g-DERIVATIVES AND GAUSS STRUCTURES ON DIFFERENTIABLE MANIFOLDS

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By

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Some results are given connecting the concepts of g-derivatives and Jacobians on differentiable manifolds. Also some general properties of Gauss structures on manifolds important for our problems are discussed here. The connection between g-derivatives and Jacobians is given by studying the following problem: Given two differentiable manifolds M_n and M_n' and a differentiable map $\phi: M_n \to M_n'$ with $|J_{U,U'}\phi(x)| > 0$* for each $x \in M_n$, find a g-function f and families of coverings (V,V') such that $f: (M_n,V) \to (M_n', V')$ generates ϕ, and for suitable Gauss structures F, F' the g-derivative DF generates a continuous function ψ, such that for all $x \in M_n$: $\psi(x) = |J_{U,U'}\phi(x)|$ for convenient local charts U,U'.

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* $J_{U,U'}\phi(x)$ means Jacobian of ϕ for local charts U,U' with $x \in U$ and $\phi(x) \in U'$.

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NOTATIONS AND SYMBOLS IN THE TEXT

$\square$	means : the empty set
$\subseteq, \subset$	means : contained or equal
$\cap$	means : intersection of sets
s. t.	means : such that
:	means : such that
$\vert$	means : such that
ε	means : belongs to
$\cup$	means : union of sets
$\exists$	means : there exists
$\forall x$	means : for all x
N.D.A.	means : Non-deterministic Analysis
int	means : interior
$\bar{A}$, clos A	means : closure of A
$\rightarrow$	means : implies
$\rightleftarrows$	means : if and only if
min	= minimum
max	= maximum
sup	= supremum
inf	= infimum
∂	= topological border
N	= natural numbers
R	= real numbers

TABLE OF CONTENTS

			Page
INTRODUCTION			
CHAPTER 1	-	DIFFERENTIABLE MANIFOLDS	14
CHAPTER 2	-	SIMPLICIAL COMPLEXES	17
CHAPTER 3	-	NON-DETERMINISTIC ANALYSIS	24
CHAPTER 4	-	INDUCED STRUCTURES IN N.D.A.	33
CHAPTER 5	-	MAIN PROBLEM	40
CHAPTER 6	-	MAIN PROBLEM FOR MANIFOLDS WITH SMOOTH TRIANGULATIONS	54
APPENDIX	-		70
REFERENCES	-		81

INTRODUCTION

This thesis studies the problem of how g-derivatives (i.e. derivatives in topological spaces, used in non-deterministic analysis) are related to Jacobians when one is dealing with differentiable manifolds. This will be a generalization of V. Buonomano's result in his Ph.D. thesis which deals specifically with the Euclidean spaces. Theorems 6.5 and 6.7 show that the correspondence is an acceptable one.

Every proof given in the thesis is my own. References are provided for the non-proved statements.

In Chapter 1, we collect the main results from the theory of differentiable manifolds for later use. Analogously in Chapter 2, we collect elements from the theory of simplicial complexes, and finally in Chapter 3, we review briefly the fundamental ideas of non-deterministic analysis relevant to this work.

Chapters 4, 5 and 6 represent the core of this work. In the last two chapters we deal with our main problem. (See page 40)

By non-deterministic analysis (N.D.A.) we mean a mathematical system in which the usual concepts of analysis (continuity, differentiability, etc.) are expressed in a non-deterministic way; namely we take as fundamental objects open sets instead of points.

Functions then operate on open sets rather than on points.[8] (We call this kind of function a generalized function, or more specifically, a g-function; cf. 3.1.)

We shall now discuss the intuitive and philosophical background of this system as an introduction to our work. The remainder of this section is taken from Sections I.2 - I.5 in reference (8).

Let us consider some fundamental concepts of mathematics such as function, continuity and derivative.

When we have a function $f: X \to Y$, where X and Y are topological spaces, we always adopt a deterministic position relative to X and Y, in the sense that we say that for each $x \in X$, f has a well-defined value, $f(x) \in Y$. A non-deterministic position would sound like this: if x is in some open set in X we can guarantee that f(x) is in some well-defined open set in Y. This is precisely what a physicist usually assumes when he is, for instance, observing the motion of a point; because in general the most he can guarantee is that in a certain small interval of time the point is in some open set of space, perhaps very small, depending on the accuracy of the experiment.

Consider the concept of continuous function $f: X \to Y$. As everyone knows this means that if y is very close to x then f(y) is very close to f(x). In terms of open sets it should be reasonable to formulate a non-deterministic concept like this: if $A \subset B$ are open sets in X then f is continuous in a non-deterministic sense if we can guarantee that $f(A) \subset f(B)$.

Therefore, if we want to formalize these non-deterministic concepts of function and continuity, we have to begin to think about functions defined on domains whose elements consist of open sets and not of points. The idea of derivative nowadays is always associated with some linear map, since to define derivative we need a certain linear structure. Because of this it is impossible to define a derivative on a general topological space without freeing the derivative from linearity.

Immediately, it occurs to us that perhaps we could make it possible to speak about derivatives in a topological space X by introducing into X a new kind of structure. This sounds reasonable because, in a few words, topological structures were invented to make it possible to generalize the idea of continuity.

1. To clarify these statements let us begin by

discussing the idea of movement. At first glance the concept of movement seems quite clear to everyone; it simply means the changes of positions of bodies in space. But if we begin to think more carefully about it, we realize that the concept is not so easy to grasp. The difficulty with this idea arises from the fact that very deep and fundamental notions are involved here, like space, continuity and so on. Indeed, when we say that something is moving, it is always understood that it moves from one place to another, and this presupposes the idea of space; and when we think about a continuous movement, namely something that is going smoothly without sudden stops and starts; this, of course, involves the idea of continuity. Moreover, the idea of measurement is present, because usually we are concerned about how far an object moves from us, and also how quickly it moves, so that the idea of speed and consequently the idea of time appears as an important element to be taken into consideration.

Therefore, a deep study of the concept of movement, necessarily will take us right into the core of fundamental and basic ideas of our intuition of the world. Now mathematics is a form of expression also involving these fundamental concepts: geometry is traditionally attached to the concept of space; algebra with computations and measurements; analysis with continuity, and so on. In this line of thought mechanics, which primarily is supposed to study movement, must have a distinguished place in the theory of knowledge.

In this way, trying to generalize the idea of movement, we were naturally led to generalize and change such fundamental concepts as continuous functions and derivatives due to their close relationship with the idea of continuous movement and speed. This is nothing new because, for instance, Newton was mainly concerned with the idea of movement when he introduced the concept of fluxion, and we are going to follow his steps here. Indeed, we have done nothing else but to generalize his ideas. So we begin, as Newton does, by trying to compare the movements of two bodies. More precisely, he was interested in determining how far the speed of one of them was greater than that of the other, as the reader can see in his work on fluxions.

In analytic language, if we have a function $y = f(x)$ graphed in the Cartesian plane, the derivative of $f(x)$ at x can be interpreted as the relation between the speed of a point y along the y-axis and the speed of a point x along the x-axis, where x and y satisfy the equation $y = f(x)$. We try now to do the same thing in a more general situation.

Suppose X and Y are topological spaces and $f: X \to Y$ is a function from X into Y. As x changes in

X in general, (suppose f is not constant) y changes in Y. How can we possibly know how much the speed of y is greater than that of x?

To do this we must have some way to measure <u>how far</u> x has moved from one given initial position to another and also in what <u>direction</u>. To solve this question, let us analyze the simple case when X is the real line. In this case, one says: let us simply use the concept of distance given by the real numbers. Indeed, that is what we do, but let us look at this situation from a slightly different point of view. Consider in X the set of intervals defined by the integers, namely all intervals of the form [n, n+1]. Now if we want to know the position of a point x after a certain time from the moment it has started from zero, let us say in the positive direction of X, we simply count how many intervals it has described. If we need to improve the accuracy of our experiment, we define a new family of intervals of length 1/2, and so on. Thus, by considering intervals of length 1/n for arbitrarily large n, we are able to improve the accuracy of our measurements as much as we want. Let us remark that this situation is exactly the one a physicist is most often concerned with. Now if we have a paper with a millimeter net printed on it, we can observe and measure the movement of a point P on it by simply counting the little squares described by P. In

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Analyzing the situation described in the examples above we conclude that:

(i) The position of the moving point can be given with the accuracy we want, if we define in advance in X (a line or a plane) a family F of collections of subsets (intervals or squares) with the following properties:

(a) If α is a collection of F, then α is made up of sets which are closures of open sets of X, namely, $G \in \alpha \rightarrow \bar{A} = G$, where A is open in X. We see this clearly if we think of α as a collection of intervals [n, n+1] in the line for n an integer.

(b) For any G_1, $G_2 \in \alpha$ we have $\text{int}(G_1) \cap \text{int}(G_2) = \square$. For instance, we see this for [n, n+1] and [m, m+1], with $m \neq n$.

(c) Given any open set A of X, there is a collection $\alpha \in F$ such that some $G \in \alpha$ is contained in A. In the line it suffices to take α as the collection of intervals of length $1/n$ for n sufficiently large.

(d) Given any point $x \in X$, there is a neighbourhood of x which intersects only a finite number of sets of α, for any α in F. In the line this is true for our previous collection of intervals.

(ii) If we consider the question of measuring the position of a moving point as a question of "counting sets", we immediately realize that no question of "homogeneity of space" is involved, and at this point we believe that since Kant, through Riemann until B. Russell there has been a continuous mistake in trying to put at the very beginning of the possibility of measure in geometry, the homogeneity of space. We think this was due to the Kantian conception of space as an intuition "a priori" which seemed to carry with it, for some unexplained reason, the idea of homogeneity of space and, therefore, the idea of measure based on it. We cannot understand why the concept of space, as an intuition "a priori", has to imply the concept of homogeneity. By accepting the space as a pure intuition there is no reason to infer the statement "the space is homogeneous", as a synthetic judgment "a priori". Neither can we maintain that statement as "a posteriori", because even in a common and everyday experiment of measuring land with a chain, it is an ideal conception to suppose that the chain is always of the same size. It is not, for many reasons: change in temperature, change in the strength we apply to it, etc. Therefore, we don't see any reason why the possibility of measurements in geometry has to start with homogeneity assumptions.

2. To advance a further step towards the general case, we now discuss more fully the concept of movement. Let us see what a physicist really does when he observes the movement of a point. First of all he needs a scale for measuring time. Let us suppose, to fix ideas, that time is running from 0 to 1 and so t is a real variable in the interval [0,1]. Now, the movement of P in space (for the moment supposed to be the usual euclidean space R^3) is given analytically by a function $f: [0,1] \to R^3$. But this is an ideal situation, of course, in the spirit of classical mechanics, because first of all we can never have precisely an instant t and secondly we cannot find an exact position for P. So really what the physicist does is to subdivide [0,1] into a finite number of intervals and then to assert that in the interval $[t_i, t_{i+1}]$ the point is in a certain open set in R^3. (We use open sets to prepare for the ideas we have to introduce later.) In a few words, our experiment suggests a study of a correspondence between open sets in [0,1] and open sets in R^3. If our physicist wants to improve the precision of his experiment,

all he has to do is to consider finer subdivisions of [0,1].

In the present case, at least theoretically, by using finer and finer subdivisions of [0,1], we could get the position of P as accurately as we want, because all points of R^3 are of countable type. A completely different situation would arise if we consider maps $f: [0,1] \to X$, where X may have points of arbitrarily large transfinite type. Let us discuss this question further.

Suppose x_0 is a point of X of transfinite type $\alpha > \aleph_0$, with $\aleph_0$ the first infinite cardinal number, namely x_0 has a base of neighborhoods whose power is equal to α. Now suppose we know that in the interval of time $[t_i, t_{i+1}]$ of [0,1] the moving point x is in a certain neighborhood $V(x_0)$ of x_0. In this case, even if we refine more and more the intervals in [0,1] we have, theoretically, always an imprecision in the position of x around x_0 because we cannot have a countable number of neighborhoods of x_0 let us say

$$V_1(x_0),\ V_2(x_0),\ \cdots, V_n(x_0), \cdots$$

such that

$$x_0 = \bigcap_{n=1}^{\infty} V_n(x_0).$$

Therefore, the non-deterministic character of the position of x around x_0 is due to a geometrical property of space and nothing else. We call attention to this point due to its relation with many situations in modern physics.

