PRESHEAF TOPOSES AND PROPOSITIONAL LOGIC

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A Thesis
Submitted to the School of Graduate Studies
in Partial Fulfilment of the Requirements
for the Degree
Doctor of Philosophy
McMaster University

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DOCTOR OF PHILOSOPHY (1989) McMaster University
(Mathematics) Hamilton, Ontario

TITLE: Presheaf Toposes and Propositional Logic

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NUMBER OF PAGES: vi, 604.

To my wife Maria

to my daughter Myrna

ABSTRACT

To each non-degenerate topos $\underline{\mathcal{E}}$ we associate $L(\underline{\mathcal{E}})$, the intermediate propositional logic (i.p.l.) consisting of those polynomials, built up from variables, 0, 1, $\wedge$, $\vee$ and $\Rightarrow$, which become valid when interpreted in the natural internal Heyting algebra structure of the subobject classifier of $\underline{\mathcal{E}}$. For various polynomials φ, we give first order characterizations for the class of small categories determined by the condition $\varphi \in L(\underline{\mathcal{S}}^{\underline{C}^{0}})$ - where $\underline{\mathcal{S}}$ is the topos of sets and $\underline{C}$ ranges over small categories. A basic example is:

$v \vee \neg v \in L(\underline{\mathcal{S}}^{\underline{C}^{0}})$ iff $\underline{C}$ is a groupoid.

The presheaf topos $\dot{\underline{\mathcal{E}}}_1$, of actions of a single idempotent, whose i.p.l. is given by:

$\varphi \in L(\dot{\underline{\mathcal{E}}}_1)$ iff $\neg u \vee v \vee (v \Rightarrow u) \vdash \varphi$,

is used to exemplify a relativization of the basic example, to toposes. For each topos $\underline{\mathcal{E}}$ we introduce $\Gamma(\underline{\mathcal{E}})$, the set of all polynomials φ such that for all internal categories $\underline{C}$ of $\underline{\mathcal{E}}$, if $\varphi \in L(\underline{\mathcal{E}}^{\underline{C}^{0}})$ then $\underline{C}$ is a groupoid. Using a theorem of Jankov's we can compute $\Gamma(\underline{\mathcal{S}}^{\underline{M}^{0}})$ when $\underline{M}$ belongs to a certain class of finite monoids that includes semilattices; in particular: $\varphi \in \Gamma(\dot{\underline{\mathcal{E}}}_1)$ iff $\varphi \vdash u \vee (u \Rightarrow (v \vee \neg v))$. This polynomial - the cogenerator of $\Gamma(\dot{\underline{\mathcal{E}}}_1)$ - is strictly weaker than the generator of $L(\dot{\underline{\mathcal{E}}}_1)$.

We use the higher order type theoretical language of a topos $\underline{\mathcal{E}}$, developed in the first half of the thesis, to establish that for an

extensive class of polynomials, the condition $\varphi \in \Gamma(\underline{\mathcal{E}})$ can be internalized; that is, we can define a formula $\varphi^{\#}$, of the language of $\underline{\mathcal{E}}$, such that: $\varphi \in \Gamma(\underline{\mathcal{E}})$ iff $\underline{\mathcal{E}} \models \varphi^{\#}$.

This theorem has as particular cases:

(1) $v \vee \neg v \in \Gamma(\underline{\mathcal{E}})$

(2) $\neg \neg v \vee \neg v \in \Gamma(\underline{\mathcal{E}})$ iff $\underline{\mathcal{E}} \models \neg \forall v(\neg \neg v \vee \neg v)$

(3) $\neg u \vee \forall v (v \Rightarrow u) \in \Gamma(\underline{\mathcal{E}})$ iff $u \vee (u \Rightarrow (v \vee \neg v)) \in \Gamma(\underline{\mathcal{E}})$ iff $\underline{\mathcal{E}} \models \neg \forall v(v \vee \neg v)$.

ACKNOWLEDGEMENTS

I wish to express my appreciation to my supervisor Professor Banaschewski. On several occassions when I had all but given up hope that my thesis would be accepted, he revived me and encouraged me to complete my work.

My greatest debt is to my wife. She has supported me financially over lengthy periods, after the financial support provided by McMaster University had run out. She has given of her own time to type most of the thesis. She has patiently endured my struggles to overcome many discouragements, obstacles, and those periods when, out of necessity, I abandoned work on my thesis. She has always encouraged my - often faltering - belief in the value of my work.

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INTRODUCTION

In this introduction we shall first legitimize the categorical constructions to be found in chapter 1 from a set theoretic viewpoint. We then trace through the main argument of the thesis using a simplest possible example. Finally we shall outline briefly the contents of each section of chapter 1. Although we touch on the language of a topos, detailed discussion will be left to the introduction to the language within chapter 0.

The possibility of an illegitimate construction arises in Gödel-Bernays set theory if we apply the functor category construction: $[\underline{C},\underline{D}]$ to categories $\underline{C}$ and $\underline{D}$ without stipulating that they be sets. For example if both $\underline{C}$ and $\underline{D}$ are equivalent, as categories, to the one object category, and $\underline{D}$ has exactly two objects then the cardinality of $|[\underline{C},\underline{D}]|$ should be 2^{γ} where γ is the cardinality of $|\underline{C}|$. Such a construction is precluded if $|\underline{C}|$ has no cardinality.

A remedy for all constructions we use is available in the axiom system for universes; the method is well-established and so details will be supressed. We shall follow the definitions of [Sch], which are a slight modification of the original definitions of [AGV]. We need first a universe $\underline{U}$ containing the natural numbers; we use both $\mathcal{S}et$ and $\underline{\mathcal{S}}$ to denote the category of sets determined by $\underline{U}$. Categories in $\mathcal{S}et$ (that is, categories whose morphism set and object set are both elements of $\underline{U}$) we call small. The universe $\underline{U}$ is an element of a larger universe $\underline{V}$. For any small category $\underline{C}$ the presheaf category $\underline{\mathcal{S}}^{\underline{C}^0}$ is in $\underline{V}$ and the whole set $\{\underline{\mathcal{S}}^{\underline{C}^0} | \underline{C} \text{ small}\}$, since it is indexed by an element of $\underline{V}$, is again an element of $\underline{V}$.

A topos in the original sense ([AGV]), was given relative to a universe $\underline{U}$, thus it was, strictly speaking, a $\underline{U}$-topos. The theorem of Giraud established that a $\underline{U}$-topos could be construed as a full reflective subcategory of $\underline{\mathcal{S}}^{\underline{C}^0}$, for some small $\underline{C}$, with a left exact reflector. Thus toposes were grounded in $\underline{\mathcal{S}}$.

The new perspective of Lawvere and Tierney in one sense, removed $\mathcal{S}et$, in that the first order definition, while including the previous toposes, contained no explicit reference to $\mathcal{S}et$; and although an actual model of the axioms may belong to a universe $\underline{V}$, no representation of Giraud's type can be inferred. Yet the new perspective, if not permitting us to view a topos as being built from $\mathcal{S}et$, recaptured "the category of sets" in a metaphorical sense. The very concepts that appeared in the definition: the subobject classifier and cartesian closedness were present in Lawvere's "elementary theory of the category of sets" cited by Lawvere in [L3] as one of the influences that gave rise to the new definition. Moreover the study of how syntactical operations within logic: "quantifiers" and "the comprehension schema" could be construed within category theory ([L2]), made clear, once the new perspective became known, how, by reversing the procedure, a logical language could be recovered from a topos (such as that developed in Chapter 0) -so that one could see exactly how the set metaphor functioned. Of course $\mathcal{S}et$ itself was included under the new definition; thus it was only when the topos deviated from $\mathcal{S}et$ that it made sense to speak of metaphor.

Introductory expositions of the new toposes have maintained a point of comparison, on questions of logic, between models of the topos axioms (in which the logic is not given explicitly) and models of the axioms for set theory: all of the latter satisfy the law of excluded middle whereas the former in general satisfy the set of axioms for the intuitionistic propositional logic, IL. In particular the free topos of Lambek and P. Scott satisfied only those formulas of the propositional logic which are deducible in IL. The recognition that the propositional logic was a feature of toposes which deserved study, appears to be present with the introduction of "Boolean" toposes, singled out early on by Lawvere. However only Johnstone's papers on the De Morgan Law [J4] (the first of four papers) recognize the possibility of developing logical consequences for a topos satisfying a strictly intermediate rule, namely the De Morgan Law.

In this thesis we intend to explore the intermediate ground of propositional logic between KL and IL. The main questions we pose have to do with what we shall call "converse logics". They are very specific questions, that actually lead us out of the strict propositional logic into a class of purely logical formulas involving in an essential way universal quantification over propostional variables. We now explain what is meant by "the propositional logic of a topos"; the topic is covered in further detail in 1.1.

Our approach to the development of the syntax of languages relies on the concept of an absolutely free algebra, of an appropiate similarity type, generated by a set of variables. The similarity type H associated with propositional logic consists of two nullary and three binary operation signs (for which we uses the infixes: $\vee$, $\wedge$, and $\Rightarrow$). From a countable set V of variables we generate the absolutely free H-algebra Poly H. For a topos $\underline{\mathcal{E}}$ the set of formulas of its language can be construed as an

H-algebra by interpreting the operation signs as applications of morphisms: $0 \longrightarrow \Omega$ and $\Omega^2 \longrightarrow \Omega$ to formulas which, informally, we may call "true", "false", "and", "or" and "implies". The universal property of the pair (V,Poly H) implies that we have an H-homomoprhism from Poly H to the set of formulas of the language determined by assigning the i-th variable of type Ω to the i-th variable of V. As long as the topos is non-degenerate this H-homomorphism will be an embedding.

Once the meaning of validity of a formula of the language of a topos $\underline{\mathcal{E}}$ is given, we can define the propositional logic of the topos $\underline{\mathcal{E}}$ to be the subset $L(\underline{\mathcal{E}}) \subset$ Poly H consisting of all those polynomials φ such that $\overline{\varphi}$, the value of φ under the embedding, is valid. When $\underline{\mathcal{E}}$ is non degenerate $L(\underline{\mathcal{E}})$ is an intermediate propositional logic lying above IL and below KL:

$$\mathrm{IL} \subset L(\underline{\mathcal{E}}) \subset \mathrm{KL}.$$

On the set Poly H there is a binary deducibility relation $\vdash$ which can be interpreted semantically by

$$\psi \vdash \varphi \qquad \text{iff}$$

φ is valid in all Heyting algebras within which ψ is valid.

The class of intermediate propostional logics which can be represented as $L(\underline{\mathcal{S}}^{\underline{C}^0})$ for some small category $\underline{C}$ includes IL as well as all tabular logics. As will become clear in 1.7 this class is the same as that determined by those Heyting algebras which can be represented as the increasing subsets of some partially ordered set. The survey paper [Kuz] states that it is unknown whether this class (determined by "Kripke frames") constitutes all intermediate propositional logics.

An observation due to Freyd [Fr] will serve as a point of departure for investigations begun in 1.2.2 and 1.2.5; namely

(1) $v \vee \neg v \in L(\underline{S}^{\underline{C}^0})$ iff $\underline{C}$ is a groupoid.

We shall, in 1.2.3, give first order characterizations of categories $\underline{C}$ for which (1) holds when $v \vee \neg v$ is replaced by a polynomial which generates a strictly intermediate logic, for several polynomials. In a second direction, we begin, in 1.2.5, an analysis of (1) when $v \vee \neg v$ is replaced by a polynomial, $\underline{S}$ is replaced by a topos and $\underline{C}$ remains a groupoid. We shall trace out this second line of thought to its culmination in 1.7 by considering a minimal example of a category which fails to be a groupoid. We take $\underline{M}_1$ to be a category with a single object and two morphisms: the identity and an idempotent; we take $\dot{\underline{\mathcal{E}}}_1$ to be the topos of presheaves on $\underline{M}_1$, so that, by (1), $v \vee \neg v \notin L(\dot{\underline{\mathcal{E}}}_1)$. The Heyting algebra structure $\underset{\sim}{\Omega}$ on the subobject classifier for $\dot{\underline{\mathcal{E}}}_1$ is determined by the external Heyting algebra of "right ideals" of $\underline{M}_1$. This is a three element chain, $\mathbf{3}$, from which we can compute: $L(\dot{\underline{\mathcal{E}}}_1) \equiv \{\varphi \mid \mathbf{3} \vdash \varphi\} \equiv \{\varphi \mid R_2 \vdash \varphi\}$ where $R_2 \equiv \neg u \vee v \vee (v \Rightarrow u)$. (The symbol "$\equiv$" denotes ordinary "external" equality, in contrast to the "internal" equality "$=$" of formal languages). Among intermediate propositional logics, this is the largest proper sublogic of the classical logic. More general computations of this kind will be carried out in 1.2.2.

We wish to pose a question that goes beyond the point of view grounded in $\underline{S}$. We want to discover to what extent the "axiom" R_2 plays a role in $\dot{\underline{\mathcal{E}}}_1$ analogous to the role played by $v \vee \neg v$ in $\underline{S}$-as shown in (1).

Before proceeding further we shall take account of a quite general fact which will lead to a refinement of what we mean by "role". We can show that for any topos $\underline{\mathcal{E}}$, internal category $\underline{C}$, and polynomial φ,

(2) if $\underline{C}$ is a groupoid and $\varphi \in L(\underline{\mathcal{E}})$ then $\varphi \in L(\underline{\mathcal{E}}^{\underline{C}^0})$

This fact implies one half of the equivalence of (1) namely:

if $\underline{C}$ is a groupoid then $v \vee \neg v \in L(\underline{\mathcal{S}}^{\underline{C}^0})$

as well as

if $\underline{C}$ is a groupoid (in $\dot{\underline{\mathcal{E}}}_1$) then $R_2 \in L(\dot{\underline{\mathcal{E}}}_1^{\underline{C}^0})$.

A first question we now pose is the analogue to the other half of the equivalence of (1): does $R_2 \in L(\dot{\underline{\mathcal{E}}}^{\underline{C}^0})$ imply that $\underline{C}$ is a groupoid? It turns out that it does, however ideas exposed in investigating this question further, by changing R_2 and $\dot{\underline{\mathcal{E}}}_1$, suggest that we should introduce, for a fixed topos $\underline{\mathcal{E}}$, the extension to all such formulas: that is define, for $\underline{C}$ ranging over all internal categories of $\underline{\mathcal{E}}$ and φ over all polynomials, a subset $\Gamma(\underline{\mathcal{E}}) \subset \text{Poly } H$ by:

$\varphi \in \Gamma(\underline{\mathcal{E}})$ iff

for all $\underline{C}$, if $\varphi \in L(\underline{\mathcal{E}}^{\underline{C}^0})$ then $\underline{C}$ is a groupoid.

The set so defined has a property enjoyed by similar sets that we shall introduce: it is closed under the converse of $\vdash$ in the sense that,

if $\varphi \in \Gamma(\underline{\mathcal{E}})$ and $\psi \vdash \varphi$ then $\psi \in \Gamma(\underline{\mathcal{E}})$.

Taking into account our description of $\dot{\underline{\mathcal{E}}}_1$ it is possible to calculate: $\Gamma(\underline{\mathcal{S}}) = \{\varphi | \varphi \vdash v \vee \neg v\}$; that is, $\Gamma(\underline{\mathcal{S}})$ is cogenerated (generated in the converse sense) by $v \vee \neg v$. However the analogy breaks down for $\dot{\underline{\mathcal{E}}}_1$: the generator of $L(\dot{\underline{\mathcal{E}}}_1)$ is not the cogenerator of $\Gamma(\dot{\underline{\mathcal{E}}}_1)$. The polynomial $\Delta^2(\underline{0}) = v \vee (v \Rightarrow (u \vee \neg u))$, which is strictly weaker than R_2 ($\Delta^2(\underline{0}) \not\vdash R_2$ and $R_2 \vdash \Delta^2(\underline{0})$), cogenerates $\Gamma(\dot{\underline{\mathcal{E}}}_1)$.

One of the changes alluded to in the last paragraph, in connection with the definition of $\Gamma(\underline{\mathcal{E}})$, is the replacement of $\dot{\underline{\mathcal{E}}}_1$ by $\underline{\mathcal{E}}_1 = \underline{\mathcal{S}}^{\cdot\rightarrow\cdot}$, the "Sierpinski topos", this leaves the propositional logic fixed: $L(\underline{\mathcal{E}}_1) = L(\dot{\underline{\mathcal{E}}}_1)$ but alters the converse logic $\Gamma(\underline{\mathcal{E}}_1) = \{\varphi \mid v \vee \neg v \vdash \varphi\}$; thus L( ) does not determine Γ( ). Such variations suggest a further question: is there a criterion which will determine when $\Delta^2(\underline{0})$ or R_2 belongs to $\Gamma(\underline{\mathcal{E}})$? The pursuit of an answer to this question is carried on through to 1.7 and has lead us to a general theory which for the particular case of $\Delta^2(\underline{0})$ and R_2 gives us the following theorem: for a topos $\underline{\mathcal{E}}$ the following are equivalent:

(a) $R_2 \in \Gamma(\underline{\mathcal{E}})$

(b) $\Delta^2(\underline{0}) \in \Gamma(\underline{\mathcal{E}})$

(c) $\underline{\mathcal{E}} \models \neg \forall p(p \vee \neg p)$.

For at least toposes $\underline{\mathcal{S}}^{\underline{M}^0}$ where $\underline{M}$ is a monoid, the last condition is easy to verify. it is equivalent to $\underline{M}$ not being a group.

We give an outline of each section of chapter 1. Further introductory remarks appear at the beginning of the two longer sections, 1.2 and 1.7.

Section 1.1 We organize the techniques needed to compute (1) the propositional logic associated with a Heyting algebra and (2) the converse logic. The latter is based on work of Jankov and Hosoi.

Section 1.2 We develop our examples of presheaf toposes, calculating $L(\underline{\mathcal{S}}^{\underline{C}^0})$ and $\Gamma(\underline{\mathcal{S}}^{\underline{C}^0})$ for several $\underline{C}$.

Section 1.3 We prove using the language developed in chapter 0 that, for $\underline{M}$ an internal monoid in a topos $\underline{\mathcal{E}}$, $\underline{M} - \underline{\mathcal{E}}$ (the topos of actions of $\underline{M}$ on $\underline{\mathcal{E}}$) is a topos. This section is independent of others in chapter 1 but guides us in developing properties of internal presheaf topos $\underline{\mathcal{E}}^{\underline{C}^0}$ in 1.5 and 1.6.

Section 1.4 To prepare for our description of the Heyting algebra structures that arise from the subobject classifier in $\underline{\mathcal{E}}^{\underline{C}^0}$, we show how an internal

Heyting algebra structure arises naturally on the object of "ideals" (i.e. increasing subsets) Idl $\mathcal{A}$ of an internal preordered object $\mathcal{A}$.

Section 1.5 We translate the diagrammatic definition of an internal presheaf topos given in Johnstone's thesis [J1] into our internal language.

Section 1.6 We establish that $\varphi \varepsilon L(\underline{\mathcal{E}}^{\underline{C}^0})$ iff φ is valid in the internal Heyting algebra $\underset{\sim}{\mathrm{Idl}}(C_1,\prec)$ where $\prec$ is the preorder of divisibility on C_1, the object of morphisms of $\underline{C}$. We also show that $\underline{\mathcal{E}} \models \neg \forall p(p \vee \neg p)$ is a sufficient condition for $\Delta^2(\underline{0})$ ε $\Gamma(\underline{\mathcal{E}})$.

Section 1.7 We solve the problem of internalizing the condition $\varphi \varepsilon \Gamma(\underline{\mathcal{E}})$ for a class of polynomials, $\mathcal{C}_{\dashv\vdash}^{\Rightarrow}$, which includes $\Delta^2(\underline{0})$ and R_2.

CHAPTER 0

THE LANGUAGE ASSOCIATED WITH A TOPOS

Section 0.1 Introduction to the Language

In this chapter, taking set theory as our starting point, we build up out of a given topos a non-classical type-theoretic language which, in its completed form - after we have interpreted it in a topos - will resemble the languages presented in Johnstone [J2] and Osius [O1].

We follow [J2] (pp. 152-161) closely in our choice of "terms" of the language; our terms are all terms of his language except those that arise from applications of the description operator $\iota x(\ldots)$ (read: "the unique x such that ..."). The language of [J1], there called the Mitchell-Bénabou language, is "designed to emphasize the convenience of $L_{\mathcal{E}}$ [the language] as a simple mathematical shorthand for $\mathcal{E}$ [the topos], rather than as a formal system for proving theorems about $\mathcal{E}$". Accordingly the description of that language is confined to the specification of terms, the interpretation of terms as morphisms of the topos, and the definition of a formula. Our objective, on the other hand, is to develop a language that can be used to prove theorems in a topos and to this end we turn to the second of our references: [O1].

In this article Osius distinguishes between two kinds of languages. The first kind, which includes our own, that of [J2] and that of an earlier article of Osius' [O2] , begins with a small topos and out of its object set and morphism set, generates the terms of the language. The second kind, which is as the intended language of [01], begins with the first order language that axiomatizes topos theory and out of its terms - those that denote objects and morphisms - builds another language. We have reason however to think of [01] as a guide to the first kind of language, since, as Osius declares at the beginning, "we sometimes pretend to work in a fixed topos ... (i.e. a model of [the elementary theory of toposes]) rather than in the [elementary theory of toposes] itself" ([01], p. 301). This is how we shall treat it.

Considered from this point of view then, the terms of Osius' language are all those terms of our language that do not arise from applications of the comprehension operator $\{x: \ldots\}$, read: "the subset of all x such that ...".

In formulating the language we shall speak of sets of terms, natural numbers and variables; of a variable being an element of the set of free variables of a term, of equality of terms as terms. We call this use of set theoretical terminology external, and the use of the terminology within the language internal. To avoid ambiguities we sometimes choose different symbols for the external and corresponding internal set theoretic notion.

Table 1

internal	external
$a \in B$	$a \varepsilon B$
$a = b$	$a \equiv b$
$\cup, \cap, \subset$	$\cup, \cap, \subset$
$\langle a,b \rangle$	$<a,b>$, (a,b)
B^A, $\underline{N}$, $\underline{\phi}$	$[A,B]$, N, ϕ
$\{x: \ldots\}$, $\{x\}$	$\{x \mid \ldots\}$, $\{x\}$
$\mathcal{P}B$	$\mathcal{P}B$.

There are also entities that can be correlated with external notions but that do not fit into the above scheme.

SubB	$\mathcal{S}_f B$
the external set of subobjects of the object B	the set of all finite subsets of the set B
$[[x \mid \varphi]]$	
the subobject determined by the formula φ	

We use round parentheses in the conventional way: to establish the order in which operations are performed. There is some overlapping with the sharp brackets: both (a,b) and <a,b>. The latter was chosen to draw a distinction between the internal and external ordered pairs.

Table 1 continued (cf. Kleene p. 225 [Kl])

internal	both read as	external
$\exists x$	there exists an x such that	(Ex)
$\exists ! x$	there exists a unique x such that	$(E!x)$
$\Rightarrow$	implies	$\rightarrow$
$\forall x$	for all x	(Ax)
$\wedge$	and	and

The formal introduction of terms of the language will be delayed until section 0.3.2. The set of terms will be a subset of expressions which we study in section 0.2, and the expressions will be a subset of the set of strings.

We shall separate the terms from the expressions by means of a type structure. Historically type theory imposed restrictions on the too free application of the comprehension operator so that, for example, $\{x: \neg(x \in U)\}$ was permitted to be a term of the language only if the type of U was higher than that of x. Our types are objects of a category, by combining morphisms we form $fg(x)$ which is taken to be a term if and only if the pair $<f,g>$ of morphisms is composable and the domain of g is the type of x. Typeless uninterpretable expressions will include both $\{x: \neg(x \in x)\}$, from which a contradiction could be derived in the presence of what would appear to be reasonable axioms and rules of inference, and $fg(x)$ where the composition $f\circ g$ is undefined in the category or the type of x is not the domain of g.

Expressions have been defined so that notions such as free and bound variable, and substitution, can be studied without having to pay attention to types. Variable binding operations and morphisms are applied freely so that the set of expressions forms an absolutely free algebra of a certain similarity type (in the sense of universal algebra). Once we introduce terms we shall be able to show without difficulty that what we have proven about expressions applies as well to terms.

Strings (we avoid at this stage putting a name to what they are strings of) are to be sort of garbled expressions such as: $\forall\neg p$ and $\forall$, $\neg$, and p all by themselves. To say "$\forall\neg p$ is a string" is to make an external statement related to the language and thus, in principal, a set theoretical one. If, that is to say, we had to work within an axiomatic system in which "everything is a set" we should have to ask: if $\forall$, $\neg$, and p are sets what set is $\forall\neg p$? A typical answer is: the ordered triple $\langle\forall, \neg, p\rangle$; or, if the natural numbers have been introduced we can take $\forall = \{\langle 0,a\rangle\}$, $\neg = \{\langle 0,b\rangle\}$, $p = \{\langle 0,c\rangle\}$ for some sets a, b, c; and then put $\forall\neg p = \{\langle 0,a\rangle,\langle 1,b\rangle,\langle 2,c\rangle\}$. In other words juxtaposing $\forall, \neg$ and p has to be interpreted as some already postulated construction of set theory.

Section 0.2 Syntax

0.2.1 Strings: $\mathbb{N}$ will denote the set of natural numbers; i, j, k, m, n, n_1, n_2, n', m' denote arbitrary natural numbers. We take $\mathbb{N}$ to be in one to one correspondence with, but disjoint from, the set of finite ordinals. We put $[n] = \{k \mid k + 1 \le n\}$ for each n; thus [n] and n are distinct (in particular $[0] = \phi$ and $0 \neq \phi$).

Let Σ be a set, elements of which we call ciphers. $\Sigma^{[n]}$ is the set of all functions with domain [n] and values in Σ. $\Sigma^* = \bigcup_{n \in \mathbb{N}} \Sigma^{[n]}$, elements of Σ^* we call strings; s, s', s" denote arbitrary strings. We define a length function $\ell: \Sigma^* \to \mathbb{N}$ by

$$\ell(s) = n \qquad \text{iff} \quad s \in \Sigma^{[n]}$$

Elements of $\Sigma^{[1]}$, that is, strings of length 1, are called signs; the string ϕ, the single element of $\Sigma^{[0]}$, is called the empty string. If s and s' are strings of lengths n and n' respectively then conc(s,s'), the concatenation of s with s', is the string of length n + n' given by

$$(\text{conc}(s,s'))\,(i) = \begin{cases} s(i) & \text{for } i+1 \le n \\ s'(i-n) & \text{for } n+1 \le i+1 \le n+n' \end{cases}$$

Equipped with the empty string and concatenation Σ^* is the free monoid generated by $\Sigma^{[1]}$ (see [Brb], Bourbaki, Algebra, p. 82). The length function is a monoid homomorphism from $\langle\Sigma^*, \phi, \text{conc}\rangle$ to

$\langle \mathbb{N}, 0, +\rangle$. From now on we shall write ss' for $\mathrm{conc}(s,s')$ and $ss's''$ for $(ss')s''$. The following is a basic property of concatenation.

<u>0.2.1.1 Proposition</u>. For any string s'' and natural number n for which $n \le \ell(s'')$ there is a uniquely determined s and a uniquely determined s' such that $\ell(s) = n$ and $ss' = s''$.

<u>Proof</u>. Let $n' = \ell(s'') - n$. Define s and s' by

(1) $s(i) = s''(i)$ for $i+1 \le n$

(2) $s'(j) = s''(j+n)$ for $j+1 \le n'$

then s and s' are of lengths n and n' respectively.

$j+1 \le n'$ iff $j+n+1 \le n+n'$

and $i = j+n$ iff $j = i-n$

hence (2) is equivalent to

(2)' $s'(i-n) = s''(i)$ for $n+1 \le i+1 \le n+n'$

(1) and (2)' imply $ss' = s''$. If $rr' = s''$ with $\ell(r) = n$ and $\ell(r') = n'$ then $i+1 \le n$ implies $r(i) = s''(i) = s(i)$ so $r = s$, also $j+1 \le n'$ implies $r'(j) = r'((j+n)-n) = s''(j+n) = s'(j)$ so $r' = s'$. □

<u>0.2.1.2 Corollary</u>. If $SR_1 = SR_2$ then $R_1 = R_2$. This follows immediately from 0.2.1.1.

<u>0.2.1.3 Corollary</u>. If $S_1S_2 = R_1R_2$ then either there exists an R' such that $S_1 = R_1R'$ or there exists an S' such that $R_1 = S_1S'$.

<u>Proof</u>. Either $\ell(S_1) \ge \ell(R_1)$ or $\ell(R_1) \ge \ell(S_1)$. Suppose $\ell(S_1) \ge \ell(R_1)$ then $S_1 = SR'$ where $\ell(S) = \ell(R_1)$, hence $SR'S_2 = R_1R_2$, hence

$S \equiv R_1$ and so $S_1 \equiv R_1R'$. Similarly if $\ell(R_1) \geq \ell(S_1)$ we have $R_1 \equiv S_1S'$. □

We introduce a property of subsets of Σ^* which is sufficient to establish that the submonoids generated by such subsets are freely generated (see [KK], Kreisel & Krivine, Elements of Mathematical Logic, p. 2).

0.2.1.3 Definition. U will be called a KK subset of Σ^* (after the above reference) if

KK-(1) $\phi \notin U$

KK-(2) $s \in U$ and $ss' \in U$ imply $s' \equiv \phi$.

It is immediate that any subset of a KK subset is again a KK subset.

0.2.1.4 Proposition. Let $\bar{U}$ denote the submonoid generated by U, U a KK subset of Σ^*, then

(1) $sr \equiv s'r'$ and $s, s' \in U$ implies $s \equiv s'$ and $r \equiv r'$.

and (2) U freely generates $\bar{U}$.

Proof (1). Suppose $sr \equiv s'r'$. Let n, n' be the lengths of s, s' respectively. Suppose $n \geq n'$ then $s \equiv r_1r_2$ with $\ell(r_1) \equiv n'$. Hence $r_1(r_2r) \equiv s'r'$ with $\ell(r_1) \equiv \ell(s')$. By 0.2.1.1 $r_1 \equiv s'$ and $r_2r \equiv r'$. But then $s \equiv s'r_1$ with s and s' elements of U; so by KK-(2) $r_1 \equiv \phi$. Hence $s \equiv s'$, hence $r \equiv r'$.

(2). We know U^* freely generates $U^{[1]}$. The bijection that sends each $\{\langle 0,s\rangle\}$ to s, $U^{[1]} \xrightarrow{\sim} U$, extends to a monoid homomorphism from U^* onto $\overline{U}$. For each $R \in U^*$ let $|R|$ be the value of R under this homomorphism, and let ℓ' be the length function of U^*. We show $|R| = \phi$ implies $R = \phi$. If $R \neq \phi$ then $\ell'(R) \geq 1$ so $R = R'R''$ with $\ell'(R') = 1$, that is $R' \in U^*$, hence $|R'| \in U$, hence $|R'| \neq \phi$. But $\ell(|R|) = \ell(|R'|) + (|R''|)$ so $|R| \neq \phi$ contradictiong the assumption.

We now prove $|R| = |R'|$ implies $R = R'$ by induction on the minimum of $\ell'(R)$ and $\ell'(R')$. If the minimum is 0, say $\ell'(R) = 0$, then $R = \phi$ so $|R'| = \phi$. Suppose $n+1$ is the minimum then $R = R_1R_2$ and $R' = R_1'R_2'$ where $|R_1|$ and $|R_1'|$ belong to U. Since $|R_1||R_2| = |R_1'||R_2'|$, by (1) $|R_1| = |R_1'|$ and $|R_2| = |R_2'|$. Hence $R_1 = R_1'$, and by induction $R_2 = R_2'$, so $R = R'$.□

<u>0.2.1.5 Notation.</u> For each KK subset U of Σ^*, $\ell'_U: U^* \longrightarrow \mathbb{N}$ is the length function on U^* and $\ell_U: \overline{U} \longrightarrow N$ is the length function induced by the isomorphism $U^* \approx \overline{U}$. For any σ we put $\hat{\sigma} = \{<0,\sigma>\}$, thus if $\sigma \in \Sigma$ then $\hat{\sigma} \in \Sigma^{[1]}$.

<u>0.2.1.6 Lifting a function f to f^*.</u> We can carry our construction of Σ^* from Σ in 0.2.1 over to functions $f: \Sigma_1 \longrightarrow \Sigma_2$ to get a monoid homorphism $f^*: \Sigma_1^* \longrightarrow \Sigma_2^*$. One way of describing f^* is to first define for each $n \in N$ the function $f^{[n]}: \Sigma_1^{[n]} \longrightarrow \Sigma_2{}^{[n]}$ by $f^{[n]}(S) = f \circ S$; just as $\Sigma_1^{[n]}$ and $\Sigma_1^{[p]}$ are disjoint for $n \neq p$ so also are $f^{[n]}$ and $f^{[p]}$ disjoint for $n \neq p$; if we define $f^* = \bigcup_{n \in \mathbb{N}} f^{[n]}$ then $f^*: \Sigma_1^* \longrightarrow \Sigma_2^*$ is given by $f^*(S) = f \circ S$. Alternatively we can define $\vec{f}: \Sigma_1^* \longrightarrow \Sigma_2^*$ to be the monoid homomorphism which on $\Sigma_1^{[1]}$ is given by

$\vec{f}(\hat{\sigma}) = (f(\sigma))^\wedge$ for each $\sigma \in \Sigma_1$.

<u>0.2.1.7 Proposition.</u> $\vec{f}(S) = f \bullet S$ for each $S \in \Sigma_1^*$.

<u>Proof.</u> We proceed by induction on $\ell(S)$. For the empty string we have $\vec{f}(\phi) = \phi = f \bullet \phi$. For $\sigma \in \Sigma_1$, $\hat{\sigma} \in \Sigma_1^{[1]}$, $\vec{f}(\hat{\sigma}) \in \Sigma_2^{[1]}$ and $f \bullet \hat{\sigma} \in \Sigma_2^{[1]}$, evaluating at $o \in [1]$ we have $(\vec{f}(\hat{\sigma}))(o) = ((f(\sigma))^\wedge)(o) = f(\sigma) = f(\hat{\sigma}(o))$ $= (f \circ \hat{\sigma})(o)$, so $\vec{f}(\hat{\sigma}) = f \bullet \hat{\sigma}$. For the induction step we let $S = S_1\hat{\sigma}$ where $S_1 \in \Sigma_1^{[n]}$, $S \in \Sigma_1^{[n+1]}$ and $\sigma \in \Sigma_1$. By definition

$$S(i) = \begin{cases} S_1(i) & \text{if } i \in [n] \\ \sigma & \text{if } i = n \end{cases}$$

Hence

$$f(S(i)) = \begin{cases} f(S_1(i)) & \text{if } i \in [n] \\ f(\sigma) & \text{if } i = n\ . \end{cases}$$

Also $\vec{f}(S) = \vec{f}(S_1\hat{\sigma}) = \vec{f}(S_1)\vec{f}(\hat{\sigma}) = (f \bullet S_1)(f \bullet \hat{\sigma})$ by induction. Since $(f \bullet S_1)(i) = f(S_1(i))$ for $i \in [n]$ and $(f \bullet \hat{\sigma})(o) = f(\sigma)$, we have $f \bullet S = (f \bullet S_1)(f \bullet \hat{\sigma}) = f(S)$.□

<u>0.2.1.8 Corollary.</u> $f \bullet (S_1S_2) = (f \bullet S_1)(f \bullet S_2)$.□

In particular 0.2.1.8 can be applied to $S \in \Sigma^*$ as follows: let $n = \ell(S)$ so that $S\colon [n] \longrightarrow \Sigma$ then $S^*\colon [n]^* \longrightarrow \Sigma^*$ and for any $\alpha_1\colon [n_1] \longrightarrow [n]$, $\alpha_2\colon [n_2] \longrightarrow [n]$ we have $S \bullet (\alpha_1\alpha_2) = (S \bullet \alpha_1)(S \bullet \alpha_2)$.

0.2.2 Expressions.

0.2.2.1 Since $\Sigma^{[1]}$ and Σ are in one-to-one correspondence via $\wedge: \Sigma \underset{\sim}{\to} \Sigma^{[1]}$, a partition of Σ can be described in terms of a partition of $\Sigma^{[1]}$. Thus we shall take the set of signs of the language to be the union of the following mutally disjoint non-empty subsets of some set $\Sigma^{[1]}$:

Sets of Signs	Name of	Arbitrary Elements
Vbls	variables	x, y, u, v
Fns	function signs	f, g, h
Q	quantifier signs	q, q_1, q_2
{*}	the basic constant	
{b}	the binary prefix	

We let Sgns = Vbls ∪ Fns ∪ Q ∪ {*,b} , and let Strgs be the monoid generated by Sgns. For any KK subset U of Strgs we let Str(U) denote the submonoid of Strgs generated by U.

0.2.2.2. Definition. Let Q(Vbls) = {qx|q ε Q, x ε Vbls}. We define a similarity type μ of algebras:

Q(Vbls) ∪ Fns ∪ {*,b} is the set of formal operations,

μ is a function, with domain the set of formal operations, assigning arities as follows:

μ(qx) = 1 for q ε Q, x ε Vbls

μ(f) = 1 for f ε Fns

μ(*) = 0, μ(b) = 2.

Let $\mathcal{A} = \langle A, \gamma \rangle$ be an algebra of similarity type μ, or briefly a μ-algebra, and let θ be a formal operation. We write $\theta_{\mathcal{A}}$ for the reaization of the formal operation as an operation $\theta_{\mathcal{A}}: A^n \longrightarrow A$ of $\mathcal{A}$ where $n = \mu(\theta)$ and $n \geq 1$; and $\theta_{\mathcal{A}} \in A$ when $\mu(\theta) = 0$. When the context is clear we shall drop the subscript and just write θ (see [Gr], p. 33). For any $U \subset A$, $[U]$ will denote the μ-subalgebra (or carrier of the μ-subalgebra) generated by U.

We turn the set of signs of the language, Strgs, into a μ-algebra $\underline{St}$ as follows:

$*_{\underline{St}} = *$

$b_{\underline{St}}(s,s') = bss'$ for each $s, s' \in$ Strgs

$f_{\underline{St}}(s) = fs$ for each $f \in$ Fns, $s \in$ Strgs

$(qx)_{\underline{St}}(s) = qxs$ for each $q \in Q$, $x \in$ Vbls, $s \in$ Strgs.

We define Expr, the set of expressions of the language to be the carrier of the μ-subalgebra, $\underline{Expr}$, of $\underline{St}$ generated by Vbls, the set of variables. We let t, t_1, t_2, t_1', t_2' denote arbitrary expressions.

<u>0.2.2.3 Proposition.</u> Let $\mathcal{A}$ be a μ-algebra with carrier A and let $U \subset A$. For any element a of $[U]$ one of the following holds:

(1) $a \in U$

(2) $a = *_{\mathcal{A}}$

(3) $(Ef)(Ea_1)(a_1 \in [U]$ and $a = f_{\mathcal{A}}(a_1))$

(4) $(Ea_1)(Ea_2)(a_1 \in [U]$, $a_2 \in [U]$ and $a = b_{\mathcal{A}}(a_1,a_2))$

(5) $(Eq)(Ex)(Ea_1)(a_1 \in [U]$ and $a = (qx)_{\mathcal{A}}(a_1))$.

Proof. Let B be the set of all elements a of [U] for which one of the above clauses holds. Since $B \subset [U]$ any operation of $\mathcal{A}$ applied to elements of B must produce an element of B so $B \equiv [B]$. But $U \subset B$, hence $B \equiv [U]$. □

0.2.2.4 Proposition. The set of expressions Expr is a KK subset of Strgs.

Proof. Expr ≡ [Vbls] so for each $t \in$ Expr one of the following holds:

(1) $t \in$ Vbls

(2) $t \equiv *$

(3) $(Ef)(Et_1)(t_1 \in \text{Expr} \text{ and } t \equiv ft_1)$

(4) $(Et_1)(Et_2)(t_1 \in \text{Expr}, t_2 \in \text{Expr} \text{ and } t \equiv bt_1t_2)$

(5) $(Eq)(Ex)(Et_1)(t_1 \in \text{Expr} \text{ and } t \equiv qxt_1)$.

Thus for each $t \in$ Expr, $\ell(t) \geq 1$; so t cannot be the empty string. Each expression t has a unique factorization $t \equiv s_1s'$ where s_1 is a sign of the language (a string of length one). Clause (n) for t implies clause (n)' for s_1 for $1 \leq n \leq 5$:

(1)' $s_1 \in$ Vbls

(2)' $s_1 \equiv *$

(3)' $s_1 \in$ Fns

(4)' $s_1 \equiv b$

(5)' $s_1 \in Q$.

Since the sets Q, {b} , {*} , Fns, Vbls are mutually disjoint, for every expression t exactly one of the clauses (1) - (5) holds for t and exactly one of the clauses (1)' - (5)' holds for the initial sign s_1 of t.

We show that if $t \in$ Expr and $ts \in$ Expr then s is the empty string, by running through cases (1) to (5) for ts and applying induction to the length of t ($t \equiv s_1 s'$):

(1) $s_1 s's \in$ Vbls implies $s_1 \in$ Vbls, $s' \equiv \phi \equiv s$.

(2) $s_1 s's \equiv *$ implies $s_1 \equiv *$, $s' \equiv \phi \equiv s$.

(3) $s_1 s's \equiv ft'_1$ implies $s_1 \equiv f \in$ Fns

$fs' \in$ Expr implies $s' \equiv t_1 \in$ Expr

hence $t_1' \equiv t_1 s$ so by induction $s \equiv \phi$.

(4) $s_1 s's \equiv bt_1't_2'$ implies $s's \equiv t_1't_2'$ and $s_1 \equiv b$

$bs' \in$ Expr implies $s' \equiv t_1 t_2$

hence $t_1't_2' \equiv t_1 t_2 s$. Either $t_1' \equiv t_1 s''$ or $t_1 \equiv t_1's'$, so by induction $t_1 \equiv t_1'$. Hence $t_2' \equiv t_2 s$ again by induction $s \equiv \phi$.

(5) $s_1 s's \equiv qxt_1'$ implies $s_1 \equiv q$ and $s's \equiv xt_1'$,

$qs' \in$ Expr implies $s' \equiv yt_1$

hence $xt_1 \equiv yt_1 s$, hence $t_1' \equiv t_1 s$ so by induction $s \equiv \phi$. □

We now give a description of the set of expressions which we will use often when we want to argue by induction on the length of expressions.

<u>0.2.2.5 Proposition</u>. For each expression t exactly one of the following holds:

(1) $t \in$ Vbls

(2) $t \equiv *$

(3) $(E!f)(E!t_1)(t \equiv ft_1)$

(4) $(E!t_1)(E!t_2)(t \equiv bt_1t_2)$

(5) $(E!q)(E!x)(E!t_1)(t \equiv qxt_1)$

<u>Proof</u>. The fact that exactly one of the clauses (1) to (5) of 0.2.2.3 holds follows from the disjointness of the sign sets (0.2.2.1). For clauses (3) and (5) the uniqueness follows from 0.2.1.2. For (4): if $t \equiv bt_1t_2 \equiv bt_1't_2'$ then $t_1t_2 \equiv t_1't_2'$ hence $t_1 \equiv t_1'$ and $t_2 \equiv t_2'$ by 0.2.2.4. □

The standard argument based on this proposition will consist of representing an expression t in each of five possible ways: y, $*$, ft_1, bt_1t_2, qyt_1 and in each case giving an appropriate argument; in the last three cases the argument will often be based on an induction hypothesis that will apply to t_1 and t_2.

The description above (0.2.2.5) does not really invoke the μ-algebra structure. <u>Expr</u> is actually absolutely freely generated as a μ-algebra by Vbls.

<u>0.2.2.6 Proposition</u>. Let $\mathcal{A}$ be a μ-algebra with carrier A. Let β: Vbls $\longrightarrow$ A, then β lifts uniquely to a μ-algebra homomorphism from <u>Expr</u> to $\mathcal{A}$.

Proof. Expr × A is the carrier of the product μ-algebra of Expr with $\mathcal{A}$, $\beta \subset$ Vbls × A and $[\beta] \subset$ Expr × A. The projection Expr × $\mathcal{A} \longrightarrow$ Expr is a μ-algebra homomorphism so the image of $[\beta]$ under the homomorphism is a μ-subalgebra which contains Vbls, hence the domain of the relation $[\beta]$ is all of Expr.

We want to show $[\beta]$ is single valued. First note that by 0.2.2.3 for any $\langle t,a \rangle$ in $[\beta]$ one of the following five clauses holds, where in each clause we have turned any equality of ordered pairs into a conjunction of an equality in Expr and an equality in A:

(1) $\beta(t) \equiv a$ with $t \in$ Vbls

(2) $t \equiv *$ and $a \equiv *_{\mathcal{A}}$

(3) $(Ef)(Et_1)(Ea_1)(\langle t_1,a_1 \rangle \in [\beta],\ t \equiv ft_1$ and $a \equiv f_{\mathcal{A}}(a_1))$

(4) $(Et_1)(Ea_1)(Et_2)(Ea_2)(\langle t_1,a_1 \rangle \in [\beta],\ \langle t_2,a_2 \rangle \in [\beta],$
$t \equiv bt_1t_2$ and $a \equiv b_{\mathcal{A}}(a_1,a_2))$

(5) $(Eq)(Ex)(Et_1)(Ea_1)(\langle t_1,a_1 \rangle \in [\beta],\ t \equiv qxt_1$ and
$a \equiv (qx)_{\mathcal{A}}(a_1))$.

If $\langle t,a \rangle \in [\beta]$ and $\langle t,a' \rangle \in [\beta]$, then, mutatis mutandis, one of the above five clauses holds for $\langle t,a' \rangle$. But 0.2.2.5 applied to t insures that whichever of the above clauses applies to $\langle t,a \rangle$, the corresponding clause must likewise apply to $\langle t,a' \rangle$, and moreover that t has one of five uniquely determined forms. We use this to argue by induction on $\ell(t)$ that $\langle t,a \rangle \in [\beta]$ and $\langle t,a' \rangle \in [\beta]$ imply $a \equiv a'$.

y $\quad a \equiv \beta(y) \equiv a'$

$*$ $\quad a \equiv *_{\mathcal{A}} \equiv a'$

ft_1 $(Ea_1)(Ea_1') <t_1,a_1> \varepsilon [\beta], <t_1,a_1'> \varepsilon [\beta], a = f_{\mathcal{A}}(a_1),$

$a' = f_{\mathcal{A}}(a_1')$; by induction $a_1 = a_1'$, hence $a = a'$

bt_1t_2 $(Ea_1)(Ea_1')(Ea_2)(Ea_2') <t_1,a_1> \varepsilon [\beta], <t_1,a_1'> \varepsilon [\beta],$

$<t_2,a_2> \varepsilon [\beta], <t_2,a_2'> \varepsilon [\beta], a = b_{\mathcal{A}}(a_1,a_2)$, and

$a' = b_{\mathcal{A}}(a_1',a_2')$; by induction $a_1 = a_1'$ and

$a_2 = a_2'$, hence $a = a'$

qxt_1 $(Ea_1)(Ea_1') <t_1,a_1> \varepsilon [\beta], <t_1,a_1'> \varepsilon [\beta],$

$a = (qx)_{\mathcal{A}}(a_1)$ and $a' = (qx)_{\mathcal{A}}(a_1')$; by induction

$a = a_1'$, hence $a = a'$.

Thus $[\beta]$ is a functional relation which is a subalgebra of <u>Expr</u> $\times \mathcal{A}$, hence it is a μ-algebra homomorphism that extends β. If β' is another homomorphism extending β then the equalizer of β' and the function $[\beta]$ is a μ-subalgebra of <u>Expr</u> containing Vbls, hence $\beta' = [\beta]$. □

<u>0.2.2.7 Expressions as Strings of Signs</u>. It is perhaps a matter of taste whether one relies on Universal Algebra to describe Expr without further ado as the absolutely free algebra of similarity type μ generated by the set Vbls, or one goes to the trouble of proving directly that this is so for a certain set of strings of signs. In the sequel we shall make good use of the absolutely free algebra point of view; for example in 0.2.4.2 and 0.3.1.2 several μ-homomorphisms with domain Expr will be defined by lifting functions that map Vbls to appropriate μ-algebras. But we will also make use of expressions as strings of signs: in 0.2.3.5 we switch the similarity type of Expr from μ to Γ ; then, in order to show that

the F-subalgebra of Expr generated by Vbls is absolutely freely generated, we rely on the representation of expressions as strings of signs - in this context the representation is a kind of common denominator of μ and F.

The string-of-signs/sequence-of-ciphers approach brings out features of expressions which our experience with written expressions confirm, features not directly accessible from Universal Algebra. Although we will not make use of the analysis of "occurance" to follow we believe it to be worth including since as a rule it is glossed over in the literature. ([RS], p. 169, [Mon], p. 69, [Gr 1] p. 226; [CK] p. 23).

We envisage $\mathbb{N}$ as counting out the positions of a tape running from an initial position, 0, indefinitely to the right (cf. [Mon], p.). An interval of this tape is a subset $I \subset \mathbb{N}$ of the form $I = \{i \mid m \leq i+1 \leq n\}$ where $m \leq n$. $\mathbb{N}$ considered as an additive monoid acts on the set, Int, of all intervals with action: $j+I = \{j+i \mid i \in I\}$. The set Σ of ciphers are positionless mathematical objects (i.e. sets). We define an occurance to be a function R with domain an interval and range contained in Σ, thus R fills each position of an interval of the tape with a cipher. The action of $\mathbb{N}$ on Int induces an action of $\mathbb{N}$ on Occ, the set of occurances: $j+R = \{\langle j+i,\sigma\rangle \mid \langle i,\sigma\rangle \in R\}$. Given a string S (i.e. an occurance whose domain is an initial segment of the tape) we define Occ(S), the set of occurances S, to be $\{j+S \mid j \in \mathbb{N}\}$. Under this definition, R is an occurance of a sign if and only if the domain of R is a singleton.

In order to include in our account the notions of free and bound occurance of a variable we will establish that the notion of scope of an occurance of a quantifier is well-defined in Expr.

<u>Proposition</u> (a) If S_1qxS_2 is an expression there exists an expression t and a string S_3 such that $S_2 \equiv tS_3$.

(b) If in (a) $S_2 \equiv t'S_3'$ then $t' \equiv t$ and $S_3' \equiv S_3$.

<u>Proof</u> (a) We proceed by induction on $\ell(S_1)$. If $\ell(S_1) \equiv 0$ then qxS_2 is an expression so $S_2 \varepsilon$ Expr by (5) of 0.2.2.5. For the induction step, when $\ell(S_1) \geq 1$, we can represent S_1 in only three possible ways: $S_1 \equiv fS'$, $S_1 \equiv bS'$ and $S_1 \equiv q'x'S'$. The cases $S_1 \equiv *S'$ and $S_1 \equiv x'S'$ are excluded since KK - (2) applied to S_1qxS_2 would in these cases imply $S'qxS_2 \equiv \phi$. If $S_1 \equiv fS'$ then by (3) of 0.2.2.5 $S'qxS_2 \varepsilon$ Expr so by induction (a) holds. If $S_1 \equiv bS'$ then by (4) of 0.2.2.5 $S'qxS_2 \equiv t_1t_2$. It is not possible for either $S'qx$ or $S'q$ to be an expression since the set (Strgs - ((Strgs) {qx,q})) of all strings not ending in qx or q contains all variables and is a μ-subalgebra of <u>St</u> hence contains all expressions. Thus either $t_1 \equiv S'qxS_2'$ for some S_2' or $t_2 \equiv S''qxS_2$ for some S'' ; in either case (a) holds by induction. If $S \equiv q'x'S'$ then $S'qxS_2 \varepsilon$ Expr so by induction (a) holds.

(b) This follows from (1) of 0.2.1.4. □

We say an occurance R_1 is <u>within</u> an occurance R_2 if $R_1 \subset R_2$. If an occurance R_1 of qx is within an occurance R_2 of an expression $\tilde{t}$ we can put $R_2 \equiv j+\tilde{t}$ and $R_1 \equiv j+i+qx$ so that $i+qx$ is within $\tilde{t}$. Using 0.2.1.1 twice we put $\tilde{t} \equiv S_1qxS_2$ where $i \equiv \ell(S_1)$. By the above Proposition we can factor S_2 so that $\tilde{t} \equiv S_1qxtS_3$. We call $R \equiv j+i+2+t$ the <u>scope</u> of R_1 in R_2; R is that occurance of t for which $R_1 \cup R$ is an occurance of qxt within $\tilde{t}$. We call an occurance R_1 of x within an occurance R_2 of $\tilde{t}$ <u>free</u> if, whenever an occurance R_3 of qx is within R_2, R_1 is not within the scope of R_3 within R_2. We say an occurance R_3 of qx <u>binds</u> an occurance R_1 of x within an occurance R_2 of $\tilde{t}$, if R_1 is within the scope R_4 of R_3 within R_2, and R_1 is free within R_4.

Lastly we reveal the source of our mysterious set-theoretical formulation of the notions "string" and "occurance". We have no occasion in future chapters to give a detailed construction of the free monoid in a topos - a nice categorical construction is presented in [J2] (Theorem 6.41) so it would not be new - but it is simple enough to carry out a construction, using our internal language, that does not resort to exponentiation in the comma category $(\underline{\mathcal{E}}, \mathbb{N})$ (see [J2], p. 180). A sketch of this alternative construction will explain the more esoteric features of our representation of strings and occurances.

The action we defined on the set of occurances makes sense when applied to all subsets of $\mathbb{N} \times \Sigma$, moreover it preserves the operations of the monoid $\langle \mathcal{P}(\mathbb{N} \times \Sigma), \cup, \phi \rangle$ so we can turn the domain of this action, $+ : \mathbb{N} \times \mathcal{P}(\mathbb{N} \times \Sigma) \to \mathcal{P}(\mathbb{N} \times \Sigma)$, into a monoid $\mathbb{N} \times_+ \mathcal{P}(\mathbb{N} \times \Sigma)$; this monoid is called the semidirect product (see [Brb], p. 66 or [Wr], p. 176), it has unit $\langle 0, \phi \rangle$ and its multiplication is given by

$$\langle n_1, R_1\rangle \cdot \langle n_2, R_2\rangle = \langle n_1+n_2,\ R_1 \cup (n_1+R_2)\rangle.$$

Embed the set of ciphers Σ into $\mathbb{N} \times \mathcal{P}(\mathbb{N} \times \Sigma)$ by sending each σ to $\langle 1, \{\langle 0,\sigma\rangle\}\rangle$, define Σ^{1*} to be the submonoid generated by these elements, Σ^* to be the image of Σ^{1*} under the second projection (the first projection induces the length map), and the map $\mathrm{Occ}\colon \Sigma^* \longrightarrow \mathcal{P}(\mathcal{P}(\mathbb{N} \times \Sigma))$ by $\mathrm{Occ}(S) = \{R\colon \exists n(R = n+S)\}$.

0.2.3 Other Languages. One intention behind the introduction of the sets Vbls and Fns in 0.2.2.1 is to eventually take them to be in one-to-one correspondence with the sets $\mathrm{Obj}(\underline{\mathcal{E}}) \times \mathbb{N}$ and $\mathrm{Morph}(\underline{\mathcal{E}})$ respectively where $\underline{\mathcal{E}}$ is a topos. With this goal in mind the reason for being indefinite about these sets and not saying what they are at the outset is only that the structure of the topos $\underline{\mathcal{E}}$ is irrelevant to the development of the syntax in 0.2 and 0.3. But there are other reasons for not specifying the sets Vbls and Fns further; these reasons will be discussed under the headings "alphagams" and "elphabets", terms we have coined for basic data of two kinds of languages.

0.2.3.1 Fixed Signs. We want to fix certain signs once and for all, and, lest there be any doubt as to how "fixed" we want them to be, we spell them out: take $\{\{\{\phi\}\}\}$ as a cipher, let

$$\star \equiv \{\{\{\phi\}\}\}^{\wedge} \quad , \quad b \equiv \hat{\hat{\star}} \; , \quad \exists \equiv \hat{\hat{b}} \; , \quad \forall \equiv \hat{\hat{\exists}} \; , \text{ and } \quad \{:\} \equiv \hat{\hat{\forall}} \; ;$$

thus encoded these signs will hopefully not turn up out of their intended context. In 0.2.3.12, 0.6.4.1 and 0.6.8 we will extend this sequence further to introduce more fixed signs.

0.2.3.2 Alphagams. If toposes $\underline{\mathcal{E}}_1$ and $\underline{\mathcal{E}}_2$ were given to us in isolation, one at a time, it would be satisfactory to take $\mathrm{Fns} \approx \mathrm{Morph}(\underline{\mathcal{E}}_1)$ on one occasion and $\mathrm{Fns} \approx \mathrm{Morph}(\underline{\mathcal{E}}_2)$ on the other. However we shall often have to consider functors $U : \underline{\mathcal{E}}_1 \longrightarrow \underline{\mathcal{E}}_2$ and we will want to know what happens to terms defined using $\underline{\mathcal{E}}_1$ when they are mapped to terms defined using $\underline{\mathcal{E}}_2$. To anticipate this situation we must alter the focus of apparent definiteness put on our sets of signs in 0.2.2.1 ("<u>the</u> ... sets ... Vbls [and] Fns") ; this is the purpose of our definitions of alphagam and alphagam map.

Most authors ([Mon] [CK], [KK]) have used the term "language" for the basic set of signs together with some classification of these signs; Rasiowa and Sikorski use the term "alphabet" for this basic data ([RS], p. 153) and reserve "language" for the triple <alphabet, terms, formulas> - the "terms" and "formulas" here being generated by (or "based on" ([RS], p. 156)) the alphabet. Although we would prefer the latter terminology we have not provided our sets of signs with enough structure yet to distinguish terms and formulas from expressions; so, to emphasize its incompleteness, we have chosen the neologism "alphagam" for our data, leaving our "alphabet" to be spelt out in 0.3.1.2 when a "type" structure is available.

<u>Definition</u>. An <u>alphagam</u> is a triple $\mathbb{A} \equiv \langle Vbls, Fns, Fix \rangle$ of pairwise disjoint sets of signs where $\{*, b\} \subset Fix \subset \{*, b, \exists, \forall, \{:\}\}$. All the other notions introduced in 0.2.2.1 and 0.2.2.2 are now to be understood as given relative to an alphagam. The set of quantifier signs of $\mathbb{A}$ is $Fix \cap \{\exists, \forall, \{:\}\}$. An <u>alphagam map</u> is a triple $\langle \alpha, \mathbb{A}_1, \mathbb{A}_2 \rangle$ where $\mathbb{A}_i \equiv \langle V_i, Fns_i, Fix_i \rangle$ are alphagams (for $i = 1,2$), and α is a function from the set $Sgns_1$ of signs of $\mathbb{A}_1$ to the set $Sgns_2$ of signs of $\mathbb{A}_2$ such that

(1) $\alpha(s) \equiv s$ for all $s \in Fix_1$

(2) $\alpha(s) \in V_2$ for all $s \in V_1$

(3) $\alpha(s) \in Fns_2$ for all $s \in Fns_1$.

<u>0.2.3.3 Lifting an alphagam map to a map of expressions</u>. We maintain the notation used in the definition of an alphagam map $\langle\alpha, \mathbb{A}_1, \mathbb{A}_2\rangle$ above. The function α lifts to a monoid homomorphism $\tilde{\alpha}$ from the set $Strgs_1$ of strings generated by $Sgns_1$ to the set $Strgs_2$ of strings generated by $Sgns_2$.

<u>Proposition</u>. Let $Expr_i$ be the set of expressions of $\mathbb{A}_i$ $(i = 1,2)$, then

(4) $\tilde{\alpha}(s) \in Expr_2$ for all $s \in Expr_1$.

<u>Proof</u>. To simplify notation a bit we let $\bar{s} = \tilde{\alpha}(s)$ for all $s \in Strgs_1$. We proceed by induction on the length of expressions of $Expr_1$:

$*$ $\quad \tilde{\alpha}(*) = \alpha(*) = *$

x $\quad \tilde{\alpha}(x) = \alpha(x) \in V_2 \subset Expr_2$.

For the induction step r and t are expressions of $\mathbb{A}_1$, f is a function sign of $\mathbb{A}_1$, q a quantifier sign of $\mathbb{A}_1$.

ft $\quad \tilde{\alpha}(ft) = \bar{f}\bar{t}$; $\bar{f} \in Fns_2$, by induction $\bar{t} \in Expr_2$
hence $\bar{f}\bar{t} \in Expr_2$.

brt $\quad \tilde{\alpha}(brt) = b\bar{r}\bar{t}$; by induction $\{\bar{r}, \bar{t}\} \subset Expr_2$
hence $b\bar{r}\bar{t} \in Expr_2$.

qrt $\quad \tilde{\alpha}(qrt) = q\bar{r}\bar{t}$; $\bar{x} \in V_2$, by induction $\bar{t} \in Expr_2$
hence $q\bar{x}\bar{t} \in Expr_2$. $\square$

<u>0.2.3.4 Elphabets</u>. As structures in their own right elphabets and alphagams are quite similar; the reason for adopting special terminology is so that we can keep straight the different roles they play. An elphabet will be an alphabet for a simple kind of syntactic language part of whose role will be that of a syntactic language in classical Model Theory ([CK] 1.3, [RS] p. 281); its less familiar role, but one more to the point, will arise from realizations, or interpretations, of the elphabet in "models in a topos". It is an ad hoc definition tailored to deal largely with equational theories of Heyting algebras ([BD] p. 177, [RS] p. 123) without being cluttered up at this stage with names for operations. It will consist essentially of a similarity type having arities no larger than 2 together with a set of variables. The theories we model will not require quantifiers; they will be a modest extension of equational theories, and thus will fall far short of the "coherent" theories of [MR] (see [MR] p. 159, p. 238, or the "finitary geometric theories" of [J2] p. 201). To some extent we will be able to draw upon the material developed in 0.2.2 by confusing roles and associating an alphagam with a given elphabet, but we have to overcome a technical problem that arises from having several nullary and several binary operations, absent from the similarity type μ (0.2.2.2) of an alphagam. Thus, to show that our polynomial algebra is freely generated by the set of variables, we shall repeat, mutatis mutandis, several of the proofs in 0.2.2.

Definition. An elphabet $\mathbb{E} = \langle V, F_0, F_1, F_2, \{*, b\} \rangle$ is a 5-tuple of pairwise disjoint sets of signs, where $V = \{\langle i, \mathbb{N}\rangle^\wedge | i \in \mathbb{N}\}$. The similarity type of $\mathbb{E}$ will be the function F with domain $F_0 \cup F_1 \cup F_2$ such that $F(f) = i$ iff $f \in F_i$ for $i \in \{0,1,2\}$.

The alphagam associated with $\mathbb{E}$ is the alphagam $\alpha(\mathbb{E}) = \langle V, F_0 \cup F_1 \cup F_2, \phi, \{*, b\} \rangle$. An interpretation of an elphabet $\mathbb{E}$ in an alphagam $\mathbb{A}$ is a function γ such that $\langle \gamma, \alpha(\mathbb{E}), \mathbb{A} \rangle$ is an alphagam map. Strings of signs and expressions of $\mathbb{E}$ will be strings of signs and expressions respectively of $\alpha(\mathbb{E})$.

0.2.3.5 The Polynomial Algebra of an Elphabet. Let $\mathbb{E} = \langle V, F_0, F_1, F_2, \{*, b\} \rangle$ be an elphabet of similarity type F, let $Fns = F_0 \cup F_1 \cup F_2$. We define an F-algebra structure $St(\mathbb{E})$ on the set Strgs of strings of signs of $\mathbb{E}$. Let e, f, and g be arbitrary elements of F_0, F_1 and F_2 respectively, let s, s_1, s_2 be arbitrary strings, and define the operations as follows:

$$e_{St(\mathbb{E})} = e*$$
$$f_{St(\mathbb{E})}(s) = fs$$
$$g_{St(\mathbb{E})}(s_1, s_2) = gbs_1s_2.$$

For $U \subset V$ define the F-polynomial algebra generated by U, $Poly_F(U)$, to be the F-subalgebra of $St(\mathbb{E})$ generated by U; define the polynomial algebra $Poly(\mathbb{E})$ of $\mathbb{E}$ by $Poly(\mathbb{E}) = Poly_F(V)$. The expressions $Expr(\mathbb{E})$ of $\mathbb{E}$ are closed under the operations of St($\mathbb{E}$) , hence Poly($\mathbb{E}$) $\subset$ Expr($\mathbb{E}$) .

0.2.3.6 Proposition. Let $\mathcal{A}$ be an F-algebra with carrier A and let $U \subset A$. For any a in [U], the subalgebra generated by U , one of the following holds:

(1) $a \varepsilon U$

(2) $(Ee)(e \varepsilon F_0$ and $a \equiv e_{\mathcal{A}})$

(3) $(Ef)(Ea_1)(f \varepsilon F_1, a_1 \varepsilon [U]$ and $a \equiv f_{\mathcal{A}}(a_1))$

(4) $(Eg)(Ea_1)(Ea_2)(g \in F_2, a_1 \varepsilon [U], a_2 \varepsilon [U]$ and $a \equiv g_{\mathcal{A}}(a_1, a_2))$.

Proof. The proof is, word for word, the same as that given for 0.2.2.3. □

0.2.3.7 Proposition. For each t in $Poly_F(U)$ exactly one of the following holds (where [U] is the carrier of $Poly_F(U)$):

(1) $t \varepsilon U$

(2) $(E!e)(e \varepsilon F_0$ and $t \equiv e^*)$

(3) $(E!f)(E!t_1)(f \varepsilon F_1, t_1 \varepsilon [U]$ and $t \equiv ft_1)$

(4) $(E!g)(E!t_1)(E!t_2)(g \varepsilon F_2, t_1 \varepsilon [U] , t_2 \varepsilon [U]$ and $t \equiv gbt_1t_2)$.

Proof. We begin with the list of possible representations of t as given in 0.2.3.6. The fact that exactly one of the four clauses must hold follows from the fact that U, F_0, F_1 and F_2 are pair-wise disjoint. Given that one of the clauses holds we argue the uniqueness of the representation. For (1) there is nothing to prove, and for (2) and (3) uniqueness follows from 0.2.1.2. For (4) suppose $gbt_1t_2 = g'bt_1't_2' = t$; by 0.2.1.1 $g' = g$ and $t_1t_2 = t_1't_2'$; by 0.2.1.4, since t_1, t_2, t_1' and t_2' are expressions, $t_1 = t_1'$ and $t_2 = t_2'$. □

<u>0.2.3.8 Proposition</u>. $Poly_F(U)$ is absolutely freely generated by U (this is essentially a repetition of 0.2.2.6).

<u>Proof</u>. Let $\mathcal{A}$ be an F-algebra with carrier A and let β be a functional relation between U and A, then $\beta \subset U \times A$ and $[\beta] \subset [U] \times A$. The projection $Poly_F(U) \times \mathcal{A} \longrightarrow Poly_F(U)$ is an F-algebra homomorphism so the image of [β] is an F-subalgebra containing U , hence the domain of [β] is [U] (the carrier of $Poly_F(U)$).

To show [β] is single valued note that by 0.2.3.6 for any $\langle t,a\rangle \in [\beta]$ one of the following clauses holds:

(1) $\beta(t) = a$

(2) $(Ee)(e \in F_0,\ t = e^* \text{ and } a = e_{\mathcal{A}})$

(3) $(Ef)(Et_1)(Ea_1)(f \in F_1,\ \langle t_1,a_1\rangle \in [\beta],\ t = ft_1$ and $a = f_{\mathcal{A}}(a_1))$

(4) $(Eg)(Et_1)(Et_2)(Ea_1)(Ea_2)(g \in F_2,\ \langle t_1,a_1\rangle \in [\beta],$ $\langle t_2,a_2\rangle \in [\beta],\ t = gbt_1t_2$ and $a = g_{\mathcal{A}}(a_1,a_2))$.

Suppose both $\langle t,a\rangle \in [\beta]$ and $\langle t,a'\rangle \in [\beta]$. We use the representations of t given in 0.2.3.7 to argue by induction on $\ell(t)$ that

$a = a'$.

y $\quad a = \beta(y)$

e^* $\quad a = e_{\mathcal{A}} = a'$

ft_1 $\quad (Ea_1)(Ea_1')\ \langle t_1,a_1\rangle \in [\beta],\ \langle t_1,a_1'\rangle \in [\beta],\ a = f_{\mathcal{A}}(a_1)$,
$a' = f_{\mathcal{A}}(a_1')$; by induction $a_1 = a_1'$, hence $a = a'$.

gbt_1t_2 $\quad (Ea_1)(Ea_1')(Ea_2)(Ea_2')\ \{\langle t_i,a_i\rangle, \langle t_i,a_i\rangle\} \subset [\beta]$ for
$i = 1, 2$; $a = g_{\mathcal{A}}(a_1,a_2)$ and $a' = g_{\mathcal{A}}(a_1',a_2')$; by
induction $a_1 = a_1'$ and $a_2 = a_2'$, hence $a = a'$.

Thus $[\beta]$ is an F-homomorphism from $Poly_F(U)$ to $\mathcal{A}$ that extends β. Another such homomorphism β' extending β must have equalizer containing U , hence $\beta' = [\beta]$. □

<u>0.2.3.9 The realization of polynomials as functions.</u> Let $\mathcal{A} = \langle A,\gamma\rangle$ be an F-algebra. A valuation (see [RS] VI §3) $\beta\colon U \longrightarrow A$ lifts to an F-homomorphism $\tilde{\beta}\colon Poly_F(U) \longrightarrow \mathcal{A}$. Let $t \in Poly_F(U)$, define $t_{\mathcal{A},U}\colon A^U \longrightarrow A$ by $t_{\mathcal{A},U}(\beta) = \tilde{\beta}(t)$; $t_{\mathcal{A},U}$ is called a polynomial function and t is said to term-define $t_{\mathcal{A},U}$ ([Mon]Ch 11, p. 195). For fixed U the set of all $t_{\mathcal{A},U}$ term-defined by some $t \in Poly_F(U)$ can be given the structure of an F-algebra as follows: (1) endow $[A^U,A]$, the set of functions from A^U to A, with the pointwise F-algebra structure $[A^U,\mathcal{A}]$; (2) define $K\colon U \longrightarrow [A^U,A]$ by $(K(x))(\beta) = \beta(x)$, then the algebra of polynomial functions $Poly(U,\mathcal{A})$ can be defined either as (3.1): the subalgebra of $[A^U,\mathcal{A}]$ generated by the set $\{K(x) \mid x \in U\}$ of projections, or equivalently as (3.2): the image of the F-homomorphism $\tilde{K}\colon Poly_F(U) \longrightarrow [A^U,\mathcal{A}]$

that extends K. The equivalence of (3.1) and (3.2) follows from the fact that $\tilde{K}(t) = t_{\mathcal{A},U}$.

A polynomial, $t \in \mathrm{Poly}_F(U)$, is simultaneously an element of any $\mathrm{Poly}_F(W)$ for which $U \subset W$. In contrast, a polynomial function in $\mathrm{Poly}(U,\mathcal{A})$ cannot belong to any $\mathrm{Poly}(W,\mathcal{A})$ where $W \neq U$. The situation is remedied by a canonical embedding $\theta_U\colon \mathrm{Poly}(U,\mathcal{A}) \longrightarrow \mathrm{Poly}(V,\mathcal{A})$ which insures that every polynomial function $t_{\mathcal{A},U}$ is induced by a "global" polynomial function $t_{\mathcal{A}}$ living in $\mathrm{Poly}(V,\mathcal{A})$. Define θ_U by $(\theta_U(t_{\mathcal{A},U}))(\beta) = t_{\mathcal{A},U}(\beta|_U)$ where $\beta \in A^V$. θ_U is well-defined since $t_{\mathcal{A},U} = s_{\mathcal{A},U}$ implies $\theta_U(t_{\mathcal{A},U}) = \theta_U(s_{\mathcal{A},U})$ (even though $t \neq s$ is possible). θ_U is an embedding: If $A \neq \phi$ then any $\beta\colon U \longrightarrow A$ can be extended to a $\beta'\colon V \longrightarrow A$ (so that $\beta = \beta'|_U$); thus if $\theta_U(t_{\mathcal{A},U}) = \theta_U(s_{\mathcal{A},U})$ then $t_{\mathcal{A},U}(\beta'|U) = s_{\mathcal{A},U}(\beta'|U)$ for all $\beta' \in A^V$, hence $t_{\mathcal{A},U}(\beta) = s_{\mathcal{A},U}(\beta)$ for all $\beta \in A^U$, hence $t_{\mathcal{A},U} = s_{\mathcal{A},U}$. If $A = \phi$ then $\mathrm{Poly}(\phi,\mathcal{A}) = \phi$ and if $W \neq \phi$ then $\mathrm{Poly}(W,\mathcal{A}) = \phi$, thus $\theta_\phi\colon \phi \longrightarrow \{\phi\}$ and for $W \neq \phi$, $\theta_W\colon \{\phi\} \longrightarrow \{\phi\}$. Thus for all $\mathcal{A}$, θ_U is an embedding.

<u>0.2.3.10 Interpreting polynomials of an elphabet as expressions of an alphagam.</u> Let γ be an interpretation of the elphabet $\mathbb{E}$ of similarity type F in the alphagam $\mathbb{A}$, and let $\tilde{\gamma}$ be the monoid homomorphism from the set Strgs_1 of strings of signs of $\mathbb{E}$ to the set Strgs_2 of strings of signs of $\mathbb{A}$, induced by γ. For each $s \in \mathrm{Strgs}_1$ we let $\bar{s} = \tilde{\gamma}(s)$. We have already given Strgs_1 an F-algebra structure $\mathrm{St}(\mathbb{E})$; using γ we turn Strgs_2 into an F-algebra $\mathrm{St}(\gamma,\mathbb{A})$ in such a way that $\tilde{\gamma}$ becomes

an F-homomorphism. For each $\theta \in (F_0 \cup F_1 \cup F_2)$ let $\dot{\theta}$ be the corresponding operation on $Strgs_2$; let e, f and g be arbitrary signs of F_0, F_1 and F_2 respectively; and let s, s_1 and s_2 be arbitrary strings of $Strgs_2$; put

$$\dot{e} = \overline{e}\,*$$

$$\dot{f}(s) = \overline{f}s$$

$$\dot{g}(s_1,s_2) = \overline{g}bs_1s_2\ .$$

<u>Proposition.</u> (1) $\tilde{\gamma}$: $St(\mathbf{E}) \longrightarrow St(\gamma,\mathbf{A})$ is an F-homomorphism. (2) The F-algebra $\tilde{\gamma}(Poly_F(U))$ is generated by $\gamma(U)$ in $St(\gamma,\mathbf{A})$, and is contained in the set of expressions of $\mathbf{A}$.

<u>Proof.</u> (1): $\tilde{\gamma}(e\,*) = \overline{e}\,* = \dot{e}$, $\tilde{\gamma}(fs) = \overline{fs} = \dot{f}(\tilde{\gamma}(s))$, and $\tilde{\gamma}(gbs_1s_2) = \overline{gb}\overline{s_1}\overline{s_2} = \dot{g}(\tilde{\gamma}(s_1),\tilde{\gamma}(s_2))$.□ (2): $U \subset Poly_F(U)$, hence $\gamma(U) \subset \tilde{\gamma}(Poly_F(U))$, hence $[\gamma(U)]_F \subset \tilde{\gamma}(Poly_F(U))$.
$U \subset \tilde{\gamma}^{-1}\gamma(U) \subset \tilde{\gamma}^{-1}[\gamma(U)]_F$, hence $Poly_F(U) \subset \tilde{\gamma}^{-1}[\gamma(U)]_F$, hence $\tilde{\gamma}(Poly_F(U)) \subset [\gamma(U)]_F$. Thus $\tilde{\gamma}(Poly_F(U)) = [\gamma(U)]_F$. Finally since every polynomial t in $Poly_F(U)$ is an expression of $\mathbf{E}$, by 0.2.3.3, $\overline{t}$ is an expression of $\mathbf{A}$.□

<u>0.2.3.11 Structures.</u> We extend the notion of a similarity type for algebras to structures. First we introduce a new fixed sign: $\underline{\delta} = \{:\}^{\wedge}$, to be interpreted as "equals". Let $\mathcal{O}_0$, $\mathcal{O}_1$, $\mathcal{O}_2$, P_1, P_2 be pairwise disjoint sets of signs with $\underline{\delta} \in P_2$, define $\mathcal{O}$: $\cup\mathcal{O}_i \longrightarrow \{0,1,2\}$ by $\mathcal{O}_i = \mathcal{O}^{-1}\{i\}$, and P: $\cup P_i \longrightarrow \{1,2\}$ by $P_i = P^{-1}\{i\}$. Elements of $\mathcal{O}_0 \cup \mathcal{O}_1 \cup \mathcal{O}_2$ we call operation signs and of $P_1 \cup P_2$ predicate or

relation signs; operation signs will be nullary, unary or binary accordingly as they are elements of $\mathcal{O}_0$, $\mathcal{O}_1$ or $\mathcal{O}_2$ respectively; predicate signs will be unary or binary accrordingly as they are elements of P_1 or P_2 respectively. By an $\langle\mathcal{O},P\rangle$ structure, or a structure of similarity type $\langle\mathcal{O},P\rangle$, we mean a triple $\mathcal{A} = \langle A,o,p\rangle$ where A is a set, $o\colon \cup\mathcal{O}_i \longrightarrow \cup[A^i,A]$ and $p\colon \cup P_i \longrightarrow \cup[A^i,[2]]$ are such that $o(\mathcal{O}_i) \subset [A^i,A]$ and $p(P_i) \subset [A^i,[2]]$, and $p(\underline{\delta})(a,b) = 1$ iff $a = b$ for $(a,b) \in A \times A$. To conform with the standard interpretation of predicate signs f we put $f^{\mathcal{A}} = (p(f))^{-1}\{1\}$, so that for $f \in P_i$, $f^{\mathcal{A}} \subset A^i$; in particualar $\underline{\delta}^{\mathcal{A}} = \{(a,a) \mid a \in A\}$.

The algebra $\mathcal{A}_{\mathcal{O}} = \langle A,o\rangle$ of similarity type $\mathcal{O}$ we call the underlying algebra of $\mathcal{A}$.

0.2.3.12 Formula elphabets. A similarity type for algebras determines an elphabet from which we can build up polynomials; a similarity type for structures allows us to build up atomic formulas by applying predicate signs to polynomials. To build up more complex formulas we introduce two more new signs: $\underline{i} = \underline{\delta}\hat{}$ and $\underline{\&} = \underline{i}\hat{}$ to be interpreted as "if...then" and "and" respectively. By a formula elphabet we mean a pair $\mathbb{P} = \langle E,P\rangle$ where $el(\mathbb{P}) = \mathbb{E} = \langle V,F_0,F_1,F_2,\{*,b\}\rangle$ is an elphabet, called the underlying elphabet of $\mathbb{P}$, $P\colon P_1 \cup P_2 \longrightarrow \{1,2\}$ is a similarity type for which $P^{-1}\{i\} = P_i \subset F_i$, and $\{\underline{i},\underline{\&}\} \subset (F_2 - P_2)$. By the operation elphabet of $\mathbb{P}$ we mean the elphabet $opn(\mathbb{P}) = \langle V,\mathcal{O}_0,\mathcal{O}_1,\mathcal{O}_2,\{*,b\}\rangle$ where $\mathcal{O}_0 = F_0$, $\mathcal{O}_1 = F_1 - P_1$, and $\mathcal{O}_2 = F_2 - (P_2 \cup \{\underline{i},\underline{\&}\})$. The elphabet determined by a similarity type for algebras $\mathcal{O}\colon \bigcup_i \mathcal{O}_i \longrightarrow \{0,1,2\}$ is $\mathbb{E}(\mathcal{O}) = \langle V,\mathcal{O}_0,\mathcal{O}_1,\mathcal{O}_2,\{*,b\}\ \rangle$, the formula elphabet determined by the similarity type $\langle\mathcal{O},\mathcal{P}\rangle$ is

$\mathbb{P}\langle\mathcal{O},\mathcal{P}\rangle = \langle\mathbb{E}\langle\mathcal{O},\mathcal{P}\rangle,\mathcal{P}\rangle$ where

$\mathbb{E}\langle\mathcal{O},\mathcal{P}\rangle = \langle V,\mathcal{O}_0,\mathcal{O}_1 \cup \mathcal{P}_1,\mathcal{O}_2 \cup \mathcal{P}_2 \cup \{\underline{i},\underline{\&}\},\{*,b\}\rangle$. For a given formula elphabet $\mathbb{P}$, we define its polynomial algebra by $\mathrm{Poly}(\mathbb{P}) = \mathrm{Poly}(\mathrm{opn}(\mathbb{P}))$, its set of unary atomic formulas by

$\mathrm{AtFml}_1(\mathbb{P}) = \{ft \mid f \in \mathcal{P}_1,\ t \in \mathrm{Poly}(\mathbb{P})\}$, its set of binary atomic formulas by $\mathrm{AtFml}_2(\mathbb{P}) = \{f/t,s/ \mid f \in \mathcal{P}_2, \{t,s\} \subset \mathrm{Poly}(\mathbb{P})\}$, its set of equations by $\mathrm{Eq}(\mathbb{P}) = \{\underline{\delta}/t,s/ \mid \{t,s\} \subset \mathrm{Poly}(\mathbb{P})\}$, (thus $\mathrm{Eq}(\mathbb{P}) \subset \mathrm{AtFml}_2(\mathbb{P})$), and its set of atomic formulas by $\mathrm{AtFml}(\mathbb{P}) = \mathrm{AtFml}_1(\mathbb{P}) \cup \mathrm{AtFml}_2(\mathbb{P})$. For $\vec{\varphi} \in \mathrm{Str}(\mathrm{AtFml}(\mathbb{P}))$, $\varphi \in \mathrm{AtFml}(\mathbb{P})$ we define the conjunction $\bigwedge(\vec{\varphi}\varphi)$ inductively as follows:

$$\bigwedge\varphi = \varphi$$

for $\mathrm{Lg}(\vec{\varphi}) \geq 1$, $\bigwedge(\vec{\varphi}\varphi) = \underline{\&}\ /\bigwedge\vec{\varphi},\varphi/$.

We also use the alternative notation $\varphi_1 \wedge \varphi_2 = \underline{\&}\ /\varphi_1,\varphi_2/$ and $\bigwedge_{i=0}^{n-1} \varphi^i$.

The set of conjunctions is

$\mathrm{Conj}(\mathbb{P}) = \{\bigwedge\vec{\varphi} \mid \mathrm{Lg}(\vec{\varphi}) \geq 1,\ \vec{\varphi} \in \mathrm{Str}(\mathrm{AtFml}(\mathbb{P}))\}$, and the set of basic Horn formulas is

$\mathrm{bHf}(\mathbb{P}) = \{\underline{i}/\theta,\varphi/ \mid \theta \in \mathrm{Conj}(\mathbb{P}),\ \varphi \in \mathrm{AtFml}(\mathbb{P})\} \cup \mathrm{AtFml}(\mathbb{P})$. We put $\theta \rightarrow \varphi = \underline{i}/\theta,\varphi/$ for $\theta \in \mathrm{Conj}(\mathbb{P})$ and $\varphi \in \mathrm{AtFml}(\mathbb{P})$. For the empty string ϕ, "$\bigwedge \phi$" remains undefined, but we shall put $(\bigwedge\phi \rightarrow \varphi) = \varphi$ for $\varphi \in \mathrm{AtFml}(\mathbb{P})$; thus $\mathrm{bHf}(\mathbb{P}) = \{\bigwedge\vec{\varphi} \rightarrow \varphi \mid \vec{\varphi}\varphi \in \mathrm{Str}(\mathrm{AtFml}(\mathbb{P}))\}$.

<u>0.2.3.13 The interpretation of basic Horn formulas in an external structure $\mathcal{A}$.</u> For each $t \in \mathrm{Poly}(\mathbb{E}(\mathcal{O}))$ we put $t_{\mathcal{A}} = t_{\mathcal{A}_0}$. For $\theta \in \mathrm{bHf}(\mathbb{P})$ and $\beta \in A^V$ we define "β satisfies θ in $\mathcal{A}$" as follows:

(1) For ft unary atomic, β satisfies ft in $\mathcal{A}$ iff $t_{\mathcal{A}}(\beta) \in f^{\mathcal{A}}$ iff $((p(f))\circ t_{\mathcal{A}})(\beta) = 1$.

(2) For $g/t,s/$ binary atomic, β satisfies $g/t,s/$ in $\mathcal{A}$ iff $(t_{\mathcal{A}}(\beta), s_{\mathcal{A}}(\beta)) \in g^{\mathcal{A}}$ iff $((p(g))\circ(t_{\mathcal{A}} \sqcap s_{\mathcal{A}}))(\beta) = 1$.

(3) Let $\vec{\varphi}$ be a string of n atomic formulas $n \geq 1$ whose $i+1$ member is φ^i $(i \in [n])$. Let $\bigwedge\vec{\varphi} \Rightarrow \psi$ be a basic Horn formula. Then β satisfies $\bigwedge\vec{\varphi} \Rightarrow \psi$ in $\mathcal{A}$ iff β satisfies φ^i in $\mathcal{A}$ for each $i \in [n]$ implies β satisfies ψ in $\mathcal{A}$. When $\theta \in \mathrm{bHf}(\mathbb{P})$ we say θ is valid in $\mathcal{A}$ or $\mathcal{A}$ is a model of θ, and write $\mathcal{A} \models \theta$, if β satisfies θ in $\mathcal{A}$ for all $\beta \in A^V$.

For the special case of the predicate sign $\underline{\delta}$, (2) becomes: β satisfies $\underline{\delta}/t,s/$ in $\mathcal{A}$ iff $t_{\mathcal{A}}(\beta) = s_{\mathcal{A}}(\beta)$; $\underline{\delta}/t,s/$ is valid in $\mathcal{A}$ iff $t_{\mathcal{A}} = s_{\mathcal{A}}$. Thus we shall abbreviate $\underline{\delta}/t,s/$ to $t = s$, so that; $\mathcal{A} \models t = s$ iff $t_{\mathcal{A}} = s_{\mathcal{A}}$.

We extend $\models$ to classes of F-structures $\mathbb{K}$ and sets of basic Horn formulas Γ, Δ from $\mathrm{bHf}(\mathbb{P})$ as in [Mon]p. 196 Definition 11.7.

$\mathbb{K} \models \varphi$ iff for each $\mathcal{A} \in \mathbb{K}$ we have $\mathcal{A} \models \varphi$

$\mathcal{A} \models \Delta$ iff for each $\varphi \in \Delta$ we have $\mathcal{A} \models \varphi$

$\mathbb{K} \models \Delta$ iff for each $\varphi \in \Delta$ we have $\mathbb{K} \models \varphi$

$\Gamma \models \varphi$ iff for each $\mathcal{A}$ such that $\mathcal{A} \models \Gamma$ we have $\mathcal{A} \models \varphi$

i.e. $\{\mathcal{A} | \mathcal{A} \models \Gamma\} \models \varphi$

$\Gamma \models \Delta$ iff for each $\varphi \in \Delta$ we have $\Gamma \models \varphi$.

<u>0.2.3.13.1 Remark:</u> Note that in [CK] the relation $\mathcal{A} \models \varphi$ is defined for φ a sentence, thus to be consistent with this definition we should extend our elphabet to include a quantifier ∀, then for φ abHf define φ' to be

$$\varphi' \equiv \forall x_1 \forall x_2 \ldots \forall x_n \varphi$$

where $s_F(\varphi) \equiv \{x_1, \ldots, x_n\}$. The translation becomes

$$\mathcal{A} \models \varphi \quad \text{iff} \quad \mathcal{A} \models \varphi'$$

Our usage is consistent with that given in [BS] for equational logic when bHf's are restricted to equations. We say (following [CK] p. 32) <u>φ is a consequence of Γ</u> if $\Gamma \models \varphi$. We call each set Γ of basic Horn formulas a basic Horn theory or simply a theory ([CK] p. 36). If $\mathcal{A}$ is an external structure and $\mathcal{A} \models \Delta$ we say $\mathcal{A}$ is an external model of the theory Δ.

<u>0.2.3.14</u> We shall sometimes use "vector" notation for strings. Let $\vec{t} \in \mathrm{Str(Expr)}$ and $\ell_{\mathrm{Expr}}(\vec{t}) \equiv n$ (0.2.1.5) define an expression $\pi(\vec{t})$ by induction on n as follows:

$$\pi(\phi) \equiv * \quad , \quad \pi(t) \equiv t \ ,$$

suppose $\pi(\vec{t})$ is defined where $\ell_{\mathrm{Expr}}(\vec{t}) \geq 1$,

$\pi(\vec{t}t_1) \equiv b(\pi(\vec{t}))t_1 \equiv /\pi(\vec{t}),t_1/$.

(For example if t_1, t_2, t_3 are expressions then $t_1t_2t_3$ is a string of expressions (an element of Str(Expr)) and

$\pi(t_1t_2t_3) \equiv //t_1,t_2/,t_3/ \equiv bbt_1t_2t_3$). Also, define $\mathrm{Lg}(\vec{t}) \equiv \ell_{\mathrm{Expr}}(\vec{t})$ for $\vec{t} \in \mathrm{Str(Expr)}$.

0.2.4 Some special μ-algebras and μ-homomorphisms.

0.2.4.1. We use table 2 to define three μ-algebras. $\mathcal{S}_f$(Vbls) the set of all finite subsets of Vbls is the carrier of both FVbls, the algebra of free variables, and BVbls, the algebra of bound variables. $\mathbb{N}$ is the carrier of Lngth the algebra of lengths. Arbitrary elements of the algebra $\mathcal{A}$ in question are denoted by c, d.

Table 2

Formal Operation	$\mathcal{A}$	Expr	FVbls	BVbls	Lngth
*	$*_{\mathcal{A}}$	*	ϕ	ϕ	1
b	$b_{\mathcal{A}}(c,d)$	bcd	$c \cup d$	$c \cup d$	c+d+1
f	$f_{\mathcal{A}}(c)$	fc	c	c	c+1
qx	$(qx)_{\mathcal{A}}(c)$	qxc	$c-\{x\}$	$c \cup \{x\}$	c+2

0.2.4.2 μ-algebra homomorphisms with domain Expr. Let $\mathcal{A}$ be a μ-algebra with carrier A. A μ-algebra homomorphism from Expr to $\mathcal{A}$ is completely determined by how it behaves on Vbls, the "basis" for Expr. If $\gamma : \text{Vbls} \longrightarrow A$ then $R(\gamma) : \text{Expr} \longrightarrow \mathcal{A}$ is the μ-homomorphism whose restriction to Vbls is γ. In table 3, in the first column we list the algebras defined in table 2, in the second column, for the algebras FVbls, BVbls and Lngth, we define a function γ by giving its value at an arbitrary $x \varepsilon$ Vbls , in the third column we give the name of the homomorphism $R(\gamma)$ thus defined, and in the fourth column we write the value of $R(\gamma)$ at an expression t together with a description of $R(\gamma)(t)$.

Table 3

$\mathcal{A}$	$\gamma(x)$	$R(\gamma)$	$R(\gamma)(t)$: value of $R(\gamma)$ at t
Expr	$\gamma(x)$	$R(\gamma)$	$R(\gamma)(t)$: expression resulting from the replacement of any occurance in t of any variable x, except those occurances immediately preceeded by a $q \varepsilon Q$, by $\gamma(x)$.
FVbls	$\{x\}$	S_F	$S_F(t)$: the set of variables x which have a free occurance in t , or, the set of free variables of t.
BVbls	$\{x\}$	S_B	$S_B(t)$: the set of variables x for which there is a $q \varepsilon Q$ such that there is an occurance of qx within t.
Lngth	1	ℓg	$\ell g(t)$: the length of t.

The descriptions above are consistent with the discussion of occurances in 0.2.2.7. For all $t \varepsilon$ Expr we let $S_{FB}(t) = S_F(t) \cup S_B(t)$. The length function for strings has the properties $\ell(*) = 1$,

$\ell(x) = 1$, $\ell(ft) = \ell(t) + 1$, $\ell(bt_1t_2) = \ell(t_1) + \ell(t_2) + 1$, and $\ell(qxt) = \ell(t) + 2$; so it is a μ-homomorphism from <u>Expr</u> to <u>Lngth</u> , hence $\ell = \ell g$.

<u>0.2.4.3.</u> The set of free variables of an expression t , without quantifiers, is the smallest set of variables U such that t belongs to the set of expressions generated by U.

<u>Proposition.</u> Let t be an expression of the alphagam <Vbls, Fns, ϕ, {*,b} > - i,e, an expression without quantifiers. For each $U \subset$ Vbls let [U] be the set of expressions (without quantifiers) generated by U . The following are equivalent:

(1) $t \varepsilon [U]$

(2) $S_F(t) \subset U$.

<u>Proof.</u> (2) → (1). We show $t' \varepsilon [S_F(t')]$. Let $B = \{t' \mid t' \varepsilon [S_F(t')]\}$. Clearly $Vbls \subset B \subset [Vbls]$. We show B is a subalgebra of [Vbls] . The constant * belongs to B since $* \varepsilon [\phi]$. If $f \varepsilon Fns$ and $t' \varepsilon B$ then $t' \varepsilon [S_F(t')]$, since $S_F(t') = S_F(ft')$ and $[S_F(t')]$ is a subalgebra, $ft' \varepsilon [S_F(ft')]$. For t_1, t_2 expressions, $S_F(t_i) \subset S_F(bt_1t_2)$ for $i = 1, 2$, hence $[S_F(t_i)] \subset [S_F(bt_1t_2)]$ for $i = 1, 2$; thus if $t_i \varepsilon B$ for $i = 1, 2$, then $t_i \varepsilon [S_F(bt_1t_2)]$ hence $bt_1t_2 \varepsilon [S_F(bt_1t_2)]$.

(1) → (2). Let $\tilde{U} = \{t' \mid t' \varepsilon [U]$ and $S_F(t') \subset U\}$. Clearly $U \subset \tilde{U} \subset [U]$. We show $\tilde{U}$ is a subalgebra of [U] . The constant

$*$ belongs to every subalgebra and $S_F(*) = \phi$ hence $* \varepsilon \tilde{U}$. If $f \varepsilon$ Fns , $t' \varepsilon [U]$ and $S_F(t) \subset U$ then $ft' \varepsilon [U]$ and $S_F(ft') \subset U$, hence $ft' \varepsilon \tilde{U}$. If $t_i \varepsilon [U]$ and $S_F(t_i) \subset U$ for $i = 1, 2$, then $bt_1t_2 \varepsilon [U]$ and $S_F(bt_1t_2) \subset U$, hence $bt_1t_2 \varepsilon \tilde{U}$. Thus $\tilde{U} = [U]$. Hence $t \varepsilon [U]$ implies $S_F(t) \subset U$. □

<u>0.2.4.4 Proposition.</u> Let t be an element of the polynomial algebra Poly($\mathbb{E}$) . For each set U of variables of the elphabet $\mathbb{E}$ let $[U]_F$ be the F-subalgebra of Poly($\mathbb{E}$) generated by U , where F is the similarity type of $\mathbb{E}$. The following are equivalent:

(1) $t \varepsilon [U]_F$

(2) $S_F(t) \subset U$.

<u>Proof</u>. (1) → (2). The set of expressions without quantifiers, $[U]$, generated by U is an F-subalgebra of St($\mathbb{E}$) hence $[U]_F \subset [U]$. If $t \varepsilon [U]$, hence by 0.2.4.3 $S_F(t) \subset U$.

(2) → (1). We show $t' \varepsilon [S_F(t')]_F$. Let $C = \{t' \mid t' \varepsilon \text{Poly}(\mathbb{E}) \text{ and } t' \varepsilon [S_F(t')]\}$.

Vbls $\subset C \subset$ Poly($\mathbb{E}$) . We show C is an F-subalgebra of Poly($\mathbb{E}$). If $e \varepsilon F_0$ then $e*$ belongs to all F-subalgegras of Poly($\mathbb{E}$) , hence it belongs to $[\phi]_F = [S_F(e*)]_F$, therefore $e* \varepsilon C$. If $f \varepsilon F_1$ and $t' \varepsilon C$ then $t' \varepsilon [S_F(t')]_F = [S_F(ft')]_F$ hence $ft' \varepsilon [S_F(ft')]_F$. For $g \varepsilon F_2$ and $t_i \varepsilon$ Poly($\mathbb{E}$) for $i = 1, 2$, we have $S_F(t_i) \subset S_F(gbt_1t_2)$ hence $[S_F(t_i)]_F \subset [S_F(gbt_1t_2)]_F$

for $i = 1, 2$: thus if $t_i \in C$ for $i = 1, 2$, $t_i \in [S_F(gbt_1t_2)]_F$ for $i = 1, 2$, so $gbt_1t_2 \in [S_F(gbt_1t_2)]_F$.□

<u>0.2.4.5 Proposition.</u> Let γ be an interpretation of $\mathbf{E}$ in $\mathbf{A}$. For each $t \in \text{Poly}(\mathbf{E})$, $\overline{S_F(t)} = S_F(\overline{t})$ where $\overline{U} \equiv \{\gamma(x) | x \in U\}$ and $\overline{s} \equiv \tilde{\gamma}(s)$.

<u>Proof.</u> Let $B = \{t \in \text{Poly}(E) | \overline{S_F(t)} = S_F(\overline{t})\}$: For each $x \in V$, $\overline{S_F(x)} = \{\gamma(x)\} = S_F(\overline{x})$, hence $V \subset B$. If $e \in F_0$ then $\overline{S_F(e *)} = \phi = S_F(\overline{e} *)$ hence B contains all constants. If $f \in F_1$ and $t \in B$ then $\overline{S_F(ft)} = \overline{S_F(t)} = S_F(\overline{t}) = S_F(\overline{f}\ \overline{t}) = S_F(\overline{ft})$, hence B is closed under the unary operations. If $g \in F_2$, $t_1 \in B$ and $t_2 \in B$ then $\overline{S_F(gbt_1t_2)} = \overline{S_F(t_1)} \cup \overline{S_F(t_2)} = S_F(\overline{t_1}) \cup S_F(\overline{t_2}) = S_F(\overline{g}\ b\ \overline{t_1}\overline{t_2})$ $= S_F(\overline{gbt_1t_2})$, hence B is closed under the binary operations. Thus B is an F-subalgebra of Poly($\mathbf{E}$) containing V, hence $B = \text{Poly}(\mathbf{E})$.□

0.2.5 Substitution: definition and connections with S_F, S_B and C

0.2.5.1 Functions from Vbls to Expr.

Let $\gamma : \text{Vbls} \longrightarrow \text{Expr}$, if $\gamma(x) \not\equiv x$ we say γ moves x and if $\gamma(x) \equiv x$ we say γ fixes x . $\text{Mov}(\gamma)$ is the set of all variables moved by γ . If $\text{Mov}(\gamma)$ is finite $n(\gamma)$ will denote its cardinality. We define γ_x as follows: $\gamma_x(x) \equiv x$ and $\gamma_x(y) \equiv \gamma(y)$ if $y \not\equiv x$; thus if $\text{Mov}(\gamma)$ is finite and $x \in \text{Mov}(\gamma)$ then $n(\gamma_x) \equiv n(\gamma) - 1$. We define $\begin{pmatrix} x \\ s \end{pmatrix} : \text{Vbls} \longrightarrow \text{Expr}$ by $\begin{pmatrix} x \\ s \end{pmatrix}(x) \equiv s$ and $\begin{pmatrix} x \\ s \end{pmatrix}(y) \equiv y$ if $y \not\equiv x$. If $x_1 \equiv x_2$ implies $s_1 \equiv s_2$ we define $\begin{pmatrix} x_1 & x_2 \\ s_1 & s_2 \end{pmatrix} : \text{Vbls} \longrightarrow \text{Expr}$ by $\begin{pmatrix} x_1 & x_2 \\ s_1 & s_2 \end{pmatrix}(x_1) \equiv s_1$, $\begin{pmatrix} x_1 & x_2 \\ s_1 & s_2 \end{pmatrix}(x_2) \equiv s_2$, and $\begin{pmatrix} x_1 & x_2 \\ s_1 & s_2 \end{pmatrix}(y) \equiv y$ if $y \notin \{x_1, x_2\}$. We let α, α', β , β' be arbitrary functions from variables to expressions.

0.2.5.2 Definition of the simultaneous substitution of expressions for variables.

For every α we define $S(\alpha)$ to be a function from Expr to Expr by induction on the length of expressions to which $S(\alpha)$ is applied:

(1) $S(\alpha)(y) \equiv \alpha(y)$

(2) $S(\alpha)(*) \equiv *$

(3) $S(\alpha)(ft_1) \equiv fS(\alpha)(t_1)$

(4) $S(\alpha)(bt_1t_2) \equiv b(S(\alpha)(t_1))(S(\alpha)(t_2))$

(5) $S(\alpha)(qyt_1) \equiv qy(S(\alpha_y)(t_1))$.

(1) and (2) define $S(\alpha)$ on the set of expressions of length 1. In (3), (4) and (5) the t_1 and t_2 are uniquely determined by the expressions to which $S(\alpha)$ is being applied on the left hand sides, as we explained in 0.2.2.5.

We shall sometimes denote $S\begin{pmatrix} s \\ x \end{pmatrix}(t)$ by $t[x|s]$ and $S\begin{pmatrix} x_1 & x_2 \\ s_1 & s_2 \end{pmatrix}(t)$ by $t[x_1,x_2|s_1,s_2]$.

The definition can be extended to strings of expresssions $\vec{t}$ by induction on $Lg(\vec{t})$:

$$S(\alpha)(\phi) \equiv \phi$$

$$S(\alpha)(\vec{t}s) \equiv (S(\alpha)(\vec{t}))(S(\alpha)(s)).$$

The length $Lg(\vec{t})$ is invariant under substitution:

$$Lg(S(\alpha)(\phi)) \equiv Lg(\phi)$$

$$\begin{aligned} Lg(S(\alpha)(\vec{t}s)) &\equiv Lg(S(\alpha)(\vec{t}))(S(\alpha)(s)) \\ &\equiv Lg(S(\alpha)(\vec{t})) + Lg(S(\alpha)(s)) \\ &\equiv Lg(\vec{t}) + 1 \\ &\equiv Lg(\vec{t}s). \end{aligned}$$

0.2.5.3 Proposition. If $x \notin S_F(t)$ then $t[x|s] \equiv t$.

Proof. By induction on the length of t.

$y \not\equiv x$: $y[x|s] \equiv y$

$* :$ $\quad *[x|s] = *$

$ft_1 :$ $\quad x \notin S_F(ft_1)$ so $x \notin S_F(t_1)$, hence

$(ft_1)[x|s] = f(t_1[x|s] = ft_1.$

$/t_1,t_2/ :$ $\quad x \notin S_F/t_1,t_2/$ so $x \notin S_F(t_1)$ and $x \notin S_F(t_2)$,

hence $/t_1,t_2/[x|s] = /t_1[x|s],t_2[x|s]/ = /t_1,t_2/.$

qyt, where $y \neq x$: $x \notin S_F(t_1) - \{y\}$ so $x \notin S_F(t_1)$, hence

$(qyt_1)[x|s] = qy(t_1[x|s]) = qyt_1.$

$qxt_1 :$ $\quad (qxt_1)[x|s] = qxt_1.$ □

<u>0.2.5.4 Proposition.</u>

$(S_F(t) - \{x\}) \subset S_F(t[x|s]) \subset (S_F(t) \cup S_F(s))$ (b)

<u>Proof</u>. By induction on the length of t. We restate (b) in terms of the indicated case.

$y \neq x :$ $\quad \{y\} \subset \{y\} \subset \{y\} \cup S_F(s)$

$x :$ $\quad \phi \subset S_F(s) \subset \{x\} \cup S_F(s)$

$* :$ $\quad \phi \subset \phi \subset S_F(s)$

$ft_1 :$ $\quad S_F(ft_1) = S_F(t_1) \quad S_F((ft_1)[x|s] = S_F(t_1[x|s])$;

(b) holds since it holds for t_1.

$/t_1,t_2/ :$ $\quad S_F/t_1,t_2/ - \{x\} = (S_F(t_1) - \{x\}) \cup (S_F(t_2) - \{x\})$

$S_F(/t_1,t_2/[x|s]) = S_F(t_1[x|s] \cup S_F(t_2[x|s])$

$S_F(/t_1,t_2/ \cup S_F(s) = (S_F(t_1) \cup S_F(s)) \cup (S_F(t_2) \cup S_F(s))$

$S_F(t_i) - \{x\} \subset S_F(t_i[x|s]) \subset S_F(t_i) \cup S_F(s)$ holds for $i = 1, 2$ by induction, hence taking unions (b) holds for $/t_1,t_2/$.

$qxt_1 :$ $\quad S_F(t_1) - \{x\} \subset S_F(t_1) - \{x\} \subset (S_F(t_1) - \{x\}) \cup S_F(s).$

qyt_1 where $y \neq x$: $S_F(qyt_1) - \{x\} = S_F(t_1) - \{x,y\}$

$S_F((qyt_1)[x|s]) = S_F(t_1[x|s] - \{y\}$

$S_F(qyt_1) \cup S_F(s) = (S_F(t_1) - \{y\}) \cup S_F(s).$

By induction we have

$$S_F(t_1) - \{x\} \subset S_F(t_1[x|s]) \subset S_F(t_1) \cup S_F(s) \quad \text{hence}$$

$$(S_F(t_1) - \{x,y\} \subset (S_F(t_1[x|s] - \{y\}) \subset ((S_F(t_1) - \{y\}) \cup S_F(s)). \quad \square$$

<u>0.2.5.5 Proposition.</u>

$$S_F(S(\alpha)(t)) \subset \left(S_F(t) \cup \bigcup_{\alpha(x) \neq x} S_F(\alpha(x)) \right) . \qquad (7)$$

<u>Proof</u>. By induction on the length of t.

y where $\alpha(y) \neq y$: $S_F(\alpha(y)) \subset \left(\{y\} \cup \bigcup_{\alpha(x) \neq x} S_F(\alpha(x)) \right)$

y where $\alpha(y) = y$: $\{y\} \subset \left(\{y\} \cup \bigcup_{\alpha(x) \neq x} S_F(\alpha(x)) \right)$

$*$: $\phi \subset \left(\phi \cup \bigcup_{\alpha(x) \neq x} S_F(\alpha(x)) \right)$

ft_1 : by induction $S_F(S(\alpha)(t_1)) \subset \left(S_F(t_1) \cup \bigcup_{\alpha(x) \neq x} S_F(\alpha(x)) \right)$

$S_F(S(\alpha)(ft_1)) = S_F(S(\alpha)(t_1))$ and

$S_F(ft_1) = S_F(t_1)$ hence (7) holds.

$\langle t_1,t_2 \rangle$: by induction for $i = 1, 2$

$S_F(S(\alpha)(t_i)) \subset S_F(t_i) \cup \left(\bigcup_{\alpha(x) \neq x} S_F(\alpha(x)) \right)$

$S_F(S(\alpha)\langle t_1,t_2 \rangle) = S_F(S(\alpha)(t_1)) \cup S_F(S(\alpha)(t_2))$

$S_F\langle t_1,t_2 \rangle = S_F(t_1) \cup S_F(t_2)$

qyt_1 where $\alpha(y) = y$:

$S_F(S(\alpha)(qyt_1)) = S_F(qy(S(\alpha_y)(t_1))) = (S_F(S(\alpha_y)(t_1)) - \{y\})$

$S_F(qyt_1) \equiv S_F(t_1) - \{y\}$

$\bigcup_{\alpha_y(x) \neq x} S_F(\alpha_y(x)) \subset \bigcup_{\alpha(x) \neq x} S_F(\alpha(x))$ since

$\alpha_y(x) \neq x$ implies $\alpha(x) \neq x$.

By induction

$$S_F(S(\alpha_y)(t_1)) \subset \left(S_F(t_1) \cup \bigcup_{\alpha_y(x) \neq x} S_F(\alpha_y(x)) \right),$$

hence (7) holds.

qyt_1 where $\alpha(y) \equiv y$:

$S_F(S(\alpha)(qyt_1)) \equiv S_F(S(\alpha)(t_1)) - \{y\}$

$S_F(qyt_1) \equiv S_F(t_1) - \{y\}$

by induction

$$S_F(S(\alpha)(t_1)) \subset \left(S_F(t_1) \cup \bigcup_{\alpha(x) \neq x} S_F(\alpha(x)) \right),$$ hence

(7) holds. □

0.2.5.6 Corollary. Let $\alpha : \text{Vbls} \longrightarrow \text{Vbls}$ and let $x_i (i \in [n])$ be a list of variables. For a given expression t we define a sequence of $n + 1$ expressions as follows:

$$t_0 \equiv t$$

$$t_{k+1} \equiv t_k[x_k|\alpha(x_k)] \ , \quad k \in [n] \text{ , then}$$

$$(S_F(t) - \{x_k|k \in [n]\}) \subset S_F(t_n) \subset (S_F(t) \cup \{\alpha(x_k)|k \in [n]\}).$$

Proof. For $n \equiv 1$ this is 0.2.5.4. For $n \geq 2$ we have again by 0.2.5.4

$(S_F(t_{n-1}) - \{x_{n-1}\}) \subset S_F(t_n) \subset (S_F(t_{n-1}) \cup \{\alpha(x_{n-1})\})$ the induction hypothesis for $n - 1$ is

$(S_F(t) - \{x_k | k \in [n-1]\} \subset S_F(t_{n-1}) \subset (S_F(t) \cup \{\alpha(x_k) | k \in [n-1]\})$

combining these two lines of inclusions gives the required result. □

<u>0.2.5.7 Proposition</u>. $S_B(t) \subset S_B(t[x|s]) \subset (S_B(t) \cup S_B(s))$. (8)

<u>Proof</u>. By induction on the length of t.

$y \neq x$: $\phi \subset \phi \subset (\phi \cup S_B(s))$

x : $\phi \subset S_B(s) \subset (\phi \cup S_B(s))$

$*$: $\phi \subset \phi \subset (\phi \cup S_B(s))$

ft_1 : $S_B(ft_1) = S_B(t_1)$ and $S_B(ft_1[x|s]) = S_B(t_1[x|s])$ so (8) follows by induction applied to t_1

$\langle t_1,t_2\rangle$: $S_B\langle t_1,t_2\rangle = S_B(t_1) \cup S_B(t_2)$,

$S_B(\langle t_1,t_2\rangle[x|s]) = S_B(t_1[x|s]) \cup S_B(t_2[x|s])$ and

$S_B(\langle t_1,t_2\rangle \cup S_B(s) = (S_B(t_1) \cup S_B(s)) \cup (S_B(t_2) \cup S_B(s))$,

and by induction, for $i = 1, 2$

$S_B(t_i) \subset S_B(t_i[x|s]) \subset (S_B(t_i) \cup S_B(s))$.

Taking unions $(i = 1, 2)$ gives (8).

qxt_1 : $(S_B(t_1) \cup \{x\}) \subset (S_B(t_1) \cup \{x\}) \subset (S_B(t_1) \cup \{x\} \cup S_B(s))$ is (8).

qyt_1 where $y \neq x$:

$S_B(qyt_1) = S_B(t_1) \cup \{y\}$,

$S_B((qyt_1)[x|s]) = S_B(t_1[x|s] \cup \{y\}$.

by induction and taking unions with $\{y\}$ gives

$$(S_B(t_1) \cup \{y\}) \subset (S_B(t_1[x|s]) \cup \{y\}) \subset (S_B(t_1) \cup \{y\} \cup S_B(s))$$

which is (8). □

<u>0.2.5.8</u> We have immediately from the above Proposition:

$$S_B(t[x|y]) = S_B(t).$$

Moreover if we apply this successively we get:

<u>Corollary</u>: Given x_i, y_i, $i \varepsilon [n]$, sequences of variables and t, define a sequence t_k, $k \varepsilon [n]$, as follows:

$$t_0 = t \ , \quad t_{k+1} = t_k[x_k|y_k] \ , \quad \text{then} \quad S_B(t_n) = S_B(t). \quad \square$$

<u>0.2.5.9 Proposition.</u> $S_B(S(\alpha)(t)) \subset_s \left(S_B(t) \cup \bigcup_{\alpha(x) \neq x} S_B(\alpha(x)) \right)$ (9)

<u>Proof</u>: by induction on the length of t.

y where $\alpha(y) \neq y$: $S_B(\alpha(y)) \subset \left(\phi \cup \bigcup_{\alpha(x) \neq x} S_B(\alpha(x)) \right)$

y where $\alpha(y) = y$: $\phi \subset \left(\phi \cup \bigcup_{\alpha(x) \neq x} S_B(\alpha(x)) \right)$

$*$: $\phi \subset \left(\phi \cup \bigcup_{\alpha(x) \neq x} S_B(\alpha(x)) \right)$

ft_1 : by induction $S_B(S(\alpha)(t_1) \subset \left(S_B(t_1) \cup \bigcup_{\alpha(x) \neq x} S_B(\alpha(x)) \right)$

$S_B(S(\alpha)(ft_1)) = S_B(S(\alpha)(t_1))$ and

$S_B(ft_1) = S_B(t_1)$ hence (9) holds.

$\langle t_1,t_2 \rangle$: by induction for $i = 1, 2$

$S_B(S(\alpha)(t_i)) \subset \left(S_B(t_i) \cup \bigcup_{\alpha(x) \neq x} S_B(\alpha(x)) \right)$

$S_B(S(\alpha)\langle t_1,t_2 \rangle = S_B(S(\alpha)(t_1)) \cup S_B(S(\alpha)(t_2))$

$S_B\langle t_1,t_2 \rangle = S_B(t_1) \cup S_B(t_2)$

qyt_1 where $\alpha(y) \neq y$: $S_B(S(\alpha)(qyt_1)) = S_B(S(\alpha_y)(t_1)) \cup \{y\}$

$$S_B(qyt_1) = S_B(t_1) \cup \{y\} .$$

$$\bigcup_{\alpha_y(x)\neq x} S_B(\alpha_y)(x)) \subset \bigcup_{\alpha(x)\neq x} S_B(\alpha(x)) ,$$

by induction

$$S_B(S(\alpha_y)(t_1)) \subset \left(S_B(t_1) \cup \bigcup_{\alpha_y(x)\neq x} S_B(\alpha_y)(x)) \right) ,$$

hence (9) holds.

qyt_1 where $\alpha(y) = y$: $S_B(S(\alpha)(qyt_1)) = S_B(S(\alpha)(t_1)) \cup \{y\}$.

$$S_B(qyt_1) = S_B(t_1) \cup \{y\}$$

by induction $S_B(S(\alpha)(t_1)) \subset \left(S_B(t_1) \cup \bigcup_{\alpha(x)\neq x} S_B(\alpha(x)) \right)$

hence (9) holds. □

<u>0.2.5.10 Proposition</u>. Length is invariant under substitution of one variable for another.

<u>Proof</u>. By induction on the length of expressions. We want to show that for t, x, y, $\ell g(t[x|y] = \ell g(t)$.

w : $\ell g(w[x|y] = 1 = \ell g(w)$

$*$: $\ell g(*[x|y] = 1 = \ell g(*)$.

ft_1 : $\ell g((ft_1)[x|y]) = \ell g(t_1[x|y]) + 1 = \ell g(t_1) + 1 = \ell g(ft_1)$

bt_1t_2 : $\ell g((bt_1t_2)[x|y]) = \ell g(t_1[x|y]) + \ell g(t_2[x|y]) + 1$

$$= \ell g(t_1) + \ell g(t_2) + 1$$

$$= \ell g(bt_1t_2)$$

qwt_1 where $w \neq x$:

$$\ell g((qwt_1)[x|y]) = \ell g(qw(t_1[x|y])) = \ell g(t_1[x|y]) + 2$$

$$= \ell g(t_1) + 2 = \ell g(qwt_1).$$

qxt_1 : $\ell g((qxt_1)[x|y]) = \ell g(t_1) + 2 = \ell g(qxt_1)$. □

We will need this Proposition for our inductive definition of $\lambda w.t$ in section 0.5.3.2.

0.2.6 The decomposition of substitution

0.2.6.1 Proposition. Suppose $x \notin S_F(\alpha(y))$ for each y such that $y \neq x$, then for all t

$$S\binom{x}{\alpha(x)} \circ S(\alpha_x)(t) = S(\alpha)(t) .$$

Proof. Let $L = S\binom{x}{\alpha(x)} \circ S(\alpha_x)$. We proceed by induction on the length of t.

x : $L(x) = \alpha(x) = S(\alpha)(x)$.

y where $y \neq x$:

$L(y) = S\binom{x}{\alpha(x)} (\alpha(y)) = \alpha(y)$ since $x \notin S_F(\alpha(y))$.

$*$: $L(*) = * = S(\alpha)(*)$

ft_1 : $L(ft_1) = f(L(t_1)) = f(S(\alpha)(t_1))$ by induction

$= S(\alpha)(ft_1)$.

$/t_1,t_2/$: $L/t_1,t_2/ = /L(t_1), L(t_2)/$

$= /S(\alpha)(t_1), S(\alpha)(t_2)/$ by induction

$= S(\alpha)/t_1,t_2/$

qxt_1 : $L(qxt_1) = S\binom{x}{\alpha(x)}(qx(S(\alpha_x)(t_1)))$

$= qx(S(\alpha_x)(t_1))$ by 0.2.5.3

$= S(\alpha)(qxt_1)$.

qyt_1 where $y \neq x$! let $\beta = \alpha_y$ then $\beta(x) = \alpha(x)$ and

$\beta(x) = (\alpha_x)_y$, so

$$\begin{aligned} L(qyt_1) &= S\binom{x}{\beta(x)}\, qyS(\beta_x)(t_1) \\ &= qyS\binom{x}{\beta(x)}\, S(\beta_x)(t_1) \\ &= qyS(\beta)(t_1) \quad \text{by induction} \\ &= S(\alpha)(qyt_1)\ . \quad \square \end{aligned}$$

<u>0.2.6.2 Definition</u>. Let α : Vbls $\longrightarrow$ Expr and x_i $(i \in [n])$ be a list of variables. We define $\underset{i=0}{\overset{k}{S}} \binom{x_i}{\alpha(x_i)}$ by induction on k:

$$\underset{i=0}{\overset{0}{S}} \binom{x_i}{\alpha(x_i)} = S\binom{x_0}{\alpha(x_0)} \qquad (k = 0)$$

$$\underset{i=0}{\overset{k+1}{S}} \binom{x_i}{\alpha(x_i)} = \left[S\binom{x_{k+1}}{\alpha(x_{k+1})}\right] \circ \left[\underset{i=0}{\overset{k}{S}} \binom{x_i}{\alpha(x_i)}\right] \qquad (k \geq 0)$$

<u>Corollary</u>. Let α : Vbls $\longrightarrow$ Expr and x_i $(i \in [n])$ be variables with $\mathrm{Mov}(\alpha) \subset \{x_i | i \in [n]\}$. If $i < j \leq n - 1$ implies $x_i \notin S_F(\alpha(x_j))$, then for all t

$$\underset{i=0}{\overset{n-1}{S}} \binom{x_i}{\alpha(x_i)} (t) = S(\alpha)(t)\ .$$

If $x_i \neq x_j$ implies $x_i \notin S_F(\alpha(x_j))$ for all i, j, and σ is a permutation of $[n]$, then for all t

$$\underset{i=0}{\overset{n-1}{S}} \binom{x_{\sigma i}}{\alpha(x_{\sigma i})} (t) = S(\alpha)(t)\ .$$

<u>Proof</u>. These are immediate consequences 0.2.6.1. $\square$

The restrictions placed on the substitutions that appear in the above corollary are necessary, for example

$$S\begin{pmatrix} x & y \\ y & x \end{pmatrix}(bxy) \equiv byx$$

but

$$S\begin{pmatrix} x \\ y \end{pmatrix} \circ S\begin{pmatrix} y \\ x \end{pmatrix}(bxy) \equiv byy.$$

0.2.6.3. Substitution is going to play a major role in the logic of our language. We will have a rule that a theorem $S(\alpha)(\varphi)$ can be derived from the theorem φ ; but it is a rule that has restrictions on it. For example we do not want to derive $S\begin{pmatrix} y \\ x \end{pmatrix} \exists x(\neg(x=y))$, which is $\exists x(\neg(x=x))$, from $\exists x(\neg(x=y))$ when $x \neq y$. The rule that we will actually verify in a topos will be that $\varphi[x|t]$ is valid if φ is valid, with certain restrictions placed on t once φ and x are given. To establish a more general rule, one where more variables are involved, we will show that $S(\alpha)(\varphi)$ can be expressed as a series of single substitutions $\mathop{S}\limits_{i=0}^{m} \begin{pmatrix} x_i \\ t_i \end{pmatrix} \varphi$; this task is one we can carry out now long before we introduce notions of validity. In the last corollary we have shown how this can be done when α has certain restrictions on it. We can remove those restrictions on α , and arrange that the individual substitutions are not of the "$\exists x \neg(x=y)$ therefore $\exists x \neg(x=x)$" kind. In the next two propositions we will present one way in which $S(\alpha)\varphi$ can be expressed as a series of single substitutions applied to φ . We take up the subject of restrictions on substitution in section 0.2.8.

A single substitution of say s for x in t can be accomplished in two steps: by first substituting a suitably chosen variable u for x and then substituting s for u ; consider the following two examples:

(1) x, y and u are distinct variables, then

$$u\,[x|y] \equiv u \quad \text{but} \quad u[x|u][u|y] \equiv y\ .$$

(2) x, y and u distinct variables, q a quantifier

$$(qux)[x|y] \equiv quy \quad \text{but} \quad (qux)[x|u][u|y] \equiv quu\ .$$

It is thus important that the chosen u not occur in the expression t ; in fact it is sufficient for (3) to hold.

<u>0.2.6.4 Proposition</u>. If u does not occur in t then

(3) $t[x|u][u|s] \equiv t[x|s]$.

<u>Proof</u>. If $x \equiv u$ then neither x nor u is a free variable of t so by 0.2.5.3 (3) holds. We suppose $x \not\equiv u$ and proceed by induction on the length of t.

x : $x[x|u][u|s] \equiv s \equiv x[x|s]$

y where $y \not\equiv x$: $y[x|u][u|s] \equiv y \equiv y[x|s]$

$*$: $*[x|u][u|s] \equiv * \equiv *[x|s]$

ft_1 : $S_{FB}(ft_1) \equiv S_{FB}(t_1)$ so $u \notin S_{FB}(t_1)$

$$(ft_1)[x|u][u|s] \equiv f(t_1[x|u][u|s]) \equiv f(t_1[x|s]) \equiv (ft_1)[x|s]$$

bt_1t_2 : $S_{FB}bt_1t_2 \equiv S_{FB}(t_1) \cup S_{FB}(t_2)$ so for $i \equiv 1, 2,\ u \notin S_{FB}(t_i)$

$$(bt_1t_2)[x|u][u|s] \equiv /t_1[x|u][u|s],t_2[x|u][u|s]/ \equiv /t_1[x|s],t_2[x|s]/$$

$$\equiv (bt_1t_2)[x|s]\ .$$

qxt_1 : $S_{FB}(qxt_1) \equiv S_{FB}(t_1) \cup \{x\}$, so $u \notin S_{FB}(t_1)$.

$(qxt_1)[x|u][u|s] \equiv qxt_1 \equiv (qxt_1)[x|s]$

qyt_1 where $y \neq x$: $S_{FB}(qyt_1) \equiv S_{FB}(t_1) \cup \{y\}$, so $u \notin S_{FB}(t_1)$

$(qyt_1)[x|u][u|s] \equiv (qy(t_1[x|u]))[u|s] \equiv qy(t_1[x|u][u|s])$

$\equiv qy(t_1[x|s]) \equiv (qyt_1)[x|s]$. □

0.2.6.6 Separating factorization of a substitution.

Definition. Let $W \subset$ Vbls and α: Vbls ⟶ Expr such that $Mov(\alpha) \subset W$. We call a pair $\langle\alpha',\tilde{u}\rangle$ where α': Vbls ⟶ Expr and $\tilde{u}$: Vbls ⟶ Vbls, a separating factorization of α through W if

(1) $(\alpha'|_{\tilde{u}(W)})\circ(\tilde{u}|_W) \equiv (\alpha|_W)$, that is, for each $x \in W$, $\alpha'(\tilde{u}(x)) \equiv \alpha(x)$.

(2) $\tilde{u}(x) \equiv x$ for each $x \notin W$.

(3) $\alpha'(x) \equiv x$ for each $x \notin \tilde{u}(W)$.

(4) $\tilde{u}|_W$ is injective.

(5) $\tilde{u}(W) \cap (W \cup \bigcup_{x \in W} S_{FB}(\alpha(x)) \equiv \phi$.

Given α and $\tilde{u}$, α' is uniquely determined by (1) and (3).

Note that we do not have $\alpha'\circ\tilde{u} \equiv \alpha$, in particular if $\alpha(y)$ is not a variable then $(\alpha'\circ\tilde{u})(\tilde{u}(y)) \equiv \alpha'(\tilde{u}(y)) \equiv \alpha(y)$ but $\alpha(\tilde{u}(y)) \equiv \tilde{u}(y)$ and $\tilde{u}(y)$ is a variable so $(\alpha'\circ\tilde{u})(\tilde{u}(y)) \neq \alpha(\tilde{u}(y))$.

Proposition. Let $\langle\alpha',\tilde{u}\rangle$ be a separating factorization of α through W, and let (6) $S_{FB}(t) \cap \tilde{u}(w) \equiv \phi$, then

(7) $S(\alpha')\circ S(\tilde{u})(t) \equiv S(\alpha)(t)$.

Proof. By induction on the length of t. Put $U \equiv \tilde{u}(w)$.

y: $S(\alpha')(S(\tilde{u})(y)) \equiv S(\alpha')(\tilde{u}(y))$ and $S(\alpha)(y) \equiv \alpha(y)$. If $y \in W$ then

$\tilde{u}(y) \varepsilon U$ and $S(\alpha')(\tilde{u}(y)) = \alpha'(u(y)) = \alpha(y)$. If $y \notin W$ then $\tilde{u}(y) = y$, $S(\alpha')(\tilde{u}(y)) = \alpha'(y) = y$ since $y \notin U$, and $\alpha(y) = y$.

$*$: $S(\alpha')\circ S(\tilde{u})(*) = * = S(\alpha)(*)$.

ft_1: $S(\alpha)\circ S(\tilde{u})(ft_1) = S(\alpha')\circ S(\tilde{u})(t_1) = S(\alpha)(t_1) = S(\alpha)(ft_1)$ by induction since $U \cap S_{FB}(t_1) = U \cap S_{FB}(ft_1) = \phi$.

$[t_1,t_2]$: $U \cap S_{FB}[t_1,t_2] = (U \cap S_{FB}(t_1)) \cup (U \cap S_{FB}(t_2)) = \phi$ hence $U \cap S_{FB}(t_i) = \phi$ for $i = 1,2$, so by induction $S(\alpha')\circ S(\tilde{u})(t_i) = S(\alpha)(t_i)$ for $i = 1,2$. Hence

$S(\alpha')S(\tilde{u})[t_1,t_2] = [S(\alpha')S(\tilde{u})(t_1),S(\alpha')S(\tilde{u})(t_2)] = S(\alpha)[t_1,t_2]$.

qyt_1: $S(\alpha)(qyt_1) = qyS(\alpha_y)(t_1)$

$S(\alpha')S(\tilde{u})(qyt_1) = S(\alpha')qy(S(\tilde{u}_y)(t_1)) = qyS((\alpha')_y)S(\tilde{u}_y)(t_1)$.

qyt_1 where $y \notin W$: $\alpha_y = \alpha$, $\tilde{u}_y = \tilde{u}$ by (6) $S_{FB}(t_1) \cap U = \phi$ and $y \notin U$ $S(\alpha')S(\tilde{u})(t_1) = S(\alpha)(t_1)$ and $(\alpha')_y = \alpha'$, hence $S(\alpha')S(\tilde{u})(qyt_1) = qyS(\alpha')S(\tilde{u})(t_1) = qyS(\alpha)(t_1) = S(\alpha)(qyt_1)$.

qyt_1 where $x \varepsilon W$: $(\alpha')_x = \alpha'$ (by (3) since $x \notin U$), put $u = \tilde{u}(x)$ α' has been defined so that it satisfies the hypothesis of the Corollary 0.2.6.2, thus for any s

$S(\alpha')(s) = S((\alpha')_u)S\binom{u}{\alpha(x)}(s)$. By 0.2.5.5 we have $S_F(S(\tilde{u}_x)(t_1)) \subset (S_F(t_1) \cup (U - \{u\}))$ hence $u \notin S_F(S(\tilde{u}_x)(t_1))$, hence $S\binom{u}{\alpha(x)}S(\tilde{u}_x)(t_1) = S(\tilde{u}_x)(t_1)$ by 0.2.5.3, hence

$$
\begin{aligned}
S(\alpha')S(\tilde{u})(qxt_1) &= qxS((\alpha')_x)S(\tilde{u}_x)(t_1) = qxS(\alpha')S(\tilde{u}_x)(t_1) \\
&= qxS((\alpha')_u)S\binom{u}{\alpha(x)}S(\tilde{u}_x)(t_1) \\
&= qxS((\alpha')_u)S(\tilde{u}_x)(t_1) \quad qxS(\alpha_x)(t_1) \\
&= S(\alpha)(qxt_1) .
\end{aligned}
$$

The second to last equality holds by virtue of the induction hypothesis since $\langle(\alpha')_u, \tilde{u}_x\rangle$ is a separating factorization of α_x through $(W - \{x\})$.□

0.2.6.7 Proposition. If u is distinct from x and u does not occur in t then $(t[x|s])[u|z] = t[x|s[u|z]]$.

Proof. by induction on the length of t.

x: $\quad (x[x|s])[u|z] = s[u|z] = x[x|s[u|z]]$

y where $y \neq x$: by hypothesis we also have $u \neq y$

$$(y[x|s])[u|z] = y[u|z] = y = y[x|s[u|z]]$$

$*$: $\quad (*[x|s])[u|z] = * = *[x|s[u|z]]$

ft_1 : $\quad S_{FB}(ft_1) = S_{FB}(t_1)$ hence u does not occur in t_1

$$((ft_1)[x|s])[u|z] = f((t_1[x|s])[u|z]) = f(t_1[x|s[u|z]]) = ft_1[x|s[u|z]].$$

$\langle t_1,t_2\rangle$: $\quad S_{FB}\langle t_1,t_2\rangle = S_{FB}(t_1) \cup S_{FB}(t_2)$ hence u does not occur in either t_1 or t_2

$$(\langle t_1,t_2\rangle[x|s])[u|z] = \langle(t_1[x|s])[u|z],\ (t_2[x|s])[u|z]\rangle$$
$$= \langle t_1[x|s[u|z]],\ t_2[x|s[u|z]]\rangle$$
$$= \langle t_1,t_2\rangle[x|s[u|z]] \ .$$

qxt_1 : $\quad S_{FB}(qxt_1) = S_{FB}(t_1) \cup \{x\}$ thus u does not occur in t,

$$((qxt_1)[x|s])[u|z] = qx(t_1[u|z]) = qxt_1 = (qxt_1)[x|s[u|z]] \ .$$

qyt_1 where $y \neq x$: $S_{FB}(qyt_1) = S_{FB}(t_1) \cup \{y\}$ thus $u \neq y$ and u does not occur in t,

$$((qyt_1)[x|s])[u|z] = (qy(t_1[x|s]))[u|z] = qy((t_1[x|s])[u|z]$$
$$= qy(t_1[x|s[u|z]]) = (qyt_1)[x|s[u|z]] \ . \ \square$$

0.2.6.8 Corollary. If u is distinct from x and u does not occur in either t or r then

$$(t[x|r[x|u]])[u|x] = t[x|r] .$$

Proof. By 0.2.6.7, taking z to be x and s to be r[x|u], we have

$$\begin{aligned}(t[x|r[x|u]])[u|x] &= t[x|(r[x|u])[u|x]] \\ &= t[x|r[x|x]] \\ &= t[x|r] .\end{aligned}$$

The second equality depends on the fact that, since u does not occur in r, we can apply 0.2.6.5. □

0.2.7 Shifting variables. We shall need, in section 0.6.9, to talk about transformations of expressions which result from unrestricted changes of variables, that is, we want to extend functions of variables such as $\begin{pmatrix} u & v & w & x \\ v & u & x & w \end{pmatrix}$ to a function of expressions under which ∀ububvw would become ∀vbvbux . We formalize this. Let γ be a function from Vbls to Vbls , extend γ to a function γ_1, of signs as follows: $\gamma_1(x) \equiv \gamma(x)$ for x a variable, $\gamma_1(s) \equiv s$ for s a sign which is not a variable i.e. an element of Fns ∪ Fix . Then γ_1 extends to an endomorphism γ_2 of Strgs.

0.2.7.1 Proposition. For any endomorphism of variables, γ , and any expression t, $\gamma_2(t)$ is an expression.

Proof. by induction on the length of expressions.

$*$: $\gamma_2(*) \equiv \gamma_1(*) \equiv *$, an expression

y : $\gamma_2(y) \equiv \gamma(y)$, a variable

ft_1 : $\gamma_2(ft_1) \equiv f\gamma_2(t_1)$

bt_1t_2 : $\gamma_2(bt_1t_2) \equiv b(\gamma_2(t_1))(\gamma_2(t_2))$

qyt_1 : $\gamma_2(qyt_1) \equiv q(\gamma(y))(\gamma_2(t_1))$. □

We denote the endomorphism of Expr which is the restriction of γ_2 by $\bar{\gamma}$. We say γ shifts t to t' where $\bar{\gamma}(t) \equiv t'$ if $\gamma|_{S_{FB}(t)}$ is bijective.

<u>0.2.7.2 Proposition</u>. Let γ : Vbls $\longrightarrow$ Vbls act bijectively on $S_{FB}(t)$ then (1) $\gamma(S_F(t)) = S_F(\bar{\gamma}(t))$ and (2) $\gamma(S_B(t)) = S_B(\bar{\gamma}(t))$.

<u>Proof</u> of (1) by induction on the length of t.

y : $\gamma(S_F(y)) = \{\gamma(y)\}$, $S_F(\bar{\gamma}(y)) = S_F(\gamma(y)) = \{\gamma(y)\}$.

$*$: $\gamma(S_F(*)) = \phi = S_F(\bar{\gamma}(*))$.

ft_1 : $\gamma(S_F(t_1)) = \gamma(S_F(t_1)) = S_F(\bar{\gamma}(t_1)) = S_F(f\bar{\gamma}(t_1)) = S_F(\bar{\gamma}(ft_1))$.

bt_1t_2 : $\gamma(S_F(bt_1t_2) = \gamma(S_F(t_1) \cup S_F(t_2)) = (\gamma(S_F(t_1)) \cup \gamma(S_F(t_2)))$
$= (S_F(\bar{\gamma}(t_1)) \cup (S_F(\bar{\gamma}(t_2))) = S_F(b(\bar{\gamma}(t_1))(\bar{\gamma}(t_2))) = S_F(\bar{\gamma}(bt_1t_2))$.

qyt_1 : $\gamma(S_F(qyt_1)) = \gamma(S_F(t_1) - \{y\} = \gamma(S_F(t_1)) - \{\gamma(y)\}$ (since γ is bijective on $S_F(t) \cup \{y\}$)
$= (S_F(\bar{\gamma}(t_1)) - \{\gamma(y)\}) = S_F(q(\gamma(y))(\bar{\gamma}(t_1))) = S_F(\bar{\gamma}(qyt_1))$.

(2): by induction on the length of t.

y : $\gamma(S_B(y)) = \phi = S_B(\bar{\gamma}(y))$

$*$, ft_1, bt_1t_2 : the proof is the same as for (1) with S_F replaced throughout by S_B.

qyt_1 : $\gamma(S_B(qyt_1)) = \gamma(S_B(t_1) \cup \{y\}) = (\gamma(S_B(t_1)) \cup \{\gamma(y)\})$
$= (S_B(\bar{\gamma}(t_1)) \cup \{\gamma(y)\}) = S_B(q(\gamma(y))(\bar{\gamma}(t_1))) = S_B(\bar{\gamma}(qyt_1))$. □

0.2.8 The condition "s is free to be substituted for y in t".

Here we define a condition on a single substitution, referred to in 0.2.6.4, which is sufficient to make the rule of substitution of 0.6.2.1 valid.

0.2.8.1 Definition. For each y we define a function $ffr(y) : \mathrm{Expr} \longrightarrow \mathcal{P}(\mathrm{Expr})$ by induction on the length of the expression t ; $ffr(y)(t)$ is called the set of expressions which are free to be substituted for y in t.

(f_1) w : $ffr(y)(w) = \mathrm{Expr}$

(f_2) $*$: $ffr(y)(*) = \mathrm{Expr}$

(f_3) ft_1 : $ffr(y)(ft_1) = ffr(y)(t_1)$

(f_4) $/t_1,t_2/$: $ffr(y)/t_1,t_2/ = ffr(y)(t_1) \cap ffr(y)(t_2)$.

(f_5) qyt_1 : $ffr(y)(qyt_1) = \mathrm{Expr}$

(f_6) qwt_1 where $y \notin S_F(t_1)$ and $w \neq y$: $ffr(y)(qwt_1) = \mathrm{Expr}$

(f_7) qwt_1 where $y \in S_F(t_1)$ and $w \neq y$:

$$ffr(y)(qwt_1) = ffr(y)(t_1) \cap \{s \mid w \notin S_F(s)\} .$$

If $s \in ffr(y)(t)$ we say s is free to be substituted for y in t . From the definition we can see that $y \in ffr(y)(t)$ always.

Clearly if we want to prove by induction on the length of t that for certain triples $\langle s,y,t \rangle$ we have $s \in ffr(y)(t)$, it suffices to examine only the induction steps (f_3), (f_4) and (f_7). This will be the case in the next two propositions.

0.2.8.2 Proposition. $y \notin S_F(t)$ implies $ffr(y)(t) = \mathrm{Expr}$.

<u>Proof</u>: by induction on the length of t.

(f_3) ft_1 : $S_F(ft_1) = S_F(t_1)$ hence $y \notin S_F(t_1)$

$ffr(y)(ft_1) = ffr(y)(t_1) = Expr$

(f_4) $\langle t_1,t_2\rangle$: $S_F\langle t_1,t_2\rangle = S_F(t_1) \cup S_F(t_2)$ hence $y \notin S_F(t_i)$, $i = 1,2$

$ffr(y)\langle t_1,t_2\rangle = (ffr(y)(t_1)) \cap (ffr(y)(t_2)) = Expr$.

(f_7) qwt_1 where $y \in S_F(t_1)$ and $w \neq y$: This is not possible since

$S_F(qwt_1) = S_F(t_1) - \{w\}$ hence $y \in S_F(qwt_1)$. □

<u>0.2.8.3 Proposition</u>. $S_F(s) \cap S_B(t) = \phi$ implies $s \in ffr(y)(t)$ for any y.

<u>Proof</u>: by induction on the length of t.

(f_3) ft_1 : $S_F(s) \cap S_B(t_1) = S_F(s) \cap S_B(ft_1) = \phi$ hence

$s \in ffr(y)(t_1) = ffr(y)(ft_1)$.

(f_4) $\langle t_1,t_2\rangle$: $S_F(s) \cap S_B\langle t_1,t_2\rangle = (S_F(s) \cap S_B(t_1)) \cup (S_F(s) \cap S_B(t_2))$

$= \phi$ hence $s \in ffr(y)(t_i)$ for $i = 1, 2$,

hence $s \in ffr(y)(t_1) \cap ffr(y)(t_2) = ffr(y)\langle t_1,t_2\rangle$.

