SKOLEM SEQUENCES: GENERALIZATIONS AND APPLICATIONS

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ABSTRACT

In this thesis the necessary conditions for the existence of near- , hooked near-, and indecomposable Skolem sequences are found and shown to be sufficient. We show also the existence of disjoint Skolem, disjoint hooked Skolem and disjoint near-Skolem sequences. Disjoint Skolem sequences are then applied to the existence problems for disjoint cyclic Steiner and Mendelsohn triple systems.

We also consider Skolem labellings of graphs: we prove that every graph with v vertices can be embedded as an induced subgraph in a Skolem labelled graph on $O(v^3)$ vertices, and show that all paths, cycles and n-windmills can be Skolem labelled or minimum hooked Skolem labelled.

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INTRODUCTION

When studying Steiner triple systems, Skolem in 1957 [S13] was led to ask the question: "Is it possible to distribute the numbers 1,2...,2n in n pairs (a_r,b_r) such that we have $b_r - a_r = r$ for r = 1,2,...,n?". He answered this question by proving that the necessary and sufficient conditions for such distribution to exist is that n must be ≡ 0,1 (mod 4); he called it a "1,+1 system". Nickerson [N2] was first to write that system in the form of a sequence. For example, if n=5 the sequence 3,5,2,3,2,4,5,1,1,4 is equivalent to the partition of the numbers 1,...,10 into the pairs (8,9),(3,5),(1,4),(6,10),(2,7); this sequence is now known as a **Skolem sequence** of order 5. In a subsequent paper [S14], Skolem showed that the existence of such a sequence of order n implies the existence of a cyclic Steiner triple system of order 6n+1. In order to show the existence of the latter also when n ≡ 2,3 (mod 4), he considered distributing the numbers 1,2,...,2n-1,2n+1, into n disjoint pairs (a_r,b_r), r=1,...,n, such that $b_r-a_r=r$, and conjectured that such distribution exists whenever n ≡ 2,3 (mod 4). O'Keefe [O] proved this conjecture to be true; the solution written in the form of Nickerson notation requires leaving a space (or zero) for the missing integer called a **hook**. For example, 4,2,5,2,4,3,6,5,3, 1,1,0,6 is a **hooked Skolem sequence** of order 6, and is equivalent to distributing the numbers 1,2,...,11,13, into the pairs (10,11),(2,4),(6,9),(1,5),(3,8),(7,13). The combined results of Skolem and O'Keefe produced a solution of Heffter's [H2] first difference problem, which implied the existence of cyclic Steiner triple systems of order $v \equiv 1 \pmod 6$ (which are known to exist for $v \equiv 1,3 \pmod 6$, $v \neq 9$). In order to prove in a similar way the existence of cyclic Steiner triple systems of order $v \equiv 3 \pmod 6$, Rosa 1966 [R2] modified the notion of Skolem sequences by inserting one extra hook in the middle. Another modification of Skolem sequences was introduced by Abrham and Kotzig [A3]: in an **extended Skolem sequence** the zero or hook may occur anywhere in the sequence. The existence problem for extended Skolem sequences with prescribed position of the zero (subject to a parity condition) remains open.

Langford [L1] noticed that his son, while playing with coloured blocks, placed them in one pile so that between the red pair there was one block, between the blue two, and between the yellow three. He expressed the case of three colours (n=3) as 3,1,2,1,3,2. This is now known as a special case of a **Langford sequence**, and adding one to each term and appending a pair of 1's results in a Skolem sequence of order n+1. Thus an **(n,d)-Langford sequence** $(l_1,l_2,\ldots,l_{2n-2d})$ is a sequence in which each of the integers $k \in \{d,d+1,\ldots,d+n-1\}$ is repeated exactly twice and whenever $l_i=l_j=k$ then $j-i=k$. The works of Priday, Davies, Bermond, Brouwer, Germa, and Simpson [P3,D1,B7,S12], proved that the necessary conditions are sufficient for the existence of Langford sequences and hooked Langford sequences (defined analogously).

Among the applications of Simpson's result on the existence of Langford sequences is the determination of the spectrum for the repeated edges in 2-factorizations of $2K_n$ by Colbourn and Rosa [C8], and determining the quadratic leaves of maximal partial triple systems [C6] (see also [C7,R3,R4] for more applications).

Skolem [S13] showed that the pairing (a_r,b_r) such that $b_r-a_r=r$ can be extended to the set of all positive integers by the formula $[n\tau,n\tau^2]$, $n=1,2,\dots$, where τ is the golden section. Earlier in 1953 Coxeter [C13] made reference to several authors who proved the same relation to be the winning combination of a modified game of Nim discovered by Wythoff in 1907 [W].

The various ways to partition the integers [1,n], and their relation to the modified Nim game and many other topics in combinatorics were discussed in Nowakowski's thesis [N3]. He also provided a survey of the various results that were found earlier. Thus we will not list them in this thesis.

The known applications of Skolem sequences to physical world include interference-free missile guidance codes [E1], and the placement of radioastronomical antennas in a linear array with distinct distances apart [B11,B12,B8].

Amar and Germa [A4] showed that the number σ_n of Skolem sequences of order n tends to infinity as n tends to infinity. Abrham [A1] used an earlier result by himself and Kotzig [A3] to show that $\sigma_n \geq 2^{[n/8]}$.

Edwards et al.(1982) [E2] used Skolem sequences to find balanced tournament designs and domino squares.

Cho [C3] (1983) employed Skolem and hooked Skolem sequences with various manipulations of the hooks and the differences to construct various designs with prescribed automorphism types.

In this thesis we are concerned primarily with developing new concepts, or expanding the known ones, that are useful to design theory. In Chapter II, we generalize the concepts of Skolem and hooked Skolem sequences to that of the near-Skolem and hooked near-Skolem sequences, that was introduced, in effect, by Stanton and Goulden (1981)[S16] and used by Billington [B9] to find several designs. We derive the necessary conditions for the existence of such sequences and prove their sufficiency for all orders.

In Chapter III, we introduce disjoint Skolem sequences, and we show that for n ≡ 0,1 (mod 4) there exist at least 4 mutually disjoint Skolem sequences, while for n ≡ 2,3 (mod 4) there exist at least 3 mutually disjoint hooked Skolem sequences. As a direct consequence of that we improve the results obtained by C.J. Colbourn and M. Colbourn [C4,C5] on the existence of disjoint cyclic Steiner and Mendelsohn triple systems. We also discuss the relationship between disjoint Skolem sequences and Room squares.

Chapter IV deals with the m-fold Skolem sequences. Again we show that the necessary conditions are sufficient for the existence of such sequences. We also introduce the concept of indecomposable m-fold Skolem sequences and we show the existence of (at least) 3-fold indecomposable Skolem sequences. Moreover, when $m \equiv 0 \pmod 6$ there exists an m-fold indecomposable Skolem sequence for all orders n.

Chapter V is different from the other chapters in that its content is the result of two joint papers of the author and E. Mendelsohn [M3,M4]. We introduce the concept of a Skolem labelling of a graph, and prove that every v-vertex graph can be embedded in a Skolem labelled graph with $O(v^3)$ vertices. We also prove the existence of a Skolem labelling of some special types of graphs, such as paths, cycles and n-windmills.

Chapter VI contains concluding remarks and open questions for further research.

CHAPTER I

BASIC CONCEPTS

I.1. Preliminaries

In this section we present the basic definitions and earliest results obtained regarding Skolem sequences.

A **Skolem sequence** of order n is a sequence $S = (s_1, s_2, \ldots, s_{2n})$ of 2n integers satisfying the following conditions:

1. for every $k \in \{1,2,\ldots,n\}$ there exists exactly two elements s_i , s_j such that $s_i = s_j = k$.

2. if $s_i = s_j = k$ then $j-i = k$.

A **hooked Skolem sequence** of order n is a sequence HS = $(s_1, s_2, \ldots, s_{2n+1})$ of 2n+1 integers satisfying conditions 1. and 2. and the following condition:

3. $s_{2n} = 0$.

Skolem [S13] found the necessary conditions for the existence of such sequences to be $n \equiv 0,1 \pmod 4$. He credited his colleague Th. Bang for finding this proof; we show it here, since similar arguments are used in finding almost all necessary conditions in this thesis.

Lemma I.1. [Bang] The necessary condition for the existence of a Skolem sequence of order n is that $n \equiv 0,1 \pmod 4$.

Proof We consider the sum of all sums, and the sum of all differences of the subscripts i,j, when $s_i=s_j=k$.

(a) $$\sum_{\substack{k=1 \\ k=s_i=s_j}}^{n} (i+j) = 2n^2 + n$$

(b) $$\sum_{\substack{k=1 \\ k=s_i=s_j}}^{n} (j-i) = n(n+1)/2$$

Adding (a) and (b) gives

$$2 \sum_{\substack{k=1 \\ k=s_i=s_j}}^{n} j = (5n^2+3n)/2$$

This implies that $(5n^2 + 3n)/4$ must be an integer, which happens only when $n \equiv 0,1 \pmod 4$. ∎

We give also the proofs of Skolem and O'Keefe [S13,O] for the existence of Skolem, and hooked Skolem sequences, respectively.

Theorem I.1. [Skolem] A necessary and sufficient condition for the existence of a Skolem sequence of order n is that $n \equiv 0,1$ (mod 4).

Proof. Necessity is shown in Lemma I.1. For sufficiency distinguish two cases.

Case 1. $n \equiv 0 \pmod 4$, let $n = 4s$.

	i	j	
1)	4s+r-1	8s-r+1	$1 \le r \le 2s$
2)	r	4s-r-1	$1 \le r \le s-2$
3)	s+r+1	3s-r	$1 \le r \le s-2$
4)	s-1	3s	----------
5)	s	s+1	----------
6)	2s	4s-1	----------
7)	2s+1	6s	----------

Skip the line 6) for the case of s=1.

To verify that the above table will give a Skolem sequence we check that every difference from 1 to n=4s is present exactly once:
The pairs in line 1) give all the even differences 2,4,...,4s. The two odd differences 2s-1 and 4s-1 are obtained from the lines 6) and 7) respectively. The smallest difference 1 is obtained from 5), the differences 3,5,...,2s-3 from 3) and the remaining odd differences 2s+1,...,4s-3 from 2).
It is easy to see that the above pairs consist of the numbers 1,...,2n.
In the subsequent proofs we will simply give the tables and skip this process of verification.

<u>Case 2.</u> $n \equiv 1 \pmod 4$, let $n = 4s+1$ $(s \ge 2)$.

i	j	
4s+r+1	8s-r+3	$1 \le r \le 2s$
r	4s-r+1	$1 \le r \le s$
s+r+2	3s-r+1	$1 \le r \le s-2$
s+1	s+2	----------
2s+1	6s+2	----------
2s+2	4s+1	----------

For n=1,5 we give the solutions:

1,1

2,4,2,3,5,4,3,1,1,5.

This completes the proof. ∎

Theorem I.2. [O'Keefe] A necessary and sufficient condition for the existence of a hooked Skolem sequence of order n is that $n \equiv 2,3 \pmod 4$.

Proof. Necessity is shown in Lemma I.1. For sufficiency distinguish two cases.

Case 1. $n \equiv 2 \pmod 4$, let $n = 4s+2$ $(s > 0)$.

i	j	
r	4s-r+2	$1 \le r \le 2s$
4s+r+3	8s-r+4	$1 \le r \le s-1$
5s+r+2	7s-r+3	$1 \le r \le s-1$
2s+1	6s+2	----------
4s+2	6s+3	----------
4s+3	8s+5	----------
7s+3	7s+4	----------

For n=2:

1,1,2,0,2.

Case 2. $n \equiv 3 \pmod 4$, let $n = 4s-1$ $(s \geq 1)$.

i	j	
4s+r	8s-r-2	$1 \leq r \leq 2s-2$
r	4s-r-1	$1 \leq r \leq s-2$
s+r+1	3s-r	$1 \leq r \leq s-2$
s-1	3s	----------
s	s+1	----------
*2s	4s-1	----------
2s+1	6s-1	----------
4s	8s-1	----------

Skip the line(*) for the case s=1.

This completes the proof. ∎

I.2. Early Applications

In this section we illustrate the earliest links between Skolem sequences and design theory.

A **Steiner triple system** of order v, STS(v), is a v-set of elements together with a set of 3-subsets of the v-set called **blocks** such that every 2-subset of the v-set appears in exactly one block. An STS(v) on the elements of Z_v is said to be **cyclic** if, whenever {a,b,c} is a block, so also is {a+1,b+1,c+1}. It is well known that STS(v) exists if and only if $v \equiv 1,3 \pmod 6$.

Heffter (1897) [H2] investigated the existence of cyclic STS(v), and reduced the question to two "difference problems":

Heffter's difference problem I. Can one partition the set

$\{1,..,3n\}$ into triples (a_i,b_i,c_i), $i=1...n$, such that in each triple either $a_i + b_i = c_i$ or $a_i+b_i+c_i \equiv 0 \pmod{6n+1}$?

Heffter's difference problem II. Can one partition the set $\{1,..., 2n, 2n+2,..., 3n+1\}$ into triples (a_i,b_i,c_i), $i=1...n$, such that in each triple either $a_i + b_i = c_i$ or $a_i+b_i+c_i \equiv 0 \pmod{6n+3}$?

Heffter observed that a solution of his first problem would give a cyclic STS(v) for $v \equiv 1 \pmod 6$, and solution to his second problem would give a cyclic STS(v) for $v \equiv 3 \pmod 6$.

Peltesohn (1939) [P1] gave a complete solution to Heffter's two problems except for the second problem when n=1 in which case a solution does not exist.

Skolem and O'Keefe also gave a solution to Heffter first problem (yielding a cyclic STS(v), $v \equiv 1 \pmod 6$). Later Rosa (1966) [R2] extended the same method to the second Heffter problem (yielding cyclic STS(v), $v \equiv 3 \pmod 6$).

The above partially answers the question: why use Skolem sequences?.

First, Skolem sequences is a subject that has appeared in many disguises in mathematics, and has several links with other topics in mathematics (see Nowakowski's thesis [N3]).

Secondly, Skolem sequences have the advantage of being easier to find, and easier to manipulate (i.e. shifting the hooks, and shifting the numbers, or adding or deleting distances) to find cyclic designs. These techniques are illustrated most clearly in Rosa's work. For instance, Rosa[R2] showed that the set

$\{1,\ldots,n,n+2,\ldots,2n+1\}$ can be partitioned into pairs (b_r,a_r) where $b_r - a_r = r$, $r = 1,\ldots,n$, if and only if $n \equiv 0,3 \pmod 4$. His solution produced a hooked Skolem sequence where the hook is in the middle; this is a special case of what is now known as the **extended Skolem sequence**, i.e. a hooked Skolem sequence in which the hook may occur anywhere in the sequence.

For example, when n =4, 1,1,3,4,0,3,2,4,2 is such a sequence. It can be obtained either directly or by finding a Skolem sequence of order n+1, where the largest number n+1 occurs in positions n+1, 2n+2, then removing this largest number, and replacing its first occurrence with 0 (in our example, such a Skolem sequence is 1,1,3,4,5,3,2,4,2,5.)

The new sequence will yield the partition of the integers {1,...,4,6,...9} into the pairs {(1,2),(7,9),(3,6),4,8)}. Every pair (b_r,a_r) will give rise to a triple (r, b_r+n, a_r+n), r=1,..n, e.g. the above set of pairs will give {(1,5,6),(2,11,13),(3,7,10),(4,8,12)} which is a solution to the second Heffter's problem with n= 4.

When replacing every triple (a,b,c) of the Heffter solution by a **base block** {0,a,a+b} such that the set {{0,a,a+b}} mod(6n+3) are blocks in the cyclic STS(6n+3), in case of problem II, Heffter observed that the extra base block {0,2n+1,4n+2}, must be added to give the STS(6n+3). For example, the above Heffter solution will give the base blocks {{0,1,6},{0,2,13},{0,3,10},{0,4,12}} mod 27; with the additional base block {0,9,18} mod 27, they will be the base blocks for a cyclic STS(27).

CHAPTER II

NEAR- AND HOOKED NEAR-SKOLEM SEQUENCES

II.1 Introduction

Let m,n be integers, $m \le n$. A **near-Skolem** sequence of **order** n and **defect** m is a sequence NS = $(s_1,s_2,\dots,s_{2n-2})$ of integers $s_i \in \{1,2,..,m-1,m+1,..n\}$ which satisfies the following conditions:

1) For every $k \in \{1,2,\dots,m-1,m+1,..n\}$ there are exactly two elements s_i,s_j, such that $s_i=s_j=k$.

2) If $s_i=s_j=k$ then $j - i = k$.

A **hooked near-Skolem** sequence of order n and defect m is a sequence HNS= $(s_1,s_2,\dots,s_{2n-1})$ of integers $s_i \in \{1,2,\dots,m-1,m+1,\dots n\}$ satisfying conditions 1), 2) and the condition:

3) $s_{2n-2} = 0$. We will refer to near-Skolem and hooked near-Skolem sequences of order n and defect m as m-near-Skolem and hooked m-near-Skolem sequences, respectively.

Note that if we did not skip the difference m in the above definitions, we would obtain the definitions of Skolem and hooked Skolem sequences, respectively.

Near-Skolem sequences were introduced, in effect, by Stanton and

Goulden [S16], in particular, for the purpose of constructing cyclic Steiner triple systems. They asked for a set of n-1 pairs P(1,n)/m with each of the integers of {1,...,2n-2} appearing exactly once and each of the integers of {1,...,m-1,m+1,..,n} occurring as a difference exactly once.

For example, the pairs (3,4),(6,8),(1,5),(2,7) form a P(1,5)/3; the corresponding 3-near-Skolem sequence of order 5 is 4,5,1,1,4,2,5,2.

Billington applied near-Skolem sequences to obtain several types of designs. (See [B9] for more details).

Recall that if we did not skip the distance m in the above definition of a hooked near-Skolem sequence, and allowed the hook (i.e. the symbol 0) to occur anywhere in the sequence, we would obtain the definition of the extended Skolem sequence. Billington [B10] conjectured that the obvious necessary conditions (which are just parity conditions, see [A3]) are sufficient for the existence of the extended Skolem sequences (with 0 in a prescribed position). This conjecture is still open.

The concept of near-Skolem sequences resembles that of the extended Skolem sequences, but the two are not likely to be equivalent. In this chapter we prove that the obvious arithmetic necessary conditions for the existence of near-Skolem, and hooked near-Skolem sequences (see Section II.2), are also sufficient.

II.2.Necessity

In this section we find the necessary conditions for the existence of near-Skolem, and hooked near-Skolem sequences, respectively.

We need to determine what possible values m can take. For example, there exist 6-,4- and 2-near-Skolem sequences of order 7:

7,1,1,2,5,2,4,7,3,5,4,3

1,1,6,3,7,5,3,2,6,2,5,7

1,1,5,6,7,3,4,5,3,6,4,7

and there exist hooked 7-,5-,3- and 1-near-Skolem sequences of order 7:

1,1,3,4,5,3,6,4,2,5,2,0,6

2,3,2,6,3,7,4,1,1,6,4,0,7

2,5,2,4,6,7,5,4,1,1,6,0,7

2,5,2,6,4,7,5,3,4,6,3,0,7,

but there is no m-near-Skolem sequence of order 7 when m is odd, and no hooked m-near-Skolem sequence of order 7 if m is even.

Lemma II.1. Necessary conditions for the existence of an m-near-Skolem sequence of order n are:

1) if $n \equiv 0,1 \pmod 4$ then m must be odd;

2) if $n \equiv 2,3 \pmod 4$ then m must be even.

Necessary conditions for the existence of a hooked m-near-Skolem sequence of order n are:

3) if $n \equiv 0,1 \pmod 4$ then m must be even;

4) if $n \equiv 2,3 \pmod 4$ then m must be odd.

