A_1A_k$, we have

$$(1.1.38) \qquad \frac{\ell}{\sin k\delta} = \frac{\ell_{k-1}}{\sin \beta_k(\ell)} = \frac{x_k(\ell)}{\sin (k\delta - \beta_k(\ell))} .$$

It follows from (1.1.38) that

$$(1.1.39) \qquad \beta_k(\ell) = \sin^{-1} \left(\frac{\ell_{k-1} \sin k\delta}{\ell}\right)$$

and

$$(1.1.40) \qquad x_k(\ell) = \frac{\ell \sin (k\delta - \beta_k(\ell))}{\sin k\delta} .$$

Now let $P_{x,k}$ be a point on $A_0P_k(\ell)$, so that $x_k(\ell) < x < a$ (cf. Figure 6). With $P_{x,k}$ as the centre, we draw a circle of radius ℓ. Since $\ell_k > \ell > \ell_{k-1}$, the circle intersects the side $A_{k-1}A_k$ of the polygon at a point $Q_k(x,\ell)$. Then from the triangle $Q_k(x,\ell)P_{x,k}D_{k-1}$, we have

$$(1.1.41) \qquad \ell = \frac{(x + d_{k-1}) \sin (\measuredangle P_{x,k}D_{k-1}Q_k(x,\ell))}{\sin (\pi - \theta_k(x,\ell) - \measuredangle P_{x,k}D_{k-1}Q_k(x,\ell))} ,$$

where $\measuredangle Q_k(x,\ell)P_{x,k}D_{k-1} = \theta_k(x,\ell)$.

Since $\measuredangle P_{x,k}D_{k-1}Q_k(x,\ell) = \pi - 2(k-1)\delta$ by (1.1.15), (1.1.41) reduces to

$$\ell = \frac{(x + d_{k-1}) \sin 2(k-1)\delta}{\sin\{2(k-1)\delta - \theta_k(x,\ell)\}} . \tag{1.1.42}$$

From (1.1.42), we have

$$\theta_k(x,\ell) = 2(k-1)\delta - \sin^{-1}\{\frac{(x + d_{k-1})}{\ell} \sin 2(k-1)\delta\}. \tag{1.1.43}$$

For a fixed $x \in [x_k(\ell), a]$, we have, in (1.1.42)

$$0 \le \theta \le \theta_k(x,\ell) \Rightarrow \sin\{2(k-1)\delta - \theta\} \ge \sin\{(2(k-1)\delta - \theta_k(x,\ell)\}$$

and consequently

$$L(x,\theta) < \ell \text{ for } 0 < \theta < \theta_k(x,\ell),\ x_k(\ell) < x < a.$$

Thus

$$(x,\theta) \in S_{k2}(\ell) \Rightarrow L(x,\theta) < \ell. \tag{1.1.44}$$

Hence

$$S_{k2}(\ell) \subset D_{11}(\ell) \tag{1.1.45}$$

Next, we show that $S_{k3}(\ell) \subset D_{11}(\ell)$ (cf. Figure 6); let $P'_{x,k}$ be a point on $A_1A_k(\ell)$ such that $0 < x < x_k(\ell)$. With $P'_{x,k}$ as the centre, we draw a circle of radius ℓ. The circle intersects $A_k A_{k+1}$ at a point $Q_k'(x,\ell)$. Extend A_0A_1 and $A_{k+1}A_k$ to meet at a point D_k. Let $\angle Q_k'(x,\ell)\, P_{k,x}'\, A_1 = \theta_k'(x,\ell)$. From the triangle $Q_k'(x,\ell)P_{x,k}'D_k$, we have

(1.1.46) $$\ell = \frac{(x + d_k)\sin 2k\delta}{\sin(2k\delta - \theta'_k(x,\ell))}.$$

From (1.1.16), it follows that

(1.1.47) $$\theta'_k(x,\ell) = 2k\delta - \sin^{-1}\left(\frac{(x + d_k)\sin 2k\delta}{\ell}\right).$$

Let $x \in [0, x_k(\ell)]$. In (1.1.46),

(1.1.48) $$0 < \theta < \theta'_k(x,\ell) \Rightarrow \sin(2k\delta - \theta) > \sin(2k\delta - \theta'_k(x,\ell)).$$

Consequently

$$L(x,\ell) < \ell \text{ for } 0 < \theta < \theta_k'(\ell),\ 0 < x < x_k(\ell).$$

Hence

(1.1.49) $$S_{k3}(\ell) \subset D_{11}(\ell).$$

Combining (1.1.37), (1.1.45) and (1.1.49), we obtain

(1.1.50) $$S_{k1}(\ell) \cup S_{k2}(\ell) \cup S_{k3}(\ell) \subset D_{11}(\ell).$$

In order to show that $D_{11}(\ell) \subset \bigcup_{i=1}^{3} S_{ki}(\ell)$, let $(x,\theta) \not\in \bigcup_{i=1}^{3} S_{ki}(\ell)$.

Then

either (i) $x \in [x_k(\ell),a]$ and $\theta > \theta_k(x,\ell)$

or (ii) $x \in [0,x_k(\ell))]$ and $\theta > \theta'_k(x,\ell)$.

In case (i), $\theta > \theta_k(x,\ell)$ and $x \in [x_k(\ell),a]$. In (1.1.42)

$$2(k-1)\delta > \theta > \theta_k(x,\ell) \Rightarrow \sin\{2(k-1)\delta - \theta\} < \sin\{2(k-1)\delta - \theta_k(x,\ell)\}.$$

Consequently

$$L(x,\theta) > \ell \text{ for } \theta_k(x,\ell) < \theta < \pi/2,\ x_k(\ell) < x < a.$$

In case (ii), $\theta > \theta'_k(x,\ell)$ and $x \in [0,x_k(\ell)]$. In (1.1.46)

$$\theta > \theta'_k(x,\ell) \Rightarrow \sin(2k\delta - \theta) < \sin\{2k\delta - \theta'_k(x,\ell)\}.$$

and consequently

$$L(x,\theta) > \ell \text{ for } \theta'_k(x,\ell) < \theta < \pi/2,\ 0 < x < x_k(\ell)$$

Hence

$$(x,\theta) \notin D_{11}(\ell) .$$

Therefore

(1.1.51) $$D_{11}(\ell) \subset \bigcup_{i=1}^{3} S_{ki} .$$

Combining (1.1.50) and (1.1.51), we obtain Lemma 4. For a graphical description of the set $D_{11}(\ell)$ for $\ell \epsilon [\ell_{k-1}, \ell_k]$, see Appendix 1.1.7A.

1.1.7A. Graphical description of the set $D_{11}(\ell)$ for $\ell\in[\ell_{k-1},\ell_k]$, $k = 2,3,\ldots,n-1$.

For $\ell\in[\ell_{k-1},\ell_k]$ we have in Lemma 4,

$$D_{11}(\ell) = S_{k1}(\ell)\cup S_{k2}(\ell)\cup S_{k3}(\ell)\,, \tag{1.1.30}$$

where

$$S_{k1}(\ell) = \{(x,\theta):\ 0\leq x\leq a,\ 0\leq\theta\leq\alpha_k(\ell)\}\quad . \tag{1.1.31}$$

$$S_{k2}(\ell) = \{(x,\theta):\ x_k(\ell)\leq x\leq a,\ \alpha_k(\ell)\leq\theta\leq\theta_{k-1}(x,\ell)\} \tag{1.1.32}$$

and

$$S_{k3}(\ell) = \{(x,\theta):\ 0\leq x\leq x_k(\ell),\ \alpha_k(\ell)\leq\theta\leq\theta_k(x,\ell)\}\,. \tag{1.1.33}$$

where $\alpha_k(\ell)$, $\theta_k(x,\ell)$ and $x_k(\ell)$ are defined in (1.1.35), (1.1.43) and (1.1.40), respectively.

Let OABC be the parameter space S (cf. Fig. 7). Let C_θ denote the point on OC which is at distance θ from O. Similarly let C'_θ be the point on AB at distance θ from A. In OABC we draw parallel lines $C_\theta C'_\theta$, $C_{2\theta}C'_{2\theta}$, ... at distances θ, 2θ, ..., respectively, from OA. Recall that for each $\ell\in[\ell_{k-1},\ell_k]$, we have a point $P_k(\ell)$ on A_0A_1 of the polygon at a distance $x_k(\ell)$ from A_1 such that $|A_kP_k(\ell)| = \ell$. Corresponding to $x_k(\ell)$, we draw in the parameter space a line $X_k(\ell)\ X'_k(\ell)$ parallel to OC

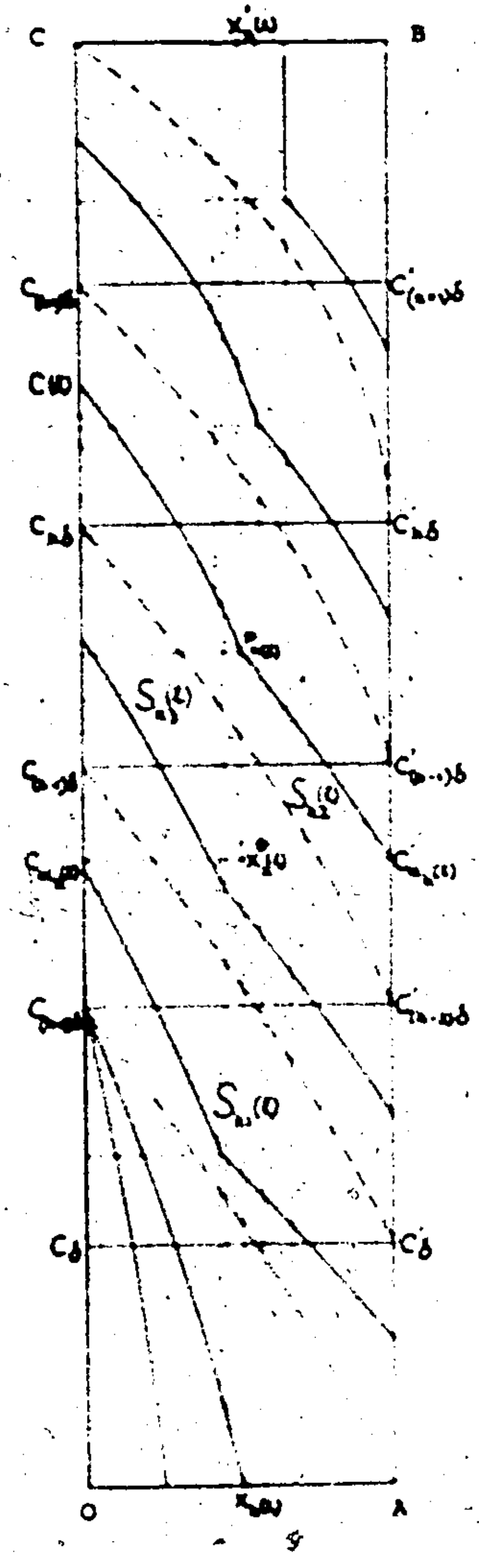

Fig. 7

where $X_k(\ell)$ and $X'_k(\ell)$ are points on OA and CB, respectively, such that $|OX_k(\ell)| = |CX'_k(\ell)| = x_k(\ell)$. Let $(x_k(\ell), \beta_k(\ell))$, where $x_k(\ell)$ and $\beta_k(\ell)$ are defined by (1.1.40) and (1.1.39), respectively, be the point $P_k(\ell)$ on $X_k(\ell)\, X'_k(\ell)$. When $\ell = \ell_{k-1}$, the line $X_k(\ell)\, X'_k(\ell)$ coincides with OC and the point $P_k(\ell)$ coincides with $C_{k\delta}$. As ℓ increases from $\ell = \ell_{k-1}$, the line $X_k(\ell)X'_k(\ell)$ moves parallel to OC further away from OC, and since $\beta_k(\ell)$ decreases as ℓ increases, the point $P_k(\ell)$ on $X_k(\ell)X'_k(\ell)$ moves downwards. When $\ell = \ell_k$, $X_k(\ell)\, X'_k(\ell)$ coincides with AB and $P_k(\ell)$ coincides with $C'_{(k-1)\delta}$. Eliminating ℓ between (1.1.39) and (1.1.40) we find that the locus of $P_k(\ell)$ is the curve

(1.1.7A.1) $$x = \ell_{k-1}(\sin k\delta \cot \theta - \cos k\delta).$$

We note that this curve passes through $(0,k\delta)$ and $(a,(k-1)\delta)$. Recall that for $\ell\epsilon[\ell_{k-1},\ell_k]$, there is a point $Q_k(\ell)$ on $A_{k-1}A_k$ of the polygon (cf. Fig. 7), such that $|Q_k(\ell)A_0| = \ell$. Corresponding to $Q_k(\ell)$ there is a point $C'_{\alpha_k(\ell)}$ on AB.

The set $S_{k1}(\ell)$ is bounded by $\theta = 0$, $\theta = \alpha_k(\ell)$, $x = 0$ and $x = a$. When $\ell = \ell_{k-1}$, $\alpha_k(\ell) = (k-2)\delta$. Consequently,

(1.1.7A.2) $$S_{k1}(\ell_{k-1}) = \{(x,\theta) : 0 \leq x \leq a,\ 0 \leq \theta \leq (k-2)\delta\}.$$

The set $S_{k1}(\ell_{k-1})$ is represented by the rectangle $OAC'_{(k-2)\delta}C_{(k-2)\delta}$.

As ℓ increases the line $C_{\alpha_k(\ell)}C'_{\alpha_k(\ell)}$ moves upwards. When $\ell = \ell_k$, $\alpha_k(\ell) = (k-1)\delta$. Consequently

(1.1.7A.3) $$S_{k1}(\ell_k) = \{(x,\theta) : 0 \leq x \leq a,\ 0 \leq \theta \leq (k-1)\delta\}.$$

The set $S_{k1}(\ell_k)$ is represented by the rectangle $OAC'_{(k-1)\delta}C_{(k-1)\delta}$.

The set $S_{k2}(\ell)$ is bounded by the curves $x = a$, $x = x_k(\ell)$, $\theta = \alpha_k(\ell)$ and

$$(1.1.7A.4) \qquad \theta = 2(k-1)\delta - \sin^{-1}\left(\left(\frac{x+d_{k-1}}{\ell}\right)\sin 2(k-1)\delta\right).$$

The set $S_{k3}(\ell)$ is bounded by $x = 0$, $x = x_k(\ell)$, $\theta = \alpha_k(\ell)$ and

$$(1.1.7A.5) \qquad \theta = 2k\delta - \sin^{-1}\left(\frac{x+d_k}{\ell}\right)\sin 2k\delta .$$

For a given $\ell\epsilon[\ell_{k-1},\ell_k]$, the line $X_k(\ell)X_k'(\ell)$ separates the two sets $S_{k2}(\ell)$ and $S_{k3}(\ell)$. The sine curves given by (1.1.7A.4) and (1.1.7A.5) intersect on $X_k(\ell)X_k'(\ell)$ at the point $P_k(\ell)$ whose coordinates are given by $(x_k(\ell), \theta_k(\ell))$, and whose locus is given by (1.1.7A.1).

Note that $\ell = \ell_1$ implies $x_k(\ell) = a$ and $P_k(\ell)$ coincides with the point $C_{(k-1)\delta}$; consequently $S_{k2}(\ell)$ vanishes and the sine curve (1.1.7A.5), which is a bounding curve of $S_{k3}(\ell)$, extends from a point on OC to a point on AB.

Now $\ell = \ell_k$ implies $x_k(\ell) = a$, and $P_k(\ell)$ coincides with $C_{k\delta}$; consequently, $S_{k3}(\ell)$ vanishes and the sine curve (1.1.8A.4) which is a bounding curve of $S_{k2}(\ell)$ extends from a point on AB to a point on OC.

The set $S_{k2}(\ell)$ is represented by $X_k^*(\ell)P_k(\ell)C'_{\gamma_k(\ell)}$ in Figure 7, where $X_k^*(\ell)$ is the point of intersection of $X_k(\ell)X_k'(\ell)$ and $C_{\gamma_k(\ell)}C'_{\gamma_k(\ell)}$.

In Figure 7,

$S_{k3}(\ell)$ is represented by $C_{a_k(\ell)}X_k^e(\ell)P_k(\ell)C(\ell)$,

$S_{k2}(\ell_k)$ is represented by the point $(a,(k-1)\delta)$ i.e. $C'_{(k-1)\delta}$,

$S_{k3}(\ell_k\ \ )$ is represented by $C_{(k+1)\delta}C_{(k-1)\delta}C'_{(k-1)\delta}$,

$S_{k2}(\ell_{k-1})$ is represented by $C_{(k-2)\delta}C'_{(k-2)\delta}C_{k\delta}$.

and

$S_{k3}(\ell_{k-1})$ is represented by $(0,k\delta)$ i.e. $C_{k\delta}$.

1.1.8. Determination of the set $D_{11}(\ell)$ for $\ell\in[\ell_{n-1},\ell_n]$, $N = 2n$.

In order to find the distribution function $F_N(\ell)$ of L for $\ell\in[\ell_{n-1},\ell_n]$ we need to find the set $\{(x,\theta): L(x,\theta)\leq\ell\}$ for $\ell\in[\ell_{n-1},\ell_n]$ in the parameter space. In the following lemma we obtain this set.

Lemma 5. Let N, the number of sides of the regular polygon, be even and equal to $2n$. Let $\ell\in[\ell_{n-1},\ell_n]$. Then

(1.1.52) $$D_{11}(\ell) = S_{n1}(\ell)\cup S_{n2}(\ell)\cup S_{n3}(\ell) ,$$

where

(1.1.53) $$S_{n1}(\ell) = \{(x,\theta): 0 \leq x \leq (\ell^2-\ell_{n-1}^2)^{\frac{1}{2}}, 0 \leq \theta \leq \frac{\pi}{2}\} .$$

(1.1.54) $$S_{n2}(\ell) = \{(x,\theta): (\ell^2-\ell_{n-1}^2)^{\frac{1}{2}} \leq x \leq a,$$

$$0 \leq \theta \leq 2(n-1)\delta - \sin^{-1} (\frac{x+d_{n-1}}{\ell}\sin 2(n-1)\delta)\} .$$

and

(1.1.55) $$S_{n3}(\ell) = \{(x,\theta): (\ell^2-\ell_{n-1}^2)^{\frac{1}{2}} \leq x \leq a,$$

$$\sin^{-1} (\frac{\ell_{n-1}}{\ell}) \leq \theta \leq \frac{\pi}{2}\}.$$

Proof. (cf. Fig. 8). First we show that $S_{ni}\subset D_{11}(\ell)$, $i = 1,2,3$.

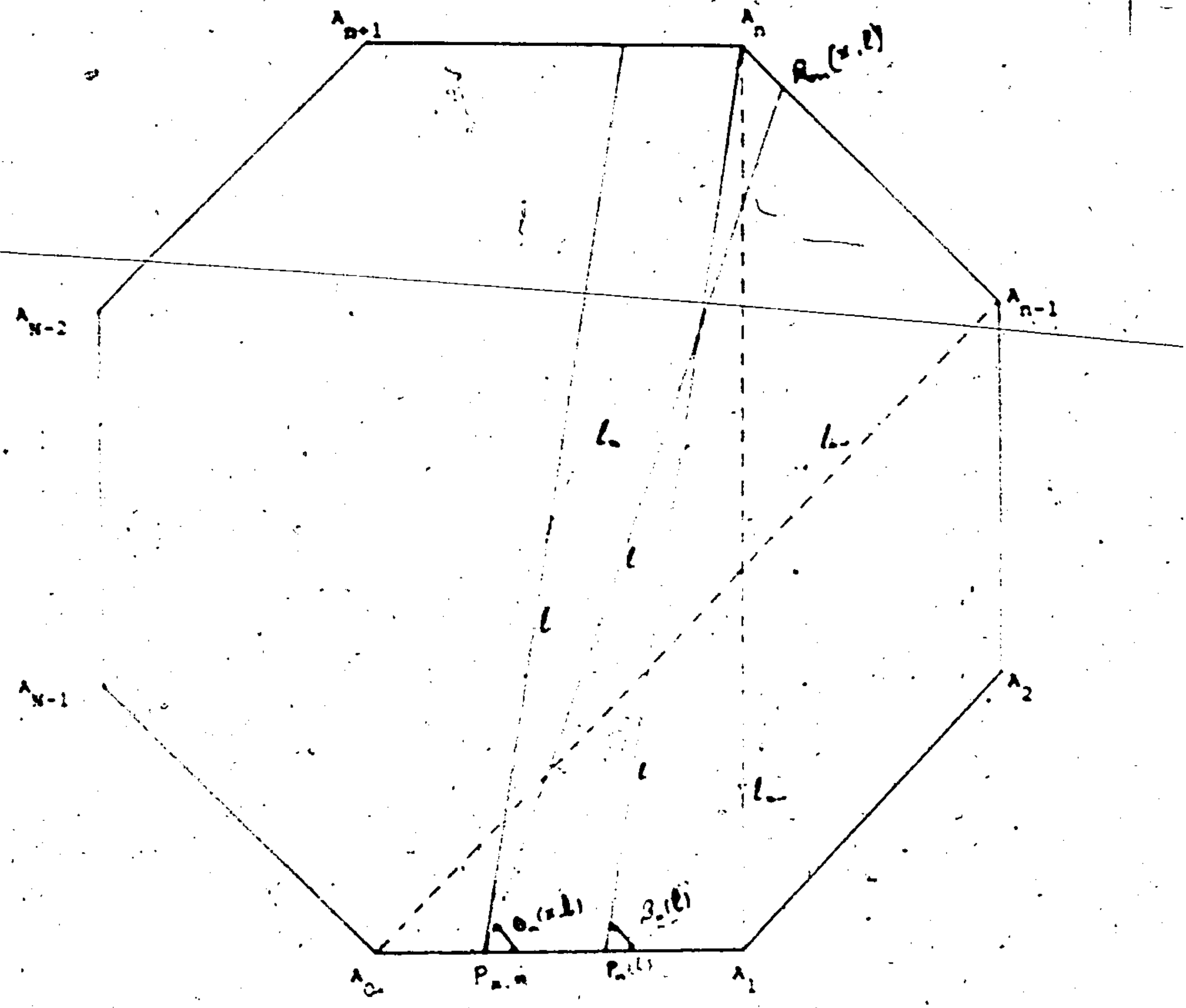

Fig. 8

With A_n as the centre a circle of radius ℓ is drawn. The circle intersects A_0A_1 at a point $P_n(\ell)$. Let $|A_1P_n(\ell)| = x_n(\ell)$ and $\measuredangle A_nP_n(\ell)A_1 = \beta_n(\ell)$. Then

(1.1.56) $$\beta_n(\ell) = \sin^{-1}\left(\frac{\ell_{n-1}}{\ell}\right),$$

(1.1.57) $$x_n(\ell) = (\ell^2-\ell_{n-1}^2)^{\frac{1}{2}}.$$

Clearly we have

(1.1.58) $$(x,\theta) \in S_{n1}(\ell) \Rightarrow L(x,\theta) \le \ell.$$

Hence

(1.1.59) $$S_{n1}(\ell) \subset D_{11}(\ell).$$

We next show that $S_{n2}(\ell) \subset D_{11}(\ell)$. Let $P_{x,n}$ be a point on $A_0P_n(\ell)$ so that $x_n(\ell) \le x \le a$. With $P_{x,n}$ as the centre draw a circle of radius ℓ. The circle intersects $A_{n-1}A_n$ at a point $Q_n(x,\ell)$. Let $\measuredangle Q_n(x,\ell)P_{x,n}A_1 = \theta_n(x,\ell)$. Then putting $k = n$ in (1.1.43) we obtain

(1.1.60) $$\theta_n(x,\ell) = 2(n-1)\delta - \sin^{-1}\left(\frac{x+d_{n-1}}{\ell}\sin 2(n-1)\delta\right).$$

For a fixed $x \in [x_n(\ell), a]$, where $x_n(\ell)$ is given by (1.1.57), by use of (1.1.42), where $k = n$, we have

$$\theta \le \theta_n(x,\ell) \Rightarrow L(x,\theta) \le \ell.$$

Therefore

$$(x,\theta) \varepsilon S_{n2}(\ell) \Rightarrow (x,\theta) \varepsilon D_{11}(\ell) \quad .$$

Thus

(1.1.61) $$S_{n2}(\ell) \subset D_{11}(\ell) \quad .$$

Finally let $x\varepsilon[x_n(\ell), a]$ and $\theta\varepsilon[\beta_n, \frac{\pi}{2}]$ where $\beta_n(\ell)$ is given by (1.1.56). Then $L(x,\theta) \leq \ell$ since A_0A_1 and A_nA_{n+1} are parallel.

Hence

(1.1.62) $$(x,\theta) \varepsilon S_{n3}(\ell) \Rightarrow (x,\theta) \varepsilon D_{11}(\ell).$$

Thus

(1.1.63) $$S_{n3}(\ell) \subset D_{11}(\ell).$$

Combining (1.1.59), (1.1.61) and (1.1.63), we obtain

(1.1.64) $$S_{n1}(\ell) \cup S_{n2}(\ell) \cup S_{n3}(\ell) \subset D_{11}(\ell).$$

To show that $D_{11}(\ell) \subset \bigcup_{i=1}^{3} S_{ni}(\ell)$, let $(x,\theta) \not\varepsilon S_{n1}(\ell) \cup S_{n2}(\ell) \cup S_{n3}(\ell)$.

Then clearly $x\varepsilon[(\ell^2-\ell^2_{n-1})^{\frac{1}{2}}, a]$ and

$$\theta\varepsilon[2(n-1)\delta - \sin^{-1}(\frac{x+d_{n-1}}{\ell} \sin 2(n-1)\delta), \sin^{-1}(\frac{\ell_{n-1}}{\ell})].$$

By the use of (1.1.42), where $k = n$, we obtain $L(x,\theta) > \ell$.

Therefore $(x,\theta) \not\varepsilon D_{11}(\ell)$ and consequently,

(1.1.65) $$D_{11}(\ell) \subset S_{n1}(\ell) \cup S_{n2}(\ell) \cup S_{n3}(\ell).$$

Combining (1.1.64) and (1.1.65), we obtain

$$D_{11}(\ell) = S_{n1}(\ell) \cup S_{n2}(\ell) \cup S_{n3}(\ell). \tag{1.1.66}$$

1.1.8A. Graphical description of the set $D_{11}(\ell)$ for $\ell\epsilon[\ell_{n-1},\ell_n]$, where $N = 2n$. For $\ell\epsilon[\ell_{n-1},\ell_n]$, we have, by Lemma 5,

$$D_{11}(\ell) = S_{n1}(\ell) \cup S_{n2}(\ell) \cup S_{n3}(\ell). \tag{1.1.52}$$

where

$$S_{n1}(\ell) = \{(x,\theta): 0 \le x \le (\ell^2-\ell^2_{n-1})^{\frac{1}{2}},\ 0 \le \theta \le \frac{\pi}{2}\}, \tag{1.1.53}$$

$$S_{n2}(\ell) = \{(x,\theta): (\ell^2-\ell^2_{n-1})^{\frac{1}{2}} \le x \le a, \tag{1.1.54}$$

$$0 \le \theta \le 2(n-1)\delta - \sin^{-1}(\frac{x+d_{n-1}}{\ell} \sin 2(n-1)\delta)\},$$

$$S_{n3}(\ell) = \{(x,\theta): (\ell^2-\ell^2_{n-1})^{\frac{1}{2}} \le x \le a, \sin^{-1}(\frac{\ell_{n-1}}{\ell}) \le \theta \le \frac{\pi}{2}\}. \tag{1.1.55}$$

In the parameter space S (cf. Fig. 9), the set $S_{n1}(\ell)$ is the rectangle $OX_n(\ell)X'_n(\ell)C$, where $|OX_n(\ell)| = x_n(\ell) = (\ell^2-\ell^2_{n-1})^{\frac{1}{2}}$. Now $\ell = \ell_{n-1} \Rightarrow x_n(\ell) = 0$. Therefore the set $S_{n1}(\ell)$ reduces to the point $C(0, \pi/2)$. Next $\ell = \ell_n \Rightarrow x_n(\ell) = a$ and the set $S_{n1}(\ell)$ is identical with S. To describe the set $S_{n2}(\ell)$ we note that for $\ell\epsilon[\ell_{n-1},\ell_n]$ we have a point $P_n(\ell)$ on $X_n(\ell)X'_n(\ell)$ whose locus is given by

$$x = \ell_{n-1} \cot \theta. \tag{1.1.8A.1}$$

As ℓ increases from ℓ_{n-1} to ℓ_n, the point $P_n(\ell)$ on $X_n(\ell)X'_n(\ell)$ moves along the locus (1.1.8A.1). From the point $P_n(\ell)$ we have a sine

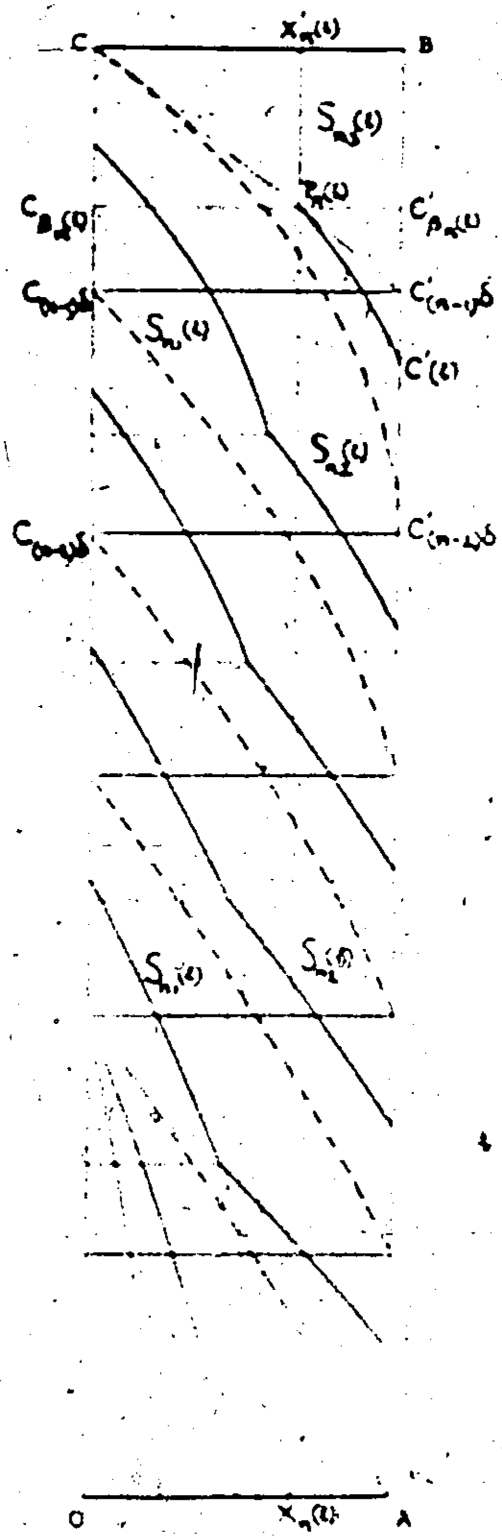

Fig. 9

curve given by

$$\sin(2(n-1)\delta - \theta) = \frac{x+d_{n-1}}{\ell} \sin 2(n-1)\delta$$

which bounds the set $S_{n2}(\ell)$. The set $S_{n2}(\ell)$ is bounded by $x = (\ell^2-\ell_{n-1}^2)^{\frac{1}{2}}$, $x = a$, $\theta = 0$, and $\theta = 2(n-1)\delta - \sin^{-1}(\frac{x+d_{n-1}}{\ell}\sin 2(n-1)\delta)$.

Now

$$\ell = \ell_{n-1} \Rightarrow S_{n2}(\ell) = \{(x,\theta): 0 \leq x \leq a,$$
$$0 \leq \theta \leq 2(n-1)\delta - \sin^{-1}(\frac{x+d_{n-1}}{\ell_{n-1}}\sin 2(n-1)\delta)\}$$

and $S_{n2}(\ell_{n-1})$ is represented by $OAC'_{(n-2)\delta}C$.

$\ell = \ell_n \Rightarrow x_n(\ell) = a$, and the set $S_{n2}(\ell)$ reduces to segment $A\,C'_{(n-1)\delta}$.

To describe the set $S_{n3}(\ell)$ we draw a line $P_n(\ell)C_{\beta_n(\ell)}$ parallel to OA, where $C_{\beta_n(\ell)}$ is on AB. The set $S_{n3}(\ell)$ is clearly the rectangle $P_n(\ell)X'_n(\ell)BC'_{\beta_n(\ell)}$ (cf. Fig. 9).

When $\ell = \ell_{n-1}$, $x_n(\ell) = 0$ and $\beta_n(\ell) = \frac{\pi}{2}$. Hence the rectangle $P_n(\ell)C'_{\beta_n(\ell)}BX'_n(\ell)$ reduces to the point $(0,\frac{\pi}{2})$. When $\ell = \ell_n$, $P_n(\ell)$ is the point $(a,(n-1)\delta)$ and the rectangle $P_n(\ell)C'_{\beta_n(\ell)}BX'_n(\ell)$ reduces to the line segment $C'_{(n-1)\delta}B$.

1.1.9. Decomposition of the interval $[\ell_{n-1}, \ell_n]$ for $N = 2n+1$

In order to find the distribution function of L for the case when the number, N, of sides of the polygon is odd and equal to 2n+1, the last interval $[\ell_{n-1}, \ell_n]$ is subdivided into the intervals $[\ell_{n-1}, \ell_{n-1,1}]$, $[\ell_{n-1,1}, \ell_{n-1,2}]$ and $[\ell_{n-1,2}, \ell_n]$,
where

(1.1.67)
$$\ell_{n-1,1} = \frac{-\ell_{n-1} \sin n\delta}{\cos (2n\delta)}$$

and

(1.1.68)
$$\ell_{n-1,2} = \frac{a \tan n\delta}{2}$$

as follows (cf. Fig. 10).

Let A_1D be orthogonal to A_0A_1, where D is on A_nA_{n+1}. Let $|A_1D| = \ell_{n-1,1}$. It follows that $\measuredangle A_1DA_n = 2n\delta - \frac{\pi}{2}$. From the triangle A_1DA_n, we have (1.1.67).

Let $\ell_{n-1,2}$ be the length of the perpendicular $A_{n+1}C$ (ref. Fig. 10) drawn from A_{n+1} to A_0A_1. From the triangle A_0CA_{n+1}, we have (1.1.68). From the construction we have

(1.1.69)
$$\ell_{n-1} \le \ell_{n-1,1} \le \ell_{n-1,2} \le \ell_n$$

The following lemma will be used in Lemmas 7, 8 and 9.

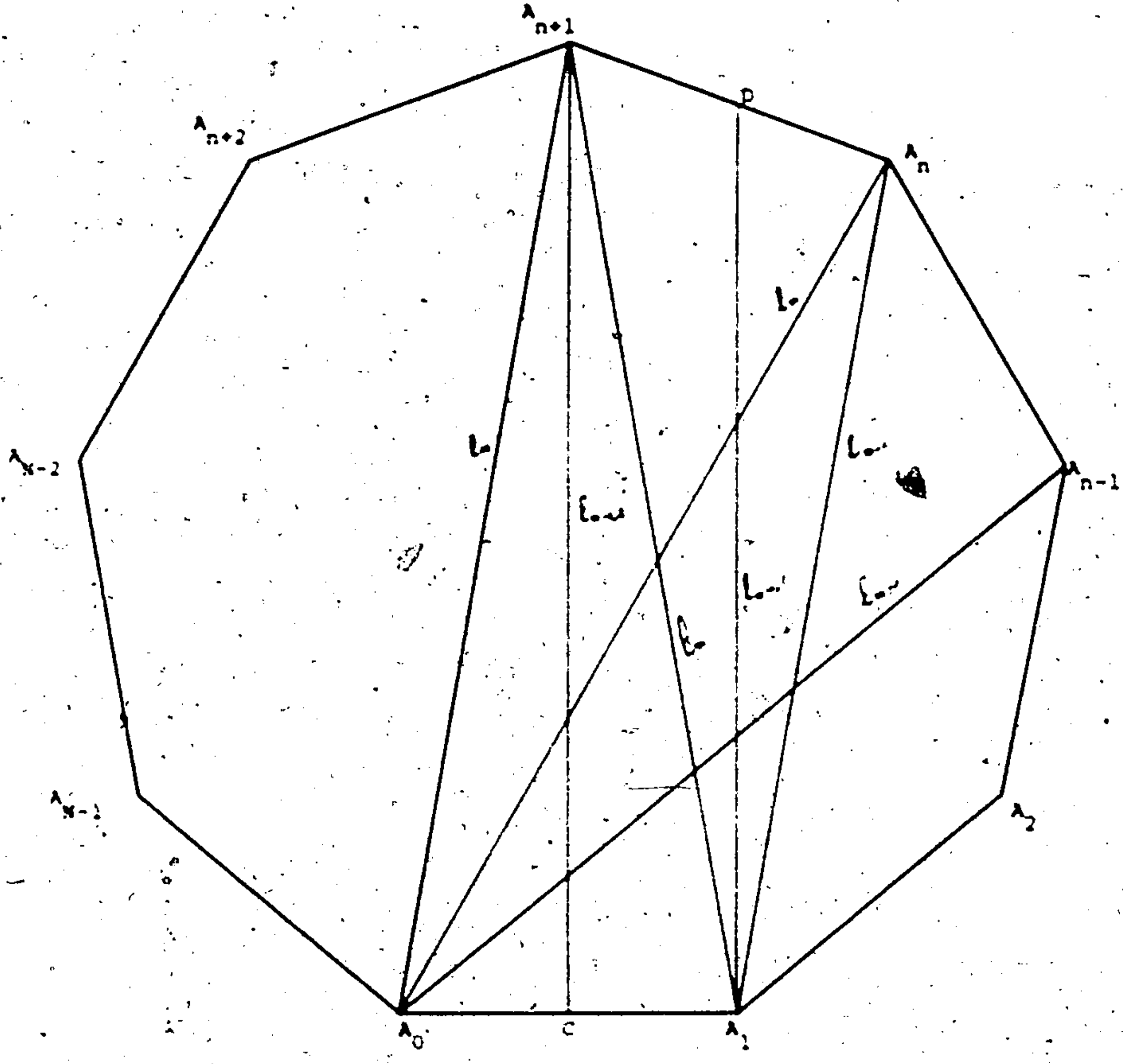

Fig. 10)

<u>Lemma 6.</u> Let $\ell\in[\ell_{n-1},\ell_n]$, where N, the number of sides of the polygon, is odd and equal to 2n+1. Then

(1.1.70) $$S'_{n1}(\ell)\cup S'_{n2}(\ell)\subset D_{11}(\ell),$$

where

(1.1.71) $$S'_{n1}(\ell)=\{(x,\theta):\ 0\leqslant x\leqslant x_n(\ell),\ 0\leqslant\theta\leqslant\beta_n(\ell)\},$$

(1.1.72) $$S'_{n2}(\ell)=\{(x,\theta):\ x_n(\ell)\leqslant x\leqslant a,\ 0\leqslant\theta\leqslant\theta_n(x,\ell)\},$$

where $x_n(\ell)$, $\beta_n(\ell)$ and $\theta_n(x,\ell)$ are given by (1.1.40), (1.1.39) and (1.1.43) with k = n.

<u>Proof</u>. (cf. Fig. 11) With A_n as the centre, we draw a circle of radius ℓ.

The circle intersects A_1A_0 at a point $P_n(\ell)$. Let $|A_1P_n(\ell)|=x_n(\ell)$, $\angle A_nP_n(\ell)A_1=\beta_n(\ell)$. Then by (1.1.39)

(1.1.73) $$\beta_n(\ell)=\sin^{-1}\left(\frac{\ell_{n-1}\sin n\delta}{\ell}\right),$$

and, by (1.1.40)

(1.1.74) $$x_n(\ell)=\frac{\ell\sin[n\delta-\beta_n(\ell)]}{\sin n\delta}.$$

Let $P_{x,n}$ be a point on $A_0P_n(\ell)$ such that $x_n(\ell)\leqslant x\leqslant a$. With $P_{x,n}$

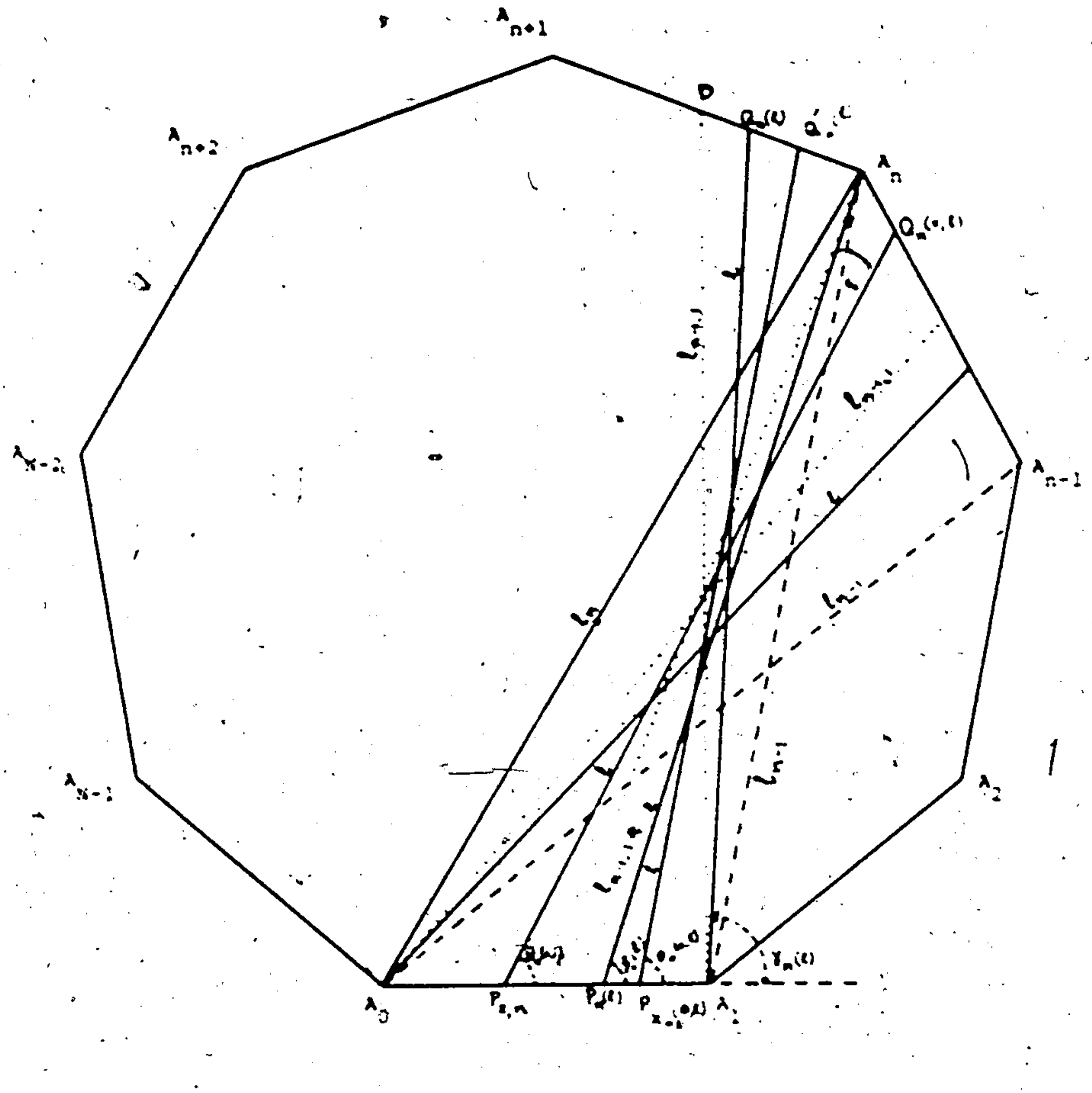

Fig. 11

as the centre we draw a circle of radius l. The circle intersects A_nA_{n-1} at a point say, $Q_n(x,l)$. Put $\angle Q_n(x,l)P_{x,n}A_1 = \theta_n(x,l)$. Then by (1.1.43) with $k = n$, we have

(1.1.75) $$\theta_n(x,l) = 2(n-1)\delta - \sin^{-1}\left[\frac{(x+d_{n-1})}{l}\sin 2(n-1)\delta\right].$$

By (1.1.42) it follows that

(1.1.76) $$0<\theta\leq\theta_n(x,l) \Rightarrow L(x,\theta)\leq l.$$

Hence

(1.1.77) $$(x,\theta)\in S'_{n2}(l) \Rightarrow L(x,\theta)\leq l.$$ Thus

(1.1.78) $$S'_{n2}(l)\subset D_{11}(l).$$

Also clearly,

(1.1.79) $$(x,\theta)\in S'_{n1}(l) \Rightarrow L(x,\theta)\leq l.$$

Hence

(1.1.80) $$S'_{n1}(l)\subset D_{11}(l).$$

Combining (1.1.78) and (1.1.80), we obtain Lemma [illegible].

1.1.10. Determination of the set $D_{11}(l)$ for $l\epsilon[l_{n-1}, l_{n-1,1}]$.

In order to find the distribution function $F_N(l)$ of L for $l\epsilon[l_{n-1}, l_{n-1,1}]$ we need the set $\{(x,\theta): L(x,\theta) \leq l,\ l\epsilon[l_{n-1}, l_{n-1,1}]\}$. In the following lemma we obtain this set.

Lemma 7. Let N be odd and equal to 2n+1. Let $l\epsilon[l_{n-1}, l_{n-1,1}]$, where

$$l_{n-1,1} = -\frac{l_{n-1}\sin n\xi}{\cos 2n\xi}.$$ Then

(1.1.81) $$D_{11}(l) = S'_{n1}(l) \cup S'_{n2}(l) \cup S'_{n3}(l),$$

where $S'_{n1}(l)$ and $S'_{n2}(l)$ are defined by (1.1.71) and (1.1.72) respectively, and

(1.1.82) $$S'_{n3}(l) = \{(x,\theta): \beta_n(l) \leq \theta \leq \gamma_n(l),\ 0 \leq x \leq x_{n2}(\theta,l)\},$$

where $\beta_n(l)$ is given by (1.1.73),

(1.1.83) $$\gamma_n(l) = 2n\xi - \sin^{-1}\left(\frac{l_{n-1}\sin n\xi}{l}\right),$$

and

(1.1.84) $$x_{n2}(\theta,l) = -d_n + \frac{l\sin(2n\xi-\theta)}{\sin 2n\xi}.$$

Proof. (cf. Fig. 11). We have already shown in the previous lemma that $S'_{n1}(\ell)\cup S'_{n2}(\ell)\subset D_{11}(\ell)$. We now show that $S'_{n3}(\ell)\subset D_{11}(\ell)$. Let $Q_n(\ell)$ be the point on A_nA_{n+1} such that $|A_1Q_n(\ell)| = \ell$. Let $\measuredangle Q_n(\ell)A_1A_n = \gamma'_n(\ell)$. Then from the triangle $Q_n(\ell)A_1A_n$ we have

$$(1.1.85) \qquad \ell = \frac{\ell_{n-1} \sin n\delta}{\sin (n\delta - \gamma'_n(\ell))} .$$

From (1.1.85) we obtain

$$\gamma'_n(\ell) = n\delta - \sin^{-1} \left(\frac{\ell_{n-1} \sin n\delta}{\ell}\right) ,$$

and

$$(1.1.84) \qquad \gamma_n(\ell) = Q_n(\ell)A_1D_n = 2n\delta - \sin^{-1} \left(\frac{\ell_{n-1} \sin n\delta}{\ell}\right).$$

Let $\theta\in[\beta_n(\ell), \gamma_n(\ell)]$, where $\beta_n(\ell)$ and $\gamma_n(\ell)$ are given by (1.1.73) and (1.1.84), respectively. Let $x_{n2}(\theta,\ell)$ be the value of X such that for a fixed θ, $L(x_{n2}(\theta,\ell),\theta) = \ell$. Let $P_{x_{n2}(\theta,\ell)}Q'_n(\ell)$ be the corresponding secant. Considering the triangle $Q'_n(\ell)P_{x_{n2}(\theta,\ell)}D_n$, we obtain

$$(1.1.86) \qquad \ell = \frac{(d_n+x_{n2}(\theta,\ell)) \sin 2n\delta}{\sin (2n\delta-\theta)} .$$

From (1.1.86) it follows that

$$(1.1.87) \qquad x_{n2}(\theta,\ell) = - d_n + \frac{\ell \sin (2n\delta-\theta)}{\sin 2n\delta} .$$

From (1.1.86) for a fixed $\theta \epsilon [\beta_n(\ell), \gamma_n(\ell)]$, it follows that

$$x \leq x_{n2}(\theta,\ell) \Rightarrow (d_n + x) \sin 2n\delta \leq \{d_n + x_{n2}(\theta,\ell)\} \sin 2n\delta \quad .$$

Consequently, $L(x,\theta) \leq \ell$. Thus

$$(x,\theta) \in S'_{n3}(\ell) \Rightarrow L(x,\theta) \leq \ell. \tag{1.1.88}$$

Therefore

$$S'_{n3}(\ell) \subset D_{11}(\ell) . \tag{1.1.89}$$

Combining (1.1.70) and (1.1.89), we obtain

$$S'_{n1}(\ell) \cup S'_{n2}(\ell) \cup S'_{n3}(\ell) \subset D_{11}(\ell) . \tag{1.1.90}$$

To prove that $D_{11}(\ell) \subset S'_{n1}(\ell) \cup S'_{n2}(\ell) \cup S'_{n3}(\ell)$,

let $(x,\theta) \notin S'_{n1}(\ell) \cup S'_{n2}(\ell) \cup S'_{n3}(\ell)$. Then either

(i) $\theta \in [\beta_n(\ell), \gamma_n(\ell)]$ and $x > x_{n2}(\theta,\ell)$

or

(ii) $x \in [x_n(\ell), a]$ and $\theta > \theta_n(x,\ell)$.

In (i) by use of (1.1.85), and in (ii) by use of (1.1.42), where $k = n$, we obtain $L(x,\theta) > \ell$. Hence $(x,\theta) \notin D_{11}(\ell)$.

Therefore

(1.1.91) $$D_{11}(\ell) \subset S'_{n1}(\ell) \cup S'_{n2}(\ell) \cup S'_{n3}(\ell).$$

Combining (1.1.90) and (1.1.91), we obtain (1.1.81).

A graphical description of the set $D_{11}(\ell)$ for $\ell \in [\ell_{n-1}, \ell_{n-1,1}]$ is given in the following section.

1.1.10A. Graphical description of the set $D_{11}(\ell)$ for $\ell \in [\ell_{n-1}, \ell_{n-1,1}]$, where $N = 2n+1$.

For $\ell \in [\ell_{n-1}, \ell_{n-1,1}]$ we have by Lemma 7,

(1.1.81) $$D_{11}(\ell) = S'_{n1}(\ell) \cup S'_{n2}(\ell) \cup S'_{n3}(\ell),$$

where

(1.1.71) $$S'_{n1}(\ell) = \{(x,\theta) : 0 \leq x \leq x_n(\ell),\ 0 \leq \theta \leq \theta_n(\ell)\},$$

(1.1.72) $$S'_{n2}(\ell) = \{(x,\theta) : x_n(\ell) \leq x \leq a,\ 0 \leq \theta \leq \theta_n(x,\ell)\},$$

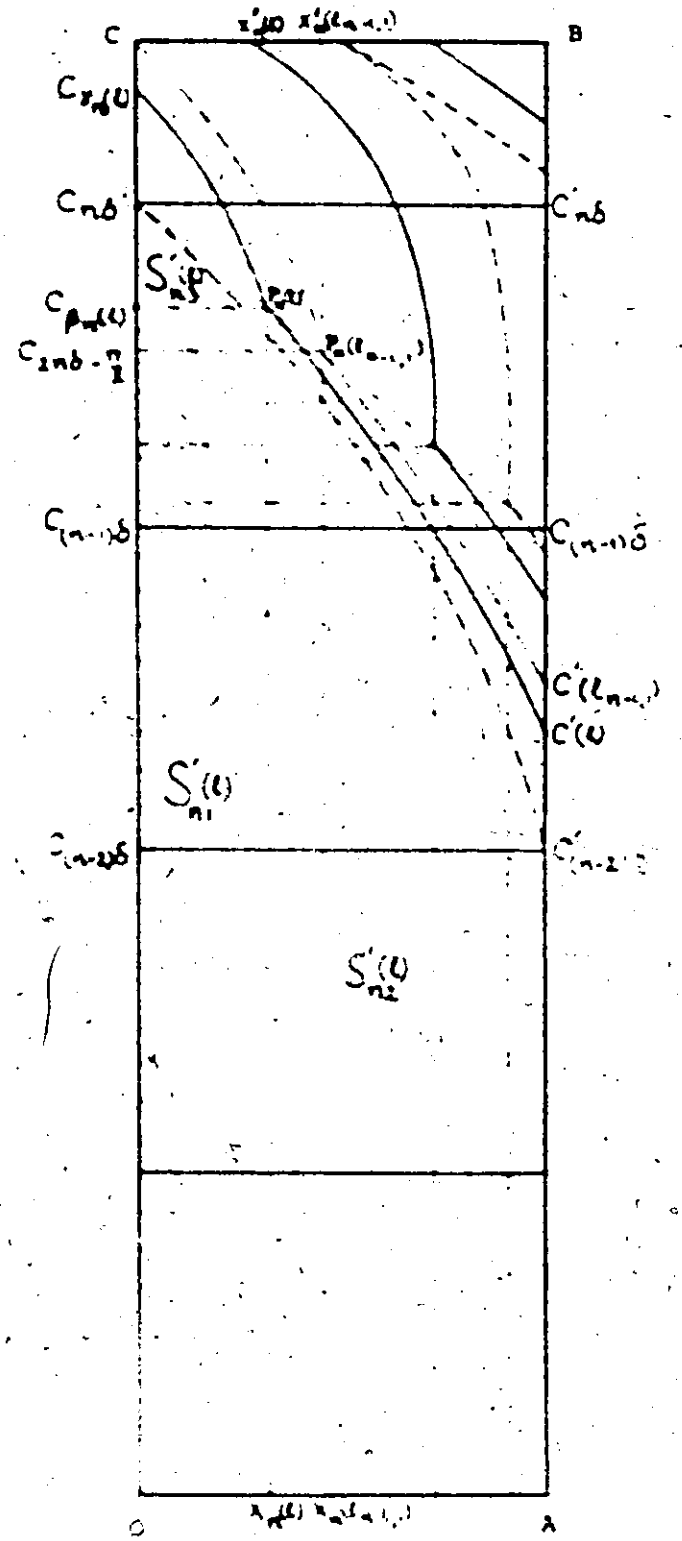

Fig. 12

(1.1.82) $\quad S'_{n3}(\ell) = \{(x,\theta): \beta_n(\ell) \leq \theta \leq \gamma_n(\ell),\ 0 \leq x \leq x_{n2}(\theta, \ell)\},$

where $x_n(\ell)$, $\beta_n(\ell)$, $\theta_n(x,\ell)$, $\gamma_n(\ell)$ and $x_{n2}(\theta,\ell)$ are given by (1.1.40), (1.1.73), (1.1.43), (1.1.83) and (1.1.84), respectively.

In the parameter space S, the set $S'_{n1}(\ell)$ is bounded by $x = 0$, $x = x_n(\ell)$, $\theta = 0$, $\theta = \beta_n(\ell)$ and is represented by the rectangle $OX_n(\ell)P_n(\ell)C_{\beta_n(\ell)}$ in Fig. 12.

Eliminating ℓ between the equations (1.1.39) and (1.1.40), we find that the locus of $P_n(\ell)$ is given by

(1.1.10A.1) $\quad x = \ell_{n-1}(\sin n\delta \cot \theta - \cos n\delta).$

We note that this curve passes through $(0, n\delta)$ and $(a, (n-1)\delta)$.

Now $\ell = \ell_{n-1}$ implies $\beta_n(\ell) = n\delta$, $x_n(\ell) = 0$. Therefore

(1.1.10A.2) $\quad S'_{n1}(\ell_{n-1}) = \{(0,\theta): 0 \leq \theta \leq n\delta\},$

i.e. the set $S'_{n1}(\ell)$ reduces to the line segment $OC_{n\delta}$ in Figure 12.

As ℓ increases from $\ell = \ell_{n-1}$, the point $P_n(\ell)$ moves along the curve (1.1.10A.1).

Next

$$\ell = \ell_{n-1,1} \text{ implies } \theta_n(\ell) = 2n\delta - \frac{\pi}{2},$$

and

$$x_n(\ell) = -d_n + \frac{\ell_{n-1,1}}{\sin 2n\delta} = \frac{-\ell_{n-1}\cos n\delta}{\cos 2n\delta}.$$

Therefore

$$(1.1.10A.3) \quad S'_{n1}(\ell_{n-1,1}) = \{(x,\theta): 0 \le \theta \le 2n\delta - \frac{\pi}{2},\ 0 \le x \le \frac{-\ell_{n-1}\cos n\delta}{\cos 2n\delta}\}.$$

The set $S'_{n1}(\ell_{n-1,1})$ is represented by $OX_n(\ell_{n-1,1})P_n(\ell_{n-1,1})C_{2n\delta - \frac{\pi}{2}}$ in Figure 12.

In the parameter space S the set $S'_{n2}(\ell)$ is bounded by the curves $x = x_n(\ell)$, $x = a$, $\theta = 0$ and $\theta = \theta_n(x,\ell)$, where $\theta = \theta_n(x,\ell)$ is a sine curve. Here $S'_{n2}(\ell)$ is represented by $AX_n(\ell)P_n(\ell)C'(\ell)$ (cf. Fig. 12).

$$\ell = \ell_{n-1} \text{ implies } \theta_n(\ell) = n\delta,\ x_n(\ell) = 0,\ \theta_n(a,\ell_{n-1}) = (n-2)\delta.$$

Therefore

$$S'_{n2}(\ell_{n-1}) = \{(x,\theta)\ 0 \le x \le a,\ 0 \le \theta \le \theta_n(x,\ell_{n-1})\}$$

The set $S'_{n2}(\ell_{n-1})$ is represented by $OAC'_{(n-2)\delta}C_{n\delta}$ in Figure 12.

Now

$$\ell = \ell_{n-1,1} \Rightarrow \begin{cases} x_n(\ell) = -\ell_{n-1} \dfrac{\cos n\delta}{\cos 2n\delta} \\ \beta_n(\ell) = 2n\delta - \dfrac{\pi}{2} \\ \theta_n(a,\ell) = \dfrac{\pi}{2} - 2\delta. \end{cases}$$

Hence

(1.1.10A.4) $$S'_{n2}(\ell_{n-1,1}) = \{(x,\theta) : -\ell_{n-1} \frac{\cos n\delta}{\cos 2n\delta} \le x \le a,$$

$$0 \le \theta \le \theta_n(x,\ell_{n-1,1})\} .$$

The set $S'_{n2}(\ell_{n-1,1})$ is represented by $X_n(\ell_{n-1,1})AC'(\ell_{n-1,1})P_n(\ell_{n-1,1})$ in Figure 12.

In the parameter space S the set $S'_{n3}(\ell)$ is bounded by $\theta = \beta_n(\ell)$, $\theta = \gamma_n(\ell)$, $x = 0$ and

(1.1.10A.5) $$x = -d_n + \frac{\ell \sin (2n\delta - \theta)}{\sin 2n\delta} \quad (= x_{n2}(\theta,\ell))$$

and is represented by $C_{\beta_n(\ell)}P_n(\ell)C_{\gamma_n(\ell)}$, where $\gamma_n(\ell)$ is defined by (1.1.83) and $C_{\gamma_n(\ell)}P_n(\ell)$ is the sine curve (1.1.10A.5).

Now $\ell = \ell_{n-1}$ implies $\beta_n(\ell) = n\delta$, $x_{n2}(\theta,\ell) = 0$. Therefore

(1.1.10A.5) $$S'_{n3}(\ell_{n-1}) = \{0, n\delta\} ,$$

i.e. $S'_{n3}(\ell)$ reduces to the point $C_{n\delta}$ for $\ell = \ell_{n-1}$.

$$\ell = \ell_{n-1,1} \Rightarrow \beta_n(\ell) = 2n\delta - \frac{\pi}{2}, \gamma_n(\ell) = \frac{\pi}{2}.$$

and so

$$x_{n2}(\beta_n(\ell),\ell) = x_n(\ell_{n-1,1}),\ x_{n2}(\gamma_n(\ell),\ell) = 0.$$

Thus the curve (1.1.10A.5) passes through $(0,\frac{\pi}{2})$. Therefore

(1.1.11A.6) $$S_{n3}(\ell_{n-1,1}) = \{(x,\theta):\ 2n\delta - \frac{\pi}{2} \le \theta \le \frac{\pi}{2},\ 0 \le x \le x_{n2}(\theta,\ell_{n-1,1})\}.$$

The set $S'_{n3}(\ell_{n-1,1})$ is represented by $C_{2n\delta - \frac{\pi}{2}} P_n(\ell_{n-1,1})$ C in Fig. 12.

1.1.11. Determination of the Set $D_{11}(\ell)$ for $\ell\in[\ell_{n-1,1},\ell_{n-1,2}]$, where $N = 2n+1$.

In order to find the distribution function $F_N(\ell)$ of L for $\ell\in[\ell_{n-1,1},\ell_{n-1,2}]$ we require the set $\{(x,\theta): L(x,\theta) \leq \ell,$ $\ell\in[\ell_{n-1,1},\ell_{n-1,2}]\}$ in the parameter space. In the following lemma we obtain this set.

Lemma 8. Let N be odd and equal to $2n+1$. Let $\ell\in[\ell_{n-1,1},\ell_{n-1,2}]$, where $\ell_{n-1,1}$ and $\ell_{n-1,2}$ are given by (1.1.67) and (1.1.68), respectively. Then

(1.1.92) $$D_{11}(\ell) = S'_{n1}(\ell)\cup S'_{n2}(\ell)\cup S'_{n4}(\ell)\cup S'_{n5}(\ell),$$

where $S'_{n1}(\ell)$, $S'_{n2}(\ell)$ are given by (1.1.71) and (1.1.72), respectively.

(1.1.93) $$S'_{n4}(\ell) = \{(x,\theta): \theta_n(\ell) \leq \theta \leq \frac{\pi}{2},\ 0 \leq x \leq x_{n3}(\theta,\ell)\},$$

and

(1.1.94) $$S'_{n5}(\ell) = \{(x,\theta): \theta_1(\ell) \leq \theta \leq \frac{\pi}{2},\ x_1(\theta,\ell) \leq x \leq a\},$$

where

(1.1.94a) $$x_{n3}(\theta,\ell) = -d_n + \frac{\ell \sin(2n\delta-\theta)}{\sin 2n\delta};$$

(1.1.94b) $$\theta_1(\ell) = \pi - 2n\delta + \sin^{-1}\left(\frac{\ell_{n-1}\sin n\delta}{\ell}\right)$$

(1.1.94c) $$x_1(\theta,\ell) = a + d_n - \frac{\ell}{\sin 2n\delta}\sin(\theta - \pi + 2n\delta).$$

<u>Proof.</u> (cf. Fig. 13). By Lemma 6, we have $S'_{n1}(\ell) \cup S'_{n2}(\ell) \subset D_{11}(\ell)$. We now show that $S'_{n4}(\ell) \subset D_{11}(\ell)$. Let $\theta \in [\beta_n(\ell), \frac{\pi}{2}]$. Let $x_{n3}(\theta,\ell)$ be the value of X such that for a fixed θ, $L(x_{n3}(\theta,\ell),\theta) = \ell$. Then if $P_{x_{n3}}(\theta,\ell)Q(x_{n3}(\theta,\ell),\ell)$ is the corresponding secant, where $Q(x_{n3}(\theta,\ell),\ell)$ lies on A_nA_{n+1}, we have, considering the triangle $P_{x_{n3}}(\theta,\ell)Q(x_{n3}(\theta,\ell),\ell)D_n$, by trigonometry,

(1.1.95) $$x_{n3}(\theta,\ell) = -d_n + \frac{\ell\sin(2n\delta - \theta)}{\sin 2n\delta}.$$

For a fixed θ in $[\beta_n(\ell), \frac{\pi}{2}]$

$$0 \le x \le x_{n3}(\theta,\ell) \Rightarrow L(x,\theta) \le \ell.$$

Hence

$$(x,\theta) \in S'_{n4}(\ell) \Rightarrow L(x,\theta) \le \ell.$$

Therefore we have

(1.1.96) $$S'_{n4}(\ell) \subset D_{11}(\ell).$$

Next we show that $S'_{n5}(\ell) \subset D_{11}(\ell)$ (cf. Fig. 13). Let R be the point on $A_{n+1}A_{n+2}$ of the polygon such that $|A_0R| = \ell$.

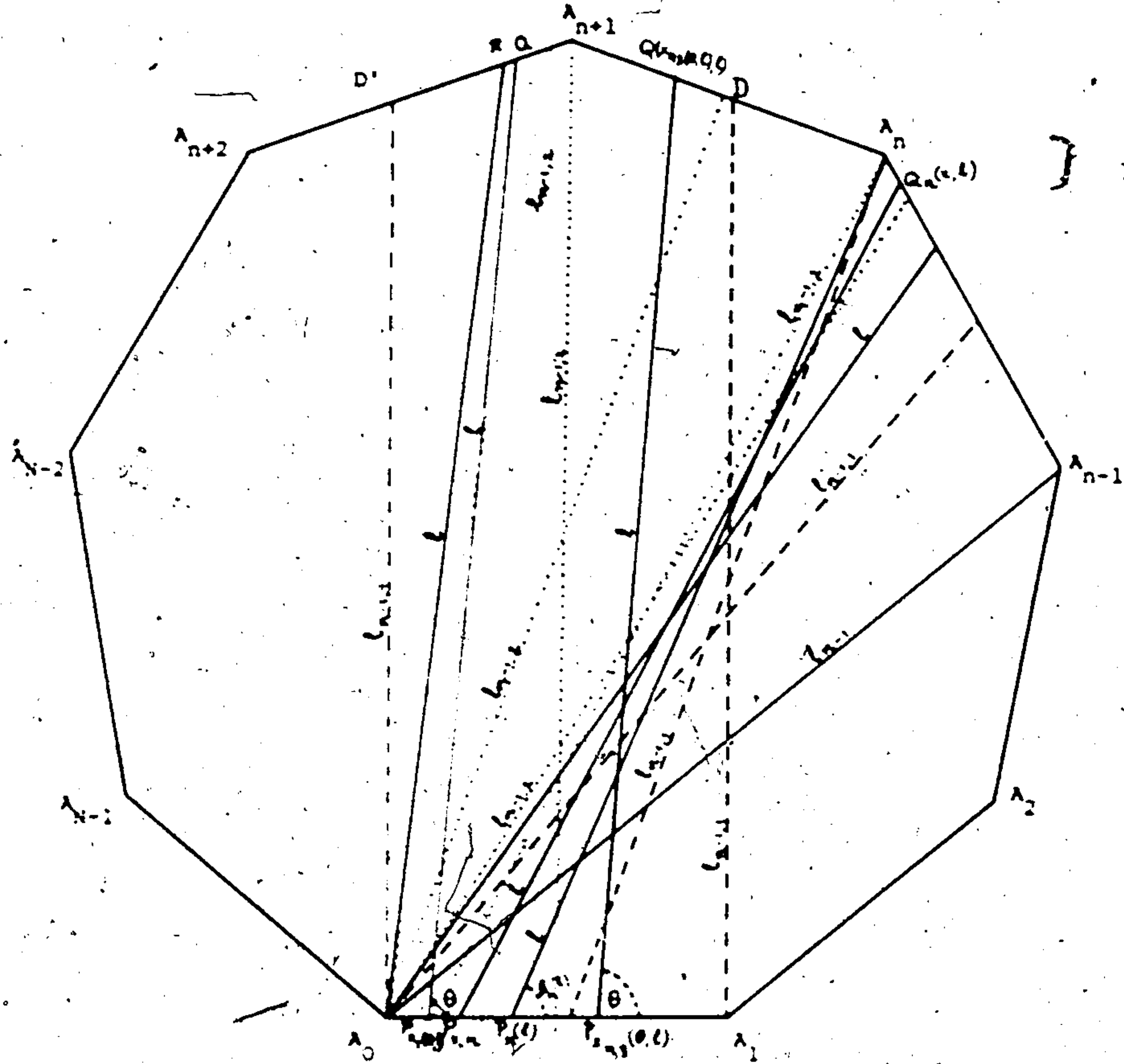

Fig. 13

Let A_0D' be drawn perpendicular to A_1A_0 and D' be the point where the perpendicular intersects $A_{n+1}A_{n+2}$. Let $\measuredangle D'A_0R = \theta'(\ell)$ and $\measuredangle RA_0A_1 = \theta_1(\ell)$. Then considering the triangle $A_0D'R$ we obtain

$$\theta'(\ell) = 2n\delta - \frac{\pi}{2} - \sin^{-1}\left(\frac{\ell_{n-1}\sin n\delta}{\ell}\right)$$

and therefore

(1.1.97) $$\theta_1(\ell) = \pi - 2n\delta + \sin^{-1}\left(\frac{\ell_{n-1}\sin n\delta}{\ell}\right).$$

Let now $\theta\epsilon[\theta_1(\ell), \frac{\pi}{2}]$. Let $x_1(\theta,\ell)$ be the value of X such that $L(x_1(\theta,\ell),\theta) = \ell$. Let $P_{x_1(\theta,\ell)}Q$ be the corresponding secant. Extend the side $A_{n+1}A_{n+2}$ of the polygon to meet A_1A_0 (extended) at D_n'. Then considering the triangle $P_{x_1(\theta,\ell)}, Q\, D_n'$, we obtain

$$x_1(\theta,\ell) = a + d_n - \frac{\ell \sin(\theta-\pi+2n\delta)}{\sin 2n\delta}.$$

For a fixed θ in $[\theta_1(\ell),\frac{\pi}{2}]$, $L(x,\theta) \leq \ell$ whenever $x_1(\theta,\ell) \leq x \leq a$ and consequently, $L(x,\theta) \leq \ell$ whenever $(x,\theta)\epsilon\, S_{n5}'(\ell)$.

Hence

(1.1.98) $$S_{n5}'(\ell) \subset D_{11}(\ell).$$

Combining (1.1.70), (1.1.90) and (1.1.98) we obtain

(1.1.99) $$S_{n1}'(\ell) \cup S_{n2}'(\ell) \cup S_{n4}'(\ell) \cup S_{n5}'(\ell) \subset D_{11}(\ell).$$

Let

$$(x,\theta) \notin S'_{n1}(\ell) \cup S'_{n2}(\ell) \cup S'_{n4}(\ell) \cup S'_{n5}(\ell).$$

Then

$$(x,\theta) \in S - \bigcup_{i=1} (S'_{ni}(\ell) \cup S'_{n4}(\ell) \cup S'_{n5}(\ell).$$

Consequently, by arguments similar to those used to prove (1.1.91) we have $(x,\theta) \notin D_{11}(\ell)$. Hence

(1.1.100) $$D_{11}(\ell) \subset S'_{n1}(\ell) \cup S'_{n2}(\ell) \cup S'_{n4}(\ell) \cup S'_{n5}(\ell).$$

(1.1.92) now follows from (1.1.99) and (1.1.100).

1.1.11A. Graphical description of the set $D_{11}(\ell)$ for $\ell\in[\ell_{n-1,1},\ell_{n-1,2}]$, when $N = 2n+1$.

For $\ell\in[\ell_{n-1,1},\ell_{n-1,2}]$ we have by Lemma 8,

$$D_{11}(\ell) = S'_{n1}(\ell)\cup S'_{n2}(\ell)\cup S'_{n4}(\ell)\cup S'_{n5}(\ell). \tag{1.1.92}$$

where $S'_{n1}(\ell)$, $S'_{n2}(\ell)$, $S'_{n4}(\ell)$ and $S'_{n5}(\ell)$ are given by (1.1.71), (1.1.72), (1.1.93), and (1.1.94), respectively.

In the parameter space S (cf. Fig. 14) the set $S'_{n1}(\ell)$ is bounded by $x = 0$, $x = x_n(\ell)$, $\theta = 0$ and $\theta = \theta_n(\ell)$ and is represented by the rectangle $OX_n(\ell)P_n(\ell)C_{\theta_n}(\ell)$.

The locus of $P_n(\ell)$ in the parameter space S is given by

$$x = \ell_{n-1}(\cot\theta \sin n\delta - \cos n\delta). \tag{1.1.11A.1}$$

Now

$$\ell = \ell_{n-1,1} \Rightarrow \begin{cases} \theta_n(\ell) = 2n\delta - \dfrac{\pi}{2}, \\ x_n(\ell) = -\ell_{n-1}\dfrac{\cos n\delta}{\cos 2n\delta} \end{cases}$$

Thus

$$S'_{n1}(\ell_{n-1,1}) = \left\{(x,\theta): \begin{cases} 0\le\theta\le 2n\delta - \dfrac{\pi}{2} \\ 0\le x\le -\ell_{n-1}\dfrac{\cos n\delta}{\cos 2n\delta} \end{cases}\right\}. \tag{1.1.11A.2}$$

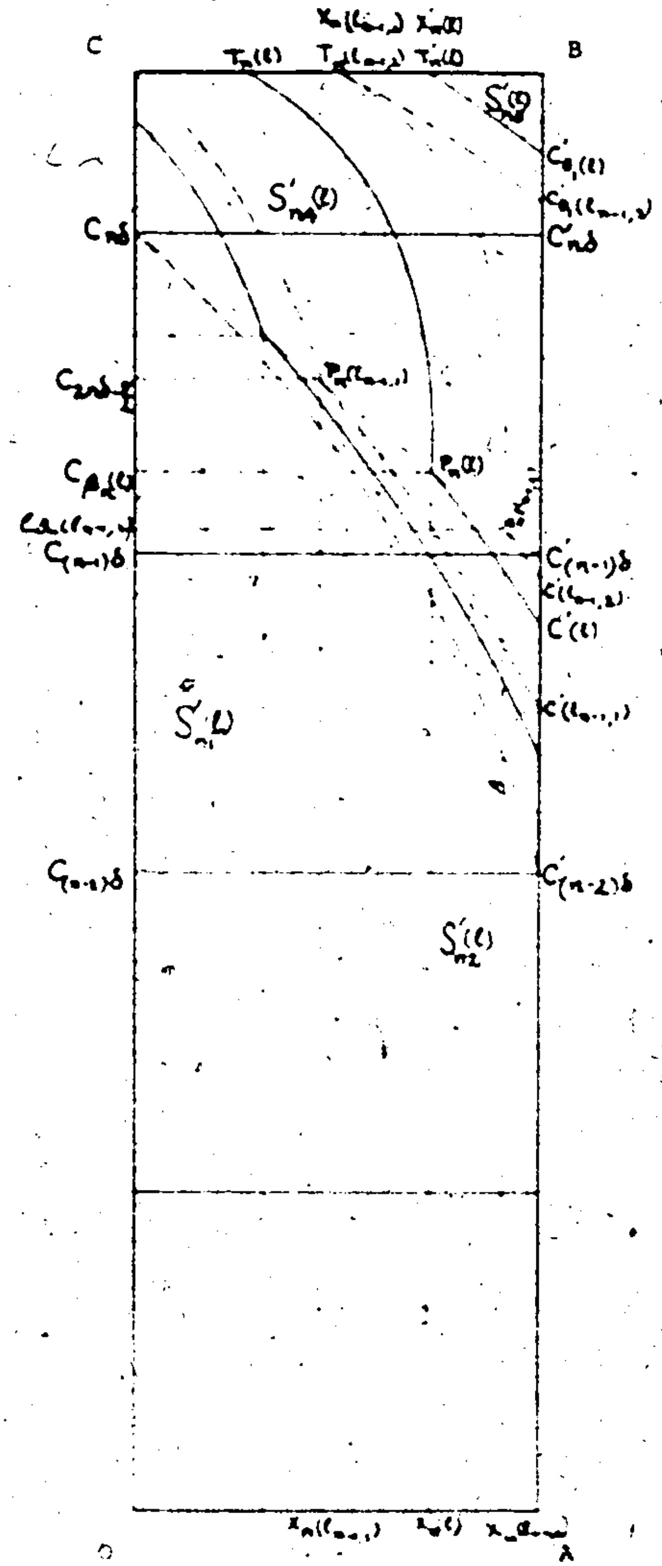

Fig. 14

Now

$$\ell = \ell_{n-1,2} \Rightarrow \begin{cases} \beta_n(\ell) = \sin^{-1}\left(\dfrac{\ell_{n-1} \sin n\delta}{\ell_{n-1,2}}\right), \\ x_n(\ell) = \ell_{n-1,2} \dfrac{\sin[n\delta - \beta_n(\ell_{n-1,2})]}{\sin n\delta}. \end{cases}$$

Thus

$$(1.1.11A.3) \qquad S'_{n1}(\ell_{n-1,2}) = \left\{(x,\theta): \begin{cases} 0 \le x \le x_n(\ell_{n-1,2}) \\ 0 \le \theta \le \beta_n(\ell_{n-1,2}) \end{cases}\right\}.$$

The set $S_{n1}(\ell_{n-1,2})$ is represented by $OX_n(\ell_{n-1,2})P_n(\ell_{n-1,2})C_{\beta_n}(\ell_{n-1,2})$ in Fig. 14.

In the parameter space S the set $S'_{n2}(\ell)$ is bounded by $x = a$, $x = x_n(\ell)$, $\theta = 0$ and $\theta = \theta_n(x,\ell)$, where $\theta = \theta_n(x,\ell)$ is a sine curve and is represented by $X_n(\ell)P_n(\ell)C'(\ell)A$.

Now

$$\ell = \ell_{n-1,1} \Rightarrow \begin{cases} x_n(\ell) = -\dfrac{\ell_{n-1} \cos n\delta}{\cos 2n\delta}, \\ \theta_n(x_n(\ell), \ell) = 2n\delta - \dfrac{\pi}{2}, \\ \theta_n(a,\ell) = \dfrac{\pi}{2} - 2\delta. \end{cases}$$

Consequently,

$$(1.1.11A.4) \qquad S_{n2}(\ell_{n-1,1}) = \left\{(x,\theta): \begin{cases} -\ell_{n-1}\dfrac{\cos n\delta}{\cos 2n\delta} \le x \le a, \\ 0 \le \theta \le \theta_n(x,\ell_{n-1,1}) \end{cases}\right\}$$

Now

$$\ell = \ell_{n-1,1} \Rightarrow \begin{cases} \beta_n(\ell) = 2n\delta - \frac{\pi}{2} , \\ x_{n3}(\theta,\ell) = -d_n + \dfrac{\ell_{n-1,1} \sin(2n\delta-\theta)}{\sin 2n\delta} , \\ x_n(\ell_{n-1,1}) = x_{n3}[\beta_n(\ell_{n-1,1}), \ell_{n-1,1}] \\ \qquad = -d + \dfrac{\ell_{n-1,1}}{\sin 2n\delta} , \\ x_{n3}(\frac{\pi}{2},\ell) = 0 . \end{cases}$$

The set $S'_{n4}(\ell_{n-1,1})$ is represented by $CC_{2n\delta-\frac{\pi}{2}}P_n(\ell_{n-1,1})$ in Fig. 14.