3. The examples above suggest that if we want to extend the concept of continuous movement to general topological spaces, we have to change fundamentally our usual conceptions of continuous functions. Also, as we show later, the concept of derivative has to be completely changed if we want to introduce the concept of speed into a topological space. Of course, besides these physical reasons, there are many mathematical reasons to suggest that we should abandon the usual concepts of continuity of functions. This leads us to the concept of generalized functions or g-functions and their continuity.

At this point one could ask how the idea of speed can be introduced if the movement of a point is defined by such a correspondence of open-sets as above?

To answer this question we take again our example of the movement of a point P in the plane. We suppose, as before, that a family F of collections α of rectangular sets is defined in it, made up of squares of smaller and smaller diameters, and let D be a subdivision of I = [0,1] consisting of intervals of the same length which covers I. So we can imagine "an ideal point" P in some open subset, A, of the plane in some time interval $J \subset I$. Now if we did this for each $J \subset I$ we would have the path of P. But instead let us fix a J and its corresponding A.

Now suppose that to each subdivision D of I we associate a collection $\alpha \in F$ such that to a subdivision D' finer than D will correspond a collection $\alpha' \in F$ finer than α. So given the sets J and A let us call n(J,D) and n(A,α) the number of sets of D and α intersecting J and A, respectively.

Then we can say that an "average speed" of P in J is

$$\frac{n(A,\alpha)}{n(J,D)}.$$

Now if we consider finer subdivisions D and correspondingly finer collections α we can consider the "limits"

$$\overline{\lim_{D}}\,\frac{n(A,\alpha)}{n(J,D)} \quad \text{and} \quad \underline{\lim_{D}}\,\frac{n(A,\alpha)}{n(J,D)}$$

as possible bounds for the average speed of P in the interval J.

This example suggests how a possible generalization of the concept of derivative could be defined. All we have to do is to define precisely, in a general situation, the several concepts sketched above.

So looking back, we see that the study of the movement of a point led us to the idea of generalizing two fundamental notions in mathematics: continuity and differentiability. From now on, in a certain sense, we can forget about the physical motivations and start creating a "generalized calculus" or a "non-deterministic analysis". In Chapter 3 this is done formally.

4. Before closing this section we want to remark that one so far has received the impression that the concept of a g-function was motivated only by certain situations in physics. This is not the case. The concept of a g-function was motivated also by certain problems in homotopy theory, which arise because of the fact that, in general, one can't map continuously a point of one transfinite type into a point of higher transfinite type.

After g-functions were first applied to homotopy theory and algebraic topology then it was thought that g-functions might be a more "natural" way to express paths, etc. This then led to the development of the g-derivative and other concepts in N.D.A.

CHAPTER 1

DIFFERENTIABLE MANIFOLDS

1.1 Definition: A C^k - differentiable manifold of dimension n is a pair (M_n, Φ) where M_n is a Hausdorff and second countable space, and Φ is a collection of maps such that the following conditions hold.

1) $\{\text{dom } \phi_i\}_{\phi_i \in \Phi}$ is an open covering of M_n.

2) each $\phi_i \in \Phi$ is a homeomorphism onto an open set in R^n.

3) for each pair ϕ_i, $\phi_j \in \Phi \ni (\text{dom } \phi_i) \cap (\text{dom } \phi_j) \neq \square$, the map $\{\phi_j(\phi_i^{-1}(u))\}$ $(u \in \phi_i(\text{dom } \phi_i \cap \text{dom } \phi_j)$ from $\phi_i(\text{dom } \phi_i \cap \text{dom } \phi_j)$ into R^n is a C^k-map into R^n (we will call it a smooth or differentiable C^k map).(*)

4) Φ is maximal relative to 2) and 3).

K may be 0, 1, 2... ∞, ω, where C^0 means continuous, C^k for k finite means all partial derivatives of order less than or equal to $k > 0$ exist and are continuous. C^∞ means C^k $\forall$ k. C^ω means real analytic. The pair (U_i, ϕ_i), where U_i is the domain of ϕ_i, is called a local chart. ϕ_i is called a local chart homeomorphism or local coordinate. Sometimes we will call U_i a local chart in itself.

1.2 Definition: Let ϕ: $M_n \to M_p$ be a continuous function

(*) For brevity, we shall sometimes use (improperly!) the notation $\phi_j \circ \phi_i^{-1}$ for $\phi_j(\phi_i^{-1}(u))$, $u \in \phi_i$ $(\text{dom } \phi_i \cap \text{dom } \phi_j)$.

from the differentiable manifold M_n to M_p. We say that ϕ is differentiable or smooth of class C^k if the function given by $\{\psi_j(\phi(\phi_i^{-1}(u)))\}$, $u \in \phi_i(U_i \cap V_j)$, from $\phi_i(U_i \cap V_j)$ into R^p is differentiable of class C^k, where (U_i, ϕ_i) and (V_j, ψ_j) are coordinate neighbourhoods of M_n and M_p, respectively.

1.3 Definition: If $A \subset M_n$, a function $\phi: A \to M_p$ is differentiable if it can be extended to a differentiable function defined on the open submanifold * induced by M_n on a neighbourhood U of A.

1.4 Definition: $\phi: M_n \to M_p$ is a diffeomorphism if ϕ and ϕ^{-1} are defined and differentiable.

1.5 Definition: If $\phi: M_n \to M_p$ the rank of ϕ at x is the rank of $J(\psi_j \phi\phi_i^{-1})$ at $\phi_i(x)$, where (U_i, ϕ_i) and (V_j, ψ_j) are local charts around x and $\phi(x)$ respectively; and J(g) represents the Jacobian of g for any $g: R^n \to R^p$.

1.6 Definition: The differentiable map $\phi: M_n \to M_p$ is an immersion if rank $\phi = n$ everywhere $(n \leq p)$. ϕ is an imbedding if it is also a homeomorphism into.

1.7 Definition: $A \subset M_n$ is a differentiable submanifold of M_n if $i: A \to M_n$ (the inclusion map) is an imbedding.

* The local charts of U are $\{(U_i', \phi_i')\}_{i\varepsilon A}$ where $\{(U_i, \phi_i)\}$ is a coordinate neighbourhood system of M_n and $U_i' = U_i \cap U$ and ϕ_i' is the restriction of ϕ_i to U_i'.

1.8 Proposition: If $\phi: M_n \to M_p$ is an imbedding then $\phi(M_n)$ is a differentiable submanifold of M_p; cf. [1].

Notation: If $x \in R^n$ and $x = (x_1 \cdots x_n)$

$$|| x || = \max | x_i |,$$

$$C^n(r) = \{x | \; || x || < r\}.$$

From now on our manifolds will be of class C^∞ for the sake of convenience; this is not a very drastic restriction in view of Whitney's result that on an n-manifold M_n, every differentiable structure of class C^k $(k > 0)$ contains a structure of class C^∞.

1.9 Theorem: Let M_n be a differentiable manifold, $\{U_\alpha\}$ an open covering of M_n. There is a collection (V_j, h_j) of coordinate systems on M_n such that:

1) $\{V_j\}$ is a locally finite refinement of $\{U_\alpha\}$
2) $h_j(V_j) = C^n(3)$
3) If $W_j = h_j^{-1}(C^n(1))$ then $\{W_j\}$ covers M_n.

Proof: See [9].

CHAPTER 2

SIMPLICIAL COMPLEXES

For the proofs of all propositions in this section, see [11].

2.1 Definition: Let R^n be the euclidean vector space over R and let C be a subset of R^n. C is convex if $c_1; c_2 \in C \rightarrow t\, c_1 + (1 - t)\, c_2 \in C \;\forall\; t \in I = [0, 1]$.

2.2 Definition: A set $\{V_o, V_1, \ldots, V_k\}$ of points in R^n is convex independent or C-independent, if the set $\{V_1 - V_o, V_2 - V_o, \ldots, V_k - V_o\}$ is linearly independent.

2.3 Proposition: Suppose $\{V_o, V_1, \ldots, V_k\}$ is a C-independent set. Let C be the convex set generated by $\{V_o, V_1, \ldots, V_k\}$; that is, C is the smallest convex set containing $\{V_o, V_1, \ldots, V_k\}$. Then C consists of all points of the form $\sum_{i=0}^{k} a_i V_i$ where $a_i \geq 0 \;\forall i$ and $\sum_{i=0}^{k} a_i = 1$. Furthermore, each $V \in C$ is uniquely expressible in this form.

2.4 Definition: Let R^n be the euclidean space. A convex set generated by C-independent vectors or points $\{V_o, V_1, \ldots, V_k\}$ is called a closed k- simplex and is denoted by $[V_o, V_1, \ldots, V_k]$, k is the dimension of the simplex. If $V \in [V_o, V_1, \ldots, V_k]$, then the coefficients a_i, with $a_i \geq 0$ and $\sum_{i=0}^{k} a_i = 1$ such that $V = \sum_{i=0}^{k} a_i V_i$, are called the <u>barycentric coordinates of V</u>.

2.5 <u>Definition</u>: Let $\{V_o, V_1, \ldots, V_k\}$ be a C-independent set. The set $\{V \in [V_o, \ldots, V_k];$ $a_i(V) > 0$ $i = 0, \ldots, k\}$ ($a_i(V)$ is the i^{th} coefficient of V) is called an open simplex and is denoted by $(V_o, V_1, \ldots, V_k)$ we will also denote an open simplex by (S) and the corresponding closed simplex by [S].

2.6 <u>Definition</u>: Let $[S] = [V_o, V_1, \ldots, V_k]$ be a closed simplex. The vertices of [S] are the points $V_o, V_1, \ldots, V_k$. The closed faces of [S] are the closed simplices $[V_{j_o}, V_{j_1}, \ldots, V_{j_n}]$ where $\{j_o, j_1, \ldots, j_n\}$ is a non-empty subset of $\{0, 1, \ldots, k\}$. The open faces of the simplex [S] are the open simplices $(V_{j_o}, V_{j_1}, \ldots, V_{j_n})$.

2.7 <u>Definition</u>: A simplicial complex K is a set of open simplices in some R^n such that

1) if (S) $\in$ K then all open faces of [S] $\in$ K;

2) if (S_1), $(S_2) \in K$ and $(S_1) \cap (S_2) \neq [\,]$, then $(S_1) = (S_2)$.

The dimension of K is the maximum dimension of the simplices of K (topological dimension).

<u>Remarks</u>: If K is a simplicial complex, let [K] denote the point set union of the open simplices

of K with weak topology(*). Then if K has only a finite number of simplices, it is compact and

$$[K] = \bigcup_{(S)\varepsilon K} (S) = \bigcup_{(S)\varepsilon K} [S].$$

If [S] is a closed simplex, the collection of its open faces is a simplicial complex which we denote by S.

2.8 <u>Definition</u>: Let K be a complex. Let r be an integer less than or equal to dim K. The <u>r- skeleton</u> K^r of K is the collection

$$K^r = \{(S) \varepsilon K \mid \dim S \leq r\}.$$

2.9 <u>Definition</u>: Let S be a k- simplex. The <u>barycenter</u> of S, denoted by b(S), is the point in (S) with barycentric coordinates $(\frac{1}{k+1}, \ldots, \frac{1}{k+1})$; that is if $(S) = (V_o, V_1, \ldots, V_k)$ then $b(S) = \frac{1}{k+1} \sum_{i=0}^{k} V_i$.

2.10 <u>Definition</u>: A <u>subdivision of a complex K</u> is a simplicial complex K^* such that: 1) $[K^*] = [K]$; 2) if $S \varepsilon K^*$ then (S) ε some open simplex of K.

2.11 <u>Definition</u>: Let K be a simplicial complex. A partial ordering is defined on K by $S_1 \leq S_2 \Leftrightarrow S_1$ is a

(*) Besides the weak topology it is also possible to consider the metric topology, but we do not discuss this here because all complexes we shall use are locally finite and in this case both topologies coincide.

face of S_2. The notation $S_1 < S_2$ means $S_1 \leq S_2$ and $S_1 \neq S_2$.

2.12 Proposition: Let K be a simplicial complex. If $S_0, S_1, \ldots, S_k \in K$ and $S_0 < S_1 < \ldots < S_k$, then $(b(S_0), b(S_1), \ldots, b(S_k))$ is C-independent and $K^1 = \{(b(S_0), \ldots, b(S_k)) \mid S_0 < S_1 < \ldots < S_k;\ S_1, S_2, \ldots, S_k \in K\}$ is a subdivision of K. Furthermore, for each $S_0, S_1, \ldots, S_r \in K$ with $S_0 < S_1 < \ldots < S_r$, $(b(S_0), \ldots, b(S_r)) \subset (S_r)$. The subdivision $K^{(1)}$ is called the first barycentric subdivision of K. Iterating: $K^{(n)} = (K^{(1)})^{(1)})\ldots)^{(1)}$ is the n^{th} barycentric subdivision of K.