Proof. The proof follows the reasoning of [S13]. Consider first the case of a near-Skolem sequence. We consider the sum of all sums, and the sum of all differences of the subscripts i,j when $s_i=s_j=k$

(a) $$\sum_{\substack{k=1 \\ k\neq m \\ k=s_i=s_j}}^{n} (i+j) = 2n^2 - 3n + 1$$

(b) $$\sum_{\substack{k=1 \\ k\neq m \\ k=s_i=s_j}}^{n} (j-i) = n(n+1)/2 - m$$

Subtracting (b) from (a) gives

$$2\sum_{\substack{k=1 \\ k\neq m \\ k=s_i=s_j}}^{n} i = (3n^2 - 7n + 2m + 2)/2$$

This implies that $(n(3n-7)+2(m+1))/4$ must be an integer.

Solving for n and m , we obtain conditions (1) and (2).

Similarly, in the case of the hooked near-Skolem sequence, we have

(a') $$\sum_{\substack{k=1 \\ k\neq m \\ k=s_i=s_j}}^{n} (i+j) = 2n^2 - 3n + 2$$

(b') $$\sum_{\substack{k=1 \\ k\neq m \\ k=s_i=s_j}}^{n} j-i = n(n+1)/2 - m$$

Subtracting (b') from (a') gives

$$2 \sum_{\substack{k=1 \\ k \neq m \\ k=s_i=s_j}}^{n} i = (3n^2 - 7n + 4 + 2m)/2$$

This implies that $(n(3n-7) + 2(m+2)/4$ must be an integer. Solving for n and m, we obtain conditions (3) and (4). ■

II.3.Sufficiency

In this section we prove our two main theorems (Theorem II.3 and Theorem II.4) showing that the above necessary conditions are sufficient for the existence of near-Skolem, and hooked near-Skolem sequences, respectively. In the process we will use some of the known results [P3,D1,B7,S12] to handle the cases when our constructions degenerate.

First we need some more definitions.

A **Langford sequence** is a generalization of a Skolem sequence. We use the same notation as in [S12]:
"A sequence $(d,d+1,\dots,d+p-1)$ of p consecutive positive integers is said to be **perfect Langford** if the integers $\{1,2,\dots 2p\}$ can be arranged in disjoint pairs $\{(a_i,b_i): \leq i \leq p\}$ so that $\{b_i-a_i: 1 \leq i \leq p\} = \{d,d+1,\dots,d+p-1\}$. A sequence is **hooked Langford** if the set $\{1,2,\dots,2p-1,2p+1\}$ can be arranged to satisfy the same condition". For example, 4,2,3,2,4,3 is a perfect Langford sequence with d=2 and p=3, and 6,4,2,5,2,4,6,3,5,0,3 is

a hooked Langford sequence with d=2 and p=5.

Theorem II.1. [S12] Necessary and sufficient conditions for the sequence $\{d,d+1,...,d+p-1\}$ to be perfect Langford are:

1) $p \geq 2d-1$.
2) $p \equiv 0,1 \pmod 4$, for d odd and,
   $p \equiv 2,3 \pmod 4$, for d even.

Theorem II.2. [S12] Necessary and sufficient conditions for the sequence $\{d,d+1,...,d+p-1\}$ to be hooked Langford are:

1) $p(p+1-2d)+2 \geq 0$.
2) $p \equiv 2,3 \pmod 4$ for d odd and,
   $p \equiv 1,2 \pmod 4$ for d even.

Lemma II.2. i) The existence of a Skolem sequence of order t ,$t \equiv 0,1 \pmod 4$ implies the existence of a (t+1)-near-Skolem sequence of order q, where $q \geq 3t + 4$, and:

a) if $t \equiv 0 \pmod 4$ then $q \equiv 0,1 \pmod 4$
b) if $t \equiv 1 \pmod 4$ then $q \equiv 2,3 \pmod 4$.

ii) The existence of a Skolem sequence of order t, $t \equiv 0,1 \pmod 4$ implies the existence of a hooked (t+1)-near-Skolem sequence of order q where $q \geq 3t+4$ and:

a) if $t \equiv 0 \pmod 4$ then $q \equiv 2,3 \pmod 4$
b) if $t \equiv 1 \pmod 4$ then $q \equiv 0,1 \pmod 4$.

iii) The existence of a hooked Skolem sequence of order t, $t \equiv 2,3 \pmod 4$ implies the existence of an (t+1)-near-Skolem sequence of order q, where $q \geq 3t + 4$, and:

a) if $t \equiv 2 \pmod 4$ then $q \equiv 0,1 \pmod 4$

b) if $t \equiv 3 \pmod 4$ then $q \equiv 2,3 \pmod 4$.

iv) The existence of a hooked Skolem sequence of order t, $t \equiv 2,3 \pmod 4$ implies the existence of a hooked (t+1)-near-Skolem sequence of order q, where $q \geq 3t+4$, and:

a) if $t \equiv 2 \pmod 4$ then $q \equiv 2,3 \pmod 4$

b) if $t \equiv 3 \pmod 4$ then $q \equiv 0,1 \pmod 4$.

Proof. The proof is a direct consequence of Theorems II.1, II.2:

For instance, in case i) there exist a Skolem sequence of order t, $t \equiv 0,1 \pmod 4$. Append to this sequence a Langford sequence with d=t+2. By condition 1) of Theorem II.1, $p \geq 2(t+2)-1$. This will give an (t+1)-near-Skolem sequence of order $q \geq 3t+4$. By applying the parity condition 2) of Theorem II.1 to p, we get the required parity for q.
The other cases are similar. ∎

For example, if we take a Skolem sequence of order 4, 1,1,4,2,3,2,4,3, and append to it a Langford sequence with d=6,p=15, say,

15,13,11,9,7,16,14,12,10,8,6,7,9,11,13,15,6,8,10,12,14,16

we get a 5-near-Skolem sequence of order 16

1,1,4,2,3,2,4,3,15,13,11,9,7,16,14,12,10,8,6,7,9,11,13,15, 6,8,10,12,14,16.

If we join the hooked Skolem sequence, 1,1,2,0,2, to a hooked Langford sequence with d=4 and p=11, say, 6,0,8,9,11,12,6,7,10,4,8,5,9,4,7,11,5,12,10, such that the last member of the first will be inserted in the hook of the second and the first member of the second will be inserted in the hook of the first. We get the 3-near-Skolem sequence of order 12: 1,1,2,6,2,8,9,11,12,6,7,10,4,8,5,9,4,7,11,5,12,10.

Finally, if we take a hooked Skolem sequence, 1,1,2,0,2, and append it to a perfect Langford sequence with d=4 and p=9, say, 9,7,5,10,8,6,4,5,7,9,4,6,8,10 we get a hooked 3-near-Skolem sequence of order 10 9,7,5,10,8,6,4,5,7,9,4,6,8,10,1,1,2,0,2.

Lemma II.2 will be used to handle some of the small cases of the main theorems. However, it is clear that this method cannot cover all cases.

Theorem II.3. An m-near-Skolem sequence of order n exists if and only if $n \equiv 0,1 \pmod 4$ and m is odd, or $n \equiv 2,3 \pmod 4$ and m is even.

Proof. Necessity was proved in Lemma II.1.

For sufficiency we have to distinguish 8 cases. In each case, solution is given in a form of a table. The first two columns of the table give the two subscripts of s_i and s_j, and the third column gives the difference k.

For m=1, the 1-near-Skolem sequence is a perfect Langford sequence with d=2, which exists by [D1], so we will omit all the subsequent cases when m=1.

<u>Case 1.</u> $n \equiv 0 \pmod 8$.

For $n \geq m > 1$, let $n = 8s$ and $m=2t+1$.

i	j	k	
8s+r-1	16s-r-2	8s-2r-1	$0 \leq r \leq 4s-t-2$
12s-t-2	12s-t-1	1	--------------
12s-t+r	12s+t-r-1	2t-2r-1	$0 \leq r \leq t-2$
4s+r	12s-r	8s-2r	$0 \leq r \leq 1$
2r-1	8s-2r-3	8s-4r-2	$1 \leq r \leq s-1$
2r	8s-2r	8s-4r	$1 \leq r \leq 2s-1$
2s-1	2s+1	2	-------------
*4s-1	8s-3	4s-2	-------------
2s+2r+1	6s-2r-1	4s-4r-2	$1 \leq r \leq s-2$

Skip the line (*) for the case n=8.

<u>Case 2.</u> $n \equiv 1 \pmod 8$.

For m=3, by Lemma II.2(iii)(a), we get all cases for all $n > 9$.
For n= 9 and $9 \geq m \geq 3$, we list them:

n=9 and m=3

7,2,8,2,5,6,9,7,4,5,8,6,4,1,1,9

n=9 and m=5

2,8,2,1,1,6,9,7,4,8,3,6,4,3,7,9

n=9 and m=7

1,1,6,2,8,2,9,4,6,5,3,4,8,3,5,9

n=9 and m=9 is a Skolem sequence of order 8.

For all the remaining cases where $n \geq m > 3$, and $n > 9$, let n = 8s+1 and m =2t+1.

i	j	k	
8s+r-2	16s-r-1	8s-2r+1	$0 \leq r \leq 4s-t-1$
12s-t-2	12s-t-1	1	------------
12s-t+r	12s+t-r-1	2t-2r-1	$0 \leq r \leq t-3$
12s-2	16s+2	4s+2	------------
4s+r	12s-r+2	8s-2r+2	$1 \leq r \leq 3$
4s+r-2	8s-r-2	4s-2r	$1 \leq r \leq 2$
*r	8s-r-4	8s-2r-4	$1 \leq r \leq 2s-4$
*2s-1	6s-1	4s	------------
*2s+r+3	6s-r-1	4s-2r-4	$1 \leq r \leq 2s-5$
2s-3	2s	3	-----------
2s-2	2s+2	4	-----------
2s+1	2s+3	2	-----------

Skip the lines (*) in the case of n=17.

<u>Case 3.</u> $n \equiv 2 \pmod 8$

For m=2, by Lemma II.2 (i)(b) the existence of the Skolem sequence 1,1 provides solutions for all $n \geq 10$.

For m=2 and n=2, obviously 1,1 is the required sequence.

For n =10 and $10 \geq m \geq 4$ we list them:

n=10 and m=4

3,9,2,3,2,7,5,10,8,6,9,5,7,1,1,6,8,10

n=10 and m=6

3,9,2,3,2,7,5,10,8,4,9,5,7,4,1,1,8,10

n=10 and m=8

3,9,2,3,2,7,5,10,6,4,9,5,7,4,6,1,1,10

n=10 and m=10 is a Skolem sequence of order 9.

For all the cases where $n \geq m > 2$ and $n > 10$, let n =8s+2 and m=2t.

i	j	k	
8s+r	16s-r+2	8s-2r+2	$0 \leq r \leq 4s-t$
12s+t	12s+t+1	1	------------
12s-t+r	12s+t-r	2t-2r	$1 \leq r \leq t-2$
4s+r-1	12s-r+2	8s-2r+3	$1 \leq r \leq 3$
1	4	3	------------
2	4s-1	4s-3	------------
3+2r	8s-2r-2	8s-4r-5	$0 \leq r \leq s-1$
6+2r	8s-2r-1	8s-4r-7	$0 \leq r \leq s-2$
6s-1	6s+1	2	-----------
2s+r+3	6s-r-2	4s-2r-5	$0 \leq r \leq 2s-5$

<u>Case 4.</u> $n \equiv 3 \pmod 8$

For m=2, by Lemma II.2 (i)(b) the existence of the sequence 1,1 provides solutions for all $n \geq 11$.

For n=3 and m=2

3,1,1,3.

For n=11 and 11 ≥ m ≥ 4 we list them:

n=11 and m=4

2,3,2,5,3,11,9,7,5,10,8,6,1,1,7,9,11,6,8,10

n=11 and m=6

2,3,2,5,3,11,9,7,5,10,8,1,1,4,7,9,11,4,8,10

n=11 and m=8

2,3,2,5,3,11,9,7,5,10,1,1,6,4,7,9,11,4,6,10

n=11 and m=10

2,3,2,5,3,11,9,7,5,1,1,8,6,4,7,9,11,4,6,8

For the remaining cases where $n \geq m > 2$ and $n > 11$, let $n = 8s+3$ and $m=2t$.

i	j	k	
8s+r+2	16s-r+4	8s-2r+2	$0 \leq r \leq 4s-t$
12s-t+r+5	12s+t-r+3	2t-2r-2	$0 \leq r \leq t-3$
12s-t+3	12s-t+4	1	-----------
4s+r	12s-r+5	8s-2r+5	$1 \leq r \leq 2$
8s	12s+5	4s+5	-----------
2+2r	8s+1-2r	8s-4r-1	$0 \leq r \leq s-2$
2s-3	2s-1	2	-----------
1+2r	8s-2r-2	8s-4r-3	$0 \leq r \leq s-3$
2s+r	6s+3-r	4s-2r+3	$0 \leq r \leq 2s$

<u>Case 5.</u> $n \equiv 4 \pmod 8$.

For m=3, by Lemma II.2 (iii)(a) the existence of the hooked Skolem sequence 1,1,2,0,2 provides solutions for all $n \geq 12$.

For n=4 and m=3

2,4,2,1,1,4.

For n=12 and 12 > m > 3 we list them:

n=12 and m=5

8,12,4,6,3,10,4,3,8,6,11,9,7,12,2,10,2,1,1,7,9,11

n=12 and m=7

8,12,4,6,3,10,4,3,8,6,11,9,5,12,2,10,2,5,1,1,9,11

n=12 and m=9

8,12,4,6,3,10,4,3,8,6,11,7,5,12,2,10,2,5,7,1,1,11

n=12 and m=11

8,12,4,6,3,10,4,3,8,6,9,7,5,12,2,10,2,5,7,9,1,1.

For all the remaining cases where n > m > 3 and n >12, let n=8s+4 and m=2t+1.

i	j	k	
8s+r+3	16s-r+6	8s-2r+3	$0 \leq r \leq 4s-t$
12s-t+4	12s-t+5	1	------------
12s-t+r+6	12s+t-r+5	2t-2r-1	$0 \leq r \leq t-3$
4s+r+2	12s-r+8	8s-2r+6	$1 \leq r \leq 3$
8s+2	12s+4	4s+2	------------
4s+r	8s+2-r	4s-2r+2	$1 \leq r \leq 2$
1+r	8s-r-1	8s-2r-2	$0 \leq r \leq 2s-3$
2s+r+5	6s-r+1	4s-2r-4	$0 \leq r \leq 2s-5$
2s-1	2s+2	3	-----------
2s	2s+4	4	-----------
2s+1	2s+3	2	-----------

<u>Case 6.</u> $n \equiv 5 \pmod 8$

We only list the case of n=5:

n=5 and m=3

4,2,5,2,4,1,1,5

n=5 and m=5 is a Skolem sequence of order 4.

For all $n \geq m > 1$ and $n > 5$, let $n = 8s+5$ and $m=2t+1$.

i	j	k	
8s+r+3	16s-r+8	8s-2r+5	$0 \leq r \leq 4s-t+1$
12s-t+5	12s-t+6	1	--------------
12s-t+r+7	12s+t-r+6	2t-2r-1	$0 \leq r \leq t-2$
4s+r+2	12s-r+8	8s-2r+6	$1 \leq r \leq 2$
2	4s+2	4s	--------------
1+2r	8s-2r+1	8s-4r	$0 \leq r \leq s-1$
4+2r	8s-2r+2	8s-4r-2	$0 \leq r \leq s-1$
2s+1	2s+3	2	--------------
2s+r+4	6s-r+2	4s-2r-2	$0 \leq r \leq 2s-3$

<u>Case 7.</u> $n \equiv 6 \pmod 8$

For m=2, by Lemma II.2 (i)(b) the existence of the sequence 1,1 provides solutions for all $n \geq 14$.

For n=6 and m=2

5,6,1,1,4,5,3,6,4,3.

For n=6 and n=4

5,1,1,6,3,5,2,3,2,6

n=6 and m=6 is a Skolem sequence of order 5.

For all the remaining cases where $n \geq m > 2$ and $n > 6$, let n =8s+6 and m=2t.

i	j	k	
8s+r+4	16s-r+10	8s+6-2r	$0 \leq r \leq 4s-t+2$
12s+t+6	12s+t+7	1	--------------
12s-t+r+7	12s+t-r+5	2t-2r-2	$0 \leq r \leq t-3$
8s+2	12s+5	4s+3	--------------
4s+r+1	12s-r+8	8s-2r+7	$1 \leq r \leq 2$
1+2r	8s-2r	8s-4r-1	$0 \leq r \leq s-2$
2+2r	8s-2r+3	8s-4r+1	$0 \leq r \leq s-1$
2s-1	2s+1	2	--------------
2s+r+2	6s-r+3	4s-2r+1	$0 \leq r \leq 2s-1$

<u>Case 8.</u> $n \equiv 7 \pmod 8$

For m=2, by Lemma II.2(i)(b), the existence of the sequence 1,1 provides solutions for all $n \geq 7$.

We list the remaining cases of n=7:

n=7 and m=4

3,7,5,3,2,6,2,5,7,1,1,6

n=7 and m=6

3,7,5,3,2,4,2,5,7,4,1,1.

For all $n > m > 2$ and $n > 7$, let $n = 8s+7$ and $m=2t$.

i	j	k	
8s+r+6	16s-r+12	8s-2r+6	$0 \le r \le 4s-t+2$
12s+t+8	12s+t+9	1	--------------
12s-t+r+9	12s+t-r+7	2t-2r-2	$0 \le r \le t-3$
8s+4	12s+7	4s+3	--------------
4s+r+1	12s-r+10	8s-2r+9	$1 \le r \le 2$
1+2r	8s-2r+2	8s-4r+1	$0 \le r \le s-1$
2+2r	8s-2r+5	8s-4r+3	$0 \le r \le s-1$
6s+2	6s+5	3	-------------
4s-1	4s+1	2	-------------
2s+2r+1	6s-2r	4s-4r-1	$0 \le r \le s-2$
2s+2r+2	6s-2r+3	4s-4r+1	$0 \le r \le s-1$

This completes the proof of Theorem II.3. ∎

Theorem II.4. A hooked m-near-Skolem sequence of order n exists if and only if $n \equiv 0,1 \pmod 4$ and m is odd, or $n \equiv 2,3 \pmod 4$ and m is even.

Proof:

Case 1. $n \equiv 0 \pmod 8$.

For m=2, by Lemma II.2 (ii) (b), the existence of the Skolem sequence 1,1 provides solutions for all $n \geq 8$.

For $n \geq m > 2$, let $n = 8s$ and $m = 2t$.

i	j	k	
1+r	8s-r+1	8s-2r	$0 \leq r \leq 4s-t-1$
4s-t+1	4s-t+2	1	--------------
4s-t+r+3	4s+t-r+1	2t-2r-2	$0 \leq r \leq t-3$
4s+3	8s+2	4s-1	--------------
4s+r	12s-r+1	8s-2r+1	$1 \leq r \leq 2$
8s+2r+4	16s-2r-1	8s-4r-5	$0 \leq r \leq s-2$
8s+2r+3	16s-2r-4	8s-4r-7	$0 \leq r \leq s-2$
14s-1	14s+1	2	-------------
10s+r+1	14s-r-2	4s-2r-3	$0 \leq r \leq 2s-3$

Case 2. $n \equiv 1 \pmod 8$.

For m=2, by Lemma II.2 (ii)(b), the sequence 1,1 provides solutions for all $n \geq 9$.

We list the remaining cases of n=9 :

n=9 and m=4

7,5,3,9,6,3,5,7,8,2,6,2,9,1,1,0,8

7n=9 and m=6

5,3,4,9,3,5,4,7,8,1,1,2,9,2,7,0,8

n=9 and m=8

5,9,7,4,2,5,2,4,6,7,9,1,1,3,6,0,3.

For all $n > m > 2$ and $n > 9$, let $n=8s+1$ and $m=2t$.

i	j	k	
1+r	8s-r+1	8s-2r	$0 \le r \le 4s-t-1$
4s-t+1	4s-t+2	1	--------------
4s-t+r+3	4s+t-r+1	2t-2r-2	$0 \le r \le t-3$
4s+3	8s+2	4s-1	--------------
4s+r	12s-r+3	8s-2r+3	$1 \le r \le 2$
8s+2r+4	16s-2r+1	8s-4r-3	$0 \le r \le s-1$
8s+2r+3	16s-2r-2	8s-4r-5	$0 \le r \le s-2$
12s+3	12s+5	2	-------------
10s+1	10s+4	3	-------------
10s+2r+3	14s-2r	4s-4r-3	$0 \le r \le s-2$
10s+2r+6	14s-2r+1	4s-4r-5	$0 \le r \le s-3$

<u>Case 3.</u> $n \equiv 2 \pmod 8$.