(f_7) qwt_1 where $y \in S_F(t_1)$ and $w \neq y$:

$\phi = S_F(s) \cap (S_B(t_1) \cup \{w\})$ hence $\phi = S_F(s) \cap S_B(t_1)$ and $w \notin S_F(s)$

hence $s \in (ffr(y)(t_1) \cap \{r | w \notin S_F(r)\}) = ffr(y)(qwt_1)$. □

<u>Corollary</u>. If $u \notin S_B(t)$ then $u \in ffr(y)(t)$ for any y . □

<u>0.2.8.4</u>. The condition $s \in ffr(y)(t)$ allows us to calculate $S_F(t[y|s])$ exactly in terms $S_F(t)$, $\{y\}$, and $S_F(s)$, thus improving on 0.2.5.4. From 0.2.5.3 the condition $y \notin S_F(t)$

implies $t[y|s] = t$, so the significant case is $y \in S_F(t)$.

<u>Proposition</u>. If $y \in S_F(t)$ and $s \in ffr(y)(t)$ then

$$S_F(t[y|s]) = (S_F(t) - \{y\}) \cup S_F(s) .$$

<u>Proof</u>: by induction on the length of t .

(f_1) y : $S_F(y[y|s]) = S_F(s) = (S_F(y) - \{y\}) \cup S_F(s)$

(f_2) $*$: in this case the hypothesis is not satisfied.

(f_3) ft_1 : $y \in S_F(t_1)$ and $s \in ffr(y)(t_1)$

$$S_F((ft_1)[y|s]) = S_F(t_1[y|s]) = (S_F(t_1) - \{y\}) \cup S_F(s)$$
$$= (S_F(ft_1) - \{y\}) \cup S_F(s)$$

(f_4) $/t_1,t_2/$: $y \in (S_F(t_1) \cup S_F(t_2))$ and $s \in ffr(y)(t_1) \cap ffr(y)(t_2)$.

$$S_F/t_1,t_2/[y|s]) = S_F(t_1[y|s]) \cup S_F(t_2[y|s])$$

$$(S_F(/t_1,t_2/) - \{y\}) \cup S_F(s) = (S_F(t_1) - \{y\}) \cup (S_F(t_2) - \{y\}) \cup S_F(s)$$

If $y \in S_F(t_i)$ then $S_F(t_i[y|s]) = (S_F(t_i) - \{y\}) \cup S_F(s)$.

If $y \notin S_F(t_i)$ then $S_F(t_i[y|s] = S_F(t_i) = (S_F(t_i) - \{y\})$.

We know that for some $i \in \{1,2\}$, $y \in S_F(t_i)$, hence

for $\{i,j\} = \{1,2\}$,

$$S_F(/t_1,t_2/[y|s]) = (S_F(t_i) - \{y\}) \cup (S_F(t_j) - \{y\}) \cup S_F(s)$$
$$= (S_F(/t_1,t_2/) - \{y\}) \cup S_F(s) .$$

(f_5) qyt_1 : hypothesis not satisfied

(f_6) qwt_1 where $y \notin S_F(t_1)$ and $w \neq y$: hypothesis not satisfied

(f_7) qwt_1 where $y \in S_F(t_1)$ and $w \neq y$: by hypothesis

$s \in ffr(y)(t_1)$ and $w \notin S_F(s)$,

hence $S_F(t_1[y|s]) = (S_F(t_1) - \{y\}) \cup S_F(s)$

$S_F((qwt_1)[y|s]) = (S_F(t_1[y|s]) - \{w\}) = ((S_F(t_1) - \{y\}) \cup S_F(s)) - \{w\}$

$= (S_F(t_1) - \{y,w\}) \cup S_F(s)$ since $w \notin S_F(s)$.

$(S_F(qwt_1) - \{y\}) \cup S_F(s) = (S_F(t_1) - \{y,w\}) \cup S_F(s)$. □

<u>0.2.8.5 Proposition</u>. If $\alpha(x) \varepsilon ffr(x)(t)$ for all x then

$S_F(S(\alpha(t)) = (S_F(t) - Mov(\alpha)) \cup \bigcup \{S_F(\alpha(x)) | x \varepsilon (Mov(\alpha) \cap S_F(t))\}$.

<u>Proof</u>: by induction on the length of t . Let $R(\alpha,t)$ denote the right hand side.

$R(\alpha,y) = (\{y\}- Mov(\alpha)) \cup \bigcup \{S_F(\alpha(x)) | x \varepsilon (Mov(\alpha) \cap \{y\})\}$

y where $\alpha(y) = y$: $R(\alpha,y) = \{y\} = S_F(S(\alpha)(y))$

y where $\alpha(y) \neq y$: $R(\alpha,y) = S_F(\alpha(y)) = S_F(S(\alpha(y))$

$*$: $R(\alpha,*) = \phi = S_F(S(\alpha(*))$.

ft_1 : $\alpha(x) \varepsilon ffr(x)(t_1)$ hence $R(\alpha,t_1) = S_F(S(\alpha(t_1))$

$R(\alpha,ft_1) = (S_F(t_1) - Mov(\alpha)) \cup \bigcup \{S_F(\alpha(x)) | x \varepsilon (Mov(\alpha) \cap S_F(t_1))\}$

$= R(\alpha,t_1) = S_F(S(\alpha)(t_1)) = S_F(S(\alpha)(ft_1))$.

$\langle t_1,t_2\rangle$: $\alpha(x) \varepsilon ffr(x)(t_1) \cap ffr(x)(t_2)$ for all x

hence $R(\alpha,t_i) = S_F(S(\alpha)(t_i))$ for $i = 1, 2$.

$R(\alpha,\langle t_1,t_2\rangle)$

$= ((S_F(t_1) \cup S_F(t_2))-Mov(\alpha)) \cup \bigcup \{S_F(\alpha(x)) | x \varepsilon Mov(\alpha) \cap (S_F(t_1) \cup S_F(t_2))\}$

$= R(\alpha,t_1) \cup R(\alpha,t_2) = S_F(S(\alpha)\langle t_1,t_2\rangle)$.

qyt_1 : by hypothesis, for all x, $\alpha(x) \varepsilon ffr(x)(qyt_1)$.

We prove that, for all x, $\alpha_y(x) \varepsilon ffr(x)(t_1)$.

If $y = x$ then $\alpha_y(y) \varepsilon ffr(y)(t_1)$, as remarked after the definition (0.2.8.1).

If $y \neq x$ and $x \notin S_F(t_1)$ then $ffr(x)(t_1) = Expr$, so

$$\alpha_y(x) \in ffr(x)(t_1) \ .$$

If $y \neq x$ and $x \in S_F(t_1)$ then (f_7) applies, so

$\alpha(x) \in ffr(x)(qyt_1)$ implies $\alpha_y(x) \in ffr(x)(t_1)$ and $y \notin S_F(\alpha(x))$.

Hence for all x , $\alpha_y(x) \in ffr(x)(t_1)$. By induction

$$R(\alpha_y,t_1) = S_F(S(\alpha_y)(t_1)) \ .$$

Hence $R(\alpha,qyt_1)$

$$= ((S_F(t_1)-\{y\})-Mov(\alpha)) \cup \bigcup\{S_F(\alpha(x)\ x\in Mov(\alpha)\cap(S_F(t_1)-\{y\})\}$$

$$= (S_F(t_1) - Mov(\alpha_y)) \cup \bigcup\{S_F(\alpha(x)) | x \in (Mov(\alpha_y) \cap S_F(t_1))\}$$

$$= R(\alpha_y,t_1) = S_F(S(\alpha_y)(t_1))$$

$$= S_F(S(\alpha)(qyt_1)) \ . \ \square$$

0.2.9 Admissible substitution. We extend the definition of 0.2.8.1 to substitutions involving more than one variable.

0.2.9.1 Definition. α is an admissible substitution for t if for all x we have $\alpha(x) \in ffr(x)(t)$.

0.2.9.2. We fix our notation: $x_i (i \in [n])$ and $u_i (i \in [n])$ are sequences of variables, t is an expression, and we define the sequence $t_i (i \in [n+1])$ as follows: $t_0 = t$, $t_{i+1} = t_i[x_i|u_i]$ for $i \in [n]$.

Proposition. If $\{u_i | i \in [n]\} \cap S_B(t) = \phi$ then $u_i \in ffr(x_i)(t_i)$ for each $i \in [n]$.

<u>Proof</u>. By the corollary of 0.2.5.8 $S_B(t) = S_B(t_i)$ so $u_i \in S_B(t_i)$, hence by the corollary of 0.2.8.3 $u_i \in ffr(x_i)(t_i)$. □

That is, at each step - t_i to t_{i+1} - along the way from t to t_n, the variable u_i is free to be substituted for x_i in t_i, provided none of the u_i is a bound variable of t. If the $x_i (i \in [n])$ are distinct from each other and from u_i, then 0.2.6.2 implies that $S(\tilde{u})(t) = \bigodot_{i=0}^{n-1} \begin{pmatrix} x_i \\ u_i \end{pmatrix} (t)$ where $\tilde{u}(x_i) = u_i$ for $i \in [n]$ and $\tilde{u}(w) = w$ otherwise. Under the same restrictions, the $S(\alpha')$ of (3) in 0.2.6.6 can be broken down in the same manner as $S(\tilde{u})$; thus

$$S(\alpha')(S(\tilde{u})(t)) = \bigodot_{i=0}^{n-1} \begin{pmatrix} u_i \\ \alpha(x_i) \end{pmatrix} (S(\tilde{u})(t)) .$$

We want to show that if α is admissible for t then for each j we have: $\alpha(x_{j+1})$ is free to be substituted for u_{j+1} in $\bigodot_{i=0}^{j} \begin{pmatrix} u_i \\ \alpha(x_i) \end{pmatrix} (S(\tilde{u})(t))$. Taken together, the decompositions of $S(\tilde{u})$ and $S(\alpha')$ will then exhibit $S(\alpha)(t)$ as the composition of a series of individual substitutions each of which is admissible.

<u>0.2.9.3 Proposition</u>. Suppose $n \geq 1$. We maintain the notation of 0.2.9.2. Under the conditions (1) $u_0 \notin S_F(t)$,

(2) $\{u_i | i \in [n]\} \cap S_B(t) = \phi$, (3) $u_0 \notin \{x_i | i \in [n]\}$, and (4) $u_0 \notin \{u_i | i \neq 0$ and $i \in [n]\}$ we have

$$x_0 \in S_F(t) \qquad \text{iff} \quad u_0 \in S_F(t_n) .$$

Proof: by induction on n.

$n = 1$ We have $u_0 \notin S_F(t)$, $u_0 \notin S_B(t)$, $u_0 \neq x_0$ and $t_1 = t[x_0|u_0]$. We want to show $x_0 \in S_F(t)$ iff $u_0 \in S_F(t[x_0|u_0])$. Suppose $x_0 \notin S_F(t)$ then $t[x_0|u_0] = t$, hence $u_0 \notin S_F(t[x_0|u_0])$. Suppose $x_0 \in S_F(t)$. Since $u_0 \notin S_B(t)$, by 0.2.8.3 we have $u_0 \in ffr(x_0)(t)$. We compute $S_F(t[x_0|u_0])$ using 0.2.8.5 : $S_F(t[x_0|u_0]) = (S_F(t) - \{x_0\}) \cup \{u_0\}$, hence $u_0 \in S_F(t[x_0|u_0])$.

$n > 1$ We assume that $x_0 \in S_F(t)$ iff $u_0 \in S_F(t_{n-1})$; we want to show that $u_0 \in S_F(t_{n-1})$ iff $u_0 \in S_F(t_{n-1}[x_n|u_n])$. If $x_n \notin S_F(t_{n-1})$ this is immediate, so assume $x_n \in S_F(t_{n-1})$. By 0.2.5.8 $S_B(t) = S_B(t_{n-1})$ so $u_n \notin S_B(t_{n-1})$, by 0.2.8.3 $u_n \in ffr(x_n)(t_{n-1})$. Computing :

$$S_F(t_{n-1}[x_n|u_n]) = \begin{cases} S_F(t_{n-1}) & \text{if } x_n = u_n \\ (S_F(t_{n-1}) - \{x_n\}) \cup \{u_n\} & \text{if } x_n \neq u_n. \end{cases}$$

Since u_0 is distinct from x_n and u_n (by (3) and (4)), we have $u_0 \in S_F(t_{n-1})$ iff $u_0 \in S_F(t_{n-1}[x_n|u_n])$. □

0.2.9.4. Suppose x is a free variable of $\bar{t}$ and u does not occur in $\bar{t}$, then, after substituting u for x in $\bar{t}$, all expressions free to be substituted for x in $\bar{t}$ are again free to be substituted for u in $\bar{t}[x|u]$.

Proposition. Given $x \neq u$ and $\bar{t}$ such that $u \notin S_{FB}(\bar{t})$ then

$$ffr(x)(\bar{t}) \subset ffr(u)(\bar{t}[x|u]) .$$

Proof: by induction on the length of $\bar{t}$. Referring to the definition of 0.2.8.1.

y : $y[x|u]$ is a variable, thus $ffr(u)(y[x|u]) = Expr$.

$*$: $ffr(u)(*[x|u]) = Expr$.

$f\bar{t}_1$: $u \notin S_{FB}(\bar{t})$ hence

$$ffr(x)(f\bar{t}_1) = ffr(x)(\bar{t}_1) \subset ffr(u)(\bar{t}_1[x|u]) = ffr(u)((f\bar{t}_1)[x|u]) .$$

$/\bar{t}_1,\bar{t}_2/$: $u \notin S_{FB}(\bar{t}_i)$ for $i = 1, 2$

$$\begin{aligned} ffr(x)/\bar{t}_1,\bar{t}_2/ &= ffr(x)(\bar{t}_1) \cap ffr(x)(\bar{t}_2) \\ &\subset ffr(x)(\bar{t}_1[x|u]) \cap ffr(x)(\bar{t}_2[x|u]) \\ &= ffr(x)(/\bar{t}_1,\bar{t}_2/[x|u]) . \end{aligned}$$

$qx\bar{t}_1$: $ffr(u)((qx\bar{t}_1)[x|u]) = ffr(u)(qx\bar{t}_1)$, $u \notin S_F(qx\bar{t}_1)$ so by 0.2.8.2 $ffr(u)(qx\bar{t}_1) = Expr$.

$qy\bar{t}_1$ where $x \neq y$: $S_{FB}(qy\bar{t}_1) = S_{FB}(\bar{t}_1) \cup \{y\}$, hence $u \neq y$ and $u \notin S_{FB}(\bar{t}_1)$, so by induction

$$ffr(x)(\bar{t}_1) \subset ffr(u)(\bar{t}_1[x|u]) .$$

If $u \notin S_F(\bar{t}_1[x|u])$ then $ffr(u)((qy\bar{t}_1)[x|u]) = Expr$.

If $u \in S_F(\bar{t}_1[x|u])$ then $x \in S_F(\bar{t}_1)$ so (f_7) applies to both sides i.e. $ffr(x)(qy\bar{t}_1) = ffr(x)(\bar{t}_1) \cap \{s|y \notin S_F(s)\}$ and

$$ffr(u)(qy(\bar{t}_1[x|u])) = ffr(u)(\bar{t}_1[x|u]) \cap \{s|y \notin S_F(s)\} ,$$

hence $ffr(x)(qy\bar{t}_1) \subset ffr(u)((qy\bar{t}_1)[x|u])$. □

0.2.9.5. Suppose x and u are free variables of $\bar{t}$ and that u is not a free variable of r, then all expressions free to be substituted for u in $\bar{t}$ are also free to be substituted for u in $\bar{t}[x|r]$.

Proposition. Given x, u, $\bar{t}$ and r such that $u \notin S_F(r)$ then

$$ffr(u)(\bar{t}) \subset ffr(u)(\bar{t}[x|r]) .$$

Proof: by induction on the length of $\bar{t}$.

x : $ffr(u)(x[x|r]) = ffr(u)(r) = Expr$ by 0.2.8.2.

y where $y \neq x$: $ffr(u)(y[x|r]) = ffr(u)(y) = Expr$.

$*$: $ffr(u)(*[x|r]) = ffr(u)(*) = Expr$.

$f\bar{t}_1$: $ffr(u)(f\bar{t}_1) = ffr(u)(\bar{t}_1) \subset ffr(u)(\bar{t}_1[x|r]) = ffr(u)((f\bar{t}_1[x|r])$.

$/\bar{t}_1,\bar{t}_2/$: $ffr(u)/\bar{t}_1,\bar{t}_2/$

$= (ffr(u)(\bar{t}_1) \cap ffr(u)(\bar{t}_2)) \subset (ffr(u)(\bar{t}_1[x|r]) \cap ffr(u)(\bar{t}_2[x|r]))$

$\subset ffr(u)(/\bar{t}_1,\bar{t}_2/[x|r])$.

$qx\bar{t}_1$: $ffr(u)(qx\bar{t}_1) = ffr(u)((qx\bar{t}_1)[x|r])$.

$qy\bar{t}_1$ where $y \neq x$: If $ffr(u)(\bar{t}[x|r]) = Expr$ then the inclusion holds. The only way that $ffr(u)(\bar{t}[x|r]) \neq Expr$ is if (f_7) holds.

$qy\bar{t}_1$ where $y \neq x$, $u \in S_F(\bar{t}_1[x|r])$, $y \neq u$:

$$ffr(u)((qy\bar{t}_1)[x|r]) = ffr(u)(qy(\bar{t}_1[x|r]))$$

$$= ffr(u)(\bar{t}_1[x|r]) \cap \{s|y \notin S_F(s)\} .$$

By 0.2.5.4 $S_F(\bar{t}_1[x|r]) \subset S_F(\bar{t}_1) \cup S_F(r)$, hence

$u \in S_F(\bar{t}_1)$, hence (f_7) applies again:
$ffr(y)(qy\bar{t}_1) = ffr(u)(\bar{t}_1) \cap \{s : y \notin S_F(s)\}$. By induction $ffr(u)(\bar{t}_1) \subset ffr(u)(\bar{t}_1[x|r])$, hence $ffr(u)((qy\bar{t}_1)[x|r]) \subset ffr(u)(qy\bar{t}_1)$. □

0.2.9.6. After each of the u_i have been substituted for the respective x_i to produce a term t_n , any term free to be substituted for x_i in t is then free to be substituted for u_i in t_n .

Proposition. Suppose $n \geq 1$, the $u_i (i \in [n])$ are distinct, the $x_i (i \in [n])$ are distinct and $\{u_i : i \in [n]\} \cap (S_{FB}(t) \cup \{x_i | i \in [n]\}) = \phi$, then for each $j \in [n]$, $ffr(x_j)(t) \subset ffr(u_j)(t_n)$.

Proof. We prove first that $ffr(x_{n-1})(t) \subset ffr(u_{n-1})(t_n)$. This follows from 0.2.9.4 provided that $u_{n-1} \notin S_{FB}(t_{n-1})$. By 0.2.5.8 $S_B(t_{n-1}) = S_B(t)$ hence $u_{n-1} \notin S_B(t_{n-1})$. By 0.2.5.5 $S_F(t_{n-1}) \subset (S_F(t) \cup \{u_i | i \in [n-1]\})$, hence $u_{n-1} \notin S_F(t_{n-1})$. We now proceed by induction. If $n = 1$ we apply what has just been proven. If $n > 1$, supposing the statement is true for $n-1$, we remove u_{n-1} and x_{n-1} from the lists of variables then for any $j \in [n-1]$ we have $ffr(x_j)(t) \subset ffr(u_j)(t_{n-1})$. By 0.2.9.5 , since $u_j \neq u_{n-1}$, $ffr(u_j)(t_{n-1}) \subset ffr(u_j)(t_{n-1}[x_{n-1}|u_{n-1}])$, hence $ffr(x_j)(t) \subset ffr(u_j)(T_n)$ for $j \in [n-1]$; for $j = n-1$ this is what we first proved. □

0.2.9.7 Proposition. Given $u_i (i \in [j+1])$ distinct, $\alpha_i (i \in [j+1])$ expressions, t' an expression, define $t_0' = t'$,

$t'_{i+1} = t_i[u_i|\alpha_i]$ $(i \in [j+1])$. If $u_j \notin S_F(\alpha_i)$ for each $i \in [j+1]$ then $ffr(u_j)(t') \subset ffr(u_j)(t_j')$.

Proof: by induction on j . For $j = 0$ the statement of the proposition reduces to : given u_0, α_0 , t', define $t_1' = t'[u_0|\alpha_0]$; if $u_0 \notin S_F(\alpha_0)$ then $ffr(u_0)(t') \subset ffr(u_0)(t'[u_0|\alpha_0])$, which follows from 0.2.9.5 . For $j > 0$, omit u_{j-1} and α_{j-1} from their respective lists, then by induction $ffr(u_j)(t') \subset ffr(u_j)(t'_{j-1})$. Applying 0.2.9.5 again we have, $u_{j-1}, u_j, \alpha_{j-1}, t'_{j-1}$ such that $u_j \notin S_F(\alpha_{j-1})$, so $ffr(u_j)(t'_{j-1}) \subset ffr(u_j)(t'_{j-1}[u_{j-1}|\alpha_{j-1}])$ that is $ffr(u_j)(t'_{j-1}) \subset ffr(u_j)(t_j')$, hence $ffr(u_j)(t') \subset ffr(u_j)(t_j')$. □

0.2.9.8 Proposition. Given x_i, u_i $(i \in [n])$, $t' = t_n$, t_i' as in 0.2.9.6. If $\{u_i|i \in [n]\} \cap S_{FB}(t) = \phi$, the u_i are distinct, $\{u_i|i \in [n]\} \cap \{x_i|i \in [n]\} = \phi_S$ with x_i distinct $(i \in [n])$, $\{u_i|i \in [n]\} \cap \bigcup_{i \in [n]} S_F(\alpha(x_i))) = \phi$, and α is admissible then for each $i \in [n]$

$$\alpha(x_i) \in ffr(u_i)(t_i') .$$

Proof. By hypothesis $\alpha(x_i) \in ffr(x_i)(t)$ for each $i \in [n]$. By 0.2.9.6 $ffr(x_i)(t) \subset ffr(u_i)(t')$. By 0.2.9.7, $ffr(u_i)(t') \subset ffr(u_i)(t_i')$. Hence

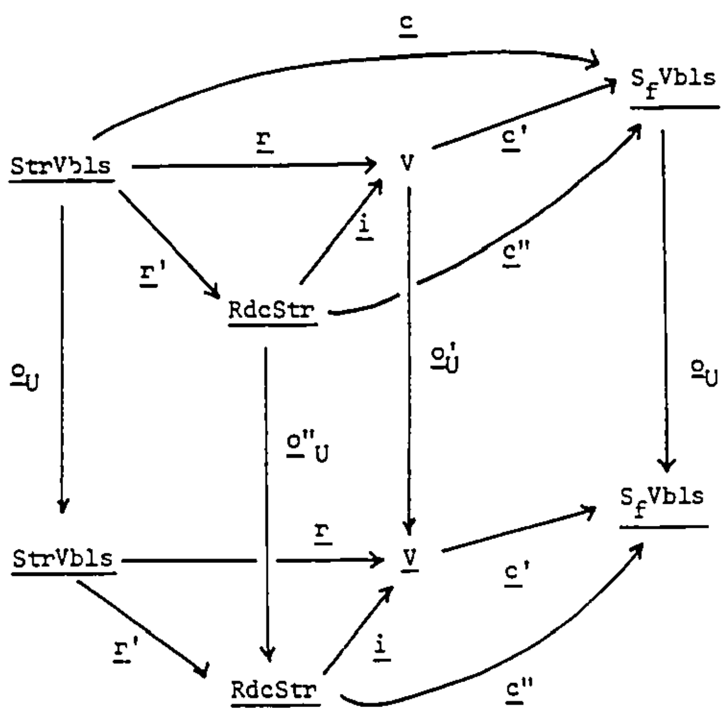
$$\alpha(x_i) \in ffr(u_i)(t_i') . \quad \square$$

0.2.10 Operations on Strings of Variables.

The material in this section does not depend in any way on the special role that the set of variables has in an alphagam, and, but for the purpose we have in mind for the operations defined here, we might just as well have placed the section after Corollary 0.2.1.3 and used an arbitrary set Σ^* instead of StrVbls .

We introduce and develop properties of five operations on StrVbls : (1) the content of a string, which maps a string to the set of variables occuring in a string; (2) the omission of a set of variables from a string; (3) the join of a pair of strings, which concatenates the first string with the string that arises from omitting the content of the first string from the second; (4) the reduction of a string, which is the omission from a string of each occurance of a variable which is to the right of some other occurance of the same variable; and (5) the replacement of one variable by another in a string. Strings that have undergone a reduction - that is, strings of distinct variables - turn up in Definition 0.3.4.1 as the first components of "augmented terms" $\langle \vec{w}, t \rangle$ where they can be thought of as indicating the arguments of a "function" of several variables.

Since the monoid StrVbls $=$ <StrVbls,ϕ,conc> is freely generated by the set Vbls , in order to define a monoid homomorphism $\underline{h}$: $\underline{\text{StrVbls}} \to \underline{M}$ it will suffice to give its values on Vbls . We choose deliberately suggestive notation for functional values, notation which we separate from the names of monoid homomorphisms; thus $\underline{c}$, $\underline{c}'$, $\underline{c}''$ will be distinct monoid homomorphisms and if $\vec{w}$ belongs to each of their domains we put $\underline{\{\vec{w}\}} = \underline{c}(\vec{w}) = \underline{c}'(\vec{w}) = \underline{c}''(\vec{w})$. We collect all homomorphisms associated with the first four of our operations in the following reference diagram (the replacement homomorphisms appear in 0.2.10.13).



<u>0.2.10.1 Definition</u>. Let $\underline{S_f\text{Vbls}} = \langle S_f\text{Vbls}, \phi, \cup \rangle$ be the monoid of finite subsets of Vbls under union. We define the monoid homomorphism $\underline{c} : \underline{\text{StrVbls}} \longrightarrow \underline{S_f\text{Vbls}}$ on Vbls by $\underline{c}(x) = \{x\}$. For each string $\vec{w}$ we put $\underline{\{\vec{w}\}} = \underline{c}(\vec{w})$ and call $\underline{\{\vec{w}\}}$ <u>the content of $\vec{w}$</u> . By definition

(1.1) $\underline{\{x\}} = \{x\}$

Since $\underline{c}$ preserves operations we have

(1.2) $\underline{\{\phi\}} = \phi$

and (1.3) $\underline{\{\vec{u}\vec{v}\}} = \underline{\{\vec{u}\}} \cup \underline{\{\vec{v}\}}$.

Behind our choice of curly brackets for $\underline{c}$ are such equations as (1.1) and $\underline{\{wxy\}} = \{w,x,y\}$, where w, x and y are variables.

<u>0.2.10.2 Proposition</u>.

(2.1) If $\ell(\vec{u}_1) \leq \ell(\vec{v}_1)$, $\ell(\vec{u}_2) \leq \ell(\vec{v}_2)$ and $\vec{u}_1\vec{u}_2 = \vec{v}_1\vec{v}_2$ then $\vec{u}_1 = \vec{v}_1$ and $\vec{u}_2 = \vec{v}_2$.

(2.2) If $\underline{\{\vec{u}\}} = \phi$ then $\vec{u} = \phi$.

(2.3) If $x \notin \underline{\{\vec{u}\}}$ and $\vec{u}x\vec{v} = \vec{u}_1x\vec{v}_1$ then $\ell(\vec{u}) \leq \ell(\vec{u}_1)$.

(2.4) If $x \notin \underline{\{\vec{u}\vec{u}_1\}}$ and $\vec{u}x\vec{v} = \vec{u}_1x\vec{v}_1$ then $\vec{u} = \vec{u}_1$ and $\vec{v} = \vec{v}_1$.

(2.5) If $x \in \underline{\{\vec{u}\}}$ then there exist $\vec{w}$ and $\vec{v}$ such that $\vec{u} = \vec{w}x\vec{v}$ and $x \notin \underline{\{\vec{w}\}}$.

<u>Proof</u> (2.1). Since $\ell(\vec{u}_1) + \ell(\vec{u}_2) = \ell(\vec{v}_1) + \ell(\vec{v}_2)$ we have $\ell(\vec{u}_1) = \ell(\vec{v}_1)$ and $\ell(\vec{u}_2) = \ell(\vec{v}_2)$. By 0.2.1.1 (with $n = \ell(\vec{u}_1)$) we have $\vec{u}_1 = \vec{v}_1$ and $\vec{u}_2 = \vec{v}_2$. □ (2.2) If $\ell(\vec{w}) = 0$ then $\vec{w} = \phi$; so suppose $\ell(\vec{w}) \geq 1$, then $\vec{w} = \vec{u}x$ for some $\vec{u}$ and some x (by 0.2.1.1 again) hence $\underline{\{\vec{w}\}} \neq \phi$ since $x \in \underline{\{\vec{w}\}}$. Thus if $\underline{\{\vec{w}\}} = \phi$ then $\ell(\vec{w}) = 0$ and $\vec{w} = \phi$. □ (2.3) Suppose $\ell(\vec{u}) > \ell(\vec{u}_1)$ then $\ell(\vec{u}) \geq \ell(\vec{u}_1x)$ so by 0.2.1.3, since

$\vec{u}(x\vec{v}) = (\vec{u}_1 x)\vec{v}_1$, there exists $\vec{w}$ such that $\vec{u} = \vec{u}_1 x\vec{w}$, hence $x \in \underline{\{\vec{u}\}}$, contradicting $x \notin \underline{\{\vec{u}\}}$, hence $\ell(\vec{u}) \leq \ell(\vec{u}_1)$. □ (2.4) By (2.3) $\ell(\vec{u}) = \ell(\vec{u}_1)$ hence, by 0.2.1.1, $\vec{u} = \vec{u}_1$ and $\vec{v} = \vec{v}_1$. □ (2.5) If $\vec{u} = \phi$ the statement is vacuously true. We continue by induction on $\ell(\vec{u})$, assuming the statement true for $\vec{u}$ and then proving it true for $\vec{u}y$. If $y = x$ and $x \notin \underline{\{\vec{u}\}}$ then we put $\vec{w} = \vec{u}$ and $\vec{v} = \phi$. If $y \neq x$ then necessarily $x \in \underline{\{\vec{u}\}}$ hence by induction there exist $\vec{w}_1$ and $\vec{v}_1$ such that $\vec{u} = \vec{w}_1 x\vec{v}_1$ and $x \notin \underline{\{\vec{w}_1\}}$; thus we put $\vec{w} = \vec{w}_1$ and $\vec{v} = \vec{v}_1 y$. □

0.2.10.3 Definition. For each finite set U of variables we define a monoid homomorphism

$\underline{0}_U$: StrVbls $\longrightarrow$ StrVbls on Vbls by $\underline{0}_U(x) = x$ if $x \in U$ and $\underline{0}_U(x) = \phi$ if $x \in (\text{Vbls} - U)$. For each $\vec{w}$ we put $\vec{w} \dot{-} U = \underline{0}_U(\vec{w})$ and call $w \dot{-} U$ the omission of U from w . By definition

$$(3.1) \quad x \dot{-} U = \begin{cases} \phi & \text{if } x \in U \\ x & \text{if } x \notin U . \end{cases}$$

Since $\underline{0}_U$ preserves operations we have

$$(3.2) \quad \phi \dot{-} U = \phi$$

$$(3.3) \quad (\vec{uv}) \dot{-} U = (\vec{u} \dot{-} U)(\vec{v} \dot{-} U) .$$

Our choice of notation for $\underline{0}_U$ is prompted by the strong connections

$\underset{\sim}{O}_U$ has with the monoid homomorphism $\underset{\sim}{O}_U : \underline{\mathcal{S}_f\text{Vbls}} \to \underline{\mathcal{S}_f\text{Vbls}}$ defined by $\underset{\sim}{O}_U(W) = W - U$; since $\underline{\mathcal{S}_f\text{Vbls}}$ is freely generated as a semilattice with zero by the set $\{\{x\} | x \in \text{Vbls}\}$, $\underset{\sim}{O}_U$ can also be defined by $\underset{\sim}{O}_U(\{x\}) = \{x\}$ if $x \in U$ and $\underset{\sim}{O}_U(\{x\}) = \phi$ if $x \notin \text{Vbls}$.

0.2.10.4 Proposition

(4.1) $\underset{\sim}{\{}\vec{w} \dot{-} U\underset{\sim}{\}} = \underset{\sim}{\{}\vec{w}\underset{\sim}{\}} - U$

(4.2) $\vec{w} \dot{-} \underset{\sim}{\{}\vec{w}\underset{\sim}{\}} = \phi$

(4.3) $\ell(\vec{w} \dot{-} U) \le \ell(\vec{w})$

(4.4) $\ell(\vec{w} \dot{-} U) = \ell(\vec{w})$ implies $\vec{w} \dot{-} U = \vec{w}$

(4.5) $U \cap \underset{\sim}{\{}\vec{w}\underset{\sim}{\}} = \phi$ implies $\vec{w} \dot{-} U = \vec{w}$

(4.6) $\vec{w} \dot{-} \phi = \vec{w}$

(4.7) $\vec{w} \dot{-} (U \cup V) = (\vec{w} \dot{-} U) \dot{-} V = (\vec{w} \dot{-} V) \dot{-} U$.

Proof. (4.1) To show $\underline{c} \circ \underset{\sim}{o}_U = \underset{\sim}{o}_U \circ \underline{c}$ it suffices to show the homomorphisms $\underline{c} \circ \underset{\sim}{o}_U$ and $\underset{\sim}{o}_U \circ \underline{c}$ are the same on Vbls ; that is, that $\underset{\sim}{\{}x \dot{-} U\underset{\sim}{\}} = \underset{\sim}{\{}x\underset{\sim}{\}} - U$. If $x \in U$ both sides become ϕ ; if $x \notin U$ both sides become $\{x\}$. □ (4.2) By (4.1) $\underset{\sim}{\{}\vec{w} \dot{-} \underset{\sim}{\{}\vec{w}\underset{\sim}{\}}\underset{\sim}{\}} = \phi$ and hence by (2.2) $\vec{w} \dot{-} \underset{\sim}{\{}\vec{w}\underset{\sim}{\}} = \phi$. □ (4.3) $\ell(\phi \dot{-} U) = 0 = \ell(\vec{w})$. We continue by induction. $\ell((\vec{wy}) \dot{-} U) = \ell(\vec{w} \dot{-} U) + \ell(y \dot{-} U) \le \ell(\vec{w}) + \ell(y) = \ell(\vec{wy})$. □ (4.4) By (3.2) $\phi \dot{-} U = \phi$. We continue by induction. Assume $\ell((\vec{wy}) \dot{-} U) = \ell(\vec{wy})$. Then $\ell(\vec{w} \dot{-} U) + \ell(y \dot{-} U) = \ell(\vec{w}) + \ell(y)$. By (4.3) we must have $\ell(\vec{w} \dot{-} U) = \ell(\vec{w})$ and $\ell(y \dot{-} U) = \ell(y)$. By induction $\vec{w} \dot{-} U = \vec{w}$ and from (3.1) $y \dot{-} U = y$. Hence $(\vec{wy}) \dot{-} U = \vec{wy}$. □ (4.5) By

(3.2) $\phi \dot{-} U = \phi$. We continue by induction. Assume $U \cap \underline{\{\vec{wy}\}} = \phi$. Then $U \cap \underline{\{\vec{w}\}} = \phi$ and $y \notin U$. By induction $\vec{w} \dot{-} U = \vec{w}$ and since $y \notin U$, $y \dot{-} U = y$. Hence $(\vec{wy}) \dot{-} U = \vec{wy}$. □ (4.6) by (4.5). □ (4.7) It suffices to show that the homomorphisms $\underline{o}_{U \cup V}$ and $\underline{o}_V \circ \underline{o}_U$ agree on Vbls . If $x \notin (U \cup V)$ then $x \dot{-} (U \cup V) = x = (X \dot{-} U) \dot{-} V$; if $x \in U$ then $x \dot{-} (U \cup V) = \phi = \phi \dot{-} V = (x \dot{-} U) \dot{-} V$; and if $x \in (V - U)$ then $x \dot{-} (U \cup V) = \phi = (x \dot{-} V) = (x \dot{-} U) \dot{-} V$. The second equality follows since $U \cup V = V \cup U$. □

<u>0.2.10.5 Definition</u>. For every pair $\langle\vec{u}, \vec{v}\rangle$ of strings we define the <u>join of $\vec{u}$ with $\vec{v}$</u> , $\vec{u} \vee \vec{v}$, by

(5.1) $\vec{u} \vee \vec{v} = \vec{u}(\vec{v} \dot{-} \underline{\{\vec{u}\}})$.

An explanation of the origin of this operation will follow the next proposition where we work out its basic properties.

<u>0.2.10.6 Proposition</u>.

(5.2) $\underline{\{\vec{u} \vee \vec{v}\}} = \underline{\{\vec{u}\}} \cup \underline{\{\vec{v}\}}$

(5.3) $\phi \vee \vec{u} = \vec{u} = \vec{u} \vee \phi$

(5.4) $(\vec{u} \vee \vec{v}) \vee \vec{w} = \vec{u} \vee (\vec{v} \vee \vec{w})$

(5.5) $(\vec{u} \vee \vec{v}) \dot{-} U = (\vec{u} \dot{-} U) \vee (\vec{v} \dot{-} U)$

(5.6) $\vec{u} \vee \vec{v} \vee \vec{u} = \vec{u} \vee \vec{v}$

<u>Proof</u>. (5.2) $\underline{\{\vec{u} \vee \vec{v}\}} = \underline{\{\vec{u}\}} \cup (\underline{\{\vec{v}\}} - \underline{\{\vec{u}\}}) = \underline{\{\vec{u}\}} \cup \underline{\{\vec{v}\}}$ by (1.3) and (4.1). □ (5.3) $\phi(\vec{u} \dot{-} \{\phi\}) = \vec{u} = \vec{u}\,(\phi \dot{-} \underline{\{\vec{u}\}})$ by (1.2), (4.6), and (3.2). □ (5.4) $\vec{u}\,(\vec{v} \dot{-} \underline{\{\vec{u}\}})(\vec{w} \dot{-} \underline{\{\vec{u}(\vec{v} \dot{-} \underline{\{\vec{u}\}})\}})$

$= \vec{u}(\vec{v} \dot{-} \underline{\{\vec{u}\}})(\vec{w} \dot{-} (\underline{\{\vec{u}\}} \cup (\underline{\{\vec{v}\}} - \underline{\{\vec{u}\}})))$ by (1.3) and (4.1).
$\underline{\{\vec{u}\}} \cup (\underline{\{\vec{v}\}} - \underline{\{\vec{u}\}}) = \underline{\{\vec{v}\}} \cup \underline{\{\vec{u}\}}$, and $\vec{w} \dot{-} (\underline{\{\vec{v}\}} \cup \underline{\{\vec{u}\}})$
$= (\vec{w} \dot{-} \underline{\{\vec{v}\}}) \dot{-} \underline{\{\vec{u}\}}$ by (4.7), hence $(\vec{u} \vee \vec{v}) \vee \vec{w}$
$= \vec{u}(\vec{v} \dot{-} \underline{\{\vec{u}\}})((\vec{w} \dot{-} \underline{\{\vec{v}\}}) \dot{-} \underline{\{\vec{u}\}}) = \vec{u}((\vec{v}(\vec{w} \dot{-} \underline{\{\vec{v}\}})) \dot{-} \underline{\{\vec{u}\}}) = \vec{u} \vee (\vec{v} \vee \vec{w})$
by using (3.3). □ (5.5) $(\vec{u} \dot{-} U) \vee (\vec{v} \dot{-} U)$
$= (\vec{u} \dot{-} U)((\vec{v} \dot{-} U) \dot{-} \underline{\{\vec{u} \dot{-} U\}})$ and $(\vec{v} \dot{-} U) \dot{-} \underline{\{\vec{u} \dot{-} U\}}$
$= \vec{v} \dot{-} (U \cup (\underline{\{\vec{u}\}} - U)) = \vec{v} \dot{-} (\underline{\{\vec{u}\}} \cup U) = (\vec{v} \dot{-} \underline{\{\vec{u}\}}) \dot{-} U$ by (4.7),
(4.1) and (4.7) again; hence $(\vec{u} \dot{-} U) \vee (\vec{v} \dot{-} U)$
$= (\vec{u} \dot{-} U)((\vec{v} \dot{-} \underline{\{\vec{u}\}}) \dot{-} U) = (\vec{u}(\vec{v} \dot{-} \underline{\{\vec{u}\}})) \dot{-} U = (\vec{u} \vee \vec{v}) \dot{-} U$ by (3.3). □
(5.6) $\vec{u} \vee (\vec{v} \vee \vec{u}) = \vec{u}((\vec{v} \vee \vec{u}) \dot{-} \underline{\{\vec{u}\}}) = \vec{u}((\vec{v} \dot{-} \underline{\{\vec{u}\}}) \vee (\vec{u} \dot{-} \underline{\{\vec{u}\}}))$
$= \vec{u} \vee \vec{v}$ by (5.5), (4.2) and (5.3). □

Equations (5.3) and (5.4) tell us that the structure $\underline{V} = \langle \mathrm{StrVbls}, \phi, \vee \rangle$ is a monoid, and equations (1.2) and (5.2) mean that taking the content of a string is a homomorphism $\underline{c}' : \underline{V} \longrightarrow \underline{\mathcal{S}_f\mathrm{Vbls}}$. We now explain how it is possible to arrive at (5.3) and (5.4) indirectly using (5.2). Equations (4.6) and (4.7) mean that omission is an action of the monoid $\underline{\mathcal{S}_f\mathrm{Vbls}}$ on StrVbls , and equations (3.2) and (3.3) mean that the operations of $\underline{\mathrm{StrVbls}}$ are compatible with the action. Thus the structure $\langle \mathcal{S}_f\mathrm{Vbls} \times \mathrm{StrVbls}, \langle \phi,\phi \rangle, \odot \rangle$, where $\odot$ is given by

$$\langle U, \vec{u} \rangle \odot \langle V, \vec{v} \rangle = \langle U \cup V, \vec{u}(\vec{v} \dot{-} U) \rangle$$

is a monoid, namely the semidirect product. The pairs $\langle U, \vec{u} \rangle$ for which U is the content of $\vec{u}$ include $\langle \phi, \phi \rangle$, by (1.2), and are closed under $\odot$, by (5.2); thus

$<\{<\{\vec{u}\}, \vec{u}> \mid \vec{u} \in \text{Str Vbls}\}, <\phi, \phi>, \cdot >$ is a monoid. This monoid, being in one-to-one correspondence (via the second projection) with Str Vbls , induces a monoid structure $\underline{V} \blacksquare <\text{Str.Vbls}, \phi, \vee>$ on Str Vbls ; thus equations (5.3) and (5.4) hold. Equation (5.5) (together with (3.2)) states that omission is a monoid homomorphism $\underset{\sim}{o}_{u} : \underline{V} \longrightarrow \underline{V}$.

<u>0.2.10.7 Definition</u>. The monoid homomorphism $\underline{r} : \underline{\text{Str Vbls}} \longrightarrow \underline{V}$ is given by $\underline{r}(x) \blacksquare x$. For each string $\vec{w}$ we put $\bigvee\vec{w} \blacksquare \underline{r}(\vec{w})$. By definition

(7.1) $\bigvee x \blacksquare x$.

Since $\underline{r}$ preserves operations we have

(7.2) $\bigvee\phi \blacksquare \phi$

and ([illegible] $\bigvee(\vec{uv}) \blacksquare (\bigvee\vec{u}) \vee (\bigvee\vec{v})$

We call $\bigvee\vec{w}$ the <u>join of $\vec{w}$</u> or the <u>reduction of $\vec{w}$</u> ; if $\bigvee\vec{w} \blacksquare \vec{w}$ we say <u>$\vec{w}$ is reduced</u>, we put $\text{RdcStr} \blacksquare \{\vec{w} \mid \bigvee\vec{w} \blacksquare \vec{w}\}$, i.e. the set of reduced strings.

<u>0.2.10.8 Proposition</u>.

(8.1) $\{\bigvee\vec{w}\} \blacksquare \{\vec{w}\}$

(8.2) $\bigvee(\vec{uv}) \blacksquare (\bigvee\vec{u})\ ((\bigvee\vec{v}) \dot{-} \{\vec{u}\})$

(8.3) $\bigvee(\vec{u}x) \blacksquare (\bigvee\vec{u})(x \dot{-} \{\vec{u}\})$

(8.4) $\bigvee(x \dot{-} U) \blacksquare x \dot{-} U$

(8.5) $\Psi(\vec{w} \dot{-} U) = (\Psi\vec{w}) \dot{-} U$

(8.6) $\Psi(\vec{u} \vee \vec{v}) = (\Psi\vec{u}) \vee (\Psi\vec{v})$

(8.7) $\Psi\Psi\vec{w} = \Psi\vec{w}$

(8.8) $\ell(\Psi\vec{w}) \leq \ell\vec{w}$

(8.9) $\ell(\Psi\vec{w}) = \ell\vec{w}$ implies $\Psi\vec{w} = \vec{w}$

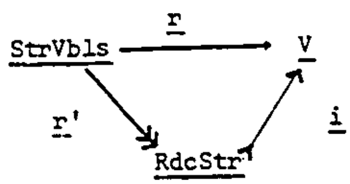
(8.10) $\Psi(\vec{u}x\vec{w}) = \vec{u}x\vec{w}$ implies $x \notin \underline{\{\vec{u}\vec{w}\}}$

(8.11) If $\vec{w}$ is such that for all $\langle\vec{u}, x, \vec{v}\rangle$, $\vec{w} = \vec{u}x\vec{v}$ implies $x \notin \underline{\{\vec{u}\vec{v}\}}$, then $\Psi\vec{w} = \vec{w}$

(8.12) If $\Psi(\vec{u}\vec{v}) = \vec{u}\vec{v}$ then $\Psi\vec{u} = \vec{u}$, $\Psi\vec{v} = \vec{v}$ and $\underline{\{\vec{u}\}} \cap \underline{\{\vec{v}\}} = \phi$.

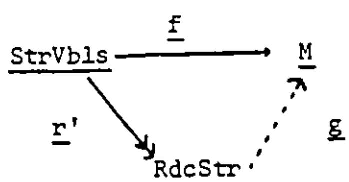
<u>Proof</u> (8.1) By (7.1) $\underline{\{\Psi x\}} = \underline{\{x\}}$; that is the monoid homomorphisms $\underline{c}' \circ r$ and $\underline{c}$ agree on Vbls , thus they agree on <u>StrVbls</u>. □ (8.2) This follows from (7.3), (5.1) and (8.1). □ (8.3) This follows from (8.2) and (7.1). □ (8.4) If $x \in U$ then by (3.1) and (7.2) $(x \dot{-} U) = \phi = \phi = x \dot{-} U$; if $x \notin U$ then by (3.1) and (7.1) $(x \dot{-} U) = x = x = x \dot{-} U$. □ (8.5) By (8.4) the homomorphism $\underline{r} \circ \underline{o}_U$ and $\underline{o}_U \circ \underline{r}$ agree on Vbls , hence $\underline{r} \circ o_U = \underline{o}_U \circ r$. □ (8.6) $\Psi(\vec{u} \vee \vec{v}) = \Psi(\vec{u}(\vec{v} \dot{-} \underline{\{\vec{u}\}}))$
$= (\Psi\vec{u}) \vee (\Psi(\vec{v} \dot{-} \underline{\{\vec{u}\}})) = (\Psi\vec{u}) \vee ((\Psi\vec{v}) \dot{-} \underline{\{\vec{u}\}})$
$= (\Psi\vec{u})(((\Psi\vec{v}) \dot{-} \underline{\{\vec{u}\}} \dot{-} \underline{\{\Psi\vec{u}\}}) = (\Psi\vec{u})((\Psi\vec{v}) \dot{-} \underline{\{\Psi\vec{u}\}}) = (\Psi\vec{u}) \vee (\Psi\vec{v})$,
by (5.1), (7.3), (8.5), (5.1), (4.7), (8.1) and (5.1) - in that order. (8.6) and (7.2) mean that there is a homomorphism $\underline{r}' : \underline{V} \longrightarrow \underline{V}$ defined by $r'(\vec{w}) = \vec{w}$. □ (8.7) The homomorphisms $\underline{r}$ and $\underline{r}' \circ \underline{r}$ agree on Vbls , that is, by (7.1) $\underline{\{\Psi x\}} = \underline{\{x\}}$,

hence $\underline{r} = \underline{r}' \circ \underline{r}$. □ (8.8) $\ell(\mathbb{V}\phi) \leq \ell(\phi)$ by (7.2). We proceed to the induction step. By (8.3) $\ell(\mathbb{V}(\overrightarrow{w}y))$ $= \ell(\mathbb{V}\vec{u}) + \ell(y \dot{-} \{\vec{u}\}) \leq \ell(\vec{u}) + \ell(y) = \ell(\vec{u}y)$. □ (8.9) For $\vec{w} = \phi$ the statement holds by (7.2). For the induction step we assume $\ell(\mathbb{V}(\overrightarrow{w}y)) = \ell(\overrightarrow{w}y)$. Then by (8.3) $\ell(\mathbb{V}\vec{w}) + \ell(y \dot{-} \{\vec{w}\}) = \ell(\vec{w}) + \ell(y)$. By (8.8) $\ell(\mathbb{V}\vec{w}) \leq \ell(\vec{w})$, and examining (3.1) we have $0 \leq \ell(y \dot{-} \{\vec{w}\}) \leq \ell(y) = 1$. Hence $\ell(\mathbb{V}\vec{w}) = \ell(\vec{w})$ and $\ell(y \dot{-} \{\vec{w}\})$ $= \ell(y)$. By induction $\mathbb{V}\vec{w} = \vec{w}$; and from (3.1), $y \dot{-} \{\vec{w}\} = y$. Hence $\mathbb{V}(\overrightarrow{w}y) = (\mathbb{V}\vec{w})(y \dot{-} \{\vec{w}\}) = \overrightarrow{w}y$. □ (8.11) Define a string $\vec{w}$ to be non-repetitive if for all $\vec{u}$, x, $\vec{v}$ such that $\vec{w} = \vec{u}x\vec{v}$ we have $x \notin \{\overrightarrow{uv}\}$. We show by induction that every non-repetitive string is reduced. The empty string is reduced so we move to the induction step and suppose $\overrightarrow{w}y$ is non-repetitive. We show $\vec{w}$ must also be non-repetitive. Let $\vec{u}$, x, $\vec{v}$ be such that $\vec{w} = \vec{u}x\vec{v}$, then $\overrightarrow{w}y = \vec{u}x(\overrightarrow{v}y)$. Since $\overrightarrow{w}y$ is non-repetitive ; $x \notin \{\overrightarrow{uvy}\}$, hence $x \notin \{\overrightarrow{uv}\}$, so $\vec{w}$ is non-repetitive. By induction $\mathbb{V}\vec{w} = \vec{w}$. Since $\overrightarrow{w}y$ is non-repetitive $y \notin \{\vec{w}\phi\}$. Thus $\mathbb{V}(\overrightarrow{w}y) = (\mathbb{V}\vec{w})(y \dot{-} \{\vec{w}\}) = \overrightarrow{w}y$. □ (8.12) Suppose $\mathbb{V}(\overrightarrow{uv}) = \overrightarrow{uv}$. Then $\ell(\mathbb{V}\vec{u}) + \ell((\mathbb{V}\vec{v}) \dot{-} \{\vec{u}\})$ $= \ell(\vec{u}) + \ell(\vec{v})$ by (8.2) . By (8.8) $\ell(\mathbb{V}\vec{u}) \leq \ell(\vec{u})$, and by (4.3) and (8.8) $\ell((\mathbb{V}\vec{v}) \dot{-} \{\vec{u}\}) \leq \ell(\mathbb{V}\vec{v}) \leq \ell(\vec{v})$. Hence $\ell(\mathbb{V}\vec{u}) = \ell(\vec{u})$ and $\ell((\mathbb{V}\vec{v}) \dot{-} \{\vec{u}\}) = \ell(\mathbb{V}\vec{v}) = \ell(\vec{v})$. By (8.9) and (4.4) , $\mathbb{V}\vec{u} = \vec{u}$, $\mathbb{V}\vec{v} = \vec{v}$ and $\vec{v} \dot{-} \{\vec{u}\} = \vec{v}$, hence $\{\vec{u}\} \cap \{\vec{v}\} = \phi$, by (4.1) . □ With equation (8.6) we established that reduction is a homomorphism $\underline{r}' : \underline{V} \longrightarrow \underline{V}$. Thus the equalizer of $\underline{r}'$ and $id_{\underline{V}}$ is a submonoid of $\underline{V}$, this is the monoid $\underline{RdcStr} = \langle RdcStr , \phi , v \rangle$. From equation (8.7) RdcStr is the image of $\underline{r}$: that is, for any $\vec{w}$ for which there is a $\vec{v}$ such that $\vec{w} = \mathbb{V}\vec{v}$, we have $\mathbb{V}\vec{w} = \mathbb{V}\mathbb{V}\vec{v}$ $= \mathbb{V}\vec{v} = \vec{w}$. Thus we have the factoration



where $\underline{i}$ is inclusion and $\underline{r}'$ is an onto map.

<u>0.2.10.9 Proposition</u>. Let $\underline{M} = \langle M, e_{\underline{M}}, \cdot \rangle$ be a monoid satisfying $a.b.a = a.b$ for all a, b in M , and let $\underline{f}$ be a monoid homomorphism from <u>StrVbls</u> to $\underline{M}$, then $\underline{f}$ factors uniquely through $\underline{r}'$.



<u>Proof</u>. Write f for the underlying function of $\underline{f}$. Suppose g and g' are functions for which $g(\mathbb{V}\vec{w}) = f(\vec{w})$ and $g'(\mathbb{V}\vec{w}) = f(\vec{w})$. Since $\underline{r}'$ is onto there can be at most one such function. If g is such a function then $g(\phi) = g(\mathbb{V}\phi) = f(\phi) = e_{\underline{M}}$ and $g(\vec{w} \vee \vec{v})$ $= g((\mathbb{V}\vec{w}) \vee (\mathbb{V}\vec{v})) = g(\mathbb{V}(\vec{w}\vec{v})) = f(\vec{w}\vec{v}) = f(\vec{w}) \cdot f(\vec{v}) = g(\mathbb{V}\vec{w}) \cdot g(\mathbb{V}\vec{v})$ $= g(\vec{w}) \cdot g(\vec{v})$; that is, g preserves the monoid operations. It remains to show that the relation $R = \{\langle \mathbb{V}\vec{w}, f(\vec{w})\rangle \mid \vec{w} \in \text{StrVbls}\}$ is functional; that is, that is $\mathbb{V}\vec{w} = \mathbb{V}\vec{v}$, then $f\vec{w} = f\vec{v}$.

We proceed by induction on $\ell(\vec{w})$ to show that $f(\mathbb{V}\vec{w}) = f(\vec{w})$.

For $\vec{w} = \phi$ we have $f(\mathbb{V}\phi) = f(\phi)$ by (7.2). The induction step requires we show $f(\mathbb{V}(\vec{w}y)) = f(\vec{w}y)$. $f(\mathbb{V}(\vec{w}y)) = f((\mathbb{V}\vec{w})(y \dot{-} \underline{\{\vec{w}\}}))$ $= f(\mathbb{V}\vec{w}) \cdot f(y \dot{-} \underline{\{\vec{w}\}}) = f(\vec{w}) \cdot f(y \dot{-} \underline{\{\vec{w}\}}) = f(\vec{w}(y \dot{-} \underline{\{\vec{w}\}})) = f(\vec{w} \vee \vec{y})$. We consider two cases $y \notin \underline{\{\vec{w}\}}$ and $y \in \underline{\{\vec{w}\}}$. If $y \notin \underline{\{\vec{w}\}}$ then $\vec{w} \vee y = \vec{w}y$, so $f(\mathbb{V}(\vec{w}y)) = f(\vec{w}y)$. If $y \in \underline{\{\vec{w}\}}$ then by (2.5) there exist $\vec{u}$ and $\vec{v}$ such that $\vec{w} = \vec{u}y\vec{v}$, so $f(\mathbb{V}(\vec{w}y))$ $= f(\vec{w}(y \dot{-} \underline{\{\vec{u}\}})) = f(\vec{u}) \cdot f(y) \cdot f(\vec{v}) = f(\vec{u}) \cdot f(y) \cdot f(\vec{v}) \cdot f(y)$ $= f(\vec{w}y)$.

It follows that R is functional since $\mathbb{V}\vec{w} = \mathbb{V}\vec{v}$ implies $f(\vec{w}) = f(\mathbb{V}\vec{w}) = f(\mathbb{V}\vec{v}) = f(\vec{v})$. □

<u>0.2.10.10 Corollary</u>. <u>RdcStr</u> is freely generated by Vbls in the variety of monoids satisfying $a \cdot b \cdot a = a \cdot b$.

<u>Proof</u>. Let $\underline{M} = \langle M, e_{\underline{M}}, \cdot \rangle$ be a monoid in the variety and let f' be a function from Vbls to M . Suppose both $\underline{g}$ and $\underline{g}'$ are extensions of f', that is, homomorphisms from <u>RdcStr</u> to $\underline{M}$ for which $\underline{g}(x) = \underline{g}'(x) = f'(x)$ for $x \in$ Vbls . Then $(\underline{r}' \circ \underline{g})(x)$ $= (\underline{r}' \circ \underline{g}')(x)$ for $x \in$ Vbls , since Vbls freely generates <u>StrVbls</u> in the variety of all monoids we have $\underline{r}' \circ \underline{g} = \underline{r}' \circ \underline{g}'$. Since $\underline{r}'$ is onto we have $\underline{g} = \underline{g}'$. We now use 0.2.10.9 to construct such a map. Let $\underline{f}$ be the extension of f' to <u>StrVbls</u> so that $\underline{f}(x) = f'(x)$ for all $x \in$ Vbls. Let $\underline{g}$ be the homomorphism of 0.2.10.9, then $\underline{g}(x) = (\underline{r}' \circ \underline{g})(x) = \underline{f}(x) = f'(x)$. Hence $\underline{g}$ extends f' . Finally by (5.6) <u>RdcStr</u> belongs to the variety. □

0.2.10.11 Definition. Let $\underline{FX} = \langle FX, e, \cdot \rangle$ be a monoid freely generated by a set X in some variety $\mathcal{V}$ of monoids. For any pair $\langle x, y \rangle$ from X we define $[x \mid y] : \underline{FX} \longrightarrow \underline{FX}$ on generators by

$$(11.1) \quad w[x \mid y] = \begin{cases} y & \text{if } x = w \\ w & \text{if } x \neq w . \end{cases}$$

Since $[x \mid y]$ preserves operations, we have

$$(11.2) \quad e[x \mid y] = e$$

and $$(11.3) \quad (a \cdot b)[x \mid y] = (a[x \mid y]) \cdot (b[x \mid y]) .$$

We call $a[x \mid y]$ the replacement of x by y in a. We shall keep the notation $[x \mid y]$ for the case $\underline{\text{StrVbls}}$, the monoid freely generated by Vbls in the variety of all monoids. For $\underline{\text{RdcStr}}$ we use $\vec{w}[x \mid y\rangle$ for the replacement of x by y in $\vec{w}$; here both $\vec{w}$ and $\vec{w}[x \mid y\rangle$ are reduced strings and we do not have $\vec{w}[x \mid y] = \vec{w}[x \mid y\rangle$ in general - for example if x and y are distinct $(xy)[x \mid y] = yy$ and $(xy)[x \mid y\rangle = y$. For $\underline{\mathcal{S}_f\text{Vbls}}$ we use $U[x \mid y\}$ for the replacement of $\{x\}$ by $\{y\}$ in the set U.

0.2.10.12 Proposition.

$$(12.1) \quad (\mathbf{V}\vec{w})[x \mid y\rangle = \mathbf{V}(\vec{w}[x \mid y]) \quad \text{for } \vec{w} \in \text{StrVbls}.$$

$$(12.2) \quad \{\vec{w}\}[x \mid y\} = \{\vec{w}[x \mid y\rangle\} \quad \text{for } \vec{w} \in \text{RdcStr}.$$

$$(12.3) \quad \{\vec{w}\}[x \mid y\} = \{\vec{w}[x \mid y]\} \quad \text{for } \vec{w} \in \text{StrVbls}.$$

<u>Proof</u>. To prove the squares (1) and (2) commute, it suffices to compare the composites for both (1) and (2) on Vbls .

$$\begin{array}{ccccc} \underline{\mathrm{StrVbls}} & \xrightarrow{\underline{r}'} & \underline{\mathrm{RdcStr}} & \xrightarrow{\underline{c}''} & \underline{\mathcal{S}_f\mathrm{Vbls}} \\ \Big\downarrow [x \mid y] & (1) & \Big\downarrow [x \mid y\rangle & (2) & \Big\downarrow [x \mid y\} \\ \underline{\mathrm{StrVbls}} & \xrightarrow[\underline{r}']{} & \underline{\mathrm{RdcStr}} & \xrightarrow[\underline{c}'']{} & \underline{\mathcal{S}_f\mathrm{Vbls}} \end{array}$$

(1) : $([x \mid y\rangle \circ \underline{r}')(x) = (\forall x)[x \mid y\rangle = y = \forall(x[x \mid y])$
$= (\underline{r}' \circ [x \mid y])(x)$. If $w \neq x$, $([x \mid y] \circ \underline{r}')(w) = w$
$= (\underline{r}' \circ [x \mid y])(w)$.

(2) : $([x \mid y\} \circ \underline{c}'')(x) = \underline{\{}x\underline{\}}[x \mid y\} = \{y\} = \underline{\{}x[x \mid y\rangle\underline{\}}$
$= (\underline{c}'' \circ [x \mid y\rangle)(x)$. If $w \neq x$, $([x \mid y\} \circ \underline{c}'')(w) = \{w\}$
$= \underline{\{}w[x \mid y\rangle\underline{\}} = (\underline{c}'' \circ [x \mid y\rangle)(w)$.

Since (1) and (2) commute, so does the outer rectangle, hence (3) holds since $\underline{c}'' \circ \underline{r}' = \underline{c}$. □

<u>0.2.10.13 Proposition</u>. Let $\underline{M} = \langle M, e, \tilde{b}\rangle$ be an algebra of similarity type $\langle 0, 2\rangle$, let X be a subset of M that generates $\underline{M}$, and let $\tilde{\underline{c}} : \underline{M} \longrightarrow \underline{\mathcal{S}_f X}$ be a homomorphism. For each $U \subset X$ let $[U]$ be the subalgebra of $\underline{M}$ generated by U . The following are equivalent

(1) $a \varepsilon [U]$

(2) $\tilde{\underline{c}}(a) \subset U$.

The proof is essentially that given in 0.2.4.3; only a few modifications are necessary: Expressions t, t_1, t_2 become elements a, a_1, a_2 of M , Fns is empty, * becomes e , s_F becomes $\underline{\tilde{c}}$, and bt_1t_2 becomes $\tilde{b}(a_1,a_2)$. □

Corollary

(13.1) $\mathrm{StrU} = \{\vec{w} \mid \{\underline{\vec{w}}\} \subset U\}$

(13.2) $\mathrm{RdcStr}(U) = \{\vec{w} \mid \{\underline{\vec{w}}\} \subset U$ and $\vec{w} \in \mathrm{RdcStr}\}$

0.2.10.14 Proposition. Let $f : Y \longrightarrow X$ be a function, $\underline{FY}$ as in 0.2.10.11 with $\mathcal{V}$ containing $\underline{\mathcal{S}_f X}$, $\bar{f} : \underline{FY} \longrightarrow \underline{\mathcal{S}_f X}$ defined by $\bar{f}(y) = \{f(y)\}$ for $y \in Y$. Let $\{y_1, y_2\} \subset Y$, $a \in FY$, then

(14.1) $f(y_1) \notin \bar{f}(a)$ implies $a[y_1 \mid y_2] = a$.

Proof. Define $\underline{\tilde{c}} : \underline{FY} \longrightarrow \underline{\mathcal{S}_f Y}$ as in 0.2.10.13 by $\underline{\tilde{c}}(y) = \{y\}$ for $y \in Y$, then $[Y - \{y_1\}] = \{a \mid \underline{\tilde{c}}(a) \subset (Y - \{y_1\})\}$ $= \{a \mid \underline{\tilde{c}}(a) \cup \underline{\tilde{c}}(y) \neq \underline{\tilde{c}}(a)\}$. Let $\tilde{f} : \underline{\mathcal{S}_f Y} \xrightarrow[\sim]{} \underline{\mathcal{S}_f X}$ to be the homomorphism defined by $\tilde{f}(\{y\}) = \{f(y)\}$ for $y \in Y$; then $\tilde{f} \circ \underline{\tilde{c}}$ and $\bar{f}$ are homomorphisms such that for any $y \in Y$, $(\tilde{f} \circ \underline{\tilde{c}})(y) = \tilde{f}(\{y\}) = \{f(y)\} = \bar{f}(y)$, thus for any $a \in FY$, $\tilde{f}(\underline{\tilde{c}}(a)) = \bar{f}(a)$.

The equalizer of $[y_1 \mid y_2]$ and $id_{\underline{FY}}$ is a submonoid of $\underline{FY}$ containing $Y - \{y_1\}$ hence $[Y - \{y_1\}] \subset \{a \mid a[y_1 \mid y_2] = a\}$. Hence $\underline{\tilde{c}}(a) \cup \underline{\tilde{c}}(y_1) \neq \underline{\tilde{c}}(a)$ implies $a[y_1 \mid y_2] = a$. If $\underline{\tilde{c}}(a) \cup \underline{\tilde{c}}(y) = \underline{\tilde{c}}(a)$ then $\tilde{f}(\underline{\tilde{c}}(a)) \cup \tilde{f}(\underline{\tilde{c}}(y_1)) = \tilde{f}(\underline{\tilde{c}}(a))$, that is, $\bar{f}(a) \cup \{f(y_1)\} = \bar{f}(a)$.

Assume $f(y_1) \notin \bar{f}(a)$ then $\bar{f}(a) \cup \{f(y_1)\} \neq \bar{f}(a)$ hence $\underline{\tilde{c}}(a) \cup \underline{\tilde{c}}(y_1) \neq \underline{\tilde{c}}(a)$, hence $a[y_1 \mid y_2] = a$. □

Corollary 0.2.10.14.1

(14.2) $y_1 \notin \{\vec{w}\}$ implies $\vec{w}[y_1 \mid y_2] = \vec{w}$

(14.3) Let $\vec{w} \in$ RdcStr . $y_1 \notin \{\vec{w}\}$ implies $\vec{w}[y_1 \mid y_2\rangle = \vec{w}$

(14.4) Let $W \in \mathcal{S}_f$Vbls . $y_1 \notin W$ implies $W[y_1 \mid y_2\} = W$.

Proof. (14.2) and (14.3) : We let $f = id_{Vbls}$ in 0.2.10.14 . □
(14.4) Let $Y = \{\{x\} \mid x \in Vbls\}$ and $f : Y \longrightarrow$ Vbls be given by $f(\{x\}) = x$, then $\bar{f}(\{x\}) = \{x\}$ where $\bar{f}$: FY $\longrightarrow \mathcal{S}_f$Vbls . We can take FY to be $\mathcal{S}_f$Vbls , thus $\bar{f} = id_{\mathcal{S}_f Vbls}$ and (14.1) becomes

$$f(\{y_1\}) \notin \bar{f}(W) \text{ implies } W[y_1 \mid y_2\} = W$$

which is (14.4). □

0.2.10.15 Proposition. If $\vec{w}$ is reduced and $y \notin \{\vec{w}\}$ then $\vec{w}[x \mid y]$ is reduced.

Proof. If $x \notin \{\vec{w}\}$ then by (14.2) $\vec{w}[x \mid y] = \vec{w}$. If $x \in \{\vec{w}\}$ then by (2.5) there must exist $\vec{u}$, $\vec{v}$ such that $\vec{w} = \vec{u}x\vec{v}$. By (8.10) and (8.12), $x \notin \{\vec{u}\vec{v}\}$, $\vec{u}$ and $\vec{v}$ are reduced, and $\{\vec{u}\} \cap \{\vec{v}\} = \phi$. Thus by (14.2) $\vec{w}[x \mid y] = \vec{u}y\vec{v}$. Since $y \notin \{\vec{u}\}$, $y \notin \{\vec{v}\}$, we have $(\vec{u}y\vec{v}) = \vec{u} \vee y \vee \vec{v} = \vec{u}y\vec{v}$ by (8.6) and (4.5). Thus $\vec{u}y\vec{v}$ is reduced. □

0.2.10.16 Proposition. For each $W \in \mathcal{S}_f$Vbls

$$W[y_1 \mid y_2] = \begin{cases} W \text{ if } y_1 \notin W \\ (W - \{y_1\}) \cup \{y_2\} \text{ if } y_1 \in W \end{cases}$$

Proof. If $y_1 \notin W$ then by (14.4) $W[y_1 \mid y_2\} = W$. If $y_1 \in W$ then $W = (W - \{y_1\}) \cup \{y_1\}$ so $W[y_1 \mid y_2\}$ $= (W - \{y_1\})[y_1 \mid y_2\} \cup \{y_1\}[y_1 \mid y_2\} = (W - \{y_1\}) \cup \{y_2\}$. □

0.2.10.17 Proposition. If $u \neq y$, $u \notin \{\vec{w}\}$ and $y \notin \{\vec{w}\}$ then $u[x \mid y] \notin \{\vec{w}[x \mid y]\}$.

Proof. We consider three cases: (1) $u = x$, (2) $u \neq x$ and $x \notin \{\vec{w}\}$, and (3) $u \neq x$ and $x \in \{\vec{w}\}$. Suppose $u \neq y$, $u \notin \{\vec{w}\}$ and $y \notin \{\vec{w}\}$.

(1) $u[x \mid y] = y$ and since $x \notin \{\vec{w}\}$, $\{\vec{w}[x \mid y]\} = \{\vec{w}\}$.

(2) $u[x \mid y] = u$ and $\{\vec{w}[x \mid y]\} = \{\vec{w}\}$.

(3) $u[x \mid y] = u$ and $\{\vec{w}[x \mid y]\} = \{\vec{w}\}[x \mid y\} = (\{\vec{w}\} - \{x\}) \cup \{y\}$. □

0.2.11 Assignment of a reduced string of variables to an expression.

Let $\mathbb{A}$ be an alphagam with similarity type μ and Vbls its set of variables. We turn StrVbls into a μ-algebra $\underline{V\mu}$ with operations $\tilde{\theta}$ corresponding to each formal operation θ as follows

$$\tilde{\star} \equiv \phi$$

$$\tilde{f}(\vec{w}) \equiv \vec{w}$$

$$\tilde{b}(\vec{u}, \vec{v}) \equiv \vec{u} \vee \vec{v}$$

$$(\widetilde{qx})(\vec{w}) \equiv \vec{w} \dot{-} \{x\} .$$

0.2.11.1 Definition. The μ-algebra homomorphism that assigns a reduced string of variables to an expression var : $\underline{Expr} \longrightarrow \underline{V\mu}$ is given by

(1) var x $\equiv$ x .

Since var preserves the operations we have

(2) var $\star \equiv \phi$

(3) var ft $\equiv$ var t

(4) var $\langle t_1, t_2 \rangle \equiv$ (var t_1) $\vee$ (var t_2)

(5) var (qxt) $\equiv$ var (t) $\dot{-}$ {x} .

These equations can also be taken to define var t by induction on the length of t .

0.2.11.2 Proposition. For each reduced string $\vec{w}$, var($\pi(\vec{w})$) $\equiv \vec{w}$, and thus the set RdcStr is a subset of the image of var.

Proof. We proceed by induction on the length of reduced strings $\vec{w}$. From the definition of the function π in 0.2.3.9 we have

$var(\pi(\phi)) = var^{*} = \phi$ and $var(\pi(x)) = var(x) = x$.

If $\vec{w}x$ is reduced so is $\vec{w}$ and $x \notin \{\vec{w}\}$ (by (30) of 0.2.10.10), hence by induction $var(\pi(\vec{w})) = \vec{w}$, hence $var(\pi(\vec{w}x))$

$= var[\pi(\vec{w}), x]$

$= (var(\pi(\vec{w}))) \vee (var\ x)$

$= \vec{w} \vee x$

$= \vec{w}(x \dot{-} \{\vec{w}\}) = \vec{w}x$.

Thus RdcStr is a subset of the imag of var . □

<u>0.2.11.3 Proposition</u>. The function that sends $\vec{w}$ to $\mathbb{V}\vec{w}$ is a μ-homomorphism from <u>Vμ</u> to <u>Vμ</u> .

<u>Proof</u> $\mathbb{V}(\tilde{*}) = \mathbb{V}\phi = \phi = \tilde{x}$.

$\mathbb{V}\tilde{f}(\vec{w}) = \mathbb{V}\vec{w} = \tilde{f}(\mathbb{V}\vec{w})$

$\mathbb{V}\tilde{b}(\vec{u}, \vec{v}) = \mathbb{V}(\vec{u} \vee \vec{v}) = (\mathbb{V}\vec{u}) \vee (\mathbb{V}\vec{v})$ by 0.2.10.9.0

$= \tilde{b}(\mathbb{V}\vec{u}, \mathbb{V}\vec{v})$.

$\mathbb{V}((\tilde{q}x)(\vec{w})) = \mathbb{V}(\vec{w} \dot{-} \{x\}) = (\mathbb{V}\vec{w})$. □

<u>0.2.11.4 Proposition</u>. For all expressions t

$\underline{r}(var\ (t)) = var\ (t)$.

<u>Proof</u>. Both $\underline{r} \circ var$ and var are μ-homomorphisms from <u>Expr</u> to <u>Vμ</u> so it suffices to note that they agree on the generating set Vbls :

$\underline{r}(var\ (x)) = x$. □

<u>0.2.11.5 Proposition</u>. RdcStr , the set of reduced strings of

variables, is the image of var .

Proof. By 0.2.11.4 every string in the image of var is reduced and by 0.2.11.2 every reduced string is in the image of var . □

0.2.11.6 Proposition. The function $\{-\} : \mathrm{StrVbls} \longrightarrow \mathcal{S}_f\mathrm{Vbls}$ is a μ-homomorphism from $\underline{V\mu}$ to the μ-algebra of free variables $\underline{\mathrm{FVbls}}$ defined in 0.2.4.1.

Proof. $\{\tilde{*}\} = \{\phi\} = \phi = *_{\underline{\mathrm{FVbls}}}$.

$$\{\tilde{f}(\vec{w})\} = \{\vec{w}\} = f_{\underline{\mathrm{FVbls}}}(\{\vec{w}\})$$

$$\{\tilde{b}(\vec{u}, \vec{v})\} = \{\vec{u} \vee \vec{v}\} = \{\vec{u}(\vec{v} \dot{-} \{\vec{u}\})\} = \{\vec{u}\} \cup (\{\vec{v}\} - \{\vec{u}\})$$

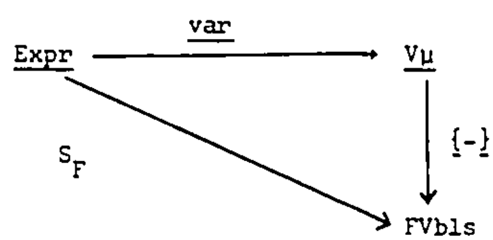
$$= \{\vec{u}\} \cup \{\vec{v}\} = b_{\underline{\mathrm{FVbls}}}(\{\vec{u}\}, \{\vec{v}\}) .$$

$$\{(qx)(\vec{w})\} = \{\vec{w} \dot{-} \{x\}\} = \{\vec{w}\} - \{x\} = (qx)_{\underline{\mathrm{FVbls}}}(\{\vec{w}\}) .$$ □

0.2.11.7 Proposition. For each expression t

$$\{\mathrm{var}\ (t)\} = S_F(t) .$$

Proof. The diagram commutes in the



category of μ-algebras, since $\underline{\mathrm{Expr}}$ freely generates Vbls , and for each $x \in \mathrm{Vbls}$, $\{\mathrm{var}\ (x)\} = \{x\} = S_F(x)$. □

<u>0.2.11.8 Proposition</u>. For each string of variables $\vec{w}$,
$S_B(\pi(\vec{w})) = \phi$.

<u>Proof</u> by induction. $S_B(\pi(\phi)) = S_B(*) = \phi$ and $S_B(\pi(x)) = S_B(x) = \phi$.
Suppose $\ell(\vec{u}) \geq 1$ and $S_B(\pi(\vec{u})) = \phi$, then $S_B(\pi(\vec{u}x)) = S_B(\langle\pi(\vec{u}),x\rangle)$
$= S_B(\pi(u)) \cup S_B(x) = \phi$. □

Section 0.3 Types and Terms.

In 0.2.2.2 we referred to μ as a "similarity type" whereas usually in Universal Algebra μ would simply be called a "type". We shall continue to use "similarity type" in the same sense, and use the word "type" - unmodified - to refer to elements of a set Tps which will be used to enrich an alphagam.

The new structure introduced in this section will be given in two stages (0.3.1): a set Tps and a structure on Tps (these are independent of the set of expressions Expr) together with a function from the set of signs Vbls $\cup$ Fns to the set Tps out of which an assignment of types to expressions can be defined, and (0.3.2): a function from the set of function signs to the set of types that finally allows us to separate out the "terms" (referred to in 0.1) from the mere expressions.

0.3.1 Types and their assignment to expressions.

0.3.1.1 Definition. An algebra of types is an algebra $\mathbb{T} = \langle Tps; \mathbb{1}, \Omega, \mathcal{P}, \times \rangle$ of similarity type $\langle 0, 0, 1, 2 \rangle$; $\mathbb{1}$ is an element of Tps called the terminal type, Ω is called the type of truth values, $\mathcal{P}$ is a unary operations called the power operation, and $\times$ is a binary operation called the product operation. The value of $\times$ at a pair $\langle A, B \rangle$ of types is written $(A \times B)$ or, if the parentheses are outermost, $A \times B$.

0.3.1.2 Definition. Let $\mathbb{A} = \langle Vbls, Fns, Fix \rangle$ be an alphagam and $\mathbb{T} = \langle Tps; \square, \Omega, \rho, \times\rangle$ be an algebra of types. A typed alphabet is a 5-tuple $\Phi = \langle \mathbb{A}, \mathbb{T}, \tau_0, cod, dom\rangle$ where $\tau_0 : Vbls \longrightarrow Tps$, $cod : Fns \longrightarrow Tps$ and $dom : Fns \longrightarrow Tps$; $\tau_0(x)$ is called the type of x , $cod(f)$ is called the codomain of f , and $dom(f)$ is called the domain of f.

For the purpose of assigning types to expressions we can ignore the domain function and work only with the structure $\langle \mathbb{A}, \mathbb{T}, \tau_0, cod\rangle$. We endow Tps with a μ-algebra structure $Tp(\Phi)$ where μ is the similarity type of $\mathbb{A}$. Among the possible formal operations are $\exists x$, $\forall x$ and $\{:\}x$ where $x \in Vbls$; in saying how these are to be realized on Tps we should keep in mind that we are saying how qx is to be realized if $q \in Fix$. We let q be an arbitrary element of $Fix \cap \{\exists, \forall, \{:\}\}$, and q' be an arbitrary element of $Fix \cap \{\exists, \forall\}$; for any expression t and variable x we let $\{x : t\} = \{:\}xt$. The μ-algebra $Tp(\Phi)$ is given by:

$$*_{Tp(\Phi)} = \square$$

$$b_{Tp(\Phi)}(A_1, A_2) = A_1 \times A_2$$

$$f_{Tp(\Phi)}(A) = cod(f)$$

$$(q'x)_{Tp(\Phi)}(A) = \Omega$$

$$(\{:\}x)_{Tp(\Phi)}(A) = \rho(\tau_0(x)) \ .$$

This provides us with an extended seventh column for table 2 of 0.2.4.1. We lift the function $\tau_0 : Vbls \longrightarrow Tps$ to the μ-algebra homomorphism $\tau : \underline{\text{Expr}} \longrightarrow \text{Tp}(\Phi)$. We call $\tau(t)$ the type of the expression t . Since τ is a μ-algebra homomorphism extending τ_0 we deduce the following six equations:

(1) $\tau(y) = \tau_0(y)$

(2) $\tau(*) =$

(3) $\tau(ft_1) = \text{cod}(f)$

(4) $\tau(\langle t_1, t_2 \rangle) = \tau(t_1) \times \tau(t_2)$

(5) $\tau(q'xt_1) = \Omega$ where $q' \in \{\exists, \forall\}$

(6) $\tau(\{x : t_1\}) = \mathcal{P}(\tau(x))$.

By 0.2.2.5 we could also take these equations to be a definition of $\tau(t)$ by induction on the length of t.

<u>0.3.1.3 Definition</u>. We say a function γ from a subset of Expr to Expr <u>preserves types</u> if whenever $\gamma(t)$ is defined $\tau(\gamma(t)) = \tau(t)$.

We show that if $\gamma : \text{Vbls} \longrightarrow \text{Expr}$ preserves types then so do $R(\gamma)$ (0.2.4.2), $S(\gamma)$ (0.2.5.2) and $\bar{\gamma}$ (when $\gamma(\text{Vbls}) \subset \text{Vbls}$) (0.2.7.1).

<u>0.3.1.4 Proposition</u>. If $\gamma : \text{Vbls} \longrightarrow \text{Expr}$ preserves types then the μ-homomorphism $R(\gamma) : \text{Expr} \longrightarrow \text{Expr}$ preserves types.

<u>Proof</u>. Both τ and $\tau \circ (R(\gamma))$ are μ-homomorphisms so $\{t \mid \tau(t) = \tau(R(\gamma)(t))\}$, their equalizer, is a μ-subalgebra of Expr. This subalgebra contains Vbls since γ preserves types, hence the equalizer is all of Expr . □

<u>0.3.1.5 Proposition.</u> If α preserves types then so does $S(\alpha)$.

<u>Proof</u>: by induction on the length of t .

y : $\tau(S(\alpha)(y)) = \tau(\alpha(y)) = \tau y$ by hypothesis

$*$: $\tau(S(\alpha)(*)) = \quad = \tau(*)$.

ft_1 : $\tau(S(\alpha)(ft_1)) = \tau(f(S(\alpha)(t_1))) = \mathrm{cod}(f) = \tau(ft_1)$

$\langle t_1,t_2\rangle$: $\tau(S(\alpha)\langle t_1,t_2\rangle = \tau(S(\alpha)(t_1)) \times \tau(S(\alpha)(t_2))$

$= \tau t_1 \times \tau t_2$ by induction

$= \tau\langle t_1,t_2\rangle$.

$q'xt_1$: $\tau(S(\alpha)(q'xt_1)) = \tau(q'x(S(\alpha_x)(t_1))) = \Omega = \tau(q'xt_1)$.

$\{x : t_1\}$: $\tau(S(\alpha)\{x : t_1\}) = \tau(\{x : S(\alpha_x)(t_1)\}) = \mathcal{P}(\tau_0(x))$

$= \tau(\{x : t_1\})$. □

<u>0.3.1.6 Proposition.</u> If $\gamma : \mathrm{Vbls} \longrightarrow \mathrm{Vbls}$ preserves types then so does $\bar{\gamma}$.

<u>Proof</u>: by induction

y : $\tau(\bar{\gamma}(y)) = \tau_0(\gamma(y)) = \tau_0(y)$ by hypothesis.

$*$: $\tau(\bar{\gamma}(*)) = \quad = \tau(*)$.

ft_1 : $\tau(\bar{\gamma}(ft_1)) = \tau(f(\bar{\gamma}(t_1))) = \mathrm{cod}f = \tau(ft_1)$

$\langle t_1,t_2\rangle$: $\tau(\bar{\gamma}\langle t_1,t_2\rangle) = \tau(\bar{\gamma}(t_1)) \times \tau(\bar{\gamma}(t_2)) = \tau(t_1) \times \tau(t_2)$

$= \tau\langle t_1,t_2\rangle$ by induction.

$q'xt_1$: $\tau(\bar{\gamma}\ (q'xt_1)) = \Omega = \tau(q'xt_1)$

$\{x : t_1\}$: $\tau(\bar{\gamma}(\{x : t_1\})) = \tau(\{(\gamma(x)) : (\bar{\gamma}(t_1))\}) = \mathcal{P}(\tau(\gamma(x)))$

$= \mathcal{P}(\tau(x)) = \tau(\{x : t_1\})$. □

0.3.2 Terms.

0.3.2.1. The domain function dom : Fns $\longrightarrow$ Tps is the final syntactic feature that we consider. We use τ together with dom to define terms.

Definition. Tms is the intersection of all subsets X of Expr for which the following hold:

(1) Vbls $\subset$ X

(2) $* \in X$

(3) $(At)(Af)((t \in X$ and $\mathrm{dom}(f) = \tau(t)) \longrightarrow ft \in X)$

(4) $(At_1)(At_2)((t_1 \in X$ and $t_2 \in X) \longrightarrow \langle t_1,t_2 \rangle \in X)$

(5) $(At)(Ax)((t \in X$ and $\tau(t) = \Omega) \longrightarrow qxt \in X)$.

Elements of Tms we call terms. It is clear from an examination of each of the above clauses that Tms is a set of expressions for which (1) through to (5) holds.

0.3.2.2 Proposition. Let t be a term, then one of the following holds:

(1) $t \in$ Vbls

(2) $t = *$

(3) $(Et_1)(Ef)(t = ft_1,\ t_1 \in$ Tms and $\mathrm{dom}(f) = \tau(t))$.

(4) $(Et_1)(Et_2)(t = \langle t_1,t_2 \rangle,\ \{t_1,t_2\} \subset$ Tms$)$.

(5) $(Ex)(Eq)(Et_1)(t = qxt_1,\ \tau(t_1) = \Omega$ and $t_1 \in$ Tms$)$.

Proof: As in 0.2.2.3 we let B be the set of all terms t' for which one of the above clauses holds. B satisfies each of the conditions stated for X in the definition above; for example for (3) of 0.3.2.1, suppose $t_1 \varepsilon B$ and $dom(f) = \tau(t_1)$ then $t_1 \varepsilon Tms$ also, hence $ft_1 \varepsilon Tms$; but then ft_1 satisfies (3) of 0.3.2.2 hence $ft_1 \varepsilon B$. Thus $B = Tms$. □

0.3.2.3 Proposition. Let t be a term, then the following hold:

(1) $t = ft_1 \longrightarrow dom(f) = \tau(t_1)$ and $t_1 \varepsilon Tms$

(2) $t = \langle t_1, t_2 \rangle \longrightarrow t_1 \varepsilon Tms$ and $t_2 \varepsilon Tms$

(3) $t = qxt_1 \longrightarrow \tau(t_1) = \Omega$ and $t_1 \varepsilon Tms$.

Proof. Since t is an expression, by 0.2.2.5, the representations given in (3), (4) and (5) of 0.3.2.2 are uniquely determined. □

0.3.2.4 Definition. We say a function γ from a subset of Expr to Expr preserves terms if whenever t is a term in the domain of γ , $\gamma(t)$ is a term.

Parallel to 0.3.1.4, 0.3.1.5 and 0.3.1.6 we show that if $\gamma : Vbls \longrightarrow Tms$ preserves types then $R(\gamma)$ and $S(\gamma)$ preserve terms, and if $\gamma : Vbls \longrightarrow Vbls$ preserves types then $\tilde{\gamma}$ preserves terms.

0.3.2.5 Proposition. If $\gamma : Vbls \longrightarrow Tms$ preserves types then the μ-homomorphism $R(\gamma)$ preserves terms.

Proof. By hypothesis γ preserves types hence so does $R(\gamma)$. We continue by induction on the length of t:

y : $R(\gamma)(y) = \gamma(y)$, a term by hypothesis.

$*$: $R(\gamma)(y) = *$, a term.

ft_1 where t_1 is a term and $\text{dom} f = \tau t_1$:

$R(\gamma)(ft_1) = f(R(\gamma)(t_1))$ since $R(\gamma)$ is a μ-homomorphism

by induction $R(\gamma)(t_1)$ is a term

also $(R(\gamma)(t_1)) = \tau(t_1) \in \text{dom} f$, by 0.3.1.4,

hence $R(\gamma)(ft_1)$ is a term.

$\langle t_1,t_2\rangle$ where t_1 and t_2 are terms:

$R(\gamma)\langle t_1,t_2\rangle = \langle R(\gamma)(t_1),R(\gamma)(t_2)\rangle$

by induction each $R(\gamma)(t_i)$ is a term $(i = 1,2)$ hence

so is $R(\gamma)\langle t_1,t_2\rangle$.

qxt_1 where t_1 is a term and $\tau t_1 = \Omega$:

$R(\gamma)(qxt_1) = qx(R(\gamma)(t_1))$

by induction $R(\gamma)(t_1)$ is a term, also

$\tau(R(\gamma)(t_1)) = \tau t_1 = \Omega$

hence $R(\gamma)(qxt_1)$ is a term. □

<u>0.3.2.6 Proposition</u>. If α : Vbls ⟶ Tms preserves types then $S(\alpha)$: Expr ⟶ Expr preserves terms.

<u>Proof</u>: By 0.3.1.5 $S(\alpha)$ preserves types. We proceed by induction on the length of t to show that if $t \in$ Tms then $S(\alpha)(t) \in$ Tms.

y : $S(\alpha)(y) = \alpha(y)$, a term.

$*$: $S(\alpha)(*) = *$, a term.

ft_1 where t_1 is a term and $\text{dom} f = \tau t_1$:

$S(\alpha)(ft_1) \equiv fS(\alpha)(t_1)$, by induction $S(\alpha)(t_1)$ is a term, also $\tau(S(\alpha)(t_1)) \equiv \tau(t_1) \equiv dom(f)$, hence $S(\alpha)(ft_1)$ is a term.

$\langle t_1,t_2 \rangle$ where t_1 and t_2 are terms:

$S(\alpha)\langle t_1,t_2\rangle \equiv \langle S(\alpha)(t_1),S(\alpha)(t_2)\rangle$.

By induction $S(\alpha)(t_i)$ is a term $(i \equiv 1,2)$ hence $S(\alpha)\langle t_1,t_2\rangle$ is a term.

qxt_1 where t_1 is a term and $\tau t_1 \quad \Omega$:

$S(\alpha)(qxt_1) \equiv qx(S(\alpha_x)(t_1))$.

If α preserves types so does α_x . By induction $S(\alpha_x)(t_1)$ is a term and $\tau(S(\alpha_x)(t_1)) \equiv \tau(t_1) \equiv \Omega$, hence $S(\alpha)(qxt_1)$ is a term. □

<u>0.3.2.7 Proposition</u>. If γ : Vbls $\longrightarrow$ Vbls preserves types then $\bar{\gamma}$ preserves terms.

<u>Proof</u>. By 0.3.1.6 $\bar{\gamma}$ preserves types. We proceed by induction on the length of terms.

y : $\bar{\gamma}(y) \equiv \gamma(y)$, a term.

$*$: $\bar{\gamma}(*) \equiv *$, a term.

ft_1 where $t_1 \in$ Tms , domf $\equiv \tau t_1$:

$\bar{\gamma}(ft_1) \equiv f(\bar{\gamma}(t_1))$. By induction $\bar{\gamma}(t_1) \in$ Tms , moreover $\tau(\bar{\gamma}(t_1)) \equiv \tau(t_1) \equiv$ domf , hence $\bar{\gamma}(ft_1)$ is a term.

$\langle t_1,t_2\rangle$ where $\{t_1,t_2\} \subset$ Tms : $\bar{\gamma}\langle t_1,t_2\rangle \equiv \langle\bar{\gamma}(t_1),\bar{\gamma}(t_2)\rangle$,

by induction each $\bar{\gamma}(t_i)$ is a term $(i \equiv 1,2)$, hence so is $\bar{\gamma}(t_1,t_2)$.

qxt_1 where $t_1 \in Tms$, $\tau t_1 = \Omega$: $\bar{\gamma}(qxt_1) = q(\gamma(x))(\bar{\gamma}(t_1))$,

by induction $\bar{\gamma}(t_1) \in Tms$ and $\tau(\bar{\gamma}(t_1)) = \tau(t_1) = \Omega$

hence $\bar{\gamma}(qxt_1)$ is a term. □

0.3.3 Interpretations in a typed alphabet.

We are now in a position to refine our notion of an interpretation of an elphabet in an alphagam. We fix a typed alphabet $\Phi = \langle \mathbb{A}, \Pi, \tau_0, cod, dom \rangle$ where $\mathbb{A} = \langle Vbls, Fns, Fix \rangle$ is an alphagam and $\Pi = \langle Tps; \mathbb{I}, \Omega, \rho, \times \rangle$ is an algebra of types.

0.3.3.1 Conventions. Unless explicitly stated otherwise the written symbols t, t_1, t', s and others which in previous sections were used to denote arbitrary expressions shall hereafter be used to denote arbitrary terms. In addition, for each type A we let V_A and T_A be the set of variables and terms, respectively, of type A , and we let $ffr[x](t)$ be the set of all terms of type $\tau_0(x)$ which are free to be substituted for x in the term t . For each type A we define A^i inductively as follows $A^0 = \mathbb{I}$, $A^1 = A$, $A^{k+1} = A^k \times A$ for $k \geq 1$. For A, B types we let $[A, B] = \{f \in Fns \mid domf = A \text{ and } codf = B\}$.

0.3.3.2 Definitions.

(1) An internal algebra of similarity type F in Φ is a pair $\langle A, \gamma \rangle$ where $A \in Tps$ and $\gamma : \bigcup_i F_i \rightarrow \bigcup_i [A^i, A]$ such that $\gamma(F_i) \subset [A^i, A]$.

(2) Let $\langle \mathcal{O}, \rho \rangle$ be a similarity type for external structures (as in 0.2.3.11). An internal $\langle \mathcal{O}, \rho \rangle$ pre-structure is a triple $\mathcal{A} = \langle A, o, p \rangle$ where $\mathcal{A}_{\mathcal{O}} = \langle A, o \rangle$ is an internal algebra of similarity type $\mathcal{O}$, and $p : \bigcup_i P_i \longrightarrow \bigcup_i [A^i, \Omega]$ such that $p(P_i) \subset [A^i, \Omega]$. (We have not specified what $p(\underline{\delta})$ should be; when we do, in 0.6.3.1, we shall call $\mathcal{A}$ an internal structure.)

(3) Let F be a similarity type for algebras and let $\mathbb{E} = \langle V, F_0, F_1, F_2, \{*,b\} \rangle$ be the elphabet of similarity type F . An interpretation of $\mathbb{E}$ in Φ of type A is an interpretation γ of $\mathbb{E}$ in $\mathbb{A}$ (see 0.2.3.4) such that $\langle A, \gamma_F \rangle$ is an internal algebra of similarity type F in Φ where γ_F is the restriction of γ to F_i , and $\gamma|_V$ is a bijection from V to V_A .

(4) Let $\langle \mathcal{O}, \rho \rangle$ be a similarity type, $\mathbb{P}$ the formula elphabet determined by $\langle \mathcal{O}, \rho \rangle$. A pre-interpretation of $\mathbb{P}$ in Φ of type A is an interpretation γ of $el(\mathbb{P})$ in $\mathbb{A}$ such that $\mathcal{A} = \langle A, \gamma_{\mathcal{O}}, \gamma_{\rho} \rangle$ is an internal pre-structure of similarity type $\langle \mathcal{O}, \rho \rangle$ where $\gamma_{\mathcal{O}}$ and γ_{ρ} are the restrictions of γ to $\mathcal{O}_i$ and ρ_i respectively, $\{\gamma(\underline{i}), \gamma(\underline{\&})\} \subset [\Omega^2, \Omega]$, and $\gamma|_V : V \to V_A$ is a bijection.

We follow the notation of 0.2.3.10 in the next Proposition: if γ is an interpretation of $\mathbb{E}$ in $\mathbb{A}$ then for any string of signs of $\mathbb{E}$, s we put $\bar{s} = \tilde{\gamma}(s)$, and for a set X we put $\bar{X} = \{\bar{s} \mid s \in X\}$.

<u>0.3.3.3 Proposition</u>. (1) Let γ be an interpretation of $\mathbb{E}$ in

Φ of type A, then $\overline{\mathrm{Poly}(\mathbb{E})} \subset T_A$. (2) Let γ be pre-interpretation of the formula elphabet $\mathbb{P}$ in Φ of type A then

(a) $\overline{\mathrm{AtFml}(\mathbb{P})} \subset T_\Omega$

(b) $\overline{\mathrm{Conj}(\mathbb{P})} \subset T_\Omega$

(c) $\overline{\mathrm{bHf}(\mathbb{P})} \subset T_\Omega$

Proof. (1) We must show that T_A is an F-subalgebra of $\mathrm{St}(\gamma, \mathbb{A})$ (see 0.2.3.10) where $\mathbb{A}$ is the alphabet of Φ and F is the similarity type of $\mathbb{E}$. If $e \varepsilon F_0$ then $\bar{e} \varepsilon [\square, A]$ hence $\dot{e} \equiv \bar{e}*$ is a term of type A . If $f \varepsilon F_1$ then $\bar{f} \varepsilon [A, A]$ so if $t \varepsilon T_A$ then $\dot{f}(t) \equiv \bar{f}t \varepsilon T_A$. If $g \varepsilon F_2$ then $\bar{g} \varepsilon [A^2, A]$ so if $t, s \varepsilon T_A$ then $\dot{g}(t, s) \equiv \bar{g}/t, s/ \varepsilon T_A$. Since $\bar{V} \subset T_A$ and $\bar{V}$ generates $\overline{\mathrm{Poly}_F(V)}$ we have $\overline{\mathrm{Poly}(\mathbb{E})} \subset T_A$. □

(2) From (1) we have $\overline{\mathrm{Poly}(\mathbb{P})} \subset T_A$.

(a) A unary atomic formula has the form ft where $f \varepsilon \rho_1$ and $t \varepsilon \mathrm{Poly}(\mathbb{P})$, hence $\bar{f} \varepsilon [A, \Omega]$, $\bar{t} \varepsilon T_A$ and $\overline{ft} \equiv \bar{f}\bar{t} \varepsilon T_\Omega$. A binary atomic formula has the form $f/t, s/$ where $f \varepsilon \rho_2$ and $\{t, s\} \subset \mathrm{Poly}(\mathbb{P})$ hence $\bar{f} \varepsilon [A^2, \Omega]$, $\{\bar{t}, \bar{s}\} \subset T_A$ and $\overline{f/t, s/} \equiv \bar{f}/\bar{t}, \bar{s}/ \varepsilon T_\Omega$. Hence $\overline{\mathrm{AtFml}(\Phi)} \subset T_\Omega$. □

(b) By (a) the image under $\tilde{\gamma}$ of a single atomic formula is a formula of Φ . For the inductions step let $\underline{\&}/\psi, \varphi/$ be a conjunction of atomic formulas where ψ is a conjunction of at least one atomic formula and φ is atomic, then by induction $\bar{\psi} \varepsilon T_\Omega$, also $\bar{\varphi} \varepsilon T_\Omega$ and $\bar{\underline{\&}} \varepsilon [\Omega^2, \Omega]$ hence $\overline{\underline{\&}/\psi, \varphi/}$ $\bar{\underline{\&}}/\bar{\psi}, \bar{\varphi}/ \varepsilon T_\Omega$. □

(c) A basic Horn formula is either atomic or is of the form $\underline{i}/\psi, \varphi/$

where ψ is a conjunction of atomic formulas and φ is atomic; by (b) $\bar{\psi} \varepsilon T_\Omega$ and $\bar{\varphi} \varepsilon T_\Omega$, since $\underline{I} \varepsilon [\Omega^2, \Omega]$ we have $\overline{\underline{i}/\psi, \varphi/} = \underline{I}/\bar{\psi}, \bar{\varphi}/ \varepsilon T_\Omega$. □

0.3.4 Typed equivalence relations on the set of augmented terms.

0.3.4.1 Definition. A pair $<\vec{w}, t>$ such that $\vec{w}$ is reduced and $S_F(t) \subset \{\vec{w}\}$ will be called an augmented term ; we let AugTms be the set of all augmented terms. The function AugTms → Str(Tms) sending $<\vec{w}, t>$ to $t\vec{w}$ is one-to-one; this provides a representation of augmented terms which is suggestive of the "extremely important convention of notation" of [CK] pp. 23, 24.

0.3.4.2 Proposition.

(1) $<\vec{w}, ft> \varepsilon$ AugTms iff both $<\vec{w}, t> \varepsilon$ AugTms and $ft \varepsilon$ Tms .

(2) $<\vec{w}, (s, t)> \varepsilon$ AugTms iff both $<\vec{w}, s> \varepsilon$ AugTms and $<\vec{w}, t> \varepsilon$ AugTms .

(3) Let $y \not\varepsilon \{\vec{w}\}$, then $<\vec{w}, qx\varphi> \varepsilon$ AugTms implies $<\vec{w}y, \varphi [x \mid y]> \varepsilon$ AugTms .

Proof. (1) and (2) follow from basic properties of S_F . (3): If $\vec{w}$ is reduced and $y \not\varepsilon \{\vec{w}\}$ then $\vec{w}y$ is reduced. If $qx\varphi$ is a term, φ is a formula; and if $\tau_0(x) = \tau_0(y)$, $\varphi[x \mid y]$ is again a formula. If $S_F(qx\varphi) \subset \{\vec{w}\}$ then

$S_F(\varphi[x \mid y] \subset (S_F(\varphi) - \{x\}) \cup \{y\} \subset \{\vec{w}\} \cup \{y\} = \{\vec{w}y\}$. □

0.3.4.3 Making choices of new variables.

In 0.5.2.1 we define for each augmented term $\langle \vec{w}, t\rangle$ a morphism $\lambda\vec{w}.t$ of a topos by induction on the length of t. In the clause in which $\lambda\vec{w}.qx\varphi$ is to be defined we choose a variable y that is in some sense independent of $\langle \vec{w}, qx\varphi\rangle$ and define $\lambda\vec{w}.qx\varphi$ using a construction of $\underline{\mathcal{E}}$ and the morphism $\lambda\vec{w}y.\varphi[x \mid y]$ which by the induction hypothesis is already defined (since $\ell(\varphi[x \mid y]) < \ell(qx\varphi))$. If instead of y another "independent" variable y' is chosen then $\lambda\vec{w}y'.\varphi[x \mid y']$ is defined but we cannot at this stage say it is the same as $\lambda\vec{w}y.\varphi[x \mid y]$ and consequently we cannot be sure that by using the same construction available in $\underline{\mathcal{E}}$, as we did on $\lambda\vec{w}y.\varphi[x \mid y]$ that we will end up with the same value for $\lambda\vec{w}.qx\varphi$. It turns out that it does not matter what choice of variable, y or y', is made; nor, as we shall see in 0.3.4.10, is it essential to be working in a topos to demonstrate this. Initially however, we must have available a choice of variables. We now introduce such a choice in the abstract context of a typed alphabet Φ.

By a <u>choice of new variables</u> for Φ we mean a function ch : Tps × $\mathcal{S}_f$Vbls → Vbls such that

(CH 1) $\qquad ch(\Lambda, U) \in (V_\Lambda - U)$

and (CH 2) $\qquad ch(\Lambda, U) = ch(\Lambda, V_\Lambda \cap U)$.

If for each $\Lambda \in$ Tps and $\Gamma \in \mathcal{S}_f V_\Lambda$ we put $ch_\Lambda(\Gamma)$

$= ch(A, F)$ then $ch_A(F) \varepsilon V_A - F$.

(If on the other hand we are given a family $ch_A : \mathcal{S}_f V_A \to V_A$ such that $ch_A(F) \not\varepsilon F$, where A runs over Tps , and we define ch by $ch(A, U) = ch_A(V_A \cap U)$ then ch is a choice of new variables.) In order for such a function ch to exist it is necessary that each V_A be infinite since if some V_A were finite (CH 1) could not hold since $ch(A, V_A) \not\varepsilon \phi$. (In the presence of the axiom of choice the condition that each V_A be infinite is clearly necessary and sufficient for a choice of new variables to exist.)

0.3.4.4 Definition. We extend the function τ that assigns types to terms to a function $\vec{\tau}$ that assigns strings of the signs for types to strings of terms. That is $\vec{\tau}$: Str Tms $\to$ Tps* is the monoid homomorphism defined on Tms by $\vec{\tau}(t) = (\tau(t))^\frown$, so that $\vec{\tau}(\phi) = \phi$ and $\vec{\tau}(\vec{t}\ \vec{s}) = (\vec{\tau}(\vec{t}))(\vec{\tau}(\vec{s}))$. In this section (0.3.4) we will only be considering $\vec{\tau}$ applied to strings of variables.

0.3.4.5. The definition of $\lambda\vec{w}.t$ (in 0.5.2.1) induces an equivalence relation $\approx$ on AugTms given by

$$\langle\vec{w}, t\rangle \approx \langle\vec{u}, s\rangle \text{ iff } \lambda\vec{w}.t = \lambda\vec{u}.s \quad .$$

From each of the five clauses defining $\lambda\vec{w}.t$ we abstract five properties of $\approx$ and introduce them in the setting of a typed alphabet with a choice of new variables.

Definition. A typed equivalence relation under a choice ch of new variables is an equivalence relation $\sim$ on the set of augmented terms satisfying

(τ1) If $\vec{u}_1 x_1 \vec{v}_1$ and $\vec{u}_2 x_2 \vec{v}_2$ are reduced strings of variables such $\ell(\vec{u}_1) = \ell(\vec{u}_2)$ and $\vec{\tau}(\vec{u}_1 x_1 \vec{v}_1) = \vec{\tau}(\vec{u}_2 x_2 \vec{v}_2)$ then $\langle \vec{u}_1 x_1 \vec{v}_1, x_1 \rangle \sim \langle \vec{u}_2 x_2 \vec{v}_2, x_2 \rangle$.

(τ2) If $\vec{w}_1$ and $\vec{w}_2$ are reduced and $\vec{\tau}(\vec{w}_1) = \vec{\tau}(\vec{w}_2)$ then $\langle \vec{w}_1, * \rangle \sim \langle \vec{w}_2, * \rangle$.

(τ3) If $\langle \vec{w}_i, ft_i \rangle \in$ Aug Tms for $i = 1, 2,$ and $\langle \vec{w}_1, t_1 \rangle \sim \langle \vec{w}_2, t_2 \rangle$, then $\langle \vec{w}_1, ft_1 \rangle \sim \langle \vec{w}_2, ft_2 \rangle$.

(τ4) If $\langle \vec{w}_1, s_1 \rangle \sim \langle \vec{w}_2, s_2 \rangle$ and $\langle \vec{w}_1, t_1 \rangle \sim \langle \vec{w}_2, t_2 \rangle$ then $\langle \vec{w}_1 , (s_1, t_1) \rangle \sim \langle \vec{w}_2, (s_2, t_2) \rangle$.

(τ5) If $\langle \vec{w}_i, qx_i\varphi_i \rangle \in$ Aug Tms and $y_i = ch(\tau_0(x_i), S_{FB}(\vec{w}_i x_i \varphi_i))$ for $i = 1, 2,$ and $\langle \vec{w}_1 y_1, \varphi_1[x_1 \mid y_1] \rangle \sim \langle \vec{w}_2 y_2, \varphi_2[x_2 \mid y_2] \rangle$, then $\langle \vec{w}_1, qx_1\varphi_1 \rangle \sim \langle \vec{w}_2, qx_2\varphi_2 \rangle$.

<u>0.3.4.6 Definition</u>. The relation $\dot{\sim}$ is defined on Aug Tms by $\alpha_1 \dot{\sim} \alpha_2$ iff $\alpha_1 \sim \alpha_2$ for all typed equivalence relations under the choice *ch* .

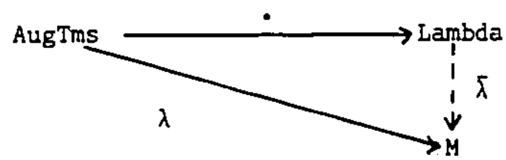
<u>0.3.4.7 Proposition</u>. The relation $\dot{\sim}$ is a typed equivalence relation under the choice *ch* , for which $\langle \vec{w}_1, t_1 \rangle \dot{\sim} \langle \vec{w}_2, t_2 \rangle$ implies $\vec{\tau}(\vec{w}_1) = \vec{\tau}(\vec{w}_2)$, $\tau(t_1) = \tau(t_2)$ and $\ell(t_1) = \ell(t_2)$.

<u>Proof</u>. An examination of each of the clauses (τ1) to (τ5) shows that $\dot{\sim}$ must satisfy (τ1) to (τ5) also, and, since equivalence relations are closed under intersection, $\dot{\sim}$ is a typed equivalence

relation. The relation $\ddot{\sim}$ defined on AugTms by $\langle \vec{w}_1, t_1 \rangle \ddot{\sim} \langle \vec{w}_2, t_2 \rangle$ iff $\langle \vec{w}_1, t_1 \rangle \dot{\sim} \langle \vec{w}_2, t_2 \rangle$, $\vec{\tau}(\vec{w}_1) = \vec{\tau}(\vec{w}_2)$, $\tau(t_1) = \tau(t_2)$ and $\ell(t_1) = \ell(t_2)$, is an equivalence relation , being the intersection of $\dot{\sim}$ with the three equivalence relation induced by the three equations. Again an examination of clauses (τ1) to (τ4) shows that $\ddot{\sim}$ must satisfy each of these clauses. We examine (τ5) . Suppose $\langle \vec{w}_i, qx_i\varphi_i \rangle \in$ Aug Tms and $y_i = ch(\tau_0(x_i), S_{FB}(\vec{w}_i x_i \varphi_i))$ for $i = 1, 2$, $\langle \vec{w}_1 y_1, \varphi_1[x_1 \mid y_1] \rangle \dot{\sim} \langle \vec{w}_2 y_2, \varphi_2[x_2 \mid y_2] \rangle$, $\vec{\tau}(\vec{w}_1 y_1) = \vec{\tau}(\vec{w}_2 y_2)$, $\tau(\varphi_1[x_1 \mid y_1]) = \tau(\varphi_2[x_2 \mid y_2])$ and $\ell(\varphi_1[x_1 \mid y_1]) = \ell(\varphi_2[x_2 \mid y_2])$. Since $\dot{\sim}$ satisfies (τ5) , $\langle \vec{w}_1, qx_1\varphi_1 \rangle \dot{\sim} \langle \vec{w}_2, qx_2\varphi_2 \rangle$. Since $(\vec{\tau}(\vec{w}_1))(\vec{\tau}(y_1)) = (\vec{\tau}(\vec{w}_2))(\vec{\tau}(y_2))$ and both $\vec{\tau}(y_1)$ and $\vec{\tau}(y_2)$ are strings of length one, $\vec{\tau}(\vec{w}_1) = \vec{\tau}(\vec{w}_2)$. Since $\tau(\varphi_i[x_i \mid y_i]) = \tau(\varphi_i)$ and $\ell(\varphi_i[x_i \mid y_i]) = \ell(\varphi_i)$ for $i = 1, 2$, we have $\tau(\varphi_1) = \tau(\varphi_2)$ and $\ell(\varphi_1) = \ell(\varphi_2)$. Now since $\ddot{\sim}$ is a typed equivalence relation under *ch* finer than $\dot{\sim}$ it must be the same as $\dot{\sim}$. □

<u>0.3.4.8</u> The relation $\dot{\sim}$ is the equivalence relation generated by the clauses τ1 to τ5 . For $\langle \vec{w}, t \rangle \in$ Aug Tms let $\vec{w}.t = \{\alpha \mid \alpha \dot{\sim} \langle \vec{w}, t \rangle\}$ be the equivalence class containing $\langle \vec{w}, t \rangle$ and let Lambda be the set of all equivalence classes. The function $. : \text{AugTms} \longrightarrow \text{Lambda}$ has the universal property that for any "interpretation" $\lambda : \text{AugTms} \longrightarrow M$ such that the induced equivalence relation $\ker\lambda$ on AugTms satisfies τ1 to τ5 , there

is a uniquely determined function $\bar{\lambda}$: Lambda $\to M$ such that



commutes, i.e. $\lambda(\langle\vec{w}, t\rangle \equiv \bar{\lambda}(\vec{w}.t)$ for each augmented term $\langle\vec{w}, t\rangle$. Proposition 0.3.4.7 implies that the following are well-defined: the string of types for the "domain" of $\vec{w}.t$ given by $\vec{\tau}(\vec{w}.t) \equiv \vec{\tau}(\vec{w})$, the type of the "codomain" of $\vec{w}.t$ given by $\tau(\vec{w}.t) \equiv \tau(t)$, and the term length of $\vec{w}.t$ given by $\ell(\vec{w}.t) \equiv \ell(t)$. We will not however be able to infer that the corresponding concepts are definable for $\lambda(\langle\vec{w}.t\rangle)$.

<u>0.3.4.9 Proposition</u>. If $\vec{w}$ is reduced, $x \in \underline{\{\vec{w}\}}$, $z \notin (\underline{\{\vec{w}\}} - \{x\})$, $\tau_0(x) \equiv \tau_0(z)$, and $\sim$ is a typed equivalence relation under *ch* , then

(1) $\langle\vec{w}, x\rangle \sim \langle\vec{w}[x \mid y] , z\rangle$ and $\vec{\tau}(\vec{w}) \equiv \vec{\tau}(\vec{w}[x \mid z])$.

(2) if $y \in (\underline{\{\vec{w}\}} - \{x\})$ then $\langle\vec{w}, y\rangle \sim \langle\vec{w}[x \mid z] , y\rangle$.

(3) $\langle\vec{w}, *\rangle \sim \langle\vec{w}[x \mid z] , *\rangle$.

<u>Proof</u>. If $z \equiv x$ then (1), (2) and (3) hold since $\vec{w}[x \mid x] \equiv \vec{w}$, thus we assume $z \not\equiv x$. Since $z \notin \underline{\{\vec{w}\}}$, by 0.2.10.15, $\vec{w}[x \mid z]$ is reduced. Since $x \in \underline{\{\vec{w}\}}$, by 0.2.10.2 there exist reduced strings $\vec{u}$ and $\vec{v}$ such that $\vec{w} \equiv \vec{u}x\vec{v}$, and by 0.2.10.8 , $x \notin \underline{\{\vec{u}\vec{v}\}}$, hence $\vec{w}[x \mid z] \equiv \vec{u}z\vec{v}$ and $z \in \underline{\{\vec{w}[x \mid z]\}}$. Thus $\langle\vec{w}[x \mid z] , z\rangle$ and $\langle\vec{w}[x \mid z] , *\rangle$ are augmented terms. (1) By (τ1) , since

$\vec{\tau}(\vec{u}x\vec{v}) \equiv (\vec{\tau}(\vec{u}))(\vec{\tau}\ (x))(\vec{\tau}(\vec{v})) \equiv (\vec{\tau}(\vec{u}))(\vec{\tau}(z))(\vec{\tau}(\vec{v})) \equiv \vec{\tau}(\vec{u}z\vec{v})$,

$<\vec{u}x\vec{v}\ , x> \sim <\vec{u}z\vec{v}\ , z>$. □

(2) Suppose $y \in (\{\vec{w}\} - \{x\})$. Put $\vec{w} \equiv \vec{u}_1 y \vec{v}_1$. Let $\vec{u}_2 \equiv \vec{u}_1[x \mid z]$ and $\vec{v}_2 \equiv \vec{v}_1[x \mid z]$ so that $\vec{w}[x \mid z] \equiv \vec{u}_2 y \vec{v}_2$. If $x \notin \{\vec{u}_1\}$ then $\vec{u}_2 \equiv \vec{u}_1$ so $\vec{\tau}(\vec{u}_1) \equiv \vec{\tau}(\vec{u}_1)$; if $x \in \{\vec{u}_1\}$ then by (1) $\vec{\tau}(\vec{u}_2) \equiv \vec{\tau}(\vec{u}_1)$ also. Similarly $\vec{\tau}(\vec{v}_1) \equiv \vec{\tau}(\vec{v}_2)$. Hence $\vec{\tau}(\vec{w}) \equiv \vec{\tau}(\vec{u}_1 y \vec{v}_1)$
$\equiv (\vec{\tau}(\vec{u}_1))(\vec{\tau}(y))(\vec{\tau}(\vec{v}_1)) \equiv (\vec{\tau}(\vec{u}_2))(\vec{\tau}(y))(\vec{\tau}(\vec{v}_2)) \equiv \vec{\tau}(\vec{u}_2 y \vec{v}_2)$
$\equiv \vec{\tau}(\vec{w}[x \mid z])$.

Since also $\ell(\vec{u}_1[x \mid z]) \equiv \ell(\vec{u}_1)$, by (τ1), $<\vec{u}_1 y \vec{v}_1\ , y> \sim <\vec{u}_2 y \vec{v}_2\ , y>$. □

(3) From (1) $\vec{\tau}(\vec{w}) \equiv \vec{\tau}(\vec{w}[x \mid z])$ and $\vec{w}[x \mid z]$ is reduced hence, by (τ2), $<\vec{w}, *> \sim <\vec{w}[x \mid z]\ , *>$. □

<u>0.3.4.10 Proposition</u>. If $<\vec{w}, t> \in$ Aug Tms , $\sim$ is a typed equivalence relation under *ch* , and $z \in (\text{ffr}[x](t) - \{\vec{w}\})$ then $<\vec{w}, t> \sim <\vec{w}[x \mid z]\ , t[x \mid z]>$.

<u>Proof</u>. If $x \notin \{\vec{w}\}$ then by 0.2.10.14, $\vec{w}[x \mid z] \equiv \vec{w}$; also, since $S_F(t) \subset \{\vec{w}\}$, by 0.2.5.3, $t[x \mid z] \equiv t$. Thus we only have to prove the proposition when $x \in \{\vec{w}\}$.

We first show that $<\vec{w}[x \mid z]\ , t[x \mid z]>$ is an augmented term if $<\vec{w}, t>$ is and $z \in (\text{ffr}[x](\cdot) - \{\vec{w}\})$. Since $z \notin \{\vec{w}\}$, by 0.2.10.15, $\vec{w}[x \mid z]$ is reduced. If $x \notin S_F(t)$ then $S_F(t[x \mid z]) \equiv S_F(t) \subset (\{\vec{w}\} - \{x\}) \subset \{\vec{w}\}[x \mid z\} \equiv \{\vec{w}[x \mid z]\}$ by 0.2.10.16 and 0.2.10.12. If $x \in S_F(t)$ then $S_F(t[x \mid z])$
$\equiv ((S_F(t) - \{x\}) \cup \{z\}) \subset \{\vec{w}\}[x \mid z\} \equiv \{\vec{w}[x \mid z]\}$.

We now proceed by induction on the length of the term t to establish the proposition when $x \varepsilon \{\vec{w}\}$.

If $t \equiv x$, or $t \equiv y$ with $y \not\equiv x$, or $t \equiv *$, we can apply (1), (2) and (3) respectively of 0.3.4.10.

Suppose $t \equiv f\bar{t}$. By induction $\langle\vec{w}, \bar{t}\rangle \sim \langle\vec{w}[x \mid z], \bar{t}[x \mid z]\rangle$ hence by (τ3), $\langle\vec{w}, f\bar{t}\rangle \sim \langle\vec{w}[x \mid z], (f\bar{t})[x \mid z]\rangle$.

Suppose $t \equiv \langle t_1, t_2\rangle$. By induction, for $i \equiv 1, 2$, $\langle\vec{w}, t_i\rangle \sim \langle\vec{w}[x \mid z], t_i\rangle$ hence by (τ4), $\langle\vec{w}, \langle t_1, t_2\rangle\rangle \sim \langle\vec{w}[x \mid z], \langle t_1, t_2\rangle[x \mid z]\rangle$.

Suppose $t \equiv qy\varphi$ and $x \not\varepsilon S_F(qy\varphi)$. 'e want to show $\langle\vec{w}, qy\varphi\rangle \sim \langle\vec{w}[x \mid z], qy\varphi\rangle$ (A_1). Let $y_1 \equiv ch(\tau_0(x), S_{FB}(\vec{w}y\varphi))$ and $y_2 \equiv ch(\tau_0(x), S_{FB}((\vec{w}[x \mid z])\, x\varphi))$. To establish A_1 we will prove

$$\langle\vec{w}y_1, \varphi[y \mid y_1]\rangle \sim \langle(\vec{w}[x \mid z])y_2, \varphi[y \mid y_2]\rangle \quad (A_2).$$

Assuming $y_1 \not\equiv y_2$ we have $y_2 \not\varepsilon \vec{w}y_1$ and $y_2 \not\varepsilon S_B(\varphi[y \mid y_1]) \equiv S_B(\varphi)$, so $\langle\vec{w}y_1, \varphi[y \mid y_1]\rangle \sim \langle\vec{w}y_2, \varphi[y \mid y_2]\rangle$ (A_3) by induction. If $y_1 \equiv y_2$ then A_3 is an equality. Thus to establish A_2 it will suffice to prove

$$\langle\vec{w}y_2, \varphi[y \mid y_2]\rangle \sim \langle(\vec{w}[x \mid z])y_2, \varphi[y \mid y_2]\rangle \quad (A_4).$$

If $y \not\equiv x$ then $x \not\varepsilon S_F(\varphi)$ and hence $x \not\varepsilon S_F(\varphi[y \mid y_2])$, so A_4 follows by induction, substituting z for x in both components. If $y \equiv x$ then since $y_2 \not\equiv x$, $x \not\varepsilon S_F(\varphi[x \mid y_2])$ hence again A_4 follows by induction.

Suppose $t \equiv qy\varphi$ and $x \varepsilon S_F(qy\varphi)$; then $x \not\equiv y$ and $x \varepsilon S_F(\varphi)$. We want to show

$$\langle \vec{w}, qy\varphi\rangle \sim \langle \vec{w}[x \mid z], qy(\varphi[x \mid z])\rangle \qquad (B_1) .$$

The hypothesis on z reduces to $z \notin \{\vec{w}\}$ (H_1) , $z \neq y$ (H_2) , and $z \in ffr[x](\varphi)$ (H_3) (this is case f_7 of 0.2.8.1). We put
$y_1 \equiv ch(\tau_0(y), S_{FB}(\vec{w}v\varphi))$ and
$y_2 \equiv ch(\tau_0(y), S_{FB}((\vec{w}[x \mid z])y(\varphi[x \mid z])))$. It will suffice, on the basis of τ_5 , to prove

$$\langle \vec{w}y_1, \varphi[y \mid y_1]\rangle \sim \langle(\vec{w}[x \mid z])y_2, (\varphi[x \mid z])[y \mid y_2]\rangle \quad (B_2) .$$

It is possible that $y_1 \equiv z$ and $y_2 \equiv x$, thus the apparently straightforward substitutions $\binom{x}{z}$ and $\binom{y_1}{y_2}$ will not achieve the relation B_2 . To overcome this obstacle we choose
$u \equiv ch(\tau_0(x), S_{FB}(\vec{w}zyy_1y_2\varphi))$ and then make the substitutions
$\binom{x}{u}$, $\binom{y_1}{y_2}$ and $\binom{u}{z}$.

We start with $\binom{x}{u}$. By our choice of u, $u \notin \{\vec{w}y_1\}$ and $u \notin S_B(\varphi) \equiv S_B(\varphi[y \mid y_1])$ hence by induction:

$$\langle \vec{w}y_1, \varphi[y \mid y_1]\rangle \sim \langle(\vec{w}[x \mid u])y_1, \varphi[x \mid u][y \mid y_1]\rangle \qquad (B_3) .$$

We are able to switch $\binom{x}{u}$ and $\binom{y}{y_1}$ since $u \neq y$ and $x \neq y_1$. (see 0.2.6.2).

We want to show the induction hypothesis applies to $\binom{y_1}{y_2}$ and the right side of (B_3) so that

$$\langle(\vec{w}[x \mid u])y_1, \varphi[x \mid u][y \mid y_1]\rangle \sim \langle(\vec{w}[x \mid u])y_2, \varphi[x \mid u][y \mid y_2]\rangle \ (B_4) .$$

If $y_1 \equiv y_2$ B_4 holds since $\sim$ is an equivalence relation, thus we assume $y_1 \neq y_2$. We have $y_2 \notin \{(\vec{w}[x \mid u])y_1\}$ and $y_2 \notin S_B(\varphi[x \mid z]) \equiv S_B(\varphi[x \mid u][y \mid y_1])$ by 0.2.5.8. Hence the substitution is legitimate. To achieve B_4 we must show $\varphi[x \mid u][y \mid y_1][y_1 \mid y_2]$

collapses to $\varphi[x \mid u][y \mid y_2]$. This will follow from 0.2.5.8 if we can show $y_1 \notin S_{FB}(\varphi[x \mid u])$. By our choice of y_1 we have $y_1 \notin S_B(\varphi) = S_B(\varphi[x \mid u])$. By our choice of u we have $y \notin (S_F(\varphi) \cup \{u\}) \supset S_F(\varphi[x \mid u])$.

The final step is to change u back to z . By 0.2.9.4, since $u \notin S_{FB}(\varphi)$

$$\mathrm{ffr}[x](\varphi) \subset \mathrm{ffr}[u](\varphi[x \mid u]) \qquad (F_1) .$$

By 0.2.9.5, since $u \neq y_2$

$$\mathrm{ffr}[u](\varphi[x \mid u]) \subset \mathrm{ffr}[u](\varphi[x \mid u][y \mid y_2]) \qquad (F_2) .$$

From H_3, F_1 and F_3 we have

$$z \in \mathrm{ffr}[u](\varphi[x \mid u][y \mid y_2]) .$$

Also $z \neq y_2$, $z \neq u$ and $z \notin \{w\}$ hence $z \notin \{(\vec{w}[x \mid u])y_2\}$. Thus by induction

$$<(\vec{w}[x \mid u])y_2, \varphi[x \mid u][y \mid y_2]> \sim <(\vec{w}[x \mid z])y_2, \varphi[x \mid u][y \mid y_2][u \mid z]> \qquad (B_5) .$$

Since $y \neq z$ and $u \neq y_2$ we can switch $\begin{pmatrix} y \\ y_2 \end{pmatrix}$ with $\begin{pmatrix} u \\ z \end{pmatrix}$. Since $u \notin S_{FB}(\varphi)$ we have $\varphi[x \mid u][u \mid z] = \varphi[x \mid z]$, thus the right side of (B_5) becomes the right side of (B_2) . From B_3, B_4 and B_5 we have B_2 . □

<u>0.3.4.11 Proposition</u>. Let $\sim$ be a typed equivalence relation on AugTms under *ch* then $\sim$ satisfies $\tau'5$.

$(\tau'5)$ If $<\vec{w}_i, qx_i\varphi_i> \in$ Aug Tms and $\bar{y}_i \in (\mathrm{ffr}[x_i](\varphi_i) - \{\vec{w}_i\})$ for $i = 1, 2,$ and $<\vec{w}_1\bar{y}_1, \varphi[x_1 \mid \bar{y}_1]> \sim <\vec{w}_2\bar{y}_2, \varphi[x_2 \mid \bar{y}_2]>$ (E_0) , then $<\vec{w}_1, qx_1\varphi_1> \sim <\vec{w}_2, qx_2\varphi_2>$.

<u>Proof</u>. Put $y_i = ch(\tau_0(x_i), S_{FB}(w_i x_i \varphi_i))$. If $y_i = \bar{y}_i$ then

$$<\vec{w}_i\bar{y}_i,\ \varphi[x_i \mid \bar{y}_i]> \sim <\vec{w}_i y_i,\ \varphi[x_i \mid y_i]> \qquad (E_1)$$

since $\sim$ is an equivalence. If $y_i \neq \bar{y}_i$ then $\bar{y}_i \notin \{\vec{w}_i y_i\}$.
We want to show

$$\bar{y}_i \in ffr[y_i](\varphi[x_i \mid y_i]) \qquad (F)$$

By 0.2.9.4, since $y_i \notin S_{FB}(\varphi_i)$,

$$ffr[x_i](\varphi_i) \subset ffr[y_i](\varphi[x_i \mid y_i]) .$$

Hence F holds. Now by 0.3.4.10

$$<\vec{w}_i y_i, \varphi[x_i \mid y_i]> \sim <\vec{w}_i\bar{y}_i,\ \varphi_i[x_i \mid y_i][y_i \mid \bar{y}_i]> \qquad (E_2) .$$

Since $y_i \notin S_{FB}(\varphi_i)$ we have, by 0.2.5.8,

$$\varphi_i[x_i \mid y_i][y_i \mid \bar{y}_i] = \varphi_i[x_i \mid \bar{y}_i] . \qquad (E_3) .$$

From E_3 and E_2 we have E_1 for $i = 1, 2$. Combining E_1 for $i = 1, 2$ with E_0 gives

$$<\vec{w}_1 y_1, \varphi[x_1 \mid y_1]> \sim <\vec{w}_2,\ y_2, \varphi[x_2 \mid y_2]> \qquad (E_4) .$$

Hence by $\tau 5$ $<\vec{w}_1,\ qx_1\varphi_1> \sim <\vec{w}_2,\ qx_2\varphi_2>$. □

<u>0.3.4.12</u>. In 0.3.4.5 the relation we defined depended on a choice of new variables *ch* , we now remove this feature from the relation and define <u>a typed equivalence relation on Aug Tms</u> to be an equivalence relation on AugTms satisfying $\tau 1$, $\tau 2$, $\tau 3$, $\tau 4$ and $\tau' 5$. Clearly $\tau' 5$ implies $\tau 5$ so a typed equivalence relation is a typed equivalence relation under *ch* . By 0.3.4.11 every typed equivalence relation under *ch* is a typed equivalence relation.

Section 0.4 The Cartesian Structure of a Category

In preparation for the interpretation of terms we collect together in this section some useful facts about the cartesian structure of a category and relate these to the terms of the language in 0.5.3.

0.4.1 The Category $\mathcal{N}$. This category is isomorphic to the category FinOrd of finite ordinals and equivalent, by way of a non-constructive choice function, to the category of finite sets (see [MacL], MacLane, p. 18). $\mathcal{N}$ has as objects all $[n]$ where $n \in \mathbb{N}$, and as morphisms all pairs $<\alpha,n>$ where $\alpha \in \mathrm{Strgs}(\mathbb{N})$, $n \in \mathbb{N}$ and $\mathrm{Rng}\ \alpha \subset [n]$;

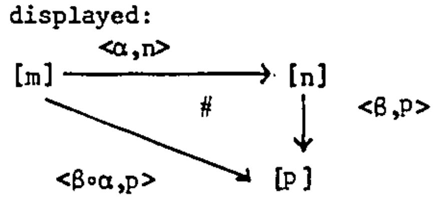
$$\mathrm{dom} <\alpha,n> = \mathrm{Dom}\ (\alpha)$$

$$\mathrm{cod} <\alpha,n> = [n].$$

(We are following the notation of Levy, [Le], p. 26; 6.9.) If $\mathrm{dom} <\beta,p> = \mathrm{cod} <\alpha,n>$, that is $\mathrm{Dom}\ (\beta) = [n]$, then composition is given by

(1) $<\beta,p>\ <\alpha,n> = <\beta\circ\alpha,p>.$

In an abstract category we label the arrows of a diagram with morphisms, thus (1) is displayed:



In concrete situations such as the above we can recover the morphism from the function α and the codomain $[n]$ so that the arrows can be labelled with functions :

$$\begin{array}{ccc} [m] & \xrightarrow{\ \alpha\ } & [n] \\ & \underset{\beta\circ\alpha}{\searrow}\ \# & \downarrow \beta \\ & & [p] \end{array}$$

We are here following the convention of [MacL] MacLane, p. 8, except that we use "function" in the set-theoretical sense and use "set morphism" for what is called a function there.

The colimit structure of $\mathcal{N}$:

1. $[0]$ is the initial object
2. There is a natural choice for the injections of the coproduct diagram

$$[n] \underset{p_1}{\hookrightarrow} [n] \cup (n + [m]) \underset{p_2}{\longleftarrow} [m]$$

where $p_1(i) = i$ for $i \in [n]$, and $p_2(i) = n+i$ for $i \in [m]$.

3. We do not make a canonical choice for the function that coequalizes two functions (even though a systematic choice is possible by the equivalence of [MacL], p. 18):

$$[m] \underset{h}{\overset{t}{\rightrightarrows}} [n] \xrightarrow[c]{} [p].$$

Geometrically $[m] \underset{h}{\overset{t}{\rightrightarrows}} [n]$ is a directed graph, c identifies vertices if and only if there is an unoriented path joining them, and p is the number of components of the graph.

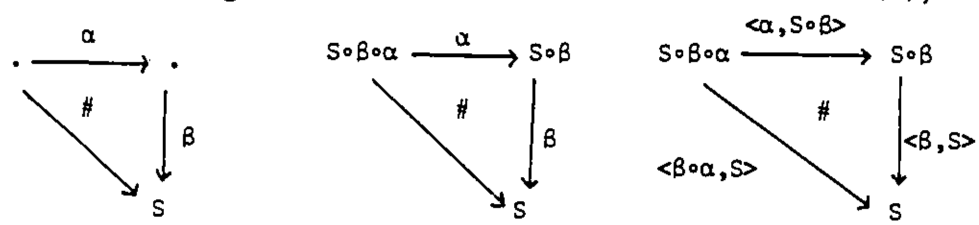
0.4.2 The cartesian category $\text{Cart}(\Sigma_0)$ generated by the set Σ_0

From the inclusion functor $\mathcal{N} \hookrightarrow \mathcal{S}et$ and the functor $1 \xrightarrow{\{\Sigma_0\}} \mathcal{S}et$ that picks out the set Σ_0 we form the comma category $(\mathcal{N}, \{\Sigma_0\})$. We define $\text{Cart}(\Sigma_0)$ to be $(\mathcal{N}, \{\Sigma_0\})^{op}$. The objects of $(\mathcal{N}, \{\Sigma_0\})$ can be identified with the set Σ_0^{*}, and the morphisms are then all pairs $\langle\beta, S\rangle$ where $\beta \in \mathbb{N}^{*}$, $S \in \Sigma_0^{*}$ and $\text{Rng}(\beta) \subset \text{Dom}(S)$;

$$\text{dom}\,\langle\beta,S\rangle = S\circ\beta$$

$$\text{cod}\,\langle\beta,S\rangle = S.$$

If $\text{dom}\,\langle\beta,S\rangle = \text{cod}\,\langle\alpha,T\rangle$ then $S\circ\beta = T$ and $\langle\beta,S\rangle\,\langle\alpha,T\rangle = \langle\beta\circ\alpha,S\rangle$. From the labelling of (1) we can recover domains (2) and from (2),



(1) (2) (3)

as noted for $\mathcal{N}$, we can recover a complete labelling (3). We define a 'forgetful' functor $|\sim| : (\mathcal{N}, \{\Sigma_0\}) \longrightarrow \mathcal{N}$ by

$$|S| = [\,\ell(S)\,] = \text{Dom } S \quad\text{and}\quad |\langle\beta,S\rangle| = \langle\beta, \ell(S)\rangle .$$

$|\sim|$ is faithful and a functor:

$$|\langle\beta,S\rangle\langle\alpha,T\rangle| = |\langle\beta\circ\alpha,S\rangle| = \langle\beta\circ\alpha, \ell S\rangle = \langle\beta, \ell(S)\rangle\langle\alpha, \ell(T)\rangle$$
$$= |\langle\beta,S\rangle|\;|\langle\alpha,T\rangle|.$$

0.4.2.1 Proposition. $|\sim|$ creates initial objects, binary coproducts, coequalizers and pushouts.

1. If $|T| = [0] = \phi$ then $T = \phi$; $\phi \xrightarrow{\phi} S$ in $(\mathcal{N}, \{\Sigma_0\})$ and if $\operatorname{dom} \langle \beta, S \rangle = \phi$ then $\beta = \phi$.

2. Let T and T' be strings (objects of $(\mathcal{N}, \{\Sigma_0\})$) of lengths p and p' respectively. We form the canonical coproduct

$$[p] \underset{p_1}{\hookrightarrow} [p] \cup (p + [p']) \underset{p_2}{\leftarrowtail} [p']$$

in $\mathcal{N}$, then

$$T \underset{p_1}{\longrightarrow} T \cup (p + T') \underset{p_2}{\longleftarrow} T'$$

is a diagram in $(\mathcal{N}, \{\Sigma_0\})$ since

for $i \in [p]$ we have $(T \cup (p + T')) \circ p_1(i)$

$= (T \cup (p + T'))\,(i) = T\,(i)$

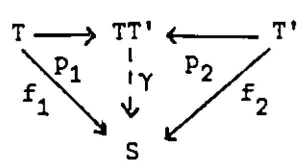
and for $i \in [p']$ we have $(T \cup (p + T')) \circ p_2(i)$

$= (T \cup (p + T'))\,(p + i) = (p + T')(p + i) = T'(i)$.

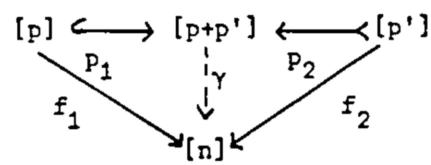
To show it is a coproduct diagram suppose

$$T \underset{f_1}{\longrightarrow} S \underset{f_2}{\longleftarrow} T' \quad \text{is a diagram in } (\mathcal{N}, \{\Sigma_0\})$$

so that $T = S \circ f_1$ and $T' = S \circ f_2$. Let n be the length of S , then the image under $|\sim|$ of (1) is (2) (without the dotted arrows).



(1)



(2)

There is exactly one function γ making (2) commute. If $\gamma\colon TT' \longrightarrow S$ is a morphism of $(\mathcal{N},\{\Sigma_0\})$ then (1) will commute, thus we have to show $TT' = S\circ\gamma$:

for $i \in [p] \subset [p+p']$ we have

$$(S\circ\gamma)\ (i) = S\circ\gamma\circ p_1(i) = S\circ f_1(i) = T(i) = (TT')\ (i)$$

for $i + p \in [p+p']$ we have

$$(S\circ\gamma)\ (i+p) = (S\circ\gamma)\circ p_1(i) = (S\circ f_2(i) = T'(i) = (p+T')(p+i)$$
$$= (TT')(p+i)$$

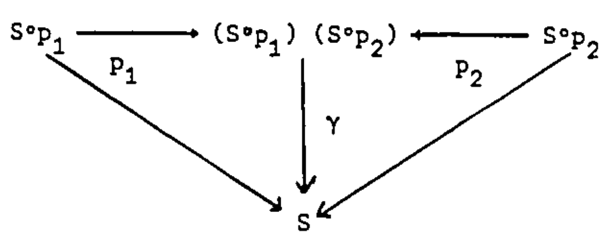
More generally if

$$[p] \underset{g_1}{\rightarrowtail} [p+p'] \underset{g_2}{\leftarrowtail} [p']$$

is a coproduct diagram in $\mathcal{N}$, there exists a permutation g of $[p+p']$ such that $gg_1 = p_1$, $gg_2 = p_2$, and $T \underset{g_1}{\longrightarrow} (TT') \underset{g_2}{\longleftarrow} T'$

is a coproduct diagram in $(\mathcal{N}, \{\Sigma_0\})$.

(Note also that if we begin with S of length $p+p'$ then we have in place of (1) the diagram (3). In diagram (2) γ is $id_{[p+p']}$

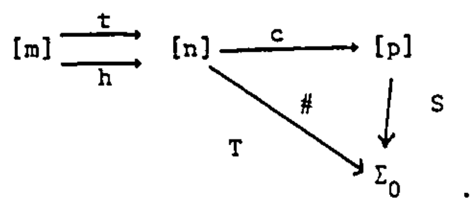


(3)

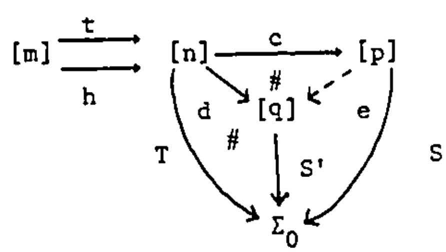
hence $S = (S\circ p_1)\ (S\circ p_2)$.)