From now on, we are going to use the terminology smooth-differentiable, interchangeably.

2.13 Definition: A smoothly triangulated manifold is a triple (M_n, K, h) where M_n is a n-dimensional C^∞ manifold, K is a simplicial complex and $h: [K] \to M_n$ is a homeomorphism such that for each simplex S of K, the map $h_{|[S]} : [S] \to M_n$ has an extension h_s to a neighbourhood U of [S] in the linear space of [S] such that $h_s: U \to M_n$ is an imbedding (or $h_s(U)$ is a differentiable submanifold).

Remark: As dim $M_n = n$ we only need to require that this last condition be satisfied for each n- simplex of K, since every simplex of K is a face of an n- simplex and since restrictions of smooth maps to submanifolds are smooth.

2.14 Proposition: Every compact differentiable (smooth) manifold can be smoothly triangulated. Note that smoothly finite triangulated manifolds are compact because [K] is compact for each (finite) simplicial complex K. See [11].

2.15 Definition: Let B be the matrix whose ith row is

$$B_i = (b_{i1}, \ldots, b_{in}) \ (i = 1, \ldots, n).$$

If $0 = (0,0,\ldots,0)$, we call $\frac{1}{n}$ $|\det B|$ the n- area of the n- simplex $(0, B_1, B_2, \ldots, B_n)$.

If the origin is not a vertex of the given n- simplex, we can transform it by a rigid motion in such a way that the origin will be a vertex of the transformed n- simplex. We then define the n- area of the original n- simplex to be that of the new n- simplex.

2.16 Definition: A simplex with faces of the same k- dimension ($0 \leq k \leq p$) all of the same k- area shall be called a standard Euclidean p- simplex Δp or a fundamental p- simplex.

As well known such simplex can be built in many ways, for instance if $x = (x_o, x_1, x_2, \ldots, x_p)$ is a point in R^{p+1}, the set

$$\Delta_p = \{(x_o, x_1, \ldots, x_p): \ x_i \geq o, \ \sum_{i=0}^{p} x_i = 1\}$$

is an example of such simplex.

2.17 Proposition: Each simplicial complex K_n of dimension n can be realized as a subcomplex of the fundamental simplex.

Proof: The proof is analogous to that of the corresponding well known result in the theory of simplicial complexes.

2.18 Lemma: Let σ_p be a p- simplex; then the barycentric subdivisions produce simplexes of the same measure in each subdivision.

Proof: If $\sigma_n = (0, B_1, B_2, \ldots, B_n)$, where $0 = (0, 0, \ldots, 0)$ and $B_i = (b_{i1}, \ldots, b_{in})$ $(i = 1, \ldots, n)$ are the rows of the matrix B, then the volume of σ_n is given by

$$\mu(\sigma_n) = \frac{1}{n!} |\det B|.$$

A simplex in the first barycentric subdivision will appear as:

$$\sigma^i = (0, B_1, \ldots, B_{i-1}, \frac{1}{n+1} \sum_{j=1}^{n} B_j, B_{i+1}, \ldots, B_n).$$

The measure of an n- face σ^i of σ_p is

$$\mu(\sigma^i) = \frac{1}{n!} \det (B_1, \ldots, B_{i-1}, \frac{1}{n+1} \sum_{j=1}^{n} B_j, B_{i+1}, \ldots, B_n)$$

$$= \frac{1}{(n+1)n!} \det (B_1, \ldots, B_{i-1}, \sum_{j=1}^{n} B_j, B_{i+1}, \ldots, B_n)$$

$$= \frac{1}{(n+1)!} \sum_{j=1}^{n} \det (B_1, \ldots, B_{i-1}, B_j, B_{i+1}, \ldots, B_n)$$

$$= \frac{1}{(n+1)!} \det B \qquad (i = 1, 2, \ldots, n).$$

If we replace 0 in $(0, B_1, \ldots, B_n)$ by $b(\sigma_n)$ = barycentre of σ_n, we then rotate and translate σ_n in such a way that some B_i coincides with 0. Then we proceed as before.

CHAPTER 3

NON-DETERMINISTIC ANALYSIS

3.1 Definition: Let X and Y be two topological spaces and V, V' be two families of open coverings of X and Y respectively.

Suppose that for each $\mu \in V$ we can associate some $\mu' \in V'$ such that each $A \in \mu$ is associated with some $A' \in \mu'$. We will call this association a g-function and denote it by:

$$f \;:\; (X, V) \to (Y, V') \quad \text{or}$$

$$f_V \;:\; \mu \in V \to \mu' \in V' \;\; (f_V(\mu) = \mu')$$

and

$$f_\mu \;:\; A \in \mu \to A' \in \mu' \;\; (f_\mu(A) = A').$$

In the special case where Y is the real line we allow $\mu' \in V'$ to be a collection of open intervals and/or points.

To differentiate between the two cases we call f a special g-function and denote it by $f\colon (X, V) \to [R, V_R]$. For convenience we will use the term g-function to include the special case too.

3.2 Definition: The g-function $f\colon (X, V) \to (Y, V')$, is continuous if for all $\mu, \lambda \in V$ such that $\lambda \geq \mu$ and for

$A \in \mu$, $B \in \lambda$ with $B \subseteq A$, then $f_\lambda(B) \subseteq f_\mu(A)$ (">" means "refines"). We know that in order to talk about continuity in a space we need a topological structure. We are going to see now that in order to talk about differentiation or velocity in a topological space we shall need what we shall call a Gauss structure or a standard family of coverings.

3.3 Definition: A standard family of coverings, F in a topological space X is a family of collections α, of subsets of X such that:

a) Any set A of $\alpha \in F$ is the closure of an open set of X.

b) Given $\alpha \in F$ and two distinct sets $A_1, A_2 \in \alpha$, then $\text{int } A_1 \cap \text{int } A_2 = \square$ (int = interior).

c) Any $\alpha \in F$ is a covering of X.

d) Given any point $x \in X$ there is a neighborhood N of x such that any α has only a finite number of sets intersecting N (each $\alpha \in F$ is locally finite).

e) Given any open set O of X there is a covering $\alpha \in F$ such that α has a set $A \subset O$.

f) Ordered by refinements, F is a directed set.

3.4 Definition: We will also call a standard family of coverings F a Gauss structure on X. A Gauss space is a topological space with a standard family of coverings.

Notation: If there is a Gauss structure F on X we call (X, F) a Gauss space.

3.5 Proposition: Each space X satisfying the T_3 axiom is a Gauss space. cf. [1].

3.6 Proposition: Each T_2 paracompact topological space is a Gauss space. cf. [1].

3.7 Corollary: Each differentiable manifold is a Gauss space.

The reason for the above nomenclature is due to the fact that a standard family of coverings is a generalization of a system of Gauss coordinates on a surface S.

3.8 Definition: Let (X, F) and (Y, F') be two Gauss spaces. A Gauss transformation is a function $G: F \to F'$ compatible with the order of refinement of F and F', i.e., if $\alpha, \beta \in F$, $\alpha < \beta$ then $G(\alpha) < G(\beta)$.

3.9 Definition: A continuous g-function $f: (X, V) \to (Y, V')$ is called g-differentiable relative to the Gauss transformation

$G: F \to F'$ and the standard family of coverings F, F' of X and Y, respectively, if for any $\mu \in V$, $\alpha \in F$, $A \in \mu$, the number of sets of α which intersect A is finite and the same holds for $\alpha' = G(\alpha), \mu' = f_V(\mu)$, $A' = f_\mu(A)$. We denote these numbers by $n(A,\alpha)$ and $n(A', \alpha')$, respectively.

3.10 Definition: Let the g-function $f: (X, V) \to (Y, V')$ be g-differentiable relative to F, F' and G. We define the g-<u>derivative</u> of f as a special g-function.

Df: $(X, V) \rightarrow [R, V_R]$. To construct Df we must have the following:

a) f: $(X, V) \rightarrow (Y, V')$ is a continuous g-function or special g-function.

b) The Gauss spaces (X, F) and (Y, F').

c) The Gauss transformation G: $(X, F) \rightarrow (Y, F')$.

Then the construction procedure is:

d) Let n (A, μ, α) denote the number of sets of α that intersect the set A, which is an element of μ ε V.

e) Let $Df_\mu(A, \alpha) = \dfrac{n(A', \mu', \alpha')}{n(A, \mu, \alpha)}$, where $f_V(\mu) = \mu'$, $f_\mu(A) = A'$ and $G(\alpha) = \alpha'$.

f) Let $\overline{Df}_\mu(A) = \overline{\lim_{\alpha \varepsilon F}}\, Df_\mu(A, \alpha)$,

$\underline{Df}_\mu(A) = \underline{\lim_{\alpha \varepsilon F}}\, Df_\mu(A, \alpha)$, where:

$\overline{\lim_{\alpha \varepsilon F}}$ = upper limit over the net F (directed set)

and $\underline{\lim_{\alpha \varepsilon F}}$ = lower limit.

g) So for each B ε λ and λ ε V we have two real numbers i.e. $\overline{Df}_\lambda(B)$ and $\underline{Df}_\lambda(B)$. Let us call $\underline{Df}_\mu(A)$ the set of all such numbers for $B \subset A$, with B ε λ and $\lambda \geq \mu$, i.e.:

$\overline{Df}_\mu$ (A) = {$\underline{Df}_\lambda$ (B), $\overline{Df}_\lambda$ (B) :

B $\subset$ A, B ε λ, $\lambda \geq \mu$, λ ε V}.

h) Finally we define Df_μ (A) to be the open interval or point in R: Df_μ (A) = (inf $\overline{Df}_\mu$ (A), sup $\overline{Df}_\mu$ (A)) = A_R. If the inf and sup are equal we understand the above to be the set consisting of that point.

i) So for each A ε μ we get a point or interval in R. We denote this collection Df_μ (A) by μ_R. As μ runs through V we get a family V_R of such collections:

$$V_R = \{\mu_R\}$$

which we call the g-derivative, Df of f. We note that by the definition of g-derivative, Df is always a <u>continuous</u> special g-function. cf. [4].

3.11 <u>Definition</u>: The g-function f: (X, V) → (Y, V') is called <u>cofinal</u> if the family $\{f_\mu (A) \mid A \; \varepsilon \; \mu\}_{\mu \varepsilon V}$ is cofinal in cov Y(*), in the sense that given any α ε Cov Y, there exists μ ε V such that for any A ε μ there is a C ε α with f_μ (A)$\subseteq$C. We do not require $\{f_\mu (A) \mid A \; \varepsilon \; \mu\}$ to be a cover of Y.

3.12 <u>Definition</u>: f: (X, V) → (Y, V') is

(*) Cov Y = family of all open coverings of Y.

pointwise cofinal if for any $x \in X$ and any $\alpha \in \text{cov}\, Y$ there exists $\mu \in V$ and $A \in \mu$ such that $x \in A$ and $f_\mu(A) \subset C$ for some $C \in \alpha$. Obviously a cofinal g-function is pointwise cofinal.

3.13 Definition: $f: (X, V) \to (Y, V')$ generates the map (continuous function) $\phi: X \to Y$ iff: for any $x \in X$ any any neighborhood W of $\phi(x)$, there exists $\mu \in V$, $A \in \mu$ such that:

$$x \in A,\ \phi(x) \in \overline{f_\mu(A)} \text{ and } f_\mu(A) \subseteq W.$$

Clearly if Y is regular T_1, then Definition 3.13 can be replaced by:

3.14 Definition: $f: (X, V) \to (Y, V')$ generates the map $\phi: X \to Y$ iff for any $x \in X$ and any neighborhood W of $\phi(x)$, there exists $\mu \in V$, $A \in \mu$ such that $x \in A \in \mu$ and $\phi(x) \in \overline{f_\mu(A)} \subseteq W$.

3.15 Proposition: If $f: (X, V) \to (Y, V')$ is pointwise cofinal, continuous and if V is cofinal, and X, Y are regular T_1, then there is a unique continuous map $\phi: X \to Y$ generated by f; cf. [7].

3.16 Definition: $f: (X, V) \to (Y, V')$ is the g-function constructed from $\phi: X \to Y$ (usual function) by the image method if: for $A \in \mu \in V$, $f_V(\mu) = \mu'$, where $\mu' = \{\text{int}\, \phi(A) \mid A \in \mu\}$ and $f_\mu(A) = \text{int}\,(\phi(A)) \neq \square$.

