ORE LOCALIZATIONS AND IRREDUCIBLE REPRESENTATIONS OF THE FIRST WEYL ALGEBRA

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Abstract

This thesis studies two problems for the first Weyl algebra $A = A_1(\mathbf{C})$, namely, Ore localizations and irreducible representations.

Our contribution to the first problem is that we find two collections of torsion theories which can be determined by Ore sets. The first consists of all torsion theories generated by classes of simple A-modules which contains either all $\mathbf{C}[q]$-torsion or all $\mathbf{C}[p]$-torsion simple A-modules, up to an automorphism of A (for instance, any torsion theory generated by all but countably many isomorphism classes of simple modules). The second consists of all torsion theories generated by classes of at most linear simple A-modules.

The second part of the thesis studies the irreducible representations of A, i.e., the structure of simple A-modules. We generalize Block's result for linear simple modules, namely, that every linear simple module can be expressed in the form $\mathbf{C}[X, \alpha^{-1}]$ for some $\alpha \in \mathbf{C}[X]$, to arbitrary simple modules which satisfies two conditions which are necessary and sufficient. The second condition is stated in terms of two invariants of the similarity class corresponding to the given simple module, which are explicitly checkable. An important tool is an index theorem which relates two different realizations of the same simple module.

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Contents

Chapter 1

Introduction

In this thesis, we study Ore localizations and irreducible representations of the first Weyl algebra.

The first Weyl algebra, denoted by $A = A_1 = A_1(\mathbf{C})$, is defined as a C-algebra with two generators q and p and a relation $[p, q] = pq - qp = 1$, where C is the field of complex numbers. It also may be realized as the algebra of differential operators with polynomial coefficients, that is, $A = \mathbf{C}[q, \frac{d}{dq}]$.

Dixmier studied the first Weyl algebra systematically in late 1960's. His four lemmas in [3] show how the multiplication in A can be expressed in terms of the multiplication of polynomials in two variables; he determined that the group of automorphisms of A is generated by $\Phi_{n,\lambda}$: $p \to p$, $q \to q + \lambda p^n$ and $\Phi'_{n,\lambda}$: $q \to q$, $p \to p + \lambda p^n$, where λ is an arbitrary complex number; an elegant proof of this fact was provided by Makar- Limanov [10]. Dixmier also classified the elements of A into five classes, and described representatives for two of those, namely the so-called "strictly nilpotent" and "strictly semisimple" elements, up to automorphisms of A.

Many aspects of the Weyl algebra A, and its generalizations A_n, were studied by numerous people; cf. eg. the results and references in [12]. Except for basic facts, Dixmier's work, and the results of Block and Goodearl (which will be described later), we could not find anything particularly applicable to our own investigation.

Ore localization in Noetherian rings has been studied by many people. Many satisfactory results have been obtained for Noetherian rings with lots of ideals, eg. FBN rings, Noetherian rings with the second layer condition, etc. However the first Weyl algebra is a simple Noetherian ring, so it has no non-trivial ideals. This makes localization in A harder since one cannot use analogues of localizations in commutative rings. Jategaonkar's second layer condition is not satisfied, either. Goodearl investigated the influence of the structure of the injective modules on localization questions for noncommutative Noetherian rings. Instead of studying the linkage between prime ideals, he defined links between uniform injective modules, and obtained that such links provide obstructions to Ore localization. For the first Weyl algebra, he proved that the elements which operate regularly on the injective hull of any simple A-module S, denoted by CES, form an Ore set, and the corresponding localization has a unique simple module. Thus he provided another example of Ore localization in A, beyond Goldie's localization which is determined by the Ore set $A \setminus \{0\}$, and the well known localizations determined by the Ore sets $\mathbb{C}[q] \setminus \{0\}$ and $\mathbb{C}[p] \setminus \{0\}$. This raises the question of finding all the Ore localizations in A.

Block determined the irreducible representations of the first Weyl algebra. He proved that every $\mathbb{C}[q]$-torsion free simple A-module can be written in the form $\frac{A}{A \cap Ba}$ for some preserving element a, where $B = \mathbb{C}(q)[p]$ is the localization of A at $\mathbb{C}[q] \setminus \{0\}$. Moreover, he gave another description for linear simple A-modules, namely, $S = \frac{A}{A \cap Ba}$ is isomorphic to the A-module $\mathbb{C}[q, \alpha^{-1}]$, where $a = \alpha p + \beta$ with $\alpha, \beta \in \mathbb{C}[q]$ is "S-regular" (An element is called *S-regular* if it operates regularly on ES' for all simple S' not isomorphic to S.). At this point two more questions arise: First, is there always an S-regular element for any $\mathbb{C}[q]$-torsion free simple A-module S of arbitrary degree n? Secondly, if such an element exists, can S be expressd in a similar form as in the linear case? In other words, can one generalize Block's theorem from linear to arbitrary $\mathbb{C}[q]$-torsion free simple A-modules?

Chapter 2 contains the basic definitions and facts about the first Weyl algebra, which we need in our investigation. In particular, we have a structure theorem for the set of torsion theories on the category of left A-modules, as an easy consequence of the fact that A has Krull dimension one, namely, that there is a one-to-one correspondence between isomorphism-closed classes of simple left A-modules and faithful torsion theories on A-Mod. This theorem transforms the problem of finding Ore localizations in A, to the study of classes of simple A-modules, where Block's classification of irreducible representations of A provides some useful information. This work of Block is described in Chapter 2, including the useful concept of indicial polynomial.

In Chapter 3, we study the Ore localization in A. Our contribution to this problem is that we find two collections of torsion theories which can be determined by Ore sets. The first contains any torsion theory generated by a class of simple A-modules closed under isomorphisms containing either all $\mathbb{C}[q]$-torsion or all $\mathbb{C}[p]$-torsion simple A-modules. (In fact, we prove that any fairly large torsion theory, for instance, any torsion theory generated by a class of simple A-modules which consists of all but countably many isomorphism classes of simple A-modules, is a torsion theory of this kind.) The second contains any torsion theory generated by a class of at most linear simple A-modules. These two collections provide a fairly large number of examples of Ore localizations in A, including all the known results.

In general, consider the torsion theory ${}_{\mathfrak{S}}\mathfrak{T}$ generated by a class $\mathfrak{S}$ of simple A-modules. The natural candidate for an Ore set determining ${}_{\mathfrak{S}}\mathfrak{T}$ is the set $\Omega(\mathfrak{S}) = \bigcap_{S \in \mathfrak{S}} CES$. In fact, if ${}_{\mathfrak{S}}\mathfrak{T}$ is determined by any left Ore set at all, then $\Omega(\mathfrak{S})$ is left Ore and determines ${}_{\mathfrak{S}}\mathfrak{T}$. We call the elements of $\Omega(\mathfrak{S})$ $\mathfrak{S}$-*regular*, and we characterize them in five different ways.

Unfortunately, the problem of finding all Ore localizations of A is far from solved, and we do not know even a single example of a torsion theory which is not an Ore localization.

In Chapter 4, we study the irreducible representations of A, namely, the structure of simple A-modules. We have given a complete answer to the two questions which arise from Block's work. First, not every similarity class contains an S-regular element for the corresponding simple module S. Such an example is given in Section 5. Secondly, not every simple module can be expressed in the form $\mathbf{C}[q,\alpha^{-1}]^n$. However, there is a large number of simple A-modules beyond linear simple modules, which can be expressed in such a form. A necessary and sufficient criterion which consists two conditions is given in Section 4. The second condition is stated in terms of two invariants of the similarity class, and are therefore checkable on any particular member. The whole machinery which we have developed to study the above two problems depends on a careful study of the regularity of elements in each similarity class. In fact, similarity classes are originally defined for simple B-modules, that is two irreducible elements b_1 and b_2 are similar if and only if $\frac{B}{Bb_1} \cong \frac{B}{Bb_2}$. The class of elements of B which are similar to b is denoted by $[b]$. By Block's classification theorem, every $\mathbf{C}[q]$-torsion free simple A-module is associated with a similarity class. Unfortunately, there is no good explicit description of the elements of $[b]$, except in the linear case. We try to find regular elements by studying the A-module $\frac{A}{Aa}$, for any element $a \in A$ in a given similarity class. The two quantities bott-tor$_\rho a$ and top-tor$_\rho a$ are defined naturally at each place $\rho \in \mathbf{C}$. A criterion for bott-tor$_\rho$ is given, in which the *associated indicial polynomials* play a crucial role. Two impotant invariants of a similarity class are found, namely, the *surplus*, and the set of roots of the indicial polynomials modulo $\mathbf{Z}$ at each place $\rho \in \mathbf{C}$.

An index theorem for the first Weyl algebra is established. It gives a quantitative analysis of why Block's result about linear simple A-modules cannot be true in general, and where the discrepancies are.

Chapter 2

Preliminaries

Throughout the thesis we use $\mathbf{Z}$ for the set of integers, $\mathbf{Z}^-$ and $\mathbf{Z}^+$ for the sets of negative and positive integers, $\mathbf{N} = \{0\} \cup \mathbf{Z}^+$ for the set of non-negative integers, and $\mathbf{C}$ for the set of complex numbers.

2.1 The first Weyl algebra

The main reference of this section is the book by McConnell and Robson [12].

Let $A = A_1 = A_1(\mathbf{C})$ denote the $\mathbf{C}$-algebra with two generators q and p and a relation $pq - qp = 1$. Note that every element a in A has a unique representation $\sum_{i,j} c_{ij} q^i p^j$, with $c_{ij} \in \mathbf{C}$ for all $i, j \in \mathbf{N}$. Define the *total degree* of a as $\max\{i + j : c_{ij} \neq 0\}$. Any $0 \neq a \in A$ can be written uniquely as $\sum_{i=0}^{n} \alpha_i(q) p^i$ or $\sum_{j=0}^{m} \beta_j(p) q^j$ with $\alpha_i(q) \in \mathbf{C}[q]$, $\beta_j(p) \in \mathbf{C}[p]$ and $\alpha_n \neq 0$, $\beta_m \neq 0$. We say that a has $p - degree\, n$ and $q - degree\, m$, denoted by $p - \deg a = n, q - \deg a = m$, respectively. This algebra first appeared in quantum mechanics as the algebra generated by position and momentum operator. The noncommutativity of the generators reflects the Heisenberg uncertainty principle. The first Weyl algebra may be realized as the algebra of differential operators with polynomial coefficients, that is, $A = \mathbb{C}[q, \frac{d}{dq}]$. Then it is easy to prove the following:

Theorem 2.1 *A is a simple Noetherian integral domain.*

There is another integral domain closely related to A, namely, $B = \mathbb{C}(q)[p]$, subject to the same relation $pq - qp = 1$. Every element of B has a unique representation $\sum_i \alpha_i(q)p^i$, where $\alpha_i(q) \in \mathbb{C}(q)$ for all i. Therefore there is an obvious embedding: $A \longrightarrow B$ which we use to identify A with a subalgebra of B.

Theorem 2.2 *B is a simple Noetherian principal left and right ideal domain.*

Since B has such a nice structure, we always try to study the problems about A that we are interested in, through their study over B.

A is a filtered ring, with the family $\{F_j \mid j \in \mathbb{N}\}$ of additive subgroups of A, where F_j is the $\mathbb{C}$-subspace generated by the $q^n p^m$ with $n + m \leq j$. This is the standard filtration of A with respect to the generators q and p. One has

$$A = \bigcup_{j \in \mathbb{N}} F_j.$$

The following lemma shows how to multiply two elements of A, in terms of the multiplication in the commutative polynomial ring in two variables.

Lemma 2.3 *Let*

$$x = \sum_{i,j} \alpha_{ij} q^i p^j, \ y = \sum_{i,j} \beta_{ij} q^i p^j, \ z = \sum_{i,j} \gamma_{ij} q^i p^j$$

in A, and

$$f = \sum_{i,j} \alpha_{ij} X^i Y^j, \ g = \sum_{i,j} \beta_{ij} X^i Y^j, \ h = \sum_{i,j} \gamma_{ij} X^i Y^j$$

in $\mathbf{C}[X,Y]$.

1) If $z = xy$, *then*

$$h = fg + \frac{\partial f}{\partial Y}\frac{\partial g}{\partial X} + \frac{1}{2!}\frac{\partial^2 f}{\partial^2 Y}\frac{\partial^2 g}{\partial^2 X} + \cdots.$$

2) If $z = xy - yx$, *then*

$$h = (\frac{\partial f}{\partial Y}\frac{\partial g}{\partial X} - \frac{\partial f}{\partial X}\frac{\partial g}{\partial Y}) + \frac{1}{2!}(\frac{\partial^2 f}{\partial^2 Y}\frac{\partial^2 g}{\partial^2 X} - \frac{\partial^2 f}{\partial^2 X}\frac{\partial^2 g}{\partial^2 Y}) + \cdots.$$

3) For any $\phi \in \mathbf{C}[X]$, *we have*

$$\phi(qp)q^n = q^n\phi(qp+n), \; p^n\phi(qp) = \phi(qp+n)p^n.$$

PROOF. Dixmier ([3], 2.2, 3.2). ∎

Theorem 2.4 *The group of automorphisms of A, denoted by $AutA$, is generated by the linear automorphisms defined by*

$$q \to c_1 q + c_2 p, \; p \to c_1' q + c_2' p$$

with $c_1 c_2' - c_1' c_2 = 1$ *where* $c_i, c_i' \in \mathbf{C}$, *and the triangular automorphisms defined by*

$$q \to q, \; p \to p + f(q)$$

with $f(q) \in \mathbf{C}[q]$.

The set of anti-automorphisms of A is the set of $\sigma \circ \tau$ with σ: $q \to p$, $p \to q$ and $\tau \in AutA$.

PROOF. Dixmier ([3], 8) and L. Makar-Limanov [10]. ∎

Theorem 2.5 *The global dimension of A is 1, and Krull dimension of A is 1.*

PROOF. See the book by McConnell and Robson ([12], 7.5.8, 6.6.15). ∎

2.2 The torsion theories on A-Mod

In this section, we first give some basic definitions. For any undefined notions and proofs, one can refer to the books by Golan [4] or Stenström [14].

Definition and Proposition 2.6 *A hereditary torsion theory on R-Mod is determined by any one of the following :*

1) A torsion class $\mathfrak{T}$: a class of R-modules closed under extensions, colimits, and submodules.

2) A torsion free class $\mathfrak{F}$: a class of R-modules closed under extensions, limits, and essential extensions.

3) A torsion radical T: a functor such that for any $M \in R$-Mod, $TM \subseteq M$, $T^2 = T$, $T(\frac{M}{TM}) = 0$ and for any $N \subseteq M$, $TN = TM \cap N$.

4) A dense filter $\mathfrak{D}$: a set of left ideals of R such that $Dx^{-1} \in \mathfrak{D}$ for any $D \in \mathfrak{D}$ and $x \in R$, and if I is a left ideal of R for which there exists $D \in \mathfrak{D}$ such that $Ia^{-1} \in \mathfrak{D}$ for all $a \in D$ then $I \in \mathfrak{D}$.

5) An equivalence class $\tilde{E}$ of injective module E: E_1 is equivalent to E_2, if there are embeddings $E_1 \longrightarrow \prod E_2$ and $E_2 \longrightarrow \prod E_1$.

Remark: For any hereditary torsion theory $\mathfrak{T}$, there is a quotient functor $\mathfrak{Q}$ on R-Mod, defined as $\mathfrak{Q}(M)=E_{\mathfrak{T}}(\frac{M}{TM})$ for any $M \in R$-Mod, where $E_{\mathfrak{T}}(M)$ is the $\mathfrak{T}$-injective hull of M. (A module X is $\mathfrak{T}$-injective if and only if every diagram of the form:

$$\begin{array}{ccccc} 0 & \longrightarrow & Y' & \longrightarrow & Y \\ & & {\scriptstyle\alpha}\big\downarrow & & \\ & & X & & \end{array}$$

can be completed commutatively by a map β: $Y \longrightarrow X$, provided $\frac{Y}{Y'} \in \mathfrak{T}$.) In particular, $\mathfrak{Q}(R) = E_{\mathfrak{T}}(\frac{R}{TR})$ is a ring. It is called the quotient ring of R with respect to $\mathfrak{T}$, and denoted by ${}_{\mathfrak{T}}R$ or Q.

In this thesis, we study hereditary torsion theories on A-Mod. "Torsion theory" will always mean "hereditary torsion theory".

Definition and Proposition 2.7 *The following are equivalent for a torsion theory $\mathfrak{T}$, with its quotient ring Q:*

1) $\mathfrak{T}$ is a perfect torsion theory.

2) Q-Mod $\subseteq \mathfrak{F}$.

3) $\mathfrak{Q} \cong Q \otimes_R -$

4) $\mathfrak{D}$ has a filter base consisting of finitely generated left ideals, and every $D \in \mathfrak{D}$ is $\mathfrak{T}$-projective, i.e., every diagram of the form:

$$\begin{tikzcd} & D \arrow[d, "\alpha"] & \\ N \arrow[r] & N' \arrow[r] & 0 \end{tikzcd}$$

can be completed commutatively by a map β: $D \longrightarrow N$, provided $N' \in \mathfrak{F}$, and $N \in \mathfrak{F}$ is $\mathfrak{T}$-injective.

Corollary 2.8 *Every torsion theory on A-Mod is perfect, and there is a one-to-one correspondence between the sets of torsion theories and their quotient rings.*

PROOF. A is a hereditary Noetherian ring, i.e., every left ideal of A is finitely generated and projective, therefore $\mathfrak{T}$-projective, for any $\mathfrak{T}$. Hence every dense filter $\mathfrak{D}$ satisfies 4) in Definition and Proposition 2.7. This proves that every torsion theory on A-Mod is perfect.

Let $Q_1 \cong Q_2$. Then for any $M \in \mathfrak{T}_1$, $Q_2 \otimes M \cong Q_1 \otimes M = 0$, thus $M \in \mathfrak{T}_2$. By symmetry, we have $\mathfrak{T}_1 = \mathfrak{T}_2$. ∎

Let us study the torsion theories $\mathfrak{T}$ on A-Mod such that $A \in \mathfrak{F}$, or equivalently that A embeds into Q. Golan [4] calls such a torsion theory *faithful*.

The following lemma holds for an arbitrary domain.

Lemma 2.9 *Let R be a domain and $\mathfrak{T}$ a torsion theory on R-Mod. If $\mathfrak{T} \neq R$-Mod, then $\mathfrak{T}$ is faithful.*

PROOF. Let $\mathfrak{T}$ be not faithful, i.e. $0 \neq T(R) \in \mathfrak{T}$. Let $0 \neq t \in T(R)$. Then there exists a dense filter $D \in \mathfrak{D}$ such that $Dt = 0$. This implies $D = 0$ because R is a domain. Since the ideal 0 annihilates everything, $\mathfrak{T} \supseteq R$-Mod, therefore we have $\mathfrak{T} = R$-Mod. ∎

Since the first Weyl algebra A is a domain, we have that A-Mod is the only non-faithful torsion theory. The corresponding quotient ring is 0. Therefore, there is no loss of generality if we study only the faithful torsion theories on A-Mod.

Definition 2.10 *Let $\mathfrak{S}$ be a class of simple left A-modules closed under isomorphisms. The torsion theory generated by $\mathfrak{S}$ on A-Mod is denoted by ${}_{\mathfrak{S}}\mathfrak{T}$.*

Similarly, let $\mathfrak{S}^*$ be a class of simple right A-modules closed under isomorphisms. The torsion theory generated by $\mathfrak{S}^*$ on Mod-A is denoted by $\mathfrak{T}_{\mathfrak{S}^*}$.

In this thesis, whenever we use a class $\mathfrak{S}$ of simple A-modules, we tacitly assume that $\mathfrak{S}$ is closed under isomorphisms.

Theorem 2.11 *There is a one-to-one correspondence between classes of simple left A-modules closed under isomorphisms, and faithful torsion theories on A-Mod, via $\mathfrak{S} \longleftrightarrow {}_{\mathfrak{S}}\mathfrak{T}$.*

PROOF. Let us first prove that for any faithful torsion theory $\mathfrak{T}$ on A-Mod there exists a class of simple left A-modules $\mathfrak{S}$ such that $\mathfrak{T} = {}_{\mathfrak{S}}\mathfrak{T}$.

Since $\mathfrak{T}$ is faithful, $D \neq 0$ for all $D \in \mathfrak{D}$.

Claim: $\frac{A}{D}$ has a finite length for any non-zero left ideal D of A.

Let $o \neq d \in D$, then $\frac{A}{Ad} \longrightarrow \frac{A}{D}$ is onto, therefore the Krull dimension of $\frac{A}{D}$ is less than or equal to the Krull dimension of $\frac{A}{Ad}$. Consider the descending chain

$$A \supseteq Ad \supseteq Ad^2 \supseteq Ad^3 \supseteq \cdots.$$

It has factors $\frac{Ad^{i-1}}{Ad^i} \cong \frac{A}{Ad}$ for any $i \in \mathbb{N}$. Since A has Krull dimension 1, $\frac{A}{Ad}$ has Krull dimension zero, i.e., $\frac{A}{Ad}$ is Artinian. Hence $\frac{A}{D}$ has a finite length.

Let $\mathfrak{S} = \{S \in \mathfrak{T} \mid \; S \text{ is simple}\}$, then $\mathfrak{S} \subseteq \mathfrak{T}$. Therefore ${}_{\mathfrak{S}}\mathfrak{T}$ is contained in $\mathfrak{T}$.

Let $Comp(\frac{A}{D})$ denote the set of composition factors of $\frac{A}{D}$. We have ${}_{\mathfrak{S}}\mathfrak{D} = \{D \mid \frac{A}{D} \in {}_{\mathfrak{S}}\mathfrak{T}\} = \{D \mid Comp(\frac{A}{D}) \subseteq \mathfrak{T}\}$, by Stenström ([14], Chapter 6, Proposition 2.5). Let $D \in \mathfrak{D}$. Then $\frac{A}{D} \in \mathfrak{T}$, therefore every composition factor of $\frac{A}{D}$ is $\mathfrak{T}$-torsion. Hence $Comp(\frac{A}{D}) \subseteq \mathfrak{S}$. This proves $\mathfrak{D} \subseteq {}_{\mathfrak{S}}\mathfrak{D}$, therefore $\mathfrak{T}$ is contained in ${}_{\mathfrak{S}}\mathfrak{T}$.

Together we have $\mathfrak{T} = {}_{\mathfrak{S}}\mathfrak{T}$.

By Stenström ([14], Chapter 6, Proposition 2.5), we have that ${}_{\mathfrak{S}}\mathfrak{T} \neq A$-Mod for any ${}_{\mathfrak{S}}\mathfrak{T}$, therefore ${}_{\mathfrak{S}}\mathfrak{T}$ is faithful by Lemma 2.9, and any two different classes of simple left A-modules generate two different faithful torsion theories. ∎

In the special case where $\mathfrak{S}$ consists only of the isomorphism class of one simple module S, we write ${}_S\mathfrak{T}$ instead of ${}_{\mathfrak{S}}\mathfrak{T}$.

Definition 2.12 *Let R be a ring, Σ a multiplicatively closed subset of R, and M be a R-module. M is called Σ-torsion, if for any $m \in M$ there exists $\sigma \in \Sigma$ such that $\sigma m = 0$.*

It is easy to check that the collection of all Σ-torsion modules forms a torsion class. It is denoted by ${}_{\Sigma}\mathfrak{T}$, and the corresponding torsion free class is given as follows:

M is Σ-torsion free if for any $0 \neq m \in M$, there exists $r \in R$ such that $\sigma r m \neq 0$ for all $\sigma \in \Sigma$.

Definition 2.13 *Let Σ be a multiplicatively closed set in a ring R. Σ is called a left Ore set in R, if for any $\sigma \in \Sigma$, $r \in R$, there exist $\sigma' \in \Sigma$ and $r' \in R$, such that $r'\sigma = \sigma' r$.*

Lemma 2.14 *Let Σ be a left Ore set in a ring R and M be a R-module. Then M is Σ-torsion free if and only if Σ operates regularly on M, i.e., $\sigma m = 0$ for some $\sigma \in \Sigma$ and $m \in M$ implies $m = 0$.*

PROOF. ($\Leftarrow$) Suppose that Σ operates regularly on M. For $0 \neq m \in M$, take $r = 1$, then $\sigma 1\ m \neq 0$ for any $\sigma \in \Sigma$. This proves that M is Σ-torsion free.

($\Rightarrow$) If M is Σ-torsion free, suppose that there exist $0 \neq m \in M$ and $\sigma \in \Sigma$ such that $\sigma m = 0$. Since $m \neq 0$, there exists $r \in R$ such that $\sigma' r m \neq 0$ for any $\sigma' \in \Sigma$. Since Σ is left Ore, there exist $r_1 \in R$ and $\sigma_1 \in \Sigma$ such that $r_1\sigma = \sigma_1 r$, therefore $0 \neq \sigma_1 r m = r_1 \sigma m = 0$. This is a contradiction. ∎

Lemma 2.15 *Let $\mathfrak{S}$ be the class of all simple A-modules, $\Sigma = A \setminus \{0\}$. Then Σ is an Ore set, and ${}_{\Sigma}\mathfrak{T} = {}_{\mathfrak{S}}\mathfrak{T}$. A finitely generated A-module is Σ-torsion if and only if it has finite length.*

PROOF. Since A is a Noetherian domain, Σ is an Ore set by Goldie's theorem.

Let $S \cong \frac{A}{I}$ be a simple A-module, where I is a maximal left ideal of A. Take any $0 \neq \sigma \in I$. We have $\sigma \in \Sigma$ and $\sigma(1 + I) = \sigma + I = I$. Hence $T(S) \neq 0$, and therefore $S = T(S)$ since S is simple. Consequently S is Σ-torsion. This proves ${}_{\mathfrak{S}}\mathfrak{T} \subseteq {}_{\Sigma}\mathfrak{T}$.

It is clear that A is not Σ-torsion, therefore ${}_{\Sigma}\mathfrak{T} \neq A$-Mod. By Lemma 2.9, ${}_{\Sigma}\mathfrak{T}$ is faithful, therefore ${}_{\Sigma}\mathfrak{T} \subseteq {}_{\mathfrak{S}}\mathfrak{T}$ by Theorem 2.11. Hence we have ${}_{\Sigma}\mathfrak{T} = {}_{\mathfrak{S}}\mathfrak{T}$.

Let M be a finitely generated Σ-torsion A-module. Then the Krull dimension of M, denoted by $|M|$, exists (cf. ([10], 6.2.3)). Write $M = \sum_{i=1}^{t} Am_i$. Then

$$|M| = \max_{1\leq i\leq t}\{|Am_i|\}.$$

For each i, we have $Am_i \cong \frac{A}{\mathrm{ann}_A m_i}$. By the claim in Theorem 2.11, Am_i is Artinian, i.e., $|Am_i| = 0$. Therefore, we have $|M| = 0$, i.e., M is Artinian. Hence M has finite length.

Conversely let M have finite length. Since every simple module is Σ-torsion, M is Σ-torsion. ∎

The torsion theory determined by $A \setminus \{0\}$ is the largest faithful torsion theory. It is called the *Goldie torsion theory*. The corresponding quotient ring is the Goldie quotient ring. It is a division ring, which we shall denote by K. It is the injective hull of A, as A-module.

The quotient ring Q, for any faithful torsion theory $\mathfrak{T}$ on A-Mod, will be identified with the subring $\{t \in K \mid Dt \subseteq A \text{ for some } D \in \mathfrak{D}\}$ of K.

Theorem 2.16 *$Hom(\_, {}_A(\frac{K}{A}))$ and $Hom(\_, (\frac{K}{A})_A)$ provide inverse dualities between the categories of left and right A-modules of finite lengths..*

Proof. By ([8], A.1.5), this holds for every hereditary Noetherian prime ring. ∎

Lemma 2.17 *$Hom_A(\_\, , \frac{K}{A}) \cong Ext^1_A(\_\, , A)$, on the category of left A-modules of finite lengths.*

Proof. The short exact sequence

$$0 \longrightarrow A \longrightarrow K \longrightarrow \frac{K}{A} \longrightarrow 0$$

gives rise the long exact sequence

$$\begin{aligned} 0 \longrightarrow\ & Hom(M\ ,A) \longrightarrow Hom(M\ ,K) \longrightarrow Hom(M\ ,\frac{K}{A}) \\ \longrightarrow\ & Ext^1(M\ ,A) \longrightarrow Ext^1(M\ ,K) \longrightarrow \cdots \end{aligned}$$

for any A-module M.