3. Let $R \underset{h}{\overset{t}{\rightrightarrows}} T$ be a diagram in $(\mathcal{N}, \{\Sigma_0\})$, with t and h functions, and R and T of lengths m and n respectively so that $R = T \circ t = T \circ h$.

Let $[m] \underset{h}{\overset{t}{\rightrightarrows}} [n] \xrightarrow{c} [p]$ be a coequalizer diagram in $\mathcal{N}$, then it is also a coequalizer diagram in $\mathcal{S}et$ so that there exists exactly one function S such that the triangle below commutes.



We show $R \underset{h}{\overset{t}{\rightrightarrows}} T \xrightarrow{c} S$ is a coequalizer diagram in $(\mathcal{N}, \{\Sigma_0\})$. Suppose $d\colon T \longrightarrow S'$ so that $T = S' \circ d$ and $d \circ t = d \circ h$. We consider the situation in $\mathcal{S}et$:



There exists exactly one e such that $e \circ c = d$. We show that $e\colon S \longrightarrow S'$ is a map, that is that $S = S' \circ e$. The defining equation for S was $S \circ c = T$, but $(S' \circ e) \circ c = S' \circ d = T$ hence $S = S' \circ e$.

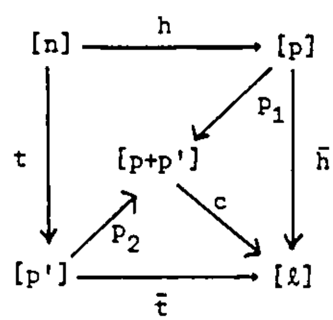
4. The creation of pushouts by $|\sim|$ can be deduced from the creation of coproducts and coequalizers.

Let (1) be a diagram in $(\mathcal{N}, \{\Sigma_0\})$ with image (2) so that

$$\begin{array}{ccc} R & \xrightarrow{t} & T \\ {\scriptstyle h}\downarrow & & \\ T' & & \end{array} \qquad\qquad \begin{array}{ccc} [m] & \xrightarrow{t} & [p] \\ {\scriptstyle h}\downarrow & & \\ [p] & & \end{array}$$

(1) (2)

$T \circ t$ ■ R ■ $T' \circ h$. Let $\bar{h}$, $\bar{t}$ be any pair that makes the square in (3) a pushout, then we can fill in the square in a unique way where (4) is a coequalizer diagram



$$[n] \overset{p_1 \circ h}{\underset{p_2 \circ t}{\rightrightarrows}} [p+p'] \xrightarrow{c} [\ell]$$

(3) (4)

From the creation of coproducts we know that $T \xrightarrow[p_1]{} TT' \xleftarrow[p_2]{} T'$ is a coproduct diagram; $R \overset{p_1 \circ h}{\underset{p_2 \circ t}{\rightrightarrows}} TT'$ is a diagram in $(\mathcal{N}, \{\Sigma_0\})$, hence S is determined such that $R \overset{p_1 \circ h}{\underset{p_2 \circ t}{\rightrightarrows}} TT' \xrightarrow{c} S$ is a coequalizer diagram. Thus we have the pushout (5)

$$\begin{array}{ccc} R & \xrightarrow{t} & T \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \bar{h}} \\ T' & \xrightarrow{\bar{t}} & S \end{array} \quad (5)$$

. □

We could have proven from scratch a more general statement in place of 0.4.2.1, namely that $|\sim|$ creates finite colimits, however we do not need this generality and furthermore we want to show explicitly the simpler constructions that we shall use.

<u>0.4.2.2</u>. There are a number of other facts that are not necessary to the development of future sections but that put the material we give proofs for and use into perspective.

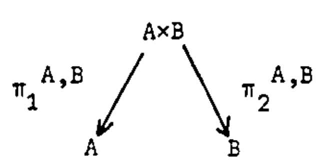
If we ignore the morphisms of the categories $(\mathcal{N},\{\Sigma_0\})$ and $\mathcal{N}$, then $|\sim|$ is essentially the function ℓ, which is a homomorphism from the free monoid on Σ_0 to $\mathbb{N}$. These monoids and the homomorphism ℓ can be "explained" as follows: The forgetful functor $U : \mathcal{Mon} \longrightarrow \mathcal{Set}$ from the category of monoids to the category of sets has a left adjoint $\text{Strgs} : \mathcal{Set} \longrightarrow \mathcal{Mon}$, the image under Strgs of the uniquely determined set morphism $\Sigma_0 \longrightarrow 1$ is, up to an isomorphism of the codomain, the length morphism $\ell : \text{Strgs}(\Sigma_0) \longrightarrow \mathbb{N}$.

For an arbitrary monoid $\underline{M}$ the natural transformation $ev : \text{Strgs} \circ U \longrightarrow I$ gives rise to a monoid homomorphism $ev_{\underline{M}} : \text{Strgs}(U(\underline{M})) \longrightarrow \underline{M}$. We are going to construct a functor which is an analogue of $ev_{\underline{M}}$. Let <u>CartCat</u> denote the category having as objects small cartesian categories - ones with finite limits - and as morphisms limit preserving functors - this we make precise in 0.4.3.1. Let $\text{Obj} : \underline{\text{CartCat}} \longrightarrow \mathcal{Set}$ be the functor that associates with a category its object set and with a functor the restriction of the functor to objects. The functor Cart : $\mathcal{Set} \longrightarrow$ CartCat associates with a set Σ the cartesian category $(\mathcal{N},\{\Sigma\})^{op}$ and with a function $f : \Sigma \longrightarrow \Sigma'$ a "cartesian" morphism $Cart(f) : (\mathcal{N},\{\Sigma\})^{op} \longrightarrow (\mathcal{N},\{\Sigma'\})^{op}$ that "extends" objects : S goes to $f \circ S$. The functor Cart can be shown to be left adjoint to Obj thus $(\mathcal{N},\{\Sigma\})^{op}$ is the free cartesian category generated by Σ and for each cartesian category $\underline{\mathcal{E}}$ there is an evaluation functor $\Pi_{\underline{\mathcal{E}}} : Cart(Obj(\underline{\mathcal{E}})) \longrightarrow \underline{\mathcal{E}}$, which preserves finite limits. We can recover something like $ev_{\underline{M}}$ by putting $\underline{M} \equiv \langle Obj(\underline{\mathcal{E}}), \ , \times\rangle$ (a monoid "up to isomorphism") and considering $\Pi_{\underline{\mathcal{E}}}$ restricted to objects.

0.4.3 The functor $\Pi : Cart(Obj(\underline{\mathcal{E}})) \longrightarrow \underline{\mathcal{E}}$.

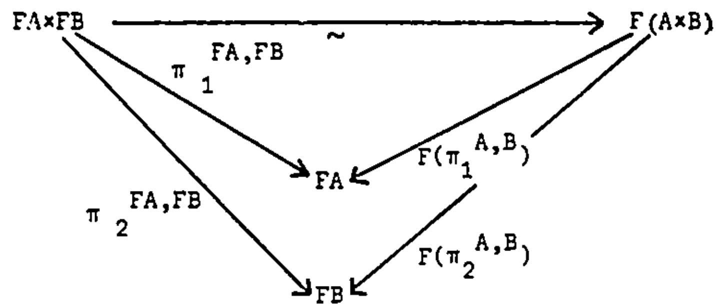
0.4.3.1. We first give a precise definition of CartCat . An object shall consist of:

1. a category $\underline{\mathcal{E}}$
2. an object $\mathbb{1}$ of $\underline{\mathcal{E}}$ which is terminal,
3. an assignment of an object $A \times B$ to pairs $\langle A, B\rangle$ of objects of $\underline{\mathcal{E}}$, and morphisms $\pi_1^{A,B} : A{\times}B \to A$ and $\pi_2^{A,B} : A{\times}B \longrightarrow B$ making



a product diagram.

A morphism is a functor $F : \underline{\mathcal{E}} \longrightarrow \underline{\mathcal{G}}$ which preserves finite limits. We do not require that the preservation be "on the nose";



the right part of the above diagram is to be a product diagram so that there is a uniquely determined isomorphism between FA×FB and F A×B . If $\mathbb{1}$ and $\mathbb{1}$ are the canonical terminal objects for $\underline{\mathcal{E}}$ and $\underline{\mathcal{G}}$ respectively then the unique morphism $F(\mathbb{1}) \longrightarrow \mathbb{1}$ is invertible. In the case of Π we will have $\Pi(\mathbb{1}) = \mathbb{1}$; but for exponentiation, $(\ )^A : \underline{\mathcal{E}} \longrightarrow \underline{\mathcal{E}}$, on cartesian closed categories $\mathbb{1}^A \longrightarrow \mathbb{1}$ will be an isomorphism only.

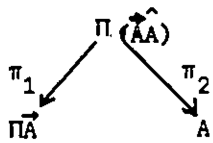
<u>0.4.3.2</u>. We define $\Pi : \mathrm{Obj}(\mathrm{Cart}(\mathrm{Obj}(\underline{\mathcal{E}}))) \longrightarrow \mathrm{Obj}(\underline{\mathcal{E}})$, by induction on the lengths of strings of objects of $\underline{\mathcal{E}}$. We let $\vec{A}$ denote arbitrary such strings.

(1) A string of length 0 is necessarily ϕ ; we put $\Pi\phi = \mathbb{1}$.

(2) For A an object of $\underline{\mathcal{E}}$ we put $\Pi(\hat{A}) = A$.

(3) A string of length n+2 can be written as $\vec{A}\hat{A}$ so that $\Pi\vec{A}$ is defined and $\ell(\vec{A}) \geq 1$; $\Pi\vec{A}\hat{A} = \Pi\vec{A} \times A$.

Secondly we equip the defined objects with projection maps. We define the projections by induction on the length of strings. If the string has length one $(\hat{A})$ we put $pr_0^{\hat{A}} = id_A$. If the string has length $n+2$ we have a product diagram



(1)

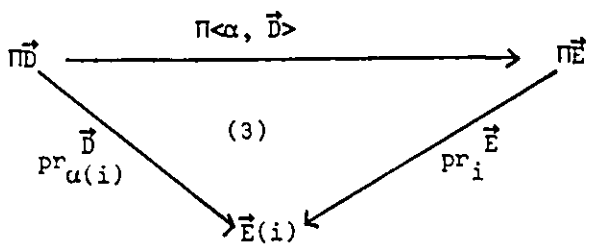
$$\pi_1 = \pi_1^{\Pi\vec{A},A}$$
$$\pi_2 = \pi_2^{\Pi\vec{A},A}$$

By induction we have projection maps $\Pi\vec{A} \xrightarrow{pr_i^{\vec{A}}} \vec{A}(i)$, $i \in [n+1]$ (2). Define $pr_{n+1}^{\vec{A}\hat{A}} = \pi_2$ and $pr_i^{\vec{A}\hat{A}} = pr_i^{\vec{A}} \circ \pi_1$ for $i \in [n+1]$.

Thirdly we show inductively that we actually have produced a product diagram. For $\Pi\hat{A} \xrightarrow{id_A} A$ this is clear. We proceed by induction, supposing the length of $\vec{A}\hat{A}$ is $n+2$ and that (2) is a product diagram. Let $E \xrightarrow{g_i} (\vec{A}\hat{A})(i)$, $i \in [n+2]$ be a family of maps. For the subfamily $E \xrightarrow{g_i} \vec{A}(i)$, $i \in [n+1]$, there exists a uniquely determined map $E \xrightarrow{g'} \Pi\vec{A}$ such that $pr_i^{\vec{A}} \circ g = g_i$ for $i \in [n+1]$. Since (1) is a product diagram we have a uniquely determined map $E \xrightarrow{g} \Pi\vec{A}\hat{A}$ such that $\pi_1 \circ g = g'$ and $\pi_2 \circ g = g_{n+1}$.

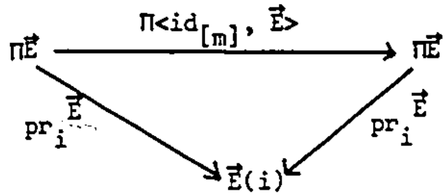
We are now able to define Π on morphisms:

Let $\vec{E} \xrightarrow{\alpha} \vec{D}$ where $[m] \xrightarrow{\alpha} [n]$ and $\vec{E} = \vec{D} \circ \alpha$. We define $\Pi\langle\alpha, \vec{D}\rangle$ by the following diagram

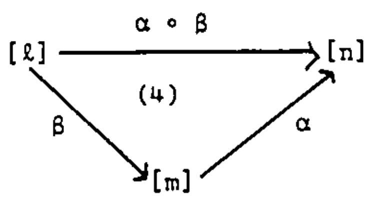


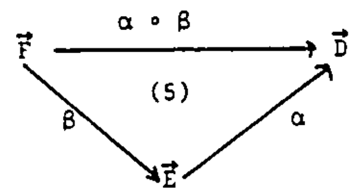
$i \in [m]$

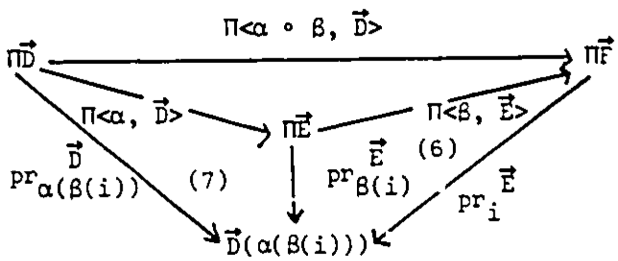
The case when $m = 0$ is included under the above definition since then there are no projections and the map $\Pi\vec{D} \xrightarrow{\Pi\langle\phi,\ \vec{D}\rangle} \mathbf{1}$ is uniquely determined. We claim that Π is a functor. Π preserves identity morphisms since for $\vec{E} \xrightarrow{id_{[m]}} \vec{E}$ the morphism for which



commutes for all $i \in [m]$ must be $id_{\Pi\vec{E}}$. Π preserves composition. From the diagrams (4) and (5) we get



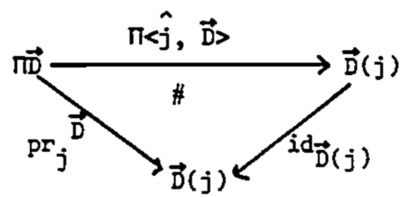




(6), (7) and the larger triangle commuting, hence (8) commutes.

<u>0.4.3.3</u>. The projection maps were used to define Π on morphisms but they are themselves the values under Π of coproduct injections in $(\mathcal{N}, \{Obj(\underline{\mathcal{E}})\})$. Specializing diagram (3) we have the

commutative triangle:



since $j = \hat{j}(0)$ and $pr_0\,(\vec{D}(j))^\wedge = id_{\vec{D}(j)}$, that is $pr_j^{\vec{D}} = \Pi\langle\hat{j},\vec{D}\rangle$. As well, the projection maps π_1 and π_2 of (1) in 0.4.3.1 are the values under Π of coproduct injections. Following the notation of 0.4.3.1 we have the coproduct diagrams (1) and (2)

(1) $[n+1] \underset{p_1}{\subset} [n+2] \xleftarrow[p_2]{} [1]$ where $p_1(i) = i$ and $p_2(0) = n+1$

(2) $\vec{A} \xrightarrow[p_1]{} \vec{A}\hat{A} \xleftarrow[p_2]{} \hat{A}$ (3) $\Pi\vec{A} \xleftarrow[\Pi\langle p_1,\vec{A}\hat{A}\rangle]{} \Pi\vec{A}\times A \xrightarrow[\Pi\langle p_2,\vec{A}\hat{A}\rangle]{} A$

and the diagram (3). Now $p_2(0) = n+1$, so

$$\Pi\langle p_2,\vec{A}\hat{A}\rangle = pr_{n+1}^{\vec{A}\hat{A}} = \pi_2 .$$

To prove $\Pi\langle p_1,\vec{A}\hat{A}\rangle = \pi_1$ we apply the projections $pr_i^{\vec{A}}$, $i \in [n+1]$:

$$pr_i^{\vec{A}}\circ\Pi\langle p_1,\vec{A}\hat{A}\rangle = \Pi\langle\hat{i},\vec{A}\rangle\circ\Pi\langle p_1,\vec{A}\hat{A}\rangle = \Pi\langle\hat{i},\vec{A}\hat{A}\rangle = pr_i^{\vec{A}\hat{A}} = pr_i^{\vec{A}}\circ\pi_1 .$$

Thus Π in case of (2) exactly preserves canonical product diagrams, whereas in general all we can prove is that Π preserves products - that the image of a canonical product diagram is merely a product diagram (not necessarily a canonical one). Introducing maps $[n+1] \xrightarrow[\alpha_1]{} [m] \xleftarrow[\alpha_2]{} [1]$, and a string $\vec{C}$ of length m , we have

$$(4) \quad \Pi\langle\alpha_1,\vec{C}\rangle \cap \Pi\langle\alpha_2,\vec{C}\rangle = \Pi\langle\alpha_1 \cup \alpha_2,\vec{C}\rangle .$$

To prove this we just apply the projections π_1 and π_2 ; (let $\alpha = \alpha_1 \sqcup \alpha_2$ so that $\alpha \circ p_i = \alpha_i$ for $i = 1,2$)

$$\pi_i \circ \Pi\langle\alpha,\vec{C}\rangle = \Pi\langle p_i,\widehat{\vec{A}\vec{A}}\rangle \circ \Pi\langle\alpha,\vec{C}\rangle = \Pi\langle\alpha_i,\vec{C}\rangle \quad \text{for } i = 1,2 .$$

<u>0.4.3.4 Proposition</u>. The functor $\Pi : \mathrm{Cart}(\mathrm{Obj}(\underline{\mathcal{E}})) \longrightarrow \underline{\mathcal{E}}$ preserves terminal objects, binary products and equalizers.

<u>Proof</u>. By definition $\Pi\phi = \square$, so Π preserves terminal objects. It will suffice to look at canonical coproducts in $(\mathcal{N},\{\mathrm{Obj}(\underline{\mathcal{E}})\})$. Let (1) be the coproduct diagram arising from the coproduct diagram (2). We want to show that (3) is a product diagram.

$$\vec{A} \xrightarrow{\;p_1\;} \vec{A}\vec{B} \xleftarrow{\;p_2\;} \vec{B} \qquad (1) \qquad\qquad [m] \overset{p_1}{\hookrightarrow} [m] \cup (m+[n]) \overset{p_2}{\hookleftarrow} [n] \qquad (2)$$

$$\Pi\vec{A} \xleftarrow{\;\Pi\langle p_1,\vec{A}\vec{B}\rangle\;} \Pi\vec{A}\vec{B} \xrightarrow{\;\Pi\langle p_2,\vec{A}\vec{B}\rangle\;} \Pi\vec{B} \qquad (3)$$

Given $\Pi\vec{A} \xleftarrow{\;f\;} C \xrightarrow{\;g\;} \Pi\vec{B}$ we want to show that the two equations (4) and (5) have a unique common solution

(4) $\Pi\langle p_1,\vec{A}\vec{B}\rangle \circ h = f$

(5) $\Pi\langle p_2,\vec{A}\vec{B}\rangle \circ h = g$

h for given $\vec{A}$, $\vec{B}$, f and g . For each $i \in [n]$ we have

$pr_i^{\vec{B}} \circ \Pi\langle p_2,\vec{A}\vec{B}\rangle = \Pi\langle p_2 \circ \hat{i},\vec{A}\vec{B}\rangle = pr_{m+i}^{\vec{A}\vec{B}}$ hence for each $j \in (m+[n])$ $pr_{j-m}^{\vec{B}} \circ \Pi\langle p_2,\vec{A}\vec{B}\rangle = pr_j^{\vec{A}\vec{B}}$. Also for each $j \in [m]$ we have $pr_j^{\vec{A}} \circ \Pi\langle p_1,\vec{A}\vec{B}\rangle = \Pi\langle p_1 \circ \hat{j},\vec{A}\vec{B}\rangle = pr_j^{\vec{A}\vec{B}}$. Applying projections to (4) we get the system of equations (6) which is equivalent to (4):

$$(6) \quad pr_j^{\vec{AB}} \circ h = pr_j^{\vec{A}} \circ f \quad \text{for each} \quad j \in [m] \ .$$

Applying projections to (5) we get the equivalent system (7):

$$(7) \quad pr_j^{\vec{AB}} \circ h = pr_{j-m}^{\vec{B}} \circ f \quad \text{for each} \quad j \in m+[n] \ .$$

Together (6) and (7) form a system of equations indexed by $j \in [m+n]$; since this system has exactly one solution the system (4) and (5) has exactly one solution.

Let (1) be a coequalizer diagram arising from the coequalizer diagram (2). We want to show that (3) is an equalizer diagram.

$$\vec{B} \underset{\alpha'}{\overset{\alpha}{\rightrightarrows}} \vec{C} \xrightarrow{\beta} \vec{D} \qquad\qquad [m] \underset{\alpha'}{\overset{\alpha}{\rightrightarrows}} [n] \xrightarrow{\beta} [p]$$

(1) (2)

$$\Pi\vec{D} \xrightarrow{\Pi<\beta,\vec{D}>} \Pi\vec{C} \underset{\Pi<\alpha',\vec{B}>}{\overset{\Pi<\alpha,\vec{B}>}{\rightrightarrows}} \Pi\vec{B}$$

(3)

Let $f : G \longrightarrow \Pi\vec{C}$ be a morphism for which

$$(4) \quad \Pi<\alpha,\vec{B}> \circ f = \Pi<\alpha',\vec{B}> \circ f \ .$$

We want to show that the equation

$$(5) \quad \Pi<\beta,\vec{D}> \circ g = f$$

has a unique solution for given β, $\vec{D}$ and f . If we apply the projections $pr_j^{\vec{C}}$ for $j \in [n]$ to (5) we get the system of equations

$$(6) \quad pr_{\beta(j)}^{\vec{D}} \circ g = pr_j^{\vec{C}} \circ f \quad \text{for all } j \in [n] .$$

which is equivalent to (5). Since β is onto it has a section σ , that is

$$(7) \quad \beta \circ \sigma = id_{[p]} .$$

If (6) holds then a consequence is

$$(8) \quad pr_k^{\vec{D}} \circ g = pr_{\sigma(k)}^{\vec{C}} \circ f \quad \text{for all } k \in [p] .$$

But (8) uniquely determines g , so if we can show (8) implies (6) we will have completed the proof.

Let $\bar{B} = \{pr_j^{\vec{C}} \circ f | j \in [n]\}$ and let $f^* : [n] \longrightarrow \bar{B}$ be the function defined by

$$(9) \quad f^*(j) = pr_j^{\vec{C}} \circ f \quad \text{for all } j \in [n] .$$

Diagram (2) is a coequalizer diagram in $\mathcal{S}et$ so if we can show that $f^* \circ \alpha = f^* \circ \alpha'$ then we have a function $\mu : [p] \longrightarrow \bar{B}$ such that $\mu \circ \beta = f^*$. For each $i \in [m]$ we have by (9) and (4)

$(f^* \circ \alpha)(i) = pr_{\alpha(i)}^{\vec{C}} \circ f = pr_i^{\vec{B}} \circ \Pi\langle\alpha,\vec{B}\rangle \circ f = pr_i^{\vec{B}} \circ \Pi\langle\alpha',\vec{B}\rangle$

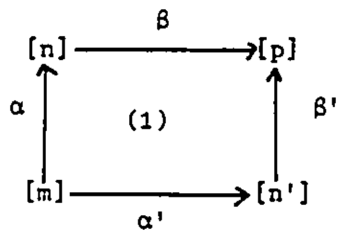
$= (f^* \circ \alpha')$. Thus μ exists so

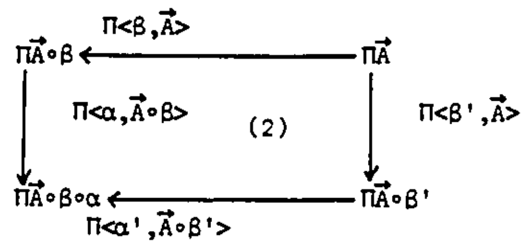
$$(10) \quad \mu(\beta(i)) = pr_j^{\vec{C}} \circ f \quad \text{for all } j \in [n] .$$

From (8), (10) and (7) we can show (6) holds: for all $j \in [n]$

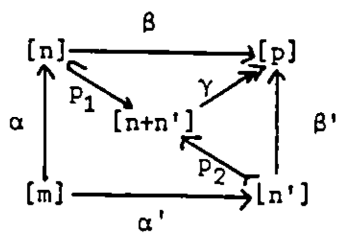
$$pr^{\vec{D}}_{\beta(j)} \circ g = pr^{\vec{C}}_{\sigma\beta(j)} \circ f = \mu(\beta\sigma\beta(j)) = \mu(\beta(j)) = pr^{\vec{C}}_{j} \circ f \ . \quad \square$$

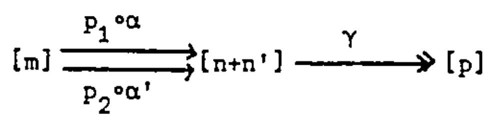
<u>0.4.3.5 Proposition</u>. Let (1) be a push-out diagram and let $\vec{A}$ be a string of objects of length p , then (2) is a pull-back diagram.





<u>Proof</u> (1) can be broken up as follows





a coequalizer diagram. $\square$

<u>0.4.4 Some commutative diagrams involving canonical isomorphisms determined by cartesian functors.</u>

We fix a cartesian functor $F\colon \underline{\mathcal{E}} \longrightarrow \underline{\mathcal{G}}$, and for each B_1, B_2 we let $\rho(B_1,B_2)\colon FB_2 \times FB_2 \longrightarrow F(B_1 \times B_2)$ be the isomorphism making the diagram of 0.4.3.1 commute, so that:

(1) $F(\pi_i^{B_1,B_2})\circ\rho(B_1,B_2) \equiv \pi_i^{FB_1,FB_2}$ for $i \equiv 1,2$.

We shall be applying the calculations of this section to powers A^n, of a single object A (in 0.6.12); however, since at this stage, there is little gain in considering the special case, we will work with products $\Pi\vec{A}$, of arbitrary finite strings $\vec{A}$, of objects of $\underline{\mathcal{E}}$.

<u>0.4.4.1 Definition.</u> For each string $\vec{A}$, of objects of $\underline{\mathcal{E}}$ we define an isomorphism

$$\sigma(\vec{A})\colon \Pi(F\circ\vec{A}) \xrightarrow[\sim]{} F(\Pi(\vec{A}))$$

as follows:

$\sigma(\phi)\colon \mathbb{1} \longrightarrow F(\mathbb{1})$ is the inverse of $u_{F(\mathbb{1})}$.

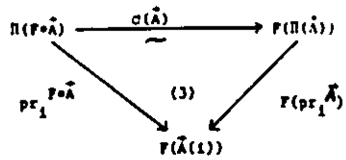
$\sigma(\hat{A}) \equiv id_{FA}\colon FA \xrightarrow[\sim]{} FA$.

For $\ell(\vec{A}) \geq 1$, we define $\sigma(\vec{A}\hat{A})$ inductively by

$$\begin{array}{ccc}
\Pi(F\circ(\vec{A}\hat{A})) & \xrightarrow[\sim]{\sigma(\vec{A}\hat{A})} & F(\Pi(\vec{A}\hat{A})) \\
\Vert & (2) & \uparrow{\scriptstyle \wr}\,\rho(\Pi\vec{A},A) \\
(\Pi(F\circ\vec{A})) \times FA & \xrightarrow[\sigma(\vec{A}) \times FA]{\sim} & F(\Pi(\vec{A})) \times FA
\end{array}$$

requiring that (2) commute. (Note that when $\vec{A} \equiv A'$ we have $\sigma(\hat{A}'\hat{A}) \equiv \rho(A',A)$)

0.4.4.2 Proposition.



(3) commutes for each $i \in [n+1]$ where $\ell(\vec{A}) \equiv n+1$.

Proof. We simplify notation by putting $\pi_i(B_1,B_2) \equiv \pi_i^{B_1,B_2}$ and $pr_i(\vec{B}) \equiv pr_i^{\vec{B}}$. If $\ell(\vec{A}) \equiv 1$ all objects in (2) become FA and all morphisms become id_{FA}. For the induction step, put

$L(i) \equiv F(pr_i(\hat{\vec{A}A}))\circ\sigma(\hat{\vec{A}A})$. By the definition of pr_i in 0.4.3.3,

$$L(n+1) \equiv F(\pi_2(\Pi\vec{A},A))\circ\rho(\Pi\vec{A},A)\circ(\sigma(\vec{A}) \times FA)$$

$$\equiv \pi_2(F(\Pi\vec{A}),FA)\circ(\sigma(\vec{A}) \times FA)$$

$$\equiv \pi_2(\Pi(F\circ\vec{A}),FA) \equiv pr_{n+1}(F\circ(\hat{\vec{A}A})).$$

For $i \in [n+1]$, again by the definition of pr_i,

$$L(i) \equiv F(pr_i(\vec{A})\circ\pi_1(\Pi\vec{A},A))\circ\rho(\Pi\vec{A},A)\circ(\sigma(\vec{A}) \times FA)$$

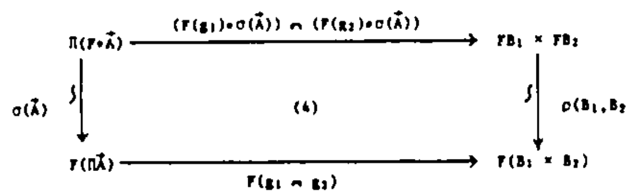
$$\equiv F(pr_i(\vec{A}))\circ\pi_1(F(\Pi\vec{A}),FA)\circ(\sigma(\vec{A}) \times FA)$$

$$\equiv F(pr_i(\vec{A}))\circ\sigma(\vec{A})\circ\pi_1(\Pi(^{-}\circ\vec{A}),FA)$$

$$\equiv pr_i(F\circ\vec{A})\circ\pi_1(\Pi(F\circ\vec{A}),FA) \qquad \text{by induction}$$

$$\equiv pr_i(F\circ(\hat{\vec{A}A})). \ \square$$

0.4.4.3 Proposition. Let $g_i : \Pi\vec{A} \longrightarrow B_i$ $(i \equiv 1,2)$, then (4) commutes.



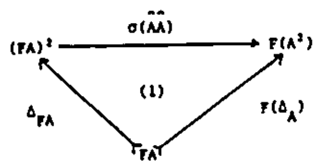
Proof. We mantain the notation of the proof of 0.4.4.2. Abbreviate the top map to k. Applying projections at $FB_1 \times FB_2$ we have:

$\pi_i(FB_1,FB_2)\circ k\circ\sigma(\vec{A})^{-1} \equiv F(g_i)\circ\sigma(\vec{A})\circ\sigma(\vec{A})^{-1} \equiv F(g_i)$ and

$$\begin{aligned}\pi_i(FB_1,FB_2)\circ(\rho(B_1,B_2))^{-1}\circ F(g_1 \sqcap g_2) &\equiv F(\pi_i(B_1,B_2))\circ F(g_1 \sqcap g_2)\\ &\equiv F(\pi_i(B_1,B_2)\circ(g_1 \sqcap g_2))\\ &\equiv F(g_i),\end{aligned}$$

for $i \equiv 1,2$, hence (4) commutes. □

<u>0.4.4.4 Proposition.</u> (1) commutes.

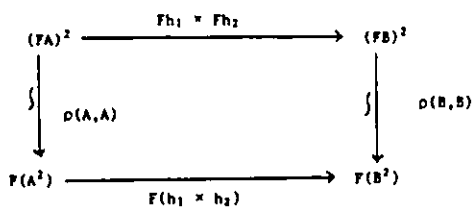


<u>Proof.</u> Let $\sigma_2 \equiv \sigma(\hat{A}\hat{A})$, $\pi_i' \equiv \pi_i^{FA,FA}$, $\pi_i \equiv \pi_i^{A,A}$ $(i \equiv 1,2)$, then

$(\pi_i'\circ\sigma_2^{-1})\circ F(\Delta_A) \equiv F(\pi_i)\circ F(\Delta_A) \equiv F(\pi_i\circ\Delta_A) \equiv id_{FA}$ and

$\pi_i'\circ\Delta_{FA} \equiv id_{FA}$ $(i \equiv 1,2)$ hence $\sigma_2^{-1}\circ F(\Delta_A) \equiv \Delta_{FA}$.□

<u>0.4.4.5 Proposition.</u> Let $h_i\colon A \longrightarrow B$ $(i \equiv 1,2)$ then



commutes.

<u>Proof.</u> For $i \equiv 1,2$ we have

$$\begin{aligned}(\pi_i(FB,FB))\circ(Fh_1 \times Fh_2)\circ(\rho(A,A))^{-1} &\equiv (Fh_i)\circ(\pi_i(FA,FA))\circ(\rho(A,A))^{-1}\\ &\equiv (F(h_i))\circ F(\pi_i(A,A)) \equiv F(h_i\circ(\pi_i(A,A)))\end{aligned}$$

and

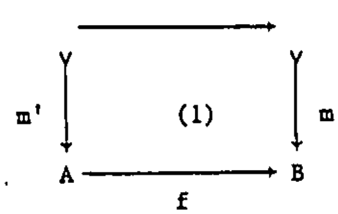
$$\begin{aligned}(\pi_i(FB,FB))\circ(\rho(B,B))^{-1}\circ F(h_1 \times h_2) &\equiv F(\pi_i(B,B))\circ F(h_1 \times h_2)\\ &\equiv F(\pi_i(B,B)\circ(h_1 \times h_2)) \equiv F(h_i\circ(\pi_i(A,A))).\ \square\end{aligned}$$

Section 0.5 The Interpretation of Terms as Morphisms of a Topos.

0.5.1 We introduce just enough structure from Topos Theory to associate a morphism with each term of our topos-based language and prove a few basic propositions. Further structure will be introduced in Section 0.6.

0.5.1.1 Subobjects ([MacL] p. 122, [Sch] p.43). Let $\mathcal{E}$ be a cartesian category. For each object A of $\mathcal{E}$ let Mono A be the full subcategory of the comma category $(\mathcal{E}, A)$ whose objects are monomorphisms. We can construe Mono A as a preordered set by taking Obj Mono A as the set and defining a preorder $\prec$ by $m \prec m'$ iff there is a (necessarily unique) morphism (of Mono A) from m to m'. We define an equivalence relation $\sim$ on this preordered set by: $m \sim m'$ iff $m \prec m'$ and $m' \prec m$. We put $[\![m]\!] = \{m' \mid m \sim m'\}$ and define a partial order on Sub A, the set of all equivalence classes induced by $\sim$, as follows: $[\![m]\!] \leq [\![m']\!]$ iff $m \prec m'$. Under this ordering $[\![id_A]\!]$ is the top element of Sub A; we put $[\![A]\!] = [\![id_A]\!]$.

Pulling back monomorphisms with codomain B along a morphism $f: A \longrightarrow B$ induces an order and top element preserving function f^{-1}: Sub B $\longrightarrow$ Sub A. The function f^{-1} may be characterized by: $m' \in f^{-1}[\![m]\!]$ iff m, f and m' form the sides of a pullback square (1).



<u>0.5.1.2 Adjoints of f^{-1}</u>. If $\mathcal{E}$ is a topos, for each morphism $f\colon A \longrightarrow B$, the order preserving function $f^{-1}\colon \text{Sub } B \longrightarrow \text{Sub } A$ has both a left and right adjoint (see [KW] pp. 30, 32; [J2] p. 144):

$$\exists_f \dashv f^{-1} \dashv \forall_f \, .$$

For subobjects U of A and W of B this means

$$\exists_f U \leq W \quad \text{iff} \quad U \leq f^{-1} W$$

and

$$f^{-1} W \leq U \quad \text{iff} \quad W \leq \forall_f U \, .$$

In a topos the Beck condition holds for both $\exists_{(\ )}$ and $\forall_{(\ )}$. For $\exists_{(\ )}$ this means (see [KW] p. 36): if

$$\begin{array}{ccc} A' & \xrightarrow{\ a\ } & A \\ {\scriptstyle f'}\downarrow & (2) & \downarrow{\scriptstyle f} \\ B' & \xrightarrow[\ b\ ]{} & B \end{array}$$

(2) is a pullback, then for any subobject U of A we have $b^{-1} \exists_f W = \exists_{f'} a^{-1} W$. We show it holds for $\forall_{(\ )}$.

<u>Proposition</u>. $b^{-1}\forall_f W = \forall_{f'} a^{-1} W$ for $W \in \text{Sub } A$.

<u>Proof</u>. For $U \in \text{Sub } B'$ we have:

$U \leq b^{-1} \forall_f W$

iff $\exists_b U \leq \forall_f W$ since $\exists_b \dashv b^{-1}$

iff $f^{-1} \exists_b U \leq W$ since $f^{-1} \dashv \forall_f$

iff $\exists_a f'^{-1} U \leq W$ by the Beck condition for $\exists_{(\ )}$

iff $f'^{-1} U \leq a^{-1} W$ since $\exists_a \dashv a^{-1}$

iff $U \leq \forall_{f'} a^{-1} W$ since $f'^{-1} \dashv \forall_{f'^{-1}}$.□

<u>0.5.1.3 The subobject classifier</u>. In a topos the subobject classifier true: $\mathbb{1} \rightarrowtail \Omega$ induces a bijection from $[A,\Omega]$ to Sub A that sends $f\colon A \to \Omega$ to the subobject $f^{-1}[\![\text{true}]\!]$. Thus $[\![m]\!] = f^{-1}[\![\text{true}]\!]$ iff m, f and true can be completed to a pullback diagram

$$\begin{array}{ccc} & \longrightarrow & \\ m \big\downarrow & & \big\downarrow \text{true} \\ A & \xrightarrow[f]{} & \Omega \end{array} .$$

For each A put $\text{true}_A = \text{true}\circ u_A$, where $u_A\colon A \longrightarrow \mathbb{1}$, then $(\text{true}_A)^{-1}[\![\text{true}]\!] = [\![A]\!]$.

<u>0.5.1.4</u> $\mathscr{E}$ is cartesian closed so the functor $(-)\times B$ has a right adjoint $(-)^B$ for each B. The front adjunction η for this adjointness is a natural transformation: For each $g\colon A_1 \longrightarrow A_2$ we have

$$\begin{array}{ccc} A_1 & \xrightarrow{g} & A_2 \\ \eta_{A_1} \big\downarrow & & \big\downarrow \eta_{A_2} \\ (A_1\times B)^B & \xrightarrow[(g\times B)^B]{} & (A_2\times B)^B \end{array}$$

commuting. For objects A, C the function $\xi_{A,C}\colon [A\times B,C] \longrightarrow [A,C^B]$, $\xi_{A,C}(f) = f^B\circ\eta_A$, is a bijection.

<u>0.5.2 The typed alphabet $\Phi(\mathscr{E})$ of a topos $\mathscr{E}$.</u>

<u>0.5.2.1 Types.</u> We take the set Tps of types of $\Phi(\mathscr{E})$ to be the set Obj$(\mathscr{E})$ of objects of $\mathscr{E}$, and the algebraic structure $\mathbb{T}$ on this set to

be the natural one arising from $\underline{\mathcal{E}}$: that is $\mathbb{1}$, the terminal type, is the canonical terminal object of $\underline{\mathcal{E}}$; Ω, the type of truth values, is the codomain of the subobject classifier of $\underline{\mathcal{E}}$; for each type/object A the type PA is Ω^A; and for each pair $\langle A,B\rangle$ of types, $A\times B$ is the object part of the product, in the category, of the pair $\langle A,B\rangle$.

<u>0.5.2.2 Variables.</u> We take Vbls for $\Phi(\underline{\mathcal{E}})$ to be in one-to-one correspondence with the set of pairs $\mathrm{Tps}\times\mathbb{N}$ by putting $\mathrm{Vbls} = (\mathrm{Tps}\times\mathbb{N})^{[1]}$. We define τ_0: Vbls $\longrightarrow$ Tps by $\tau_0(\langle A,n\rangle^\wedge) = A$ and an index function ind: Vbls $\longrightarrow N$ by $\mathrm{ind}(\langle A,n\rangle^\wedge) = n$ so that $\tau_0 \sqcap \mathrm{ind}$: Vbls $\longrightarrow \mathrm{Tps}\times\mathbb{N}$ is the bijection sending $\langle A,n\rangle^\wedge$ to $\langle A,n\rangle$; if $\mathrm{ind}(y) = n$ and $\tau_0(y) = A$ we say y is the n-th variable of type A. For a finite subset U of Vbls and a type A the set of variables of type A which are not members of U is nonempty thus there is one with minimum index; we let $ch(A,U)$ be this variable, explicitly $ch(A,U) = \langle A,\min\{n \mid \langle A,n\rangle^\wedge \notin U\}\rangle^\wedge$; then ch is a choice of new variables for $\Phi(\underline{\mathcal{E}})$. Let τ_1: $\mathrm{Tps}\times\mathbb{N} \longrightarrow \mathrm{Tps}$ be the first projection so that $\tau_1\circ(\tau_0 \sqcap \mathrm{ind}) = \tau_0$. Let $\vec{w}$ be a string of variables of length n; that is

$$\vec{w} \in (\mathrm{Tps}\times\mathbb{N})^{[n]} \subset \bigcup_{k\in N} (\mathrm{Tps}\times\mathbb{N})^{[k]} = (\mathrm{Tps}\times\mathbb{N})^* = \mathrm{Str(Vbls)},\ \text{then}$$

$$\tau_1\circ\vec{w} \in (\mathrm{Tps})^{[n]} \subset (\mathrm{Tps})^* \quad \text{so} \quad \ell(\tau_1\circ\vec{w}) = n = \ell(\vec{w}).$$

<u>0.5.2.3 Function signs.</u> We assume Morph($\underline{\mathcal{E}}$) is already a set of signs, and take it to be the set of function signs of $\Phi(\underline{\mathcal{E}})$; the functions dom and cod for $\Phi(\underline{\mathcal{E}})$ will be the same as for the category $\underline{\mathcal{E}}$. If Σ is the set of morphisms of a category we can always construct an isomorphic

category fulfilling this assumption by replacing Σ by $\Sigma^{[1]}$ as morphism set and redefining domain, codomain and composition on $\Sigma^{[1]}$. Thus if f and g are morphisms of $\underline{\mathcal{E}}$ we can form fg, their concatenation, and f∘g their composition as morphisms. This will mean that fg is a string of signs of length 2 -and thus not a morphism of $\underline{\mathcal{E}}$ - and f∘g is a string of length 1 (i.e. a sign) as well as a morphism of $\underline{\mathcal{E}}$. We must be careful with our usage of the infix ∘ because it will be doing double duty within 0.5. Its other role will be that of set theoretical composition; for example a string of variables $\vec{w}$ of length m is a function with domain [m] and so is composable with any function $\alpha:[k] \longrightarrow [m]$ to produce a string of variables $\vec{w}\circ\alpha$ of length k. If we were to compose f and g in this set-theoretic sense we would produce either ϕ or f. As we do not want or need this latter kind of composition for morphisms of $\underline{\mathcal{E}}$, we declare that, for morphisms f and g, f∘g shall always have the category theoretic meaning.

<u>0.5.2.4 Proposition.</u> For any strings $\vec{u}$ and $\vec{w}$, and variable y, we have

(1) $\tau_1\circ y = (\tau_0(y))^\wedge$

(2) $\tau_1\circ\vec{w} = \vec{\tau}(\vec{w})$

(3) $\tau_1\circ(\vec{w}\ \vec{u}) = (\tau_1\circ\vec{w})(\tau_1\circ\vec{w})$

(4) $\ell(\vec{\tau}(\vec{w})) = \ell(\vec{w})$.

<u>Proof</u> (1) Let $y = \langle A,n\rangle^\wedge$, $0 \in [1]$, $(\tau_1\circ y)(0) = \tau_1(\langle A,n\rangle) = A$ and $(\tau_0(y))^\wedge(0) = \hat{A}(0) = A$. (2): by 0.2.1.5.1 and 0.3.4.5. (3): by (2). (4): by (2).□

<u>0.5.2.5 Proposition.</u> Let $\vec{t}$ be a string of terms then

$$\Pi(\vec{\tau}(\vec{t})) = \tau(\pi(\vec{t})).$$

<u>Proof.</u> By induction on $Lg(t)$. We apply the definitions of π and Lg (0.2.3.9), τ (0.3.1.2), $\vec{\tau}$ (0.3.4.5), and Π (0.4.3.1). For the empty string of terms we have $\Pi(\vec{\tau}(\phi)) = \Pi(\phi) = \mathbb{1} = \tau(*) = \tau(\pi(\phi))$. For a single term t we have $\Pi(\vec{\tau}(t)) = \Pi((\tau(t))^\wedge) = \tau(t) = \tau(\pi(t))$. For the inductive clause we have $\Pi(\vec{\tau}(\vec{t}\ t)) = \Pi((\vec{\tau}(\vec{t}))(\tau(t))^\wedge) = (\Pi(\vec{\tau}(\vec{t}))) \times (\tau(t))$ $= (\tau(\pi(\vec{t}))) \times (\tau(t)) = \tau(\langle\pi(\vec{t}),t\rangle) = \tau(\pi(\vec{t}\ t))$.□

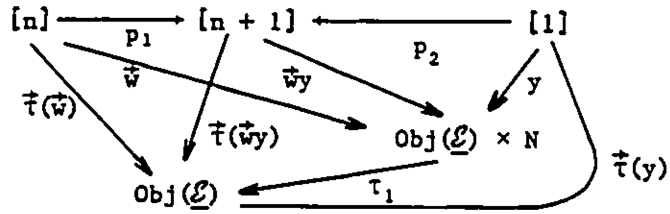
<u>0.5.2.6 Definition.</u> We define the type of a string $\vec{t}$ by $\tau(\vec{t}) = \tau(\pi(\vec{t}))$, the set of free variables of $\vec{t}$ by $s_F(\vec{t}) = s_F(\pi(\vec{t}))$ and the set of bound variables of $\vec{t}$ by $s_B(\vec{t}) = s_B(\pi(\vec{t}))$.

<u>0.5.3 Interpreting an augmented term $\langle\vec{w},t\rangle$ as a morphism $\lambda\vec{w}.t$.</u>

<u>0.5.3.1</u> Before giving the definition of $\lambda\vec{w}.t$ we introduce some notation to cover an awkward case, namely $\vec{w} = \phi$. As long as $\vec{w} \neq \phi$ we have $\tau(\vec{w}\ y) = \tau(\vec{w}) \times \tau_0(y)$, but for $\vec{w} = \phi$ we have $\tau(\phi y) = \tau_0(y)$ and $\tau(\phi) \times \tau_0(y) = \mathbb{1} \times \tau_0(y)$, so in general all we have is an isomorphism from $\tau(\vec{w}) \times \tau(y)$ to $\tau(\vec{w}y)$. We make use of the extension of Π to morphism of $Cart(Obj(\mathcal{E}))$ given in 0.4.3.1, to define this isomorphism and the projections from $\tau(\vec{w}\ y)$ to $\tau(\vec{w})$ and $\tau_0(y)$. Let $\vec{w}$ have length n. The diagram

$$[n] \xhookrightarrow[p_1]{} [n+1] \xleftarrow[p_2]{} [1]$$

is a coproduct in $\mathcal{N}$ where $p_1(i) = i$ for $i \in [n]$ and $p_2(0) = n$. All triangles commute in the diagram



so $$\vec{w} \xrightarrow{\langle p_1, \vec{w}y\rangle} \vec{w}y \xleftarrow{\langle p_2, y\rangle} y \quad . \qquad \text{and}$$

$$\vec{\tau}(\vec{w}) \xrightarrow{\langle p_1, \vec{\tau}(\vec{w}y)\rangle} \vec{\tau}(\vec{w}y) \xleftarrow{\langle p_2, \vec{\tau}(y)\rangle} \vec{\tau}(y)$$

are coproduct diagrams in $(\mathcal{N}, \{\mathrm{Obj}(\underline{\mathcal{E}}) \times N\})$ and $(\mathcal{N}, \{\mathrm{Obj}(\underline{\mathcal{E}})\})$ respectively. Applying the functor Π to the last diagram, we have by 0.4.3.3 that

$$\tau(\vec{w}) \xleftarrow{\Pi\langle p_1, \vec{\tau}(\vec{w}y)\rangle} \tau(\vec{w}y) \xrightarrow{\Pi\langle p_2, \vec{\tau}(\vec{w}y)\rangle} \tau(y)$$

is a product diagram in $\underline{\mathcal{E}}$. Let

$$\tau(\vec{w}) \xleftarrow[\pi_1]{} \tau(\vec{w}) \times \tau(y) \xrightarrow[\pi_2]{} \tau(y)$$

be the canonical product diagram. Put $\tilde{p}_i = \Pi\langle p_i, \vec{\tau}(\vec{w}y)\rangle$ for $i = 1, 2$ and $\sigma = (\tilde{p}_1 \sqcap \tilde{p}_2)^{-1}$, so that $\tilde{p}_i \circ \sigma = \pi_i$ for $i = 1, 2$.

<u>0.5.3.2 Definition.</u> We define a function λ: Aug Tms $\longrightarrow$ Morph$(\underline{\mathcal{E}})$ such that $\lambda(\langle \vec{w}, t\rangle)$ has domain $\tau(\vec{w})$ and codomain $\tau(t)$, by induction on the length of t. To conform with the notation of the lambda-calculus (see for example Curry [Cu] p. 166) we will write $\lambda(\langle \vec{w}, t\rangle)$ as $\lambda\vec{w}.t$.

x: If $\langle \vec{w}, x\rangle \in$ Aug Tms we can put $\vec{w} = \vec{u}x\vec{v}$; we let $i = \ell(\vec{u})$. We put $\lambda\vec{w}.x = pr_i^{\vec{\tau}(\vec{w})}$. This map has domain $\Pi\vec{\tau}(\vec{w}) = \tau(\vec{w})$ and codomain $\tau_0(x)$ (see 0.4.3.1).

*: For $\langle \vec{w}, *\rangle \in$ Aug Tms we let $\lambda\vec{w}.*$ be the morphism from $\tau(\vec{w})$ to $[]$.

ft_1: If $\langle \vec{w}, ft_1 \rangle \in$ Aug Tms we have $\langle \vec{w}, t_1 \rangle \in$ Aug Tms and $\mathrm{dom}(f) = \tau(t_1)$. By induction $\lambda\vec{w}.t_1$ is defined and has domain $\tau(\vec{w})$ and codomain $\tau(t_1)$. We put $\lambda\vec{w}.ft_1 = f \circ \lambda\vec{w}.t_1$.

$qx\varphi$: If $\langle \vec{w}, qx\varphi \rangle \in$ Aug Tms and $y = ch(\tau_0(x), s_{FB}(\vec{w}x))$ then $\langle \vec{w}y, \varphi[x|y] \rangle \in$ Aug Tms. By induction $\lambda\vec{w}y.\varphi[x|y]$ is defined and has domain $\tau(\vec{w}y)$ and codomain Ω; let $[\![\vec{w}y|\ \varphi[x|y]]\!]$ be the subobject of $\tau(\vec{w}y)$ which $\lambda\vec{w}y.\varphi[x|y]$ classifies. Define $\tilde{p}_1 : \tau(\vec{w}y) \longrightarrow \tau(\vec{w})$ and $\sigma: \tau(\vec{w}) \times \tau(y) \longrightarrow \tau(\vec{w}y)$ as in 0.5.3.1, and consider two cases for q: (1) $q \in \{\forall, \exists\}$ and (2) $q = \{:\}$.

(1) $q \in \{\forall, \exists\}$: $\tilde{p}_1 : \tau(\vec{w}y) \longrightarrow \tau(\vec{w})$ induces a function $q_{p_1} : \mathrm{Sub}(\tau(\vec{w}y)) \longrightarrow \mathrm{Sub}(\tau(\vec{w}))$. We first define $[\![\vec{w}|qx\varphi]\!] = q_{p_1}[\![\vec{w}y|\varphi[x|y]]\!]$ and then let $\lambda\vec{w}.qx\varphi$ be the morphism with domain $\tau(\vec{w})$ and codomain Ω that classifies the subobject $[\![\vec{w}|qx\varphi]\!]$.

(2) $q = \{:\}$: We define $\lambda\vec{w}.\{x:\varphi\}$ via the adjointness:

$$\frac{\tau(\vec{w}) \xrightarrow{\lambda\vec{w}.\{x:\varphi\}} \Omega^{\tau(x)}}{\dfrac{\tau(\vec{w}) \times \tau(x) \xrightarrow{\quad f \quad} \Omega}{\tau(\vec{w}x) \xrightarrow[\lambda\vec{w}y.\varphi[x|y]]{} \Omega}} \qquad \begin{matrix}(A)\\ \\ (B)\end{matrix}$$

where $f = (\lambda\vec{w}y.\varphi[x|y]) \circ \sigma$. (A) and (B) are interdefinable by the adjointness of 0.5.1.4 with

$$\lambda\vec{w}.\{x:\varphi\} = f^{\tau(x)} \circ \eta_{\tau(\vec{w})} = (\lambda\vec{w}y.\varphi[x|y])^{\tau(x)} \circ \sigma^{\tau(x)} \circ \eta_{\tau(\vec{w})}.$$

<u>0.5.3.3 Definition.</u>

(1) $\approx$ is the relation on Aug Tms for $\Phi(\mathcal{E})$ given by

$<\vec{w},t> \approx <\vec{u},s>$ iff $\lambda\vec{w}.t = \lambda\vec{u}.s$.

(2) $\lambda\vec{w}.\vec{t} = \lambda\vec{w}.\pi(\vec{t})$ where $\{\vec{w}\} \subset s_F(\vec{t})$.

(3) $|\vec{t}| = \lambda var(\pi(\vec{t})).\vec{t}$.

(4) $\|\varphi\| = [\![var(\varphi)|\varphi]\!]$ where φ is a formula.

We call $|\vec{t}|$ and $\|\varphi\|$ the canonical interpretation of $\vec{t}$ and φ respectively.

<u>0.5.3.4 Proposition.</u>

(1) $\approx$ is a typed equivalence relation on Aug Tms.

(2) If $<\vec{w},t> \in$ Aug Tms and $z \in (ffr[x](t) - \{\vec{w}\})$ then

$$\lambda\vec{w}.t = \lambda\vec{w}[x|z].\ t[x|z].$$

<u>Proof.</u> (1): Each clause - $(\tau 1)$ to $(\tau 5)$ - of 0.3.4.6 is immediately verifiable for $\approx$, hence $\approx$ is a typed equivalence relation under ch; by 0.3.4.13 $\approx$ is a typed equivalence relation.□ (2): by (1) and 0.3.4.11.□

<u>0.5.4 Change of domain for $\lambda\vec{w}.\vec{t}$.</u>

<u>0.5.4.1 Proposition.</u> Let $\vec{u}$ be a reduced string of variables of length m, and let α: $[k] \longrightarrow [m]$ be a function, then

$$\lambda\vec{u}.(\vec{u}\circ\alpha) = \Pi<\alpha,\vec{\tau}(\vec{u})>.$$

<u>Proof.</u> By induction on k.

$k = 0$: $\lambda\vec{u}.\phi$: $\tau(\vec{u}) \longrightarrow [\!]$ and $\Pi<\phi,\vec{\tau}(\vec{u})>$: $\Pi\vec{\tau}(\vec{u}) \longrightarrow [\!]$. Since $\tau(\vec{u}) = \Pi\vec{\tau}(\vec{u})$, $\lambda\vec{u}.(\vec{u}\circ\phi) = \Pi<\phi,\vec{\tau}(\vec{u})>$.

$k = 1$: For some $i \in [m]$, $\alpha = \hat{i}$. Put $u_i = (\vec{u}(i))^\wedge = \vec{u}\circ\hat{i}$, then $\lambda\vec{u}.u_i = pr_i^{\vec{\tau}(\vec{u})} = \Pi<\hat{i},\vec{\tau}(\vec{u})>$.

$k = n + 2$: α is itself a string of length $n + 2$. Following o.4.2.1 we put $\alpha_i = \alpha \circ p_i$ for $i = 1, 2$, then $\ell(\vec{u} \circ \alpha_1) = n + i$, $\ell(\vec{u} \circ \alpha_2) = 1$, and $\vec{u} \circ \alpha = (\vec{u} \circ \alpha \circ p_1)(\vec{u} \circ \alpha \circ p_2)$, so

$$\begin{aligned} \lambda\vec{u}.(\vec{u}\circ\alpha) &= \lambda\vec{u}.(\vec{u}\circ\alpha_1)(\vec{u}\circ\alpha_2) = (\lambda\vec{u}.(\vec{u}\circ\alpha_1)) \sqcap (\lambda\vec{u}.(\vec{u}\circ\alpha_2)) \\ &= \Pi\langle\alpha_1,\vec{\tau}(\vec{u})\rangle \sqcap \Pi\langle\alpha_2,\vec{\tau}(\vec{u})\rangle \qquad \text{by induction} \\ &= \Pi\langle\alpha_1 \sqcup \alpha_2,\vec{\tau}(\vec{u})\rangle \qquad \text{by 0.4.3.2 (4)} \\ &= \Pi\langle\alpha,\vec{\tau}(\vec{u})\rangle. \square \end{aligned}$$

<u>0.5.4.2 Proposition.</u>

(1) If $\vec{u}$ and $\vec{w}$ are reduced, $\vec{\tau}(\vec{u}) = \vec{\tau}(\vec{w})$, $\ell(\vec{u}) = m$, and $\alpha: [k] \longrightarrow [m]$, then $\lambda\vec{u}.(\vec{u}\circ\alpha) = \lambda\vec{w}.(\vec{w}\circ\alpha)$.

(2) If $\vec{u}$ is reduced and $\{\vec{w}\} \subset \{\vec{u}\}$, with $k = \ell(\vec{w})$ and $m = \ell(\vec{u})$ then there is a unique function $\alpha: [k] \longrightarrow [m]$ such that $\lambda\vec{u}.\vec{w} = \Pi\langle\alpha,\vec{\tau}(\vec{u})\rangle$.

(3) If $\vec{u}$ and $\vec{v}$ are reduced and $\{\vec{w}\} \subset \{\vec{u}\} \subset \{\vec{v}\}$, then $(\lambda\vec{u}.\vec{w})\circ(\lambda\vec{v}.\vec{u}) = \lambda\vec{v}.\vec{w}$.

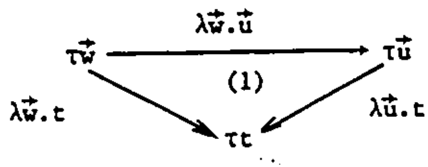
(4) $\lambda\vec{u}.\vec{u} = id_{\tau(\vec{u})}$.

(5) If $f(\pi(\vec{u}))$ is a term then $\lambda\vec{u}.f(\pi(\vec{u})) = f$; in particular, $|f*| = f$, $|fx| = f$ and $|f/x,y/| = f$ where $x \neq y$.

<u>Proof.</u> (1) By 0.5.4.1 $\lambda\vec{u}.(\vec{u}\circ\alpha) = \Pi\langle\alpha,\vec{\tau}(\vec{u})\rangle = \Pi\langle\alpha,\vec{\tau}(w)\rangle = \lambda\vec{w}.(\vec{w}\circ\alpha)$.$\square$

(2) Let $\alpha = \{\langle i,j\rangle \mid \vec{w}(i) = \vec{u}(j)\}$; the domain of α is $[k]$ since $\{\vec{w}\} \subset \{\vec{u}\}$, α is single-valued since $\vec{u}$ is one-to-one, and $(\vec{u}\circ\alpha)(i) = \vec{u}(\alpha(i)) = \vec{w}(i)$ for $i \in [k]$, hence $\Pi\langle\alpha,\vec{\tau}(\vec{u})\rangle = \lambda\vec{u}.(\vec{u}\circ\alpha) = \lambda\vec{u}.\vec{w}$.$\square$ (3) Let α and β be such that $\vec{u}\circ\alpha = \vec{w}$ and $\vec{v}\circ\beta = \vec{u}$ respectively, then $(\lambda\vec{u}.\vec{w})\circ(\lambda\vec{v}.\vec{u}) = \Pi\langle\alpha,\vec{\tau}(\vec{u})\rangle\circ\Pi\langle\beta,\vec{\tau}(\vec{v})\rangle = \Pi\langle\beta\circ\alpha,\vec{\tau}(\vec{v})\rangle$

<u>0.5.4.5 Proposition.</u> Let $\vec{u}$ and $\vec{w}$ be reduced strings and let t be a term such that $s_F(t) \subset \{\vec{u}\} \subset \{\vec{w}\}$, then $\lambda\vec{w}.t = (\lambda\vec{u}.t)\circ(\lambda\vec{w}.\vec{u})$, that is (1) commutes



<u>Proof.</u> By cases and induction on the length of t.

$*$: $\tau * = \mathbb{1}$ so (1) must commute.

x: $\lambda\vec{w}.x = (\lambda\vec{u}.x)\circ(\lambda\vec{w}.\vec{u})$ by 0.5.4.2.

$\langle t_1,t_2\rangle$: $(\lambda\vec{u}\langle t_1,t_2\rangle)\circ(\lambda\vec{w}.\vec{u}) = ((\lambda\vec{u}.t_1) \sqcap (\lambda\vec{u}.t_2))\circ(\lambda\vec{w}.\vec{u})$

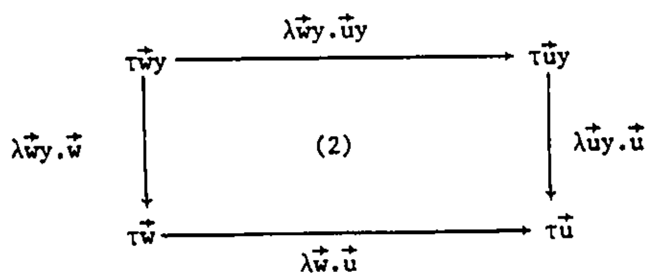
$$= ((\lambda\vec{u}.t_1)\circ(\lambda\vec{w}.\vec{u})) \sqcap ((\lambda\vec{u}.t_2)\circ(\lambda\vec{w}.\vec{u}))$$

$$= (\lambda\vec{w}.t_1) \sqcap (\lambda\vec{w}.t_2) = \lambda\vec{w}.\langle t_1,t_2\rangle.$$

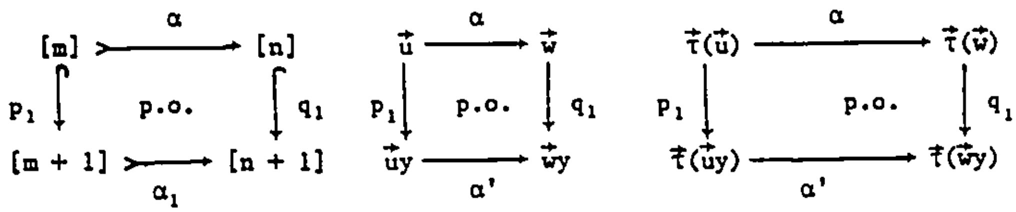
ft_1: $(\lambda\vec{u}.ft_1)\circ(\lambda\vec{w}.\vec{u}) = f\circ((\lambda\vec{u}.t_1)\circ(\lambda\vec{w}.\vec{u})) = f\circ(\lambda\vec{w}.t_1) = \lambda\vec{w}.ft_1$.

$qx\varphi$: We want to show $\lambda\vec{w}.qx\varphi = (\lambda\vec{u}.qx\varphi)\circ(\lambda\vec{w}.\vec{u})$ or equivalently $(\lambda\vec{w}.\vec{u})^{-1}[\![\vec{u}\,|\,qx\varphi]\!] = [\![\vec{w}\,|\,qx\varphi]\!]$. By virtue of 0.5.3.4 we can select y such that $\lambda\vec{w}.qx\varphi$ is defined from $\lambda\vec{w}y.\varphi[x|y]$, $\lambda\vec{u}.qx\varphi$ is defined from $\lambda\vec{u}y.\varphi[x|y]$, y is of the same type as x and $y \notin s_{FB}(\varphi x\vec{w})$. By induction $\lambda\vec{w}y.\varphi[x|y] = (\lambda\vec{u}y.\varphi[x|y])\circ(\lambda\vec{w}y.\vec{u}y)$ or equivalently $(\lambda\vec{w}y.\vec{u}y)^{-1}[\![\vec{u}y\,|\,\varphi[x|y]]\!] = [\![\vec{w}y\,|\,\varphi[x|y]]\!]$ since $s_F(\varphi[x|y]) \subset \{\vec{u}y\} \subset \{\vec{w}y\}$.

The morphism $\lambda\vec{w}y.\vec{u}y$ fits into the pullback diagram(2)



To see that this is a pullback diagram, let α be the function for which $\vec{w}\circ\alpha = \vec{u}$, $m = \ell(\vec{u})$ and $n = \ell(\vec{w})$ then we have push-out diagrams



Thus (2) is a pullback diagram. Let $p = \lambda\vec{w}y.\vec{w}$ and $p' = \lambda\vec{u}y.\vec{u}$. If $q \in \{\exists,\forall\}$ then by the Beck condition we have:

$$(\lambda\vec{w}.\vec{u})^{-1}[\![\vec{u}\,|\,qx\varphi]\!] = (\lambda\vec{w}.\vec{u})^{-1}\, q_p[\![\vec{u}y\,|\,\varphi[x|y]]\!] \quad (3)$$

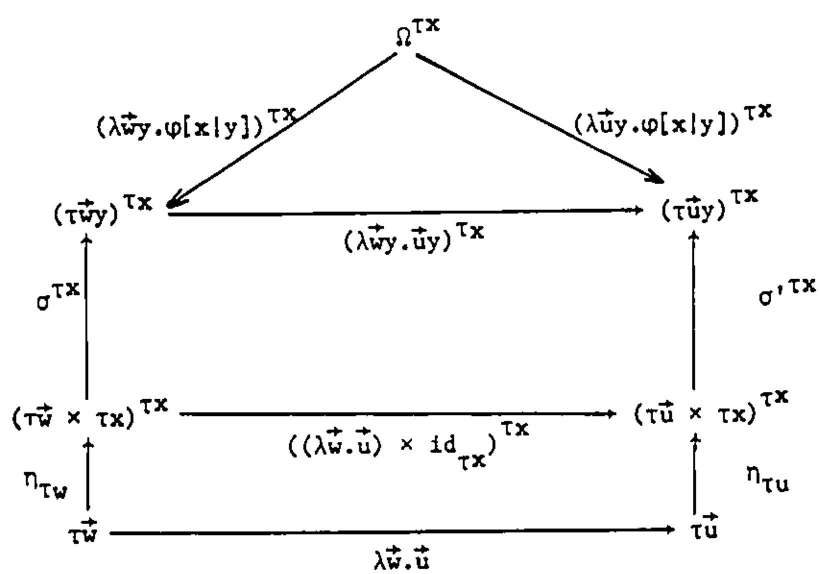
$$= q_{p'}(\lambda\vec{w}y.\vec{u}y)^{-1}[\![\vec{u}y\,|\,\varphi[x|y]]\!] \quad (4)$$

$$= q_{p'}[\![\vec{w}y\,|\,\varphi[x|y]]\!] \quad (5)$$

$$= [\![\vec{w}\,|\,qx\varphi]\!] \quad (6)$$

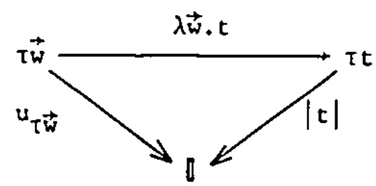
(3) by definition 0.5.2.7, (4) by the Beck condition, (5) by induction, (6) by definition.

If $q = \{:\}$, that is $qx\varphi = \{x:\varphi\}$, we consider the following diagram:



The upper triangle arises from an application of the functor $(\ )^{\tau x}$ to the induction hypothesis. The middle square commutes by 0.5.4.4 and an application of $(\ )^{\tau x}$. The bottom square commutes by the naturality of the front adjunction η for the adjointness $(\ ) \times \tau x \dashv (\ )^{\tau x}$ (see 0.5.1.4). By the last clause in Definition 0.5.3.2
$\lambda\vec{w}.\{x:\varphi\} \equiv (\lambda\vec{w}y.\varphi[x|y])^{\tau x}\circ\sigma^{\tau x}\circ\eta_{\tau\vec{w}}$ and
$\lambda\vec{u}.\{x:\varphi\} \equiv (\lambda\vec{u}y.\varphi[x|y])^{\tau x}\circ(\sigma')^{\tau x}\circ\eta_{\tau\vec{u}}$. Hence by the commutativity of the above diagram, $\lambda\vec{u}.\{x:\varphi\} \equiv (\lambda\vec{w}.\{x:\varphi\})\circ(\lambda\vec{w}.\vec{u})$.□

If $s_F(t) \subset \{\vec{w}\}$ with $\vec{w}$ reduced then, putting $\vec{u} \equiv \mathrm{var}(t)$, we have $\lambda\vec{w}.t \equiv |t|\circ\lambda\vec{w}.\mathrm{var}(t)$. If $s_F(t) \equiv \phi$ then, putting $\vec{u} \equiv \phi$, we have $\lambda\vec{w}.t \equiv (\lambda\phi.t)\circ(\lambda\vec{w}.\phi)$; since $\tau\phi \equiv \mathbb{1}$, $\lambda\vec{w}.\phi \equiv u_{\tau\vec{w}}$ and $\tau\phi.t \equiv |t|$, i.e.



commutes.

We extend this proposition to strings of terms.

<u>0.5.4.6 Proposition.</u> Let $s_F(\vec{t}) \subset \{\vec{u}\} \subset \{\vec{w}\}$ with $\vec{u}$ and $\vec{w}$ reduced, then $\lambda\vec{w}.\vec{t} \equiv (\lambda\vec{u}.\vec{t})\circ(\lambda\vec{w}.\vec{u})$.

<u>Proof.</u> By induction on $Lg(\vec{t})$. If the length is 0 we have $(\lambda\vec{u}.\phi)\circ(\lambda\vec{w}.\vec{u}) \equiv (\lambda\vec{u}.*)\circ(\lambda\vec{w}.\vec{u}) \equiv \lambda\vec{w}.* \equiv \lambda\vec{w}.\phi$. If the length is 1 we have $(\lambda\vec{u}.t)\circ(\lambda\vec{w}.\vec{u}) \equiv \lambda\vec{w}.t$. Let $\vec{t}s$ be a string of terms with $Lg(\vec{t}) \geq 1$ then

$$(\lambda\vec{u}.(\vec{t}s))\circ(\lambda\vec{w}.\vec{u}) \equiv ((\lambda\vec{u}.\vec{t}) \sqcap (\lambda\vec{u}.s))\circ(\lambda\vec{w}.\vec{u}) \equiv (\lambda\vec{u}.\vec{t})\circ(\lambda\vec{w}.\vec{u}) \sqcap (\lambda\vec{u}.s)\circ(\lambda\vec{w}.\vec{u})$$
$$\equiv (\lambda\vec{w}.\vec{t}) \sqcap (\lambda\vec{w}.s) \equiv \lambda\vec{w}.\vec{t}s.\square$$

<u>0.5.4.7 Definition.</u> Let ρ: Str Tms $\xrightarrow{\sim}$ Tms* be the monoid isomorphism which on Tms is given by $\rho(t) = \hat{t}$. Let $n = \ell(\rho(\vec{t})) = L_g(\vec{t})$ and let α: $[m] \longrightarrow [n]$ so that $(\rho(\vec{t}))\bullet\alpha$ is defined. We put $\vec{t} \circledast \alpha = \rho^{-1}((\rho(\vec{t}))\bullet\alpha)$, so that $\rho(\vec{t})\bullet\alpha = \rho(\vec{t} \circledast \alpha)$ and $Lg(\vec{t} \circledast \alpha) = \ell(\alpha) = m$. The function $\vec{t} \circledast -:[n]^* \longrightarrow$ Str Tms is then a monoid homomorphism: clearly $\vec{t} \circledast \phi = \phi$,

$$\vec{t} \circledast (\alpha_1\alpha_2) = \rho^{-1}(((\rho(\vec{t}))\bullet\alpha_1)((\rho(\vec{t})\bullet\alpha_2) \quad \text{by } 0.2.1.5.2$$
$$= (\vec{t} \circledast \alpha_1)(\vec{t} \circledast \alpha_2) \quad \text{since } \rho^{-1} \text{ is a homomorphism.}$$

<u>0.5.4.8 Proposition.</u> Let $\vec{t}$ be a string of terms and let $n = \ell(\rho(\vec{t})) = Lg(\vec{t})$. For each $i \in [n]$ put $t_i = (\rho(\vec{t}))(i) = \vec{t} \circledast \hat{i}$. Let $\vec{u}$ be a reduced string of variables of length n. For each $i \in [n]$ put $u_i = (\rho(\vec{u}))(i) = \vec{u}\bullet\hat{i} = (\vec{u}(i))^\wedge$. Suppose $\vec{\tau}(\vec{t}) = \vec{\tau}(\vec{u})$, or equivalently $\tau(t_i) = \tau_0(u_i)$ for each $i \in [n]$. Let $\vec{v}$ be a reduced string such that $s_F(\vec{t}) \subset \{\vec{v}\}$. The following hold:

(1) $(\lambda\vec{u}.\phi)\circ(\lambda\vec{v}.\vec{t}) = \lambda\vec{v}.\phi$

(2) For each $i \in [n]$, $(\lambda\vec{u}.u_i)\circ(\lambda\vec{v}.\vec{t}) = \lambda\vec{v}.t_i$.

Let $\alpha:[k] \longrightarrow [n]$, then:

(3) $s_F(\vec{t} \circledast \alpha) \subset s_F(\vec{t})$

(4) $(\lambda\vec{u}.(\vec{u}\bullet\alpha))\circ(\lambda\vec{v}.\vec{t}) = \lambda\vec{v}.\vec{t} \circledast \alpha$.

<u>Proof.</u> (1) $cod(\lambda\vec{v}.\vec{t}) = \tau(\pi(\vec{t})) = \Pi\vec{\tau}(\vec{t}) = \Pi\vec{\tau}(\vec{u}) = \tau(\vec{u}) = dom(\lambda\vec{u}.\phi)$ hence $\lambda\vec{u}.\phi$ and $\lambda\vec{v}.\vec{t}$ are composable, also dom $\lambda\vec{v}.\vec{t} = \tau(\vec{v}) =$ dom $\lambda\vec{v}.\phi$ and $cod(\lambda\vec{u}.\phi) = [\,] = cod(\lambda\vec{v}.\phi)$. Since there is only one map from $\tau\vec{v}$ to $[\,]$, (1) holds.□ (2) We proceed by induction on n, $n \geq 1$. If $n = 1$ we have $(\lambda u_0.u_0)\circ(\lambda\vec{v}.t_0) = \lambda\vec{v}.t_0$. If $\vec{t} = \vec{s}r$ with $\ell(\vec{s}) \geq 1$ and $\vec{u} = \vec{w}x$, then

if $\ell(\vec{t}) = m + 1$, $x = u_m$, $\vec{w}$ is reduced and $\vec{\tau}(\vec{w}) = \vec{\tau}(\vec{s})$ and $s_F(\vec{s}) \subset s_F(\vec{t}) \subset \{\vec{v}\}$ as well as $s_F(r) \subset \{\vec{v}\}$ so we can apply the induction hypothesis. By (2) $(\lambda\vec{u}.u_m)\bullet(\lambda\vec{v}.\vec{t}) = \lambda\vec{v}.t_m$; for $i \in [m]$ we have

$$(\lambda\vec{u}.u_i)\bullet(\lambda\vec{v}.\vec{t}) = (\lambda\vec{w}.u_i)\bullet(\lambda\vec{w}u_m.\vec{w})\bullet(\lambda\vec{v}.\vec{s} \sqcap \lambda\vec{v}.r) = (\lambda\vec{w}.u_i)\bullet\lambda\vec{v}.\vec{s}$$
$$= \lambda\vec{v}.t_i$$

since $s_i = (\rho(\vec{s}))(i) = ((\rho(\vec{s}))(\rho(t_m)))(i) = (\rho(\vec{t}))(i) = t_i$.□

(3) $s_F(\vec{t} \otimes \alpha) = \bigcup_{i \in [k]} s_F(t_{\alpha(i)}) \subset \bigcup_{j \in [n]} s_F(t_j) = s_F(\vec{t})$. □

(4): By induction on k. For $k = 0$ we have $\alpha = \phi$ hence the equation holds by (1). For $k = 1$, let $\alpha = \hat{i}$ then $\vec{u}\bullet\alpha = u_i$ and $\vec{t} \otimes \alpha = t_i$ so the equation holds by (2). For $k = \ell + 2$ we can put $\alpha = \beta\hat{i}$ where $i = \alpha(\ell + 1)$, then $\vec{u}\bullet\alpha = (\vec{u}\bullet\beta)u_i$ and $\vec{t} \otimes \alpha = (\vec{t} \otimes \beta)(t_i)$ so

$$(\lambda\vec{u}.(\vec{u}\,\alpha))\bullet(\lambda\vec{v}.\vec{t}) = ((\lambda\vec{u}.(\vec{u}\bullet\beta)) \sqcap (\lambda\vec{u}.u_i))\bullet(\lambda\vec{v}.\vec{t})$$
$$= ((\lambda\vec{u}.\vec{u}\,\beta)\bullet(\lambda\vec{v}.\vec{t})) \sqcap ((\lambda\vec{u}.u_i)\circ(\lambda\vec{v}.\vec{t}))$$
$$= (\lambda\vec{v}.(\vec{t} \otimes \beta)) \sqcap (\lambda\vec{v}.t_i) = \lambda\vec{v}.(\vec{t} \otimes \beta)(t \otimes \hat{i})$$
$$= \lambda\vec{v}.(\vec{t} \otimes \alpha) \;.\square$$

<u>0.5.4.9 Proposition.</u> Let $\vec{\tau}(\vec{u}) = \vec{\tau}(\vec{t})$, $s_F(\vec{t}) \subset \{\vec{w}\}$, with $\vec{w}y$ and $\vec{u}y$ reduced, then (1) commutes.

$$\begin{array}{ccc}
\tau(\vec{w}y) & \xrightarrow{\lambda\vec{w}y.\vec{t}y} & \tau(\vec{u}y) \\
\Big\downarrow{\scriptstyle \lambda\vec{w}y.\langle\pi(\vec{w}),y\rangle} & (1) & \Big\downarrow{\scriptstyle \lambda\vec{u}y.\langle\pi(\vec{u}),y\rangle} \\
\tau(\vec{w}) \times \tau(y) & \xrightarrow[(\lambda\vec{w}.\vec{t}) \times id_{\tau(y)}]{} & \tau(\vec{u}) \times \tau(y)
\end{array}$$

Proof. Let π_1', π_2' be the projections from the product of $\tau(\vec{u})$ with $\tau(y)$, and π_1, π_2 those from the product of $\tau(\vec{w})$ with $\tau(y)$. Put $f = \lambda\vec{w}y.\vec{t}y$, $g = (\lambda\vec{w}.\vec{t}) \times id_{\tau(y)}$, $h = \lambda\vec{w}y.\langle\pi(\vec{w}),y\rangle$ and $\ell = \lambda\vec{u}y.\langle\pi(\vec{u}),y\rangle$. We show $\pi_i'\circ(\ell\circ f) = \pi_i'\circ(g\circ h)$ for $i = 1, 2$, from which it follows that (1) commutes.

$\pi_1'\circ(\ell\circ f) = (\lambda\vec{u}y.\vec{u})\circ(\lambda\vec{w}y.\vec{t}y) = \lambda\vec{w}y.\vec{t}$ by 0.5.4.8 (4),

$\pi_1'\circ(g\circ h) = ((\lambda\vec{w}.\vec{t})\circ\pi_1)\circ h = (\lambda\vec{w}.\vec{t})\circ(\lambda\vec{w}y.\vec{w}) = \lambda\vec{w}y.\vec{t}$

$\pi_2'\circ(\ell\circ f) = (\lambda\vec{u}y.y)\circ(\lambda\vec{w}y.\vec{t}y) = \lambda\vec{w}y.y$ by 0.5.4.8

$\pi_2'\circ(g\circ h) = \pi_2\circ h = \lambda\vec{w}y.y$.□

0.5.5 Substitution and composition.

0.5.5.1 Proposition. Let $\langle\vec{w}x,t\rangle$ and $\langle\vec{v},\pi(\vec{w}s)\rangle$ be augmented terms such that $s \in ffr\ [x](t)$ then $\langle\vec{v},t[x|s]\rangle$ is an augmented term and $\lambda\vec{v}.t[x|s] = (\lambda\vec{w}x.t)\circ(\lambda\vec{v}.\vec{w}s)$.

Proof. $s_F(t[x|s]) \subset (s_F(t) - \{x\}) \cup s_F(s) \subset \{\vec{w}\} \cup s_F(s) \subset \{\vec{v}\}$ by 0.2.8.4. hence $\langle\vec{v},t[x|s]\rangle \in$ Aug Tms. Assume first that $x \notin s_F(t)$, then

$(\lambda\vec{w}x,t)\circ(\lambda\vec{v}.\vec{w}s) = (\lambda\vec{w}.t)\circ(\lambda\vec{w}x.\vec{w})\circ(\lambda\vec{v}.\vec{w}s)$ by 0.5.4.5

$= (\lambda\vec{w}.t)\circ(\lambda\vec{v}.\vec{w})$ by 0.5.4.8

$= \lambda\vec{v}.t$ by 0.5.4.5.

We now proceed by induction on $\ell(t)$ (passing over those cases where $x \notin s_F(t)$):

If $t = x$, we have $(\lambda\vec{w}x.x)\circ(\lambda\vec{v}.\vec{w}s) = \lambda\vec{v}.s$ by 0.5.4.8. If $t = \langle t_1,t_2\rangle$, we have $(\lambda\vec{w}x.\langle t_1,t_2\rangle)\circ(\lambda\vec{v}.\vec{w}s) = ((\lambda\vec{w}x.t_1)\circ(\lambda\vec{v}.\vec{w}s)) \sqcap ((\lambda\vec{w}x.t_2)\circ(\lambda\vec{v}.\vec{w}s))$

$= (\lambda\vec{v}.t_1[x|s]) \sqcap (\lambda\vec{v}.t_2[x|s])$

$= \lambda\vec{v}.\langle t_1,t_2\rangle[x|s]$.

If $t \equiv qy\varphi$ with $y \not\equiv x$ and $x \in s_F(\varphi)$ then by (f_7)

$ffr[x](qy\varphi) \equiv ffr[x](\varphi) \cap \{r \mid y \notin s_F(r)\}$ so $s \in ffr[x](\varphi)$ and

$y \notin s_F(s)$. Choose $u \notin s_{FB}(\varphi sxy\vec{v})$ such that $\tau_0(u) \equiv \tau_0(y)$. Since

$\{\vec{w}ux\} \equiv \{\vec{w}xu\}$ so by 0.3.4.3 $\langle \vec{w}ux, \varphi[y|u]\rangle \in$ Aug Tms. By our choice of u

$s_F(\vec{w}us) \subset \{\vec{v}u\}$ and $ffr[x](\varphi) \subset ffr[x](\varphi[y|u])$ by 0.2.9.5. Hence by

induction $\lambda\vec{v}u.(\varphi[y|u])[x|s] \equiv (\lambda\vec{w}ux.\varphi[y|u]) \circ (\lambda\vec{v}u.\vec{w}us)$

$\equiv (\lambda\vec{w}xu.\varphi[y|u]) \circ (\lambda\vec{w}ux.\vec{w}xv) \circ (\lambda\vec{v}u.\vec{w}us)$ by 0.5.4.5

$\equiv (\lambda\vec{w}xu.\varphi[y|u]) \circ (\lambda\vec{v}u.\vec{w}su)$ by 0.5.4.8.

By 0.2.6.1. $\varphi[y|u][x|s] \equiv \varphi[x|s][y|u]$. Hence

(1) $\lambda\vec{v}u.(\varphi[x|s])[y|u] \equiv (\lambda\vec{w}xu.\varphi[y|u]) \circ (\lambda\vec{v}u.\vec{w}su)$ or equivalently

(2) $(\lambda\vec{v}u.\vec{w}su)^{-1}[\![\vec{w}xu \mid \varphi[y|u]]\!] \equiv [\![\vec{v}u \mid \varphi[x|s][y|u]]\!]$.

Suppose $q \in \{\exists, \forall\}$. We use (2) to show

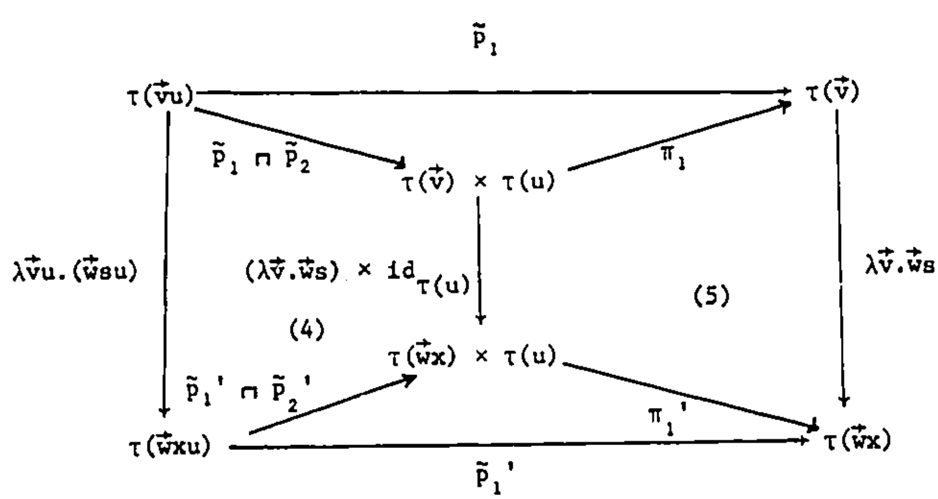
(3) $(\lambda\vec{v}.\vec{w}s)^{-1}[\![\vec{w}x \mid qy\varphi]\!] \equiv [\![\vec{v} \mid qy(\varphi[x|s])]\!]$.

Let $\tilde{p}_1 \equiv \lambda\vec{v}u.\vec{v}$, $\tilde{p}_2 \equiv \lambda\vec{v}u.u$, $\tilde{p}_1' \equiv \lambda\vec{w}xu, \vec{w}x$, $\tilde{p}_2' \equiv \lambda\vec{w}xu.u$,

$\sigma \equiv (\tilde{p}_1 \sqcap \tilde{p}_2)^{-1}$ and $\sigma' \equiv (\tilde{p}_1' \sqcap p_2')^{-1}$. By 0.5.4.9 (4) commutes, and

since $\tilde{p}_1 \sqcap \tilde{p}_2$ and $\tilde{p}_1' \sqcap \tilde{p}_2'$ are isomorphisms (4) is a pullback. Since

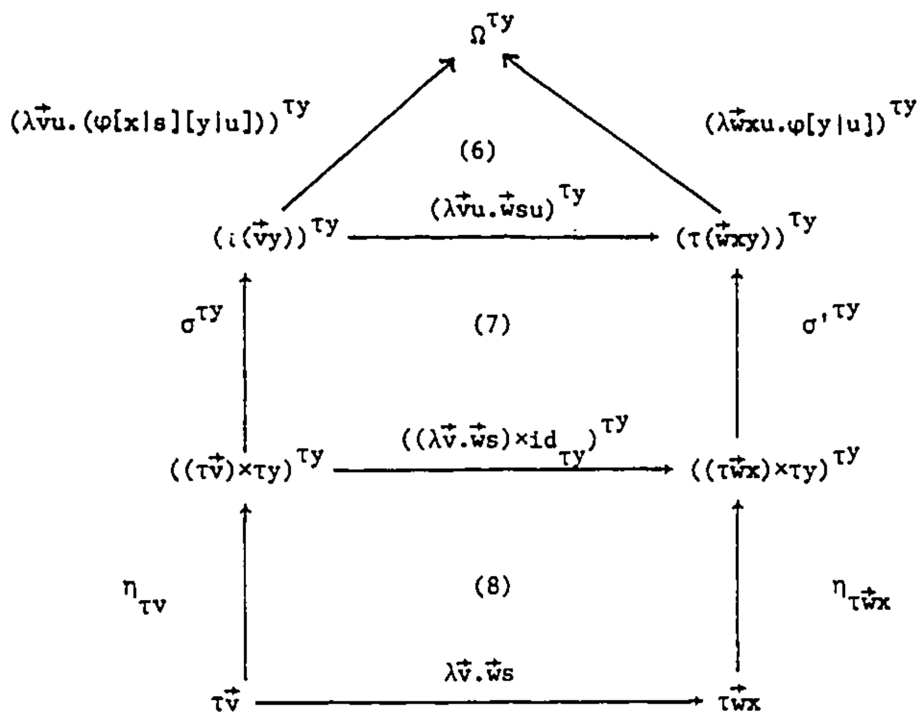
(5) is a pullback the outer rectangle is a pullback.



Applying the Beck condition we have

$$(\lambda\vec{v}.\vec{w}s)^{-1}[\![\vec{w}x|qy\varphi]\!] = (\lambda\vec{v}.\vec{w}s)^{-1}(q_{\tilde{p}_1'}[\![\vec{w}xu|\varphi[y|u]]\!])$$
$$= q_{\tilde{p}_1'}((\lambda\vec{v}u.\vec{w}su)^{-1}[\![\vec{w}xu|\varphi[y|u]]\!]$$
$$= q_{\tilde{p}_1'}[\![\vec{v}u|\varphi[x|s][y|u]]\!] = [\![\vec{v}|qy(\varphi[x|s])]\!] .$$

Suppose $qy\varphi = \{y:\varphi\}$. Applying the functor $(\ )^{\tau y}$ first to (1) then to (4), we get (6) and (7) commuting respectively. By the naturality of η, (8) commutes.



Thus $(\lambda\vec{w}x.\{y:\varphi\})\circ(\lambda\vec{v}.\vec{w}s) = \lambda\vec{v}.\{y:\varphi[x|s]\}$.□

In the special case when $\vec{w} = \phi$ the proposition reduces to: if (1) $s_F(t) \subset \{x\}$, (2) $s_F(s) \subset \{v\}$ and (3) $s \ \varepsilon \ ffr[x](t)$, then $\lambda\vec{v}.t[x|s] = (\lambda x.t)\circ(\lambda\vec{v}.s)$.□

0.5.5.2 Definition. Let $\vec{v}$ be a string of distinct variables of length n. Let $i + 1 \varepsilon [n + 1]$ define $\vec{v}^i$ to be the string of length $i + 1$ such that $\vec{v} = \vec{v}^i\vec{w}$ for some $\vec{w}$.

We have $\vec{v}^{n-1} = \vec{v}$, $\vec{v}^{-1} = \phi$ and if we put $v_i = (\vec{v}(i))^\wedge$ then $\vec{v}^0 = v_0$, and $\vec{v}^{i+1} = \vec{v}^i v_{i+1}$.

0.5.5.3 Proposition. Let $\vec{v}$ be a string of distinct variables, $\ell(\vec{v}) = n$, $n \geq 1$. Let $h: \tau_0 y \longrightarrow \tau\vec{v}$ where $y \notin s_F\vec{v}$. Define $h_i = (\lambda\vec{v}.v_i)\bullet h$ and $f_i = \lambda y\vec{v}^{i-1}.y\vec{v}^{i-1}h_i(y)$ for each $i \varepsilon [n]$. Define F_j by $F_0 = f_0$ and $F_{i+1} = f_{i+1}\circ F_i$ for $i \varepsilon [n - 1]$, then

(1) $(\lambda y\vec{v}.y)\bullet F_{n-1} = id_{\tau y}$ and (2) $(\lambda y\vec{v}.\vec{v})\bullet F_{n-1} = h$.

Proof. F_n is well-defined: dom $f_{i+1} = \tau y\vec{v}^i = \tau((y\vec{v}^{i-1})h_i(y)) =$ cod f_i for $i \varepsilon [n - 1]$ hence $f_{i+1}\bullet f_i$ is well-defined. F_0 is well-defined and cod $F_0 =$ cod f_0, suppose F_i is well-defined and cod $F_i =$ cod f_i then dom $f_{i+1} =$ cod F_i so F_{i+1} is well-defined and cod $F_{i+1} =$ cod f_{i+1} thus by induction F_{n-1} is well-defined and cod $F_{n-1} =$ cod f_{n-1}.

To prove (1) and (2) we proceed by induction on n. If $n = 1$ $F_0 = f_0 = \lambda y.yh_0(y) = id_{\tau y} \sqcap h$ so (1) and (2) hold. If $n > 1$, we put $g = (\lambda\vec{v}.\vec{v}^{n-2})\bullet h$, $g: \tau_0 y \longrightarrow \tau v^{n-2}$, $g_i = (\lambda\vec{v}^{n-2}.v_i)\bullet g = h_i$ for $i \varepsilon [n - 1]$, the "f_i" and "F_i" are the same ($i \varepsilon [n - 2]$). By induction,

(1)' $(\lambda y\vec{v}^{n-2}.y)\bullet F_{n-2} = id_{\tau y}$ and (2)' $(\lambda y\vec{v}^{n-2}.v^{n-2})\bullet F_{n-2} = g$.

To show (1) and (2):

$$\begin{aligned}(1):\ (\lambda y\vec{v}.y)\bullet F_{n-1} &= (\lambda y\vec{v}.y)\bullet(\lambda y\vec{v}^{n-2}.y\vec{v}^{n-2}h_{n-1}(y))\bullet F_{n-2}\\ &= (\lambda y\vec{v}^{n-2}.y)\bullet F_{n-2} \quad \text{by 0.5.5.1}\\ &= id_{\tau y} \quad \text{by (1)'}.\end{aligned}$$

$$(2):\quad (\lambda y\vec{v}.\vec{v})\ F_{n-1} = (\lambda y\vec{v}.\vec{v})\circ(\ y\vec{v}^{n-2}.y\vec{v}^{n-2}h_{n-1}(y))\bullet F_{n-2}$$
$$= (\lambda y\vec{v}^{n-2}.\vec{v}^{n-2}h_{n-1}(y))\bullet F_{n-2} \qquad \text{by } 0.5.4.8$$
$$= (\lambda y\vec{v}^{n-2}.\vec{v}^{n-2})\bullet F_{n-2} - h_{n-1}\circ(\lambda y\vec{v}^{n-2}.y)\bullet F_{n-2}$$
$$= (\lambda\vec{v}.\vec{v}^{n-2})\circ h \sqcap (\lambda\vec{v}.v_{n-1})\bullet h \qquad \text{by (1)' and (2)'}$$
$$= h\ .\square$$

<u>0.5.5.4 Proposition.</u> Let $\vec{v}$, y and h be as in the previous proposition 0.5.5.3. Let t be a term for which $s_F(t) \subset \{\vec{v}\}$ and $y \notin s_{FB}(t)$. Define t_j by $t_0 = t$, and $t_{j+1} = t_j[v_{n-j-1} \mid h_{n-j-1}(y)]$ for $j \in [n]$, then

$$\lambda y.t_n = (\lambda\vec{v}.t)\bullet h.$$

<u>Proof.</u> Let h_i, f_i and F_i ($i \in [n]$) be as in 0.5.5.3. We prove by induction on j that

$$\lambda y.t_n = (\lambda y\vec{v}^j.t_{n-(j+1)})\bullet F_j \qquad j \in [n].$$

For $j = 0$ we have $(\lambda yv_0.t_{n-1})\circ(\lambda y.yh_0(y)) = \lambda y.t_{n-1}[v_0 \mid h_0(y)] = \lambda y.t_n$.

Induction step

$$(\lambda y\vec{v}^{i+1}.t_{n-(i+2)})\bullet F_{i+1} = (\lambda y\vec{v}^{i+1}.t_{n-(i+2)})\bullet(\lambda y\vec{v}^{i}.y\vec{v}^{i}h_{i+1}(y))\bullet F_i$$
$$= (\lambda y\vec{v}^{i}.(t_{n-(i+2)}[v_{i+1} \mid h_{i+1}(y)]))\bullet F_i \quad \text{by } 0.5.5.1$$
$$= (\lambda y\vec{v}^{i}.t_{n-(i+1)})\bullet F_i$$
$$= \lambda y.t_n \qquad \text{by induction.}$$

Hence putting $j = n - 1$

$$\lambda y.t_n = (\lambda y\vec{v}.t)\bullet F_{n-1} = (\lambda\vec{v}.t)\circ(\lambda y\vec{v}.\vec{v})\bullet F_{n-1} = (\lambda\vec{v}.t)\bullet h\ ,$$

the last equality by 0.5.5.3.□

<u>0.5.5.5 Corollary.</u> Let $\langle\vec{v},t\rangle$ be an augmented term, $\ell(\vec{v}) = n \geq 1$, $v_i = (\vec{v}(i))^\wedge$ ($i \in [n]$), y a variable of type $\tau(\vec{v})$, $y \notin \{\vec{v}\}$, put $\pi_i = pr_i^{\vec{\tau}(\vec{v})}$ where $pr_i^{\vec{\tau}(\vec{v})}: \Pi\vec{\tau}(\vec{v}) \longrightarrow \tau_0(v_i)$ is the i-th projection

(see 0.4.3.1). Define α: Vbls $\longrightarrow$ Tms by $\alpha(x) \equiv x$ for $x \notin \{\vec{v}\}$ and $\alpha(v_i) \equiv \pi_i y$ for $i \in [n]$.

$$\lambda y.S(\alpha)(t) \equiv \lambda\vec{v}.t \ .$$

<u>Proof.</u> Define t_{j+1} for $j \in [n]$ as in 0.5.5.4. By 0.2.6.2 $S(\alpha)(t) \equiv \mathop{S}_{j=1}^{n} \begin{pmatrix} v_{n-j-1} \\ \pi_{n-j-1} y \end{pmatrix} (t) \equiv t_n$. Take $h \equiv id_{\tau(\vec{v})}$, then $h_i \equiv (\lambda\vec{v}.v_i) \circ id_{\tau(\vec{v})} \equiv \pi_i$ for $i \in [n]$, so by 0.5.5.4 $\lambda y.t_n \equiv \lambda\vec{v}.t$.□

Section 0.6 Validity in $\Phi(\mathcal{E})$

0.6.1 Definitions and basic properties.

0.6.1.1 A formula φ is said to be *valid* if

$$|\varphi| = \mathrm{true}_{\tau(\mathrm{var}(\varphi))}.$$

We write $\models\varphi$ to indicate φ is valid and for Σ a set of formulas we write $\models\Sigma$ to indicate that each φ in Σ is valid.

0.6.1.2 Proposition. The following are equivalent

(1) $\models\varphi$

(2) $\|\varphi\| = [\![\tau(\mathrm{var}(\varphi))]\!]$

(3) For all augmented formulas $\langle\vec{v},\varphi\rangle$ (0.3.4.1) we have $\lambda\vec{v}.\varphi = \mathrm{true}_{\tau(\vec{v})}$.

(4) For all augmented formulas $\langle\vec{v},\varphi\rangle$ we have $[\![v|\varphi]\!] = [\![\tau(\vec{v})]\!]$.

Proof. The implications (1) $\leftrightarrow$ (2), (3) $\leftrightarrow$ (4), (3) $\rightarrow$ (1), (4) $\rightarrow$ (2) are immediate. We show (1) $\rightarrow$ (3). Since $S_F(\varphi) = \{\mathrm{var}(\varphi)\}$ (by 0.2.11.7), $\lambda\vec{v}.\mathrm{var}(\varphi)$ is well-defined and

$\lambda\vec{v}.\varphi = (\lambda\mathrm{var}(\varphi).\varphi)\bullet(\lambda\vec{v}.\mathrm{var}\varphi) = \mathrm{true}_{\tau(\mathrm{var}(\varphi))}\bullet(\lambda\vec{v}.\mathrm{var}\varphi) = \mathrm{true}_{\tau(\vec{v})}$.□

0.6.2 Substitution

0.6.2.1 Proposition. If $s \in \mathrm{ffr}[x](\varphi)$ and $\models\varphi$ then $\models\varphi[x|s]$.

Proof. If $x \notin S_F(\varphi)$ there is nothing to prove, so we assume $x \in S_F(\varphi)$. Let $\vec{w}x$ be a string of distinct variables for which $S_F(\vec{w}x) = S_F(\varphi)$ (for example $\vec{w} = \text{omit}(x)(\text{var}(\varphi))$), then $[\![\vec{w}x|\varphi]\!] = [\![\tau(\vec{w}x)]\!]$. Let $\vec{v} = \text{var}(\varphi[x|s])$ then $S_F(\vec{v}) = S_F(\varphi[x|s]) = (S_F(\varphi) - \{x\}) \cup S_F(s)$ so $S_F(\vec{w}x) \subset S_F(\vec{v})$; by 0.5.5.1 $(\lambda\vec{v}.\vec{w}s)^{-1}[\![\vec{w}x|\varphi]\!] = [\![\vec{v}|\varphi[x|s]]\!]$ hence $[\![\vec{v}|\varphi[x|s]]\!] = [\![\tau(\vec{v})]\!]$ that is $\models \varphi[x|s]$.□

0.6.2.2 Proposition. If α is an admissible substitution for φ then $\models \varphi$ implies $\models S(\alpha)(\varphi)$.

Proof. Let $n = n(\alpha)$, $x_i (i \in [n])$ a listing of the variables moved by α , and $u_i (i \in [n])$ variables chosen in such a way that the set $U = \{u_i | i \in [n]\}$ satisfies the conditions of the proposition of 0.2.6.6 then

$$S(\alpha)(\varphi) = \mathop{S}_{i=0}^{n-1} \begin{pmatrix} u_i \\ \alpha(x_i) \end{pmatrix} \mathop{S}_{i=0}^{n-1} \begin{pmatrix} x_i \\ u_i \end{pmatrix} (\varphi).$$

By 0.2.9.2 and 0.2.9.8 each of the $2n$ substitutions is admissible hence $\models S(\alpha)(\varphi)$.□

0.6.2.3 If after a single substitution a formula becomes valid then the relationship between the term substituted and the original formula is as follows.

Proposition. Let $s \in \text{ffr}[x](\varphi)$, $\vec{w}x$ distinct variables for which $S_F(\vec{w}x) = S_F(\varphi)$, $\vec{v} = \text{var}(\varphi[x|s])$, and $m \in [\![\vec{w}x|\varphi]\!]$ then

$$\models \varphi[x|s] \text{ iff } \lambda\vec{v}.\vec{w}s \text{ factors through } m.$$

Proof. By 0.5.5.1 we have

$$(\lambda\vec{v}.\vec{w}s)^{-1}\,[\![m]\!] = [\![\vec{v}|\varphi[x|s]]\!].$$

If $\models \varphi[x|s]$ then $(\lambda\vec{v}.\vec{w}s)^{-1}\,[\![m]\!] = [\![\tau(\vec{v})]\!]$ so (1) is a

$$\begin{array}{ccc} \tau(\vec{v}) & \longrightarrow & \cdot \\ {\scriptstyle id_{\tau(\vec{v})}}\Big\| & (1) & \Big\downarrow{\scriptstyle m} \\ \tau(\vec{v}) & \xrightarrow[\lambda\vec{v}.\vec{w}s]{} & \tau(\vec{w}x) \end{array}$$

pullback diagram so it commutes, that is, $\lambda\vec{v}.\vec{w}s$ factors through m. On the other hand if (1) commutes it is a pullback hence $[\![\vec{v}|\varphi[x|s]]\!] = (\lambda\vec{v}.\vec{w}s)^{-1}\,[\![m]\!] = [\![\tau(\vec{v})]\!]$, so $\models \varphi[x|s]$.□

We state some special cases of this proposition:

If $S_P(\varphi) = \{x\}$, $s \in ffr[x](\varphi)$, $[\![m]\!] = \|\varphi\|$ then

$$\models \varphi[x|s] \text{ iff } |s| \text{ factors through } m.$$

Specializing further, with the above hypothesis, if $s = f\star$, $s = fy$ or $s = f\langle y_1,y_2\rangle$ (with $y_1 \neq y_2$) then

$$f \text{ factors through } m$$

(i.e. there exists a unique g such that $m \circ g = f$) is equivalent to $\models \varphi[x|f\star]$, $\models \varphi[x|fy]$ or $\models \varphi[x|f\langle y_1,y_2\rangle]$ respectively.

0.6.3 A simple metatheorem for $\Phi(\mathcal{E})$.

Let $\Sigma \cup \{\varphi\}$ be a set of atomic formulas of a formula alphabet $\mathbb{P}$, and suppose that φ is a consequence of Σ, that is: $\Sigma \models \varphi$. In this section we shall prove that if γ is an interpretation (to be defined in

0.6.3.1) of $\mathbb{P}$ in $\Phi(\underline{\mathcal{E}})$ of type A such that $\models \tilde{\gamma}(\Sigma)$, then $\models \tilde{\gamma}(\varphi)$. Thus φ is a consequence of Σ, relative to interpretations in $\Phi(\underline{\mathcal{E}})$. (In 0.6.6.5 we will generalize this to the case when $\Sigma \cup \{\varphi\}$ is a set of basic Horn formulas of $\mathbb{P}$).

<u>0.6.3.1 Internal structures and interpretations in $\Phi(\underline{\mathcal{E}})$.</u> Now that $\Phi(\underline{\mathcal{E}})$ has been defined from a topos $\underline{\mathcal{E}}$, we can complete the definitions given in 0.3.3.2. An <u>internal structure</u> of similarity type <O,P> is an internal pre-structure $\mathcal{A} = \langle A, o, p\rangle$ such that $p(\underline{\delta}) = \delta_A$, where δ_A is the morphism of $\underline{\mathcal{E}}$ classifying the diagonal:

$$\begin{array}{ccc} A & \xrightarrow{\quad u_A \quad} & 1 \\ {\scriptstyle \Delta_A}\big\downarrow & \text{pb} & \big\downarrow{\scriptstyle \text{true}} \\ A \times A & \xrightarrow[\quad \delta_A \quad]{} & \Omega \end{array}$$

Let $\mathbb{P} = \mathbb{P}\langle O,P\rangle$ be the formula elphabet determined by the similarity type <O,P>. An <u>interpretation of $\mathbb{P}$ in $\Phi(\underline{\mathcal{E}})$ of type A</u> is a pre-interpretation γ of $\mathbb{P}$ in $\Phi(\underline{\mathcal{E}})$ of type A such that the following conditions on γ hold:

(1) $\gamma|_V$ is the bijection from V to V_A that assigns the i-th variable of V to the i-th variable of V_A, i.e. $\gamma(\langle i,N\rangle^\wedge) = \langle i,A\rangle^\wedge$ for $i \in N$.

(2) $\mathcal{A} = \langle A, \gamma_O, \gamma_P\rangle$ is an internal structure, i.e. $\gamma(\underline{\delta}) = \delta_A$.

(3) $\gamma(\underline{\&}) = \wedge_\Omega$, the morphism classifying true $\sqcap$ true:

$$\begin{array}{ccc} 1 & \longrightarrow & 1 \\ {\scriptstyle true \sqcap true}\downarrow & \text{pb} & \downarrow{\scriptstyle true} \\ \Omega \times \Omega & \xrightarrow[\wedge_\Omega]{} & \Omega \end{array}$$

(4) $\gamma(\underline{i}) = \Rightarrow_\Omega$, the classifying map of the equalizer ℓ_Ω of $\wedge_\Omega$ with the first projection:

$$\underline{\leq} \overset{\ell_\Omega}{\rightarrowtail} \Omega \times \Omega \underset{\pi_1}{\overset{\wedge_\Omega}{\rightrightarrows}} \Omega \qquad\qquad \begin{array}{ccc} \underline{\leq} & \longrightarrow & 1 \\ {\scriptstyle \ell_\Omega}\downarrow & \text{pb} & \downarrow{\scriptstyle true} \\ \Omega \times \Omega & \xrightarrow[\Rightarrow_\Omega]{} & \Omega \end{array}$$

equalizer

(For the above maps see [J2] p. 127, [KW] p. 34, and [Fr] p. 28).
For any formula φ of $\mathbb{P}$ - atomic, conjuction or basic Horn - we let $\overline{\varphi}$ be the value of φ under $\tilde{\gamma}$, the extension of $\overline{\varphi}\gamma$ to strings of el($\mathbb{P}$). We say <u>φ is valid in $\mathcal{A}$</u> and write $\mathcal{A} \models \varphi$ if $\models \overline{\varphi}$; if Σ is a set of such formulas, we say <u>$\mathcal{A}$ is an internal model of Σ</u> and write $\mathcal{A} \models \Sigma$ if $\models \overline{\Sigma}$.

<u>0.6.3.2 External structures induced by applying the hom functors [B,-] to an internal structure.</u> What we call external structures are in fact internal structures in the topos of sets $\underline{\mathcal{S}}$ (in, say, some universe); since internal structures are describable using only the cartesian structures of $\underline{\mathcal{E}}$ and $\underline{\mathcal{S}}$, and since moreover [B,-] preserves cartesian structure, the image under [B,-] of an internal structure $\mathcal{A}$, must be an external structure [B,$\mathcal{A}$]. While this gives us some insight into the externalizing process we shall avoid the, by now, excessive notational complications which would be forced upon us if we were to work with $\Phi(\underline{\mathcal{S}})$.

Definitions. (1) Let $\mathcal{A} = \langle A,o\rangle$ be an internal algebra in $\Phi(\underline{\mathcal{E}})$ of similarity type O. For each operation sign θ we let $\overline{\theta} = o(\theta)$ the interpretation of θ in $\Phi(\underline{\mathcal{E}})$, and let $\theta_{[B,\mathcal{A}]}$ be the interpretation of θ as an operation on the set $[B,A]$; e, f and g will denote arbitrary nullary, unary and binary operations respectively, and k, k_1, k_2 are morphisms from B to A. The external algebra $[B,\mathcal{A}]$ is given by:

$$e_{[B,\mathcal{A}]} = \overline{e} \cdot u_B$$

$$f_{[B,\mathcal{A}]}(k) = \overline{f} \cdot k$$

$$g_{[B,\mathcal{A}]}(k_1,k_2) = \overline{g} \cdot (k_1 \sqcap k_2).$$

(2) Let $\mathcal{A} = \langle A,o,p\rangle$ be an internal structure in $\Phi(\underline{\mathcal{E}})$ of similarity type $\langle O,P\rangle$. For each $\theta \in P_1 \cup P_2$ we put $\overline{\theta} = p(\theta)$ and let $\theta^{[B,\mathcal{A}]}$ be the realization of θ as an external relation on the set $[B,A]$; f and g are arbitrary unary and binary predicate signs respectively; and k, k_1, k_2 are arbitrary morphisms from B to A. The external structure $[B,\mathcal{A}]$ has $[B, \mathcal{A}_O]$ as its underlying external algebra. Its relations are:

$$f^{[B,\mathcal{A}]} = \{k \mid \overline{f} \cdot k = true_B\}$$

$$g^{[B,\mathcal{A}]} = \{(k_1,k_2) \mid \overline{g} \cdot (k_1 \sqcap k_2) = true_B\} \quad \text{for } g \neq \underline{\delta}$$

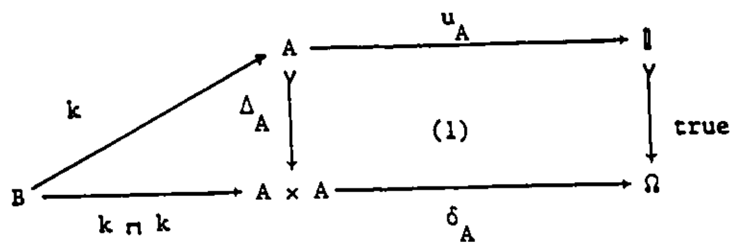
$$\underline{\delta}^{[B,\mathcal{A}]} = \{(k,k) \mid k \in [B,\mathcal{A}]\} .$$

The morphism δ_A, which is the interpretation of $\underline{\delta}$ in $\Phi(\underline{\mathcal{E}})$, does not appear in the definition of $[B,\mathcal{A}]$ in (2). We now establish that the stipulation that $\underline{\delta}^{[B,\mathcal{A}]}$ be the diagonal of $[B,A]$ - a requirement of our definition of external structure given in 0.2.3.1 - is redundant, in that, by dropping the restriction $g \neq \underline{\delta}$ in our definition of $g^{[B,\mathcal{A}]}$, the equality of $\underline{\delta}^{[B,\mathcal{A}]}$ and the diagonal of $[B,A]$ can be deduced.

<u>0.6.3.3 Proposition.</u> Let $k_1, k_2 : B \longrightarrow A$, then

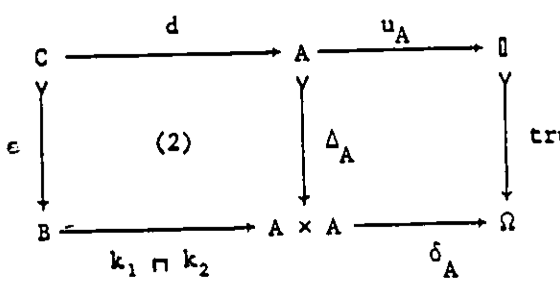
$$k_1 = k_2 \quad \text{iff} \quad \delta_A \circ (k_1 \sqcap k_2) = true_B \ .$$

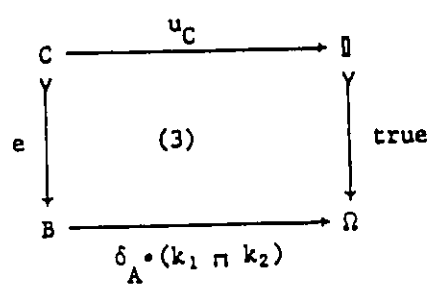
<u>Proof.</u> We first prove that, for $k: B \longrightarrow A$, $\delta_A \circ (k \sqcap k) = true_B$. The square (1) commutes, hence



$\delta_A \circ (k \sqcap k) = \delta_A \circ \Delta_A \circ k = (true) \circ u_A \circ k = true_B$.□

For the converse we introduce the pullback (2). By combining (1), which is also a pullback, with (2) we get the pullback (3).





Since $\delta_A \circ (k_1 \sqcap k_2) = true_B$ we must have $[\![e]\!] = [\![id_B]\!]$, hence e is an isomorphism. From (2) we have $(k_1 \sqcap k_2) \circ e = \Delta_A \circ d$, hence $k_1 \sqcap k_2 = \Delta_A \circ d \circ e^{-1}$. Hence for both $i = 1, 2$ we have $k_i = \pi_i \circ (k_1 \sqcap k_2) = \pi_i \circ \Delta_A \circ d \circ e^{-1} = d \circ e^{-1}$, hence $k_1 = k_2$.

<u>Corollary.</u> $\underline{\delta}^{[B,\mathcal{A}]} = \{(k_1,k_2) \mid \overline{\underline{\delta}} \circ (k_1 \sqcap k_2) = true_B\}$.□

<u>0.6.3.4 Linking polynomial functions on the algebra $[B,\mathcal{A}]$ to the λ-calculus.</u> Let $t \in Poly(E)$ where $E = E(0)$ the elphabet of similarity type 0, and $\mathcal{A} = \langle A,o\rangle$ an internal algebra of similarity type 0 in $\Phi(\mathcal{E})$. Each polynomial term defines a polynomial function $t_{[B,\mathcal{A}]} : [B,A]^V \longrightarrow [B,A]$. Let $\beta \in [B,A]^V$ and let $y \in V_B$; define β_y: Vbls $\longrightarrow$ Tms by $\beta_y(x) = x$ for $x \notin V_A$, let x_i be the i-th variable of type A and let v_i be the i-th variable of V, put $\beta_y(x_i) = (\beta(v_i))y$.

<u>Proposition.</u> $t_{[B,\mathcal{A}]}(\beta) = \lambda y.S(\overline{\beta})(\overline{t})$, where $y \in V_B$ and $\overline{\beta} = \beta_y$.
<u>Proof.</u> We define $\tilde{\beta}$: Poly(E) $\longrightarrow$ [B,A] by $\tilde{\beta}(t) = \lambda y.S(\overline{\beta})(\overline{t})$, then $\tilde{\beta}(v_i) = \lambda y.\overline{\beta}(\overline{v}_i) = \lambda y.(\beta(v_i))y = \beta(v_i)$, that is, $\tilde{\beta}$ restricted to the generating set V of Poly(E) is β. We show $\tilde{\beta}$ is an 0-homomorphism.
$\tilde{\beta}(e^*) = \overline{e}\circ\lambda y.* = \overline{e}\circ u_B = e_{[B,\mathcal{A}]}$
$\tilde{\beta}(ft) = \overline{f}\circ\lambda y.S(\overline{\beta})(\overline{t}) = \overline{f}\circ\tilde{\beta}(t) = f_{[B,\mathcal{A}]}(\tilde{\beta}(t))$.
$\tilde{\beta}(g/t,s/) = \overline{g}\circ\lambda y/S(\overline{\beta})(\overline{t}),S(\overline{\beta})(\overline{s})/ = \overline{g}\ (\tilde{\beta}(t) \sqcap \tilde{\beta}(s)) = g_{[B,\mathcal{A}]}(\tilde{\beta}(t),\tilde{\beta}(s))$.
Hence by definition of $t_{[B,\mathcal{A}]}$, $t_{[B,\mathcal{A}]}(\beta) = \tilde{\beta}(t)$.□

<u>0.6.3.5 Canonical valuations.</u>

<u>Definition.</u> Let A be a type of $\Phi(\mathcal{E})$, $\vec{x} \in$ Rdc Str(V_A) the class of canonical valuations for $\vec{x}$ is ($\overline{v}$ is the image of v under $V \doteq V_A$)
$Val_A(\vec{x}) = \{k \in [\tau(\vec{x}),A]^V \mid$ for all $\overline{v} \in \{\vec{x}\}$, $k(v) = \lambda\vec{x}.\overline{v}\}$.
If $\vec{x} \neq \phi$ then $Val_A(\vec{x}) \neq \phi$. If $\vec{x} = \phi$ then $Val_A(\phi) \neq \phi$ iff $[0,A] \neq \phi$.

<u>Proposition.</u> Let $\mathcal{A} = \langle A,\gamma\rangle$ be an internal algebra and let E be the elphabet determined by $\mathcal{A}$. If k is a canonical valuation for $\vec{x}$,

$t \in Poly(\mathbb{E})$ and $<x,\overline{t}>$ is an augmented term, then $t_{[\tau\vec{x},\mathcal{A}]}(k) = \lambda\vec{x}.\overline{t}$.

Proof. If $\vec{x} = \phi$ then $s_F(t) = \phi = s_F(\overline{t})$ and we want to show $t_{[\mathbb{1},\mathcal{A}]}(k) = |\overline{t}|$. Let $y \in V_{\mathbb{1}}$ then

$t_{[\mathbb{1},\mathcal{A}]}(k) = \lambda y.S(\overline{k})(\overline{t}) = \lambda y.\overline{t} = (\lambda\phi.\overline{t})\bullet(\lambda y.\phi) = |\overline{t}|$.

If $\vec{x} \neq \phi$, define α: Vbls $\longrightarrow$ Tms by $\alpha(x) = x$ for $x \notin \{\vec{x}\}$ and $\alpha(x) = (\lambda\vec{x}.x)y$ for $x \in \{\vec{x}\}$. By 0.5.5.5 $\lambda y.S(\alpha)(\overline{t}) = \lambda\vec{x}.\overline{t}$. For $x \in s_F(\overline{t})$ we have $\alpha(x) = (\lambda\vec{x}.x)y = \overline{k}(x)$ for some $k \in Val_A(\vec{x})$, hence $S(\alpha)(\overline{t}) = S(\overline{k})(t)$ by 0.2.5.3. Thus $t_{[\tau\vec{x},\ ]}(k) = \lambda y.S(k)(t) = \lambda y.S(\alpha)(t) = \lambda\vec{x}.t.\square$

0.6.3.6 Proposition. Let $\mathcal{A} = <A,o,p>$ be an internal structure of similarity type $<O,P>$, $\mathbb{P} = \mathbb{P}<O,P>$, $\varphi \in AtFml(\mathbb{P})$, $\beta \in [B,A]^V$, $y \in V_B$, and $\overline{\beta} = \beta_y$, then (1) β satisfies φ in $[B,\mathcal{A}]$ iff $\lambda y.S(\overline{\beta})(\overline{\varphi}) = true_B$, and (2) $\mathcal{A} \models \varphi$ implies $[B,\mathcal{A}] \models \varphi$.

Proof. (1) Suppose $\varphi = ft$; β satisfies ft in $[B,\mathcal{A}]$ iff $\overline{f}\bullet t_{[B,\mathcal{A}]}(\beta) = true_B$ iff $\overline{f}\bullet\lambda y.S(\overline{\beta})(\overline{t}) = true_B$ iff $\lambda y.S(\overline{\beta})(\overline{ft}) = true_B$ by 0.2.3.13, 0.6.3.2 and 0.6.3.4. Suppose $\varphi = g/t,s/$; β satisfies $g/t,s/$ in $[B,\mathcal{A}]$ iff $\overline{g}\bullet(t_{[B,\mathcal{A}]}(\beta) \sqcap s_{[B,\mathcal{A}]}(\beta)) = true_B$ iff $\overline{g}\bullet(\lambda y.S(\overline{\beta})(\overline{t}) \sqcap \lambda y.S(\overline{\beta})(\overline{s})) = true_B$ iff $\lambda y.S(\overline{\beta})(\overline{g/t,s/}) = true_B.\square$

(2) $\mathcal{A} \models \varphi$ means $\models \overline{\varphi}$. By 0.6.1.2 $\models S(\overline{\beta})(\overline{\varphi})$, hence by 0.6.2.2 $\lambda y.S(\overline{\beta})(\overline{\varphi}) = true_B$, by (1) β satisfies φ in $[B,\mathcal{A}]$ for all $\beta \in [B,A]^V$, hence $[B,\mathcal{A}] \models \varphi$.$\square$

0.6.3.7 Proposition. Let $\mathcal{A} = <A,o,p>$ be an internal structure, $\vec{x} = var(\overline{\varphi})$, and k a canonical valuation for $\vec{x}$. If k satisfies φ in $[\tau\vec{x},\mathcal{A}]$, then $\mathcal{A} \models \varphi$.

Proof. If $\varphi = ft$ then $\overline{f}\bullet t_{[\tau\vec{x},\mathcal{A}]}(k) = true_{\tau\vec{x}}$, by 0.6.3 $\overline{f}\bullet\lambda\vec{x}.\overline{t} = true_{\tau\vec{x}}$, hence $\models \overline{ft}$, hence $\mathcal{A} \models ft$. If $\varphi = g/t,s/$ then

$\overline{g}\bullet(t_{[\tau\vec{x},\mathcal{A}]}(k) \sqcap s_{[\tau\vec{x},\mathcal{A}]}(k)) = true_{\tau\vec{x}}$, hence

$true_{\tau\vec{x}} = \overline{g} \cdot (\lambda\vec{x}.\overline{t} \sqcap \lambda\vec{x}.\overline{s}) = \lambda\vec{x}.\overline{g/t,s/} = \lambda\vec{x}.\overline{g/t,s/}$, hence $\mathcal{A} \models g/t,s/$.□

<u>0.6.3.8 Metatheorem.</u> Let $\mathcal{A} = \langle A,o,p\rangle$ be an internal structure in $\Phi(\mathcal{E})$, $\mathbb{P}$ the formula elphabet determined by $\mathcal{A}$, $\Sigma \cup \{\varphi\} \subset AtFml(\mathbb{P})$. If $\mathcal{A} \models \Sigma$ and $\Sigma \models\!\!\!\equiv \varphi$ then $\mathcal{A} \models \varphi$.

<u>Proof.</u> Since $\mathcal{A} \models \Sigma$, by 0.6.3.6 we have $[A^n,\mathcal{A}] \models\!\!\!\equiv \Sigma$ for each n; hence $[A^n,\mathcal{A}] \models\!\!\!\equiv \varphi$ for each n, in particular for k a canonical valuation for var(φ), k satisfies φ in $[\tau(var(\varphi)),\mathcal{A}]$. If $s_F(\varphi) = \phi$ then $\mathbb{P}$ must have a nullary operation e, thus $\overline{e} \in [\mathbb{I},A]$ hence there is a $k \in Val(\phi)$ hence $\mathcal{A} \models \varphi$. If $s_F(\varphi) \neq \phi$ then there is a $k \in Val_A(var(\varphi))$ hence $\mathcal{A} \models \varphi$.□

<u>0.6.4 Propositional Logic</u>

In this section we translate propositions of topos theory, from [J2], [KW] and [Fr], into our language, and apply 0.6.3.7, to establish the validity in $\Phi(\mathcal{E})$ of tautologies of the intuitionistic propositional calculus as presented in [RS] and [Mc & T 3].

<u>0.6.4.1 The similarity type h.</u> H is to have two nullary operations signs: $\underline{0}$ and $\underline{1}$, and three binary operations signs: $\underline{\wedge}$, $\underline{\vee}$ and $\underline{\Rightarrow}$; these new fixed signs we define by iterating our sign building procedure ($s \rightsquigarrow s^{\wedge}$) beginning with $\underline{0} = \underline{\&}^{\wedge}$. We introduce abbreviations in notation on the H-polynomial algebra Poly($\mathbb{E}$(H)) of the elphabet $\mathbb{E}$(H) determined by H as follows: $t \wedge s = \underline{\wedge}/t,s/$, $t \vee s = \underline{\vee}/t,s/$, $t \Rightarrow s = \underline{\Rightarrow}/t,s/$, $t \Leftrightarrow s = (t \Rightarrow s) \wedge (s \Rightarrow t)$ and $\neg t = t \Rightarrow \underline{0}$, for t, s polynomials.

<u>0.6.4.2 The internal H-algebra $\underline{\Omega}$.</u> In addition to the morphism: true (0.5.1.3), $\wedge_\Omega$ and $\Rightarrow_\Omega$ (0.6.3.1) we will need the morphisms: false and

v_Ω ([J2] p.134, 5.1; p.94. 3.51; and [Fr] p.28); they are given as follows: false: $\mathbb{0} \rightarrowtail \Omega$ classifies $0 \rightarrowtail \Omega$ (0 is the initial object of $\underline{\mathcal{E}}$), and v_Ω: $\Omega \times \Omega \longrightarrow \Omega$ classifies the monomorphism $\Omega \vee \Omega \rightarrowtail \Omega \times \Omega$ of an epi-mono factorization of $(id_\Omega \sqcap true_\Omega) \sqcup (true_\Omega \sqcap id_\Omega)$: $\Omega + \Omega \longrightarrow \Omega \times \Omega$. We define the internal $\mathbb{H}$-algebra $\underline{\Omega} = \langle\Omega,h\rangle$ as follows: $h(\underline{1}) = true$, $h(\underline{0}) = false$, $h(\underline{\wedge}) = \wedge_\Omega$, $h(\underline{v}) = v_\Omega$, $h(\underline{\Rightarrow}) = \Rightarrow_\Omega$.

<u>0.6.4.3 The external H-algebra Sub(A).</u> Each Sub(A) is a Heyting algebra with operations compatible with the partial order $\leq$ of 0.5.1.1 ([KW] p.30, Proposition 1.28). Thus the set Sub(A), together with a self-evident interpretation of signs is an H-algebra $\underline{\text{Sub}}$(A). Moreover the bijection Sub(A) $\approx$ [A,Ω] of 0.5.1.3 is an $\mathbb{H}$-algebra isomorphism: $\underline{\text{Sub}}$(A) $\approx$ [A,$\underline{\Omega}$] ([KW] p.34, 1.35; [Fr] p.27, 2.41).

<u>0.6.4.4 The formula elphabet $\mathbb{H}$eq and its interpretations.</u> $\mathbb{H}$eq is to have exactly one predicate sign $\underline{\delta}$, and operation elphabet $\mathbb{E}(\mathbb{H})$ (0.2.3.12); so that its atomic formulas are just its equations. We define the external structure $\underline{\text{Sub}}$(A) to be $\underline{\text{Sub}}$(A) together with the interpretation of $\underline{\delta}$ as the diagonal of Sub(A), and the internal structure $\underline{\Omega}$ to be $\underline{\Omega}$ together with the interpretation of $\underline{\delta}$ as δ_Ω. If φ is an equation of $\mathbb{H}$eq and φ is valid in each $\underline{\text{Sub}}$(A) then [A,$\underline{\Omega}$] $\models\varphi$ for each A, hence $\underline{\Omega} \models \varphi$, and hence $\overline{\varphi}$ is valid in $\Phi(\underline{\mathcal{E}})$. Thus an equation valid in every Heyting algebra is internally valid in $\underline{\Omega}$.

<u>0.6.4.5 Equations valid in every Heyting algebra.</u> The following equations, taken from [RS], [BD] and [Macn] , are valid in every Heyting al-

gebra. We let u, v, w be the first three distinct variables of V.

Equations from [RS]

idempotent	$u \wedge u = u$	$u \vee u = u$
commutative	$u \wedge v = v \wedge u$	$u \vee v = v \vee u$
associative	$(u \wedge v) \wedge w = u \wedge (w \wedge w)$	$(u \vee v) \vee w = u \vee (v \vee w)$
absorption	$(u \wedge v) \vee v = v$	$u \wedge (u \vee v) = u$
unit element	$u \wedge \underline{1} = u$	$u \vee \underline{1} = \underline{1}$
zero element	$u \wedge \underline{0} = \underline{0}$	$u \vee \underline{0} = u$
distributive	$u \wedge (v \vee w) = (u \wedge v) \vee (u \wedge w)$	
	$u \vee (v \wedge w) = (u \vee v) \wedge (u \vee w)$	

(1) $\neg \underline{0} = \underline{1}$

(2) $\neg \underline{1} = \underline{0}$

(3) $u \wedge \neg u = \underline{0}$

(4) $u \Rightarrow u = \underline{1}$

(5) $\underline{1} \Rightarrow u = u$

(6) $\underline{0} \Rightarrow u = \underline{1}$

(7) $\neg \neg \neg u = \neg u$

(8) $\neg (u \wedge \neg u) = \underline{1}$

(9) $\neg (u \vee v) = \neg u \wedge \neg v$

(10) $u \Rightarrow \neg v = \neg (u \wedge v)$

(11) $v \Rightarrow \neg u = u \Rightarrow \neg v$

(12) $u \Rightarrow \neg v = \neg \neg (u \Rightarrow \neg v)$

(13) $(u \Rightarrow v) \wedge v = v$

(14) $u \wedge (u \Rightarrow v) = u \wedge v$

(15) $(u \Rightarrow v) \wedge (u \Rightarrow w) = u \Rightarrow (v \wedge w)$

(16) $(u \Rightarrow v) \wedge (w \Rightarrow v) = (u \vee w) \Rightarrow v$

(17) $u \Rightarrow (v \Rightarrow w) = (u \wedge v) \Rightarrow w$

(18) $u \wedge ((u \wedge v) \Rightarrow (u \wedge w)) = u \wedge (v \Rightarrow w)$

A set of equations which suffices to axiomatize Heyting algebras is: both commutative laws, both associative laws, both absorption laws, either unit element law, either zero element law, (4), (13), (14), and (15).

Equations from [BD]

(19) $\neg\neg u \wedge \neg\neg v = \neg\neg(u \wedge v)$

(20) $\neg\neg(\neg\neg u \vee \neg\neg v) = \neg\neg(u \vee v)$

(21) $\neg(u \vee \neg u) = \underline{0}$

(22) $u \wedge \neg(u \wedge v) = u \wedge \neg v$

(23) $\neg(u \Rightarrow v) = \neg\neg u \wedge \neg v$

(24) $u \wedge ((v \wedge w) \Rightarrow v) = u$

Equations from [Macn]

(25) $u \Rightarrow \neg u = \neg u$

(26) $\neg u \Rightarrow u = \neg\neg u$

(27) $u \Rightarrow (u \Rightarrow v) = u \Rightarrow v$

(28) $(u \Rightarrow v) \Rightarrow (u \Rightarrow w) = u \Rightarrow (v \Rightarrow w)$

(29) $((u \wedge v) \Rightarrow w) \Rightarrow w = ((u \Rightarrow w) \Rightarrow w) \wedge ((v \Rightarrow w) \Rightarrow w)$

Many equations in [Macn] involve a "modal" operator on a Heyting algebra; (27) is equivalent to the fact that a function U_a on a Heyting algebra defined by $U_a(b) = a \Rightarrow b$ is idempotent for each a, and (29) is equivalent to each W_a preserving meets where $W_a(b) = (b \Rightarrow a) \Rightarrow a$.

0.6.4.6 Abbreviations for propositional connectives in $\Phi(\mathcal{E})$. For any formulas φ, ψ of $\Phi(\mathcal{E})$ we put

$$\varphi \vee \psi = \vee_\Omega(\varphi,\psi) \qquad \varphi \wedge \psi = \wedge_\Omega(\varphi,\psi)$$

$$\varphi \Rightarrow \psi = \Rightarrow_\Omega(\varphi,\psi) \qquad \top = \text{true}\ * \qquad \bot = \text{false}\ *$$

and define

$$\varphi \Leftrightarrow \psi = (\varphi \Rightarrow \psi) \wedge (\psi \Rightarrow \varphi) \qquad \neg\varphi = \varphi \Rightarrow \bot$$

Let $\langle\vec{v},\varphi\rangle$ and $\langle\vec{v},\psi\rangle$ be augmented formulas of $\Phi(\mathcal{E})$, let $A = \tau(\vec{v})$, and $0_A\colon 0 \rightarrowtail A$ then from the H-isomorphism $\underline{\text{Sub}}(A) \approx [A,\underline{\Omega}]$ we have

$$[\vec{v}|\top] = [\![A]\!] \qquad [\vec{v}|\bot] = [\![0_A]\!]$$

$$[\![\vec{v}|\varphi \vee \psi]\!] = [\![\vec{v}|\varphi]\!] \vee [\![\vec{v}|\psi]\!] \qquad [\![\vec{v}|\varphi \wedge \psi]\!] = [\![\vec{v}|\varphi]\!] \wedge [\![\vec{v}|\psi]\!]$$

$$[\![\vec{v}|\varphi \Rightarrow \psi]\!] = [\![\vec{v}|\varphi]\!] \Rightarrow [\![\vec{v}|\psi]\!] \qquad [\![\vec{v}|\varphi \Leftrightarrow \psi]\!] = [\![\vec{v}|\varphi]\!] \Leftrightarrow [\![\vec{v}|\psi]\!]$$

$$[\![\vec{v}|\neg\, \varphi]\!] = \neg\, [\![\vec{v}|\varphi]\!].$$

<u>0.6.4.7 Proposition.</u> Let φ and ψ be formulas, and t and s be terms of type A, in $\Phi(\mathcal{L})$.

(1) If $\langle\vec{u},t\rangle$ and $\langle\vec{u},s\rangle$ are augmented terms, $\{\vec{u}\} \subset \{\vec{w}\}$ and $\lambda\vec{u}.t = \lambda\vec{u}.s$, then $\lambda\vec{w}.t = \lambda\vec{w}.s$.

(2) $\models t = s$ iff $\lambda var(ts).t = \lambda var(ts).s$.

(3) If $\{\vec{w}\} = s_F(ts)$ then: $\models t = s$ iff $\lambda\vec{w}.t = \lambda\vec{w}.s$.

(4) $\models t = s$ iff $\models s = t$.

(5) $\models \varphi$ iff $\models \varphi = \top$.

(6) $\models \varphi = \psi$ iff $\models \varphi \Leftrightarrow \psi$.

(7) $\models \varphi \Leftrightarrow \psi$ iff both $\models \varphi \Rightarrow \psi$ and $\models \psi \Rightarrow \varphi$.

