ADVANCES IN LIFE TESTING:

PROGRESSIVE CENSORING AND GENERALIZED DISTRIBUTIONS

By

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to my loving father: as you leave a wonderful career, I begin.

to my beautiful mother: you give me the strength I need to succeed.

Preface

This thesis contains material which has been published or sent for publication under joint authorship between myself and my supervisor, Dr. N. Balakrishnan. My contribution to each of these publications was to perform all of the mathematical derivations, write all the computer programs for simulation and calculation, and write up and type most of the material, including tables and figures. Dr. Balakrishnan gave me ideas and helpful suggestions when needed, and edited all of the publications.

Abstract

This thesis is presented in three sections: Section One (Progressive Type-II Censoring), Section Two (Generalized Distributions) and Section Three (Conclusions).

In Section One of the thesis, a method of censoring known as progressive Type-II censoring is presented. Mathematical properties of the progressive Type-II censored order statistics arising from this type of censoring are established for particular as well as arbitrary distributions. Applications to inference, including best linear unbiased estimation and maximum likelihood estimation, are discussed, as well as simulation, and the question of optimal censoring patterns is also addressed through an extensive computational study.

Section Two of the thesis concerns generalized distributions. Here, we introduce a shape parameter to the logistic and half logistic distributions and discuss the properties of the resulting distributions. Many recurrence relations for single and product moments of order statistics from these distributions are established. Best linear unbiased, maximum likelihood and moment estimation of

parameters arising from these distributions are considered, and truncated versions of the distributions are also examined.

Finally, we conclude with a number of questions and open problems which have yet to be addressed in the future.

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Section One: Progressive Type-II Censoring

1 Introduction

1.1 General Introduction

The importance of product reliability cannot be overstated. As more products are introduced to the market, consumers have the luxury of demanding high quality and long life in the products which they purchase. For example, one way in which manufacturers attract consumers to their products is by providing warranties on product life times. In order to design a cost-effective warranty, a manufacturer must have knowledge about product failure time distributions. To gain this knowledge, life testing and reliability experiments are carried out before (and while) products are put on the market. Of course, the information gained through life testing experiments is used for many purposes in addition to providing warranties: in pharmaceutical applications, for example, the life times of certain drugs need to be studied for dosage administration. Furthermore, the continuous improvement of products is desirable in all competitive industries. Life testing experimentation is one way in which product improvement is gauged.

In Section One of this thesis, a versatile censoring method known as "Progressive Type-II censoring," which may be used in life testing and reliability experimentation, is discussed both mathematically and empirically. In addition to many interesting mathematical properties and inferential issues that arise, it is shown that in using this method of censoring, which includes as a subset "conventional" methods of censoring, the efficiency of certain estimators increases by as much as 600 per cent as opposed to conventional methods alone.

1.2 The Need For Progressive Censoring

In life testing and reliability experiments, units are often lost or removed from experimentation before failure. This loss may occur unintentionally, or it may be designed into the study. Unintentional loss may occur, for example, in the case of accidental breakage of an experimental unit, or if the experimenter loses contact with an individual under study, or if, due to some unforeseen circumstance, such as depletion of funds, experimentation must cease. More often, however, units are removed from an experiment intentionally, perhaps in order to free up testing facilities for other experimentation, or to save time and cost if, for example, experimental facilities are being rented on a per unit time basis. In some cases, when there are live units on test, intentional removal of items or termination of an experiment may be due to ethical considerations. It is this case of intentional censoring which we will be interested in.

Conventional Type-I and Type-II one-stage right censoring has been studied in detail by many authors, including, Cohen (1991), Lawless (1982), Nelson (1982) and London (1988), who consider life time studies in industrial as well as actuarial contexts, in both parametric and non-parametric cases. Consider a sample of n units put on test at time 0. In Type-I right censoring, a time T, independent of the failure times, is fixed such that beyond this time no failures are observed, that is, experimentation terminates. Thus, the number of complete life times (and therefore the number of partial life times) observed is a random variable. Type-II censoring differs in that the number of observed failures is fixed, say m, so that at the time of the m^{th} failure, experimentation terminates, leaving n-m partially observed failure times. Here, the time of termination of the experiment is random.

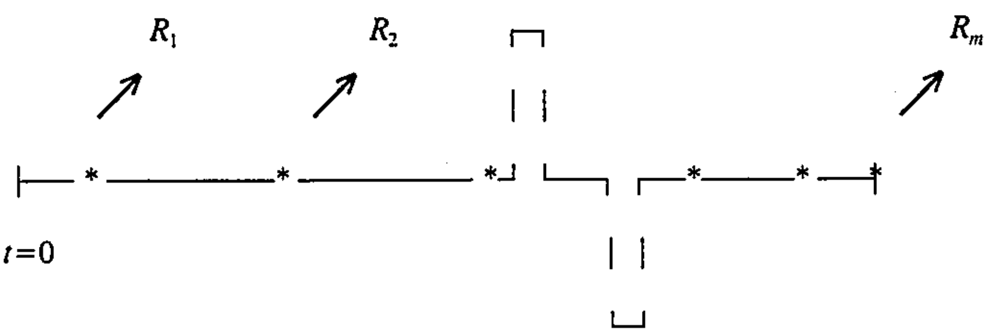
Both Type-I and Type-II one-stage right censoring described above have been generalized to the case of double (left and right) censoring, where, in the case of Type-I censoring, observation begins at a fixed time T_L rather than at time 0, and in Type-II censoring, observation of failures begins with the $(r+1)^{th}$ failure where r is fixed, rather than at time 0. These situations arise when, for example, the first few failures occur too quickly to be observed [see Nelson (1982)].

Neither of the censoring schemes discussed above allow for units to be lost or removed from the test at points other than the final termination point. This allowance may be desirable, as in the case of studies of wear, in which the study of the actual aging process requires items to be fully disassembled at various stages in the experiment. Intermediate removal may also be desirable when a compromise between reduced time of experimentation and the observation of at least some extreme values is desired, or when some of the surviving items in an experiment which are removed early on, perhaps items which are very difficult to obtain or very expensive, can be sent for subsequent testing. As mentioned earlier, the loss of items at points other than the final termination point may also be unavoidable, as in the case of accidental breakage of experimental units or loss of contact with individuals under study. It may be mentioned here that in cases of unintentional censoring, an analysis assuming censoring was intentional may be carried out, keeping in mind that the variances of estimators may be low.

1.3 Literature Review

Cohen (1963, 1966, 1975, 1976), Cohen and Whitten (1988), Cohen (1991), Ringer and Sprinkle (1972), Wingo (1973), Cohen and Norgaard (1977), Sherif and Tan (1978), Gibbons and Vance (1983), Wingo (1993) and Wong (1993), have all considered maximum likelihood estimation for parameters of various life time distributions in the case where m times, $T_1, \ldots, T_m$ are fixed such that at each of these times, $R_1, \ldots, R_m$ surviving units are randomly removed from the test, respectively. Here, the R_i's are fixed, with the provision that there are R_i surviving items at time T_i, $i = 1, 2, \ldots, m$. This is a generalization of the Type-I one-stage right censoring discussed above, and is referred to as *Progressive Type-I right censoring*, or *Type-I multi-stage right censoring*. Cohen (1963) has mentioned that even if the T_i's are not pre-determined (as in the case of accidental breakage), the likelihood function is unchanged apart from a function which is independent of the parameters of interest, as long as these T_i's have distribution independent of the life times under study. This may be a questionable assumption in some cases. Sampford (1952) and London (1988) have considered parametric and non-parametric models for this case of random loss, where patients withdraw from study before its termination for various reasons. Here, both R_i's ant T_i's are random. Robinson (1983) has considered a case in which the R_i's are random but the T_i's are fixed. Gajjar and Khatri (1969) have considered a case of progressive Type-I right censoring in which at each time of removal T_i, the population parameters change (for example, due to adjustment of temperature settings).

A generalization of the conventional Type-II one-stage right censoring is considered in this thesis. It is referred to as *Progressive Type-II right censoring*. In this censoring scheme, n units are placed on test at time zero. Immediately following the first failure, R_1 surviving items are removed from the test at random. Then, immediately following the second observed failure, R_2 surviving items are removed from the test at random. This process continues until, at the time of the m^{th} observed failure, the remaining $R_m = n - R_1 - R_2 - \dots - R_{m-1} - m$ items are all removed from the experiment. This censoring scheme may be depicted pictorially as follows:



(* = failure)

In this scheme, R_1, R_2, ..., R_m (and therefore m) are pre-determined. Thus, here the censoring times (T_i's) are random, but the numbers of items to fail before each censoring time are fixed. The resulting m ordered values which are obtained as a consequence of this type of censoring are referred to as *progressive Type-II right censored order statistics*. Notice that if $R_1 = R_2 = \dots = R_{m-1} = 0$, so that $R_m = n - m$, this scheme reduces to the conventional Type-II one-stage right censoring scheme discussed above, that is, the first m usual order statistics are observed. Also notice that if $R_1 = R_2 = \dots = R_m = 0$ and therefore $m =$

n, the progressive Type-II right censoring scheme reduces to the case of no censoring, so that all n usual order statistics are observed. By "usual order statistics," we refer to those discussed, for example, in Hogg and Craig (1995), David (1981) or Arnold, Balakrishnan and Nagaraja (1992): if $X_1, X_2, ..., X_n$ is a *random sample* of size n from some distribution $F(x)$ (that is, X_i, $i = 1, 2, ..., n$ are *independent* and *identically distributed* with cumulative distribution function $F(x)$), then the ordered sample $X_{1:n}, X_{2:n}, ..., X_{n:n}$ are the usual order statistics for a sample of size n from $F(x)$. Thus, usual order statistics are a special case of progressive Type-II right censored order statistics, so that any results obtained for progressive Type-II right censored order statistics are generalizations of corresponding results for usual order statistics.

Mann (1969, 1971), Thomas and Wilson (1972), Cacciari and Montanari (1987), and Viveros and Balakrishnan (1994) have all discussed linear inference for the case of progressive Type-II right censoring when life time distributions are Weibull and exponential. It may be mentioned here that the maximum likelihood estimates obtained for progressive Type-I right censoring (mentioned above) may be easily modified to progressive Type-II right censoring by replacing the T_i's with the appropriate failure times. However, unconditional linear inference can only be carried out for Type-II censored data, since it is only in this case that the number of observed failures and partial life times are known to the experimenter in advance; see Arnold, Balakrishnan and Nagaraja (1992). Mann (1971) and

Viveros and Balakrishnan (1994) among others, have used Type-II linear inference as an approximation for Type-I data with some success. An interesting real application of progressive Type-II right censoring has been carried out by Montanari and Cacciari (1988), where a study of wear for insulated cable is carried out assuming a Weibull life time distribution for the cable.

Progressive Type-II right censoring may be generalized to the case of left and right censoring as well, where observation of failures begins at the time of the $(r+1)^{th}$ failure, at which time R_{r+1} items are removed from the sample, and so on. Progressive Type-I censoring may be similarly generalized by starting observation at a fixed time $T_L > 0$.

As one can see from the discussion above, most of the work which has been done thus far on progressive censored order statistics has involved solving estimating equations numerically for maximum likelihood estimates. The inference done by Mann (1969, 1971) also involved cumbersome expressions and tables, as well as numerical integration. Thomas and Wilson (1972) gave an algorithm to compute moments of progressive Type-II right censored order statistics from arbitrary continuous distributions based on moments of corresponding usual order statistics, as well as an independence result for progressive Type-II censored order statistics involving exponential spacings. Viveros and Balakrishnan (1994) presented some closed form results for BLUE's of the two-parameter exponential distribution. As mentioned earlier, it is only in the Type-II case that the number of failures to be observed is known in advance; as such, we may discuss interesting mathematical properties of progressive Type-II right censored order statistics. However, in all, very few closed form results involving progressive Type-II censored order statistics have been obtained, and virtually no work has been done on the mathematical properties of these progressive Type-II censored order statistics. This is an area which we concentrate on in this thesis.

1.4 Mathematical Notations

We begin with the following assumptions and notations: Suppose n independent items are put on test with continuous, identically distributed failure times $X_1, X_2, \ldots, X_n$, where each X_i, $i = 1, 2, \ldots, n$, has cumulative distribution function $F(x)$ and probability density function $f(x)$. Suppose further that the pre-determined progressive Type-II right censoring scheme is $(R_1, R_2, \ldots, R_m)$. Then, we can denote the m completely observed failure times as:

$$X_{i:m:n}^{(R_1,R_2,\ldots,R_m)} \;, \quad i = 1, 2, \ldots, m \;.$$

For convenience, when it is clear as to what the censoring scheme is, we will use the simplified notation $X_{i:m:n}$, $i = 1, 2, \ldots, m$, to denote these failure times. We call these completely observed failure times the progressive Type-II right censored order statistics from $F(x)$ for a sample of size n with censoring scheme $(R_1, \ldots, R_m)$. Note that $X_{i:m:n}$ is not the same as $X_{i:n}$, the i^{th} usual order statistic from a sample of size n, for $i \geq 2$. This can be easily seen since there is a possibility that the item corresponding to the i^{th} ordered failure time from the original sample of size n may have been censored by the time of the observation of the i^{th} progressive Type-II right censored order statistic. However, $X_{1:m:n} = X_{1:n}$, since no items have been censored prior to the first failure.

Unlike the special case of usual order statistics from continuous distributions, where the marginal probability density function of the i^{th} order statistic has the simple form [see David (1981) or Arnold, Balakrishnan and Nagaraja (1992)]

$$f_{X_{i:n}}(x_i) = \frac{n!}{(i-1)!\,(n-i)!}\,[F(x_i)]^{i-1}\,[1-F(x_i)]^{n-i} f(x_i) \ , \tag{1.4.1}$$

and joint marginal distributions have similar forms, it is not immediately evident as to what the marginal probability density function of the i^{th} progressive Type-II right censored order statistic should look like. However, one can write the joint probability density function of all m progressive Type-II right censored order statistics as follows:

$$f_{X_{1:m:n},\ldots,X_{m:m:n}}(x_1,\ldots,x_m) = c\prod_{i=1}^{m} f(x_i)[1-F(x_i)]^{R_i}, \quad 0 < x_1 < \ldots < x_m < \infty$$

where $c = n(n-R_1-1)(n-R_1-R_2-2)\ \ldots\ (n-R_1-R_2-\ldots-R_{m-1}-m+1).$

(1.4.2)

Notice that immediately preceding the first observed failure, n items are still on test; immediately preceding the second observed failure, n-R_1-1 items are still on test, and so on; immediately preceding the m^{th} observed failure, n-R_1-R_2-...-R_{m-1}-m+1 are still on test. Thus, the constant c, in addition to being simply the normalizing constant to make the joint pdf integrate to one, is also the number of ways in which the m progressive Type-II censored failures may occur if the observed failure times are x_1, ..., x_m. The constant c may also be written as:

$$c = \frac{n!}{({}_{n-1}P_{R_1})({}_{n-R_1-2}P_{R_2})\ldots({}_{n-R_1-R_2-\ldots-R_{m-1}-m}P_{R_m})} \quad \text{where} \quad ({}_aP_b) = \frac{a!}{(a-b)!},$$

which may be interpreted as follows: At the time of the first removal of R_1 items, n-1 surviving items are on test; at the time of the second removal of R_2 items, n - R_1 - 2 items are on test, and so on; at the time of the m^{th} removal of R_m items, n -R_1 - ... - R_{m-1} - m items remain on test. In yet another light, c is the number of identical terms added together if a Jacobian argument with a c to 1 transformation is used. [see Hogg and Craig (1995, ch. 4)].

Having motivated the study of progressive Type-II right censored order statistics, we may now just regard them as a set of special order statistics, and allow the support of the distribution $F(x)$ to range over the entire real line. In fact, even in the realm of reliability studies, data which take on only positive values are sometimes modelled by, for example, normal and extreme-value distributions, with location and scale parameters suitably adjusted so that the probability of obtaining a negative value is virtually zero [see Bain and Engelhardt (1991)].

As mentioned earlier, any property which holds for progressive Type-II right censored order statistics will also hold for usual order statistics. Incidentally, we may go one step further to say that any property which holds for general progressive Type-II censored order statistics, where both right and left censoring are implemented so that observation begins at the $(r+1)^{\text{th}}$ ordered failure, will also hold for progressive Type-II right censored order statistics, which is the special case when $r = 0$. Thus, many of the topics which we will discuss will be for the general progressive Type-II censoring case, which we will now describe in detail.

Suppose n randomly selected units with independent failure time distribution $F(x)$ were placed on a life test; the failure times of the first r units to fail were not observed; at the time of the $(r+1)^{\text{th}}$ failure, R_{r+1} number of surviving units are withdrawn from the test randomly, and so on; at the time of the $(r+i)^{\text{th}}$ failure $(1 \leq i \leq m - r)$, R_{r+i} surviving units are randomly withdrawn from the test. We will then denote the life-times of the $m - r$ completely observed units to fail by ${}_rX_{r+1:m:n} \leq {}_rX_{r+2:m:n} \leq \dots \leq {}_rX_{m:m:n}$. We will call these completely observed life times the general progressive Type-II censored order statistics from $F(x)$ for a sample of size n with censoring scheme $(R_{r+1}, \dots, R_m)$. [When it is unclear as to what the censoring scheme is, the superscript $(R_{r+1}, R_{r+2}, \dots, R_m)$ will be added to the notation of each general progressive Type-II censored order statistic.] It is clear that $n = m + R_{r+1} + R_{r+2} + \dots + R_m$, and that ${}_rX_{r+1:m:n} = X_{r+1:n}$, the $(r+1)^{\text{th}}$ usual order statistic for a sample of size n from $F(x)$, since there has been no random removal of items before this point. If the failure times are from a continuous population with cumulative distribution function $F(x)$ and probability density function $f(x)$, the joint probability density function of ${}_rX_{r+1:m:n} \leq {}_rX_{r+2:m:n} \leq \dots \leq {}_rX_{m:m:n}$ is given by

$$f_{{}_rX_{r+1:m:n},{}_rX_{r+2:m:n},\dots,{}_rX_{m:m:n}}(x_{r+1},\dots,x_m) = c'[F(x_{r+1})]^r \prod_{i=r+1}^{m} f(x_i)[1-F(x_i)]^{R_i} ,$$

$$\begin{aligned} \text{where} \quad c' &= \binom{n}{r}(n-r)(n-r-R_{r+1}-1)(n-r-R_{r+1}-R_{r+2}-2)\dots \\ &\qquad \times(n-r-R_{r+1}-R_{r+2}-\dots-R_{m-1}-(m-r)+1) \\ &= \binom{n}{r}(n-r)\prod_{j=r+2}^{m}\left[n-\sum_{i=r+1}^{j-1}R_i-j+1\right] \end{aligned} \qquad (1.4.3)$$

It may be mentioned here that general progressive Type-II censoring is not usually carried out in practice. However, it is not uncommon to approximate Type-I censored data with Type-II censoring methods; see Mann (1971), Nelson

(1982) and Viveros and Balakrishnan (1994) for examples. Therefore, results obtained for general progressive Type-II censored order statistics may be used to approximately analyze data obtained from general progressive Type-I censoring.

In Section One of this thesis, properties of progressive Type-II censored order statistics from both particular and general continuous distributions are investigated. The particular distributions which are considered include the uniform, exponential, truncated exponential, Laplace, extreme-value, log-normal and normal distributions. Determination of single and product moments either directly or through recursive algorithms is possible for a number of these distributions, while for others it is possible to utilize a general algorithm which requires knowledge of single and product moments of usual order statistics only. The problem of efficiently generating progressive Type-II censored order statistics from arbitrary continuous distributions is also seen to be tractable. Many of the results obtained aid in tackling inferential issues such as efficient estimation of parameters. All of the mathematical results obtained generalize those for conventional Type-II censoring.

2 Some Mathematical Results for the Uniform Distribution

2.1 Simulation For Progressive Type-II Right Censoring

2.1.1 Introduction

In Monte Carlo studies relating to many statistical problems, one may wish to generate usual order statistics from some continuous population with cumulative distribution function $F(x)$. A direct way of simulating usual order statistics is to generate a pseudo-random sample from the distribution $F(x)$, and then sort the sample to produce the required order statistics. This direct method will be time consuming and inefficient due to the sorting involved, and will also produce some unwanted order statistics (if only a partial set of order statistics is required for the problem, like a conventional censored sample). For this reason, many authors, including Schucany (1972), Lurie and Hartley (1972), Lurie and Mason (1973), Ramberg and Tadikamalla (1978), and Horn and Schlipf (1986) have discussed various efficient algorithms for generating either a complete or a partial set of usual order statistics without requiring any sorting routines. A brief review of all these developments may be found in Arnold, Balakrishnan and Nagaraja (1992, pp. 95-97).

In this section, we consider a progressive Type-II right censored sample from the standard uniform distribution and establish an independence result. This result is then used to develop a simple simulational algorithm to generate a progressive Type-II right censored sample (without involving sorting routines) from the uniform distribution. This algorithm, together with the inverse cdf transformation, will enable one to simulate progressive Type-II right censored order statistics from any continuous distribution in a simple and efficient manner.

2.1.2 Main Result

Malmquist (1950) established the following property of usual order statistics from the uniform (0,1) distribution, which we will denote, for a random sample of size n, by $U_{1:n}, U_{2:n}, \ldots, U_{n:n}$.

Theorem [Malmquist (1950)]: *The random variables*

$$V_1 = \frac{1-U_{n:n}}{1-U_{n-1:n}}, \quad V_2 = \frac{1-U_{n-1:n}}{1-U_{n-2:n}}, \quad \ldots, \quad V_{n-1} = \frac{1-U_{2:n}}{1-U_{1:n}}, \; V_n = 1-U_{1:n}$$

are independent Beta random variables, distributed as follows:

$V_1 \sim Beta(1,1)$, $V_2 \sim Beta(2,1)$, ..., $V_n \sim Beta(n,1)$.

[See also David (1981) and Balakrishnan and Cohen (1991).]

This result was later exploited by Lurie and Hartley (1972) for the computer generation of conventional Type-II one-stage right censored samples from uniform and thus from arbitrary continuous distributions.

A natural question which arises is whether this elegant result generalizes to the case of progressive Type-II right censored order statistics. That is, how are the variables

$$V_1 = \frac{1-U_{m:m:n}}{1-U_{m-1:m:n}} , \quad V_2 = \frac{1-U_{m-1:m:n}}{1-U_{m-2:m:n}} , \quad \ldots, \quad V_{m-1} = \frac{1-U_{2:m:n}}{1-U_{1:m:n}} , \quad V_m = 1-U_{1:m:n} \tag{2.1.1}$$

distributed? Here, $U_{i:m:n}$, $i=1, 2, \ldots, m$, represent the progressive Type-II right censored order statistics from the uniform (0,1) distribution for a sample of size n with censoring scheme $(R_1, R_2, \ldots, R_m)$. Distributional results on these V_i's can then be used to generalize the Lurie-Hartley algorithm to progressive Type-II right censored order statistics, as follows.

Theorem 2.1.1: *Let $U_{i:m:n}$, $i=1, 2, \ldots, m$, denote a progressive Type-II right* *censored sample from the Uniform (0,1) distribution for a sample of size n with censoring scheme ($R_1, R_2, \ldots, R_m$). Let V_i, $i = 1, 2, \ldots, m$, be as defined in Eq. (2.1.1). Further, let*

$$W_i = V_i^{i+R_m+R_{m-1}+\ldots+R_{m-i+1}} , \quad i = 1, 2, \ldots, m . \tag{2.1.2}$$

Then W_i, $i = 1, 2, \ldots, m$, are independent and identically distributed Uniform (0,1) random variables.

Proof: From Eq. (1.4.2), the joint density of $U_{i:m:n}$, $i=1, 2, \ldots, m$, is given by

$$f_{U_{1:m:n},\ldots,U_{m:m:n}}(u_1,\ldots,u_m) = c \prod_{i=1}^{m} (1-u_i)^{R_i} , \quad 0 < u_1 < \ldots < u_m < 1 \tag{2.1.3}$$

where $c = n(n-R_1-1)(n-R_1-R_2-2) \ldots (n-R_1-R_2-\ldots-R_{m-1}-m+1)$.

From Eq. (2.1.1) we have

$$U_{i:m:n} = 1 - \prod_{j=m-i+1}^{m} V_j \quad , \; i = 1, 2, \ldots, m , \tag{2.1.4}$$

and the Jacobian of the transformation is $\prod_{i=2}^{m} V_i^{i-1}$. From Eq. (2.1.3), we then obtain the joint density function of $V_1, V_2, \ldots, V_m$ as

$$f_{V_1,V_2,\ldots,V_m}(v_1,v_2,\ldots,v_m) = c \prod_{i=1}^{m} v_i^{\,i-1+\sum_{j=m-i+1}^{m} R_j} , \quad 0 < v_1,v_2,\ldots,v_m < 1 . \tag{2.1.5}$$

From Eq. (2.1.5) it is clear that V_i has a Beta ($i+\sum_{j=m-i+1}^{m} R_j$, 1) distribution, and that the variables $V_1, V_2, \ldots, V_m$ are statistically independent. Then it immediately follows that the variables W_i, $i = 1, 2, \ldots, m$, defined in Eq. (2.1.2) are iid uniform (0,1) variables. □

2.1.3 Simulational Algorithm

The result established in the last subsection can be utilized to present a simple and efficient simulational algorithm to generate a progressive Type-II right censored sample from any continuous distribution as follows:

1. Generate m independent uniform (0,1) observations $W_1, W_2, \ldots, W_m$.

2. From Eq. (2.1.2) we set $V_i = W_i^{1/(i+R_m+R_{m-1}+\cdots+R_{m-i+1})}$ for $i = 1, 2, \ldots, m$.

3. From Eq. (2.1.4) we set $U_{i:m:n} = 1 - V_m V_{m-1} \ldots V_{m-i+1}$ for $i = 1, 2, \ldots, m$. Then $U_{1:m:n}, U_{2:m:n}, \ldots, U_{m:m:n}$ is the required progressive Type-II right censored sample from the uniform (0,1) distribution.

4. Finally, we set $X_{i:m:n} = F^{-1}(U_{i:m:n})$, for $i = 1, 2, \ldots, m$, where $F^{-1}(\cdot)$ is the inverse cdf of the distribution under consideration. Then $X_{1:m:n}, X_{2:m:n}, \ldots, X_{m:m:n}$ is the required progressive Type-II right censored sample from the distribution $F(\cdot)$.

It is important to point out here that the above simulational algorithm requires exactly m pseudo-random uniform observations, and does not require any sorting. We may also point out that in the special case $R_1 = R_2 = \ldots = R_{m-1} = 0$, the above algorithm reduces to that given by Lurie and Hartley (1972) for the efficient generation of conventional Type-II right censored samples from an arbitrary continuous distribution $F(\cdot)$.

2.2 Generalization: General Progressive Type-II Censoring

2.2.1 Introduction

In this section, we generalize the results of the last section to the case of general progressive Type-II censoring, described in section 1.4. The independence result established in this section will enable one to efficiently generate general progressive Type-II censored samples from the Uniform (0,1) distribution, and thus, in light of the probability integral transformation, from any continuous distribution.

2.2.2 Main Result

Let us denote the general progressive Type-II censored order statistics described in section 1.4 by ${}_rU_{r+1:m:n}, {}_rU_{r+2:m:n}, \ldots, {}_rU_{m:m:n}$ when they are from the uniform (0,1) distribution. Then, using (1.4.3), the joint probability density function of ${}_rU_{r+1:m:n}, {}_rU_{r+2:m:n}, \ldots, {}_rU_{m:m:n}$ is given by

$$f_{{}_rU_{r+1:m:n}, {}_rU_{r+2:m:n}, \ldots, {}_rU_{m:m:n}}(u_{r+1}, u_{r+2}, \ldots, u_m) = c'(u_{r+1})^r \prod_{i=r+1}^{m} (1-u_i)^{R_i}, \qquad 0 \le u_{r+1} \le u_{r+2} \le \ldots \le u_m \tag{2.2.1}$$

where the constant c' is given in (1.4.3).

Now, consider the following one-to-one transformation, which is a generalization of the transformation given in subsection 2.1.2:

$$V_{r+i} = \frac{1-{}_rU_{m-i+1:m:n}}{1-{}_rU_{m-i:m:n}}, \quad i = 1, \ldots, m-r-1, \quad V_m = 1-{}_rU_{r+1:m:n}. \tag{2.2.2}$$

Then

$${}_rU_{r+i:m:n} = 1-\prod_{j=m-i+1}^{m} V_j, \quad i = 1, 2, \ldots, m-r, \tag{2.2.3}$$

and the Jacobian of this transformation is

$$\prod_{i=2}^{m-r} V_{r+i}^{i-1}.$$

Thus, the joint distribution of V_{r+i}, $i = 1, 2, \ldots, m-r$, is given by

$$f_{V_{r+1},V_{r+2},\ldots,V_m}(v_{r+1},v_{r+2},\ldots,v_m) = c'(1-v_m)^r \prod_{i=1}^{m-r} v_{r+i}^{i-1+R_{m-i+1}+R_{m-i+2}+\cdots+R_m} ,$$

$$0 < v_{r+1},v_{r+2},\ldots,v_m < 1 ,$$

and we find that V_{r+1}, V_{r+2}, ..., V_m are independent beta random variables, distributed as follows:

$$V_{r+i} \sim \text{Beta}\,(i+\sum_{j=m-i+1}^{m} R_j ,1) ,\; i = 1,\ldots,m-r-1 \;,\; V_m \sim \text{Beta}\,(m-r+\sum_{j=r+1}^{m} R_j ,r+1) , \tag{2.2.4}$$

which is a generalization of the result obtained in subsection 2.1.2 for the special case $r = 0$. Now, the random variables

$$W_{r+i} = V_{r+i}^{a_{r+i}} \;,\; i = 1,\, 2,\, \ldots,\, m-r-1 ,$$
$$\text{where } a_{r+i} = i+\sum_{j=m-i+1}^{m} R_j \;,\; i = 1,\, 2,\, \ldots,\, m-r-1 , \tag{2.2.5}$$

are i. i. d. uniform (0,1) random variables, (independent of V_m, of course). Notice that the first parameter in the distribution of V_m is simply n - r, as we would expect, since $V_m = 1 - {}_rU_{r+1:m:n} = 1 - U_{r+1:n}$, and $U_{r+1:n}$ has Beta $(r+1, n - r)$ distribution, as can be seen directly from Eq. (1.4.1). We now state this result formally.

Theorem 2.2.1: *Let ${}_rU_{r+i:m:n}$, $i=1, 2, \ldots, m-r$, denote a general progressive Type-II censored sample from the Uniform (0,1) distribution. Let V_{r+i}, $i = 1, 2, \ldots, m-r$, be the random variables as defined in Eq. (2.2.2). Further, let*

$$W_{r+i} = V_{r+i}^{a_{r+i}}, \quad i = 1, 2, \ldots, m - r - 1,$$

where a_{r+i}, $i = 1, 2, \ldots, m\text{-}r\text{-}1$, be as defined in (2.2.5). Then W_{r+i}, $i = 1, 2, \ldots, m\text{-}r\text{-}1$, are independent and identically distributed Uniform (0,1) random variables, also independent of V_m, which has Beta $(n - r, r+1)$ distribution.

2.2.3 Simulational Algorithm

We can now generate a general progressive Type-II censored sample from any arbitrary continuous distribution $F(x)$ as follows:

1. Generate a Beta $(n - r, r+1)$ random variable V_m. This may be done using any efficient Beta-generating algorithm; see, for example, Johnson, Kotz and Balakrishnan (1995).

2. Independently generate m-r-1 independent Uniform (0,1) observations $W_{r+1}, \ldots, W_{m-1}$.

3. Set $V_{r+i} = W_{r+i}^{1/a_{r+i}}$, $i = 1,\ldots,m-r-1$.

4. Set ${}_rU_{r+i:m:n} = 1\text{-}V_{m-i+1}V_{m-i+2}\ldots V_m$, $i = 1, 2, \ldots, m - r$. Then ${}_rU_{r+i:m:n}$, $i = 1, 2, \ldots, m - r$, is the required general progressive Type-II censored sample

from the Uniform (0,1) distribution.

5. Finally, set ${}_rX_{r+i:m:n} = F^{-1}({}_rU_{r+i:m:n})$ for $i = 1, 2, \ldots, m - r$, where $F^{-1}(x)$ is the inverse cdf of the distribution from which it is desired to generate the sample. Then ${}_rX_{r+1:m:n}$, ${}_rX_{r+2:m:n}$, $\ldots$, ${}_rX_{m:m:n}$ is the required general progressive Type-II censored sample from the distribution $F(\cdot)$.

For the special case $r = 0$, this algorithm will reduce to that given in subsection 2.1.3, since in that case, an efficient Beta-generating algorithm for V_m, where V_m has Beta $(n,1)$ distribution, is to take the n^{th} power of a Uniform (0,1) random variable.

2.2.4 Conditioning Method of Simulation

Perhaps a more intuitive approach to the problem of efficiently simulating general progressive Type-II censored order statistics from the uniform distribution is to consider the following conditional distribution of ${}_rU_{r+2:m:n}$, $\ldots$, ${}_rU_{m:m:n}$ given ${}_rU_{r+1:m:n}$:

$$f_{{}_rU_{r+2:m:n},\ldots,{}_rU_{m:m:n} \mid {}_rU_{r+1:m:n}}(u_{r+2},\ldots,u_m \mid u_{r+1}) = \frac{\binom{n}{r}(n-r)\prod_{j=r+2}^{m}\left[n-\sum_{i=r+1}^{j-1} R_i - j+1\right] u_{r+1}^{r} \prod_{i=r+1}^{m}[1-u_i]^{R_i}}{\frac{n!}{r!(n-r-1)!} u_{r+1}^{r}(1-u_{r+1})^{n-r-1}}$$

$$= \left[\prod_{j=r+2}^{m}\left(n-\sum_{i=r+1}^{j-1} R_i - j+1\right)\right] \left[\prod_{k=r+2}^{m}\frac{1}{1-u_{r+1}}\left(\frac{1-u_k}{1-u_{r+1}}\right)^{R_k}\right].$$

Thus, given ${}_rU_{r+1:m:n} = u_{r+1}$, ${}_rU_{r+2:m:n}$, …, ${}_rU_{m:m:n}$ are jointly distributed as a progressive Type-II right censored sample of size m-r-1 from n-R_1-r-1 uniform (u_{r+1}, 1) variates with progressive censoring scheme (R_{r+2}, …, R_m).

We can therefore generate a general progressive Type-II censored sample from an *arbitrary* continuous distribution with cumulative distribution function $F(x)$ and progressive censoring scheme (R_{r+1}, R_{r+2}, …, R_m) using the following algorithm:

1) Generate ${}_rU_{r+1:m:n}$ as a Beta ($r+1$, n - r) random variate.

2) Generate a progressive Type-II right censored sample from the uniform (0,1) distribution of size m-r-1, which we will denote $U_{(r+2)}$, …, $U_{(m)}$, with censoring scheme (R_{r+2}, …, R_m) using the algorithm given in subsection 2.1.3.

3) Set ${}_rU_{i:m:n} = U_{(i)}(1 - {}_rU_{r+1:m:n}) + {}_rU_{r+1:m:n}$, $i = r+2, \ldots, m$, where $U_{(i)}$ ($i = r+2, \ldots, m$) is as defined in step 2 of this algorithm.

4) Set ${}_rX_{i:m:n} = F^{-1}({}_rU_{i:m:n})$, $i = r+1, \ldots, m$. This is the desired general progressive Type-II censored sample.

Remark: The simulational algorithm developed in this subsection generalizes the one given by Horn and Schlipf (1986) for generating conventional doubly censored samples.

Remark: The distributional result obtained in this subsection may have been obtained directly by applying a well known result of Scheffé and Tukey (1945), which states that usual order statistics from continuous distributions form Markov chains. Perhaps a more interesting question to consider is whether progressive Type-II censored order statistics from continuous distributions form Markov chains. This question will be addressed in Chapter 4 of this thesis.

Remark: The simulational algorithm given in this subsection, although arrived at in a completely different manner, is mathematically equivalent to that given in subsection 2.2.3. This may be seen simply by noting that if X has a Beta distribution with parameters a and b, then $1 - X$ has a Beta distribution with parameters b and a.

2.3 Moments of General Progressive Type-II Censored Order Statistics From the Uniform (0,1) Distribution

Making use of the independence results obtained in section 2.2, we will be able to obtain exact expressions for the moments of general progressive Type-II censored order statistics from the Uniform (0,1) distribution.

Recall from Eq. (2.2.3) that

$$ {}_rU_{r+i:m:n} = 1 - \prod_{j=m-i+1}^{m} V_j \ , \quad i = 1, 2, \ldots, m-r \, , $$

where V_j, $j = r+1, \ldots, m$, are independent random variables with Beta distributions. Using this result, we can obtain the single and product moments of ${}_rU_{r+1:m:n}, \ldots, {}_rU_{m:m:n}$. In doing so, we adopt the following notation:

1) $a_{i+r} = i + \sum_{j=m-i+1}^{m} R_j \ , \quad i = 1, \ldots, m-r$

2) $\alpha_j = \dfrac{a_j}{1+a_j} \ , \quad j = r+1, \ldots, m-1 \ , \quad \alpha_m = \dfrac{a_m}{1+r+a_m}$

3) $\beta_j = \dfrac{1}{(a_j+2)(a_j+1)} \ , \quad j = r+1, \ldots, m-1 \ , \quad \beta_m = \dfrac{r+1}{(a_m+r+2)(a_m+r+1)}$

4) $\gamma_j = \alpha_j + \beta_j \ , \quad j = r+1, \ldots, m.$ (2.3.1)

Notice that $E(V_j) = \alpha_j$ and $Var(V_j) = \alpha_j\beta_j$, $j = r+1, \ldots, m$. Now, beginning with

$$E({}_rU_{r+i:m:n}) = 1 - \prod_{j=m-i+1}^{m} E(V_j) \ , \quad i = 1,\ 2,\ \ldots,\ m-r$$

and

$$Cov({}_rU_{r+i:m:n}, {}_rU_{r+k:m:n}) = Cov\Big(\prod_{j=m-i+1}^{m} V_j, \prod_{j=m-k+1}^{m} V_j\Big) \ ,$$

$$i = 1, \ldots, m-r \ , \ k = 1, \ldots, m-r \ ,$$

we obtain, after some algebra, the following expressions:

$$E({}_rU_{r+i:m:n}) = 1 - \prod_{j=m-i+1}^{m} \alpha_j \ , \quad i = 1,\ \ldots,\ m-r \tag{2.3.2}$$

$$Var({}_rU_{r+i:m:n}) = \left[\prod_{j=m-i+1}^{m} \alpha_j\right] \left[\prod_{j=m-i+1}^{m} \gamma_j - \prod_{j=m-i+1}^{m} \alpha_j\right] \ , \quad i = 1,\ldots,m-r \tag{2.3.3}$$

and

$$Cov({}_rU_{r+i:m:n}, {}_rU_{r+k:m:n}) = \left[\prod_{j=m-i+1}^{m} \alpha_j\right] \left[\prod_{j=m-k+1}^{m} \gamma_j - \prod_{j=m-k+1}^{m} \alpha_j\right] \ , \tag{2.3.4}$$

$$k < i \ , \quad i = 2,\ldots,m-r.$$

These expressions may be used, for example, in developing inference for uniform distributions under moment and best linear unbiased estimation. We will discuss various types of inference in relation to progressive Type-II censoring in Chapter 5 of this thesis.

3 The Exponential Distribution: New Mathematical Results

3.1 Independent Spacings

3.1.1 Introduction

To facilitate the study of usual order statistics from the exponential distribution, one may use the following result due to Sukhatme (1937).

Theorem [Sukhatme (1937)]: *Let $X_{1:n}$, $X_{2:n}$, …, $X_{n:n}$ denote the usual order statistics from the standard exponential distribution. Then the "spacings"*

$$Z_1 = nX_{1:n}\ ,\quad Z_2 = (n-1)(X_{2:n}-X_{1:n})\ ,\ \ldots,\quad Z_n = X_{n:n}-X_{n-1:n}$$

are independent standard exponential random variables.

This result has been used to derive moments of usual order statistics from the exponential distribution, and in developing efficient algorithms for generating conventional Type-II censored samples from the exponential distribution. [See, for example, Arnold, Balakrishnan and Nagaraja (1992).]

Thomas and Wilson (1972) derived a similar set of independent standard exponential spacings for progressive Type-II right censored order statistics, viz.,

$$Z_1 = nX_{1:m:n}\,,\ Z_2 = (n-R_1-1)(X_{2:m:n}-X_{1:m:n})\,,\ Z_3 = (n-R_1-R_2-2)(X_{3:m:n}-X_{2:m:n})\,,\ldots,$$
$$Z_m = (n-R_1-\ldots-R_{m-1}-m+1)(X_{m:m:n}-X_{m-1:m:n})\ ,$$

where $X_{1:m:n}, \ldots, X_{m:m:n}$ denote a progressive Type-II right censored sample from the standard exponential distribution, with censoring scheme $(R_1, \ldots, R_m)$. This result has facilitated the derivation of best linear unbiased and maximum likelihood estimates of parameters from exponential distributions. [See, for example, Viveros and Balakrishnan (1994).]

In this section, we further generalize the results of Sukhatme (1937) and of Thomas and Wilson (1972) by considering a general progressive Type-II censored sample of size m - r from the standard exponential distribution, with censoring scheme $(R_{r+1}, R_{r+2}, \ldots, R_m)$, which we will denote ${}_rX_{r+1:m:n}, {}_rX_{r+2:m:n}, \ldots, {}_rX_{m:m:n}$.

3.1.2 Generalized Spacings

For the standard exponential distribution let us define the following generalized spacings:

$$Z_{r+1} = (n-r)\,{}_rX_{r+1:m:n},\ Z_{r+2} = (n-r-R_{r+1}-1)({}_rX_{r+2:m:n} - {}_rX_{r+1:m:n}),$$

$$Z_{r+3} = (n-r-R_{r+1}-R_{r+2}-2)({}_rX_{r+3:m:n} - {}_rX_{r+2:m:n}),\ \ldots,$$

$$Z_m = (n-r-R_{r+1}-R_{r+2}-\ldots-R_{m-1}-(m-r-1))({}_rX_{m:m:n} - {}_rX_{m-1:m:n}).$$

The inverse transformation is then

$$\frac{Z_{r+1}}{n-r} = {}_rX_{r+1:m:n},\ \frac{Z_{r+2}}{n-r-R_{r+1}-1} + \frac{Z_{r+1}}{n-r} = {}_rX_{r+2:m:n},\ \frac{Z_{r+3}}{n-r-R_{r+1}-R_{r+2}-2} + \frac{Z_{r+2}}{n-r-R_{r+1}-1}$$
$$+ \frac{Z_{r+1}}{n-r} = {}_rX_{r+3:m:n},\ \ldots,\ \frac{Z_m}{n-r-R_{r+1}-\ldots-R_{m-1}-(m-r-1)} + \ldots + \frac{Z_{r+1}}{n-r} = {}_rX_{m:m:n}. \tag{3.1.1}$$

The Jacobian of this transformation is therefore

$$\frac{1}{(n-r)(n-r-R_{r+1}-1)(n-r-R_{r+1}-R_{r+2}-2)\ldots(n-r-R_{r+1}-\ldots-R_{m-1}-(m-r-1))}.$$

From (1.4.3), the joint distribution of the generalized spacings is then obtained as

$$f_{Z_{r+1},Z_{r+2},\ldots,Z_m}(z_{r+1},z_{r+2},\ldots,z_m) = \binom{n}{r}[1-e^{-z_{r+1}/(n-r)}]^r e^{-z_{r+1}} e^{-\sum_{i=r+2}^{m} z_i},$$
$$0 \le z_{r+1}, z_{r+2}, \ldots, z_m < \infty.$$

Thus, we have the following theorem.

Theorem 3.1.1: *The generalized spacings $Z_{r+2}, Z_{r+3}, \ldots, Z_m$ are independent standard exponential random variables, independent of $Z_{r+1}/(n-r) = {}_rX_{r+1:m:n}$.*

Remark: As we have mentioned previously, $Z_{r+1}/(n-r) = {}_rX_{r+1:m:n}$ is distributed as the $(r+1)^{th}$ usual order statistic, $X_{r+1:n}$, for a sample of size n from standard exponential distribution.

Remark: From (3.1.1), it is evident that the general progressive Type-II censored order statistics from an exponential distribution form an *additive Markov chain*, as noted by Renyi (1953) for usual order statistics.

The generalized spacings introduced in this section will facilitate us in developing efficient algorithms for simulating general progressive Type-II censored samples from exponential distributions, and will also help in obtaining moments of general progressive Type-II censored order statistics from exponential distributions.

3.2 Simulational Algorithm

We are now able to efficiently generate general progressive Type-II censored samples from the exponential distribution using an alternative method to those presented in section 2.2. Since ${}_rX_{r+1:m:n}$ is simply $X_{r+1:n}$, we may use the result of Sukhatme (1937) and write ${}_rX_{r+1:m:n} = \sum_{i=1}^{r+1} \frac{W_i}{n-i+1}$, where W_i, $i = 1, 2, \ldots, r+1$ are independent standard exponential random variables. We therefore have the following simulational algorithm:

1. Simulate $r + 1$ independent exponential random variables, $W_1, W_2, \ldots, W_{r+1}$. (This may be done simply by using the inverse transformation $W_i = -\ln(1-U_i)$ where U_i are independent Uniform (0,1) random variables, or by using some other efficient algorithm.)

2. Set ${}_rX_{r+1:m:n} = \sum_{i=1}^{r+1} \frac{W_i}{n-i+1}$.

3. Simulate $m - r - 1$ independent exponential random variables, $Z_{r+2}, Z_{r+3}, \ldots, Z_m$.

4. Set

$$X_{r+i:m:n} = {}_rX_{r+1:m:n} + \frac{Z_{r+2}}{n-r-R_{r+1}-1} + \frac{Z_{r+3}}{n-r-R_{r+1}-R_{r+2}-2} + \ldots + \frac{Z_{r+i}}{n-r-R_{r+1}-\ldots-R_{r+i-1}-(i-1)}, \quad i = 2,3,\ldots,m-r .$$

Remark: In comparing this method of simulation to those given in Chapter 2, it is seen that we do not need to simulate a Beta random variable, and exponents of uniform random variables need not be taken. However, we do need to simulate m uniform (0,1) random variables and perform a number of divisions.

3.3 Obtaining Moments Using Independent Spacings

The results of section 3.1 may also be used to obtain moments of general progressive Type-II censored order statistics from the standard exponential distribution. We begin with the fact that ${}_{r}X_{r+1:m:n} = \sum_{i=1}^{r+1} \frac{W_i}{n-i+1}$, where W_i ($i = 1, 2, \ldots, r+1$) are independent standard exponential random variables, so that, in accordance with Sukhatme (1937),

$$E({}_{r}X_{r+1:m:n}) = E(X_{r+1:n}) = \sum_{i=1}^{r+1} \frac{1}{n-i+1} = \alpha_{r+1:n}$$

and

$$Var({}_{r}X_{r+1:m:n}) = Var(X_{r+1:n}) = \sum_{i=1}^{r+1} \left[\frac{1}{n-i+1}\right]^2 = \beta_{r+1:n} .$$

Now, upon using (3.1.1), we may obtain the means, variances and covariances of the generalized progressive Type-II censored order statistics from the standard exponential distribution as follows:

$$E(X_{r+1:m:n}) = \sum_{i=1}^{r+1} \frac{1}{n-i+1} = \alpha_{r+1:n} ,$$

$$E(X_{j:m:n}) = \alpha_{r+1:n} + \sum_{i=r+2}^{j} \frac{1}{n - \sum_{k=r+1}^{i-1} R_k - i + 1} , \quad j = r+2, \ldots, m ,$$

$$Var(X_{r+1:m:n}) = \sum_{i=1}^{r+1} \left[\frac{1}{n-i+1} \right]^2 = \beta_{r+1:n} ,$$

$$Var(X_{j:m:n}) = \beta_{r+1:n} + \sum_{i=r+2}^{j} \frac{1}{\left[n - \sum_{k=r+1}^{i-1} R_k - i + 1 \right]^2} , \quad j = r+2, \ldots, m ,$$

and, in light of the independence of spacings [see also Renyi (1953)],

$$Cov(X_{i:m:n}, X_{j:m:n}) = Var(X_{i:m:n}) , \quad r+1 \le i < j \le m .$$

The single and product moments obtained in this section will enable us to develop inference for one- and two-parameter exponential distributions when progressive Type-II censored samples are obtained. This will be discussed in Chapter 5.

3.4 Recursive Relationships for Moments

3.4.1 Introduction

The idea of obtaining moments of usual order statistics recursively has been explored for a number of distributions; for example, see Balakrishnan, Malik and Ahmed (1988). In this section, we establish several recurrence relations satisfied by the single and product moments of progressive Type-II right censored order statistics from an exponential distribution. These relations may then be used, for example, to compute all the means, variances and covariances of exponential progressive Type-II right censored order statistics for all sample sizes n and all censoring schemes $(R_1, R_2, \ldots, R_m)$, $m \leq n$. The results presented generalize the results given by Joshi (1978, 1982) for the single moments and product moments of usual order statistics from the exponential distribution.

To further generalize these results, we also consider the right truncated exponential distribution. Recurrence relations for the single and product moments are established for progressive Type-II right censored order statistics from the right truncated exponential distribution which are again complete, in the sense that they may be used in a simple recursive manner in order to compute the single and product moments of all progressive Type-II right censored order statistics from truncated exponential distributions for all sample sizes and all censoring schemes. Recurrence relations established in this case also generalize the results given by Joshi (1978, 1982) for the single and product moments of usual order statistics from the right truncated exponential distribution.

3.4.2 Recurrence Relations for Single Moments - Exponential Distribution

Let $X_1, X_2, \ldots, X_n$ denote a random sample from the standard exponential distribution, that is, with probability density function $f(x) = e^{-x}$ and cumulative distribution function $F(x) = 1-e^{-x}$. Then the corresponding progressive Type-II right censored order statistics with censoring scheme $(R_1, R_2, \ldots, R_m)$ will be denoted by $X_{1:m:n}^{(R_1,R_2,\ldots,R_m)}$, $X_{2:m:n}^{(R_1,R_2,\ldots,R_m)}$, ... , $X_{m:m:n}^{(R_1,R_2,\ldots,R_m)}$. The characterizing differential equation for the exponential distribution is $f(x) = 1 - F(x)$. This relationship will be exploited in this section to derive recurrence relations for the single moments of exponential progressive Type-II right censored order statistics, which can be written from (1.4.2) as

$$\mu_{i:m:n}^{(R_1,R_2,\ldots,R_m)^{(k)}} = E[X_{i:m:n}^{(R_1,R_2,\ldots,R_m)}]^k = A(n,m-1) \int\int_{0<x_1<\ldots<x_m<\infty}\cdots\int x_i^k f(x_1)[1-F(x_1)]^{R_1}$$

$$\cdot f(x_2)[1-F(x_2)]^{R_2}\ldots f(x_m)[1-F(x_m)]^{R_m} dx_1\ldots dx_m, \tag{3.4.1}$$

where we have denoted

$$A(p,q) = p(p-R_1-1)(p-R_1-R_2-2)\ \ldots\ (p-R_1-R_2-\ldots-R_q-q) \quad \text{for } q = 0,\ 1,\ldots,\ p-1\ . \tag{3.4.2}$$

When $k = 1$, the superscript in the notation of the mean of the progressive Type-II right censored order statistic may be omitted without any confusion.

Theorem 3.4.1: *For $2 \leq m \leq n$ and $k \geq 0$,*

$$\mu_{1:m:n}^{(R_1,\ldots,R_m)^{(k+1)}} = \frac{1}{R_1+1}\left[(k+1)\mu_{1:m:n}^{(R_1,\ldots,R_m)^{(k)}} - (n-R_1-1)\mu_{1:m-1:n}^{(R_1+R_2+1,R_3,\ldots,R_m)^{(k+1)}}\right], \tag{3.4.3}$$

and for $m = 1$, $n = 1, 2, \ldots$ and $k \geq 0$,

$$\mu_{1:1:n}^{(n-1)^{(k+1)}} = \frac{k+1}{n}\mu_{1:1:n}^{(n-1)^{(k)}} . \tag{3.4.4}$$

Proof: Let us consider, for $i = 1$ in (3.4.1), upon employing the characterizing differential equation $f(x_1) = 1 - F(x_1)$,

$$\mu_{1:m:n}^{(R_1,R_2,\ldots,R_m)^{(k)}} = A(n,m-1) \int\int_{0 < x_2 < x_3 < \ldots < x_m < \infty}\cdots\int \left\{ \int_0^{x_2} x_1^k \, [1-F(x_1)]^{R_1+1} dx_1 \right\}$$

$$\cdot f(x_2)[1-F(x_2)]^{R_2} \ldots f(x_m)[1-F(x_m)]^{R_m} dx_2 \ldots dx_m .$$

(3.4.5)

Now, integrating the innermost integral by parts, we have

$$\int_0^{x_2} x_1^k\,[1-F(x_1)]^{R_1+1}dx_1 = \frac{1}{k+1}\Bigg\{ x_2^{k+1}\,[1-F(x_2)]^{R_1+1}$$

$$+ (R_1+1)\int_0^{x_2} x_1^{k+1}[1-F(x_1)]^{R_1}f(x_1)dx_1 \Bigg\} .$$

Substituting the resulting expression into Eq. (3.4.5), and simplifying the resulting expression, we arrive at the following equation:

$$\mu_{1:m:n}^{(R_1,\dots,R_m)^{(k)}} = \frac{n-R_1-1}{k+1}\mu_{1:m-1:n}^{(R_1+R_2+1,R_3,\dots,R_m)^{(k+1)}} + \frac{R_1+1}{k+1}\mu_{1:m:n}^{(R_1,\dots,R_m)^{(k+1)}} ,$$

which, upon rearrangement, gives the relation in (3.4.3). Eq. (3.4.4) is obtained similarly, noting that the integral in (3.4.5) will be a single integral, ranging over the entire positive real line, and R_1 must be simply n - 1, since the equation $R_1 + R_2 + \dots + R_m + m = n$ must be satisfied. □

Remark: As mentioned in earlier chapters, it is true that the first progressive Type-II right censored order statistic is the same as the first usual order statistic from a sample of size n, regardless of the censoring scheme employed. However, these recurrence relations have been included in order to establish completeness even without this knowledge.

Theorem 3.4.2: *For $2 \leq i \leq m - 1$, $m \leq n$ and $k \geq 0$,*

$$\mu_{i:m:n}^{(R_1,\ldots,R_m)^{(k+1)}} = \frac{1}{R_i+1}\Big[(k+1)\mu_{i:m:n}^{(R_1,\ldots,R_m)^{(k)}} - (n-R_1-R_2-\ldots-R_i-i)\mu_{i:m-1:n}^{(R_1,\ldots,R_{i-1},R_i+R_{i+1}+1,R_{i+2},\ldots,R_m)^{(k+1)}}$$

$$+(n-R_1-R_2-\ldots-R_{i-1}-i+1)\mu_{i-1:m-1:n}^{(R_1,\ldots,R_{i-2},R_{i-1}+R_i+1,R_{i+1},\ldots,R_m)^{(k+1)}}\Big]. \tag{3.4.6}$$

Proof: Let us consider, substituting $f(x_i) = 1 - F(x_i)$ into (3.4.1),

$$\mu_{i:m:n}^{(R_1,R_2,\ldots,R_m)^{(k)}} = A(n,m-1) \int_{0<x_1<\ldots<x_{i-1}<x_{i+1}<\ldots<x_m<\infty} \cdots \int \cdots \int \left\{ \int_{x_{i-1}}^{x_{i+1}} x_i^k [1-F(x_i)]^{R_i+1} dx_i \right\}$$

$$\cdot f(x_1)[1-F(x_1)]^{R_1} \ldots f(x_{i-1})[1-F(x_{i-1})]^{R_{i-1}} f(x_{i+1})[1-F(x_{i+1})]^{R_{i+1}} \ldots$$

$$\cdot f(x_m)[1-F(x_m)]^{R_m} dx_1 \ldots dx_{i-1} dx_{i+1} \ldots dx_m. \tag{3.4.7}$$

Integrating the innermost integral by parts, we obtain upon simplification,

$$\int_{x_{i-1}}^{x_{i+1}} x_i^k \, [1-F(x_i)]^{R_i+1} dx_i = \frac{1}{k+1} \Bigg\{ x_{i+1}^{k+1}[1-F(x_{i+1})]^{R_i+1} - x_{i-1}^{k+1}[1-F(x_{i-1})]^{R_i+1}$$

$$+ (R_i+1) \int_{x_{i-1}}^{x_{i+1}} x_i^{k+1} [1-F(x_i)]^{R_i} f(x_i) dx_i \Bigg\}.$$

Substituting this into Eq. (3.4.7), we obtain

$$\mu_{i:m:n}^{(R_1,\ldots,R_m)^{(k)}} = \frac{n-R_1-\ldots-R_i-i}{k+1}\mu_{i:m-1:n}^{(R_1,\ldots,R_{i-1},R_i+R_{i+1}+1,R_{i+2},\ldots,R_m)^{(k+1)}}$$

$$-\frac{n-R_1-\ldots-R_{i-1}-i+1}{k+1}\mu_{i-1:m-1:n}^{(R_1,\ldots,R_{i-2},R_{i-1}+R_i+1,R_{i+1},\ldots,R_m)^{(k+1)}} + \frac{R_i+1}{k+1}\mu_{i:m:n}^{(R_1,\ldots,R_m)^{(k+1)}} .$$

Rewriting the above equation, we obtain the relation in (3.4.6). □

Theorem 3.4.3: For $2 \leq m \leq n$ and $k \geq 0$,

$$\mu_{m:m:n}^{(R_1,\ldots,R_m)^{(k+1)}} = \frac{k+1}{R_m+1}\mu_{m:m:n}^{(R_1,\ldots,R_m)^{(k)}} + \mu_{m-1:m-1:n}^{(R_1,\ldots,R_{m-2},R_{m-1}+R_m+1)^{(k+1)}} . \qquad (3.4.8)$$

Proof: Let us write, from (3.4.1) and the characterizing differential equation $f(x_m) = 1 - F(x_m)$,

$$\mu_{m:m:n}^{(R_1,R_2,\ldots,R_m)^{(k)}} = A(n,m-1) \int\int \cdots \int_{0<x_1<x_2<\ldots<x_{m-1}<\infty} \left\{ \int_{x_{m-1}}^{\infty} x_m^k \, [1-F(x_m)]^{R_m+1} dx_m \right\}$$

$$\cdot f(x_1)[1-F(x_1)]^{R_1} \ldots f(x_{m-1})[1-F(x_{m-1})]^{R_{m-1}} dx_1 \ldots dx_{m-1} .$$

(3.4.9)

Integrating the innermost integral by parts, we arrive at

$$\int_{x_{m-1}}^{\infty} x_m^k \,[1-F(x_m)]^{R_m+1} dx_m = \frac{1}{k+1}\Bigg\{ -x_{m-1}^{k+1}[1-F(x_{m-1})]^{R_m+1}$$

$$+ (R_m+1)\int_{x_{m-1}}^{\infty} x_m^{k+1}[1-F(x_m)]^{R_m} f(x_m) dx_m \Bigg\} .$$

Upon substituting this expression into (3.4.9), and simplifying, we get

$$\mu_{m:m:n}^{(R_1,\ldots,R_m)^{(k)}} = -\frac{n-R_1-\ldots-R_{m-1}-m+1}{k+1}\mu_{m-1:m-1:n}^{(R_1,\ldots,R_{m-2},R_{m-1}+R_m+1)^{(k+1)}} + \frac{R_m+1}{k+1}\mu_{m:m:n}^{(R_1,\ldots,R_m)^{(k+1)}} .$$

After rewriting this equation, we obtain the relation in (3.4.8). □

Remark: Using these recurrence relations, we can obtain all the single moments of all progressive Type-II right censored order statistics for all sample sizes and all censoring schemes (R_1, ..., R_m) in a simple recursive manner. The recursive algorithm will be described in detail in subsection 3.4.4.

3.4.3 Recurrence Relations for Product Moments - Exponential Distribution

Using (1.4.2) we can write the $(i,j)^{\text{th}}$ product moment of the progressive Type-II right censored order statistics as follows:

$$\mu_{i,j:m:n}^{(R_1,R_2,\ldots,R_m)} = E[X_{i:m:n}^{(R_1,R_2,\ldots,R_m)} X_{j:m:n}^{(R_1,R_2,\ldots,R_m)}]$$

$$= A(n,m-1) \int\int_{0<x_1<\ldots<x_m<\infty}\cdots\int x_i x_j f(x_1)[1-F(x_1)]^{R_1} f(x_2)[1-F(x_2)]^{R_2} \ldots$$

$$\cdot f(x_m)[1-F(x_m)]^{R_m} dx_1 \ldots dx_m \ . \quad (3.4.10)$$

In this section, assuming a standard exponential distribution for the failure times, we will again exploit the characterizing differential equation $f(x) = 1 - F(x)$ to obtain recurrence relations for these product moments which will enable us to compute all the product moments of progressive Type-II right censored order statistics for all sample sizes and all possible censoring schemes.

Theorem 3.4.4: *For $1 \leq i < j \leq m - 1$ and $m \leq n$,*

$$\mu_{i,j:m:n}^{(R_1,\ldots,R_m)} = \frac{1}{R_j+1}\Big[\mu_{i:m:n}^{(R_1,\ldots,R_m)} - (n-R_1-R_2-\ldots-R_j-j)\mu_{i,j:m-1:n}^{(R_1,\ldots,R_{j-1},R_j+R_{j+1}+1,R_{j+2},\ldots,R_m)}$$

$$+ (n-R_1-R_2-\ldots-R_{j-1}-j+1)\mu_{i,j-1:m-1:n}^{(R_1,\ldots,R_{j-2},R_{j-1}+R_j+1,R_{j+1},\ldots,R_m)}\Big] \ . \quad (3.4.11)$$

Proof: For $1 \leq i < j \leq m - 1$, from (3.4.1) and the fact that $f(x_j) = 1 - F(x_j)$,

$$\mu_{i:m:n}^{(R_1,R_2,\ldots,R_m)} = A(n,m-1) \int_{0<x_1<\ldots<x_{j-1}<x_{j+1}<\ldots<x_m<\infty} \cdots \int \cdots \int x_i \left\{ \int_{x_{j-1}}^{x_{j+1}} [1-F(x_j)]^{R_j+1} dx_j \right\}$$

$$\cdot f(x_1)[1-F(x_1)]^{R_1} \ldots f(x_{j-1})[1-F(x_{j-1})]^{R_{j-1}} f(x_{j+1})[1-F(x_{j+1})]^{R_{j+1}} \ldots$$

$$\cdot f(x_m)[1-F(x_m)]^{R_m} dx_1 \ldots dx_{j-1} dx_{j+1} \ldots dx_m . \qquad (3.4.12)$$

Integrating the innermost integral by parts, we obtain

$$\int_{x_{j-1}}^{x_{j+1}} [1-F(x_j)]^{R_j+1} dx_j = x_{j+1}[1-F(x_{j+1})]^{R_j+1} - x_{j-1}[1-F(x_{j-1})]^{R_j+1}$$

$$+ (R_j+1) \int_{x_{j-1}}^{x_{j+1}} x_j [1-F(x_j)]^{R_j} f(x_j) dx_j ,$$

which, when substituted into Eq. (3.4.12), results in the following:

$$\mu_{i:m:n}^{(R_1,\ldots,R_m)} = (n-R_1-\ldots-R_j-j)\mu_{i,j:m-1:n}^{(R_1,\ldots,R_{j-1},R_j+R_{j+1}+1,R_{j+2},\ldots,R_m)}$$

$$-(n-R_1-\ldots-R_{j-1}-j+1)\mu_{i,j-1:m-1:n}^{(R_1,\ldots,R_{j-2},R_{j-1}+R_j+1,R_{j+1},\ldots,R_m)} + (R_j+1)\mu_{i,j:m:n}^{(R_1,\ldots,R_m)} .$$

Upon rearrangement of this equation, we obtain the relation in (3.4.11). □

Remark: Notice that Theorem 3.4.4 holds even for $j = i + 1$, without altering the proof, provided we realize that $\mu_{i,i:m:n}^{(R_1,\ldots,R_m)} = \mu_{i:m:n}^{(R_1,\ldots,R_m)(2)}$.

Theorem 3.4.5: *For $1 \le i \le m-1$ and $m \le n$,*

$$\mu_{i,m:m:n}^{(R_1,\ldots,R_m)} = \frac{1}{R_m+1}\left[\mu_{i:m:n}^{(R_1,\ldots,R_m)} + (n-R_1-R_2-\ldots-R_{m-1}-m+1)\mu_{i,m-1:m-1:n}^{(R_1,R_2,\ldots,R_{m-2},R_{m-1}+R_m+1)}\right].$$

(3.4.13)

Proof: From (3.4.1) and the fact that $f(x_m) = 1 - F(x_m)$,

$$\mu_{i:m:n}^{(R_1,R_2,\ldots,R_m)} = A(n,m-1) \int\int\cdots\int_{0<x_1<x_2<\ldots<x_{m-1}<\infty} x_i \left\{ \int_{x_{m-1}}^{\infty} [1-F(x_m)]^{R_m+1} dx_m \right\}$$
$$\times f(x_1)[1-F(x_1)]^{R_1}\ldots f(x_{m-1})[1-F(x_{m-1})]^{R_{m-1}} dx_1\ldots dx_{m-1}.$$

(3.4.14)

Integrating the innermost integral by parts, we obtain

$$\int_{x_{m-1}}^{\infty} [1-F(x_m)]^{R_m+1} dx_m = -x_{m-1}[1-F(x_{m-1})]^{R_m+1} + (R_m+1)\int_{x_{m-1}}^{\infty} x_m[1-F(x_m)]^{R_m} f(x_m)dx_m ,$$

which, upon substituting into Eq. (3.4.14), yields

$$\mu_{i:m:n}^{(R_1,\ldots,R_m)} = -(n-R_1-\ldots-R_{m-1}-m+1)\mu_{i,m-1:m-1:n}^{(R_1,\ldots,R_{m-2},R_{m-1}+R_m+1)} + (R_m+1)\mu_{i,m:m:n}^{(R_1,\ldots,R_m)} .$$

Rewriting this equation, we immediately obtain the relation in (3.4.13). □

Remark: Using these recurrence relations, we can obtain all the product moments of progressive Type-II right censored order statistics for all sample sizes and all censoring schemes $(R_1, \ldots, R_m)$.