3.17 Proposition: Let $\phi: R^n \to R^n$ such that J_ϕ (= Jacobian of ϕ) has absolute value > 0 ($|J\phi| > 0$). Let V be the family of all open coverings of R^n such that each set is bounded. Let V' be the family of open sets constructed by the image method, f the g-function $f: (R^n, V) \to (R^n, V')$ built by the image method. Then f generates ϕ; cf. [4].

3.18 Definition: The standard family of coverings we will be using on the real line R will be denoted by F_R and called the Canonical standard family of coverings on R. It is defined by:

$$F_R = \{\alpha_i\}_{i=1}^{\infty} \quad \alpha_i = \{I_{ij}\}_{j=1}^{\infty}$$

$$I_{ij} = \{x \in R: \frac{j}{2^{i-1}} \leq x \leq \frac{j+1}{2^{i-1}},\ j = 0, \pm 1, \pm 2, \ldots\}.$$

This is simply the family of closed intervals of length $\frac{1}{2^{i-1}}$, starting from the origin, which have only their end points in common.

3.19 Definition: The Canonical standard family of coverings on R^n will be denoted by F_R^n and:

$$F_R^n = \{\{\prod_{i=1}^{n} A_i : A_i \in \alpha\} : \alpha \in F_R\}.$$

G_R^n will denote the Canonical Gauss transformation which is the identity Gauss transformation, i.e.,

$G_R^n : F_R^n \to F_R^n$ s.t: $G_R^n(\alpha_i) = \alpha_i$.

3.20 Definition: Let (X, F) be a Gauss space and $\emptyset \neq A \subset X$. If the set $F_A = \{\{A \cap B : B \in \alpha\} : \alpha \in F\}$ is a standard family of coverings on A with the relative topology, we call (A, F_A) a Gauss subspace of (X, F).

3.21 Proposition: Before completely stating this important proposition let us fix the following:

a) Q: $R^n \to R^n$ is continuously differentiable and $|J\,Q\,(x)\,| > 0 \;\forall\, x \in R^n$.

b) V is the family of all open coverings of R^n such that any set A in any cover has a finite Jordan measure.

c) f is the g-function constructed from Q and V by the image method.

d) We speak of the g-derivative of f relative to the canonical Gauss space (R^n, F_R^n) and its identity Gauss transformation G_R^n: $F_R^n \to F_R^n$.

Then there exists a g-function f which generates Q such that Df generates |JQ|. (f is the g-function defined in c); cf. [3].

3.22 Proposition: If U is a non-empty open subset of X then (U, F_U) is a Gauss subspace of (X, F). If U is not open this is not necessarily true, cf. [1], Example 1.

3.23 Definition: If (X, F) is a Gauss space and α, $\beta \in F$, with $\alpha < \beta$ let us define $\overline{n}(\alpha, \beta)$ as the supremum of the number of elements of β contained in some element of α, and $\underline{n}(\alpha, \beta)$ as the infimum of the number of elements of β contained in some element of α.

If these numbers are finite we say the Gauss space is of finite type. If moreover $\overline{n}(\alpha, \beta) = \underline{n}(\alpha, \beta)$ for any α, $\beta \in F$ with $\alpha < \beta$ we say that the Gauss space is equitable and we use the notation $n(\alpha, \beta)$ for either one of the numbers defined above.

Sometimes we will use $n(A, \alpha) = n(A, \mu, \alpha)$ for $A \in \mu \in V$ whenever it is clear what the covering μ is.

CHAPTER 4

INDUCED STRUCTURES IN N.D.A.

4.1 Definition: Let $\phi: X \to Y$ be a homeomorphism, and F be a Gauss structure in X. We are going to define the elements of a Gauss structure in Y, which we will call the image of the Gauss structure F by ϕ or Gauss structure induced by ϕ in the following way.

Let $A \in \alpha \in F$ and $A = \text{clos } 0$, 0 = open in X. Then we build F' in Y by putting

$$A' = \phi(A) = \phi(\text{clos } 0) = \text{clos } \phi(0),$$

$$\alpha' = \{A' = \phi(A) \mid A \in F\}, \quad F' = \{\alpha' \mid \alpha \in F\}.$$

It is trivial to check that the six conditions for a Gauss space in Y are fulfilled by F'. We will write $F' = \phi(F)$ or $\alpha' = \phi(\alpha)$ for a covering $\alpha \in F$.

4.2 Definition: The image of a g-function by a pair of homeomorphisms. Let

$f : (X, V) \to (Y, V')$ be a g-function,

$\phi_1 : X \to \tilde{X}$

and

$\phi_2 : Y \to \tilde{Y}$ a pair of homeomorphisms (ϕ_1, ϕ_2).

Let $\tilde{f} : (\tilde{X}, \tilde{V}) \to (\tilde{Y}, \tilde{V}')$

be the g-function defined as follows:

$$\text{If } \tilde{A} \in \tilde{\mu} \in \tilde{V} \text{ then } \tilde{A} = \phi_1(A) \text{ for } A \in \mu \in V,$$

where $\tilde{\mu}$ is the image of μ by ϕ_1 in the canonical way.

We then define:

$$\tilde{f}_{\tilde{\mu}}(\tilde{A}) \underline{\text{ def }} \phi_2(f_\mu(A)).$$

$\tilde{f}$ is called the g-function induced by the pair (ϕ_1, ϕ_2) and $\tilde{G}$: $\phi_1(F) \to \phi_2(F')$ defined in an obvious way is called the Gauss transformation induced by the pair (ϕ_1, ϕ_2) and the transformation

G: F → F':

$$(\tilde{G}(\phi_1(\alpha)) = \phi_2(G(\alpha))).$$

4.3 Lemma: Let f: (X, V) → (Y, V') be a continuous g-function and let (ϕ_1, ϕ_2) be a pair of homeomorphisms

$$\phi_1: X \to \tilde{X}$$

$$\phi_2: Y \to \tilde{Y}.$$

Then for any $\sigma \in V$ and any $A \in \sigma$ we have

$$D f_\sigma(A) = D \tilde{f}_{\tilde{\sigma}}(\tilde{A}),$$

where $\tilde{\sigma} = \phi_1(\sigma)$, $\tilde{A} = \phi_1(A)$ and the derivatives are taken relative to G: F → F' and $\tilde{G}$: $\tilde{F} \to \tilde{F}'$, respectively

$(\tilde{F} = \phi_1(F) \quad \tilde{F}' = \phi_2(F'))$.

Proof: It is immediate from Definition 4.2.

4.4 Corollary: Df_μ (A) is pointwise cofinal $\Leftrightarrow$ $D\tilde{f}_{\tilde{\mu}}(\tilde{A})$ is pointwise cofinal.

4.5 Corollary: If f generates ϕ: X → Y $\tilde{f}$ generates $\tilde{\phi}$: $\tilde{X} \to \tilde{Y}$, with $\tilde{\phi} = \phi_2 \cdot \phi \cdot \phi_1^{-1}$, whenever this composition is defined.

4.6 Definition: Let F be a Gauss structure in open set $X \subset R^n$ satisfying the following conditions:

i) F is countable and the coverings of F are such that $\alpha_1 < \alpha_2 \cdots \alpha_m < \cdots$.

ii) For each α_i all $F \in \alpha_i$ are Jordan measurable and have all the same measure.

iii) If $||\alpha_i||$ is the maximum of the diameters of the elements of α_i, then $\lim_{i \to \infty} ||\alpha_i|| = 0$.

A Gauss structure as above will be referred to as admissible.

4.7 Remarks: a) At first glance it seems that (iii) of definition above is a consequence of (i) and (ii). However that is not the case as we can show with the following example. Let X be the subset of the plane defined as

$$X = \{(x,y) : x > 1, o < x^2y < 1\}$$

which is the figure bounded by the curve $x^2y = 1$, the line $x = 1$ and the real axis. Define a collection β_1 of two sets by considering a point (x_1, o) such that the area of the subset of X lying between the lines $x = 1$ and $x = x_1$ is equal to the area of the subset of X, given by $x \geq x_1$. Define β_2 by considering a point (x_2, o) with $1 < x_2 < x_1$ and a point $x_3 > x_1$ such that the sets

$$A_1 = \{(x, y) \in X : 1 < x \leq x_2\}$$

$$A_2 = \{(x, y) \in X : x_2 \leq x \leq x_1\}$$

$$A_3 = \{(x, y) \in X : x_1 \leq x \leq x_3\}$$

$$A_4 = \{(x, y) \in X : x \geq x_3\}$$

have all the same area. Proceeding in the same way for $\beta_3, \beta_4, \cdots, \beta_n, \cdots$, we have a family $\{\beta_i\}$ satisfying (i) and (ii).

To define $\{\alpha_i\}$ we start with β_1 and consider a subdivision of the two sets in β_1 by a line parallel to the x axis for the set

$$\{(x, y) : 1 < x \leq x_1\}$$

such that the upper and lower part have the same area and for the set

$$\{(x, y) : x \geq x_1\}$$

we consider a similar curve such that the upper and lower part have also the same area. Define α_1 to be the collection of these four sets. In the same manner we define $\alpha_2, \alpha_3, \ldots$.

As easily seen α_i so defined satisfies (i) and (ii) but not (iii).

b) Another important point to observe is that in R^n or in open sets of R^n we cannot have admissible structures with more than countable coverings, namely

$$\alpha_1 < \alpha_2 < \ \ldots \ < \alpha_\omega < \alpha_{\omega+1} < \ \ldots \ < \alpha_\chi < \ldots,$$

where χ is a non-countable ordinal, assuming of course that all α_i are distinct. Indeed, suppose the contrary and take a point $x \in R^n$ together with a sequence of balls $B_i(x)$ with center x and radius $1/i$, $i \geq 1$. Inside $B_i(x)$ take $A_{j(i)} \in \alpha_{j(i)}$ such that

$$A_{j(i)} \subset B_i(x) \qquad (1)$$

$$A_{j(i)} \neq A_{j(k)} \text{ if } i \neq k$$

and $j(i)$ is the first index satisfying (1). This is possible by condition (iii).

Now as α_χ refines all $a_{j(i)}$, due to the way we select $j(i)$, this implies that in each $B_i(x)$ there is some $c_i \in \alpha_\chi$ and this clearly contradicts condition d) of definition 3.3.

Therefore, there is no loss of generality assuming that we are always dealing with countable families of coverings.

4.8 Lemma: Let A and A' be two Jordan measurable sets in R^n with A having non-zero Jordan measure $\mu(A)$.

Suppose (R^n, F) and (R^n, F') are Gauss spaces with F and F' admissible Gauss structures. Let $G: F \to F'$ be a Gauss transformation such that if $\alpha'_m = G(\alpha_m)$, $\alpha_m \in F$, $\alpha'_m \in F'$, then the common measure of the elements of α_m is equal to the common measure of the elements of α'_m. In this case we have

$$\overline{\lim_{\alpha_m \in F}} \frac{n(A', \alpha'_m)}{n(A, \alpha_m)} = \underline{\lim_{\alpha_m \in F}} \frac{n(A', \alpha'_m)}{n(A, \alpha_m)} = \frac{\mu(A')}{\mu(A)} .$$

Proof: Consider the sequence $\{n(A', \alpha'_m) \cdot \mu_m\}$, where μ_m is the common measure of the elements of α_m and α'_m.

By definition of Jordan measure: for any $\varepsilon > 0$ we can choose an N such that for all $m > N$

$$|n(A', \alpha'_m) \mu_m - \mu(A')| < \varepsilon.$$

So $\lim_{\alpha'_m \in F'} n(A', \alpha'_m) \mu_m = \mu(A')$ which implies

$$\underline{\lim_{\alpha'_m}} n(A', \alpha'_m) \mu_m = \overline{\lim_{\alpha'_m}} n(A', \alpha'_m) \mu_m = \mu(A').$$

In the same way we have the analogous result for Λ.
Then since:

$$\frac{\lim n(A', \alpha'_m)\,\mu_m}{\lim n(A, \alpha_m)\,\mu_m} = \lim \frac{n(A', \alpha'_m)\,\mu_m}{n(A, \alpha_m)\,\mu_m} = \frac{\mu(A')}{\mu(A)},$$

we have $\dfrac{\mu(A')}{\mu(A)} = \varlimsup\limits_{\alpha_m \in F} \dfrac{n(A', \alpha'_m)}{n(A, \alpha_m)} = \varliminf\limits_{\alpha_m \in F} \dfrac{n(A', \alpha'_m)}{n(A, \alpha_m)}$.