We list all cases for n=10

n=10 and m=3

5,8,4,9,10,5,4,6,7,8,1,1,9,6,10,7,2,0,2

n=10 and m=5

9,7,10,3,1,1,3,8,7,9,6,2,10,2,4,8,6,0,4

n=10 and m=7

9,6,3,5,8,3,10,6,5,9,1,1,8,2,4,2,10,0,4

n=10 and m=9

8,6,10,3,1,1,3,6,8,7,5,2,10,2,4,5,7,0,4

For all $n > m > 1$ and $n > 10$, let $n=8s+2$ and $m=2t+1$.

i	j	k	
1+r	8s-r+2	8s-2r+1	$0 \le r \le 4s-t-1$
4s-t+1	4s-t+2	1	--------------
4s-t+r+3	4s+t-r+2	2t-2r-1	$0 \le r \le t-3$
4s+1	8s+3	4s+2	--------------
4s+r+1	12s-r+5	8s-2r+4	$1 \le r \le 3$
8s+2r+5	16s-2r+1	8s-4r-4	$0 \le r \le s-2$
8s+2r+4	16s-2r-2	8s-4r-6	$0 \le r \le s-3$
10s	10s+2	2	-------------
16s	16s+3	3	-------------
10s+r+3	14s-r+3	4s-2r	$0 \le r \le 2s-2$

<u>Case 4.</u> $n \equiv 3 \pmod 8$.

We list all cases for n=11 :

n=11 and m=3

7,5,6,9,11,8,5,7,6,2,10,2,9,8,4,11,1,1,4,0,10

n=11 and m=5

8,10,7,3,11,9,3,6,8,7,2,10,2,6,9,11,4,1,1,0,4

n=11 and m=7

5,3,4,10,3,5,4,11,6,8,2,9,2,10,6,1,1,8,11,0,9

n=11 and m=9

7,10,6,3,11,8,3,7,6,5,2,10,2,8,5,11,4,1,1,0,4

n=11 and m=11 is a hooked Skolem sequence of order 10.

For all $n \geq m > 1$ and $n>11$, let $n=8s+3$ and $m=2t+1$.

i	j	k	
1+r	8s-r+4	8s-2r+3	$0 \leq r \leq 4s-t$
4s-t+2	4s-t+3	1	--------------
4s-t+r+4	4s+t-r+3	2t-2r-1	$0 \leq r \leq t-2$
8s+5	12s+3	4s-2	--------------
4s+r+2	12s-r+6	8s-2r+4	$1 \leq r \leq 2$
8s+2r+7	16s-2r+5	8s-4r-2	$0 \leq r \leq s-1$
8s+2r+6	16s-2r+2	8s-4r-4	$0 \leq r \leq s-1$
14s+3	14s+5	2	-------------
10s+r+6	14s-r+2	4s-2r-4	$0 \leq r \leq 2s-4$

<u>Case 5.</u> $n \equiv 4 \pmod 8$.

For m=2, by Lemma II.2(ii)(b), the existence of the sequence 1,1 gives solutions for all $n \geq 12$. We list the remaining cases of n=4,12 :

n=4 and m=2

4,,1,1,3,4,0,3

n=4 and m=4 is a hooked Skolem sequence of order 3.

n=12 and m=4

12,10,8,6,1,1,7,11,9,6,8,10,12,7,2,5,2,9,11,3,5,0,3

n=12 and m=6

12,10,8,1,1,4,7,11,9,4,8,10,12,7,2,5,2,9,11,3,5,0,3

n=12 and m=8

12,10,1,1,6,4,7,11,9,4,6,10,12,7,2,5,2,9,11,3,5,0,3

n=12 and m=10

12,1,1,8,6,4,7,11,9,4,6,8,12,7,2,5,2,9,11,3,5,0,3

n=12 and m=12 is a hooked Skolem sequence of order 11.

For $n \geq m > 2$ and $n > 12$, let $n = 8s+4$ and $m = 2t$.

i	j	k	
1+r	8s-r+5	8s-2r+4	$0 \leq r \leq 4s-t+1$
4s-t+3	4s-t+4	1	--------------
4s-t+r+5	4s+t-r+3	2t-2r-2	$0 \leq r \leq t-3$
8s+6	12s+3	4s-3	--------------
4s+r+2	12s-r+7	8s-2r+5	$1 \leq r \leq 3$
8s+2r+8	16s-2r+5	8s-4r-3	$0 \leq r \leq s-1$
8s+2r+7	16s-2r+2	8s-4r-5	$0 \leq r \leq s-1$
14s+3	14s+5	2	-------------
16s+4	16s+7	3	-------------
10s+2r+7	14s-2r+2	4s-4r-5	$0 \leq r \leq 2s-5$

<u>Case 6.</u> $n \equiv 5 \pmod 8$.

For m=2, Lemma II.2(ii)(b) similar to above will give solutions for all $n \geq 13$.

We need only to list all cases for $n=5$:

$n=5$ and $m=2$

4,5,1,1,4,3,5,0,3

$n=5$ and $m=4$

2,5,2,1,1,3,5,0,3

For all $n > m > 2$ and $n > 5$, let $n=8s+5$ and $m=2t$.

i	j	k	
1+r	8s-r+5	8s-2r+4	$0 \le r \le 4s-t+1$
4s-t+3	4s-t+4	1	--------------
4s-t+r+5	4s+t-r+3	2t-2r-2	$0 \le r \le t-3$
4s+3	8s+6	4s+3	--------------
4s+r+3	12s-r+10	8s-2r+7	$1 \le r \le 2$
8s+2r+8	16s-2r+9	8s-4r+1	$0 \le r \le s-1$
8s+2r+7	16s-2r+6	8s-4r-1	$0 \le r \le s-2$
10s+5	10s+7	2	-------------
10s+r+8	14s-r+9	4s-2r+1	$0 \le r \le 2s-1$

<u>Case 7.</u> $n \equiv 6 \pmod 8$. We list all cases of $n=6$.

$n=6$ and $m=3$

5,1,1,4,6,5,2,4,2,0,6

$n=6$ and $m=5$

1,1,3,4,6,3,2,4,2,0,6.

For all n,m where $n > m > 1$ and $n > 6$, let $n=8s+6$ and $m=2t+1$.

i	j	k	
1+r	8s-r+6	8s-2r+5	$0 \le r \le 4s-t+1$
4s-t+3	4s-t+4	1	--------------
4s-t+r+5	4s+t-r+4	2t-2r-1	$0 \le r \le t-2$
12s+7	16s+11	4s+4	--------------
4s+r+3	12s-r+11	8s-2r+8	$1 \le r \le 2$
10s+2r+7	14s-2r+9	4s-4r+2	$0 \le r \le s-1$
10s+2r+10	14s-2r+10	4s-4r	$0 \le r \le s-1$
10s+6	10s+8	2	-------------
8s+r+7	16s-r+9	8s-2r+2	$0 \le r \le 2s-2$

<u>Case 8.</u> $n \equiv 7 \pmod 8$.

For m=3, by Lemma II.2(iv)(a), the existence of the hooked Skolem sequence 1,1,2,0,2, will give solutions for all $n \ge 15$.

We list all the remaining cases for n=7,15 :

n=7 and m=3

4,2,7,2,4,5,6,1,1,7,5,0,6

n=7 and m=5

6,4,2,7,2,4,6,1,1,3,7,0,3

n=7 and m=7 is a hooked Skolem sequence of order 6.

n=15 and m=5

14,12,10,15,7,4,2,13,2,4,11,7,10,12,14,8,3,9,15,3,13,11,6,8,11,
9,0,6

n=15 and m=7

8,6,11,12,13,14,15,6,8,9,10,3,4,11,3,12,4,13,9,14,10,15,2,5,2,1,
1,0,5

n=15 and m=9

8,10,11,12,13,14,15,7,8,3,6,10,3,11,7,12,6,13,4,14,5,15,4,1,1,5,
2,0,2

n=15 and m=11

4,7,3,9,4,3,10,13,7,15,5,6,9,12,14,5,10,6,8,2,13,2,1,1,15,12,8,0,
14

n=15 and m=13

10,8,12,4,7,3,14,4,3,8,10,7,9,15,12,11,6,1,1,5,14,9,6,2,5,2,1,1,
0,15

n=15 and m=15 is a hooked Skolem sequence of order 14

For all n ≥ m > 3 and n > 15, let n=8s+7 and m=2t+1.

i	j	k	
*1+r	8s-r+8	8s-2r+7	0 ≤ r ≤ 4s-t+2
*4s-t+4	4s-t+5	1	--------------
*4s-t+r+6	4s+t-r+5	2t-2r-1	0 ≤ r ≤ t-3
*4s+7	8s+9	4s+2	-------------
8s+10	12s+6	4s-4	-------------
8s+11	12s+11	4s	-------------
8s+2r+13	16s-2r+13	8s-4r	0 ≤ r ≤ s-1
8s+2r+12	16s-2r+10	8s-4r-2	0 ≤ r ≤ s-2
*4s+r+3	12s-r+11	8s-2r+8	1 ≤ r ≤ 3
10s+10	14s+8	4s-2	-------------
10s+2r+13	14s-2r+7	4s-4r-6	0 ≤ r ≤ s-3
10s+2r+12	14s-2r+4	4s-4r-8	0 ≤ r ≤ s-4
14s+6	14s+10	4	------------
14s+9	14s+12	3	------------
*14s+11	14s+13	2	------------

For n=23 use only the lines (*), and then add the following (i,j) pairs:

(26,38),(27,31),(28,42),(29,45),(30,36),(35,43),(37,40),(39,41).

This completes the proof of Theorem II.4. ■

II.4. Possible Extensions.

The obvious generalization of the near-Skolem sequence is a near-Skolem sequence with more than one defect. For instance, a near-Skolem sequence of order n and defects m_1, m_2 ($m_1 < m_2 \leq n$) is a sequence $S' = (s_1, s_2, \ldots, s_{2n-4})$ of integers $s_i \in \{1,2,\ldots,m_1-1, m_1+1,\ldots, m_2-1, m_2+1,\ldots,n\}$, which satisfies the following conditions:

1) For every $k \in \{1,2,\ldots,m_1-1,m_1+1,\ldots,m_2-1,m_2+1,\ldots n\}$ there are exactly two elements $s_i, s_j \in S'$, such that $s_i=s_j=k$.

2) If $s_i=s_j=k$ then $j - i = k$.

For example, 7,5,2,6,2,3,5,7,3,6 is a (1,4)-near-Skolem sequence of order 7, and 1,1,7,8,3,5,2,3,2,7,5,8 is a (4,6)-near-Skolem sequence of order 8.

It is easy to see that the necessary conditions for the existence of such sequences are:

If $n \equiv 0,1 \pmod 4$ then m_1, m_2 must either be both odd or be both even, and if $n \equiv 2,3 \pmod 4$ then m_1 and m_2 must be of opposite parity. However, to prove that this condition is sufficient does not appear to be feasible with our methods.

Another interesting generalization is to consider adding rather than deleting one more difference. So an **excess-Skolem sequence** of order n and **surplus** h is a sequence $S''= (s_1,s_2,\ldots,s_{2n+2})$ of integers $s_i \in \{1,2,\ldots n\}$ which satisfies the following conditions:

1) For every $k \in \{1,2,\ldots,h-1,h+1,..n\}$ there are exactly two elements $s_i,s_j \in S''$, such that $s_i=s_j=k$ and $j - i = k$.

2) for k=h, there are exactly 4 elements $s_i,s_j,s_a,s_b \in S''$, such that $s_i=s_j=s_a=s_b=h$, and $j-i = b-a = h$.

For example, 5,7,1,1,3,5,7,3,7,8,6,4,2,7,2,4,6,8 is a 7-excess-Skolem sequence of order 8.

It is also easy to see that the necessary conditions for the existence of such sequences are:

For $n \equiv 0,1 \pmod 4$, h must be odd, and for $n \equiv 2,3 \pmod 4$, h must be even.

It is also interesting to note that the settling of the existence question for the extended Skolem sequences will imply the sufficiency of the above conditions for the excess-Skolem sequence.

CHAPTER III

DISJOINT SKOLEM AND HOOKED SKOLEM SEQUENCES

III.1 Introduction

In this chapter we show the existence of disjoint Skolem, hooked Skolem, (n,2)-Langford and near-Skolem sequences. We apply the results obtained to the problems of disjoint cyclic Steiner triple systems and Mendelsohn triple systems. We also discuss the relation between disjoint Skolem sequences and Room squares.

Two Skolem sequences S and S' (or two hooked Skolem sequences that have the same zero location, HS and HS') of order n are said to be **disjoint** if whenever $s_i=s_j=k$ and $s'_l=s'_t=k$, then $\{s_i,s_j\} \neq \{s'_l,s'_t\}$, for all $k=1,\ldots,n$.

If whenever $s_i=s_j=k$ and $s'_l=s'_t=k$, then $\{s_i,s_j\} \cap \{s'_l,s'_t\}=\emptyset$, for all $k=1,\ldots,n$. then the sequences are called **completely disjoint**.

Given a Skolem sequence $S=(s_1,s_2,\ldots,s_{2n})$, the **reverse** of S is $S_r = (s_{2n},s_{2n-1},\ldots,s_2,s_1)$, i.e. S_r is obtained by reading S backward as a Skolem sequence. If S and S_r are disjoint, then we call S **reverse-disjoint.** These definitions can be extended to the near-Skolem and hooked near-Skolem sequences.

Two m-near-Skolem sequences NS and NS' (or two hooked near-Skolem sequences that have the same zero location, HNS and HNS') of order n are said to be **disjoint** if whenever $s_i=s_j=k$ and $s'_l=s'_t=k$, then $\{s_i,s_j\} \neq \{s'_1,s'_t\}$, for all k=1,..,m-1,m+1,...,n.

If whenever $s_i=s_j=k$ and $s'_1=s'_t=k$, then $\{s_i,s_j\} \cap \{s'_1,s'_t\} = \emptyset$, for all k=1,..,m-1,m+1,...,n. then the two near-Skolem sequences are called **completely disjoint**.

Given a near-Skolem sequences NS $=(s_1,s_2,...,s_{2n-2})$, the reverse of NS is $NS_r = (s_{2n-2},s_{2n-3},...,s_2,s_1)$, If NS and NS_r are disjoint, then we call NS **reverse-disjoint.**

The above definitions can also be extended in the obvious manner to the perfect and hooked Langford sequences.

We give several examples to illustrate these definitions:

Two disjoint Skolem sequences of order 4:

3,4,2,3,2,4,1,1

2,3,2,4,3,1,1,4;

note that both are reverse-disjoint.

Two completely disjoint Skolem sequences of order 8:

4,2,6,2,4,8,5,7,6,1,1,5,3,8,7,3

6,1,1,7,8,3,6,4,3,5,7,4,8,2,5,2.

Two disjoint hooked Skolem sequences of order 7:

5,7,1,1,6,5,3,4,7,3,6,4,2,0,2

6,1,1,5.7,2,6,2,5,3,4,7,3,0,4.

Two disjoint 4-near-Skolem sequences of order 6:

6,3,5,2,3,2,6,5,1,1

5,1,1,6,3,5,2,3,2,6;

note that the second is reverse-disjoint but first is not.

Two completely disjoint hooked 3-near-Skolem sequences of order 6:

5,2,4,2,6,5,4,1,1,0,6

6,4,5,1,1,4,6,5,2,0,2.

Finally, two disjoint hooked Langford sequences of order 7 and d=2:

5,7,4,6,3,5,4,3,7,6,2,0,2

3,6,7,3,2,5,2,6,4,7,5,0,4.

It is natural to ask the question: what is the maximum number of mutually disjoint Skolem and hooked Skolem sequences of a given order n? It is easy to see that the maximum cannot exceed n. Since all the largest two numbers in any set of mutually disjoint Skolem sequences must occupy distinct positions, but the number of available positions is 2n (2n+1 and 2n-2 in cases of hooked and near-Skolem sequences, respectively), we cannot have more than n mutually disjoint (hooked) Skolem sequences or n-1 disjoint near-Skolem sequences.

Lemma III.1. The maximum number of mutually disjoint (hooked) Skolem sequences of order n is at most n. For near-Skolem sequences of order n, the maximum is at most n-1. ∎

Initial investigations suggest that this upper bound is probably attainable.

III.2 Main Results

In this section we show the existence of disjoint Skolem and disjoint hooked Skolem sequences by producing constructions that yield disjoint Skolem (hooked Skolem) sequences, or sequences disjoint to some of the known constructions of Skolem (hooked Skolem) sequences.

Theorem III.1 For all $n \equiv 0,1 \pmod 4$, $n \geq 4$, there exist at least 4 mutually disjoint Skolem sequences.

Proof: We present here two reverse-disjoint solutions that are disjoint with each other, thus producing 4 disjoint Skolem sequences.

Case 1. $n \equiv 0 \pmod 4$; there are two subcases.

Case 1.(a) $n \equiv 0 \pmod 8$, $n > 8$, let $n = 8s$.

Solution 1:

i	j	
r	8s-r+1	$1 \leq r \leq 4s-1$
4s	8s+2	----------
4s+1	12s+1	----------
12s+2	12s+3	----------
8s+2r-1	16s-2r+1	$1 \leq r \leq s-1$
10s-1	10s+1	---------
8s+2r+2	16s-2r+2	$1 \leq r \leq s-1$
10s+r+1	14s-r+3	$1 \leq r \leq 2s-1$

Solution 2:

i	j	
r	8s-r-1	1 ≤ r ≤ 4s-2
4s+r-2	12s-r+3	1 ≤ r ≤ 2
8s-1	16s-2	----------
8s+r-1	16s-r-3	1 ≤ r ≤ 2s-2
10s-2	10s-1	-----------
10s+r-1	14s-r-1	1 ≤ r ≤ 2s-2
12s	16s	-----------
16s-3	16s-1	-----------

For n=8 we give the solutions:

(3,4),(10,12),(2,5),(9,13),(11,16),(1,7),(8,15),(6,14).

(14,15),(9,11),(3,6),(12,16),(2,7),(4,10),(1,8),(5,13).

Case 1.(b) n ≡ 4 (mod 8), n > 4, let n = 8s+4.

Solution 1:

i	j	
r	8s-r+5	1 ≤ r ≤ 4s+1
4s+2	8s+6	-----------
4s+3	12s+7	----------
12s+5	12s+6	----------
8s+2r+3	16s-2r+9	1 ≤ r ≤ s
14s+8	14s+10	---------
8s+2r+6	16s-2r+10	1 ≤ r ≤ s-1
10s+r+4	14s-r+8	1 ≤ r ≤ 2s

Solution 2:

i	j	
1	4s+3	------------
r+1	8s-r+4	$1 \le r \le 4s+1$
4s+4	12s+8	-----------
8s+r+5	16s-r+9	$1 \le r \le 2s$
10s+6	10s+7	----------
10s+7+r	14s-r+9	$1 \le r \le 2s$

for n=4:

(1,2),(4,6),(5,8),(3,7).

(2,3),(6,8),(4,7),(1,5).

Case 2. $n \equiv 1 \pmod 4$, $n > 5$, let $n=4s+1$.

Solution 1:

i	j	
r	4s-r+3	$1 \le r \le 2s$
2s+1	4s+3	----------
2s+2	6s+2	----------
3s+r+3	8s-r+3	$1 \le r \le s-2$
7s+4	7s+3	----------
5s+r+1	7s-r+3	$1 \le r \le s$

Solution 2:

i	j	
r	4s-r+1	$1 \le r \le 2s-1$
2s	6s+1	-----------
2s+1	4s+1	-----------
4s+r+1	8s-r+3	$1 \le r \le s$
5s+r+1	7s-r+1	$1 \le r \le s-1$
7s+1	7s+2	------------

For n=5 we give the solutions:

(1,2),(7,9),(3,6),(4,8),(5,10).