Remark: For the special case $R_1 = \ldots = R_m = 0$, so that $m = n$ and all n usual order statistics $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$, whose k^{th} moments are denoted $\mu_{i:n}^{(k)}$ for $1 \leq i \leq n$ and whose product moments are denoted $\mu_{i,j:n}$ for $1 \leq i < j \leq n$, are obtained, the relations in subsections 3.4.2 and 3.4.3 reduce to the following:

From Eq. (3.4.3): For $k \geq 0$,

$$\mu_{1:n}^{(k+1)} = (k+1)\mu_{1:n}^{(k)} - (n-1)\mu_{1:n-1:n}^{(1,0,\ldots,0)^{(k+1)}} .$$

From Eq. (3.4.6): For $2 \leq i \leq n$-1 and $k \geq 0$,

$$\mu_{i:n}^{(k+1)} = (k+1)\mu_{i:n}^{(k)} - (n-i)\mu_{i:n-1:n}^{(0,\ldots,0,1,0,\ldots,0)^{(k+1)}} + (n-i+1)\mu_{i-1:n-1:n}^{(0,\ldots,0,1,0,\ldots,0)^{(k+1)}} ,$$

where, in the superscript of the second term on the right hand side, the 1 is in the i^{th} position, and in the superscript of the third term on the right hand side, the 1 is in the $(i - 1)^{th}$ position.

From Eq. (3.4.8): For $k \geq 0$,

$$\mu_{n:n}^{(k+1)} = (k+1)\mu_{n:n}^{(k)} + \mu_{n-1:n-1:n}^{(0,\ldots,0,1)^{(k+1)}} .$$

Now, realize that if $R_1 = R_2 = \ldots = R_{j-1} = 0$, so that there is no censoring before the time of the j^{th} failure, then the first j progressive Type-II right censored order statistics are simply the first j usual order statistics. Thus, the relations above reduce to:

From Eq. (3.4.3): For $k \geq 0$,

$$\mu_{1:n}^{(k+1)} = (k+1)\mu_{1:n}^{(k)} - (n-1)\mu_{1:n}^{(k+1)} ,$$

which simplifies to

$$n\mu_{1:n}^{(k+1)} = (k+1)\mu_{1:n}^{(k)} .$$

From Eq. (3.4.6): For $2 \leq i \leq n\text{-}1$ and $k \geq 0$,

$$\mu_{i:n}^{(k+1)} = (k+1)\mu_{i:n}^{(k)} - (n-i)\mu_{i:n}^{(k+1)} + (n-i+1)\mu_{i-1:n}^{(k+1)} ,$$

which simplifies to

$$(n-i+1)\mu_{i:n}^{(k+1)} = (k+1)\mu_{i:n}^{(k)} + (n-i+1)\mu_{i-1:n}^{(k+1)} .$$

From Eq. (3.4.8): For $k \geq 0$,

$$\mu_{n:n}^{(k+1)} = (k+1)\mu_{n:n}^{(k)} + \mu_{n-1:n}^{(k+1)} .$$

These recurrence relations are equivalent to those given by Joshi (1978).

Following similar arguments, substituting $R_1 = R_2 = \dots = R_m = 0$ into Eqs. (3.4.11) and (3.4.13) gives, for $1 \leq i < j \leq n$,

$$\mu_{i,j:n} = \frac{1}{n-j+1}\mu_{i:n} + \mu_{i,j-1:n} ,$$

which is the relation given in Joshi (1982).

3.4.4. Recursive Algorithm - Exponential Distribution

Using the recurrence relations established in subsections 3.4.2 and 3.4.3, the means, variances and covariances of all progressive Type-II right censored order statistics from the standard exponential distribution can be readily computed as follows.

Setting $k = 0$, Eq. (3.4.4) will give us the values $\mu_{1:1:n}^{(n-1)} = 1/n$, $n = 1$, 2, ...which in turn, again using (3.4.4) with $k = 1$, will give us the values $\mu_{1:1:n}^{(n-1)^{(2)}} = 2/n^2$, $n = 1, 2, \ldots$. Thus, all first and second moments with m = 1 for all sample sizes n will be obtained. Next, using Eq. (3.4.3), we can determine all moments of the form $\mu_{1:2:n}^{(R_1,R_2)}$, $n = 2, 3, \ldots$, which can in turn be used, with (3.4.3), to determine all moments of the form $\mu_{1:2:n}^{(R_1,R_2)^{(2)}}$, $n = 2, 3, \ldots$. Eq. (3.4.8) can then be used to obtain $\mu_{2:2:n}^{(R_1,R_2)}$ for all R_1, R_2, and $n \geq 2$, and these values can be used to obtain all moments of the form $\mu_{2:2:n}^{(R_1,R_2)^{(2)}}$ using (3.4.8) again. Eq. (3.4.3) can now be used again to obtain $\mu_{1:3:n}^{(R_1,R_2,R_3)}$, $\mu_{1:3:n}^{(R_1,R_2,R_3)^{(2)}}$ for all n, R_1, R_2, and R_3 and (3.4.6) can be used next to obtain all moments of the form $\mu_{2:3:n}^{(R_1,R_2,R_3)}$, $\mu_{2:3:n}^{(R_1,R_2,R_3)^{(2)}}$. Finally, (3.4.8) can be used to obtain all moments of the form $\mu_{3:3:n}^{(R_1,R_2,R_3)}$, $\mu_{3:3:n}^{(R_1,R_2,R_3)(2)}$. This process can be continued until all desired first and second order moments (and therefore all variances) are obtained.

From Eq. (3.4.13), all moments of the form $\mu_{m-1,m:m:n}^{(R_1,\ldots,R_m)}$, $m = 2, 3, \ldots, n$ can be determined, since only single moments, which have already been obtained, are needed to calculate them. Then, using (3.4.11), all moments of the form $\mu_{i-1,i:m:n}^{(R_1,\ldots,R_m)}$, $2 \le i < m$ can be obtained. From this point, using (3.4.13), we can obtain all moments of the form $\mu_{m-2,m:m:n}^{(R_1,\ldots,R_m)}$, $m = 3, 4, \ldots, n$ and, subsequently, using (3.4.11), all moments of the form $\mu_{i-2,i:m:n}^{(R_1,\ldots,R_m)}$, $3 \le i < m$. Continuing this way, all the desired product moments (and therefore all covariances) can be obtained.

Remark: As we can see by setting $r = 0$ in section 3.3, means, variances and covariances of progressive Type-II right censored order statistics from the standard exponential distribution can be obtained explicitly. [See also Thomas and Wilson (1972) and Viveros and Balakrishnan (1994)]. Thus, this method of recursion is an alternative method, and can be used for moments of *any* order. However, for many distributions, such as the right truncated exponential distributions whose results we will present now, explicit expressions for single and product moments of progressive Type-II right censored order statistics are not easily obtained, and this recursive method of computation will be very useful in such cases. This is the primary reason why such recursive methods have been developed for a variety of distributions for the usual order statistics; for example, see Arnold and Balakrishnan (1989).

3.4.5. Recurrence Relations for Single Moments - Right Truncated Exponential Distribution

In this and the following subsection, we will present recurrence relations for the single and product moments of progressive Type-II right censored order statistics from a right truncated exponential distribution which generalize the results presented in subsections 3.4.2 and 3.4.3. The results developed will, as mentioned earlier, generalize the results given by Joshi (1978, 1982) for usual order statistics. Proofs of the relations are similar to those presented earlier for the exponential distribution and are therefore omitted.

The probability density function for the right truncated exponential distribution is

$$f(x) = \frac{e^{-x}}{P}, \quad 0 < x < P_1, \quad \text{where } P_1 = -\ln(1-P).$$

Here, 1-P is the proportion of right truncation of the standard exponential distribution. Thus, the characterizing differential equation for this distribution is

$$f(x) = \frac{1}{P} - F(x) = \left[\frac{1}{P} - 1\right] + 1 - F(x) .$$

This equation will be exploited in order to derive the complete recurrence relations for the single and product moments of progressive Type-II right censored order statistics from a right truncated exponential distribution.

Theorem 3.4.6: *For $k \geq 0$,*

$$\mu_{1:1:1}^{(0)^{(k+1)}} = (k+1)\mu_{1:1:1}^{(0)^{(k)}} - \left[\frac{1}{P} - 1\right] P_1^{k+1} . \tag{3.4.15}$$

Theorem 3.4.7: *For $n \geq 2$ and $k \geq 0$,*

$$\mu_{1:1:n}^{(n-1)^{(k+1)}} = \frac{k+1}{n}\mu_{1:1:n}^{(n-1)^{(k)}} - \left[\frac{1}{P} - 1\right] \mu_{1:1:n-1}^{(n-2)^{(k+1)}} . \tag{3.4.16}$$

Theorem 3.4.8: *For $2 \leq m \leq n - 1$, $R_1 \geq 1$ and $k \geq 0$,*

$$\mu_{1:m:n}^{(R_1,\ldots,R_m)^{(k+1)}} = \frac{1}{R_1+1}\left\{(k+1)\mu_{1:m:n}^{(R_1,\ldots,R_m)^{(k)}} - \left[\frac{1}{P} - 1\right]\left[\frac{n(n-R_1-1)}{n-1}\mu_{1:m-1:n-1}^{(R_1+R_2,R_3,\ldots,R_m)^{(k+1)}} + \frac{n}{n-1}R_1\mu_{1:m:n-1}^{(R_1-1,R_2,\ldots,R_m)^{(k+1)}}\right] - (n-R_1-1)\mu_{1:m-1:n}^{(R_1+R_2+1,R_3,\ldots,R_m)^{(k+1)}}\right\} . \tag{3.4.17}$$

Theorem 3.4.9: *For* $2 \le m \le n$, $R_1 = 0$ *and* $k \ge 0$,

$$\mu_{1:m:n}^{(0,R_2,\ldots,R_m)^{(k+1)}} = (k+1)\mu_{1:m:n}^{(0,R_2,\ldots,R_m)^{(k)}} - n\left(\frac{1}{P}-1\right)\mu_{1:m-1:n-1}^{(R_2,\ldots,R_m)^{(k+1)}} - (n-1)\mu_{1:m-1:n}^{(R_2+1,R_3,\ldots,R_m)^{(k+1)}} . \tag{3.4.18}$$

Theorem 3.4.10: *For* $2 \le i \le m-1$, $m \le n-1$, $R_i \ge 1$ *and* $k \ge 0$,

$$\begin{aligned}\mu_{i:m:n}^{(R_1,\ldots,R_m)^{(k+1)}} = \frac{1}{R_i+1}\Bigg\{&(k+1)\mu_{i:m:n}^{(R_1,\ldots,R_m)^{(k)}} - \left(\frac{1}{P}-1\right)\Bigg[\frac{A(n,i)}{A(n-1,i-1)}\mu_{i:m-1:n-1}^{(R_1,\ldots,R_{i-1},R_i+R_{i+1},R_{i+2},\ldots,R_m)^{(k+1)}}\\ &- \frac{A(n,i-1)}{A(n-1,i-2)}\mu_{i-1:m-1:n-1}^{(R_1,\ldots,R_{i-2},R_{i-1}+R_i,R_{i+1},\ldots,R_m)^{(k+1)}} + \frac{A(n,i-1)}{A(n-1,i-1)}R_i\mu_{i:m:n-1}^{(R_1,\ldots,R_{i-1},R_i-1,R_{i+1},\ldots,R_m)^{(k+1)}}\Bigg]\\ &-(n-R_1-\ldots-R_i-i)\mu_{i:m-1:n}^{(R_1,\ldots,R_{i-1},R_i+R_{i+1}+1,R_{i+2},\ldots,R_m)^{(k+1)}}\\ &+(n-R_1-\ldots-R_{i-1}-i+1)\mu_{i-1:m-1:n}^{(R_1,\ldots,R_{i-2},R_{i-1}+R_i+1,R_{i+1},\ldots,R_m)^{(k+1)}}\Bigg\} .\end{aligned} \tag{3.4.19}$$

Theorem 3.4.11: *For* $2 \le i \le m-1$, $m \le n$, $R_i = 0$ *and* $k \ge 0$,

$$\begin{aligned}\mu_{i:m:n}^{(R_1,\ldots,R_{i-1},0,R_{i+1},\ldots,R_m)^{(k+1)}} &= (k+1)\mu_{i:m:n}^{(R_1,\ldots,R_{i-1},0,R_{i+1},\ldots,R_m)^{(k)}}\\ &- \left(\frac{1}{P}-1\right)\frac{A(n,i-1)}{A(n-1,i-2)}\left[\mu_{i:m-1:n-1}^{(R_1,\ldots,R_{i-1},R_{i+1},\ldots,R_m)^{(k+1)}} - \mu_{i-1:m-1:n-1}^{(R_1,\ldots,R_{i-1},R_{i+1},\ldots,R_m)^{(k+1)}}\right]\\ &-(n-R_1-\ldots-R_{i-1}-i)\mu_{i:m-1:n}^{(R_1,\ldots,R_{i-1},R_{i+1}+1,R_{i+2},\ldots,R_m)^{(k+1)}}\\ &+(n-R_1-\ldots-R_{i-1}-i+1)\mu_{i-1:m-1:n}^{(R_1,\ldots,R_{i-2},R_{i-1}+1,R_{i+1},\ldots,R_m)^{(k+1)}} .\end{aligned} \tag{3.4.20}$$

Theorem 3.4.12: *For $2 \leq m \leq n-1$, $R_m \geq 1$ and $k \geq 0$,*

$$\mu_{m:m:n}^{(R_1,\ldots,R_m)^{(k+1)}} = \frac{1}{R_m+1}\left\{(k+1)\mu_{m:m:n}^{(R_1,\ldots,R_m)^{(k)}} + \left(\frac{1}{P}-1\right)\left[\frac{A(n,m-1)}{A(n-1,m-2)}\mu_{m-1:m-1:n-1}^{(R_1,\ldots,R_{m-2},R_{m-1}+R_m)^{(k+1)}}\right.\right.$$

$$\left.\left.-\frac{A(n,m-1)}{A(n-1,m-1)}R_m\mu_{m:m:n-1}^{(R_1,\ldots,R_{m-1},R_m-1)^{(k+1)}}\right] + (n-R_1-\ldots-R_{m-1}-m+1)\mu_{m-1:m-1:n}^{(R_1,\ldots,R_{m-2},R_{m-1}+R_m+1)^{(k+1)}}\right\}. \tag{3.4.21}$$

Theorem 3.4.13: *For $2 \leq m \leq n$, $R_m = 0$ and $k \geq 0$,*

$$\mu_{m:m:n}^{(R_1,\ldots,R_{m-1},0)^{(k+1)}} = (k+1)\mu_{m:m:n}^{(R_1,\ldots,R_{m-1},0)^{(k)}} + (n-R_1-\ldots-R_{m-1}-m+1)\mu_{m-1:m-1:n}^{(R_1,\ldots,R_{m-2},R_{m-1}+1)^{(k+1)}}$$

$$-\left(\frac{1}{P}-1\right)\frac{A(n,m-1)}{A(n-1,m-2)}\left[P_1^{k+1} - \mu_{m-1:m-1:n-1}^{(R_1,\ldots,R_{m-1})^{(k+1)}}\right]. \tag{3.4.22}$$

Remark: The relations presented in this subsection are complete in the sense that they will enable one to compute all the single moments of progressive Type-II right censored order statistics from right truncated exponential distributions for all sample sizes and all censoring schemes. The recursive algorithm is presented in subsection 3.4.7.

Remark: It is easily seen that by letting $P \to 1$, these results readily reduce to the recurrence relations given in subsection 3.4.2 for the standard exponential distribution.

3.4.6. Recurrence Relations for Product Moments - Truncated Exponential Distribution

Following steps similar to those in subsection 3.4.3 for product moments of progressive Type-II right censored order statistics from the exponential distribution, we may obtain the following recurrence relations for product moments of progressive Type-II right censored order statistics from the right truncated exponential distribution.

Theorem 3.4.14: *For* $1 \le i < j \le m-1$, $m \le n-1$ *and* $R_j \ge 1$,

$$
\begin{aligned}
\mu_{i,j:m:n}^{(R_1,\ldots,R_m)} = \frac{1}{R_j+1}\Bigg\{ & \mu_{i:m:n}^{(R_1,\ldots,R_m)} - \left(\frac{1}{P}-1\right) \Bigg[\frac{A(n,j)}{A(n-1,j-1)}\mu_{i,j:m-1:n-1}^{(R_1,\ldots,R_{j-1},R_j+R_{j+1},R_{j+2},\ldots,R_m)} \\
& - \frac{A(n,j-1)}{A(n-1,j-2)}\mu_{i,j-1:m-1:n-1}^{(R_1,\ldots,R_{j-2},R_{j-1}+R_j,R_{j+1},\ldots,R_m)} + \frac{A(n,j-1)}{A(n-1,j-1)}R_j\mu_{i,j:m:n-1}^{(R_1,\ldots,R_{j-1},R_j-1,R_{j+1},\ldots,R_m)}\Bigg] \\
& - (n-R_1-\ldots-R_j-j)\mu_{i,j:m-1:n}^{(R_1,\ldots,R_{j-1},R_j+R_{j+1}+1,R_{j+2},\ldots,R_m)} \\
& + (n-R_1-\ldots-R_{j-1}-j+1)\mu_{i,j-1:m-1:n}^{(R_1,\ldots,R_{j-2},R_{j-1}+R_j+1,R_{j+1},\ldots,R_m)} \Bigg\}.
\end{aligned}
\tag{3.4.23}
$$

Theorem 3.4.15: *For $1 \leq i < j \leq m-1$, $m \leq n$ and $R_j = 0$,*

$$\mu_{i,j:m:n}^{(R_1,\ldots,R_{j-1},0,R_{j+1},\ldots,R_m)} = \mu_{i:m:n}^{(R_1,\ldots,R_{j-1},0,R_{j+1},\ldots,R_m)} - \left(\frac{1}{P}-1\right) \frac{A(n,j-1)}{A(n-1,j-2)}\Big[\mu_{i,j:m-1:n-1}^{(R_1,\ldots,R_{j-1},R_{j+1},\ldots,R_m)}$$
$$- \mu_{i,j-1:m-1:n-1}^{(R_1,\ldots,R_{j-1},R_{j+1},\ldots,R_m)}\Big] - (n-R_1-\ldots-R_{j-1}-j)\mu_{i,j:m-1:n}^{(R_1,\ldots,R_{j-1},R_{j+1}+1,R_{j+2},\ldots,R_m)}$$
$$+ (n-R_1-\ldots-R_{j-1}-j+1)\mu_{i,j-1:m-1:n}^{(R_1,\ldots,R_{j-2},R_{j-1}+1,R_{j+1},\ldots,R_m)} . \tag{3.4.24}$$

Theorem 3.4.16: *For $1 \leq i \leq m-1$, $m \leq n-1$ and $R_m \geq 1$,*

$$\mu_{i,m:m:n}^{(R_1,\ldots,R_m)} = \frac{1}{R_m+1}\left\{\mu_{i:m:n}^{(R_1,\ldots,R_m)} - \left(\frac{1}{P}-1\right) \left[-\frac{A(n,m-1)}{A(n-1,m-2)}\mu_{i,m-1:m-1:n-1}^{(R_1,\ldots,R_{m-2},R_{m-1}+R_m)}\right.\right.$$
$$\left.\left. + \frac{A(n,m-1)}{A(n-1,m-1)}R_m\mu_{i,m:m:n-1}^{(R_1,\ldots,R_{m-1},R_m-1)}\right] + (n-R_1-\ldots-R_{m-1}-m+1)\mu_{i,m-1:m-1:n}^{(R_1,\ldots,R_{m-2},R_{m-1}+R_m+1)}\right\} . \tag{3.4.25}$$

Theorem 3.4.17: *For $1 \leq i \leq m-1$, $m \leq n$ and $R_m = 0$,*

$$\mu_{i,m:m:n}^{(R_1,\ldots,R_{m-1},0)} = \mu_{i:m:n}^{(R_1,\ldots,R_{m-1},0)} - \left(\frac{1}{P}-1\right) \frac{A(n,m-1)}{A(n-1,m-2)}\left[P_1\mu_{i:m-1:n-1}^{(R_1,\ldots,R_{m-1})} - \mu_{i,m-1:m-1:n-1}^{(R_1,\ldots,R_{m-2},R_{m-1})}\right]$$
$$+ (n-R_1-\ldots-R_{m-1}-m+1)\mu_{i,m-1:m-1:n}^{(R_1,\ldots,R_{m-2},R_{m-1}+1)} . \tag{3.4.26}$$

***Remark*:** Using these recurrence relations and those in subsection 3.4.5, the means, variances and covariances of all progressive Type-II right censored order statistics from a right truncated exponential distribution can be calculated for all sample sizes and all possible censoring schemes. This is described in detail in subsection 3.4.7.

Remark: Letting $P \to 1$ in the above recurrence relations, we readily obtain the relations derived in subsection 3.4.3 for the standard exponential distribution.

Remark: Following arguments similar to those in the last remark of subsection 3.4.3, we may show that for the special case $R_1 = R_2 = \ldots = R_m = 0$, the recurrence relations in subsections 3.4.5 and 3.4.6 reduce to the following which are equivalent to those given by Joshi (1978, 1982) for the usual order statistics from the right truncated exponential distribution:

From Eq. (3.4.15), we obtain for $k \geq 0$,

$$\mu_{1:1}^{(k+1)} = (k+1)\mu_{1:1}^{(k)} - \left(\frac{1}{P}-1\right) P_1^{k+1} .$$

From Eqs. (3.4.16) and (3.4.18), we have for $n \geq 2$ and $k \geq 0$,

$$\mu_{1:n}^{(k+1)} = \frac{k+1}{n}\mu_{1:n}^{(k)} - \left(\frac{1}{P}-1\right) \mu_{1:n-1}^{(k+1)} .$$

From Eq. (3.4.20): For $2 \leq i \leq n - 1$ and $k \geq 0$,

$$\mu_{i:n}^{(k+1)} = \mu_{i-1:n}^{(k+1)} + \frac{1}{n-i+1}\left\{ (k+1)\mu_{i:n}^{(k)} - n\left[\frac{1}{P}-1\right]\left[\mu_{i:n-1}^{(k+1)} - \mu_{i-1:n-1}^{(k+1)}\right]\right\} ,$$

and from Eq. (3.4.22): For $n \geq 2$ and $k \geq 0$,

$$\mu_{n:n}^{(k+1)} = (k+1)\mu_{n:n}^{(k)} + \mu_{n-1:n}^{(k+1)} - n\left[\frac{1}{P}-1\right]\left[P_1^{k+1} - \mu_{n-1:n-1}^{(k+1)}\right] .$$

These recurrence relations are equivalent to those given by Joshi (1978) for the single moments of usual order statistics from the right truncated exponential distribution.

From Eq. (3.4.24), we obtain, for $1 \leq i < j \leq n - 1$,

$$\mu_{i,j:n} = \mu_{i,j-1:n} + \frac{1}{n-j+1}\left\{ \mu_{i:n} - n\left[\frac{1}{P}-1\right]\left[\mu_{i,j:n-1} - \mu_{i,j-1:n-1}\right]\right\} ,$$

and from Eq. (3.4.26): For $1 \leq i \leq n - 1$,

$$\mu_{i,n:n} = \mu_{i,n-1:n} + \mu_{i:n} - n\left[\frac{1}{P}-1\right]\left[P_1\mu_{i:n-1} - \mu_{i,n-1:n-1}\right] .$$

These recurrence relations are equivalent to those given by Joshi (1982) for the product moments of usual order statistics from the right truncated exponential distribution.

3.4.7. Recursive Algorithm - Truncated Exponential Distribution

Using the recurrence relations developed in subsections 3.4.5 and 3.4.6, the means, variances and covariances of all progressive Type-II right censored order statistics can be readily computed as follows:

Setting $k = 0$, Eq. (3.4.15) will give us the value $\mu_{1:1:1}^{(0)}$, which in turn, again using (3.4.15) with $k = 1$, will give us the value $\mu_{1:1:1}^{(0)^{(2)}}$. From these values, we can recursively compute $\mu_{1:1:n}^{(n-1)}$ and $\mu_{1:1:n}^{(n-1)^{(2)}}$ for n = 2, 3. ... using (3.4.16). Thus, all first and second moments with $m = 1$ for all sample sizes n will be obtained. Next, using (3.4.18), we can determine all moments of the form $\mu_{1:2:n}^{(0,n-2)}$, $n = 2, 3, \ldots$which can in turn be used, with (3.4.18), to determine all moments of the form $\mu_{1:2:n}^{(0,n-2)^{(2)}}$, $n = 2, 3, \ldots$. Eq. (3.4.17) can then be used to obtain $\mu_{1:2:n}^{(R_1,R_2)}$ for $R_1 = 1, 2, \ldots$ and $n \geq 3$, and these values can be used to obtain all moments of the form $\mu_{1:2:n}^{(R_1,R_2)^{(2)}}$ using (3.4.17) again. Now, Eq. (3.4.22) can be used again to obtain $\mu_{2:2:n}^{(n-2,0)}$ and $\mu_{2:2:n}^{(n-2,0)^{(2)}}$ for all n and (3.4.21) can be used next to obtain, for $R_1, R_2 = 1, 2, \ldots$ and $n \geq 3$, all moments of the form $\mu_{2:2:n}^{(R_1,R_2)}$ and $\mu_{2:2:n}^{(R_1,R_2)^{(2)}}$. This process can be continued until all desired first and second order moments (and therefore all variances) are obtained.

From (3.4.26), all moments of the form $\mu_{m-1,m:m:n}^{(R_1,\ldots,R_{m-1},0)}$, $m = 2, 3, \ldots, n$ can be determined, since only the single moments, which have already been computed, are needed to calculate them. Eq. (3.4.25) can then be used to obtain $\mu_{m-1,m:m:n}^{(R_1,\ldots,R_m)}$, for $R_m = 1, 2, \ldots$. Then, using (3.4.24), all moments of the form $\mu_{j-1,j:m:n}^{(R_1,\ldots,R_{j-1},0,R_{j+1},\ldots,R_m)}$ $(j < m)$ can be obtained, and using (3.4.23), all moments of the form $\mu_{j-1,j:m:n}^{(R_1,\ldots,R_m)}$, $j < m$, $R_j \geq 1$, can be calculated. From this point, using (3.4.26) and (3.4.25), we can obtain all moments of the form $\mu_{m-2,m:m:n}^{(R_1,\ldots,R_m)}$ and, subsequently, using (3.4.24) and (3.4.23), all moments of the form $\mu_{j-2,j:m:n}^{(R_1,\ldots,R_m)}$, $j < m$. Continuing this way, all the desired product moments (and therefore all covariances) can be obtained.

Remark: Along the lines of Joshi (1979), all the results presented in this section may also be generalized to the case of progressive Type-II right censored order statistics from a doubly truncated exponential distribution. However, we abstain from presenting those results for brevity.

4 General Continuous Distributions: Some Mathematical Results

4.1 Properties of General Progressive Type-II Censored Order Statistics

4.1.1 Introduction

A well known property of usual order statistics from arbitrary continuous distributions is the following Markovian property, which is given in David (1981) and Arnold, Balakrishnan and Nagaraja (1992).

Theorem (Markovian Property of Usual Order Statistics): *Given* $X_{i:n} = x_i$, $X_{j:n}$ *($j > i$) is independent of the first i-1 usual order statistics,* $X_{1:n}, \ldots, X_{i-1:n}$.

It will be of interest to examine whether a similar result can be established for general progressive Type-II censored (and therefore progressive Type-II right censored) order statistics. That is, given ${}_rX_{r+i:m:n} = x_i$, is ${}_rX_{r+j:m:n}$ independent of the first i-1 general progressive Type-II censored order statistics? In obtaining this result for progressive Type-II censored order statistics, one will wish to derive the joint density of the first i general progressive Type-II censored order statistics, ${}_rX_{r+1:m:n}, \ldots, {}_rX_{r+i:m:n}$.

Scheffé and Tukey (1945) presented the following result for usual order statistics $X_{1:n}$, $X_{2:n}$, ..., $X_{n:n}$ with cumulative distribution function $F(x)$ from arbitrary continuous distributions.

Theorem [Scheffé and Tukey (1945)]: Given $X_{i:n} = x_i$, $X_{j:n}$ $(j > i)$ is distributed as the $(j\text{-}i)^{\text{th}}$ usual order statistic for a sample of size n-i from $F(x)$ left truncated at x_i, that is, from $G(y) = \dfrac{F(y)-F(x)}{1-F(x)}$ $(y > x)$; see also David (1981) and Arnold, Balakrishnan and Nagaraja (1992). This result was later used by Horn and Schlipf (1986) to develop an efficient algorithm for the generation of conventional Type-II doubly censored samples.

Thus, a natural question which arises is the following: Does a similar result hold for general progressive Type-II censored order statistics? That is, given ${}_rX_{r+i:m:n} = x_i$, how are ${}_rX_{r+i+1:m:n}, {}_rX_{r+i+2:m:n}, \ldots, {}_rX_{m:m:n}$ distributed? Can we, using this result, develop a method generalizing that of Horn and Schlipf to efficiently generate "trimmed" progressive Type-II censored samples?

Notice that, in the case of usual order statistics, arbitrary joint marginal densities may be easily written down; however, for progressive Type-II censored order statistics, we can, at this point, only readily write joint densities of all m order statistics. Thus, proofs of some theorems will be much different and are more complicated in this case than in the case of usual order statistics.

In this section, we establish three useful results for general progressive Type-II censored order statistics ${}_rX_{r+1:m:n}$, ${}_rX_{r+2:m:n}$, $\ldots$, ${}_rX_{m:m:n}$. We will use the notation presented in section 1.4 for general progressive Type-II censored samples from arbitrary continuous distributions with censoring scheme (R_{r+1}, R_{r+2}, $\ldots$, R_m).

4.1.2 Results

Theorem 4.1.1: *(i) The distribution of ${}_rX_{r+i:m:n}$, $1 \leq i \leq m-r$, is free of R_{r+i}, R_{r+i+1}, $\ldots$, R_m.*

(ii) ${}_rX_{r+1:m:n}$, ${}_rX_{r+2:m:n}$, $\ldots$, ${}_rX_{r+i:m:n}$ form a general progressive Type-II censored sample of size i from n items put on test with censoring scheme (R_{r+1}, $\ldots$, R_{r+i-1}, $n - R_{r+1} - \ldots - R_{r+i-1} - r - i$).

Proof: We will repeatedly use the following:

$$\int_a^\infty f(x)[1-F(x)]^k \, dx = -\frac{[1-F(x)]^{k+1}}{k+1}\Big|_a^\infty = \frac{[1-F(a)]^{k+1}}{k+1} . \tag{4.1.1}$$

We will now start with the joint probability density function of ${}_rX_{r+1:m:n}$, ${}_rX_{r+2:m:n}$, ..., ${}_rX_{r+i:m:n}$, which is given by

$$f_{{}_rX_{r+1:m:n},{}_rX_{r+2:m:n},\dots,{}_rX_{r+i:m:n}}(x_{r+1},x_{r+2},\dots,x_{r+i}) = c'[F(x_{r+1})]^r \prod_{k=r+1}^{r+i} f(x_k)[1-F(x_k)]^{R_k}$$

$$\cdot \int_{x_{r+i}}^\infty f(x_{r+i+1})[1-F(x_{r+i+1})]^{R_{r+i+1}} \dots \int_{x_{m-1}}^\infty f(x_m)[1-F(x_m)]^{R_m} dx_m \dots dx_{r+i+1} ,$$

where c' is as defined in (1.4.3). By repeated use of (4.1.1), we obtain the joint density function of ${}_rX_{r+1:m:n}$, ${}_rX_{r+2:m:n}$, ..., ${}_rX_{r+i:m:n}$ to be

$$f_{{}_rX_{r+1:m:n},\dots,{}_rX_{r+i:m:n}}(x_{r+1},\dots,x_{r+i}) = c'[F(x_{r+1})]^r \prod_{j=r+1}^{r+i-1} f(x_j)[1-F(x_j)]^{R_j}$$

$$\cdot f(x_{r+i}) \frac{[1-F(x_{r+i})]^{\sum_{k=r+i}^{m} R_k + m - r - i}}{\prod_{k=1}^{m-r-i} \left[\left(\sum_{l=1}^{k} R_{m-l+1} \right) + k \right]} . \tag{4.1.2}$$

The factors in the denominator of (4.1.2) cancel with some of the factors in the constant c', due to the fact that $n = m + R_{r+1} + \dots + R_m$. Furthermore, the summation in the exponent of $[1-F(x_{r+i})]$ is simply $n - R_{r+1} - \dots - R_{r+i-1} - r - i$. We therefore obtain from (4.1.2) that

$$f_{{}_rX_{r+1:m:n},\ldots,{}_rX_{r+i:m:n}}(x_{r+1},\ldots,x_{r+i}) = \binom{n}{r}(n-r)(n-r-R_{r+1}-1)(n-r-R_{r+1}-R_{r+2}-2)\ldots$$
$$\cdot\,(n-r-R_{r+1}-\ldots-R_{r+i-1}-i+1)\prod_{j=r+1}^{r+i-1} f(x_j)[1-F(x_j)]^{R_j}$$
$$\cdot\,[F(x_{r+1})]^r f(x_{r+i})[1-F(x_{r+i})]^{n-R_{r+1}-R_{r+2}-\ldots-R_{r+i-1}-r-i}. \qquad (4.1.3)$$

Thus, the joint pdf of ${}_rX_{r+1:m:n}$, ${}_rX_{r+2:m:n}$, …, ${}_rX_{r+i:m:n}$ is completely free of the constants R_{r+i}, …, R_m. As a matter of fact, a look at (4.1.3) reveals that these progressive censored order statistics form a general progressive Type-II censored sample of size i from n items put on test (where the first r failure times are not observed) with censoring scheme (R_{r+1}, …, R_{r+i-1}, n - R_{r+1} - … - R_{r+i-1} - r - i).

□

Remark: Theorem 4.1.1 will be useful in simplifications; for example, for $r = 0$, using this theorem, we have that $X_{i:m:n}^{(R_1,\ldots,R_m)} \overset{d}{=} X_{i:i:n}^{(R_1,\ldots,R_{i-1},n-R_1-\ldots-R_{i-1}-i)}$. It is also intuitively easy to understand: what happens after the observation of the i^{th} progressive Type-II right censored order statistic should not affect its distribution.

Theorem 4.1.2: *The general progressive Type-II censored order statistics from an arbitrary continuous distribution form a Markov chain; that is, given* ${}_rX_{r+i:m:n}$, ${}_rX_{r+j:m:n}$ *($j > i$) is independent of* ${}_rX_{r+1:m:n}$, ${}_rX_{r+2:m:n}$, …, ${}_rX_{r+i-1:m:n}$.

Proof: Let us consider the conditional distribution

$$f_{X_{r+j:m:n}|X_{r+i:m:n},X_{r+i-1:m:n},\ldots,X_{r+1:m:n}}(x_{r+j}|x_{r+i},x_{r+i-1},\ldots,x_{r+1})$$

$$= \frac{f_{X_{r+j:m:n},X_{r+i:m:n},X_{r+i-1:m:n},\ldots,X_{r+1:m:n}}(x_{r+j},x_{r+i},x_{r+i-1},\ldots,x_{r+1})}{f_{X_{r+i:m:n},X_{r+i-1:m:n},\ldots,X_{r+1:m:n}}(x_{r+i},x_{r+i-1},\ldots,x_{r+1})}. \tag{4.1.4}$$

The joint distribution in the denominator of (4.1.4) is precisely the expression in (4.1.3). The joint distribution in the numerator of (4.1.4) is given by

$$f_{X_{r+1:m:n},X_{r+2:m:n},\ldots,X_{r+i:m:n},X_{r+j:m:n}}(x_{r+1},x_{r+2},\ldots,x_{r+i},x_{r+j}) = c[F(x_{r+1})]^r \prod_{i=r+1}^{r+i} f(x_i)[1-F(x_i)]^{R_i}$$

$$\cdot \int_{x_{r+i}}^{x_{r+j}} f(x_{r+j-1})[1-F(x_{r+j-1})]^{R_{r+j-1}} \int_{x_{r+i}}^{x_{r+j-1}} f(x_{r+j-2})[1-F(x_{r+j-2})]^{R_{r+j-2}} \ldots$$

$$\cdot \int_{x_{r+i}}^{x_{r+i+2}} f(x_{r+i+1})[1-F(x_{r+i+1})]^{R_{r+i+1}} dx_{r+i+1} \ldots dx_{r+j-2} dx_{r+j-1}$$

$$\cdot f(x_{r+j})[1-F(x_{r+j})]^{R_{r+j}}$$

$$\cdot \int_{x_{r+j}}^{\infty} f(x_{r+j+1})[1-F(x_{r+j+1})]^{R_{r+j+1}} \ldots \int_{x_{m-1}}^{\infty} f(x_m)[1-F(x_m)]^{R_m} dx_m \ldots dx_{r+j+1}.$$

The last multiple integral can be evaluated by repeated use of (4.1.1). The multiple integral in the second and third lines is clearly a function of only x_{r+i} and x_{r+j}, which we shall denote $K(x_{r+i}, x_{r+j})$. Upon simplification, we obtain the conditional density of interest to be

$$f_{{}_rX_{r+j:m:n}|{}_rX_{r+i:m:n},{}_rX_{r+i-1:m:n},\ldots,{}_rX_{r+1:m:n}}(x_{r+j}|x_{r+i},x_{r+i-1},\ldots,x_{r+1})$$

$$= \prod_{k=r+i+1}^{r+j} \left[\left(\sum_{q=k}^{m} R_q \right) + m - k + 1 \right] K(x_{r+i},x_{r+j}) f(x_{r+j}) \frac{[1-F(x_{r+j})]^{R_{r+j}+R_{r+j+1}+\ldots+R_m+m-r-j}}{[1-F(x_{r+i})]^{R_{r+i+1}+R_{r+i+2}+\ldots+R_m+m-r-i}},$$

$$x_{r+j} > x_{r+i}.$$

This expression is clearly completely independent of x_{r+1}, x_{r+2}, $\ldots$, x_{r+i-1}, so that it depends only on x_{r+i} and x_{r+j}. Thus, for continuous distributions, we have

$$f_{{}_rX_{r+j:m:n}|{}_rX_{r+i:m:n},{}_rX_{r+i-1:m:n},\ldots,{}_rX_{r+1:m:n}}(x_{r+j}|x_{r+i},x_{r+i-1},\ldots,x_{r+1}) = f_{{}_rX_{r+j:m:n}|{}_rX_{r+i:m:n}}(x_{r+j}|x_{r+i}).$$

Hence, the general progressive Type-II censored order statistics ${}_rX_{r+1:m:n}$, $\ldots$, ${}_rX_{m:m:n}$ form a Markov chain. □

Remark: It should be mentioned here that Theorem 4.1.2 is a generalization of a well known result for usual order statistics given, for example, in David (1981) and Arnold, Balakrishnan and Nagaraja (1991, p. 24) and discussed in the introduction of section 4.1. We will also use this theorem to establish the following theorem.

Theorem 4.1.3: *Given ${}_rX_{r+i:m:n} = x_{r+i}$, ${}_rX_{r+i+1:m:n}$, $\ldots$, ${}_rX_{m:m:n}$ $(i \geq 1)$ are jointly distributed as a progressive Type-II right censored sample of size m-r-i from n - R_{r+1} - R_{r+2} - $\ldots$ - R_{r+i} - r - i identically distributed random variables from the density f(x) left-truncated at x_{r+i}, that is, with density $f(x)/[1-F(x_{r+i})]$ $(x > x_{r+i})$, and with progressive censoring scheme R_{r+i+1}, $\ldots$, R_m.*

Proof: Given ${}_rX_{r+i:m:n} = x_{r+i}$, the conditional joint distribution of ${}_rX_{r+i+1:m:n}, \ldots, {}_rX_{m:m:n}$ is given by, upon invoking the Markov property established in Theorem 4.1.2,

$$\begin{aligned} & f_{{}_rX_{r+i+1:m:n},\ldots,{}_rX_{m:m:n} | {}_rX_{r+i:m:n}}(x_{r+i+1},\ldots,x_m | x_{r+i}) \\ &= f_{{}_rX_{r+i+1:m:n},\ldots,{}_rX_{m:m:n} | {}_rX_{r+i:m:n},{}_rX_{r+i-1:m:n},\ldots,{}_rX_{r+1:m:n}}(x_{r+i+1},\ldots,x_m | x_{r+i},x_{r+i-1},\ldots,x_{r+1}) \\ &= \frac{f_{{}_rX_{r+1:m:n},\ldots,{}_rX_{m:m:n}}(x_{r+1},\ldots,x_m)}{f_{{}_rX_{r+1:m:n},\ldots,{}_rX_{r+i:m:n}}(x_{r+1},\ldots,x_{r+i})} . \end{aligned} \tag{4.1.5}$$

The joint density in the numerator of (4.1.5) is given in (1.4.3), and the joint density in the denominator is given in (4.1.3). Thus, upon simplification, we obtain the conditional distribution to be

$$f_{{}_rX_{r+i+1:m:n},\ldots,{}_rX_{m:m:n} | {}_rX_{r+i:m:n}}(x_{r+i+1},\ldots,x_m | x_{r+i})$$

$$= \left[\prod_{j=r+i+1}^{m} \left[n - \sum_{k=r+1}^{j-1} R_k - j + 1 \right] \right] \left[\prod_{k=r+i+1}^{m} \frac{f(x_k)}{1-F(x_{r+i})} \left[\frac{1-F(x_k)}{1-F(x_{r+i})} \right]^{R_k} \right] .$$

This is seen simply to be the joint density of m-r-i progressive Type-II right censored order statistics from $n - R_{r+1} - R_{r+2} - \ldots - R_{r+i} - r - i$ identically distributed random variables with probability density function $f(x)/[1-F(x_{r+i})]$, and with progressive censoring scheme $R_{r+i+1}, \ldots, R_m$. □

Remark: It should be mentioned here that Theorem 4.1.3 is a generalization of a result presented originally by Scheffé and Tukey (1945) and discussed in the introduction to section 4.1; see also David (1981) and Balakrishnan and Cohen

(1991).

Remark: Using this result, one may, for example, generalize the simulational algorithm given in subsection 2.4.4 for the generation of general progressive Type-II censored samples to the generation of "trimmed" general progressive Type-II censored samples, since we now know we may condition on *any* of the progressive censored order statistics.

Remark: Notice that the converse of Theorem 4.1.3 will not be true in general. This is because items removed before the observation of the j^{th} order statistic, which is assumed to be given, say x_j , may have had life times larger than x_j .

4.2 Symmetric Distributions

4.2.1 Introduction

A number of results for usual order statistics have been discussed and established for the special case when the random sample is from a symmetric distribution. See, for example, Arnold, Balakrishnan and Nagaraja (1992, pp. 26, 119, 123-126, 128, 136-137, 173-174, 187). In this section, we establish results which enable one to compute the moments of progressive Type-II right censored order statistics from an arbitrary symmetric distribution when the moments of progressive Type-II right censored order statistics and progressive Type-II left withdrawn order statistics from the corresponding folded distribution are known. The idea of progressive Type-II left withdrawal is introduced in order to arrive at this result.

We begin with the assumption of an underlying distribution which is symmetric (about 0, without loss of generality), with probability density function $f(x)$ $(-\infty < x < \infty)$ and cumulative distribution function $F(x)$ $(-\infty < x < \infty)$. Next, we denote the probability density function for the corresponding folded distribution by $g(x) = 2f(x)$, $0 < x < \infty$, and the cumulative distribution function by $G(x) = 2F(x) - 1$, $0 < x < \infty$.

The main results established here generalize two classical results for usual order statistics given by Govindarajulu (1963) which state that if the (single or product) moments of order statistics from a folded distribution are known, then the (single or product) moments of order statistics from the corresponding symmetric distribution can be obtained.

4.2.2 Progressive Withdrawal

It is a well known property of usual order statistics from a random sample from a symmetric distribution (symmetric about 0), $F(x)$, which we have previously denoted by $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$, that $X_{i:n} \stackrel{d}{=} -X_{n-i+1:n}$, $i = 1, 2, \ldots, n$ or that jointly, $(X_{1:n}, X_{2:n}, \ldots, X_{n:n}) \stackrel{d}{=} (-X_{n:n}, -X_{n-1:n}, \ldots -X_{1:n})$; for example, see David (1981) and Arnold, Balakrishnan and Nagaraja (1992). Thus, the negatives of the usual order statistics are again usual order statistics. It will be interesting to see whether a similar result holds for progressive Type-II right censored order statistics from a symmetric distribution, as it will facilitate the handling of these random variables.

In the case of usual order statistics, the result for individual order statistics is easily obtained by simply rewriting the marginal density of the i^{th} order statistic and using the fact the $f(x) = f(-x)$, and $F(x) = 1-F(-x)$:

$$\frac{n!}{(i-1)!(n-i)!} f(x)[F(x)]^{i-1}[1-F(x)]^{n-i}$$

$$= \frac{n!}{(n-(n-i+1))!((n-i+1)-1)!} f(-x)[1-F(-x)]^{i-1}[F(-x)]^{n-i}.$$

However, in working with progressive Type-II right censored order statistics, arbitrary marginal densities are not easily dealt with, and so we begin with the joint distribution of all m progressive Type-II right censored order statistics.

As earlier, we will denote the sample of progressive Type-II right censored order statistics of size m with progressive censoring scheme $(R_1, R_2, \ldots, R_m)$ from a random sample of size n from a symmetric distribution with probability density function $f(x)$ be $X_{1:m:n}^{(R_1,R_2,\ldots,R_m)}$, $X_{2:m:n}^{(R_1,R_2,\ldots,R_m)}$, ..., $X_{m:m:n}^{(R_1,R_2,\ldots,R_m)}$. Then we consider the following transformation:

$$Z_1 = -X_{m:m:n}^{(R_1,\ldots,R_m)}, \quad Z_2 = -X_{m-1:m:n}^{(R_1,\ldots,R_m)}, \quad \ldots, \quad Z_m = -X_{1:m:n}^{(R_1,\ldots,R_m)}.$$

The Jacobian of this transformation is 1, and the joint distribution of $(Z_1, Z_2, \ldots, Z_m)$ is given by

$$h_{1,2,\ldots,m}(z_1,z_2,\ldots,z_m) = c\ [F(z_1)]^{R_m} f(z_1)\ [F(z_2)]^{R_{m-1}} f(z_2) \ldots [F(z_m)]^{R_1} f(z_m),$$
$$-\infty < z_1 < z_2 < \ldots < z_m < \infty \qquad (4.2.1)$$

where $c = n\ (n-R_1-1)\ (n-R_1-R_2-2) \ldots (n-R_1-R_2-\ldots-R_{m-1}-m+1)$.

Thus, the negatives of progressive Type-II right censored order statistics are not jointly distributed as progressive Type-II right censored order statistics. If we interpret (4.2.1) in terms of live items on test, we would be withdrawing, at the failure times $z_1, z_2, \ldots, z_m$, items which have not been observed to fail but are known to have failed already. This may seem to be an impractical idea, however mathematical consideration of this set of random variables will give rise to useful results involving progressive Type-II right censored order statistics.

To formalize the above discussion, we introduce the following notation:

if m random variables have a joint density given by (4.2.1), we call them a *progressive Type-II left withdrawn sample* of size m from a random sample of size n with withdrawal scheme $(R_m, R_{m-1}, \ldots, R_1)$ from the distribution with probability density function $f(x)$, and denote the corresponding progressive Type-II left withdrawn order statistics by:

$$^{(R_m,\ldots,R_1)}Z_{1:m:n}\ ,\quad ^{(R_m,\ldots,R_1)}Z_{2:m:n}\ ,\ \ldots\ ,\quad ^{(R_m,\ldots,R_1)}Z_{m:m:n}\ .$$

Notice that this definition applies to *arbitrary* continuous distributions, and the constant c remains unchanged. It is clear that $n = m + R_1 + R_2 + \ldots + R_m$, and that if $R_1 = R_2 = \ldots = R_m = 0$ so that $m = n$, there is no withdrawal and the usual order statistics are obtained. We may also note here that, the largest progressive Type-II withdrawn order statistic is distributed as the largest standard order statistic from a sample of size n, since we know that $R_1 + R_2 + \ldots + R_m + m - 1 = n - 1$ items have already failed by this time. Also, if $R_1 = R_2 = \ldots = R_{m-1} = 0$ so that $R_m = n - m$, we are left with the case of conventional Type-II left censoring (hence the name progressive Type-II left withdrawal) wherein the largest m usual order statistics are observed.

4.2.3 Properties of Progressive Type-II Left Withdrawn Order Statistics

The following properties, which parallel those given in Theorems 4.1.1, 4.1.2 and 4.1.3, hold for progressive Type-II left withdrawn order statistics. We have removed the preceding superscript $(R_m, R_{m-1}, \ldots, R_1)$ from the notation for progressive Type-II left withdrawn order statistics, and for convenience, we have denoted R_i' for the term $n - R_1 - \ldots - R_{i-1} - i$.

Theorem 4.2.1: *(i) The distribution of* $Z_{m-i+1:m:n}$ *is free of* $R_i, R_{i+1}, \ldots, R_m$.

(ii) $Z_{m-i+1:m:n}, Z_{m-i+2:m:n}, \ldots, Z_{m:m:n}$ *form a progressive Type-II left withdrawn sample of size i from n items with withdrawal scheme* $(n-R_1-\ldots-R_{i-1}-i, R_{i-1}, \ldots, R_1) = (R_i', R_{i-1}, \ldots, R_1)$.

Theorem 4.2.2: *The progressive Type-II left withdrawn order statistics from an arbitrary continuous distribution form a "reverse" Markov chain; that is, given* $Z_{i:m:n}$, $Z_{j:m:n}$ $(j < i)$ *is independent of* $Z_{i+1:m:n}, Z_{i+2:m:n}, \ldots, Z_{m:m:n}$.

Theorem 4.2.3: *Given* $Z_{m-i+1:m:n} = z_{m-i+1}$, $Z_{1:m:n}, \ldots, Z_{m-i:m:n}$ $(i \geq 1)$ *are jointly distributed as a progressive Type-II left withdrawn sample of size m-i from* $n-R_1-R_2-\ldots-R_i-i$ *identically distributed random variables from the density f(x) right-truncated at* z_{m-i+1}, *that is, with density* $f(z)/F(z_{m-i+1})$ $(z < z_{m-i+1})$, *and with progressive withdrawal scheme* $(R_m, \ldots, R_{i+1})$.

Remark: Again, notice that the converse of Theorem 4.2.3 will not be true in general, since, even after the observation of the i^{th} progressive Type-II left withdrawn order statistic, say z_i, we may withdraw items which failed before the time z_i. This is because we only know that the withdrawn items failed at some point since time 0.

Remark: The proofs for Theorems 4.2.1, 4.2.2 and 4.2.3 parallel those for Theorems 4.1.1, 4.1.2 and 4.1.3, and hence are not presented here for brevity.

4.2.4. Moments of Progressive Type-II Right Censored Order Statistics From Symmetric Distributions

Govindarajulu (1963) established the following result for usual order statistics.

Theorem [Govindarajulu (1963)]: *We denote the order statistics for a random sample of size n from a population with cumulative distribution function F(x) and probability density function f(x) symmetric about 0 (without loss of generality) by* $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$. *Further, we denote the order statistics for a sample of size n from the folded population (folded at zero) with cumulative distribution function* $G(x) = 2F(x) - 1$ $(x \geq 0)$ *and probability density function* $g(x) = 2f(x)$ $(x \geq 0)$ *by* $Y_{1:n}, Y_{2:n}, \ldots, Y_{n:n}$. *Then, denoting* $E(X_{i:n}^k)$ *by* $\mu_{i:n}^{(k)}$ *and* $E(Y_{i:n}^k)$ *by* $\alpha_{i:n}^{(k)}$, *for* $i = 1, 2, \ldots, n$ *and* $k = 1, 2, \ldots,$

$$2^n \mu_{i:n}^{(k)} = \sum_{r=0}^{i-1} \binom{n}{r} \alpha_{i-r:n-r}^{(k)} + (-1)^k \sum_{r=i}^{n} \binom{n}{r} \alpha_{r-i+1:r}^{(k)} . \tag{4.2.2}$$

This result is proved by integrating the marginal probability density function of the i^{th} order statistic multiplied by x^k over the entire real line, splitting the integral into its positive and negative components, using the relationships

$$f(x) = f(-x) , \quad F(x) = 1 - F(-x) , \quad g(x) = 2f(x) , \quad G(x) = 2F(x) - 1 , \tag{4.2.3}$$

and then expanding certain terms binomially. A simple and interesting probabilistic proof of this result has also been given recently be Balakrishnan, Govindarajulu and Balasubramanian (1993).

We will generalize this result for progressive Type-II right censoring. Denote the sample of progressive Type-II right censored order statistics of size m with censoring scheme $(R_1, R_2, \ldots, R_m)$ from a sample of size n from a symmetric distribution by

$$X_{1:m:n}^{(R_1,R_2,\ldots,R_m)} , X_{2:m:n}^{(R_1,R_2,\ldots,R_m)} , \ldots , X_{m:m:n}^{(R_1,R_2,\ldots,R_m)} .$$

Denote the sample of progressive Type-II right censored order statistics of size m with censoring scheme $(R_1, R_2, \ldots, R_m)$ from a sample of size n from the distribution folded at zero by $Y^{(R_1,R_2,\ldots,R_m)}_{1:m:n}$, $Y^{(R_1,R_2,\ldots,R_m)}_{2:m:n}$, ..., $Y^{(R_1,R_2,\ldots,R_m)}_{m:m:n}$ and the sample of progressive Type-II left withdrawn order statistics of size m with withdrawal scheme $(R_m, R_{m-1}, \ldots, R_1)$ from a sample of size n from this distribution by ${}^{(R_m,\ldots,R_1)}Y_{1:m:n}$, ${}^{(R_m,\ldots,R_1)}Y_{2:m:n}$, ..., ${}^{(R_m,\ldots,R_1)}Y_{m:m:n}$. Further, let

$$E[X^{(R_1,\ldots,R_m)}_{i:m:n}]^k = \mu^{(R_1,\ldots,R_m)^{(k)}}_{i:m:n}, \quad E[Y^{(R_1,\ldots,R_m)}_{i:m:n}]^k = \alpha^{(R_1,\ldots,R_m)^{(k)}}_{i:m:n},$$

$$\text{and } E[{}^{(R_m,\ldots,R_1)}Y_{i:m:n}]^k = {}^{(R_m,\ldots,R_1)}\alpha^{(k)}_{i:m:n}.$$

Finally, the constant c in (1.4.3) and (4.2.1) can be rewritten as $(R_m + 1)(R_m + R_{m-1} + 2)\ldots(R_m + R_{m-1} + \ldots + R_1 + m)$. Thus, we will denote this quantity by $c_{R_m,R_{m-1},\ldots,R_1}$, and similarly $c_{R_m,R_{m-1},\ldots,R_i} = (R_m + 1)(R_m + R_{m-1} + 2) \ldots (R_m + R_{m-1} + \ldots + R_i + m - i + 1)$. The following result can now be derived.

Theorem 4.2.4: *For $i = 1, 2, \ldots, m \leq n$, and $k \geq 0$,*

$$2^n \mu_{i:m:n}^{(R_1,R_2,\ldots,R_m)^{(k)}} = c_{R_i',R_{i-1},\ldots,R_1} \left\{ \sum_{j=0}^{i-1} \left\{ \left[\sum_{l_j=0}^{R_j} \sum_{l_{j-1}=0}^{R_{j-1}} \cdots \sum_{l_1=0}^{R_1} \binom{R_j}{l_j} \binom{R_{j-1}}{l_{j-1}} \cdots \binom{R_1}{l_1} \frac{1}{c_{l_j,l_{j-1},\ldots,l_1}} \right] \cdot \frac{\alpha_{i-j:i-j:n-R_1-\ldots-R_j-j}^{(R_{j+1},\ldots,R_{i-1},R_i')^{(k)}}}{c_{R_i',R_{i-1},\ldots,R_{j+1}}} \right\} + (-1)^k \sum_{l_i=0}^{R_i'} \sum_{l_{i-1}}^{R_{i-1}} \cdots \sum_{l_1=0}^{R_1} \binom{R_i'}{l_i} \binom{R_{i-1}}{l_{i-1}} \cdots \binom{R_1}{l_1} \frac{{}^{(l_i,l_{i-1},\ldots,l_1)}\alpha_{1:i:l_1+\ldots+l_i+i}^{(k)}}{c_{l_i,l_{i-1},\ldots,l_1}} \right\}, \tag{4.2.4}$$

where $R_i' = n - R_1 - R_2 - \ldots - R_{i-1} - i$.

Proof: Using Theorem 4.1.1, we have $\mu_{i:m:n}^{(R_1,\ldots,R_m)^{(k)}} = \mu_{i:i:n}^{(R_1,\ldots,R_{i-1},R_i')^{(k)}}$. Thus, we begin with

$$\mu_{i:m:n}^{(R_1,\ldots,R_m)^{(k)}} = c_{R_i',R_{i-1},\ldots,R_1} \int \int \cdots \int_{-\infty < x_1 < \ldots < x_i < \infty} x_i^k f(x_1) [1-F(x_1)]^{R_1} \ldots \cdot f(x_{i-1}) [1-F(x_{i-1})]^{R_{i-1}} f(x_i) [1-F(x_i)]^{R_i'} dx_i \, dx_{i-1} \ldots dx_1 . \tag{4.2.5}$$

Now, the area of integration, $-\infty < x_1 < \ldots < x_i < \infty$, can be divided into $i + 1$ mutually exclusive and collectively exhaustive regions as follows:

$$\{-\infty < x_1 < x_2 < \ldots < x_i < \infty\}$$
$$= \{0 < x_1 < \ldots < x_i < \infty\}$$
$$\bigcup \{-\infty < x_1 < 0 \bigcap 0 < x_2 < \ldots < x_i < \infty\}$$
$$\bigcup \{-\infty < x_1 < x_2 < 0 \bigcap 0 < x_3 < \ldots x_i < \infty\}$$
$$\vdots$$
$$\bigcup \{-\infty < x_1 < \ldots < x_{i-1} < 0 \bigcap 0 < x_i < \infty\}$$
$$\bigcup \{-\infty < x_1 < \ldots < x_i < 0\} .$$

Thus, the right hand side of Eq. (4.2.5) can be written as, using (4.2.3),

$$c_{R_i', R_{i-1}, \ldots, R_1} \left[\left\{ \int_{0 \le x_1 < \ldots <} \int \cdots \int_{x_i \le \infty} x_i^k \, \frac{g(x_1)}{2} \left[\frac{1-G(x_1)}{2} \right]^{R_1} \cdots \frac{g(x_{i-1})}{2} \left[\frac{1-G(x_{i-1})}{2} \right]^{R_{i-1}} \right. \right.$$

$$\left. \cdot \, \frac{g(x_i)}{2} \left[\frac{1-G(x_i)}{2} \right]^{R_i'} dx_1 \ldots dx_i \right\}$$

$$+ \left\{ \int_{-\infty}^{0} f(x_1)[1-F(x_1)]^{R_1} dx_1 \cdot \int_{0 \le x_2 < \ldots <} \int \cdots \int_{x_i \le \infty} x_i^k \, \frac{g(x_2)}{2} \left[\frac{1-G(x_2)}{2} \right]^{R_2} \cdots \right.$$

$$\left. \cdot \, \frac{g(x_{i-1})}{2} \left[\frac{1-G(x_{i-1})}{2} \right]^{R_{i-1}} \frac{g(x_i)}{2} \left[\frac{1-G(x_i)}{2} \right]^{R_i'} dx_2 \ldots dx_i \right\}$$

$$+\left\{\int_{-\infty<x_1<x_2<0}\int f(x_1)[1-F(x_1)]^{R_1}f(x_2)[1-F(x_2)]^{R_2}dx_1dx_2\right.$$

$$\cdot\int_{0<x_3<\dots<x_i<\infty}\int\cdots\int x_i^k\,\frac{g(x_3)}{2}\left[\frac{1-G(x_3)}{2}\right]^{R_3}\cdots\frac{g(x_{i-1})}{2}\left[\frac{1-G(x_{i-1})}{2}\right]^{R_{i-1}}$$

$$\left.\cdot\,\frac{g(x_i)}{2}\left[\frac{1-G(x_i)}{2}\right]^{R_i'}dx_3\dots dx_i\right\}$$

$$+\ \dots$$

$$+\left\{\int_{-\infty<x_1<\dots<x_{i-1}<0}\int\cdots\int f(x_1)[1-F(x_1)]^{R_1}\dots f(x_{i-1})[1-F(x_{i-1})]^{R_{i-1}}dx_1\dots dx_{i-1}\right.$$

$$\left.\cdot\int_0^\infty x_i^k\,\frac{g(x_i)}{2}\left[\frac{1-G(x_i)}{2}\right]^{R_i'}dx_i\right\}$$

$$+\left\{\int_{-\infty<x_1<\dots<x_i<0}\int\cdots\int x_i^k f(x_1)[1-F(x_1)]^{R_1}\dots f(x_{i-1})[1-F(x_{i-1})]^{R_{i-1}}\right.$$

$$\left.\left.\cdot f(x_i)[1-F(x_i)]^{R_i'}dx_1\dots dx_i\right\}\right]$$

(4.2.6)

Now, making the substitution $y_p = -x_1$, $y_{p-1} = -x_2$, ..., $y_1 = -x_p$ in the integrals containing the probability density function $f(\cdot)$, where p is the dimension of the integral, (4.2.6) becomes

$$c_{R',R_{i-1},\ldots,R_1}\Bigg[\Bigg\{\frac{1}{2^n}\int_{0<x_1<}\int_{\ldots<}\cdots\int_{x_i<\infty} x_i^k\, g(x_1)\,[1-G(x_1)]^{R_1}\ldots g(x_{i-1})\,[1-G(x_{i-1})]^{R_{i-1}}$$

$$\cdot\; g(x_i)\,[1-G(x_i)]^{R'}dx_1\ldots dx_i\Bigg\}$$

$$+\;\Bigg\{\frac{1}{2^{n-R_1-1}}\int_0^\infty f(y_1)\,[F(y_1)]^{R_1}dy_1\;\cdot\int_{0<x_2<}\int_{\ldots<}\cdots\int_{x_i<\infty} x_i^k\, g(x_2)\,[1-G(x_2)]^{R_2}\ldots$$

$$\cdot\; g(x_{i-1})\,[1-G(x_{i-1})]^{R_{i-1}}g(x_i)\,[1-G(x_i)]^{R'}dx_2\ldots dx_i\Bigg\}$$

$$+\;\Bigg\{\frac{1}{2^{n-R_1-R_2-2}}\int_{0<y_1<}\int_{y_2<\infty} f(y_1)\,[F(y_1)]^{R_2}f(y_2)\,[F(y_2)]^{R_1}dy_1\,dy_2$$

$$\cdot\int_{0<x_3<}\int_{\ldots<}\cdots\int_{x_i<\infty} x_i^k\, g(x_3)\,[1-G(x_3)]^{R_3}\ldots g(x_{i-1})\,[1-G(x_{i-1})]^{R_{i-1}}$$

$$\cdot\; g(x_i)\,[1-G(x_i)]^{R'}dx_3\ldots dx_i\Bigg\}$$

$$+\;\ldots$$

$$+\;\Bigg\{\frac{1}{2^{n-R_1-\ldots-R_{i-1}-i+1}}\int_{0<y_1<}\int_{\ldots}\cdots\int_{<y_{i-1}<\infty} f(y_1)\,[F(y_1)]^{R_{i-1}}\ldots f(y_{i-1})\,[F(y_{i-1})]^{R_1}dy_1\ldots dy_{i-1}$$

$$\cdot\int_0^\infty x_i^k g(x_i)[1-G(x_i)]^{R'}dx_i\Bigg\}$$

$$+\;\Bigg\{\int_{0<y_1<}\int_{\ldots}\cdots\int_{<y_i<\infty}(-1)^k y_1^k f(y_1)\,[F(y_1)]^{R'}f(y_2)[F(y_2)]^{R_{i-1}}\ldots$$

$$\cdot\, f(y_i)\,[F(y_i)]^{R_1}dy_1\ldots dy_i\Bigg\}\Bigg]\;.$$

Finally, using (4.2.3), we write, for all terms containing the cumulative distribution function $F(\cdot)$,

$$[F(y_s)]^{R_s} = \left[\frac{1+G(y_s)}{2}\right]^{R_s} = \frac{1}{2^{R_s}} \sum_{l_s=0}^{R_s} \binom{R_s}{l_s} [G(y_s)]^{l_s} .$$

Upon interchanging the order of summations and integrations, the relation in (4.2.4) is obtained. □

Remark: Notice that, upon setting $R_1 = R_2 = \ldots = R_m = 0$, Eq. (4.2.4) simplifies to (4.2.2), since $\mu_{i:n}^{(k)} = \mu_{i:n:n}^{(0,0,\ldots,0)^{(k)}} = \mu_{i:i:n}^{(0,0,\ldots,0,n-i)^{(k)}}$, so that $R_i' = n-i$.

Similar arguments are used to generalize Govindarajulu's (1963) result for the product moments of usual order statistics from symmetric distributions given by

$$2^n \mu_{i,j:n} = \sum_{l=0}^{i-1} \binom{n}{l} \alpha_{i-l,j-l:n-l} - \sum_{l=i}^{j-1} \binom{n}{l} \alpha_{l-i+1:l}\alpha_{j-l:n-l} + \sum_{l=j}^{n} \binom{n}{l} \alpha_{l-j+1,l-i+1:l} , \tag{4.2.7}$$

where $\mu_{i,j:n} = E(X_{i:n}X_{j:n})$ is the $(i,j)^{th}$ product moment of the order statistics from the symmetric distribution, and $\alpha_{i,j:n} = E(Y_{i:n}Y_{j:n})$ is the $(i,j)^{th}$ product moment of the order statistics from the corresponding folded distribution.

Thus, for the progressive Type-II censored and withdrawn order statistics described earlier in this section, let us denote

$E[X_{i:m:n}^{(R_1,\ldots,R_m)} X_{j:m:n}^{(R_1,\ldots,R_m)}] = \mu_{i,j:m:n}^{(R_1,\ldots,R_m)}$, $E[Y_{i:m:n}^{(R_1,\ldots,R_m)} Y_{j:m:n}^{(R_1,\ldots,R_m)}] = \alpha_{i,j:m:n}^{(R_1,\ldots,R_m)}$

and $E[{}^{(R_m,\ldots,R_1)}Y_{i:m:n}\,{}^{(R_m,\ldots,R_1)}Y_{j:m:n}] = {}^{(R_m,\ldots,R_1)}\alpha_{i,j:m:n}$.

The result which generalizes the relation in Eq. (4.2.7) for the product moments of progressive Type-II censored order statistics is as follows.

