N-COMPACT FRAMES AND APPLICATIONS

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Abstract

This thesis is a study of what it means for a frame to be N-compact. We find that the frame analogues of equivalent conditions defining N-compact spaces are no longer equivalent in the frame context; one must be careful in deciding what the appropriate frame notion is. We show that it is the assumption of a choice principle (the Axiom of Countable Choice) which provokes this departure from the spatial situation.

We analyze the several possibilities and show how it is the 'H-N-compact' frames which best embody the notion of N-compactness. We develop the theory and construct the H-N-compactification, which uses a frame-theoretic version of the classical ultrafilter formulation of the spatial N-compactification. We use this compactification to show how these frames relate to the other 'N-compact' frames. Along the way we construct a 0-dimensional Lindelöf co-reflection, and show how this relates to the H-N-compactification.

Recent works in Abelian group theory have employed the groups $C(X, \mathbb{Z})$ in the study of reflexivity and duality. The N-compact spaces are important in this regard because of a theorem of Mrówka which shows how a group homomorphism from $C(X, \mathbb{Z})$ to $\mathbb{Z}$ is determined on a small part of X, if X is N-compact. We use the H-N-compact frames to lift this to a result about any group of global sections of a sheaf of Abelian groups. We then are able to give a sufficient condition for the local reflexivity of a sheaf to lift to global reflexivity; it is enough that the frame is H-N-compact. We show that the groups known to be reflexive (in ZFC) each appear as a group of global sections of some sheaf on an N-compact frame, or as the dual

of such a group of sections. We can then use our generalized Mrówka's Theorem to establish their reflexivity.

In the final chapter we apply the techniques of Chapter 1 to the study of realcompact frames. These have been studied, but the definition usually given is quite restrictive. We construct the H-realcompactification and develop enough of the basic theory of H-realcompact frames to justify proposing that these be thought of as the realcompact frames.

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Contents

Introduction

0.0.1 Frames

There are several different aspects to the study of frames, each corresponding to motivations arising from topology, logic, and algebra. From the viewpoint of a topologist, frame theory is topology with the lattice of open sets taken as the primitive notion. P. Johnstone explicates this idea well in his article [Jo1] from which we draw the following history.

The use of lattices in the study of topological spaces began with Stone's duality theorems in 1936-1937. There for the first time an explicit connection was made between certain algebraic objects (Boolean algebras) and certain topological objects, (0-dimensional compact Hausdorff spaces.) In an important sense, they are the same thing. The idea of applying lattice theory to topology was soon developed further; the Stone-Cěch compactification (in its ultrafilter formulation) was one of the fruits.

In the 1950's Ehressman and his students began to take the equation between Boolean algebras and Boolean spaces seriously; proposing that lattices with the right properties (frames) be viewed as 'topological spaces' in their own right, regardless of whether the lattice was in fact the lattice of open subsets of some topological space. Much of topology can be extended to these generalized spaces, and where it cannot, it is often in the frame context that the situation is better. For example, by a result of Dowker and Strauss [Do,St], the Lindelöf property is preserved under coproducts of regular frames (the analogue of products of spaces,) whereas it is well known that even binary products of (regular) Lindelöf spaces need not be Lindelöf. One even has a Lindelöf co-reflection for frames, (a 'Lindelöf-ification'.) We shall spend some time considering these facts in Chapter 1, where we shall prove some results which bear upon the study of frames in its logical aspect, which we consider next.

It is well-known that the statement of the classical Tychonoff Theorem is equivalent to the Axiom of Choice, and that even when the statement is restricted to Hausdorff spaces one still requires the Boolean Ultrafilter Theorem. With frames however the situation is different; there are precise analogues of these preservation results, but they can be established without the use of any choice principles.

It turns out that many of the results in classical topology which depend on nonconstructive principles no longer need such principles in their frame-theoretic versions. Often it is only in showing that these more general results about frames imply the classical results that one requires the choice principle; usually it is found in the proof that a particular frame has 'enough points', that it is representable as a lattice of open subsets of some topological space. If one is content to work with 'pointless spaces' then one can even use the Tychonoff theorem in a constructive context, [Ve]. Much of frame theory has been developed with a constructive approach to topology in mind.

0.0.2 An Outline

In Chapter 1, we investigate what it could mean for a frame to be N-compact. We see that equivalent notions of N-compactness for spaces separate into inequivalent notions in the frame setting, primarily due to the preservation of the Lindelöf property under coproducts. We show how one of these notions, Herrlich-N-compactness, is really the 'right' notion for frames, develop an H-N-compactification, and explore some interesting consequences.

Members of the other class of 'N-compact' frames we call Stone-N-compact; we show that these are exactly the 0-dimensional Lindelöf frames, and that this fact is equivalent in logical strength to the Axiom of Countable Choice. We construct an S-N-compactification, and show that it is the co-reflection to the category of 0-dimensional Lindelöf spaces if and only if the Axiom of Countable Choice holds. We prove that the S-N-compactification coincides with the H-N-compactification after a spatial co-reflection.

The class of H-N-compact frames includes many non-spatial examples, but remains in a connection with some important properties of the groups and rings $C(X, \mathbb{Z})$ first established for N-compact spaces. We prove some of these facts in Chapter 1, and leave the proof of others until the next chapter.

Chapter 2 contains some applications of our work in Chapter 1. This thesis was originally motivated by some results in Abelian group theory concerning the groups $C(X, \mathbb{Z})$, in which N-compact spaces play a significant role. Such groups $C(X, \mathbb{Z})$ occur as (particularly simple) examples of groups of global sections of sheaves on frames. If one is to generalize the results alluded to, one needs a notion of an 'N-compact frame.'

We show how H-N-compact frames can be used to lift one such result of Mrówka up to a result about arbitrary groups of global sections, (adding further support for our assertion that these are the N-compact frames.) We note how one can use this generalized theorem to prove some recent results concerning Boolean powers and the groups $C(X, \mathbb{Z})$. As another corollary we obtain a 'sheaf-theoretic' result; we show how the property of local reflexivity implies global reflexivity.

The book [Ek,Me] contains some important new methods of constructing reflexive groups; constructions via 'tree sums' and 'tree products.' We show how a tree product may be obtained as a group of global sections, and a tree sum as its dual. Our generalized Mrówka's theorem can then be used to establish the associated reflexivity results in [Ek,Me]. This brings all of the known reflexive groups under one umbrella; groups of global sections of certain sheaves on N-compact spaces.

In Chapter 3 we take some of the lessons learned in our investigation of N-compact frames and apply them to an investigation of frame realcompactness. As we did in Chapter 1, we show how the 'H-realcompact' frames 'lift' the idea of spatial realcompactness properly, something the 'S-realcompact' frames (which have already been studied) fail to do. We develop the basic theory, including the H-realcompactification.

Chapter 0

Preliminaries

0.1 Introduction

We briefly review the basic definitions and results we shall need in subsequent chapters. Most of the introductory material concerning frames is well-known, except perhaps for Theorem 0.2.2. The book [Jo] is a good source for the details and background we cannot provide here. We give explicit references for anything not covered there.

In Section 0.3 we introduce the basic notions concerning sheaves of Abelian groups. The book [Te] is an excellent reference for this. Section 0.4 contains the few definitions and results regarding properties of Abelian groups we shall need. The books [Fu] are the best resource for general information about Abelian groups, and the book [Ek,Me], soon to appear, contains most of what is known about reflexivity.

0.2 Frames

A frame is a complete lattice L which is $\wedge$-continuous; $a \wedge \bigvee S = \bigvee_{s \in S} a \wedge s$ for $a \in L$ and $S \subseteq L$. We will denote the bottom of L by 0 and the top by e. A homomorphism of frames is a map $L \xrightarrow{h} M$ which preserves finitary meets and arbitrary joins, (and hence 0 and e.) The category of all frames and frame morphisms is denoted **Frm**. Note that the $\wedge$-continuity of a frame implies that it is a (complete) Heyting algebra. As is usual, we shall write the Heyting implication by $\rightarrow$. The associated psuedocomplement operator takes u to $u^* = u \rightarrow 0$. We shall usually denote frames by upper-case letters, typically L and M, and elements of frames by lower-case letters such as u and v. If S is a subset of a frame L and ϕ is a frame map with domain L we will denote the image of S under ϕ by $\phi[S]$. To avoid complicated notation, we shall often supress the mention of an index set for joins taken in a frame, writing $\bigvee_L u_\alpha$ for the join of a family of elements $\{u_\alpha\}_\alpha$ of L.

The canonical example of a frame is the **open subset lattice** $\mathcal{O}(X)$ of some topological space X. Frame of this sort are called **spatial**. A complete Boolean algebra which is not atomic is an example of a non-spatial frame; it cannot be represented as the open subset lattice of a topological space.

0.2.1 The Connection with Topological Spaces

The connection between spaces X and frames $\mathcal{O}(X)$ is part of a deeper relationship; there are contravariant functors

$$\mathbf{Top} \underset{\mathcal{O}}{\overset{\Sigma}{\rightleftarrows}} \mathbf{Frm}$$

which are adjoint on the right. For a space X, $\mathcal{O}(X)$ is the frame of open subsets of X. A continuous map $X \xrightarrow{f} Y$ is taken to the map $\mathcal{O}(Y) \xrightarrow{\mathcal{O}(f)} \mathcal{O}(X)$, which sends the open set U to $f^{-1}U$. The functor Σ takes L to the **spectrum** of L, the space of all completely prime filters on L; a completely prime filter F has the property that $\bigvee S \in F$ iff $S \cap F \neq \emptyset$, for any $S \subseteq L$. The sets $\{F \in \Sigma L \mid u \in F\}$ for $u \in L$ are the open subsets of ΣL. For any frame homomorphism $L \xrightarrow{h} M$ the corresponding continuous map $\Sigma(h)$ takes a completely prime filter F of M to $f^{-1}F$, a completely prime filter in L. Completely prime filters correspond to prime elements of L; those elements u with the property that $u = v \wedge w$ only if $u = v$ or $u = w$. Thus the spectrum ΣL can be seen as the space of all prime elements of L; it has a base for the topology consisting of the sets $\{p \text{ prime} \mid u \not\leq p\}$, for $u \in L$.

The functors $\mathcal{O}$ and Σ are adjoint on the right; the corresponding adjunctions are $X \xrightarrow{\sigma_L} \Sigma\mathcal{O}(X)$ and $L \xrightarrow{o_L} \mathcal{O}\Sigma L$. The first of these takes an element x to $\mathcal{O}(x) = \{U \in \mathcal{O}(X) \mid x \in U\}$, and the second takes u to $\{F \in \Sigma L \mid u \in F\}$. The elements of Fix($o_-$), those frames for which o_L is an isomorphism, are called **spatial** or said to **have enough points**. These are exactly the frames of the form $\mathcal{O}(X)$ mentioned above. The elements of Fix(σ_-) are the **sober spaces**. Sobriety is a property intermediate in strength between the T_2 and T_0 separation properties.

0.2.2 Separation and Covering Properties

A frame L is **compact** (**Lindelöf**) if $\bigvee_L S = e$ for some subset $S \subseteq L$ implies that S has a finite (countable) subset with the same join.

The **Boolean part** of a frame L consists of all the complemented elements of L, a sublattice which is a Boolean algebra. It is denoted BL. A frame is **0-dimensional** if any element of the frame is the join of the complemented elements below it, or equivalently if the Boolean part generates the frame.

Given two elements u, v of L, we say $u \prec v$ ('u **is rather below** v') if there is a third element w such that $u \wedge w = 0$ and $w \vee v = e$. A frame is **regular** if $u = \bigvee_{v \prec u} v$ for any u in L. It is a fact that **compact regular frames are spatial**, a statement equivalent in logical strength to the Boolean Ultrafilter Theorem. It is often useful to know that the **prime elements of a regular frame are exactly the maximal** elements.

The **completely below relation** $\prec\!\prec$ is a strengthening of the $\prec$-relation. We say that $u\prec\!\prec v$ if there is a family $(x_{i,k})$ in L, for $i \in \omega$ and k ranging from $0, \cdots, 2^i$, depending on i, with the following properties.

(i) $x_{0,0} = u$ and $x_{0,1} = v$,

(ii) $x_{i,k} \prec x_{i,k+1}$, and

(iii) $x_{i,k} = x_{i+1,2k}$.

This is exactly the sort of situation constructed in the proof of Urysohn's Lemma. Indeed one can use the lemma's argument to show that $U\prec\!\prec V$ for $U,V \in \mathcal{O}(X)$, iff there is a continuous function $X \xrightarrow{\phi} [0,1]$ such that ϕ takes the value 0 on U and the value 1 outside V.

A frame is **completely regular** if for any element u, $u = \bigvee_{v\prec\prec u} v$. One can check without difficulty (using the methods of Urysohn's Lemma) that a space X is completely regular iff $\mathcal{O}(X)$ is completely regular.

We list some simple but important properties of the $\prec\!\prec$-relation.

Lemma 0.2.1 (i) *$u\prec\!\prec u$ iff u has a complement.*

(ii) *$u\prec\!\prec v$ implies $u \prec v$ and hence $u \leq v$.*

(iii) *$u \leq v\prec\!\prec w \leq y$ implies $u\prec\!\prec y$.*

(iv) *For any $u \in L$, the set $\{v \in L \mid v\prec\!\prec u\}$ is an ideal and the set $\{v \in L \mid u\prec\!\prec v\}$ is a filter.*

(v) *If $u\prec\!\prec v$ then there exists a w with $u\prec\!\prec w\prec\!\prec v$. (We say that the $\prec\!\prec$-relation* interpolates.*)*

0.2.3 Compactifications

For any distributive lattice D with zero and unit the collection of all the ideals on D, $\mathfrak{J}D$, is a compact frame.

For any frame L the frame $\mathfrak{J}BL$ is the **universal 0-dimensional compactification** of L, with coreflection map $\mathfrak{J}BL \xrightarrow{j} L$ taking I to $\bigvee_L I$. This is the constructive analogue of the Stone-Banaschewski compactification of a 0-dimensional topological space. The functor $\mathfrak{J}B$ takes a frame morphism $L \xrightarrow{\phi} M$ to the morphism $\mathfrak{J}B\phi$, which itself takes an ideal I to $\langle\phi[I]\rangle$, the ideal in BM generated by the image of I

An ideal I of L is **completely regular** if for any $u \in I$ there is some $v \in I$ with $u \prec\!\prec v$. For any frame L, the subframe of $\mathfrak{J}L$ consisting of all completely regular ideals is the **universal completely regular compactification** of L, or the **Stone-Cěch compactification**, written βL. The coreflection map $\beta L \longrightarrow L$ takes an ideal to its join in L. The proof of these facts is entirely constructive. The functor β takes $L \xrightarrow{\phi} M$ to $\beta(\phi)$, the frame homomorphism defined as follows.

$$\begin{array}{ccc} \beta L & \xrightarrow{\beta(\phi)} & \beta M \\ I & \longrightarrow & \{u \in M \mid u \leq \phi(v), \text{ some } v \in I\}. \end{array}$$

Given a space X, the frame $\beta\mathcal{O}(X)$ is compact and completely regular, and therefore spatial. (Here the Boolean Ultrafilter Theorem is used.) A simple category theory argument then shows that the space $\Sigma\beta\mathcal{O}(X)$ is the usual Stone-Cěch compactification of X. In Chapter 3 we will need a more explicit proof of this, one which we present after first supplying a few necessary definitions.

If D is a distributive lattice with 0 and unit, Max$D$ denotes the topological space consisting of all maximal ideals of D with a base for open sets consisting of the sets $\{I \in \text{Max}D \mid u \notin I\}$ for $u \in L$.

A familiar construction of the Stone-Cěch compactification βX proceeds as follows. Given X, form Coz(X), the (distributive) lattice of co-zero sets in X. (These are the sets $f^{-1}([0,1]-\{0\})$, for continuous $X\xrightarrow{f}[0,1]$.) Then βX is MaxCoz(X). (A yet more familiar version of this is the 'dual' construction, using zero sets and filters.) We connect this with the Stone-Cěch compactification $\beta\mathcal{O}(X)$ via the following result.

Theorem 0.2.2 *For any space X the spaces $\Sigma\beta\mathcal{O}(X)$ and MaxCoz(X) are homeomorphic via the maps*

$$\begin{array}{rcl} \Sigma\beta\mathcal{O}(X) & \underset{\phi}{\overset{\psi}{\rightleftarrows}} & MaxCoz(X) \\ I & \longrightarrow & \{V \in Coz(X) \mid \Downarrow(V) \vee (I \cap Coz(X)) \neq E_{3Coz(X)}\} \\ \{U \mid U \prec\!\prec V,\ some\ V \in J\} & \longleftarrow & J \end{array}$$

Before we begin the proof we establish the following simple result.

Lemma 0.2.3 *For any space X, and $f, g \in C(X,[0,1])$, such that $Coz(f) \cup Coz(g) = X$, there are continuous functions f_1, g_1 so that*

- *$Coz(f_1) \prec\!\prec Coz(f)$, as elements of $\mathcal{O}(X)$.*
- *$Coz(g_1) \prec\!\prec Coz(g)$,*
- *$Coz(f_1) \cap Coz(g_1) = \emptyset$,*
- *$Coz(f_1) \cup Coz(g) = X$, and*
- *$Coz(g_1) \cup Coz(f) = X$.*

Proof Set $f_1 = \max\{0, f-g\}$, so that $\mathrm{Coz}(f_1) = \{x \mid f(x) > g(x)\}$, and $g_1 = \max\{0, g-f\}$. Then obviously $\mathrm{Coz}(f_1) \cap \mathrm{Coz}(g_1) = \emptyset$. As $\mathrm{Coz}(f) \cup \mathrm{Coz}(g) = X$, we have $\mathrm{Coz}(f_1) \cup \mathrm{Coz}(g) = X$, and similarly for g_1.

Now $\mathrm{Coz}(f_1) \prec\!\prec \mathrm{Coz}(f)$ since the continous function $f/\mathrm{Max}\{f,g\}$ is 1 on $\mathrm{Coz}(f_1)$ and 0 off $\mathrm{Coz}(f)$. (The hypothesis ensures that the function is defined everywhere.) Of course we can use a similar function to show that $\mathrm{Coz}(g_1) \prec\!\prec \mathrm{Coz}(g)$. □

Proof (Theorem 0.2.2) Our main task will be to see that the definition of the maps in Theorem 0.2.2 makes sense. Once this is done it is straightforward to show that they are continuous and inverse to each other.

Let I be an element of $\Sigma\beta\mathcal{O}(X)$, a maximal completely regular ideal in $\mathcal{O}(X)$. If we can show that $\phi(I)$ is an ideal, it will clearly be a maximal ideal. To do this we show that $\phi(I)$ is closed under joins. Suppose that

$$\downarrow\mathrm{Coz}(f) \vee (I \cap \mathrm{Coz}(X)) \neq E \text{ and}$$

$$\downarrow\mathrm{Coz}(g) \vee (I \cap \mathrm{Coz}(X)) \neq E,$$

and towards a contradiction, that $\mathrm{Coz}(f)\cup\mathrm{Coz}(g)\cup\mathrm{Coz}(h) = X$, for some co-zero set $\mathrm{Coz}(h)$ in I. Note that $\mathrm{Coz}(f)\cup\mathrm{Coz}(h)\cup\mathrm{Coz}(k) \neq X$ for any $\mathrm{Coz}(k) \in I\cap\mathrm{Coz}(X)$. Note also that if $U \in I$, the complete regularity of I implies that there is a co-zero set in I which is above U. These two observations together imply that $\mathrm{Coz}(f)\cup \mathrm{Coz}(h) \cup U \neq X$ for any $U \in I$. It follows that any V in $\mathcal{O}(X)$ which is completely below $\mathrm{Coz}(f)\cup\mathrm{Coz}(h)$ is in I, by the maximality of I.

However, by the previous lemma, we know that there is a co-zero set $\mathrm{Coz}(k)$, completely below $\mathrm{Coz}(f)\cup\mathrm{Coz}(h)$, and hence in I, such that $\mathrm{Coz}(k)\cup\mathrm{Coz}(g) = X$. This contradicts the hypothesis on $\mathrm{Coz}(g)$. Thus $\phi(I)$ must be an ideal, so that ϕ is a map as shown.

To see that the same is true for ψ, fix J an element of $\mathrm{MaxCoz}(X)$. Then $\psi(J)$ is clearly an element of $\beta\mathcal{O}(X)$ and we must show that it is a maximal such element. Towards this, suppose that $\psi(J) \subsetneq K$, for $K \in \beta\mathcal{O}(X)$, and fix $w \in K\backslash\psi(J)$. There is a u in K so that $w \prec\!\prec u$, and therefore a cozero set $\mathrm{Coz}(f)$ so that $w \prec\!\prec \mathrm{Coz}(f) \leq u$. It follows that $\mathrm{Coz}(f)$ is not in J. Since J is maximal, there is a continuous function g such that $\mathrm{Coz}(g) \in J$ and $\mathrm{Coz}(g)\cup\mathrm{Coz}(f) = X$. By Lemma 0.2.3 there exist f_1 and g_1 so that

$$\mathrm{Coz}(f_1) \;\prec\!\prec\; \mathrm{Coz}(f),$$

$$\mathrm{Coz}(g_1) \prec\!\prec \mathrm{Coz}(g) \text{ and,}$$
$$\mathrm{Coz}(f) \cup \mathrm{Coz}(g_1) = X.$$

Then $\mathrm{Coz}(g_1) \in \psi(J) \subseteq K$, and $\mathrm{Coz}(f) \cup \mathrm{Coz}(g_1) = X$, so that K is not a proper ideal. Thus $\psi(J)$ must be maximal.

It is straightforward to show that ϕ and ψ are continuous and inverses one of the other. □

0.2.4 Quotient Frames

A subspace inclusion $X \subseteq Y$ gives rise to a surjective frame map $\mathcal{O}(Y) \to \mathcal{O}(X)$, and so it is important to understand frame quotients. This is best done with the use of nuclei.

A nucleus r on a frame L is a (set) map $L \to L$ such that

(i) $u \leq r(u)$,

(ii) $r(u) \wedge r(v) = r(u \wedge v)$, and

(iii) $r^2(u) = r(u)$.

Nuclei correspond to frame congruences: a nucleus r takes $u \in L$ to the largest element in the u-block of the corresponding congruence. The quotient frame of L mod r is written $[L]_r$, and consists of the r-closed elements ($r(u) = u$,) with finite meets as in L and $\bigvee_{[L]_r} u_\alpha = r(\bigvee_L u_\alpha)$.

Corresponding to special sorts of subspace inclusions there are special sorts of nuclei:

(i) A nucleus $L \xrightarrow{r} L$ is dense if $r(0) = 0$.

(ii) r is **co-dense** if $r(u) = e$ implies $u = e$.

(iii) r is **open** if it is of the form $u \to (-)$ for some $u \in L$. The frame $[L]_r$ is isomorphic to $\Downarrow(u)$.