Now,

$$\ell = \ell_{n-1,2} \Rightarrow \begin{cases} \beta_n(\ell) = \beta_n(\ell_{n-1,2}) \\ x_n(\theta,\ell_{n-1,2}) = x_{n3}(\beta(\ell_{n-1,2}), \ell_{n-1,2}) , \\ x_{n3}(\frac{\pi}{2}, \ell_{n-1,2}) = a/2. \end{cases}$$

Thus

(1.1.11A.7)

$$S'_{n4}(\ell_{n-1,2}) = \left\{(x,\theta) : \begin{array}{l} \beta_n(\ell_{n-1,2}) \le \theta \le \frac{\pi}{2} \\ 0 \le x \le x_n(\theta,\ell_{n-1,2}) \end{array}\right\} .$$

The set $S'_{n4}(\ell_{n-1,2})$ is represented by $C_{\beta_n(\ell_{n-1,2})}P_n(\ell_{n-1,2})T_n(\ell_{n-1,2})C$ in Fig. 14.

The set $S'_{n2}(\ell_{n-1,1})$ is represented by $X_n(\ell_{n-1,1})P_n(\ell_{n-1,1})C'(\ell_{n-1,1})A$.

Now

$$\ell = \ell_{n-1,2} \Rightarrow \begin{cases} x_n(\ell) = \ell_{n-1,2}\dfrac{\sin\{n\delta-\beta_n(\ell_{n-1,2})\}}{\sin n\delta}, \\ \theta_n(x_n(\ell),\ell) = \beta_n(\ell_{n-1,2}), \\ \theta_n(a,\ell) = 2(n-1)\delta - \sin^{-1}\{\dfrac{(a+d_{n-1})\sin 2(n-1)\delta}{\ell_{n-1,2}}\}. \end{cases}$$

Hence

$$(1.1.11A.5)\quad S'_{n2}(\ell_{n-1,2}) = \{(x,\theta) \begin{Bmatrix} x_n(\ell_{n-1,2}) < x < a, \\ 0 < \theta < \theta_n(x,\ell_{n-1,2}) \end{Bmatrix}\}.$$

The set $S'_{n2}(\ell_{n-1,2})$ is represented by $X_n(\ell_{n-1,2})P_n(\ell_{n-1,2})C'(\ell_{n-1,2})A$ in Figure 14.

In the parameter space S, the set $S'_{n4}(\ell)$ is bounded by $\theta = \beta_n(\ell)$, $\theta = \frac{\pi}{2}$, $x = 0$ and

$$(1.1.11A.6)\qquad x = x_{n3}(\theta,\ell)\ ,$$

where $x_{n3}(\cdot,\ell)$ is given by (1.1.94A). We note that the equation (1.1.11A.6) represents a sine curve.

The set $S'_{n4}(\ell)$ is represented by $C_{\beta_n(\ell)}P_n(\ell)T_n(\ell)C$ in Fig. 14.

In S the set $S'_{n5}(\ell)$ is bounded by $\theta = \theta_1(\ell)$, where $\theta_1(\ell) = \pi - 2n\delta + \sin^{-1}(\frac{\ell_{n-1} \sin n\delta}{\ell})$, $\theta = \frac{\pi}{2}$, $x = a$ and

(1.1.11A.8) $$x = x_1(\theta,\ell) = a + d_n + \frac{\ell \sin(\theta+2n\delta)}{\sin 2n\delta},$$

The set $S'_{n5}(\ell)$ is represented by $T'_n(\ell)$ B $C''(\ell)$ in Fig. 14. Now

$$\ell = \ell_{n-1,1} \Rightarrow \theta_1(\ell) = \frac{\pi}{2} \text{ and } x_1(\frac{\pi}{2}, \ell_{n-1,1}) = a .$$

Consequently,

(1.1.11A.9) $$S'_{n5}(\ell_{n-1,1}) = \{a, \frac{\pi}{2}\},$$

Thus the set $S'_{n5}(\ell_{n-1,1})$ is represented by the point B in Fig. 14.

Next

$$\ell = \ell_{n-1,2} \Rightarrow \begin{cases} \theta_1(\ell) = \pi - 2n\delta + \sin^{-1}\left(\frac{2\sin(n-1)\delta \cos n\delta}{\sin \delta}\right), \\ x_1(\theta_1(\ell),\ell) = a . \end{cases}$$

$$\ell = \ell_{n-1,2}, \ \theta = \frac{\pi}{2} \Rightarrow x_1(\theta_1(\ell),\ell),\ell) = a/2 .$$

Thus

(1.1.11A.10) $$S'_{n5}(\ell_{n-1,2}) = \{(x,\theta): \theta_1(\ell_{n-1,2}) \leq \theta \leq \frac{\pi}{2}, x_1(\theta,\ell_{n-1,2}) \leq x \leq a.$$

The set $S'_{n5}(\ell_{n-1,2})$ is represented by $T'_n(\ell_{n-1,2})C'_{\theta_1(\ell_{n-1,2})}$ B in Fig. 14.

1.1.12. Determination of the set $D_{11}(\ell)$ for $\ell\epsilon[\ell_{n-1,2},\ell_n]$.

In order to find the distribution function $F_N(\ell)$ of L for $\ell\epsilon[\ell_{n-1,2},\ell_n]$, we require the set $\{(x,\theta): L(x,\theta) \leq \ell,\ \ell\epsilon[\ell_{n-1,2},\ \ell_n]\}$ in the parameter space S. In the following lemma we obtain this set.

Lemma 9. Let N be odd and equal to 2n+1. Let $\ell\epsilon[\ell_{n-1,2},\ell_n]$ where $\ell_{n-1,2}$ is given by (1.1.68). Then

(1.1.101) $$D_{11}(\ell) = \bigcup_{i=1}^{7} S_i(\ell) \cup S'(\ell),$$

where

(1.1.102) $$S'(\ell) = \{(x,\theta): 0\leq x\leq a,\ 0\leq\theta\leq\theta_1(\ell)\}.$$

(1.1.103) $$S_1(\ell) = \{(x,\theta): \theta_1(\ell)\leq\theta\leq\theta_n(\ell),\ 0\leq x\leq x_{n4}(\theta,\ell)\},$$

(1.1.104) $$S_2(\ell) = \{(x,\theta): \theta_n(\ell)\leq\theta\leq\theta_2(\ell),\ 0\leq x\leq x_{n5}(\theta,\ell)\},$$

(1.1.105) $$S_3(\ell) = \{(x,\theta): \theta_2(\ell)\leq\theta\leq\theta_3(\ell),\ 0\leq x\leq a\},$$

(1.1.106) $$S_4(\ell) = \{(x,\theta): \theta_3(\ell)\leq\theta\leq\theta_4(\ell),\ 0\leq x\leq x_{n5}(\theta,\ell)\},$$

(1.1.107) $$S_5(\ell) = \{(x,\theta): \theta_4(\ell)\leq\theta\leq\theta_5(\ell),\ 0\leq x\leq x_{n5}(\theta,\ell)\},$$

(1.1.108) $$S_6(\ell) = \{(x,\theta): \theta_4(\ell)\leq\theta\leq\theta_5(\ell),\ x_{n6}(\theta,\ell)\leq x\leq a\},$$

(1.1.109) $$S_7(\ell) = \{(x,\theta): \theta_5(\ell)\leq\theta\leq\frac{\pi}{2},\ 0\leq x\leq a\},$$

where

(1.1.109a) $\theta_1(\ell) = 2(n-1)\delta - \sin^{-1}\left[\frac{a+d_{n-2}}{\ell}\sin 2(n-1)\delta\right],$

(1.1.109b) $\beta_n(\ell) = \sin^{-1}\left(\frac{\ell_{n-1}\sin n\delta}{\ell}\right),$

(1.1.109c) $\theta_2(\ell) = (n-1)\delta + \frac{1}{2}\{\delta - 2\cos^{-1}\left(\frac{\ell_{n-1,2}}{\ell}\right)\},$

(1.1.109d) $\theta_3(\ell) = \frac{\pi}{2} - \delta + \cos^{-1}\left(\frac{\ell_{n-1,2}}{\ell}\right),$

(1.1.109e) $\theta_4(\ell) = \delta + \sin^{-1}\left(\frac{a+d_{n-1}}{\ell}\sin 2(n-1)\delta\right),$

(1.1.109f) $\theta_5(\ell) = \sin^{-1}\left(\frac{\ell_{n-1,2}}{\ell}\right),$

(1.1.109g) $x_{n4}(\theta,\ell) = -d_{n-1} + \frac{\ell\sin[2(n-1)\delta-\theta]}{\sin 2(n-1)\delta},$

(1.1.109h) $x_{n5}(\theta,\ell) = -d_n + \frac{\ell\sin(2n\delta-\theta)}{\sin 2n\delta},$

(1.1.109i) $x_{n6}(\theta,\ell) = a + d_n + \frac{\ell\sin(\theta+2n\delta)}{\sin 2n\delta},$

<u>Proof</u>. (cf. Fig. 15A). By (1.1.37) where $k = n$ we have $S_{n1}(\ell) = S'(\ell)CD_{11}(\ell)$. We now show that $S_i(\ell)CD_{11}(\ell)$, $i = 1,2,\ldots,7$.

With A_0 as the centre and ℓ as the radius we draw a circle.

The circle intersects $A_{n-1}A_n$ at a point B_1, A_nA_{n+1} at points B_2, B_3 and $A_{n+1}A_{n+2}$ at a point B_4. From trigonometric considerations the angles $\angle B_iA_0A_1 = \theta_i$, $i = 2,3,4,5$ are found to be

(1.1.109a) $\theta_1(\ell) = 2(n-1)\delta - \sin^{-1}\left[\frac{a+d_{n-1}}{\ell}\sin 2(n-1)\delta\right],$

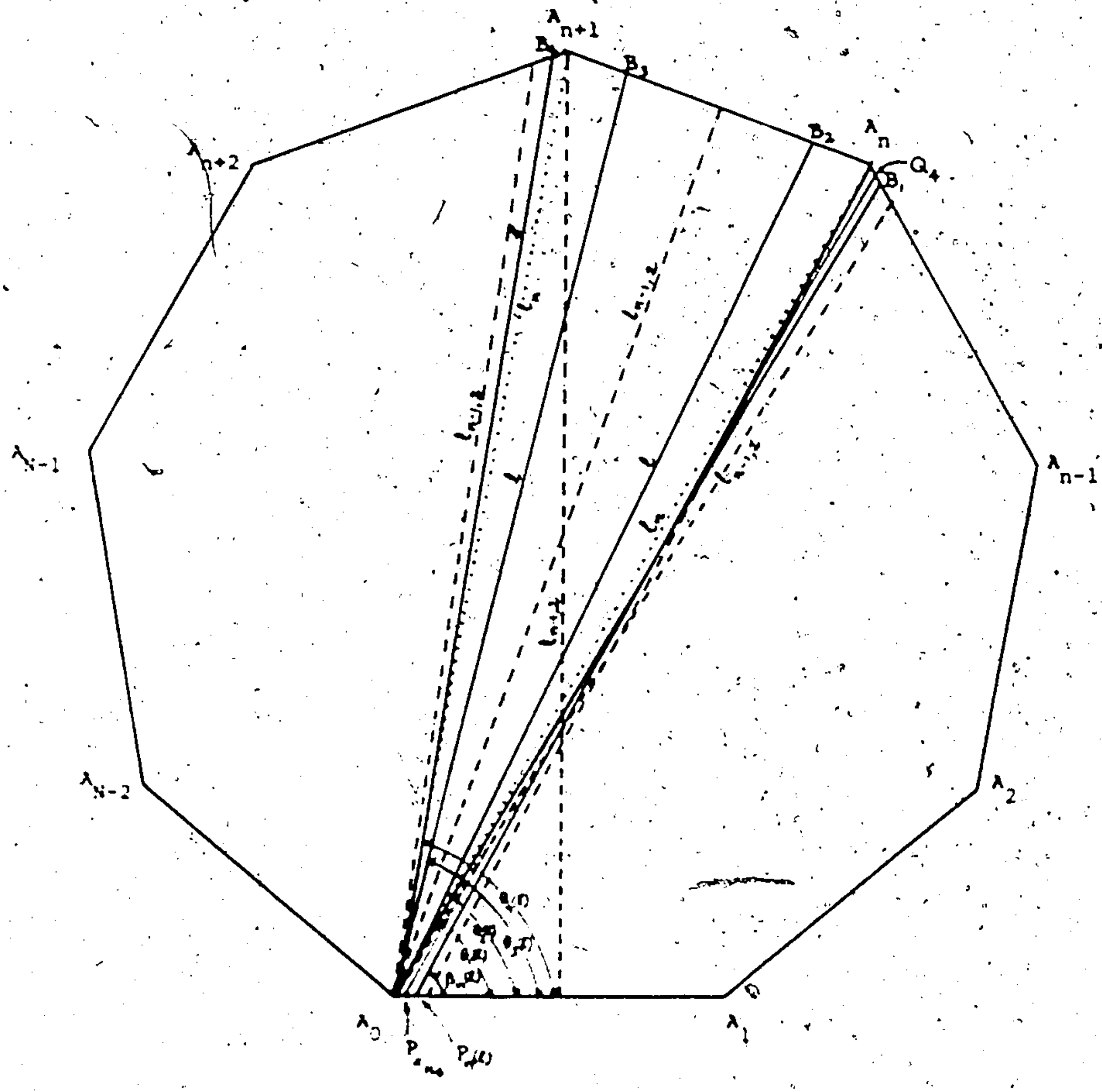

FIG. 15A

(1.1.109c) $$\theta_2(\ell) = (n-1)\delta + \frac{1}{2}[\delta - 2\cos^{-1}(\frac{\ell_{n-1,2}}{\ell})],$$

(1.1.109d) $$\theta_3(\ell) = \frac{\pi}{2} - \delta + \cos^{-1}(\frac{\ell_{n-1,2}}{\ell}),$$

and

(1.1.109e) $$\theta_4(\ell) = \delta + \sin^{-1}[\frac{a+d_{n-1}}{\ell}\sin 2(n-1)\delta].$$

With A_n as the centre we draw a circle of radius ℓ. The circle intersects A_0A_1 at a point $P_n(\ell)$. Let $|A_1P_n(\ell)| = x_n(\ell)$ and $\measuredangle A_nP_n(\ell)A_1 = \beta_n(\ell)$. Then by (1.1.73) we have

(1.1.109b) $$\beta_n(\ell) = \sin^{-1}(\frac{\ell_{n-1}\sin n\delta}{\ell})$$

and by (1.1.74)

$$x_n(\ell) = \frac{\ell\sin[n\delta - \beta_n(\ell)]}{\sin n\delta}.$$

Let $\theta \in [\theta_1(\ell), \beta_n(\ell)]$. Let $x_{n4}(\theta,\ell)$ be the value of X such that $L(x_{n4}(\theta,\ell),\theta) = \ell$. Let $P_{x_{n4}(\theta,\ell)}Q_4$ be the corresponding secant (cf. Fig. 15A) of length ℓ. Then considering the triangle $P_{x_{n4}}Q_4D_{n-1}$, we obtain

(1.1.110) $$\ell = \frac{\sin(2(n-1)\delta)\,(x_{n4}+d_{n-1})}{\sin[2(n-1)\delta-\theta]}.$$

From (1.1.110) we have

(1.1.111) $$x_{n4}(\theta,\ell) = -d_{n-1} + \frac{\ell \sin[2(n-1)\delta - \theta]}{\sin 2(n-1)\delta} .$$

For every fixed $\theta \in [\theta_1(\ell), \beta_n(\ell)]$ we have by (1.1.110), $L(x,\theta) \leq \ell$,

whenever $0 \leq x \leq x_{n4}(\theta,\ell)$

Therefore

(1.1.112) $$S_1(\ell) \subset D_{11}(\ell) .$$

Let $\theta \in [\beta_n(\ell), \theta_2(\ell)]$. Let $P_{x_{n5}(\theta,\ell)}Q_5$ be the corresponding secant of length ℓ (cf. Fig. 15B). Then from the triangle $P_{x_{n5}(\theta,\ell)}Q_5D_n$ we have

(1.1.113) $$\ell = \frac{(x_{n5}(\theta,\ell) + d_n)\sin 2n\delta}{\sin(2n\delta - \theta)} .$$

From (1.1.113) we obtain

(1.1.114) $$x_{n5}(\theta,\ell) = -d_n + \frac{\ell \sin(2n\delta - \theta)}{\sin 2n\delta} .$$

By the use of (1.1.113), we find that $x \leq x_{n5}(\theta,\ell)$ implies $L(x,\theta) \leq \ell$ for $\theta \in [\beta_n(\ell), \theta_2(\ell)]$. Hence

(1.1.115) $$S_2(\ell) \subset D_{11}(\ell).$$

It is easy to check that

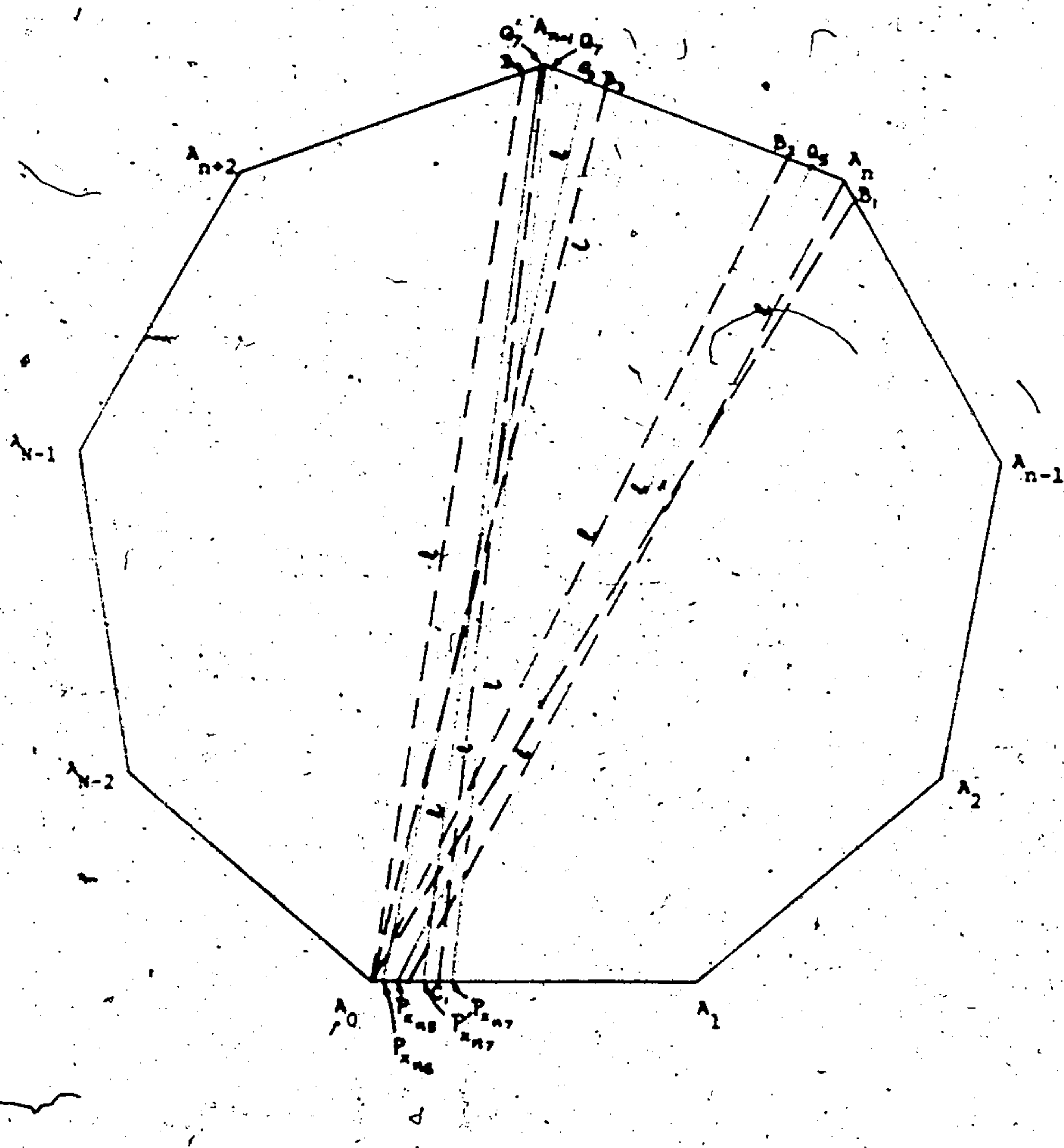

Fig. 15B

$L(x,\theta) \leq \ell$ whenever

$$(x,\theta) \in S_3(\ell) = \{(x,\theta) : \theta_2(\ell) \leq \theta \leq \theta_3(\ell),\ 0 \leq x \leq a\} .$$

Hence

(1.1.116) $$S_3(\ell) \subset D_{11}(\ell) .$$

Let $\theta \in [\theta_3(\ell), \theta_4(\ell)]$. Let $P_{x_{n6}(\ell)}\ Q_6$ be the secant with $L(x_{n6}(\ell), \theta) = \ell$. Then from the triangle $P_{x_{n6}(\ell)}\ Q_6 A_1$ we obtain

(1.1.117) $$x_{n6}(\ell) = -d_n + \frac{\ell \sin (2n\delta - \theta)}{\sin 2n\delta}.$$

It follows by arguments similar to those for showing (1.1.115) that

(1.1.118) $$S_4(\ell) \subset D_{11}(\ell) .$$

With vertex A_{n+1} as the centre, a circle of radius ℓ is drawn. The circle intersects A_0A_1 at points C_1 and C_2.

Now let $\angle A_{n+1}C_1A_1 = \sin^{-1}\left(\frac{\ell_{n-1,2}}{\ell}\right) = \theta_5(\ell)$. Let $\theta \in [\theta_4(\ell), \theta_5(\ell)]$. Then there are two points $P_{x_{n7}(\ell)}$ and $P_{x'_{n7}(\ell)}$ on A_0A_1 such that $L(x_{n7}(\ell),\theta) = L(x'_{n7}(\ell),\theta) = \ell$. Let $P_{x_{n7}(\ell)}Q_7$ and $P_{x'_{n7}(\ell)}Q'_7$ be the corresponding secants of length ℓ. Then from the triangles $Q_7P_{x_{n7}(\ell)}D_n$ and $Q'_7P_{x'_{n7}(\ell)}D'_n$, we obtain

(1.1.119) $$x_{n7}(\ell) = -d_n + \frac{\ell \sin(2n\delta - \theta)}{\sin 2n\delta},$$

(1.1.120) $$x'_{n7}(\ell) = a + d_n + \frac{\ell \sin(\theta + 2n\delta)}{\sin 2n\delta}.$$

It follows by arguments similar to those used to show (1.1.115) that

(1.1.121) $$S_5(\ell) \cup S_6(\ell) \subset D_{11}(\ell).$$

It also follows that

(1.1.122) $$S_7(\ell) \subset D_{11}(\ell).$$

Clearly

(1.1.123) $$(x,\theta) \notin \bigcup_{i=1}^{7} S_i(\ell) \cup S'(\ell) \Rightarrow L(x,\theta) > \ell.$$

By the use of (1.1.37), (1.1.112), (1.1.115), (1.1.116), (1.1.118), (1.1.121), (1.1.122) and (1.1.123) we obtain the Lemma.

A graphical description of the set $D_{11}(\ell)$ for $\ell \in [\ell_{n-1,2}, \ell_n]$ is given in 1.1.12A.

1.1.12A. Graphical description of the set $D_{11}(\ell)$ for $\ell \in [\ell_{n-1,2}, \ell_n]$ when $N = 2n+1$.

For $\ell \in [\ell_{n-1,2}, \ell_n]$ we have by Lemma 9,

(1.1.101) $$D_{11}(\ell) = S'(\ell) \bigcup_{i=1}^{7} S_i(\ell),$$

where $S'(\ell)$, $S_1(\ell)$, ..., $S_7(\ell)$ are given by (1.1.102),........(1.1.109), respectively.

In the parameter space S the set $S'(\ell)$ is bounded by $x = 0$, $x = a$, $\theta = 0$, and $\theta = \theta_1(\ell) = 2(n-1)\delta - \sin^{-1} [\frac{a+d_{n-1}}{\ell} \sin 2(n-1)\delta]$ and is represented by the rectangle $OAC'_{\alpha_n(\ell)}C_{\alpha_n(\ell)}$ in Figure 16.

Now

$$\ell = \ell_{n-1,2} \Rightarrow \theta_1(\ell) = \alpha_n(\ell_{n-1,2}).$$

Consequently,

(1.1.12A.1) $$S'(\ell_{n-1,2}) = \{(x,\theta): 0 \leq x \leq a,\ 0 \leq \theta \leq \alpha_n(\ell_{n-1,2})\}.$$

The set $S'(\ell_{n-1,2})$ is the rectangle $OAC'_{\theta_1(\ell_{n-1,2})}C_{\theta_1(\ell_{n-1,2})}$ in Figure 16.

Now

$$\ell = \ell_n \Rightarrow \alpha_n(\ell) = (n-1)\delta .$$

Therefore

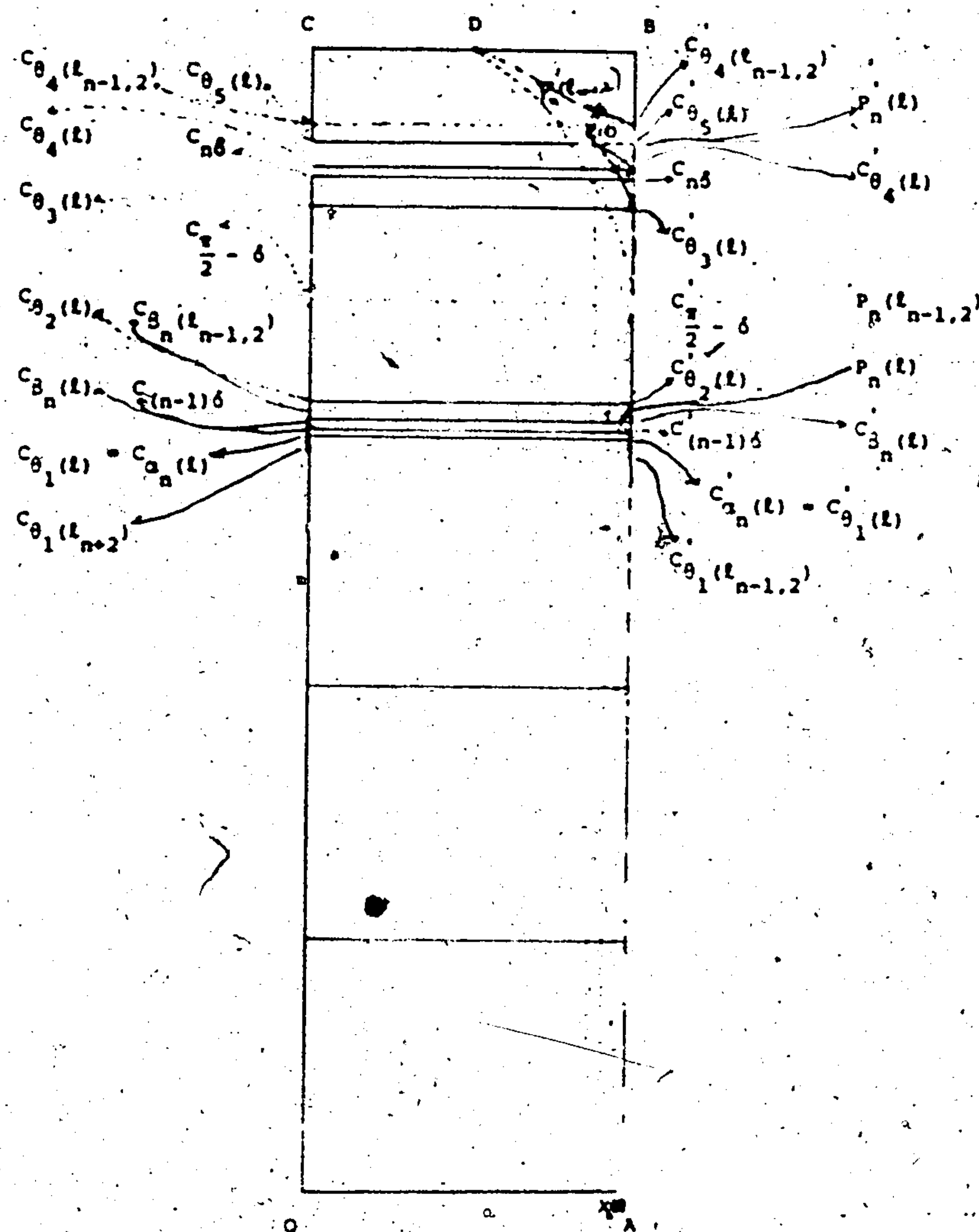

Fig. 16

(1.1.12A.2) $S'(\ell_n) = \{(x,\theta):\ 0\leq x\leq a,\ 0\leq\theta\leq (n-1)\delta\}$.

The set $S'(\ell_n)$ is the rectangle $OAC'_{(n-1)\delta}C_{(n-1)\delta}$ in Figure 16. In S the set $S_1(\ell)$ is bounded by $\theta = \theta_1(\ell)$, $\theta = \beta_n(\ell)$, $x = 0$ and

(1.1.12A.3) $x = x_{n4}(\theta,\ell)$,

where

$$x_{n4}(\theta,\ell) = -d_{n-1} + \frac{\ell \sin[2(n-1)\delta-\theta]}{\sin 2(n-1)\delta} .$$

The equation (1.1.12A.3) represents a sine curve.

The set $S_1(\ell)$ is represented by $C_{\theta_1(\ell)}C_{\beta_n(\ell)}P_n(\ell)C'_{\theta_1(\ell)}$ in Figure 16.

$$\ell = \ell_{n-1,2} \Rightarrow \begin{cases} \theta_1(\ell) = \alpha_n(\ell_{n-1,2}),\ x = a . \\ \beta_n(\ell) = \beta_n(\ell_{n-1,2}),\ x = x_{n4}[\beta_n(\ell_{n-1,2}),\ \ell_{n-1,2}] . \end{cases}$$

(1.1.12A.4) $S_1(\ell_{n-1,2})$

$$= \{(x,\theta):\ \theta_1(\ell_{n-1,2}) \leq\theta\leq \beta_n(\ell_{n-1,2}),\ 0\leq x\leq x_{n4}[\theta,\ell_{n-1,2}]\} .$$

The set $S_1(\ell_{n-1,2})$ is represented by

$C_{\theta_1(\ell_{n-1,2})}C_{\beta_n(\ell_{n-1,2})}P_n(\ell_{n-1,2})C'_{\theta_1(\ell_{n-1,2})}$ in Figure 16.

Next

$$\ell = \ell_n \Rightarrow \begin{cases} \theta_1(\ell) = (n-1)\delta, \ \beta_n(\ell) = (n-1)\delta, \\ x_{n4}(\theta,\ell) = a \quad x_{n4}(\theta,\ell)_p = a . \end{cases}$$

Thus,

(1.1.12A.5) $$S_1(\ell_n) = \{(x,\theta): \theta = (n-1)\delta, \ 0<x<a\}.$$

The set $S_1(\ell_n)$ is represented by the line $C_{(n-1)\delta}C'_{(n-1)\delta}$ in Fig. 16.

In the parameter space S the set $S_2(\ell)$ is bounded by $\theta = \beta_n(\ell)$, $\theta = \theta_2(\ell)$, $x = 0$ and

(1.1.12A.6) $$x = x_{n5}(\theta,\ell) ,$$

where $x_{n5}(\theta,\ell) = -d_n + \frac{\ell \sin(2n\delta - \theta)}{\sin 2n\delta}$ and $\theta_2(\ell)$ is given by (1.1.109c).

The equation (1.1.13A.6) represents a sine curve.

The set $S_2(\ell)$ is represented by $C_{\beta_n(\ell)}P_n(\ell)C'_{\theta_2(\ell)}C_{\theta_2(\ell)}$ in Fig. 16.

$$\ell = \ell_{n-1,2} \Rightarrow \begin{cases} \beta_n(\ell) = \beta_n(\ell_{n-1,2}), \\ x = x_{n3}[\beta_n(\ell_{n-1,2}), \ell_{n-1,2}] . \\ \theta_2(\ell) = \frac{\pi}{2} - \delta \text{ and } x = a. \end{cases}$$

Hence

(1.1.12A.7) $S_2(\ell_{n-1,2})$

$$= \{(x,\theta):\ \beta_n(\ell_{n-1,2}) \le \theta \le \theta_2(\ell_{n-1,2}),\ 0 \le x \le x_{n5}(\theta,\ell_{n-1})\}.$$

The set $S_2(\ell_{n-1,2})$ is represented by $C_{\beta_n(\ell_{n-1,2})} C_{\frac{\pi}{2}-\delta} C'_{\frac{\pi}{2}-\delta} P_n(\ell_{n-1,2})$ in Fig. 16.

$$\ell = \ell_n \Rightarrow \begin{cases} \beta_n(\ell) = (n-1)\delta,\ x = a, \\ \theta_2(\ell) = (n-1)\delta,\ x = a. \end{cases}$$

Hence

(1.1.12A.8) $S_2(\ell_n) = \{(x,\theta):\ (n-1)\delta = \theta \text{ and } 0 \le x \le a\}$.

The set $S_2(\ell_n)$ is represented by $C_{(n-1)\delta}\, C'_{(n-1)\delta}$ in Figure 16.

In the parameter space S the set $S_3(\ell)$ is represented by $\theta = \theta_2(\ell)$, $\theta = \theta_3(\ell)$, $x = 0$ and $x = a$, where $\theta_2(\ell)$ and $\theta_3(\ell)$ are given by (1.1.109c) and (1.1.109d), respectively.

The set $S_3(\ell)$ is represented by the rectangle $C_{\theta_2(\ell)} C'_{\theta_2(\ell)} C_{\theta_3(\ell)} C'_{\theta_3(\ell)}$ in Fig. 16.

Now

$$\ell = \ell_{n-1,2} \Rightarrow \begin{cases} \theta_2(\ell) = \frac{\pi}{2} - \delta,\ x = a, \\ \theta_3(\ell) = \frac{\pi}{2} - \delta,\ x = a. \end{cases}$$

Hence

(1.1.12A.9) $$S_3(\ell_{n-1,2}) = \{(x,\theta): 0<x<a,\ \theta = \frac{\pi}{2} - \delta\}.$$

The set $S_3(\ell_{n-1,2})$ is represented by the line $C'_{\frac{\pi}{2}-\delta}\ C_{\frac{\pi}{2}-\delta}$.

Next

$$\ell = \ell_n \Rightarrow \begin{cases} \theta_2(\ell) = (n-1)\delta,\ x = a, \\ \theta_3(\ell) = n\delta, \quad x = a. \end{cases}$$

Hence

(1.1.12A.10) $$S_3(\ell_n) = \{(x,\theta): 0<x<a,\ (n-1)\delta<\theta<n\delta\}.$$

The set $S_3(\ell_n)$ is represented by $C_{(n-1)\delta}C'_{(n-1)\delta}C'_{n\delta}C_{n\delta}$.

In the parameter space S the set $S_4(\ell)$ is bounded by $\theta = \theta_3(\ell)$, $\theta = \theta_4(\ell)$, $x = 0$ and $x = x_{n5}(\theta,\ell)$, where $x_{n5}(\theta,\ell) = -d_n + \frac{\ell \sin(2n\delta - \theta)}{\sin 2n\delta}$.

The set $S_4(\ell)$ is represented by $C_{\theta_4(\ell)}P'_n(\ell)C'_{\theta_3(\ell)}C_{\theta_3(\ell)}$ in Figure 16.

Now

$$\ell = \ell_{n-1,2} \Rightarrow \begin{cases} \theta_3(\ell) = \frac{\pi}{2} - \delta,\ x = a, \\ \theta_4(\ell) = \pi - 2\delta - \alpha_n(\ell_{n-1,2}) \\ \quad x = x_{n5}[\ell_{n-1,2},\ \theta_4(\ell_{n-1,2})] \end{cases}$$

Hence

(1.1.12A.11)
$$S_4(\ell_{n-1,2})$$

$$= \{(x,\theta):\ \theta_3(\ell_{n-1,2}) < \theta < \theta_4(\ell_{n-1,2}),\ 0<x<x_{n5}[\ell_{n-1,2},\ \theta_4(\ell_{n-1,2})]\}.$$

The set $S_4(\ell_{n-1,2})$ is represented by $C_{\frac{\pi}{2}-\delta} C'_{\frac{\pi}{2}-\delta} P'_n(\ell_{n-1,2}) C_{\theta_4(\ell_{n-1,2})}$ in Fig. 16.

Now

$$\ell = \ell_n \Rightarrow \theta_3(\ell) = n\delta,\ x = a;\ \theta_4(\ell) = n\delta,\ x = a .$$

Hence
$$S_4(\ell_n) = \{(x,\theta):\ 0 \le x \le a,\ \theta = n\delta\} .$$

The set $S_4(\ell_n)$ is the line $C_{n\delta}C'_{n\delta}$.

In the parameter space S, the set $S_5(\ell)$ is bounded by $\theta = \theta_4(\ell)$, $\theta = \theta_5(\ell)$, $x = 0$ and $x = x_{n5}(\theta,\ell)$, where $x_{n5}(\theta,\ell)$, $\theta_5(\ell)$ and $\theta_4(\ell)$ are given by (1.1.109.h), (1.1.109.f) and (1.1.109.e), respectively.

The set $S_5(\ell)$ is represented by $C_{\theta_4(\ell)} C_{\theta_5(\ell)} P''_n(\ell) P'_n(\ell)$ in Fig. 16.

Now

$$\ell = \ell_{n-1,2} \Rightarrow \begin{cases} \theta_4(\ell) = \pi - 2\delta - \alpha_n(\ell_{n-1,2}), \\ x_{n5}(\theta,\ell) = x_{n6}[\theta_4(\ell_{n-1,2}),\ \ell_{n-1,2}] , \\ \theta_5(\ell) = \frac{\pi}{2},\ x = \frac{a}{2} . \end{cases}$$

Hence

(1.1.12A.12) $S_5(\ell_{n-1,2})$

$$= \{(x,\theta): \theta_4(\ell_{n-1,2}) \le \theta \le \frac{\pi}{2},\ 0 \le x \le x_{n6}[\theta_4(\ell_{n-1,2}), \ell_{n-1,2}]\}.$$

The set $S_5(\ell_{n-1,2})$ is represented by $C_{\theta_4(\ell_{n-1,2})}P'_n(\ell_{n-1,2})DC$ in Fig. 16, where D is the mid-point of BC.

$$\ell = \ell_n \Rightarrow \begin{cases} \theta_4(\ell) = n\delta, & x = a, \\ \theta_5(\ell) = n\delta, & x = a. \end{cases}$$

Hence

(1.1.12A.13) $S_5(\ell_n) = \{(x,\theta): 0 \le x \le a,\ \theta = n\delta\}$.

$S_5(\ell_n)$ is thus the line $C_{n\delta}C'_{n\delta}$.

In the parameter space S the set $S_6(\ell)$ is bounded by $\theta = \theta_4(\ell)$, $\theta = \theta_5(\ell)$, $x = a$ and $x = x_{n6}(\theta,\ell)$, where $x_{n6}(\theta,\ell)$, $\theta_4(\ell)$ and $\theta_5(\ell)$ are defined by (1.1.109.i), (1.1.109.e) and (1.1.109.f), respectively.

Now

$$\ell = \ell_{n-1,2} \Rightarrow \begin{cases} \theta_4(\ell) = \pi - 2\delta - \beta_n(\ell_{n-1,2}), \\ x_{n7}(\theta_4(\ell), \ell) = a, \\ \theta_5(\ell) = \frac{\pi}{2},\ x_{n5}(\theta_5(\ell), \ell) = \frac{a}{2}. \end{cases}$$

The set $S_6(\ell)$ is represented by $P''_n(\ell)C'_{\theta_4(\ell)}C'_{\theta_5(\ell)}$.

The set $S_6(\ell_{n-1,2})$ is represented by $DBC'_{\theta_4(\ell_{n-1,2})}$ in Fig. 16.

Now

$$\ell = \ell_n \Rightarrow \begin{cases} \theta_4(\ell) = n\delta, \ x_{n6}(n\delta,\ \ell_n) = a; \\ \theta_5(\ell) = n\delta, \ x_{n6}(n\delta,\ \ell_n) = a. \end{cases}$$

Hence

$$S_6(\ell_n) = (a,\ n\delta).$$

The set $S_6(\ell_n)$ is the point $C'_{n\delta}$ in the parameter space S.

In the parameter space S the set $S_7(\ell)$ is bounded by $\theta = \theta_5(\ell)$, $\theta = \frac{\pi}{2}$, $x = 0$ and $x = a$.

The set $S_7(\ell)$ is represented by the rectangle $C_{\theta_5(\ell)}C'_{\theta_5(\ell)}BC$ in Fig. 16.

$$\ell = \ell_{n-1,2} \Rightarrow \theta_5(\ell) = \frac{\pi}{2},\ 0\leq x\leq a.$$

Hence

$$S_7(\ell_{n-1,2}) = \{(x,\ \frac{\pi}{2}):\ 0\leq x\leq a\}.$$

The set $S_7(\ell_{n-1,2})$ is represented by the line BC in the parameter space.

$$\ell = \ell_n \Rightarrow \theta_5(\ell) = n\delta,\ x = a.$$

$$S_7(\ell_n) = (a, n\delta)\ .$$

1.1.13. Proof of Theorem 1.

Now that we have obtained the appropriate sets in the parameter space for $\ell\epsilon[\ell_{k-1},\ell_k]$, $k = 1,2,\ldots,n$, in Lemmas 3 to 9, we are in a position to prove the theorem. The result (1.1.1) is obtained by finding the areas of the appropriate sets.

(A) For $\ell\epsilon[0,\ell_1]$, $F_N(\ell) = Pr(L\leq\ell) = \frac{2}{a\pi}\int_{D_{11}(\ell)} dxd\theta = \frac{2}{a\pi}\int_{S_{11}(\ell)} dxd\theta$, by (1.1.22) ,

$$= \frac{2}{a\pi}\int_0^{2\delta - \sin^{-1}\left(\frac{x\sin 2\delta}{\ell}\right)} d\theta dx$$

$$= \frac{2}{a\pi}\,\ell\tan\delta\ .$$

This proves part (A) of (1.1.1).

(B) For $\ell\epsilon[\ell_{k-1},\ell_k]$, $F_N(\ell) = \frac{2}{a\pi}\int_{D_{11}(\ell)} dxd\theta$

$$= \frac{2}{a\pi}\int_{S_{k1}(\ell)\cup S_{k2}(\ell)\cup S_{k3}(\ell)} dxd\theta, \text{ by Lemma 3,}$$

where $S_{k1}(\ell)$, $S_{k2}(\ell)$ and, $S_{k3}(\ell)$ are given by (1.1.31), (1.1.32) and (1.1.33), respectively:

$$= \frac{2}{a\pi}\left[\int_0^a \int_0^{\alpha_k(\ell)} d\theta dx + \int_{x_k(\ell)}^a \int_{\alpha_k(\ell)}^{\theta_k(x,\ell)} d\theta dx + \int_0^{x_k(\ell)} \int_{\alpha_k(\ell)}^{\theta_k'(x,\ell)} d\theta dx\right].$$

where $a_k(\ell)$, $x_k(\ell)$, $\theta_k(x,\ell)$ and $\theta_k'(x,\ell)$ are given by (1.1.35), (1.1.40), (1.1.43) and (1.1.47), respectively.

$$= \frac{2}{\pi a}\Big[2k\delta a - 2k\delta(a-x_k(\ell)) + (x_k(\ell)+d_{k-1})\sin^{-1}\Big[\frac{x_k(\ell)+d_{k-1}}{\ell}\sin 2(k-1)\delta\Big]$$

$$-(a+d_{k-1})\sin^{-1}\Big[\frac{(a+d_{k-1})}{\ell}\sin 2(k-1)\delta\Big] +$$

$$+\frac{\ell}{\sin 2(k-1)\delta}\Big\{\Big(1-\Big(\frac{x_k(\ell)+d_{k-1}}{\ell}\Big)^2\sin^2 2(k-1)\delta\Big)^{\frac{1}{2}}$$

$$-\Big(1-\Big(\frac{a+d_{k-1}}{\ell}\Big)^2\sin^2 2(k-1)\delta\Big)^{\frac{1}{2}}\Big\} + d_k\sin^{-1}\Big[\frac{d_k}{\ell}\sin 2k\delta\Big]$$

$$-(x_k(\ell)+d_k)\sin^{-1}\Big(\frac{x_k(\ell)+d_k}{\ell}\sin 2k\delta\Big) + \frac{\ell}{\sin 2k\delta}\Big\{\Big[1-\Big(\frac{d_k}{\ell}\Big)^2\sin^2 2k\delta\Big]^{\frac{1}{2}}$$

$$-\Big[1-\Big(\frac{x_k(\ell)+d_k}{\ell}\Big)^2\sin^2 2k\delta\Big]^{1/2}\Big\}\Big] .$$

This proves part (B) of (1.1.1).

(C) For $\ell\epsilon[\ell_{n-1},\ell_n]$, $N = 2n$ we have

$$F_N(\ell) = \frac{2}{a\pi}\int_{D_{11}(\ell)} dx d\theta = \frac{2}{a\pi}\int_{\bigcup_{i=1}^{3} S_{ni}(\ell)} dx d\theta ,$$

by Lemma 5, where $S_{ni}(\ell)$, $i = 1,2,3$, are given by (1.1.53), (1.1.54), and (1.1.55), respectively.

$$= \frac{2}{a\pi}\Bigg[\int_0^{(\ell^2-\ell_{n-1}^2)^{\frac{1}{2}}}\int_0^{\frac{\pi}{2}} d\theta dx + \int_{(\ell^2-\ell_{n-1}^2)^{\frac{1}{2}}}^{a}\int_0^{\theta_n(x,\ell)} d\theta dx$$

$$+ \int_{(\ell^2-\ell_{n-1}^2)^{\frac{1}{2}}}^{a} \int_{\sin^{-1}(\frac{\ell_{n-1}}{\ell})}^{\frac{\pi}{2}} d\theta dx \Bigg] ,$$

where $\theta_n\ (x,\ell)$ is given by (1.1.43) with $k = n$,

$$= \frac{2}{a\pi}\Big[a\ (\frac{\pi}{2} - \theta_n) + x_n(\ell)\theta_n + 2(n-1)\delta\ (a - x_n(\ell)) +$$

$$(x_n(\ell) + d_{n-1})\ \sin^{-1}\ (\frac{x_n(\ell)+d_{n-1}}{\ell}\ \sin 2(n-1)\delta)$$

$$- (a+d_{n-1})\ \sin^{-1}(\frac{a+d_{n-1}}{\ell}\ \sin 2(n-1)\delta) +$$

$$\frac{\ell}{\sin 2(n-1)\delta}\ \{[1 - (\frac{x_n(\ell)+d_{n-1}}{\ell})^2\ \sin^2 2(n-1)\delta]^{\frac{1}{2}} -$$

$$- [1 - (\frac{a+d_{n-1}}{\ell})^2\ \sin^2 2(n-1)\delta]^{\frac{1}{2}}\}] .$$

This proves part (C) of (1.1.1).

(D) For $\ell \epsilon [\ell_{n-1}, \ell_{n-1,1}]$, $N = 2n+1$ we have

$$F_N(\ell) = \frac{2}{a\pi} \int_{\bigcup_{i=1}^{3} S'_{ni}(\ell)} d\theta dx, \quad \text{by Lemma 7,}$$

where $S'_{ni}(\ell)$, $i = 1,2,3$ are defined in (1.1.71), (1.1.72) and (1.1.82), respectively,

$$= \frac{2}{a^2}\left[\int_0^{x_n(\ell)}\int_0^{\beta_n(\ell)} d\theta dx + \int_{x_n(\ell)}^{a}\int_0^{\theta_n(x,\ell)} d\theta dx + \int_{\beta_n(\ell)}^{\gamma_n(\ell)}\int_0^{x_{n2}(\theta,\ell)} dx d\theta\right],$$

where $x_n(\ell)$, $\beta_n(\ell)$, $\gamma_n(\ell)$, $\theta_n(x,\ell)$ and

$x_{n2}(\theta,\ell)$ are, respectively, defined by (1.1.40), (1.1.73), (1.1.83), (1.1.75) and (1.1.84).

$$= \frac{2}{a^2}\,[x_n(\ell)\beta_n + 2(n-1)\delta\,(a-x_n(\ell)) + 2d_n\,(\beta_n-n\delta) +$$

$$\ell\{\cos\beta_n - \cos(2n\delta-\beta_n)\}/S(n) + W(n)]\,,$$

where

$$S(n) = \sin 2n\delta,$$

$$W(n) = (x_n(\ell)+d_{n-1})\phi(x_n(\ell),\,n-1) - (a+d_{n-1})\phi(a,n-1)$$

$$+ \frac{\ell}{S(n-1)}\,\{U(x_n(\ell),\,n-1) - U(a,n-1)\}\,,$$

and

$$U(x_k(\ell),i) = [1 - (\frac{x_k(\ell)+d_i}{\ell})^2 \sin^2(2i\delta)]^{\frac{1}{2}}\,,$$

$$k = 1, 2, \ldots n,\quad i = 1, 2, \ldots n;$$

$$\phi(x_k(\ell),\, i) = \sin^{-1}\{(x_k(\ell)+d_i)\sin(2i\delta)/\ell\},$$

for $k = 1,2,\ldots,n,\ i = 1,2,\ldots,n.$

This proves part (D) of (1.1.1).

(E) For $\ell\in[\ell_{n-1,1},\ell_{n-1,2}]$, $N = 2n+1$, we have

$$F_N(\ell) = \frac{2}{a\pi}\int_{D_{\lambda 1}(\ell)} d\theta dx = \frac{2}{a\pi}\int_{S'_{n1}(\ell)\cup S'_{n2}(\ell)\cup S'_{n4}(\ell)\cup S'_{n5}(\ell)} d\theta dx$$

where $S'_{n1}(\ell)$, $S'_{n2}(\ell)$, $S'_{n4}(\ell)$ and $S'_{n5}(\ell)$ are given by (1.1.71), (1.1.72), (1.1.93) and (1.1.94), respectively,

$$= \frac{2}{a\pi}\left[\int_0^{x_n(\ell)}\int_0^{\beta_n(\ell)} d\beta dx + \int_{x_n(\ell)}^{a}\int_0^{\theta_n(x,\ell)} d\theta dx\right.$$

$$\left. + \int_{\beta_n(\ell)}^{\frac{\pi}{2}}\int_0^{x_{n3}(\theta,\ell)} dx d\theta + \int_{\theta_1(\ell)}^{\frac{\pi}{2}}\int_{x_1(\theta)}^{a} dx d\theta\right],$$

where $x_n(\ell)$, $\theta_n(x,\ell)$, $x_{n3}(\theta,\ell)$, $\theta_1(\ell)$ and $x_1(\theta)$ are given by (1.1.74) (1.1.75), (1.1.94a), (1.1.94b) and (1.1.94c), respectively,

$$= \frac{2}{a\pi}[\beta_n\, x_n(\ell) + 2(n-1)\delta\, (a-x_n(\ell)) + 2(\beta_n - n\delta)d_n + \ell(\cos 2n\delta$$

$$- \cos(2n\delta-\beta_n)\,/S(n) + W(n)],$$ where $S(n)$ and $W(n)$ are defined in (D).

This proves part (E) of (1.1.1).

(F). For $Lc[\ell_{n-1,2},\ell_n]$, $N = 2n+1$, we have

$$F_N(\ell) = \frac{2}{a\pi}\int_{D_{11}(\ell)} dx\,d\theta = \frac{2}{a\pi}\int_{\bigcup_{i=1}^{7} S_i(\ell)\cup S'(\ell)} dx\,d\theta$$

where $S'(\ell)$, $S_i(\ell)$, $i = 1,2,\ldots,7$, are given by (1.1.102) to (1.1.109).

$$= \frac{2}{a\pi}\Bigg[\int_0^a\int_0^{\theta_1(\ell)} d\theta\,dx + \int_{\theta_1(\ell)}^{\beta_n(\ell)}\int_0^{x_{n4}(\theta,\ell)} dx\,d\theta$$

$$+ \int_{\beta_n(\ell)}^{\theta_2(\ell)}\int_0^{x_{n5}(\theta,\ell)} dx\,d\theta \int_{\theta_2(\ell)}^{\theta_3(\ell)}\int_0^{a} dx\,d\theta +$$

$$\int_{\theta_3(\ell)}^{\theta_4(\ell)}\int_0^{x_{n5}(\theta,\ell)} dx\,d\theta \int_{\theta_4(\ell)}^{\theta_5(\ell)}\int_0^{x_{n5}(\theta,\ell)} dx\,d\theta +$$

$$\int_{\theta_4(\ell)}^{\theta_5(\ell)}\int_{x_{n_6}(\theta,\ell)}^{a} dx\,d\theta + \int_{\theta_5(\ell)}^{\frac{\pi}{2}}\int_0^{a} dx\,d\theta\Bigg],$$ where

$\theta_i(\ell)$, $i = 1,2,3,4,5$, $x_{nj}(\theta,\ell)$, $j = 4,5,6$, are given by (1.1.109a), (1.1.109c),...,(1.1.109i), respectively,

$$= \frac{2}{a\pi}[a(\theta_1 - \theta_2 + \theta_3 - \theta_5 + \frac{\pi}{2}) + d_{n-1}(\theta_1 - \theta_n) +$$

$$d_n(\theta_n - \theta_2 + \theta_3 + \theta_4 - 2\theta_5) + \frac{\ell}{S(n)}\{\cos(2n\delta - \theta_2)$$

$$- \cos(2n\delta - \theta_n) + \cos(2n\delta - \theta_4) - \cos(2n\delta - \theta_3) +$$

$$\cos(2n\delta - \theta_5) - \cos(2n\delta - \theta_4) + \cos(2n\delta + \theta_5) - \cos(2n\delta + \theta_4)\}$$

$$+ \frac{\ell}{S(n-1)}\{\cos(2(n-1)\delta - \theta_n) - \cos(2(n-1)\delta - \theta_1)\}] .$$

This proves part (F) of (1.1.1).

Thus Theorem 1 is proved.

Corollary 1. For the case $N = 4$, we have from (1.1.1),

$$F(\ell) = \frac{2\ell}{\pi a} \qquad \text{for } \ell \in [0,a].$$

$$= \frac{2}{\pi a}\Big[a\cos^{-1}(a/\ell) + \sqrt{\ell^2-a^2}\,\sin^{-1}(a/\ell) +$$

$$\frac{\pi}{2}(a - \sqrt{\ell^2-a^2}) + \sqrt{\ell^2-a^2}\,\sin^{-1}\sqrt{1 - a^2/\ell^2}$$

$$- a\sin^{-1}(a/\ell) + a - \sqrt{\ell^2-a^2}\,;$$

$$= \frac{2}{\pi a}[2a\cos^{-1}(a/\ell) + a - \sqrt{\ell^2-a^2}\,] \qquad \text{for } \ell \in [a, a\sqrt{2}].$$

Differentiating $F(\ell)$ with respect to ℓ, we obtain the following density $f(\ell)$ of L:

$$f(\ell) = F'(\ell) = \begin{cases} \dfrac{2}{\pi a} & \text{for} \quad \ell \epsilon [0, a] \\ \dfrac{4a}{\pi \ell \sqrt{\ell^2 - a^2}} - \dfrac{2}{\pi a \sqrt{\ell^2 - a^2}} & \text{for} \quad \ell \epsilon [a, a\sqrt{2}] \end{cases}$$

which is the same result as given by Horowitz [27].

SECTION TWO

1.2 DISTRIBUTIONS OF LENGTHS OF S_1-RANDOM SECANTS OF A TRIANGLE

1.2.0. Introduction.

In this section we consider the distributions of the lengths of S_1-random secants of triangles. We consider here three different types of triangles. In Section 1.2.3, we illustrate the general procedure offered in this thesis for a case of an irregular triangle. In Sections 1.2.4 and 1.2.5 we obtain the distributions of lengths of S_1-random secants of triangles in the other two cases without going through the details of the geometric arguments, as the arguments are repetitive. The case of the regular triangle can be obtained from the formula (1.1.1) for N odd by substituting $N = 3$. It is also a special case of the result given by (1.2.2) where $a = b = c$ and $p_1 = p_2 = p_3$. However, it is interesting and instructive to derive the formula for the distribution function directly by the use of the geometrical argument used by Horowitz (7) and Coleman (8) to the case of a rectangle and Section 1.2.6 is devoted to finding this distribution function.

1.2.1 General Triangle.

Let ABC be an arbitrary triangle. Let P be a random point on the perimeter of the triangle. A secant of the triangle through P in a random direction intersects a side of the triangle at another point Q. We are interested in the probability distribution of the length $L = |PQ|$.

In order to find the distribution function of L we consider the following three types of triangles, since each of these types must be treated separately: (for notations, cf. Section 1.2.2)

(A) $a > b > c$, $\alpha > 90°$, (B) $a>b>c>p_3>p_2>p_1$ and

(C) $a>b>p_3>c>p_2>p_1$. For each of (A), (B) and (C), we have three cases since the rays emanate from points on (1) BC, (2) CA or (3) AB of the triangle ABC. We shall consider each case and find the appropriate set contributing to $Pr(L \leq l)$. We note that the range of the random variable L is $[0,a]$. To find the distribution we proceed as follows.

Parametrization.

In order to find the probability distribution of L we need to parametrize the secants. Parametrization of the secants is done as follows. For any point P on the perimeter of the triangle ABC let X be the distance of P from C measured in the direction from C→B→A→C. Let θ be the angle which is measured in the counter clockwise direction that a secant makes with the side of the triangle on which P lies. We assume that (1) X is uniformly distributed on $[0, a+b+c]$, where a, b, c are the lengths of the sides of the triangle, (2) θ is uniformly distributed on $[0,\pi]$, and (3) X and θ are independent. It follows that the joint density of X and θ is given by

$$(1.2.0.)\qquad P(x,\theta)=\begin{cases}\frac{1}{a+b+c}\cdot\frac{1}{\pi} & \text{if } 0 \le x \le a+b+c,\ 0 \le \theta \le \pi\\ 0, & \text{elsewhere in the } (x,\theta)\text{-plane.}\end{cases}$$

The set

$$S = \{(x,\theta):\ 0 \le x \le a+b+c,\ 0 \le \theta \le \pi\}$$

is the parameter space.

L is a function of X and Θ and the density $P(x,\theta)$, given by (1.2.0) of X and Θ induces a distribution on L. We find the distribution of L in the following section.

1.2.2. Distribution of L. The following theorem provides the distribution of L.

Theorem 2. Let a, b, c denote the lengths of the sides of the triangle ABC (cf. Fig. 17), α, β and γ be respectively the angles A, B and C. Let p_1, p_2 and p_3 be the lengths of the perpendiculars drawn from A, B and C to, respectively, the sides BC, CA and AB of the triangle ABC. Then for :

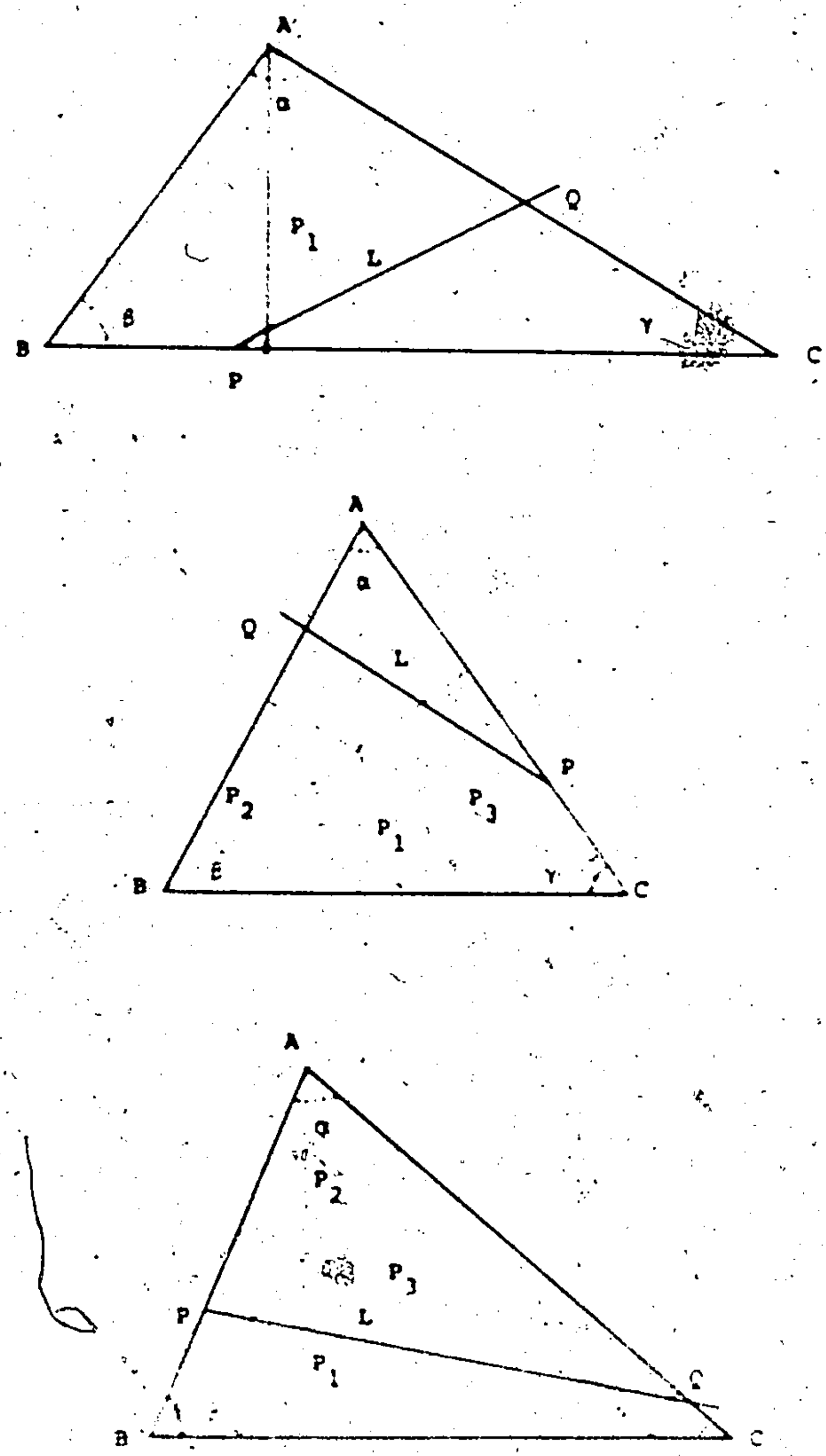

Fig. 17

(A) Triangle I: $a > b > c$, $\alpha > 90°$,

$$
(1.2.1) \qquad F(\ell) = \begin{cases} d\phi_1(\ell) & \text{for } \ell \epsilon [0, p_1] , & (A_1) \\ d\phi_2(\ell) & \text{for } \ell \epsilon [p_1, c] , & (A_2) \\ d\phi_3(\ell) & \text{for } \ell \epsilon [c, b] , & (A_3) \\ d\phi_4(\ell) & \text{for } \ell \epsilon [b, a] , & (A_4) \end{cases}
$$

where $d = \frac{1}{\pi(a+b+c)}$.

$$\phi_1(\ell) = 2\ell(\cot \alpha/2 + \cot \beta/2 + \cot \gamma/2) ,$$

$$\phi_2(\ell) = 2\ell \left(\frac{1}{\sin \alpha} + \frac{1}{\sin \beta} + \frac{1}{\sin \gamma} + \cot \alpha + \cot \beta + \cot \gamma\right)$$

$$+ 2(a+b+c) \cos^{-1}\left(\frac{b \sin \gamma}{\ell}\right)$$

$$- 2 \cot \gamma(\ell^2-b^2 \sin^2\gamma)^{\frac{1}{2}} - 2\cot \beta(\ell^2-b^2 \sin^2\gamma)^{\frac{1}{2}}$$

$$- \frac{2}{\sin \beta} (\ell^2-b^2 \sin^2\gamma)^{\frac{1}{2}} - \frac{2}{\sin \gamma}(\ell^2-b^2 \sin^2\gamma)^{\frac{1}{2}} ,$$

$$\phi_3(\ell) = [(a+b+c)\gamma + \frac{2\ell}{\sin \gamma} + 2\ell \cot \gamma + \ell \cot \beta$$

$$+ \ell \cot \alpha + \ell\left(\frac{1}{\sin \alpha} + \frac{1}{\sin \beta}\right) + 2a \cos^{-1}\left(\frac{a \sin \gamma}{\ell}\right)$$

$$+ 2b \cos^{-1}\left(\frac{b \sin \gamma}{\ell}\right) + c \cos^{-1}\left(\frac{c \sin \beta}{\ell}\right) + c \cos^{-1}\left(\frac{c \sin \alpha}{\ell}\right)$$

$$- \cot\gamma(\ell^2-b^2\sin^2\gamma)^{1/2} - \cot\beta(\ell^2-b^2\sin^2\gamma)^{1/2}$$

$$- \cot\gamma(\ell^2-a^2\sin^2\gamma)^{1/2} - \cot\alpha(\ell^2-a^2\sin^2\gamma)^{1/2}$$

$$- \frac{1}{\sin\gamma}(\ell^2-a^2\sin^2\gamma)^{1/2} - \frac{1}{\sin\gamma}(\ell^2-b^2\sin^2\gamma)^{1/2}$$

$$- \frac{1}{\sin\beta}(\ell^2-c^2\sin^2\beta)^{1/2} - \frac{1}{\sin\alpha}(\ell^2-c^2\sin^2\alpha)^{1/2} .$$

$$\phi_4(\ell) = (a+b+c)(\beta+\gamma)$$

$$+ \ell\left(\cot\beta + \cot\gamma + \frac{1}{\sin\gamma} + \frac{1}{\sin\beta}\right) - \cot\alpha(\ell^2-a^2\sin^2\beta)^{1/2}$$

$$- \cot\alpha(\ell^2-a^2\sin^2\gamma)^{1/2} - \cot\beta(\ell^2-a^2\sin^2\beta)^{1/2}$$

$$- \cot\gamma(\ell^2-a^2\sin^2\gamma)^{1/2} + (a+2b+c)\cos^{-1}\left(\frac{a\sin\beta}{\ell}\right)$$

$$+ (a+c)\cos^{-1}\left(\frac{a\sin\gamma}{\ell}\right)$$

$$- \frac{1}{\sin\alpha}\left((\ell^2-c^2\sin^2\alpha)^{1/2} + (\ell^2-b^2\sin^2\alpha)^{1/2}\right)$$

$$- \frac{1}{\sin\beta}(\ell^2-a^2\sin^2\beta)^{1/2} - \frac{1}{\sin\gamma}(\ell^2-a^2\sin^2\gamma)^{1/2} .$$

(B) Triangle II: $a > b > c > p_3 > p_2 > p_1$.

$$(1.2.1)\quad F(\ell) = \begin{cases} d\phi_1(\ell) & \text{for } \ell\epsilon[0,\ p_1] , & (B_1) \\ d\phi_2(\ell) & \text{for } \ell\epsilon[p_1,\ p_2] , & (B_2) \\ d\phi_3(\ell) & \text{for } \ell\epsilon[p_2,\ p_3] , & (B_3) \\ d\phi_4(\ell) & \text{for } \ell\epsilon[p_3,\ c] , & (B_4) \\ d\phi_5(\ell) & \text{for } \ell\epsilon[c,\ b] , & (B_5) \\ d\phi_6(\ell) & \text{for } \ell\epsilon[b,\ a] , & (B_6) \end{cases}$$

where $\phi_1(\ell) = 2\ell\{\frac{1}{\sin\alpha} + \frac{1}{\sin\beta} + \frac{1}{\sin\gamma} + \cot\alpha + \cot\beta + \cot\gamma\}$,

$$\phi_2(\ell) = 2(a+b+c)\cos^{-1}(\frac{c\sin\beta}{\ell})$$

$$+ 2\ell\{\frac{1}{\sin\alpha} + \frac{1}{\sin\beta} + \frac{1}{\sin\gamma} + \cot\alpha + \cot\beta + \cot\gamma\}$$

$$- 2\ell\{(\frac{1}{\sin\beta} + \frac{1}{\sin\gamma})\sqrt{1 - \frac{b^2\sin^2\gamma}{\ell^2}}$$

$$+ (\cot\beta + \cot\gamma)\sqrt{1 - \frac{b^2\sin^2\gamma}{\ell^2}}\}$$.

$$\phi_3(\ell) = 2(a+b+c)(\cos^{-1}(\frac{c\sin\beta}{\ell}) + \cos^{-1}(\frac{c\sin\alpha}{\ell}))$$

$$+ 2\ell(\frac{1}{\sin\alpha} + \frac{1}{\sin\beta} + \frac{1}{\sin\gamma} + \cot\alpha + \cot\beta + \cot\gamma)$$

$$- 2(\frac{1}{\sin\gamma}(\ell^2 - a^2\sin^2\gamma)^{\frac{1}{2}} + \frac{1}{\sin\gamma}(\ell^2 - b^2\sin^2\gamma)^{\frac{1}{2}}$$

$$+ \frac{1}{\sin\beta}(\ell^2 - c^2 \sin^2\gamma)^{\frac{1}{2}} + \frac{1}{\sin\alpha}(\ell^2 - c^2\sin^2\alpha)^{\frac{1}{2}}\}$$

$$- 2\{\cot\gamma(\ell^2 - b^2\sin^2\gamma)^{\frac{1}{2}} + \cot\alpha(\ell^2 - c^2\sin^2\alpha)^{\frac{1}{2}}$$

$$+ \cot\beta(\ell^2 - b^2\sin^2\gamma)^{\frac{1}{2}} + \cot\gamma(\ell^2 - c^2\sin^2\alpha)^{\frac{1}{2}}\} ,$$

$$\phi_4(\ell) = 2(a+c+b)\cos^{-1}(\frac{c\sin\beta}{\ell})$$

$$+ 2(a+b+c)\cos^{-1}(\frac{c\sin\alpha}{\ell}) + 2(a+b+c)\cos^{-1}(\frac{a\sin\beta}{\ell})$$

$$+ \ell(\frac{2}{\sin\alpha} + \frac{2}{\sin\beta} + \frac{2}{\sin\gamma} + 2\cot\alpha + 2\cot\beta + 2\cot\gamma)$$

$$- 2\{\frac{1}{\sin\gamma}\sqrt{\ell^2 - a^2\sin^2\gamma} + \frac{1}{\sin\beta}\sqrt{\ell^2 - a^2\sin^2\beta}$$

$$+ \frac{1}{\sin\alpha}\sqrt{\ell^2 - c^2\sin^2\alpha} + \frac{1}{\sin\beta}\sqrt{\ell^2 - c^2\sin^2\beta}$$

$$+ \frac{1}{\sin\alpha}\sqrt{\ell^2 - b^2\sin^2\alpha} + \frac{1}{\sin\gamma}\sqrt{\ell^2 - b^2\sin^2\gamma}\}$$

$$- 2\{\cot\alpha\sqrt{\ell^2 - a^2\sin^2\beta} + \cot\beta\sqrt{\ell^2 - a^2\sin^2\beta}$$

$$+ \cot\gamma\sqrt{\ell^2 - c^2\sin^2\alpha} + \cot\alpha\sqrt{\ell^2 - c^2\sin^2\alpha}$$

$$+ \cot\beta\sqrt{\ell^2 - b^2\sin^2\gamma} + \cot\gamma\sqrt{\ell^2 - b^2\sin^2\gamma}\} .$$

$$\phi_5(\ell) = (a+b+c)\gamma + 2(a+b+c)\cos^{-1}(\frac{a\sin\beta}{\ell})$$

$$+ (a+b+c)\cos^{-1}(\frac{c\sin\beta}{\ell}) + (a+b+c)\cos^{-1}(\frac{c\sin\alpha}{\ell})$$

$$+ \ell(\cot\alpha + \cot\beta + 2\cot\gamma) + \frac{1}{\sin\alpha} + \frac{1}{\sin\beta} + \frac{1}{\sin\gamma}$$

$$- 2\left(\frac{1}{\sin\beta}\sqrt{\ell^2 - a^2\sin^2\beta} + \frac{1}{\sin\alpha}\sqrt{\ell^2 - b^2\sin^2\alpha}\right)$$

$$- \frac{1}{\sin\gamma}\sqrt{\ell^2 - a^2\sin^2\gamma} - \frac{1}{\sin\beta}\sqrt{\ell^2 - c^2\sin^2\beta}$$

$$- \frac{1}{\sin\gamma}\sqrt{\ell^2 - c^2\sin^2\beta} - \frac{1}{\sin\alpha}\sqrt{\ell^2 - c^2\sin^2\alpha}$$

$$- (\cot\beta + \cot\gamma)\sqrt{\ell^2 - c^2\sin^2\beta}$$

$$- (\cot\alpha + \cot\gamma)\sqrt{\ell^2 - c^2\sin^2\alpha}$$

$$- 2(\cot\beta + \cot\alpha)\sqrt{\ell^2 - a^2\sin^2\beta} .$$

$$\phi_6(\ell) = (a+b+c)(\beta+\gamma)$$

$$+ (a+b+c)\left(\cos^{-1}\left(\frac{b\sin\alpha}{\ell}\right) + \cos^{-1}\left(\frac{c\sin\alpha}{\ell}\right)\right)$$

$$+ \ell\left(\frac{1}{\sin\gamma} + \frac{1}{\sin\beta} + \cot\beta + \cot\gamma\right)$$

$$- \left(\frac{1}{\sin\alpha} + \frac{1}{\sin\beta}\right)\sqrt{\ell^2 - b^2\sin^2\alpha}$$

$$- \left(\frac{1}{\sin\gamma} + \frac{1}{\sin\alpha}\right)\sqrt{\ell^2 - c^2\sin^2\alpha}$$

$$- (\cot\alpha + \cot\beta)\sqrt{\ell^2 - b^2\sin^2\alpha}$$

$$- (\cot\alpha + \cot\gamma)\sqrt{\ell^2 - c^2\sin^2\alpha} .$$

(C) Triangle III: $a > b > p_3 > c > p_2 > p_1$.

$$
(1.2.1)\quad F(\ell) = \begin{cases} d\xi_1(\ell) & \text{for } \ell \in [0,\, p_1] \,, & (C_1) \\ d\xi_2(\ell) & \text{for } \ell \in [p_1,\, p_2] \,, & (C_2) \\ d\xi_3(\ell) & \text{for } \ell \in [p_2,\, c] \,, & (C_3) \\ d\xi_4(\ell) & \text{for } \ell \in [c,\, p_3] \,, & (C_4) \\ d\xi_5(\ell) & \text{for } \ell \in [p_3,\, b] \,, & (C_5) \\ d\xi_6(\ell) & \text{for } \ell \in [b,\, a] \,, & (C_6) \end{cases}
$$

where

$$
\xi_1(\ell) = 2\ell\left(\frac{1}{\sin\alpha} + \frac{1}{\sin\beta} + \frac{1}{\sin\gamma} + \cot\alpha + \cot\beta + \cot\gamma\right) ,
$$

$$
\xi_2(\ell) = \ell\left[\frac{1}{\sin\gamma} + \frac{1}{\sin\beta} + \frac{1}{\sin\alpha} + \frac{1}{\sin\gamma} + \frac{1}{\sin\alpha} + \frac{1}{\sin\beta} + 2\cot\alpha\right.
$$

$$
\left. + 2\cot\beta + 2\cot\gamma\right] + 2a\cos^{-1}\left(\frac{b\sin\gamma}{\ell}\right) + 2b\cos^{-1}\left(\frac{b\sin\gamma}{\ell}\right)
$$

$$
+ 2c\cos^{-1}\left(\frac{c\sin\beta}{\ell}\right)
$$

$$
+ \ell\left[-\frac{2}{\sin\beta}\sqrt{1 - \frac{c^2\sin^2\beta}{\ell^2}} - \frac{2}{\sin\gamma}\sqrt{1 - \frac{b^2\sin^2\gamma}{\ell^2}}\right.
$$

$$
\left. - 2\cot\beta\sqrt{1 - \frac{b^2\sin^2\gamma}{\ell^2}} - 2\cot\gamma\sqrt{1 - \frac{b^2\sin^2\gamma}{\ell^2}}\right] ,
$$

$$
\xi_3(\ell) = \ell\left[\cot\gamma + \cot\gamma + \cot\alpha + \cot\gamma + \cot\beta\right.
$$

$$
\left. + \frac{2}{\sin\alpha} + \frac{2}{\sin\beta} + \frac{2}{\sin\gamma}\right]
$$

$$+ 2a\cos^{-1}\left(\frac{b\sin\gamma}{\ell}\right) + 2b\cos^{-1}\left(\frac{b\sin\gamma}{\ell}\right) + 2b\cos^{-1}\left(\frac{c\sin\alpha}{\ell}\right)$$

$$+ 2a\cos^{-1}\left(\frac{a\sin\gamma}{\ell}\right) + 2c\cos^{-1}\left(\frac{c\sin\alpha}{\ell}\right) + 2c\cos^{-1}\left(\frac{c\sin\beta}{\ell}\right)$$

$$- 2\cot\alpha\sqrt{\ell^2 - c^2\sin^2\alpha} - 2\cot\gamma\sqrt{\ell^2 - c^2\sin^2\alpha}$$

$$- 2\cot\beta\sqrt{\ell^2 - b^2\sin^2\gamma} - 2\cot\gamma\sqrt{\ell^2 - b^2\sin^2\gamma}$$

$$- \frac{2}{\sin\alpha}\sqrt{\ell^2 - c^2\sin^2\alpha} - \frac{2}{\sin\beta}\sqrt{\ell^2 - c^2\sin^2\beta}$$

$$- \frac{2}{\sin\gamma}\sqrt{\ell^2 - b^2\sin^2\gamma} \quad ,$$

$$\zeta_4(\ell) = (a+b+c)\gamma$$

$$+ \ell\left[\frac{1}{\sin\alpha} + \frac{2}{\sin\gamma} + \frac{1}{\sin\beta} + \cot\alpha + \cot\beta + 2\cot\gamma\right]$$

$$+ (a+b+c)\left[\cos^{-1}\left(\frac{c\sin\beta}{\ell}\right) + \cos^{-1}\left(\frac{c\sin\alpha}{\ell}\right)\right]$$

$$- \frac{1}{\sin\gamma}\sqrt{\ell^2 - a^2\sin^2\gamma} - \frac{1}{\sin\gamma}\sqrt{\ell^2 - c^2\sin^2\beta}$$

$$- \frac{1}{\sin\gamma}\sqrt{\ell^2 - c^2\sin^2\beta} - \frac{1}{\sin\beta}\sqrt{\ell^2 - c^2\sin^2\beta}$$

$$- \frac{1}{\sin\alpha}\sqrt{\ell^2 - c^2\sin^2\alpha} - \cot\alpha\sqrt{\ell^2 - c^2\sin^2\alpha}$$

$$- \cot\beta\sqrt{\ell^2 - c^2\sin^2\beta} - \cot\gamma\sqrt{\ell^2 - c^2\sin^2\gamma} \quad ,$$

$$\xi_5(\ell) = (a+b+c)\gamma - c\pi$$

$$+ \ell[\cot\alpha + \cot\beta + 2\cot\gamma + \frac{1}{\sin\alpha} + \frac{1}{\sin\beta} + \frac{1}{\sin\gamma}]$$

$$+ 2a\cos^{-1}(\frac{a\sin\beta}{\ell}) + 2b\cos^{-1}(\frac{b\sin\alpha}{\ell}) + 2c\cos^{-1}(\frac{a\sin\beta}{\ell})$$

$$+ (a+b+c)\cos^{-1}(\frac{c\sin\alpha}{\ell}) + (a+b+c)\cos^{-1}(\frac{c\sin\beta}{\ell})$$

$$- \frac{2}{\sin\alpha}(\sqrt{\ell^2 - b^2\sin^2\alpha}) - \frac{2}{\sin\beta}(\sqrt{\ell^2 - a^2\sin^2\beta})$$

$$- \frac{1}{\sin\beta}(\sqrt{\ell^2 - c^2\sin^2\beta}) - \frac{1}{\sin\gamma}(\sqrt{\ell^2 - a^2\sin^2\gamma})$$

$$- \frac{1}{\sin\gamma}(\sqrt{\ell^2 - c^2\sin^2\beta}) - \frac{1}{\sin\alpha}(\sqrt{\ell^2 - c^2\sin^2\alpha})$$

$$- \cot\alpha\sqrt{\ell^2 - c^2\sin^2\alpha} - \cot\beta\sqrt{\ell^2 - c^2\sin^2\beta}$$

$$- \cot\gamma\sqrt{\ell^2 - c^2\sin^2\beta} - \cot\gamma\sqrt{\ell^2 - c^2\sin^2\alpha}$$

$$- 2\cot\beta\sqrt{\ell^2 - a^2\sin^2\beta} - 2\cot\alpha\sqrt{\ell^2 - a^2\sin^2\beta} ,$$

$$\xi_6(\ell) = (a+b+c)(\beta+\gamma)$$

$$+ \ell[\cot\beta + \cot\gamma + \frac{1}{\sin\beta} + \frac{1}{\sin\gamma}]$$

$$+ (a+b+c)\cos^{-1}(\frac{c\sin\alpha}{\ell}) + (a+b+c)\cos^{-1}(\frac{b\sin\alpha}{\ell})$$

$$- \frac{1}{\sin \alpha} \sqrt{\ell^2 - c^2 \sin^2 \alpha} - \frac{1}{\sin \gamma} \sqrt{\ell^2 - b^2 \sin^2 \alpha}$$

$$- \frac{1}{\sin \beta} \sqrt{\ell^2 - b^2 \sin^2 \alpha} - \frac{1}{\sin \gamma} \sqrt{\ell^2 - a^2 \sin^2 \gamma}$$

$$- \cot \alpha \sqrt{\ell^2 - b^2 \sin^2 \alpha} - \cot \beta \sqrt{\ell^2 - b^2 \sin^2 \alpha}$$

$$- \cot \gamma \sqrt{\ell^2 - c^2 \sin^2 \alpha} - \cot \alpha \sqrt{\ell^2 - c^2 \sin^2 \alpha} \; .$$

Corollary. The probability distribution function $F(\ell)$ of the S_1-random secant of a regular triangle of side a is given by

$$F(\ell) = \begin{cases} \dfrac{2\sqrt{3}\ell}{\pi a} & \text{for } 0 \le \ell \le \dfrac{\sqrt{3}a}{2} \; , \\[2ex] \dfrac{2\sqrt{3}\ell}{\pi a}\left[1 - \dfrac{(4\ell^2 - 3a^2)^{1/2}}{\ell}\right] + \dfrac{6}{\pi} \cos^{-1}\left(\dfrac{\sqrt{3}a}{2\ell}\right) & \text{for } \dfrac{a\sqrt{3}}{2} \le \ell \le a \; . \end{cases}$$

Proof: From the case: Triangle II, putting $a = b = c$, $\alpha = \beta = \gamma$ in (1.2.1) (B_1) and (1.2.1) (B_4), we obtain this result.