<u>Proof.</u> (1) From 0.5.4.5, if $\lambda\vec{u}.t = \lambda\vec{u}.s$ then $\lambda\vec{w}.t = (\lambda\vec{u}.t)\circ(\lambda\vec{w}.\vec{u}) = (\lambda\vec{u}.s)\circ(\lambda\vec{w}.\vec{u}) = \lambda\vec{w}.s$.□

(2) Put $\vec{v} = var(ts)$. $\models t = s$ iff $\lambda\vec{v}.\delta_A\langle t,s\rangle = true_{\tau(\vec{v})}$ iff $\delta_A\circ(\lambda\vec{v}.t \sqcap \lambda\vec{v}.s) = true_{\tau(\vec{v})}$ iff $\lambda\vec{v}.t = \lambda\vec{v}.s$ by 0.6.3.3.□ (3) Put $\vec{v} = var(ts)$, then $\{\vec{v}\} = s_F(ts) = \{\vec{w}\}$, hence by (1) $\lambda\vec{v}.t = \lambda\vec{v}.s$ iff $\lambda\vec{w}.t = \lambda\vec{w}.s$. By (2), $\models t = s$ iff $\lambda\vec{w}.t = \lambda\vec{w}.s$.□ (4) Let $\vec{v} = var(ts)$ so that $\{\vec{v}\} = s_F(st)$. $\models t = s$ iff $\lambda\vec{v}.t = \lambda\vec{v}.s$ iff $\models s = t$ by (3).□
(5) Put $\vec{v} = var(\varphi)$, then $\lambda\vec{v}.\varphi = true_{\tau(\vec{v})}$. $\models \varphi$ iff $\lambda\vec{v}.\varphi = true_{\tau(\vec{v})}$ iff $\models \varphi = \top$ by (2).□ (6) Put $\vec{v} = var(\varphi\psi)$ and $[\![\tau(\vec{v})]\!] = 1$. $\models \varphi = \psi$ iff $\lambda\vec{v}.\varphi = \lambda\vec{v}.\psi$ iff $[\![\vec{v}|\varphi]\!] = [\![\vec{v}|\psi]\!]$ iff

$[\![\vec{v}|\varphi]\!] \Leftrightarrow [\![\vec{v}|\psi]\!] = 1$ iff $[\![\vec{v}|\varphi \Leftrightarrow \psi]\!] = 1$ iff $\models \varphi \Leftrightarrow \psi$ □ (7) Put $\vec{v} = \mathrm{var}(\varphi\psi)$ and $\vec{w} = \mathrm{var}(\psi\varphi)$. $\models \varphi \Leftrightarrow \psi$ iff $[\![\vec{v}|\varphi \Rightarrow \psi]\!] = [\![\tau(\vec{v})]\!]$ and $[\![\vec{v}|\psi \Rightarrow \varphi]\!] = [\![\tau(\vec{v})]\!]$ iff $\models \varphi \Rightarrow \psi$ and $\lambda\vec{v}.(\psi \Rightarrow \varphi) = \lambda\vec{v}.\top$ iff $\models \varphi \Rightarrow \psi$ and $\lambda\vec{w}.(\psi \Rightarrow \varphi) = \lambda\vec{w}.\top$ iff $\models \varphi \Rightarrow \psi$ and $\models \psi \Rightarrow \varphi$.□

<u>0.6.4.8 Tautologies of the intuitionistic propositional calculus.</u> We formulate the tautologies using formulas of $\Phi(\mathcal{E})$. If $t \in \mathrm{Poly}(\mathrm{Heq})$ and t is a tautology of the intuitionistic propositional calculus, then $t = \underline{1}$ is valid in every Heyting algebra, hence in each $\underline{\mathrm{Sub}}(A)$, hence $t = \underline{1}$ is internally valid in $\underline{\Omega}$, which means $\overline{t = \underline{1}}$ is valid in $\Phi(\mathcal{E})$, hence $\models \overline{t} = \top$, so by (5) of 0.6.4.7, $\models \overline{t}$. When we pass over to $\Phi(\mathcal{E})$ we let p, q, r be the first three distinct variables of type Ω, so that if u, v and w are as in 0.6.4.5, then $p = \overline{u}$, $q = \overline{v}$ and $r = \overline{w}$. Thus for example the following formulas, taken from [RS] (p.379), which are there taken as the logical axioms for the intuitionistic propositional calculus, are valid in $\Phi(\mathcal{E})$:

(T_1) $(p \Rightarrow q) \Rightarrow ((q \Rightarrow r) \Rightarrow (p \Rightarrow r))$ (syllogism law)

(T_2) $p \Rightarrow (p \vee q)$

(T_3) $q \Rightarrow (p \vee q)$

(T_4) $(p \Rightarrow r) \Rightarrow ((q \Rightarrow r) \Rightarrow ((p \vee q) \Rightarrow r)$

(T_5) $(p \wedge q) \Rightarrow p$

(T_6) $(p \wedge q) \Rightarrow q$

(T_7) $(r \Rightarrow p) \Rightarrow ((r \Rightarrow q) \Rightarrow (r \Rightarrow (p \wedge q)))$

(T_8) $(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \wedge q) \Rightarrow r)$ (importation law)

(T_9) $((p \wedge q) \Rightarrow r) \Rightarrow (p \Rightarrow (q \Rightarrow r))$ (exportation law)

(T_{10}) $(p \wedge \neg p) \Rightarrow q$ (Duns Scotus law)

(T_{11}) $(p \Rightarrow (p \wedge \neg p)) \Rightarrow \neg p$

$[\![\vec{v}|\varphi]\!] \Leftrightarrow [\![\vec{v}|\psi]\!] = 1$ iff $[\![\vec{v}|\varphi \Leftrightarrow \psi]\!] = 1$ iff $\models \varphi \Leftrightarrow \psi$ □ (7) Put $\vec{v} = \mathrm{var}(\varphi\psi)$ and $\vec{w} = \mathrm{var}(\psi\varphi)$. $\models \varphi \Leftrightarrow \psi$ iff $[\![\vec{v}|\varphi \Rightarrow \psi]\!] = [\![\tau(\vec{v})]\!]$ and $[\![\vec{v}|\psi \Rightarrow \varphi]\!] = [\![\tau(\vec{v})]\!]$ iff $\models \varphi \Rightarrow \psi$ and $\lambda\vec{v}.(\psi \Rightarrow \varphi) = \lambda\vec{v}.\top$ iff $\models \varphi \Rightarrow \psi$ and $\lambda\vec{w}.(\psi \Rightarrow \varphi) = \lambda\vec{w}.\top$ iff $\models \varphi \Rightarrow \psi$ and $\models \psi \Rightarrow \varphi$.□

<u>0.6.4.8 Tautologies of the intuitionistic propositional calculus.</u> We formulate the tautologies using formulas of $\Phi(\mathcal{E})$. If $t \in \mathrm{Poly(Heq)}$ and t is a tautology of the intuitionistic propositional calculus, then $t = \underline{1}$ is valid in every Heyting algebra, hence in each $\underset{\sim}{\mathrm{Sub}}(A)$, hence $t = \underline{1}$ is internally valid in $\underset{\sim}{\Omega}$, which means $\overline{t = \underline{1}}$ is valid in $\Phi(\mathcal{E})$, hence $\models \overline{t} = \top$, so by (5) of 0.6.4.7, $\models \overline{t}$. When we pass over to $\Phi(\mathcal{E})$ we let p, q, r be the first three distinct variables of type Ω, so that if u, v and w are as in 0.6.4.5, then $p = \overline{u}$, $q = \overline{v}$ and $r = \overline{w}$. Thus for example the following formulas, taken from [RS] (p.379), which are there taken as the logical axioms for the intuitionistic propositional calculus, are valid in $\Phi(\mathcal{E})$:

(T_1) $(p \Rightarrow q) \Rightarrow ((q \Rightarrow r) \Rightarrow (p \Rightarrow r))$ (syllogism law)

(T_2) $p \Rightarrow (p \vee q)$

(T_3) $q \Rightarrow (p \vee q)$

(T_4) $(p \Rightarrow r) \Rightarrow ((q \Rightarrow r) \Rightarrow ((p \vee q) \Rightarrow r)$

(T_5) $(p \wedge q) \Rightarrow p$

(T_6) $(p \wedge q) \Rightarrow q$

(T_7) $(r \Rightarrow p) \Rightarrow ((r \Rightarrow q) \Rightarrow (r \Rightarrow (p \wedge q)))$

(T_8) $(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \wedge q) \Rightarrow r)$ (importation law)

(T_9) $((p \wedge q) \Rightarrow r) \Rightarrow (p \Rightarrow (q \Rightarrow r))$ (exportation law)

(T_{10}) $(p \wedge \neg p) \Rightarrow q$ (Duns Scotus law)

(T_{11}) $(p \Rightarrow (p \wedge \neg p)) \Rightarrow \neg p$

In addition the following are valid ([RS] p.388):

(T_0) $p \to p$

(T_{13}) $p \to (q \to p)$ (simplification law)

(T_{14}) $p \to (q \to (p \wedge q))$

(T_{15}) $(p \to (q \to r)) \to (q \to (p \to r))$ (law of interchange of premises)

(T_{16}) $(p \to (q \to r)) \to ((p \to q) \to (p \to r))$ (Frege law)

(T_{17}) $p \to \neg \neg p$ (law of double negation)

(T_{19}) $\neg (p \wedge \neg p)$ (denial of contradiction)

(T_{21}) $\neg (p \vee q) \to (p \to q)$

(T_{22}) $\neg (p \vee q) \to (\neg p \wedge \neg q)$

(T_{23}) $(\neg p \wedge \neg q) \to \neg (p \vee q)$ (De Morgan laws)

(T_{25}) $(\neg p \vee \neg q) \to \neg (p \wedge q)$

(T_{26}) $(p \to q) \to (\neg q \to \neg p)$

(T_{27}) $(p \to \neg q) \to (q \to \neg p)$ (contrapositions laws)

(T_{65}) $\neg \neg \neg p \to \neg p$

(T_{66}) $\neg p \to \neg \neg \neg p$

(T_{67}) $\neg \neg (p \to q) \to (q \to \neg \neg p)$

(T_{68}) $(p \to q) \to ((p \to (q \to r)) \to (p \to r))$

In [Mc & T 3] a list of eleven logical axioms for the "Heyting calculus", (H1) to (H11), is given (H5) is (T_{13}), (H7) is (T_2); the remaining axioms are:

(H1) $p \to (p \wedge p)$

(H2) $(p \wedge q) \to (q \wedge p)$

(H3) $(p \to q) \to [(p \wedge r) \to (q \wedge r)]$

(H4) $[(p \to q) \wedge (q \to r)] \to (p \to r)$

(H6) $[p \wedge (p \Rightarrow q)] \Rightarrow q$

(H8) $(p \vee q) \Rightarrow (q \vee p)$

(H9) $[(p \Rightarrow r) \wedge (q \Rightarrow r)] \Rightarrow [(p \vee q) \Rightarrow r]$

Tautologies can also be derived from equations, $t = s$, valid in every Heyting algebra: from $\underset{\sim}{\Omega} \models t = s$ we have $\models \overline{t} = \overline{s}$, hence by (6) and (7) of 0.6.4.7 we have $\models \overline{t} \Leftrightarrow \overline{s}$, $\models \overline{t} \Rightarrow \overline{s}$ and $\models \overline{s} \Rightarrow \overline{t}$. For example from $\underset{\sim}{\Omega} \models u \wedge u = u$ we have $\models p \wedge p = p$ hence $\models p \Rightarrow (p \wedge p)$. By this method we derive the validity of the following formulas of $\Phi(\underset{\sim}{\mathcal{E}})$:

(I1) $(p \wedge (p \Rightarrow q)) \Rightarrow (p \wedge q)$

(I2) $(p \wedge q) \Rightarrow (p \wedge (p \Rightarrow q))$

(I3) $((p \wedge q) \wedge r) \Rightarrow (p \wedge (q \wedge r))$

(I4) $(p \wedge (q \wedge r)) \Rightarrow ((p \wedge q) \wedge r)$

(I5) $((p \Rightarrow q) \wedge (p \Rightarrow r)) \Rightarrow (p \Rightarrow (q \wedge r))$

(I6) $(p \Rightarrow (q \wedge r)) \Rightarrow ((p \Rightarrow q) \wedge (p \Rightarrow r))$.

<u>0.6.4.9 Rules of inference for $\Phi(\underset{\sim}{\mathcal{E}})$.</u> (For a discussion see [RS] p. 173-179). Just as we have done for signs we shall take rules to be sets of a special kind. We define a rule to be either (1) a partial function from Fmls to Fmls or (2) a partial function from Fmls^2 to Fmls. That is, as a relation R, either: (1) $R \subset \text{Fmls}^2$ such that $(\varphi,\psi_1) \in R$ and $(\varphi,\psi_2) \in R$ imply $\psi_1 = \psi_2$, or: (2) $R \subset \text{Fmls}^2 \times \text{Fmls}$ such that $((\varphi,\varphi'),\psi_i) \in R$ for $i = 1, 2$, implies $\psi_1 = \psi_2$. Thus when we speak of applying a rule (1) to a formula φ or (2) to formulas φ, φ', to derive a formula ψ we shall mean (1) $R(\varphi) = \psi$ and (2) $R(\varphi,\varphi') = \psi$. By a valid rule we mean one which maps valid formulas to valid formulas. Thus

for example let $R' = \{(\varphi \wedge \psi, \psi) \mid s_F(\varphi) \subset s_F(\psi)\}$, $R'' = \{((\varphi, \varphi \Rightarrow \psi), \psi) \mid s_F(\varphi) \subset s_F(\psi)\}$ and $R''' = \{((\varphi, \psi), \varphi \wedge \psi) \mid \varphi \text{ and } \psi \text{ formulas}\}$; R' valid means for all formulas φ and ψ, if $\models \varphi \wedge \psi$ and $s_F(\varphi) \subset s_F(\psi)$, then $\models \psi$; R'' valid means for all φ and ψ, if $\models \varphi$, $\models \varphi \Rightarrow \psi$, and $s_F(\varphi) \subset s_F(\psi)$, then $\models \psi$; R''' valid means for all φ, ψ, if $\models \varphi$ and $\models \psi$, then $\models \varphi \wedge \psi$.
We shall take as an unconventional alternative notation for the sets R', R'' and R''', the following:

$$\frac{\varphi \wedge \psi}{\psi} \quad s_F(\varphi) \subset s_F(\psi) \quad ,$$

$$\frac{\varphi, \varphi \Rightarrow \psi}{\psi} \quad s_F(\varphi) \subset s_F(\psi) \quad ,$$

and

$$\frac{\varphi, \psi}{\varphi \wedge \psi} \quad ; \text{ respectively.}$$

That is, we convert the schema

$$\{((\_{1}, \_{2}), \_{3}) \mid \_{4}\}$$

to the schema

$$\frac{\_{1}, \_{2}}{\_{3}} \quad \_{4} \; .$$

To indicate the intersection of a rule with its converse we use a double line where for the rule we would use a single line; thus the rule $\frac{\varphi}{\varphi = T}$ has converse $\frac{\varphi = T}{\varphi}$, and the intersection of the rule and its converse is $\frac{\varphi}{\varphi = T}$.

We have already established the validity of several rules. For each type preserving function α there is a valid rule of substitution R_α,

(1) $\dfrac{\varphi}{S(\alpha)(\varphi)}$ α admissable for φ.

From 0.6.4.7 the following are also valid rules:

(2) $\dfrac{\varphi}{\varphi = \top}$ (3) $\dfrac{\varphi = \psi}{\varphi \Leftrightarrow \psi}$ (4) $\dfrac{t = s}{s = t}$

(5) $\dfrac{\varphi \Rightarrow \psi, \psi \Rightarrow \varphi}{\varphi \Leftrightarrow \psi}$ (6) $\dfrac{\varphi \Leftrightarrow \psi}{\varphi \Rightarrow \psi}$ (7) $\dfrac{\varphi \Leftrightarrow \psi}{\psi \Rightarrow \varphi}$.

<u>Proposition.</u> The following are valid rules of inference:

(8) $\dfrac{\varphi, \psi}{\varphi \wedge \psi}$ (9) $\dfrac{\varphi, \varphi \Rightarrow \psi}{\psi}$ $s_F(\varphi) \subset s_F(\psi)$.

<u>Proof.</u> (8) Put $\vec{v} = var(\varphi\psi)$ and $1 = [\tau(\vec{v})]$. From $[\![\vec{v}|\varphi]\!] = 1$ and $[\![\vec{v}|\psi]\!] = 1$ we deduce $1 = [\![\vec{v}|\varphi]\!] \wedge [\![\vec{v}|\psi]\!] = [\![\vec{v}|\varphi \wedge \psi]\!]$, hence $\models \varphi \wedge \psi$.□

(9) Put $\vec{v} = var(\psi)$ and $1 = [\![\tau(\vec{v})]\!]$. From $1 = [\![\vec{v}|\varphi \Rightarrow \psi]\!] = [\![\vec{v}|\varphi]\!] \Rightarrow [\![\vec{v}|\psi]\!]$ and $[\![\vec{v}|\varphi]\!] = 1$ we deduce $[\![\vec{v}|\psi]\!] = 1$, hence $\models \psi$.□

<u>Corollary.</u> The following are valid rules of inference:

(10) $\dfrac{\varphi \wedge (\varphi \Rightarrow \psi)}{\varphi \wedge \psi}$ (11) $\dfrac{\varphi \wedge \psi}{\psi}$ $s_F(\varphi) \subset s_F(\psi)$

(12) $\dfrac{\varphi \wedge \psi}{\psi \wedge \varphi}$ (13) $\dfrac{(\varphi_1 \wedge \varphi_2) \wedge \varphi_3}{\varphi_1 \wedge (\varphi_2 \wedge \varphi_3)}$ (14) $\dfrac{(\varphi_1 \wedge \varphi_2) \Rightarrow \varphi_3}{\varphi_1 \Rightarrow (\varphi_2 \Rightarrow \varphi_3)}$

(15) $\dfrac{\varphi_1 \Rightarrow (\varphi_2 \Rightarrow \varphi_3)}{\varphi_2 \Rightarrow (\varphi_1 \Rightarrow \varphi_3)}$ (16) $\dfrac{\varphi_1 \Rightarrow \varphi_2}{(\varphi_2 \Rightarrow \varphi_3) \Rightarrow (\varphi_1 \Rightarrow \varphi_3)}$

(17) $\dfrac{\varphi_1 \Rightarrow \varphi_2, \varphi_2 \Rightarrow \varphi_3}{\varphi_1 \Rightarrow \varphi_3}$ $s_F(\varphi_2) \subset s_F(\varphi_1\varphi_3)$

(18) $\dfrac{(\varphi_1 \Rightarrow \varphi_2) \wedge (\varphi_1 \Rightarrow \varphi_3)}{\varphi_1 \Rightarrow \varphi_2 \wedge \varphi_3}$ (19) $\dfrac{\varphi_1 \Rightarrow \varphi_2}{(\varphi_1 \wedge \varphi_3) \Rightarrow (\varphi_2 \wedge \varphi_3)}$

(20) $\dfrac{\varphi}{\psi \Rightarrow (\varphi \wedge \psi)}$ (21) $\dfrac{\varphi}{\psi \Rightarrow \varphi}$

<u>Proof.</u> There are no bound variables in the valid formulas of 0.6.4.8, hence any substitution of formulas for variables of type Ω is admissible.

(10) Substitute $\begin{pmatrix} p & q \\ \varphi & \psi \end{pmatrix}$ in (I1) and (I2) and apply (9).□ The rest are proven similarly; we give the rule numbers followed by the tautologies used. (11) by (T_6); (12) by (H2); (13) by (I3) and (I4); (14) by (T_8) and (T_9); (15) by (T_{15}); (16) by (T_1); (17) by (H4); (18) by (I5) and (I6); (19) by (H3); (20) by (T_{14}); (21) by (T_{13}).□

0.6.4.10 To each string of terms we associate a valid formula which has the same free variables as the string of terms.

Definition. Let $\vec{t}$ be a string of terms.

$$\nabla(\vec{t}) \equiv (\text{true}_{\tau(\pi(\text{var}(\pi(\vec{t}))))})(\pi(\text{var}(\pi(\vec{t})))).$$

Proposition. Basic properties of ∇.

(1) $\nabla(\phi) \equiv \top$

(2) For $\vec{w}$ [illegible]uced, $\nabla(\vec{w}) \equiv (\text{true}_{\tau(\vec{w})})(\pi(\vec{w}))$

(3) $\nabla(t) \equiv (\text{true}_{\tau(\text{var}(t))})(\pi(\text{var}(t)))$

(4) $s_B(\nabla(\vec{t})) \equiv \phi$

(5) $\text{var}(\nabla(\vec{t})) \equiv \text{var}(\pi(\vec{t}))$

(6) $s_F(\nabla(\vec{t})) \equiv s_F(\pi(\vec{t}))$

(7) $\models \nabla(\vec{t})$

(8) For $\langle\vec{w},\varphi\rangle$ an augmented formula we have:

(8.1) $\text{var}(\nabla(\vec{w}) \wedge \varphi) \equiv \vec{w}$, and

(8.2) $\lambda\vec{w}.\varphi \equiv \text{true}_{\tau(\vec{w})}$ iff $\models \nabla(\vec{w}) \wedge \varphi$

(9) For $s_F(\vec{t}) \subset \{\vec{w}\}$, $\lambda\vec{w}.\nabla(\vec{t}) \equiv \text{true}_{\tau(\vec{w})}$.

The following rules of inference are valid

(10) $\dfrac{\nabla(\vec{t}) \wedge \psi}{\nabla(\vec{s}) \wedge \psi}$ $\quad s_F(\vec{t}) \subset s_F(\vec{s}\psi)$

(11) $\dfrac{(\varphi \Rightarrow \varphi_1) \wedge (\varphi \Rightarrow \varphi_2)}{\nabla(\varphi_2) \wedge (\varphi \Rightarrow \varphi_1)}$

(12) $\dfrac{\nabla(\varphi_2) \wedge (\varphi \Rightarrow \varphi_1), \nabla(\varphi_1) \wedge (\varphi \Rightarrow \varphi_2)}{(\varphi \Rightarrow \varphi_1) \wedge (\varphi \Rightarrow \varphi_2)}$

(13) $\dfrac{\nabla(\vec{t}) \wedge \varphi}{\nabla(\vec{t}) \Rightarrow \varphi}$

(14) $\dfrac{\nabla(\vec{w}) \wedge \varphi}{\varphi}$ $\quad \{\vec{w}\} \subset s_F(\varphi)$.

<u>Proofs.</u> (1) $\nabla(\phi) = (\text{true})^* = \top$.□ (2) Since $\vec{w}$ is reduced, by 0.2.11.2, $\vec{w} = \text{var}\pi(\vec{w})$.□ (3) Clear.□ (4) $s_B(\nabla(\vec{t})) = s_B(\pi(\text{var}(\pi(\vec{t})))) = \phi$ by 0.2.11.8.□ (5) $\text{var}(\nabla(\vec{t})) = \text{var}(\pi(\text{var}(\pi(\vec{t})))) = \text{var}(\pi(\vec{t}))$ by 0.2.11.2.□ (6) $s_F(\nabla(\vec{t})) = s_F(\pi(\text{var}(\pi(\vec{t})))) = \{\text{var}(\pi(\text{var}(\pi(\vec{t}))))\} = \{\text{var}(\pi(\vec{t}))\} = s_F(\pi(\vec{t}))$ by 0.2.11.7 and 0.2.11.2.□
(7) $|\nabla(t)| = (\text{true}_{\tau(\text{var}(\pi(\vec{t})))})\bullet|\pi(\text{var}(\pi(\vec{t})))| = \text{true}_{\tau(\text{var}(\pi(\vec{t})))}$.□
(8) We first prove a statement that more properly belongs in 0.2.11: If $\vec{w}$ and $\vec{v}$ are reduced and $\{\vec{v}\} \subset \{\vec{w}\}$ then $\vec{w} \vee \vec{v} = \vec{w}$. By definition ((5.1) of 0.2.10.5), $\vec{w} \vee \vec{v} = \vec{w}(\vec{v} \dot{-} \{\vec{w}\})$. By (4.1) of 0.2.10.4, $\{\vec{v} \dot{-} \{\vec{w}\}\} = \{\vec{v}\} \dot{-} \{\vec{w}\} = \phi$, hence by (2.2) 3f 0.2.10.2, $\vec{v} \dot{-} \{\vec{w}\} = \phi$. Hence $\vec{w} \vee \vec{v} = \vec{w}\phi = \vec{w}$.□ (8.1) $\text{var}(\nabla(\vec{w}) \wedge \varphi) = \text{var}(\nabla(\vec{w})) \vee \text{var}(\varphi)$. By (5) (above) and 0.2.11.2, $\text{var}(\nabla(\vec{w})) = \text{var}(\pi(\vec{w})) = \vec{w}$. By 0.2.11.7, $\{\text{var}(\varphi)\} = s_F(\varphi) \subset \{\vec{w}\}$. Now we take $\vec{v} = \text{var}(\varphi)$ and get $\text{var}(\nabla(\vec{w}) \wedge \varphi) = \vec{w} \vee \vec{v} = \vec{w}$.□ (8.2) $\models \nabla(\vec{w}) \wedge \varphi$ iff $[\![\tau(\vec{w})]\!] = [\![\vec{w}|\nabla(\vec{w}) \wedge \varphi]\!] = [\![\vec{w}|\nabla(\vec{w})]\!] \wedge [\![\vec{w}|\varphi]\!] = [\![\vec{w}|\varphi]\!]$ iff $\lambda\vec{w}.\varphi = \text{true}_{\tau(w)}$.□ (9) By (7) $\models \nabla(\vec{t})$ and $\models \nabla(\vec{w})$ hence, by the valid rule (8) of 0.6.4.9, $\models \nabla(\vec{w}) \wedge \nabla(\vec{t})$. Thus by (8.1), $\lambda\vec{w}.\nabla(\vec{t}) = \text{true}_{\tau(\vec{w})}$.□

(10) Put $\vec{w} = var(\nabla(\vec{s}) \wedge \psi)$. By 0.2.11.7 and (6), $\{\vec{w}\} = s_F(\nabla(\vec{s}) \wedge \psi) = (s_F(\vec{s}) \cup s_F(\psi)) \supset (s_F(\vec{t}) \cup s_F(\psi)) = s_F(\nabla(\vec{t}) \wedge \psi)$, $[\![\tau(\vec{w})]\!] = [\![\vec{w}|\nabla(\vec{t}) \wedge \psi]\!] = [\![\vec{w}|\nabla(\vec{t})]\!] \wedge [\![\vec{w}|\psi]\!] = [\![\vec{w}|\psi]\!]$, hence $\|\nabla(\vec{s}) \wedge \psi\| = [\![\vec{w}|\nabla(\vec{s})]\!] \wedge [\![\vec{w}|\psi]\!] = [\![\tau(\vec{w})]\!]$, hence $\models \nabla(\vec{s}) \wedge \psi$.□ (11) Put $\vec{w} = var(\nabla(\varphi_2) \wedge (\varphi \rightarrow \varphi_1))$. By 0.2.11.7 and (6), $\{\vec{w}\} = s_F(\varphi_2\varphi\varphi_1)$. Hence $[\![\tau(\vec{w})]\!] = [\![\vec{w}|\varphi \rightarrow \varphi_1]\!] \wedge [\![w|\varphi \rightarrow \varphi_2]\!] = [\![\vec{w}|\varphi \rightarrow \varphi_1]\!] = [\![\vec{w}|\nabla(\varphi_2)]\!] \wedge [\![\vec{w}|\varphi \rightarrow \varphi_1]\!] = \|\nabla(\varphi_2) \wedge (\varphi \rightarrow \varphi_1)\|$.□ (12) Apply rules (8), (12), (13) and (11) of 0.6.4.9. (13) Put $\vec{w} = var(\pi(\vec{t}\varphi))$. $\|\nabla(\vec{t}) \wedge \varphi\| = [\![\vec{w}|\nabla(\vec{t})]\!] \wedge [\![\vec{w}|\varphi]\!] = [\![\vec{w}|\varphi]\!] = [\![\vec{w}|\nabla(\vec{t})]\!] \rightarrow [\![\vec{w}|\varphi]\!] = \|\nabla(\vec{t}) \rightarrow \varphi\|$.□ (14) From rule (11) of 0.6.4.9.□

<u>0.6.4.11</u> To each term we associate a simpler term which has no bound variables; the two terms will have the same free variables and the same type and will moreover, in the internal sense, denote the same element of their common universe of discourse.

<u>Definition.</u> For each term t we define $smpl(t)$, the simplification of t, by

$$smpl(t) = |t|\pi(var(t)).$$

<u>Proposition.</u> Basic properties of simplification.

(1) $s_B(smpl(t)) = \phi$

(2) $var(smpl(t)) = var(t)$

(3) $\tau(smpl(t)) = \tau(t)$

(4) If $\langle\vec{w},t\rangle$ is an augmented term then $\lambda\vec{w}.smpl(t) = \lambda\vec{w}.t$.

(5) If $\langle\vec{w},f/t,s/\rangle$ is an augmented term then

$$\lambda\vec{w}.f/smpl(t),smpl(s)/ = \lambda\vec{w}.f/t,s/$$

(6) $\models smpl(t) = t$

(6) $\models$ smpl(t) = t

(7) smpl(smpl(t)) = smpl(t)

Proof. Put $\vec{v}$ = var(t), t' = smpl(t) and s' = smpl(s). (1) $s_B(t')$ = $s_B(\pi(\vec{v})) = \phi$ by 0.2.11.8.□ (2) var(t') = var(π(var(t))) = var(t) by 0.2.11.2.□ (3) τ(t') = cod|t| = τ(t).□ (4) $s_F(t')$ = $\{var(t')\}$ = $\{var(t)\}$ = $s_F(t)$ hence $s_F(t') \subset \{\vec{w}\}$. $\lambda\vec{w}.t' = (\lambda\vec{v}.t)\cdot(\lambda\vec{w}.\vec{v}) = \lambda\vec{w}.t$ by (4) of 0.5.4.2.□ (5) $\lambda\vec{w}.f\langle t',s'\rangle = f\cdot((\lambda\vec{w}.t') \sqcap (\lambda\vec{w}.s'))$ = $f\cdot(\lambda\vec{w}.t \sqcap \lambda\vec{w}.s) = \lambda\vec{w}.f\langle t,s\rangle$.□ (6) var(t' = t) = var(t') v var(t) = $\vec{v}$ and $\lambda\vec{v}.t' = \lambda\vec{v}.t$ hence $\models$ t' = t by (3) of 0.6.4.7.□ (7) smpl(t') = $|t'|\pi(var(t')) = |t|\pi(\vec{v})$ = t' by (2) and (4) above.□

0.6.4.12 The absorption of variables. Apart from the rule of substitution ((1) of 0.6.4.9), the valid rules presented so far have the property that the free variables of the premise form a subset of the free variables of the conclusion. In [O1] (p.315, 3.17), Osius weakens this condition by requiring only that the types of the free variables of the premise form a subset of the types of the free variables of the conclusion. We shall make use of the rule of substitution to define an even weaker relation $\prec$ on the subsets of variables (so that $\tau_0(U) \subset \tau_0(W)$ will imply $U \prec W$); when this relation is used in place of the inclusion condition stated in rules (9), (11) and (17) of 0.6.4.9, and (10) and (14) of 0.6.4.10 they will remain valid. Some condition is necessary, for example the rule

$$\frac{\varphi \wedge \psi}{\psi}$$

is not valid: if x is a variable of type 0, then, since 0 has just the one subobject [[0]], the formula $\nabla(x) \wedge \bot$ is valid whereas $\bot$ by itself is not valid since it is interpreted as the bottom subobject $[0_{\Omega}]$ of Ω which is distinct from $[\![0]\!]$ unless the topos itself is degenerate.

Definition. (1) We say a set Σ of variables absorbs a variable x if there exists a term t such that $\tau_0(x) = \tau(t)$ and $s_F(t) \subset \Sigma$. We say Σ absorbs a set Σ' of variables if Σ absorbs every variable in Σ'; in this case we write $\Sigma' \prec \Sigma$.

Definition. (2) Let Σ and Σ' be subsets of Vbls we write $\alpha\colon \Sigma' \prec \Sigma$ and say Σ absorbs Σ' through α if $\alpha\colon \text{Vbls} \longrightarrow \text{Tms}$ is a function such that

(a) α preserves types

(b) for all $x \notin (\Sigma' - \Sigma)$, $\alpha(x) = x$

(c) for all $x \in (\Sigma' - \Sigma)$, $s_B(\alpha(x)) = \phi$ and $s_F(\alpha(x)) \subset \Sigma$.

Proposition.

(1) $\{x\} \prec \Sigma$ iff there exists a term t such that $\tau(t) = \tau_0(x)$, $s_B(t) = \phi$ and $s_F(t) \subset \Sigma$.

(2) $\Sigma' \prec \Sigma$ iff there exists an $\alpha\colon \Sigma' \prec \Sigma$.

(3) If $\alpha\colon \Sigma' \prec \Sigma$, then both $(\Sigma' - \text{Mov}(\alpha)) \subset \Sigma$ and $\text{Mov}(\alpha) \subset \Sigma' - \Sigma$.

(4) If $\alpha\colon s_F(t) \prec \Sigma$ and $s_B(t) = \phi$, then $s_F(S(\alpha)(t)) \subset \Sigma$.

(5) If $\Sigma' \subset \Sigma$ then $\Sigma' \prec \Sigma$.

(6) If $\Sigma'' \prec \Sigma'$ and $\Sigma' \prec \Sigma$ then $\Sigma'' \prec \Sigma$.

(7) If $\Sigma_i' \prec \Sigma_i$ for $i = 1, 2$, then $(\Sigma_1' \cup \Sigma_2') \prec (\Sigma_1 \cup \Sigma_2)$.

(8) If $[\![0,\tau_0(x)]\!] \neq \phi$ then $\{x\} \prec \phi$.

(9) If $\tau_0(x) = \tau(\pi(\vec{v}))$ then $\{x\} \prec \{\vec{v}\}$.

Proof. (1) By definition, if $\{x\} \prec \Sigma$ there is a term t such that $\tau(t) = \tau_0(x)$ and $s_F(t) \subset \Sigma$. By 0.6.4.11 we can replace t by t' having the additional property $s_B(t') = \phi$. The converse is immediate.□

(2) Suppose $\Sigma' \prec \Sigma$. Define α on $\Sigma' - \Sigma$ so that (b) holds. For $x \in (\Sigma' - \Sigma)$ select $\alpha(x)$ as in (1), then (c) holds and (a) holds. For the converse let $x \in \Sigma'$ then $t = \alpha(x)$ satisfies condition (1) of the definition.□

(3) Suppose $\alpha: \Sigma' \prec \Sigma$ and $x \in (\Sigma' - \mathrm{Mov}(\alpha))$ we show $x \in \Sigma$. Assume $x \notin \Sigma$, then $x \in (\Sigma' - \Sigma)$ hence $s_F(\alpha(x)) \subset \Sigma$ by (c); but $x \in \mathrm{Mov}(\alpha)$ hence $\alpha(x) = x$, hence $x \in \Sigma$ – a contradiction.□ (4) Put $s = S(\alpha)(t)$. Since $s_B(t) = \phi$, by 0.2.8.3, $\alpha(x) \in ff[x](t)$ for all x. Thus by 0.2.8.5 $s_F(s) = (s_F(t) - \mathrm{Mov}(\alpha)) \cup \bigcup\{s_F(\alpha(x)) \mid x \in (\mathrm{Mov}(\alpha) \cap s_F(t))\}$. By (3), $(s_F(t) - \mathrm{Mov}(\alpha)) \subset \Sigma$ and $\mathrm{Mov}(\alpha) \subset (s_F(t) - \Sigma)$. Hence by (c) if $x \in (\mathrm{Mov}(\alpha) \cap s_F(t))$ then $s_F(\alpha(x)) \subset \Sigma$. Thus $s_F(s) \subset \Sigma$.□ (5) Let $x \in \Sigma'$ then $x \in \Sigma'$ also hence $\{x\} \prec \Sigma$, hence $\Sigma' \prec \Sigma$.□ (6) Let $x \in \Sigma''$ and let t be a term as in (1): $\tau(t) = \tau_0(x)$, $s_B(t) = \phi$ and $s_F(t) \subset \Sigma'$. From $\Sigma' \prec \Sigma$ we have $s_F(t) \prec \Sigma$ hence for some α, $\alpha: s_F(t) \prec \Sigma$. Put $s = S(\alpha)(t)$, then by (4) $s_F(s) \subset \Sigma$; also $\tau(s) = \tau(t) = \tau_0(x)$. Hence Σ absorbs x.□ (7) Let $x \in \Sigma_1' \cup \Sigma_2'$. If $x \in \Sigma_1'$ then Σ_1 absorbs x hence $\Sigma_1 \cup \Sigma_2$ absorbs x.□ (8) Let $a \in [\mathbb{0},\tau_0(x)]$ then $\tau(a^*) = \tau_0(x)$ and $s_F(a^*) = \phi$.□ (9) $s_F(\pi(\vec{v})) = \{\vec{v}\}$.□

0.6.4.13 We strengthen the valid rules (9), (11) and (17) of 0.6.4.9 and (10) and (14) of 0.6.4.10.

Proposition. The following rules are valid

(1) $\dfrac{\varphi \wedge \psi}{\psi} \quad s_F(\varphi) \prec s_F(\psi)$ (2) $\dfrac{\nabla(\vec{t}) \wedge \psi}{\nabla(\vec{s}) \wedge \psi} \quad s_F(\vec{t}) \prec s_F(\vec{s}\psi)$

(3) $\dfrac{\varphi, \varphi \Rightarrow \psi}{\psi} \quad s_F(\varphi) \prec s_F(\psi)$ (4) $\dfrac{\varphi, \varphi \Rightarrow \psi}{\nabla(\vec{v}) \Rightarrow \psi} \quad s_F(\varphi) \prec s_F(\vec{v}\psi)$.

(5) $\dfrac{\varphi \Rightarrow \psi, \psi \Rightarrow \theta}{\varphi \Rightarrow \theta} \quad s_F(\psi) \prec s_F(\varphi\theta)$

Proof. Put $\varphi' = \mathrm{smpl}(\varphi)$ and $\psi' = \mathrm{smpl}(\psi)$. (1) By 0.6.4.11, $\mathrm{var}(\varphi' \wedge \psi') = \mathrm{var}(\varphi \wedge \psi)$, $s_F(\varphi') \prec s_F(\psi')$ and $|\varphi' \wedge \psi'| = |\varphi \wedge \psi|$. Let α: $s_F(\varphi') \prec s_F(\psi')$, then by (3) of 0.6.4.12, $\mathrm{Mov}(\alpha) \cap s_F(\psi') = \phi$ hence $S(\alpha)(\psi') = \psi'$ by 0.2.5.3; by (4) of 0.6.4.12 $s_F(S(\alpha)(\varphi')) \subset s_F(\psi')$. From $\models \varphi \wedge \psi$ we deduce $\models \varphi' \wedge \psi'$, hence $\models S(\alpha)(\varphi') \wedge \psi'$, hence, by (9) of 0.6.4.9, $\models \psi'$ thus $\models \psi$.□ (2) Let $\vec{v} = \mathrm{var}(\pi(\vec{t}))$ and $\vec{w} = \mathrm{var}(\pi(\vec{s}))$ then $\nabla(\vec{v}) = \nabla(\vec{t})$, $\nabla(\vec{w}) = \nabla(\vec{s})$, $\{\vec{v}\} = s_F(\vec{t})$ and $s_F(\vec{w}\psi) = s_F(\vec{s}\psi)$, so the rule whose validity we wish to establish is

$$\frac{\nabla(\vec{v}) \wedge \psi}{\nabla(\vec{w}) \wedge \psi} \quad \{\vec{v}\} \prec s_F(\vec{w}\psi).$$

From $\models \nabla(\vec{v}) \wedge \psi$ and (8), (12) and (13) of 0.6.4.9 we have $\models \nabla(\vec{v}) \wedge (\nabla(\vec{w}) \wedge \psi)$, hence by (2) of 0.6.4.13, $\models \nabla(\vec{w}) \wedge \psi$.□ (3) Assume $\models \varphi$ and $\models \varphi \Rightarrow \psi$. By (8) of 0.6.4.9, $\models \varphi \wedge (\varphi \Rightarrow \psi)$; by (10) of 0.6.4.9 $\models \varphi \wedge \psi$; by (1) above $\models \psi$. (4) Assume $\models \varphi$, $\models \varphi \Rightarrow \psi$ and $s_F(\varphi) \prec s_F(\vec{v}\varphi)$. By (8), (10), (12) and (13) of 0.6.4.9, $\models \varphi \wedge (\nabla(\vec{v}) \wedge \psi)$; by (2) of 0.6.4.13, $\models \nabla(\vec{v}) \wedge \psi$; hence by (13) of 0.6.4.10, $\models \nabla(\vec{v}) \Rightarrow \psi$.□ (5) Assume $\models \varphi \Rightarrow \psi$, $\models \psi \Rightarrow \theta$ and $s_F(\psi) \prec s_F(\varphi\theta)$. By (8) of 0.6.4.9, $\models (\varphi \Rightarrow \psi) \wedge (\psi \Rightarrow \theta)$. By (H4) of 0.6.4.7 $\models ((\varphi \Rightarrow \psi) \wedge (\psi \Rightarrow \theta)) \Rightarrow (\varphi \Rightarrow \theta)$. From (7) of 0.6.4.12, $s_F(\varphi\psi\theta) \prec s_F(\varphi\theta)$. Hence by (3), $\models \varphi \Rightarrow \theta$.□

Note. Rule (3) is known as modus ponens.

<u>0.6.5 An analogue to the deduction theorem.</u> The first proposition we prove in this section will have the same form and role as the deduction theorem for syntactic systems (cf. [RS] 9.1, Ch. X, p. 432). We state a simple case of what we shall prove. Let φ and ψ be formulas with $s_F(\varphi) = s_F(\psi) = \{x\}$; the following are equivalent:

(1) $\models \varphi \Rightarrow \psi$

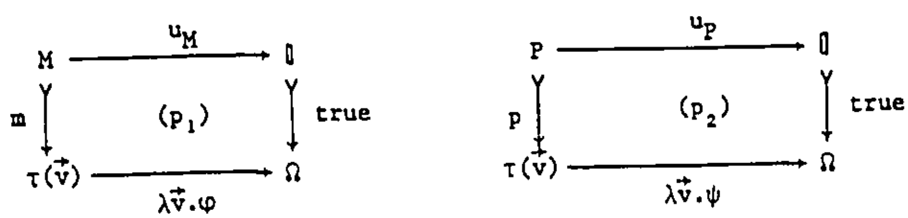
(2) $\models \varphi[x|t] \longrightarrow \models \psi[x|t]$ for all terms t which are free to be substituted for x in $\varphi \Rightarrow \psi$.

This case of the proposition is not difficult to prove, however, technical difficulties arise when we have to sort through several variables. These have been dealt with largely in 0.5.5.3 and 0.5.5.4.

<u>0.6.5.1 Proposition.</u> Let φ and ψ be formulas, $\vec{v}$ a string of distinct variables such that

$$\emptyset \neq s_F(\varphi) \;,\quad s_F(\varphi) \subset s_F(\psi) \;,\quad s_F(\psi) = \underline{\{\vec{v}\}} \;.$$

Let (p_1) and (p_2) be pullback diagrams



Let $n = lg(\vec{v})$; for each $i \in [n]$ let v_i be the $i + 1$ variable in the string $\vec{v}$ ($v_i = (\vec{v}(i))^{\wedge}$ in the set-theoretic representation). Choose a variable, y, of type M which is neither free nor bound in $\varphi \Rightarrow \psi$. Let $m_i = (\lambda\vec{v}.v_i)\circ m$ for each $i \in [n]$. Define β: Vbls $\longrightarrow$ Tms by

$\beta(v_i) = m_i y \qquad$ for $i \in [n]$

$\beta(x) = x \qquad$ for $x \notin \{\vec{v}\}$.

The following are equivalent:

(1) m factors through p

(2) $\models \varphi \Rightarrow \psi$

(3) If α is any substitution admissable for $\varphi \Rightarrow \psi$ then $\models S(\alpha)(\varphi)$ implies $\models S(\alpha)(\psi)$.

(4) $\models S(\beta)(\psi)$.

Proof. (1) iff $[\![m]\!] \le [\![p]\!]$ iff $[\![\vec{v}|\varphi]\!] \le [\![\vec{v}|\psi]\!]$ iff $[\![\vec{v}|\varphi \Rightarrow \psi]\!] = [\![\tau(\vec{v})]\!]$ iff (2).□

(2) ⟶ (3): Let α be as in (3) and suppose $\models S(\alpha)(\varphi)$. From (2) we have $\models S(\alpha)(\varphi) \Rightarrow S(\alpha)(\psi)$, since $s_F(S(\alpha)(\varphi)) \subset s_F(S(\alpha)(\psi)$ we have $\models S(\alpha)(\psi)$.□

(3) ⟶ (4). We prove $\models S(\beta)(\varphi)$. Define φ_j ($j \in [n+1]$) as follows

$\varphi_0 = \varphi \qquad \varphi_{j+1} = \varphi_j[v_{n-j+1} | m_{n-j+1}y] \quad$ for $j \in [n]$

then $\lambda y.\varphi_n = (\lambda\vec{v}.\varphi)\circ m$ by 0.5.5.4

hence $\lambda y.S(\beta)(\varphi) = \lambda y.\varphi_n$ by Corollary 0.2.6.2 since $y \notin \{\vec{v}\}$

$= true_M$ by diagram (p_1)

hence $\models \nabla(y) \wedge S(\beta)(\varphi)$

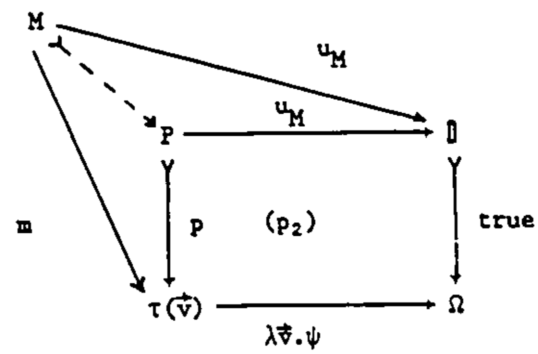
hence $\models S(\beta)(\varphi)$ since $s_F(\varphi) \neq \phi$, $s_F(S(\beta)(\varphi)) = \{y\}$.

By (3) we have $\models S(\beta)(\psi)$.□

(4) ⟶ (1). Define ψ_j ($j \in [n+1]$) by $\psi_0 = \psi$ and

$\psi_{j+1} = \psi_j[v_{n-j+1} | m_{n-j+1}y]$ for $j \in [n]$

then $S(\beta)(\psi) = \psi_n$ by 0.2.6.2 since $y \notin \{\vec{v}\}$ and $\lambda y.\psi_n = (\lambda\vec{v}.\psi)\circ m$ by 0.5.5.4 hence $(\lambda\vec{v}.\psi)\circ m = true_M$ by (4)



Since (p_2) is a pullback diagram m factors through p .□

0.6.5.2. We call β in 0.6.5.1 a normalized substitution for $\varphi \Rightarrow \psi$. More precisely we define (1) a triple $\langle y,m,\vec{v}\rangle$ to be compatible if $\vec{v}$ is a reduced string of variables and $m: \tau_0(y) \rightarrowtail \tau(\vec{v})$ is a mono, and

(2) Sbs $\langle y,m,\vec{v}\rangle$: Vbls ⟶ Tms by $(\text{Sbs}\langle y,m,\vec{v}\rangle)(x) = x$ for $x \notin \{\vec{v}\}$

$(\text{Sbs}\langle y,m,\vec{v}\rangle)(x) = ((\lambda\vec{v}.x)\circ m)y$ for $x \in \{\vec{v}\}$.

(3) A compatible triple $\langle y,m,\vec{v}\rangle$ will be called normal for a formula φ if

(a) $s_F(\varphi) \subset \{\vec{v}\}$

(b) $m \in [\![\vec{v}|\varphi]\!]$

(c) $y \notin s_{FB}(\vec{v})$.

(4) A normalized substitution for $\varphi \Rightarrow \psi$ is then a function Sbs $\langle y,m,\vec{v}\rangle$ where $\langle y,m,\vec{v}\rangle$ is a normal compatible triple for φ and

(d) $\{\vec{v}\} = s_F(\varphi \Rightarrow \psi)$

(e) $y \notin s_{FB}(\varphi \Rightarrow \psi)$.

0.6.5.3 Proposition. Let $\vec{v}$ be reduced, $\{\vec{v}\} = s_F(\varphi\psi)$, $s_F(\psi) \neq \phi$, $\beta = \text{Sbs}\langle y,m,\vec{v}\rangle$ a normalized substitution for $\varphi \Rightarrow \psi$, and $p \in [\![\vec{v}|\psi]\!]$. Then the following are equivalent:

(1) m factors through p

(2) $\models \varphi \Rightarrow \psi$

(3) $\models S(\beta)(\psi)$

Proof. Put $\varphi' = \varphi \wedge \nabla(\vec{v})$ and $\psi' = \psi \wedge \nabla(\vec{v})$ and apply 0.6.5.1 to $\varphi' \Rightarrow \psi'$.□

0.6.5.4 Proposition. Let $s_F(\psi) \neq \phi$, $\{\vec{v}\} = s_F(\psi)$, $A = \tau(\vec{v})$, $\langle y,id_A,\vec{v}\rangle$ a compatible triple, $\beta = \mathrm{Sbs}\langle y,id_A,\vec{v}\rangle$, $y \notin s_{FB}(\psi)$. The following are equivalent:

(1) $\models \psi$

(2) $\models S(\beta)(\psi)$.

Proof. Put $\varphi = \nabla(v_0)$, where $v_0 = (\vec{v}(0))^\wedge$, and apply 0.6.5.3.□

(Notu that the β in 0.6.5.4 has a particularly simple form: for $x \in \{\vec{v}\}$, $\beta(x) = (\lambda\vec{v}.x)y)$.

0.6.5.5. In 0.6.5.1 the hypotheses on φ and ψ are necessary to mantain the equivalence of (2) and (3). For example if the topos has a global section $1 \longrightarrow \Omega$ which is neither true nor false and hence a simplified term e for which $|e|$ is the global section then $\not\models e \Rightarrow \bot$, whereas $\models e$ implies $\models \bot$; thus (3) may hold while (2) is false. To show (2) may hold while (3) is false take x to be a variable of type 0 then $\models (x = x) \Rightarrow \bot$, but $\models x = x$ and $\not\models \bot$. If we modify these two examples by putting $\psi = \neg (y = y)$ where $\tau_0(y) = 1$ (instead of $\psi = \bot$), we still have $(3) \not\Rightarrow (2)$ and $(2) \not\Rightarrow (3)$.

0.6.5.6. In the Corollary of 0.6.4.9 we established rules of inference by applying modus ponens (rule (9) of 0.6.4.9) to tautologies of 0.6.4.8. By using 0.6.5.1 we reverse the process, establishing the validity of formulas $\varphi \Rightarrow \psi$ given that $\models S(\alpha)(\varphi)$ implies $\models S(\alpha)(\psi)$ for all α admissable for $\varphi \Rightarrow \psi$.

Proposition. Let p and q be distinct variables of type Ω. The following are valid formulas of $\Phi(\mathcal{E})$:

(1) $p \Leftrightarrow (p = \top)$

(2) $(p = q) \Leftrightarrow (p \Leftrightarrow q)$

(3) $p = (p = \top)$

(4) $(p = q) = (p \Leftrightarrow q)$

Proof. (1): by rules (2) and (5) of 0.6.4.9 and 0.6.5.1.□ (2): by rules (3) and (5) of 0.6.4.9 and 0.6.5.1.□ (3): by rule (2) of 0.6.4.9 applied to (1).□ (4): by rule (2) of 0.6.4.9 applied to (2).□

0.6.6 Extending the metatheorem. So far the metatheorem of 0.6.3.8 only appears to have been useful in establishing the validity of equations in $\Phi(\mathcal{E})$; that is, equality is the only predicate of our formula elphabet Heq and the only formulas used are atomic. By using 0.6.5.1, the analogue of the deduction theorem, we will be able to extend the metatheorem to basic Horn formulas and thus to other predicates such as $\leq$ (partial order) which require basic Horn formulas to axiomatize.

0.6.6.1 Proposition. Let $\mathcal{A} = \langle A,o,p\rangle$ be an internal structure of similarity type $\langle O,P\rangle$, $\mathbb{P} = \mathbb{P}\langle O,P\rangle$ and $\varphi \in bHf(\mathbb{P})$, then $\mathcal{A} \models \varphi$ implies for all B, $[B,\mathcal{A}] \models \varphi$.

Proof. By 0.6.3.6 (2) the proposition holds for $\varphi \in \mathrm{At\,Fml}(\mathbb{P})$ so we assume $\varphi = \left[\bigwedge_{i=0}^{n-1} \varphi^i \Rightarrow \psi\right]$ with $n \geq 1$, where $\varphi^i \in \mathrm{At\,Fml}(\mathbb{P})$ for $0 \leq i \leq n-1$, and $\psi \in \mathrm{At\,Fml}(\mathbb{P})$. By hypothesis $\models \bigwedge_{i=0}^{n-1} \overline{\varphi^i} \Rightarrow \overline{\psi}$.

For B an object we want to show $[B,\mathcal{A}] \models \varphi$; that is: for each valuation $\beta \in [B,A]^V$, if for $0 \leq i \leq n-1$ β satisfies φ^i in $[B,\mathcal{A}]$, then β satisfies ψ in $[B,\mathcal{A}]$.

Choose $y \in V_B$ and put $\overline{\beta} = \beta_y$, then our assumption on β and each φ^i is equivalent, by 0.6.3.6 (1) and 0.6.4.10 (8.2), to:

$$\models \nabla(y) \wedge S(\overline{\beta})(\overline{\varphi^i}) \qquad \text{for } 0 \leq i \leq n-1.$$

Applying rules (8), (13), (12), (11) of 0.6.4.9 repeatedely we get

$$\models S(\overline{\beta}) \bigwedge_{i=0}^{n-1} \overline{\varphi^i} \wedge \nabla(y).$$

From our original hypothesis we have, by (19) of 0.6.4.9,

$$\models S(\overline{\beta}) \bigwedge_{i=0}^{n-1} \overline{\varphi^i} \wedge \nabla(y) \Rightarrow S(\overline{\beta})(\overline{\psi}) \wedge \nabla(y).$$

Hence by modus ponens $\models S(\overline{\beta})(\psi) \wedge \nabla(y)$. Thus β satisfies ψ in $[B,\mathcal{A}]$.□

0.6.6.2 Proposition. Let $\mathbb{P}$ be a formula elphabet with O_0 its set of nullary operation signs. Then the following are equivalent:

(1) $O_0 = \phi$

(2) $s_F(t) \neq \phi$ for all $t \in \mathrm{Poly}(\mathbb{P})$

(3) $s_F(\varphi) \neq \phi$ for all $\varphi \in \mathrm{AtFml}(\mathbb{P})$

(4) $s_F(\varphi) \neq \phi$ for all $\varphi \in \mathrm{bHf}(\mathbb{P})$.

Proof. (1) $\rightarrow$ (2). We proceed by induction on the length of polynomials. For $t = v \in V$, $s_F(t) \neq \phi$. For t of the form either fr or $g/r,s/$, by induction, $s_F(r) \neq \phi$, hence $s_F(t) = \phi$.□

(2) → (1). Suppose $O_0 \neq \phi$ then we have a term e^* where $e \in O_0$, and $s_F(e^*) = \phi$.□

(2) → (3). If φ is of the form either fr or $g(r,s)$ where f is a unary predicate sign and g is a binary predicate sign then since $s_F(r) \neq \phi$, $s_F(\varphi) \neq \phi$.□

(3) → (2). For any $t \in \mathrm{Poly}(\mathbb{P})$, $(t = t) \in \mathrm{AtFml}(\mathbb{P})$, hence $s_F(t) = s_F(t = t) \neq \phi$.□

(3) → (4). Let $\bigwedge\vec{\varphi} \to \varphi$ be a basic Horn formula then its free variables include those of φ hence $s_F(\bigwedge\vec{\varphi} \to \varphi) \neq \phi$.□

(4) → (3). $\mathrm{AtFml}(\mathbb{P}) \subset \mathrm{bHf}(\mathbb{P})$.□

<u>0.6.6.3.</u> Let $\mathcal{A} = \langle A,o,p\rangle$ be an internal structure of similarity type $\langle O,P\rangle$, $\mathbb{P} = \mathbb{P}\langle O,P\rangle$. By 0.6.6.2 if there is an atomic formula ψ having no free variables then there must be a nullary operation in O_0, hence $[\emptyset,A] \neq \phi$. We shall prove that basic Horn formulas having no free variables are in some sense equivalent to bHf's which do have free variables.

<u>Definition.</u> With the above notation, let $v \in V$, for $\psi \in \mathrm{AtFml}(\mathbb{P})$ put $\mathrm{Eq}(v)(\psi) = ((v = v) \to \psi)$, for $\left[\bigwedge_{i=0}^{n-1} \varphi^i \to \psi\right] \in \mathrm{bHf}(P)$ put

$$\mathrm{Eq}(v)\left[\bigwedge_{i=0}^{n-1} \varphi^i \to \psi\right] = \left[\left[\bigwedge_{i=0}^{n-1} \varphi^i\right] \wedge (v = v)\right] \to \psi$$

<u>Proposition.</u> (1) Let $\mathcal{A} = \langle A,o,p\rangle$ be an <u>external</u> structure of similarity type $\langle O,P\rangle$, β a valuation in $\mathcal{A}$, $\varphi \in \mathrm{bHf}(\mathbb{P}\langle O,P\rangle)$; then β satisfies φ in $\mathcal{A}$ iff β satisfies $\mathrm{Eq}(v)(\varphi)$ in $\mathcal{A}$.

(2) Let $\mathcal{A} = \langle A, o, p\rangle$ be an internal structure of similarity type $\langle O, P\rangle$, $\varphi \in bHf(\mathbb{P}\langle O,P\rangle)$; then $\mathcal{A} \models \varphi$ iff $\mathcal{A} \models Eq(v)(\varphi)$.

Proof. For each bHf φ we put $\varphi' = Eq(v)(\varphi)$.

(1) Suppose φ is atomic so that $\varphi' = (v = v) \Rightarrow \varphi$, and let $\beta \in A^V$. β satisfies $(v = v) \Rightarrow \varphi$ in $\mathcal{A}$ iff $\beta(v) = \beta(v)$ implies β satisfies φ in $\mathcal{A}$ iff β satisfies φ in $\mathcal{A}$. For non-atomic bHf's we have β satisfies $\bigwedge\vec{\varphi} \wedge (v = v) \Rightarrow \psi$ in $\mathcal{A}$ iff both β satisfies $\bigwedge\vec{\varphi}$ in $\mathcal{A}$ and $\beta(v) = \beta(v)$ imply β satisfies ψ in $\mathcal{A}$ iff β satisfies $\bigwedge\vec{\varphi} \Rightarrow \psi$ in $\mathcal{A}$.□

(2) Suppose φ is atomic and let $x = \bar{v}$. We claim $s_F(\bar{\varphi})$ absorbs x. If $s_F(\bar{\varphi}) \neq \phi$ then any $y \in s_F(\varphi)$ has type A -the same type as that of x- hence $\{x\} \prec s_F(\bar{\varphi})$. If $s_F(\bar{\varphi}) = \phi$ then there must be a nullary operation sign e in the similarity type O, and thus a term $\overline{e^*}$ of type A having no free variables, hence $\{x\} \prec s_F(\bar{\varphi})$ by (8) of 0.6.4.12. Thus $\mathcal{A} \models \varphi'$ iff $\models (x = x) \Rightarrow \bar{\varphi}$ iff $\models \bar{\varphi}$ iff $\mathcal{A} \models \varphi$, by (3) of 0.6.4.13 since $\models x = x$ (from (2) of 0.6.4.7), and by (21) of 0.6.4.9. For non-atomic bHf's we have $\models \left[\bigwedge_{i=0}^{n-1} \overline{\varphi^i}\right] \wedge (x = x) \Rightarrow \bar{\psi}$ iff $\models (x = x) \Rightarrow \left[\bigwedge_{i=0}^{n-1} \overline{\varphi^i} \Rightarrow \bar{\psi}\right]$, by (14) and (15), iff $\models \bigwedge_{i=0}^{n-1} \overline{\varphi^i} \Rightarrow \bar{\psi}$ for the same reason as for the atomic case.□

0.6.6.4 Proposition. Let $\varphi \in bHf(\mathbb{P}\langle O,P\rangle)$, $\mathcal{A} = \langle A,o,p\rangle$, an internal algebra of similarity type $\langle O,P\rangle$. If $[B,\mathcal{A}] \models \varphi$ for each B, then $\mathcal{A} \models \varphi$.

Proof. By 0.6.6.3 it suffices to consider basic Horn formulas

$\varphi = \bigwedge_{i=0}^{n-1} \varphi^i \Rightarrow \psi$ which are not atomic ($n \geq 1$), and for which φ^{n-1} has at least one free variable. Let $\vec{u} = \mathrm{var}(\varphi)$, $\vec{v} = \mathrm{var}(\overline{\varphi})$ so that if $u_i = (\vec{u}(i))^\wedge$ and $\vec{v} = (\vec{v}(i))^\wedge$ for $1 \leq i+1 \leq \ell(\vec{u})$ then $\overline{u}_i = v_i$. Let $\psi_1 = \left[\bigwedge_{i=0}^{n-1} \varphi^i\right] \wedge \nabla(\vec{v})$ and $\psi_2 = \overline{\psi} \wedge \nabla(\vec{v})$. We shall prove $\models \psi_1 \Rightarrow \psi_2$.

Let $m \in [\![\vec{v}|\psi_1]\!]$, $m\colon M \rightarrowtail \tau(\vec{v})$, $y \in V_M$ so that $y \notin \{\vec{v}\}$, $m_i = (\lambda\vec{v}.v_i)\circ m$, and $\beta\colon \mathrm{Vbls} \longrightarrow \mathrm{Tms}$ given by $\beta(\omega) = \omega$ for $\omega \notin \{\vec{v}\}$, $\beta(v_i) = m_i y$ for $1 \leq i+1 \leq \ell(\vec{v})$ -that is β is a normalized substitution for $\psi_1 \Rightarrow \psi_2$. Define $k\colon V \rightarrow [M,\mathcal{A}]$ by $k(u_i) = m_i$ for $1 \leq i+1 \leq \ell(\vec{u})$ and $k(u) = m_0$ for $u \notin \{\vec{u}\}$. Let $\overline{k} = k_y\colon \mathrm{Vbls} \longrightarrow \mathrm{Tms}$, as in 0.6.3.4, that is: $\overline{k}(\overline{u}) = (k(u))y$ for $u \in V$ and $\overline{k}(\omega) = \omega$ for $\omega \notin V_A$. Then $\beta(v_i) = m_i y = (k(u_i))y = \overline{k}(v_i)$ for $v_i \in \{\vec{v}\}$, hence $\beta|\{\vec{v}\} = \overline{k}|\{\vec{v}\}$, hence by 0.2.5.3, $S(\beta)(\psi_i) = S(\overline{k})(\psi_i)$ for $i = 1, 2$. Since β is normalized, $\models S(\beta)(\psi_1)$, that is $\models S(\overline{k})(\psi_1)$. Since ψ_1 has no bound variables we have, by 0.2.8.5,

$$s_F(S(\overline{k})(\psi_1)) = (s_F(\psi_1) - \mathrm{Mov}(\overline{k})) \cup \bigcup\{s_F(\overline{k}(x)) \mid x \in (\mathrm{Mov}(\overline{k}) \cap s_F(\psi_1))\}$$
$$= (\{\vec{v}\} - V_A) \cup \{y\} = \{y\}.$$

By (8) of 0.6.4.9 $\models S(\overline{k})(\psi_1) \wedge \nabla(y)$. Expanding $S(\overline{k})(\psi_1)$ we have $\models \left[\bigwedge_{i=0}^{n-1} S(\overline{k})(\varphi^i)\right] \wedge (S(\overline{k})(\nabla(\vec{v}))) \wedge \nabla(y)$ hence $\models \nabla(y) \wedge S(\overline{k})(\varphi^i)$ for $0 \leq i \leq n-1$, by (11), (12) and (13) of 0.6.4.9. Hence by 0.6.3.6 and 0.6.4.10 (8.2), for each $i \in [n]$ k satisfies φ^i in $[M,\mathcal{A}]$. Since by hypothesis k satisfies $\bigwedge_{i=0}^{n-1} \varphi^i \Rightarrow \psi$ in $[M,\mathcal{A}]$, we infer k satisfies ψ in $[M,\mathcal{A}]$, hence $\models \nabla(y) \wedge S(\overline{k})(\overline{\psi})$, hence $\models \nabla(y) \wedge S(\beta)(\overline{\psi})$, hence $\models S(\beta)(\psi_2)$. By 0.6.5.3, $\models \psi_1 \Rightarrow \psi_2$, hence $\models \overline{\bigwedge_{i=0}^{n-1} \varphi^i} \Rightarrow \overline{\psi}$, that is $\mathcal{A} \models \varphi$.□

<u>Corollary.</u> Let $\Sigma \subset bHf(\mathbb{P}\langle O,P\rangle)$ and let $\mathcal{A}$ be an internal structure of similarity type $\langle O,P\rangle$. If $[B,\mathcal{A}] \models \Sigma$ for each B, then $\models \overline{\Sigma}$.□

<u>0.6.6.5 Metatheorem for basic Horn formulas.</u> Let $\Sigma \cup \Sigma' \subset bHf(\mathbb{P}\langle O,P\rangle)$ and let $\mathcal{A}$ be an internal structure of similarity type $\langle O,P\rangle$, then

(1) $\models \overline{\Sigma}$ iff for all B, $[B,\mathcal{A}] \models \Sigma$.

(2) if $\Sigma \models \Sigma'$ and $\models \overline{\Sigma}$ then $\models \overline{\Sigma'}$

<u>Proof.</u> (1): by 0.6.6.1 and 0.6.6.4. (2) Let $\varphi \in \Sigma'$. Suppose $\Sigma \models \Sigma'$ and $\models \overline{\Sigma}$. Then $\Sigma \models \varphi$ and, by (1), $[B,\mathcal{A}] \models \Sigma$ for all P, hence $[B,\mathcal{A}] \models \varphi$ for all B. By (1) $\models \overline{\varphi}$. Thus $\models \overline{\Sigma'}$.□

<u>0.6.6.6.</u> Using basic Horn formulas we can express the definability of one operation or predicate in terms of others. For example partial order is uniquely determined in models of meet semilattices by the addition to this theory of the two axioms:

$$u \leq v \rightarrow u \wedge v = v$$

and $$u \wedge v = v \rightarrow u \leq v.$$

In order to save on notation we introduce equivalences of atomic formulas:

$$u \leq v \leftrightarrow u \wedge v = v,$$

and show that by adding such formulas to the set of basic Horn formulas, the metatheorem still holds.

<u>Definitions.</u> (1) For φ, ψ atomic formulas of $\mathbb{P}$ we call the string $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$, which we abbreviate to $\varphi \leftrightarrow \psi$, an equivalence of atomic formulas; we let $AtEqv(\mathbb{P}) = \{\varphi \leftrightarrow \psi \mid \varphi, \psi \in AtFml(\mathbb{P})\}$ be the set of all such formulas. (2) For $\varphi \leftrightarrow \psi \in AtEqv(\mathbb{P})$ we let

$\tilde{S}(\varphi \leftrightarrow \psi) = \{\varphi \rightarrow \psi, \psi \rightarrow \varphi\}$, and for $\varphi \in bHf(\mathbb{P})$ we let $\tilde{S}(\varphi) = \{\varphi\}$. Put $bHf^*(\mathbb{P}) = (AtEqv(\mathbb{P}) \cup bHf(\mathbb{P}))$ then for $\Sigma \subset bHf^*(\mathbb{P})$ we let $\tilde{S}(\Sigma) = \cup\{S(\varphi) \mid \varphi \in \Sigma\}$, so that $\tilde{S}(\Sigma) \subset bHf(\mathbb{P})$.

The interpretation in an external structure $\mathcal{A}$ of the connective $\wedge$ has already been given for conjuctions of atomic formulas in 0.2.3.13; it is the same for equivalences of atomic formulas, that is: for every valuation β in $\mathcal{A}$, β satisfies $\varphi \leftrightarrow \psi$ in $\mathcal{A}$ iff both β satisfies $\varphi \rightarrow \psi$ in $\mathcal{A}$ and β satisfies $\psi \rightarrow \varphi$ in $\mathcal{A}$. It follows that $\mathcal{A} \models \varphi \leftrightarrow \psi$ iff both $\mathcal{A} \models \varphi \rightarrow \psi$ and $\mathcal{A} \models \psi \rightarrow \varphi$, and, for $\Sigma \subset bHf^*(\mathbb{P})$, $\mathcal{A} \models \Sigma$ iff $\mathcal{A} \models \tilde{S}(\Sigma)$. The internal interpretation works the same way: for $\mathcal{B}$ an internal structure and $\varphi \leftrightarrow \psi \in AtEqv(\mathbb{P})$, $\overline{\varphi \leftrightarrow \psi} = \overline{\varphi} \leftrightarrow \overline{\psi}$ is a formula, and $\mathcal{B} \models \varphi \leftrightarrow \psi$ iff both $\mathcal{B} \models \varphi \rightarrow \psi$ and $\mathcal{B} \models \psi \rightarrow \varphi$, by (8) and (11) of 0.6.4.9, hence for $\Sigma \subset bHf^*(P)$, $\mathcal{B} \models \Sigma$ iff $\mathcal{B} \models \tilde{S}(\Sigma)$. It is now clear that the metatheorem must hold with $bHf^*(\mathbb{P})$ in place of $bHf(\mathbb{P})$:

(1) $\models \overline{\Sigma}$ iff $\models \overline{\tilde{S}(\Sigma)}$ iff for all B, $[B,\mathcal{A}] \models S(\Sigma)$ iff for all B, $[B,\mathcal{A}] \models \Sigma$.

(2) The proof is exactly the same as in 0.6.6.5.□

<u>0.6.7 Change of similarity type.</u> Suppose L is an object in a topos and $\wedge_L$, $\vee_L$ and $\vee'_L$ are morphisms from $L \times L$ to L, such that both the internal structure determined by $\langle L, \wedge_L, \vee_L \rangle$ and the internal structure determined by $\langle L, \wedge_L, \vee'_L \rangle$ satisfy the lattice axioms. For external structures we know that $\vee_L$ and $\vee'_L$ will coincide, and thus that the lattice axioms define $\vee$ implicitly (in the terminology of Model Theory ([CK] p. 87)). In order to prove that for internal lattices we also have $\vee_L = \vee'_L$, we shall set up an "expansion" of the similarity type for lattices, so that if Σ' is the set of equations produced by replacing each occurance of $\vee$ by $\vee'$, in the equations belonging to the set Σ of lattice identities, then, in the expanded similarity type,
$\Sigma \cup \Sigma' \models u \vee v = u \vee' v$, hence by the metatheorem $\models x \vee_L y = x \vee'_L y$, so $\vee_L = \vee'_L$.

In this section we shall set up the necessary notational machinery to apply the metatheorem to implicity definable operations and predicates in full generality. As we proceed we shall be making some adjustments to our original notation; in particular we must be explicit about which polynomial algebra a valuation β is extended to when a set has two or more possible algebraic structures on it. We now introduce some abbreviations.

We call similarity types $\mathcal{O}$, for algebras, algebraic similarity types, and we put $|\mathcal{O}| = \mathcal{O}_0 \cup \mathcal{O}_1 \cup \mathcal{O}_2$ where $\mathcal{O}_i = \mathcal{O}^{-1}\{i\}$ $(i = 0, 1, 2)$; we call similarity types $\langle \mathcal{O}, \mathcal{P} \rangle$, for structures, structural similarity types; we put $|\mathcal{P}| = \mathcal{P}_1 \cup \mathcal{P}_2$ where $\mathcal{P}_j = \mathcal{P}^{-1}\{j\}$ $(j = 1, 2)$, and $|\langle \mathcal{O}, \mathcal{P} \rangle| = |\mathcal{O}| \cup |\mathcal{P}|$. For B a set we put $\mathrm{Opn}(B) = B \cup [B,B] \cup [B^2,B]$,

$Prd(B) = [B,[2]] \cup [B^2,[2]]$, and $OpPr(B) = Opn(B) \cup Prd(B)$. For $\mathcal{B}$ an object we put $Opn(B) = [1,B] \cup [B,B] \cup [B^2,B]$, $Prd(B) = [B,\Omega] \cup [B^2,\Omega]$, and $OpPr(B) = Opn(B) \cup Prd(B)$.

0.6.7.1 Maps of similarity types.

Definition. (1) Let $\mathcal{O}$ and $\overline{\mathcal{O}}$ be algebraic similarity types, for each $i \in [3]$ let $\mathcal{O}_i$ and $\overline{\mathcal{O}}_i$ be the i-ary operation signs of $\mathcal{O}$ and $\overline{\mathcal{O}}$ respectively, and let $\theta: |\mathcal{O}| \longrightarrow |\overline{\mathcal{O}}|$ be a function such that for each $i \in [3]$, $\theta(\mathcal{O}_i) \subset \overline{\mathcal{O}}_i$ (i.e. $\overline{\mathcal{O}} \cdot \theta = \mathcal{O}$); $\langle\theta,\mathcal{O},\overline{\mathcal{O}}\rangle$ will be called a map of algebraic similarity types.

(2) Let $\langle\mathcal{O},\mathcal{P}\rangle$ and $\langle\overline{\mathcal{O}},\overline{\mathcal{P}}\rangle$ be structural similarity types, for each $j \in \{1, 2\}$ let $\mathcal{P}_j$ and $\overline{\mathcal{P}}_j$ be the j-ary predicate sign sets of $\mathcal{P}$ and $\overline{\mathcal{P}}$ respecively, and let $\theta: |\langle\mathcal{O},\mathcal{P}\rangle| \longrightarrow |\langle\overline{\mathcal{O}},\overline{\mathcal{P}}\rangle|$ be a function such that for each $i \in [3]$, $\theta(\mathcal{O}_i) \subset \overline{\mathcal{O}}_i$, and for each $j \in \{1, 2\}$, $\theta(\mathcal{P}_j) \subset \overline{\mathcal{P}}_j$; then $\langle\theta,\langle\mathcal{O},\mathcal{P}\rangle,\langle\overline{\mathcal{O}},\overline{\mathcal{P}}\rangle\rangle$ will be called a map of structural similarity types. We let $\theta_1: |\mathcal{O}| \longrightarrow |\overline{\mathcal{O}}|$ be the restriction of θ to $|\mathcal{O}|$ then $\langle\theta_1,\mathcal{O},\overline{\mathcal{O}}\rangle$ is a map of algebraic similarity types.

(3) Let $\mathcal{B} = \langle B,\gamma\rangle$ be an $\overline{\mathcal{O}}$-algebra. This means $\gamma: |\overline{\mathcal{O}}| \longrightarrow Opn(B)$ is a function such that, if $\mathcal{B}$ is external then $\gamma(\overline{\mathcal{O}}_0) \subset B$ and $\gamma(\overline{\mathcal{O}}_i) \subset [B^i,B]$ for each $i \in \{1, 2\}$, and if $\mathcal{B}$ is internal then $\gamma(\overline{\mathcal{O}}_i) \subset [B^i,B]$ for each $i \in [3]$. Let $\langle\theta,\mathcal{O},\overline{\mathcal{O}}\rangle$ be a map of algebraic similarity types we call $\mathcal{B}^\theta = \langle B,\gamma\cdot\theta\rangle$ the $\mathcal{O}$-algebra θ-induced by $\mathcal{B}$.

(4) Let $\mathcal{B} = \langle B,\gamma\rangle$ be an $\langle\overline{\mathcal{O}},\overline{\mathcal{P}}\rangle$-structure. If $\mathcal{B}$ is external this means γ is a function with domain $|\langle\overline{\mathcal{O}},\overline{\mathcal{P}}\rangle|$ which when restricted to operation signs behaves as in (3), and for predicate signs $\gamma(\mathcal{P}_j) \subset [B^j,[2]]$

($j = 1, 2$). If $\mathcal{B}$ is internal, γ has domain $|\langle\overline{\mathcal{O}},\overline{P}\rangle|$, behaves as in (3) when restricted to operation signs, and for predicate signs $\gamma(P_j) \subset [A^j,\Omega]$ ($j = 1, 2$). In both internal and external cases we call $\mathcal{B}^\theta = \langle B,\gamma\cdot\theta\rangle$, the $\langle\mathcal{O},P\rangle$-structure θ-induced by $\mathcal{B}$.

(5) The map $\langle\theta,\mathcal{O},\overline{\mathcal{O}}\rangle$ of algebraic similarity types induces an $\mathcal{O}$-homomorphism $\underline{\theta}$: $\mathrm{Poly}_{\mathcal{O}}(V) \longrightarrow (\mathrm{Poly}_{\overline{\mathcal{O}}}(V))^\theta$ by extending the inclusion $V \subset |(\mathrm{Poly}_{\overline{\mathcal{O}}}(V))^\theta|$.

(6) For $\langle\theta,\langle\mathcal{O},P\rangle,\langle\overline{\mathcal{O}},\overline{P}\rangle\rangle$ a map of structural similarity types we let θ_1 be the restriction of θ to operation signs, and define the extension of $\underline{\theta}_1$ to formulas, $\underline{\theta}$: $\mathrm{bHf}^*(P\langle\mathcal{O},P\rangle) \longrightarrow \mathrm{bHf}^*(P\langle\overline{\mathcal{O}},\overline{P}\rangle)$, as follows: For ft atomic, $f \in P_1$, $t \in \mathrm{Poly}_{\mathcal{O}}(V)$, $\underline{\theta}(ft) = (\theta(f))(\underline{\theta}_1(t))$; for $f/t,s/$ atomic; $f \in P_2$, $\{t,s\} \subset \mathrm{Poly}_{\mathcal{O}}(V)$, $\underline{\theta}(f/t,s/) = (\theta(f))/\underline{\theta}_1(t),\underline{\theta}_1(s)/$; for φ^i and ψ atomic, $\bigwedge_i \varphi^i \longrightarrow \psi$ a basic Horn formula,

$\underline{\theta}(\bigwedge_i \varphi^i \longrightarrow \psi) = \bigwedge_i \underline{\theta}(\varphi^i) \longrightarrow \underline{\theta}(\psi)$; for φ^1, φ^2 atomic,

$\underline{\theta}(\varphi^1 \leftrightarrow \varphi^2) = \underline{\theta}(\varphi^1) \leftrightarrow \underline{\theta}(\varphi^2)$.

<u>0.6.7.2 Proposition.</u> Let $\underline{h}$: $\mathcal{B} \longrightarrow \mathcal{C}$ be an $\overline{\mathcal{O}}$-homomorphism of external $\overline{\mathcal{O}}$-algebras, $\mathcal{B} = \langle B,\gamma\rangle$ and $\mathcal{C} = \langle C,\mu\rangle$ with underlying function h: $B \longrightarrow C$; let $\langle\theta,\mathcal{O},\overline{\mathcal{O}}\rangle$ be a map of algebraic similarity types, then h is the carrier of an $\mathcal{O}$-homomorphism $\underline{h}^\theta$: $\mathcal{B}^\theta \longrightarrow \mathcal{C}^\theta$, where $\mathcal{B}^\theta = \langle B,\gamma\cdot\theta\rangle$ and $\mathcal{C}^\theta = \langle C,\mu\cdot\theta\rangle$.

<u>Proof.</u> Let $f \in \mathcal{O}_0$ then $\underline{h}^\theta((\gamma\cdot\theta)(f)) = h(\gamma(\theta(f))) = \mu(\theta(f)) = (\mu\cdot\theta)(f)$. Let $f \in \mathcal{O}_1$ and $a \in B$, then $\underline{h}^\theta(((\gamma\cdot\theta)(f))(a)) = h((\gamma(\theta(f)))(a)) = (\mu(\theta(f)))(h(a)) = ((\mu\cdot\theta)(f))(\underline{h}^\theta(a))$. Let $f \in \mathcal{O}_2$, $\{a,b\} \subset B$, then

$$\underline{h}^{\theta}(((\gamma\cdot\theta)(f))(a,b)) = h((\gamma(\theta(f)))(a,b)) = (\mu(\theta(f)))(h(a),h(b))$$
$$= ((\mu\cdot\theta)(f))(\underline{h}^{\theta}(a),\underline{h}^{\theta}(b)).\square$$

<u>0.6.7.3 Corollary (1).</u> Let $\mathcal{B} = \langle B,\gamma\rangle$ be an external $\overline{\mathcal{O}}$-algebra $\beta \in B^V$, $\beta_{\gamma\cdot\theta} : \mathrm{Poly}_{\mathcal{O}}(V) \longrightarrow \mathcal{B}^{\theta}$ the $\mathcal{O}$-homomorphism extending β, and $\beta_{\gamma} : \mathrm{Poly}_{\overline{\mathcal{O}}}(V) \longrightarrow \mathcal{B}$ the $\overline{\mathcal{O}}$-homomorphism extending β, then

$$(\beta_{\gamma})^{\theta}\cdot\underline{\theta} = \beta_{\gamma\cdot\theta} \quad \text{(i.e. } \beta_{\gamma}\cdot\underline{\theta} = \beta_{\gamma\cdot\theta} \text{ as functions).}$$

<u>Proof.</u> It suffices to show that the $\mathcal{O}$-homomophisms $(\beta_{\gamma})^{\theta}\cdot\underline{\theta}$ and $\beta_{\gamma\cdot\theta}$ agree on the generating set V. Let $u \in V$, then

$\beta_{\gamma}(\underline{\theta}(u)) = \beta_{\gamma}(u) = \beta(u) = \beta_{\gamma\cdot\theta}(u)\square$

<u>0.6.7.4 Corollary (2).</u> Let $\mathcal{A} = \langle A,\gamma\rangle$ be an internal $\overline{\mathcal{O}}$-algebra, $\overline{\gamma}$ its natural extension to an interpretation of the elphabet $\mathbb{E}(\overline{\mathcal{O}})$ of type A in $\Phi(\mathcal{E})$, and $\tilde{\overline{\gamma}} : \mathrm{Poly}(E(\overline{\mathcal{O}})) \longrightarrow T_A$ the $\overline{\mathcal{O}}$-homomophism that extends the canonical monomorphism $i: V \rightarrowtail T_A$ which sends the n-th variable of V to the n-th variable of type A, with T_A construed as an $\overline{\mathcal{O}}$-algebra via γ (see 0.2.3.10 and 0.3.3.3). Let $\langle\theta,\mathcal{O},\overline{\mathcal{O}}\rangle$ be a map of algebraic inter-similarity types, let $\overline{\gamma\cdot\theta}$ be the natural extension of $\gamma\cdot\theta$ to an interpretation of $\mathbb{E}(\mathcal{O})$ of type A in $\Phi(\mathcal{E})$, and $\widetilde{\overline{\gamma\cdot\theta}} : \mathrm{Poly}(\mathbb{E}(\overline{\mathcal{O}})) \longrightarrow T_A$ the $\mathcal{O}$-homomorphism extending $i: V \rightarrowtail T_A$ with T_A construed as an $\mathcal{O}$-algebra via $\gamma\cdot\theta$. Then

$$\tilde{\overline{\gamma}}^{\theta}\circ\underline{\theta} = \widetilde{\overline{\gamma\cdot\theta}}$$

and for each polynomial $t \in \mathrm{Poly}\ (V)$

$$\tilde{\overline{\gamma}}(\underline{\theta}(t)) = \widetilde{\overline{\gamma\cdot\theta}}(t).$$

<u>Proof.</u> Put $\beta = i$ in Corollary (1), then $i_{\gamma} = \tilde{\overline{\gamma}}$ and $i_{\gamma\cdot\theta} = \widetilde{\overline{\gamma\cdot\theta}}$.$\square$

<u>0.6.7.5 Proposition.</u> Let $\mathcal{B} = \langle B,\gamma\rangle$ be an external $\langle\bar{\mathcal{O}},\bar{\mathcal{P}}\rangle$ structure, $\langle\theta,\langle\mathcal{O},\mathcal{P}\rangle,\langle\bar{\mathcal{O}},\bar{\mathcal{P}}\rangle\rangle$ a map of structural similarity types, $\varphi \in \mathrm{AtFml}(\mathbb{P}\langle\mathcal{O},\mathcal{P}\rangle)$, and $\beta \in B^V$. Then, β satisfies $\underline{\theta}(\varphi)$ in $\mathcal{B}$ iff β satisfies φ in $\mathcal{B}^\theta$.

<u>Proof.</u> Suppose $\varphi = f/t,s/ \in \mathrm{AtFml}_2(\mathbb{P}\langle\mathcal{O},\mathcal{P}\rangle)$ (we omit the case when φ is unary atomic), then $\underline{\theta}(f/t,s/) = (\theta(f))/\underline{\theta}_1(t),\underline{\theta}_1(s)/$. Let γ_1 be the restriction of γ to operation signs, put $\beta_{\gamma_1} = \tilde{\beta}$ and $\beta_{\gamma_1\cdot\theta_1} = \tilde{\tilde{\beta}}$.

β satisfies $(\theta(f))/\underline{\theta}_1(t),\underline{\theta}_1(s)/$ in $\mathcal{B}$

iff $(\theta(f))(\tilde{\tilde{\beta}}(\underline{\theta}_1(t)),\tilde{\tilde{\beta}}(\underline{\theta}_1(s))) = 1$ by definition

iff $((\gamma\cdot\theta)(f))(\tilde{\beta}(t),\tilde{\beta}(s)) = 1$ by 0.6.7.3

iff β satisfies $f/t,s/$ in $\mathcal{B}^\theta$.□

<u>0.6.7.6 Proposition.</u> For each $\varphi \in \mathrm{bHf}^*(\mathbb{P}\langle\mathcal{O},\mathcal{P}\rangle)$

$$\mathcal{B} \models \underline{\theta}(\varphi) \quad \text{iff} \quad \mathcal{B}^\theta \models \varphi .$$

<u>Proof.</u> For φ atomic this follows directly from 0.6.7.5. For $\varphi = \bigwedge_i \varphi^i \longrightarrow \psi$ with φ^i, ψ atomic we have:

$\mathcal{B} \models \bigwedge_i \underline{\theta}(\varphi^i) \longrightarrow \underline{\theta}(\psi)$

iff $\beta((i(\beta$ satisfies $\underline{\theta}(\varphi^i)$ in $\mathcal{B})) \longrightarrow (\beta$ satisfies $\underline{\theta}(\psi)$ in $\mathcal{B}))$

iff $\beta((i(\beta$ satisfies φ^i in $\mathcal{B}^\theta)) \longrightarrow (\beta$ satisfies ψ in $\mathcal{B}^\theta))$

iff $\mathcal{B}^\theta \models \bigwedge_i \varphi^i \longrightarrow \psi$.

For $\varphi = \varphi^1 \leftrightarrow \varphi^2$ with φ^i atomic we have

$\mathcal{B} \models \underline{\theta}(\varphi^1) \leftrightarrow \underline{\theta}(\varphi^2)$

iff $\beta((\beta$ satisfies $\underline{\theta}(\varphi^1)$ in $\mathcal{B}) \leftrightarrow (\beta$ satisfies $\underline{\theta}(\varphi^2)$ in $\mathcal{B}))$

iff $\beta((\beta$ satisfies φ^1 in $\mathcal{B}) \leftrightarrow (\beta$ satisfies φ^2 in $\mathcal{B}))$

iff $\mathcal{B}^\theta \models \varphi^1 \leftrightarrow \varphi^2$.□

<u>0.6.7.7 Proposition.</u> Let $\mathcal{A} = \langle A,\gamma\rangle$ be an internal $\langle\overline{\mathcal{O}},\overline{P}\rangle$ structure, $\langle A,\gamma_1\rangle$ the internal $\overline{\mathcal{O}}$-algebra formed by restricting γ to operation signs, $\bar{\gamma}$ the extension of γ to an interpretation of the formula elphabet $\mathbf{P}\langle\overline{\mathcal{O}},\overline{P}\rangle$ of type A in $\Phi(\mathcal{E})$, (so that $\bar{\gamma}$ is an extension of $\overline{\gamma_1}$) and $\tilde{\bar{\gamma}}$ the extension of $\tilde{\bar{\gamma}}_1$ and $\bar{\gamma}$ to basic Horn formulas, $\tilde{\bar{\gamma}} : \mathrm{bHf}^*(\mathbf{P}\langle\overline{\mathcal{O}},\overline{P}\rangle) \longrightarrow T_\Omega$. Let $\langle\theta,\langle\mathcal{O},P\rangle,\langle\overline{\mathcal{O}},\overline{P}\rangle\rangle$ be a map of similarity types, then for all $\varphi \in \mathrm{bHf}^*(\mathbf{P}\langle\mathcal{O},P\rangle)$,

$$\tilde{\bar{\gamma}}(\underline{\theta}(\varphi)) = \widetilde{\overline{\gamma\cdot\theta}}(\varphi) \; .$$

<u>Proof.</u> For $f/t,s/ \in \mathrm{AtFml}_2(\mathbf{P}\langle\mathcal{O},P\rangle)$ (omitting the unary case):

$$\begin{aligned}
\tilde{\bar{\gamma}}(\underline{\theta}(f/t,s/)) &= \tilde{\bar{\gamma}}((\theta(f))/\underline{\theta}_1(t),\underline{\theta}_1(s)/) \\
&= (\gamma(\theta(f)))/\tilde{\bar{\gamma}}_1(\underline{\theta}_1(t)),\tilde{\bar{\gamma}}_1(\underline{\theta}_1(s))/ \\
&= ((\gamma\cdot\theta)(f))/(\widetilde{\overline{\gamma_1\cdot\theta_1}})(t),(\widetilde{\overline{\gamma_1\cdot\theta_1}})(s)/ \quad \text{by } 0.6.7.4 \\
&= \widetilde{\overline{\gamma\cdot\theta}}(f/t,s/) \qquad \text{since } (\gamma\cdot\theta)_1 = \gamma_1\cdot\theta_1 \; .
\end{aligned}$$

For $\bigwedge\varphi^i \longrightarrow \psi$ a basic Horn formula:

$$\begin{aligned}
\tilde{\bar{\gamma}}(\underline{\theta}(\bigwedge\varphi^i \longrightarrow \psi)) &= \tilde{\bar{\gamma}}(\bigwedge\underline{\theta}(\varphi^i) \longrightarrow \underline{\theta}(\psi)) \\
&= \bigwedge\tilde{\bar{\gamma}}(\underline{\theta}(\varphi^i)) \longrightarrow \tilde{\bar{\gamma}}(\underline{\theta}(\psi)) \\
&= \bigwedge\widetilde{\overline{\gamma\cdot\theta}}(\varphi^i) \longrightarrow \widetilde{\overline{\gamma\cdot\theta}}(\psi) \\
&= \widetilde{\overline{\gamma\cdot\theta}}\;(\bigwedge\varphi_i \longrightarrow \psi).
\end{aligned}$$

Similarly $\tilde{\bar{\gamma}}(\underline{\theta}(\varphi^1 \leftrightarrow \varphi^2)) = \widetilde{\overline{\gamma\cdot\theta}}(\varphi^1 \leftrightarrow \varphi^2)$.□

<u>0.6.7.8 Proposition.</u> Let $\mathcal{A} = \langle A,\gamma\rangle$ be an internal $\langle\overline{\mathcal{O}},\overline{P}\rangle$-structure, $\langle\theta,\langle\mathcal{O},P\rangle,\langle\overline{\mathcal{O}},\overline{P}\rangle\rangle$ a map of similarity types, and $\varphi \in \mathrm{bHf}^*(\mathbf{P}\langle\mathcal{O},P\rangle)$, then

$$\mathcal{A}^\theta \models \varphi \quad \text{iff} \quad \mathcal{A} \models \underline{\theta}(\varphi).$$

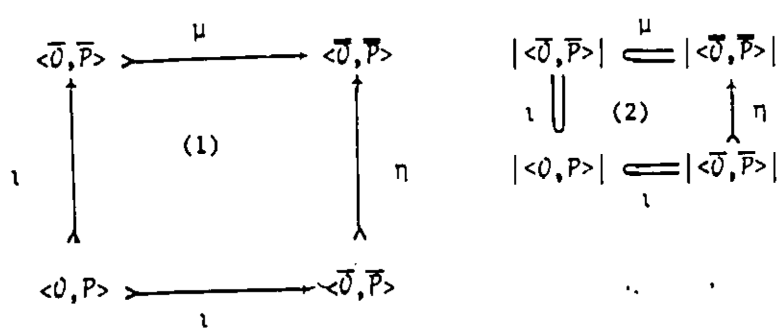
<u>Proof.</u> $\mathcal{A}^\theta \models \varphi$ (where $\mathcal{A}^\theta = \langle A,\gamma\cdot\theta\rangle$) iff $\models \widetilde{\overline{\gamma\cdot\theta}}(\varphi)$ iff $\models \tilde{\bar{\gamma}}(\underline{\theta}(\varphi))$ by 0.6.7.7 iff $\mathcal{A} \models \underline{\theta}(\varphi)$.□

<u>0.6.7.9 Amalgamation of similarity types.</u> We keep the notation for operation and predicate signs introduced in 0.6.7.1. We call a map of similarity types $\langle \iota, \langle \mathcal{O}, P\rangle, \langle \bar{\mathcal{O}}, \bar{P}\rangle\rangle$ an inclusion map if $|\langle \mathcal{O}, P\rangle| \subset |\langle \bar{\mathcal{O}}, \bar{P}\rangle|$ with $\iota(f) = f$ for all $f \in |\langle \mathcal{O}, P\rangle|$. Let $W_\iota = |\langle \bar{\mathcal{O}}, \bar{P}\rangle| - |\langle \mathcal{O}, P\rangle|$. For each $f \in W_\iota$ put $f' = \langle f, \iota\rangle^{\wedge}$, and for each $U \subset W_\iota$ put $U' = \{f' \mid f \in U\}_\iota$, then $|\langle \bar{\mathcal{O}}, \bar{P}\rangle| \cap W'_\iota = \phi$ and the assignment $f \rightsquigarrow f'$ sets up a one-to-one correspondence between W_ι and W'_ι.

We define a new similarity type $\langle \bar{\bar{\mathcal{O}}}, \bar{\bar{P}}\rangle$ with operation signs $\bar{\bar{\mathcal{O}}}_i = \bar{\mathcal{O}}_i \cup (\bar{\mathcal{O}}_i - \mathcal{O}_i)'$ $(i \in [3])$ and predicate signs $\bar{\bar{P}}_j = \bar{P}_j \cup (\bar{P}_j - P_j)'$ $(j \in \{1, 2\})$. We let $\langle \mu, \langle \bar{\mathcal{O}}, \bar{P}\rangle, \langle \bar{\bar{\mathcal{O}}}, \bar{\bar{P}}\rangle\rangle$ be the inclusion map and let $\langle \eta, \langle \bar{\mathcal{O}}, \bar{P}\rangle, \langle \bar{\bar{\mathcal{O}}}, \bar{\bar{P}}\rangle\rangle$ be the map defined by

$$\eta(f) = \begin{cases} f & \text{for } f \in |\langle \mathcal{O}, P\rangle| \\ f' & \text{for } f \in W_\iota \end{cases}$$

For each $f \in |\langle \mathcal{O}, P\rangle|$ we have $(\eta \circ \iota)(f) = f = (\mu \circ \iota)(f)$, that is (1) commutes in the category Sets/[3] × Sets/{1,2}.



$\mu(|\langle \bar{\mathcal{O}}, \bar{P}\rangle|) \cup \eta(|\langle \bar{\mathcal{O}}, \bar{P}\rangle|) = |\langle \bar{\mathcal{O}}, \bar{P}\rangle| \cup |\langle \mathcal{O}, P\rangle| \cup W_\iota = |\langle \bar{\bar{\mathcal{O}}}, \bar{\bar{P}}\rangle|$ and

$\mu(|\langle \bar{\mathcal{O}}, \bar{P}\rangle|) \cap \eta(|\langle \bar{\mathcal{O}}, \bar{P}\rangle|) = |\langle \bar{\mathcal{O}}, \bar{P}\rangle| \cap (|\langle \mathcal{O}, P\rangle| \cup W_\iota) = |\langle \mathcal{O}, P\rangle|$, thus

(1) and (2) are pushouts or amalgamated sums.

<u>0.6.7.10 Amalgamation of structures.</u> Let $\mathcal{A}_k = \langle A,\gamma^k\rangle$ $(k = 1,2)$ be internal $\langle\mathcal{O},\overline{P}\rangle$- structures such that $\gamma^1\cdot\iota = \gamma^2\cdot\iota$, that is $\mathcal{A}_1 = \mathcal{A}_2$, then there is a uniquely determined $\langle\overline{\mathcal{O}},\overline{P}\rangle$- structure $\mathcal{A} = \langle A,\gamma\rangle$ such that (1) $\gamma\cdot\mu = \gamma^1$ and (2) $\gamma\cdot\eta = \gamma^2$, i.e. $\mathcal{A}^{\mu} = \mathcal{A}_1$ and $\mathcal{A}^{\eta} = \mathcal{A}_2$. Using (1) and (2) we can describe γ explicitly.

(1) holds iff $\gamma(f) = \gamma^1(f)$ for all $f \in |\langle\overline{\mathcal{O}},\overline{P}\rangle|$.

(2) holds iff $\gamma(\eta(f)) = \gamma^2(f)$ for all $f \in |\langle\overline{\mathcal{O}},\overline{P}\rangle|$, iff both $\gamma(f) = \gamma^2(f)$ for all $f \in |\langle\mathcal{O},P\rangle|$ and $\gamma(f') = \gamma^2(f)$ for all $f \in W_\iota$.

By hypothesis $\gamma^1(f) = \gamma^2(f)$ for all $f \in |\langle\mathcal{O},P\rangle|$. Thus γ is uniquely determined by

$$\gamma(f) = \begin{cases} \gamma^1(f) & \text{for } f \in W_\iota \\ \gamma^1(f) = \gamma^2(f) & \text{for } f \in |\langle\mathcal{O},P\rangle| \end{cases}$$

and $\gamma(f') = \gamma^2(f)$ for $f' \in W_\iota$.

<u>0.6.7.11 Definition of eq.</u> We keep the notation of 0.6.7.9. Let u, v be the first two variables of V. We define $eq: W_\iota \longrightarrow bHf^{*}(P\langle\overline{\mathcal{O}},\overline{P}\rangle)$ as follows: $eq(k) = (k* = k'*)$ if $k \in \overline{\mathcal{O}}_0 \cap W_\iota$, $eq(k) = (ku = k'u)$ if $k \in \overline{\mathcal{O}}_1 \cap W_\iota$, $eq(k) = (k(u,v) = k'(u,v))$ if $k \in \overline{\mathcal{O}}_2 \cap W_\iota$, $eq(k) = (ku \leftrightarrow k'u)$ if $k \in \overline{P}_1 \cap W_\iota$, $eq(k) = (k(u,v) \leftrightarrow k'(u,v))$ if $k \in \overline{P}_2 \cap W_\iota$.

<u>0.6.7.12 Proposition.</u> Let $\mathcal{B} = \langle B,\gamma\rangle$ be an external $\langle\overline{\mathcal{O}},\overline{P}\rangle$ structure. For all $k \in W_\iota$, $\mathcal{B} \models eq(k)$ iff $\gamma(k) = \gamma(k')$.

<u>Proof.</u> We consider only the cases (1) $k \in \overline{\mathcal{O}}_2 \cap W_\iota$ and (2) $k \in \overline{P}_2 \cap W_\iota$.

(1) For each valuation $\beta \in B^V$, $\tilde{\beta}: Poly_{\overline{\mathcal{O}}}(V) \longrightarrow B$ is an $\overline{\mathcal{O}}$-homomorphism thus for $f \in \overline{\mathcal{O}}_2$, $\tilde{\beta}(f(u,v)) = (\gamma(f))(\beta(u),\beta(v))$ (see 0.2.3.10). For

$k \in \overline{O}_2 \cap W_1$, $\mathcal{B} \models k/u,v/ = k'/u,v/$ iff $\gamma(k)(\beta(u),\beta(v)) = \gamma(k')(\beta(u),\beta(v))$ for all $\beta \in B^V$ iff $\gamma(k) = \gamma(k')$.

(2) For each $\beta \in B^V$, if $f \in \overline{P}_2$, β satisfies $f/u,v/$ in $\mathcal{B}$ iff $(\gamma(f))(\beta(u),\beta(v)) = 1$ (see 0.2.3.13). For $k \in \overline{P}_2 \cap W_2$, $\mathcal{B} \models k/u,v/ \leftrightarrow k'/u,v/$ iff $(\gamma(k)(\beta(u),\beta(v)) = 1$ iff $\gamma(k')(\beta(u),\beta(v)) = 1)$ for all $\beta \in B^V$ iff $\gamma(k) = \gamma(k')$.□

<u>0.6.7.13 Proposition.</u> Let $\mathcal{A} = \langle A,\gamma\rangle$ be an internal $\langle\overline{O},\overline{P}\rangle$ structure. For each $k \in W_1$, $\mathcal{A} \models eq(k)$ iff $\gamma(k) = \gamma(k')$.

<u>Proof.</u> Again we consider only the cases (1) $k \in \overline{O}_2 \cap W_1$ and (2) $k \in \overline{P}_2 \cap W_1$. Let x, y be the first two variables of type A.
(1) $\mathcal{A} \models k/u,v/ = k'/u,v/$ iff $\models (\gamma(k))/x,y/ = (\gamma(k'))/x,y/$ iff $\gamma(k) = |\gamma(k)/x,y/| = |\gamma(k')/x,y/| = \gamma(k')$ by 0.6.4.7-(3) and 0.5.4.2-(5). (2) $\mathcal{A} \models k/u,v/ \leftrightarrow k'/u,v/$ iff $\models \gamma(k)/x,y/ \leftrightarrow \gamma(k')/x,y/$ iff $\models \gamma(k)/x,y/ = \gamma(k')/x,y/$ (by 0.6.4.7-(6)) iff $\gamma(k) = \gamma(k')$.□

<u>0.6.7.14 Proposition.</u> Let $\Sigma \subset bHf^*(P\langle\overline{O},\overline{P}\rangle)$, $k \in |\langle\overline{O},\overline{P}\rangle|$, and $\langle\iota;\langle O,P\rangle,\langle\overline{O},\overline{P}\rangle\rangle$ be an inclusion map, such that for all $\langle\overline{O},\overline{P}\rangle$-structures $\mathcal{B}_1$ and $\mathcal{B}_2$ which are models of Σ and for which $\mathcal{B}_1^\iota = \mathcal{B}_2^\iota$, we have $k_{\mathcal{B}_1} = k_{\mathcal{B}_2}$. If $\mathcal{A}_1$ and $\mathcal{A}_2$ are internal models of Σ for which $\mathcal{A}_1^\iota = \mathcal{A}_2^\iota$, then $k_{\mathcal{A}_1} = k_{\mathcal{A}_2}$.

<u>Proof.</u> Let $\mathcal{A}$ be the $\langle\overline{O},\overline{P}\rangle$ structure formed by amalgamating $\mathcal{A}_1$ and $\mathcal{A}_2$ as in 0.6.7.10. For each $\varphi \in \Sigma$ we have $\mathcal{A}^\eta \models \varphi$ and $\mathcal{A}^\mu \models \varphi$ hence $\mathcal{A} \models \underline{\eta}(\varphi)$ and $\mathcal{A} \models \underline{\mu}(\varphi)$, hence $\mathcal{A} \models \underline{\eta}(\Sigma) \cup \underline{\mu}(\Sigma)$. Let $\mathcal{B}$ be any external $\langle\overline{O},\overline{P}\rangle$ model of $\underline{\eta}(\Sigma) \cup \underline{\mu}(\Sigma)$, then $\mathcal{B}^\mu \models \Sigma$ and $\mathcal{B}^\eta \models \Sigma$ and $(\mathcal{B}^\mu)^\iota = (\mathcal{B}^\eta)^\iota$; thus, by hypothesis, $k_{\mathcal{B}^\mu} = k_{\mathcal{B}^\eta}$. If $k \in |\langle O,P\rangle|$ then

$k_{\mathcal{A}_1} = k_{\mathcal{A}_1^\iota} = k_{\mathcal{A}_2^\iota} = k_{\mathcal{A}_2}$. If $k \in W_\iota$ then $k_{\mathcal{B}} = k_{\mathcal{B}\mu} = k_{\mathcal{B}\eta} = k'_{\mathcal{B}}$, hence $\mathcal{B} \models$ eq(k) (by 0.6.7.12). By the metatheorem $\mathcal{A} \models$ eq(k), hence $k_{\mathcal{A}} = k'_{\mathcal{A}}$, (by 0.6.7.11) hence $k_{\mathcal{A}_1} = k_{\mathcal{A}_2}$.□

<u>0.6.7.15 Independence of validity from extraneous signs.</u> Let $\langle\iota,\langle\mathcal{O},\mathcal{P}\rangle,\langle\overline{\mathcal{O}},\overline{\mathcal{P}}\rangle\rangle$ be an inclusion map, let $\mathcal{A}$ be an $\langle\overline{\mathcal{O}},\overline{\mathcal{P}}\rangle$-structure and let $\Sigma \subset bHf^*(\mathbb{P}\langle\mathcal{O},\mathcal{P}\rangle)$, then $\underline{\iota}(\Sigma) = \Sigma$, and so, by 0.6.7.6, $\mathcal{A}^\iota \models \Sigma$ iff $\mathcal{A} \models \Sigma$ for $\mathcal{A}$ external, and, by 0.6.7.8, $\mathcal{A}^\iota \models \Sigma$ iff $\mathcal{A} \models \Sigma$ for $\mathcal{A}$ internal. This justifies the obvious: that the validity of a formula in a structure is independent of the interpretation in the structure of operation and predicate signs that do not occur in the formula.

<u>0.6.7.16 An application to implicitly defined operations of an internal Heyting algebra.</u> Define Eql = $\{\langle\underline{\delta},2\rangle\}$, take $\langle\mathcal{O},\mathcal{P}\rangle = \langle\{\langle\underline{\wedge},2\rangle\},\text{Eql}\rangle$. and $\langle\overline{\mathcal{O}},\overline{\mathcal{P}}\rangle = \langle H,\text{Eql}\rangle$ (H as in 0.6.4.1). Let Σ(Heyt.) be the set of equations which we indicated in 0.6.4.5 suffice to axiomatize Heyting algebras. In $\mathcal{S}et$ it is a fact that if $\mathcal{B}$ and $\mathcal{C}$ are Heyting algebras with a common underlying set and a common interpretation of the meet operation then $\mathcal{B} = \mathcal{C}$. Hence for internal Heyting algebras, $\mathcal{B}$ and $\mathcal{C}$ with a common interpretation of $\underline{\wedge}$, by 0.6.7.14, $\mathcal{B} = \mathcal{C}$.

Some of the force of the metatheorem is lost if we do not grant such external facts. Thus if we demand a complete proof in $\mathcal{S}et$ that $\mathcal{B} = \mathcal{C}$, with minor adjustments we can produce a satisfactory internal proof -this will become evident once 0.6 is complete .

<u>0.6.8 Partial order.</u> We introduce a new fixed sign $\textcircled{\leq}$, the <u>partial order sign</u>, which be take to be $(\textcircled{=})^{\wedge}$, where $\underline{=}$, defined in 0.6.4.1, is the last such sign. We require that all formula elphabets $\mathbb{P}\langle\mathcal{O},\mathcal{P}\rangle$, in this and future sections, are such that, if $\textcircled{\leq} \in |\langle\mathcal{O},\mathcal{P}\rangle|$ then $\textcircled{\leq}$ is a binary predicate sign of $\mathbb{P}\langle\mathcal{O},\mathcal{P}\rangle$. We define $\text{PartOrd} = \{\langle\underline{\delta},2\rangle,\langle\textcircled{\leq},2\rangle\}$ $\mathcal{P}_{\leq} = \mathcal{P} \cup \text{PartOrd}$ and, if $\mathbb{P} = \mathbb{P}\langle\mathcal{O},\mathcal{P}\rangle$, define $\mathbb{P}_{\leq} = \mathbb{P}\langle\mathcal{O},\mathcal{P}_{\leq}\rangle$. If t and s are polynomials of $\mathbb{P}_{\leq}$ we abbreviate $\textcircled{\leq}/t,s/$ to $t \leq s$. Let u, v, w be the first three distinct variables of V, and take the theory of partial order, $\Sigma(\leq)$, to consist of the three basic Horn formulas:

$$u \leq u$$

$$(u \leq v) \wedge (v \leq u) \rightarrow (u = v)$$

$$(u \leq v) \wedge (v \leq w) \rightarrow (u \leq w).$$

Internal models of $\Sigma(\leq)$ will be called internal partially ordered structures, and when the equality and partial order signs are the only signs in the similarity type, partially ordered objects. If $\mathcal{A} = \langle A,o,p\rangle$ is an internal $\langle\mathcal{O},\mathcal{P}_{\leq}\rangle$-structure and t and s are terms of type A we put $t \leq s = (p(\textcircled{\leq}))/t,s/$.

<u>0.6.8.1 Extensions of the theory of partial order, in which operations are uniquely defined relative to the equality and partial order signs.</u>

1. $\Sigma(\leq,\underline{1}) = \Sigma(\leq) \cup \{u \leq 1\}$
2. $\Sigma(\leq,\underline{0}) = \Sigma(\leq) \cup \{0 \leq u\}$
3. $\Sigma(\leq,\text{sup}) = \Sigma(\leq) \cup \{u \leq u \vee v,\ v \leq u \vee v,\ (u \leq w) \wedge (v \leq w) \rightarrow (u \vee v \leq w)\}$

4. $\Sigma(\underline{\leq},\text{inf}) \equiv \Sigma(\underline{\leq}) \cup \{u \wedge v \leq u,\ u \wedge v \leq v,\ (w \leq u) \wedge (w \leq v) \rightarrow (w \leq u \wedge v)\}$

5. $\Sigma(\underline{\leq},\text{inf},\text{sup}) \equiv \Sigma(\underline{\leq},\text{inf}) \cup \Sigma(\underline{\leq},\text{sup})$

6. $\Sigma(\underline{\leq},\text{inf closed}) \equiv \Sigma(\underline{\leq},\text{inf}) \cup \{(u \wedge v) \leq w \leftrightarrow u \leq (v \Rightarrow w)\}$

7. $\Sigma(\underline{\leq},\text{Heyt.}) \equiv \Sigma(\underline{\leq},\text{inf closed}) \cup \Sigma(\underline{\leq},\text{sup}) \cup \Sigma(\underline{\leq},\underline{0}) \cup \Sigma(\underline{\leq},\underline{1})$.

Each of the above theories, Σ, contains $\Sigma(\underline{\leq})$ and is a subset of some $bHf^*(\mathbb{P}\langle\mathcal{O},\text{PartOrd}\rangle)$, and for each operation sign k introduced we have $\Sigma \models eq(k)$. That is, given an interpretation of $\textcircled{\leq}$, the interpretation of k, which gives rise to a model of Σ, is uniquely determined. Note that in cases 6. and 7. we are not strictly speaking saying that k is implicitly definable ([CK] p. 87) relative to $\textcircled{\leq}$ because two new operation signs are involved. Of course $\underline{\wedge}$ is already fixed by the theory $\Sigma(\underline{\leq},\text{inf})$, but we do not need to make this observation to apply 0.6.7.14.

Define atomic equivalences $e(\vee)$, $e(\wedge)$ and $e(\Rightarrow)$:

$$e(\vee) \equiv ((u \leq v) \leftrightarrow (u \vee v = v))$$

$$e(\wedge) \equiv ((u \leq v) \leftrightarrow (u \wedge v = v))$$

$$e(\Rightarrow) \equiv ((u \leq v) \leftrightarrow ((u \Rightarrow v) = \underline{1})).$$

Then we have the following consequences:

8. $\Sigma(\underline{\leq},\text{sup}) \models e(\vee)$

9. $\Sigma(\underline{\leq},\text{inf}) \models e(\wedge)$

10. $\Sigma(\underline{\leq},\text{inf closed}) \cup \Sigma(\underline{\leq},\underline{1}) \models e(\Rightarrow)$.

Now we reverse the definability procedure. Let $\Sigma(\vee)$ and $\Sigma(\wedge)$ be the equational theories of join and meet semilattices respectively and

Σ(Heyt.) the equational theory of Heyting algebras (see 0.6.4.5). The theories $\Sigma(\vee) \cup \{e(\vee)\}$, $\Sigma(\wedge) \cup \{e(\wedge)\}$ and $\Sigma(\text{Heyt.}) \cup \{e(\Rightarrow)\}$ implicitly define $\underline{\leq}$ relative to an inclusion $<\theta,<\mathcal{O},\text{Eql}>,<\mathcal{O},\text{PartOrd}>>$, moreover $\Sigma(\underline{\leq})$ is a consequence of each of these theories.

<u>0.6.8.2 Ω as a partially ordered object.</u> We extend the <H,Eql>-structure $\underset{\sim}{\Omega}$ (0.6.4.2) to a <H,PartOrd>-structure $\underset{\sim}{\Omega}_{\underline{<}}$ by interpreting $\textcircled{\leq}$ as $\Rightarrow_{\Omega}$, so that $\underset{\sim}{\Omega} = (\underset{\sim}{\Omega}_{\underline{<}})^{\iota}$, where ι is the natural inclusion of the two similarity types. The equivalence of atomic formulas (which we have already abbreviated as $e(\Rightarrow)$)

$$(u \underline{<} v) \leftrightarrow ((u \Rightarrow v) = \underline{1})$$

is internaly valid in $\underset{\sim}{\Omega}_{\underline{<}}$ since its interpretation in $\Phi(\mathcal{L})$ is the formula

$$(p \Rightarrow q) \Leftrightarrow ((p \Rightarrow q) = \top);$$

that this is valid follows from the substitution $\binom{p}{p \Rightarrow q}$ in (1) of 0.6.5.6.

We began in 0.6.4.4 by noting that $\underset{\sim}{\Omega}$ was an internal Heyting algebra, thus $\underset{\sim}{\Omega} \models \Sigma(\text{Heyt.})$, and so $\underset{\sim}{\Omega}_{\underline{<}} \models \Sigma(\text{Heyt.}) \cup \{e(\Rightarrow)\}$. By the metatheory, since $\Sigma(\text{Heyt.}) \cup \{e(\Rightarrow)\} \models \Sigma(\underline{<})$, we have $\underset{\sim}{\Omega}_{\underline{<}} \models \Sigma(\underline{<})$. Since $\Sigma(\underline{<}) \subset \text{bHf}^{*}(\mathbb{P}<\phi,\text{PartOrd}>)$, we have $\Omega_{\underline{<}} \models \Sigma(\underline{<})$ where $\Omega_{\underline{<}}$ is the <ϕ,PartOrd>-structure induced by the inclusion of <ϕ,PartOrd> in <H,PartOrd>. That is, $\underset{\sim}{\Omega}_{\underline{<}}$ is an internal partially ordered Heyting algebra and $\Omega_{\underline{<}}$ is a partially ordered object.

0.6.9 Further properties of internal equality.

0.6.9.1. Some immediate conclusions can be drawn about internal equality by using the metatheory; in any structure the following are valid:

(1) $u = u$

(2) $(u = v) \rightarrow (v = u)$

(3) $(u = v) \wedge (v = w) \rightarrow (u = w)$

(4) $(u = fv) \wedge (v = w) \rightarrow (u = fw)$

(5) $(u = v) \rightarrow (fu = fv)$

(6) $(u_1 = v_1) \wedge (u_2 = v_2) \rightarrow (f\langle u_1,u_2\rangle = f\langle v_1,v_2\rangle)$

(7) $(u = v) \wedge fu \rightarrow fv$

(8) $(u_1 = v_1) \wedge (u_2 = v_2) \wedge f\langle u_1,u_2\rangle \rightarrow f\langle v_1,v_2\rangle$,

and thus certain rules are valid; for example

(9) $\dfrac{r = s\ ,\ s = t}{(r = t) \wedge \nabla(s)}$ and (10) $\dfrac{r = s\ ,\ fs}{fr \wedge \nabla(s)}$

can be derived from (3) (the transitivity axiom), (7), rules (8), (10), (11) of 0.6.4.9 and (7) of 0.6.4.10, where r, s and t are terms of $\Phi(\mathcal{E})$ of the same type. Our metatheory has limited our interpretations to a single type of $\Phi(\mathcal{E})$ and while this is appropriate for the formulas (1), (2), (3) and (7), interpretations of the other formulas suggest valid formulas involving more than one type; these generalizations will be proven in this section.

0.6.9.2. In order to incorporate equality into our metatheorem we proved, in 0.6.3.3, a basic fact about toposes, namely that the external equality of the pair of morphisms

$$B \underset{k_2}{\overset{k_1}{\rightrightarrows}} A$$

is equivalent to the external equality of the pair

$$B \underset{\delta_A \bullet (k_1 \sqcap k_2)}{\overset{true_B}{\rightrightarrows}} \Omega \quad .$$

We used this fact again in (2) of 0.6.4.7 in order to pass (in 0.6.4.8) from valid equations involving variables of type Ω to tautologies not involving equality. We shall restate this same fact in the language $\Phi(\mathcal{E})$ but this time we keep extraneous variables so that we can readily establish further properties of equality.

<u>Proposition.</u> Let $\langle \vec{v}, t = s \rangle$ be an augmented term, the following are equivalent:

(1) $\lambda\vec{v}.t \equiv \lambda\vec{v}.s$

(2) $\lambda\vec{v}.(t = s) \equiv true_{\tau(\vec{v})}$

(3) $\models \nabla(\vec{v}) \wedge (t = s)$.

<u>Proof.</u> (1) $\leftrightarrow$ (2): $\lambda\vec{v}.(t = s) \equiv \delta_{\tau(t)} \circ ((\lambda\vec{v}.t) \sqcap (\lambda\vec{v}.s))$; now apply 0.6.3.3.□ (2) $\leftrightarrow$ (3): by (8.2) of 0.6.4.10.□

<u>Corollary.</u> Let $\langle \vec{v}, t \rangle$ be an augmented term and $f\colon \tau(\vec{v}) \to \tau(t)$, then

$$\lambda\vec{v}.t \equiv f \qquad \text{iff} \qquad \models t = f\pi(\vec{v}).$$

<u>Proof.</u> Put $s \equiv f\pi(\vec{v})$ in the Proposition.□

This corollary <u>defines</u> the morphism f by requiring that $f\pi(\vec{v}) = t$ be valid. For t a formula f is defined by requiring that $f\pi(\vec{v}) \leftrightarrow t$ be valid (by (6) of 0.6.4.7).

<u>0.6.9.3</u> We translate various external equations involving morphisms of $\underline{\mathcal{E}}$ into internal equaltions of $\Phi(\underline{\mathcal{E}})$.

<u>Proposition.</u>

(1) $f_1 = f_2$ iff $\models f_1x = f_2x$; where $f_i: \tau_0(x) \longrightarrow A$ $(i = 1,2)$.

(2) $f = id_{\tau(x)}$ iff $\models fx = x$; where $f: \tau_0(x) \longrightarrow \tau(x)$.

(3) $f = g \circ h$ iff $\models fx = ghx$; where $h: \tau_0(x) \longrightarrow B$, $g: B \longrightarrow A$, and $f: \tau(x) \longrightarrow A$.

(4) $f_i = \pi_i$ iff $\models f_i\langle x_1, x_2\rangle = x_i$ (for $i = 1,2$); where $x_1 \neq x_2$, $f_i: \tau\langle x_1, x_2\rangle \longrightarrow \tau_0(x_i)$, and π_i are projections $(i = 1,2)$.

(5) $f_1 = \pi_1$ and $f_2 = \pi_2$ iff $\models w = \langle f_1w, f_2w\rangle$; where $\pi_i, f_i: A_1 \times A_2 \longrightarrow A_i$ $(i = 1,2)$, and $\tau_0(w) = A_1 \times A_2$.

(6) $g = f_1 \sqcap f_2$ iff $\models gy = \langle f_1y, f_2y\rangle$; where $f_i: \tau_0(y) \longrightarrow A_i$ $(i = 1,2)$ and $g: B \longrightarrow A_1 \times A_2$.

(7) $g = f_1 \times f_2$ iff $\models g\langle x_1, x_2\rangle = \langle f_1x_1, f_2x_2\rangle$; where $f_i: \tau_0(x_i) \longrightarrow A_i$ $(i = 1,2)$ and $g: B_1 \times B_2 \longrightarrow A_1 \times A_2$.

(8) Let $A = \tau_0(x) = \tau_0(y)$ with $x \neq y$. u_A is a mono iff $\models x = y$.

<u>Proof.</u> Each equivalence is a direct application of 0.6.9.2. We show only (5) and (8). (5): $\models w = \langle f_1w, f_2w\rangle$ iff $\lambda w.w = \lambda w.\langle f_1w, f_2w\rangle$ (by (1) $\leftrightarrow$ (3) of 0.6.9.2) iff $id_{A_1 \times A_2} = f_1 \sqcap f_2$ iff $\pi_i \circ id_{A_1 \times A_2} = \pi_i \circ (f_1 \sqcap f_2)$ for both $i = 1$ and $i = 2$ iff $\pi_1 = f_1$ and $\pi_2 = f_2$.□ (8): $(\rightarrow)$: $u_A \circ \pi_1 = u_{A\times A} = u_A \circ \pi_2$ hence $\pi_1 = \pi_2$, that is $\lambda xy.x = \lambda xy.y$, so $\models x = y$.□ (8): $(\leftarrow)$: Let $f_1, f_2: B \longrightarrow A$. $u_A \circ f_1 = u_B = u_A \circ f_2$, so we must show $f_1 = f_2$. Let $w \in V_B$ then, substituing $\begin{pmatrix} x & y \\ f_1w & f_2w \end{pmatrix}$, we have $\models f_1w = f_2w$, hence (by (1) above) $f_1 = f_2$.□

<u>0.6.9.4.</u> Monomorphisms can be characterized using 0.6.5.1.

<u>Proposition.</u> Let $\tau_0(x_1) \equiv \tau_0(x_2) \equiv \text{dom}(f)$ with $x_1 \not\equiv x_2$; then f is a monomorphism iff $\models fx_1 = fx_2 \Rightarrow x_1 = x_2$.

<u>Proof.</u> ($\leftarrow$): Suppose $f \circ g_1 \equiv f \circ g_2$. By (1) and (3) of 0.6.9.3, and (9) of 0.6.9.1, $\models fg_1y = fg_2y$, where $\tau_0(y) \equiv \text{dom } g_1 \equiv \text{dom } g_2$. Substituting $\begin{pmatrix} x_1 & x_2 \\ g_1y & g_2y \end{pmatrix}$ yields $\models fg_1y = fg_2y \Rightarrow g_1y = g_2y$. By modus ponens $\models g_1y = g_2y$ hence, by (1) of 0.6.9.3, $g_1 \equiv g_2$.□ ($\rightarrow$): Let $\begin{pmatrix} x_1 & x_2 \\ \ell_1u & \ell_2u \end{pmatrix}$ be a normalized substitution for $fx_1 = fx_2 \Rightarrow x_1 = x_2$. Then $\models f\ell_1u = f\ell_2u$ hence $f \circ \ell_1 \equiv f \circ \ell_2$. Since f is a mono $\ell_1 \equiv \ell_2$ hence $\models \ell_1u = \ell_2u$. By 0.6.5.1, $\models fx_1 = fx_2 \Rightarrow x_1 = x_2$.□

<u>0.6.9.5. Proposition.</u> Let x_1, x_2, y_1, y_2 be distinct variables, $\tau(x_1) \equiv \tau(x_2) \equiv \text{dom } f$, and $\tau(y_1) \equiv \tau(y_2)$.

(1) $\models (x_1 = x_2) \Rightarrow fx_1 = fx_2$

(2) $\models (x_1 = x_2) \wedge (y_1 = y_2) \Rightarrow \langle x_1, y_1 \rangle = \langle x_2, y_2 \rangle$

(3) $\models \langle x_1, y_1 \rangle = \langle x_2, y_2 \rangle \Rightarrow (x_1 = x_2) \wedge (y_1 = y_2)$.

<u>Proof.</u> In each case we begin by making a normalized substitution.

(1): $\models \ell_1u = \ell_2u$, then $\ell_1 \equiv \ell_2$ so $\models f\ell_1u = f\ell_2u$. Hence (1) holds.□
(2): $\models (\ell_1u = \ell_2u) \wedge (\ell_3u = \ell_4u)$ hence $\models \ell_1u = \ell_2u$ and $\models \ell_3u = \ell_4$ hence $\ell_1 \equiv \ell_2$ and $\ell_3 \equiv \ell_4$, hence $\models \langle \ell_1u, \ell_3u \rangle = \langle \ell_2u, \ell_4u \rangle$; hence (2) holds.□ (3): $\models \langle \ell_1u, \ell_3u \rangle = \langle \ell_2u, \ell_4u \rangle$. Let $\pi_1 \equiv \lambda x_1y_1.x_1 \equiv \lambda x_2y_2.x_2$ and $\pi_2 \equiv \lambda x_1y_1.y_1 \equiv \lambda x_2y_2.y_2$; $\ell_1 \equiv \pi_1 \circ (\ell_1 \sqcap \ell_3) \equiv \pi_1 \circ |\langle \ell_1u, \ell_3u \rangle|$ $\equiv \pi_1 \circ |\langle \ell_2u, \ell_4u \rangle| \equiv \pi_1 \circ (\ell_2 \sqcap \ell_4) \equiv \ell_2$, hence $\models \ell_1u = \ell_2u$, similarly using π_2 we have $\models \ell_3u = \ell_4u$, hence

$\models (\ell_1 u = \ell_2 u) \wedge (\ell_3 u = \ell_4 u)$. Hence (3) holds.□

Corollary. The following are valid rules, where $\tau(t_1) = \tau(t_2)$ and $\tau(s_1) = \tau(s_2)$,

$$\frac{t_1 = t_2}{ft_1 = ft_2} \qquad \frac{(t_1 = t_2) \wedge (s_1 = s_2)}{\langle t_1, s_1\rangle = \langle t_2, s_2\rangle} .□$$

0.6.9.6 Valid equations arising from substitutions and the shifting of variables. Under an admissable substitution $S\binom{x}{y}$, in a term t, we do not, in general, get a valid equation $t = t[x|y]$; for example, unless (see (8) of 0.6.9.3) $u_{\tau(x)}$ is a mono, $\not\models x = x[x|y]$. If however we only shift bound variables, we maintain the equality; for example $\overline{\binom{x}{y}}\{x: x = 0\} \equiv \{y: y = 0\}$, where $\overline{\binom{x}{y}}$ is defined as in 0.2.7.1, and $\models \{x: x = 0\} = \{y: y = 0\}$. A corollary to this example is that if $P: 1 \longrightarrow \Omega^A$ is defined by $\models P_* = \{x: x = 0\}$ (see 0.6.9.2) then also $\models P_* = \{y: y = 0\}$. More generally a term t may have both free and bound variables; for example, still applying $\overline{\binom{x}{y}}$ we have $\models \{x: x = u\} = \{y: y = u\}$, and we can define $P: A \longrightarrow \Omega^A$ by $\models Pu = \{x: x = u\}$ or equivalently by $\models Pu = \{y: y = u\}$. If a variable x, is both bound and free in a term t, for example if $t \equiv \{x: x = 0\} \cup \{u: u = x\}$, then $t = \overline{\binom{x}{y}}(t)$ is not in general valid, i.e. $\not\models \{x: x = 0\} \cup \{u: u = x\} = \{y: y = 0\} \cup \{u: u = y\}$; but we can compensate for the change of a free variable by making the substitution $S\binom{x}{y}$ thus: $\models t[x|y] = \overline{\binom{x}{y}}(t)$. We develop this minor theme in 0.6.9.6 through to 0.6.9.10.

<u>0.6.9.7 Proposition.</u> If x and u are the same type and $u \notin S_{FB}(t)$ then $\models t[x|u] = \overline{\binom{x}{u}}(t)$.

<u>Proof.</u> By induction on the complexity of t.

$*$: $*[x|u] \equiv * \equiv \overline{\binom{x}{u}}(*)$.

x: $x[x|u] \equiv u \equiv \overline{\binom{x}{u}}(x)$

y where $y \not\equiv x$: $y[x|u] \equiv y \equiv \overline{\binom{x}{u}}(y)$

inductive clauses:

ft_1: $(ft_1)[x|u] \equiv f(t_1[x|u])$ and $\overline{\binom{x}{u}}(ft_1) \equiv f\left(\overline{\binom{x}{u}}(t_1)\right)$, by induction $\models t_1[x|u] = \overline{\binom{x}{u}}(t_1)$ hence $\models (ft_1)[x|u] = \overline{\binom{x}{u}}(ft_1)$.

$\langle t_1,t_2\rangle$: $\langle t_1,t_2\rangle[x|u] \equiv \langle t_1[x|u],t_2[x|u]\rangle$ and $\overline{\binom{x}{u}}\langle t_1,t_2\rangle \equiv \left\langle\overline{\binom{x}{u}}(t_1), \overline{\binom{x}{u}}(t_2)\right\rangle$. By induction $\models t_i[x|u] = \overline{\binom{x}{u}}(t_i)$ for $i = 1,2$ hence $\models \langle t_1,t_2\rangle[x|u] = \overline{\binom{x}{u}}\langle t_1,t_2\rangle$.

For the remainder we put $r' \equiv \overline{\binom{x}{u}}(r)$ for each term r.

$qx\varphi$: $(qx\varphi)[x|u] \equiv qx\varphi$ and $\overline{\binom{x}{u}}(qx\varphi) \equiv qu\varphi'$. By induction $\models \varphi[x|u] = \varphi'$. We want to show that $\models qx\varphi = qu\varphi'$, which is equivalent to $\lambda\vec{u}.qx\varphi \equiv \lambda\vec{u}.qu\varphi'$ where $\vec{u} \equiv \text{var}(qx\varphi = qu\varphi')$. By 0.3.4.11 ($\tau$'5) this follows from $\lambda\vec{u}z.\varphi[x|z] \equiv \lambda\vec{u}z.\varphi'[u|z]$ where z is chosen so that $z \notin S_{FB}(\varphi xu)$. Thus it is sufficient to prove $\models \varphi[x|u][u|z] = \varphi'[u|z]$. By the induction hypothesis, substituting $\binom{u}{z}$, we have $\models \varphi[x|u][u|z] = \varphi'[u|z]$. By 0.2.6.4 $\varphi[x|u][u|z] \equiv \varphi[x|z]$ hence $\models \varphi[x|z] = \varphi'[x|z]$.

$qy\varphi$ where $y \neq x$: $(qy\varphi)[x|u] \equiv qy(\varphi[x|u])$ and $(qy\varphi)' \equiv qy\varphi'$. By induction $\models \varphi[x|u] = \varphi'$. We want to show $\models qy(\varphi[x|u]) = qy\varphi'$, which is equivalent to $\lambda\vec{u}.qy\varphi[x|u] \equiv \lambda\vec{u}.qy\varphi'$ where $\vec{u} \equiv var(qy(\varphi[x|u]) = qy\varphi')$. By 0.3.4.11 this follows from $\lambda\vec{u}z.\varphi[x|u][y|z] \equiv \lambda\vec{u}z.\varphi'[y|z]$ where $z \notin S_{FB}(\varphi xuy)$. It suffices to prove $\models \varphi[x|u][y|z] = \varphi'[y|z]$. This follows immediately from the induction hypothesis by substituting $\binom{y}{z}$.□

<u>0.6.9.8 Corollary.</u> If $u \notin S_{FB}(\varphi)$ the following are equivalent:

(1) $\models \varphi$ (2) $\models \varphi[x|u]$ (3) $\models \overline{\binom{x}{u}}(\varphi)$.

<u>Proof.</u> We have proven $\models \varphi[x|u] = \overline{\binom{x}{u}}(\varphi)$ hence by (6) and (7) of 0.6.4.7 $\models \varphi[x|u] \Rightarrow \overline{\binom{x}{u}}(\varphi)$ and $\models \overline{\binom{x}{u}}(\varphi) \Rightarrow \varphi[x|u]$.

If $x \notin S_F\varphi$, $\varphi[x|u] \equiv \varphi$. By (1) of 0.2.7.2 we have $S_F\left(\overline{\binom{x}{u}}(\varphi)\right) \equiv \binom{x}{u}(S_F(\varphi)) \equiv S_F(\varphi)$. Hence by modus ponens $\models \varphi$ iff $\models \overline{\binom{x}{u}}(\varphi)$.

If $x \in S_F\varphi$, $S_F\left(\overline{\binom{x}{u}}(\varphi)\right) \equiv \binom{x}{u} S_F(\varphi) \equiv (S_F(\varphi) - \{x\}) \cup \{u\} \equiv S_F(\varphi[x|u])$, hence by modus ponens $\models \varphi[x|u]$ iff $\models \overline{\binom{x}{u}}(\varphi)$. By substitution $\models \varphi$ implies $\models \varphi[x|u]$ and $\models \varphi[x|u]$ implies $\models \varphi[x|u][u|x]$; but $\varphi \equiv \varphi[x|u][u|x]$ so (1) and (2) are equivalent.□

<u>0.6.9.9 Proposition.</u> Let t be a term and let Υ, Υ' be a separating decomposition of id_{Vbls} through $S_{FB}(t)$, then

$$\models S(\Upsilon)(t) = \overline{\Upsilon}(t).$$

<u>Proof.</u> The hypothesis means (1) for all $x \notin S_{FB}(t)$, $\gamma(x) = x$, (2) $\gamma|_{S_{FB}(t)}$ is injective and (3) $\gamma(S_{FB}(t)) \cap S_{FB}(t) = \phi$. Let x_i $(i \in [n])$ be a listing of the distinct elements of $S_{FB}(t)$, put $u_i = (x_i)$ $(i \in [n])$ define

$$t_0 = t \ , \qquad t_{i+1} = t_i[x_i|u_i] \qquad \text{for } i \in [n]$$

so that $S(\gamma)(t) = t_n$, define

$$\overline{t}_0 = t \ , \qquad \overline{t}_{i+1} = \overline{\binom{x_i}{u_i}}(\overline{t}_i) \qquad \text{for } i \in [n]$$

so that $\overline{\gamma}(t) = \overline{t}_n$. We prove $\models t_n = \overline{t}_n$ by induction. By reflexivity of equality $\models t_0 = \overline{t}_0$; suppose $\models t_i = \overline{t}_i$ for $i \in [n-1]$, $u_i \notin S_{FB}(t_i)$ since $S_B(t_i) = S_B(t)$ (0.2.5.8) and $S_F(t_i) \subset S_F(t) \cup \{u_j | j \in [i]\}$ (0.2.5.6) hence $\models t_i[x_i|u_i] = \overline{\binom{x_i}{u_i}}(t_i)$ (by 0.6.9.7). Define γ_i inductively by $\gamma_0 = id_{Vbls}$, $\gamma_{i+1} = \binom{x_i}{u_i} \circ \gamma_i$ then $\gamma_n = \gamma$ and $\overline{t}_i = \overline{\gamma}_i(t)$ so $S_{FB}(\overline{t}_i) = \gamma_i(S_{FB}(t))$ hence $u_i \notin S_{FB}(\overline{t}_i)$ therefore $u_i \notin S_{FB}(t_i = \overline{t}_i)$ hence $\models \overline{\binom{x_i}{u_i}}(t_i) = \overline{\binom{x_i}{u_i}}(\overline{t}_i)$ (0.6.9.7). By transitivity of equality and since we loose no variables in the inference $\left(S_F\left(\overline{\binom{x_i}{u_i}}(t_i)\right) = \binom{x_i}{u_i} S_F(t_i) = S_F(t_i[x_i|u_i])\right)$ we have $\models t_i[x_i|u_i] = \overline{\binom{x_i}{u_i}}(\overline{t}_i)$ that is $\models t_{i+1} = \overline{t}_{i+1}$, hence $\models t_n = \overline{t}_n$, that is $\models S(\gamma)(t) = \overline{\gamma}(t)$.□

<u>0.6.9.10 Proposition.</u> Let α: Vbls $\longrightarrow$ Vbls be type preserving, t a term for which (1) $\alpha(x) = x$ for all $x \notin S_{FB}(t)$, (2) $\alpha|_{S_{FB}(t)}$ is injective and (3) α is admissable for t, then

$$(4) \quad \models S(\alpha)(t) = \overline{\alpha}(t).$$

Proof. Let γ, β be a separating decomposition of α through $S_{FB}(t)$. The hypothesis of 0.6.9.9 applies to γ and t so $\models S(\gamma)(t) = \overline{\gamma}(t)$. We verify that the hypothesis also applies to β and $\overline{\gamma}(t)$.

$S_{FB}(\overline{\gamma}(t)) = \gamma(S_{FB}(t)) = \{u_i | \ i \in [n]\}$ hence (5.1) $x \notin S_{FB}(\overline{\gamma}(t))$ implies $\beta(x) = x$, (5.2) $\beta(u_i) = \beta(u_j)$ implies $\alpha(x_i) = \beta\cdot\gamma(x_i) = \beta\ \gamma(x_j) = \alpha(x_j)$, hence by (2) $\beta|_{S_{FB}(\gamma(t))}$ is injective,

$$(5.3) \quad (S_{FB}(\overline{\gamma}(t))) \cap S_{FB}(\overline{\gamma}(t))) = (\beta\cdot\gamma)(S_{FB}(t)) \cap \gamma(S_{FB}(t))$$
$$= \alpha(S_{FB}(t)) \cap \gamma(S_{FB}(t))$$

which is empty by definition (0.2.6.6). Hence $\models S(\beta)(\overline{\gamma}(t)) = \overline{\beta}(\overline{\gamma}(t))$. Since α is admissable for t, β is admissable for $S(\gamma)(t)$, by 0.2.9.9 (see 0.2.9.2), hence $\models S(\beta)S(\gamma)(t) = S(\beta)\overline{\gamma}(t)$, by transitivity since $S_F(S(\beta)(\overline{\gamma}(t))) = \alpha(S_F(t)) = S_F(\overline{\beta\gamma}(t))$ we have $\models S(\beta)S(\gamma)(t) = \overline{\beta\cdot\gamma}(t)$ thus $\models S(\alpha)(t) = \overline{\alpha}(t)$.□

Note that $S_F(S(\alpha)(t)) = (S_F(t) - Mov(\alpha)) \cup \alpha(S_F(t)) = \alpha(S_F(t)) = S_F(\overline{\alpha}(t))$ and that if α fixes $S_F(t)$ then $\models t = \overline{\alpha}(t)$.

<u>0.6.9.11 Corollary</u>. Let α be as above with φ a formula in place of t then

$$\models S(\alpha)(\varphi) \quad \text{iff} \quad \models \overline{\alpha}(\varphi)$$

Proof. Since $S_F(S(\alpha)(t)) = S_F(\overline{\alpha}(t))$ we have $\models S(\alpha)(\varphi)$ iff $\models \overline{\alpha}(\varphi)$.□

<u>0.6.9.12 Proposition</u>. Let γ_i: Vbls $\longrightarrow$ Tms be functions for which $\tau(\gamma_1(x)) = \tau(\gamma_2(x))$ and $\models \gamma_1(x) = \gamma_2(x)$ for all x and let t be a term, then $\models R(\gamma_1)(t) = R(\gamma_2)(t)$.

Proof. By induction on the complexity of t.

$*$: $R(\gamma_i)(*) = *$ for $i = 1,2$ and $\models * = *$.

y: $R(\gamma_i)(y) = \gamma_i(y)$ for $i = 1,2$ and we are given $\models \gamma_1(y) = \gamma_2(y)$.

Induction steps.

ft : $R(\gamma_i)(ft_1) = f(R(\gamma_i)(t_i))$ for $i = 1,2$, by induction $\models R(\gamma_1)(t_1) = R(\gamma_2)(t_1)$ hence $\models f(R(\gamma_1)(t_1)) = f(R(\gamma_2)(t_1))$.

$\langle t_1,t_2\rangle$: $R(\gamma_i)\langle t_1,t_2\rangle = \langle R(\gamma_i)(t_1),R(\gamma_i)(t_2)\rangle$ for $i = 1,2$, by induction $\models R(\gamma_1)(t_i) = R(\gamma_2)(t_i)$ for $i = 1,2$ hence $\models \langle R(\gamma_1)(t_1),R(\gamma_1)(t_2)\rangle = \langle R(\gamma_2)(t_1),R(\gamma_2)(t_2)\rangle$.

$qy\varphi$: $R(\gamma_i)(qy\varphi) = qy(R(\gamma_i)(\varphi))$ for $i = 1,2$. Put $\varphi_i = R(\gamma_i)(\varphi)$. By induction $\models \varphi_1 = \varphi_2$. Let $\vec{u} = var(qy\varphi_1 = qy\varphi_2)$ and choose z such that $\tau_0 z = \tau_0 y$, $z \notin \{\vec{u}\}$, $z \notin s_{FB}(\varphi_1 = \varphi_2)$. $\lambda\vec{u}.qy\varphi_i$ is uniquely determined by $\lambda\vec{u}z.\varphi_i[y|z]$. Substituting $\binom{y}{z}$ we get $\models \varphi_1[y|z] = \varphi_2[y|z]$ hence $\lambda\vec{u}z.\varphi_1[y|z] = \lambda\vec{u}z.\varphi_2[y|z]$ hence $\lambda\vec{u}.qy\varphi_1 = \lambda\vec{u}.qy\varphi_2$ hence $\models qy\varphi_1 = qy\varphi_2$ that is $\models R(\gamma_1)(qy\varphi) = R(\gamma_2)(qy\varphi)$.□

<u>0.6.9.13 Corollary.</u> Let γ_i: Vbls $\longrightarrow$ Tms $(i = 1,2)$ be type preserving functions. Suppose $\models \gamma_1(x) = \gamma_2(x)$ for all x. Let φ be a formula. We have:

(1) $\models R(\gamma_1)(\varphi) \leftrightarrow R(\gamma_2)(\varphi)$

(2) If $\models R(\gamma_1)(\varphi)$, and $R(\gamma_2)(\varphi)$ absorbs $R(\gamma_1)(\varphi)$, then $\models R(\gamma_2)(\varphi)$.□

<u>0.6.9.14 Proposition.</u> Suppose $\tau_0(x) = \tau_0(y)$ and $x \neq y$. φ a formula then (1) $\models (\varphi \wedge (x = y)) \rightarrow \varphi[x|y]$.

<u>Proof.</u> Suppose $\{x,y\} \subset s_F(\varphi)$. Let $\vec{w}$ be a string of distinct variables not containing x or y such that $\{\vec{w}xy\} = s_F\varphi$. Let $P = \lambda\vec{w}xy.\varphi$, then

$\models P/\pi(\vec{w}),x,y/ = \varphi$ so

$\models P/\pi(\vec{w}),y,y/ = \varphi[x|y]$.

We prove (2) $\models (P/\pi(\vec{w}),x,y/ \wedge (x = y)) \Rightarrow P/\pi(\vec{w}),y,y/$ (1) follows by replacement (0.6.9.12). Let $\tau_0(w) = \tau(\vec{w})$. Then (2) follows from

(3) $\models (P/w,x,y/ \wedge (x = y)) \Rightarrow P/w,y,y/$ by substituting $\pi(\vec{w})$ for w.

To prove (3) we let $\begin{pmatrix} w & x & y \\ m_1u & m_2u & m_3u \end{pmatrix}$ be a normalized substitution for (3) so that (4) $\models P/m_1u,m_2u,m_3u/ \wedge (m_2u = m_3u)$ thus

(5) $\models P/m_1u,m_2u,m_3u/$ and (6) $\models m_2u = m_3u$. From (6) we have $m_2 = m_3$ hence (4) implies $\models P/m_1u,m_2u,m_3u/$.