(iv) r is **closed** if it is of the form $u \vee (-)$, for some $u \in L$. In this case $[L]_u$ is $\Uparrow(u)$.

In a similar way we define a frame homomorphism $L \xrightarrow{\phi} M$ to be **dense** if $\phi(u) = 0$ implies $u = 0$, and ϕ to be **co-dense** if $\phi u = e$ implies $u = e$.

We shall frequently make use of the following result. The proof is easy.

Lemma 0.2.4 *Suppose L is a regular frame, and $L \xrightarrow{\phi} M$ is a frame homomorphism. Then the following hold:*

(i) *If ϕ is dense then it is monic in the category of regular frames, and also 1-1 on the Boolean part of L.*

(ii) *ϕ is 1-1 iff it is co-dense.*

(iii) *If M is compact and ϕ is dense, then ϕ is 1-1.*

Frame Coproducts We shall need to know nothing more about frame coproducts than their existence, and so we shall not describe their construction here. We denote the coproduct of a family of frames $\{L_\alpha\}_\alpha$ by $\bigoplus_\alpha L_\alpha$, and a **copower** $\bigoplus_{\alpha \in I} L$ by $L^{(I)}$. It is a fact that the coproduct of regular frames is regular, and likewise for 0-dimensional frames. The **Tychonoff Theorem** for frames says that the coproduct of a family of compact frames is compact. There is a constructive proof of this in [Ve].

0.3 Sheaves of Abelian Groups

A frame L can be viewed as a category, with objects the elements of L and morphisms the pairs (U, V), for $U \leq V$, with domain U and codomain V. Notice that we write the frame elements in upper-case, the usual practice when one considers sheaves. A **presheaf of Abelian groups on a frame** L is a contravariant functor from L to Ab, the category of Abelian groups. More descriptively, A attaches to each $U \in L$ an Abelian group AU, and to each pair (U, V) a group homomorphism $AV \rightarrow AU$ (when $U \leq V$.) We describe the action of such a map as taking $a \in AV$ to $a \mid U$, the **restriction of** a **to** U. The properties of a functor imply that

(i) If $a \in AU$ then $a \mid U = a$.

(ii) If $U \leq V \leq W$ and $a \in AW$, then $(a \mid V) \mid U = a \mid U$.

A morphism of preheaves $A \longrightarrow B$ is just a natural transformation. The category of all presheaves of Abelian groups is denoted **AbPSh**L.

A presheaf $A \in$AbShL is a **sheaf** of Abelian groups if it satisfies the following two properties.

(i) If $U = \bigvee_\alpha U_\alpha$ in L, a and b are elements of AU and $a \mid U_\alpha = b \mid U_\alpha$ for all α, then $a = b$. This is the **separation** condition.

(ii) Suppose that $U = \bigvee_\alpha U_\alpha$, and $a_\alpha \in AU_\alpha$ for each α. If the a_α are **compatible**, that is, if $a_\alpha \mid U_\alpha \wedge U_\beta = a_\beta \mid U_\alpha \wedge U_\beta$ for any α and β, then there is an element $a \in AU$ such that $a \mid U_\alpha = a_\alpha$ for any α. This is the **patching** condition.

Note that that the conditions (i) and (ii) imply that $A0 = \{0\}$. The group AE is called the **group of global sections**. The full subcategory of sheaves of Abelian groups is denoted **AbSh**L.

The following is a useful result we shall often employ in Chapter 2.

Lemma 0.3.1 *If A is a sheaf on a frame L, and $U \in L$ is the join of a family of pairwise disjoint elements U_α, then $AU \cong \prod_\alpha AU_\alpha$ via an isomorphism ϕ which commutes with the projection maps π_α, and restriction maps as shown, for any α.*

$$\begin{array}{ccc} AU & \xrightarrow{\ \phi\ } & \prod_\alpha AU_\alpha \\ {\scriptstyle restriction}\downarrow & \swarrow{\scriptstyle \pi_\alpha} & \\ AU_\alpha & & \end{array}$$

Proof Omitted.

The subcategory AbSh$L \subseteq$ AbPShL is a reflective subcategory, that is, there is a left adjoint to the inclusion functor. We construct the left adjoint $(\tilde{-})$ as follows. For an $A \in$ AbPShL, and a cover $\{C_\alpha\}_\alpha$ of $U \in L$, let A_C be the equalizer of the maps f and g shown, which are determined by the restriction maps in the obvious manner.

$$A_C \longrightarrow \prod_\alpha AU_\alpha \underset{g}{\overset{f}{\rightrightarrows}} \prod_{\alpha,\beta} A(U_\alpha \wedge U_\beta)$$

The covers of U form a directed set when ordered under refinement, and we can form a direct system of groups, using the obvious maps $A_C \longrightarrow A_D$ present when D refines C. Then $\tilde{A}U$ is defined to have value equal to the direct limit of this system. Since $\{U\}$ is a cover of U, there is a map $AU \to \tilde{A}U$, for any $U \in L$, and the presheaf morphism $A \to \tilde{A}$ defined in this way is the adjunction for the sheaf reflection.

For any frame L, we denote by $\mathbf{Z}_L$ the 'constant sheaf on L', defined to be the sheaf reflection of the presheaf A which has values $AU = \mathbf{Z}$ for all $U \in L$, with constant restriction maps. We can also view $\mathbf{Z}_L$ as the sheaf defined thus: for any $U \in L$, $\mathbf{Z}_L U$ is the group of all frame homomorphisms $\mathcal{O}(\mathbf{Z}) \longrightarrow \Downarrow(U)$, where $\mathbf{Z}$ is given the discrete topology and the group operations are defined in the obvious ('pointwise') way. Note that any such frame homomorphism is completely determined by its values on the atoms of $\mathcal{O}(\mathbf{Z})$; the sets $\{n\}$ for $n \in \mathbf{Z}$.

If $V \leq U$, the restriction map $\mathbf{Z}_L U \longrightarrow \mathbf{Z}_L V$ of the sheaf $\mathbf{Z}_L$ takes an element ξ of $\mathbf{Z}_L U$ to that element of $\mathbf{Z}_L V$ which takes value $\xi(\{n\}) \wedge V$ at n, for any $n \in \mathbf{Z}$.

Since $\mathbf{Z}$ is also a ring, we can view $\mathbf{Z}_L$ as a sheaf of commutative rings with identity. Note that when L is spatial ($= \mathcal{O}(X)$ say,) then $\mathbf{Z}_L E$ is (isomorphic to) the group (ring) of continuous functions $C(X, \mathbf{Z})$, via the map taking a continuous function $f \in C(X, \mathbf{Z})$ to the element $f' \in \mathbf{Z}_L E$ with values $f'(\{n\}) = f^{-1}(\{n\})$ for any $n \in \mathbf{Z}$.

0.4 Abelian Groups

If A is an Abelian group, then $A^* = \text{Hom}(A, \mathbf{Z})$ is the **dual** of A, the group of all homomorphisms, under pointwise addition. There is a canonical map $A \xrightarrow{\sigma_A} A^{**}$ with value at a the homomorphism $\sigma_A(a)$, itself with action $\sigma_A(a)(f) = f(a)$, (the **evaluation map** at a.) The group A is **torsionless** if σ_A is 1-1 (the homomorphisms of A^* separate points,) and **reflexive** if σ_A is an isomorphism.

If $A \xrightarrow{h} B$ is a group homomorphism, then the **dual map** h^* takes an element ϕ of B^* to the element $h^*(\phi)$ of A^* which itself takes ψ to $h \circ \psi$.

It follows from a classical theorem of Loś that the direct sum and direct product of a family of reflexive groups is reflexive, provided the family is of nonmeasurable cardinality. This fact depends heavily on the **slenderness** of $\mathbf{Z}$: Any homomorphism $\mathbf{Z}^\omega \xrightarrow{h} \mathbf{Z}$ has $h(e_n) = 0$ for almost all $n \in \omega$ (ie, all but a finite number.) Here e_n denotes the element with values $e_n(m) = \delta_{n,m}$.

It is an easy exercise to show that a direct summand of a reflexive group is reflexive, and that the dual of any reflexive group is again reflexive.

Chapter 1

N-Compact Frames

1.1 Introduction

Since the work of Stone it has been known that the compact completely regular spaces are exactly those spaces which are (homeomorphic to) closed subsets of $[0,1]^I$, for I some index set. Soon thereafter [Hew] the realcompact spaces were defined and it was shown that the realcompact spaces are the spaces homeomorphic to closed subspaces of $\mathbb{R}^I$, (today this property is commonly used as a definition of realcompactness.) It was natural step then for Engelking and Mrówka [En,Mr] to define an E-compact space to be a closed subspace of E^I, for a topological space E. The $\mathbb{N}$-compact spaces (the case $E = \mathbb{N}$, the discrete space of natural numbers) are important in the study of the groups and rings $C(X, \mathbb{Z})$. The material of [Ed,Oh] is an example we will discuss in Chapter 2.

In [Her] and in [Mr1] there were established a number of conditions equivalent to $\mathbb{N}$-compactness, analogous to those earlier obtained in the case of realcompactness. We introduce some basic concepts before stating their results, the first being the 0-dimensional compactification, constructed by Banaschewski in [Ba].

Definition 1.1.1 *If X is a 0-dimensional T_2 space, then β_0 denotes the evaluation embedding of X into $2^{C(X,2)}$ and $\beta_0 X$ the closure of $\beta_0[X]$, which is the* **universal 0-dimensional compactification.**

Remark In the literature this compactification is often written ζX and termed the 'Stone-Banaschewski compactification', or just the 'Banaschewski compactification'

Definition 1.1.2 ([En,Mr]) *A topological space is* **N-compact** *if it is homeomorphic to a closed subspace of the product space $\mathbb{N}^I$ for some index set I.*

Definition 1.1.3 *For a given space X, a* **clopen ultrafilter** *on X is an ultrafilter in $BO(X)$, the lattice of clopen subsets of X. Such an ultrafilter $\mathcal{F}$ has the* **countable intersection property** *(cip) if $\bigcap S \neq \emptyset$ for any countable subset $S \subset \mathcal{F}$, and is* **fixed** *if $\bigcap \mathcal{F} \neq \emptyset$.*

In the following, $\mathbb{N}^*$ denotes the one-point compactification of the discrete space of integers $\mathbb{N}$. A subspace X of Y is $C_{\mathbb{Z}}$-embedded if any continuous function from X to $\mathbb{Z}$ extends to Y.

Theorem 1.1.4 *Suppose X is a 0-dimensional T_2 space. Then these are equivalent:*

1. *X is $\mathbb{N}$-compact.*
2. *If X is a dense $C_{\mathbb{Z}}$-embedded subspace of a 0-dimensional T_2 space Y, then $X = Y$.*
3. *For any point $x \in \beta_0 X \setminus \beta_0[X]$ there is continuous function $\beta_0 X \xrightarrow{h} \mathbb{N}^*$ such that $h \restriction \beta_0[X] \subseteq \mathbb{N}$ and $h(x) = \infty$.*
4. *Every clopen ultrafilter with the countable intersection property is fixed.*
5. *Any ring homomorphism $C(X,Z) \xrightarrow{h} \mathbb{Z}$ is the evaluation map at some point $x_0 \in X$, i.e., $h(f) = f(x_0)$ for all $f \in C(X, \mathbb{Z})$.*

The equivalences (1)↔(2)↔(5) are due to Mrówka and Engelking [En,Mr] and [Mr1] respectively, and (1) ↔(3) ↔(4) to Herrlich, in [Her] and Chew in [Ch].

Of these conditions, (1) and (4) are arguably the most important, and certainly the most well known. In applications of the theory of N-compact spaces, and in the study of compactifications, it is the formulation (4) that is most often used. Our definition of H-N-compact frames is based on (4).

The known proofs of the equivalence of (1) and (4) pass through the intermediate statement (3), and use the Boolean Ultrafilter Theorem in a strong way for the implication (1)→(4). The implication (4)→(1) holds in ZF; see the proof in [Her].

We list a few more basic facts about N-compact spaces in Section 1.2.

The definition 1.1.2 of an N-compact space has an obvious interpretation for frames, and these are our 'Stone-N-compact' frames defined in Section 1.3 below, following the route hitherto taken for realcompact frames. While this is the canonical translation of the definition into the language of frames, the proof of Theorem 1.3.5 shows that there is a radical departure from the spatial situation; all such 'N-compact' frames are Lindelöf, given the Axiom of Countable Choice. (Curiously, it is exactly the assumption of a choice principle which provokes this departure, as we shall see in Theorem 1.4.2) For similar reasons, the equivalences in Theorem 1.1.4 break down for frames. We shall show in Section 1.5 that our definition based on (4) of 1.1.4 captures the 'right' notion of what an N-compact frame should be. It is interesting to note that some of the other conditions in Theorem 1.1.4 have natural interpretations for frames. The analogous frame statements separate into two logical equivalence classes, (1) ↔(2) (Theorem 1.3.5) and (4) ↔(5), (Theorem 1.5.21.) The statement (3) does not seem to have a natural frame counterpart.

The reasons for this departure from the spatial situation are quite interesting. As we have mentioned, the proofs of the equivalences in 1.1.4 pass via statement (3) there, which makes a seemingly inescapable use of points. Thus the usual proofs break down in the frame setting. However it is the preservation of the Lindelöf property under frame coproducts that make such proofs impossible. We show in Section 1.4 that the statement of a particular case of this result is equivalent in logical strength to the Axiom of Countable Choice.

In Section 1.5 we develop the theory of H-N-compact frames including the H-N-compactification, and discuss the relations between H-N-compact frames and N-compact spaces. Along the the way we find a way to express N-compactness for topological spaces as a cover condition, see Corollary 1.5.4.

In Chapter 2 we will discuss applications of the theory of H-N-compact frames to the study of reflexive Abelian groups.

In the spirit of one of the major motivations of frame theory, (see the Introduction) we pay particular attention to the use of choice principles, following Johnstone in [Jo] by marking propositions using these with a '*'.

1.2 Basic Facts about N-compact Spaces

We here collect a few notions which we shall need later. Further material may be found in [Her].

Proposition 1.2.1 *Any 0-dimensional Lindelöf space is* N*-compact.*

Definition 1.2.2 *A cardinal κ is said to be* measurable *if there is an ultrafilter on* $\mathsf{P}(\kappa)$ *which has the countable intersection property but is not principal.*

Proposition 1.2.3 ([Je]) *The first measurable cardinal is strongly inaccessible ($2^\lambda < \kappa$ if $\lambda < \kappa$), and therefore one cannot show that such cardinals exist in ZFC.*

No one has shown that measurable cardinals do not exist.

Proposition 1.2.4 *A discrete space X is N-compact iff its cardinality is non-measurable.*

Proof In this setting a fixed ultrafilter is a principal ultrafilter.□

The N-compactification of a 0-dimensional Hausdorff space X is denoted νX, and may be constructed in these two ways:

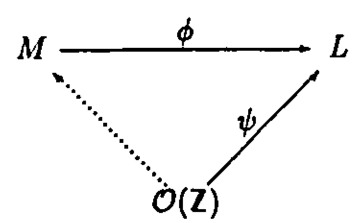
1. Embed the space via the evaluation map into $\mathsf{N}^{C(X,\mathsf{N})}$ and take the closure of its image.
2. Form the space of all clopen ultrafilters with the countable intersection property, with the usual ultrafilter space topology.

1.3 Stone-N-Compact Frames

The definition of an N-compact topological space has a natural translation into the language of frames. We recall that an I-indexed copower of a frame L is denoted $L^{(I)}$ and make the following definitions.

Definition 1.3.1 *A frame L is* **Stone-N-compact** *('S-N-compact') if it is a closed quotient of the frame $\mathcal{O}(\mathsf{N})^{(I)}$ for some index set I.*

Definition 1.3.2 *A frame L is a C_{Z}-***quotient** *of a frame M, via the quotient map $M \xrightarrow{\phi} L$, if any $\mathcal{O}(\mathsf{Z}) \xrightarrow{\psi} L$ factors through ϕ as shown.*



(Of course, this is the analogous property to one space being $C_{\mathbb{Z}}$-embedded in another.)

Definition 1.3.3 *If L is a frame, and $I \in \mathfrak{J}BL$, we say that I is* **σ-proper** *if $\bigvee_L S \neq e_L$ for any countable $S \subseteq I$, and that I is* **completely proper** *if $\bigvee_L I \neq e_L$.*

We will need the following result of [Do,St] below.

Proposition 1.3.4 *(*) A coproduct of regular Lindelöf frames is Lindelöf.*

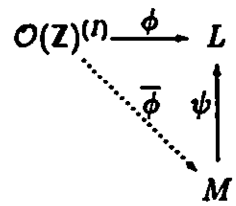
The following result, similar to Theorem 2.1 of [Ma,Ve], shows that the 'liftings' of statements 1 and 2 in Theorem 1.1.4 are still equivalent for frames, and establishes the equivalence $2 \leftrightarrow 3$, which we discuss at greater length below.

Theorem 1.3.5 *(*) For a 0-dimensional frame L, the following are equivalent:*

1. *If L is a dense $C_{\mathbb{Z}}$-quotient of a 0-dimensional frame M, then $L = M$, (ie. the map is an isomorphism.)*
2. *L is S-N-compact.*
3. *L is Lindelöf.*
4. *If $I \in \mathfrak{J}BL$ is σ-proper, then it is completely proper.*

Proof ($1 \rightarrow 2$) The canonical evaluation map, $\mathcal{O}(\mathbb{Z})^{(\mathbb{Z}_L E)} \xrightarrow{F} L$ is surjective since L is 0-dimensional. (Here we think of $\mathbb{Z}_L E$ as the set of all frame homomorphisms from $\mathcal{O}(\mathbb{Z})$ to L, see Section 0.3.) Then L is clearly a $C_{\mathbb{Z}}$-quotient of $\mathcal{O}(\mathbb{Z})^{(\mathbb{Z}_L E)}$ and therefore of its closure, with which, as a dense quotient, it must coincide. Since $\mathcal{O}(\mathbb{Z}) \cong \mathcal{O}(\mathbb{N})$, L is S-N-compact.

($2 \rightarrow 1$) Suppose that L is S-N-compact, and therefore a closed quotient of $\mathcal{O}(\mathbb{Z})^{(I)}$ for some set I, via ϕ say. If L is a dense quotient of a 0-dimensional frame M via ψ, using the $C_{\mathbb{Z}}$-quotient property of ψ we can factor ϕ through ψ as shown.



Now L is a closed quotient of $\text{Im}(\overline{\phi})$ via a dense map ψ, so that the restriction of ψ to $\text{Im}(\overline{\phi})$ is an isomorphism. Using the density of ψ again, it is not hard to show that ψ is an isomorphism.

($2 \to 3$) follows from Proposition 1.3.4 above, and the obvious fact that a closed quotient of a Lindelöf frame is again Lindelöf.

($3 \to 4$) Trivial.

($4 \to 3$) If $e = \bigvee_L u_\alpha$, consider I, the ideal in BL generated by $\{\Downarrow(u_\alpha)\}_\alpha$. This is not completely proper and therefore not σ-proper. Thus there are $w_n \in I$ so that $\bigvee_L w_n = e$. Each w_n is dominated by a finite join $u_{\alpha_1} \vee \cdots \vee u_{\alpha_N}$, so that some countable subset of the u_α's covers e.

($3 \to 1$) Suppose that L is a dense $C_{\mathbb{Z}}$-quotient of a 0-dimensional frame M, via $M \xrightarrow{\phi} L$. Since M is 0-dimensional and hence regular, it suffices to show by Lemma 0.2.4 that ϕ is co-dense.

Suppose that $\phi(u) = e$. Since $u = \bigvee_\alpha v_\alpha$ for some $v_\alpha \in BM$, $e = \phi(u) = \bigvee_\alpha \phi(v_\alpha)$. By hypothesis, there is a countable subfamily $\{v_{\alpha_n}\}_{n \in \mathbb{Z}}$ so that $\bigvee_{\mathbb{Z}} \phi(v_{\alpha_n}) = e$. We can suppose that this is an increasing list, and by subtracting off common intersections, produce countably many $w_n \in BM$, ($n \in \mathbb{Z}$), which are pairwise disjoint with $\bigvee_{\mathbb{Z}} w_n = \bigvee_{\mathbb{Z}} v_{\alpha_n} \leq u$.

Now let $\mathcal{O}(\mathbb{Z}) \xrightarrow{\psi} L$ be the map determined by requiring $\psi(\{n\}) = \phi(w_n)$. By hypothesis there is a map $\mathcal{O}(\mathbb{Z}) \xrightarrow{\overline{\psi}} L$ so that $\phi\overline{\psi} = \psi$. Since $\phi\overline{\psi}(\{n\}) = \psi(\{n\}) = \phi(w_n)$, we must have $\overline{\psi}(n) = w_n$, since the dense map ϕ is 1-1 on complemented elements (Lemma 0.2.4.) Then $e = \bigvee_{\mathbb{Z}} \overline{\psi}(n) = \bigvee_{\mathbb{Z}} w_n \leq u$, so that $u = e$. □

Remark The implications which use choice principles are ($4 \to 3$), which uses Countable Choice, (see Definition 1.4.1) and ($2 \to 3$), in that Proposition 1.3.4 uses Countable Choice. In the next section we will show that statement of the implication $2 \to 3$ is in fact equivalent to Countable Choice.

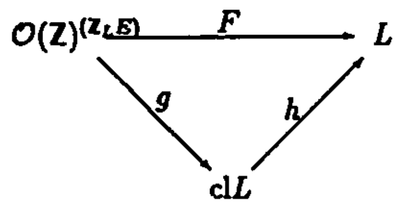
Example The frame $\mathcal{O}(\mathbf{Q}) \oplus \mathcal{O}(\mathbf{Q})$ is a nonspatial frame which is Lindelöf (as it is countably generated) and is therefore S-N-compact.

We have seen that although it is a natural notion, S-N-compactness is somewhat restrictive. For when one admits the choice principle CC, all S-N-compact frames are Lindelöf. Thus one can have an N-compact space X, (for instance ω_1 with the discrete topology,) such that $\mathcal{O}(X)$ is not S-N-compact, and the notion hence fails to be a 'conservative' one. That is to say, the spatial notion is not preserved under the passage to frames, and the concept is not properly lifted (or co-lifted!) from the class of topological spaces to the larger class of frames. As we will see, (Theorem 1.5.21) the statement that S-N-compact frames are Lindelöf depends upon CC, so that in ZF it is *consistent* that there are more S-N-compact frames than Lindelöf frames. But of course one could show nothing more in ZF about S-N-compact frames than about Lindelöf frames. It is in this sense that the obvious (and historically accurate) notion of frame N-compactness is 'restrictive.'

Of course this all follows from the preservation of the Lindelöf property under frame coproducts, a desirable thing to have. But it demands a change in what one views as the fundamental notion of N-compactness. There are other alternatives available; the statements of Theorem 1.1.4. We return to this in Section 1.5.