(8,9),(3,5),(1,4),(6,10),(2,7).

To see that these solutions are reverse-disjoint, we check that all the values of (i,j) and their new positions in the reverse sequence (2n+1-j,2n+1-i) are distinct componentwise.

For example, the values (r,4s-r+3) will be become (8s+3-(4s-r+3),8s+3-r); these are unequal componentwise for all r = 1,2...,2s. ∎

Theorem III.2 For all $n \equiv 2,3 \pmod 4$, $n \ge 6$, there are at least 3 mutually disjoint hooked Skolem sequences of order n.

Proof.

Case 1. $n \equiv 2 \pmod 4$, $n > 10$.

We present below a solution which is disjoint with O'Keefe's solution (see Theorem I.2. Case 1.), and from a solution obtained from Stanton-Goulden solution [S16] for the 2-near-Skolem sequence of order n by appending to it the hooked sequence 2,0,2. We distinguish two subcases (a) and (b).

Case 1.(a) $n \equiv 2 \pmod 8$, let $n = 8s+2$ and $n > 10$.

i	j	
r	4s+3	------------
r+1	8s-r+4	$1 \le r \le 4s+2$
4s+2	12s+4	----------
8s+2r+2	16s-2r+4	$1 \le r \le s-1$
8s+2r+3	16s-2r+7	$1 \le r \le s$
10s+r+4	14s-r+6	$1 \le r \le 2s-1$
10s+2	10s+4	------------
12s+5	12s+6	-----------

For n = 10, we give the following 3 mutually disjoint hooked sequences:

5,3,8,10,3,5,2,7,2,6,8,9,4,10,7,6,4,1,1,0,9.

4,8,3,7,4,3,9,5,10,8,7,2,5,2,6,9,1,1,10,0,6.

9,6,10,1,1,5,7,6,8,9,5,4,10,7,3,4,8,3,2,0,2.

Case 1.(b) $n \equiv 6 \pmod 8$, let $n=8s+6$ and $n > 6$.

i	j	
1	4s+5	------------
r+1	8s-r+8	$1 \le r \le 4s+2$
4s+4	12s+10	----------
8s+2r+6	16s-2r+12	$1 \le r \le s$
8s+2r+7	16s-2r+15	$1 \le r \le s$
10s+r+7	14s-r+11	$1 \le r \le 2s$
11s+9	11s+10	------------
14s+11	14s+13	-----------

For n=6, we give the following 3 mutually disjoint hooked sequences:

5,2,4,2,6,5,4,1,1,3,6,0,3.

1,1,2,5,2,4,6,3,5,4,3,0,6.

2,3,2,6,3,5,1,1,4,6,5,0,4.

Case 2. $n \equiv 3 \pmod 4$.

In this case we give two solutions which are disjoint, and are also disjoint from O'Keefe's solution (Theorem I.2 Case 2) for $n \equiv 3 \pmod 4$, let $n = 4s+3$ and $n > 7$.

Solution 1:

i	j	
r	4s-r+4	$1 \le r \le 2s+1$
2s+2	6s+5	---------
4s+r+3	8s-r+6	$1 \le r \le s$
6s+6	8s+7	------------
5s+4	5s+5	------------
5s+r+5	7s+6-r	$1 \le r \le s-1$.

Solution 2:

i	j	
1+r	4s-r+4	$1 \le r \le s$
1	2s+2	---------
s+r+1	3s-r+2	$1 \le r \le s-1$
3s+2	3s+3	---------
2s+1	6s+4	--------
4s+2r+2	8s-2r+6	$1 \le r \le s$
4s+2r+3	8s-2r+9	$1 \le r \le s+1$

For n=7 we give the following 3 mutually disjoint hooked sequences:

4,2,6,2,4,5,3,7,6,3,5,1,1,0,7.

3,6,2,3,2,7,5,6,1,1,4,5,7,0,4.

6,4,7,5,3,4,6,3,5,7,1,1,2,0,2.

This completes the proof. ∎

As shown above, we checked the previously known results to see if the new properties we introduce are included in them. One of the interesting cases is Davies' solution of the perfect Langford sequences of order $n \equiv 0,1 \pmod 4$ and d=2. We find that Davies' solution for $n \equiv 0 \pmod 4$ is reverse-disjoint but for $n \equiv 1 \pmod 4$ is not. We include Davies' solution in the proof of the following theorem.

__Theorem III.3.__ For $n \equiv 0,1 \pmod 4$, there exists a reverse-disjoint perfect Langford sequence of order n and d=2.

__Proof.__ We distinguish two cases:

Case 1. $n \equiv 0 \pmod 4$, $n > 8$. (Davies) [D1].

We list here Davies' solution for $n \equiv 0 \pmod 4$, let $n = 4s$ and $n > 8$.

i	j	
r	4s-r-1	$1 \le r \le s-1$
s	5s-1	----------
s+r	3s-r	$1 \le r \le s$
2s	6s	----------
4s-1	6s-1	----------
4s+r-1	8s-r-1	$1 \le r \le s-1$
5s+r-1	7s-r	$1 \le r \le s-1$

For n=4 and 8 we give the following solutions:

3,4,2,3,2,4.

5,7,4,6,8,5,4,3,7,6,3,2,8,2.

Case 2. $n \equiv 1 \pmod 4$. We distinguish two subcases:

Case 2.(a) $n \equiv 1 \pmod 8$, let $n = 8s+1$.

i	j	
2r-1	8s-2r-3	$1 \le r \le s-1$
2r	8s-2r	$1 \le r \le s$
2s-1	2s+1	----------
2s+r+1	6s-r-1	$1 \le r \le 2s-3$
4s-1	8s-3	----------
4s+r-1	12s-r+1	$1 \le r \le 2$
8s-r	16s-r+1	$1 \le r \le 4s+1$

Case 2.(b) $n \equiv 5 \pmod 8$, $n > 5$, let $n = 8s+5$.

i	j	
2r-1	8s-2r+5	$1 \le r \le s+1$
2	4s+2	----------
2r+1	8s-2r+6	$1 \le r \le s$
2s+r+2	6s-r+2	$1 \le r \le 2s-2$
4s+1	12s+6	---------
4s+3	12s+5	----------
6s+2	6s+4	-----------
8s+r+4	16s-r+9	$1 \le r \le 4s+1$

For $n = 5$ we have the solution

5,2,4,2,3,5,4,3.

This completes the proof. ■

Theorem III.3 is not only valuable in showing the existence of disjoint sequences for a special case of Langford sequences but also will assist us in proving the following theorem for the near-Skolem sequences.

Theorem III.4. For all $n \geq m$, $n \geq 4$, there exists a reverse disjoint near-Skolem sequence of order n and defect m.

Proof. We observe that 6 of the 8 constructions given in Theorem II.3 are reverse-disjoint, thus providing solutions for the cases $n \equiv 1,2,4,5,6,7 \pmod 8$, and m either ≥ 2 or 3. We give below solutions for the remaining unsettled cases.

We note that for m=1, the 1-near-Skolem sequence is a perfect Langford sequence with d=2; by Theorem III.3. we proved the existence of such reverse-disjoint sequences for all $n \equiv 0,1 \pmod 4$, so we will omit all cases for m=1.

Case 1. $n \equiv 0 \pmod 8$.

The construction of Theorem II.3. Case.1. is not reverse-disjoint , so we present an alternative construction which is reverse-disjoint for all $n \geq m \geq 1$. Let n=8s, m=2t+1.

i	j	
8s+r-1	16s-r-2	$0 \le r \le 4s-t-2$
12s+t-2	12s+t-1	-------------
12s-t+r-2	12s+t-r-3	$0 \le r \le t-2$
4s+r-2	12s-r-2	$0 \le r \le 1$
2r-1	8s-2r-1	$1 \le r \le s$
2	4s	----------
2r+2	8s-2r	$1 \le r \le s-1$
2s+r	6s-r-2	$1 \le r \le 2s-3$
6s-2	6s	-----------

<u>Case 2.</u> $n \equiv 1 \pmod 8$.

All constructions and solutions for small cases given for Case 2. of Theorem II.3 are reverse-disjoint.

<u>Case 3.</u> $n \equiv 2 \pmod 8$.

All constructions and solutions for small cases given for Case 3. of Theorem II.3 are reverse-disjoint except in the cases n=10 and $10 \ge m \ge 4$; we list reverse-disjoint sequences here:

n=10 and m=4

8,6,7,2,10,2,9,6,8,7,1,1,5,3,10,9,3,5.

n=10 and m=6

8,9,4,10,1,1,4,7,8,3,9,5,3,10,7,2,5,2.

n=10 and m=8

7,9,4,10,1,1,4,7,6,3,9,5,3,10,6,2,5,2.

<u>Case 4.</u> $n \equiv 3 \pmod 8$.

The constructions given for all small cases in Case 4. of Theorem II.3. are reverse-disjoint except for n=11 and m=10 we give it here:

6,7,9,11,3,5,6,3,7,8,5,9,1,1,11,4,2,8,2,4.

Neither is the construction given for the cases $n \geq m > 2$ and $n > 11$, reverse disjoint. We give here an alternative construction which is reverse disjoint. Let n=8s+3 and m=2t.

i	j	
8s+r+2	16s-r+4	$0 \leq r \leq 4s-t$
12s+t+2	12s+t+3	------------
12s-t+r+3	12s+t-r+1	$0 \leq r \leq t-3$
4s+r-2	12s-r+3	$1 \leq r \leq 2$
8s	12s+3	-----------
2r	8s-2r+3	$1 \leq r \leq s-1$
2r-1	8s-2r	$1 \leq r \leq s$
2s+r-1	6s-r	$1 \leq r \leq 2s-1$
6s+1	6s+3	----------

<u>Case 5.</u> $n \equiv 4 \pmod 8$.

All the constructions given in Case 5. of Theorem II.3 are reverse-disjoint.

<u>Case 6.</u> $n \equiv 5 \pmod 8$.

All the constructions given in Case 6. of Theorem II.3 are reverse-disjoint.

Case 7. $n \equiv 6 \pmod 8$.

All the constructions given in Case 7. of Theorem II.3 are reverse-disjoint.

Case 8. $n \equiv 7 \pmod 8$.

All the constructions given in Case 8. of Theorem II.3 are reverse-disjoint. ■

III.3 Applications.

III.3.1 Disjoint Cyclic Steiner and Mendelsohn Triple Systems

Recall from Chapter I that a Skolem sequence yields a solution of (a restricted version of) Heffter's first problem since, given a partition of the set $\{1,2,..,2n\}$ into distinct pairs (a_i,b_i) such that $b_i=a_i+i$, $i=1,....n$, the triples (i,a_i+n,b_i+n), $i=1,...,n$, give the required solution to this problem. Then $\{0,i,b_i+n\}$, $i = 1,...,n$ will be the base blocks for a cyclic STS(6n+1).

One may observe that also $\{0,a_i+n,b_i+n\}$ $i=1,...n$, is (another) set of base blocks of an STS(6n+1). Two STS(v)'s are called **disjoint** if they have no blocks in common. It follows that two cyclic STS(v)'s are disjoint if they have no base block in common. Thus the existence of a Skolem or a hooked Skolem sequence of order n implies the existence of two mutually disjoint STS(6n+1) [C4]. Denote the number of mutually disjoint STS(v)'s and cyclic STS(v)'s by $n(v)$ and $nc(v)$ respectively. Observe that a Skolem sequence of order n disjoint from the given one, will produce another two sets of base blocks $\{0,i,b'_i+n\}$ and $\{0,a'_i+n,b'_i+n\}$, $i = 1,2,\ldots n$, thus another two cyclic STS(6n+1), and all four systems are disjoint. Thus we have two corollaries of Theorems III. 1, III.2.

Corollary III. . For all $v \geq 25$ and $v \equiv 1,7 \pmod{24}$, $nc(v) \geq 8$.

Corollary III.2 For all $v \geq 37$ and $v \equiv 13,19 \pmod{24}$, $nc(v) \geq 6$.

A **Mendelsohn triple system** of order v, MTS(v), is a pair (V,B) where V is a v-set and B is a collection of cyclic triples on V (i.e. a triple <a,b,c> that contains the ordered pairs (a,b), (b,c), and (c,a)); such that every ordered pair of distinct elements from V appears in precisely one triple. Cyclic MTS(v) is defined similarly to cyclic STS(v). We observe that a cyclic Mendelsohn triple system can be obtained from a cyclic STS(v) by replacing any base block {a,b,c} of the cyclic STS(v) by the two base cyclic triples <a,b,c> and <a,c,b>. Thus we can also improve on the best known bounds for the numbers of disjoint cyclic Mendelsohn triple systems [C5].

Denote the number of mutually disjoint Mendelsohn triple systems, and disjoint cyclic Mendelsohn triple systems of order v by *m(v)* and *mc(v)*, respectively. Theorems III.1 and III.2, give us also another improvement on the results obtained in [C5].

Corollary III.3. For all $v \geq 25$ and $v \equiv 1,7 \pmod{24}$, $mc(v) \geq 8$.

Corollary III.4 For all $v \geq 37$ and $v \equiv 13,19 \pmod{24}$, $mc(v) \geq 6$.

For example, the sequence 1,1,4,2,3,2,4,3 will give the solution for the first Heffter problem:

(1,5,6),(2,8,10),(3,9,12),(4,7,11).

This will give the base blocks for two disjoint cyclic STS(25)

1) {0,1,6},{0,2,10},{0,3,12},{0,4,11} (mod 25)

2) {0,5,6},{0,8,10},{0,9,12},{0,7,11} (mod 25).

Now the sequence 3,4,2,3,2,4,1,1 is disjoint with the first one (reverse-disjoint) and will give another two disjoint cyclic STS(25)

3) {0,1,12},{0,2,9},{0,3,8},{0,4,10} (mod 25).

4) {0,11,12},{0,7,9},{0,5,8},{0,6,10} (mod 25).

III.3.2 Disjoint Skolem sequences and Room Squares.

Let S be a set of n+1 elements. A **Room square** of side n is an n by n array, R, which satisfies the following conditions:

1) every cell of R is either empty or contains an unordered pair of symbols from S,
2) each element of S occurs once in each row and column of R,
3) every unordered pair of elements occurs in precisely one cell.

Room squares of side n are known to exist for all odd n, $n \neq 3$ or 5.

It is also known that the existence of a Room square of side n is equivalent to the existence of two orthogonal symmetric latin squares of side n, and to the existence of two orthogonal one-factorizations of the complete graph K_{n+1}.

In this part we establish the relationship between Skolem sequences and Room squares. In 1968 Stanton and Mullin [S17] introduced the orthogonal starters method (or equivalently the starter-adder method). Let G be an abelian group of order 2n+1, A **starter** in G is a set of unordered pairs $S = \{ \{s_i,t_i\} : 1 \le i \le n \}$ such that:

1) $\{ s_i : 1 \le i \le n \} \cup \{ t_i : 1 \le i \le n\} = G \setminus \{0\}$

2) $\{ \pm(s_i - t_i) : 1 \le i \le n\} = G \setminus \{0\}$.

A starter $S = \{ \{s_i,t_i\} : 1 \le i \le n \}$ is said to be **strong** if $s_i+t_i=s_j+t_j$ implies i=j. Let $S = \{ \{s_i,t_i\} : 1 \le i \le n \}$ and $T = \{ \{u_i,v_i\} : 1 \le i \le n \}$ be two starters in G. Without loss of generality we may assume that

$s_i-t_i=u_i-v_i$ for all i. Then S and T are said to be **orthogonal** starters if $u_i-s_i=u_j-s_j$ implies $i = j$, and if $u_i \ne s_i$ for all i. Let $S = \{ \{s_i,t_i\} : 1 \le i \le n \}$ be a starter; a set $A = \{ \{a_i\} : 1 \le i \le n \}$ is said to be an **adder** for S if the elements of A are non-zero and distinct, and the set $S + A = \{\{s_i + a_i, t_i + a_i \} : 1 \le i \le n \}$ is again a starter. Recall that a Skolem sequence of order n is equivalent to a partition of the numbers $\{1,2,\ldots,2n\}$ into pairs (a_i,b_i) such that $b_i-a_i = i$ for $i = 1,\ldots,n$. Hence if we take the abelian group G to be Z_{2n+1} then the set $SS=\{ \{a_i, b_i\} : 1 \le i \le n \}$ will form a starter for Z_{2n+1}. We call it **Skolem starter**. As for all $n \equiv 0,1 \pmod 4$ there exists a Skolem sequence, we have:

Lemma III.2 [Skolem] For all $n \equiv 0,1 \pmod 4$, there exists a Skolem starter for the group Z_{2n+1}. ∎

For the cases $n \equiv 2,3 \pmod 4$, we define a **pseudo-Skolem sequence** PS of order n to be a sequence $(b_1,b_2, \ldots, b_{2n})$ of length 2n, in which each of the elements of the set $\{a_1,a_2,\ldots,a_n\}$, where $a_i = i$ or $-i \pmod{2n+1}$, occurs exactly twice, and if $b_i=b_j=k$ then $j-i=k$. Let the **defect D** of a pseudo-Skolem sequence be $|\{ k: k \in PS, k > n\}|$; obviously $0 \le D \le n$.

Theorem III.5 For every n there exists a pseudo-Skolem sequence of order n and defect $D \le 1$.

Proof. If $n \equiv 0,1 \pmod 4$, there is a Skolem sequence which has a defect 0.

If $n \equiv 2,3 \pmod 4$, a hooked Skolem sequence $b_1,b_2,\ldots, b_{2n-1},b_{2n+1}$ exists. If $b_{2n+1} = k$, then the sequence will also contain $b_{2n+1-k} = k$ (cf. [R2]). Then the hooked sequence can be transformed to $2n+1-k,b_1,b_2,\ldots,b_{2n-k},2n+1-k,b_{2n-k+2},\ldots,b_{2n-1}$, which is a pseudo-Skolem sequence with defect 1. ∎

We also notice that two disjoint Skolem sequences of order n will give two different partitions for the numbers $\{1,2,\ldots,2n\}$ into pairs (a_i,b_i) and (u_i,v_i), $i=1,\ldots n$. This will give two **orthogonal Skolem starters** if the following condition is satisfied :

if $a_i-u_i= a_j-u_j$ then $i=j$, and $a_i \neq u_i$ for all $1 \le i,j \le n$. For example, the disjoint Skolem sequences

1,1,4,2,3,2,4,3

4,1,1,3,4,2,3,2

will yield the two orthogonal Skolem starters

$S = \{\{1,2\},\{4,6\},\{5,8\},\{3,7\}\}$

$T = \{\{2,3\},\{6,8\},\{4,7\},\{1,5\}\}$

which will give the following Room square of side 9

{0,∞}	{5,8}	{3,7}					{4,6}	{1,2}
{2,3}	{1,∞}	{6,0}	{4,8}					{5,7}
{6,8}	{3,4}	{2,∞}	{7,1}	{5,0}				
	{7,0}	{4,5}	{3,∞}	{8,2}	{6,1}			
		{8,1}	{5,6}	{4,∞}	{0,3}	{7,2}		
			{0,2}	{6,7}	{5,∞}	{1,4}	{3,8}	
				{1,3}	{7,8}	{6,∞}	{2,5}	{0,4}
{1,5}					{2,4}	{8,0}	{7,∞}	{3,6}
{4,7}	{2,6}					{3,5}	{0,1}	{8,∞}

This leads to.

Lemma III.3 The existence of two disjoint Skolem sequences of order n with the added condition: if $a_i - u_i = a_j - u_j$ then $i=j$, and $a_i \neq u_i$ for all $1 \le i,j \le n$, implies the existence of a Room square of side $2n+1$.

We call two disjoint Skolem sequences with the above condition **orthogonal** Skolem sequences.

Conjecture: For all $n \equiv 0,1 \pmod 4$, there exist two orthogonal Skolem sequences of order n.

It is also obvious that a Skolem sequence with the added condition:

$$a_i+b_i = a_j+b_j \text{ implies } i = j \text{ , for } 1 \le i,j \le n,$$

gives a **strong Skolem starter**.