If $\{x,y\} \not\subset s_F(\varphi)$ we put $\varphi' = \varphi \wedge \nabla(xy)$ then $\models (\varphi' \wedge (x = y)) \Rightarrow \varphi'[x|y]$ hence $\models (\varphi \wedge (x = y)) \Rightarrow \varphi[x|y]$. □

<u>0.6.9.15 Proposition.</u> Suppose x, y and z are distinct variables of the same type

(1) $\models y = z \Rightarrow t[x|y] = t[x|z]$

(2) $\models \psi[x|t] \wedge (t = x) \Rightarrow \psi$, where $t \in ffr[x](\psi)$.

<u>Proof.</u> (1) Put $\varphi = (t = t[x|z])$ then $\varphi[x|y] = (t[x|y] = t[x|z])$, by 0.6.9.14 we have

$$\models (t = t[x|z]) \wedge (x = y) \Rightarrow t[x|y] = t[x|z].$$

Substituting z for x we have

$$\models (t[x|z] = t[x|z]) \wedge (z = y) \Rightarrow (t[x|y] = t[x|z]).$$

By propositional logic, since $\models t[x|z] = t[x|z]$, we have (1).□

(2) Choose $w \notin s_{FB}(\psi)$, $w \not\equiv x$ and make the substitutuion $\begin{pmatrix} x & w \\ t & x \end{pmatrix}$ in the valid formula $\psi \wedge (x = w) \Rightarrow \psi[x|w]$ to get (2).□

0.6.9.16 Proposition. Let $\vec{v}$ be a reduced string, $\ell(\vec{v}) \geq 1$, $A = \tau_0(y) = \tau(v)$, $y \notin \{\vec{v}\}$. Then

(1) $\models y = S(\beta)(\pi(\vec{v}))$, where $\beta = \text{Sbs}\langle y, id_A, \vec{v}\rangle$; and

(2) $\models \psi$ iff $\models \psi[y|\pi(\vec{v})]$, where $\{\vec{v}\} \cap S_F(\psi) = \phi$ and $y \in S_F(\psi)$.

Proof. (1) The substitution $S(\beta)$ has been used already in 0.5.5.5 and 0.6.5.4. Put $t = \pi(\vec{v})$ in 0.5.5.5, then $\lambda y.S(\beta)(\pi(\vec{v})) = \lambda\vec{v}.\vec{v} = id_A = \lambda y.y$, hence by 0.6.9.2, (1) holds.□

(2) ($\rightarrow$): By substitution.□ ($\leftarrow$): Suppose $\models \psi[y|\pi(\vec{v})]$. By 0.6.9.15 (2), we have $\models \psi[y|\pi(\vec{v})] \wedge (y = \pi(\vec{v})) \rightarrow \psi$, hence $\models (y = \pi(\vec{v})) \rightarrow \psi$. Substituting $S(\beta)$, where $\beta = \text{Sbs}\langle y, id_A, \vec{v}\rangle$, we get $\models (y = S(\beta)(\pi(\vec{v}))) \rightarrow \psi$. Hence by (1), since $y \in S_F(\psi)$, $\models \psi$.□

0.6.9.17 Abbreviations in prcofs involving equality. Given a sequence t_i of terms of the same type such that $\models t_i = t_{i+1}$ for $0 \leq i \leq n-2$ and $n \geq 2$, we can deduce $\models \bigwedge_{i=0}^{n-2} (t_i = t_{i+1})$; then applying transitivity we have $\models (t_0 = t_{n-1}) \wedge \nabla(t_1 \cdots t_{n-2})$. If furthermore $s_F(t_0 t_{n-1})$ absorbs $\bigcup_{i=1}^{n-2} s_F(t_i)$, then $\models t_0 = t_{n-1}$. Let us take $t_0 = t_1 = \cdots\cdots = t_{n-1} = t_{n-2}$ to be the formula $\bigwedge_{i=0}^{n-2} (t_i = t_{i+1})$, where $n \geq 2$. In future sections the ellipsis gets filled in as n takes on particular values; the formula may also be rearranged further for readability, viz.

$$\begin{aligned} t_0 &= t_1 = t_2 \\ &= t_3 \\ &= t_4 = t_5 \,. \end{aligned}$$

In this format we must take care to observe the conditions under which we can conclude the validity of $t_1 = t_5$; for example if $\tau_0(z) = 0$ and $o_\Omega : 0 \longrightarrow \Omega$ then $\models \top = o_\Omega z = \bot$, but $\not\models \top = \bot$. Caution is also necessary when the terms are of type Ω since bracketing can change the meaning of a formula completely: $\bot = \bot = \bot$ is valid whereas $(\bot = \bot) = \bot$ is not valid. Similar conventions will also apply to equivalences $\varphi_i \Leftrightarrow \varphi_{i+1}$ and implications $\varphi_i \Rightarrow \varphi_{i+1}$. In the latter case the formula $\bigwedge_{i=0}^{n-2} (\varphi_i \Rightarrow \varphi_{i+1})$ will only be given its alternative presentation vertically:

$$
\begin{aligned}
\varphi_0 &\Rightarrow \varphi_1 \\
&\Rightarrow \varphi_2 \\
&\;\;\vdots \\
&\Rightarrow \varphi_{n-1}.
\end{aligned}
$$

0.6.10 Valid formulas and rules of inference involving the quantifiers $\forall$ and $\exists$.

0.6.10.1 Proposition. If $x \in S_F\varphi$ then $\models (\forall x\varphi) \rightarrow \varphi$.

Proof. Let $\vec{u}x$ be a string of distinct variables for which $\{\vec{u}x\} = S_F\varphi$. We want to show $[\![\vec{u}x|\forall x\varphi]\!] \leq [\![\vec{u}x|\varphi]\!]$. Let $\pi = \lambda\vec{u}x.\vec{u}: \tau\vec{u}x \longrightarrow \tau\vec{u}$ then $\lambda\vec{u}x.\forall x\varphi = (\lambda\vec{u}.\forall x\varphi)\circ(\lambda\vec{u}x.\vec{u})$ so $[\![\vec{u}x|\forall x\varphi]\!] = \pi^{-1}[\![\vec{u}|\forall x\varphi]\!]$. For some $y \notin S_{FB}(\vec{u}x\varphi)$, $\tau_0 y = \tau_0 x$, we have $[\![\vec{u}|\forall x\varphi]\!] = \forall_\pi[\![\vec{u}y|\varphi[x|y]]\!]$ but by 0.5.3.4 (2) since $S_F(\varphi) \subset \{\vec{u}x\}$ we have $[\![\vec{u}x|\varphi]\!] = [\![\vec{u}y|\varphi[x|y]]\!]$, hence $[\![\vec{u}x|\forall x\varphi]\!] = \pi^{-1}\forall_\pi[\![\vec{u}x|\varphi]\!] \leq [\![\vec{u}x|\varphi]\!]$. The inequality follows form the adjointness $\pi^{-1} \dashv \forall_\pi$.□

From this proposition we get the rule

$$\frac{\forall x\varphi}{\varphi} \quad x \in S_F\varphi.$$

In [J1] p. 155 $\forall x\varphi$ is first interpreted only when $x \in S_F\varphi$ (in 5.42 (e)) and then $\forall x\varphi$, when $x \notin S_F\varphi$, is defined as $\forall x(x = x \wedge \varphi)$ (in 5.42 (f)). This definition is followed by a rule: $\models \forall x\varphi$ implies $\models \varphi$, without conditions on x and φ (5.44 (iv)). This is not correct, as the following example shows: take x to be of type 0 and φ to be $\perp$, let $0_{\mathbb{1}}$ be the map $0 \longrightarrow \mathbb{1}$, then $\|\perp\| = [\![0_{\mathbb{1}}]\!]$ and $\|\forall x((x = x) \wedge \perp)\| = \forall_{0_{\mathbb{1}}}([\![x|x = x]\!] \wedge [\![x|\perp]\!]) = [\![\mathbb{1}]\!]$, thus $\models \forall x((x = x) \wedge \perp)$ but $\not\models \perp$.

0.6.10.2 Proposition. If $x \in S_F\varphi$ then $\models \varphi \rightarrow (\exists x\varphi)$.

Proof. Let $\vec{u}x$ be a string of distinct variables for which $\{\vec{u}x\} = S_F\varphi$. We want to show $[\![\vec{u}x|\varphi]\!] \leq [\![\vec{u}x|\exists x\varphi]\!]$. Let $\pi = \lambda\vec{u}x.\vec{u}: \tau\vec{u}x \longrightarrow \tau\vec{u}$ then $[\![\vec{u}x|\exists x\varphi]\!] = \pi^{-1}[\![\vec{u}|\exists x\varphi]\!] = \pi^{-1}\exists_\pi[\![\vec{u}x|\varphi]\!]$, by the adjointness $\exists_\pi \dashv \pi^{-1}$ the result follows.□

As was the case in 0.6.10.1 we cannot drop the condition $x \in S_F\varphi$; for example if x is of type 0 then

$\|T \Rightarrow \exists x\| = (\|T\| \Rightarrow \exists_{0_1}[\![x|T]\!]) = ([\![1]\!] \Rightarrow \exists_{0_1}[\![0_0]\!] = [\![1]\!] \Rightarrow [\![0_1]\!] = [\![0_0]\!]$,

and so $\not\models T \Rightarrow \exists x$.

<u>0.6.10.3 Proposition.</u> Let $\tau_0 x \equiv \tau t$, $x \notin S_F(t)$ then $\models \exists x(x = t)$.

<u>Proof.</u> Choose y such that $\tau_0 x \equiv \tau_0 y$, $x \not\equiv y$. We have

$\models x = y \Rightarrow \exists x(x = y)$ (0.6.10.2) hence

$\models (y = y) \Rightarrow \exists x(x = y)$ hence

$\models \exists x(x = y)$ therefore

$\models \exists x(x = t)$.□

<u>0.6.10.4 Proposition.</u> The following are valid:

(1) $\dfrac{\exists x\varphi \Rightarrow \psi}{\varphi \Rightarrow \psi}$ $x \notin S_F\psi$ and $x \in S_F\varphi$

(2) $\dfrac{\varphi \Rightarrow \psi}{\exists x\varphi \Rightarrow \psi}$ $x \notin S_F\psi$

(3) $\dfrac{\neg \exists x\varphi}{\neg \varphi}$ $x \in S_F\varphi$

(4) $\dfrac{\neg \varphi}{\neg \exists x\varphi}$

(5) $\exists x\psi \Rightarrow \psi$ where $x \notin S_F\psi$.

<u>Proof.</u> We first assume $x \notin S_F$ and $x \in S_F\varphi$, and prove that $\models \varphi \Rightarrow \psi$ iff $\models \exists x\varphi \Rightarrow \psi$. Let $\vec{u} \equiv \mathrm{var}(\exists x\varphi \Rightarrow \psi)$ so that $S_F(\varphi \Rightarrow \psi) \equiv \{\vec{u}x\}$, and let $\pi \equiv \lambda\vec{u}x.\vec{u}$. We have $\models \varphi \Rightarrow \psi$ iff $[\![\vec{u}x|\varphi]\!] \leq [\![\vec{u}x|\psi]\!]$ iff $[\![\vec{u}x|\varphi]\!] \leq \pi^{-1}[\![\vec{u}|\psi]\!]$ iff $\exists_\pi[\![\vec{u}x|\varphi]\!] \leq [\![\vec{u}|\psi]\!]$ iff $\models \exists x\varphi \Rightarrow \psi$. This establishes (1) and its converse.□ (2): Assume $\models \varphi \Rightarrow \psi$ with $x \notin S_F\psi$. Let $\varphi' \equiv \nabla(x) \wedge \varphi$, then $\models \varphi \Leftrightarrow \varphi'$ and $\models \varphi' \Rightarrow \psi$. Since $x \in S_F(\varphi')$ we

have $\models \exists x\varphi' \Rightarrow \psi$, then by 0.6.9.13 $\models \exists x\varphi \Rightarrow \psi$.□ (3) Put $\psi \equiv \bot$ in (1)□ (4) $\psi \equiv \bot$ in (2).□ (5) Put $\varphi \equiv \psi$ in (2).□

<u>0.6.10.5 Proposition.</u> The following are valid:

(1) $\dfrac{\psi \Rightarrow \forall x\varphi}{\psi \Rightarrow \varphi}$ $x \notin S_F\psi$ and $x \in S_F\varphi$

(2) $\dfrac{\psi \Rightarrow \varphi}{\psi \Rightarrow \forall x\varphi}$ $x \notin S_F(\psi)$

(3) $\dfrac{\varphi}{\forall x\varphi}$

(4) $\psi \Rightarrow \forall x\psi$ where $x \notin S_F\psi$.

<u>Proof.</u> We assume $x \notin S_F\psi$ and $x \in S_F\varphi$, and prove that $\models \psi \Rightarrow \varphi$ iff $\models \psi \Rightarrow \forall x\varphi$. Let $\vec{u} \equiv \mathrm{var}(\psi \Rightarrow \forall x\varphi)$ so that $S_F(\psi \Rightarrow \varphi) \equiv \{\vec{u}x\}$, and let $\pi \equiv \lambda\vec{u}x.\vec{u}$; then $\models \psi \Rightarrow \varphi$ iff $[\![\vec{u}x|\psi]\!] \leq [\![\vec{u}x|\varphi]\!]$ iff $\pi^{-1}[\![\vec{u}|\psi]\!] \leq [\![\vec{u}x|\varphi]\!]$ iff $[\![\vec{u}|\psi]\!] \leq \forall_\pi[\![\vec{u}x|\varphi]\!]$ iff $\models \psi \Rightarrow \forall x\varphi$. This establishes (1) and its converse.□ (2): Assume $\models \psi \Rightarrow \varphi$ with $x \notin S_F\psi$. Let $\varphi' \equiv \nabla(x) \wedge \varphi$, then $\models \varphi \Leftrightarrow \varphi'$ and $\models \psi \Rightarrow \varphi'$. Since $x \in S_F(\varphi')$, $\models \psi \Rightarrow \forall x\varphi'$; by 0.6.9.13 $\models \psi \Rightarrow \forall x\varphi$.□ (3) Put $\psi \equiv \top$ in (2).□ (4) Put $\varphi \equiv \psi$ in (2).□

As we noted, in 0.6.10.1, for the rule $\dfrac{\forall x\varphi}{\varphi}$ $x \in S_F(\varphi)$, 0.6.10.4 (1) and 0.6.10.5 (1) do not hold without the restriction $x \in S_F(\varphi)$: If $\tau_0(x) \equiv 0$

then $\models \exists x\top \Rightarrow \bot$ but $\not\models \top \Rightarrow \bot$

and $\models \top \Rightarrow \forall x\bot$ but $\not\models \top \Rightarrow \bot$.

<u>0.6.10.6 Definition.</u> For each string of distinct varibles we define derived operations on the μ-algebra of expression by applying $\exists$ and $\forall$ repeatedly as follows:

(∃) $\exists\langle\phi\rangle \equiv \phi$ $\exists\langle\vec{v}x\rangle \equiv (\exists\langle\vec{v}\rangle)\exists x$

(∀) $\forall\langle\phi\rangle \equiv \phi$ $\forall\langle\vec{v}x\rangle \equiv (\forall\langle\vec{v}\rangle)\forall x.$

For example $(\exists\langle wxy\rangle)\varphi \equiv \exists w\exists x\exists y\varphi$.

<u>0.6.10.6.1 Proposition.</u> The following rule is valid.

$$\frac{(\exists\langle\vec{w}\rangle\varphi, \varphi \to \psi}{\psi} \quad (\{\vec{w}\} \cap S_F(\psi)) = \phi,\ S_F(\varphi) \subset \{\vec{w}\} .$$

<u>Proof.</u> We proceed by induction on the length of $\vec{w}$. $\exists\langle\phi\rangle \equiv \phi$ (see 0.6.4.10 (1)). From $\models \varphi$ and $\models \varphi \to \psi$, by modus ponens, we deduce $\models \psi$. Now assume $x \notin S_F(\psi)$, $S_F(\varphi) \subset \{x\}$, $\models \exists x\varphi$ and $\models \varphi \to \psi$. By 0.6.11.4 $\models \exists x\varphi \to \psi$. Hence by modus ponens $\models \psi$. For the induction step assume $\models (\exists\langle\vec{v}\rangle)(\exists x\varphi)$, $\models \varphi \to \psi$, $S_F(\varphi) \subset \{\vec{v}x\}$ and $\{\vec{v}x\} \cap S_F(\psi) = \phi$, where $\ell(\vec{v}) \geq 1$. By 0.6.10.4 $\models (\exists x\varphi) \to \psi$, so by induction, $\models \psi$.□

<u>0.6.10.7 Proposition.</u> If $x \in S_F\varphi$, $y \in ffr[x](\varphi)$, and $x \neq y$, then $\models \exists x((x = y) \wedge \varphi) \leftrightarrow \varphi[x|y]$.

<u>Proof.</u> By 0.6.9.14 $\models (x = y) \wedge \varphi \to \varphi[x|y]$, and by 0.6.10.4 $\models \exists x((x = y) \wedge \varphi) \to \varphi[x|y]$. By 0.6.10.2 $\models ((x = y) \wedge \varphi) \to \exists x((x = y) \wedge \varphi)$, substituting y for x we get $\models \varphi[x|y] \to \exists x((x = y) \wedge \varphi)$.□

<u>0.6.10.8 Proposition.</u> The following are valid rules.

$$(1) \quad \frac{\varphi \to \psi}{\forall x\varphi \to \forall x\psi} \qquad (2) \quad \frac{\varphi \to \psi}{\exists x\varphi \to \exists x\psi}$$

<u>Proof.</u> We first prove both rules under the additional assumption $x \in (S_F(\varphi) \cap S_F(\psi))$. (1): By 0.6.10.1 $\models (\forall x\varphi) \to \varphi$. By hypothesis $\models \varphi \to \psi$, hence $\models \forall x\varphi \to \psi$. By 0.6.10.5 $\models \forall x\varphi \to \forall x\psi$.□ (2): By 0.6.10.2 $\models \psi \to \exists x\psi$, and by hypothesis $\models \varphi \to \psi$, hence $\models \varphi \to \exists x\psi$. By 0.6.10.4

$\models \exists x\varphi \Rightarrow \exists x\psi$.□ Now we assume only $\models \varphi \Rightarrow \psi$ and put $\varphi' \equiv \varphi \wedge \nabla(x)$, $\psi' \equiv \psi \wedge \nabla(x)$ so that $\models \varphi' \Leftrightarrow \varphi$, $\models \psi' \Leftrightarrow \psi$ and $\models \varphi' \Rightarrow \psi'$. From what we have just proven, $\models \forall x\varphi' \Rightarrow \forall x\psi'$ and $\models \exists x\varphi' \Rightarrow \exists x\psi'$; hence, replacing φ' with φ and ψ' with ψ we have $\models \forall x\varphi \Rightarrow \forall x\psi$ and $\models \exists x\varphi \Rightarrow \exists x\psi$.□

<u>0.6.10.9 Osius' list of valid formulas involving the quantifiers $\forall$ and $\exists$.</u> (See [O1] Proposition 3.21 p. 217). We keep the numbering of [O1].

(1) $\models \exists x\exists y\varphi \Leftrightarrow \exists y\exists x\varphi$

<u>Proof.</u> We first assume $x \in S_F\varphi$ and $y \in S_F\varphi$. By 0.6.10.2 $\models \varphi \Rightarrow \exists x\varphi$, by 0.6.10.8 $\models \exists y\varphi \Rightarrow \exists y\exists x\varphi$, by 0.6.10.4 since $x \notin S_F(\exists y\exists x\varphi)$ we have $\models \exists x\exists y\varphi \Rightarrow \exists y\exists x\varphi$. Similarly $\models \exists y\exists x\varphi \Rightarrow \exists x\exists y\varphi$. Hence, $\models \exists x\exists y\varphi \Leftrightarrow \exists y\exists x\varphi$ if $x \in S_F\varphi$ and $y \in S_F\varphi$. Now we drop the conditions on φ but let $\varphi' \equiv \nabla(xy) \wedge \varphi$ so that $\models \varphi' \equiv \varphi$ and $\models \exists x\exists y\varphi' \Leftrightarrow \exists y\exists x\varphi'$. By 0.6.9.3 $\models \exists x\exists y\varphi \Leftrightarrow \exists y\exists x\varphi$.□

In (1), (2) and (3) the formula to be proven valid can be put in the form $R\binom{p}{\varphi}(\theta)$ where p is a proposition variable which occurs free but not bound in θ. After showing that $R\binom{p}{\varphi'}(\theta)$ is valid for $\{x,y\} \subset S_F\varphi'$ $\varphi' \equiv \nabla(xy) \wedge \varphi$, since in each of (1), (2) and (3) we have $S_F R\binom{p}{\varphi'}(\theta) \equiv S_F\varphi - \{x,y\} \equiv S_F R\binom{p}{\varphi}(\theta)$, we have by 0.6.9.13 $\models R\binom{p}{\varphi}(\theta)$.

(2) $\models \forall x\forall y\varphi \Leftrightarrow \forall y\forall x\varphi$

The proof is similar to that for (1).

(3) $\models \exists x \forall y \varphi \Rightarrow \forall y \exists x \varphi$

Proof. Assume $x \in S_F\varphi$ and $y \in S_F\varphi$. By 0.6.10.2 $\models \varphi \Rightarrow \exists x\varphi$, by 0.6.10.8 $\models \forall y\varphi \Rightarrow \forall y \exists x\varphi$, and by 0.6.10.4 since $x \notin S_F(\forall y \exists x\varphi)$ we have $\models \exists x \forall y\varphi \Rightarrow \forall y \exists x\varphi$. The condition $\{x,y\} \subset S_F\varphi$ is eliminated by the argument following (1).□

(There is a misprint in [01] since clearly the converse is false. In a two element set $\not\models \forall y \exists x(x = y) \Rightarrow \exists x \forall y(x = y)$).

The method of eliminating hypotheses involving the occurance of free variables which we belaboured in order to establish (1), (2) and (3) will now be applied to (4), (5), (6), (7), (8), (8.1), (9). In these cases we prove $\models R\begin{pmatrix} p_1 & p_2 \\ \varphi_1' & \varphi_2' \end{pmatrix}(\theta)$ where p_1, p_2 are proposition variables which occur free and not bound in θ, and φ_1' and φ_2' are formulas in which x occurs freely. Then for φ_1 and φ_2 we put $\varphi_i' = \nabla(x) \wedge \varphi_i$ for $i = 1,2$ so that $\models \varphi_i' = \varphi_i$ for $i = 1,2$.

Since in each case $S_F(R\begin{pmatrix} p_1 & p_2 \\ \varphi_1' & \varphi_2' \end{pmatrix}(\theta)) = S_F(R\begin{pmatrix} p_1 & p_2 \\ \varphi_1 & \varphi_2 \end{pmatrix}(\theta))$

$= S_F(\varphi_1\varphi_2) - \{x\}$, by 0.6.9.13 we have $\models R\begin{pmatrix} p_1 & p_2 \\ \varphi_1 & \varphi_2 \end{pmatrix}(\theta)$.□

(4) $\models \forall x\varphi_1 \wedge \forall x\varphi_2 \Leftrightarrow \forall x(\varphi_1 \wedge \varphi_2)$

Proof. We can assume without loss of generality that $x \in (S_F\varphi_1 \cap S_F\varphi_2)$. From $\models (\varphi_1 \wedge \varphi_2) \Rightarrow \varphi_i$ for $i = 1,2$ we deduce by 0.6.10.8 $\models \forall x(\varphi_1 \wedge \varphi_2) \Rightarrow \forall x\varphi_i$ for $i = 1,2$, hence $\models \forall x(\varphi_1 \wedge \varphi_2) \Rightarrow \forall x\varphi_1 \wedge \forall x\varphi_2$. From $\models (\forall x\varphi_i) \Rightarrow \varphi_i$ for $i = 1,2$ we have $\models (\forall x\varphi_1) \wedge (\forall x\varphi_2) \Rightarrow (\varphi_1 \wedge \varphi_2)$, since $x \notin S_F(\forall x\varphi_1 \wedge \forall x\varphi_2)$ we have $\models \forall x\varphi_1 \wedge \forall x\varphi_2 \Rightarrow \forall x(\varphi_1 \wedge \varphi_2)$.□

(5) $\models \exists x(\varphi_1 \vee \varphi_2) \Leftrightarrow \exists x\varphi_1 \vee \exists x\varphi_2$.

The proof is similar to (4).□

(6) $\models (\forall x\varphi_1 \vee \forall x\varphi_2) \Rightarrow \forall x(\varphi_1 \vee \varphi_2)$.

Proof. Without loss of generality assume $x \varepsilon (S_F\varphi_1 \cap S_F\varphi_2)$. From $\models \varphi_i \Rightarrow (\varphi_1 \vee \varphi_2)$ for $i = 1,2$ we have $\models \forall x\varphi_i \Rightarrow \forall x(\varphi_1 \vee \varphi_2)$ for $i = 1,2$. (6) follows by propositional logic.□

(7) $\models \exists x(\varphi_1 \wedge \varphi_2) \Rightarrow \exists x\varphi_1 \wedge \exists x\varphi_2$.

The proof is similar to (6).□

(8) $\models \forall x(\varphi_1 \Rightarrow \varphi_2) \Rightarrow (\forall x\varphi_1 \Rightarrow \forall x\varphi_2)$.

Proof. Without loss of generality $x \varepsilon S_F\varphi_1 \cap S_F\varphi_2$. By (4) $\models ((\forall x\varphi_1) \wedge (\forall x(\varphi_1 \Rightarrow \varphi_2))) \Rightarrow (\forall x(\varphi_1 \wedge (\varphi_1 \Rightarrow \varphi_2)))$. By 0.6.1.0.8 since $\models (\varphi_1 \wedge (\varphi_1 \Rightarrow \varphi_2)) \Rightarrow \varphi_2$ $\models \forall x(\varphi_1 \wedge (\varphi_1 \Rightarrow \varphi_2)) \Rightarrow \forall x\varphi_2$. Hence $\models ((\forall x\varphi_1) \wedge (\forall x(\varphi_1 \Rightarrow \varphi_2))) \Rightarrow \forall x\varphi_2$, (8) follows from propositional logic.□

(9) $\models \forall x(\varphi_1 \Rightarrow \varphi_2) \Rightarrow (\exists x\varphi_1 \Rightarrow \exists x\varphi_2)$

Proof. Without loss of generality $x \in S_F\varphi_1 \cap S_F\varphi_2$. By 0.6.10.2 $\models \varphi_2 \Rightarrow \exists x\varphi_2$ hence $\models (\varphi_1 \Rightarrow \varphi_2) \Rightarrow (\varphi_1 \Rightarrow \exists x\varphi_2)$. By 0.6.10.1 $\models \forall x(\varphi_1 \Rightarrow \varphi_2) \Rightarrow (\varphi_1 \Rightarrow \varphi_2)$. Since $x \in S_F\varphi_1$, $S_F(\varphi_1 \Rightarrow \exists x\varphi_2) = S_F\varphi_1 \cup ((S_F\varphi_2) - \{x\}) = S_F\varphi_1 \cup S_F\varphi_2 = S_F(\varphi_1 \Rightarrow \varphi_2)$, hence $\models \forall x(\varphi_1 \Rightarrow \varphi_2) \Rightarrow (\varphi_1 \Rightarrow \exists x\varphi_2)$, $\models \varphi_1 \Rightarrow (\forall x(\varphi_1 \Rightarrow \varphi_2) \Rightarrow \exists x\varphi_2)$ and $\models \exists x\varphi_1 \Rightarrow (\forall x(\varphi_1 \Rightarrow \varphi_2) \Rightarrow \exists x\varphi_2)$, hence (9).□

For the next set of formulas (13) to (20) we drop the hypothesis $x \in S_F\varphi$ as follows: we prove $\models R\binom{p}{\varphi'}(\theta)$ where p is a propositional variable that occurs free and not bound in θ and φ' is a formula in which x occurs freely. We put $\varphi' = \nabla(x) \wedge \varphi$ so that $\models \varphi' = \varphi$. Then $S_F(R\binom{p}{\varphi'}(\theta)) = S_F(R\binom{p}{\varphi}(\theta) = (S_F\varphi - \{x\}) \cup S_F\psi$ so by 0.6.9.12 (2) $\models R\binom{p}{\varphi}(\theta)$.

In each case, (13) to (20), we will assume $x \notin S_F(\psi)$.

(13) $\models (\psi \wedge \forall x\varphi) \Rightarrow \forall x(\psi \wedge \varphi)$, $x \notin S_F\psi$.

Proof. Without loss of generality $x \in S_F\varphi$. From $\models \forall x\varphi \Rightarrow \varphi$ we get $\models \psi \wedge \forall x\varphi \Rightarrow \psi \wedge \varphi$. Since $x \notin S_F(\psi \wedge \forall x\varphi)$ we get $\models \psi \wedge \forall x\varphi \Rightarrow \forall x(\psi \wedge \varphi)$.□

(14) $\models \psi \vee \forall x\varphi \Rightarrow. \forall x(\psi \vee \varphi)$, where $x \notin S_F(\psi)$.

Proof. Without loss of generality $x \in S_F\varphi$. From $\models \psi \Rightarrow (\psi \vee \varphi)$ we have $\models \psi \Rightarrow \forall x(\psi \vee \varphi)$. From $\models \varphi \Rightarrow (\psi \vee \varphi)$ we get $\models \forall x\varphi \Rightarrow \forall x(\psi \vee \varphi)$ hence $\models (\psi \vee \forall x\varphi) \Rightarrow \forall x(\psi \vee \varphi)$.□

(15) $\models \exists x(\psi \vee \varphi) \rightarrow \psi \vee \exists x\varphi$, where $x \notin S_F(\psi)$.

The proof is similar to that for (13).□

(16) $\models \exists x(\psi \wedge \varphi) \leftrightarrow (\psi \wedge \exists x\varphi)$, where $x \notin S_F(\psi)$.

Proof. Without loss of generality $x \in S_F\varphi$. From $\models \varphi \rightarrow \exists x\varphi$ we have $\models \psi \wedge \varphi \rightarrow \psi \wedge \exists x\varphi$, since $x \notin S_F(\psi \wedge \exists x\varphi)$ we have $\models \exists x(\psi \wedge \varphi) \rightarrow (\psi \wedge \exists x\varphi)$. Conversely, by 0.6.10.2, since $x \in S_F(\psi \wedge \varphi)$, we have $\models (\psi \wedge \varphi) \rightarrow \exists x(\psi \wedge \varphi)$, hence $\models \varphi \rightarrow (\psi \rightarrow \exists x(\psi \wedge \varphi))$. By 0.6.10.4, $\models \exists x\varphi \rightarrow (\psi \rightarrow \exists x(\psi \wedge \varphi))$, hence $\models \psi \wedge \exists x\varphi \rightarrow \exists x(\psi \wedge \varphi)$.□

(17) $\models \exists x(\varphi \rightarrow \psi) \rightarrow (\forall x\varphi \rightarrow \psi)$, where $x \notin S_F(\psi)$.

Proof. Without loss of generality $x \in S_F\varphi$. From $\models \forall x\varphi \rightarrow \varphi$ we have $\models (\varphi \rightarrow \psi) \rightarrow (\forall x\varphi \rightarrow \psi)$, hence (17).□

(18) $\models \exists x(\psi \rightarrow \varphi) \rightarrow (\psi \rightarrow \exists x\varphi)$, where $x \notin S_F(\psi)$.

Proof. Without loss of generality $x \in S_F\varphi$. From $\models \varphi \rightarrow \exists x\varphi$ we have $\models (\psi \rightarrow \varphi) \rightarrow (\psi \rightarrow \exists x\varphi)$, hence (18).□

(19) $\models \forall x(\psi \rightarrow \varphi) \leftrightarrow (\psi \rightarrow \forall x\varphi)$, where $x \notin S_F(\psi)$.

Proof. Without loss of generality $x \in S_F\varphi$. We have $\models \forall x(\psi \rightarrow \varphi) \rightarrow (\psi \rightarrow \varphi)$ hence $\models (\forall x(\psi \rightarrow \varphi)) \wedge \psi \rightarrow \varphi$, hence $\models (\forall x(\psi \rightarrow \varphi)) \wedge \psi \rightarrow \forall x\varphi$, hence $\models \forall x(\psi \rightarrow \varphi) \rightarrow (\psi \rightarrow \forall x\varphi)$. From $\models \forall x\varphi \rightarrow \varphi$ we get $\models (\psi \rightarrow \forall x\varphi) \rightarrow (\psi \rightarrow \varphi)$ hence $\models (\psi \rightarrow \forall x\varphi) \rightarrow \forall x(\psi \rightarrow \varphi)$.□

(20) $\models \forall x(\varphi \rightarrow \psi) \leftrightarrow ((\exists x\varphi) \rightarrow \psi)$, where $x \notin S_F(\psi)$.

Proof. Without loss of generality $x \in S_F\varphi$. We have $\models \forall x(\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \psi)$ hence $\models \varphi \rightarrow (\forall x(\varphi \rightarrow \psi) \rightarrow \psi)$, hence $\models \exists x\varphi \rightarrow (\forall x(\varphi \rightarrow \psi) \rightarrow \psi)$, hence

$\models\ \forall x(\varphi \rightarrow \psi) \rightarrow (\exists x\varphi \rightarrow \psi)$. Conversely from $\models \varphi \rightarrow \exists x\varphi$ we get $\models (\exists x\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \psi)$ hence $\models (\exists x\varphi \rightarrow \psi) \rightarrow \forall x(\varphi \rightarrow \psi)$.□

From (20) by putting $\psi \equiv \perp$ we get $\models \forall x \neg \varphi \leftrightarrow \neg \exists x\varphi$, which yields (11) and (12) of Osius' list and from (17) we get (10) $\models \exists x \neg \varphi \rightarrow \neg \forall x\varphi$.□

<u>0.6.10.10 Proposition.</u> If $x \in s_F\varphi$, $y \in ffr[x](\varphi)$, and $x \not\equiv y$, then $\models \forall x((x = y) \rightarrow \varphi) \leftrightarrow \varphi[x|y]$.

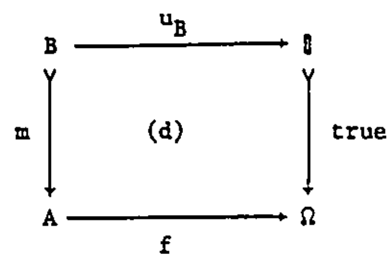
<u>Proof.</u> By 0.6.9.15 $\models ((x = y) \wedge \varphi[x|y]) \rightarrow \varphi$, hence $\models \varphi[x|y] \rightarrow ((x = y) \rightarrow \varphi)$, hence by 0.6.10.5 $\models \varphi[x|y] \rightarrow\ x((x = y) \rightarrow \varphi)$. By 0.6.10.1, $\models \forall x((x = y) \rightarrow \varphi) \rightarrow ((x = y) \rightarrow \varphi)$, substituting y for x we get $\models \forall x((x = y) \rightarrow \varphi) \rightarrow \varphi[x|y]$. □

0.6.11 Translations of simple external statements about morphisms of $\mathcal{E}$ into internally valid statements of $\Phi(\mathcal{E})$ in which $\exists$ is used.

0.6.11.1 Proposition. Let $m: B \rightarrowtail A$ be a monomorphism, $x \not\equiv y$, $x \in V_A$, $y \in V_B$, then

$$f \text{ classifies } m \text{ iff } (1) \models fx \Leftrightarrow \exists y(my \equiv x).$$

Proof. ($\rightarrow$) We assume (d) is a pullback diagram



Since (d) commutes $\models fmy$, hence $\models (my \equiv x) \Rightarrow fx$, hence (2) $\models \exists y(my \equiv x) \Rightarrow fx$. To show (3) $\models fx \Rightarrow \exists y(my \equiv x)$, let $z \in (V_B - \{x,y\})$ and note that $\binom{x}{mz} \equiv Sbs\langle z,m,x\rangle$ is a normalized substitution for $fx \Rightarrow \exists y(my \equiv x)$, thus, by 0.6.5.3, it suffices to prove (4) $\models \exists y(my \equiv mz)$. But (4) is a tautology since from $\models (my \equiv mz) \Rightarrow \exists y(my \equiv mz)$ we deduce $\models (mz \equiv mz) \Rightarrow \exists y(my \equiv mz)$ and hence (4) by modus ponens. From (2) and (3) we have (1).□

($\leftarrow$) We assume (1) and let f' be the morphism classifying m. By what we have just proven ($\rightarrow$), (5) $\models f'y \Leftrightarrow \exists y(my \equiv x)$. Combining (1) and (5) we have $\models f'y \Leftrightarrow fy$, hence $f \equiv f'$.□

0.6.11.2 Corollary. Let $\vec{v}$ be a reduced string of variables, $m: B \rightarrowtail \tau(\vec{v})$ a monomorphism, and $y \in (V_B - \{\vec{v}\})$.

(1): f classifies m iff $\models f\pi(\vec{v}) \Leftrightarrow \exists y(my \equiv \pi(\vec{v}))$.

(2): If, in addition, φ is a formula for which $\{\vec{v}\} = S_F(\varphi)$, then, $m \in [\![v|\varphi]\!]$ iff $\models \varphi \leftrightarrow \exists y(my = \pi(\vec{v}))$.

Proof. (1): By 0.6.11.1 and 0.6.9.16 (2).□ (2) Put $f = \lambda\vec{v}.\varphi$ in (1); then $\models f\pi(\vec{v}) \leftrightarrow \varphi$, so $f\pi(\vec{v})$ can be replaced by φ.□

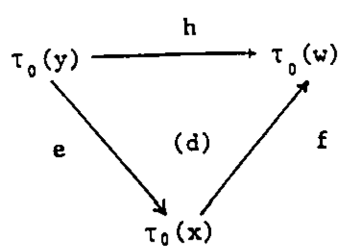
0.6.11.3 Proposition. Let $f\colon A \longrightarrow B$, $\pi\colon B \times A \longrightarrow B$ (the first projection), $\tau_0(x) = \tau_0(w) = A$, $\tau_0(y) = B$, x, w and y distinct. Then

(1) $f \sqcap id_A \in [\![yx|y = fx]\!]$

(2) f is an epimorphism iff $\models \exists x(y = fx)$.

Proof. (1). By 0.6.10.7, $\models (y = fx) \leftrightarrow \exists w((x = x) \wedge (fw = y))$. By 0.6.9.4 (2) and (3) $\models (w = x) \wedge (fw = y) \leftrightarrow \langle fw,w\rangle = \langle y,x\rangle$. By 0.6.9.3 $\models (f \sqcap id_A)w = \langle fw,w\rangle$, hence $\models (y = fx) \leftrightarrow \exists w((f - id_A)w = \langle y,x\rangle$. By 0.6.11.2 (2), $f \sqcap id_A \in [yx|y = fx]$.□ (2) f is an epimorphism iff the pair $\langle f, id_B\rangle$ is an epi-mono factorization of the morphism $f = \pi_*(f \sqcap id_A)$ iff the value under $\exists_\pi\colon Sub(B \times A) \longrightarrow Sub(B)$ of the subobject $[\![f - id_A]\!]$ is $[\![id_B]\!]$ iff $\exists_\pi[yx|y = fx] = [B]$ by (1) above, iff $\models \exists x(y = fx)$.□

0.6.11.4 Proposition. Let x, y, and w be distinct variables and let (d) commute. Then



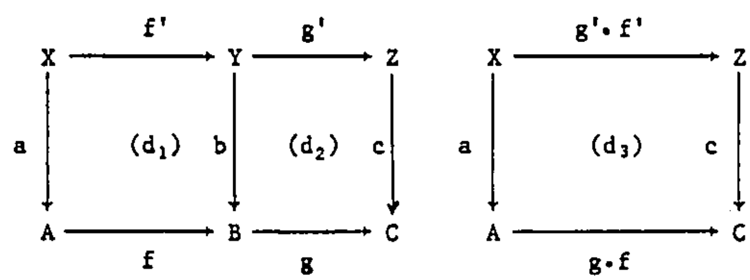
(1) $\models \exists y(hy = w) \Rightarrow \exists x(fx = w)$

(2) If e is an epimorphism then $\models \exists y(hy = w) \Leftrightarrow \exists x(fx = w)$.

Proof. (1) By 0.6.10.2 $\models fx = w \Rightarrow \exists x(fx = w)$ hence $\models fey = w \Rightarrow \exists x(fx = w)$. Since $\models fey = hy$, $\models (hy = w) \Rightarrow \exists x(fx = w)$ therefor (1) holds.

(2) We want to show $\models \exists x(fx = w) \Rightarrow \exists y(fey = w)$. By 0.6.10.2 $\models (fey = w) \Rightarrow \exists y(fey = w)$, by 0.6.9.14 $\models (ey = x) \wedge (fx = w) \Rightarrow (fey = w)$, hence $\models (ey = x) \Rightarrow [(fx = w) \Rightarrow \exists y(fey = w)]$. Since e is an epimorphism, by 0.6.11.3 (2), $\models \exists y(x = ey)$. Hence, by modus ponens, $\models fx = w \Rightarrow \exists y(fey = w)$. By 0.6.10.4 $\models \exists x(fx = w) \Rightarrow \exists y(fey = w)$.□

0.6.11.5 Pullbacks of monomorphisms. Proposition 7.8.4 of [Sch] states that under the assumption that (d_2) is a pullback, (d_1) is a pullback iff (d_3) is a pullback. If we take a and b to be monomorphisms and c to be



the subobject classifier true: $1 \longrightarrow \Omega$, we can restate this as: under the assumption that g classifies b, (d_1) is a pullback iff $g \cdot f$ classifies a.

Proposition. Let a and b be monomorphisms, then (d_1) is a pullback iff $\models \exists v(av = u) \Leftrightarrow \exists y(by = fu)$, where $\tau_0(v) = X$, $\tau_0(u) = A$, $\tau_0(y) = Y$, $\tau_0(z) = B$.

Proof. Let g classify b; by 0.6.11.1 this is equivalent to $\models gz \leftrightarrow \exists y(by = z)$. Then (d_1) is a pullback iff $g \cdot f$ classifies a iff $\models (g \cdot f)u \leftrightarrow \exists v(av = u)$ iff $\models \exists y(by = fu) \leftrightarrow \exists v(av = u)$, since $\models gfu \leftrightarrow \exists y(by = fu)$.□

0.6.11.6 Proposition. Let $\tau_0(x) = \mathbf{1}$, then $\models \exists x\varphi \leftrightarrow \varphi$.

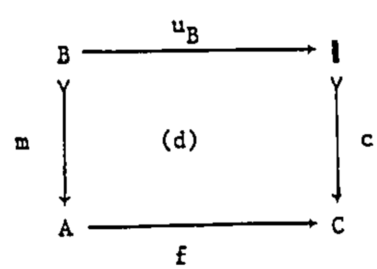
Proof. Let $\varphi' = \nabla(x) \wedge \varphi$ and choose $y \in (V_{\mathbf{1}} - S_F(\varphi'))$. By 0.6.10.7, $\models \exists x((x = y) \wedge \varphi') \leftrightarrow \varphi'[x|y]$. Substitute $*$ for y to get $\models \exists x((x = *) \wedge \nabla(x) \wedge \varphi) \leftrightarrow \nabla(*) \wedge \varphi[x|*]$, hence $\models \exists x\varphi \leftrightarrow \varphi[x|*]$. By 0.6.9.15, $\models (x = *) \rightarrow (\varphi \leftrightarrow \varphi[x|*])$, hence $\models \varphi \leftrightarrow \varphi[x|*]$, hence $\models \exists x\varphi \leftrightarrow \varphi[x|*]$. By 0.6.4.12, $\{x\} \prec \phi$ and $\{y\} \prec \phi$, hence $\models \exists x\varphi \leftrightarrow \varphi$.□

0.6.11.7 Proposition. Let $m: \mathbf{1} \rightarrowtail A$, $x \in V_A$, then

$$f \text{ classifies } m \text{ iff } \models fx \leftrightarrow (m* = x).$$

Proof. Let $y \in (V_{\mathbf{1}} - \{x\})$. By 0.6.11.6, $\models \exists y(my = x) \leftrightarrow (my = x)$, hence $\models \exists y(my - x) \leftrightarrow (m* = x)$. By 0.6.11.1, f classifies m iff $\models fx \leftrightarrow \exists y(my = z)$ iff $\models fx \leftrightarrow (m* = x)$.□

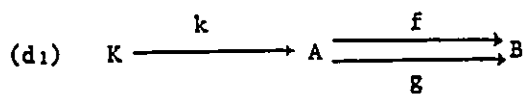
0.6.11.8 Proposition. Let $\tau_0(x) = B$, $\tau_0(y) = A$, m, c monos, then (d) is a pullback iff $\models fy = c* \leftrightarrow \exists x(mx = y)$.

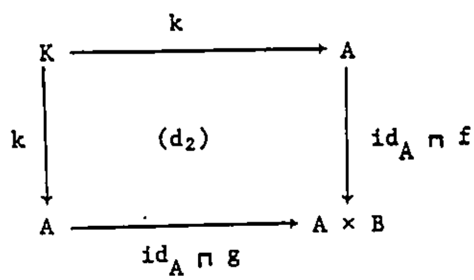


Proof. By 0.6.11.5, (d) is a pullback iff $\models \exists x(mx = y) \leftrightarrow \exists z(cz = fy)$ where $\tau_0(z) = \mathbf{1}$.

By 0.6.11.6, $\models \exists z(cz = fy) \leftrightarrow (cz = fy)$, hence substituting $*$ for z, $\models \exists z(cz = fy) \leftrightarrow (c* = fy)$. Thus (d) is a pullback iff $\models \exists x(mx = y) \leftrightarrow (c* = fy)$.□

<u>0.6.11.9 Characterization of equalizers.</u> By 7.8.7 of [Sch] p. 60, (d_1) is an equalizer diagram iff (d_2) is a pullback diagram.





Using this equivalence and 0.6.11.5 we can prove:

<u>Proposition.</u> The following are equivalent

(1) (d_1) is an equalizer diagram

(2) k is a monomorphism and $\models (fu = gu) \leftrightarrow \exists v(kv = u)$, where $\tau_0(u) = A$ and $\tau_0(v) = K$.

<u>Proof.</u> We assume throughout that k is a monomorphism; if it were not then clearly $(1) \leftrightarrow (2)$. By 0.6.11.5 (1) is equivalent to

(3) $\models \exists v(kv = u) \leftrightarrow \exists y((id_A \sqcap f)y = (id_A \sqcap g)\, u)$, where $\tau_0(y) = A$.

Since $\models [(id_A \sqcap f)y = (id_A \sqcap g)\, u] \leftrightarrow [(y = u) \wedge (fy = gu)]$,

(3) is equivalent to (4) $\models \exists v(kv = u) \leftrightarrow \exists y((y = u) \wedge (fy = gu))$. By 0.6.10.7 $\models (fu = gu) \leftrightarrow \exists y((y = u) \wedge (fy = gu))$, hence (4) is equivalent to (2).□

<u>0.6.11.10.</u> Let $\pi_i: A_1 \times A_2 \longrightarrow A_i$ $(i = 1,2)$ be projections, $m: M \rightarrowtail A_1 \times A_2$ a mono, and $f_i: A_i \longrightarrow B$ $(i = 1,2)$ morphisms.

$$(d_1) \qquad M \overset{m}{\rightarrowtail} A_1 \times A_2 \underset{f_2 \circ \pi_2}{\overset{f_1 \circ \pi_1}{\rightrightarrows}} B$$

$$\begin{array}{ccc} M & \xrightarrow{\pi_2 \circ m} & A_2 \\ {\scriptstyle \pi_1 \circ m}\downarrow & (d_2) & \downarrow{\scriptstyle f_2} \\ A_1 & \xrightarrow[f_1]{} & B \end{array}$$

If (d_1) is an equalizer then (d_2) is a pullback, (see [HS] 21.3). The converse also holds: Let (d_2) be a pullback and let $k: P \longrightarrow A_1 \times A_2$ be such that (1) $f_1 \circ \pi_1 \circ k = f_2 \circ \pi_2 \circ k$. Since (d_2) is a pullback the system of equations

(2) $\quad \pi_i \circ m \circ p = \pi_i \circ k \qquad (i = 1,2)$

has a unique solution $p: P \longrightarrow M$. But (2) is equivalent to the single equation

(3) $\quad m \circ p = k$.

Hence (d_1) is an equalizer. Thus (d_1) is an equalizer iff (d_2) is a pullback.

<u>0.6.11.11 Proposition.</u> Given π_i, f_i $(i = 1,2)$ and m a mono, as in 0.6.11.10, then (d_2) is a pullback iff

(1) $\quad \models (f_1x_1 = f_2x_2) \leftrightarrow \exists w(mw = \langle x_1x_2 \rangle)$,

where $\tau_0(x_i) = A_i$ $(i = 1,2)$, $\tau_0(w) = M$.

<u>Proof.</u> (d_2) is a pullback iff (d_1) is an equalizer iff, by 0.6.11.9,

$\models ((f_1 \circ \pi_1)u = (f_2 \circ \pi_2)u) \leftrightarrow \exists w(mw = u)$, where $\tau_0(u) \equiv A_1 \times A_2$. Substitute $\langle x_1, x_2 \rangle$ for u, then, since $\models (f_i \circ \pi_i)\langle x_1, x_2 \rangle = f_i x_i$ $(i \equiv 1,2)$,

(1) holds.□

0.6.11.12 Proposition. (1) Let $m: \tau_0(y) \longrightarrow \tau_0(x)$ be a monomorphism, $x \not\equiv y$. The following are equivlaent.

(a) m is the equalizer of f_1 and f_2

(b) $\models (f_1 x = f_2 x) \leftrightarrow \exists y(my = x)$

(c) $|f_1 x = f_2 x|$ classifies m.

(2) Let $m: \tau_0(y) \longrightarrow \tau(\vec{v})$, be a mono, $y \notin \{\vec{v}\}$, $\langle \vec{v}, t_1 = t_2 \rangle$ an augmented formula. The following are equivalent.

(d) m is the equalizer of $\lambda\vec{v}.t_1$ and $\lambda\vec{v}.t_2$

(e) $\models (t_1 = t_2) \leftrightarrow \exists y(my = \pi(\vec{v}))$

(f) $\lambda\vec{v}.(t_1 = t_2)$ classifies m.

Proof. (1) follows from 0.6.11.2 (2) and 0.6.11.9.□ (2) Put $f_i \equiv \lambda\vec{v}.t_i$ for $i \equiv 1,2$. From (1), (d) is equivalent to (b), which, by 0.6.9.16 (2), is equivlaent to

$$\models (f_1\pi(\vec{v}) = f_2\pi(\vec{v})) \Leftrightarrow \exists y(my = \pi(\vec{v}))$$

which is equivalent to (e). By 0.6.11.2 (2), (e) is equivalent to (f).□

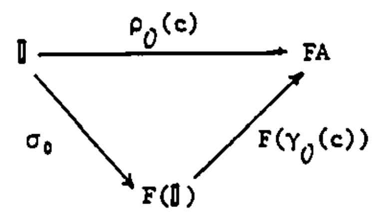
<u>0.6.12 Cartesian functors and basic Horn formulas.</u> Let $\mathcal{A}$ be an internal $\langle\mathcal{O},\mathcal{P}\rangle$-structure in a topos $\underline{\mathcal{E}}$ and let $F\colon \underline{\mathcal{E}} \longrightarrow \underline{\mathcal{G}}$ be a cartesian functor of toposes. In this section we shall show that F induces an internal $\langle\mathcal{O},\mathcal{P}\rangle$-structure $F(\mathcal{A})$ in the topos $\underline{\mathcal{G}}$, and that $F(\mathcal{A})$ is a model of all basic Horn formulas and equivalences of atomic formulas of which $\mathcal{A}$ itself is a model. We have already exhaustively investigated a special case of this theorem in 0.6.3 and 0.6.6; there we showed that for each object B of $\underline{\mathcal{E}}$, the cartesian functor $[B,-] : \underline{\mathcal{E}} \longrightarrow \underline{\mathcal{S}}$ induces an external model $[B,\mathcal{A}]$ of all basic Horn formulas which are valid in $\mathcal{A}$. The cartesian functors to which we shall apply the more general theory of this section are: internal exponentiation $(-)^B : \underline{\mathcal{E}} \longrightarrow \underline{\mathcal{G}}$ (in 0.6.13) and the "forgetful" functor $\underline{\mathcal{E}}^{\underline{C}^0} \longrightarrow \underline{\mathcal{E}}$ (in 1.6.2.).

<u>0.6.12.1 Notation for canonical maps.</u> For each $n \geq 1$, A^n and $(FA)^n$ (defined in 0.3.3.1) come equipped with projections $pr_i : A^n \longrightarrow A$ and $pr_i'\colon (FA)^n \longrightarrow FA$ where $i \in [n]$ (defined in 0.4.3.2). By 0.4.4.2 there is a canonical isomorphism $\sigma_n\colon (FA)^n \longrightarrow F(A^n)$ such that $F(pr_i)\circ\sigma_n = pr_i'$ for each $i \in [n]$. For $n = 0$ we also have the isomorphism $\sigma_0\colon \mathbb{I} \longrightarrow F\mathbb{I}$ where $\mathbb{I}$ and $\mathbb{I}$ are the canonical terminal objects of $\underline{\mathcal{E}}$ and $\underline{\mathcal{G}}$ respectively. Throughout 0.6.12, σ_n^{-1} will be the inverse image function $\sigma_n^{-1}\colon \mathrm{Sub}\, F(A^n) \longrightarrow \mathrm{Sub}((Fa)^n)$ induced by σ_n (as in 0.5.1.1), not the inverse morphism of σ_n having domain $F(A^n)$ and codomain $(FA)^n$ -which was used explicitly in 0.4.4.3 and 0.4.4.4.

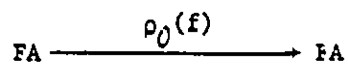
<u>0.6.12.2 The internal algebra $F(\mathcal{A}_{\mathcal{O}})$ induced by F.</u> Let $\mathcal{A}_{\mathcal{O}} = \langle A,\gamma_{\mathcal{O}}\rangle$ be an internal algebra in $\underline{\mathcal{E}}$ of similarity type $\mathcal{O}$. We define an internal

$\mathcal{O}$-algebra $F(\mathcal{A}_{\mathcal{O}}) = \langle FA, \rho_{\mathcal{O}} \rangle$ in $\underline{\mathcal{G}}$ as follows:

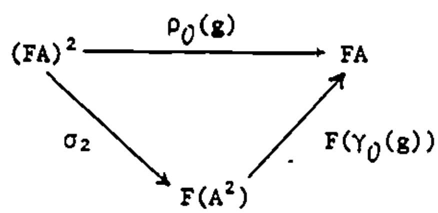
For $c \in \mathcal{O}_0$ we put $\rho_{\mathcal{O}}(c) = F(\gamma_{\mathcal{O}}(c)) \circ \sigma_0$



For $f \in \mathcal{O}_1$ we put $\rho_{\mathcal{O}}(f) = F(\gamma_{\mathcal{O}}(f))$



For $g \in \mathcal{O}_2$ we put $\rho_{\mathcal{O}}(g) = F(\gamma_{\mathcal{O}}(g)) \circ \sigma_2$



The functions $\gamma_{\mathcal{O}}$ and $\rho_{\mathcal{O}}$ determine interpretations γ' and ρ' of $\mathbb{E}(\mathcal{O})$ in $\Phi(\underline{\mathcal{S}})$ of type A and in $\Phi(\underline{\mathcal{G}})$ of type FA respectively (see 0.3.3.2 (2)); moreover we can uniquely determine γ' and ρ' by having them map the i-th variable of V to the i-th variables of V_A and V_{FA} respectively (so that they are compatible with interpretations of formula elphabets defined in 0.6.3.1). For each string of signs of $\mathbb{E}(\mathcal{O})$ we put $\bar{s} = \tilde{\gamma}'(s)$ and $\bar{\bar{s}} = \tilde{\rho}'(s)$. By 0.3.3.3 (1) it follows that if $t \in \text{Poly}(\mathbb{E}(\mathcal{O}))$ then $\bar{t} \in T_A$ and $\bar{\bar{t}} \in T_{FA}$. Since γ' and ρ' are bijective on variables, $\tilde{\gamma}'$ and $\tilde{\rho}'$ restrict to monoid join-isomorphisms of reduced strings.

<u>0.6.12.3 Proposition.</u> Let $t \in \mathrm{Poly}(\mathbb{E}(\mathcal{O}))$ and let $\vec{v}$ be a reduced string of variables of V such that $s_F(t) \subset \{\vec{v}\}$. Put $\vec{u} = \tilde{\gamma}'(\vec{v})$, $\vec{w} = \tilde{\rho}'(\vec{v})$ and $n = \ell(\vec{v})$. Then

$$F(\lambda\vec{u}.\overline{t})\circ\sigma_n = \lambda\vec{w}.\overline{\overline{t}}.$$

<u>Proof.</u> We show that the set H of all polynomials t, for which the statement holds, is an $\mathcal{O}$-subalgebra of $\mathrm{Poly}(\mathbb{E}(\mathcal{O}))$ containing V.

Let $x \in V$. For $x \in \{\vec{v}\}$ we have $\overline{x} \in \{\vec{u}\}$, $\overline{\overline{x}} \in \{\vec{w}\}$, and for some $i \in [n]$, $(\vec{v}(i))\hat{} = x$. Hence

$$pr_i = \lambda\vec{u}.\overline{x} \quad \text{and} \quad pr_i' = \lambda\vec{w}.\overline{\overline{x}}\ ,$$

therefore $F(\lambda\vec{u}.\overline{x})\circ\sigma_n = \lambda\vec{w}.\overline{\overline{x}}$ (by 0.4.4.2). Thus $V \subset H$.

Let $c \in \mathcal{O}_0$. Then

$\overline{\overline{c*}} = \tilde{\rho}'(c*) = (\tilde{\rho}'(c))(\tilde{\rho}'(*)) = (\rho'(c))(\rho'(*)) = (\rho_{\mathcal{O}}(c))*$

$= ((F(\gamma_{\mathcal{O}}(c)))\circ\sigma_0)*$, hence $\lambda\vec{w}.\overline{\overline{c*}} = (F(\overline{c}))\circ\sigma_0\circ(\lambda\vec{w}.*)$. Also $F(\lambda\vec{u}.\overline{c*})\circ\sigma_n = F(\overline{c})\circ F(\lambda\vec{u}.*)\circ\sigma_n$. But (1) must commute since $F(\mathbb{1})$ is terminal,

$$\begin{array}{ccc} (FA)^n & \xrightarrow{\ \sigma_n\ } & F(A^n) \\ {\scriptstyle \lambda\vec{w}.*}\downarrow & (1) & \downarrow{\scriptstyle F(\lambda\vec{u}.*)} \\ \mathbb{1} & \xrightarrow{\ \sigma_0\ } & F(\mathbb{1}) \end{array}$$

hence $F(\lambda\vec{u}.\overline{c*})\circ\sigma_n = \lambda\vec{w}.\overline{\overline{c*}}$. Thus $\overline{c*} \in H$ for each $c \in \mathcal{O}_0$.

We omit the proof for $f \in \mathcal{O}_1$. It is straightforward. Let $g \in \mathcal{O}_2$ and $\langle t_1,t_2\rangle \in H \times H$. Suppose $s_F(g/t_1,t_2/) \subset \{\vec{v}\}$, then $s_F(t_i) \subset \{\vec{v}\}$, hence $F(\lambda\vec{u}.\overline{t_i})\circ\sigma_n = \lambda\vec{w}.t_i$ for $i = 1,2$. Therefore, using 0.4.4.3, we have:

$$F(\lambda\vec{u}.\overline{g/t_1,t_2/})\circ\sigma_n = F(\bar{g})\circ F((\lambda\vec{u}.\bar{t}_1) \sqcap (\lambda\vec{u}.\bar{t}_2))\circ\sigma_n$$
$$= F(\bar{g})\circ\sigma_2\circ((F(\lambda\vec{u}.\bar{t}_1)\circ\sigma_n) \sqcap (F(\lambda\vec{u}.\bar{t}_2)\circ\sigma_n))$$
$$= \bar{\bar{g}}\circ(\lambda\vec{w}.\bar{\bar{t}}_1 \sqcap \lambda\vec{w}.\bar{\bar{t}}_2)$$
$$= \lambda\vec{w}.\overline{\overline{g/t_1,t_2/}} .$$

Hence $g/t_1,t_2/ \in H$. Thus $H = \mathrm{Poly}_{\mathcal{O}}(V) = \mathrm{Poly}(\mathbb{E}(\mathcal{O}))$.□

In order to extend the functor F to structures we have to explain how F induces an interpretation of predicate signs in $\Phi(\mathcal{G})$ from a given interpretation in $\Phi(\mathcal{E})$. For this we first have to define F on subobjects.

<u>0.6.12.4 Proposition.</u> For each object B of $\mathcal{E}$, F induces a function $\tilde{F} : \mathrm{Sub}(B) \longrightarrow \mathrm{Sub}(FB)$ such that

(1) for each mono $b: B_0 \rightarrowtail B$, $\tilde{F}([\![b]\!]) = [\![F(b)]\!]$

(2) $\tilde{F}[\![B]\!] = [\![FB]\!]$

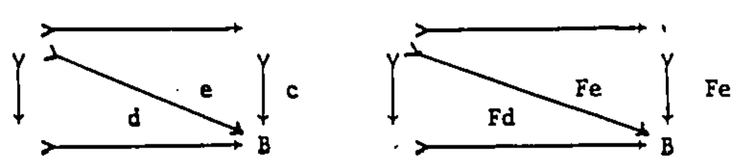
(3) for any $\mathcal{C}, \mathcal{D} \in \mathrm{Sub}(B)$, $\tilde{F}(\mathcal{C} \wedge \mathcal{D}) = \tilde{F}(\mathcal{C}) \wedge \tilde{F}(\mathcal{D})$

(4) $\mathcal{C} \leq \mathcal{D}$ implies $\tilde{F}(\mathcal{C}) \leq \tilde{F}(\mathcal{D})$.

<u>Proof.</u> Since F preserves monos and composition, $b_1 \prec b_2$ implies $F(b_1) \prec F(b_2)$ (see 0.5.1.1), hence $b_1 \sim b_2$ implies $F(b_1) \sim F(b_2)$; thus there is a well-defined function $\tilde{F}$ such that $\tilde{F}[\![b]\!] = [\![Fb]\!]$.□

(2) Follows from (1) since $F(id_B) = id_{FB}$.□

(3) Let $\mathcal{C} = [\![c]\!]$, $\mathcal{D} = [\![d]\!]$, $[\![e]\!] = \mathcal{C} \wedge \mathcal{D}$ then the square (s) is a pullback, hence its image under F,



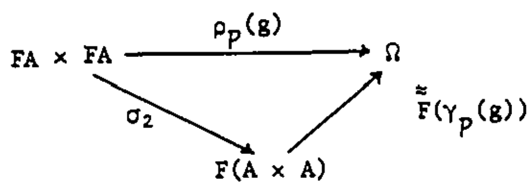
(s) (s')

the square (s'), is a pullback. Thus

$\tilde{F}(\mathcal{C} \wedge \mathcal{D}) = [\![Fe]\!] = [\![Fc]\!] \wedge [\![Fd]\!] = \tilde{F}(\mathcal{C}) \wedge \tilde{F}(\mathcal{D})$.□ (4) Follows from (3).□

<u>0.6.12.5 Definition.</u> For each $f: B \longrightarrow \varOmega$ we let $\tilde{\tilde{F}}(f): FB \longrightarrow \Omega$ be the characteristic map of the subobject $\tilde{F}(f^{-1}[\![true]\!])$ of $\mathrm{Sub}(FB)$.
If $f: B \longrightarrow \varOmega$ classifies $d: D \rightarrowtail B$ then $\tilde{F}(f^{-1}[\![true]\!]) = \tilde{F}[\![d]\!] = [\![Fd]\!]$, and $\tilde{\tilde{F}}(f)$ classifies Fd, that is $(\tilde{\tilde{F}}(f))^{-1}[\![true]\!] = [\![Fd]\!] = \tilde{F}(f^{-1}[\![true]\!])$.

<u>0.6.12.6 The internal structure $F(\mathcal{A})$ induced by F.</u> Let $\mathcal{A} = \langle A, \gamma_{\mathcal{O}}, \gamma_{\mathcal{P}} \rangle$ be an internal $\langle \mathcal{O}, \mathcal{P} \rangle$-structure in $\underline{\mathcal{E}}$. We define an internal $\langle \mathcal{O}, \mathcal{P} \rangle$-pre-structure $F(\mathcal{A}) = \langle FA, \rho_{\mathcal{O}}, \rho_{\mathcal{P}} \rangle$ in $\underline{\mathcal{G}}$ as follows.
For $\mathcal{A}_{\mathcal{O}} = \langle A, \gamma_{\mathcal{O}} \rangle$ we put $(F(\mathcal{A}))_{\mathcal{O}} = F(\mathcal{A}_{\mathcal{O}}) = \langle FA, \rho_{\mathcal{O}} \rangle$, as in 0.6.12.2.
For $f \in P_1$ we put $\rho_{\mathcal{P}}(f) = \tilde{\tilde{F}}(\gamma_{\mathcal{P}}(f))$.
For $g \in P_2$ we put $\rho_{\mathcal{P}}(g) = (\tilde{\tilde{F}}(\gamma_{\mathcal{P}}(g))) \circ \sigma_2$; that is



commutes.

We now show that $F(\mathcal{A})$ is actually an internal structure (see 0.6.3.1).

<u>0.6.12.7 Proposition.</u> $\rho_{\mathcal{P}}(\underline{\delta}) = \delta_{FA}$.

<u>Proof.</u> By definition $\rho_{\mathcal{P}}(\underline{\delta}) = (\tilde{\tilde{F}}(\gamma_{\mathcal{P}}(\underline{\delta}))) \circ \sigma_2 = \tilde{\tilde{F}}(\delta_A) \circ \sigma_2$,

$$(\tilde{\tilde{F}}(\delta_A) \circ \sigma_2)^{-1}[\![true]\!] = \sigma_2^{-1}((\tilde{\tilde{F}}(\delta_A))^{-1}[\![true]\!]) = \sigma_2^{-1}(\tilde{F}(\delta_A^{-1}[\![true]\!]))$$
$$= \sigma_2^{-1}(\tilde{F}[\![\Delta_A]\!]) = \sigma_2^{-1}[\![F(\Delta_A)]\!],$$

and $\delta_{FA}^{-1}[\![true]\!] = [\![\Delta_{FA}]\!]$.

By 0.4.4.2, (1) commutes, hence it is a pullback

$$
\begin{array}{ccc}
FA & \xrightarrow[\;=\;]{\text{id}} & FA \\
{\scriptstyle \Delta_{FA}} \big\downarrow & (1) & \big\downarrow {\scriptstyle F(\Delta_A)} \\
(FA)^2 & \xrightarrow[\sigma_2]{\sim} & F(A^2)
\end{array}
$$

diagram, thus $\sigma_2^{-1}[\![F(\Delta_A)]\!] = [\![\Delta_{FA}]\!]$, hence $(\rho_P(\underline{\delta}))^{-1}[\![\mathrm{true}]\!] = \delta_{FA}^{-1}[\![\mathrm{true}]\!]$.□

<u>0.6.12.8 Notation.</u> The function $\gamma_{\mathcal{O}} \cup \gamma_{\mathcal{P}}$ extends uniquely to an interpretation (0.6.3.1) γ of $\mathbb{P}\langle\mathcal{O},\mathcal{P}\rangle$ in $\Phi(\underline{\mathcal{E}})$ of type A, and the function $\rho_{\mathcal{O}} \cup \rho_{\mathcal{P}}$ extends to an interpretation ρ of $\mathbb{P}\langle\mathcal{O},\mathcal{P}\rangle$ in $\Phi(\underline{\mathcal{G}})$ of type FA. The restrictions of γ and ρ to the signs of the operation alphabet $\mathbb{E}(\mathcal{O})$ are γ' and ρ' respectively (see 0.6.12.2). Since γ and ρ are bijective on variables, $\tilde{\gamma}$ and $\tilde{\rho}$ are isomorphisms from the join monoid <u>RdcStr</u>(V) of reduced strings of variables of V to the monoids <u>RdcStr</u>(V_A) and <u>RdcStr</u>(V_{FA}) respectively. For each string, s, of signs of $\mathbb{P}\langle\mathcal{O},\mathcal{P}\rangle$, we put $\bar{s} = \tilde{\gamma}(s)$ and $\bar{\bar{s}} = \tilde{\rho}(s)$.

<u>0.6.12.9 Proposition.</u> Let $\varphi \in \mathrm{AtFml}(\mathbb{P}\langle\mathcal{O},\mathcal{P}\rangle)$ and $\vec{v} \in \mathrm{RdcStr}(V)$ such that $s_F(\varphi) \subset \{\vec{v}\}$. Put $n = \ell(\vec{v})$, $\vec{u} = \tilde{\gamma}(\vec{v})$ and $\vec{w} = \tilde{\rho}(\vec{v})$. Then

$$\sigma_n^{-1}(\tilde{F}([\![\vec{u}|\bar{\varphi}]\!])) = [\![\vec{w}|\bar{\bar{\varphi}}]\!] .$$

<u>Proof.</u> We shall prove the proposition for binary atomic formulas only; that is we take $\varphi = g\langle t_1,t_2\rangle$ where $g \in \mathcal{P}_2$ and t_1, $t_2 \in \mathrm{Poly}(\mathbb{P}\langle\mathcal{O},\mathcal{P}\rangle)$. We have that $\bar{\varphi} = \bar{g}\langle\bar{t}_1,\bar{t}_2\rangle$, $\bar{\bar{\varphi}} = \bar{\bar{g}}\langle\bar{\bar{t}}_1,\bar{\bar{t}}_2\rangle = (\tilde{F}(\bar{g})\cdot\sigma_2)\langle\bar{\bar{t}}_1,\bar{\bar{t}}_2\rangle$ and $\lambda u.\bar{\varphi} = \bar{g}\cdot\lambda u.\langle\bar{t}\ ,\bar{t}\ \rangle$.

Let $k = \lambda\vec{u}.\langle\bar{t}_1,\bar{t}_2\rangle$ and let a be a mono classified by $\bar{g}$. We form a pullback (1)

$$\begin{array}{ccc} Y & \longrightarrow & Y \\ {\scriptstyle b}\downarrow & (1) & \downarrow{\scriptstyle a} \\ A^n & \underset{k}{\longrightarrow} & A^2 \end{array}$$

then $\bar{g}\circ k$ must classify the mono b. Applying F to (1) yields a pullback (2),

$$\begin{array}{ccc} Y & \longrightarrow & Y \\ {\scriptstyle Fb}\downarrow & (2) & \downarrow{\scriptstyle Fa} \\ F(A^n) & \underset{Fk}{\longrightarrow} & F(A^2) \end{array}$$

then $\tilde{F}(\bar{g})$ classifies Fa, so $\tilde{F}(\bar{g})\circ Fk$ classifies Fb. We have

$$\begin{aligned} \lambda\vec{w}.\bar{\varphi} &= (\tilde{F}(\bar{g})\circ\sigma_2)\circ(\lambda\vec{w}.\bar{t}_1 \sqcap \lambda\vec{w}.\bar{t}_2) \\ &= (\tilde{F}(\bar{g})\circ\sigma_2)\circ\left[((F(\lambda\vec{u}.\bar{t}_1))\circ\sigma_n) \sqcap ((F(\lambda\vec{u}.\bar{t}_2))\circ\sigma_n)\right] && \text{by 0.6.12.3} \\ &= \tilde{F}(\bar{g})\circ(F(\lambda\vec{u}.\bar{t}_1 \sqcap \lambda\vec{u}.\bar{t}_2))\circ\sigma_n && \text{by 0.4.4.3} \\ &= \tilde{F}(\bar{g})\circ F(k)\circ\sigma_n \quad \text{, hence} \end{aligned}$$

$$\begin{aligned} [\![\vec{w}|\bar{\varphi}]\!] &= \sigma_n^{-1}((\tilde{F}(\bar{g})\circ F(k))^{-1}[\![\mathrm{true}]\!]) = \sigma_n^{-1}[\![Fb]\!] = \sigma_n^{-1}(\tilde{F}[\![b]\!]) \\ &= \sigma_n^{-1}(\tilde{F}(\bar{g}\circ k)^{-1}[\![\mathrm{true}]\!])) = \sigma_n^{-1}(\tilde{F}[\![\vec{u}|\bar{\varphi}]\!])).\square \end{aligned}$$

<u>0.6.12.10 Proposition.</u> If φ is a basic Horn formula or an equivalence of atomic formulas and φ is valid in $\mathcal{A}$, then φ is valid in $F(\mathcal{A})$.

<u>Proof.</u> Let $\vec{v} = \mathrm{var}(\varphi)$. We use the notation introduced in 0.6.12.6, and we put $n = \ell(\vec{v})$, $\vec{u} = \tilde{\gamma}(\vec{v})$, $\vec{w} = \tilde{\rho}(\vec{v})$.

(1) Suppose φ is atomic and $\mathcal{A} \models \varphi$. We have $[\![A^n]\!] = [\![\vec{u}|\bar{\varphi}]\!]$. Thus $[\![\vec{w}|\bar{\varphi}]\!] = \sigma_n^{-1}(F[\![\vec{u}|\bar{\varphi}]\!]) = \sigma_n^{-1}([\![F(A^n)]\!]) = [\![(FA)^n]\!]$. That is $F(\mathcal{A}) \models \varphi$.□

(2) Suppose $\varphi = \bigwedge_{i=0}^{n-1} \varphi_i \Rightarrow \psi$ where φ_i $(i \in [n]; n \geq 1)$ and ψ are all atomic.

$$[\![A^n]\!] = [\![\vec{u}|\overline{\varphi}]\!] = (\bigwedge_{i=0}^{n-1} [\![\vec{u}|\overline{\varphi}_i]\!]) \Rightarrow [\![\vec{u}|\overline{\psi}]\!]$$

hence $\bigwedge_{i=0}^{n-1} [\![\vec{u}|\overline{\varphi}_i]\!] \leq [\![\vec{u}|\overline{\psi}]\!]$

hence $\bigwedge_{i=0}^{n-1} (\sigma_n^{-1}(\tilde{F}[\![\vec{u}|\overline{\varphi}_i]\!])) \leq (\sigma_n^{-1}(\tilde{F}[\![\vec{u}|\overline{\psi}]\!]))$ by 0.6.12.4 (3) and (4),

hence, by 0.6.12.9,

$$\bigwedge_{i=0}^{n-1} [\![\vec{v}|\overline{\varphi}_i]\!] \leq [\![\vec{v}|\overline{\psi}]\!]$$

hence $[\![\vec{v}|\overline{\varphi}]\!] = [\![\vec{v}|\bigwedge_{i=0}^{n-1} \overline{\varphi}_i \Rightarrow \overline{\psi}]\!] = [\![(FA)^n]\!]$, thus $F(\mathcal{A}) \models \varphi$.□

(3) Suppose $\varphi = \varphi_1 \Leftrightarrow \varphi_2$ where φ_1 and φ_2 are atomic. For $\{i,j\} = \{1,2\}$ we have $\mathcal{A} \models \varphi_i \Rightarrow \varphi_j$ hence by (2) $F(\mathcal{A}) \models \varphi_i \Rightarrow \varphi_j$. Thus $F(\mathcal{A}) \models \varphi$. □

<u>0.6.12.11 Proposition.</u> If F is in addition faithful, and $F(\mathcal{A}) \models t_1 = t_2$ where $t_1, t_2 \in \text{Poly}(E(\mathcal{U}))$, then

$$\mathcal{A} \models t_1 = t_2.$$

<u>Proof.</u> Let $\vec{v} = \text{var}(t_1 t_2)$. We maintain the notation of 0.6.12.10. We have $\models \overline{t}_1 = \overline{t}_2$, hence $\lambda\vec{w}.\overline{t}_1 = \lambda\vec{w}.\overline{t}_2$, hence by 0.6.12.3, $F(\lambda\vec{u}.\overline{t}\ )\circ\sigma_n = F(\lambda\vec{u}.\overline{t}\ )\circ\sigma_n$, then since σ_n is an isomorphism, $F(\lambda\vec{u}.\overline{t}_1) = F(\lambda\vec{u}.\overline{t}_2)$, hence since F is faithful, $\lambda\vec{u}.\overline{t}_1 = \lambda\vec{u}.\overline{t}_2$, hence $\models \overline{t}_1 = \overline{t}_2$, hence $\mathcal{A} \models t_1 = t_2$.□

Note that we cannot extend 0.6.12.11 to basic Horn formulas because faithful ness of F does not imply F is one-to-one on subobjects -that is monos a and b with common codomain A^n can be incomparable even though Fa and Fb are comparable.

<u>0.6.13 Translations into $\Phi(\mathcal{E})$ of basic properties of a cartesian closed category.</u> Some of the valid formulas we prove appear in [O1] p. 322. For the category theory involved see [MacL] Chapter IV.

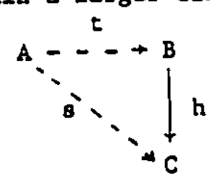
<u>0.6.13.1 Notation for "internal morphisms".</u> We use the notation

$$t\colon A \dashrightarrow B$$

or

$$A \overset{t}{\dashrightarrow} B$$

for triples (t,A,B) where A and B are objects and t is a term of type B^A. We call $t\colon A \dashrightarrow B$ an internal morphism. The alternative notation may appear within a larger diagram, for example



would mean $t\colon A \dashrightarrow B$ and $s\colon A \dashrightarrow C$ are internal morphism. Sometimes, to denote the term part of an internal morphism, we shall use script variants of the latin letters used to denote morphisms of $\mathcal{E}$, thus when $f_1\colon B \longrightarrow C$ we may use $\mathcal{f}_1\colon B \dashrightarrow C$ for an internal morphism.

<u>0.6.13.2 "Composing" a morphism $h\colon B \longrightarrow C$ with an internal morphism $t\colon A \dashrightarrow B$.</u> Applying the functor $(-)^A$ to the morphism $h\colon B \longrightarrow C$ yields a morphism $h^A\colon B^A \longrightarrow C^A$. If $t\colon A \dashrightarrow B$ is an internal morphism, i.e. t is a term of type B^A, then $h^A t$ is a term of type C^A. We put

$$h{\bullet}t \equiv h^A t$$

and call the internal morphism $h{\bullet}t\colon A \dashrightarrow C$ the composite of the internal morphism $t\colon A \dashrightarrow B$ with the morphism h.

Since $(-)^A$ preserves identity morphisms we have, for $id_B: B \longrightarrow B$

$$(id_B)^A \equiv id_{B^A}$$

hence

$$id_B \bullet t \equiv id_{B^A} t$$

hence

$$(1) \qquad \models id_B \bullet t = t \qquad \text{, by 0.6.9.3 (2).}$$

Since $(-)^A$ preserves composition, if $k: C \longrightarrow D$, then $(k \bullet h)^A \equiv k^A \bullet h^A$. By 0.6.9.3 (3), $\models (k^A \bullet h^A)t = k^A(h^A t)$ hence

$$(2) \models (k \bullet h) \bullet t = k \bullet (h \bullet t).$$

<u>0.6.13.3 Consequences of the adjointness $(-) \times A - (-)^A$.</u> For each B, $\epsilon_B^A: B^A \times A \longrightarrow B$ is the back adjunction at B and for each C, $\eta_C^A: C \longrightarrow (C \times A)^A$ is the front adjunction at C (cf. 0.5.1.4). In what follows we supress the superscripts "A" in "ϵ_B^A" and "η_C^A" to avoid confusion with the role of A as a functor e.g. "the image under $(-)^A$ of f is f^A". For each term s of type A and internal morphism $t: A - - \rightarrow B$ we put

$$t[s] \equiv \epsilon_B \langle t,s \rangle \ .$$

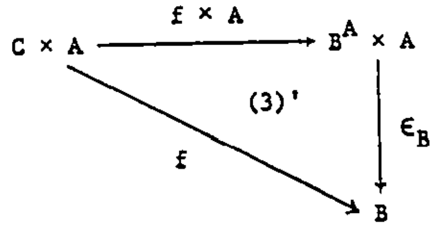
For each term r of type C we put

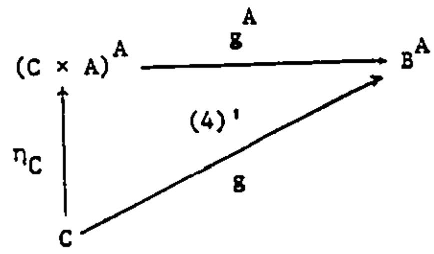
$$\hat{r} \equiv \eta_C \, r \quad ;$$

If the morphisms f and g correspond under the adjointness $(-) \times A - (-)^A$:

$$\frac{C \xrightarrow{\quad f \quad} B^A}{C \times A \xrightarrow[\quad g \quad]{} B}$$

then we have commuting triangles





which translate into the valid equations:

(3) $\models g\langle c,a\rangle = (fc)[a]$

(4) $\models (fc) = g \circ \check{c}$

where $\tau_0 c = C$ and $\tau_0 a = A$.

The correspondence

$$\frac{C \xrightarrow{\eta_C} (C \times A)^A}{C \times A \xrightarrow[id_{C \times A}]{} C \times A}$$

gives us

(5) $\models \langle c,a\rangle = \check{c}[a]$,

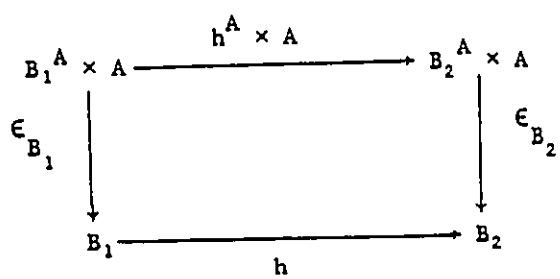
and

$$\frac{B^A \xrightarrow{id_{B^A}} B^A}{B^A \times A \xrightarrow[\epsilon_B]{} B}$$

yields

(6) $\models t = \epsilon_B \check{t}$ where $t\colon A \dashrightarrow B$.

<u>0.6.13.4</u> The naturality of ϵ means that we have a commutative square



which translates into

(7) $\models (h \circ t_1)[a] = h(t_1[a])$ where $t_1: A \dashrightarrow B$.

The naturality of η:

$$\begin{array}{ccc} C_1 & \xrightarrow{\quad k \quad} & C_2 \\ {\scriptstyle \eta_{C_1}} \downarrow & & \downarrow {\scriptstyle \eta_{C_2}} \\ (C_1 \times A)^A & \xrightarrow[(k \times A)^A]{} & (C_2 \times A)^A \end{array}$$

yields

(8) $\models (kc)^{\hat{}} = (k \times A) \circ \hat{c}$ where $\tau_0 c = C_1$.

<u>0.6.13.5 Proposition.</u> Let $t: A \dashrightarrow B$ be an internal morphism and $s_F(t) \subset \{a,c\}$ where $\tau_0(a) = A$, $\tau_0(c) = C$, and $a \neq c$. The following are equivalent for a morphism $f: C \longrightarrow B^A$:

(1) $\models (fc)[a] = t$

(2) $\models (fc) = (\lambda ca.t) \circ \hat{c}$

(3) $\models f = \lambda c.((\lambda ca.t) \circ \hat{c})$.

<u>Proof.</u> Let $g = \lambda ca.t$ so that $\models g/c,a/ = t$. (1) is equivalent to $\models (fc)[a] = g/c,a/$, which is 0.6.13.3 (3), hence f and g correspond under the adjointness

$$\frac{C \xrightarrow{\quad f \quad} B^A}{C \times A \xrightarrow[\quad g \quad]{} B}$$

hence (1) is equivalent to $\models fc = g \circ \hat{c}$ (i.e. 0.6.13.3 (4)). The equivalence of (2) and (3) follows from 0.6.9.2.□

Just as we noted for the corollary of 0.6.9.2, this proposition defines a morphism f by requiring that $(fc)[a] = t$ be valid.

The next two propositions prepare the way for a generalization of 0.6.13.5. We first generalize 0.6.10.1 and 0.6.10.5 (3).

<u>0.6.13.6 Proposition.</u> The following rule is valid

$$\frac{\varphi}{(\forall\langle\vec{u}\rangle)(\varphi)} \qquad \{\vec{u}\} \subset s_F(\varphi).$$

<u>Proof.</u> By induction. If $\ell(\vec{u}) = 0$ then $(\forall\langle\vec{u}\rangle)(\varphi) = \varphi$.
For the induction step we suppose $\{\vec{u}x\} \subset s_F(\varphi)$, then $\models (\forall\langle\vec{u}x\rangle)(\varphi)$ iff $\models (\forall\langle\vec{u}\rangle)(\forall x\varphi)$. Since $\{\vec{u}\} \subset s_F(\forall x\varphi)$ and $\ell(\vec{u}) < \ell(\vec{u}x)$ induction applies hence

$$\models (\forall\langle\vec{u}\rangle)(\forall x\varphi) \quad \text{iff} \quad \models \forall x\varphi.$$

By 0.6.10.1 and 0.6.10.5 (3), $\models \forall x\varphi$ iff $\models \varphi$.□

We generalize 0.6.5.4 by relaxing the condition $\{\vec{v}\} = s_F(\varphi)$.

<u>0.6.13.7 Proposition.</u> Let $\ell(\vec{v}) \geq 1$, $\{\vec{v}\} \subset s_F(\varphi)$, $A = \tau(\vec{v})$, $\langle y, id_A, \vec{v}\rangle$ a compatible triple, $\beta = Sbs\langle y, id_A, \vec{v}\rangle$, $y \notin s_{F\beta}(\varphi)$. The following are equivalent:

(1) $\models \varphi$

(2) $\models S(\beta)(\varphi)$.

<u>Proof.</u> Let $\vec{u} = (var(\varphi) \dot{-} \{\vec{v}\})$ so that $\{\vec{u}\vec{v}\} = s_F(\varphi)$ and $\{\vec{u}\} \cap \{\vec{v}\} = \phi$

By 0.6.13.6, (1) is equivalent to

(3) $\models (\forall\langle\vec{u}\rangle)\varphi$.

$s_F((\forall\langle\vec{u}\rangle)\varphi) = (s_F(\varphi) - \{\vec{u}\}) = \{\vec{v}\}$, hence we can apply 0.6.5.2 to get (3) equivalent to

(4) $\models S(\beta)((\forall\langle\vec{u}\rangle)\varphi)$,

which is the same as

(5) $\models (\forall\langle\vec{u}\rangle)((S(\beta))(\varphi))$.

Again by 0.6.13.6, (5) is equivalent to

(6) $\models (S(\beta))(\varphi)$.□

We now generalize 0.6.13.5. Define $t[\vec{s}] = t[\pi(\vec{s})]$ for $t\vec{s}$ a string of terms.

<u>0.6.13.8 Proposition.</u> Let $\langle\vec{u}\vec{v},t\rangle$ be an augmented term, $\tau(t) = B$, $\tau(\vec{u}) = C$, $\tau(\vec{v}) = A$. There is exactly one morphism $f\colon C \longrightarrow B^A$ such that

(1) $\models (f(\pi(\vec{u})))[\vec{v}] = t$.

<u>Proof.</u> Let x and y be distinct variables of types C and A respectively not occurring in $\vec{u}\vec{v}t$.

We assume first that $\vec{u} \neq \phi$ and $\vec{v} \neq \phi$. Let $\beta = \mathrm{Sbs}\langle y,\mathrm{id}_A,\vec{v}\rangle$; by 0.6.13.7, (1) is equivalent to

(2) $\models (f\pi(\vec{u}))\big[S(\beta)(\pi(\vec{v}))\big] = (S(\beta))(t)$.

Let $\alpha = \mathrm{Sbs}\langle x,\mathrm{id}_C,\vec{u}\rangle$; by 0.6.13.7 again, (2) is equivalent to

(3) $\models (f(S(\alpha)(\pi(\vec{u}))))\big[S(\beta)(\pi(\vec{v}))\big] = S(\alpha)(S(\beta)(t))$.

By 0.6.9.16 (1) we have $\models y = S(\beta)(\pi(\vec{v}))$ and $\models x = S(\alpha)(\pi(\vec{u}))$. Thus we can apply 0.6.9.13 to (3), replacing terms, to get the equivalent assertion

(4) $\models (fx)[y] = S(\alpha)(S(\beta)(t))$.

Now by 0.6.14.7, (4), and hence (1), uniquely determines $f\colon C \longrightarrow B^A$.

There remain three cases: (i) $\vec{u} = \vec{v} = \phi$, (ii) $\vec{u} = \phi$ and $\vec{v} \neq \phi$, and (iii) $\vec{u} \neq \phi$ and $\vec{v} = \phi$. The application of what we have proven to the three augmented terms which arise under the three cases

(i) $\langle xy,t\rangle$ when $\models (f_1x)[y] = t$

(ii) $\langle x\vec{v},t\rangle$ when $\models (f_2x)[\vec{v}] = t$

(iii) $\langle\vec{u}y,t\rangle$ when $\models (f_3(\pi(\vec{u})))[y] = t$

gives rise to uniquely determined morphisms $f_k\colon C \longrightarrow B^A$ $(k = 1,2,3)$. For (i) we have $\models x = *$ and $\models y = *$, for (ii) $\models x = *$ and for (iii) $\models y = *$. Thus the defining equations for the f_k are:

$$(f_1 *)[*] = t \quad , \quad (f_2 *)[\vec{v}] = t \quad \text{and} \quad (f_3(\pi(\vec{u})))[*] = t \ ,$$

respectively. Since $* = \pi(\phi)$, (1) holds for cases (i), (ii), (iii).□

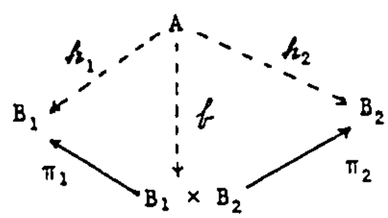
<u>0.6.13.9 Corollary.</u> Let $\tau(\vec{v}) = B$, $\tau(t) = \tau(s) = B^A$; we have the following valid rule

$$\frac{t[\vec{v}] = s[\vec{v}]}{t = s} \qquad s_F(ts) \cap \{\vec{v}\} = \phi$$

<u>Proof.</u> Let $\vec{w} = \text{var}\,(t,s)$. Define $f, g\colon \tau(\vec{w}) \longrightarrow B^A$ by $\models (f\pi(\vec{w})) = t$ and $\models (g\pi(\vec{w})) = s$ then $\models (f\pi(\vec{w}))[\vec{v}] = t[\vec{v}]$ and, by hypothesis, $\models (g\pi(\vec{w}))[\vec{v}] = s[\vec{v}] = t[\vec{v}]$, hence by 0.6.13.8, $f = g$, hence $\models t = s$.□

<u>0.6.13.10 The canonical isomorphism :</u> $(B_1 \times B_2)^A \approx B_1^A \times B_2^A$.

Let $\pi_i\colon B_1 \times B_2 \longrightarrow B_i$ $(i = 1,2)$ be projections. Let a, f, h_1, h_2 be distinct variables of types A, $(B_1 \times B_2)^A$, B_1^A and B_2^A respectively, i.e. we have a diagram



Define $\sigma\colon B_1^A \times B_2^A \longrightarrow (B_1 \times B_2)^A$ by

(1) $\qquad \models (\sigma(h_1,h_2))[a] = (h_1[a],h_2[a])$.

For all $t_i\colon A \dashrightarrow B$ $(i = 1,2)$ put $\sigma(t_1,t_2) = t_1 \sqcap t_2$.

Define $\alpha\colon (B_1 \times B_2)^A \longrightarrow B_1^A \times B_2^A$ by

(2) $\qquad \models \alpha f = (\pi_1 \circ f, \pi_2 \circ f)$.

By 0.6.9.3 (6) we have $\alpha = (\pi_1)^A \sqcap (\pi_2)^A$. We will show directly that σ is the inverse of α, confirming that, for the functor $(-)^A$, σ is the isomorphism $\rho(B_1,B_2)$ or $\sigma(\hat{B}_1\hat{B}_2)$ of 0.4.3.1 and 0.4.4.1.

<u>0.6.13.11 Proposition.</u> (1) $\models (\sigma\circ\alpha)\mathcal{f} = \mathcal{f}$ (2) $\models \pi_1\circ(\mathcal{h}_1 \sqcap \mathcal{h}_2) = \mathcal{h}_1$
(3) $\models (\alpha\circ\sigma)\langle\mathcal{h}_1,\mathcal{h}_2\rangle = \langle\mathcal{h}_1,\mathcal{h}_2\rangle$.

<u>Proof.</u> (1) $\models ((\sigma\circ\alpha)\mathcal{f})[a] = (\sigma(\alpha\mathcal{f}))[a] = (\sigma\langle\pi_1\circ\mathcal{f},\pi_2\circ\mathcal{f}\rangle)[a]$

$$= \langle(\pi_1\circ\mathcal{f})[a],\ (\pi_2\circ\mathcal{f})[a]\rangle = \langle\pi_1(\mathcal{f}[a]),\pi_2(\mathcal{f}[a])\rangle$$
$$= \mathcal{f}[a]\ .$$

The above equalities are justified by 0.6.9.3 (3), the definitions of α β and $\pi_1\circ\mathcal{f}$, and 0.6.9.3 (5).□

(2): $\models (\pi_1\circ(\mathcal{h}_1 \sqcap \mathcal{h}_2))[a] = \pi_1((\mathcal{h}_1 \sqcap \mathcal{h}_2)[a]) = \pi_1\langle\mathcal{h}_1[a],\mathcal{h}_2[a]\rangle$

$$= \mathcal{h}_1[a].$$

The last equality follows from 0.6.9.3 (4).

(3): $\models (\alpha\circ\sigma)\langle\mathcal{h}_1,\mathcal{h}_2\rangle = \alpha\langle\mathcal{h}_1 \sqcap \mathcal{h}_2\rangle = \langle(\pi_1\ (\mathcal{h}_1 \sqcap \mathcal{h}_2))\ ,\ \ (\pi_2\circ(\mathcal{h}_1 \sqcap \mathcal{h}_2))\rangle$

$$= \langle\mathcal{h}_1,\mathcal{h}_2\rangle\ .$$

The last equality follows from (2) above. □

<u>0.6.13.12 Definition.</u> Let $\tau_0(a) = A$. For each morphism $f: A \longrightarrow B$ we define $\ulcorner f\urcorner : \mathbf{1} \longrightarrow B^A$ by

$$\models (\ulcorner f\urcorner\star)[a] = fa$$

and put $\langle f\rangle = \ulcorner f\urcorner\star$, so that $|\langle f\rangle| = \ulcorner f\urcorner$ and

$$\models \langle f\rangle[a] = fa.$$

We call $\langle f\rangle: A \dashrightarrow B$ the constant internal morphism induced by f.

<u>0.6.13.13 Proposition.</u> Let $\tau_0(a) = A$, $f: A \longrightarrow B$, $g: B \longrightarrow C$, $h: A \longrightarrow B$, then

(1) $\models g \bullet \lceil f \rceil = \lceil g \bullet f \rceil$

(2) $g^A \bullet \ulcorner f \urcorner = \ulcorner g \bullet f \urcorner$

(3) $\models g \bullet \lceil id_B \rceil = \lceil g \rceil$

(4) $g^B \bullet \ulcorner id_B \urcorner = \ulcorner g \urcorner$

(5) $f = h$ iff $\ulcorner f \urcorner = \ulcorner h \urcorner$ iff $\models \lceil f \rceil = \lceil h \rceil$

Proof. (1) $\models (g \bullet \lceil f \rceil)[a] = g(\lceil f \rceil [a]) = g(fa) = (g \bullet f)a = \lceil g \bullet f \rceil [a]$,

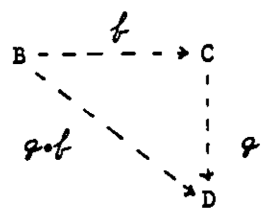
hence $\models g \bullet \lceil f \rceil = \lceil g \bullet f \rceil$ by 0.6.13.11.□

(2) $g^A \bullet \ulcorner f \urcorner = g^A | \lceil f \rceil | = | g \bullet \lceil f \rceil | = | \lceil g \bullet f \rceil | = \ulcorner g \bullet f \urcorner$.□

(3) and (4): Put $f = id_B$ in (1) and (2) respectively.□

(5) Suppose $\models \lceil f \rceil = \lceil h \rceil$ then $\models fa = ha$, hence $f = h$. This is the only non-trivial implication.□

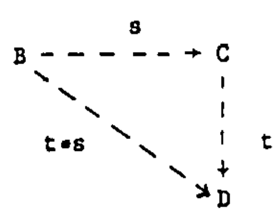
0.6.13.14 Definition. Let $\mathcal{f}$, $\mathcal{g}$ and b be distinct variables, $\tau_0(b) = B$, $\mathcal{f}$ and $\mathcal{g}$ as in the diagram



Define $\mu^C_{BD}: D^C \times C^B \longrightarrow D^B$ by

$$\models (\mu^C_{BD}(\mathcal{g}, \mathcal{f}))[b] = \mathcal{g}[\mathcal{f}[b]] \ .$$

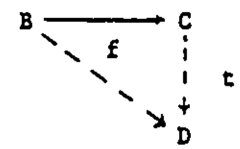
For all internal morphisms as represented in the diagram



we put $t \circ s = \mu^{C}_{BD}(t,s)$. The defining equation becomes

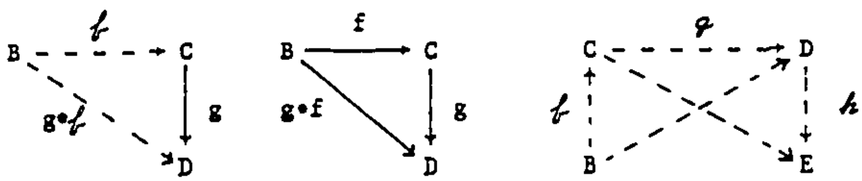
$$\models (t \circ s)[b] = t[s[b]] \ .$$

For



we define $t \circ f = t \circ \ulcorner f \urcorner$.

<u>0.6.13.15 Proposition.</u> The following formulas are valid for morphisms and internal morphisms as represented in the diagrams.



(1) $\quad g \circ \mathit{f} = \ulcorner g \urcorner \circ \mathit{f}$

(2) $\quad \ulcorner g \urcorner \circ \ulcorner f \urcorner = \ulcorner g \circ f \urcorner$

(3) $\quad (\mathit{h} \circ \mathit{g}) \circ \mathit{f} = \mathit{h} \circ (\mathit{g} \circ \mathit{f})$.

<u>Proof.</u> (1) $\models (g \circ \mathit{f})[b] = g(\mathit{f}[b]) = \ulcorner g \urcorner[\mathit{f}[b]] = (\ulcorner g \urcorner \circ \mathit{f})[b]$. □

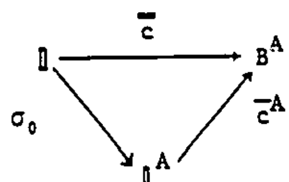
(2) $\models \ulcorner g \urcorner \circ \ulcorner f \urcorner = g \circ \ulcorner f \urcorner = \ulcorner g \circ f \urcorner$. □

(3) $\models ((\mathit{h} \circ \mathit{f}) \circ \mathit{f})[b] = (\mathit{h} \circ \mathit{g})[\mathit{f}[b]] = \mathit{h}[\mathit{g}[\mathit{f}[b]]] = \mathit{h}[(\mathit{g} \circ \mathit{f})[b]]$

$$= (\mathit{h} \circ (\mathit{g} \circ \mathit{f}))[b] \ . \square$$

<u>0.6.14 Exponentiating structure.</u> For each internal $\langle \mathcal{O},\mathcal{P}\rangle$-structure $\mathcal{B} = \langle B,\gamma_{\mathcal{O}},\gamma_{\mathcal{P}}\rangle$ and each object A, the cartesian functor $(-)^A$ induces an $\langle \mathcal{O},\mathcal{P}\rangle$-structure $\mathcal{B}^A = \langle B^A,\rho_{\mathcal{O}},\rho_{\mathcal{P}}\rangle$ as defined in 0.6.12. Thus if $\varphi \in \mathrm{bHf}^*(\mathbb{P}(\mathcal{O},\mathcal{P}))$, and $\mathcal{B} \models \varphi$ then $\mathcal{B}^A \models \varphi$, by 0.6.12.10. In this section we shall show how the operations and relations of $\mathcal{B}^A$ are given "pointwise" in terms of the operations and relations of $\mathcal{B}$.

<u>0.6.14.1 Operations of the internal structure $\mathcal{B}^A$.</u> We follow the conventions of 0.6.13.2. Let $\tau_0(\varphi) = \tau_0(\mathscr{f}_1) = B^A$ and $\tau_0(a) = A$. For $c \in \mathcal{O}_0$, $\bar{\bar{c}} = \bar{c}^A \circ \sigma_0$



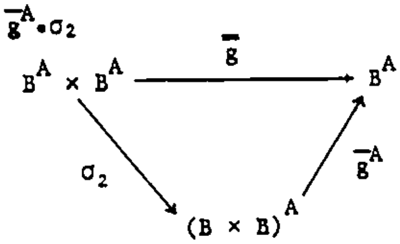
Thus $\models \bar{\bar{c}}^* = \bar{c}\circ\ulcorner u_A\urcorner = \ulcorner \bar{c}\circ u_A\urcorner$, so $\models (\bar{\bar{c}}^*)[a] = \ulcorner \bar{c}\circ u_A\urcorner[a] = \bar{c}u_A a = \bar{c}^*$, i.e.

(1) $\models (\bar{\bar{c}}^*)[a] = \bar{c}^*$.

For $f \in \mathcal{O}_1$, $\bar{\bar{f}} = \bar{f}^A$. Thus $\models \bar{\bar{f}}\varphi = \bar{f}\,\varphi$, so

(2) $\models (\bar{\bar{f}}\varphi)[a] = \bar{f}(\varphi[a])$.

For $g \in \mathcal{O}_2$, $\bar{\bar{g}} = \bar{g}^A \circ \sigma_2$

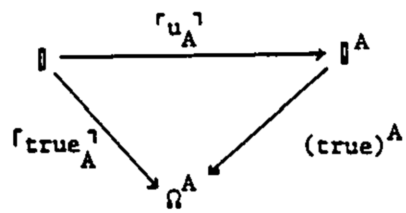


Thus $\models \bar{\bar{g}}(\mathscr{f}_1,\mathscr{f}_2) = \bar{g}\circ(\mathscr{f}_1 \sqcap \mathscr{f}_2)$, so

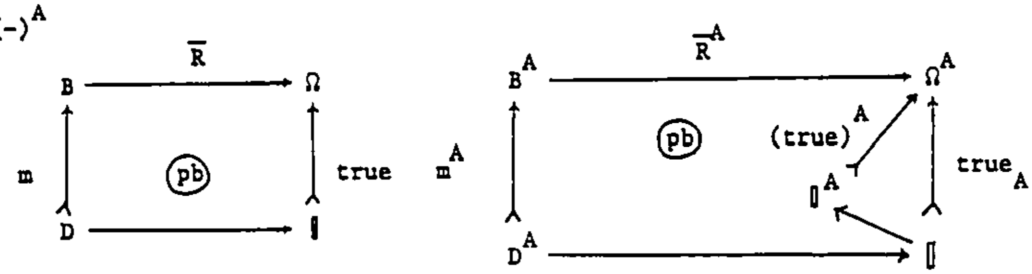
(3) $\models (\bar{\bar{g}}(\mathscr{f}_1,\mathscr{f}_2))[a] = \bar{g}((\mathscr{f}_1 \sqcap \mathscr{f}_2)[a]) = \bar{g}(\mathscr{f}_1[a],\mathscr{f}_2[a])$.

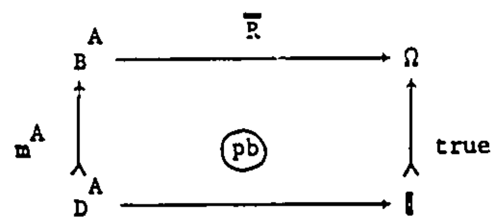
<u>0.6.14.2 Relations on the internal structure $\mathcal{B}^A$.</u> From 0.6.13.13 (1) and (2), for $u_A\colon A \longrightarrow 1$ and $\mathrm{true}\colon 1 \longrightarrow \Omega$ we have

$\models \mathrm{true}\circ\ulcorner u_A\urcorner = \ulcorner \mathrm{true}_A\urcorner$ and



commutes. Let $\tau_0(\varphi) = \tau_0(\varphi_1) = B^A$, $\tau_0(x) = \tau_0(x_1) = B$, and $\tau_0(a) = A$. For $R \in P_1$, we introduce a mono m such that $[\![m]\!] = [\![x|Rx]\!]$, then apply $(-)^A$





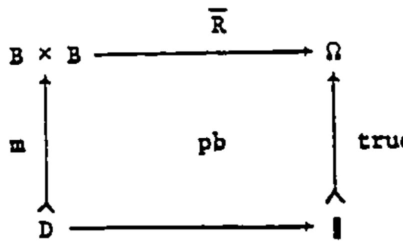
By 0.6.11.8 we have $\models (\overline{R}^A\varphi = \ulcorner\mathrm{true}_A\urcorner) \leftrightarrow \exists h(m^A h = \varphi) \leftrightarrow \overline{R}\,\varphi$.

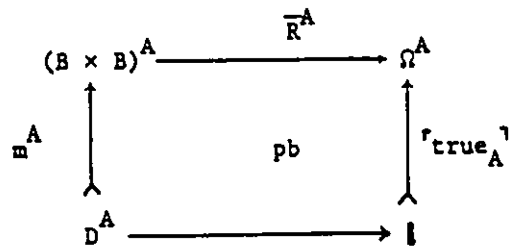
Hence $\models \overline{R}\varphi \leftrightarrow \forall a(\overline{R}(\varphi[a]) \leftrightarrow (\ulcorner\mathrm{true}_A\urcorner[a]))$ therefore

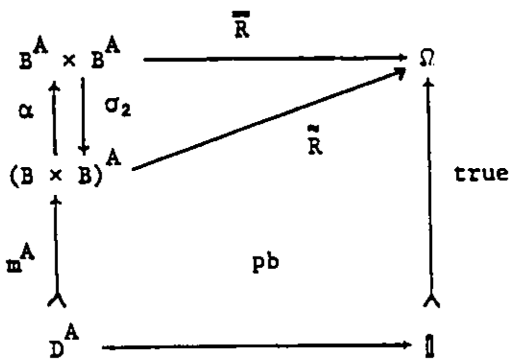
(4) $\models \overline{R}\varphi \leftrightarrow \forall a(\overline{R}(\varphi[a]))$

since $\models \ulcorner\mathrm{true}_A\urcorner[a] = \mathrm{true}(u_A a) = (\mathrm{true})^*$.

For $R \in P_1$ we have the pullbacks







Let $\tau_0(k) = (B \times B)^A$. By 0.6.11.8 we have

$\models (\overline{R}^A k = \ulcorner \mathrm{true}_A \urcorner) \leftrightarrow \exists h(m^A h = k)$.

By definition 0.6.12.6, $\overline{\overline{R}} = \tilde{R} \circ \sigma_2$, where $\tilde{R}$ is the morphism classifying m^A (see 0.6.12.5). But σ_2 is the canonical isomorphism σ with inverse α given in 0.6.13.10, hence $\tilde{\tilde{R}} = \overline{\overline{R}}\,\alpha$, hence

$\models \exists h(m^A h = k) \leftrightarrow \tilde{\tilde{R}}k \leftrightarrow (\overline{\overline{R}}\alpha k)$. Thus $\models \overline{\overline{R}}\alpha k \leftrightarrow (\overline{R}^A k = \ulcorner \mathrm{true}_A \urcorner)$, hence

$\models \overline{\overline{R}}(f_1, f_2) \leftrightarrow (\overline{\overline{R}}\alpha\sigma(f_1, f_2)) \leftrightarrow (\overline{R}^A(\sigma(f_1, f_2)) = \ulcorner \mathrm{true}_A \urcorner)$ therefore

(5) $\models \overline{\overline{R}}(f_1, f_2) \leftrightarrow \forall a(\overline{R}(f_1[a], f_2[a]))$.

In particular we have:

<u>0.6.14.3 Corollary.</u> $\models (f_1 = f_2) \leftrightarrow \forall a(f_1[a] = f_2[a])$.

<u>Proof.</u> Put $R = \underline{\delta}$ in (5) of 0.6.14.2.□

<u>0.6.14.4 "Pointwise" definition.</u> Equations (1), (2) and (3) of 0.6.14.1 and the equivalences (4), (5) of 0.6.14.2 can be taken to define new operations and relations; the terms and formulas appearing on the right hand side are instances of a more general procedure which we now describe.

For a fixed type A we define a function

$$(-)^A\colon \mathrm{Vbls} \longrightarrow \mathrm{Vbls}$$

as follows: if x is the i-th variable of type B then x^A is the i-th variable of type B^A. Now for a fixed variable a of type A we define a substitution

$$\mathrm{Sbs}[a]\colon \mathrm{Tms} \longrightarrow \mathrm{Tms}$$

by extending to terms the function which on variables is given by

$$(\mathrm{Sbs}[a])(x) = x^A[a].$$

Thus one could define a morphism f pointwise given a term t with $s_B(t) = \phi$ and $a \notin \{x^A \mid x \in s_F(t)\}$ by letting $\vec{v}^A$ be the extension to a string $\vec{v}$ of distinct variables of the function $(-)^A$ and requiring

$$\models (f(\vec{v}^A))[0] = \mathrm{Sbs}[a](t).$$

For φ a formula we could define a predicate R by

$$\models R(\vec{v}^A) \leftrightarrow \forall a(\mathrm{Sbs}[a](\varphi)).$$

<u>0.6.14.5 Proposition.</u> Let $t \in \mathrm{Poly}(E(\mathcal{O}))$. Then

$$\models \bar{t}[a] = \mathrm{Sbs}[a](\bar{t}).$$

<u>Proof.</u> For $t = x$, $\models \bar{x}[a] = \overline{x^A}[a] = \mathrm{Sbs}[a](\bar{x})$.

$\models (\overline{c^*})[a] = \overline{c^*}$ by 0.6.14.1 (1)

$= \mathrm{Sbs}[a](\overline{c^*})$.

$\models \overline{\overline{ft}}[a] = (\overline{\overline{f}}\ \overline{\overline{t}})[a] = \overline{f}(\overline{\overline{t}}[a])$ by 0.6.14.1 (2)

$= \overline{f}(\mathrm{Sbs}[a](\overline{t}))$ by induction

$= \mathrm{Sbs}[a](\overline{ft})$.

$\models \overline{\overline{g/t,s/}}[a] = (\overline{\overline{g}}/\overline{\overline{t}},\overline{\overline{s}}/)[a] = \overline{g}/\overline{\overline{t}}[a],\overline{\overline{s}}[a]/$ by 0.6.14.1 (3)

$= \overline{g}/\mathrm{Sbs}[a](\overline{t}),\ \mathrm{Sbs}[a](\overline{s})/$ by induction

$= \mathrm{Sbs}[a](\overline{g/t,s/})$.□

<u>0.6.14.6 Proposition.</u> Let $\varphi \in \mathrm{AtFml}(\mathbb{P}\langle\mathcal{O},P\rangle)$. Then

$$\models \overline{\overline{\varphi}} \leftrightarrow \forall a(\mathrm{Sbs}[a](\overline{\varphi})).$$

<u>Proof.</u> For $t \in \mathrm{Poly}\,(\mathbb{E}(\mathcal{O}))$ and $R \in P_1$ we have

$\models \overline{\overline{Rt}} \leftrightarrow \overline{\overline{R}}\ \overline{\overline{t}} \leftrightarrow \forall a(\overline{R}(\overline{\overline{t}}[a]))$ by 0.6.14.2 (4)

$\leftrightarrow \forall a(\overline{R}((\mathrm{Sbs}[a])(\overline{t})))$ by 0.6.14.4

$\leftrightarrow \forall a(\mathrm{Sbs}[a](\overline{Rt}))$.

$\models \overline{\overline{R/t,s/}} \leftrightarrow \overline{\overline{R}}/\overline{\overline{t}},\overline{\overline{s}}/ \leftrightarrow \forall a(\overline{R}/\overline{\overline{t}}[a],\overline{\overline{s}}[a]/)$ by 0.6.14.2 (5).

$\leftrightarrow \forall a(\overline{R}/\mathrm{Sbs}[a](\overline{t}),\mathrm{Sbs}[a](\overline{s})/)$ by 0.6.14.4

$\leftrightarrow \forall a(\mathrm{Sbs}[a](\overline{R/t,s/}))$.□

<u>0.6.15 Valid formulas involving the comprehension operator.</u>

<u>0.6.15.1 Notation.</u> For each object A, term s of type A, and term t of type Ω^A, we put

$$s \in t = \epsilon_\Omega^A\langle t,s\rangle = t[s]$$

where $\epsilon_\Omega^A\colon \Omega^A \times A \longrightarrow \Omega$ is, as in 0.6.13.3, the back adjunction at Ω.

<u>0.6.15.2 Proposition.</u> Let $\tau_0(a) = A$, then

$$\models \varphi \leftrightarrow a \in \{a{:}\varphi\} \ .$$

<u>Proof.</u> We first suppose $a \in s_F(\varphi)$ and $s_F(\varphi) - \{a\} \neq \phi$. Let $\vec{w} = var(\{a{:}\varphi\})$, then $\tau(\vec{w}a) = \tau(\vec{w}) \times A$. $|\{a{:}\varphi\}|$ is defined in 0.5.3.2 through the correspondence

$$\frac{\tau(\vec{w}) \xrightarrow{\quad |\{a:\varphi\}| \quad} \Omega^A}{\tau(\vec{w}) \times A \xrightarrow[\lambda\vec{w}y.\varphi[a|y]]{\qquad} \Omega}$$

where $y \notin \{\vec{w}a\}$ and $\tau_0(y) = A$. Let $f = |\{a{:}\varphi\}|$ and $g = \lambda\vec{w}y.\varphi[a|y]$. By 0.6.9.2, $\models \{a{:}\varphi\} = f(\pi(\vec{w}))$ and $\models \varphi[a|y] = g(\pi(\vec{w}y))$. Let u be a variable of type $\tau(\vec{w})$ not occurring in $\varphi\vec{w}y$. By 0.6.13.3 (3)

$$\models g\langle u,y\rangle = (fu)[y] \ .$$

Substituting $\pi(\vec{w})$ for u we have

$$\models g(\pi(\vec{w}y)) = (f(\pi\vec{w}))[y]$$

hence $\qquad \models \varphi[a|y] = (\{a{:}\varphi\})[y]$

so $\qquad \models \varphi[a|y] \leftrightarrow y \in \{a{:}\varphi\}$

or equivalently, by 0.2.6.4,

$$\models \varphi \leftrightarrow a \in \{a{:}\varphi\} \ .$$

Now we drop the hypothesis, $a \in s_F(\varphi)$ and $s_F(\varphi) - \{a\} \neq \phi$. Let $\varphi' = \varphi \wedge (a = a) \wedge (z = z)$, where $\tau_0(z) = \mathbf{1}$, then

$$\models \varphi \leftrightarrow a \in \{a:\varphi'\} .$$

But $\models \varphi' \leftrightarrow \varphi$, hence by 0.6.9.13 (1)

$$\models \varphi \leftrightarrow a \in \{a:\varphi\} .\square$$

<u>0.6.15.3 Corollary.</u> Let $t \in \mathrm{ffr}[y](\varphi)$.

$$\models \varphi[y|t] \leftrightarrow t \in \{y:\varphi\} .$$

<u>0.6.15.4 Proposition.</u> Let $\tau_0(U) = \tau_0(W) = \Omega^A$, $\tau_0(a) = A$.

(1) $\models U = W \leftrightarrow \forall a((a \in U) \leftrightarrow (a \in W))$

(2) $\models U = \{a:\varphi\} \leftrightarrow \forall a((a \in U) \leftrightarrow \varphi)$

(3) $\models U = \{a:a \in U\}$.

<u>Proof.</u> (1) By 0.6.14.3.□ (2) Substitute $\{a:\varphi\}$ for W in (1), then apply 0.6.15.2.□ (3) Take $\varphi = (a \in U)$ in (2).□

<u>0.6.15.5 Exponentiating</u> $\underline{\Omega}_{\leq}$. The internal <H,PartOrd>-structure $\underline{\Omega}_{\leq}$ is a model of $\sum(\leq,\text{Heyt.})$, that is $\underline{\Omega}_{\leq}$ is an internal partially ordered Heyting algebra (see 0.6.4.4, 0.6.8.1 (7) and 0.6.8.2). Since $(-)^A$ is cartesian we have $(\underline{\Omega}_{\leq})^A \models \sum(\leq,\text{Heyt})$ also. For each operation or predicate sign s of $\mathbb{P}$<H,PartOrd> we let $\bar{s}$ be its interpreation in $(\underline{\Omega}_{\leq})^A$. We put

$$\phi_A = \overline{\underline{(0)}}^* \quad \text{and} \quad \ulcorner A \urcorner = \overline{\underline{(1)}}^* .$$

When, from the context, it is clear that these are terms of type Ω^A, we write:

$$\phi \text{ for } \phi_A \quad \text{and} \quad A \text{ for } \ulcorner A \urcorner .$$

For terms r and t of type Ω^A we put:

$$r \cap t = \overline{\underline{(\wedge)}}(r,t) \qquad r \cup t = \overline{\underline{(\vee)}}(r,t)$$

$r \Rightarrow t \equiv \overline{(\Leftrightarrow)}/r,t/ \quad r \subset t \equiv \overline{(\leq)}/r,t/ \quad \neg r \equiv r \Rightarrow \phi$.

<u>0.6.15.6 Proposition.</u> Let U, V, W be distinct variables of type Ω^A and let $\tau_0(a) \equiv A$. The following formulas are valid:

(1) $\neg\ (a \in \phi)$

(2) $a \in A$

(3) $U \cap W = \{a\colon a \in U \wedge a \in W\}$

(4) $U \cup W = \{a\colon a \in U \vee a \in W\}$

(5) $U \Rightarrow W = \{a\colon a \in U \Rightarrow a \in W\}$

(6) $\neg\ U = \{a\colon \neg\ (a \in U)\}$

(7) $(U \subset W) \Leftrightarrow \forall a(a \in U \Rightarrow a \in W)$.

<u>Proof.</u> We apply 0.6.14.1 and 0.6.14.2. First note that $(a \in t) \equiv t[a]$ for t of type Ω^A.

(1) From 0.6.14.1 (1) we have $\models (\overline{(0)}^*)[a] = \overline{(0)}^*$, hence $\models (a \in \phi) \Leftrightarrow \bot$, hence $\models \neg\ (a \in \phi)$.□

(2) As in (1), $\models (a \in A) \Leftrightarrow \top$, hence $\models a \in A$.□

(3) From 0.6.14.1 (3) we have $\models (\overline{(\wedge)}/U,W/)[a] = \overline{(\wedge)}/U[a],W[a]/$, hence $\models\ a \in (U \cap W) \Leftrightarrow (a \in U) \wedge (a \in W)$, hence $\models\ U \cap\ W = \{a\colon a \in U \wedge a \in W\}$, by 0.6.15.4 (2).□

(5), (5). Similar to (3).□

(6) From (5), $\models \neg\ U = \{a\colon a \in U \Rightarrow a \in \phi\} = \{a\colon \neg\ (a \in U)\}$, by (1).□

(7) From 0.6.14.2 (5), $\models \overline{(\leq)}/U,W/ \Leftrightarrow \forall a(\overline{(\leq)}/U[a],W[a]/$, hence $\models\ U \subset W \Leftrightarrow \forall a(a \in U \Rightarrow a \in W)$.□

<u>0.6.16 The singleton and unique existentiation.</u> For each A we define a singleton map, $\{-\}_A : A \longrightarrow \Omega^A$, by

$$\models \{-\}_A x = \{y : y = x\}$$

where x and y are distinct variables of type A. For each term t we put

$$\{t\} \equiv \{-\}_{\tau(t)}(t).$$

<u>0.6.16.1 Proposition.</u> Let x, y, w, U, W be distinct variables

(1) $\models \{x\} = \{y : y = x\}$

(2) $\models y \in \{x\} \Leftrightarrow y = x$

(3) $\models \{w\} \subset \{x\} \Rightarrow w = x$

(4) $\models \{w\} = \{x\} \Rightarrow w = x$

(5) $\models y \in U \Leftrightarrow \{y\} \subset U$

(6) $\models \exists x(U = \{x\}) \Leftrightarrow (\exists x(x \in U) \wedge \forall y \forall w((y \in U) \wedge (w \in U) \Rightarrow (y = w)))$.

<u>Proof.</u> (1) By definition.□

(2) $\models y \in \{x\} \Leftrightarrow y \in \{y : y = x\} \Leftrightarrow y = x$ by 0.6.15.2.□

(3) $\models \{w\} \subset \{x\} \Rightarrow \forall y(y \in \{w\} \Rightarrow y \in \{x\})$

$$\Rightarrow (y = w) \Rightarrow (y = x)$$

therefore $\models \{w\} \subset \{x\} \Rightarrow (w = w \Rightarrow w = x)$

$$\Rightarrow w = x. \ \square$$

(4) $\models U = W \Rightarrow U \subset W$, by 0.6.15.5; now we combine this with (3).□

(5) $\models \{y\} \subset U \Leftrightarrow \forall x(x \in \ y \ \Rightarrow x \in U) \Leftrightarrow \forall x((x = y) \Rightarrow (x \in U))$, therefore

$$\models (\{y\} \subset U) \Rightarrow ((x = y) \Rightarrow (x \in U))$$
$$\Rightarrow ((y = y) \Rightarrow (y \in U))$$
$$\Rightarrow y \in U. \ \square$$

By 0.6.9.14 (1), $\models (y \in U) \wedge (x = y) \Rightarrow (x \in U)$, hence

$$\models y \in U \Rightarrow \forall x((x = y) \Rightarrow (x \in U))$$
$$\Rightarrow \{y\} \subset U. \square$$

(6) To show $\models \exists x(U = \{x\}) \Rightarrow \exists x(x \in U)$, apply the rule 0.6.10.8 (2) to

$\models (U = \{x\}) \Rightarrow (x \in U)$. The latter follows by combining (5) with $\models U = \{x\} \Rightarrow \{x\} \subset U$. We show

$\models \exists x(U = \{x\}) \Rightarrow \forall y \forall w((y \in U) \wedge (w \in U) \Rightarrow (y = w))$.

$$\begin{aligned}\models (U = \{x\}) \wedge (y \in U) \wedge (w \in U) &\Rightarrow (U = \{x\}) \wedge (\{y\} \subset U) \wedge (\{w\} \subset U)\\ &\Rightarrow (\{y\} \subset \{x\}) \wedge (\{w\} \subset \{x\})\\ &\Rightarrow (y = x) \wedge (w = x)\\ &\Rightarrow (y = x),\end{aligned}$$

hence $\models (U = \{x\}) \Rightarrow ((y \in U) \wedge (w \in U) \Rightarrow (y = w))$. Thus

$\models \exists x(U = \{x\}) \Rightarrow (\exists x(x \in U) \wedge \forall y \forall w((y \in U) \wedge (w \in U) \Rightarrow (y = w)))$.

For the converse, let $\begin{pmatrix} x & U \\ a & L \end{pmatrix}$ be a normalized substitution for

$(x \in U) \wedge \forall y \forall w((y \in U) \wedge (w \in U) \Rightarrow (y = w)) \Rightarrow \exists x(U = \{x\})$.

Then

(6.1) $\models a \in L$ and (6.2) $\models (y \in L) \wedge (w \in L) \Rightarrow (y = w)$.

Substitution a for y in (6.2) we have, by (6.1),

(6.3) $\models w \in L \Rightarrow w = a$. Combining (6.3) with (2) yields

(6.4) $\models w \in L \Rightarrow w \in \{a\}$, hence (6.5) $\models L \subset \{a\}$. From (6.1) and (5) we have (6.6) $\models \{a\} \subset L$. By (6.5) and (6.6) $\models L = \{a\}$, hence $\models \exists x(L = \{x\})$. Thus the formula to which we applied the normalization is valid, hence (6) is valid.□

0.6.16.2 Unique existentiation.

Definition. For every formula φ and variable x we put $\exists! x\varphi = \exists x(\{x{:}\varphi\} = \{x\})$. If we put $\varphi = x \in U$ we have from 0.6.12.3 that

$$\models \exists! x(x \in U) \Leftrightarrow \exists x(U = \{x\}).$$

0.6.16.3 Proposition. Suppose $x \in s_F(\varphi)$, $y \notin s_{FB}(\varphi)$ then

$$\models \exists! x\varphi \leftrightarrow \exists y \forall x(\varphi \leftrightarrow x = y).$$

Proof. Let $\tau_0(U) = P(\tau_0(x))$ then

$$\models U = \{x\} \leftrightarrow \forall y(y \in U \leftrightarrow y \in \{x\}).$$

By (2) of 0.6.13.1, replacing $y \in \{x\}$ by $y = x$, and applying $\exists x$ (0.6.10.8), we have

$$\models \exists x(U = \{x\}) \leftrightarrow \exists x \forall y(y \in U \leftrightarrow y = x)$$

shifting variables and replacing equivalent formulas yields

$$\models \exists x(U = \{x\}) \leftrightarrow \exists y \forall x(x \in U \leftrightarrow x = y).$$

Substituting $\{x:\varphi\}$ for U yields

$$\models \exists x(\{x:\varphi\} = \{x\}) \leftrightarrow \exists y \forall x(x \in \{x:\varphi\} \leftrightarrow x = y),$$

since $\models (x \in \{x:\varphi\}) \leftrightarrow \varphi$ we have

$$\models \exists! x\varphi \leftrightarrow \exists y \forall x(\varphi \leftrightarrow x = y).\square$$

0.6.16.4 Proposition. Suppose x_1 and x_2 are not free variables of φ.

$$\models \exists! x\varphi \leftrightarrow \exists x\varphi \wedge \forall x_1 \forall x_2(\varphi[x|x_1] \wedge \varphi[x|x_2] \rightarrow x_1 = x_2).$$

Proof. Substitute $\{x:\varphi\}$ for U is (5). Since $\models x \in \{x:\varphi\} \leftrightarrow \varphi$ and $\models x_i \in \{x:\varphi\} \leftrightarrow \varphi[x|x_i]$ for $i = 1,2$, the result follows by a replacement of formulas.□

0.6.16.5. We use the singleton morphism and unique existentiation to establish that a binary relation, $R: A \times B \longrightarrow \Omega$ which in the internal language is a functional relation, actually rises from a uniquely determined morphism $G: A \longrightarrow B$.

Definition. We call $R: A \times B \longrightarrow \Omega$ functional if

$$\models \forall x \exists! y R(x,y) \text{ , where } \tau_0(x) = A,\ \tau_0(y) = B \text{ and } x \neq y.$$

<u>0.6.16.6 Proposition.</u> If $R\colon A \times B \longrightarrow \Omega$ is functional there is a uniquely determined morphism $G\colon A \longrightarrow B$ such that

$$\models R\langle x,y\rangle \leftrightarrow Gx = y \qquad (x \neq y).$$

<u>Proof.</u> Define $F\colon A \longrightarrow \Omega^B$ by $\models Fx = \{y\colon R\langle x,y\rangle\}$, so that $\models y \in Fx \leftrightarrow R\langle x,y\rangle$. From the hypothesis $\models \exists! y(y \in Fx)$ hence $\models \exists y(Fx = \{y\})$ which is the same as $\models (\exists y(U = \{y\}))[U|Fx]$. By 0.6.2.3, F factors through m, where $[\![m]\!] = [\![\exists y(U = \{y\})]\!]$. But by 0.6.11.2, $\{-\}_B \in \|\exists y(U = \{y\})\|$, so F factors through $\{-\}_B$. Let $G\colon A \longrightarrow B$ be the factor: $\{-\}_B \circ G = F$ it is uniquely determined since $\{-\}_B$ is a mono. The external equation is equivalent to $\models \{Gx\} = Fx$ hence $\models y \in \{Gx\} \leftrightarrow R\langle x,y\rangle$, so $\models R\langle x,y\rangle \leftrightarrow (Gx = y)$. To establish uniqueness, suppose $\models R\langle x,y\rangle \leftrightarrow G_1x = y$ then $\models Gx = y \leftrightarrow G_1x = y$, substituting $\binom{y}{Gx}$ yields $\models G_1x = Gx$, hence $G_1 = G$. □

<u>0.6.16.7 Proposition.</u> If $R\colon A \times B \longrightarrow \Omega$ satisfies $\models \forall x \exists! y R\langle x,y\rangle$ and $\models \forall y \exists! x R\langle x,y\rangle$, with $x \neq y$, then there is a uniquely determined isomorphism $G\colon A \longrightarrow B$ such that $\models R\langle x,y\rangle \leftrightarrow Gx = y$.

<u>Proof.</u> From 0.6.16.6 we know there is a morphism $G\colon A \longrightarrow B$ which is uniquely determined by the condition $\models R\langle x,y\rangle \leftrightarrow Gx = y$. Define $\overline{R}\colon B \times A \longrightarrow \Omega$ by $\models \overline{R}\langle y,x\rangle \leftrightarrow R\langle x,y\rangle$, then there is a uniquely determined morphism $\overline{G}\colon B \longrightarrow A$ for which $\models \overline{R}\langle y,x\rangle \leftrightarrow \overline{G}y = x$. Hence $\models (\overline{G}y = x) \leftrightarrow (Gx = y)$. Substituting first $\binom{y}{Gx}$ then $\binom{x}{\overline{G}y}$ we get $\models \overline{G}Gx = x$ and $\models G\overline{G}y = y$, hence $\overline{G} \circ G = id_A$ and $G \circ \overline{G} = id_B$. □

<u>0.6.16.8 Proposition</u> (Osius [O1] 5.9). Suppose (d) commutes then the following are equivalent

$$\begin{array}{ccc} C & \xrightarrow{g_1} & B_1 \\ {\scriptstyle g_2}\downarrow & (d) & \downarrow{\scriptstyle f_1} \\ B_2 & \xrightarrow[f_2]{} & A \end{array}$$

(1) (d) is a pullback

(2) $\models (f_1x_1 = f_2x_2) \leftrightarrow \exists! w((g_1w = x_1) \wedge (g_2w = x_2))$.

<u>Proof.</u> Let $m = g_1 \sqcap g_2$ then (3) $\models mw = \langle g_1w, g_2w\rangle$. Since (d) commutes (4) $\models f_1g_1w = f_2g_2w$. The condition that m be a mono is

$$\models (mu_1 = mu_2) \rightarrow (u_1 = u_2)$$

which is equivalent to

$\models \forall u_1 \forall u_2((mu_1 = \langle x_1,x_2\rangle) \wedge (mu_2 = \langle x_1,x_2\rangle) \rightarrow (u_1 = u_2))$.

(1) $\rightarrow$ (2): From (1) we have m is a mono and hence by 0.6.11.11 and 0.6.16.4 , $\models (f_1x_1 = f_2x_2) \leftrightarrow \exists! w(mw = \langle x_1,x_2\rangle)$.

(2) $\rightarrow$ (1) From (2) we have

$\models (f_1x_1 = f_2x_2) \rightarrow [(mu_1 = \langle x_1,x_2\rangle \wedge (mu_2 = \langle x_1,x_2\rangle) \rightarrow (u_1 = u_2)]$.

Substituting g_1u_1 for x_1 and g_2u_1 for x_2 yields

$$\models (mu_2 = mu_1) \rightarrow (u_1 = u_2)$$

hence m is a mono. Hence by 0.6.16.4, (2) reduces to (1) of 0.6.11.11, hence (d) is a pullback. □

<u>0.6.16.9 Proposition.</u> If $x \notin s_F(t)$ and $m: \tau_0(x) \rightarrowtail \tau(t)$ is a monomorphism then

$$\models \exists! x(mx = t) \leftrightarrow \exists x(mx = t).$$

Proof. By 0.6.16.4, for $\{x_1, x_2\} \cap s_F(tx) \equiv \phi$, $x_1 \not\equiv x_2$,

$\models \ \exists! x(mx = t) \Leftrightarrow (\exists x(mx = t) \wedge \forall x_1 \forall x_2((mx_1 = t) \wedge (mx_2 = t) \Rightarrow (x_1 = x_2)))$.

Since m is a mono

$$\models ((mx_1 = t) \wedge (mx_2 = t)) \Rightarrow (mx_1 = mx_2)$$

$$\Rightarrow (x_1 = x_2)$$

hence $\models \forall x_1 \forall x_2((mx_1 = t) \wedge (mx_2 = t) \Rightarrow (x_1 = x_2))$.□

0.6.17 Homomorphisms between internal algebras.

0.6.17.1 Definition. Let $\mathcal{O}$ be a similarity type for algebras, let $\mathcal{A} = \langle A,\gamma\rangle$ and $\mathcal{B} = \langle B,\rho\rangle$ be internal $\mathcal{O}$-algebras. A morphism $h: A \longrightarrow B$ is an $\mathcal{O}$-homomorphism from $\mathcal{A}$ to $\mathcal{B}$ if

(1) for each $c \in \mathcal{O}_0$, $\models h\overline{c^*} = \overline{c^*}$

(2) for each $f \in \mathcal{O}_1$, $\models h\overline{f}a = \overline{f}ha$

(3) for each $g \in \mathcal{O}_2$, $\models h\overline{g}\langle a,a'\rangle = \overline{g}\langle ha,ha'\rangle$

where a, a' are distinct variables of type A ; we shall indicate this situation by $h: \mathcal{A} \longrightarrow \mathcal{B}$.

0.6.17.2 Definition. From the proof of 0.3.3.3 (1) it follows that T_B is an $\mathcal{O}$-subalgebra of $St(\rho,A)$ whose operations are given in 0.2.3.10:

(1) for each $c \in \mathcal{O}_0$, $\dot{c} = \overline{c^*}$

(2) for each $f \in \mathcal{O}_1$, $\dot{f}(t) = \overline{f}t$

(3) for rach $g \in \mathcal{O}_2$, $\dot{g}(t,r) = \overline{g}\langle t,r\rangle$

where t and r are terms of type B.

Since $Poly(\mathbb{E}(\mathcal{O})) = Poly_{\mathcal{O}}(V)$ is the absolutely free $\mathcal{O}$-algebra generated by V, there is an $\mathcal{O}$-homomorphism

$$\theta_h: Poly(\mathbb{E}(\mathcal{O})) \longrightarrow T_B$$

given on V by

$$\theta_h(v_i) = hx_i$$

where v_i is the i-th variable of V and x_i is the i-th variable of type A.

The fact that θ_h is an $\mathcal{O}$-homomorphism can be stated as

(1) for $c \in \mathcal{O}_0$: $\theta_h(c^*) = \overline{c}^*$,

(2) for $f \in \mathcal{O}_1$, $t \in Poly(\mathbb{E}(\mathcal{O}))$: $\theta_h(ft) = \overline{f}(\theta_h(t))$,

(3) for $g \in \mathcal{O}_2$, $\{t,r\} \subset Poly(\mathbb{E}(\mathcal{O}))$: $\theta_h(g\langle t,r\rangle) = \overline{g}\langle\theta_h(t),\theta_h(r)\rangle$.

<u>0.6.17.3 Proposition.</u> For each $t \in \mathrm{Poly}(\mathbb{E}(\mathcal{O}))$,

$$\models \theta_h(t) = h\overline{t}\ .$$

<u>Proof.</u> Let $Q = \{t \in \mathrm{Poly}(\mathbb{E}(\mathcal{O})) \mid\ \models \theta_h(t) = h\overline{t}\}$. For each $v \in V$, $\theta_h(v) = h\overline{v}$. Hence $V \subset Q$. We show Q is closed under the operations of $\mathrm{Poly}(\mathbb{E}(\mathcal{O}))$.

(1) For $c \in \mathcal{O}_0$, $\models \theta_h(c^*) = \overline{c^*} = h\overline{c^*}$.

(2) For $f \in \mathcal{O}_1$, $t \in Q$, $\models \theta_h(ft) = \overline{f}(\theta_h(t)) = \overline{f}(h\overline{t}) = h\overline{ft}$.

(3) For $g \in \mathcal{O}_2$, $\{t,r\} \subset Q$,

$\models \theta_h(g\langle t,r\rangle) = \overline{g}\langle\theta_h(t),\theta_h(r)\rangle = \overline{g}\langle h\overline{t},h\overline{r}\rangle = h\overline{g}\langle\overline{t},\overline{r}\rangle = h\overline{g\langle t,r\rangle}$.

Thus $Q = \mathrm{Poly}(\mathbb{E}(\mathcal{O}))$.□

<u>0.6.17.4 Corollary.</u> Let $t,\ r \in \mathrm{Poly}(\mathbb{E}(\mathcal{O}))$. If $\models \overline{t} = \overline{r}$, then $\models \theta_h(t) = \theta_h(r)$.

<u>Proof.</u> By the Corollary of 0.6.9.5, $\models h\overline{t} = h\overline{r}$, hence by 0.6.17.3 $\models \theta_h(t) = \theta_h(r)$.□

<u>0.6.17.5 Definition.</u> We define a function

$$\alpha_h\colon \mathrm{Vbls} \longrightarrow \mathrm{Tms}$$

by $\alpha_h(y_i) = hx_i$ where y_i is the i-th variable of type B and x_i is the i-th variable of type A

$\alpha_h(w) = w$ for $w \notin V_B$.

<u>0.6.17.6 Proposition.</u> For $t \in \mathrm{Poly}(\mathbb{E}(\mathcal{O}))$, $\theta_h(t) = S(\alpha_h)(\overline{t})$.

<u>Proof.</u> Let $Q' = \{t \in \mathrm{Poly}(\mathbb{E}(\mathcal{O})) \mid \theta_h(t) = S(\alpha_h)(\overline{t})\ \}$. For each $v \in V$, $(S(\alpha_h))(\overline{v}) = \alpha_h(\overline{v}) = h\overline{v} = \theta_h(v)$. Hence $V \subset Q'$. We show Q' is closed under the operations of $\mathrm{Poly}(\mathbb{E}(\mathcal{O}))$.

$\theta_h(c^*) \equiv \overline{c}^* \equiv S(\alpha_n)(\overline{c^*})$ hence $c^* \in Q'$. If $t \in Q'$ and $f \in \mathcal{O}_1$ then $\theta_h(ft) \equiv \overline{f}(\theta_h(t)) \equiv \overline{f}(S(\alpha_h)(\overline{t})) \equiv S)\alpha_h)(\overline{ft})$ hence $ft \in Q'$.

If $\{t_1,t_2\} \subset Q'$ and $g \in \mathcal{O}_2$ then

$$\theta_h(g/t_1,t_2/) \equiv \overline{g}/\theta_h(t_1),\theta_h(t_2)/$$

$$\equiv \overline{g}/S(\alpha_h)(\overline{t}_1),S)\alpha_h)(\overline{t}_2)/$$

$$S(\alpha_h)/(\overline{g/t_1,t_2/})/ .$$

Hence $g/t_1,t_2/ \in Q'$.□

<u>0.6.17.7 Corollary.</u> Let $t, r \in \mathrm{Poly}(E(\mathcal{O}))$, $h: \mathcal{A} \longrightarrow \mathcal{B}$ an $\mathcal{O}$-homomorphism from $\mathcal{A}$ to $\mathcal{B}$.

(1) If $\mathcal{B} \models t = r$, then $\models h\overline{t} = h\overline{r}$

(2) If h is a mono and $\mathcal{B} \models t = r$, then $\mathcal{A} \models t = r$.