The connecting homomorphisms

$$Hom(M\ ,\frac{K}{A}) \longrightarrow Ext^1(M\ ,K)$$

constitute a natural transformation (cf. [6] Chapter 3, Theorem 5.2).

If M is Goldie torsion, then $Hom(M,K) = 0$ since K is Goldie torsion free. $Ext^1(M\ ,K) = 0$ since K is injective. This proves that the connecting homomorphism is one-one and onto. ∎

Corollary 2.18 *There is a one-to-one correspondence between the class of faithful torsion theories on A-Mod and the class of faithful torsion theories on Mod-A, where* ${}_{\mathfrak{S}}\mathfrak{T}$ *corresponds to* $\mathfrak{T}_{\mathfrak{S}^*}$, *with* $\mathfrak{S}^* = \{S^* \mid S^* \cong Hom(S, \frac{K}{A}),\ S \in \mathfrak{S}\}$. *Moreover the quotient ring of* ${}_{\mathfrak{S}}\mathfrak{T}$ *coincides with the quotient ring of* $\mathfrak{T}_{\mathfrak{S}^*}$.

PROOF. By Theorem 2.16, $\mathfrak{S} \longrightarrow \mathfrak{S}^*$ is a one-to-one correspondence between isomorphism -closed classes of simple left and right A-modules. Together with Theorem 2.11 this establishes the one-one correspondence between faithful torsion theories on A-Mod and Mod-A.

Let I be a nonzero left ideal of A. Define $I^* = \{x \in K \mid Ix \subseteq A\}$.

For any $\alpha \in Hom(I,\ A)$, there exists an extension $\overline{\alpha}$: $A \longrightarrow K$, since K is injective. Therefore $I\overline{\alpha}(1) = \overline{\alpha}(I) = \alpha(I) \subseteq A$, hence $\overline{\alpha}(1) \in I^*$. This construction defines a map $Hom(I,\ A) \longrightarrow I^*$.

For any $t \in I^*$, define $\beta : I \longrightarrow A$ via $x \rightarrow xt$ for any $x \in I$. Then $\beta \in$ Hom$(I,\ A)$, and we have constructed another map $I^* \longrightarrow Hom(I,\ A)$.

It is easy to check that these two maps are inverse isomorphisms, and that, if $I_1 \supseteq I_2$, the square

$$\begin{array}{ccc} Hom(I_1,\ A) & \xrightarrow{r} & Hom(I_2,\ A) \\ \cong & & \cong \\ I_1^* & \subseteq & I_2^* \end{array}$$

(where r is the restriction map) commutes. Consequently

$$\frac{Hom(I_2,\ A)}{r(Hom(I_1,\ A))} \cong \frac{I_2^*}{I_1^*}.$$

Let t be an element of the quotient ring of $\mathfrak{S}\mathfrak{T}$. Then there exists $D \in \mathfrak{D}$ such that $Dt \subseteq A$, therefore $t \in D^*$.

Since $\frac{A}{D}$ has finite length, we have a descending chain:

$$A = X_0 \supseteq X_1 \supseteq \cdots \supseteq X_n = D$$

with $\frac{X_{i-1}}{X_i} \in \mathfrak{S}$, for $1 \leq i \leq n$.

Therefore we obtain the ascending chain:

$$A = X_0^* \subseteq X_1^* \subseteq \cdots \subseteq X_n^* = D^*.$$

For each i, the short exact sequence

$$0 \longrightarrow X_i \longrightarrow X_{i-1} \longrightarrow \frac{X_{i-1}}{X_i} \longrightarrow 0$$

gives rise to the long exact sequence

$$\begin{aligned} 0 \ \longrightarrow\ & Hom(X_{i-1}/X_i,\ A) \longrightarrow Hom(X_{i-1},\ A) \xrightarrow{r} Hom(X_i,\ A) \\ \longrightarrow\ & Ext^1(X_{i-1}/X_i,\ A) \longrightarrow Ext^1(X_{i-1},\ A) \longrightarrow \cdots. \end{aligned}$$

We have $Hom(X_{i-1}/X_i,\ A) = 0$ since X_{i-1}/X_i is simple, and $Ext^1(X_{i-1},\ A) = 0$ since X_{i-1} is projective (since A is hereditary).

Therefore we have

$$Ext^1(\frac{X_{i-1}}{X_i},\ A) \cong \frac{Hom(X_i,\ A)}{r(Hom(X_{i-1},\ A))}.$$

Using Lemma 2.17 we have

$$\mathfrak{S}^* \ni Hom(\frac{X_{i-1}}{X_i},\ \frac{K}{A}) \cong Ext^1(\frac{X_{i-1}}{X_i},\ A) \cong \frac{Hom(X_i,\ A)}{r(Hom(X_{i-1},\ A))} \cong \frac{X_i^*}{X_{i-1}^*}.$$

This proves that $\frac{D^*}{A}$ has finite length, with composition factors in $\mathfrak{S}^*$. Therefore $\frac{D^*}{A} \in \mathfrak{T}_{\mathfrak{S}^*}$, hence D^* is contained in the quotient ring of $\mathfrak{T}_{\mathfrak{S}^*}$. In particular, t belongs to this quotient ring.

By symmetry, we have that the quotient ring of $\mathfrak{T}_{\mathfrak{S}^*}$is contained in the quotient ring of $_{\mathfrak{S}}\mathfrak{T}$. ∎

2.3 Indicial polynomials

Since the concept of an indicial polynomial plays a crucial role in Block's classification of the simple modules of the Weyl algebra, we will study it in detail in this section. Some generalizations which are used in our work are listed as well.

Recall that for any $\rho \in \mathbb{C}$, the ***valuation*** of $\mathbb{C}(q)$ at ρ, denoted by ν_ρ, is defined as the map ν_ρ: $\mathbb{C}(q) \longrightarrow \mathbb{Z}$, $\nu_\rho\phi = m$, where m is the precise power of $q-\rho$ in $\phi(q)$. This valuation extends to $B = \mathbb{C}(q)[p]$ as follows:

Definition 2.19 *Let $b = \sum_i b_i p^i \in B$, where $b_i \in \mathbb{C}(q)$. For $\rho \in \mathbb{C}$, define*

$$\nu_\rho b = \min_i \{\nu_\rho b_i - i\}.$$

From now on, we state the definitions and results which involve a single place $\rho \in \mathbb{C}$, only for $\rho = 0$ (and we usually omit the index ρ). Corresponding statements for arbitrary ρ follow immediately by applying the automorphism of A defined by: $q \to q - \rho$, $p \to p$.

Lemma 2.20 *ν is a valuation on B.*

PROOF. We must show that 1) $\nu(a+b) \geq \min\{\nu a,\ \nu b\}$, and 2) $\nu ab = \nu a + \nu b$ for any $a,\ b \in B$.

Let m be the maximum of the p-degree of $a = \sum_k a_k p^k$ and $b = \sum_k b_k p^k$. Thus

$$a + b = \sum_{k=0}^{m} (a_k + b_k) p^k.$$

Therefore,

$$\begin{aligned}
\nu(a+b) &= \min_k \{\nu(a_k + b_k) - k\} \\
&\geq \min_k \{\min\{\nu a_k - k, \nu b_k - k\}\} \\
&\geq \min\{\nu a,\ \nu b\}.
\end{aligned}$$

This proves 1).

For 2),

$$\begin{aligned}
ab &= \sum_i a_i p^i \sum_j b_j p^j = \sum_i a_i (\sum_j p^i b_j p^j) \\
&= \sum_i \sum_j (\sum_{k=0}^{i} \begin{pmatrix} i \\ k \end{pmatrix} a_i b_j^{(k)} p^{i+j-k}) \\
&= \sum_l (\sum_i \sum_{k=0}^{i} \begin{pmatrix} i \\ k \end{pmatrix} a_i b_{l+k-i}^{(k)}) p^l = \sum_{l=0}^{n+m} c_l p^l,
\end{aligned}$$

where $b_j^{(k)} = \frac{\partial^k}{\partial q^k}(b_j)$. For each l, $\nu c_l \geq \nu a + \nu b + l$ because of $\nu a_i + \nu b_{l+k-i} - k \geq \nu a + i + \nu b + (l + k - i) - k = \nu a + \nu b + l$ for all i and k. Hence $\nu ab \geq \nu a + \nu b$.

Let l_1 and l_2 be the largest integers such that $\nu a = \nu a_{l_1} - l_1$, $\nu b = \nu b_{l_2} - l_2$, respectively. We are going to establish the equality $\nu ab = \nu a + \nu b$ by showing $\nu c_{l_1+l_2} - (l_1 + l_2) = \nu a + \nu b$.

We have

$$
\begin{aligned}
c_{l_1+l_2} &= \sum_{i=0}^{l_1-1}\sum_{k=0}^{i}\begin{pmatrix} i \\ k \end{pmatrix} a_i b^{(k)}_{l_1+l_2+k-i} + \sum_{k=1}^{l_1}\begin{pmatrix} l_1 \\ k \end{pmatrix} a_{l_1} b^{(k)}_{l_2+k} + \\
&+ \ a_{l_1} b_{l_2} + \sum_{i=l_1+1}^{n}\sum_{k=0}^{i}\begin{pmatrix} i \\ k \end{pmatrix} a_i b^{(k)}_{l_1+l_2+k-i}.
\end{aligned}
$$

For $i \leq l_1 - 1$ and $k \geq 0$, since $l_1 + l_2 + k - i > l_1 + l_2 + k - l_1 = l_2 + k \geq l_2$, $\nu a_i + \nu b_{l_1+l_2+k-i} - k > \nu a + i + \nu b + (l_1 + l_2 + k - i) - k = \nu a + \nu b + l_1 + l_2$.

For $i = l_1$ and $k > 0$, since $l_1 + l_2 + k - i > l_1 + l_2 - i \geq l_1 + l_2 - l_1 = l_2$, $\nu a_i + \nu b_{l_1+l_2+k-i} - k > \nu a + i + \nu b + (l_1 + l_2 + k - i) - k = \nu a + \nu b + l_1 + l_2$.

For $i > l_1$ and $k \geq 0$, since $\nu a_i > \nu a + i$, $\nu a_i + \nu b_{l_1+l_2+k-i} - k > \nu a + i + \nu b + (l_1 + l_2 + k - i) - k = \nu a + \nu b + l_1 + l_2$.

But, for the last remaining term, $\nu a_{l_1} b_{l_2} = \nu a_{l_1} + \nu b_{l_2} = \nu a + l_1 + \nu b + l_2$.

Note that $\nu(\alpha + \beta) = \min\{\nu\alpha,\ \nu\beta\} = \nu\alpha$ if $\nu\beta > \nu\alpha$, for any α and β in $\mathrm{C}(q)$. Therefore the above observations yield $\nu c_{l_1+l_2} = \nu a + \nu b + l_1 + l_2$ as required. ∎

Lemma 2.21 *Let $b \in B$, and*

$$
b = \sum_{k=0}^{n} b_k p^k = \sum_{k=0}^{n} p^k b^*_k,
$$

where b_k and b^*_k are in $\mathrm{C}(q)$ for $0 \leq k \leq n$. Define $\nu^* b = \nu\sigma b$, where σ is the anti-automorphism of A defined by $q \to q$, $p \to -p$. Then $\nu^* b = \min_{0 \leq k \leq n}\{\nu b^*_k - k\}$ and $\nu b = \nu^* b$.

PROOF.

$$
b \ = \ \sum_{k=0}^{n} b_k p^k = \sum_{k=0}^{n}\sum_{i=0}^{k}(-1)^i \begin{pmatrix} k \\ i \end{pmatrix} p^{k-i} b^{(i)}_k
$$

$$
\begin{aligned}
&= \sum_{k=0}^{n} p^k [b_k - \begin{pmatrix} k+1 \\ 1 \end{pmatrix} b_{k+1}^{(1)} + \cdots + (-1)^{n-k} \begin{pmatrix} n \\ n-k \end{pmatrix} b_n^{(n-k)}] \\
&= \sum_{k=0}^{n} p^k b_k^*
\end{aligned}
$$

where $b_i^{(j)} = \frac{d^j b_i}{dq^j}$ for $0 \le k \le n$, $i, j \in \mathrm{N}$.

Note that

$$
\begin{aligned}
\sigma b &= \sigma(\sum_{k=0}^{n} p^k b_k^*) = \sum_{k=0}^{n} \sigma(p^k b_k^*) \\
&= \sum_{k=0}^{k} \sigma(b_k^*)\sigma(p^k) = \sum_{k=0}^{n} b_k^*(-p)^k = \sum_{k=0}^{n} (-1)^k b_k^* p^k.
\end{aligned}
$$

Therefore we have

$$\nu^* b = \min_{0 \le k \le n} \{\nu b_k^* - k\}.$$

Since

$$
\begin{aligned}
\nu b_k^* &\ge \min_{0 \le j \le n-k} \{\nu b_{k+j} - j\} \\
&\ge \min_{0 \le j \le n-k} \{\nu b + (k+j) - j\} \\
&= \nu b + k,
\end{aligned}
$$

$\nu^* b \ge \nu b$. Since σ^2 is the identity map on A, this inequality implies

$$\nu^* b = \nu \sigma b \le \nu^* \sigma b = \nu \sigma^2 b = \nu b.$$

Therefore we have $\nu^* b = \nu b$. ∎

Lemma 2.22 *1) For $j \in \mathrm{N}$, $(qp)^j = \sum_{k=1}^{j} d_{jk} q^k p^k$, where $d_{jk} \in \mathrm{N}$ for $1 \le k \le j$, and $d_{jj} = 1$.*

2) The family $\{q^i (qp)^j\}_{i,\, j \in \mathrm{N}}$ (or $\{(qp)^j q^i\}$) is linearly independent over C.

3) For any $k \in \mathbf{N}$, we have

$$\begin{aligned} q^k p^k &= qp(qp-1)\cdots(qp-k+1) \\ p^k q^k &= pq(pq+1)\cdots(pq+k-1). \end{aligned}$$

PROOF. 1) Induction on j.

For $j = 1$, we have $(qp)^1 = qp$, where $d_{11} = 1 \in \mathbf{N}$.

Suppose that $(qp)^j = \sum_{k=1}^{j} d_{jk} q^k p^k$ with $d_{jk} \in \mathbf{N}$. Then

$$\begin{aligned} (qp)^{j+1} &= \left(\sum_{k=1}^{j} d_{jk} q^k p^k\right) qp \\ &= \sum_{k=1}^{j} d_{jk}(q^{k+1}p^{k+1} + kq^k p^k) \\ &= \sum_{k=1}^{j+1} (d_{j,k-1} + kd_{jk}) q^k p^k \\ &= \sum_{k=1}^{j+1} d_{j+1,k} q^k p^k \end{aligned}$$

where $d_{j+1,k} = d_{j,k-1} + kd_{jk} \in \mathbf{N}$, in particular, $d_{j+1,j+1} = d_{jj} + 0 = 1$.

2) Suppose we have

$$X = \sum_{ij} c_{ij} q^i (qp)^j = \sum_j (\sum_i c_{ij} q^i)(qp)^j = 0,$$

where $c_{ij} \in \mathbf{C}$. We have to show $c_{ij} = 0$ for all i and j.

Suppose not, let m be the largest integer such that $\sum_i c_{im} q^i = c_m(q) \neq 0$. Therefore $c_m(q) q^m \neq 0$. On the other hand,

$$\begin{aligned} c_m(q)(qp)^m &= c_m(q) \sum_k d_{mk} q^k p^k = c_m(q)(q^m p^m + \text{lower terms in } p) \\ &= c_m(q) q^m p^m + \text{lower terms in } p, \end{aligned}$$

and therefore

$$X = (c_m(q) q^m) p^m + \text{lower terms in } p.$$

We conclude $c_m q^m = 0$. This is a contradiction.

3) (cf. [3], 3.2). ∎

Lemma 2.23 *Let $a \in A$. Write $a = \sum_{k=0}^{n} a_k p^k = \sum_{k=0}^{n} p^k a_k^*$ with $a_k = \sum_{l\geq 0} \alpha_{kl} q^l$ and $a_k^* = \sum_{l\geq 0} \alpha_{kl}^* q^l \in \mathbf{C}[q]$, for $0 \leq k \leq n$. Then*

$$\begin{aligned} q^{-\nu a} a &= \sum_{j\geq 0} q^j \Theta_j(qp), \\ a q^{-\nu a} &= \sum_{j\geq 0} \Theta_j^*(-pq) q^j, \end{aligned}$$

where

$$\begin{aligned} \Theta_j(\xi) &= \sum_{k=0}^{n} \alpha_{k,k+\nu a+j} \xi(\xi-1)\cdots(\xi-k+1), \\ \Theta_j^*(\xi) &= \sum_{k=0}^{n} (-1)^k \alpha_{k,k+\nu a+j}^* \xi(\xi-1)\cdots(\xi-k+1) \end{aligned}$$

are polynomials which are uniquely determined by a. Furthermore,

$$\Theta_j^*(\xi) = \Theta_j(-\xi-(\nu a+j+1)).$$

PROOF. We have

$$\begin{aligned} q^{-\nu a} a &= \sum_{k=0}^{n} (q^{-\nu a-k} a_k) q^k p^k \\ &= \sum_{k=0}^{n} (q^{-\nu a-k} a_k) qp(qp-1)\cdots(qp-k+1). \end{aligned}$$

Note that $\nu(q^{-\nu a-k} a_k) = -\nu a - k + \nu a_k \geq -\nu a + \nu a = 0$ for each k. This implies $\alpha_{kl} = 0$ for any $l < \nu a + k$. Therefore $q^{-\nu a-k} a_k = \sum_{l\geq 0} \alpha_{kl} q^{-\nu a-k+l} = \sum_{j\geq 0} \alpha_{k,k+\nu a+j} q^j \in \mathbf{C}[q]$. Hence we have

$$\begin{aligned} q^{-\nu a} a &= \sum_{j\geq 0} q^j \left(\sum_{k=0}^{n} \alpha_{k,k+\nu a+j} qp(qp-1)\cdots(qp-k+1)\right) \\ &= \sum_{j\geq 0} q^j \Theta_j(qp), \end{aligned}$$

where

$$\Theta_j(\xi) = \sum_{k=0}^{n} \alpha_{k,k+\nu a+j}\xi(\xi-1)\cdots(\xi-k+1)$$

for all $j \geq 0$. By Lemma 2.22, such an expression is unique.

Similarly we have

$$\begin{aligned} aq^{-\nu a} &= \sum_{k=0}^{n} p^k q^k (a_k^* q^{-k-\nu a}) \\ &= \sum_{j\geq 0} (pq)(pq+1)\cdots(pq+k-1)(a_k^* q^{-k-\nu a}). \end{aligned}$$

Note that $\nu a_k^* q^{-\nu a-k} = \nu a_k^* - \nu a - k \geq \nu^* a - \nu a = \nu a - \nu a = 0$ because $\nu a = \nu^* a$ by Lemma 2.21. This implies that $\alpha_{kl}^* = 0$ for any $l < \nu a + k$. Therefore $q^{\nu a-k} a_k^* = \sum_{l\geq 0} \alpha_{kl}^* q^{-\nu a-k+l} = \sum_{j\geq 0} \alpha_{k,k+\nu a+j}^* q^j$. Hence we have

$$\begin{aligned} aq^{-\nu a} &= \sum_{j\geq 0}\Big(\sum_{k=0}^{n} (pq)(pq+1)\cdots(pq+k-1)\alpha_{k,k+\nu a+j}^*\Big)q^j \\ &= \sum_{j\geq 0}\Big(\sum_{k=0}^{n} (-1)^k(-pq)(-pq-1)\cdots(-pq-k+1)\alpha_{k,k+\nu a+j}^*\Big)q^j \\ &= \sum_{j\geq 0} \Theta_j^*(-pq)q^j, \end{aligned}$$

where

$$\Theta_j^*(\xi) = \sum_{k=0}^{n} (-1)^k \alpha_{k,k+\nu a+j}^* \xi(\xi-1)\cdots(\xi-k+1)$$

for all $j \geq 0$. By Lemma 2.22, such an expression is unique.

For each $j \geq 0$, since

$$\begin{aligned} \sum_{j\geq 0} \Theta_j^*(-pq)q^j &= aq^{-\nu a} \\ &= q^{\nu a}(q^{-\nu a}a)q^{-\nu a} \\ &= q^{\nu a}\Big(\sum_{j\geq 0} q^j \Theta_j(qp)\Big)q^{-\nu a} \\ &= \sum_{j\geq 0} q^{\nu a+j}\Theta_j(qp)q^{-\nu a} \\ &= \sum_{j\geq 0} \Theta_j(qp-(\nu a+j))q^j \\ &= \sum_{j\geq 0} \Theta_j(pq-(\nu a+j+1))q^j, \end{aligned}$$

we have

$$\Theta_j^*(-pq) = \Theta_j(pq - (\nu a + j + 1)),$$

by Lemma 2.22, therefore

$$\Theta_j^*(\xi) = \Theta_j(-\xi - (\nu a + j + 1)),$$

for all $j \geq 0$. ∎

Obviously, the definitions of $\Theta_j(\xi)$ and $\Theta_j^*(\xi)$ depend on the choice of the element $a \in A$. If necessary, we shall indicate this quantity by an extra subscript.

The following is Block's definition of the indicial polynomial. (In (3.2) of [1], he states this definition for more general algebras. The special case of our algebra B is written out in [2], except for a change of variable, $\lambda = -\xi$.)

Definition 2.24 *Let $b \in B$, $b = \sum_{k\geq 0} b_k p^k$. The polynomial*

$$\Theta_b(\xi) = \sum_{k\geq 0} (q^{-\nu b - k} b_k)(0)\xi(\xi - 1)\cdots(\xi - k + 1)$$

is called the indicial polynomial of b.

Note that $(q^{-\nu b - k} b_k)(0)$ is well-defined because $\nu b_k \geq \nu b + k$, for all k.

Lemma 2.25 *For $a \in A$, the indicial polynomial $\Theta_a(\xi)$ coincides with the polynomial $\Theta_0(\xi)$ of Lemma 2.23.*

PROOF. By definition,

$$\begin{aligned}\Theta_a(\xi) &= \sum_{k\geq 0} (q^{-\nu a - k} a_k)(0)\xi(\xi - 1)\cdots(\xi - k + 1) \\ &= \sum_{k\geq 0} \alpha_{k,\nu a + k}\xi(\xi - 1)\cdots(\xi - k + 1) = \Theta_0(\xi).\end{aligned}$$

∎

We shall call the $\Theta_j(\xi)$ of Lemma 2.23 the *associated polynomials* of the element $a \in A$.

The following is Block's product formula for indicial polynomials.

Lemma 2.26 *Let $a,\ b \in B$. Then*

$$\Theta_{ab}(\xi) = \Theta_a(\xi + \nu b)\Theta_b(\xi).$$

PROOF. ([1], Lemma 3.3.2) ∎

The expressions obtained in Lemma 2.23 for an element $a \in A$, are very useful later on. As a first application, we give an easy proof of Lemma 2.26, for the special case of elements of A:

Let $\Theta_{x,j}(\xi)$ denote the associated polynomials for $x \in A$, defined in Lemma 2.23. By Lemma 2.23 and Lemma 2.3, we have

$$\begin{aligned} ab &= (q^{\nu a}\sum_{j\geq 0} q^j\Theta_{a,j}(qp))(q^{\nu b}\sum_{k\geq 0} q^k\Theta_{b,k}(qp)) \\ &= \sum_{j,k\geq 0} q^{\nu a+j}(\Theta_{a,j}(qp)q^{\nu b+k})\Theta_{b,k}(qp) \\ &= \sum_{j,k\geq 0} q^{\nu a+j+\nu b+k}\Theta_{a,j}(qp+\nu b+k)\Theta_{b,k}(qp). \end{aligned}$$

On the other hand, we have

$$ab = q^{\nu ab}\sum_{l\geq 0} q^l\Theta_{ab,l}(qp).$$

By comparing the coefficient of $q^{\nu ab}$, we obtain

$$\Theta_{ab,0}(\xi) = \Theta_{a,0}(\xi + \nu b)\Theta_{b,0}(\xi).$$

2.4 The irreducible representations of A

In this section, we will briefly introduce Block's work on the irreducible representations. The results from 2.31 to 2.34 are his. Most proofs provided here are simpler than his, because we only need the special case of his results referring to the Weyl algebra. One may refer to [1] for the general statements and proofs.

Lemma 2.27 *Let $\Sigma = \mathbb{C}[q]\setminus\{0\}$. Then Σ is a two-sided Ore set in A and the quotient ring of $_{\Sigma}\mathfrak{T}$ is $B = \mathbb{C}(q)[p]$. Furthermore, every Σ-torsion module is semisimple.*

PROOF. Let $a \in A$ with p-degree n, $0 \neq \phi \in \mathbb{C}[q]$. Then

$$\phi^{n+1}a = \phi^{n+1}\sum_{k=0}^{n} a_k p^k = \sum_{k=0}^{n}(\sum_{j=0}^{k}\binom{k}{j}p^{k-j}(\phi^{n+1})^{(j)}(-1)^j).$$

Note that $(\phi^{n+1})^{(j)} = \phi\psi_j$ for some $\psi_j \in \mathbb{C}[q]$, where $j \leq k \leq n$. Therefore $\phi^{n+1}a = b\phi$ for some $b \in A$. This proves that Σ is a left Ore set. Similarly, one can show that Σ is a right Ore set, hence Σ is a two sided Ore set in A. It is clear that the quotient ring is B.

Let M be a $\mathbb{C}[q]$-torsion module. Let $0 \neq m \in M$. There is $\phi \in \mathbb{C}[q]$ such that $\phi m = 0$. By ([11], Theorem 5.7), we have that $\frac{A}{A\phi}$ is a direct sum of $\mathbb{C}[q]$-torsion simple A-modules. Therefore $Am \cong \frac{A}{\mathrm{ann}_A m}$ is semisimple as a factor module of $\frac{A}{A\phi}$. Hence $Am \subseteq \mathrm{Soc}M$. This implies $m \in \mathrm{Soc}M$. Hence $M = \mathrm{Soc}M$ is semisimple. ∎

We shall simply call a $\mathbb{C}[q] \setminus \{0\}$-torsion (or torsion free) module $\mathbb{C}[q]$-torsion (or torsion free).

It is easy to show the following lemma:

Lemma 2.28 *Let $\mathfrak{T}$ be any torsion theory on A-Mod and S a simple A-module. Then S is either $\mathfrak{T}$-torsion or $\mathfrak{T}$-torsion free.*

PROOF. Consider the $\mathfrak{T}$-torsion radical $T(S)$ of S. $T(S)$ is a submodule of S, therefore $T(S) = 0$ or $T(S) = S$ because S is a simple module. Hence S is either torsion free or torsion. ∎

We will characterize the set of simple A-modules according to whether they are $\mathbb{C}[q]$-torsion or $\mathbb{C}[q]$-torsion free. The structure of a $\mathbb{C}[q]$-torsion simple A-module is explicitly known.

Proposition 2.29 *1) For $\rho \in \mathbb{C}$, $\frac{A}{A(q-\rho)}$ is a simple $\mathbb{C}[q]$-torsion A-module.*

2) If M is a simple $\mathbb{C}[q]$-torsion A-module, then there exists $\rho \in \mathbb{C}$ such that $M \cong \frac{A}{A(q-\rho)}$.

3) $\frac{A}{A(q-\rho_1)} \cong \frac{A}{A(q-\rho_2)}$ if and only if $\rho_1 = \rho_2$.

PROOF. 1) Without loss of generality, let us assume $\rho = 0$. Let M be a nonzero submodule of $\frac{A}{Aq}$, and $0 \neq m \in M$. Write $m = (\sum_{i=0}^{n} c_i p^i) + Aq$ with $c_i \in \mathbb{C}$, $0 \leq i \leq n$, and $c_n \neq 0$. Then

$$\begin{aligned} q^n m &= (\sum_{i=0}^{n} c_i q^n p^i) + Aq \\ &= \sum_{i=0}^{n} c_i (\sum_{k=0}^{i} \begin{pmatrix} i \\ k \end{pmatrix} p^{i-k} n(n-1)\cdots(n-k+1) q^{n-k}(-1)^k) \ + Aq \\ &= c_n n!(-1)^n \ + Aq \ \in M. \end{aligned}$$

Hence we have $1 + Aq \in M$, that is, $M = \frac{A}{Aq}$. This proves that $\frac{A}{Aq}$ is simple.