Theorem 4.2.5: *For* $1 \le i < j \le m \le n$,

$$\begin{aligned}
2^n \mu_{i,j:m:n}^{(R_1,\ldots,R_m)} = c_{R_j',R_{j-1},\ldots,R_1} \Bigg\{ \sum_{l=0}^{i-1} \Bigg\{ \Bigg[\sum_{k_l=0}^{R_l} \sum_{k_{l-1}=0}^{R_{l-1}} \cdots \sum_{k_1=0}^{R_1} \binom{R_l}{k_l} \binom{R_{l-1}}{k_{l-1}} \cdots \binom{R_1}{k_1} \frac{1}{c_{k_l,k_{l-1},\ldots,k_1}} \Bigg] \\
\cdot \frac{\alpha_{i-l,j-l:j-l:n-R_1-\ldots-R_l-l}^{(R_{l+1},\ldots,R_{j-1},R_j')}}{c_{R_j',R_{j-1},\ldots,R_{l+1}}} \Bigg\} \\
- \sum_{l=i}^{j-1} \Bigg\{ \Bigg[\sum_{k_l=0}^{R_l} \sum_{k_{l-1}}^{R_{l-1}} \cdots \sum_{k_1=0}^{R_1} \binom{R_l}{k_l} \binom{R_{l-1}}{k_{l-1}} \cdots \binom{R_1}{k_1} \frac{{}^{(k_l,k_{l-1},\ldots,k_1)}\alpha_{l-i+1:l:k_1+\ldots+k_l+l}}{c_{k_l,k_{l-1},\ldots,k_1}} \Bigg] \\
\cdot \frac{\alpha_{j-l:j-l:n-R_1-\ldots-R_l-l}^{(R_{l+1},\ldots,R_{j-1},R_j')}}{c_{R_j',R_{j-1},\ldots,R_{l+1}}} \Bigg\} \\
+ \sum_{k_j=0}^{R_j'} \sum_{k_{j-1}=0}^{R_{j-1}} \cdots \sum_{k_1=0}^{R_1} \binom{R_j'}{k_j} \binom{R_{j-1}}{k_{j-1}} \cdots \binom{R_1}{k_1} \frac{{}^{(k_j,\ldots,k_1)}\alpha_{1,j-i+1:j:k_1+\ldots+k_j+j}}{c_{k_j,\ldots,k_1}} \Bigg\} .
\end{aligned} \tag{4.2.8}$$

Remark: The proof of this theorem is quite similar to that of Theorem 4.2.4; here, using Theorem 4.1.1, we have $\mu_{i,j:m:n}^{(R_1,\ldots,R_m)^{(k)}} = \mu_{i,j:j:n}^{(R_1,\ldots,R_{j-1},R_j')^{(k)}}$, and the area of integration, $-\infty < x_1 < \ldots < x_j < \infty$, is divided into $j + 1$ mutually exclusive and collectively exhaustive regions.

4.3 Method of Obtaining Moments

In this section, we give results for an arbitrary continuous distribution which allow us to obtain the moments of progressive Type-II left withdrawn order statistics if the moments of the progressive Type-II right censored order statistics are known, and vice versa. These results may be used if no better alternative method, such as recursion or explicit expressions, is available. Remember that in what follows, all expressions for single and product moments are for any *arbitrary* distribution, and not just symmetric distributions.

Theorem 4.3.1: *For $1 \leq i \leq m \leq n$,*

$$\mu_{i:m:n}^{(R_1,\ldots,R_m)^{(k)}} = c_{R_i',R_{i-1},\ldots,R_1} \sum_{j_i=0}^{R_i'} \sum_{j_{i-1}=0}^{R_{i-1}} \cdots \sum_{j_1=0}^{R_1} \binom{R_i'}{j_i} \binom{R_{i-1}}{j_{i-1}} \cdots \binom{R_1}{j_1} (-1)^{j_1+\cdots+j_i} \cdot \left[\frac{\mu_{j_1+\cdots+j_i+i:j_1+\cdots+j_i+i}^{(k)}}{c_{j_1,\ldots,j_i}} \right] . \tag{4.3.1}$$

Proof: Using the fact that $\mu_{i:m:n}^{(R_1,\ldots,R_m)} = \mu_{i:i:n}^{(R_1,\ldots,R_{i-1},R_i')}$ and expanding the terms containing 1 - $F(\cdot)$ in (4.2.5) binomially, upon interchanging the order of integrations and summations and simplifying, we have

$$\mu_{i:m:n}^{(R_1,\ldots,R_m)^{(k)}} = \mu_{i:i:n}^{(R_1,\ldots,R_{i-1},R_i')} = c_{R_i',R_{i-1},\ldots,R_1} \sum_{j_i=0}^{R_i'} \sum_{j_{i-1}=0}^{R_{i-1}} \cdots \sum_{j_1=0}^{R_1} \binom{R_i'}{j_i} \binom{R_{i-1}}{j_{i-1}} \cdots \binom{R_1}{j_1} (-1)^{j_1+\cdots+j_i}$$

$$\cdot \left[\frac{{}^{(j_1,\ldots,j_i)}\mu_{i:i:j_1+\cdots+j_i+i}^{(k)}}{c_{j_1,\ldots,j_i}} \right] .$$

But the largest progressive Type-II left withdrawn order statistic from a sample of size n is also the largest order statistic from a sample of size n, since we know that n-1 items have failed before this time. Eq. (4.3.1) then readily follows. □

Remark: In the special case when there is no censoring, this result reduces to a relation given by Srikantan (1962) for usual order statistics which is as follows:

$$\mu_{i:n}^{(k)} = \sum_{j=i}^{n} \binom{n}{j} \binom{j-1}{i-1} (-1)^{j-i} \mu_{j:j}^{(k)} ;$$

see also David (1981) and Arnold, Balakrishnan and Nagaraja (1992).

Proceeding in a manner similar to that of Theorem 4.3.1, we obtain the following result for product moments of progressive Type-II right censored order statistics.

Theorem 4.3.2: *For $1 \leq i < j \leq m \leq n$,*

$$\mu_{i,j:m:n}^{(R_1,\ldots,R_m)} = c_{R_j',R_{j-1},\ldots,R_1} \sum_{l_j=0}^{R_j'} \sum_{l_{j-1}=0}^{R_{j-1}} \cdots \sum_{l_1=0}^{R_1} \binom{R_j'}{l_j} \binom{R_{j-1}}{l_{j-1}} \cdots \binom{R_1}{l_1} (-1)^{l_1+\cdots+l_j} \cdot \left[\frac{{}^{(l_1,\ldots,l_j)}\mu_{i,j:j:l_1+\cdots+l_j+j}^{(k)}}{c_{l_1,\ldots,l_j}} \right] . \tag{4.3.2}$$

Thus, if we know the product moments of progressive Type-II left withdrawn order statistics from an arbitrary continuous distribution, we can calculate the product moments of progressive Type-II right censored order statistics.

Remark: For the special case when there is no censoring, Eq. (4.3.2) reduces to the following identity for usual order statistics:

$$\mu_{i,j:n} = \sum_{l=0}^{n-j} \sum_{k=0}^{l} (-1)^{l+k} \frac{n!}{(n-j-l)!\,(l-k)!\,(k+j)!} \mu_{i,j:k+j} .$$

Similar identities for usual order statistics have been established by, for example, Arnold and Balakrishnan (1989).

The following two theorems consider the converse result of Theorems 4.3.1 and 4.3.2.

Theorem 4.3.3: *For $1 \leq i \leq m \leq n$,*

$$
{}^{(R_m, R_{m-1}, \dots, R_1)}\mu_{i:m:n}^{(k)} = c_{R'_{m-i+1}, R_{m-i}, \dots, R_1} \sum_{j_{m-i+1}=0}^{R'_{m-i+1}} \sum_{j_{m-i}=0}^{R_{m-i}} \cdots \sum_{j_1=0}^{R_1} (-1)^{j_1+\dots+j_{m-i+1}} \begin{bmatrix} R'_{m-i+1} \\ j_{m-i+1} \end{bmatrix} \begin{bmatrix} R_{m-i} \\ j_{m-i} \end{bmatrix} \cdots \begin{bmatrix} R_1 \\ j_1 \end{bmatrix}
$$

$$
\cdot \left[\frac{\mu_{1:j_1+\dots+j_{m-i+1}+m-i+1}^{(k)}}{c_{j_1, \dots, j_{m-i+1}}} \right] . \qquad (4.3.3)
$$

Proof: From Theorem 4.1.1, we have ${}^{(R_m, \dots, R_1)}\mu_{i:m:n}^{(k)} = {}^{(R'_{m-i+1}, R_{m-i}, \dots, R_1)}\mu_{1:m-i+1:n}$.

Thus, we begin with

$$
{}^{(R_m, \dots, R_1)}\mu_{i:m:n}^{(k)} = c_{R'_{m-i+1}, R_{m-i}, \dots, R_1} \int\int \cdots \int_{-\infty < x_1 < \dots < x_{m-i+1} < \infty} x_1^k f(x_1)[F(x_1)]^{R'_{m-i+1}} f(x_2)[F(x_2)]^{R_{m-i}} \cdots
$$

$$
\cdot f(x_{m-i+1})[F(x_{m-i+1})]^{R_1} dx_1 \dots dx_{m-i+1} .
$$

Now, writing $F(x) = 1-[1-F(x)]$, expanding these terms binomially, interchanging the order of summations and integrations, and realizing that the first progressive Type-II right censored order statistic is also the first usual order statistic from a sample of size n, we obtain Eq. (4.3.3). □

Remark: If we consider the case of no censoring, Eq. (4.3.3) simplifies to a result for usual order statistics due to Srikantan (1962) given by

$$\mu_{i:n}^{(k)} = \sum_{j=n-i+1}^{n} \binom{n}{j} \binom{j-1}{n-i} (-1)^{j-n+i-1} \mu_{1:j}^{(k)} \ ;$$

see also David (1981) and Arnold, Balakrishnan and Nagaraja (1992).

Following lines similar to that of Theorem 4.3.3, we establish the following theorem for product moments of progressive Type-II left withdrawn order statistics.

Theorem 4.3.4: *For $1 \leq i < j \leq m \leq n$,*

$$^{(R_m,R_{m-1},\ldots,R_1)}\mu_{i\,j:m:n} = c_{R'_{m-i+1},R_{m-i},\ldots,R_1} \sum_{l_{m-i+1}=0}^{R'_{m-i+1}} \sum_{l_{m-i}=0}^{R_{m-i}} \cdots \sum_{l_1=0}^{R_1} (-1)^{l_1+\cdots+l_{m-i+1}} \binom{R'_{m-i+1}}{l_{m-i+1}} \binom{R_{m-i}}{l_{m-i}} \cdots \binom{R_1}{l_1} \cdot \left[\frac{\mu_{1\,j-i+1:m-i+1:l_1+\cdots+l_{m-i+1}+m-i+1}^{(l_{m-i+1},l_{m-i},\ldots,l_1)}}{c_{l_1,\ldots,l_{m-i+1}}} \right] . \tag{4.3.4}$$

Remark: For the special case of no censoring, Eq. (4.3.4) reduces to the following identity for usual order statistics:

$$\mu_{i,j:n} = \sum_{l=0}^{i-1} \sum_{k=0}^{l} (-1)^{l+k} \frac{n!}{(i-l-1)!\,(l-k)!\,(n-i+k-1)!} \mu_{k+1\,j-i+k+1:n-i+k+1} \, .$$

Final Remark: Many of the results obtained to this point generalize known results for usual order statistics to the case of progressive Type-II censored order statistics; however, it should be kept in mind that these are not just trivial generalizations of already known results. The methods of proof are quite different, and in some cases, the result for progressive Type-II censored order statistics was obtained not with the intention to generalize some particular result known for usual order statistics, but through necessity to proceed with the problem in mind. Often, the result in the case of usual order statistics is quite obvious, whereas in the progressive Type-II censored case it must be derived and proved before it is believed, at which point an intuitive argument may be made. This may have been how pioneers of the usual order statistics literature first perceived usual order statistics!

5 Applications to Inference

5.1 Best Linear Unbiased Estimation

5.1.1 Introduction

Consider an arbitrary "standard" continuous distribution $F(x)$. Then, if the single and product moments of the progressive Type-II censored order statistics from F are known, the best linear unbiased estimators for location and/or scale parameters may be obtained following the steps in, for example, Arnold, Balakrishnan and Nagaraja (1992, pp. 171-173). We will briefly outline the results.

If we are to consider the linear transformation $Y = \theta X$, where the vector X represents a vector of progressive Type-II censored order statistics from the standard distribution $F(x)$ (which we will assume to be of length p), then the best linear unbiased estimator of θ is obtained by minimizing with respect to θ the generalized variance

$$Q(\theta) = (Y-\theta\mu)'\Sigma^{-1}(Y-\theta\mu),$$

where μ is the mean vector of X and Σ is the variance - covariance matrix of X. The minimum occurs when $\theta^* = \frac{\mu'\Sigma^{-1}}{\mu'\Sigma^{-1}\mu}Y$. This is the best linear unbiased estimator of θ.

Suppose we believe the data to be represented by the linear transformation $Y = \theta_1 + \theta_2 X$. Then the best linear unbiased estimators of θ_1 and θ_2 will be obtained by minimizing the generalized variance

$$Q(\theta) = (Y-A\theta)'\Sigma^{-1}(Y-A\theta)$$

with respect to θ where $\theta = (\theta_1, \theta_2)'$, A is the $p \times 2$ matrix $(\mathbf{1}, \mu)$ and $\mathbf{1}$ is the $p \times 1$ vector with components all 1's. The minimum occurs when

$$\theta_1^* = -\mu'\Gamma Y \text{ and } \theta_2^* = \mathbf{1}'\Gamma Y,$$

where $\Gamma = \Sigma^{-1}(\mathbf{1}\mu' - \mu\mathbf{1}')\Sigma^{-1}/\Delta$ and $\Delta = (\mathbf{1}'\Sigma^{-1}\mathbf{1})(\mu'\Sigma^{-1}\mu) - (\mathbf{1}'\Sigma^{-1}\mu)^2$.

From these expressions for best linear unbiased estimators, variances and covariances of the estimators are readily obtained.

5.1.2 The Uniform Distribution: One-parameter Case

For a general progressive Type-II censored sample from the Uniform (0,1) distribution, as described in Chapter 2, let us denote the corresponding progressive Type-II censored order statistics by ${}_rU_{r+1:m:n}, {}_rU_{r+2:m:n}, \ldots, {}_rU_{m:m:n}$.

In Chapter 2, we found that

$$E({}_rU_{r+i:m:n}) = 1 - \prod_{j=m-i+1}^{m} \alpha_j \,, \quad i = 1, \ldots, m-r$$

$$Var({}_rU_{r+i:m:n}) = \left[\prod_{j=m-i+1}^{m} \alpha_j\right] \left[\prod_{j=m-i+1}^{m} \gamma_j - \prod_{j=m-i+1}^{m} \alpha_j\right], \quad i = 1, \ldots, m-r$$

and

$$Cov({}_rU_{r+i:m:n}, {}_rU_{r+k:m:n}) = \left[\prod_{j=m-i+1}^{m} \alpha_j\right] \left[\prod_{j=m-k+1}^{m} \gamma_j - \prod_{j=m-k+1}^{m} \alpha_j\right],$$

$$k > i \,, \quad i = 2, \ldots, m-r,$$

where α_j and γ_j, $j = r+1, \ldots, m$, are as defined in (2.3.1).

Upon making use of the above expressions, we see that the $(m\text{-}r) \times (m\text{-}r)$ variance-covariance matrix is of the special form $\sigma_{ik} = s_i t_k$ which can be inverted explicitly as given in Graybill (1983, pp. 198). The elements of the symmetric tri-diagonal inverted matrix (c_{ik}) are as follows:

$$c_{ii} = \frac{\gamma_{m-i}\gamma_{m-i+1} - \alpha_{m-i}\alpha_{m-i+1}}{\beta_{m-i}\beta_{m-i+1} \prod_{j=m-i+1}^{m} \alpha_j \gamma_j}, \quad i = 2, \ldots, m-r-1$$

$$c_{m-r,m-r} = \frac{1}{\beta_{r+1} \prod_{j=r+1}^{m} \alpha_j \prod_{j=r+2}^{m} \gamma_j} \tag{5.1.1}$$

$$c_{i,i+1} = \frac{-1}{\beta_{m-i} \prod_{j=m-i+1}^{m} \alpha_j \gamma_j}, \quad i = 1, 2, \ldots, m-r-1,$$

where β_j, $j = r+1, \ldots, m$, are as defined in (2.3.1).

Now, let us represent the general progressive Type-II censored order statistics with the same censoring scheme from the Uniform $(0,\theta)$ distribution by ${}_rY_{r+i:m:n}$, $i = 1, 2, \ldots, m-r$. Using (5.1.1), we can obtain the exact best linear unbiased estimator (BLUE) of θ. Explicit expressions of the BLUE [illegible] its variance are as follows:

$$\theta^* = \left\{ \sum_{i=1}^{m-r-1} \frac{\gamma_{m-i}\gamma_{m-i+1} - \alpha_{m-i}\alpha_{m-i+1} - \beta_{m-i+1} - \alpha_{m-i+1}\gamma_{m-i+1}\beta_{m-i}}{\beta_{m-i}\beta_{m-i+1}\prod_{j=m-i+1}^{m} \alpha_j\gamma_j} \, Y_{r+i:m:n} + \frac{(1-\alpha_{r+1})}{\beta_{r+1}\alpha_{r+1}\prod_{j=r+2}^{m} \alpha_j\gamma_j} \, Y_{m:m:n} \right\}$$

$$\times \left\{ \sum_{i=1}^{m-r-1} \frac{\gamma_{m-i}\gamma_{m-i+1} - \alpha_{m-i}\alpha_{m-i+1} - \beta_{m-i+1} - \alpha_{m-i+1}\gamma_{m-i+1}\beta_{m-i}}{\beta_{m-i}\beta_{m-i+1}\prod_{j=m-i+1}^{m} \alpha_j\gamma_j} (1 - \prod_{j=m-i+1}^{m} \alpha_j) + \frac{(1-\alpha_{r+1})(1-\prod_{j=r+1}^{m} \alpha_j)}{\beta_{r+1}\alpha_{r+1}\prod_{j=r+2}^{m} \alpha_j\gamma_j} \right\}^{-1} \quad (5.1.2)$$

and

$$Var(\theta^*) = \theta^2 \left\{ \sum_{i=1}^{m-r-1} \frac{\gamma_{m-i}\gamma_{m-i+1} - \alpha_{m-i}\alpha_{m-i+1} - \beta_{m-i+1} - \alpha_{m-i+1}\gamma_{m-i+1}\beta_{m-i}}{\beta_{m-i}\beta_{m-i+1}\prod_{j=m-i+1}^{m} \alpha_j\gamma_j} (1 - \prod_{j=m-i+1}^{m} \alpha_j) + \frac{(1-\alpha_{r+1})(1-\prod_{j=r+1}^{m} \alpha_j)}{\beta_{r+1}\alpha_{r+1}\prod_{j=r+2}^{m} \alpha_j\gamma_j} \right\}^{-1} . \quad (5.1.3)$$

Remark: It is of interest to note that the precision of θ^* in (5.1.3) does depend on the censoring scheme used, unlike in the one-parameter exponential case, as we will see shortly.

Remark: In the special case when there is no censoring, (that is, $m = n$, $r = 0$, $R_1 = \ldots = R_n = 0$), the following expressions are obtained:

$$\alpha_i = \frac{i}{i+1} \ , \quad \gamma_i = \frac{i+1}{i+2} \ , \quad \beta_i = \frac{1}{(i+2)(i+1)} \ , \quad i = 1, 2, \ldots, n \qquad (5.1.4)$$

so that

$$c_{ii} = 2(n+1)(n+2) \ , \quad i = 1, \ldots, n \ ,$$

$$c_{i,i+1} = -(n+1)(n+2) \ , \quad i = 1, \ldots, n-1 \ ,$$

and we have

$$\theta^{*} = \frac{n+1}{n} Y_{n:n} \qquad \text{and} \qquad Var(\theta^{*}) = \frac{\theta^2}{n(n+2)} \ ,$$

which is a well known result; see Arnold, Balakrishnan and Nagaraja (1992).

5.1.3 The Uniform Distribution: Two-parameter case

Exact expressions for the BLUE's of μ and σ in the two-parameter Uniform $(\mu, \mu+\sigma)$ distribution can also be obtained, as well as their variances and covariance. Let us denote the general progressive Type-II censored order statistics from the Uniform $(\mu,\mu+\sigma)$ distribution by ${}_rY_{r+1:m:n}, {}_rY_{r+2:m:n}, \ldots, {}_rY_{m:m:n}$.

Letting

$$\Delta = \left\{ \sum_{i=1}^{m-r-1} \frac{\gamma_{m-i}\gamma_{m-i+1} - \alpha_{m-i}\alpha_{m-i+1} - \beta_{m-i+1} - \alpha_{m-i+1}\gamma_{m-i+1}\beta_{m-i}}{\beta_{m-i}\beta_{m-i+1}\prod_{j=m-i+1}^{m}\alpha_j\gamma_j}(1 - \prod_{j=m-i+1}^{m}\alpha_j) \right.$$

$$\left. + \frac{(1-\alpha_{r+1})(1-\prod_{j=r+1}^{m}\alpha_j)}{\beta_{r+1}\alpha_{r+1}\prod_{j=r+2}^{m}\alpha_j\gamma_j} \right\}$$

$$\cdot \left\{ \sum_{i=1}^{m-r-1} \frac{\gamma_{m-i}\gamma_{m-i+1} - \alpha_{m-i}\alpha_{m-i+1} - 2\beta_{m-i+1}}{\beta_{m-i}\beta_{m-i+1}\prod_{j=m-i+1}^{m}\alpha_j\gamma_j} + \frac{1}{\beta_{r+1}\alpha_{r+1}\prod_{j=r+2}^{m}\alpha_j\gamma_j} \right\}$$

$$- \left\{ \sum_{i=1}^{m-r-1} \frac{\gamma_{m-i}\gamma_{m-i+1} - \alpha_{m-i}\alpha_{m-i+1} - \beta_{m-i+1} - \alpha_{m-i+1}\gamma_{m-i+1}\beta_{m-i}}{\beta_{m-i}\beta_{m-i+1}\prod_{j=m-i+1}^{m}\alpha_j\gamma_j} + \frac{1-\alpha_{r+1}}{\beta_{r+1}\alpha_{r+1}\prod_{j=r+2}^{m}\alpha_j\gamma_j} \right\}^2, \tag{5.1.5}$$

we obtain (after a great deal of algebra)

$$\sigma^*\Delta = \sum_{i=2}^{m-r-1} \frac{\gamma_{m-i}\gamma_{m-i+1}-\alpha_{m-i}\alpha_{m-i+1}-\beta_{m-i+1}-\alpha_{m-i+1}\gamma_{m-i+1}\beta_{m-i}}{\beta_{m-i}\beta_{m-i+1}\beta_m \prod_{j=m-i+1}^{m} \alpha_j\gamma_j} \, .Y_{r+i:m:n}$$

$$+ \frac{1-\alpha_{r+1}}{\beta_{r+1}\beta_m\alpha_{r+1}\prod_{j=r+2}^{m} \alpha_j\gamma_j} \, .Y_{m:m:n}$$

$$- \left\{ \sum_{i=2}^{m-r-1} \frac{\gamma_{m-i}\gamma_{m-i+1}-\alpha_{m-i}\alpha_{m-i+1}-\beta_{m-i+1}-\alpha_{m-i+1}\gamma_{m-i+1}\beta_{m-i}}{\beta_{m-i}\beta_{m-i+1}\beta_m \prod_{j=m-i+1}^{m} \alpha_j\gamma_j} \right.$$

$$\left. + \frac{1-\alpha_{r+1}}{\beta_{r+1}\beta_m\alpha_{r+1}\prod_{j=r+2}^{m} \alpha_j\gamma_j} \right\} .Y_{r+1:m:n} \tag{5.1.6}$$

and

$$Var(\sigma^*) = \frac{\sigma^2}{\Delta} \left\{ \sum_{i=1}^{m-r-1} \frac{\gamma_{m-i}\gamma_{m-i+1}-\alpha_{m-i}\alpha_{m-i+1}-2\beta_{m-i+1}}{\beta_{m-i}\beta_{m-i+1}\prod_{j=m-i+1}^{m} \alpha_j\gamma_j} + \frac{1}{\beta_{r+1}\alpha_{r+1}\prod_{j=r+2}^{m} \alpha_j\gamma_j} \right\} , \tag{5.1.7}$$

$$\mu^*\Delta = \left\{ \sum_{i=2}^{m-r-1} \frac{\gamma_{m-i}\gamma_{m-i+1} - \alpha_{m-i}\alpha_{m-i+1} - \beta_{m-i+1} - \alpha_{m-i+1}\gamma_{m-i+1}\beta_{m-i}}{\beta_{m-i}\beta_{m-i+1}\beta_m \prod_{j=m-i+1}^{m} \alpha_j\gamma_j} (1 - \prod_{j=m-i+1}^{m} \alpha_j) \right.$$

$$\left. + \frac{(1-\alpha_{r+1})(1 - \prod_{j=r+1}^{m} \alpha_j)}{\beta_{r+1}\beta_m\alpha_{r+1} \prod_{j=r+2}^{m} \alpha_j\gamma_j} \right\} , Y_{r+1:m:n}$$

$$- \left\{ \sum_{i=2}^{m-r-1} \frac{\gamma_{m-i}\gamma_{m-i+1} - \alpha_{m-i}\alpha_{m-i+1} - \beta_{m-i+1} - \alpha_{m-i+1}\gamma_{m-i+1}\beta_{m-i}}{\beta_{m-i}\beta_{m-i+1}\beta_m \prod_{j=m-i+1}^{m} \alpha_j\gamma_j} , Y_{r+i:m:n} + \right.$$

$$\left. \frac{1-\alpha_{r+1}}{\beta_{r+1}\beta_m\alpha_{r+1} \prod_{j=r+2}^{m} \alpha_j\gamma_j} , Y_{m:m:n} \right\} (1-\alpha_m) \tag{5.1.8}$$

and

$$Var(\mu^*) = \frac{\sigma^2}{\Delta} \left\{ \sum_{i=1}^{m-r-1} \frac{\gamma_{m-i}\gamma_{m-i+1} - \alpha_{m-i}\alpha_{m-i+1} - \beta_{m-i+1} - \alpha_{m-i+1}\gamma_{m-i+1}\beta_{m-i}}{\beta_{m-i}\beta_{m-i+1} \prod_{j=m-i+1}^{m} \alpha_j\gamma_j} (1 - \prod_{j=m-i+1}^{m} \alpha_j) \right.$$

$$\left. + \frac{(1-\alpha_{r+1})(1 - \prod_{j=r+1}^{m} \alpha_j)}{\beta_{r+1}\alpha_{r+1} \prod_{j=r+2}^{m} \alpha_j\gamma_j} \right\} . \tag{5.1.9}$$

Finally,

$$Cov(\mu^{*},\sigma^{*}) =$$

$$-\frac{\sigma^2}{\Delta}\left\{\sum_{i=1}^{m-r-1}\frac{\gamma_{m-i}\gamma_{m-i+1}-\alpha_{m-i}\alpha_{m-i+1}-\beta_{m-i+1}-\alpha_{m-i+1}\gamma_{m-i+1}\beta_{m-i}}{\beta_{m-i}\beta_{m-i+1}\prod_{j=m-i+1}^{m}\alpha_j\gamma_j}+\frac{1-\alpha_{r+1}}{\beta_{r+1}\alpha_{r+1}\prod_{j=r+2}^{m}\alpha_j\gamma_j}\right\}. \tag{5.1.10}$$

Remark: For the case of no censoring, upon using (5.1.4), the expressions given above simplify to the well known results

$$\sigma^{*} = \frac{n+1}{n-1}(Y_{n:n}-Y_{1:n}) \ , \quad \mu^{*} = \frac{1}{n-1}(nY_{1:n}-Y_{n:n}) \ ,$$

$$Var(\sigma^{*}) = \frac{2\sigma^2}{(n+2)(n-1)} \ , \quad Var(\mu^{*}) = \frac{n\sigma^2}{(n+1)(n+2)(n-1)} \ ,$$

$$Cov(\mu^{*},\sigma^{*}) = \frac{-\sigma^2}{(n+2)(n-1)} \ .$$

5.1.4 The Exponential Distribution: One-parameter case

For a general progressive Type-II censored sample from the standard exponential distribution, as described in Chapter 3, let us denote the corresponding progressive Type-II censored order statistics by ${}_rX_{r+1:m:n}$, ${}_rX_{r+2:m:n}$, $\ldots$, ${}_rX_{m:m:n}$.

Now, let us represent the general progressive Type-II censored order statistics with the same censoring scheme from the exponential (σ) distribution, that is, with probability density function

$$f(y;\sigma) = \frac{1}{\sigma} e^{-y/\sigma} , \quad y \geq 0 , \quad \sigma > 0 \tag{5.1.11}$$

by ${}_rY_{r+i:m:n}$, $i = 1, 2, ..., m-r$. Using the expressions for the means, variances, and covariances of ${}_rX_{r+1:m:n}, {}_rX_{r+2:m:n}, ..., {}_rX_{m:m:n}$ obtained in section 3.3, we see that the $(m-r) \times (m-r)$ variance-covariance matrix is again of the special form $\sigma_{ij} = s_i t_j$ (where $t_j = 1$ for all j) which can be inverted explicitly as given in Graybill (1983, pp. 198). We can therefore obtain the exact best linear unbiased estimator (BLUE) of σ. Explicit expressions of the BLUE and its variance are as follows:

$$\sigma^* = \frac{1}{m-r-1+\dfrac{\alpha_{r+1}^2}{\beta_{r+1}}} \left[\sum_{j=r+2}^{m} (R_j+1) \left({}_rY_{j:m:n} - {}_rY_{r+1:m:n} \right) + \left(\frac{\alpha_{r+1}}{\beta_{r+1}} \right) {}_rY_{r+1:m:n} \right] \tag{5.1.12}$$

and

$$Var(\sigma^*) = \frac{\sigma^2}{m-r-1+\dfrac{\alpha_{r+1}^2}{\beta_{r+1}}} , \tag{5.1.13}$$

where, as in section 3.3,

$$\alpha_{r+1} = \sum_{i=1}^{r+1} \frac{1}{n-i+1} \qquad \text{and} \qquad \beta_{r+1} = \sum_{i=1}^{r+1} \frac{1}{(n-i+1)^2} \ .$$

Remark: It is of interest to observe that the precision of the BLUE of σ in (5.1.13) depends only on r, m and n, and not on the progressive censoring scheme $(R_{r+1}, \ldots, R_m)$.

5.1.5 The Exponential Distribution: Two-parameter case

Consider the two-parameter exponential distribution with probability density function

$$f(y;\mu,\sigma) = \frac{1}{\sigma} e^{-\left(\frac{y-\mu}{\sigma}\right)} , \quad y \geq \mu , \quad \sigma > 0 . \tag{5.1.14}$$

Let us denote the general progressive Type-II censored sample from this distribution by ${}_rY_{r+i:m:n}$, $i = 1, 2, \ldots, m-r$. Then, again using the moments for the standard exponential distribution obtained in section 3.3 and the formulae for BLUE's presented in subsection 5.1.1, we derive, after much algebraic simplification, the BLUE's of μ and σ to be

$$\mu^* = {}_rY_{r+1:m:n} - \frac{\alpha_{r+1}}{m-r-1} \sum_{i=r+2}^{m} (R_i+1) \left({}_rY_{i:m:n} - {}_rY_{r+1:m:n}\right) \tag{5.1.15}$$

and

$$\sigma^* = \frac{1}{m-r-1}\sum_{i=r+2}^{m}(R_i+1)\ ({}_rY_{i:m:n} - {}_rY_{r+1:m:n})\ , \qquad (5.1.16)$$

respectively. Further, the variances and covariance of these estimators are given by

$$Var(\mu^*) = \sigma^2\left[\frac{\alpha_{r+1}^2}{m-r-1}+\beta_{r+1}\right]\ , \qquad (5.1.17)$$

$$Var(\sigma^*) = \frac{\sigma^2}{m-r-1}\ , \qquad (5.1.18)$$

and

$$Cov(\mu^*,\sigma^*) = -\sigma^2\frac{\alpha_{r+1}}{m-r-1}\ . \qquad (5.1.19)$$

Remark: By writing σ^* in (5.1.16) equivalently as

$$\sigma^* = \frac{1}{m-r-1}\sum_{j=2}^{m-r}(n-r-R_{r+1}-\ldots-R_{r+j-1}-j+1)({}_rY_{r+j:m:n} - {}_rY_{r+j-1:m:n}) \qquad (5.1.20)$$

and making use of the fact that the spacing $(n\text{-}r\text{-}R_{r+1}\text{-}R_{r+2}\text{-}\ldots\text{-}R_{r+j-1}\text{-}j+1)({}_rY_{r+j:m:n}\text{-}{}_rY_{r+j-1:m:n})/\sigma, j = 2, \ldots, m$ are independent standard exponential random variables, we have $2(m\text{-}r\text{-}1)\sigma^*/\sigma$ to be distributed exactly as a central chi-square random variable with $2(m\text{-}r\text{-}1)$ degrees of freedom. This fact can be used to develop confidence intervals or tests of hypothesis about σ. However, no such exact distributional result can be obtained for the pivotal quantity $(\mu^*\text{-}\mu)/\sigma^*$ in this general case; for the special case $r = 0$, this pivotal quantity can be shown to have an F-distribution; see Viveros and Balakrishnan (1994).

Remark: It is important to note here that the precision of the BLUE's of μ and σ, as given by (5.1.17)-(5.1.19), depend only on r, m, and n, and not on the progressive censoring scheme $(R_{r+1}, \ldots, R_m)$.

Numerical Example 5.1.1:

For the purpose of illustration, let us consider Nelson's data (1982, p. 228, Table 6.1) which gives observations on times to breakdown of an insulating fluid in an accelerated test conducted at various test voltages. In analyzing the complete data, Nelson assumed a scaled Weibull distribution for the times to breakdown (from the 90% confidence interval [0.459, 1.381] that he determined for the shape parameter, it is quite clear that an exponential model is also appropriate for this data). For the purpose of illustrating the methods of inference presented in this article, we generated the following general progressive Type-II censored sample from the $n = 19$ observations recorded at 34 kV in Nelson's Table 6.1 (with one smallest observation censored and three stages of progressive censoring).

i	1	2	3	4	5	6	7	8
x_i	-	0.78	0.96	1.31	2.78	4.85	6.50	7.35
R_i	-	0	3	0	3	0	0	5

If we assume a one-parameter exponential distribution for the data at hand, we get from (5.1.12) and (5.1.13) that $\sigma^* = 9.110$ and $SE(\sigma^*) = \sigma^*/\sqrt{7.99837} = 3.221$. If, instead, we assume a two-parameter exponential distribution for the data at hand, we get from (5.1.15) and (5.1.16) that $\mu^* = -0.274$ and $\sigma^* = 9.743$, and their standard errors to be $SE(\mu^*) = \sigma^*\sqrt{0.0078} = 0.861$ and $SE(\sigma^*) = \sigma^*/\sqrt{6} = 3.978$. From these values, we get the BLUE of the expected time to breakdown as $\mu^* + \sigma^* = 9.469$ with its standard error being 3.735. We may note that the standard error in this case is larger than in the one-parameter case (the negative value of μ^* should probably indicate here that the two-parameter model is not quite appropriate). □

Numerical Example 5.1.2:

Using the simulation algorithm described in subsection 2.2.4, the following general progressive Type-II censored sample was generated (with $\mu = 25$ and $\sigma = 10$):

$n=50$; $r=5$; $R_6=2$; $R_7=1$; $R_8=0$; $R_9=2$; $R_{10}=1$; $R_{11}=3$; $R_{12}=2$; $R_{13}=3$;

$R_{14}=4$; $R_{15}=3$; $R_{16}=0$; $R_{17}=2$; $R_{18}=1$; $R_{19}=2$; $R_{20}=4$.

The sample observed (of size 15) was:

25.99609, 26.17323, 26.55884, 26.65558, 27.32842, 27.52826,
28.58114, 28.58350, 28.68850, 29.09515, 29.17521, 29.47387,
29.61337, 33.44267, 35.74206.

The BLUE's are obtained as $\mu^* = 24.66292$ and $\sigma^* = 10.54061$ with standard errors $\text{SE}(\mu^*) = 10.54061\sqrt{.00381} = 0.65082$ and $\text{SE}(\sigma^*) = 2.81710$. These estimated values are very close to the corresponding actual values. □

5.1.6 The Laplace Distribution

Using the results presented in section 4.2 for the single and product moments of progressive Type-II right censored order statistics from arbitrary continuous symmetric distributions, one may determine the best linear unbiased estimators (BLUE's) for parameters of scale and location-scale shifted distributions when samples are progressive Type-II right censored from symmetric distributions, provided the single and product moments of progressive Type-II right censored and progressive Type-II left withdrawn order statistics from the corresponding folded distribution are known.

We can therefore consider the Laplace (or double exponential) distribution with location parameter μ and scale parameter σ, whose cumulative distribution function is given by

$$F(x) = \frac{1}{2} e^{(x-\mu)/\sigma}, \qquad x \leq \mu,$$
$$= 1 - \frac{1}{2} e^{-(x-\mu)/\sigma}, \quad x \geq \mu. \qquad (5.1.21)$$

This distribution has been used to model certain real life-test data [see Bain and Engelhardt (1991)].

Numerical Example 5.1.3:

A progressive Type-II right censored sample of size $m = 10$ from a sample of size $n = 20$ from the distribution given in (5.1.21) with $\mu = 25$ and $\sigma = 5$ was simulated, with censoring scheme $\boldsymbol{R} = (2,0,0,2,0,0,0,2,0,4)$. This sample was simulated using the algorithm given in subsection 2.1.3. The simulated progressive Type-II right censored sample is as follows:

19.21167876, 21.97364262, 23.41776818, 23.66253070, 23.80222832, 24.23017797, 25.62072188, 25.86990938, 26.47997028, 27.55344134.

Using the results for moments of progressive Type-II right censored order statistics from the exponential distribution, which can be obtained by setting $r = 0$ in the expressions for moments given in section 3.3, as well as (4.3.3) and (4.3.4), moments of progressive Type-II left withdrawn order statistics from the exponential distribution were calculated for the censoring scheme $\boldsymbol{R}$. It may be noted here that an alternate method of obtaining these moments is through recursive algorithms. Recurrence relations for moments of progressive Type-II left withdrawn order statistics will be very similar to those for moments of progressive Type-II right censored order statistics given in section 3.4.

Using (4.2.4) and (4.2.8), we are now able to calculate the desired single and product moments, and therefore variances and covariances, of progressive Type-II right censored order statistics with censoring scheme $\boldsymbol{R}$ from the Laplace distribution. This was done using Maple V, Release 3, with exact results. Using these results and the formulae for best linear unbiased estimators for location and scale parameters given earlier, the best linear unbiased estimators for μ and σ were obtained. The coefficients $a_{i:m:n}$ for the BLUE of μ, given to 10 decimal places for convenience, are given by

-0.0215281009, 0.0098155747, 0.0069570375, -0.0901429654, 0.2068718600,
0.0772377616, 0.1039355790, 0.2099103064, 0.1371049958, 0.3598379513,

and the coefficients $b_{i:m:n}$ for the BLUE of σ are given by

-0.1310403616, -0.0930984026, -0.0957542723, -0.2299462758, 0.0918178668,

-0.0494235976, -0.0299327584, 0.0453859038, 0.0594679977, 0.4325239001.

The variance of the BLUE of μ is $0.0750292328\sigma^2$, the variance of the BLUE of σ is $0.1097132433\sigma^2$, and the covariance of the BLUE's is $0.0108583470\sigma^2$.

Thus, for the simulated sample, the BLUE's of μ and σ and their standard errors are given by:

$\mu^* = 26.26607179$, $SE(\mu^*) = 0.7233296358$;

$\sigma^* = 2.640711801$, $SE(\sigma^*) = 0.2897210562$. □

Remark: Using the results given in Govindarajulu (1966), the variances and covariance of the BLUE's for a conventional Type-II right censored sample of size 10 from a sample of size $n = 20$ from a Laplace distribution were also calculated. In this case, the censoring pattern can be written as (0,0,0,0,0,0,0,0,0,10). The variance of the BLUE of μ is $0.0700\sigma^2$, the variance of the BLUE of σ is $0.1095\sigma^2$ and the covariance of the two BLUE's is $0.0133\sigma^2$. These variances are only slightly more favourable than the variances given above for the censoring pattern (2,0,0,2,0,0,0,2,0,4). However, for this censoring scheme (R), items censored early on may be of use to the experimenter. The question of an optimal censoring pattern in terms of the variances of BLUE's will be addressed further in Chapter 6 of this thesis.

5.2 Maximum Likelihood Estimation

5.2.1 Introduction

Maximum likelihood estimation for parameters when samples are progressive Type-II right censored has been discussed for a number of distributions by, for example, Cohen (1963, 1966, 1975, 1976) and Cohen and Norgaard (1977). In the case of general progressive Type-II censored samples, the likelihood equation to be maximized will be that given in (1.4.3) for an arbitrary continuous distribution $F(x)$, namely,

$$L(\theta) = c'[F(X_{r+1:m:n})]^r \prod_{i=r+1}^{m} f(X_{i:m:n})[1-F(X_{i:m:n})]^{R_i}, \tag{5.2.1}$$

where c' is given in (1.4.3) and θ is the vector of parameter(s) whose maximum likelihood estimates we would like to find.

5.2.2 The Uniform Distribution

In subsection 5.1.2, we considered the best linear unbiased estimation of the one-parameter Uniform $(0,\theta)$ distribution, which we denoted by ${}_rY_{r+i:m:n}$, $i = 1, 2, \ldots, m\text{-}r$. In this case, the maximum likelihood estimator (MLE) of θ does not exist in an explicit form and has to be determined from the likelihood equation by a numerical method. The equation to be solved numerically for θ is given by

$$\sum_{i=r+1}^{m} \frac{R_i\,{}_rY_{i:m:n}}{\theta - {}_rY_{i:m:n}} = m\ , \qquad \theta \geq {}_rY_{m:m:n}\ , \tag{5.2.2}$$

Notice that at $\theta = {}_rY_{m:m:n}$, the left-hand side of (5.2.2) is infinite, while as θ goes to infinity, the left-hand side goes to 0. Furthermore, the left-hand side of (5.2.2) is monotonically decreasing in θ between ${}_rY_{m:m:n}$ and ∞. Thus, there must be a unique solution to this equation in $({}_rY_{m:m:n}, \infty)$.

In the two-parameter Uniform $(\mu, \mu+\sigma)$ distribution considered in subsection 5.1.3, the two equations to be solved simultaneously are:

$$\begin{aligned} \sum_{i=r+1}^{m} \frac{R_i}{\sigma+\mu-{}_rY_{i:m:n}} &= \frac{r}{{}_rY_{r+1:m:n}-\mu}\ , \qquad \mu \leq {}_rY_{r+1:m:n}\ , \\ \sum_{i=r+1}^{m} \frac{R_i\,{}_rY_{i:m:n}}{\sigma+\mu-{}_rY_{i:m:n}} &= \frac{\mu r}{{}_rY_{r+1:m:n}-\mu}+m\ , \qquad \mu+\sigma \geq {}_rY_{m:m:n}\ , \end{aligned} \tag{5.2.3}$$

where ${}_rY_{r+1:m:n}$, ${}_rY_{r+2:m:n}$, $\ldots$, ${}_rY_{m:m:n}$ denote the general progressive Type-II censored order statistics from the Uniform (μ, $\mu + \sigma$) distribution. However, in the special case when $r = 0$, so that we are left with a progressive Type-II right censored sample $Y_{i:m:n}$ ($i = 1, ..., m$), the MLE of μ is given by $\hat{\mu} = {}_rY_{r+1:m:n} = Y_{1:m:n} = Y_{1:n}$, and the following equation must then be solved for σ:

$$\sum_{i=2}^{m} \frac{R_i(Y_{i:m:n}-Y_{1:m:n})}{\sigma-(Y_{i:m:n}-Y_{1:m:n})} = m \ , \quad \sigma \geq Y_{m:m:n}-Y_{1:m:n} \ ,$$

which is very similar to (5.2.2). Using similar arguments then, we conclude that there must be exactly one solution to this equation in $((Y_{m:m:n} - Y_{1:m:n}), \infty)$

5.2.3 The Exponential Distribution

In subsection 5.1.4, best linear unbiased estimation for the one-parameter exponential distribution with probability density function given by (5.1.11) was considered. For the general progressive Type-II censored sample from the one-parameter exponential distribution (${}_rY_{r+i:m:n}$, $i = 1, ..., m-r$), the MLE of σ does not exist in an explicit form and has to be determined from the likelihood equation by a numerical method. The equation to be solved numerically for σ is

$$ {}_rY_{r+1:m:n} \frac{r e^{\frac{-{}_rY_{r+1:m:n}}{\sigma}}}{1-e^{\frac{-{}_rY_{r+1:m:n}}{\sigma}}} = -\sigma(m-r)+\sum_{i=r+1}^{m} (R_i+1)\,{}_rY_{i:m:n} \ . \qquad (5.2.4)$$

In the special case when $r = 0$, however, the MLE of σ is given explicitly by

$$\hat{\sigma} = \frac{1}{m}\left[nY_{1:m:n} + \sum_{j=1}^{m}(R_j+1)\ (Y_{j:m:n} - Y_{1:m:n})\right] = \frac{1}{m}\sum_{j=1}^{m}(R_j+1)Y_{j:m:n}$$

which becomes identical with the BLUE of σ in (5.1.12).

In subsection 5.1.5, best linear unbiased estimation for the two-parameter exponential distribution with probability density function given by (5.1.14) was considered. Denoting the general progressive Type-II censored order statistics from the two-parameter exponential distribution again by ${}_rY_{r+i:m:n}$, $i = 1, \ldots, m-r$, the log-likelihood function based on this general progressive Type-II censored sample is given by

$$\ln L = Constant - (m-r)\ln\sigma + r\ln\left[1 - e^{-\frac{{}_rY_{r+1:m:n}-\mu}{\sigma}}\right] - \sum_{i=r+1}^{m}(R_i+1)\left[\frac{{}_rY_{i:m:n}-\mu}{\sigma}\right] \cdot \quad (5.2.5)$$

For the special case when $r = 0$, the log-likelihood function in (5.2.5) is monotonically increasing in μ so that the MLE of μ is $\hat{\mu} = Y_{1:n}$. In this case, the MLE of σ is given by

$$\hat{\sigma} = \frac{1}{m}\sum_{i=2}^{m}(R_i+1)\ \left(Y_{i:m:n} - Y_{1:m:n}\right)\ .$$

It may be noted that the MLE's $\hat{\mu}$ and $\hat{\sigma}$ are both biased.

For the case when $r > 0$, even though it initially appears from (5.2.5) that the MLE's of μ and σ may not exist explicitly, it turns out that the MLE's of μ and σ take on simple linear forms. To see this, we have the likelihood equations for μ and σ from (5.2.5) to be

$$\frac{\partial \ln L}{\partial \mu} = -\frac{r}{\sigma}\left[\frac{e^{\frac{-({}_rY_{r+1:m:n}-\mu)}{\sigma}}}{1-e^{\frac{-({}_rY_{r+1:m:n}-\mu)}{\sigma}}}\right] + \frac{1}{\sigma}\sum_{i=r+1}^{m}(R_i+1) = 0 \tag{5.2.6}$$

and

$$\frac{\partial \ln L}{\partial \sigma} = -\frac{m-r}{\sigma} - \frac{r}{\sigma}\left[\frac{{}_rY_{r+1:m:n}-\mu}{\sigma}\right]\left[\frac{e^{\frac{-({}_rY_{r+1:m:n}-\mu)}{\sigma}}}{1-e^{\frac{-({}_rY_{r+1:m:n}-\mu)}{\sigma}}}\right] - \frac{(n-r)\mu}{\sigma^2} + \frac{1}{\sigma^2}\sum_{i=r+1}^{m}(R_i+1)\,{}_rY_{i:m:n} = 0. \tag{5.2.7}$$

Eq. (5.2.6) simplifies to

$$r\frac{e^{\frac{-({}_rY_{r+1:m:n}-\mu)}{\sigma}}}{1-e^{\frac{-({}_rY_{r+1:m:n}-\mu)}{\sigma}}} = n-r$$

which immediately yields the MLE of μ to be

$$\hat{\mu} = {}_rY_{r+1:m:n} + \sigma \ln\left[1-\frac{r}{n}\right]. \tag{5.2.8}$$

Eq. (5.2.7), in conjunction with the MLE of μ in (5.2.8), reduces to

$$-\frac{m-r}{\sigma} - \frac{n-r}{\sigma^2}\,{}_rY_{r+1:m:n} + \frac{1}{\sigma^2}\sum_{i=r+1}^{m}(R_i+1)\,{}_rY_{i:m:n} = 0$$

which readily yields the MLE of σ to be

$$\hat{\sigma} = \frac{1}{m-r} \sum_{i=r+2}^{m} (R_i+1) \left({}_rY_{i:m:n} - {}_rY_{r+1:m:n} \right) . \tag{5.2.9}$$

Remark: The derivation of the MLE's of μ and σ in (5.2.8) and (5.2.9) for the general progressive Type-II censored sample case generalizes the results of Cohen for the special case of progressive Type-II right-censored sample ($r=0$). It is also of interest to mention here that the results of Kambo (1978) for the case of doubly Type-II censored samples may be deduced as a special case from (5.2.8) and (5.2.9).

Remark: Comparing the MLE's in (5.2.8) and (5.2.9) with the BLUE's in (5.1.15) and (5.1.16), it is quite clear that the BLUE's are simply the MLE's adjusted for their bias. As a result, the expressions of the variances and covariance of the BLUE's presented in (5.1.17) - (5.1.19) may suitably be used to derive expressions for the variances and covariance of the MLE's of μ and σ given in (5.2.8) and (5.2.9). These quantities turn out to be

$$Var(\hat{\mu}) = \left\{ \beta_{r+1} + \left[\ln\left(1-\frac{r}{n}\right) / (m-r) \right]^2 (m-r-1) \right\} \sigma^2 \text{ and } Var(\hat{\sigma}) = \frac{m-r-1}{(m-r)^2}\sigma^2$$

respectively.

Remark: Due to the equivalence of the MLE's and the BLUE's in this case, the inference developed in subsection 5.1.5 for parameters μ and σ using pivotal quantities based on the BLUE's will be the same as those based on the MLE's. For the special case of progressive Type-II right-censored samples, this has been noted by Viveros and Balakrishnan (1994).

Remark: For large m, it is clear that the MLE $\hat{\sigma}$ in (5.2.9) approaches the BLUE σ^* in (5.1.16). Furthermore, since

$$\alpha_{r+1} = \sum_{i=1}^{r+1} \frac{1}{n-i+1} = \sum_{i=1}^{n} \frac{1}{i} - \sum_{i=1}^{n-r-1} \frac{1}{i}$$

and

$$\lim_{N \to \infty} \left[\sum_{k=1}^{n} \frac{1}{k} - \ln N \right] = \gamma \quad \text{(Euler's constant)}$$

[see Abramowitz and Stegun (1965, p. 68)], the MLE $\hat{\mu}$ in (5.2.8) becomes identical with the BLUE μ^* in (5.1.15) for large m.

Numerical Example 5.2.1

In Numerical Example 5.1.1, we considered Nelson's data on times to breakdown, where a one-parameter exponential model seemed appropriate. The one-parameter BLUE in that case was found to be $\sigma^* = 9.110$ with a standard error of 3.221. Using (5.2.4), the MLE of σ is found numerically to be $\hat{\sigma} = 9.111$. □

Numerical Example 5.2.2

In Numerical Example 5.1.2, a general progressive Type-II censored sample from the two-parameter exponential distribution was generated using the simulational algorithm given in subsection 2.2.4. The BLUE's in this case were obtained as $\mu^* = 24.66292$ and $\sigma^* = 10.54061$ with standard errors $SE(\mu^*) = 0.65082$ and $SE(\sigma^*) = 2.81710$. Using (5.1.15) and (5.1.16), the MLE's are obtained as $\hat{\mu} = 24.95956$ and $\hat{\sigma} = 9.83791$ with standard errors $SE(\hat{\mu}) = 10.54061\sqrt{.00360} = 0.61102$ and $SE(\hat{\sigma}) = 2.62929$. In both cases, the estimated values are very close to the corresponding actual values. Furthermore, the two methods of estimation, BLUE and MLE, reveal very similar estimates and standard errors. □

5.2.4 The Laplace Distribution

For the Laplace distribution with cdf given by (5.1.21) and pdf given by

$$f(x) = \frac{1}{2\sigma} e^{-|x-\mu|/\sigma}, \quad -\infty < x < \infty, \ -\infty < \mu < \infty, \ \sigma > 0, \qquad (5.2.10)$$

the maximum likelihood estimation based on complete samples has been discussed by a number of authors; see, for example, Johnson, Kotz and Balakrishnan (1995). Recently, Balakrishnan and Cutler (1995) have discussed maximum likelihood estimation for parameters of the Laplace distribution based on conventional Type-II censored samples. They consider both symmetric and onesided (right) censoring.

In this subsection, we derive the maximum likelihood estimators of the location and scale parameters of a Laplace distribution based on progressive Type-II right censored samples. The results obtained are generalizations of those given in Balakrishnan and Cutler (1995), where it is shown that for conventional Type-II right censored samples, where only the first m failure times are observed, the maximum likelihood estimator of μ is simply the usual sample median based on the full sample, provided $m \geq n/2$. For $m < n/2$, the MLE of μ turns out to be a linear function of the observed order statistics. In both cases, the MLE of σ is a linear function of the observed order statistics. The results presented in this subsection for maximum likelihood estimation based on progressive Type-II right censored samples from the Laplace distribution simplify to those presented by Balakrishnan and Cutler (1995) for the special case when $R_1 = R_2 = \ldots = R_{m-1} = 0$ and $R_m = n - m$, in which case we are left with a conventional progressive Type-II right censored sample.

To begin, consider a progressive Type-II right censored sample of size m with censoring scheme $(R_1, R_2, \ldots, R_m)$ from a random sample of size n from the Laplace distribution. Denote this progressive Type-II right censored sample $X_{i:m:n}$, $i = 1, \ldots, m$.

The likelihood function, L, for a progressive Type-II right censored sample is given by

$$L(\mu,\sigma) = c\prod_{i=1}^{m} f(X_{i:m:n})[1-F(X_{i:m:n})]^{R_i}, \tag{5.2.11}$$

where c is given in (1.4.2), and $f(\cdot)$ and $F(\cdot)$ are given in (5.2.10) and (5.1.21), respectively. We will first maximize with respect to μ.

Notice that, for values of $\mu < X_{1:m:n}$, the likelihood function reduces to an expression that is proportional to

$$L_0(\mu,\sigma) = \frac{1}{\sigma^m} e^{-\sum_{i=1}^{m}(R_i + 1)(X_{i:m:n} - \mu)/\sigma}, \tag{5.2.12}$$

which is a monotonically increasing function of μ. Next, we consider values of $\mu > X_{m:m:n}$. For these values, the likelihood function reduces to an expression which is proportional to

$$L_m(\mu,\sigma) = \frac{1}{\sigma^m} e^{\sum_{i=1}^{m}(X_{i:m:n} - \mu)/\sigma} \prod_{i=1}^{m}\left[1 - \frac{1}{2}e^{(X_{i:m:n} - \mu)/\sigma}\right]^{R_i}. \tag{5.2.13}$$

Upon taking the logarithm of L_m and differentiating with respect to μ, we obtain

$$\frac{\partial L_m}{\partial \mu} = \frac{-m}{\sigma} + \sum_{i=1}^{m} \frac{R_i e^{(X_{i:m:n}-\mu)/\sigma}}{2\sigma\left[1 - \frac{1}{2}e^{(X_{i:m:n}-\mu)/\sigma}\right]}. \tag{5.2.14}$$

Now, if $R_1 = R_2 = \dots = R_m = 0$, then $m = n$, so that the right hand side of

(5.2.14) is simply $-n/\sigma$ which is strictly less than 0. If some $R_i > 0$, $i = 1, 2, \ldots, m$, then

$$\frac{\partial \ln L_m}{\partial \mu} < \frac{-m}{\sigma} + \frac{\sum_{i=1}^{m} R_i}{\sigma} \le 0 \text{ if } \sum_{i=1}^{m} R_i \le m, \text{ ie. } n-m \le m, \text{ ie. } m \ge \frac{n}{2},$$

so that L_m is monotonically decreasing for these values of m. Thus, if the observed number of failures $m \ge n/2$, the maximum likelihood estimator of μ lies in the interval $[X_{1:m:n}, X_{m:m:n}]$.

Consider now the values of μ such that $X_{j:m:n} < \mu < X_{j+1:m:n}$ for $j \in \{1, 2, \ldots, m-1\}$. In this case, the likelihood function reduces to an expression which is proportional to

$$L_j(\mu,\sigma) = \frac{1}{\sigma^m} e^{\sum_{i=1}^{j} (X_{i:m:n}-\mu)/\sigma} \prod_{i=1}^{j} \left[1 - \frac{1}{2} e^{(X_{i:m:n}-\mu)/\sigma}\right]^{R_i} e^{-\sum_{i=j+1}^{m} (X_{i:m:n}-\mu)/\sigma} \prod_{i=j+1}^{m} \left[e^{-(X_{i:m:n}-\mu)/\sigma}\right]^{R_i}. \tag{5.2.15}$$

Upon taking the logarithm of L_j and differentiating with respect to μ, we obtain

$$\frac{\partial \ln L_j}{\partial \mu} = \frac{1}{\sigma} \left[\sum_{i=1}^{j} \frac{R_i e^{(X_{i:m:n}-\mu)/\sigma}}{2 - e^{(X_{i:m:n}-\mu)/\sigma}} + n - \sum_{i=1}^{j} R_i - 2j \right]. \tag{5.2.16}$$

Now, if $R_i = 0$, $i = 1, 2, \ldots, j$, then the right hand side of (5.2.16) becomes simply $(n-2j)/\sigma$, which is strictly negative, provided $j > n/2$. If some $R_i > 0$, $i = 1, 2, \ldots, j$, then the right hand side of (5.2.16) is strictly less that $(n-2j)/\sigma \le 0$ if $j \ge n/2$. Thus, in general,

$$\frac{\partial L_j}{\partial \theta} < 0 \text{ provided } j > \frac{n}{2},$$

so that L_j is monotonically decreasing for these values of j. Thus, if the observed number of failures, $m > n/2$, the maximum likelihood estimator of μ lies in the interval $[X_{1:m:n}, X_{[n/2]+1:m:n}]$.

Upon further inspection of (5.2.16), it is evident that if $n - R_1 - R_2 - \ldots - R_j - 2j > 0$, then the right hand side is strictly positive. This can only be possible if $j < n/2$, in which case $n - R_1 - R_2 - \ldots - R_j - 2j > n/2 - j - R_1 - R_2 - \ldots - R_j \geq 0$ if $R_1 + R_2 + \ldots + R_j + j \leq n/2$. Thus,

$$\frac{\partial L_j}{\partial \mu} > 0 \text{ provided } j < \frac{n}{2} \text{ and } R_1 + R_2 + \ldots + R_j + j \leq \frac{n}{2},$$

so that L_j is monotonically increasing for these values of j.

At this point, we can formulate the following algorithm for obtaining maximum likelihood estimates of the location and scale parameters from a Laplace distribution when a progressive Type-II right censored sample of size m from a sample of size n is observed, with censoring scheme $(R_1, R_2, \ldots, R_m)$. We will denote the progressive Type-II right censored order statistics from this sample by $X_{i:m:n}$, $i = 1, 2, \ldots, m$, and the corresponding observed values of the order statistics by x_i, $i = 1, 2, \ldots, m$.

1) Find the largest number $k < n/2$ such that $R_1 + R_2 + \ldots + R_k + k \leq n/2$, assuming $R_0 = 0$.

2) a) If $m < n/2$, the maximum likelihood estimates are those corresponding to

$$\max\left\{ \max_{x_{k+1} < \theta < x_{k+2},\, 0 < \sigma < \infty} \left(L_{k+1}\right),\ \max_{x_{k+2} < \theta < x_{k+3},\, 0 < \sigma < \infty} \left(L_{k+2}\right),\ \ldots,\ \max_{\theta > x_m,\, 0 < \sigma < \infty} \left(L_m\right) \right\}.$$

b) If $m > n/2$, the maximum likelihood estimates are those corresponding to

$$\max\left\{ \max_{x_{k+1} < \mu < x_{k+2},\, 0 < \sigma < \infty} \left(L_{k+1}\right),\ \max_{x_{k+2} < \mu < x_{k+3},\, 0 < \sigma < \infty} \left(L_{k+2}\right),\ \ldots,\ \max_{x_{n/2} < \mu < x_{(n/2)+1},\, 0 < \sigma < \infty} \left(L_{n/2}\right) \right\} \text{ if } n \text{ is even and}$$

$$\max\left\{ \max_{x_{k+1} < \mu < x_{k+2},\, 0 < \sigma < \infty} \left(L_{k+1}\right),\ \max_{x_{k+2} < \mu < x_{k+3},\, 0 < \sigma < \infty} \left(L_{k+2}\right),\ \ldots,\ \max_{x_{(n-1)/2} < \mu < x_{(n+1)/2},\, 0 < \sigma < \infty} \left(L_{(n-1)/2}\right) \right\} \text{ if } n \text{ is odd}.$$

For the case n is odd, notice that we may have $k = (n - 1)/2$. This simply means that the likelihood function is monotonically increasing for $\mu < X_{(n+1)/2:m:n}$ and the likelihood function is monotonically decreasing for $\mu > X_{(n+1)/2:m:n}$. Thus, the maximum likelihood estimator of μ is $X_{(n+1)/2:m:n}$, which we can use to solve for the maximum likelihood estimator of σ.

c) If $m = n/2$, the maximum likelihood estimates are those corresponding to

$$\max\left\{ \max_{x_{k+1} < \theta < x_{k+2},\ 0 < \sigma < \infty} (L_{k+1}),\ \max_{x_{k+2} < \theta < x_{k+3},\ 0 < \sigma < \infty} (L_{k+2}),\ \ldots,\ \max_{x_{(n/2)-1} < \theta < x_{(n/2)},\ 0 < \sigma < \infty} (L_{(n/2)-1}) \right\}.$$

Notice that here, it is possible to obtain $k = (n/2)$-1. This means that for $\mu < X_{n/2:m:n}$, the likelihood function is monotonically increasing and for $\mu > X_{n/2:m:n}$, the likelihood function is monotonically decreasing. Therefore, the maximum likelihood estimator of μ is $X_{n/2:m:n}$. This can be used to solve for the maximum likelihood estimator of σ.

Remark: For the special case of conventional Type-II right censoring, where $R_1 = R_2 = \ldots = R_{m-1} = 0$ and $R_m = n - m$, this algorithm reduces to that given in Balakrishnan and Cutler (1995): for $m < n/2$, $k = m - 1$, and we just maximize L_m. For $m > n/2$, for n odd, $k = (n\text{-}1)/2$, and the maximum likelihood estimator for μ is $X_{(n+1)/2:m:n}$. For n even, $k = (n/2) - 1$, and we must maximize $L_{n/2}$. From (5.2.16), $\dfrac{\partial \ln L_j}{\partial \mu}$ is obviously zero, so the maximum likelihood estimator of μ is any value in $[X_{n/2:m:n}, X_{(n/2)+1:m:n}]$. Finally, for $m = n/2$, $k = (n/2) - 1$, so that the maximum likelihood estimator of μ is $X_{n/2:m:n}$. These estimates of μ may then be used to obtain maximum likelihood estimates of σ.

Numerical Example 5.2.3

In Numerical Example 5.1.3, a progressive Type-II right censored sample of size $m = 10$ from a sample of size $n = 20$ from the Laplace distribution with $\mu = 25$ and $\sigma = 5$ was simulated, with censoring scheme $R = (2,0,0,2,0,0,0,2,0,4)$ using the algorithm presented in subsection 2.1.3. The simulated progressive Type-II right censored sample is as follows:

19.21167876, 21.97364262, 23.41776818, 23.66253070, 23.80222832,
24.23017797, 25.62072188, 25.86990938, 26.47997028, 27.55344134.

From part (1) of the algorithm presented above, we find $k = 6$. Thus, from part (2c), we must find μ and σ which correspond to

$$\max\left\{ \max_{x_7 < \mu < x_8,\, 0 < \sigma < \infty} (L_7),\ \max_{x_8 < \mu < x_9,\, 0 < \sigma < \infty} (L_8),\ \ldots,\ \max_{x_9 < \mu < x_{10},\, 0 < \sigma < \infty} (L_9) \right\}.$$

Using Maple V Release 3, the maximum value of the likelihood function is obtained when we maximize $L_8(\mu,\sigma)$ over the region specified above. The corresponding maximum likelihood estimates are $\hat{\sigma} = 2.67091$ and $\hat{\mu} = 26.31069$. In Numerical Example 5.1.3, it was shown that the best linear unbiased estimates and their standard errors for the two parameters in this case are

$\mu^* = 26.26607 \quad SE(\mu^*) = 0.72333 \qquad \sigma^* = 2.64071 \quad SE(\sigma^*) = 0.28972.$

These values agree well with the MLE's which we have obtained. It should be noted here that to obtain standard errors of the MLE's, a simulational study needs be conducted. Furthermore, since the class of distributions under study does not possess "regularity" properties, due to its lack of differentiability, it may not be appropriate to approximate standard errors using the method of inverting the matrix of second derivatives. Simulational studies to obtain standard errors of estimates when samples are from "generalized distributions" will be discussed later on in Section Two of this thesis. □

6 Optimal Censoring Schemes

6.1 Introduction

The question of choosing optimal values of R_1, R_2, ..., R_m when considering a progressive Type-II right censoring scheme has not been addressed even in the finite sample case. Here, optimality may be defined, for example, in terms of the variance (for the one-parameter case) and the trace or determinant of the variance-covariance matrix (for the two-parameter case) of the best linear unbiased estimators. For the finite sample case, one may list all possible choices of censoring schemes and corresponding efficiencies, and determine the best from this list, much as one determines a best regression from a list of Cp statistics for all possible regressions computed using SAS.

For the exponential distribution, it has been shown in Chapter 5 that the variance (for the one-parameter case) and the trace and determinant of the variance-covariance matrix (for the two-parameter case) of the BLUE's are independent of the censoring scheme employed. Hence, there is no need for the determination of optimal censoring; if we are concerned with completing the experiment quickly, we may employ a conventional Type-II right censoring scheme and simply observe the first m failures, whereas if we are concerned with saving experimental units for subsequent testing, we may employ a progressive Type-II right censoring scheme with $R_1 = n - m$ and all subsequent R_i's $= 0$. However, for other distributions, such as normal, log-normal and extreme-value that are commonly used as life-time models, we aim in this chapter to determine the optimal progressive censoring scheme for some practical choices of n and m. This work will be especially useful for practitioners who intend to implement the progressive Type-II right censoring scheme in their experimentation but may be concerned with the loss of efficiency.