<u>Proof.</u> (1) Suppose $\mathcal{B} \models t = r$, that is, $\models \overline{\overline{t}} = \overline{\overline{r}}$. Then $\models S(\alpha_h)(\overline{\overline{t}}) = S(\alpha_h)(\overline{\overline{r}})$, hence by 0.6.17.6, $\models \theta_h(t) = \theta_h(r)$, hence $\models h\overline{t} = h\overline{r}$. □

(2) By 0.6.9.4, from $\models h\overline{t} = h\overline{r}$ we deduce $\models \overline{t} = \overline{r}$. □

<u>0.6.17.8 The diagonal homomorphism.</u> Let $\mathcal{B} = \langle B, \gamma \rangle$ be an $\mathcal{O}$-algebra and let $\mathcal{B}^A = \langle B^A, \rho \rangle$. Let $\tau_0(b) = B = \tau_0(b')$, $\tau_0(a) = A$; b, b' and a distinct. Define $\Delta: B \longrightarrow B^A$ by $\models (\Delta b)[a] = b$. We show Δ is an $\mathcal{O}$-homorphism from $\mathcal{B}$ to $\mathcal{B}^A$.

(1) For $c \in \mathcal{O}_0$, $\models (\Delta(\overline{c^*}))[a] = \overline{c^*} = \overline{\overline{c^*}}[a]$ (by 0.6.15.1 (1)), hence $\models \Delta(\overline{c^*}) = \overline{\overline{c^*}}$.

(2) For $f \in \mathcal{O}_1$, $\models (\Delta(\overline{f}b))[a] = \overline{f}b = \overline{f}((\Delta b)[a])$

$$= (\overline{\overline{f}}\Delta b)[a] \quad \text{(by 0.6.15.1 (2))},$$

hence $\models \Delta\overline{f}b = \overline{\overline{f}}\Delta b$.

(3) For $g \in \mathcal{O}_2$, $\models (\Delta\overline{g}\langle b,b'\rangle)[a] = \overline{g}\langle b,b'\rangle = \overline{g}\langle(\Delta b)[a],(\Delta b')[a]\rangle$

$$= (\overline{\overline{g}}\langle\Delta b,\Delta b'\rangle)[a] \quad \text{(by 0.6.15.1 (3))},$$

hence $\models \Delta\overline{g}\langle b,b'\rangle = \overline{\overline{g}}\langle\Delta b,\Delta b'\rangle$. □

<u>0.6.17.9 Proposition.</u> If $u_A: A \longrightarrow []$ is an epi then $\Delta: B \longrightarrow B^A$ is a mono.

<u>Proof.</u> By 0.6.13.3 (2), u_A is an epi iff $\models \exists a(y = u_A a)$ where

$\tau_0(y) = [\,]$. Thus u_A is an epi iff $\models \exists a(u_A a = *)$ iff $\models \exists a(a = a)$.
By 0.6.14.3 $\models (\Delta b_1 = \Delta b_2) \to (\Delta b_1)[a] = (\Delta b_2)[a]$, hence
$\models (a = a) \to \left[(\Delta b_1 = \Delta b_2) \to ((\Delta b_1)[a] = (\Delta b_2)[a])\right]$, now by replacement (0.6.9.13 (2)) and the definition of Δ we have
$\models (a = a) \to \left[(\Delta b_1 = \Delta b_2) \to (b_1 = b_2)\right]$. Since u_A is an epi
$\models \Delta b_1 = \Delta b_2 \to b_1 = b_2$, hence Δ is a mono.□

In order to prove that the homomorphic image of an internal algebra satisfying an equation $t = r$ again satisfies $t = r$ we need to generalize 0.6.14.

<u>0.6.17.10 Proposition</u>. Let α be an admissible substitution for t, $\langle\vec{y},t\rangle$ an augmented term, $y_i \equiv (\vec{y}(i))^\wedge$ and $\ell(\vec{y}) \equiv n$, then

$$\models \bigwedge_{i \equiv 0}^{n-1} (\alpha(y_i) = y_i) \to S(\alpha)(t) = t$$

<u>Proof.</u> By induction on $\ell(t)$:

$t \equiv *$, then $S(\alpha)(t) \equiv y$.

$t \equiv y_j$, then $\models \bigwedge_{i \equiv 0}^{n-1} (\alpha(y_i) = y_i) \to \alpha(y_j) = y_j$

$$\to S(\alpha)(y_j) = y_j$$

$t \equiv fr$, then $\models \bigwedge_{i \equiv 0}^{n-1} (\alpha(y_i) = y_i) \to (S(\alpha)(r) = r)$

$$\to f(S(\alpha)(r)) = fr$$

$$\to S(\alpha)(fr) = fr.$$

$t \equiv \langle r_1,r_2\rangle$, then $\models \bigwedge_{i \equiv 0}^{n-1} (\alpha(y_i) = y_i) \to (S(\alpha)(r_j) = r_j)$ for $j \equiv 1,2$

hence $\models \bigwedge_{i \equiv 0}^{n-1} (\alpha(y_i) = y_i) \to ((S(\alpha)(r_1) = r_1) \wedge (S(\alpha)(r_2) = r_2))$

$$\to (S(\alpha)\langle r_1,r_2\rangle = \langle r_1,r_2\rangle).$$ □

0.6.17.11 Proposition. Let $t, r \in Poly(E(\mathcal{O}))$, $A \not\equiv B$, $h: A \twoheadrightarrow B$ an epimorphism which is an $\mathcal{O}$-homomorphism from $\mathcal{A}$ to $\mathcal{B}$. If

$\mathcal{A} \models t = r$ then $\mathcal{B} \models t = r$.

Proof. We interpret the polynomials t and r as terms $\bar{t}$ and $\bar{r}$ of type A in $\mathcal{A}$ and as terms $\bar{\bar{t}}$ and $\bar{\bar{r}}$ of type B in $\mathcal{B}$.

We have $\models \bar{t} = \bar{r}$, hence $\models h\bar{t} = h\bar{r}$. Since h is an $\mathcal{O}$-homomorphism, by 0.6.17.3 and 0.6.17.6, we have

$$S(\alpha_h)(\bar{\bar{t}}) = S(\alpha_h)(\bar{\bar{r}}).$$

Let $y \equiv var(\overline{\overline{tr}})$, $(\vec{y}(i))^\wedge \equiv y_i$, $\ell(\vec{y}) \equiv n$, $\vec{x} \equiv var(\overline{tr})$, $(\vec{x}(i))^\wedge \equiv x_i$.

Then $\alpha_h(y_i) \equiv hx_i$. Hence by 0.6.17.10,

$\models \bigwedge_{i \equiv 0}^{n-1} (hx_i = y_i) \Rightarrow \bar{\bar{t}} = \bar{\bar{r}}$. Since $x_i \notin s_F(\overline{\overline{tr}})$ we can introduce existencial quantifiers: $\models \exists x_j \bigwedge_{i \equiv 0}^{n-1} (hx_i = y_i) \Rightarrow (\bar{\bar{t}} = \bar{\bar{r}})$ $j \in [n]$

hence $\models \bigwedge_{i \equiv 0}^{n-1} \Big((\exists x_i(hx_i = y_i))\Big) \Rightarrow (\bar{\bar{t}} = \bar{\bar{r}})$.

Since h is an epi, $\models \exists x_i(hx_i = y_i)$. Also $y_i \in s_F(\overline{\overline{tr}})$. Hence $\models \bar{\bar{t}} = \bar{\bar{r}}$. □

0.6.17.12 Proposition. Let $h: \mathcal{A} \longrightarrow \mathcal{B}$ be an $\mathcal{O}$-homomorphism, in a topos $\underline{\mathcal{E}}$, and let $F: \underline{\mathcal{E}} \longrightarrow \underline{\mathcal{G}}$ be a catesian functor of toposes, then $F(h): F(\mathcal{A}) \longrightarrow F(\mathcal{B})$ is an $\mathcal{O}$-homomorphism in $\underline{\mathcal{E}}$.

Proof. For each sign f of $\mathcal{O}$ we let $\bar{f}$, $\bar{\bar{f}}$, f' and f'' denote their respective interpretations in $\mathcal{A}$, $\mathcal{B}$, $F(\mathcal{A})$ and $F(\mathcal{B})$.

(1) $c \in \mathcal{O}_0$, $\sigma: \underline{1} \longrightarrow F(1)$ the canonical isomorphism in $\underline{\mathcal{G}}$.

$F(h)\ c' \equiv F(h)\ F(\bar{c})\ \sigma$ by definition of c' (0.6.12.2)

$\equiv F(\bar{\bar{c}})\ \sigma$ since h preserves c

$\equiv c''$.

(2) $f \in \mathcal{O}_1$.

$$F(h)\, f' \equiv F(h)\bullet F(\overline{f}) \equiv F(h\bullet\overline{f})$$

$$\equiv F(\overline{\overline{f}}\bullet h) \qquad \text{since } h \text{ preserves } f$$

$$\equiv F(\overline{\overline{f}})\bullet F(h) \equiv f''\bullet F(h).$$

The product structure induces a natural transformation .

$$\begin{array}{ccc} \underline{\mathcal{E}} \times \underline{\mathcal{E}} & \xrightarrow{F \times F} & \underline{\mathcal{G}} \times \underline{\mathcal{G}} \\ \times \downarrow & \eta \nearrow & \downarrow \times \\ \underline{\mathcal{E}} & \xrightarrow{F} & \mathcal{G} \end{array} \qquad F(A \times D) \xrightarrow{\eta_{A,D}} FA \times FD$$

Since F is cartesian each component of η is an isomorphism, hence η is a natural isomorphism with inverse σ.

Putting $\sigma' \equiv \sigma_{A,A}$ and $\sigma'' \equiv \sigma_{B,B}$ we have a commutative diagram:

$$\begin{array}{ccc} FA \times FA & \xrightarrow{\sigma'} & F(A \times A) \\ F(h) \times F(h) \downarrow & & \downarrow F(h \times h) \\ FB \times FB & \xrightarrow{\sigma''} & F(B \times B) \end{array}$$

(3) $g \in \mathcal{O}_2$.

$$F(h)\; g' \equiv F(h)\bullet F(\overline{g})\bullet\sigma' \equiv F(h\bullet\overline{g})\; \sigma'$$

$$\equiv F(\overline{\overline{g}})\bullet F(h \times h)\bullet\sigma' \qquad \text{since } h \text{ preserves } g$$

$$\equiv F(\overline{\overline{g}})\bullet\sigma''\bullet(F(h) \times F(h)) \qquad \text{from the above}$$

$$\equiv g''\bullet(F(h) \times F(h)).\square$$

<u>0.6.17.13 Proposition.</u> Let $h: \mathcal{A} \longrightarrow \mathcal{B}$ and $k: \mathcal{B} \longrightarrow \mathcal{C}$ be $\mathcal{O}$-homomorphisms, in a topos $\underline{\mathcal{E}}$, then $k \bullet h$ is an $\mathcal{O}$-homomorphism from $\mathcal{A}$ to $\mathcal{C}$.

<u>Proof.</u> For each sign f of $\mathcal{O}$ we let $\bar{f}$, $\bar{\bar{f}}$ and f' denote their respective interpretations in $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$.

(1) $c \in \mathcal{O}_0$

$\models (k \bullet h)(\overline{c^*}) = k(h\, \overline{c^*}) = k\overline{\overline{c^*}} = c'^*$ se 0.6.12.2

(2) $f \in \mathcal{O}_1$

$(k \bullet h)\, \bar{f} \equiv k\, \bar{\bar{f}}\, h \equiv f'\, (k\, h)$

(3) $g \in \mathcal{O}_2$

$\models (k \bullet h)(\bar{g}\langle a_1, a_2\rangle) = k(h(\bar{g}\langle a_1, a_2\rangle)) = kg\langle ha_1, ha_2\rangle$

$= g'\langle kha_1, kha_2\rangle$. □

<u>0.6.17.14 Proposition.</u> Let $h: A \longrightarrow B$ be an isomorphism in $\mathcal{E}$ with inverse $h^{-1}: B \longrightarrow A$. Let $\mathcal{A}$ and $\mathcal{B}$ be any $\mathcal{O}$-algebra structures on A and B respectively and for each sign $f \in |\mathcal{O}|$ let $\bar{f}$ and $\bar{\bar{f}}$ be its interpretations in $\mathcal{A}$ and $\mathcal{B}$ respectively. Let $c \in \mathcal{O}_0$, $f \in \mathcal{O}_1$, $g \in \mathcal{O}_2$.

(1) The following are equivalent

(1.1) $h\overline{c^*} = \overline{\overline{c^*}}$

(1.2) $h^{-1}\overline{\overline{c^*}} = \overline{c^*}$

(2) The following are eouiavlent

(2.1) $\bar{\bar{f}}b = h\bar{f}h^{-1}b$

(2.2) $h^{-1}\bar{\bar{f}}b = \bar{f}h^{-1}b$

(2.3) $\bar{\bar{f}}ha = h\bar{f}a$

(2.4) $\bar{f}a = h^{-1}\bar{\bar{f}}ha$

(3) The following are equivalent

(3.1) $g(b_1,b_2) = hg(h^{-1}b_1,h^{-1}b_2)$

(3.2) $h^{-1}\bar{g}(b_1,b_2) = \bar{g}(h^{-1}b_1,h^{-1}b_2)$

(3.3) $\bar{g}(ha_1,ha_2) = h\bar{g}(a_1,a_2)$

(3.4) $\bar{g}(a_1,a_2) = h^{-1}\bar{g}(ha_1,ha_2)$.

Proof. Use the fact that h^{-1} is the inverse of h repeatedly. For example (3.4) → (3.2): Substitute $h^{-1}b_i$ for a_i ($i \equiv 1,2$) in (3.4)
$h^{-1}\bar{g}(b_1,b_2) = h^{-1}\bar{g}(hh^{-1}b_1,hh^{-1}b_2) = \bar{g}(h^{-1}b_1,h^{-1}b_2)$.□

0.6.17.14.1 Corollary. Let $h: A \longrightarrow B$ be an isomorphism and let $\mathcal{A}$ be an $\mathcal{O}$-algebra structure on A.

(1) There is exactly one $\mathcal{O}$-algebra structure, $\mathcal{B}$, on B which makes $h: \mathcal{A} \longrightarrow \mathcal{B}$ on $\mathcal{O}$-homomorphism.

(2) If $h: \mathcal{A} \longrightarrow \mathcal{B}$ is an $\mathcal{O}$-homomorphism then so is $h^{-1}: \mathcal{B} \longrightarrow \mathcal{A}$.

Proof. (1) Define $\mathcal{B}$ by the equations (1.1), (2.1) and (3.1) in 0.6.17.15, then (2.3) and (3.3) hold so $h: \mathcal{A} \longrightarrow \mathcal{B}$ is an $\mathcal{O}$-homomorphism Suppose that $\mathcal{B}'$ is an $\mathcal{O}$-algebra structure and $h: \mathcal{A} \longrightarrow \mathcal{B}'$ is an $\mathcal{O}$-homomorphism, so that (1.1), (2.3) and (3.3) hold, then (2.1) and (3.1) hold, hence $\mathcal{B}' \equiv \mathcal{B}$.

(2) In 0.6.17.15, (1.1) implies (1.2), (2.3) implies (2.2), and (3.3) implies (3.2).

We will use this corollary in 1.7.1.24 to show that there is an H-isomorphism between $\mathrm{Idl}(\mathcal{A} \times B)$ and $(\mathrm{Idl}\mathcal{A})^B$ where $\mathcal{A}$ is an internal partially ordered object.

CHAPTER 1

THE PROPOSITIONAL LOGIC AND CONVERSE LOGICS OF A TOPOS

Section 1.1 Intermediate Propositional Logic.

Every topos satisfies the axioms of the intuitionistic propositional logic, as given in 0.6.4; but a particular topos may satisfy formulas, definable within this system of logic, which are not consequences of the axioms of intuitionistic propositional logic. Such formulas are associated with intermediate propositional logics. Here we shall present material—relevant to the presheaf toposes we shall introduce—on such logics, derived from a number of sources: [Ba], [BS], [BD], [Du], [Gö], [Ho2], [Ho3], [HO], [Ja2], [Ja3], [Ma1], [Ma3],[Mc & T 2], [Ra][Th]. In addition we shall prove several results which are either only stated or do not appear in the literature cited.

1.1.1. Intermediate propositional logics and their models. We shall abbreviate Poly($\mathbb{E}$(H)) to Poly(H); $|A|$ will denote the carrier of an H-algebra A, and for any function $\gamma: V \longrightarrow |A|$, $\tilde{\gamma}$: Poly(H) $\longrightarrow A$ will denote the H-homomorphism extending γ.

1.1.1.1 The intermediate propositional logic determined by a non-degenerate topos. By a non degenerate topos $\underline{\mathcal{E}}$ we mean one for which the morphisms, true: $1 \longrightarrow \Omega$, and, false: $1 \longrightarrow \Omega$, are distinct. Let $\rho: V \xrightarrow[\approx]{} V_\Omega$ be the natural bijection, then, since the operations of the internal H-algebra $\underline{\Omega}$ are distinct, $\tilde{\rho}$: Poly(H) $\longrightarrow T_\Omega$ is a mono. We abbreviate $\tilde{\rho}(\varphi)$ to $\overline{\varphi}$ for each $\varphi \in$ Poly(H), and let $\overline{\text{Poly(H)}}$ be the image of

Poly(H) under $\tilde{\rho}$; thus $\tilde{\rho}$: Poly(H) $\xrightarrow{\sim}$ $\overline{\text{Poly(H)}}$ is an H-isomorphism and $\overline{\text{Poly(H)}}$ is the free H-algebra generated by V_Ω.

We let $L(\mathcal{E}) \equiv \{\varphi \in \text{Poly(H)} \mid \mathcal{E} \models \overline{\varphi}\}$. The set $\Sigma \equiv L(\mathcal{E})$ has the following properties:

(1) If $\varphi \in \Sigma$ and $\alpha \in (\text{Poly(H)})^V$ then $\tilde{\alpha}(\varphi) \in \Sigma$.

(2) If $\varphi \in \Sigma$ and $(\varphi \Rightarrow \psi) \in \Sigma$ then $\psi \in \Sigma$.

(3) $\varphi \in \Sigma$ for each φ in the list (T_1) to (T_{11}) given in 0.6.4.8; where the letters p, q and r are replaced by the letters u, v and w.

(4) $\underline{0} \notin \Sigma$.

Any subset of Poly(H) satisfying (1) and (2) is called a propositional logic ([Ba]). The smallest propositional logic satisfying (3), i.e. generated by (T_1) to (T_{11}), is the intuitionistic propositional logic IL, the largest propositional logic satisfying (3) and (4), i.e. the largest consistent propositional logic, is the classical propositional logic KL. If Σ satisfies all four conditions then $IL \subset \Sigma \subset KL$; such a set is therefore called an intermediate propositional logic which we abbreviate to i.p.l. We shall now justify the claim we have made for $L(\mathcal{E})$.

1.1.1.1.1. Proposition. $L(\mathcal{E})$ is an i.p.l.

Proof. (1) We must verify that the substitution we have developed for $\Phi(\mathcal{E})$ corresponds to the operations $\tilde{\alpha}$ defined on Poly(H). Given any $\alpha: V \longrightarrow \text{Poly(H)}$, $\tilde{\rho}\cdot\tilde{\alpha}\cdot(\tilde{\rho})^{-1}$: Poly(H) $\longrightarrow$ Poly(H) is an H-homomorphism. Let α': Vbls $\longrightarrow$ Tms be any extension of $\tilde{\rho}\cdot\alpha\circ\rho^{-1}: V_\Omega \longrightarrow \text{Poly(H)}$ to a type preserving function. From the definition (0 ˆ ˉ.2) of substitution, $S(\alpha')$ preserves the H-operations defined on Fmls. Also for $p \in V_\Omega$,

$S(\alpha')(p) \equiv (\tilde{\rho}\circ\alpha\circ\rho^{-1})(p) \equiv (\tilde{\rho}\circ\tilde{\alpha}\circ(\tilde{\rho})^{-1})(p)$, hence $S(\alpha')(\psi) \equiv (\tilde{\rho}\circ\tilde{\alpha}\circ(\tilde{\rho})^{-1})(\psi)$ for $\psi \in \overline{\mathrm{Poly}(\mathbb{H})}$. Now if $\varphi \in L(\underline{\mathcal{E}})$ then $\models \tilde{\rho}(\varphi)$, hence $\models S(\alpha')(\tilde{\rho}(\varphi))$, i.e. $\models \tilde{\rho}(\tilde{\alpha}(\varphi))$, so $\tilde{\alpha}(\varphi) \in L(\underline{\mathcal{E}})$. □

(2) If $\varphi \in \Sigma$ and $\varphi \Rightarrow \psi \in \Sigma$ then $\models \tilde{\rho}(\varphi)$ and $\models \tilde{\rho}(\varphi) \Rightarrow \tilde{\rho}(\psi)$, hence by 0.6.4.13 (3), $\models \tilde{\rho}(\psi)$.□

(3) Proven in 0.6.4.8.□

(4) Since $\underline{\mathcal{E}}$ is non-degenerate.□

<u>1.1.1.2 Connections with Varieties of Heyting algebras.</u>

There is a one-to-one correspondence between consistent equational theories Γ such that $\Sigma(\mathrm{Heyt.}) \subset \Gamma \subset \mathrm{Eq}(\mathbb{H}\mathrm{eq})$, and i.p.l.'s Σ. The correspondence is given by $\Gamma \equiv \{t_1 = t_2 \mid (t_1 \Leftrightarrow t_2) \in \Sigma\}$ and $\Sigma \equiv \{\varphi \mid (\varphi = 1) \in \Gamma\}$. Thus the correspondence made in Universal Algebra between equational theories and their models can be transferred to a correspondence between i.p.l.'s and non-trivial varieties of Heyting algebras. We shall denote by $\mathcal{Heyt}$ the class of all Heyting algebras i.e.

$$A \in \mathcal{Heyt} \text{ iff } A \models \Sigma(\mathrm{Heyt.}).$$

<u>1.1.1.2.1 Definition.</u> We shall carry over the definition of $\models$, as given in 0.2.3.13, to a relation $\vdash$. Throughout $A \in \mathcal{Heyt}$, $\mathbb{K} \subset \mathcal{Heyt}$, $\varphi \in \mathrm{Poly}(\mathbb{H})$ and $\Sigma \subset \mathrm{Poly}(\mathbb{H})$.

$A \vdash \varphi$ iff $A \models \varphi = \underline{1}$

$A \vdash \Sigma$ iff $A \vdash \varphi$ for all $\varphi \in \Sigma$

$\mathbb{K} \vdash \varphi$ iff $A \vdash \varphi$ for all $A \in \mathbb{K}$

$\mathbb{K} \vdash \Sigma$ iff $\mathbb{K} \vdash \varphi$ for all $\varphi \in \Sigma$.

$\Sigma \vdash \varphi$ iff for each A such that $A \vdash \Sigma$ we have $A \vdash \varphi$, i.e.

$$\{A \mid A \vdash \Sigma\} \vdash \varphi$$

$\Sigma' \vdash \Sigma$ iff for each $\varphi \in \Sigma$ we have $\Sigma' \vdash \varphi$

$\psi \vdash \varphi$ iff $\{\psi\} \vdash \varphi$, $\vdash \varphi$ iff $\phi \vdash \varphi$, and $\psi \dashv\vdash \varphi$ iff both $\psi \vdash \varphi$ and $\varphi \vdash \psi$.

Our usage of the "single turnstile" $\vdash$ is that of [Ja2]. Jankov calls the relation $\{(\psi,\varphi) | \psi \vdash \varphi\}$ deducibility with substitution; that is he defines the relation syntactically. Birkhoff's completeness theorem for equational logic (see [BS]) applied to the equational theories which include Σ(Heyt.) implies that the syntactic definition is equivalent to the semantic one given above (for each Heyting algebra A, if $A \models \psi = \underline{1}$ then $A \models \varphi = \underline{1}$). There is a direct proof of completeness for i.p.l. -that does not use the correspondence between equational theories and propositional logic- in [Ra] (thoerem 7.1 p. 185).

The qualifier "with substitution" is there to distinguish $\vdash$ from the relation $\vdash'$ of Kleene [Kl] for which we have

$$\psi \vdash' \varphi \quad \text{iff} \quad \vdash' \psi \to \varphi ,$$

in our case $v \vee \neg v \vdash u \vee \neg u$ but $\not\vdash (v \vee \neg v) \to (u \vee \neg u)$. (To see the latter consider the three element Heyting algebra $0 < e < 1$ for which $(1 \vee \neg 1) \to (e \vee \neg e) \equiv e$.).

We put $\mathrm{Mod}(\Sigma) \equiv \{A | A \vdash \Sigma\}$, the class of Heyting algebras which are models of Σ; $\mathrm{Val}(\mathbb{K}) \equiv \{\varphi | \mathbb{K} \vdash \varphi\}$, the set of polynomials valid in all Heyting algebras of $\mathbb{K}$; $\mathbb{K}^e \equiv \mathrm{Mod}(\mathrm{Val}(\mathbb{K}))$, the variety of Heyting algebras generated by $\mathbb{K}$; and $\Sigma^m \equiv \mathrm{Val}(\mathrm{Mod}(\Sigma))$ the propositional logic generated IL $\cup$ Σ. We put $\Sigma_{\dashv\vdash} \equiv \{\varphi |$ there exists $\psi \in \Sigma$ such that $\varphi \dashv\vdash \psi\}$.

Using this notation we have for example $\mathrm{Mod}(\phi) \equiv \mathcal{H}eyt \equiv \mathrm{Mod}(\mathrm{IL})$, Mod(v ∨ ¬ v) ≡ Mod(KL) ≡ $\mathcal{Bool}$ (the class of Boolean algebras considered as algebras of similarity type H).

We let H(𝕂) be the class of homomorphic images of algebras in 𝕂, and S(𝕂) the class of subalgebras of algebras in 𝕂 and I(𝕂) the class of algebras isomorphic to algebras in 𝕂.

In future section "H-algebra" will be used as we have defined it, but "algebra", unqualified, will mean "Heyting algebra", i.e. an H-algebra satisfying Σ(Heyt.).

1.1.1.3 Polynomials with variable exponents.

1.1.1.3.1 Definition. We let Φ be the set of all $\Sigma \subset$ Poly(H) such that

1) $\{\underline{0}, \underline{1}\} \subset \Sigma$

2) $V \subset \Sigma$

3) if $\{\varphi,\psi\} \subset \Sigma$ then $\{\varphi \wedge \psi, \varphi \vee \psi\} \subset \Sigma$

4) if $v \in V$ and $\varphi \in \Sigma$ then $(v \to \varphi) \in \Sigma$.

We put $\mathcal{C} \equiv \bigcap\Phi$. It is straightforward to verify that $\mathcal{C} \in \Phi$. We put

$$\mathcal{C}^{\to} \equiv \mathcal{C} \cup \{\varphi \to \psi \mid \{\varphi,\psi\} \subset \mathcal{C}\}.$$

We shall call members of $\mathcal{C}$ polynomials with variable exponents.

1.1.1.3.2 Polynomials with variable exponents arise in 1.7.3.7 and $\mathcal{C}^{\to}$ arises in 1.7.3.8. The theory of 1.7 is applicable to $\mathcal{C}^{\to}_{\dashv\vdash} \equiv (\mathcal{C}^{\to})_{\dashv\vdash}$.

In 1.1.2 and 1.1.3 we shall show that three sequences of polynomials, used in defining certain i.p.l.'s, belong to $\mathcal{C}^{\to}_{\dashv\vdash}$.

1.1.2 Subdirectly irreducible algebras.

We shall let $\mathcal{Si}$ denote the class of subdirectly irreducible algebras ([BD] p.13) and call any member of $\mathcal{Si}$ an $\mathcal{Si}$-algebra. These algebras will turn up naturally in our study of the logic of presheaf categories, thus a detailed examination of the propositional logic associated with these algebras will be useful to us.

1.1.2.1 Proposition. ([Mc & T 2] Theorem 1.7 and [BD] Theorem 2.8 p. 175) If $\mathcal{B}$ is an algebra and $a \in |\mathcal{B}|$, then $(\leftarrow, a]$, with the order induced by $\mathcal{B}$, has an algebra structure $\mathcal{B}_a$. Moreover there is an onto homomorphism $\theta_a \colon \mathcal{B} \twoheadrightarrow \mathcal{B}_a$ given by $\theta_a(x) \equiv x \wedge a$; θ_a has as kernel the filter $[a, \rightarrow)$ in $\mathcal{B}$.□

1.1.2.2 Definition. We call an element e of an algebra, having top element 1, penultimate ([Ma 3]) if $e < 1$ and for each $x < 1$ we have $x \leq e$; it is necessarily unique in an algebra if it exists.

If A is an algebra with top element e we can always construct an algebra $A \oplus 1$ with e as its penultimate element and $(A \oplus 1)_e \equiv A$. We just let 1 be any element not in $|A|$ and define a partial order $\leq_2$ on the set $|A| \cup \{1\}$ in terms of the partial order $\leq_1$ of A by:

$$x \leq_2 y \text{ iff either } y \equiv 1 \text{ or } x \leq_1 y.$$

In [Mc & T 1] Theorem 1.9 it is verified that there is a uniquely determined algebra structure $A \oplus 1$ compatible with $\leq_2$.

1.1.2.3 Proposition. ([BD] p. 179 Theorem 4.5) $\mathcal{Si}$ is the class of algebras with penultimate elements.□

It follows that $\mathcal{Si} \equiv \{A \oplus 1 \mid A \in \mathcal{Heyt}\}$. [By allowing 1 to be any element not in $|A|$ we can avoid requiring that $\mathcal{Si}$ consist of algebras isomorphic to algebras of the form $A \oplus 1$.]

<u>1.1.2.4 Δ- and ▲- projections.</u> ([Ho 3] Definition 7.1) We first take $\ell : \mathrm{Poly}(\mathbb{H}) \longrightarrow V$ to be such that $\ell(\varphi) \notin S_F(\varphi)$ for any $\varphi \in \mathrm{Poly}(\mathbb{H})$. Any such ℓ would do but for definiteness we shall let $\ell(\varphi)$ be the first variable of V not in $S_F(\varphi)$.

We define $\Delta : \mathrm{Poly}(\mathbb{H}) \longrightarrow \mathrm{Poly}(\mathbb{H})$ by

$\Delta(\varphi) \equiv ((\ell(\varphi) \Rightarrow \varphi) \Rightarrow \ell(\varphi)) \Rightarrow \ell(\varphi)$ for all $\varphi \in \mathrm{Poly}(\mathbb{H})$, thus $\Delta(\varphi) \equiv ((v \Rightarrow \varphi) \Rightarrow v) \Rightarrow v$ iff $v \equiv \ell(\varphi)$. We define $\blacktriangle : \mathrm{Poly}(\mathbb{H}) \longrightarrow \mathrm{Poly}(\mathbb{H})$ by $\blacktriangle(\varphi) \equiv \ell(\varphi) \vee (\ell(\varphi) \Rightarrow \varphi)$ for all $\varphi \in \mathrm{Poly}(\mathbb{H})$, thus $\blacktriangle(\varphi) \equiv v \vee . (v \Rightarrow \varphi)$ iff $v = \ell(\varphi)$. In future we shall not explicitly mention ℓ.

The function Δ is defined by Hosoi and called the Δ-projection. The sequence $\blacktriangle^n(\underline{0})$, $n \equiv 1, 2, \ldots$ appears in [Ma 1].

<u>1.1.2.4.1 Proposition.</u> $\mathcal{C}$ is closed under $\blacktriangle$ and $\blacktriangle^n(\underline{0}) \in \mathcal{C}$ for each n.

<u>Proof.</u> We use the defining clauses in 1.1.1.3.1. If $\varphi \in \mathcal{C}$ then $(v \Rightarrow \varphi) \in \mathcal{C}$, by 4); by 2), $v \in \mathcal{C}$, hence $(v \vee (v \Rightarrow \varphi)) \in \mathcal{C}$ by 3); so $\mathcal{C}$ is closed under $\blacktriangle$. By 1), $\underline{0} \in \mathcal{C}$, hence for each n, $\blacktriangle^n(\underline{0}) \in \mathcal{C}$.□

<u>1.1.2.5 Proposition.</u> Let $\varphi \in \mathrm{Poly}(\mathbb{H})$ and $A \in \mathcal{H}eyt$. The following are equivalent:

(1) $A \vdash \varphi$

(2) $A \oplus 1 \vdash \blacktriangle(\varphi)$

(3) $A \oplus 1 \vdash \Delta(\varphi)$

<u>Proof.</u> Let $v \equiv \ell(\varphi)$ and let $i : |A| \rightarrowtail |A \oplus 1|$ be the inclusion map. For each $\beta : V \longrightarrow |A|$ and each $\alpha : V \longrightarrow |A \oplus 1|$ we let $\overline{\beta} : \mathrm{Poly}(\mathbb{H}) \longrightarrow A$ and $\tilde{\alpha} : \mathrm{Poly}(\mathbb{H}) \longrightarrow A \oplus 1$ be their respective extensions. For all $x \in V$ we have

$(\theta\circ\tilde{\alpha})(x) \equiv (\theta\circ\alpha)(x) \equiv \overline{(\theta\circ\alpha)}(x)$, hence $\theta\circ\tilde{\alpha} \equiv \overline{\theta\circ\alpha}$; furthermore, since $\theta\circ i\circ\beta \equiv \beta$, $\theta\circ(\widetilde{i\circ\beta}) \equiv \overline{\beta}$.

(1) → (2). Let $\alpha\colon V \longrightarrow |A \oplus 1|$ be any valuation, then $\theta\circ\alpha$ is a valuation for the algebra A. By (1) $e \equiv \overline{(\theta\circ\alpha)}(\varphi) \equiv \tilde{\alpha}(\varphi) \wedge e$ hence $e \leq \tilde{\alpha}(\varphi)$. We want to show $1 \equiv \tilde{\alpha}(\Delta(\varphi))$ where $\tilde{\alpha}(\Delta(\varphi)) \equiv \alpha(v) \vee (\alpha(v) \Rightarrow \tilde{\alpha}(\varphi))$. If $\alpha(v) \equiv 1$ we are finished; so suppose $\alpha(v) \leq e$, then $\tilde{\alpha}(v) \leq \tilde{\alpha}(\varphi)$, hence $\tilde{\alpha}(v \Rightarrow \varphi) \equiv 1$. Hence $\tilde{\alpha}(\Delta(\varphi)) \equiv 1$.□

(2) → (3). In any Heyting algebra we have

$$a \vee b \leq (b \Rightarrow a) \Rightarrow a.$$

Hence for any $\alpha\colon V \longrightarrow |A \oplus 1|$, by (2)

$1 \equiv \tilde{\alpha}(\Delta(\varphi)) \equiv \alpha(v) \vee \tilde{\alpha}(v \Rightarrow \varphi) \leq (\tilde{\alpha}(v \Rightarrow \varphi) \Rightarrow \alpha(v)) \Rightarrow \alpha(v) \equiv \tilde{\alpha}(\Delta(\varphi))$.□

(3) ⟶ (1) Let $\beta\colon V \longrightarrow |A|$ be any valuation and let $\gamma \equiv i\circ\beta\colon V \longrightarrow |A \oplus 1|$. Define $\beta_e\colon V \longrightarrow |A|$ by $\beta_e(v) \equiv e$ and $\beta_e(x) \equiv \beta(x)$ for each $x \in V - \{v\}$ and let $\gamma_e \equiv i\circ\beta_e\colon V \longrightarrow |A \oplus 1|$. By 0.2.4.4, since $v \notin S_F(\varphi)$, $\varphi \in [V - \{v\}]_{\mathbb{H}}$. Since γ and γ_e agree on $V - \{v\}$, we have $\tilde{\gamma}_e(\varphi) \equiv \tilde{\gamma}(\varphi)$. By (3), $1 \equiv \tilde{\gamma}(\Delta(\varphi)) \equiv ((e \Rightarrow \tilde{\gamma}(\varphi)) \Rightarrow e) \Rightarrow e$, hence $(e \Rightarrow \tilde{\gamma}(\varphi)) \Rightarrow e \leq e$, hence $e \Rightarrow \tilde{\gamma}(\varphi) \not\leq e$, hence $e \leq \tilde{\gamma}(\varphi)$. Thus $\overline{\beta}(\varphi) \equiv \theta(\tilde{\gamma}(\varphi)) \equiv \tilde{\gamma}(\varphi) \wedge e \equiv e$.□

<u>1.1.2.6 Corollary</u>. 1) If $\varphi \dashv\vdash \psi$ then $\Delta(\varphi) \dashv\vdash \Delta(\psi)$

2) $\Delta(\varphi) \dashv\vdash \Delta(\varphi)$ and 3) $\Delta^n(\underline{0}) \in \mathcal{C}_{\dashv\vdash}$ for each n.

<u>Proof.</u> 1) By 1.1.2.5.□

2) For $\mathbb{K}$ a variety of algebras $(\mathcal{SL} \cap \mathbb{K})^e \equiv \mathbb{K}$ ([BD] p. 13 by Birkhoff's subdirect representation theorem). Thus

$$\mathrm{Mod}\{\Delta(\varphi)\} \equiv (\mathcal{SL} \cap \mathrm{Mod}\{\Delta(\varphi)\})^e \equiv (\mathcal{SL} \cap \mathrm{Mod}\{\Delta(\varphi)\})^e \quad \text{by 1.1.2.5}$$

$$\equiv \mathrm{Mod}\{\Delta(\varphi)\} .\square$$

3) By 1) and 2). □

<u>1.1.2.6.1</u> We can now characterize the subdirectly irreducible models of $\mathbb{A}(\underline{0})$ and $\mathbb{A}^2(\underline{0})$.

$\mathcal{S}i \cap \mathrm{Mod}\{\mathbb{A}(\underline{0})\} \equiv \{A \oplus 1 \mid A \oplus 1 \vdash \Delta(\underline{0})\} \equiv \{A \oplus 1 \mid A \vdash \underline{0}\}$

$\equiv \{A \oplus 1 \mid A$ is a one element algebra$\}$

$\equiv \{B \mid B$ is a two element algebra$\}$

$\equiv I\{\mathbf{2}\}$ where $\mathbf{2}$ is the two element Boolean algebra.

$\mathrm{Mod}\{\mathbb{A}(\underline{0})\} \equiv (\mathcal{S}i \cap \mathrm{Mod}\{\mathbb{A}(\underline{0})\})^e \equiv \{\mathbf{2}\}^e \equiv \mathcal{B}ool.$

Thus $\mathbb{A}(\underline{0}) \dashv\vdash v \vee \neg v$.

$\mathcal{S}i \cap \mathrm{Mod}\{\mathbb{A}^2(\underline{0})\} \equiv \{A \oplus 1 \mid A \vdash v \vee \neg v\}$

$\equiv \{A \oplus 1 \mid A$ is a Boolean algebra$\}$.

<u>1.1.2.7 Proposition.</u> $\varphi \vdash \mathbb{A}(\varphi)$.

<u>Proof</u>. Suppose $A \oplus 1 \vdash \varphi$, then, since $A \in H\{A \oplus 1\}$, $A \vdash \varphi$, hence, by 1.1.2.5, $A \oplus 1 \vdash \mathbb{A}(\varphi)$. Thus $\mathcal{S}i \cap \mathrm{Mod}\{\varphi\} \subset \mathcal{S}i \cap \mathrm{Mod}\{\mathbb{A}(\varphi)\}$ hence $\mathrm{Mod}\{\varphi\} \subset \mathrm{Mod}\{\mathbb{A}(\varphi)\}$. □

1.1.3 A survey of i.p.l.'s by slicing.

The purpose of this section is to explain an approach to i.p.l.'s taken by Hosoi in a series of papers, in particular we shall be interested in the results of a paper by Hosoi and Ono [HO].

1.1.3.1 Background. We take ω to be the first infinite ordinal. We let S_ω be the Heyting algebra having carrier $|S_\omega| \equiv \{0\} \cup \{e_i \mid i < \omega\}$, $0 \leq i < \omega$ and ordering $0 < e_i$ for $i < \omega$ and $e_j < e_i$ for $i < j$. Thus S_ω is a chain with top element $1 \equiv e_0$, and a uniquely determined Heyting algebra structure. Define algebras S_n $(1 \leq n < \omega)$ as follows $|S_n| \equiv \{e_i \mid 0 \leq i < n\}$ and put $0 \equiv e_n$, thus $0 \equiv e_n < e_{n-1} < \dots < e_1 < e_0 \equiv 1$, making S_n an n+1 element chain. $S_1 \subset S_2 \subset \dots \subset S_\omega$ is a chain of subalgebras. Let $L_n \equiv Val(S_n)$ for each $n \leq \omega$. Then we have a chain of i.p.l.'s $L_\omega \subset \dots \subset L_2 \subset L_1$.

In the first paper on i.p.l.'s [Gö] it was established, by introducing polynomials F_k, that for $i < j < \omega$, $L_i \not\equiv L_j$, thus each inclusion is proper.

In [Du] Dummett studied the logic L_ω. Define $Z \equiv (u_1 \to u_2) \vee (u_2 \to u_1)$, then $\{Z\}^m \equiv L_\omega$ and $Z \notin IL$ so that L_ω properly contains IL. Furthermore $\{Z\}^m \equiv Val\{S_n \mid n < \omega\}$, thus if Σ is any i.p.l. such that $L_\omega \subset \Sigma \subset L_n$ for each $n < \omega$ then $L_\omega \equiv \Sigma$, thus $\bigcap\{L_n \mid n < \omega\} \equiv L_\omega$.

In [Ho 2] Hosoi established that if $\varphi \in L_k$ but $\varphi \notin L_{k-1}$ then $\{Z,\varphi\}^m \equiv L_k$, from this it follows, as noted in [Ho 3], that if Σ is a proper extension of L_ω which contains $L_1 \equiv KL$ then for some k (namely the greatest k such that $\Sigma \subset L_k$) $\Sigma \equiv L_k$. Thus the L_k constitute all the extensions of L_ω (if we take S_0 to be the trivial algebra $Poly(H) \equiv Val(S_0) \equiv L_0$).

The logics L_k, $k < \infty$, were first axiomatized by I. Thomas [Th] in 1961. It follows from Hosoi's characterization that we can use the formulas F_k of Gödel to axiomatize the L_k, $L_k \equiv \{Z, F_k\}^m$. In [Ho 3] Hosoi shows that another sequence of formulas R_k, first introduced by Umezawa, also axiomatize the L_k. First define

$$R'_k \equiv \bigvee_{i \equiv 0}^{k} (u_i \to u_{i+1})$$

Let $R_k \equiv R'_k[u_0|\underline{1}][u_{k+1}|\underline{0}]$, then $R'_k \dashv\vdash R_k$.

For example $R'_1 \equiv (u_0 \to u_1) \vee (u_1 \to u_2)$, $R_1 \equiv u_1 \vee \neg u_1$

$R'_2 \equiv (u_0 \to u_1) \vee (u_1 \to u_2) \vee (u_2 \to u_3)$

$R_2 \equiv u_1 \vee (u_1 \to u_2) \vee \neg u_2$.

Then $L_k \equiv \{R_k\}^m$ for $1 \leq k < \infty$.

1.1.3.1.1. Proposition. $R_k \in \mathcal{C}_{\dashv\vdash}$.

Proof. It is clear that $R'_k \in \mathcal{C}$; since $R'_k \dashv\vdash R_k$, $R_k \in \mathcal{C}_{\dashv\vdash}$. □

1.1.3.2. Slicing of i.p.l.'s.

In [Ho 3] Hosoi introduces a classification of i.p.l.'s based on the fact that the extensions of L_∞ form a chain. For any intermediate logic Σ, $(\Sigma \cup \{Z\})^m$ is an extension of L_∞, thus there is a uniquely determined n $(1 \leq n \leq \infty)$ such that $(\Sigma \cup \{Z\})^m \equiv L_n$.

Define $\mathit{Slice}_n \equiv \{\Sigma |\ \Sigma$ is an i.p.l. and $(\Sigma \cup \{Z\})^m \equiv L_n\}$, ([Ho 3] Definition 2.4.). It is immediate that $\bigcup_{1 \leq n \leq \infty} \mathit{Slice}_n$ is the set of all i.p.l.'s, that $i \neq j$ implies $\mathit{Slice}_i \cap \mathit{Slice}_j \equiv \phi$, and that $L_i \in \mathit{Slice}_i$. Furthermore

$\mathcal{Slice}_1 \equiv \{KL\}$ and $\mathcal{Slice}_0 \equiv \{Poly(\mathbb{H})\}$.

Clearly, within $\mathcal{Slice}_i$, L_i is the maximal i.p.l.; Hosoi establishes that there is a minimal i.p.l. within $\mathcal{Slice}_i$. Let $u \in V$, $P_0 \equiv u$, $P_n \equiv \Delta^n(u)$ for $n \geq 1$; by 1.1.2.5, $P_n \dashv\vdash \Delta^n(\underline{0})$. In Corollary 4.7 ([Ho 3] p. 304) it is shown that $\{P_n\}^m$ is the minimal i.p.l. in $\mathcal{Slice}_n$ $(1 \leq n < \infty)$.

1.1.3.3. The second slice.

In [HO] Hosoi and Ono describe $\mathcal{Slice}_2$. Let $\mathbf{2} \equiv S_1$, let $\mathbf{2}^n$ be the 2^n element Boolean algebra, then $\mathbf{2}^n \oplus 1$ is in $\mathcal{SI} \cap \mathcal{Heyt}$. Define $M_n \equiv Val\{\mathbf{2}^n \oplus 1\}$ for $0 \leq n \leq \infty$. Then $M_0 \equiv L_1 \equiv KL$ and $M_1 \equiv L_2$. In Corollary 1.3 of [H O], they prove that M_{n+1} is a proper subset of M_n and, in Theorem 1.6 of [H O], that

$$\mathcal{Slice}_2 \equiv \{M_n \mid 1 \leq n < \infty\} \cup \{\{P_2\}^m\}.$$

One can see also that $M_\infty \equiv \{P_2\}^m$ where $M_\infty \equiv Val\{\mathbf{2}^N \oplus 1\}$, (infact $\mathbf{2}^N$ could be replaced by any infinite Boolean algebra).

In [HO] polynomials φ_n are introduced for which $M_n \equiv \{P_2, \varphi_n\}^m$. There is another, more familiar sequence ([BD] p. 162) which works, namely the polynomials E_n first introduced by K. B. Lee in the context of pseudocomplemented distributive lattices. Let $v_i \in V$ $(i \geq 1)$ be distinct. Define $E_n \equiv \neg\left(\bigwedge_{i \leq n} v_i\right) \vee \bigvee_{j \leq n} \neg\left(\left(\bigwedge_{i \neq j, i \leq n} v_i\right) \wedge \neg v_j\right)$

then $\mathbf{2}^n \oplus 1 \vdash E_n$ and $\mathbf{2}^{n+1} \oplus 1 \not\vdash E_n$ (Theorem 6.2 p. 162 [DL]). It follows that $\{P_2, E_n\}^m \equiv \{\varphi \mid \mathbf{2}^n \oplus 1 \vdash \varphi\} \equiv M_n$.

1.1.3.4 Polynomials that generate the same i.p.l. as E_n.

We shall give two other ways of generating the same i.p.l. as E_n. We first introduce $B_n \in \mathrm{Poly}\,\mathbb{H}$ for which $E_n \dashv\vdash B_n$ and then intrduce $C_n \in \mathrm{Poly}\,\mathbb{H}$ for which we have the stronger equivalence $\vdash B_n \leftrightarrow C_n$. We shall show that $C_n \in \mathcal{C}^{\rightarrow}$ and hence that $E_n \in \mathcal{C}^{\rightarrow}_{\dashv\vdash}$. The first member of the sequence E_n is $E_1 \equiv \neg v_1 \vee \neg\neg v_1$; for the other two sequences the first members will be

$B_1 \equiv \neg(v_1 \wedge v_1) \rightarrow (\neg v_1 \vee \neg v_2)$ and $C_1 \equiv (v_1 \rightarrow \neg v_2) \rightarrow (\neg v_1 \vee \neg v_2)$.

1.1.3.4.1 Definition of B_n $(n \geq 1)$.

$$B_n \equiv \neg\left(\bigwedge_{i \leq n+1} v_i\right) \rightarrow \bigvee_{j \leq n+1} \neg\left(\bigwedge_{i \neq j, i \leq n+1} v_i\right), \quad n \geq 1$$

where the variables v_i $(i \geq 1)$ are the same as for E_n, and $j \geq 1$.

1.1.3.4.2 Proposition. $E_n \dashv\vdash B_n$ for each $n \geq 1$.

Proof. We will show that for $\mathcal{S}i$ algebras

(I) $\quad \mathcal{A} \oplus 1 \vdash E_n$ iff $\mathcal{A} \oplus 1 \vdash B_n$.

($\rightarrow$) Assume $\mathcal{A} \oplus 1 \vdash E_n$. Let $\alpha: V \longrightarrow |\mathcal{A} \oplus 1|$; we want to show that $\tilde{\alpha}(B_n) \equiv 1$ in $\mathcal{A} \oplus 1$. Let $a_i \equiv \alpha(v_i)$ for $1 \leq i \leq n+1$.

We have $\tilde{\alpha}(E_n) \equiv 1$, hence

$$\bigwedge_{i \leq n} \neg\neg a_i \equiv 0, \text{ or, for some } j \leq n, \left(\bigwedge_{i \neq j, i \leq n} \neg\neg a_i\right) \leq \neg\neg a_j$$

hence for some $j \leq n+1$,

$$\bigwedge_{i \neq j, i \leq n+1} \neg\neg a_i \leq \neg\neg a_j$$

hence $\bigwedge_{i \neq j, i \leq n+1} \neg\neg a_i \equiv \bigwedge_{i \leq n+1} \neg\neg a_i$,

hence $\neg\left(\bigwedge_{i \leq n+1} a_i\right) \equiv \neg\left(\bigwedge_{i \leq n+1} \neg\neg a_i\right) \equiv \neg\left(\bigwedge_{i \neq j, i \leq n+1} a_i\right)$

hence $\tilde{\alpha}(B_n) \equiv 1$.

($\leftarrow$) Assume $\mathcal{A} \oplus 1 \vdash B_n$. Let $\alpha: V \longrightarrow |\mathcal{A} \oplus 1|$; we want to show that

$\tilde{\alpha}(E_n) \equiv 1$ in $\mathcal{A}$ 1. Let $a_i \equiv \alpha(v_i)$ for $1 \le i \le n$. We have $\mathcal{A}$ 1

$1 \vdash B_n[v_{n+1} | \neg (\bigwedge_{i \le n} v_i)]$, hence $\tilde{\alpha}(B_n[v_{n+1} | \neg(\bigwedge_{i \le n} v_i)]) \equiv 1$.

Hence

$$\neg \left[\left(\bigwedge_{i \le n} a_i\right) \wedge \neg\left(\bigwedge_{i \le n} a_i\right)\right] \le \bigvee_{j \le n} \neg\left[\left(\bigwedge_{i \ne j, i \le n} a_i\right) \wedge \neg\left(\bigwedge_{i \le n} a_i\right)\right] \vee \neg\left(\bigwedge_{i \le n} a_i\right)$$

hence either $\bigwedge_{i \le n} a_i \equiv 0$ or for some $j \le n$,

$$\left(\bigwedge_{i \ne j, i \le n} a_i\right) \wedge \neg\left(\bigwedge_{i \le n} a_i\right) = 0.$$

We have $\bigwedge_{i \le n} a_i \le a_j$ for $j \le n$,

hence $\neg a_j \le \neg \left(\bigwedge_{i \le n} a_i\right)$

hence $\left(\bigwedge_{i \ne j, i \le n} a_i\right) \wedge \neg a_j \quad \left(\bigwedge_{i \ne j, i \le n} a_i\right) \wedge \neg\left(\bigwedge_{i \le n} a_i\right) \le 0.$

Thus either $\neg\left(\bigwedge_{i \le n} a_i\right) \equiv 1$ or for some $j \le n$,

$$\neg \left[\left(\bigwedge_{i \ne j, i \le n} a_i\right) \wedge \neg a_j\right] \equiv 1.$$

Hence $\tilde{\alpha}(E_n) \equiv 1$. □

<u>1.1.3.4.3 Definition.</u> For all $n \in \mathbb{N}$ and for all sequences $t_0, t_1, \ldots, t_p$ from Poly $\mathbb{H}$ with $n \le p$ we define $\left(\overset{n}{\underset{i \equiv 0}{\Longrightarrow}} t_i\right) \in$ Poly $\mathbb{H}$ by induction on n:

$$\left(\overset{0}{\underset{i \equiv 0}{\Longrightarrow}} t_i\right) \equiv t_0$$

$$\left(\overset{k+1}{\underset{i \equiv 0}{\Longrightarrow}} t_i\right) \equiv \left(t_{k+1} \Rightarrow \left(\overset{k}{\underset{i \equiv 0}{\Longrightarrow}} t_i\right)\right)$$

For example $\left(\overset{3}{\underset{i \equiv 0}{\Longrightarrow}} t_i\right) \equiv t_3 \Rightarrow (t_2 \Rightarrow (t_1 \Rightarrow t_0))$.

We define a function $\overset{n}{\underset{i \equiv 1}{\Rrightarrow}} t_i$: Poly $\mathbb{H} \longrightarrow$ Poly $\mathbb{H}$ by

$$\left(\overset{n}{\underset{i \equiv 1}{\Rrightarrow}} t_i\right)(t_0) \equiv \left(\overset{n}{\underset{i \equiv 0}{\Longrightarrow}} t_i\right)$$

<u>1.1.3.4.4 Proposition.</u> $\models \left(\overset{n}{\underset{i \equiv 1}{\Longrightarrow}} t_i\right)(t) = \left(\overset{n}{\underset{i \equiv 1}{\bigwedge}} t_i\right) \Rightarrow t.$

<u>Proof.</u> By induction $\models \left(\overset{1}{\underset{i \equiv 1}{\Longrightarrow}} t_i\right)(t) = (t_1 \Rightarrow t)$

$$= \left(\left(\overset{1}{\underset{i \equiv 1}{\bigwedge}} t_i\right) \Rightarrow t\right)$$

$$\models \left(\overset{k+1}{\underset{i \equiv 1}{\Longrightarrow}} t_i\right)(t) = \left(t_{k+1} \Rightarrow \left(\left(\overset{n}{\underset{i \equiv 1}{\Longrightarrow}} t_i\right) \Rightarrow (t)\right)\right)$$

$$= \left(t_{k+1} \Rightarrow \left(\left(\overset{k}{\underset{i \equiv 1}{\bigwedge}} t_i\right) \Rightarrow t\right)\right)$$

$$\left(\overset{k+1}{\underset{i \equiv 1}{\bigwedge}} t_i\right) \Rightarrow t).$$

<u>1.1.3.4.5 Corollary.</u> $\models \left(\overset{n}{\underset{i \equiv 1}{\Longrightarrow}} t_i\right)(0) = \neg\left(\overset{n}{\underset{i \equiv 1}{\bigwedge}} t_i\right).$

<u>1.1.3.4.6</u> For $1 \leq j \leq n$, $1 < n$, it is straightforward to define $\overset{n}{\underset{\substack{i \equiv 1 \\ i \not\equiv j}}{\Longrightarrow}} t_i$:

define a new list of polynomials

$$s_1, s_2, \ldots, s_{n-1} \qquad \text{by} \qquad s_k \equiv \begin{cases} t_k & \text{if } k+1 \leq j \\ t_{k+1} & \text{if } k \geq j \end{cases}$$

then $\left(\overset{n}{\underset{i \equiv 1, i \not\equiv j}{\Longrightarrow}} t_i\right) \equiv \left(\overset{n-1}{\underset{i \equiv 1}{\Longrightarrow}} s_i\right)$; by 1.1.3.4.4

$$\models \left(\overset{n}{\underset{i \equiv 1, i \not\equiv j}{\Longrightarrow}} t_i\right)(t) = \left(\overset{n}{\underset{i \equiv 1, i \not\equiv j}{\bigwedge}} t_i\right) \Rightarrow t.$$

<u>1.1.3.4.7 Definition of C_n $(n \geq 1)$.</u>

$$C_n \equiv \left(\overset{n+1}{\underset{i \equiv 1}{\Longrightarrow}} v_i\right)(\underline{0}) \Rightarrow \overset{n+1}{\underset{i \equiv 1}{\bigvee}} \left(\overset{n+1}{\underset{j \equiv 1, j \not\equiv i}{\Longrightarrow}} v_j\right)(\underline{0}), \quad n \geq 1.$$

<u>1.1.3.4.8 Proposition.</u> $B_n \dashv\vdash C_n$.

<u>Proof.</u> By 1.1.3.4.5.

The significance of the new form we have found for B_n is:

<u>1.1.3.4.9 Proposition.</u> $C_n \varepsilon \mathcal{C}^{\to}$.

<u>Proof.</u> We show that $\left(\overset{n+1}{\underset{i\,\equiv\,1}{\Longrightarrow}} v_i\right)(\underline{0}) \varepsilon \mathcal{C}$ for $1 \leq i \leq n+1$, by induction on n.

$n \equiv 1$: $\underline{0} \varepsilon \mathcal{C}$ hence $\left(\overset{1}{\underset{i\,\equiv\,1}{\Longrightarrow}} v_i\right)(\underline{0}) \varepsilon \mathcal{C}$ since

$\left(\overset{1}{\underset{i\,\equiv\,1}{\Longrightarrow}} v_i\right)(\underline{0}) \equiv (v \to \underline{0})$.

Suppose $\left(\overset{k}{\underset{i\,\equiv\,1}{\Longrightarrow}} v_i\right)(\underline{0}) \varepsilon \mathcal{C}$ then

$$\left(\overset{k+1}{\underset{i\,\equiv\,1}{\Longrightarrow}} v_i\right)(\underline{0}) \equiv \left(v_{k+1} \to \left(\overset{k}{\underset{i\,\equiv\,1}{\Longrightarrow}} v_i\right)(\underline{0})\right) \varepsilon \mathcal{C} .$$

We have $\left(\overset{n}{\underset{j\,\equiv\,1,j\,\not\equiv\,i}{\Longrightarrow}} v_j\right)(\underline{0}) \varepsilon \mathcal{C}$ for each i, $1 \leq i \leq n$, hence by

$\bigvee_{i\,\equiv\,1}^{n} \left(\overset{n}{\underset{j\,\equiv\,1,j\,\not\equiv\,i}{\Longrightarrow}} v_j\right)(\underline{0}) \varepsilon \mathcal{C}$. Hence $C_n \varepsilon \mathcal{C}^{\to}$.□

<u>1.1.3.4.10 Corollary.</u> $E_n \varepsilon \mathcal{C}^{\to}_{\dashv\vdash}$ and $B_n \varepsilon \mathcal{C}^{\to}_{\dashv\vdash}$.□

1.1.4 I.p.l.'s defined by a class of algebras which do not have a particular finite $\mathcal{S}\iota$-algebra as a subalgebra.

We give several results from a paper of Jankov's [Ja 3]. No proofs appear in the paper; one of the main results (1.1.4.3) is proven in [Ba] using Jonson's work on congruence distributive equational classes.

1.1.4.1 Jankov's characteristic formula. Let $\alpha \equiv \mathcal{L} \oplus 1$ be a finite algebra with penultimate element e. Let $\chi: |\alpha| \rightarrowtail V$ be an injection of the algebra into the set of variables for the elphabet $\mathbb{H}$. For each operation sign g of $\mathbb{H}$ we let $\bar{g}$ be its interpretation as an operation of α. For f nullary ($\underline{1}$ or $\underline{0}$) we put $F(f) \equiv \chi(\bar{f}) \leftrightarrow f$.

For g binary ($\underline{\wedge}$, $\underline{\vee}$ or $\underline{\Rightarrow}$) we put

$$F(g) \equiv \bigwedge_{(a,b) \in |\alpha|^2} (\chi(\bar{g}(a,b)) \leftrightarrow g(\chi(a),\chi(b))),$$

where $\bigwedge$ must involve a particular enumeration of $|\alpha|^2$ which we gloss over notationally.

We put $F_0 \equiv \chi(e)$

$$F_1 \equiv F(\underline{1}) \wedge F(\underline{0}) \wedge F(\underline{\Rightarrow}) \wedge F(\underline{\wedge}) \wedge F(\underline{\vee})$$

$$X_\alpha \equiv F_1 \Rightarrow F_0 .$$

The validity of X_α in any algebra is clearly independent of χ and of any enumeration of $|\alpha|^2$. We shall abuse language and call X_α the characteristic formula of α.

With the above conventions we have:

1.1.4.2 Proposition. $\alpha \not\models X_\alpha$.

Proof. Let $r: V \twoheadrightarrow |\alpha|$ be a retraction of χ so that $r(\chi(a)) \equiv a$ for each $a \in |\alpha|$. Let $\tilde{r}$: Poly(H) $\longrightarrow \alpha$ be its extension to an H-homomorphism. We shall show $\tilde{r}(X_\alpha) \not\equiv 1$.

For f nullary we have:

$\tilde{r}(\chi(\overline{f}) \leftrightarrow f) \equiv (\tilde{r}\ \chi(\overline{f}) \leftrightarrow \tilde{r}(f)) \equiv (\overline{f} \leftrightarrow \overline{f}) \equiv 1.$

For g binary we have:

$$\begin{aligned}\tilde{r}(\chi(\overline{g}(a,b)) \leftrightarrow g/\chi(a),\chi(b)/) &\equiv \tilde{r}(\chi(\overline{g}(a,b))) \leftrightarrow \tilde{r}(g/\chi(a),\chi(b)/)\\ &\equiv \overline{g}(a,b) \leftrightarrow \overline{g}(\tilde{r}(\chi(a),\tilde{r}\ \chi(b))\\ &\equiv \overline{g}(a,b) \leftrightarrow \overline{g}(a,b)\\ &\equiv 1\end{aligned}$$

hence $\tilde{r}(F(g)) \equiv \bigwedge_{(a,b)\ \varepsilon\ |\alpha|^2} \tilde{r}(\ (\overline{g}(a,b)) \leftrightarrow g/\chi(a),\chi(b)/) \equiv 1.$

Hence $F_1 \equiv 1$, $\tilde{r}(F_0) \equiv \tilde{r}(\chi(e)) \equiv e$, hence $\tilde{r}(X_\alpha) \equiv (1 \leftrightarrow e) \equiv e.$□

<u>1.1.4.2.1 Corollary.</u> If $\varphi \vdash X_\alpha$ then $\alpha \not\vdash \varphi$.

<u>Proof.</u> Suppose to the contrary that $\alpha \vdash \varphi$, then $\alpha \vdash X_\alpha$ -contradicting 1.1.4.2.□

The next Proposition will lead us to the converse of 1.1.4.2.

<u>1.1.4.3 Proposition.</u> If A is an algebra and $A \not\vdash X_\alpha$ then $\alpha\ \varepsilon\ SH\{A\}$.

<u>Proof.</u> Since $A \not\vdash X_\alpha$ there is a valuation $\beta\colon V \longrightarrow |A|$ such that $\beta(X_\alpha) \not\equiv 1$ in A, where $\tilde{\beta}\colon \text{Poly}\,\mathbb{H} \longrightarrow A$ is the extension of β. Let $c \equiv \tilde{\beta}(F_1)$ and $d \equiv \tilde{\beta}(F_0)$, then $c \not\leq d$ in A. Let $\theta\colon A \twoheadrightarrow A_c$ be the canonical homomorphism onto A_c, the quotient via the principal filter $[c,\rightarrow)$; then, for all a, b in A, $\theta(a) \equiv \theta(b)$ iff $c \leq (a \leftrightarrow b)$; $\theta(c) \equiv 1$ in A_c and $\theta(d) \not\equiv 1$ in A_c. Define $k\colon |\alpha| \longrightarrow |A_c|$ by $k(a) \equiv \theta(\beta(\chi(a)))$ for all $a\ \varepsilon\ |\alpha|$. We shall show that $k\colon \alpha \longrightarrow A_c$ is a homomorphism.

$1 \equiv \theta(c) \equiv \theta(\tilde{\beta}(F_1)) \equiv \bigwedge_g \theta(\tilde{\beta}(F(g)))$ where g ranges of the operation signs of $\mathbb{H}$, hence for each g, $\theta(\tilde{\beta}(F(g))) \equiv 1.$

Let $\bar{g}$ be the interpretation of g in A_c. For each nullary operation sign we have $1 \equiv \theta\tilde{\beta}(F(f)) \equiv (\theta(\tilde{\beta}(\chi(\bar{f}))) \leftrightarrow \theta(\tilde{\beta}(f))) \equiv (k(\bar{f}) \leftrightarrow \bar{\bar{f}}$, hence $k(\bar{f}) \equiv \bar{\bar{f}}$. For each binary operation sign we have

$$1 \equiv \theta(\tilde{\beta}(F(g))) \equiv \bigwedge_{(a,b) \in |\alpha|^2} (\theta(\tilde{\beta}((\bar{g}(a,b))) \leftrightarrow \theta(\tilde{\beta}(g/X(a),X(b)/)))$$

hence
$$k(\bar{g}(a,b)) \equiv \theta(\tilde{\beta}(X(\bar{g}(a,b))) \equiv \theta(\tilde{\beta}(g/X(a),X(b)/))$$
$$\equiv \bar{\bar{g}}/\theta(\tilde{\beta}(X(a))),\theta(\tilde{\beta}(X(b)))/ \equiv \bar{\bar{g}}/k(a),k(b)/.$$

Thus $k: \alpha \longrightarrow A_c$ is a homomorphism. We shall show that k is a mono. $\theta(d) \equiv \theta(\tilde{\beta}(F_0)) \equiv \theta(\tilde{\beta}(X(e))) \equiv k(e)$, hence $k(e) \not\equiv 1$ in A_c. If k were not a mono, $k^{-1}\{1\}$ would be a non-trivial filter in A and hence would contain the penultimate element e; but $e \notin k^{-1}\{1\}$, hence k is a mono. Thus $\alpha \in SH\{A\}$, since $SH \equiv ISH$.□

<u>1.1.4.3.1 Corollary.</u> For $\alpha \equiv \mathcal{L} \oplus 1$ finite, $\mathcal{L} \vdash X_\alpha$.

<u>Proof.</u> Suppose $\mathcal{L} \vdash X_\alpha$; by 1.1.4.3, $\mathcal{L} \oplus 1 \in SH\{\mathcal{L}\}$ which impliesthat the cardinality of $\mathcal{L} \oplus 1$ is bounded above by the cardinality of $\mathcal{L}$ -a contradiction.□

<u>1.1.4.4. Proposition.</u> X_α is characterized syntactically by

(1) $\varphi \vdash X_\alpha$ iff $\alpha \not\vdash \varphi$

and in terms of its models by:

(2) $A \vdash X_\alpha$ iff $\alpha \notin SH\{A\}$; as a consequence:

(3) $Mod\{X_\alpha\}$ is the largest subvariety of *Heyt* of which α is not a member.

<u>Proof.</u> (1) (→): By 1.1.4.2.1.□ (←) Assume $\varphi \not\vdash X_\alpha$ and $\alpha \not\vdash \varphi$; then there is an algebra A such that $A \vdash \varphi$ but $A \not\vdash X_\alpha$. By 1.1.4.3, $\alpha \in SH\{A\}$, hence $\alpha \vdash \varphi$ -a contradiction.□

(2) (→): Suppose $\alpha \in SH\{A\}$ and $A \vdash X_\alpha$; then $\alpha \vdash X_\alpha$ -a contradiction, by 1.1.4.2.□ (←): By 1.1.4.3.□

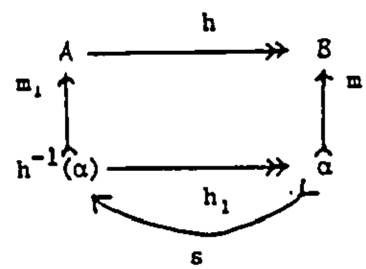
(3) By 1.1.4.2, $\alpha \not\models X_\alpha$, hence $\alpha \notin \mathrm{Mod}\{X_\alpha\}$. Suppose $\mathbb{K}$ is a subvariety of $\mathcal{H}eyt$ and $\alpha \notin \mathbb{K}$. To show $\mathbb{K} \subset \mathrm{Mod}\{X_\alpha\}$ let $A \in \mathbb{K}$, then $SH\{A\} \subset \mathbb{K}$, hence $\alpha \notin SH\{A\}$, hence $A \in \mathrm{Mod}\{X_\alpha\}$, by (2).□

Let $\mathcal{H}eyt$ be the category of Heyting algebras; α is throughout this section a finite $\mathcal{SI}$ algebra.

<u>1.1.4.5 Proposition.</u> If α is projective in $\mathcal{H}eyt$ then $\mathrm{Mod}\{X_\alpha\} \equiv \{A \mid \alpha \notin IS\{A\}\}$.

<u>Proof.</u> One inclusion holds for any α. We have $IS\{A\} \subset SH\{A\}$, hence $\alpha \notin SH\{A\}$ implies $\alpha \notin IS\{A\}$, hence $\mathrm{Mod}\{X_\alpha\} \subset \{A \mid \alpha \notin IS\{A\}\}$.

To show the converse inclusion we puppose $\alpha \in SH\{A\}$ and show $\alpha \in IS\{A\}$. We have



an onto homomorphism h and an inclusion m; $h^{-1}(\alpha)$ is the inverse image under h of the subalgebra α and m_1 is the inclusion of $h^{-1}(\alpha)$ in A; since $h(h^{-1}(\alpha)) \equiv \alpha$, we have an onto homomorphism h_1 making the above diagram commute. Since α is projective, there exists a section $s\colon \alpha \rightarrowtail h^{-1}(\alpha)$ of h_1; $m_1 \circ s\colon \alpha \rightarrowtail A$ is a mono, hence $\alpha \in IS\{A\}$. □

<u>1.1.4.6 Projectives in $\mathcal{H}eyt$</u>

We first explain a minor point in terminology. We could have used the term "weakly projective" in 1.1.4.5. in place of "projective"; it is the former term which is used in [BD] to describe the algebras we are interested in. In fact the two concepts are the same. Projectives are defined in terms of epis and monos, whereas weak projectives are defined in terms of onto maps and one-to-one maps. The only reason that the concepts might be different is if epis were not onto; but in $\mathcal{H}eyt$ they are onto. This follows from the amalgamation property which holds in $\mathcal{H}eyt$ ([Ma 3]).

In order to define the finite projective algebras we introduce the ordered sum $\mathcal{L}_1 \dagger \mathcal{L}_2$ of the algebras $\mathcal{L}_1$ and $\mathcal{L}_2$ ([Ma 3] and [BD]).
We take $\mathcal{L}_2'$ to be isomorphic to $\mathcal{L}_2$ with bottom element 0_2 and such that

$$|\mathcal{L}_1| \cap |\mathcal{L}_2| \equiv \{1_1\} \equiv \{o_2\}$$

where 1_1 is the top element of $\mathcal{L}_1$. Let $\leq_1$ and $\leq_2$ be the respective orderings of $\mathcal{L}_1$ and $\mathcal{L}_2'$.
We take $\mathcal{L}_1 \dagger \mathcal{L}_2$ to have carrier $|\mathcal{L}_1| \cup |\mathcal{L}_2'|$ and ordering $\leq$ given by: $x \leq y$ iff either $x \leq_1 y$ or $x \leq_2 y$ or both $x \in \mathcal{L}_1$ and $y \in \mathcal{L}_2$.
Balbes and Horn characterized the finite projectives as follows:

<u>1.1.4.6.1 Proposition.</u> ([BD] p. 190) $\mathcal{L}$ is a finite projective iff there is a finite (possibly empty) sequence $\mathcal{L}_1, \ldots, \mathcal{L}_n$ from $\{\mathbf{2},\mathbf{2}^2\}$ such that $\mathcal{L} \approx (\mathcal{L}_1 \dagger \cdots \dagger \mathcal{L}_n) \dagger \mathbf{2}$.□

In case the above sequence is empty we interpret the statement as meaning $\mathcal{L} \approx \mathbf{2}$, if it is nonempty we have $\mathcal{L} \approx (\mathcal{L}_1 \dagger \cdots \dagger \mathcal{L}_n) \oplus 1$. Thus $\mathcal{L} \in \mathcal{SI}$.
We can now restate 1.1.4.5 as:

<u>1.1.4.6.2 Corollary.</u> If $\mathcal{L}_1, \ldots, \mathcal{L}_n$ is a sequence of algebras from $\{\mathbf{2},\mathbf{2}^2\}$ then

$\mathrm{Mod}\{X_{\mathcal{L}_1 \dagger \cdots \dagger \mathcal{L}_n \oplus 1}\} \equiv \{A \mid (\mathcal{L}_1 \dagger \cdots \dagger \mathcal{L}_n \oplus 1) \notin \mathrm{IS}(A)\}$.□

We shall be interested in particular in the three finite projective algebras:

$$\mathbf{3} \approx \mathbf{2} \oplus 1 \;,\; \mathbf{4} \approx (\mathbf{2} \dagger \mathbf{2}) \oplus 1 \;, \quad \text{and} \quad \mathbf{2}^2 \oplus \mathbf{2} \approx \mathbf{2}^2 \dagger \mathbf{2} \dagger \mathbf{2}$$

<u>1.1.4.6.3 Proposition.</u> $X_3 \dashv\vdash v \vee \neg v$.

<u>Proof.</u> $\mathcal{SL} \cap \mathrm{Mod}\{X_3\} \equiv \{\mathcal{L} \oplus 1 \mid \mathbf{3} \notin \mathrm{IS}\{\mathcal{L} \oplus 1\}\}$. Let e be the penultimate element of $\mathcal{L} \oplus 1$, then $\{0,e,1\}$ is a subalgebra of $\mathcal{L} \oplus 1$; if this is not isomorphic to $\mathbf{3}$ then $0 \equiv e$, hence

$$\mathcal{SL} \cap \mathrm{Mod}\{X_3\} \equiv \mathrm{I}\{\mathbf{2}\}$$

Thus $\mathrm{Mod}\{X_3\} \equiv \mathcal{Bool}$, the variety of Boolean algebras, hence $X_3 \dashv\vdash v \vee \neg v$.□

We shall show that X_n can be similarly simplified where $\mathbf{n} \equiv S_{n-1}$ $(n \geq 4)$.

<u>1.1.4.6.4 Proposition.</u> Let e be the penultimate element of the algebra $\mathcal{L} \oplus 1$. For each $y < e$, $(e \Rightarrow y) \equiv y$.

<u>Proof.</u> Since $y < e$, $(e \Rightarrow y) \not\equiv 1$, hence $(e \Rightarrow y) \leq e$, hence $(e \Rightarrow y) \equiv (e \wedge (e \Rightarrow y)) \leq y$; but $y \leq (e \Rightarrow y)$, hence $(e \Rightarrow y) \equiv y$.□

We shall need the following consequence of 1.1.4.6.4 in 1.2.4.1.

<u>1.1.4.6.5 Proposition.</u> Let $A \equiv |\mathcal{A}|$ and let $\mathcal{A}$ be a subalgebra of $\mathcal{L} \oplus 1$, then $A \cup \{e\}$ forms a subalgebra of $\mathcal{L} \oplus 1$.

<u>Proof.</u> If $e \in A$ we have nothing to prove, so we suppose $e \notin A$. The sets A and $\{0,e,1\}$ form subalgebras. To see that $A \cup \{e\}$ is closed under $\wedge$, $\vee$ and $\Rightarrow$, it suffices to take $x \in (A - \{0,e,1\})$ and $e \in (\{0,e,1\} - A)$ and apply the operations to these elements. Since $x < e$, we have

$$\{x \vee e,\; x \wedge e,\; x \Rightarrow e,\; e \Rightarrow x\} \equiv \{e,x,1,x\} \subset (A \cup \{e\}).□$$

<u>1.1.4.6.6 Proposition.</u> $S_n \in IS\{A\}$ iff $S_{n+1} \in IS\{A \oplus 1\}$

<u>Proof.</u> Let $\overline{S}_n$ and $\overline{S}_{n+1}$ be copies of S_n and S_{n+1} respectively which are subalgebras of A and $A \oplus 1$ respectively. Let e and 1 be the top elements of A and $A \oplus 1$ respectively. Let $\theta: A \oplus 1 \twoheadrightarrow A$ be the canonical homomorphism.

(→) $\theta^{-1}(\overline{S}_n) \equiv \overline{S}_{n+1}$.□

(←) If $e \in \overline{S}_{n+1}$ then $\theta(\overline{S}_{n+1}) \equiv \overline{S}_n$. If $e \notin \overline{S}_{n+1}$ then $\theta(\overline{S}_{n+1}) \equiv \overline{S}_{n+1}$; but $S_n \in IS\{\overline{S}_{n+1}\}$, hence $S_n \in IS\{A\}$. □

<u>1.1.4.6.7 Proposition.</u> $X_{n\oplus 2} \dashv\vdash \Delta^n(\underline{0})$.

<u>Proof.</u> We proceed by induction. By 1.1.4.6.3 $X_3 \dashv\vdash \Delta(\underline{0})$. Suppose $X_{n\oplus 1} \dashv\vdash \Delta^{n-1}(\underline{0})$. We have

$A \oplus 1 \vdash X_{n\oplus 2}$	iff $S_{n+1} \notin IS\{A \oplus 1\}$	by 1.1.4.6.2
	iff $S_n \notin IS\{A\}$	by 1.1.4.6.6
	iff $A \vdash X_{n\oplus 1}$	by 1.1.4.6.2
	iff $A \vdash \Delta^{n-1}(\underline{0})$	by induction
	iff $A \oplus 1 \vdash \Delta^n(\underline{0})$	by 1.1.2.5

hence $\mathcal{S}\!\iota \cap Mod\{X_{n\oplus 2}\} \equiv \mathcal{S}\!\iota \cap Mod\{\Delta^n(\underline{0})\}$.□

We use the next proposition to compare the deductive strengths of X_4 and $X_{2^2 \oplus 2}$.

<u>1.1.4.7 Proposition.</u> If α and α' are finite $\mathcal{S}\!\iota$ algebras and α is a proper subalgebra of α' ($\alpha \rightarrowtail \alpha'$) then

(1) $X_\alpha \vdash X_{\alpha'}$

(2) $X_{\alpha'} \nvdash X_\alpha$

<u>Proof.</u> (1) By 1.1.4.2, $\alpha \nvdash X_\alpha$. Since $\alpha \rightarrowtail \alpha'$, $\alpha' \vdash \varphi$ implies $\alpha \vdash \varphi$; hence $\alpha' \nvdash X_\alpha$. By 1.1.4.4 (1), $X_\alpha \vdash X_{\alpha'}$.□

(2) Since α' is finite and α has fewer elements than α', $\alpha' \notin SH\{\alpha\}$.

By 1.1.4.4 (2), $\alpha \vdash X_{\alpha'}$, so by 1.1.4.4 (1), $X_{\alpha'} \nvdash X_{\alpha}$. □

<u>1.1.4.7.1</u> Since $\mathbf{4}$ is a proper subalgebra of $\mathbf{2}^2 \oplus \mathbf{2}$, we have, by 1.1.4.6.6. (2),

(1) $u \vee (u \Rightarrow (v \vee \neg v)) \vdash X_{\mathbf{2}^2 \oplus \mathbf{2}}$

(2) $X_{\mathbf{2}^2 \oplus \mathbf{2}} \nvdash u \vee (u \Rightarrow (v \vee \neg v))$.

Section 1.2 Presheaf Toposes over $\underline{\mathcal{S}}$ and their Propositional and Converse Logics

In 1.2.1 to 1.2.5 we study presheaf toposes $\underline{\mathcal{S}}^{\underline{C}^0}$ for $\underline{C}$ a small category. In summary these subsections are as follows:

1.2.1 A discussion of how a structure defined by finite limits in a presheaf topos $\underline{\mathcal{S}}^{\underline{C}^0}$ can be viewed as a functor from $\underline{C}$ into the category of external structures of the same kind.

1.2.2 The development of concepts using in describing "logical features" of presheaf toposes $\underline{\mathcal{S}}^{\underline{C}^0}$; in partucular the algebraic structure on Ω and on that of the (external) set of "ideals" of $\underline{C}$.

1.2.3 The presentation of examples of presheaf toposes $\underline{\mathcal{S}}^{\underline{C}^0}$, showing their propositional logic and making further logical distinctions between presheaf toposes having the same propositional logic.

1.2.4 A presentation of a construction which allows us to view the internal presheaf topos $(\underline{\mathcal{S}}^{\underline{\Lambda}^0})^{\underline{C}^0}$ directly as an external presheaf topos $\underline{\mathcal{S}}^{(\underline{\Lambda} \propto \underline{C})^0}$.

1.2.5 The presentation of converse logics and the calculation of various converse logics associated with various presheaf toposes.

The last subsection, 1.2.6, explains a break in style of discourse.

<u>1.2.1 Preliminary remarks.</u> In future sections (i.e. after 1.2) we work with internal categories in a topos and with the topos of presheaves defined over the internal categories. To say that, as an application, the statements we prove there hold for the topos $\underline{\mathcal{S}}$ and for presheaf toposes valued in $\underline{\mathcal{S}}$ would run counter to the dictum of Lawvere ([K W] p.3): "an important technique is to lift constructions first understood for 'the' category $\underline{\mathcal{S}}$ of abstract sets to an arbitrary topos." On the other hand it seems to us redundant to always exhibit proofs first for $\underline{\mathcal{S}}$ of the very statements which we will prove using the set theoretical language developed in chapter 0. We are therefor going to adopt the policy, within 1.2, of indicating where, in future sections, general facts about toposes, which we use herein, will be proven, and only proving those statements whose proofs are either not to be found in future sections or are relatively simple in $\underline{\mathcal{S}}$.

Although we cannot always locate sources many of the features of presheaves over $\underline{\mathcal{S}}$ are by now standard; in particular, that "structures defined by finite limits" may be viewed as a presheaf into the set based category of such structures or such a structure in presheaves over $\underline{\mathcal{S}}$. What needs explication is the quoted phrase. This is done in various ways: using sketches or small categories to "present" the finite limit theory. In particular Gabriel and Ulmer give presentations for our "internal categories".

1.2.1.1 Pointwise calculations in presheaf categories.

We quote from [MacL] Chapter 5, section 3, "In a functor category, limits may be calculated pointwise (provided the pointwise limits exists)". This is the conclusion to a theorem which we restate in 1.2.1.1.1 in our own notation for the category of sets. For each $A \in |\underline{\Lambda}|$, $\underline{\Lambda}$ a small category, $E_A: [\underline{\Lambda}^0, \underline{\mathcal{S}}] \longrightarrow \underline{\mathcal{S}}$ is the functor "evaluate at A", given for $\theta: F \longrightarrow G$ in $[\underline{\Lambda}^0, \underline{\mathcal{S}}]$ by

$$E_A(F) = F(A) \quad \text{and} \quad E_A(\theta) = \theta_A.$$

1.2.1.1.1. Proposition. If $D: \underline{J} \longrightarrow [\underline{\Lambda}^0, \underline{\mathcal{S}}]$ is such that for each $A \in |\underline{\Lambda}|$ the composite $E_A \cdot D: \underline{J} \longrightarrow \underline{\mathcal{S}}$ has a limit L_A with limiting cone $\tau_A: L_A \longrightarrow E_A \cdot D$, then there is a unique functor $L: \underline{\Lambda}^0 \longrightarrow \underline{\mathcal{S}}$ with $L(A) = L_A$ such that $\bar{\tau}: L \longrightarrow D$ is a limiting cone where $\bar{\tau}_j: L \longrightarrow D(j)$ for $j \in (\underline{J})$ is defined by $\bar{\tau}_{jA} = \tau_{Aj}: L_A \longrightarrow D(j)(A)$ for $A \in |\underline{\Lambda}|$.

Proof. [MacL] Chapter 5, section 3, Theorem 1, page 111.□

1.2.1.1.2 There is another kind of pointwise calculation for $[\underline{\Lambda}^0, \underline{\mathcal{S}}]$ which we can view as derived from the pointwise limit calculations. It is also derivable from a more general point of view, presented in section 8 of [GU]. We summarize without giving full definitions, the material we need.

In 8.1 [GU], for $\underline{U}$ a small category, Σ a set of morphisms of $[\underline{U}^0, \underline{\mathcal{S}}]$, and $\underline{B}$ an arbitrary category, a full subcategory $\mathrm{Cont}_\Sigma[\underline{U}^0, \underline{B}]$ of $[\underline{U}^0, \underline{B}]$ is defined whose objects are called Σ-continuous functors. In 8.2 (a), Σ is restricted to morphisms arising from some family of cones defined over $\underline{U}^0$. In 8.2 (c) it is shown how to choose $\underline{U}$ and Σ, as in (a), so that, when $\underline{B}$ has pullbacks.

(1) $\text{Cont}_\Sigma[\underline{U}^0, \underline{B}] \approx \underline{\text{Cat}}(\underline{B})$,

the category of internal categories and functors in $\underline{B}$. In 8.2 (d), by choosing $\underline{U}$ to be an algebraic theory, in the sense of Lawvere; taking Σ to arise from a family of discrete co-cones, determined by $\underline{U}$ (so that we can supress mention of Σ); and letting $\underline{B}$ be a category with finite products, we have

(2) $\text{Cont}[\underline{U}^0, \underline{B}] \approx \underline{U}\text{-}\mathcal{Alg}(\underline{B})$

the category of internal $\underline{U}$-algebras in $\underline{B}$.

1.2.1.1.3 The importance of these representations for our purposes lies in the possibility of interchanging structures given in:

8.8 Theorem [G U]. Let $\underline{U}$ and $\underline{\Lambda}$ be small categories and Σ and T morphisms sets in $[\underline{U}^0, \underline{\mathcal{S}}]$ and $[\underline{\Lambda}^0, \underline{\mathcal{S}}]$ such that $\{\text{id}_{[-,U]} \mid U \in |\underline{U}|\} \subset \Sigma$ and $\{\text{id}_{[-,V]} \mid V \in |\underline{\Lambda}|\} \subset T$. Then the isomorphism $[(\underline{U} \times \underline{\Lambda})^0, \underline{\mathcal{S}}] \rightleftarrows [\underline{U}^0, [\underline{\Lambda}^0, \underline{\mathcal{S}}]]$ induces an isomorphism of full subcategories $\text{Cont}_{\Sigma \times T}[(\underline{U} \times \underline{\Lambda})^0, \underline{\mathcal{S}}] \rightleftarrows \text{Cont}_\Sigma[\underline{U}^0, \text{Cont}_T[\underline{\Lambda}^0, \underline{\mathcal{S}}]]$.

As a corollary, by taking $T = \{\text{id}_{[-,V]} \mid V \in |\underline{\Lambda}|\}$, and using the isomorphism $\underline{U} \times \underline{\Lambda} \rightleftarrows \underline{\Lambda} \times \underline{U}$, we get

$\text{Cont}_\Sigma[\underline{U}^0, [\underline{\Lambda}^0, \underline{\mathcal{S}}]] \rightleftarrows [\underline{\Lambda}^0, \text{Cont}_\Sigma[\underline{U}^0, \underline{\mathcal{S}}]]$.

Thus in particular we have

(1) $\underline{\text{Cat}}([\underline{\Lambda}^0, \underline{\mathcal{S}}]) \rightleftarrows [\underline{\Lambda}^0, \underline{\text{Cat}}(\underline{\mathcal{S}})]$

and

(2) $\underline{U}\text{-}\mathcal{Alg}([\underline{\Lambda}^0, \underline{\mathcal{S}}]) \rightleftarrows [\underline{\Lambda}^0, \underline{U}\text{-}\mathcal{Alg}(\underline{\mathcal{S}})]$.

Now in practice we shall work with the right side of these isomorphism and we shall not be explicit about $\underline{U}$ or Σ since we have more direct ways available for describing $\underline{\text{Cat}}(\underline{\mathcal{S}})$ and $\underline{U}\text{-}\mathcal{Alg}(\underline{\mathcal{S}})$. The former is the category of categories with underlying morphism sets in $\underline{\mathcal{S}}$ and the latter is, in general, a variety of algebras of some similarity type.

Without introducing a presentation $(\underline{U},\Sigma)$ for groupoids we can, from the definition 1.5.1.2 view $\underline{Gpd}([\underline{\Lambda}^0,\underline{\mathcal{S}}])$, the category of internal groupoids in $[\underline{\Lambda}^0,\underline{\mathcal{S}}]$, as the full subcategory of $\underline{Cat}([\underline{\Lambda}^0,\underline{\mathcal{S}}])$ consisting of an internal category $\underline{C}$ together with a uniquely determined involution I, (which is necessarily preserved by internal functors). It is clear, by evaluating I that the pair $(\underline{C},I)$ corresponds to a functor in $[\underline{\Lambda}^0,\underline{Gpd}(\underline{\mathcal{S}})]$ and that any such functor produces such a pair. What we want out of this is :

(1.1) an internal category in $[\underline{\Lambda}^0,\underline{\mathcal{S}}]$ is a functor $\underline{C}$: $\underline{\Lambda}^0 \longrightarrow \underline{Cat}(\underline{\mathcal{S}})$

and

(1.2) $\underline{C}$ is an internal groupoid iff $\underline{C}(A)$ is a groupoid for each $A \in |\underline{\Lambda}|$.

<u>1.2.1.2 The Yoneda Lemma</u> ([MacL]Chapter III, section 2, page 59).

Let $[-,\ ]$: $\underline{C} \longrightarrow [\underline{C}^0,\underline{\mathcal{S}}]$ be the Yoneda embedding. Its value on $f: A \longrightarrow B$ is the natural transformation $[-,f]: [-,A] \longrightarrow [-,B]$ which for $D \in C_0$ and $g \in [D,A]$ is given by $[D,f](g) = f \cdot g$.

<u>1.2.1.2.1</u> By a representation of a functor $\underline{C} \xrightarrow{K} \underline{\mathcal{S}}$ we mean a pair $\langle G,\psi \rangle$ where ψ: $[G,-] \approx K$; G is called the representing object and K the representable functor.

<u>1.2.1.2.2</u> The Yoneda lemma states that for each $A \in C_0$ the evaluation functor $[\underline{C}^0,\underline{\mathcal{S}}] \xrightarrow{E_A} \underline{\mathcal{S}}$ is representable with representation $\langle [-,A],\psi_A \rangle$ where for each F: $\underline{C}^0 \longrightarrow \underline{\mathcal{S}}$

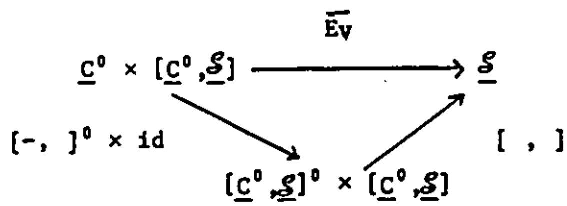
$$\psi_{A,F}:\ [[-,A],F] \approx F(A)$$

is given by

$$\psi_{A,F}(\theta) = \theta_A(\mathrm{id}_A).$$

By [CWM] Chapter III, section 2 addendum to Yoneda Lemma (p.61), the isomorphism is natural in both A and F. We formulate this isomorphism as follows. Define $\mathrm{Ev}\colon \underline{C}^0 \times [\underline{C}^0, \underline{\mathcal{S}}] \longrightarrow \underline{\mathcal{S}}$ to be the evaluation functor $(\mathrm{Ev}(A,F) = F(A))$ which corresponds under the adjointness $\underline{C}^0 \times - \dashv [\underline{C}^0, -]$ to the identity functor $\mathrm{id}\colon [\underline{C}^0, \underline{\mathcal{S}}] \longrightarrow [\underline{C}^0, \underline{\mathcal{S}}]$

Define $\overline{\mathrm{Ev}}$ to be the composite



Then $\psi\colon \overline{\mathrm{Ev}} \rightleftarrows \mathrm{Ev}$ is a natural equivalence. The functor $\overline{\mathrm{Ev}}$ corresponds under the same adjointness to a functor $J\colon [\underline{C}^0, \underline{\mathcal{S}}]^0 \longrightarrow \underline{\mathcal{S}}^0$, so we have an equivalence

$$\tilde{\psi}\colon J \rightleftarrows \mathrm{id} .$$

Explicitly $(J(F))(A) = [[-,A],F]$ and $J(F) = [-,F], \circ [-,]^0$; J is an adjoint equivalence with both adjoints being id.

If $\mathcal{G}$ is an algebra with carrier F, then we have

$$\tilde{\psi}_F\colon \ J(\mathcal{G}) \rightleftarrows \mathcal{G}$$

is an isomorphism of algebras.

Evaluating at $A \in C_0$ gives an algebra isomorphism

$$\tilde{\psi}_{A,F}\colon [[-,A],\mathcal{G}] \rightleftarrows \mathcal{G}(A).$$

<u>1.2.1.2.3</u> The correpondence (2) in 1.2.1.13 suggests a more exact relationship, a proof of which we sketch. Let $\mathcal{O}$ be a similarity type for algebras and let $\mathcal{A}$ be an internal $\mathcal{O}$-algebra in $[\underline{V}^0, \underline{\mathcal{S}}]$. For each $B \in |\underline{V}|$ let $E_B: [\underline{V}^0, \underline{\mathcal{S}}] \longrightarrow \underline{\mathcal{S}}$ be the cartesian functor "evaluate at B", and put $\bar{\mathcal{A}}(B) = E_B(\mathcal{A})$. Let $t_1, t_2 \in \text{Poly}\, \mathcal{O}$ then $\mathcal{A} \models t_1 = t_2$ iff for all B, $\bar{\mathcal{A}}(B) \models t_1 = t_2$.

<u>Sketch of proof</u>. ($\rightarrow$): Since E_B are cartesian, by 0.6.12.11.□

($\leftarrow$) $\bar{\mathcal{A}}$ is actually a functor from $\underline{V}^0$ to $\mathcal{O}\text{-}\mathcal{A}\ell g$; this is the correspondence given by (2) for $\underline{U}$ the algebraic theory of $\mathcal{O}$-algebra. Let $D: \underline{J} \longrightarrow [\underline{V}^0, \underline{\mathcal{S}}]$ be a diagram of representable functors with $D(i) = [-, D_i]$, $D_i \in |\underline{V}|$. We have $\bar{\mathcal{A}}(D_i) \models t_1 = t_2$ for each $i \in |\underline{J}|$. $\bar{\mathcal{A}}(D_i) \approx [[-, D_i], \mathcal{A}]$, by the "enriched" version of the Yoneda Lemma; hence $[\lim_{\to \underline{J}} [-, D_i], \mathcal{A}] \approx \lim_{\leftarrow \underline{J}} \bar{\mathcal{A}}(D_i) \models t_1 = t_2$. Now for every $F: \underline{V}^0 \longrightarrow \underline{\mathcal{S}}$ there is a diagram D such that $F = \lim_{\to \underline{J}} [-, D_i]$, hence for each F in $[\underline{V}^0, \underline{\mathcal{S}}]$, $[F, \mathcal{A}] \models t_1 = t_2$; hence by 0.6.6.4 or 0.6.3.7 $\mathcal{A} \models t_1 = t_2$.□

1.2.2 Preorders and ideals associated with presheaves

In this subsection we study those structures which are relevant to the logic of the topos of presheaves over a small category $\underset{\sim}{C}$. We shall extract external structures from internal structures by a combination of colimits and limits; in particular we use (1) the coproduct functor $\coprod_{A \in C_0}$ (1.2.2.3), (2) the product functor $\prod_{A \in C_0}$ (1.2.2.5), and (3) the limit functor $\varprojlim_{\underset{\sim}{C}^0}$ (1.2.2.9).

Of central importance to this study is the subobject classifier true: $1 \longrightarrow \Omega$ (1.2.2.6) with its internal partially ordered structure $\Omega_{\underline{<}}$. The structure $\Omega_{\underline{<}}$ is usually construed as the composite $\text{Sub}_{\underline{<}} \circ [-,]^0$ where $[-,]^0$ is the Yoneda embedding $\underline{C}^0 \rightarrowtail (\mathcal{S}^{\underline{C}^0})^0$ $\text{Sub}_{\underline{<}}$: $(\mathcal{S}^{\underline{C}^0})^0 \longrightarrow$ PreOrd is the contravariant functor associating to each presheaf the preordered set of its subpresheaves, and to each morphism of presheaves the inverse image function derived from pulling back subpresheaves. Our first construction of external structures concentrates on a factorization of $\text{Sub}_{\underline{<}}$ (1.2.2.4).

The coproduct functor, after a slight adjustment of domain (1.2.2.3) can be constructrued as a union, on objects: $\overline{F} \equiv \bigcup_{A \in C_0} F(A)$, with a natural extension $\overline{\eta}: \overline{G} \longrightarrow \overline{F}$ to presheaf morphisms $\eta: G \longrightarrow F$. When applied to subpresheaves of F the coproduct functor yields subsets of $\overline{F}$ and the inverse image map: $\eta^*: \text{Sub}_{\underline{<}}\, F \longrightarrow \text{Sub}_{\underline{<}}\, G$ carries over to a genuine external inverse image map $\overline{\eta}^*: \mathcal{P}(\overline{F}) \longrightarrow \mathcal{P}(\overline{G})$.

We impose on each $\overline{F}$ a preorder $\prec_F$ (1.2.2.3) having the following properties:

a) each $\overline{\eta}: \overline{G} \longrightarrow \overline{F}$ preserves the preorder, so that we have a functor Pr: $\mathcal{S}^{\underline{C}^0} \longrightarrow$ PreOrd,

(b) the values of subpresheaves of F, under Pr, coincide with the "ideals" (that is, increasing, subsets) of the preordered set $(\overline{F}, \prec_F)$ (1.2.2.4), both (c) the inverse image $\overline{\eta}^*$ and (d) the direct image $\exists_{\overline{\eta}}$ preserve ideals.

We study separately (1.2.2.0) the category PreOrd with attention to the features associated with the functor Pr. Such a separate study is necessary in the context of a topos $\mathcal{E}$ in place of *Set*. We need to work with the simplest possible structures in $\mathcal{E}$ because of the complications that arise when interpreting in its language, the propositional logic of $\mathcal{E}^{\underline{C}^*}$

We can define a functor $\text{Idl}_{\underline{<}} : (\text{PreOrd})^0 \longrightarrow$ PreOrd quite naturally as essentially: exponentiation of **2**, the two element chain in the cartesian closed category PreOrd (1.2.2.1). Each poset, $\text{Idl}(\mathcal{A})_{\underline{<}}$, of ideals of $\mathcal{A}$, inherits from **2** a completely distributive lattice structure (in $\mathcal{E}$ this is only a localic structure inherited from $\Omega_{\underline{<}}$), and thus possesses both the Heyting algebra operation, $\Rightarrow$, and the dual operation $\nLeftarrow$, (1.2.2.1). The inverse image map $f^* = \text{Idl}_{\underline{<}}(f)$ preserves joins and meets but fails in general to preserve either $\Rightarrow$ or $\nLeftarrow$.

Maps, f, in PreOrd whose direct image, $\exists_f$, preserves ideals constitute a subcategory h-PreOrd (1.2.2.2) of PreOrd through which the functor Pr: $\mathcal{S}^{\underline{C}^0} \longrightarrow$ PreOrd factors. Such maps, f, have the property that f^* does preserve $\Rightarrow$. Thus the functor $\text{Idl}_{\underline{<}}$ when restricted to h-PreOrd is valued in *Heyt*, so the factorization of $\text{Sub}_{\underline{<}}$ is really for the Heyting algebra enriched functors: $\text{Sub} \approx \text{Idl} \circ \text{Pr}^0 : (\mathcal{S}^{\underline{C}^0})^0 \longrightarrow$ *Heyt* (1.2.2.4), as is the factorization of $\Omega_{\Leftarrow}$ (1.2.2.6):

$\underset{\sim}{\Omega} \approx \underset{\sim}{\text{Sub}}\circ[-,\ ]^0 \approx \underset{\sim}{\text{Idl}}\circ(\text{Pr}\circ[-,\ ])^0 : \underline{C}^0 \longrightarrow \mathit{Heyt}$. An examination of $\Omega_{\underline{<}}$ shows, however that $\dashrightarrow$ can only be preserved by each $\Omega_{\underline{<}}(f)$ if $\Omega_{\underline{<}}$ is complemented (1.2.2.1.9).

The second way of passing from internal to external structures, the product functor, $\prod_{A \in C^0}$, has a representing presheaf Y (1.2.2.5). The functor $\prod_{A \in C_0} \approx [Y,-]$, being left exact, lifts to algebras; thus for the internal Heyting algebra $\underset{\sim}{\Omega}$, there are external isomorphisms: $\prod_{A \in C_0} \underset{\sim}{\Omega}(A) \approx [Y,\underset{\sim}{\Omega}] \approx \underset{\sim}{\text{Sub}}(Y) \approx \underset{\sim}{\text{Idl}}\ (C_1,\prec)$ (1.2.2.7). The last of these algebras consists of subsets of C_1, the set of morphisms of $\underline{C}$, which are closed under composition from the right, that is "right ideals". The isomorphism allows us to conclude that for $\varphi \in$ Poly H

$$\underset{\sim}{\Omega} \vdash \varphi \quad \text{iff} \quad \underset{\sim}{\text{Idl}}\ (C_1,\prec) \vdash \varphi \quad ;$$

thus $\underset{\sim}{\text{Idl}}\ (C_1,\prec)$ becomes our external replacement for $\underset{\sim}{\Omega}$.

Our third externalization: $\lim\limits_{\to \underline{C}^0} : \mathcal{S}^{\underline{C}^0} \longrightarrow \mathcal{S}$ is the global sections functor (1.2.2.9) $\lim\limits_{\to \underline{C}^0} \approx [\mathbb{1},-]$ which when applied to $\underset{\sim}{\Omega}$ gives:

$\lim\limits_{\to \underline{C}^0} (\underset{\sim}{\Omega}) \approx [\mathbb{1},\underset{\sim}{\Omega}] \approx \underset{\sim}{\text{Sub}}(\mathbb{1}) \approx \underset{\sim}{\text{Idl}}\ (C_0,\prec_0)$ where the preorder, $\prec_0$, is given by: $A \prec_0 B$ iff there exists a morphism from B to A (1.2.2.3.3). The morphism $u: Y \twoheadrightarrow \mathbb{1}$ induces a mono $[u,-]: [\mathbb{1},-] \rightarrowtail [Y,-]$ which when applied to $\underset{\sim}{\Omega}$ gives a Heyting algebra monomorphism

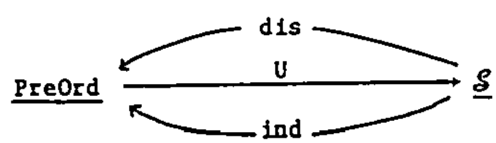
$$\underset{\sim}{\text{Idl}}(C_0,\prec_0) \rightarrowtail \underset{\sim}{\text{Idl}}(C_1,\prec).$$

Of particular interest,when determining whether or not a small category is agroupoid,is the ideal of nonretractions of the category (1.2.2.8) (a slight variation of N will be considered internally). The ideal N is empty iff the category is a groupoid iff N belongs to the subalgebra $\underset{\sim}{\text{Idl}}(C_0,\prec_0)$.

<u>1.2.2.0 The category PreOrd.</u> We are going to introduce a category within which we shall work internally in 1.4.3 and subsequent sections. The techniques we develop in this setting will greatly simplify our analysis of the propositional logic of presheaf toposes.

By a preorder (0.5.1.1 and 1.4.3.1), $\prec$, on a set A we mean a reflexive transitive relation; we call $\mathcal{A} = (A,\prec)$ a preordered set ([Sch] p. 4) and we let $|\mathcal{A}| = A$, the carrier of $\mathcal{A}$. Preordered sets are the objects of a category <u>PreOrd</u> which has as morphisms the preorder preserving functions (1.7.1.1 (2)) ; such morphisms will also be said to preserve order.

We can view $\underline{\mathcal{S}}$ as a full subcategory of <u>PreOrd</u> in two ways: as the category of discrete preordered sets it is a full reflective subcategory, and as the category of indiscrete preordered sets it is a full coreflective subcategory.



The functors are determined by their values on objects: $U(A,\prec) = A$, $dis(A) = (A, \prec_{dis})$, $ind(A) = (A, \prec_{ind})$ where $x \prec_{dis} y$ iff $x = y$ and $x \prec_{ind} y$ for all x, y.

U is the reflector of dis and the coreflector of ind: $dis \dashv U \dashv ind$; U is a retraction of both its adjoints: $U \circ dis = id_{\underline{\mathcal{S}}} = U \circ ind$.

The category <u>PreOrd</u> has all small limits and colimits. The limit of a diagram $\mathcal{D}: \underline{J} \longrightarrow$ <u>PreOrd</u> is constructed by taking the limit of

$U \cdot \mathcal{D} : \underline{J} \longrightarrow \underline{\mathcal{S}}$ and endowing the set L of the limit $L \xrightarrow{\lambda} U \cdot \mathcal{D}$ with the strongest preorder $\prec$ making all the $\lambda_i : (L, \prec) \longrightarrow \mathcal{D}(i)$ $(i \in |\underline{J}|)$ order preserving. It is not difficult to verify the existence of such a preorder; it is given by $x \prec y$ iff for all i $(\lambda_i(x) \prec_i \lambda_i(y))$ where $\prec_i$ is the preorder of $\mathcal{D}(i)$. Dually colimits are constructed by giving the vertex C of the limiting cocone $U \cdot \mathcal{D} \longrightarrow C$ the weakest preorder making the λ_i preorder preserving. The relation $R \equiv \{(\lambda_i(x), \lambda_i(y)) | \ x \prec_i y$ in $\mathcal{D}(i)$, $i \in |\underline{J}|\}$ is contained in the indiscrete preorder, and preorders are closed under intersection so there is a weakest preorder containing R.

In the case of coproducts the relation R is itself a preorder; if we assume $\mathcal{D}(i) \equiv (D_i, \prec_i)$ are such that the D_i are pairwise disjoint then

$$\coprod_{i \in J} \mathcal{D}(i) \equiv (\bigcup_{i \in J} D_i, \prec)$$

where $x \prec y$ iff for some i $(x \prec_i y)$.

Monos and epis in PreOrd are morphisms whose underlying functions are respectively monos and epis in $\underline{\mathcal{S}}$. PreOrd is not balanced; we do have epi-mono factorization but they are not unique. We do have unique factorizations: (extremal epi, mono) and (epi, extremal mono).

PreOrd is cartesian closed, inheriting this structure from Cat. For $\mathcal{A}$ and $\mathcal{B}$ in PreOrd, $[\mathcal{A},\mathcal{B}] \equiv ([\mathcal{A},\mathcal{B}], \prec)$ will denote exponentiation, where $\prec$ is given pointwise:

$$f \prec g \text{ iff for all } a \in |\mathcal{A}| \ (f(a) \prec g(a)).$$

1.2.2.1 Operations on the set of ideals.

We have chosen the term "ideals" (1.2.2.1.1 (2)) for what, in the context of "posets", are usually called "increasing subsets", partly because of their prominent role as "truth values" and also because, in the context of $\underline{M}^{o}$ - *Set* those very truth values coincide with the "right ideals" of the monoid $\underline{M}$.

In 1.2.2.1.3 to 1.2.2.1.7 we examine both the Heyting algebra operations and the dual Heyting algebra operations on the set of ideals of a preordered set. In 1.2.2.1.8 and 1.2.2.1.9 we establish a limitation on the use of the dual operations by showing they cannot exist on $\Omega_{\underline{<}}$ unless $\Omega_{\underline{<}}$ is complemented. In 1.2.2.1.10, 1.2.2.1.11 and 1.2.2.1.12 we show that certain properties of the set of ideals which are expressible using these operations are also expressable in a first order sense, in terms of elements of the preordered set. In 1.2.2.2.1.13, relying in part on our internalized version of ideals in 1.7.1, we introduce the inverse image maps f^* of ideals induced by order preserving maps f. In 1.2.2.1.14 we introduce adjointness between preordered sets; and in 1.2.2.1.15 we show that the passage from f to f^* reverses the adjointness.

1.2.2.1.1 The ideal classifier **2**. We let $\mathbf{2} = (\{0,1\}, \leq)$. For any $f: \mathcal{A} \longrightarrow \mathbf{2}$ in PreOrd, $f^{-1}\{1\}$ satisfies

$$((x \in f^{-1}\{1\}) \text{ and } (x \prec y)) \text{ imply } (y \in f^{-1}\{1\})$$

moreover, if $F \subset A$ satisfies

(2) $((x \in F)$ and $(x \prec y))$ imply $(y \in F)$

then the function $\theta_F: A \longrightarrow \{0,1\}$ defined by

$$\theta_F(x) = 1 \quad \text{iff} \quad x \in F$$

is preorder preserving, $\theta_F: \mathcal{A} \longrightarrow \mathbf{2}$.

We have

$$(\theta_F)^{-1}\{1\} = F \quad \text{and} \quad \theta_{(f^{-1}\{1\})} = f.$$

We call sets F satisfying (2) ideals of $\mathcal{A}$; we let $\mathrm{Idl}\,\mathcal{A}$ be the set of all ideals of $\mathcal{A}$, and let $\mathrm{Idl}\,\mathcal{A}_{\leq} = (\mathrm{Idl}\,\mathcal{A}, \subset)$ be the preordered structure induced by the bijection

(3) $\mathrm{Idl}\,\mathcal{A}_{\leq} \approx [\![\mathcal{A}, \mathbf{2}]\!]$

The preordering is inclusion since

$F \subset G$ iff for all x $((x \in F)$ implies $(x \in G))$

iff for all x $(\theta_F(x) = 1$ implies $\theta_G(x) = 1)$

iff $\theta_F \leq \theta_G$.

1.2.2.1.2 Proposition. Let $\mathcal{B}_i \equiv (B_i, \prec_i)$ $(i \in J)$ be preordered sets with B_i $(i \in I)$ pairwise disjoint, then there is an isomorphism

$$\theta: \mathrm{Idl}(\coprod_{i \in I} \mathcal{B}_i)_{\leq} \xrightarrow{\sim} \prod_{i \in I} (\mathrm{Idl}(\mathcal{B}_i)_{\leq})$$

given by $\theta(U) \equiv (U \cap B_i)_{i \in I}$ with inverse $\theta^{-1}((U_i)_{i \in I}) \equiv \bigcup_{i \in I} U_i$.

Proof. $\theta^{-1}(\theta(U)) \equiv \bigcup_{i \in I} (U \cap B_i) \equiv U \cap (\bigcup_{i \in I} B_i) \equiv U$, and $\theta(\theta^{-1}(U_i)_{i \in I})) \equiv \theta(\bigcup_{j \in I} U_j) \equiv ((\bigcup_{j \in I} U_j) \cap B_i)_{i \in I} \equiv (U_i)_{i \in I}$. □

Note that this isomorphism can be viewed as arising from (3) and the cartesian closed properties of PreOrd

$$\mathrm{Idl}(\coprod_{i \in I} \mathcal{B}_i)_{\leq} \approx [\coprod_{i \in I} \mathcal{B}_i, \mathbf{2}] \approx \prod_{i \in I} [\mathcal{B}_i, \mathbf{2}] \approx \prod_{i \in I} \mathrm{Idl}(\mathcal{B}_i)_{\leq}$$

1.2.2.1.3 Preordered sets as categories. We construe a preordered set $\mathcal{A} = (A, \prec)$ as a category by taking A as the set of objects, the set of all pairs $(a,b) \in A^2$ such that $a \prec b$ to be the morphism set, and for (a,b) a morphism, $\mathrm{dom}(a,b) = a$ and $\mathrm{cod}(a,b) = b$. Note that in 1.7.2.1, for technical reasons, we will associate with a preordered object the opposite of the internal version of the category just described. We can regard $\mathbf{2}$ as a closed monoidal category with monoidal structure given by 1 and $\wedge$, and internal hom by $\Rightarrow$. From the point of view of [Ll], PreOrd is a $\mathbf{2}$-valued category in the sense that the usual hom functor for PreOrd may be construed as having values in $\mathbf{2}$:

$$\mathcal{A}^{o} \times \mathcal{A} \xrightarrow{[\ ,\ ]} \mathbf{2}$$

If we embed $\mathbf{2}$ into $\mathcal{S}$ by preserving the initial and terminal objects then the composite is naturally equivalent to the usual hom functor. Passing to ideals, the Yoneda embedding is the upper segment

$$\mathcal{A}^{o} \xrightarrow{[\ ,\rightarrow)} \mathrm{Idl}(\mathcal{A})_{\leq}$$

$$a \sim b \quad \text{iff} \quad [a,\rightarrow) = [b,\rightarrow)$$

where $a \sim b$ says that a and b are isomorphic:

$$a \sim b \quad \text{iff both} \quad a \prec b \quad \text{and} \quad b \prec a;$$

and the Yoneda Lemma is

$$a \in F \quad \text{iff} \quad [a,\rightarrow) \subset F.$$

1.2.2.1.4 $\mathrm{Idl}(\mathcal{A})_{\leq}$ as a Heyting algebra. We can also consider $\mathbf{2}$ as a locale, then $\mathrm{Idl}(\mathcal{A})_{\leq}$ inherits, "pointwise ", its own localic structure from $\mathbf{2}$. We are interested only in the Heyting algebra structure on $\mathrm{Idl}(\mathcal{A})_{\leq}$ arising from this localic structure. The bounded lattice operations on $\mathrm{Idl}(\mathcal{A})_{\leq}$ are the same as for $\mathcal{P}(A)$, the set of all subsets of A: top and bottom elements are A and ϕ respectively, meet is intersection, and join is union. For F and G ideals we have

$$x \in (F \Rightarrow G) \text{ iff } [x,\rightarrow) \subset (F \Rightarrow G)$$
$$\text{iff } [x,\rightarrow) \cap F \subset G$$

iff for all y such that $x \prec y$, if $y \in F$ then $y \in G$. Thus

$$(F \Rightarrow G) = \{x \mid ([x,\rightarrow) \cap F) \subset G\}.$$

We let $\underset{\sim}{\mathrm{Idl}}(\mathcal{A})$ be the Heyting algebra structure defined on $\mathrm{Idl}(\mathcal{A})_{\leq}$.

1.2.2.1.5 $(\mathrm{Idl}(\mathcal{A})_{\leq})^{0}$ is a Heyting algebra.

The set of ideals with the opposite ordering $(\mathrm{Idl}(\mathcal{A})_{\leq})^{0}$ also has a compatible Heyting algebra structure. The operation signs $\underline{0}$, $\underline{1}$, $\underline{\vee}$ and $\underline{\wedge}$ are interpreted in $\mathrm{Idl}(\mathcal{A})$ as A, ϕ, $\cap$ and $\cup$ respectively, but $\underline{\Rightarrow}$ has a new interpretation which is not even a derived operation of $\underset{\sim}{\mathrm{Idl}}(\mathcal{A})$ (see 1.2.2.1.9, 1.2.3.6.2.11).

We first show that the new structure can be construed as arising from the Heyting algebra structure of $\underset{\sim}{\mathrm{Idl}}(\mathcal{A}^{0})$.

1.2.2.1.6 Proposition. Subset complementation within $\mathcal{P}(|\mathcal{A}|)$ sets up a bijection between $\mathrm{Idl}(\mathcal{A})$ and $\mathrm{Idl}(\mathcal{A}^{0})$ in such a way that the natural orderings, $\mathrm{Idl}(\mathcal{A})_{\leq}$ and $\mathrm{Idl}(\mathcal{A}^{0})_{\leq}$, become reversed.

Proof. F is an ideal of $\mathcal{A}$ iff

$(\mathcal{A}x)(\mathcal{A}y)(((x \prec y) \text{ and } (x \in F)) \rightarrow (y \in F))$ iff

$(\mathcal{A}x)(\mathcal{A}y)(((x \prec y) \text{ and } (y \in \sim F)) \rightarrow (x \in \sim F))$ iff

$\sim F$ is an ideal of $\mathcal{A}^0$.

Dually, if G is an ideal of $\mathcal{A}^0$ then $\sim G$ is an ideal of $\mathcal{A}$. The ordering of $(\mathrm{Idl}(\mathcal{A})_{\leq})^0$ is $\supset$ and of $\mathrm{Idl}(\mathcal{A}^0)_{\leq}$ is $\subset$; for all ideals F, E of $\mathcal{A}$

$$F \supset E \quad \text{iff} \quad \sim F \subset \sim E. \square$$

<u>1.2.2.1.7 The operations $\dashv\!\!\!\!\!\Rightarrow$ and $\urcorner$ on $\mathrm{Idl}(\mathcal{A})$.</u> Let $\underset{\sim}{\mathrm{Idl}}^{\bullet}(\mathcal{A})$ be the Heyting algebra structure on $\mathrm{Idl}(\mathcal{A})$ induced by the bijection with the Heyting algebra $\underset{\sim}{\mathrm{Idl}}(\mathcal{A}^0)$. We use $\Rightarrow^0$ for the natural interpretation of $\underline{\Rightarrow}$ in $\underset{\sim}{\mathrm{Idl}}(\mathcal{A}^0)$ and $\dashv\!\!\!\!\!\Rightarrow$ for the corresponding operation of $\underset{\sim}{\mathrm{Idl}}^0(\mathcal{A})$.

$$\begin{aligned}(U \dashv\!\!\!\!\!\Rightarrow W) &= \sim ((\sim U) \Rightarrow^0 (\sim W)) \\ &= \sim \{x \mid ((\leftarrow,x] \cap (\sim U)) \subset (\sim W)\} \\ &= \{x \mid W \not\subset (\leftarrow,x] \cup U)\} \\ &= \{x \mid (Ey)((y \in W) \text{ and } (y \notin U) \text{ and } (y \prec x))\}\end{aligned}$$

The adjointness of $- \dashv\!\!\!\!\!\Rightarrow W$ becomes

$$(U \dashv\!\!\!\!\!\Rightarrow W) \subset V \quad \text{iff} \quad W \subset (U \cup V)$$

for all ideals U, V and W of $\mathcal{A}$.

We specialize to the dual of pseudocomplements. For F and ideal of $\mathcal{A}^0$ we put $\neg^0 F = F \Rightarrow^0 \phi$

and for U an ideal of $\mathcal{A}$ we put $\urcorner U = (U \dashv\!\!\!\!\!\Rightarrow A)$

$$\begin{aligned}\urcorner U &= (U \dashv\!\!\!\!\!\Rightarrow A) \\ &= \sim (\neg^0 (\sim U)) \\ &= \{x \mid (Ey)((y \prec x) \text{ and } (y \notin U)\}.\end{aligned}$$

The universal property of $\urcorner$ becomes

$$\urcorner U \subset V \quad \text{iff} \quad (U \cup V) = A$$

for all ideals U, V of $\mathcal{A}$.

1.2.2.1.8 Algebraic structure compatible with $(\Omega_{\leq})^{0}$.

Let $(\Omega_{\leq})^{0}$ be the codomain of true in a topos $\mathcal{E}$ with the partial order opposite to its natural ordering. If $(\Omega_{\leq})^{0}$ has the algebraic structure of either a Heyting algebra, a pseudo complemented lattice, or a Boolean algebra in each case compatible with the given partial order, then that algebraic structure is uniquely determined, by 1.7.1.24. The metatheory can be used efficiently to establish the following equivalences.

1.2.2.1.9 Proposition. In a topos, if $(\Omega_{\leq})^{0}$ has one of the following structures compatible with its ordering, then it has the other structures

(1) Heyting algebra

(2) Pseudocomplemented lattice

(3) Boolean algebra.

Proof. (1) → (2) and (3) → (1) are clear.

(2) → (3). In any pseudocomplemented distributive lattice A the bHf $(\neg\neg x = \underline{1}) \wedge (x = \underline{0}) \Rightarrow (\underline{1} = \underline{0})$ is valid, hence in $(\Omega_{\leq})^{0}$, with pseudocomplemented $\urcorner\!\urcorner$, we have

$$\models ((\urcorner\!\urcorner\urcorner\!\urcorner p = \bot) \wedge (p = \top)) \Rightarrow (\bot = \top)$$

hence $\models (\urcorner\!\urcorner\urcorner\!\urcorner p = \bot) \Rightarrow (p = \bot)$.

If in A, $(\neg\neg x = \underline{1}) \rightarrow (x = \underline{1})$ is valid, then A is a Boolean algebra, since $\neg\neg(x \vee \neg x) = \underline{1}$ is valid in A. Thus $\models \urcorner\!\urcorner p \wedge p = \bot$. Since $\models \urcorner\!\urcorner p \vee p = \top$, $\urcorner\!\urcorner$ must be the pseudocomplement of $\Omega_{\leq}$, because the lattice structure of both $\Omega_{\leq}$ and $(\Omega_{\leq})^{0}$ is distributive. □

Note that we might add to the list of equivalent structures on $(\Omega_{\leq})^{0}$ that of a locale; $\dashv\!\!\!\supset$ can be defined on Ω by

$$p \dashv\!\!\!\supset q = \bigwedge\{r : q \leq (r \vee p)\}.$$

Furthermore $\Omega_{\leq}$ and hence $(\Omega_{\leq})^0$ are complete so if $(\Omega_{\leq})^0$ has a Heyting algebra structure we could show $(\Omega_{\leq})^0$ has a localic structure. To actually prove the equivalence, since we cannot use the metatheory, we would have to develop theory for which we would have no other use but to make this point. We shall not do this.

We now characterize the $\neg\neg$ - dense and the complemented ideals.

1.2.2.1.10 Proposition. Let F be an ideal of $\mathcal{A}$ then

(1) $\neg\neg F = A$ iff $(Ax)(Ay)((x \prec y)$ and $(y \in F))$

(2) $\neg F \cup F = A$ iff F is an ideal of $\mathcal{A}^0$.

Proof. (1): $\neg\neg F = A$ iff $\neg F = \phi$ iff there is no x such that $(([x,\to) \cap F) = \phi)$ iff $(Ax)(Ey)((x \prec y)$ and $(y \in F))$.□

(2) Suppose $\neg F \cup F = A$, then $\neg F = {\sim}F$, hence ${\sim}F$ is an ideal of $\mathcal{A}$ hence F is an ideal of $\mathcal{A}^0$.

Suppose F is an ideal of $\mathcal{A}^0$, then ${\sim}F$ is an ideal of $\mathcal{A}$. Now ${\sim}F \cap F = \phi$, hence ${\sim}F \subset \neg F$, but $\neg F \cap F = \phi$, hence ${\sim}F = \neg F$, hence $F \cup \neg F = F \cup {\sim}F = A$. □

The dual of $\neg\neg$ -density is $\urcorner\urcorner$-codensity which we can characterize, for the ideals of $\mathcal{A} = (A,\prec)$, as follows.

1.2.2.1.11 Proposition. Let F be an ideal of $\mathcal{A}$. The following are equivalent

(1) $\urcorner\urcorner F = \phi$

(2) For all ideals G of $\mathcal{A}$

$$((F \cup G) = A) \to (G = A)$$

(3) $(Ax)(Ey)((y \prec x)$ and $(y \notin F))$.

Proof. (1) → (2). Assume (1), then $A = \urcorner\phi = \urcorner\urcorner\urcorner F = \urcorner F$, hence

$((F \cup G) = A) \rightarrow (\neg F \subset G)$

$\rightarrow A = G.\square$

(2) $\rightarrow$ (1). $(F \cup \neg F) = A$, hence, by (2), $\neg F = A$, hence $\neg\neg F = \phi.\square$

(1) $\leftrightarrow$ (3) $\neg\neg F = \phi$ iff $\neg^0 \neg^0 (\sim F) = A$ iff

$(\forall x)(\forall y)((x \prec y)$ and $(y \notin F)).\square$

1.2.2.1.12 Proposition. Let $\mathcal{A} = (A,<)$ be a preordered set then the following are equivalent

(1) $\neg F \cup F = A$ for all ideals F of $\mathcal{A}$

(2) $\prec$ is an equivalence relation

Proof. (1) $\rightarrow$ (2). For each $x \in A$, $[x,\rightarrow)$ is an ideal of $\mathcal{A}^0$; thus $y \prec x$ and $x \in [x,\rightarrow)$ imply $y \in [x,\rightarrow)$ so $y \prec x$ implies $x \prec y.\square$

(2) $\rightarrow$ (1) Let F be an ideal of $\mathcal{A}$, since $\mathcal{A} = \mathcal{A}^0$, F is an ideal of $\mathcal{A}^0$, hence $\neg F \cup F = A$ (by 1.2.2.1.8 (2)).$\square$

1.2.2.1.13 Maps for ideals. The contravariant functor

$$2^{(\ )}: \underline{\text{PreOrd}} \longrightarrow (\underline{\text{PreOrd}})^0$$

has right adjoint

$$2^{(\ )}: (\underline{\text{PreOrd}})^0 \longrightarrow \underline{\text{PreOrd}}$$

Thus $2^{(\ )}$ sends colimits in PreOrd to limits in PreOrd, hence epis are sent to monos. Applied to a map $f: \mathcal{A} \longrightarrow \mathcal{B}$ of $\underline{\text{PreOrd}}^0$, 2^f post-composes with maps $k: \mathcal{B} \longrightarrow 2$ to give maps $k \circ f: \mathcal{A} \longrightarrow 2$. Transferring to ideals 2^f becomes the inverse image map

$$\begin{array}{ccc} \mathcal{P}_B & \xrightarrow{\ f^{-1}\ } & \mathcal{P}_A \\ \uparrow & & \uparrow \\ \mathrm{Idl}(\mathcal{B}) & \xrightarrow{\ f^{*}\ } & \mathrm{Idl}(\mathcal{A}) \end{array}$$

Since f^{-1} preserves arbitrary meets and joins of subsets we have $\exists_f \dashv f^{-1} \dashv \forall_f$. Since the operations of joins and meets of ideals are just unions and intersections, f^* preserves them, so we have

$$f_! \dashv f^* \dashv f_* \ .$$

f^{-1} preserves the implication operation on subsets, an operation which is not the same for ideals, so we cannot conclude f^* preserves implication. We will show in 1.7.1 that for any f we have

(1) f^{-1} preserves ideals iff

$\exists_f[a,\rightarrow) \subset [fa,\rightarrow)$ for all $a \in A$

(2) $\exists_f$ preserves ideals iff

$[fa,\rightarrow) \subset \exists_f[a,\rightarrow)$ for all $a \in A$.

From the topological viewpoint, explained following 1.7.1.2, f is continuous when (1) holds and open when (2) holds. If f is both continuous and open then f^* is an H-algebra homomorphism (1.7.1.6), and $\exists_f = f_!$.

1.2.2.1.14 Internal adjointness between partially ordered objects is defined in 1.4.1.1. There is one instance however where we shall use external adjointness between preordered sets. For f, g order preserving maps between preordered sets $\mathcal{A}$ and $\mathcal{B}$:

$$\mathcal{A} \underset{f}{\overset{g}{\rightleftarrows}} \mathcal{B}$$

we say g is left adjoint to f and write $g \dashv f$ $(\mathcal{A},\mathcal{B})$ if for all $a \in |\mathcal{A}|$ and $b \in |\mathcal{B}|$

$$g(b) \prec a \text{ iff } b \prec f(a).$$

<u>1.2.2.1.15 Proposition</u>. If $g \dashv f$ $(\mathcal{A},\mathcal{B})$ then $f^* \dashv g^*$ $(\mathrm{Idl}\,\mathcal{B}_{\leq}, \mathrm{Idl}\,\mathcal{A}_{\leq})$.

<u>Proof</u>. $(U \subset g^*(W))$ iff $(\mathcal{A}b)((b \in U) \to (g(b) \in W))$. Since W is an ideal

$(g(B) \in W) \leftrightarrow (\mathcal{A}a)((g(b) \prec a) \to (a \in W))$

$\leftrightarrow (\mathcal{A}a)((b \prec f(a)) \to (a \in W))$, hence

$(U \subset g^*(W))$ iff $(\mathcal{A}a)(\mathcal{A}b)(((b \in U) \text{ and } (b \prec f(a))) \to (a \in W))$

iff $(\mathcal{A}a)(((\mathcal{E}b)((b \in U) \text{ and } (b \prec f(a))) \to (a \in W))$.

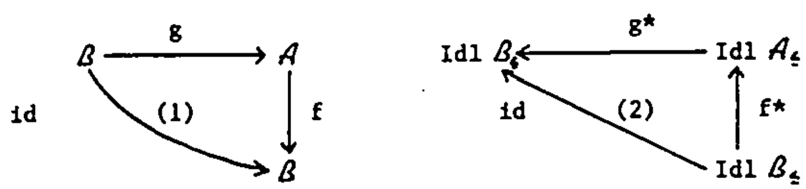
Since U is an ideal $(f(a) \in U) \leftrightarrow (\mathcal{E}b)((b \in U) \text{ and } (b \prec f(a)))$, hence

$(U \subset g^*(W))$ iff $(\mathcal{A}a)((f(a) \in U) \to (a \in W))$

iff $f^*(U) \subset W$. □

Since $(\ )^*: (\underline{\mathrm{PreOrd}})^{\circ} \longrightarrow \underline{\mathrm{PreOrd}}$ is a functor we have also:

<u>1.2.2.1.16 Proposition</u>. If (1) commutes then so does (2)



<u>Proof</u>. Let U be an ideal of $\mathcal{B}$, then

$b \in g^*(f^*(U))$ iff $g(b) \in f^*(U)$

iff $f(g(b)) \in U$

iff $b \in U$.□

<u>1.2.2.2 The category h-PreOrd</u>.

<u>1.2.2.2.1 Definition</u>. Let $\mathcal{A}$ and $\mathcal{B}$ be preordered sets; a function $f: |\mathcal{A}| \longrightarrow |\mathcal{B}|$ is called an h-map from $\mathcal{A}$ to $\mathcal{B}$ if for all $a \in |\mathcal{A}|$

$$[fa,\to) = \exists_f[a,\to).$$

We let <u>h-PreOrd</u> be the subcategory of <u>PreOrd</u> having preordered sets as objects and h-maps as morphism. (The definition of h-map appears in [AKS]).

The operation of taking ideals of a preordered set extends to functors making (1) commute

$$\begin{array}{ccc} \underline{\text{PreOrd}}^{o} & \xrightarrow{\underline{\text{Idl}}} & \underline{\mathcal{D}}_{01} \\ \uparrow & (1) & \uparrow \\ \underline{\text{h-PreOrd}}^{o} & \xrightarrow[\underset{\sim}{\text{Idl}}]{} & \underline{\mathcal{H}eyt} \end{array}$$

$\underline{\mathcal{D}}_{01}$ is the category of bounded distributive lattices with homomorphisms preserving 0, 1, $\wedge$ and $\vee$; and the right vertical functor is the reduct which ignores the operation $\Rightarrow$.

<u>1.2.2.2.2 Ideals as subobjects in h-PreOrd</u>. An ideal W of $\mathcal{B} = (B,\prec)$ gives rise to a map $f\colon \mathcal{W} \longrightarrow \mathcal{B}$ where $\mathcal{W} = (W,\prec)$ has the ordering induced from $\mathcal{B}$ and f is the inclusion map given by $f(x) = x$.

<u>1.2.2.2.3 Proposition</u>. For W and ideal of $\mathcal{B}$, $f\colon \mathcal{W} \longrightarrow \mathcal{B}$ is a mono in <u>h-PreOrd</u>.

<u>Proof</u>. It is clear that f preserves order, so we must show

$[f(u),\rightarrow) \subset \exists_f[u,\rightarrow)$ where $u \in W$.

$(b \in [f(u),\rightarrow)) \rightarrow (f(u) \prec b)$

$\rightarrow (u \prec b)$ with $(u \in W)$

$\rightarrow (b \in W)$ since W is an ideal

$\rightarrow (b \in \exists_f[u,\rightarrow))$.

Thus f is an h-map. Since the forgetful functor U: <u>h-PreOrd</u> $\longrightarrow \underline{\mathcal{S}}$

is faithful and U(f) is a mono, f is a mono in h-PreOrd. □

The converse satement, that if $g: \mathcal{A} \longrightarrow \mathcal{B}$ is a mono in h-PreOrd then g is equivalent to an ideal of $\mathcal{B}$, in the sense that there is a uniquely determined ideal W such that for the mono $f: \mathcal{W} \longrightarrow \mathcal{B}$ defined above there is a uniquely determined isomorphism $\gamma: \mathcal{A} \longrightarrow \mathcal{W}$ such that $f \cdot \gamma \equiv g$, is also true, but more difficult to get at. What does have a straightforward proof is the following.

1.2.2.2.4 Proposition. If $g: \mathcal{A} \longrightarrow \mathcal{B}$ is a map in h-PreOrd and U(g) is a mono, then g is equivalent to an ideal.

Proof. Without loss of generality we can let $|\mathcal{A}| \subset |\mathcal{B}|$ and let g be the inclusion map. Let $\prec'$ and $\prec$ be the orderings of $\mathcal{A}$ and $\mathcal{B}$ respectively. We want to show

(1) $a_1 \prec' a_2$ iff $a_1 \prec a_2$ for all a_1, a_2 in $|\mathcal{A}|$

and

(2) $a \in |\mathcal{A}|$ and $a \prec b$ imply $b \in |\mathcal{A}|$

Since g is order preserving, we have

(3) $(a_1 \prec' a_2)$ implies $(a_1 \prec a_2)$ for all a_1, a_2 in $|\mathcal{A}|$.

Since g is an h-map

$$[g(a), \rightarrow) \subset \exists_g [a, \rightarrow') \quad \text{for all } a \in |\mathcal{A}|$$

thus

$$(a \prec b) \text{ implies } (a \prec' b) \quad \text{for all } a \in |\mathcal{A}|,\ b \in |\mathcal{B}|$$

thus

$$(a_1 \prec a_2) \text{ implies } (a_1 \prec' a_2) \quad \text{for all } a_1, a_2 \in |\mathcal{A}|$$

so (1) holds and $a \prec b$ implies $b \in |\mathcal{A}|$ for all $a \in |\mathcal{A}|$, $b \in |\mathcal{B}|$ so (2) holds. □

The difficult part is to establish that f mono in h-PreOrd implies U(f) mono; and what makes the corresponding proof for PreOrd unavailable is that we cannot use $\mathbb{1}$, the terminal object, to separate elements of $U(\mathcal{A})$. $\mathbb{1} \equiv (\{*\},\equiv)$ is the terminal object in h-PreOrd, but an h-map $\mathbb{1} \xrightarrow{a} \mathcal{A}$ which picks out $a \in |\mathcal{A}|$ satisfies

$$\{a\} \equiv \exists_a [a,\rightarrow) \equiv [a,\rightarrow)$$

and thus a must be maximal. Thus only when $\mathcal{A}$ is discrete can we use $\mathbb{1}$ to separate elements of $U(\mathcal{A})$.

1.2.2.2.5 Proposition. If $\mathcal{A} \xrightarrow{f} \mathcal{B}$ is a mono in h-PreOrd then $|\mathcal{A}| \xrightarrow{f} |\mathcal{B}|$ is a mono in $\underline{\mathcal{S}}$.

Proof. Let $\mathcal{A} \xrightarrow{f} \mathcal{B}$ be a mono in h-PreOrd and suppose $f(a_1) \equiv f(a_2) \equiv b$. We shall show $a_1 \equiv a_2$.

We take the product in PreOrd of $\mathcal{A}$ with itself. $\mathcal{A} \times \mathcal{A} \equiv (A \times A, \prec)$ where, for all c_1, c_2, b_1, b_2 in A,

$$(c_1,c_2) \prec (b_1,b_2) \quad \text{iff both} \quad c_1 \prec b_1 \text{ and } c_2 \prec b_2.$$

We let $\mathcal{C}$ be the preorder having carrier the following subset of $A \times A$:

$|\mathcal{C}| \equiv \{(x_1,x_2) \in A^2 \mid \ a_1 \prec x_1,\ a_2 \prec x_2 \text{ and } f(x_1) \equiv f(x_2)\}$

we give $\mathcal{C}$ the order induced from $\mathcal{A} \times \mathcal{A}$, and we let $m\colon \mathcal{C} \rightarrowtail \mathcal{A} \times \mathcal{A}$ be the order embedding (note that m is not in general an h-map).

Let $\pi_i\colon \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$ be the projections in PreOrd and let $\bar{\pi}_i \equiv \pi_i \circ m\colon |\mathcal{C}| \longrightarrow |\mathcal{A}|$ for $i \equiv 1,2$. We claim $\bar{\pi}_i$ is an h-map ($i \equiv 1,2$).

The symmetry of the definitions ($i \equiv 1,2$) allows us to concentrate in one map $\bar{\pi}_1$. For $(x_1,x_2) \in |\mathcal{C}|$ and $y \in |\mathcal{A}|$ we have

$y \in \exists_{\bar{\pi}_1}[(x_1,x_2),\rightarrow)$

iff $(Ey_1)(Ey_2)((x_1,x_2) \prec (y_1,y_2),(y_1,y_2) \in |\mathcal{C}|$ and $(\bar{\pi}_1(y_1,y_2) \equiv y))$

iff $(Ey_2)((x_1 \prec y),(x_2 \prec y_2),(a_1 \prec y),(a_2 \prec y_2)$ and $f(y) \equiv f(y_2))$

we are assuming $(x_1,x_2) \in |\mathcal{C}|$ so $f(x_1) \equiv f(x_2)$ and

$y \in \exists_{\pi_1}[(x_1,x_2),\rightarrow)$

iff $(Ey_2)((x_1 \prec y),(x_2 \prec y_2)$ and $f(y) \equiv f(y_2))$

iff $(x_1 \prec y)$ and $f(y) \in \exists_f[x_2,\rightarrow)$

iff $(x_1 \prec y)$ and $f(y) \in [f(x_2),\rightarrow)$ (since f is an h-map).

But $f(x_1) \equiv f(x_2)$ hence

$$(x_1 \prec y) \text{ implies } f(x_2) \prec f(y)$$

hence $y \in \exists_{\bar{\pi}_1}[(x_1,x_2),\rightarrow)$ iff $x_1 \prec y$ iff $y \in [\bar{\pi}_1(x_1,x_2),\rightarrow)$.

Thus the maps in the diagram

$$\mathcal{C} \underset{\bar{\pi}_2}{\overset{\bar{\pi}_1}{\rightrightarrows}} \mathcal{A} \xrightarrow{\quad f \quad} \mathcal{B}$$

are h-maps. Let $(x_1,x_2) \in |\mathcal{C}|$ then $f(x_1) \equiv f(x_2)$ hence $(f\circ\bar{\pi}_1)(x_1,x_2) \equiv (f\circ\bar{\pi}_2)(x_1,x_2)$. Since f is a mono in <u>h-PreOrd</u>, hence $a_1 \equiv \pi_1(a_1,a_2) \equiv \pi_2(a_1,a_2) \equiv a_2$.

Thus $f\colon |\mathcal{A}| \longrightarrow |\mathcal{B}|$ is a mono. □

Let $\mathrm{Sub}_h(\mathcal{A})$ be the set of subobjects in <u>h-PreOrd</u> of $\mathcal{A}$, then $\mathrm{Sub}_h(\mathcal{A}) \approx \mathrm{Idl}(\mathcal{A})$. We shall show that the natural ordering of ideals is the same as that for subobjects within <u>h-PreOrd</u>.

<u>1.2.2.2.6 Lemma</u>. Let $g\colon \mathcal{B} \longrightarrow \mathcal{C}$ be an order preserving mono. If $k\colon A \longrightarrow B$ such that $g\circ k$ is open, then k is open.

<u>Proof</u>. $b \in [ka,\rightarrow) \rightarrow k(a) \prec b$

$\rightarrow (g\circ k)(a) \prec g(b)$ since g preserves order

$\rightarrow (Ea')((a \prec a')$ and $(g\circ k)(a') \equiv g(b))$ since $g\circ k$ is open

$\rightarrow b \in \exists_h[a,\rightarrow)$

Hence k is open.□

1.2.2.2.7 Corollary. Let $g: \mathcal{B} \longrightarrow \mathcal{C}$ and $f: \mathcal{A} \longrightarrow \mathcal{C}$ be monos in h-PreOrd such that f factors through g in $\underline{\mathcal{S}}$; that is there a map $k: A \longrightarrow B$ in $\underline{\mathcal{S}}$ such that $g \circ k \equiv f$. Then k is a mono in h-PreOrd.