There are 6 Skolem sequences of order 4 but, of course, there is no strong Skolem starter for Z_9 (in fact, there is no strong starter for Z_9, see [D3]). However, there exist strong Skolem starters for all admissible orders $n \le 28$.

Lemma III.4 For all $5 \le n \le 28$, $n \equiv 0,1 \pmod 4$, there exist a strong Skolem starter for the group Z_{2n+1}.

Proof

We present below the Skolem sequences that give the strong Skolem starters.

n=5

5,2,4,2,3,5,4,3,1,1

n=8

5,6,7,8,2,5,2,6,4,7,3,8,4,3,1,1

n=9

1,1,9,6,2,5,2,7,8,6,5,9,4,7,3,8,4

n=12

4,5,11,8,4,10,5,7,9,12,2,8,2,11,7,10,6,9,1,1,3,12,6,3

n=13

8,9,10,11,12,13,7,4,8,6,9,4,10,7,11,6,12,5,13,2,3,2,5,3,1,1

n=16

11,12,13,14,15,16,6,7,8,9,10,11,6,12,7,13,8,14,9,15,10,16,5,2,4,
2,3,5,4,3,1,1

n=17

11,12,13,14,15,16,17,2,6,2,8,11,10,12,6,13,9,14,8,15,7,16,10,17,
5,9,4,7,3,5,4,3,1,1

n=20

13,14,15,16,17,18,19,20,8,3,12,7,3,13,10,14,8,15,7,16,12,17,12,
18,10,19,6,20,5,9,4,11,6,5,4,1,1,2,9,2

n=21

13,14,15,16,17,18,19,20,21,11,12,5,6,13,10,14,5,15,6,16,11,17,12,
18,10,19,9,20,8,21,3,7,2,3,2,9,8,4,7,1,1,4

n=24

16,17,18,19,20,21,22,23,24,6,7,8,11,1,15,6,16,7,17,8,18,14,19,11,
20,12,21,10,22,15,23,5,24,9,13,14,5,10,2,4,2,3,9,4,3,1,1,13

n=25

16,17,18,19,20,21,22,23,24,5,9,25,3,10,5,3,16,4,17,9,18,4,19,10,
20,15,21,8,22,14,23,7,24,11,13,8,25,12,7,6,15,1,1,14,11,6,2,13,
2,12

n=28

18,19,20,21,22,23,24,25,26,27,28,6,7,3,15,8,3,6,18,7,19,16,20,8,
21,10,22,14,23,15,24,9,25,13,26,12,27,16,28,11,9,14,17,5,10,4,13,
12,5,4,11,1,1,2,10. ■

Strong Skolem starters are likely to exist for all $n \geq 5$.

CHAPTER IV

INDECOMPOSABLE SKOLEM SEQUENCES

IV.1 Introduction

A **2-fold Skolem sequence** of order n is a sequence $(s_1, s_2, \ldots, s_{4n})$ of 4n integers satisfying the following conditions:

1. for every $k \in \{1,2,\ldots,n\}$ there exist exactly 4 elements $s_i, s_j, s_l, s_m \in \{s_1, s_2, \ldots, s_{4n}\}$ such that $s_i = s_j = s_l = s_m = k$.
2. if $s_i = s_j = s_l = s_m = k$ $(i < j < l < m)$ then $|i-j| = |l-m| = k$ or $|i-l| = |j-m| = k$.

An **m-fold Skolem sequence** of order n is a sequence $(s_1, s_2, \ldots, s_{2mn})$ of 2mn integers satisfying the following conditions:

1. for every $k \in \{1,2,\ldots,n\}$ there exist exactly 2m elements $s_{i1}, s_{i2}, \ldots, s_{i(2m)} \in \{s_1, s_2, \ldots, s_{2mn}\}$ such that $s_{i1} = s_{i2} = \ldots = s_{i(2m)} = k$.
2. if $s_{ij} = k$ for $j = 1,2,\ldots,2m$, then there exist a perfect matching $M = \{\{t_q, t'_q\} : q = 1,2\ldots,m;\ \cup^{m}_{q=1} \{t_q, t'_q\} = \{s_{ij} : j = 1,2,\ldots,2m\}\}$ such that $|t_q - t'_q| = k$, for $q = 1,\ldots,m$.

Similarly, an **m-fold extended Skolem sequence** of order n is a sequence $(s_1, s_2, \ldots, s_{2mn+1})$ of 2mn+1 integers satisfying the following conditions:

1. For every $k \in \{1,2,\ldots,n\}$ there exist exactly 2m elements $s_{i1}, s_{i2}, \ldots, s_{i(2m)} \in \{s_1, s_2, \ldots, s_{2mn+1}\}$ such that $s_{i1} = s_{i2} = \ldots =$

$s_{i(2m)} = k$.

2. if $s_{ij} = k$ for $j = 1,2,\ldots,2m$, then there exist a perfect matching $M = \{\{t_q,t'_q\} : q = 1,2\ldots,m;\ \cup^{m}_{q=1} \{t_q,t'_q\} = \{s_{ij} : j = 1,2,\ldots,2m\}\}$ such that $|t_q - t'_q| = k$, for $q = 1,\ldots,m$ $(1 \le k \le n)$.

3. There is exactly one subscript x such that $s_x=0$.

An **m-fold hooked Skolem sequence** satisfies conditions 1. and 2. for an m-fold extended Skolem sequence and the following condition:

3'. $s_{2mn} = 0$ (i.e. there is exactly one subscript, namely $x = 2mn$ such that $s_x = 0$).

A **t-indecomposable m-fold (extended)(hooked) Skolem sequence** is an m-fold (extended)(hooked) Skolem sequence such that, for all subscripts i,j , $(1 \le i < j \le 2mn)$, the subsequence $(s_i,s_{i+1},\ldots,s_j)$ is not a t-fold (extended)(hooked) Skolem sequence of order L $(n \ge L > 1)$, for some $t < m$.

If an m-fold (extended) (hooked) Skolem sequence is t-indecomposable for all $t < m$, then it is called a **totally indecomposable m-fold** (extended)(hooked) Skolem sequence or simply **indecomposable**. The above definitions could be extended to Langford sequences (perfect or hooked) in the obvious manner.

Necessary conditions for the existence of m-fold Skolem sequences

We use the notation (a_{ij},b_{ij}) for (t_q,t'_q).

Thus (1) $$\sum_{\substack{i=1\\ j=1}}^{n,\,2m} (b_{ij} - a_{ij}) = mn(n+1)/2$$

$$(2) \quad \sum_{\substack{i=1\\j=1}}^{n,2m} (b_{ij} + a_{ij}) = 2mn(2mn+1)/2$$

adding (1) and (2) then dividing by 2, we get

$$(3) \quad \sum_{\substack{i=1\\j=1}}^{n,2m} b_{ij} = mn(4mn+n+3)/4$$

The left side of (3) is always an integer, thus the necessary condition is:

If m is even then m-fold Skolem sequences may exist for all n, and if m is odd, n must be $\equiv 0,1 \pmod 4$.

The existence of m-fold (hooked) (extended) Skolem sequences

a) m-fold Skolem sequences

For $n \equiv 0,1 \pmod 4$ and all m, an m-fold Skolem sequence trivially exists, by simply placing m Skolem sequences side by side.

We need to consider the case $n \equiv 2,3 \pmod 4$, m even. This case is also easy; we build an m-fold Skolem sequence from m/2 hooked Skolem sequences and their reverses, always inserting the first member of the reverse sequence in the hook of the sequence. The last member of the sequence will fit in the hook of the reverse. Then we place the m/2 combinations of sequence and reverse side by side to form the m-fold Skolem sequence.

For example, the hooked sequence 1,1,2,3,2,0,3 and its reverse 3,0,2,3,2,1,1 will form a 2-fold Skolem sequence 1,1,2,3,2,3,3,2,3,2,1,1.

b) m-fold hooked Skolem sequences

These only exist if $n \equiv 2,3 \pmod 4$, and m is odd. We can easily build them by using the same m/2 combinations of a hooked sequence and its reverse, used in part a), and add at the end a hooked sequence to make a (2m+1)-fold hooked Skolem sequence.

The example of part a) will give the 3-fold hooked Skolem sequence 1,1,2,3,2,3,3,2,3,2,1,1,1,1,2,3,2,0,3, when the hooked sequence is added at the end.

c) m-fold extended Skolem sequences

We know that extended Skolem sequences exist for all n. An easy construction given in [A3] is as follows:

$(P_n, P_n-2, \ldots, 2, 0, 2, \ldots, P_n-2, P_n, Q_n, Q_n-2, \ldots, 3, 1, 1, 3, \ldots, Q_n-2, Q_n)$

where P_n, Q_n are the largest even and odd number not exceeding n, respectively. For $n \equiv 0,1 \pmod 4$ and all m, an m-fold extended Skolem sequence could be formed by taking an (m-1)-fold Skolem sequence from part a), followed by construction of [AK] given above.

For $n \equiv 2,3 \pmod 4$ and m odd, b) is a special case of the m-fold extended Skolem sequence.

For $n \equiv 2,3 \pmod 4$ and m even we need the following construction for a 2-fold extended Skolem sequence:

if n is even, take

n,n-2,....,4,2,n,2,4,...,n-2,n,n-2,n-4,...,4,2,n,2,4,...,n-4, n-2,n-1,n-3,...,3,1,1,3,..,n-3,n-1,0,n-1,n-3,...,3,1,1,3,..., n-3,n-1.

and if n is odd, take

n-1,n-3,...,4,2,n-1,2,4,...,n-3,n-1,n-3,n-5,...,4,2,n-1,2,4,...,
n-5,n-3,n,n-2,...,3,1,1,3,...,n-2,n,0,n,n-2,...,3,1,1,3,...,n-2,
n.

For higher m keep adding copies of this 2-fold Skolem sequence as many times as required.

IV.2 The Existence of Indecomposable m-fold Skolem Sequences.

It is easy to show the existence of 1-indecomposable m-fold Skolem sequences if m is even: we use the construction for m=2, given at the end of the last section. For higher m, repeat adding the even part of the above construction at the beginning of the sequence and the odd part at the end as many times as necessary. Thus we have:

Theorem IV.1. For all even m and all $n \geq 2$, there exists a 1-indecomposable m-fold Skolem sequence. ∎

For odd m, we need to consider first the case m=3.

Theorem IV.2. For m = 3 and $n \equiv 0,1 \pmod 4$, $n \geq 4$, there exist a 3-fold 2-indecomposable Skolem sequence.

Case 1. $n \equiv 0 \pmod 4$; let n=4s

a_{1i}	b_{1i}	
i	4s-i+2	$1 \leq i \leq 2s$
8s+i+1	12s-i-1	$1 \leq i \leq 2s-1$
20s-2	24s-1	----------

a_{2i}	b_{2i}	
2s+1	6s+1	---------
4s+i+1	8s-i+1	$1 \le i \le 2s-1$
12s+i-2	16s-i-1	$1 \le i \le 2s$

a_{3i}	b_{3i}	
21s	24s	---------
20s+i	24s-i-1	$1 \le i \le 2s-2$
18s	22s-1	----------
16s+i-2	20s-i-2	$1 \le i \le 2s-2$
18s-1	18s+1	----------
18s-3	18s-2	----------

Case 2. $n \equiv 1 \pmod 4$; let $n = 4s+1$.

a_{1i}	b_{1i}	
1	4s-i+2	$1 \le i \le 2s$
8s+i	12s-i-3	$1 \le i \le s$
19s+1	21s	----------
20s+i-1	24s-i+2	$1 \le i \le s$

a_{2i}	b_{2i}	
2s+1	6s+1	---------
4s+i+1	8s-i+1	$1 \le i \le 2s-1$
12s+i-4	16s-i-1	$1 \le i \le 2s+1$

a_{3i}	b_{3i}	
16s+i+2	18s-i-1	$1 \le i \le s$
22s+i+1	24s-i+7	$1 \le i \le s$
21s+i+1	23s-i+7	$1 \le i \le s+1$
18s+i-1	20s-i	$1 \le i \le s$

■

Corollary IV.1. For odd m and $n \equiv 0,1 \pmod 4$, $n \ge 4$, there exists a 2-indecomposable m-fold Skolem sequence.

Proof. Take the even part of the construction of Theorem IV.1 and place corresponding sequence before the 3-fold indecomposable solution given in Theorem IV.2, and place the odd part at the end; we obtain an m-fold 2-indecomposable Skolem sequence for all odd m. ■

Theorem IV.3 For $m \equiv 0 \pmod 6$ and all n, there exists an indecomposable m-fold (extended) Skolem sequence.

Proof

Let $m = 6s$, and let E_n, O_n be the largest even and odd numbers in the sequence, respectively. Arrange the numbers in the following manner:

$E_n, E_n-2, \ldots, 4, 2, E_n, 2, 4, \ldots, E_n-2, E_n, E_n-2, E_n-4, \ldots, 4, 2, E_n, 2, 4, \ldots, E_n-4, E_n-2$

repeated 3s times, followed by

$O_n, O_n-2, \ldots, 5, 3, O_n, O_n, 3, 5, \ldots, O_n, O_n-2, \ldots, 5, 3, O_n, O_n, 3, 5, \ldots, O_n-2, O_n$

repeated 2s times, followed by

$1,1,O_n-2,O_n-4,\ldots,3,1,1,3,\ldots,O_n-4,O_n-2,1,1$

repeated 2s times.

This will give an indecomposable m-fold Skolem sequence. To find an m-fold indecomposable extended Skolem sequence, we insert a zero following any one of the above segments. ■

Billington in an earlier work [M5] modified the notion of pairings (cf. [S16]) to include hooks and m-fold sequences. For example, $p^2(1,4)/4 - \{5,13\}$ denotes a 2-fold Skolem sequence of order 4 and defect 4 with two hooks in the locations 5 and 13:

1,1,1,1,0,4,2,3,2,4,3,3,0,2,3,2.

She proved that $p^2(1,n)/n - \{n+1,n+2\}$ (i.e. a 2-fold sequence of order n with one missing distance n, and two hooks at locations n+1 and n+2) exists if and only if n is even, and used this result to show the existence of balanced ternary designs with block size three (designs that allow repeated elements to exist in one block).

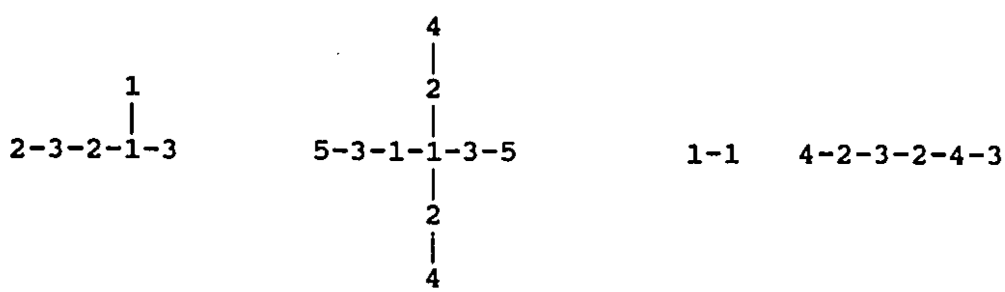
Thus there are numerous possibilities of combining the ideas of the previous chapters and this one, and finding or creating new applications to them.

CHAPTER V

SKOLEM LABELLING OF GRAPHS

V.1. Introduction

Given a Skolem sequence of order 4, say, 4,1,1,3,4,2,3,2, it is natural to think of 4-1-1-3-4-2-3-2 as a labelling of a 7-path. Fig V.1, shows that the idea can be extended to graphs.



(a) (b) (c)

Fig V.1

Note that Fig V.1(c) shows the necessity of some degeneracy conditions - we wish a valid labelling of the disconnected graph which is the union of a 1-path and 5-path to be distinct from the labelling of a 7-path.

Analogous to the use of BIBD's in experimental designs this labelling problem can be seen as one of testing reliability of a network where error propagation rate is dependent on number of nodes used but not necessarily directly. We place test equipment at fixed distances within the network one per node and each is sending to/or receiving from one specific location. We want all distances represented and all nodes used exactly once. Further, each edge should be tested.

Definition V.1 Let n and d be integers, n > d > 0. A **d-Skolem labelled graph** is a triple (G,L,d), where

(a) G = (V,E) is an undirected graph

(b) L : V ---> {d,d+1,...,d+n-1}

(c) L(v)=L(w)=d+i exactly once for i=0,1,...d+n-1 and d(v,w) = d+i

(d) If G'=(V,E') and E'⊂ E then (G',L,d) violates (c).

Definition V.2 Let n and d be integers, n > d > 0. A **d-hooked Skolem labelled graph** is a triple (G,L,d) satisfying (a) and (d) of definition V.1 and

(b') L:V ---> {{0} U {d,d+1,...,d+n-1}}

(c') If L(v) ≠ 0 then (c).

That is, hooked graphs can have some vertices labelled 0. Note that condition (d) in the above definitions requires that for any edge e, G-{e} is not Skolem labelled. Given a graph G and a labelling of its vertices. An edge is **essential** if the labelling is a valid labelling of G but not of G-{e}. The condition (d) stipulates that in a Skolem labelled graph every edge is essential. If condition (d) is not satisfied but conditions (a) to (c) are, the triple (G,L,d) will be called a **weak** (hooked) Skolem labelled graph. A **minimum** hooked labelling of G is a one with as few hooks as possible.

In this chapter we show the following:

(1) Any tree can be embedded in a Skolem labelled tree with O(2V) vertices.

(2) Any graph can be embedded as an induced subgraph in a Skolem labelled graph on $O(v^3)$ vertices.

(3) For d=1, we exhibit a Skolem or a minimum hooked Skolem (with as few hooks as possible) labelling for paths and cycles. There are at most two hooks.

(4) For d=1, we exhibit the minimum Skolem labelled graph containing a path or a cycle of length n as an induced subgraph.

(5) For d=1, we exhibit a Skolem or a minimum hooked Skolem labelling of all n-windmills.

We first summarize the main known results in terms of d-Skolem labelled graphs. Note that d=1 is understood if not explicitly stated otherwise. Property (d) must be checked as it is not part of the original quoted theorems.

Theorem V.1 Skolem[S13] If $n \equiv 0,1 \pmod 4$, a path of length $2n-1$ can be Skolem labelled.

Theorem V.2 Davies[D1] If $n \equiv 2,3 \pmod 4$, $n \neq 2$ then a path of length $2n-3$ $\{v_1, v_2, .., v_{2n-3}, v_{2n-2}\}$ can be 2-hooked Skolem labelled with $L(v_{2n-3})=0$

Theorem V.3 Davies[D1] If $n \equiv 0,1 \pmod 4$, $n \neq 2$ then a path of length $2n-3$ can be 2-Skolem labelled.

Corollary Davies[D1] If $n \equiv 2,3 \pmod 4$ then the disjoint union of a path of length $2n-3$ with a path of length 1 can be Skolem labelled.

Theorem V.4 Simpson[S12] A path of length $2m-1$ can be d-Skolem labelled if and only if

(i) $m \geq 2d-1$

(ii) $m \equiv 0,1 \pmod 4$ for d odd

(iii) $m \equiv 0,3 \pmod 4$ for d even.

Theorem V.5 Simpson[S12] For $d > 2$, a path of length $2m$ with a label 0 only at the next to the last vertex can be d-hooked Skolem labelled if and only if

(i) $m(m+1-2d) + 2 \geq 0$

(ii) $m \equiv 2,3 \pmod 4$, d odd

(iii) $m \equiv 1,2 \pmod 4$, d even.

Theorem V.6 Rosa[R2] For $n \equiv 0,3 \pmod 4$ a path of length 2n+1 has a minimum hooked Skolem labelling with label 0 at the middle.

Theorem V.7 O'Keefe [O] For $n \equiv 2 \pmod 4$ a path of length 2n+1 has a minimum hooked Skolem labelling with label 0 on the vertex next to the last.

We shall prove the following:

Theorem V.8 Every tree is an induced subgraph of a Skolem labelled tree on O(2v) vertices.

Theorem V.9 Every tree is an induced subgraph of d-Skolem labelled tree on O(2v+4m') vertices where m' is the minimum value of m obtainable from d in Theorems V.4 and V.5.