The S-N-compactification can be modelled on the usual spatial compactification ((1) following Proposition 1.2.4.)

Given a 0-dimensional frame L, we form the evaluation morphism $\mathcal{O}(\mathbf{Z})^{(Z_L E)} \xrightarrow{F} L$, which is a quotient mapping as L is 0-dimensional, and then the closure of this mapping, g. We obtain the diagram



with h a dense map. (See Section 0.2.4.)

Proposition 1.3.6 *For a 0-dimensional frame L, the frame $\nu_S L = cl(L)$ defined above is the* **Stone-N-compactification**, *with coreflection map h.*

The proof of this fact proceeds as for the spatial case, and requires no choice principles.□

Theorem 1.3.7 *Let X be a 0-dimensional space. Then $\Sigma(\nu_S \mathcal{O}(X)) \cong \nu X$.*

Proof The spectrum functor transfers coproducts to products. □

Remark In Theorem 1.4.12 we will give another version of this co-reflection.

1.4 S-N-Compact Frames and Countable Choice

By the result of Dowker and Strauss mentioned above, (Proposition 1.3.4,) any S-N-compact frame is Lindelöf. In this section we show that this equation of S-N-compact frames with Lindelöf frames is equivalent in logical strength to the Axiom of Countable Choice.

We begin by recalling the definition of the Axiom of Countable Choice.

Definition 1.4.1 *The Axiom of Countable Choice (CC) states that, given a countable family $(X_n)_{n\in\omega}$ of arbitrary non-empty sets, there is a choice function; a function $\omega \xrightarrow{F} \bigcup_{n\in\omega} X_n$ such that $F(n) \in X_n$ for all n.*

Theorem 1.4.2 *Every S-N-compact frame is Lindelöf iff the Axiom of Countable Choice holds.*

We will establish Theorem 1.4.2 in a series of lemmas, beginning with the following observation.

Lemma 1.4.3 *Countable Choice holds iff*

(i) Given $(X_n)_{n\in\omega}$, a countable list of non-empty sets, there is a function F so that $F(n)$ =a non-empty countable subset of X_n, for $n \in \omega$.

(ii) A countable union of countable sets is countable.

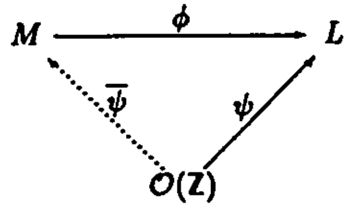
Proof (Lemma) Necessity is clear. Given X_n, a list of non-empty sets, let F be a function as hypothesized in (i). By (ii) the set $\bigcup_\omega F(n)$ is countable, and so there is a function $\omega \xrightarrow{G} \bigcup_\omega F(n)$ which is a bijection. Define $\omega \xrightarrow{H} \bigcup_\omega X_n$ by setting $H(n) = G(m)$ where m is the first number such that $G(m) \in F(n)$. □

The following is a corollary of Theorem 1.3.5.

Lemma 1.4.4 *Any 0-dimensional frame L whose unit is the join of a countable set of compact elements is S-N-compact.*

Proof We will verify Condition 1 of Theorem 1.3.5.

Suppose that L is a dense $C_{\mathbb{Z}}$-quotient of 0-dimensional M, via $M \xrightarrow{\phi} L$. We must show that ϕ is an isomorphism; it suffices by Lemma 0.2.4 to show that ϕ is codense. Now the unit of L, e, is the join of a set of countably many compact elements, say $\{t_n \mid n \in \mathbb{Z}\}$. Since L is 0-dimensional each t_n is complemented, and by a simple argument we can assume that they are pairwise disjoint. Define $\mathcal{O}(\mathbb{Z}) \xrightarrow{\psi} L$ by $\psi(S) = \bigvee_{n\in S} t_n$. Then by hypothesis there is a map $\overline{\psi}$ making the diagram commute.

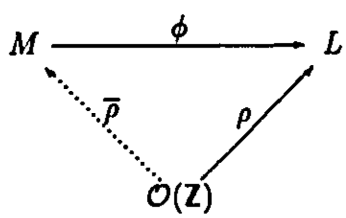


We show first that $\phi' = \phi \restriction \Downarrow(\overline{\psi}\{(n)\})$ is a co-dense map into $\Downarrow(t_n)$. Without a loss of generality we can assume that $n = 0$.

Suppose that $\phi'(u) = t_0$ for some $u \leq \overline{\psi}(\{0\})$. Now $u = \bigvee_M u_\alpha$ for some $u_\alpha \in BM$, so that $\phi'(u) = \bigvee_L \phi'(u_\alpha)$. Since t_0 is compact, there is a finite subcover $\phi'(u_{\alpha_1}) \vee \cdots \vee \phi'(u_{\alpha_N}) = t_0$. By subtracting off intersections if necessary, we can suppose that the the u_{α_i} are pairwise disjoint, with the same join. Now let $\mathcal{O}(\mathbb{Z}) \xrightarrow{\rho} L$ be determined by requiring

$$\rho(\{i\}) = \begin{cases} \phi'(u_{\alpha_i}) \text{ if } 0 \leq i \leq N \\ t_i \text{ if } i < 0 \\ t_{i-N} \text{ otherwise} \end{cases}$$

and denote the resulting extension by $\overline{\rho}$, which exists by hypothesis.



Then $\phi\overline{\rho}(\{i\}) = \rho(\{i\}) = \phi'(u_{\alpha_i}) = \phi(u_{\alpha_i})$, for any $0 \leq i \leq N$. Since ϕ is dense it is 1-1 on complemented elements, so that $\overline{\rho}(\{i\}) = u_{\alpha_i}$ for $0 \leq i \leq N$. Now

$$\phi\left(\bigvee_{0 \leq i \leq N} \overline{\rho}(\{i\})\right) = t_0 = \phi(\overline{\psi}(\{0\})), \text{ so}$$

$$\bigvee_{0 \leq i \leq N} \overline{\rho}(\{i\}) = \overline{\psi}(\{0\})$$

as they are both complemented. Also,

$$\bigvee_{0\leq i\leq N} \overline{p}(\{i\}) = \bigvee_{0\leq i\leq N} u_{\alpha_i} \leq u.$$

Thus $\overline{\psi}(\{0\}) \leq u$, and hence $\overline{\psi}(\{0\}) = u$, so that ϕ' is co-dense.

Now towards showing that ϕ is co-dense, suppose that $\phi(u) = e_L$. Then $u = u \wedge \bigvee_{\mathbf{Z}} \overline{\psi}(n)$, so that $e_L = \phi u = \bigvee_{\mathbf{Z}} \phi(u \wedge \overline{\psi}(n))$. Since the t_n are pairwise disjoint, $\phi(u \wedge \overline{\psi}(n)) = t_n$ for all n, so that $u \wedge \overline{\psi}(n) = \overline{\psi}(n)$ since each $\phi \restriction \overline{\psi}(n)$ is co-dense. Thus $u \geq \overline{\psi}(n)$ for all n, which implies that $u = \bigvee_{\mathbf{Z}} \overline{\psi}(n) = e_M$. Hence ϕ is co-dense and thus an isomorphism.□

We recall the definition of an open nucleus from Section 0.2 and make the following *ad hoc* definition.

Definition 1.4.5 *For a 0-dimensional frame L, a* **countable-cover nucleus** *k on $\mathfrak{J}BL$ is the open nucleus $H \to (-)$ determined by some countably generated ideal $H \in \mathfrak{J}BL$ with $\bigvee_L H = e$.*

Remark As a simple consequence of Lemma 1.4.4, we see that for any countable cover nucleus k on $\mathfrak{J}BL$, $[\mathfrak{J}BL]_k$ is S-N-compact. For if H is the ideal corresponding to k, then $\downarrow H$ is a frame satisfying the hypotheses of the Lemma.

Remark For any frame, the collection of all the nuclei on the frame is itself a frame, when given the pointwise order. See [Jo, pages 51-52] for details.

Definition 1.4.6 *We will denote by C the collection of all countable cover nuclei on $\mathfrak{J}BL$, and its join in the frame of all nuclei on $\mathfrak{J}BL$ by r.*

Lemma 1.4.7 *If every S-N-compact frame is Lindelöf, then for any 0-dimensional frame L, $[\mathfrak{J}BL]_r$ is Lindelöf.*

Proof It follows immediately from the definition of r that $[\mathfrak{J}BL]_r$ is the colimit of the diagram in Frm with vertices $[\mathfrak{J}BL]_d$ for $d \in C$ and maps the canonical

$[\mathfrak{J}BL]_d \longrightarrow [\mathfrak{J}BL]_e$ obtained when $d \leq e$. By a familiar categorical argument, this colimit can be seen as a coequalizer of a pair of maps

$$\rightrightarrows \bigoplus_{d \in C} [\mathfrak{J}BL]_d \longrightarrow [\mathfrak{J}BL]_r$$

in which the domain of the coequalized pair is another coproduct involving only the $[\mathfrak{J}BL]_d$. (See [Mac, page 109]) By Corollary III.1.3 of [Jo], a coequalizer of a pair of maps with regular domain is closed, so that $[\mathfrak{J}BL]_r$ is a closed quotient of the coproduct in the diagram. By Lemma 1.4.4, all of the $[\mathfrak{J}BL]_d$ are S-N-compact, and it follows from Proposition 1.3.6 that S-N-compact frames are closed under frame coproducts. So the coproduct in the diagram is Lindelöf by hypothesis, and as a closed quotient, $[\mathfrak{J}BL]_d$ is also Lindelöf.□

We will need the following two results:

Lemma 1.4.8 *Let j be the join nucleus on $\mathfrak{J}BL$, given by $jI = \{u \in BL \mid u \leq \bigvee_L I\}$. Then $r \leq j$.*

Proof We first remark that j is indeed a nucleus, in fact the nucleus corresponding to the homomorphism $\mathfrak{J}BL \xrightarrow{l} L$ with action $lI = \bigvee_L I$.

Let $d = H \rightarrow (-)$ be some countable cover nucleus, for H an ideal generated by a countable set S with $\bigvee_L S = e$. Then if $u \in H \rightarrow J$, we have $\Downarrow(u) \wedge H \subseteq J$, so that $u \wedge s \in J$ for all $s \in S$. Then

$$u = u \wedge e = u \wedge \bigvee_L S = \bigvee_{s \in S} u \wedge s \in jJ.$$

Thus $d \leq j$ and hence $r \leq j$.□

Corollary 1.4.9 *Any principal ideal $\downarrow u$ in $\mathfrak{J}BL$ is r-closed.*

Proof This follows immediately from Lemma 1.4.8. □

We construct two examples to show that (i) and (ii) of Lemma 1.4.3 hold.

Lemma 1.4.10 *If S-N-compact frames are Lindelöf, then CC holds.*

Proof Construction (i) Given $(X_n)_{n\in\omega}$ a list of nonempty sets, (which we can assume are disjoint,) form $X = \bigcup_{n\in\omega} X_n$ and then $Y = \bigcup_{n\in\omega} X \times \{n\}$. We will consider the frame $[\mathfrak{I}\mathsf{P}(Y)]_r$, which we know to be Lindelöf by Lemma 1.4.7.

For each $n \in \omega$, define a 1-1 function f_n by

$$X_n \xrightarrow{f_n} \mathfrak{I}\mathsf{P}(Y)$$
$$x \longmapsto \Downarrow((X \times \{n\}) \cup \{(x, n+1)\})$$

Note that any element of $f_n(x)$ is principal, and therefore r-closed, by Corollary 1.4.9.

The ideal

$$\bigvee_{\mathfrak{I}\mathsf{P}(Y)} \bigcup_{n\in\omega} f_n[X_n]$$

has a countable subset $P = \{X \times \{n\} \mid n \in \omega\}$ whose join in $\mathsf{P}(Y)$ is Y. Let d be the countable cover nucleus $< P > \rightarrow (-)$, where $< P >$ is the ideal generated by P. Then

$$d\left(\bigvee_{\mathfrak{I}\mathsf{P}(Y)} \bigcup_{n\in\omega} f_n[X_n]\right) = \mathsf{P}(Y),$$

and thus

$$\bigvee_{[\mathfrak{I}\mathsf{P}(Y)]_r} \bigcup_{n\in\omega} f_n[X_n] = r\left(\bigvee_{\mathfrak{I}\mathsf{P}(Y)} \bigcup_{n\in\omega} f_n[X_n]\right) = \mathsf{P}(Y).$$

As $[\mathfrak{I}\mathsf{P}(Y)]_r$ is Lindelöf, there is a countable subset

$$\mathfrak{S} \subseteq \bigcup_{n\in\omega} f_n[X_n], \text{ so that } \bigvee_{[\mathfrak{I}\mathsf{P}(Y)]_r} \mathfrak{S} = \mathsf{P}(Y).$$

Claim 1.4.11 $\mathfrak{S} \cap f_n[X_n] \neq \emptyset$ *for every* n.

Proof (Claim) Suppose this were not the case for some nonzero n, (the case $n = 0$ is similar,) and let x be some element of X_n. The definition of the f_n and the fact that the X_i are pairwise disjoint ensures that $\Downarrow(\{(x,n)\}) \not\subseteq f_m(y)$ for any $m \neq n$ and $y \in X_m$. Then $\Downarrow(\{(x,n)\}) \cap I = \{0\}$ for any $I \in \mathfrak{S}$, and it follows that

$$r(\Downarrow\{(x,n)\}) \cap r(\bigvee_{\mathfrak{I}\mathsf{P}(Y)} \mathfrak{S}) = r(\{0\}) = \{0\}$$

with the last equality holding since $r \leq j$, by Lemma 1.4.8. But then

$$\Downarrow(\{(x,n)\}) \cap \mathsf{P}(Y) = \{0\}$$

which cannot be the case. We have proved the claim, and we can now define our desired function F by $F(n) = f_n^{-1}(\mathfrak{S})$, which picks out a nonempty countable subset of X_n for each $n \in \omega$. So statement (i) of the Lemma 1.4.3 holds.

Construction (ii) To show that (ii) of Lemma 1.4.3 holds, it suffices to show that a union of a countable set of pairwise disjoint countable sets is countable. Given a list $(X_n)_{n\in\omega}$ of such sets, form $X = \bigcup_\omega X_n$, and then the Lindelöf frame $[\mathfrak{I}\mathsf{P}(X)]_r$. Consider the collection $\mathfrak{G} = \{\Downarrow\{x\} : x \in X\}$, a subset of $[\mathfrak{I}\mathsf{P}(X)]_r$ by Corollary 1.4.9. We will show that $r(\bigvee_{\mathfrak{I}\mathsf{P}(X}\mathfrak{G}) = \mathsf{P}(X)$.

For any $n \in \omega$, let $H_n \in \mathfrak{I}\ \mathsf{P}(X)$ be the ideal generated by the sets $\bigcup_{m\neq n} X_m$ and $\{x\}$ for all $x \in X_n$. Then H_n is a countably generated ideal with $\bigcup_{\mathsf{P}(X)} H_n = X$. We have

$$\Downarrow(X_n) \leq H_n \rightarrow \bigvee_{\mathfrak{I}\ \mathsf{P}(X)} \mathfrak{G} \quad \text{since } \Downarrow(X_n) \cap H_n \subseteq \bigvee_{\mathfrak{I}\ \mathsf{P}(X)} \mathfrak{G}$$

as the sets X_n are pairwise disjoint. Let $d_n = H_n \rightarrow (-)$ be the countable cover nucleus corresponding to H_n. Then

$$X_n \in d_n(\bigvee_{\mathfrak{I}\ \mathsf{P}(X)} \mathfrak{G}) \text{ for all } n, \text{ so}$$

$$X_n \in r(\bigvee_{\mathfrak{I}\ \mathsf{P}(X)} \mathfrak{G}) \text{ for all } n.$$

Now let H be the ideal in $\mathfrak{I}\ \mathsf{P}(X)$ generated by the X_n for $n \in \omega$; a countably generated ideal with $\bigcup_{\mathsf{P}(X)} H = X$. Then $d = H \rightarrow (-)$ is a countable cover nucleus. It follows that

$$r(\bigvee_{\mathfrak{I}\ \mathsf{P}(X)} \mathfrak{G}) = dr(\bigvee_{\mathfrak{I}\ \mathsf{P}(X)} \mathfrak{G}) = \mathsf{P}(X)$$

since $H \subseteq r(\bigvee_{\mathfrak{I}\,\mathsf{P}(X)}\mathfrak{G})$, by our work in the previous paragraph. Thus $\bigvee_{[\mathfrak{I}\,\mathsf{P}(X)]_r}\mathfrak{G} = \mathsf{P}(X)$.

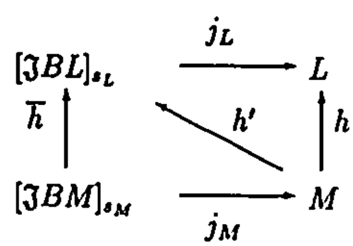
Since $[\mathfrak{I}\ \mathsf{P}(X)]_r$ is Lindelöf, there is a countable subset $\mathfrak{T} \subseteq \mathfrak{G}$ such that $\bigvee_{[\mathfrak{I}\,\mathsf{P}(X)]_r}\mathfrak{T} = \mathsf{P}(X)$. By reasoning as we did for Construction (i), $\mathfrak{T}$ must be *all* of $\mathfrak{G}$ so that $\mathfrak{G}$ and hence X are countable. □

Theorem 1.4.12 *(*) The sub-category* 0-L-Frm *of 0-dimensional Lindelöf spaces is co-reflective in* 0-Frm.

Proof For a 0-dimensional frame L, define a map $\mathfrak{I}BL \xrightarrow{s} \mathfrak{I}BL$ by $s(I) = \{u \in BL \mid u \le \bigvee_L S, S \subseteq I \text{ countable }\}$. It is not difficult to see (using CC) that s is a nucleus, and that $[\mathfrak{I}BL]_s \xrightarrow{j_L} L$ defined by $j_L(I) = \bigvee_L I$ is a frame homomorphism.

If L is already Lindelöf, then j_L is an isomorphism, for if $j_L I = \bigvee_L I = E$ for some $I \in [\mathfrak{I}BL]_s$, then there is a countable subset $S \subseteq I$ so that $\bigvee_L S = e$, and hence $e \in sI = I$. So j_L is co-dense with 0-dimensional domain and is clearly onto and is thus an isomorphism, by Lemma 0.2.4.

Given a Lindelöf M and a map $M \xrightarrow{h} L$ define $[\mathfrak{I}BM]_{s_M} \xrightarrow{\overline{h}} [\mathfrak{I}BL]_{s_L}$ by $\overline{h}I = s_L(\mathcal{J}B\phi(I))$. It is straightforward, (but tedious) to verify that $\overline{h}$ is a frame homomorphism. The outer square of the diagram



commutes, and since j_M is an isomorphism (as M is Lindelöf), we obtain a map h' as shown, which is unique since j_L is dense and therefore monic. □

Remarks One can show without trouble that s is equal to r, the nucleus defined in Definition 1.4.6. Note also that Theorem 1.3.5 implies that the co-reflection of Theorem 1.4.12 is equal to the S-N-compactification, (given CC.)

We can now prove Theorem 1.4.2, which we restate for convenience.

Theorem 1.4.2 *All S-N-compact frames are Lindelöf iff CC holds.*

Proof The ($\rightarrow$) direction is Lemma 1.4.10, and the other follows from Theorem 1.4.12, since a coreflective subcategory is closed under all colimits, and colimits in 0-Frm are the same as those in Frm. □

1.5 Herrlich-N-Compact Frames

In a search for a natural definition of an N-compact frame, one may take one's cue from the several spatially equivalent statements of Theorem 1.1.4. Phrased in the language of frames, statement (1) is our S-N-compactness, which is equivalent to the frame version of (2) (Theorem 1.3.5.) The statement (4) suggests the definition of our 'Herrlich-N-compact' frames defined in 1.5.2 below, which turns out to be equivalent to the rendering of (5) for frames (Theorem 1.5.21.) We will see that this notion is not equivalent to Stone-N-compactness, and that it serves better as a definition of an N-compact frame. It is conservative, in the sense of our comments following Theorem 1.3.5, possesses a N-compactification, (Theorem 1.5.12) and consequently is closed under coproducts and closed quotients. Moreover this notion of N-compactness preserves a connection with a theorem of Mrõwka regarding the groups $C(X, \mathbb{Z})$, (Theorem 2.2.2) with interesting consequences we explore in the next chapter.

In Section 1.5.1 we introduce the basic definitions and consider a few examples, and in the next section develop the compactification. As a corollary of the existence of the compactification, we show that the class of H-N-compact frames includes the class of S-N-compact frames. We end the chapter by considering some aspects of the relation between N-compact frames and N-compact spaces.

1.5.1 Definitions

We begin with an extension of Definition 1.3.3.

Definition 1.5.1 *An ideal $I \in \mathfrak{J}BL$ is* **super-σ-proper** *if any proper ideal $I' \supseteq I$ is σ-proper.*

We remark that the improper ideal is super-σ-proper, since the condition is vacuously fulfilled.

Definition 1.5.2 *A 0-dimensional frame L is said to be* **Herrlich-N-compact ('H-N -compact')** *if any proper $I \in \mathfrak{J}BL$ which is super-σ-proper is completely proper.*

Remark The definition looks less mysterious if we for the moment assume the Boolean Ultrafilter Theorem. Then a frame L is H-N-compact iff every maximal ideal in BL which is σ-proper is completely proper. This then resembles the statement (3) of Theorem 1.1.4, and indeed we have

Lemma 1.5.3 *(*) For a space X, $\mathcal{O}(X)$ is H-N-compact iff X is N -compact.*

Proof ($\rightarrow$) Suppose that $\mathcal{F}$ is a ultrafilter with the countable intersection property in $\mathcal{BO}(X)$. Then $\mathcal{F}^* = \{U^* : U \in \mathcal{F}\}$ is a maximal ideal in $\mathcal{BO}(X)$ which is σ-proper. So $\mathcal{F}^*$ is completely proper, implying that $\mathcal{F}$ is fixed.

($\leftarrow$) (*) Suppose that $I \in \mathcal{JBO}(X)$ has the property of the definition. If $I' \supseteq I$ is some maximal ideal extending I, it is σ-proper by hypothesis, and therefore completely proper, by considerations like those in ($\rightarrow$). This implies that I is completely proper. □

Remark Lemma 1.5.3 tells us that this notion of an N-compact frame is a conservative one. It is not surprising that we need the Boolean Ultrafilter Theorem to show this, since the definition of H-N-compact frames is based on statement (4) of Theorem 1.1.4, which explicitly mentions ultrafilters.

Note that if one is willing to accept a reference to maximal ideals (no existence statements are required,) then one could define a frame to be 'N-compact' if every maximal ideal in its Boolean part which is σ-proper is completely proper. In this case one does not require any choice principles to prove that the notion is conservative.