It may be mentioned here that all of the computation for this chapter was performed by programming in the Fortran language.

6.2 Moments of Progressive Type-II Right Censored Order Statistics From Arbitrary Continuous Distributions [Thomas and Wilson (1972)]

A method for obtaining single and product moments of progressive Type-II right censored order statistics from a sample of size n from arbitrary continuous distributions, provided the mean vector and variance-covariance matrix for moments of usual order statistics up to the sample size n are known, is given in

Thomas and Wilson (1972). This method will be used in this chapter to obtain optimal censoring schemes when samples are from normal, log-normal or extreme-value distributions, as we have not yet determined more elegant methods of obtaining moments for these distributions. In fact, moments of *usual* order statistics for these frequently used distributions are quite difficult to derive explicitly, and are therefore tabulated for various sample sizes using numerical integration.

Thomas and Wilson's method for obtaining single and product moments of progressive Type-II right censored order statistics from arbitrary continuous distributions is outlined as follows:

Suppose we denote the usual order statistics from a sample of size n from the standard distribution of interest by $Z_{i:n}$, $i = 1, 2, \ldots, n$. Suppose further that we denote the m progressive Type-II right censored order statistics from the sample of size n obtained with censoring scheme $(R_1, R_2, \ldots, R_m)$ by $Z_{j:m:n}$, $j = 1, 2, \ldots, m$. Then each progressive Type-II right censored order statistic corresponds to some usual order statistic from the original n items on test, that is, $Z_{j:m:n} = Z_{K_j:n}$ where the rank of $Z_{j:m:n}$, K_j, can take on the values $K_{j-1}+1, K_{j-1}+2, \ldots, j+R_1+R_2+\ldots+R_{j-1}$ for $j = 2, 3, \ldots, m$ and $K_1 = 1$. Thomas and Wilson (1972) have shown that the joint probability function of the rank vector is given by

$$P(K_1,\ldots,K_m) = P(K_1)\prod_{i=2}^{m} P(K_i|K_1,\ldots,K_{i-1}) \text{ where } P(K_1=1)=1, \text{ and}$$

$$P(K_i|K_1,\ldots,K_{i-1}) = \frac{\binom{n-K_i}{\sum_{j=1}^{i-1}(R_j+1)-K_i+1}}{\binom{n-K_{i-1}}{\sum_{j=1}^{i-1}(R_j+1)-K_{i-1}}}, \; i=2,\ldots,k.$$

Thus, if all possible rank vectors (say there are M of them) can be listed for a particular censoring scheme $(R_1, R_2, \ldots, R_m)$, then for each rank vector, we can define an $m \times n$ indicator matrix D_l, $l = 1, 2, \ldots, M$, whose $(r,s)^{th}$ element is 1 if $s = K_r$ and 0 otherwise, so that $Z_{ps} = D_l\, Z_u$ for some l, where Z_{ps} is the $m \times 1$ vector of progressive Type-II right censored order statistics and Z_u is the $n \times 1$ vector of usual order statistics from the standard distribution of interest. Then, denoting $E(Z_u)$ by μ_u, we can obtain the first moments of the progressive Type-II right censored order statistics as follows:

$$E(Z_{ps}) = \mu = EE(Z_{ps}|D_l) = E(D_l\mu_u) = \left(\sum_{l=1}^{M} D_l p_l\right)\mu_u \tag{6.2.1}$$

where p_l is the probability function of the rank vector corresponding to D_l, $l = 1, 2, \ldots, M$. Similarly, denoting the variance-covariance matrix of Z_u by Σ_u, we can obtain the variance-covariance matrix of Z_{ps} as follows:

$$
\begin{aligned}
Var(Z_{ps}) = \Sigma = E(Z_{ps}Z_{ps})' - \mu\mu' \\
&= EE(D_l Z_u Z_u' D_l' | D_l) - \mu\mu' \\
&= E[D_l(\Sigma_u + \mu_u\mu_u')D_l'] - \mu\mu' \\
&= \sum_{l=1}^{M} D_l(\Sigma_u + \mu_u\mu_u')D_l' p_l - \mu\mu' . \qquad (6.2.2)
\end{aligned}
$$

Tables of single and product moments of usual order statistics for standard normal, log-normal and extreme-value distributions are available for various sample sizes. Thus, using the results from subsection 5.1.1 for obtaining best linear unbiased estimators for two-parameter distributions and Eqs. (6.2.1) and (6.2.2), we will be able to determine the best linear unbiased estimators and their variances and covariance for any progressive censoring scheme. We will also be able to compare efficiencies of estimators obtained using different censoring schemes. Furthermore, if *all* possible censoring schemes can be listed for fixed values of m and n, (there will be $\binom{n-1}{m-1}$ possibilities), we can determine the optimal censoring scheme to employ in terms of the trace or determinant of the variance-covariance matrix of the BLUE's. This work will be of practical interest, and may point us towards new and interesting results for progressive censored samples.

Notice that the optimal progressive Type-II right censoring scheme can never be less optimal than a conventional Type-II right censoring scheme, as the conventional scheme is just a special case of progressive censoring.

6.3 The Normal Distribution

6.3.1 Introduction

The standard normal distribution, with pdf given by

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \qquad (6.3.1)$$

is occasionally used in reliability studies to model life-time distributions [see Bain and Engelhardt (1991)] perhaps due to the vast amount of knowledge that has accumulated regarding this distribution. It is also commonly found that if certain types of data are transformed in some fashion, the resulting values will follow an approximate normal distribution, and so it is a common practice to analyse the transformed data using the normality assumption.

Consider a progressive Type-II right censored sample of size m from a sample of size n from the two-parameter normal distribution, which we will denote $Y_{i:m:n}^{(R_1,R_2,\ldots,R_m)}$. Here, we can write $Y_{i:m:n}^{(R_1,\ldots,R_m)} = \mu + \sigma X_{i:m:n}^{(R_1,\ldots,R_m)}$, where $X_{i:m:n}^{(R_1,\ldots,R_m)}$ represents a progressive Type-II right censored sample with the same censoring scheme from the standard normal distribution in (6.3.1).

Tables of means, variances and covariances of usual order statistics from the standard normal distribution are tabulated up to sample size $n = 50$ in Tietjen, Kahaner and Beckman (1977). We have used these values and Eqs. (6.2.1) and (6.2.2) to obtain optimal censoring schemes for selected values of m and n when progressive Type-II right censored samples are from two-parameter normal distributions.

6.3.2 Results for Optimal Censoring

For various choices of m and n, all possible censoring schemes were considered and the variances and covariance of the BLUE's (scaled by the population variance) arising from these censoring schemes were determined. The optimal censoring schemes and their efficiencies with respect to the minimum trace and determinant of the variance-covariance matrix, along with the least efficient censoring schemes are presented in Tables 6.3.1 to 6.3.6. Efficiencies have been calculated in the usual way, that is, for example, trace efficiency (scheme A, scheme B) = trace (scheme B) / trace (scheme A), so that high values of efficiency are desirable. Table 6.3.7 gives coefficients of the (ordered) progressive Type-II right censored order statistics for BLUE's of the location and scale parameters for various schemes considered, up to $m = 6$, along with the variances and covariance of the BLUE's.

The values of m and n when $m < n/2$ are such that the total number of censoring schemes to be considered is not larger than 11 628, the number of possible schemes when $n = 20$ and $m = 6$. For m $n/2$, a few selected values were also considered, although in practice only a small fraction of the n failure times will be generally observed.

The observations we will make here will be mainly concerned with trace efficiency, although a similar analysis based on determinant efficiency (or some other measure of efficiency involving variances and covariance of BLUE's) may be made. As mentioned already, we have included tables for determinant efficiencies also (Tables 6.3.3, 6.3.4 and 6.3.6).

Consider first Table 6.3.1. Efficiencies for the trace-optimal schemes (that is, the scheme for which minimum trace of the variance - covariance matrix of BLUE's is attained for some values of m and n) with respect to the conventional Type-II right censoring schemes are shown in regular font, and the actual traces corresponding to these optimal schemes are shown in italics. From the table, as one would expect, as m increases (with n being held constant), the gain in efficiency over the conventional scheme decreases, that is, the smaller the proportion of failure times an experimenter would like to observe, the more precision can be gained by employing progressive Type-II right censoring over conventional Type-II right censoring. Also, as n increases (with m being constant), the gain in trace-efficiency increases. For small values of m, the increase in precision over conventional censoring can be as much as 360% for the different cases tabulated.

Looking at the trace-optimal schemes presented in Table 6.3.2, it is quite evident that for small values of m, a scheme where all but m-1 surviving units are randomly removed after the first observed failure (ie. $R_1 = n - m$) will result in the best precision of BLUE's. For $n=15$, where a complete analysis was possible, the optimal censoring scheme starts to shift at about $m = 7$, with censoring occurring at points other than immediately following the first failure. However, from Table 6.3.1, the gain in efficiency for optimal schemes for $m > 7$, $n = 15$ diminishes quite rapidly.

An interesting question that arises here is the following: Suppose an experimenter wants to observe m failures. Is it wise to begin with a large number (n) of experimental units, in terms of precision? Intuitively, one would immediately say yes, since more experimental units, even if they are censored, results in more knowledge and should therefore lower variances of estimators. However, looking at Table 6.3.1, it seems that for $m = 2$ and $m = 3$, the values of the optimal traces actually *increase* with n. (Notice that this is not so in Table 6.3.3 for determinants.) However, for $m \geq 4$, our intuition prevails: for fixed values of m precision increases with n, and more drastically for larger m. For populations with large variance, however, the gain in precision by starting with a larger value of n will be quite substantial, even for moderate values of m. Furthermore, if "time on test" is of concern to an experimenter, quite often, depending upon the censoring schemes employed, a large value of n will result in a smaller expected time on test for a fixed value of m.

Tables 6.3.5 and 6.3.6 show least precise censoring schemes with respect to the trace and determinant of the variance-covariance matrix of BLUE's. It is very interesting to note that for most of the entries displayed in Table 6.3.5 and all of the entries displayed in Table 6.3.6, the commonly used conventional Type-II right censoring scheme is actually the *least* efficient of all progressive Type-II right censoring schemes!

Remark: While obtaining the results for $n = 50$, large values of m were not considered due to computational difficulties. For this reason, only small values of m were considered. As mentioned earlier, it is only small values of m which are generally considered in practice.

Table 6.3.1: Efficiencies and *Traces* for Trace-Optimal Censoring Schemes: Normal Distribution (Line 1: Efficiency; Line 2: *Trace*)

$n \setminus m$	2	3	4	5	6	7	8	9	10	11	12	n-2	n-1	n
15	2.2757 *1.0163*	1.8746 *0.5015*	1.5827 *0.3455*	1.3849 *0.2700*	1.2509 *0.2248*	1.1656 *0.1934*	1.1104 *0.1703*	1.0711 *0.1532*	1.0454 *0.1398*	1.0277 *0.1292*	1.0159 *0.1207*	1.0082 *0.1137*	1.0033 *0.1080*	1.0000 *0.1033*
20	2.5622 *1.0371*	2.1533 *0.5031*	1.8305 *0.3423*	1.5987 *0.2649*	1.4312 *0.2189*							1.0051 *0.0823*	1.0022 *0.0794*	1.0000 *0.0768*
25	2.7976 *1.0513*	2.3881 *0.5048*	2.0464 *0.3409*	1.7920 *0.2621*								1.0036 *0.0645*	1.0016 *0.0627*	1.0000 *0.0612*
30	2.9981 *1.0614*	2.5909 *0.5063*	2.2363 *0.3401*										1.0013 *0.0519*	1.0000 *0.0508*
40	3.3288 *1.0748*	2.9289 *0.5085*	2.5583 *0.3393*										1.0009 *0.0385*	1.0000 *0.0380*
50	3.6010 *1.0819*	3.2088 *0.5094*												

Table 6.3.2: Trace-Optimal Censoring Schemes: Normal Distribution

$n \setminus m$	2	3	4	5	6	7	8	9	10	11	12	n-2	n-1
15	13,0	12,0,0	11,0,0,0	10,0,0,0, 0	8,1,0,0, 0,0	2,6,0,0, 0,0,0	0,7,0,0, 0,0,0,0	0,3,3,0, 0,0,0,0,0	0,0,5,0, 0,0,0,0, 0,0	0,0,0,4, 0,0,0,0, 0,0,0	0,0,0,1, 2,0,0,0, 0,0,0,0	$R_5=2$, $R_i=0$, $i \neq 5$	$R_6=1$, $R_i=0$, $i \neq 6$
20	18,0	17,0,0	16,0,0,0	15,0,0,0, 0	14,0,0,0, 0,0							$R_8=2$, $R_i=0$, $i \neq 8$	$R_9=1$, $R_i=0$, $i \neq 9$
25	23,0	22,0,0	21,0,0,0	20,0,0,0, 0								$R_{11}=2$, $R_i=0$, $i \neq 11$	$R_{12}=1$, $R_i=0$, $i \neq 12$
30	28,0	27,0,0	26,0,0,0										$R_{15}=1$, $R_i=0$, $i \neq 15$
40	38,0	37,0,0	36,0,0,0										$R_{20}=1$, $R_i=0$, $i \neq 20$
50	48,0	47,0,0											

Table 6.3.3: Efficiencies and *Determinants* for Determinant-Optimal Censoring Schemes: Normal Distribution (Line 1: Efficiency; Line 2: *Determinant*)

$n \setminus m$	2	3	4	5	6	7	8	9	10	11	12	n-2	n-1	n
15	1.9029 *0.0706*	1.6024 *0.0306*	1.4241 *0.0187*	1.3494 *0.0127*	1.2951 *0.0094*	1.2442 *0.0074*	1.1994 *0.0060*	1.1638 *0.0051*	1.1331 *0.0043*	1.1040 *0.0038*	1.0765 *0.0033*	1.0505 *0.0030*	1.0254 *0.0027*	1.0000 *0.0024*
20	2.0448 *0.0599*	1.7070 *0.0260*	1.5054 *0.0159*	1.4431 *0.0107*	1.3847 *0.0109*							1.0382 *0.0015*	1.0195 *0.0014*	1.0000 *0.0013*
25	2.1585 *0.0530*	1.7889 *0.0230*	1.5676 *0.0141*	1.5176 *0.0093*								1.0310 *0.0009*	1.0159 *0.0009*	1.0000 *0.0008*
30	2.2539 *0.0481*	1.8564 *0.0210*	1.6284 *0.0128*										1.0134 *0.0006*	1.0000 *0.0006*
40	2.4089 *0.0416*	1.9641 *0.0181*	1.7339 *0.0109*										1.0103 *0.0003*	1.0000 *0.0003*
50	2.5358 *0.0372*	2.0515 *0.0163*												

Table 6.3.4: Determinant-Optimal Censoring Schemes: Normal Distribution

$n \setminus m$	2	3	4	5	6	7	8	9	10	11	12	n-2	n-1
15	13,0	12,0,0	11,0,0,0	0,10,0,0, 0	0,9,0,0, 0,0	0,8,0,0, 0,0,0	0,7,0,0, 0,0,0,0	0,0,6,0, 0,0,0,0, 0	0,0,5,0, 0,0,0,0, 0,0	0,0,4,0, 0,0,0,0, 0,0,0	0,0,3,0, 0,0,0,0, 0,0,0,0	$R_3=2$, $R_i=0$, $i \neq 3$	$R_4=1$, $R_i=0$, $i \neq 4$
20	18,0	17,0,0	16,0,0,0	0,15,0,0, 0	0,14,0,0, 0,0							$R_5=2$, $R_i=0$, $i \neq 5$	$R_5=1$, $R_i=0$, $i \neq 5$
25	23,0	22,0,0	21,0,0,0	0,20,0,0, 0								$R_6=2$, $R_i=0$, $i \neq 6$	$R_6=1$, $R_i=0$, $i \neq 6$
30	28,0	27,0,0	0,26,0,0										$R_7=1$, $R_i=0$, $i \neq 7$
40	38,0	37,0,0	0,36,0,0										$R_9=1$, $R_i=0$, $i \neq 9$
50	48,0	47,0,0											

Table 6.3.5: Least Precise Censoring Schemes (With Respect to Trace): Normal Distribution

$n \setminus m$	2	3	4	5	6	7	8	9	10	11	12	n-2	n-1
15	0,13	0,0,12	0,0,0,11	0,0,0,0,10	0,0,0,0,0,9	0,0,0,0,0,0,8	0,0,0,0,0,0,0,7	0,0,0,0,0,0,0,0,6	0,0,0,0,0,0,0,0,0,5	0,0,0,0,0,0,0,0,0,0,4	3,0,0,0,0,0,0,0,0,0,0,0	$R_1=2$, $R_i=0$, $i\neq1$	$R_1=1$, $R_i=0$, $i\neq1$
20	0,18	0,0,17	0,0,0,16	0,0,0,0,15	0,0,0,0,0,14							$R_1=2$, $R_i=0$, $i\neq1$	$R_1=1$, $R_i=0$, $i\neq1$
25	0,23	0,0,22	0,0,0,21	0,0,0,0,20								$R_1=2$, $R_i=0$, $i\neq1$	$R_1=1$, $R_i=0$, $i\neq1$
30	0,28	0,0,27	0,0,0,26										$R_1=1$, $R_i=0$, $i\neq1$
40	0,38	0,0,37	0,0,0,36										$R_1=1$, $R_i=0$, $i\neq1$
50	0,48	0,0,47											

Table 6.3.6: Least Precise Censoring Schemes (With Respect to Determinant): Normal Distribution												
$n \setminus m$	2	3	4	5	6	7	8	9	10	11	12	n-2	n-1
15	0,13	0,0,12	0,0,0,11	0,0,0,0, 10	0,0,0,0, 0,9	0,0,0,0, 0,0,8	0,0,0,0, 0,0,0,7	0,0,0,0, 0,0,0,0, 6	0,0,0,0, 0,0,0,0, 0,5	0,0,0,0, 0,0,0,0, 0,0,4	0,0,0,0, 0,0,0,0, 0,0,0,3	$R_{13}=2$, $R_i=0$, $i \neq 13$	$R_{14}=1$, $R_i=0$, $i \neq 14$
20	0,18	0,0,17	0,0,0,16	0,0,0,0, 15	0,0,0,0, 0,14							$R_{18}=2$, $R_i=0$, $i \neq 18$	$R_{19}=1$, $R_i=0$, $i \neq 19$
25	0,23	0,0,22	0,0,0,21	0,0,0,0, 20								$R_{23}=2$, $R_i=0$, $i \neq 23$	$R_{24}=1$, $R_i=0$, $i \neq 24$
30	0,28	0,0,27	0,0,0,26										$R_{29}=1$, $R_i=0$, $i \neq 29$
40	0,38	0,0,37	0,0,0,36										$R_{39}=1$, $R_i=0$, $i \neq 39$
50	0,48	0,0,47											

Table 6.3.7: Table of Coefficients, Variances and Covariance of BLUE's for selected schemes in Tables 6.3.1 - 6.3.6

Scheme	Coefficients (μ^*)	Coefficients (σ^*)	$Var(\mu^*)/\sigma^2$	$Var(\sigma^*)/\sigma^2$	$Cov(\mu^*,\sigma^*)/\sigma^2$
0,13	-2.5574,3.5574	-2.049,2.049	1.5404	0.7723	1.0273
13,0	0.0667,0.9333	-0.5377,0.5377	0.7213	0.2950	0.3771
0,0,12	-1.0242,-0.4676,2.4918	-1.0601,-0.5477,1.6077	0.5713	0.3689	0.4020
12,0,0	0.1656,0.2361,0.5983	-0.4338,0.0267,0.4071	0.3472	0.1543	0.1514
0,0,0,11	-0.5462,-0.2448,-0.1148,1.9058	-0.7201,-0.3892,-0.2458,1.3552	0.3101	0.2368	0.2163
11,0,0,0	0.1840,0.1401,0.2504,0.4255	-0.3787,-0.0554,0.1028,0.3313	0.2354	0.1101	0.0849
0,0,0,0,10	-0.3217,-0.1364,-0.0560,0.0043,1.5097	-0.5459,-0.3050,-0.2002,-0.1211,1.1722	0.2022	0.1718	0.1325
10,0,0,0,0	0.1821,0.1101,0.1625,0.2215,0.3231	-0.3390,-0.0842,0.0196,0.1194,0.2842	0.1818	0.0882	0.0542
0,10,0,0,0	-0.1170,0.4216,0.1271,0.2109,0.3574	-0.3240,-0.0928,-0.0072,0.1125,0.3115	0.1816	0.1007	0.0747
0,0,0,0,0,9	-0.1950,-0.0732,-0.0203,0.0196,0.0531, 1.2157	-0.4390,-0.2518,-0.1700,-0.1082,-0.0562, 1.0252	0.1480	0.1332	0.0868
8,1,0,0,0,0	0.1342,0.1410,0.1207,0.1534,0.1909, 0.2597	-0.3056,-0.0951,-0.0226,0.0466,0.1211, 0.2555	0.1485	0.0763	0.0392
0,9,0,0,0,0	-0.0528,0.3402,0.0996,0.1412,0.1910, 0.2808	-0.2715,-0.1221,-0.0370,0.0388,0.1222, 0.2695	0.1462	0.0812	0.0497
0,18	-3.0609,4.0609	-2.1745,2.1745	1.8710	0.7863	1.1613
18,0	0.0500,0.9500	-0.5087,0.5087	0.7668	0.2703	0.3839
0,0,17	-1.2872,-0.6610,2.9482	-1.1244,-0.6212,1.7456	0.7056	0.3778	0.4713
17,0,0	0.1523,0.2296,0.6181	-0.4153,0.0303,0.3850	0.3628	0.1403	0.1578

Table 6.3.7: Table of Coefficients, Variances and Covariance of BLUE's for selected schemes in Tables 6.3.1 - 6.3.6

Scheme	Coefficients (μ^*)	Coefficients (σ^*)	$Var(\mu^*)/\sigma^2$	$Var(\sigma^*)/\sigma^2$	$Cov(\mu^*,\sigma^*)/\sigma^2$
0,0,0,16	-0.7284,-0.3766,-0.2298,2.3348	-0.7645,-0.4380,-0.3014,1.5039	0.3828	0.2438	0.2634
16,0,0,0	0.1748,0.1287,0.2529,0.4436	-0.3666,-0.0471,0.1019,0.3118	0.2424	0.0999	0.0911
0,0,0,0,15	-0.4632,-0.2378,-0.1433,-0.0738,1.9180	-0.5804,-0.3417,-0.2414,-0.1673,1.3308	0.2456	0.1778	0.1682
15,0,0,0,0	0.1769,0.0970,0.1589,0.2271,0.3401	-0.3316,-0.0744,0.0235,0.1160,0.2664	0.1850	0.0799	0.0600
0,15,0,0,0	-0.1671,0.4537,0.1149,0.2148,0.3837	-0.3079,-0.0924,-0.0063,0.1095,0.2971	0.1908	0.0929	0.0841
0,0,0,0,0,14	-0.3117,-0.1565,-0.0914,-0.0433,-0.0037, 1.6066	-0.4679,-0.2813,-0.2028,-0.1447,-0.0967, 1.1934	0.1747	0.1386	0.1155
14,0,0,0,0,0	0.1714,0.0837,0.1184,0.1564,0.1978, 0.2723	-0.3033,-0.0862,-0.0130,0.0502,0.1167, 0.2357	0.1512	0.0677	0.0425
0,14,0,0,0,0	-0.0961,0.3711,0.0856,0.1373,0.1979, 0.3042	-0.2560,-0.1235,-0.0339,0.0391,0.1186, 0.2557	0.1504	0.0745	0.0579
0,23	-3.4564,4.4564	-2.2675,2.2675	2.1451	0.7960	1.2621
23,0	0.0400,0.9600	-0.4885,0.4885	0.7984	0.2529	0.3859
0,0,22	-1.4931,-0.8173,3.3104	-1.1719,-0.6754,1.8474	0.8218	0.3839	0.5237
22,0,0	0.1435,0.2261,0.6304	-0.4023,0.0330,0.3693	0.3741	0.1307	0.1608
0,0,0,21	-0.8707,-0.4828,-0.3245,2.6780	-0.7970,-0.4740,-0.3419,1.6129	0.4489	0.2486	0.2991
21,0,0,0	0.1682,0.1218,0.2553,0.4547	-0.3579,-0.0412,0.1012,0.2979	0.2480	0.0928	0.0943
0,0,0,0,20	-0.5733,-0.3192,-0.2151,-0.1393,2.2469	-0.6056,-0.3687,-0.2714,-0.2006,1.4463	0.2878	0.1819	0.1954
20,0,0,0,0	0.1727,0.0888,0.1573,0.2307,0.3505	-0.3261,-0.0675,0.0263,0.1135,0.2539	0.1879	0.0742	0.0633

Table 6.3.7: Table of Coefficients, Variances and Covariance of BLUE's for selected schemes in Tables 6.3.1 - 6.3.6

Scheme	Coefficients (μ^*)	Coefficients (σ^*)	$Var(\mu^*)/\sigma^2$	$Var(\sigma^*)/\sigma^2$	$Cov(\mu^*,\sigma^*)/\sigma^2$
0,20,0,0,0	-0.1991,0.4730,0.1071,0.2181,0.4008	-0.2954,-0.0934,-0.0053,0.1077,0.2865	0.1988	0.0872	0.0895
0,28	-3.7822,4.7822	-2.3410,2.3410	2.3790	0.8032	1.3425
28,0	0.0333,0.9667	-0.4732,0.4732	0.8216	0.2398	0.3859
0,0,27	-0.1663,-0.9485,3.6107	-1.2094,-0.7182,1.9275	0.9234	0.3884	0.5655
27,0,0	0.1371,0.2241,0.6388	-0.3925,0.0351,0.3574	0.3828	0.1235	0.1623
0,0,0,26	-0.9873,-0.5718,-0.4050,2.9640	-0.8226,-0.5023,-0.3735,1.6984	0.5084	0.2521	0.3277
26,0,0,0	0.1632,0.1172,0.2574,0.4622	-0.3513,-0.0367,0.1006,0.2874	0.2525	0.0876	0.0962
0,26,0,0	-0.3773,0.6446,0.1928,0.5400	-0.3815,-0.0181,0.0616,0.3380	0.2971	0.1156	0.1469
0,38	-4.3004,5.3004	-2.4530,2.4530	2.7642	0.8137	1.4660
38,0	0.0250,0.9750	-0.4512,0.4512	0.8538	0.2210	0.3836
0,0,37	-1.9307,-1.1611,4.0918	-1.2663,-0.7829,2.0492	1.0944	0.3949	0.6298
37,0,0	0.1283,0.2222,0.6495	-0.3783,0.0382,0.3401	0.3952	0.1133	0.1632
0,0,0,36	-1.1719,-0.7158,-0.5369,3.4245	-0.8613,-0.5453,-0.4212,1.8277	0.6109	0.2572	0.3717
36,0,0,0	0.1558,0.1116,0.2610,0.4717	-0.3415,-0.0303,0.0998,0.2720	0.2592	0.0802	0.0980
0,36,0,0	-0.4156,0.6649,0.1904,0.5603	-0.3643,-0.0210,0.0604,0.3250	0.3157	0.1075	0.1517
0,48	-4.7054,5.7054	-2.5368,2.5368	3.0749	0.8211	1.5590
48,0	0.0200,0.9800	-04357,0.4357	0.8743	0.2076	0.3799
0,0,47	-2.1399,-1.3302,4.4701	-1.3087,-0.8312,2.1399	1.2350	0.3995	0.6782

Table 6.3.7: Table of Coefficients, Variances and Covariance of BLUE's for selected schemes in Tables 6.3.1 - 6.3.6

Scheme	Coefficients (μ^*)	Coefficients (σ^*)	$Var(\mu^*)/\sigma^2$	$Var(\sigma^*)/\sigma^2$	$Cov(\mu^*,\sigma^*)/\sigma^2$
47,0,0	0.1224,0.2212,0.6564	-0.3680,0.0400,0.3280	0.4033	0.1061	0.1628

6.4 The Extreme-Value Distribution

6.4.1 Introduction

The standard extreme-value distribution has pdf given by

$$f(x) = e^{x} e^{-e^{x}}, \; -\infty < x < \infty . \tag{6.4.1}$$

This distribution has been used extensively in modelling failure times of items [see, for example, Nelson (1982), Viveros and Balakrishnan (1994).] The extreme-value random variable may also be seen as a transformed Weibull random variable [see, for example, Johnson, Kotz and Balakrishnan (1995)], and will therefore be an important distribution to consider. Furthermore, many asymptotic results for extreme order statistics from arbitrary distributions (more specifically, for the largest order statistic) involve the extreme-value distribution [see Arnold, Balakrishnan and Nagaraja (1992)].

Tables of means, variances and covariances of order statistics for the standard extreme-value distribution given in (6.4.1) are available in Balakrishnan and Chan (1992) for sample sizes up to $n = 30$. We have used these values and Eqs. (6.2.1) and (6.2.2) to obtain optimal censoring schemes for selected values of m and n when progressive Type-II right censored samples are from two-parameter (location-scale) extreme-value distributions.

6.4.2 Results for Optimal Censoring

Tables 6.4.1 to 6.4.7 parallel Tables 6.3.1 to 6.3.7 for the normal distribution for selected values of m and n. We will consider first Table 6.4.1 for optimal trace efficiencies. Again we see that for small values of m the improvement in precision of estimates with respect to the trace of the variance-covariance matrix of BLUE's is remarkable, as high as 575% for $m = 2$ and $n = 30$. Here, even for values of m/n as high as 2/3 (which we can only observe for $n = 15$), the gain in precision is still 21%. Again, in practice, experimenters are generally interested in values of $m/n < \frac{1}{2}$. The results here look very promising, and if possible, further theoretical work to confirm our intuitions should be carried out in the future.

Again, if one poses the question of whether or not a larger value of n will be beneficial if m is fixed, we find that for $m = 2$ or 3, the values of trace are actually increasing with n, indicating a decrease in precision, whereas for $m \geq 4$, the values decrease, indicating an increase in precision. This is an anomaly which would be very interesting to consider further in a theoretical sense. (From Table 6.4.3, we find again that this observation does not hold if one is considering determinants.)

The actual schemes corresponding to optimal censoring with respect to trace efficiency are shown in Table 6.4.2. Once again, this table looks similar to Table 6.3.2 for the normal distribution, although the extreme-value distribution has very different physical properties than the normal distribution. For small values of m, the optimal scheme is to randomly remove all but m-1 live units from experimentation immediately following the first failure. At this point, one may try to find an argument for this pattern in general. Further sections will show, however that this censoring scheme is not optimal in general.

Table 6.4.5 shows that, for *all* cases considered, the least efficient censoring scheme with respect to the trace of the variance-covariance matrix of BLUE's is the conventional Type-II right censoring scheme.

Again, tables for determinant-efficiencies are provided and the experimenter using this as a criterion may carry out a similar analysis using these tables (Tables 6.4.3, 6.4.4, 6.4.6).

Table 6.4.1: Efficiencies and *Traces* for Trace-Optimal Censoring Schemes: Extreme-Value Distribution (Line 1: Efficiency; Line 2: *Trace*)

$n \setminus m$	2	3	4	5	6	7	8	9	10	11	12	n-2	n-1	n
15	3.9670 *1.6588*	3.3962 *0.6836*	2.7037 *0.4438*	2.1875 *0.3380*	1.8255 *0.2776*	1.5872 *0.2357*	1.4220 *0.2053*	1.2972 *0.1835*	1.2093 *0.1662*	1.1425 *0.1529*	1.0847 *0.1420*	1.0553 *0.1333*	1.0260 *0.1261*	1.0000 *0.1201*
20	4.6589 *1.7188*	4.1973 *0.6886*	3.4382 *0.4402*	2.8225 *0.3319*	2.3645 *0.2707*							1.0409 *0.0958*	1.0202 *0.0922*	1.0000 *0.0891*
25	5.2419 *1.7607*	4.8953 *0.6924*	4.0989 *0.4381*	3.4122 *0.3282*								1.0330 *0.0748*	1.0168 *0.0727*	1.0000 *0.0707*
30	5.7497 *1.7915*	5.5159 *0.6951*	4.6976 *0.4368*										1.0145 *0.0600*	1.0000 *0.0587*

Table 6.4.2: Trace-Optimal Censoring Schemes: Exteme-Value Distribution													
$n \setminus m$	2	3	4	5	6	7	8	9	10	11	12	n-2	n-1
15	13,0	12,0,0	11,0,0,0	10,0,0,0,0	9,0,0,0,0,0	1,7,0,0,0,0,0	0,7,0,0,0,0,0,0	0,5,1,0,0,0,0,0,0	0,0,5,0,0,0,0,0,0,0	0,0,4,0,0,0,0,0,0,0,0	0,0,0,3,0,0,0,0,0,0,0,0	$R_5=2$, $R_i=0$, $i \neq 5$	$R_5=1$, $R_i=0$, $i \neq 5$
20	18,0	17,0,0	16,0,0,0	15,0,0,0,0	14,0,0,0,0,0							$R_7=2$, $R_i=0$, $i \neq 7$	$R_7=1$, $R_i=0$, $i \neq 7$
25	23,0	22,0,0	21,0,0,0	20,0,0,0,0								$R_9=2$, $R_i=0$, $i \neq 9$	$R_9=1$, $R_i=0$, $i \neq 9$
30	28,0	27,0,0	26,0,0,0										$R_{11}=1$, $R_i=0$, $i \neq 11$

Table 6.4.3: Efficiencies and *Determinants* for Determinant-Optimal Censoring Schemes: Extreme-Value Distribution (Line 1: Efficiency; Line 2: *Determinant*)

$n \setminus m$	2	3	4	5	6	7	8	9	10	11	12	n-2	n-1	n
15	2.9835 *0.2089*	2.6736 *0.0701*	2.3606 *0.0373*	2.1174 *0.0238*	1.9924 *0.0162*	1.8555 *0.0120*	1.7242 *0.0094*	1.6030 *0.0076*	1.4916 *0.0063*	1.3927 *0.0053*	1.2756 *0.0046*	1.1997 *0.0040*	1.1041 *0.0035*	1.0000 *0.0031*
20	3.3404 *0.1882*	3.0255 *0.0628*	2.6794 *0.0335*	2.4434 *0.0212*	2.3216 *0.0144*							1.1603 *0.0020*	1.0842 *0.0019*	1.0000 *0.0017*
25	3.6362 *0.1738*	3.3149 *0.0578*	2.9380 *0.0309*	2.7145 *0.0193*								1.1348 *0.0012*	1.0711 *0.0011*	1.0000 *0.0011*
30	3.8911 *0.1630*	3.5629 *0.0540*	3.1571 *0.0289*										1.0619 *0.0008*	1.0000 *0.0007*

Table 6.4.4: Determinant-Optimal Censoring Schemes: Extreme-Value Distribution

$n \setminus m$	2	3	4	5	6	7	8	9	10	11	12	n-2	n-1
15	13,0	12,0,0	11,0,0,0	0,10,0,0,0	0,9,0,0,0,0	0,8,0,0,0,0,0	0,7,0,0,0,0,0,0	0,6,0,0,0,0,0,0,0	0,0,5,0,0,0,0,0,0,0	0,0,4,0,0,0,0,0,0,0,0	0,0,3,0,0,0,0,0,0,0,0,0	$R_3=2$, $R_i=0$, $i \neq 3$	$R_3=1$, $R_i=0$, $i \neq 3$
20	18,0	17,0,0	16,0,0,0	0,15,0,0,0	0,14,0,0,0,0							$R_4=2$, $R_i=0$, $i \neq 4$	$R_4=1$, $R_i=0$, $i \neq 4$
25	23,0	22,0,0	21,0,0,0	0,20,0,0,0								$R_5=2$, $R_i=0$, $i \neq 5$	$R_5=1$, $R_i=0$, $i \neq 5$
30	28,0	27,0,0	26,0,0,0										$R_6=1$, $R_i=0$, $i \neq 6$

Table 6.4.5: Least Precise Censoring Schemes (With Respect to Trace): Extreme-Value Distribution												
$n \setminus m$	2	3	4	5	6	7	8	9	10	11	12	n-2	n-1
15	0,13	0,0,12	0,0,0,11	0,0,0,0, 10	0,0,0,0, 0,9	0,0,0,0, 0,0,8	0,0,0,0, 0,0,0,7	0,0,0,0, 0,0,0,0, 6	0,0,0,0, 0,0,0,0, 0,5	0,0,0,0, 0,0,0,0, 0,0,4	0,0,0,0, 0,0,0,0, 0,0,0,3	$R_{13}=2$, $R_i=0$, $i \neq 13$	$R_{14}=1$, $R_i=0$, $i \neq 14$
20	0,18	0,0,17	0,0,0,16	0,0,0,0, 15	0,0,0,0, 0,14							$R_{18}=2$, $R_i=0$, $i \neq 18$	$R_{19}=1$, $R_i=0$, $i \neq 19$
25	0,23	0,0,22	0,0,0,21	0,0,0,0, 20								$R_{23}=2$, $R_i=0$, $i \neq 23$	$R_{24}=1$, $R_i=0$, $i \neq 24$
30	0,28	0,0,27	0,0,0,26										$R_{29}=1$, $R_i=0$, $i \neq 29$

Table 6.4.6: Least Precise Censoring Schemes (With Respect to Determinant): Extreme-Value Distribution

$n \setminus m$	2	3	4	5	6	7	8	9	10	11	12	n-2	n-1
15	0,13	0,0,12	0,0,0,11	0,0,0,0, 10	0,0,0,0, 0,9	0,0,0,0, 0,0,8	0,0,0,0, 0,0,0,7	0,0,0,0, 0,0,0,0, 6	0,0,0,0, 0,0,0,0, 0,5	0,0,0,0, 0,0,0,0, 0,0,4	0,0,0,0, 0,0,0,0, 0,0,0,3	$R_{13}=2$, $R_i=0$, $i \neq 13$	$R_{14}=1$, $R_i=0$, $i \neq 14$
20	0,18	0,0,17	0,0,0,16	0,0,0,0, 15	0,0,0,0, 0,14							$R_{18}=2$, $R_i=0$, $i \neq 18$	$R_{19}=1$, $R_i=0$, $i \neq 19$
25	0,23	0,0,22	0,0,0,21	0,0,0,0, 20								$R_{23}=2$, $R_i=0$, $i \neq 23$	$R_{24}=1$, $R_i=0$, $i \neq 24$
30	0,28	0,0,27	0,0,0,26										$R_{29}=1$, $R_i=0$, $i \neq 29$

Table 6.4.7: Table of Coefficients, Variances and Covariance of BLUE's for selected schemes in Tables 6.4.1 - 6.4 6					
Scheme	Coefficients (μ^*)	Coefficients (σ^*)	$Var(\mu^*)/\sigma^2$	$Var(\sigma^*)/\sigma^2$	$Cov(\mu^*,\sigma^*)/\sigma^2$
0,13	-2.1745,3.1745	-0.9663,0.9663	5.6144	0.9661	2.1911
13,0	-0.1323,1.1323	-0.3447,0.3447	1.3716	0.2872	0.4302
0,0,12	-0.8285,-0.7651,2.5936	-0.4799,-0.4573,0.9372	1.8474	0.4743	0.8299
12,0,0	0.0300,0.0978,0.8792	-0.2507,-0.1109,0.3617	0.5211	0.1625	0.1208
0,0,0,11	-0.4341,-0.4035,-0.3457,2.1833	-0.3166,-0.3076,-0.2798,0.9040	0.8898	0.3101	0.4334
11,0,0,0	0.0555,0.0542,0.1960,0.6944	-0.2044,-0.1199,-0.0353,0.3596	0.3204	0.1234	0.0473
0,0,0,0,10	-0.2576,-0.2398,-0.2054,-0.1590,1.8618	-0.2343,-0.2312,-0.2145,-0.1880,0.8680	0.5115	0.2278	0.2570
10,0,0,0,0	0.0623,0.0462,0.1060,0.2228,0.5627	-0.1727,-0.1124,-0.0759,0.0086,0.3524	0.2351	0.1029	0.0182
0,10,0,0,0	-0.0484,0.1683,0.0607,0.1858,0.6335	-0.1073,-0.1730,-0.0773,0.0018,0.3557	0.2592	0.1026	0.0524
0,0,0,0,0,9	-0.1618,-0.1499,-0.1272,-0.0971,-0.0598, 1.5958	-0.1845,-0.1845,-0.1738,-0.1558,-0.1312, 0.8297	0.3284	0.1783	0.1618
9,0,0,0,0,0	0.0610,0.0431,0.0754,0.1316,0.2218, 0.4671	-0.1486,-0.1033,-0.0836,-0.0447,0.0364, 0.3438	0.1881	0.0895	0.0039
0,9,0,0,0,0	-0.0210,0.1413,0.0477,0.1022,0.2067, 0.5231	-0.0908,-0.1592,-0.0800,-0.0452,0.0338, 0.3414	0.1956	0.0863	0.0250
0,18	-2.4828,3.4828	-0.9748,0.9748	7.0332	0.9747	2.4953
18,0	-0.1330,1.1330	-0.3171,0.3171	1.4638	0.2550	0.4302
0,0,17	-0.9844,-0.9322,2.9167	-0.4851,-0.4682,0.9532	2.4096	0.4808	0.9842
17,0,0	0.0192,0.0870,0.8938	-0.2360,-0.0970,0.3330	0.5449	0.1438	0.1247

Table 6.4.7: Table of Coefficients, Variances and Covariance of BLUE's for selected schemes in Tables 6.4.1 - 6.4.6

Scheme	Coefficients (μ^*)	Coefficients (σ^*)	$Var(\mu^*)/\sigma^2$	$Var(\sigma^*)/\sigma^2$	$Cov(\mu^*,\sigma^*)/\sigma^2$
0,0,0.16	-0.5393,-0.5145,-0.4660,2.5198	-0.3210,-0.3141,-0.2938,0.9289	1.1973	0.3161	0.5373
16,0,0,0	0.5287,0.0440,0.1899,0.7132	-0.1961,-0.1059,-0.0290,0.3309	0.3305	0.1097	0.0524
0,0,0,0,15	-0.3374,-0.3234,-0.2943,-0.2553,2.2104	-0.2385,-0.2361,-0.2236,-0.2045,0.9028	0.7031	0.2337	0.3355
15,0,0,0,0	0.0614,0.0369,0.0982,0.2204,0.5831	-0.1686,-0.1000,-0.0664,0.0105,0.0325	0.2401	0.0918	0.0236
0,15,0,0,0	-0.0584,0.1689,0.0464,0.1779,0.6652	-0.0936,-0.1699,-0.0676,0.0038,0.3274	0.2757	0.0903	0.0610
0,0,0,0,0,14	-0.2264,-0.2173,-0.1984,-0.1723,-0.1408, 1.9551	-0.1888,-0.1886,-0.1807,-0.1674,-0.1500, 0.8755	0.4560	0.1841	0.2248
14,0,0,0,0,0	0.0618,0.0349,0.0674,0.1263,0.2215, 0.4881	-0.1475,-0.0925,-0.0741,-0.0383,0.0354, 0.3170	0.1905	0.0802	0.0092
0,14,0,0,0,0	-0.0302,0.1449,0.0349,0.0916,0.2040, 0.5548	-0.0789,-0.1590,-0.0699,-0.0385,0.0330, 0.3133	0.2042	0.0757	0.0328
0,23	-2.7197,3.7197	-0.9799,0.9799	8.2493	0.9799	2.7296
23,0	-0.1321,1.1321	-0.2982,0.2982	1.5277	0.2330	0.4268
0,0,22	-1.1039,-1.0595,3.1633	-0.4881,-0.4747,0.9628	2.9046	0.4847	1.1030
22,0,0	0.0169,0.0804,0.9027	-0.2256,-0.0876,0.3132	0.5613	0.1311	0.1258
0,0,0,21	-0.6198,-0.5988,-0.5568,2.7754	-0.3235,-0.3181,-0.3019,0.9435	1.4762	0.3196	0.6173
21,0,0,0	0.0509,0.0376,0.1868,0.7247	-0.1900,-0.0965,-0.0246,0.3111	0.3377	0.1004	0.0549
0,0,0,0,20	-0.3985,-0.3868,-0.3616,-0.3271,2.4740	-0.2410,-0.2390,-0.2291,-0.2139,0.9230	0.8828	0.2371	0.3960
20,0,0,0,0	0.0604,0.0310,0.0934,0.2196,0.5955	-0.1653,-0.0916,-0.0601,0.0119,0.3051	0.2438	0.0844	0.0265

Table 6.4.7: Table of Coefficients, Variances and Covariance of BLUE's for selected schemes in Tables 6.4.1 - 6.4 6

Scheme	Coefficients (μ^*)	Coefficients (σ^*)	$Var(\mu^*)/\sigma^2$	$Var(\sigma^*)/\sigma^2$	$Cov(\mu^*,\sigma^*)/\sigma^2$
0,20,0,0,0	-0.0635,0.1673,0.0376,0.1737,0.6850	-0.0844,-0.1673,-0.0612,0.0053,0.3076	0.2882	0.0819	0.0653
0,28	-2.9118,3.9118	-0.9832,0.9832	9.3175	0.9832	2.9201
28,0	-0.1307,1.1307	-0.2842,0.2842	1.5748	0.2167	0.4222
0,0,27	-1.2007,-1.1617,3.3624	-0.4901,-0.4790,0.9691	3.3470	0.4873	1.1994
27,0,0	0.0153,0.0760,0.9087	-0.2177,-0.0808,0.2985	0.5733	0.1218	0.1258
0,0,0,26	-0.6850,-0.6668,-0.6292,2.9810	-0.3252,-0.3207,-0.3072,0.9531	1.7297	0.3220	0.6823
26,0,0,0	0.0493,0.0332,0.1849,0.7325	-0.1852,-0.0896,-0.0215,0.2962	0.3432	0.0936	0.0563

6.5 The Extreme-Value (II) Distribution

6.5.1 Introduction

Suppose X has the extreme-value distribution discussed in section 6.4. If we make the transformation $Y = -X$, then Y is said to have the extreme-value distribution of Type II or the extreme-value (II) distribution, with pdf

$$g(y) = e^{-y}e^{-e^{-y}}, \ -\infty < y < \infty . \tag{6.5.1}$$

This distribution is again seen in results involving extreme (specifically smallest) order statistics from many distributions. Moments for usual order statistics from this distribution can be easily obtained from the moments of usual order statistics for the extreme-value distribution. From a theoretical point of view, it will be interesting to see if results obtained for optimal censoring with respect to the trace of the variance-covariance matrix of two-parameter BLUE's for the extreme-value (II) distribution will be somehow "opposite" to those obtained for the extreme-value distribution in section 6.4.

6.5.2 Results for Optimal Censoring

Tables 6.5.1 to 6.5.7 deal with results for the extreme-value (II) distribution. Considering the trace efficiencies in Table 6.5.1, it is evident that for this distribution, although the gains in precision over the conventional censoring scheme are still respectable for small values of n, they are no where near those for the extreme-value distribution. Furthermore, from Table 6.5.2, for larger values of m, the optimal censoring scheme is seen to be the conventional Type-II right censoring scheme. However, for small values of m, the optimal schemes tend to have censoring at the beginning and end of the experimentation, but not much in between. We again find that for fixed values of $m > 3$, the value of the trace decreases with the sample size n employed.

Table 6.5.5 reveals that the least efficient censoring schemes with respect to the trace for values of $m > 5$ is the scheme in which all but m - 1 units are removed from experimentation immediately following the first observed failure. Although this is in contrast to the extreme-value distribution discussed in section 6.4, for small values of m the least efficient scheme is still seen to be in many cases the conventional scheme. Table 6.5.7 enables us to compare the efficiency of the worst and the best censoring schemes, for example, for $m = 4$ and $n = 15$, the least precise censoring scheme (0,11,0,0) is 87% as precise as the most trace-efficient censoring scheme (7,0,0,4) and 90% as precise as the conventional censoring scheme, whereas for $m = 4$ and $n = 25$, the least precise censoring scheme (0,0,0,21) is only 78% as precise as the most precise censoring scheme (17,0,0,4).

Table 6.5.1: Efficiencies and *Traces* for Trace-Optimal Censoring Schemes: Extreme-Value (II) Distribution (Line 1: Efficiency; Line 2: *Trace*)

$n \setminus m$	2	3	4	5	6	7	8	9	10	11	12	n-2	n-1	n
15	1.3747 *1.0056*	1.2061 *0.5117*	1.1060 *0.3529*	1.0347 *0.2748*	1.0153 *0.2282*	1.0018 *0.1968*	1.0000 *0.1744*	1.0000 *0.1585*	1.0000 *0.1470*	1.0000 *0.1384*	1.0000 *0.1319*	1.0000 *0.1269*	1.0000 *0.1230*	1.0000 *0.1201*
20	1.5030 *1.0130*	1.3221 *0.5064*	1.1991 *0.3455*	1.1181 *0.2664*	1.0652 *0.2193*							1.0000 *0.0925*	1.0000 *0.0906*	1.0000 *0.0891*
25	1.6155 *1.0141*	1.4256 *0.5026*	1.2884 *0.3406*	1.1925 *0.2612*								1.0000 *0.0728*	1.0000 *0.0717*	1.0000 *0.0707*
30	1.7132 *1.0135*	1.5172 *0.4995*	1.3705 *0.3371*										1.0000 *0.0593*	1.0000 *0.0587*

Table 6.5.2: Trace-Optimal Censoring Schemes: Exteme-Value (II) Distribution

$n \backslash m$	2	3	4	5	6	7	8	9	10	11	12	n-2	n-1
15	12,1	9,0,3	7,0,0,4	5,0,0,0, 5	3,0,0,0, 0,6	1,0,0,0, 0,0,7	0,0,0,0, 0,0,0,7	0,0,0,0, 0,0,0,0, 6	0,0,0,0, 0,0,0,0, 0,5	0,0,0,0, 0,0,0,0, 0,0,4	0,0,0,0, 0,0,0,0, 0,0,0,3	$R_{13}=2$, $R_i=0$, $i \neq 13$	$R_{14}=1$, $R_i=0$, $i \neq 14$
20	16,2	14,0,3	12,0,0,4	10,0,0,0, 5	8,0,0,0, 0,6							$R_{18}=2$, $R_i=0$, $i \neq 18$	$R_{19}=1$, $R_i=0$, $i \neq 19$
25	21,2	19,0,3	17,0,0,4	15,0,0,0, 5								$R_{23}=2$, $R_i=0$, $i \neq 23$	$R_{24}=1$, $R_i=0$, $i \neq 24$
30	26,2	24,0,3	22,0,0,4										$R_{29}=1$, $R_i=0$, $i \neq 29$

Table 6.5.3: Efficiencies and *Determinants* for Determinant-Optimal Censoring Schemes: Extreme-Value (II) Distribution (Line 1: Efficiency; Line 2: *Determinant*)

$n \setminus m$	2	3	4	5	6	7	8	9	10	11	12	n-2	n-1	n
15	1.2443 *0.0548*	1.1067 *0.0256*	1.0435 *0.0162*	1.0042 *0.0116*	1.0050 *0.0090*	1.0001 *0.0073*	1.0000 *0.0061*	1.0000 *0.0053*	1.0000 *0.0047*	1.0000 *0.0042*	1.0000 *0.0038*	1.0000 *0.0036*	1.0000 *0.0033*	1.0000 *0.0031*
20	1.3143 *0.0433*	1.1538 *0.0201*	1.0740 *0.0127*	1.0418 *0.0090*	1.0230 *0.0069*							1.0000 *0.0019*	1.0000 *0.0018*	1.0000 *0.0017*
25	1.3719 *0.0364*	1.1941 *0.0168*	1.1031 *0.0106*	1.0659 *0.0075*								1.0000 *0.0011*	1.0000 *0.0011*	1.0000 *0.0011*
30	1.4209 *0.0318*	1.2285 *0.0146*	1.1286 *0.0092*										1.0000 *0.0008*	1.0000 *0.0007*

Table 6.5.4: Determinant-Optimal Censoring Schemes: Exteme-Value (II) Distribution

$n \setminus m$	2	3	4	5	6	7	8	9	10	11	12	n-2	n-1
15	11,2	9,0,3	6,0,0,5	0,0,0,3, 7	0,3,0,0, 0,6	0,0,0,1, 0,0,7	0,0,0,0, 0,0,0,7	0,0,0,0, 0,0,0,0, 6	0,0,0,0, 0,0,0,0, 0,5	0,0,0,0, 0,0,0,0, 0,0,4	0,0,0,0, 0,0,0,0, 0,0,0,3	$R_{13}=2$, $R_i=0$, $i \neq 13$	$R_{14}=1$, $R_i=0$, $i \neq 14$
20	16,2	13,0,4	9,2,0,5	0,10,0,0, 5	0,7,0,0, 0,7							$R_{18}=2$, $R_i=0$, $i \neq 18$	$R_{19}=1$, $R_i=0$, $i \neq 19$
25	21,2	18,0,4	14,1,0,6	0,14,0,0, 6								$R_{23}=2$, $R_i=0$, $i \neq 23$	$R_{24}=1$, $R_i=0$, $i \neq 24$
30	26,2	22,0,5	18,1,0,7										$R_{29}=1$, $R_i=0$, $i \neq 29$

Table 6.5.5: Least Precise Censoring Schemes (With Respect to Trace): Extreme-Value (II) Distribution

$n \setminus m$	2	3	4	5	6	7	8	9	10	11	12	n-2	n-1
15	0,13	0,0,12	0,11,0,0	10,0,0,0,0	9,0,0,0,0,0	8,0,0,0,0,0,0	7,0,0,0,0,0,0,0	6,0,0,0,0,0,0,0,0	5,0,0,0,0,0,0,0,0,0	4,0,0,0,0,0,0,0,0,0,0	3,0,0,0,0,0,0,0,0,0,0,0	$R_1=2$, $R_i=0$, $i \neq 1$	$R_1=1$, $R_i=0$, $i \neq 1$
20	0,18	0,0,17	0,0,16,0	0,15,0,0,0	14,0,0,0,0,0							$R_1=2$, $R_i=0$, $i \neq 1$	$R_1=1$, $R_i=0$, $i \neq 1$
25	0,23	0,0,22	0,0,0,21	0,0,20,0,0								$R_1=2$, $R_i=0$, $i \neq 1$	$R_1=1$, $R_i=0$, $i \neq 1$
30	0,28	0,0,27	0,0,0,26										$R_1=1$, $R_i=0$, $i \neq 1$

Table 6.5.6: Least Precise Censoring Schemes (With Respect to Determinant): Extreme-Value (II) Distribution

$n \setminus m$	2	3	4	5	6	7	8	9	10	11	12	n-2	n-1
15	0,13	12,0,0	11,0,0,0	10,0,0,0, 0	9,0,0,0, 0,0	8,0,0,0, 0,0,0	7,0,0,0, 0,0,0,0	6,0,0,0, 0,0,0,0, 0	5,0,0,0, 0,0,0,0, 0,0	4,0,0,0, 0,0,0,0, 0,0,0	3,0,0,0, 0,0,0,0, 0,0,0,0	$R_1=2$, $R_i=0$, $i \neq 1$	$R_1=1$, $R_i=0$, $i \neq 1$
20	0,18	17,0,0	16,0,0,0	15,0,0,0, 0	14,0,0,0, 0,0							$R_1=2$, $R_i=0$, $i \neq 1$	$R_1=1$, $R_i=0$, $i \neq 1$
25	0,23	22,0,0	21,0,0,0	20,0,0,0, 0								$R_1=2$, $R_i=0$, $i \neq 1$	$R_1=1$, $R_i=0$, $i \neq 1$
30	0,28	0,0,27	26,0,0,0										$R_1=1$, $R_i=0$, $i \neq 1$

Table 6.5.7: Table of Coefficients, Variances and Covariance of BLUE's for selected schemes in Tables 6.5.1 - 6.5.6

Scheme	Coefficients (μ^*)	Coefficients (σ^*)	$Var(\mu^*)/\sigma^2$	$Var(\sigma^*)/\sigma^2$	$Cov(\mu^*,\sigma^*)/\sigma^2$
0,13	-2.2897,3.2897	-2.9044,2.9044	0.6754	0.7070	0.6397
12,1	0.0345,0.9655	-0.8524,0.8524	0.5487	0.4569	0.4422
11,2	-0.2429,1.2429	-1.0973,1.0973	0.5352	0.4732	0.4455
0,0,12	-0.8766,-0.1520,2.0286	-1.5323,-0.4373,1.9697	0.2815	0.3356	0.2573
9,0,3	-0.0469,0.1601,0.8868	-0.8577,-0.0146,0.8724	0.2649	0.2468	0.1995
12,0,0	0.3750,0.3218,0.3032	-0.5171,0.2187,0.2985	0.3310	0.2459	0.2279
0,0,0,11	-0.4284,-0.0444,0.0539,1.4188	-1.0582,-0.3234,-0.1193,1.5010	0.1745	0.2159	0.1441
7,0,0,4	0.0054,0.1020,0.1645,0.7281	-0.7435,-0.1048,0.0653,0.7830	0.1798	0.1732	0.1220
0,11,0,0	-0.3687,0.9472,0.2217,0.1998	-0.8797,0.4617,0.1874,0.2306	0.2122	0.1949	0.1531
6,0,0,5	-0.0671,0.0768,0.1558,0.8344	-0.7960,-0.1433,0.0444,0.8950	0.1762	0.1788	0.1237
11,0,0,0	0.3623,0.2186,0.2325,0.1886	-0.4871,0.0847,0.2008,0.2015	0.2338	0.1693	0.1430
0,0,0,0,10	-0.2096,0.0118,-0.0575,0.3192,0.9361	-0.8104,-0.2599,-0.2455,0.2555,1.0602	0.1280	0.1563	0.0915
5,0,0,0,5	0.0202,0.0851,0.1154,0.1441,0.6352	-0.6646,-0.1417,-0.0124,0.0787,0.7401	0.1383	0.1365	0.0846
10,0,0,0,0	0.3450,0.1755,0.1834,0.1668,0.1293	-0.4597,0.0181,0.1268,0.1624,0.1524	0.1848	0.1307	0.1010
0,0,0,3,7	-0.2051,0.0137,-0.0064,0.4301,0.7677	-0.8006,-0.2564,-0.1889,0.3603,0.8857	0.1286	0.1547	0.0911
0,0,0,0,0,9	-0.0949,0.0434,0.0595,0.1090,0.1175, 0.7655	-0.6652,-0.2199,-0.0973,-0.0106,0.0237, 0.9693	0.1078	0.1238	0.0658
3,0,0,0,0,6	0.0202,0.0767,0.0983,0.1142,0.1133, 0.5772	-0.6020,-0.1630,-0.0469,0.0221,0.0646, 0.7252	0.1138	0.1143	0.0629
9,0,0,0,0,0	0.3254,0.1545,0.1549,0.1441,0.1250, 0.0960	-0.4348,-0.0203,0.0777,0.1206,0.1337, 0.1231	0.1552	0.1074	0.0761
0,3,0,0,0,6	-0.0761,0.1797,0.0902,0.1105,0.1132, 0.5825	-0.6254,-0.1337,-0.0576,0.0172,0.0613, 0.7381	0.1121	0.1179	0.0652
0,18	-2.9299,3.9299	-3.2127,3.2127	0.8055	0.7171	0.7215
16,2	-0.2686,1.2686	-1.0371,1.0371	0.5755	0.4376	0.4566

Table 6.5.7: Table of Coefficients, Variances and Covariance of BLUE's for selected schemes in Tables 6.5.1 - 6.5.6					
Scheme	Coefficients (μ^*)	Coefficients (σ^*)	$Var(\mu^*)/\sigma^2$	$Var(\sigma^*)/\sigma^2$	$Cov(\mu^*,\sigma^*)/\sigma^2$
0,0,17	-1.2170,-0.3252,2.5422	-1.6926,-0.5636,2.2562	0.3283	0.3412	0.2981
14,0,3	-0.0662,0.1483,0.9180	-0.8177,-0.0084,0.8261	0.2798	0.2266	0.2080
13,0,4	-0.1775,0.1076,1.0699	-0.9020,-0.0601,-0.9621	0.2765	0.2342	0.2113
17,0,0	0.3510,0.3295,0.3194	-0.5027,0.2157,0.2869	0.3496	0.2319	0.2395
0,0,0,16	-0.6673,-0.1647,-0.0307,1.8627	-1.1681,-0.4105,-0.1988,1.7773	0.1947	0.2196	0.1706
12,0,0,4	-0.0085,0.0895,0.1557,0.7633	-0.7158,-0.0885,0.0627,0.7415	0.1870	0.1584	0.1298
0,0,16,0	-0.6285,-0.1381,1.5432,0.2234	-1.0651,-0.3640,1.1910,0.2382	0.2113	0.2082	0.1733
9,2,0,5	-0.1872,0.1797,0.1309,0.8766	-0.8236,-0.0595,0.0260,0.8571	0.1837	0.1691	0.1354
16,0,0,0	0.3432,0.2151,0.2434,0.1983	-0.4758,0.0864,0.1968,0.1926	0.2423	0.1594	0.1523
0,0,0,0,15	-0.4028,-0.0834,0.0003,0.0497,1.4363	-0.8987,-0.3277,-0.1672,-0.0695,1.4631	0.1376	0.1603	0.1124
10,0,0,0,5	0.0123,0.0682,0.1116,0.1330,0.6749	-0.6457,-0.1258,0.0008,0.0696,0.7012	0.1419	0.1246	0.0919
0,15,0,0,0	-0.3179,0.8208,0.1658,0.1809,0.1504	-0.7066,0.2764,0.0971,0.1678,0.1653	0.1735	0.1393	0.1180
0,10,0,0,5	-0.3047,0.3958,0.0885,0.1222,0.6983	-0.7524,-0.0128,-0.0303,0.0559,0.7395	0.1412	0.1370	0.1019
15,0,0,0,0	0.3310,0.1658,0.1880,0.1763,0.1389	-0.4513,0.0220,0.1267,0.1577,0.1450	0.1885	0.1229	0.1091
0,0,0,0,0,14	-0.2500,-0.0350,0.0206,0.0528,0.0739, 1.1378	-0.7333,-0.2752,-0.1453,-0.0662,-0.0118, 1.2317	0.1080	0.1256	0.0803
8,0,0,0,0,6	0.0183,0.0591,0.0890,0.1048,0.1124, 0.6164	-0.5901,-0.1462,-0.0363,0.0263 0.0656, 0.6806	0.1152	0.1041	0.0697
14,0,0,0,0,0	0.3166,0.1412,0.1544,0.1501,0.1334, 0.1043	-0.4290,-0.0152,0.0796,0.1188,0.1291, 0.1168	0.1561	0.1008	0.0834
0,7,0,0,0,7	-0.1955,0.2323,0.0666,0.0898,0.1037, 0.7031	-0.6471,-0.1185,-0.0698,0.0027,0.0505, 0.7821	0.1120	0.1128	0.0754
0,23	-3.4458,4.4458	-3.4510,3.4510	0.9135	0.7248	0.7824
21,2	-0.2515,1.2515	-0.9947,0.9947	0.6021	0.4120	0.4601
0,0,22	-1.4907,-0.4731,2.9639	-1.8166,-0.6613,2.4778	0.3710	0.3456	0.3288

Table 6.5.7: Table of Coefficients, Variances and Covariance of BLUE's for selected schemes in Tables 6.5.1 - 6.5.6					
Scheme	Coefficients (μ^*)	Coefficients (σ^*)	$Var(\mu^*)/\sigma^2$	$Var(\sigma^*)/\sigma^2$	$Cov(\mu^*,\sigma^*)/\sigma^2$
19,0,3	-0.0777,0.1432,0.9345	-0.7904,-0.0019,0.7922	0.2903	0.2123	0.2115
18,0,4	-0.1868,0.1013,1.0855	-0.8693,-0.0506,0.9199	0.2870	0.2186	0.2143
22,0,0	0.3350,0.3354,0.3296	-0.4924,0.2139,0.2785	0.3639	0.2219	0.2459
0,0,0,21	-0.8584,-0.2685,-0.1068,2.2337	-1.2527,-0.4788,-0.2602,1.9918	0.2163	0.2226	0.1909
17,0,0,4	-0.0170,0.0806,0.1557,0.7806	-0.6965,-0.0788,0.0662,0.7091	0.1926	0.1480	0.1333
14,1,0,6	-0.2020,0.0883,0.1166,0.9971	-0.8129,-0.1082,0.0136,0.9075	0.1882	0.1587	0.1388
21,0,0,0	0.3300,0.2136,0.2510,0.2054	-0.4679,0.0880,0.1939,0.1859	0.2497	0.1523	0.1576
0,0,0,0,20	-0.5521,-0.1641,-0.0586,0.0029,1.7720	-0.9634,-0.3802,-0.2147,-0.1152,1.6736	0.1490	0.1626	0.1273
15,0,0,0,5	0.0070,0.0575,0.1084,0.1338,0.6933	-0.6324,-0.1147,0.0068,0.0714,0.6690	0.1451	0.1161	0.0954
0,0,20,0,0	-0.4921,-0.1227,1.2696,0.1777,0.1675	-0.8245,-0.3128,0.8093,0.1502,0.1778	0.1702	0.1483	0.1300
0,14,0,0,6	-0.4041,0.4140,0.0656,0.1097,0.8148	-0.7743,-0.0178,-0.0476,0.0401,0.7996	0.1456	0.1335	0.1093
20,0,0,0,0	0.3209,0.1602,0.1918,0.1825,0.1446	-0.4455,0.0252,0.1266,0.1542,0.1396	0.1925	0.1173	0.1138
0,28	-3.8786,4.8786	-3.6450,3.6450	1.0052	0.7310	0.8305
26,2	-0.2886,1.2886	-0.9627,0.9627	0.6210	0.3925	0.4604
0,0,27	-1.7192,-0.6045,3.3237	-1.9168,-0.7431,2.6599	0.4089	0.3490	0.3532
24,0,3	-0.0850,0.1405,0.9445	-0.7698,0.0034,0.7664	0.2980	0.2015	0.2129
22,0,5	-0.2909,0.0592,1.2317	-0.9138,-0.0850,0.9989	0.2943	0.2130	0.2192
0,0,0,26	-1.0180,-0.3584,-0.1774,2.5538	-1.3215,-0.5341,-0.3124,2.1680	0.2369	0.2251	0.2072
22,0,0,4	-0.0228,0.0752,0.1565,0.7912	-0.6820,-0.0712,0.0690,0.6843	0.1969	0.1402	0.1350
18,1,0,7	-0.2610,0.0561,0.0997,1.1052	-0.8302,-0.1256,-0.0015,0.9572	0.1924	0.1533	0.1425
26,0,0,0	0.3201,0.2132,0.2566,0.2101	-0.4619,0.0895,0.1916,0.1808	0.2558	0.1468	0.1610

6.6 The Log-Normal Distribution

6.6.1 Introduction

The standard log-normal distribution has pdf

$$f(x) = \frac{1}{x\sqrt{2\pi}} e^{-(\ln x)^2/2}, \; 0 < x < \infty . \qquad (6.6.1)$$

This distribution has been used widely in modelling real data, including life-testing data. [see Aitchison and Brown (1969)]. Single and product moments of usual order statistics from this distribution are tabulated in Gupta, McDonald and Galarneau (1974) for sample sizes up to $n = 20$. These will be used in the following subsection, along with Eqs. (6.2.1) and (6.2.2), to obtain optimal progressive censoring patterns when data are from the log-normal distribution.