Theorem V.10 Every graph is an induced subgraph of a Skolem labelled graph on $O(v^3)$ edges.

Theorem V.11 Every graph is an induced subgraph of a d-Skolem labelled graph on $O((v+d)^3)$ vertices, where v is the number of vertices.

Theorem V.12 Every path of length m, with the exception of {2,3,4,5,6}, can be

(a) Skolem labelled if and only if $m \equiv 1,7 \pmod 8$.

(b) hooked Skolem labelled if and only if m is even.

(c) minimum hooked Skolem labelled with two hooks if and only if $m \equiv 3,5 \pmod 8$.

Theorem V.13 Every cycle of length $m \geq 13$ can be

(a) Skolem labelled if and only if $m \equiv 0,2 \pmod 8$

(b) hooked Skolem labelled if and only if m is odd.

(c) minimum hooked Skolem labelled with two hooks if and only if $m \equiv 4,6 \pmod 8$.

Theorem V.14 The graph on the fewest vertices containing an induced path of length m which is Skolem labelled has m vertices if $m \equiv 1,7 \pmod 8$, m+1 vertices if m even, and m+2 vertices if $m \equiv 3,5 \pmod 8$ (for $m \leq 8$ see Fig.V.2).

Theorem V.15 The graph with the fewest vertices that contains an induced m-cycle, $m \geq 14$, and is Skolem labelled is:

(a) the cycle with one pendant edge added if m is odd

(b) the cycle itself if $m \equiv 2,4 \pmod 8$.

(We conjecture that the only 2-connected Skolem labelled graphs aɪe these cycles).

(c) the cycle with two pendant edges added

if $m \equiv 0,6 \pmod 8$.

(for $3 \leq m \leq 13$ see Fig V.3).

Theorem V.16 The necessary conditions are sufficient for the Skolem or minimum hooked Skolem labelling of all n-windmills.

Embeddings of paths of length n, $n \leq 8$

<table>
<tr><td>n = 1

1 − 1</td><td>n = 5

1
|
4 − 1 − 3 − 2 − 4 − 3
|
2</td></tr>
<tr><td>n = 2

2 − 1 − 1
|
2</td><td>n = 6

2 − 3 − 2 − 4 − 3 − 1 − 1 − 4</td></tr>
<tr><td>n = 3

2
|
3 − 1 − 1 − 3
|
2</td><td>n = 7

2 − 3 − 2 − 4 − 3 − 1 − 1 − 4</td></tr>
<tr><td>n = 4

2 − 3 − 1 − 1 − 3
|
2</td><td>n = 8

2 − 4 − 2 − 3 − 5 − 4 − 3 − 1 − 1 − 5</td></tr>
</table>

Fig. V.2

Embeddings of cycles of length n, $3 \leq n \leq 13$:

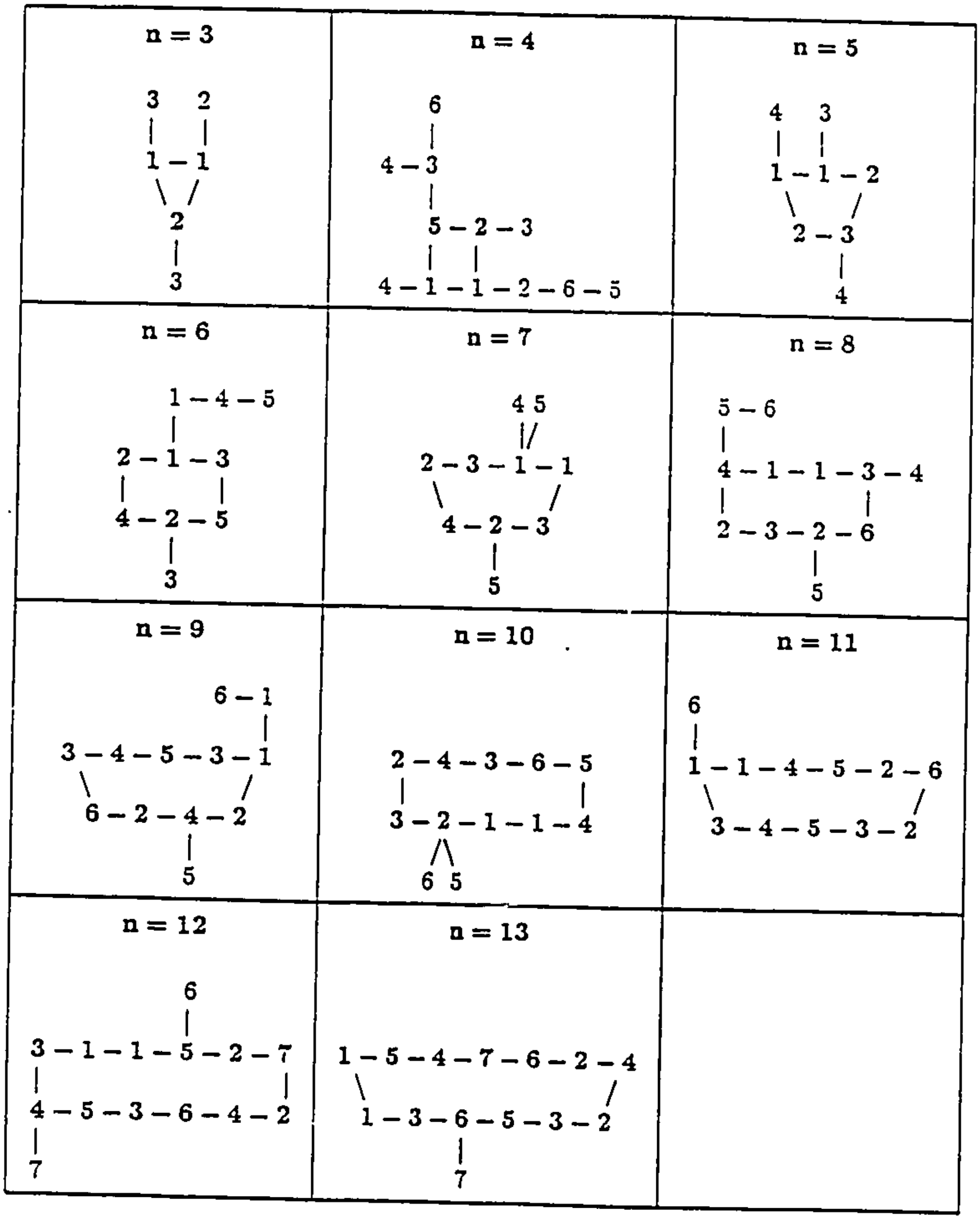


Fig. V.3

V.2. Embedding of Trees and Graphs.

In this section we shall embed trees and graphs into Skolem labelled graphs. We present the proofs in the form of algorithms, the correctness of these algorithms is easily seen [M3]. We shall construct trees containing other trees by the process of adding a new leaf to a given vertex.

We first present an algorithm which proves Theorem V.8.

Algorithm 1 Embed a tree in a Skolem labelled tree with 2v vertices.

```
Let T be a tree (rooted at vertex v_o),
LAST VERTEX=v_0, d(v,v_0)=LEVEL=0,
LEAST LABEL=1
*WHILE: There is an unlabelled vertex v, d(v,v_0)=LEVEL
        DO
     Add a leaf to LAST VERTEX
     Label this unlabelled vertex and the
     new vertex by LEAST LABEL
     LEAST LABEL = LEAST LABEL+1
     LAST VERTEX = the new leaf
        END;
     LEVEL=LEVEL+1
     LAST VERTEX= FATHER(LAST VERTEX)
        END;
```

A modification of this algorithm will give Theorem V.9.

Proof of Theorem V.9.

Algorithm 2 Embed a tree into a d-Skolem tree.

Input: a Tree T and integer d.

Root T at v_0

Add a path of length 2m-1=4d-3 to the tree T beginning at v_0 and label it with this labelling which satisfies Simpson's bounds:

3d-2,3d-4,...,d+4,d+2,d,3d-3,3d-5,...,d+3,d+1,d,d+2,...,3d+2, d+1,d+3,...,3d-3.

Set LEAST LABEL = d+m

LAST VERTEX = $(d+m-1)^{th}$ VERTEX from root of new path

Proceed from * of Algorithm 1.

We note that if a partially labelled tree is given with labelled vertices ≤ d, then Algorithm 2 is robust enough to continue this labelling provided that any edge from a labelled leaf to its father is declared essential. Note that the minimum distance from an unlabelled vertex to the end of the added path exceeds d. We shall use an analogue of this in the proof of

Proof Of Theorem V.10.

We shall first assume that the graph G=(V,E) is not a tree and is connected (if it is not connected we may add edges. We shall see from the proof that this addition of edges does not change the $O(v^3)$ bound on the embedding). We shall use two algorithms to embed G, the first embeds a graph as an induced subgraph of a hooked-Skolem labelled graph , the second embeds a hooked Skolem labelled graph as an induced subgraph of a Skolem labelled graph preserving the non-zero labels.

We shall need these auxiliary labelled graphs. Let P_i be a path of length i with ends labelled i and internal vertices labelled 0. Let D_d be a path of length 3d-3 labelled by: 3d-2,3d-4,...,d+4,d+2,d,3d-3,3d+5,...,d+3,d+1,d,d+2,...,3d+2, d+1,d+3,...,3d-3. (see Algorithm 2).

If S is an induced subgraph of both G and H, then by $G \cup_s H$ we mean a graph with vertices $(V(G)\setminus V(S)) \cup (V(H)\setminus V(S)) \cup V(S)$ and whose edges are $(E(G)\setminus E(S)) \cup (E(H)\setminus E(S)) \cup E(S)$. If G and H have labellings then the label on $v \in S$ is the larger of the two labels of v (its label in G, and in H, respectively).

The idea of algorithms 3 and 4 is to add to G one path for each vertex $w \in G$ then to label the vertices adjacent to w and the corresponding unlabelled vertices on P_i's with the same labels such that every edge incident to w will become essential. We then add a long path D_d (meets Simpson's bound) to the last path (the longest) and label the unlabelled vertices (in P_i's) and attache pendant edges to D_d and label their end vertices with the same labels.

Algorithm 3 To embed a graph as an induced subgraph of a hooked Skolem labelled graph.

Input: A graph G.

Initialize: Let the edges of the graph G be $e_1, e_2, \ldots, e_v, e_{v+1}, \ldots, e_e$; the vertex v_j is an end of e_j, for $j = 1,2,\ldots,v$. (This can be done since G is connected and not a tree). Set I (the unessential edges) to be $e_1, e_2, \ldots, e_v, e_{v+1}, \ldots, e_e$. Set the (hooked) graph H to be G, and i=0.

End;

While ($I \neq \emptyset$) do

Let e be the first edge in I.

Let $H := H \cup_S P_i$ (here S is as large as possible subject to the condition that $e \in S$ if e is among the first v edges; we also require that the end of e which is v_j get label i, i.e. the label of the end of P_i).

Remove e and all $e_j \in S$, from I, where $j>v$.

i:= i+ 1.

End;

Output: H.

End.

We note that H has every edge essential and further, every vertex of G has a non-zero label, and the largest label is e. We pipe H into:

Algorithm 4 To embed a hooked Skolem labelled graph with largest label d into a Skolem labelled graph preserving non-zero labels.

Input: A hooked Skolem labelled graph H with largest label d.

Initialize: Hooks:= {the set of all hooks of H}.

d:= 3 (the largest label of G) + 1.

V(G'):= V(H) U V(D_d).

E(G'):= E(H) U E(D_d) U {f}, where f is an edge connecting a vertex of G labelled d to an end of D_d.

Order Hooks by increasing distance from the end of f in G.

End;

While (Hooks ≠ ∅) do

Label the first member h of Hooks by d+1.

Find a vertex x∈G' so that d(h,x)=d (this can be done since D_d is long enough).

V(G'):= V(G') U {t_d} where t_d is a new vertex labelled d+1.

E(G'):= E(G') U {t_d,x}.

d:=d+1.

Hooks := Hooks - {h}.

End;

Output: G'.

End.

This completes the embedding.

We only now calculate the bound. To do this we need to find the total number of unlabelled vertices just before starting Algorithm 4. For each vertex x of G' processed, the number of unlabelled vertices added is = (the last label used) + 1 - 2. When we have processed all the edges through x, we will have used labels = (last label used before edges containing x) + deg(x) + 1. This gives us the following bounds. The vertex i has last label $\leq (i-1)(v-i/2)$, and uses new labels $\leq v-i$. Thus introducing $(i-1)(v-i/2)-1$ unlabelled vertices. Thus the total number of unlabelled vertices is

$$\leq \sum_{i=1}^{v} (i-1)(v-i/2) -1 = v(v+1)(2v-5)/6$$

Thus the total number of vertices in the embedding is

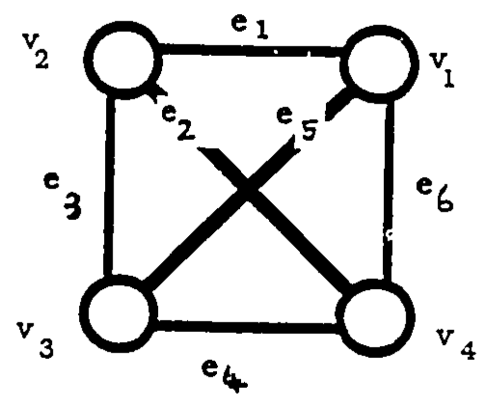
$$\leq v(v+1)(2v-5)/6+3(v(v-1))+2$$

Thus the embedding is $O(v^3)$. ∎

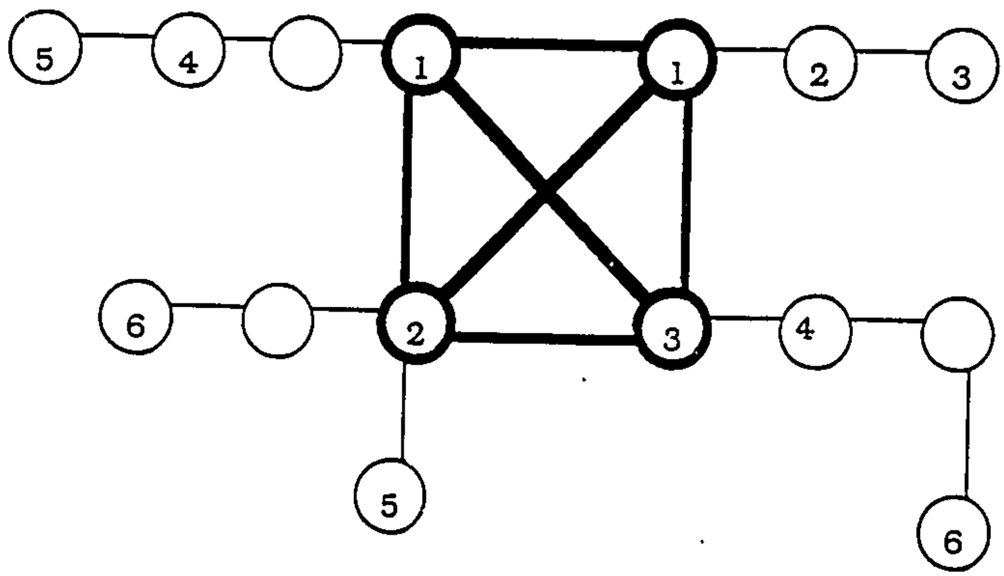
The obvious modification gives the proof of Theorem V.11.

Note that the embedding of K_4 shown in Fig.V.4 and Fig.V.5. is similar to but does not illustrate algorithms 3 and 4.

An Embedding of K_4

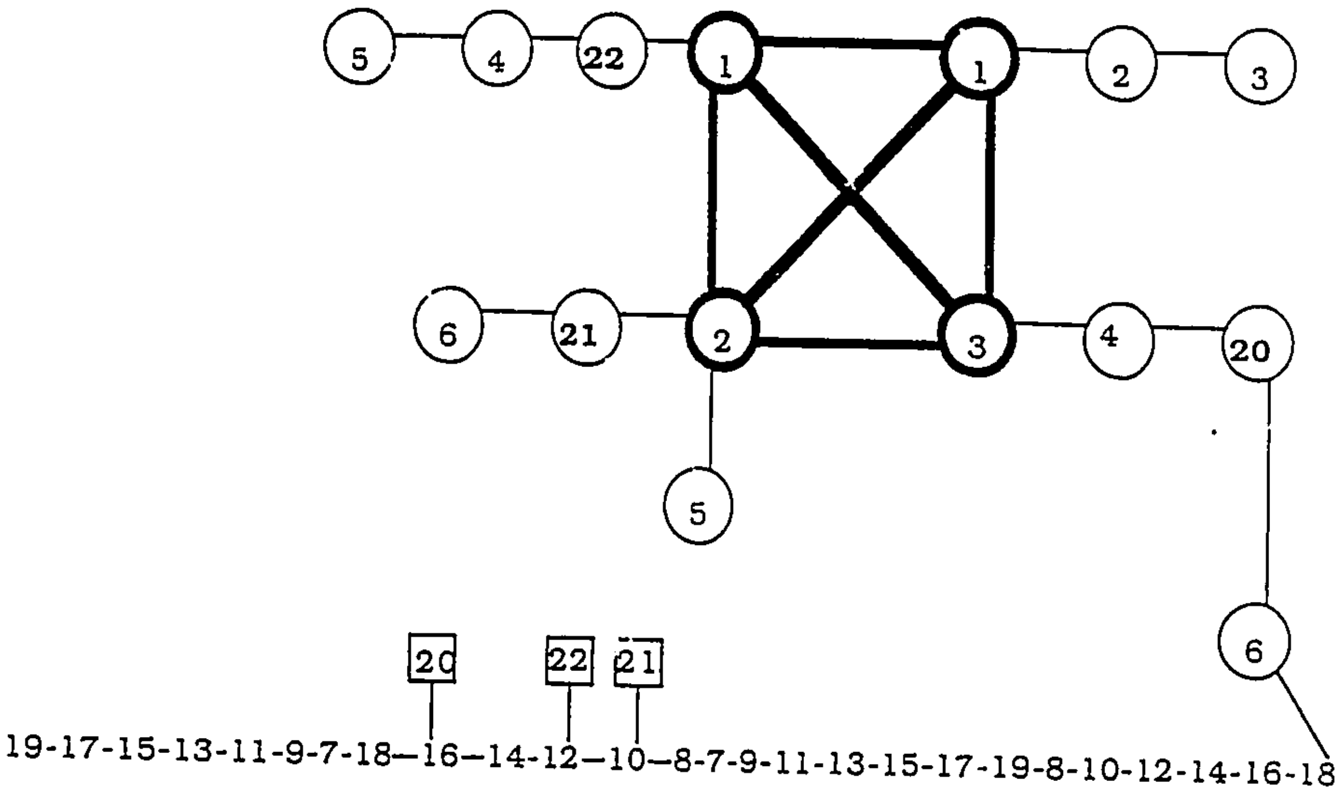


K_4



EMBEDDING STAGE 1

Fig. V.4



K_4 EMBEDDED

Fig. V.5

V.3. Skolem Labellings of Paths and Cycles

In this section we will exhibit the Skolem (minimum hooked Skolem) labelling of paths and cycles with as few "hooks" as possible.

V.3.1. Paths

Proof of Theorem V.12

The necessity follows from a simple computation similar to [S13].

(a) paths of length $m = 8s+1$ or $8s+7$

The number of vertices to be labelled is $8s+2$ and $8s+8$, respectively, hence $n \equiv 0,1 \pmod 4$, thus apply Theorem V.1.

(b) 1. paths of length $m = 8s$ or $8s+6$

Use Theorem V.3.

2. paths of length $m = 8s+2$ or $8s+4$.

For $m=8s+2$ apply Theorem V.7.

For $m=8s+4$, $s \geq 2$, use the following construction:

Label the vertices $1,2,\ldots.8s+5$

$(r, 4s+4-r)$ $\quad r = 1,2\ldots.2s+1$

$(4s+3+r, 8s+4-r)$ $\quad r = 1,2,\ldots s-2$

$(5s+1+r, 7s+4-r)$ $\quad r = 1,2,\ldots s$

$(2s+2, 6s+3), (6s+2, 8s+5), (7s+4, 7s+5)$

For s= 1, the solution is:

(1,7),(3,4),(5,8),(6,11),(10,12),(9,13).