This result is independently obtained following a different argument (cf. [27]) in the next section.

1.2.3. Triangle 1: $a \geq b \geq c$, $\alpha \geq 90°$

Since $0 \leq p_1 \leq c \leq b \leq a$, where $p_1 = b \sin C = c \sin B$ and the derivation of the set $D(\ell)$ in the parameter space $S = [0, a+b+c] \times [0,\pi]$ which contributes to $F(\ell)$ needs separate treatment for ℓ lying in different intervals we decompose the interval $[0,a]$ into the following appropriate subintervals (i) $[0, p_1]$, (ii) $[p_1,c]$, (iii) $[c,b]$ and (iv) $[b,a]$.

Determination of the set $D(\ell)$.

In order to find the distribution function of L we require the set $D(\ell) = \{(x,\theta): L(x,\theta) \leq \ell\}$ in the parameter space. In this section we obtain this set for different intervals of L. In order to determine $D(\ell)$ we consider the three cases in which the secants arise from random points on the three sides of the triangle.

Case 1. Secants arising from random points on BC.

Part (i) $\quad 0 \leq \ell < p_1$.

(cf. Fig. 18). We first consider secants intersecting CA. Let $\theta \in [0,\pi-\gamma]$. Let PQ be the corresponding secant such that $|PQ| = L(x_1(\theta,\ell),\theta) = \ell$. Considering the triangle QPC we have

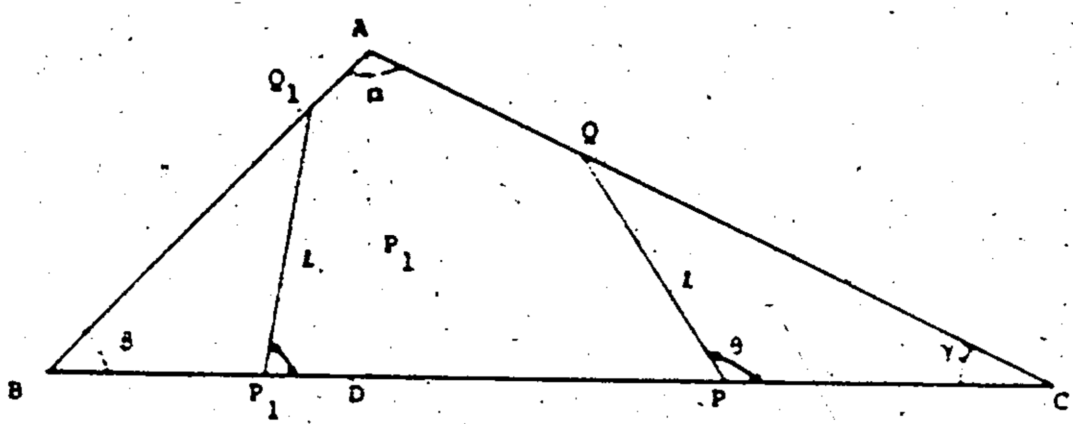

Fig. 18

$$x_1(\theta,\ell) = \frac{\ell \sin(\gamma+\theta)}{\sin \gamma} . \tag{1.2.2}$$

In (1.2.2) we note that

$$0 \le x \le x_1(\theta,\ell) \Rightarrow L(x,\theta) \le L(x_1(\theta,\ell),\theta) = \ell . \tag{1.2.3}$$

Hence the set

$$S_{11}(a,\ell) = \{(x,\theta):\ 0 \le \theta \le \pi - \gamma,\ 0 \le x \le \frac{\ell \sin(\theta+\gamma)}{\sin}\} \tag{1.2.4}$$

contributes to $P(\ell)$.

Now we consider secants intersecting BA. Let $\theta \in [\beta,\pi]$. Let P_1Q_1 be the corresponding secant such that $|P_1Q_1| = L(x_2(\theta,\ell),\ \theta) = \ell$. Considering the triangle BP_1Q_1 we have

$$x_2(\theta,\ell) = a - \frac{\ell \sin(\theta-\beta)}{\sin \beta} . \tag{1.2.5}$$

It is clear that (cf. Fig. 18)

$$x_2(\theta,\ell) \le x \le a \Rightarrow L(x,\theta) \le \ell .$$

Hence the set

$$S_{12}(a,\ell) = \{(x,\theta):\ \beta \le \theta \le \pi,\ x_2(\theta,\ell) \le x \le a\} \tag{1.2.6}$$

contributes to $P(\ell)$;

Part (ii). $p_1 \leq \ell \leq c$.

With A as the centre and ℓ as the radius we draw a circle intersecting BC at D_1 and D_2. Let $\angle AD_iC = \theta_i$, $i = 1, 2$. (cf. Fig. 19). Then $\ell \sin \theta_1 = b \sin \gamma$, so that $\theta_1 = \sin^{-1} (\frac{b \sin \gamma}{\ell})$ and $\theta_2 = \pi - \theta_1 = \pi - \sin^{-1} (\frac{b \sin \gamma}{\ell})$. Let $\theta \in [0, \theta_1]$. Then we find by arguments similar to those used to show (1.2.3) that

$$0 \leq x \leq x_1(\theta, \ell) \Rightarrow L(x, \theta) \leq \ell.$$

Hence

(1.2.7) $$S_{21}(a, \ell) = \{(x, \theta): 0 \leq \theta \leq \theta_1, 0 \leq x \leq x_1(\theta, \ell)\}$$

contributes to $Pr(L \leq \ell)$.

Let $\theta \in [\theta_1, \theta_2]$. Then clearly $L(x, \theta) \geq \ell$ for $0 \leq x \leq a$. Hence the set

(1.2.8) $$S_{22}(a, \ell) = \{(x, \theta): \theta_1 \leq \theta \leq \theta_2, 0 \leq x \leq a\}$$

contributes to $F(\ell)$.

Let $\theta \in [\theta_2, \pi - \gamma]$. Then $0 \leq x \leq x_1(\theta, \ell) \Rightarrow L(x, \theta) \leq \ell$. Hence the set

(1.2.9) $$S_{23}(a, \ell) = \{(x, \theta): \theta_2 \leq \theta \leq \pi - \gamma, 0 \leq x \leq x_1(\theta, \ell)\}$$

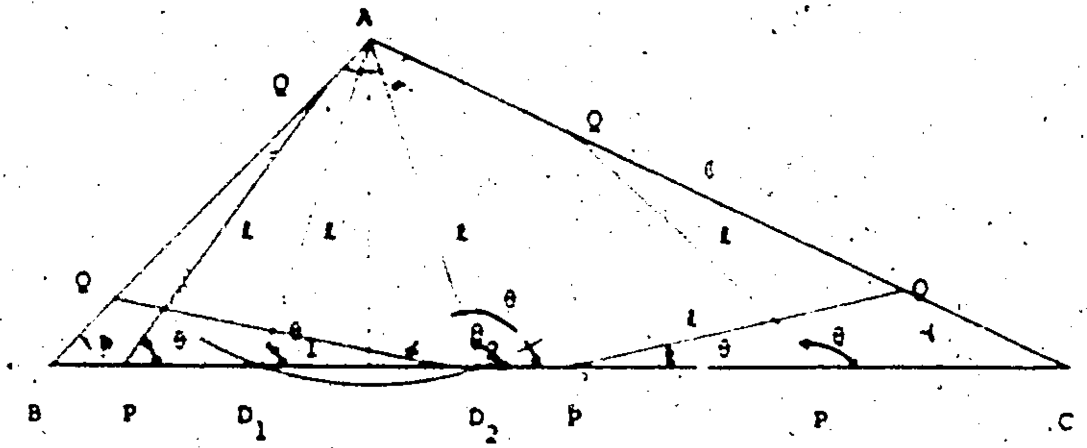

Fig. 19

contributes to $F(\ell)$.

Now we consider the secants intersecting AB. Let $\theta \in [\beta, \theta_1]$. Then we find that

$$x_2(\theta,\ell) \leq x \leq a \Rightarrow L(x,\theta) \leq \ell.$$

Hence

(1.2.10) $S_{24}(a,\ell) = \{(x,\theta): \beta \leq \theta \leq \theta_1, x_2(\theta,\ell) \leq x \leq a\}$

contributes to $F(\ell)$. Also the set

(1.2.11) $S_{25}(a,\ell) = \{(x,\theta): \theta_2 \leq \theta \leq \pi, x_2(\theta,\ell) \leq x \leq a\}$

contributes to $F(\ell)$.

<u>Part (iii)</u>: $c \leq \ell \leq b$.

With A as the centre we draw a circle of radius ℓ (cf. Fig. 20). The circle intersects BC at D". Let $\measuredangle$AD"C = θ". Then $\theta'' = -\sin^{-1}\left(\frac{b \sin\gamma}{\ell}\right) + \pi$. With B as the centre and ℓ as the radius we draw a circle. The circle intersects CA at D'. Let $\measuredangle$D'BC = θ'. Then $\theta' = -\gamma + \sin^{-1}\left(\frac{a \sin\gamma}{\ell}\right)$.

We first consider secants intersecting AC. Let $\theta \in [0, \theta']$. Then for $L(x,(\theta,\ell),\theta) = \ell$, we have

$$0 \leq x \leq x_1(\theta,\ell) \Rightarrow L(x,\theta) \leq \ell.$$

Hence

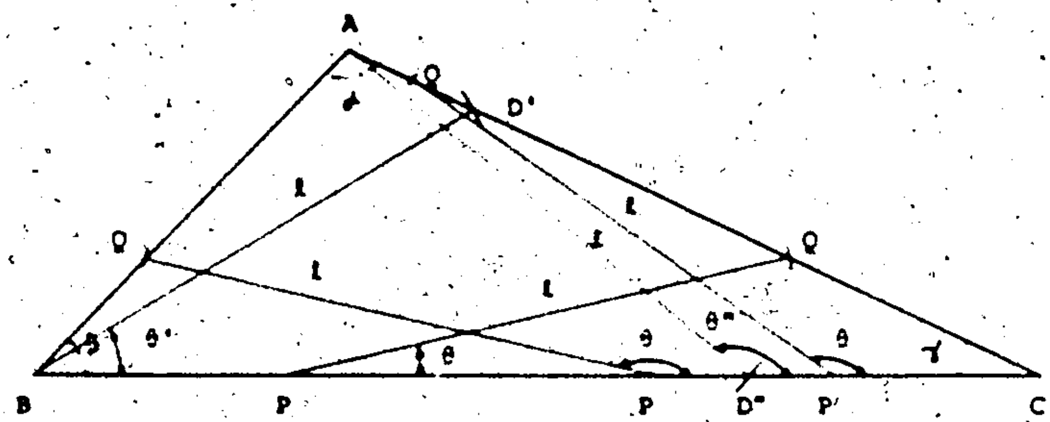

Fig. 20

(1.2.12) $S_{31}(a,\ell) = \{(x,\theta): 0 \leq \theta \leq \theta', 0 \leq x \leq x_1(\theta,\ell)\}$

contributes to $F(\ell)$.

For secants intersecting either AC or AB we find $\theta \in [\theta', \theta"]$ and $0 \leq x \leq a \Rightarrow L(x,\theta) \leq \ell$.

Hence

(1.2.13) $S_{32}(a,\ell) = \{(x,\theta): \theta' \leq \theta \leq \theta", 0 \leq x \leq a\}$

contributes to $F(\ell)$.

For secants intersecting AC. Let $\theta \in [\theta", \pi - \gamma]$. Then $0 \leq x \leq x_1(\theta,\ell) \Rightarrow L(x,\theta) \leq \ell$.

Hence

(1.2.14) $S_{33}(a,\ell) = \{(x,\theta): \theta" \leq \theta \leq \pi - \gamma: 0 \leq x \leq x_1(\theta,\ell)\}$

contributes to $F(\ell)$.

We now consider secants intersecting AB.

Let $\theta \in [\theta", \pi]$. Then

(1.2.15) $x_2(\theta,\ell) \leq x \leq a \Rightarrow L(x,\theta) \leq \ell$.

Hence

(1.2.16) $S_{34}(a,\ell) = \{(x,\theta) : \theta^* \leq \theta \leq \pi,\ x_2(\theta,\ell) \leq x \leq a\}$,

contributes to $F(\ell)$.

Part (iv). $h \leq \ell \leq a$.

With B as the centre and ℓ as the radius we draw a circle (cf. Fig. 21). The circle intersects CA at E_1. With C as the centre and ℓ as the radius we draw a circle. The circle intersects BA at E_2. Let $\angle E_1BC = \theta_1$ and $\angle E_2CB = \theta_2$. Then $\theta_1 = -\gamma + \sin^{-1}\left(\frac{a \sin \gamma}{\ell}\right)$ and $\theta_2 = -\beta + \sin^{-1}\left(\frac{a \sin \beta}{\ell}\right)$.

Let $\theta \in [0, \theta_1]$. Then $0 \leq x \leq x_1(\theta,\ell) \Rightarrow L(x,\theta) \leq \ell$.

Hence

(1.2.17) $S_{41}(a,\ell) = \{(x,\theta) : 0 \leq \theta \leq \theta_1,\ 0 \leq x \leq x_1(\theta,\ell)\}$

contributes to $F(\ell)$.

Let $\theta \in [\theta_1, \pi - \theta_2]$. Then $0 \leq x \leq a \Rightarrow L(x,\theta) \leq \ell$.

Hence

(1.2.18) $S_{42}(a,\ell) = \{(x,\theta) : \theta_1 \leq \theta \leq \pi - \theta_2,\ 0 \leq x \leq a\}$

contributes to $F(\ell)$.

Let $\theta \in [\pi - \theta_2, \pi]$. Then $x_2(\theta,\ell) \leq x \leq a \Rightarrow L(x,\theta) \leq \ell$.

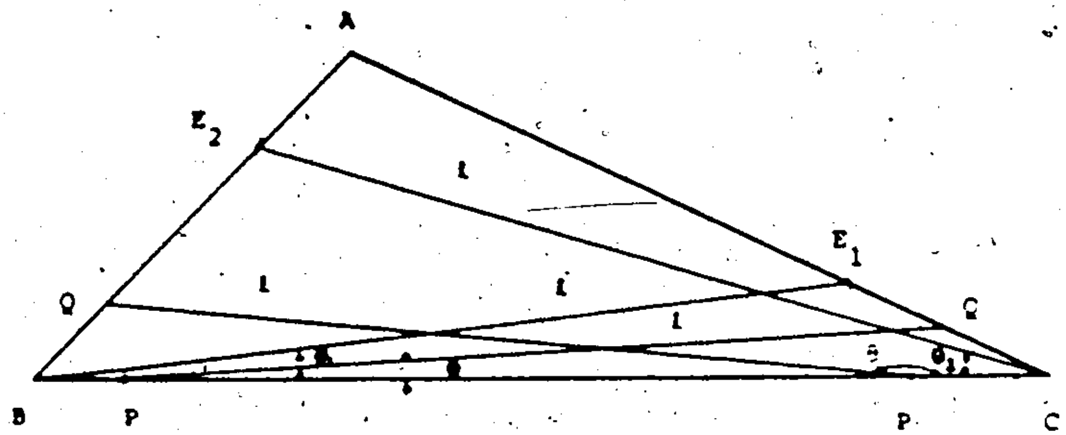

Fig. 21

Hence

(1.2.19) $S_{43}(a,\ell) = \{(x,\theta): \pi - \theta_2 \leq \theta \leq \pi,\ x_2(\theta,\ell) \leq x \leq a\}$,

contributes to $F(\ell)$.

Case 2. Secants arising from random points on CA.

Part (i) $0 \leq \ell \leq b \sin \gamma$.

(cf. Fig. 22) we consider secants intersecting BC. Let $\theta \in [0,\pi-\gamma]$. Then there exists a value $x_1(\theta,\ell)$ of x such that $L(x_1(\theta,\ell),\theta) = \ell$. Clearly

$$x_1(\theta,\ell) \leq x \leq a + b + c \Rightarrow L(x,\theta) \leq \ell.$$

Hence

(1.2.20) $S_{11}(b,\ell) = \{(x,\theta): 0 \leq \theta \leq \pi - \gamma,\ x_1(\theta,\ell) \leq x \leq a + b + c\}$

contributes to $F(\ell)$. Putting $a + b + c - x = y$ and $a + b + c - x_1(\theta,\ell) = y_1(\theta,\ell)$ we find that (1.2.20) reduces to

(1.2.20) $S_{11}(b,\ell) = \{(x,\theta): 0 \leq \theta \leq \pi - \gamma,\ 0 \leq y \leq y_1(\theta,\ell)\}$, where

$$y_1(\theta,\ell) = \frac{\ell \sin(\gamma+\theta)}{\sin\gamma}.$$

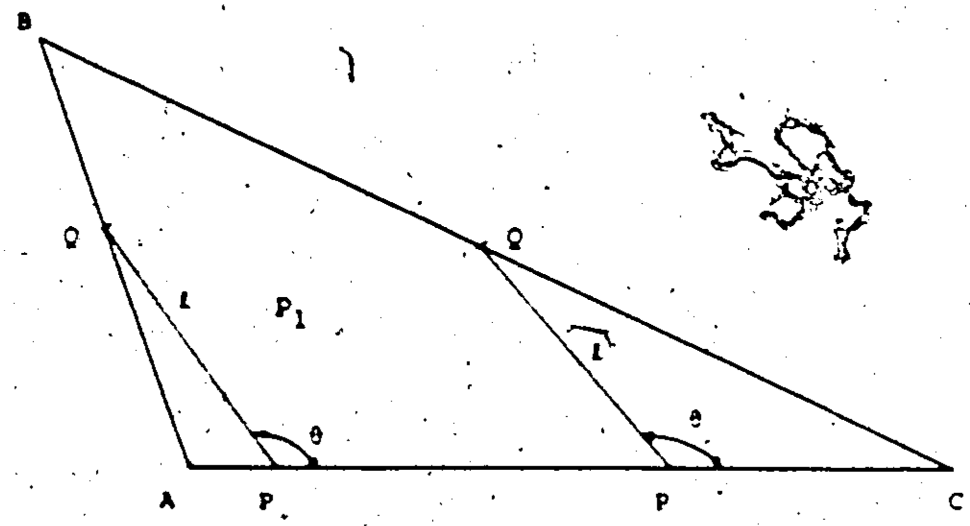

Fig. 22

We consider now secants intersecting AB. Let $\theta \in [\alpha, \pi]$. Then there exists a value $x_2(\theta,\ell)$ of X such that $L(x_2(\theta,\ell),\theta) = \ell$. Clearly $x_2(\theta,\ell) \geq x \geq a + c \Rightarrow L(x,\theta) \leq \ell$. Putting

$$a + b + c - x = y \text{ and } a + b + c - x_2(\theta,\ell) = y_2(\theta,\ell) = a - \frac{\ell \sin (\theta - \beta)}{\sin \beta}$$

we find that

(1.2.21) $S_{12}(b,\ell) = \{(y,\theta) : \alpha \leq \theta \leq \pi,\ y_2(\theta,\ell) \leq y \leq b\}$

contributes to $F(\ell)$.

Part II: $b \sin\gamma \le \ell \le c.$

(cf. Fig.23). With A as the centre we draw a circle of radius ℓ. The circle intersects BC at D_1 and D_2. Let $\angle D_iAC = \phi_i$, $i = 1,2$. Then

$$\phi_1 = \frac{\pi}{2} - \gamma - \cos^{-1}\left(\frac{b\sin\gamma}{\ell}\right) \text{ and } \phi_2 = -\gamma + \cos^{-1}\left(\frac{b\sin\gamma}{\ell}\right) + \frac{\pi}{2} .$$

We consider rays intersecting CB.

Let $\theta \in [0,\phi_1]$, then there exists a value $x_1(\theta,\ell)$ of X such that $L(x_1(\theta,\ell),\theta)=\ell$. Clearly $0 \le y \le y_1(\theta,\ell) = \frac{\ell\sin(\gamma+\theta)}{\sin\gamma} \Rightarrow L(y,\theta) \le \ell$, where $y = a + b + c - x$ and $y_1(\theta,\ell) = a + b + c - x_1(\theta,\ell)$.

Hence

(1.2.22) $\quad S_{21}(b,\ell) = \{(y,\theta) : 0 \le \theta \le \phi_1,\ 0 \le y \le y_1(\theta,\ell)\}$

contributes to $F(\ell)$.

Let $\theta \in [\phi_2,\pi-\gamma]$, $y = a + b + c - x$, $y_1(\theta,\ell) = a + b + c - x_1(\theta,\ell)$, where $L(x_1(\theta,\ell),\theta) = \ell$. Then

$$0 \le y \le y_1(\theta,\ell) \Rightarrow L(y,\theta) \le \ell.$$

Hence

(1.2.23) $\quad S_{22}(b,\ell) = \{(y,\theta) : \phi_2 \le \theta \le \pi-\gamma,\ 0 \le y \le y_1(\theta,\ell)\}$

contributes to $F(\ell)$.

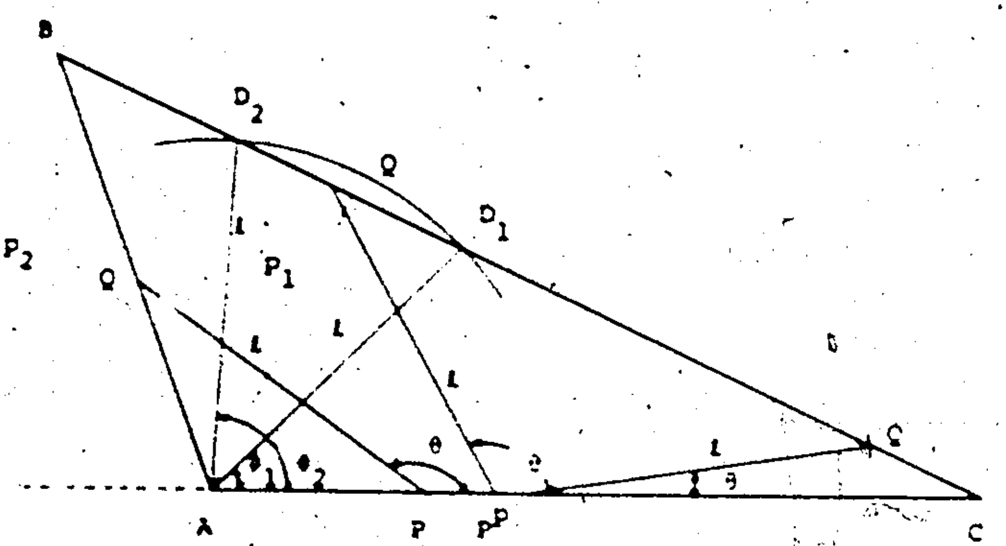

Fig. 23

Let

$$\theta \in [\phi_1, \phi_2],\ y = a + b + c - x.$$

Then

$$y \in [0, b] \Rightarrow L(y, \theta) \leq \ell.$$

Hence

(1.2.24) $$S_{23}(b, \ell) = \{(y, \theta) : \phi_1 \leq \theta \leq \phi_2,\ 0 \leq y \leq b\}$$

contributes to $F(\ell)$.

<u>We now consider secants intersecting AB.</u>

(cf. Fig.23). Let $\theta \in [\alpha, \pi]$, $y = a + b + c - x$,

$$y_2(\theta, \ell) = a + b + c - x_2(\theta, \ell).$$

Then

$$y \in [y_2(\theta, \ell), b] \Rightarrow L(y, \theta) \leq \ell.$$

Hence

(1.2.25) $$S_{24}(b, \ell) = \{(y, \theta) : \alpha \leq \theta \leq \pi,\ y_2(\theta, \ell) \leq y \leq b\}$$

contributes to $F(\ell)$.

Part III: $c \leq \ell \leq b$.

(cf. Fig. 24). With A as the centre we draw a circle of radius ℓ. The circle intersects BC at D'. Let $\angle D'AC = \phi' = -\gamma + \sin^{-1}\left(\frac{b \sin \gamma}{\ell}\right)$.

We first consider secants intersecting CB.

Let

$$\theta \in [0,\phi'],\ y = a + b + c - x,\ y_1(\theta,\ell) = a + b + c - x_1(\theta,\ell)$$

where $L(x_1(\theta,\ell),\theta) = \ell$. Then

$$0 \leq y \leq y_1(\theta,\ell) \Rightarrow L(y,\theta) \leq \ell.$$

Hence

(11.2.26) $$S_{31}(b,\ell) = \{(y,\theta): 0 \leq \theta \leq \phi',\ 0 \leq y \leq y_1(\theta,\ell)\}$$

contributes to $F(\ell)$.

(cf. Fig. 24). With B as the centre we draw a circle of radius ℓ. The circle intersects AC at D''. Let $\angle BD''C = \phi''$. Then $\phi'' = -\sin^{-1}\left(\frac{a \sin \gamma}{\ell}\right) + \pi$.

Let $\theta \in [\phi',\phi'']$. Then

$$0 \leq y \leq b \Rightarrow L(y,\theta) \leq \ell.$$

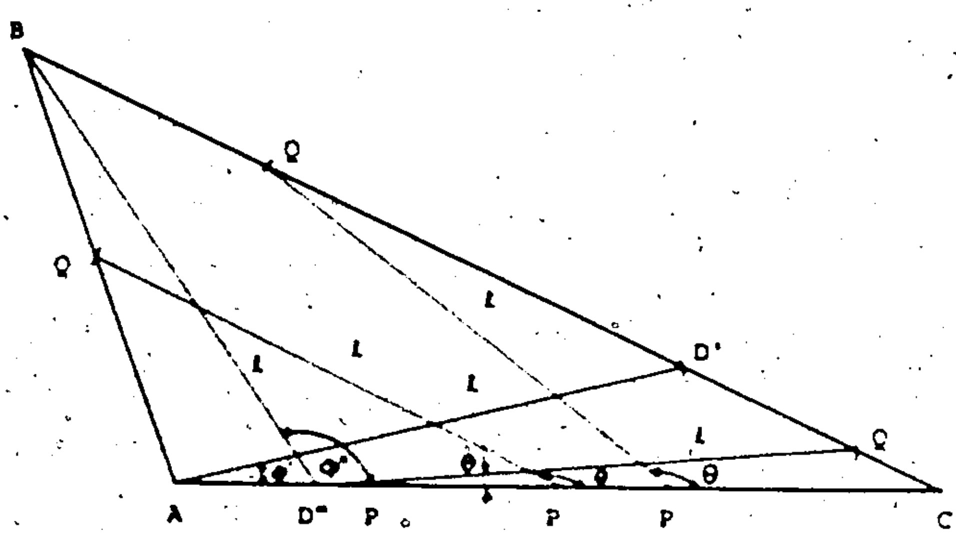

Fig. 24

Hence

(1.2.27) $$S_{32}(b,\ell) = \{(y,\theta): \phi^{*} \le \theta \le \phi^{**},\ 0 \le y \le b\}$$

contributes to $F(\ell)$ (note that secants here intersect AB or BC).

Let

$$\theta \in [\phi^{**}, \pi-\gamma],\ y = a+b+c-x,\ y_1(\theta,\ell) = a+b+c-x_1(\theta,\ell),$$

where $L(x_1(\theta,\ell),\theta) = \ell$. Then

$$0 \le y \le y_1(\theta,\ell) \Rightarrow L(y,\theta) \le \ell.$$

Hence

(1.2.28) $$S_{33}(b,\ell) = \{(y,\theta): \phi^{**} \le \theta \le \pi-\gamma,\ 0 \le y \le y_1(\theta,\ell)\}$$

contributes to $F(\ell)$.

We consider now the secants intersecting AB. (cf. Fig. 24).

Let

$$\theta \in [\theta^{*}, \pi],\ y = a+b+c-x,\ y_2(\theta,\ell) = a+b+c-x_2(\theta,\ell),$$

where $L(x_2(\theta,\ell),\theta) = \ell$. Then

$$y_2(\theta,\ell) \le y \le b \Rightarrow L(y,\theta) \le \ell.$$

Hence

(1.2.29) $\quad S_{34}^{c}(b,\ell) = \{(y,\theta): \phi^{*} \le \theta \le \pi,\ y_2(\theta,\ell) \le y \le b\}$

contributes to $F(\ell)$.

Part IV: $\quad b \le \ell \le a.$

(cf. Fig.25). With B as the centre and ℓ as the radius we draw a circle. The circle intersects AC at D_1. Let $\angle BD_1C = \phi$. Then it follows from the triangle BD_1C that $\phi = \pi - \sin^{-1}(\frac{a \sin \gamma}{\ell})$.

Let $\theta \in [0,\phi]$ and $a + b + c - x = y$. Then

$$0 \le y \le b \Rightarrow L(y,\theta) \le \ell.$$

Hence

(1.2.30) $\quad S_{41}(b,\ell) = \{(y,\theta): 0 \le \theta \le \phi,\ 0 \le y \le b\}$

contributes to $F(\ell)$.

We consider the secants intersecting BC (cf. Fig. 25).

Let

$$\theta \in [\phi, \pi - \gamma],\ a + b + c - x_1(\theta,\ell) = y_1(\theta,\ell).$$

Then

$$0 \le y \le y_1(\theta,\ell) \Rightarrow L(y,\theta) \le \ell.$$

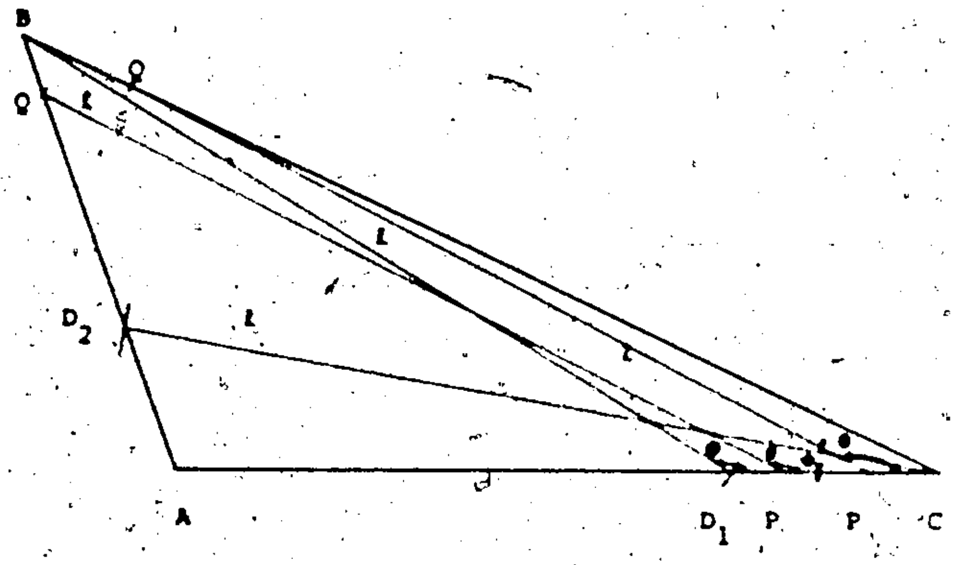

Fig. 25

Hence

(1.2.31) $$S_{42}(b,\ell) = \{(y,\theta): \phi \leq \theta \leq \pi - \gamma,\ 0 \leq y \leq y_1(\theta,\ell)\}$$

contributes to $F(\ell)$.

With C as the centre we draw a circle of radius ℓ (cf. Fig. 25). The circle intersects AB at D_2. Let $\angle D_2CA = \phi_0$. Then

$$\frac{\ell}{\sin \alpha} = \frac{b}{\sin(\pi-\alpha-\phi_0)}, \text{ so that } \phi_0 = \pi-\alpha-\sin^{-1}\left(\frac{b \sin \alpha}{\ell}\right).$$

Let

$$\theta \in [\phi, \pi-\phi_0],\ a + b + c - x = y \text{ and } a + b + c - x_2(\theta,\ell) = y_2(\theta,\ell),$$

where $L(x_2(\theta,\ell),\theta) = \ell$. Then

$$y_2(\theta,\ell) \leq y \leq a \Rightarrow L(y,\theta) \leq \ell.$$

Hence

(1.2.32) $$S_{43}(b,\ell) = \{(y,\theta): \phi \leq \theta \leq \pi - \theta_0,\ y_2(\theta,\ell) \leq y \leq b\}$$

contributes to $F(\ell)$.

Let

$$\theta \in [\pi-\phi_0, \pi] \text{ and } a + b + c - x = y.$$

Then

$$0 \le y \le b \Rightarrow L(y,\theta) \le \ell .$$

Hence

(1.2.33) $$S_{44}(b,\ell) = \{(y,\theta) : \pi - \phi_0 \le \theta \le \pi,\ 0 \le y \le b\}$$

contributes to $F(\ell)$.

Case 3. Secants arising from random points on AB.

Part (i) $0 \le \ell \le b \sin \gamma$.

We consider secants intersecting BC.

(cf. Fig.26). Let $\theta \in [0,\pi-\beta]$, $a + y = x$ and $a + y_1(\theta,\ell) = x_1(\theta,\ell)$, where $L(x_1(\theta,\ell),\theta) = \ell$. Then $0 \le y \le y_1(\theta,\ell) \Rightarrow L(y,\theta) \le \ell$.

Hence

(1.2.34) $$S_{11}(c,\ell) = \{(y,\theta) : 0 \le \theta \le \pi-\beta,\ 0 \le y \le y_1(\theta,\ell)\}$$

contributes to $F(\ell)$.

Next, We consider secants intersecting AC (cf. Fig 26).

Let $\theta \in [\alpha, \pi]$, $a + y = x$ and $a + y_2(\theta,\ell) = x_2(\theta,\ell)$, where $L(x_2(\theta,\ell),\theta) = \ell$. Then

$$y_2(\theta,\ell) \le y \le c \Rightarrow L(y_2(\theta,\ell)) \le \ell .$$

Hence

(1.2.35) $$S_{12}(c,\ell) = \{(y,\theta) : \alpha \le \theta \le \pi,\ y_2(\theta,\ell) \le y \le c\}$$

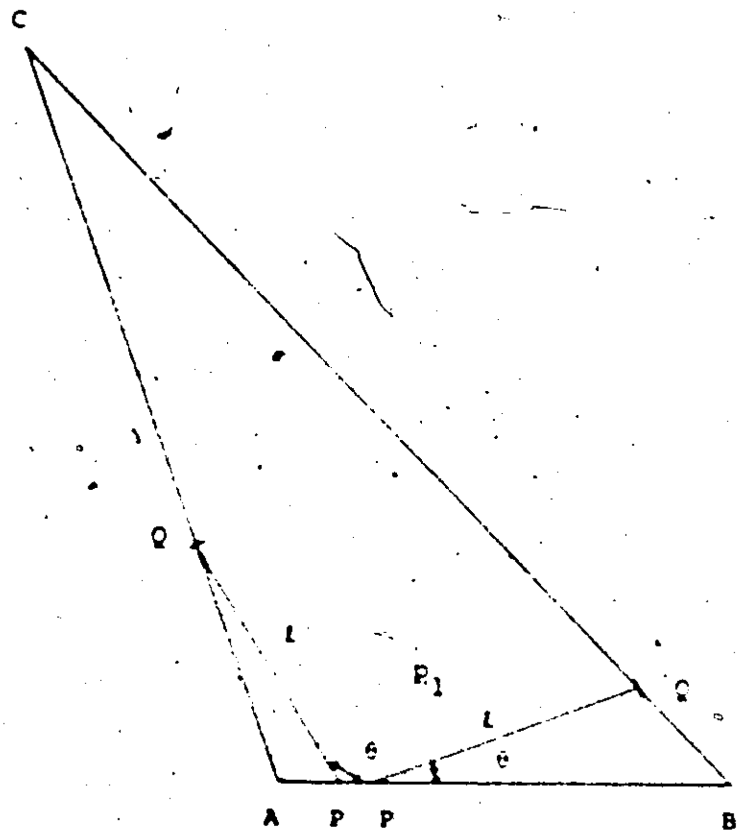

Fig. 26

contributes to $F(\ell)$.

Part (ii): $b \sin\gamma \le \ell \le c$.

(cf. Fig. 27). With A as the centre and ℓ as the radius we draw a circle. The circle intersects BC at D_1 and D_2. Let $\angle D_iAB = \alpha_i$, $i=1,2$. Then

$$\alpha_1 = \pi/2 - \beta - \cos^{-1}\left(\frac{b \sin\gamma}{\ell}\right), \text{ and}$$

$$\alpha_2 = \pi/2 - \beta + \cos^{-1}\left(\frac{b \sin\gamma}{\ell}\right).$$

We now consider secants intersecting BC.

Let $y = x - a$, $y_1(\theta,\ell) = x_1(\theta,\ell) - a$, where $L(x_1(\theta,\ell),\theta) = \ell$.

Then the sets

(1.2.36) $$S_{21}(c,\ell) = \{(y,\theta) : 0 \le \theta \le \alpha_1,\ 0 \le y \le y_1(\theta,\ell)\},$$

(1.2.37) $$S_{22}(c,\ell) = \{(y,\theta) : \alpha_1 \le \theta \le \alpha_2,\ 0 \le y \le c\},$$

and

(1.2.38) $$S_{23}(c,\ell) = \{(y,\theta) : \alpha_2 \le \theta \le \pi - \beta,\ 0 \le y \le y_1(\theta,\ell)\}$$

contribute to $F(\ell)$.

We next consider the secants intersecting AC (cf. Fig. 27).

Let $\theta \in [\alpha,\pi]$, $y = x - a$, $y_2(\theta,\ell) = x_2(\theta,\ell) - a$, where $L(x_2(\theta,\ell),\theta) = \ell$. Then $y_2(\theta,\ell) \le y \le c \Rightarrow L(y,\theta) \le \ell$.

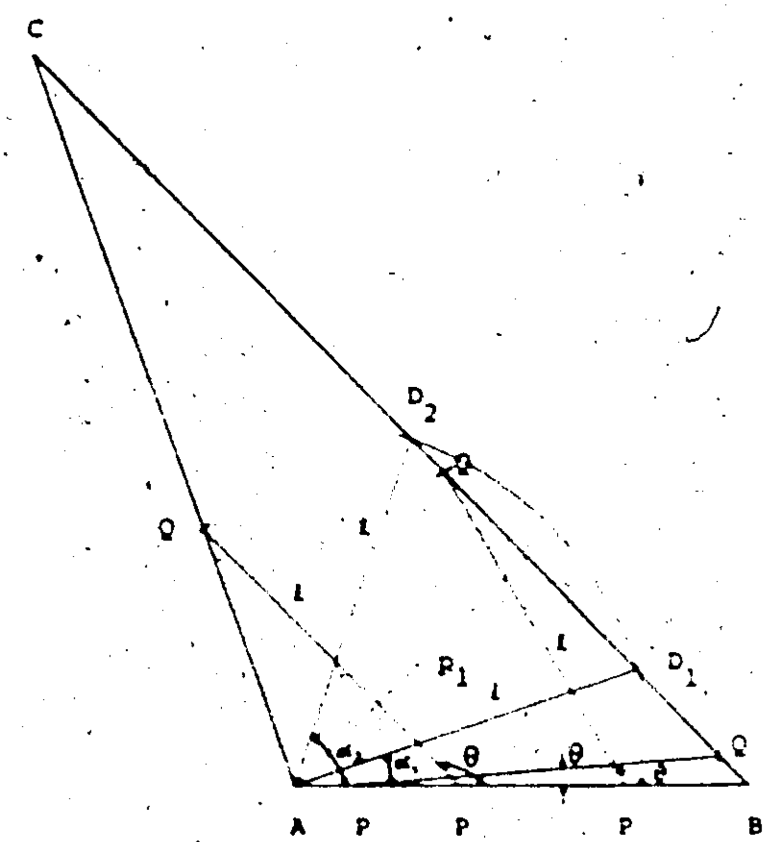

Fig. 27

Hence

(1.2.39) $S_{24}(c,\ell) = \{(y,\theta) : \alpha \le \theta \le \pi,\ y_2(\theta,\ell) \le y \le c\}$

contributes to $F(\ell)$.

Part (iii): $c \le \ell \le b.$

(cf. Fig. 28). With A as the centre and ℓ as the radius we draw a circle. The circle intersects BC at a point D. Let $\measuredangle$ DAB = δ.

Then

$$\delta = \pi - \beta - \sin^{-1}\left(\frac{c \sin \beta}{\ell}\right).$$

Let $\theta \epsilon [0,\delta]$, $y = x - a$, $y_1(\theta,\ell) = x_1(\theta,\ell) - a$, where $L(x_1(\theta,\ell),\theta) = \ell$.

Then $0 \le y \le c \Rightarrow L(y,\theta) \le \ell.$

Let $\theta \in [\delta,\pi-\beta]$. Then $0 \le y \le y_1(\theta,\ell) \Rightarrow L(y,\theta) \le \ell.$

Hence the following sets contribute to $F(\ell)$:

(1.2.40) $S_{31}(c,\ell) = \{(y,\theta) : 0 \le \theta \le \delta,\ 0 \le y \le c\}$ and

(1.2.41) $S_{32}(c,\ell) = \{(y,\theta) : \delta \le \theta \le \pi-\beta,\ 0 \le y \le y_1(\theta,\ell)\}$,

where $y = x - a$ and $y_1(\theta,\ell) = x_1(\theta,\ell) - a$.

We now consider secants intersecting AC.

With B as the centre we draw a circle of radius ℓ. The circle intersects AC at D_1. Let $\measuredangle D_1BA = \pi-\delta_1$. Then $\delta_1 = \alpha + \sin^{-1}\left(\frac{c \sin \alpha}{\ell}\right)$.

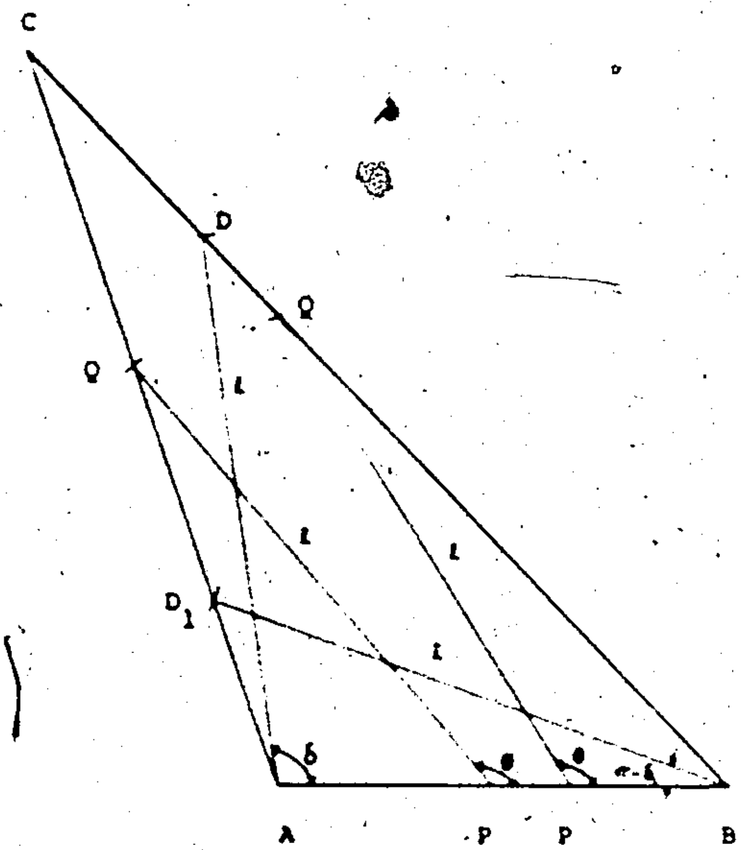

Fig. 28

Let $\theta \in [\alpha, \delta_1]$, $y_2(\theta,\ell) = x_2(\theta,\ell) - a$, where $L(x_2(\theta,\ell),\theta) = \ell$.

Then

$$y_2(\theta,\ell) \le y \le c \Rightarrow L(y,\theta) \le \ell .$$

Hence the sets

(1.2.42) $$S_{33}(c,\ell) = \{(y,\theta) : \alpha \le \theta \le \delta_1,\ y_2(\theta,\ell) \le y \le c\} ,$$

(1.2.43) $$S_{34}(c,\ell) = \{(y,\theta) : \delta_1 \le \theta \le \pi,\ 0 \le y \le c\}$$

contribute to $F(\ell)$.

Part (iv) $\qquad b \le \ell \le a$.

(cf. Fig. 29). With B and C as the centres and ℓ as the radius, we draw circles C_1 and C_2 respectively. The circles intersect AC and AB respectively at points D_2 and D_1. Let $\angle CD_1B = \alpha'$, $\angle D_2BA = \alpha''$.

Then

$$\alpha' = \pi - \sin^{-1}\left(\frac{a \sin \theta}{\ell}\right) \text{ and } \alpha'' = -\alpha - \sin^{-1}\left(\frac{c \sin \alpha}{\ell}\right) + \pi .$$

Then the following sets contribute to $F(\ell)$:

(1.2.44) $$S_{41}(c,\ell) = \{(y,\theta) : 0 \le \theta \le \alpha',\ 0 \le y \le c\} ,$$

(1.2.45) $$S_{42}(c,\ell) = \{(y,\theta) : \alpha' \le \theta \le \pi-\theta,\ 0 \le y \le y_1(\theta,\ell)\} ,$$

(1.2.46) $$S_{43}(c,\ell) = \{(y,\theta) : \alpha' \le \theta \le \pi-\alpha'',\ y_2(\theta,\ell) \le y \le c\}$$

and

(1.2.47) $$S_{44}(c,\ell) = \{(y,\theta) : \pi-\alpha'' \le \theta \le \pi,\ 0 \le y \le c\},$$

where $y = x - a$, $y_1(\theta,\ell) = x_1(\theta,\ell) - a$, $y_2(\theta,\ell) = x_2(\theta,\ell) - a$ and $L(x_1(\theta,\ell),\theta) = L(x_2(\theta,\ell),\theta) = \ell$.

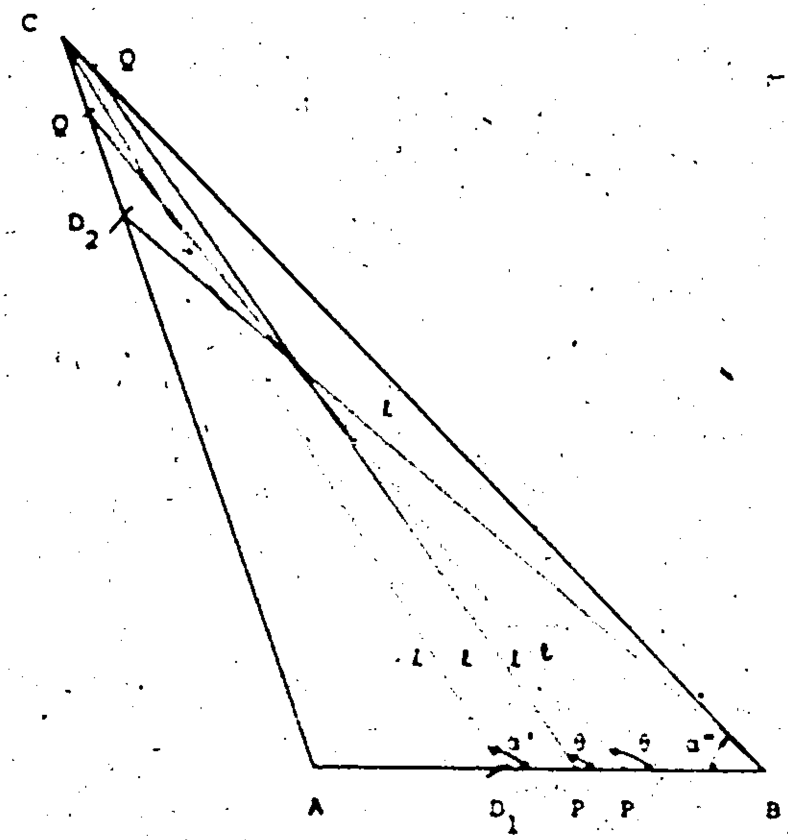

Fig. 29

1.2.4. Triangle II: $a > b > c > p_3 > p_2 > p_1$.

Let ABC be a triangle with $a > b > c > p_3 > p_2 > p_1$. We decompose $[0,a]$ into subintervals $[0,p_1]$, $[p_1,p_2]$, $[p_2,p_3]$, $[p_3,c]$, $[c,b]$, and $[b,a]$.

Case 1: Secants arising from random points on BC.

Part (i) $0 \le \ell \le p_1$.

Using the same arguments as in the preceeding section we have the following sets in the parameter space contributing to $F(\ell)$ (cf. Fig. 38).

(1.2b.1)

$$S_1(a,\ell) = \{(x,\theta): \ 0 \le \theta \le \pi-\gamma, \ 0 \le x \le x_1(a,\ell)\} ,$$

(1.2b.2)

$$S_2(a,\ell) = \{(x,\theta): \ \beta \le \theta \le \pi, \ x_2(a,\ell) \le x \le a\} ,$$

where

(1.2b.3)

$$x_1(a,\ell) = \frac{\ell \sin(\theta+\gamma)}{\sin\gamma} ,$$

(1.2b.4)

$$x_2(a,\ell) = a - \frac{\ell \sin(\theta-\beta)}{\sin\beta} .$$

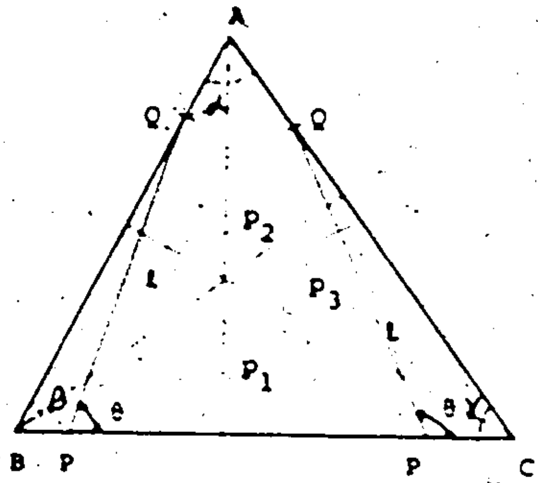

Fig. 30

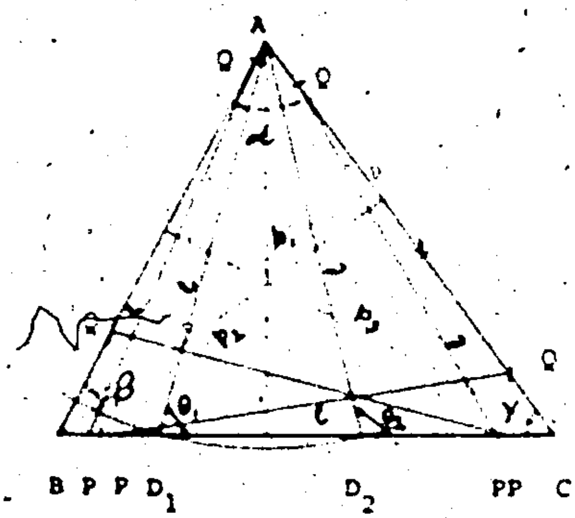

Fig. 31

Part (ii): $p_1 \le \ell \le p_2$.

In this case we have the following sets in the parameter space contributing to $F(\ell)$ (cf. Fig. 31).

(1.2b.5)

$$S_3(a,\ell) = \{(x,\theta): \ 0 \le \theta \le \theta_1(a,\ell),\ 0 \le x \le x_1(a,\ell)\} ,$$

(1.2b.6)

$$S_4(a,\ell) = \{(x,\theta): \ \theta_1(a,\ell) \le \theta \le \theta_2(a,\ell),\ 0 \le x \le a\} ,$$

(1.2b.7)

$$S_5(a,\ell) = \{(x,\theta): \ \theta_2(a,\ell) \le \theta \le \pi-\gamma,\ 0 \le x \le x_1(a,\ell)\} ,$$

(1.2b.8)

$$S_6(a,\ell) = \{(x,\theta): \ \beta \le \theta \le \theta_1(a,\ell),\ x_2(a,\ell) \le x \le a\} ,$$

(1.2b.9)

$$S_7(a,\ell) = \{(x,\theta): \ \theta_2(a,\ell) \le \theta \le \pi,\ x_2(a,\ell) \le x \le a\} ,$$

where

(1.2b.10)

$$\theta_1(a,\ell) = \sin^{-1}\left(\frac{b \sin \gamma}{\ell}\right) ,$$

(1.2b.11)

$$\theta_2(a,\ell) = \pi - \sin^{-1}\left(\frac{b \sin \gamma}{\ell}\right) ,$$

and $x_1(a,\ell)$ and $x_2(a,\ell)$ are given by (1.2b.3) and (1.2b.4) respectively.

Part (iii): $p_2 \le \ell \le p_3$.

In this case we have the following sets in the parameter space contributing to $F(\ell)$ (cf. Fig. 32).

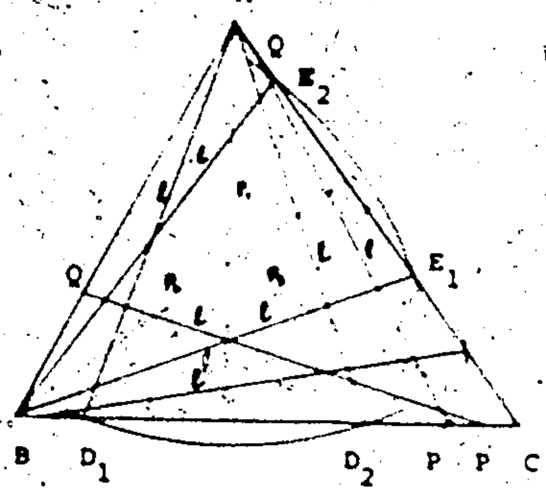

Fig. 32

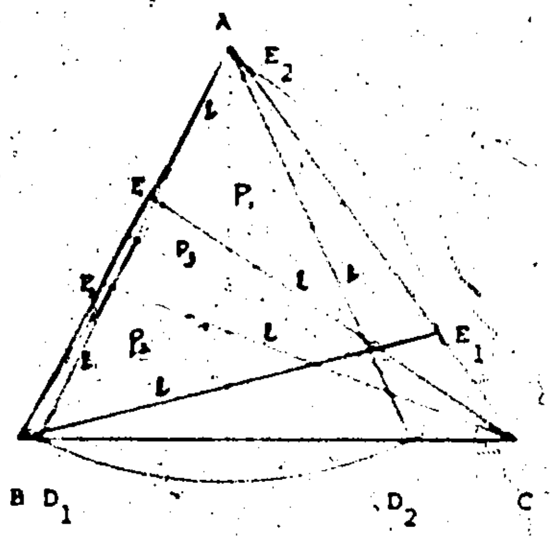

Fig. 33

(1.2b.12)

$$S_8(a,\ell) = \{(x,\theta): \; 0 \le \theta \le \alpha_1(a,\ell), \; 0 \le x \le x_1(a,\ell)\} ,$$

(1.2b.13)

$$S_9(a,\ell) = \{(x,\theta): \; \alpha_1(a,\ell) \le \theta \le \alpha_2(a,\ell), \; 0 \le x \le a\} .$$

(1.2b.14)

$$S_{10}(a,\ell) = \{(x,\theta): \; \alpha_2(a,\ell) \le \theta \le \theta_1(a,\ell),$$

$$0 \le x \le x_1(a,\ell)\} .$$

(1.2b.15)

$$\bigcup_{i=4}^{7} S_i(a,\ell) ,$$

where

(1.2b.16)

$$\alpha_1(a,\ell) = -\gamma + \sin^{-1}\left(\frac{a \sin \gamma}{\ell}\right) ,$$

(1.2b.17)

$$\alpha_2(a,\ell) = \frac{\pi}{2} - \gamma + \cos^{-1}\left(\frac{a \sin \gamma}{\ell}\right) ,$$

and $x_1(a,\ell)$, $x_2(a,\ell)$, $\theta_1(a,\ell)$, $\theta_2(a,\ell)$, and $S_4(a,\ell)$, ..., $S_7(a,\ell)$ are given by (1.2b.3), (1.2b.4), (1.2b.10), (1.2b.11), and (1.2b.6), ..., (1.2b.9), respectively.

Part (iv). $p_3 \le \ell \le c$:

In this case, the sets in the parameter space that contribute to $F(\ell)$ are the following (cf. Fig. 33).

(1.2b.18)

$$\left(\bigcup_{i=4}^{6} S_i(a,\ell)\right) \bigcup_{j=8}^{10} S_j(a,\ell)$$

(1.2b.19)

$$S_{11}(a,l) = \{(x,\theta):\ \theta_2(a,l) \le \theta \le \theta_1(a,l),$$

$$x_2(a,l) \le x \le a\} \ .$$

(1.2b.20)

$$S_{12}(a,l) = \{(x,\theta):\ \theta_1(a,l) \le \theta \le \theta_2(a,l),\ 0 \le x \le a\} \ .$$

(1.2b.21)

$$S_{13}(a,l) = \{(x,\theta):\ \theta_2(a,l) \le \theta \le \pi,\ x_2(a,l) \le x \le a\} \ ,$$

where

(1.2b.22)

$$\theta_1(a,l) = \beta + \sin^{-1}\left(\frac{a \sin \beta}{l}\right) \ .$$

(1.2b.23)

$$\theta_2(a,l) = \frac{\pi}{2} + \beta + \cos^{-1}\left(\frac{a \sin \beta}{l}\right) \ ,$$

and $S_4(a,l)$, ..., $S_6(a,l)$, $S_8(a,l)$, ..., $S_{10}(a,l)$, $x_1(a,l)$, $x_2(a,l)$, $\alpha_1(a,l)$, $\alpha_2(a,l)$, $\theta_1(a,l)$, and $\theta_2(a,l)$ are given by (1.2b.6), ..., (1.2b.8), (1.2b.12), ..., (1.2b.14), (1.2b.3), (1.2b.4), (1.2b.16), (1.2b.17), (1.2b.10), and (1.2b.11), respectively.

Part (v): $c \le l \le b$.

In this case we have the following sets in the parameter space contributing to $F(l)$ (cf. Fig. 34).

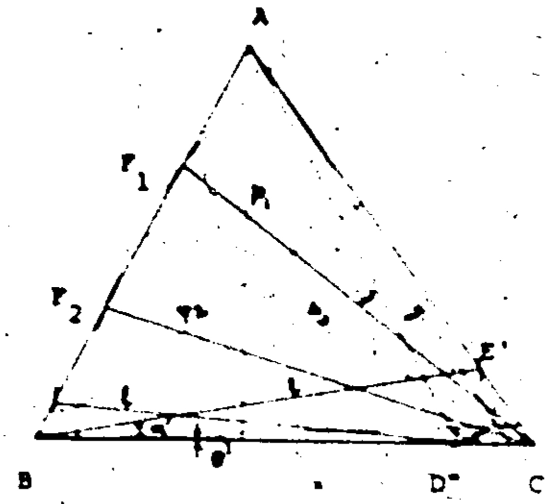

Fig. 34

Fig. 35

(1.2b.24)

$$S_{14}(a,\ell) = \{(x,\theta): \ 0 \le \theta \le \alpha'(a,\ell), \ 0 \le x \le x_1(a,\ell)\} \ ,$$

(1.2b.25)

$$S_{15}(a,\ell) = \{(x,\theta): \ \alpha'(a,\ell) \le \theta \le \theta''(a,\ell), \ 0 \le x \le a\} \ ,$$

(1.2b.26)

$$S_{16}(a,\ell) = \{(x,\theta): \ \theta''(a,\ell) \le \theta \le \pi-\gamma, \ 0 \le x \le x_1(a,\ell)\} \ ,$$

(1.2b.27)

$$S_{17}(a,\ell) = \{(x,\theta): \ \theta''(a,\ell) \le \theta \le \beta_1(a,\ell),$$

$$x_2(a,\ell) \le x \le a\} \ ,$$

(1.2b.28)

$$S_{12}(a,\ell) \cup S_{13}(a,\ell) \ ,$$

where

(1.2b.29)

$$\alpha'(a,\ell) = -\gamma + \sin^{-1}\left(\frac{a \sin \gamma}{\ell}\right) \ ,$$

(1.2b.30)

$$\theta''(a,\ell) = \pi - \sin^{-1}\left(\frac{c \sin \beta}{\ell}\right) \ ,$$

and $x_1(a,\ell)$, $x_2(a,\ell)$, $\beta_1(a,\ell)$, $\beta_2(a,\ell)$, $S_{12}(a,\ell)$, and $S_{13}(a,\ell)$ are given by (1.2b.3), (1.2b.4), (1.2b.22), (1.2b.23), (1.2b.20), and (1.2b.21) respectively.

Part (vi) $b \le \ell \le a$.

In this case we have the following sets in the parameter space contributing to $F(\ell)$ (cf. Fig. 35).

(1.2b.31)

$$S_{14}(a,\ell) = \{(x,\theta):\ 0 \le \theta \le \alpha'(a,\ell),\ 0 \le x \le x_1(a,\ell)\},$$

(1.2b.32)

$$S_{18}(a,\ell) = \{(x,\theta):\ \alpha'(a,\ell) \le \theta \le \beta''(a,\ell),\ 0 \le x \le a\}\ ,$$

(1.2b.33)

$$S_{19}(a,\ell) = \{(x,\theta):\ \beta''(a,\ell) \le \theta \le \pi,\ x_2(a,\ell) \le x \le a\}\ ,$$

where

(1.2b.34)

$$\beta''(a,\ell) = \beta + \sin^{-1}\left(\frac{b \sin \alpha}{\ell}\right)\quad ,$$

and $\alpha'(a,\ell)$, $x_1(a,\ell)$, and $x_2(a,\ell)$, are given by (1.2b.29), (1.2b.3), and (1.2b.4) respectively.

Case 2: Secants arising from random points on CA.

Using the same geometrical arguments as in the previous case of a triangle, we obtain the following sets in the parameter space S contributing to $F(l)$ for l lying in different appropriate intervals of $[0,a]$.

Part (i): $0 \leq l \leq p_1$.

In this case we have the following sets in the parameter space contributing to $F(l)$ (cf. Fig. 36).

(1.2b. 35)

$$S_1(b,l) = \{(x,\theta):\ 0 \leq \theta \leq \pi-\alpha,\ 0 \leq x \leq x_1(b,l)\}$$

(1.2b. 36)

$$S_2(b,l) = \{(x,\theta):\ \gamma \leq \theta \leq \pi,\ x_2(b,l) \leq x \leq b\},$$

where

(1.2b. 37)

$$x_1(b,l) = \frac{l \sin(\theta+\alpha)}{\sin \alpha},$$

(1.2b. 38)

$$x_2(b,l) = b - \frac{l \sin(\theta-\gamma)}{\sin \gamma}.$$

Part (ii): $p_1 \leq l \leq p_2$.

In this case we have the following sets in the parameter space S contributing to $F(l)$ (cf. Fig. 37):

(1.2b. 39)

$$S_1(b,l) = \{(x,\theta):\ 0 \leq \theta \leq \pi-\alpha,\ 0 \leq x \leq x_1(b,l)\},$$

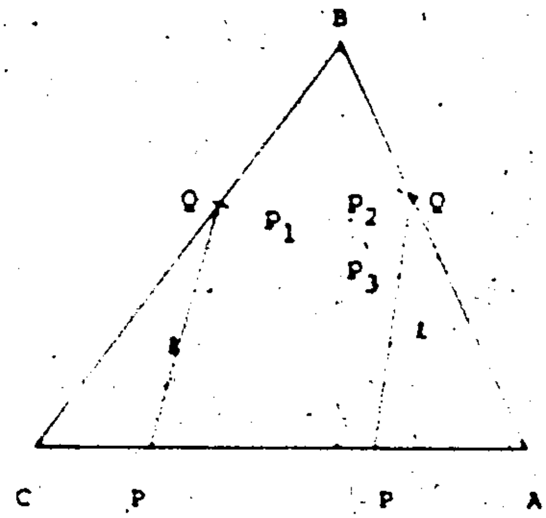

Fig. 36

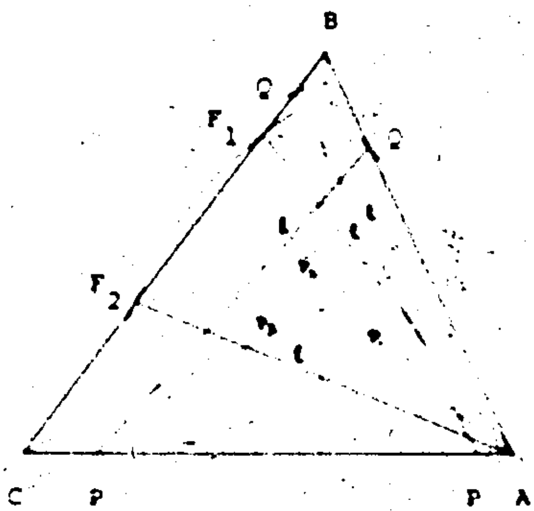

Fig. 37

(1.2b.40)

$$S_3(b,\ell) = \{(x,\theta): \ \gamma \le \theta \le \theta_1(b,\ell),\ x_2(b,\ell) \le x \le b\} ,$$

(1.2b.41)

$$S_4(b,\ell) = \{(x,\theta): \ \theta_1(b,\ell) \le \theta \le \theta_2(b,\ell),\ 0 \le x \le b\} ,$$

(1.2b.42)

$$S_5(b,\ell) = \{(x,\theta): \ \theta_2(b,\ell) \le \theta \le \pi,\ x_2(b,\ell) \le x \le b\} ,$$

where

(1.2b.43)

$$\theta_1(b,\ell) = \gamma + \sin^{-1}\left(\frac{b \sin \gamma}{\ell}\right) ,$$

(1.2b.44)

$$\theta_2(b,\ell) = \frac{\pi}{2} + \gamma + \cos^{-1}\left(\frac{b \sin \gamma}{\ell}\right) ,$$

and $x_1(b,\ell)$ and $x_2(b,\ell)$ are given by (1.2b.37) and (1.2b.38), respectively.

Part (iii): $p_2 \le \ell \le p_3$.

In this case we have the following sets in the parameter space contributing to $F(\ell)$ (cf. Fig. 38).

(1.2b.45)

$$S_6(b,\ell) = \{(x,\theta): \ 0 \le \theta \le \theta_1(b,\ell),\ 0 \le x \le x_1(b,\ell)\} ,$$

(1.2b.46)

$$S_7(b,\ell) = \{(x,\theta): \ \theta_1(b,\ell) \le \theta \le \theta_2(b,\ell),\ 0 \le x \le b\} ,$$

(1.2b.47)

$$S_8(b,\ell) = \{(x,\theta): \ \theta_2(b,\ell) \le \theta \le \pi - \alpha,\ 0 \le x \le x_1(b,\ell)\} ,$$

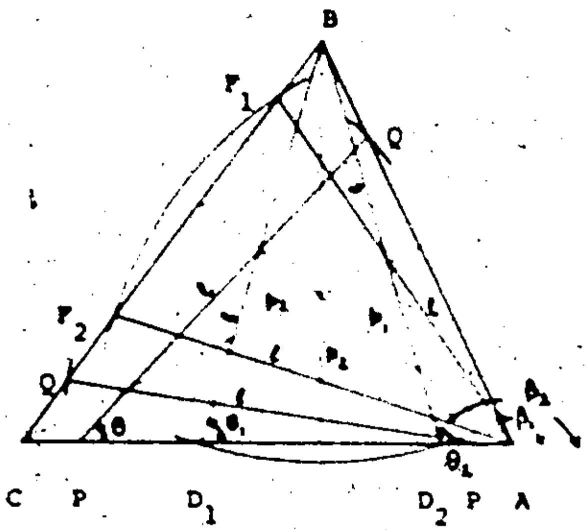

Fig. 38

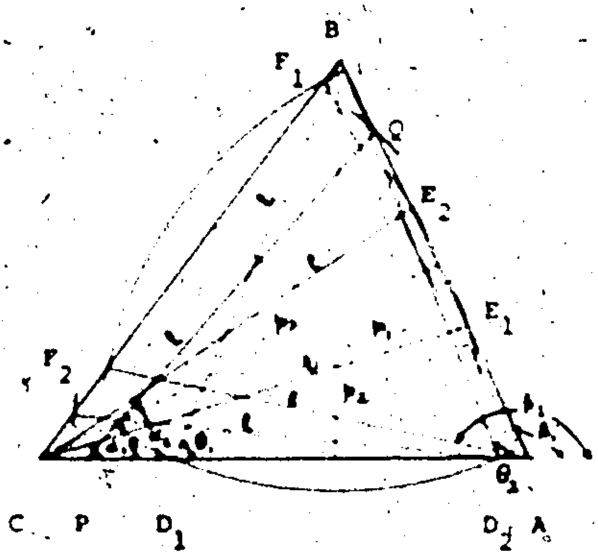

Fig. 39

(1.2b.48)

$$S_9(b,\ell) = \{(x,\theta):\ \gamma \le \theta \le \theta_1(b,\ell),\ x_2(b,\ell) \le x \le b\}\ .$$

(1.2b.49)

$$S_{10}(b,\ell) = \{(x,\theta):\ \theta_2(b,\ell) \le \theta \le \theta_1(b,\ell),$$

$$x_2(b,\ell) \le x \le b\}\ .$$

(1.2b.50)

$$S_4(b,\ell) \cup S_5(b,\ell)\ ,$$

where

(1.2b.51)

$$\theta_1(b,\ell) = \sin^{-1}\left(\frac{c\ \sin\ \alpha}{\ell}\right)\ .$$

(1.2b.52)

$$\theta_2(b,\ell) = \pi - \sin^{-1}\left(\frac{c\ \sin\ \alpha}{\ell}\right)\ .$$

and $S_4(b,\ell)$, $S_5(b,\ell)$, $x_1(b,\ell)$, $x_2(b,\ell)$, $\theta_1(b,\ell)$ and $\theta_2(b,\ell)$ are given by (1.2b.41), (1.2b.42), (1.2b.37), (1.2b.38), (1.2b.43) and (1.2b.44), respectively.

Part (iv): $p_3 \le \ell \le c$.

In this case we have the following sets in the parameter space contributing to $F(\ell)$ (cf. Fig. 39):

(1.2b.53)

$$S_{11}(b,\ell) = \{(x,\theta):\ 0 \le \theta \le \alpha_1(b,\ell),\ 0 \le x \le x_1(b,\ell)\}\ .$$

(1.2b.54)

$$S_{12}(b,\ell) = \{(x,\theta):\ \alpha_1(b,\ell) \le \theta \le \alpha_2(b,\ell),\ 0 \le x \le b\}\ .$$

(1.2b. 55)

$$S_{13}(b,\ell) = \{(x,\theta): \alpha_2(b,\ell) \le \theta \le \theta_1(b,\ell),$$

$$0 \le x \le x_1(b,\ell)\}$$

(1.2b. 56)

$$\bigcup_{i=y}^{10} S_i(b,\ell) \cup S_4(b,\ell) \cup S_5(b,\ell)$$

where

(1.2b. 57)

$$\alpha_1(b,\ell) = -\alpha + \sin^{-1}\left(\frac{b \sin \alpha}{\ell}\right) ,$$

(1.2b. 58)

$$\alpha_2(b,\ell) = \frac{\pi}{2} - \alpha + \cos^{-1}\left(\frac{b \sin \alpha}{\ell}\right) ,$$

and $S_i(b,\ell)$, $i = 7, \ldots, 10$, $S_4(b,\ell)$, $S_5(b,\ell)$, $x_1(b,\ell)$, $x_2(b,\ell)$, $\theta_1(b,\ell)$, $\theta_2(b,\ell)$, $\beta_1(b,\ell)$ and $\beta_2(b,\ell)$ are given by (1.2b. 46), ..., (1.2b.49), (1.2b.41), (1.2b.42), (1.2b.37), (1.2b. 38), (1.2b.51), (1.2b.52), (1.2b.43) and (1.2b.44), respectively.

Part (v): $c \le \ell \le b$.

In this case we have the following sets in the parameter space contributing to $F(\ell)$ (cf. Fig. 40).

(1.2b. 59)

$$S_{14}(b,\ell) = \{(x,\theta): 0 \le \theta \le \alpha_1(b,\ell), 0 \le x \le x_1(b,\ell)\} ,$$

(1.2b. 60)

$$S_{15}(b,\ell) = \{(x,\theta): \alpha_1(b,\ell) \le \theta \le \alpha_2(b,\ell), 0 \le x \le b\} .$$

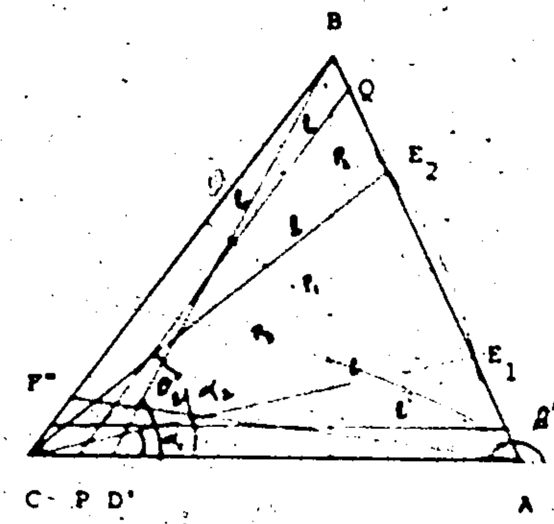

Fig. 40

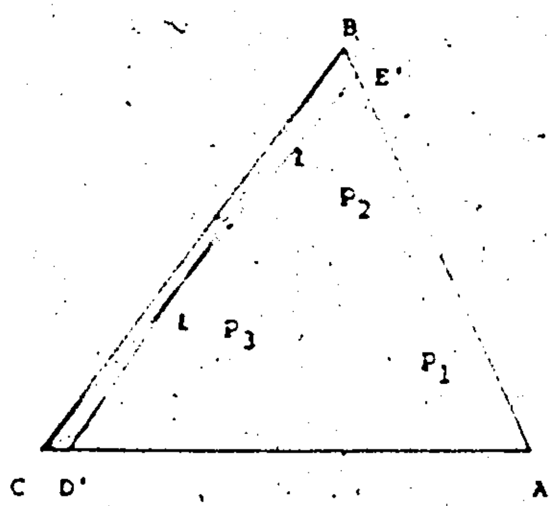

Fig. 41

(1.2b.61)

$$S_{16}(b,\ell) = \{(x,\theta):\ \alpha_2(b,\ell) \le \theta \le \theta'(b,\ell),$$

$$0 \le x \le x_1(b,\ell)\}\ ,$$

(1.2b.62)

$$S_{17}(b,\ell) = \{(x,\theta):\ \theta'(b,\ell) \le \theta \le \theta''(b,\ell);\ 0 \le x \le b\}\ ,$$

(1.2b.63)

$$S_{18}(b,\ell) = \{(x,\theta):\ \gamma \le \theta \le \theta'(b,\ell),\ x_2(b,\ell) \le x \le b\}\ ,$$

(1.2b.64)

$$S_{19}(b,\ell) = \{(x,\theta):\ \theta''(b,\ell) \le \theta \le \pi,\ x_2(b,\ell) \le x \le b\}\ .$$

where

(1.2b.65)

$$\theta'(b,\ell) = \sin^{-1}\left(\frac{c \sin \alpha}{\ell}\right)\ ,$$

(1.2b.66)

$$\theta''(b,\ell) = \gamma - \sin^{-1}\left(\frac{c \sin \beta}{\ell}\right) + \pi\ ,$$

and $\alpha_1(b,\ell)$, $\alpha_2(b,\ell)$, $x_1(b,\ell)$ and $x_2(b,\ell)$ are given by (1.2b.57), (1.2b.58), (1.2b.37), and (1.2b.38), respectively.