Proof. Follows from the Lemma. □

1.2.2.2.8 It follows from 1.2.2.2.7 that the isomorphism between $Sub_h(\mathcal{A})$ and $Idl(\mathcal{A})$ is an order isomorphism and hence a H-algebra isomorphism:

$$\underset{\sim}{Sub}_h(\mathcal{A}) \approx \underset{\sim}{Idl}(\mathcal{A}).$$

1.2.2.2.9 Products in PreOrd. Let $\mathcal{A}_i = (A_i, \prec_i)$ $(i = 1,2)$ be objects of PreOrd; their product in PreOrd is the structure $\mathcal{A}_1 \times \mathcal{A}_2 \equiv (A_1 \times A_2, \prec)$ where $(a,b) \prec (c,d)$ iff both $a \prec_1 c$ and $b \prec_2 d$. Projections $\pi_i: \mathcal{A}_1 \times \mathcal{A}_2 \longrightarrow \mathcal{A}_i$ $(i = 1,2)$ are given by $\pi_i(a_1,a_2) = a_i$ $(i = 1,2)$ for all $a_i \in A_i$ $(i = 1,2)$.

1.2.2.2.10 Proposition. The projections $\pi_i: \mathcal{A}_1 \times \mathcal{A}_2 \longrightarrow \mathcal{A}_i$ are in h-PreOrd.

Proof. $[\pi_1(a_1,a_2),\rightarrow) = \exists_{\pi_1}[(a_1,a_2),\rightarrow)$ iff for all b

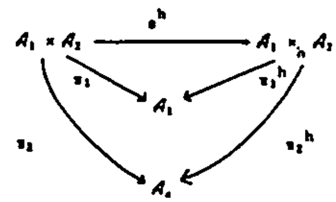
$a_1 \prec_1 b$ iff there exists $(b_1,b_2) \in A_1 \times A_2$ such that $a_1 \prec_1 b_1$, $a_2 \prec_2 b_2$ and $\pi_1(b_1,b_2) = b$

iff there exists $b_2 \in A_2$ such tat $a_1 \prec_1 b$ and $a_2 \prec_2 b_2$

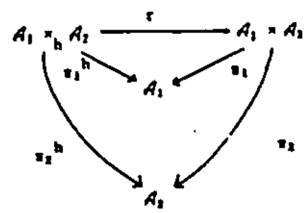
iff $a_1 \prec_1 b$. □

1.2.2.2.11 Products in h-PreOrd. Let $\mathcal{A}_i = (A_i, \prec_i)$ $(i = 1,2)$ be preordered sets for which their product exists: $\pi_i^h: \mathcal{A}_1 \times_h \mathcal{A}_2 \longrightarrow \mathcal{A}_i$ $(i = 1,2)$.

Let $s^h \equiv \pi_1 \sqcap_h \pi_2$, the h-map induced



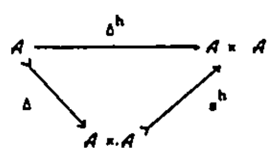
by the h-maps π_1 and π_2 ; and let $r \equiv \pi_1{}^h \sqcap \pi_2{}^h$, the preorder preserving maps induced by the maps $\pi_1{}^h$ and $\pi_2{}^h$ in <u>PreOrd</u>.



Then $\pi_i \circ r \circ s^h \equiv \pi_i{}^h \circ s^h \equiv \pi_i$ $(i \equiv 1,2)$ hence $r \circ s^h \equiv id_{\mathcal{A}_1 \times \mathcal{A}_2}$.
Thus s^h order embeds $\mathcal{A}_1 \times \mathcal{A}_2$ in $\mathcal{A}_1 \times_h \mathcal{A}_2$ in such a way that $\exists_s h\ (A_1 \times A_2)$ is an ideal of $\mathcal{A}_1 \times_h \mathcal{A}_2$.

<u>1.2.2.2.12</u>. If the product in <u>h-PreOrd</u> of $\mathcal{A}$ with itself exists then $\mathcal{A}$ is discrete; that is, $x \prec y$ iff $x \equiv y$ for all x, y in $|\mathcal{A}|$.

<u>Proof</u>. That the triangle (1) commutes,



where $\Delta^h \equiv id_{\mathcal{A}} \sqcap_h id_{\mathcal{A}}$ in <u>h-PreOrd</u>, $s^h \equiv \pi_1 \sqcap_h \pi_2$ in <u>h-PreOrd</u> and Δ is in <u>PreOrd</u>, follows since $\pi_i{}^h \circ (s^h \circ \Delta) \equiv \pi_i \circ \Delta \equiv id_{\mathcal{A}} \equiv \pi_i{}^h \circ \Delta^h$ for $i \equiv 1,2$. Since both Δ^h and s^h are open, we have, by that Δ must be an h-map. Since Δ is both a mono in $\underset{\sim}{\mathcal{S}}$ and an h-map, Δ is a mono in h-PreOrd, thus $\exists_\Delta(|\mathcal{A}|)$ is an ideal of $\mathcal{A} \times \mathcal{A}$.

Suppose $x \prec y$; we have $(x,x) \in \exists_\Delta(|\mathcal{A}|)$ and $(x,x) \prec (x,y)$, hence $x \equiv y$. □

<u>1.2.2.3 Disjoint presheaves and their associated preorders.</u> Let $\underline{C}$ be a small category having C_0 as its set of objects and C_1 as its set of morphisms. We call a presheaf $F: \underline{C}^0 \longrightarrow \underline{\mathcal{S}}$ disjoint if for all A, B in C_0,

$$\text{if } F(A) \cap F(B) \neq \phi \text{ then } A = B.$$

<u>1.2.2.3.1 Proposition</u>. The following are equivalent for a presheaf $F: \underline{C}^0 \longrightarrow \underline{\mathcal{S}}$:

(1) F is disjoint

(2) $(\mathcal{E}x)(F(f)(x) = F(g)(x)) \longrightarrow [(\text{dom } f = \text{dom } g) \text{ and } (\text{cod } f = \text{cod } g)]$.

<u>Proof.</u> (2) $\rightarrow$ (1) Put $f = id_A$ and $g = id_B$. □

(1) $\rightarrow$ (2) Let $F(f)(x) = F(g)(x)$. Then $x \in (F(\text{dom } f) \cap F(\text{dom } g))$ and $F(f)(x) \in (F(\text{cod } f) \cap F(\text{cod } g))$ hence $\text{dom } f = \text{dom } g$ and $\text{cod } f = \text{cod } g$. □

<u>1.2.2.3.2 The equivalence between presheaves and disjoint presheaves</u>.

Let $\mathbb{1}: \underline{C}^0 \longrightarrow \underline{\mathcal{S}}$ be the presheaf defined by

$$\mathbb{1}(A) = \{A\} \quad \text{for all } A \in C_0$$

and for $f: B \longrightarrow A$

$$\mathbb{1}(f): \{A\} \longrightarrow \{B\} \text{ is uniquely defined.}$$

This presheaf is both terminal and disjoint, moreover for any presheaf G there is an isomorphism $\varphi_G: G \times \mathbb{1} \rightleftarrows G$ where $G \times \mathbb{1}$ is the disjoint presheaf defined pointwise. Let $\underline{\mathcal{S}}^{\underline{C}^0}$ be the full subcategory of $[\underline{C}^0, \underline{\mathcal{S}}]$ whose objects are disjoint presheaves,

$$\underline{\mathcal{S}}^{\underline{C}^0} \underset{I}{\overset{- \times \mathbb{1}}{\rightleftarrows}} [\underline{C}^0, \underline{\mathcal{S}}]$$

We can consider $\mathbb{1} \times -$ as a functor with values in the full subcategory, then $\mathbb{1} \times -$ and I, the inclusion functor, form an adjoint equivalence with units, the natural equivalences $\varphi_G: I(G \times \mathbb{1}) \rightleftarrows G$ for any G, and $\varphi_F: I(F) \times \mathbb{1} \rightleftarrows F$ for F disjoint.

<u>1.2.2.3.3 The associated preorder.</u> For each disjoint presheaf F we let

$$\overline{F} = \bigcup_{A \in C_0} F(A)$$

thus

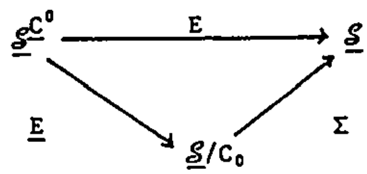
$$\overline{\mathbb{1}} = \bigcup_{A \in C_0} \{A\} = C_0.$$

We let $p_F: \overline{F} \longrightarrow \overline{\mathbb{1}}$ be defined by $p_F(x) = A$ iff $x \in F(A)$.

For each morphism $\eta: F \longrightarrow G$ of disjoint presheaves we define $\overline{\eta}: \overline{F} \longrightarrow \overline{G}$ by $\overline{\eta}(x) = \eta_A(x)$ where $A = p_F(x)$.

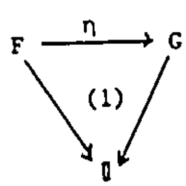
In particular for $u^F: F \longrightarrow \mathbb{1}$, abbreviated $u = u^F$, we have for $x \in F(A)$, $\overline{u}(x) = u_A(x) = A = p_F(x)$, thus $\overline{u^F} = p_F$.

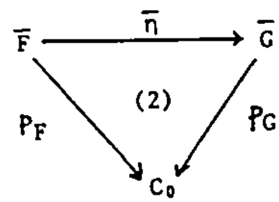
These operations on morphisms and objects of $\underline{\mathcal{S}}^{\underline{C}^0}$ give functors:



On objects E is given by $E(F) = \overline{F}$, and on morphisms by $E(\eta) = \overline{\eta}$.

We apply E to (1) to get (2) commuting.





Thus we can extend E to $\underline{E}$, putting $\underline{E}(F) = (\overline{F}, p_F)$ and $\underline{E}(\eta) = (\overline{\eta}, \underline{E}(F), \underline{E}(G))$.

We enrich the sets $\overline{F}$, F a disjoint presheaf, with a preorder $\prec_F$ defined by $x \prec_F y$ iff there exists f such that $F(f)(x) = y$. Note that by the disjointness of F we must have $\operatorname{dom} f = p_F(x)$ and $\operatorname{cod}(f) = p_F(y)$.

We put $Pr(F) = (\overline{F}, \prec_F)$, the preordered set determined by F. In particular, then, $Pr(\mathbb{1}) = (C_0, \prec_{\mathbb{1}})$ where

$$A \prec_{\mathbb{1}} B \text{ iff } [B,A] \neq \phi .$$

<u>1.2.2.3.4 Proposition.</u> Let $\eta\colon F \longrightarrow G$ be a morphism of $\mathcal{S}^{\underline{C}^0}$, then $\overline{\eta}\colon (\overline{F}, \prec_F) \longrightarrow (\overline{G}, \prec_G)$ is a morphism of <u>h-PreOrd</u>.

<u>Proof.</u> We must show that for any $A \in C_0$, $x \in F(A)$ $B \in C_0$, and $z \in G(B)$ we have

(2) $(z \in \exists_{\overline{\eta}}(x,\rightarrow))$ iff $(\overline{\eta}(x) \prec z)$.

We have $(\overline{\eta}(x) \prec z)$ iff $(\mathcal{E}f)(G(f)(\eta_A(x)) = z)$.

On the other hand $z \in \exists_{\overline{\eta}}[x,\rightarrow)$ iff $[(\mathcal{E}y)((x \prec y)$ and $(\overline{\eta}(y) = z))]$ iff $[(\mathcal{E}y)(\mathcal{E}g)((F(g)(x) = y)$ and $(\eta_B(y) = z))]$ iff $(\mathcal{E}g)(\eta_B(F(g)(x)) = z)$.

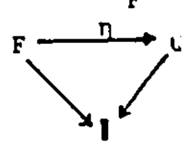
By the naturality of η, $\eta_B(F(g)(x)) = G(f)(\eta_A(x))$, hence (2) holds.□

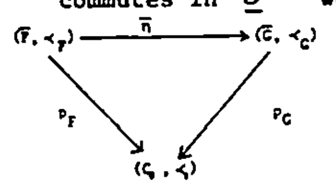
<u>1.2.2.3.5</u> We have defined a functor

$$Pr\colon \mathcal{S}^{\underline{C}^0} \longrightarrow \underline{\text{h-PreOrd}}$$

which on morphisms $\eta : F \longrightarrow G$ is given by

$Pr(\eta) = \overline{\eta}\colon (\overline{F}, \prec_F) \longrightarrow (\overline{G}, \prec_G)$.

Since 

commutes in $\mathcal{S}^{\underline{C}^0}$ we have


commuting in <u>h-PreOrd</u>.

Thus Pr extends to a functor

$$\underline{Pr} : \mathcal{S}^{\underline{C}^0} \longrightarrow \underline{\text{h-PreOrd}}/(C_0, \prec_{\mathbb{1}})$$

defined on objects by $\underline{Pr}(F) = (Pr(F), p_F)$ and on morphisms $\eta\colon F \longrightarrow G$ by $\underline{Pr}(\eta) = (\overline{\eta}, \underline{Pr}(F), \underline{Pr}(G))$.

<u>1.2.2.3.6 $\mathbb{P}$r as a functor of categories.</u> We define

$\mathbb{P}r$: <u>Cat</u> $\longrightarrow$ <u>PreOrd</u> on categories Λ by $Pr(\Lambda) = (\Lambda_1, \prec)$ where $f \prec g$ iff $(Ek)(f \cdot k = g)$; for functors $F: \underline{\Lambda} \longrightarrow \underline{\Gamma}$ we define $\mathbb{P}r(F)$ by $\mathbb{P}r(F) = F: (\Lambda_1, \prec) \longrightarrow (\Gamma_1, \prec)$; that is $\mathbb{P}r(F)(f) = F(f)$.

$(f \prec g) \rightarrow (Ek)(f \cdot k = g)$

$\rightarrow (Ek)(F(f) \cdot F(k) = F(g))$

$\rightarrow F(f) \prec F(g)$

<u>1.2.2.3.7</u> The functor <u>Pr</u> is derivable, not from $\mathbb{P}r$, but from the functor $\mathbb{P}r_o$: <u>Cat</u> $\longrightarrow$ <u>Pre Ord</u> given by $\mathbb{P}r_o(\underline{C}) = (C_o, \prec_o)$ where $A \prec_o B$ iff $[B,A] \not\equiv \phi$. $\mathbb{P}r_o$ induces $\underline{\mathbb{P}r_o}$: <u>Cat/C</u> $\longrightarrow$ <u>Pre Ord</u>$/(C_o, \prec_o)$, and <u>Pr</u> is the restriction of $\underline{\mathbb{P}r_o}$ to discrete fibration via $i: \underline{\mathcal{E}}^{C^o} \longrightarrow \underline{Cat/C}$ ($i(F)$ is $r_F : \underline{C} \ltimes F \longrightarrow \underline{C}$ defined in 1.2.3).

<u>1.2.2.3.8 Definition.</u> $F: \underline{\Lambda} \longrightarrow \underline{\Gamma}$ is an h-functor if $F: (\Lambda_1, \prec) \longrightarrow (\Gamma_1, \prec)$ is an h-map.

<u>1.2.2.3.9 Proposition.</u> $K: \underline{\Lambda} \longrightarrow \underline{\Gamma}$ is an h-functor iff for all $f \in \Lambda_1$ and $t \in \Gamma_1$ for which $K(f) \cdot t$ is defined, there exists $h \in \Lambda_1$ such that $K(f \cdot h) = K(f) \cdot t$.

<u>Proof.</u> K is an h-functor iff $K: (\Lambda_1, \prec) \longrightarrow (\Gamma_1, \prec)$ is an h-map iff

$$(Af)([K(f), \rightarrow) \subset \exists_K[f, \rightarrow)) \qquad (1)$$

$(\ell \in [K(f), \rightarrow))$ iff $(K(f) \prec \ell)$ iff $(Et)(K(f) \cdot t = \ell)$

and $(\ell \in \exists_K[f, \rightarrow))$ iff $(Eg)((f \prec g)$ and $(K(g) = \ell))$

iff $(Eg)((Eh)(f \cdot h = g)$ and $(K(g) = \ell))$

iff $(Eh)(K(f \cdot h) = \ell)$.

Thus (1) iff

$(Af)(A\ell)((Et)(K(f)\ t = \ell) \rightarrow (Eh)(K(f \cdot h) = \ell))$ iff

$(Af)(At)(A\ell)((K(f)\ t = \ell) \rightarrow (Eh)(K(f \cdot h) = \ell))$ iff

$(Af)(At)(Eh)(K(f \cdot h) = K(f) \cdot t)$. □

1.2.2.3.10 Restrictions of K. For each $A \varepsilon |\underline{\Lambda}|$ a functor $K: \underline{\Lambda} \longrightarrow \underline{\Gamma}$ maps $f \varepsilon P_A$ to $K(f) \varepsilon P_{K(A)}$, and for any f, g in P_A,

$f \prec_A g \rightarrow f \prec g$

$\rightarrow K(f) \prec K(g)$

$\rightarrow K(f) \prec_{K(A)} K(g)$

thus $K_A: (P_A, \prec_A) \longrightarrow (P_{K(A)}, \prec_{K(A)})$,

where $K_A = K|P_A$, is preorder preserving.

1.2.2.3.11 Proposition. Let $K: \underline{\Lambda} \longrightarrow \underline{\Gamma}$ be a functor. K is an h-functor iff for each $A \varepsilon |\underline{\Lambda}|$, $K_A: (P_A, \prec_A) \longrightarrow (P_{K(A)}, \prec_{K(A)})$ is an h-map.

Proof. K is an h-functor iff for each $A \varepsilon |\underline{\Lambda}|$, each $f \varepsilon P_A$ and each $\ell \varepsilon P_{K(A)}$

(1) $(K(f) \prec \ell) \rightarrow (Eg)((f \prec g)$ and $(K(g) = \ell))$.

But (1) is equivalent to

(2) $(K_A(f) \prec_{K(A)} \ell) \rightarrow (Eg)((f \prec_A g)$ and $(K_A(g) = \ell))$.□

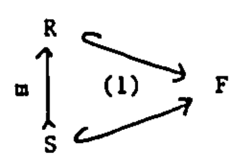
1.2.2.4 The correspondence between subpresheaves and ideals.

1.2.2.4.1 Subpresheaves instead of subobjects. Let $\theta : G \rightarrowtail F$ be a monomorphism in $\underline{\mathcal{S}}^{\underline{C}^0}$. We call θ an inclusion, or inclusion morphism, if for all $A \varepsilon C_0$ and all $x \varepsilon G(A)$ we have $\theta_A(x) = x$.

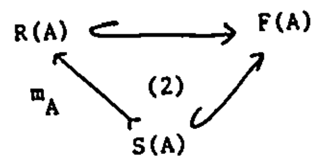
We will use "$G \hookrightarrow F$" both as the name of the inclusion, θ , and as the assertion that θ is an inclusion. Thus, if $G \hookrightarrow F$, then $G(A) \hookrightarrow F(A)$ for each $A \in C_0$. If $G \hookrightarrow F$, we call G a subpresheaf of F. Inclusion morphisms are the morphisms of a category whose objects are presheaves; this category is a subcategory of $\underline{\text{Mono}}(\mathcal{S}^{\underline{C}^0})$, the category of monopresheaves. The subcategory is not equivalent to $\underline{\text{Mono}}(\mathcal{S}^{\underline{C}^0})$ but if we take the comma category at a presheaf F we do get an equivalence. That is, within an equivalence class of monos with codomain F, there is exactly one subpresheaf; it is obtained by taking the image of the components of a mono with codomain F. The presence of this canonical choice allows us to drop the awkward subobject construction.

We shall mantain the notation while changing the meaning, thus for F a presheaf $\text{Sub}(F)$ will be the set of subpresheaves of F and for $\langle\vec{w},\varphi\rangle$ an augmented formula, $\tau(\vec{w}) = F$, $[\![\vec{w}|\varphi]\!] \hookrightarrow F$ will be the subpresheaf classified by $\lambda w.\varphi : F \longrightarrow \Omega$.

To see that the ordering of subobjects of F corresponds to that of subpresheaves consider first the situation in $\underline{\mathcal{S}}$: If (1) commutes



then m must be inclusion $S \hookrightarrow R$. If (1) represents a diagram of presheaves then evaluation at $A \in |\underline{C}|$ yields (2), hence each m_A is



an inclusion hence m is a subpresheaf $S \hookrightarrow R$. $Sub(F)_{\leq}$ is $Sub(F)$ with the above partial ordering.

We can cut down the information needed to determine a subpresheaf: let $|Sub|(F)$ be the set of all $K: C_0 \longrightarrow |\underline{\mathcal{S}}|$ such that

(P_1) $K(A) \subset F(A)$ for all $A \in C_0$, and

(P_2) For all A, B, f, x with $f: B \longrightarrow A$ and $x \in F(A)$,

if $x \in K(A)$ then $F(f)(x) \in K(B)$.

For each subfunctor $G \hookrightarrow F$, the map $|G|: |C_0| \longrightarrow |\underline{\mathcal{S}}|$ given by $|G|(A) = G(A)$ satisfies (P_1), and the naturality of $G \hookrightarrow F$ implies $|G|$ satisfies (P_2), thus we have a function

$$Sub(F) \xrightarrow{\;|\;\;|\;} |Sub|(F).$$

(P_2) gives the way of extending $K: C_0 \longrightarrow |\underline{\mathcal{S}}|$ to morphisms: by defining

$$K(f)(x) = F(f)(x) \text{ for all } x \in K(A).$$

Thus $|\;\;|$ is an isomorphism.

<u>1.2.2.4.2 Defining maps between Sub(F) and Idl Pr(F).</u> When F is a disjoint presheaf each subpresheaf of F is disjoint. We apply $Pr: \underline{S}^{\underline{C}^0} \longrightarrow$ <u>h-PreOrd</u> to $G \hookrightarrow F$ to get an h-map which is a set inclusion $\overline{G} \hookrightarrow \overline{F}$ and thus, by 1.2.2.2.4, an ideal of $(\overline{F}, \prec_F)$. Thus we have ψ_F: Sub(F) $\longrightarrow$ Idl$(\overline{F}, \prec_F)$ where $\psi_F(G) = \overline{G} = \bigcup_{A \in C^0} G(A)$. We define φ_F: Idl$(\overline{F}, \prec_F) \longrightarrow$ Sub(F) as follows: for U an ideal of $(\overline{F}, \prec_F)$ let $\tilde{U}: C_0 \longrightarrow |\underline{S}|$ be given by $\tilde{U}(A) = U \cap F(A)$, then $\tilde{U}$ satisfies (P_2): for $f: B \longrightarrow A$ and $x \in \tilde{U}(A)$ we have $x \prec F(f)(x)$ in $(\overline{F}, \prec)$; since U is an ideal, $F(f)(x) \in U$, and since $F(f)(x) \in F(B)$, $F(f)(x) \in \tilde{U}(B)$; thus there is exactly one subpresheaf $\varphi_F(U)$ of F such that $\varphi_F(U)(A) = \tilde{U}(A)$ for all $A \in C_0$.

<u>1.2.2.4.3 Proposition.</u> ψ_F: Sub$(F)_{\leq} \longrightarrow$ Idl$(\overline{F}, \prec_F)_{\leq}$ is an order isomorphism with inverse φ_F.

<u>Proof.</u> Let $G \hookrightarrow F$ and $K \hookrightarrow F$. $G \leq K$ in Sub$(F)_{\leq}$

iff for all $A \in C_0$, $G(A) \subset K(A)$

iff $\overline{G} \subset \overline{K}$ in Sub$_h(\overline{F}, \prec_F)$

iff $\psi_F(G) \subset \psi_F(K)$ in Idl$(\overline{F}, \prec_F)_{\leq}$

This shows that ψ_F is an order embedding. Let U be an ideal of $(\overline{F}, \prec_F)$ then $(\psi_F \circ \varphi_F)(U) = \overline{\varphi_F(U)} = \bigcup_{A \in C_0} (\varphi_F(U))(A)$

$$= \bigcup_{A \in C_0} (U \cap F(A)) = U \cap \overline{F} = U.$$

Since ψ_F is a mono and a retraction it is an isomorphism with $\psi_F^{-1} = \varphi_F$; moreover φ_F is an order embedding since ψ_F is.□

1.2.2.4.3.1 The order isomorphism ψ_F induces on $\mathrm{Sub}(F)_{\leq}$ a Heyting algebra structure $\mathrm{Sub}(F)$ from the Heyting algebra structure $\mathrm{Idl}(\hat{F}, \prec_F)$ described in 1.2.2.1.4.

1.2.2.4.4 The functor Sub: $(\mathcal{S}^{\underline{C}^0})^{\bullet} \longrightarrow \mathcal{S}$.

Let $\theta: K \longrightarrow F$ in $\mathcal{S}^{\underline{C}^0}$. We describe the inverse image function $\theta^*: \mathrm{Sub}(F) \longrightarrow \mathrm{Sub}(K)$ for presheaves. (We use "θ^*" here instead of "θ^{-1}" and on components, "θ^*_A" instead of "θ^{-1}_A" to avoid confusion with the inverse of maps which appear in 1.2.2.4.6.

Let $G \in \mathrm{Sub}(F)$; for each $A \in C_0$ we take the inverse image under θ_A:

$$\begin{array}{ccc} \theta_A^*(G(A)) & \longrightarrow & G(A) \\ \downarrow & \text{pb} & \downarrow \\ K(A) & \xrightarrow[\theta_A]{} & F(A) \end{array} \qquad (1)$$

Let $(\theta^*(G))(A) = \theta_A^*(G(A))$ for each $A \in |\underline{C}|$. We show that $\theta^*(G)$ is a subpresheaf of K. Suppose $f: A \longrightarrow B$ and $x \in \theta_B^*(G(B))$, then $\theta_B(x) \in G(B)$, hence $F(f)(\theta_B(x)) \in G(A)$, thus, by the naturality of θ, $\theta_A(K(f)(x)) \in G(A)$, hence $K(f)(x) \in \theta_A^*(G(A))$. Thus $\theta^*(G)$ satisfies (P2). Since, moreover $\theta_A^*: \mathcal{P}(F(A)) \longrightarrow \mathcal{P}(K(A))$ preserves order, so does $\theta^*: \mathrm{Sub}(F)_{\leq} \longrightarrow \mathrm{Sub}(K)_{\leq}$.

1.2.2.4.5 Proposition. Idl•Pr and Sub: $(\mathcal{S}^{\underline{C}^0})^{\bullet} \longrightarrow \mathcal{S}$ are naturally equivalent via ψ and φ.

Proof. We shall show that the ψ_K (as K ranges over disjoint presheaves) constitute a natural transformation ψ: Sub $\longrightarrow$ Idl$\circ$Pr. We want to show that for $\theta: K \longrightarrow F$ in $\mathcal{S}^{C^0}$ we have (1) commuting

$$\begin{array}{ccc} \mathrm{Sub}(K) & \xrightarrow{\psi_K} & \mathrm{Idl}(\bar{K}, \prec_K) \\ \downarrow \theta^* & (1) & \downarrow \bar{\theta}^* \\ \mathrm{Sub}(F) & \xrightarrow[\psi_F]{} & \mathrm{Idl}(\bar{F}, \prec_F) \end{array}$$

Let G be a subpresheaf of K

$$(\psi_F \circ \theta^*)(G) \equiv \overline{\theta^*(G)} \equiv \bigcup_{A \in C_0} (\theta^*(G))(A) \equiv \bigcup_{A \in C_0} \theta_A{}^*(G(A))$$

$$(\bar{\theta}^* \circ \psi_K)(G) \equiv \bar{\theta}^*(\bar{G}) \equiv \bar{\theta}^*(\bigcup_{A \in C_0} G(A)) \equiv \bigcup_{A \in C_0} (\bar{\theta}^*(G(A)))$$

$x \in \bar{\theta}^*(G(A))$ iff $\bar{\theta}(x) \in G(A)$

iff $\theta_A(x) \in G(A)$

iff $x \in \theta_A{}^*(G(A))$

thus $(\psi_F \circ \theta^*)(G) \equiv (\bar{\theta}^* \circ \psi_K)(G)$. Since $\psi_F \circ \theta^* \equiv \bar{\theta}^* \circ \psi_K$, we have

$$\varphi_F \circ \bar{\theta}^* \equiv \varphi_F \circ \bar{\theta}^* \circ \psi_K \circ \varphi_K$$

$$\equiv \varphi_F \circ \psi_F \circ \theta^* \circ \varphi_K \equiv \theta^* \circ \varphi_K \;. \square$$

Let U: *Heyt* $\longrightarrow \mathcal{S}$ be the forgetful functor.

1.2.2.4.6 Proposition. There is a uniquely determined functor Sub: $(\mathcal{S}^{C^0})^0 \longrightarrow$ *Heyt* such that

(1) $U \circ$ Sub $\equiv$ Sub and

(2) ψ: Idl$\circ$Pr $\longrightarrow$ Sub is a natural equivalence.

Proof. By definition (1.2.2.4.3.1) Sub(F) is defined so that ψ_F: Idl$(F, \prec_F) \longrightarrow$ Sub(F) is an H-isomorphism. Since for each $\theta: K \longrightarrow F$, $\theta^* \equiv \psi_F{}^{-1} \circ \theta^* \circ \psi_K$ is an H-homomorphism, we can put Sub$(\theta) \equiv \theta^*$ so that Sub is a functor satisfying (1) and (2).$\square$

<u>1.2.2.4.7 Principal subpresheaves.</u> Let $F: \underline{C}^0 \longrightarrow \underline{\mathcal{S}}$ be disjoint

We let $[\ ]_F$ be the composite

$$\begin{array}{ccc} (\bar{F},\prec_F) & \xrightarrow{\quad [\ ]_F \quad} & Sub(F) \\ & {\scriptstyle [\ ,\rightarrow)} \searrow \quad \nearrow {\scriptstyle \varphi_F} & \\ & Idl(\bar{F},\prec_F) & \end{array}$$

For $x \in \bar{F}$, we put $[x] \equiv [x]_F$. For all $A \in C_0$,

$[x](A) \equiv (\varphi_F([x,\rightarrow)))(A) \equiv [x,\rightarrow) \cap F(A)$

$\equiv \{y \in F(A) \mid (\exists F)((f: A \longrightarrow p_F(x))$ and $(F(f)(x) \equiv y)\}$

Since F is disjoint it is unnecessary to specify the domain and codomain of f:

$[x](A) \equiv \{y \in F(A) \mid (\exists f)(F(f)(x) \equiv y)\}$.

<u>1.2.2.4.8 Proposition.</u> Let $G \hookrightarrow F$ be disjoint and let $x \in F(A)$.
$x \in G(A)$ iff $(\forall B)(\forall f)((F(f)(x) \in F(B)) \rightarrow (F(f)(x) \in G(B)))$.

<u>Proof.</u> $x \in G(A)$ iff $[x,\rightarrow) \subset \bar{G}$ iff $[x] \leq G$ in $Sub(F)$

iff $(\forall B)([x](B) \subset G(B))$

iff $(\forall B)(\forall y)([(y \in F(B))$ and $(\exists f)(F(f)(x) \equiv y)] \rightarrow [y \in G(B)])$

iff $(\forall B)(\forall y)(\forall f)([(F(f)(x) \equiv y) \rightarrow ((F(f)(x) \equiv F(B)) \rightarrow (F(f)(x) \in G(B))])$

iff $(\forall B)(\forall f)((F(f)(x) \in F(B)) \rightarrow (F(f)(x) \in G(B)))$.□

<u>1.2.2.4.9 Proposition.</u> Let F be disjoint, let G and K be subpresheaves of F, and let $x \in F(A)$. Then

$(x \in (G \Rightarrow K)(A))$ iff

$(\forall B)(\forall f)(F(f)(x) \in G(B)) \rightarrow (F(f)(x) \in K(B)))$.

<u>Proof.</u> $(x \in (G \Rightarrow K)(A))$ iff $(([x] \wedge G) \leq K)$

iff $(\forall B)(([x](B) \cap G(B)) \subset K(B))$

iff $(\forall B)(\forall y)(((y \in [x](B))$ and $(y \in G(B))) \rightarrow (y \in K(B)))$

iff $(\forall B)(\forall y)(((\exists f)(F(f)(x) = y) \text{ and } (y \in G(B))) \rightarrow (y \in K(B)))$

iff $(\forall B)(\forall f)((F(f)(x) \in G(B)) \rightarrow (F(f)(x) \in K(B))$.□

1.2.2.4.10 Heyting algebra operations on $Sub(F)_{\leq}$.

The bottom 0 and top 1 of Sub(F) are given by $0(A) = \phi$ for all A and $1 = F$. For $G \hookrightarrow F$ and $K \hookrightarrow F$:

$(G \wedge K)(A) = G(A) \cap K(A)$

$(G \vee K)(A) = G(A) \cup K(A)$

$(G \Rightarrow K)(A) = \{x \in F(A) | (\forall B)(\forall f)((F(f)(x) \in G(B)) \rightarrow (F(f)(x) \in K(B)))\}$

1.2.2.4.11 In particular we are interested in the condition that $G \hookrightarrow F$ is $\neg\neg$ - dense in F.

$(\neg G)(A) = \{x \in F(A) | (\forall B)(\forall f)(\sim(F(f)(x) \in G(B)))\}$

$\neg\neg G = F$ iff $\neg G = 0$ iff $(\forall A)(\neg G)(A) = \phi)$

iff $(\forall A)(\sim[(\exists x)\{(x \in F(A)) \,\&\, (\forall B)(\forall f)(\sim(F(f)(x) \in G(B)))\}])$

iff $(\forall A)(\forall x)((x \in F(A)) \rightarrow (\exists B)(\exists f)(F(f)(x) \in G(B)))$

1.2.2.5 Representable presheaves and their coproduct.

Let $[-,\ ]: \underline{C} \longrightarrow \mathcal{E}^{\underline{C}^0}$ be the Yoneda embedding.

1.2.2.5.1 Definition of the disjoint presheaf $Y: \underline{C}^0 \longrightarrow \mathcal{E}$.

Let Y be the coproduct of the $[-,A]$ with injections $i_B: [-,B] \hookrightarrow \coprod_{A \in C_0} [-,A]$. Because of the disjointness of the hom sets we can take $Y(D) = \bigcup_{A \in C_0} [D,A] = \mathrm{dom}^{-1}\{D\}$.

Each functor $[-,A]$ has value on $D_1 \xrightarrow{k} D_2$ in $\underline{C}$, $[k,A]: [D_2,A] \longrightarrow [D_1,A]$ given by $[k,A](h) = h \circ k$ where $h: D_2 \longrightarrow A$. Thus $Y(k): \mathrm{dom}^{-1}\{D_2\} \longrightarrow \mathrm{dom}^{-1}\{D_1\}$ is given by $Y(k)(h) = h \circ k$ where $\mathrm{dom}(h) = D_2$. The injections, i_B, are natural transformations which at D are given by $i_{B,D}(\ell) = \ell$, for $\ell: D \longrightarrow B$, $i_{B,D}: [D,B] \hookrightarrow \mathrm{dom}^{-1}\{B\}$.

1.2.2.5.2 Y is the representing presheaf for the product functor.

For any H-algebra $\mathcal{G}$ in the presheaf topos $\mathcal{E}^{\underline{C}^0}$ we have H-isomorphisms:

$$\prod_{A \in C_0} \mathcal{G}(A) \approx \prod_{A \in C_0} [[-,A],\mathcal{G}]$$
$$\approx [\coprod_{A \in C_0} [-,A],\mathcal{G}]$$
$$= [Y,\mathcal{G}]$$

Thus Y is a representing presheaf for the product functor $\Pi: \underline{\mathcal{Heyt}}^{\underline{C}^0} \longrightarrow \underline{\mathcal{Heyt}}$.

We can regard this situation as a special case of a more familiar general fact: Any functor $J: \underline{D} \longrightarrow \underline{C}$ (in the case at hand J is the inclusion $C_0 \hookrightarrow \underline{C}$) gives rise to an adjoint pair $J_! \dashv J^*$

$$\underline{\underline{S}}^{\underline{C}^0} \underset{J_!}{\overset{J^*}{\rightleftarrows}} \underline{\underline{S}}^{\underline{D}^0}$$

so for any H-algebra $\mathcal{G}$ in $\underline{\underline{S}}^{\underline{C}^0}$,

$(\varprojlim J^*)(\mathcal{G}) \approx [1, \varprojlim J^*(\mathcal{G})] \approx [J_! \Delta(1), \mathcal{G}] \approx [J_!(\mathbb{1}), \mathcal{G}]$, where 1 and $\mathbb{1}$ are terminal in $\underline{\underline{S}}$ and $\underline{\underline{S}}^{\underline{D}^0}$ respectively. In our case $\Pi = \varprojlim \circ J^*$ and $Y = J_!(\mathbb{1})$.

<u>1.2.2.5.3 Preorders determined by representable presheaves.</u> Let A be an object of $\underline{C}$; when used as a subscript we shall abbreviate "$[-,A]$" to "A" thus "$\prec_A$" and "p_A" abbreviate "$\prec_{[-,A]}$" and "$p_{[-,A]}$" respectively. We put $P_A = \text{cod}^{-1}\{A\}$.

$Pr([-,A]) = (\bigcup_{A \, \varepsilon \, C^0} [D,A], \prec_A) = (\text{cod}^{-1}\{A\}, \prec_A) = (P_A, \prec_A)$.

$p_A \colon P_A \longrightarrow C_0$ is given by

$p_A(f) = B$ iff $f \; \varepsilon \; [B,A]$

iff $\text{dom}(f) = B$ and $\text{cod } f = A$

thus $p_A = \text{dom}\big|_{P_A}$.

For any f, g with codomain A,

$f \prec_A g$ iff $(\exists k)([k,A](f) = g)$

iff $(\exists k)(f \circ k = g)$.

Thus for all g with codomain A, $\text{id}_A \prec_A g$; hence $[\text{id}_A, \rightarrow) = \text{cod}^{-1}\{A\} = P_A$, the principal ideal generated by id_A.

<u>1.2.2.5.4 The preorder determined by $Y = \coprod_{A \, \varepsilon \, C_0} [-,A]$.</u>

$E(Y) = \bar{Y} = \bigcup_{D \, \varepsilon \, C_0} Y(D) = \bigcup_{D \, \varepsilon \, C_0} \text{dom}^{-1}\{D\} = C_1$

$p_Y \colon C_1 \longrightarrow C_0$ is given by $p_Y(f) = B$ iff $f \; \varepsilon \; \text{dom}^{-1}\{B\}$

thus $p_Y = \text{dom}$,

$Pr(Y) = (C_1, \prec_Y)$ where for any f, g in C_1

$f \prec_Y g$ iff $(\exists k)(Y(k)(f) = g)$

iff $(\exists k)(f \cdot k = g)$.

1.2.2.6 The subobject classifier in $\mathcal{S}^{\mathcal{C}^{0}}$.

Define Ω to be the composite

$$\begin{array}{ccc} \mathcal{C}^{0} & \xrightarrow{\quad \underline{H} \quad} & \underline{Heyt} \\ \big\downarrow{\scriptstyle [-,]^{0}} & & \big\uparrow{\scriptstyle \underline{Idl}} \\ (\mathcal{S}^{\mathcal{C}^{0}})^{0} & \xrightarrow{\quad Pr^{0} \quad} & (\underline{h\text{-}PreOrd})^{0} \end{array}$$

For any $f: B \longrightarrow A$

$$(f\cdot -)^{*}: \underline{Idl}(P_A, \prec_A) \longrightarrow \underline{Idl}(P_B, \prec_B)$$

is a Heyting algebra homomorphism , thus - as we have noted in 1.2.1 - any C_0-indexed family of Heyting algebra operations constitutes a natural transformation, and thus an operation of Ω.

In particular $\{A\} \xrightarrow{true_A} Idl(P_A, \prec_A)$ given by $true_A(A) = P_A$ for each $A \in C_0$, lifts to a morphism $1 \xrightarrow{true} \Omega$.

The pullback of any $F \longrightarrow \Omega$ along true picks out a subpresheaf K of F.

At each $A \in C_0$ (1) $K(A) = \theta_A{}^{*}\{P_A\}$.

Thus $\psi: [F, \Omega] \longrightarrow Sub(F)$ is defined

by (2) $\psi(\theta) = K$ where K is given by (1).

1.2.2.6.1 Proposition. $\psi: [F, \Omega] \longrightarrow Sub\ F$ is a bijection.

Proof. To show that ψ is injective we will show that θ is uniquely determined by (2). The naturality of θ implies that for each $f: B \longrightarrow A$, (3) commutes, and hence that for each $x \in FA$, each C, and each $g: C \longrightarrow B$,

(4) $g \in \theta_B(F(f)(x))$ iff $f \cdot g \in \theta_A(x)$,

hence for $g = id_B$ we have

(5) $(id_B \in \theta_B(F(f)(x))$ iff $(f \in \theta_A(x))$

But $\theta_B(F(f)(x))$ is an ideal of $(P_B, \prec_B)$, so $K(B) = P_B = \theta_B(F(f)(x))$, thus

(6) $\theta_A(x) = \{f \mid (\text{cod } f = A) \text{ and } (F(f)(x) \in K(\text{dom } f))\}$.

Thus θ is uniquely determined by (2).

To show that ψ is surjective we show that for any $K \hookrightarrow F$,

(a): (6) defines an ideal of $(P_A, \prec_A)$, (b): the family $(\theta_A)_{A \in C_0}$ defined by (6) is a natural transformation from F to Ω, and

(c): $\psi(\theta) = K$.

(a): Let $f \in \theta_A(x)$, $B = \text{dom } f$, and $g: C \longrightarrow B$.

From (6), $F(f)(x) \in K(B)$, and since $K \hookrightarrow F$, $F(g)(F(f)(x)) \in K(C)$.

Hence $(\text{cod}(f \bullet g) = A)$ and $(F(f \bullet g)(x) \in K(C))$, hence $f \bullet g \in \theta_A(x)$.

Thus $\theta_A(x)$ is an ideal of $(P_A, \prec_A)$.

(b): Let $f: B \longrightarrow A$, $x \in FA$, $g: C \longrightarrow B$. We show (4) holds

$(f \bullet g \in \theta_A(x))$ iff both $(\text{cod}(f \bullet g) = A)$ and $(F(f \bullet g)(x) \in K(\text{dom}(f \bullet g)))$

iff both $(\text{cod } g = B)$ and $(F(g)(F(f)(x)) \in K(\text{dom } g))$ iff $g \in \theta_B(F(f)(x))$.

(c) Let $x \in F(A)$. We show (1) holds. $x \in \theta_A^{-1}\{P_A\}$ iff $(id_A \in \theta_A(x))$ iff both $(\text{cod } id_A = A)$ and $(F(id_A)(x) \in K(\text{dom } id_A))$ iff $(x \in K(A))$.□

<u>1.2.2.7 Ideals of $\underline{C}$</u>. We shall show in 1.2.2.7.2 that the validity of $\varphi \in$ Poly H, interpreted as a formula $\bar{\varphi}$ of the language of the presheaf topos $\underline{\mathcal{S}}^{\underline{C}^0}$, is equivalent to the external validity of φ in the algebra $\underset{\sim}{\text{Idl}}(C_1, \prec)$ of ideals of $\underline{C}$.

There are two contrasting ways of viewing the algebra $\underset{\sim}{\text{Idl}}(C_1, \prec)$. From 1.2.2.4.3 we have an H-isomorphism

(1) $\underset{\sim}{\text{Sub}}(Y) \approx \underset{\sim}{\text{Idl}}(C_1,\prec)$

which associates to each ideal U of $(C_1,\prec)$ the subpresheaf

$\tilde{U} \hookrightarrow Y$, given by

(1)' $\tilde{U}(A) = U \cap \text{dom}^{-1}\{A\}$ for each $A \in C_0$.

We can arrive at a second isomorphism using (1) as follows

$\underset{\sim}{\text{Sub}}(Y) \approx [Y,\underset{\sim}{\Omega}]$ by 1.2.2.6.1

$\approx \prod_{A \in C_0} \underset{\sim}{\Omega}(A)$ by 1.2.2.5.2

Alternately we can give a more direct description of the relationship between $\underset{\sim}{\text{Idl}}(C_1,\prec)$ and $\prod_{A \in C} \underset{\sim}{\Omega}(A)$:

1.2.2.7.1 Proposition. There is an H-isomorphism

(2) $\underset{\sim}{\text{Idl}}(C_1,\prec) \doteq \prod_{A \in C_0} \underset{\sim}{\text{Idl}}(P_A,\prec_A)$

which maps each ideal U of $(C_1,\prec)$ to the C_0-indexed family of ideals whose A-th component U_A is given by

(2)' $U_A = U \cap \text{cod}^{-1}\{A\}$ for each $A \in C_0$.

Proof. For each $A \in C_0$, the natural injection $i_A: [-,A] \hookrightarrow Y$ in $\mathcal{S}^{\underline{C}^0}$, induces an h-map $(P_A,\prec_A) \xrightarrow{\bar{i}_A} (C_1,\prec)$, where $Pr([-,A]) = (P_A,\prec_A)$ and $Pr(Y) = (C_1,\prec)$ and $\bar{i}_A(f) = f$ for each $f \in P_A$. If U is an ideal of $(C_1,\prec)$ then its inverse image under $\bar{i}_A{}^*$ is $\bar{i}_A{}^*(U) = U \cap P_A$. The functor $\underset{\sim}{\text{Idl}}$: h-$\underline{\text{PreOrd}} \longrightarrow \underline{\mathit{Heyt}}$ applied to $\bar{i}_A$, gives an onto Heyting algebra homomorphism $\bar{i}_A{}^*: \underset{\sim}{\text{Idl}}(C_1,\prec) \longrightarrow \underset{\sim}{\text{Idl}}(P_A,\prec_A)$. We lift the family $\bar{i}_A{}^*(A \in C_0)$ to a single H-homomorphism

$$\theta: \underset{\sim}{\text{Idl}}(C_1,\prec) \longrightarrow \prod_{A \in C_0} \underset{\sim}{\text{Idl}}(P_A,\prec_A)$$

given by $\theta(U) = (U \cap P_A)_{A \in C_0}$.

Since $(C_1, \prec) = \coprod_{A \in C_0} (P_A, \prec_A)$, as a coproduct in PreOrd, by 1.2.2.1.2, θ is a bijection. Since θ is also an H-homomorphism, θ is an H-isomorphism.

1.2.2.7.2 Corollary. Let $\varphi \in \text{Poly } H$ and let $\overline{\varphi}$ be its interpretation in $\underset{\sim}{\Omega}$ in $\mathcal{S}^{\underline{C}^0}$. The following are equivalent

(1) $\mathcal{S}^{\underline{C}^0} \models \overline{\varphi}$

(2) $\underset{\sim}{\Omega} \models \varphi = \underline{1}$

(3) $\underset{\sim}{\text{Idl}}(P_A, \prec_A) \vdash \varphi$ for each $A \in C_0$

(4) $\underset{\sim}{\text{Idl}}(C_1, \prec) \vdash \varphi$

Proof. (1) $\leftrightarrow$ (2): From the definition of internal validity.□

(2) $\leftrightarrow$ (3): By 1.2.1.2.3, $\underset{\sim}{\Omega} \models \varphi = \underline{1}$ iff for all $A \in C_0$, $\underset{\sim}{\Omega}(A) \models \varphi = \underline{1}$.□

(3) $\leftrightarrow$ (4): By 1.2.2.7.1, (4) is equivalent to

(4)' $\prod_{A \in C_0} \underset{\sim}{\text{Idl}}(P_A, \prec_A) \vdash \varphi$. Each factor is non-empty and hence is a homomorphic image of the product, hence (4)' holds iff $\underset{\sim}{\text{Idl}}(P_A, \prec_A) \vdash \varphi$ for each $A \in C_0$.□

1.2.2.8 The ideal N of non-retractions of $\underline{C}$.

A morphism f of $\underline{C}$ with codomain A is a retraction if there exists $h \in C_1$ such that $f \cdot h = id_A$. The set $Retr(\underline{C})$ of retractions is right cancellative:

$$((f \cdot g) \cdot h = id_A) \to (f \cdot (g \cdot h) = id_A)$$

if $f \cdot g \in Retr(\underline{C})$ then $f \in Retr(\underline{C})$. Let $N(\underline{C}) = (C_1 \setminus Retr(\underline{C})$, the set of non-retractions of $\underline{C}$, then by contraposition, $N(\underline{C})$ is closed under composition on the right, in other words $N(\underline{C})$ is an ideal of $(C_1, \prec_Y)$. We abbreviate: $N = N(\underline{C})$. Under the above isomorphisms, we have

(1) $\underset{\sim}{Idl}(C_1, \prec_Y) \cong \underset{\sim}{Sub}(Y)$, $\tilde{N} \hookrightarrow Y$ where $\tilde{N}(A) = N \cap dom^{-1}\{A\}$, and projections:

(2) $\underset{\sim}{Idl}(C_1, \prec_Y) \xrightarrow{i_A^*} \underset{\sim}{\Omega}(A)$, where $N_A = N \cap P_A$ is an element of $\Omega(A)$.

1.2.2.8.1 Proposition. For each $A \in C_0$, N_A is the penultimate element of $\underset{\sim}{\Omega}(A)$.

Proof. The identity morphism id_A is a retraction, hence N_A is a proper ideal of $(P_A, \prec_{[-,A]})$: $N_A \subsetneqq P_A$. Consider any proper ideal U. If U contained a retraction f, then $f \cdot h = id_A$ for some h, so $id_A \in U$, but then $P_A = [id_A, \to) \subset U \subset P_A$ -a contradiction. Hence if U is a proper ideal, $U \subset N_A$. So N_A is penultimate.□

1.2.2.8.2 Notation. We let L_A be the set of U in $\Omega(A)$ contained in N_A, we let $\mathcal{L}_A$ be the corresponding partially ordered set with order induced from $\underset{\sim}{\Omega}(A)$ and we let $\underset{\sim}{\mathcal{L}}_A$ be the Heyting algebra structure defined via the quotient $\underset{\sim}{\Omega}(A) \twoheadrightarrow \underset{\sim}{\mathcal{L}}_A$; $U \rightsquigarrow (U \cap N_A)$, homomoprhism. We have

$\Omega_{\leq}(A) \approx \mathcal{L}_A \oplus 1$ and $\underset{\sim}{\Omega}(A) \approx \underline{\mathcal{L}}_A \oplus 1$.

1.2.2.9 Global subpresheaves.

By a global subpresheaf $G \hookrightarrow F$ we mean a subpresheaf for which the characteristic map $\chi_G: F \longrightarrow \Omega$ factors through $\mathbb{1}$. There is just one map $u: F \longrightarrow \mathbb{1}$ so the factorization depends on the choice of $\mathbb{1} \longrightarrow \Omega$. For F disjoint, $u: F \longrightarrow \mathbb{1}$ induces Heyting algebra homomorphisms:

$$\begin{array}{ccc}
[\mathbb{1},\underline{\Omega}] & \xrightarrow{[u,\Omega]} & [F,\Omega] \\
\downarrow\wr & & \downarrow\wr \\
\underline{\text{Sub}}\,\mathbb{1} & \xrightarrow{u^{-1}} & \underline{\text{Sub}}\,F \\
\downarrow\wr & & \downarrow\wr \\
\underline{\text{Idl}}(C_0, \prec_1) & \xrightarrow{\bar{u}^*} & \underline{\text{Idl}}(F, \prec_F)
\end{array}$$

the global subpresheaves are those $G \hookrightarrow F$ in the image of u^{-1}.

1.2.2.9.1 Proposition. Let F be a disjoint presheaf and let $G \hookrightarrow F$. G is a global subpresheaf of F iff for all $A \in C_0$, if $G(A) \neq \phi$ then $G(A) = F(A)$.

Proof. $u: F \longrightarrow \mathbb{1}$ induces an adjoint pair $\exists_u \dashv u^{-1}$

$$\text{Sub}\,\mathbb{1}_{\leq} \underset{\exists_u}{\overset{u^{-1}}{\rightleftarrows}} \text{Sub}\,F_{\leq}$$

For any $W \hookrightarrow \mathbb{1}$ we have $u^{-1}\exists_u u^{-1}(W) = u^{-1}(W)$.

Thus $G \hookrightarrow F$ is global iff for some $W \hookrightarrow \mathbb{1}$, $G = u^{-1}(W)$ iff $u^{-1}\exists_u G = G$. But $G \leq u^{-1}\exists_u(G)$ in Sub $F_{\leq}$ is always true. Thus $G \hookrightarrow F$ is global iff $u^{-1}\exists_u(G) \leq G$

iff $(\forall B)(u_B^{-1}(\exists_{u_B}(G(B))) \subset G(B))$

iff $(\forall B)(\forall x)((u_B(x) \in \exists_{u_B}(G(B))) \rightarrow (x \in G(B)))$

iff $(\forall B)(\forall x)((\exists y)((y \in G(B))$ and $(u_B(y) = u_B(x))) \rightarrow (x \in G(B)))$

iff $(\forall B)(\forall x)((G(B) \neq \phi) \rightarrow (x \in G(B)))$

iff $(\forall B)((G(B) \neq \phi) \rightarrow (G(B) = F(B)))$.□

<u>1.2.2.9.2 Corollary</u>. $G \hookrightarrow Y$ is global iff for all $A \in C_0$ and all $f \in C_1$ with domain A, if $f \in G(A)$ then $(G(A) = \mathrm{dom}^{-1}\{A\})$.□

<u>1.2.2.9.3 Proposition.</u> $\bar{N} \hookrightarrow Y$ is global iff $N = \phi$.

<u>Proof</u>. (←) if $N = \phi$ then $\bar{N} = 0$. (→) Suppose $\bar{N}$ is global but that $N \neq \phi$; that is there exists $f \in N$ which is not a retraction; let $A = \mathrm{dom}(f)$, $f \in \bar{N}(A)$ hence $\bar{N}(A) = \mathrm{dom}^{-1}\{A\}$, hence $id_A \in \bar{N}(A)$ - which is false. Hence $N = \phi$. □

<u>1.2.2.9.4 Proposition.</u> If $C_1 = \mathrm{Retr}(\underline{C}) \cup \mathrm{Retr}(\underline{C}^{\circ})$ then $N(\underline{C}) = \phi$.

<u>Proof.</u> Suppose to the contrary that $f \in N(\underline{C})$; let $A = \mathrm{dom}\ f$ and $B = \mathrm{cod}\ f$. Since $f \in \mathrm{Retr}(\underline{C}^{\circ})$, for some $h: B \longrightarrow A$, $h \cdot f = id_A$. If $f \cdot h \in \mathrm{Retr}(\underline{C})$ then for some g, $f \cdot h \cdot g = id_B$; but then $f \notin N(\underline{C})$. If $f \cdot h \in \mathrm{Retr}(\underline{C}^{\circ})$ then for some g, $g \cdot f \cdot h = id_B$, hence $g \cdot f = g \cdot f \cdot h \cdot f = f$, hence $f \cdot h = id_B$, so $f \notin N(\underline{C})$.□

1.2.3 Examples of presheaf toposes and their propositional logic.

1.2.3.0 Summary. In 1.2.3.1 we explain how the i.p.l.'s $L(\underline{\mathcal{S}}^{\underline{C}^0})$ for $\underline{C}$ finite and non empty coincide with the finite logics - that is, those propositional logics determined by some finite non-trivial Heyting algebra. In contrast to these toposes are toposes $\underline{\mathcal{E}}$ for which $L(\underline{\mathcal{E}}) = IL$; we shall provide examples of such toposes in 1.2.3.2. There are interesting examples of presheaf toposes $\underline{\mathcal{S}}^{\underline{C}^0}$ where $\underline{C}$ is not a monoid but where $\underline{\mathcal{S}}^{\underline{C}^0}$ is equivalent to $\underline{M} - \underline{\mathcal{S}}$ for some monoid $\underline{M}$; the latter presentation allows a simplification of calculations; we shall show in 1.2.3.3 how such a reduction is possible. In 1.2.3.4 we introduce two sequences of non-Boolean presheaf toposes I: $\dot{\underline{\mathcal{E}}}_n$ $(n \in N^+)$ and II: $\underline{\mathcal{E}}_n$ $(n \in N^+)$ such that $L(\dot{\underline{\mathcal{E}}}_n) = L(\underline{\mathcal{E}}_n) = \{\varphi \mid E_n \wedge \Delta^2(\underline{0}) \vdash \varphi\}$ for each $n \in N^+$. In 1.2.3.4.4 we characterize $\underline{C}$ for which $E_n \in L(\underline{\mathcal{S}}^{\underline{C}^0})$ and for which $\Delta^2(\underline{0}) \in L(\underline{\mathcal{S}}^{\underline{C}^0})$. In 1.2.3.5 we introduce another sequence $\underline{\mathcal{G}}_n$ $(n \in N^+)$, for which $L(\underline{\mathcal{G}}_n) = \{\varphi \mid R_{n+1} \vdash \varphi\}$, and in 1.2.3.5.1 we characterize $\underline{C}$ for which $R_n \in L(\underline{\mathcal{S}}^{\underline{C}^0})$.

In 1.2.3.6 we begin an examination of other logical criteria that go beyond that part of the propositional logic which is built up from $\vee$, $\wedge$, $\Rightarrow$ and $\perp$. The criteria will allow us to distinguish $\dot{\underline{\mathcal{E}}}_n$ from $\underline{\mathcal{E}}_n$. We will show that for I, $\dot{\underline{\mathcal{E}}}_n \models \neg \forall p(p \vee \neg p)$, and for II, $\underline{\mathcal{E}}_n \models \neg \neg \forall p(p \vee \neg p)$ and $\urcorner [\![\forall p(p \vee \neg p)]\!] = [\![0]\!]$ (where $\urcorner$ is the dual pseudocomplement defined in 1.2.2.1.7). These properties of I and II will be used in 1.6 and 1.7.

1.2.3.1 Finite toposes.

A topos is a finite topos ([AGV]I, Exposé IV, Exercice 9.1.12) if it is equivalent to a presheaf topos $\mathcal{S}^{\underline{C}^{o}}$ with $\underline{C}$ a finite category. The Heyting algebra $\mathcal{A} = \underline{\mathrm{Idl}}(C_1,\prec)$ of ideals of a finite category $\underline{C}$, where $\prec$ is the preorder of divisibility on the morphism set C_1, is again finite; hence $L(\mathcal{S}^{\underline{C}^{o}}) = \{\varphi \mid \mathcal{A} \vdash \varphi\}$ is a finite logic in the sense of McKay [McK], or a tabular logic (having a "finite truth table") in the sense of Maksimova [Ma1].

The paper [Gö], mentioned in the context of "slicing" of i.p.l.'s (1.1.3.1), established that IL is not tabular as follows: For $\mathcal{A}$ a finite algebra of cardinality n, the formula $F_n = \bigvee_{0 \leq i < j \leq n} (x_i \leftrightarrow x_j)$ is valid on $\mathcal{A}$ since for any sequence of elements of $\mathcal{A}$ of length n + 1, two different members must be equal elements. On the other hand F_n fails on the n + 1 element chain S_n, since $\bigvee_{0 \leq i < j \leq n} (e_i \leftrightarrow e_j) = \bigvee_{1 \leq j \leq n} e_j = e_1 < 1$ (using again the notation of [Ho 3]). Thus $\mathcal{A} \vdash F_n$ but $F_n \notin$ IL. Hence for a finite topos $\mathcal{S}^{\underline{C}^{o}}$, there exists n such that $F_n \in L(\mathcal{S}^{\underline{C}^{o}})$ and $F_n \notin$ IL. In [McK] and in [Ba] it is further established (in [McK] with the aid of F_n) that for a finite algebra $\mathcal{A}$ there is a polynomial $\psi_{\mathcal{A}}$ such that $\{\varphi \mid \mathcal{A} \vdash \varphi\} = \{\varphi \mid \psi_{\mathcal{A}} \vdash \varphi\}$. We can thus sharpen our description of the propositional logic of a finite topos $\mathcal{Q}$ to: there exists a finite algebra $\mathcal{A}$ and a polynomial $\psi \notin$ IL such that

$$L(\mathcal{Q}) = \{\varphi \mid \mathcal{A} \vdash \varphi\} = \{\varphi \mid \psi \vdash \varphi\}.$$

Conversely we can show that every tabular logic arises as the propositional logic of some finite topos. Let $\mathcal{A}$ be a finite Heyting algebra, let P be the set of meet irreducible elements of $\mathcal{A}$ and let $\underline{P} = (P,\leq)$ be the poset with ordering induced from $\mathcal{A}$. The two maps $\sigma\colon |\mathcal{A}| \longrightarrow \mathrm{Idl}(\underline{P})$ and $f\colon \mathrm{Idl}(\underline{P}) \longrightarrow |\mathcal{A}|$ defined by $\sigma(x) = \{p \in P \mid x \leq p\}$ and $f(U) = \bigwedge U$ establish a lattice isomorphism ([Gr 2]p. 61) and thus a Heyting algebra isomorphism:

$$\underset{\sim}{\mathrm{Idl}}(\underline{P}) \approx \mathcal{A}.$$

Define $\underline{C}$, as in 1.7.2.1, to be the category with objects $C_0 = P$ and morphisms $C_1 = \{(q,p) \in P^2 \mid p \leq q\}$. Then, by 1.7.2.2.9, for all $\varphi \in$ Poly H, $\underset{\sim}{\mathrm{Idl}}(C_1,<) \vdash \varphi$ iff $\mathrm{Idl}(\underline{P}) \vdash \varphi$. Thus $L(\mathcal{S}^{\underline{C}^0}) = \{\varphi \mid \mathcal{A} \vdash \varphi\}$.

<u>1.2.3.2 Toposes for which $L(\mathcal{E}) = \underline{\mathrm{IL}}$.</u> Toposes for which <u>only</u> intuitionistic propositional tautologies hold include Sh(ℝ). We will briefly discuss such toposes.

<u>1.2.3.2.1 The Jaskowski Sequence.</u> [Ro] . By the Jaskowski sequence we mean the sequence $\{\underline{J}_n \mid n \in N\}$ of finite Heyting algebras defined as follows

$$\underline{J}_0 = \mathbf{2} = S_1 \quad , \quad \underline{J}_{k+1} = \underline{J}_k^{\,k+1} \oplus 1.$$

In [Ro] it is shown that $\varphi \in$ IL iff $\{\underline{J}_n \mid n \in \mathbb{N}\} \vdash \varphi$, thus $\mathcal{H}eyt = \{\underline{J}_n \mid n \in \mathbb{N}\}^e$.

By the duality between finite posets and finite distributive lattices ([Gr 2]p.61) we have, for $\underline{P}_k$ the poset of meet irreducible elements of $\underline{J}_k$, $\underline{J}_k \approx \mathbf{2}^{\underline{P}_k} \approx \underset{\sim}{\mathrm{Idl}}\ P_k$.

Let $\underline{P}$ be the coproduct of the $\underline{P}_k$ ($k \in N$), then

$\underset{\sim}{\text{Idl}}\ \underline{P} = \underset{\sim}{\text{Idl}} \coprod_k \underline{P}_k \approx \prod_k \underline{J}_k$. Thus

$$\varphi \in \text{IL} \text{ iff } \underset{\sim}{\text{Idl}}\ \underline{P} \vdash \varphi.$$

If $\underline{C}$ is the category associated with $\underline{P}$ then by 1.7.2.2.9 and 1.6.2.5 (their external versions)

$$\varphi \in L(\underline{\mathcal{S}}^{\underline{C}^0}) \text{ iff } \underset{\sim}{\text{Idl}}\ \underline{P} \vdash \varphi ,$$

hence $L(\underline{\mathcal{S}}^{\underline{C}^0}) = \text{IL}$. This example can be turned into one of sheaves over a topological space: $\text{Sh}(\underline{P}) \approx \underline{\mathcal{S}}^{\underline{C}^0}$ by taking $\underline{P}$ to have P as its underlying set and $\text{Idl}(\underline{P})_{\underline{\leq}}$ as open sets. We pass to more natural topological examples.

<u>1.2.3.2.2 Spaces for which $L(\text{Sh}(\underline{X})) = \text{IL}$.</u> From the three articles by McKinsey and Tarski [Mc & T-i] (i = 1,2,3), it follows that if $\underline{X} = (X,\mathcal{O})$ is normal, dense-in-itself and has a countable base then for each finite Heyting algebra $\mathcal{A}$ there exists an open set U such that $\mathcal{A}$ can be embedded in the Heyting algebra $\mathcal{O}|U$ of open subsets of U. In particular for each member of the Jaskowski sequence $\underline{J}_k$ there is an open U_k such that $\underline{J}_k$ can be embedded in $\mathcal{O}|U_k$. Thus $\varphi \in \text{IL}$ iff for all $U \in \mathcal{O}$ $(\mathcal{O}|U \vdash \varphi)$. The subobject classifier with its algebraic structure, $\underset{\sim}{\Omega}$, for $\text{Sh}(\underline{X})$ is given by $\underset{\sim}{\Omega}(U) = \mathcal{O}|U$ for all $U \in \mathcal{O}$ thus $L(\text{Sh}(\underline{X})) = \text{IL}$.

<u>1.2.3.3 Splitting idempotents.</u>

The theory developed herein is designed to capture a particular example of a presheaf topos relevant details of which we present first.

<u>1.2.3.3.1 n-truncated simplicial sets.</u>

Let $\underline{\Delta}$ be the category having as objects all finite totally ordered sets $[n] = \{0,1,\ldots,n-1\}$ where $n \in \mathbb{N}^+$, and as morphisms all order preserving functions $f\colon [n] \longrightarrow [m]$: for all $i, j \in [n]$ if $i \leq j$ then $f(i) \leq f(j)$. $\underline{\Delta}_n$ is the full subcategory of $\underline{\Delta}$ with objects all $[k]$ such that $k \leq n$. $\underline{\text{End}}\ [n]$ is the full subcategory of $\underline{\Delta}_n$ whose only object is $[n]$.

We claim that each $[k]$ in $\underline{\Delta}_n$ is a retract of $[n]$. For each $k \leq n$ define $s_k\colon [k] \rightarrowtail [n]$ by $s_k(i) = i$ for all $i \in [k]$, and define $r_k\colon [n] \twoheadrightarrow [k]$ by

$$r_k(i) = \begin{cases} i & \text{if } i \leq k-1 \\ k-1 & \text{if } k-1 \leq i \end{cases}$$

then both s_k and r_k are maps of $\underline{\Delta}_n$ and $r_k \cdot s_k = id_{[k]}$.

We shall show that this property of $\underline{\Delta}_n$ implies that the inclusion functor

$$J\colon (\underline{\text{End}}\ [n])^0 \hookrightarrow (\underline{\Delta}_n)^0$$

induces an equivalence of categories

$$J^*\colon \underline{\mathcal{S}}^{(\underline{\Delta}_n)^0} \xrightarrow{\ \approx\ } \underline{\mathcal{S}}^{(\underline{\text{End}}\ [n])^0}.$$

This gives us two ways of viewing essentially the same topos. The category $\underline{\mathcal{S}}^{(\underline{\Delta}_n)^0}$ is the category of n-truncated simplicial sets, the objects of which can be presented as k-simplices for each dimension $k \leq n$ together with face and degeneracy maps. The codomain of J^* is a variety of unary algebras with operations from $\underline{\text{End}}\ [n]$.

<u>1.2.3.3.2</u> Although we shall apply the theory only to monoids the situation is actually more general and there is no advantage to restricting ourselves to that special case. The general situation is as follows: $J\colon \underline{R} \hookrightarrow \underline{C}$ is a full inclusion functor such that for each $A \in |\underline{C}|$ there exists $B \in |\underline{R}|$ such that A is a retract of B. We call such a functor a

retractor. $J: \underline{R}^0 \hookrightarrow \underline{C}^0$ induces, by composition, a functor

$J^*: [\underline{C}^0, \underline{S}] \longrightarrow [\underline{R}^0, \underline{S}]$, $J^*(F) \equiv F \cdot J$, which restricts to

$$J^*: \underline{S}^{\underline{C}^0} \longrightarrow \underline{S}^{\underline{R}^0}.$$

We shall show that J^* is full, faithful and surjective on objects (which are disjoint presheaves).

1.2.3.3.3 Choosing retractions.

Let $\underline{R} \hookrightarrow \underline{C}$ be a retractor. Let ρ be a choice function that assigns to each $A \in (|\underline{C}| - |\underline{R}|)$ a triple $\rho(A) \equiv (\tilde{A}, s_A, r_A)$ such that $\tilde{A} \in |\underline{R}|$, dom $r_A \equiv$ cod $s_A \equiv \tilde{A}$, and $r_A \circ s_A \equiv id_A$. We extend ρ to all of $|\underline{C}|$ by putting $\rho(B) \equiv (B, id_B, id_B)$ for each $B \in |\underline{R}|$. We call ρ a choice of retractions for $\underline{R} \hookrightarrow \underline{C}$, and fix ρ throughout.

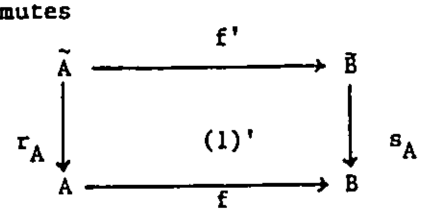
1.2.3.3.3.1 The semifunctor induced by ρ.

We define a function of morphisms $\bar{\rho}: \underline{C} \longrightarrow \underline{C}$, abbreviating values to $f' \equiv \bar{\rho}(f)$ for all morphisms f.

Given $f: A \longrightarrow B$ in $\underline{C}$ we put

(1) $\quad f' \equiv s \circ f \circ r$

so that (1)' commutes



as a consequence we have

(2) $r_B \circ f' \equiv f \circ r_A$, (3) $f' \circ s_A \equiv s_B \circ f$, and (4) $f \equiv r_B \circ f' \circ s_A$.

1.2.3.3.3.2 Proposition. The function $\bar{\rho}: \underline{C} \longrightarrow \underline{C}$ preserves composition.

Proof. Given

$$\begin{array}{ccc} \tilde{A} & \xrightarrow{f} & B \\ & {\scriptstyle g\circ f} \searrow & \downarrow {\scriptstyle g} \\ & & C \end{array}$$

in $\underline{C}$ we have $g'\circ f' = s_C\circ g\circ r_B\circ s_B\circ f\circ r_A = s_C\circ g\circ f\circ r_A = (g\circ f)'$.□

<u>1.2.3.3.3.3 Idempotents</u>. For each $A \in |\underline{C}|$ we let $\theta_A\colon \tilde{A} \longrightarrow \tilde{A}$ be given by

$$\theta_A = (id_A)' = s_A\circ r_A \qquad (5)$$

so that $\theta_A\circ\theta_A = \theta_A$. For any functor $W\colon \underline{R}^0 \longrightarrow \underline{\mathcal{S}}$, $W(\theta_A)$ will be an idempotent. It is this function which we will "split".

<u>1.2.3.3.4 Proposition</u>. $J^*\colon \underline{\mathcal{S}}^{\underline{C}^0} \longrightarrow \underline{\mathcal{S}}^{\underline{R}^0}$ is full and faithful if $J\colon \underline{R} \hookrightarrow \underline{C}$ is a retractor.

<u>Proof</u>. Given F, G in $\underline{\mathcal{S}}^{\underline{C}^0}$ we shall show that J^* maps $[F,G]$ one-to-one onto $[J^*F,J^*G]$. Let $\alpha\colon F\circ J \longrightarrow G\circ J$; for each $h\colon E \longrightarrow D$ in $\underline{R}$ we have $G(h)\circ\alpha_D = \alpha_E\circ F(h)$

$$\begin{array}{ccc} FE & \xrightarrow{\alpha_E} & GE \\ {\scriptstyle Fh}\downarrow & (1) & \downarrow{\scriptstyle Gh} \\ Fd & \xrightarrow[\alpha_D]{} & GD \end{array}$$

Suppose $\beta\colon F \longrightarrow G$ is such that $J^*(\beta) = \alpha$; that is for each $E \in |\underline{R}|$, $\beta_E = \alpha_E$. By the naturality of β, for each $A \in |\underline{C}|$, (2) commutes

$$\begin{array}{ccc} F\tilde{A} & \xrightarrow{\alpha_{\tilde{A}}} & G\tilde{A} \\ {\scriptstyle Fs_A}\downarrow & (2) & \downarrow{\scriptstyle Gs_A} \\ FA & \xrightarrow[\beta_A]{} & GA \end{array}$$

hence $\beta_A = \beta_A \circ id_{FA} = \beta_A \circ F(s_A) \circ F(r_A) = G(s_A) \circ \alpha_{\tilde{A}} \circ F(r_A)$.

Thus J^* is faithful.

To show J^* is full it suffices to show that β, given by

(3) $\beta_A = G(s_A) \circ \alpha_{\tilde{A}} \circ F(r_A)$ for each $A \in |\underline{C}|$,

is natural; that is, for each $f: A \longrightarrow B$, (4) commutes

$$\begin{array}{ccc} FA & \xrightarrow{\ \beta_A\ } & GA \\ {\scriptstyle Ff}\uparrow & (4) & \uparrow{\scriptstyle Gf} \\ FB & \xrightarrow[\ \beta_B\ ]{} & GB \end{array}$$

$$\begin{aligned} G(f) \circ \beta_B &= G(f) \circ G(s_B) \circ \alpha_{\tilde{B}} \circ F(r_B) \\ &= G(s_B \circ f) \circ \alpha_{\tilde{B}} \circ F(r_B) \\ &= C(f' \circ s_A) \circ \alpha_{\tilde{B}} \circ F(r_B) && \text{by (3)} \\ &= G(s_A) \circ G(f') \circ \alpha_{\tilde{B}} \circ F(r_B) \\ &= G(s_A) \circ \alpha_{\tilde{A}} \circ F(f') \circ F(r_B) \\ &= G(s_A) \circ \alpha_{\tilde{A}} \circ F(r_A) \circ F(f) && \text{by (2)} \\ &= \beta_A \circ F(f). \end{aligned}$$

Hence J^* is full.□

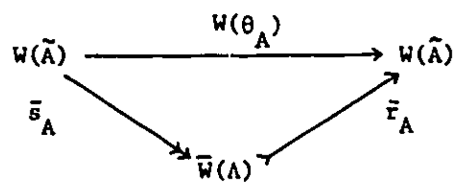
<u>1.2.3.3.5 Proposition.</u> $J^*: \mathcal{S}^{\underline{C}^0} \longrightarrow \mathcal{S}^{\underline{R}^0}$ is surjective if $J: \underline{R}^0 \hookrightarrow \underline{C}^0$ is a retractor.

<u>Proof</u>. Given $W: \underline{R}^0 \longrightarrow \underline{\mathcal{S}}$ we shall construct $\overline{W}: \underline{C}^0 \longrightarrow \underline{\mathcal{S}}$ such that $J^*(\overline{W}) = W$. For each $A \in \underline{C}$ we let

$$\overline{W}(A) = \{x \in W(\tilde{A}) \mid \theta_A(x) = x\}.$$

For $B \in |\underline{R}|$ we have $\tilde{B} = B$ and $\theta_B = id_B$, hence $W(\overline{B}) = W(B)$.

For $A \in |\underline{C}|$, $\theta_A = s_A \circ r_A : \tilde{A} \longrightarrow \tilde{A}$ is an idempotent morphism in $\underline{R}$, hence $W(\theta_A): W(\tilde{A}) \longrightarrow W(\tilde{A})$ is also idempotent. We define $\bar{r}_A: \bar{W}(A) \hookrightarrow W(\tilde{A})$ to be the inclusion function. Then $W(\theta_A)$ splits through $\bar{r}_A$:



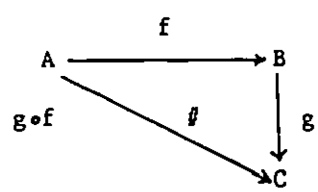
$\bar{s}_A$ is given by $\bar{s}_A(y) = W(\theta_A)(y)$ for $y \in W(\tilde{A})$. Hence $\bar{s}_A \circ \bar{r}_A = id_{\bar{W}(A)}$ and $\bar{r}_A \circ \bar{s}_A = W(\theta_A)$; and for $B \in |\underline{R}|$, $\bar{s}_B = \bar{r}_B = id_{W(B)}$.

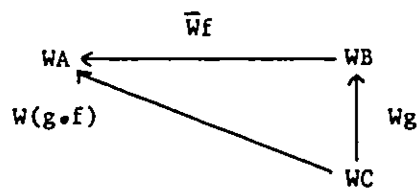
For each $f: A \longrightarrow B$ in $\underline{C}^0$ we put $\bar{W}(f) = \bar{s}_A \circ W(f') \circ \bar{r}_B$.

$\bar{W}$ preserves identity morphisms:

$$
\begin{aligned}
W(id_A) &= \bar{s}_A \circ W((id_A)') \circ \bar{r}_A \\
&= \bar{s}_A \circ W(\theta_A) \circ \bar{r}_A \\
&= \bar{s}_A \circ \bar{r}_A \circ \bar{s}_A \circ \bar{r}_A = id_{\bar{W}(A)} .
\end{aligned}
$$

$\bar{W}$ preserves composition





$$
\begin{aligned}
\bar{W}(f) \circ \bar{W}(g) &= \bar{s}_A \circ W(f') \circ \bar{r}_B \circ \bar{s}_B \circ W(g') \circ \bar{r}_C \\
&= \bar{s}_A \circ W(f') \circ W((id_A)') \circ W(g') \circ \bar{r}_C \\
&= \bar{s}_A \circ W((g \circ f)') \circ r_C \\
&= \bar{W}(g \circ f).
\end{aligned}
$$

Thus $\bar{W}: \underline{C}^0 \longrightarrow \underline{\mathcal{S}}$ is a functor.

For $f: A \longrightarrow B$ in $\underline{R}$ we have

$W(f) = \bar{s}_A \bullet W(f') \bullet \bar{r}_B = W(F')$

$= W(s_B \bullet f \bullet r_A) = W(f)$.

This establishes that $J^*(\bar{W}) = W$.

Now let $\eta: W_1 \longrightarrow W_2$ in $\underline{\mathcal{S}}^{\underline{R}^0}$. We have $J^*(\bar{W}_1) = W_1$ and $J^*(\bar{W}_2) = W_2$. Since J^* is full and faithful there is a uniquely determined morphism $\bar{\eta}: \bar{W}_1 \longrightarrow \bar{W}_2$ in $\underline{\mathcal{S}}^{\underline{C}^0}$ such that $J^*(\bar{\eta}) = \eta$. Hence J^* is surjective.□

1.2.3.3.6 Since $J^*: \underline{\mathcal{S}}^{\underline{C}^0} \longrightarrow \underline{\mathcal{S}}^{\underline{R}^0}$ is full, faithful and surjective it is an equivalence. In fact the functor $J_*: \underline{\mathcal{S}}^{\underline{R}^0} \longrightarrow \underline{\mathcal{S}}^{\underline{C}^0}$ defined by$J_*(W) = \bar{W}$ and $J_*(\eta) = \bar{\eta}$ (as in 1.2.3.3.5) is a right inverse: $J^* \circ J_*$ is the identity functor, but $J_* J^*$ is only isomorphic to the identity functor.

1.2.3.4 Simplest examples of non-Boolean presheaf toposes.

In order that $\underline{\mathcal{S}}^{\underline{\Lambda}^0}$ be not Boolean it is necessary and sufficient that $\underline{\Lambda}$ contain a morphism ℓ which is neither a retraction nor a section. We distinguish two cases:

(I): dom ℓ = cod ℓ and (II): dom $\ell \neq$ cod ℓ ,

each of which we consider to be representative of two classes constructed as follows.

1.2.3.4.1 Classes I and II. Let $L \neq \phi$. If L is finite we let n_L be the cardinality of L, if L is infinite we put $n_L = \infty$. We abbreviate n_L to n.

(I). We let $\underline{M}_n$ be the category with a single object A, and morphisms $[A,A] = L \cup \{id_A\}$ where $id_A \notin L$. Composition is definied on $[A,A]$ by

$$f \circ id_A = id_A = id_A \circ f \quad \text{for each} \quad f \in [A,A], \quad \text{and}$$

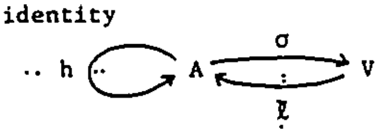
$$f \circ g = f \quad \text{for each} \quad \{f,g\} \subset L .$$

We put $\dot{\underline{\mathcal{E}}}_n = \mathcal{S}^{\underline{M}_n^{\circ}}$.

(I') Using 1.2.3.3 we can define equivalent toposes $\tilde{\underline{\mathcal{E}}}_n \approx \dot{\underline{\mathcal{E}}}_n$ as follows. Let $\tilde{L} = \{\tilde{\ell} \mid \ell \in L\}$ be a disjoint copy of L. Define the category $\underline{\tilde{M}}_n$ so that $\underline{M}_n \hookrightarrow \underline{\tilde{M}}_n$ is a retractor, as follows:

$|\underline{M}_n| = \{A,V\}$ with $A \neq V$.

$[A,A] = L \cup \{id_A\}$, $[V,V] = \{id_V\}$, $[A,V] = \{\sigma\}$ and $[V,A] = \tilde{L}$. We exhibit typical non identity



$\tilde{\ell} \in \tilde{L}$, $h \in L$

morphisms. Certain rules of composition are forced:

1) $\sigma \circ \tilde{\ell} = id_V$ for $\tilde{\ell} \in [V,A]$

2) $\sigma \circ h = \sigma$ for $h \in [A,A]$.

We require that composition in [A,A] be the same as for $\underline{M}_n$:

3) $h \circ t = h$ for $\{h,t\} \subset L$.

We want [A,A] to be generated by $[A,V] \cup [V,A]$ so we put

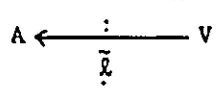
4) $\tilde{\ell} \circ \sigma = \ell$ for $\ell \in L$.

As a consequence of these rules we have

5) $h \circ \tilde{\ell} = \tilde{h} \circ \sigma \circ \tilde{\ell} = \tilde{h}$ for $\{h,\ell\} \subset L$.

We put $\tilde{\underline{\mathcal{E}}}_n = \mathcal{S}^{\underline{\tilde{M}}_n^{\circ}}$. Since the inclusion $\underline{M}_n \hookrightarrow \underline{\tilde{M}}_n$ is a retractor, there is an equivalence $\tilde{\underline{\mathcal{E}}}_n \simeq \dot{\underline{\mathcal{E}}}_n$.

(II) We let $\underline{\Lambda}_n$ be the subcategory of $\underline{\tilde{M}}_n$ having $|\underline{\Lambda}_n| = \{V,A\}$, $[A,V] = \phi$, $[V,A] = \tilde{L}$, $[A,A] = \{id_A\}$, $[V,V] = \{id_V\}$.



$\tilde{\ell} \in \tilde{L}$

We put $\underline{\mathcal{E}}_n = \underline{\mathcal{S}}^{\underline{\Lambda}_n^{\circ}}$.

The inclusion $\underline{\Lambda}_n \hookrightarrow \underline{\tilde{M}}_n$ induces a forgetful functor $\tilde{\underline{\mathcal{E}}}_n \longrightarrow \underline{\mathcal{E}}_n$.

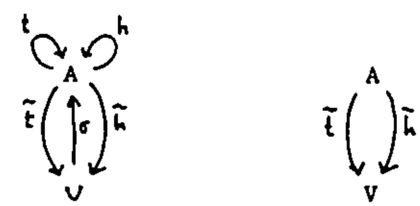
At least two members of these classes have been given names:

$\underline{\mathcal{E}}_1 \equiv$ *Sierp* : "Sheaves over the Sierpinski space" or "the Sierpinski topos"

$\underline{\mathcal{E}}_2 \equiv$ *Dgph* : "Directed graphs" (In [MacL] p.49 $\underline{\mathcal{E}}_2 \equiv$ Grph "the category of small grphs").

$\dot{\underline{\mathcal{E}}}_2 \equiv \dot{\mathit{Dgph}}$, $\tilde{\underline{\mathcal{E}}}_2 \equiv \tilde{\mathit{Dgph}}$

The figures which display the non-identity morphisms of $\underline{\tilde{M}}_2^{\circ}$ and $\underline{\Lambda}_2^{\circ}$:



can also be viewed as diagrams in $\underline{\mathcal{S}}$ indexed by $\underline{\tilde{M}}_2^{\circ}$ and $\underline{\Lambda}_2^{\circ}$; that is, as objects of $\tilde{\mathit{Dgph}}$ and *Dgph* respectively. The functor $\tilde{\mathit{Dgph}} \longrightarrow \mathit{Dgph}$, induced by the inclusion $\underline{\Lambda}_2 \hookrightarrow \underline{\tilde{M}}_2$, "forgets" which loops of A are specified by σ.

1.2.3.4.2 Description of Ω for classes I and II.

(I) For $\underline{\mathcal{E}}_n$, Ω(A) is the set of right ideals of $\underline{M}_n$ which has at least {ϕ,N,[A,A]} as a subalgebra, where N = L. We claim every U ⊂ L is a right ideal:

If u ε U then $u \circ id_A = u$ ε U and for every ℓ ε L, u∘ℓ = u ε U. Thus

$$\Omega(A) = \mathcal{P}(L) \cup \{L \cup \{id_A\}\},$$

where $P_A = [A,A] = L \cup \{id_A\}$. Thus $\Omega(A) \approx 2^n \oplus 1$.

(II) For $\underline{\mathcal{E}}_n$, $P_A = [V,A] \cup [A,A] = L \cup \{id_A\}$ and $P_V = [V,V] = \{id_V\}$.

$$\Omega(V) = \{U \mid U \subset \{id_V\} \text{ and } U \varepsilon R(\underline{\Lambda}_n)\}$$

$$= \{\phi, id_V\}$$

$\Omega(A) = \{U \mid U \subset (L \cup \{id_A\})$ and $U \varepsilon R(\underline{\Lambda}_n)\}$

$= \mathcal{P}(L) \cup \{L \cup \{id_A\}\}$.

Thus $\underset{\sim}{\Omega}(V) \approx 2$ and $\underset{\sim}{\Omega}(A) \approx 2^n \oplus 1$.

1.2.3.4.3 Propositional logic for classes I and II.

For both classes we have

$$\underset{\sim}{\Omega}(A) \vdash \Delta^2(\underline{0}) \quad \text{, by 1.1.2.6.1}$$

For $n = \infty$

$$\underset{\sim}{\Omega}(A) \vdash \varphi \text{ iff } \Delta^2(\underline{0}) \vdash \varphi \text{ for all } \varphi \varepsilon \text{ Poly}(H).$$

For $n < \infty$

$$\underset{\sim}{\Omega}(A) \vdash \varphi \text{ iff } \Delta^2(\underline{0}) \wedge E_n \vdash \varphi$$

for all $\varphi \varepsilon$ Poly(H).

Both follow from 1.1.3.3 and 1.1.2.6.

For I we have $L(\underline{\mathcal{E}}_n) = \{\varphi \mid \Delta^2(\underline{0}) \wedge E_n \vdash \varphi\}$. For II, since $\underset{\sim}{\Omega}(V)$ is a homomorphic image of $\underset{\sim}{\Omega}(A)$,

$\{\varphi \varepsilon \text{ Poly}(H) \mid \underset{\sim}{\Omega}(A) \vdash \varphi\} \subset \{\varphi \varepsilon \text{ Poly}(H) \mid \underset{\sim}{\Omega}(V) \vdash \varphi\}$

$\varphi \varepsilon L(\underline{\mathcal{E}}_n)$ iff both $\underset{\sim}{\Omega}(A) \vdash \varphi$ and $\underset{\sim}{\Omega}(V) \vdash \varphi$

iff $\underset{\sim}{\Omega}(A) \vdash \varphi$

iff $\varphi \varepsilon L(\dot{\underline{\mathcal{E}}}_n)$.

Since $R_2 \dashv \vdash \Delta^2(\underline{0}) \wedge E_2$ we have

$L(\underline{\mathcal{S}ierp}) = L(\underline{\mathcal{E}}_1) = \{\varphi \mid \neg u \vee v \vee (v \rightarrow u) \vdash \varphi\}$.

1.2.3.4.4 Characterizations of $\underline{\Lambda}$ for which $\varphi \varepsilon L(\underline{\mathcal{S}}^{\underline{\Lambda}^0})$ when $\varphi \varepsilon \{E_n \mid n < \infty\} \cup \{\Delta^2(\underline{0})\}$.

We will use "$\underline{\Lambda}$" for the morphism set of the category of the same name.

1.2.3.4.4.1 Proposition. The following are equivalent

(1) $u \vee (u \Rightarrow (v \vee \neg v)) \in L(\underline{\mathcal{S}}^{\Lambda^0})$

(2) For all f and k, if f is not a retraction and dom(f) = cod(k) then there exists g such that $f \cdot k \cdot g = f$.

Proof. (1) → (2). Let dom(f) = cod(k) = A, then

$N_A \cup (N_A \Rightarrow ([f \cdot k, \rightarrow) \cup \neg [f \cdot k, \rightarrow)) = \underline{\Lambda}$.

Since $id_A \in \underline{\Lambda}$ and $id_A \notin N_A$ we have $N_A \subset ([f \cdot k, \rightarrow) \cup \neg [f \cdot k, \rightarrow))$.

Since $f \in N_A$, either $f \in [f \cdot k, \rightarrow)$ or $[f, \rightarrow) \cap [f \circ k, \rightarrow) = \phi$. But

$f \cdot k \in [f, \rightarrow)$, hence $f \in [f \circ k, \rightarrow)$. Hence there exists g such that $f \cdot k \cdot g = f$.□

(2) → (1). Suppose (1) is false. Let U and W be ideals such that

$$U \cup (U \Rightarrow (W \cup \neg W)) \subsetneq \underline{\Lambda}$$

Let $Q = U \cup (U \Rightarrow (W \cup \neg W))$. If $id_A \in Q$ for all $A \in |\underline{\Lambda}|$ then

$\underline{\Lambda} = \bigcup_{A \in |\underline{\Lambda}|} P_A = Q$, hence, for some A, $id_A \notin Q$. We have

(3) $id_A \notin U$ and (4) $P_A \cap U \not\subset (W \cup \neg W)$.

By (4) there exists f such that

(5) $f \in P_A \cap U$ and (6) $f \notin (W \cup \neg W)$.

From (3) and (5) we have

(7) $f \in N_A$.

From (6) we have

(8) $f \notin W$ and (9) $[f, \rightarrow) \cap W \neq \phi$.

From (9) there exists k with cod k = dom f such that (10) $f \circ k \in W$.

Now by (2), there is a g such that $f \cdot k \cdot g = f$, hence $f \in W$ -contradicting (8).□

We characterize $\underline{\Lambda}$ for which the K.B. Lee polynomials E_n are valid. We let $n \geq 1$; i, j, k, range over $\{0,1,2,\ldots,n-1\} = [n]$.

<u>1.2.3.4.4.2 Proposition</u>. The following are equivalent

(1) $[\neg \bigwedge_i v_i \Rightarrow \bigvee_j \neg (\bigwedge_{i \neq j} v_i)] \in L(\underline{\mathcal{S}}^{\Lambda^0})$

(2) For all $B \in |\underline{\Lambda}|$ and all $b\colon [n] \longrightarrow P_B$ there exists $k \neq \ell$ and f and g such that

$$\begin{array}{ccc} \cdot & \xleftarrow{\;f\;} & \cdot \\ {\scriptstyle b_k}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ B & \xleftarrow[\;b_\ell\;]{} & \cdot \end{array}$$

commutes.

<u>Proof.</u> (1) $\rightarrow$ (2). We have, for all ideals U_i $(0 \leq i \leq n-1)$,

$$\neg (\bigcap_i U_i) \subset \bigcup_j (\neg(\bigcap_{i \neq j} U_i)).$$

Suppose (2) is false. Let $B \in |\underline{\Lambda}|$ and $b\colon [n] \longrightarrow P_B$ be such that for all $k \neq \ell$, $[b_k, \rightarrow) \cap [b_\ell, \rightarrow) = \phi$.

Let $W_j = \bigcup_{i \neq j} [b_i, \rightarrow)$ for each j.

For $i \neq j$, $b_i \in W_j$, hence $b_i \in \bigcap_{j \neq i} W_j$, hence $id_B \notin \neg (\bigcap_{j \neq i} W_j)$ for each i, hence $id_B \notin \bigcup_i \neg (\bigcap_{j \neq i} W_j)$.

By (1), $id_B \notin \neg (\bigcap_i W_i)$, hence there exists $f \in (P_B \cap (\bigcap_i W_i))$, hence $f \in P_B$ and $f \in W_i$ for each i, hence for each i there exists $j \neq i$ such that $f \in [b_j, \rightarrow)$. Since $n \geq 1$, we have for $i = 1$ there is a $j \neq 1$ such that $f \in [b_j, \rightarrow)$, and for j there is a $k \neq j$ such that $f \in [b_k, \rightarrow)$, hence $f \in [b_j, \rightarrow) \cap [b_k, \rightarrow)$ -contradiction.□

(2) $\rightarrow$ (1). Suppose (1) is false then there exists $B \in |\underline{\Lambda}|$ such that

(3) $id_B \in \neg(\bigcap_i U_i)$ and (4) $id_B \notin \bigcup_j \neg(\bigcap_{i \neq j} U_i)$.

(3) is equivalent to $P_B \cap (\bigcap_i U_i) = \phi$.

(4) iff for all j $(P_B \cap (\bigcap_{i \neq j} U_i) \neq \phi)$

iff for all j there exists b_j such that $b_j \in P_B$ and for all i, if $i \neq j$, then $b_j \in U_i$. Thus we have $b: [n] \longrightarrow P_B$ such that for all j, $b_j \notin U_j$, by (3) and (4). By (2) there exists k and ℓ, and f, such that $k \neq \ell$ and $f \in [b_k,\rightarrow) \cap [b_\ell,\rightarrow)$. Since $[b_k,\rightarrow) \subset \bigcap_{i \neq k} U_i$ and $[b_\ell,\rightarrow) \subset \bigcap_{i \neq \ell} U_i$, $f \in \bigcap_i U_i$; since $f \in [b_k,\rightarrow) \subset P_B$, $f \in P_B \cap (\bigcap_i U_i)$ -which contradicts (3).□

We show that E_n determines the dimension of k-truncated simplicial sets.

<u>1.2.3.4.4.3 Proposition</u>. Let $k \geq 1$ and $n \geq 2$.

$$E_n \in L(\underline{\underline{S}}^{\Delta^\circ}_k) \text{ iff } k + 1 \leq n.$$

<u>Proof</u>. ($\rightarrow$) Suppose $n < k + 1$, then we can define $b_i: [k] \longrightarrow [k]$ by $b_i(j) = i$ for each $j \in [k]$ and $i \in [k]$, so that $b_0, b_1, \ldots, b_{n-1}$ are distinct maps and for each $h, \ell \in [k]$ such that $h \neq \ell$,

$$b_h \circ f = b_\ell \circ g$$

is not possible. Hence by 1.2.3.4.4.2 $E_n \notin L(\underline{\underline{S}}^{\Delta^\circ}_n)$.□

($\leftarrow$) Suppose $E_n \notin L(\underline{\underline{S}}^{\Delta^\circ}_k)$, then there exists $\{b_0, b_1, \ldots, b_{n-1}\} \subset \mathrm{End}[k]$ such that for all $h \neq \ell$ and all f, g with codomain $[k]$,

$$b_h \circ f \neq b_\ell \circ g.$$

The images, im b_i, of the functions $b_i: [k] \longrightarrow [k]$ are nonempty subsets of $[k]$. We will show the im b_i are pairwise disjoint. Suppose to the contrary that there exists h, ℓ such that $\mathrm{im}\, b_h \cap \mathrm{im}\, b_\ell \neq \phi$. Let $\bar{h}, \bar{\ell} \in [k]$ be such that $b_h(\bar{h}) = b_\ell(\bar{\ell})$. Define $\hat{h}, \hat{\ell}: [k] \longrightarrow [k]$ by $\hat{h}(j) = \bar{h}$ for all $j \in [k]$ and $\hat{\ell}(j) = \bar{\ell}$ for all $j \in [k]$. Then $b_h \circ \hat{h} = b_\ell \circ \hat{\ell}$ -a contradiction. Hence im $b_0, \ldots,$ im b_{n-1} are non-empty pairwise disjoint subsets of $[k]$. Hence $n - 1 \leq k - 1$, hence $n < k + 1$. □

1.2.3.5 Presheaf toposes in which R_k is valid.

1.2.3.5.1 Example. Consider the $k+1$ element chain S_k of 1.1.3.1. Put $e_0 = 1$ and $e_k = 0$, so that $S_k = \{e_i \mid 0 \leq i \leq k\}$. Define $\underset{\sim}{S}_k$ to be the category with one object A and composition given by

$$e_i \circ e_j = e_i \wedge e_j = e_{\max\{i,j\}}.$$

Define $\mathcal{G}_k = \mathcal{S}^{\underset{\sim}{S}_k^{\circ}}$. The preorder of divisibility on $\underset{\sim}{S}_k$ is given by

$$\begin{aligned} e_i \prec e_j \quad &\text{iff} \quad (\mathcal{E}n)(e_i \circ e_n = e_j) \\ &\text{iff} \quad i \leq j \\ &\text{iff} \quad e_j \leq e_i \end{aligned}$$

where the last ordering, $\leq$, is the natural order on the chain S_k. The non-empty ideals of $(\underset{\sim}{S}_k, \prec)$ are principal; thus

$\mathrm{Idl}(\underset{\sim}{S}_k, \prec) = \{\phi\} \cup \{[e_i, \rightarrow) \mid 0 \leq i \leq k\}$, (where $[e_i, \rightarrow) = \{e_j \mid i \leq j \leq k\}$)

and, as Heyting algebras, $\underset{\sim}{\mathrm{Idl}}(S_k, \prec)_{\leq} \approx S_{k+1}$, the $k+2$ element chain. Hence $L(\mathcal{G}_k) = \{\varphi \mid S_{k+1} \vdash \varphi\} = \{\varphi \mid R_{k+1} \vdash \varphi\} = L_{k+1}$.

1.2.3.5.2 Characterization of $\underline{C}$ such that $R_k \in L(\mathcal{S}^{\underline{C}^{\circ}})$.

This condition on $\underline{C}$ is equivalent to: $\underset{\sim}{\mathrm{Idl}}(P_A; \prec_A) \vdash R_k$ for all $A \in C_0$. We begin by describing a relationship between $\mathrm{Idl}(\mathcal{A})_{\leq}$ and $\mathcal{A}$.

1.2.3.5.2.1 Proposition. Let $\mathcal{A} = (A, \prec)$ be a non empty preordered set, $k \geq 1$. The following are equivalent.

(1) $\mathrm{Idl}\,(\mathcal{A})_{\leq}$ is a chain of length at most $k+1$

(2) There exists a sequence $b_0, b_1, \ldots, b_{k-1}$ in A such that for each i such that $(0 \leq i \leq k-2)$ we have $b_i \prec b_{i+1}$, and for each $b \in A$ there exists j such that $0 \leq j \leq k-1$ and $b \sim b_j$.

(3) For any sequence $a_0, a_1, \ldots, a_k$ in A there exists i such that $0 \leq i \leq k-1$ and $a_{i+1} \prec a_i$.

Proof. (1) $\to$ (2) Let $U_k = \phi$ and $\mathrm{Idl}(\mathcal{A})_{\leq} = \{U_i \mid 0 \leq i \leq k\}$ such that $U_{i+1} \subset U_i$ for $0 \leq i \leq k-1$. Let $b_i \in (U_i \setminus U_{i+1})$ if $U_i \setminus U_{i+1} \neq \phi$ If $U_i = U_{i+1}$ let $b_i = b_j$ where $j = \min\{\ell \mid U_\ell = U_{i+1}\}$. Then $[b_i,\to) = U_i$ for each i, thus $b_i \prec b_{i+1}$ and for any b, there exists j such that $[b_j,\to) = [b,\to)$, hence $b_j \sim b$.□

(2) $\to$ (3). Fix $b_0, b_1, \ldots, b_{k-1}$ satisfying (2), and suppose $a_0, a_1, \ldots, a_k$ is a sequence for which (3) fails; define $f\colon \{0,1,..,k\} \longrightarrow \{0,1,..,k-1\}$ by $f(j) = \min\{i \mid b_i \sim a_j\}$ so that $b_{f(j)} \sim a_j$ for $0 \leq j \leq k$. For any a_i and a_j we have $a_i \sim b_{f(i)}$, $a_j \sim b_{f(j)}$ and either $b_{f(i)} \prec b_{f(j)}$ or $b_{f(j)} \prec b_{f(i)}$ hence either $a_i \prec a_j$ or $a_j \prec a_i$. Since $a_{i+1} \not\prec a_i$ for $0 \leq i \leq k-1$, $a_i \prec a_{i+1}$, thus $a_0 \prec a_1 \ldots \prec a_{k-1} \prec a_k$; f cannot be a mono so there exists $r < s$ such that $f(r) = f(s)$, hence $a_r \sim a_s$, then $a_r \prec a_{r+1} \prec a_s \prec a_r$ so $a_{r+1} \prec a_r$ -a contradiction.□

(3) $\to$ (1) We first show that $\mathrm{Idl}(\mathcal{A})_{\leq}$ is a chain. Suppose $U \not\subset W$ and $W \not\subset U$ are ideals, then there exists $b_0 \in U \setminus W$ and $b_1 \in W \setminus U$. Define $b_{2j} = b_0$ for $2j \leq k$ and $b_{2i+1} = b_1$ for $2i+1 \leq k$. By (3), either $b_0 \prec b_1$ or $b_1 \prec b_0$, thus either $b_1 \in U$ or $b_0 \in W$ -a contradiction. Thus $\mathrm{Idl}(\mathcal{A})_{\leq}$ is a chain. Now suppose the chain is longer than $k+1$. There exist U_i $(0 \leq i \leq k+1)$ with $U_{i+1} \subsetneq U_i$ $(0 \leq i \leq k)$. Let $a_i \in (U_i \setminus U_{i+1})$, $(0 \leq i \leq k)$. By hypothesis (3) there exists j such that $a_{j+1} \prec a_j$, hence $a_j \in U_{j+1}$ -a contradiction.□

1.2.3.5.2.2 Corollary. Let $\mathcal{A} = (A,\prec)$ be a preordered set with an element 0 such that $0 \prec x$ for all x; let $k \geq 1$. The following are equivalent.

(1)' $\mathrm{Idl}(\mathcal{A})_{\leq}$ is a chain of length at most $k + 1$

(2)' There exists a sequence $b_1, \ldots, b_{k-1}$ in A such that

(p)' For each i such that $1 \leq i \leq k - 2$ we have

$$b_i \prec b_{i-1}.$$

and

(p)' For each b such that $b \not\sim 0$ there exists j such that $1 \leq j \leq k - 1$ and $b \sim b_j$.

(3)' For any sequence $a_1, \ldots, a_k$ such that for each i, $a_i \not\sim 0$, there exists j such that $1 \leq j \leq k - 1$ and $a_{j+1} \prec a_j$.

<u>Proof</u>. (1) $\leftrightarrow$ (1)'.□ (2) $\rightarrow$ (2)'. Using the sequence of (2) we have that $b_1, \ldots, b_{k-1}$ satisfies $(p_1)'$. For $b = 0$, from (p_2), there exists j such that $0 \leq j \leq k - 1$ and $0 \sim b_j$. Since $b_0 \prec b_1 \prec \ldots \prec b_j$ we have $b_0 \sim 0$. Thus if $b \not\sim 0$ there exists i such that $j < i \leq k - 1$ and $b \sim b_i$. Hence $(p_2)'$ holds.□ (2)' $\rightarrow$ (2). Let $b_0 = 0$, then the sequence $b_0, b_1, \ldots, b_{k-1}$ satisfies (2) clearly.□

(3) $\rightarrow$ (3)' Let $a_1, \ldots, a_k$ be such that for each i, $a_i \not\sim 0$. Put $a_0 = 0$. By (3) there exists j such that $0 \leq j \leq k - 1$ and $a_{j+1} \prec a_j$. □ However if $a_1 \prec 0$ then $a_1 \sim 0$ contradicting the hypothesis for $a_1, \ldots, a_k$. Thus there exists j such $1 \leq j \leq k - 1$ and $a_{j+1} \prec a_j$.□

(3)' $\rightarrow$ (3) Let $a_0, a_1, \ldots, a_k$ be any sequence. If $a_i \sim 0$ for some i such that $1 \leq i \leq k$, then $a_i \prec a_{i-1}$; thus (3) would hold for $a_0, \ldots, a_k$. If $a_i \not\sim 0$ for each i such that $1 \leq i \leq k$, then (3)' implies that for some j such that $1 \leq j \leq k - 1$ we have $a_{j+1} \prec a_j$. □

The algebras $\underset{\sim}{Idl}(P_A, \prec_A)$ to which we shall apply the above Proposition,are subdirectly irreducible and are thus of the form $\mathcal{L} \oplus 1$. For them we have :

<u>1.2.3.5.2.3 Proposition</u>. The following are equivalent.

(1) $\mathcal{L} \oplus 1$ is a chain of length at most $k + 1$

(2) $\mathcal{L} \oplus 1 \vdash R_k$.

<u>Proof</u>. (2) $\to$ (1) From R_k we deduce Z hence $L \oplus 1$ is a chain. Suppose (1) is false, then there exists a_i $(0 \le i \le k + 1)$ such that $a_{i+1} < a_i$ $(0 \le i \le k)$: a chain of length $k+ 2$. We have $(a_i \Rightarrow a_{i+1}) = a_{i+1}$ for $(0 \le i \le k)$. From (2)

$$1 = \bigvee_{i=0}^{k} (a_i \Rightarrow a_{i+1}) = \bigvee_{i=0}^{k} a_{i+1} = a_1 < a_0$$

a contradiction. □

(1) $\longrightarrow$ (2) Let $\alpha: V \longrightarrow L \oplus 1$, $\tilde{\alpha}$: Poly $H \longrightarrow \mathcal{L} \oplus 1$,

$R_k = \bigvee_{i=0}^{k} (u_i \Rightarrow u_{i+1})$, $\tilde{\alpha}(R_k) = \bigvee_{i=0}^{k} (\alpha(u_i) \Rightarrow \alpha(u_{i+1}))$ and let $a_i = \alpha(u_i)$ $(0 \le i \le k + 1)$. Suppose (2) is false, then $\tilde{\alpha}(R\ ) \le e < 1$, hence $(a_i \Rightarrow a_{i+1}) \le e$ for $(0 \le i \le k)$. Since $\mathcal{L} \oplus 1$ is a chain and $a_i \not\le a_{i+1}$ for $(0 \le i \le k)$ then $a_{i+1} < a_i$ for $0 \le i \le k$, thus

$$a_{k+1} < a_k < \dots < a_1 < a_0$$

a chain of length $k + 2$ - a contradiction.□

Now we combine the above Propositions to get our characterization:

<u>1.2.3.5.2.4 Proposition</u>. Let $\underline{C}$ be a small category, $k \ge 1$ and $R'_k = \bigvee_{i=0}^{k} (u_i \Rightarrow u_{i+1})$. The following are equivalent.

(1) $R_k \in L(\underset{\sim}{S}^{\underline{C}^0})$

(2) For each $A \in C_0$ there exists a sequence of morphisms $b_0, b_1, \dots, b_{k-2}$

with codomain A having the following two properties

(p1)' For each $i \in [k-2]$, there exists a morphism g_{i+1} such that

$b_i \circ g_{i+1} = b_{i+1}$.

(p2)' For each $b \in N_A$ there exists $j \in [k-1]$ and morphisms ℓ and ℓ' such that $b_j \circ \ell = b$ and $b \circ \ell' = b_j$.

(3) For each $A \in C_0$ and each sequence of morphisms $a_0, a_1, \ldots, a_{k-1}$ in N_A with codomain A there exists $i \in [k-1]$ and a morphism f such that $a_{i+1} \circ f = a_i$.

Proof. By 1.2.3.5.2.2 and 1.2.3.5.2.3.□

1.2.3.5.2.5 For the case $k = 2$ we have

$$(u_1 \vee (u_2 \Rightarrow u_1) \vee \neg u_2) \in L(\underline{\mathcal{S}}^{\underline{C}^0}) \quad \text{iff}$$

For each $A \in C_0$ and each a_1, a_2 in N_A there exists ℓ and ℓ' such that $a_1 \circ \ell = a_2$ and $a_2 \circ \ell' = a_1$. Thus either $N_A = \phi$ or N_A is both principal and minimal non-empty.

1.2.3.6 Distinguishing between the classes I and II by their logic.

Since $L(\dot{\underline{\mathcal{E}}}_n) = L(\underline{\mathcal{E}}_n)$ for each n, we have to move beyond that part of the propositional logic built up from $\wedge$, $\vee$, $\Rightarrow$ and $\perp$, in order to make a "logical" distinction between $\dot{\underline{\mathcal{E}}}_n$ and $\underline{\mathcal{E}}_n$.

1.2.3.6.1 The algebra of propositional constants.

By the algebra of propositional constants for a topos with Heyting algebra $\underline{\Omega}$ as subobject classifier, we mean the external Heyting algebra $[1, \underline{\Omega}]$. We shall use the term "propositional constant" to refer both to morphisms $1 \longrightarrow \Omega$, and to formulas with no free variables.

1.2.3.6.1.1 Definition. A Heyting algebra $\mathcal{A}$ is called well-connected [Mc & T-1] if, for all a, b in $\mathcal{A}$, if $a \vee b = 1$ then $a = 1$ or $b = 1$.

1.2.3.6.1.2 We shall investigate the following four properties of a Heyting algebra $\mathcal{A}$ when $\mathcal{A}$ is the algebra of propositional constants of a topos of presheaves over $\underline{\mathcal{S}}$:

(1) $\mathcal{A} \approx \mathbf{2}$

(2) $\mathcal{A}$ is a Boolean algebra

(3) $\mathcal{A}$ is subdirectly irreducible

(4) $\mathcal{A}$ is well-connected.

The obvious relationship between these properties are (for a general $\mathcal{A}$), (1) iff ((2) $\wedge$ (3)) iff ((2) $\wedge$ (4)); and (3) implies (4).

1.2.3.6.1.3 Proposition. Let $\underline{\mathcal{S}}^{\underline{\Lambda}^0}$ be a presheaf topos and let $\underline{\Omega}$ be the Heyting algebra structure of its subobject classifier.

(1) The following are equivalent

(1.1) $[\mathbf{1}, \underline{\Omega}] \approx \mathbf{2}$

(1.2) For all propositional constants a, either $\underline{\mathcal{S}}^{\underline{\Lambda}^0} \models a$ or $\underline{\mathcal{S}}^{\underline{\Lambda}^0} \models \neg a$

(1.3) For all A, B in $|\underline{\Lambda}|$, $[A,B] \neq \phi$.

(2) The following are equivalent

(2.1) $[\mathbf{1}, \underline{\Omega}]$ is a Boolean algebra

(2.2) For all propositional constants a,

$$\underline{\mathcal{S}}^{\underline{\Lambda}^0} \models a \vee \neg a$$

(2.3) For all A, B in $|\underline{\Lambda}|$,

if $[A,B] \neq \phi$ then $[B,A] \neq \phi$

(3) The following are equivalent

(3.1) $[\mathbb{I},\underline{\Omega}]$ is subdirectly irreducible

(3.2) There exists a propositional constant e such that for all propositional constants a,

$$\underline{\mathcal{S}}^{\underline{\Lambda}^0} \models a \Rightarrow e \text{ iff } \underline{\mathcal{S}}^{\underline{\Lambda}^0} \models a$$

(3.3) There exists $E \varepsilon |\underline{\Lambda}|$ such that for all $A \varepsilon |\underline{\Lambda}|$, $[A,E] \neq \phi$

(4) The following are equivalent

(4.1) $[\mathbb{I},\underline{\Omega}]$ is well-connected

(4.2) For all propositional constants a and b, if $\underline{\mathcal{S}}^{\underline{\Lambda}^0} \models a \vee b$ then either $\underline{\mathcal{S}}^{\underline{\Lambda}^0} \models a$ or $\underline{\mathcal{S}}^{\underline{\Lambda}^0} \models b$

(4.3) For all A, B in $|\underline{\Lambda}|$ there exists C in $|\underline{\Lambda}|$ such that both $[A,C] \neq \phi$ and $[B,C] \neq \phi$.

Proof. The Heyting algebra isomorphisms $[\mathbb{I},\underline{\Omega}] \approx \underline{\text{Sub}}(\mathbb{I}) \approx \underline{\text{Idl}}(|\underline{\Lambda}|,\prec)$ allow us to replace constant maps by ideals of $(|\underline{\Lambda}|,\prec)$ where the preorder $\prec = \prec_{\mathbb{I}}$ is given by: $A \prec B$ iff $[B,A] \neq \phi$.
The equivalences (i.1) $\leftrightarrow$ (i.2) for i = 1,2,3,4 are clear. We will establish the equivalences (i.1) $\leftrightarrow$ (i.3) for i = 1,2,3,4.

(1.1) $\longleftrightarrow$ (1.3). (1.1) iff $\text{Idl}(|\underline{\Lambda}|,\prec) = \{\phi, |\underline{\Lambda}|\}$ iff (1.3).□

(2.1) $\longleftrightarrow$ (2.3). (2.1) iff $\underline{\text{Idl}}(|\underline{\Lambda}|,\prec) \vdash v \vee \neg v$ iff $\prec$ is an equivalence relation iff (2.3).□

(3.1) $\longrightarrow$ (3.3) $\underline{\text{Idl}}(|\underline{\Lambda}|,\prec)$ contains a penultimate ideal Σ. Let $E \not\in \Sigma$, then $[E,\rightarrow) = |\underline{\Lambda}|$. □

(3.3) $\longrightarrow$ (3.1). Let $\Sigma = \{B \varepsilon |\underline{\Lambda}| \mid [B,\rightarrow) \neq |\underline{\Lambda}|\}$, then there exists $E \varepsilon (|\underline{\Lambda}| \setminus \Sigma)$. We show that Σ is an ideal by showing that $\sim\Sigma = \{B \mid [B,\rightarrow) = |\underline{\Lambda}|\}$ is an ideal of $(|\underline{\Lambda}|,\prec)^0$. Suppose $B \varepsilon \sim\Sigma$ and

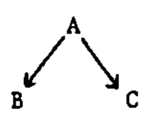
$C \prec B$, then $[B,\to) \subset [C,\to)$ and $[B,\to) = |\underline{\Lambda}|$, hence $[C,\to) = |\underline{\Lambda}|$, hence $C \in \sim \Sigma$. We show Σ is penultimate in $\underset{\sim}{\mathrm{Idl}}(|\underline{\Lambda}|\prec)$. Suppose U is an ideal of $(|\underline{\Lambda}|,\prec)$, and $\Sigma \subset U$. Let $A \in U \setminus \Sigma$, then $[A,\to) = |\underline{\Lambda}|$, hence $U = |\underline{\Lambda}|$. □

(4.1) → (4.3). Suppose (4.3) is false, then there exists A, B in $|\underline{\Lambda}|$ such that for each C either $[A,C] = \phi$ or $[B,C] = \phi$. Let $U = \{C|\ [A,C] = \phi\}$ and $W = \{C|\ [B,C] = \phi\}$. Each is an ideal of $(|\underline{\Lambda}|,\prec_0)$, and $U \cup W = |\underline{\Lambda}|$. Since $\mathrm{Idl}(|\underline{\Lambda}|,\prec_0)$ is well connected either $U = |\underline{\Lambda}|$ or $W = |\underline{\Lambda}|$; but $A \notin U$ and $B \notin W$ -a contradiction.□

(4.3) → (4.1) Suppose (4.1) is false, then there exists ideals U and W of $(|\underline{\Lambda}|,\prec_0)$ such that $U \cup W = |\underline{\Lambda}|$ but both $U \neq |\underline{\Lambda}|$ and $W \neq |\underline{\Lambda}|$ hence there exists $A \in U \setminus W$ and $B \in W \setminus U$. By (4.3) there exists C such that both $[A,C] \neq \phi$ and $[B,C] \neq \phi$.

Since $C \in U \cup W$, either $C \in U$ or $C \in W$; but then either $\{A,B\} \subset U$ or $\{A,B\} \subset W$ -a contradiction.□

<u>1.2.3.6.1.4</u> No two of the four properties above are equivalent. We exhibit the non-identity morphisms of three categories that show the distinctness of the properties (of 1.2.3.6.1.2)



satisfies (4) but not (3), (2) or (1)

$A \longrightarrow B$ satisfies (3) but not (2) or (1)

A B satisfies (2) but not (3) or (1)

In the depicted categories A, B and C are distinct objects.

1.2.3.6.1.5 Example of $\underline{M}$-sets for $\underline{M}$ a proper monoid.

Let $\underline{M}$ be a monoid with $|\underline{M}| = \{\Lambda\}$ and $\underline{M}$ not a group. Let $\underline{\Omega}$ be the Heyting algebra structure of the subobject classifier of $\mathcal{S}^{\underline{M}^0}$, then

$$[\mathbf{1},\underline{\Omega}] \approx \underset{\sim}{\mathrm{Idl}}(\{A\},\prec) = \mathbf{2}.$$

Thus (1) and hence also the other three properties of $[\mathbf{1},\underline{\Omega}]$ given in 1.2.3.6.1.2 hold. Since $\mathcal{S}^{\underline{M}^0} \not\models \forall p(p \vee \neg p)$ and (2) hold we must have, from (2.2),

$$\mathcal{S}^{\underline{M}^0} \models \neg \forall p(p \vee \neg p).$$

1.2.3.6.1.6 Example of the class II.

For the toposes $\mathcal{E}_{\underline{n}} = \mathcal{S}^{\underline{\Lambda}_n^0}$ where $n = n_L$ and $L \neq \phi$ we also have $\mathcal{E}_{\underline{n}} \not\models \forall p(p \vee \neg p)$. Properties (1) and (2) fail, but (3), and hence also (4), hold, since (3.3) holds for $\underline{\Lambda}_{\underline{n}}$:

$$V \xrightarrow{\ \ell\ } A \qquad \ell \in L.$$

Thus for any propositional constant a, if $\mathcal{E}_{\underline{n}} \models a \vee \forall p(p \vee \neg p)$ then $\mathcal{E}_{\underline{n}} \models a$. That is, we have the rule of inference

$$\frac{\varphi \vee \forall p(p \vee \neg p)}{\varphi} \qquad s_F(\varphi) = \phi$$

holding in $\mathcal{E}_{\underline{n}}$.

1.2.3.6.2 Logical conditions involving $[\![\forall p(p \vee \neg p)]\!] \hookrightarrow \mathbf{1}$.

We need to establish one further property of class II, namely that $\mathcal{E}_{\underline{n}} \models \neg\neg \forall p(p \vee \neg p)$; this will be a consequence of 1.2.3.6.2.8.

We shall work out an explicit description of the subpresheaf, $[\![\forall p(p \vee \neg p)]\!] \hookrightarrow \mathbf{1}$., in the context of presheaf toposes $\mathcal{S}^{\underline{\Lambda}^0}$, in 1.2.3.6.2.5. Using this description we give necessary and sufficient conditions on $\underline{\Lambda}$ for the validity in $\mathcal{S}^{\underline{\Lambda}^0}$ of $\neg \forall p(p \vee \neg p)$

(in 1.2.3.6.2.7), of $\neg\neg\forall p(p \vee \neg p)$ (in 1.2.3.6.2.8), and of $\neg\neg[\![\forall p(p \vee \neg p)]\!] = [\![\mathbb{0}]\!]$ (in 1.2.3.6.2.10).

<u>1.2.3.6.2.1 The subpresheaf $[\![p \vee \neg p]\!] \hookrightarrow \Omega$</u>

$[\![p \mid p \vee \neg p]\!] = [\![\top]\!] \vee [\![\bot]\!]$

$[\![\top]\!]\ (A) = \{U \in \Omega(A) \mid (\mathcal{E}x)((x \in \mathbb{1}(A)) \text{ and } (\tau_A(x) = U))\}$

$\qquad = \{U \in \Omega(A) \mid P_A = U\} = \{P_A\}$

$[\![\bot]\!]\ (A) = \{\phi\}$

$[\![p \mid p \vee \neg p]\!](A) = \{\phi, P_A\}$ for all $A \in |\underline{A}|$.

<u>1.2.3.6.2.2 Description of $\forall_u(W)$.</u> The morphism $u: \Omega \longrightarrow \mathbb{1}$ induces the adjoint pair $u^* \dashv \forall_u$ thus

$$\mathrm{Sub}(\mathbb{1}) \underset{\forall_u}{\overset{u^*}{\rightleftarrows}} \mathrm{Sub}(\Omega)$$

$u^*(U) \leq W$ iff $U \leq \forall_u(W)$ for all $U \hookrightarrow \mathbb{1}$ and all $W \hookrightarrow \Omega$.

For each $A \in |\underline{A}|$

$(u^*(U))(A) = u_A^{-1}(U(A)) = \{V \in \Omega(A) \mid u_A(V) \in U(A)\}$

$$= \begin{cases} \Omega(A) & \text{if } U(A) = \{A\} \\ \phi & \text{if } U(A) = \phi. \end{cases}$$

<u>1.2.3.6.2.4 Proposition.</u> For each $A \in |\underline{A}|$, $(\forall_u(W))(A) = \{A\}$ iff $(\mathcal{A}B)(([B,A] \neq \phi) \rightarrow (\Omega(B) = W(B)))$.

<u>Proof.</u> $(\forall_u(W))(A) = \{A\}$ iff $\mathbb{1}(A) \subset (\forall_u(W))(A)$

iff $[A] \leq \forall_u(W)$

iff $(u^*([A])) \leq W$

iff $(\mathcal{A}B)(u_B^{-1}([A](B)) \subset W(B))$

iff $(\mathcal{A}B)(([A](B) = \{B\}) \rightarrow (\Omega(B) = W(B))$

iff $(\mathcal{A}B)(([B,A] \neq \phi) \rightarrow (\Omega(B) = W(B)))$. □

1.2.3.6.2.5 Proposition.

$$[\![\forall p(p \vee \neg p)]\!](A) = \begin{cases} \{A\} & \text{if } \underline{\Lambda}/A \text{ is a groupoid} \\ \phi & \text{otherwise} \end{cases}$$

Proof. $[\![\forall p(p \vee \neg p)]\!] = \forall_u([\![\top]\!] \vee [\![\bot]\!])$. For any $A \in |\underline{\Lambda}|$

$[\![\forall p(p \vee \neg p)]\!](A) = \{A\}$ iff

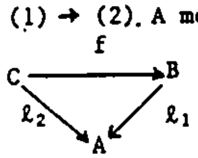
$(\forall B)(([B,A] \neq \phi) \to (\Omega(B) = \{\phi, P_B\}))$ iff

(1): $(\forall B)(([B,A] \neq \phi) \to (N_B = \phi))$.

We want to show that (1) is equivalent to:

(2) $\underline{\Lambda}/A$ is a groupoid.

(1) → (2). A morphism of $\underline{\Lambda}/A$ is a triple: (ℓ_1, f, ℓ_2) such that



commutes.

Since $N_B = \phi$, f is a retraction in $\underline{\Lambda}$, with $f \circ s = id_B$ but then $\ell_2 \circ s = \ell_1 \circ f \circ s = \ell_1$ so (ℓ_1, s, ℓ_1) is a section for (ℓ_1, f, ℓ_2). Thus $\underline{\Lambda}/A$ is a groupoid.□

(2) → (1). Let $\ell_1 \in [B,A]$, and let $f \in P_B$, then $(\ell_1, f, \ell_1 \circ f)$ is a morphism of $\underline{\Lambda}/A$ with codomain ℓ_1, hence is invertible. Thus there is a map $(\ell_1 \circ f, s, \ell_1)$ such that $f \circ s = id_B$, thus $N_B = \phi$. □

1.2.3.6.2.6 Proposition. Let $U \hookrightarrow \Omega$, then

$\neg U = 0$ iff $(\forall A)(\exists B)(([B,A] \neq \phi)$ and $(U(B) = \{B\}))$.

Proof. $\neg U = 0$ iff $(\forall A)((\neg U)(A) = \phi)$.

$A \in (\neg U)(A)$ iff $[A] \leq \neg U$

iff $[A] \wedge U = 0$

iff $(\forall B)([A](B) \cap U(B) = \phi)$

iff $(\forall B)(([B,A] = \phi)$ or $(U(B) = \phi))$.

Thus $(\neg U)(A) = \phi$ iff $(\mathcal{E}B)(([B,A] \neq \phi)$ and $(U(B) = \{B\}))$. □

1.2.3.6.2.7 Proposition. $\mathcal{S}^{\underline{\Lambda}^0} \models \neg \forall p(p \vee \neg p)$ iff $(\mathcal{A}A)(\mathcal{E}B)(([B,A] \neq \phi)$ and $(N_B \neq \phi))$.

Proof. $\mathcal{S}^{\underline{\Lambda}^0} \models \neg \forall p(p \vee \neg p)$ iff $[\![\forall p(p \vee \neg p)]\!] = 0$

iff $(\mathcal{A}A)([\![\forall p(p \vee \neg p)]\!](A) = \phi)$ iff $(\mathcal{A}A)(\sim (\mathcal{A}B)(([B,A] \neq \phi) \rightarrow (N_B = \phi)))$

iff $(\mathcal{A}A)((\mathcal{E}B)(([B,A] \neq \phi)$ and $(N_B \neq \phi))$. □

1.2.3.6.2.8 Proposition. $\mathcal{S}^{\underline{\Lambda}^0} \models \neg\neg \forall p(p \vee \neg p)$ iff $(\mathcal{A}A)(\mathcal{E}B)(([B,A] \neq \phi)$ and $(\underline{\Lambda}/B$ is a groupoid)).

Proof. $\mathcal{S}^{\underline{\Lambda}^0} \models \neg\neg \forall p(p \vee \neg p)$ iff $\neg [\![\forall p(p \vee \neg p)]\!] = 0$

iff $(\mathcal{A}A)(\mathcal{E}B)(([B,A] \neq \phi)$ and $([\![\forall p(p \vee \neg p)]\!](B) = \{B\}))$

iff $(\mathcal{A}A)(\mathcal{E}B)(([B,A] \neq \phi)$ and $(\underline{\Lambda}/B$ is a groupoid)). □

1.2.3.6.2.9 $\neg\neg$ - codensity in $\mathrm{Sub}(\mathbb{1})_{\tau}$.

A comparison of the rule of inference for $\mathcal{L}_n$ given in 1.2.3.6.1.6, with the characterization of $\neg\neg$ - codensity given in 1.2.2.1.11 suggests an alternate way of viewing this rule.

Let F be an ideal of $(|\underline{\Lambda}|, <_1)$, and, under the isomorphisms $\mathrm{Idl}(|\underline{\Lambda}|, <_1) \approx \underline{\mathrm{Sub}}(\mathbb{1}) \approx [\mathbb{1}, \underline{\Omega}]$, let $\overline{F} \hookrightarrow \mathbb{1}$ and $f: \mathbb{1} \longrightarrow \Omega$ be the corresponding subpresheaf of $\mathbb{1}$ and global section of Ω, respectively. Let $\overline{f} = f^*$. Restated in terms of the preordered structure $(|\underline{\Lambda}|, <_1)$, and in the language of the topos $\mathcal{S}^{\underline{\Lambda}^0}$, 1.2.2.1.11 becomes:

The following are equivalent.

(1) $\neg F = |\underline{\Lambda}|$

(2) For all constant propositions a

if $\mathcal{S}^{\underline{\Lambda}^0} \models a \vee \overline{f}$ then $\mathcal{S}^{\underline{\Lambda}^0} \models a$

(3) $(\mathcal{A}A)(\mathcal{E}B)(([B,A] \neq \phi)$ and $(\overline{F}(A) = \phi))$.

In (3) we have used 1.2.2.9.1 to convert $A \notin F$ to $\overline{F}(A) = \phi$ under the isomorphism of 1.2.4.2.

Now we replace $\overline{F}$ by $[\![\forall p(p \vee \neg p)]\!]$.

1.2.3.6.2.10 Proposition. For a presheaf topos $\underline{\mathcal{S}}^{\Lambda^0}$ the following are equivalent.

(1) $\pi[\![\forall p(p \vee \neg p)]\!] = [\![0]\!]$

(2) The rule of deduction

$$\frac{\varphi \vee \forall p(p \vee \neg p)}{\varphi} \qquad s_F(\varphi) = \phi$$

holds

(3) $(\mathcal{A}A)(\mathcal{E}B)(([A,B] \neq \phi)$ and $(\underline{\Lambda}/B$ is not a groupoid)).

Proof. From 1.2.3.6.2.9 and 1.2.3.6.2. □

1.2.3.6.2.11 Comment on the operation π. We cannot turn (1) above into an internal statment of the language. A formula $\pi\, \forall p(p \vee \neg p)$ can only make sense for us if there is an internal pseudocomplement $\pi\colon \Omega \longrightarrow \Omega$; but this can only happen if Ω is a Boolean algebra, or equivalently if $\underline{\Lambda}$ is a groupoid. On each $\mathrm{Sub}(F)_{\leq} \approx \mathrm{Idl}(\overline{F},\prec_F)_{\leq}$, for $F\colon \underline{\Lambda}^0 \longrightarrow \underline{\mathcal{S}}$ there is a dual pseudocomplement. Hence for each representable presheaf, $[-,A]$, there is a map $\pi_A\colon \Omega(A) \longrightarrow \Omega(A)$, but the family $\pi_A (A \in |\underline{\Lambda}|)$ fails to be natural.

1.2.4 Semidirect products: the Grothendieck construction.

Our account of this construction is derived from [Wr] and [Mo] . The presheaf categories $\mathcal{S}^{\Lambda^0}$ are toposes and thus it is possible to model categories and discrete fibrations in $\mathcal{S}^{\Lambda^0}$. In $\mathcal{S}$ discrete fibrations over $\underline{\Lambda}$ together with mediating functors form a category equivalent to the presheaf topos $\mathcal{S}^{\Lambda^0}$. Thus within a topos, and in particular in the topos $\mathcal{S}^{\Lambda^0}$, we can take $(\mathcal{S}^{\Lambda^0})^{\underline{C}^0}$, for $\underline{C}$ an internal category in $\mathcal{S}^{\Lambda^0}$, to be the category of discrete fibrations over $\underline{C}$ in $\mathcal{S}^{\Lambda^0}$. Examination of $(\mathcal{S}^{\Lambda^0})^{\underline{C}^0}$ reveals an alternate, more direct, way of construing this category.

The internal category $\underline{C}$, we take to be a functor

$$\underline{C}\colon \underline{\Lambda}^0 \longrightarrow \underline{\mathrm{Cat}}\ ,$$

out of which we build a split fibration

$$p\colon \mathcal{G}_{\underline{\Lambda}}(\underline{C}) \longrightarrow \underline{\Lambda} \quad \text{in} \quad \underline{\mathrm{Cat}}.$$

(The construction applied to groups gives the semidirect product; we shall use the notation of groups for the "total category" of the split fibration:

$$\underline{\Lambda} \ltimes \underline{C} \equiv \mathcal{G}_{\underline{\Lambda}}(\underline{C})).$$

The discrete fibrations over $\underline{C}$, in $\underline{\mathrm{Cat}}(\mathcal{S}^{\Lambda^0})$, correspond to discrete fibrations over $\underline{\Lambda} \ltimes \underline{C}$ in $\underline{\mathrm{Cat}}(\mathcal{S})$ ($\equiv\underline{\mathrm{Cat}}$) in such a way that there is an equivalence of categories:

$$(\mathcal{S}^{\Lambda^0})^{\underline{C}^0} \approx \mathcal{S}^{(\underline{\Lambda} \ltimes \underline{C})^0}.$$

In this section we shall denote by "$|\underline{\Lambda}|$" the object set of a small category $\underline{\Lambda}$. For the values of the functor $\underline{C}\colon \underline{\Lambda}^0 \longrightarrow \underline{\mathrm{Cat}}$ we shall use "$C_0(A)$" for the object set of the category $\underline{C}(A)$.

<u>1.2.4.1 Definition of the "total category" $\underline{\Lambda} \ltimes \underline{C}$.</u>

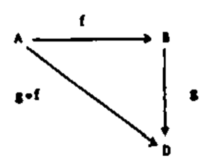
$|\underline{\Lambda} \ltimes \underline{C}|$ consists of all (A,a) such that

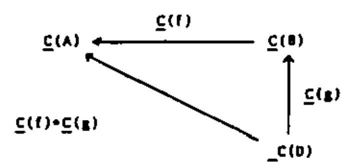
$$A \in |\underline{\Lambda}| \quad \text{and} \quad a \in C_0(A).$$

A morphism of $\underline{\Lambda} \ltimes \underline{C}$ from (A,a) to (B,b) is a triple (f,γ,b) where $f\colon A \longrightarrow B$ and $\gamma\colon a \longrightarrow \underline{C}(f)(b)$. Once we have specified the codomain we only need the ordered pair (f,γ) to specify the morphism, but note that (f,γ) does not in general determine the codomain: we may have $b \neq b'$ in $C_0(B)$ with $\underline{C}(f)(b) = \underline{C}(f)(b')$. We further simplify the notation for morphisms with codomain (B,b) where $b \in C_0(B)$ by putting: $(B,\gamma) \equiv (id_B,\gamma)$ and $(f,a) \equiv (f,id_a)$, where $a \equiv \underline{C}(f)(b)$.

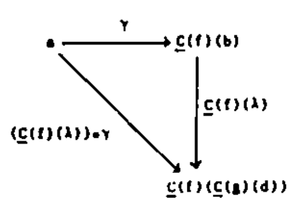
Identities of $\underline{\Lambda} \ltimes \underline{C}$. $(id_A,a)\colon (A,a) \longrightarrow (A,a)$ where $a \in C_0(A)$ and $A \in |\underline{\Lambda}|$.

Composition in $\underline{\Lambda} \ltimes \underline{C}$. Let $(f,\gamma)\colon (A,a) \longrightarrow (B,b)$ and $(g,\lambda)\colon (B,b) \longrightarrow (D,d)$. We compose f with g and apply the functor $\underline{C}$:





We apply $\underline{C}(f)$ to the morphism $\lambda\colon b \longrightarrow \underline{C}(g)(d)$ of $\underline{C}(B)$ and compose with the morhism $\gamma\colon a \longrightarrow \underline{C}(f)(b)$ of $\underline{C}(A)$



We put $(g,\lambda)\circ(f,\gamma) \equiv (g\cdot f,(\underline{C}(f)(\lambda))\cdot\gamma)$.

<u>1.2.4.1.2.</u> The split fibration $p\colon \underline{\Lambda} \ltimes \underline{C} \longrightarrow \underline{\Lambda}$ is the projection functor: $p(A,a) \equiv A$, $p(f,\gamma) \equiv f$.

The "splitting" is the choice, for every pair $(f,(B,b))$ where $f\colon A \longrightarrow B$ and $b \in C_0(B)$ of the cartesian morphism

$$(f,\underline{A}(f)(b))\colon (A,\underline{C}(f)(b)) \longrightarrow (B,b).$$

<u>1.2.4.1.3 Proposition</u>. Let $\underline{C}\colon \underline{\Lambda}^0 \longrightarrow \underline{Cat}$, then the split fibration $p\colon \underline{\Lambda} \ltimes \underline{C} \longrightarrow \underline{\Lambda}$ is an h-functor.

<u>Proof</u>. Let $(f,\gamma)\colon (A,a) \longrightarrow (B,b)$ in $\underline{\Lambda} \ltimes \underline{C}$ and let $\ell\colon D \longrightarrow p(A,a) \equiv A$ in $\underline{\Lambda}$. Put $d \equiv \underline{C}(\ell)(a)$, then $p((f,\gamma)\circ(\ell,d)) \equiv p(f\cdot\ell,\underline{C}(\ell)(\gamma)) \equiv f\circ\ell \equiv p(f,\gamma)\circ\ell$ hence, by 1.2.2.3.9, p is an h-functor.□

<u>1.2.4.1.4</u> By 1.2.2.3.11, the functor $p\colon \underline{\Lambda} \ltimes \underline{C} \longrightarrow \underline{\Lambda}$ induces, for each $a \in C_0(A)$ an h-map $p_a \equiv p_{(A,a)}$

$$p_a\colon (P_{(A,a)},\prec_{(A,a)}) \longrightarrow (P_A,\prec_A).$$

Moerdjik has observed [Mo] that for each $(A,a) \in |\underline{\Lambda} \ltimes \underline{C}|$ the functor $p_{(A,a)}\colon (\underline{\Lambda} \ltimes \underline{C})/(A,a) \longrightarrow \underline{\Lambda}/A$ has a right adjoint. We shall only need to consider these comma categories as preordered sets; $(P_A,\prec_A)$ arises from $\underline{\Lambda}/A$ by: $f \prec_A g$ iff $[g,f] \neq \phi$. The reversal of directions in passing to the preorder means that a right adjoint becomes a left adjoint. We will show that this adjoint $t_a\colon P_A \longrightarrow P_{(A,a)}$ is given on $f\colon B \longrightarrow A$ by

$t_a(f) \equiv (f,\underline{C}(f)(a))\colon (B,\underline{C}(f)(a)) \longrightarrow (A,a)$. □

<u>1.2.4.1.5 Proposition.</u> Let $a \in C_0(A)$, the function

$t_a: P_A \longrightarrow P_{(A,a)}$ has the following properties

(1) It is a section of $p_a = p_{(A,a)}: P_{(A,a)} \longrightarrow P_A$

(2) $t_a \dashv p_a$

(3) t_a is preorder preserving.

<u>Proof</u>. (1) $p_a(t_a(f)) = p_a(f, \underline{C}(f)(a)) = f$. □

(2) $t_a(f) \prec (g,\delta)$

iff $(E\ell)(E\gamma)((f,\underline{C}(f)(a))\circ(\ell,\gamma) = (g,\delta))$

iff $(E\ell)(E\gamma)((f\circ\ell = g)$ and $(\gamma = \delta))$

iff $f \prec g$

iff $f \prec p_a(g,\delta)$.□

(3) $f \prec g$ iff $f \prec p_a(t_a(g))$ by (1)

iff $t_a(f) \prec t_a(g)$ by (2). □

<u>1.2.4.2 Connecting the subobject classifiers of $\underline{\mathcal{S}}^{\underline{\Lambda}^0}$ and $\underline{\mathcal{S}}^{(\underline{\Lambda} \ltimes \underline{C})^0}$.</u>

Let $\mathrm{true}: 1 \rightarrowtail \Omega$ and $\mathbf{true}: \mathbf{1} \rightarrowtail \boldsymbol{\Omega}$ be the subobject classifiers of $\underline{\mathcal{S}}^{\underline{\Lambda}^0}$ and $\underline{\mathcal{S}}^{(\underline{\Lambda} \ltimes \underline{C})^0}$ respectively. Fix $A \in |\underline{\Lambda}|$ and $a \in C_0(A)$. Define functions θ_a, η_a, μ_a by

$\theta_a(U) = \{(f,\gamma) \in P_{(A,a)} \mid f \in U\}$ for all ideals U of $(P_A, \prec_A)$, and

$\eta_a(W) = \{f \in P_A \mid (E\gamma)((f,\gamma) \in W)\}$,

$\mu_\alpha(W) = \{f \in P_A \mid (f,\underline{C}(f)(a)) \in W\}$, for all ideals W of $(P_{(A,a)}, \prec_{(A,a)})$.

<u>1.2.4.2.1 Proposition.</u> The functions θ_a, η_a and μ_a map ideals to ideals:

$$\mathrm{Idl}(P_A,\prec_A)_{\leq} \; \begin{matrix} \xleftarrow{\ \eta_a\ } \\ \xrightarrow{\ \theta_a\ } \\ \xleftarrow[\ \mu_a\ ]{} \end{matrix} \; \mathrm{Idl}(P_{(A,a)},\prec_{(A,a)})_{\leq}$$

$\eta_a \dashv \theta_a \dashv \mu_a$, and $\eta_a \circ \theta_a \equiv id \equiv \mu_a \circ \theta_a$, where id is the identity map on $\Omega(A) \equiv Idl(P_A, \prec_A)$.