4.9 <u>Corollary</u>: If $f: (R^n, V) \to (R^n, V')$ is a continuous g-function, for $\forall\, \omega \in V$ and any $A \in \omega$, Jordan measurable such that $A' = f_\omega(A)$ is Jordan measurable, we have

$$\overline{Df}_\omega(A) = \underline{Df}_\omega(A) = \frac{\mu(A')}{\mu(A)}.$$

4.10 <u>Theorem</u>: Under the condition of Proposition 3.21, except that we will now deal with our new Gauss structures F, F' used in Definition 4.6, if Q is a continuously differentiable function from R^n to R^n, then Df generates $|J_Q|$.

<u>Proof</u>: It is analogous to that of Theorem 1, Section 4.2 in [4], where only the canonical Gauss structure is considered.

CHAPTER 5

THE MAIN PROBLEM

5.1 In this chapter we start the discussion of the main problem of this work, namely:

Given differentiable manifolds M_n and M'_n and a differentiable map

$$\phi: M_n \to M'_n$$

with

$$|J_{U,U'}\phi(x)| > 0,$$

where $J_{U,U'}\phi(x)$ is the value of the Jacobian of ϕ at a point $x \in M_n$ for the local charts (U,h) in M_n and (U',h') in M'_n, we want to find a continuous g-function

$$f: (M_n,V) \to (M'_n,V')$$

such that

(i) f generates ϕ

(ii) Df generates a function $\psi: M_n \to R$ (reals) such that for every $x \in M_n$ there exist local charts (U,h) at x and (U',h') at $\phi(x)$ such that

$$\psi(x) = |J_{U,U'}\phi(x)|.$$

In this work we solve this problem for the case where the manifolds involved have triangulations which satisfy certain conditions to be specified. However without any additional condition we are able to prove the following theorem:

5.2 Theorem: The main problem has a solution if instead of condition (ii) we consider the weaker condition.

(ii)': Df generates a function $\psi: (M_n - H) \to R$ where H is a meagre(*) set, such that for every $x \in M_n - H$, there exist local charts (U, h) at x and (U', h') at $\phi(x)$ such that

$$\psi(x) = |J_{U,U'}\phi(x)| .$$

Proof: Without loss of generality we can assume that ϕ is surjective, because as the Jacobian is never zero by hypothesis, ϕ is locally a diffeomorphism and by the theorem of invariance of open sets for manifolds, the image of local charts by ϕ are open and therefore their union is a submanifold of M_n' with the induced differentiable structure.

Let us begin by introducing the notations:

$F\ \overline{C^n(1)}$ = Gauss structure in $\overline{C^n(1)}$ defined by means of intersecting the elements of the Canonical Gauss structure in R^n with $\overline{C^n(1)}$; thus

$$F\ C^n(1) = \{\beta_j \cap C^n(1) \ / \ \beta_j \in F_{R^n}\}.$$

(*) A meagre set is a set A : $\mathring{\bar{A}} = \emptyset$.

Now we define a Gauss structure in M_n, called the <u>Canonical Gauss structure on M_n</u>.

We suppose our manifolds to be connected without losing generality.

If the manifold is not connected we work in each component. Then having built the Gauss structure in each component we have it in the whole manifold. We also suppose in what follows, that M_n is not covered by only one local chart, which case has already been considered by V. Buonomano. We can suppose that our manifold is covered by a countable family and locally finite collection of local charts $(V_i, \phi_i)_{i \in N}$ such that

1) $\phi_i (V_i) = C^n (3)$ and

2) $\{U_j = \phi_j^{-1} (C^n(1))\}$

are also local charts covering M_n as stated in Theorem 1.9. We are now going to re-index our set of local charts: Suppose we have the sequence of charts in some order. Then we index them in the following way: We call U_1 the first local chart in the sequence, U_2 will be the first local chart in the original sequence different from U_1 such that $U_2 \cap U_1 \neq \square$ and $U_2 - U_1 \neq \square$. U_3 will be the first different from U_1 and U_2 that intersects U_1 and U_2, and $U_3 - U_1 \cup U_2 \neq \square$, etc.

This process might end in a finite number of steps or not.

We can construct a Gauss structure in the closure of each of these local charts in a canonical way. Take for instance U_1: we build the Gauss structure in $\overline{U}_1$ taking as members of it the image by means of homeomorphism of local charts ϕ_1^{-1} of the elements of the Gauss structure $F\ C^n(1)$ in $\overline{C^n(1)}$ (For each covering in $F\ C^n(1)$ we have a covering in $\overline{U}_1$.) As we said, this can be done in every local chart U_i.

This means that we have a Gauss structure in the closure of each local chart, but they are superimposed inside the intersections of the different local charts. We are going to avoid this in order to have a nice and suitable Gauss structure in the whole manifold. Let us denote by $\beta_j' = \beta_j \cap \overline{C^n(1)}$ the covering in $\overline{C^n(1)}$ obtained from $\beta_j \in F_{R^n}$ by intersecting the elements of β_j with $\overline{C^n(1)}$. The elements in $\overline{U}_\ell$ which come from β_j' through ϕ_ℓ^{-1} are denoted by $A_{\ell j}$.

We denote also by $\phi_1^{-1}(F\ C^n(1))$ the Gauss structure in $\overline{U}_1$ obtained by taking the inverse images of the coverings in $F\ C^n(1)$ by means of ϕ_1^{-1}.

The elements in $F\ C^n(1)$ shall be denoted by $\tilde{A}_{1j}, \tilde{A}_{2j}, \ldots$. Now to build a Gauss structure in $\bar{U}_1 \cup \bar{U}_2$ we will proceed in the following way.

We have the elements A_{1j} in $\bar{U}_1$ belonging to the Gauss structure in $\bar{U}_1 \cup \bar{U}_2$. Then we preserve in $\bar{U}_2$ those elements of the Gauss structure on it which do not intersect U_1, and then we add the elements of the form $\overline{A_{2j} - \bar{U}_1}$ for any A_{2j} belonging to the Gauss structure in $\bar{U}_2$ which intersect U_1. We denote by ϕ_{12}^{-1} $(F\ C^n(1))$ the family of coverings in $\bar{U}_1 \cup \bar{U}_2$ obtained in this way.

Claim: ϕ_{12}^{-1} $(F\ C^n(1))$ is a Gauss structure in $\bar{U}_1 \cup \bar{U}_2$. Indeed, if ϕ_{12}^{-1} (β_j') denotes the covering in ϕ_{12}^{-1} $(F\ C^n(1))$ which comes from β_j' in $C^n(1)$, it is composed of:

a) those elements $A_{1j} = \phi_1^{-1}$ $(\tilde{A}_{1j})$ which come by means of ϕ_1^{-1} from $\tilde{A}_{1j} \in \beta_j'$.

b) those elements which are of the form $\overline{\phi_2^{-1}(\tilde{A}_{2j}) - \bar{U}_1}$ for $\tilde{A}_{2j} \in \beta_j'$.

Now we check all conditions of a Gauss structure as defined in 3.3.

a) Any set $A_{ij} \in \phi_{12}^{-1}$ $(\beta_j') \in \phi_{12}^{-1}$ $(F\ C^n(1))$ is the closure of an open set.

Indeed suppose the set we are dealing with is of the form $A_{1j} = \phi_1^{-1}(\tilde{A}_{1j})$ where $\tilde{A}_{1j} \in \beta_j' \in F\, C^n(1) =$ Gauss structure in $C^n(1)$. $\tilde{A}_{1j}$ is the closure of an open set in $C^n(1)$. The same is true for $A_{1j} = \phi_1^{-1}(\tilde{A}_{1j})$ because ϕ_1 is a homeomorphism and the theorem of invariance of open sets guarantees that A_{1j} has non-empty interior in $\bar{U}_1 \cup \bar{U}_2$.

The same could be said about an element $A_{2j} \ni A_{2j} \cap U_1 = \square$. The other kind of element we are dealing with is of the form $\overline{\phi_2^{-1}(\tilde{A}_{2j}) - \bar{U}_1} = \overline{A_{2j} - \bar{U}_1}$, with $A_{2j} \cap U_1 \neq \square$. We have to prove now that any of these elements is the closure of an open set.

$\phi_2^{-1}(\tilde{A}_{2j}) = A_{2j}$ is the closure of an open set in $\bar{U}_1 \cup \bar{U}_2$, so $A_{2j} = \bar{A}$. A is an open set in $\bar{U}_1 \cup \bar{U}_2$. This is true because $\overline{\bar{A} - \bar{U}_1} = \overline{A - \bar{U}_1}$ (A is open and $\bar{U}_1$ is closed so $A - \bar{U}_1$ is open).

b) Given $\phi_{12}^{-1}(\beta_j') \in \phi_{12}^{-1} F\, C^n(1)$ and two distinct sets $A_1, A_2 \in \phi_{12}^{-1}(\beta_j')$ then int $A_1 \cap$ int $A_2 = \square$ (int A denotes the interior of A). We have to deal with several cases:

1) Suppose $A_1 = \phi_1^{-1}(\tilde{A}_{1j})$ and $A_2 = \phi_1^{-1}(\tilde{A}_{1j}')$,

$\tilde{A}_{1j} \neq \tilde{A}'_{1j}$, $\tilde{A}_{1j}$ and $\tilde{A}'_{1j} \varepsilon \beta'_j \varepsilon F C^n(1)$.

As we know, in $C^n(1)$, int $\tilde{A}_{1j} \cap$ int $\tilde{A}'_{1j} = \square$,

so taking inverse images by ϕ_1^{-1}, we have

$$\text{int } (\phi_1^{-1} (\tilde{A}_{1j})) \cap \text{int } (\phi_1^{-1} (\tilde{A}'_{1j}))$$

$$= \text{int } (A_1) \cap \text{int } (A_2) = \square \quad (\phi_1 \text{ is homeomorphism}).$$

ii) Suppose $A_1 = \phi_1^{-1} (\tilde{A}_{1j})$ and $A_2 = \overline{\phi_2^{-1} (\tilde{A}_{2j}) - \bar{U}_1}$.

In this case clearly int $(A_1) \cap$ int $(A_2) = \square$.

iii) Suppose $A_1 = \overline{\phi_2^{-1} (\tilde{A}_{2j}) - \bar{U}_1}$ and

$A_2 = \overline{\phi_1^{-1} (\tilde{A}'_{2j}) - \bar{U}_1}$. Then

$$\text{int } (\overline{\phi_2^{-1} (\tilde{A}_{2j}) - \bar{U}_1}) \cap \text{int } (\overline{\phi_2^{-1} (\tilde{A}'_{2j}) - \bar{U}_1})$$

$$= (\text{int } (A_{2j}) - \bar{U}_1) \cap (\text{int } (A'_{2j}) - \bar{U}_1) = \square.$$

c) Any $\phi_{12}^{-1} (\beta'_j) \varepsilon \phi_{12}^{-1} F C^n(1)$ is a covering of

$\bar{U}_1 \cup \bar{U}_2$. Take any point $x \in \bar{U}_1 \cup \bar{U}_2$. If $x \in \bar{U}_1$, then $\phi_1(x) \in \overline{C^n(1)}$, and $\exists$ $\tilde{A}_{1j}$ s.t. $\phi_1(x) \in \tilde{A}_{1j}$, which obviously implies that $x \in \phi_1^{-1}(\tilde{A}_{1j})$.

If $x \in \bar{U}_2 - U_1$, then $\phi_2(x) \in \overline{C^n(1)}$ which $\rightarrow \phi_2(x) \in \tilde{A}_{2j} \subset \overline{C^n(1)} \rightarrow x \in \overline{\phi_2^{-1}(\tilde{A}_{2j}) - \bar{U}_1}$.

d) We have to verify now that, given any point $x \in \bar{U}_1 \cup \bar{U}_2$, there is a neighbourhood N of x such that any $\phi_{12}^{-1}(\beta_j')$ has just a finite number of sets intersecting N.

That is obvious, because for any $\phi_{12}^{-1}(\beta_j')$ there are a finite number of elements belonging to $\phi_{12}^{-1}(\beta_j')$ in $\bar{U}_1$ and $\bar{U}_2$ so any neighborhood of $x \in \bar{U}_1 \cup \bar{U}_2$ intersects a finite number of elements of $\phi_{12}^{-1}(\beta_j')$.

e) Given any open set $0 \subset \bar{U}_1 \cup \bar{U}_2$, there is a covering $\phi_{12}^{-1}(\beta_j') \in \phi_{12}^{-1}(F\, C^n(1))$ such that $\phi_{12}^{-1}(\beta_j')$ has a set $A \subset 0$.