Part (vi): $b \le \ell \le a$.

In this case we have the following sets in the parameter space contributing to $F(\ell)$ (cf. Fig. 41).

(1.2b.67).

$$S_{20}(b,\ell) = \{(x,\theta):\ 0 \le \theta \le \alpha''(b,\ell),\ 0 \le x \le b\}\ .$$

(1.2b.68)

$$S_{21}(b,\ell) = \{(x,\theta): \alpha''(b,\ell) \le \theta \le \theta'(b,\ell),$$

$$0 \le x \le x_1(b,\ell)\}.$$

(1.2b.69)

$$S_{22}(b,\ell) = \{(x,\theta): \theta'(b,\ell) \le \theta \le \pi, 0 \le x \le b\}.$$

(1.2b.70)

$$S_{23}(b,\ell) = \{(x,\theta): \gamma \le \theta \le \theta'(b,\ell), x_2(b,\ell) \le x \le b\}.$$

where

(1.2b.71)

$$\alpha''(b,\ell) = -\alpha + \sin^{-1}\left(\frac{b \sin \alpha}{\ell}\right),$$

and $\theta'(b,\ell)$, $x_1(b,\ell)$, and $x_2(b,\ell)$ are given by (1.2b.65), (1.2b.37), and (1.2b.38), respectively.

Case 3: Secants arising from random points on AB.

Using the same geometrical arguments as in the previous case of a triangle, we obtain the following sets in the parameter space S contributing to $F(\ell)$ for ℓ lying in different appropriate intervals of $[0,a]$.

Part (i): $0 \le \ell \le p_1$.

In this case we have the following sets in the parameter space S contributing to $F(\ell)$ (cf. Fig. 42).

(1.2b.72)

$$S_1(c,\ell) = \{(x,\theta):\ 0 \le \theta \le \pi-\beta,\ 0 \le x \le x_1(c,\ell)\}\ ,$$

(1.2b.73)

$$S_2(c,\ell) = \{(x,\theta):\ \alpha \le \theta \le \pi,\ x_2(c,\ell) \le x \le c\}\ ,$$

where

(1.2b.74)

$$x_1(c,\ell) = \frac{\ell \sin(\theta+\beta)}{\sin\beta}\ ,$$

(1.2b.75)

$$x_2(c,\ell) = b - \frac{\ell \sin(\theta-\alpha)}{\sin\alpha}$$

Part (ii): $p_1 \le \ell \le p_2$.

In this case we have the following sets in the parameter space S contributing to $F(\ell)$ (cf. Fig. 43).

(1.2b.76)

$$S_3(c,\ell) = \{(x,\theta):\ 0 \le \theta \le \alpha_1(c,\ell),\ 0 \le x \le x_1(c,\ell)\}\ ,$$

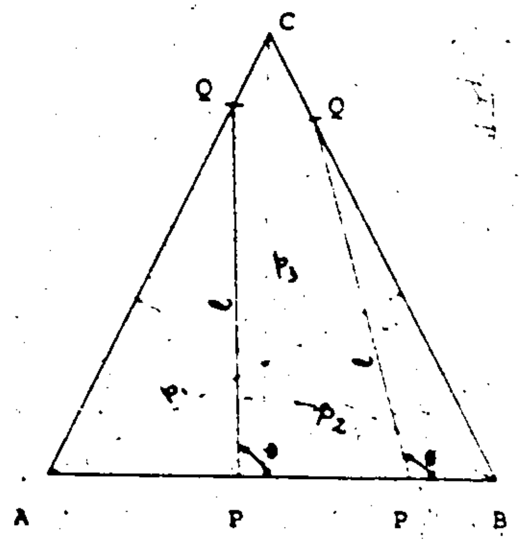

Fig. 42

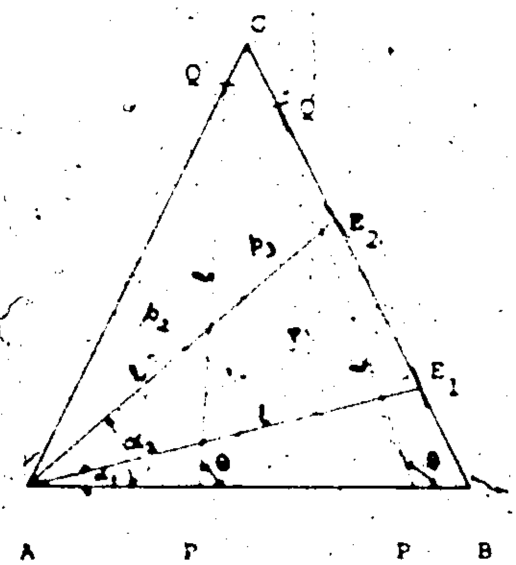

Fig. 43

(1.2b.77)

$$S_4(c,\ell) = \{(x,\theta): \ \alpha_1(c,\ell) \le \theta \le \alpha_2(c,\ell), \ 0 \le x \le c\}$$

(1.2b.78)

$$S_5(c,\ell) = \{(x,\theta): \ \alpha_2(c,\ell) \le \theta \le \pi-\beta, \ 0 \le x \le x_1(c,\ell)\}$$

(1.2b.79)

$$S_2(c,\ell) = \{(x,\theta): \ \alpha \le \theta \le \pi, \ x_2(c,\ell) \le x \le c\},$$

where

(1.2b.80)

$$\alpha_1(c,\ell) = -\beta + \sin^{-1}\left(\frac{c \sin \beta}{\ell}\right)$$

(1.2b.81)

$$\alpha_2(c,\ell) = \frac{\pi}{2} - \beta + \cos^{-1}\left(\frac{c \sin \beta}{\ell}\right)$$

and $x_1(c,\ell)$ and $x_2(c,\ell)$ are given by (1.2b.74) and (1.2b.75), respectively.

Part (iii) : $p_2 \le \ell \le p_3$.

In this case we have the following sets in the parameter space S contributing to $F(\ell)$ (cf. Fig. 44).

(1.2b.82)

$$\bigcup_{i=3}^{5} S_i(c,\ell) \ .$$

(1.2b.83)

$$S_6(c,\ell) = \{(x,\theta): \ \alpha \le \theta \le \beta_1(c,\ell), \ x_2(c,\ell) \le x \le c\} \ ,$$

(1.2b.84)

$$S_7(c,\ell) = \{(x,\theta): \ \beta_1(c,\ell) \le \theta \le \beta_2(c,\ell), \ 0 \le x \le c\} \ ,$$

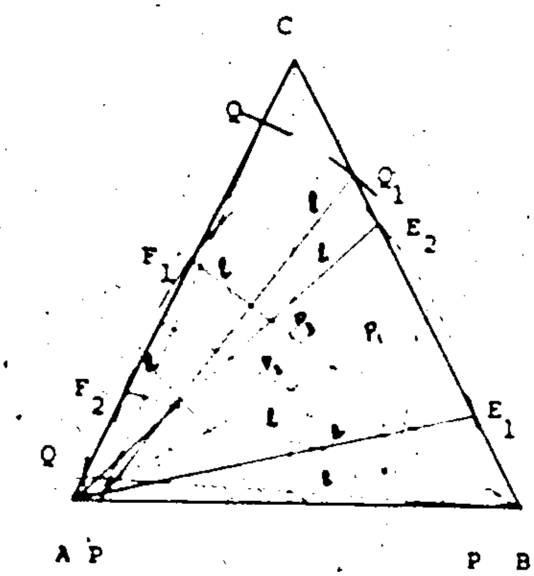

Fig. 44

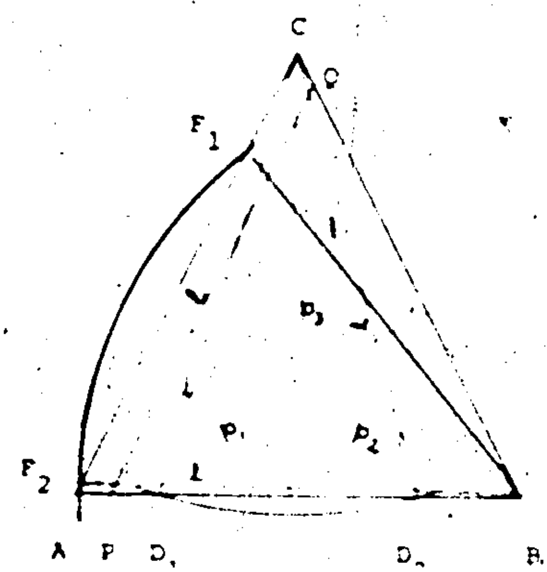

Fig. 45

(1.2b.85)

$$S_8(c,\ell) = \{\beta_2(c,\ell) \le \theta \le \pi,\ x_2(c,\ell) \le x \le c\} ,$$

where

(1.2b.86)

$$\beta_1(c,\ell) = \alpha + \sin^{-1}\left(\frac{c \sin \alpha}{\ell}\right) ,$$

(1.2b.87)

$$\beta_2(c,\ell) = \frac{\pi}{2} + \alpha + \cos^{-1}\left(\frac{c \sin \alpha}{\ell}\right) ,$$

and $S_i(c,\ell)$, $i = 3, 4, 5$, $x_1(c,\ell)$, $x_2(c,\ell)$, $\alpha_1(c,\ell)$, and $\alpha_2(c,\ell)$ are given by (1.2b.76), ..., (1.2b.78), (1.2b.74), (1.2b.75), (1.2b.80), and (1.2b.81), respectively.

Part (iv): $p_3 \le \ell \le c$.

In this case we have the following sets in the parameter space S contributing to $F(\ell)$ (cf. Fig. 45).

(1.2b.88)

$$S_3(c,\ell) \cup S_4(c,\ell) \cup S_7(c,\ell) \cup S_8(c,\ell) .$$

(1.2b.89)

$$S_9(c,\ell) = \{(x,\theta):\ \alpha_2(c,\ell) \le \theta \le \theta_1(c,\ell),$$

$$0 \le x \le x_1(c,\ell)\} .$$

(1.2b.[illegible])

$$S_{10}(c,\ell) = \{(x,\theta):\ \theta_1(c,\ell) \le \theta \le \theta_2(c,\ell),\ 0 \le x \le c\} ,$$

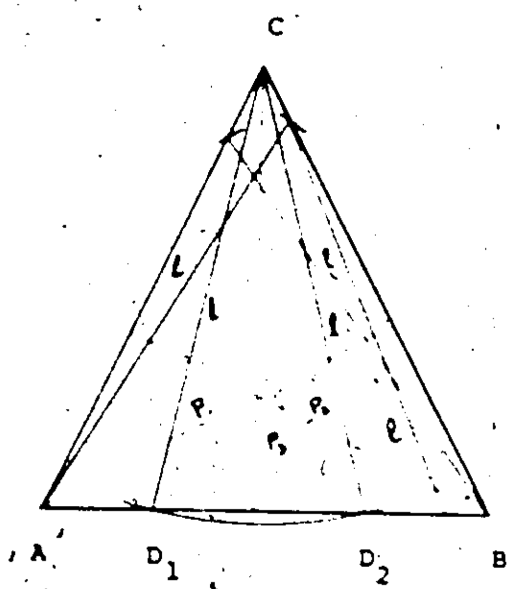

Fig. 46

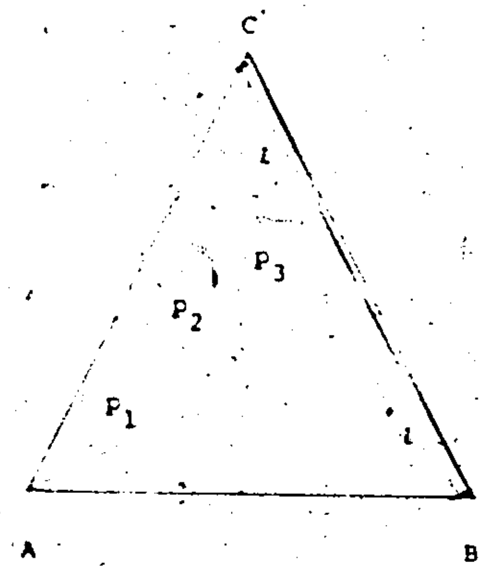

Fig. 47

(1.2b. 91)

$$S_{11}(c,\ell) = \{(x,\theta): \theta_2(c,\ell) \le \theta \le \pi-\beta,\ 0 \le x \le x_1(c,\ell)\},$$

(1.2b. 92)

$$S_{12}(c,\ell) = \{(x,\theta): \alpha \le \theta \le \theta_1(c,\ell),\ x_2(c,\ell) \le x \le c\},$$

(1.2b. 93)

$$S_{13}(c,\ell) = \{(x,\theta): \theta_2(c,\ell) \le \theta \le \beta_1(c,\ell),$$

$$x_2(c,\ell) \le x \le c\},$$

where

(1.2b. 94)

$$\theta_1(c,\ell) = \sin^{-1}\left(\frac{a \sin \beta}{\ell}\right),$$

(1.2b. 95)

$$\theta_2(c,\ell) = \pi - \sin^{-1}\left(\frac{a \sin \beta}{\ell}\right),$$

and $S_3(c,\ell)$, $S_4(c,\ell)$, $S_7(c,\ell)$, $S_8(c,\ell)$, $x_1(c,\ell)$, $x_2(c,\ell)$, $\alpha_1(c,\ell)$, $\alpha_2(c,\ell)$, $\beta_1(c,\ell)$, and $\beta_2(c,\ell)$ are given by (1.2b. 76), (1.2b. 77), (1.2b. 84), (1.2b. 85), (1.2b. 74), (1.2b. 75), (1.2b. 80), (1.2b. 81), (1.2b. 86), and (1.2b. 87), respectively.

Part (v): $c \le \ell \le b$.

In this case we have the following sets in the parameter space S contributing to $F(\ell)$ (cf. Fig. 46).

(1.2b. 96)

$$S_{14}(c,\ell) = \{(x,\theta): 0 \le \theta \le \alpha^*(c,\ell),\ 0 \le x \le c,$$

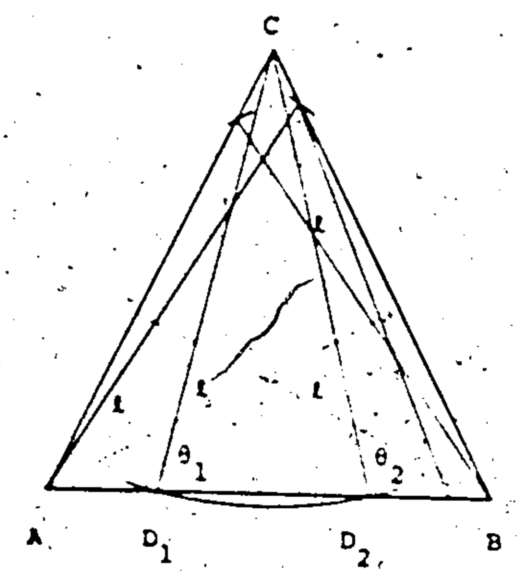

Fig. 46

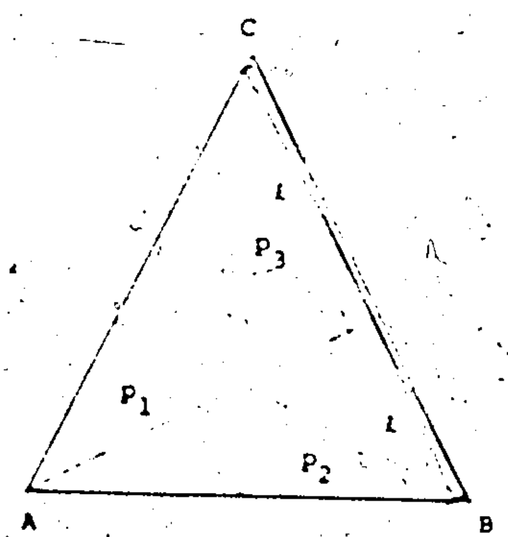

Fig. 47

(1.2b.97)

$$S_{15}(c,\ell) = \{(x,\theta): \alpha''(c,\ell) \le \theta \le \theta_1(c,\ell),$$

$$0 \le x \le x_1(c,\ell)\} .$$

(1.2b.98)

$$S_{16}(c,\ell) = \{(x,\theta): \theta_1(c,\ell) \le \theta \le \theta_2(c,\ell),\ 0 \le x \le c\} ,$$

(1.2b.99)

$$S_{17}(c,\ell) = \{(x,\theta): \theta_2(c,\ell) \le \theta \le \pi-\beta,\ 0 \le x \le x_1(c,\ell)\} ,$$

(1.2b.100)

$$S_{18}(c,\ell) = \{(x,\theta): \alpha \le \theta \le \theta_1(c,\ell),\ x_2(c,\ell) \le x \le c\} ,$$

(1.2b.101)

$$S_{19}(c,\ell) = \{(x,\theta): \theta_2(c,\ell) \le \theta \le \beta'(c,\ell),$$

$$x_2(c,\ell) \le x \le c\} .$$

(1.2b.102)

$$S_{20}(c,\ell) = \{(x,\theta): \beta'(c,\ell) \le \theta \le \pi,\ 0 \le x \le c\} ,$$

where

(1.2b.103)

$$\alpha''(c,\ell) = \pi - \beta + \sin^{-1}\left(\frac{c \sin \beta}{\ell}\right) ,$$

(1.2b.104)

$$\beta'(c,\ell) = \alpha + \sin^{-1}\left(\frac{c \sin \alpha}{\ell}\right) ,$$

and $x_1(c,\ell)$, $x_2(c,\ell)$, $\theta_1(c,\ell)$, and $\theta_2(c,\ell)$ are given by (1.2b.74), (1.2b.75), (1.2b.94), and (1.2b.95) respectively.

Part (vi): $b \le \ell \le a$.

In this case we have the following sets in the parameter space S contributing to $F(\ell)$ (cf. Fig. 47).

(1.2b.105)

$$S_{21}(c,\ell) = \{(x,\theta): \ 0 \le \theta \le \theta''(c,\ell), \ 0 \le x \le c\} \quad ,$$

(1.2b.106)

$$S_{22}(c,\ell) = \{(x,\theta): \ \theta''(c,\ell) \le \theta \le \pi-\beta, \ 0 \le x \le x_1(c,\ell)\} \quad ,$$

(1.2b.107)

$$S_{23}(c,\ell) = \{(x,\theta): \ \theta''(c,\ell) \le \theta \le \beta'(c,\ell),$$

$$x_2(c,\ell) \le x \le c\} \quad ,$$

(1.2b.108)

$$S_{24}(c,\ell) = \{(x,\theta): \ \beta'(c,\ell) \le \theta \le \pi, \ 0 \le x \le c\} \quad ,$$

where

(1.2b.109)

$$\theta''(c,\ell) = \pi - \sin^{-1}\left(\frac{b \sin \alpha}{\ell}\right) \quad ,$$

(1.2b.110)

$$\beta'(c, \ ) = \alpha + \sin^{-1}\left(\frac{c \sin \alpha}{\ell}\right) \quad ,$$

and $x_1(c,\ell)$ and $x_2(c,\ell)$ are given by (1.2b.74) and (1.2b.75), respectively.

1.2.5. Triangle III: $a > b > p_3 > c > p_2 > p_1$.

The corresponding sets that contribute to $p_\gamma(L \le \ell)$ for different intervals of ℓ are as follows (cf. Fig. 48):

For Part (i) : $0 \le \ell \le p_1$.

(1) $S_1(a,\ell) = \{(x,\theta):\ 0 \le \theta \le \pi-\gamma,\ 0 \le x \le x_1(a,\ell)\}$,

(2) $S_2(a,\ell) = \{(x,\theta):\ \beta \le \theta \le \pi,\ x_2(a,\ell) \le x \le a\}$,

(3) $S_3(b,\ell) = \{(x,\theta):\ 0 \le \theta \le \pi-\alpha,\ 0 \le x \le x_1(b,\ell)\}$,

(4) $S_4(b,\ell) = \{(x,\theta):\ \gamma \le \theta \le \pi,\ x_2(b,\ell) \le x \le b\}$;

(5) $S_5(c,\ell) = \{(x,\theta):\ 0 \le \theta \le \pi-\beta,\ 0 \le x \le x_1(c,\ell)\}$,

(6) $S_6(c,\ell) = \{(x,\theta):\ \alpha \le \theta \le \pi,\ x_2(c,\ell) \le x \le c\}$;

For Part (ii) : $p_1 \le \ell \le p_2$.

(1) $S_7(a,\ell) = \{(x,\theta):\ 0 \le \theta \le \theta_1(a,\ell),\ 0 \le x \le x_1(a,\ell)\}$,

(2) $S_8(a,\ell) = \{(x,\theta):\ \theta_1(a,\ell) \le \theta \le \theta_2(a,\ell),\ 0 \le x \le a\}$,

(3) $S_9(a,\ell) = \{(x,\theta):\ \theta_2(a,\ell) \le \theta \le \pi-\gamma,\ 0 \le x \le x_1(a,\ell)\}$,

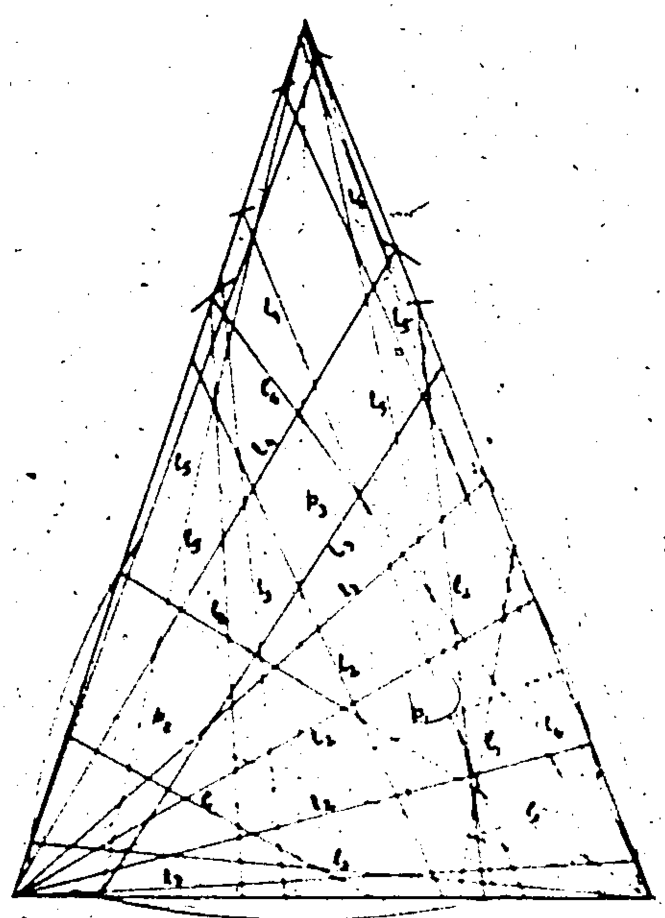

Fig. 48

$\ell_1 \subset [0,p_1]$, $\ell_2 \subset [p_1,p_2]$, $\ell_3 \subset [p_2,c]$,

$\ell_4 \subset [c,p_3]$, $\ell_5 \subset [p_3,b]$, $\ell_6 \subset [b,a]$

(4) $S_{10}(a,\ell) = \{(x,\theta): \beta \le \theta \le \theta_1(a,\ell),\ x_2(a,\ell) \le x \le a\}$,

(5) $S_{11}(a,\ell) = \{(x,\theta): \theta_2(a,\ell) \le \theta \le \pi,\ x_2(a,\ell) \le x \le a\}$,

(6) $S_{12}(b,\ell) = \{(x,\theta): 0 \le \theta \le \pi-\alpha,\ 0 \le x \le x_1(b,\ell)\}$,

(7) $S_{13}(b,\ell) = \{(x,\theta): \gamma \le \theta \le \beta_1(b,\ell),\ x_2(b,\ell) \le x \le b\}$,

(8) $S_{14}(b,\ell) = \{(x,\theta): \beta_1(b,\ell) \le \theta \le \beta_2(b,\ell),\ 0 \le x \le b\}$,

(9) $S_{15}(b,\ell) = \{(x,\theta): \beta_2(b,\ell) \le \theta \le \pi,\ x_2(b,\ell) \le x \le b\}$,

(10) $S_{16}(c,\ell) = \{(x,\theta): 0 \le \theta \le \alpha_1(c,\ell),\ 0 \le x \le x_1(c,\ell)\}$,

(11) $S_{17}(c,\ell) = \{(x,\theta): \alpha_1(c,\ell) \le \theta \le \alpha_2(c,\ell),\ 0 \le x \le c\}$,

(12) $S_{18}(c,\ell) = \{(x,\theta): \alpha_2(c,\ell) \le \theta \le \pi-\beta,\ 0 \le x \le x_1(c,\ell)\}$,

(13) $S_{19}(c,\ell) = \{(x,\theta): \alpha \le \theta \le \pi,\ x_2(c,\ell) \le x \le c\}$.

For Part (iii): $p_2 \le \ell \le c$.

(1) $S_{20}(a,\ell) = \{(x,\theta): 0 \le \theta \le \alpha_1(a,\ell),\ 0 \le x \le x_1(a,\ell)\}$,

(2) $S_{21}(a,\ell) = \{(x,\theta): \alpha_1(a,\ell) \le \theta \le \alpha_2(a,\ell),\ 0 \le x \le a\}$,

(3) $S_{22}(a,\ell) = \{(x,\theta):\ \alpha_2(a,\ell) \leq \theta \leq \theta_1(a,\ell),$

$0 \leq x \leq x_1(a,\ell)\}$,

(4) $S_{23}(a,\ell) = \{(x,\theta):\ \theta_1(a,\ell) \leq \theta \leq \theta_2(a,\ell),\ 0 \leq x \leq a\}$,

(5) $S_{24}(a,\ell) = \{(x,\theta):\ \theta_2(a,\ell) \leq \theta \leq \pi-\gamma,\ 0 \leq x \leq x_1(a,\ell)\}$,

(6) $S_{25}(a,\ell) = \{(x,\theta):\ \beta \leq \theta \leq \theta_1(a,\ell),\ x_2(a,\ell) \leq x \leq a\}$,

(7) $S_{26}(a,\ell) = \{(x,\theta):\ \theta_2(a,\ell) \leq \theta \leq \pi,\ x_2(a,\ell) \leq x \leq a\}$,

(8) $S_{27}(b,\ell) = \{(x,\theta):\ 0 \leq \theta \leq \theta_1(b,\ell),\ 0 \leq x \leq x_1(b,\ell)\}$,

(9) $S_{28}(b,\ell) = \{(x,\theta):\ \theta_1(b,\ell) \leq \theta \leq \theta_2(b,\ell),\ 0 \leq x \leq b\}$,

(10) $S_{29}(b,\ell) = \{(x,\theta):\ \theta_2(b,\ell) \leq \theta \leq \pi-\alpha,\ 0 \leq x \leq x_1(b,\ell)\}$,

(11) $S_{30}(b,\ell) = \{(x,\theta):\ \gamma \leq \theta \leq \theta_1(b,\ell),\ x_2(b,\ell) \leq x \leq b\}$,

(12) $S_{31}(b,\ell) = \{(x,\theta):\ \theta_2(b,\ell) \leq \theta \leq \beta_1(b,\ell),\ x_2(b,\ell) \leq x \leq b\}$,

(13) $S_{32}(b,\ell) = \{(x,\theta):\ \beta_1(b,\ell) \leq \theta \leq \beta_2(b,\ell),\ 0 \leq x \leq b\}$,

(14) $S_{33}(b,\ell) = \{(x,\theta):\ \beta_2(b,\ell) \leq \theta \leq \pi,\ x_2(b,\ell) \leq x \leq b\}$,

(15) $S_{34}(c,\ell) = \{(x,\theta): 0 \le \theta \le \alpha_1(c,\ell),\ 0 \le x \le x_1(c,\ell)\}$,

(16) $S_{35}(c,\ell) = \{(x,\theta): \alpha_1(c,\ell) \le \theta \le \alpha_2(c,\ell),\ 0 \le x \le c\}$.

(17) $S_{36}(c,\ell) = \{(x,\theta): \alpha_2(c,\ell) \le \theta \le \pi-\beta,\ 0 \le x \le x_1(c,\ell)\}$,

(18) $S_{37}(c,\ell) = \{(x,\theta): \alpha \le \theta \le \beta_1(c,\ell),\ x_2(c,\ell) \le x \le c\}$.

(19) $S_{38}(c,\ell) = \{(x,\theta): \beta_1(c,\ell) \le \theta \le \beta_2(c,\ell),\ 0 \le x \le c\}$,

(20) $S_{39}(c,\ell) = \{(x,\theta): \beta_2(c,\ell) \le \theta \le \pi,\ x_2(c,\ell) \le x \le c\}$.

For Part (iv) : $c \le \ell \le p_3$.

(1) $S_{40}(a,\ell) = \{(x,\theta): 0 \le \theta \le \alpha'(a,\ell),\ 0 \le x \le x_1(a,\ell)\}$.

(2) $S_{41}(a,\ell) = \{(x,\theta): \alpha'(a,\ell) \le \theta \le \theta''(a,\ell),\ 0 \le x \le a\}$.

(3) $S_{42}(a,\ell) = \{(x,\theta): \theta''(a,\ell) \le \theta \le \pi-\gamma,\ 0 \le x \le x_1(a,\ell)\}$.

(4) $S_{43}(a,\ell) = \{(x,\theta): \theta''(a,\ell) \le \theta \le \pi,\ x_2(a,\ell) \le x \le a\}$.

(5) $S_{44}(b,\ell) = \{(x,\theta): 0 \le \theta \le \theta'(b,\ell),\ 0 \le x \le x_1(b,\ell)\}$.

(6) $S_{45}(b,\ell) = \{(x,\theta): \theta'(b,\ell) \le \theta \le \theta''(b,\ell),\ 0 \le x \le b\}$.

(7) $S_{46}(b,\ell) = \{(x,\theta): \gamma \leq \theta \leq \theta'(b,\ell),\ x_2(b,\ell) \leq x \leq b\}$.

(8) $S_{47}(b,\ell) = \{(x,\theta): \beta''(b,\ell) \leq \theta \leq \pi,\ x_2(b,\ell) \leq x \leq b\}$.

(9) $S_{48}(c,\ell) = \{(x,\theta): 0 \leq \theta \leq \alpha''(c,\ell),\ 0 \leq x \leq c\}$.

(10) $S_{49}(c,\ell) = \{(x,\theta): \alpha''(c,\ell) \leq \theta \leq \pi-\beta,\ 0 \leq x \leq x_1(c,\ell)\}$.

(11) $S_{50}(c,\ell) = \{(x,\theta): \alpha \leq \theta \leq \beta'(c,\ell),\ x_2(c,\ell) \leq x \leq c\}$.

(12) $S_{51}(c,\ell) = \{(x,\theta): \beta'(c,\ell) \leq \theta \leq \pi,\ 0 \leq x \leq c\}$.

For Part (v) : $p_3 \leq \ell \leq b$.

(1) $S_{52}(a,\ell) = \{(x,\theta): 0 \leq \theta \leq \alpha'(a,\ell),\ 0 \leq x \leq x_1(a,\ell)\}$.

(2) $S_{53}(a,\ell) = \{(x,\theta): \alpha'(a,\ell) \leq \theta \leq \theta''(a,\ell),\ 0 \leq x \leq a\}$.

(3) $S_{54}(a,\ell) = \{(x,\theta): \theta''(a,\ell) \leq \theta \leq \pi-\gamma,\ 0 \leq x \leq x_1(a,\ell)\}$.

(4) $S_{55}(a,\ell) = \{(x,\theta): \theta''(a,\ell) \leq \theta \leq \beta_1(a,\ell),$

$x_2(a,\ell) \leq x \leq a\}$.

(5) $S_{56}(a,\ell) = \{(x,\theta): \beta_1(a,\ell) \leq \theta \leq \beta_2(a,\ell),\ 0 \leq x \leq a\}$,

(6) $S_{57}(a,\ell) = \{(x,\theta): \beta_2(a,\ell) \le \theta \le \pi,\ x_2(a,\ell) \le x \le a\}$,

(7) $S_{58}(b,\ell) = \{(x,\theta): 0 \le \theta \le \alpha_1(b,\ell),\ 0 \le x \le x_1(b,\ell)\}$,

(8) $S_{59}(b,\ell) = \{(x,\theta): \alpha_1(b,\ell) \le \theta \le \alpha_2(b,\ell),\ 0 \le x \le b\}$,

(9) $S_{60}(b,\ell) = \{(x,\theta): \alpha_2(b,\ell) \le \theta \le \theta'(b,\ell),$

$0 \le x \le x_1(b,\ell)\}$,

(10) $S_{61}(b,\ell) = \{(x,\theta): \theta'(b,\ell) \le \theta \le \beta''(b,\ell),\ 0 \le x \le b\}$,

(11) $S_{62}(b,\ell) = \{(x,\theta): \gamma \le \theta \le \theta'(b,\ell),\ x_2(b,\ell) \le x \le b\}$,

(12) $S_{63}(b,\ell) = \{(x,\theta): \beta''(b,\ell) \le \theta \le \pi,\ x_2(b,\ell) \le x \le b\}$,

(13) $S_{64}(c,\ell) = \{(x,\theta): 0 \le \theta \le \alpha''(c,\ell),\ 0 \le x \le c\}$,

(14) $S_{65}(c,\ell) = \{(x,\theta): \alpha''(c,\ell) \le \theta \le \theta_1(c,\ell),$

$0 \le x \le x_1(c,\ell)\}$,

(15) $S_{66}(c,\ell) = \{(x,\theta): \theta_1(c,\ell) \le \theta \le \theta_2(c,\ell),\ 0 \le x \le c\}$,

(16) $S_{67}(c,\ell) = \{(x,\theta): \theta_2(c,\ell) \le \theta \le \pi-\beta,\ 0 \le x \le x_1(c,\ell)\}$.

(17) $S_{68}(c,\ell) = \{(x,\theta): \alpha \le \theta \le \theta_1(c,\ell),\ x_2(c,\ell) \le x \le c\}$,

(18) $S_{69}(c,\ell) = \{(x,\theta): \theta_2(c,\ell) \le \theta \le \beta'(c,\ell),$

$x_2(c,\ell) \le x \le c\}$,

(19) $S_{70}(c,\ell) = \{(x,\theta): \beta'(c,\ell) \le \theta \le \pi,\ 0 \le x \le c\}$.

For Part (vi) : $b \le \ell \le a$.

(1) $S_{71}(a,\ell) = \{(x,\theta): 0 \le \theta \le \alpha'(a,\ell),\ 0 \le x \le x_1(a,\ell)\}$,

(2) $S_{72}(a,\ell) = \{(x,\theta): \alpha'(a,\ell) \le \theta \le \beta''(a,\ell),\ 0 \le x \le a\}$,

(3) $S_{73}(a,\ell) = \{(x,\theta): \beta''(a,\ell) \le \theta \le \pi,\ x_2(a,\ell) \le x \le a\}$,

(4) $S_{74}(b,\ell) = \{(x,\theta): 0 \le \theta \le \alpha''(b,\ell),\ 0 \le x \le b\}$,

(5) $S_{75}(b,\ell) = \{(x,\theta): \alpha''(b,\ell) \le \theta \le \theta'(b,\ell),$

$0 \le x \le x_1(b,\ell)\}$.

(6) $S_{76}(b,\ell) = \{(x,\theta): \theta'(b,\ell) \le \theta \le \pi,\ 0 \le x \le b\}$,

(7) $S_{77}(b,\ell) = \{(x,\theta): \gamma \le \theta \le \theta'(b,\ell),\ x_2(b,\ell) \le x \le b\}$,

(8) $S_{78}(c,\ell) = \{(x,\theta):\ 0 \le \theta \le \theta''(c,\ell),\ 0 \le x \le c\}$,

(9) $S_{79}(c,\ell) = \{(x,\theta):\ \theta''(c,\ell) \le \theta \le \pi-\beta,\ 0 \le x \le x_1(c,\ell)\}$,

(10) $S_{80}(c,\ell) = \{(x,\theta):\ \theta''(c,\ell) \le \theta \le \beta'(c,\ell),$

$$x_2(c,\ell) \le x \le c\} ,$$

(11) $S_{81}(c,\ell) = \{(x,\theta):\ \beta'(c,\ell) \le \theta \le \pi,\ 0 \le x \le c\}$.

where $x_1(a,\ell)$, $x_2(a,\ell)$, $x_1(b,\ell)$, $x_2(b,\ell)$, $x_1(c,\ell)$, $x_2(c,\ell)$, $\theta_1(a,\ell)$, $\theta_2(a,\ell)$, $\theta_1(b,\ell)$, $\theta_2(b,\ell)$, $\theta_1(c,\ell)$, $\theta_2(c,\ell)$, $\alpha_1(a,\ell)$, $\alpha_2(a,\ell)$, $\alpha_1(b,\ell)$, $\alpha_2(b,\ell)$, $\alpha_1(c,\ell)$, $\alpha_2(c,\ell)$, $\beta_1(a,\ell)$, $\beta_2(a,\ell)$, $\beta_1(b,\ell)$, $\beta_2(b,\ell)$, $\beta_1(c,\ell)$, $\beta_2(c,\ell)$, $\theta''(a,\ell)$, $\theta'(b,\ell)$, $\theta''(c,\ell)$, $\alpha'(a,\ell)$, $\alpha''(b,\ell)$, $\alpha''(c,\ell)$, $\beta''(a,\ell)$, $\beta''(b,\ell)$, $\beta'(c,\ell)$ are given by (1.2b.3), (1.2b.4), (1.2b.37), (1.2b.38), (1.2b.74), (1.2b.75), (1.2b.10), (1.2b.11), (1.2b.51), (1.2b.52), (1.2b.94), (1.2b.95), (1.2b.16), (1.2b.17), (1.2b.57), (1.2b.58), (1.2b.80), (1.2b.81), (1.2b.22), (1.2b.23), (1.2b.43), (1.2b.44), (1.2b.86), (1.2b.87), (1.2b.30), (1.2b.65), (1.2b.109), (1.2b.29), (1.2b.71), (1.2b.103), (1.2b.34), (1.2b.66), (1.2b.104), respectively.

Proof of Theorem 2.

Now that we have determined the sets that contribute to $F(\ell)$ for ℓ lying in different intervals of $[0, a]$ and for the random point P (source) lying on any side of the triangle, we proceed to obtain the result (1.2.1).

For the case (A): Triangle 1: $a > b > c$, $\alpha > 90°$

Combining all the sets that contribute to $F(\ell)$ for $\ell \epsilon [0, p_1]$, we have

$$F(\ell) = \frac{1}{\pi(a+b+c)} \int_{D(\ell)} dx d\theta$$

$$= \frac{1}{\pi(a+b+c)} \int_{\substack{S_{11}(a,\ell) \cup S_{12}(a,\ell) \cup S_{11}(b,\ell) \cup S_{12}(b,\ell) \\ \cup S_{11}(c,\ell) \cup S_{12}(c,\ell)}} dx d\theta$$

where $S_{11}(a,\ell)$, $S_{12}(a,\ell)$, $S_{11}(b,\ell)$, $S_{12}(b,\ell)$, $S_{11}(c,\ell)$ and $S_{12}(c,\ell)$ are given by (1.2.4), (1.2.5), (1.2.20), (1.2.21), (1.2.34) and (1.2.35), respectively. On evaluating the integral, we have

$$F(\ell) = \frac{1}{\pi(a+b+c)} (\cot \alpha/2 + \cot \beta/2 + \cot \gamma/2), \text{ for } \ell \epsilon [0, p_1]$$

This proves Part (A_1) of (1.2.1).

Combining all the sets that contribute to $F(\ell)$ for $\ell \epsilon [p_1, c]$, we have

$$F(\ell) = \frac{1}{\pi(a+b+c)} \int_{D(\ell)} dx d\theta$$

$$= \frac{1}{\pi(a+b+c)} \int\limits_{\bigcup_{i=1}^{5} S_{2i}(a,\ell)\ \bigcup_{i=1}^{4} S_{2i}(b,\ell)\ \bigcup_{i=1}^{4} S_{2i}(c,\ell)} dx\,d\theta$$

where $S_{2i}(a,\ell)$, $i = 1,\ldots,5$; $S_{2i}(b,\ell)$, $i = 1,\ldots,4$; and $S_{2i}(c,\ell)$, $i = 1,\ldots,4$ are given by (1.2.7) to (1.2.11), (1.2.22) to (1.2.25), and (1.2.36) to (1.2.39), respectively.

$$F(\ell) = \frac{1}{\pi(a+b+c)} \Big[2\ell\left(\frac{1}{\sin\alpha} + \frac{1}{\sin\beta} + \frac{1}{\sin\gamma}\right)$$

$$+ 2\ell(\cot\alpha + \cot\beta + \cot\gamma) + 2(a+b+c)\cos^{-1}\left(\frac{b\sin\gamma}{\ell}\right)$$

$$- 2\cot\gamma(\ell^2-b^2\sin^2\gamma)^{\frac{1}{2}} - 2\cot\beta(\ell^2-b^2\sin^2\gamma)^{\frac{1}{2}}$$

$$- \frac{2}{\sin\beta}(\ell^2-b^2\sin^2\gamma)^{\frac{1}{2}} - \frac{2}{\sin\gamma}(\ell^2-b^2\sin^2\gamma)^{\frac{1}{2}}\Big], \text{for } \ell\in[p_1, c] .$$

This proves Part (A_2) of (1.2.1).

Combining all the sets that contribute to $F(\ell)$ for $\ell\in[c, b]$, we have on evaluating the integrals

$$F(\ell) = \frac{1}{\pi(a+b+c)} \int\limits_{\bigcup_{i=1}^{4} S_{3i}(a,\ell)\ \bigcup_{i=1}^{4} S_{3i}(b,\ell)\ \bigcup_{i=1}^{4} S_{3i}(c,\ell)} dx\,d\theta$$

where $S_{3i}(a,\ell)$, $i = 1,\ldots,4$; $S_{3i}(b,\ell)$, $i = 1,\ldots,4$; $S_{3i}(c,\ell)$, $i = 1,\ldots,4$ are given by (1.2.12), (1.2.13), (1.2.14) and (1.2.16); (1.2.26) to (1.2.29); and (1.2.40) to (1.2.43), respectively.

$$F(\ell) = \frac{1}{\pi(a+b+c)}\Big[(a+b+c)\gamma + \frac{2\ell}{\sin\gamma} + 2\ell\cot\gamma + \ell\cot\beta$$

$$+ \ell\cot\alpha + \ell\Big(\frac{1}{\sin\alpha} + \frac{1}{\sin\beta}\Big) + 2a\cos^{-1}\Big(\frac{a\sin\gamma}{\ell}\Big)$$

$$+ 2b\cos^{-1}\Big(\frac{b\sin\gamma}{\ell}\Big) + c\cos^{-1}\Big(\frac{c\sin\beta}{\ell}\Big)$$

$$+ c\cos^{-1}\Big(\frac{c\sin\alpha}{\ell}\Big) - \cot\gamma(\ell^2-b^2\sin^2\gamma)^{\frac{1}{2}} - \cot\beta(\ell^2-b^2\sin^2\gamma)^{\frac{1}{2}}$$

$$- \cot\gamma(\ell^2-a^2\sin^2\gamma)^{\frac{1}{2}} - \cot\alpha(\ell^2-a^2\sin^2\gamma)^{\frac{1}{2}}$$

$$- \frac{1}{\sin\gamma}(\ell^2-a^2\sin^2\gamma)^{\frac{1}{2}} - \frac{1}{\sin\gamma}(\ell^2-b^2\sin^2\gamma)^{\frac{1}{2}}$$

$$- \frac{1}{\sin\beta}(\ell^2-c^2\sin^2\beta)^{\frac{1}{2}} - \frac{1}{\sin\alpha}(\ell^2-c^2\sin^2\alpha)^{\frac{1}{2}}\Big], \text{ for } \ell\in[c, b].$$

This proves Part (A_3) of (1.2.1).

Combining all the sets that contribute to $F(\ell)$ for $\ell\in[b, a]$, we have on evaluating the integral

$$F(\ell) = \frac{1}{\pi(a+b+c)} \int\limits_{\bigcup_{i=1}^{4} S_{4i}(a,\ell)\ \bigcup_{i=1}^{4} S_{4i}(b,\ell)\ \bigcup_{i=1}^{4} S_{4i}(c,\ell)} dx\,d\theta$$

where $S_{4i}(a,\ell)$, $i = 1,2,3$; $S_{4i}(b,\ell)$, $i = 1,\ldots,4$; $S_{4i}(c,\ell)$, $i = 1,\ldots,4$ are given by (1.2.17), (1.2.18), (1.2.19); (1.2.30) to (1.2.33); and (1.2.44) to (1.2.47), respectively.

$$F(\ell) = \frac{1}{\pi(a+b+c)} \Big[(a+b+c)\ (\beta+\gamma)$$

$$+ \ell\left(\cot\beta + \cot\gamma + \frac{1}{\sin\gamma} + \frac{1}{\sin\beta}\right) - \cot\alpha(\ell^2-a^2\sin^2\beta)^{\frac{1}{2}}$$

$$- \cot\alpha(\ell^2-a^2\sin^2\gamma)^{\frac{1}{2}} - \cot\beta(\ell^2-a^2\sin^2\beta)^{\frac{1}{2}}$$

$$- \cot\gamma(\ell^2-a^2\sin^2\gamma)^{\frac{1}{2}} + (a+2b+c)\cos^{-1}\left(\frac{a\sin\beta}{\ell}\right) +$$

$$(a+c)\cos^{-1}\left(\frac{a\sin\gamma}{\ell}\right) - \frac{1}{\sin\alpha}\left((\ell^2-c^2\sin^2\alpha)^{\frac{1}{2}} + (\ell^2-b^2\sin^2\alpha)^{\frac{1}{2}}\right)$$

$$- \frac{1}{\sin\beta}(\ell^2-a^2\sin^2\gamma)^{\frac{1}{2}} - \frac{1}{\sin\gamma}(\ell^2-a^2\sin^2\gamma)^{\frac{1}{2}}\Big], \text{ for } \ell\in[b,\ a]\ .$$

This proves Part (A_4) of (1.2.1).

For the case B: Triangle II: $a > b > c > p_3 > p_2 > p_1$.

Combining all the sets that contribute to $F(\ell)$ for $\ell \epsilon [0, p_1]$, we have

$$F(\ell) = \frac{1}{\pi(a+b+c)} \int_{D(\ell)} dx d\theta$$

$$= \frac{1}{\pi(a+b+c)} \int_{\substack{S_1(a,\ell) \cup S_2(a,\ell) \cup S_1(b,\ell) \\ \cup S_2(b,\ell) \cup S_1(c,\ell) \cup S_2(c,\ell)}} dx d\theta$$

where $S_i(a,\ell)$, $i = 1, 2$, $S_i(b,\ell)$, $i = 1, 2$, $S_i(c,\ell)$, $i = 1, 2$, are given by (1.2b.1), (1.2b.2), (1.2b.35), (1.2b.36), (1.2b.72), and (1.2b.73), respectively. On evaluating the integral, we obtain

$$F(\ell) = d\phi_1(\ell) \quad \text{for} \quad \ell \epsilon [0, p_1] ;$$

where $d = \dfrac{1}{\pi(a+b+c)}$

and

$\phi_1(\ell)$ is given by Part (B) of (1.2.1).

Combining all the sets in the parameter space that contribute to $F(\ell)$ for $\ell \epsilon [p_1, p_2]$, we have

$$F(\ell) = \frac{1}{\pi(a+b+c)} \int_{D(\ell)} dx d\theta = \frac{1}{\pi(a+b+c)} \int_{\substack{(\bigcup_{i=3}^{7} S_i(a,\ell)) \cup S_1(b,\ell) \bigcup_{i=3}^{5} S_i(b,\ell) \\ \bigcup_{i=2}^{5} S_i(c,\ell)}} dx d\theta$$

where $S_i(a,\ell)$, $i = 3$ to 7, $S_1(b,\ell)$, $S_i(b,\ell)$, $i = 3$ to 5, $S_i(c,\ell)$, $i = 2$ to 5 are given by (1.2b.6) to (1.2b.9), (1.2b.35), (1.2b.40) to (1.2b.42), (1.2b.73), (1.2b.76) to (1.2b.78), respectively. On evaluating the integrals, we obtain

$$F(\ell) = d\phi_2(\ell) \quad \text{for } \ell \epsilon [p_1, p_2] \quad ,$$

where

$\phi_2(\ell)$ is given by Part (B) of (1.2.1).

This proves Part (B_2) of (1.2.1).

Combining all the sets in the parameter space that contribute to $F(\ell)$ for $\ell \epsilon [p_2, p_3]$, we have

$$F(\ell) = \frac{1}{\pi(a+b+c)} \int_{D(\ell)} dx d\theta$$

$$= \frac{1}{\pi(a+b+c)} \int_{\left(\bigcup_{i=4}^{10} S_i(a,\ell)\right) \bigcup_{i=4}^{10} S_i(b,\ell) \bigcup_{i=3}^{8} S_i(c,\ell)} dx d\theta \quad ,$$

where $S_i(a,\ell)$, $i = 4$ to 10, $S_i(b,\ell)$, $i = 4$ to 10, $S_i(c,\ell)$, $i = 3$ to 8, are given by (1.2b.6) to (1.2b.9), (1.2b.12) to (1.2b.14), (1.2b.41), (1.2b.42), (1.2b.45) to (1.2b.49), (1.2b.76) to (1.2b.78), (1.2b.83) to (1.2b.85), respectively. On evaluating the integral, we obtain

$$F(\ell) = d\phi_3(\ell) \quad \text{for } \ell \epsilon [p_2, p_3] \quad ,$$

where

$\phi_3(\ell)$ is given by Part (B) of (1.2.1).

This proves Part (B_3) of (1.2.1).

Combining all the sets in the parameter space that contribute to $F(\ell)$ for $\ell\epsilon[p_3, c]$, we have

$$F(\ell) = \frac{1}{\pi(a+b+c)} \int_{D(\ell)} dx d\theta$$

$$= \frac{1}{\pi(a+b+c)} \int_{\bigcup_{i=4}^{6} S_i(a,\ell) \bigcup_{i=8}^{13} S_i(a,\ell) \cup S_4(b,\ell) \cup S_5(b,\ell) \bigcup_{i=7}^{13} S_i(b,\ell) \cup S_3(c,\ell) \cup S_4(c,\ell) \bigcup_{i=9}^{13} S_i(c,\ell)} dx d\theta$$

where $S_i(a,\ell)$, i = 4 to 6, 8 to 13, $S_i(b,\ell)$, i = 4, 5, 7 to 13, $S_3(c,\ell)$, $S_4(c,\ell)$, $S_i(c,\ell)$, i = 7 to 13, are given by (1.2b.7) to (1.2b.9), (1.2b.12) to (1.2b.14), (1.2b.19) to (1.2b.21), (1.2b.41), (1.2b.42), (1.2b.46) to (1.2b.49), (1.2b.53) to (1.2b.55), (1.2b.76), (1.2b.77), (1.2b.84), (1.2b.85), (1.2b.89) to (1.2b.93), respectively. On evaluating the integral, we obtain after simplification

$$F(\ell) = d\phi_4(\ell) \quad \text{for } \ell\epsilon[p_3, c] \quad ,$$

where

$\phi_4(\ell)$ is given by Part (B) of (1.2.1).

This proves Part (B_4) of (1.2.1).

Combining all the sets that contribute to $F(\ell)$ for $\ell\epsilon[c, b]$, we have

$$F(\ell) = \frac{1}{\pi(a+b+c)} \int_{D(\ell)} dxd\theta ,$$

$$= \frac{1}{\pi(a+b+c)} \int_{\bigcup_{i=12}^{17} S_i(a,\ell) \bigcup_{i=14}^{19} S_i(b,\ell) \bigcup_{i=14}^{20} S_i(c,\ell)} dxd\theta$$

where $S_i(a,\ell)$, $i = 12$ to 17, $S_i(b,\ell)$, $i = 14$ to 19, $S_i(c,\ell)$, $i = 14$ to 20, are given by (1.2b.20), (1.2b.21), (1.2b.24) to (1.2b.27), (1.2b.59) to (1.2b.64), (1.2b.96) to (1.2b.102), respectively. On evaluating the integral, we obtain

$$F(\ell) = \phi_5(\ell) \quad \text{for } \ell\epsilon[c, b] ,$$

where

$\phi_5(\ell)$ is given on page 116.

This proves Part (B_5) of (1.2.1).

Combining all the sets in the parameter space that contribute to $F(\ell)$, we have for $\ell\epsilon[b, a]$,

$$F(\ell) = \frac{1}{\pi(a+b+c)} \int_{D(\ell)} dxd\theta$$

$$= \frac{1}{\pi(a+b+c)} \int_{\bigcup_{i=14,18,19} S_i(a,\ell) \bigcup_{i=20}^{23} S_i(b,\ell) \bigcup_{i=21}^{24} S_i(c,\ell)} dxd\theta ,$$

where $S_i(a,\ell)$, $i = 14, 18, 19$, $S_i(b,\ell)$, $i = 20$ to 23, $S_i(c,\ell)$, $i = 20$ to 24, are given by (1.2b.31) to (1.2b.33), (1.2b.67) to (1.2b.71), (1.2b.105) to (1.2b.108), respectively. On evaluating the integral, we obtain

$$F(\ell) = d\phi_6(\ell) \quad \text{for } \ell \in [b, a] \quad ,$$

where

$\phi_6(\ell)$ is given on page 117.

This proves Part (B_6) of (1.2.1).

For the case (C): Triangle III: $a > b > p_3 > c > p_2 > p_1$.

Combining the corresponding sets (given in Section 1.2.5) in the parameter space contributing to the distribution function $F(\ell)$ for $\ell\epsilon[0, p_1]$, $\ell\epsilon[p_1, p_2]$, $\ell\epsilon[p_3, c]$, $\ell\epsilon[c, b]$, and $\ell\epsilon[b, a]$, and evaluating the integrals, we obtain:

$$
(1.2.1)\quad F(\ell) = \begin{cases}
d\xi_1(\ell) \text{ for } \ell\epsilon[0, p_1] , & (C_1) \\
d\xi_2(\ell) \text{ for } \ell\epsilon[p_1, p_2] , & (C_2) \\
d\xi_3(\ell) \text{ for } \ell\epsilon[p_2, c] , & (C_3) \\
d\xi_4(\ell) \text{ for } \ell\epsilon[c, p_3] , & (C_4) \\
d\xi_5(\ell) \text{ for } \ell\epsilon[p_3, b] , & (C_5) \\
d\xi_6(\ell) \text{ for } \ell\epsilon[b, a] , & (C_6)
\end{cases}
$$

where the ξ_i ($i = 1, \ldots, 6$) are given on pages 118-120.

1.2.6. DISTRIBUTIONS OF LENGTHS OF S_1-RANDOM SECANTS OF A REGULAR TRIANGLE.

Introduction.

In this section, the exact probability distribution of the length of the S_1-random secant of a regular triangle is obtained. A secant here is specified by a perimeter point and a direction. To find this distribution we shall use the geometric argument used by Horowitz to derive the density function of L in the case of a rectangle.

Statement of the problem: Let ABC be a triangle with each side of length a. For any point P on the perimeter of this triangle ABC, let X be the distance of P from A measured in the direction from $A \to B \to C \to A$. Let θ be the angle, which is measured in the anti-clockwise direction, that a secant makes with the side of the triangle on which P lies.

We assume that

(1) X is uniformly distributed on $[0, 3a]$,

(2) θ is uniformly distributed on $[0, \pi]$, and

(3) X and θ are independent.

It follows that the joint density of X and θ is given by

(1.2c.1)
$$p(x,y) \begin{cases} = \frac{1}{3a\pi}, \text{ if } 0 \leq x \leq 3a,\ 0 \leq \theta \leq \pi, \\ = 0, \text{ elsewhere in the } (x,\theta) \text{ plane.} \end{cases}$$

The set

$$S = \{(x,\theta) : 0 \leq x \leq 3a,\ 0 \leq \theta < \pi\}$$

is the parameter space.

The Distribution of L.

We provide the distribution of L in the following theorem.

Theorem 3. The distribution function $F(\ell)$ of the S_1-random secant length L of a regular triangle of side a is given by

(1.2c.2)
$$F(\ell) = \begin{cases} \frac{2\sqrt{3}}{\pi a}\ell, & \text{for } 0 \leq \ell \leq \frac{\sqrt{3}a}{2}; \\ \frac{2\sqrt{3}\ell}{\pi a}\left[1 - \frac{(4\ell^2-3a^2)^{\frac{1}{2}}}{\ell}\right] + \frac{6}{\pi}\cos^{-1}\left(\frac{\sqrt{3}a}{2\ell}\right) & \text{for } \frac{\sqrt{3}a}{2} \leq \ell \leq a. \end{cases}$$

Proof. We observe certain symmetries. For every x such that $0 \leq x \leq a$, we have $L(x,\theta) = L(a+x,\theta) = L(2a+x,\theta)$ for every θ such that $0 \leq \theta \leq \pi$. Since X is uniformly distributed over $[0,3a]$, elementary considerations show that we can compute $F(\ell)$ by restricting X to be uniformly distributed over [0,a]. Now since for each x, where $0 \leq x \leq a$, $L(x,\theta) = L(a-x,\pi-\theta)$ for $0 \leq \theta \leq \frac{\pi}{2}$, similar considerations show that θ can be restricted to be uniformly distributed over $[0, \frac{\pi}{2}]$. Hence we can consider the parameter space to be the set

$$S = \{(x,\theta): 0 \leq x \leq a,\ 0 \leq \theta \leq \frac{\pi}{2}\}. \tag{1.2c.3}$$

The density of X and θ is given by

$$f(x,\theta) \begin{cases} = \frac{2}{\pi a}, & 0 \leq x \leq a,\ 0 \leq \theta \leq \frac{\pi}{2}, \\ = 0 & \text{elsewhere}. \end{cases}$$

We now explicitly determine the mapping $L(x,\theta) = L$ for all possible values $(x,\theta) \in S$.

Let the random secant PQ intersect AB of the triangle ABC at P and one of the other sides at Q. Let $|PQ| = L(x,\theta)$. Let D be the midpoint of AB. We then have the following transformations from (x,θ)-space to $\mathcal{L}$, the sample space of secant lengths:

(i) P is on the segment DB, $\theta \in [0,\frac{\pi}{2}]$.

Referring to Fig. 49a we obtain, by the sine law

$$L(x,\theta) = \frac{\sqrt{3}}{2} \frac{a-x}{\sin(\frac{\pi}{3}+\theta)} \quad \text{for } (x,\theta):\ 0 \leq x \leq \frac{a}{2}. \tag{1.2c.4}$$

(ii) P is on AD, Q is on BC.

From Figure 49b, using trigonometry we obtain

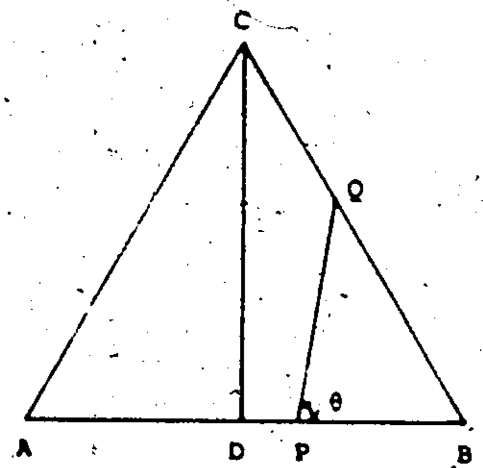

Fig. 49A

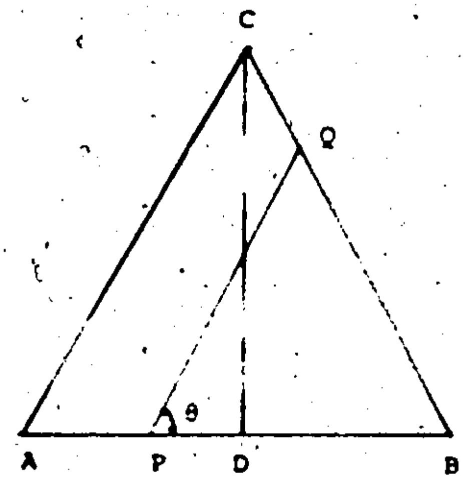

Fig. 49B

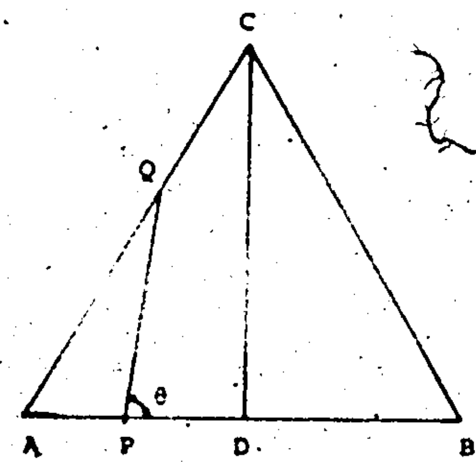

Fig. 49C

(1.2c.5) $L(x,\theta) = \frac{\sqrt{3}}{2} \frac{a - x}{\sin(\frac{\pi}{3} + \theta)}$, for $0 \le x \le \frac{a}{2}$, $0 \le \theta \le \tan^{-1} \frac{\sqrt{3}a}{a-2x}$.

(iii) P is on AD, Q is on AC.

From Figure 49c, using trigonometry, we have

(1.2c.6) $L(x,\theta) = \frac{\sqrt{3}x}{2 \sin(\theta - \frac{\pi}{3})}$, for $\tan^{-1} \frac{\sqrt{3}a}{a-2x} \le \frac{\pi}{2}$, $0 \le x \le \frac{a}{2}$.

In order to obtain $F(\ell)$ we need to observe the set of points (x,θ) in the parameter space S that are mapped onto ℓ for different increasing values of ℓ satisfying $0 \le \ell \le a$) under the transformations defined by (1.2c.4) ... (1.2c.6) from (x,θ) space to , the sample space of secant lengths. We now obtain $F(\ell)$ for $\ell \in [0, \frac{\sqrt{3}a}{2}]$ in part (i) and for $\ell \in [\frac{\sqrt{3}a}{2}, a]$ in part (ii), below.

<u>Part (i)</u>: $0 \le \ell \le \frac{\sqrt{3}a}{2}$.

We observe that $\ell = 0$ corresponds to $\{(x,\theta): x = a, 0 \le \theta \le \frac{\pi}{2}\}$. From the equation $\ell = \frac{\sqrt{3}}{2} \frac{(a - x)}{\sin(\frac{\pi}{3} + \theta)}$ or $x = a - \frac{2\ell}{3} \sin(\frac{\pi}{3} + \theta)$ for different values of ℓ, we note that as ℓ increases from $\ell = 0$ to $\ell = \frac{\sqrt{3}a}{2}$, the corresponding curves preserve the same shape (by linear transformation the equation reduces to $y = c \sin \beta$, with c changing) and the Lebesgue measure of the set

$$\{(x,\theta): a - \frac{2\ell}{3}\sin\left(\frac{\pi}{3} + \theta\right) \le x \le a,\ 0 \le \theta \le \frac{\pi}{2}\}$$

increases with ℓ. This feature of the system of curves for increasing ℓ provides the basis for the solution of the problem. When $0 \le \ell \le \frac{\sqrt{3a}}{2}$ we thus have an area A_1 of the set S_1 bounded by $x = a$, $x = a - \frac{2\ell}{3}\sin\left(\theta + \frac{\pi}{3}\right)$, $\theta = 0$, $\theta = \frac{\pi}{2}$, contributing to $Pr(L \le \ell)$, which increases as ℓ increases.

It may be further observed that in the equation

$$L(x,\theta) = \ell = \frac{\sqrt{3}}{2}\,\frac{x}{\sin\left(\theta - \frac{\pi}{3}\right)}\ ;\ 0 \le x ,a/2,\ \tan^{-1}\frac{3a}{a-2x} \le \theta \le \frac{\pi}{2}\ ,$$

to $x = 0$ corresponds $\ell = 0$. As ℓ increases continuously from $\ell = 0$ to $\ell = \frac{\sqrt{3a}}{2}$, the shape of the corresponding curve does not change and the area A_2 of the set S_2 bounded by $x = 0$, $\theta = \frac{\pi}{2}$ and $x = \frac{2\ell}{\sqrt{3}}\sin\left(\theta - \frac{\pi}{3}\right)$ and $x = \frac{a - \sqrt{3a}\cot\theta}{2}$ increases continuously. When $0 \le \ell \le \frac{\sqrt{3a}}{2}$ we thus have an area A_2 contributing to $Pr(L \le \ell)$. Thus we have $\{(x,\theta): L(x,\theta) \le \ell\} = S_1 \quad S_2$. Now the probability that the random secant length $L(x,\theta)$ is less than a given quantity ℓ is the ratio of the Lebesgue measure of the set $\{(x,\theta): L(x,\theta) \le \ell\}$ to the Lebesgue measure of the parameter space S which is $\frac{a\pi}{2}$.

Now the area A_1 of the set $S_1 = \int_0^{\frac{\pi}{2}} \frac{2\ell}{\sqrt{3}}\sin\left(\frac{\pi}{3} + \theta\right) d\theta = \frac{\ell}{\sqrt{3}}\left(1 + \sqrt{3}\right)$,

and the

area A_2 of the set $S_2 = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{2\ell}{\sqrt{3}} \sin(\theta - \frac{\pi}{3})\, d\theta = \ell(\frac{2}{\sqrt{3}} - 1)$.

Hence

$$F(\ell) = \frac{2\ell\sqrt{3}}{\pi a} \qquad \text{for } 0 \leq \ell \leq \frac{a\sqrt{3}}{2} .$$

Part (ii): $\frac{a\sqrt{3}}{2} \le \ell \le a.$

Now we consider the case $\frac{a\sqrt{3}}{2} \le \ell \le a$ for the probability distribution of L. Geometrical considerations reveal that this case arises only when

$$L(x,\theta) = \begin{cases} \frac{\sqrt{3}}{2} \dfrac{(a-x)}{\sin(\frac{\pi}{3}+\theta)} & 0 \le \theta \le \tan^{-1}\frac{a\sqrt{3}}{a-2x},\ 0 \le x \le \frac{a}{2}, \\ \frac{\sqrt{3}}{2} \dfrac{x}{\sin(\theta-\frac{\pi}{3})} & \tan^{-1}\frac{a\sqrt{3}}{a-2x} \le \theta \le \frac{\pi}{2},\ 0 \le x \le \frac{a}{2}. \end{cases}$$

If $\frac{a\sqrt{3}}{2} \le \ell \le a$, there is an area A_3 bounded by

$$\ell = \frac{\sqrt{3}}{2} \frac{(a-x)}{\sin(\frac{\pi}{3}+\theta)},\ x = 0,\ \theta = 0,\ \theta = \frac{\pi}{3}$$

and

$$x = a - a \sin(\frac{\pi}{3} + \theta)$$

contributing to $Pr(L \le \ell)$ which increases as ℓ increases. Area A_3 may be decomposed into two equal areas A_{31} and A_{32}. So $A_3 = 2A_{31}$. The solution θ_0 of $\ell = \frac{\sqrt{3}}{2} \frac{(a-x)}{\sin(\frac{\pi}{3}+\theta)}$ and $x = 0$ for θ is given by

$$\theta_0 = \sin^{-1}\frac{a\sqrt{3}}{2\ell} - \frac{\pi}{3}.$$

Now,

$$A_{31} = \int_0^{\sin^{-1}\frac{a\sqrt{3}}{2\ell} - \frac{\pi}{3}} \left(\frac{2\ell}{\sqrt{3}} - a\right)\left(\sin\left(\frac{\pi}{3} + \theta\right)\right) d\theta + \int_{\sin^{-1}\frac{a\sqrt{3}}{2} - \frac{\pi}{3}}^{\frac{\pi}{2}} \left(a - a\sin\left(\frac{\pi}{3}+\theta\right)\right) d\theta$$

$$= a\left(\frac{\pi}{2} - \sin^{-1}\frac{a\sqrt{3}}{2\ell}\right) - \frac{\ell}{\sqrt{3}}\left(4\ell^2 - 3a^2\right)^{\frac{1}{2}} \frac{1}{2}\left(a - \frac{2\ell}{\sqrt{3}}\right).$$

Hence

$$A_3 = 2a\left(\frac{\pi}{2} - \sin^{-1}\frac{a\sqrt{3}}{2\ell}\right) - \frac{2}{\sqrt{3}}\left(4\ell^2 - 3a^2\right)^{\frac{1}{2}} - \left(a + \frac{2\ell}{\sqrt{3}}\right).$$

When $\frac{a\sqrt{3}}{2} \leq \ell \leq a$, there is an area A_4 bounded by $a - x = \frac{2\ell}{\sqrt{3}}\sin\left(\theta + \frac{\pi}{3}\right)$, $a - x = a\sin\left(\theta + \frac{\pi}{3}\right)$, $\theta = \frac{\pi}{3}$, $\theta = \tan^{-1}\frac{a\sqrt{3}}{a-2x}$. The solution of $a - x = \frac{2\ell}{\sqrt{3}}\sin\left(\theta + \frac{\pi}{3}\right)$ and $\theta = \tan^{-1}\frac{a\sqrt{3}}{a-2x}$ for θ is $\sin^{-1}\frac{a\sqrt{3}}{2\ell}$. There is further an area A_5 bounded by $a - x = a - a\sin\left(\theta - \frac{\pi}{3}\right)$, $a - x = a\sin\left(\frac{\pi}{3} + \theta\right)$, $\theta = \sin^{-1}\frac{a\sqrt{3}}{2\ell}$ and $\theta = \frac{\pi}{2}$, and an area A_6 bounded by $a - x = a - a\sin\left(\theta - \frac{\pi}{3}\right)$, $a - x = a - \frac{2\ell}{\sqrt{3}}\sin\left(\theta - \frac{\pi}{3}\right)$, $\theta = \frac{\pi}{3}$ and $\theta = \sin^{-1}\frac{a\sqrt{3}}{2\ell}$, which contributes to $Pr(L \leq \ell)$.

As ℓ increases the total area $B = A_4 + A_5 + A_6$ increases.

Area B = $A_4 + A_5 + A_6$

$$= a\left(\frac{\pi}{2} - \sin^{-1}\frac{a\sqrt{3}}{2\ell}\right) - \frac{\ell}{\sqrt{3}}(4\ell^2 - 3a^2)^{\frac{1}{2}} - \frac{1}{2}\left(a - \frac{2\ell}{\sqrt{3}}\right).$$

Also to $\Pr(L \leq \ell)$, for $\frac{a\sqrt{3}}{2} \leq \ell \leq a$, there is a contribution $\frac{3}{\pi} = \Pr\left(L \leq \frac{a\sqrt{3}}{2}\right)$.

Hence

$$F(\ell) = \frac{4\ell}{a\pi\sqrt{3}}\left[\frac{3}{2} - \frac{3}{2}\,\frac{\sqrt{4\ell^2 - 3a^2}}{\ell}\right] + \frac{6}{\pi}\cos^{-1}\frac{a\sqrt{3}}{2\ell}, \text{ for } \frac{a\sqrt{3}}{2} \leq \ell \leq a.$$

This proves Theorem 3.

SECTION THREE

1.3. DISTRIBUTION OF LENGTHS OF S_1-RANDOM SECANTS OF A RECTANGLE.

1.3.0. Introduction.

In this section the probability distribution of the length of a random secant of a rectangle under S_1-randomness is obtained. The case of the rectangle was considered first by Horowitz [27] and later by Coleman [8]. We present the distribution function of the random secant length here derived by a direct geometrical argument which differs from the method used by Horowitz and Coleman.

Let ABCD be a rectangle with sides a and b, where $|AB| = a$, $|BC| = b$ and $a > b$ (cf. Fig. 50A). A point P is chosen at random on the perimeter of the rectangle and a straight line passing through P in a random direction intersects one of the remaining sides of the rectangle at a point Q. We are interested in the probability distribution of $L = |PQ|$.

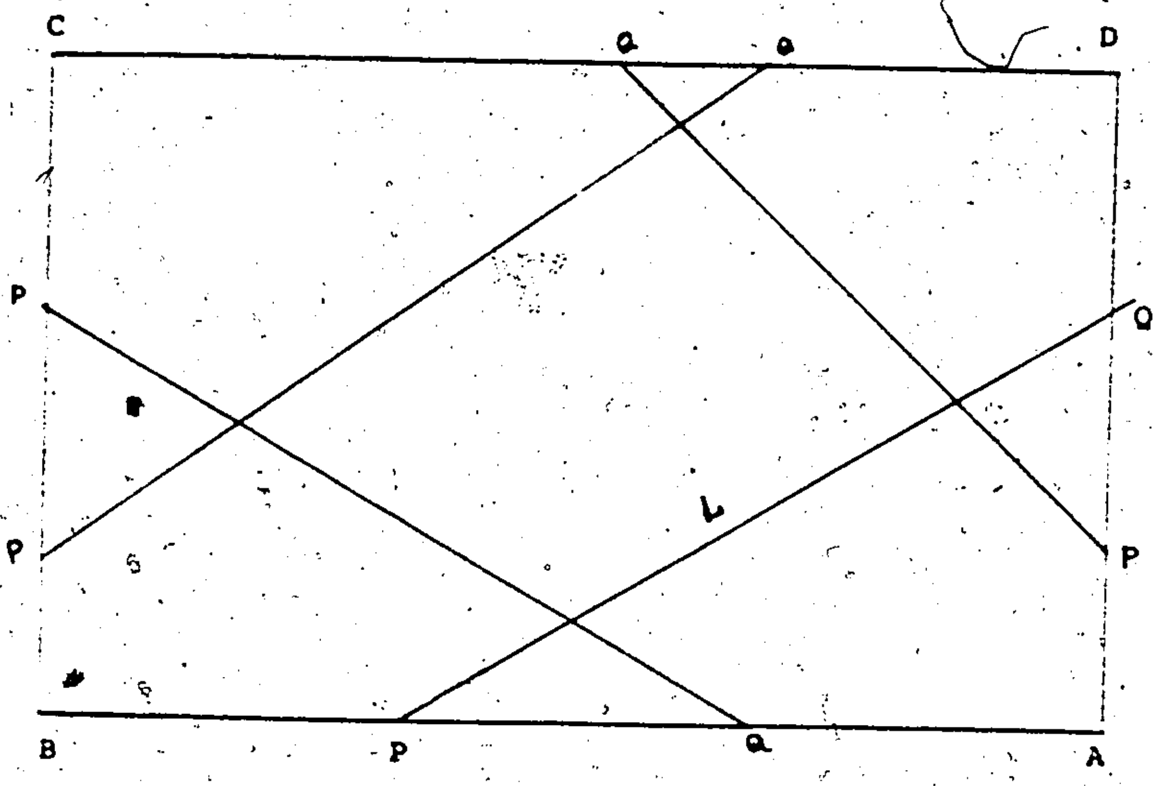

Fig. 50A

1.3.1. Distribution of L.

The following theorem provides the distribution of L.

Theorem 4. The probability distribution function $F(\ell)$ of L under S_1-randomness is given by

$$(1.3.1)\quad F(\ell)\begin{cases} = \dfrac{4\ell}{\pi(a+b)} , & \text{for } 0 \le \ell \le b , \quad (A) \\ = \dfrac{2}{\pi(a+b)}\,\psi(\ell) , & \text{for } b \le \ell \le a , \quad (B) \\ = \dfrac{2}{\pi(a+b)}\,\phi(\ell) , & \text{for } a \le \ell \le (a^2+b^2)^{1/2} , \quad (C) \end{cases}$$

where

$$\psi(\ell) = b + a\cos^{-1}(\tfrac{b}{\ell}) + b\cos^{-1}(\tfrac{b}{\ell}) + \ell - \sqrt{\ell^2-b^2} ,$$

$$\phi(\ell) = a + b + (a+b)(\cos^{-1}\tfrac{a}{\ell} + \cos^{-1}\tfrac{b}{\ell})$$

$$- \sqrt{\ell^2-a^2} - \sqrt{\ell^2-b^2} .$$

To prove this result we proceed as follows.

1.3.2. Parametrization of the Secants.

Let X measure along the perimeter of the rectangle the random distance of the random point P from the reference point A and let θ measure, following an anti-clockwise direction (cf. Fig. 50B), the random angle that PQ makes with the side of the rectangle on which P lies.

We assume that:

(i) X is uniformly distributed on $[0, 2(a+b)]$,

(ii) θ is uniformly distributed on $[0, \pi]$, and

(iii) the distributions of X and θ are independent.

The joint density of X and θ is given by

$$(1.3.2)\quad P(x,\theta) = \begin{cases} \dfrac{1}{2(a+b)\pi} & \text{for } \{(x,\theta):\ x\varepsilon[0,2(a+b)],\ \theta\varepsilon[0,\pi]\}, \\ 0 & \text{elsewhere in the } (x-\theta) \text{ plane.} \end{cases}$$

L is a function of X and θ and the random variables X and θ induce a distribution for L.