One can use the Lemma and the Boolean Ultrafilter Theorem to provide a formulation of spatial N-compactness which is a cover condition. We have not seen this mentioned in the literature, but it is probably not new.

Corollary 1.5.4 *(*) A 0-dimensional space X is N-compact iff for every cover S of X by clopen sets which has no finite subcovers, there is a larger (clopen) cover $T \supseteq S$ which also contains no finite subcovers, but does contain a countable subcover.*

Remark That any Lindelöf frame is H-N-compact is easy to see. Then by Theorem 1.3.5 any S-N-compact frame is H-N-compact , if we assume CC. As a corollary of Theorem 1.5.12 we will obtain this result without any additional set-theoretic assumptions.

One can show that there are H-N-compact frames which are not Lindelöf in any model of ZF:

Example 1 *Let $L = \mathsf{P}(2^\omega)$. Then L is not Lindelöf, but is H-N-compact .*

The frame L is clearly not Lindelöf. Towards seeing that it is H-N-compact , let I be a proper ideal in $BL = \mathsf{P}(2^\omega)$ which is super-σ-proper . We define a sequence of ideals

$$I \subseteq I_0 \subseteq I_1 \subseteq \cdots \subseteq I_n \subseteq \cdots$$

as follows;

For $n = 0$, let $X_0 := \{f \in 2^\omega : f(0) = 0\}$ or $\{f \in 2^\omega : f(0) = 1\}$ depending on whether

(i) $I \vee \downarrow \{f \in 2^\omega : f(0) = 0\}$ is proper, or

(ii) $I \vee \downarrow \{f \in 2^\omega : f(0) = 1\}$ is proper,

and if *both* are proper, let X_0 be the first of the two sets. (Clearly at least one must be proper since I is proper).

Let ϵ_0 be 1 or 0 accordingly, and set $I_0 = I \vee \downarrow X_0$, a proper ideal. Proceed in this way to define I_n and ϵ_n for every $n \in \omega$, and let $\epsilon \in 2^\omega$ be defined by $\epsilon(n) = \epsilon_n$. Now the I_n are all proper, so their union is a proper ideal containing I and is therefore a σ-proper ideal. Thus $I \vee \downarrow (\bigcup_{n\in\omega} X_n) \neq E$, and since $\bigcup_{n\in\omega} X_n = 2^\omega - \{\epsilon\}$, we have $I \subseteq \Downarrow(2^\omega - \{\epsilon\})$, so that I is completely proper.

Example 2 *(*)* $\mathsf{P}(2^\omega)$ *is an H-*N*-compact frame which is not S-*N*-compact. For by Theorem 1.3.5 any S-*N*-compact frame is Lindelöf.*

We do not know if one can find an H-N-compact frame which is not S-N-compact working only in ZF.

Lemma 1.5.5 *(*) If B is a complete Boolean algebra, then*

(i) B is S-N*-compact iff any antichain in B is countable.*

(ii) B is H-N*-compact if any antichain in B is of non-measurable cardinality.*

Proof (i) If $S \subseteq B$ is an antichain which is uncountable, then by adjoining another element of B if necessary we have a cover with no countable subcover, so that B is not Lindelöf and therefore not S-N-compact. This is necessity. Towards sufficiency, we suppose that $(u_\alpha)_{\alpha\in\kappa}$ is a cover of e_B, for some cardinal κ. Define $v_\beta = \bigvee_{\alpha\leq\beta} u_\alpha$, and then set $w_\beta = v_{\beta+1} \wedge v_\beta^*$ for $\beta > 0$ and $w_0 = w_0$. Then the w_β are pairwise disjoint, so that there is a $\gamma \in \kappa$ with $w_\alpha = 0$ if $\alpha > \gamma$, for γ some countable ordinal. This implies that $v_\beta = v_\alpha$ if $\beta > \alpha$, so that there is a countable subcover of the cover $(u_\alpha)_{\alpha\in\kappa}$. By Theorem 1.3.5 (3→2), B is S-N-compact.

Proof (ii) Suppose B is not H-N-compact . Then there is a maximal ideal I in B so that I is σ-proper but not completely proper. Using Zorn's Lemma, we can find a maximal antichain S in I, and we clearly must have $\bigvee S = e$. Let $\mathcal{F} \subseteq \mathbf{P}(S)$ be defined by requiring $X \in \mathcal{F}$ iff $\bigvee X \notin I$. Then $\mathcal{F}$ is a non-principal ultrafilter on S with the countable intersection property, so that $|S|$ must be measurable. □

We know of no counterexample to necessity for (ii), but are unable to show that it holds.

1.5.2 The H-N-Compactification

We construct the coreflection from the category of frames to the subcategory of H-N-compact frames. Since a compact frame is H-N-compact, there should be a frame map from $\mathfrak{J}BL$ to the H-N-compactification of L. We use this observation to see the H-N-compactification as a quotient of $\mathfrak{J}BL$.

For any frame L, define $\mathfrak{J}BL \xrightarrow{h} \mathfrak{J}BL$ by

$$hI = \{u \in BL \cap \Downarrow(\textstyle\bigvee_L I) \mid I \subseteq J, J \text{ super-}\sigma\text{-proper } \Rightarrow u \in J\}$$

Lemma 1.5.6 *The map h is a nucleus.*

Proof (i) Clearly $I \subseteq hI$.

(ii) We have only to show that $hI \cap hK \subseteq h(I \cap K)$, since h is order preserving. Towards this, fix $u \in hI \cap hK$. Then $u \leq \bigvee_L I \wedge \bigvee_L K = \bigvee_L (I \cap K)$, so u satisfies the first criterion for membership in $h(I \cap K)$.

Suppose that $J \supseteq I \cap K$ is a super-σ-proper ideal. We must show that $u \in J$. Note that $J \vee I$ is an ideal containing J and is therefore itself super-σ-proper. Since $J \vee I$ contains I, $u \in J \vee I$, as $u \in h(I)$. We can similarly show that $u \in J \vee K$, so that $u \in J = (J \vee K) \wedge (J \vee I)$, as desired.

(iii) Towards showing that $h^2 I \subseteq hI$, note first that if $u \in h^2 I$, then $u \leq \bigvee_L hI \leq \bigvee_L I$.

Now suppose $u \in h^2 I$, and $I \subseteq J$, J super-σ-proper. Then $hI \subseteq J$, by definition of hI. As $u \in h^2 I$, $u \in J$, and altogether we have $u \in hI$. □

We will eventually define the H-N-compactification of L to be $[\mathfrak{J}BL]_h$. The coreflection map $[\mathfrak{J}BL]_h \xrightarrow{j_L} L$ will be the join map, defined by $j_L I = \bigvee_L I$, a frame homomorphism by an easy argument. When we must be careful to distinguish among nuclei, we will subscript them appropriately.

Lemma 1.5.7 *If L is H-N-compact then j_L is an isomorphism.*

Proof If $u \in L$ then $\Downarrow(u) \cap BL$ is h_L-closed, so that j_L is onto. We thus need only to show that j_L is co-dense, by Lemma 0.2.4.

Suppose that $I \in [\mathfrak{J}BL]_h$ and $j_L I = \bigvee_L I = e$, so that I is not completely proper. If I were proper, since it is h-closed, there would be a proper super-σ-proper ideal $J \supseteq I$. But since L is H-N-compact such a J would be completely proper, which is impossible as the sub-ideal I is not. Thus I is not proper, so that j_L is co-dense. □

Lemma 1.5.8 *The frame $[\mathfrak{J}BL]_h$ is H-N-compact.*

Proof We first show that

$$\begin{array}{ccc} B[\mathfrak{J}BL]_h & \cong & BL \quad \text{via,} \\ J & \xrightarrow{\alpha} & \bigvee_L J \end{array}$$

$$\mathfrak{J}(u) \cap BL \overset{\beta}{\longleftarrow} u$$

Note that the range of α is indeed as claimed, since if $J \in B[\mathfrak{J}BL]_h$ with complement J^*, then $J \cap J^* = 0_{\mathfrak{J}BL}$, so that

$$\alpha J \wedge \alpha J^* = \bigvee_L J \wedge \bigvee_L J^* = \bigvee_L J \cap J^* = 0_L,$$

and $h(J \vee J^*) = E_{\mathfrak{J}BL}$, so that

$$e_L = \bigvee_L (J \vee J^*) = \bigvee_L J \vee \bigvee_L J^* = \alpha J \vee \alpha J^*.$$

One can show in similar fashion that

(i) α, β are both Boolean algebra homomorphisms, and

(ii) $\alpha\beta = \mathrm{id}_L$.

It is also easy to see that $\beta\alpha = \mathrm{id}_{B[\mathfrak{J}BL]_h}$ For suppose $J \in B[\mathfrak{J}BL]_h$. Then $\bigvee_L J$ is complemented (see the paragraph above,) and we must show that $\bigvee_L J \in J$. Towards this, assume that $J \subseteq K$ and K is super-σ-proper . Then $K \vee J^* = E_{\mathfrak{J}BL}$ so that $v \vee \bigvee_L J^* = e$ for some $v \in K$. Since K is an ideal, it follows that $\bigvee_L J$ is in K, since $\bigvee_L J^* = (\bigvee_L J)^*$.

Now towards our goal of showing that $[\mathfrak{J}BL]_h$ is H-N-compact , suppose that $\mathsf{I} \in \mathfrak{J}B[\mathfrak{J}BL]_h$ is a super-σ-proper ideal. We must show that I is completely proper.

First note that the image of I under α is an ideal in BL, $\alpha[\mathsf{I}]$.

Claim 1.5.9 $\alpha[\mathsf{I}]$ *is h-closed.*

Proof It is enough to show that $\alpha[\mathsf{I}]$ is super-σ-proper .

Towards this, suppose that $\alpha[\mathsf{I}] \subseteq K$ and that K is proper. We must show that K is σ-proper. First note that $\mathsf{I} \subseteq \beta[K]$, and since $\beta[K]$ is proper, it is σ-proper.

Let $\{u_n \mid n \in \omega\}$ be a countable subset of K. We know that

$$\bigvee_{[\mathfrak{J}BL]_h} \beta(u_n) \neq E_{[\mathfrak{J}BL]_h} \qquad \text{(Inequality 1.5.9)}$$

since $\beta[K]$ is σ-proper. Now if it were the case that $\bigvee_L u_n = e_L$, we could reason as follows.

$$\bigvee_{[\mathfrak{J}BL]_h} \beta(u_n) = h\left(\bigvee_{\mathfrak{J}BL} \beta(u_n)\right) =$$
$$\{v \in BL \cap \downarrow\bigvee_L \bigvee_{\mathfrak{J}BL} \beta(u_n) \mid \bigvee_{\mathfrak{J}BL} \beta(u_n) \subseteq H, \text{ H super-}\sigma\text{-proper} \Rightarrow v \in H\} =$$
$$\{v \in BL \mid \bigvee_{\mathfrak{J}BL} \beta(u_n) \subseteq H, \text{ H super-}\sigma\text{-proper} \Rightarrow v \in H\}$$

so that because of Inequality 1.5.9, there must be some proper super-σ-proper ideal H which contains $\bigvee_{\mathfrak{J}BL} \beta(u_n)$. But such an H could not be σ-proper, as $\bigvee_L u_n = e_L$. This is a contradiction, so that we must have $\bigvee_L u_n \neq e_L$. Hence K is σ-proper, so that $\alpha[\mathsf{I}]$ is super-σ-proper , and thus h-closed.

We finish by observing that $\mathsf{I} \subseteq B[\mathfrak{J}BL]_h \cap \downarrow\alpha[\mathsf{I}]$, which implies that I is completely proper, since $\alpha[\mathsf{I}] \neq E_{[\mathfrak{J}BL]_h}$ We have shown that $[\mathfrak{J}BL]_h$ is H-N-compact . □

Towards showing that the map $[\mathfrak{J}BL]_h \xrightarrow{j_L} L$ is a universal as a map from H-N-compact frames to L, we prove the following lemma. Recall the definition of the functor $\mathfrak{J}B$ from Section 0.2.3.

Lemma 1.5.10 *If $M \xrightarrow{\phi} L$ is a frame homomorphism, then*

$$\begin{array}{ccc} [\mathfrak{J}BM]_{h_M} & \xrightarrow{\overline{\phi}} & [\mathfrak{J}BL]_{h_L} \quad \textit{defined by,} \\ I & \longmapsto & h_L(\mathfrak{J}B\phi(I)) \end{array}$$

is a frame homomorphism.

Proof It is clear that $\overline{\phi}$ preserves finite meets and is thus order-preserving. To see that it transfers arbitary joins, it is enough to see that, given a collection of elements $\{I_\alpha\}_\alpha$ of $[\mathfrak{J}BM]_{h_M}$, we have

$$\overline{\phi}\left(\bigvee_{[\mathfrak{J}BM]_h} I_\alpha\right) \subseteq \bigvee_{[\mathfrak{J}BL]_h} \overline{\phi} I_\alpha.$$

(We will supress mention of an index set for the indices α to avoid complicating the notation.) We begin by noting that

$$\begin{aligned}\bigvee_{[\mathfrak{J}BL]_h} \overline{\phi} I_\alpha &= \bigvee_{[\mathfrak{J}BL]_h} h_L \mathfrak{J}B\phi(I_\alpha)\\ &= h_L(\bigvee_{\mathfrak{J}BL} h_L \mathfrak{J}B\phi(I_\alpha))\\ &= h_L(\bigvee_{\mathfrak{J}BL} \mathfrak{J}B\phi(I_\alpha))\end{aligned}$$

so it is enough to show that

$$h_L(\mathfrak{J}B\phi(\bigvee_{[\mathfrak{J}BM]_h} I_\alpha)) \subseteq h_L(\bigvee_{\mathfrak{J}BL} \mathfrak{J}B\phi(I_\alpha)) \qquad \text{(Inequality 1.5.10)}$$

Fix v in the left-hand side of the Inequality 1.5.10. From the definition of h, we see that we have two criteria to verify in order to see that v is in the right-hand side. Towards the first, we have

$$\begin{aligned} v &\in h_L(\mathfrak{J}B\phi(\bigvee_{[\mathfrak{J}BM]_h} I_\alpha))\\ &= h_L(\mathfrak{J}B\phi(h_M \bigvee_{\mathfrak{J}BM} I_\alpha)), \quad \text{so that,}\\ v &\leq \bigvee_{L} \mathfrak{J}B\phi(h_M \bigvee_{\mathfrak{J}BM} I_\alpha)\\ &= \bigvee_{L} \phi[h_M \bigvee_{\mathfrak{J}BM} I_\alpha]\\ &= \phi(\bigvee_{M} h_M(\bigvee_{\mathfrak{J}BM} I_\alpha))\end{aligned}$$

$$\begin{aligned}
&= \phi(\bigvee_{M}\bigvee_{\mathfrak{J}BM} I_\alpha) \text{ since } h_M(-) \subseteq \Downarrow\bigvee_{M}(-) \\
&= \bigvee_{L}\phi[\bigvee_{\mathfrak{J}BM} I_\alpha] \\
&= \bigvee_{L}\mathfrak{J}B\phi(\bigvee_{\mathfrak{J}BM} I_\alpha) \\
&= \bigvee_{L}\bigvee_{\mathfrak{J}BL}\mathfrak{J}B\phi(I_\alpha),
\end{aligned}$$

so v satisfies the first criterion for membership in the right hand side of Inequality 1.5.10. To see that it satisfies the second, suppose that $\bigvee_{\mathfrak{J}BL}\mathfrak{J}B\phi(I_\alpha) \subseteq H$, for H some super-σ-proper element of $\mathfrak{J}BL$. We must show that $v \in H$. If we can show that $\phi[h_M(\bigvee_{\mathfrak{J}BM} I_\alpha)] \subseteq H$, then $\mathfrak{J}B\phi(h_M(\bigvee_{\mathfrak{J}BM} I_\alpha)) \subseteq H$, so that $v \in H$, by hypothesis on v.

Claim 1.5.11 $\phi[h_M(\bigvee_{\mathfrak{J}BM} I_\alpha)] \subseteq H$.

Proof If H is improper, we are done. Otherwise, fix $u \in h_M(\bigvee_{\mathfrak{J}BM} I_\alpha)$ and let $K = \bigvee_{\mathfrak{J}BM}\{L \mid \mathfrak{J}B\phi(L) \subseteq H\}$. Then K is proper since H is.

We assert that if $K' \in \mathfrak{J}BM$ is a proper ideal which contains K, then $\mathfrak{J}B\phi(K') \vee H$ is proper. For if not, there are elements $w \in K'$ and $p \in H$ so that $\phi w \vee p = e_L$. Then

$$\begin{aligned}
\phi(w^{**}) \vee p &= e_L \text{ so,} \\
\phi(w^{*}) &= \phi(w^{*}) \wedge (\phi(w^{**}) \vee p) \\
&= \phi(w^{*}) \wedge p, \text{ so} \\
\phi(w^{*}) &\leq p.
\end{aligned}$$

Thus $\phi(w^*) \in H$. Then $\Downarrow(w^*) \subseteq K \subseteq K'$, so that both w and w^* are in K', so that $e \in K'$ (since $w \in BM$,) contradicting the propriety of K'.

Thus $\mathfrak{J}B\phi(K') \vee H$ is proper for any such K', and since it contains H it is σ-proper. But this implies that K' is also σ-proper and hence that K is super-σ-proper .

Now the definition of K and the hypothesis on H imply that $\bigvee_{\mathfrak{I}BM} I_\alpha \subseteq K$, since $\mathfrak{I}B\phi(\bigvee_{\mathfrak{I}BM} I_\alpha) = \bigvee_{\mathfrak{I}BB}\mathfrak{I}B\phi(I_\alpha)$, and so by hypothesis on u, we have $u \in K$. Thus $\phi u \in \phi[K] \subseteq H$, as desired. □ (Claim) □ (Lemma)

We can now prove the

Theorem 1.5.12 *For an arbitrary frame L, the map $[\mathfrak{I}BL]_h \xrightarrow{j_L} L$ is universal as a map from H-N-compact frames to L.*

Proof Suppose that we are given an H-N-compact frame M and a frame homomorphism $M \xrightarrow{\phi} L$. We can form the diagram

$$\begin{array}{ccc} [\mathfrak{I}BL]_{h_L} & \xrightarrow{j_L} & L \\ \uparrow{\scriptstyle \overline{\phi}} & \nwarrow{\scriptstyle \phi'} & \uparrow{\scriptstyle \phi} \\ [\mathfrak{I}BM]_{h_M} & \xrightarrow{j_M} & M \end{array}$$

and by considering the form of $\overline{\phi}$, see without trouble that the outer square commutes. Since j_M is an isomorphism, we can find a map ϕ' making the upper triangle commute. This map is unique since j_L is dense and therefore monic. □

Definition 1.5.13 *The full subcategory of H-N-compact frames will be denoted H-N-Frm. The coreflection from Frm to H-N-Frm supplied by Theorem 1.5.12 we denote by ν_H.*

Corollary 1.5.14 *The subcategory H-N-Frm is closed under frame coproducts and closed quotients.*

Proof Any coreflective subcategory is closed under all colimits, and hence by Theorem 1.5.12 H-N-Frm is closed under frame coproducts. Towards seeing the second assertion, we first show that it holds for 'clopen'-quotients; those of the form $\Uparrow(u)$ for some complemented u.

Let L be an H-N-compact frame and $u \in BL$. If I is a proper super-σ-proper ideal in $B(\Uparrow(u))$, then $I' = \{v \in BL \mid v \vee u \in I\}$ is a proper super-σ-proper ideal in BL. By hypothesis I' is completely proper, and as it contains I, I is also.

Now if L is H-N-compact and $u \in L$ is any element, we know that $u = \bigvee v_\alpha$ for v_α some elements in BL. It follows that the frame $\Uparrow(u)$ is the colimit in Frm of the diagram with vertices the frames $\Uparrow(v_\alpha)$ and maps the canonical $\Uparrow(v_\alpha) \rightarrow \Uparrow(v_\beta)$ obtained when $v_\alpha \leq v_\beta$. Since H-N-Frm is closed under colimits, $\Uparrow(u)$ is H-N-compact . □

Corollary 1.5.15 *Any S-N-compact frame is H-N-compact .*

Proof By Corollary 1.5.14 we need only show that $\mathcal{O}(\mathbb{N})$ is H-N-compact . This follows easily (and in ZF) from the fact that any cover of $\mathcal{O}(\mathbb{N})$ has a countable refinement. □.

Theorem 1.5.16 *(*) The H-N-compactification is conservative, so that $\nu_H \mathcal{O}(X) \cong \mathcal{O}(\nu X)$ for any 0-dimensional Hausdorff space X.*

Proof If we can show that $\nu_H \mathcal{O}(X)$ is spatial, then the co-universal properties of $\nu_H \mathcal{O}(X) \xrightarrow{j} \mathcal{O}(X)$ and the natural map $\mathcal{O}(\nu X) \longrightarrow \mathcal{O}(X)$ together imply the existence of the isomorphism. Since $\nu_H \mathcal{O}(X)$ is regular, we can see that it suffices to show that any proper element I is dominated by a maximal element. Now one of the following holds;

(i) $\bigvee_{\mathcal{O}(X)} I = e_L$. In this case, since I is $h_{\mathcal{O}(X)}$-closed, there must be a proper super-σ-proper $J \in \mathfrak{J}B\mathcal{O}(X)$ so that $I \subseteq J$. Then J can be expanded to a maximal element of $\mathfrak{J}B\mathcal{O}(X)$, which is then also super-σ-proper , and hence $h_{\mathcal{O}(X)}$-closed.

(ii) $\bigvee_{\mathcal{O}(X)} I \neq e_L$. In this case there is a maximal element P of $\mathcal{O}(X)$ such that $\bigvee_{\mathcal{O}(X)} I \leq P$. Then $\Downarrow(P) \cap B\mathcal{O}(X)$ is $h_{\mathcal{O}(X)}$-closed and maximal in $\mathfrak{J}B\mathcal{O}(X)$, and is hence a maximal element of $\nu_H \mathcal{O}(X)$ containing I. □

Now it is clear that the S-N-compactification of a frame will in general differ from the H-N-compactification, since the first of these will always be a Lindelöf frame. However we can show that after a spatial reflection, the two compactifications coincide;

Lemma 1.5.17 *(*) For any frame L, $\Sigma\nu_S L = \Sigma\nu_H L$.*

Proof We know that $\nu_S L = [\mathfrak{J}BL]_{s_L}$ and $\nu_H L = [\mathfrak{J}BL]_{h_L}$, where s_L is the nucleus of 1.4.13. (It was in showing that s_L is a nucleus that we used choice principles.) It is not difficult to show that the the maximal elements of $[\mathfrak{J}BL]_{s_L}$ and $[\mathfrak{J}BL]_{h_L}$ are maximal in $\mathfrak{J}BL$, and it is easy to see that the s_L-closed maximal ideals are exactly the h_L-closed maximal ideals, so that the spectrums coincide. The topologies coincide since they both have a base consisting of sets of the form $\{P \text{ maximal} \mid u \notin P\}$, for $u \in BL$. □

Corollary 1.5.18 *For any frame L, $\Sigma\nu_S L$ ($= \Sigma\nu_H L$) is* N*-compact.*

Proof We know that $\nu_S L$ is a closed quotient of $\mathcal{O}(\mathsf{N})^{(I)}$ for some index set I. It follows that $\sum \nu_S L$ is a closed subspace of $\Sigma\mathcal{O}(\mathsf{N})^{(I)} = \mathsf{N}^I$, and is hence N-compact. □

An important property of the spatial N-compactification ν is that the rings $C(X, \mathbf{Z})$ and $C(\nu X, \mathbf{Z})$ are isomorphic, via the restriction map. We have the analogous property for frames, and for either N-compactification. The analogue of the ring $C(X, \mathbf{Z})$ is the ring $\mathbf{Z}_L E$, defined in Section 0.3.