6.6.2 Results for Optimal Censoring

Tables 6.6.1 to 6.6.7 summarize the results obtained for the log-normal distribution, and follow the same sequence as tables for the other three distributions which we have considered. From Tables 6.6.1 to 6.6.4, it is evident that for the log-normal distribution, the optimal progressive Type-II right censoring pattern (for both trace and determinant criterion) is the conventional Type-II right censoring for the vast majority of cases, and in those cases where it is not, the efficiency of the optimal censoring pattern with respect to the conventional censoring pattern is very slightly favourable. Furthermore, from Tables 6.6.5 and 6.6.6, without exception, the least precise censoring scheme for the cases considered is that in which all censoring is carried out immediately following the first observed failure. Thus, our suspicions from very early on have been answered: the optimal censoring pattern is not the one in which $R_1 = n - m$ in general.

It is interesting to note that for the log-normal distribution, a larger value of n for fixed m results in a decrease in the optimal value of the trace, even for $m = 2$ and $m = 3$. The increase in precision is quite significant.

Again, if one is interested in considering the trace efficiency of the least precise and most precise censoring schemes with respect to the trace of the variance-covariance matrix of BLUE's, Table 6.6.7 will be of help. For example, for $m = 5$ and $n = 15$, the least precise censoring scheme (10,0,0,0,0) is about 58% as precise as the most precise censoring scheme (0,0,0,0,10). Thus, if an experimenter is concerned with saving units for further testing, this factor should be taken into account before embarking on the progressive censoring scheme (10,0,0,0,0). Similar calculations for determinant analyses can also be made using Table 6.6.7.

The results obtained for the extreme-value (II) distribution and the log-normal distribution were similar, in the sense that these distributions tended to result in optimal censoring schemes being the conventional Type-II right censoring scheme. One immediate similarity of these two distributions is that they are both right-skewed distributions. Further study into patterns associated with optimal censoring schemes is an area which will be interesting to pursue in the future.

Remark: The tables of covariances of usual order statistics from the log-normal distribution for $n = 20$ which were used in this section were given to three-decimal accuracy, and as such, there were problems in confirming that the variance-covariance matrices of progressive Type-II right censored order statistics from the log-normal distribution for large values of m were positive definite. For this reason, only small values of m were considered for $n = 20$. Furthermore, although the results for $n = 20$ may not be as precise as others presented in this chapter, they should point us in the right direction for future study.

Table 6.6.1: Efficiencies and *Traces* for Trace-Optimal Censoring Schemes: Log-normal Distribution (Line 1: Efficiency; Line 2: *Trace*)													
$n \setminus m$	2	3	4	5	6	7	8	9	10	11	12	n-2	n-1	n
15	1.0018 *0.8366*	1.0000 *0.4325*	1.0000 *0.3019*	1.0000 *0.2385*	1.0000 *0.2014*	1.0000 *0.1774*	1.0000 *0.1607*	1.0000 *0.1486*	1.0000 *0.1396*	1.0000 *0.1327*	1.0000 *0.1274*	1.0000 *0.1234*	1.0000 *0.1205*	1.0000 *0.1187*
20	1.0000 *0.6751*	1.0007 *0.3756*	1.0000 *0.2697*	1.0253 *0.2192*	1.0000 *0.1739*									

Table 6.6.2: Trace-Optimal Censoring Schemes: Log-normal Distribution												
$n \setminus m$	2	3	4	5	6	7	8	9	10	11	12	n-2	n-1
15	2,11	0,0,12	0,0,0,11	0,0,0,0, 10	0,0,0,0, 0,9	0,0,0,0, 0,0,8	0,0,0,0, 0,0,0,7	0,0,0,0, 0,0,0,0, 6	0,0,0,0, 0,0,0 0, 0,5	0,0,0,0, 0,0,0,0, 0,0,4	0,0,0,0, 0,0,0,0, 0,0,0,3	$R_{13}=2$, $R_i=0$, $i \neq 13$	$R_{14}=1$, $R_i=0$, $i \neq 14$
20	0,18	0,1,16	0,0,0,16	0,0,0,9, 6	0,0,0,0, 0,14								

Table 6.6.3: Efficiencies and *Determinants* for Determinant-Optimal Censoring Schemes: Log-normal Distribution (Line 1: Efficiency; Line 2: *Determinant*)

$n \setminus m$	2	3	4	5	6	7	8	9	10	11	12	n-2	n-1	n
15	1.0000 *0.0086*	1.0000 *0.0043*	1.0000 *0.0030*	1.0000 *0.0023*	1.0000 *0.0020*	1.0000 *0.0017*	1.0000 *0.0015*	1.0000 *0.0014*	1.0000 *0.0013*	1.0000 *0.0013*	1.0000 *0.0012*	1.0000 *0.0012*	1.0000 *0.0011*	1.0000 *0.0011*
20	1.0000 *0.0048*	1.0000 *0.0026*	1.0000 *0.0018*	1.0171 *0.0015*	1.0000 *0.0011*									

Table 6.6.4: Determinant-Optimal Censoring Schemes: Log-normal Distribution

$n \setminus m$	2	3	4	5	6	7	8	9	10	11	12	n-2	n-1
15	2,11	0,0,12	0,0,0,11	0,0,0,0,10	0,0,0,0,0,9	0,0,0,0,0,0,8	0,0,0,0,0,0,0,7	0,0,0,0,0,0,0,0,6	0,0,0,0,0,0,0,0,0,5	0,0,0,0,0,0,0,0,0,0,4	0,0,0,0,0,0,0,0,0,0,0,3	$R_{13}=2$, $R_i=0$, $i \neq 13$	$R_{14}=1$, $R_i=0$, $i \neq 14$
20	0,18	0,1,16	0,0,0,16	0,0,0,9,6	0,0,0,0,0,14								

Table 6.6.5: Least Precise Censoring Schemes (With Respect to Trace): Log-normal Distribution

$n \setminus m$	2	3	4	5	6	7	8	9	10	11	12	n-2	n-1
15	13,0	12,0,0	11,0,0,0	10,0,0,0,0	9,0,0,0,0,0	8,0,0,0,0,0,0	7,0,0,0,0,0,0,0	6,0,0,0,0,0,0,0,0	5,0,0,0,0,0,0,0,0,0	4,0,0,0,0,0,0,0,0,0,0	3,0,0,0,0,0,0,0,0,0,0,0	$R_1=2$, $R_i=0$, $i \neq 1$	$R_1=1$, $R_i=0$, $i \neq 1$
20	18,0	17,0,0	16,0,0,0	15,0,0,0,0	14,0,0,0,0,0								

Table 6.6.6: Least Precise Censoring Schemes (With Respect to Determinant): Log-normal Distribution

$n \setminus m$	2	3	4	5	6	7	8	9	10	11	12	n-2	n-1
15	13,0	12,0,0	11,0,0,0	10,0,0,0,0	9,0,0,0,0,0	8,0,0,0,0,0,0	7,0,0,0,0,0,0,0	6,0,0,0,0,0,0,0,0	5,0,0,0,0,0,0,0,0,0	4,0,0,0,0,0,0,0,0,0,0	3,0,0,0,0,0,0,0,0,0,0,0	$R_1=2$, $R_i=0$, $i \neq 1$	$R_1=1$, $R_i=0$, $i \neq 1$
20	18,0	17,0,0	16,0,0,0	15,0,0,0,0	14,0,0,0,0,0								

Table 6.6.7: Table of Coefficients, Variances and Covariance of BLUE's for selected schemes in Tables 6.6.1 - 6.6.6

Scheme	Coefficients (μ^*)	Coefficients (σ^*)	$Var(\mu^*)/\sigma^2$	$Var(\sigma^*)/\sigma^2$	$Cov(\mu^*,\sigma^*)/\sigma^2$
0,13	2.8333,-1.8333	-9.0515,9.0515	0.0502	0.7879	-0.1759
2,11	2.6045,-1.6045	-7.9216,7.9216	0.0496	0.7870	-0.1744
13,0	1.1307,-0.1307	-0.6454,0.6454	0.0939	2.0131	-0.4081
0,0,12	1.9257,0.0547,-0.9804	-4.8240,0.2576,4.5664	0.0328	0.4002	-0.0927
12,0,0	1.1959,-0.1559,-0.0400	-0.9784,0.7854,0.1930	0.0467	0.8522	-0.1740
0,0,0,11	1.6020,0.0732,-0.0333,-0.6418	-3.3829,0.1754,0.3505,2.8570	0.0260	0.2759	-0.0647
11,0,0,0	1.2129,-0.1163,-0.0756,-0.0210	-1.0925,0.6302,0.3630,0.0993	0.0338	0.5332	-0.1097
0,0,0,0,10	1.4313,0.0819,-0.0161,-0.0437,-0.4533	-2.6500,0.1378,0.2765,0.2894,1.9463	0.0227	0.2157	-0.0507
10,0,0,0,0	1.2143,-0.0807,-0.0741,-0.0459,-0.0136	-1.1438,0.5038,0.3626,0.2142,0.0631	0.0278	0.3869	-0.0803
0,0,0,0,0,9	1.3258,0.0860,-0.0053,-0.0331,-0.0442, -0.3292	-2.2088,0.1206,0.2314,0.2450,0.2349, 1.3770	0.0207	0.1807	-0.0423
9,0,0,0,0,0	1.2084,-0.0509,-0.0657,-0.0503,-0.0317, -0.0098	-1.1695,0.4091,0.3361,0.2351,0.1445, 0.0448	0.0244	0.3032	-0.0634
0,18	2.9753,-1.9753	-11.2580,11.2580	0.0311	0.6440	-0.1236
18,0	1.1131,-0.1131	-0.6448,0.6448	0.0691	1.9918	-0.3501
0,0,17	2.1821,-0.2837,-0.8985	-6.7690,1.6841,5.0849	0.0220	0.3539	-0.0723
0,1,16	2.1736,-0.3122,-0.8614	-6.7289,1.8622,4.8667	0.0220	0.3536	-0.0722
17,0,0	1.1700,-1.3589,-0.0342	-0.9791,0.7884,0.1907	0.0339	0.8391	-0.1487
0,0,0,16	1.7962,-0.0965,-0.0480,-0.6518	-4.7148,0.6877,0.5573,3.4698	0.0184	0.2513	-0.0530

Table 6.6.7: Table of Coefficients, Variances and Covariance of BLUE's for selected schemes in Tables 6.6.1 - 6.6.6

Scheme	Coefficients (μ^*)	Coefficients (σ^*)	$Var(\mu^*)/\sigma^2$	$Var(\sigma^*)/\sigma^2$	$Cov(\mu^*,\sigma^*)/\sigma^2$
16,0,0,0	1.1859,-0.1033,-0.0648,-0.0178	-1.0958,0.6376,0.3603,0.0978	0.0243	0.5241	-0.0937
0,0,0,0,15	1.5430,0.0329,-0.0125,-0.0636,-0.4999	-3.3032,-0.0335,0.3592,0.1909,2.7865	0.0170	0.2077	-0.0452
0,0,0,9,6	1.5860,-0.0490,-0.0095,-0.2954,-0.2322	-3.6876,0.4555,0.3694,1.7284,1.1344	0.0164	0.2028	-0.0431
15,0,0,0,0	1.1888,-0.0743,-0.0638,-0.0391,-0.0115	-1.1499,0.5144,0.3617,0.2117,0.0620	0.0199	0.3798	-0.0685
0,0,0,0,0,14	1.4997,-0.1558,0.0627,-0.0876,0.2712, -0.5903	-3.1263,0.7367,0.0523,0.2888,-0.3612, 2.4098	0.0141	0.1597	-0.0335
14,0,0,0,0,0	1.1857,-0.0511,-0.0562,-0.0434,-0.0268, -0.0083	-1.1786,0.4223,0.3363,0.2337,0.1423, 0.0440	0.0174	0.2975	-0.0541

Section Two: Generalized Distributions

7 Introduction

Many data sets are modelled using the logistic distribution, which has a support ranging over the entire real line. Hosking (1986) introduced a shape parameter to the logistic distribution which restricts the support to only a portion of the real line. In some cases, this new distribution, known as the *generalized logistic distribution*, may describe the process of interest more satisfactorily. The generalization considered simply replaces the terms e^{-x} in the logistic distribution by the term $(1-kx)^{1/k}$, and modifies the support accordingly:

$$F(x) = \begin{cases} \dfrac{1}{1+(1-kx)^{1/k}}, & -\infty < x < \dfrac{1}{k} \quad \text{when } k > 0, \\ & \dfrac{1}{k} < x < \infty \quad \text{when } k < 0, \\ \dfrac{1}{1+e^{-x}}, & -\infty < x < \infty \quad \text{when } k = 0. \end{cases} \tag{7.1}$$

Thus, as $k \to 0$, this distribution becomes the well known logistic distribution.

Properties of this generalized distribution have not been discussed to great extent, and as such, the distribution can be studied in more detail: Determination of the values of k which result in an increasing failure rate distribution will aid in deciding which members of this family of distributions are plausible models as failure time distributions, and plots of skewness versus kurtosis as a function of the shape parameter k, for example, may aid in determining what value of k to use for a particular data set. Best linear unbiased estimation, maximum likelihood estimation and moment estimation may also be addressed based on full or censored samples.

Recurrence relations for single moments and product moments of order statistics from the generalized logistic distribution will generalize results given by Shah (1966, 1970) for the logistic distribution. In the case of the logistic distribution, expressions for the i^{th} moment of the r^{th} order statistic, $E(X_{r:n}^{i})$ for $r = 1$ and $r > 1$ were manipulated. However, for the generalized logistic distribution, we consider expressions like $E(X_{r:n}^{i}) - kE(X_{r:n}^{i+1})$. This is due to the characterizing differential equation for this distribution, $(1\text{-}kx)f(x) = F(x)[1\text{-}F(x)]$. Because of this added complication, more recurrence relations must be introduced in order to make the recurrence relations complete in the sense that the single and product moments of all order statistics from all sample sizes and all choices of the shape parameter k may be determined using these recurrence relations, provided a few starting values for the recursive process are known.

For theoretical interest, the recurrence relations established for the generalized logistic distribution will be generalized to the doubly truncated generalized logistic distribution, which has a cumulative distribution function given by

$$F(x)=\begin{cases}\dfrac{1}{(P-Q)\left[1+(1-kx)^{1/k}\right]}, & Q_1<x<P_1<\dfrac{1}{k} \quad \text{when } k>0,\\[2ex] & \dfrac{1}{k}<Q_1<x<P_1 \quad \text{when } k<0,\\[2ex] \dfrac{1}{(P-Q)\left[1+e^{-x}\right]}, & Q_1<x<P_1 \quad \text{when } k=0,\end{cases} \tag{7.2}$$

where 1-P is the proportion of right truncation and Q is the proportion of left truncation of the generalized logistic distribution, thereby uniquely defining P_1 and Q_1. Recurrence relations established here will generalize the results of Tarter (1966) and Balakrishnan and Kocherlakota (1986) for the doubly truncated logistic distribution.

Another distribution which is used in modelling life time distributions is the half logistic distribution. This distribution can also be generalized by introducing a shape parameter, and the following family of distributions is then obtained, which includes the half logistic distribution as $k \to 0$:

$$F(x) = \frac{1-(1-kx)^{\frac{1}{k}}}{1+(1-kx)^{\frac{1}{k}}}, \quad 0 \le x \le \frac{1}{k}, \quad k \ge 0 \tag{7.3}$$

This distribution is an increasing failure rate family for all values of k and as such will be a plausible model for data on failure times. Properties of this distribution and inference based on samples from this distribution will therefore be of interest.

Recurrence relations for single moments and product moments of order statistics from the generalized half logistic distribution will generalize results given by Balakrishnan (1985) for the half logistic distribution. Again, the recurrence relations obtained will be complete in the sense described above for the generalized logistic distribution, and they may be generalized to the right truncated generalized half logistic distribution, with cumulative distribution function given by .

$$F(x) = \frac{1}{P}\left[\frac{1-(1-kx)^{\frac{1}{k}}}{1+(1-kx)^{\frac{1}{k}}}\right], \quad 0 \le x \le P_1, \tag{7.4}$$

where 1-P is the proportion of truncation on the right of the generalized half logistic distribution, and P_1 is thereby uniquely defined. This distribution will also be of theoretical and practical interest.

It may be mentioned here, in light of the material discussed in Section One of this thesis, that inference based on *conventional* Type-II right censored samples from some of these generalized distributions will be considered in Section Two. Results for progressive Type-II censored samples from these distributions are certainly something we wish to explore in the future.

8 The Generalized Logistic Distribution

8.1 Introduction

Zelterman and Balakrishnan (1992) discussed four types of generalizations of the logistic distribution. The generalized logistic distribution in (7.1), however, is not one of those types; for this reason, the distribution in (7.1) may be referred to as *Type V Generalized Logistic Distribution*. This distribution has been proposed, discussed and applied by Hosking (1986, 1990).

In this chapter, we establish several recurrence relations satisfied by single and product moments of order statistics from this generalized logistic distribution. These generalize the corresponding results for the logistic distribution proved by Shah (1966, 1970). The relations established in this chapter are complete in the sense that they may be used in a simple recursive manner in order to compute single and product moments of all order statistics for all sample sizes; this may be done for any choice of the shape parameter k. These moments are then used to determine the best linear unbiased estimators of location and scale parameters from complete as well as Type-II censored samples. Maximum likelihood estimation for the two-parameter and the three-parameter models are also discussed based on Type-II right censored samples.

To further generalize these results, we consider in Chapter 9 the doubly truncated generalized logistic distribution, as Balakrishnan and Kocherlakota (1986) have considered the doubly truncated logistic distribution. Recurrence relations for single and product moments are established for the doubly truncated generalized logistic distribution which are again complete, and will enable one to compute the single and product moments of all order statistics for all sample sizes in a simple recursive manner; this may be done for any choice of the shape parameter k and for any choice of the truncation parameters P and Q.

8.2 Properties of the Generalized Logistic Distribution

Let X_1, X_2, ..., X_n be a random sample of size n from the generalized logistic population with cumulative distribution function given by (7.1) and with probability density function

$$f(x) = \begin{cases} \dfrac{(1-kx)^{(1/k)-1}}{[1+(1-kx)^{1/k}]^2}, & -\infty < x < \dfrac{1}{k} \quad \text{when } k > 0, \\ & \dfrac{1}{k} < x < \infty \quad \text{when } k < 0, \\ \dfrac{e^{-x}}{(1+e^{-x})^2}, & -\infty < x < \infty \quad \text{when } k = 0. \end{cases} \tag{8.2.1}$$

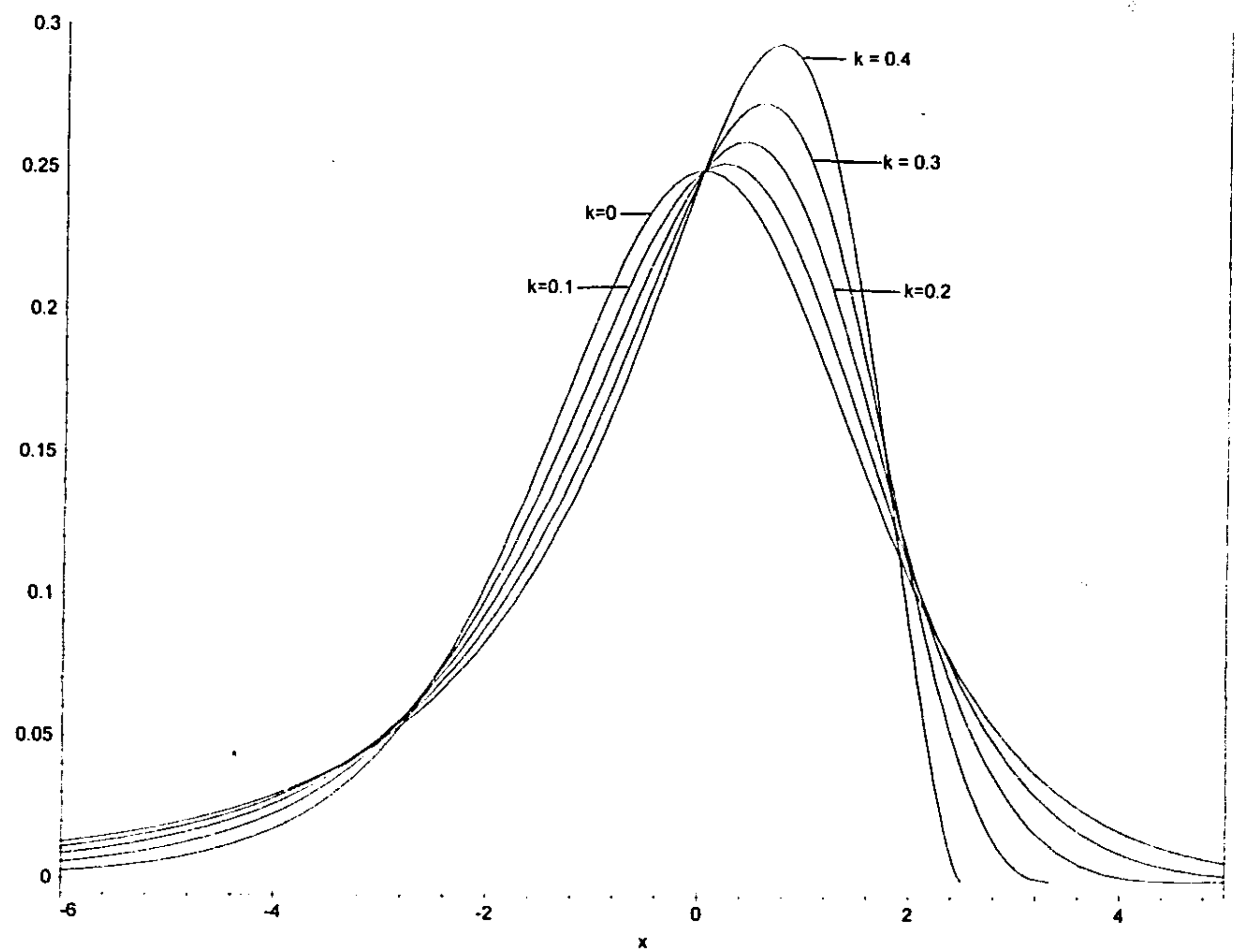
Plots of the density function for selected values of k are given in Figure 8.2.1. Only positive values of k are plotted, since the corresponding plots for $-k$ can be obtained simply by reflection across the vertical axis.

From (7.1) and (8.2.1), we observe easily that the hazard function for this family of distributions is given by

$$h(x) = \frac{f(x)}{1-F(x)} = \begin{cases} \dfrac{1}{(1-kx)[1+(1-kx)^{1/k}]}, & -\infty < x < \dfrac{1}{k} \quad \text{when } k > 0, \\ & \dfrac{1}{k} < x < \infty \quad \text{when } k < 0, \\ \dfrac{1}{1+e^{-x}}, & -\infty < x < \infty \quad \text{when } k = 0. \end{cases} \tag{8.2.2}$$

It is clear from the expression of $h(x)$ in (8.2.2) that the family of generalized logistic distributions in (7.1) is an IFR (Increasing Failure Rate) family for $k \geq 0$ and a DFR (Decreasing Failure Rate) family for $k < 0$. Hence, the generalized logistic family discussed in this chapter will be quite useful as a life span model;

Figure 8.2.1: pdf's for generalized logistic distribution



see Cohen and Whitten (1988) for an excellent treatment of many such life span models.

The first four moments of the generalized logistic random variable are as follows:

$$E(X) = \frac{1}{k}[1-\beta(1-k,1+k)] = \frac{1}{k}\left[1-\frac{k\pi}{\sin(k\pi)}\right], \qquad |k| < 1 \tag{8.2.3}$$

$$E(X^2) = \frac{1}{k^2}[1-2\beta(1-k,1+k)+\beta(1-2k,1+2k)]$$
$$= \frac{1}{k^2}\left[1-\frac{2k\pi}{\sin(k\pi)}+\frac{2k\pi}{\sin(2k\pi)}\right], \qquad |k| < \frac{1}{2} \tag{8.2.4}$$

$$E(X^3) = \frac{1}{k^3}[1-3\beta(1-k,1+k)+3\beta(1-2k,1+2k)-\beta(1-3k,1+3k)]$$
$$= \frac{1}{k^3}\left[1-\frac{3k\pi}{\sin(k\pi)}+\frac{6k\pi}{\sin(2k\pi)}-\frac{3k\pi}{\sin(3k\pi)}\right], \qquad |k| < \frac{1}{3} \tag{8.2.5}$$

$$E(X^4) = \frac{1}{k^4}[1-4\beta(1-k,1+k)+6\beta(1-2k,1+2k)-4\beta(1-3k,1+3k)$$
$$+\beta(1-4k,1+4k)]$$
$$= \frac{1}{k^4}\left[1-\frac{4k\pi}{\sin(k\pi)}+\frac{12k\pi}{\sin(2k\pi)}-\frac{12k\pi}{\sin(3k\pi)}+\frac{4k\pi}{\sin(4k\pi)}\right], \quad |k| < \frac{1}{4} \tag{8.2.6}$$

As a result, the coefficients of skewness and kurtosis ($\sqrt{\beta_1}$, β_2) may be calculated for any value of k in the specified regions. Selected values of these two measures are presented in Table 8.2.1.

Table 8.2.1: Values of coefficients of skewness and kurtosis for selected values of k

k	skewness, $\sqrt{\beta_1}$	kurtosis, β_2
.02	-0.17461	4.27530
.04	-0.35216	4.50848
.06	-0.53582	4.92294
.08	-0.72919	5.56385
.10	-0.93667	6.51021
.12	-1.16392	7.89982
.14	-1.41853	9.98345
.16	-1.71133	13.25416
.18	-2.05842	18.80437
.20	-2.48528	29.55619
.22	-3.03497	56.47212
.24	-3.78713	199.73337

Note that only positive values of k are considered in Table 8.2.1, since $\sqrt{\beta_1}\ (X;k) = -\ \sqrt{\beta_1}\ (X;-k)$ and $\beta_2(X;k) = \beta_2(X;-k)$.

A parametric plot of kurtosis versus skewness for the range $-0.20 \le k \le 0.20$ is given in Figure 8.2.2. Figure 8.2.3 displays plots of skewness versus and

Figure 8.2.2: Parametric plot of kurtosis vs skewness

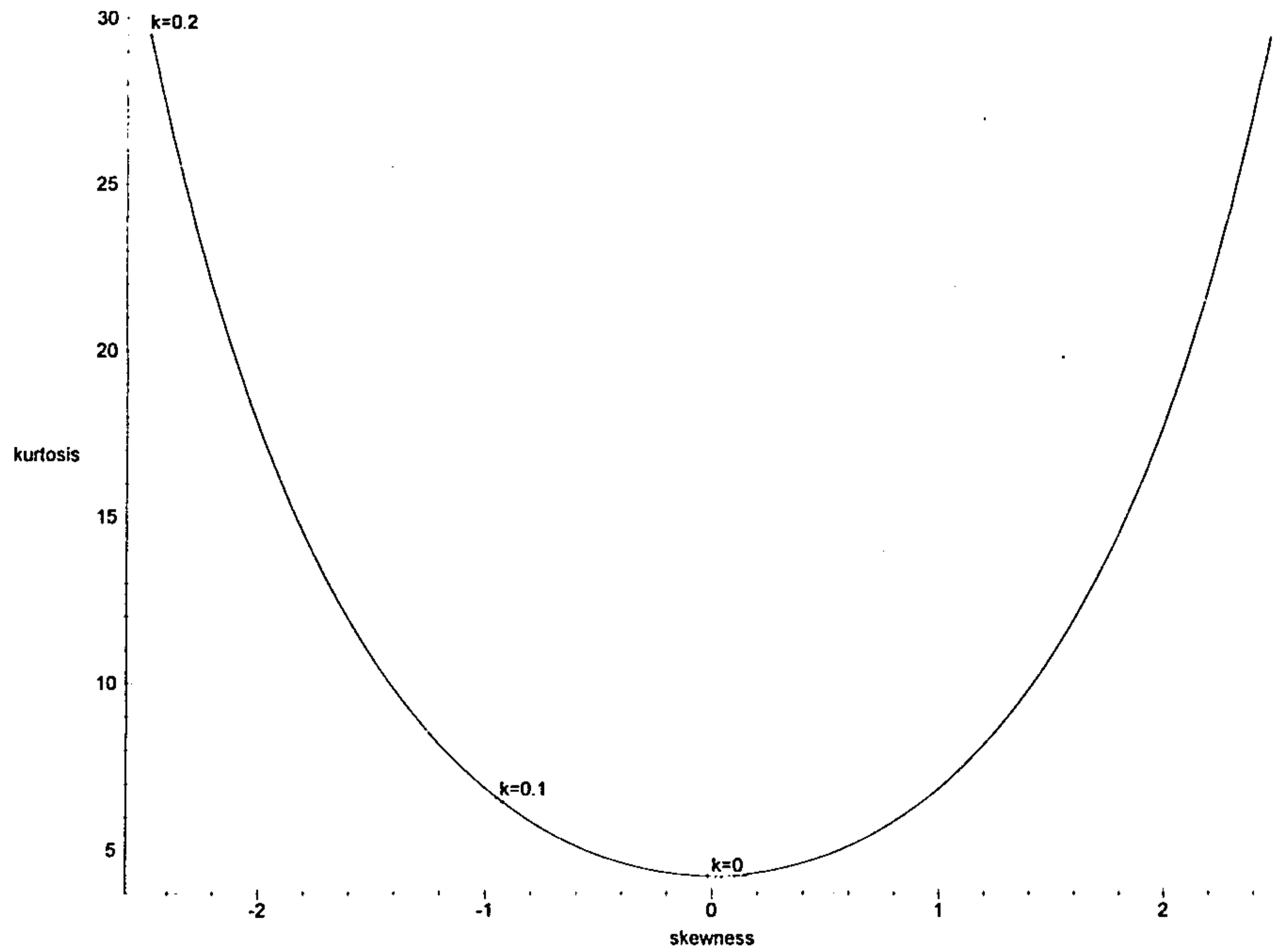
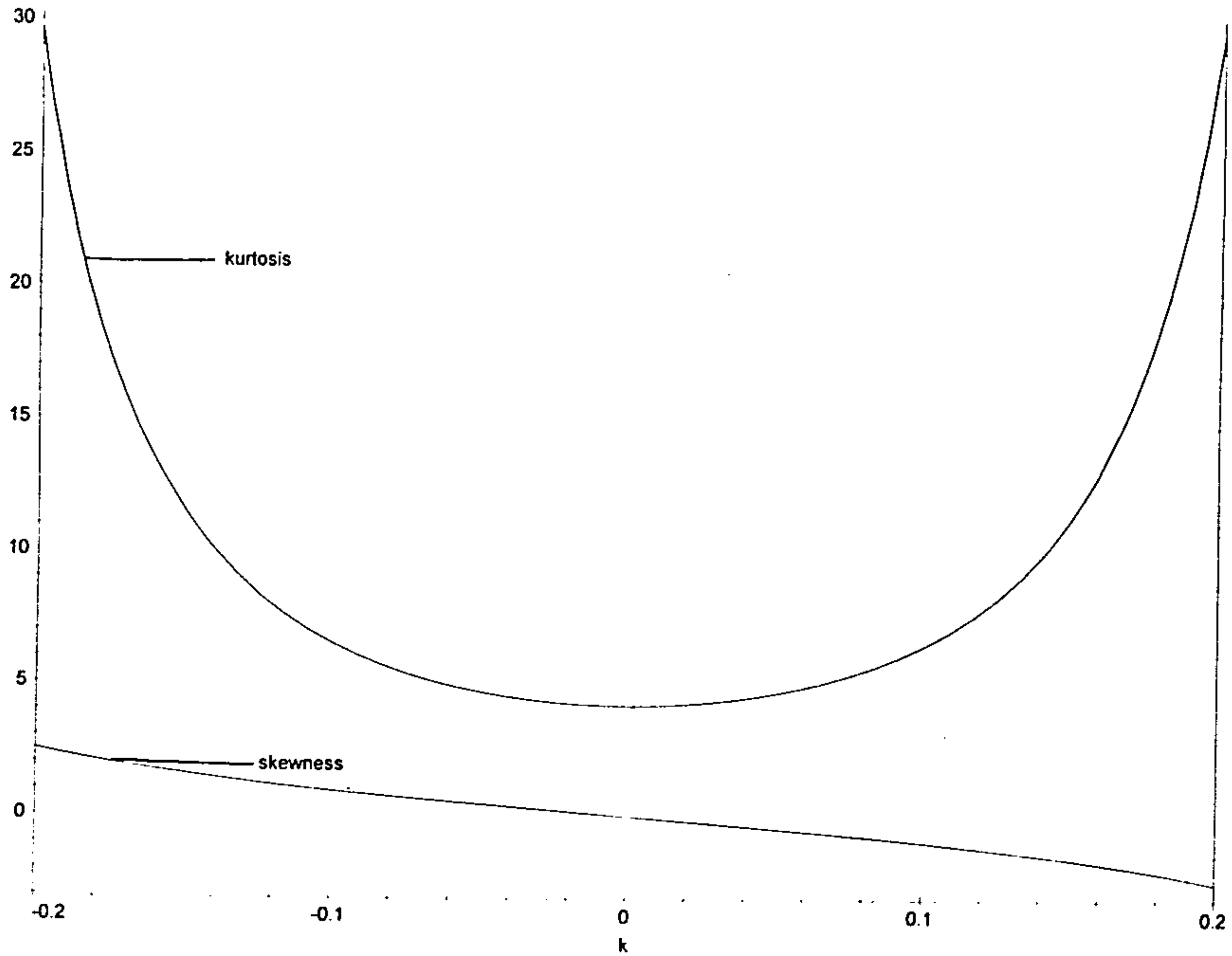


Figure 8.2.3: skewness and kurtosis plots



kurtosis versus k for the same range. These plots may be useful in the initial determination of the value of k for a given sample. The chosen range for these plots does not extend to (-0.25, 0.25), as the values for kurtosis tend to increase drastically due to its poles at $k = -0.25$ and $k = 0.25$.

From (7.1) and (8.2.1), we observe that the characterizing differential equation for this distribution is

$$(1-kx)f(x) = F(x)[1-F(x)]. \qquad (8.2.7)$$

As Shah (1966, 1970) exploited the differential equation $f(x) = F(x)[1-F(x)]$ [case when $k = 0$ in (8.2.7)] of the standard logistic distribution in order to derive several recurrence relations for single and product moments of order statistics, we shall use the differential equation in (8.2.7) in the following sections to make similar developments for the generalized logistic distribution in (7.1).

8.3 Recurrence Relations for Single Moments

Let $X_{1:n} \leq X_{2:n} \leq ... \leq X_{n:n}$ denote the order statistics obtained by arranging the n X_i's in increasing order of magnitude. Let $\mu_{r:n}^{(i)}$ denote the single moments $E(X_{r:n}^i)$ for $1 \leq r \leq n$ and $i \geq 1$, and $\mu_{r,s:n}$ denote the product moments $E(X_{r:n}X_{s:n})$ for $1 \leq r < s \leq n$. Let us further denote $Var(X_{r:n})$ by $\sigma_{r,r:n}$ and $Cov(X_{r:n}, X_{s:n})$ by $\sigma_{r,s:n}$.

For simplicity, we shall also use the notation $\mu_{r:n}$ for $\mu_{r:n}^{(i)}$.

In this and the next section, we establish several recurrence relations satisfied by the single moments $\mu_{r:n}^{(i)}$ and the product moments $\mu_{r,s:n}$. These recurrence relations will enable one to compute all the single and product moments of order statistics for all sample sizes in a simple recursive manner. If we let the shape parameter $k \to 0$, the recurrence relations reduce to the corresponding results for the logistic distribution established by Shah (1966, 1970); see also Balakrishnan (1992).

By starting with the values of $E(X) = \mu_{1:1}$, $E(X^2) = \mu_{1:1}^{(2)}$, and $\mu_{1,2:2} = \mu_{1:1}^{2}$, for example, we have determined the means, variances and covariances of order statistics and have tabulated them for samples of size up to 8 for $k = 0.1(0.1)0.4$. These quantities have then been used to determine the best linear unbiased estimators (BLUE's) of the location and scale parameters of the generalized logistic distribution, and the necessary tables of coefficients and the variances and covariance of the BLUE's have been tabulated for sample sizes $n = 5(5)20$ for $k = 0.1(0.1)0.4$.

The density function of $X_{r:n}$ $(1 \le r \le n)$ is given by [David (1981, p. 9), Arnold, Balakrishnan and Nagaraja (1992, p. 10)]

$$f_{r:n}(x) = \frac{n!}{(r-1)!(n-r)!} [F(x)]^{r-1} [1-F(x)]^{n-r} f(x). \tag{8.3.1}$$

In this section, we establish some recurrence relations for the single moments of order statistics. It is of interest to point out here that similar results are available for many other distributions; for example, see Arnold and Balakrishnan (1989).

Theorem 8.3.1: *For $n \geq 1$ and $i = 0,1,2,\ldots,$*

$$\mu_{1:n+1}^{(i+1)} = \left\{1 + \frac{k(i+1)}{n}\right\} \mu_{1:n}^{(i+1)} - \frac{i+1}{n}\mu_{1:n}^{(i)}. \tag{8.3.2}$$

Proof: For $n \geq 1$ and $i = 0,1,2,\ldots,$ let us consider from (8.3.1)

$$\begin{aligned} \mu_{1:n}^{(i)} - k\mu_{1:n}^{(i+1)} &= n\int_x (x^i - kx^{i+1})[1-F(x)]^{n-1}f(x)dx \\ &= n\int_x x^i F(x)[1-F(x)]^n dx \end{aligned}$$

upon using (8.2.7). Integrating now by parts treating x^i for integration and the rest of the integrand for differentiation, we obtain

$$\begin{aligned} \mu_{1:n}^{(i)} - k\mu_{1:n}^{(i+1)} &= \frac{n}{i+1}\left[n\int_x x^{i+1}[1-F(x)]^{n-1}f(x)dx \; - \; (n+1)\int_x x^{i+1}[1-F(x)]^n f(x)dx\right] \\ &= \frac{n}{i+1}\left[\mu_{1:n}^{(i+1)} - \mu_{1:n+1}^{(i+1)}\right]. \end{aligned} \tag{8.3.3}$$

The relation in (8.3.2) follows simply by rewriting (8.3.3). □

Theorem 8.3.2: *For $1 \le r \le n$ and $i = 0,1,2,\ldots,$*

$$\mu_{r+1:n+1}^{(i+1)} = \mu_{r:n+1}^{(i+1)} + \frac{(i+1)(n+1)}{r(n-r+1)}\left[\mu_{r:n}^{(i)} - k\mu_{r:n}^{(i+1)}\right]. \tag{8.3.4}$$

Proof: From (8.3.1), let us consider for $1 \le r \le n$ and $i = 0,1,2,\ldots,$

$$\begin{aligned}\mu_{r:n}^{(i)} - k\mu_{r:n}^{(i+1)} &= \frac{n!}{(r-1)!(n-r)!}\int_x (x^i - kx^{i+1})[F(x)]^{r-1}[1-F(x)]^{n-r}f(x)dx \\ &= \frac{n!}{(r-1)!(n-r)!}\int_x x^i[F(x)]^r[1-F(x)]^{n-r+1}dx\end{aligned}$$

upon using (8.2.7). Integrating now by parts treating x^i for integration and the rest of the integrand for differentiation, we obtain

$$\begin{aligned}\mu_{r:n}^{(i)} - k\mu_{r:n}^{(i+1)} &= \frac{n!}{(r-1)!(n-r)!(i+1)}\Bigg[(n-r+1)\int_x x^{i+1}[F(x)]^r[1-F(x)]^{n-r}f(x)dx \\ &\qquad - r\int_x x^{i+1}[F(x)]^{r-1}[1-F(x)]^{n-r+1}f(x)dx\Bigg] \\ &= \frac{r(n-r+1)}{(i+1)(n+1)}\left[\mu_{r+1:n+1}^{(i+1)} - \mu_{r:n+1}^{(i+1)}\right].\end{aligned} \tag{8.3.5}$$

The relation in (8.3.4) follows upon simply rewriting (8.3.5). □

Remark: Letting the shape parameter $k \to 0$ in Theorems 8.3.1 and 8.3.2, we deduce the recurrence relations

$$\mu_{1:n+1}^{(i+1)} = \mu_{1:n}^{(i+1)} - \frac{i+1}{n}\mu_{1:n}^{(i)}, \quad n \ge 1, \quad i = 0,1,2,\ldots, \tag{8.3.6}$$

and

$$\mu_{r+1:n+1}^{(i+1)} = \mu_{r:n+1}^{(i+1)} + \frac{(i+1)(n+1)}{r(n-r+1)}\mu_{r:n}^{(i)}, \quad 1 \leq r \leq n, \quad i = 0,1,2,\ldots, \tag{8.3.7}$$

for the single moments of order statistics from the standard logistic distribution. These relations were originally derived by Shah (1970); see also Balakrishnan (1992).

Remark: The relations established in Theorems 8.3.1 and 8.3.2 will enable one to compute the single moments of all order statistics for all sample sizes in a simple recursive manner. For example, by starting with the values of $\mu_{1:1} = E(X)$ and $\mu_{1:1}^{(2)} = E(X^2)$, one can use the recurrence relations in (8.3.2) and (8.3.4) in order to determine the first two single moments (or the means and variances) of order statistics for all sample sizes. This may be done for any choice of the shape parameter k that is of interest.

The recursive computational algorithm is explained in detail in section 8.5.

8.4 Recurrence Relations for Product Moments

The joint density function of $X_{r:n}$ and $X_{s:n}$ $(1 \leq r < s \leq n)$ is given by [David (1981, p. 10), Arnold, Balakrishnan and Nagaraja (1992, p. 16)]

$$f_{r,s:n}(x,y)=\frac{n!}{(r-1)!(s-r-1)!(n-s)!}[F(x)]^{r-1}[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s}f(x)f(y),$$

$$x < y \tag{8.4.1}$$

In this section, we establish some recurrence relations for the product moments of order statistics from the generalized logistic distribution in (7.1).

Theorem 8.4.1: *For $1 \le r \le n-1$,*

$$\mu_{r,r+1:n+1} = \frac{n+1}{n-r+1}\left\{\left[1+\frac{k}{n-r}\right]\mu_{r,r+1:n}-\frac{r}{n+1}\mu^{(2)}_{r+1:n+1}-\frac{1}{n-r}\mu_{r:n}\right\}. \tag{8.4.2}$$

Proof: For $1 \le r \le n-1$, let us consider from (8.4.1)

$$\begin{aligned}\mu_{r:n}-k\mu_{r,r+1:n} &= E\left[X_{r:n}X^{0}_{r+1:n}-kX_{r:n}X_{r+1:n}\right]\\ &=\frac{n!}{(r-1)!(n-r-1)!}\iint_{x<y}(x-kxy)[F(x)]^{r-1}[1-F(y)]^{n-r-1}f(x)f(y)dydx\\ &=\frac{n!}{(r-1)!(n-r-1)!}\int_x x[F(x)]^{r-1}f(x)I_1(x)dx \end{aligned} \tag{8.4.3}$$

where, upon using (8.2.7),

$$I_1(x)=\int_y F(y)[1-F(y)]^{n-r}dy.$$

Integrating by parts treating dy for integration and the rest of the integrand for differentiation, we get

$$I_1(x) = -xF(x)[1-F(x)]^{n-r} + (n-r)\int_y y[1-F(y)]^{n-r-1}f(y)dy \\ -(n-r+1)\int_y y[1-F(y)]^{n-r}f(y)dy. \quad (8.4.4)$$

Upon substituting the expression of $I_1(x)$ in (8.4.4) into (8.4.3) and simplifying the resulting expression, we obtain

$$\mu_{r:n} - k\mu_{r,r+1:n} = -\frac{r(n-r)}{n+1}\mu^{(2)}_{r+1:n+1} + (n-r)\mu_{r,r+1:n} - \frac{(n-r)(n-r+1)}{n+1}\mu_{r,r+1:n+1}.$$

The recurrence relation in (8.4.2) is derived simply by rewriting the above equation. □

Theorem 8.4.2: *For $1 \le r < s \le n$ and $s - r \ge 2$,*

$$\mu_{r,s:n+1} = \mu_{r,s-1:n+1} + \frac{n+1}{n-s+2}\left\{\left[1+\frac{k}{n-s+1}\right]\mu_{r,s:n} - \mu_{r,s-1:n} - \frac{1}{n-s+1}\mu_{r:n}\right\}. \quad (8.4.5)$$

Proof: For $1 \le r < s \le n$ and $s - r \ge 2$, let us consider from (8.4.1)

$$\mu_{r:n} - k\mu_{r,s:n} = E\left[X_{r:n}X^0_{s:n} - kX_{r:n}X_{s:n}\right]$$

$$= \frac{n!}{(r-1)!(s-r-1)!(n-s)!}\int\int_{x<y}(x-kxy)[F(x)]^{r-1}[F(y)-F(x)]^{s-r-1}$$

$$\cdot[1-F(y)]^{n-s}\, f(x)f(y)dydx$$

$$= \frac{n!}{(r-1)!(s-r-1)!(n-s)!}\int_x x[F(x)]^{r-1}f(x)I_2(x)dx \qquad (8.4.6)$$

where, upon using (8.2.7),

$$I_2(x) = \int_y [F(y)-F(x)]^{s-r-1}F(y)[1-F(y)]^{n-s+1}dy.$$

Integrating now by parts treating dy for integration and the rest of the integrand for differentiation, we get

$$I_2(x) = -(s-r-1)\int_y y[F(y)-F(x)]^{s-r-2}[1-F(y)]^{n-s+1}f(y)dy$$

$$+(s-r-1)\int_y y[F(y)-F(x)]^{s-r-2}[1-F(y)]^{n-s+2}f(y)dy$$

$$-(n-s+2)\int_y y[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s+1}f(y)dy$$

$$+(n-s+1)\int_y y[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s}f(y)dy. \qquad (8.4.7)$$

Upon substituting the expression of $I_2(x)$ in (8.4.7) into (8.4.6) and simplifying the resulting expression, we obtain

$$\mu_{r:n}-k\mu_{r,s:n} = -(n-s+1)\mu_{r,s-1:n}+\frac{(n-s+1)(n-s+2)}{n+1}\mu_{r,s-1:n+1}$$
$$-\frac{(n-s+1)(n-s+2)}{n+1}\mu_{r,s:n+1}+(n-s+1)\mu_{r,s:n}.$$

The recurrence relation in (8.4.5) is derived simply by rewriting the above equation. □

Theorem 8.4.3: *For $1 \le r \le n-1$,*

$$\mu_{r+1,r+2:n+1}=\frac{n+1}{r+1}\left\{\frac{1}{r}\mu_{r+1:n}+\left[1-\frac{k}{r}\right]\mu_{r,r+1:n}-\frac{n-r}{n+1}\mu^{(2)}_{r+1:n+1}\right\}. \tag{8.4.8}$$

Proof: For $1 \le r \le n-1$, let us consider from (8.4.1)

$$\mu_{r+1:n}-k\mu_{r,r+1:n}=E\left[X^{0}_{r:n}X_{r+1:n}-kX_{r:n}X_{r+1:n}\right]$$
$$=\frac{n!}{(r-1)!(n-r-1)!}\int\int_{x<y}(y-kxy)[F(x)]^{r-1}[1-F(y)]^{n-r-1}f(x)f(y)dxdy$$
$$=\frac{n!}{(r-1)!(n-r-1)!}\int_{y}y[1-F(y)]^{n-r-1}f(y)J_1(y)dy \tag{8.4.9}$$

where, upon using (8.2.7),

$$J_1(y) = \int_{x}[F(x)]^{r}[1-F(x)]dx.$$

Integrating by parts treating dx for integration and the rest of the integrand for differentiation, we get

$$J_1(y) = y[F(y)]^r[1-F(y)] - r\int_x x[F(x)]^{r-1}f(x)dx + (r+1)\int_x x[F(x)]^r f(x)dx. \tag{8.4.10}$$

Upon substituting the expression of $J_1(y)$ in (8.4.10) into (8.4.9) and simplifying the resulting expression, we obtain

$$\mu_{r+1:n} - k\mu_{r,r+1:n} = \frac{r(n-r)}{n+1}\mu^{(2)}_{r+1:n+1} - r\mu_{r,r+1:n} + \frac{r(r+1)}{n+1}\mu_{r+1,r+2:n+1}.$$

The recurrence relation in (8.4.8) follows simply by rewriting the above equation.

□

Corollary 8.4.1: *Setting $r = n - 1$ in (8.4.8), we obtain the relation*

$$\mu_{n,n+1:n+1} = \frac{n+1}{n}\left\{\frac{1}{n-1}\mu_{n:n} + \left[1 - \frac{k}{n-1}\right]\mu_{n-1,n:n} - \frac{1}{n+1}\mu^{(2)}_{n:n+1}\right\}, \quad n \geq 2. \tag{8.4.11}$$

Theorem 8.4.4: *For $1 \leq r < s \leq n$ and $s - r \geq 2$,*

$$\mu_{r+1,s+1:n+1} = \mu_{r+2,s+1:n+1} + \frac{n+1}{r+1}\left\{\frac{1}{r}\mu_{s:n} + \left[1 - \frac{k}{r}\right]\mu_{r,s:n} - \mu_{r+1,s:n}\right\}. \tag{8.4.12}$$

Proof: For $1 \le r < s \le n$ and $s - r \ge 2$, let us consider from (8.4.1)

$$\mu_{s:n} - k\mu_{r,s:n} = E\left[X_{r:n}^{0}X_{s:n} - kX_{r:n}X_{s:n}\right]$$

$$= \frac{n!}{(r-1)!(s-r-1)!(n-s)!}\int\int_{x<y}(y-kxy)[F(x)]^{r-1}[F(y)-F(x)]^{s-r-1}$$

$$\cdot\ [1-F(y)]^{n-s}f(x)f(y)dxdy$$

$$= \frac{n!}{(r-1)!(s-r-1)!(n-s)!}\int_{y} y[1-F(y)]^{n-s}f(y)J_2(y)dy \qquad (8.4.13)$$

where, upon using (8.2.7),

$$J_2(y) = \int_{x}[F(x)]^{r}[1-F(x)][F(y)-F(x)]^{s-r-1}dx.$$

Integrating by parts treating dx for integration and the rest of the integrand for differentiation, we get

$$\begin{aligned} J_2(y) = & -r\int_{x} x[F(x)]^{r-1}[F(y)-F(x)]^{s-r-1}f(x)dx \\ & +(r+1)\int_{x} x[F(x)]^{r}[F(y)-F(x)]^{s-r-1}f(x)dx \\ & +(s-r-1)\int_{x} x[F(x)]^{r}[F(y)-F(x)]^{s-r-2}f(x)dx \\ & -(s-r-1)\int_{x} x[F(x)]^{r+1}[F(y)-F(x)]^{s-r-2}f(x)dx. \end{aligned} \qquad (8.4.14)$$

Upon substituting the expression of $J_2(y)$ in (8.4.14) into (8.4.13) and simplifying the resulting expression, we obtain

$$\mu_{s:n}-k\mu_{r,s:n} = -r\mu_{r,s:n}+\frac{r(r+1)}{n+1}\mu_{r+1,s+1:n+1}+r\mu_{r+1,s:n}-\frac{r(r+1)}{n+1}\mu_{r+2,s+1:n+1}.$$

The recurrence relation in (8.4.12) follows simply by rewriting the above equation.

□

Corollary 8.4.2: *Setting s = n in (8.4.12), we obtain the relation*

$$\mu_{r+1,n+1:n+1} = \mu_{r+2,n+1:n+1}+\frac{n+1}{r+1}\left\{\frac{1}{r}\mu_{n:n}+\left[1-\frac{k}{r}\right]\mu_{r,n:n}-\mu_{r+1,n:n}\right\},\ 1 \le r \le n-2. \tag{8.4.15}$$

Theorem 8.4.5: *For $1 \le r \le n-1$,*

$$\mu_{r,r+2:n+1} = \mu_{r,r+1:n+1}-r\left(\mu_{r+1,r+2:n+1}-\mu^{(2)}_{r+1:n+1}\right)+\frac{n+1}{n-r}\left(\mu_{r:n}-k\mu_{r,r+1:n}\right). \tag{8.4.16}$$

Proof: From (8.4.3), we have for $1 \le r \le n-1$,

$$\mu_{r:n}-k\mu_{r,r+1:n}=\frac{n!}{(r-1)!(n-r-1)!}\int_x x[F(x)]^{r-1}f(x)I_1(x)dx\,, \tag{8.4.17}$$

where

$$I_1(x) = \int_y F(y)[1-F(y)]^{n-r} dy.$$

Upon integrating by parts treating dy for integration and the rest of the integrand for differentiation, and splitting the integral into two by writing $F(y)$ as $F(x)$ + $[F(y) - F(x)]$, we get

$$\begin{aligned} I_1(x) = & -xF(x)[1-F(x)]^{n-r} - \int_y y[1-F(y)]^{n-r} f(y) dy \\ & +(n-r)\int_y y[F(y)-F(x)][1-F(y)]^{n-r-1} f(y) dy \\ & +(n-r)F(x)\int_y y[1-F(y)]^{n-r-1} f(y) dy. \end{aligned} \tag{8.4.18}$$

Upon substituting the expression of $I_1(x)$ in (8.4.18) into (8.4.17) and simplifying the resulting expression, we obtain

$$\begin{aligned} \mu_{r:n} - k\mu_{r,r+1:n} = & -\frac{r(n-r)}{n+1}\mu^{(2)}_{r+1:n+1} - \frac{n-r}{n+1}\mu_{r,r+1:n+1} + \frac{n-r}{n+1}\mu_{r,r+2:n+1} \\ & + \frac{r(n-r)}{n+1}\mu_{r+1,r+2:n+1} \end{aligned}$$

which, when rewritten, yields the recurrence relation in (8.4.16). □

Corollary 8.4.3: *Upon setting $n = 2$ (and $r = 1$) in (8.4.16), we obtain the relation*

$$\mu_{1,3:3} = \mu_{1,2:3} - \mu_{2,3:3} + \mu^{(2)}_{2:3} + 3\left(\mu_{1:2} - k\mu_{1,2:2}\right). \tag{8.4.19}$$

Theorem 8.4.6: *For $1 \le r < s \le n$ and $s - r \ge 2$,*

$$\mu_{r,s+1:n+1} = \mu_{r,s:n+1} - \frac{r}{s-r}\left(\mu_{r+1,s+1:n+1} - \mu_{r+1,s:n+1}\right) + \frac{n+1}{(n-s+1)(s-r)}\left(\mu_{r:n} - k\mu_{r,s:n}\right).$$

(8.4.20)

Proof: From (8.4.6), we have for $1 \le r < s \le n$ and $s - r \ge 2$,

$$\mu_{r:n} - k\mu_{r,s:n} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}\int_x x[F(x)]^{r-1}f(x)I_2(x)dx, \tag{8.4.21}$$

where

$$I_2(x) = \int_y [F(y)-F(x)]^{s-r-1}F(y)[1-F(y)]^{n-s+1}dy.$$

Upon integrating by parts treating dy for integration and the rest of the integrand for differentiation, and splitting the integrals into two by writing $F(y)$ as $F(x)$ + $[F(y) - F(x)]$, we get

$$\begin{aligned} I_2(x) = & -(s-r)\int_y y[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s+1}f(y)dy \\ & -(s-r-1)F(x)\int_y y[F(y)-F(x)]^{s-r-2}[1-F(y)]^{n-s+1}f(y)dy \\ & +(n-s+1)\int_y y[F(y)-F(x)]^{s-r}[1-F(y)]^{n-s}f(y)dy \\ & +(n-s+1)F(x)\int_y y[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s}f(y)dy. \end{aligned} \tag{8.4.22}$$

Upon substituting the expression of $I_2(x)$ in (8.4.22) into (8.4.21) and simplifying the resulting expression, we obtain

$$\mu_{r:n}-k\mu_{r,s:n}=\frac{(s-r)(n-s+1)}{n+1}\left(\mu_{r,s+1:n+1}-\mu_{r,s:n+1}\right)+\frac{r(n-s+1)}{n+1}\left(\mu_{r+1,s+1:n+1}-\mu_{r+1,s:n+1}\right)$$

which, when rewritten, yields the recurrence relation in (8.4.20). □

Corollary 8.4.4: *Setting r = 1 and s = n in (8.4.20), we obtain the relation*

$$\mu_{1,n+1:n+1} = \mu_{1,n:n+1}-\frac{1}{n-1}\left(\mu_{2,n+1:n+1}-\mu_{2,n:n+1}\right)+\frac{n+1}{n-1}\left(\mu_{1:n}-k\mu_{1,n:n}\right), \quad n \geq 3. \tag{8.4.23}$$

Remark: Letting the shape parameter $k \to 0$ in Theorems 8.4.1 - 8.4.6, we deduce the recurrence relations for the product moments of order statistics from the logistic distribution established by Shah (1966); see also Balakrishnan (1992).

Remark: The relations established in Theorems 8.4.1 - 8.4.6 are complete in the sense that they will enable one to compute the product moments of order statistics for all sample sizes in a simple recursive manner. This may be done for any choice of the shape parameter k that is of interest. The recursive computational algorithm is explained in the next section in detail.

8.5. Recursive Computational Algorithm

Starting with the values of $\mu_{1:1} = E(X)$ and $\mu_{1:1}^{(2)} = E(X^2)$, relations in (8.3.2) and (8.3.4) can be used to determine $\mu_{r:n}$ and $\mu_{r:n}^{(2)}$ for $r = 1,2,\ldots,n$ and for $n = 2,3,4,\ldots$. From these values, variances of all order statistics can be readily computed.

By starting with the fact that $\mu_{1,2:2} = \mu_{1:1}^2$ [see Arnold and Balakrishnan (1989)], $\mu_{1,2:3}$ and $\mu_{2,3:3}$ can be determined from relations in (8.4.2) and (8.4.11), respectively; $\mu_{1,3:3}$ can then be determined from (8.4.19). For the sample of size 4, $\mu_{1,2:4}$ and $\mu_{2,3:4}$ can be determined from (8.4.2), $\mu_{3,4:4}$ from (8.4.11), $\mu_{1,3:4}$ from (8.4.5), $\mu_{2,4:4}$ from (8.4.15), and finally $\mu_{1,4:4}$ from (8.4.23). This process may be followed similarly to determine $\mu_{r,s:n}$ for $1 \le r < s \le n$ and for $n = 5,6,\ldots$. From these values, one can readily compute all the covariances of order statistics.

Table 8.5.1 gives the means of order statistics for selected values of $k = 0.1(0.1)0.4$ up to sample size $n = 8$. Table 8.5.2 gives the values of variances and covariances for the same choices of n and k. Only positive values for k are considered, since $E(X_{i:n}; k) = -E(X_{n-i+1:n}; -k)$ and $Cov(X_{i:n}, X_{j:n}; k) = Cov(X_{n-j+1:n}, X_{n-i+1:n}; -k)$, $1 \le i \le j \le n$, where $Cov(X_{i:n}, X_{i:n}) = Var(X_{i:n})$.

TABLE 8.5.1: MEANS OF ORDER STATISTICS, $\mu_{i:n}$, FOR GENERALIZED LOGISTIC DISTRIBUTION

i	*n*	*k*=0.1	*k*=0.2	*k*=0.3	*k*=0.4
1	1	-0.16641	-0.34480	-0.54989	-0.80327
1	2	-1.18305	-1.41376	-1.71486	-2.12457
2	2	0.85023	0.72416	0.61508	0.51804
1	3	-1.74220	-2.05513	-2.47208	-3.04949
2	3	-0.06474	-0.13100	-0.20040	-0.27474
3	3	1.30772	1.15175	1.02282	0.91443
1	4	-2.13361	-2.52547	-3.05263	-3.78942
2	4	-0.56798	-0.64411	-0.73046	-0.82969
3	4	0.43849	0.38210	0.32966	0.28021
4	4	1.59746	1.40830	1.25387	1.12584
1	5	-2.43695	-2.90175	-3.53157	-4.41836
2	5	-0.92025	-1.02038	-1.13684	-1.27365
3	5	-0.03958	-0.07969	-0.12089	-0.16375
4	5	0.75721	0.68996	0.63003	0.57618
5	5	1.80753	1.58788	1.40983	1.26326
1	6	-2.68569	-3.21782	-3.94347	-4.97183
2	6	-1.19325	-1.32140	-1.47210	-1.65102
3	6	-0.37423	-0.41834	-0.46631	-0.51892
4	6	0.29507	0.25895	0.22453	0.19141
5	6	0.98828	0.90546	0.83278	0.76856
6	6	1.97138	1.72437	1.52524	1.36220
1	7	-2.89711	-3.49174	-4.30731	-5.46995
2	7	-1.41712	-1.57425	-1.76043	-1.98310
3	7	-0.63359	-0.68926	-0.75129	-0.82081
4	7	-0.02843	-0.05712	-0.08635	-0.11640
5	7	0.53769	0.49600	0.45769	0.42227
6	7	1.16851	1.06924	0.98281	0.90708
7	7	2.10519	1.83355	1.61564	1.43805
1	8	-3.08136	-3.73437	-4.63476	-5.92538
2	8	-1.60740	-1.79340	-2.01511	-2.28197
3	8	-0.84626	-0.91683	-0.99636	-1.08648
4	8	-0.27914	-0.30997	-0.34282	-0.37804
5	8	0.22228	0.19574	0.17013	0.14524
6	8	0.72694	0.67616	0.63023	0.58849
7	8	1.31571	1.20027	1.10034	1.01327
8	8	2.21797	1.92402	1.68926	1.49873

TABLE 8.5.2: VARIANCES AND COVARIANCES, $\sigma_{i,j:n}$, FOR GENERALIZED LOGISTIC DISTRIBUTION

i	*j*	*n*	*k*=0.1	*k*=0.2	*k*=0.3	*k*=0.4
1	1	1	3.54009	4.46581	6.94236	15.81242
1	1	2	3.21455	5.10946	9.75063	26.71650
1	2	2	1.03356	1.14267	1.35715	1.74585
2	2	2	1.79852	1.53681	1.41980	1.41663
1	1	3	3.22336	5.71999	12.10242	36.54763
1	2	3	1.04485	1.34670	1.84310	2.70557
1	3	3	0.55362	0.59750	0.68042	0.82155
2	2	3	1.32104	1.42022	1.60667	1.92143
2	3	3	0.73478	0.66110	0.62167	0.61055
3	3	3	1.40937	1.04662	0.82761	0.69285
1	1	4	3.28505	6.26143	14.18588	45.74617
1	2	4	1.06283	1.49483	2.22352	3.53490
1	3	4	0.58503	0.71809	0.92185	1.24534
1	4	4	0.36921	0.39491	0.44289	0.52306
2	2	4	1.19989	1.44100	1.80771	2.38204
2	3	4	0.68084	0.71372	0.77263	0.86465
2	4	4	0.43740	0.39942	0.37751	0.36902
3	3	4	0.93569	0.87291	0.84371	0.84489
3	4	4	0.61883	0.50235	0.42332	0.36962
4	4	4	1.23146	0.84125	0.60870	0.46340
1	1	5	3.35729	6.74599	16.08437	54.49984
1	2	5	1.08387	1.62022	2.55631	4.30274
1	3	5	0.60343	0.79929	1.10391	1.59998
1	4	5	0.39789	0.47839	0.59832	0.78270
1	5	5	0.27466	0.29245	0.32549	0.38027
2	2	5	1.15580	1.49155	2.00411	2.82015
2	3	5	0.65767	0.75234	0.88499	1.07213
2	4	5	0.43910	0.45590	0.48550	0.53059
2	5	5	0.30559	0.28093	0.26615	0.25965
3	3	5	0.80070	0.83424	0.89382	0.98575
3	4	5	0.54420	0.51443	0.49868	0.49570
3	5	5	0.38336	0.32074	0.27644	0.24510
4	4	5	0.77173	0.66174	0.58475	0.53199
4	5	5	0.55482	0.42064	0.33005	0.26743
5	5	5	1.12576	0.72488	0.49307	0.35184
1	1	6	3.42971	7.18577	17.84484	62.91348
1	2	6	1.10535	1.73146	2.85917	5.03126
1	3	6	0.61859	0.86542	1.26044	1.92369
1	4	6	0.41378	0.53337	0.71330	0.99537
1	5	6	0.29890	0.35537	0.43831	0.56382
1	6	6	0.21788	0.23138	0.25637	0.29761
2	2	6	1.13905	1.55009	2.19232	3.24177
2	3	6	0.64835	0.78834	0.98360	1.26138

i	*j*	*n*	*k*=0.1	*k*=0.2	*k*=0.3	*k*=0.4
2	4	6	0.43784	0.49052	0.56187	0.65859
2	5	6	0.31824	0.32881	0.34729	0.37514
2	6	6	0.23298	0.21499	0.20396	0.19877
3	3	6	0.74209	0.83079	0.95327	1.12247
3	4	6	0.50749	0.52342	0.55122	0.59293
3	5	6	0.37197	0.35375	0.34340	0.34024
3	6	6	0.27398	0.23266	0.20279	0.18121
4	4	6	0.63533	0.60833	0.59574	0.59674
4	5	6	0.47142	0.41604	0.37534	0.34603
4	6	6	0.35041	0.27603	0.22349	0.18570
5	5	6	0.67975	0.54913	0.45593	0.38858
5	6	6	0.51319	0.36970	0.27517	0.21111
6	6	6	1.05388	0.64826	0.42058	0.28575
1	1	7	3.49933	7.58979	19.49694	71.05382
1	2	7	1.12628	1.83248	3.14047	5.73102
1	3	7	0.63213	0.92303	1.40176	2.22871
1	4	7	0.42579	0.57732	0.81102	1.18728
1	5	7	0.31232	0.39642	0.52073	0.71178
1	6	7	0.23838	0.28145	0.34417	0.43820
1	7	7	0.18024	0.19107	0.21109	0.24404
2	2	7	1.13450	1.61016	2.37227	3.65018
2	3	7	0.64561	0.82268	1.07430	1.44019
2	4	7	0.43823	0.51855	0.62632	0.77291
2	5	7	0.32303	0.35780	0.40405	0.46546
2	6	7	0.24741	0.25489	0.26792	0.28743
2	7	7	0.18754	0.17347	0.16471	0.16043
3	3	7	0.71193	0.84047	1.01505	1.25583
3	4	7	0.48793	0.53490	0.59741	0.68014
3	5	7	0.36195	0.37139	0.38773	0.41193
3	6	7	0.27847	0.26574	0.25819	0.25538
3	7	7	0.21181	0.18145	0.15922	0.14295
4	4	7	0.57303	0.58954	0.61824	0.66112
4	5	7	0.42873	0.41276	0.40448	0.40344
4	6	7	0.33191	0.29713	0.27089	0.25145
4	7	7	0.25367	0.20382	0.16779	0.14132
5	5	7	0.54469	0.49130	0.45202	0.42410
5	6	7	0.42561	0.35681	0.30526	0.26637
5	7	7	0.32766	0.24646	0.19030	0.15059
6	6	7	0.62007	0.47837	0.37871	0.30722
6	7	7	0.48341	0.33434	0.23865	0.17538
7	7	7	1.00085	0.59312	0.37035	0.24190
1	1	8	3.56536	7.96462	21.06088	78.96685
1	2	8	1.14633	1.92557	3.40504	6.40817
1	3	8	0.64455	0.97487	1.53250	2.52060
1	4	8	0.43594	0.61517	0.89864	1.36693
1	5	8	0.32219	0.42892	0.59017	0.84404
1	6	8	0.24980	0.31394	0.40770	0.55007
1	7	8	0.19781	0.23245	0.28261	0.35734

i	*j*	*n*	*k*=0.1	*k*=0.2	*k*=0.3	*k*=0.4
1	8	8	0.15355	0.16257	0.17923	0.20661
2	2	8	1.13611	1.66948	2.54463	4.04755
2	3	8	0.64630	0.85547	1.15941	1.61182
2	4	8	0.43994	0.54334	0.68426	0.87957
2	5	8	0.32649	0.38038	0.45117	0.54517
2	6	8	0.25385	0.27919	0.31251	0.35620
2	7	8	0.20144	0.20715	0.21706	0.23182
2	8	8	0.15662	0.14510	0.13786	0.13422
3	3	8	0.69517	0.85592	1.07680	1.38618
3	4	8	0.47691	0.54789	0.64042	0.76211
3	5	8	0.35572	0.38550	0.42433	0.47456
3	6	8	0.27757	0.28394	0.29491	0.31105
3	7	8	0.22086	0.21122	0.20534	0.20291
3	8	8	0.17207	0.14825	0.13067	0.11769
4	4	8	0.53883	0.58455	0.64520	0.72489
4	5	8	0.40451	0.41392	0.43013	0.45401
4	6	8	0.31711	0.30627	0.30025	0.29880
4	7	8	0.25322	0.22861	0.20974	0.19551
4	8	8	0.19783	0.16089	0.13381	0.11366
5	5	8	0.48152	0.46665	0.45973	0.46044
5	6	8	0.37990	0.34741	0.32280	0.30468
5	7	8	0.30485	0.26056	0.22650	0.20018
5	8	8	0.23911	0.18407	0.14502	0.11676
6	6	8	0.48709	0.41954	0.36801	0.32861
6	7	8	0.39377	0.31687	0.25993	0.21720
6	8	8	0.31073	0.22514	0.16731	0.12730
7	7	8	0.57774	0.42931	0.32702	0.25498
7	8	8	0.46072	0.30803	0.21239	0.15065
8	8	8	0.95953	0.55104	0.33319	0.21057

8.6. Best Linear Unbiased Estimators

With the use of the recursive algorithms to compute means, variances and covariances of order statistics from the generalized logistic distribution, we can develop best linear unbiased estimators of location (θ_1) and scale (θ_2) parameters for a given value of k when available samples are either complete or conventional Type-II censored either from the left, the right or both, given prior knowledge of which order statistics were observed. Following the steps for obtaining BLUE's in Chapter 5 (section 5.1), [see also Balakrishnan and Cohen (1991) and David (1981)], we have

$$\theta_1^* = -\mu'\Gamma Y \qquad \text{and} \qquad \theta_2^* = \mathbf{1}'\Gamma Y \qquad (8.6.1)$$

where Y is the vector of observed order statistics, μ is the corresponding vector of means of order statistics for the standard (location 0, scale 1) distribution, Σ is the corresponding variance-covariance matrix for the standard distribution, $\mathbf{1}$ is a vector of ones of appropriate dimension, and Γ and Δ have the same form as described in Chapter 5, namely,

$$\Gamma = \frac{\Sigma^{-1}(\mathbf{1}\mu' - \mu\mathbf{1}')\Sigma^{-1}}{\Delta} \qquad \text{and} \qquad \Delta = (\mathbf{1}'\Sigma^{-1}\mathbf{1})(\mu'\Sigma^{-1}\mu) - (\mathbf{1}'\Sigma^{-1}\mu)^2.$$

From this, one also obtains

$$Var(\theta_1^*) = \theta_2^2\frac{\mu'\Sigma^{-1}\mu}{\Delta}, \quad Var(\theta_2^*) = \theta_2^2\frac{\mathbf{1}'\Sigma^{-1}\mathbf{1}}{\Delta}, \quad cov(\theta_1^*, \theta_2^*) = -\theta_2^2\frac{\mu'\Sigma^{-1}\mathbf{1}}{\Delta}. \qquad (8.6.2)$$

Table 8.6.1 gives coefficients for the observed order statistics obtained using (8.6.1) for the BLUE's of the location and scale parameters for selected values of k when the entire sample of size n is observed. Table 8.6.2 gives corresponding variances and the covariance of the estimators obtained using (8.6.2). A short table of coefficients, variances and covariance of BLUE's is also given for $k = 0.1$ and $n = 20$ for right censored samples, where the number observed is $r = 5$, 10, and 15 (Table 8.6.3).