(c) 1. paths of length $m = 8s+3$

Use Rosa's construction [R2] for $n \equiv 1 \pmod 4$.

2. paths of length $m = 8s+5$ $(s \geq 3)$

Label the vertices 1,2,...8s+6

$(r,4s+4-r)$	$r= 1,2,.....2s+1$
$(5s+4+r,7s+7-r)$	$r=1,2,...\ s+1$
$(4s+2+2r,8s+7-2r)$	$r=1,2...\ \lfloor s/2 \rfloor$
$(4s+5+2r,8s+8-2r)$	$r=1,2...\ \lfloor (s-1)/2 \rfloor$
$(2s+2,4s+5)$.	

For s=1 the solution is:

(2,3),(11,13),(7,10),(1,5),(4,9),(8,14).

For s=2 the solution is:

(1,11),(2,10),(3,9),(4,8),(5,7),(6,13),(12,21),(14,15),(16,19),(17,22).

∎

V.3.2. Cycles

Lemma V.3.2.1 A necessary condition for an m-cycle to be Skolem labelled is $m \equiv 0,2 \pmod 8$.

Proof

Let us assume that the vertices of an m-cycle are the residues $\{1,2,...,m\}$ in Z_m, and the edges are $(i,i+1) \pmod m$. Define $|j| = \min(j,m-j)$, and $d(i,j) = |i-j|$. We need only to consider cycles of even length, since odd cycles will have a hook.

Assume that every two vertices a_i, b_i of an m-cycle are labelled by i, $1 \le i \le m/2 = n$.

$$\text{Note that } b_i - a_i = \begin{cases} d(a_i,b_i) & \text{if } b_i > a_i \\ d(a_i,b_i) \pm 2n & \text{otherwise.} \end{cases}$$

Thus

1) $\sum_{i=1}^{n} b_i - a_i = n(n+1)/2 \pm 2sn$, where s is an integer.

but,

2) $\sum_{i=1}^{n} b_i + a_i = 2n(2n+1)/2$

Adding (1) and (2) gives

$$\sum_{i=1}^{n} b_i \pm sn = 1/4\ n(5n+3)$$

which is an integer when $n \equiv 0,1 \pmod 4$. ∎

Thus it may be possible to label a cycle of length $m \equiv 0,2 \pmod 8$ by m/2 labels, but if the cycle length $m \equiv 4,6 \pmod 8$ then the labelled cycle must have at least two hooks.

Proof of Theorem V.13

We distinguish 3 cases:

Case(a). Skolem labelling of cycles of length $m \equiv 0,2 \pmod 8$.

(a)1. Cycles of length m=8s.

Label the vertices 1,2,...,2m.

a_i	b_i	$\leq i \leq$	$d(a_i,b_i)$
1	4s+1	----------	4s
s+2	s+3	----------	1
2s+2	6s+1	----------	4s - 1
2s+3	2	----------	2s+1
2+i	4s+1-i	$1 \leq i \leq s-2$	4s-1-2i
s+1	7s+1	----------	2s
8s+1-i	4s+1+i	$1 \leq i \leq s-1$	4s-2i
6s+1+i	6s+1-i	$1 \leq i \leq s-1$	2i
3s+2	5s+1	----------	2s-1
s+3+i	3s+2-i	$1 \leq i \leq s-2$	2s-1-2i

(a) 2. Cycles of length m = 8s+2.

We subdivide into two cases:

(a) 2.1. Let m = 16r+2, where $r \geq 1$.

a_i	b_i	$\leq i \leq$	$d(a_i,b_i)$
i	8r+3-i	$1 \leq i \leq 4r$	8r+3-2i
4r+1	12r+3	----------	8r+2
8r+1+2i	16r+3-2i	$1 \leq i \leq r-1$	8r+2-4i
8r+4+2i	16r+4-2i	$1 \leq i \leq r-1$	8r-4i
4r+2	8r+4	----------	4r+2
12r+4	12r+5	----------	1
10r+1	10r+3	----------	2
10r+3+i	14r+5-i	$1 \leq i \leq 2r-1$	4r+2-2i

(a)2.2. Cycles of length $m = 16r+10$, where $r \geq 1$.

a_i	b_i	$\leq i \leq$	$d(a_i,b_i)$
i	$8r+7-i$	$1 \leq i \leq 4r+2$	$8r+7-2i$
$4r+3$	$12r+9$	----------	$8r+4$
$8r+5+2i$	$16r+11-2i$	$1 \leq i \leq r$	$8r+6-4i$
$8r+8+2i$	$16r+12-2i$	$1 \leq i \leq r-1$	$8r-4i$
$4r+4$	$8r+8$	---------	$4r+4$
$12r+7$	$12r+8$	---------	1
$14r+10$	$14r+12$	---------	2
$10r+6+i$	$14r+10-i$	$1 \leq i \leq 2r$	$4r+4-2i$

<u>Case (b).</u> Hooked Skolem labelling of odd length m. There are 4 cases to consider.

(b).1. $m=8s+1$, $s \geq 2$.

We subdivide into three subcases.

(b).1.1 Let $m=24r+1$, $r \geq 1$.

a_i	b_i	$\leq i \leq$	$d(a_i,b_i)$
i	$12r-1-i$	$1 \leq i \leq 4r-2$	$12r-1-2i$
$24r$	$12r-1$	----------	$12r$
$4r-1$	$4r$	----------	1
$4r+i$	$8r+1-i$	$1 \leq i \leq 2r-1$	$4r+1-2i$
$6r+1$	$18r$	-----------	$12r-1$
$20r$	$24r+1$	---------	$4r+1$
$12r$	$16r$	---------	$4r$
$12r+i$	$24r-i$	$1 \leq i \leq 4r-1$	$12r-2i$
$16r+i$	$20r-i$	$1 \leq i \leq 2r-1$	$4r-2i$

(b).1.2 Let m= 24r+9.

a_i	b_i	≤ i ≤	$d(a_i,b_i)$
i	12r+3-i	1 ≤ i ≤ 4r	12r+3-2i
24r+8	12r+3	----------	12r+4
4r+1	4r+2	----------	1
4r+2+i	8r+3-i	1 ≤ i ≤ 2r-1	4r+1-2i
6r+3	18r+6	-----------	12r+3
20r+7	24r+9	---------	4r+2
12r+4	16r+5	---------	4r+1
12r+4+i	24r+8-i	1 ≤ i ≤ 4r	12r+4-2i
16r+5+i	20r+7-i	1 ≤ i ≤ 2r	4r+2-2i

(b).1.3. Let m=24r +17.

a_i	b_i	≤ i ≤	$d(a_i,b_i)$
i	12r+7-i	1 ≤ i ≤ 4r+1	12r+7-2i
24r+16	12r+7	----------	12r+8
4r+2	4r+3	----------	1
4r+3+i	8r+6-i	1 ≤ i ≤ 2r	4r+3-2i
6r+5	18r+12	-----------	12r+7
20r+13	24r+17	-----------	4r+4
12r+8	16r+10	-----------	4r+2
12r+8+i	24r+16-i	1 ≤ i ≤ 4r+1	12r+8-2i
16r+11+i	20r+13-i	1 ≤ i ≤ 2r	4r+2-2i
16r+11	20r+14	------------	4r+3

(b).2. Cycles of length m=8s+3, s ≥ 2. We subdivide into 3 subcases.

(b).2.1. Let m=24r+3, r ≥ 1.

a_i	b_i	≤ i ≤	$d(a_i,b_i)$
1	12r+1	----------	12r
2	20r+4	----------	4r+1
2i+1	12r+1-2i	1 ≤ i ≤ 2r-1	12r-4i
2i+2	12r+4-2i	1 ≤ i ≤ 2r	12r+2-4i
4r+1	4r+3	----------	2
4r+3+i	8r+3-i	1 ≤ i ≤ 2r-2	4r-2i
6r+3	6r+4	----------	1
6r+2	18r+3	----------	12r+1
12r+3	16r+3	----------	4r
12r+3+i	24r+4-i	1 ≤ i ≤ 4r-1	12r+1-2i
16r+3+i	20r+4-i	1 ≤ i ≤ 2r-1	4r+1-2i

(b).2.2. Let m=24r+11, r ≥ 1.

a_i	b_i	≤ i ≤	$d(a_i,b_i)$
1	12r+5	----------	12r+4
2	20r+10	----------	4r+3
2i+1	12r+5-2i	1 ≤ i ≤ 2r	12r+4-4i
2i+2	12r+8-2i	1 ≤ i ≤ 2r	12r+6-2i
8r+4	8r+6	----------	2
4r+2+i	8r+4-i	1 ≤ i ≤ 2r-1	4r+2-2i
6r+2	6r+3	----------	1
6r+4	18r+9	----------	12r+5

12r+7	16r+8	----------	4r+1
12r+7+i	24r+12-i	1 ≤ i ≤ 4r	12r+5-2i
16r+9+i	20r+10-i	1 ≤ i ≤ 2r-1	4r+1-2i
16r+9	20r+11	----------	4r+2

(b).2.3. Let m=24r+19, r ≥ 0.

a_i	b_i	≤ i ≤	$d(a_i,b_i)$
1	12r+9	---------	12r+8
2	20r+17	---------	4r+4
21+1	12r+9-2i	1 ≤ i ≤ 2r	12r+8-4i
2i+2	12r+12-2i	1 ≤ i ≤ 2r+1	12r+10-4i
4r+3	4r+5	---------	2
4r+5+i	8r+9-i	1 ≤ i ≤ 2r	4r+4-2i
6r+7	6r+8	---------	1
6r+6	18r+15	---------	12r+9
12r+11	16r+14	---------	4r+3
12r+11	24r+20-i	1 ≤ i ≤ 4r+2	12r+9-2i
16r+14+i	20r+17-i	1 ≤ i ≤ 2r	4r+3-2i

(b).3. m=8s+5, s ≥ 2. We subdivide into two cases

(b).3.1. Let m=16r+5, r ≥ 1.

a_i	b_i	$\le i \le$	$d(a_i,b_i)$
i	8r+3-i	$1 \le i \le 4r$	8r+3-2i
4r+1	16r+4	----------	4r+2
4r+2	12r+4	----------	8r+2
8r+3+2i	16r+7-2i	$1 \le i \le r$	8r+4-4i
8r+2+2i	16r+4-2i	$1 \le i \le r-1$	8r+2-4i
10r+2	10r+4	----------	2
10r+4+i	14r+6-i	----------	4r+2-2i
12r+5	12r+6	----------	1

(b).3.2. Let m=16r+13, $r \ge 1$.

a_i	b_i	$\le i \le$	$d(a_i,b_i)$
i	8r+7-i	$1 \le i \le 4r+2$	8r+7-2i
4r+3	16r+12	-----------	4r+4
4r+4	12r+10	-----------	8r+6
8r+7+2i	16r+15-2i	$1 \le i \le r$	8r+8-4i
8r+6+2i	16r+10-2i	$1 \le i \le r$	8r+6-4i
14r+11	14r+13	-----------	2
10r+7+i	14r+11-i	$1 \le i \le 2r$	4r+4-2i
12r+9	12r+8	-----------	1

For m= 16, the solution is:

(8,9),(2,4),(10,13),(3,12),(6,11),(1,7).

(b).4. Cycles of length m =8s+7, $s \ge 1$.

a_i	b_i	$\le i \le$	$d(a_i,b_i)$
1	4s+5	----------	4s+3
1+i	4s+4-i	$1 \le i \le 2s$	4s+3-2i
2s+2	8s+7	----------	2s+2
2s+3	6s+5	----------	4s+2
4s+5+i	8s+7-i	$1 \le i \le s-1$	4s+2-2i
7s+6	7s+7	----------	1
5s+4+i	7s+6-i	$1 \le i \le s$	2s+2-2i

__Case (c).__ **Minimum hooked Skolem labelling of cycles of length $m \equiv 4,6 \pmod 8$.**

(c).1. Cycles of length m= 8s+4. We subdivide into two cases:

(c).1.1. Let m=16r+4, $r \ge 2$.

a_i	b_i	$\le i \le$	$d(a_i,b_i)$
i	8r+3-i	$1 \le i \le 4r$	8r+3-2i
4r+1	16r+3	----------	4r+2
4r+2	12r+2	----------	8r
12r+3	12r+4	----------	1
8r+4+2i	16r+6-2i	$1 \le i \le r-1$	8r+2-4i
8r+3+2i	16i+3-2i	$1 \le i \le r-1$	8r-4i
14r+4	14r+6	----------	2
10r+2+i	14r+4-i	$1 \le i \le 2r-1$	4r+2-2i

For m=20 the solution is:

(13,14),(4,6),(5,8),(11,15),(2,7),(12,18),(3,16),(9,17),(1,10).

(c)1.2. Let m=16r+12, r ≥ 2.

a_i	b_i	≤ i ≤	$d(a_i,b_i)$
i	8r+7-i	1 ≤ i ≤ 4r+2	8r+7-2i
4r+3	16r+9	-----------	4r+6
4r+4	12r+8	-----------	8r+4
12r+6	12r+7	-----------	1
8r+6+2i	16r+12-2i	1 ≤ i ≤ r-1	8r+6-4i
8r+5+2i	16r+9-2i	1 ≤ i ≤ r-1	8r+4-4i
14r+10	14r+12	-----------	2
10r+4+i	14r+10-i	1 ≤ i ≤ 2r+1	4r+6-2i

For m=28 (i.e. r=1) the solution is:
(17,18),(6,8),(23,26),(5,9),(16,21),(4,10),(20,27),(3,11),
(19,28),(2,12),(7,24),(13,25),(1,14).

(c).2. Cycles of length m=8s+6, s ≥ 2.

a_i	b_i	≤ i ≤	$d(a_i,b_i)$
i	4s+3-i	1 ≤ i ≤ 2s	4s+3-2i
2s+2	6s+4	-----------	4s+2
2s+1	6s+9	-----------	4s-2
4s+3	3s+7	-----------	4
4s+4	4s+5	-----------	1
4s+6	8s+6	-----------	4s
4s+6+2i	8s+4-2i	1 ≤ i ≤ s-2	4s-2-4i
4s+7+2i	8s+7-2i	1 ≤ i ≤ s-2	4s-4i
6s+5	6s+7	------------	2

For s=1 the solution is:

(8,9),(3,5),(10,13),(12,2),(6,11),(1,7).

This concludes the proof of Theorem V.13. ∎

V.4. Minimum Embedding of Paths and Cycles

In this section we find the minimum Skolem labelled graph containing a path or a cycle, thus proving Theorems V.14, V.15.

Proof of Theorem V.14

a) For paths of length $m \equiv 1,7 \pmod 8$ part (a) of Theorem V.12 gives the required minimum path.

b) For paths of even lengths, we use the constructions of Theorem V.12 part (b), then label the hook by $(m/2)+1$, then travel towards the other end of the path and attach a pendant vertex and label it by $(m/2)+1$.

c) For paths of length $m \equiv 3,5 \pmod 8$, we similarly use the constructions of part (c) of Theorem V.12, then apply the same procedure of part (b) above to fill the two hooks. ∎

Proof of Theorem V.15

a) Use constructions of part (b) of Theorem V.13, then add the extra label in the same manner we did in Theorem V.14 (b).

b) Use the constructions of part (a) of Theorem V.13.

c) Use the constructions of part (c) of Theorem V.13, then add the two new labels in the same manner it was done in Theorem V.14. (c). ∎

V.5. Skolem Labelling of Trees

Initial investigations of Skolem labellings of trees seem to lead to a conjecture, that all trees can be Skolem labelled if they satisfy the necessary parity and degeneracy conditions. This conjecture seems to be as difficult as Ringel-Kotzig conjecture on the graceful labelling of trees (introduced by Rosa under the name ß-valuation, and later called graceful labelling by Golomb); for details see [G1].

In this section we study the Skolem labelling of trees (for example see Fig. V.6). We show the existence of a necessary parity condition for all trees, and we show the existence of a degeneracy condition for a special kind of trees (n-windmills). We then show that these two necessary conditions are sufficient for the existence of a Skolem labelling or minimum hooked Skolem labelling for all n-windmills.

An **n-windmill** is a tree with n leaves (vertices of degree one) which are equidistant from a unique vertex of degree > 2 (the centre). The paths from the centre to the leaves are called **vanes**.

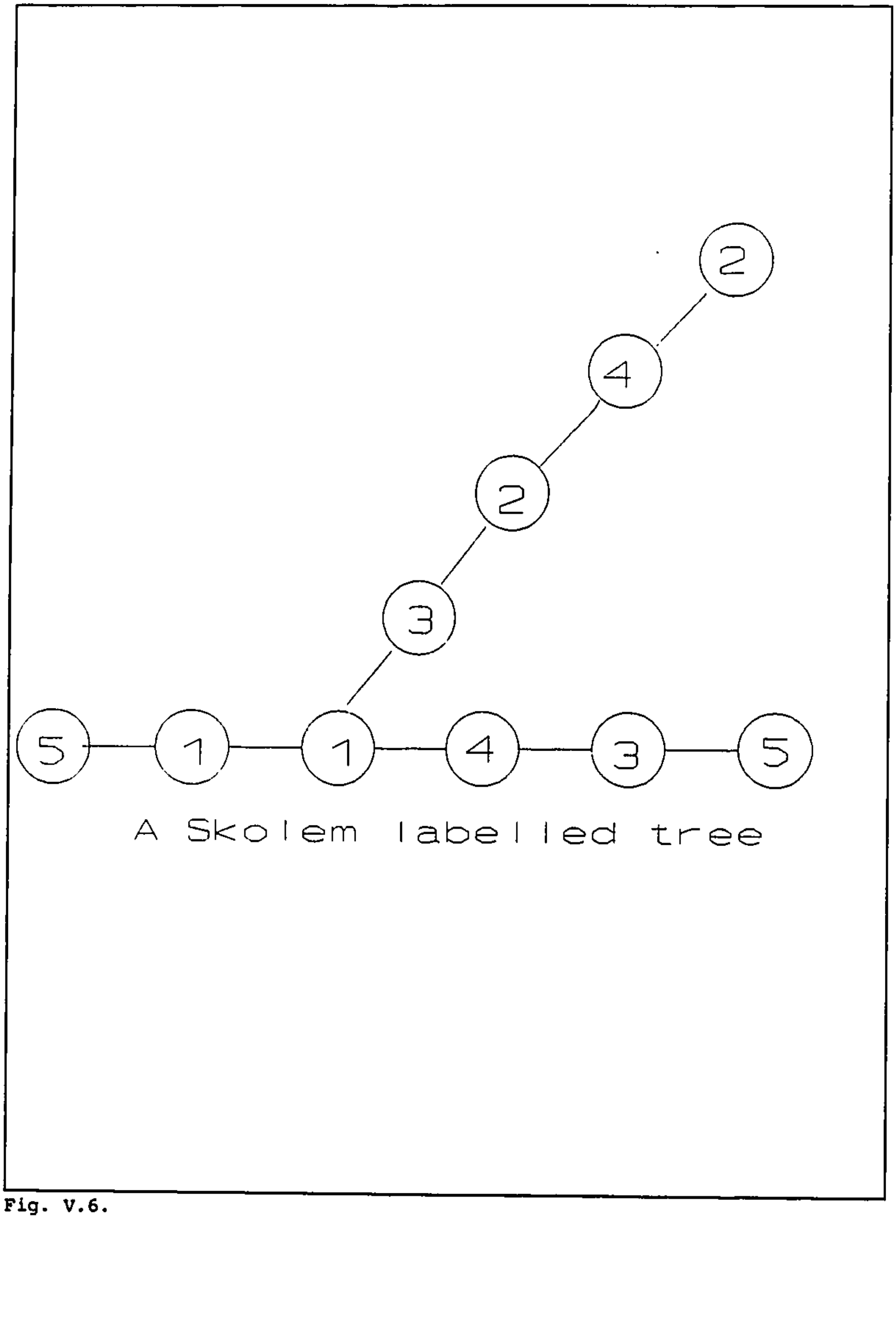


Fig. V.6.