Lemma 1.5.19 *For a frame L, there are ring isomorphisms such that $\mathbf{Z}_L E \cong \mathbf{Z}_{\nu_S L} E$ and $\mathbf{Z}_L E \cong \mathbf{Z}_{\nu_H L} E$.*

Proof Define $\mathbf{Z}_{\nu_S L}E \xrightarrow{f} \mathbf{Z}_L E$ by $f(\xi) = h \circ \xi$, where $\nu_S L \xrightarrow{h} L$ is the coreflection map of 1.3.6. It is not difficult to show that f is a ring homomorphism. We know from Section 1.3 that h is both a dense map and a $C_{\mathbf{Z}}$-quotient map which implies that f is 1-1 and onto respectively.

For the second isomorphism, we similarly define $\mathbf{Z}_{\nu_H L}E \xrightarrow{g} \mathbf{Z}_L E$ by $g(\xi) = j_L \circ \xi$, where j_L is the coreflection map of 1.5.12. Then g is a ring homomorphism which is 1-1 since j_L is dense. The map g is onto, for given $\rho \in \mathbf{Z}_L E$, define $\mathcal{O}(\mathbf{Z}) \xrightarrow{\overline{\rho}} \nu_H L$ by $\overline{\rho}(n) = \Downarrow(\overline{\rho}(n)) \cap BL$. Then $\overline{\rho}(n) \cap \overline{\rho}(m) = \{0\}$ if $n \neq m$, and $\bigvee_{n \in \mathbf{Z}} \overline{\rho}(n) = h_L(\bigvee_{n \in \mathbf{Z}} \overline{\rho}(n)) = E_{\nu_H L}$ since $\bigvee_{n \in \mathbf{Z}} \rho(n) = e_L$. Thus $\overline{\rho} \in \mathbf{Z}_{\nu_H L}E$ and clearly $g(\overline{\rho}) = \rho$. $\square$.

Remark The previous lemma has an interesting consequence. By means of (5) in Theorem 1.1.4, one can recover any N-compact space X from its ring of continuous functions $C(X, \mathbf{Z})$. But if we take any H-N-compact frame L which is not Stone-N-compact, then $\nu_S L \ncong L$, $\nu_S L$ is H-N-compact (Corollary 1.5.15), and $\mathbf{Z}_L E \cong \mathbf{Z}_{\nu_S L}E$. So we cannot hope to recover H-N-compact frames from ring theoretic information about their 'rings of continuous functions' $\mathbf{Z}_L E$. This stands in contrast to the situation with compact frames: given a compact 0-dimensional frame M, one can take the Boolean algebra B of idempotent elements in the ring $\mathbf{Z}_M E$ and then form $\mathfrak{J}B$, a frame isomorphic to M.

We do not know if one can recover an S-N-compact frame L, from its ring of "integer valued continuous functions," $\mathbf{Z}_L E$. We conjecture that the answer is no.

Even though one cannot recover H-N-compact frames from the the rings $\mathbf{Z}_L E$, an intimate connection with group and ring homomorphisms from $\mathbf{Z}_L E$ to $\mathbf{Z}$ is preserved. In the next chapter we shall prove Theorem 2.2.2, which extends a classical theorem of Mrówka's concerning N-compact spaces. We will then be able to apply that result and prove Theorem 1.5.21 below.

Definition 1.5.20 *For a frame L, a ring homomorphism $\mathbf{Z}_L E \xrightarrow{h} \mathbf{Z}$ is* evaluation at a prime element *p of L if $h(\xi) = n$ iff $\xi(\{n\}) \not\leq p$.*

Remark One can easily check that for an arbitrary prime p, a map defined in this way is indeed a ring homomorphism. For a spatial frame $L = \mathcal{O}(X)$ these correspond to the homomorphisms $C(X, \mathbb{Z}) \longrightarrow \mathbb{Z}$ which are the evaluation maps for various points of X.

Theorem 1.5.21 *(*) A 0-dimensional frame L is H-$\mathbb{N}$-compact iff any ring homomorphism $\mathbb{Z}_L E \longrightarrow \mathbb{Z}$ is the evaluation map for some prime element of L.*

Proof See page 59.

Chapter 2

Applications to Abelian Group Theory

2.1 Introduction

If V is a finite-dimensional vector space over some field, then V is isomorphic to V^{**} via the familiar natural map $V \longrightarrow V^{**}$, taking an element of $v \in V$ to the evaluation map at v. Changing the underlying ring to $\mathbf{Z}$, the integers, one studys the $\mathbf{Z}$-modules (Abelian groups) A for which $A \cong A^{**}$ via the analogous map. These are the **reflexive** groups. One also studies the the **non-reflexive dual groups**, those groups A^* which are not reflexive. Inspired by some questions of Reid in [Re], there has been a considerable amount of research into the structure of these sorts of groups, with most of the important results being fairly recent. The book [Ek,Me], soon to appear, contains most of what is known.

One of Reid's questions was whether there are reflexive groups which are not in the (subsequently named) **Reid class**, the smallest class containing the integers and closed under (non-measurably indexed) direct sums and direct products. (It follows from the results mentioned in Section 0.4 that all the members of the Reid class are reflexive.) There were some consistency results in the 1970's but the question was not answered until 1987, when K Eda and H. Ohta published [Ed,Oh]. There they established some interesting connections between properties of a topological space X and the behaviour of the group $C(X, \mathbf{Z})$ and its duals. In particular, the group $C(\mathbf{Q}, \mathbf{Z})$ (where $\mathbf{Q}$ denotes the rationals with the subspace topology induced from the reals) was the first to be shown (in ZFC) reflexive, but not in the Reid class. Another group $C(X, \mathbf{Z})$ was shown to have a non-reflexive dual. Since then, there have been many examples of reflexive groups and non-reflexive dual groups constructed, mostly due to the work of Mekler and Eklof. The **tree sums** and **tree products** (defined in Section 2.3) of reflexive groups are all reflexive, and in [Ek,Me] it is shown that there is a non-free $\aleph_1$-separable group of cardinality $\aleph_1$ which is a **tree group**, (a tree sum of free groups,) and hence reflexive. (A group is called $\aleph_1$-separable if every countable subgroup is contained in a free direct summand.) A non-free group with this separability property cannot be a group of continuous functions, and therefore falls outside the class of groups considered in [Ed,Oh]. (In fact it is consistent that *all* $\aleph_1$-separable group of cardinality of $\aleph_1$ are tree groups.) Thus the groups known to be reflexive fall into two classes, (with some overlap,) the groups $C(X, \mathbf{Z})$ (which contains the Reid class ([Ed2]) and the tree groups.[1]

The results of Eda and Ohta made use of an important theorem proved by Mrówka in [Mr3]:

Definition 2.1.1 *A group homomorphism* $h : C(X, \mathbf{Z}) \longrightarrow \mathbf{Z}$ *has* **compact support** K *for* K *a compact subset of* X *if* $f \restriction K = 0$ *implies* $h(f) = 0$ *for any* $f \in C(X, \mathbf{Z})$.

Theorem 2.1.2 ([Mr2]) *A 0-dimensional topological space* X *is* $\mathbf{N}$*-compact iff any homomorphism* $h : C(X, \mathbf{Z}) \longrightarrow \mathbf{Z}$ *has compact support.*

In Section 1, we are able to use our work on $\mathbf{N}$-compact frames to 'lift' this theorem to groups of global sections of a sheaf of Abelian groups (Theorem 2.2.2.)

[1] We have recently been told that there are now some other constructions of reflexive groups, but we have not seen these.

As we noticed in Chapter 0, this class includes the groups $C(X, \mathbb{Z})$, but also much more, (see Corollary 2.2.7.) We show in Section 2 that the tree products of [Ek, Me] can be seen as groups of global sections and tree sums as their duals. We can then use our generalized Mrówka's theorem to show both of these sorts of groups reflexive. Since Eda's and Ohta's work depends on Theorem 2.1.2, a special case of 2.2.2, the scope of the latter includes all the groups known to be reflexive in ZFC, and it is powerful enough to prove their reflexivity. These results come via a Corollary to 2.2.2 which allows us obtain the 'global reflexivity' of a sheaf of Abelian groups from its 'local reflexivity.' We expect that this will find application in the construction of new reflexive groups.

In recent years, the behaviour of the duals of global sections of groups in a Boolean valued universe $V^{(B)}$ has been studied. (See [Ed3, Ed4] for example.) A complete Boolean algebra B is a frame, and groups in $V^{(B)}$ can be seen as groups in AbShB, (first established in [Hi], see also [Go].) Theorem 2.2.2 can be used to prove some the results in [Ed3] and [Ed4], we present Corollary 2.2.7 as a simple preliminary example.

Thus groups of global sections of sheaves on *frames* (= complete Heyting algebras) include both the groups $C(X, \mathbb{Z})$ and the global sections of groups in a Boolean-valued universe, which have been recently objects of considerable interest. Theorem 2.2.2 seems the tool to unify and extend the study of these two classes.

In this chapter all the groups considered will be Abelian, unless otherwise mentioned. We will write the elements of a frame L in upper-case; the usual practice when one considers sheaves. To avoid confusion between elements of frames and ideals we will denote ideals with a Fraktur font.

2.2 Mrówka's Theorem

We generalize Theorem 2.1.2 of the Introduction to a theorem which applies to the class consisting of all groups of global sections of sheaves of Abelian groups on frames.

We remind the reader of the material in Section 0.3 regarding sheaves. Section 0.4 contains the few facts about Abelian groups we will need.

The notion of support in a topological space becomes that of 'co-support' in the frame setting, and that of a compact subset 'co-compact element.'

Definition 2.2.1 *Let L be a frame, $A \in AbShL$, and $\phi \in Hom(AE, \mathbb{Z}) = (AE)^*$. We say that $S \in L$ is a* **co-support** *of ϕ if whenever both $V \vee S = E$ and $\xi \mid V = 0$ hold, then $\phi(\xi) = 0$. (For $\xi \in AE$.)*

An element S of L is **co-compact** *if $\uparrow S$ is a compact frame.*

Theorem 2.2.2 *Let L be a 0-dimensional frame. Then L is H-$\mathbb{N}$-compact iff for any $A \in AbShL$ and group homomorphism $AE \xrightarrow{h} \mathbb{Z}$, h has co-compact co-support.*

Proof ($\rightarrow$) The proof proceeds in a series of claims. We begin by sketching the general plan:

Let A and h be as given. We employ the frame homomorphism $\mathfrak{J}BL \xrightarrow{j} L$ of Section 0.2.3 to form the sheaf $j_*(A) \in AbSh\mathfrak{J}BL$ defined by

$$j_*(A)I = A(jI), \text{ for any } I \in \mathfrak{J}BL,$$

with the obvious restriction maps. It is easy to show that this is indeed a sheaf (see [Te] for details.) Note that $j_*(A)E_{\mathfrak{J}BL} = A(jE_{\mathfrak{J}BL}) = AE_L$. We will exploit this fact throughout the proof, for example sometimes viewing h as a homomorphism with domain $j_*(A)E_{\mathfrak{J}BL}$. Claim 2.2.3 will be that h, viewed in this way, has *largest* co-support in $\mathfrak{J}BL$, say K, and the remainder of the argument will show that jK is a co-support for $AE \xrightarrow{h} \mathbb{Z}$, (Claim 2.2.6) and co-compact (Claim 2.2.5.)

The following general result is the sheaf theoretic analogue of a classical result for the groups $C(X, \mathbb{Z})$, see [Mr3].

Claim 2.2.3 *Let G be a sheaf of Abelian groups on a compact 0-dimensional frame M. Then any homomorphism $GE \xrightarrow{g} \mathbb{Z}$ has a largest co-support in M.*

Proof We begin by setting $\mathfrak{S} = \{U \in BM \mid \text{U is a co-support for } g\}$. Note that 0_M is always a co-support, so $\mathfrak{S} \neq \emptyset$. In fact $\mathfrak{S}$ is an ideal in BM. For suppose $U_1, U_2 \in \mathfrak{S}$, $V \vee (U_1 \vee U_2) = E$ and $\xi \in GE$ has $\xi \mid V = 0$. We show that $g(\xi) = 0$.

Since $V \geq U_1^* \wedge U_2^*$, we have $\xi \mid U_1^* \wedge U_2^* = 0$. Let ρ be the unique patch in $G(U_1^* \vee U_2^*)$ of $\xi \mid U_1^*$ and $0 \in GU_2^*$. Since $U_1^* \vee U_2^*$ is complemented, ρ extends to $\overline{\rho} \in GE$. Then

$$\xi \mid U_1^* = \overline{\rho} \mid U_1^* \text{ and } U_1 \vee U_1^* = E, \text{ so } g(\xi) = g(\overline{\rho})$$

since U_1 is a co-support, and,

$$0 \mid U_2^* = \overline{\rho} \mid U_2^* \text{ and } U_2^* \vee U_2 = E, \text{ so } g(0) = 0 = g(\overline{\rho})$$

since U_2 is a co-support. Thus $g(\xi) = 0$ and so $U_1 \vee U_2$ is a co-support for g. Since S is clearly an order-ideal, it is an ideal.

Now let $T = \bigvee_M \mathfrak{S}$, our candidate for the largest co-support. Suppose that $\xi \in GE$, $\xi \mid V = 0$, and $V \vee T = E$, for some $V \in M$. Since M is compact, there is a set $\{U_1, U_2, \cdots, U_N\} \subseteq \mathfrak{S}$ such that $V \vee (U_1 \vee U_2 \vee \cdots \vee U_N) = E$. Since $\mathfrak{S}$ is an ideal, $U_1 \vee U_2 \vee \cdots \vee U_N \in \mathfrak{S}$, so that $g(\xi) = 0$. Thus T is a co-support for g.

Now if S is a co-support, S is a join of complemented elements, which, being below a co-support are themselves co-supports, and are thus in $\mathfrak{S}$. So $S \leq T$, and hence T is the *largest* co-support for g. □ (Claim 2.2.3.)

Now viewing h as a map from $j_*(A)E = AE$ to $\mathbb{Z}$, let $\mathfrak{K} \in \mathfrak{J}BL$ be the largest co-support, supplied by Claim 2.2.3.

Claim 2.2.4 *For any $\mathfrak{S} \in \mathfrak{J}BL$, $j\mathfrak{S} = E_L$ implies $\mathfrak{K} \vee \mathfrak{S} = E_{\mathfrak{J}BL}$*

Proof Towards a contradiction, assume that $j\mathfrak{S} = E_L$, but $\mathfrak{K} \vee \mathfrak{S} \neq E_{\mathfrak{J}BL}$ Then $\mathfrak{K} \vee \mathfrak{S}$ is not completely-proper. Since L is H-N-compact , there is a proper ideal $\mathfrak{T}$ extending $\mathfrak{K} \vee \mathfrak{S}$ which is not σ-proper. So there are $V_1, V_2, \cdots$ in $\mathfrak{T}$ with $\bigvee_L V_n = E$. We may assume that this sequence is increasing.

For any $V \in BL$, set $k_L V = \downarrow V \cap BL \in \mathfrak{J}BL$. As $\mathfrak{K} \subseteq \mathfrak{T}$, $k_L V_n \vee \mathfrak{K} \neq E_{\mathfrak{J}BL}$for all n, so that $V_n^* \notin \mathfrak{K}$ and hence $k_L V_n^* \not\subseteq \mathfrak{K}$, for all n.

Then since $\mathfrak{K}$ is the *largest* co-support in $\mathfrak{J}BL$, for any n, $\mathfrak{K} \vee k_L V_n^*$ is not a co-support for h, so that there are $\mathfrak{W}_n \in \mathfrak{J}BL$ so that $\mathfrak{W}_n \vee (\mathfrak{K} \vee k_L V_n^*) = E$ and $\xi_n \in j_*(A)E$ with $\xi_n \mid \mathfrak{W}_n = 0$, but $h(\xi_n) \neq 0$. For each n define a new element $\xi_n^\flat \in j_*(A)E$ by requiring $\xi_n^\flat \mid k_L V_n = 0$ and $\xi_n^\flat \mid k_L V_n^* = \xi_n \mid k_L V_n^*$, using the patching property of the sheaf j_*A. Then

$$\xi_n^\flat \mid \mathfrak{W}_n \wedge k_L V_n = 0 = \xi_n \mid \mathfrak{W}_n \wedge k_L V_n$$

and,

$$\xi_n^\flat \mid \mathfrak{W}_n \wedge k_L V_n^* = \xi_n \mid \mathfrak{W}_n \wedge k_L V_n^*,$$

for all n, so that by the separation property, $\xi_n^\flat \mid \mathfrak{W}_n = \xi_n \mid \mathfrak{W}_n$. Now $\xi_n^\flat \mid k_L V_n^* = \xi_n \mid k_L V_n^*$ by construction, so that $\xi_n^\flat \mid \mathfrak{W}_n \vee k_L V_n^* = \xi_n \mid \mathfrak{W}_n \vee k_L V_n^*$ for all n, again by the separation property of the sheaf $j_*(A)$. Since $\mathfrak{K} \vee \mathfrak{W}_n \vee k_L V_n^* = E$ and $\mathfrak{K}$ is a co-support, we know that $h(\xi_n^\flat) = h(\xi_n) \neq 0$, for all n.

Thinking now of the $\xi_n^\flat$ as being in AE, we have in L an increasing sequence $\{V_n\}_{n\in\omega}$ of complemented elements such that $\bigvee_L V_n = E$ and $\xi_n^\flat \mid V_n = 0$, (since in $j_*(A)$, $\xi_n^\flat \mid k_L V_n = 0$.) We will show that $h(\xi_n^\flat) = 0$ for almost all n, contradicting the results of the last paragraph.

If N is some integer, define $\sum_{k>N}^{\infty} \xi_k^\flat$ to be the patch in AE of the elements $\sum_{N+1}^{k} \xi_i^\flat \mid V_k \in AV_k$, for $k > N$. (These are compatible, for if $N < k \leq l$ then

$$\left[\sum_{N+1}^{l}(\xi_i^\flat \mid V_l)\right] \mid V_k \;=\; \sum_{N+1}^{k}(\xi_i^\flat \mid V_k) + \sum_{k+1}^{l}(\xi_i^\flat \mid V_k)$$

$$= \sum_{N+1}^{k} \xi_i^b \mid V_k + 0.)$$

Set $\rho_n = \sum_{k>n}^{\infty} \xi_k^b$. Then $\rho_n \mid V_n = 0$, and so given any $f \in \mathbf{Z}^{\mathbf{N}}$ we can define $\sum_1^\infty f(n)\rho_n$ as above. We thus obtain a homomorphism

$$\begin{array}{ccccc} \mathbf{Z}^{\mathbf{N}} & \longrightarrow & AE & \longrightarrow & \mathbf{Z} \\ f & \longrightarrow & \sum_1^\infty f(n)\rho_n & \longrightarrow & h(\sum_1^\infty f(n)\rho_n), \end{array}$$

which we denote by ϕ. It follows from the definition that $\phi(e_n) = h(\rho_n)$, for all n. But recall from Section 0.4 that $\mathbf{Z}$ is *slender*, so that $\phi(e_n) = 0$ for almost all n. Then $h(\rho_n) = 0$ for almost all n, so that $h(\xi_n^b) = 0$ for almost all n, since $\rho_n = \xi_{n+1}^b + \rho_{n+1}$ for all n. (To see this, note that both sides of the equation restrict to the same element of V_k for $k > N$.) This is the promised contradiction with the earlier statement $h(\xi_n^b) \neq 0$ for all n, above. So it must be the case that $\mathfrak{K} \vee \mathfrak{S} = E$. □ (Claim 2.2.4.)

Claim 2.2.5 *The frame* $\uparrow(j\mathfrak{K})$ *is compact, ie,* $j\mathfrak{K}$ *is co-compact.*

Proof Let $\mathfrak{S} \subseteq \uparrow (j\mathfrak{K})$ be some set of elements, and suppose that $\bigvee_L \mathfrak{S} = E_L$. Let $\mathsf{S} \subseteq \mathfrak{J}BL$ be a collection of ideals so that $j[\mathsf{S}] = \mathfrak{S}$, which is possible since L is 0-dimensional. Then $j(\bigvee_{\mathfrak{J}BL}\mathsf{S}) = \bigvee_L \mathfrak{S} = E_L$, so that $\mathfrak{K} \vee \bigvee_{\mathfrak{J}BL}\mathsf{S} = E_{\mathfrak{J}BL}$ by Claim 2.2.4. Since $\mathfrak{J}BL$ is compact, there is a finite subset $\mathsf{T} \subseteq \mathsf{S}$ so that $\mathfrak{K} \vee \bigvee_{\mathfrak{J}BL}\mathsf{T} = E_{\mathfrak{J}BL}$ Then $j(\mathfrak{K} \vee \bigvee_{\mathfrak{J}BL}\mathsf{T}) = j\mathfrak{K} \vee \bigvee_L j[\mathsf{T}] = \bigvee_L j[\mathsf{T}] = E$, and so $j[\mathsf{T}]$ is a finite subcover of $\mathfrak{S}$. Thus $j\mathfrak{K}$ is co-compact. □

Claim 2.2.6 *The element* $j\mathfrak{K}$ *is a co-support for* h *in* L.

Proof Suppose the $V \vee j\mathfrak{K} = E_L$ in L, and $\xi \in AE$ has $\xi \mid V = 0$. Fix $\mathfrak{V} \in \mathfrak{J}BL$ so that $j\mathfrak{V} = V$. Then $j(\mathfrak{V} \vee \mathfrak{K}) = j(\mathfrak{V}) \vee j\mathfrak{K} = V \vee j\mathfrak{K} = E_L$. Then by Claim 2.2.4, $\mathfrak{K} \vee \mathfrak{V} \vee \mathfrak{K} = \mathfrak{V} \vee \mathfrak{K} = E_{\mathfrak{J}BL}$ Now $\xi \mid \mathfrak{V} = \xi \mid j\mathfrak{V} = \xi \mid V = 0$, (recall that ξ is in $j_*(A)E_{\mathfrak{J}BL} = AE_L$.) Thus $h(\xi) = 0$, since $\mathfrak{K}$ is a co-support for $j_*(A)E_{\mathfrak{J}BL} \xrightarrow{h} \mathbf{Z}$. □

Claims 2.2.5 and 2.2.6 together provide a co-compact co-support for h, and we have established 'necessity' for Theorem 2.2.2.