Take any $0_i \subset \bar{U}_i$, 0_i open in $\bar{U}_1 \cup \bar{U}_2$ and $0_i \subset 0$, $i = 1$ or 2 (at least one of these choices of i is possible).

Suppose $i = 2$, and take $\phi_2(O_2) = O_2' \subset \overline{C^n(1)}$.

Then as $\overline{C^n(1)}$ has a standard family of coverings $F\ C^n(1)$, there exists $\tilde{A}_{2\ell} \in \beta_\ell' \supseteq \tilde{A}_{2\ell} \subset O_2'$. Then

$\phi_2^{-1}(\tilde{A}_{2\ell}) \subset \phi_2^{-1}(O_2') = O_2 \subset O$; and so

$\phi_2^{-1}(\tilde{A}_{2\ell}) - \overline{U}_1 \quad O$.

f) Ordered by refinement $\phi_{12}^{-1}\ F\ (C^n(1)$ is a directed set: if $\phi_{12}^{-1}(\beta_j')$ and $\phi_{12}^{-1}(\beta_\ell')$ are members of $\phi_{12}^{-1}(F\ C^n(1))$, then there exists $\phi_{12}^{-1}(\beta_m') \ni \phi_{12}^{-1}(\beta_m') > \phi_{12}^{-1}(\beta_j')$ and $\phi_{12}^{-1}(\beta_m') > \phi_{12}^{-1}(\beta_\ell')$, where $m \geq j$ and $m \geq \ell$.

Now we can construct a Gauss structure for the whole manifold by induction.

Suppose that the Gauss structure is already defined for $\overline{U}_1 \cup \overline{U}_2 \cup \cdots \cup \overline{U}_h$. We take now U_{h+1} such that

$\overline{U}_{h+1} - (\overline{U}_1 \cup \overline{U}_2 \cup \cdots \cup \overline{U}_h) \neq \Box$, $U_{h+1} \cap (U_1 \cup U_2 \cup \cdots \cup U_h) \neq \Box$.

If after U_h all the remaining U_i are such that

$\overline{U}_i - (U_1 \cup U_2 \cup \cdots \cup U_h) = \Box$, then $U_1, U_2, \ldots, U_h$ cover the whole manifold and we don't need to go ahead with the construction. If this is not the case, we continue our construction by induction in the following way:

We restrict the Gauss structure, which we have in $\bar{U}_{h+1}$ to $\bar{U}_{h+1} - (\bar{U}_1 \cup \bar{U}_2 \cup \cdots \cup \bar{U}_h)$, as we have done in the steps of building the Gauss structure from $\bar{U}_1$ to $\bar{U}_1 \cup \bar{U}_2$, and then all the steps of the proof are the same as for $\bar{U}_1 \cup \bar{U}_2$, except for some changes in notation; for instance, we will have $\phi^{-1}_{12\ldots(h+1)}$ instead of ϕ^{-1}_{12}. We call this Gauss Structure the canonical Gauss structure in M_n: $F(M^n)$; $\phi^{-1}_{123\ldots h\ldots}(\beta'_j)$ will denote the covering in $F(M^n)$ which comes from β'_j in $C^n(1)$.

As said above, we suppose that we are dealing with a function $\phi: M_n \to M'_n$ which is onto and so we can assume that the locally finite covering by closed local charts $\bar{U}'_i = \phi'^{-1}_i \overline{(C^n(1))}$ is pairwise diffeomorphic to the covering $\{\bar{U}_i\}$ we already have in M_n. So in the same way as we have a Gauss structure in M_n, we also have one in $M'_n = \phi(M_n)$, using this family of pairwise diffeomorphic local charts.

Let us call Border of the Gauss Structure in M_n with respect to $\{U_i\}$, the meagre set

$$\partial(GS) = \partial\,\bar{U}_1 \cup [\partial\bar{U}_2 - \bar{U}_1] \cup [\partial\bar{U}_3 - (\bar{U}_1 \cup \bar{U}_2)]$$ (*)

$$\cup [\partial\bar{U}_4 - (\bar{U}_1 \cup \bar{U}_2 \cup \bar{U}_3)] \quad \cdots .$$

The same sort of set can be defined in M'_n:

$$\partial(GS') = \partial\bar{U}'_1 \cup [\partial\bar{U}'_2 - \bar{U}'_1] \cup [\partial\bar{U}'_3 - (\bar{U}'_1 \cup \bar{U}'_2]$$

$$\cup [\partial\bar{U}'_4 - (\bar{U}'_1 \cup \bar{U}'_2 \cup \bar{U}'_3] \cup \cdots .$$

Let us now introduce the following <u>Gauss transformation</u> $G_1 : F(M_n) \rightarrow F(M_n)$,

$$G_1\,(\phi^{-1}_{12...h...}(\beta'_j)) = \phi'^{-1}_{12...h...}(\beta'_j)\ (\phi'^{-1}_{1.2..h..}$$

plays the same role in M'_n as $\phi^{-1}_{1.2..h...}$ in M_n).

We are going now to define our <u>family of open coverings</u> in M_n (M'_n).

We take as family V (V') of open coverings in M_n (M'_n) the one built in the following way:

Let us consider a family $W\,[C^n(1)]$ of open coverings of $C^n(1)$ each of which is composed of open, finite Jordan measurable sets, and $\phi'_i \circ \phi \circ \phi^{-1}_i$ is injective

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(*) (∂ means topological border).

in each one of these open sets in which this composition is defined, for all pairs of local charts (U_i, ϕ_i), (U'_i, ϕ'_i) in M_n and M'_n, respectively. After that we take the image by each local chart homeomorphism ϕ_i^{-1} $(\phi_i'^{-1})$ of these coverings and we obtain a family of coverings in M_n (M'_n).

Now the required g-function f is defined by the image method, i.e., $\mu \to \mu' = \{\text{int } \phi(A) \mid A \in \mu\}$, $f_\mu(A) = \text{int } (\phi(A))$ for $A \in \mu \in V$, and V is the family of open coverings in M_n already defined.

Observe that, int $(\phi(A)) \neq \square$ by the theorem of invariance of open sets because A is inside a local chart, and int $(\phi(A))$ is the image by homeomorphism of local charts, of an open, finite Jordan measurable set.

(If $\phi_i^{-1}(A_1) = A$, then $\phi'_i \circ \phi \circ \phi_i^{-1}$ is a diffeomorphism, and diffeomorphisms preserve all these properties.) Obviously f generates ϕ.

The Gauss structure we have now in $M_n - \partial(GS)$ $(M'_n - \partial(GS'))$ is the one obtained by restricting the canonical Gauss structure in M_n (M'_n) to $M_n - \partial(GS)$ $(M'_n - \partial(GS'))$.

We can place this restriction on $M_n - \partial(GS)$ because this set is obviously open. The family of open coverings is the same as in $M_n(M'_n)$ but this time the open sets are restricted to $M_n - \partial(GS)$ $(M'_n - \partial(GS'))$.

Then as ϕ^{-1} $[\partial(GS')]$ is meagre in M_n, considering the pair $M_n - \partial(GS) - \phi^{-1}[\partial(GS')]$ and $M'_n - \partial(GS')$, we can apply V. Buonomano's result; cf. [3], Theorem 15, Chapter $\overline{VI}$.

Our theorem is proved and so the Main Problem is solved on M_n except for a set of first category or a meagre set namely:

$$\partial\bar{U}_1 \cup (\partial\bar{U}_2 - \bar{U}_1) \cup [\partial\bar{U}_3 - [\bar{U}_1 \cup \bar{U}_2]] \cup \cdots \cup \phi^{-1}[\partial(GS')].$$

We finish this chapter by proving a lemma which will be useful later.

5.3 Lemma: The covering V (V') defined in the manifold M_n (M'_n) is cofinal.

Proof: Let λ be any covering of M_n. We wish to find $\mu \in V \ni \mu > \lambda$ (μ refines λ).

Let us take the intersection of λ with the local charts, namely:

$$\lambda \wedge \{U_i\}_{i\in n}^{(*)} .$$

This is a refinement of λ, each of whose elements is contained in a local chart.

Next, we take the images by homeomorphism of local charts of these open sets into the cube $C^n(1)$. In this cube

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(*) $\lambda \wedge \{U_i\} = \{A \cap U_i \mid A \in \lambda,\ U_i \in \{U_i\}\}$.

we refine the image of these open sets by means of small cubes or spheres, which are finite Jordan measurable. Besides these properties $\phi_i' \circ \phi \circ \phi_i^{-1}$ is one to one in these cubes or spheres for each i.

After this, we take the images of these cubes or spheres back to M_n and with those images of cubes or spheres we have our desired covering μ which refines λ. ($\mu > \lambda$).

Note: The same can be said when V_1 is the family of coverings V restricted to $M_n - \partial$ (GS). Interchanging M_n by M_n', V by V', V_1 by V_1', we see that our result is also true in (M_n', V') and ($M_n' - \partial$(GS'), V_1').

CHAPTER 6

MAIN PROBLEM FOR MANIFOLDS WITH SMOOTH TRIANGULATIONS

6.1 Definition of a Parametric Manifold:

Let X be a subset of an euclidean space E of dimension p. We call an admissible parametric representation of X of class C^r and dimension n:

a map $\psi: U \to \psi(U)$ of an open set U in R^n into E

with the following properties:

1) ψ is a homeomorphism from U onto X of class C^r.

2) rank of $\psi = n$ at each point of U. (ψ is an immersion).

A parametric manifold is a differentiable manifold imbedded in R^p where local charts have parametric representations given by the homeomorphisms attached to those charts. More precisely if (U, ϕ_i) is a local chart of M_n, then ϕ_i^{-1} is a parametric representation of U.

6.2 Measure in a parametric manifold M_n.

Let (V, ψ) be a local chart in M_n defined in a parametric way, $\psi: U \to \psi(U) = V$, U open in R^n. Let us define the measure of V in M_n as its n-area given by

$$\mu_\psi(V) = \int_U \sqrt{\det_{i,j} \left(\frac{\partial\psi}{\partial x_i}(u) \mid \frac{\partial\psi(u)}{\partial x_j}\right)}^{\,(*)} \, du.$$

and in the same manner we define the measure of any subset of V which is image by ψ of Jordan measurable sets in U. According to[10] we have the following result:

6.3 Theorem: Let M_n be a parametric manifold. Then there exists on M_n a measure μ, and only one, such that in in each local chart (V, ψ), μ coincides with μ_ψ: this measure is called "n-dimensional area".

Proof: See [10].

6.4 In this section we solve the main problem stated in 5.1 for the case where the manifolds involved have triangulations which satisfy a certain condition to be specified. This is done in Theorem 6.7. First we solve the main problem for a particular case which is Theorem 6.5 and then, we obtain the main Theorem 6.7 by using Theorem 6.5.

(*) $\left(\frac{\partial\psi(u)}{\partial x_i} \mid \frac{\partial\psi(u)}{\partial x_j} \right) = \sum_{k=1}^{p} \frac{\partial\psi_k(u)}{\partial x_i} \cdot \frac{\partial\psi_k(u)}{\partial x_j}$,

where $\psi(u) = (\psi_1(u), \cdots \psi_p(u))$.

We believe that Theorem 6.7 is a reasonable solution of the main problem in the sense that it includes many important cases commonly found in the applications.

6.5 <u>Theorem</u>: The main problem has a solution under the following two hypotheses:

1) There is an imbedding $\psi: M_n \to R^p$ and Gauss structure F, such that for any $\alpha \in F$ all sets $A \in \alpha$ have the same n-dimensional area. Similarly for M_n' we have an imbedding $\psi': M_n' \to R^{p'}$, such that all sets of any $\alpha' \in F'$ have the same n-area. The Gauss transformation $G: F \to F'$ takes a covering $\alpha_i \in F$ into another $\alpha_i' \in F'$ such that the sets of α_i' have the same n-area as those of α_i. Also we require that the standard family of coverings are countable and that for $m > n$, $\alpha_m > \alpha_n$ $(\alpha_m' > \alpha_n')$.

2) For any $x \in M_n$ there is a local chart (U, h) of the atlas of M_n such that h(x) belongs to the closure of an open set $W \subset h(U)$ whose Gauss structure induced by h is admissible. The same for M_n' and local chart (U', h'). Moreover, if $y = \phi(x)$ and $\tilde{G}$ is the Gauss transformation induced by the pair (h, h'), then the sets of $\tilde{\alpha}$ in W and those of $\tilde{G}(\tilde{\alpha})$ in $W' \subset h'(U')$ have all the same n-area. (Here, ϕ is the mapping of 5.1.)