2) Let M be a simple $\mathbb{C}[q]$-torsion A-module. Let $0 \neq m \in M$. There exists $0 \neq f \in \mathbb{C}[q]$ such that $fm = 0$. Therefore $0 \neq I = \text{ann}_A m \cap \mathbb{C}[q] \subseteq \mathbb{C}[q]$. Let $\frac{J}{I}$ be a minimal ideal contained in $\frac{\mathbb{C}[q]}{I}$. Then $\text{ann}_{\mathbb{C}[q]}\frac{J}{I}$ is a maximal ideal of $\mathbb{C}[q]$, therefore it has the form $\mathbb{C}[q](q-\rho)$. Note that $\frac{J}{I} \subseteq \frac{\mathbb{C}[q]}{I} \cong \mathbb{C}[q]m$. Let $0 \neq x \in \mathbb{C}[q]m$ be the image of some element $\frac{J}{I}$. Then $(q-\rho)x = 0$, therefore $A(q-\rho) \subseteq \text{ann}_A x$. But $A(q-\rho)$ is maximal by 1). Therefore we have $\text{ann}_A x = A(q-\rho)$, hence $M \cong Ax \cong \frac{A}{A(q-\rho)}$.

3) ([11], Proposition 5.6). ∎

Now let us characterize the $\mathbb{C}[q]$-torsion free simple A-modules.

Theorem 2.30 *The two maps $S \longrightarrow B \otimes_A S$, $T \longrightarrow Soc_A T$ establish a one-one correspondence between the isomorphism classes of simple* $\mathbb{C}[q]$*-torsion free A-modules S and of simple B-modules T.*

PROOF. By Corollary 2.8 the $\mathbb{C}[q]$-torsion theory is perfect. Therefore, in particular, its quotient functor is $\Omega \cong B \otimes_A -$. If S is a $\mathbb{C}[q]$-torsion free simple A-module, then $\Omega(S)$ is a simple B-module.

If T is a simple B-module, since B is a principal ideal domain, there exists some irreducible element $b \in B$ such that $T \cong \frac{B}{Bb} \supseteq \frac{A+Bb}{Bb} \cong \frac{A}{A\cap Bb}$. As $\frac{A}{A\cap Bb}$ is nonzero and of finite length, we obtain $\mathrm{Soc}_A(T) \neq 0$. Then $\Omega(\mathrm{Soc}_A T) \cong T$, and consequently $\mathrm{Soc}_A T$ is simple.

It follows immediately that the maps are inverse of each other, up to isomorphisms. ∎

Two irreducible elements $b_1,\ b_2 \in B$ are called similar, $b_1 \sim b_2$, if $\frac{B}{Bb_1} \cong \frac{B}{Bb_2}$. We denote the similarity class of b by $[b]$; and also by $[S]$ where S is the $\mathbb{C}[q]$-torsion free simple A-module corresponding to $\frac{B}{Bb}$ by Theorem 2.30.

We note that the p-degree of b is positive, and is an invariant of $[b]$ (since it equals the $\mathbb{C}(q)$-dimension of $\frac{B}{Bb}$). Again we transfer this concept to the corresponding S, and talk of the degree of S. In particular, we call S linear or quadratic, if this degree is 1 or 2. It is natural to extend this degree to $\mathbb{C}[q]$-torsion simple A-modules, by giving them the degree zero.

There is a stronger equivalence relation, $b_1 \simeq b_2$, defined by $Bb_1 = Bb_2$, or equivalently $\phi b_1 = b_2$ for some $0 \neq \phi \in \mathbb{C}(q)$. We note that the indicial polynomials $\Theta_{b_1}(\xi)$ and $\Theta_{b_2}(\xi)$, for such strongly similar elements, differ only by the nonzero scalar factor $(q^{-\nu_\phi}\phi)(0)$, and therefore have the same roots.

For any irreducible $b \in B$ we can find $a \in A$ such that $a \simeq b$ (just multiply b by a common denominator of its coefficients). We may choose a to be left-normalized, i.e. $a \in A \setminus fA$ for any nonconstant $f \in \mathbb{C}[q]$ (remove the greatest common divisor of the coefficients). Such a left-normalized $a \in A$ is uniquely determined, up to a nonzero scalar, by being strongly similar to a given $b \in B$. This construction will be used frequently, to relate an irreducible element of B to a strongly similar element of A. In particular, $[b] = (\mathbb{C}(q) \setminus \{0\}) \cdot ([b] \cap A)$.

Definition 2.31 *Let $b \in B$, $\rho \in \mathbb{C}$. b is called ρ-preserving if $\Theta_{b,\rho}(\xi) = 0$ has no solutions in $\mathbb{Z}^-$. b is called preserving if b is ρ-preserving for all ρ.*

Remark: 1) b is preserving if it is ρ-preserving for a certain finite set of ρ's, namely, the set of zeros of the leading coefficient and the set of poles of the coefficients of b (cf. [1], 3.4).

2) If b is regarded as the differential operator $\sum_{j=0}^{n} \beta_j (\frac{\partial}{\partial q})^j$ of order n, then $\Theta_{b,\rho}(\xi)$ coincides with the indicial equation relative to the singularity ρ (cf. [1], 3.2).

3) There is always a preserving element in each similarity class (cf. [1], Lemma 3.4). Obviously one can then also find such an element which lies on A.

Lemma 2.32 *Let I be a maximal left ideal of B. If for every $\mathbb{C}[q]$-torsion simple A-module S, there exists an element of $A \cap I$ acting injectively on S, then the left ideal $A \cap I$ of A is maximal.*

PROOF. Let $J \supseteq A \cap I$, where J is a maximal left ideal of A. The simple module $\frac{A}{J}$ is either $\mathbb{C}[q]$-torsion or $\mathbb{C}[q]$-torsion free. In case $\frac{A}{J}$ is $\mathbb{C}[q]$-torsion free, $B \otimes \frac{A}{J} \cong \frac{B}{BJ}$ is a simple B-module. We have $A \cap BJ = J \supseteq A \cap I$, and this implies $I \subseteq BJ$. But I is maximal, thus $I = BJ$ and $A \cap I = J$. Thus $A \cap I$ is maximal in this case.

Now let $\frac{A}{J}$ be $\mathbb{C}[q]$-torsion. By the assumption, there exists $a \in A \cap I$ such that a acts injectively on $\frac{A}{J}$. We have therefore $a(1 + J) \neq 0$. But since $a \in A \cap I \subseteq J$, $a(1 + J) = a + J = \overline{0}$. This is a contradiction, so this second case does not occur. ∎

Lemma 2.33 *Let $a \in A$. Then a is 0-preserving if and only if the element $q^{-\nu a}a$ of A acts injectively on $\frac{A}{Aq}$.*

PROOF. Write $a = \sum_{j=0}^{n} a_j p^j$ with $a_j \in \mathbb{C}[q]$. Since $\nu a \leq \nu a_j - j$ for all j, we have $q^{-\nu a - j} a_j \in \mathbb{C}[q]$, and in particular $q^{-\nu a} a \in A$. For any $k \in \mathbb{N}$, we obtain

$$\begin{aligned}
q^{-\nu a} a \cdot p^k &= q^{-\nu a} \sum_{j=0}^{n} a_j p^j \cdot p^k \\
&= \sum_{j=0}^{n} q^{-\nu a - j} a_j (q^j p^j) p^k \\
&= \sum_{j=0}^{n} q^{-\nu a - j} a_j (qp)(qp-1)\cdots(qp-j+1) p^k \\
&= \sum_{j=0}^{n} q^{-\nu a - j} a_j (qp)\cdots(qp-j+2) p^k (pq-k-j) \\
&\equiv \sum_{j=0}^{n} q^{-\nu a - j} a_j (qp)\cdots(qp-j+2) p^k (-k-j) \pmod{Aq} \\
&\equiv \cdots \\
&\equiv \sum_{j=0}^{n} q^{-\nu a - j} a_j p^k (-k-1)(-k-2)\cdots(-k-j) \pmod{Aq} \\
&\equiv \sum_{j=0}^{n} (q^{-\nu a - j} a_j)(0)(-k-1)(-k-2)\cdots(-k-j) p^k + \\
&\quad + \text{lower terms in } p \\
&\equiv \Theta_a(-k-1) p^k + \text{lower terms in } p.
\end{aligned}$$

Let a be 0-preserving. Any nonzero element of $\frac{A}{Aq}$ has a representative $0 \neq x \in \mathbb{C}[p]$. If $x = x_0 p^k + \text{lower terms in } p$, where $0 \neq x_0 \in \mathbb{C}$, then the above observation shows

$$q^{-\nu a} a \cdot x \equiv x_0 \Theta_a(-k-1) p^k + \text{lower terms in } p \pmod{Aq}.$$

Since a is 0-preserving, $\Theta_a(-k-1) \neq 0$, and therefore $q^{-\nu a} a \cdot x + Aq \neq Aq$, i.e., $q^{-\nu a} a$ acts injectively on $\frac{A}{Aq}$.

Conversely, suppose that a is not 0-preserving, then there exists $k \in \mathbb{N}$ such that $\Theta_a(-k-1) = 0$. The above observation shows that $q^{-\nu a} a$ maps the $(k+1)$-dimensional vector space spanned by $p^k + Aq,\ p^{k-1} + Aq, \cdots,\ p + Aq,\ 1 + Aq$ into the k-dimensional vector space spanned by $p^{k-1} + Aq,\ \cdots,\ 1 + Aq$. This proves that $q^{-\nu a} a$ does not act injectively on $\frac{A}{Aq}$. ∎

Theorem 2.34 *Let $a \in A$ be irreducible and preserving. Then $\frac{A}{A\cap Ba}$ is simple (and therefore $\frac{A}{A\cap Ba} \cong \frac{A+Ba}{Ba} = Soc_A \frac{B}{Ba}$).*

PROOF. By Lemma 2.33, $(q-\rho)^{-\nu_\rho a} a \in A$ acts injectively on $\frac{A}{A(q-\rho)}$, for any ρ, and obvious $(q-\rho)^{-\nu_\rho a} a \in A \cap Ba$. Therefore, by Proposition 2.29 and Lemma 2.32, $A \cap Ba$ is a maximal left ideal of A. This proves that $\frac{A}{A\cap Ba}$ is simple. ∎

Chapter 3

Ore localizations in the Weyl algebra

From Chapter 1 we know that any hereditary torsion theory on the category of left A-modules is generated by a class of simple A-modules, and is a perfect torsion theory. An interesting question arises: which of these torsion theories can be determined by left Ore sets? Or, equivalently, for which torsion theories does the corresponding quotient ring have the form ${}_{\Sigma}A$ for some left Ore set Σ in A? We will call such a torsion theory *Ore*.

In this chapter, we study Ore localization in A. We first state the known results about this problem, some direct consequences, and a characterization of Ore localization in terms of *regular elements*. We then give our own contributions. We find that two collections of torsion theories can be determined by left Ore sets. The first consists of all torsion theories generated by classes of simple A-modules containing either all C$[q]$-torsion simple A-modules or all C$[p]$-torsion simple A-modules. In fact, we prove that any fairly large torsion theory, for instance, any torsion theory generated by all but countably many isomorphism classes of simple A-modules, is a torsion theory of this kind, up to an automorphism of A. The second consists of all torsion theories generated by classes of at most linear simple A-modules. These two collections of torsion theories provide a fairly large number of examples of Ore localizations in A.

3.1 Definitions and Ore torsion theories

Let R be a domain and $0 \notin \Sigma$ be multiplicatively closed subset of R. The left ring of fractions of R with respect to Σ is a ring ${}_{\Sigma}R$ containing R as a subring such that 1) σ is invertible in ${}_{\Sigma}R$ for every $\sigma \in \Sigma$; 2) every element of ${}_{\Sigma}R$ is of the form $\sigma^{-1}a$ for $\sigma \in \Sigma$ and $a \in R$.

Proposition 3.1 *Let R be a left Noetherian domain and Σ be a multiplicatively closed subset of R. Then ${}_{\Sigma}R$ exists if and only if Σ is a left Ore set.*

PROOF. Stenström ([14], Chapter 11, Proposition 6.4). ∎

The following are well-known examples of Ore localization in A.

Example 3.2 *Goldie's torsion theory (cf. 2.15).*

Example 3.3 *For $\rho \in C$, let $\Sigma_\rho = \{(q-\rho)^n\}_{n \in N}$. Then Σ_ρ is an Ore set. The corresponding quotient ring is ${}_{\Sigma_\rho}A = C[q,(q-\rho)^{-1}][p]$. The torsion theory ${}_{\Sigma_\rho}\mathfrak{T}$ is generated by the isomorphism class of the simple A-module $\frac{A}{A(q-\rho)}$. One can obtain a similar result for $p-\rho$, for instance, by using the A-automorphism defined by $q \to p, p \to -q$.*

PROOF. See the proof of Lemma 2.27, letting $\phi = q - \rho$. ∎

Example 3.4 *Let S be any simple A-module. Then CES, the set of elements in A which operate regularly on the injective hull ES of S, is an Ore set. The simple modules that generate this torsion theory are the ones <u>not</u> isomorphic to S.*

PROOF. Goodearl ([5], Theorem 3.2). ∎

Lemma 3.5 *Let $\mathfrak{S}$ be an isomorphism-closed class of simple A-modules, and let $x \in A$. Then following statements are equivalent:*

1) $x \in CET$ for every $T \notin \mathfrak{S}$.

2) $Hom(\frac{A}{Ax}, ET) = 0$ for every $T \notin \mathfrak{S}$.

3) $x \neq 0$, and every composition factor of $\frac{A}{Ax}$ is contained in $\mathfrak{S}$.

4) $\frac{A}{Ax} \in_{\mathfrak{S}} \mathfrak{T}$.

5) x is invertible in the quotient ring of ${}_{\mathfrak{S}}\mathfrak{T}$.

PROOF. 1) $\Rightarrow$ 2): We are given $x \in CET$ for all $T \notin \mathfrak{S}$. Let $\alpha \in \mathrm{Hom}(\frac{A}{Ax}, ET)$ with $\alpha(\overline{1}) = t$. Since $xt = \alpha(\overline{x}) = \alpha(\overline{0}) = 0$ and $x \in CET$, we conclude $t = 0$. Hence $\alpha = 0$ and $\mathrm{Hom}(\frac{A}{Ax}, ET) = 0$.

2) $\Rightarrow$ 3) We are given $\mathrm{Hom}(\frac{A}{Ax}, ET) = 0$ for every $T \notin \mathfrak{S}$. Let

$$Ax = X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n = A$$

be a composition series of $\frac{A}{Ax}$. For each i, since the homomorphism $\frac{A}{Ax} \longrightarrow \frac{A}{X_i}$ is onto and $\mathrm{Hom}(\frac{A}{Ax}, ET) = 0$, we have $\mathrm{Hom}(\frac{A}{X_i}, ET) = 0$. Since $\frac{X_{i+1}}{X_i} \subseteq \frac{A}{X_i}$ and ET is injective, we have $\mathrm{Hom}(\frac{X_{i+1}}{X_i}, ET) = 0$. Hence $\frac{X_{i+1}}{X_i} \ncong T$. This proves that each composition factor of $\frac{A}{Ax}$ is contained in $\mathfrak{S}$.

3) $\Rightarrow$ 4): This implication follows from the fact that any torsion class is closed under extensions.

4) $\Rightarrow$ 5): Let Q be the quotient ring of ${}_{\mathfrak{S}}\mathfrak{T}$. Since $\frac{A}{Ax} \in {}_{\mathfrak{S}}\mathfrak{T}$ and ${}_{\mathfrak{S}}\mathfrak{T}$ is a perfect torsion theory, we have $Q \otimes \frac{A}{Ax} = 0$. It is easy to show $Q \otimes \frac{A}{Ax} \cong \frac{Q}{Qx}$. Hence $Q = Qx$, i.e., there exists $y \in Q$ such that $yx = 1$. Note that $(xy-1)x = xyx - x = x(yx) - x = x - x = 0$ and $Q \subseteq K$ is a domain; therefore $xy = 1$. Hence x is invertible in the quotient ring Q of ${}_{\mathfrak{S}}\mathfrak{T}$.

5) $\Rightarrow$ 1): Let T be any simple A-module such that $T \notin \mathfrak{S}$. Then we have $T \in_{\mathfrak{S}} \mathfrak{F}$. Since T is essential in its injective hull ET and ${}_{\mathfrak{S}}\mathfrak{F}$ is closed under essential extension, $ET \in_{\mathfrak{S}} \mathfrak{F}$. Therefore ET is a Q-module because ${}_{\mathfrak{S}}\mathfrak{T}$ is perfect. For any $0 \neq t \in ET$, we have $t = (yx)t = y(xt)$, where y is the inverse element of x in Q. Hence $xt \neq 0$ for any $0 \neq t \in ET$. This proves that x operates regularly on ET. ∎

Definition 3.6 *An element x of A is called $\mathfrak{S}$-regular if it satisfies the five equivalent conditions of Lemma 3.5. The set of $\mathfrak{S}$-regular elements of A is denoted by $\Omega(\mathfrak{S})$.*

We call these elements $\mathfrak{S}$-regular, in spite of the fact that they operate regularly on the injective hulls ET of the simple A-modules T which <u>are not</u> contained in $\mathfrak{S}$. An $\mathfrak{S}$-regular element x belongs to $\mathfrak{S}$ in the sense that the module $\frac{A}{Ax} \in_{\mathfrak{S}} \mathfrak{T}$, and that it is invertible in the quotient ring of ${}_{\mathfrak{S}}\mathfrak{T}$. We also find this terminology useful later on. In particular, if $\mathfrak{S}$ contains the isomorphism class of one simple module S only, we shall call the $\mathfrak{S}$-regular elements S-regular, (or $[a]$-regular, where $[a]$ is the corresponding similarity class), and the set of S-regular elements will be denoted by $\Omega(S)$.

Corollary 3.7 *Let $x \in A$ be irreducible in B, and $S \cong Soc_A \frac{B}{Bx}$. Then x is S-regular if and only if $\frac{A}{Ax} \cong S$.*

PROOF. Since x is irreducible in B, any composition series of $\frac{A}{Ax}$ has precisely one $\mathbb{C}[q]$-torsion free factor. Therefore x is S-regular, if and only if all composition factors of $\frac{A}{Ax}$ are isomorphic to S, if and only if $\frac{A}{Ax} \cong S$. ∎

The following remark provides some examples of S-regular elements:

Remark: Let $x \in A$, irreducible in B. If x is of the form cp^n+lower terms in p, where $c \in \mathbb{C} \setminus \{0\}$, then $S = \frac{A}{Ax}$ is simple.

Indeed, since the leading coefficient of x is a nonzero scalar, $\Theta_{a,\rho}(\xi) = c\xi(\xi - 1)\cdots(\xi - n + 1)$ for every ρ, and therefore it has roots $0, 1, \cdots, n-1$. This means

x is preserving. Hence $\frac{A}{A\cap Bx}$ is a simple A-module by Theorem 2.34. Suppose there exists an element $b \in B$ such that $bx \in A \cap Bx$ and $b \notin A$. Write $b = \sum_{j=0}^{t} b_j p^j$. Let t_0 be the largest integer such that $b_{t_0} \notin \mathbb{C}[q]$. Therefore $\sum_{j=t_0+1}^{t} b_j p^j \in A$. Hence we have $(\sum_{j=0}^{t_0} b_j p^j)x \in A$. But the leading coefficient of this element is $cb_{t_0} \notin \mathbb{C}[q]$. This is a contradiction. We conclude that such an element does not exist, therefore $A \cap Bx = Ax$. Hence $\frac{A}{Ax} = \frac{A}{A\cap Bx}$ is simple.

It is easy to prove the following:

Proposition 3.8 *Let Σ be a multiplicatively closed set in a ring R. The following are equivalent:*

1) Σ is left Ore.

2) Σ operates regularly on every (uniform injective) Σ-torsion free module.

PROOF. 1) $\Rightarrow$ 2): (cf. Lemma 2.14).

2) $\Rightarrow$ 1): Claim: $\frac{R}{Rx}$ is Σ-torsion for any $x \in \Sigma$.

Suppose not. Let us consider the torsion radical $T(\frac{R}{Rx})$ of $\frac{R}{Rx}$. Write $T(\frac{R}{Rx}) = \frac{T}{Rx}$, then $T \neq R$, therefore $\frac{R}{Rx}/T(\frac{R}{Rx}) \cong \frac{R}{T} \neq 0$, and is Σ-torsion free. Consider $\frac{R}{Rx} \longrightarrow \frac{R}{T} \longrightarrow E(\frac{R}{T}) = \bigoplus_{i\in I} E_i$ where each E_i is indecomposable injective. Since $\frac{R}{T}$ is essential in $E(\frac{R}{T})$, $E_i \cap \frac{R}{T} \neq 0$ for every i. We have that $E_i \cap \frac{R}{T}$ is Σ-torsion free as a submodule of $\frac{R}{T}$. Therefore $T(E_i) \cap (E_i \cap \frac{R}{T}) = 0$. We conclude $T(E_i) = 0$ for each i, since E_i is uniform. Hence each E_i is Σ-torsion free, and is therefore Σ-regular by the assumption. Let $\frac{R}{Rx} \ni \overline{1} \rightarrow (e_i)_{i\in I} \in \bigoplus_{i\in I} E_i$. There exists $i \in I$ such that $e_i \neq 0$. Since $\overline{0} = \overline{x} \rightarrow (xe_i)_i = (0)_i$, we have $xe_i = 0$ for every $i \in I$. Since x operates regularly on E_i, $e_i = 0$ for every $i \in I$. This is a contradiction. Hence $\frac{R}{Rx}$ is Σ-torsion. We have proved the claim.

Now, it follows easily that Σ is a left Ore set in R: for any elements $x \in \Sigma$ and $r \in R$, since $\frac{R}{Rx}$ is Σ-torsion, there exists $y \in \Sigma$ such that $yr \in Rx$, i.e., $yr = r'x$ for some $r' \in R$. ∎

Lemma 3.9 *The indecomposable injective A-modules are precisely the injective hulls of simple A-modules and the quotient division ring K of A.*

PROOF. It is clear that ES is indecomposable injective for any simple A-module S. K is indecomposable because it is the injective hull of A and A is uniform.

Let E be an indecomposable injective module, and X any finitely generated submodule of E. Then $|X| \leq |A| = 1$. Therefore X has either a 0-critical or 1-critical submodule, say Y, and certainly $EY = E$. If Y is 0-critical, Y is simple. If Y is 1-critical, then Y is Goldie torsion free by Lemma 2.15. Therefore E is Goldie torsion free and hence a K-module. Since K is a division ring, E is a vector space over K. We conclude $E \cong K$, because E is indecomposable. ∎

Now we are ready to characterize Ore localization in terms of regular elements.

Proposition 3.10 *Let $\mathfrak{S}$ be an isomorphism-closed class of simple left A-modules. Then ${}_{\mathfrak{S}}\mathfrak{T}$ is Ore if and only if ${}_{\mathfrak{S}}\mathfrak{T} = {}_{\Omega(\mathfrak{S})}\mathfrak{T}$. Furthermore if ${}_{\mathfrak{S}}\mathfrak{T} = {}_{\Sigma}\mathfrak{T}$ for some left Ore set Σ in A, then $\Sigma \subseteq \Omega(\mathfrak{S})$ and $\Omega(\mathfrak{S})$ is a left Ore set in A.*

PROOF. ($\Rightarrow$) Suppose that ${}_{\mathfrak{S}}\mathfrak{T}$ is Ore, i.e. we have ${}_{\mathfrak{S}}\mathfrak{T} = {}_{\Sigma}\mathfrak{T}$ for some left Ore set Σ in A. Let $x \in \Sigma$. Since Σ is left Ore, $\frac{A}{Ax}$ is Σ-torsion, and therefore $\frac{A}{Ax} \in {}_{\mathfrak{S}}\mathfrak{T}$. We conclude $x \in \Omega(\mathfrak{S})$ by Lemma 3.5. This proves $\Sigma \subseteq \Omega(\mathfrak{S})$, and therefore ${}_{\mathfrak{S}}\mathfrak{T} = {}_{\Sigma}\mathfrak{T} \subseteq {}_{\Omega(\mathfrak{S})}\mathfrak{T}$. On the other hand, for every simple A-module $T \notin \mathfrak{S}$, we have $T \in {}_{\Omega(\mathfrak{S})}\mathfrak{F}$ since $\Omega(\mathfrak{S})$ operates regularly on ET. Therefore ${}_{\Omega(\mathfrak{S})}\mathfrak{T} \subseteq {}_{\mathfrak{S}}\mathfrak{T}$, and we conclude ${}_{\mathfrak{S}}\mathfrak{T} = {}_{\Omega(\mathfrak{S})}\mathfrak{T}$.

($\Leftarrow$) Let ${}_{\mathfrak{S}}\mathfrak{T} = {}_{\Omega(\mathfrak{S})}\mathfrak{T}$. We show that $\Omega(\mathfrak{S})$ is a left Ore set in A. By Proposition 3.8, it is sufficient to show that $\Omega(\mathfrak{S})$ operates regularly on every uniform injective $\Omega(\mathfrak{S})$-torsion free A-module E. By Lemma 3.9, $E \cong ET$ for some simple A-module T or $E \cong K$. In case $E \cong ET$, we have $T \notin {}_{\Omega(\mathfrak{S})}\mathfrak{T} = {}_{\mathfrak{S}}\mathfrak{T}$ hence $T \notin \mathfrak{S}$; therefore $\Omega(\mathfrak{S})$ operates regularly on ET, hence on E. In case $E \cong K$, $\Omega(\mathfrak{S})$ trivialy operates regularly. This proves that $\Omega(\mathfrak{S})$ is a left Ore set in A, and consequently ${}_{\mathfrak{S}}\mathfrak{T}$ is Ore.

Now let ${}_{\mathfrak{S}}\mathfrak{T} =_{\Sigma} \mathfrak{T}$ for some left Ore set Σ in A. By the first half of the proof, we know $\Sigma \subseteq \Omega(\mathfrak{S})$ and ${}_{\mathfrak{S}}\mathfrak{T} =_{\Omega(\mathfrak{S})} \mathfrak{T}$. The second half of the proof shows that $\Omega(\mathfrak{S})$ is a left Ore set. ∎

Remark: The above proposition tells us that $\Omega(\mathfrak{S})$ is a natural candidate for a left Ore set, to determine the torsion theory ${}_{\mathfrak{S}}\mathfrak{T}$. In fact, if ${}_{\mathfrak{S}}\mathfrak{T}$ is Ore, then $\Omega(\mathfrak{S})$ is the *saturation* of any left Ore set Σ such that ${}_{\mathfrak{S}}\mathfrak{T} =_{\Sigma} \mathfrak{T}$, i.e., $\Omega(\mathfrak{S}) = \{x \in A \mid x^{-1} \in {}_{\Sigma}A\}$.

The following lemma is known.

Lemma 3.11 *Let $\{\Sigma_\alpha\}_{\alpha\in\Lambda}$ be a family of left Ore set in a ring R. Then the smallest multiplicatively closed set Σ containing all Σ_α, is a left Ore set, and*

$$_{\Sigma}\mathfrak{T} = \bigvee_{\alpha\in\Lambda} ({}_{\Sigma_\alpha}\mathfrak{T}).$$

Proof. Let $\sigma \in \Sigma$, $r \in R$. Write $\sigma = \sigma_n\sigma_{n-1}\cdots\sigma_1$ with $\sigma_i \in \Sigma_{\alpha_i}$ for some $\alpha_i \in \Lambda$. Since Σ_{α_1} is left Ore, there is exist $\sigma_1' \in \Sigma_{\alpha_1}$ and $r_1 \in R$ such that $r_1\sigma_1 = \sigma_1' r$. Since Σ_{α_2} is left Ore, there exist $\sigma_2' \in \Sigma_{\alpha_2}$ and $r_2 \in R$ such that $r_2\sigma_2 = \sigma_2' r_1$. Therefore $r_2(\sigma_2\sigma_1) = \sigma_2' r_1\sigma_1 = (\sigma_2'\sigma_1')r$. If one continues this way, one obtains $r_n \in R$, $\sigma_n' \in \Sigma_n$ such that $r_n\sigma = (\sigma_n'\sigma_{n-1}'\cdots\sigma_1')r$. Hence Σ is a left Ore set.