Proof ($\leftarrow$) This implication is proved in a manner similar to Mrówka's original result. Towards the contrapositive, suppose that L is 0-dimensional but not H-N-compact. Then there is an $\mathfrak{I} \in \mathfrak{J}BL$ which is maximal and σ-proper, but not completely proper, by the Boolean Ultrafilter Theorem. We recall the definition of the group $\mathbf{Z}_L E$ from Section 0.3, and define a map $\mathbf{Z}_L E \xrightarrow{h} \mathbf{Z}$ by setting $h(\xi) = n$ if $\xi(\{n\}) \notin \mathfrak{I}$. Using the maximality of $\mathfrak{I}$ it is not difficult to see that h is a group homomorphism. (This is the the frame analogue to 'evaluation at a point of νX.')

We claim that h does not have co-compact co-support in L. For if K were such, then whenever $V \vee K = E_L$ and $\xi \mid V = 0$, we would have $h(\xi) = 0$. In other words, whenever $V \vee K = E_L$ and $V \leq \xi(0)$, $\xi(0) \notin \mathfrak{I}$. Now let P be any element of BL, and define $\xi_P \in \mathbf{Z}_L E$ by

$$\xi_P(\{n\}) = \begin{cases} P \text{ if } n = 0 \\ P^* \text{ if } n = 1 \\ 0_L \text{ otherwise} \end{cases}$$

Applied to this element of $\mathbf{Z}_L E$, the statements above imply that if $V \vee K = E_L$ and $V \leq P$, then $P \notin \mathfrak{I}$, and as a special case, if $P \in BL$ and $P \vee K = E_L$ then $P \notin \mathfrak{I}$.

Now since $\bigvee_L \mathfrak{I} = E_L$, $K \vee \bigvee_L \mathfrak{I} = E_L$. As K is co-compact, there is a $W \in \mathfrak{I}$ so that $W \vee K = E_L$. But this contradicts the results of the previous paragraph, so that h can not have co-compact co-support. □ (Theorem 2.2.2.)

Remark It is in the use of the sheaf $j_* A$ that our proof differs substantively in nature from Mrówka's proof of 2.1.2, which was essentially for the case $A = \mathbf{Z}_L$, $L = \mathcal{O}(X)$, X N-compact . The object which performs a similar duty there can be seen as $\mathbf{Z}_{\mathcal{J}BL}$, whereas our proof in this case uses $j_*(\mathbf{Z}_L)$.

Remark Note that the proof of the $\rightarrow$ implication in Theorem 2.2.2 took place in ZF, but the converse used the Boolean Ultrafilter Theorem.

We can use Theorem 2.2.2 to prove a result of Eda in [Ed3], and obtain a plentiful supply of groups A such that A is $\aleph_1$-free (every countable subgroup is free) but $A^* = 0$. Recall the definition of the 'constant sheaf' $\mathbf{Z}_L$ from Section 0.3.

Corollary 2.2.7 *If B is a complete Boolean algebra without atoms, and with no antichains of measurable cardinality, then $\mathbf{Z}_B E$ is $\aleph_1$-free but $(\mathbf{Z}_B E\ )^* = 0$.*

Proof Towards showing $(\mathbf{Z}_B E\ )^* = 0$, suppose that $\mathbf{Z}_B E \xrightarrow{h} \mathbf{Z}$ is a homomorphism. We know by Lemma 1.5.5 that B is H-N-compact , and so by Theorem 2.2.2 that h has co-compact co-support K. Now K^* is compact, and therefore a finite join of atoms. Since B has no atoms, this must be the empty join, so that $K^* = 0$, $K = E_B$, and thus $h = 0$.

We can actually show that for *any* frame L, $\mathbf{Z}_L E$ is $\aleph_1$-free. To do this, we show that for any fixed countable subgroup $S \subseteq \mathbf{Z}_L E$, if $F \subseteq S$ is a subgroup of finite rank, then F is free. By a lemma of Pontryagin [Fu, page 93], this implies that S is free.

Let $F \subseteq S$ be as given, and fix $\{f_1, \cdots, f_N\} \subseteq F$ a maximal independent subset. Each f_i induces a certain countable cover of E_L, say P_i. (See Section 0.3.) Let P be a common refinement of the P_i. Then we can see $f_1, \cdots, f_N$ as elements of a group C which arises as an equalizer:

$$C \rightarrowtail \prod_{U \in P} \mathbf{Z} \rightrightarrows \prod_{U,V \in P} \mathbf{Z}$$

Now if $f \in F$, since the f_i's form a maximal independent subset, there are integers $n, n_1, n_2, \cdots, n_N$ such that $nf = n_1 f_1 + n_2 f_2 + \cdots + n_N f_N$, hence $nf \in C$ and so $f \in C$, by a simple argument. So $F \subseteq C \subseteq \prod_{U \in P} \mathbf{Z}$ and since a product of copies of the integers is $\aleph_1$-free, ([Fu, page 94]), F is free. □

Remark $\mathbf{Z}_B E$ is the same thing as the Boolean power $\mathbf{Z}^{(B)}$ (see [Ed3].)

Remark Such groups $\mathbf{Z}_B E$ are clearly not of the form $C(X, \mathbf{Z})$.

We introduce the notion of local reflexivity, and give a condition for passage to global reflexivity.

Definition 2.2.8 *If $A \in AbShL$, we say that A is* **locally reflexive** *if there is a cover $E_L = \bigvee_L U_\alpha$ so that AU_α is reflexive for each α. The sheaf A is* **globally reflexive** *if AE is reflexive.*

Theorem 2.2.9 *If $A \in AbShL$ is locally reflexive and L is H-$\aleph$-compact then A is globally reflexive.*

Proof First note that if $U \leq V$ and $U \in BL$, then U is complemented in $\downarrow V$, so that AU is a direct summand of AV, by the sheaf laws. It is straightforward to show that a summand of a reflexive group is reflexive, and so we can assume that the cover of E_L in the hypothesis consists of complemented elements.

Let D be the directed set formed by taking all finite joins of the U_α. We assert that if $V \in D$ then AV is reflexive. This follows by induction from the following claim.

Claim 2.2.10 *If $U, V \in BL$ and AU and AV are reflexive, then so is $A(U \vee V)$.*

Proof Since $U \vee V$ is the disjoint join of U and $V \wedge U^*$, we know that $A(U \vee V) \cong AU \oplus A(V \wedge U^*)$, (see Lemma 0.3.1.) Now $V \wedge U^*$ is complemented and below V so that $A(V \wedge U^*)$ is a summand of AV and hence reflexive. Then $A(U \vee V)$ is a sum of two reflexive groups, and is thus itself reflexive. □ (Claim).

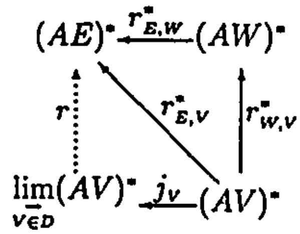
On the directed set D, we form the inverse system of groups; $\langle AW,\ W \in D,\ r_{W,V} \rangle$, with maps $AW \xrightarrow{r_{W,V}} AV$, for $V \leq W$ the sheaf restriction maps. It follows easily from the sheaf patching conditions that we have an isomorphism;

$$\begin{aligned} AE &\cong \varprojlim_{W \in D} AW \\ \xi &\to (\xi \mid W)_{W \in D} \end{aligned}$$

Taking dual groups and dual maps, we obtain a direct system $((AW)^*,\ W \in D,\ r^*_{W,V})$. (See [Fu, Thm. 44.2].)

Claim 2.2.11 *The group $(AE)^*$ is canonically isomorphic to $\varinjlim (AW)^*$.*

Proof (Claim.) The upper triangle in the diagram below commutes, so there is a unique map r making the lower commute, where j_V is the V-th colimit map.



This r is our candidate for the claimed isomorphism. Towards showing it 1-1, suppose that $r(g) = 0$. Now $g = j_V(g_V)$ for some $V \in D$ and $g_V \in (AV)^*$, (a general property of direct limits,) so that $r^*_{E,V}(g_V) = 0$. Thus $r^*_{E,V}(g_V)(h) = g_V(r_{E,V}(h)) = 0$ for all $h \in AE$. Since V is complemented, any $\xi \in AV$ can be lifted to a $\xi' \in AE$ with $\xi' \mid V = \xi$. Then $g_V(r_{E,V}(\xi')) = g_V(\xi) = 0$, so that $g_V = 0$ and hence $g = 0$.

Now r is also onto. For given $h \in (AE)^*$ we know that by Theorem 2.2.2 that it has co-compact co-support K. Since $\bigvee_L D = E_L$, we must have $K \vee V = E_L$ for some $V \in D$, since K is co-compact. Define $h' \in (AV)^*$ by $h'(g) = h(\overline{g})$ where $\overline{g}$ is some element of AE extending g (which exists since V is complemented.) Then h' is well-defined, for if $\overline{\overline{g}}$ is another such extension, then $\overline{g} - \overline{\overline{g}} \mid V = 0$, so that $h(\overline{g}) = h(\overline{\overline{g}})$ since K is a co-support for h. Finally,

$$r^*_{E,V}(h')(g) = h'(r_{E,V}(g)) = h'(g \mid V) = h(g),$$

so that $r^*_{E,V}(h') = h$, thus $rj_V(h') = h$, and hence r is onto. □ (Claim)

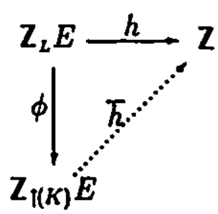
A purely categorical argument implies that $(AE)^{**} \cong \varprojlim (AW)^{**}$, so that $(AE)^{**} \cong \varprojlim AW \cong AE$, since the AW are reflexive and hence isomorphic to $(AW)^{**}$

via the canonical map. It is a straightforward (but tedious) matter to check that this isomorphism $AE \cong (AE)^{**}$ is itself the canonical map. Thus AE is reflexive. □

We can now prove Theorem 1.5.21, which we restate here.

Theorem 1.5.21 *A 0-dimensional frame L is H-N-compact iff any ring homomorphism $\mathbb{Z}_L E \rightarrow \mathbb{Z}$ is the evaluation map for some prime element of L.*

Proof ($\rightarrow$) Suppose that L is H-N-compact, and let $\mathbb{Z}_L E \xrightarrow{h} \mathbb{Z}$ be a ring homomorphism. Let K be the co-compact co-support of h supplied by Theorem 2.2.2. Define $\mathbb{Z}_L E \xrightarrow{\phi} \mathbb{Z}_{\Uparrow(K)} E$ by $\phi(\xi)(S) = \xi(S) \vee K$ for $S \in \mathcal{O}(\mathbb{Z})$. One can verify without difficulty that ϕ is a ring homomorphism. (This is the analogue of the map restricting a continuous function to a subspace.) We claim that h factors through ϕ as shown.



To see this, we proceed as follows. First observe that if $\rho \in \mathbb{Z}_{\Uparrow(K)} E$, there is a $\rho' \in \mathbb{Z}_L E$ with $\phi(\rho') = \rho$. (This follows from an argument just like the one which shows that a continuous $\mathbb{Z}$-valued function on a compact subset of a 0-dimensional space extends to the whole space.) Now define $\overline{h}(\rho) = h(\rho')$. The definition is independent of the choice of ρ' since K is a co-support, and defines a homomorphism for the same reason.

Since $\Uparrow(K)$ is a compact and 0-dimensional, it is spatial, (see Section 0.2.2,) and as we noticed in Section 0.3, this means that there is a ring isomorphism $\mathbb{Z}_{\Uparrow(K)} E \cong C(\Sigma \uparrow (K), \mathbb{Z})$. Thus $\overline{h}$ can be seen as a map from $C(\Sigma \uparrow (K), \mathbb{Z})$ to $\mathbb{Z}$. Theorem 1.1.4 implies that $\overline{h}$ viewed in this way is the evaluation map at some point p of $\Sigma \uparrow (K)$, and thus, passing via the ring isomorphism just mentioned, $\overline{h}$ is evaluation at the prime element P of $\uparrow (K)$ corresponding to p. Thus $\overline{h}(\rho) = n$ iff $\rho(\{n\}) \not\leq P$. It follows that

$$\overline{h}\phi(\xi) = n \text{ iff } \phi(\xi)(\{n\}) = \xi(\{n\}) \vee K \not\leq P,$$

and hence that $h(\xi) = n$ iff $\xi(\{n\}) \not\leq P$, since $K \leq P$. Thus h is the evaluation map at the prime element P of L.

($\leftarrow$) Towards the contrapositive, suppose that L is not H-N-compact. In the proof of Theorem 2.2.2 ($\leftarrow$) we found a group homomorphism $\mathbb{Z}_L E \xrightarrow{h} \mathbb{Z}$ without a co-compact co-support. But h is also a ring homomorphism (a fact easy to establish,) and if it were the evaluation map at a prime P of L, P would be a co-compact co-support. So L is not H-N-compact.□

2.3 Tree groups and Mrówka's Theorem

In [Ek,Me] the tree product and the tree sum of a 'tree of groups' are defined. It is shown there that both the tree product and the tree sum of a tree of reflexive groups are reflexive, and that there is a non-free $\aleph_1$-separable group of cardinality $\aleph_1$ which is a 'tree group': a tree sum of a tree of free groups. Here we show how a sheaf of groups can be made from any tree of groups so that the group of global sections is the tree product (Claim 2.3.7.) We can then employ Theorem 2.2.2 to establish the reflexivity results of [Ek,Me].

Definition 2.3.1 *A tree T is a partially ordered set in which $\Downarrow\nu = \{\mu \in T \mid \mu < \nu\}$ is well-ordered, for any $\nu \in T$. A* **branch** *is a maximal well-ordered subset of T. If $\nu \in T$, the* **height** *of ν is the order type of $\Downarrow\nu$, and the set of elements of height α is denoted T_α. The tree T is said to be of* **height** *β if β is the least ordinal such that $T_\beta = \emptyset$.*

We will always restrict ourselves to trees such that $| T_0 | = 1$ and every element is contained in a branch of height $\omega + 1$. Note that any tree can be regarded as a $\wedge$-semilattice, (if we exclude the meet of the empty set.) See [Ku] for more information about trees.

Definition 2.3.2 ([Ek,Me]) *If T is a tree of height $\omega + 1$, then a* **tree of groups** *is a collection $\{A_\nu \mid \nu \in T\}$ of groups indexed by T, along with homomorphisms $\{\iota_{\nu,\eta}, \pi_{\eta,\nu} \mid \nu \le \eta \in T\}$, such that, when $\nu \le \eta \le \tau$,*

(i) $A_\nu \xrightarrow{\iota_{\nu,\eta}} A_\eta$ *and* $\iota_{\nu,\tau} = \iota_{\eta,\tau} \circ \iota_{\nu,\eta}$,

(ii) $A_\eta \xrightarrow{\pi_{\eta,\nu}} A_\nu$ *and* $\pi_{\tau,\nu} = \pi_{\eta,\nu} \circ \pi_{\tau,\eta}$, *and*

(iii) $\pi_{\eta,\nu} \circ \iota_{\nu,\eta} = id_{A_\nu}$.

Remark Note that condition (iii) implies that A_ν is a summand of A_η when $\nu \le \eta$. Indeed, we have $A_\eta \cong A_\nu \oplus \mathrm{Ker}\pi_{\eta,\nu}$. We can thus think of a tree of groups as a collection of nested groups, each a summand of any supergroup, but in the coherent manner which follows from the conditions (i) and (ii).

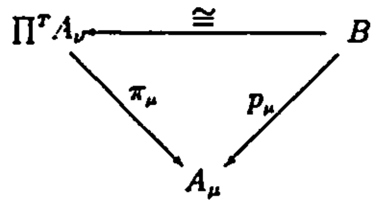
We can view the tree T as a category, with objects the elements of T and morphisms having the properties (i), (ii), (iii) of 3.3.2. (For example, $\iota_{\nu,\eta}$ has domain ν and range η.) Then any tree of groups on T corresponds to a functor from T to **Ab**, the category of Abelian groups. We will use this approach in the following definition, which is basically Theorem 2.6 of Chapter XIV in [Ek,Me].

Definition 2.3.3 *Given a tree of groups $\{A_\nu \mid \nu \in T\}$, the* **tree product** *is the limit of the diagram in* **Ab** *consisting of the groups A_ν and maps $\pi_{\eta,\nu}$ for $\nu \le \eta \in T$. We denote this by $\prod^T A_\nu$, with projection (limit) maps $\prod^T A_\nu \xrightarrow{\pi_\nu} A_\nu$.*

The **tree sum** *is the colimit of the diagram consisting of the groups A_ν and maps $\iota_{\nu,\eta}$ for $\nu \le \eta$. We will denote this by $\sum^T A_\nu$, and the colimit maps by $A_\nu \xrightarrow{\iota_\nu} \sum^T A_\nu$.*

Remark Tree products and sums are defined in a different but equivalent way in [Ek,Me]. Our definition is for our purposes easier to work with, but we will also need to use the original definition of a tree product, which is expressed in the following lemma.

Lemma 2.3.4 *Given a tree of groups $\{A_\nu \mid \nu \in T\}$, denote by B the subgroup of $\prod_{\nu \in T_\omega} A_\nu$ consisting of elements $(a_\nu)_{\nu \in T_\omega}$ such that $\pi_{\nu,\nu\wedge\mu}(a_\nu) = \pi_{\mu,\nu\wedge\mu}(a_\mu)$ for any $\nu, \mu \in T_\omega$. For any $\mu \in T$, denote by p_μ the map $B \longrightarrow A_\mu$ defined by $p_\mu((a_\nu)_{\nu \in T_\omega}) = \pi_{\nu,\mu}(a_\nu)$ where $\mu \leq \nu$ for some $\nu \in T_\omega$. (This is well-defined by the definition of B.) Then there is an isomorphism making the diagram commute;*



Proof Straightforward. □

We show how a sheaf of groups A may be constructed from a tree of groups $\{A_\nu \mid \nu \in T\}$ so that the group of global sections AE is isomorphic to the tree product $\prod^T A_\nu$.

The first step is to define a topological space X_T with the same underlying set as T, and with the **tree topology**; the collection of intervals (α, β) for $\beta \in T$ and $\alpha \in T \cup \{-\infty\}$ serves as a basis. See S. Todorčević's article in [To] for more information about tree topologies.

Theorem 2.3.5 *If T is a tree of height $\omega + 1$ of non-measurable cardinality, then X_T is an $\mathbb{N}$-compact space.*

Proof Suppose that $\mathcal{F}$ is a clopen ultrafilter on X_T with the countable intersection property. We must show that $\mathcal{F}$ is fixed. We have two cases;

Case 1 There is some $U \in \mathcal{F}$ such that $U \cap T_\omega = \emptyset$.

The ultrafilter $\mathcal{F}$ induces a clopen ultrafilter $\mathcal{F} \cap \downarrow U$ on the subspace U, and this induced ultrafilter also has the countable intersection property. Since every point in $T\setminus T_\omega$ is isolated in X_T, U has the discrete topology. Hence by the cardinality assumption, $\mathcal{F} \cap \Downarrow U$ is principal, so that $\mathcal{F}$ is fixed.

Case 2 For every $U \in \mathcal{F}$, $U \cap T_\omega \neq \emptyset$.

We first show that for any $N \leq \omega$, there is a unique $\nu \in T_N$ so that $\Uparrow\nu = \{\eta \in T \mid \eta \geq \nu\}$ is in $\mathcal{F}$. Towards this, fix an N, and consider the clopen partition of X_T into the two sets $\bigcup_{m<N} T_n$ and $\bigcup_{\nu \in T_N} \Uparrow\nu$. By the Case 2 hypothesis the first of these is not in $\mathcal{F}$ so the second must be. Towards a contradiction, suppose that for no $\nu \in T_N$ is the clopen set $\Uparrow\nu$ in $\mathcal{F}$. Define a filter $\mathcal{E} \subseteq \mathsf{P}(T_N)$ by declaring A to be in $\mathcal{E}$ iff the clopen set $\bigcup_{\nu \in A} \Uparrow\nu$ is in $\mathcal{F}$. One can easily check that $\mathcal{E}$ is indeed a proper filter. In fact it is maximal, since if $A \notin \mathcal{E}$, $\bigcup_{\nu \in A} \Uparrow\nu \notin \mathcal{F}$, so that the complement $\bigcup_{\nu \in \complement A}(\Uparrow\nu) \cup \bigcup_{m<N} T_m$ is in $\mathcal{F}$. Since the second set of this disjoint union is not in $\mathcal{F}$ by hypothesis, by maximality the first is. Thus $\complement A$ is in $\mathcal{E}$, and hence $\mathcal{E}$ is maximal. Moreover, the ultrafilter $\mathcal{E}$ has the countable intersection property. For if $A_n \in \mathcal{E}$ for $n \in \omega$ one easily checks that

$$\bigcup_{\nu \in A} \Uparrow\nu = \bigcap_{n \in \omega} \bigcup_{\nu \in A_n} \Uparrow\nu \qquad \text{Equation 2.1}$$

where $A = \bigcap_n A_n$. Since $\mathcal{F}$ has the countable intersection property, the right-hand-side of Eqn 2.1 is nonempty, so that A must be as well.

Thus $\mathcal{E}$ is an ultrafilter with the countable intersection property in $\mathcal{P}(T_N)$. But it cannot be principal since for no $\nu \in T_N$ is $\Uparrow\nu$ in $\mathcal{F}$, by hypothesis. This contradicts the non-measurability of $\mid T \mid$.

So for every level $N \in \omega$ there is a (necessarily unique) $\nu_N \in T_N$ such that $\Uparrow\nu_N \in \mathcal{F}$, and clearly if $N > M$ then $\nu_N > \nu_M$. Since $\mathcal{F}$ has the countable intersection property, $\bigcap_N \Uparrow\nu_N$ is nonempty, and thus must contain the one element $\nu \in T_\omega$ which is greater than ν_N for every N. But then *every* set in $\mathcal{F}$ must contain ν, for if $U \in \mathcal{F}$ did not, $U \cap \bigcap_N \Uparrow\nu_N$ would be an empty intersection of countably many elements of $\mathcal{F}$. Thus $\mathcal{F}$ is fixed, and hence X_T is N-compact. □

Recall that we wish to construct a sheaf of groups A on $\mathcal{O}(X_T)$ from the tree of groups $\{A_\nu \mid \nu \in T\}$. We begin with the special case $T = \omega + 1$, so that X_T is the usual (interval) topology.