For a given $\ell\varepsilon[0, (a^2+b^2)^{1/2}]$, we are interested in

$$(1.3.3)\quad F(\ell) = \Pr(L \le \ell) = \frac{1}{2(a+b)\pi}\int_{D(\ell)} dx\,d\theta ,$$

where

$$(1.3.4)\quad D(\ell) = \{(x,\theta):\ L(x,\theta) \le \ell,\ 0 \le x \le 2(a+b),\ 0 \le \theta \le \pi\} .$$

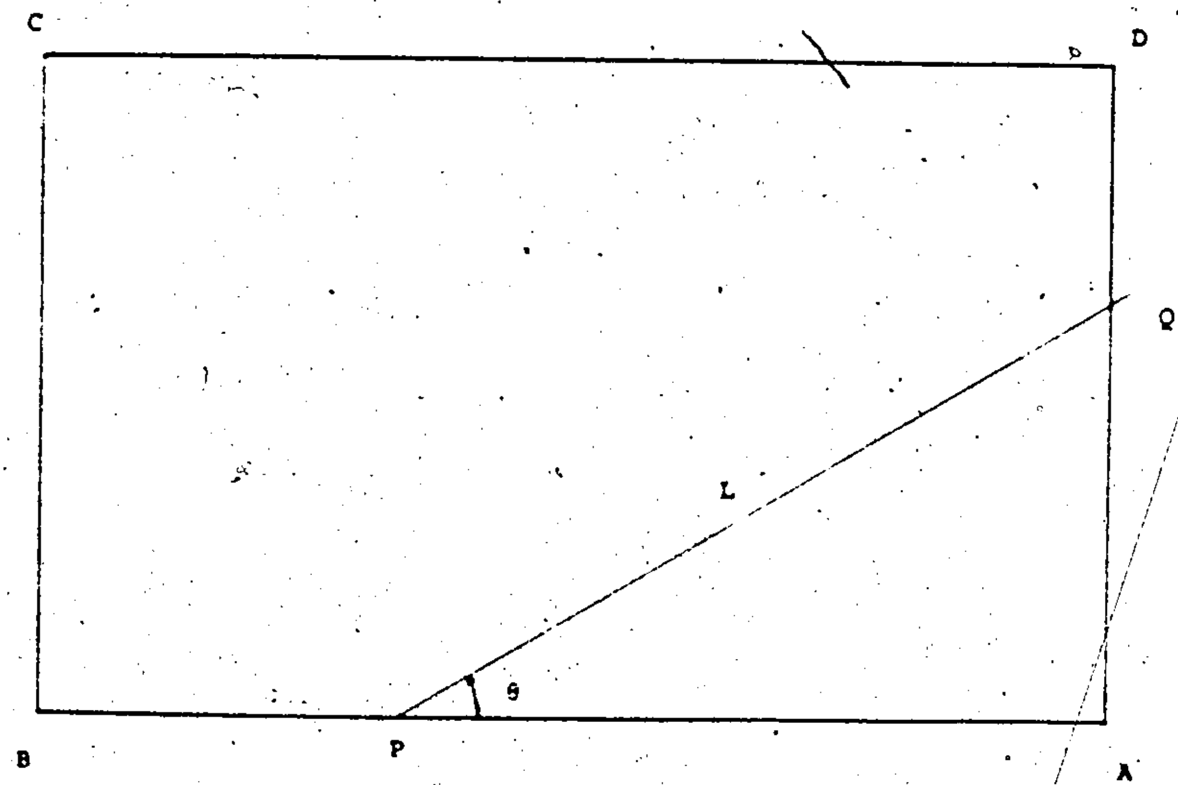

Fig. 50B

The <u>parameter space</u> is the set

$$S' = \{(x,\theta): \ 0 \le x \le 2(a+b), \ 0 \le \theta \le \pi\} .$$

1.3.3. Reduction of the Parameter Space.

Using symmetry we reduce the parameter space S' as follows: we decompose $D(\ell)$ into the following four sets which are disjoint except possibly for boundary points:

(1.3.5) $D_1(\ell) = \{(x,\theta): L(x,\theta) \le \ell,\ 0 \le x \le a + b,\ 0 \le \theta \le \frac{\pi}{2}\}$,

(1.3.6) $D_2(\ell) = \{(x,\theta): L(x,\theta) \le \ell,\ 0 \le x \le a + b,\ \frac{\pi}{2} \le \theta \le \pi\}$,

(1.3.7) $D_3(\ell) = \{(x,\theta): L(x,\theta) \le \ell,\ a + b \le x \le 2(a + b),$

$0 \le \theta \le \frac{\pi}{2}\}$,

and

(1.3.8) $D_4(\ell) = \{(x,\theta): L(x,\theta) \le \ell,\ a + b \le x \le 2(a + b),$

$\frac{\pi}{2} \le \theta \le \pi\}$.

It may easily be verified that

(1.3.9) $\int_{D_1(\ell)} dx\,d\theta = \int_{D_i(\ell)} dx\,d\theta \quad , \quad i = 2, 3, 4$.

Hence we have

(1.3.10)

$$F(\ell) = \frac{1}{2(a+b)\pi} \int_{D(\ell)} dx\,d\theta$$

$$= \frac{1}{2(a+b)\pi} \int_{\bigcup_{i=1}^{4} D_i(\ell)} dx\,d\theta$$

$$= \frac{2}{(a+b)\pi} \int_{D_1(\ell)} dx\,d\theta \quad .$$

The set

$$S = \{(x,\theta): \quad 0 \le x \le a+b, \ 0 \le \theta \le \frac{\pi}{2}\}$$

is the <u>reduced parameter space</u>.

The appropriate decomposition of the interval $[0, (a^2+b^2)^{1/2}]$, the range of L consists of the subintervals:

$$[0,b] \quad , \quad [b,a] \quad \text{and} \quad [a, (a^2+b^2)^{1/2}] \quad .$$

1.3.4. Proof of Theorem 4.

We now obtain the set $D_1(\ell)$ and the distribution function $F(\ell)$ for $\ell \in [0,b]$ in part 1, $\ell \in [b,a]$ in part 2 and $\ell \in [a, (a^2+b^2)^{1/2}]$ in part 3, below.

Part 1: $0 \le \ell \le b$. The set $D_1(\ell)$ and the distribution function $F(\ell)$ for $\ell \in [0,b]$ are determined as follows: Fix ℓ in $[0,b]$. Fix also θ in $[0, \frac{\pi}{2}]$. If the secant of length ℓ is determined by a point $P_1(x)$ on AB (cf. Fig. 51), then the corresponding x such that $L(x,\theta) = \ell$ is $\ell \cos \theta$, and $L(x,\theta) \le \ell$ for $0 \le x \le \ell \cos \theta$.

The set

$$S_{11}(\ell) = \{(x,\theta): \ 0 \le x \le \ell \cos \theta, \ 0 \le \theta \le \frac{\pi}{2}\}$$

thus contributes to $F(\ell)$. If the secant of length ℓ is determined by a point $P_2(x)$ on BC (cf. Fig. 51) then the corresponding x such that $L(x,\theta) = \ell$ is $a + \ell \cos \theta$ and $L(x,\theta) \le \ell$ whenever $a \le x \le a + \ell \cos \theta$.

The set

$$S_{12}(\ell) = \{(x,\theta): \ a \le x \le a + \ell \cos \theta, \ 0 \le \theta \le \frac{\pi}{2}\}$$

also contributes to $F(\ell)$.

These are the only ways in which a secant of length $L(x,\theta)$, where $L(x,\theta) \le \ell$ can arise. Hence

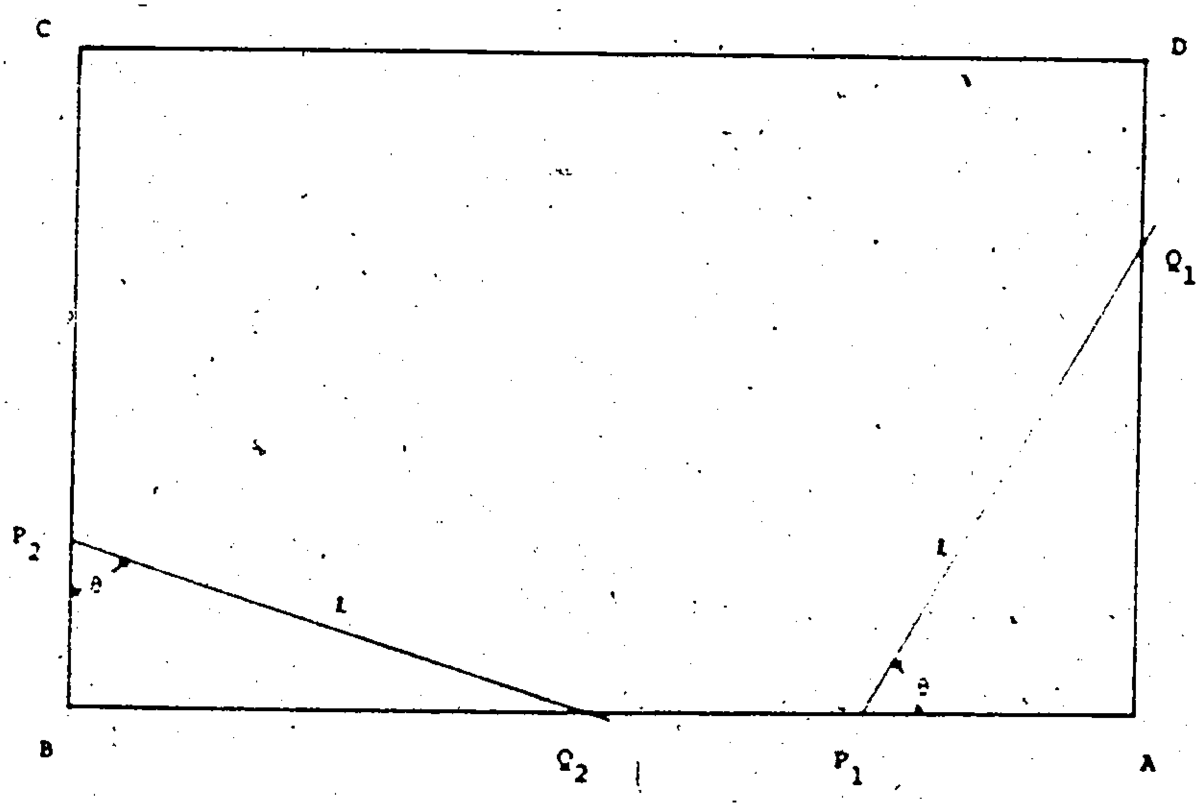

Fig. 51

$$D_1(\ell) = S_{11}(\ell) \cup S_{12}(\ell)$$

and evaluating (1.3.10), we have for $0 \leq \ell \leq b$,

$$F(\ell) = \frac{2}{(a+b)\pi} \int_{D_1(\ell)} dx d\theta$$

$$= \frac{2}{(a+b)\pi} \int_{S_{11}(\ell) \cup S_{12}(\ell)} dx d\theta = \frac{4\ell}{(a+b)\pi} .$$

This proves part (A) of (1.3.1).

Part 2: $b \leq \ell \leq a$. Let ℓ be fixed in $[b,a]$. Let PQ be the random chord of length ℓ. We have two cases:

(1) P is on AB, and

(2) P is on BC.

Case 1: Let P be on AB. Then we have further two possibilities: (i) Q is on AD and (ii) Q is on DC.

(i) Let P be on AB and Q be on AD, (cf. Fig. 52). For $L(x,\theta) = \ell$, the maximum value θ_1 of θ corresponds to Q coinciding with D in which case we have $\frac{b}{\ell} = \sin \theta_1$, giving $\theta_1 = \sin^{-1} \frac{b}{\ell}$. Thus for a fixed $\ell \in [b,a]$, θ varies from 0 to $\sin^{-1} \frac{b}{\ell}$. Now fix θ in $[0, \sin^{-1} \frac{b}{\ell}]$. Then the corresponding value of x with $L(x,\theta) = \ell$

is $\ell \cos\theta$ and $L(x,\theta) \le \ell$ for $0 \le x \le \ell \cos\theta$. Thus the set

(1.3.11)

$$S_{21}(\ell) = \{(x,\theta): \ 0 \le x \le \ell \cos\theta, \ 0 \le \theta \le \sin^{-1} \frac{b}{\ell}\}$$

contributes to $\Pr(L \le \ell)$.

(ii) Let P be on AB and Q be on DC (cf. Fig. 52). When Q coincides with D, the angle θ' that PQ makes with BA is given by $\theta' = \sin^{-1} \frac{b}{\ell}$. A secant determined by θ where $\sin^{-1} \frac{b}{\ell} \le \theta \le \frac{\pi}{2}$ has length $L(x,\theta) \le \ell$. A secant specified by (x,θ) where $\sin^{-1} \frac{b}{\ell} \le \theta \le \frac{\pi}{2}$, $0 \le x \le a$ has the corresponding length $L(x,\theta) \le \ell$. Hence we have the set

(1.3.12)

$$S_{22}(\ell) = \{(x,\theta): \ 0 \le x \le a, \ \sin^{-1} \frac{b}{\ell} \le \theta \le \frac{\pi}{2}\}$$

contributing to $\Pr(L \le \ell)$. We write

$$D_1(\ell) = \{(x,\theta): \ L(x,\theta) \le \ell,$$

$$0 \le x \le a + b, \ 0 \le \theta \le \frac{\pi}{2}\}$$

$$= D_{11}(\ell) \cup D_{12}(\ell) \quad , \quad \text{for} \quad b \le \ell \le a \quad ,$$

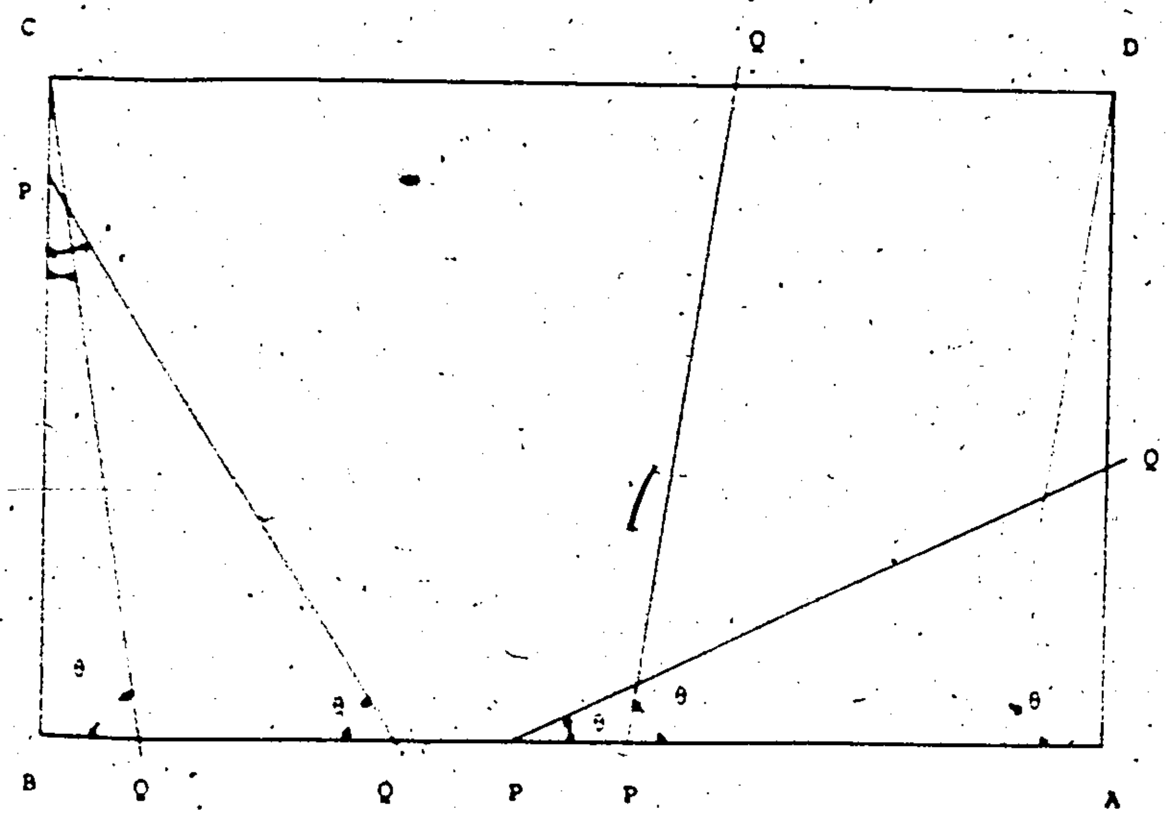

Fig. 52

where

$$D_{11}(\ell) = \{(x,\theta): \ L(x,\theta) \le \ell,$$

$$0 \le x \le a, \ 0 \le \theta \le \frac{\pi}{2}\} ,$$

and

$$D_{12}(\ell) = \{(x,\theta): \ L(x,\theta) \le \ell,$$

$$a \le x \le a + b, \ 0 \le \theta \le \frac{\pi}{2}\} .$$

Combining (i) and (ii), we have

(1.3.13)

$$D_{11}(\ell) = S_{21}(\ell) \cup S_{22}(\ell)$$

$$= \{(x,\theta): \ 0 \le \theta \le \sin^{-1}\frac{b}{\ell}, \ 0 \le x \le \ell\cos\theta\}$$

$$\cup \{(x,\theta): \ 0 \le x \le a, \ \sin^{-1}\frac{b}{\ell} \le \theta \le \frac{\pi}{2}\} .$$

Therefore for P lying on AB and for ℓ satisfying $b \le \ell \le a$, we have

(1.3.14)

$$F_1(\ell) = \frac{2}{a\pi}\int_{D_{11}(\ell)} dx d\theta = \frac{2}{a\pi}\{\int_0^{\sin^{-1}\frac{b}{\ell}}\int_0^{\ell\cos\theta} dx d\theta + \int_{\sin^{-1}\frac{b}{\ell}}^{\frac{\pi}{2}}\int_0^a dx d\theta\}$$

$$= \frac{2}{a\pi} (b + a \cos^{-1} \frac{b}{\ell}) \quad ,$$

where $P_1(\ell)$ is the (conditional) probability that L is less than or equal to ℓ, given that P is on AB.

Now the probability that P is on AB given that P is on AB ∪ BC is $\frac{a}{a+b}$. Hence for $b \le \ell \le a$, the probability that P is on AB and $L \le \ell$ is

$$\frac{a}{(a+b)} \cdot \frac{2}{a\pi} \int_{D_{11}(\ell)} dx d\theta = \frac{2}{(a+b)\pi} (b + a \cos^{-1} \frac{b}{\ell})$$

by use of (1.3.14).

Case 2: Let P be on BC. As in the proceeding case we have the following decomposition of the set

(1.3.15)

$$D_{12}(\ell) = \{(x,\theta): \ L(x,\theta) \le \ell, \ a \le x \le a + b, \ 0 \le \theta \le \frac{\pi}{2}\}$$

$$= \{(x,\theta): \ 0 \le \theta \le \cos^{-1} \frac{b}{\ell}, \ a \le x \le a + b\}$$

$$\cup \{(x,\theta): \ \cos^{-1} \frac{b}{\ell} \le \theta \le \frac{\pi}{2}, \ a \le x \le a + \ell \cos \theta\} \quad .$$

Therefore for P lying on BC and for ℓ satisfying $b \le \ell \le a$, we have

(1.3.16)

$$P_2(\ell) = \frac{2}{b\pi} \int_{D_{12}(\ell)} dx d\theta = \frac{2}{b\pi} \{ \int_0^{\cos^{-1} \frac{b}{\ell}} \int_a^{a + b} dx d\theta + \int_{\cos^{-1} \frac{b}{\ell}}^{\frac{\pi}{2}} \int_a^{a + \ell \cos \theta} dx d\theta \}$$

$$= \frac{2}{b\pi} [b \cos^{-1} \frac{b}{\ell} + \ell - (\ell^2 - b^2)^{1/2}] ,$$

where $F_2(\ell)$ is the (conditional) probability that L is less than or equal to ℓ, given that P is on BC.

Now the probability that P is on BC given that P is on AB $\cup$ BC is $\frac{b}{a+b}$. Hence for $b \le \ell \le a$, the probability that P is on BC and $L \le \ell$ is

$$\frac{b}{(a+b)} \cdot \frac{2}{b\pi} \int_{D_{12}(\ell)} dx d\theta = \frac{2}{\pi(a+b)} [b \cos^{-1} \frac{b}{\ell} + \ell - (\ell^2 - b^2)^{1/2}] .$$

Combining both cases (1) and (2), we have

$$F(\ell) = \Pr(L \le \ell)$$

$$= \frac{2}{\pi(a+b)} [b + a \cos^{-1} \frac{b}{\ell} + b \cos^{-1} \frac{b}{\ell} + \ell - (\ell^2 - b^2)^{1/2}],$$

$$\text{for } b \le \ell \le a .$$

Thus we have proved part (B) of (1.3.1).

Part 3: $a \le \ell \le (a^2+b^2)^{1/2}$. Let P be on AB (cf. Fig. 53). The minimum angle that the secant PQ of length ℓ makes with BA is given by $\cos \theta_1 = \frac{a}{\ell}$ or $\theta_1 = \cos^{-1} \frac{a}{\ell}$. The maximum angle that the secant of length ℓ makes with BA is given by $\theta_2 = \sin^{-1} \frac{b}{\ell}$. Thus for the random secant PQ of length

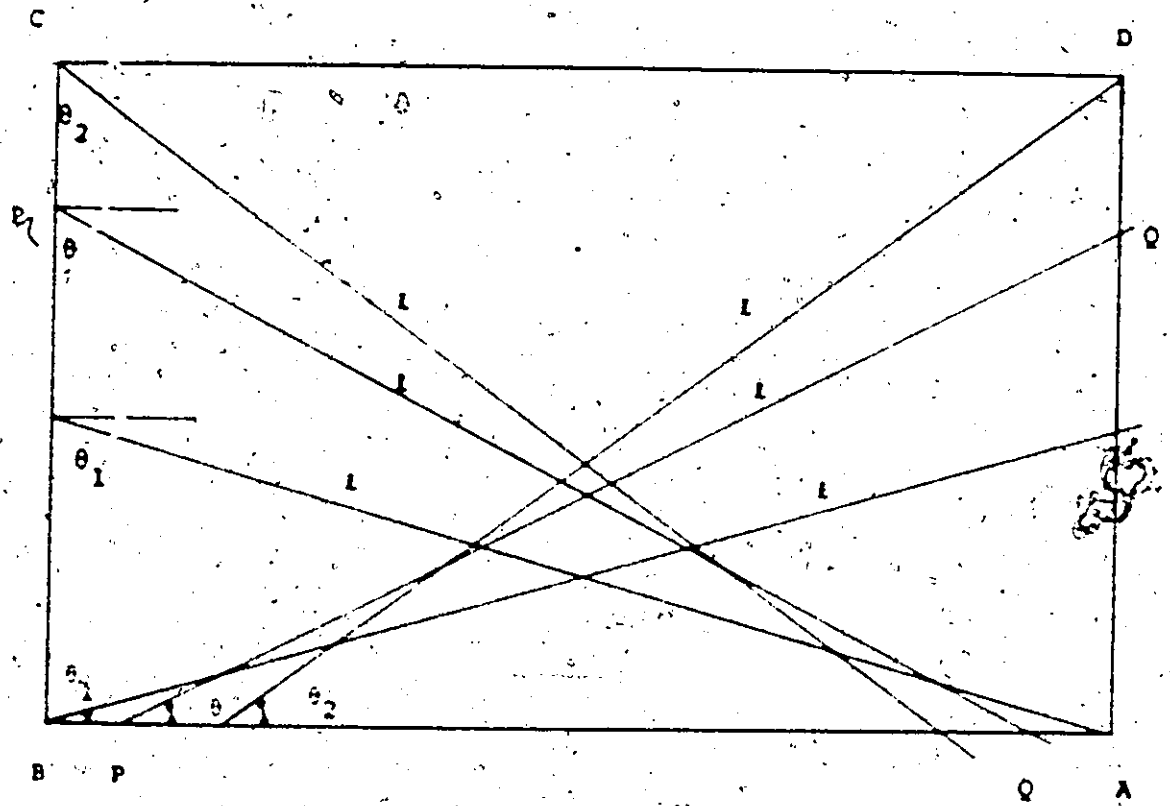

Fig. 53

$L(x,\theta) = \ell$, θ satisfies $\cos^{-1}(\frac{a}{\ell}) \le \theta \le \sin^{-1}(\frac{b}{\ell})$ and for a fixed θ in $[\cos^{-1}\frac{a}{\ell}, \sin^{-1}\frac{b}{\ell}]$, the corresponding x is $\ell \cos\theta$. Since $0 \le x \le \ell\cos\theta \Rightarrow L(x,\theta) \le \ell$, we have the set

(1.3.17)

$$S_{31}(\ell) = \{(x,\theta): \cos^{-1}\frac{a}{\ell} \le \theta \le \sin^{-1}\frac{b}{\ell}, \; 0 \le x \le \ell\cos\theta\}$$

contributing to $F(\ell)$. Now $0 \le x \le a$ and $0 \le \theta \le \cos^{-1}\frac{a}{\ell}$ implies $L(x,\theta) \le \ell$. Hence the set

(1.3.18)

$$S_{32}(\ell) = \{(x,\theta): 0 \le x \le a, \; 0 \le \theta \le \cos^{-1}\frac{a}{\ell}\}$$

contributes to $Pr(L \le \ell)$. Also $0 \le x \le a$ and $\sin^{-1}\frac{b}{\ell} \le \theta \le \frac{\pi}{2}$ implies $L(x,\theta) \le \ell$. Hence the set

(1.3.19)

$$S_{33}(\ell) = \{(x,\theta): 0 \le x \le a, \; \sin^{-1}\frac{b}{\ell} \le \theta \le \frac{\pi}{2}\}$$

also contributes to $F(\ell)$.

Let P be on BC. The minimum angle that the secant PQ of length ℓ makes with BC is given by $\theta_1 = \cos^{-1}\frac{b}{\ell}$ and the maximum angle that it makes with BC is given by $\sin^{-1}\frac{b}{\ell}$. For a fixed θ in $[\cos^{-1}\frac{b}{\ell}, \sin^{-1}\frac{a}{\ell}]$, the corresponding x of the random secant PQ of length $L(x,\theta) = \ell$ is given by $a + \ell\cos\theta$. Clearly $a \le x \le a + \ell\cos\theta$ implies $L(x,\theta) \le \ell$. Hence the set

(1.3.20)

$$S'_{31}(\ell) = \{(x,\theta): \ 0 \le x \le \ell \cos\theta, \ \cos^{-1}\frac{b}{\ell} \le \theta \le \sin^{-1}\frac{b}{\ell}\}$$

contributes to $F(\ell)$. The sets

(1.3.21)

$$S'_{32}(\ell) = \{(x,0): \ 0 \le \theta \le \cos^{-1}\frac{b}{\ell}, \ 0 \le x \le b\}$$

and

(1.3.22)

$$S'_{33}(\ell) = \{(x,\theta): \ 0 \le x \le b, \ \sin^{-1}\frac{a}{\ell} \le \theta \le \frac{\pi}{2}\}$$

also contribute to $F(\ell)$.

Hence combining (1.3.17) to (1.3.22), and noticing that

$$(x,\theta) \not\in D_1(\ell)$$

implies

$$(x,\theta) \not\in \bigcup_{i=1}^{3} S_{3i}(\ell) \ \bigcup_{i=1}^{3} S'_{3i}(\ell) \quad ,$$

we obtain

$$D_1(\ell) = \bigcup_{i=1}^{3} S_{3i}(\ell) \ \bigcup_{i=1}^{3} S'_{3i}(\ell) \quad .$$

Hence we have

$$F(\ell) = \Pr(L \le \ell) = \frac{2}{\pi(a+b)} \int_{D_1(\ell)} dx\,d\theta$$

$$= \frac{2}{\pi(a+b)} \int_{\bigcup_{i=1}^{3} S_{3i}(\ell)\ \bigcup_{i=1}^{3} S'_{3i}(\ell)} dx\,d\theta \qquad \text{(by 1.3.23)}$$

$$= \frac{2}{\pi(a+b)} [a + b + (a+b)(\cos^{-1}\frac{a}{\ell} + \cos^{-1}\frac{b}{\ell})$$

$$- (\ell^2-a^2)^{1/2} - (\ell^2-b^2)^{1/2}] ,$$

$$\text{for } a \le \ell \le (a^2+b^2)^{1/2} .$$

This proves part (C) of (1.3.1).

SECTION FOUR

1.4. DISTRIBUTIONS OF LENGTHS OF RANDOM SECANTS OF A CIRCLE.

1.4.0. Introduction.

In this section the probability distribution of the length L of a random secant of a circle is obtained. A secant is specified by a perimeter point and a direction in 1.4.1 and by an interior point and a direction in 1.4.2. The results presented in this section are known (cf. [8] [27]). They have been obtained independently and are presented here for completeness and interest.

1.4.1. Probability distribution of L under S_1-randomness or S_2-randomness.

The probability distribution of L under S_1-randomness is given in the following theorem.

Theorem 5. The probability distribution of the length L of a random secant of a circle of radius a is given by

$$(1.4.1) \qquad F(l) = Pr(L \le l) = \frac{2}{\pi} \sin^{-1} \frac{l}{2a}, \quad 0 \le l \le 2a ,$$

when the secant is specified by a perimeter point and a direction or by two perimeter points.

The proof of this result is simple.

Corollary: Since $F(\ell)$ is absolutely continuous on $[0, 2a]$, the density of L is given by

(1.4.2) $$f(\ell) = F'(\ell) = \frac{2}{\pi(4a^2-\ell^2)^{\frac{1}{2}}}, \quad 0 \leq \ell \leq 2a,$$

which is the result given by Coleman [8].

1.4.2. The probability distribution of L under I_1-randomness.

The distribution of L under I_1-randomness is given in the following theorem.

Theorem 6. The probability distribution of the length L of a random secant of a circle of radius a is given by

(1.4.3) $$F(\ell) = 1 - \frac{2}{\pi}\sin^{-1}\sqrt{1 - \frac{\ell^2}{4a^2}} - \frac{\ell}{\pi a}\sqrt{1 - \frac{\ell^2}{4a^2}}, \text{for } 0 \leq \ell \leq 2a ,$$

when the secant is specified by an interior point and a direction.

Proof. A point P is chosen at random in the interior of a circle of radius a and centre O and then a straight line is passing through P and making a random angle θ $(-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2})$ with the radius vector OP (Fig. 54). The random line through the point P is intercepted by the circle at points A and B giving a length L. The problem is to determine the probability distribution of $L = |AB|$.

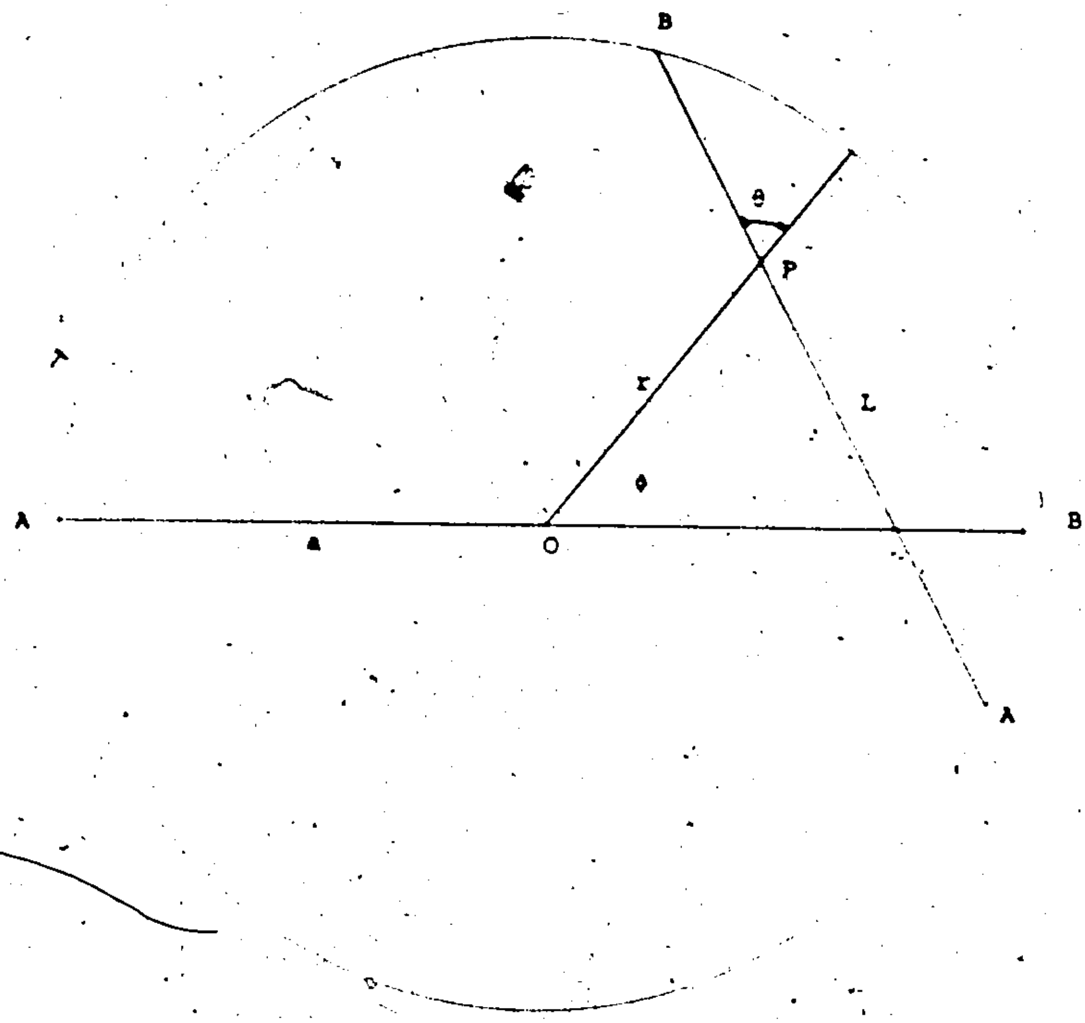

Fig. 54

Let the interior point P have plane polar coordinates r, ϕ with reference to the centre as origin and a fixed axis through the centre. An element of area containing P is given by $r dr d\phi$. The area of the circle is πa^2 and θ is distributed uniformly on $[-\frac{\pi}{2}, \frac{\pi}{2}]$. The probability that P is in the element of area $r d\phi dr$ and the secant through P falls in $[\theta, \theta + d\theta]$ is $\frac{r d\phi dr d\theta}{\pi^2 a^2}$.

Therefore the probability density is given by

$$f(r,\theta,\phi) = \begin{cases} \frac{r}{\pi^2 a^2}, & 0 \le r \le a, -\pi \le \phi \le \pi, \ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}, \\ 0, & \text{elsewhere}. \end{cases}$$

We first find the secant length $L(r,\theta,\phi)$ in terms of r, θ and ϕ. Let OD be drawn perpendicular to the secant AB (cf. Figure 55). We have

$$OD = r \sin\theta ,$$

or

$$a^2 - \left(\frac{L(r,\theta,\phi)}{2}\right)^2 = r^2 \sin^2\theta .$$

Therefore,

$$L(r,\theta,\phi) = 2(a^2 - r^2 \sin^2\theta)^{\frac{1}{2}}.$$

We note that $L(r,\theta,\phi)$ is independent of ϕ and depends on $\sin^2\theta$. We can, therefore, limit our consideration to $\phi = 0$, $0 \le \theta \le \frac{\pi}{2}$. Then

$$f(r,\theta) = \begin{cases} \frac{4r}{\pi a^2}, & 0 \le r < a, \ 0 \le \theta \le \frac{\pi}{2}, \\ 0, & \text{elsewhere}. \end{cases}$$

We are interested in

$$\Pr(L \le \ell) = \int_D \frac{4r}{\pi a^2} dr d\theta ,$$

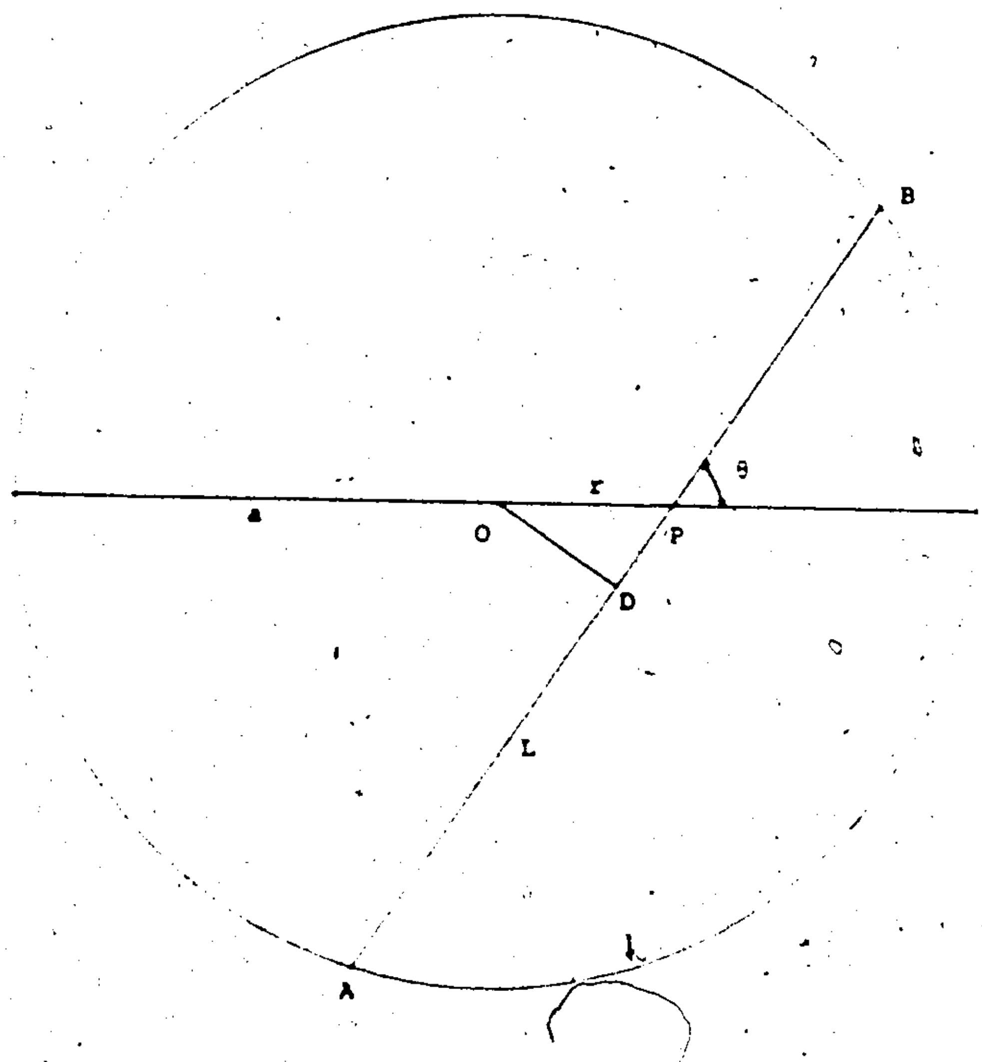

Fig. 55

where $D = \{(r,\theta) : L(r,\theta) \leq l\}$.

Let $l \in [0, 2a]$.

We have

$$r^2 \sin^2\theta = a^2 - \frac{L^2(r,\theta)}{4}.$$

We determine the ranges for θ and r. The minimum value of θ corresponds to the maximum value a of r and the maximum value l of $L(r,\theta)$. Therefore θ must satisfy

$$\sin^{-1}\sqrt{1 - \frac{l^2}{4a^2}} \leq \theta \leq \frac{\pi}{2}.$$

Let

$$\theta \in [\sin^{-1}\sqrt{1 - \frac{l^2}{4a^2}}, \frac{\pi}{2}].$$

For the minimum value of r we have

$$\frac{\sqrt{a^2 - \frac{l^2}{4}}}{r} = \sin\theta.$$

The maximum value of r is a.

It is easy to see that for

$$\theta \in [\sin^{-1}\sqrt{1 - \frac{l^2}{4a^2}}, \frac{\pi}{2}]$$

and

$$rc\left[\frac{\sqrt{a^2 - \frac{\ell^2}{4}}}{\sin\theta}, a\right] ,$$

we have

$$L(r,\theta) \leq \ell .$$

Therefore, we have

$$F(\ell) = 1 - \frac{2}{\pi}\sin^{-1}\sqrt{1 - \frac{\ell^2}{4a^2}} - \frac{\ell}{\pi a}\sqrt{1 - \frac{\ell^2}{4a^2}} , \quad 0 \leq \ell \leq 2a .$$

Corollary. Since $F(\ell)$ is absolutely continuous on $[0, 2a]$, the density of L is given by

$$(1.4.4)\quad f(\ell) = F'(\ell) = \frac{\ell^2}{\pi a^2\sqrt{4a^2 - \ell^2}} , \quad 0 \leq \ell \leq 2a .$$

Putting $a = 1$, in (1.4.4), we find that the result of Coleman [8] is verified.

SECTION FIVE

1.5. DISTRIBUTION OF LENGTHS OF S_1-RANDOM SECANTS OF A SPHERE.

1.5.0. Introduction.

In this section the probability distribution of the length of a random secant of a sphere of radius a, under S_I-randomness, is obtained. Here a secant is specified by a random point on the surface of the sphere and a random direction. A random direction is defined by a random point on the surface of a unit sphere. The question of the density function of the S_1-random secant length was considered by Horowitz [27]. Although Horowitz provides the solution, the method adopted here for obtaining the solution is interesting in itself and the solution here was obtained independently.

Let S be a sphere of radius a. Let P be a random point on the surface of the sphere. A line through P in a random direction intersects the surface of the sphere at another point Q and determines a secant length $L = |PQ|$. Our object is to find the distribution of L.

1.5.1. Distribution of L.

In the following theorem, we obtain the distribution of L.

Theorem 5: The probability distribution of the length L of S_1-random secant of a sphere of radius a is given by

$$F(\ell) = \frac{\ell}{2a}, \qquad 0 \leq \ell \leq 2a \quad (1.5.1)$$

Proof. To the position of the chord PQ and the tangent plane to the sphere at P corresponds a solid angle Ω as follows: Let PO be the perpendicular to the tangent plane at P. The chord PQ makes an angle α with PO. The solid angle subtended by a cone at its vertex with a semi vertical angle α is denoted by Ω.

Now let ℓ be fixed in $[0,2a]$. Since a point on the sphere generates the same set of lengths as those of any other point on the sphere we can, therefore, fix P on the surface of the sphere and consider only the randomness of the direction. We find a transformation from Ω to ℓ. Let a secant of length ℓ make an angle α with the diametral line PD through P. All chords through P making the same angle α with PD have the same secant length ℓ. They form a cone with P as the vertex and semi-vertical angle α. Let Ω be the solid angle at P. We express Ω in terms of α. To find Ω we are to find the area of the portion of the surface on a unit sphere subtended by Ω. Now this area on the surface S of the unit sphere subtended by Ω is

$$\iint_S \left[1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right]^{\frac{1}{2}} dxdy \quad (1.5.2)$$

where $\bar{S}$ is the projection of S on the xy-plane.

Since $z^2 = 1 - x^2 - y^2$, the integrand of (1.5.2) becomes $\frac{1}{(1-x^2-y^2)^{\frac{1}{2}}}$.
Converting to polar coordinates by the relations $x = r\cos\theta$, $y = r\sin\theta$, we find that (1.5.2) reduces to

$$\int_0^{2\pi}\int_0^{\sin\alpha}(1-r^2)^{-\frac{1}{2}}\, r\,dr\,d\theta\ ;$$

which on integration reduces to $2\pi\,(1-\cos\alpha)$.

Thus

$$\Omega = 2\pi\,(1-\cos\alpha).$$

We note that $\cos\alpha = \frac{\ell}{2a}$, that is, $\alpha = \cos^{-1}\frac{\ell}{2a}$,

and to the set of points Q such that $|PQ| = L \geq \ell$ corresponds the solid angle Ω.

Thus

$$\Pr(L \geq \ell) = 1 - F(\ell) = \frac{1}{2\pi}\int_0^{\cos^{-1}\frac{\ell}{2a}} d\Omega(\alpha)$$

$$= \frac{1}{2\pi}\int_0^{\cos^{-1}\frac{\ell}{2a}} d[2\pi(1-\cos\alpha)] = 1 - \frac{\ell}{2a}.$$

Hence

$$F(\ell) = \frac{\ell}{2a}, \quad 0 \le \ell \le 2a .$$

Corollary. Since $F(\ell)$ is absolutely continuous on $[0,2a]$ the density $f(\ell)$ of L is given by

$$f(\ell) = F'(\ell) = \frac{1}{2a}, \qquad 0 \le \ell \le 2a,$$

which is the result given by Horowitz [27].

SECTION SIX

1.6 LIMIT OF THE SEQUENCE OF DISTRIBUTION FUNCTIONS OF LENGTHS OF S_1-RANDOM SECANTS OF REGULAR POLYGONS INSCRIBED IN A CIRCLE

1.6.0. Introduction.

It is clear that the limit of a sequence of regular polygons of N sides inscribed in a circle of constant radius r as $N \to \infty$ is the circle itself. In this section we show that the corresponding sequence of distribution functions of the lengths of S_1-random secants of the sequence of regular N-sided polygons inscribed in the circle also converges to the corresponding distribution function of the length L of an S_1-random secant of the circle. In the following theorem we prove this result.

1.6.1. Limiting distribution.

Theorem 8. Let C be a circle of radius r. Let P_N be a regular polygon of N sides inscribed within the circle C. Let $F_N(\ell)$ and $F(\ell)$ be the corresponding distribution functions of the lengths of S_1-random secants of P_N and C respectively. Then

$$\lim_{N \to \infty} F_N(\ell) = F(\ell) \quad \text{for } \ell \in [0, 2r]. \tag{1.6.1}$$

Proof. Without any loss of generality we take $r = 1$. Let $\ell \in [0, 2]$. Let a be the length of a side of the regular polygon of N sides. Then $a = 2\sin\delta$, where $\delta = \frac{\pi}{N}$.

Let P_N be the polygon $A_0 A_1 \ldots A_{N-1}$ inscribed within the circle C

of radius 1. Let A_0 be a fixed point on the perimeter of the circle. For a fixed $\ell \in [0,2]$, let A_ℓ be a point on the perimeter of the circle such that $|A_0A_\ell| = \ell$. As N increases indefinitely let the closest vertex A_k of the polygon P_N tend to coincide with the point A_ℓ. Then as $N \to \infty$, $k \to \infty$, where k depends on N, and the length $2k\delta$ of the arc A_0A_k tends to the length $2\sin^{-1}\frac{\ell}{2}$ of the arc A_0A_ℓ. Therefore as $N \to \infty$

(1.6.2) $$k\delta \to \sin^{-1}\left(\frac{\ell}{2}\right)$$

and

(1.6.3) $$\ell_k \to \ell.$$

The distribution function $F_N(\ell)$ for $\ell \in [\ell_{k-1}, \ell_k]$ is, by (1.1.1), the following:

(1.6.4) $$F_N(\ell) = \frac{2}{a\pi}\Big\{2k\delta a - 2\delta(a - x_k(\ell))$$

$$+ (x_k(\ell) + d_{k-1})\sin^{-1}\left(\frac{(x_k(\ell) + d_{k-1})\sin 2(k-1)\delta}{\ell}\right)$$

$$- (a + d_{k-1})\sin^{-1}\left(\frac{(a+d_{k-1})\sin 2(k-1)\delta}{\ell}\right)$$

$$+ \frac{1}{\sin 2(k-1)\delta}\Big\{\Big[1 - \frac{(x_k(\ell) + d_{k-1})^2}{\ell^2}\sin^2 2(k-1)\delta\Big]^{1/2}$$

$$- \Big[1 - \frac{(a+d_{k-1})^2}{\ell^2}\sin^2 2(k-1)\delta\Big]^{1/2}\Big\} + d_k\sin^{-1}\left(\frac{d_k \sin 2k\delta}{\ell}\right)$$

$$- (x_k(\ell) + d_k)\sin^{-1}\left(\frac{d_k + x_k(\ell)}{\ell}\sin 2k\delta\right)$$

$$+ \frac{1}{\sin 2k\delta}\Big\{\Big[1 - \frac{d_k^2}{\ell^2}\sin^2 2k\delta\Big]^{1/2} - \Big[1 - \left(\frac{x_k(\ell) + d_k}{\ell}\right)^2\sin^2 2k\delta\Big]^{1/2}\Big\}\Big\}$$

$$= \left(\Xi + \frac{2N(\delta)}{D(\delta)}\right), \text{ say}$$

where $\delta = \frac{\pi}{N}$, $n = \frac{N}{2}$ or $\frac{N-1}{2}$ for N even or odd respectively;

$$\ell_k = \frac{a \sin k\delta}{\sin\delta} ,$$

$$\beta_k(\ell) = \sin^{-1}\left(\frac{\ell_{k-1} \sin k\delta}{\ell}\right) ,$$

$$x_k(\ell) = \frac{\ell \sin (k\delta - \beta_k(\ell))}{\sin k\delta} ,$$

$$d_k = \ell_{k-1} \frac{\sin k\delta}{\sin 2k\delta} ,$$

$D(\delta) = a\pi$, $N(\delta)$ is the remaining expression.

Now to consider the limit of $F_N(\ell)$ as $N\to\infty$, we note that

(1.6.5) $$a = 2 \sin \delta \to 0 \quad , \quad \text{as } N \to \infty \ ;$$

(1.6.6) $$\beta_k(\ell) = \sin^{-1}\left(\frac{\ell_{k-1} \sin k\delta}{\ell}\right) \to (k-1)\delta \to \sin^{-1}\frac{\ell}{2} \quad \text{as } N \to \infty \ ;$$

(1.6.7) $$d_k = \frac{\ell_{k-1} \sin k\delta}{\sin 2k\delta} \to \frac{\ell}{[4-\ell^2]^{1/2}} , \quad \text{as } N \to \infty \ ;$$

(1.6.8) $$x_k(\ell) = \frac{\ell \sin (k\delta - \beta_k(\ell))}{\sin k\delta} \to 0, \quad \text{as } N \to \infty .$$

and, by (1.6.2)

$$k\delta \to \sin^{-1}\left(\frac{\ell}{2}\right) .$$

Consequently both $N(\delta)$ and $D(\delta)$ tend to zero as $N \to \infty$ and reduce the limit of $\frac{2N(\delta)}{D(\delta)}$ to $\frac{0}{0}$ form. We then apply L'Hospital's rule to the corresponding expression of $\frac{2N(\delta)}{D(\delta)}$, where δ is replaced by a real variable x.

We note:

$$k\delta \to \sin^{-1}\frac{\ell}{2} \quad , \quad \text{as } N \to \infty ,$$

$$d_k(x) \to \frac{\quad}{\sqrt{4-\ell^2}} \quad \text{as } x \to 0, \text{ where}$$

$$d_k(\delta) = d_k = \frac{\ell}{\sqrt{4-\ell^2}} \cos\delta - \sin\delta$$

$$\frac{dd_k(x)}{dx} \to -1 \ , \quad \text{as } x \to 0 \ ,$$

$$\frac{dD(x)}{dx} \to 2\pi \ , \quad \text{as } x \to 0.$$

and (1.6.5) to (1.6.8), and obtain:

$$\lim_{N\to\infty} F_N(\ell) = \frac{2}{\pi} \sin^{-1} \frac{\ell}{2} \ ,$$

which is the distribution function $F(\ell)$ of the S_1-random secant length L of a circle of radius 1 (cf. Theorem 4).

CHAPTER TWO

2. DISTRIBUTIONS OF LENGTHS OF S_2-RANDOM SECANTS.

2.0. Introduction.

This thesis is devoted to investigating certain regular features of random secants or 'rays' of convex geometrical configurations. In the last chapter, we obtained the probability laws of the lengths of S_1-random secants of some convex geometrical configurations. In S_1-randomness a secant is defined by a random point on the perimeter and a random direction. A natural question in geometrical probability to ask is: What will be the probability laws of the lengths of random secants, when a secant is specified by two points on the perimeter?

In this chapter we deal with this question and obtain the probability distributions of the lengths of S_2-random secants of regular polygons. The cases of linear segments, rectangles and circles were considered by Matern [43]. The results for the polygon under S_1-randomness and S_2-randomness will differ and may appear "paradoxical" as in the case of the well-known Bertrands "paradoxical" results for the probability that a 'random' chord of a circle of unit radius has a length greater than $\sqrt{3}$.

Both solutions are correct, but they really refer to different problems. We note that in the case of S_1-randomness, the point on the perimeter and the direction defining the random secant have uniform distributions and the probability distributions of the length of a S_1-random secant is the one induced by the probability laws for the perimeter point and the direction, whereas in the case of S_2-randomness, the two perimeter points defining the random secant have uniform distributions over "appropriate" ranges and the probability distribution of the length of a S_2-random secant is the one induced by the distributions of the two points. It is of further interest to us to see how these two probability laws for the random length of a secant, of a geometrical configuration, a regular polygon, for example, differ.

In Section 2 of this chapter, we will show that as a sequence of regular polygons P_N, N = 3, 4, ..., of N sides inscribed within a circle C of constant radius r tends to the circle C, the corresponding sequence of distribution functions $F_N(\ell)$ of the random secant length L of polygons also converges to the distribution function $F(\ell)$ of the random secant length L of the circle C.

SECTION ONE

2.1. DISTRIBUTIONS OF LENGTHS OF S_2-RANDOM SECANTS OF A REGULAR POLYGON.

2.1.0. Introduction.

In this section the probability distribution of the length of a regular polygon of N sides, where a side is of length a, under S_2-randomness is obtained. Here a random secant is defined by two random points on the perimeter. A precise statement of the problem is the following:

Let $A_0A_1 \ldots A_{N-1}$ be a regular polygon of N sides (cf. Fig. 56). Let two points P and Q be chosen at random on the perimeter but not on the same side of the polygon. Let L denote the distance $|PQ|$. Then L is a random variable. The problem is to determine the probability distribution of L.

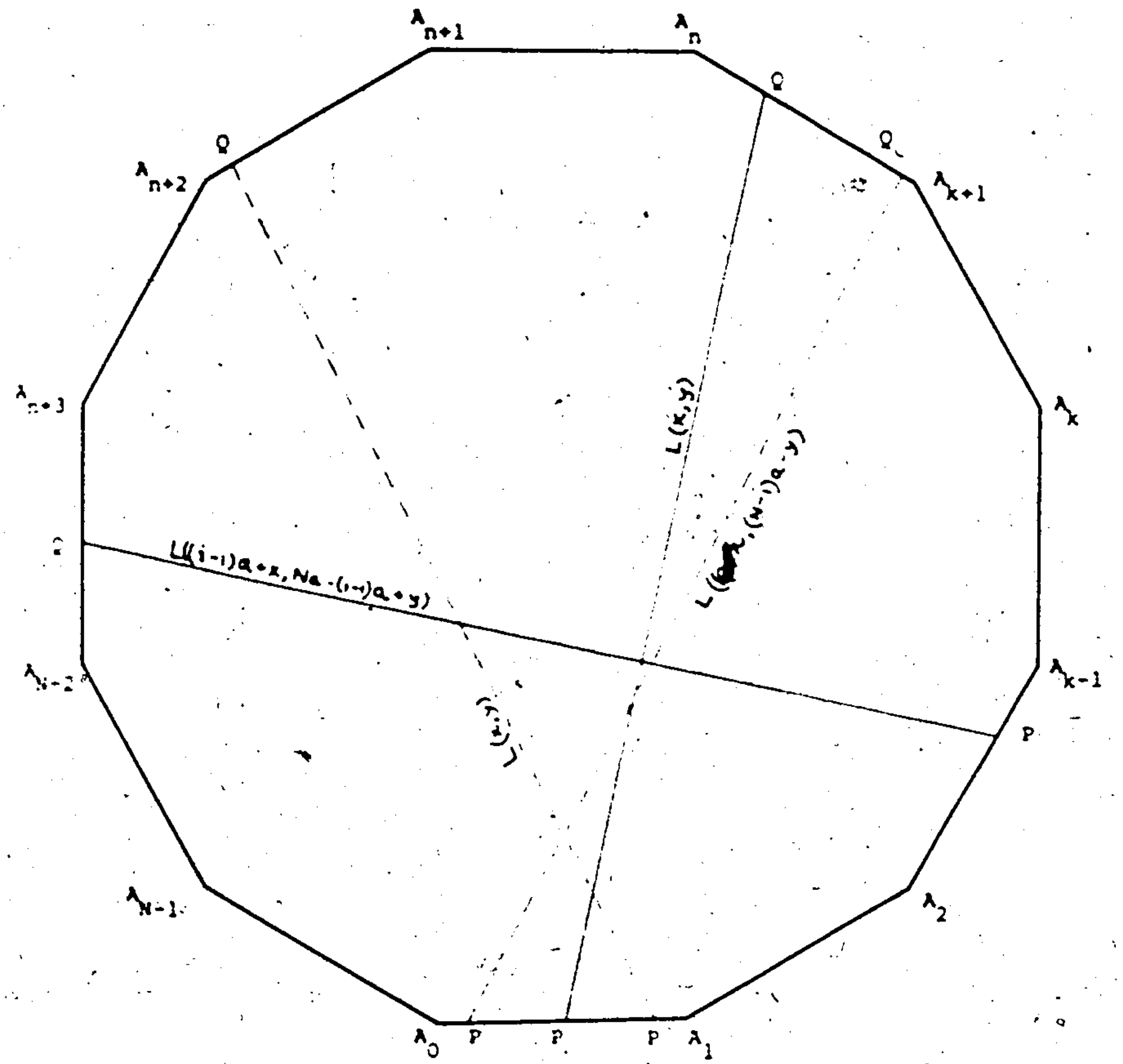

Fig. 56

2.1.1. The probability distribution of L.

The probability distribution of L is given in the following:

Theorem: The probability distribution function $F_N(\ell)$ of the random secant length L of a regular polygon of N sides each of length a under S_2-randomness is given by

$$
(2.1.1)\quad F_N(\ell)\begin{cases}
= c_0\,\phi_1(\ell) & \text{for } \ell\epsilon[0,\ell_1],\ N \text{ even} & \text{(A)}\\
= c_0'\,\phi_1(\ell) & \text{for } \ell\epsilon[0,\ell_1],\ N \text{ odd} & \text{(B)}\\
= c_0\,\phi_2(\ell) & \text{for } \ell\epsilon[\ell_{k-1},\ell_k],\ k=2,3,\ldots,n-1, N \text{ even} & \text{(C)}\\
= c_0'\,\phi_2(\ell) & \text{for } \ell\epsilon[\ell_{k-1},\ell_k],\ k=2,3,\ldots,n-1, N \text{ odd} & \text{(D)}\\
= c_0\,\phi_3(\ell) & \text{for } \ell\epsilon[\ell_{n-1},\ell_c],\ N \text{ even} & \text{(E)}\\
= c_0\,\phi_4(\ell) & \text{for } \ell\epsilon[\ell_c,\ell_b],\ N \text{ even} & \text{(F)}\\
= c_0\,\phi_5(\ell) & \text{for } \ell\epsilon[\ell_b,\ell_n],\ N \text{ even} & \text{(G)}\\
= c_0'\,\phi_6(\ell) & \text{for } \ell\epsilon[\ell_{n-1},\ell_{n-1,1}],\ N \text{ odd} & \text{(H)}\\
= c_0'\,\phi_7(\ell) & \text{for } \ell\epsilon[\ell_{n-1,1},\ell_{n-1,2}],\ N \text{ odd} & \text{(I)}\\
= c_0'\,\phi_8(\ell) & \text{for } \ell\epsilon[\ell_{n-1,2},\ell_n],\ N \text{ odd}, & \text{(J)}
\end{cases}
$$

where $n = \frac{N}{2}$ or $\frac{N-1}{2}$ according as N is even or odd,

$\delta = \frac{\pi}{N}$,

(2.1.1a) $c_0 = \frac{2}{a^2(2n-1)}$,

(2.1.1b) $c_0' = \frac{1}{na^2}$,

(2.1.1c) $\phi_1(\ell) = \frac{\delta\ell^2}{\sin 2\delta}$, (2.1.1d) $$\phi_2(\ell) = \{(k-2)a-d_{k-1}\}(a-x_k(\ell)) + \frac{(a+d_{k-1})}{2}[\ell^2-(a+d_{k-1})^2\sin^2 2(k-1)\delta]^{\frac{1}{2}}$$

$$- \frac{(x_k(\ell) + d_{k-1})}{2}[\ell^2-(x_k(\ell) + d_{k-1})^2 \sin^2 2(k-1)\delta]^{\frac{1}{2}}$$

$$+ \frac{\ell^2}{2 \sin 2(k-1)\delta}[\sin^{-1}\{\frac{(a+d_{k-1})}{\ell}\sin 2(k-1)\delta\}$$

$$- \sin^{-1}\{\frac{x_k(\ell)+d_{k-1}}{\ell}\sin 2(k-1)\delta\}]$$

$$- \frac{\cos 2(k-1)\delta}{2}[(a+d_{k-1})^2 - (x_k(\ell) + d_{k-1})^2] + [(k-1)a-d_k]x_k(\ell)$$

$$+ \frac{1}{2}[x_k(\ell) + d_k][\ell^2-(x_k(\ell) + d_k)^2\sin^2 2k\delta]^{\frac{1}{2}}$$

$$- \frac{1}{2}d_k[\ell^2 - d_k^2 \sin^2 2k\delta]^{\frac{1}{2}} + \frac{\ell^2}{2 \sin 2k\delta}[\sin^{-1}\{(\frac{x_k(\ell)+d_k}{\ell}) \sin 2k\delta\}$$

$$- \sin^{-1}(\frac{d_k}{\ell}\sin 2k\delta)\} - \frac{1}{2}\cos 2k\delta[(x_k(\ell) + d_k)^2 - d_k^2],$$

(2.1.1e) $$\phi_3(\ell) = \frac{1}{2}(\ell_{n-1}^2-\ell^2) + a[\ell^2-\ell_{n-1}^2]^{\frac{1}{2}} + (x_n(\ell) - a)(y_{n1}(\ell) + d_{n-1})$$

$$+ \frac{a+d_{n-1}}{2}[\ell^2 - \sin^2(2(n-1)\delta)(a+d_{n-1})^2]^{\frac{1}{2}}$$

$$- \frac{x_n(\ell) + d_{n-1}}{2}[\ell^2 - (x_n(\ell) + d_{n-1})^2 \sin^2 2(n-1)\delta]^{\frac{1}{2}}$$

$$+ \frac{\ell^2}{2 \sin 2(n-1)\delta}[\sin^{-1}\{\frac{a+d_{n-1}}{\ell}\sin 2(n-1)\delta\}$$

$$- \sin^{-1}\left\{\frac{x_n(\ell)+d_{n-1}}{\ell}\sin 2(n-1)\delta\right\}]$$

$$+ \frac{\cos 2(n-1)\delta}{2}\left[(x_n(\ell)+d_{n-1})^2 - (a+d_{n-1})^2\right]$$

$$+ x_n(\ell)\ (a-y_{n1}(\ell)) \ + a\{y_{n1}(\ell) + (n-2)\ a\},$$

(2.1.1f) $$\phi_4(\ell) = \frac{1}{2}(\ell_{n-1}^2-\ell^2) + a[\ell^2-\ell_{n-1}^2]^{\frac{1}{2}} + (x_n(\ell) - a)(y_{n1}(\ell) + d_{n-1})$$

$$+ \frac{a+d_{n-1}}{2}[\ell^2 - \sin^2(2(n-1)\delta)(a+d_{n-1})^2]^{\frac{1}{2}}$$

$$- \frac{x_n(\ell)+d_{n-1}}{2}[\ell^2-(x_n(\ell)+d_{n-1})^2\sin^2 2(n-1)\delta]^{\frac{1}{2}}$$

$$+ \frac{\ell^2}{2\sin 2(n-1)\delta}\left[\sin^{-1}\left(\frac{a+d_{n-1}}{\ell}\sin 2(n-1)\delta\right)\right.$$

$$\left.- \sin^{-1}\left(\frac{x_n(\ell)+d_{n-1}}{\ell}\sin 2(n-1)\delta\right)\right]$$

$$+ \frac{1}{2}\cos 2(n-1)\delta\ \left[(x_n(\ell)+d_{n-1})^2 - (a+d_{n-1})^2\right]$$

$$+ x_n(\ell)\ (a - y_{n1}(\ell)) + a\{y_{n1}(\ell) + (n-2)a\}.$$

(2.1.1g) $$\phi_5(\ell) = \frac{1}{2}(\ell_{n-1}^2-\ell^2) + a[\ell^2-\ell_{n-1}^2]^{\frac{1}{2}} + (x_n(\ell) - a)(y_{n1}(\ell) + d_{n-1})$$

$$+ \frac{a+d_{n-1}}{2}[\ell^2 - \sin^2(2(n-1)\delta)(a+d_{n-1})^2]^{\frac{1}{2}}$$

$$- \frac{x_n(\ell)+d_{n-1}}{2} [\ell^2 - (x_n(\ell) + d_{n-1})^2 \sin^2 2(n-1)\delta]^{\frac{1}{2}}$$

$$+ \frac{\ell^2}{2 \sin 2(n-1)\delta} [\sin^{-1} (\frac{a+d_{n-1}}{\ell} \sin 2(n-1)\delta)$$

$$- \sin^{-1} (\frac{x_n(\ell)+d_{n-1}}{\ell} \sin 2(n-1)\delta)]$$

$$+ \frac{1}{2} \cos 2(n-1)\delta[(x_n(\ell) + d_{n-1})^2 - (a+d_{n-1})^2]$$

$$+ x_n(\ell) (a-y_{n1}(\ell)) + a(y_{n1}(\ell) + (n-2)a),$$

(2.1.1h) $$\phi_6(\ell) = (- d_n + (n-1)a) c + \frac{d_n+c}{2} [\ell^2-(d_n+c)^2 \sin^2 2n\delta]^{\frac{1}{2}}$$

$$- \frac{d_n}{2} [\ell^2-d_n^2 \sin^2 2n\delta]^{\frac{1}{2}} + \frac{\ell^2}{2 \sin 2n\delta} [\sin^{-1} (\frac{d_n+c}{\ell} \sin 2n\delta)$$

$$- \sin^{-1} (\frac{d_n}{\ell} \sin 2n\delta)] - \frac{\cos 2n\delta}{2}[(d_n+c)^2 - d_n^2]$$

$$+ (a-c) \{(n-2)a-d_{n-1}\} + \frac{d_{n-1}+a}{2} [\ell^2-(d_{n-1}+a)^2 \sin^2 2(n-1)\delta]^{\frac{1}{2}}$$

$$- \frac{d_{n-1}+c}{2} [\ell^2 - (d_{n-1} + c)^2 \sin^2 2(n-1)\delta]^{\frac{1}{2}}$$

$$+ \frac{\ell^2}{2 \sin 2(n-1)\delta} [\sin^{-1} (\frac{d_{n-1}+a}{\ell} \sin 2(n-1)\delta)$$

$$- \sin^{-1}(\frac{d_{n-1}+c}{\ell} \sin 2(n-1)\delta)] - \frac{\cos 2(n-1)\delta}{2} [(a+d_{n-1})^2-(c+d_{n-1})^2],$$

(2.1.11) $$\phi_7(\ell) = (-d_n + (n-1)a)c + \frac{d_n+c}{2}[\ell^2-(d_n+c)^2 \sin^2 2n\delta]^{\frac{1}{2}}$$

$$- \frac{d_n}{2}[\ell^2 - d_n^2 \sin^2 2n\delta]^{\frac{1}{2}} + \frac{\ell^2}{2 \sin 2n\delta}[\sin^{-1}(\frac{d_n+c}{\ell} \sin 2n\delta)$$

$$- \sin^{-1}(\frac{d_n}{\ell} \sin 2n\delta)] - \frac{\cos 2n\delta}{2}[(d_n+c)^2 - d_n^2]$$

$$+ (a-c)((n-2)a-d_{n-1})$$

$$+ \frac{d_{n-1}+a}{2}[\ell^2-(d_{n-1}+a)^2 \sin^2 (2(n-1)\delta)]^{\frac{1}{2}}$$

$$- \frac{d_{n-1}+c}{2}[\ell^2-(d_{n-1}+c)^2 \sin^2(2(n-1)\delta)]^{\frac{1}{2}}$$

$$+ \frac{\ell^2}{2 \sin(2(n-1)\delta)}[\sin^{-1}(\frac{d_{n-1}+a}{\ell} \sin (2(n-1)\delta))$$

$$- \sin^{-1}(\frac{d_{n-1}+c}{\ell} \sin(2(n-1)\delta))] + \frac{\cos (2(n-1)\delta)}{2}[(c+d_{n-1})^2 - (a+d_{n-1})^2]$$

$$+ (d_n+x_1)[\ell^2 - (d_n+x_1)^2 \sin^2 2n\delta]^{\frac{1}{2}}$$

$$- (d_n+c)[\ell^2 - (d_n+c)^2 \sin^2 2n\delta]^{\frac{1}{2}}$$

$$+ \frac{\ell^2}{\sin 2n\delta}[\sin^{-1}(\frac{d_n+x_1}{\ell} \sin 2n\delta)$$

$$- \sin^{-1}(\frac{d_n+c}{\ell} \sin 2n\delta)] \ .$$

$$(2.1.1j)\quad \phi_8(\ell) = \{-d_n+(n-1)a\}c_1 + \frac{d_n+c_1}{2}\left[\ell^2-(d_n+c_1)^2\sin^2 2n\delta\right]^{1/2}$$

$$-\frac{d_n}{2}\left[\ell^2-d_n^2\sin^2 2n\delta\right]^{1/2} + \frac{\ell^2}{2\sin 2n\delta}\Big[\sin^{-1}\left(\frac{d_n+c_1}{\ell}\sin 2n\delta\right)$$

$$-\sin^{-1}\left(\frac{d_n}{\ell}\sin 2n\delta\right)\Big] - \frac{\cos 2n\delta}{2}\left[(d_n+c_1)^2 - d_n^2\right]$$

$$+\, na(c_2-c_1)$$

$$+\{-d_n+(n-1)a\}(c-c_2) + \frac{d_n+c}{2}\left[\ell^2-(d_n+c)^2\sin^2 2n\delta\right]^{1/2}$$

$$-\frac{d_n+c_2}{2}\left[\ell^2-(d_n+c_2)^2\sin^2 2n\delta\right]^{1/2}$$

$$+\frac{\ell^2}{2\sin 2n\delta}\left[\sin^{-1}\left(\frac{d_n+c}{\ell}\sin 2n\delta\right) - \sin^{-1}\left(\frac{d_n+c_2}{\ell}\sin 2n\delta\right)\right]$$

$$-\frac{\cos 2n\delta}{2}\left[(d_n+c)^2 - (d_n+c_2)^2\right]$$

$$+(a-c)\{(n-2)a-d_{n-1}\} + \frac{d_{n-1}+a}{2}\left[\ell^2-(d_{n-1}+a)^2\sin^2 2(n-1)\delta\right]^{1/2}$$

$$-\frac{d_{n-1}+c}{2}\left[\ell^2-(d_{n-1}+c)^2\sin^2 2(n-1)\delta\right]^{1/2}$$

$$+\frac{\ell^2}{2\sin 2(n-1)\delta}\Big[\sin^{-1}\left(\frac{d_{n-1}+a}{\ell}\sin 2(n-1)\delta\right)$$

$$-\sin^{-1}\left(\frac{d_{n-1}+c}{\ell}\sin 2(n-1)\delta\right)\Big]$$

$$+\frac{\cos 2(n-1)\delta}{2}\left[(c+d_{n-1})^2 - (a+d_{n-1})^2\right]$$

$$+ (d_n+a)[\ell^2-(d_n+a)^2 \sin^2 2n\delta]^{1/2}$$

$$- (d_n+c)[\ell^2-(d_n+c)^2 \sin^2 2n\delta]^{1/2}$$

$$+ \frac{\ell^2}{\sin 2n\delta}\left[\sin^{-1}\left(\frac{d_n+a}{\ell}\sin 2n\delta\right) - \sin^{-1}\left(\frac{d_n+c}{\ell}\sin 2n\delta\right)\right] ,$$

where $\ell_k = \frac{a \sin k\delta}{\sin \delta}$, $k = 1, 2, \ldots, n-1$; $d_k = \ell_{k-1}\frac{\sin k\delta}{\sin 2k\delta}$,

$k = 1, 2, \ldots, n$.

$$c = \frac{\ell}{\sin n\delta}\sin\left(n\delta - \sin^{-1}\left(\frac{\ell_{n-1}}{\ell}\sin n\delta\right)\right) ,$$

$$x_1 = -d_n + \frac{\ell}{\sin 2n\delta} , \qquad y_{n1}(\ell) = \frac{\ell \sin(\alpha_n(\ell) - (n-2)\delta)}{\sin n\delta}$$

$$c_1 = \frac{a}{2} - [\ell^2 - \ell_{n-1}^2]^{1/2} , \qquad \ell_{n-1,1} = -d_n \tan 2n\delta ,$$

$$c_2 = \frac{a}{2} + [\ell^2 - \ell_{n-1}^2]^{1/2} , \qquad \ell_{n-1,2} = (\ell_n^2 - \frac{a^2}{4})^{1/2} ,$$

$$\ell_b = (\ell_{n-1}^2 + (\frac{a}{2})^2)^{1/2} ,$$

$$\ell_c = (\ell_{n-1}^2 + (\frac{a}{4})^2)^{1/2} .$$

To prove this theorem we proceed as follows.

2.1.2. Parameterization of the random secants.

The secants are parametrized as follows. Let $A_0A_1 \ldots A_{N-1}$ be a regular polygon. Let A_1 be the reference point and PQ a random chord (cf. Fig. 56). Let X be the distance of the point P measured along the perimeter from the reference point A_1 in the direction A_1A_0 of the polygon. Let Y be the distance of the point Q from the point A_1 measured along the perimeter following the direction $\overrightarrow{A_1A_2}$.

We assume that X and Y have uniform distributions on the perimeter of the polygon subject to the condition that with probability zero the random variables X and Y assume values which correspond to points lying on the same side of the polygon.

The joint density of X and Y is given by

$$(2.1.2) \qquad P(x,y) = \begin{cases} \dfrac{1}{N(N-1)a^2}, & \text{for } (x,y) \in S', \\ 0, & \text{elsewhere in the } (x,y)\text{-plane,} \end{cases}$$

where

(2.1.3)
$$S' = \bigcup_{\substack{i,j=1,2,\ldots,N \\ i\neq j}} S'_{ij} ,$$

where

(2.1.4)
$$S'_{ij} = \{(x,y):\ (i-1)a \leq x \leq ia,\ (j-1)a \leq y \leq ja\},$$

$$i,\ j = 1,\ 2,\ \ldots,\ N\ .$$

Clearly L is a function of X and Y. To emphasize this fact we will write L(x,y) instead of L.

We need to evaluate

(2.1.5)
$$F_N(\ell) = \Pr(L \leq \ell) = \frac{1}{N(N-1)a^2} \int_{D(\ell)} dxdy\ ,$$

where

(2.1.6)
$$D(\ell) = \{(x,y):\ L(x,y) \leq \ell,\ (x,y) \in S'\}\ .$$

The set S', defined in (2.1.3), is the <u>parameter space</u>.

2.1.3. Reduction of the Parameter space.

Using symmetry we can reduce the parameter space. We write $D(\ell)$ defined in (2.1.6) as the union of N 'equal sets', disjoint except possibly for boundary points, as

(2.1.7) $$D(\ell) = \bigcup_{i=1}^{N} D_i'(\ell) ,$$

where

(2.1.8) $$D_i'(\ell) = \{(x,y): L(x,y) \le \ell,\ (x,y) \in \bigcup_{\substack{j=1,2,\ldots,N \\ i \ne j}} S_{ij}'\},\ i=1,2,\ldots,N ,$$

We have the following:

Lemma 1. Let $F_N(\ell)$ be the distribution function of L. Then

(2.1.9) $$F_N(\ell) = \frac{1}{N(N-1)a^2} \int_{D(\ell)} dxdy = \frac{1}{(N-1)a^2} \int_{D_1'(\ell)} dxdy ,$$

where $D_i'(\ell)$ is given by (2.1.8).

Proof. Since $L(x,y) = \begin{cases} L((i-1)a+x,\ -(i-1)a+y), \text{ if } -(i-1)a+y \geq 0\ , \\ L((i-1)a+x,\ Na-(i-1)a+y), \text{ if } -(i-1)a+y < 0\ , \end{cases}$

for $(x,y) \in \bigcup_{j=2}^{N} S'_{1j} = S'_1$ (say), we have

$$\int_{D'_1(\ell)} dxdy = \int_{D'_i(\ell)} dxdy, \text{ for } i = 1,2,\ldots,N\ .$$

Therefore,

$$F_N(\ell) = \frac{1}{N(N-1)a^2} \int_{D(\ell)} dxdy = \frac{1}{(N-1)a^2} \int_{D'_1(\ell)} dxdy\ .$$

The parameter space S' given by (2.1.3) is now reduced to

(2.1.10) $$S'_1 = \bigcup_{i=2}^{N} S'_{1i}\ .$$

We reduce S'_1 further using the following lemma.

Lemma 2. Let $D'_1(\ell)$ be defined by (2.1.8). Then

(2.1.11) $$\frac{1}{(N-1)a^2} \int_{D'_1(\ell)} dxdy = \begin{cases} \dfrac{1}{na^2} \displaystyle\int_{D_1(\ell)} dxdy, \text{ when } N \text{ is odd and equal to } 2n+1\ , \\ \dfrac{2}{(2n-1)a^2} \displaystyle\int_{D_1(\ell)} dxdy, \text{ when } N \text{ is even and equal to } 2n\ , \end{cases}$$

where

$$(2.1.12)\quad D_1(\ell) = \begin{cases} \{(x,y):\ L(x,y) \le \ell,\ 0 \le x \le a,\ 0 \le y \le na\}, & \text{for } N \text{ odd}, \\ \{(x,y):\ L(x,y) \le \ell,\ 0 \le x \le a,\ 0 \le y \le na-a/2\}, & \text{for } N \text{ even}. \end{cases}$$

__Proof__. Clearly $L(x,y) = L(a-x, (N-1)a-y)$ for $0 \le x \le a$,

$$\text{and } \begin{cases} 0 \le y \le na - \frac{a}{2}, & \text{for } N \text{ even}, \\ 0 \le y \le na, & \text{for } N \text{ odd}, \end{cases}$$

(cf. ~~Fig.~~ 56).

Lemma 2 is now obvious. The parameter space S_1' is finally reduced to

$$(2.1.13)\quad S = \begin{cases} \{(x,y):\ 0 \le x \le a,\ 0 \le y \le na-a/2\}, & \text{when } N \text{ is even}, \\ \{(x,y):\ 0 \le x \le a,\ 0 \le y \le na\}, & \text{when } N \text{ is odd}. \end{cases}$$

The reduced parameter space S will be referred to as the __parameter space__.

We decompose the range $[0, \ell_n]$ of L into the appropriate subintervals as follows:

$$[\ell_0, \ell_n] = \bigcup_{k=1}^{n} [\ell_{k-1}, \ell_k],$$

where $N = 2n$ or $2n+1$ for n even or odd respectively, and ℓ_k is given by (1.1.16).

2.1.4. Determination of the set $D_1(\ell)$ for $\ell \varepsilon [0,\ell_1]$.

In order to find the distribution function $F_N(\ell)$ of L for $\ell \varepsilon [0,\ell_1]$ we require the set $\{(x,\theta): L(x,\theta) \leq \ell,\ \ell \varepsilon [0,\ell_1]\}$ in the parameter space. In the following lemma we obtain this set.