Since $\Sigma_\alpha \subseteq \Sigma$ for every $\alpha \in \Lambda$, ${}_{\Sigma_\alpha}\mathfrak{T} \subseteq {}_{\Sigma}\mathfrak{T}$, and therefore $\bigvee_{\alpha\in\Lambda} ({}_{\Sigma_\alpha}\mathfrak{T}) \subseteq_{\Sigma} \mathfrak{T}$.

By Stenström ([14], Chapter 6, Propositions 2.5 and 3.3) and the fact that Σ_α are Ore sets, we have ${}_{\Sigma}\mathfrak{T} \subseteq \bigvee_{\alpha\in\Lambda} ({}_{\Sigma_\alpha}\mathfrak{T})$.

Together we have the equality. ∎

The above lemma enables us to construct new left Ore sets from known ones. For instance, we can apply this to the Σ_p's defined in 3.3.

3.2 Large classes of simple modules

We have seen some examples of Ore torsion theories on A-Mod with the property that $\mathfrak{S}$ is large: the Goldie torsion theory where $\mathfrak{S}$ is the set of all simple modules (Example 3.2); and Goodearl's torsion theories where $\mathfrak{S}$ is the set of all simple modules except for one isomorphism class (Example 3.4). One can ask whether $_{\mathfrak{S}}\mathfrak{T}$ is Ore if $\mathfrak{S}$ is a set of all simple modules except for finitely many or even countablely many isomorphism classes. The answer is Yes! We prove that if $\mathfrak{S}$ is sufficiently large, then $_{\mathfrak{S}}\mathfrak{T}$ is Ore.

The following lemma enables us to construct a left Ore set from another point of view.

Lemma 3.12 *Let R be a Noetherian domain, Φ a left Ore set in R, and $T =_{\Phi} R$. If Σ is a left Ore set in T, then $(\Phi * \Sigma) \cap R$ is a left Ore set in R and $_{(\Phi*\Sigma)\cap R}R =_{\Sigma} T$, where $\Phi * \Sigma$ is the smallest multiplicatively closed set containing Φ and Σ.*

PROOF. **Claim 1:** Φ is a left Ore set in T.

Let $\psi \in \Phi$ and $t \in T$. Write $t = \phi^{-1}r$ with $\phi \in \Phi$ and $r \in R$. Since Φ is left Ore, there exist $r_1 \in R$ and $\psi_1 \in \Phi$ such that $r_1\psi = \psi_1 r$. Therefore $\phi t = r = (\psi_1^{-1}r_1)\psi$, $\psi_1^{-1}r_1 \in T$. Hence Φ is a left Ore set in T.

Claim 2: $(\Phi * \Sigma) \cap R$ is a left Ore set in R.

First $\Phi * \Sigma$ is a left Ore set in T by Lemma 3.11. Let $x \in (\Phi * \Sigma) \cap R$ and $r \in R$. Since $\Phi * \Sigma$ is a left Ore set in T, there exist $y \in \Phi * \Sigma$ and $t \in T$ such that $yr = tx$. Write $y = \phi^{-1}y_1$ and $t = \phi^{-1}t_1$ where $\phi \in \Phi$, y_1 and $t_1 \in R$. We have $\phi^{-1}y_1 r = \phi^{-1}t_1 x$, i.e., $y_1 r = t_1 x$. It is clear that $y_1 = \phi y \in (\Phi * \Sigma) \cap R$. Hence $(\Phi * \Sigma) \cap R$ is a left Ore set in R.

Claim 3: $_{\Sigma}(_{\Phi}R)$ coincides with $_{(\Phi*\Sigma)\cap R}R$.

Let $\sigma^{-1}(\phi^{-1}r) \in_{\Sigma} (_{\Phi}R)$ with $\sigma \in \Sigma$, $\phi \in \Phi$ and $r \in R$. $\sigma^{-1}(\phi^{-1}r) = (\phi\sigma)^{-1}r$

and $\phi\sigma \in \Phi * \Sigma \subseteq T$. There exists $\phi_1 \in \Phi$ such that $\phi_1(\phi\sigma) \in R$. Then $(\phi\sigma)^{-1}r = (\phi\sigma)^{-1}\phi_1^{-1}(\phi_1 r) = (\phi_1\phi\sigma)^{-1}(\phi_1 r) \in_{(\Phi*\Sigma)\cap R} R$. This proves ${}_{\Sigma}({}_{\Phi}R) \subseteq_{(\Phi*\Sigma)\cap R} R$.

Let $x^{-1}r \in_{(\Phi*\Sigma)\cap R} R$, where $x \in (\Phi*\Sigma)\cap R$ and $r \in R$. Write $x = \sigma_k\phi_k \cdots \sigma_1\phi_1$ with $\phi_i \in \Phi$ and $\sigma_i \in \Sigma$ for $1 \leq i \leq k$. $x^{-1}r = \phi_1^{-1}\sigma_1^{-1}\cdots\phi_k^{-1}\sigma_k^{-1}r$. Since ϕ_i^{-1}, $\sigma_i^{-1} \in_{\Sigma} ({}_{\Phi}R)$ for all i, we obtain $x^{-1}r \in_{\Sigma} ({}_{\Phi}R)$. This proves ${}_{(\Phi*\Sigma)\cap R}R \subseteq_{\Sigma} ({}_{\Phi}R)$. ∎

The following lemma is well-known.

Lemma 3.13 *Let R be a principal left ideal domain. Then every torsion theory on R-Mod can be determined by a left Ore set.*

PROOF. See Stenström ([14], Chapter 11, Proposition 6.1). ∎

As an immediate consequence of Lemma 3.12 and Lemma 3.13 we have the following:

Theorem 3.14 *Let $\mathfrak{S}$ be an isomorphism-closed class of simple A-modules. If $\mathfrak{S}$ contains all $\mathbb{C}[q]$-torsion (or $\mathbb{C}[p]$-torsion) simple A-modules, then ${}_{\mathfrak{S}}\mathfrak{T}$ is Ore.*

PROOF. Without loss of generality, let us assume that $\mathfrak{S}$ contains all $\mathbb{C}[q]$-torsion simple A-modules. Therefore ${}_{\mathfrak{S}}\mathfrak{T} \supseteq {}_{\mathbb{C}[q]\setminus\{0\}}\mathfrak{T}$. Let $\Phi = \mathbb{C}[q] \setminus \{0\}$. It is a left Ore set in A, and ${}_{\Phi}A = B = \mathbb{C}(q)[p]$.

Consider the torsion theory on B-Mod generated by $\{B \otimes_A S \mid S \in \mathfrak{S}\}$. By Lemma 3.13, this torsion theory can be determined by a left Ore set, say Σ, because B is a left principal ideal domain. We conclude ${}_{\Sigma}({}_{\Phi}A) =_{(\Phi*\Sigma)\cap A} A$, i.e., ${}_{\mathfrak{S}}\mathfrak{T} =_{(\Phi*\Sigma)\cap A} \mathfrak{T}$ by Corollary 2.8, and $(\Phi * \Sigma) \cap A$ is a left Ore set in A by Lemma 3.12. ∎

Note that Goodearl's theorem (Example 3.4) is a special case of Theorem 3.14 since the class of all simple A-modules but one isomorphism class certainly contains all

C[q]-torsion or C[p]-torsion simple A-modules (since no simple module is simultaneous C[q]-torsion and C[p]-torsion).

In order to generalize Theorem 3.14, we have to use automorphisms of A.

Lemma 3.15 *Let σ be an automorphism of A. Then*

1) $\sigma(I)$ is a maximal left ideal of A for any maximal left ideal I of A.

2) Define $\sigma(\frac{A}{I}) = \frac{A}{\sigma(I)}$ for any left ideal I of A. This induces a well defined map of isomorphisms types. In particular, simple modules are mapped to simple modules.

3) $\sigma(\Sigma)$ is a left Ore set for any left Ore set Σ in A.

PROOF. 1) and 3) are straightforward.

2) Let I_1 and I_2 be two left ideals of A such that $\frac{A}{I_1} \overset{\alpha}{\cong} \frac{A}{I_2}$; we have to show $\frac{A}{\sigma(I_1)} \cong \frac{A}{\sigma(I_2)}$.

Define a map β: $\frac{A}{\sigma(I_1)} \xrightarrow{\sigma^{-1}} \frac{A}{I_1} \xrightarrow{\alpha} \frac{A}{I_2} \xrightarrow{\sigma} \frac{A}{\sigma(I_2)}$.

It is clear that β is well-defined, one-one and onto.

Since σ^{-1} is σ^{-1}-linear, σ is σ-linear, and α is an A-homomorphism, β is an A-homomorphism.

Together we proved $\frac{A}{\sigma(I_1)} \cong \frac{A}{\sigma(I_2)}$ as A-modules.

If $\frac{A}{I}$ is a simple A-module, then I is maximal. Hence $\sigma(I)$ is maximal by 1), and therefore $\sigma(\frac{A}{I}) = \frac{A}{\sigma(I)}$ is a simple A-module. ∎

Lemma 3.16 *Let $\sigma \in$ Aut A, and $\mathfrak{S}$ an isomorphism-closed classes of simple A-modules. Define $\sigma(\mathfrak{S})$ to be the isomorphism-closure of $\{\sigma(\frac{A}{I}) \mid \frac{A}{I} \in \mathfrak{S}\}$. Then ${}_{\mathfrak{S}}\mathfrak{T}$ is Ore if and only if ${}_{\sigma(\mathfrak{S})}\mathfrak{T}$ is Ore.*

PROOF. Let ${}_{\mathfrak{S}}\mathfrak{T}$ be Ore, i.e., there exists a left Ore set Σ such that ${}_{\mathfrak{S}}\mathfrak{T} = {}_{\Sigma}\mathfrak{T}$. Clearly $\sigma(\Sigma)$ is also a left Ore set. We show ${}_{\sigma(\mathfrak{S})}\mathfrak{T} = {}_{\sigma(\Sigma)}\mathfrak{T}$. It is clear that if S is Σ-torsion, then $\sigma(S)$ is $\sigma(\Sigma)$-torsion. This proves ${}_{\sigma(\mathfrak{S})}\mathfrak{T} \subseteq {}_{\sigma(\Sigma)}\mathfrak{T}$.

Conversely, let S be $\sigma(\Sigma)$-torsion. Then $\sigma^{-1}(S)$ is Σ-torsion, and therefore belongs to $\mathfrak{S}$ since ${}_{\Sigma}\mathfrak{T} = {}_{\mathfrak{S}}\mathfrak{T}$. This implies $S \cong \sigma(\sigma^{-1}S) \in \sigma(\mathfrak{S})$. Hence ${}_{\sigma(\Sigma)}\mathfrak{T} \subseteq {}_{\sigma(\mathfrak{S})}\mathfrak{T}$. We conclude that ${}_{\sigma(\Sigma)}\mathfrak{T} = {}_{\sigma(\mathfrak{S})}\mathfrak{T}$ holds.

Using $\sigma^{-1} \in \mathrm{Aut}A$, one proves similarly that if ${}_{\sigma(\mathfrak{S})}\mathfrak{T}$ is Ore, then ${}_{\mathfrak{S}}\mathfrak{T}$ is Ore. ∎

In order to show that ${}_{\mathfrak{S}}\mathfrak{T}$ is Ore, our next strategy is to find an automorphism σ in A such that $\sigma(\mathfrak{S})$ contains all $\mathbf{C}[p]$ (or $\mathbf{C}[q]$) torsion simple A-modules. If this can be done, then by Theorem 3.14 ${}_{\sigma(\mathfrak{S})}\mathfrak{T}$ is Ore, and hence ${}_{\mathfrak{S}}\mathfrak{T}$ is Ore by Lemma 3.16.

Let $f \in q\mathbf{C}[q]$. Define $\mathfrak{S}_f$ to be the class of simple A-modules which are isomorphic to simple modules of the form $\frac{A}{A(p+f+\rho)}$ for some $\rho \in \mathbf{C}$.

Theorem 3.17 *Let $\mathfrak{S}$ be an isomorphism-closed class of simple A-modules. If there exists an $f \in q\mathbf{C}[q]$ such that $\mathfrak{S}_f \subseteq \mathfrak{S}$, then ${}_{\mathfrak{S}}\mathfrak{T}$ is Ore. In particular, if $\mathfrak{S}$ contains all but countably many isomorphism classes of simple modules, then ${}_{\mathfrak{S}}\mathfrak{T}$ is Ore.*

PROOF. Define $\sigma \in \mathrm{Aut}\,A$ by $q \to q$, $p \to p - f(q)$ (cf. 2.4). Since $\mathfrak{S}_f \subseteq \mathfrak{S}$, $\sigma(\mathfrak{S}_f) \subseteq \sigma(\mathfrak{S})$. Note that $\sigma(\mathfrak{S}_f)$ consists of all $\mathbf{C}[p]$-torsion simple A-modules, this implies that ${}_{\sigma(\mathfrak{S})}\mathfrak{T}$ is Ore by Theorem 3.14. Hence ${}_{\mathfrak{S}}\mathfrak{T}$ is Ore by Lemma 3.16.

Claim: If $f, g \in q\mathbf{C}[q]$ are different, then $\mathfrak{S}_f \cap \mathfrak{S}_g = \emptyset$.

It is a well-known fact that two linear elements, say $\alpha p - \beta$ and $\alpha_1 p - \beta_1$, are similar if and only if $\beta/\alpha - \beta_1/\alpha_1$ is a logarithmic derivative ϕ'/ϕ of an element $\phi \in \mathbf{C}(q)\setminus\{0\}$ (cf. [11], Proposition 4.5). Note that $\phi'/\phi = \sum_\rho \frac{n_\rho}{q-\rho}$, if $\phi = \prod(q-\rho)^{n_\rho}$. In particular, $\phi'/\phi \in \mathbf{C}[q]$ if and only if $n_\rho = 0$ for all ρ if and only if $\phi \in \mathbf{C}\setminus\{0\}$.

If $\frac{A}{A(p+f+\rho_1)} \cong \frac{A}{A(p+g+\rho_2)}$ for some $\rho_i \in \mathbf{C}$, where $i = 1, 2$, then $p + f + \rho_1 \sim p + g + \rho_2$, therefore $p + f + \rho_1 = p + g + \rho_2 + \frac{\phi'}{\phi}$ for some $\phi \in \mathbf{C}(q)$. This implies

$\frac{\phi'}{\phi} = f - g + \rho_1 - \rho_2 \in \mathbb{C}[q]$, and therefore $\phi \in \mathbb{C}$. Hence $p + f + \rho_1 = p + g + \rho_2$, this implies $f - g = \rho_2 - \rho_1 \in \mathbb{C} \cap q\mathbb{C}[q] = 0$ which is a contradiction. We have proved the claim.

Suppose that $\mathfrak{S}$ contains all but countably many isomorphism classes of simple A-modules. Since there are uncountably many $f \in q\mathbb{C}[q]$, and the $\mathfrak{S}_f$ are pairwise disjoint, $\mathfrak{S}_f \not\subseteq \mathfrak{S}$ for all f implies that $\mathfrak{S}$ misses uncountably many isomorphism classes. Thus if $\mathfrak{S}$ contains all but countably many isomorphism classes, there is f such that $\mathfrak{S}_f \subseteq \mathfrak{S}$. Hence ${}_{\mathfrak{S}}\mathfrak{T}$ is Ore by the above argument. ∎

Remark: We could replace $\mathfrak{S}_f$ by $\{\frac{A}{A(\sigma(p)+\rho)} \mid \rho \in \mathbb{C}\}$ in Theorem 3.17, where σ is any automorphism of A, and the statement still holds. The problem is that we cannot describe the automorphism images $\sigma(p)$ of p explicitly.

The method obviously requires that $\mathfrak{S}$ contains 2^{ω} isomorphism classes. If this holds, can one always find an automorphism σ of A such that $\sigma(\mathfrak{S})$ contains all $\mathbb{C}[p]$-torsion simple A-modules? We do not know at this stage.

3.3 Small classes of simple modules

In the previous section, we have seen that if $\mathfrak{S}$ is sufficiently large, then ${}_{\mathfrak{S}}\mathfrak{T}$ is Ore. In this section, we will study ${}_S\mathfrak{T}$ where S is a simple A-module. These torsion theories are the most interesting and difficulty ones for the following reason: we recall that it is unknown whether every torsion theory on A-Mod is Ore. To show that this is true, it is sufficient to prove that every ${}_S\mathfrak{T}$ is Ore, because any arbitrary torsion theory ${}_{\mathfrak{S}}\mathfrak{T}$ is a join of the ${}_S\mathfrak{T}$'s, where $S \in \mathfrak{S}$ and because of Lemma 3.11. On the other hand, if a counter example exists, there will be one among the ${}_S\mathfrak{T}$'s.

Unfortunately, we cannot decide this question. What we can show is that if S is a linear simple A-module, then ${}_S\mathfrak{T}$ is Ore.

Recall that a simple A-module S is linear if S is corresponding to a similarity class of linear elements in B. Any linear element of A is always irreducible in B. Let us take a close look of the indicial polynomials of a linear element.

Lemma 3.18 *Let $a = \alpha p - \beta \in A$ such that $(\alpha,\ \beta) = 1$, where $\alpha = \sum_i \alpha_i q^i$, $\beta = \sum_j \beta_j q^j \in \mathbb{C}[q]$.*

1) If 0 is not a root of α, then $\Theta_a(\xi) = \alpha_0 \xi$ has the root 0.

2) If 0 is a simple root of α, then $\Theta_a(\xi) = \alpha_1 \xi - \beta_0$ has the root $Res_0 \beta/\alpha$ (the residue of β/α at 0).

3) If 0 is a multiple root of α, then $\Theta_a(\xi) = -\beta_0 \neq 0$ has no root.

4) Let 0 be a root of α. If $\Theta_a(N) = 0$ for some integer N, then there exists $b \in A$ such that $aq^N = q^{N+1}b$.

PROOF. 1) If 0 is not a root of α, then $\alpha_0 \neq 0$, and therefore $\Theta_a(\xi) = \alpha_0 \xi$.

2) Let 0 be a simple root of α. Then $\alpha_0 = 0$, $\alpha_1 \neq 0$, and $\beta_0 \neq 0$ since $(\alpha,\ \beta) = 1$. Hence $\Theta_a(\xi) = \alpha_1 \xi - \beta_0$ has the root β_0/α_1, which is the residue of β/α at 0.

3) Let 0 be a multiple root of α. Then $\alpha_0 = \alpha_1 = 0$, and $\beta_0 \neq 0$ since $(\alpha,\ \beta) = 1$. Hence $\Theta_a(\xi) = -\beta_0 \neq 0$.

4) Since 0 is a root of α, if $\Theta_a(N) = 0$ for some integer N, then 0 must be a simple root of α, and therefore $N = \beta_0/\alpha_1$ by 2). Note $\alpha q^{-1} \in A$. We have

$$\begin{aligned}
aq^N &= (\alpha p - \beta)q^N = \alpha(q^N p + Nq^{N-1}) - \beta q^N \\
&= q^N(\alpha p + (\alpha q^{-1} N - \beta)) \\
&= q^N(\alpha p + [(N\alpha_1 - \beta_0) + \text{higher terms in } q] \\
&= q^N(\alpha p - (0 + \phi q)) \quad (\text{for some } \phi \in \mathbb{C}[q]) \\
&= q^{N+1}[(\alpha q^{-1})p - \phi].
\end{aligned}$$

Let $b = (\alpha q^{-1})p - \phi$. Then $b \in A$ and $aq^N = q^{N+1}b$. ∎

Lemma 3.19 *Let $a = \alpha p - \beta \in A$ such that $(\alpha, \beta) = 1$. Let S be the simple A-module corresponding to $[a]$. Then following statements are equivalent.*

1) a is S-regular.

2) $\Theta_{a,\rho}(\xi) = 0$ has no solutions in $\mathbf{Z}$ for every root ρ of α.

3) $Res_\rho \beta/\alpha \notin \mathbf{Z}$ for any simple pole ρ of β/α.

PROOF. 1) $\Rightarrow$ 3): Given that a is S-regular, that is $S \cong \frac{A}{Aa}$ by Corollary 3.7. Let ρ be a simple pole of β/α. Since $(\alpha, \beta) = 1$, $\nu_\rho\alpha = 1$ and $\nu_\rho\beta = 0$, therefore ρ is a simple root of α. By Lemma 3.18 2), $\Theta_{a,\rho}(\xi)$ has the root $\mathrm{Res}_\rho\beta/\alpha$.

Suppose there exists an integer N such that $\Theta_{a,\rho}(N) = 0$. By Lemma 3.18, there exists $b \in A$ such that $a(q-\rho)^N = (q-\rho)^{N+1}b$. If $N \in \mathbf{Z}^+$, then the equality implies that $\frac{A}{Aa}$ has a factor which is isomorphic to $\frac{A}{A(q-\rho)}$. This is a contradiction. If $N \in \mathbf{Z}^-$, again we have $a(q-\rho)^N = (q-\rho)^{N+1}b$ for some element $b \in A$, therefore $(q-\rho)^{-N-1}a = b(q-\rho)^{-N}$. Let $M = -N-1$. Then $M \in \mathbf{Z}^+$, $-N = M+1$ and $(q-\rho)^M a = b(q-\rho)^{M+1}$. The equality implies that $\frac{A}{Aa}$ has a factor which is isomorphic to $\frac{A}{A(q-\rho)}$. This is a contradiction.

Together we have that the root $\mathrm{Res}_\rho\beta/\alpha$ of $\Theta_{a,\rho}(\xi) = 0$ is not an integer for any root ρ of α.

3) $\Rightarrow$ 2) Let 0 be a multiple root of α. By Lemma 3.18 3), $\Theta_a(\xi) = -\beta_0 \neq 0$ has no roots.

Let 0 be a simple root of α. Then $\nu\alpha = 1$ and $\nu\beta = 0$, and therefore $\nu(\beta/\alpha) = -1$, i.e., 0 is a simple pole of α. Hence $\mathrm{Res}_0\beta/\alpha \notin \mathbf{Z}$ by 3). This implies that the root $\mathrm{Res}_0\beta/\alpha$ of $\Theta_a(\xi)$ is not in $\mathbf{Z}$.

Similarly, one can show that, for an arbitrary root ρ of α, $\Theta_{a,\rho}(\xi)$ has no root in $\mathbf{Z}$.

2) $\Rightarrow$ 1) By the remark after Definition 2.31, the fact that $\Theta_{a,\rho}(\xi) = 0$ has no solution in $\mathbf{Z}^-$ for every root ρ of α means that a is preserving, thus $\frac{A}{A\cap Ba}$ is simple by Theorem 2.34.

We show $A \cap Ba = Aa$. Since $\frac{A\cap Ba}{Aa}$ is $\mathbf{C}[q]$-torsion, it is semisimple by ([11], Theorem 5.7). It is sufficient to show that $\frac{A\cap Ba}{Aa}$ has no submodule which is isomorphic to $\frac{A}{A(q-\rho)}$ for all ρ. (The proof provided here is a specialization of the proof for Theorem 4.10 in Chapter 4.)

Suppose there is a $\mathbf{C}[q]$-torsion simple A-module T that is isomorphic to a submodule of $\frac{A\cap Ba}{Aa}$. Without loss of generality, let us assume $T = \frac{A}{Aq}$. The image of $1+Aq$ yields an element $x \in A\setminus Aa$ such that $qx \in Aa$. Write $qx = ra$ for some $r \in A$. Hence $ra \equiv 0 \pmod{qA}$. Note that $r \notin qA$, because $r = q\hat{r}$ for some $\hat{r} \in A$ implies $qx = q\hat{r}a$ hence $x \in Aa$. Let $r \equiv \sum_{j=0}^{t} r_j p^j \pmod{qA}$, where $r_j \in \mathbf{C}$, $r_t \neq 0$. We have $0 \equiv ra \equiv \sum_{j=0}^{t} r_j p^j(\alpha p - \beta) = (r_t\alpha)p^{t+1} + (r_{t-1}\alpha + r_t(\alpha' t - \beta))p^t +$ lower terms in p. This implies that $r_t\alpha$ is divisible by q, therefore 0 is a root of α. We also have that $\alpha' t - \beta$ is divisible by q, hence $\alpha_1 t - \beta_0 = 0$. If 0 is a simple root of α, then by Lemma 3.18 2) $\Theta_a(\xi) = \alpha_1\xi - \beta_0$. Therefore the integer t is a root of $\Theta_a(\xi)$. This is a contradiction. If 0 is a multiple root of α, then by Lemma 3.18 3) $\Theta_a(\xi) = -\beta_0 \neq 0$. But $0 = \alpha_1 t - \beta_0 = -\beta_0$, this is a contradiction again.

Hence we have $A \cap Ba = Aa$. This proves that a is S-regular. ∎

The following is Block's structure theorem for a linear simple A-module (cf. [1], Theorem 7.1).

Theorem 3.20 *Let S be a linear simple A-module with the corresponding similarity class $[b]$. Then there exists a S-regular element $a = \alpha p - \beta \in A \cap [b]$ such that S is isomorphic to the A-module*

$$\left(\mathbf{C}[q,\alpha^{-1}]; q \textit{ acts by multiplication, } p \textit{ acts as } \beta/\alpha + d/dq\right).$$

PROOF. By Lemma 3.19, an element a is S-regular if and only if a satisfies the condition:

$$\nu_\rho \beta/\alpha = -1 \Rightarrow \mathrm{Res}_\rho \beta/\alpha \notin \mathbf{Z} \quad (*)$$

where $\rho \in \mathbf{C}$. Block proves that there is an element a in each linear similarity class which satisfies the condition $(*)$, and $S \cong \mathbf{C}[q, \alpha^{-1}]$, where q acts by multiplication, p acts as $\beta/\alpha + d/dq$ (cf. [1], Theorem 7.1). ∎

Theorem 3.21 *If S is a linear simple A-module, then ${}_S\mathfrak{T}$ is Ore.*

PROOF. Let S be the linear simple A-module corresponding to the similarity class $[b]$. By Theorem 3.20, there exists an S-regular element $a = \alpha p - \beta \in A \cap [b]$ such that

$$S \cong \frac{A}{Aa} \cong \mathbf{C}[q, \alpha^{-1}]$$

where q acts as multiplication and p acts as $D + t$, where $D = \frac{d}{dq}$ and $t = \frac{\beta}{\alpha}$.

(Since p acts as $D + t$, $a = \alpha p - \beta$ acts as $\alpha D + \beta - \beta = \alpha D$. Hence $\alpha^{-1}a$ should act as D, and $(\alpha^{-1}a)^{k+1}$ should act as D^{k+1} and therefore should annihilate q^k.)

Claim 1: For every $k \in \mathbf{Z}$, the element $a - k\alpha'$ belongs to $A \cap [a]$ and is S-regular. Moreover $\alpha^k a = (a - k\alpha')\alpha^k$.

We have $\alpha^k a = \alpha^k(\alpha p - \beta) = \alpha(p\alpha^k - k\alpha^{k-1}\alpha') - \alpha^k\beta = (\alpha p - \beta - k\alpha')\alpha^k = (a - k\alpha')\alpha^k$. If $k \geq 0$, from the chain $A\alpha^k a \subseteq Aa \subseteq A$, we have that $\frac{A}{A\alpha^k a}$ has the factors $\frac{A}{Aa}$ and $\frac{A}{A\alpha^k}$. From the chain $A(a - k\alpha')\alpha^k \subseteq A\alpha^k \subseteq A$, we have that $\frac{A}{A\alpha^k a}$ has the factors $\frac{A}{A(a-k\alpha')}$ and $\frac{A}{A\alpha^k}$. By the Jordan-Hölder theorem, we have $\frac{A}{Aa} \cong \frac{A}{A(a-k\alpha')}$. Hence $a - k\alpha'$ is similar to a and S-regular. If $k < 0$, we have $a\alpha^{-k} = \alpha^{-k}(a - k\alpha')$, and the same type of argument applies.