The idea is that A should be a (the) sheaf on $\mathcal{O}(\omega + 1)$ such that

$$A\{0, \cdots, \alpha\} \cong A_\alpha \qquad \text{Equation 2.2}$$

for $\alpha \leq \omega$, with the restriction maps inherited from the tree of groups. If A is to be such a sheaf, the sheaf laws stipulate certain constraints on the values of AU for arbitrary $U \in \mathcal{O}(\omega + 1)$, and on the restriction maps. The conditions of Definition 3.3.2 will then be exactly what we need to find groups and maps satisfying such constraints, and hence build a sheaf. To avoid some otherwise necessary intricate notation and tedious calculations, we shall frequently ignore the difference between two isomorphic groups.

We proceed as follows.

(i) As for any sheaf, we require $A\emptyset = \{0\}$.

(ii) The set $\{0, \cdots, n+1\}$ is the disjoint union of $\{0, \cdots, n\}$ and $\{n+1\}$, so a sheaf A satisfying Equation 2.2 would have to have $A\{n+1\} \oplus A\{0, \cdots, n\} \cong A\{0, \cdots n+1\}$, and hence $A\{n+1\} \oplus A_n \cong A_{n+1}$. (See Section 0.3.) From the remark after Definition 3.3.2 we see that setting $A\{n+1\} = \mathrm{Ker}(\pi_{n+1,n})$ will do this job, for any $n > 0$. The patching laws of a sheaf then entirely determine the value of AS for any $S \subseteq \omega$; $AS \cong \prod_{n \in S} A\{n\}$ with restriction maps $AT \longrightarrow AS$ for $S \subseteq T \subseteq \omega$ isomorphic to the appropriate projection maps. (See Lemma 0.3.1.) So we let AS have exactly these values for these open sets S, with the corresponding maps.

(iii) For any $n \in \omega$ the set $\{0, \cdots, \omega\}$ is the disjoint union of $\{0, \cdots, n\}$ and $\{n + 1, \cdots, \omega\}$. A sheaf A satisfying 2.2 must then have $A\{n+1, \cdots, \omega\} \oplus A_n = A_\omega$. The group $\mathrm{Ker}(\pi_{\omega,n})$ is such a group and so we set $A\{n+1, \cdots, \omega\} = \mathrm{Ker}(\pi_{\omega,n})$ for any n.

Now if $S \in \mathcal{O}(\omega + 1)$ is *any* open set such that $\omega \in S$, we can write S as the disjoint union of $(n, \omega]$ and R for some $n \in \omega$ and $R \subseteq [0, n]$. Then if A is to be sheaf it must have $AS \cong AR \oplus A(n,\omega]$. Of course, it should not matter which n and R we choose, and the following Lemma ensures that this is indeed the case.

Lemma 2.3.6 *Suppose that $n < m < \omega$. Then*

$$Ker(\pi_{\omega,n}) \cong Ker(\pi_{\omega,m}) \oplus \prod_{i=n}^{m-1} Ker(\pi_{i+1,i})$$

Proof By induction on $m - n$ using

$$\begin{array}{ccc} \mathrm{Ker}(\pi_{\omega,n}) & \cong & \mathrm{Ker}(\pi_{\omega,n+1}) \oplus \mathrm{Ker}(\pi_{n+1,n}) \text{ via} \\ a + \iota_{n+1,\omega}(b) & \longleftarrow & (a,b) \end{array} \qquad \square$$

We have determined AS for any $S \in \mathcal{O}(\omega + 1)$. The Lemma will let us fix the remaining undetermined restriction maps.

Suppose that $S, T \in \mathcal{O}(\omega + 1)$, with $S \subseteq T$. We determine what the map $AT \longrightarrow AS$ must be, as follows.

Case a; $\omega \notin T$. This was described in paragraph (ii) above.

Case b; $\omega \notin S, \omega \in T$. If $S = \emptyset$ we must have the 0-map. Otherwise, if n is some integer in S, we can write T as the disjoint union of $(m,\omega]$ and R for some integer m and set R such that $n \in R$. Then the restriction map $AT \longrightarrow A\{n\}$ will be (isomorphic to) the appropriate projection map through AR, and hence the map $AT \longrightarrow AS$ will be the product of all these maps. (Recall that $AS \cong \prod_{n \in S} A\{n\}$.)

Case c; $\omega \in S$. There is an integer n so that S and T are the disjoint union of $(n,\omega]$ and some finite sets R_S and R_T, respectively. Then the restriction map $AT \longrightarrow AS$ must be (isomorphic to) the combination of the appropriate projection map $A_{R_S} \longrightarrow A_{R_T}$ and the identity.

We have shown how any sheaf A on $\mathcal{O}(\omega+1)$ is completely determined by asserting that A satisfy Equation 2.2, and one can soon convince oneself that we have in the process actually built such a sheaf A.

Now for an *arbitrary* tree T of height $\omega+1$ we define a presheaf A on X_T as follows. For any $\epsilon \in T_\omega$ denote the sheaf defined on $[0,\epsilon]$ (as above) by A_ϵ. Then set

$$AU = \begin{cases} A_\epsilon U \text{ if } U \subseteq [0,\epsilon] \text{ for some } \epsilon \in T_\omega \\ \{0\} \text{ otherwise} \end{cases}$$

with the same restriction maps as the sheaf A_ϵ below $[0,\epsilon]$ and the trivial maps elsewhere. The definition of the A_ϵ ensures that this definition makes sense (i.e. that $A_\epsilon U = A_\theta U$ if $U \subseteq [0,\epsilon] \cap [0,\theta]$.) Then $\tilde{A}$, the corresponding sheaf on X_T has $\tilde{A}[0,\epsilon] = A_\epsilon$ for $\epsilon \in T_\omega$, since each A_ϵ is already a sheaf on $[0,\epsilon]$.

Claim 2.3.7 *The group of global sections $\tilde{A}X_T$ is isomorphic to the tree product of $\{A_\nu \mid \nu \in T\}$.*

Proof This is quite clear. We know that X_T is the union of all the open sets $[0,\epsilon]$ for $\epsilon \in T_\omega$ and hence that $\tilde{A}X_T$ can be seen as the subgroup of $\prod_{\epsilon \in T_\omega} \tilde{A}[0,\epsilon]$ which consists of those elements $(\xi_\epsilon)_{\epsilon \in T_\omega} \in \tilde{A}X_T$, such that

$$\xi_\epsilon \mid [0,\epsilon] \cap [0,\nu] = \xi_\nu \mid [0,\epsilon] \cap [0,\nu].$$

(See Section 0.3.) It follows from our construction of the sheaves $\tilde{A}$ and A_ϵ that this is (isomorphic to) the subgroup of $\prod_{\epsilon \in T_\omega} A_\epsilon$ consisting of elements $(a_\epsilon)_{\epsilon \in T_\omega}$ such that $\pi_{\epsilon,\epsilon\wedge\nu}(a_\epsilon) = \pi_{\nu,\epsilon\wedge\nu}(a_\nu)$. But this is exactly the tree product of $\{A_\nu \mid \nu \in T\}$, by Lemma 2.3.4.□

We can now present a simple proof of Mekler's result [Ek,Me, Thm XIV.2.8(i)].

Corollary 2.3.8 *If T is a tree of nonmeasurable cardinality, then any tree of reflexive groups on T has a reflexive tree product.*

Proof If $\{A_\nu \mid \nu \in T\}$ is a tree of reflexive groups then $\tilde{A}$, the corresponding sheaf of groups is locally reflexive (with cover $X_T = \bigcup_\epsilon [0,\epsilon],$) on an $\mathbb{N}$-compact frame $\mathcal{O}(X_T)$. Theorem 2.2.9 tells us that the tree product ($= \tilde{A}X_T$) is reflexive. □

With a little more work we can also prove the following theorem, (Theorem XIV.2.8(ii) of [Ek,Me].

Theorem 2.3.9 *If T is a tree of nonmeasurable cardinality, then the tree sum of any tree of reflexive groups on T is itself reflexive.*

Before giving a proof of Theorem 3.3.9 we establish a couple of preliminary results.

Definition 2.3.10 ([Ek,Me]) *If $\{A_\nu \mid \nu \in T\}$ is a tree of groups with maps $\{\iota_{\nu,\eta}, \pi_{\eta,\nu} \mid \nu < \eta \in T\}$, the* **dual tree of groups** *is the collection $\{A^*_\nu \mid \nu \in T\}$ with maps $\{\pi^*_{\eta,\nu}, \iota^*_{\nu,\eta} \mid \nu < \eta\}$.*

Remark It is not difficult to show that the dual tree of a tree of groups is itself a tree of groups.

Lemma 2.3.11 *If T is a tree of height $\omega + 1$ and $S \subseteq X_T$ is compact, then $S \subseteq \downarrow\{\nu_1, \cdots, \nu_N\}$ for some finite set $\{\nu_1, \cdots, \nu_N\}$ contained in T_ω.*

Proof Given S, form $\bigcap\{\Downarrow R \mid S \subseteq \Downarrow R, R \subseteq T_\omega\}$, which must be $\Downarrow R$ for some $R \subseteq T_\omega$, and is thus the smallest such set containing S. Now $\Downarrow R = \bigcup_{\nu \in R} \Downarrow\nu$, so $S = \bigcup_{\nu\in R} S \cap \Downarrow\nu$. This is a cover of S with no proper subcover, by the construction of $\Downarrow R$, and since S is compact R must be finite. □

Lemma 2.3.12 *If T is a tree of height $\omega+1$, then $\downarrow\{\nu_1, \cdots, \nu_N\}$ is a compact, clopen subset of T for any set $\{\nu_1, \cdots, \nu_N\} \subseteq T$.*

Proof Clear. □

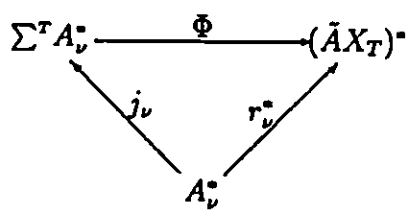
Towards a proof of Theorem 2.3.9 we will prove the following, which improves somewhat on Theorem XIV.2.7 of [Ek,Me].

Theorem 2.3.13 (i) *The dual of the tree sum of a tree of groups is the tree product of the dual tree of groups.*

(ii) *The dual of the tree product of a tree of groups (of nonmeasurable cardinality) is the tree sum of the dual tree of groups.*

Proof (i) This is just a case of the familiar categorical fact stating that the 'dual' of a colimit of a diagram is the limit of the 'dual' diagram.

(ii) We first form the associated tree topology X_T and sheaf of groups $\tilde{A}$, as above. For any $\nu \in T$, we have the sheaf restriction maps r_ν taking $\tilde{A}X_T$ to $\tilde{A}[0,\nu]$ ($\cong A_\nu$). Since $[0,\nu]$ is clopen in the tree toplogy, the map r_ν is surjective. We denote the coproduct maps $A_\nu^* \longrightarrow \sum^T A_\nu^*$, by j_ν. We may use these to obtain a map Φ making the diagram below commute, for any ν.



We wish to show that Φ is an isomorphism.

Each map r_ν^* is injective since the r_ν are surjective. It follows that Φ is injective.

Towards seeing that Φ is surjective, fix an h in $\tilde{A}X_T$. We know from Theorem 2.2.2 that h has co-compact co-support in $\mathcal{O}(X_T)$, say K. Since $\complement K$ is a compact set in X_T, via Lemma 2.3.11 we can find $\nu_1, \cdots \nu_N$, elements of T_ω such that $K \subseteq \downarrow\{\nu_1, \cdots, \nu_N\}$. Then $\downarrow\{\nu_1, \cdots, \nu_N\}$ is a clopen set which is a support for h, (if $f \mid \downarrow\{\nu_1, \cdots, \nu_N\} = 0$ for $f \in \tilde{A}E$ then $h(f) = 0$.)

Now if ν is any element of T, the set $[0,\nu]$ is clopen in X_T, and so there is a canonical map k_ν taking $\tilde{A}[0,\nu](=A_\nu)$ to $\tilde{A}E$; $k_\nu(a)$ is the element of $\tilde{A}E$ which restricts to a on $[0,\nu]$ and to 0 on the complement of $[0,\nu]$. We consider the following element of $(\tilde{A}E)^*$;

$$\sum_{i=1}^{N} \sum_{\substack{S\subsetneq\{\nu_1,\cdots,\nu_N\}\\|S|=i}} (-1)^{i+1} r^*_{\wedge S} \circ k^*_{\wedge S}(h) = \qquad \text{Equation 2.3}$$

$$\sum_{i=1}^{N} \sum_{\substack{S\subsetneq\{\nu_1,\cdots,\nu_N\}\\|S|=i}} (-1)^{i+1} h \circ k_{\wedge S} \circ r_{\wedge S} =$$

$$h \circ \left(\sum_{i=1}^{N} \sum_{\substack{S\subsetneq\{\nu_1,\cdots,\nu_N\}\\|S|=i}} (-1)^{i+1} k_{\wedge S} \circ r_{\wedge S} \right) \qquad \text{Equation 2.4}$$

Now it is not difficult to see that for any $f \in \tilde{A}E$, the element of $\tilde{A}E$,

$$\sum_{i=1}^{N} \sum_{\substack{S\subsetneq\{\nu_1,\cdots,\nu_N\}\\|S|=i}} (-1)^{i+1} k_{\wedge S} \circ r_{\wedge S}(f)$$

agrees with f on the clopen set $\downarrow\{\nu_1,\cdots,\nu_N\}$. (This is entirely analogous to the 'inclusion-exclusion' argument used to count the number of elements in a union of a finite number of finite sets!) Since $\downarrow\{\nu_1,\cdots,\nu_N\}$ is a support for h, it follows that the element of $(\tilde{A}E)^*$ in Equation 2.3 is just h, and hence that h is a sum of elements of the form $r^*_\nu(-)$. The commutivity of the diagram above then implies that Φ is surjective, and hence an isomorphism. □

Remark Statement (i) is exactly Theorem XIV2.7(i) of [Ek,Mc], but statement (ii) there requires that the constituent groups be reflexive.

We are now able to prove Theorem 2.3.9, which we restate here.

Theorem 2.3.14 *If T is a tree of nonmeasurable cardinality, then the tree sum of any tree of reflexive groups on T is itself reflexive.*

Proof The dual of the tree sum is the tree product of the dual tree, by Theorem 2.3.13 (i). By what we have just proved, if we take the dual again, we obtain the tree sum of the original tree of groups, since the constituent groups are reflexive. □.

2.4 Conclusions and Questions

It is an interesting observation that reflexive groups seem to be associated with $\mathbb{N}$-compact spaces. For as we noticed in the introduction to this chapter, there are two classes of groups known to be reflexive (in ZFC). In the first are certain groups of the form $C(X, \mathbb{Z})$ (and their duals.) The spaces X are $\mathbb{N}$-compact and Mrówka's Theorem 2.1.2 is used to establish the reflexivity of the group. In the second there are the tree sums of free groups (and their duals.) We have just seen how the duals of the groups in this second class can be seen as groups of global sections of a sheaf on an $\mathbb{N}$-compact space. In both cases the essential element in the proof of the reflexivity is Mrówka's Theorem or it's generalized version Theorem 2.2.2. We wonder if this connection reflects some deeper facts!

We do not know if one gains any more generality when nonspatial frames are used to construct reflexive groups as groups of global sections. One can show, although we do not do this here, that if $\mathbb{Z}_L E$ is a reflexive group for a H-$\mathbb{N}$-compact frame L, then it is isomorphic to the group of continuous functions, $C(\Sigma L, \mathbb{Z})$, and is hence a group of global sections on a spatial frame. But what about arbitrary sheaves $A \in \text{AbSh}L$? If AE is reflexive can it be seen as a group of global sections of a sheaf on a spatial frame (in a non-trivial way?)

Eda and Ohta proved that when X is an $\mathbb{N}$-compact $k_{\mathbb{N}}$-space, the group $C(X, \mathbb{Z})$ is reflexive, which can be viewed as result about the group of global sections of the sheaf $\mathbb{Z}_{\mathcal{O}(X)}$. It is not difficult to use Theorem 2.2.2 to show that given a space X of this sort, any *subsheaf* of $\mathbb{Z}_{\mathcal{O}(X)}$ has reflexive group of global sections. We do not know if there are groups of global sections of such subsheaves which are not groups of continuous functions. We conjecture that there are.

Chapter 3

Realcompact Frames

3.1 Introduction

In this chapter we take some of the lessons learned in our earlier investigation of N-compact frames and apply them to the study of realcompactness for frames. This has been discussed in the literature, in which 'realcompact' means the analogue of 'Stone-N-compact.' However, as we have seen in Chapter 1 Herrlich-N-compactness is the notion better suited for discussions of frame N-compactness. Here we pursue a course parallel to that of the earlier chapter, developing enough of the theory of 'H-realcompact' frames to justify proposing that these be thought of as the realcompact frames.

We begin with a discussion of realcompact spaces. The following is perhaps the most familiar of several definitions of these.

Definition 3.1.1 *A realcompact space is one homeomorphic to a closed subspace of* $\mathbb{R}^I$ *for some index set I.*

The study of realcompact spaces goes back to the independent works of Hewitt and Nachbin in 1947-1948, (hence the fairly common term 'Hewitt-Nachbin space,') and has been furthered by many authors since then. These spaces important in several regards; they play a role in the study of the rings $C(X)$ analogous to that played by compact spaces in the study of $C^*(X)$, and moreover, (barring the existence of a measurable cardinal) they are exactly the spaces admitting a complete uniformity. For a proof of this last statement and a thorough look at realcompact spaces, see [We]. We list a few of the more important statements equivalent to the definition.

Definition 3.1.2 *Let X be a topological space. An ideal in Coz(X) is* σ-**proper** *if it contains no countable covers of X, and* **completely proper** *if it does not cover X.*

Theorem 3.1.3 (cf. [We]) *If X is a Tychonoff space, these are equivalent;*

(i) *X is realcompact.*

(ii) *If X is dense and C-embedded in a Tychonoff space Y, then $X = Y$.*

(iii) *Every maximal ideal in $Coz(X)$ which is σ-proper is completely proper.*

(iv) *If $C(X) \xrightarrow{h} \mathbb{R}$ is a ring homomorphism, then h is the evaluation map for some point $p \in X$; ($h(\phi) = \phi(p)$ for any $\phi \in C(X)$.)*

The study of realcompact *frames* got its start in Reynolds' 1979 paper [Rey], entitled 'On the spectrum of a real representable ring.' Reynolds was primarily interested in realcompact topoi, but certain statements given there restrict to statements about frames; for a given frame L, the topos $Sh(L)$ is realcompact iff $L \cong \sigma - \mathrm{Idl}(\mathrm{Coz}(L))$. (Theorem B and Theorem 2.5 of [Re].) In [Ma,Ve], Madden and Vermeer proved Theorem 3.1.5, for which we need the following definition.

Definition 3.1.4 *A frame L is a* $C_{\mathbb{R}}$-**quotient** *of a frame M, via a quotient map ϕ, if any frame homomorphism $\mathcal{O}(\mathbb{R}) \xrightarrow{\rho} L$ factors through ϕ.*

Theorem 3.1.5 *(*) For a completely regular frame L, the following are equivalent;*

(i) *If L is a dense $C_{\mathbb{R}}$-quotient of a completely regular frame M, then the quotient map is an isomorphism.*

(ii) *L is a closed quotient of $\mathcal{O}(\mathbb{R})^{(I)}$ for some index set I.*

(iii) *L is Lindelöf.*

(iv) *$L \cong \sigma\text{-}Idl(Coz(L))$.*

The statements (i) and (ii) in Theorem 3.1.5 are clearly the natural frame-theoretic analogues of (i) and (ii) in Theorem 3.1.3.

We pause to note that of the equivalences in 3.1.5, (iv)→(iii) and (ii)→(iii) require choice principles, the latter because it employs Dowker and Strauss' result [Do,St] that a coproduct of regular Lindelöf frames is Lindelöf. (See Proposition 1.3.4.) In fact (ii) → (iii) is *equivalent* to Countable Choice. For any $\mathcal{O}(\mathbb{N})$ is a closed quotient of $\mathcal{O}(\mathbb{R})$, and hence any S-N-compact frame is a closed quotient of $\mathcal{O}(\mathbb{R})^{(I)}$ for some I. Thus if the latter frames are Lindelöf, then so is any S-N-compact frame, a statement which implies the Axiom of Countable Choice, by Theorem 1.4.2.

Altogether then, property (ii) of Theorem 3.1.5 seems a natural frame theoretic notion of realcompactness. We will refer to this notion as **Stone-realcompactness.** However as in the N-compact case, this notion is not a conservative one: A discrete uncountable (non-measurable) space X is realcompact, but one cannot prove (in ZF) that $\mathcal{O}(X)$ is Stone-realcompact, since assuming CC, it would then be Lindelöf.

Of course this is because of the preservation of the Lindelöf property under coproducts, which is a desirable thing, but just as for we saw for N-compact frames, its consequences necessitate a change in what one views as the fundamental notion of realcompactness. (We note that one could make a definition using $L(\mathbb{R})$, the constructive frame-theoretic version of the real-numbers. But one would still have the same problem, since this frame is also Lindelöf.)

We propose a new notion of realcompactness in Section 3.2, one based on (iii) of Thm 3.1.3. We show that (assuming some choice principles) it is conservative, (Theorem 3.2.4) and go on to develop a conservative realcompactification in Section 3.3.

3.2 H-Realcompact Frames

We define the H-realcompact frames, and show that their definition generalizes the spatial definition.

We first recall the basic definitions and constructions discussed in Section 0.2.3, and make the following definitions.

Definition 3.2.1 *For a frame L, and subsets $S, T \subseteq L$, we write $S \prec\!\prec T$ if any $s \in S$ is completely below ($\prec\!\prec$) some $t \in T$.*

Definition 3.2.2 *For any frame L, an ideal $I \in \beta L$ is* **σ-proper** *if for any sequence*

$$T_1 \succ\!\succ T_2 \succ\!\succ T_3 \succ\!\succ \cdots$$

of countable subsets of I, there is some $n \in \omega$ so that $\bigvee_L T_n \neq e_L$. An ideal I is **super-σ-proper** *if any ideal $J \supseteq I$ in βL which is proper is σ-proper. The ideal I is* **completely proper** *if $\bigvee_L I \neq e_L$.*

Remark Note that the improper ideal qualifies as a super-σ-proper ideal since the condition is vacuously satisfied.