V.4.2.The Necessary Parity Conditions for Trees

In this section we prove the parity conditions for all trees and the degeneracy condition for the n-windmills.

We define the **Skolem parity of a vertex $u \in T$** in a tree T to be $\sum_i d(u,v_i) \pmod 2$ for all $v_i \in T$.

If all vertices of T have the same parity, we speak about the **Skolem parity** of a tree T or simply the **parity** of T.

Lemma V.5.2.1 Let T be a tree with 2n vertices, then all $u \in T$ have the same parity.

Proof Let u and w be any two adjacent vertices in T, and let v be any other vertex in T. Since T is a tree then:

either $d(u,v)-d(w,v)=1$ or $d(w,v)-d(u,v)=1$.

Thus partition the vertices of T into two sets L and M, such that:

$L = \{ v \in T : d(u,v) - d(w,v) =1\}$ and

$M = \{ v \in T : d(w,v) - d(u,v) =1\}$.

Thus for all $v_i \in T$

$$\sum_{i=1}^{2n-1} d(w,v_i) = \sum_{i=1}^{2n-1} d(u,v_i) + |L| - |M|$$

But T has an even number of vertices. Thus |L| and |M| are either both even or both odd, hence u have w the same parity.

Since T is connected all vertices have the same parity. ∎

Note that in case of a tree with an odd number of vertices the number added to $\Sigma d(u,v_i)$ is an odd number, so the parity of the vertices will depend only on which colour class the vertex is in.

Hence for a trees with even number of vertices the (Skolem) parity of T is the parity of any of its vertices. If T has an odd number of vertices the parity of a vertex depends on which colour class of the bipartition it is in.

Let $D(x,\{u,v\}) = \Sigma_{v\in T\setminus\{u,v\}} d(x,v) \pmod 2$, and we call $D(x,\{u,v\})$ to be the **parity of $T\setminus\{u,v\}$.**

Lemma V.5.2.2 Let T be a tree with 2n vertices, and let the vertices $u,w,x \in T$ then

1) $D(x,\{u,w\}) = \Sigma_{v\in T\setminus\{u,w\}} d(x,v) \pmod 2$, does not depend on x for all $x \in T\setminus\{u,w\}$

2) $D(x,\{u,v\})$ differs from the parity of T if $d(u,w)$ is odd, otherwise it has the same parity as T.

Proof We observe than

$$\Sigma_{v\in T\setminus\{u,w\}} d(x,v) = \Sigma_{v\in T} d(x,v) - (d(x,u)+d(x,w))$$
$$= \Sigma_{v\in T} d(x,v) - d(u,w) - 2A$$

where $A = \min\{d(x,u),d(x,w),d(x,z)\}$, where z is any vertex on the u,w path with the shortest distance to x.

We see that $\Sigma_{v\in T\setminus\{v,w\}} d(x,v) \pmod 2$ is different from the parity of T by d(u,w) (mod 2). ∎

We can extend the notation D(x,{u,v}) to D(x,{v_1,v_2},...,{v_{j-1},v_j}) to define the parity of T\{v_1,v_2,...,v_j} in the obvious manner.

Now we can prove the necessary parity condition for all trees

Lemma V.5.2.3 A necessary conditions for Skolem labelling of any tree with 2n vertices are:

1) If $n \equiv 0,3 \pmod 4$ the parity of T must be even
2) if $n \equiv 1,2 \pmod 4$ the parity of T must be odd

Proof By applying Lemma V.5.2.2 n times we obtain this result. Since the parity of T is independent of choice of vertices, we know that parity of T = D(x,{a_n,b_n}) + n (mod 2), x ≠ a_n or b_n, is the same as D(x,{a_n,b_n},{a_{n-1},b_{n-1}}) + n + (n-1) (mod 2). Hence the parity of T is the same as parity of (n(n+1))/2.

∎

An obvious degeneracy condition for a Skolem labelling of a tree is that the tree must have 2n vertices and a path of at least length n. Thus all windmills with more than four vanes can not have a Skolem labelling. However, these windmills can have a minimum hooked Skolem labelling if they satisfy the following condition:

$$(*) \qquad k \leq (n^2-n+2)/2m$$

where k is the number of vanes of the windmill of length m (m>1). Since the maximum distance to be covered by all labels combined is n + (n-2) + (n-3) + ... + 1 = n + (n-1)(n-2)/2, which should exceed the sum of all lengths of all vanes mk.

V.5.3.Sufficiency for all n-windmills

In this section we show that the above necessary conditions are sufficient for obtaining the minimum Skolem labelling for all n-windmills. For a k-windmill with k vanes we arbitrarily number the vanes (say clockwise) 1 to k, let m denote the length of the vane of the windmill, then to every vertex v we associate two coordinates (i,j) where i is the vane number and j is its distance from the centre, denote the vertex by v_{ij}.

3-windmills

Lemma V.5.3.1 All 3-windmills with vane length ≡ 1,7 (mod 8) have a Skolem labelling.

Proof

Case (1) m ≡ 1 (mod 8) , m > 1

a_{ij}	b_{ij}	$\le r \le$	label
(1,m-r+1)	(3,½(m+1)-r)	1 ≤ r ≤ ¼(m-1)	½(3m+3)-2r
(1,3/4(m-1)-r+2)	(2,3/4(m-1)-r+2)	1 ≤ r ≤ ½(m+3)	½(3m+5)-2r
(1,r+1)	(3,r)	1 ≤ r ≤ ¼(m-5)	2r+1
(2,r-1)	(2,m-r+1)	1 ≤ r ≤ ¼(m-1)	m-2r+2
(3,1)	(3,½(m+3))	-------------	½(m+1)
(3,½(m+1))	(3,3/4(m-1)+1)	-------------	¼(m-1)
(3,½(m+3)+r)	(3,m-r+1)	1 ≤ r ≤ 1/8(m-9)	½(m-1)-2r
(3,¼(3m+1)-r)	(3,¼(3m+1)+r)	1 ≤ r ≤ 1/8(m-9)	2r
(3,1/8(7m+9))	(3,1/8(7m+1))	--------------	1

For m=1

label (1,1),(1,2) by 2

(0,0),(1,3) by 1.

Case (2) m ≡ 7 (mod 8) , m > 7

a_{ij}	b_{ij}	$\le r \le$	label
(1,m-r+1)	(3,½(m+3)-r)	1 ≤ r ≤ ¼(m+1)	½(3m+5)-2r
(1,3/4(m+1)-r)	(2,3/4(m+1)-r)	1 ≤ r ≤ ½(m+1)	½(3m+3)-2r
(1,r)	(3,r)	1 ≤ r ≤ ¼(m-3)	2r
(2,r-1)	(2,m-r+1)	1 ≤ r ≤ ¼(m+1)	m-2r+2
(3,¼(m+1))	(3,¼(3m-1))	-------------	½(m-1)
(3,½(m+3))	(3,¼(3m+3))	-------------	¼(m-3)
(3,½(m+3)+r)	(3,m-r+1)	1 ≤ r ≤ 1/8(m-7)	½(m-1)-2r
(3,¼(3m-1)-r)	(3,¼(3m+3)+r)	1 ≤r ≤ 1/8(m-15)	2r+1
(3,7/8(m+1))	(3,1/8(7m-1))	--------------	1

For m=7

label (1,7),(3,4) by 11 ;(1,5),(2,5) by 10;
(1,6),(3,3) by 9 ;(1,4),(2,4) by 8;
(0,0),(2,7) by 7; (1,3),(2,3) by 6;
(2,1),(2,6) by 5; (1,2),(2,2) by 4;
(3,2),(3,5) by 3; (1,1),(3,1) by 2;
(3,6),(3,7) by 1. ∎

Lemma V.5.3.2 For all 3-windmills with vane length $m \equiv 0,2,4,6 \pmod 8$, there is a minimum hooked Skolem labelling (i.e. with one hook), with the exception of m = 2.

Proof

Case 1. $m \equiv 2,6 \pmod 8$, $m > 2$

a_{ij}	b_{ij}	$\le r \le$	label
(1,1)	(1,m)	---------------	m-1
(1,m-r)	$(1,\frac{1}{2}m+r)$	$1 \le r \le \frac{1}{4}(m-6)$	$\frac{1}{2}m-2r$
(1,r+1)	$(3,\frac{1}{2}m+r-1)$	$1 \le r \le \frac{1}{4}(m-6)$	$\frac{1}{2}m+2r$
$(1,\frac{1}{4}(m+2))$	$(1,\frac{1}{4}(3m+2))$	---------------	$\frac{1}{2}m$
$(1,\frac{1}{4}(m+2)+r)$	(3,m-r+1)	$1 \le r \le \frac{1}{4}(m-2)$	$\frac{1}{2}(3m-4r+4)$
$(3,\frac{1}{4}(3m+2))$	$(2,\frac{1}{4}(m-2))$	---------------	m+1
$(2,\frac{1}{4}(m-2)-r)$	$(2,\frac{1}{4}(m-2)+r)$	$1 \le r \le \frac{1}{4}(m+2)$	2r
(3,r)	$(2,\frac{1}{2}m+r+1)$	$1 \le r \le \frac{1}{2}m-1$	$\frac{1}{2}m+2r+1$
$(3,\frac{1}{4}(3m-6))$	$(3,\frac{1}{4}(3m-2))$	---------------	1

note that the case m=2 is degenerate by the necessary condition (*).

Case 2. $m \equiv 0,4 \pmod 8$, $m > 4$

a_{ij}	b_{ij}	$\le r \le$	label
$(1,r)$	$(3,\frac{1}{2}m+r+1)$	$1 \le r \le \frac{1}{4}m-1$	$\frac{1}{2}m+2r+1$
$(3,\frac{1}{2}m+1)$	$(3,m-1)$	---------------	$\frac{1}{2}m-2$
$(2,1)$	$(3,m)$	---------------	$m+1$
$(1,m)$	$(2,\frac{1}{2}m-1)$	---------------	$3m/2-1$
$(3,m-r-1)$	$(2,\frac{1}{2}m-r)$	$1 \le r \le \frac{1}{4}m-2$	$3m/2-2r-1$
$(2,\frac{1}{4}m-r)$	$(2,\frac{1}{4}m+r)$	$1 \le r \le \frac{1}{4}m-2$	$2r$
$(3,r)$	$(2,\frac{1}{2}m+r)$	$1 \le r \le \frac{1}{2}m+1$	$\frac{1}{2}m+2r$
$(1,\frac{1}{4}m+r-1)$	$(1,3m/4-r)$	$1 \le r \le 2$	$\frac{1}{2}m-2r+3$
$(1,3m/4+r+1)$	$(1,3m/4-r)$	$1 \le r \le \frac{1}{4}m-2$	$2r+1$
$(1,\frac{1}{2}m)$	$(1,\frac{1}{2}m+1)$	---------------	1

For m=4

label (2,2),(1,4) by 6;(2,4),(3,1) by 5;

(0,0),(3,4) by 4;(3,2),(1,1) by 3;

(2,1),2,3) by 2; (1,2),(1,3) by 1. ∎

Lemma V.5.3.3 All 3-windmills with vane length $\equiv 3,5 \pmod 8$, have a minimum hooked Skolem labelling with 2 hooks, with the exception of m=3.

Proof

In this case m=3 also is degenerate by the necessary condition (*)

For m > 5.

a_{ij}	b_{ij}	$\le r \le$	label
$(2,\frac{1}{4}(m+1)-r)$	$(2,\frac{1}{4}(m+1)+r)$	$1 \le r \le \frac{1}{4}(m-3)$	$2r$
$(1,\frac{1}{2}(3m-1))$	$(2,\frac{1}{4}(m+1))$	---------------	m
$(2,m-r+1)$	$(3,\frac{1}{2}(m+1)-r)$	$1 \le r \le \frac{1}{2}m$	$3(m+1)/2-2r$
$(1,1)$	$(1,\frac{1}{2}(m+5))$	---------------	$\frac{1}{2}(m+3)$
$(1,1+r)$	$(3,\frac{1}{2}(m+5)+r)$	$1 \le r \le \frac{1}{4}(m-3)$	$(m+3)/3-2r$
$(3,m-r+1)$	$(1,\frac{1}{2}(m-1)-r)$	$1 \le r \le \frac{1}{4}(m-3)$	$3(m-1)/2-2r$
$(1,m-r-2)$	$(1,\frac{1}{2}(m+3)-r)$	$1 \le r \le \frac{1}{4}(m-3)$	$\frac{1}{2}(m-3)-2r$
$(3,\frac{1}{4}(3m-1))$	$(3,\frac{1}{4}(3m+3))$	---------------	1

For m=5

label (1,5),(2,2) by 7;(2,5),(3,1) by 6;
(0,0),(3,5) by 5;(1,1),(2,3) by 4;
(2,1),(3,2) by 2;(3,3),(3,4) by 1. ∎

4-windmills

All 4-windmills have odd number of vertices, the only case that is degenerate by the necessary condition (*) is m=1. So the minimum hooked Skolem labelling in this case has at least one hook.

Lemma V.5.3.4 All 4-windmills with m > 1, have a hooked Skolem labelling (one hook).

Proof All the following cases have this construction in common:

In vanes 2,4 we distribute the even numbers as follows

a_{ij}	b_{ij}	≤ r ≤	label
(4,r)	(2,r)	1 ≤ r ≤ m	2r

Case 1. m ≡ 0 (mod 3)

a_{ij}	b_{ij}	≤ r ≤	label
(3,m-r+1)	(1,m-r)	1 ≤r ≤2m/3 -1	2m-2r+1
(1,m)	(1,(m+3)/3)	---------------	(2m+3)/3
(1,r-2)	(3,1+r)	1 ≤ r ≤ m/3	2r-1

Case 2. m ≡ 1 (mod 3)

a_{ij}	b_{ij}	≤ r ≤	label
(3,m-r+1)	(1,m-r)	1 ≤r ≤2(m-1)/3	2m-2r+1
(1,m)	(1,(m+3)/3)	---------------	(2m+1)/3
(1,r-2)	(3,1+r)	1 ≤ r ≤(m-1)/3	2r-1

Case 3. $m \equiv 2 \pmod 6$

a_{ij}	b_{ij}	$\le r \le$	label
$(3,m-r+1)$	$(1,m-r)$	$1 \le r \le m/2$	$2m-2r+1$
$(1,m)$	$(1,1)$	---------------	$m-1$
$(3,\frac{1}{2}m-1)$	$(3,\frac{1}{2}m)$	---------------	1
$(3,r)$	$(1,r+2)$	$1 \le r \le \frac{1}{2}m-2$	$2r+1$

Case 4. $m \equiv 5 \pmod 6$

a_{ij}	b_{ij}	$\le r \le$	label
$(3,m-r+1)$	$(1,m-r)$	$1 \le r \le \frac{1}{2}(m-1)$	$2m-2r+1$
$(1,m)$	$(0,0)$	---------------	m
$(3,\frac{1}{2}(m-1))$	$(3,\frac{1}{2}(m+1))$	---------------	1
$(3,r+1)$	$(1,r)$	$1 \le r \le \frac{1}{2}(m-3)$	$2r+1$

k-windmills, k > 4

In this case there is no Skolem labelling, thus the only possibility is a minimum hooked Skolem labelling. Note that the maximum possible label is 2m; substituting in

$k \le (n^2-n+2)/2m$ the necessary condition will be $k \le m-1+1/m$

i.e. $k \le 2m-1$

Lemma V.5.3.5 For any k-windmill the condition $k \le 2m-1$ is sufficient for a minimum hooked Skolem labelling.

Proof Fix m

Case 1. $k=2t$, $k+1 \le 2m$

Label the vanes $L_1, L_{2m}, L_2, L_{2m-1}, \ldots L_t, L_{2m+1-t}$.

a_{ij}	b_{ij}	$\le r \le$	label
(2m-r,m)	(r+1,m-r)	$0 \le r \le t$	2m-r
(r,m)	(r,m-r)	$2 \le r \le t$	r

The distances not used are 1,t+1,... m-t+1

So for

(2m,r)	(1,r)	$t+1 \le 2r \le m-t+1$	2r
(2m-1,r)	(2m-2,r+1)	$t+1 \le 2r+1 \le m-t+1$	2r+1
(0,0)	(m-1,1)	----------------	1

Case 2. $k=2t+1$, $k \ne 2m$.

Label the vanes $L_1, L_{2m}, L_2, L_{2m-1}, \ldots L_t, L_{2m+1-t}, L_{2m-t}$.

a_{ij}	b_{ij}	$\le r \le$	label
(2m-r,m)	(r+1,m-r)	$0 \le r \le t$	2m-r
(r,m)	(r,m-r)	$2 \le r \le t$	r
(2m-t+1,m)	(2,m-t+1)	---------------	2m-t+1

The distances not used are 1,t+1,... m-t

So for

(2m,r)	(1,r)	$t+1 \le 2r \le m-t$	2r
(2m-1,r)	(2m-2,r+1)	$t+1 \le 2r+1 \le m-t$	2r+1
(0,0)	(2m-1,1)	----------------	1

Case 3. k+1=2m

Label the vanes $L_1, L_{2m}, L_2, L_{2m-1}, \ldots . L_{m-1}, L_{m+2}, L_m$.

a_{ij}	b_{ij}	$\le r \le$	label
(2m-r,m)	(r+1,m-r)	$0 \le r \le m-1$	2m-r
(r,m)	(r,m-r)	$2 \le r \le m-1$	r
(m+2,m)	(2,2)	---------------	m+2

The distances not used are m and 1

(0,0)	(m,m)	----------------	m
(m,m-1)	(m,m-2)	----------------	1

This completes the proof of Theorem V.16. ∎

We have many scattered results on special classes of trees such as caterpillars and trees with exactly one vertex of maximum degree 3. But this is the first class to be completely settled.

CHAPTER VI

CONCLUDING REMARKS AND OPEN QUESTIONS

The idea of this thesis originated through discussions with Alex Rosa concerning the extended Skolem sequence conjecture. Although this conjecture remains unsettled, the results in this thesis, in particular those concerning the near- and hooked near-Skolem sequences, are due to the various techniques that were employed in the -unsuccessful- attempt to solve the extended Skolem sequence conjecture.

In Chapter II. we proved the existence of near-Skolem and hooked near-Skolem sequences. In addition to several open questions mentioned here, for instance, the existence of sequences with two or more defects, and of excess-Skolem sequences, several other generalizations are possible. For instance, one may want to increase the number of hooks and/or the distances (see for example, Cho [C3]).

In Chapter III. we have shown the existence of 4 mutually disjoint Skolem sequences, and of 3 mutually disjoint hooked Skolem sequences, respectively. It may be not too difficult to improve these results somewhat, but to find a general construction to show the existence of the maximum number of mutually disjoint or hooked disjoint Skolem sequences does not appear feasible at this point. The relation between Skolem sequences and Room squares suggests the use of the former as an additional tool for solving some of the problems in this area.

In Chapter IV. we showed the existence of an indecomposable m-fold Skolem sequence of order n, $m \equiv 0 \pmod 6$. The problem in the other cases is still open, as is the existence problem for indecomposable m-fold hooked Skolem sequences. As shown at the end of the chapter, there are numerous possibilities of combining ideas from Chapters II and IV to create new concepts that have applications in design theory [M5].

Chapter V. is, in our opinion, of interest in that it provides a link between Skolem sequences and graph theory, the largest branch in combinatorial theory. It brings a new animal to the graph labelling "zoo" [G2], namely the Skolem labelling of graphs. We obtained several results concerning the embedding of any graph in a Skolem labelled graph, and we found the Skolem labelling or a minimum Skolem labelling (with as few hooks as possible) of paths, cycles and n-windmills. There still remain many open questions concerning the Skolem labellings of trees in general, or even for special types of graphs or trees.

Index of Definitions and Symbols

Bibliography

In this bibliography we list what are, to the best of our knowledge, all references related to Skolem sequences, their generalizations and applications. These references go well beyond the scope of this thesis. However, it is hoped that they will serve as a basis for a future more comprehensive bibliography.

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