Claim 2 For every $k \in \mathbf{N}$, the element $\alpha^{k+1}(\alpha^{-1}a)^{k+1}$ belongs to A. Furthermore

$$\alpha^{k+1}(\alpha^{-1}a)^{k+1} = (a - k\alpha') \cdots (a - 2\alpha')(a - \alpha')a$$

is S-regular and annihilates q^k, and therefore also annihilates q^{k-1}, $\cdots$, q and 1.

Induction on k.

For $k = 0$, $\alpha(\alpha^{-1}a) = a \in A$ and $a \cdot 1 = \alpha D \cdot 1 = 0$

Suppose the statements hold for $k-1$.

For k, using the induction hypothesis and the claim 1 we have

$$\begin{aligned}
\alpha^{k+1}(\alpha^{-1}a)^{k+1} &= \alpha(\alpha^k(\alpha^{-1}a)^k)(\alpha^{-1}a) \\
&= \alpha[(a-(k-1)\alpha')\cdots(a-2\alpha')(a-\alpha')a](\alpha^{-1}a) \\
&= \alpha[(a-(k-1)\alpha')\cdots(a-2\alpha')(a-\alpha')a\alpha^{-1}]a \\
&= \alpha[\alpha^{-1}(a-k\alpha')\cdots(a-2\alpha')(a-\alpha')]a \\
&= (a-k\alpha')\cdots(a-2\alpha')(a-\alpha')a
\end{aligned}$$

and

$$\begin{aligned}
&(a-k\alpha')\cdots(a-2\alpha')(a-\alpha')a\cdot q^k \\
= \;& (a-k\alpha')\cdots(a-2\alpha')(a-\alpha')[\alpha D\cdot q^k] \\
= \;& (a-k\alpha')\cdots(a-2\alpha')(a-\alpha')[k\alpha q^{k-1}] \\
= \;& k[(a-k\alpha')\cdots(a-2\alpha')(a-\alpha')\alpha]\cdot q^{k-1} \\
= \;& k\alpha[(a-(k-1)\alpha')\cdots(a-2\alpha')(a-\alpha')a\cdot q^{k-1}] = 0
\end{aligned}$$

Such elements are S-regular because they are products of S-regular elements.

Now take an arbitrary element from $\mathbf{C}[q,\alpha^{-1}]$, it has the form $\alpha^{-l}\phi$, where $l \in \mathbf{N}$ and $\phi \in \mathbf{C}[q]$. Let $\deg\phi = k$. By the claim 2, we have

$$(a-k\alpha')\cdots(a-\alpha')a\alpha^l\cdot\alpha^{-l}\phi = 0.$$

By the claim 1, we have

$$(a-k\alpha')\cdots(a-\alpha')a\alpha^l = \alpha^l(a-(k-l)\alpha')\cdots(a-(1-l)\alpha')(a-(-l)\alpha').$$

Since $\mathbb{C}[q,\alpha^{-1}]$ is $\mathbb{C}[q]$-torsion free, α^l acts injectively on $\mathbb{C}[q,\alpha^{-1}]$. Hence the element $(a-(k-l)\alpha')\cdots(a-(1-l)\alpha')(a-(-l)\alpha')$, which is S-regular, annihilates $\alpha^{-l}\phi$.

Now we have proved $S \in {}_{\Omega(S)}\mathfrak{T}$. For any simple $T \ncong S$, $\Omega(S)$ operates regularly on the injective hull ET of T, therefore on T. Hence T is $\Omega(S)$-torsion free. This proves that, up to the isomorphism, the simple module S is the only $\Omega(S)$-torsion simple A-module. By Theorem 2.11, we have ${}_S\mathfrak{T} = {}_{\Omega(S)}\mathfrak{T}$. This establishes that ${}_S\mathfrak{T}$ is Ore, by Proposition 3.10. ∎

It is known that if S is a $\mathbb{C}[q]$-torsion simple A-module, ${}_S\mathfrak{T}$ is Ore (Example 3.3). We deduce the following:

Corollary 3.22 *Let $\mathfrak{S}$ be an isomorphism-closed class of simple A-modules containing $\mathbb{C}[q]$-torsion and linear modules only. Then ${}_{\mathfrak{S}}\mathfrak{T}$ is Ore, and $\Omega(\mathfrak{S})$ is a left Ore set in A.*

PROOF. This follows from Theorem 3.21 and Lemma 3.11. ∎

The following discussion will establish that $\Omega(\mathfrak{S})$ in Corollary 3.22 is actually a two sided Ore set.

Lemma 3.23 *For any $a \in A$, $Hom(\frac{A}{Aa}, \frac{K}{A}) \cong Ext^1(\frac{A}{Aa}, A) \cong \frac{A}{aA}$.*

PROOF. By the proof of Corollary 2.18, we have

$$\mathrm{Ext}^1(\frac{A}{Aa}, A) \cong \frac{\mathrm{Hom}(Aa, A)}{r(\mathrm{Hom}(A, A))} \cong \frac{(Aa)^*}{A}.$$

Note that $(Aa)^* = \{t \in K \mid Aat \subseteq A\} = \{t \in K \mid t \in a^{-1}A\} = a^{-1}A$. It is easy to check $\frac{a^{-1}A}{A} \cong \frac{A}{aA}$ via the left multiplication by a. Hence we have proved $\mathrm{Hom}(\frac{A}{Aa}, \frac{K}{A}) \cong \frac{A}{aA}$. ∎

Proposition 3.24 *Let $\mathfrak{S}$ be an isomorphism-closed class of simple A-modules containing* $\mathbb{C}[q]$*-torsion and linear simple A-modules only, then* $\Omega(\mathfrak{S})$ *is a two sided Ore set.*

PROOF. For any simple A-module S, $S' = \mathrm{Ext}^1(S, A)$ is a simple right A-module, by Theorem 2.16. By Goodearl (cf. [5], Theorem 3.2), we have $CES = CES'$. Therefore, we have

$$\Omega(\mathfrak{S}) = \bigcap_{S \in \mathfrak{S}} CES = \bigcap_{S' \in \mathfrak{S}'} CES' = \Omega(\mathfrak{S}'),$$

where $\mathfrak{S}'$ is the isomorphism closure of $\{S' \mid S \in \mathfrak{S}\}$.

Since each $S \in \mathfrak{S}$ is either $\mathbb{C}[q]$-torsion or linear, by Proposition 2.29 and Theorem 3.20 there exists $a_s \in A$ with $p-\deg a \leq 1$ such that $S \cong A/Aa_s$. Therefore $S' \cong A/a_sA$ by Lemma 3.23. Hence $\mathfrak{S}'$ a class of simple right A-modules containing $\mathbb{C}[q]$-torsion and linear modules only. By the right-module analogue of Corollary 3.22, $\Omega(\mathfrak{S}')$ is a right Ore set.

By the above equality, we obtain that $\Omega(\mathfrak{S})$ is a two sided Ore set. ∎

In general, we do not know whether a left Ore set in A is always two sided.

The above is more or less what we know when $\mathfrak{S}$ is small. For quadratic simple A-modules, we had an example such that ${}_S\mathfrak{T}$ is Ore, namely $S = \mathrm{Soc}_A \frac{B}{Ba} = \frac{A}{Aa}$, where $a = p^2 - q^2$. We found later that a is the image of a linear element under an automorphism of A.

Dixmier classified the elements of A into five classes. The *strictly semisimple* and *strictly nilpotent* elements form two of them. He describes them in the following way: an element $x \in A$ is strictly nilpotent if and only if there exists an automorphism σ of A such that $\sigma(x) = \prod_i (q - \alpha_i)$, where $\alpha_i \in \mathbb{C}$; an element $x \in A$ is strictly semisimple if and only if there is an automorphism σ of A such that $\sigma(x) = \lambda qp + \mu$ with $\lambda \neq 0$, $\mu \in \mathbb{C}$. (cf. [3], 9.1 and 9.3)

Proposition 3.25 *Let S be a simple A-module corresponding to the similarity class*

$[a]$, *where* $p-\deg a \geq 2$. *If there exists* $x \in [a] \cap A$ *such that 1) the total degree of* x *is two, or 2)* x *is strictly nilpotent, or 3)* x *is strictly semisimple, then* ${}_S\mathfrak{T}$ *is Ore. Moreover,* x *is an* S*-regular element in* $[a]$.

PROOF. 1) Let $x = \alpha p^2 + \beta p + \gamma$. Since the total degree of x is two, we have $\alpha \in \mathbb{C}$, $\deg \beta \leq 1$ and $\deg \gamma \leq 2$. Without loss generality, we may assume $\alpha = 1$, hence $x = p^2 + \beta p + \gamma$.

Claim: There exists σ, an automorphism of A, such that $p - \deg \sigma(x) \leq 1$.

According to Theorem 2.4, $q \to c_{11}q + c_{12}p$, $p \to c_{21}q + c_{22}p$ such that $c_{11}c_{22} - c_{12}c_{21} = 1$, where $c_{ij} \in \mathbb{C}$ defines an automorphism σ of A. Then

$$\begin{aligned}\sigma(x) &= (c_{21}q + c_{22}p)^2 + (\beta_0 + \beta_1(c_{11}q + c_{12}p))(c_{21}q + c_{22}p) + \\ &\quad + (\gamma_0 + \gamma_1(c_{11}q + c_{12}p) + \gamma_2(c_{11}q + c_{12}p)^2) \\ &= (c_{22}^2 + \beta_1 c_{12}c_{22} + \gamma_2 c_{12}^2)p^2 + \text{lower terms in } p.\end{aligned}$$

Hence $p - \deg \sigma(x) \leq 1$ if and only if

$$c_{22}^2 + \beta_1 c_{12}c_{22} + \gamma_2 c_{12}^2 = 0.$$

Chosse $c_{11} = 0$, $c_{21} = 1$ $c_{12} = -1$ and c_{22} a solution of $c_{22}^2 - \beta_1 c_{22} + \gamma_2 = 0$. We obtain $c_{11}c_{22} - c_{12}c_{21} = -(-1) = 1$ and $c_{22}^2 + \beta_1 c_{12}c_{22} + \gamma_2 c_{12}^2 = c_{22}^2 - \beta_1 c_{22} + \gamma_2 = 0$.

By the remark after Corollary 3.7, $\frac{A}{Ax}$ is simple. Hence $S \cong \frac{A}{Ax}$, and therefore $\sigma(S) \cong \sigma(\frac{A}{Ax}) = \frac{A}{A\sigma(x)}$ is simple by Lemma 3.15. By Corollary 3.22, we have that ${}_{\sigma(S)}\mathfrak{T}$ is Ore. Hence ${}_S\mathfrak{T}$ is Ore by Lemma 3.16. Note that the element x is S-regular.

2) Let x be strictly nilpotent. Then there exists an automorphism σ of A such that $\sigma(x) \in \mathbb{C}[q]$. Write $\sigma(x) = \prod_i (q - \alpha_i)$, where $\alpha_i \in \mathbb{C}$. Then $x = \prod_i (\sigma^{-1}q - \alpha_i)$. Note that x is irreducible in B. Therefore all factors but one, $\sigma^{-1}q - \alpha_{i_0}$ say, are units in B. Hence if x has more than one factor, then for $i \neq i_0$, $\sigma^{-1}q - \alpha_i \in \mathbb{C}[q]$ and $\sigma^{-1}q \in \mathbb{C}[q]$. But this implies $\sigma^{-1}q - \alpha_{i_0} \in \mathbb{C}[q]$ which contradicts $p-\deg x \geq 2$. Hence we have $x = \sigma^{-1}(q - \alpha)$ for some $\alpha \in \mathbb{C}$, and $\sigma(x) = q - \alpha$. Since $\sigma(S) = \frac{A}{A\sigma(x)} = \frac{A}{A(q-\alpha)}$

is simple and ${}_{\sigma(S)}\mathfrak{T}$ is Ore, $\sigma^{-1}(\frac{A}{A\sigma x}) = \frac{A}{Ax}$ is simple. Hence x is S-regular and ${}_S\mathfrak{T}$ is Ore.

3) Let x be strictly semisimple. Then there exists an automorphism σ of A such that $\sigma(x) = \lambda qp + \mu$ with $\lambda \neq 0$, $\mu \in \mathbb{C}$. We can assume $\lambda = 1$. By Lemma 3.19, $\sigma(x) = qp + \mu$ is $\sigma(S)$-regular iff $\Theta_{\sigma(x),0}(\xi) = \xi + \mu$ has no integer root (iff $\mu \notin \mathbb{Z}$).

If μ is not an integer, then $\sigma(x)$ is $\sigma(S)$-regular. Therefore $\frac{A}{A\sigma(x)}$ is simple, and ${}_{\sigma(S)}\mathfrak{T}$ is Ore because $\sigma(x)$ is a linear element. Hence ${}_S\mathfrak{T}$ is Ore by Lemma 3.16. Moreover, since $\sigma^{-1}(\frac{A}{A\sigma(x)}) = \frac{A}{Ax}$, we have $\frac{A}{Ax}$ is simple. Hence $S \cong \frac{A}{Ax}$, i.e., x is S-regular.

If μ is an integer, $\mu = N$, then $q^N(qp+N) = (qpq^N - Nq^N) + Nq^N = (qp)q^N$. If $N \geq 0$, by applying σ^{-1}, we obtain $\sigma^{-1}(q^N)x = \sigma^{-1}(qp)\sigma^{-1}(q^N)$. Let $y = \sigma^{-1}(qp)$. Consider factors of $A/A\sigma^{-1}(q^N)x = A/Ay\sigma^{-1}(q^N)$; we obtain $B/Bx \cong B/By$; from this we can conclude $y \sim x$. Since $y = \sigma^{-1}(q)\sigma^{-1}(p)$ is irreducible in B, we have either $\sigma^{-1}(q) \in \mathbb{C}[q]$ or $\sigma^{-1}(p) \in \mathbb{C}[q]$. Suppose $\sigma^{-1}(q) \in \mathbb{C}[q]$. Write $\sigma^{-1}(q) = c\prod_i(q-\alpha_i)$ where $c \neq 0$, $\alpha_i \in \mathbb{C}$. Hence $q = c\prod_i(\sigma q - \alpha_i)$. But q is irreducible in A. Hence $i = 1$, and therefore $\sigma^{-1}q = c(q-\alpha)$ for some $\alpha \in \mathbb{C}$. Let $\sigma^{-1}(p) = \sum_i a_i(q)p^i$. We have

$$\begin{aligned} 1 &= \sigma^{-1}(1) = [\sigma^{-1}p,\ \sigma^{-1}q] = c[\sum_i a_i p^i,\ q-\alpha] \\ &= c\sum_i a_i[p^i,\ q-\alpha] = c\sum_i a_i i p^{i-1}. \end{aligned}$$

This implies $a_i = 0$ for all $i > 1$, and $a_1 = c^{-1}$. Hence $\sigma^{-1}(p) = c^{-1}p + a_0$. Therefore $y = c(q-\alpha)(c^{-1}p + a_0)$ is linear. But $y \sim x$ implies $p-\deg y = p-\deg x \geq 2$; contradiction. In the same way, one can derive a contradiction if $\sigma^{-1}(p) \in \mathbb{C}[q]$.

If $N < 0$, a similar argument applies. Thus the case $\mu = N$ cannot occur. ∎

Chapter 4

The structure of simple A-modules

In this chapter, we study the structure of simple A-modules. From Block's representation theorem, we know that there exists a preserving element a in each similarity class, and for such an element a the corresponding $\mathbb{C}[q]$-torsion free simple A-module has the form $\frac{A}{A\cap Ba}$.

The way we study this problem is the following:

First, we consider the larger module, $\frac{A}{Aa}$, for an arbitrary element $a \in A$ in the given similarity class corresponding to a simple module S. We know that it has a composition series of finite length with one $\mathbb{C}[q]$-torsion free factor, isomorphic to S, and possibly some $\mathbb{C}[q]$-torsion factors. We define top-tor$_\rho a$ and bott-tor$_\rho a$ according to the location of $\frac{A}{A(q-\rho)}$ in $\frac{A}{Aa}$. In particular, $\frac{A}{Aa}$ is simple if and only if $\frac{A}{Aa} \cong S$ if and only if a is S-regular. These regular elements play an important role not only in the study of Ore localization in A but also in the study of the structure of simple A-modules.

Secondly, we establish a criterion for bott-tor$_\rho a$ in which the associated polynomials play the crucial role. We find that if bott-tor$_\rho a \neq 0$, then ρ must be a root of the leading coefficient α of a, and $\Theta_{a,\rho}(\xi)$ has a positive integer root.

Thirdly, we develop some new machinery to calculate $\text{tor}_\rho a$. We find two important invariants of the similarity class $[a]$, namely, the *surplus* and the set of roots of indicial polynomials modulo $\mathbf{Z}$ at each place $\rho \in \mathbf{C}$. The ρ-surplus is non-zero only for finite many ρ's, a subset of the set of roots of the leading coefficient of a. One can first calculate the ρ-surplus, for a preserving element, by the criterion. Then the ρ-surplus of an arbitrary element can be found by reading off the valuation of the leading coefficient at ρ. Furthermore, we prove that the ρ-surplus of $[a]$ is always larger than or equal to zero.

Finally, we establish an index theorem for the Weyl algebra. An immediate consequence is a generalization of Block's result (Theorem 3.20). It gives a quantitative analysis for why his result about linear elements cannot be true in general and where the discrepancies are. Some examples are given at the end.

4.1 Definitions

Definition 4.1 *An element $a \in A$, irreducible in B, is called admissible, if $A \cap Ba$ is a maximal left ideal of A.*

If $a \in A$, irreducible in B, is preserving, then it is admissible, by Theorem 2.34. On the other hand, we shall give an example of an admissible element which is not preserving, in Section 5.

Let S be a $\mathbf{C}[q]$-torsion free simple A-module with a corresponding similarity class $[a]$. Then we have $S \cong \text{Soc}_A B/Ba \subseteq B/Ba$. For any $0 \neq s \in S$, $\text{ann}_B s$ is a maximal left ideal of B, hence it is of the form Bb for some $b \in [a]$, because B is a principal ideal domain. We call b is a *minimal annihilator* of S.

Proposition 4.2 *The set of admissible elements in $[b]$ coincides with the set of minimal annihilators of $S = Soc_A \frac{B}{Bb}$ which lie in A.*

PROOF. Let a be a admissible in $[b]$, i.e., $A \cap Ba$ is a maximal left ideal of A. Then $\frac{A}{A\cap Ba}$ is simple, hence isomorphic to S. Let s be the image of $1 + (A \cap Ba)$ in S, then $as = 0$. Since a is irreducible in B, a is a minimal annihilator of $s \in S$.

Conversely, let $a \in A$ be a minimal annihilator for some $s \in S$. Since S is simple, we have $As = S \cong \frac{A}{\mathrm{ann}_A s}$. Obviously, $\mathrm{ann}_A s = A \cap \mathrm{ann}_B s$ and $\mathrm{ann}_B s = Ba$. Since Ba is maximal in B, $A \cap Ba = \mathrm{ann}_A s$ is maximal in A. This implies that a is admissible. ∎

Now, we study the module $\frac{A}{Aa}$, where $a \in A$, irreducible in B. $\frac{A}{Aa}$ has a finite composition series with a unique $\mathbf{C}[q]$-torsion free factor which is isomorphic to $\mathrm{Soc}_A \frac{B}{Ba}$, and possibly some $\mathbf{C}[q]$-torsion factors $\frac{A}{A(q-\rho)}$.

Definition 4.3 *1) Let M be an A-module of finite length. For any $\rho \in \mathbf{C}$, the number of composition factors isomorphic to $\frac{A}{A(q-\rho)}$, in any composition series of M, is denoted by $tor_\rho M$. In particular, for any element $a \in A$, $tor_\rho \frac{A}{Aa}$ is denoted by $tor_\rho a$.*

2) For $0 \neq b \in B$, choose $\phi \in \mathbf{C}[q]$ (or $\psi \in \mathbf{C}[q]$) such that $\phi b \in A$ ($b\psi \in A$), and define $tor_\rho b = tor_\rho \phi b - \nu_\rho \phi$ ($tor_\rho b = tor_\rho b\psi - \nu_\rho \psi$).

We will deal with $\rho = 0$ only, and omit the index 0 whenever convenient.

Remark: 1) Definition 4.3 2) is well-defined.

2) For $a,\ b \in B$, $\mathrm{tor} ab = \mathrm{tor} a + \mathrm{tor} b$.

PROOF. 1) Note that for any $a,\ a' \in A$, $\mathrm{tor} aa' = \mathrm{tor} a + \mathrm{tor} a'$, since $Aaa' \subseteq Aa' \subseteq A$ and $\frac{Aa'}{Aaa'} \cong \frac{A}{Aa}$. Let $\psi_i \in \mathbf{C}[q]$, $i = 1,2$ such that $\psi_1 b,\ b\psi_2 \in A$. We have the following:

$$\begin{aligned} \mathrm{tor}\psi_1\phi b &= \mathrm{tor}\psi_1(\phi b) = \mathrm{tor}\psi_1 + \mathrm{tor}\phi b = \nu\psi_1 + \mathrm{tor}\phi b, \\ \mathrm{tor}\psi_1\phi b &= \mathrm{tor}\phi(\psi_1 b) = \mathrm{tor}\phi + \mathrm{tor}\psi_1 b = \nu\phi + \mathrm{tor}\psi_1 b. \end{aligned}$$

Therefore,

$$\text{tor}\phi b - \nu\phi = \text{tor}\psi_1 b - \nu\psi_1.$$

Similarly,

$$\text{tor}b\psi - \nu\psi = \text{tor}b\psi_2 - \nu\psi_2.$$

2) For $b \in B$, choose $\phi \in \mathbf{C}[q]$ such that $\phi b \in A$. For $a\phi^{-1} \in B$, choose $\psi \in \mathbf{C}[q]$ such that $\psi a\phi^{-1} \in A$. Therefore, $\psi ab = (\psi a\phi^{-1})(\phi b) \in A$.

$$\begin{aligned} \text{tor}ab &= \text{tor}\psi ab - \text{tor}\psi = \text{tor}(\psi a\phi^{-1})(\phi b) - \text{tor}\psi \\ &= \text{tor}\psi(a\phi^{-1}) + \text{tor}\phi b - \nu\psi = \text{tor}a\phi^{-1} + \text{tor}\psi + \text{tor}b + \nu\phi - \nu\psi \\ &= \text{tor}a\phi^{-1} + \text{tor}b + \nu\phi. \end{aligned}$$

For ϕ^{-1}, choose $\phi_1 \in \mathbf{C}[q]$ such that both $\phi^{-1}\phi_1$ and $a\phi^{-1}\phi_1 \in A$. Therefore

$$\begin{aligned} \text{tor}a\phi^{-1} &= \text{tor}a(\phi^{-1}\phi_1) - \text{tor}\phi_1 = \text{tor}a + \text{tor}\phi^{-1}\phi_1 - \text{tor}\phi_1 \\ &= \text{tor}a + \nu\phi^{-1}\phi_1 - \nu\phi_1 = \text{tor}a + \nu\phi^{-1} + \nu\phi_1 - \nu\phi_1 \\ &= \text{tor}a - \nu\phi. \end{aligned}$$

Hence

$$\begin{aligned} \text{tor}ab &= \text{tor}a\phi^{-1} + \text{tor}b + \nu\phi \\ &= \text{tor}a - \nu\phi + \text{tor}b + \nu\phi = \text{tor}a + \text{tor}b. \end{aligned}$$

∎

We know that for any $b \in A$, there exists a unique left-normalized element $a \in A$ (up to a nonzero scalar in $\mathbf{C}$) such that a is strongly similar to b, i.e., $b = \phi(q)a$, for some $\phi(q) \in \mathbf{C}[q]$. Since $\text{tor}b = \text{tor}\phi + \text{tor}a$, it is enough to determine $\text{tor}a$. Another reason why we prefer left-normalized elements is that one cannot distinguish b from the corresponding left-normalized a in terms of roots of its indicial polynomials, because we have

$$\Theta_{b,\rho}(\xi) = \left(\frac{\phi(q)}{(q-\rho)^{\nu_\rho\phi}}\right)(\rho)\Theta_{a,\rho}(\xi)$$

by Lemma 2.26.

Let $a \in A$ be left-normalized. Write $a = \sum_{k=0}^{n} \alpha_k(q)p^k$, where $\alpha_k = \sum_j \alpha_{kj}q^j$ with $\alpha_{kj} \in \mathbf{C}$ for $0 \le k \le n$. We call

$$\begin{pmatrix} \alpha_{n0} & 0 & 0 & \cdots & 0 \\ \alpha_{n1} & \alpha_{n-1,0} & 0 & \cdots & 0 \\ \alpha_{n2} & \alpha_{n-1,1} & \alpha_{n-2,0} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_{nn} & \alpha_{n-1,n-1} & \alpha_{n-2,n-2} & \cdots & \alpha_{00} \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

as the *coefficient matrix* of a at $\rho = 0$.

Note that $-n \le \nu a \le 0$ since a is left-normalized. The first <u>non-zero</u> row of the coefficient matrix of a at 0 is $(\alpha_{n,n-|\nu a|}, \cdots, \alpha_{k,k-|\nu a|}, \cdots, \alpha_{|\nu a|,0}, 0, \cdots, 0)$, and it displays exactly the coefficients of $\Theta_0(\xi)$.

Let $a \in A$ be left-normalized, and $\Theta_j(\xi)$ be the polynomials defined in Lemma 2.23 for $j \ge 0$. Define $\Theta_j(\xi) = 0$ if $j < 0$. For each $N \in \mathbf{N}$, define the $(N+1) \times (N + |\nu a| + 1)$ matrix $\mathrm{M}_a(N) = (\Theta_{|\nu a|+i-j}(j))_{ij}$, with $0 \le i \le N$ and $0 \le j \le N + |\nu a|$, namely

$$\mathrm{M}_a(N) = \begin{pmatrix} \Theta_{|\nu a|}(0) & \cdots & \Theta_0(|\nu a|) & 0 & \cdots & 0 \\ \vdots & & \vdots & \ddots & & \vdots \\ \vdots & & \vdots & & \ddots & \vdots \\ \Theta_{N+|\nu a|}(0) & \cdots & \Theta_N(|\nu a|) & \cdots & \cdots & \Theta_0(N+|\nu a|) \end{pmatrix}.$$

We shall call the vector

$$(\ \Theta_0(|\nu a|),\ \Theta_0(1+|\nu a|),\ \cdots,\ \Theta_0(N+|\nu a|)\ )$$

the Θ_0-diagonal of $\mathrm{M}_a(N)$.

4.2 A criterion for bott-tor$_\rho$

By the definition, $a \in A$ is not 0-preserving if $\Theta_a(\xi) = 0$ has negative integer solutions. We are interested in the meaning of $\Theta_a(\xi) = 0$ having non-negative integer solutions. We find that if a is left normalized, then the positive integer solutions which are strictly larger than $|\nu a|$ give some information about bott-tor$a$.

Proposition 4.4 *Let $a \in A$ be irreducible in B. Then $\frac{A\cap Ba}{Aa}$ is $\mathbf{C}[q]$-torsion and $\frac{A}{A\cap Ba}$ is $\mathbf{C}[q]$-torsion free.*

PROOF. Since $B \otimes_A \frac{A\cap Ba}{Aa} \cong \frac{Ba}{Ba} = 0$, $\frac{A\cap Ba}{Aa}$ is $\mathbf{C}[q]$-torsion. Since $\frac{A}{A\cap Ba} \cong \frac{A+Ba}{Ba} \subseteq \frac{B}{Ba}$ and $\frac{B}{Ba}$ is $\mathbf{C}[q]$-torsion free, $\frac{A}{A\cap Ba}$ is $\mathbf{C}[q]$-torsion free. ∎

In the previous section we defined tor$a$. Now we refine that definition by separating the contributions coming from the top and the bottom of $\frac{A}{Aa}$.

Definition 4.5 *Let $a \in A$ be irreducible in B and $\rho \in \mathbf{C}$. The number of composition factors isomorphic to $\frac{A}{A(q-\rho)}$, in any composition series of $\frac{A}{A\cap Ba}$, is denoted by top-tor$_\rho a$. The number of composition factors isomorphic to $\frac{A}{A(q-\rho)}$, in any composition series of $\frac{A\cap Ba}{Aa}$, is denoted by $bott-tor_\rho a$.*

It is clear that tor$_\rho a$ = top-tor$_\rho a$ + bott-tor$_\rho a$. In particular: a is $[a]$-regular iff tor$_\rho a = 0$ for all $\rho \in \mathbf{C}$. a is admissible iff top-tor$_\rho a = 0$ for all $\rho \in \mathbf{C}$, and then tor$_\rho a$ = bott-tor$_\rho a$.