Remark This property has a more complex definition than one might suppose necessary (cf. our definition of σ-proper in 1.3.3.) However we need this to avoid an otherwise necessary use of choice principles in the construction of the realcompactification. If we assume CC, then the definition given coincides with the more obvious notion: $\bigvee_L S \neq e_L$ for any countable $S \subseteq I$. To see the non-trivial half of this, suppose that $I \in \beta L$ is σ-proper and let $S \subseteq I$ be some countable subset. Using CC, for each $u \in S$ pick an element $v_u \succ\!\succ u$ which is in I, and call the set of these T_1. Since the $\prec\!\prec$ relation interpolates, we can find a sequence of sets T_n as in Definition 3.2.2, all completely above S. As the join of one of them is not e_L, the join of S cannot be.

Definition 3.2.3 *A completely regular frame L is* **Herrlich-realcompact** *('H-realcompact') if any proper super-σ-proper ideal $I \in \beta L$ is completely proper. In other words, if $\bigvee_L I = e_L$ for a proper ideal $I \in \beta L$, there is a proper ideal $J \supseteq I$ in βL which is not σ-proper.*

Remark This definition looks more familiar if one assumes some choice principles. With CC and the BUT it would say: A frame L is H-realcompact if the following holds for any maximal $I \in \beta L$: If $\bigvee_L S \neq e_L$ for all countable $S \subseteq I$, then I is completely proper. This then resembles condition (iii) of Theorem 3.1.3.

Theorem 3.2.4 *(*) A space X is realcompact iff $\mathcal{O}(X)$ is H-realcompact.*

Proof Recall from Theorem 0.2.2 that we have an isomorphism

$$\begin{array}{ccc} \mathrm{Max}\beta\mathcal{O}(X) & \overset{\cong}{\longleftrightarrow} & \mathrm{Max}(\mathrm{Coz}(X)) \\ I & \overset{\phi}{\longrightarrow} & \{V \in \mathrm{Coz}(X) \mid \Downarrow(V) \vee (I \cap \mathrm{Coz}(X)) \neq E_{\mathfrak{I}\mathrm{Coz}(X)}\} \\ \{U \mid U \prec\!\prec V, \text{ some } V \in J\} & \overset{\psi}{\longleftarrow} & J \end{array}$$

Now suppose that X is realcompact, and that I is a proper super-σ-proper element of $\beta\mathcal{O}(X)$. There is a maximal $J \in \beta\mathcal{O}(X)$ containing I, and J is then σ-proper. It follows that $\phi(J)$ is σ-proper. For suppose otherwise, so that there are cozero sets $V_n \in \phi(J)$ with $\bigcup_n V_n = X$. By an easy argument, any co-zero set is a countable union of co-zero sets completely below it. Now, using CC to pick these covers and to see that a countable union of countable sets is countable, there is a countable subset of $\psi\phi(J) = J$ which covers X. Using CC along with our observations after Definition 3.2.2, we see that J is not σ-proper. This is a contradiction, so that $\phi(J)$ must be σ-proper and hence by hypothesis completely proper. It follows that $\psi\phi(J) = J$ is completely proper, and hence that I is completely proper.

For the converse, suppose that $\mathcal{O}(X)$ is H-realcompact and let J be a maximal σ-proper ideal in Coz(X). If $T \subseteq \psi(J)$ is countable, using CC we can find a countable $T' \subseteq J$ such that $T \prec\!\prec T'$. By hypothesis, $\bigvee_{\mathcal{O}(X)} T' \neq E_{\mathcal{O}(X)} = X$, and so $\bigvee_{\mathcal{O}(X)} T \neq X$. Thus $\psi(J)$ is σ-proper, and since $\psi(J)$ is maximal, it is super-σ-proper and thus completely proper, by hypothesis. The complete regularity of X implies that J is completely proper, so that X is realcompact. □

Thus the H-realcompact frames 'include' the realcompact spaces, at least as long as we assume some choice principles. It is not surprising that we need the Boolean Ultrafilter Theorem to show this, since the definition of H-realcompact frames is based on (iii) of Theorem 3.1.3, which explicitly mentions ultrafilters. However it is interesting that we require the Axiom of Countable Choice in such a strong way.

We will be able to show that any S-realcompact frame is H-realcompact, after we have developed the H-compactification.

3.3 The H-Real-Compactification

We show how to construct the H-realcompactification of a frame, and discuss various corollaries. Our arguments, on the whole, follow those of Section 1.5.2.

Definition 3.3.1 *Given a frame L, define*

(i) $\beta L \xrightarrow{r} \beta L$ *by,*

$$rI = \{u \leq \bigvee_L I \mid I \subseteq J \in \beta L, \textit{super-}\sigma\textit{-proper} \Rightarrow u \in J\}, \textit{ and}$$

(ii) $\beta L \xrightarrow{t} \beta L$ *by*

$$tI = \{u \mid u \prec\!\prec v, \text{some } v \in rI\}.$$

The methods of Lemma 1.5.6 apply to show that

(i) $I \subseteq rI$,

(ii) $rI \cap rJ = r(I \cap J)$, and

(iii) $r^2 I \subseteq rI$.

Using these facts one can easily show that t is a nucleus. The frame $[\beta L]_t$ is our candidate for the realcompactification, with co-reflection map $[\beta L]_t \xrightarrow{j_L} L$ the frame homomorphism taking an ideal to its join in L.

Definition 3.3.2 *For a frame L, and $u \in L$, $k_L u$ is the ideal $\{v \mid v \prec\!\prec u\}$, an element of βL.*

It is a fact that the map k_L preserves the $\prec\!\prec$-relation. (See [Ba,Mu] or [Jo] for a proof.)

Lemma 3.3.3 *If L is H-realcompact then j_L is an isomorphism.*

Proof Since L is regular, it suffices by Lemma 0.2.4 to show that j_L is onto and co-dense. The first property holds since for any $u \in L$, $k_L u$ is t-closed, and $j_L k_L u = u$ by complete regularity. Towards seeing the second, suppose it does not hold, so that $j_L I = \bigvee_L I = e_L$, but I is proper. Since I is t-closed, there must be a proper super-σ-proper $J \in \beta L$ containing I, by the definition of t. Since L is H-realcompact, J and thus I are completely proper, a contradiction. So j_L is co-dense and hence an isomorphism. □

Lemma 3.3.4 *For any frame L, $[\beta L]_t$ is H-realcompact.*

Proof We first show that $\beta[\beta L]_t$ and βL are isomorphic, via,

$$\beta[\beta L]_t \underset{\lambda_L}{\overset{\beta(j_L)}{\rightleftarrows}} \beta L$$

where

(i) $\beta(j_L)(\mathfrak{J}) = \{w \in L \mid w \leq j_L(V), \text{some } V \in \mathfrak{J}\}$ and

(ii) $\lambda_L(I) = \{V \in [\beta L]_t \mid V \leq k_L u, \text{some } u \in I\}$.

That $\lambda_L(I)$ is indeed in $\beta[\beta L]_t$ follows quickly from the following observations:

- If $u \in L$, $k_L u$ is t-closed and thus in $[\beta L]_t$.
- For any u,$v \in L$, $k_L u \vee k_L v \leq k_L(u \vee v)$.
- The map k_L preserves the $\prec\!\prec$ relation, as does the frame homomorphism $\beta L \xrightarrow{t} [\beta L]_t$.

An easy argument using the complete regularity of I establishes that $\beta(j_L) \circ \lambda_L(I) = I$, for any $I \in \beta L$, so that $\beta(j_L)$ is onto. Since $\beta(j_L)$ is a dense map with a compact range, it is 1-1, by Lemma 0.2.4. So $\beta(j_L)$ is an isomorphism with inverse λ_L.

Now towards our goal of showing that $[\beta L]_t$ is H-realcompact, suppose that $\mathfrak{J} \in \beta[\beta L]_t$ is proper and super-σ-proper . We first show that $\beta(j_L)(\mathfrak{J})$ is t-closed and proper, and then that $\mathfrak{J} \subseteq k_{[\beta L]_t}(\beta(j_L)(\mathfrak{J}))$. We can then conclude that $\mathfrak{J}$ is completely proper.

Since $\mathfrak{J}$ is proper, $\beta(j_L)(\mathfrak{J})$ is proper, as $\beta(j_L)$ is an isomorphism. To show that $\beta(j_L)(\mathfrak{J})$ is t-closed, it suffices to show that it is super-σ-proper . Towards this, let

$K \supseteq \beta(j_L)(\mathfrak{J})$ be some proper ideal in βL. Then $\lambda_L(K) \supseteq \mathfrak{J}$ is a proper and therefore super-σ-proper ideal.

Towards a contradiction, suppose that K is not σ-proper. Then there is a sequence

$$T_1 \succ\!\succ T_2 \succ\!\succ T_3 \cdots$$

of countable subsets of K such that $\bigvee_L T_n = e_L$ for every n. Since k_L preserves the $\succ\!\succ$-relation, we obtain a sequence

$$\mathfrak{S}_1 \succ\!\succ \mathfrak{S}_2 \succ\!\succ \mathfrak{S}_3 \cdots$$

of countable subsets of $\lambda_L(K)$, where $\mathfrak{S}_n = \{k_L u \mid u \in T_n\}$. Since $\lambda_L(K)$ is σ-proper, for some $N \in \omega$, $\bigvee_{[\beta L]_t} \mathfrak{S}_N \neq E_{[\beta L]_t}$. Then

$$\begin{aligned}
E_{[\beta L]_t} &\neq t(\bigvee_{\beta L} \mathfrak{S}_N) \\
&= \{u \mid u \prec\!\prec v, v \in r(\bigvee_{\beta L} \mathfrak{S}_N)\}, \text{so that,} \\
E_{[\beta L]} &\neq r(\bigvee_{\beta L} \mathfrak{S}_N) \\
&= \{u \leq \bigvee_L \bigvee_{\beta L} \mathfrak{S}_N \mid \bigvee_{\beta L} \mathfrak{S}_N \subseteq G \text{ super-}\sigma\text{-proper } \Rightarrow u \in G\} \\
&= \{u \in L \mid \bigvee_{\beta L} \mathfrak{S}_N \subseteq G \text{ super-}\sigma\text{-proper } \Rightarrow u \in G\}
\end{aligned}$$

since $\bigvee_L T_{N+1} = E$ and $T_{N+1} \subseteq \bigvee_{\beta L} \mathfrak{S}_N$. Thus there must be a proper super-σ-proper (and therefore σ-proper) ideal G so that $\bigvee_{\beta L} \mathfrak{S}_N \subseteq G$. But $\bigvee_{\beta L} \mathfrak{S}_N \supseteq T_l$ for $l \geq N+1$, so that G cannot be σ-proper. This is a contradiction, so that K must be σ-proper, as we had hoped. Hence $\beta j_L(\mathfrak{J})$ is t-closed.

Finally, we must show that $\mathfrak{J} \subseteq k_{[\beta L]_t}(\beta(j_L)(\mathfrak{J}))$. Towards this, suppose that $I \in \mathfrak{J}$. We must show that $I \prec\!\prec \beta(j_L)(\mathfrak{J})$. There is a $K \in \mathfrak{J}$ with $I \prec\!\prec K$, and if $w \in K$ then $w \leq j_L(K)$, so that $K \leq \beta(j_L)(\mathfrak{J})$. Thus $I \prec\!\prec \beta(j_L)(\mathfrak{J})$. □ (Lemma 2.3.4.)

We now have a way of making arbitrary frames L into H-realcompact frames $[\beta L]_t$. In order to make this functorial, we show the following:

Lemma 3.3.5 *For any frame morphism $M \xrightarrow{\phi} L$, the map*

$$\begin{array}{rcl}[\beta M]_{t_M} & \xrightarrow{\overline{\phi}} & [\beta L]_{t_L} \\ I & \longmapsto & t_L(\beta\phi(I))\end{array}$$

is a frame homomorphism.

Proof It is obvious that $\overline{\phi}$ preserves finite meets, since both t_L and $\beta\phi$ do. To see that it transfers arbitrary joins, we have only to show that

$$\overline{\phi}I \subseteq \bigvee_{[\beta L]_t} \overline{\phi}(I_\alpha),$$

when $I = \bigvee_{[\beta M]_t} I_\alpha$, since $\overline{\phi}$ is order-preserving. (To simplify the notation we will suppress mention of an index set for the indices α.) First note that

$$\begin{aligned}\bigvee_{[\beta L]_t} \overline{\phi}(I_\alpha) &= \bigvee_{[\beta L]_t} t_L(\beta\phi(I_\alpha)) \\ &= t_L(\bigvee_{\beta L} t_L(\beta\phi(I_\alpha)) \\ &= t_L(\bigvee_{\beta L}(\beta\phi(I_\alpha)) \\ &= \{u \mid u \prec\!\prec v \text{ some } v \in r_L(\bigvee_{\beta L} \beta\phi(I_\alpha))\}\end{aligned}$$

On the other hand,

$$\begin{aligned}\overline{\phi}I &= t_L(\beta\phi(\bigvee_{[\beta M]_t} I_\alpha)) \\ &= \{u \mid u \prec\!\prec v, \text{ some } v \in r_L(\beta\phi(\bigvee_{[\beta M]_t} I_\alpha))\}\end{aligned}$$

so it enough to show that

$$r_L(\beta\phi(\bigvee_{[\beta M]_t} I_\alpha)) \subseteq r_L(\bigvee_{\beta L} \beta\phi(I_\alpha)) \tag{3.1}$$

Fix v in the left-hand side of Inequality 3.1. From the definition of r we see that we have two criteria to verify in order to see that v is in the right-hand side. For the first, we have

$$\begin{aligned}
v &\in r_L\beta\phi(\bigvee_{[\beta M]_t} I_\alpha) \\
&= r_L(\beta\phi(t_M(\bigvee_{\beta M} I_\alpha)). \text{ Then,} \\
v &\leq \bigvee_L \beta\phi(t_M(\bigvee_{\beta M} I_\alpha)) \\
&= \bigvee_L \phi[t_M(\bigvee_{\beta M} I_\alpha)] \\
&= \phi(\bigvee_M t_M(\bigvee_{\beta M} I_\alpha)) \\
&= \phi(\bigvee_M(\bigvee_{\beta M} I_\alpha)) \text{ (since } t_M(-) \subseteq \downarrow\bigvee_M(-)) \\
&= \bigvee_L \phi[\bigvee_{\beta M} I_\alpha] \\
&= \bigvee_L \beta\phi(\bigvee_{\beta M} I_\alpha) \\
&= \bigvee_L \bigvee_{\beta L} \beta\phi(I_\alpha),
\end{aligned}$$

so v satisifies the first criterion for membership in the right-hand side of Inequality 3.1. To see that it satisfies the second, suppose that $\bigvee_{\beta L}\beta\phi(I_\alpha) \subseteq H$, a super-$\sigma$-proper element of βL. We must show that $v \in H$. If we can show that $\phi[r_M(\bigvee_{\beta M} I_\alpha)] \subseteq H$, then $\beta\phi(r_M(\bigvee_{\beta M} I_\alpha)) \subseteq H$, so that $v \in H$, by hypothesis on v.

To obtain $\phi[r_M(\bigvee_{\beta M} I_\alpha)] \subseteq H$, we assume in the non-trivial case that H is proper. Fix $u \in r_M(\bigvee_{\beta M} I_\alpha)$ and let $K = \bigvee_{\beta M}\{J \mid \beta\phi(J) \subseteq H\}$. Then K is proper as H is. We claim that if $K' \in \beta M$ is proper and contains K, then $\beta\phi(K') \vee H$ is proper. For otherwise there are $w \in K'$ and $p \in H$ so that $\phi(w) \vee p = e_L$. Then

$$\begin{aligned}\phi(w^{**}) \vee p &= e_L, \text{ so that,}\\ \phi(w^{*}) &= \phi(w^{*}) \wedge (\phi(w^{**}) \vee p)\\ &= \phi(w^{*}) \wedge p \text{ so,}\\ \phi(w^{*}) &\leq p.\end{aligned}$$

Thus $\phi(w^*) \in H$. This implies $k_M w^* \subseteq K$, by definition of K. Now since K' is completely regular, we can find $\overline{w} \in K'$ with $\overline{w} \succ\!\succ w$. Then $\overline{w}^* \prec\!\prec w^*$, so that $\overline{w}^* \in k_M w^*$ and hence $\overline{w}^* \in K \subseteq K'$. So both $\overline{w}$ and $\overline{w}^*$ are in K'. But this cannot happen in our proper completely regular ideal K'. For there is an $s \in K'$ with $\overline{w} \prec\!\prec s$, so that $\overline{w} \prec s$, with separating element c, say. Then $c \wedge \overline{w} = 0$, so that $c \leq \overline{w}^*$, and hence $s \vee \overline{w}^* = e$, contradicting our hypothesis that K' is proper.

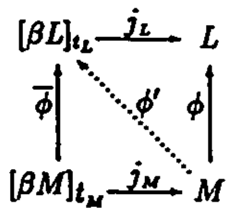
So it must be that $\beta\phi(K') \vee H$ is proper. Since it contains H it is σ-proper, which then implies that K' is σ-proper as well, by a straightforward argument, and altogether that K is super-σ-proper . Now by the definition of K and the hypothesis on H, $K \supseteq \bigvee_{\rho M} I_\alpha$. Thus $u \in K$, by the hypothesis on u, so that $\phi u \in \phi[K] \subseteq H$, as desired.

We have shown that Inequality 3.1 holds, which finishes the proof. □

We can now prove the

Theorem 3.3.6 *For a given frame L, the map $[\beta L]_t \xrightarrow{j_L} L$ is universal as a map from H-realcompact frames to L.*

Proof Suppose we are given an H-realcompact M and frame homomorphism $M \xrightarrow{\phi} L$. We can form



and using the form of $\overline{\phi}$ given in Lemma 3.3.5, see without difficulty that the outer square commutes. Since j_M is an isomorphism (Lemma 3.3.3,) we can find a map ϕ' making the upper triangle commute. This map is unique since j_L is dense, and therefore monic (Lemma 0.2.4.) □

Definition 3.3.7 *The full sub-category of H-realcompact frames is denoted $H\mathbf{R}K\mathsf{Frm}$.*

Theorem 3.3.6 provides us with a coreflection from Frm to $H\mathbf{R}K\mathsf{Frm}$, which we denote v_H.

Corollary 3.3.8 *The subcategory $H\mathbf{R}K\mathsf{Frm}$ is closed under coproducts and closed quotients.*

Proof Coreflective subcategories are closed under all colimits, and in particular under coproducts. To see that $H\mathbf{R}K\mathsf{Frm}$ is closed under closed quotients, we reason as follows.

Let L be an H-realcompact frame, and u some element of L. Denote the map $L \to \uparrow u$ taking v to $v \vee u$ by ϕ. Recall that the induced map $\beta L \to \beta\uparrow(u)$ is written $\beta\phi$. Let H be some proper super-σ-proper ideal in $\beta\uparrow(u)$. We must show that H is completely proper.

Let $K = \bigvee_{\beta L}\{J \mid \beta\phi(J) \subseteq H\}$, a proper ideal, since H is proper. By reasoning just as we did in the latter half of Lemma 2.3.5, we can conclude that K is super-σ-proper , and thus completely proper, by hypothesis on L. Now if $v \in H$, the map ϕ takes an element of $k_L v = \{w \in L \mid w \prec\!\prec_L v\}$ into H. It follows that $k_L v \subseteq K$, for any $v \in H$. Since K is completely proper and L is completely regular, H must be completely proper. □

Corollary 3.3.9 *Any Stone-realcompact frame is Herrlich-realcompact.*

Proof By the previous corollary we need only to show that $\mathcal{O}(\mathbf{R})$ is H-realcompact. Suppose that $I \in \beta\mathcal{O}(\mathbf{R})$ is proper but not completely proper, so that $\bigvee_{\mathcal{O}(\mathbf{R})} I = e_{\mathcal{O}(\mathbf{R})} = \mathbf{R}$. Since $\mathcal{O}(\mathbf{R})$ has a basis consisting of the open intervals with rational endpoints, there is a countable subset $T_1 = \{s_n \mid n \in \omega\} \subseteq I$, consisting of such intervals, such that $\bigvee_{\mathcal{O}(\mathbf{R})} T_1 = \mathbf{R}$. Now each s_n is an open subset of $\mathbf{R}$ and therefore the union of the intervals with rational endpoints which are completely inside it. These form a countable set, (say R_n.) We can then take $T_2 = \dot{\bigcup}_n R_n$, which is countable as it is a subset of the countable set of all intervals with rational endpoints. Repeating this process, we obtain

$$T_1 \succ\!\succ T_2 \succ\!\succ T_3 \succ\!\succ \cdots$$

with $\bigvee_{\mathcal{O}(\mathbf{R})} T_n = e_{\mathcal{O}(\mathbf{R})}$ for any n, so that I is not σ-proper. Since it is proper, it is not super-σ-proper . □

Remark It is easy to find H-realcompact frames which are not S-realcompact. One need only take an space X which is realcompact but not Lindelöf, for example ω_1 with the discrete topology. (Theorems 3.1.5 and 3.2.4.)

We show that the H-realcompactification is conservative.

Theorem 3.3.10 *(*) The frames $v_H\mathcal{O}(X)$ and $\mathcal{O}(vX)$ are isomorphic.*

Proof If we can show that $v_H\mathcal{O}(X)$ is spatial, then the co-universal properties of $v_H\mathcal{O}(X) \longrightarrow \mathcal{O}(X)$ and $\mathcal{O}(vX) \longrightarrow \mathcal{O}(X)$ imply the result.

Since $v_H\mathcal{O}(X)$ is regular we need only show that any nonzero element is dominated by a maximal element (= prime element.) If $E \neq I \in v_H\mathcal{O}(X)$ then one of the following holds:

(i) $\bigvee_L I = e$. In this case, since I is t-closed, $I \subseteq J$ for some proper super-σ-proper $J \in \beta\mathcal{O}(X)$. We can expand J to J', a maximal element of $\beta\mathcal{O}(X)$ which is then also σ-proper and thus t-closed, and so a maximal element of $v_H\mathcal{O}(X)$.

(ii) $\bigvee_L I \neq E$. Then $\bigvee_L I \leq p$ for p some maximal element of $\mathcal{O}(X)$. Then $I \subseteq \{v \mid v \prec\!\prec p\}$, a maximal element of $v_H\mathcal{O}(X)$. □

3.4 Unsolved Problems

In Chapter 1 we saw how the conditions of Theorem 1.1.4 separate into two logical equivalence classes, (1)↔(2), and (4)↔(5). We suspect that the analogous result is true for realcompact frames. In fact we have already seen that conditions (i) and (ii) of Theorem 3.1.3 are equivalent in the frame setting, (Theorem 3.1.5.) Is the following the case?

Conjecture 3.4.1 *A completely regular frame L is H-realcompact iff any ring homomorphism $\mathsf{R}_L E \xrightarrow{\phi} \mathsf{R}$ is 'evaluation at a prime element p' of L. (Where the definition of 'evaluation at a prime' is the analogue of Definition 1.5.20.)*

We furthermore conjecture that there are results for realcompact frames entirely analogous to the propositions 1.5.17, 1.5.18, and 1.5.19.

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