8.7 Maximum Likelihood Estimation

We consider in this section maximum likelihood estimation for the two- and three-parameter models. For the two-parameter case, we assume that the shape parameter k is a known quantity, and we estimate the location (θ_1) and scale (θ_2) parameters using the maximum likelihood method. For the three-parameter case, the shape parameter k, as well as the scale and location parameters are assumed to be unknown. In both cases, for a right censored sample of size r from n independent random variables from the generalized logistic distribution with location parameter θ_1, scale parameter θ_2 and shape parameter k, the likelihood function to be maximized is given by

TABLE 8.6.1: COEFFICIENTS FOR OBSERVED ORDER STATISTICS IN BLUE's FOR GENERALIZED LOGISTIC DISTRIBUTION

n = 5:

θ_1^*				θ_2^*			
k=0.1	k=0.2	k=0.3	k=0.4	k=0.1	k=0.2	k=0.3	k=0.4
.07486	.04985	.02744	.01054	-.12093	-.07036	-.03522	-.01274
.23328	.21701	.19840	.17613	-.16055	-.17684	-.17932	-.17095
.30752	.31589	.32719	.33854	-.07005	-.13730	-.19949	-.25513
.27178	.30212	.34389	.39800	.07676	.00675	-.08201	-.18900
.11257	.11513	.10309	.07679	.27477	.37774	.49604	.62782

n = 10:

θ_1^*				θ_2^*			
k=0.1	k=0.2	k=0.3	k=0.4	k=0.1	k=0.2	k=0.3	k=0.4
.01937	.01168	.00562	.00185	-.04482	-.02052	-.00797	-.00224
.06451	.05401	.04310	.03270	-.07419	-.05909	-.04451	-.03222
.10293	.09363	.08338	.07245	-.07547	-.07399	-.06831	-.06083
.13161	.12611	.11979	.11223	-.06378	-.07564	-.08169	-.08361
.14887	.14861	.14829	.14701	-.04250	-.06536	-.08345	-.09718
.15363	.15903	.16569	.17259	-.01412	-.04400	-.07240	-.09845
.14502	.15544	.16885	.18461	.01914	-.01215	-.04697	-.08369
.12218	.13569	.15394	.17743	.05500	.03000	-.00441	-.04699
.08421	.09678	.11502	.14157	.09083	.08365	.06205	.02472
.02767	.01901	-.00370	-.04245	.14992	.23711	.34766	.48049

n = 15:

θ_1^*				θ_2^*			
k=0.1	k=0.2	k=0.3	k=0.4	k=0.1	k=0.2	k=0.3	k=0.4
.00884	.00508	.00228	.00070	-.02566	-.01041	-.00355	-.00088
.02940	.02345	.01756	.01241	-.04369	-.03032	-.01992	-.01266
.04850	.04185	.03491	.02820	-.04885	-.04043	-.03197	-.02475
.06530	.05908	.05236	.04539	-.04901	-.04628	-.04145	-.03617
.07928	.07431	.06876	.06263	-.04539	-.04840	-.04812	-.04611
.09013	.08698	.08335	.07900	-.03869	-.04708	-.05179	-.05398
.09760	.09664	.09549	.09368	-.02947	-.04252	-.05226	-.05925
.10150	.10292	.10460	.10594	-.01817	-.03489	-.04932	-.06139
.10165	.10546	.11011	.11500	-.00519	-.02427	-.04269	-.05979
.09789	.10387	.11139	.12001	.00907	-.01074	-.03203	-.05367
.09004	.09773	.10772	.11990	.02422	.00569	-.01679	-.04194
.07790	.08651	.09814	.11322	.03982	.02519	.00398	-.02282
.06124	.06948	.08118	.09766	.05544	.04829	.03224	.00723
.03968	.04533	.05405	.06842	.07074	.07692	.07324	.05727
.01104	.00131	-.02191	-.06216	.10483	.17926	.28045	.40891

θ_1^*				θ_2^*			
n = 20:							
k=0.1	*k*=0.2	*k*=0.3	*k*=0.4	*k*=0.1	*k*=0.2	*k*=0.3	*k*=0.4
.00510	.00284	.00122	.00035	-.01742	-.00652	-.00204	-.00046
.01677	.01296	.00931	.00628	-.02976	-.01893	-.01137	-.00660
.02795	.02332	.01866	.01439	-.03431	-.02572	-.01847	-.01305
.03833	.03348	.02842	.02348	-.03625	-.03049	-.02456	-.01948
.04770	.04306	.03808	.03299	-.03624	-.03357	-.02959	-.02555
.05589	.05181	.04734	.04255	-.03464	-.03512	-.03350	-.03105
.06280	.05957	.05594	.05185	-.03169	-.03524	-.03623	-.03579
.06837	.06618	.06367	.06064	-.02758	-.03400	-.03773	-.03964
.07251	.07153	.07036	.06870	-.02246	-.03145	-.03793	-.04242
.07518	.07549	.07582	.07579	-.01647	-.02764	-.03676	-.04399
.07631	.07794	.07987	.08168	-.00973	-.02261	-.03415	-.04416
.07586	.07879	.08235	.08613	-.00237	-.01637	-.03001	-.04274
.07376	.07790	.08305	.08886	.00551	-.00895	-.02421	-.03945
.06997	.07513	.08173	.08954	.01377	-.00032	-.01658	-.03399
.06442	.07035	.07813	.08777	.02230	.00953	-.00687	-.02585
.05703	.06334	.07187	.08298	.03097	.02072	.00536	-.01428
.04772	.05384	.06243	.07435	.03966	.03349	.02086	.00203
.03636	.04143	.04893	.06041	.04827	.04845	.04129	.02587
.02274	.02530	.02952	.03782	.05695	.06749	.07112	.06483
.00523	-.00425	-.02669	-.06656	.08150	.14725	.24137	.36579

TABLE 8.6.2: VARIANCES AND COVARIANCE OF BLUE's FOR GENERALIZED LOGISTIC DISTRIBUTION

n	k	$\frac{Var(\theta_1^*)}{\theta_2^2}$	$\frac{Var(\theta_2^*)}{\theta_2^2}$	$\frac{Cov(\theta_1^*,\theta_2^*)}{\theta_2^2}$
5	0.1	0.63269	0.17883	-0.08046
	0.2	0.64600	0.20481	-0.16286
	0.3	0.66815	0.25010	-0.24942
	0.4	0.69936	0.31783	-0.34292
10	0.1	0.30861	0.07901	-0.03739
	0.2	0.31196	0.08603	-0.07447
	0.3	0.31714	0.09879	-0.11123
	0.4	0.32380	0.11845	-0.14805
15	0.1	0.20393	0.05063	-0.02426
	0.2	0.20553	0.05413	-0.04813
	0.3	0.20790	0.06083	-0.07144
	0.4	0.21082	0.07163	-0.09434
20	0.1	0.15226	0.03724	-0.01794
	0.2	0.15323	0.03941	-0.03553
	0.3	0.15464	0.04377	-0.05258
	0.4	0.15631	0.05104	-0.06919

TABLE 8.6.3: COEFFICIENTS FOR OBSERVED ORDER STATISTICS IN BLUE's BASED ON RIGHT-CENSORED SAMPLES FOR GENERALIZED LOGISTIC DISTRIBUTION (SAMPLE SIZE $n=20$, r=number observed, $k=0.1$)

r	θ_1^*	θ_2^*	$\frac{Var(\theta_1^*)}{\theta_2^2}$	$\frac{Var(\theta_2^*)}{\theta_2^2}$	$\frac{Cov(\theta_1^*,\theta_2^*)}{\theta_2^2}$
5	-0.19604	-0.13848	0.86691	0.27064	0.36583
	-0.27905	-0.21007			
	-0.26271	-0.21422			
	-0.21836	-0.19832			
	1.95617	0.76109			
10	-0.02280	-0.05057	0.20775	0.10373	0.03760
	-0.02639	-0.08158			
	-0.01702	-0.08893			
	-0.00432	-0.08877			
	-0.01014	-0.08329			
	0.02553	-0.07368			
	0.04119	-0.06079			
	0.05667	-0.04525			
	0.07155	-0.02763			
	0.86546	0.60050			
15	0.00145	-0.02744	0.15567	0.05780	-0.01048
	0.01092	-0.04591			
	0.02162	-0.05190			
	0.03207	-0.05379			
	0.04188	-0.05266			
	0.05081	-0.04913			
	0.05871	-0.04357			
	0.06543	-0.03635			
	0.07091	-0.02765			
	0.07499	-0.01779			
	0.07764	-0.00692			
	0.07874	0.00472			
	0.07820	0.01691			
	0.07592	0.02943			
	0.26073	0.36204			

$$L_{X_{1:n},X_{2:n},\ldots,X_{r:n}}(x_1,x_2,\ldots,x_r) \propto \frac{\left\{\left[1-k\left(\frac{x_r-\theta_1}{\theta_2}\right)\right]^{\frac{1}{k}}\right\}^{n-r} \prod_{i=1}^{r}\left[1-k\left(\frac{x_i-\theta_1}{\theta_2}\right)\right]^{\frac{1}{k}-1}}{\theta_2^{\,r}\left\{1+\left[1-k\left(\frac{x_r-\theta_1}{\theta_2}\right)\right]^{\frac{1}{k}}\right\}^{n-r} \prod_{i=1}^{r}\left\{1+\left[1-k\left(\frac{x_i-\theta_1}{\theta_2}\right)\right]^{\frac{1}{k}}\right\}^{2}},$$

(8.7.1)

and its logarithm is given by

$$\ln L = constant - r\ln\theta_2 + \left(\frac{1}{k}-1\right)\sum_{i=1}^{r}\ln\left[1-k\left(\frac{x_i-\theta_1}{\theta_2}\right)\right] - 2\sum_{i=1}^{r}\ln\left\{1+\left[1-k\left(\frac{x_i-\theta_1}{\theta_2}\right)\right]^{\frac{1}{k}}\right\}$$
$$+\left(\frac{n-r}{k}\right)\ln\left[1-k\left(\frac{x_r-\theta_1}{\theta_2}\right)\right] - (n-r)\ln\left\{1+\left[1-k\left(\frac{x_r-\theta_1}{\theta_2}\right)\right]^{\frac{1}{k}}\right\}.$$

(8.7.2)

It should be noted here that maximization of the likelihood function is subject to the constraint $x_r \le \theta_1 + \theta_2/k$ when $k > 0$ and $x_1 \ge \theta_1 + \theta_2/k$ when $k < 0$. Notice that in both cases, at the boundary x_r (or x_1) $= \theta_1 + \theta_2/k$, the likelihood function takes the value 0. Thus, the maximum likelihood estimates must be subject to the strict inequalities $x_r < \theta_1 + \theta_2/k$ when $k > 0$ and $x_1 > \theta_1 + \theta_2/k$ when $k < 0$.

In the two-parameter case, upon differentiation of $\ln L$ with respect to θ_1 and θ_2, the maximum likelihood estimates for θ_1 and θ_2 are obtained by simultaneously solving the following two equations:

$$\frac{\partial \ln L}{\partial \theta_1} = \left(\frac{1}{k}-1\right)\sum_{i=1}^{r}\frac{\frac{k}{\theta_2}}{1-k\left(\frac{x_i-\theta_1}{\theta_2}\right)} - 2\sum_{i=1}^{r}\frac{\frac{1}{k}\left[1-k\left(\frac{x_i-\theta_1}{\theta_2}\right)\right]^{\frac{1}{k}-1}\left(\frac{k}{\theta_2}\right)}{1+\left[1-k\left(\frac{x_i-\theta_1}{\theta_2}\right)\right]^{\frac{1}{k}}}$$

$$+\left(\frac{n-r}{k}\right)\frac{\frac{k}{\theta_2}}{1-k\left(\frac{x_r-\theta_1}{\theta_2}\right)} - (n-r)\frac{\frac{1}{k}\left[1-k\left(\frac{x_r-\theta_1}{\theta_2}\right)\right]^{\frac{1}{k}-1}\left(\frac{k}{\theta_2}\right)}{1+\left[1-k\left(\frac{x_r-\theta_1}{\theta_2}\right)\right]^{\frac{1}{k}}} = 0 \tag{8.7.3}$$

and

$$\frac{\partial \ln L}{\partial \theta_2} = -\frac{r}{\theta_2} + \left(\frac{1}{k}-1\right)\sum_{i=1}^{r}\frac{k\left(\frac{x_i-\theta_1}{\theta_2^2}\right)}{1-k\left(\frac{x_i-\theta_1}{\theta_2}\right)} - 2\sum_{i-1}^{r}\frac{\frac{1}{k}\left[1-k\left(\frac{x_i-\theta_1}{\theta_2}\right)\right]^{\frac{1}{k}-1}\left[k\left(\frac{x_i-\theta_1}{\theta_2^2}\right)\right]}{1+\left[1-k\left(\frac{x_i-\theta_1}{\theta_2}\right)\right]^{\frac{1}{k}}}$$

$$+\left(\frac{n-r}{k}\right)\frac{k\left(\frac{x_r-\theta_1}{\theta_2^2}\right)}{1-k\left(\frac{x_r-\theta_1}{\theta_2}\right)} - (n-r)\frac{\frac{1}{k}\left[1-k\left(\frac{x_r-\theta_1}{\theta_2}\right)\right]^{\frac{1}{k}-1}\left[k\left(\frac{x_r-\theta_1}{\theta_2^2}\right)\right]}{1+\left[1-k\left(\frac{x_r-\theta_1}{\theta_2}\right)\right]^{\frac{1}{k}}} = 0\ . \tag{8.7.4}$$

Upon simplification, these equations become

$$\left(\frac{1}{k}-1\right)\sum_{i=1}^{r}\frac{k}{1-k\left(\frac{x_i-\theta_1}{\theta_2}\right)}-2\sum_{i=1}^{r}\frac{\left[1-k\left(\frac{x_i-\theta_1}{\theta_2}\right)\right]^{\frac{1}{k}-1}}{1+\left[1-k\left(\frac{x_i-\theta_1}{\theta_2}\right)\right]^{\frac{1}{k}}}+\frac{n-r}{1-k\left(\frac{x_r-\theta_1}{\theta_2}\right)}$$

$$-(n-r)\frac{\left[1-k\left(\frac{x_r-\theta_1}{\theta_2}\right)\right]^{\frac{1}{k}-1}}{1+\left[1-k\left(\frac{x_r-\theta_1}{\theta_2}\right)\right]^{\frac{1}{k}}}=0$$

(8.7.5)

and

$$-r\theta_2+\left(\frac{1}{k}-1\right)\sum_{i=1}^{r}\frac{kx_i}{1-k\left(\frac{x_i-\theta_1}{\theta_2}\right)}-2\sum_{i=1}^{r}\frac{x_i\left[1-k\left(\frac{x_i-\theta_1}{\theta_2}\right)\right]^{\frac{1}{k}-1}}{1+\left[1-k\left(\frac{x_i-\theta_1}{\theta_2}\right)\right]^{\frac{1}{k}}}+\frac{x_r(n-r)}{1-k\left(\frac{x_r-\theta_1}{\theta_2}\right)}$$

$$-(n-r)\frac{x_r\left[1-k\left(\frac{x_r-\theta_1}{\theta_2}\right)\right]^{\frac{1}{k}-1}}{1+\left[1-k\left(\frac{x_r-\theta_1}{\theta_2}\right)\right]^{\frac{1}{k}}} = 0 .$$

(8.7.6)

These two equations must be solved numerically for θ_1 and θ_2.

In the three-parameter case, in addition to solving these two equations, we must simultaneously solve the following equation:

$$\frac{\partial \ln L}{\partial k} = -\frac{1}{k^2}\sum_{i=1}^{r} \ln\left[1-k\left(\frac{x_i-\theta_1}{\theta_2}\right)\right] - \left(\frac{1}{k}-1\right)\sum_{i=1}^{r} \frac{\frac{x_i-\theta_1}{\theta_2}}{1-k\left(\frac{x_i-\theta_1}{\theta_2}\right)}$$

$$-\left(\frac{n-r}{k^2}\right)\ln\left[1-k\left(\frac{x_r-\theta_1}{\theta_2}\right)\right] - \left(\frac{n-r}{k}\right)\frac{\frac{x_r-\theta_1}{\theta_2}}{1-k\left(\frac{x_r-\theta_1}{\theta_2}\right)}$$

$$-2\sum_{i=1}^{r} \frac{\left[1-k\left(\frac{x_i-\theta_1}{\theta_2}\right)\right]^{\frac{1}{k}}}{1+\left[1-k\left(\frac{x_i-\theta_1}{\theta_2}\right)\right]^{\frac{1}{k}}}\left\{-\frac{1}{k^2}\ln\left[1-k\left(\frac{x_i-\theta_1}{\theta_2}\right)\right] - \frac{1}{k}\frac{\left(\frac{x_i-\theta_1}{\theta_2}\right)}{\left[1-k\left(\frac{x_i-\theta_1}{\theta_2}\right)\right]}\right\}$$

$$-(n-r)\frac{\left[1-k\left(\frac{x_r-\theta_1}{\theta_2}\right)\right]^{\frac{1}{k}}}{1+\left[1-k\left(\frac{x_r-\theta_1}{\theta_2}\right)\right]^{\frac{1}{k}}}\left\{-\frac{1}{k^2}\ln\left[1-k\left(\frac{x_r-\theta_1}{\theta_2}\right)\right] - \frac{1}{k}\frac{\left(\frac{x_r-\theta_1}{\theta_2}\right)}{\left[1-k\left(\frac{x_r-\theta_1}{\theta_2}\right)\right]}\right\} = 0 \,.$$

(8.7.7)

Using (8.7.5) and (8.7.6), this likelihood equation simplifies to

$$-rk-\sum_{i=1}^{r}\ln\left[1-k\left(\frac{x_i-\theta_1}{\theta_2}\right)\right]-(n-r)\ln\left[1-k\left(\frac{x_r-\theta_1}{\theta_2}\right)\right]$$

$$+2\sum_{i=1}^{r}\frac{\left[1-k\left(\frac{x_i-\theta_1}{\theta_2}\right)\right]^{\frac{1}{k}}}{1+\left[1-k\left(\frac{x_i-\theta_1}{\theta_2}\right)\right]^{\frac{1}{k}}}\ln\left[1-k\left(\frac{x_i-\theta_1}{\theta_2}\right)\right]$$

$$-(n-r)\frac{\left[1-k\left(\frac{x_r-\theta_1}{\theta_2}\right)\right]^{\frac{1}{k}}}{1+\left[1-k\left(\frac{x_r-\theta_1}{\theta_2}\right)\right]^{\frac{1}{k}}}\ln\left[1-k\left(\frac{x_r-\theta_1}{\theta_2}\right)\right]=0. \tag{8.7.8}$$

Thus, in the three-parameter case, one must solve Eqs. (8.7.5), (8.7.6), and (8.7.8) simultaneously for θ_1, θ_2 and k.

Remark: Notice that, as $k \to 0$, the left hand side of (8.7.8) approaches 0. Thus, $k = 0$ is always a solution to the three Eqs. (8.7.5), (8.7.6) and (8.7.8). However, for any particular sample, the probability is 0 that the maximum likelihood estimate of k is exactly 0. One should be aware of this when using computer algorithms to solve (8.7.5), (8.7.6) and (8.7.8) simultaneously.

8.8 A Numerical Example and Simulation Study

In this example, we will consider maximum likelihood and best linear unbiased estimation for the 2-parameter generalized logistic distribution in (7.1), and we will consider maximum likelihood and moment estimation for the 3-parameter generalized logistic distribution.

A sample of size $n = 20$ was generated using APL with $k = 0.1$, $\theta_1 = 50$ and $\theta_2 = 10$, using the probability transform method, that is, setting

$$F(x;\theta_1;\theta_2) = \frac{1}{1 + \left[1 - k\left(\frac{x-\theta_1}{\theta_2}\right)\right]^{\frac{1}{k}}} = U,$$

a uniform (0,1) random variable. The ordered sample obtained was:

11.98487, 21.04888, 26.85949, 30.21810, 33.59806, 35.34005, 40.67467, 45.92994, 47.88610, 48.10389, 49.12268, 54.51988, 55.83412, 59.00270, 59.64955, 62.58630, 63.72846, 63.88126, 69.40010, 72.72817.

A calculation of the correlation between the ordered sample values $Y_{i:n}$ and the quantiles $F^{-1}[i/(20+1)]$, $i = 1,2,\ldots,20$, was made assuming $k = 0.1$ and found to be 0.99255. Similar calculations assuming $k = 0.05$, $k = 0.15$, $k = 0.20$ and $k = 0.25$ were obtained, giving the results 0.98924, 0.99363, 0.99255, and

0.98940 respectively, indicating that perhaps any value of k between 0.1 and 0.2 describes the data well. Figures 8.8.1 and 8.8.2 show quantile plots for $k = 0.1$ and $k = 0.2$.

Using 5000 samples each time, p-values for the significance of correlations between $Y_{i:n}$ and $F^{-1}[i/(20+1)]$ were calculated to test the values of $k = 0.1$, $k = 0.2$, $k = 0.3$ and $k = 0.4$ and were found to be 0.9648, 0.9686, 0.7638 and 0.4894, respectively. None of these were significant, which was to be expected, considering the nature of the test. However, the p-values for $k = 0.1$ and $k = 0.2$ were much higher than the others, confirming the region of best choice for k.

The BLUE's obtained using the full sample were as follows:

(SE = standard error, obtained using Table 8.6.2)

Using $k = 0.1$,

$\theta_1^* = 49.54979$, $SE(\theta_1^*) = 3.72811$, $\theta_2^* = 9.55430$, $SE(\theta_2^*) = 1.84375$;

and using $k = 0.2$,

$\theta_1^* = 50.33276$, $SE(\theta_1^*) = 3.78815$, $\theta_2^* = 9.67733$, $SE(\theta_2^*) = 1.92114$.

FIGURE 8.8.1: QUANTILE PLOT OF GENERATED VALUES, Y, FROM GENERALIZED LOGISTIC DISTRIBUTION WITH QUANTILES FROM GENERALIZED LOGISTIC DISTRIBUTION, k=0.1

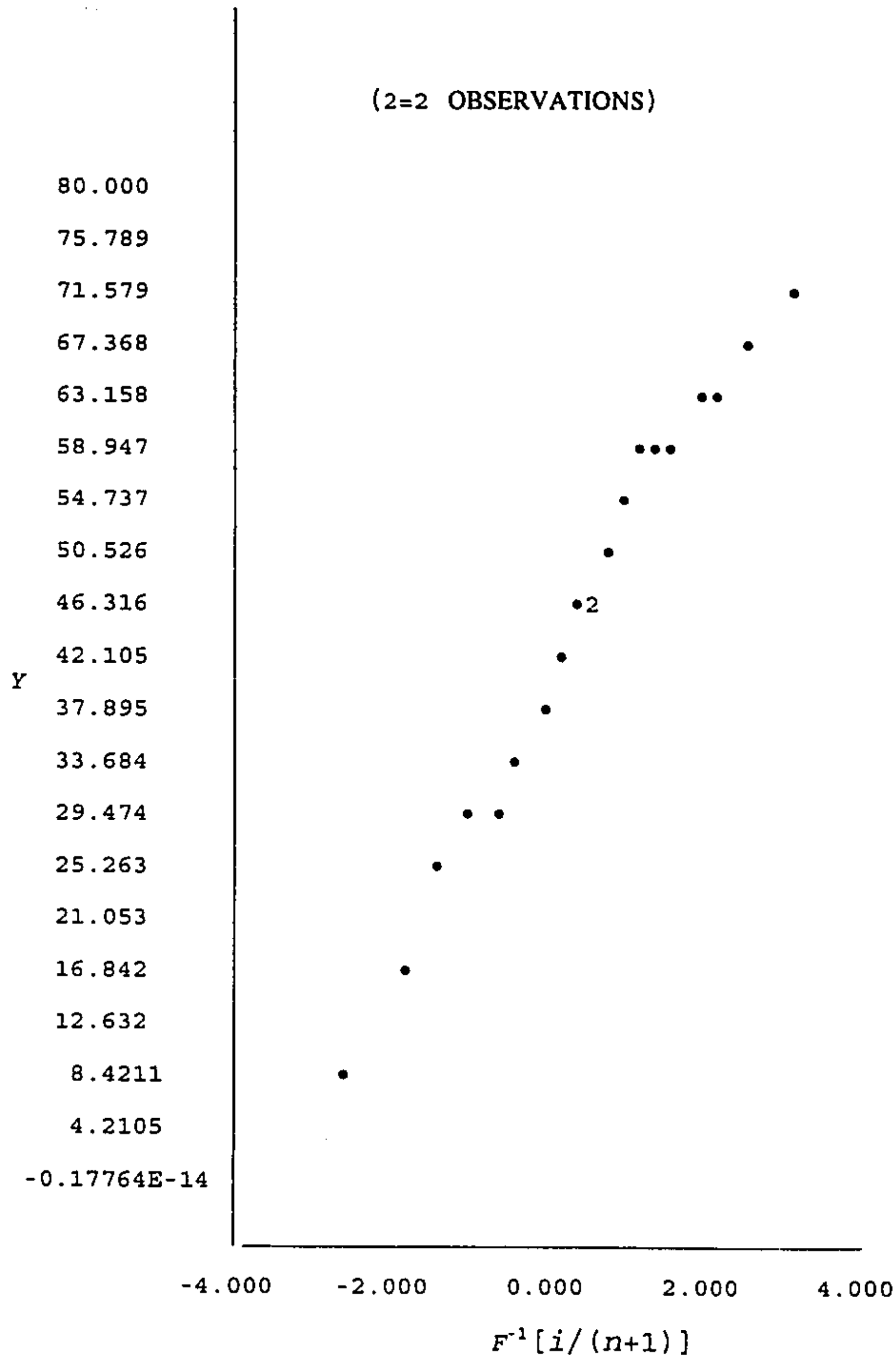
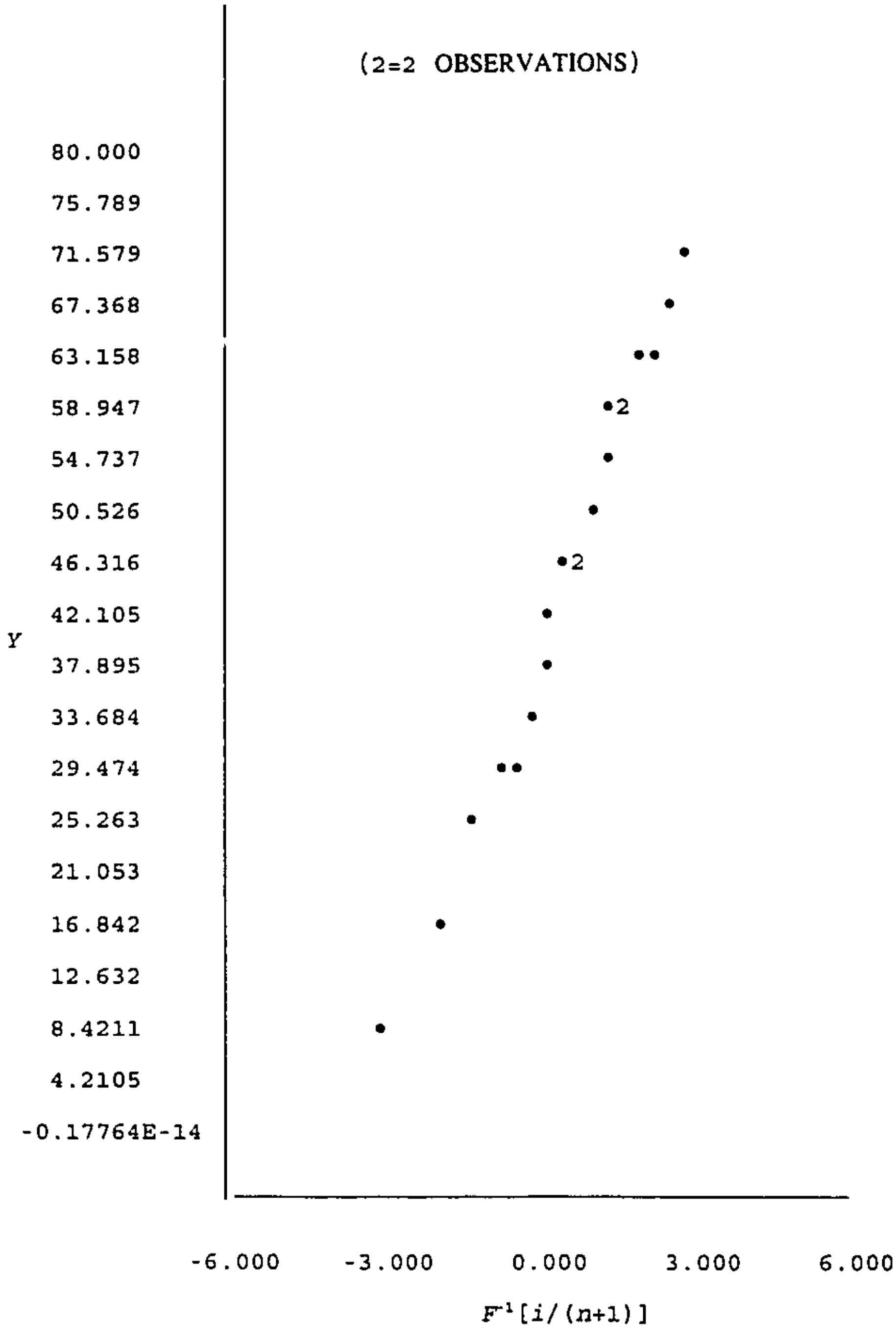


FIGURE 8.8.2: QUANTILE PLOT OF GENERATED VALUES, *Y*, FROM GENERALIZED LOGISTIC DISTRIBUTION WITH QUANTILES FROM GENERALIZED LOGISTIC DISTRIBUTION, k=0.2



In both cases, the estimates of θ_1 and θ_2 obtained are very close to the true values.

Next, the BLUE's were calculated for (conventional) Type-II right-censored samples using the coefficients given in Table 8.6.3, with $k = 0.1$. The results were as follows: (r = number observed)

for $r = 5$:

$\theta_1^* = 43.84540$, $SE(\theta_1^*) = 7.20936$, $\theta_2^* = 7.74302$, $SE(\theta_2^*) = 4.02812$

for $r = 10$:

$\theta_1^* = 49.16279$, $SE(\theta_1^*) = 4.65610$, $\theta_2^* = 10.21530$, $SE(\theta_2^*) = 3.29010$

for $r = 15$:

$\theta_1^* = 49.90012$, $SE(\theta_1^*) = 4.24223$, $\theta_2^* = 10.75206$, $SE(\theta_2^*) = 2.58488$

for $r = 20$:

$\theta_1^* = 49.54979$, $SE(\theta_1^*) = 3.72811$, $\theta_2^* = 9.55430$, $SE(\theta_2^*) = 1.84375$

As expected, for small values of r, the standard errors increase drastically. The $r = 20$ line is, of course, the full sample, re-displayed for comparison. Again, the estimates are all close to the true values of θ_1 and θ_2.

Using Maple V Release 3, maximum likelihood estimates were also obtained for the full sample as follows:

Using $k = 0.1$, $\hat{\theta}_1 = 49.44142$ and $\hat{\theta}_2 = 9.30316$.

Using $k = 0.2$, $\hat{\theta}_1 = 50.18144$ and $\hat{\theta}_2 = 9.28734$.

A two-parameter maximum likelihood analysis was conducted for $k = 0.1$, and $r = 5(5)20$. The results are as follows:

for $r = 5$:	$\hat{\theta}_1 = 41.02149$	$\hat{\theta}_2 = 6.38566$
for $r = 10$:	$\hat{\theta}_1 = 48.25653$	$\hat{\theta}_2 = 9.41583$
for $r = 15$:	$\hat{\theta}_1 = 49.58711$	$\hat{\theta}_2 = 10.21531$
for $r = 20$:	$\hat{\theta}_1 = 49.44142$	$\hat{\theta}_2 = 9.30316$

In this case, we had to ensure that these were the best possible estimates in the permissible region with respect to maximization of the likelihood function. In fact, they turned out to be the only permissible estimates. One may also obtain approximate standard errors of these estimates. However, due to the non-regularity of the two-parameter distribution, this should be done via a simulation study, rather than by the method of approximation of the information matrix. Thus, 1000 random samples of size 20 from the generalized logistic distribution with shape parameter $k = 0.1$ were generated, and the biases, variances, and covariance of the maximum likelihood estimates were obtained as follows:

r	$\frac{bias(\hat{\theta}_1)}{\theta_2}$	$\frac{bias(\hat{\theta}_2)}{\theta_2}$	$\frac{Var(\hat{\theta}_1)}{\theta_2^2}$	$\frac{Var(\hat{\theta}_2)}{\theta_2^2}$	$\frac{cov(\hat{\theta}_1,\hat{\theta}_2)}{\theta_2^2}$
5	-0.383334	-0.174934	0.634252	0.185203	0.218841
10	-0.093311	-0.066467	0.193798	0.088359	0.028468
15	-0.037775	-0.040353	0.148690	0.054835	-0.008588
20	-0.020462	-0.021671	0.144961	0.035865	-0.015321

For comparison purposes, 1000 random samples of size 20 from the generalized logistic distribution with shape parameter $k = 0.2$ were generated, and the biases, variances, and covariance of the maximum likelihood estimates for the full sample were obtained as follows:

r	$\frac{bias(\hat{\theta}_1)}{\theta_2}$	$\frac{bias(\hat{\theta}_2)}{\theta_2}$	$\frac{Var(\hat{\theta}_1)}{\theta_2^2}$	$\frac{Var(\hat{\theta}_2)}{\theta_2^2}$	$\frac{cov(\hat{\theta}_1,\hat{\theta}_2)}{\theta_2^2}$
20	-0.024896	-0.034524	0.159742	0.034619	-0.035709

In obtaining these values, we ensured that each estimate from the simulation was a permissible one.

In correcting for the bias of the maximum likelihood estimates, since these are only simulated and not exact values, we must first determine whether or not the calculated bias is significantly different from zero. Since the estimated bias is just an average of 1000 maximum likelihood estimates, a simple z-test will suffice. Thus, to test H_0: (estimated bias of θ_i)/$\theta_2 = 0$ for $i = 1, 2$, the following p-values are obtained:

For $k = 0.1$:

r:	5	10	15	20
$i = 1$:	0.000	0.000	0.002	0.089
$i = 2$:	0.000	0.000	0.000	0.000

and for $k = 0.2$ ($r = 20$ only):

$i = 1$:	0.049
$i = 2$:	0.000

All of these values are at least somewhat significant, even if dependence between the estimates of θ_1 and θ_2 is considered, and Type-I errors are summed using a Bonferroni argument. As such, we have corrected for the bias of all maximum likelihood estimates of the original simulated sample. The resulting estimates, corrected for bias, and their variances are given by the following:

For $k = 0.1$:

r	$\hat{\theta}_1^{(\bullet)}$	$SE\left[\hat{\theta}_1^{(\bullet)}\right]$	$\hat{\theta}_2^{(\bullet)}$	$SE\left[\hat{\theta}_2^{(\bullet)}\right]$
5	43.9883	7.2504	7.7396	4.0369
10	49.1977	4.5149	10.0862	3.2116
15	49.9892	4.0965	10.6449	2.5975
20	49.6360	3.6127	9.5092	1.8408

and for $k = 0.2$:

r	$\hat{\theta}_1^{(\bullet)}$	$SE\left[\hat{\theta}_1^{(\bullet)}\right]$	$\hat{\theta}_2^{(\bullet)}$	$SE\left[\hat{\theta}_2^{(\bullet)}\right]$
20	50.4209	3.8227	9.6194	1.8538

All of these values compare very well with the best linear unbiased estimates presented earlier.

Using these unbiased estimates of θ_2, we may also now obtain standard errors and an estimate of the covariance of the original (not corrected for bias) maximum likelihood estimates:

For $k = 0.1$:

r	$\hat{\theta}_1$	$SE(\hat{\theta}_1)$	$\hat{\theta}_2$	$SE(\hat{\theta}_2)$	$\hat{cov}(\hat{\theta}_1,\hat{\theta}_2)$
5	41.02149	6.1638	6.38566	3.3308	13.1089
10	48.25653	4.4402	9.41583	2.9981	2.8961
15	49.58711	4.1047	10.21531	2.4927	-0.9731
20	49.44142	3.6205	9.30316	1.8009	-1.3854

and for $k = 0.2$:

r	$\hat{\theta}_1$	$SE(\hat{\theta}_1)$	$\hat{\theta}_2$	$SE(\hat{\theta}_2)$	$\hat{cov}(\hat{\theta}_1,\hat{\theta}_2)$
20	50.18144	3.8447	9.28734	1.7898	-3.3043

Incidentally, the values which are obtained when using the method of approximating the information matrix, that is, substituting the original maximum likelihood estimates (not corrected for bias) into the matrix of negatives of second derivatives of the log-likelihood and inverting, agree quite well with these variances and covariance for larger values of r, and are given by the following:

For $k = 0.1$:

r	$\sqrt{\hat{I}_{(1,1)}}$	$\sqrt{\hat{I}_{(2,2)}}$	$\hat{I}_{(1,2)}$
5	5.04440	2.90077	10.04024
10	4.14645	2.77845	2.07025
15	4.03852	2.30928	-1.29874
20	3.71267	1.76191	-1.80124

and for k = 0.2:

r	$\sqrt{\hat{I}_{(1,1)}}$	$\sqrt{\hat{I}_{(2,2)}}$	$\hat{I}_{(1,2)}$
20	3.70088	1.80608	-3.14516

Next, we consider a three-parameter analysis. If the shape parameter k is assumed to be unknown, best linear unbiased estimates for the three parameters can not be obtained. However maximum likelihood estimation is still possible.

The maximum likelihood estimates for r = 10, 15 and 20 were as follows:

for r = 10: $\hat{\theta}_1 = 49.06638$ $\hat{\theta}_2 = 14.03007$ $\hat{k} = -0.23045$

for r = 15: $\hat{\theta}_1 = 48.82512$ $\hat{\theta}_2 = 11.13237$ $\hat{k} = -0.05129$

for r = 20: $\hat{\theta}_1 = 50.51735$ $\hat{\theta}_2 = 9.40326$ $\hat{k} = 0.24861$.

In light of the close agreement between simulated results and the approximated information matrix for the two-parameter case, the approximate information matrix for this three-parameter case was also obtained, with the following results (3rd parameter is k):

r	$\hat{I}_{(1,1)}$	$\hat{I}_{(2,2)}$	$\hat{I}_{(3,3)}$	$\hat{I}_{(1,2)}$	$\hat{I}_{(1,3)}$	$\hat{I}_{(2,3)}$
10	40.08580	66.89431	0.13555	26.90281	-0.69739	-2.74171
15	21.10314	10.44796	0.07841	- 1.45586	0.37938	-0.62508
20	15.60632	3.81140	0.03745	- 3.01869	0.25180	0.12295

A simulation study was also conducted for the full sample case. 500 samples of size 20 from the generalized logistic distribution with shape parameter $k = 0.1$ were simulated, and the following values were obtained:

$\frac{bias(\hat{\theta}_1)}{\theta_2}$	$\frac{bias(\hat{\theta}_2)}{\theta_2}$	$bias(\hat{k})$	$\frac{Var(\hat{\theta}_1)}{\theta_2^2}$	$\frac{Var(\hat{\theta}_2)}{\theta_2^2}$	$Var(\hat{k})$
-0.032107	-0.031482	-0.005367	0.169703	0.037129	0.023955

$\frac{cov(\hat{\theta}_1,\hat{\theta}_2)}{\theta_2^2}$	$\frac{cov(\hat{\theta}_1,\hat{k})}{\theta_2}$	$\frac{cov(\hat{\theta}_2,\hat{k})}{\theta_2}$
-0.015203	0.011412	0.004379

For comparison purposes, 500 samples of size 20 from the generalized logistic distribution with shape parameter $k = 0.2$ were simulated, and the following values for the full sample case were obtained:

$\frac{bias(\hat{\theta}_1)}{\theta_2}$	$\frac{bias(\hat{\theta}_2)}{\theta_2}$	$bias(\hat{k})$	$\frac{Var(\hat{\theta}_1)}{\theta_2^2}$	$\frac{Var(\hat{\theta}_2)}{\theta_2^2}$	$Var(\hat{k})$
-0.026821	-0.050732	-0.025533	0.146539	0.043898	0.022385

$\frac{cov(\hat{\theta}_1,\hat{\theta}_2)}{\theta_2^2}$	$\frac{cov(\hat{\theta}_1,\hat{k})}{\theta_2}$	$\frac{cov(\hat{\theta}_2,\hat{k})}{\theta_2}$
-0.023482	0.006827	0.012748

Again, in this case, in order to correct for bias, we first tested whether the bias was significantly different from zero. The p-values obtained for the test H_0: bias (parameter$_i$) = 0, i = 1, 2, 3, where parameter$_1$ = θ_1, parameter$_2$ = θ_2, parameter$_3$ = k, were as follows:

For k = 0.1:

i:	1	2	3
p-val:	0.081	0.000	0.438

and for k = 0.2:

i:	1	2	3
p-val:	0.117	0.000	0.000

We have chosen here to correct estimates of θ_1 and θ_2 for bias in the simulation with $k = 0.1$, and to correct estimates of θ_2 and k for bias in the simulation with $k = 0.2$. Thus, the unbiased estimates and their standard errors are given in the following table (k_{sim} = value of k used for corresponding simulation):

k_{sim}	$\hat{\theta}_1^{(\cdot)}$	$SE\left[\hat{\theta}_1^{(\cdot)}\right]$	$\hat{\theta}_2^{(\cdot)}$	$SE\left[\hat{\theta}_2^{(\cdot)}\right]$	$\hat{k}^{(\cdot)}$	$SE\left[\hat{k}^{(\cdot)}\right]$
0.1	50.8291	3.9882	9.7089	1.9316	0.2486	0.1548
0.2	50.5174	3.7920	9.9058	2.1864	0.2741	0.1496

Finally, we are able to approximate the variances and covariances of the original maximum likelihood estimators using the unbiased estimate for θ_2:

k_{sim}	$\hat{Var}(\hat{\theta}_1)$	$\hat{Var}(\hat{\theta}_2)$	$\hat{Var}(\hat{k})$	$\hat{cov}(\hat{\theta}_1,\hat{\theta}_2)$	$\hat{cov}(\hat{\theta}_1,k)$	$\hat{cov}(\hat{\theta}_2,k)$
0.1	15.9967	3.4999	0.0240	-1.4331	0.1108	0.0425
0.2	14.3791	4.3075	0.0224	-2.3042	0.0676	0.1263

For the most part, these agree well with the values given above for the approximate information matrix ($r = 20$).

Although best linear unbiased estimates can not be obtained for the three-parameter analysis, the method of moment estimation, where expected and observed sample mean, variance and skewness are equated, is possible in this case when the full sample is observed.

The estimates obtained using the method of moments for the full sample were as follows:

for $r=20$: $\hat{\theta}_1 = 48.38003$ $\hat{\theta}_2 = 9.16606$ $\hat{k} = 0.05126$.

A simulation was conducted, and 1000 samples of size 20 with shape parameter $k = 0.1$ were generated . Again, for comparison purposes, 1000 samples of size 20 with shape parameter $k = 0.2$ were also generated. The approximate biases, variances and covariances of the moment estimators were obtained as follows:

k_{sim}	$\frac{bias(\hat{\theta}_1)}{\theta_2}$	$\frac{bias(\hat{\theta}_2)}{\theta_2}$	$bias(\hat{k})$	$\frac{Var(\hat{\theta}_1)}{\theta_2^2}$	$\frac{Var(\hat{\theta}_2)}{\theta_2^2}$	$Var(\hat{k})$
0.1	-0.097042	-0.023411	-0.056100	0.148594	0.042713	0.004154
0.2	-0.188451	0.045116	-0.113374	0.190247	0.079860	0.003862

k_{sim}	$\frac{cov(\hat{\theta}_1,\hat{\theta}_2)}{\theta_2^2}$	$\frac{cov(\hat{\theta}_1,\hat{k})}{\theta_2}$	$\frac{cov(\hat{\theta}_2,\hat{k})}{\theta_2}$
0.1	-0.017226	0.001375	0.004774
0.2	-0.052578	0.000397	0.009752

Once again, in order to correct for bias, we first tested whether the biases were significantly different from zero. The p-values obtained for the test H_0: bias (parameter$_i$) = 0, i = 1, 2, 3, where parameter$_1$ = θ_1, parameter$_2$ = θ_2, parameter$_3$ = k, were as follows:

For k = 0.1:

i:	1	2	3
p-val:	0.000	0.000	0.000

and for k = 0.2:

i:	1	2	3
p-val:	0.000	0.000	0.000

All of these values are highly significant, and as such, all three parameters may be corrected for bias:

k_{sim}	$\hat{\theta}_1^{(\cdot)}$	$SE\left[\hat{\theta}_1^{(\cdot)}\right]$	$\hat{\theta}_2^{(\cdot)}$	$SE\left[\hat{\theta}_2^{(\cdot)}\right]$	$\hat{k}^{(\cdot)}$	$SE\left[\hat{k}^{(\cdot)}\right]$
0.1	49.2908	3.5813	9.38579	1.9863	0.1074	0.0645
0.2	50.0328	3.6572	8.77038	2.3715	0.1646	0.0621

For the most part, these estimates are comparable to those obtained using the maximum likelihood method. Therefore, due to the comparable ease of computation for the method of moments, one may prefer to use this method for obtaining estimates of parameters in the case when all three parameters of the generalized logistic distribution are unknown.

9 The Doubly Truncated Generalized Logistic Distribution

9.1 Introduction

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from the doubly truncated generalized logistic population with cumulative distribution function given by

$$F(x) = \begin{cases} \dfrac{1}{(P-Q)\left[1+(1-kx)^{1/k}\right]}, & Q_1 < x < P_1 < \dfrac{1}{k} \quad when\ k > 0, \\ & \dfrac{1}{k} < Q_1 < x < P_1 \quad when\ k < 0, \\ \dfrac{1}{(P-Q)\left[1+e^{-x}\right]}, & Q_1 < x < P_1 \quad when\ k = 0, \end{cases} \tag{9.1.1}$$

and with probability density function

$$f(x)=\begin{cases}\dfrac{(1-kx)^{\frac{1}{k}-1}}{(P-Q)\{[1+(1-kx)^{1/k}]^2\}}, & Q_1 < x < P_1 < \dfrac{1}{k} \quad when\ k > 0,\\ & \dfrac{1}{k} < Q_1 < x < P_1 \quad when\ k < 0,\\ \dfrac{e^{-x}}{(P-Q)[(1+e^{-x})^2]}, & Q_1 < x < P_1 \quad when\ k = 0,\end{cases} \tag{9.1.2}$$

where 1-P is the proportion of right truncation and Q is the proportion of left truncation of the generalized logistic distribution with cumulative distribution function as given by (7.1):

$$G(x)=\begin{cases}\dfrac{1}{1+(1-kx)^{1/k}}, & -\infty < x < \dfrac{1}{k} \quad when\ k > 0,\\ & \dfrac{1}{k} < x < \infty \quad when\ k < 0,\\ \dfrac{1}{1+e^{-x}}, & -\infty < x < \infty \quad when\ k = 0.\end{cases}$$

Thus, $P = G(P_1)$ and $Q = G(Q_1)$.

Let $X_{1:n} \le X_{2:n} \le \ldots \le X_{n:n}$ denote the order statistics obtained by arranging the n X_i's in increasing order of magnitude. Let us use $\mu_{r:n}^{(i)}$ to denote the single moments $E(X_{r:n}^i)$ for $1 \le r \le n$ and $i \ge 1$, and $\mu_{r,s:n}$ to denote the product moments $E(X_{r:n}X_{s:n})$ for $1 \le r < s \le n$. Let us further denote $Var(X_{r:n})$ by $\sigma_{r,r:n}$ and

$Cov(X_{r:n},X_{s:n})$ by $\sigma_{r,s:n}$. For simplicity, we shall also use $\mu_{r:n}$ for $\mu_{r:n}^{(1)}$.

In this chapter, we establish several recurrence relations satisfied by the single moments $\mu_{r:n}^{(i)}$ and the product moments $\mu_{r,s:n}$ from the doubly truncated generalized logistic distribution in (9.1.1). These relations will enable one to compute all the single and product moments of order statistics for all sample sizes in a simple recursive manner. If we let the shape parameter $k \to 0$, the recurrence relations reduce to the corresponding results for the doubly truncated logistic distribution established by Balakrishnan and Kocherlakota (1986). By starting with the values $E(X) = \mu_{1:1}$, $E(X^2) = \mu_{1:1}^{(2)}$, and $\mu_{1,2:2} = \mu_{1:1}^2$, one can determine the means, variances and covariances of all order statistics for all sample sizes through this recursive computational procedure. Work of this nature has been carried out by Joshi (1978, 1979, 1982) for truncated exponential distributions and by Balakrishnan and Malik (1987) and Ali and Khan (1987) for truncated log-logistic distributions.

From (9.1.1) and (9.1.2), we observe that the characterizing differential equation for the doubly truncated generalized logistic population is

$$(1-kx)f(x) = F(x)-(P-Q)F^2(x) \tag{9.1.3}$$

$$= (1-P+Q)F(x)+(P-Q)F(x)[1-F(x)]. \tag{9.1.4}$$

As Balakrishnan and Kocherlakota (1986) exploited these equations for the standard doubly truncated logistic distribution (case when $k = 0$) in order to derive several recurrence relations for the single and the product moments of order statistics, we shall use (9.1.3) and (9.1.4) in the following sections to establish similar results for the doubly truncated generalized logistic distribution in (9.1.1). These generalized recurrence relations are then shown to be complete in the sense that they will enable one to compute all the single and product moments of order statistics for all sample sizes in a simple recursive manner.

9.2. Recurrence Relations for Single Moments

The density function of $X_{r:n}$ $(1 \le r \le n)$ is given by [David (1981, p. 9), Arnold, Balakrishnan and Nagaraja (1992, p. 10)]

$$f_{r:n}(x) = \frac{n!}{(r-1)!(n-r)!}[F(x)]^{r-1}[1-F(x)]^{n-r}f(x), \quad Q_1 \le x \le P_1 . \tag{9.2.1}$$

In this section, we establish some recurrence relations for the single moments of order statistics from the doubly truncated generalized logistic distribution.

Theorem 9.2.1: *For $1 \le r \le n - 1$, $n \ge 2$ and $i = 0,1,2,\ldots,$*

$$\mu_{r+1:n+1}^{(i+1)} = \mu_{r:n+1}^{(i+1)} + \frac{n+1}{P-Q}\left[\frac{i+1}{r(n-r+1)}\left(\mu_{r:n}^{(i)} - k\mu_{r:n}^{(i+1)}\right) + \frac{1-P+Q}{n-r+1}\left(\mu_{r:n}^{(i+1)} - \mu_{r+1:n}^{(i+1)}\right)\right]; \quad (9.2.2)$$

and for $n \ge 1$ and $i = 0,1,2,\ldots,$

$$\mu_{n+1:n+1}^{(i+1)} = \mu_{n:n+1}^{(i+1)} + \frac{n+1}{P-Q}\left[\frac{i+1}{n}\left(\mu_{n:n}^{(i)} - k\mu_{n:n}^{(i+1)}\right) + (1-P+Q)\left(\mu_{n:n}^{(i+1)} - P_1^{i+1}\right)\right]. \quad (9.2.3)$$

Proof: For $1 \le r \le n - 1$ and $i = 0,1,2,\ldots,$ let us consider from (9.2.1) and (9.1.4)

$$\mu_{r:n}^{(i)} - k\mu_{r:n}^{(i+1)} = \frac{n!}{(r-1)!(n-r)!}\left\{(1-P+Q)\int_{Q_1}^{P_1} x^i\,[F(x)]^r\,[1-F(x)]^{n-r}\,dx + (P-Q)\int_{Q_1}^{P_1} x^i\,[F(x)]^r\,[1-F(x)]^{n-r+1}\,dx\right\}.$$

Integrating now by parts treating x^i for integration and the rest of the integrands for differentiation, we obtain

$$\mu_{r:n}^{(i)} - k\mu_{r:n}^{(i+1)} = \frac{r(1-P+Q)}{i+1}\left[\mu_{r+1:n}^{(i+1)} - \mu_{r:n}^{(i+1)}\right] + \frac{r(P-Q)(n-r+1)}{(i+1)(n+1)}\left[\mu_{r+1:n+1}^{(i+1)} - \mu_{r:n+1}^{(i+1)}\right]. \quad (9.2.4)$$

The relation in (9.2.2) follows simply by rewriting (9.2.4). The relation in (9.2.3) is obtained by setting $r = n$ in the above proof and simplifying. □

Theorem 9.2.2: *For $n \geq 2$ and $i = 0, 1, 2, \ldots,$*

$$\mu_{1:n+1}^{(i+1)} = (1-P+Q)\mu_{1:n-1}^{(i+1)} - (1-2P+2Q)\mu_{1:n}^{(i+1)} - \frac{i+1}{n}\left(\mu_{1:n}^{(i)} - k\mu_{1:n}^{(i+1)}\right) ; \qquad (9.2.5)$$

and for $i = 0, 1, 2, \ldots,$

$$\mu_{1:2}^{(i+1)} = (1-P+Q)P_1^{i+1} - (1-2P+2Q)\mu_{1:1}^{(i+1)} - (i+1)\left(\mu_{1:1}^{(i)} - k\mu_{1:1}^{(i+1)}\right) . \qquad (9.2.6)$$

Proof: For $n \geq 1$ and $i = 0,1,2,\ldots,$ let us consider from (9.2.1) and (9.1.4)

$$\mu_{1:n}^{(i)} - k\mu_{1:n}^{(i+1)} = n\left\{(1-P+Q)\int_{Q_1}^{P_1} x^i\, [F(x)][1-F(x)]^{n-1}\, dx \right.$$

$$\left. +(P-Q)\int_{Q_1}^{P_1} x^i\, [F(x)][1-F(x)]^{n}\, dx\right\}.$$

Integrating by parts treating x^i for integration and the rest of the integrands for differentiation, and then writing $F(x)$ as $1 - [1 - F(x)]$, we obtain

$$\mu_{1:n}^{(i)} - k\mu_{1:n}^{(i+1)} = \frac{n}{i+1}\left\{(1-P+Q)\left[\mu_{1:n-1}^{(i+1)} - \mu_{1:n}^{(i+1)}\right] + (P-Q)\left[\mu_{1:n}^{(i+1)} - \mu_{1:n+1}^{(i+1)}\right]\right\} . \qquad (9.2.7)$$

The relation in (9.2.5) follows simply by rewriting (9.2.7). The relation in (9.2.6) is obtained by setting $n = 1$ in the above proof and simplifying. □

Remark: Letting the shape parameter $k \to 0$ in (9.2.2) - (9.2.5), we deduce the recurrence relations established by Balakrishnan and Kocherlakota (1986) for the single moments of order statistics from the doubly truncated logistic distribution. Furthermore, letting $P \to 1$ and $Q \to 0$, we deduce the recurrence relations for the generalized logistic distribution, established in Chapter 8.

Remark: The recurrence relations in (9.2.2) - (9.2.5) will enable one to compute all the single moments of all order statistics for all sample sizes in a simple recursive manner. This is explained in detail in section 9.4.

9.3 Recurrence Relations for Product Moments

The joint density function of $X_{i:n}$ and $X_{j:n}$ $(1 \le i < j \le n)$ is given by [David (1981, p. 10), Arnold, Balakrishnan and Nagaraja (1992, p. 16)]

$$f_{i,j:n}(x,y) = \frac{n!}{(i-1)!(j-i-1)!(n-j)!}[F(x)]^{i-1}[F(y)-F(x)]^{j-i-1}[1-F(y)]^{n-j}f(x)f(y),$$
$$Q_1 \le x < y \le P_1 . \tag{9.3.1}$$

In this section, we establish several recurrence relations for the product moments of order statistics, from the doubly truncated generalized logistic distribution.