Lemma 3. Let $\ell \varepsilon [0,\ell_1]$. Then

(2.1.14) $$D_1(\ell) = S_{11}(\ell) = \{(x,y):\ 0 \leq x \leq \ell,\ 0 \leq y \leq \frac{\ell}{\sin 2\delta} \sin[2\delta - \sin^{-1}(\frac{x \sin 2\delta}{\ell}$$

Proof. Let P_x be a point on A_0A_1 such that $0 \leq x \leq \ell$. With P_x as the centre we draw a circle of radius ℓ. The circle intersects A_1A_2 at a point Q. To this Q will correspond a y. Let $|A_1Q| = y_1(x,\ell)$. For emphasis, we write Q as $Q(y_1)$. From the triangle $Q(y_1)P_xA_1$ we have, by the sine law, denoting $\measuredangle Q(y_1)P_xA_1$ by $\theta_1(x,\ell)$,

(2.1.15) $$\frac{\ell}{\sin 2\delta} = \frac{y_1(x,\ell)}{\sin \theta_1(x,\ell)} = \frac{x}{\sin[2\delta - \theta_1(x,\ell)]},\ \text{(cf. Fig. 57)}.$$

From (2.1.15) it follows that

(2.1.16) $$\theta_1(x,\ell) = 2\delta - \sin^{-1}(\frac{x \sin 2\delta}{\ell}),$$

and

(2.1.17) $$y_1(x,\ell) = \frac{\ell \sin \theta_1(x,\ell)}{\sin 2\delta}.$$

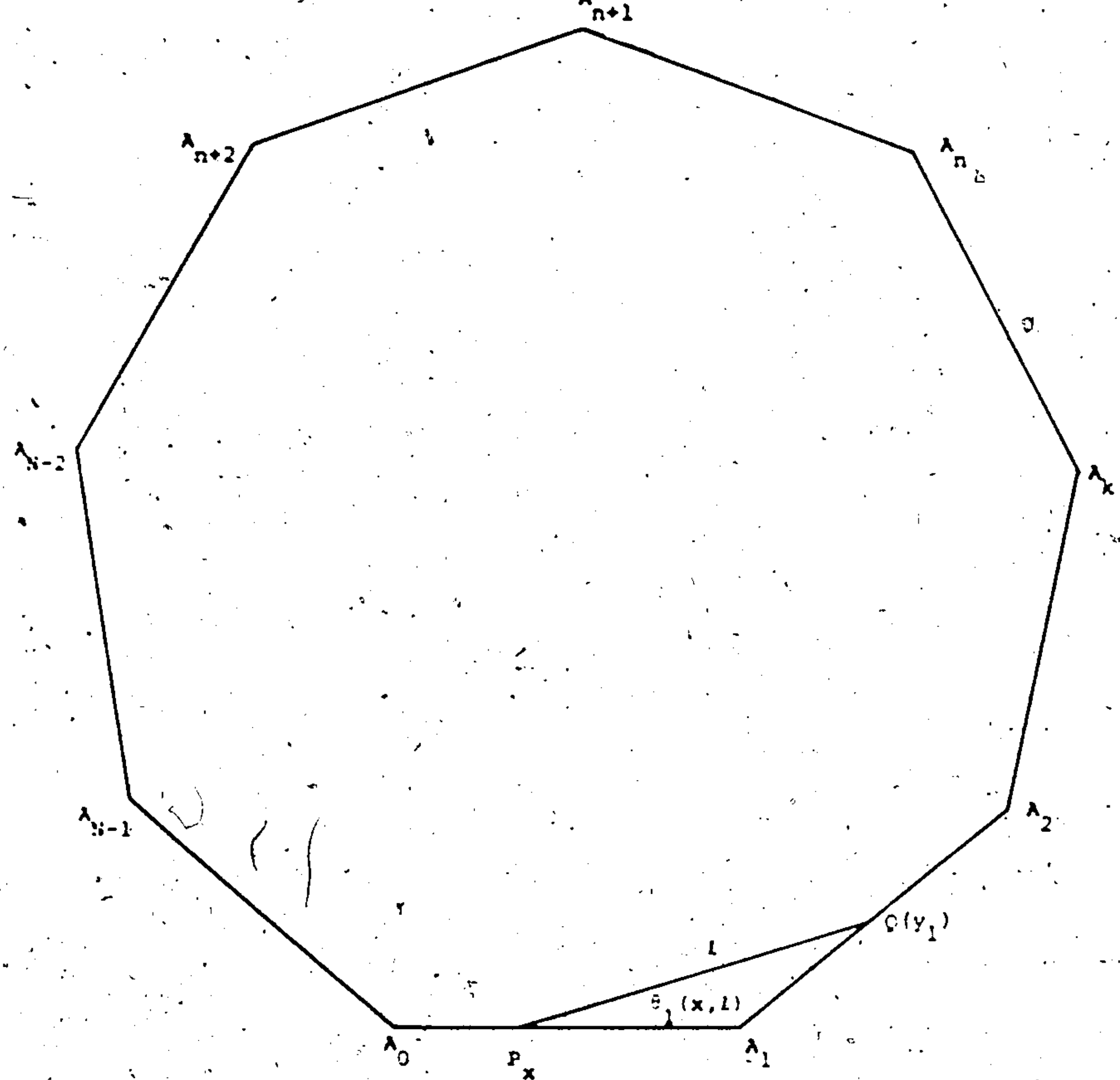

Fig. 57

By using (2.1.15) and observing that $0 \leq x \leq \ell$, we obtain, by a similar argument as in the proof of (1.1.26),

(2.1.18) $$(x,y) \in \left\{(x,y):\ 0 \leq x \leq \ell,\ 0 \leq y \leq \frac{\ell \sin(2\delta - \sin^{-1}(\frac{x \sin 2\delta}{\ell}))}{\sin 2\delta}\right\}$$

$$\Rightarrow L(x,y) \leq \ell.$$

Hence

(2.1.19) $$S_{11}(\ell) \subset D_1(\ell).$$

We note, by the use of (2.1.15), that

(2.1.20) $$0 \leq x \leq \ell,\ y > y_1(x,\ell) \Rightarrow L(x,y) > \ell\ ,$$

(2.1.21) $$x > \ell,\ 0 \leq y \leq y_1(x,\ell) \Rightarrow L(x,y) > \ell$$

and

(2.1.22) $$x > \ell,\qquad y > y_1(x,\ell) \Rightarrow L(x,y) > \ell\ .$$

Hence by (2.1.20), (2.1.21) and (2.1.22), we obtain

(2.1.23) $$D_1(\ell) \subset S_{11}(\ell).$$

Combining (2.1.19) and (2.1.23), we obtain

(2.1.24) $$D_1(\ell) = S_{11}(\ell).$$

2.1.4A. Graphical description of the set $D_1(\ell)$ for $\ell\varepsilon[0,\ell_1]$.

For $\ell\varepsilon[0,\ell_1]$ we have in Lemma 3 of Section 2.1.4,

(2.1.13)
$$D_1(\ell) = S_{11}(\ell) = \{(x,y): 0 \le x \le \ell, \quad 0 \le y \le \frac{\ell \sin[2\delta - \sin^{-1}(\frac{x \sin 2\delta}{\ell})]}{\sin 2\delta}\}$$

The set $A_{11}(\ell)$ of points in the parameter space S which are mapped onto ℓ are those that satisfy the equation

(2.1.4A.1)
$$y = \frac{\ell}{\sin 2\delta} \sin\{2\delta - \sin^{-1}(\frac{x \sin 2\delta}{\ell})\}$$

(which is obtained by equating y with $\frac{\ell \sin\{2\delta - \sin^{-1}(\frac{x \sin 2\delta}{\ell})\}}{\sin 2\delta}$ in (2.1.13)).

Simplifying (2.1.4A.1), we obtain

(2.1.4A.2)
$$A_{11}(\ell) = \{(x,y): x^2 + y^2 + 2xy \cos 2\delta = \ell^2\}\cap S.$$

Since $x^2 + y^2 + 2xy \cos 2\delta$ is a non-negative quadratic form, the equation

(2.1.4A.3)
$$x^2 + y^2 + 2xy \cos 2\delta = \ell^2$$

represents an ellipse. By suitable change of axes,(2.1.4A.3) is reduced to the following form:

$$\frac{u_1^2}{a_1^2} + \frac{v_1^2}{b_1^2} = 1 \quad , \tag{2.1.4A.4}$$

where $u_1 = \frac{x+y}{\sqrt{2}}$, $v_1 = \frac{x-y}{\sqrt{2}}$, $a_1 = \frac{\ell}{\sqrt{2}\cos\delta}$, and $b_1 = \frac{\ell}{\sqrt{2}\sin\delta}$.

We note that for different values of ℓ, (2.1.4A.4) represents concentric coaxial ellipses with different lengths of major and minor axes. The centre of each ellipse is (0,0), and the equations of the major and minor axes are $x+y = 0$ and $x-y = 0$, respectively. In the parameter space S, given by OABC in Fig. 58, the curve (2.1.4A.3) passes through the points $(\ell,0)$ and $(0,\ell)$, which we denote by P_ℓ and C_ℓ, respectively. It may be noted that as ℓ increases from 0 to a, the ellipse gets bigger and P_ℓ and C_ℓ both move further away from (0,0) to A and C_a; respectively (cf. Fig. 58) (a is taken to be 1 for S, the parameter space).

The set $S_{11}(\ell)$ is bounded by $x = 0$, $y = 0$ and the curve $A_{11}(\ell)$ in the parameter space S and is represented by $0\,P_\ell C_\ell$ in Fig. 58. We now describe the set $S_{11}(\ell)$ for extreme values of ℓ satisfying $0 \le \ell \le a$. We note that

$$\ell = 0 \Rightarrow x = 0, \quad y = 0 \quad .$$

Consequently,

$$S_{11}(0) = \{(0,0)\} \quad ,$$

and

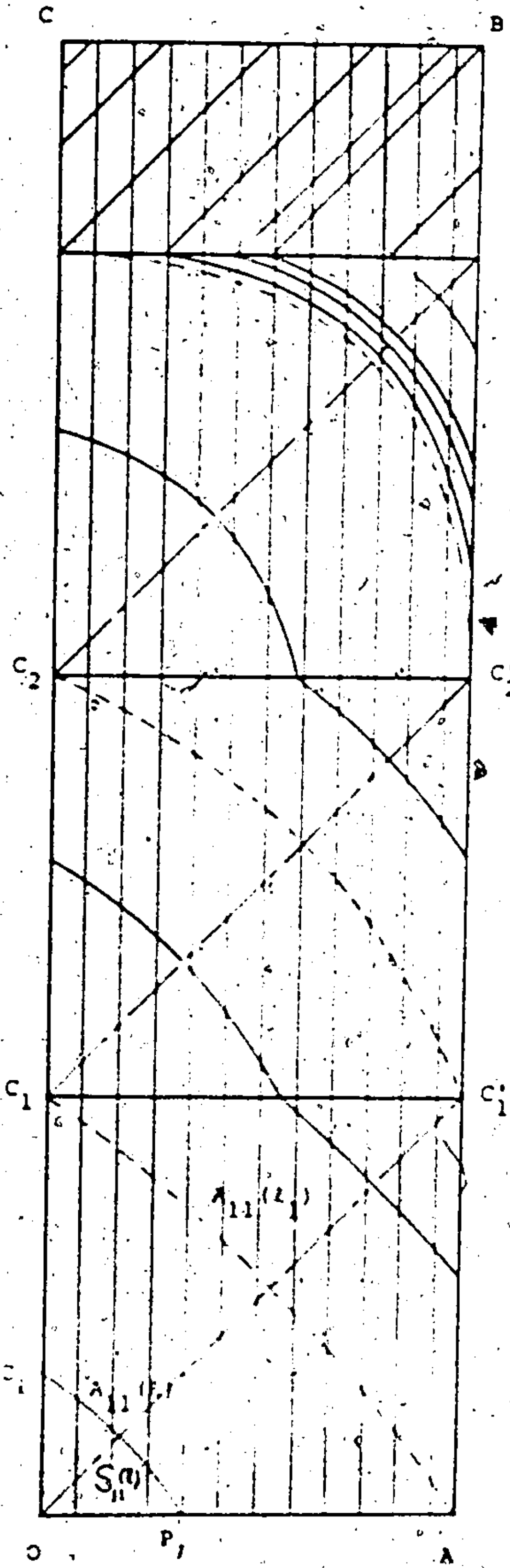

Fig. 58

$S_{11}(0)$ is represented by the origin O.

When $\ell = a$, P_ℓ coincides with A and C_ℓ coincides with C_a. The set $S_{11}(a)$ is bounded by $x = 0$, $y = 0$ and the ellipse $x^2 + y^2 + 2xy \cos 2\delta = a^2$; it is represented by OAC_1 in Fig. 58, where $a = 1$.

2.1.5. Determination of the set $D_1(\ell)$ for $\ell\epsilon[\ell_{k-1},\ell_k]$.

In order to find the distribution function $F_N(\ell)$ of L for $\ell\epsilon[\ell_{k-1},\ell_k]$ we require the set $\{(x,\theta): \quad L(x,\theta) \leq \ell,\ \ell\epsilon[\ell_{k-1},\ell_k]\}$ in the parameter space. In the following lemma we obtain this set.

Lemma 4. Let $\ell\epsilon[\ell_{k-1},\ell_k]$, $k = 2,3,\ldots,n-1$. Then

(2.1.25)
$$D_1(\ell) = S_{k1}(\ell) \cup S_{k2}(\ell) \cup S_{k3}(\ell) \quad ,$$

where

(2.1.26)
$$S_{k1}(\ell) = \{(x,y): \quad 0 \leq x \leq a,\ 0 \leq y \leq y_k(\ell)\},$$

(2.1.27)
$$S_{k2}(\ell) = \{(x,y): x_k(\ell) < x \leq a,\ y_k(\ell) \leq y \leq y_{k1}(x,\ell)\} \ ,$$

(2.1.28)
$$S_{k3}(\ell) = \{(x,y): 0 \leq x \leq x_k(\ell),\ y_k(\ell) \quad \leq y \leq y_{k2}(x,\ell),$$

where

(2.1.29)
$$x_k(\ell) = \frac{\ell \sin\{k\delta-\beta_k(\ell)\}}{\sin k\delta} \ ,$$

(2.1.30)
$$\beta_k(\ell) = \sin^{-1}\left(\frac{\ell_{k-1}\sin k\delta}{\ell}\right) \ ,$$

(2.1.31)
$$\alpha_k(\ell) = 2(k-1)\delta - \beta_k(\ell) \ ,$$

(2.1.32) $$y_k(\ell) = \frac{\ell \sin\{\alpha_k(\ell) - (k-2)\delta\}}{\sin k\delta} + (k-2)a \quad ,$$

(2.1.33) $$y_{k1}(x,\ell) = -d_{k-1} + \frac{\ell \sin(\theta_{k-1}(x,\ell))}{\sin 2(k-1)\delta} + (k-2)a \; ,$$

(2.1.34) $$\theta_{k-1}(x,\ell) = 2(k-1)\delta - \sin^{-1}\{\frac{x+d_{k-1}}{\ell}\sin 2(k-1)\delta\} \; ,$$

(2.1.35) $$y_{k2}(x,\ell) = -d_k + \frac{\ell \sin\theta_k(x,\ell)}{\text{sine } 2k\delta} + (k-1)a \quad ,$$

(2.1.36) $$\theta_k(x,\ell) = 2k\delta - \sin^{-1}(\frac{x+d_k}{\ell}\sin 2k\delta) \; .$$

Proof. (cf. Fig. 59) Let Q_k be the point on $A_{k-1}A_k$ of the polygon such that $|A_0Q_k| = \ell$. Let $\angle Q_kA_0A_1 = \alpha_k$ and $|A_{k-1}Q_k| = y_k(\ell) - (k-2)a$. Then considering the triangle $A_0A_{k-1}Q_k$ we have by the sine law,

(2.1.37) $$\frac{\ell}{\sin k\delta} = \frac{\ell_{k-1}}{\sin(k\delta-\alpha'_k(\ell))} = \frac{y_k(\ell)}{\sin \alpha'_k(\ell)} \quad ,$$

where

$$\alpha'_k(\ell) = \alpha_k(\ell) - (k-2)\delta .$$

From (2.1.37), we obtain (2.1.31) and (2.1.32).

Since (i) $L(x,y) \leq \ell_{k-1} \leq \ell$, for $0 \leq x \leq a$, $0 \leq y \leq (k-2)a$, and (ii) $(k-2)a \leq y \leq y_k(\ell)$ implies $L(x,y) \leq \ell$, by virtue of (2.1.36), it follows that

(2.1.38) $$S_{k1}(\ell) \subset D_1(\ell)$$

Next we show that $S_{k2}(\ell) \subset D_{11}(\ell)$.

With A_k as the centre we draw a circle of radius ℓ (cf. Fig. 59). Since $|A_0A_k| = \ell_k$, $|A_1A_k| = \ell_{k-1}$ and $\ell \in [\ell_{k-1}, \ell_k]$, the circle intersects A_0A_1 at a point $P_k(\ell)$. Let $|A_1P_k(\ell)| = x_k(\ell)$ and $\measuredangle A_kP_k(\ell)A_1 = \beta_k(\ell)$. We recall (1.1.39) and (1.1.40) of Section One of Chapter One and note that

$$(2.1.30) \qquad \beta_k(\ell) = \sin^{-1}\left(\frac{\ell_{k-1}\sin k\delta}{\ell}\right)$$

and

$$(2.1.29) \qquad x_k(\ell) = \frac{\ell \sin\{k\delta - \beta_k(\ell)\}}{\sin k\delta}.$$

Now consider a point $P_{x,k}$ on $A_0P_k(\ell)$ such that $x_k(\ell) \leq x \leq a$ (cf. Fig. 59). Let $Q_k(x,\ell)$ be the point on $A_{k-1}A_k$ whose distance from $P_{x,k}$ is ℓ.

Let $|A_{k-1}Q_k(x,\ell)| = y_{k1}(x,\ell) - (k-2)a$ and $\gamma_{k-1}(x,\ell) = \measuredangle Q_k(x,\ell)P_{x,k}D_{k-1}$, where D_{k-1} is the intersection of the lines A_0A_1 and A_kA_{k-1} produced. Then from the triangle $Q_k(x,\ell)P_{x,k}D_{k-1}$, we have

$$(2.1.39) \qquad \frac{\ell}{\sin 2(k-1)\delta} = \frac{d_{k-1}+x}{\sin[2(k-1)\delta - \gamma_{k-1}(x,\ell)]} = \frac{d_{k-1}+y_{k1}(x,\ell)-(k-1)a}{\sin[\gamma_{k-1}(x,\ell)]}.$$

Simplifying, we obtain (2.1.34) and (2.1.33) from (2.1.39).

By the use of (2.1.39), we also obtain

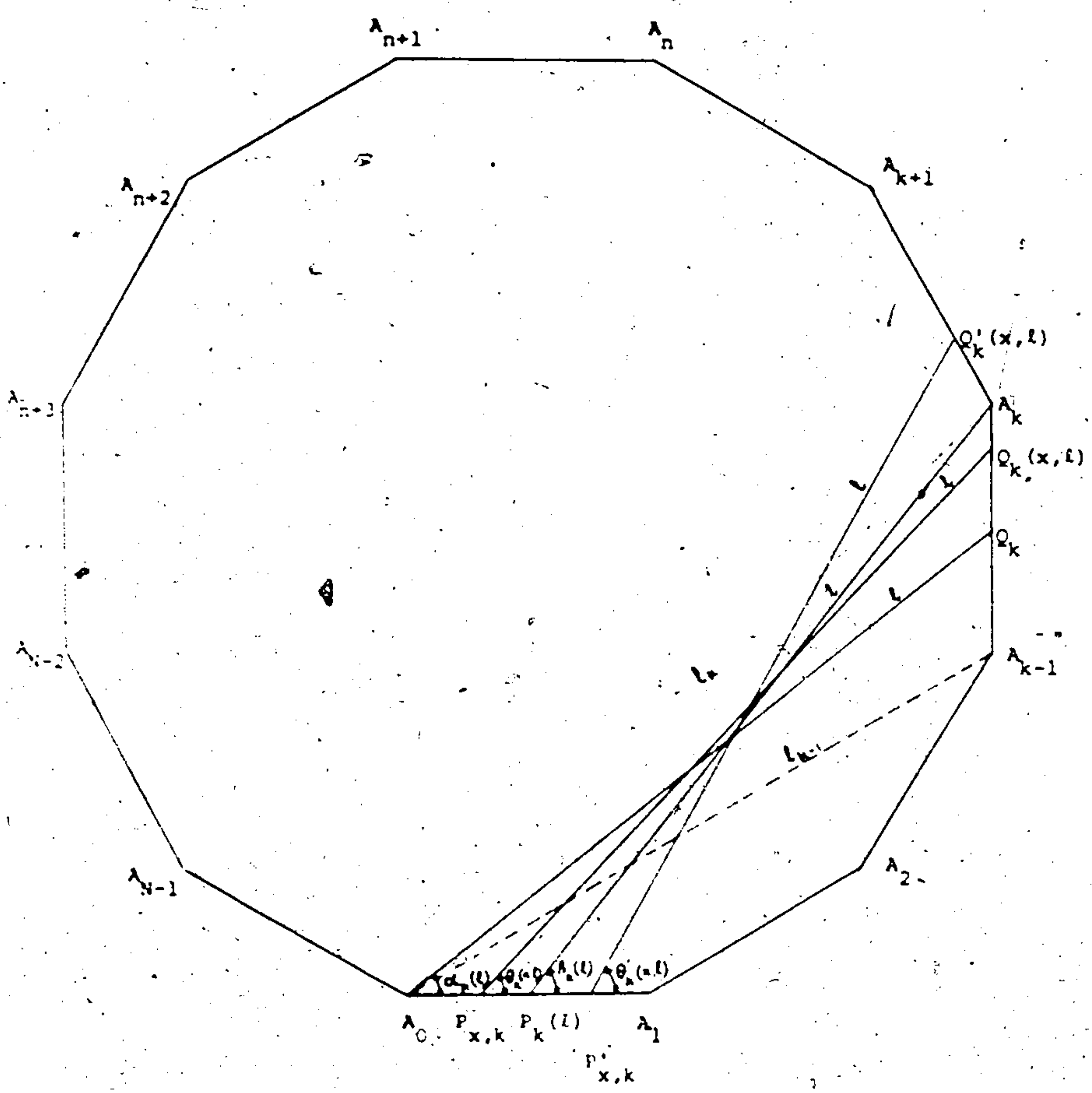

Fig. 59

(2.1.40) $$S_{k2}(\ell) \subset D_1(\ell).$$

Let $P'_{x,k}$ be a point on A_0A_1 such that $0 \leq x \leq x_k(\ell)$. Let $Q'_k(x;\ell)$ be the point on A_kA_{k+1} whose distance from $P'_{x,k}$ is ℓ. Let $\theta_k(x,\ell) = \measuredangle Q'_k(x,\ell)P'_{x,k}D_k$. Let $y_{k2}(x,\ell) = |A_kQ'_k(x,\ell)| + (k-1)a$. Then by considering the triangle $Q_k(x,\ell)P'_{x,k}D_k$ we have by the sine law,

(2.1.41) $$\frac{\ell}{\sin 2k\delta} = \frac{d_k + x}{\sin[2k\delta - \theta_k(x,\ell)]} = \frac{d_k + y_{k2}(x,\ell) - (k-1)a}{\sin \theta_k(x,\ell)}.$$

We now obtain (2.1.35) and (2.1.36) from (2.1.41).

From (2.1.41) it follows that for a fixed $x \in [0, x_k(\ell)]$, $0 \leq y \leq y_{k2}(x,\ell)$ implies $L(x,y) \leq \ell$. Hence

(2.1.42) $$S_{k3}(\ell) \subset D_1(\ell).$$

Combining (2.1.38), (2.1.40) and (2.1.42), we obtain

(2.1.43) $$\bigcup_{i=1}^{3} S_{ki}(\ell) \subset D_1(\ell)$$

In order to show that $D_1(\ell) \subset \bigcup_{i=1}^{3} S_{ki}(\ell)$, let $(x,y) \notin \bigcup_{i=1}^{3} S_{ki}(\ell)$.

Then either (i) $0 \leq x \leq x_k(\ell)$ and $y > (k-1)a + y_{k2}(x,\ell)$

or (ii) $a \geq x > x_k(\ell)$ and $y > (k-2)a + y_{n1}(x,\ell)$.

In both cases (i) and (ii), by the use of (2.1.41), we obtain $L(x,y) > \ell$. It follows that

$$D_1(\ell) \subset \bigcup_{i=1}^{3} S_{ki}(\ell) . \tag{2.1.44}$$

Combining (2.1.43) and (2.1.44), we obtain (2.1.25).

We provide a graphical description of the set $D_1(\ell)$ for $\ell\varepsilon[\ell_{k-1},\ell_k]$ in the following section.

2.1.5A. Graphical description of the set $D_1(\ell)$ for $\ell\varepsilon[\ell_{k-1},\ell_k]$, $k = 2,3,\ldots, n-1$.

For $\ell\varepsilon[_{k-1},\ell_k]$, we have in Lemma 4 of Section 2.1.5

$$D_1(\ell) = S_{k1}(\ell)\cup S_{k2}(\ell)\cup S_{k3}(\ell) ,$$

where

$$S_{k1}(\ell) = \{(x,y): 0 \leq x \leq a,\ 0 \leq y \leq y_k(\ell)\} ,$$

$$S_{k2}(\ell) = \{(x,y): x_k(\ell) \leq x \leq a,\ y_k(\ell) \leq y \leq y_{k1}(x,\ell)\} ,$$

and

$$S_{k3}(\ell) = \{(x,y): 0 \leq x \leq x_k(\ell),\ y_k(\ell) \leq y \leq y_{k2}(x,\ell)\} .$$

Let OABC be the parameter space S (cf. Fig. 60) with a=1. Let C_y denote

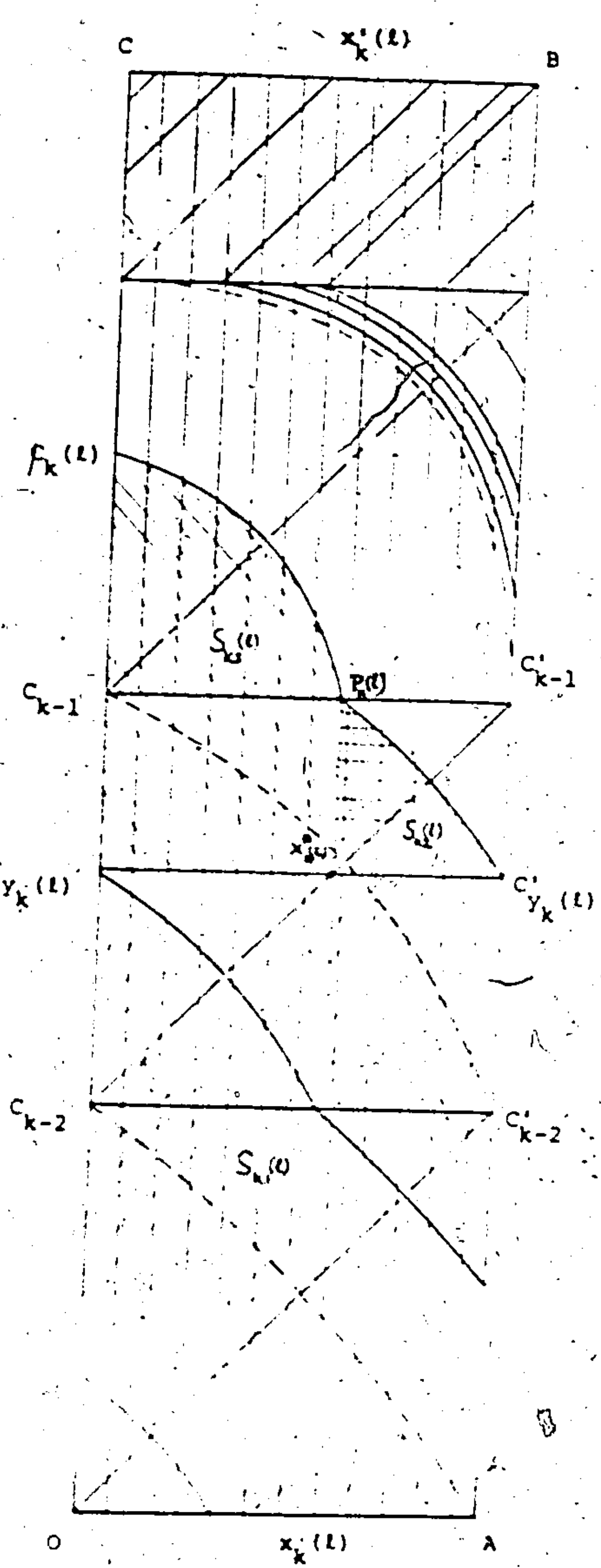

Fig. 60

the point on OC whose distance from O is y. Similarly, let C'_y denote the point on AB whose distance from A is y. We draw parallel lines $C_1C'_1$, $C_2C'_2$ .... at distances 1,2,3,........ units respectively. For each $\ell \in [\ell_{k-1}, \ell_k]$ we have a point $P_k(\ell)$ on A_0A_1 of the polygon (whose distance from A_0 is denoted by $x_k(\ell)$) such that the distance of $P_k(\ell)$ from A_k is ℓ. Corresponding to $x_k(\ell)$, we draw, in the parameter space, a line $X_k(\ell)X'_k(\ell)$ parallel to OC where $X_k(\ell)$ and $X'_k(\ell)$ are points on OA and CB, respectively and such that $|OX_k(\ell)| = |CX_k(\ell)| = x_k(\ell)$. Let $(x_k(\ell), y_k(\ell))$ be the point $P_k(\ell)$ on $X_k(\ell)X'_k(\ell)$, where $x_k(\ell)$ and $y_k(\ell)$ are defined, respectively, by (2.1.29) and (2.1.32). When $\ell = \ell_{k-1}$, the line $X_k(\ell)X'_k(\ell)$ coincides with OC and the point $P_k(\ell)$ with C_{k-1}. As ℓ increases from $\ell = \ell_{k-1}$, the line $X_k(\ell)X'_k(\ell)$ moves parallel to OC away from OC, and since $y_k(\ell)$ remains the same, the point $P_k(\ell)$ on $X_k(\ell)X'_k(\ell)$ moves along $C_{k-1}C'_{k-1}$. When $\ell = \ell_k$, $X_k(\ell)X'_k(\ell)$ coincides with AB and $P_k(\ell)$ coincides with $C'_{(k-1)}$. The locus of $P_k(\ell)$ is the line $C_{k-1}C'_{k-1}$ in the parameter space S. (Note that in S_1-case the corresponding locus of $P_k(\ell)$ is a curve which is given by (1.1.7A.1) of 1.1.)

The set $S_{k1}(\ell)$ is bounded by $x = 0$, $x = a$, $y = 0$ and $y = y_k(\ell)$, and is represented by $OAC'_{y_k(\ell)}C_{y_k(\ell)}$ in Fig. 60.

When $\ell = \ell_{k-1}$, $y_k(\ell) = (k-2)a$. Consequently the set $S_{k1}(\ell_{k-1})$ is represented by $OAC'_{k-2}C_{k-2}$ in Fig. 60.

When $\ell = \ell_k$, $y_k(\ell) = (k-1)a$.

Consequently,

$$S_{k1}(\ell_k) = \{(x,\theta):\ 0 \le x \le a,\ 0 \le y \le (k-1)\,a\}\quad ,$$

which is the rectangle $0\ AC'_{k-1}C_{k-1}$ in the parameter space S, when a=1.

The set $S_{k2}(\ell)$ is bounded by the curves $x = a$, $x = x_k(\ell)$, $y = y_k(\ell)$ and $y = y_{k1}(x,\ell)$, where, from (2.1.33),

$$y_{k1}(x,\ell) = -\,d_{k-1} + \frac{\ell\,\sin\{2(k-1)\delta - \sin^{-1}\left(\frac{x+d_{k-1}}{\ell}\sin 2(k-1)\delta\right)\}}{\sin 2(k-1)\delta}$$

$$+\ (k-2)a\ .$$

The equation $y = y_{k1}(x,\ell)$ reduces to

$$(2.1.5A.1)\quad (x+d_{k-1})^2 + \{y+d_{k-1}-(k-2)a\}^2 + 2\{y+d_{k-1}-(k-2)a\}(x+d_{k-1})\cos 2(k-1)\delta = \ell^2\ .$$

The left hand expression of (2.1.5A.1) is a non-negative quadratic form and therefore represents the equation of an ellipse. The equation (2.1.5A.1) in matrix form is

$$\left(x+d_{k-1}\quad y+d_{k-1}-(k-2)a\right)\begin{pmatrix}1 & \cos 2(k-1)\delta\\ \cos 2(k-1)\delta & 1\end{pmatrix}\begin{pmatrix}x+d_{k-1}\\ y+d_{k-1}-(k-2)a\end{pmatrix} = \ell^2$$

or $N'_{k-1}M_{k-1}N_{k-1} = \ell^2$, writing $N'_{k-1} = (x+d_{k-1}\ \ y+d_{k-1}-(k-2)a)$,

and

$$M_{k-1} = \begin{pmatrix} 1 & \cos 2(k-1)\delta \\ \cos 2(k-1)\delta & 1 \end{pmatrix} .$$

Since M_{k-1} is a non-negative symmetric matrix, there exists a non-singular matrix P such that $PM_{k-1}P'$ is a diagonal matrix D_{k-1}. We choose P to be

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} .$$

Then (with a = 1)

(2.1.5A2) $$\lambda^2 = W'_{k-1} D_{k-1} W_{k-1} .$$

where

$$W_{k-1} = PN_{k-1} = \begin{pmatrix} u_{k-1} \\ v_{k-1} \end{pmatrix} = \begin{pmatrix} \frac{x+y+2d_{k-1}+2-k}{\sqrt{2}} \\ \frac{-x+y+2-k}{\sqrt{2}} \end{pmatrix}$$

and

$$D_{k-1} = \begin{pmatrix} 1 + \cos 2(k-1)\delta & 0 \\ 0 & 1 - \cos 2(k-1)\delta \end{pmatrix}.$$

Therefore the reduced equation (2.1.5A.2) represents the standard form of the equation of the ellipse given by (2.1.5A.1). The equation (2.1.5A.2) can be written as

$$\frac{u_{k-1}^2}{\frac{\ell^2}{1+\cos 2(k-1)\delta}} + \frac{v_{k-1}^2}{\frac{\ell^2}{1-\cos 2(k-1)\delta}} = 1 .$$

The lengths of the major and minor axes are, respectively,

$$\sqrt{\frac{\ell^2}{1-\cos 2(k-1)\delta}} \text{ and } \sqrt{\frac{\ell^2}{1+\cos 2(k-1)\delta}} .$$

The equations of the axes are given by the equations

$$u_{k-1} \equiv x + y + 2d_{k-1} + 2 - k = 0,$$

and

$$v_{k-1} \equiv - x + y + 2 - k = 0.$$

Note that the transformed major axis $v_{k-1} = 0$ passes through $(0,k-2)$ and $(1,k-1)$.

The set $S_{k3}(\ell)$ is bounded by the curves $x = 0$, $x = x_k(\ell)$,

$y = y_k(\ell)$ and $y = y_{k2}(x,\ell)$, where, by (2.1.35),

$$y_{k2}(x,\ell) = -d_k + \frac{\ell \sin \theta_k(x,\ell)}{\sin 2k\delta} + (k-1)a.$$

The equation $y = y_{k2}(\ell)$ reduces to

(2.1.5A.3) $$(x+d_k)^2 + \{y+d_k+(1-k)a\}^2 + 2\,(x+d_k)\,(y+d_k-(k-1)a)\cos 2k\delta = \ell^2.$$

The equation (2.1.5A.3) reduces to

(2.1.5A4) $$\ell^2 = W_k D_k W_k,$$

where

$$W_k = PN_k = \begin{pmatrix} u_k \\ \\ v_k \end{pmatrix} = \begin{pmatrix} \dfrac{x+y+2d_k+1-k}{\sqrt{2}} \\ \\ \dfrac{-x+y+1-k}{\sqrt{2}} \end{pmatrix}$$

and

$$D_k = \begin{pmatrix} 1+\cos 2k\delta & 0 \\ 0 & 1-\cos 2k\delta \end{pmatrix}.$$

The equation (2.1.5A.4) represents an ellipse. The lengths of the axes are $\sqrt{\dfrac{\ell^2}{1-\cos 2k\delta}}$ and $\sqrt{\dfrac{\ell^2}{1+\cos 2k\delta}}$. The equations of the axes are given by

$$u_k = x + y + 2d_k + 1 - k = 0,$$

and

$$v_k = -x + y + 1 - k = 0 .$$

We note that the transformed axis $v_k = 0$ passes through $(0,k-1)$ and $(1,k)$.

For a given $\ell \in [\ell_{k-1}, \ell_k]$, the line $X_k(\ell)X_k'(\ell)$ separates the two sets $S_{k2}(\ell)$ and $S_{k3}(\ell)$. The elliptic curves given by (2.1.5A.1) and (2.1.5A.3) intersect on $X_k(\ell)X_k'(\ell)$ at the point $P_k(\ell)$ whose coordinates are given by $(x_k(\ell), k-1)$ and whose locus is the line $C_{k-1}C_{k-1}'$ in Fig. 60. Now $\ell = \ell_k \Rightarrow x_k(\ell) = a = 1$ and $P_k(\ell)$ coincides with the point C_{k-1}'. Consequently, $S_{k2}(\ell)$ reduces to $(1,k-1)$, and the elliptic curve (2.1.5A.3) which is a bounding curve of $S_{k3}(\ell)$ extends from C_k to C_{k-1}'.

Also, $\ell = \ell_{k-1} \Rightarrow x_k(\ell) = 0$ and $P_k(\ell)$ coincides with C_{k-1}. Consequently, $S_{k3}(\ell)$ reduces to the point $(0,k-1)$, and the elliptic curve (2.1.5A.1) which is a bounding curve of $S_{k2}(\ell)$ extends from a point on AB to a point on OC (i.e., C_{k-2}' to C_{k-1}).

In Fig. 60:

the set $S_{k2}(\ell)$ is represented by $X_k^*(\ell)P_k(\ell)C_{y_k(\ell)}$ where $X_k^*(\ell)$ is the point of intersection of $X_k(\ell)X_k'(\ell)$ and $C_{y_k(\ell)}C_{y_k(\ell)}'$,

the set $S_{k3}(\ell)$ is represented by $C_{y_k(\ell)}X_k^*(\ell)P_k(\ell)C_k(\ell)$, the set $S_{k2}(\ell_k)$ is represented by the point C'_{k-1},

the set $S_{k3}(\ell_k)$ is represented by $C_{k-1}C'_{k-1}C_k$,

the set $S_{k2}(\ell_{k-1})$ is represented by $C_{k-2}C'_{k-2}C_{k-1}$,

and

the set $S_{k3}(\ell_{k-1})$ is represented by C_{k-1}.

2.1.6. Determination of the set $D_1(\ell)$ for $\ell\epsilon[\ell_{n-1},\ell_n]$ where $N = 2n$.

In order to find the distribution $F_N(\ell)$ of L we need the set $\{(x,y): L(x,y) \leq \ell\}$ for $\ell\epsilon[\ell_{n-1},\ell_n]$. In the following lemma we obtain this set.

Lemma 5. Let N, the number of sides of the regular polygon, be even and equal to 2n. Let $\ell\epsilon[\ell_{n-1},\ell_n]$. Then

$$(2.1.45)\quad D_1(\ell) = \begin{cases} (\bigcup_{i=1}^{3} S_i(\ell)) \cup S_{n1}(\ell) \cup S_{n2}(\ell) \cup \dot{S}_{n3}(\ell) \text{ for } \ell\epsilon[\ell_{n-1},\ell_c], \ldots (A) \\ (\bigcup_{i=1}^{3} S_i'(\ell)) \cup S_{n1}(\ell) \cup S_{n2}(\ell) \cup \dot{S}_{n3}(\ell), \text{ for } \ell\epsilon[\ell_c,\ell_b] \ldots\ldots (B) \\ (\bigcup_{i=1}^{2} S_i''(\ell)) \cup S_{n1}(\ell) \cup S_{n2}(\ell) \cup \dot{S}_{n3}(\ell), \text{ for } \ell\epsilon[\ell_b,\ell_n] \ldots\ldots (C), \end{cases}$$

where

$$(2.1.46)\quad \ell_b = \{\ell^2_{n-1} + (\tfrac{a}{2})^2\}^{\frac{1}{2}},\ \ell_c = \{\ell^2_{n-1} + (\tfrac{a}{4})^2\}^{\frac{1}{2}},$$

$$(2.1.47)\quad S_1(\ell) = \{(x,y): 0 \leq x \leq \sqrt{\ell^2-\ell^2_{n-1}},\ (n-1)a \leq y \leq (n-1)a + x + \sqrt{\ell^2-\ell^2_{n-1}}\},$$

$$(2.1.48)\quad S_2(\ell) = \{(x,y): x_n(\ell) \leq x \leq \tfrac{a}{2} - x_n(\ell),\ x - x_n(\ell) + (n-1)a \leq y \leq x + (n-1)a + x_n(\ell)\}$$

(2.1.49) $S_3(\ell) = \{(x,y): \frac{a}{2} - x_n(\ell) \leq x \leq \frac{a}{2} + x_n(\ell),$
$(n-1)a + x - x_n(\ell) \leq y \leq na - \frac{a}{2}\},$

(2.1.50) $S_{n1}(\ell) = \{(x,y): 0 \leq x \leq a,\ 0 \leq y \leq y_{n1}(\ell)\},$

(2.1.51) $S_{n2}(\ell) = \{(x,y): x_n(\ell) \leq x \leq a,\ y_{n1}(\ell) \leq y \leq y_{n2}(\ell,1)\}$.

(2.1.52) $S_{n3}(\ell) = \{(x,y): 0 \leq x \leq x_n(\ell),\ y_{n1}(\ell) \leq y \leq (n-1)a\}$,

(2.1.53) $S_1'(\ell) = \{(x,y): 0 \leq x \leq \frac{a}{2} - x_n(\ell),\ (n-1)a \leq y \leq (n-1)\ a +$
$x + x_n(\ell)\}$,

(2.1.54) $S_2'(\ell) = \{(x,y): \frac{a}{2} - x_n(\ell) \leq x \leq x_n(\ell),\ (n-1)a \leq y \leq (n-1)a + \frac{a}{2}\}$,

(2.1.55) $S_3'(\ell) = \{(x,y): x_n(\ell) \leq x \leq \frac{a}{2} + x_n(\ell),$
$(n-1)a + x - x_n(\ell) \leq y \leq (n - \frac{1}{2})a\}$,

(2.1.56) $S_1''(\ell) = \{(x,y): 0 \leq x \leq x_n(\ell),\ (n-1)a \leq y \leq (n - \frac{1}{2})a\}$,

(2.1.57) $S_2''(\ell) = \{(x,y): x_n(\ell) \leq x \leq a,\ (n-1)a + x - x_n(\ell)$
$\leq y \leq (n - \frac{1}{2})a\}$,

where

(2.1.58) $x_n(\ell) = [\ell^2 - \ell_{n-1}^2]^{\frac{1}{2}}$, for $\ell\epsilon[\ell_{n-1}, \ell_n]$,

$$(2.1.59) \quad y_{n1}(\ell) = \frac{\ell \sin\{\alpha_n(\ell) - (n-2)\delta\}}{\sin n\delta},$$

$$(2.1.60) \quad \alpha_n(\ell) = 2(n-1)\delta - \sin^{-1}\left(\frac{\ell_{n-1}\sin n\delta}{\ell}\right),$$

$$(2.1.61) \quad y_{n2}(x,\ell) = -d_{n-1} + \frac{\ell \sin[\theta_{n-1}(x,\ell)]}{\sin 2(n-1)\delta}, \quad \text{and}$$

$$(2.1.62) \quad \theta_{n-1}(x,\ell) = 2(n-1)\delta - \sin\left[\frac{x+d_{n-1}}{\ell}\sin 2(n-1)\delta\right].$$

Proof. We subdivide the interval $[\ell_{n-1}, \ell_n]$ as follows (cf. Fig. 61). Let B, B' be the mid-points of A_0A_1 and A_nA_{n+1}, respectively. Let C bisect A_1B. Then

$$|A_nC| = [\ell_{n-1}^2 + (\tfrac{a}{4})^2]^{\frac{1}{2}} = |B'C| = \ell_c$$

and

$$|A_nB| = |A_1B'| = [\ell_{n-1}^2 + (\tfrac{a}{2})^2]^{\frac{1}{2}} = \ell_b.$$

Clearly,

$$\ell_{n-1} \le \ell_c \le \ell_b \le \ell_n,$$

and

$$[\ell_{n-1}, \ell_n] = [\ell_{n-1}, \ell_c] \cup [\ell_c, \ell_b] \cup [\ell_b, \ell_n]$$

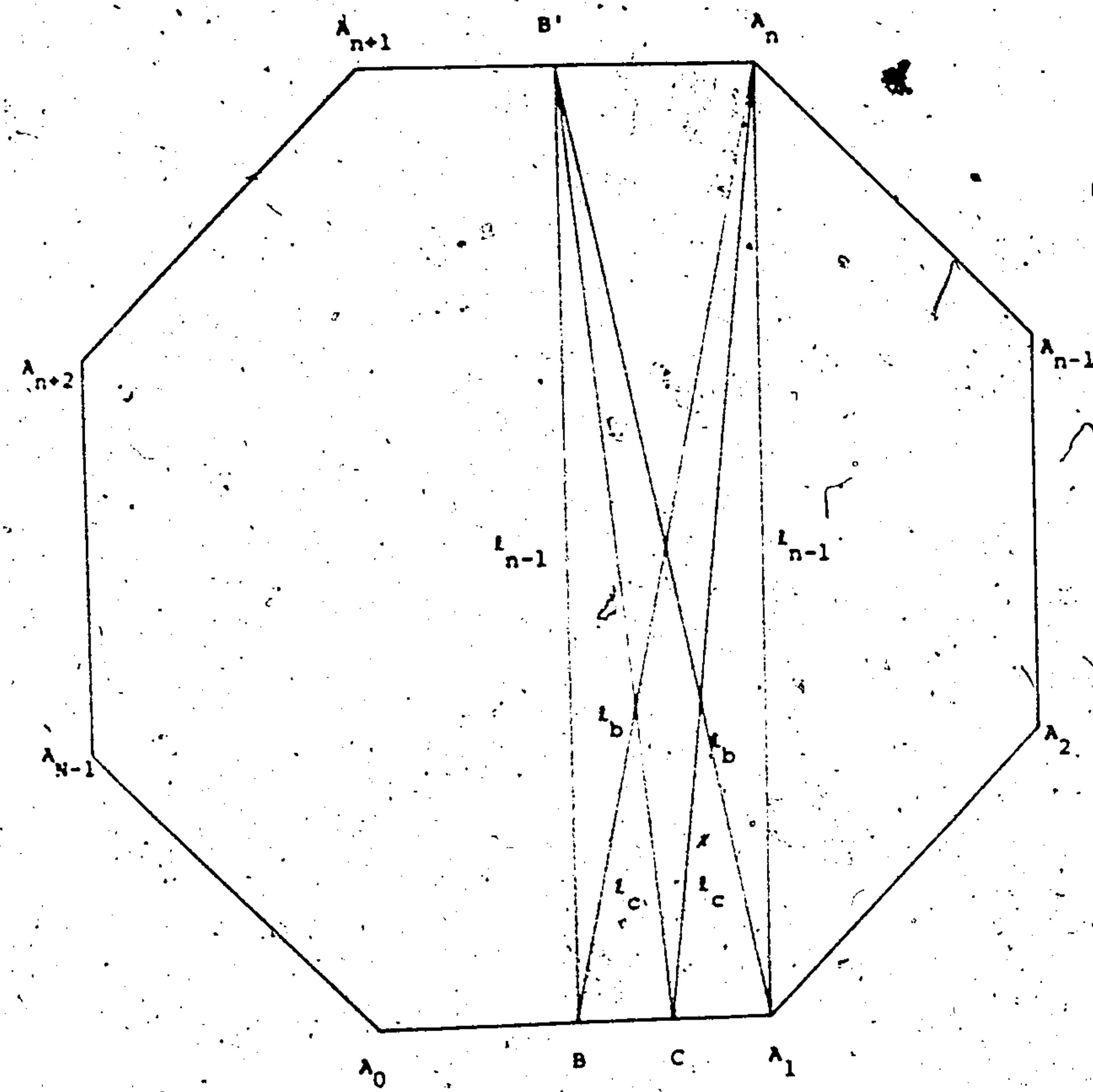

Fig. 61

<u>Proof of (A) of (2.1.1)</u>. Let $\ell \in [\ell_{n-1}, \ell_c]$. Let $P_n(\ell)$ be the point on A_1C such that $|A_nP_n(\ell)| = \ell$. Then

$$|A_1P_n(\ell)| = [\ell^2 - \ell_{n-1}]^{\frac{1}{2}} = x_n(\ell) .$$

Let $P_{x,n}$ be a point on $A_1P_n(\ell)$ such that $0 \leq x \leq x_n(\ell)$ (cf. Fig. 62). With $P_{x,n}$ as the centre we draw a circle C_ℓ of radius ℓ.

The circle intersects A_nA_{n+1} at a point $Q(x,\ell)$. Then

$$|A_nQ(x,\ell)| = x + x_n(\ell) .$$

Since $0 \leq x \leq x_n(\ell)$, A_n lies within the circle C_ℓ. Consequently,

(2.1.63) $$(x,y) \in S_1(\ell) \Rightarrow L(x,y) \leq \ell .$$

Assertion (2.1.63) implies that

(2.1.64) $$S_1(\ell) \subset D_1(\ell).$$

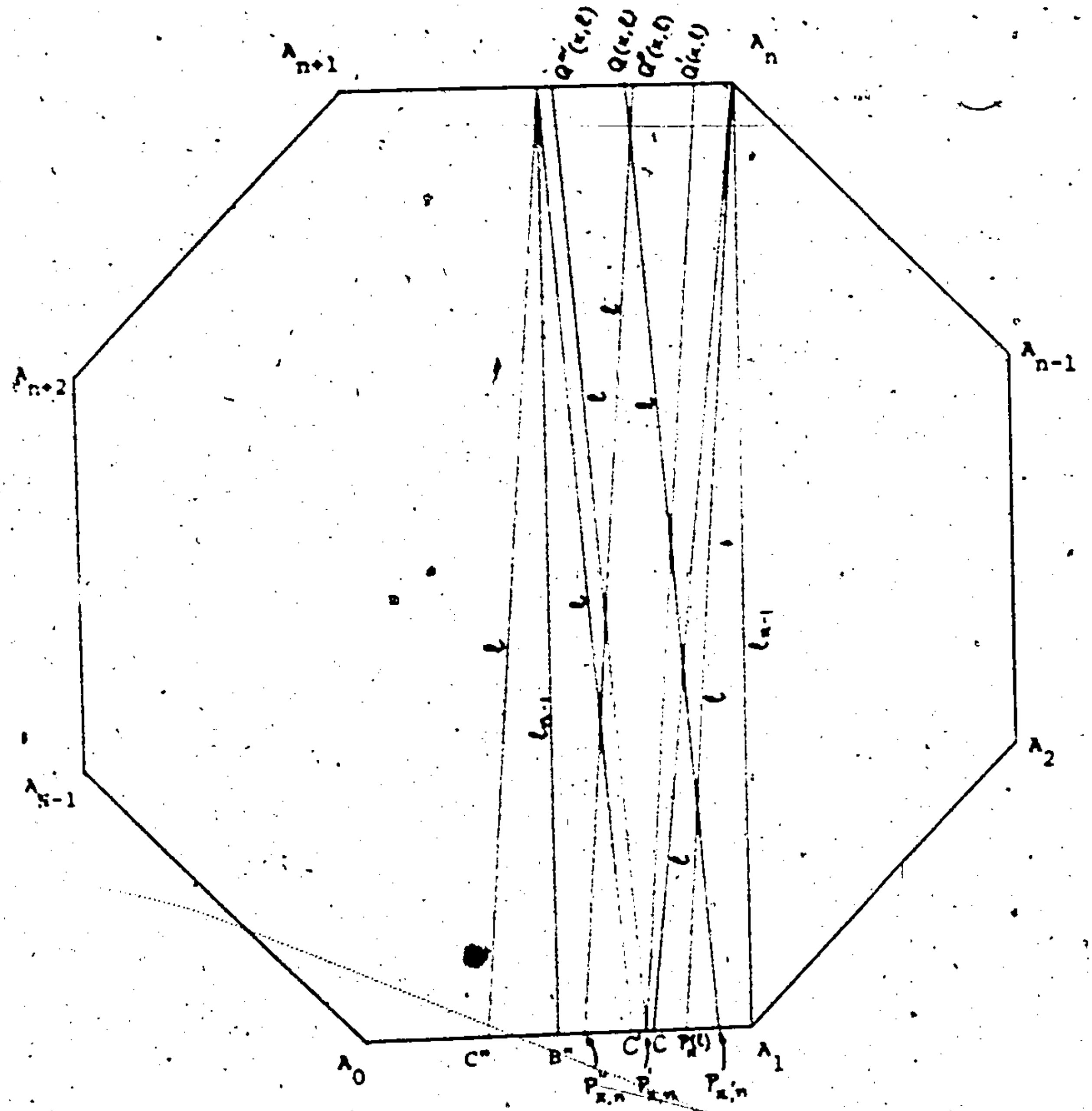

Fig. 62

With B' as the centre draw a circle of radius ℓ. Let C', C" be the points where the circle intersects the side A_0A_1 of the polygon. Then

$$|A_1C'| = \frac{a}{2} - [\ell^2 - \ell_{n-1}^2]^{\frac{1}{2}}, \quad \text{and} \quad |A_1C''| = \frac{a}{2} + [\ell^2 - \ell_{n-1}^2]^{\frac{1}{2}}.$$

Let $P'_{x,n}$ be a point on A_1A_0 of the polygon such that $[\ell^2-\ell_{n-1}^2]^{\frac{1}{2}} \le x \le \frac{a}{2} - [\ell^2-\ell_{n-1}^2]^{\frac{1}{2}}$.

With $P'_{x,n}$ as the centre we draw a circle of radius ℓ. The circle intersects the side A_nA_{n+1} of the polygon at points $Q'(x,\ell)$ and $Q''(x,\ell)$. The points $Q'(x,\ell)$ and $Q''(x,\ell)$ are, respectively, given by

$$y = (n-1)a + x - [\ell^2-\ell_{n-1}^2]^{\frac{1}{2}} \text{ and } (n-1)\,a + x + [\ell^2-\ell_{n-1}^2]^{\frac{1}{2}}.$$

Since the segment $Q'(x,\ell)Q''(x,\ell)$ is within the interior of the circle, evidently $L(x,y) \le \ell$, whenever $(x,y) \in S_2(\ell)$, where

$$(2.1.65) \quad S_2(\ell) = \{(x,y) : x_n(\ell) \le x \le \frac{a}{2} - x_n(\ell),\ (n-1)a + x - x_n(\ell) \le y \le x + x_n(\ell) + (n-1)a\}$$

where $x_n(\ell) = [\ell^2-\ell_{n-1}^2]^{\frac{1}{2}}$.

Thus

(2.1.66) $$S_2(\ell) \subset D_1(\ell).$$

Let $P''_{x,n}$ be a point on A_0A_1 of the polygon such that $\frac{a}{2} - x_n(\ell) \le x \le \frac{a}{2} + x_n(\ell)$, where $x_n(\ell) = [\ell^2 - \ell^2_{n-1}]^{\frac{1}{2}}$. With $P''_{x,n}$ as the centre we draw a circle of radius ℓ. The circle intersects $B'A_n$ at a point $Q''(x,\ell)$. The distance of $Q''(x,\ell)$ from A_n is given by $x - x_n(\ell)$ where $x_n(\ell) = [\ell^2 - \ell^2_{n-1}]^{\frac{1}{2}}$. Clearly $(x,y) \in S_3(\ell)$, where

(2.1.67) $$S_3(\ell) = \{(x,y) : \frac{a}{2} - x_n(\ell) \le x \le \frac{a}{2} + x_n(\ell),$$
$$-(n-1)a + x - x_n(\ell) \le y \le na - \frac{a}{2}\} \Rightarrow L(x,y) \le \ell.$$

Hence

(2.1.68) $$S_3(\ell) \subset D_1(\ell).$$

Also the sets $S_{n1}(\ell)$ and $S_{n2}(\ell)$ defined in (2.1.26) and (2.1.27) (putting $k = n$) are subsets of $D_1(\ell)$, i.e.,

(2.1.69) $$S_{n1}(\ell) \cup S_{n2}(\ell) \subset D_1(\ell).$$

Finally,

(2.1.70) $$S_{n3}(\ell) \subset D_1(\ell),$$

where

(2.1.71) $\quad S_{n3}(\ell) = \{(x,y):\ 0 \le x \le x_n(\ell),\ y_{n1}(\ell) \le y \le (n-1)a\}.$

Since $(x,y)\notin\left(\bigcup_{i=1}^{3} S_i(\ell)\right)\cup S_{n1}(\ell)\cup S_{n2}(\ell)\cup S_{n3}(\ell) \Rightarrow L(x,y) > \ell$, we have

(2.1.72) $\quad D_1(\ell)\subset\left(\bigcup_{i=1}^{3} S_i(\ell)\right)\cup S_{n1}(\ell)\cup S_{n2}(\ell)\cup S_{n3}(\ell).$

Combining (2.1.64), (2.1.66), (2.1.68), (2.1.69), (2.1.70) and (2.1.71), we obtain (A) of (2.1.45).

<u>Proof of (B) of (2.1.1)</u>. Let $\ell\in[\ell_c,\ell_b]$.

Let D_1, D_1', D_2 be the points on A_0A_1 such that $|A_nD_2| = |B'D_1| = |B'D_1'| = \ell$ (cf Fig. 63). Then $|A_1D_2| = [\ell^2-\ell_{n-1}^2]^{\frac{1}{2}} = x_n(\ell)$ and $|A_1D_1| = \frac{a}{2} - x_n(\ell)$.

Let $0 \le x \le \frac{a}{2} - x_n(\ell)$. With P_x as the centre we draw a circle of radius ℓ. The circle intersects A_nB' at a point denoted by $Q(y)$, which is given by $y = x + x_n(\ell) + (n-1)a$. Clearly $(x,y)\in S_1'(\ell)$, where

(2.1.73) $\quad S_1'(\ell) = \{(x,y):\ 0 \le x \le \frac{a}{2} - x_n(\ell),\ (n-1)a \le y \le (n-1)a + x + x_n(\ell)\} \Rightarrow L(x,y) \le \ell.$

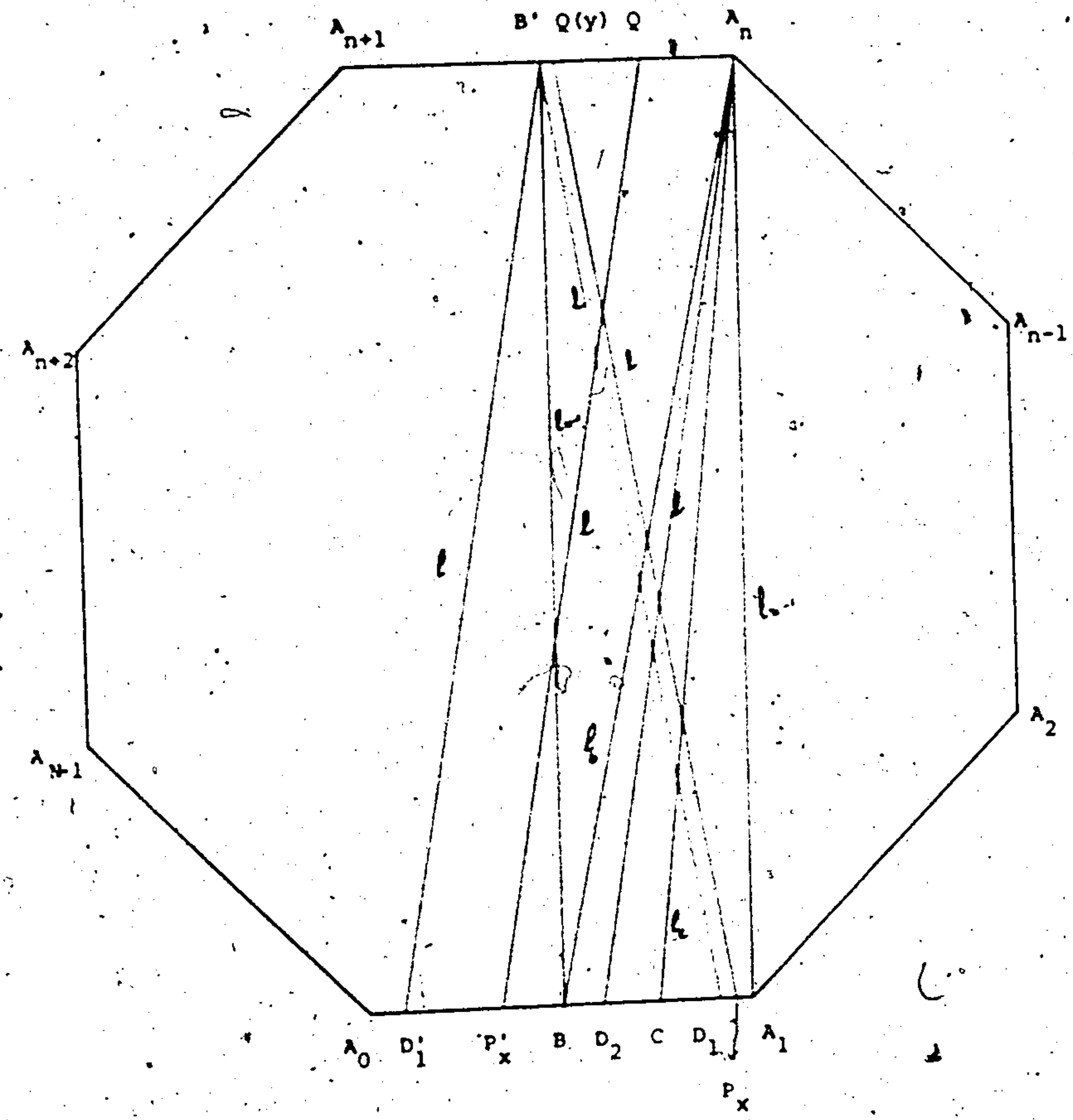

Fig. 63

Hence

$$(2.1.74) \qquad S_2'(\ell) \subset D_1(\ell).$$

Let P_x' be a point within $D_1'D_2$ so that $x_n(\ell) \leq x \leq \frac{a}{2} + x_n(\ell)$. Then the point Q on A_nB' whose distance from P_x' is ℓ, is given by $y = (n-1)a + x - x_n(\ell)$. Hence

$$(2.1.75) \qquad S_3'(\ell) = \{(x,y): x_n(\ell) \leq x \leq \frac{a}{2} + x_n(\ell), \; (n-1)a + x - x_n(\ell) \leq y \leq (n - \frac{1}{2})a\} \subset D_1(\ell).$$

Further, by (2.1.69) and (2.1.70),

$$(2.1.76) \qquad S_{n1}(\ell) \cup S_{n2}(\ell) \cup S_{n3}(\ell) \subset D_1(\ell).$$

Hence using (2.1.73), (2.1.74), (2.1.75) and (2.1.76), we obtain (2.1.45)(B), i.e.,

$$\left(\bigcup_{i=1}^{3} S_i'(\ell)\right) \cup S_{n1}(\ell) \cup S_{n2}(\ell) \cup S_{n3}(\ell) = D_1(\ell).$$

<u>Proof of (C) of (2.1.1).</u> Let $\ell \epsilon [\ell_b, \ell_n]$.

Let D_3 be the point on A_0A_1 such that $|A_nD_3| = \ell$. (cf. Fig. 64).

Let $|A_1D_3| = [\ell^2-\ell_{n-1}^2]^{\frac{1}{2}} = x_n(\ell)$. Clearly

$$(x,y) \in S_1^{*}(\ell) = \{(x,y): 0 \leq x \leq x_n(\ell),\ (n-1)a \leq y \leq (n-\tfrac{1}{2})a\} \Rightarrow L(x,y) \leq \ell.$$

Hence,

(2.1.77) $$S_1^{*}(\ell) \subset D_1(\ell).$$

Let P_x''' be a point on A_0D_3 such that $x_n(\ell) \leq x \leq a$.

Let Q_3 be the corresponding point on A_nB' such that $|Q_3P_x'''| = \ell$. Then Q_3 is given by $y = x - x_n(\ell) + (n-1)a$. Clearly

(2.1.78) $$S_2^{*}(\ell) = \{(x,y): x_n(\ell) \leq x \leq a,\ (n-1)a + x - x_n(\ell) \leq y \leq (n-\tfrac{1}{2})a\}$$

$$\subset D_1(\ell).$$

Thus using (2.1.77), (2.1.78) and (2.1.76) we obtain (2.1.45)(C), i.e.,

$$S_1^{*}(\ell) \cup S_2^{*}(\ell) \cup S_{n1}(\ell) \cup S_{n2}(\ell) \cup S_{n3}(\ell) = D_1(\ell).$$

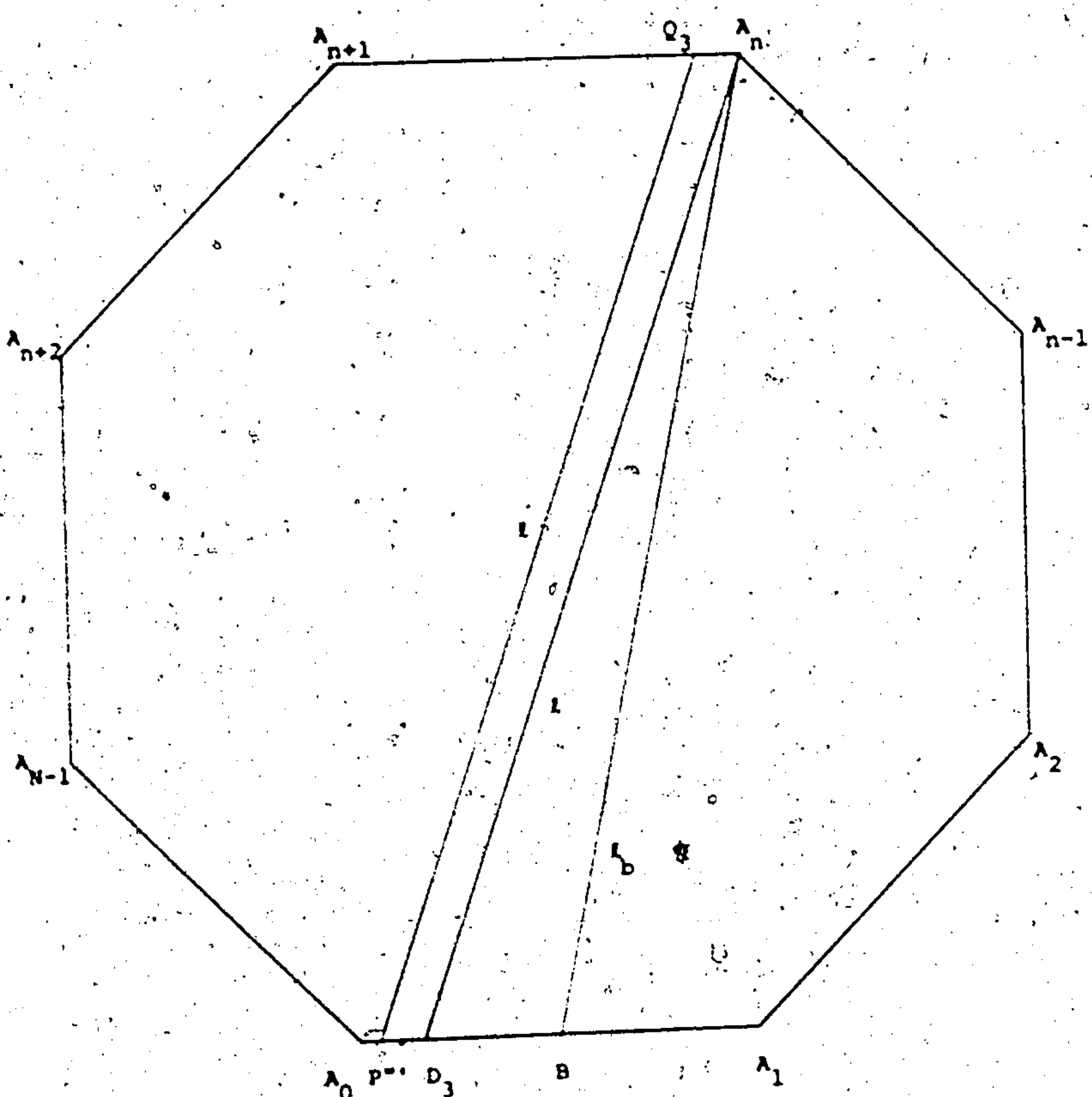

Fig. 64

2.1.6A. Graphical description of the set $D_1(\ell)$ for $\ell\epsilon[\ell_{n-1},\ell_n]$, $N = 2n$.

According to (2.1.45) of Lemma 5, we have

$$D_1(\ell) = \begin{cases} \bigcup_{i=1}^{3} S_i(\ell) \cup S_{n1}(\ell) \cup S_{n2}(\ell) \cup \dot{S}_{n3}(\ell) \quad \ldots\ldots (A) \\ \bigcup_{i=1}^{3} S_i'(\ell) \cup S_{n1}(\ell) \cup S_{n2}(\ell) \cup \dot{S}_{n3}(\ell) \quad \ldots\ldots (B) \\ \bigcup_{i=1}^{2} S_i^*(\ell) \cup S_{n1}(\ell) \cup S_{n2}(\ell) \cup \dot{S}_{n3}(\ell) \quad \ldots\ldots (C) \end{cases}$$

where $S_i(\ell)$, $i = 1, 2, 3$, $S_{ni}(\ell)$, $i = 1, 2$, $\dot{S}_{n3}(\ell)$, $S_{ni}'(\ell)$, $i = 1, 2, 3$, $S_{ni}^*(\ell)$, $i = 1, 2$, are defined by (2.1.47) to (2.1.57), respectively.

We have the following general description of the set $D_1(\ell)$ for $\ell\epsilon[\ell_{n-1},\ell_n]$ in the parameter space S (cf. Fig. 65) with $a = 1$.

For any $\ell\epsilon[\ell_{n-1},\ell_n]$, there is a point $P_n(\ell)$ on A_0A_1 of the polygon such that $|A_nP_n(\ell)| = \ell$. Let $|AP_n(\ell)| = x_n(\ell)$. Let $X_n(\ell)X_n'(\ell)$ be the corresponding line in the parameter space parallel to OC. Let $P_n(\ell)$ denote the point $(x_n(\ell), (n-1))$ in the parameter space. When $\ell = \ell_{n-1}$, $P_n(\ell)$ coincides with the point $C_{(n-1)}$ in S. As ℓ increases from ℓ_{n-1} to ℓ_n, $P_n(\ell)$ moves along $C_{(n-1)}C'_{(n-1)}$ from $C_{(n-1)}$ to $C'_{(n-1)}$. The set $D_1(\ell)$ for $\ell = \ell_{n-1}$, consists of the set $S^*(\ell_{n-1})$ bounded by the elliptic curve $y = y_{n2}(\ell_{n-1})$, which extends from $C_{(n-1)}$ to $C'_{(n-1)}$, $x = 0$, $x = a$ and $y = 0$ and the set $S_1^*(\ell_{n-1})$, forming the linear segment joining $C_{(n-1)}$ to D where D is the mid-point of BC in the parameter space S (cf. Fig. 65). Let $a = 1$. The set

$S^*(\ell_{n-1})$ is represented by $OC_{n-1}C'_{n-2}A$ and $S_1^*(\ell_{n-1})$ is represented by $C_{(n-1)}D$ in Fig. 65.

As ℓ increases from ℓ_{n-1} to ℓ_b, $P_n(\ell)$ moves from $C_{(n-1)}$ to $P_n(\ell_b)$, the line segment $C_{(n-1)}D$ separates into two line segments $P_n(\ell)T_1(\ell)$ and $C(\ell)T_2(\ell)$ which move parallel to $C_{(n-1)}D$ away from it. When $\ell = \ell_b$, $C(\ell)$ and $T_2(\ell)$ coincide with C and $T_1(\ell)$ coincides with B. For a fixed $\ell \epsilon [\ell_{n-1}, \ell_b]$, the set $D_1(\ell)$ is the union of the set $S^*(\ell)$ bounded by $y = 0$, $y = y_{n2}(\ell)$, $x = 0$, $x = a$, and $y = 0$ and the set $S_1^*(\ell)$ bounded by two parallel lines $P_n(\ell)T_1(\ell)$ and $C(\ell)T_2(\ell)$. When $\ell = \ell_b$, the set $D_1(\ell)$ is represented by $S^*(\ell_b)$ and $S_1^*(\ell_b)$ in Fig. 65.

As ℓ increases from ℓ_b to ℓ_n, $P_n(\ell)$ moves towards $C'_{(n-1)}$ and the line segment $P_n(\ell)C'(\ell)$ moves parallel to $P_n(\ell_b)B$. For $\ell = \ell_n$, the line segment $P_n(\ell)C'(\ell)$ is just the point $C'_{(n-1)}$ in Fig. 65.

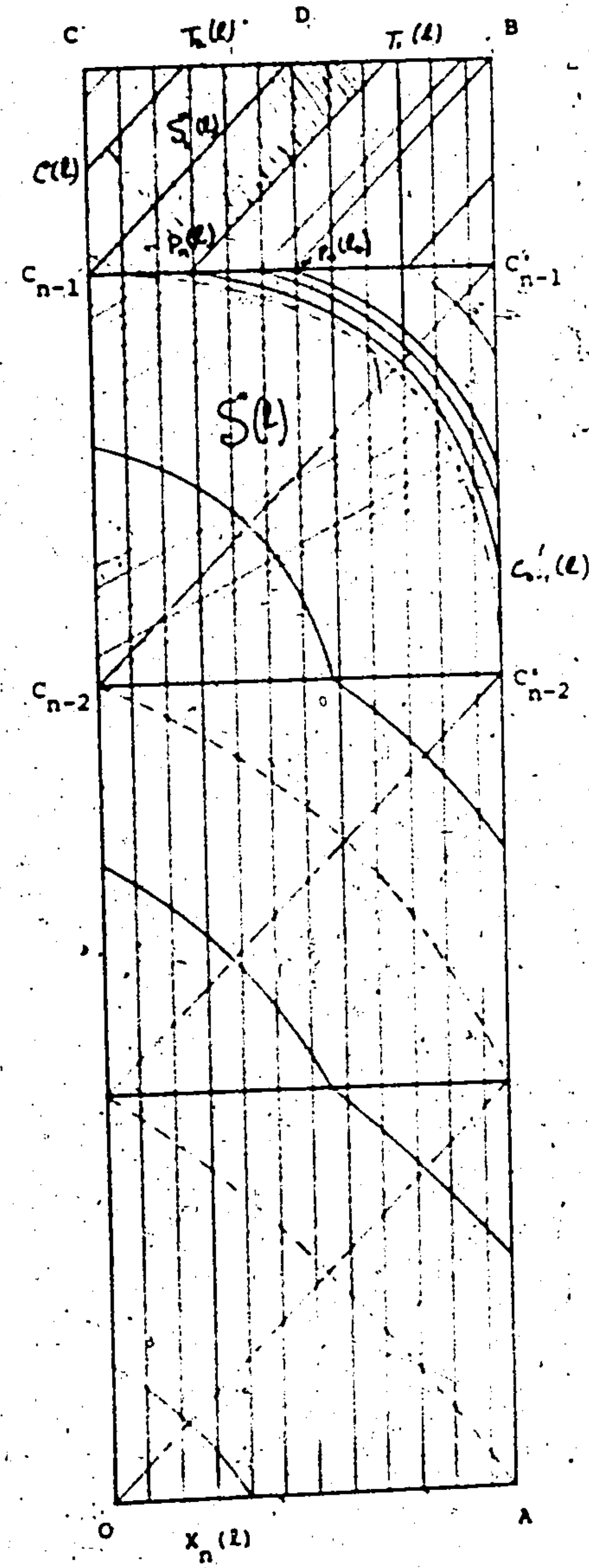

Fig. 65

2.1.7. Determination of the set $D_1(\ell)$ for $\ell \varepsilon [\ell_{n-1}, \ell_n]$, where $N = 2n + 1$.

In order to find the distribution function of L, we need the set $D_1(\ell)$ and to find $D_1(\ell)$ for the case when N, the number of sides of the polygon, is odd, the last interval is subdivided as follows:

$$(2.1.79) \quad [\ell_{n-1}, \ell_n] = [\ell_{n-1}, \ell_{n-1,1}] \cup [\ell_{n-1,1}, \ell_{n-1,2}] \cup [\ell_{n-1,2}, \ell_n]$$

where

$$(2.1.80) \quad \ell_{n-1,1} = - d_n \tan 2n\delta$$

and

$$(2.1.81) \quad \ell_{n-1,2} = [\ell_n^2 - \frac{a^2}{4}]^{\frac{1}{2}}.$$

Draw a line perpendicular to $A_{n+1}A_n$ through A_n (cf. Fig. 66). The perpendicular line intersects A_1A_0 at a point B. The length $|BA_n| = \ell_{n-1,1}$ is now determined by considering the triangle A_nBD_n. The length of the perpendicular (drawn from A_{n+1} to A_0A_1) $|A_{n+1}C| = \ell_{n-1,2}$ is clearly given by $[\ell_n^2 - \frac{a^2}{4}]^{\frac{1}{2}}$.