<u>Proof</u>:

1) <u>Required family of coverings (V, V')</u>

We will choose as family of coverings V in M_n (the same can be done in M'_n) the family of all coverings such that each covering is formed of inverse images by local charts homeomorphisms of open, finite Jordan measurable sets in R^n, exactly as considered in Theorem 5.2.

2) <u>Required g-function</u>

The g-function f will be the one obtained from ϕ by the image method: $\mu \to \mu'$, where $\mu' = \{\phi(A) | A \in \mu\}$, $f_\mu(A) = \phi(A)$, $A \in \mu \in V$.

$\phi(A)$ is open because $|J_{U,U'}\phi(x)| \neq 0$ for all $x \in M_n$ exactly as in Theorem 5.2.

3) <u>Required Gauss structure or standard family of coverings</u>

We will take as Gauss structures F in M_n and F' in M'_n any of those satisfying hypotheses 1) and 2) in M_n, M'_n respectively, and the Gauss transformation is the one satisfying hypothesis 1), as in the hypothesis of the theorem. Now we proceed as follows: Suppose all the requirements 1) to 3) are fulfilled. Recall the definition of measure as given in 6.2, i.e.,

$$\mu(B) = \int_{\phi_i(B)} \sqrt{\det_{k,\ell} \left(\frac{\partial \phi_i^{-1}(u)}{\partial x_\ell} \;\Big|\; \frac{\partial \phi_i^{-1}(u)}{\partial x_k} \right)} \, d\,u, \text{ where}$$

$B \subset U_i$ is an open set in M_n (the same can be done in M_n').

If

$$M(u) = \sqrt{\det_{k,j}\left(\frac{\partial\phi_i^{-1}(u)}{\partial x_j} \mid \frac{\partial\phi_i^{-1}(u)}{\partial x_k}\right)},$$

this function is $> o$, whenever $u \in U_i$, by the definition of a parametric manifold. By the mean value theorem of integral calculus:

$$\mu(B) = \int_{\phi_i(B)} M(u)\, d\, u = M(\bar{P})\, \bar{\mu}\, (\bar{B}),$$

where $\bar{P} \in \bar{B} = \phi_i(B)$, and $\bar{\mu}(\bar{B})$, is the measure of $\bar{B} = \phi_i(B)$.

We have to remember that according to the way the family of open coverings in M_n is defined, we will not deal with open sets B which are outside the local chart in question.

Now let B, B' be two open sets belonging to the coverings $\omega \in V$, $\omega' = f_V(\omega) \in V'$, in V, V' of M_n and M_n', respectively. Suppose $B \subset U_\ell$, $B' \subset U_\ell'$; U_ℓ, U_ℓ', local charts in M_n and M_n', respectively,

Then:

$$\frac{\mu(B')}{\mu(B)} = \frac{\int_{\overline{B}'} M'(Q)\, du}{\int_{\overline{B}} M(P)\, du} = \frac{M'(\overline{Q})}{M(\overline{P})} \cdot \frac{\overline{\mu}(\overline{B}')}{\overline{\mu}(\overline{B})} =$$

$$= \frac{M'(\overline{Q})}{M(\overline{P})} \left| J_{U_\ell, U'_\ell} \tilde{\phi}(\overline{P}) \right|^{(\cdot)}; \quad \bar{\bar{P}} \in \overline{B},\ \overline{P} \in \overline{B},\ \overline{Q} \in \overline{B}'.$$ Indeed,

$\dfrac{\mu(\overline{B}')}{\mu(\overline{B})} = \left| J_{U_\ell, U'_\ell} \tilde{\phi}(\bar{\bar{P}}) \right|$ by the mean value theorem of integral calculus in R^n, $\bar{\bar{P}} \in \phi_\ell(U_\ell)$ and ϕ_ℓ, ϕ'_ℓ are imbeddings.

Now we have, exactly as in 4.3,

$$\frac{\mu(B')}{\mu(B)} = \overline{\lim_{\alpha_m \in F}} \frac{n(B', \alpha'_m)}{n(B, \alpha_m)} = \underline{\lim}_{\alpha_m \in F} \frac{n(B', \alpha'_m)}{n(B, \alpha_m)}.$$

This implies that we have:

$$\lim \frac{n(B', \alpha'_m)}{n(B, \alpha_m)} = \underline{D}f_\omega(B) = \overline{D}f_\omega(B) = \frac{\mu(B')}{\mu(B)} = \frac{M'(\overline{Q})}{M(\overline{P})} \left| J_{U_\ell, U'_\ell} \tilde{\phi}(\bar{\bar{P}}) \right|$$

and $F(\bar{\bar{P}}, \overline{P}, \overline{Q}) = \dfrac{M'(\overline{Q})}{M(\overline{P})} \left| J_{U_\ell, U'_\ell} \tilde{\phi}(\bar{\bar{P}}) \right|$ is a continuous function from $\overline{B} \times \overline{B} \times \overline{B}' \to R$ in the variables $\overline{Q}$, $\overline{P}$, $\bar{\bar{P}}$, because $M(\overline{P}) \neq 0$. Moreover as M_n and M'_n are locally compact, we can assume that F is uniformly continuous in $\overline{B} \times \overline{B} \times \overline{B}'$.

(·) $\tilde{\phi} = \phi'_\ell \circ \phi \circ \phi_\ell^{-1}$.

Let us show now that given $\varepsilon > 0$ and $x \in M^n$, there is an open set A in M_n with $x \in A$ such that $\forall\, \omega \in V$ and $\forall\, B \in \omega$, $B \subset A$, we have diam $[Df_\omega (B)] < \varepsilon$. Indeed, select A contained in some local chart (U_i, ϕ_i) at x such that whenever we take points $\ddot{P}, \overline{P} \in \phi_i\ (A)$ and $\overline{Q} \in \phi_i' \circ \phi\ (A)$, with (U_i, ϕ_i') the corresponding chart at $\phi(x)$, we have

$$|F\ (\ddot{P}, \overline{P}, \overline{Q}) - F\ (\ddot{P}', \overline{P}', \overline{Q}')| < \varepsilon$$

for $\ddot{P}', \overline{P}' \in \phi_i\ (A)$ and $\overline{Q}' \in \phi_i' \circ \phi\ (A)$, as well. This is possible due to the uniform continuity of F in A.

Therefore by the definition of g-derivative for any $\omega \in V$ and $B \in \omega$ with $B \subset A$ and any $\gamma > \omega$, $\gamma \in V$ we have

$$|\overline{Df}_\gamma\ (B_1) - \underline{Df}_\gamma\ (B_2)| < \varepsilon$$

for any $B_1, B_2 \in \gamma$ and $B_1, B_2 \subset B$. So by the definition of g-derivative again, we have

$$\text{diam}\ [Df_\omega\ (B)] < \varepsilon\ .$$

Let us now show that Df is pointwise cofinal.

According to what we just proved we have $B \in \omega \to \text{diam}\ [Df_\omega(B)] < \varepsilon$, and so given any covering Γ of the set R of real numbers, we restrict it to the interval $\overline{Df_\omega(B)}$ = clos $(DF_\omega(B))$. This interval is compact, and so we have a Lebesgue number for the covering Γ restricted to this interval. Let this number be denoted by ε_1.

And now let us put $\varepsilon' = \min(\varepsilon, \varepsilon_1)$. Then by what we have seen above, there exists $B_1 \subset B$ $(B_1 \in \gamma \in V,\ \gamma > \omega)$ $x \in B_1$ such that, by the continuity of Df, diam $[Df_\gamma(B_1)] < \varepsilon' \rightarrow Df_\gamma(B_1)$ $\subset C$ for some $C \in \Gamma \bigcap (\overline{Df_\omega(B)})$.

Finally, we are going to prove that Df generates a function ψ which coincides with the Jacobian $|J_{U_\ell, U'_\ell}\ \phi|$ on each point x for suitable local charts $U_\ell(x)$, $U'_\ell\ (\phi(x))$.

As Df is pointwise cofinal, it generates a function ψ: $M_n \rightarrow R$, as V is cofinal. When ϕ_ℓ takes $x \in U_\ell$ into the interior of a set in R^n whose induced Gauss structure by ϕ_ℓ is admissible, then we can prove that Df generates $|J_{U_\ell}, U'_\ell\ \phi(x)|$ as in Theorem 3.21. Otherwise, by the hypothesis of the theorem, the point x in question is such that $\phi_\ell(x)$ belongs to the boundary of a set E, whose Gauss structure (induced by ϕ_ℓ) in clos (E) is admissible. Now Df generates a unique continuous function ψ: $M_n \rightarrow R$ (Reals), and in U_ℓ we have that $\psi \circ \phi_\ell^{-1}$ and $|J_{U_\ell}, U'_\ell\ \phi| \circ \phi_\ell^{-1}$ coincide in all points of the interior of E. As int $E \subset \phi_\ell(U_\ell)$ and both $\psi \circ \phi_\ell^{-1}$ and $|J_{U_\ell}, U'_\ell\ \phi| \circ \phi_\ell^{-1}$ coincide in int E, and are defined in $\phi_\ell(U_\ell) \supset E$, they also coincide in $x \in \partial(E)$.

Therefore

$$\psi \circ \phi_\ell^{-1} = |J_{U_\ell}, U'_\ell \; \phi| \circ \phi_\ell^{-1} ,$$

in $E \cap \phi_\ell (U_\ell)$, which implies

$$\psi(x) = |J_{U_\ell}, U'_\ell \; \phi(x)| ,$$

because $x \in E \cap \phi_\ell (U_\ell)$.

This completes the proof of the theorem.

6.6 Before we prove the next theorem, let us discuss a few questions.

Let M_n be a differentiable manifold of dimension n and assume M_n has a triangulation T satisfying the following property:

(α) for a convenient subdivision T' of T the star St(a) of every vertex a of T' is simplicially equivalent to an n-complex K in R^n having all n-simplices with the same n-area.

We call T a <u>balanced triangulation</u> if it has property (α).

Related to this concept, we can prove the proposition which follows, whose statement depends on the concept of n-complex inscribed in the n-ball. A n-complex K is inscribed in the n-ball B^n if one vertex of K is at the center of B^n and all the others lie on the boundary S^{n-1} of B^n. Now our proposition is:

(β) There exists for every $n \geq 0$ and every $k \geq n + 1$ an n-complex inscribed in the n-ball B^n with k vertices having all n-simplices with the same n-area.

Proof: We proceed by induction on n. It is true for $n = 2$ and $k \geq 3$ as is easily checked. Suppose the proposition is true for $n - 1 \geq 2$ and $k \geq n$. For n and $k = n + 1$ the proposition is true because we can take as K an n-simplex with all edges of the same length. If $k > n + 1$, consider $k - 2 \geq n$. Then by the induction hypothesis there is a (n-1) - complex inscribed in B^{n-1} with k-2 vertices. Consider B^{n-1} as the equatorial hyperplane of B^n, and put the remaining two points from those k points given before at the north and south pole of B^n, respectively. Now, joining the poles with all the k-2 points in B^{n-1}, we obtain the required n-complex.

6.7 Theorem: Let M_n, M'_n and $\phi: M_n \to M'_n$, be as in the main problem, and assume that both M_n and M'_n are compact and have smooth triangulations satisfying condition α) stated above. Then the main problem has a solution.

Proof: If we analyse the proof of Theorem 6.5, we notice that its fundamental steps are:

1) The proof that Df is pointwise cofinal and therefore generates a continuous function $\psi: M_n \to R$.

2) The proof that ψ coincides with the absolute value of the Jacobian for suitable local charts.

So, roughly speaking, to prove our theorem we should somehow obtain 1) and 2) above, and that is what we take as guide for the proof of our theorem.

Let us analyse 1). The property of being pointwise cofinal at x is a local property, i.e., it depends only on what is going on in the neighborhood of x, and due to Lemma 4.3, it is invariant under a homeomorphic transformation, in the sense of that lemma. Now as M_n and M'_n have balanced triangulations, by hypothesis, property α just says that $x \in M_n$ and $\phi(x) \in M'_n$ have small neighborhoods which are homeomorphic to the interior of a certain n-complex L in R^n. Now the set of barycentric subdivisions of the n-complex L is clearly an admissible Gauss structure in the sense of 4.6 and so, locally, ϕ induces a map $\tilde{\phi}: L \to L$ which is differentiable with Jacobian different from zero. For such $\tilde{\phi}$ according to what we said before, and [4], we can find a g-function $\tilde{f}$ generating $\mathfrak{F}$ where g-derivative $D\tilde{f}$ generates the absolute value of the Jacobian of $\mathfrak{F}$. So all we have to do according to Lemma 4.6 is to provide for M_n and M'_n, Gauss structures, having local images in L, which are precisely the barycentric subdivisions of L .