Theorem 9.3.1: *For $1 \le i < j \le n$ and $j - i \ge 2$,*

$$\mu_{i+1,j+1:n+1} = \mu_{i+2,j+1:n+1} + \frac{n+1}{(i+1)(P-Q)}\left[\mu_{i,j:n} - \mu_{i+1,j:n} + \frac{1}{i}\left(\mu_{j:n} - k\mu_{i,j:n}\right)\right] ; \qquad (9.3.2)$$

and for $1 \le i \le n - 1$,

$$\mu_{i+1,i+2:n+1} = \mu^{(2)}_{i+2:n+1} + \frac{n+1}{(i+1)(P-Q)}\left[\mu_{i,i+1:n} - \mu^{(2)}_{i+1:n} + \frac{1}{i}\left(\mu_{i+1:n} - k\mu_{i,i+1:n}\right)\right] . \qquad (9.3.3)$$

Proof: For $1 \le i < j \le n$ and $j - i \ge 2$, let us consider from (9.3.1) and (9.1.3)

$$\begin{aligned}\mu_{j:n} - k\mu_{i,j:n} &= E\left(X^0_{i:n}X_{j:n} - kX_{i:n}X_{j:n}\right)\\ &= \frac{n!}{(i-1)!(j-i-1)!(n-j)!}\int_{Q_1}^{P_1} y[1-F(y)]^{n-j}f(y)I_1(y)dy \ , \qquad (9.3.4)\end{aligned}$$

where

$$I_1(y) = \int_{Q_1}^{y} [F(x)]^i[F(y)-F(x)]^{j-i-1}dx - (P-Q)\int_{Q_1}^{y} [F(x)]^{i+1}[F(y)-F(x)]^{j-i-1}dx.$$

Integrating by parts treating dx for integration and the rest of the integrands for differentiation, we obtain

$$I_1(y) = \int_{Q_1}^{y} x(j-i-1)[F(x)]^i[F(y)-F(x)]^{j-i-2}f(x)dx - \int_{Q_1}^{y} xi[F(x)]^{i-1}[F(y)-F(x)]^{j-i-1}f(x)dx$$
$$-(P-Q)\int_{Q_1}^{y} x(j-i-1)[F(x)]^{i+1}[F(y)-F(x)]^{j-i-2}f(x)dx$$
$$-\int_{Q_1}^{y} x(i+1)[F(x)]^i[F(y)-F(x)]^{j-i-1}f(x)dx.$$

Upon substituting this expression of $I_1(y)$ into (9.3.4) and simplifying the resulting expression, we obtain

$$\mu_{j:n} - k\mu_{i,j:n} = i\left[\mu_{i+1,j:n} - \mu_{i,j:n}\right] + \frac{i(i+1)(P-Q)}{n+1}\left[\mu_{i+1,j+1:n+1} - \mu_{i+2,j+1:n+1}\right].$$

The recurrence relation in (9.3.2) is derived simply by rewriting the above equation. The relation in (9.3.3) is obtained by setting $j = n$ in the above proof and simplifying. □

Theorem 9.3.2: *For $1 \le i < j \le n-1$ and $j-i \ge 2$,*

$$\mu_{i,j:n+1} = \mu_{i,j-1:n+1} + \frac{n+1}{(n-j+2)(P-Q)}\Big[(1-2P+2Q)\left(\mu_{i,j-1:n} - \mu_{i,j:n}\right)$$
$$-\frac{n(1-P+Q)}{n-j+1}\left(\mu_{i,j-1:n-1} - \mu_{i,j:n-1}\right) - \frac{1}{n-j+1}\left(\mu_{i:n} - k\mu_{i,j:n}\right)\Big]; \tag{9.3.5}$$

and for $1 \le i \le n$-2,

$$\mu_{i,n:n+1} = \mu_{i,n-1:n+1} + \frac{n+1}{2(P-Q)}\Big[(1-2P+2Q)\big(\mu_{i,n-1:n}-\mu_{i,n:n}\big)$$
$$-n(1-P+Q)\big(\mu_{i,n-1:n-1}-P_1\mu_{i:n-1}\big)-\big(\mu_{i:n}-k\mu_{i,n:n}\big)\Big] \cdot \qquad (9.3.6)$$

Proof: For $1 \le i < j \le n-1$ and $j - i \ge 2$, let us consider from (9.3.1) and (9.1.4)

$$\mu_{i:n}-k\mu_{i,j:n} = E\big(X_{i:n}X^0_{j:n}-kX_{i:n}X_{j:n}\big)$$
$$= \frac{n!}{(i-1)!(j-i-1)!(n-j)!}\int_{Q_1}^{P_1} x[F(x)]^{i-1}f(x)J_1(x)dx \ , \qquad (9.3.7)$$

where

$$J_1(x) = (1-P+Q)\int_x^{P_1}[F(y)-F(x)]^{j-i-1}[1-F(y)]^{n-j}F(y)dy$$
$$+(P-Q)\int_x^{P_1}[F(y)-F(x)]^{j-i-1}[1-F(y)]^{n-j+1}F(y)dy.$$

Writing $F(y)$ as 1 - [1 - $F(y)$] in the above integrals and then integrating by parts treating dy for integration and the rest of the integrands for differentiation, we obtain an expression for $J_1(x)$ which, when substituted into (9.3.7) and simplified, gives

$$\mu_{i:n}-k\mu_{i,j:n} = (1-P+Q)\Big\{n\big[\mu_{i,j:n-1}-\mu_{i,j-1:n-1}\big]+(n-j+1)\big[\mu_{i,j-1:n}-\mu_{i,j:n}\big]\Big\}$$
$$+(P-Q)\left\{(n-j+1)\big[\mu_{i,j:n}-\mu_{i,j-1:n}\big]+\frac{(n-j+1)(n-j+2)}{n+1}\big[\mu_{i,j-1:n+1}-\mu_{i,j:n+1}\big]\right\} .$$

The recurrence relation in (9.3.5) is derived simply by rewriting the above equation. The relation in (9.3.6) is obtained by setting $j = n$ in the above proof and simplifying. □

Theorem 9.3.3: *For $1 \le i \le n-2$,*

$$\mu_{i,i+1:n+1} = \frac{n+1}{(n-i+1)(P-Q)}\Big[(1-P+Q)\big(n\mu_{i,i+1:n-1} - i\mu^{(2)}_{i+1:n}\big) - (1-2P+2Q)\mu_{i,i+1:n} - \frac{1}{n-i}\big(\mu_{i:n} - k\mu_{i,i+1:n}\big)\Big] - \frac{i}{n-i+1}\mu^{(2)}_{i+1:n+1}\ ; \tag{9.3.8}$$

and for $n \ge 2$,

$$\mu_{n-1,n:n+1} = \frac{n+1}{2(P-Q)}\Big[(1-P+Q)\big(nP_1\mu_{n-1:n-1} - (n-1)\mu^{(2)}_{n:n}\big) - (1-2P+2Q)\mu_{n-1,n:n} - \big(\mu_{n-1:n} - k\mu_{n-1,n:n}\big)\Big] - \frac{n-1}{2}\mu^{(2)}_{n:n+1}\ . \tag{9.3.9}$$

Proof: For $1 \le i \le n-2$, let us consider from (9.3.1) and (9.1.4)

$$\begin{aligned}\mu_{i:n} - k\mu_{i,i+1:n} &= E\big(X_{i:n}X^0_{i+1:n} - kX_{i:n}X_{i+1:n}\big)\\ &= \frac{n!}{(i-1)!(n-i-1)!}\int_{Q_1}^{P_1} x[F(x)]^{i-1}f(x)J_2(x)dx\ ,\end{aligned} \tag{9.3.10}$$

where

$$J_2(x) = (1-P+Q)\int_x^{P_1}[1-F(y)]^{n-i-1}F(y)dy+(P-Q)\int_x^{P_1}[1-F(y)]^{n-i}F(y)dy.$$

Writing $F(y)$ as 1 - [1 - $F(y)$] in the above integrals and then integrating by parts treating dy for integration and the rest of the integrands for differentiation, we obtain an expression for $J_2(x)$ which, when substituted into (9.3.10) and simplified by combining integrands containing $(1-P+Q)x^2$ and then combining integrands containing $(P-Q)x^2$, gives

$$\begin{aligned}\mu_{i:n}-k\mu_{i,i+1:n} &= (1-P+Q)\left[-i\mu^{(2)}_{i+1:n}+n\mu_{i,i+1:n-1}-(n-i)\mu_{i,i+1:n}\right]\\ &\quad +(P-Q)(n-i)\left[\frac{-i}{n+1}\mu^{(2)}_{i+1:n+1}+\mu_{i,i+1:n}-\frac{n-i+1}{n+1}\mu_{i,i+1:n+1}\right].\end{aligned}$$

The recurrence relation in (9.3.8) and is derived simply by rewriting the above equation. The relation in (9.3.9) is derived by setting $i = n - 1$ in the above proof and simplifying. □

Theorem 9.3.4: *For $1 \le i < j \le n-1$ and $j - i \ge 2$,*

$$\begin{aligned}\mu_{i,j+1:n+1} &= \mu_{i,j:n+1}+\frac{i}{j-i}\left(\mu_{i+1,j:n+1}-\mu_{i+1,j+1:n+1}\right)\\ &\quad +\frac{n+1}{(n-j+1)(P-Q)}\left\{(1-P+Q)\left[\frac{i}{j-i}\left(\mu_{i+1,j:n}-\mu_{i+1,j+1:n}\right)+\left(\mu_{i,j:n}-\mu_{i,j+1:n}\right)\right]+\frac{1}{j-i}\left(\mu_{i:n}-k\mu_{i,j:n}\right)\right\}\end{aligned}$$

(9.3.11)

and for 1 ≤ i ≤ n-2,

$$\mu_{i,i+2:n+1} = \mu_{i,i+1:n+1} + i\left(\mu^{(2)}_{i+1:n+1} - \mu_{i+1,i+2:n+1}\right)$$

$$+ \frac{n+1}{(n-i)(P-Q)}\left\{(1-P+Q)\left[i\left(\mu^{(2)}_{i+1:n} - \mu_{i+1,i+2:n}\right) + \left(\mu_{i,i+1:n} - \mu_{i,i+2:n}\right)\right] + \left(\mu_{i:n} - k\mu_{i,i+1:n}\right)\right\} .$$

(9.3.12)

Proof: Following the proof of Theorem 9.3.2, writing $F(y)$ as $F(x) + [F(y) - F(x)]$ in $J_1(x)$ and then integrating by parts treating dy for integration and the rest of the integrands for differentiation, we obtain an expression for $J_1(x)$ which, when substituted into (9.3.7) and simplified, gives, for $1 \le i < j \le n - 1$ and $j - i \ge 2$,

$$\mu_{i:n} - k\mu_{i,j:n} = (1-P+Q)\left\{i\left[\mu_{i+1,j+1:n} - \mu_{i+1,j:n}\right] + (j-i)\left[\mu_{i,j+1:n} - \mu_{i,j:n}\right]\right\}$$

$$+ \frac{(P-Q)(n-j+1)}{n+1}\left\{i\left[\mu_{i+1,j+1:n+1} - \mu_{i+1,j:n+1}\right] + (j-i)\left[\mu_{i,j+1:n+1} - \mu_{i,j:n+1}\right]\right\} .$$

The recurrence relation in (9.3.11) is derived simply by rewriting the above equation. The relation in (9.3.12) is obtained by setting $j = i + 1$ in the above proof and simplifying. □

Theorem 9.3.5: *For $1 \le i \le n-2$,*

$$\mu_{i,n+1:n+1} = \mu_{i,n:n+1} + \frac{i}{n-i}\left(\mu_{i+1,n:n+1} - \mu_{i+1,n+1:n+1}\right) + \frac{n+1}{P-Q}\left[\frac{1-P+Q}{n-i}\left(i\mu_{i+1,n:n} - nP_1\mu_{i:n-1} + \mu_{i,n:n}\right) + \frac{1}{n-i}\left(\mu_{i:n} - k\mu_{i,n:n}\right)\right] ; \tag{9.3.13}$$

and for $n \ge 2$,

$$\mu_{n-1,n+1:n+1} = \mu_{n-1,n:n+1} + (n-1)\left(\mu^{(2)}_{n:n+1} - \mu_{n,n+1:n+1}\right) + \frac{n+1}{P-Q}\left\{(1-P+Q)\left[(n-1)\mu^{(2)}_{n:n} - nP_1\mu_{n-1:n-1} + \mu_{n-1,n:n}\right] + \left(\mu_{n-1:n} - k\mu_{n-1,n:n}\right)\right\} . \tag{9.3.14}$$

Proof: For $1 \le i \le n-2$, let us consider from (9.3.1) and (9.1.4)

$$\begin{aligned} \mu_{i:n} - k\mu_{i,n:n} &= E\left(X_{i:n}X^0_{n:n} - kX_{i:n}X_{n:n}\right) \\ &= \frac{n!}{(i-1)!(n-i-1)!}\int_{Q_1}^{P_1} x[F(x)]^{i-1}f(x)J_3(x)\,dx \ , \end{aligned} \tag{9.3.15}$$

where

$$J_3(x) = (1-P+Q)\int_x^{P_1}[F(y)-F(x)]^{n-i-1}F(y)dy$$

$$+(P-Q)\int_x^{P_1}[F(y)-F(x)]^{n-i-1}[1-F(y)]F(y)dy.$$

Writing $F(y)$ as $F(x) + [F(y) - F(x)]$ in the above integrals and then integrating by parts treating dy for integration and the rest of the integrands for differentiation, we obtain an expression for $J_3(x)$ which, when substituted into (9.3.15) and simplified by combining integrands containing P_1, gives

$$\mu_{i:n}-k\mu_{i,n:n} = (1-P+Q)\left[nP_1\mu_{i:n-1}-i\mu_{i+1,n:n}-(n-i)\mu_{i,n:n}\right]$$
$$+\frac{P-Q}{n+1}\left\{i\left[\mu_{i+1,n+1:n+1}-\mu_{i+1,n:n+1}\right]+(n-i)\left[\mu_{i,n+1:n+1}-\mu_{i,n:n+1}\right]\right\} .$$

The recurrence relation in (9.3.13) follows readily upon simplifying the above equation. The relation in (9.3.14) is obtained by setting $i = n - 1$ in the above proof and simplifying. □

Remark: Letting the shape parameter $k \to 0$ in the above results, we deduce the recurrence relations for the product moments of order statistics from the doubly truncated logistic distribution established by Balakrishnan and Kocherlakota (1986). Furthermore, letting $P \to 1$ and $Q \to 0$, we deduce the recurrence relations for product moments of order statistics from the generalized logistic distribution, which were established in Chapter 8.

Remark: The recurrence relations established in this section are complete in the sense that they will enable one to compute all the product moments of all order statistics for all sample sizes in a simple recursive manner. This can be done for any choice of the shape parameter k; the recursive computational algorithm is explained in section 9.4 in detail.

9.4 Recursive Algorithm

Starting with the value of $\mu_{1:1} = E(X)$, we can use (9.2.6) to obtain $\mu_{1:2}$ and (9.2.5) to obtain $\mu_{1:3}, \mu_{1:4}, \ldots, \mu_{1:n}$. Then, using (9.2.3), we can obtain $\mu_{2:2}$ and using (9.2.2), we obtain $\mu_{2:3}$. Again, using (9.2.3), we can obtain $\mu_{3:3}$ and using (9.2.2), we obtain $\mu_{2:4}$ and $\mu_{3:4}$. Proceeding in this way, we can obtain all the first-order moments $\mu_{r:n}$ for $r = 1, 2, \ldots, n$ and $n = 2, 3, 4, \ldots$. Starting with the value of $\mu_{1:1}^{(2)} = E(X^2)$, we proceed exactly on similar lines to obtain all the second-order moments $\mu_{r:n}^{(2)}$ for $r = 1, 2, \ldots, n$ and $n = 2, 3, 4, \ldots$. From these values, variances of all order statistics can be computed.

By starting with the fact that $\mu_{1,2:2} = \mu_{1:1}^2$ [see Arnold and Balakrishnan

(1989)], $\mu_{2,3:3}$, $\mu_{3,4:4}$, ..., $\mu_{n-1,n:n}$ can be determined from (9.3.3). Then $\mu_{1,2:3}$ can be determined using (9.3.9), and $\mu_{1,2:4}$ $\mu_{1,2:5}$, ..., $\mu_{1,2:n}$ from (9.3.8). $\mu_{1,3:3}$ can then be determined using (9.3.14) with $n = 2$ and $\mu_{2,4:4}$, $\mu_{3,5:5}$, ..., $\mu_{n-2,n:n}$ can be found using (9.3.2). $\mu_{1,3:4}$ can then be found using (9.3.5), and $\mu_{1,3:5}$, $\mu_{1,3:6}$, ..., $\mu_{1,3:n}$ can be determined using (9.3.5). $\mu_{2,3:4}$ is then determined using (9.3.9) and $\mu_{2,3:5}$, $\mu_{2,3:6}$, ..., $\mu_{2,3:n}$ from (9.3.8). Next, $\mu_{1,4:4}$ can be determined using (9.3.13) and following the steps above, we may similarly determine $\mu_{r,s:n}$ for $1 \leq r < s \leq n$ and for $n =$ 5, 6, ... From these values, covariances of order statistics can be readily computed.

Thus, by starting just with the values of $E(X)$ and $E(X^2)$, we may compute the means, variances and covariances of order statistics for all sample sizes in a simple recursive manner. This may be done for any value of the shape parameter k and the truncation parameters P and Q.

10 The Generalized Half Logistic Distribution

10.1 Introduction and Properties of the Distribution

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from the generalized half logistic population with cumulative distribution function

$$F(x) = \frac{1-(1-kx)^{\frac{1}{k}}}{1+(1-kx)^{\frac{1}{k}}}, \quad 0 \le x \le \frac{1}{k}, \quad k \ge 0, \tag{10.1.1}$$

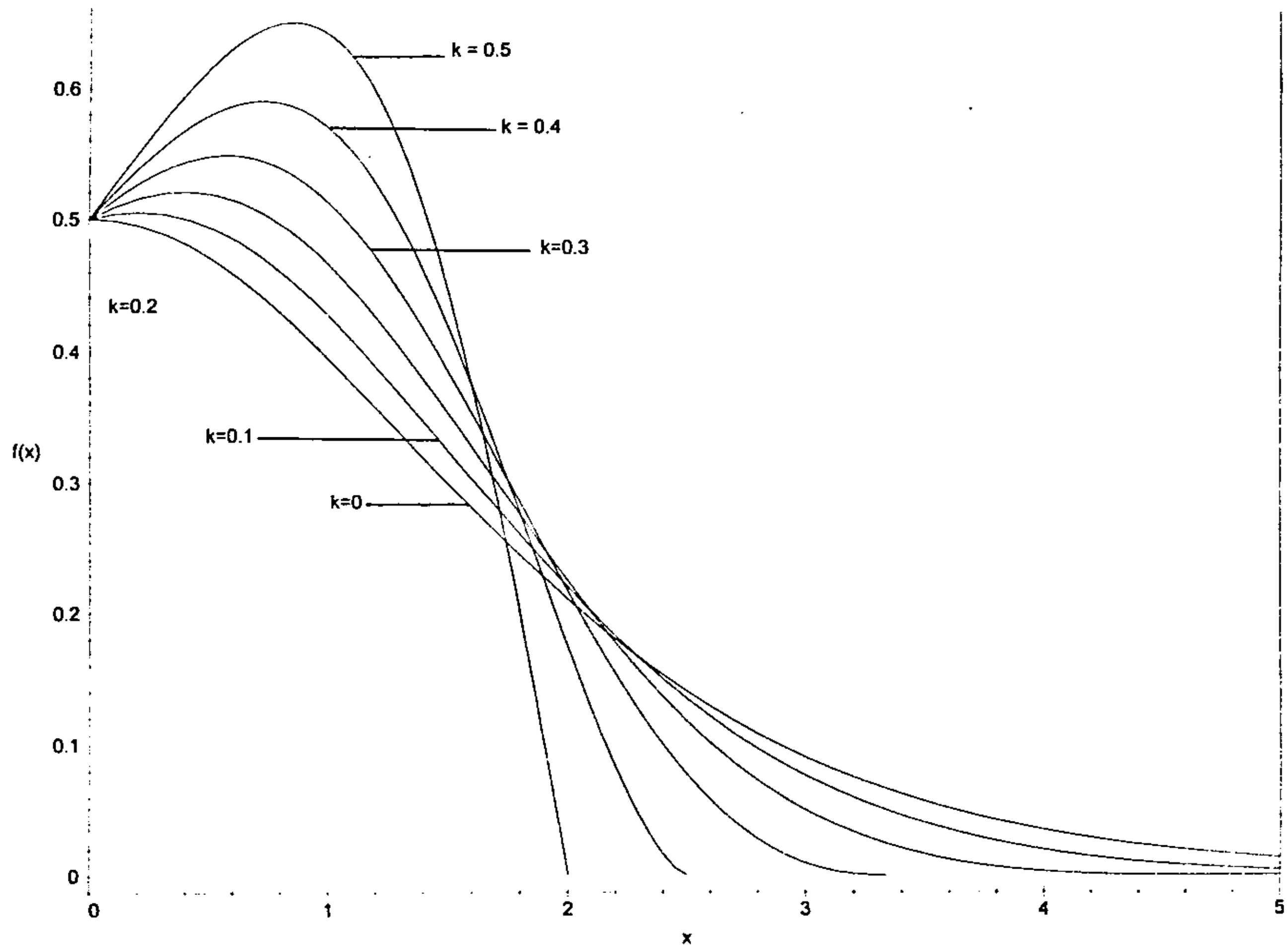
and with probability density function

$$f(x) = \frac{2(1-kx)^{\frac{1}{k}-1}}{\left[1+(1-kx)^{\frac{1}{k}}\right]^2}, \quad 0 \le x \le \frac{1}{k}, \quad k \ge 0. \tag{10.1.2}$$

Plots of the density function for selected values of k are given in Figure 10.1.1.

Let $X_{1:n} \le X_{2:n} \le \ldots \le X_{n:n}$ denote the order statistics obtained by arranging the n X_i's in increasing order of magnitude. As in Chapters 8 and 9, we will use $\mu_{r:n}^{(i)}$ to denote the single moments $E(X_{r:n}^i)$ for $1 \le r \le n$ and $i \ge 1$, $\mu_{r,s:n}$ to denote the

Figure 10.1.1: pdf's for generalized half logistic



product moments $E(X_{r:n}X_{s:n})$ for $1 \le r < s \le n$, $\sigma_{r,r:n}$ to denote $Var(X_{r:n})$, and $\sigma_{r,s:n}$ to denote $cov(X_{r:n}, X_{s:n})$. We shall also use $\mu_{r:n}$ for $\mu_{r:n}^{(1)}$.

In this chapter, we establish several recurrence relations satisfied by the single moments $\mu_{r:n}^{(i)}$ and the product moments $\mu_{r,s:n}$ for the generalized half logistic distribution. These relations will enable one to compute all single and product moments of order statistics for all sample sizes in a simple recursive manner. If we let the shape parameter $k \to 0$, the recurrence relations reduce to the corresponding results for the half logistic distribution established by Balakrishnan (1985).

By starting with the values $E(X) = \mu_{1:1}$, $E(X^2) = \mu_{1:1}^{(2)}$, and $E(X_{1,2:2}) = \mu_{1:1}^2$, one can determine the means, variances and covariances of all order statistics for all sample sizes through this recursive computational procedure. These quantities have then been used to determine the best linear unbiased estimators (BLUE's) of the location and scale parameters of the generalized half logistic distribution, and the necessary tables of coefficients and the variances and covariance of the BLUE's have been tabulated for sample sizes 5(5)20 for $k = 0.1(0.1)0.5$. Finally, we have presented an example to illustrate the method of inference discussed in this chapter.

With $f(x)$ as in (10.1.2) and

$$1-F(x) = \frac{2(1-kx)^{\frac{1}{k}}}{1+(1-kx)^{\frac{1}{k}}}, \quad 0 \leq x \leq \frac{1}{k}, \tag{10.1.3}$$

the hazard function of this generalized half logistic distribution is given by

$$h(x) = \frac{f(x)}{1-F(x)} = \frac{1}{(1-kx)\left[1+(1-kx)^{\frac{1}{k}}\right]}, \quad 0 \leq x \leq \frac{1}{k}, \quad k > 0. \tag{10.1.4}$$

It is clear from the above expression of $h(x)$ that the family of generalized half logistic distributions is an IFR (Increasing Failure Rate) family. Hence, the generalized half logistic distribution proposed and discussed in this chapter will be quite useful as a life-span model; see Cohen and Whitten (1988) for an excellent treatment of many such life-span models.

If we let the shape parameter $k \to 0$, the cdf in (10.1.1) and the pdf in (10.1.2) become

$$\frac{1-e^{-x}}{1+e^{-x}} \qquad \text{and} \qquad \frac{2e^{-x}}{\left[1+e^{-x}\right]^2} \tag{10.1.5}$$

respectively, for $x \geq 0$. This is the standard half logistic distribution discussed in detail by Balakrishnan (1985); see Balakrishnan (1992) for a review of these developments. The results established by Balakrishnan (1985) for order statistics from the standard half logistic distribution are extended here for the case of the generalized half logistic distribution in (10.1.1).

The first four moments of the generalized half logistic random variable are as follows:

$$E(X) = \frac{1}{k}\left\{1 - \frac{[{}_2F_1(2,1;2+k;0.5)]}{2(1+k)}\right\} \tag{10.1.6}$$

$$E(X^2) = \frac{1}{k^2}\left\{1 - \frac{[{}_2F_1(2,1;2+k;0.5)]}{1+k} + \frac{[{}_2F_1(2,1;2+2k;0.5)]}{2(1+2k)}\right\} \tag{10.1.7}$$

$$E(X^3) = \frac{1}{k^3}\left\{1 - \frac{3[{}_2F_1(2,1;2+k;0.5)]}{2(1+k)} + \frac{3[{}_2F_1(2,1;2+2k;0.5)]}{2(1+2k)} - \frac{[{}_2F_1(2,1;2+3k;0.5)]}{2(1+3k)}\right\} \tag{10.1.8}$$

$$E(X^4) = \frac{1}{k^4}\left\{1 - \frac{2[{}_2F_1(2,1;2+k;0.5)]}{(1+k)} + \frac{3[{}_2F_1(2,1;2+2k;0.5)]}{(1+2k)} - \frac{2[{}_2F_1(2,1;2+3k;0.5)]}{(1+3k)} + \frac{[{}_2F_1(2,1;2+4k;0.5)]}{2(1+4k)}\right\}, \tag{10.1.9}$$

where ${}_2F_1(a,b;c;z)$ is the Gaussian hypergeometric function, defined by:

$${}_2F_1(a,b;c;z) = 1 + \frac{ab}{1!c}z + \frac{a(a+1)b(b+1)}{2!c(c+1)}z^2 + \dots = \sum_{j=0}^{\infty} \frac{(a)_j(b)_j}{j!(c)_j}z^j .$$

As a result, the coefficients of skewness and kurtosis ($\sqrt{\beta_1}$, β_2) may be obtained for any value of k. Selected values of these two measures are presented in Table 10.1.1.

Table 10.1.1: Values of coefficients of skewness and kurtosis of the generalized half logistic distribution for selected values of k

k	skewness, $\sqrt{\beta_1}$	kurtosis, β_2
0	1.54033	6.58374
0.1	1.08140	4.21887
0.2	0.76580	3.11242
0.3	0.50658	2.52782
0.4	0.30268	2.20554
0.5	0.13005	2.03261

These values were obtained using Maple V Release 3. A parametric plot of kurtosis versus skewness for the range [0,0.5] is given in Figure 10.1.2. Figure 10.1.3 displays plots of skewness versus and kurtosis versus k for the same range. These plots may be useful in the initial determination of the value of k for a given sample.

From (10.1.1) and (10.1.2), we observe that the characterizing differential equation for the generalized half logistic population is

Figure 10.1.2: Parametric plot of kurtosis vs skewness

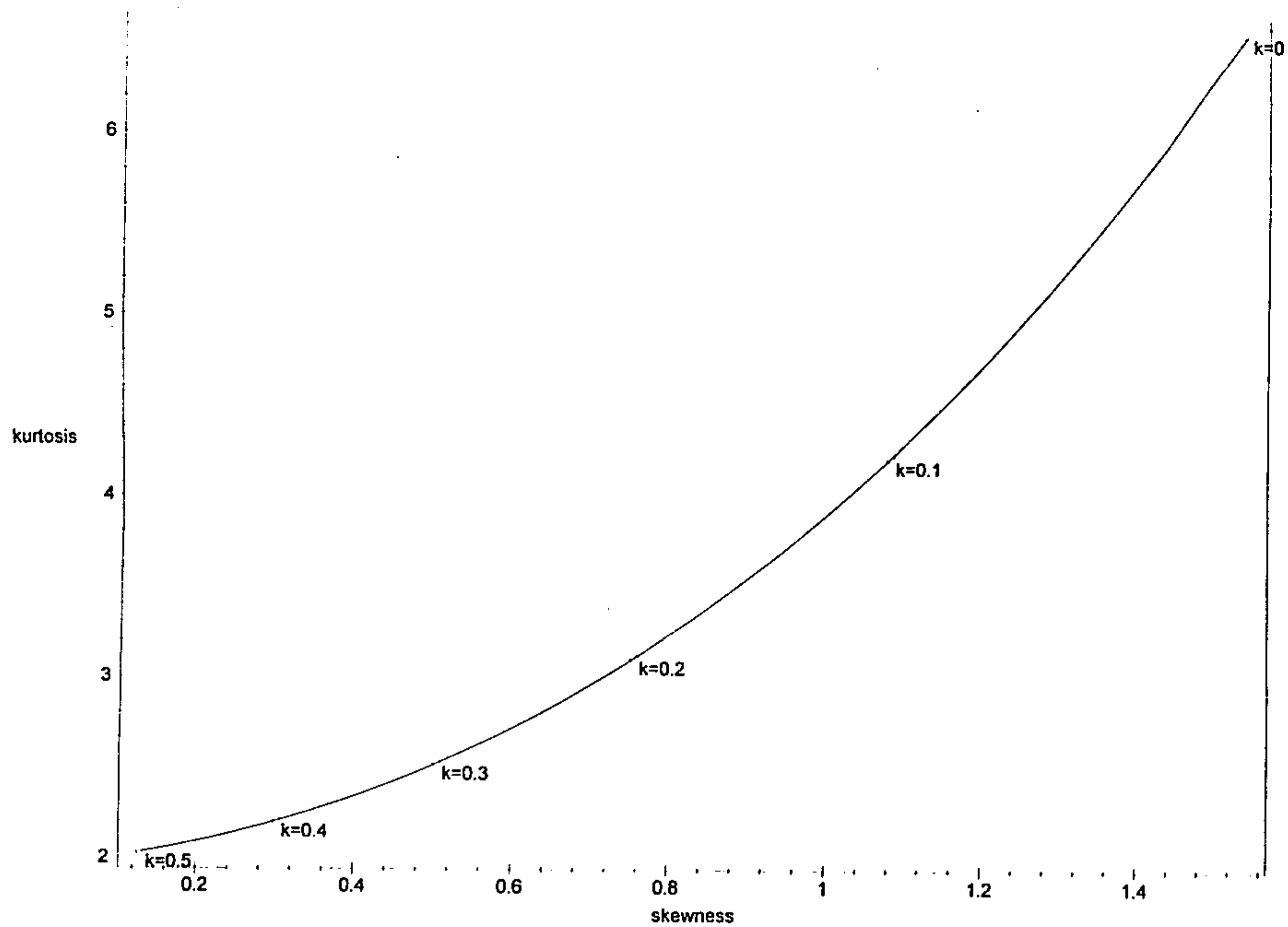
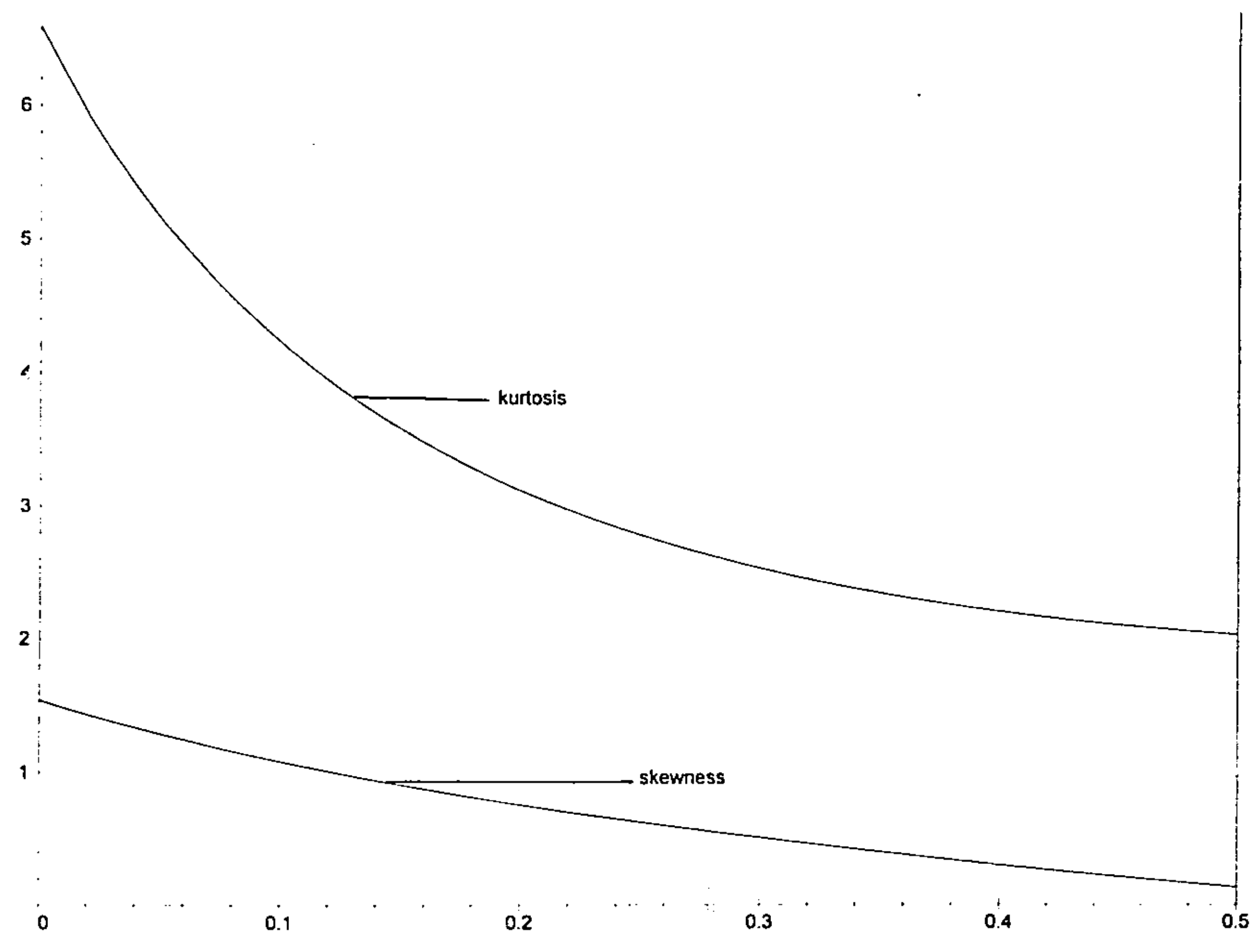


Figure 10.1.3: skewness and kurtosis plots



$$(1-kx)f(x) = \frac{1}{2}\left[1-F^2(x)\right] \qquad (10.1.10)$$

$$= \frac{1}{2}\left\{1-F(x)+F(x)[1-F(x)]\right\} \qquad (10.1.11)$$

$$= 1-F(x)-\frac{1}{2}[1-F(x)]^2 \qquad (10.1.12)$$

for $0 \le x \le 1/k$. As Balakrishnan (1985) exploited these equations for the standard half logistic distribution (case when $k = 0$) in order to derive several recurrence relations for the single and the product moments of order statistics, we shall use (10.1.10) - (10.1.12) in the following sections to establish similar results for the generalized half logistic distribution in (10.1.1).

10.2 Recurrence Relations for Single Moments

The density function of $X_{r:n}$ $(1 \le r \le n)$ is given by [David (1981, p. 9), Arnold, Balakrishnan and Nagaraja (1992, p. 10)]

$$f_{r:n}(x) = \frac{n!}{(r-1)!(n-r)!}[F(x)]^{r-1}[1-F(x)]^{n-r}f(x), \quad 0 \le x \le \frac{1}{k}. \qquad (10.2.1)$$

In this section, we establish some recurrence relations for the single moments of order statistics from the generalized half logistic distribution. It should be mentioned here that similar results are available for many other distributions; see, for example, Arnold and Balakrishnan (1989).

Theorem 10.2.1: *For $n \geq 1$ and $i = 0,1,2,\ldots,$*

$$\mu_{1:n+1}^{(i+1)} = 2\left[\mu_{1:n}^{(i+1)} - \frac{i+1}{n}\left(\mu_{1:n}^{(i)} - k\mu_{1:n}^{(i+1)}\right)\right]. \tag{10.2.2}$$

Proof: For $n \geq 1$ and $i = 0,1,2,\ldots,$ let us consider from (10.2.1)

$$\begin{aligned}\mu_{1:n}^{(i)} - k\mu_{1:n}^{(i+1)} &= n\int_0^{1/k}(x^i - kx^{i+1})[1-F(x)]^{n-1}f(x)dx \\ &= n\left\{\int_0^{1/k} x^i\,[1-F(x)]^n dx - \frac{1}{2}\int_0^{1/k} x^i\,[1-F(x)]^{n+1}dx\right\}\end{aligned}$$

upon using (10.1.12). Integrating now by parts treating x^i for integration and the rest of the integrands for differentiation, we obtain

$$\begin{aligned}\mu_{1:n}^{(i)} - k\mu_{1:n}^{(i+1)} &= \frac{n}{i+1}\left\{n\int_0^{1/k} x^{i+1}\,[1-F(x)]^{n-1}f(x)dx - \frac{n+1}{2}\int_0^{1/k} x^{i+1}\,[1-F(x)]^n f(x)dx\right\} \\ &= \frac{n}{i+1}\left[\mu_{1:n}^{(i+1)} - \frac{1}{2}\mu_{1:n+1}^{(i+1)}\right].\end{aligned} \tag{10.2.3}$$

The relation in (10.2.2) follows simply by rewriting (10.2.3). □

Theorem 10.2.2: *For $1 \leq r \leq n$ and $i = 0,1,2,\ldots,$*

$$\mu_{r+1:n+1}^{(i+1)} = \mu_{r:n+1}^{(i+1)} + \frac{n+1}{r}\left[\frac{2(i+1)}{n-r+1}\left(\mu_{r:n}^{(i)} - k\mu_{r:n}^{(i+1)}\right) - \left(\mu_{r:n}^{(i+1)} - \mu_{r-1:n}^{(i+1)}\right)\right] \tag{10.2.4}$$

with the convention that $\mu^{(i)}_{0:n} = 0$ *for* $n \geq 1$ *and* $i = 0,1,\ldots$.

Proof: For $1 \leq r \leq n$ and $i = 0,1,2,\ldots,$ let us consider from (10.2.1)

$$\mu^{(i)}_{r:n} - k\mu^{(i+1)}_{r:n} = \frac{n!}{(r-1)!(n-r)!}\int_0^{1/k}(x^i - kx^{i+1})[F(x)]^{r-1}[1-F(x)]^{n-r}f(x)dx$$

$$= \frac{n!}{(r-1)!(n-r)!\ 2}\left\{\int_0^{1/k} x^i [F(x)]^{r-1}[1-F(x)]^{n-r+1}dx + \int_0^{1/k} x^i [F(x)]^r[1-F(x)]^{n-r+1}dx\right\}$$

upon using (10.1.11). Integrating by parts treating x^i for integration and the rest of the integrands for differentiation, we obtain

$$\mu^{(i)}_{r:n} - k\mu^{(i+1)}_{r:n} = \frac{n!}{(r-1)!(n-r)!\ 2(i+1)}\left\{-(r-1)\int_0^{1/k} x^{i+1}\,[F(x)]^{r-2}[1-F(x)]^{n-r+1}f(x)dx\right.$$

$$+(n-r+1)\int_0^{1/k} x^{i+1}\,[F(x)]^{r-1}[1-F(x)]^{n-r}f(x)dx$$

$$-r\int_0^{1/k} x^{i+1}\,[F(x)]^{r-1}[1-F(x)]^{n-r+1}f(x)dx$$

$$\left.+(n-r+1)\int_0^{1/k} x^{i+1}\,[F(x)]^{r}[1-F(x)]^{n-r}f(x)dx\right\}$$

$$= \frac{n-r+1}{2(i+1)}\left[\mu^{(i+1)}_{r:n} - \mu^{(i+1)}_{r-1:n} + \frac{r}{n+1}\left(\mu^{(i+1)}_{r+1:n+1} - \mu^{(i+1)}_{r:n+1}\right)\right] .$$

(10.2.5)

The relation in (10.2.4) follows simply by rewriting (10.2.5). □

Remark: Letting the shape parameter $k \to 0$ in Theorems 10.2.1 and 10.2.2, we deduce the recurrence relations established by Balakrishnan (1985) for the single moments of order statistics from the standard half logistic distribution.

Remark: The recurrence relations established in Theorems 10.2.1 and 10.2.2 will enable one to compute all the single moments of all order statistics for all sample sizes in a simple recursive manner. For example, by starting with the values of $\mu_{1:1} = E(X)$ and $\mu_{1:1}^{(2)} = E(X^2)$, one can use the recurrence relations in (10.2.2) and (10.2.4) in order to determine the first two single moments (or the means and variances) of all order statistics for all sample sizes. This can be done for any choice of the shape parameter k. The recursive algorithm will be discussed in detail in section 10.4.

10.3 Recurrence Relations for Product Moments

The joint density function of $X_{r:n}$ and $X_{s:n}$ $(1 \le r < s \le n)$ is given by [David (1981, p. 10), Arnold, Balakrishnan and Nagaraja (1992, p. 16)]

$$f_{r,s:n}(x,y) = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}[F(x)]^{r-1}[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s}f(x)f(y),$$
$$0 \le x < y \le \frac{1}{k}. \tag{10.3.1}$$

In this section, we establish several recurrence relations for the product moments of order statistics from the generalized half logistic distribution.

Theorem 10.3.1: *For $1 \leq r \leq n-1$,*

$$\mu_{r,r+1:n+1} = \mu^{(2)}_{r:n+1} + \frac{2(n+1)}{n-r+1}\left[\mu_{r,r+1:n} - \mu^{(2)}_{r:n} + \frac{1}{n-r}\left(k\mu_{r,r+1:n} - \mu_{r:n}\right)\right]. \tag{10.3.2}$$

Proof: For $1 \leq r \leq n$-1, let us consider from (10.3.1)

$$\begin{aligned}\mu_{r:n} - k\mu_{r,r+1:n} &= E\left(X_{r:n}X^0_{r+1:n} - kX_{r:n}X_{r+1:n}\right) \\ &= \frac{n!}{(r-1)!(n-r-1)!}\int_0^{1/k}\int_x^{1/k} x(1-ky)[F(x)]^{r-1}[1-F(y)]^{n-r-1}f(x)f(y)dydx \\ &= \frac{n!}{(r-1)!(n-r-1)!}\int_0^{1/k} x[F(x)]^{r-1}f(x)I_1(x)dx\end{aligned}$$

(10.3.3)

upon using (10.1.12), where

$$I_1(x) = \int_x^{1/k}[1-F(y)]^{n-r}dy - \frac{1}{2}\int_x^{1/k}[1-F(y)]^{n-r+1}dy.$$

Integrating by parts treating dy for integration and the rest of the integrands for differentiation, we obtain

$$I_1(x) = -x[1-F(x)]^{n-r}+(n-r)\int_x^{1/k} y[1-F(y)]^{n-r-1}f(y)dy$$
$$+\frac{1}{2}x[1-F(x)]^{n-r+1}-\frac{n-r+1}{2}\int_x^{1/k} y[1-F(y)]^{n-r}f(y)dy. \qquad (10.3.4)$$

Upon substituting the expression of $I_1(x)$ in (10.3.4) into (10.3.3) and simplifying the resulting expression, we obtain

$$\mu_{r:n}-k\mu_{r,r+1:n} = (n-r)\left[\mu_{r,r+1:n}-\mu_{r:n}^{(2)}\right]+\frac{(n-r)(n-r+1)}{2(n+1)}\left[\mu_{r:n+1}^{(2)}-\mu_{r,r+1:n+1}\right].$$

The recurrence relation in (10.3.2) is derived simply by rewriting the above equation. □

Theorem 10.3.2: *For $1 \le r < s \le n$ and $s - r \ge 2$,*

$$\mu_{r,s:n+1} = \mu_{r,s-1:n+1}+\frac{2(n+1)}{n-s+2}\left[\mu_{r,s:n}-\mu_{r,s-1:n}+\frac{1}{n-s+1}\left(k\mu_{r,s:n}-\mu_{r:n}\right)\right]. \qquad (10.3.5)$$

Proof: For $1 \le r < s \le n$ and $s - r \ge 2$, let us consider from (10.3.1)

$$\begin{aligned}
\mu_{r:n} - k\mu_{r,s:n} &= E\left(X_{r:n}X^{0}_{s:n} - kX_{r:n}X_{s:n}\right) \\
&= \frac{n!}{(r-1)!(s-r-1)!(n-s)!}\int_{0}^{1/k}\int_{x}^{1/k} x(1-ky)[F(x)]^{r-1}[F(y)-F(x)]^{s-r-1} \\
&\qquad \cdot [1-F(y)]^{n-s} f(x) f(y)\, dy\, dx \\
&= \frac{n!}{(r-1)!(s-r-1)!(n-s)!}\int_{0}^{1/k} x[F(x)]^{r-1} f(x) I_2(x)\, dx
\end{aligned} \tag{10.3.6}$$

upon using (10.1.12), where

$$I_2(x) = \int_{x}^{1/k}[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s+1}dy - \frac{1}{2}\int_{x}^{1/k}[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s+2}dy.$$

Integrating by parts treating dy for integration and the rest of the integrands for differentiation, we obtain

$$I_2(x) = -(s-r-1)\int_x^{1/k} y[F(y)-F(x)]^{s-r-2}[1-F(y)]^{n-s+1}f(y)dy$$

$$+(n-s+1)\int_x^{1/k} y[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s}f(y)dy$$

$$+\frac{s-r-1}{2}\int_x^{1/k} y[F(y)-F(x)]^{s-r-2}[1-F(y)]^{n-s+2}f(y)dy$$

$$-\frac{n-s+2}{2}\int_x^{1/k} y[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s+1}f(y)dy. \qquad (10.3.7)$$

Upon substituting the expression of $I_2(x)$ in (10.3.7) into (10.3.6) and simplifying the resulting expression, we obtain

$$\mu_{r:n}-k\mu_{r,s:n} = (n-s+1)\left[\mu_{r,s:n}-\mu_{r,s-1:n}\right]+\frac{(n-s+1)(n-s+2)}{2(n+1)}\left[\mu_{r,s-1:n+1}-\mu_{r,s:n+1}\right].$$

The recurrence relation in (10.3.5) is derived simply by rewriting the above equation. □

Theorem 10.3.3: *For $2 \le r \le n-1$,*

$$\mu_{r+1,r+2:n+1} = \mu^{(2)}_{r+2:n+1}+\frac{2(n+1)}{r(r+1)}\left[\mu_{r+1:n}-k\mu_{r,r+1:n}-\frac{n}{2}\left(\mu^{(2)}_{r:n-1}-\mu_{r-1,r:n-1}\right)\right]. \qquad (10.3.8)$$

Proof: For $2 \le r \le n-1$, let us consider from (10.3.1)

$$\mu_{r+1:n} - k\mu_{r,r+1:n} = E\left(X^0_{r:n} X_{r+1:n} - kX_{r:n} X_{r+1:n}\right)$$

$$= \frac{n!}{(r-1)!(n-r-1)!} \int_0^{1/k} \int_0^{y} y(1-kx)[F(x)]^{r-1}[1-F(y)]^{n-r-1} f(x) f(y)\, dx\, dy$$

$$= \frac{n!}{(r-1)!(n-r-1)!} \int_0^{1/k} y[1-F(y)]^{n-r-1} f(y) J_1(y)\, dy$$

(10.3.9)

upon using (10.1.10), where

$$J_1(y) = \frac{1}{2}\int_0^y [F(x)]^{r-1} dx - \frac{1}{2}\int_0^y [F(x)]^{r+1} dx.$$

Integrating by parts treating dx for integration and the rest of the integrands for differentiation, we obtain

$$J_1(y) = \frac{1}{2} y[F(y)]^{r-1} - \frac{r-1}{2}\int_0^y x[F(x)]^{r-2} f(x)\, dx - \frac{1}{2} y[F(y)]^{r+1} + \frac{r+1}{2}\int_0^y x[F(x)]^{r} f(x)\, dx.$$

(10.3.10)

Upon substituting the expression of $J_1(y)$ in (10.3.10) into (10.3.9) and simplifying the resulting expression, we obtain

$$\mu_{r+1:n} - k\mu_{r,r+1:n} = \frac{n}{2}\left[\mu^{(2)}_{r:n-1} - \mu_{r-1,r:n-1}\right] + \frac{r(r+1)}{2(n+1)}\left[\mu_{r+1,r+2:n+1} - \mu^{(2)}_{r+2:n+1}\right].$$

The recurrence relation in (10.3.8) is derived simply by rewriting the above equation. □

Corollary 10.3.1: *Setting $r = n - 1$ in Theorem 10.3.3, we get the relation*

$$\mu_{n,n+1:n+1} = \mu^{(2)}_{n+1:n+1} + \frac{2(n+1)}{n(n-1)}\left[\mu_{n:n} - k\mu_{n-1,n:n} - \frac{n}{2}\left(\mu^{(2)}_{n-1:n-1} - \mu_{n-2,n-1:n-1}\right)\right], \quad n \geq 3.$$

(10.3.11)

Theorem 10.3.4: *For $2 \leq r < s \leq n$ and $s - r \geq 2$,*

$$\mu_{r+1,s+1:n+1} = \mu_{r+2,s+1:n+1} + \frac{2(n+1)}{r(r+1)}\left[\mu_{s:n} - k\mu_{r,s:n} - \frac{n}{2}\left(\mu_{r,s-1:n-1} - \mu_{r-1,s-1:n-1}\right)\right].$$

(10.3.12)

Proof: For $2 \leq r < s \leq n$ and $s - r \geq 2$, let us consider from (10.3.1)

$$\begin{aligned}\mu_{s:n} - k\mu_{r,s:n} &= E\left(X^0_{r:n}X_{s:n} - kX_{r:n}X_{s:n}\right)\\ &= \frac{n!}{(r-1)!(s-r-1)!(n-s)!}\int_0^{1/ky}\int_0 y(1-kx)[F(x)]^{r-1}[F(y)-F(x)]^{s-r-1}\\ &\qquad \cdot[1-F(y)]^{n-s}f(x)f(y)dxdy\\ &= \frac{n!}{(r-1)!(s-r-1)!(n-s)!}\int_0^{1/k} y[1-F(y)]^{n-s}f(y)J_2(y)dy\end{aligned}$$

(10.3.13)

upon using (10.1.10), where

$$J_2(y) = \frac{1}{2}\int_0^y [F(x)]^{r-1}[F(y)-F(x)]^{s-r-1}dx - \frac{1}{2}\int_0^y [F(x)]^{r+1}[F(y)-F(x)]^{s-r-1}dx.$$

Integrating by parts treating dx for integration and the rest of the integrands for differentiation, we obtain

$$\begin{aligned} J_2(y) = & -\frac{r-1}{2}\int_0^y x[F(x)]^{r-2}[F(y)-F(x)]^{s-r-1}f(x)dx \\ & +\frac{s-r-1}{2}\int_0^y x[F(x)]^{r-1}[F(y)-F(x)]^{s-r-2}f(x)dx \\ & +\frac{r+1}{2}\int_0^y x[F(x)]^{r}[F(y)-F(x)]^{s-r-1}f(x)dx \\ & -\frac{s-r-1}{2}\int_0^y x[F(x)]^{r+1}[F(y)-F(x)]^{s-r-2}f(x)dx. \end{aligned} \tag{10.3.14}$$

Upon substituting the expression of $J_2(y)$ in (10.3.14) into (10.3.13) and simplifying the resulting expression, we obtain

$$\mu_{s:n}-k\mu_{r,s:n} = \frac{n}{2}\left[\mu_{r,s-1:n-1}-\mu_{r-1,s-1:n-1}\right]+\frac{r(r+1)}{2(n+1)}\left[\mu_{r+1,s+1:n+1}-\mu_{r+2,s+1:n+1}\right].$$

The recurrence relation in (10.3.12) is derived simply by rewriting the above equation. □

Corollary 10.3.2: *Setting $s = n$ in Theorem 10.3.4, we get the relation*

$$\mu_{r+1,n+1:n+1}=\mu_{r+2,n+1:n+1}+\frac{2(n+1)}{r(r+1)}\left[\mu_{n:n}-k\mu_{r,n:n}-\frac{n}{2}\left(\mu_{r,n-1:n-1}-\mu_{r-1:n-1:n-1}\right)\right],$$

$$2 \le r \le n-2. \tag{10.3.15}$$

Similarly, by setting $r = 1$ in the proof of Theorem 10.3.4 and simplifying, we obtain the following result.

Corollary 10.3.3: *For 3 $\le s \le n$,*

$$\mu_{2,s+1:n+1} = \mu_{3,s+1:n+1} + (n+1)\left[\mu_{s:n} - k\mu_{1,s:n} - \frac{n}{2}\mu_{1,s-1:n-1}\right]. \tag{10.3.16}$$

Theorem 10.3.5: *For $n \ge 3$,*

$$\mu_{1,n+1:n+1} = \frac{2(n+1)}{n(n-1)}\left[(n-1+k)\mu_{1,n:n} + \mu_{1,n-1:n} - \frac{n-1}{n+1}\mu_{1,n:n+1} - \frac{1}{n+1}\mu_{1,n-1:n+1} - \mu_{n:n}\right]. \tag{10.3.17}$$

Proof: For $n \ge 3$, let us consider from (10.3.1)

$$\begin{aligned}\mu_{n:n} - k\mu_{1,n:n} &= E\left(X_{1:n}^{0}X_{n:n} - kX_{1:n}X_{n:n}\right)\\ &= n(n-1)\int_0^{1/k}\int_0^{y} y(1-kx)[F(y)-F(x)]^{n-2}f(x)f(y)\,dx\,dy\\ &= n(n-1)\int_0^{1/k} yf(y)J_3(y)\,dy\end{aligned} \tag{10.3.18}$$

upon using (10.1.12), where

$$\begin{aligned}J_3(y) &= \int_0^y [F(y)-F(x)]^{n-2}[1-F(x)]\,dx - \frac{1}{2}\int_0^y [F(y)-F(x)]^{n-2}[1-F(x)]^2\,dx\\ &= \int_0^y [F(y)-F(x)]^{n-1}\,dx + [1-F(y)]\int_0^y [F(y)-F(x)]^{n-2}\,dx - \frac{1}{2}\int_0^y [F(y)-F(x)]^{n}\,dx\\ &\quad -[1-F(y)]\int_0^y [F(y)-F(x)]^{n-1}\,dx - \frac{1}{2}[1-F(y)]^2\int_0^y [F(y)-F(x)]^{n-2}\,dx.\end{aligned}$$

Integrating by parts treating dx for integration and the rest of the integrands for differentiation, we obtain

$$\begin{aligned} J_3(y) &= (n-1)\int_0^y x[F(y)-F(x)]^{n-2}f(x)dx+(n-2)[1-F(y)]\int_0^y x[F(y)-F(x)]^{n-3}f(x)dx \\ &-\frac{n}{2}\int_0^y x[F(y)-F(x)]^{n-1}f(x)dx-(n-1)[1-F(y)]\int_0^y x[F(y)-F(x)]^{n-2}f(x)dx \\ &-\frac{n-2}{2}[1-F(y)]^2\int_0^y x[F(y)-F(x)]^{n-3}f(x)dx. \end{aligned} \tag{10.3.19}$$

Upon substituting the expression of $J_3(y)$ in (10.3.19) into (10.3.18) and simplifying the resulting expression, we obtain

$$\mu_{n:n}-k\mu_{1,n:n} = (n-1)\mu_{1,n:n}+\mu_{1,n-1:n}-\frac{n(n-1)}{2(n+1)}\mu_{1,n+1:n+1}-\frac{n-1}{n+1}\mu_{1,n:n+1}-\frac{1}{n+1}\mu_{1,n-1:n+1}.$$

The recurrence relation in (10.3.17) follows readily upon simplifying the above equation. □

For the case when $n = 2$, upon simplifying the proof of Theorem 10.3.5, we obtain the following result.

Corollary 10.3.4:

$$\mu_{1,3:3} = 3\left[(1+k)\mu_{1,2:2}+\mu_{1:2}^{(2)}-\frac{1}{3}\mu_{1,2:3}-\frac{1}{3}\mu_{1:3}^{(2)}-\mu_{2:2}\right]. \tag{10.3.20}$$

Proceeding along the lines of Corollary 10.3.3 for the case when $s = 2$, we establish the following result.

Corollary 10.3.5: *For $n \ge 2$,*

$$\mu_{2,3:n+1} = \mu^{(2)}_{3:n+1} + (n+1)\left[\mu_{2:n} - k\mu_{1,2:n} - \frac{n}{2}\mu^{(2)}_{1:n-1}\right]. \tag{10.3.21}$$

***Remark*:** Letting the shape parameter $k \to 0$ in the above results, we deduce the recurrence relations for the product moments of order statistics from the standard half logistic distribution established by Balakrishnan (1985).

***Remark*:** The recurrence relations established in this section are complete in the sense that they will enable one to compute all the product moments of all order statistics for all sample sizes in a simple recursive manner. This can be done for any choice of the shape parameter k; the recursive computational algorithm is explained in the next section in detail.

10.4 Recursive Algorithm

Starting with the values of $\mu_{1:1} = E(X)$ and, $\mu^{(2)}_{1:1} = E(X^2)$, relations in (10.2.2) and (10.2.4) can be used to compute $\mu_{r:n}$ and $\mu^{(2)}_{r:n}$ for $r = 1, 2, \ldots, n$ and $n = 2, 3, 4, \ldots$. From these values, variances of all order statistics can be computed.

By starting with the fact that $\mu_{1,2:2} = \mu_{1:1}^2$, $\mu_{1,2:3}$ can be determined from (10.3.2), $\mu_{2,3:3}$ from (10.3.21), and $\mu_{1,3:3}$ from (10.3.20). For the sample of size 4, $\mu_{1,2:4}$ and $\mu_{2,3:4}$ can be determined from (10.3.2), $\mu_{3,4:4}$ from (10.3.11), $\mu_{1,3:4}$ from (10.3.5), $\mu_{2,4:4}$ from (10.3.16), and finally $\mu_{1,4:4}$ from (10.3.17). This process may be followed similarly to determine $\mu_{r,s:n}$ for $1 \le r < s \le n$ and for $n = 5, 6, \ldots$. From these values, covariances of order statistics can be readily computed.

Table 10.4.1 gives the means of order statistics for values of k=0.1(0.1)0.5 up to sample size $n = 8$. Table 10.4.2 gives the values of variances and covariances for the same choices of n and k.

10.5 Best Linear Unbiased Estimators

With the use of the recursive algorithms to compute means, variances and covariances of order statistics from the generalized half logistic distribution, and then using (8.6.1) and (8.6.2), we can develop best linear unbiased estimators of location (θ_1) and scale (θ_2) parameters for a given value of k when available samples are either complete or conventional Type-II censored either from the left, the right or both, given prior knowledge of which order statistics were observed.

TABLE 10.4.1: MEANS OF ORDER STATISTICS, $\mu_{i:n}$, FOR GENERALIZED HALF LOGISTIC DISTRIBUTION

i	*n*	*k*=0.1	*k*=0.2	*k*=0.3	*k*=0.4	*k*=0.5
1	1	1.23811	1.11686	1.01602	.93098	.85841
1	2	.72385	.68047	.64166	.60675	.57522
2	2	1.75238	1.55325	1.39039	1.25521	1.14159
1	3	.52009	.49704	.47582	.45621	.43806
2	3	1.13138	1.04734	.97335	.90784	.84956
3	3	2.06288	1.80621	1.59891	1.42890	1.28761
1	4	.40818	.39369	.38013	.36741	.35546
2	4	.85581	.80710	.76288	.72261	.68583
3	4	1.40695	1.28758	1.18381	1.09307	1.01328
4	4	2.28152	1.97909	1.73727	1.54084	1.37906
1	5	.33677	.32675	.31728	.30830	.29979
2	5	.69382	.66145	.63154	.60384	.57815
3	5	1.09879	1.02557	.95989	.90076	.84736
4	5	1.61239	1.46225	1.33309	1.22128	1.12389
5	5	2.44880	2.10830	1.83832	1.62073	1.44285
1	6	.28702	.27965	.27263	.26593	.25954
2	6	.58556	.56228	.54052	.52015	.50105
3	6	.91034	.85977	.81356	.77122	.73235
4	6	1.28723	1.19137	1.10622	1.03031	.96237
5	6	1.77497	1.59768	1.44653	1.31677	1.20465
6	6	2.58357	2.21043	1.91667	1.68152	1.49049
1	7	.25026	.24461	.23919	.23399	.22900
2	7	.50752	.48990	.47328	.45760	.44278
3	7	.78065	.74324	.70863	.67654	.64673
4	7	1.08327	1.01515	.95347	.89747	.84650
5	7	1.44020	1.32354	1.22079	1.12993	1.04926
6	7	1.90888	1.70734	1.53682	1.39150	1.26681
7	7	2.69602	2.29428	1.97998	1.72986	1.52777
1	8	.22197	.21748	.21316	.20900	.20499
2	8	.44836	.43451	.42137	.40889	.39703
3	8	.68500	.65605	.62901	.60372	.58003
4	8	.94006	.88856	.84132	.79790	.75790
5	8	1.22649	1.14173	1.06561	.99704	.93511
6	8	1.56844	1.43262	1.31390	1.20967	1.11776
7	8	2.02236	1.79892	1.61113	1.45211	1.31649
8	8	2.79225	2.36504	2.03268	1.76954	1.55795

TABLE 10.4.2: VARIANCES AND COVARIANCES, $\sigma_{i,j:n}$, FOR GENERALIZED HALF LOGISTIC DISTRIBUTION

i	*j*	*n*	*k*=0.1	*k*=0.2	*k*=0.3	*k*=0.4	*k*=0.5
1	1	1	.89207	.61142	.43573	.32048	.24194
1	1	2	.34358	.27415	.22189	.18186	.15072
1	2	2	.26447	.19044	.14015	.10512	.08019
2	2	2	.91163	.56781	.36928	.24884	.17278
1	1	3	.19040	.16128	.13769	.11839	.10246
1	2	3	.15695	.12533	.10110	.08233	.06762
1	3	3	.12502	.09043	.06656	.04977	.03773
2	2	3	.40083	.29802	.22528	.17282	.13435
2	3	3	.32162	.21635	.14903	.10484	.07514
3	3	3	.87780	.51074	.31083	.19636	.12803
1	1	4	.12317	.10783	.09489	.08390	.07451
1	2	4	.10500	.08810	.07440	.06320	.05398
1	3	4	.08895	.07052	.05640	.04547	.03693
1	4	4	.07256	.05223	.03819	.02833	.02128
2	2	4	.24181	.19344	.15620	.12722	.10445
2	3	4	.20551	.15526	.11866	.09166	.07151
2	4	4	.16814	.11527	.08051	.05718	.04124
3	3	4	.40797	.28718	.20576	.14979	.11064
3	4	4	.33591	.21437	.14024	.09378	.06396
4	4	4	.84320	.46572	.26928	.16176	.10038
1	1	5	.08695	.07776	.06980	.06287	.05681
1	2	5	.07576	.06552	.05692	.04965	.04348
1	3	5	.06601	.05481	.04576	.03841	.03240
1	4	5	.05700	.04480	.03548	.02830	.02273
1	5	5	.04729	.03380	.02451	.01801	.01340
2	2	5	.16608	.13851	.11625	.09813	.08329
2	3	5	.14495	.11602	.09356	.07597	.06208
2	4	5	.12537	.09495	.07260	.05601	.04357
2	5	5	.10417	.07172	.05020	.03567	.02569
3	3	5	.25700	.19628	.15144	.11797	.09271
3	4	5	.22295	.16104	.11777	.08712	.06514
3	5	5	.18576	.12193	.08159	.05557	.03845
4	4	5	.40310	.27150	.18626	.12992	.09200
4	5	5	.33770	.20654	.12957	.08314	.05444
5	5	5	.81330	.43079	.23898	.13780	.08213
1	1	6	.06495	.05897	.05369	.04902	.04486
1	2	6	.05751	.05078	.04498	.03997	.03563
1	3	6	.05101	.04361	.03744	.03226	.02790
1	4	6	.04515	.03711	.03065	.02545	.02122
1	5	6	.03954	.03080	.02416	.01908	.01517
1	6	6	.03322	.02355	.01694	.01234	.00909
2	2	6	.12263	.10512	.09052	.07829	.06799
2	3	6	.10887	.09035	.07538	.06320	.05324

i	*j*	*n*	*k*=0.1	*k*=0.2	*k*=0.3	*k*=0.4	*k*=0.5
2	4	6	.09645	.07694	.06176	.04987	.04050
2	5	6	.08455	.06390	.04870	.03741	.02895
2	6	6	.07109	.04890	.03416	.02419	.01735
3	3	6	.18267	.14630	.11799	.09578	.07824
3	4	6	.16210	.12475	.09678	.07564	.05955
3	5	6	.14231	.10374	.07639	.05678	.04259
3	6	6	.11983	.07948	.05363	.03675	.02554
4	4	6	.26031	.19127	.14207	.10659	.08073
4	5	6	.22913	.15942	.11236	.08014	.05781
4	6	6	.19341	.12242	.07904	.05195	.03471
5	5	6	.39520	.25659	.16975	.11422	.07807
5	6	6	.33519	.19786	.11984	.07426	.04699
6	6	6	.78795	.40306	.21599	.12035	.06933
1	1	7	.05051	.04638	.04268	.03936	.03637
1	2	7	.04528	.04059	.03648	.03287	.02969
1	3	7	.04068	.03550	.03108	.02729	.02403
1	4	7	.03656	.03093	.02627	.02238	.01914
1	5	7	.03275	.02666	.02181	.01792	.01479
1	6	7	.02899	.02238	.01740	.01362	.01072
1	7	7	.02459	.01730	.01234	.00891	.00651
2	2	7	.09489	.08296	.07279	.06408	.05659
2	3	7	.08531	.07261	.06204	.05321	.04580
2	4	7	.07672	.06328	.05244	.04365	.03648
2	5	7	.06875	.05456	.04355	.03494	.02818
2	6	7	.06089	.04583	.03476	.02656	.02043
2	7	7	.05168	.03543	.02465	.01738	.01241
3	3	7	.13868	.11466	.09530	.07959	.06678
3	4	7	.12483	.10002	.08060	.06531	.05320
3	5	7	.11197	.08631	.06698	.05231	.04111
3	6	7	.09926	.07254	.05349	.03978	.02981
3	7	7	.08430	.05612	.03796	.02605	.01811
4	4	7	.18900	.14624	.11400	.08948	.07071
4	5	7	.16978	.12635	.09483	.07174	.05468
4	6	7	.15071	.10633	.07581	.05459	.03967
4	7	7	.12817	.08236	.05385	.03577	.02411
5	5	7	.25918	.18429	.13250	.09627	.07062
5	6	7	.23061	.15541	.10613	.07336	.05130
5	7	7	.19656	.12061	.07551	.04814	.03121
6	6	7	.38685	.24342	.15611	.10186	.06752
6	7	7	.33113	.18962	.11144	.06703	.04117
7	7	7	.76629	.38045	.19792	.10707	.05990
1	1	8	.04048	.03750	.03480	.03235	.03012
1	2	8	.03664	.03323	.03021	.02751	.02510
1	3	8	.03325	.02947	.02619	.02332	.02082
1	4	8	.03022	.02610	.02260	.01963	.01710
1	5	8	.02743	.02299	.01933	.01630	.01380
1	6	8	.02480	.02001	.01622	.01320	.01079
1	7	8	.02214	.01695	.01307	.01014	.00791

i	*j*	*n*	*k*=0.1	*k*=0.2	*k*=0.3	*k*=0.4	*k*=0.5
1	8	8	.01892	.01321	.00935	.00670	.00485
2	2	8	.07592	.06737	.05996	.05350	.04786
2	3	8	.06893	.05977	.05199	.04536	.03969
2	4	8	.06265	.05293	.04488	.03818	.03260
2	5	8	.05691	.04664	.03838	.03171	.02631
2	6	8	.05146	.04061	.03221	.02568	.02057
2	7	8	.04595	.03441	.02596	.01972	.01508
2	8	8	.03929	.02682	.01858	.01303	.00925
3	3	8	.10981	.09293	.07895	.06733	.05763
3	4	8	.09988	.08234	.06818	.05669	.04734
3	5	8	.09077	.07258	.05833	.04710	.03821
3	6	8	.08213	.06323	.04897	.03815	.02988
3	7	8	.07337	.05360	.03948	.02930	.02191
3	8	8	.06276	.04180	.02826	.01937	.01344
4	4	8	.14613	.11711	.09437	.07645	.06224
4	5	8	.13292	.10331	.08078	.06354	.05026
4	6	8	.12036	.09006	.06787	.05149	.03932
4	7	8	.10762	.07641	.05475	.03957	.02884
4	8	8	.09213	.05962	.03921	.02616	.01769
5	5	8	.19085	.14333	.10847	.08269	.06348
5	6	8	.17305	.12510	.09122	.06706	.04969
5	7	8	.15493	.10625	.07365	.05158	.03647
5	8	8	.13278	.08299	.05279	.03413	.02239
6	6	8	.25634	.17714	.12381	.08746	.06240
6	7	8	.22997	.15074	.10013	.06736	.04584
6	8	8	.19750	.11795	.07189	.04463	.02817
7	7	8	.37885	.23197	.14479	.09196	.05936
7	8	8	.32657	.18212	.10425	.06108	.03655
8	8	8	.74755	.36160	.18330	.09663	.05269

Table 10.5.1 gives coefficients for the observed order statistics obtained using (8.6.1) for the BLUE's of the location and scale parameters for selected values of k when the entire sample of size n is observed. Table 10.5.2 gives corresponding variances and covariance of the estimators obtained using (8.6.2). Three short tables of coefficients, variances and covariance of BLUE's are also given for $k = 0.1$, 0.3 and 0.5 when $n = 20$ for right-censored samples, where the number observed is $r = 10$ and 15 (Tables 10.5.3, 10.5.4 and 10.5.5).