As before, whenever convenient, we shall deal with $\rho = 0$ only, and omit the subscript.

Lemma 4.6 *Let $a \in A$ be irreducible in B. Then the following statements are equivalent.*

1) bott-tor$a \neq 0$.

2) There is $x \in A \setminus Aa$ such that $qx \in Aa$.

3) There is $r \in A \setminus qA$ such that $ra \in qA$.

PROOF. 1) $\Rightarrow$ 2) Let bott-tor$a > 0$. By Lemma 2.27, $\frac{A\cap Ba}{Aa}$ is a direct sum of $\mathbb{C}[q]$-torsion simple A-modules. Therefore $\frac{A}{Aq}$ is isomorphic to a direct summand of $\frac{A\cap Ba}{Aa}$. If $\overline{x} \in \frac{A\cap Ba}{Aa} \subseteq \frac{A}{Aa}$ is the image of $1 + Aq$, then $x \in A \setminus Aa$ and $qx \in Aa$.

2) $\Rightarrow$ 1) Let $x \in A \setminus Aa$ such that $qx \in Aa$. Since $A\overline{x} \cong \frac{A}{\mathrm{ann}\overline{x}}$ and $Aq \subseteq \mathrm{ann}_A\overline{x}$, but Aq is maximal, we conclude $\mathrm{ann}_A\overline{x} = Aq$ and $A\overline{x} \cong \frac{A}{Aq}$. This implies that bott-tor$a > 0$.

2) $\Rightarrow$ 3) We are given that there is $x \in A \setminus Aa$ such that $qx \in Aa$. This means that there exists $r \in A$ such that $qx = ra$. If $r \in qA$, i.e. $r = q\hat{r}$ for some $\hat{r} \in A$, then we have $qx = ra = q\hat{r}a$. Since A is a domain, this implies $x = \hat{r}a \in Aa$ which contradicts the fact $x \notin Aa$.

3) $\Rightarrow$ 2) We are given that there is $r \in A \setminus qA$ such that $ra \in qA$. This means that there exists $x \in A$ such that $ra = qx$. If $x \in Aa$, i.e. $x = \hat{x}a$ for some $\hat{x} \in A$, then $ra = qx = q\hat{x}a$. Since A is a domain, this implies $r = \hat{x}a \in Aa$ which contradicts the fact $r \notin Aa$. ∎

Lemma 4.7 *Let $a \in A$ be irreducible in B. Then*

$$\begin{aligned} \textit{bott-tor}_\rho a &= \mathbb{C} - \dim\left(\mathit{ann}_{\frac{A\cap Ba}{Aa}}(q-\rho)\right) \\ &= \mathbb{C} - \dim\left(\mathit{ann}_{\frac{A}{(q-\rho)A}}(a)\right). \end{aligned}$$

PROOF. By Lemma 2.27, any $\mathbb{C}[q]$-torsion A-module is a direct sum of $\mathbb{C}[q]$-torsion simple A-modules, hence $\frac{A\cap Ba}{Aa}$ has the form $\oplus_\rho(\frac{A}{A(q-\rho)})^{(n_\rho)}$, where $n_\rho \in \mathbb{N}$. Consider $\rho = 0$. Note that

$$\mathrm{ann}_{\frac{A}{A(q-\rho)}} q = \begin{cases} \mathbb{C}\cdot\overline{1} & \text{if } \rho = 0 \\ 0 & \text{if } \rho \neq 0 \end{cases}$$

Therefore,

$$\operatorname{ann}_{\frac{A\cap B_a}{Aa}} q = \bigoplus_{\rho} (\operatorname{ann}_{\frac{A}{A(q-\rho)}} q)^{(n_\rho)} \cong \mathbf{C}^{(n_0)}.$$

That is, bott-tor$_0 a = \mathbf{C} - \dim\ (\operatorname{ann}_{\frac{A\cap B_a}{Aa}} q)$.

By Lemma 4.6, we have the second equality. ∎

Lemma 4.8 *Let $a \in A$ be irreducible in B and left normalized. Then the first row of the matrix $\mathrm{M}_a(N)$ is a non-zero vector for any $N \in \mathbf{N}$.*

PROOF. The first row of $\mathrm{M}_a(N)$ is $(\Theta_{|\nu a|}(0),\ \cdots,\ \Theta_0(|\nu a|)\ )$. Write $a = \sum_j^n a_j p^j$, where $a_j = \sum_l \alpha_{jl} q^l$. Since a is left normalized, we have $-n \leq \nu a \leq 0$, and therefore $\nu a = -|\nu a|$. Recall from Lemma 2.23,

$$\begin{aligned}\Theta_j(\xi) &= \sum_{k\geq 0} \alpha_{k,k-|\nu a|+j} \xi(\xi-1)\cdots(\xi-k+1) \\ &= \sum_{k\geq|\nu a|-j} \alpha_{k,k-|\nu a|+j} \xi(\xi-1)\cdots(\xi-k+1)\end{aligned}$$

for all $0 \leq j \leq |\nu a|$.

If $\xi = |\nu a| - j$, then $\xi(\xi-1)\cdots(\xi-k+1)$ equals zero for $k > |\nu a| - j$, and equals $(|\nu a| - j)!$ for $k = |\nu a| - j$. Therefore $\Theta_j(|\nu a| - j) = \alpha_{|\nu a|-j,0}(|\nu a| - j)!$.

Therefore the first row of $\mathrm{M}_a(N)$ is zero if and only if $(\ \alpha_{00},\ \alpha_{10}, \cdots,\ \alpha_{|\nu a|,0})$ is a zero vector. But if $\alpha_{j0} = 0$, then $\nu a_j > 0$ for all $0 \leq j \leq |\nu a|$. Note $\nu a_j \geq \nu a + j \geq \nu a + |\nu a| + 1 = 1$, for any $j \geq |\nu a| + 1$. Together we have $\nu a_j > 0$ for all j, hence $a = q\hat{a}$ for some $\hat{a} \in A$. This contradicts the fact that a is left-normalized. ∎

Now, let us study the equation $ra \equiv 0$ modulo qA.

Lemma 4.9 *Let $a = \sum_{j=0}^n (\sum_{i\geq 0} \alpha_{ji} q^i)\ p^j \in A$ be left-normalized, and $r \in A$ with $r \equiv \sum_{k=0}^N \frac{r_k}{k!} p^k$ (mod qA) where $r_k \in \mathbf{C}$ for $0 \leq k \leq N$. Then $ra \equiv 0$ (mod qA) if and* *only if*

$$\begin{pmatrix} r_0 & r_1 & \cdots & r_N \end{pmatrix} \begin{pmatrix} \Theta_{|\nu a|}(0) & \cdots & \Theta_0(|\nu a|) & 0 & \cdots & 0 \\ \vdots & & \vdots & \ddots & & \vdots \\ \vdots & & \vdots & & \ddots & \vdots \\ \Theta_{N+|\nu a|}(0) & \cdots & \Theta_N(|\nu a|) & \cdots & \cdots & \Theta_0(N+|\nu a|) \end{pmatrix} = 0.$$

Proof. By using induction on i, one can show

$$p^k q^i \equiv k(k-1)\cdots(k-i+1)p^{k-i} \pmod{qA}$$

for any $k, i \in \mathrm{N}$. In particular, if $i > k$, then $p^k q^i \equiv 0 \pmod{qA}$.

$$\begin{aligned} ra &\equiv \sum_{k=0}^{N} \frac{r_k}{k!} p^k \sum_{j=0}^{n} \sum_{i\geq 0} \alpha_{ji} q^i\, p^j \pmod{qA} \\ &= \sum_{k=0}^{N}\sum_{j=0}^{n} \frac{r_k}{k!} (\sum_{i\geq 0} \alpha_{ji} p^k q^i) p^j \\ &\equiv \sum_{k=0}^{N}\sum_{j=0}^{n} \frac{r_k}{k!} (\sum_{i=0}^{k} \alpha_{ji} k(k-1)\cdots(k-i+1)p^{k-i}) p^j \pmod{qA} \\ &= \sum_{k=0}^{N}\sum_{j=0}^{n} \frac{r_k}{k!} \sum_{i=0}^{k} \alpha_{ji} \frac{k!}{(k-i)!}\, p^{k-i+j} \\ &= \sum_{m=0}^{n+N} (\sum_{k=0}^{N}\sum_{j=0}^{n} r_k \alpha_{j,k-m+j} \frac{1}{(m-j)!}\, p^m \quad (\text{where } m = k-i+j) \end{aligned}$$

Hence $ra \equiv 0$ if and only if

$$\sum_{k=0}^{N}\sum_{j=0}^{n} r_k \alpha_{j,k-m+j} \frac{1}{(m-j)!} = 0$$

for all $0 \leq m \leq n+N$.

For $m > N + |\nu a|$, we have $k - m + j < N - (N + |\nu a|) + n \leq 0$, and so $\alpha_{j,k-m+j} = 0$. The above expression vanishes automatically.

For $0 \leq m \leq N + |\nu a|$, we have

$$m! \sum_{k=0}^{N}\sum_{j=0}^{n} r_k \alpha_{j,k-m+j} \frac{1}{(m-j)!}$$

$$
\begin{aligned}
&= \sum_{k=0}^{N}\sum_{j=0}^{n} r_k \alpha_{j,k-m+j} m(m-1)\cdots(m-j+1) \\
&= \sum_{k=0}^{N} r_k \Theta_{k-m+|\nu a|}(m)
\end{aligned}
$$

Hence the vanishing of the expressions means that the product of $(r_0, \cdots, r_N)$ with the m-th column of $\mathrm{M}_a(N)$ is zero.

Hence we have

$$
\begin{pmatrix} r_0 & r_1 & \cdots & r_N \end{pmatrix}
\begin{pmatrix}
\Theta_{|\nu a|}(0) & \cdots & \Theta_0(|\nu a|) & 0 & \cdots & 0 \\
\vdots & & \vdots & \ddots & & \vdots \\
\vdots & & \vdots & & \ddots & \vdots \\
\Theta_{N+|\nu a|}(0) & \cdots & \Theta_N(|\nu a|) & \cdots & \cdots & \Theta_0(N+|\nu a|)
\end{pmatrix} = 0.
$$

iff $ra \equiv 0 \pmod{qA}$. ∎

The following is the criterion for bottom torsion.

Theorem 4.10 *Let $a = \alpha p^n +$ (lower terms in p) in A, irreducible in B, and left-normalized.*

1) If the indicial polynomial $\Theta_a(\xi)$ has no integer root strictly larger than $|\nu a|$, then bott-tor$a = 0$. (In particular, bott-tor$_\rho a = 0$ if $\alpha(\rho) \neq 0$.)

2) If $\Theta_a(\xi)$ has integer roots strictly larger than $|\nu a|$, and if $N + |\nu a|$ is the largest one, then bott-tor$a = N + 1 -$ rank $\mathrm{M}_a(N)$. Moreover bott-tor$a \leq t \leq n$, where t is the number of distinct integer roots of $\Theta_a(\xi)$ strictly larger than $|\nu a|$. In particular, if $\nu a = 0$, and $\Theta_a(\xi)$ has a root strictly larger than 0, then bott-tor$a > 0$.

PROOF. By Lemma 4.6 and 4.9, bott-tor$a \neq 0$ iff there is $r \in A \setminus qA$ such that $ra \equiv 0 \pmod{qA}$ iff there is $r \in A \setminus qA$ such that

$$\begin{pmatrix} r_0 & r_1 & \cdots & r_N \end{pmatrix} \begin{pmatrix} \Theta_{|\nu a|}(0) & \cdots & \Theta_0(|\nu a|) & 0 & \cdots & 0 \\ \vdots & & \vdots & \ddots & & \vdots \\ \vdots & & \vdots & & \ddots & \vdots \\ \Theta_{N+|\nu a|}(0) & \cdots & \Theta_N(|\nu a|) & \cdots & \cdots & \Theta_0(N+|\nu a|) \end{pmatrix} = 0,$$

where $r \equiv \sum_{k=0}^{N} \frac{r_k}{k!} p^k$ modulo qA, $r_k \in \mathbf{C}$.

1) Since a is left-normalized, by Lemma 4.8 the first row of $\mathbf{M}_a(N)$ is non-zero. Moreover $\Theta_a(|\nu a| + l) = \Theta_0(|\nu a| + l) \neq 0$ for all $l > 0$. Therefore the row vectors of $\mathbf{M}_a(N)$ are linearly independent, hence rank$\mathbf{M}_a(N) = N + 1$. This implies that the system has no nontrivial solutions for any choice of N. Consequently bott-tor$a = 0$.

In particular, this applies if ρ is not a root of α, because then $\nu_\rho a = -n$, and $\Theta_a(\xi) = \alpha(\rho)\xi(\xi-1)\cdots(\xi-n+1)$ has no solutions larger than $|\nu a| = n$.

2) Let $r \equiv \sum_{k=0}^{N} \frac{r_k}{k!} p^k$ be a solution of $ra \equiv 0$ modulo qA. If $r_N \neq 0$, then the product of $(r_0, \cdots, r_N)$ with the last column of $\mathbf{M}_a(N)$ produces the equation

$$r_N \Theta_0(N + |\nu a|) = 0,$$

and therefore $N + |\nu a|$ is a root of $\Theta_0(\xi)$.

We conclude from this observation that, if $\Theta_a(\xi) = 0$ has integer roots strictly larger than $|\nu a|$, and if $N + |\nu a|$ is the largest one, then every solution r of $ra \equiv 0$ modulo qA is of the form $r \equiv \sum_{k=0}^{N} \frac{r_k}{k!} p^k$ modulo qA (where r_N could be 0).

Now, let us consider the rank of $\mathbf{M}_a(N)$. For $1 \leq i \leq N$, if $\Theta_0(i + |\nu a|) \neq 0$, then the i-th row of $\mathbf{M}_a(N)$ has length $i + |\nu a| + 1$. For $i = 0$, the 0-th row of $\mathbf{M}_a(N)$ is non-zero by Lemma 4.8, and has length at most $|\nu a| + 1$. Therefore, the number of linearly independent rows of $\mathbf{M}_a(N)$ is at least $N - t + 1$, where t is the number of distinct positive integer solutions of $\Theta_a(\xi)$ which are strictly larger than $|\nu a|$. This implies bott-tor$a = N + 1 - \text{rank}\mathbf{M}_a(N) \leq N + 1 - (N + 1 - t) = t$. Since $\Theta_a(\xi)$ has degree less than or equal to n, we have $t \leq n$.

In particular, if $\nu a = 0$ and $\Theta_a(\xi)$ has a root strictly larger than 0, then $\mathrm{M}_a(N)$ is an $N+1$ by $N+1$ square lower triangular matrix, and therefore the determinant is $\prod_{j=0}^{N}\Theta_0(j) = 0$. Hence rank $\mathrm{M}_a(N) < N+1$, and therefore bott-tor$a > 0$. ∎

Corollary 4.11 *Let $b = \beta p^n +$ (lower terms in p) $\in A$ be admissible. Then $\nu\beta -$ tor$b \geq 0$.*

PROOF. Write $b = f(q)a$, where $f(q) \in \mathbb{C}[q]$ and $a = \alpha p^n +$ (lower terms in p) is left-normalized. a is also admissible, and therefore tor$a$ = bott-tor$a$. Note $\Theta_a(\xi) = \xi(\xi-1)\cdots(\xi-|\nu a|+1)\hat{\Theta}$ for some $\hat{\Theta} \in \mathbb{C}[\xi]$ with degree less than or equal to $n-|\nu a|$. Hence $\Theta_a(\xi) = 0$ can have at most $n - |\nu a|$ many integer solutions which are larger than $|\nu a|$. Hence by Theorem 4.10, we have

$$\begin{aligned} \nu\beta - \mathrm{tor}b &= \nu(\beta f^{-1}) + \nu f - \mathrm{tor}f - \mathrm{tor}a = \nu\alpha - \text{bott-tor}a \\ &\geq (n - |\nu a|) - (n - |\nu a|) = 0. \end{aligned}$$

∎

4.3 Invariants of $[a]$

First, let us extend the definition of similarity, to not necessary irreducible elements of B.

Definition 4.12 *Let a, $b \in B$ be nonzero. We write $a \sim b$ if $\frac{B}{Ba}$ and $\frac{B}{Bb}$ have the same composition factors (up to order).*

It is clear that if a and b are irreducible, then $a \sim b$ iff $\frac{B}{Ba} \cong \frac{B}{Bb}$.

Next, let us prove a trivial but useful lemma.

Lemma 4.13 *Let $a \sim b$, both irreducible in B. Then there exist u and v in B such that $au = vb$. Furthermore, u and v can be chosen such that $p - \deg v = p - \deg u < p - \deg b = p - \deg a$.*

PROOF. Let $u + Bb$ be the image of $1 + Ba$ under the isomorphism $\frac{B}{Ba} \cong \frac{B}{Bb}$. Then $au \in Bb$, hence $au = vb$ for some $v \in B$. It is clear that one can choose a representative u of $u + Bb$ such that its p-degree is strictly less than $p - \deg b$. Since we know $p - \deg b = p - \deg a$, we conclude

$$p - \deg v = p - \deg u < p - \deg b = p - \deg a.$$

∎

Theorem 4.14 *Let a and b in B, $a \sim b$, $a = \alpha p^n +$ (lower terms in p) and $b = \beta p^n +$ (lower terms in p). Then*

$$\nu\alpha - tora = \nu\beta - torb.$$

PROOF. We proceed by induction on $n = p - \deg a$.

1) For $n = 1$. Since a and b are linear elements, they are irreducible in B. By Lemma 4.13, there exist $u = f(q)$ and $v = g(q)$ such that $af = gb$. Note that $\alpha f = g\beta$ and $\mathrm{tor}a + \nu f = \nu g + \mathrm{tor}b$. Therefore,

$$\nu\alpha - \mathrm{tor}a = (\nu g + \nu\beta - \nu f) - (\nu g + \mathrm{tor}b - \nu f) = \nu\beta - \mathrm{tor}b.$$

2) Suppose that the statement is true for elements with p-degree less than n. Let $p - \deg a = n$ and $a \sim b$.

(i) Suppose that a is irreducible in B. By Lemma 4.13, there exist u and v in B such that $au = vb$ and $p - \deg u \leq n - 1$. It is clear that $u \sim v$ by Definition 4.12. Let μ and λ be the leading coefficients of u and v respectively. Then by the induction hypothesis

$$\nu\mu - \mathrm{tor}u = \nu\lambda - \mathrm{tor}v.$$

Note $\mathrm{tor}a + \mathrm{tor}u = \mathrm{tor}v + \mathrm{tor}b$ and $\alpha\mu = \lambda\beta$. Therefore

$$\begin{aligned}\nu\alpha - \mathrm{tor}a &= (\nu\lambda + \nu\beta - \nu\mu) - (\mathrm{tor}v + \mathrm{tor}b - \mathrm{tor}u) \\ &= (\nu\beta - \mathrm{tor}b) + (\nu\lambda - \mathrm{tor}v) - (\nu\mu - \mathrm{tor}u) = \nu\beta - \mathrm{tor}b.\end{aligned}$$

(ii) Suppose that a is not irreducible in B. Write

$$a = a_1a_2\cdots a_t, \quad b = b_1b_2\cdots b_s,$$

where a_i and b_j are in B and irreducible. Since $a \sim b$, the Jordan-Hölder Theorem implies that $t = s$ and for each i there is j such that $a_i \sim b_j$. Hence a_i and b_j have the same $p-\deg$, say m_i, write $a_i = \alpha_i p^{m_i}$ + lower terms, $b_j = \beta_j p^{m_i}$ + lower terms. Then by induction hypothesis, we have

$$\nu\alpha_i - \mathrm{tor}a_i = \nu\beta_j - \mathrm{tor}b_j.$$

Therefore

$$\sum_{i=1}^{t}(\nu\alpha_i - \mathrm{tor}a_i) = \sum_{j=1}^{t}(\nu\beta_j - \mathrm{tor}b_j).$$

But

$$\sum_{i=1}^{t}\nu\alpha_i - \sum_{i=1}^{t}\mathrm{tor}a_i = \nu\alpha - \mathrm{tor}a, \ \sum_{j=1}^{t}\nu\beta_j - \sum_{j=1}^{t}\mathrm{tor}b_j = \nu\beta - \mathrm{tor}b.$$

Hence

$$\nu\alpha - \mathrm{tor}a = \nu\beta - \mathrm{tor}b.$$

∎

For an irreducible element $a \in B$, with corresponding simple A-module S, Theorem 4.14 shows that the difference between the valuation of the leading coefficient of a at ρ and $\mathrm{tor}_\rho a$ is an invariant of the similarity class $[a]$. Let us denote this invariant by $\sigma_\rho[a]$, or $\sigma_\rho[S]$, and call it the *ρ-surplus* of $[a]$, or of S.

An immediate consequence of Corollary 4.11 and Theorem 4.14 is the following:

Corollary 4.15 *For any $a = \alpha p^n$ + (lower terms in p), irreducible in B, we have $\sigma[a] \geq 0$. In particular, all the $[a]$-regular elements have the same leading coefficient, up to non-zero scalars.*

PROOF. Without loss of generality, let us assume that a is irreducible in B. By Remark 3) after Definition 2.31, $[a]\cap A$ contains a preserving, hence admissible element, say $b = \beta p^n +$ lower terms in p. By Corollary 4.11, $\nu\beta - \mathrm{tor} b \geq 0$. By Theorem 4.14, we have $\sigma[a] = \nu\alpha - \mathrm{tor} a = \nu\beta - \mathrm{tor} b \geq 0$.

Now let $a \in A$ be irreducible in B. Let $b = \beta p^n +$ (lower terms in p) be S-regular. Then $\mathrm{tor}_\rho b = 0$ for all ρ. Hence $\sigma_\rho[a] = \nu_\rho\beta - \mathrm{tor}_\rho b = \nu_\rho\beta$, and therefore $\beta = c\prod_\rho (q-\rho)^{\sigma_\rho[a]}$, where c is a non-zero scalar. ∎

It is clear that $\sigma_\rho[a] \neq 0$ for only finite many $\rho \in \mathbf{C}$, since $\sigma_\rho[a] \neq 0$ implies $\alpha(\rho) = 0$.

Theorem 4.16 *Let $a \sim b$, irreducible in B, let the degree of Θ_a be d ($\leq p - \deg a$), and let $\xi_1, \xi_2, \cdots, \xi_d$ be the roots of Θ_a. Then the degree of Θ_b is also d, and the roots of Θ_b have the form $\xi_1 + n_1, \xi_2 + n_2, \cdots, \xi_d + n_d$, where $n_1, n_2, \cdots, n_d \in \mathbf{Z}$.*

PROOF. We proceed by induction on $n = p - \deg a$.

(i) Let $p - \deg a = 1$. By Lemma 4.13, there exists $f(q)$ and $g(q)$ in $\mathbf{C}[q]$ such that $af = gb$. Note that $\Theta_a(\xi + \nu f) = c\Theta_b(\xi)$ by Lemma 2.26, where $0 \neq c \in \mathbf{C}$. Therefore the roots of Θ_a are obtained from roots of Θ_b by shifting with the integer νf. This proves the theorem for the case $n = 1$.

(ii) Suppose the statement is true for p-degree less than n. Let $p - \deg a = n$. Again by Lemma 4.13, there exist u and v in B with $p - \deg v = p - \deg u \leq n - 1$ such that $au = vb$. We have $\Theta_a(\xi + \nu u)\Theta_u(\xi) = \Theta_v(\xi + \nu b)\Theta_b(\xi)$ by Lemma 2.26. By using the induction hypothesis on u and v and the equality, we have proved the claim for n. ∎

Now, we have obtained another set of invariants of $[a]$, namely, the set of roots modulo $\mathbf{Z}$ of indicial polynomials at any place $\rho \in \mathbf{C}$. We will call them *indicial roots* of $[a]$ or of S, at ρ.

4.4 An index theorem for the Weyl algebra

Block proved the following (cf. Theorem 3.20):

1) Each linear similarity class $[S]$ has a S-regular representative.

2) If $a = \alpha p + \beta \in A \cap [S]$ is S-regular, then

$$S \cong (\mathbb{C}[X, \alpha^{-1}],\ q \to X,\ p \to D - \frac{\beta}{\alpha}).$$

One can ask whether there always is a S-regular element in an arbitrary similarity class $[S]$; if such a S-regular element exists, whether the simple A-module S has a similar representation as in the linear case.

It is easy to show that every left ideal of A can be generated by at most two elements. In fact, Stafford proved in [13] that every left ideal of the n-th Weyl algebra A_n can be generated by at most two elements. The first question is equivalent to asking whether any simple module is cyclically presented. The answer is negative, and a quadratic counter example will be given in Section 5. Our index theorem will give a complete analysis to the second problem.

Lemma 4.17 *Let $a \in B$, irreducible in B. Write $a = \alpha p^n - \alpha_{n-1}p^{n-1} - \cdots - \alpha_0$. Then $\frac{B}{Ba}$ is isomorphic to the B-module*

$$\{\mathbb{C}(X)^n,\ q \to X,\ p \to \mathbf{P}_a\} = \mathbb{C}(X)^n_a,$$

where the elements of $\mathbb{C}(X)^n$ are written as column vectors, and

$$\mathbf{P}_a = \begin{pmatrix} D & 0 & \cdots & 0 & \frac{\alpha_0}{\alpha} \\ 1 & D & \cdots & 0 & \frac{\alpha_1}{\alpha} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & D & \frac{\alpha_{n-2}}{\alpha} \\ 0 & 0 & \cdots & 1 & D + \frac{\alpha_{n-1}}{\alpha} \end{pmatrix}$$

with $D = \frac{d}{dX}$.

PROOF. Every element of $\frac{B}{Ba}$ has a unique representative $\sum_{i=0}^{n-1}\beta_i(q)p^i$ of $p-\deg\leq n-1$, and the sum of two such representatives is again such a representative. Therefore

$$\tau=\tau_a:\ \sum_{i=0}^{n-1}\beta_i(q)p^i+Ba\to\begin{pmatrix}\beta_0\\ \beta_1\\ \vdots\\ \beta_{n-1}\end{pmatrix}=\vec{\beta}$$

is a well-defined group homomorphism from $\frac{B}{Ba}$ to $\mathrm{C}(X)^n$. Obviously it is one-one and onto.