We proceed as follows:

Reasoning with M_n, we have a homeomorphism $h: K \to M_n$, where K is a n-complex, because M_n is triangulable and for each n-simplex s_i, $h_{|s_i}: s_i \to M_n$ has an extension h_{s_i} to a neighborhood U_i of s_i in R^n (plane of s_i) such that $h_{s_i}: U_i \to M_n$ is an imbedding (i.e. $h_{s_i}(U_i)$ is a smooth submanifold), because the triangulation of M_n is smooth.

So $(h_{s_i}(U_i), h_{s_i}^{-1})$ are local charts; $h_{s_i}^{-1}$ is differentiable because it is the inverse image of an imbedding and $h_{s_i}(U_i)$ is open, because of the theorem of invariance of domain or open sets *.

(*) Brower's theorem on the invariance of domain: Let X be an arbitrary subset of R^n and h a homeomorphism of X on another subset h(X) of R^n. If x is an interior point of X, then h (x) is an interior point of h (X). In particular, if A, B are homeomorphic subsets in R^n and A is open, then B is open.

We also know that K is simplicially equivalent to a subcomplex of the fundamental simplex H_p, $p \geq n$,

Let $H_p^{(o)} = H_p$,

$H_p^{(1)}$ = the first barycentric subdivision of H_p,

....

$H_p^{(m)}$ = the m^{th} barycentric subdivision of H_p, and

so on.

Call α^o the family of all simplices in H_p,

α^1 " " " " " " $H_p^{(1)}$,

α^2 " " " " " " $H_p^{(2)}$,

α^m " " " " " " $H_p^{(m)}$, etc.

Then it is obvious that the family of all α^i, is a standard family of coverings or a Gauss structure in H_p. This implies that $\{h(\alpha_i)\}$, $i \geq 1$, is a Gauss structure (standard family of coverings) in M_n.

Therefore we can identify M_n and M_n' with subcomplexes of H_p with structures of differentiable manifolds induced by M_n and M_n'. Now ϕ: $M_n \to M_n'$ will be regarded as a differentiable map of a subset of H_p into another.

Now we are able to prove step 1) referred to above, i.e., Df is pointwise cofinal and so generates a function $\psi: M_n \to R$.

Indeed, let $x \in M_n$ be arbitrary and take a star $St(a) \ni x \in int\ St(a)$. Then according to α) we can suppose that St(a) is simplicially equivalent to a complex L in R^n defined by some barycentric subdivision of an n-simplex L. Again by Brower's invariance theorem, the open star (int St(a)) is taken into an open set in R^n, the interior of the complex referred to above, which is a differentiable manifold in R^n with differentiable structure induced by the differentiable structure of int [St(a)]. This is an easy consequence of the implicit function theorem. We do the same for M_n' obtaining a complex L' and a star $St(a') \ni$ $\phi(x) \in int\ (St\ (a'))$.

Let $h_1 : St(a) \to L$

$h_1' : St(a') \to L'$ be the corresponding pair of homeomorphisms.

We have reduced the problem to the interior of St(a) because pointwise - cofinality is a local property, as said before. By Corollary 4.4, (Df is pointwise cofinal $\Leftrightarrow$ $\tilde{D}f$ is pointwise cofinal), the pair of homeomorphisms h_1, h_1' induce a g-function $\tilde{f} \ni \tilde{D}f$

is pointwise cofinal, by restricting ourselves to the interior of the star St(a) and St(a'), as discussed above. Therefore, as $\tilde{Df}$ is pointwise cofinal, this implies that Df is also pointwise cofinal.

Our second step is to prove that ψ, generated by Df, coincides with the absolute value of the Jacobian of ϕ. Again, analysing the proof of Theorem 6.5, we conclude that this depends essentially on condition 2 of that theorem, and so we are going to show that this condition is also true in the present situation.

Indeed; let $x \in M_n$ and $\phi(x) \in M_n'$. As the triangulation of M_n is smooth, we can take as local chart (U_ℓ, h_ℓ) at x, the open set U_ℓ which contains the simplex s_ℓ, to which x belongs, and is in the hyperplane defined by s_ℓ. We take as h_ℓ the restriction of the homeomorphism $h: K \to M_n$ to s_ℓ together with its extension to U_ℓ. We do the same for $\phi(x)$.

Clearly (U_ℓ, h_ℓ) and (U_ℓ', h_ℓ') satisfy condition 2) of Theorem 6.5, where the W required in that theorem is given by the interior of s_ℓ in U_ℓ.

This completes the proof of the theorem.

Remark: Condition 2) is introduced to obtain locally an imbedding in R^n. The more general situation, where we have a manifold M^n homeomorphically imbedded in R^p and we want to know if it is also diffeomorphically imbedded in R^p, is studied by B. Hajduk in [5].

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APPENDIX

We want to finish our work by pointing out several possibilities open to further research suggested by what we have done so far. We believe this might indicate that our ideas could produce new achievements in non-deterministic mathematics in the future.

I

Measure Induced by Gauss Structures

1. The reader certainly noticed that frequently we have approached the concept of derivation of g-functions with the concept of measure. This is clear, for instance in the proof of Theorem 6.5. We believe that it is possible to define a measure every time we have a Gauss space satisfying certain conditions. This we want to discuss briefly at this point.

Let (X, F) be a Gauss space. Call a <u>figure</u> in X the union of a countable set (possibly finite) of elements of coverings of F (such that no two of them have common interior points). We emphasize that if H is a figure of X, the sets whose union is H are not necessarily taken from the same covering of F; they might belong to different coverings.

Let us assume that F satisfies the following conditions:

C_I) F has a cofinal, countable, well-ordered set of coverings

$$\alpha_1 < \alpha_2 < \cdots < \alpha_i < \alpha_{i+1} < \cdots .$$

C_{II}) There is a monotone sequence $\{m_i\}_{i>1}$ of positive integers with $\lim m_i = \infty$, such that the number of sets of α_{i+1} contained in one set of α_i is precisely $m_i = n(\alpha_i, \alpha_{i+1})$.

We observe that F being directed by refinements, we can drop the word "well-ordered" from C_I because it is a consequence of cofinal + countable; however we believe it is good to have it explicitly stated.

The collection $\{\alpha_i\}_{i>1}$ of C_I will be called a base for F.

We believe F can generate a measure in X as follows: first select as measure of any set in α_1 an arbitrary number K > 0, called the gauge of F. Then

as measure of each set in α_2 take $\frac{K}{m_1} = \frac{K}{n(\alpha_1, \alpha_2)}$,

and for any set A in α_i we take

$$\mu(A) = \frac{K}{m_1, m_2 \cdots m_{i-1}} = \frac{K}{n(\alpha_1, \alpha_2), n(\alpha_2, \alpha_3) \cdots n(\alpha_{i-1}, \alpha_i)}$$

Clearly the measure of each set in α_i tends to zero as $i \to \infty$.

Now let $H = \bigcup_1^\infty A_i$ be a figure in X. We put

$m(H) = \sum_1^\infty m(A_i)$, which may be finite or infinite.

Finally if E is an arbitrary set in X we define the <u>exterior measure of E</u> as the number

$$m_e(E) = \inf_{E \subset H} m(H), \quad H \text{ a figure in } X.$$

To have a measure in X we have only to verify that $m_e(E)$ so defined satisfies the requirements of an exterior measure in the sense of Caratheodory. Unfortunately by the time this thesis was written we did not check these points, but what we have said is enough to show how a theory of measure can be built in a Gauss space. Certainly many interesting connections with derivatives must exist and there is also the question of deciding if every open set is measurable when X has a countable basis of open sets. We plan to investigate all these questions in our future work.

2. Another question suggested to us by our work connected with measure is the following: Suppose ϕ is a function of R^n into R^n which maps measurable sets in R^n into measurable sets, taking measure, for instance, in the Jordan sense. Suppose there is a real continuous function f defined in R^n such that for any measurable set A there is a point $x \in A$ such that

$$\frac{\mu\ \phi(A)}{\mu\ (A)} = f\ (x). \qquad (1)$$

We ask if ϕ is then necessarily differentiable, or if that is not the case, what conditions we have to assume for ϕ such that f as above does exist.

If ϕ is differentiable with Jacobian different from zero, we know that a function like f does exist: precisely, f is the Jacobian of ϕ and also relation (1) is true.

If we can answer these questions, other theorems similar to 6.5 and 6.7 can be established. So far, we do not know anything about them. The classical theorem of Radon-Nikodym is connected with these questions but in general, f, the Radon-Nikodym derivative, is not continuous.

II

Brownian Motion and N.D.A.

Just to make references to possibilities of applications of g-derivatives in physics, we recall the work of Wiener about the Brownian motion of particles in a fluid.

Roughly speaking, the result is that the path of such particles is mathematically represented by a continuous curve without tangent at any point. Now, classically the speed of a particle is always connected with the tangent to the trajectory, which in the present case cannot be represented at all. However, following the ideas discussed in this thesis, a reasonable approach to this problem could be as follows:

Let $\phi: [0, 1] \rightarrow R^3$ be the path of the particle in the Brownian motion. So ϕ is continuous but nowhere differentiable. Our aim is then to find a continuous g-function

$$f: ([0, 1], V) \rightarrow (R^3, V')$$

which generates ϕ and whose g-derivative Df is pointwise cofinal. If we can do this we can regard Df as the velocity of the particle, because under these circumstances Df generates a usual function $\psi : [0, 1] \to R$ which is continuous. For more details about other applications of non-deterministic mathematics to physics, see V. Buonomano's Ph.D. thesis [3].

III

Complexes Inscribed in the n-Ball

We wish to make a few comments about property α) used in Theorem 6. 7.

Let K be a n-complex inscribed in the ball B^n. Consider the following statement: β)"K is always simplicially isomorphic to another n-complex L inscribed in B^n having all n-simplices with same n-area".

So far we do not know any rigorous proof of this fact, even though we believe it to be clearly true on intuitive grounds, i.e., one might be able to "move" the vertices of K lying on the boundary S^{n-1} of B^n until we get the desired result. As a matter of fact this is evident for $n = 2$.

Assuming β) to be true, it is often helpful to use it to prove property α). For instance, if a manifold M_n has a triangulation such that the star of each vertex is simplicially isomorphic to a n-complex inscribed in the n-ball, then it will satisfy also property α). Usually that is the situation we encounter in the applications and so it is of some interest to investigate the property β) above.

IV.

Extension of N.D.A. to Infinite Dimensions

Another natural question is the extension of our results to infinite dimensions. More precisely, let

$$\phi: X \to Y$$

be a differentiable map with X, Y Banach spaces.

We ask if there is a g-function

$$f: (X, V) \to (Y, V')$$

for convenient families V, V' such that f generates ϕ and its derivative Df is pointwise cofinal. If such an f exists, Df generates a real function ψ defined in X. Then what is the meaning of ψ? More precisely, when we deal with a finite dimension, the derivative is connected with two things: a linear map and a real function which is the value of the Jacobian at each point. Now, for infinite dimensions the derivative is still associated with linear maps, but what replaces the Jacobian?

So far we do not know any reasonable answer to this question, but it certainly deserves attention and the function ψ referred above might tell us something about it.

V

Possible Intrinsic Study of Manifolds

Analysing Theorems 6.5 and 6.7, we realize that they show how, in some cases, the Gauss structure does, in certain sense, reproduce the usual differentiable structure. However, we do not need necessarily to restrict ourselves to these cases; we might just start the study of manifolds directly with g-functions and Gauss structures. More precisely, instead of looking to the category D of differentiable manifolds and differentiable maps, we look at the category P of pairs (M, V), where M is a topological manifold with Gauss structure and the morphisms are continuous differentiable g-functions.

From this point of view, our theorems merely state conditions under which objects and morphisms of P can be identified with objects and morphisms of D.

The advantage of starting with P is that most of the concepts defined will be topological invariants, due to Lemmas 4.3 and 4.4, and this avoids one of the serious problems with manifolds, namely, every time we define something using local charts it is not usually easy to verify the topological invariance of the concept introduced. In few words, we have always to worry about local charts which were really introduced because we did not have any other way of speaking about derivatives. Now that an intrinsic theory of derivatives does exist, it seems to us that the natural thing to do is to use Gauss structure instead of local charts. We agree that it is perhaps too early to decide which philosophy to adopt, but the idea of reconstructing all of differential topology and differential geometry in terms of Gauss structure seems to us a fascinating enterprise.

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