Numerical Example 10.5.1

A sample of size $n = 20$ was generated using APL with $k = 0.5$, $\theta_1 = 20$ and $\theta_2 = 5$, using the probability transform method, that is, setting

$$F(x;\theta_1;\theta_2) = \frac{1 - \left[1 - k\left(\frac{x-\theta_1}{\theta_2}\right)\right]^{\frac{1}{k}}}{1 + \left[1 - k\left(\frac{x-\theta_1}{\theta_2}\right)\right]^{\frac{1}{k}}} = U,$$

a uniform (0,1) random variable. The ordered sample obtained was:

20.54501, 20.60763, 20.68928, 21.08791, 21.13221, 22.26578, 22.56018, 22.78290, 22.87524, 23.92760, 24.45981, 24.64939, 25.03210, 25.55358,

TABLE 10.5.1: COEFFICIENTS FOR OBSERVED ORDER STATISTICS IN BLUE'S FOR GENERALIZED HALF LOGISTIC DISTRIBUTION

$n = 5$:

θ_1^* k=0.1	k=0.2	k=0.3	k=0.4	k=0.5	θ_2^* k=0.1	k=0.2	k=0.3	k=0.4	k=0.5
1.20097	1.19922	1.20298	1.21176	1.22497	-.69895	-.72548	-.76791	-.82603	-.89924
-.00162	.01047	.02011	.02774	.03377	.07939	.04980	.02504	.00440	-.01284
-.02710	-.01261	.00070	.01275	.02359	13243	.10191	.07272	.04520	.01948
-.05770	-.04592	-.03235	-.01754	-.00191	18903	.17039	.14633	.11786	.08585
-.11456	-.15117	-.19143	-.23471	-.28042	.29811	.40338	.52383	.65856	.80675

$n = 10$

θ_1^* k=0.1	k=0.2	k=0.3	k=0.4	k=0.5	θ_2^* k=0.1	k=0.2	k=0.3	k=0.4	k=0.5
1.07698	1.07178	1.06992	1.07150	1.07637	-.61542	-.61789	-.63642	-.67189	-.72397
.00761	.00979	.01132	.01233	.01296	.01534	.00544	-.00178	-.00686	-.01033
.00478	.00770	.00998	.01171	.01299	.02662	.01468	.00524	-.00196	-.00733
.00149	.00502	.00803	.01052	.01253	.03867	.02528	.01390	.00457	-.00289
-.00224	.00170	.00534	.00858	.01141	.05131	.03734	.02450	.01318	.00351
-.00638	-.00234	.00173	.00567	.00935	.06432	.05100	.03752	.02461	.01273
-.01092	-.00723	-.00307	.00136	.00568	.07751	.06662	.05387	.04020	.02636
-.01583	-.01326	-.00964	-.00515	.00000	.09074	.08509	.07553	.06285	.04793
-.02125	-.02120	-.01949	-.01606	-.01097	.10430	.10917	.10804	.10070	.08747
-.03425	-.05197	-.07414	-.10047	-.13052	.14662	.22326	.31959	.43462	.56653

$n = 15$

θ_1^* k=0.1	k=0.2	k=0.3	k=0.4	k=0.5	θ_2^* k=0.1	k=0.2	k=0.3	k=0.4	k=0.5
1.04370	1.03898	1.03657	1.03672	1.03944	-.58784	-.58426	-.59655	-.62633	-.67359
.00490	.00570	.00621	.00650	.00663	.00547	.00028	-.00317	-.00528	-.00647
.00409	.00516	.00593	.00645	.00677	.01024	.00396	-.00056	-.00362	-.00557
.00318	.00450	.00552	.00629	.00683	.01526	.00801	.00245	-.00158	-.00437
.00217	.00371	.00499	.00601	.00680	.02049	.01243	.00590	.00088	-.00281
.00106	.00278	.00429	.00558	.00663	.02589	.01725	.00985	.00385	-.00080
-.00014	.00172	.00346	.00500	.00634	.03142	.02246	.01433	.00740	.00175
-.00143	.00050	.00240	.00420	.00582	.03704	.02810	.01948	.01169	.00501
-.00280	-.00087	.00116	.00317	.00509	.04271	.03421	.02538	.01688	.00918
-.00426	-.00245	-.00039	.00180	.00401	.04835	.04089	.03228	.02332	.01465
-.00579	-.00424	-.00226	.00002	.00248	.05406	.04829	.04052	.03151	.02203
-.00741	-.00634	-.00464	-.00241	.00023	.05973	.05677	.05082	.04248	.03256
-.00915	-.00889	-.00782	-.00594	-.00330	.06553	.06710	.06473	.05851	.04901
-.01113	-.01234	-.01265	-.01180	-.00968	.07203	.08158	.08666	.08612	.07961
-.01396	-.02794	-.04279	-.06159	-.08410	.09956	.16294	.24789	.35419	.47981

$n = 20$

θ_1^*					θ_2^*				
k=0.1	k=0.2	k=0.3	k=0.4	k=0.5	k=0.1	k=0.2	k=0.3	k=0.4	k=0.5
1.02930	1.02565	1.02477	1.02311	1.02513	-.57415	-.56815	-.57846	-.60480	-.65014
.00325	.00219	-.00246	.00192	.00101	.00239	-.00008	.00076	-.00275	-.00264
.00293	.00581	.01423	.00734	.00909	.00501	-.00020	-.00722	-.00501	-.00737
.00254	.00047	-.00839	.00014	-.00164	.00775	.00460	.00641	-.00019	.00027
.00213	.00552	.01439	.00762	.00984	.01058	.00393	-.00466	-.00343	-.00640
.00178	-.00021	-.00612	-.00001	-.00140	.01346	.00941	.00872	.00249	.00132
.00105	.00603	.01313	.00881	.01140	.01657	.00828	-.00024	-.00143	-.00577
.00105	-.00404	-.01086	-.00423	-.00640	.01934	.01634	.01524	.00775	.00655
-.00032	.00902	.02016	.01385	.01836	.02289	.01174	.00009	-.00111	-.00782
.00026	-.00799	-.01884	-.00974	-.01302	.02538	.02393	.02444	.01478	.01352
-.00146	.00862	.02200	.01506	.02033	.02914	.01766	.00427	.00250	-.00588
-.00124	-.00714	-.01661	-.00838	-.01081	.03181	.02947	.02891	.01891	.01600
-.00248	.00315	.01248	.00881	.01246	.03529	.02706	.01599	.01192	.00388
-.00271	-.00416	-.00811	-.00369	-.00354	.03818	.03462	.03136	.02296	.01710
-.00360	-.00170	.00208	.00183	.00424	.04139	.03713	.03012	.02422	.01635
-.00432	-.00409	-.00414	-.00213	-.00097	.04451	.04276	.03897	.03232	.02516
-.00508	-.00476	-.00370	-.00268	-.00089	.04769	.04819	.04558	.04044	.03288
-.00590	-.00626	-.00614	-.00524	-.00378	.05110	.05547	.05649	.05359	.04714
-.00690	-.00819	-.00896	-.00898	-.00802	.05532	.06617	.07378	.07648	.07339
-.01028	-.01793	-.02888	-.04340	-.06139	.07636	.13165	.20945	.31037	.43245

TABLE 10.5.2: VARIANCES AND COVARIANCE OF BLUE's FOR GENERALIZED HALF LOGISTIC DISTRIBUTION

$\underline{n}$	$\underline{k}$	$\frac{Var(\theta_1^*)}{\theta_2^2}$	$\frac{Var(\theta_2^*)}{\theta_2^2}$	$\frac{Cov(\theta_1^*,\theta_2^*)}{\theta_2^2}$
5	0.1	0.11849	0.15494	-0.07332
	0.2	0.10826	0.12727	-0.06789
	0.3	0.09989	0.10738	-0.06489
	0.4	0.09297	0.09381	-0.06370
	0.5	0.08716	0.08535	-0.06384
10	0.1	0.03232	0.06496	-0.01879
	0.2	0.03063	0.05098	-0.01778
	0.3	0.02918	0.04030	-0.01735
	0.4	0.02792	0.03262	-0.01744
	0.5	0.02683	0.02754	-0.01795
15	0.1	0.01512	0.04070	-0.00858
	0.2	0.01455	0.03125	-0.00819
	0.3	0.01404	0.02384	-0.00806
	0.4	0.01359	0.01841	-0.00818
	0.5	0.01320	0.01474	-0.00852
20	0.1	0.00878	0.02955	-0.00491
	0.2	0.00852	0.02239	-0.00471
	0.3	0.00828	0.01669	-0.00466
	0.4	0.00807	0.01247	-0.00475
	0.5	0.00788	0.00959	-0.00498

TABLE 10.5.3: COEFFICIENTS FOR OBSERVED ORDER STATISTICS IN BLUE's BASED ON RIGHT-CENSORED SAMPLES FOR GENERALIZED HALF LOGISTIC DISTRIBUTION (SAMPLE SIZE n=20, r=number observed, k=0.1)

r	θ_1^*	θ_2^*	$\frac{Var(\theta_1^*)}{\theta_2^2}$	$\frac{Var(\theta_2^*)}{\theta_2^2}$	$\frac{Cov(\theta_1^*,\theta_2^*)}{\theta_2^2}$
10	1.10251	-1.20713	0.00941	0.07670	-0.01034
	.00283	.00594			
	.00143	.01836			
	.00146	.01676			
	.00000	.02906			
	-.00084	.03636			
	-.00202	.04322			
	-.00257	.05082			
	-.00453	.05947			
	-.09828	.94713			
15	1.05610	- .78649	0.00901	0.04396	-0.00673
	.00300	.00441			
	.00265	.00728			
	.00196	.01233			
	.00163	.01443			
	.00074	.02185			
	.00010	.02412			
	-.00020	.02922			
	-.00173	.03405			
	-.00133	.03796			
	-.00325	.04339			
	-.00319	.04733			
	-.00464	.05241			
	-.00503	.05657			
	-.04680	.40113			

TABLE 10.5.4: COEFFICIENTS FOR OBSERVED ORDER STATISTICS IN BLUE's BASED ON RIGHT-CENSORED SAMPLES FOR GENERALIZED HALF LOGISTIC DISTRIBUTION (SAMPLE SIZE n=20, r=number observed, k=0.3)

r	θ_1^*	θ_2^*	$\frac{Var(\theta_1^*)}{\theta_2^2}$	$\frac{Var(\theta_2^*)}{\theta_2^2}$	$\frac{Cov(\theta_1^*,\theta_2^*)}{\theta_2^2}$
10	1.10708	-1.29612	.00893	.06431	-.01018
	-.00513	.06911			
	.01778	-.11503			
	-.01008	.11070			
	.01236	-.06263			
	-.00602	.05556			
	.00990	.00274			
	-.01427	.05964			
	.01765	.00500			
	-.12929	1.17102			
15	1.05732	-.82909	.00854	.03260	-.00672
	-.00042	-.01576			
	.01133	.01641			
	-.00380	-.03096			
	.00792	.04769			
	-.00145	-.02938			
	.00905	.03255			
	-.01033	.01035			
	.01773	.01938			
	-.01964	.02998			
	.01839	.03295			
	-.01738	.03395			
	.00763	.05424			
	-.01030	.04768			
	-.06604	.58001			

TABLE 10.5.5: COEFFICIENTS FOR OBSERVED ORDER STATISTICS IN BLUE's BASED ON RIGHT-CENSORED SAMPLES FOR GENERALIZED HALF LOGISTIC DISTRIBUTION (SAMPLE SIZE n=20, r=number observed, k=0.5)

r	θ_1^*	θ_2^*	$\frac{Var(\theta_1^*)}{\theta_2^2}$	$\frac{Var(\theta_2^*)}{\theta_2^2}$	$\frac{Cov(\theta_1^*,\theta_2^*)}{\theta_2^2}$
10	1.11010	-1.37332	0.00851	0.05474	-0.01029
	-.00043	.03249			
	.01546	-.10084			
	-.00461	.07045			
	.01147	-.05700			
	-.00104	.02106			
	.01072	-.01129			
	-.00865	.03179			
	.01745	-.00647			
	-.15047	1.39313			
15	1.06004	-.91319	0.00815	0.02466	-0.00698
	.00255	-.01477			
	.00651	.01383			
	.00342	-.04048			
	.00616	.02346			
	.00185	-.02490			
	.00843	.01818			
	-.00555	-.00078			
	.01688	.00390			
	-.01383	.01906			
	.01771	.01467			
	-.01220	.02573			
	.00820	.03676			
	-.00666	.04012			
	-.09350	.79843			

26.40186, 26.74783, 27.79711, 28.79056, 29.45665, 29.59182

Calculations of the correlation between the ordered sample values $Y_{i:n}$ and the quantiles $F^{-1}[i/(20+1)]$, $i = 1,2,\ldots,20$, was made assuming $k = 0(0.1)0.7$ and are tabulated as follows:

k	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7
correlation	.97849	.98637	.99110	.99290	.99205	.98889	.98377	.97701.

These values indicate that perhaps any value of k between 0.1 and 0.5 will describe the data well.

Using 5000 samples each time, p-values for the significance of correlations between $Y_{i:n}$ and $F^{-1}[i/(20+1)]$ were calculated to test the values of $k = 0(0.1)0.7$ and are tabulated as follows:

k	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7
p-value	.4732	.6910	.8646	.9388	.8822	.7044	.4506	.2504

None of these were significant, which was to be expected, considering the nature of the test. However, the p-values for $k = 0.1$ to $k = 0.5$ were much higher than the others, confirming the region of best choice for k. Figures 10.5.1 and 10.5.2 show quantile plots for $k = 0.3$ and $k = 0.5$.

The BLUE's obtained using the full sample were as follows::

(SE = standard error, obtained using Table 10.5.2)

k	θ_1^*	$SE(\theta_1^*)$	θ_2^*	$SE(\theta_2^*)$
0.1	20.23262	0.31053	3.31401	0.56968
0.2	20.18740	0.34501	3.73776	0.55929
0.3	20.12688	0.38776	4.26219	0.55068
0.4	20.06608	0.43892	4.88593	0.54561
0.5	19.98395	0.49885	5.61896	0.55024

In general, the estimates of θ_1 and θ_2 obtained are close to the true values.

Finally, BLUE's were calculated using conventional Type-II right-censored samples and using the coefficients given in Tables 10.5.4 and 10.5.5, with $k = 0.3$ and 0.5. The results were as follows: (r = number observed)

FIGURE 10.5.1: QUANTILE PLOT OF GENERATED VALUES, Y, FROM GENERALIZED HALF LOGISTIC DISTRIBUTION WITH QUANTILES FROM GENERALIZED HALF LOGISTIC DISTRIBUTION, k=0.3

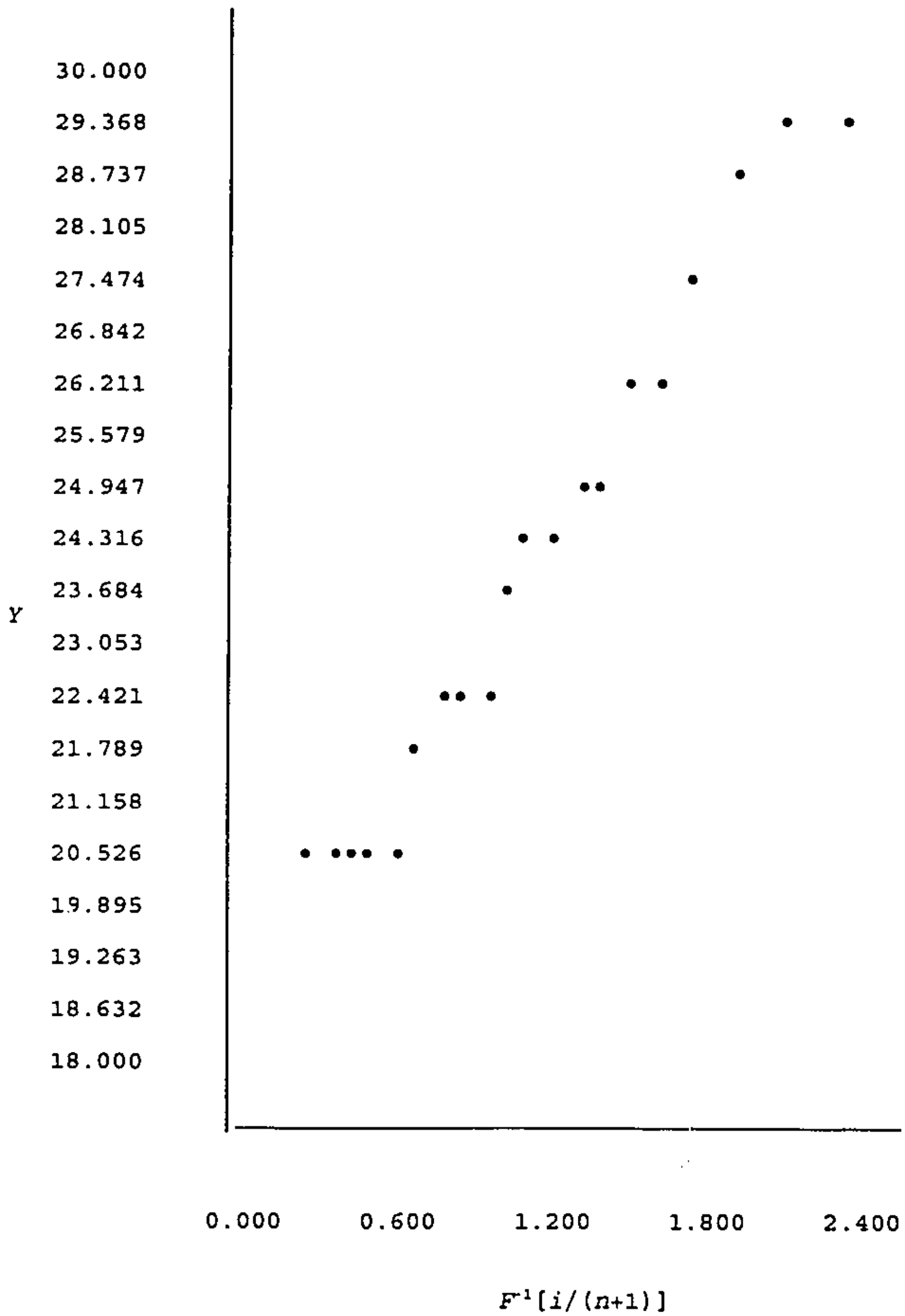
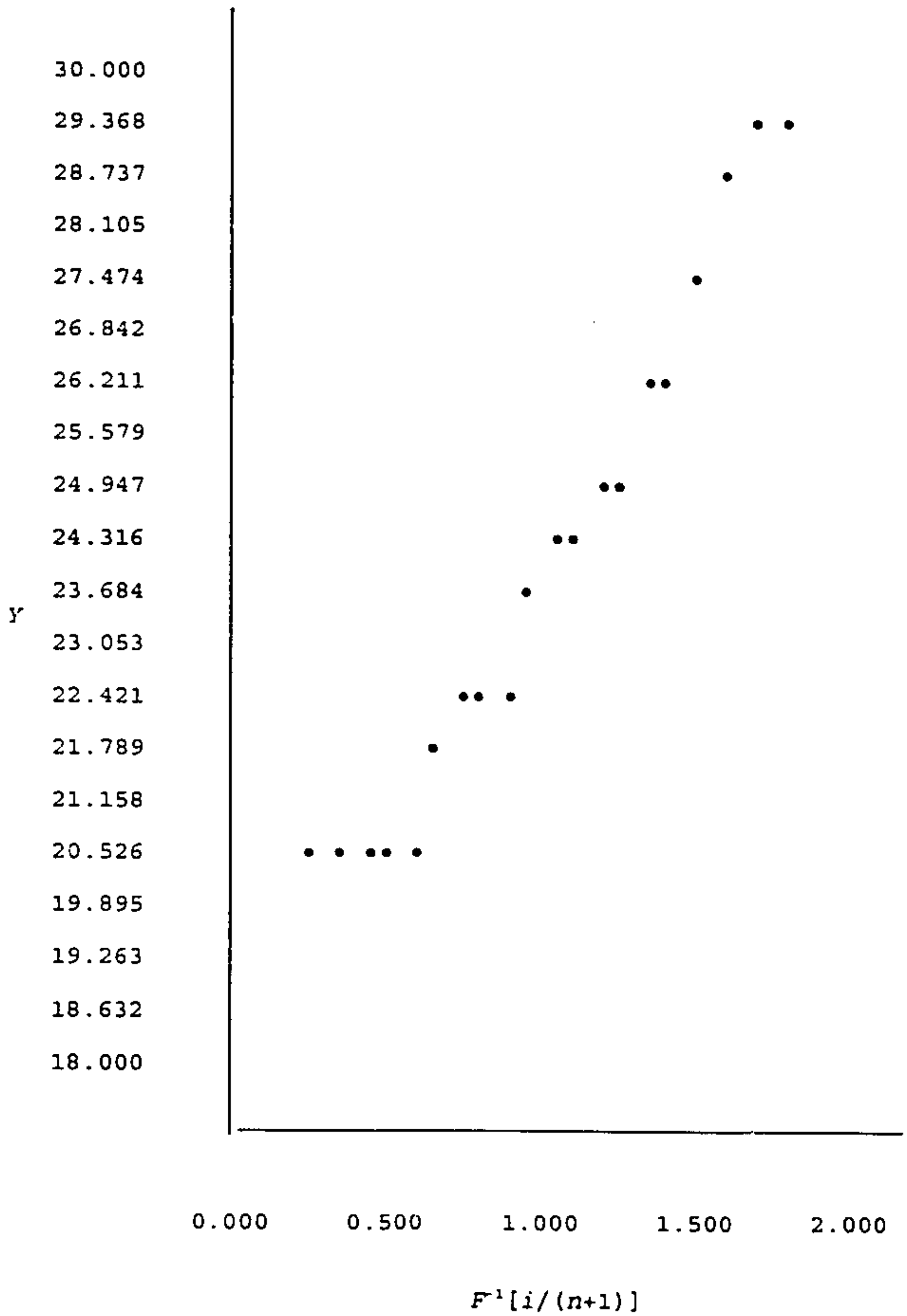


FIGURE 10.5.2: QUANTILE PLOT OF GENERATED VALUES, *Y*, FROM ENERALIZED HALF LOGISTIC DISTRIBUTION WITH QUANTILES FROM GENERALIZED HALF LOGISTIC DISTRIBUTION, k=0.5



k	r	θ_1^*	$SE(\theta_1^*)$	θ_2^*	$SE(\theta_2^*)$
0.3	10	20.13051	0.39852	4.21839	1.06979
	15	20.11322	0.40161	4.34491	0.78447
	20	20.12688	0.38776	4.26219	0.55068
0.5	10	20.08357	0.44048	4.77420	1.11702
	15	20.02692	0.47508	5.26362	0.82652
	20	19.98395	0.49885	5.61896	0.55024

As expected, for low values of r, the standard errors increase drastically. The $r = 20$ line is, of course, the whole sample, re-displayed for comparison. Again, the estimates are all close to the true values of θ_1 and θ_2.

Although not presented here, the question of maximum likelihood estimation will again (as in Chapter 8 for the generalized logistic distribution) involve numerical solutions of equations and should therefore be addressed through large simulations. Efficiencies of the estimators discussed in this chapter relative to maximum likelihood estimators can then be determined. It should be mentioned here that similar work for the one- and two-parameter half logistic distributions have been carried out by Balakrishnan and Chan (1992) and Balakrishnan and Puthenpura (1986), respectively.

11 The Right-Truncated Generalized Half Logistic Distribution

11.1 Introduction and Properties of the Distribution

In this chapter, we generalize the results obtained in Chapter 10 for the generalized half logistic distribution by introducing a truncation parameter P. These results will be of both theoretical and practical interest, as the truncated distribution is a natural choice for practical life-testing situations. Several recurrence relations satisfied by the single and product moments of order statistics from the right-truncated generalized half logistic distribution are established. These relationships may be used in a simple recursive manner in order to compute the single and product moments of all order statistics for all sample sizes and for any choice of the truncation parameter P.

Let us denote the cdf of the generalized half logistic distribution discussed in Chapter 10 by

$$G(x) = \frac{1-(1-kx)^{1/k}}{1+(1-kx)^{1/k}}, \quad 0 \le x \le \frac{1}{k}, \quad k \ge 0 \tag{11.1.1}$$

and the pdf by

$$g(x) = \frac{2(1-kx)^{\frac{1}{k}-1}}{[1+(1-kx)^{1/k}]^2}, \quad 0 \le x \le \frac{1}{k}, \quad k \ge 0. \tag{11.1.2}$$

Recall from section 10.1 that this distribution is an IFR (Increasing Failure Rate) family. Hence, it will be quite useful as a life-span model. A natural further generalization is to truncate this distribution on the right. The right-truncated forms of life-span models are often of great interest in reliability studies; for example, see Cohen (1991).

In this chapter, we therefore consider the right-truncated generalized half logistic distribution with pdf

$$f(x) = \begin{cases} \dfrac{1}{P}\,\dfrac{2(1-kx)^{\frac{1}{k}-1}}{[1+(1-kx)^{1/k}]^2}, & 0 \le x \le P_1 \\ 0, & \text{otherwise} \end{cases} \tag{11.1.3}$$

and cdf

$$F(x) = \frac{1}{P}\,\frac{1-(1-kx)^{1/k}}{1+(1-kx)^{1/k}}, \quad 0 \le x \le P_1; \tag{11.1.4}$$

here, $1 - P$ $(0 < P \le 1)$ is the proportion of truncation on the right of the standard generalized half logistic distribution in (11.1.1) and

$$P_1 = \frac{1}{k}\left[1-\left(\frac{1-P}{1+P}\right)^k\right], \; k \geq 0 \qquad (11.1.5)$$

is the point of truncation on the right.

It is easy to note that $P = G\,(P_1)$ $(k \geq 0,\; 0 < P_1 \leq 1/k)$ is monotonic increasing in P_1 and hence, for fixed values of k and P, P_1 is uniquely determined. Similarly, we also note that $P_1 = G^{-1}(P)$ $(k \geq 0,\; 0 < P \leq 1)$ is monotonic increasing in P so that, for fixed values of k and P_1, P is uniquely determined. Next, we consider P_1 as a function of k. If we can show that, for fixed P, P_1 as a function of k is monotonic, it will immediately imply that k is uniquely determined for fixed values of P and P_1. Thus, by denoting $(1-P)/(1+P)$ by a, we would like to show that

$$\frac{dP_1}{dk} = \frac{1 - a^k + ka^k \ln a}{-k^2}$$

is either positive or negative for all $k \geq 0$. Since $0 < a < 1$ for $0 < P < 1$ and $k > 0$, we may use the expansion

$$\begin{aligned}
-\ln a^k &= -\ln\left[1-\left(1-a^k\right)\right] \\
&= \left(1-a^k\right) + \frac{1}{2}\left(1-a^k\right)^2 + \frac{1}{3}\left(1-a^k\right)^3 + \ldots \\
&< \left(1-a^k\right) + \left(1-a^k\right)^2 + \left(1-a^k\right)^3 + \ldots \\
&= \frac{1-a^k}{a^k}
\end{aligned}$$

which simply implies that $1 - a^k + ka^k \ln a > 0$ and, hence,

$$\frac{dP_1}{dk} < 0 \cdot$$

Also,

$$\lim_{k \to 0} \frac{dP_1}{dk} = \frac{-(\ln a)^2}{2} < 0 \ .$$

Therefore, for $0 < P < 1$, we have established that P_1 is a monotonic decreasing function of k. Further, for the special case when $P = 1$, we have $P_1 = 1/k$ which is again a monotonic decreasing function of k. Consequently, there is a unique choice of k for fixed values of P and P_1.

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from the right-truncated generalized half logistic distribution in (11.1.4), and $X_{1:n} \le X_{2:n} \le \ldots \le X_{n:n}$ be the corresponding order statistics. As we have previously done, we will denote the single moments $E(X_{r:n}^i)$ by $\mu_{r:n}^{(i)}$ for $1 \le r \le n$ and $i = 0, 1, 2, \ldots$, the product moments $E(X_{r:n}X_{s:n})$ by $\mu_{r,s:n}$ for $1 \le r < s \le n$, and for convenience, we will also use $\mu_{r:n}$ for $\mu_{r:n}^{(1)}$ and $\mu_{r,r:n}$ for $\mu_{r:n}^{(2)}$.

From (11.1.3) and (11.1.4), we observe that the characterizing differential equation for the right-truncated generalized half logistic distribution is

$$(1-kx)f(x) = \frac{1}{2P}[1-P^2F^2(x)] \tag{11.1.6}$$

$$=\frac{1}{2P}\left\{1-P^2+2P^2[1-F(x)]-P^2[1-F(x)]^2\right\} \tag{11.1.7}$$

$$= \frac{1}{2P}\{1-P+P[1-F(x)]+P(1-P)F(x)$$

$$+P^2F(x)[1-F(x)]\Big\} \tag{11.1.8}$$

for $0 \le x \le P_1$. We shall use Eqs. (11.1.6) - (11.1.8) in the following sections to establish several recurrence relations satisfied by the single and product moments of order statistics. These relations will enable one to compute all the single and product moments of all order statistics for all sample sizes in a simple recursive manner. For example, by starting with the values of $E(X) = \mu_{1:1}$ and $E(X^2) = \mu_{1:1}^{(2)}$, one can determine the means, variances and covariances of ail order statistics for all sample sizes through this recursive computational procedure.

Since the values of $E(X)$ and $E(X^2)$ are needed as initial values for the recursive process, we next derive exact explicit expressions for these two quantities. Consider

$$E(X) = \int_0^1 F^{-1}(u)du = \frac{1}{k}\int_0^1 \left[1 - \left(\frac{1-Pu}{1+Pu}\right)^k\right]du$$

$$= \frac{1}{k}\left[1 - \frac{2}{P}\int_{\frac{1-P}{1+P}}^{1} \frac{y^k}{(1+y)^2}dy\right] \left(\text{ setting } y = \frac{1-Pu}{1+Pu}\right)$$

$$= \frac{1}{k} - \frac{2}{kP}\left[\int_0^1 \frac{y^k}{(1+y)^2}dy - \int_0^{\frac{1-P}{1+P}} \frac{y^k}{(1+y)^2}dy\right]$$

$$= \frac{1}{k} - \frac{2}{kP}\left\{\frac{1}{4(1+k)}\, {}_2F_1\left[2, 1; 2+k; \tfrac{1}{2}\right]\right.$$

$$\left. - \frac{1}{1+k}\left(\frac{1-P}{1+P}\right)^{1+k} {}_2F_1\left[1, 1+k; 2+k; \tfrac{1-P}{1+P}\right]\right\}, \qquad (11.1.9)$$

where

$${}_2F_1[a, b; c; x] = \sum_{i=0}^{\infty} \frac{(a)_i (b)_i x^i}{(c)_i i!}, \; c \neq 0, -1, -2, \ldots \qquad (11.1.10)$$

is the Gaussian hypergeometric function. Proceeding similarly, we obtain

$$E(X^2) = \frac{1}{k^2} - \frac{4}{k^2 P}\left\{\frac{1}{4(1+k)}\,{}_2F_1\left[2,1;2+k;\tfrac{1}{2}\right]\right.$$

$$\left. - \frac{1}{1+k}\left(\frac{1-P}{1+P}\right)^{1+k} {}_2F_1\left[2,1+k;2+k;-\tfrac{1-P}{1+P}\right]\right\}$$

$$+ \frac{2}{k^2 P}\left\{\frac{1}{4(1+2k)}\,{}_2F_1\left[2,1;2+2k;\tfrac{1}{2}\right]\right.$$

$$\left. - \frac{1}{1+2k}\left(\frac{1-P}{1+P}\right)^{1+2k} {}_2F_1\left[2,1+2k;2+2k;-\tfrac{1-P}{1+P}\right]\right\}. \qquad (11.1.11)$$

Higher order moments may also be similarly derived.

The Gaussian hypergeometric function in (11.1.10) may be calculated to any desired accuracy by using, for example, the *hypergeom* command in MAPLE V, Release 3. In order to facilitate the easy usage of the recursive algorithm developed in this paper, we have computed the values of the mean and variance of the right-truncated generalized half logistic distribution from Eqs. (11.1.9) and (11.1.11) and have presented them in Tables 11.1.1 and 11.1.2, respectively, for some selected choices of k and P. These values were calculated to 20 digit accuracy and are correct to all 8 decimal places reported in the tables.

The results established in this chapter generalize the corresponding results for the generalized half logistic distribution presented in Chapter 10. Similar recurrence relations for moments of order statistics from exponential and truncated

TABLE 11.1.1: MEAN OF RIGHT TRUNCATED GENERALIZED HALF LOGISTIC DISTRIBUTION FOR SELECTED VALUES OF *k* AND *P*

P \ *k*	0.05	0.10	0.20	0.30	0.40	0.50
.50	.51406922	.50513286	.48795723	.47166276	.45619501	.44150328
.55	.57037461	.55932774	.53819504	.51826939	.49946767	.48171312
.60	.62854366	.61506220	.58939987	.56536005	.54281886	.52166297
.65	.68899982	.67270410	.64184933	.61314529	.58641064	.56148109
.70	.75229522	.73272875	.69589451	.66188430	.63043563	.60131332
.75	.81918113	.79577715	.75199925	.71191078	.67513376	.64133358
.80	.89073828	.86276277	.81081098	.76367907	.72082146	.68176195
.85	.96864634	.93509069	.87330364	.81785536	.76794854	.72289887
.90	1.05584620	1.01519025	.94111768	.87553030	.81722758	.76520131
.95	1.15874725	1.10826237	1.01764090	.93887393	.87002068	.80950358
1.00	1.30833018	1.23811426	1.11686427	1.01602278	.93098366	.85840735

TABLE 11.1.2: VARIANCE OF RIGHT TRUNCATED GENERALIZED HALF LOGISTIC DISTRIBUTION FOR SELECTED VALUES OF *k* AND *P*

P \ *k*	0.05	0.10	0.20	0.30	0.40	0.50
.50	.09318739	.08835339	.07951988	.07168372	.06472247	.05852971
.55	.11654737	.10980655	.97619784	.08696054	.07762065	.06942212
.60	.14419122	.13494501	.11841794	.10417733	.09187922	.08123468
.65	.17710827	.16456461	.14241634	.12363652	.10766796	.09405152
.70	.21671050	.19979344	.17031961	.14576191	.12522814	.10799843
.75	.26510137	.24229675	.20315206	.17116372	.14490741	.12326064
.80	.32560576	.29467625	.24248077	.20076013	.16722389	.14011511
.85	.40395028	.36134645	.29089176	.23603012	.19299650	.15899518
.90	.51142677	.45084026	.35319853	.27963374	.22365499	.18064008
.95	.67669998	.58418383	.44060923	.33744876	.26221222	.20654458
1.00	1.09690889	.89207280	.61142036	.43573174	.32047569	.24194366

exponential distributions were derived by Joshi (1978, 1979, 1982). Results of this nature are also available for a number of other distributions, and interested readers may refer to the monograph on this topic by Arnold and Balakrishnan (1989).

11.2 Relationships for Single Moments

The density function of $X_{r:n}$ is given by [David (1981, p. 9); Arnold, Balakrishnan and Nagaraja (1992, p. 10)]

$$f_{r:n}(x) = \frac{n!}{(r-1)!\,(n-r)!}[F(x)]^{r-1}[1-F(x)]^{n-r}f(x), \quad 0 \le x \le P_1, \tag{11.2.1}$$

where $f(x)$, $F(x)$ and P_1 are given in Eqs. (11.1.3), (11.1.4) and (11.1.5), respectively.

Then, by making use of the characterizing differential equations in (11.1.6) - (11.1.8), we establish in this section several recurrence relations for the single moments of order statistics from the right-truncated generalized half logistic distribution.

Theorem 11.2.1: *For $i = 0, 1, 2, \ldots$,*

$$\mu_{1:2}^{(i+1)} = \frac{1}{P^2}\Big[(1-P^2)P_1^{i+1} + 2P^2\mu_{1:1}^{(i+1)} - 2P(i+1)(\mu_{1:1}^{(i)} - k\mu_{1:1}^{(i+1)})\Big], \tag{11.2.2}$$

and for n ≥ 2

$$\mu_{1:n+1}^{(i+1)} = \frac{1}{P^2}\left[(1-P^2)\mu_{1:n-1}^{(i+1)} + 2P^2\mu_{1:n}^{(i+1)} - \frac{2P(i+1)}{n}(\mu_{1:n}^{(i)} - k\mu_{1:n}^{(i+1)})\right] . \qquad (11.2.3)$$

Proof: For $n \geq 1$ and $i = 0, 1, 2, \ldots$, let us consider

$$\mu_{1:n}^{(i)} - k\mu_{1:n}^{(i+1)} = n\int_0^{P_1} x^i(1-kx)[1-F(x)]^{n-1}f(x)dx$$

$$= \frac{n}{2P}\int_0^{P_1} x^i\{1-P^2+2P^2[1-F(x)]-P^2[1-F(x)]^2\}$$

$$\cdot[1-F(x)]^{n-1}dx$$

$$= \frac{n}{2P}[(1-P^2)I_{1,n-1} + 2P^2 I_{1,n} - P^2 I_{1,n+1}] , \qquad (11.2.4)$$

upon using (11.1.7). Integration by parts directly gives

$$I_{1,n-1} = \frac{P_1^{i+1}}{i+1} \text{ when } n = 1$$

$$= \frac{\mu_{1:n-1}^{(i+1)}}{i+1} \text{ when } n \geq 2,$$

$$I_{1,n} = \frac{\mu_{1:n}^{(i+1)}}{i+1}$$

and

$$I_{1,n+1} = \frac{\mu_{1:n+1}^{(i+1)}}{i+1} .$$

Substituting these expressions in (11.2.4) and simplifying the resulting equations, we derive the recurrence relations in (11.2.2) and (11.2.3). □

Theorem 11.2.2: For $i = 0, 1, 2, \ldots,$

$$\mu_{2:2}^{(i+1)} = \frac{1}{P^2}\left[-(1-P^2)P_1^{i+1} + 2P(i+1)(\mu_{1:1}^{(i)} - k\mu_{1:1}^{(i+1)})\right] , \tag{11.2.5}$$

and for $n \geq 2$

$$\mu_{n+1:n+1}^{(i+1)} = \frac{1}{P^2}\left[\mu_{n-1:n-1}^{(i+1)} - (1-P^2)P_1^{i+1} + \frac{2P(i+1)}{n}(\mu_{n:n}^{(i)} - k\mu_{n:n}^{(i+1)})\right] . \tag{11.2.6}$$

Proof: For $n \geq 1$ and $i = 0, 1, 2, \ldots,$ let us consider

$$\mu_{n:n}^{(i)} - k\mu_{n:n}^{(i+1)} = n\int_0^{P_1} x^i(1-kx)[F(x)]^{n-1}f(x)dx$$

$$= \frac{n}{2P}\int_0^{P_1} x^i[F(x)]^{n-1}[1-P^2F^2(x)]dx$$

$$= \frac{n}{2P}[I_{2,n-1} - P^2 I_{2,n+1}] , \tag{11.2.7}$$

upon using (11.1.6). Integration by parts now gives

$$I_{2,n-1} = \frac{P_1^{i+1}}{i+1} \qquad \text{when } n = 1$$

$$= \frac{P_1^{i+1} - \mu_{n-1:n-1}^{(i+1)}}{i+1} \qquad \text{when } n \geq 2$$

and

$$I_{2,n+1} = \frac{P_1^{i+1} - \mu_{n+1:n+1}^{(i+1)}}{i+1} \; .$$

Upon substituting these expressions in (11.2.7) and simplifying the resulting equations, we derive the recurrence relations in (11.2.5) and (11.2.6). □

Theorem 11.2.3: *For $2 \leq r \leq n - 1$ and $i = 0, 1, 2, \ldots,$*

$$\mu_{r:n+1}^{(i+1)} = \mu_{r-1:n+1}^{(i+1)} + \frac{n+1}{(n-r+1)(n-r+2)P^2}\Big[n(1-P^2)(\mu_{r:n-1}^{(i+1)} - \mu_{r-1:n-1}^{(i+1)})$$
$$+2(n-r+1)P^2(\mu_{r:n}^{(i+1)} - \mu_{r-1:n}^{(i+1)}) - 2(i+1)P(\mu_{r:n}^{(i)} - k\mu_{r:n}^{(i+1)})\Big] \; ,$$

and for $n \geq 2$

$$\mu_{n:n+1}^{(i+1)} = \mu_{n-1:n+1}^{(i+1)} + \frac{n+1}{2P^2}\Big[n(1-P^2)(P_1^{i+1} - \mu_{n-1:n-1}^{(i+1)})$$
$$+2P^2(\mu_{n:n}^{(i+1)} - \mu_{n-1:n}^{(i+1)}) - 2(i+1)P(\mu_{n:n}^{(i)} - k\mu_{n:n}^{(i+1)})\Big] .$$

Proof: For $2 \le r \le n$ and $i = 0, 1, 2, \ldots$, let us consider

$$\mu_{r:n}^{(i)} - k\mu_{r:n}^{(i+1)} = \frac{n!}{(r-1)!(n-r)!}\int_0^{P_1} x^i(1-kx)[F(x)]^{r-1}[1-F(x)]^{n-r}f(x)dx$$

$$= \frac{n!}{(r-1)!(n-r)!2P}\int_0^{P_1} x^i[F(x)]^{r-1}[1-F(x)]^{n-r}$$

$$\cdot\{1-P^2+2P^2[1-F(x)]-P^2[1-F(x)]^2\}dx$$

$$= \frac{n!}{(r-1)!(n-r)!2P}\left[(1-P^2)I_{3,n-r} + 2P^2I_{3,n-r+1} - P^2I_{3,n-r+2}\right], \qquad (11.2.10)$$

upon using (11.1.7). Integration by parts directly yields

$$I_{3,n-r} = \frac{(r-1)!(n-r)!}{(n-1)!(i+1)}\left(\mu_{r:n-1}^{(i+1)} - \mu_{r-1:n-1}^{(i+1)}\right) \quad \text{when } 2 \le r \le n-1$$

$$= \frac{1}{i+1}\left(P_1^{i+1} - \mu_{n-1:n-1}^{(i+1)}\right) \quad \text{when } r = n,$$

$$I_{3,n-r+1} = \frac{(r-1)!(n-r+1)!}{n!(i+1)}\left(\mu_{r:n}^{(i+1)} - \mu_{r-1:n}^{(i+1)}\right)$$

and

$$I_{3,n-r+2} = \frac{(r-1)!(n-r+2)!}{(n+1)!(i+1)}\left(\mu_{r:n+1}^{(i+1)} - \mu_{r-1:n+1}^{(i+1)}\right).$$

Upon substituting these expressions into (11.2.10) and simplifying the resulting equations, we derive the recurrence relations in (11.2.8) and (11.2.9). □

Remark: By letting the proportion of truncation $1 - P \to 0$ (and hence, $P_1 \to 1/k$) in Theorems 11.2.1 - 11.2.3, we deduce the recurrence relations established in Chapter 10 for the single moments of order statistics from the standard generalized half logistic distribution.

Remark: The relationships proved in Theorems 11.2.1 - 11.2.3 will enable one to compute all the single moments of all order statistics for all sample sizes in a simple recursive manner. By starting with the values of $\mu_{1:1} = E(X)$ and $\mu_{1:1}^{(2)} = E(X^2)$ (see Tables 11.1.1 and 11.1.2), for example, one can use these results to determine the first two single moments (or the means and variances) of all order statistics for all sample sizes n. This can be done for any choice of the truncation parameter P and the shape parameter k.

11.3 Relationships for Product Moments

The joint density function of $X_{r:n}$ and $X_{s:n}$ $(1 \le r < s \le n)$ is given by [(David (1981, p. 10); Arnold, Balakrishnan and Nagaraja (1992, p. 16)]

$$f_{r,s:n}(x,y) = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}[F(x)]^{r-1}[F(y)-F(x)]^{s-r-1} [1-F(y)]^{n-s}f(x)f(y),\ 0 \le x < y \le P_1, \tag{11.3.1}$$

where $f(x)$, $F(x)$ and P_1 are as given in Eqs. (11.1.3), (11.1.4) and (11.1.5), respectively.

Then, by making use of the characterizing differential equations in (11.1.6) - (11.1.8), we establish in this section several recurrence relations for the product moments of order statistics from the right truncated generalized half logistic distribution.

Theorem 11.3.1 *For $1 \le r \le n-2$,*

$$\mu_{r,r+1:n+1} = \mu^{(2)}_{r:n+1} + \frac{2(n+1)}{n-r+1}\Big[\mu_{r,r+1:n} - \mu^{(2)}_{r:n} + \frac{1}{P(n-r)}(k\mu_{r,r+1:n} - \mu_{r:n}) + \frac{n(1-P^2)}{2(n-r)P^2}(\mu_{r,r+1:n-1} - \mu^{(2)}_{r:n-1})\Big], \qquad (11.3.2)$$

and for $n \ge 2$

$$\mu_{n-1,n:n+1} = \mu^{(2)}_{n-1:n+1} + (n+1)\Big[\mu_{n-1,n:n} - \mu^{(2)}_{n-1:n} + \frac{1}{P}(k\mu_{n-1,n:n} - \mu_{n-1:n}) + \frac{n(1-P^2)}{2P^2}(P_1\mu_{n-1:n-1} - \mu^{(2)}_{n-1:n-1})\Big]. \qquad (11.3.3)$$

Proof: For $1 \le r \le n-1$, let us consider from (11.3.1)

$$\mu_{r:n} - k\mu_{r,r+1:n} = E(X_{r:n}X^{0}_{r+1:n} - kX_{r:n}X_{r+1:n})$$

$$= \frac{n!}{(r-1)!(n-r-1)!}\int_0^{P_1}\int_x^{P_1} x(1-ky)[F(x)]^{r-1}[1-F(y)]^{n-r-1}$$

$$\cdot f(x)f(y)\,dy\,dx$$

$$= \frac{n!}{2P(r-1)!(n-r-1)!}\int_0^{P_1} x[F(x)]^{r-1}f(x)J_1(x)dx \qquad (11.3.4)$$

upon using (11.1.7), where

$$J_1(x) = (1-P^2)\int_x^{P_1}[1-F(y)]^{n-r-1}dy + 2P^2\int_x^{P_1}[1-F(y)]^{n-r}dy$$

$$- P^2\int_x^{P_1}[1-F(y)]^{n-r+1}dy$$

$$= (1-P^2)J_{1,n-r-1} + 2P^2 J_{1,n-r} - P^2 J_{1,n-r+1}\,. \qquad (11.3.5)$$

Integration by parts directly gives

$$J_{1,n-r-1} = x[1-F(x)]^{n-r-1} + (n-r-1)\int_x^{P_1} y[1-F(y)]^{n-r-2}f(y)dy$$

when $1 \le r \le n-2$

$$= P_1 - x \qquad \text{when } r = n-1\,,$$

$$J_{1,n-r} = -x[1-F(x)]^{n-r} + (n-r)\int_x^{P_1} y[1-F(y)]^{n-r-1}f(y)dy\,,$$

and

$$J_{1,n-r+1} = -x[1-F(x)]^{n-r+1} + (n-r+1)\int_x^{P_1} y[1-F(y)]^{n-r} f(y)dy\,.$$

Substituting these expressions in (11.3.5) and the resulting expression of $J_1(x)$ in (11.3.4) and then simplifying the resulting equation, we derive the recurrence relations in (11.3.2) and (11.3.3). □

Theorem 11.3.2: For *$1 \le r < s \le n-1$ and $s-r \ge 2$,*

$$\mu_{r,s:n+1} = \mu_{r,s-1:n+1} + \frac{2(n+1)}{n-s+2}\Big[\mu_{r,s:n} - \mu_{r,s-1:n} + \frac{1}{P(n-s+1)}(k\mu_{r,s:n} - \mu_{r:n}) + \frac{n(1-P^2)}{2(n-s+1)P^2}(\mu_{r,s:n-1} - \mu_{r,s:n-1})\Big], \qquad (11.3.6)$$

and for $1 \le r \le n-2$

$$\mu_{r,n:n+1} = \mu_{r,n-1:n+1} + (n+1)\Big[\mu_{r,n:n}\mu_{r,n-1:n} + \frac{1}{P}(k\mu_{r,n:n} - \mu_{r:n}) + \frac{n(1-P^2)}{2P^2}(P_1\mu_{r:n-1} - \mu_{r,n-1:n-1})\Big]. \qquad (11.3.7)$$

Proof: For $1 \le r < s \le n$ and $s - r \ge 2$, let us consider from (11.3.1)

$$\mu_{r:n} - k\mu_{r,s:n} = E(X_{r:n} - kX_{r:n}X_{s:n})$$

$$= \frac{n!}{(r-1)!(s-r-1)!(n-s)!} \int_0^{P_1}\int_x^{P_1} x(1-ky)[F(x)]^{r-1}[F(y)-F(x)]^{s-r-1}$$

$$\cdot [1-F(y)]^{n-s} f(x) f(y)\, dy dx$$

$$= \frac{n!}{2P(r-1)!(s-r-1)!(n-s)!} \int_0^{P_1} x[F(x)]^{r-1} f(x) J_2(x)\, dx \qquad (11.3.8)$$

upon using (11.1.7), where

$$J_2(x) = (1-P^2)\int_x^{P_1} [F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s} dy$$

$$+2P^2 \int_x^{P_1} [F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s+1} dy$$

$$-P^2 \int_x^{P_1} [F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s+2} dy$$

$$= (1-P^2) J_{2,n-s} + 2P^2 J_{2,n-s+1} - P^2 J_{2,n-s+2} . \qquad (11.3.9)$$

Integration by parts yields

$$J_{2,n-s} = -(s-r-1)\int_x^{P_1} y[F(y)-F(x)]^{s-r-2}[1-F(y)]^{n-s}f(y)dy$$

$$+(n-s)\int_x^{P_1} y[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s-1}f(y)dy$$

when $s \leq n-1$

$$= P_1[1-F(x)]^{n-r-1}-(n-r-1)\int_x^{P_1} y[F(y)-F(x)]^{n-r-2}f(y)dy$$

when $s = n$,

$$J_{2,n-s+1} = -(s-r-1)\int_x^{P_1} y[F(y)-F(x)]^{s-r-2}[1-F(y)]^{n-s+1}f(y)dy$$

$$+(n-s+1)\int_x^{P_1} y[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s}f(y)dy,$$

and

$$J_{2,n-s+2} = -(s-r-1)\int_x^{P_1} y[F(y)-F(x)]^{s-r-2}[1-F(y)]^{n-s+2}f(y)dy$$

$$+(n-s+2)\int_x^{P_1} y[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s+1}f(y)dy.$$

Upon substituting these expressions in (11.3.9) and the resulting expression of $J_2(x)$ in (11.3.8) and then simplifying the resulting equation, we derive the recurrence relations in (11.3.6) and (11.3.7). □

Theorem 11.3.3: *For $n \geq 2$,*

$$\mu_{2,3:n+1} = \mu_{3:n+1}^{(2)} + (n+1)\left[\frac{1}{P}(\mu_{2:n} - k\mu_{1,2:n}) - \frac{n}{2P^2}\mu_{1:n-1}^{(2)}\right], \tag{11.3.10}$$

and for $2 \leq r \leq n - 1$,

$$\mu_{r+1,r+2:n+1} = \mu_{r+2:n+1}^{(2)} + \frac{2(n+1)}{r(r+1)}\left[\frac{1}{P}(\mu_{r+1:n} - k\mu_{r,r+1:n}) - \frac{n}{2P^2}(\mu_{r:n-1}^{(2)} - \mu_{r-1,r:n-1})\right]. \tag{11.3.11}$$

Proof: For $1 \leq r \leq n - 1$, let us consider from (11.3.1)

$$\begin{aligned} \mu_{r+1:n} - k\mu_{r,r+1:n} &= E(X_{r:n}^{0}X_{r+1:n} - kX_{r:n}X_{r+1:n}) \\ &= \frac{n!}{(r-1)!(n-r-1)!}\int_0^{P_1}\int_0^{y} y(1-kx)\,[F(x)]^{r-1}\,[1-F(y)]^{n-r-1} f(x)f(y)\,dxdy \end{aligned}$$

$$= \frac{n!}{2P(r-1)!(n-r-1)!}\int_0^{P_1} y\,[1-F(y)]^{n-r-1} f(y)\,K_1(y)dy \tag{11.3.12}$$

upon using (11.1.6), where

$$\begin{aligned} K_1(y) &= \int_0^{y}[F(x)]^{r-1}dx - P^2\int_0^{y}[F(x)]^{r+1}dx \\ &= K_{1,r-1} - P^2K_{1,r+1}\,. \end{aligned} \tag{11.3.13}$$

Integration by parts yields

$$K_{1,r-1} = y \qquad \text{when } r = 1$$

$$= y[F(y)]^{r-1} - (r-1)\int_0^y x[F(x)]^{r-2} f(x)dx \qquad \text{when } r \geq 2,$$

and

$$K_{1,r+1} = y[F(y)]^{r+1} - (r+1)\int_0^y x[F(x)]^{r} f(x)dx .$$

Upon substituting these expressions in (11.3.13) and the resulting expression of $K_1(y)$ in (11.3.12) and then simplifying the resulting equations, we derive the recurrence relations in (11.3.10) and (11.3.11). □

Corollary 11.3.1: By *setting* $r = n - 1$ *in (11.3.11), we obtain for* $n \geq 3$

$$\mu_{n,n+1:n+1} = \mu^{(2)}_{n+1:n+1} + \frac{2(n+1)}{(n-1)n}\left[\frac{1}{P}(\mu_{n:n} - k\mu_{n-1,n:n}) - \frac{n}{2P^2}(\mu^{(2)}_{n-1:n-1} - \mu_{n-2,n-1:n-1})\right]. \qquad (11.3.14)$$

Theorem 3.4: *For* $3 \leq s \leq n$,

$$\mu_{2,s+1:n+1} = \mu_{3,s+1:n+1} + (n+1)\left[\frac{1}{P}(\mu_{s:n} - k\mu_{1,s:n}) - \frac{n}{2P^2}\mu_{1,s-1:n-1}\right], \qquad (11.3.15)$$

and for $2 \leq r < s \leq n$ *and* $s - r \geq 2$

$$\mu_{r+1,s+1:n+1} = \mu_{r+2,s+1:n+1} + \frac{2(n+1)}{r(r+1)}\left[\frac{1}{P}(\mu_{s:n} - k\mu_{r,s:n}) - \frac{n}{2P_2}(\mu_{r,s-1:n-1} - \mu_{r-1,s-1:n-1})\right]. \tag{11.3.16}$$

Proof: For $1 \le r < s \le n$ and $s - r \ge 2$, let us consider from (11.3.1)

$$\mu_{s:n} - k\mu_{r,s:n} = E(X^0_{r:n}X_{s:n} - kX_{r:n}X_{s:n})$$

$$= \frac{n!}{(r-1)!\,(s-r-1)!\,(n-s)!}\int_0^{P_1}\int_0^{y} y(1-kx)\,[F(x)]^{r-1}\,[F(y)-F(x)]^{s-r-1} \cdot [1-F(y)]^{n-s} f(x) f(y)\,dx dy$$

$$= \frac{n!}{2P(r-1)!\,(s-r-1)!\,(n-s)!}\int_0^{P_1} y\,[1-F(y)]^{n-s} f(y)\,K_2(y)\,dy \tag{11.3.17}$$

upon using (11.1.6), where

$$K_2(y) = \int_0^{y}[F(x)]^{r-1}\,[F(y)-F(x)]^{s-r-1}\,dx - P^2\int_0^{y}[F(x)]^{r+1}\,[F(y)-F(x)]^{s-r-1}dx$$

$$= K_{2,r-1} - P^2 K_{2,r+1}\,. \tag{11.3.18}$$

Integration by parts yields

$$K_{2,r-1} = (s-2)\int_0^y x\,[F(y)-F(x)]^{s-3} f(x)\,dx \qquad \text{when } r = 1$$

$$= -(r-1)\int_0^y x\,[F(x)]^{r-2}\,[F(y)-F(x)]^{s-r-1} f(x)\,dx$$

$$+(s-r-1)\int_0^y x\,[F(x)]^{r-1}\,[F(y)-F(x)]^{s-r-2} f(x)\,dx \qquad \text{when } r \geq 2\,,$$

and

$$K_{2,r+1} = -(r+1)\int_0^y x\,[F(x)]^{r} F(y)-F(x)]^{s-r-1} f(x)\,dx$$

$$+(s-r-1)\int_0^y x\,[F(x)]^{r+1}[F(y)-F(x)]^{s-r-2} f(x)\,dx\,.$$

Upon substituting these expressions in (11.3.18) and the resulting expression of $K_2(y)$ in (11.3.17) and then simplifying the resulting equations, we derive the recurrence relations in (11.3.15) and (11.3.16). □

Corollary 11.3.2: *Upon setting $s = n$ in (11.3.16), we obtain for $2 \leq r \leq n - 2$*

$$\mu_{r+1,n+1:n+1} = \mu_{r+2,n+1:n+1} + \frac{2(n+1)}{r(r+1)}\left[\frac{1}{P}(\mu_{n:n} - k\mu_{r,n:n}) - \frac{n}{2P^2}(\mu_{r,n-1:n-1} - \mu_{r-1,n-1:n-1})\right]. \qquad (11.3.19)$$

Theorem 11.3.5: *We have*

$$\mu_{1,3:3} = 3\left[\left(1+\frac{k}{P}\right)\mu_{1,2:2} + \mu_{1:2}^{(2)} - \frac{1}{3}\mu_{1,2:3} - \frac{1}{3}\mu_{1:3}^{(2)} - \frac{1}{P}\mu_{2:2} + \frac{1-P^2}{P^2}\mu_{1:1}^{(2)}\right], \tag{11.3.20}$$

and for $n \geq 3$

$$\mu_{1,n+1:n+1} = \frac{2(n+1)}{n(n+1)}\left[\left(n-1+\frac{k}{P}\right)\mu_{1,n:n} + \mu_{1,n-1:n} - \frac{n-1}{n+1}\mu_{1,n:n+1}\right.$$

$$\left. - \frac{1}{n+1}\mu_{1,n-1:n+1} - \frac{1}{P}\mu_{n:n} + \frac{n(1-P^2)}{2P^2}\mu_{1,n-1:n-1}\right]. \tag{11.3.21}$$

Proof: For $n \geq 2$, let us consider from (11.3.1)

$$\mu_{n:n} - k\mu_{1,n:n} = E(X_{1:n}^0 X_{n:n} - kX_{1:n}X_{n:n})$$

$$= n(n-1)\int_0^{P_1}\int_0^{y} y(1-kx)\,[F(y)-F(x)]^{n-2}f(x)f(y)\,dx\,dy$$

$$= \frac{n(n-1)}{2P}\int_0^{P_1} yf(y)L(y)\,dy \tag{11.3.22}$$

upon using (11.1.7), where

$$L(y) = (1-P^2)\int_0^y [F(y)-F(x)]^{n-2}dx + 2P^2\int_0^y [F(y)-F(x)]^{n-2}[1-F(x)]dx$$

$$-P^2\int_0^y [F(y)-F(x)]^{n-2}[1-F(x)]^2dx$$

$$= (1-P^2)L_{n-2} + 2P^2\{L_{n-1} + [1-F(y)]L_{n-2}\}$$

$$-P^2\{L_n + 2[1-F(y)]L_{n-1} + [1-F(y)]^2L_{n-2}\}$$

$$= \{1 - P^2 + 2P^2[1-F(y)] - P^2[1-F(y)]^2\}L_{n-2}$$

$$+ \{2P^2 - 2P^2[1-F(y)]\}L_{n-1} - P^2L_n . \qquad (11.3.23)$$

Integration by parts yields

$$L_{n-2} = y \qquad \text{when } n = 2$$

$$= (n-2)\int_0^y x[F(y)-F(x)]^{n-3}f(x)dx \qquad \text{when } n \geq 3$$

$$L_{n-1} = (n-1)\int_0^y x[F(y)-F(x)]^{n-2}f(x)dx ,$$

and

$$L_n = n\int_0^y x[F(y)-F(x)]^{n-1}f(x)dx .$$

Upon substituting these expressions in (11.3.23) and the resulting expression of $L(y)$ in (11.3.22) and then simplifying the resulting equations, we derive the recurrence relations in (11.3.20) and (11.3.21).

Remark: As mentioned earlier in the remark immediately following Theorem 11.2.3, if we let the proportion of truncation $1 - P \to 0$ in Theorems 11.3.1 - 11.3.5, we deduce the recurrence relations for the product moments of order statistics from the standard generalized half logistic distribution established earlier in Chapter 10.

Remark: The relationships established in Theorems 11.3.1 - 11.3.5 will enable one to compute all the product moments of all order statistics for all sample sizes in a simple recursive manner. This can be done for any choice of the shape parameter k and the truncation parameter P.

11.4 Recursive Computational Algorithm

By starting with the values of $\mu_{1:1} = E(X)$ and $\mu_{1:1}^{(2)} = E(X^2)$ (see Tables 11.1.1 and 11.1.2), $\mu_{1:2}$ and $\mu_{1:2}^{(2)}$ can be determined from (11.2.2) while $\mu_{2:2}$ and $\mu_{2:2}^{(2)}$ can be computed from (11.2.5). For the sample of size 3, $\mu_{1:3}$ and $\mu_{1:3}^{(2)}$ can be determined from (11.2.3), $\mu_{2:3}$ and $\mu_{2:3}^{(2)}$ from (11.2.9) and finally $\mu_{3:3}$ and $\mu_{3:3}^{(2)}$ from (11.2.6). Similarly, for the sample of size 4, $\mu_{1:4}$ and $\mu_{1:4}^{(2)}$ can be determined from (11.2.3), $\mu_{2:4}$ and $\mu_{2:4}^{(2)}$ from (11.2.8), $\mu_{3:4}$ and $\mu_{3:4}^{(2)}$ from (11.2.9) and finally, $\mu_{4:4}$ and $\mu_{4:4}^{(2)}$ from (11.2.6). This process may be followed similarly to determine $\mu_{r:n}$ and $\mu_{r:n}^{(2)}$ for $1 \leq r \leq n$ and for $n = 5, 6, \ldots$ From these values, variances of order statistics can be readily computed.

By starting with the fact that $\mu_{1,2:2} = \mu_{1:1}^{2}$, $\mu_{1,2:3}$ can be determined from (11.3.3), $\mu_{2,3:3}$ from (11.3.10), and then $\mu_{1,3:3}$ from (11.3.20) For the sample of size 4, $\mu_{1,2:4}$ can be determined from (11.3.2), $\mu_{2,3:4}$ from (11.3.3), $\mu_{3,4:4}$ from (11.3.14), $\mu_{1,3:4}$ from (11.3.7), $\mu_{2,4:4}$ from (11.3.15), and finally $\mu_{1,4:4}$ from (11.3.21). For the sample of size 5, $\mu_{1,2:5}$ and $\mu_{2,3:5}$ can be determined from (11.3.2), $\mu_{3,4:5}$ from (11.3.3), $\mu_{4,5:5}$ from (11.3.14), $\mu_{1,3:5}$ from (11.3.6), $\mu_{2,4:5}$ from (11.3.15), $\mu_{3,5:5}$ from (11.3.19), $\mu_{1,4:5}$ from (11.3.7), $\mu_{2,5:5}$ from (11.3.15) and finally $\mu_{1,5:5}$ from (11.3.21). This process may be followed similarly to determine $\mu_{r,s:n}$ for $1 \leq r < s \leq n$ and for $n = 6, 7, \ldots$. From these values, covariances of order statistics can be readily computed.

Section Three: Conclusions

12 Conclusions and Discussion - Section One

Progressive censoring is a versatile method of censoring which gives experimenters many options in their study of product reliability. In Section One of this thesis, we have established a number of interesting and useful mathematical properties for progressive censored order statistics from some specific as well as arbitrary continuous distributions. We have also considered optimal censoring schemes for some commonly encountered life-time distributions, in the finite sample case. From this point, there are still many potential problems which can be tackled, and we discuss some of them here.

The recursive algorithm for obtaining moments of progressive Type-II censored order statistics was explored in Chapter 3 for the exponential and truncated exponential distributions. Similar results may also be established for moments of progressive Type-II censored order statistics from the power function and Pareto distributions and truncated forms of these distributions, where explicit expressions for single and product moments may not be easily obtained. Again, the method of proof for these recurrence relations will be very different here as compared to the usual order statistics case, since we do not begin with marginal densities, but rather with the joint density of all m progressive Type-II right censored order statistics. It will be of particular interest to establish such relationships for the logistic distribution, where, in the case of usual order statistics, the symmetry of the moments order statistics was used to make the relations complete [see Shah (1966, 1970)]. We know, however, from our discussions in Chapter 4, that the progressive Type-II censored order statistics from symmetric distributions do not have these same features.

The optimal censoring schemes which were discussed in Chapter 6 were obtained through computation for various sample sizes. Theoretical work in determining optimal censoring schemes for these and other commonly used distributions has yet to be considered. Furthermore, work of an asymptotic nature will be very useful to the practitioner, that is, if a large number of items n are to be included in a test, and it is desired that m complete failure times be observed, is it possible to determine the approximate proportions of units which should be censored immediately following each observed failure so as to maximize efficiency? Finally, the criterion for optimal censoring may be varied, for example, so as to minimize some cost function involving the progressive Type-II censored order statistics.

In Chapter 2, moments of progressive Type-II censored order statistics from the Uniform (0,1) distribution were obtained. In light of the probability integral transform for continuous distributions, one will now be able to use these moments to estimate moments of progressive Type-II censored order statistics from arbitrary continuous distributions, using the method of Taylor's series expansion approximations. This has been discussed for usual order statistics in, for example, Arnold, Balakrishnan and Nagaraja (1992).

The problems of studying progressive Type-II censored order statistics from discrete distributions has yet to be addressed, as well as the non-iid case, where items on test may have dependencies on one another and/or may not be identically distributed. In the case where items are independent but not identically distributed, the joint distribution of the progressive Type-II right censored order statistics will be obtained by considering permanents of matrices (ie., all possible outcomes of the censoring).

The question of outlier analysis when samples are progressively censored will also be of potential interest, and, again results obtained here will generalize results for outliers in the case of usual order statistics. Another area which may benefit from the implementation of progressive censoring is that of quantile estimation; since there is a possibility of observing extreme order statistics when progressive censoring is carried out, extreme quantiles may be better estimated under these schemes.

As mentioned throughout these sections, the results we obtain for progressive Type-II right censored order statistics will reduce to results for usual order statistics if we simply set $R_1 = R_2 = \dots = R_m = 0$. In many situations, we begin with something which seems very formidable, and after considerable work, arrive at something quite reasonable and most often elegant. This is very satisfying, and is certainly of practical and theoretical interest.

13 Conclusions and Discussion - Section Two

In Section Two of this thesis we discussed generalized distributions and developed several recurrence relations to determine the moments of order statistics from such distributions. We have also considered best linear unbiased, moment and maximum likelihood estimation of location, scale and shape parameters pertaining to these distributions, when samples are complete and when they are conventional Type-II censored. These distributions are useful in modelling life-time data, and are more versatile than the distributions which they generalize due to the addition of a shape parameter.

In light of our work in Section One, a natural question which arises is whether mathematical results based on progressive censoring are tractable for these distributions. This would generalize nicely the results presented. However, as mentioned in the previous chapter, obtaining recurrence relations for moments of progressive censored order statistics from the logistic distribution will be a first step in this endeavour.

Other distributions containing exponential terms may be similarly generalized. These distributions may result in a better fit to many data sets and may thereby assist in increasing the knowledge of system life times.

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