We use τ to carry the B-module structure of $\frac{B}{Ba}$ over to $\mathrm{C}(X)^n$, by $b\cdot\vec{\beta}=\tau b\tau^{-1}\vec{\beta}$, where $b\in B$ and $\vec{\beta}\in\mathrm{C}(X)^n$. Then, in particular,

$$\begin{aligned}
q\cdot\vec{\beta} &= \tau q\tau^{-1}\cdot\vec{\beta}=\tau q(\sum_{i=0}^{n-1}\beta_ip^i)=\tau(\sum_{i=0}^{n-1}q\beta_ip^i)=X\vec{\beta}\\
p\cdot\vec{\beta} &= \tau p\tau^{-1}\vec{\beta}=\tau(\sum_{i=0}^{n-1}(p\beta_ip^i)=\tau(\sum_{i=0}^{n-1}(\beta_ip+\beta_i')p^i)\\
&= \tau(\sum_{i=0}^{n-1}(\beta_{i-1}+\beta_i')p^i+\beta_{n-1}p^n)=\tau(\sum_{i=0}^{n-1}(\beta_{i-1}+\beta_i'+\beta_{n-1}\frac{\alpha_i}{\alpha})p^i)\\
&= \begin{pmatrix}\beta_0'+\beta_{n-1}\frac{\alpha_0}{\alpha}\\ \beta_0+\beta_1'+\beta_{n-1}\frac{\alpha_1}{\alpha}\\ \vdots\\ \beta_{n-2}+\beta_{n-1}'+\beta_{n-1}\frac{\alpha_{n-1}}{\alpha}\end{pmatrix}\\
&= \begin{pmatrix}D & 0 & \cdots & 0 & \frac{\alpha_0}{\alpha}\\ 1 & D & \cdots & 0 & \frac{\alpha_1}{\alpha}\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & 1 & D+\frac{\alpha_{n-1}}{\alpha}\end{pmatrix}\begin{pmatrix}\beta_0\\ \beta_1\\ \vdots\\ \beta_{n-1}\end{pmatrix}=\mathbf{P}_a\vec{\beta}
\end{aligned}$$

where $\beta_j'=\frac{d}{dX}(\beta_j)$ for all j. ∎

τ induces an A-isomorphism

$$\frac{B}{A+Ba}\cong\frac{B}{Ba}/\frac{A+Ba}{Ba}\overset{\tau}{\cong}\mathrm{C}(X)^n_a/\tau(\frac{A+Ba}{Ba}).$$

As $\frac{B}{A+Ba}$ is C[q]-torsion (every element of B can be bought down to A by multiply some polynomial in C[q]), and any C[q]-torsion A-module is semisimple (cf. Lemma 2.27), we can write

$$\frac{\mathrm{C}(X)^n_a}{\tau(\frac{A+Ba}{Ba})} \cong \bigoplus_\rho \left(\frac{A}{A(q-\rho)}\right)^{(\gamma_\rho a)} \tag{4.1}$$

Now let $a \in A$. We will study the map

$$\hat{\tau}: \ \frac{A}{Aa} \twoheadrightarrow \frac{A}{A \cap Ba} \cong \frac{A+Ba}{Ba} \xrightarrow{\tau} \mathrm{C}(X)^n_a$$

It is clear that the kernel of $\hat{\tau}$ is isomorphic to $\frac{A\cap Ba}{Aa} \cong \oplus_\rho(\frac{A}{A(q-\rho)})^{\text{bott-tor}_\rho a}$, and the cokernel of $\hat{\tau}$ is $\oplus_\rho(\frac{A}{A(q-\rho)})^{(\gamma_\rho a)}$.

Now let us study the two exponents bott-tor$_\rho a$ and $\gamma_\rho a$ for any $\rho \in$ C. Note that bott-tor$_\rho a$ can be calculated by the machinery developed in Section 2 and Section 3 (in particular, Theorem 4.10). However, $\gamma_\rho a$ is hard to calculate directly, even for linear elements a. We will prove that these two quantities are related by $\nu_\rho a$.

Without loss of generality, let us restrict our attention to $\rho = 0$, and we omit the index whenever convenient.

Lemma 4.18 *Let $a = \alpha p^n - \alpha_{n-1}p^{n-1} - \cdots - \alpha_0 \in A$, where $\alpha = \alpha_n$ and $\alpha_j = \sum_{k\geq 0} \alpha_{jk}q^k$ for each $0 \leq j \leq n$, irreducible in B. If the indicial polynomial of a at 0 has no roots in N, then $\gamma_0 a = 0$.*

Proof. If 0 is not a root of α, then the indicial polynomial of a at 0 always has non-negative integer roots; the claim of the theorem is vacuous. Hence we can assume that 0 is a root of α.

For any $0 \neq f \in$ C[q], we have $Bfa = Ba$. Therefore by (4.1) we have $\gamma fa = \gamma a$. Hence we can, without loss of generality, assume that a is left-normalized, and therefore $-n \leq \nu a \leq 0$. Recall from Lemma 2.25, $\Theta_a(\xi) = \sum_{k\geq 0} \alpha_{k,k+\nu a}\xi(\xi-1)\cdots(\xi-k+1)$. If $\nu a < 0$, then $\Theta_a(0) = 0$. This contradicts the fact that $\Theta_a(\xi)$ has no roots in

N. Hence $\nu a = 0$. This means $\nu\alpha_i \geq i$, and therefore $\alpha_i = \alpha_{ii}q^i +$ (higher terms in q) for each $0 \leq i \leq n$. We decompose $\alpha = q^t\hat{\alpha}$, where $\hat{\alpha}(0) \neq 0$; then $t = \nu\alpha = \alpha_n \geq n$. Note that if $\nu\alpha_0 > 0$, then $a = q\hat{a}$ for some $\hat{a} \in A$. This contradicts the fact that a is left-normalized. Hence $\nu\alpha_0 = 0$, i.e. $\alpha_{00} \neq 0$.

By localizing

$$0 \longrightarrow \frac{A+Ba}{Ba} \xrightarrow{\tau} \mathrm{C}(X)^n_a \longrightarrow \bigoplus_\rho \left(\frac{A}{A(q-\rho)}\right)^{(\gamma_\rho a)} \longrightarrow 0$$

at $\star = \mathrm{C}[q] \setminus (q)$, where (q) is the ideal generated by q, we have the exact sequence

$$0 \longrightarrow \frac{A_\star + Ba}{Ba} \xrightarrow{\tau} \mathrm{C}(X)^n_a \longrightarrow (\frac{A_\star}{A_\star q})^{(\gamma a)} \longrightarrow 0.$$

Therefore $\gamma a = 0$ iff $\mathrm{C}(X)^n_a = \tau(\frac{A_\star+Ba}{Ba})$. We abbreviate $\tau(\frac{A_\star+Ba}{Ba})$ by M.

Claim: M contains the set $\{b_{1k}, b_{2k}, \cdots b_{nk}\}_{k\in\mathbf{Z}}$, where

$$b_{10} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \; b_{20} = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \; \cdots, \; b_{n0} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix},$$

and $b_{ik} = X^{-k}b_{i0}$ for $1 \leq i \leq n$, $k \in \mathbf{Z}$.

It is clear that $b_{ik} = X^{-k}b_{i0} = \tau(q^{-k}p^{i-1} + Ba) \in M$ for any $k \in \mathbf{Z}^-$.

For $i < n$, and $k \in \mathrm{N}$, we have

$$\mathbf{P}_a b_{ik} \;=\; \begin{pmatrix} D & 0 & \cdots & 0 & \frac{\alpha_0}{\alpha} \\ 1 & D & \cdots & 0 & \frac{\alpha_1}{\alpha} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & D + \frac{\alpha_{n-1}}{\alpha} \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ X^{-k} \\ \vdots \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \vdots \\ -kX^{-k-1} \\ X^{-k} \\ \vdots \end{pmatrix} = (-k)b_{i,k+1} + b_{i+1,k}.$$

We prove the claim by using induction on $l = i + k$, where $1 \leq i \leq n$ and $k \in \mathrm{N}$.

For $l = 1$, it is clear that $b_{10} = \tau(1 + Ba) \in M$.

Suppose $b_{ik} \in M$ for $i + k \leq l$. We have to show $b_{1,l}, b_{2,l-1}, \cdots, b_{l,1} \in M$.

The arguments are slightly different according to whether $l \leq n$ or $l > n$.

Let us deal with $l \leq n$ first.

Since M is an A-module, $t - l \geq 0$, and $\alpha(X) = X^t\hat{\alpha}(X)$, we have

$$\begin{aligned} M \ni b &= X^{t-l}\hat{\alpha}\mathrm{P}_a^n b_{10} = X^{-l}\alpha \mathrm{P}_a b_{n0} \\ &= X^{-l}\begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{n-1} \end{pmatrix} = \alpha_0 X^{-l} b_{10} + \cdots + \alpha_{n-1}X^{-l}b_{n0}. \end{aligned}$$

For $0 \leq i \leq l-1$, since $\nu\alpha_i \geq i$, we obtain

$$\begin{aligned} \alpha_i X^{-l} b_{i+1,0} &= \sum_{j\geq i} \alpha_{ij} X^{-l+j} b_{i+1,0} \\ &= \sum_{j \geq i} \alpha_{ij} b_{i+1,l-j} = \alpha_{ii} b_{i+1,l-i} + \sum_{j>i} \alpha_{ij} b_{i+1,l-j}. \end{aligned}$$

Since $(i+1) + (l-j) < (i+1) + (l-i) = l+1$ for all $j > i$, by the induction hypothesis, we have $\sum_{j>i} \alpha_{ij} b_{i+1,l-j} \in M$.

For $i \geq l$, since $-l + j \geq -l + i \geq 0$, we have

$$\alpha_i X^{-l} b_{i+1,0} = \sum_{j \geq i} \alpha_{ij} X^{-l+j} b_{i+1,0} \in M.$$

Together we have

$$b = \sum_{0 \leq i \leq l-1} \alpha_{ii} b_{i+1,l-i} + m_l$$

for some $m_l \in M$.

We obtain a system of l linear equations (modulo M):

$$\begin{aligned}
0 &\equiv \mathbf{P}_a b_{1,l-1} = -(l-1)b_{1,l} + b_{2,l-1} \\
0 &\equiv \mathbf{P}_a b_{2,l-2} = -(l-2)b_{2,l-1} + b_{3,l-2} \\
&\vdots \\
0 &\equiv \mathbf{P}_a b_{l-1,1} = -b_{l-1,2} + b_{l,1} \\
0 &\equiv b - m_l = \alpha_{00} b_{1,l} + \alpha_{11} b_{2,l-1} \cdots + \alpha_{l-1,l-1} b_{l,1}.
\end{aligned}$$

Since

$$\det \begin{pmatrix} \alpha_{00} & -(l-1) & 0 & \cdots & 0 \\ \alpha_{11} & 1 & -(l-2) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_{l-2,l-2} & 0 & 0 & \cdots & -1 \\ \alpha_{l-1,l-1} & 0 & 0 & \cdots & 1 \end{pmatrix}$$

$$\begin{aligned}
&= \alpha_{00} + (-\alpha_{11})(-l+1) + \cdots + (-1)^{l-1}\alpha_{l-1,l-1}(-l+1)(-l+2)\cdots(-1) \\
&= \alpha_{00} + \alpha_{11}(l-1) + \cdots + \alpha_{l-1,l-1}(l-1)! = -\Theta_a(l-1) \neq 0,
\end{aligned}$$

this implies that $b_{1,l},\ b_{2,l-1},\ \cdots\ b_{l-1,2},\ b_{l,1} \in M$.

Now, let us deal with $l > n$. Note that, since $t - n \geq 0$,

$$\begin{aligned}
M \ni \hat{b} &= \hat{\alpha} X^{t-n} \mathbf{P}_a b_{n,l-n} = \alpha X^{-n} \mathbf{P}_a b_{n,l-n} \\
&= \begin{pmatrix} \alpha_0 X^{-l} \\ \alpha_1 X^{-l} \\ \vdots \\ \alpha_{n-1} X^{-l} + (-l+n)\alpha_n X^{-l-1} \end{pmatrix} \\
&= \sum_{i=0}^{n-2} \alpha_i X^{-l} b_{i+1,0} + (\alpha_{n-1} X^{-l} + (-l+n)\alpha_n X^{-l-1}) b_{n,0}.
\end{aligned}$$

For $0 \le i \le n-2$, note that

$$\alpha_i X^{-l} b_{i+1,0} = \sum_{j \ge i} \alpha_{ij} X^{-l+j} b_{i+1,0} = \alpha_{ii} b_{i+1,l-i} + \sum_{j>i} \alpha_{ij} b_{i+1,l-j}.$$

Since $(i+1)+(l-j) < (i+1)+(l-i) < l+1$ for all $j > i$, we have $\sum_{j>i} \alpha_{ij} b_{i+1,l-j} \in M$ by the induction hypothesis.

For $i = n-1$,

$$\begin{aligned} & \alpha_{n-1} X^{-l} b_{n0} + (-l+n) X^{-l-1} \alpha_n b_{n0} \\ = \; & (\alpha_{n-1,n-1} + (-l+n)\alpha_{nn}) b_{n,l-n+1} + \sum_{j>n} (\alpha_{n-1,j-1} + (-l+n)\alpha_{n,j}) b_{n,l+1-j}. \end{aligned}$$

Since $n + (l+1-j) < n + (l+1-n) = l+1$ for all $j > n$, by the induction hypothesis, we have $\sum_{j>n}(\alpha_{n-1,j-1} + (-l+n)\alpha_{n,j}) b_{n,l+1-n} \in M$.

Hence

$$\hat{b} = \sum_{i=0}^{n-2} \alpha_{ii} b_{i+1,l-i} + (\alpha_{n-1,n-1} + (-l+n)\alpha_{nn}) b_{n,l+1-n} + \hat{m}_l$$

for some $\hat{m}_l \in M$.

Hence we obtain the system n linear equations (modulo M):

$$\begin{aligned} 0 &\equiv \mathbf{P}_a b_{1,l-1} = (-l+1) b_{1,l} + b_{2,l-1} \\ &\vdots \\ 0 &\equiv \mathbf{P}_a b_{n-1,l-n+1} = (-l+n-1) b_{n-1,l-n+2} + b_{n,l-n+2} \\ 0 &\equiv \hat{b} - \hat{m}_l = \alpha_{00} b_{1,l} + \cdots + \alpha_{n-2,n-2} b_{n-1,l-n+2} + (\alpha_{n-1,n-1} + (-l+n)\alpha_{nn}) b_{n,l-n+1} \end{aligned}$$

Since

$$\det \begin{pmatrix} \alpha_{00} & -l+1 & 0 & \cdots & 0 \\ \alpha_{11} & 1 & -l+2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_{n-2,n-2} & 0 & 0 & \cdots & -l+n-1 \\ \alpha_{n-1,n-1} + (-l+n)\alpha_{nn} & 0 & 0 & \cdots & 1 \end{pmatrix}$$

$$\begin{aligned} &= \alpha_{00} + (-\alpha_{11})(-l+1) + \cdots + (-1)^{n+1}\alpha_{n-1,n-1}(-l+1)\cdots(-l+n-1)+ \\ &\quad +(-1)^{n+1}(-l+n)\alpha_{nn}(-l+1)(-l+2)\cdots(-l+n-1) \\ &= -\Theta_a(l-1) \neq 0, \end{aligned}$$

we have $b_{1,l}, b_{2,l-1}, \cdots, b_{n,l-n+1} \in M$.

Now we deduce

$$M \supseteq \sum_{i=1}^{n}\sum_{k\geq 0} A_* b_{ik} \supseteq \sum_{i=1}^{n}(\sum_{k\geq 0} \mathrm{C}[X]_* X^{-k})b_{i0} = \sum_{i=1}^{n} \mathrm{C}(X)b_{i0} = \mathrm{C}(X)_a^n.$$

Hence $M = \mathrm{C}(X)_a^n$, i.e., $\gamma a = 0$. ∎

Lemma 4.19 *Let $a \in A$, irreducible in B, and $b = aq$. Then*

$$\gamma a + \textit{top-tor}a = \gamma b + \textit{top-tor}b.$$

PROOF. Define r_q: $\frac{B}{Ba} \longrightarrow \frac{B}{Bb}$ via the right multiplication by q, namely, $\overline{u} \to \overline{uq}$. It is clear that r_q is a B-isomorphism. Therefore, it is also an A-homomorphism.

Claim:

$$\begin{array}{ccc} \frac{B}{Ba} & \xrightarrow{r_q} & \frac{B}{Bb} \\ \tau\downarrow & & \downarrow\tau \\ \mathrm{C}(X)_a^n & \xrightarrow{\mathbf{W}} & \mathrm{C}(X)_b^n \end{array}$$

is a commuting square, where

$$\mathbf{W} = \begin{pmatrix} X & 1 & 0 & \cdots & 0 \\ 0 & X & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & (n-1) \\ 0 & 0 & 0 & \cdots & X \end{pmatrix}$$

and $\mathbf{W}$ acts on the left. Moreover $\mathbf{W}$ is a B-isomorphism.

Let $z = \sum_{i=0}^{n-1} b_i(q)p^i + Ba$ be an arbitrary element in $\frac{B}{Ba}$. Then

$$\begin{aligned}
\tau r_q \cdot z &= \tau z\, q = \tau(\sum_{i=0}^{n-1} b_i(qp^i + ip^{i-1}) + Ba) \\
&= \tau(\sum_{i=0}^{n-1}(b_i q + (i+1)b_{i+1})\, p^i + Ba) \\
&= \begin{pmatrix} b_0 X + b_1 \\ b_1 X + 2b_2 \\ \vdots \\ b_{n-2}X + (n-1)b_{n-1} \\ b_{n-1}X \end{pmatrix}.
\end{aligned}$$

On the other hand,

$$\mathbf{W}\tau \cdot z = \mathbf{W}\begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-1} \end{pmatrix} = \begin{pmatrix} b_0 X + b_1 \\ b_1 X + 2b_2 \\ \cdots \\ b_{n-2}X + (n-1)b_{n-1} \\ b_{n-1}X \end{pmatrix}.$$

Thus $\tau r_q = \mathbf{W}\tau$. It follows that $\mathbf{W}$ is a B-isomorphism. (Alternatively, one can show that $\mathbf{W}$ is a B-homomorphism, by checking $\mathbf{W}\mathbf{P}_a = \mathbf{P}_a\mathbf{W}$.)

It is easy to check that the diagram

$$\begin{array}{ccccccc}
\frac{A}{A\cap Ba} & \cong & \frac{A+Ba}{Ba} & \subseteq & \frac{B}{Ba} & \cong & \mathrm{C}(X)^n_a \\
 & & r_q \downarrow & & r_q \downarrow & & \downarrow \mathbf{W} \\
\frac{A}{A\cap Bb} & \cong & \frac{A+Bb}{Bb} & \subseteq & \frac{B}{Bb} & \cong & \mathrm{C}(X)^n_b
\end{array}$$

commutes. Hence

$$\mathrm{C}(X)^n_a/\tau_a(\frac{A+Ba}{Ba}) \cong \mathrm{C}(X)^n_b/\tau_b r_q(\frac{A+Ba}{Ba}).$$

This implies

$$\begin{aligned}
\gamma a &= \gamma b + \mathrm{tor}\frac{A+Bb}{Bb} - \mathrm{tor}\frac{A+Ba}{Ba} \\
&= \gamma b + \mathrm{tor}\frac{A}{A\cap Bb} - \mathrm{tor}\frac{A}{A\cap Ba} \\
&= \gamma b + \text{top-tor}b - \text{top-tor}a.
\end{aligned}$$

Hence

$$\gamma a + \text{top-tor}a = \gamma b + \text{top-tor}b.$$

∎

We prove now our index theorem.

Theorem 4.20 *Let $a = \alpha p^n +$ (lower terms in p) $\in A$, irreducible in B. Then for any $\rho \in \mathbf{C}$, we have*

$$\textit{bott-tor}_\rho a - \gamma_\rho a = \nu_\rho a.$$

PROOF. Without loss of generality, let us prove the claim for $\rho = 0$.

Suppose that $\Theta_a(\xi) = 0$ has roots in $\mathbf{N}$, let N be the largest one. Let $fb = aq^{N+1}$ for some $f \in \mathbf{C}[q]$ such that $b \in A$ is left-normalized. Then $\Theta_b(\xi) = 0$ has no roots in $\mathbf{N}$ since $\Theta_b(\xi) = c\Theta_{fb}(\xi) = c\Theta_a(\xi + N + 1)$ for some non-zero scalar c, and therefore bott-tor$b = 0$ by Theorem 4.10. Since $\Theta_{fb}(\xi)$ has no roots in $\mathbf{N}$, by Lemma 4.18, we have $\gamma fb = 0$.

Let $b_1 = aq$, $b_2 = b_1 q$, $\cdots$, $fb = b_N = b_{N-1}q$. By Lemma 4.19, we have

$$\begin{aligned}
\gamma a &= \gamma b_1 + (\text{top-tor}b_1 - \text{top-tor}a) \\
&= \gamma b_2 + (\text{top-tor}b_2 - \text{top-tor}b_1) + (\text{top-tor}b_1 + \text{top-tor}a) \\
&= \gamma b_2 + (\text{top-tor}b_2 - \text{top-tor}a) \\
&= \cdots \\
&= \gamma b_N + (\text{top-tor}b_N - \text{top-tor}a) \\
&= \gamma fb + \text{top-tor}fb - \text{top-tor}a \\
&= \text{top-tor}fb - \text{top-tor}a
\end{aligned}$$

Since $Bfb = Bb$, top-torfb = top-tor$b$. Since bott-tor$b = 0$, tor$b$ = top-tor$b$, and therefore tor$fb = \nu f +$ tor$b = \nu f +$ top-tor$b$.

From $fb = aq^{N+1}$, we have torfb = tor$a + N + 1$ and $\nu b + \nu f = \nu a + N + 1$. Hence γa = top-tor$b$−top-tor$a$ = tor$fb - \nu f$−top-tor$a$ = tor$a + N + 1 - \nu f$−top-tor$a$ = bott-tor$a + (N + 1 - \nu f)$. Since b is left-normalized and 0 is not a root of $\Theta_b(\xi)$, $\nu b = 0$ (cf. the beginning proof of Lemma 4.18), and therefore $N + 1 - \nu f = \nu b - \nu a = -\nu a$. Hence we have bott-tor$a - \gamma a = \nu a$. ∎

Now let us consider the A-submodule $\mathbb{C}[X, \alpha^{-1}]^n$ of $\mathbb{C}(X)^n_a$ for an element $a = \alpha p^n$ + (lower terms in p) $\in A$. It is clear that $\tau(A + Ba/Ba)$ is contained in $\mathbb{C}[X, \alpha^{-1}]^n$. Moreover $\mathbb{C}[X, \alpha^{-1}]^n/\tau(\frac{A+Ba}{Ba})$ is isomorphic to a $\mathbb{C}[q]$-torsion A-module, hence it is semisimple.

Lemma 4.21 *Let $a = \alpha p^n$ + (lower terms in p) $\in A$, irreducible in B. Then*

$$\mathbb{C}[X, \alpha^{-1}]^n/\tau(\frac{A + Ba}{Ba}) \cong \bigoplus_{\rho, \alpha(\rho)=0} (\frac{A}{A(q - \rho)})^{(\gamma_\rho a)}.$$

PROOF. It is clear that if $\frac{A}{A(q-\rho)}$ is isomorphic to summand of $\mathbb{C}[X, \alpha^{-1}]^n/\tau(\frac{A+Ba}{Ba})$, then $\alpha(\rho) = 0$. Write

$$\mathbb{C}[X, \alpha^{-1}]^n/\tau(\frac{A + Ba}{Ba}) \cong \bigoplus_{\rho, \alpha(\rho)=0} (\frac{A}{A(q - \rho)})^{(m_\rho)}.$$

By localizing this expression, and (4.1) at $\star = \mathbb{C}[q] \setminus (q - \rho)$, where ρ is a root of α, we have

$$(\frac{A_\star}{A_\star(q - \rho)})^{(m_\rho)} \cong \mathbb{C}(X)^n_a/\tau(\frac{A_\star + Ba}{Ba}) \cong (\frac{A_\star}{A_\star(q - \rho)})^{(\gamma_\rho a)}.$$

Hence $m_\rho = \gamma_\rho a$ for any root of α. ∎

Since $\mathrm{Soc}_A \frac{B}{Ba} \subseteq \frac{A+Ba}{Ba}$, and $\tau(\frac{A+Ba}{Ba}) \subseteq \mathbb{C}[X, \alpha^{-1}]^n$, we deduce that $\tau(\mathrm{Soc}_A \frac{B}{Ba})$ is contained in $\mathbb{C}[X, \alpha^{-1}]^n$.

Definition 4.22 *Let $a = \alpha p^n$ + (lower terms in p) $\in A$, irreducible in B, with corresponding simple A-module $S = Soc_A \frac{B}{Ba}$. Define the ρ-discrepancy of S as $\delta_\rho a = tor_\rho(\mathbb{C}[X, \alpha^{-1}]^n/\tau(S))$.*

Note that the $\delta_\rho a$ measure how far the simple module S is from being isomorphic to $\mathbf{C}[X, \alpha^{-1}]^n$.

Corollary 4.23 *Let a be as in Definition 4.22. Then for $\rho \in \mathbf{C}$, if $\alpha(\rho) \neq 0$, $\delta_\rho a = 0$; if $\alpha(\rho) = 0$,*

$$\delta_\rho a = tor_\rho a - \nu_\rho a.$$

PROOF. Let $\rho \in \mathbf{C}$. Then

$$\begin{aligned} \delta_\rho a &= \mathrm{tor}_\rho(\mathbf{C}[X, \alpha^{-1}]^n / \tau S) \\ &= \mathrm{tor}_\rho \mathbf{C}[X, \alpha^{-1}]^n / \tau(\frac{A + Ba}{Ba}) + \mathrm{tor}_\rho(\frac{A + Ba}{Ba} / S) \end{aligned}$$

If $\alpha(\rho) \neq 0$, since the map $\mathbf{C}[X, \alpha^{-1}]^n / \mathbf{C}[X]^N \to \mathbf{C}[X, \alpha^{-1}]^n / \tau(\frac{A+Ba}{Ba})$ is onto, and $\mathrm{tor}_\rho \mathbf{C}[X, \alpha^{-1}]^n / \mathbf{C}[X]^n = 0$, we have $\mathrm{tor}_\rho \mathbf{C}[X, \alpha^{-1}]^n / \tau(\frac{A+Ba}{Ba}) = 0$; since $\mathrm{tor}_\rho(\frac{A+Ba}{Ba}/S) = \text{top-tor}_\rho a \leq \mathrm{tor}_\rho a$, and $\mathrm{tor}_\rho a = 0$ by Corollary 4.15, we have $\mathrm{tor}_\rho(\frac{A+Ba}{Ba}/S) = 0$. Hence we obtain $\delta_\rho a = 0$.

If $\alpha(\rho) = 0$, by Lemma 4.21 and the above equality, we have $\delta_\rho a = \gamma_\rho a + \text{top-tor}_\rho a$. By the index theorem, $\gamma_\rho a = \text{bott-tor}_\rho a - \nu_\rho a$. Therefore we have $\delta_\rho a = \text{bott-tor}_\rho a - \nu_\rho a + \text{top-tor}_\rho a = \mathrm{tor}_\rho a - \nu_\rho a$. ∎

An immediate consequence of the above corollary is the following:

Corollary 4.24 *Let $a = \alpha p^n +$ (lower terms in p) $\in A$, irreducible in B. If a is left-normalized, then $\delta_\rho a = 0$ for all roots ρ of α iff a is $[a]$-regular and $\nu_\rho a = 0$ for all roots of α.*

PROOF. ($\Rightarrow$): Let ρ be a root of α. Since a is left-normalized, $-\nu_\rho a \geq 0$. By Corollary 4.23, $\delta_\rho a = 0$ iff $\mathrm{tor}_\rho a = 0$ and $\nu_\rho a = 0$. Since $\mathrm{tor}_\rho a = 0$ always holds for any ρ which is not a root of the leading coefficient α of a by Corollary 4.15, we conclude $\mathrm{tor}_\rho a = 0$ for all ρ, this means that a is $[a]$-regular.

($\Leftarrow$): We are given that a is $[a]$-regular, and $\nu_\rho a = 0$ for any root ρ of α. Therefore $\mathrm{tor}_\rho a = 0$ for any ρ. Hence $\delta_\rho a = \mathrm{tor}_\rho a - \nu_\rho a = 0$ by Corollary 4.23. ∎

Corollary 4.25 *Let S be a $\mathbf{C}[q]$-torsion free simple A-module. If there exists an element $a \in [S] \cap A$ with the leading coefficient α and p-degree n such that S is isomorphic to the A-submodule $\mathbf{C}[X, \alpha^{-1}]^n$ of $\mathbf{C}(X)^n_a$, then a is S-regular and $\nu_\rho a = 0$ for every root ρ of α. Moreover α is uniquely determined by S, up to a non-zero scalar.*

PROOF. We are given that $S \cong \mathbf{C}[X, \alpha^{-1}]^n$. Hence $\mathbf{C}[X, \alpha^{-1}]^n$ is a simple A-module, and therefore $\mathbf{C}[X, \alpha^{-1}]^n = \tau(\mathrm{Soc}_A \frac{B}{Ba})$, i.e., $\delta_\rho a = 0$ for any root ρ of α by Definition 4.22. By Corollary 4.24, we have that a is S-regular (i.e. $\mathrm{tor}_\rho a = 0$ for all ρ) and $\nu_\rho a = 0$ for any root ρ of α.

Moreover by Corollary 4.15, α as the leading coefficient of the S-regular element a, is uniquely determined by S, up to a non-zero scalar. ∎

The following theorem is the main application of the index theorem. It generalizes Block's representation theorem (Theorem 3.20). The necessary and sufficient conditions for $S \cong \mathbf{C}[X, \alpha^{-1}]^n$ are (roughly speaking) that there is an S-regular element in $[S]$ and the indicial roots of $[S]$ at each place $\rho \in \mathbf{C}$ fall into two extreme cases, namely, either all roots are integers or no roots are integers.