<u>Lemma 6a.</u> Let N be odd and equal to 2n+1. Let $\ell \varepsilon [\ell_{n-1}, \ell_{n-1,1}]$, where $\ell_{n-1,1}$ is given by (2.1.80). Then

$$(2.1.82) \quad D_1(\ell) = S_1(\ell) \cup S_2(\ell)$$

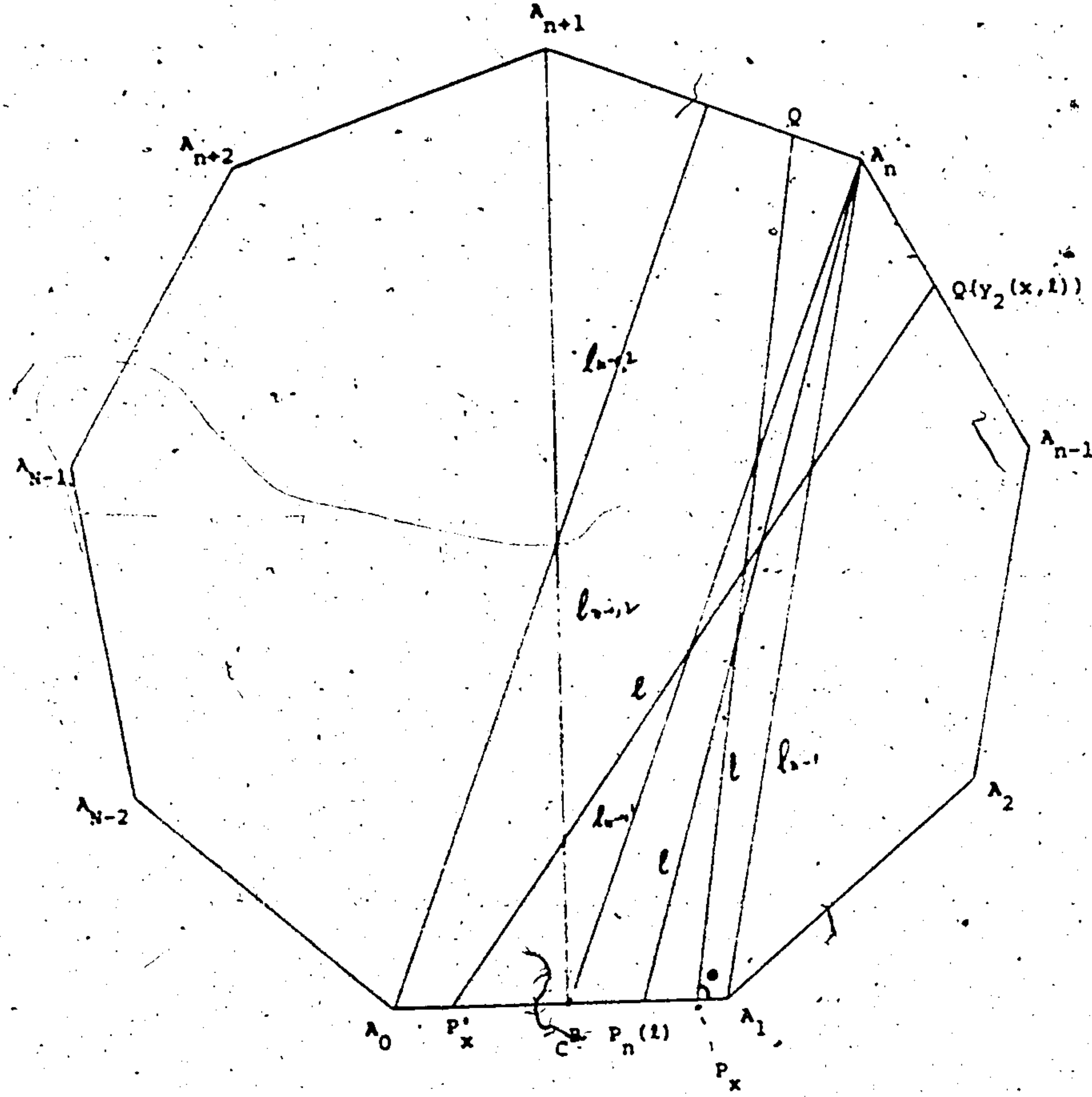

Fig. 66

where

$$S_1(\ell) = \{(x,y): 0 \le x \le c,\ 0 \le y \le y_1(x,\ell)\}\ , \tag{2.1.83}$$

$$S_2(\ell) = \{(x,y): c \le x \le a,\ 0 \le y \le y_2(x,\ell)\}\ , \tag{2.1.84}$$

where

$$c = \frac{\ell}{\sin n\delta} \sin\{ n\delta - \sin^{-1}(\frac{\ell_{n-1}}{\ell} \sin n\delta)\}\ , \tag{2.1.85}$$

$$y_1(x,\ell) = -d_n + (n-1)a + \frac{\ell \sin[2n\delta - \sin^{-1}\{\frac{(d_n+x)\sin 2n\delta}{\ell}\}]}{\sin 2n\delta} \tag{2.1.86}$$

$$y_2(x,\ell) = -d_{n-1} + (n-2)a + \frac{\ell \sin[2(n-1)\delta - \sin^{-1}\{\frac{(x+d_{n-1})\sin 2(n-1)\delta}{\ell}\}]}{\sin 2(n-1)\delta} \tag{2.1.87}$$

<u>Proof.</u> Let $0 \le x \le c$. With P_x on A_0A_1 as the centre and ℓ as the radius we draw a circle (cf. Fig. 66). The circle intersects A_nA_{n+1} at a point Q. Let $y_1(x,\ell)$ be the distance of Q from A_1 along the perimeter of the polygon.

Then considering the triangle QP_xD_n, we have

$$\frac{\ell}{\sin 2n\delta} = \frac{x+d_n}{\sin(2n\delta-\theta)} = \frac{y_1(x,\ell) - (n-1)a}{\sin\theta}\ , \tag{2.1.88}$$

where $\measuredangle QP_xA_1 = \theta$.

It follows from (2.1.88) that

(2.1.89) $$y_1(x,\ell) = -d_n + (n-1)a + \frac{\ell \sin[2n\delta - \sin^{-1}\{\frac{(d_n+x)}{\ell}\sin 2n\delta\}]}{\sin 2n\delta}$$

It follows from (2.1.88) that

(2.1.90) $y\ [0,y_1(x,\ell)] \Rightarrow L(x,y) \leq \ell.$

Hence

(2.1.91) $$S_1(\ell) \subset D_1(\ell).$$

Let $c \leq x \leq a$. With P'_x as the centre we draw a circle of radius ℓ. The circle intersects A_nA_{n-1} at a point $Q(y_2(x,\ell))$, where $y_2(x,\ell)$ is given by

(2.1.87) $$y_2(x,\ell) = -d_{n-1} + (n-2)a + \frac{\ell \sin[2(n-1)\delta - \sin^{-1}\{\frac{x+d_n}{\ell}\sin 2(n-1)\delta\}]}{\sin 2(n-1)\delta}.$$

Recollecting (2.1.26) and (2.1.27) for $k = n$, we obtain

(2.1.92) $$S_2(\ell) \subset D_1(\ell).$$

where $S_2(\ell)$ is defined in (2.1.84). Further

(2.1.93) $$(x,y) \notin D_1(\ell) \Rightarrow L(x,y) > \ell.$$

Hence

$$D_1(\ell) \subset S_1(\ell) \cup S_2(\ell).$$

Using (2.1.91), (2.1.92) and (2.1.93), we obtain (2.1.82).

<u>Lemma 6b.</u> Let $N = 2n+1$. Let $\ell \epsilon [\ell_{n-1,1}, \ell_{n-1,2}]$, where $\ell_{n-1,1}$ and $\ell_{n-1,2}$ are given by (2.1.80) and (2.1.81), respectively. Then

(2.1.94) $$D_1(\ell) = S_1'(\ell) \cup S_2'(\ell) \cup S_3'(\ell) ,$$

where

(2.1.95) $$S_1'(\ell) = \{(x,y): 0 \le x \le c, \ 0 \le y \le y_1(x,\ell)\} ,$$

(2.1.96) $$S_2'(\ell) = \{(x,y): c \le x \le a, \ 0 \le y \le y_2(x,\ell)\} ,$$

(2.1.97) $$S_3'(\ell) = \{(x,y): c \le x \le x_1(\ell), \ y_3(x,\ell) \le y \le y_4(x,\ell)\} ,$$

where $x_1(\ell) = - d_n + \frac{\ell}{\sin 2n\delta}$

and

$$y_4(x,\ell) - y_3(x,\ell) = 2\ell[1 - (\frac{d_n+x}{\ell})^2 \sin^2 2n\delta]^{\frac{1}{2}} .$$

<u>Proof</u>. That the sets $S_1'(\ell) \cup S_2'(\ell) \subset D_1(\ell)$ follows from (2.1.91) and (2.1.92) (cf. Fig. 67).

Perpendicular to A_nA_{n+1}, we draw the secant $P_{x_1(\ell)}Q$ that has

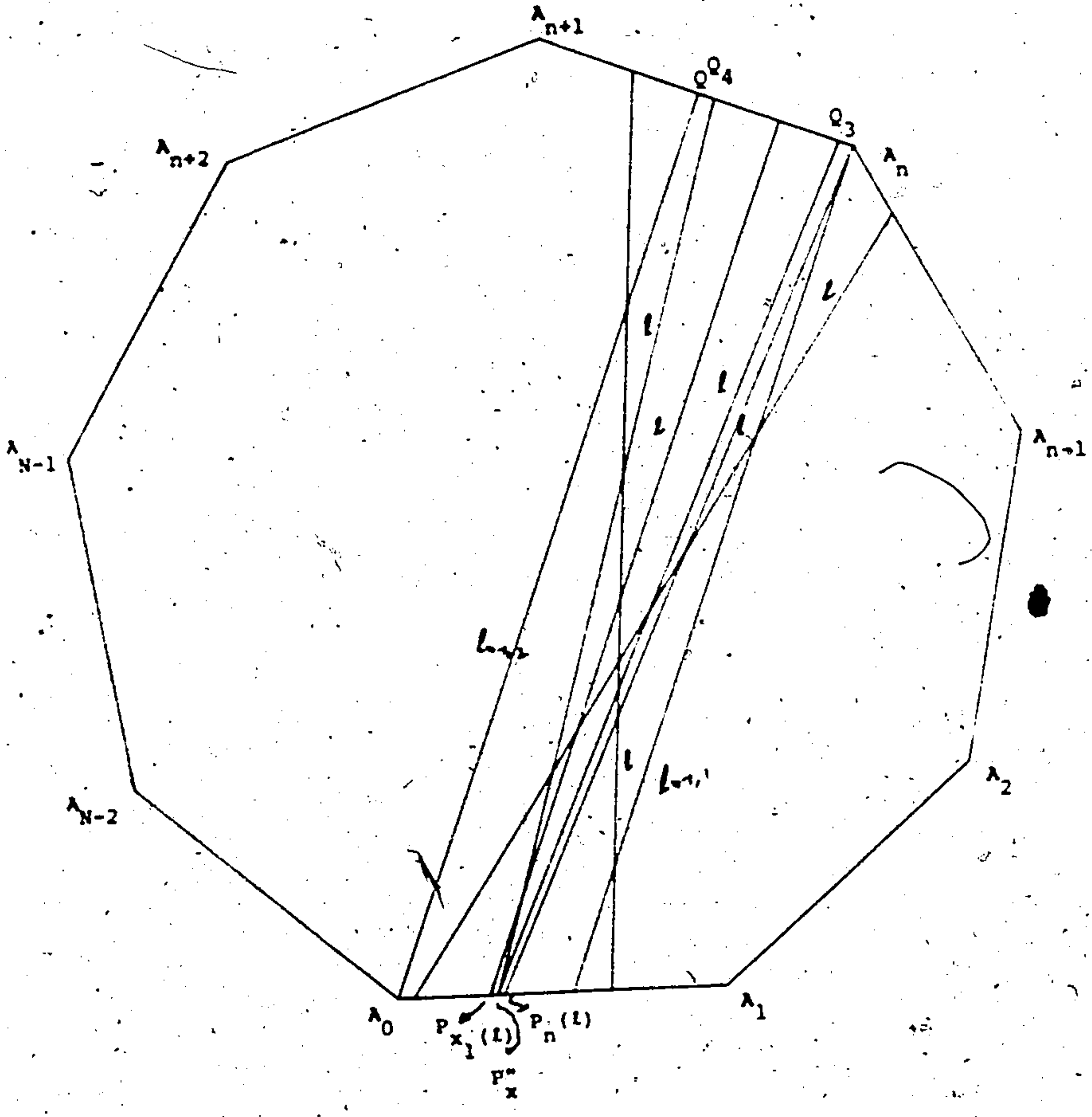

Fig. 67

length ℓ. From the triangle $P_{x_1(\ell)}\, Q\, D_n$, we obtain

(2.1.98) $$x_1(\ell) = -d_n + \frac{\ell}{\sin 2n\delta}.$$

Let $c \le x \le x_1(\ell)$. With $P_x^{"}$ as the centre we draw a circle of radius ℓ. The circle intersects A_nA_{n+1} at Q_3 and Q_4. Let $y_3(x,\ell)$ and $y_4(x,\ell)$ be the y's corresponding to the positions of Q_3 and Q_4, respectively. Then

(2.1.99) $$y_4(x,\ell) - y_3(x,\ell) = |Q_4\, Q_3|$$

$$= 2\ell \left|\cos\left(\sin^{-1}\left(\frac{d_n+x}{\ell}\sin 2n\delta\right)\right)\right|$$

$$= 2\ell \left[1 - \left(\frac{d_n+x}{\ell}\right)^2 \sin^2 2n\delta\right]^{\frac{1}{2}}.$$

Since the line segment Q_4Q_3 is in the interior of the circle $C(P_x^{"},\ell)$, we have

(2.1.100) $$L(x,y) \le \ell, \text{ for } y \in [y_3(x,\ell),\, y_4(x,\ell)].$$

Hence

(2.1.101) $$S_3'(\ell) \subset D_1(\ell).$$

Also, clearly,

(2.1.102) $$D_1(\ell) \subset S_1'(\ell) \cup S_2'(\ell) \cup S_3'(\ell).$$

Lemma 6b now follows from (2.1.91), (2.1.92), (2.1.101) and (2.1.102).

<u>Lemma 6 c.</u> Let $N = 2n+1$. Let $\ell \varepsilon [\ell_{n-1,2}, \ell_n]$.

Then

(2.1.103) $$D_1(\ell) = S_1(\ell) \cup S_2'(\ell) \cup S_3'(\ell) \cup S_4(\ell) \;.$$

where

(2.1.104) $$S_1(\ell) = \{(x,y): 0 \le x \le c_1,\ 0 \le y \le (n-1)a - d_n + \frac{\ell \sin(\theta_n(x))}{\sin 2n\delta}\} ,$$

(2.1.105) $$S_2'(\ell) = \{(x,y): c_1 \le x \le c_2,\ 0 \le y \le na\} ,$$

(2.1.106) $$S_3'(\ell) = \{(x,y): c_2 \le x \le c,\ 0 \le y \le y_{n2}(x,\ell)\} ,$$

(2.1.107) $$S_4(\ell) = \{(x,y): c \le x \le a,\ 0 \le y \le y_{n1}(x,\ell)\}$$

$$\cup \{(x,y): c \le x \le a,\ y_2(x,\ell) \le y \le y_3(x,\ell)\} ,$$

where $c_1 = \frac{a}{2} - [\ell^2 - \ell^2_{n-1}]^{\frac{1}{2}}$, $c_2 = \frac{a}{2} + [\ell^2 - \ell^2_{n-1,2}]^{\frac{1}{2}}$.

and

$y_{n2}(x,\ell)$, $y_{n1}(x,\ell)$ are given by (2.1.35) and (2.1.33), respectively,

(2.1.108) $$y_3(x,\ell) - y_2(x,\ell) = 2\ell[1 - (\frac{d_n + x}{\ell} \sin 2n\delta)^2]^{\frac{1}{2}} ,$$

<u>Proof</u>. With A_n as the centre, we draw a circle of radius ℓ (cf. Fig. 68).

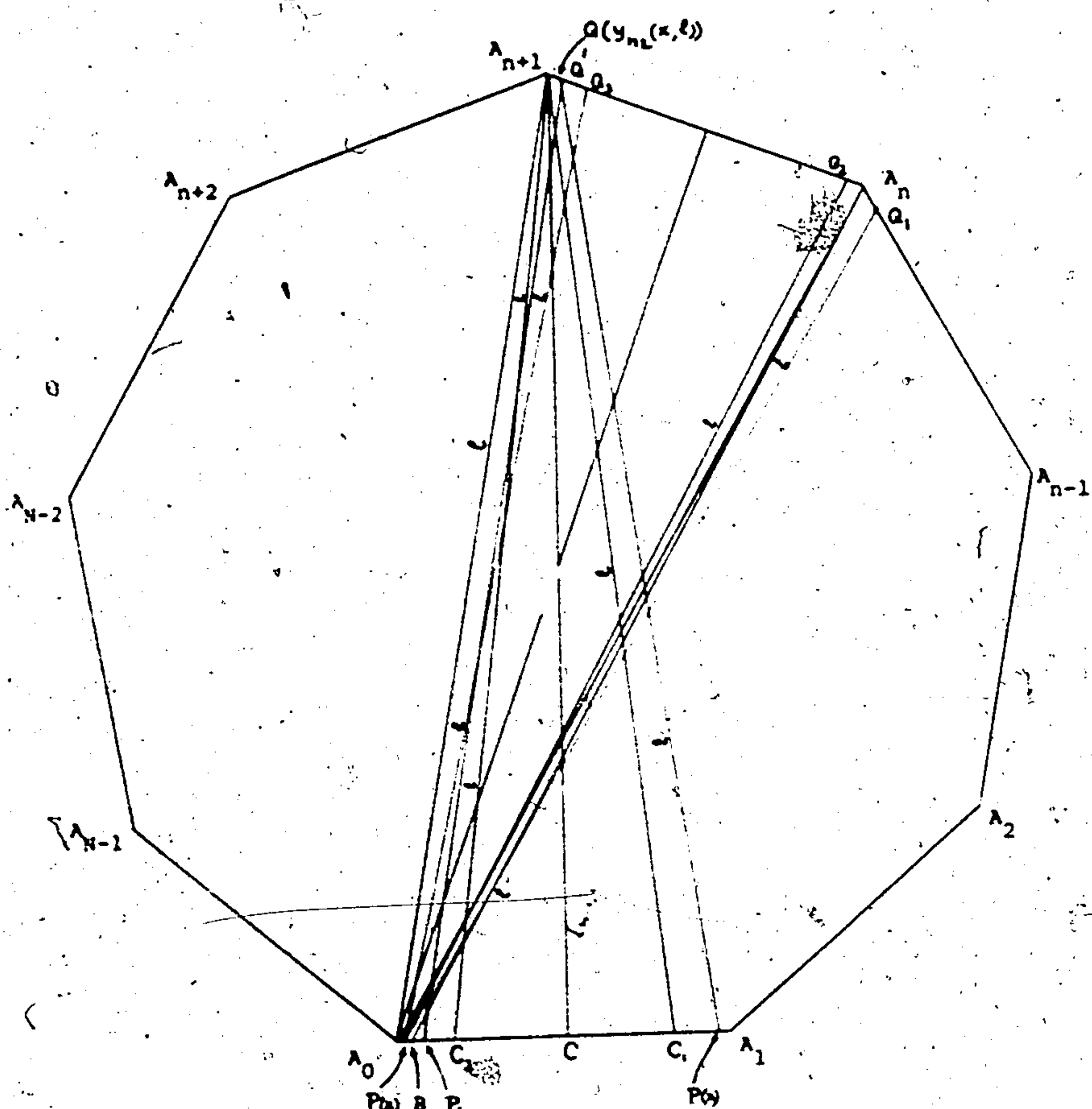

Fig. 68

Let the circle intersect A_0A_1 at B_1. Let $|A_1B_1| = c$. With A_{n+1} as the centre draw a circle of radius ℓ. Let this circle intersect A_0A_1 at C_1 and C_2. It follows that $c > c_2 > c_1$, where $|A_1C_1| = c_1$ and $|A_1C_2| = c_2$. From the triangles A_nCC_1 and A_nCC_2, we have

(2.1.109) $$c_1 = \frac{a}{2} - [\ell^2 - \ell^2_{n-1,2}]^{\frac{1}{2}} ,$$

(2.1.110) $$c_2 = \frac{a}{2} + [\ell^2 - \ell^2_{n-1,2}]^{\frac{1}{2}} .$$

Let $0 \le x \le c_1$. With $P(x)$ as the centre we draw a circle of radius ℓ. The circle will intersect A_nA_{n+1} at a point $Q(y_{n2}(x,\ell))$, given by

$$y_{n2}(x,\ell) = (n-1)a - d_n + \frac{\ell \sin (\theta_n(x))}{\sin 2n\delta} .$$

Then

(2.1.111) $$S_1(\ell) = \{(x,y) : 0 \le x \le c_1, \ 0 \le y \le (n-1)a - d_n + \frac{\ell \sin (\theta_n(x))}{\sin 2n\delta}\} \subset D_1(\ell) ,$$

by (2.1.91).

Let $c_1 \le x \le c_2$, where c_1 and c_2 are given by (2.1.109) and (2.1.110). Then since $c > c_2$,

$$y \in [0,na] \Rightarrow L(x,y) \leq \ell.$$

Thus

(2.1.112) $S_2'(\ell) = \{(x,y): c_1 \leq x \leq c_2,\ 0 \leq y \leq na\} \subset D_1(\ell).$

Let $c_2 \leq x \leq c$.

By (2.1.83),

(2.1.113) $S_3'(\ell) = \{(x,y): c_2 \leq x \leq c,\ 0 \leq y \leq y_{n2}(x,\ell)\} \subset D_1(\ell)$.

Let $c \leq x \leq a$.

With $P(x)$ as the centre, we draw a circle of radius ℓ. The circle intersects $A_{n-1}A_n$ and A_nA_{n+1} at points Q_1, Q_2, Q_3, giving $y_{n1}(x,\ell)$, $y_2(x,\ell)$, $y_3(x,\ell)$, where $y_{nP}(x,\ell)$ is given by (2.1.33).

To find $y_2(x,\ell)$ and $y_3(x,\ell)$ we consider the triangle $P(x)Q_2Q_3$.

Clearly

$$\angle PQ_3D_n = 2n\delta - \theta_n(x), \text{ where } \angle Q_3PD_n = \theta_n(x). \quad (P \equiv P(x)).$$

From the triangle Q_3PD_n we have

(2.1.114) $$\ell = \frac{(d_n+x)\,\sin 2n\delta}{\sin(2n\delta - \theta_n(x))},$$

so that

$$\theta_n(x) = 2n\delta - \sin^{-1}\left(\frac{d_n+x}{\ell}\sin 2n\delta\right).$$

$$\measuredangle PQ_2D_n = \pi - \sin^{-1}\left(\frac{d_n+x}{\ell}\sin 2n\delta\right).$$

(2.1.115) $$y_3(x,\ell)-(y_2(x,\ell) = |Q_2Q_3| = 2\ell\cos\left[\quad\sin^{-1}\left(\frac{d_n+x}{\ell}\sin 2n\delta\right)\right]$$

$$= 2\ell\left[1 - \left(\frac{d_n+x}{\ell}\sin 2n\delta\right)^2\right]^{\frac{1}{2}}.$$

Hence we have the set

(2.1.107) $$S_4(\ell) = S_{31}(\ell)\cup S_{32}(\ell)$$

$$= \{(x,y) : c \le x \le a,\ 0 \le y \le y_{n1}(x,\ell)\}$$

$$\cup\ \{(x,y) : c \le x \le a,\ y_2(x,\ell) \le y \le y_3(x,\ell)\},$$

where $y_3(x,\ell) - y_2(x,\ell) = 2\ell[1 - (\frac{d_n+x}{\ell}\sin 2n\delta)^2]^{\frac{1}{2}}$. Clearly it follows that

(2.1.116) $S_4(\ell) \subset D_1(\ell)$

and

(2.1.117) $D_1(\ell) \subset S_1(\ell)\cup S_2'(\ell)\cup S_3'(\ell)\cup S_4(\ell)$.

Using (2.1.111), (2.1.112), (2.1.113), (2.1.114), (2.1.115), and (2.1.116), we obtain (2.1.103).

2.1.7A. Graphical description of the set $D_1(\ell)$ for $\ell\in[\ell_{n-1},\ell_{n-1,1}]$ where $N = 2n+1$.

For $\ell\in[\ell_{n-1},\ell_{n-1,1}]$, we have by Lemma 6a,

$$D_1(\ell) = S_1(\ell)\cup S_2(\ell) \quad , \tag{2.1.82}$$

where

$$S_1(\ell) = \{(x,y): \ 0 \le x \le c,\ 0 \le y \le y_1(x,\ell)\} \tag{2.1.83}$$

and

$$S_2(\ell) = \{(x,y): \ c \le x \le a,\ 0 \le y \le y_2(x,\ell)\} \quad , \tag{2.1.84}$$

where

$$c = \frac{1}{\sin n\delta}\sin[n\delta - \sin^{-1}\{(\frac{\ell_{n-1}}{\ell}\sin n\delta)\}] \quad , \tag{2.1.85}$$

$$y_1(x,\ell) = -d_n + (n-1)a + \frac{\ell}{\sin 2n\delta}\quad \sin[2n\delta - \sin^{-1}\frac{(d_n+x)\sin 2n\delta}{\ell}] \, , \tag{2.1.86}$$

and

$$y_2(x,\ell) = -d_{n-1} + (n-2)a + \frac{\ell}{\sin 2(n-1)\delta}\sin[2(n-1)\delta \tag{2.1.87}$$

$$- \sin^{-1}\{\frac{(x+d_{n-1})\sin 2(n-1)\delta}{\ell}\}] \quad .$$

For $\ell\in[\ell_{n-1},\ell_n]$, there is a point $X_n(\ell)$ on A_0A_1 of the polygon and a corresponding line $X_n(\ell)X_n'(\ell)$ parallel to OC in the parameter space S (cf. Fig. 69A). In the parameter space S, for $\ell\in[\ell_{n-1},\ell_{n-1,1}]$, the set $S_1(\ell)$ is bounded by $x = 0$, $x = x_n(\ell)$, $y = 0$ and $y = y_1(x,\ell)$ which represents the ellipse $y = y_1(x,\ell)$.

In S, the set $S_2(\ell)$ is bounded by $x = a$, $x = x_n(\ell)$, $y = 0$ and $y = y_2(x,\ell)$ which represents the ellipse

$$(x+d_{n-1})^2 + (y+d_{n-1} - (n-2)a)^2 + 2(x+d_{n-1})(y+d_{n-1} - (n-2)a)\cos 2(n-1)\delta = \ell^2 .$$

The set $S_2(\ell)$ is represented by $X_n(\ell)P_n(\ell)C'_{n-1}(\ell)A$ in Fig. 69A. When $\ell = \ell_{n-1}$, $S_1(\ell)$ reduces to the line segment OC_{n-1} and $S_2(\ell)$ is represented by $OC_{n-1}C'_{n-2}$ (in Fig. 69A, C'_{n-2} is the same point as A). In S, the set $S_1(\ell)$ is represented by $OX_n(\ell)P_n(\ell)C_n(\ell)$.

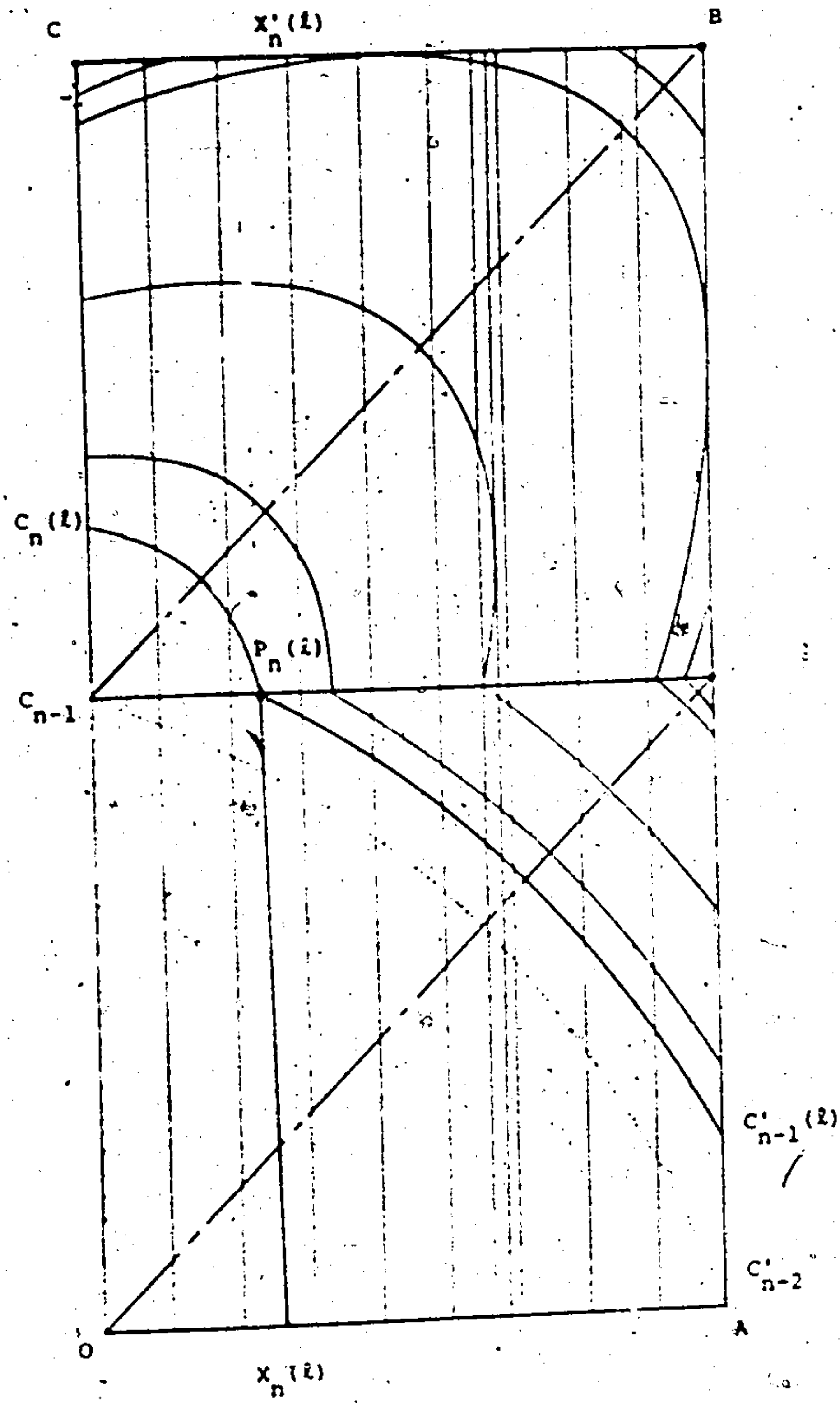

Fig. 69A

2.1.7B. Graphical description of the set $D_1(\ell)$ for $\ell\in[\ell_{n-1,1},\ell_{n-1,2}]$.

For $\ell\in[\ell_{n-1,1},\ell_{n-1,2}]$, we have by Lemma 6b of this chapter,

$$D_1(\ell) = S_1'(\ell)\cup S_2'(\ell)\cup S_3'(\ell) , \tag{2.1.94}$$

where

$$S_1'(\ell) = \{(x,y):\ 0 \le x \le c,\ 0 \le y \le y_1(x,\ell)\} , \tag{2.1.95}$$

$$S_2'(\ell) = \{(x,y):\ c \le x \le a,\ 0 \le y \le y_2(x,\ell)\} , \tag{2.1.96}$$

$$S_3'(\ell) = \{(x,y):\ c \le x \le x_1(x,\ell),\ y_3(x,\ell) \le y \le y_4(x,\ell)\} , \tag{2.1.97}$$

where,

$$x_1(\ell) = - d_n + \frac{\ell}{\sin 2n\delta}$$

and

$$y_4(x,\ell) - y_3(x,\ell) = 2\ell[1 - (\frac{d_n+x}{\ell})^2 \sin^2 2n\delta]^{\frac{1}{2}} .$$

In the parameter space S (cf. Fig. 69B) the set $D_1(\ell)$ for $\ell\in[\ell_{n-1,1},\ell_{n-1,2}]$ is bounded by the elliptic curves $y = y_1(x,\ell)$ and $y = y_2(x,\ell)$, $x = 0$, $x = a$ and $y = 0$ and is represented by $OC_n(\ell)P_n(\ell)C'_{n-1}(\ell)A$. We recall that for $\ell\in[\ell_{n-1,1},\ell_{n-1,2}]$, there exists a point P on A_0A_1 of the polygon such that a secant PQ is perpendicular to A_nA_{n+1} and $|PQ| = \ell$. We take $|A_1P| = x_1(\ell)$. We note that for $c \le x \le x_1(\ell)$, the line $X(\ell)X'(\ell)$ which is parallel to OC and such that $|OX(\ell)| = x$, intersects the ellipse $y = y_1(x,\ell)$ at two points $(x,y_3(x,\ell))$ and $(x,y_4(x,\ell))$. When $x = x_1(\ell)$, the line $X(\ell)X'(\ell)$ is tangent to the ellipse. When $\ell = \ell_{n-1,1}$ the two points $(x,y_3(x,\ell))$ and $(x,y_4'(x,\ell))$ coincide with $(-d_n + \frac{d_n}{\cos 2n\delta}, (n-1)a)$. For $\ell = \ell_{n-1,2}$, the lines $x = a$ and $y = na$ are tangent to $y = y_1(x,\ell)$ at the points $(a, (n-1)a + a/2)$ and $(a/2, na)$, respectively (cf. Fig. 69B, where $a = 1$).

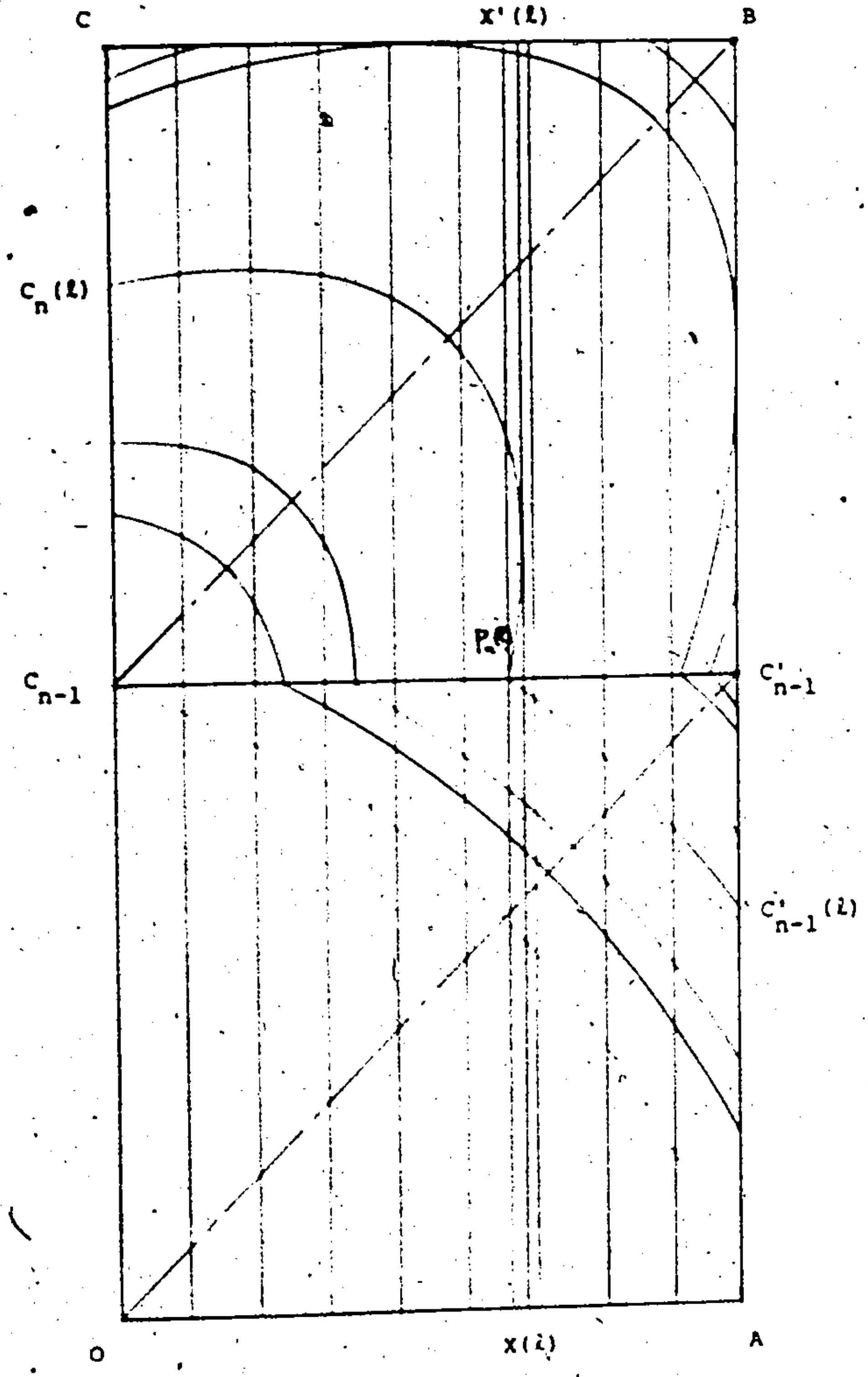

Fig. 69B

2.1.7C. Graphical description of the set $D_1(\ell)$ for $\ell\in[\ell_{n-1,2},\ell_n]$, when $N = 2n + 1$.

For $\ell\in[\ell_{n-1,2},\ell_n]$, we have, by 2.1.6, Lemma 6c,

(2.1.103) $D_1(\ell) = S_1(\ell)\cup S_2'(\ell)\cup S_3'(\ell)\cup S_4(\ell)$,

where

(2.1.104) $S_1(\ell) = \{(x,y):\ 0 \le x \le c_1,\ 0 \le y \le y_{n2}(x,\ell)\}$,

(2.1.105) $S_2'(\ell) = \{(x,y):\ c_1 \le x \le c_2,\ 0 \le y \le na\}$,

(2.1.106) $S_3'(\ell) = \{(x,y):\ c_2 \le x \le c,\ 0 \le y \le y_{n2}(x,\ell)\}$,

(2.1.107) $S_4(\ell) = \{(x,y):\ c \le x \le a,\ 0 \le y \le y_{n1}(x,\ell)\}$

$$\cup\{(x,y):\ c \le x \le a,\ y_2(x,\ell) \le y \le y_3(x,\ell)\}\ ,$$

where

$$c_1 = \frac{a}{2} - [\ell^2 - \ell_{n-1}^2]^{\frac{1}{2}},\quad c_2 = \frac{a}{2} + [\ell^2 - \ell_{n-1,2}^2]^{\frac{1}{2}}\ ;$$

$$y_3(x,\ell) - y_2(x,\ell) = 2\ell[1 - (\frac{d_n+x}{\ell}\sin 2n\delta)^2]^{\frac{1}{2}}\ .$$

In the parameter space S (cf. Fig. 70), the set

$S_1(\ell)$ is represented by $OA_1C_1C_n^*(\ell)$, where $|OA_1| = c_1 = |CC_1|$,

$S_2'(\ell)$ is represented by $A_1C_1C_2A_2$, where $|OA_2| = c_2 = |CC_2|$,

$S_3(\ell)$ is represented by $A_2C_2RX_n(\ell)$.

$S_4(\ell)$ is represented by $X_n(\ell)AC'_{n-1}(\ell)P_n(\ell)$ and $P_n(\ell)C'_n(\ell)C''_n(\ell)R$.

When $\ell = \ell_n$, clearly $D_1(\ell)$ represents the points (a, (n-1)a), (a, na), and (0, na) in S (cf. Fig. 70), where a = 1.

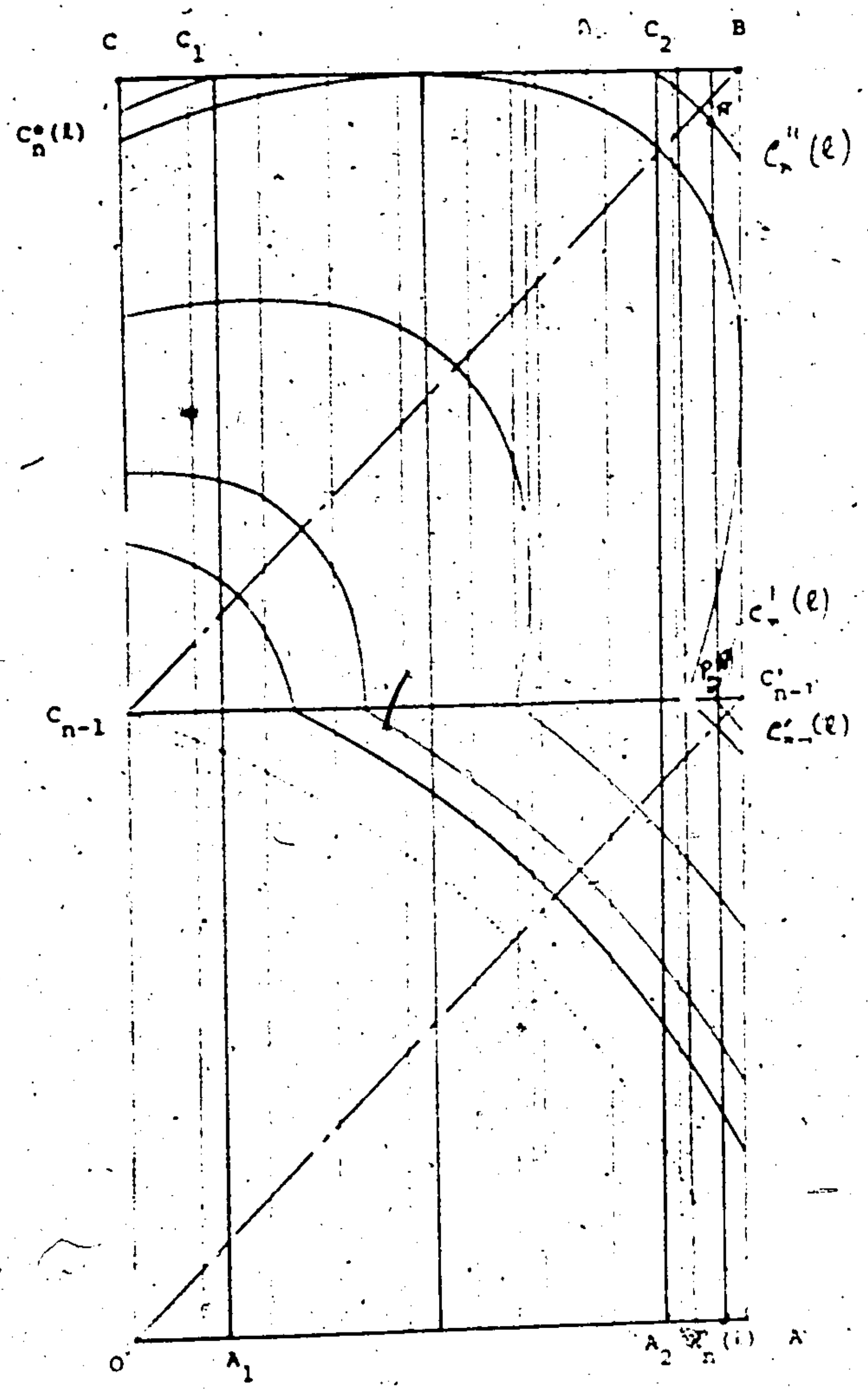

Fig. 70

2.1.8. Proof of the theorem.

By Lemma 3 for $\ell\varepsilon[0,\ell_1]$, we have

$$D_1(\ell) = \{(x,y): 0 \le x \le \ell,\ 0 \le y \le \frac{\ell \sin\{(2\delta \sin^{-1}(\frac{x \sin}{\ell})\delta)\}}{\sin 2\delta}\}.$$

Now the area of $D_1(\ell)$ is

$$= \int_0^{\ell} \int_0^{\frac{\ell \sin\{(2\delta - \sin^{-1}(\frac{x \sin 2\delta}{\ell}))\}}{\sin 2\delta}} dydx$$

$$= \frac{\delta \ell^2}{\sin 2\delta} = \phi_1(\ell)\ .$$

Hence

$$F_N(\ell) = \begin{cases} c_0\ \phi_1(\ell) & \text{for } \ell\varepsilon[0,\ell_1],\ N \text{ even}, \quad (A) \\ c_0'\ \phi_1(\ell) & \text{for } \ell\varepsilon[0,\ell_1],\ N \text{ odd} \quad (B) \end{cases}$$

By Lemma 4, we have

$$D_1(\ell) = S_{k1}(\ell) \cup S_{k3}(\ell) \cup S_{k3}(\ell),$$

where $S_{k1}(\ell)$, $S_{k2}(\ell)$, $S_{k3}(\ell)$ are given by (2.1.26), (2.1.27) and (2.1.28), respectively.

The area of $D_1(\ell)$ is

$$\int_0^a \int_0^{y_k(\ell)} dydx + \int_{x_k(\ell)}^a \int_{y_k(\ell)}^{y_{k1}(x,\ell)} dydx + \int_0^{x_k(\ell)} \int_{y_k(\ell)}^{y_{k2}(x,\ell)} dy\ dx'$$

$$= ((k-2)a - d_{k-1})\ (a - x_k(\ell)) + \left(\frac{a + d_{k-1}}{2}\right) [\ell^2 - (a + d_{k-1})^2 \sin^2 2(k-1)\delta]^{\frac{1}{2}}$$

$$- \frac{(x_k(\ell) + d_{k-1})}{2} [\ell^2 - (x_k(\ell) + d_{k-1})^2 \sin^2 2(k-1)\delta]^{\frac{1}{2}}$$

$$+ \frac{\ell^2}{2 \sin 2(k-1)\delta} [\sin^{-1}\{(\frac{a+d_{k-1}}{\ell}) \sin 2(k-1)\delta\} - \sin^{-1}\{\frac{x_k(\ell) + d_{k-1}}{\ell} \sin 2(k-1)\delta\}]$$

$$- \frac{\cos 2(k-1)\delta}{2} [(a+d_{k-1})^2 - (x_k(\ell) + d_{k-1})^2] + [(k-1)a - d_k]x_k(\ell)$$

$$+ \frac{1}{2} (x_k(\ell) + d_k) [\ell^2 - (x_k(\ell) + d_k)^2 \sin^2 2k\delta]^{\frac{1}{2}}$$

$$- \frac{1}{2} d_k [\ell^2 - d_k^2 \sin^2 2k\delta]^{\frac{1}{2}} + \frac{\ell^2}{2 \sin 2k\delta} [\sin^{-1}\{(\frac{x_k(\ell)+d_k}{\ell}) \sin 2k\delta\}$$

$$- \sin^{-1} (\frac{d_k}{\ell} \sin 2k\delta)] - \frac{1}{2} \cos 2k\delta[(x_k(\ell) + d_k)^2 - d_k^2] ,$$

$$\equiv \phi_2(\ell) .$$

Hence

$$F_N(\ell) = \begin{cases} c_0\phi_2(\ell) , & \text{for } \ell\epsilon[\ell_{k-1},\ell_k], \ N \text{ even} , \qquad \text{(C)} \\ c_0'\phi_2(\ell) , & \text{for } \ell\epsilon[\ell_{k-1},\ell_k], \ N \text{ odd} . \qquad \text{(D)} \end{cases}$$

By Lemma 5 for $N = 2n$, we have

$$D_1(\ell) = \begin{cases} \bigcup_{i=1}^{3} S_i(\ell) \cup S_{n1}(\ell) \cup S_{n2}(\ell) \cup \dot{S}_{n3}(\ell) & \text{for } \ell\varepsilon[\ell_{n-1},\ell_c] \\ \bigcup_{i=1}^{3} S_i'(\ell) \cup S_{n1}(\ell) \cup S_{n2}(\ell) \cup \dot{S}_{n3}(\ell) & \text{for } \ell\varepsilon[\ell_c,\ell_b] \\ \bigcup_{i=1,2} S_i''(\ell) \cup S_{n1}(\ell) \cup S_{n2}(\ell) \cup \dot{S}_{n3}(\ell) & \text{for } \ell\varepsilon[\ell_b,\ell_n] \end{cases},$$

where ℓ_b and ℓ_c, $S_i(\ell)$, $S_{ni}(\ell)$, $\dot{S}_{n3}(\ell)$, $S_i'(\ell), i = 1,2,3$, $S_i''(\ell), i = 1,2$, are given by (2.1.46) to (2.1.57), respectively.

The area of $D_1(\ell)$ for $\ell\varepsilon[\ell_{n-1},\ell_c]$ is

$$\frac{1}{2}(\ell_{n-1}^2 - \ell^2) + a[\ell^2 - \ell_{n-1}^2]^{\frac{1}{2}}$$

$$+ (x_n(\ell) - a)\ (y_{n1}(\ell) + d_{n-1}) + \frac{a + d_{n-1}}{2}\,[\ell^2 - \sin^2(2(n-1)\delta)(a+d_{n-1})^2]^{\frac{1}{2}}$$

$$- \frac{x_n(\ell) + d_{n-1}}{2}\,[\ell^2 - \{x_n(\ell) + d_{n-1}\}^2 \sin^2 2(n-1)\delta]^{\frac{1}{2}}$$

$$+ \frac{\ell^2}{2\sin 2(n-1)\delta}\ \{\sin^{-1}\left(\frac{a + d_{n-1}}{\ell}\sin 2(n-1)\delta\right)$$

$$- \sin^{-1}\left(\frac{x_n(\ell) + d_{n-1}}{\ell}\sin 2(n-1)\delta\right)\}$$

$$+ \frac{\cos 2(n-1)\delta}{2}\ \{(x_n(\ell) + d_{n-1})^2 - (a + d_{n-1})^2\}$$

$$+ x_n(\ell)\ \{a - y_{n1}(\ell)\} + a\{y_{n1}(\ell) + (n-2)a\}$$

$$\equiv \phi_3(\ell).$$

The area of $D_1(\ell)$ for $\ell\epsilon[\ell_c,\ell_b]$ is

$$\frac{1}{2}(\ell^2_{n-1} - \ell^2) + a[\ell^2 - \ell^2_{n-1}]^{\frac{1}{2}}$$

$$+ (x_n(\ell) - a)(y_{n1}(\ell) + d_{n-1}) + \frac{a + d_{n-1}}{2}[\ell^2 - \sin^2(2(n-1)\delta)(a + d_{n-1})^2]^{\frac{1}{2}}$$

$$- \frac{x_n(\ell) + d_{n-1}}{2}[\ell^2 - (x_n(\ell) + d_{n-1})^2 \sin^2 2(n-1)\delta]^{\frac{1}{2}}$$

$$+ \frac{\ell^2}{2 \sin 2(n-1)\delta}[\sin^{-1}(\frac{a + d_{n-1}}{\ell} \sin 2(n-1)\delta)$$

$$- \sin^{-1}(\frac{x_n(\ell) + d_{n-1}}{\ell} \sin 2(n-1)\delta)]$$

$$+ \frac{1}{2}\cos 2(n-1)\delta\,[(x_n(\ell) + d_{n-1})^2 - (a + d_{n-1})^2]$$

$$+ x_n(\ell)(a - y_{n1}(\ell)) + a(y_{n1}(\ell) + (n-2)a)$$

$$\equiv \phi_4(\ell).$$

The area of $D_1(\ell)$ for $\ell\epsilon[\ell_b,\ell_n]$ is

$$\frac{1}{2}(\ell^2_{n-1} - \ell^2) + a(\ell^2 - \ell^2_{n-1})^{\frac{1}{2}}$$

$$+ (x_n(\ell) - a)(y_{n1}(\ell) + d_{n-1}) + \frac{a + d_{n-1}}{2}[\ell^2 - \sin^2(2(n-1)\delta)(a+d_{n-1})^2]^{\frac{1}{2}}$$

$$- \frac{x_n(\ell) + d_{n-1}}{2}[\ell^2 - (x_n(\ell) + d_{n-1})^2 \sin^2 2(n-1)\delta]^{\frac{1}{2}}$$

$$+ \frac{\ell^2}{2 \sin 2(n-1)\delta} [\sin^{-1} (\frac{a + d_{n-1}}{\ell} \sin 2(n-1)\delta)$$

$$- \sin^{-1} (\frac{x_n(\ell) + d_{n-1}}{\ell} \sin 2(n-1)\delta)]$$

$$+ \frac{1}{2} \cos 2(n-1)\delta \; [(x_n(\ell) + d_{n-1})^2 - (a + d_{n-1})^2]$$

$$+ x_n(\ell) \; (a - y_{n1}(\ell)) + a\{y_{n1}(\ell) + (n-2)a\}$$

$$\equiv \phi_5(\ell).$$

Hence when $N = 2n$,

$$F_N(\ell) = \begin{cases} C_0 \; \phi_3(\ell), & \text{for } \ell\epsilon[\ell_{n-1}, \ell_c] \; , & \text{(E)} \\ C_0 \; \phi(\ell), & \text{for } \ell\epsilon[\ell_c, \ell_b] \; , & \text{(F)} \\ C_0 \; \phi_5(\ell), & \text{for } \ell\epsilon[\ell_b, \ell_n] \; . & \text{(H)} \end{cases}$$

By Lemma 6a for $N = 2n + 1$, $\ell\epsilon[\ell_{n-1}, \ell_{n-1,1}]$, we have

(2.1.82) $$D_1(\ell) = S_1(\ell) \cup S_2(\ell) \; ,$$

where

$S_1(\ell)$ and $S_2(\ell)$ are given by (2.1.83) and (2.1.84), respectively.

The area of $D_1(\ell)$ for $\ell\varepsilon[\ell_n, \ell_{n-1,1}]$ is

$$\{-d_n + (n-1)a\}c + \frac{d_n+c}{2}[\ell^2 - (d_n+c)^2 \sin^2 2n\delta]^{\frac{1}{2}}$$

$$- \frac{d_n}{2}[\ell^2 - d_n^2 \sin^2 2n\delta]^{\frac{1}{2}} + \frac{\ell^2}{2 \sin 2n\delta}[\sin^{-1}(\frac{d_n+c}{\ell} \sin 2n\delta)$$

$$- \sin^{-1}(\frac{d_n}{\ell} \sin 2n\delta)] - \frac{\cos 2n\delta}{2}[(d_n+c)^2 - d_n^2]$$

$$+ (a - c)\{(n-2)a - d_{n-1}\} + \frac{d_{n-1}+a}{2}[\ell^2 - (d_{n-1}+a)^2 \sin^2 2(n-1)\delta]^{\frac{1}{2}}$$

$$- \frac{d_{n-1}+c}{2}[\ell^2 - (d_{n-1}+c)^2 \sin^2 2(n-1)\delta]^{\frac{1}{2}}$$

$$+ \frac{\ell^2}{2 \sin 2(n-1)\delta}[\sin^{-1}(\frac{d_{n-1}+a}{\ell} \sin 2(n-1)\delta)$$

$$- \sin^{-1}(\frac{d_{n-1}+c}{\ell} \sin 2(n-1)\delta)] - \frac{\cos 2(n-1)\delta}{2}[(a + d_{n-1})^2 - (c + d_{n-1})^2]$$

$$\equiv \phi_6(\ell).$$

Hence $F_N(\ell) = C_0 \phi_6(\ell)$, for $\ell\varepsilon[\ell_{n-1}, \ell_{n-1,1}]$ (H)

By Lemma 6b, for $N = 2n + 1$, $\ell\varepsilon[\ell_{n-1,1}, \ell_{n-1,2}]$, we have

(2.1.94) $$D_1(\ell) = S_1'(\ell) \cup S_2'(\ell) \cup S_3'(\ell),$$

where $S_1'(\ell)$, $S_2'(\ell)$ and $S_3'(\ell)$ are given by (2.1.95), (2.1.96) and (2.1.97).

respectively.

The area of $D_1(\ell)$ for $\ell\varepsilon[\ell_{n-1,1},\ell_{n-1,2}]$ is

$$\{-d_n + (n-1)a\} c + \frac{d_n+c}{2} \{\ell^2 - (d_n+c)^2 \sin^2 2n\delta\}^{\frac{1}{2}}$$

$$- \frac{d_n}{2} (\ell^2 - d_n^2 \sin^2 2n\delta)^{\frac{1}{2}} + \frac{\ell^2}{2 \sin 2n\delta}[\sin^{-1} (\frac{d_n+c}{\ell} \sin 2n\delta)$$

$$- \sin^{-1} (\frac{d_n}{\ell} \sin 2n\delta)] - \frac{\cos 2n\delta}{2} \{(d_n+c)^2 - d_n^2\} + (a-c) \{(n-2)a - d_{n-1}\}$$

$$+ \frac{d_{n-1}+a}{2} \{\ell^2 - (d_{n-1}+a)^2 \sin^2 2(n-1)\delta\}^{\frac{1}{2}}$$

$$- \frac{d_{n-1}+c}{2} \{\ell^2 - (d_{n-1}+c)^2 \sin^2 2(n-1)\delta\}^{\frac{1}{2}}$$

$$+ \frac{\ell^2}{2 \sin 2(n-1)\delta} [\sin^{-1} (\frac{d_{n-1}+a}{\ell} \sin 2(n-1)\delta)$$

$$- \sin^{-1} (\frac{d_{n-1}+c}{\ell} \sin 2(n-1)\delta)] + \frac{\cos 2(n-1)\delta}{2}[(c + d_{n-1})^2 - (a+d_{n-1})^2]$$

$$+ (d_n + x_1) \{\ell^2 - (d_n + x_1)^2 \sin^2 2n\delta\}^{\frac{1}{2}}$$

$$- (d_n+c) \{\ell^2 - (d_n+c)^2 \sin^2 2n\delta\}$$

$$+ \frac{\ell^2}{\sin 2n\delta}[\sin^{-1} (\frac{d_n+x_1}{\ell} \sin 2n\delta)$$

$$- \sin^{-1} (\frac{d_n+c}{\ell} \sin 2n\delta)] , \quad \text{where } x_1 = -d_n + \frac{\ell}{\sin 2n\delta}$$

$$\equiv \phi(\ell) .$$

Hence

$$F_N(\ell) = c_0\phi_7(\ell) \text{ for } \ell\epsilon[\ell_{n-1,1},\ell_{n-1,2}].$$

By Lemma 6c, for $N = 2n+1$, $\ell\epsilon[\ell_{n-1,2},\ell_n]$, $D_1(\ell) = S_1'(\ell)\cup S_2'(\ell)\cup S_3'(\ell)\cup S_4(\ell)$, where $S_1(\ell)$, $S_2'(\ell)$, $S_3(\ell)$, $S_4(\ell)$ are given by (2.1.104), (2.1.105), (2.1.106) and (2.1.107), respectively.

The area of $D_1(\ell)$ for $\ell\epsilon[\ell_{n-1,2},\ell_n]$ is

$$\{ - d_n + (n-1)a\} c_1 + \frac{d_n+c_1}{2}\{\ell^2 - (d_n+c_1)^2 \sin^2 2n\delta\}^{\frac{1}{2}}$$

$$- \frac{d_n}{2}\{\ell^2 - d_n^2 \sin^2 2n\delta\}^{\frac{1}{2}} + \frac{\ell^2}{2 \sin 2n\delta}[\sin^{-1}(\frac{d_n+c_1}{\ell} \sin 2n\delta)$$

$$- \sin^{-1}(\frac{d_n}{\ell} \sin 2n\delta)] - \frac{\cos 2n\delta}{2}[(d_n + c_1)^2 - d_n^2] + na(c_2 - c_1)$$

$$+ \{ - d_n + (n-1)a\}(c-c_2) + \frac{d_n+c}{2}\{\ell^2 - (d_n+c)^2 \sin^2 2n\delta\}^{\frac{1}{2}}$$

$$- \frac{d_n+c_2}{2}\{\ell^2 - (d_n + c_2)^2 \sin^2 2n\delta\}^{\frac{1}{2}}$$

$$+ \frac{\ell^2}{2 \sin 2n\delta}\{\sin^{-1}(\frac{d_n+c}{\ell} \sin 2n\delta) - \sin^{-1}(\frac{d_n+c_2}{\ell} \sin 2n\delta)\}$$

$$- \frac{\cos 2n\delta}{2} \{(d_n+c)^2 - (d_n+c_2)^2\}$$

$$+ \delta(a-c)\ ((n-2)a-d_{n-1}) + \frac{d_{n-1}+a}{2} [\ell^2 - (d_{n-1}+a)^2 \sin^2 2(n-1)\delta]^{\frac{1}{2}}$$

$$- \frac{d_{n-1}+c}{2} \{\ell^2 - (d_{n-1}+c)^2 \sin^2 2(n-1)\delta\}^{\frac{1}{2}}$$

$$+ \frac{\ell^2}{2 \sin 2(n-1)\delta} [\sin^{-1} (\frac{d_{n-1}+a}{\ell} \sin 2(n-1)\delta)$$

$$- \sin^{-1} (\frac{d_{n-1}+c}{\ell} \sin 2(n-1)\delta)]$$

$$+ \frac{\cos 2(n-1)\delta}{2} [(c+d_{n-1})^2 - (a+d_{n-1})^2]$$

$$+ (d_n+a)\ \{\ell^2 - (d_n+a)^2 \sin^2 2n\delta\}^{\frac{1}{2}}$$

$$- (d_n+c)\ \{\ell^2-(d_n+c)^2 \sin^2 2n\delta\}^{\frac{1}{2}}$$

$$+ \frac{\ell^2}{\sin 2n\delta} [\sin^{-1} (\frac{d_n+a}{\ell} \sin 2n\delta) - \sin^{-1} (\frac{d_n+c}{\ell} \sin 2n\delta)]$$

$$\equiv \phi_8(\ell)\ .$$

Hence $F_N(\ell) = c_0\phi_8(\ell)$, for $\ell\varepsilon[\ell_{n-1,2},\ell_n]$, when $N = 2n+1$.

SECTION TWO

2.2. LIMIT OF THE SEQUENCE OF DISTRIBUTION FUNCTIONS OF LENGTHS OF S_2-RANDOM SECANTS OF REGULAR POLYGONS INSCRIBED IN A CIRCLE

2.2.0. Introduction.

We have seen in 1.1.6 that if a regular polygon of N sides is always inscribed within a circle of constant radius r for any N, N = 3,4,... then the corresponding sequence of distribution functions $F_N(\ell)$, N = 3,4,... of the random secant length L of the regular polygons under S_1-randomness converges to the distribution function $F(\ell)$ of the random secant length L of the circle under S_1-randomness. We prove in the following theorem that under S_2-randomness this is also the case. Since the distributions of the lengths of 'random' secants of a circle under S_1-randomness and S_2-randomness are the same, we have discovered an instance where two different sequences of distribution functions $F_N(\ell)$ (S_1-randomness) and $F_N(\ell)$ (S_2-randomness), N = 3,4,... (which are 'paradoxical' for a fixed N according to Bertrand) of the 'random' secant length L of a polygon under S_1-randomness and S_2-randomness converge to the same distribution function $F(\ell)$ of the random secant length L of a circle under S_1 or S_2-randomness.

2.2.1 Limiting Distribution.

Theorem 10. Let C be a circle of radius r. Let P_N be a regular polygon of N sides - inscribed within the circle C. Let $F_N(\ell)$ and $F(\ell)$ be the corresponding distribution functions of the lengths of S_2-random secants of

P_N and C respectively. Then

(2.2.1) $$\lim_{N\to\infty} F_N(\ell) = F(\ell) \quad \text{for } \ell \ [0,2r].$$

Proof. As in the proof of Theorem 8, without loss of generality, we consider $r = 1$. Let $\ell \in [0,2]$. Let a be the length of a side of the regular polygon of N sides. Then $a = 2\sin\delta$, where $\delta = \frac{\pi}{N}$.

Let P_N be the polygon $A_0, A_1 \ldots A_{N-1}$ inscribed within the circle C of radius 1. Let A_0 be a fixed point on the perimeter of the circle. For a fixed $\ell \in [0,2]$, let A_ℓ be a point on the perimeter of the circle such that $|A_0 A_\ell| = \ell$. As N increases indefinitely let the closest vertex A_k of the polygon P_N tends to coincide with the point A_ℓ. Then as $N\to\infty$, $k\to\infty$, where k depends on N, and the length $2k\delta$ of the arc A_0A_k tends to the length $2\sin^{-1}\frac{\ell}{2}$ of the arc A_0A_ℓ. Therefore as $N\to\infty$

(2.2.2) $$k\delta \to \sin^{-1}\left(\frac{\ell}{2}\right)$$

and

(2.2.3) $$\ell_k \to \ell$$

The distribution function $F_N(\ell)$ for $\ell \in [\ell_{k-1}, \ell_k]$ is, by (2.1.1), the following:

(2.2.4) $$F_N(\ell) = \frac{\phi_2(\ell)}{c_i}, \qquad \begin{array}{l} i = 0 \text{ for N even} \\ \ \ = 1 \text{ for N odd,} \end{array}$$

where $$\frac{1}{c_0} = \frac{2}{a^2(2n-1)}, \quad \frac{1}{c_1} = \frac{1}{a^2 n}, \quad \text{and}$$

$$\phi_2(\ell) = \{(k-2)a-d_{k-1}\}(a-x_k(\ell)) + \frac{a+d_{k-1}}{2}\left[\ell^2-(a+d_{k-1})^2\sin^2 2(k-1)\delta\right]^{1/2}$$
$$-\frac{(x_k(\ell)+d_{k-1})}{2}\left[\ell^2-(x_k(\ell)+d_{k-1})^2\sin^2 2(k-1)\delta\right]^{1/2}$$
$$+\frac{\ell^2}{2\sin 2(k-1)\delta}\left[\sin^{-1}\left\{\left(\frac{a+d_{k-1}}{\ell}\right)\sin 2(k-1)\delta\right\}-\sin^{-1}\left\{\frac{(x_k(\ell)+d_{k-1})}{\ell}\sin 2(k-1)\delta\right\}\right]$$

$$- \frac{\cos 2(k-1)\delta}{2} [(a+d_{k-1})^2 - (x_k(\ell)+d_{k-1})^2] + \{(k-1)a-d_k\}x_k(\ell)$$

$$+ \frac{1}{2} \{x_k(\ell)+d_k\} [\ell^2 - \{x_k(\ell) + d_k\}^2 \sin^2 2k\delta]^{1/2}$$

$$- \frac{1}{2} d_k [\ell^2-d_k^2 \sin^2 2k\delta]^{1/2} + \frac{\ell^2}{2 \sin 2k\delta} [\sin^{-1}\{(\frac{x_k(\ell)+d_k}{\ell}) \sin 2k\delta\}$$

$$- \sin^{-1} (\frac{d_k}{\ell} \sin 2k\delta)] - \frac{1}{2} \cos 2k\delta [\{x_k(\ell)+d_k\}^2 - d_k^2]$$

where

$$\delta = \frac{\pi}{N}, \quad n = \frac{N}{2} \text{ or } \frac{N-1}{2} \text{ for } N \text{ even or odd respectively;}$$

$$\ell_k = \frac{a \sin k\delta}{\sin\delta}$$

$$\beta_k(\ell) = \sin^{-1} (\frac{\ell_{k-1} \sin k\delta}{\ell})$$

$$x_k(\ell) = \frac{\ell \sin\{k\delta-\beta_k(\ell)\}}{\sin 2k\delta}$$

and

$$d_k = \ell_{k-1} \frac{\sin k\delta}{\sin 2k\delta} .$$

Now to consider the limit of $F_N(\ell)$ as $N \to \infty$ we note that:

$$a = 2 \sin\delta \to 0, \qquad \text{as } N \to \infty ,$$

$$\beta_k(\ell) = \sin^{-1} (\frac{\ell_{k-1} \sin k\delta}{\ell}) \to (k-1)\delta + \sin^{-1} \frac{\ell}{2} \qquad \text{as } N \to \infty ,$$

$$d_k = \frac{\ell_{k-1} \sin k\delta}{\sin 2k\delta} \to \frac{\ell}{(4-\ell^2)^{1/2}} \qquad \text{as } N \to \infty ,$$

$$x_k(\ell) = \frac{\ell \sin(k\delta-\beta_k(\ell))}{\sin k\delta} \to 0 \qquad \text{as } N \to \infty ,$$

and

$$c_i \to 0, \quad i = 0,1 \qquad \text{as } N \to \infty$$

in (2.2.3).

Consequently both the numerator and the denominator of $F_N(\ell)$ defined in (2.2.3) tend to zero as $N \to \infty$ and reduce the limit to $\frac{0}{0}$ form. Applying L'Hospital's rule to the corresponding expression of $F_N(\ell)$, where δ is replaced by a real variable x, we obtain as in (1.1.6):

$$\lim_{N\to\infty} F_N(\ell) = \frac{2}{\pi} \sin^{-1} (\frac{\ell}{2}) ,$$

which is the distribution function $F(\ell)$ of the S_1-random secant length L of a circle of radius 1.

CHAPTER THREE

3. DISTRIBUTIONS OF LENGTHS OF RAYS.

3.0. Introduction.

In this chapter, we consider some typical problems arising in an area in geometrical probability which has remained unexplored. The general problem when stated precisely is as follows.

Consider a compact convex body K in Euclidean n-space with non-empty interior. Choose a point P at random in the interior of K and consider a ray G originating at P in a random direction. The ray will intersect the surface of K at a unique point Q. Let $L = |PQ|$. The probability law of L is of our interest in geometric probability. Let P be denoted by $\underline{X}$ and the direction of G be determined by $\underline{\theta}$ with reference to a set of axes. Assume that $\underline{X}$ and $\underline{\theta}$ are independently distributed. The probability law of L is the one induced by the distributions of $\underline{X}$ and $\underline{\theta}$. Problems on random rays have not been considered before.

In this chapter, we shall consider some typical problems concerning lengths of random rays. We shall consider the distributions of the random ray-length L in the cases of rectangles, circles, and spheres in 3.1, 3.2, and 3.3, respectively. In 3.2 and 3.3, the source P is restricted to a diameter.

SECTION ONE

3.1. DISTRIBUTION OF LENGTHS OF RANDOM RAYS FROM RANDOM SOURCES IN THE INTERIOR OF A RECTANGLE.

3.1.0. Introduction.

In this section we obtain the probability distribution of the length of a random ray emanating from a random source in the interior of a rectangle of sides a and b $(a > b)$. Here a 'random ray' is defined by a random point in the interior and a random direction.

Let ABCD be the rectangle (cf. Fig. 71) and P be a random point in the interior of the rectangle. A ray in a random direction emanates from P and is intercepted by a side of the rectangle at a point Q. We are interested in the probability distribution of $L = |PQ|$.

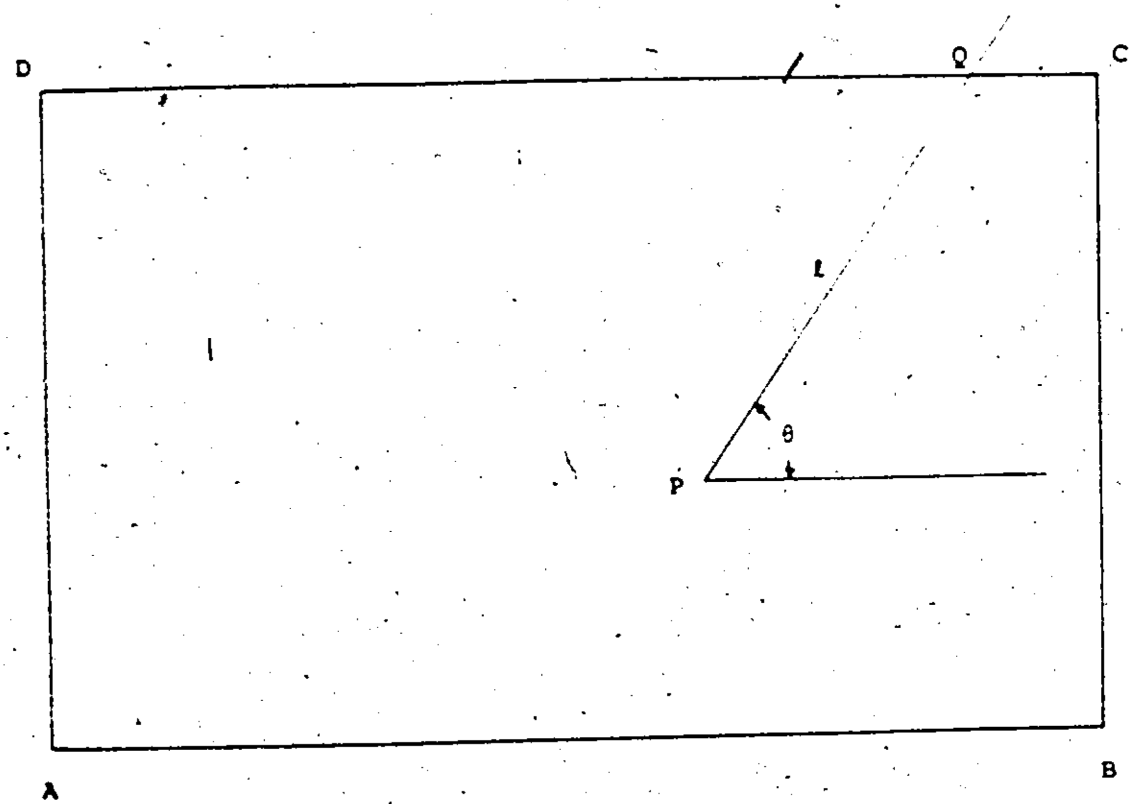

Fig. 71

3.1.1. Distribution of L.

The distribution of L is presented in the following theorem.

Theorem 11 The probability distribution function $F(\ell)$ of the random length L of a random ray emanating from a random source in the interior of a rectangle of sides a and b, where a > b, is given by

$$(3.2.1)\qquad F(\ell) = \begin{cases} \frac{2}{\pi ab}\,\psi_1(\ell), \text{for } \ell\varepsilon[0,b], & (A) \\ \frac{2}{\pi ab}\,\psi_2(\ell), \text{for } \ell\varepsilon[b,a], & (B) \\ \frac{2}{\pi ab}\,\psi_3(\ell), \text{for } \ell\varepsilon[a, (a^2+b^2)^{\frac{1}{2}}], & (C) \end{cases}$$

where $\psi_1(\ell) = (a+b)\,\ell - \frac{\ell^2}{2}$,

$$\psi_2(\ell) = \frac{b^2}{2} + \ell^2\,(1 - \frac{\pi^2}{16}) - \ell(\ell^2-b^2)^{\frac{1}{2}} + \frac{\pi\ell^2}{8}\{\cos^{-1}\frac{b}{\ell} - \sin^{-1}\frac{b}{\ell}\} + b\ell\cos^{-1}\frac{b}{\ell} + \frac{\pi^2\ell^2}{8} - \frac{\pi\ell^2}{4}\cos^{-1}\frac{b}{\ell} + (a-\ell)\{\ell + b\cos^{-1}\frac{b}{\ell} - (\ell^2-b^2)^{\frac{1}{2}}\},$$

$$\psi_3(\ell) = \frac{a^2+b^2+\ell^2}{2} - a(\ell^2-b^2)^{\frac{1}{2}} - b(\ell^2-a^2)^{\frac{1}{2}} + ab(\cos^{-1}\frac{b}{\ell} + \cos^{-1}\frac{a}{\ell}) + \frac{\pi\ell^2}{8}(\cos^{-1}\frac{a}{\ell} - \sin^{-1}\frac{b}{\ell}).$$

To prove this theorem we proceed as follows.

3.1.2. Parametrization of the Rays.

Let A be the origin, AB the X-axis and AD the Y-axis. A source P in the interior of ABCD is determined by (x,y) where $0 \leq x \leq a$, $0 \leq y \leq b$. Let θ be the angle that a ray makes with a line paralled to AB. $P(x,y)$ has uniform distribution in the interior of the rectangle and θ has uniform distribution over $[0, 2\pi]$.

The joint density of X, Y and θ is given by

$$(3.1.2) \qquad p(x,y,\theta) = \begin{cases} \frac{1}{2\pi ab}, & 0 \leq x \leq a,\ 0 \leq y \leq b,\ 0 \leq \theta < 2\pi \\ 0, & \text{elsewhere.} \end{cases}$$

The set

$$S' = \{(x,y,\ ): 0 \leq x \leq a,\ 0 \leq y \leq b,\ 0 \leq \theta \leq 2\pi\}$$

is the <u>parameter space</u>.

Each triple $(x,y,\theta) \in S'$ determines a ray and hence a length $L(x,y,\theta) = \ell$.

For convenience, we can consider L, X, Y, θ as functions defined on the probability space S' with normalized Lebesgue measure: X, Y, θ being the projections. We are interested in

$$(3.1.3) \qquad F(\ell) = Pr(L < \ell) = \frac{1}{2\pi ab} \int_{D(\ell)} dx\,dy\,d\theta$$

where

$$(3.1.4) \qquad D(\ell) = \{(x,y,\theta): L(x,y,\theta) \leq \ell,\ (x,y,\theta) \in S'\}.$$

Now

$$(3.1.5)\qquad F(\ell) = \frac{1}{2\pi ab}\int_{D(\ell)} dx\,dy\,d\theta$$

$$= \frac{1}{2\pi ab}\left[\int_{D_1(\ell)} dx\,dy\,d\theta + \int_{D_2(\ell)} dx\,dy\,d\theta + \int_{D_3(\ell)} dx\,dy\,d\theta + \int_{D_4(\ell)} dx\,dy\,d\theta\right],$$

where

$$(3.1.6)\quad D_1(\ell) = \{(x,y,\theta): L(x,y,\theta) \le \ell,\ 0 \le x \le a,\ 0 \le y \le b,\ 0 \le \theta \le \tfrac{\pi}{2}\},$$

$$(3.1.7)\quad D_2(\ell) = \{(a-x,y,\pi-\theta): L(a-x,y,\pi-\theta) \le \ell,\ (x,y,\theta) \in S\},$$

$$(3.1.8)\quad D_3(\ell) = \{(a-x,b-y,\pi+\theta): L(a-x,b-y,\pi+\theta) \le \ell,\ (x,y,\theta) \in S\}$$

and

$$(3.1.9)\quad D_4(\ell) = \{(x,b-y,2\pi-\theta): L(x,b-y,2\pi-\theta) \le \ell,\ (x,y,\theta) \in S\},$$

where

$$(3.1.10)\qquad S = \{(x,y,\theta): 0 \le x \le a,\ 0 \le y \le b,\ 0 \le \theta \le \tfrac{\pi}{2}\}.$$

Since each coordinate of the point $(a-x,\ b-y,\ \pi+\theta) \in D_3(\ell)$ is a linear translate of the corresponding coordinate of the point $(x,y,\theta) \in D_1(\ell)$, we conclude that

$$(3.1.11)\qquad \int_{D_1(\ell)} dx\,dy\,d\theta = \int_{D_3(\ell)} dx\,dy\,d\theta$$

Similarly,

(3.1.12)
$$\int_{D_1(\ell)} dxdyd\theta = \int_{D_2(\ell)} dxdyd\theta = \int_{D_4(\ell)} dxdyd\theta .$$

By (3.2.11) and (3.2.12), (3.2.3) reduces to

(3.1.13)
$$F(\ell) = \frac{2}{\pi ab} \int_{D_1(\ell)} dxdyd\theta .$$

The set S defined in (3.1.10) is the reduced parameter space.

For the evaluation of (3.1.13) we split the interval $[0,(a^2+b^2)^{\frac{1}{2}}]$, the range of L, into the three subintervals:

(1) $[0,b]$, (2) $[b,a]$, (3) $[a,(a^2+b^2)^{\frac{1}{2}}]$.

Now we proceed to determine the set $D_1(\ell)$ for ℓ lying in the above different subintervals.

3.1.3. Determination of the Set $D_1(\ell)$ for $\ell \epsilon [0,b]$.

In the following lemma, we obtain the set $D_1(\ell)$ for $\ell \epsilon [0,b]$.

Lemma 1. Let $\ell \epsilon [0,b]$.