Theorem 4.26 *Let S be a $\mathbf{C}[q]$-torsion free simple A-module. Then S is isomorphic to the A- submodule $\mathbf{C}[X, \alpha^{-1}]^n$ of $\mathbf{C}(X)^n_a$ for some $a = \alpha p^n +$ (lower terms in p) $\in [S] \cap A$, iff*

1) there exists an S-regular element in $[S]$, and

2) for every $\rho \in \mathbf{C}$, either all the indicial roots of $[S]$ are integers and $\sigma_\rho[S] = 0$, or none of the indicial roots is an integer.

If so, the elements a giving such isomorphisms, are precisely the S-regular elements in $[S] \cap A$.

PROOF. ($\Rightarrow$) By Corollary 4.25, we have that a is S-regular and $\nu_\rho a = 0$ for any root of α, in particular 1) holds.

First let us consider any ρ which is a root of α. Since a is S-regular and $\nu_\rho a = 0$, the indicial polynomial of a at ρ has no roots in $\mathbf{N}$.

We show that the indicial polynomial has no roots in $\mathbf{Z}^-$ for any roots of α. Since $\nu_\rho a = 0$, by Lemma 2.23 we have $\nu_\rho^* a = 0$ and $\Theta^*_{a,\rho}(\xi) = \Theta_{a,\rho}(-\xi-1)$. If $a = \hat{a}f$ for some $\hat{a} \in A$ and $f \in \mathbf{C}[q] \setminus \{\mathbf{C}\}$, then a is not S-regular. Hence a must be also normalized from the right. Therefore, $\nu_\rho^* a = 0$ implies $\Theta^*_{a,\rho}(0) \neq 0$, i.e., $\Theta_{a,\rho}(-1) \neq 0$. Any negative integer solution of $\Theta_{a,\rho}(\xi)$ which is smaller than -1, will produce a positive integer solution of $\Theta^*_{a,\rho}(\xi) = 0$. By the right-module analogue of Theorem 4.10 this means bott-tor$_\rho \frac{A}{aA} \neq 0$. By duality, top-tor$_\rho \frac{A}{Aa} \neq 0$; a contradiction to a being S-regular. Hence $\Theta_{a,\rho}(\xi) = 0$ has no negative integer solutions for any root of α.

Next let us consider any ρ which is not a root of α. Then the solution of $\Theta_{a,\rho}(\xi) = 0$ are always the integers $0, 1, \cdots, n-1$, and $\sigma_\rho[S] = \nu_\rho\phi - \text{tor}_\rho a = 0 - 0 = 0$. Thus 2) holds.

($\Leftarrow$) Consider any S-regular element $a = \alpha p^n +$ (lower terms in p) $\in [S] \cap A$. We have $\text{tor}_\rho a = 0$ for any ρ.

Let ρ be a root of α. Since a is left-normalized, $\nu_\rho a \leq 0$. If $\nu_\rho a < 0$, then 0 is a solution of the indicial polynomial of a at ρ; therefore $\sigma_\rho[S] = 0$ by the assumption 2). But $\sigma_\rho a = \nu_\rho \alpha - \text{tor}_\rho a = \nu_\rho \alpha > 0$. This is a contradiction. Hence $\nu_\rho a = 0$.

By Corollary 4.24, we have $\delta_\rho a = 0$. Hence by Definition 4.22 we have $\mathbf{C}[X, \alpha^{-1}]^n = \tau(\text{Soc}_A \frac{B}{Ba})$, and $\tau(\text{Soc}_A \frac{B}{Ba})$ is isomorphic to S. ∎

Remark: Note that the second condition in Theorem 4.26 is stated in terms of two invariants of the similarity class $[S]$; hence it can be checked by considering any element in $[S]$, and it does not matter whether this element is $[S]$-regular. The first condition is not easy to check in general. In fact, there are some similarity classes $[a]$

which contain no $[a]$-regular element; such an example is constructed in Section 5.

The two conditions are independent of each other. That the first condition does not imply the second one can be seen easily. That the second condition does not imply the first one is not obvious (cf. Example 4.29).

We now obtain Block's result (cf. Theorem 3.20) as a special case of Theorem 4.26.

Corollary 4.27 *For each linear similarity class $[S]$, the corresponding simple A-module S is of the form $(\mathbf{C}[X,\alpha^{-1}],\ q \to X,\ p \to \mathbf{P}_a = \frac{d}{dX} + \frac{\beta}{\alpha})$, for some $a = \alpha p - \beta \in [S] \cap A$.*

PROOF. We need to verify the two conditions in Theorem 4.26. There is always an S-regular element in each linear similarity class (cf. Theorem 3.20). Therefore, the first condition is satisfied.

Let $a = \alpha p - \beta$ be such an S-regular element in $[S]$. If ρ is a root of α, then $\Theta_\rho(\xi) = 0$ has no integer solutions by Lemma 3.19. If ρ be not a root of α, then $\Theta_\rho(\xi) = 0$ has the solution zero, and $\sigma_\rho[a] = \nu_\rho\alpha - \mathrm{tor}_\rho a = 0 - 0 = 0$, as desired. ∎

4.5 Examples

In every linear similarity class $[a]$, there is an $[a]$-regular element, every admissible element is preserving, and there is an easy description for similar elements. The following examples show that all of this is no longer true for quadratic similarity classes.

Example 4.28 *Let $a = q^2p^2 + q - 2$. Then a is irreducible in B, and admissible. But a is not preserving.*

PROOF. First, a is irreducible in B by the Eisenstein criterion ([9], Proposition 5).

Since $\Theta_{a,0}(\xi) = \xi(\xi-1)+(-2) = \xi^2-\xi-2 = (\xi-2)(\xi+1)$ has the solution -1, a is not 0-preserving, and therefore not preserving by the definition.

Note that $\nu_0 a = \min\{\nu_0 q^2 - 2,\ \nu_0 0 - 1,\ \nu_0(q-2) - 0\} = 0$. Thus $\Theta_{a,0}(\xi)$ has the root $2 > 0 = \nu_0 a$. By the criterion (Theorem 4.10), we have bott-tor$_0 a = 1$. Our aim is to show that top-tor$_0 a = 0$.

Now let us calculate the invariant $\sigma_0[a]$ using another element in $[a]$. Let

$$\begin{aligned} \hat{a} &= aq^{-1} = q^2p^2q^{-1} + (q-2)q^{-1} \\ &= q^2(q^{-1}p^2 + 2(-1)q^{-2}p + 2q^{-3}) + 1 - 2q^{-1} \\ &= qp^2 - 2p + 1. \end{aligned}$$

Then $\hat{a}$ is similar to a. We show that $\hat{a}$ is $[a]$-regular. By Remark 1) after Definition 2.31, in order to show that $\hat{a}$ is preserving, it is sufficient to show that $\hat{a}$ is 0-preserving. Since $\Theta_{\hat{a},0}(\xi) = \xi(\xi-1) - 2\xi = (\xi-3)\xi$ has no negative integer solutions, $\hat{a}$ is 0-preserving. Hence $\frac{A}{A\cap B\hat{a}}$ is simple by Theorem 2.34, i.e., top-tor$_\rho \hat{a} = 0$ for all ρ.

Now we show bott-tor$_\rho \hat{a} = 0$, for all ρ. By Theorem 4.10, we only need to check for $\rho = 0$. Since $\Theta_{\hat{a},0}(\xi)$ has the root 3 strictly larger than $|\nu_0\hat{a}| = 1$, by Theorem 4.10, bott-tor$_0\hat{a} = (3+1) - \mathrm{rank}\mathbf{M}_{\hat{a}}(3)$, where

$$\mathbf{M}_{\hat{a}}(3) = \begin{pmatrix} 1 & -2 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 4 \end{pmatrix}.$$

Since $\mathbf{M}_{\hat{a}}(3)$ has rank 4, bott-tor$_0\hat{a} = 0$.

Together we obtain that $\hat{a}$ is $[a]$-regular. Therefore $\sigma_0[a] = \sigma_0[\hat{a}] = \nu_0 q -$ tor$_0\hat{a} = 1 - 0 = 1$. Hence, and since bott-tor$_0 a = 1$,

$$1 = \sigma_0[a] = \nu_0 q^2 - \mathrm{tor}_0 a = 2 - (\text{bott-tor}_0 a + \text{top-tor}_0 a) = 1 - \text{top-tor}_0 a,$$

and consequently, top-tor$_0 a = 0$. This proves that $A \cap Ba$ is maximal, i.e. a is admissible. ∎

The following is an example of a similarity class which contains no regular element.

Example 4.29 *Let $a = qp^2 - 2p + \gamma$ with $0 \neq \gamma \in \mathbb{C}[q]$. (Then at $\rho = 0$, the indicial polynomial Θ_a has roots 0 and 3.) Choose γ with degree even and larger than 2, such that the rank of the 4 by 5 matrix $\mathrm{M}_a(3)$ is 3 (for instance, $\gamma = q + q^4$). Then a is irreducible in B, and there is no $[a]$-regular element in $[a]$.*

PROOF. **Claim 1:** a is irreducible in B.

Suppose there exist u and t in $\mathbb{C}(q)$ such that

$$a = q(p+t)(p+u) = q(p^2 + (u+t)p + (u' + tu)).$$

Therefore,

$$\begin{cases} -2 &= q(u+t) \\ \gamma &= q(u' + tu). \end{cases}$$

We obtain

$$t = \frac{-2}{q} - u,$$

therefore,

$$\gamma = qu' + qu(\frac{-2}{q} - u) = qu' - 2u - qu^2.$$

If $u = 0$, then $\gamma = 0$. This is a contradiction. Hence $u \neq 0$. Write $u = f/g$, with $0 \neq f, g \in \mathbb{C}[q]$, and $(f, g) = 1$. Then $u' = (f'g - fg')/g^2$, and therefore

$$g^2\gamma = [q(f'g - fg') - 2fg] - qf^2.$$

Let $\deg f = m$, $\deg g = n$, $\deg \gamma = l$. Now, let us compare the degrees $\deg g^2\gamma = 2n + l$, $\deg qf^2 = 2m + 1$ and $\deg(q(f'g - fg') - 2fg)$.

Observe that $\deg g^2\gamma = 2n + l \neq 2m + 1 = \deg qf^2$, since l is even. Therefore $q(f'g - fg') - 2fg \neq 0$, hence has degree $m + n - \delta$ for some $\delta \in \mathbf{N}$. Moreover $m + n - \delta$ equals $2n + l$ or $2m + 1$, whichever is larger.

In the first case, $2m + 1 < 2n + l = m + n - \delta$, we obtain the contradiction

$$n < n + l + \delta = m < m + 1 + \delta < n.$$

In the second case, $2n + l < 2m + 1 = m + n - \delta$, we obtain a contradiction

$$n < n + l + \delta < m = n - 1 - \delta < n.$$

This proves that a is irreducible in B.

Claim 2: There is no $[a]$-regular element in $[a]$.

Note that $\nu_0 a = -1$, and $\Theta_a(\xi) = \xi(\xi - 3)$ has a root $3 > |\nu_0 a| = 1$. By the assumption we know that the rank of $\mathrm{M}_a(3)$ is 3, therefore bott-tor$_0 a = 4 - 3 = 1$ by Theorem 4.10. Since tor$_0 a \leq \nu_0 q = 1$ by Corollary 4.15, tor$_0 a =$ bott-tor$_0 a = 1$. Therefore $\sigma_0[a] = 1 - 1 = 0$. For any $\rho \neq 0$, $\sigma_\rho[a] = \nu_\rho q -$ tor$_\rho a = 0$. Hence if there exists a $[a]$-regular element, say b, at all, then $\sigma_\rho[b] = 0$ for all ρ, and therefore it must have the form $b = p^2 + Dp + C$, where D and C are in $\mathbf{C}[q]$. Since b is S-regular, $\frac{A}{Ab} \cong S$ by Corollary 3.7. Since a is preserving, $\frac{A}{A \cap Ba} \cong S$ by Theorem 2.24. Together we have $\frac{A}{A \cap Ba} \cong \frac{A}{Ab}$. Let $\overline{u}$ be the image of $1 + (A \cap Ba)$. Since $\overline{a} = \overline{0}$, $a\overline{u} = \overline{0}$ in $\frac{A}{Ab}$. Since b has leading coefficient 1, $\overline{u}$ has a representative of the form $u = g + fp$ with $f, g \in \mathbf{C}[q]$. Since $\overline{u} \neq 0$, f and g cannot both be zero. We obtain $au \in Ab$, namely ,

$$au = vb$$

for some $v \in A$.

If $f = 0$, then $u, v \in \mathbf{C}[q]$, and $vb = au = qup^2 + (2qu' - 2u)p + (u'' - 2u' + \gamma u)$. Hence $v = qu$, $vD = 2qu' - 2u$ and $vC = qu'' - 2u' + \gamma u$. This implies $quD = 2qu' - 2u$. Since quD would have degree strictly larger than the right hand side, we conclude $D = 0$; therefore $u = cq$ for some non-zero scalar c. We now have $vC = cq^2C = -2c + c\gamma q$. But this equation has no solution in $\mathbf{C}[q]$; contradiction.

Thus $f \neq 0$. It is clear that $v \sim u$. Therefore

$$v = qfp + qg - qf\frac{\sigma'}{\sigma}$$

for some $\sigma \in \mathbf{C}(q)$ (cf. the proof of the claim in Theorem 3.17). By substituting in the equation $au = vb$, we have

$$(qfp + qg + -qf\frac{\sigma'}{\sigma})(p^2 + Dp + C) = (qp^2 - 2p + \gamma)(fp + g),$$

and

$$\begin{aligned}
L.H.S. &= qfp^3 + qfDp^2 + qfCp + (qg - f\frac{\sigma'}{\sigma})p^2 + \\
&\quad +(qg + qf\frac{\sigma'}{\sigma})Dp + (qg - qf\frac{\sigma'}{\sigma})C + qfD' + qfC' \\
&= qfp^3 + [qfD + (qg - qf\frac{\sigma'}{\sigma})]p^2 + \\
&\quad +[qfC + (qg - qf\frac{\sigma'}{\sigma})D + qfD']p + \\
&\quad +[(qg - qf\frac{\sigma'}{\sigma})C + qfC'],
\end{aligned}$$

and

$$\begin{aligned}
R.H.S. &= qfp^3 + 2qf'p^2 + qf''p + qgp^2 + 2qg' + qg' + \\
&\quad -2fp^2 - 2f'p - 2gp - 2g' + \gamma fp + \gamma g \\
&= qfp^3 + [2qf' + qg - 2f]p^2 + \\
&\quad +[qf'' + 2qg' - 2f' - 2g + \gamma f]p + \\
&\quad +[qg'' - 2g' + \gamma g].
\end{aligned}$$

Hence, we have

$$qfD + (qg - qf\frac{\sigma'}{\sigma}) = 2qf' + qg - 2f \tag{4.2}$$

$$qfC + (qg - \frac{\sigma'}{\sigma})D + qfD' = qf'' + 2qg' - 2f' - 2g + \gamma f \tag{4.3}$$

$$(qg - qf\frac{\sigma'}{\sigma})C + qfC' = \gamma g + qg'' - 2g'. \tag{4.4}$$

From (4.2)

$$\begin{aligned} D &= \frac{1}{qf}[2qf' + qg - 2f - (qg - qf\frac{\sigma'}{\sigma})] \\ &= \frac{2f'}{f} - \frac{2}{q} + \frac{\sigma'}{\sigma}, \end{aligned}$$

and by solving this differential equation of order one, we conclude $D = 0$ and $\sigma = (q/f)^2$, and therefore $qf\frac{\sigma'}{\sigma} = 2f - 2qf'$. Therefore $b = p^2 + C$. Since b is irreducible in B, $C \neq 0$.

If $g = 0$, then (4.4) implies $C = c_1\sigma$ for some non-zero scalar c_1. But $C \in \mathbb{C}[q]$, hence $\sigma = (q/f)^2 \in \mathbb{C}[q]$. This implies $f = c_2^{-1}$ or $f = c_3^{-1}q$, for some non-zero scalars c_2 and c_3, and therefore $\sigma = (c_2q)^2$ or $\sigma = c_3^2$. Hence $C = c_1\sigma = c_1(c_2q)^2$ or $C = c_1c_3^2$. By substituting in (4.3), we deduce $c_1c_2^{-1}(c_2q)^2q = c_2^{-1}\gamma$ or $c_3^{-1}c_1c_3^2q^2 = -2c_3^{-1} + c_3q\gamma$, that is $c_1c_2^2q^3 = \gamma$ or $c_1c_3^2q^2 = -2 + q\gamma$. Since $\deg\gamma$ is even, we have a contradiction in both cases.

Consequently f and g are both nonzero. From (4.4) we obtain

$$(qg - 2f + 2qf')qfC = qf\gamma g + q^2fg'' - 2qfg' - (qf)^2C'. \tag{4.5}$$

As $D = 0$, (4.3) reads

$$qfC = qf'' + 2qg' - 2f' - 2g + \gamma f.$$

We solve this equation for C, and obtain an expression for C' by differentiation.

Then we substitute these expressions for qfC and C' into (4.5). We obtain an equation:

$$0 = X_1 + X_2 + X_3 + X_4,$$

where

$$\begin{aligned} X_1 &= qf^2\gamma' + 2q\gamma ff' + (-3)\gamma f^2 \\ X_2 &= q^2fg'' + q^2gf'' + 2q^2f'g' + 6fg - 4qfg' - 4qf'g \\ X_3 &= 2q^2gg' + (-2)qg^2 \\ X_4 &= q^2ff^{(3)} + q^2f'f'' + 6ff' - 4qff'' - 2qf'f'. \end{aligned}$$

Note that

$$\begin{aligned}
\deg X_1 &\leq 2\deg f + \deg \gamma = 2m + l,\\
\deg X_2 &\leq \deg f + \deg g = m + n,\\
\deg X_3 &\leq 2\deg g + 1 = 2n + 1,\\
\deg X_4 &< 2m.
\end{aligned}$$

Let f_m, g_n, and γ_l be the leading terms of f, g, and γ, respectively.

If $\deg X_1 < 2m + l$, then

$$0 = lf_m^2\gamma_l + 2m\gamma_l f_m^2 - 3\gamma_l f_m^2 = f_m^2\gamma_l(l + 2m - 3),$$

which is a contradiction since l is even. We conclude that $\deg X_1 = 2m + l$, and hence $\deg X_4 < \deg X_1$.

Write $\deg X_2 = m + n - \delta_1$, where $\delta_1 \in \mathrm{N}$. If $\delta_1 \neq 0$, then

$$\begin{aligned}
0 &= n(n-1)f_m g_n + m(m-1)f_m g_n + 2mnf_m g_n + 6f_m g_n\\
&= f_m g_n(n^2 + (2m-5)n + (m^2 - 5m + 6)).
\end{aligned}$$

This implies that $n = m - 2$ or $m - 3$.

Write $\deg X_3 = 2n + 1 - \delta_2$, where $\delta_2 \in \mathrm{N}$. If $\delta_2 \neq 0$, then

$$0 = 2ng_n^2 - 2g_n^2 = g_n^2(2n - 2),$$

hence $n = 1$.

The following two cases could arise:

Case 1: $\deg X_1 \geq \max\{\deg X_2,\ \deg X_3\}$, to say,

$$2m + l \geq \max\{m + n - \delta_1,\ 2n + 1 - \delta_2\}.$$

If $\delta_2 = 0$, we have $2m + l = m + n - \delta_1$ since l is even, i.e., $n = m + l + \delta_1$. We also have $2m + l > 2n + 1$. Therefore $2m + l > 2n + 1 = 2m + 2l + 2\delta_1 + 1 > 2m + l$. This is a contradiction. So $\delta_2 \neq 0$, and therefore $\deg g = n = 1$, and $2n + 1 - \delta_2 \leq 3$. Since $l \geq 4$, we have $2m + l = m + n - \delta_1 = m + 1 - \delta_1$. This implies $1 = n = m + l + \delta_1 \geq 3$. This is a contradiction. We conclude that Case 1 cannot occur.

Case 2: $\deg X_3 \geq \max\{\deg X_1,\ \deg X_2\}$, to say,

$$2n + 1 - \delta_2 \geq \max\{2m + l,\ m + n - \delta_1\}.$$

If $\delta_2 = 0$, we have $2n + 1 = m + n - \delta_1$, and therefore $m = n + \delta_1 + 1$. We also have $n - \delta_1 \geq m + l = n + \delta_1 + 1 + l > n$. This is a contradiction. So $\delta_2 \neq 0$, and therefore $\deg g = 1$. Obviously, $2 - \delta_2 \leq 2m + l$ since $l \geq 4$. We conclude that Case 2 cannot occur either. This complete the proof of Claim 2. ∎

The following example gives a description of all elements in the similarity class of the quadratic element $a = p^2 - \gamma$.

Example 4.30 *Let $a = p^2 - \gamma \in A$, irreducible in B (for instance $\gamma = q^3$). Then $b \in [a]$ if and only if $b = f(up^2 + vp + w)$, where $0 \neq f \in \mathrm{C}(q)$, and*

$$\begin{aligned} u &= \phi^2 + \phi\psi' - \phi'\psi - \psi^2\gamma \\ v &= -u' \\ w &= (\phi' + \psi\gamma)(2\phi' + \psi'' + \psi\gamma) + \\ &\quad -(\phi + \psi')(\phi'' + \psi\gamma' + \phi\gamma + 2\psi'\gamma) \end{aligned}$$

for some ϕ, $\psi \in \mathrm{C}(q)$, not both zero.

In particular, $b \in [a] \cap A$ is admissible if and only if $b = f(up^2 + vp + w)$ with u, v and w defined as above, where ϕ, $\psi \in \mathrm{C}[q]$ and $f \in d^{-1}\mathrm{C}[q]$, where d is the greatest common divisor of u, v and w.

Moreover, if $\deg\gamma$ is odd, then a is the only $[a]$-regular element in $[a]$, up to a non-zero scalar.

Proof. Let $b \sim a$, i.e., $\frac{B}{Bb} \cong \frac{B}{Ba}$. Let $\overline{s}$ be the image of $1 + Bb$. Since $\overline{p^2} = \overline{\gamma}$ in $\frac{B}{Ba}$, the coset $\overline{s}$ has a representative of the form $s = \phi + \psi p$ for some ϕ, $\psi \in \mathrm{C}(q)$. Since

$1 + Bb \neq \overline{0}$, ϕ and ψ cannot both be zero. Write $b = up^2 + vp + w$ with u, v, and $w \in \mathbf{C}(q)$. Then we have (modulo Ba)

$$\begin{aligned} 0 &\equiv (up^2 + vp + w)(\phi + \psi p) \\ &= (u\psi)p^3 + (u\phi + v\psi + 2u\psi')p^2 + \\ &\quad +(v\phi + w\psi + 2u\phi' + v\psi' + u\psi'')p + (w\phi + v\phi' + u\phi''). \end{aligned}$$

By substituting $p^2 \equiv \gamma$, $p^3 \equiv p\gamma = \gamma p + \gamma'$, we have

$$0 = u(2\phi' + \psi'' + \psi\gamma) + v(\phi + \psi') + w\psi \tag{4.6}$$

$$0 = u(\phi'' + \psi\gamma' + \phi\gamma + 2\psi'\gamma) + v(\phi' + \psi\gamma) + w\phi. \tag{4.7}$$

We deduce

$$\begin{aligned} 0 &= -(4.6)\phi + (4.7)\psi \\ &= u(\psi\phi'' + \psi^2\gamma' - 2\phi\phi' - \phi\psi'' + 2\psi\psi'\gamma) + \\ &\quad +v(\psi\phi' + \psi^2\gamma - \phi^2 - \phi\psi'). \end{aligned}$$

Note that if $\phi^2 + \phi\psi' - \psi\phi' - \psi^2\gamma = 0$, then $\phi \neq 0$, $\psi \neq 0$, and therefore $a = p^2 - \gamma = (p - \phi/\psi)(p + \phi/\psi)$. This contradicts the fact that a is irreducible in B. Hence we have, up to a factor $0 \neq f \in \mathbf{C}(q)$,

$$u = \phi^2 + \phi\psi' - \psi\phi' - \psi^2\gamma, \quad v = -u'.$$

We know that ϕ, ψ cannot both be zero. If $\psi \neq 0$, by substituting the expressions for u and v into (4.6), we obtain the stated expression for w; If we then substitute the expressions for u, v and w into (4.7), we obtain an equality. Similarly, we obtain the same expression for w from (4.7), if $\phi \neq 0$.

It is clear that $fb \sim b$ for any $0 \neq f \in \mathbf{C}(q)$.

Conversely, if b is of such a form $f(up^2 + vp + w)$, then by inverting the above calculation, one obtains $b(\phi + \psi p) \equiv 0$ (modulo Ba). Hence the map $1 + Bb \to (\phi + \psi p) + Ba$ establish an isomorphism between $\frac{B}{Bb}$ and $\frac{B}{Ba}$, i.e. $b \sim a$.

Let $b \in [a] \cap A$ be admissible. Note that a is $[a]$-regular, i.e., $\frac{A}{Aa}$ is simple. By Proposition 4.2, b is admissible implies that b is a minimal annihilator $\overline{s} \in \frac{A}{Aa}$. Since $\overline{p^2} = \overline{\gamma}$ in $\frac{A}{Aa}$, the coset $\overline{s}$ has a representative of the form $\phi + \psi p$, for some ϕ and $\psi \in \mathrm{C}[q]$. Then we obtain the same expression for b in terms of ϕ and ψ, by the same calculation.

Let $d \in \mathrm{C}[q]$ be the greatest common divisor of u, v and w. Then $fb \in A$ for any $0 \neq f \in d^{-1}\mathrm{C}[q]$, and it also annihilates $\phi + \psi p$. Hence fb is admissible.

Conversely, if $b \in A$ is of the such form $f(up^2 + vp + w)$ for some $f \in d^{-1}\mathrm{C}[q]$, then by inverting the above calculation one obtain $b(\phi + \psi p) \equiv 0$ (modulo Aa), hence b is a minimal annihilator of S, hence it is admissible by Proposition 4.2.

Now if $\deg \gamma$ is odd, we show that a is the only S-regular element in $[S]$, up to a non-zero scalar. Let b be a S-regular element in $[S]$. By Corollary 4.15 and the fact that the S-regular element a has leading coefficient 1, the leading coefficient of b is equal to a non-zero scalar. Without loss of generality, we may then assume that b has the leading coefficient 1. Since every S-regular element is in particular admissible, b is of the form $fd^{-1}(up^2 + vp + w)$ with u, v, w and d defined as above, for some $0 \neq f \in \mathrm{C}[q]$. Hence $db = f(up^2 + vp + w)$. Since b has leading coefficient 1, we obtain $d = fu$. Since d divides u, f must be a scalar, say c. We also have that $d = cu$ divides both v and w. But $v = -u'$ has lower degree, hence $v = 0$, and therefore u is a non-zero scalar.

Recall that

$$u = \phi^2 + \phi\psi' - \psi\phi' - \psi^2\gamma \tag{4.8}$$

for some ϕ and $\psi \in \mathrm{C}[q]$.

If $\phi = 0$, then $\psi \neq 0$, and $u = -\psi^2\gamma$. This is a contradiction since u is a nonzero scalar. If $\psi = 0$, then $u = \phi^2$. This implies $\phi = u^{1/2}$, and therefore $w = -u\gamma$ from (4.7). Hence $b = p^2 - \gamma = a$.

Now let us assume that both ϕ and ψ are non-zero. Let $\deg\phi = n$, $\deg\psi = m$. We shall compare degrees in (4.8). If $n > m$, then $\deg(\phi)^2 = 2n > n + m - 1 = \deg(\phi\psi' - \psi\phi')$, and $2n \neq \deg(-\psi^2\gamma)$ since $\deg\gamma$ is odd. This is a contradiction. If $n \leq m$, then the largest degree in (4.8) will be $2m + \deg\gamma = \deg(-\psi^2\gamma)$. Hence it must be equal to the degree of u which is zero. This is a contradiction since $\deg\gamma \neq 0$. ∎

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