Then

(3.1.14) $$D_1(\ell) = S_{11}(\ell) \cup S_{12}(\ell) \cup S_{13}(\ell) \cup S_{14}(\ell) \quad ,$$

where

(3.1.15) $$S_{11}(\ell)$$

$$= \{(x,y,\theta): a - \ell \leq x \leq a,\ 0 \leq y \leq b - [\ell^2-(a-x)^2]^{\frac{1}{2}},\ 0 \leq \theta \leq \cos^{-1} (\frac{a-x}{\ell})\},$$

(3.1.16) $$S_{12}(\ell)$$

$$= \{(x,y,\theta): a - \ell \leq x \leq a,\ b - [\ell^2-(a-x)^2]^{\frac{1}{2}} \leq y \leq b,\ 0 \leq \theta \leq \frac{\pi}{2}\} ,$$

(3.1.17) $$S_{13}(\ell)$$

$$= \{(x,y,\theta): a - \ell \leq x \leq a,\ b - \ell \leq y \leq b - [\ell^2-(a-x)^2]^{\frac{1}{2}},\ \sin^{-1} \frac{b-y}{\ell} \leq \theta \leq \frac{\pi}{2}\}$$

and

(3.1.18) $$S_{14}(\ell)$$

$= \{(x,y,\theta): 0 \leq x \leq a - \ell,\ b - \ell \leq y \leq b,\ \sin^{-1}(\frac{b-y}{\ell}) \leq \theta \leq \frac{\pi}{2}\}.$

Proof (cf. Fig. 72). We draw a circle of radius ℓ with C as the centre intersecting BC and CD at K_1 and K_2, respectively. The following are the possible ways in which $L(x,y,\theta) \leq \ell$ can arise.

Case I (cf. Fig. 72).

a. Let $T(x_1,y_1) \in C_1$,

where

$$C_1 = \{(x,y): (x-a)^2 + (y-b)^2 = \ell^2\} \cap \{(x,y): 0 \leq x < a,\ 0 \leq y \leq b\}.$$

Draw TS_1 and TS_2 orthogonal to AB and BC respectively. Let $\angle CTS_2 = \theta_1$. Then from the triangle CTS_2 we have

(3.1.19) $$\ell = \frac{a-x_1}{\cos\theta_1}.$$

From the relation (3.1.19), it follows that

(3.1.20) $$0 \leq \theta < \theta_1 \Rightarrow L(x,y,\theta) \leq \ell.$$

Let $P(x,y)$ be a point on TS_1 such that $x = x_1$, $0 \leq y \leq y_1$. Clearly [by (3.1.19)] there follows

(3.1.21) $$0 \leq \theta \leq \theta_1,\ x = x_1 \text{ and } 0 \leq y \leq y_1 \Rightarrow L(x,y,\theta) \leq \ell.$$

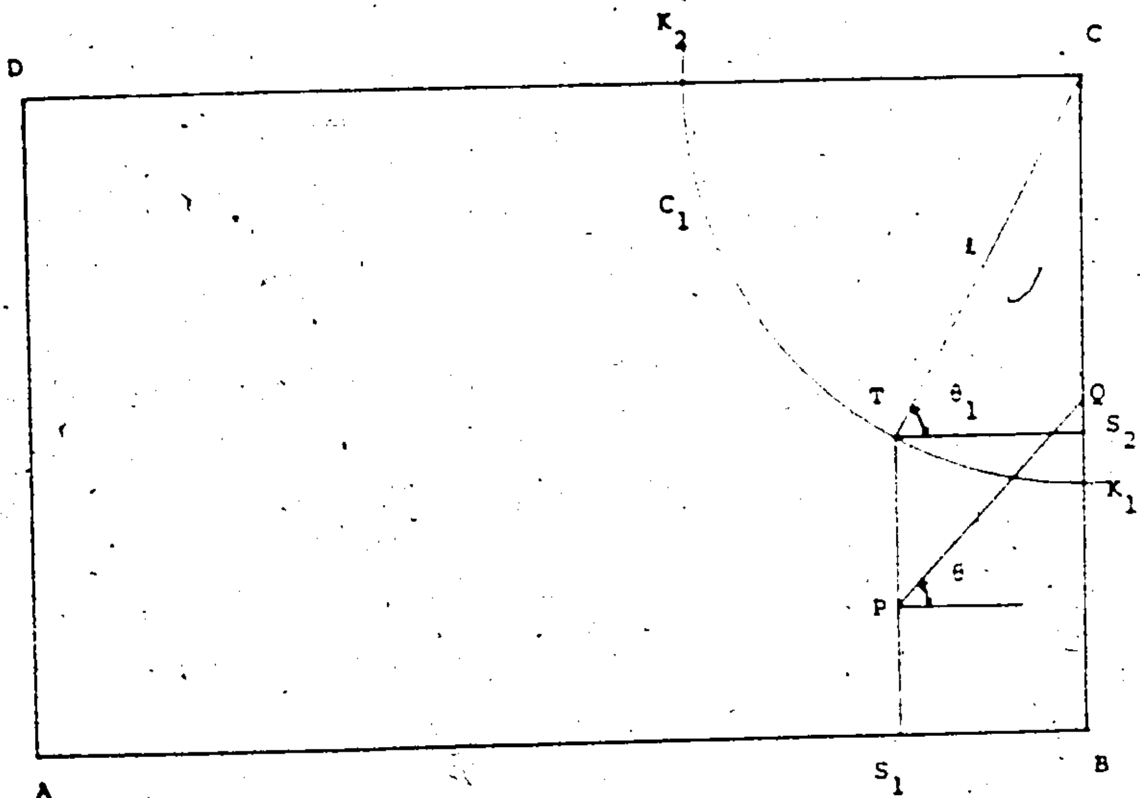

Fig. 72

Allowing $T(x_1, y_1)$ to vary over the segment K_1K_2 of the circle, we find that $L(x,y,\theta) \leq \ell$ whenever $(x,y,\theta) \in S_{11}(\ell)$, where $S_{11}(\ell)$ is defined by (3.1.15).

Thus there is a contribution by $S_{11}(\ell)$ to $Pr(L \leq \ell)$.

Hence

(3.1.22) $$S_{11}(\ell) \subset D_1(\ell).$$

<u>Case II</u> (cf. Fig. 73).

Let $P(x,y) \in C_2$ where

$$C_2 = \{(x,y): (x-a)^2 + (y-b)^2 \leq \ell^2\} \cap \{(x,y): 0 \leq x \leq a, \ 0 \leq y \leq b\}.$$

Then clearly $L(x,y,\theta) \leq \ell$ for $0 \leq \theta \leq \frac{\pi}{2}$.

Consequently $S_{12}(\ell)$ defined by (3.1.16), contributes to $Pr(L \leq \ell)$.

Hence

(3.1.23) $$S_{12}(\ell) \subset D_1(\ell).$$

<u>Case III</u> (cf. Fig. 74).

Let $P(x_1, y_1)$ be a point such that

$$a - \ell \leq x_1 \leq a, \quad b - \ell \leq y_1 \leq b - (\ell^2 - (a-x)^2)^{\frac{1}{2}}.$$

We draw a circle with ℓ as the radius and $P(x_1, y_1)$ as the centre. The

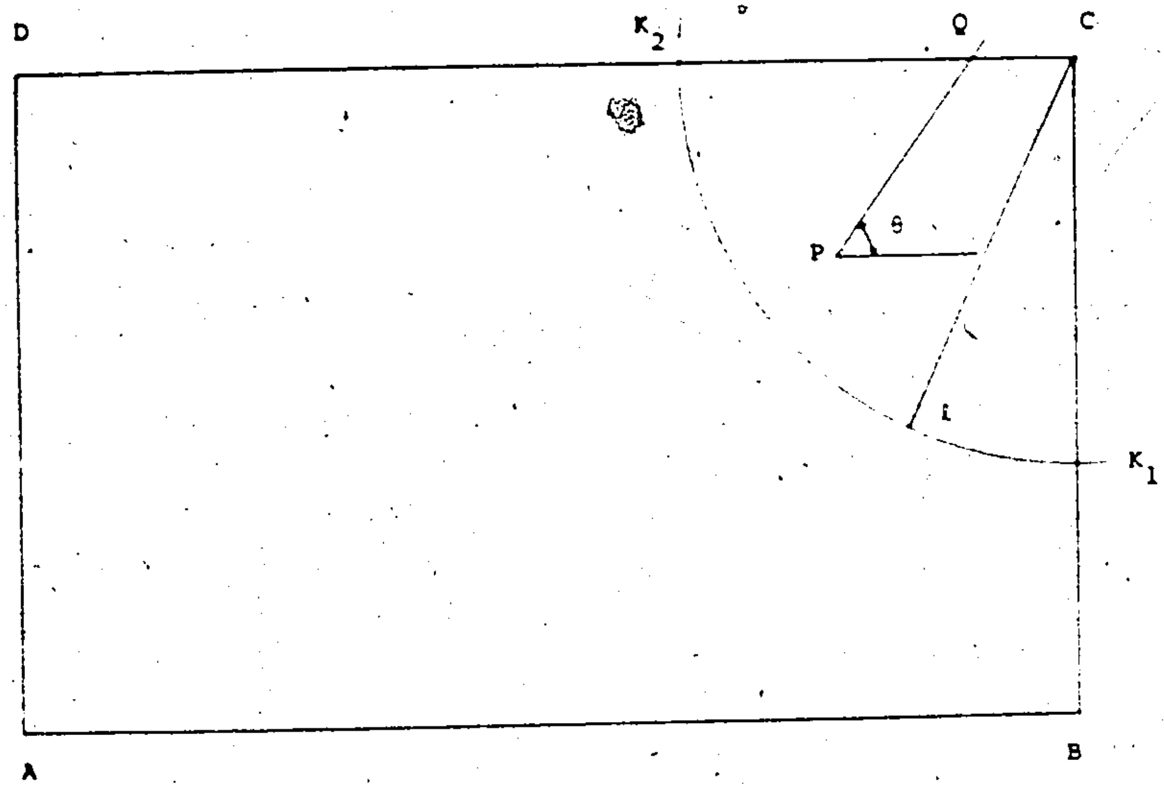

Fig. 73

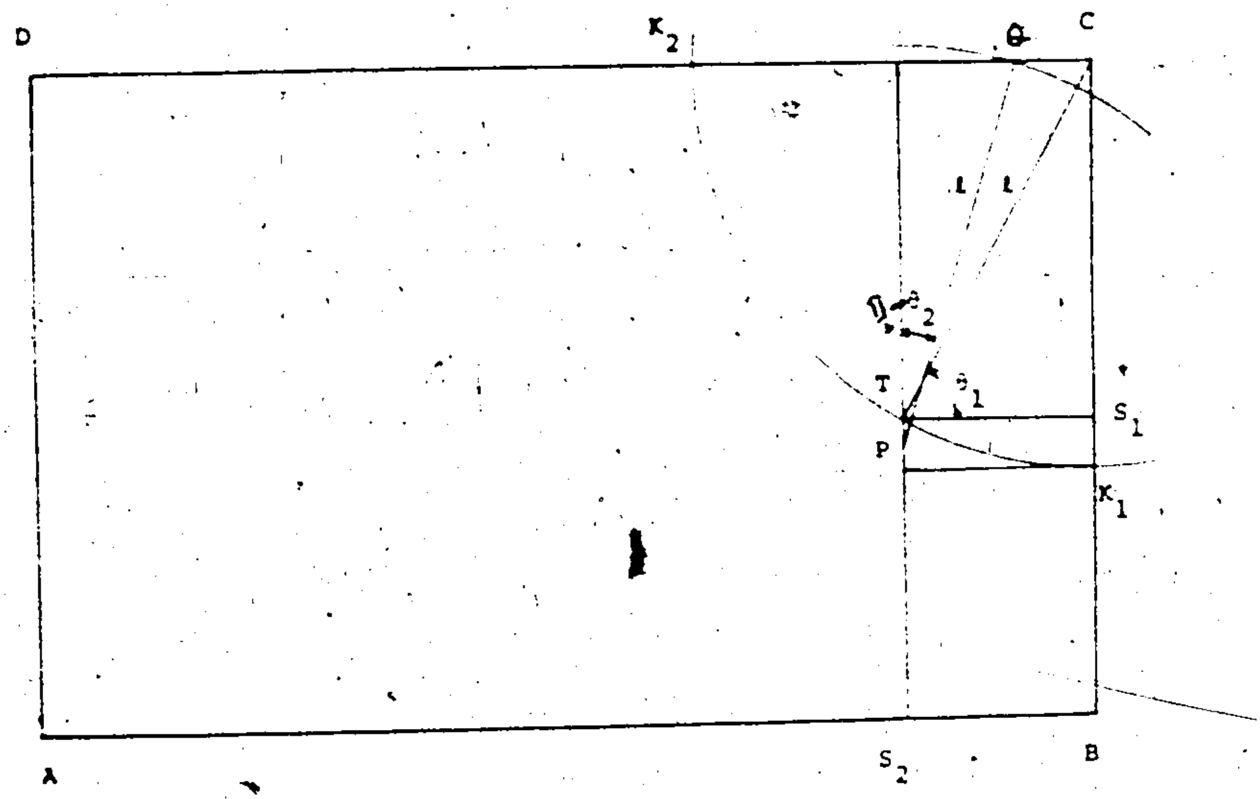

Fig. 74

circle intersects CD at Q so that the angle θ_2 that QP makes with AB is less than or equal to $\frac{\pi}{2}$.

We have, using the sine ratio,

(3.1.24) $$\sin\theta_2 = \frac{b-y_1}{\ell}.$$

From (3.2.24) it follows that

(3.1.25) $$\sin^{-1}\frac{b-y_1}{\ell} \le \theta \le \frac{\pi}{2} \Rightarrow L(x_1,y_1,\theta) \le \ell.$$

Therefore $S_{13}(\ell)$ defined by (3.1.17) contributes to $Pr(L \le \ell)$.

Hence

(3.1.26) $$S_{13}(\ell) \subset D_1(\ell).$$

<u>Case IV</u> (cf. Fig. 75).

Let $P(x_1,y_1)$ be a point such that

$$0 \le x_1 \le a - \ell, \; b - \ell \le y_1 \le b.$$

We draw a circle with $P(x_1,y_1)$ as the centre and ℓ as the radius. The circle intersects CD at a point Q such that the angle θ_3 subtended by QP with AB is less than or equal to $\frac{\pi}{2}$. The angle θ_3 is given by

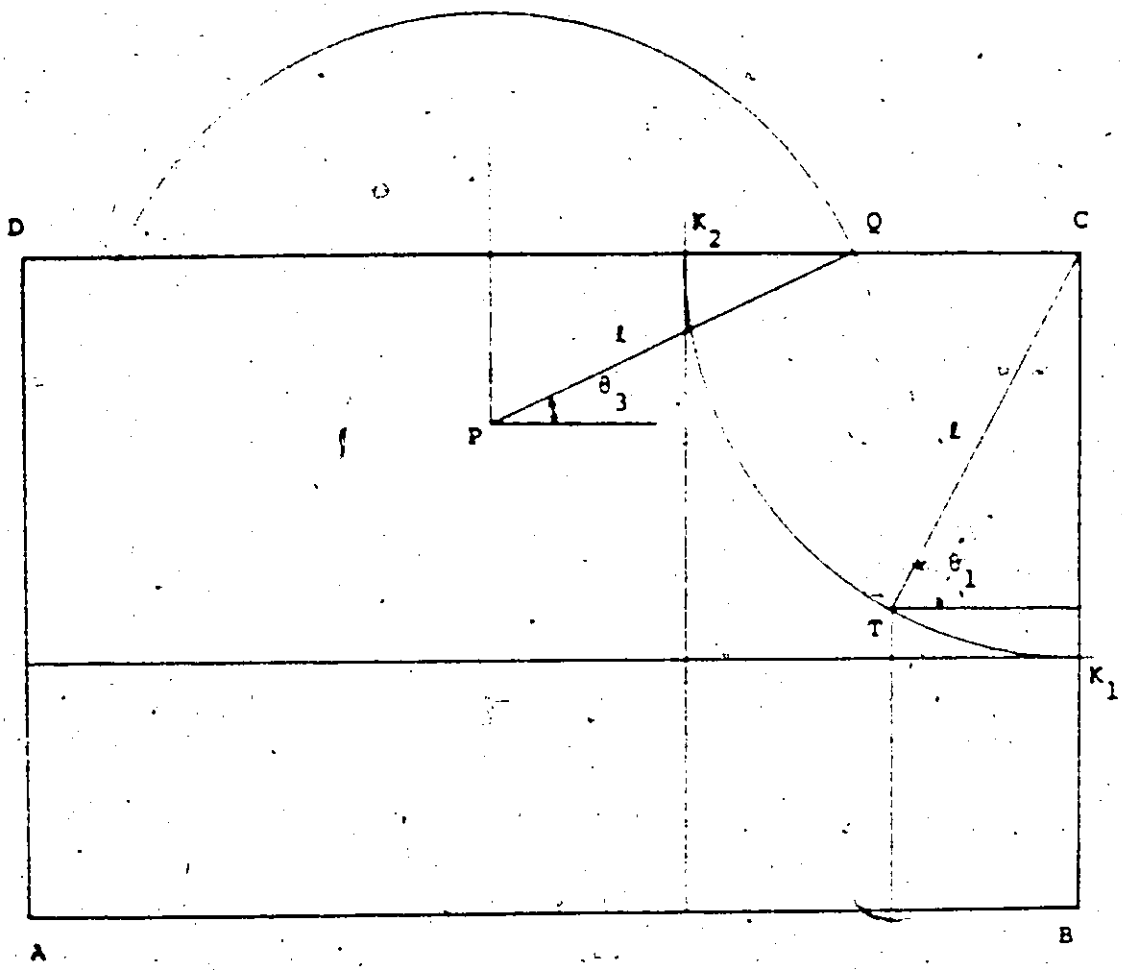

Fig. 75

(3.1.27) $$\sin\theta_3 = \frac{b-y_1}{\ell}.$$

From (3.1.27) it follows that

(3.1.28) $$\sin^{-1}\left(\frac{b-y_1}{\ell}\right) \le \theta \le \frac{\pi}{2} \Rightarrow L(x_1,y_1,\theta) \le \ell.$$

Hence $S_{14}(\ell)$ defined by (3.1.18) contributes to $Pr(L \le \ell)$.

Thus

(3.1.29) $$S_{14}(\ell) \subset D_1(\ell).$$

Clearly, a random ray specified by $(x,y,\theta) \in S$ with $L(x,y,\theta) \le \ell$ must intersect either side BC or side CD of the rectangle.

Consequently,

(3.1.30) $$(x,y,\theta) \notin D_1(\ell) \Rightarrow (x,y,\theta) \notin \bigcup_{i=1}^{4} S_{1i}(\ell).$$

Hence

$$D_1(\ell) \subset \bigcup_{i=1}^{4} S_{1i}(\ell).$$

By virtue of (3.1.22), (3.1.23), (3.1.26), (3.1.29) and (3.1.30) we obtain (3.1.14).

3.1.4. Determination of the set $D_1(\ell)$ for $\ell\in[b,a]$.

In the following lemma we obtain the set $D_1(\ell)$ for $\ell\in[b,a]$.

Lemma 2. Let $\ell\in[b,a]$. Then

(3.1.31) $$D_1(\ell) = S_{21}(\ell)\cup S_{22}(\ell)\cup S_{23}(\ell)\cup S_{24}(\ell)\cup S_{25}(\ell),$$

where,

(3.1.32) $$S_{21}(\ell) = \{(x,y,\theta): a - \ell \le x \le a - (\ell^2-b^2)^{1/2}, 0 \le y \le b - \{\ell^2-(a-x)^2\}^{1/2}, 0 \le \theta \le \cos^{-1}(\tfrac{a-x}{\ell})\},$$

(3.1.33) $$S_{22}(\ell) = \{(x,y,\theta): a - \ell \le x \le a - (\ell^2-b^2)^{1/2}, 0 \le y \le b - \{\ell^2-(a-x)^2\}^{1/2}, \sin^{-1}(\tfrac{b-y}{\ell}) \le \theta \le \tfrac{\pi}{2}\},$$

(3.1.34) $$S_{23}(\ell) = \{(x,y,\theta): a-\ell\le x \le a-(\ell^2-b^2)^{1/2}; b - \{\ell^2-(a-x)^2\}^{1/2} \le y \le b, 0 \le \theta \le \tfrac{\pi}{2}\},$$

(3.1.35) $$S_{24}(\ell) = \{(x,y,\theta): a - (\ell^2-b^2)^{1/2} \le x \le a, 0 \le y \le b, 0 \le \theta \le \tfrac{\pi}{2}\}$$

and

(3.1.36) $S_{25}(\ell) = \{(x,y,\theta): 0 \le x \le a - \ell,\ 0 \le y \le b,\ \sin^{-1}(\frac{b-y}{\ell}) \le \theta \le \frac{\pi}{2}\}$.

Proof. We show that $S_{2i}(\ell) \subset D_1(\ell)$, $i = 1,2$ (cf. Fig. 76). Let $P(x,y)$ be a point in the rectangle such that

$$a - \ell \le x \le a - \sqrt{\ell^2-b^2},\ 0 \le y \le b - [\ell^2-(a-x)^2]^{\frac{1}{2}}.$$

With $P(x,y)$ as the centre, we draw a circle of radius ℓ intersecting BC and CD at Q_1 and Q_2, respectively, such that the angles θ_1 and θ_2 that Q_1P and Q_2P, respectively, hold with AB are each between 0 and $\frac{\pi}{2}$. Obviously the angles θ_1 and θ_2 are given by

(3.1.37) $$\cos\theta_1 = \frac{a-x}{\ell}$$

and

(3.1.38) $$\sin\theta_2 = \frac{b-y}{\ell}.$$

From (3.1.27) and (3.1.28) it follows that the corresponding length of the ray determined by $P(x,y)$ and θ where $0 \le \theta \le \theta_1$ or $\theta_2 \le \theta \le \frac{\pi}{2}$ is less than or equal to ℓ.

Thus the set $S_{21}(\ell) \cup S_{22}(\ell)$ where $S_{21}(\ell)$ and $S_{22}(\ell)$ and defined by (3.1.32), and (3.1.33) respectively, contributes to $Pr(L \le \ell)$. Consequently,

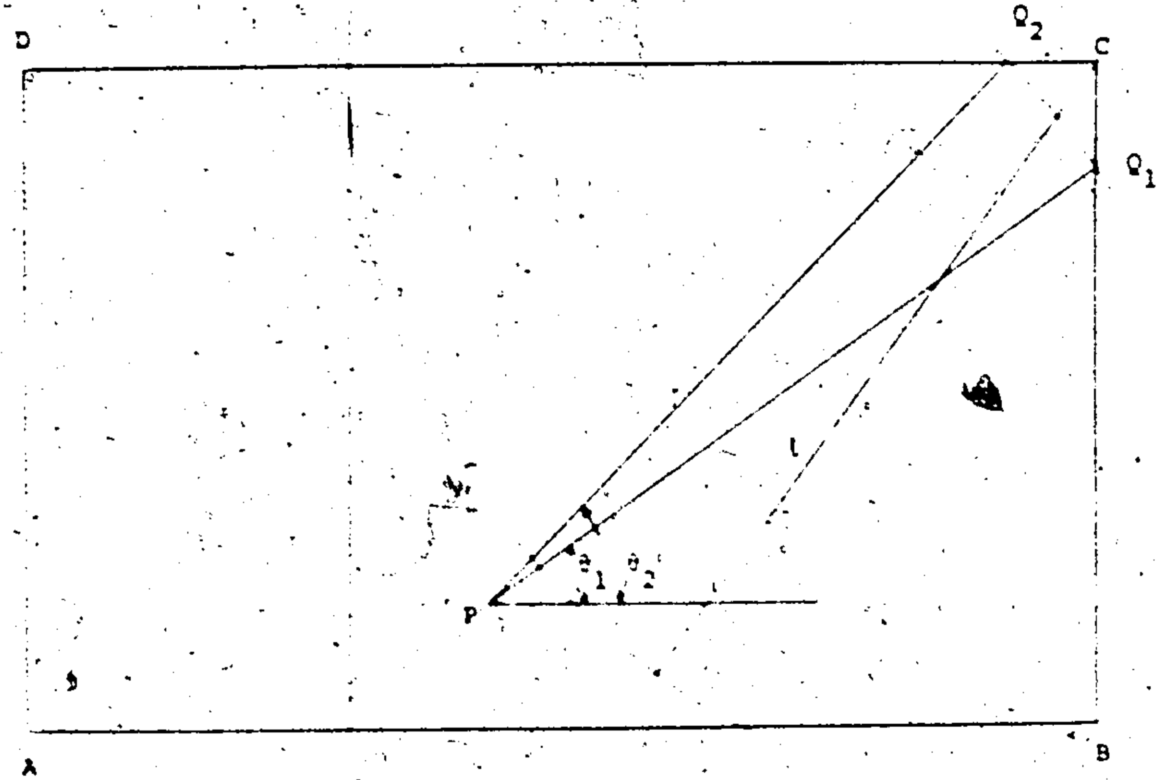

Fig. 76

(3.1.39) $$S_{21}(\ell)\cup S_{22}(\ell)\subset D_1(\ell)$$

We next show that $S_{2i}(\ell)\subset D_1(\ell)$, $i = 3,4$ (cf. Fig. 77).

Let $P(x,y)$ be a point in the interior of the rectangle such that $(x-a)^2 + (y-b)^2 \leq \ell^2$. Then $L(x,y,\theta) \leq \ell$ whenever $0 \leq \theta \leq \frac{\pi}{2}$. Hence we have a contribution to $Pr(L \leq \ell)$ by sets $S_{23}(\ell)$ and $S_{24}(\ell)$ which are defined by (3.1.34) and (3.1.35), respectively. Thus

(3.1.40) $$S_{23}(\ell)\cup S_{24}(\ell) \subset D_1(\ell) .$$

We finally show that $S_{25}(\ell)\subset D_1(\ell)$ (cf. Fig. 78).

Let $P(x,y)$ be a point in the interior of the rectangle such that

$$0 \leq x \leq a - \ell \text{ and } 0 \leq y \leq b ,$$

With $P(x,y)$ as the centre we draw a circle of radius ℓ intersecting BC at a point Q such that the angle θ_4 that QP makes with AB is $\leq \frac{\pi}{2}$. The angle θ_4 is given by

(3.1.41) $$\sin\theta_4 = \frac{b-y}{\ell} .$$

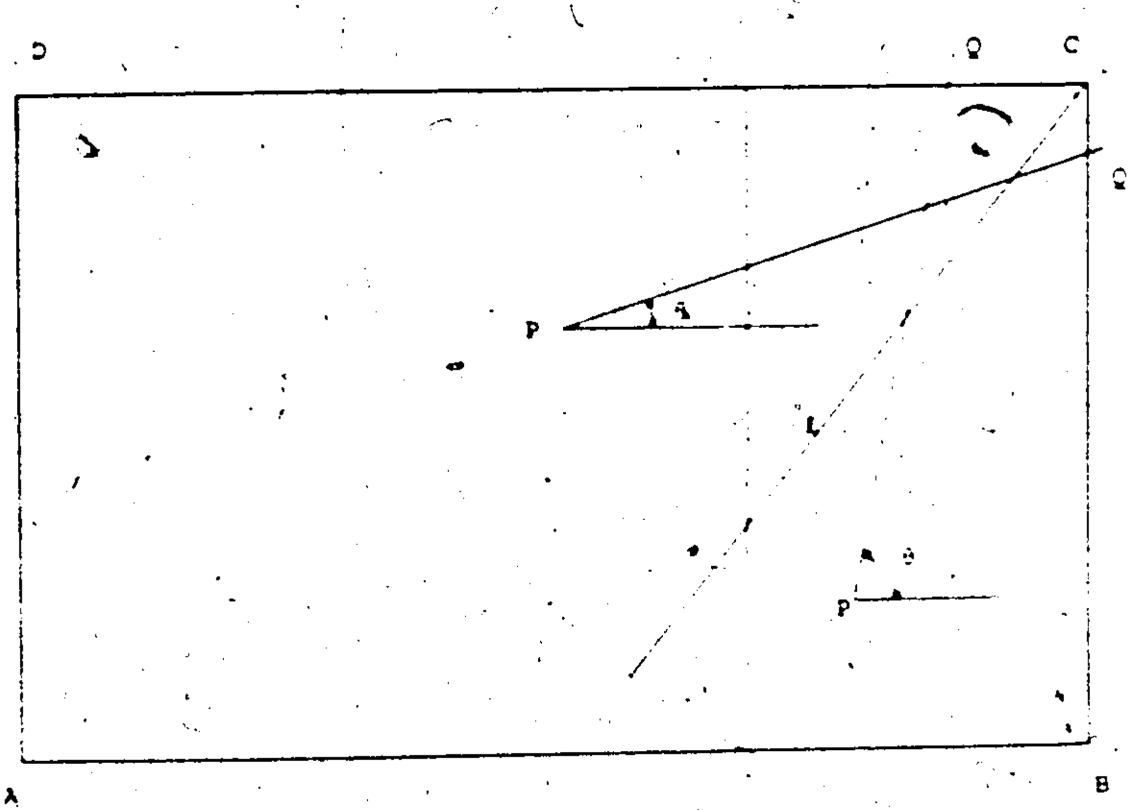

Fig. 77

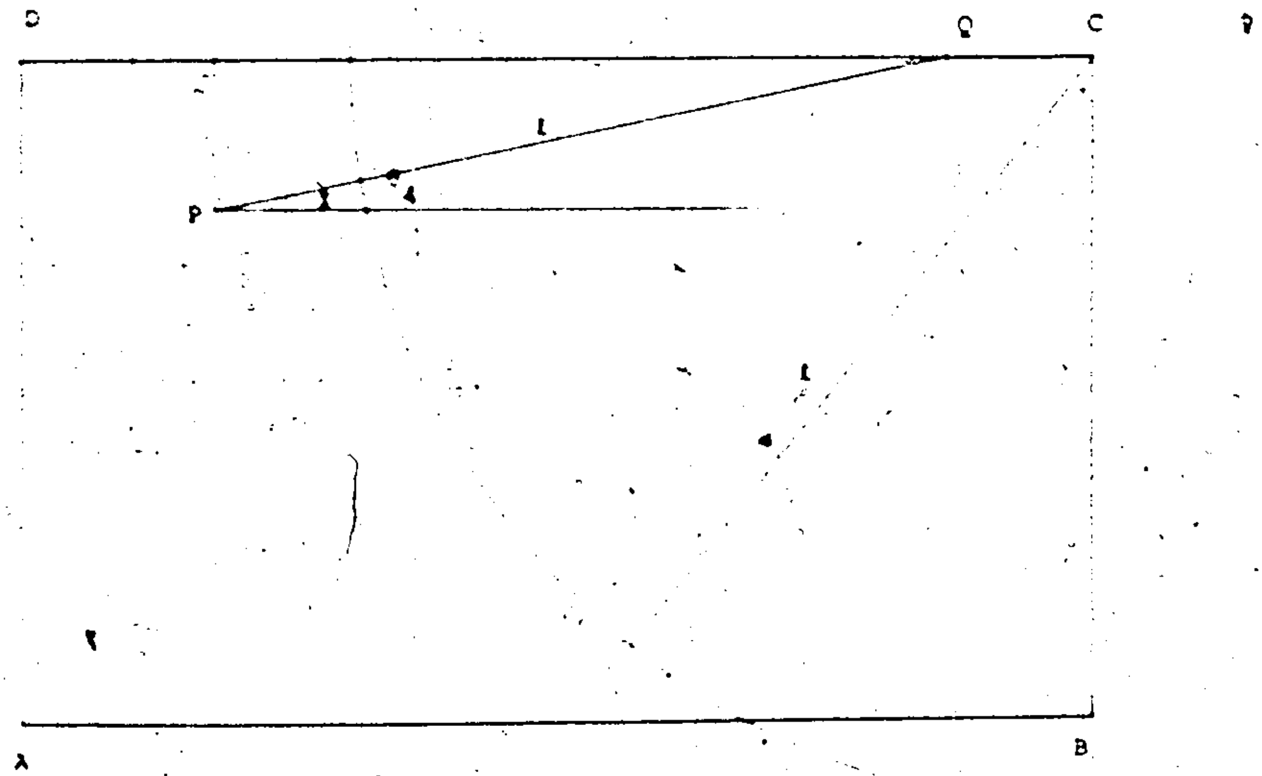

Fig. 78

From (3.2.41) it follows that $L(x,y,\theta) \leq \ell$ whenever $\sin^{-1} \frac{b-y}{\ell} \leq \theta \leq \frac{\pi}{2}$.

Thus the set $S_{25}(\ell)$ defined by (3.1.36) contributes to $F(\ell)$.

Hence

(3.1.42) $$S_{25}(\ell) \subset D_1(\ell).$$

Clearly

(3.1.43) $$(x,y,\theta) \notin D_1(\ell) \Rightarrow (x,y,\theta) \notin S_{2i}(\ell) \; , \; i = 1, 2, \ldots, 5 \; .$$

Hence by use of (3.1.30) (3.1.40), (3.1.42) and (3.1.43), we obtain (3.1.31).

3.1.5. Determination of the Set $D_1(\ell)$ for $\ell\in[a,(a^2+b^2)^{1/2}]$.

In the following lemma, we obtain the set $D_1(\ell)$ for $\ell\in[a,\ (a^2+b^2)^{1/2}]$.

Lemma 3. Let $\ell\in[a,\ (a^2+b^2)^{1/2}]$. Then

(3.1.44) $$D_1(\ell) = S_{31}(\ell) \cup S_{32}(\ell),$$

where

(3.1.45) $$S_{31}(\ell)$$

$$= \{(x,y,\theta):\ 0 \le x \le a - (\ell^2-b^2)^{1/2},\ b - \sqrt{\ell^2-(a-x)^2} \le y \le b,\ 0 \le \theta \le \tfrac{\pi}{2}\}$$

$$\cup \{(x,y,\theta):\ a - (\ell^2-b^2)^{1/2} \le x \le a,\ 0 \le y \le b,\ 0 \le \theta \le \tfrac{\pi}{2}\},$$

and

(3.1.46) $$S_{32}(\ell)$$

$$= \{(x,y,\theta):\ 0 \le x \le a - (\ell^2-b^2)^{1/2},\ 0 \le y \le b - (\ell^2-(a-x)^2)^{1/2},\ 0 \le \theta \le \cos^{-1}\left(\tfrac{a-x}{\ell}\right)\}$$

$$\cup \{(x,y,\theta):\ 0 \le x \le a - (\ell^2-b^2)^{1/2},\ 0 \le y \le b - (\ell^2-(a-x)^2)^{1/2},\ \sin^{-1}\left(\tfrac{b-y}{\ell}\right) \le \theta \le \tfrac{\pi}{2}\}.$$

Proof. (cf. Fig. [illegible]) Let $P(x,y)$ be a point in the interior of the rectangle such that $(a-x)^2 + (b-y)^2 \le \ell^2$.

Now

$$L(x,y,\theta) \le \ell, \text{ for } (x,y,\theta) \in S_{31}(\ell).$$

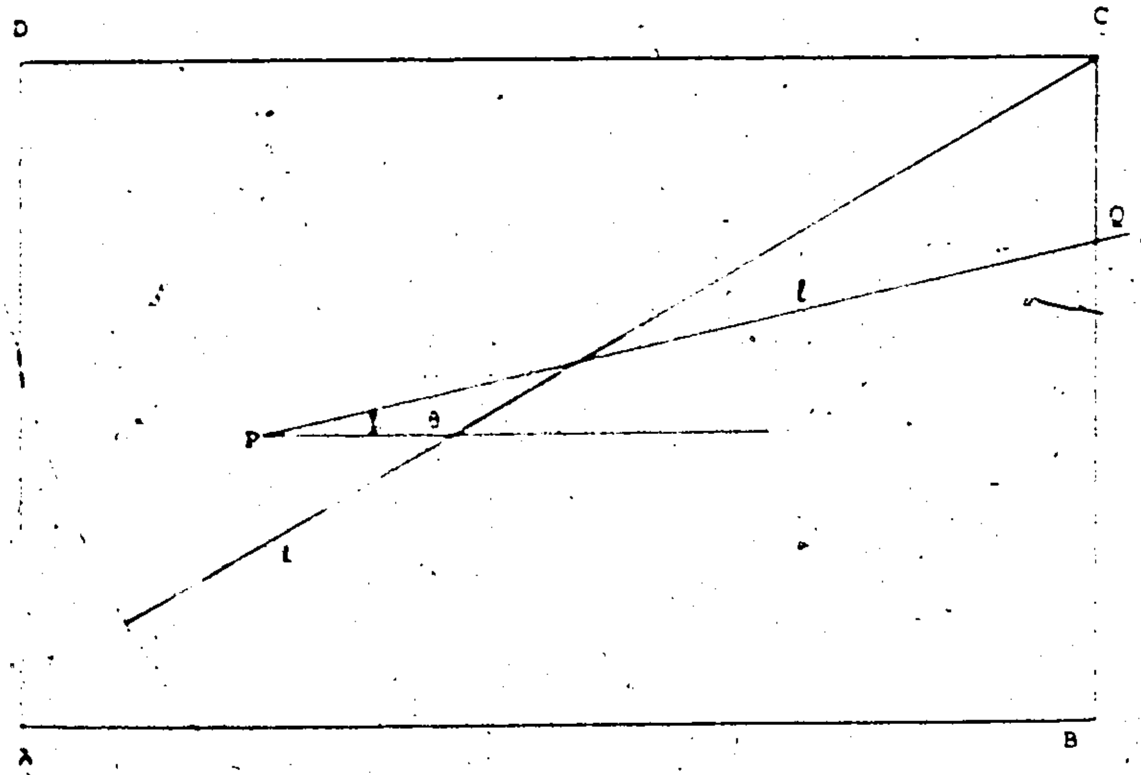

Fig. 79

Hence

(3.1.47) $$S_{31}(\ell) \subset D_1(\ell).$$

To show that $S_{32}(\ell) \subset D_1(\ell)$, let $P(x,y)$ be a point in the interior of the rectangle such that $(a-x)^2 + (b-y)^2 \geq \ell^2$ (cf. Fig. 80). With $P(x,y)$ as the centre we draw a circle of radius ℓ intersecting BC and CD at N_1 and N_2, respectively, such that the angles θ_5 and θ_6 that N_1P and N_2P, respectively, make with the line AB are each less than or equal to $\frac{\pi}{2}$. The angles θ_5 and θ_6 are given by

(3.1.48) $$\cos \theta_5 = \frac{a-x}{\ell}$$

and

(3.1.49) $$\sin \theta_6 = \frac{b-y}{\ell}.$$

From (3.1.48) and (3.1.49) it follows that

$$L(x,y,\theta) \leq \ell \text{ for } \theta \in [0, \cos^{-1}(\tfrac{a-x}{\ell})] \text{ or } \theta \in [\sin^{-1}(\tfrac{b-y}{\ell}), \tfrac{\pi}{2}].$$

Hence the set

$S_{32}(\ell)$ defined in [3.1.46] contributes to $Pr(L \leq \ell)$.

Thus

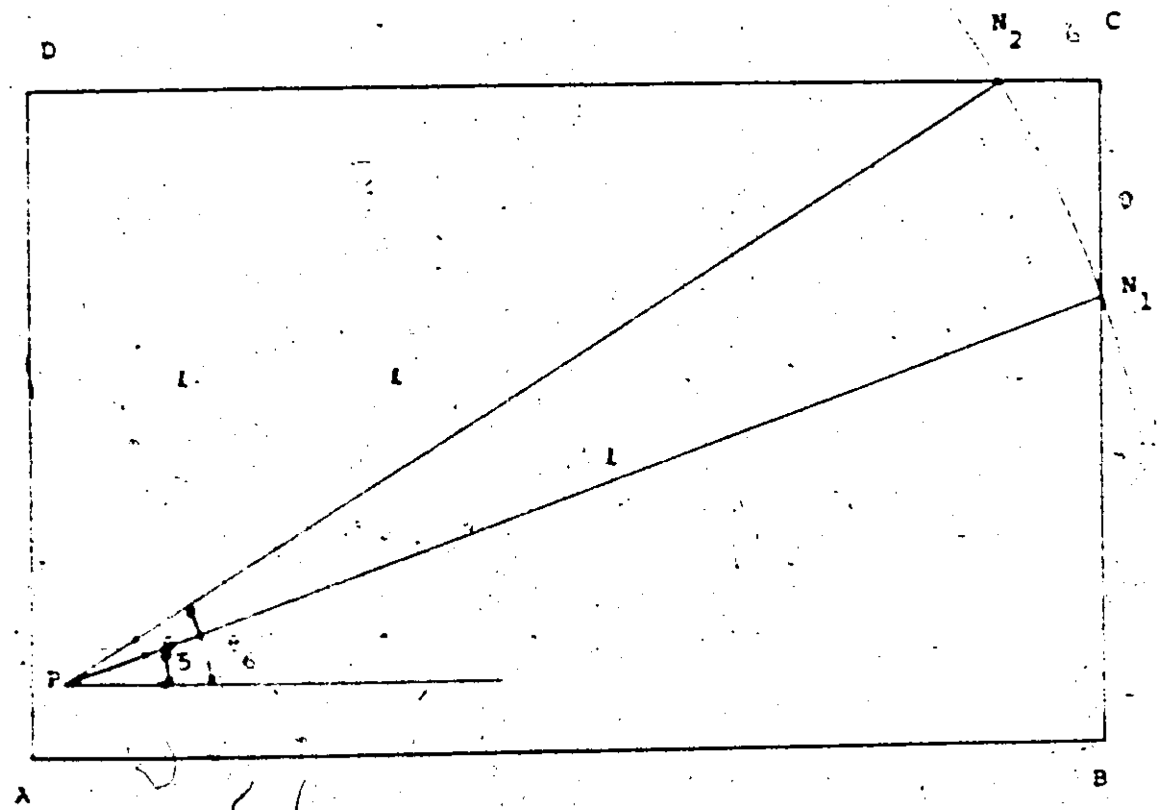

Fig.80

(3.1.50) $$S_{32}(\ell)\subset D_1(\ell).$$

By use of (3.1.48) and (3.1.49), we further obtain that

(3.1.51) $$D_1(\ell)\subset S_{31}(\ell)\cup S_{32}(\ell).$$

Now from (3.1.47), (3.1.50) and (3.1.51) we derive (3.1.44).

We are now ready to prove the theorem.

3.1.6. Proof of Theorem 11.

By Lemma 1, for $\ell\in[0,b]$, we have

(3.1.52) $$\int_{D_1(\ell)} dxdyd\theta = \int_{\bigcup_{i=1}^{4} S_{1i}(\ell)} dxdyd\theta \quad ,$$

where S_{1i}, $i = 1,2,3,4$ are defined by (3.2.15), (3.2.16), (3.2.17), and (3.2.18), respectively. Evaluating the integral (3.2.52) we obtain (A) of (3.2.1).

By Lemma 2, for $\ell\in[L,a]$, we have

(3.1.53) $$\int_{D_1(\ell)} dxdyd\theta = \int_{\bigcup_{i=1}^{5} S_{2i}(\ell)} dxdyd\theta \quad ,$$

where $S_{2i}(\ell)$, $i = 1,2,3,4,5$ are defined by (3.1.32), (3.1.33), (3.1.34), (3.1.35), and (3.1.36), respectively. Evaluating the integral (3.1.53), we obtain (B) of (3.1.1).

By Lemma 3, for $\ell \in [a,(a^2+b^2)^{\frac{1}{2}}]$, we have

(3.1.54)
$$\int_{D_1(\ell)} dxdyd\theta = \int_{S_{31}(\ell) \cup S_{32}(\ell)} dxdyd\theta ,$$

where $S_{31}(\ell)$ and $S_{32}(\ell)$ are defined by (3.1.45) and (3.1.46), respectively. Evaluating the integral (3.1.54), we obtain (C) of (3.1.1I).

SECTION TWO

3.2. PROBABILITY DISTRIBUTIONS OF LENGTHS OF RANDOM RAYS FROM THE INTERIOR OF A CIRCLE - THE RANDOM SOURCE IS ON A DIAMETRAL LINE.

3.2.1. Introduction.

In this section, we shall obtain the probability distribution of the random length of a ray emanating in a random direction from a random source on a diametral line of a circle. The result for the distribution is obtained in terms of integrals which cannot be exactly evaluated, but which can be reduced to elliptic integrals of the first and second kinds. Since tables for elliptic integrals are readily available, one can calculate the desired probabilities approximately with the help of the tables. A precise statement of the problem is the following: Let C be a circle of radius a and AOB is a diametral line with O as the centre of the circle (cf. Fig. 81). Suppose a ray emanates from a random source P on AOB in a random direction. The random ray is intercepted by the circle at a point Q. We are interested in the probability distribution of the length $L = PQ$.

3.2.2. Probability distribution of the random ray length L.

The following theorem provides the probability distribution of L.

<u>Theorem 12</u>. Let a be the radius of a circle. Then the probability distribution of the length ℓ of a random ray in a random direction emanating from a random source on a diametral line of the circle is given by

$$
(3.2.1)\quad F(\ell) = \begin{cases} \dfrac{1}{\pi a}\left[\displaystyle\int_{a-\ell}^{\sqrt{a^2-\ell^2}} \int_{0}^{\cos^{-1}\left(\frac{a^2-\ell^2-x^2}{2\ell x}\right)} d\theta dx + \int_{-a}^{-\sqrt{a^2-\ell^2}} \int_{\cos^{-1}\left(\frac{x^2-a^2+\ell^2}{2\ell x}\right)}^{\frac{\pi}{2}} d\theta dx + \frac{\pi}{2}\left(a-\sqrt{a^2-\ell^2}\right)\right] & \text{for } \ell\epsilon[0, a] , \\ \dfrac{1}{\pi a}\left[\displaystyle\int_{-a}^{a-\ell} \int_{\cos^{-1}\left(\frac{\ell^2-a^2+x^2}{2\ell x}\right)}^{\frac{\pi}{2}} d\theta dx + \frac{\pi}{2}\ell\right] , & \text{for } \ell\epsilon[a, 2a] \end{cases}
$$

Proof. We parametrize the rays as follows. Let AOB be the x-axis and O be the origin. Let (x,θ) be the parameters that specify a ray PQ, where x is measured along the diameter AOB and θ is the angle (measured in the anti-clockwise direction) that PQ makes with the diameter AOB. We note that: X is uniformly distributed on $[-a,a]$, Θ is uniformly distributed on $[0,2\pi]$ and the distributions of X and Θ are independent. The joint density of X and Θ is given by

$$(3.2.2) \qquad P(x,\theta) = \begin{cases} \frac{1}{4a\pi}, & -a \le x \le a,\ 0 \le \theta \le 2\pi, \\ 0, & \text{elsewhere.} \end{cases}$$

The parameter space is given by the set $S = \{(x,\theta): -a \le x \le a,\ 0 \le \theta \le 2\pi\}$. L is a random variable defined on S. Our object is to find

$$(3.2.3) \qquad F(\ell) = \Pr(L \le \ell) = \frac{1}{4a\pi} \int_{D(\ell)} dx\, d\theta,$$

where

$$D(\ell) = \{(x,\theta): L(x,\theta) \le \ell,\ -a \le x \le a,\ 0 \le \theta \le 2\pi\},$$

and $L(x,\theta)$ is the length of the ray determined by (x,θ). From symmetry, the set D can be decomposed into the following four "equivalent" sets:

$$(3.2.4) \qquad D_1(\ell) = \{(x,\theta): L(x,\theta) \le \ell,\ -a \le x \le a,\ 0 \le \theta \le \frac{\pi}{2}\},$$

$$(3.2.5) \qquad D_2(\ell) = \{(x,\theta): L(x,\theta) \le \ell,\ -a \le x \le a,\ \frac{\pi}{2} \le \theta \le \pi\},$$

(3.2.6) $D_3(\ell) = \{(x,\theta): L(x,\theta) \le \ell, -a \le x \le a, \pi \le \theta \le \frac{3\pi}{2}\}$,

and

(3.2.7) $D_4(\ell) = \{(x,\theta): L(x,\theta) \le \ell, -a \le x \le a, \frac{3\pi}{2} \le \theta \le 2\pi\}$.

It follows that

$$\int_{D_1(\ell)} dx d\theta = \int_{D_i(\ell)} dx d\theta \ , \ i = 2,3,4.$$

Hence (3.2.3) reduces to

(3.2.8) $$F(\ell) = \frac{1}{4a^2} \int_{D_1(\ell)} dx d\theta,$$

where $D_1(\ell)$ is given by (3.2.4). In the following two lemmas we determine the set $D_1(\ell)$ for $0 \le \ell \le a$ and $a \le \ell \le 2a$.

Lemma 1. For $\ell \in [0,a]$,

(3.2.9) $$D_1(\ell) = S_{11}(\ell) \cup S_{12}(\ell),$$

where

(3.2.10) $$S_{11}(\ell) = \{(x,\theta): a - \ell \le x \le \sqrt{a^2-\ell^2},$$
$$0 \le \theta \le \cos^{-1}\left(\frac{a^2-\ell^2-x^2}{2\ell x}\right)\}$$
$$\cup \{(x,\theta): \sqrt{a^2-\ell^2} \le x \le a, \ 0 \le \theta \le \frac{\pi}{2}\} .$$

and

(3.2.11) $$S_{12}(\ell) = \{(x,\theta): = \{(x,\theta): -a \le x \le -\sqrt{a^2-\ell^2}\,,$$

$$\cos^{-1}\left(\frac{x^2-a^2+\ell^2}{2\ell x}\right) \le \theta \le \frac{\pi}{2}\} \,.$$

Proof. (cf. Fig. 81). Let K be the point on AB such that $|BK| = \ell$. Let T_1 and T_2 be points where a line parallel to AB and at distance ℓ from AB intersect the circle C. Draw T_1M and T_2N perpendicular to AB. Then $OM = \sqrt{a^2-\ell^2}$ and $ON = -\sqrt{a^2-\ell^2}$. Let $P(x_1)$ (P determined by x_1) be a point on KM such that $a - \ell \le x_1 \le (a^2-\ell^2)^{\frac{1}{2}}$. With $P(x_1)$ as the centre, we draw a circle C_1 of radius ℓ. Since $|BP(x_1)| \le \ell \le |A\,P(x_1)|$, the circle C_1 intersects C at two points Q and Q'. Let $\angle Q\,P(x_1)B = \theta_1$. Then from the triangle $OQP(x_1)$, we have $x_1^{\,2}+\ell^2+2\ell x_1 \cos\theta_1 = a^2$.

Thus $\theta_1 = \cos^{-1}\dfrac{a^2-x_1^2-\ell^2}{2\ell x_1}$. Since $|BP(x_1)| \le \ell$ and $|P(x_1)Q| = |P(x_1)Q'| = \ell$, the arc QBQ' lies within the circle C_1. Therefore the join of $P(x_1)$ with a point on QBQ' lies within the circle. Thus we have

(3.2.12) $$x\in[a-\ell,(a^2-\ell^2)^{\frac{1}{2}}]$$
$$\text{and}\quad \theta\in[0,\cos^{-1}\left(\frac{a^2-\ell^2x^2}{2\ell x}\right)] \Rightarrow L(x,\theta) \le \ell.$$

Let now $x\in[(a^2-\ell^2)^{\frac{1}{2}}, a]$. Clearly a circle of radius ℓ with centre $P(x)$ contains within it the lengths $L(x,\theta)$ for $0 \le \theta \le \frac{\pi}{2}$.

Thus, we have

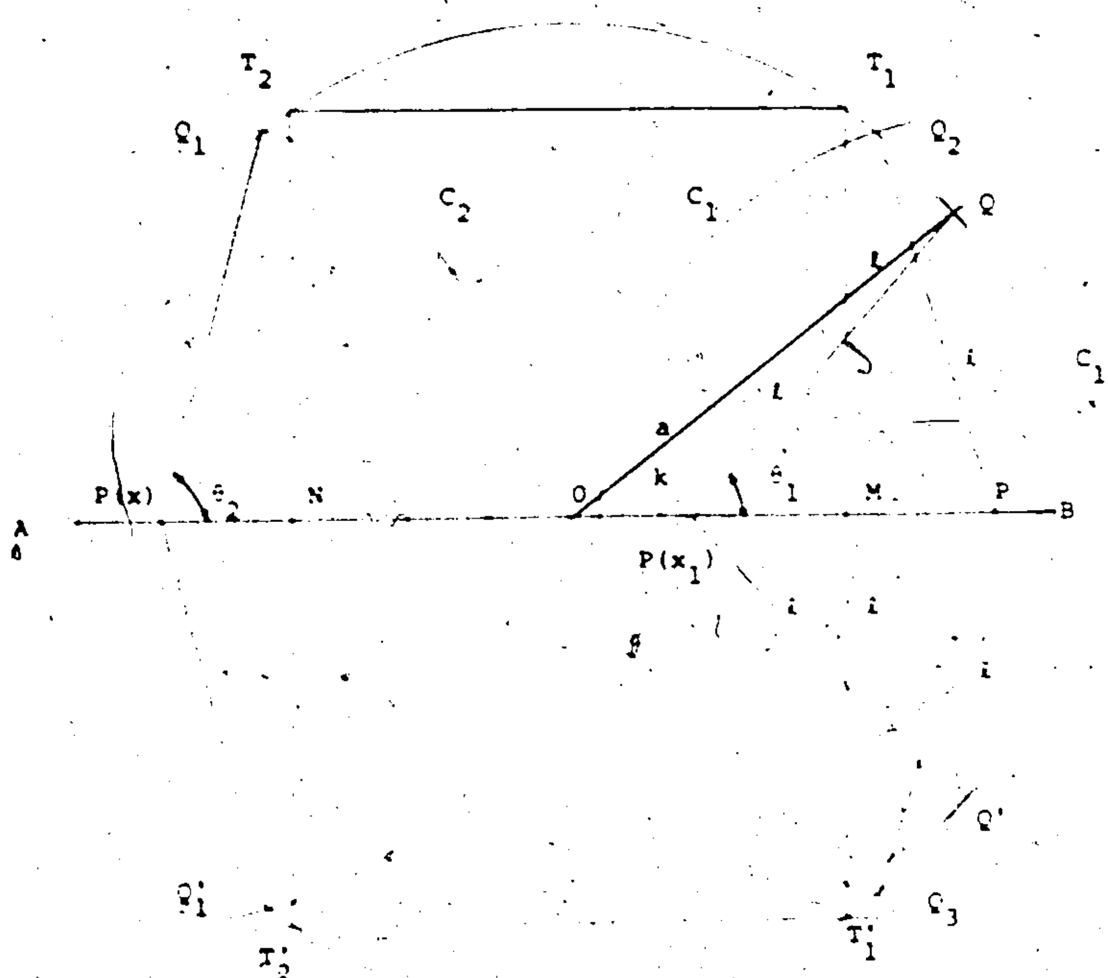

Fig. 81

(3.2.13) $\qquad x\epsilon[(a^2-\ell^2)^{\frac{1}{2}}, a]$ and $\theta\epsilon[0, \frac{\pi}{2}] \Rightarrow L(x,\theta) \leq \ell.$

Combining (3.2.12) and (3.2.13) we obtain that $S_{11}(\ell)$, defined by (3.2.10), is a subset of $D_1(\ell)$.

To show that $S_{12}(\ell)\subset D_1(\ell)$, let $x\epsilon[-a, -(a^2-\ell^2)^{\frac{1}{2}}]$. With $P(x)$ as the centre we draw a circle C_2' of radius ℓ (cf. Fig. 81). The circle C_2' intersects the circle C at points Q_1 and Q_1'. Let $\measuredangle Q_1P(x)O = \theta_2$. Then considering the triangle $Q_1P(x)O$ we obtain $a^2 = x^2 + \ell^2 - 2\ell x \cos\theta_2$, giving $\theta_2 = \cos^{-1}(\frac{x^2+\ell^2-a^2}{2\ell x})$. Clearly $\theta_2 \leq \theta \leq \frac{\pi}{2} \Rightarrow L(x,\theta) \leq \ell$. Thus $S_{12}(\ell)\subset D_1(\ell)$. Therefore

(3.2.14) $$S_{11}(\ell)\cup S_{12}(\ell)\subset D_1(\ell).$$

The complement of the set $S_{11}(\ell)\cup S_{12}(\ell)$ in the parameter space S is the set

$$\overline{S - S_{11}(\ell)\cup S_{12}(\ell)}$$

$$= \{(x,\theta): 0 \leq x \leq a - \ell,\ 0 \leq \theta \leq \frac{\pi}{2}\}\cup\{a-\ell \leq x \leq (a^2-\ell^2)^{\frac{1}{2}},$$

$$\cos^{-1}(\frac{a^2-\ell^2-x^2}{2\ell x}) \leq \theta \leq \frac{\pi}{2}\}$$

$$\cup\{(x,\theta): -(a^2-\ell^2)^{\frac{1}{2}} \leq x \leq 0,\ 0 \leq \theta \leq \frac{\pi}{2}\}\cup\{(x,\theta): -a \leq x \leq -(a^2-\ell^2)^{\frac{1}{2}},$$

$$0 \leq \theta \leq \cos^{-1}(\frac{x^2+\ell^2-a^2}{2\ell x})\}.$$

It is clear from geometrical considerations that

$$(3.2.15) \quad (x,\theta) \in [S - S_{11}(\ell) \cup S_{12}(\ell)] \Rightarrow L(x,\theta) > \ell.$$

Combining (3.2.14) and (3.2.15) we obtain (3.2.9).

Lemma 2. Let $\ell \in [a,2a]$. Then

$$(3.2.16) \qquad D_1(\ell) = S_{21}(\ell) \cup S_{22}(\ell).$$

where,

$$(3.2.17) \qquad S_{21}(\ell) = \{(x,\theta): -a \le x \le a - \ell, \quad \cos^{-1}\left(\frac{\ell^2-a^2+x^2}{2\ell x}\right) \le \theta \le \frac{\pi}{2}\}$$

$$(3.2.18) \qquad S_{22}(\ell) = \{(x,\theta): a - \ell \le x \le a, \ 0 \le \theta \le \frac{\pi}{2}\}.$$

Proof. (cf. Fig. 82). Let $P(x)$ be a point on AOB such that $-a \le x \le a - \ell$. With $P(x)$ as the centre we draw a circle C_2 of radius ℓ. Since $B\,P(x) \ge \ell$, the circle C_2 intersects the circle C at two points Q' and Q". The angle $\theta_1 = \measuredangle Q'P(x)B$ is given by $\theta_1 = \cos^{-1}\left(\frac{\ell^2-a^2+x^2}{2\ell x}\right)$. Clearly $L(x,\theta) \le \ell$ whenever $\left(\frac{\ell^2-a^2+x^2}{2\ell x}\right) \le \theta \le \frac{\pi}{2}$. Hence $S_{21}(\ell) \subset D_1(\ell)$. Also $L(x,\theta) \le \ell$, whenever $(x,\theta) \in \{(x,\theta): a - \ell \le x \le a, \ 0 \le \theta \le \frac{\pi}{2}\}$. Thus $S_{22}(\ell) \subset D_1(\ell)$. Whence

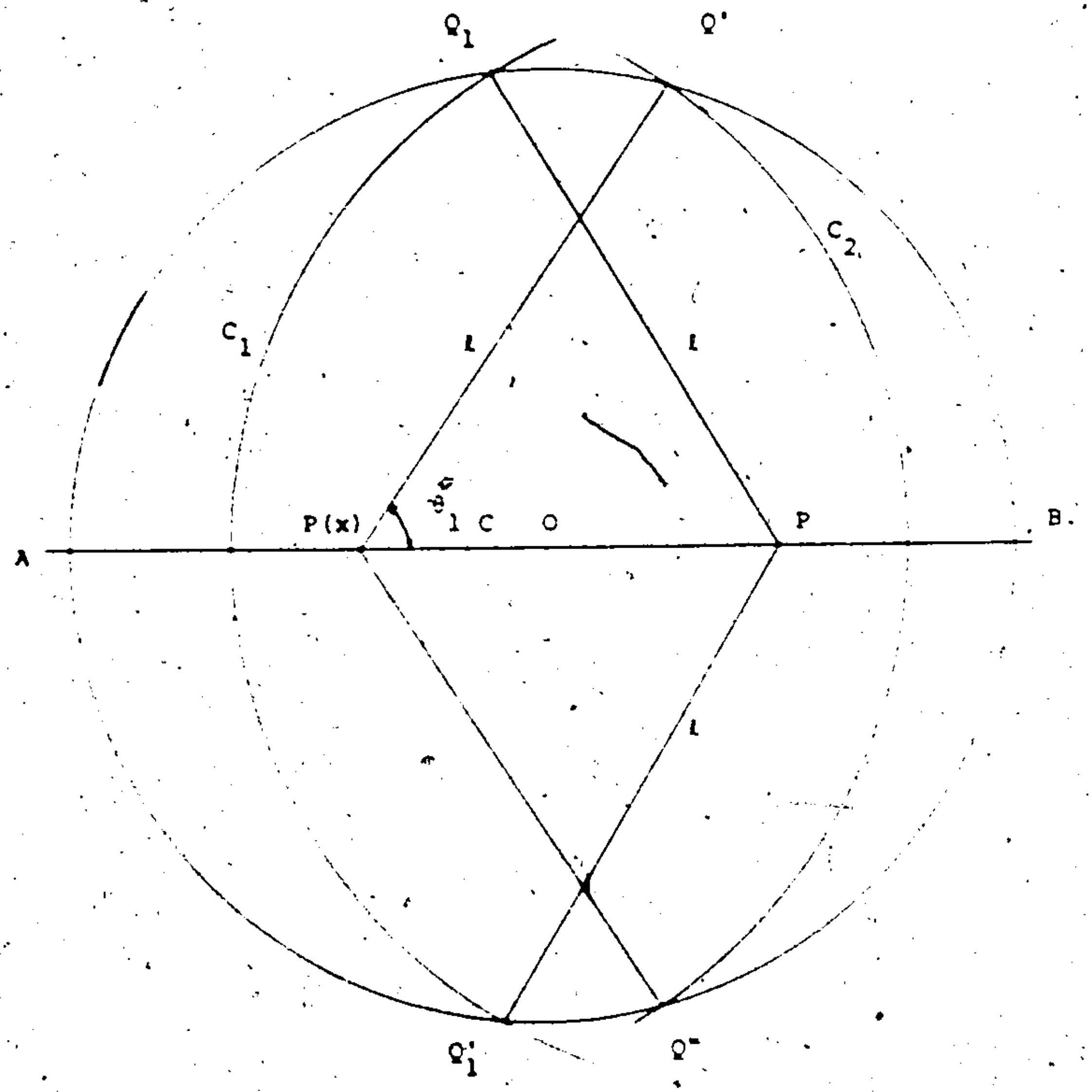

Fig. 82

(3.2.19) $$S_{21}(\ell) \cup S_{22}(\ell) \subset D_1(\ell).$$

The complement of the set $S_{21}(\ell) \cup S_{22}(\ell)$ in S is the set

$$S' = \{(x,\theta): -a \leq x \leq a - \ell,\ 0 \leq \theta \leq \cos^{-1}\left(\frac{\ell^2 - a^2 + x^2}{2\ell x}\right)\}.$$

Clearly $L(x,\theta) > \ell$ whenever $(x,\theta) \in S'$.

Therefore we have

(3.2.20) $$D_1(\ell) \subset S_{21}(\ell) \cup S_{22}(\ell).$$

Combining (3.2.19) and (3.2.20) we obtain (3.2.16).

Now that we have determined the set $D_1(\ell)$ for $0 \leq \ell \leq a$ and $a < \ell \leq 2a$ in Lemma 1 and Lemma 2 we can proceed to evaluate (3.2.8).

SECTION THREE

3.3 PROBABILITY DISTRIBUTIONS OF LENGTHS OF RANDOM RAYS FROM THE INTERIOR OF A SPHERE - THE RANDOM SOURCE IS ON A DIAMETRAL LINE.

3.3.0 Introduction.

In this section we consider the probability distribution of the random length of a random ray emanating in a random direction from a random source on a diametral line of a sphere. The result for the distribution in the case of the sphere is, contrary to our expectation, simpler than the one for the corresponding case of the circle. A precise formulation of the problem is the following.

Let $S(0,a) = \{(x,y,z): x^2+y^2+z^2 = a^2\}$ be a sphere of radius a and A'0A be the x-axis. Let P be a random point on A'0A. From the point P let there be a ray in a random direction. A random direction is determined by the position of a random point Z on the surface of a unit sphere. The ray through P in a random direction intersects the surface of the sphere S(0,a) at a point Q. The probability distribution of the length $L = |PQ|$ is of interest to us.

3.3.1. Probability distribution of the random ray length L.

The following theorem provides the probability distribution of L.

Theorem 13. Let a be the radius of a sphere. Then the probability distribution of the length L of a random ray in a random direction emanating from a random source on a diametral line of the sphere is given by

$$(3.3.2)\qquad F(\ell) = \frac{1}{4} + \frac{3\ell}{8a} + \frac{1}{4a\ell}(a^2 - \ell^2)\ln\frac{|a-\ell|}{a}, \text{for } \begin{cases} 0 \le \ell < a, \\ a < \ell \le 2a. \end{cases}$$

Proof. Let X = (x,0,0) be the position vector of the point P. Then X has uniform distribution on [-a, a]. By the uniform distribution of the direction Σ of a ray we mean that if S is any region of area A on the surface of the unit sphere, the probability that a ray passes through S is $\frac{A}{4\pi}$. The distribution for the direction Σ of the ray and the distribution for the source X of the ray are independent. Our object is to find

$$(3.3.1)\qquad F(\ell) = \Pr(L \le \ell) = \frac{1}{8a\pi}\int_{D(\ell)} dx\, ds(n),$$

where ds(n) denotes an element of surface area on the unit sphere at the point n and $D(\ell) = \{(x, n): x \in [-a,a],\ n \in S(x,1) \text{ and } L(x,n) \le \ell\}$.

We note that

$$S(x,1) = \{X: (X - x)\cdot(X - x) = 1\}$$

is the sphere of radius 1 and centre

$\underset{\sim}{x} = (x,0,0)$ and $L(\underset{\sim}{x},\underset{\sim}{n})$ is the length of the ray determined by $\underset{\sim}{x} = (x,0,0)$ and direction $\underset{\sim}{n} = (n_1, n_2, n_3)$ with $\sum_{i=1}^{3} n_i^2 = 1$.

To find the probability distribution of L, we need to find the set

$$D(\ell) = \{(\underset{\sim}{x},\underset{\sim}{n}) : x \varepsilon [-a, a], \; \underset{\sim}{n} \; \varepsilon \; S(\underset{\sim}{x},1) \text{ and } L(\underset{\sim}{x},\underset{\sim}{n}) \leq \ell\} .$$

We deduce this set from geometrical considerations.

We have two cases : $\ell \varepsilon [0,a]$ and $\ell \varepsilon [a,2a]$.

Case 1: $0 \le \ell \le$. Let A_1 be a point on OA such that $|A_1A| = \ell$. Draw a line CD parallel to A'A at a distance ℓ from O and contained in the (x,y)-plane. Then the point D, which is on the circle of intersection of the sphere S(0,a) and the plane z = 0 is $(\sqrt{a^2-\ell^2}, \ell, 0)$. Let $P(x_1)$ be a point on OA such that $a - \ell \le x_1 \le (a^2-\ell^2)^{1/2}$. With $P(x_1)$ as the centre we draw a sphere $S(P,\ell)$ of radius ℓ. The x-coordinate of the centre of the circle of intersection of the spheres $S(P,\ell)$ and S(o,a) is given by $x = \frac{a^2-\ell^2+x_1^2}{2x_1}$. The semi-vertical angle α_1, subtended at the point $P(x_1)$ of the cone formed by the joins of $P(x_1)$ with the points of the circle $C = S(P,\ell) \cap S(o,a)$ is given by $\cos \alpha_1 = \frac{x-x_1}{\ell} = \frac{a^2-\ell^2-x_1^2}{2\ell x_1}$. The measure of the solid angle $W(\alpha_1)$ at the vertex $P(x_1)$ of the cone of semi-vertical angle α_1 is given by the area of the surface subtended by the cone on a unit sphere with $P(x_1)$ as the centre. This surface area is $2\pi(1 - \cos \alpha_1)$.

Now $L(x_1,n) \le \ell$ if $(x_1,n) \in S'$, where

$$S' = \{(x,n) : x \in [a-\ell, (a^2-\ell^2)^{1/2}],\ n \in S(x, 1) \cap C(x,\alpha), \text{for } 0 \le \alpha \le \alpha_1\}, \text{ and}$$

where $x = (x,0,0)$ and $C(x,\alpha)$ is the cone with vertex x and semi-vertical angle α. The Volume $V(S')$ of the set S' is given by

$$V(S') = \int_{a-\ell}^{(a^2-b^2)^{1/2}} \int_{\alpha=0}^{\alpha=\alpha_1} dw(\alpha)dx = \int_{a-\ell}^{(a^2-\ell^2)^{1/2}} \int_{\alpha=0}^{\alpha=\alpha_1} d[2\pi(1-\cos\alpha)]dx .$$

Thus

$$V(S') = 2\pi[(a^2-\ell^2)^{1/2} - \frac{a^2-\ell^2}{2\ell} \ln\frac{(a^2-\ell^2)^{1/2}}{a-\ell} - \frac{a-\ell}{2}]. \tag{3.3.3}$$

Now let $P(x_1)$ be a point on OA such that $(a^2-\ell^2)^{\frac{1}{2}} \leq x_1 \leq a$. With $P(x_1)$ as the centre draw a sphere $S(P,\ell)$ of radius ℓ. The centre of the circle of intersection $C_1 = S(P,\ell) \cap S(o,a)$ of the spheres $S(P,\ell)$ and $S(o,a)$ is given by $(\frac{a^2-\ell^2+x_1^2}{2x_1}, 0, 0)$. The semi-vertical angle α_1 of the cone formed by the joins of the circle C_1 to $P(x_1)$ is given by $\cos\alpha = \frac{x_1-x}{\ell} = \frac{x_1^2-a^2+\ell^2}{2\ell x_1}$. Then the solid angle $W'(\alpha_1)$, subtended by the cone $C(P,\alpha_1)$ and the unit sphere $S(x_1,1)$, is given by $2\pi(1-\cos\alpha_1) = 2\pi(1 - \frac{x_1^2-a^2+\ell^2}{2\ell x_1})$. The complement of $W'(\alpha_1)$ is $4\pi - 2\pi(1-\cos\alpha_1) = 2\pi + \pi[\frac{x_1}{\ell} - \frac{(a^2-\ell^2)}{\ell} \cdot \frac{1}{x_1}]$. Now $L(x,n) \leq \ell$, whenever $(x,n) \in S''$, where $S'' = \{(x,n): x\in[(a^2-\ell^2)^{\frac{1}{2}}, a]$ and $n \in S(x,1)\} \cap$ (complement of $C(x,\alpha)$).

The volume $V(S'')$ of S'' is given by

$$(3.3.4) \qquad V(S'') = \int_{\sqrt{a^2-\ell^2}}^{a} 2\pi + \pi\{\frac{x}{\ell} - \frac{(a^2-\ell^2)}{\ell} \cdot \frac{1}{x}\}\, dx$$

$$= 2\pi a + \frac{\pi\ell}{2} - 2\pi(a^2-\ell^2)^{\frac{1}{2}} + \pi(\frac{a^2-\ell^2}{\ell}) \ln \frac{(a^2-\ell^2)^{\frac{1}{2}}}{a} .$$

From symmetric situations we have similarly two other sets S_1' and S_1'' 'equivalent' respectively to S' and S'' such that $L(x,n) \leq \ell$ whenever $(x,n) \in S_1' \cup S_1''$, and such that volume of S_1' = volume of S' and volume of S_1'' = volume of S''.

Therefore f... $\leq a$,

$$F(\ell) = \frac{1}{8a\pi} \cdot 2\,[\text{volume of } (S' \cup S'')]$$
$$= \frac{1}{4} + \frac{3\ell}{8a} + \frac{1}{4a\ell}(a^2-\ell^2) \ln (\frac{a-\ell}{a}) \text{ ; by (3.3.3) and (3.3.4)}$$

Case 2: $a \leq \ell \leq 2a$. Let B' and B be the points on A'A such that $|OB'| = |OB| = \ell - a$. Let $(\underset{\sim}{x}, \underset{\sim}{n}) \in S_{21}$, where $S_{21} = \{(\underset{\sim}{x}, \underset{\sim}{n}): a-\ell \leq x \leq \ell - a,\ \underset{\sim}{n} \in S(\underset{\sim}{x}, \ell)\}$. Then $L(\underset{\sim}{x}, \underset{\sim}{n}) \leq \ell$.

The volume of the set S_{21} is given by

(3.3.5) $$V(S_{21}) = 8\pi(\ell - a).$$

Let $-a \leq x_1 \leq a-\ell$. With $P(x_1)$ as the centre we draw a sphere $S(P,\ell)$ of radius ℓ. The intersection $C_2 = S(P,\ell) \cap S(o,a)$ is a circle with centre

$$\left(\frac{a^2-\ell^2+x_1^2}{2x_1},\ 0,\ 0\right).$$

The semi-vertical angle α_1 of the cone formed by the circle C_2 and the vertex $P(x_1)$ is given by $\cos\alpha_1 = \frac{a^2-\ell^2-x_1^2}{2\ell x_1}$. The solid angle at the vertex of the cone $C(P,\alpha_1)$ is given by $2\pi(1 - \cos\alpha_1)$.

Now $L(\underset{\sim}{x}, \underset{\sim}{n}) \leq \ell$ whenever $(\underset{\sim}{x}, \underset{\sim}{n}) \in S_{22}$, where

$$S_{22} = \{(\underset{\sim}{x}, \underset{\sim}{n}): x \in [-a, a-\ell] \text{ and } \underset{\sim}{n} \in S(\underset{\sim}{x},1) - S(\underset{\sim}{x},1) \cap C(P,\alpha_1)\}$$

The area of $S(\underset{\sim}{x},1) - S(\underset{\sim}{x},1) \cap C(P,\alpha_1)$ is $2\pi(1 + \cos\alpha_1)$. The volume of S_{22} is

$$V(S_{22}) = \int_{-a}^{a-\ell} 2\pi\left(1 + \frac{a^2-\ell^2-x_1^2}{2\ell x_1}\right) dx_1 .$$

Therefore

(3.4.6) $$V(s_{22}) = 2\pi\left[\frac{5a}{2} - \frac{5\ell}{4} + \frac{a^2-\ell^2}{2\ell}\ln\left(\frac{\ell-a}{a}\right)\right].$$

From symmetric situations we have similarly a set S_{22} 'equivalent' to S_{22} such that

$$(\underset{\sim}{x},\underset{\sim}{n}) \in S'_{22} \Rightarrow L(\underset{\sim}{x},\underset{\sim}{n}) \leq \ell$$

and

(3.3.7) $$V(S_{22}) = V(S'_{22}).$$

Therefore for $\ell\epsilon[a, 2a]$,

$$F(\ell) = \frac{1}{8a\pi}\{V(S_{21} \cup S_{22} \cup S'_{22})\} = \frac{1}{4} + \frac{3\ell}{8a} + \frac{1}{4a\ell}(a^2-\ell^2)\ln\left(\frac{\ell-a}{a}\right)$$

by (3.3.5), (3.3.6) and (3.3.7).

APPLICATIONS.

There are some practical situations where one requires the statistical properties of random path lengths through convex configurations. Coleman* [9] in his paper on random paths through rectangles and cubes discusses applications of his work in metallurgy and makes a number of references to applications in other fields, e.g., Hensler [26], Exner [16], Nicholson [54], Watson [72].

It is mentioned in the paper by Coleman [9], that for the problem in metallurgy for estimating distribution of sizes of particles randomly embedded in an opaque material one method in use is that of placing a straight line probe through the material and observing the lengths of the segments intersecting the particles. The distribution of the lengths of these line segments is related to the distributions of the lengths of paths through the particles arising out of other random mechanisms. We are thus free to use the easiest mathematical analysis and obtain the theoretical results for the stereological probe by a simple transformation.

Moran [49] discusses in detail the geometric probability basis of the theory of estimating the properties of the distributions of one material within another by investigating the observed patterns of plane and line intersects. He considers the theory of linear probes and the estimation of spatial properties on a plane section.

Instead of making estimates of the volume structure from observations on plane sections (cf. Bach [3] , Nicholson [54]) one can

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* I am grateful to Dr. Coleman of the Department of Mathematics, Imperial College, London, for sending me some references concerning applications.

also consider estimates from linear traverses, i.e., by observing the material along randomly positioned straight lines.

The problem of random lines through convex bodies occurs in technologies other than metallurgy: e.g., in acoustics [5], [36], in atomic reactor design [58], in radiology [30], and in hydrology [38]. Work in these fields has also appeared in proceedings of conferences on Stereology (cf. [15] and Journal of Microscopy 95 (1972) 1-396).

Hawksley [25] and Phillips [57] discussed the use of the equivalent problem of taking a random line across the projected image of particles in a transparent section. Primak [58] considered the distribution functions of all possible path lengths from the interior of the bodies considered in the sections; Geometric Probabilities for Emergence and Self-Absorption Calculations and Geometric Probabilities for calculating the Local Flux. In his paper he considers the source fixed in the interior of the configuration, whereas in Chapter Three of the thesis we are considering the source to be "random" in the interior and thus we are dealing with a more general situation.

Kendall and Moran [31] investigated the "mean free path" of secants of a convex body K and found it to be 4v/s, where v is the volume and s is the surface area of K. The problem is connected with the study of "mean free paths". Among others who considered the geometrical quantity "mean free paths" and its applications are Kingman [34], Bate and Pillow [5] and Kosten [36].

The results of the thesis could be of use in the determination of probabilities in some theoretical situations such as the following:

1. Consider an object randomly situated on a regular polygonal

path, a triangular path, a rectangular path or a circular path. Suppose the object moves in a random direction and crosses a side of the path. We can calculate the probability of its travelling a distance ≤ d before crossing the path again from the results in Chapter One of the thesis.

2. Let two objects be randomly situated on a polygonal path. We can determine the probability of the distance d between the two objects falling in a given interval from the results in Chapter Two of the thesis.

3. Suppose that a piloted object is moving in a straight path in a plane or in space and that a warning is issued to the effect that a bomb with a specified effective radius will explode after a given interval of time where the initial position of the object is within the range of the bomb. Suppose the object moves in a random direction and traverses a random distance d from the initial position in the period of time before the explosion of the bomb. What is the probability that the object is then outside of the bomb's range? Sections Two and Three of Chapter Three provide answers to this question.

4. Consider the hypothetical situation where a person is dropped into a rectangular pool at a random position and he swims in a straight path to reach an edge of the pool without knowing where the edges are. What is the probability that he reaches the edge of the pool before swimming a specified distance? If his capacity to swim is limited to a distance d, what is the probability that he survives? Section One of Chapter Three provides answers to these questions.

The author has made some computational studies on some of the distribution functions obtained in this thesis. The graphs of the cumulative distribution functions (given by (1.1.1) and (2.1.1)) of the length of random secants of a regular polygon of N sides have been plotted for (i) N = 10,12,20,25,40,50,60,75,99,100 and 2^k for k = 2,3,6...,8 under S_1-randomness. (ii) N = 4,8,10,12,16,18,20 under S_2-randomness. The graphs of the distribution function (given by 3.2.1)) of the length of a random ray emanating in a random from a random source in a rectangle of sides a and b have been plotted for a = b and a > b.

Tables of $Pr(L \leq \ell)$ for a large number of sample values ℓ of L have been made in each of the above cases.

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