<u>Proof</u>. The projection functor $p: \underline{\Lambda} \ltimes \underline{C} \longrightarrow \underline{\Lambda}$ induces a map p_a which has a splitting t_a such that $t_a \dashv p_a$ and $p_a \circ t_a \equiv id_{P_{(A,a)}}$ by 1.2.4.1.5. From 1.2.2.1.9 and 1.2.2.1.10, by appl ying the contravariant fuctor $Idl_{\leq}: (\underline{PreOrd})^0 \longrightarrow \underline{PreOrd}$ to these maps we get $(p_a)^* \dashv (t_a)^*$ and $(t_a)^* \circ (p_a)^* \equiv id$.

For U and ideal of $(P_A, \prec_A)$,

$(p_a)^*(U) \equiv \{(f,\gamma) \in P_{(A,a)} | \; p_a(f,\gamma) \; \varepsilon \; U\} \equiv \theta_a(U)$ and for W an ideal of $(P_{(A,a)}, \prec_{(A,a)})$,

$(t_a)^*(W) \equiv \{f \; \varepsilon \; P_A | \; (f, \underline{C}(f)(a)) \; \varepsilon \; W\} \equiv \mu_a(W)$, thus $\theta_a \dashv \mu_a$ and $\mu_a \circ \theta_a \equiv id$.

Since p_a is an h-map we can use the description of $(p_a)_!$ given in 1.7.1.4:

$$m_A \circ (p_a)_! \equiv (\exists_{p_a}) \circ m_{(A,a)}$$

where m_A and $m_{(A,a)}$ are just inclusion maps. Thus for each ideal W of $(P_{(A,a)}, \prec_{(A,a)})$,

$(p_a)_!(W) \equiv (\exists_{p_a})(W) \equiv \{f \; \varepsilon \; P_A | \; (E\gamma)((f,\gamma) \; \varepsilon \; W)\} \equiv \eta_a(W)$.

Thus $\eta_a \dashv \theta_a$. Since θ_a is a mono, by 1.4.1.2 (9. → 10.), $\eta_a \circ \theta_a \equiv id$.□

<u>1.2.4.2.2 Proposition</u>. Let $\underline{C}: \underline{\Lambda}^0 \longrightarrow \underline{Cat}$, $A \; \varepsilon \; |\underline{\Lambda}|$ and $a \; \varepsilon \; C_0(A)$. The following are equivalent.

(1) $\theta_a: \Omega(A) \rightarrowtail \Omega(A,a)$ is an isomorphism

(2) $\eta_a = \mu_a$

(3) $(AW)((W \in \Omega(A,a)) \rightarrow (\eta_a(W) \subset \mu_a(W))$

(4) $(Ag)(A\gamma)(((g,\gamma) \in P_{(A,a)}) \rightarrow ((g,\gamma) \prec_{(A,a)} (g,\underline{C}(g)(a))))$.

<u>Proof.</u> (1) → (2). $\eta_a \dashv \theta_a$, hence by 1.4.1.2 (9. → 10.) $\eta_a \circ \theta_a = id_{\Omega(A)}$; $\theta_a \dashv \mu_a$, hence by 1.4.1.2 (12. → 13.) $\mu_a \circ \theta_a = id_{\Omega(A)}$. Hence $\eta_a = \mu_a$. □

(2) → (1). $\eta_a \dashv \theta_a$ and $\theta_a \dashv \eta_a$, hence by 1.4.1.2 (1. and 2.), $\eta_a(\theta_a(U)) \subset U$, $W \subset \theta_a(\eta_a(W))$, $\theta_a(\eta_a(W)) \subset W$ and $U \subset \eta_a(\theta_a(U))$ for all $U \in \Omega(A)$ and all $W \in \Omega(A,a)$. Hence $\eta_a \circ \theta_a = id_{\Omega(A)}$ and $\theta_a \circ \eta_a = id_{\Omega(A,a)}$. □

(2) → (3). Clear.□ (3) → (2). We show that for $W \in \Omega(A,a)$, $\mu_a(W) \subset \eta_a(W)$. By 1.4.1.2 (1.), since $\theta_a \dashv \mu_a$, $\theta_a\mu_a(W) \subset W$; by 1.4.1.2 (2.), since $\eta_a \dashv \theta_a$, $W \subset \theta_a\eta_a(W)$. Hence $\theta_a\mu_a(W) \subset \theta_a\eta_a(W)$. Since $\eta_a \circ \theta_a = id_{\Omega(A)}$, by 1.4.1.2 (10. → 11.), $\mu_a(W) \subset \eta_a(W)$. Thus if $\eta_a(W) \subset \mu_a(W)$, then $\eta_a(W) = \mu_a(W)$. □

(3) → (4). Let $(g,\gamma) \in P_{(A,a)}$, then

$g \in \eta_a([(g,\gamma),\rightarrow)) = \{f \in P_A | (E\lambda)((f,\lambda) \in [(g,\gamma),\rightarrow)\}$

hence $g \in \mu_a([(g,\gamma),\rightarrow)) = \{f \in P_A | (g,\gamma) \prec_{(A,a)} (f,\underline{C}(f)(a))\}$

hence $(g,\gamma) \prec_{(A,a)} (g,\underline{C}(g)(a))$. □

(4) → (3). Let $W \in \Omega(A,a)$ and $g \in \eta_a(W) = \{f \in P_A | (E\lambda)((f,\lambda) \in W)\}$, then for some λ, $(g,\lambda) \in W$. By (4), $(g,\lambda) \prec_{(A,a)} (g,\underline{C}(g)(a))$, hence $(g,\underline{C}(g)(a)) \in W$, hence $g \in \mu_a(W) = \{f \in P_A | (f,\underline{C}(f)(a)) \in W\}$. □

<u>1.2.4.2.3 Proposition.</u> Let $\underline{C}$: $\underline{\Lambda}^0 \longrightarrow \underline{Cat}$.

(1) Let $A \in |\underline{\Lambda}|$ and $a \in C_0(A)$. If $\theta_a: \Omega(A) \rightarrowtail \Omega(A,a)$ is an isomorphism then $N_a = \phi$, where N_a is the set of non retractions of $\underline{C}(A)$ having codomain a.

(2) Let $A \in |\underline{\Lambda}|$. If for all $a \in C_0(A)$, θ_a is an isomorphism, then $\underline{C}(A)$ is a groupoid.

(3) The following are equivalent

(3.1) For all $A \in |\underline{\Lambda}|$ and all $a \in C_0(A)$, θ_a is an isomorphism.

(3.2) $\underline{C}$ is an internal groupoid in $\underline{\mathcal{S}}^{\underline{\Lambda}^0}$.

<u>Proof.</u> (1) Let $\gamma \in P_a$, $\gamma: a' \longrightarrow a$ in $\underline{C}(A)$, then $(A,\gamma): (A,a') \longrightarrow (A,a)$ is a morphism of $\underline{\Lambda} \ltimes \underline{C}$. If θ_a is an isomorphism, by 1.2.4.2.3 (4), since $(A,\gamma) \in P_{(A,a)}$,

$$(A,\gamma) \prec_{(A,a)} (A,id_a)$$

that is, $(\mathcal{E}f)(\mathcal{E}\lambda)(((f,\lambda) \in P_{(A,a')})$ and $((A,\gamma)\bullet(f,\lambda) = (A,id_a)))$ hence $(\mathcal{E}\lambda)((\lambda \in P_{a'})$ and $(\gamma\bullet\lambda = id_a))$ hence $\gamma \notin N_a$. Thus $N_a = \phi$. □

(2) If for each $a \in C_0(A)$, θ_a is an isomorphism, then, by (1), each $N_a = \phi$; hence $N(\underline{C}(A)) = \phi$, hence $\underline{C}$ is a groupoid.□

(3) (3.1) → (3.2). The hypothesis implies that $\underline{C}(A)$ is a groupoid for each $A \in |\underline{\Lambda}|$. Hence $\underline{C}$ is an internal groupoid.□

(3.2) → (3.1) Let $A \in |\underline{\Lambda}|$, $a \in C_0(A)$, and $(g,\gamma) \in P_{(A,a)}$. Let (b,b) be the domain of (g,γ) and let $d = \underline{C}(g)(a)$, then $g: B \longrightarrow A$ and $\gamma: b \longrightarrow d$. Since $\underline{C}(B)$ is a groupoid, γ is an isomorphism, $\gamma^{-1}: d \longrightarrow b$, and $(B,\gamma^{-1}): (B,d) \longrightarrow (B,b)$ is a morphism of $\underline{\Lambda} \times \underline{C}$. $(g,\gamma)\bullet(B,\gamma^{-1}) = (g,\gamma\bullet\gamma^{-1}) = (g,\underline{C}(g)(a))$, hence $(g,\gamma) \prec_{(A,a)} (g,\underline{C}(g)(a))$. Thus 1.2.4.2.3 (4) holds, so θ_a is an isomorphism.□

<u>1.2.4.2.4 Proposition</u>. $N_d = \phi$ iff $\theta_d(N_D) = N_{(D,d)}$.

<u>Proof</u>. (→) Assume $N_d = \phi$. We shall show $N_{(D,d)} \subset \theta_d(N_D)$. Let $(f,\gamma)\colon (B,b) \longrightarrow (D,d)$ be in $N_{(D,d)}$ and suppose $(f,\gamma) \notin \theta_d(N_D)$, then $f \notin N_D$, hence there exists $g\colon D \longrightarrow B$ such that $f \circ g = id_D$. We have $\gamma\colon b \longrightarrow \underline{C}(f)(d)$ in $\underline{C}(B)$, hence $\underline{C}(g)(\gamma)\colon \underline{C}(g)(b) \longrightarrow d$. Since $N_d = \phi$, there exists $\delta\colon d \longrightarrow \underline{C}(g)(b)$ such that $(\underline{C}(g)(\gamma))\circ\delta = id_D$: but then $(f,\gamma)\circ(g,\delta) = (D,id_d) = id_{(D,d)}$, hence $(f,\gamma) \notin N_{(D,d)}$, a con tradiction.□

(←) Suppose $N_d \neq \phi$, then there exists $\gamma\colon b \longrightarrow d$ which is not a retraction. Suppose (D,γ) were a retraction, then we would have

$(D,id_d) = (D,\gamma)\circ(f,\delta) = (f,\underline{C}(f)(\gamma)\ \delta)$

hence $f = id_D$, hence $\gamma\circ\delta = id_d$ –a contradiction. Thus $(D,\gamma) \in N_{(D,d)}$. But $(D,\gamma) \notin \theta_d(N_D)$. Hence $N_{(D,d)} \not\subset \theta_d(N_d)$.□

<u>1.2.4.2.5 Proposition</u>. Let $\underline{C}\colon \underline{\Lambda}^0 \longrightarrow \underline{Cat}$ and $\varphi \in$ Poly H.

(1) if $\underline{C}$ is inhabited and $\varphi \in L((\underline{S}^{\underline{\Lambda}^0})^{\underline{C}^0})$ then $\varphi \in L(\underline{S}^{\underline{\Lambda}^0})$.

(2) If $\underline{C}$ is a groupoid and $\varphi \in L(\underline{S}^{\underline{\Lambda}^0})$ then $\varphi \in L((\underline{S}^{\underline{\Lambda}^0})^{\underline{C}^0})$.

<u>Proof</u>. (1) Let $A \in |\underline{\Lambda}|$ and take $a \in C_0(A)$, then $\underset{\sim}{\Omega}(A,a) \vdash \varphi$. Since $\underset{\sim}{\Omega}(A)$ is isomorphic to a subalgebra of $\underset{\sim}{\Omega}(A,a)$, $\underset{\sim}{\Omega}(A) \vdash \varphi$. Hence $\varphi \in L(\underline{S}^{\underline{\Lambda}^0})$.□

(2) Let $A \in |\underline{\Lambda}|$ and let $a \in C_0(A)$. Since $\underline{C}$ is a groupoid $\theta_a\colon \underset{\sim}{\Omega}(A) \xrightarrow{\sim} \underset{\sim}{\Omega}(A,a)$ is an isomorphism. Since $\underset{\sim}{\Omega}(A) \vdash \varphi$, $\underset{\sim}{\Omega}(A,a) \vdash \varphi$. Hence $\varphi \in L((\underline{S}^{\underline{\Lambda}^0})^{\underline{C}^0})$. □

Each part of 1.2.4.2.5 is proven later, (1) in 1.6.2.9 and (2) in 1.6.4.12.

1.2.5 Converse Logic.

1.2.5.1 The deducibility relation, $\vdash$ as a preorder on Poly H.

Intermediate propositional logics Σ are closed, within $\mathcal{P}$(Poly H), under deduction, in the sense that

(1) $((U \vdash \varphi)$ and $(U \subset \Sigma)) \rightarrow (\varphi \in \Sigma)$ holds for all $U \subset$ Poly H and all $\varphi \in$ Poly H.

By restricting U to a single formula, $\vdash$ appears as a binary relation, in fact a preorder, on Poly H; thus

(2) $((\psi \vdash \varphi)$ and $(\psi \in \Sigma)) \rightarrow (\varphi \in \Sigma)$ for all $\psi, \varphi \in$ Poly H, is just the statement that Σ is an ideal of the preordered set (Poly H, $\vdash$).

The union of subsets of Poly H, each of which satisfies (1), does not necessarily satisfy (1). For example, both sets $\Sigma_1 = \{\varphi \mid 2^2 \oplus 1 \vdash \varphi\}$ and $\Sigma_2 = \{\varphi \mid 4 \vdash \varphi\}$ satisfy (1), however $(\{\Delta^2(\underline{0}), R_3\} \vdash R_2)$ and $\{\Delta^2(\underline{0}), R_3\} \subset (\Sigma_1 \cup \Sigma_2)$ but $R_2 \notin (\Sigma_1 \cup \Sigma_2)$. We denote by $\sim U$ the complement of $U \subset$ Poly H within Poly H: $\sim U =$ Poly H $\setminus U$. Idl(Poly H, $\vdash)_{\leq}$ is closed under unions,and set complementation is a bijection :

$$(\text{Idl}(\text{Poly H}, \vdash)_{\leq})^0 \underset{\sim(\ )}{\overset{\sim(\ )}{\rightleftarrows}} \text{Idl}(\text{Poly H}, \vdash)^0_{\leq}.$$

Ideals of (Poly H, $\vdash)^0$ are subsets Σ satisfying

(3) $((\varphi \vdash \psi)$ and $(\psi \in \Sigma)) \rightarrow (\varphi \in \Sigma)$ for all $\psi, \varphi \in$ Poly H. Since (3) arises from (2) by replacing "$\psi \vdash \varphi$" by its converse:"$\varphi \vdash \psi$", we shall call Σ a converse logic.

Let $\mathbf{\Sigma}$ be a nonempty family of i.p.l.'s then for each $\Sigma \in \mathbf{\Sigma}$, $IL \subset \Sigma \subset KL$, hence $IL \subset \bigcup\mathbf{\Sigma} \subset KL$, hence

(4) $\sim KL \subset \sim\bigcup\mathbf{\Sigma} \subset \sim IL$,

that is $\{\varphi \mid \varphi \vdash \underline{0}\} \subset \sim\bigcup\mathbf{\Sigma} \subset \{\varphi \mid \underline{1} \nvdash \varphi\}$. The first inclusion says that all φ which are not valid on $\mathbf{2}$, i.e. which for some valuation become 0, are contained in $\sim\bigcup\Sigma$; the second says that no tautology of IL belongs to $\sim\bigcup\Sigma$.

We shall call a converse logic between ~KL and ~IL an intermediate converse logic.

If $\Sigma' \subset \Sigma$ are both non-empty families of i.p.l.'s then $\sim\bigcup\Sigma \subset \sim\bigcup\Sigma'$.

1.2.5.2 The inconsistent propositional logic and the degenerate topos.

Intermediate propositional logics exclude the inconsistent propositional logic Poly H itself. A topos $\underline{\mathcal{E}}$ is degenerate if true = false; all such toposes are equivalent to a one object topos $\underline{\xi}$, moreover the definition of $L(\underline{\mathcal{E}})$ still makes sense: $L(\underline{\xi}) = \{\varphi \mid \underline{\xi} \models \varphi\} = \text{Poly H}$ and conversely for any topos $\underline{\mathcal{G}}$, $L(\underline{\mathcal{G}}) = \text{Poly H}$ implies $\underline{\mathcal{G}}$ is degenerate.

1.2.5.2.1 Proposition. $\underline{\mathcal{E}}^{\underline{C}^0}$ is a degenerate topos iff $\underline{C}$ is an initial internal category.

Proof. We define $\underline{\mathcal{E}}^{\underline{C}^0}$ in 1.5.1 . Let $\underline{A}$ be the internal category for which $A_0 = A_1 = A_2 = C_0$ and $d_0' = d_1' = i' = m' = id_{C_0}$. There is an internal functor $J: \underline{A} \longrightarrow \underline{C}$ given on objects by id_{C_0} and on morphisms by $i: C_0 \rightarrowtail C_1$. This functor induces a functor $J^*: \underline{\mathcal{E}}^{\underline{C}^0} \longrightarrow \underline{\mathcal{E}}/C_0 = \underline{\mathcal{E}}^{\underline{A}}$ which has both left and right adjoints ([J1] Theorem 2.34) and thus preserves initial and terminal objects.

($\rightarrow$) If $\underline{\mathcal{E}}^{\underline{C}^0} \approx \underline{\xi}$ then $J^*: \underline{\xi} \longrightarrow \underline{\mathcal{E}}/C_0$ forces the initial and terminal objects of $\underline{\mathcal{E}}/C_0$ to be the same, that is $0 \rightarrowtail C_0$ must be isomorphic to $C_0 \xrightarrow{id} C_0$; but then $C_0 \approx 0$, so $\underline{C}$ is initial.□

($\leftarrow$) If $\underline{C}$ is initial then $C_0 \approx 0$, hence $\underline{\mathcal{E}}/C_0 \approx \underline{\xi}$.

The adjunction $J_! \dashv J^*$ gives rise to a morphism $\mathbb{1} \longrightarrow J_! J^*(\mathbb{1})$ in $\underline{\mathcal{E}}^{\underline{C}^0}$; $\mathbb{1} \approx 0$ in $\underline{\mathcal{E}}/C_0$ hence $J_! J^*(\mathbb{1}) \approx J_! J^*(0) \approx J_!(0) \approx \theta$, hence $\underline{\mathcal{E}}^{\underline{C}^0}$ is degenerate.□

1.2.5.2.2 Corollary. $L(\underline{\mathcal{E}}^{\underline{C}^0}) = \text{Poly } H$ iff $\underline{C}$ is initial.

Proof. $L(\underline{\mathcal{E}}^{\underline{C}^0}) = \text{Poly } H$ iff $\underline{\mathcal{E}}^{\underline{C}^0} \approx \underline{\mathcal{E}}$ iff $\underline{C}$ is initial.□

1.2.5.3 Definition. Let $\mathcal{C}$ be a set of internal categories of a topos $\underline{\mathcal{E}}$. We call $\mathcal{C}$ an intermediate class if (a) $\mathcal{C}$ does not contain all internal categories, and (b) $\mathcal{C}$ contains all initial internal categories. Each $\underline{\mathcal{E}}^{\underline{C}^0}$, for $\underline{C} \notin \mathcal{C}$, is nondegenerate hence $IL \subset L(\underline{\mathcal{E}}^{\underline{C}^0}) \subset KL$. Define

$$\gamma(\mathcal{C}) = \sim \bigcup\{L(\underline{\mathcal{E}}^{\underline{C}^0}) \mid \underline{C} \notin \mathcal{C}\},$$

then this set of H-polynomials is an intermediate converse logic. Adopting the conventions that $\underline{C}$ ranges over internal categories of $\underline{\mathcal{E}}$, and φ ranges over Poly H, membership in $\gamma(\mathcal{C})$ can be expressed as:

$\varphi \in \gamma(\mathcal{C})$ iff

for all $\underline{C}$ if $\varphi \in L(\underline{\mathcal{E}}^{\underline{C}^0})$ then $\underline{C} \in \mathcal{C}$.

1.2.5.3.1 Proposition. γ is order preserving.

Proof. If $\mathcal{C} \subset \mathcal{C}'$ then $\{L(\underline{\mathcal{E}}^{\underline{C}^0}) \mid \underline{C} \notin \mathcal{C}'\} \subset \{L(\underline{\mathcal{E}}^{\underline{C}^0}) \mid \underline{C} \notin \mathcal{C}\}$ hence $\gamma(\mathcal{C}) \subset \gamma(\mathcal{C}')$.□

1.2.5.3.2 We define four intermediate classes for a nondegenerate topos $\underline{\mathcal{E}}$.

1. gpds$(\underline{\mathcal{E}})$ is the set of internal groupoids ; $\Gamma(\underline{\mathcal{E}}) = \gamma(\text{gpds}(\underline{\mathcal{E}}))$.

$\varphi \in \Gamma(\underline{\mathcal{E}})$ iff for all $\underline{C}$ if $\varphi \in L(\underline{\mathcal{E}}^{\underline{C}^0})$ then $\underline{C}$ is a groupoid.

2. [mon → gps]$(\underline{\mathcal{E}})$ is the set of all internal categories which are either not monoids or are groups ; $\Gamma_{mon}(\underline{\mathcal{E}}) = \gamma([\text{mon} \to \text{gps}](\underline{\mathcal{E}}))$.

$\varphi \in \Gamma_{mon}(\underline{\mathcal{E}})$ iff for all $\underline{C}$, if $\underline{C}$ is a monoid and $\varphi \in L(\underline{\mathcal{E}}^{\underline{C}^0})$ then $\underline{C}$ is a group.

3. $\mathrm{init}(\underline{\mathcal{E}})$ is the set of all initial internal categories ;

$\Gamma_{\max}(\underline{\mathcal{E}}) = \gamma(\mathrm{init}(\underline{\mathcal{E}}))$.

$\varphi \in \Gamma_{\max}(\underline{\mathcal{E}})$ iff for all $\underline{C}$ if $\varphi \in L(\underline{\mathcal{E}}^{\underline{C}^0})$ then $\underline{C}$ is initial.

4. $[\mathrm{inh} \to \mathrm{gpds}](\underline{\mathcal{E}})$ is the set of all internal categories $\underline{C}$ such that if $\underline{C}$ is inhabited then $\underline{C}$ is a groupoid;

$\Gamma_{\mathrm{inh}}(\underline{\mathcal{E}}) = \gamma([\mathrm{inh} \to \mathrm{gpds}](\underline{\mathcal{E}}))$.

$\varphi \in \Gamma_{\mathrm{inh}}(\underline{\mathcal{E}})$ iff for all $\underline{C}$ if $\underline{C}$ is inhabited and $\varphi \in L(\underline{\mathcal{E}}^{\underline{C}^0})$ then $\underline{C}$ is a groupoid.

The inclusions:

$\mathrm{init}(\underline{\mathcal{E}}) \subset \mathrm{gpds}(\underline{\mathcal{E}}) \subset [\mathrm{inh} \to \mathrm{gpds}](\underline{\mathcal{E}}) \subset [\mathrm{mon} \to \mathrm{gps}](\underline{\mathcal{E}})$ imply

$\Gamma_{\max}(\underline{\mathcal{E}}) \subset \Gamma(\underline{\mathcal{E}}) \subset \Gamma_{\mathrm{inh}}(\underline{\mathcal{E}}) \subset \Gamma_{\mathrm{mon}}(\underline{\mathcal{E}})$.

1.2.5.4 Γ as a function of toposes.

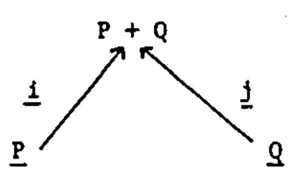
It is not difficult to see that for any toposes $\underline{\mathcal{E}}$ and $\underline{\mathcal{F}}$ $L(\underline{\mathcal{E}} \times \underline{\mathcal{F}}) = L(\underline{\mathcal{E}}) \cap L(\underline{\mathcal{F}})$; the same relationship holds for Γ but a proof would involve us in somewhat detailed internal category theory. What we shall prove is that the relationship holds for presheaf toposes over $\underline{\mathcal{S}}$.

1.2.5.5 Proposition. Let $\underline{P}$ and $\underline{Q}$ be small categories then

$$\Gamma(\underline{\mathcal{S}}^{\underline{P}^0} \times \underline{\mathcal{S}}^{\underline{Q}^0}) = \Gamma(\underline{\mathcal{S}}^{\underline{P}^0}) \cap \Gamma(\underline{\mathcal{S}}^{\underline{Q}^0}).$$

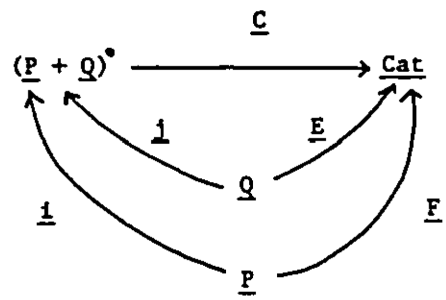
Proof. Let (1) be a coproduct diagram in Cat,

(1)



then opposite, $(\ )^{o}: \underline{Cat} \longrightarrow \underline{Cat}$, involution, applied to (1) is again a coproduct diagram. There is a one-to-one correspondence between pairs of functors $(\underline{E},\underline{F})$ and functors $\underline{C}$ in (2)

(2)



given by $\underline{C} = \underline{F} \sqcup \underline{E}$

where

$$(\underline{F} \sqcup \underline{E})(g) = \begin{cases} \underline{F}(g) & \text{for } g \text{ in } \underline{P} \\ \underline{E}(g) & \text{for } g \text{ in } \underline{Q} \end{cases}$$

and by both $\underline{E} = \underline{C} \cdot \underline{j}$ and $\underline{F} = \underline{C} \cdot \underline{i}$.

For the sake of simplicity we shall assume the morphism sets of $\underline{P}$ and $\underline{Q}$ to be disjoint so that the morphism set of $\underline{P} + \underline{Q}$ is actually the union of the component morphism sets. We will show that $((\underline{P} + \underline{Q}) \ltimes \underline{C}) = (\underline{P} \ltimes \underline{F}) + (\underline{Q} \ltimes \underline{E})$ where $\underline{F} = \underline{C} \cdot \underline{i}$ and $\underline{E} = \underline{C} \cdot \underline{i}$. Let $(g,\gamma): (A,a) \longrightarrow (B,b)$ be a morphism of $(\underline{P} + \underline{Q}) \ltimes \underline{C}$ and suppose $g: A \longrightarrow B$ lies in $\underline{P}$, then $\gamma: a \longrightarrow \underline{C}(g)(b)$ lies in $\underline{C}(A) = (\underline{C} \cdot \underline{i})(A) = \underline{F}(A)$ and $\underline{C}(g) = \underline{F}(g)$ thus (g,γ) is in $\underline{P} \ltimes \underline{F}$. From this argument it is clear that $(\underline{P} + \underline{Q}) \ltimes \underline{C}$ is just the union of the categories $\underline{P} \ltimes \underline{F}$ and $\underline{Q} \ltimes \underline{E}$.

The property of being a groupoid also "distributes" over $\underline{F}$ and $\underline{E}$: $\underline{E} \sqcup \underline{F}$ is a groupoid in $\underline{g}^{(\underline{P} + \underline{Q})^{o}}$

$(\forall B)((B \in |\underline{P} + \underline{Q}|) \rightarrow ((\underline{E} \cup \underline{F})(B)$ is a groupoid)) iff

$(\forall B)(((B \in |\underline{P}|) \rightarrow (\underline{E}(B)$ is a groupoid)) and $((B \in |\underline{Q}|) \rightarrow (\underline{F}(B)$ is a groupoid))) iff

$(\forall B)((B \in |\underline{P}|) \rightarrow (\underline{E}(B)$ is a groupoid)) and $(\forall D)((D \in |\underline{Q}|) \rightarrow (\underline{F}(D)$ is a groupoid)) iff

$\underline{E}$ is a groupoid in $\underline{\mathcal{S}}^{\underline{P}^0}$ and $\underline{F}$ is a groupoid in $\underline{\mathcal{S}}^{\underline{Q}^0}$.

Since $\underline{\mathcal{S}}^{(\underline{P} + \underline{Q})^0} \approx \underline{\mathcal{S}}^{\underline{P}^0} \times \underline{\mathcal{S}}^{\underline{Q}^0}$, it suffices to show

$[\varphi \in \Gamma(\underline{\mathcal{S}}^{(\underline{P} + \underline{Q})^0})] \leftrightarrow$ both $[\varphi \in \Gamma(\underline{\mathcal{S}}^{\underline{P}^0})$ and $\varphi \in \Gamma(\underline{\mathcal{S}}^{\underline{Q}^0})]$.

$(\rightarrow)$ Suppose $\varphi \in \Gamma(\underline{\mathcal{S}}^{(\underline{P} + \underline{Q})^0})$; we will show $\varphi \in \Gamma(\underline{\mathcal{S}}^{\underline{P}^0})$.

Let $\underline{F}$ be any internal category in $\underline{\mathcal{S}}^{\underline{P}^0}$ and suppose $\varphi \in L((\underline{\mathcal{S}}^{\underline{P}^0})^{\underline{F}^0})$.

Let $\underline{0}$ be the initial internal category of $\underline{\mathcal{S}}^{\underline{Q}^0}$ (with values the empty category $\underline{0}$).

$$(\underline{\mathcal{S}}^{\underline{P}^0})^{\underline{F}^0} \approx \underline{\mathcal{S}}^{(\underline{P} \ltimes \underline{F})^0} \times \underline{\mathcal{S}} \approx \underline{\mathcal{S}}^{(\underline{P} \ltimes \underline{F})^0} \times \underline{\mathcal{S}}^{\underline{0}}$$

$$\approx \underline{\mathcal{S}}^{(\underline{P} \ltimes \underline{F})^0} \times \underline{\mathcal{S}}^{(\underline{Q} \ltimes 0)^0} \approx \underline{\mathcal{S}}^{(\underline{P} \ltimes \underline{F} + \underline{Q} \ltimes \underline{0})^0}$$

$$\approx (\underline{\mathcal{S}}^{(\underline{P} + \underline{Q})^0})^{(\underline{F} + \underline{0})^0}$$

thus $\varphi \in L((\underline{\mathcal{S}}^{(\underline{P} + \underline{Q})^0})^{(\underline{F} + \underline{0})^0})$, hence $\underline{F} + \underline{0}$ is a groupoid in $\underline{\mathcal{S}}^{(\underline{P} + \underline{Q})^0}$, hence $\underline{F}$ is a groupoid in $\underline{\mathcal{S}}^{\underline{P}^0}$. Thus $\varphi \in \Gamma(\underline{\mathcal{S}}^{\underline{P}^0})$, and similarly $\varphi \in \Gamma(\underline{\mathcal{S}}^{\underline{Q}^0})$.

$(\leftarrow)$ Suppose both $\varphi \in \Gamma(\underline{\mathcal{S}}^{\underline{P}^0})$ and $\varphi \in \Gamma(\underline{\mathcal{S}}^{\underline{Q}^0})$.

Let $\underline{C}$ be an internal category in $\underline{\mathcal{S}}^{(\underline{P} + \underline{Q})^0}$ such that $\varphi \in L((\underline{\mathcal{S}}^{(\underline{P} + \underline{Q})^0})^{\underline{C}^0})$. Let $\underline{E} \equiv \underline{C} \cdot \underline{i}$ and $\underline{F} \equiv \underline{C} \cdot \underline{i}$.

$(\underline{P} + \underline{Q}) \ltimes \underline{C} \equiv \underline{P} \ltimes \underline{E} + \underline{Q} \ltimes \underline{F}$, hence $\varphi \in L((\underline{\mathcal{S}}^{\underline{P}^0})^{\underline{E}^0} \times (\underline{\mathcal{S}}^{\underline{Q}^0})^{\underline{F}^0})$, hence both $\varphi \in L((\underline{\mathcal{S}}^{\underline{P}^0})^{\underline{E}^0})$ and $\varphi \in L((\underline{\mathcal{S}}^{\underline{Q}^0})^{\underline{F}^0})$, hence both $\underline{E}$ and $\underline{F}$ are groupoids, hence $\underline{C} \equiv \underline{E} \cup \underline{F}$ is a groupoid. □

1.2.5.6 A calculation of $\Gamma(\mathcal{S}^{\underline{M}^0})$.

We show that under certain conditions on a monoid $\underline{M}$ we can give a polynomial ψ such that

$$\varphi \in \Gamma(\mathcal{S}^{\underline{M}^0}) \text{ iff } \varphi \vdash \psi$$

We take $\underline{M}$ to be a category with one object, $|\underline{M}| = \{A\}$, and we let $\alpha = \underline{\Omega}(A) \oplus 1$ where $\underline{\Omega}(A)$ is the algebra of right ideals of $\underline{M}$. Throughout φ and ψ will be polynomials.

1.2.5.6.1 Proposition If $\underline{M}$ has finitely many right ideals then

$$\varphi \vdash X_\alpha \quad \text{implies} \quad \varphi \in \Gamma(\mathcal{S}^{\underline{M}^0}).$$

Proof. We must show that for all $\underline{C}: \underline{M}^0 \longrightarrow \underline{Cat}$, if for all $a \in C_0(A)$, $\underline{\Omega}(A,a) \vdash X_\alpha$, then for all $a \in C_0(A)$, $\theta_a: \underline{\Omega}(A) \longrightarrow \underline{\Omega}(A,a)$ is an isomorphism. Let $\underline{C}: \underline{M}^0 \longrightarrow \underline{Cat}$ and assume $\theta_b: \underline{\Omega}(A) \rightarrowtail \underline{\Omega}(A,b)$ is not an isomorphism. By 1.1.4.6.5 $\underline{\Omega}(A) \oplus 1$ is isomorphic to a subalgebra of $\underline{\Omega}(A,b)$; that is, $\alpha \in IS\{\underline{\Omega}(A,b)\}$. Hence $\alpha \in SH\{\underline{\Omega}(A,b)\}$, hence, by 1.1.4.4. (2), $\underline{\Omega}(A,b) \not\vdash X_\alpha$.□

1.2.5.6.2 We shall introduce sufficient conditions on $\underline{M}$ which allow the converse: $\varphi \not\vdash X_\alpha$ implies $\varphi \notin \Gamma(\mathcal{S}^{\underline{M}^0})$.

By 1.1.4.4 this is: if $\alpha \vdash \varphi$ then for some $\underline{C}: \underline{M}^0 \longrightarrow \underline{Cat}$ we have: for all $a \in C_0(A)$, $\underline{\Omega}(A,a) \vdash \varphi$ but for some $b \in C_0(A)$ $\theta_b: \underline{\Omega}(A) \rightarrowtail \underline{\Omega}(A,b)$ is not an isomorphism.

If we can define $\underline{C}$ so that $C_0(A) \neq \phi$ and for all $a \in C_0(A)$ $\underline{\Omega}(A,a) \approx \alpha$, this will give us our converse. We shall make $\underline{C}(A)$ as small as possible, without being a groupoid, in order to make $N_{(A,a)}$ the only ideal of $\underline{\Omega}(A,b)$ not in the image of θ_b; thus we take $\underline{C}(A)$ to be isomorphic to the monoid $\underline{M}_1$. In order to extend $\underline{C}$ to a functor we shall need N_A in $\underline{\Omega}(A)$ to be a two sided ideal of $\underline{M}$.

1.2.5.6.3 Proposition. Let N be the right ideal of non-retractions of $\underline{M}$. The following are equivalent.

(1) N is a left ideal.

(2) Every section is a retraction.

(3) Sections and retractions are isomorphisms.

Proof. (1)$\rightarrow$(2). Let $r \cdot s = id_A$, then $s \in N$ is not possible, hence s is a retraction. □

(2)$\rightarrow$(3) If $r \cdot s = id_A$, s is an isomorphism with $s^{-1} = r$, hence $r^{-1} = s$. □

(3)$\rightarrow$(1) Let $f \in N$ and suppose $g \cdot f \notin N$. Then $(g \cdot f)^{-1} \cdot (g \cdot f) = id_A$, hence f is a section, hence a retraction -contradicting $f \in N$.□

For such a monoid $(M \setminus N)$ is the group of units and N the two sided ideal of morphisms which are neither left nor right invertible. We give a sufficient condition on N for it to be a left ideal.

1.2.5.6.4 Proposition. If for all $f \in N$, $f \in fN$, then N is a left ideal.

Proof. Suppose to the contrary that there exist $g \in M$ and $f \in N$ such that $gf \notin N$. There exists $k \in N$ such that $f = f \cdot k$ and there exists s such that $gfs = 1$. We will show that both $s \notin N$ and $s \in N$ lead to contradictions. If $s \notin N$ then for some t, $st = 1$.

$t = (gfs)t = (gf)st = gf$ hence

$sgf = 1$

$1 = sgf = sgfk = k$ but $k \in N$ -a contradiction.

If $s \in N$ then for some u, $u \in N$ and $su = s$.

$1 = gfs = gfsu = u$ -a contradiction.□

1.2.5.6.5 Construction of "the minimal counterexample" $\underline{C}: \underline{M}^0 \longrightarrow \underline{Cat}$ for N a left ideal. We let $|\underline{C}(A)| = \{a\}$, $\underline{C}(A) = \{id_a, \sigma\}$ with $\sigma^2 = \sigma$. For r a unit we let

$$\underline{C}(r) = id_{\underline{C}(A)}$$

then it is clear that $\underline{C}$ preserves (contravariantly) composition of units. For $f \in N_A$ we let

$$\underline{C}(f)(\sigma) = \underline{C}(f)(id_a) = id_a$$

that is, $\underline{C}(f)$ is the constant functor on $\underline{C}(A)$ with value id_a on morphisms. Clearly $\underline{C}$ co-preserves composition of non-units.

For $f \in N_A$ and r a unit, $f \cdot r \in N_A$ and $r \cdot f \in N_A$.

$\underline{C}(r \cdot f) = \underline{C}(f) = \underline{C}(f) \cdot \underline{C}(r)$

$\underline{C}(f \cdot r) = \underline{C}(f) = \underline{C}(r) \cdot \underline{C}(f)$.

To emphasize the dependency of this internal category on $\underline{M}$ we will denote it by "$\underline{C}_{\underline{M}}$".

1.2.5.6.6 Proposition. Let $\underline{M}$ be a category with one object A, and suppose that for all morphisms f, if $f \in N_A$ then $f \in f \cdot N_A$. Then

$$\Gamma(\underline{\mathcal{S}}^{\underline{M}^0}) = \sim\mathcal{L}((\underline{\mathcal{S}}^{\underline{M}^0})^{\underline{C}^0_{\underline{M}}}).$$

Proof We define $\underline{C}: \underline{M}^0 \longrightarrow \underline{Cat}$ to be the minimal counterexample of 1.2.5.6.5, thus $\underline{C}$ is not a groupoid. We shall show that $\Omega(A,a) = \theta_a(\Omega(A)) \cup \{N_{(A,a)}\}$. We do this in two steps: (a) we show that $N_{(A,a)} = [(r,\sigma),\rightarrow)$ for all units r. From (a) it follows that any proper right ideal properly contained in $N_{(A,a)}$ must be contained in $\theta_a(N_A)$. (b) we show that $R = \theta_a(\eta_a(R))$ for $R \subset \theta_a(N_A)$.

(a) We have $(r,\sigma) \cdot (r^{-1},\sigma) = (id_A, \underline{C}(r^{-1})(\sigma) \cdot \sigma) = (id_A, \sigma)$, for r a unit, hence for r a unit, $[(r,\sigma),\rightarrow) = [(id_A,\sigma),\rightarrow)$. We show $(id_A,\sigma) \in N_{(A,a)}$:

$(id_A,\sigma)\circ(f,\gamma) = (id_A,id_a)$ iff $f = id_A$ and $\sigma = id_a$; $\sigma \neq id_a$, hence $(id_A,\sigma) \in N_{(A,a)}$. Hence $[(id_A,\sigma),\rightarrow) \subset N_{(A,a)}$

$$(id_A,\sigma)\cdot(f,\gamma) = (f,\underline{C}(f)(\sigma)\cdot\gamma) = \begin{cases}(f,\sigma) & \text{if } f \text{ is a unit}\\ (f,\gamma) & \text{if } f \in N_A\end{cases}$$

$\theta_a(N_A) = N_A \times \{id_a,\sigma\}$ hence $[(id_A,\sigma),\rightarrow) = \theta_a(N_A) \cup ((M \smallsetminus N_A) \times \{\sigma\})$

Thus $(f,\gamma) \notin [(id_A,\sigma),\rightarrow)$ iff $(f,\gamma) \in ((M \smallsetminus N_A) \times \{id_a\})$. But for r a unit, $(r,id_a)\cdot(r^{-1},id_a) = (id_A, id_a)$, thus $(f,\gamma) \notin [(id_A,\sigma),\rightarrow)$ implies $(f,\gamma) \notin N_{(A,a)}$. Thus $N_{(A,a)} = [(r,\sigma),\rightarrow)$ for all units r.

(b) Let $R \subsetneq N_{(A,a)}$, then for r a unit, $(r,\sigma) \notin R$. Thus $R \subset \theta_a(N_A)$.

We have
$$R \subset \theta_a\eta_a(R)$$
we want to show
$$\theta_a\eta_a(R) \subset R.$$

$(f,\gamma) \in \theta_a\eta_a(R)$ iff

$f \in \eta_a(R)$ iff

$(f,id_a) \in R$ or $(f,\sigma) \in R$.

We have $(f,id_a)\cdot(id_A,\sigma) = (f,\sigma)$, thus $(f,id_a) \in R$ implies $(f,\sigma) \in R$.

Thus $(f,\gamma) \in \theta_a\eta_a(R)$ iff $(f,\sigma) \in R$.

We thus want to show

$$(f,\sigma) \in R \text{ implies } (f,id_a) \in R.$$

$(f,\sigma) \in R$ implies $f \in N_A$. We invoke the hypothesis: there exists $g \in N_A$ such that $f\cdot g = f$, hence

$(f,\sigma)\cdot(g,id_a) = (f\cdot g,(\underline{C}(g)(\sigma))\cdot id_a) = (f,id_a)$.

Hence $R = \theta_a(\eta_a(R))$.

Now let $\underline{D}: \underline{M}^0 \longrightarrow \underline{Cat}$ be any non-groupoid of $\underline{\mathcal{S}}^{\underline{M}^0}$ and let $\underline{\Omega}'$ be the subobject classifier of $\underline{\mathcal{S}}^{(\underline{M} \ltimes \underline{D})^0} \approx (\underline{\mathcal{S}}^{\underline{M}^0})^{\underline{D}^0}$. There exists $d \in D_0(A)$ such that $\theta'_d: \underline{\Omega}(A) \rightarrowtail \underline{\Omega}'(A,d)$ is not an isomorphism, hence $\theta_d'(\Omega(A)) \cup \{N'_{(A,d)}\} \approx \underline{\Omega}(A) \oplus 1 \approx \underline{\Omega}(A,a)$ is a subalgebra of $\underline{\Omega}'(A,d)$.

Hence $\varphi \in L((\underline{\mathcal{S}}^{\underline{M}^0})^{\underline{D}^0}) \to \underline{\Omega}'(A,d) \vdash \varphi$

$$\to \underline{\Omega}(A,a) \vdash \varphi$$

$$\to \varphi \in L((\underline{\mathcal{S}}^{\underline{M}^0})^{\underline{C}^0_{\underline{M}}}).$$

Thus $$\Gamma(\underline{\mathcal{S}}^{\underline{M}^0}) \equiv \sim \bigcup\{L((\underline{\mathcal{S}}^{\underline{M}^0})^{\underline{D}^0}) \mid \underline{D} \notin \mathrm{gpd}(\underline{\mathcal{S}}^{\underline{M}^0})\} \equiv \sim L((\underline{\mathcal{S}}^{\underline{M}^0})^{\underline{C}^0_{\underline{M}}}).\square$$

<u>1.2.5.6.7 Corollary</u>. Let $\underline{M}$ be a category with one object A, having finitely many right ideals and such that for all morphisms f, if $f \in N_A$ then $f \in f \cdot N_A$. Let $\alpha \equiv \underline{\Omega}(A) \oplus 1$. Then

$$\Gamma(\underline{\mathcal{S}}^{\underline{M}^0}) \equiv \{\varphi \mid \varphi \vdash \chi_\alpha\}$$

<u>Proof</u>. By 1.2.5.6.6, $\varphi \in \Gamma(\underline{\mathcal{S}}^{\underline{M}^0})$ iff $\underline{\Omega}(A,a) \not\vdash \varphi$ in $\underline{\mathcal{S}}^{(\underline{M} \times \underline{C}_{\underline{M}})^0}$. iff $\alpha \not\vdash \varphi$ iff $\varphi \vdash \chi_\alpha$, by 1.1.4.4.$\square$

<u>1.2.5.7 Examples of $\Gamma(\underline{\mathcal{S}}^{\underline{\Lambda}^0})$.</u>

A monoid $\underline{M}$ which is a finite band satisfies the hypothesis of 1.2.5.6.7: $N_A \equiv (M - \{1\})$ and $(f \neq 1) \to (f \circ f \equiv f)$ Our first two examples are of this kind.

<u>1.2.5.7.1 Finite chains</u>. Let $\underline{S}_n$ be the $n+1$ element chain considered as a monoid, with meet as multiplication (as defined in 1.2.3.3.4.1) then

$$\alpha \equiv \underline{\Omega}(A) \oplus 1 \approx \underline{\underline{\mathrm{Idl}}}(S_n, <_A) \oplus 1$$

$$\approx S_{n+1} \oplus 1 \approx S_{n+2} \approx \mathbf{n} \oplus \mathbf{3}$$

$\chi_\alpha \equiv \chi_{\mathbf{n} \oplus \mathbf{3}} \dashv\vdash \Delta^{n+1}(Q)$ by 1.1.4.6.7, hence

$$\Gamma(\underline{\mathcal{S}}_n) \equiv \{\varphi \mid \varphi \vdash \Delta^{n+1}(\underline{Q})\} \equiv \{\varphi \mid \mathbf{n} \oplus \mathbf{3} \not\vdash \varphi\}.$$

1.2.5.7.2 Let $\underline{M}_n$ $(n < \infty)$ be the monoids of 1.2.3.2. (I). We put $N_A = L$, where card $L = n$, then $\alpha = \Omega(A) \oplus 1 \approx 2^n \oplus 2$ hence

$$\Gamma(\dot{\mathcal{E}}_n) = \{\varphi \mid \varphi \vdash X_{2^n \oplus 2}\}$$

1.2.5.7.3 A category $\underline{\Lambda}$ which is a disjoint union of finitely many monoids $\underline{P}_i$ each of which satisfies the hypothesis of 1.2.5.5.7 gives rise to a converse logic

$$\Gamma(\underline{\mathcal{S}}^{\Lambda^0}) = \bigcap_{i=1}^{n} \Gamma(\underline{\mathcal{S}}^{\underline{P}_i^0}) = \bigcap_{i=1}^{n} \{\varphi \mid \varphi \vdash X_{\alpha_i}\}.$$

1.2.5.7.4 n-truncated simplicial sets.

Let $[n]$ be the totally ordered set $\{0,1,.., n-1\}$. We show that $\underline{\text{End}}\ [n]$, the full subcategory of $\underline{\Delta}_n$ whose only object is $[n]$, satisfies the hypothesis of 1.2.5.5.7.

If $n = 1$ then $\underline{\text{End}}\ [n]$ is the trivial group.

If $n > 0$ then $N_{[n]} \neq \phi$ (within $\underline{\text{End}}\ [n]$).

For each $i \in [n-1]$ define $g_i\colon [n] \longrightarrow [n]$ by

$$g_i(j) = \begin{cases} i+1 & \text{if } j = i \\ j & \text{if } j \neq i \end{cases}$$

then $g_i(i) = i + 1 = g_i(i+1)$, hence $g_i \in N_{[n]}$. For each $f \in N_{[n]}$, since f cannot be a mono, there exists $i \in [n-1]$ such that $f(i) = f(i+1)$; hence $(f \circ g_i)(i) = f(i+1) = f(i)$ and for $j \neq i$, $(f \circ g_i)(j) = f(j)$, hence $f \circ g_i = f$.

Thus $\underline{\text{End}}\ [n]$ satisfies the hypothesis of 1.2.5.5.7.

If $\alpha(n)$ is the algebra of right ideals of $\underline{\text{End}}\ [n]$, then

$$\Gamma(\underline{\mathcal{S}}^{\underline{\Delta}_n^0}) = \Gamma(\underline{\mathcal{S}}^{\underline{\text{End}}[n]^0}) = \{\varphi \mid \varphi \vdash X_{\alpha(n) \oplus 1}\}.$$

1.2.5.7.5 A topos $\mathcal{E}$ for which $\sim\Gamma(\mathcal{E})$ is not an i.p.l.

$\Gamma(\mathcal{G}ph \times \mathcal{G}_2) = \Gamma(\mathcal{S}^{M_2^0}) \cap \Gamma(\mathcal{S}^{S_2^0})$

$= \{\varphi \mid \varphi \vdash X_{2^2 \oplus 2}\} \cap \{\varphi \mid \varphi \vdash \Delta^3(\underline{0})\}$

$\sim\Gamma(\mathcal{G}ph \times \mathcal{G}_2) = \{\varphi \mid 2^2 \oplus 2 \vdash \varphi\} \cup \{\varphi \mid \mathbf{5} \vdash \varphi\} = \Sigma$

$2^2 \oplus 2 \vdash \Delta^3(\underline{0})$

$\mathbf{5} \vdash Z$

If Σ were an i.p.l. we would have, since $\{\Delta^3(\underline{0}), Z\} \subset \Sigma$, and $\Delta^3(\underline{0}) \wedge Z \vdash R_3$ (1.1.3.2), that $R_3 \in \Sigma$. But $2^2 \oplus 2 \nvdash R_3$ since this algebra is not a chain, and $\mathbf{5} \nvdash R_3$ since the length of a chain on which R_3 is valid, is at most 4 (1.1.3.1). Hence $R_3 \notin \Sigma$. Thus Σ is not an i.p.l.

1.2.5.7.6 The converse logics, $\Gamma(\mathcal{E}_n)$.

We shall show in 1.6.4.1 that for any topos $\mathcal{E}$, $v \vee \neg v \in \Gamma(\mathcal{E})$. Here we shall show that $\Gamma(\mathcal{E}_n) = \{\varphi \mid \varphi \vdash v \vee \neg v\}$, thus $\Gamma(\mathcal{E}_n)$ is the smallest possible converse logic. This also establishes the independence of Γ from L: $L(\mathcal{E}_n) = L(\dot{\mathcal{E}}_n)$ but $\Gamma(\mathcal{E}_n) \subsetneq \Gamma(\dot{\mathcal{E}}_n)$.

1.2.5.7.6.1 Proposition. $\Gamma(\mathcal{E}_n) = \{\varphi \mid \varphi \vdash v \vee \neg v\}$.

Proof. We have $v \vee \neg v \dashv\vdash \Delta(\underline{0}) \dashv\vdash X_{\mathbf{3}}$, hence $\varphi \vdash v \vee \neg v$ iff $\varphi \vdash X_{\mathbf{3}}$ iff $\mathbf{3} \nvdash \varphi$. We shall show $\sim\Gamma(\mathcal{E}_n) = \{\varphi \mid \mathbf{3} \vdash \varphi\}$.

($\supset$): We want to show $\{\varphi \mid \mathbf{3} \vdash \varphi\} \subset \sim\Gamma(\mathcal{E}_n)$. Define $\underline{D}: \Lambda_n^{\ 0} \longrightarrow \underline{Cat}$ by $|\underline{D}(V)| = \{v\}$, $\underline{D}(V) = \{id_v, \mu\}$ and $\mu \circ \mu = \mu$, so that $\underline{D}(V)$ is isomorphic to $\underline{M}_1$; $\underline{D}(A) = \phi$, so for any $\ell: V \longrightarrow A$, we have $\underline{D}(\ell): \phi \longrightarrow \underline{D}(V)$.

We compute $\Lambda_n \ltimes \underline{D}$. $|\Lambda_n \ltimes \underline{D}| = \{(v,v)\}$ and $\Lambda_n \ltimes \underline{D} = \{id_{(v,v)}, (V,\mu)\}$ with $(V,\mu) \circ (V,\mu) = (V,\mu)$ so that $\Lambda_n \ltimes \underline{D}$ is also isomorphic to $\underline{M}_1$.

Hence $\{\varphi \mid \mathbf{3} \vdash \varphi\} = L(\underline{\mathcal{E}}_1) = L(\underline{\mathcal{E}}_n^{\underline{D}^0}) \subset \sim \Gamma(\underline{\mathcal{E}}_n)$.

($\subset$): We show $\sim \Gamma(\underline{\mathcal{E}}_n) \subset \{\varphi \mid \mathbf{3} \vdash \varphi\}$. Suppose $\varphi \in \sim \Gamma(\underline{\mathcal{E}}_n)$, then for some non-groupoid $\underline{C}: \underline{\Lambda}_n{}^0 \longrightarrow \underline{Cat}$ we have $\varphi \in L(\underline{\mathcal{E}}_n^{\underline{C}^0})$. First we assume $\underline{C}(A) \neq \phi$, and let $a \in |\underline{C}(A)|$. We have an embedding

$$\theta_a: \underset{\sim}{\Omega}(A) \rightarrowtail \underset{\sim}{\Omega}(A,a)$$

where $\underset{\sim}{\Omega}(A) \approx \mathbf{2}^n \oplus 1$. Since $\mathbf{3}$ is a subalgebra of $\mathbf{2}^n \oplus 1$, it is also a subalgebra of $\underset{\sim}{\Omega}(A,a)$, hence $\mathbf{3} \vdash \varphi$.

Now we suppose $\underline{C}(A) = \phi$. For each $\ell: V \longrightarrow A$ we have $\underline{C}(\ell): \phi \longrightarrow \underline{C}(V)$. Since $\underline{C}$ is not a groupoid, $\underline{C}(V)$ is not a groupoid. Hence for some $v \in |\underline{C}(V)|$, we have $N_v \neq \phi$. By 1.2.3.2.5, $\theta_v: \underset{\sim}{\Omega}(V) \longrightarrow \underset{\sim}{\Omega}(V,v)$ is not an isomorphism. Since $\underset{\sim}{\Omega}(V) \approx \mathbf{2}$, $\mathbf{3}$ must be a subalgebra of $\underset{\sim}{\Omega}(V,v)$, hence $\mathbf{3} \vdash \varphi$. □

<u>1.2.5.8 Examples: $\Gamma_{mon}(\mathcal{E}_n)$.</u>

The calculation of $\Gamma(\mathcal{E}_n)$ in 1.2.5.7.5.1 implies that even though $L(\mathcal{Sierp}) = \{\varphi \mid R_2 \vdash \varphi\}$, so that R_2 appears to play a role analogous to that of $v \vee \neg v$ for $\mathcal{S}$, yet there must exist a non groupoid $\underline{C}$ in $\mathcal{Sierp}$ such that R_2 remains valid in $\mathcal{Sierp}^{\underline{C}^0}$. The calculations we now make will show that $\Gamma_{mon}(\mathcal{Sierp}) = \{\varphi \mid \varphi \vdash \Delta^2(\underline{0})\}$; and hence that for any monoid $\underline{M}$ in $\mathcal{Sierp}$, if R_2 is valid in $\underline{M}$-$\mathcal{Sierp}$ then $\underline{M}$ is a group. Thus we have $\Gamma(\mathcal{Sierp}) \subsetneq \Gamma_{mon}(\mathcal{Sierp})$ and $\Gamma_{mon}(\mathcal{Sierp}) = \Gamma(\mathcal{E}_1)$.

<u>1.2.5.8.1 Proposition.</u> For any n, $X_{\mathbf{4}} \in \Gamma_{mon}(\mathcal{E}_n)$.

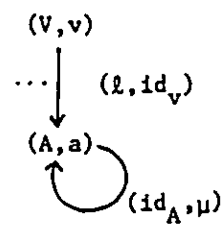
<u>Proof</u>. Let $\underline{M}\colon \underline{\Lambda}_n{}^0 \longrightarrow \underline{Cat}$ be an internal monoid which is not a group; we shall show that $X_{\mathbf{4}} \notin L(\mathcal{E}_n^{\underline{M}^0})$. Let $|\underline{M}(A)| = \{a\}$ and $|\underline{M}(V)| = \{v\}$. Either $\underline{M}(A)$ or $\underline{M}(V)$ is not a group. Suppose first that $\underline{M}(A)$ is not a group. By 1.2.4.2.3 (1) and 1.2.4.2.4, $\theta_a(N_A)$ is properly contained in $N_{(A,a)}$, hence $\{\phi, \theta_a(N_A), N_{(A,a)}, P_{(A,a)}\}$ constitutes a subalgebra of $\underline{\Omega}(A,a)$, thus $X_{\mathbf{4}} \in SH\{\underline{\Omega}(A,a)\}$. By 1.1.4.4, $\underline{\Omega}(A,a) \not\models X_{\mathbf{4}}$, hence $X_{\mathbf{4}} \notin L(\mathcal{E}^{\underline{M}^0})$. Suppose now that $\underline{M}(V)$ is not a group, so that $N_v \neq \phi$. We claim $L \times N_v$ is in $\Omega(A,a)$. Let $(\ell,\gamma) \in L \times N_v$ then $(\ell,\gamma)\colon (V,v) \longrightarrow (A,a)$ and $\gamma\colon v \longrightarrow \underline{M}(\ell)(a)$, so $(\ell,\gamma) \in P_{(A,a)}$. The morphisms composable on the right with (ℓ,γ) are of the form $(id_V,\delta)\colon (V,v) \longrightarrow (V,v)$; composing, we get $(\ell,\gamma)\circ(id_V,\delta) = (\ell,\gamma\circ\delta)$. Thus $L \times N_v$ is an ideal. Since $(\ell, id_v)\colon (V,v) \longrightarrow (A,a)$ is in $N_{(A,a)}$ but not in $L \times N_v$, $\{\phi, L \times N_v, N_{(A,a)}, P_{(A,a)}\}$ is a four element subalgebra of $\underline{\Omega}(A,a)$, thus $\mathbf{4} \in SH\{\underline{\Omega}(A,a)\}$, so $X_{\mathbf{4}} \notin L(\mathcal{E}_n^{\underline{M}^0})$.□

The above proof does not have a simple generalization to $X_{\mathbf{2}^n \oplus \mathbf{1}} \in \Gamma_{mon}(\mathcal{E}_n)$. If $\underline{M}(A)$ is not a group then we have

$2^n \oplus 2 \approx \theta_a(\underline{\Omega}(A)) \cup \{N_{(A,a)}\}$ and this is a subalgebra of $\underline{\Omega}(A,a)$ so $X_{2^n \oplus 2} \notin L(\underline{\mathcal{E}}_n^{\underline{M}^o})$. But if $\underline{M}(A)$ is a group and $\underline{M}(V)$ is not, we cannot replace $\{\phi \times N_V, L \times N_V\}$ by $\{U \times N_V \mid U \subset L\}$ and still have a subalgebra (unless $|L| = 1$).

<u>1.2.5.8.2 Proposition</u>. $\varphi \in \Gamma_{mon}(\underline{\mathcal{E}}_n)$ implies $\varphi \vdash X_{2^n \oplus 2}$ for $n < \infty$.

<u>Proof</u>. We suppose $\varphi \not\vdash X_{2^n \oplus 2}$ so that, by 1.1.4.4, $2^n \oplus 2 \vdash \varphi$. We shall exhibit amonoid $\bar{\underline{M}}: \underline{\Lambda}_n{}^o \longrightarrow \underline{Cat}$, which is not a group, and for which $\varphi \in L(\underline{\mathcal{E}}_n^{\bar{\underline{M}}^o})$. We let $|\bar{\underline{M}}(A)| = \{a\}$, $|\bar{\underline{M}}(V)| = \{v\}$, $\bar{\underline{M}}(A) = \{id_a, \mu\}$ $\mu \circ \mu = \mu$, and $\bar{\underline{M}}(V) = \{id_v\}$. For $\ell \in L = [V,A]$ we put $\bar{\underline{M}}(\ell)(\mu) = \bar{\underline{M}}(\ell)(id_a) = id_v$. We exhibit the non-identity morphisms of $\underline{\Lambda}_n \ltimes \bar{\underline{M}}$.



$\ell \in [V,A]$

The non-trivial compositions are:

$(id_A,\mu)\circ(id_A,\mu) = (id_A,\mu)$ and $(id_A,\mu)\circ(\ell,id_V) = (\ell,(\bar{\underline{M}}(\ell)(\mu))\circ id_V) = (\ell,id_V)$.

Thus $[(id_A,\mu),\to) = \{(id_A,\mu)\} \cup (L \times \{id_V\})$ and for each $U \subset L$, $U \times \{id_V\}$ is an ideal. Thus $\underline{\Omega}(A,a) \approx 2^n \oplus 2$. Also $\Omega(V,v) = \{\phi,\{(id_V,id_V)\}\}$, hence $\underline{\Omega}(V,v) \approx 2$. Since $2^n \oplus 2 \vdash \varphi$, we have φ internally valid in $\underline{\Omega}$, thus $\varphi \in L(\underline{\mathcal{E}}_n^{\bar{\underline{M}}^o})$. □

<u>1.2.5.8.3 Corollary</u>. For $n < \infty$, $\{\varphi \mid \varphi \vdash X_{\frac{1}{4}}\} \subset \Gamma_{mon}(\underline{\mathcal{E}}_n) \subset \{\varphi \mid \varphi \vdash X_{2^n \oplus 2}\}$, and so in particular, $\Gamma_{mon}(\underline{\mathcal{E}}_1) = \{\varphi \mid \varphi \vdash X_{\frac{1}{4}}\}$ □

1.2.5.9 Examples: $\Gamma_{inh}(\underline{\mathcal{E}_n})$.

Since this converse logic lies above $\Gamma(\underline{\mathcal{E}_n})$ and, for $n < \infty$, lies below $\Gamma_{mon}(\underline{\mathcal{E}_n})$, we have:

$$\{\varphi | \varphi \vdash v \vee \neg v\} \subset \Gamma_{inh}(\underline{\mathcal{E}_n}) \subset \{\varphi | \varphi \vdash X_{2^n \oplus 2}\} .$$

We shall show that $\Gamma_{inh}(\underline{\mathcal{S}ierp}) = \{\varphi | \varphi \vdash v \vee \neg v\} = \Gamma(\underline{\mathcal{S}ierp})$.

We have shown $\Gamma_{mon}(\underline{\mathcal{S}ierp}) = \{\varphi | \varphi \vdash X_4\}$. Thus the degeneracy of the converse logic $\Gamma(\underline{\mathcal{S}ierp})$ is not remedied by simply requiring that the internal categories, used to define the converse logic, be inhabited.

1.2.5.9.1 Proposition. For $n < \infty$, $\Gamma_{inh}(\underline{\mathcal{E}_n}) \subset \{\varphi | \varphi \vdash X_{2^n \oplus 1}\}$.

Proof. Let $\underline{D}: \underline{\Lambda_n}^0 \longrightarrow \underline{Cat}$ be the internal category described in the proof of 1.2.5.7.5.1. Let $\underline{1}: \underline{\Lambda_n}^0 \longrightarrow \underline{Cat}$ be a terminal internal category: $|\underline{1}(V)| = \{u\}$, $|\underline{1}(A)| = \{a\}$, with u, a and v distinct, and for each $\ell: V \longrightarrow A$, $\underline{1}(\ell)$ is the uniquely determined map from $\{a\}$ to $\{u\}$. Let $\bar{\underline{D}}$ be the coproduct $\underline{D} + \underline{1}$ in $\underline{Cat}(\underline{\mathcal{E}_n})$, so that $|\bar{\underline{D}}(A)| = \{a\}$ (since $\underline{D}(A) = \phi$), $\bar{\underline{D}}(A) = \{id_a\}$, $|\bar{\underline{D}}(V)| = \{v,u\}$, and $\bar{\underline{D}}(V) = \{id_v, id_u, \mu\}$, where $\mu \cdot \mu = \mu$. Then $\bar{\underline{D}}$ is inhabited.

$\underline{\Lambda_n} \ltimes (\underline{D} + \underline{1}) = (\underline{\Lambda_n} \ltimes \underline{D}) + (\underline{\Lambda_n} \ltimes \underline{1})$, hence $\underline{\mathcal{E}_n}^{\bar{\underline{D}}^0} \approx \underline{\mathcal{E}_n}^{\underline{D}^0} \times \underline{\mathcal{E}_n}$. But from the proof of 1.2.5.7.5.1, $\underline{\mathcal{E}_n}^{\underline{D}^0} \approx \dot{\underline{\mathcal{E}}}_1$, hence $\underline{\mathcal{E}_n}^{\bar{\underline{D}}_n^0} \approx \dot{\underline{\mathcal{E}}}_1 \times \underline{\mathcal{E}_n}$. Therefore

$$L(\underline{\mathcal{E}_n}^{\bar{\underline{D}}^0}) = L(\dot{\underline{\mathcal{E}}}_1) \cap L(\underline{\mathcal{E}_n}) = \{\varphi | 3 \vdash \varphi\} \cap \{\varphi | 2^n \oplus 1 \vdash \varphi\}$$
$$= \{\varphi | 2^n \oplus 1 \vdash \varphi \}$$

hence $\{\varphi | 2^n \oplus 1 \vdash \varphi\} \subset \sim \Gamma_{inh}(\underline{\mathcal{E}_n})$, hence $\Gamma_{inh}(\underline{\mathcal{E}_n}) \subset \{\varphi | \varphi \vdash X_{2^n \oplus 1}\}$.□

1.2.5.9.2 Corollary. $\Gamma_{inh}(\underline{\mathcal{S}ierp}) = \{\varphi | \varphi \vdash v \vee \neg v\}$.□

<u>1.2.5.9 Examples: $\Gamma_{inh}(\underline{\mathcal{E}}_n)$.</u>

Since this converse logic lies above $\Gamma(\underline{\mathcal{E}}_n)$ and, for $n < \infty$, lies below $\Gamma_{mon}(\underline{\mathcal{E}}_n)$, we have:

$$\{\varphi|\varphi \vdash v \vee \neg v\} \subset \Gamma_{inh}(\underline{\mathcal{E}}_n) \subset \{\varphi| \varphi \vdash X_{2^n \oplus 2}\} .$$

We shall show that $\Gamma_{inh}(\underline{\mathit{Sierp}}) = \{\varphi| \varphi \vdash v \vee \neg v\} = \Gamma(\underline{\mathit{Sierp}})$.
We have shown $\Gamma_{mon}(\underline{\mathit{Sierp}}) = \{\varphi| \varphi \vdash X_4\}$. Thus the degeneracy of the converse logic $\Gamma(\underline{\mathit{Sierp}})$ is not remedied by simply requiring that the internal categories, used to define the converse logic, be inhabited.

<u>1.2.5.9.1 Proposition.</u> For $n < \infty$, $\Gamma_{inh}(\underline{\mathcal{E}}_n) \subset \{\varphi| \varphi \vdash X_{2^n \oplus 1}\}$.

<u>Proof.</u> Let $\underline{D}$: $\underline{\Lambda}_n{}^0 \longrightarrow \underline{Cat}$ be the internal category described in the proof of 1.2.5.7.5.1. Let $\underline{\mathbb{1}}$: $\underline{\Lambda}_n{}^0 \longrightarrow \underline{Cat}$ be a terminal internal category: $|\underline{\mathbb{1}}(V)| = \{u\}$, $|\mathbb{1}(A)| = \{a\}$, with u, a and v distinct, and for each ℓ: $V \longrightarrow A$, $\underline{\mathbb{1}}(\ell)$ is the uniquely determined map from $\{a\}$ to $\{u\}$. Let $\bar{\underline{D}}$ be the coproduct $\underline{D} + \underline{\mathbb{1}}$ in $\underline{Cat}(\underline{\mathcal{E}}_n)$, so that $|\bar{\underline{D}}(A)| = \{a\}$ (since $\underline{D}(A) = \phi$), $\bar{\underline{D}}(A) = \{id_a\}$, $|\bar{\underline{D}}(V)| = \{v,u\}$, and $\bar{\underline{D}}(V) = \{id_v, id_u, \mu\}$, where $\mu \cdot \mu = \mu$. Then $\bar{\underline{D}}$ is inhabited.
$\underline{\Lambda}_n \ltimes (\underline{D} + \underline{\mathbb{1}}) = (\underline{\Lambda}_n \ltimes \underline{D}) + (\underline{\Lambda}_n \ltimes \underline{\mathbb{1}})$, hence $\underline{\mathcal{E}}_n{}^{\bar{\underline{D}}^0} \approx \underline{\mathcal{E}}_n{}^{\underline{D}^0} \times \underline{\mathcal{E}}_n$. But from the proof of 1.2.5.7.5.1, $\underline{\mathcal{E}}_n{}^{\underline{D}^0} \approx \dot{\underline{\mathcal{E}}}_1$, hence $\underline{\mathcal{E}}_n{}^{\bar{\underline{D}}_n{}^0} \approx \dot{\underline{\mathcal{E}}}_1 \times \underline{\mathcal{E}}_n$. Therefore
$L(\underline{\mathcal{E}}_n{}^{\bar{\underline{D}}^0}) = L(\dot{\underline{\mathcal{E}}}_1) \cap L(\underline{\mathcal{E}}_n) = \{\varphi| 3 \vdash \varphi\} \cap \{\varphi| 2^n \oplus 1 \vdash \varphi\}$
$= \{\varphi| 2^n \oplus 1 \vdash \varphi\}$

hence $\{\varphi| 2^n \oplus 1 \vdash \varphi\} \subset \sim\Gamma_{inh}(\underline{\mathcal{E}}_n)$, hence $\Gamma_{inh}(\underline{\mathcal{E}}_n) \subset \{\varphi| \varphi \vdash X_{2^n \oplus 1}\}$.□

<u>1.2.5.9.2 Corollary.</u> $\Gamma_{inh}(\underline{\mathit{Sierp}}) = \{\varphi| \varphi \vdash v \vee \neg v\}$.□

1.2.6 A new convention for validity

In future sections we have dropped the turnstile "$\models$" on occasions which, following the conventions of chapter 0, we would appear to require it. We shall explain a new convention which will make our new discourse understandable.

1.2.6.1 A revision of languages associated with a topos.

Under our construction of sets of terms from a topos it is possible that a term may arise from two distinct toposes and that a formula may be valid in one topos and invalid in another. To facilitate our discussion of terms, formulas and validity, we shall, for each topos $\underline{\mathcal{E}}$, replace the set $\Sigma^{[1]}$ of signs of the typed alphabet $\Phi(\underline{\mathcal{E}})$, by the set $(\Sigma \times \{\underline{\mathcal{E}}\})^{[1]}$, and build up new terms and new formulas from the new set of signs. From this point on a term, respectively formula, will mean one so constructed.

1.2.6.2 Object language versus metalanguage

A formal language, or object language, apart from its intended interpretation in a topos, is built up as a theory to reflect actual linguistic practice. Each of the words: sign, variable, concatenate, string of signs, expression, substitute, term, formula, equation, valid, is selected deliberately to reflect an informal and well-recognized usage which guides us in developing the theory (that is, the object language). This informal usage is applicable to the discourse of both chapters 0 and 1; but we must take precautions to distinguish the informal from the formal senses of the words. We shall use the words "symbolic expression" to indicate the informal sense of "expression", and we shall place in double quotes a symbolic expression when we wish it to be considered as such. When, outside of 2.6, we first introduce a symbolic expression and state how it is to function, we are, at the same time recognizing it for what it is, and yet the conventions of mathematical discourse do not require that on such an occassion the symbolic expression be placed in quotes. These latter conventions determine the denotation of the symbolic expression, on those occassions that we wish to refer to "objects" of our theory. The additional use of the informal predicate symbols: "ε", "$=$" and "$\models$" indicate that an (atomic) assertion is being made.

Apart from the presence within a symbolic expression of one of these predicate symbols, the position itself, occupied by a symbolic expression within a sentence, will imply that the symbolic expression denotes an assertion rather than an object. In the list of samples of such contexts the blanks are forced to be occupied by symbolic expressions which denote assertions

(1) Suppose [that] —— .

(2) Show [that] —— .

(3) We have [that] —— .

(4) . . . therefore —— , since —— .

(5) by (a), —— , hence by (b), ——

(6) —— and —— imply ——

(7) We define the morphism f by —— .

(8) —— and —— are equivalent.

These samples are intended to capture mathematical practice; they are not explicit about finer distinctions of meaning: for example in (6), (7) and (8), before each blank we could insert "the statement".

1.2.6.3 Elimination of the turnstile "$\models$"

If we insert the symbolic expression "$\neg\forall p(p \vee \neg p)$" into the blank of (1) to produce

(1)' Suppose $\neg\forall p(p \vee \neg p)$,

then (1)' appears to be incoherent since $\neg\forall p(p \vee \neg p)$ is a formula and hence "$\neg\forall p(p \vee \neg p)$" should function as a noun within a sentence of discourse.

We now make the convention that whenever a symbolic expression denotes a formula and appears in the position within a sentence which would normally be held by a statement (as in (1) to (5)) or even the name of a statement (as in (6) to (8)), then the discourse will be understood as asserting the validity of the formula denoted by the mathematical expression.

We draw attention to an application of this rule, the result of which deviates from an informal construal. When we complete (6) and (8) using formulas φ_1, φ_2 and φ_3 to

(6)' φ_1 and φ_2 imply φ_3

this is to be interpreted as

$\models \varphi_1$ and $\models \varphi_2$ imply $\models \varphi_3$

and not as

$$\models (\varphi_1 \wedge \varphi_2) \rightarrow \varphi_3$$

and

(8)' φ_1 and φ_2 are equivalent

becomes

$\models \varphi_1$ iff $\models \varphi_2$

and not

$$\models \varphi_1 \leftrightarrow \varphi_2 \ .$$

Section 1.3 The Topos $\underline{M} - \mathcal{E}$

We give a proof, in the spirit of the proof that, for $\underline{M}$ a monoid, $\underline{M}$- Sets is a topos (c.f. for example [Gb] (p.100) and [Fr] (pp. 8, 19), and using the typed alphabet $\Phi(\mathcal{E})$, that, for $\underline{M}$ an internal monoid, $\underline{M} - \mathcal{E}$ is a topos. In [J2] and [KW] a more general theorem is proven: namely that if $\underline{C}$ is an internal category in a topos $\mathcal{E}$ then $\mathcal{E}^{\underline{C}^0}$ is a topos. In these latter references the proofs are dependent on the theory of triples - a topic we shall not touch on. Of the constructions needed to describe $\underline{M} - \mathcal{E}$ we shall be especially interested in the subobject classifier; it turns out that its object part can be described, using $\Phi(\mathcal{E})$, as the object of left ideals of $\underline{M}$.

We adopt the following conventions throughout. M_0, A, B, C and D are objects of $\mathcal{E}$. The variables w, x, y, a, a', b, c and d are distinct; w, x and y are of type M_0; a and a' are of type A; b, c and d are of types B, C and D respectively.

1.3.1 The Category M_0-$\mathcal{E}$

1.3.1.1 Definition of the category M_0-obj-$\mathcal{E}$ and notation.

(1) A morphism $\alpha: M_0 \times A \longrightarrow A$ is called an <u>action of M_0 on A.</u> An <u>M_0-object</u> is a pair $\langle A,\alpha\rangle$ where α is an action of M_0 on A; A is called <u>the underlying object of $\langle A,\alpha\rangle$</u> and α is called the <u>action of $\langle A,\alpha\rangle$</u> . $\underline{A}$, $\underline{B}$, $\underline{C}$ and $\underline{D}$ will denote arbitrary M_0 -objects whose underlying objects are A, B, C and D respectively, throughout.

(2) We define a category M_0-Obj-$\mathcal{E}$ as follows.

- The objects are M_0-objects.

- The morphisms are all triples $\langle \underline{A}, f, \underline{B} \rangle$ where $f: A \longrightarrow B$.

-The domain and codomain of a morphism $\langle \underline{A}, f, \underline{B} \rangle$ are $\underline{A}$ and $\underline{B}$ respectively.

-The identity morphism $id_{\underline{A}}$ for $\underline{A}$ is $\langle \underline{A}, id_A, \underline{A} \rangle$.

- The composition of morphisms is given by $\langle \underline{A}, f, \underline{B} \rangle \cdot \langle \underline{C}, g, \underline{A} \rangle = \langle \underline{C}, f \cdot g, \underline{B} \rangle$. This composition inherits the left and right identity and associativity laws from the compostiion in $\underline{\mathcal{E}}$.

(3) We define a faithful functor $U: M_0\text{-obj-}\underline{\mathcal{E}} \longrightarrow \underline{\mathcal{E}}$ by $U(\langle A, \alpha \rangle) = A$ and $U(\langle \underline{A}, f, \underline{B} \rangle) = f$. We call f the undelying morphism of $\langle \underline{A}, f, \underline{B} \rangle$. Throughout $\underline{f}$, $\underline{g}$, $\underline{\ell}$ and $\underline{k}$ will denote arbitrary morphisms of $M_0\text{-obj-}\underline{\mathcal{E}}$ whose underlying morphisms are f, g, ℓ and k respectively.

(4) There is a one to one correspondence between M_0-objects and actions of M_0. On some occasions we shall suppress the symbol for the action of an M_0-object. Thus if $\underline{A} = \langle A, \alpha \rangle$ we may for all terms r of type M_0 and all terms s of type A put

$$(r \cdot s) = \alpha(r, s) \ ,$$

dropping the parentheses "(" and ")" when this does not lead to ambiguities. This convention applies in particular in all those occasions when no individual symbol has explicitly been intoduced for the action of the object M_0-object.

<u>1.3.1.2 Definition.</u> A morphism $f: A \longrightarrow B$ is called equivariant from $\underline{A}$ to $\underline{B}$ if

$$f(x \cdot a) = x \cdot (fa).$$

A morphism $\underline{f}: \underline{A} \longrightarrow \underline{B}$ of $M_0\text{-obj-}\underline{\mathcal{E}}$ for which f is equivariant from $\underline{A}$ to $\underline{B}$ is called an <u>M_0-morphism.</u>

1.3.1.3 Proposition. The M_0-objects and M_0-morphisms of M_0-obj-$\underline{\mathcal{E}}$ form a subcategory M_0-$\underline{\mathcal{E}}$ of M_0-obj-$\underline{\mathcal{E}}$.

Proof. (1) Each $id_{\underline{A}}$ is an M_0-morphism: $x\cdot(id_A a) = x\cdot a = id_A(x\cdot a)$.□

(2) If $\underline{f}: \underline{A} \longrightarrow \underline{B}$ and $\underline{g}: \underline{C} \longrightarrow \underline{A}$ are M_0-morphisms then $\underline{f}\circ\underline{g}$ is a M_0-morphism:

$x\cdot((f\circ g)c) = x\cdot(f(gc)) = f(x\cdot(gc)) = f(g(x\cdot c)) = (f\circ g)(x\cdot c)$.□

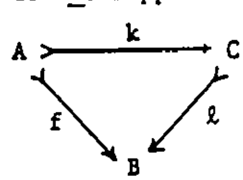
We let $U': M_0$-$\underline{\mathcal{E}} \longrightarrow \underline{\mathcal{E}}$ be the restriction of U to the subcategory M_0-$\underline{\mathcal{E}}$.

1.3.1.4 Equivariant Monomorphisms

Proposition. 1. Let $f: A \rightarrowtail B$ be a monomorphism in $\underline{\mathcal{E}}$ such that f is equivariant from $\underline{A}$ to $\underline{B}$. If $\underline{A} = \langle A,\alpha\rangle$ and $\underline{A}' = \langle A,\alpha'\rangle$ and m is equivariant from $\underline{A}'$ to $\underline{B}$, then $\alpha = \alpha'$.

2. Let $\underline{f}: \underline{A} \longrightarrow \underline{B}$ in M_0-$\underline{\mathcal{E}}$, then $\underline{f}$ is a monomorphism if the underlying morphism $f: A \longrightarrow B$ is a monomorphism.

3. Let f and ℓ be monomorphisms, f equivariant from $\underline{A}$ to $\underline{B}$, and ℓ equivariant from $\underline{C}$ to $\underline{B}$. Suppose



commutes, then k is equivariant from $\underline{A}$ to $\underline{C}$.

4. Let $\sigma: A \longrightarrow B$ be an isomorphism in $\underline{\mathcal{E}}$ such that σ is equivariant from $\underline{A}$ to $\underline{B}$, then σ^{-1} is equivariant from $\underline{B}$ to $\underline{A}$.

5. Let $\sigma: A \longrightarrow B$ be an isomorphism in $\underline{\mathcal{E}}$ and $\underline{A}$ an M_0-object then there is a uniquely determined M_0-object $\underline{B}$ such that σ is equivariant from $\underline{A}$ to $\underline{B}$.

Proof. 1. $f\alpha\langle x,a\rangle = x\cdot(fa) = f\alpha'\langle x,a\rangle$, hence $\alpha\langle x,a\rangle = \alpha'\langle x,a\rangle$, hence $\alpha = \alpha'$.□

2. Since U is faithful, if f is a mono then so is $\underline{f}$.□

3. $\ell(x\cdot(ka)) = x\cdot(\ell ka) = x\cdot(fa) = f(x\cdot a) = \ell(k(x\cdot a))$, hence $x\cdot(ka) = k(x\cdot a)$.□

4. $\sigma(x\cdot(\sigma^{-1}b)) = x\cdot((\sigma\,\sigma^{-1})b) = x\cdot b = \sigma^{-1}(\sigma(x\cdot b))$, hence $x\cdot(\sigma^{-1}b) = \sigma(x\cdot b)$.□

5. Define an action on $\underline{B}$ by $x\cdot b = \sigma(x\cdot(\sigma^{-1}b))$ then $\sigma^{-1}(x\cdot b) = x\cdot(\sigma^{-1}b)$, hence σ^{-1} is equivariant from $\underline{B}$ to $\underline{A}$, hence σ is equivariant from $\underline{A}$ to $\underline{B}$.□

In view of 1. we shall say a monomorphism f is equivariant to $\underline{B}$ if there is an M_0-object $\underline{A}$ such that f is equivariant from $\underline{A}$ to $\underline{B}$.

1.3.1.5 Proposition. Let $\underline{B}$ be an M_0-object, $f: A \longrightarrow B$ a monomorphism, and $h: B \longrightarrow \Omega$ the morphism which classifies f. The following are equivalent:

(1) f is equivariant to $\underline{B}$

(2) $h(x\cdot(fa))$

(3) $hb \Leftrightarrow h(x\ b)$.

Proof. (1) → (2). Suppose $x\cdot(fa) = f(x\cdot a)$. Then $h(x\cdot(fa)) \Leftrightarrow h(f(x\cdot a)) \Leftrightarrow T$.□

(2) ↔ (3). Put $\varphi \equiv hb$ and $\psi \equiv \forall x h(x\cdot b)$ in 0.6.51.□ (2) → (1). Define $\alpha: M_0 \times A \longrightarrow B$ by $\alpha\langle x,a\rangle = x\cdot(fa)$. Then, by (2), $h\alpha\langle x,a\rangle$. Hence α factors through f; let $\alpha': M_0 \times A \longrightarrow A$ be this factor: $f\circ\alpha' = \alpha$, i.e. $f\alpha'\langle x.a\rangle = \alpha\langle x,a\rangle$. Let $\underline{A} = \langle A,\alpha'\rangle$ then $f(x\cdot a) = x\cdot(fa)$.□

1.3.1.6 Notation and definition of limit in $\underline{\mathcal{E}}$ from [MacL]

Let $\underline{J}$ be a finite category. We let $\Delta: \underline{\mathcal{E}} \longrightarrow \underline{\mathcal{E}}^{\underline{J}}$ be the diagonal functor; that is, for each object A of $\underline{\mathcal{E}}$, $\Delta A: \underline{J} \longrightarrow \underline{\mathcal{E}}$ is a functor given by

$(\Delta A)(i) = A$ for each object i of $\underline{J}$

$(\Delta A)(\alpha) = id_A$ for each morphism α of $\underline{J}$, and for each morphism $h: A \longrightarrow B$ of $\underline{\mathcal{E}}$, $\Delta h: \Delta A \longrightarrow \Delta B$ is the natural transformation which is given by

$(\Delta h)_i = h$ for each object i of $\underline{J}$.

Let $G: \underline{J} \longrightarrow \underline{\mathcal{E}}$. By a cone with vertex C and base G we mean a natural transformation $g: \Delta C \longrightarrow G$. By a limiting cone for G we mean a cone $f: \Delta A \longrightarrow G$ such that for any $g: \Delta C \longrightarrow G$ there exists a uniquely determined $h: C \longrightarrow A$ such that $f \circ (\Delta h) = g$.

1.3.1.7 Proposition. Let $f: \Delta A \longrightarrow G$ be a limiting cone for $G: \underline{J} \longrightarrow \underline{\mathcal{E}}$. Let s and t be terms of type A. The following are equivalent:

(1) $f_i s = f_i t$ for all objects i of $\underline{J}$

(2) $s = t$

Proof. Let $\vec{v} = var(s,t)$, $\tau(\vec{v}) = C$, $h = \lambda\vec{v}.s$, $h' = \lambda\vec{v}.t$, $g = f \circ (\Delta h)$ and $g' = f \circ (\Delta h')$; then g and g' are both cones with base G and vertex C. (1) holds iff $f_i \circ h = f_i \circ h'$ for all objects i of $\underline{J}$ iff $g = g'$ iff $h = h'$ iff $s = t$.□

1.3.1.8 Proposition. $U': M_0\text{-}\underline{\mathcal{E}} \longrightarrow \underline{\mathcal{E}}$ creates finite limits.

Proof. Let $F: \underline{J} \longrightarrow M_0\text{-}\underline{\mathcal{E}}$, and let $f: \Delta A \longrightarrow U' \circ F$ be a limiting cone for $U' \circ F$ with vertex A in $\underline{\mathcal{E}}$. Let $F_i = (U' \circ F)(i)$ for each object i of $\underline{J}$, let $\gamma_i: M_0 \times F_i \longrightarrow F_i$ be the action of $F(i)$ so that

$F(i) = \langle F_i, \gamma_i \rangle$, and let $\alpha_F = (U' \circ F)(\alpha)$ for each morphism α of $\underline{J}$.
For each i define $g_i: M_0 \times A \longrightarrow F_i$ by

$$g_i/x,a/ = \gamma_i/x, f_i a/ .$$

We suppress "γ_i" since "$f_i a$" identifies the object, thus

$$g_i/x,a/ = x\cdot(f_i a) .$$

We will show there is a cone $g: \Delta(M_0 \times A) \longrightarrow U' \circ F$ by showing that for each $\alpha: i \longrightarrow j$ in $\underline{J}$, $\alpha_F \circ g_i = g_j$.

$(\alpha_F \circ g_i)/x,a/ = \alpha_F(x\cdot(f_i a))$

$= x\cdot((\alpha_F \circ f_i)a)$ since α_F is equivariant from $F(j)$ to $F(j)$

$= x\cdot(f_j a)$ since $f: \Delta A \longrightarrow U' \circ F$ is a cone

$= g_j/x,a/$.□

Since $f: \Delta A \longrightarrow U' \circ F$ is a limiting cone there is a uniquely determined $\gamma: M_0 \times A \longrightarrow A$ such that $f_i \circ \gamma = g_i$ for all i.
Suppressing "γ" for the M_0-object $\underline{A} = \langle A, \gamma \rangle$ this means

$$f_i(x\ a) = g_i/x,a/ = x\cdot(f_i a) \quad \text{for all } i.$$

Thus we have morphisms $\underline{f_i} = \langle \underline{A}, f_i, F(i) \rangle$ for each i, and a cone $\underline{f}: \Delta\underline{A} \longrightarrow F$ since $\alpha_F \circ f_i = f_j$ for each $\alpha: i \longrightarrow j$ implies $F(\alpha) \circ \underline{f_i} = \underline{f_j}$ for each $\alpha: i \longrightarrow j$.

We show $\underline{f}: \Delta\underline{A} \longrightarrow F$ is a limiting cone in M_0-$\underline{\mathcal{E}}$. Let $\underline{g}: \Delta\underline{C} \longrightarrow F$ be a cone in M_0-$\underline{\mathcal{E}}$. This gives rise to a cone $g: \Delta C \longrightarrow U' \circ F$ in $\underline{\mathcal{E}}$ where $C = U'(\underline{C})$ and $U'(g_i) = g_i$. Thus there is a uniquely determined morphism $h: C \longrightarrow A$ in $\underline{\mathcal{E}}$ such that $f\ (\Delta h) = g$. We wish to show that h is equivariant from $\underline{C}$ to $\underline{A}$ i.e. that $x\cdot(hc) = h(x\cdot c)$.

For each i we have

$f_i(x\cdot(hc)) = x\cdot(f_i hc)$ since f_i is equivariant from $\underline{A}$ to $F(i)$

$= x\cdot(g_i c) = g_i(x\cdot c)$ since g_i is equivariant from $\underline{C}$ to $F(i)$

$= f_i(h(x\cdot c))$.

Thus by 1.3.1.7 we have $x\cdot(hc) = h(x\cdot c)$. Hence there is a uniquely determined morphism $\underline{h}: \underline{C} \longrightarrow \underline{A}$ in M_0-$\mathcal{E}$ such that $\underline{f}\,(\Delta h) = \underline{g}$. It also follows that $\underline{f}: \Delta\underline{A} \longrightarrow F$ is uniquely determined by $f: \Delta A \longrightarrow U'\cdot F$. For suppose that $\underline{g}: \Delta\underline{C} \longrightarrow F$ is a cone in M_0-$\mathcal{E}$ such that $U(\underline{C}) = A$ and for each i $U(\underline{g}_i) = f_i$. Let $\gamma': M_0 \times A \longrightarrow A$ be the action of $\underline{C}$. Since $\underline{f}: \Delta\underline{A} \longrightarrow F$ is a limiting cone and $f: \Delta A \longrightarrow U'\cdot F$ is a limiting cone there is a uniquely determined morphism $\underline{h}: \underline{C} \longrightarrow \underline{A}$ with $U(\underline{h}) = id_A$ such that $\underline{f}\circ(\Delta\underline{h}) = \underline{g}$; id_A equivariant from $\underline{C}$ to $\underline{A}$ means $x\cdot(id_A a) = id_A(\gamma'\langle x,a\rangle)$ i.e. $\gamma'\langle x,a\rangle = x\cdot a$ thus $\underline{C} = \underline{A}$, $\underline{h} = id_A$ and $\underline{g} = \underline{f}$.□

1.3.2 The category $\underline{M}$ - $\mathcal{E}$, $\underline{M}$ a monoid.

1.3.2.1 The monoid $\underline{M}$. We introduce a structure on the object M_0. First: $\underline{M}$ is an M_0-object with underlying object M_0; we introduce no symbol for the action morphism: $M_0 \times M_0 \longrightarrow M_0$. Second: e is a simplified term of type M_0 with no free variables; thus $|e|: 1 \longrightarrow M_0$ and $e = |e|^*$. We assume throughout that the following three equations hold.

$x\cdot e = x$	(RU)	(right unit)
$e\cdot x = x$	(LU)	(left unit)
$(w\cdot x)\cdot y = w\cdot(x\cdot y)$	(AS)	(associativity).

We can now apply our metatheory (introduced primarily to study internal Heyting algebras) to internal monoids. We define $\underline{M}$ to be an internal model in $\mathcal{E}$ of type M_0, whose similarity type consists of a single nullary operation sign which we interpret as $|e|$, a single binary operation sign which we interpret as the action of $\underline{M}$, and equality as its only predicate sign (we take the operation signs to be the successors of the sign $\leq$). The three equations (LU), (RU) and (AS) are just the interpretation in $\underline{\mathcal{E}}$ of three equations, built up from the elphabet of this similarity type, which axiomatize monoids. By the metatheorem 0.6.6.5 we can now freely use general facts about monoids without further justification as long as they are formulated as basic Horn formulas - for example: $(w\cdot x = e \wedge x\cdot y = e) \Rightarrow w = y$.

1.3.2.2 Definition. An M_0-object $\underline{A}$ will be called an $\underline{M}$-object if it satisfies the equations

$e\cdot a = a$ (ACT-LU)

$(w\cdot x)\cdot a = w\cdot(x\cdot a)$ (ACT-AS).

For all $\underline{M}$-objects $\underline{A}$, for all terms r and s of type M_0, and all terms t of type A, we put

$$r\cdot s\cdot t \equiv (r\cdot s)\cdot t.$$

We let $\underline{M} - \underline{\mathcal{E}}$ be the full subcategory of $M_0\text{-}\underline{\mathcal{E}}$ whose objects are $\underline{M}$-objects. We let $U'': \underline{M} - \underline{\mathcal{E}} \longrightarrow \underline{\mathcal{E}}$ be the restriction of U' to $\underline{M} - \underline{\mathcal{E}}$.

1.3.2.3 Proposition. Let $F: \underline{J} \longrightarrow M_0\text{-}\underline{\mathcal{E}}$ with values in $\underline{M} - \underline{\mathcal{E}}$ and let $\underline{f}: \Delta\underline{A} \longrightarrow F$ be a limiting cone in $M_0\text{-}\underline{\mathcal{E}}$, then $\underline{A}$ is an $\underline{M}$-object.

Proof. For each object i of $\underline{J}$, $f_i(e\cdot a) = e\cdot(f_i a) = f_i a$ hence $e\cdot a = a$.

For each i $f_i(x\cdot(y\cdot a)) = x\cdot(f_i(y\cdot a)) = x\cdot(y\cdot(f_i a)) = (x\cdot y)\cdot(f_i a)$

$$= f_i((x\cdot y)\cdot a)$$

hence $x\cdot(y\cdot a) = (x\cdot y)\cdot a.$ □

1.3.2.4 Proposition. Let $f: A \rightarrowtail B$ be a monomorphism equivariant from $\underline{A}$ to $\underline{B}$. If $\underline{B}$ is an $\underline{M}$-object then $\underline{A}$ is an $\underline{M}$-object.

Proof. $f(e\cdot a) = e\cdot(fa)$ hence $e\cdot a = a$.

$f((x\cdot y)\cdot a) = (x\cdot y)\cdot(fa) = x\cdot(y\cdot(fa)) = x\cdot(f(y\cdot a)) = f(x\cdot(y\cdot a))$ hence $(x\cdot y)\cdot a = x\cdot(y\cdot a)$.□

1.3.3 The cartesian closed structure of $\underline{M} - \mathcal{E}$.

1.3.3.1 Object of equivariant morphisms. Let $\underline{A}$ and $\underline{B}$ be M_0-objects. We internalize the notion of an equivariant morphism by defining a predicate $\mathrm{equiv}(\underline{A},\underline{B})$ on B^A:

$$(\mathrm{equiv}(\underline{A},\underline{B}))(f) \leftrightarrow \forall x \forall a(x\cdot(f[a]) = f[x\cdot a]).$$

Let $\theta(\underline{A},\underline{B})\colon \mathrm{Eqv}(\underline{A},\underline{B}) \rightarrowtail B^A$ be a monomorphism classified by $\mathrm{equiv}(\underline{A},\underline{B})$ (we suppose that we choose one for each pair $(\underline{A},\underline{B})$.) For each pair $(\underline{A},\underline{B})$ and term r of type $\mathrm{Eqv}(\underline{A},\underline{B})$ we put

$$\overline{r} \equiv (\theta(\underline{A},\underline{B}))r.$$

Thus we have for g a variable of type $\mathrm{Eqv}(\underline{A},\underline{B})$,

(1) $$x\cdot(\overline{g}[a]) = \overline{g}[x\cdot a] .$$

1.3.3.2 Definition. For any objects A, B of $\mathcal{E}$ we define $B^{\underline{M} \times A}$ to be the M_0-object with underlying object $B^{M_0 \times A}$ and action

(2) $$(x\cdot f)[y,a] = f[y\cdot x,a]$$

where $\tau_0(f) \equiv B^{M_0 \times A}$.

1.3.3.3 Proposition. $B^{\underline{M} \times A}$ is an $\underline{M}$-object.

Proof. (ACT-LU): $(e\cdot f)[y,a] = f[y\cdot e,a] = f[y,a]$, hence $e\cdot f = f$.

(ACT-AS): $((w\cdot x)\cdot f)[y,a] = f[y\cdot w\cdot x,a] = (x\cdot f)[y\cdot w,a] = (w\cdot(x\cdot f))[y,a]$, hence $(w\cdot x)\cdot \mathcal{g} = w\cdot(x\cdot \mathcal{g})$. □

<u>1.3.3.4 Proposition.</u> $\theta(\underline{M} \times \underline{A},\underline{B})$: $Eqv(\underline{M} \times \underline{A},\underline{B}) \rightarrowtail B^{M_0 \times A}$ is equivariant to $B^{\underline{M} \times A}$.

<u>Proof.</u> By 1.3.1.5 it suffices to prove

$(equiv(\underline{M} \times \underline{A},\underline{B}))(x\cdot((\theta(\underline{M} \times \underline{A},\underline{B}))g)$ where $\tau_0(g) = Eqv(\underline{M} \times \underline{A},\underline{B})$.

$$\begin{aligned} w\cdot((x\cdot\overline{g})[y,a]) &= w\cdot(\overline{g}[y\cdot x,a]) && \text{by (2)} \\ &= \overline{g}[w\cdot y\cdot x,w\cdot a] && \text{by (1)} \\ &= (x\cdot\overline{g})[w\cdot y,w\cdot a] && \text{by (2)} \\ &= (x\cdot\overline{g})\,[w\cdot/y,a/] \end{aligned}$$.□

<u>1.3.3.5 Definition.</u> For any $\underline{M}$-objects $\underline{A}$ and $\underline{B}$, $\underline{B}^{\underline{A}}$ is the M_0-object with underlying object $Eqv(\underline{M} \times \underline{A},\underline{B})$ and with action compatible with $\theta(\underline{M} \times \underline{A},\underline{B})$. Thus for g of type $Eqv(\underline{M} \times \underline{A},\underline{B})$:

(3) $\qquad x\cdot\overline{g} = \overline{x\cdot g}$,

and, by 1.3.2.4 and 1.3.3.3, we have $\underline{B}^{\underline{A}}$ is an $\underline{M}$-object.

<u>1.3.3.6 Proposition.</u> $\underline{M} - \mathcal{E}$ is cartesian closed.

<u>Proof.</u> We introduce our candidate for the evaluation morphism. Define ρ: $Eqv(\underline{M} \times \underline{A},\underline{B}) \times A \longrightarrow B$ by

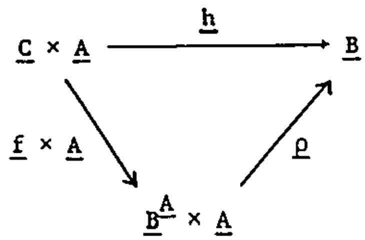
(4) $\qquad \rho/g,a/ = \overline{g}[e,a]$.

We show ρ is equivariant from $\underline{B}^{\underline{A}} \times \underline{A}$ to $\underline{B}$:

$$\begin{aligned} x\cdot(\rho/g,a/) &= x\cdot(\overline{g}[e,a]) && \text{by (4)} \\ &= \overline{g}[x,x\cdot a] && \text{by (1)} \\ &= (x\cdot\overline{g})[e,x\cdot a] && \text{by (2)} \\ &= \overline{x\cdot g}[e,x\cdot a] && \text{by (3)} \\ &= \rho/x\cdot g,x\cdot a/ && \text{by (4)} \\ &= \rho(x\cdot/g,a/). \end{aligned}$$

Put $\underline{\rho} = \langle \underline{B}^{\underline{A}} \times \underline{A}, \rho, \underline{B} \rangle$. We show that each h equivariant from $\underline{C} \times \underline{A}$ to $\underline{B}$ uniquely determines a morphism f equivariant from $\underline{C}$ to $\underline{B}^{\underline{A}}$ such that (5) commutes.

(5)



h equivariant means

(6) $x\cdot(h/c,a/) = h/x\cdot c, x\cdot a/$.

We require a morphism f which is equivariant; i.e. $x\cdot(fc) = f(x\cdot c)$, which is equivalent to $\overline{x\cdot(fc)}[y,a] = \overline{f(x\cdot c)}[y,a]$ which is equivalent to

(7) $\overline{fc}[y\cdot x,a] = \overline{f(x\cdot c)}[y,a]$.

The morphism f must satisfy (5) which is equivalent to

$\rho(f \times A)/c,a/ = h/c,a/$ that is

(5)' $\overline{(fc)}[e,a] = h/c,a/$.

From (7) and (5)' we have $\overline{fc}[x,a] = \overline{f(x\cdot c)}[e,a]$ by (7) hence

(8) $\overline{fc}[x,a] = h/x\cdot c,a/$ by (5).

From (8) we can deduce $\overline{fc}[y\cdot x,a] = h/y\cdot x\cdot c,a/ = \overline{f(x\cdot c)}[y,a]$ and $\overline{fc}[e,a] = h/e\cdot c,a/ = h/c,a/$.

Thus f satisfies (8) iff both f is equivariant from $\underline{C}$ to $\underline{B}^{\underline{A}}$ and (5) commutes. If f and f' both satisfy (8) then $f'c[x,a] = h/x\cdot c,a/ = \overline{fc}[x,a]$ hence $\overline{f'c} = \overline{fc}$, hence $f' = f$. Thus (8) uniquely determines f.

We have only to show the existance of such a morphism f satisfying (8).

Define $g\colon C \longrightarrow B^{Mo \times A}$ by

(9) $(gc)[x,a] = h/x\cdot c,a/$.

To show g factors through θ it suffices to show that gc is internally equivariant that is: eqv(gc).

$$x\cdot((gc))[y,a]) = x\cdot(h/y\cdot c,a/) \quad \text{by (9)}$$
$$= h/x\cdot y\cdot c,x\cdot a/ \quad \text{by (6)}$$
$$= (gc)[x\cdot y,x\ a] \quad \text{by (9)}$$
$$= (gc)[x\cdot/y,a/]\ .$$

Since eqv(gc), there exists a morphism $f\colon C \longrightarrow B^{M_0} \times A$ such that $\theta\circ f = g$, hence $\overline{fc}[y,a] = (gc)[y,a] = h/x\cdot c,a/$.□

1.3.4 The subobject classifier for $\underline{M} - \mathcal{E}$.

1.3.4.1 Definition of $\Omega^{\underline{M}}$. We define $\Omega^{\underline{M}}$ to be the M_0-object with underlying object Ω^{M_0} and action $(x\cdot U)[y] = U[y\cdot x]$; using the notation appropriate to Ω this becomes

(1) $y \in (x\cdot U) \Leftrightarrow y\cdot x \in U.$

1.3.4.2 Proposition. $\Omega^{\underline{M}}$ is an $\underline{M}$-object.

Proof. $(y \in e\cdot U) \Leftrightarrow (y \in U)$, hence $e\cdot U = U$.

$(y \in (w\cdot x)\cdot U) \Leftrightarrow (y\cdot w\cdot x \in U) \Leftrightarrow (y\cdot w) \in (x\cdot U) \Leftrightarrow y \in w\cdot(x\cdot U)$, hence $(w\cdot x)\cdot U = w\cdot(x\cdot U)$. □

1.3.4.3 Definitions. We define a morphism $\mathit{lclsd}_{\underline{M}}\colon \Omega^{M_0} \longrightarrow \Omega$ by

(2) $\mathit{lclsd}_{\underline{M}}\ U \Leftrightarrow \forall x \forall y(x \in U \rightarrow y\cdot x \in U).$

We say r is left closed if $\mathit{lclsd}_{\underline{M}}\ r$. We select a monomorphism

$$\ell_{\underline{M}}\colon \mathcal{L}(\underline{M}) \rightarrowtail \Omega^{M_0}$$

which classifies lclsd. For each term t of type $\mathcal{L}(\underline{M})$ we put $\bar{t} = \ell_{\underline{M}} t$.

Throughout we let L, L', L_1, L_2, L_3 be distinct variables of type $\mathcal{L}(\underline{M})$.

From (2) we have

(3) $x \in \overline{L} \Rightarrow y\cdot x \in \overline{L}$.

<u>1.3.4.4 Proposition.</u> $\ell_{\underline{M}}$ is equivavariant to $\Omega^{\underline{M}}$.

<u>Proof.</u> By 1.3.1.5 it suffices to show $w\cdot\overline{L}$ is left closed:

$x \in (w\cdot\overline{L}) \Rightarrow x\cdot w \in \overline{L}$ by (1)

$\Rightarrow y\cdot x\cdot w \in \overline{L}$ by (3)

$\Rightarrow (y\cdot x) \in (w\cdot\overline{L})$ by (1)

By 1.3.2.3 the action on $\mathcal{L}(\underline{M})$, for which $\ell_{\underline{M}}$ is equivariant to $\Omega^{\underline{M}}$, makes $\mathcal{L}(\underline{M})$ into an $\underline{M}$-object. We let $\underline{\Omega}$ denote $\mathcal{L}(\underline{M})$ with this action. The equivariance of $\ell_{\underline{M}}$ can be stated as

(4) $x\cdot\overline{L} = \overline{x\cdot L}$.

<u>1.3.4.5 Proposition.</u> $|\{x:T\}|: \mathbb{1} \longrightarrow \Omega^{M_0}$ factors through $\ell_{\underline{M}}$.

<u>Proof.</u> $w \in \{x:T\}$ and $T \Rightarrow y\cdot w \in \{x:T\}$, hence $\{x:T\}$ is left closed. By 0.6.2.3 there exists a morphism $f: \mathbb{1} \longrightarrow \mathcal{L}(\underline{M})$ such that $|\{x:T\}| = \ell_{\underline{M}} f$.□

<u>1.3.4.6 Definition.</u> We denote the constant term of type $\mathcal{L}(\underline{M})$ which is determined by $|\{x:T\}|$ and $\ell_{\underline{M}}$, by M. Thus $|\{x:T\}| = \ell_{\underline{M}} |M|$ and $M = |M|^*$, hence M is defined by (5) and

(6) $\overline{M} = \{x:T\}$

which is equivalent to

(7) $x \in \overline{M}$.

<u>1.3.4.7 Proposition.</u> $|M|$ is equivariant from $\underline{\mathbb{1}}$ to $\underline{\Omega}$, and

(8) $y\cdot M = M$.

Proof. $x\cdot y \in \overline{M}$ by (7); $x \in y\cdot\overline{M}$ by (1), therefore

$\overline{y \cdot M} = y \cdot \overline{M} = \{x:T\} = \overline{M}$ by (4) and (6). Let u be a variable of type $\mathbb{I}$, then $* = u = y \cdot u$ and therefore $y.(|M|u) = y.M = M = |M|(y \cdot u)$.□

<u>1.3.4.8 Definition.</u> <u>true</u> $\equiv \langle \underline{\mathbb{I}}, |M|, \underline{\Omega} \rangle$.

<u>1.3.4.9 Proposition.</u> (9) $(L = M) \Leftrightarrow (e \in \overline{L})$

<u>Proof.</u> $L = M \Rightarrow \forall x(x \in \overline{L})$

$\Rightarrow e \in \overline{L}.$

$e \in \overline{L} \Rightarrow x \in \overline{L}$

$\Rightarrow \forall x(x \in \overline{L})$

$\Rightarrow \overline{L} = \overline{M}$

$\Rightarrow L = M.$ □

<u>1.3.4.10 Proposition.</u> Let $g\colon B \longrightarrow \mathcal{L}(\underline{M})$ and let $\underline{B}$ be an $\underline{M}$-object. The following are equivalent

(10) $x \cdot (gb) = g(x \cdot b)$

(11) $x \in \overline{gb} \Leftrightarrow g(x \cdot b) = M$

<u>Proof.</u> (10) implies (11).

$x \in \overline{gb} \Leftrightarrow e \cdot x \in \overline{gb} \Leftrightarrow e \in \overline{x \cdot (gb)}$

$\Leftrightarrow e \in \overline{g(x \cdot b)}$ by (10)

$\Leftrightarrow g(x \cdot b) = M.$

(11) implies (10)

$y \in \overline{x \cdot (gb)} \Leftrightarrow y \cdot x \in \overline{gb}$

$\Leftrightarrow g(y \cdot x \cdot b) = M$ by (11)

$\Leftrightarrow y \in \overline{g(x \cdot b)}$ by (11)

hence $\overline{x \cdot (gb)} = \overline{g(x \cdot b)}$, hence $x \cdot (gb) = g(x \cdot b)$.□

<u>1.3.4.11 Propostition.</u> <u>true</u>: $\underline{1} \longrightarrow \underline{\Omega}$ classifies subobjects in $\underline{M} - \underline{\mathcal{E}}$.

<u>Proof.</u> In $\underline{M} - \underline{\mathcal{E}}$ we wish to show that for each $\underline{f}$ there is a uniquely determined $\underline{g}$ making (d_1) a pullback

$$\begin{array}{ccc} \underline{A} & \longrightarrow & \underline{1} \\ \underline{f} \downarrow & (d_1) & \downarrow \underline{true} \\ \underline{B} & \xrightarrow[\underline{g}]{} & \underline{\Omega} \end{array}$$

Since U" creates and preserves pullbacks this is equivalent to showing the following. For each f, equivariant to $\underline{B}$, there is a uniquely determined g, equivariant from $\underline{B}$ to $\underline{\Omega}$, making (d_2) a pullback

$$\begin{array}{ccc} A & \longrightarrow & 1 \\ f \downarrow & (d_2) & \downarrow |M| \\ B & \xrightarrow[g]{} & \mathcal{L}(\underline{M}) \end{array} .$$

Let $h: B \longrightarrow \Omega$ be the classifying morphism for f, so that

(12) $\quad hb \leftrightarrow \exists a(fa = b)$.

We first show that there is at most one g which is equivariant from $\underline{B}$ to $\underline{\Omega}$ and for which (d_2) is a pullback.

The condition on g that (d_2) be a pullback is, by 0.6.11.8,

(13) $\quad gb = M \leftrightarrow \exists a(fa = b)$,

by (12), this is equivalent to

(14) $\quad (gb = M) \leftrightarrow hb$.

Since g is to be equivariant, by 1.3.4.10, we have

(15) $x \in \overline{gb} \Leftrightarrow h(x \cdot b)$.

Now if g' also makes (d_2) a pullback and is equivariant, then it also satisfies (15), thus $[x \in \overline{gb}] \Leftrightarrow [x \in \overline{g' \cdot b}]$, hence $\overline{gb} = \overline{g'b}$, hence $g = g'$.

To show the existance of such a morphism define $k\colon B \longrightarrow \Omega^{M_0}$ by

(16) $x \in kb \Leftrightarrow h(x\ b)$.

Since f is equivariant to $\underline{B}$ we have

(17) $hb \Rightarrow h(x\ b)$

hence

(18) $h(x \cdot b) \Rightarrow h(y \cdot x \cdot b)$

hence

(19) $x \in kb \Rightarrow y \cdot x \in kb$.

Thus k factor through $\ell_{\underline{M}}$, i.e. there is a morphism $g\colon B \longrightarrow \mathcal{L}(\underline{M})$ such that $\ell_{\underline{M}}\, g = k$, i.e.

(20) $\overline{gb} = kb$.

Hence $gb = M \Leftrightarrow e \in \overline{gb} \Leftrightarrow e \in kb$ by (20)

$\Leftrightarrow hb$ by (16).

Thus by (14), (d_2) is a pullback.

We show equivariance:

$x \in \overline{gb} \Leftrightarrow x \in kb$ by (20)

$\Leftrightarrow h(x \cdot b)$ by (16)

$\Leftrightarrow e \in k(x \cdot b)$ by (16)

$\Leftrightarrow e \in \overline{g(x \cdot b)}$ by (20)

$\Leftrightarrow \overline{g(x \cdot b)} = M$.

Thus, by 1.3.4.10, g is equivariant.□

Section 1.4 Heyting Algebra Structures Induced by Internal Adjoints.

1.4.1 Internal adjoints. ([J2], 5.3, p. 147) Johnstone defines internal adjointness between "internal posets" (our "preordered objects" (1.4.2.1)). We shall restrict ourselves to adjointness between partially ordered objects (his "antisymmetric posets"). Let $\mathcal{A}$ and $\mathcal{B}$ be partially ordered objects with carriers A and B respectively; a, a', b and b' are distinct variables, a and a' are of type A, and b and b' are of type B. We use $\leq$ as an infix for both partial orders. $\mathcal{A}^{o}$ and $\mathcal{B}^{o}$ will denote the structures with carriers A and B respectively and the opposite ordering, for which we use the infix $\geq$.

1.4.1.1 Definition. Let $f: A \longrightarrow B$ and $g: B \longrightarrow A$. We say that f is internally left adjoint to g with respect to $\mathcal{A}$ and $\mathcal{B}$ if

$$(fa \leq b) \Leftrightarrow (a \leq gb).$$

When the ordering is understood we write $f \dashv g$ and say (f,g) is an adjoint pair. If we wish to be explicit we write $f \dashv g\ (\mathcal{A},\mathcal{B})$. Thus we have

$$f \dashv g\ (\mathcal{A},\mathcal{B}) \text{ iff } g \dashv f\ (\mathcal{A}^{o},\mathcal{B}^{o}).$$

We show that the above definition is equivalent to that given in [J1](5.3, p. 147).

1.4.1.2 Proposition. $f \dashv g$ iff (1), (2), (3) and (4) hold.

1. $fgb \leq b$
2. $a \leq gfa$
3. $(b \leq b') \Rightarrow (gb \leq gb')$
4. $(a \leq a') \Rightarrow (fa \leq fa')$.

If $f \dashv g$ then (5) and (6) hold.

5. $fgfa = fa$

6. $gfgb = gb$.

Each component uniquely determines the other component of an adjoint pair.

7. $f \dashv g$ and $f \dashv g'$ imply $g = g'$

8. $f \dashv g$ and $f' \dashv g$ imply $f = f'$.

If $f \dashv g$ then (9), (10) and (11) are equivalent.

9. $(gb = gb') \Rightarrow (b = b')$

10. $fgb = b$

11. $(gb \le gb') \Rightarrow (b \le b')$

If $f \dashv g$ then (12), (13) and (14) are equivalent.

12. $(fa = fa') \Rightarrow (a = a')$

13. $gfa = a$

14. $(fa \le fa') \Rightarrow (a \le a')$.

Proof. Suppose $(fa \le b) \Leftrightarrow (a \le gb)$. Substitute fa for b to get (2), and gb for a to get (1). We show (3):

$$(b \le b') \Rightarrow (fgb \le b) \wedge (b \le b') \quad \text{by (1)}$$
$$\Rightarrow fgb \le b' \quad \text{by transitivity}$$
$$\Rightarrow gb \le gb'.$$

From $f \dashv g\ (\mathcal{A},\mathcal{B})$ we have $g \dashv f\ (\mathcal{A}^o,\mathcal{B}^o)$ hence $(a \ge a') \Rightarrow (fa \ge fa')$, thus (4) holds.□

For the converse we show that (2) and (3) imply $(fa \le b) \Rightarrow (a \le gb)$:

$$(fa \le b) \Rightarrow (gfa \le gb) \quad \text{by (3)}$$
$$\Rightarrow (a \le gb) \quad \text{by (2).}$$

Dually (1) and (4) imply $(a \le gb) \Rightarrow (fa \le b)$.□

Suppose $f \dashv g$. We show (5). Substitute fa for b in (1) to get $fgfa \le fa$. Apply (4) to (2) to get $fa \le fgfa$. Thus (5) holds. By duality from $g \dashv f\ (\mathcal{A}^o,\mathcal{B}^o)$ we get (6). □

To show (7) note that $f \dashv g$ and $f \dashv g'$ together imply $(a \leq gb) \Leftrightarrow (a \leq g'b)$. Substitute first gb for a then $g'b$ for a to get $gb \leq g'b$ and $g'b \leq gb$, thus $gb = g'b$, hence $g = g'$. (8) follows by duality. □

Suppose $f \dashv g$.

(9) implies (10): Applying (9) to (6) yields (10).

(10) implies (11): $(gb \leq gb') \Rightarrow (fgb \leq b')$ since $f \dashv g$

$\Rightarrow (b \leq b')$ by (10).

(11) implies (9):

$(gb = gb') \Rightarrow (gb \leq gb') \wedge (gb' \leq gb)$ by reflexivity

$\Rightarrow (b \leq b') \wedge (b' \leq b)$ by (11)

$\Rightarrow b = b'$ by transitivity. □

From $f \dashv g\ (\mathcal{A},\mathcal{B})$ we have $g \dashv f\ (\mathcal{A}^o,\mathcal{B}^o)$; thus the equivalence of (12), (13) and (14) follows from the equivalence of (9), (10) and (11). □

In the next proposition $\mathcal{A}$ and $\mathcal{B}$ are internal models of extensions of the theory of partial orders as described in 0.6.8.1. The second and fourth statements are duals of the first and third and are included for future reference.

1.4.1.3 Proposition. Let $f \dashv g\ (\mathcal{A},\mathcal{B})$.

1. If both $\mathcal{A}$ and $\mathcal{B}$ have a minimum, 0, and a binary supremum operation, $\vee$, then f preserves 0 and $\vee$.

2. If both $\mathcal{A}$ and $\mathcal{B}$ have a maximum, 1, and a binary infimum operation, $\wedge$, then g preserves 1 and $\wedge$.

3. Suppose g is a monomorphism, and $\mathcal{A}$ has 1 and a binary supremem operation $\vee$. The element $f1$ is a maximum element of $\mathcal{B}$. Define $\vee$ on B by $b \vee b' = f((gb) \vee (gb'))$, then $\vee$ is a binary operation on $\mathcal{B}$.

4. Suppose f is a monomorphism, and $\mathcal{B}$ has 0 and a binary infimum operation $\wedge$. The element $g0$ is a minimum element for $\mathcal{A}$, Define $\wedge$ on A by $a \wedge a' = g((fa) \wedge (fa'))$, then $\wedge$ is a binary infimum operation of $\mathcal{A}$.

<u>Proof.</u> $0 \leq gb$ hence $\forall b(f0 \leq b)$.

$$((fa \vee fa') \leq b) \leftrightarrow ((fa \leq b) \wedge (fa' \leq b)) \leftrightarrow ((a \leq gb) \wedge (a' \leq gb))$$
$$\leftrightarrow ((a \vee a') \leq gb) \leftrightarrow (f(a \vee a') \leq b)$$

hence $(fa \vee fa') = f(a \vee a')$. □

2. This is dual to (1). □

3. $gb \leq 1$ hence $fgb \leq f1$ (by 1.4.1.2 (4)), hence $b \leq f1$ (by 1.4.1.2 (10)). □

$$((b \vee b') \leq c) \leftrightarrow (f((gb) \vee (gb')) \leq c) \leftrightarrow ((gb) \vee (gb') \leq gc) \quad \text{since } f \dashv g$$
$$\leftrightarrow (gb \leq gc) \wedge (gb' \leq gc)$$
$$\leftrightarrow (b \leq c) \wedge (b' \leq c) \quad \text{since } g \text{ is a mono,}$$

hence we have defined a binary supremum operation on $\mathcal{B}$.□

4. Dual to (3).□

<u>1.4.1.4 Corollary.</u> Let $f \dashv m$ $(\mathcal{A},\mathcal{B})$ and $m \dashv g$ $(\mathcal{B},\mathcal{A})$, where $m: B \rightarrowtail A$ is a monomorphism. If $\mathcal{A}$ has a bounded lattice structure (compatible with its partial order) then $\mathcal{B}$ has a bounded lattice structure and m preserves this structure.

<u>Proof.</u> From (3) and (4) of 1.4.1.3, $\mathcal{B}$ can be given the structure of a bounded lattice. From (1) and (2) m preserves this structure.□

1.4.1.5 Proposition. Let $f \dashv m$ $(\mathcal{A},\mathcal{B})$, $m \dashv g$ $(\mathcal{B},\mathcal{A})$ with m a monomorphism. If $\mathcal{A}$ has a Heyting algebra structure then $\mathcal{B}$ has a Heyting algebra structure with implication defined on B by

$$b \Rightarrow b' = g(mb \Rightarrow mb').$$

Proof. From the corollary, $\mathcal{B}$ has a bounded lattice structure.

$(c \leq (b \Rightarrow b')) \Leftrightarrow mc \leq (mb \Rightarrow mb')$ since $m \dashv g$

$\Leftrightarrow (mc \wedge mb) \leq mb'$

$\Leftrightarrow m(c \wedge b) \leq mb'$

$\Leftrightarrow (c \wedge b) \leq b'$. □

1.4.1.6 Definition. Let $f\colon A \longrightarrow B$. We define three morphisms $\exists_f\colon \Omega^A \longrightarrow \Omega^B$, $f^{-1}\colon \Omega^B \longrightarrow \Omega^A$ and $\forall_f\colon \Omega^A \longrightarrow \Omega^B$, internal versions of functions between Sub A and Sub B, by

$(b \in \exists_f W) \Leftrightarrow \exists a((b = fa) \wedge (. \in W))$

$(a \in f^{-1}U) \Leftrightarrow (fa \in U)$

$(b \in \forall_f W) \Leftrightarrow \forall a((b = fa) \Rightarrow (a \in W)$.

1.4.1.7 Proposition. With respect to inclusion on Ω^A and Ω^B we have

$$\exists_f \dashv f^{-1} \dashv \forall_f$$

Proof. $\exists_f \dashv f^{-1}$:

$(\exists_f W \subset U) \Leftrightarrow \forall b((b \in \exists_f W) \Rightarrow (b \in U))$

$\Leftrightarrow \forall b((\exists a((b = fa) \wedge (a \in W))) \Rightarrow (b \in U))$

$\Leftrightarrow \forall b \forall a(((b = fa) \wedge (a \in W)) \Rightarrow (b \in U))$

$\Leftrightarrow \forall a((a \in W) \Rightarrow \forall b((b = fa) \Rightarrow (b \in U)))$

$\Leftrightarrow \forall a((a \in W) \Rightarrow (fa \in U))$ by 0.6.10.10

$\Leftrightarrow (W \subset f^{-1}U)$

$f^{-1} \dashv \forall_f$:

$$(U \subset \forall_f W) \leftrightarrow \forall b((b \in U) \rightarrow \forall a((b = fa) \rightarrow (a \in W)))$$
$$\leftrightarrow \forall a \forall b(((b = fa) \wedge (b \in U)) \rightarrow (a \in W))$$
$$\leftrightarrow \forall a((\exists b((b = fa) \wedge (b \in U))) \rightarrow (a \in W))$$
$$\leftrightarrow \forall a((fa \in U) \rightarrow (a \in W)) \qquad \text{by 0.6.10.7}$$
$$\leftrightarrow (f^{-1}U \subset W).\square$$

An important special case occurs when $f = u_A: A \longrightarrow \mathbb{1}$. In this case the codomain is isomorphic to Ω. We formulate this case separately. Define $\exists_A: \Omega^A \longrightarrow \Omega$ and $\forall_A: \Omega^A \longrightarrow \Omega$ by

$$\exists_A U \leftrightarrow \exists a(a \in U) \quad \text{and} \quad \forall_A U \leftrightarrow \forall a(a \in U).$$

<u>1.4.1.8 Proposition.</u> $\exists_A \dashv \Delta \dashv \forall_A$.

<u>Proof.</u> $(\exists_A U \leq p) \leftrightarrow (\exists a(a \in U) \rightarrow p)$

$$\leftrightarrow \forall a((a \in U) \rightarrow (a \in \Delta p)$$
$$\leftrightarrow (U \subset \Delta p).$$

$$(p \leq \forall_A U) \leftrightarrow (p \rightarrow \forall a(a \in U))$$
$$\leftrightarrow \forall a(p \rightarrow (a \in U))$$
$$\leftrightarrow \forall a((a \in \ \ p) \rightarrow (a \in U))$$
$$\leftrightarrow (\Delta p \subset U).\ \square$$

<u>1.4.2 Closure operators.</u>

<u>1.4.2.1 Definition.</u> Let $\mathcal{A}$ be a partially ordered object with carrier A. A morphism $j: A \longrightarrow A$ will be called a closure operator on $\mathcal{A}$ (see [BD] p. 47) if

(1) $(a \leq a') \rightarrow (ja \leq ja')$

(2) $a \leq ja$

(3) $jja \leq ja$.

From (2) and (3) we deduce

(4) $jja = ja$.

We call $j: A \longrightarrow A$ an interior operator on $\mathcal{A}$ if j is a closure operator on $\mathcal{A}^o$.

Given an adjoint situation $f \dashv g$ $(\mathcal{A},\mathcal{B})$ we can derive

(1) $(a \leq a') \Rightarrow (gfa \leq gfa')$ by (3) and (4) of 1.4.1.2

(2) $a \leq gfa$ by (2) of 1.4.1.2

(3) $gfgfa \leq gfa$ by (5) of 1.4.1.2.

Thus $gf: A \longrightarrow A$ is a closure orperator on $\mathcal{A}$. We shall show that every closure operator can be defined this way with g a monomorphism. To this end we show first that a monomorphism into A induces a partial ordering on its domain such that the mono is an order embedding into $\mathcal{A}$.

<u>1.4.2.2 Proposition.</u> Let $m: B \rightarrowtail A$ be a monomorphism. Given a partial order $\leq$ on A we define a relation $\leq'$ on B by

$$(b \leq b') \Leftrightarrow (mb \leq mb')$$

then $\leq'$ is a partial order.

<u>Proof.</u> Reflexivity: $(b \leq' b) \Leftrightarrow (mb \leq mb) \Leftrightarrow \top$.

Antisymmetry:

$$\begin{aligned}(b \leq' b') \wedge (b' \leq' b) &\Rightarrow (mb \leq mb') \wedge (mb' \leq mb) \\ &\Rightarrow mb = mb' \\ &\Rightarrow b = b'.\end{aligned}$$

Transitivity:

$$\begin{aligned}(a \leq' b) \wedge (b \leq' c) &\Rightarrow (ma \leq mb) \wedge (mb \leq mc) \\ &\Rightarrow ma \leq mc \\ &\Rightarrow a \leq' c. \ \square\end{aligned}$$

In future we shall use the same infix for the relation defined via the monomorphism as we use for the given relation.

<u>1.4.2.3 Proposition.</u> Let $j: A \longrightarrow A$ be a closure operator on $\mathcal{A}$ and let $m: B \rightarrowtail A$ be a monomorphism classified by $|ja = a|$ then there is a uniquely determined morphism $f: A \longrightarrow B$ such that $j = m \cdot f$. For the partial order $\mathcal{B}$ induced by m in B we have

$$f \dashv m \ (\mathcal{A}, \mathcal{B}).$$

If $f' \dashv m' \ (\mathcal{A}, \mathcal{B}')$ with $m': B' \rightarrowtail A$ a mono and $j = f' \cdot m$, then $(f' \cdot m) \cdot (f \cdot m') = id_{B'}$ and $(f \cdot m') \cdot (f' \cdot m) = id_B$.

<u>Proof.</u> Substitution of the term ja for the variable a in the formula $ja = a$ yields the valid formula $jja = ja$, hence by 0.6.2.1, we have a morphism f such that $ja = mfa$.

We show f is a retraction. $mfmb = jmb = mb$, hence $fmb = b$.

We show $f \dashv m$:

$(fa \leq b) \leftrightarrow (mfa \leq mb)$

$\leftrightarrow ja \leq mb$

$\rightarrow a \leq mb$.

Conversely

$a \leq mb \rightarrow ja \leq jmb$

$\rightarrow ja \leq mb$

If $f' \dashv m'$ with m' a mono and $j = m' \cdot f'$ then $f'm'b' = b'$

$((f' \cdot m) \cdot (f \cdot m'))b' = f'jm'b' = f'm'f'm'b' = b'$

$((f \cdot m') \cdot (f'm))b = fjmb = fmfmb = b.$□

<u>1.4.2.4 Proposition.</u> Let $\mathcal{A}$ be a Heyting algebra object, let k be a constant of type A, and let $m\colon A_k \rightarrowtail A$ be a monomorphism classified by $|a \leq k|$. Then

(1) m has a right adjoint, g, with respect to the induced order.

(2) m induces on A_k a Heyting algebra structure $\mathcal{A}_k$ and m preserves $\wedge$.

(3) g is a Heyting algebra homomorphism.

(4) $gx = 1 \Leftrightarrow k \leq x$.

<u>Proof.</u> (1) Define $\hat{k}\colon A \longrightarrow A$ by $\hat{k}a = a \wedge k$, then $\hat{k}$ is an interior operator on $\mathcal{A}$ since $(a \leq a') \Rightarrow ((a \wedge k) \leq (a' \wedge k))$, $(a \wedge k) \leq a$, and $(a \wedge k) \leq ((a \wedge k) \wedge k)$ are all bHf's which hold in a meet semilattice with a constant. Thus $\hat{k}$ can be split into a pair (m,g) where $g\colon A \longrightarrow A_k$, $m \dashv g$ $(\mathcal{A}_k, \mathcal{A})$, and $mga = a \wedge k$, by the dual of 1.4.2.3.□

(2) By 1.4.1.3 (4) we can define 0 and $\wedge$ on A_k by $0 = g0$ and $b \wedge b' = g((mb) \wedge (mb'))$. We show $1 = g1$ is a correct definition for 1 on A_k. From $mb \leq 1$ on A we have $b \leq g1$ on A_k, hence $1 = g1$. We show $b \vee b' = g(mb \vee mb')$ is a correct definition of $\vee$ on A_k:

$$
\begin{aligned}
g(mb \vee mb') \leq c &\Leftrightarrow mg(mb \vee mb') \leq mc \\
&\Leftrightarrow (mb \vee mb') \wedge k \leq mc \\
&\Leftrightarrow (mb \wedge k) \vee (mb' \wedge k) \leq mc \\
&\Leftrightarrow (mb \vee mb') \leq mc \\
&\Leftrightarrow (mb \leq mc) \wedge (mb' \leq mc) \\
&\Leftrightarrow (b \leq c) \wedge (b' \leq c).
\end{aligned}
$$

We show m preserves $\wedge$:

$$
\begin{aligned}
m(b \wedge b') &= mg((mb) \wedge (mb')) = (mb \wedge mb') \wedge k \\
&= ((mb) \wedge k) \wedge ((mb') \wedge k) = mb \wedge mb'.
\end{aligned}
$$

We show $(b \Rightarrow d) = g(mb \Rightarrow md)$ is a correct definition of $\Rightarrow$ on A_k:

$$(c \leq g(mb \Rightarrow md)) \Leftrightarrow (mc \leq (mb \Rightarrow md))$$
$$\Leftrightarrow ((mc \Rightarrow mb) \leq md)$$
$$\Leftrightarrow (m(c \wedge b) \leq md)$$
$$\Leftrightarrow ((c \wedge b) \leq d).\square$$

(3) From (2) we have $g0 = 0$ and $g1 = 1$. From (1), $m \dashv g$, hence $g(a \wedge a') = g(a \wedge a')$. We show g preserves $\vee$:

$$m(ga \vee ga') = m(g(mga \vee mga'))$$
$$= ((a \wedge k) \vee (a' \wedge k)) \wedge k$$
$$= (a \vee a') \wedge k$$
$$= mg(a \vee a')$$

hence $ga \vee ga' = g(a \vee a')$.

We show g preserves $\Rightarrow$:

$$m(ga \Rightarrow ga') = mg(mga \Rightarrow mga')$$
$$= ((a \wedge k) \Rightarrow (a' \wedge k)) \wedge k$$
$$= (a \Rightarrow a') \wedge k$$
$$= mg(a \Rightarrow a')$$

hence $(ga \Rightarrow ga') = g(a \Rightarrow a').\square$

(4) $$gx = 1 \Leftrightarrow gx = g1$$
$$\Leftrightarrow mgx = mg1$$
$$\Leftrightarrow x \wedge k = 1 \wedge k$$
$$\Leftrightarrow k \leq x.\square$$

1.4.3 Preorders.

1.4.3.1 Definition. Let $\mathcal{A}$ be an internal structure with object part A and a single binary relation for which we used the infix $\prec$. We call $\mathcal{A}$

a preordered object and the relation a preorder if the following hold

$a \prec a$ reflexivity

$(a \prec b) \wedge (b \prec c) \Rightarrow (a \prec c)$ transitivity.

1.4.3.2 Example. Let $\underline{A}$ be an $\underline{M}$-object of $\underline{M} - \mathcal{E}$, define $\prec$ on A by $(a \prec b) \Leftrightarrow \exists x(x \cdot a = b)$, then $\prec$ is a preorder on A. Reflexivity follows since $e \cdot a = a$. For transitivity

$$
\begin{aligned}
(x \cdot a = b) \wedge (y \cdot b = c) &\Rightarrow y \cdot (x \cdot a) = c \\
&\Rightarrow (y \cdot x) \cdot a = c \\
&\Rightarrow \exists w(w \cdot a = c)
\end{aligned}
$$

hence $\exists x(x \cdot a = b) \wedge \exists y(y \cdot b = c) \Rightarrow \exists w(w \cdot a = c)$. □

In this example the preorder arises from the divisibility of b by a. We shall be applying the theory we develop for preorders to another divisibility relation in 1.5.3.2 : the relation of a morphism factoring through another morphism defined on the object of morphisms of an internal category. A special case of this factorizarion is that of divisibility on an internal monoid - a situation covered by our example above.

1.4.3.3 Definition. Let $\mathcal{A}$ be a preordered object. The morphism $\uparrow: \Omega^A \longrightarrow \Omega^A$ is given by

$$(a \in \uparrow U) \Leftrightarrow \exists b((b \in U) \wedge (b \prec a)).$$

The morphism $(\ )^{\circ}: \Omega^A \longrightarrow \Omega^A$ is given by

$$(a \in U^{\circ}) \Leftrightarrow \forall b((a \prec b) \Rightarrow (b \in U)).$$

The predicate $\mathrm{idl}: \Omega^A \longrightarrow \Omega$

$$\mathrm{idl}(U) \Leftrightarrow \forall a \forall b((a \in U) \wedge (a \prec b) \Rightarrow (b \in U)).$$

The upper segment $[\ ,\rightarrow): A \longrightarrow \Omega^A$ is given by

$$(b \in [a,\rightarrow)) \Leftrightarrow (a < b).$$

The lower segment $(\leftarrow,\ ]: A \longrightarrow \Omega^A$ is given by

$$(b \in (\leftarrow,a]) \Leftrightarrow (b < a).$$

<u>1.4.3.3.1 Proposition.</u> $\mathrm{idl}([a,\rightarrow))$.

<u>Proof.</u> $((c \in [a,\rightarrow)) \wedge (c < b)) \Rightarrow ((a < c) \wedge (c < b))$

$$\Rightarrow (a < b).\square$$

<u>1.4.3.4 Proposition.</u> $\uparrow$ is a closure operator on $(\Omega^A,)$.

<u>Proof.</u> We show first

(1) $(U_1 \subset U_2) \Rightarrow (\uparrow U_1 \subset \uparrow U_2)$.

Suppose $\overline{U}_1 \subset \overline{U}_2$ and $\overline{a} \in \uparrow\overline{U}_1$, then $\exists b((b \in \overline{U}_1) \wedge (b < \overline{a}))$, hence $\exists b((b \in \overline{U}_2) \wedge (b < \overline{a}))$, hence $\overline{a} \in \uparrow\overline{U}_2$ $\square$

Seondly $\uparrow$ satisfies.

(2) $U \subset \uparrow U$.

We have $((b \in U) \wedge (b < a)) \Rightarrow \exists b((b \in U) \wedge (b < a))$. Substituting a for b yields $(a \in U) \Rightarrow \exists b((b \in U) \wedge (b < a))$.$\square$

We want to show

(3) $\uparrow\uparrow U \subset \uparrow U$.

By transitivity we have

$(b_2 \in U) \wedge (b_2 < b_1) \wedge (b_1 < a) \Rightarrow (b_2 \in U) \wedge (b_2 < a)$ hence by 0.6. and 0.6

$(\exists b_2((b_2 \in U) \wedge (b_2 < b_1))) \wedge (b_1 < a) \Rightarrow \exists b_2((b_2 \in U) \wedge (b_2 < a)$

i.e. $(b_1 \in \uparrow U) \wedge (b_1 < a) \Rightarrow (a \in \uparrow U)$ hence

$\exists b\ ((b_1 \in \uparrow U) \wedge (b_1 < a)) \Rightarrow (a \in \uparrow U)$ hence $a \in \uparrow\uparrow U \Rightarrow a \in \uparrow U$.$\square$

<u>1.4.3.5 Proposition.</u> $(\ )^{o}$ is an interior operator with respect to on Ω^{A}.

<u>Proof.</u> (1): Suppose $\overline{U}_1 \subset \overline{U}_2$ and $\overline{a} \in (\overline{U}_1)^{o}$, so that $(\overline{a} \prec b) \Rightarrow (b \in \overline{U}_1)$. Since $(b \in \overline{U}_1) \Rightarrow (b \in \overline{U}_2)$ we have $(\overline{a} \prec b) \Rightarrow (b \in \overline{U}_2)$ hence $\overline{a} \in (\overline{U}_2)^{o}$.□

(2) Suppose $\overline{a} \in (\overline{U})^{o}$, so that $(\overline{a} \prec b) \Rightarrow (b \in \overline{U})$. Substituting $\overline{a}$ for b yields $\overline{a} \in \overline{U}$.□

(3) We want to show $(a \in U^{o}) \Rightarrow (a \in (U^{o})^{o})$ that is

$$(a \in U^{o}) \Rightarrow \forall b((a \prec b) \Rightarrow (b \in U^{o}))$$

which is equivalent to

$$((a \in U^{o}) \wedge (a \prec b)) \Rightarrow (b \in U^{o}).$$

We suppose $\overline{a} \prec \overline{b}$ and $\overline{a} \in (\overline{U})^{o}$ so that $(\overline{a} \prec c) \Rightarrow (c \in \overline{U})$. We want to show $(\overline{b} \prec c) \Rightarrow (c \prec \overline{U})$. We have

$$\begin{aligned} \overline{b} \prec c &\Rightarrow (\overline{a} \prec \overline{b}) \wedge \overline{b} \prec c) \\ &\Rightarrow \overline{a} \prec c \\ &\Rightarrow c \in \overline{U}. \ \square \end{aligned}$$

<u>1.4.3.6 Proposition.</u> Let t be a term of type Ω^{A}. The following are equivalent.

(1) $\uparrow t \subset t$

(2) $\mathrm{idl}(t)$

(3) $t \subset t^{o}$.

<u>Proof.</u> (1) iff (2).

(1) iff $a \in \uparrow t \Rightarrow a \in t$

iff $\exists b((b \in t) \wedge (b \prec a)) \Rightarrow (a \in t)$

iff $(b \in t) \wedge (b \prec a) \Rightarrow (a \in t)$. □

(3) iff (2)

(3) iff $(a \in t) \Rightarrow (a \in t^{o})$

iff $(a \in t) \Rightarrow \forall b((a \prec b) \Rightarrow (b \in t))$

iff $(a \in t) \wedge (a \prec b) \Rightarrow (b \in t)$.

<u>1.4.3.7 Corollary</u>.

(1) $\mathrm{idl}(U) \Leftrightarrow (\uparrow U = U)$

(2) $\mathrm{idl}(U) \Leftrightarrow (U^{o} = U)$

(3) $\mathrm{idl}(\uparrow U)$

(4) $\mathrm{idl}(U^{o})$.□

<u>1.4.3.8 The object of ideals of a pre-ordered object.</u>

Let $\mathrm{Idl}(\mathcal{A}) \overset{m}{\rightarrowtail} \Omega^{A}$ be the monomorphism classified by the predicate $\mathrm{idl}: \Omega^{A} \longrightarrow \Omega$. By 1.4.3.7 m classifies $|\uparrow U = U|$ where $\uparrow$ is a closure operator on the partially ordered object $(\Omega_{\leq})^{A}$ (i.e. Ω^{A} with inclusion as partial order). By 1.4.2.3 there is a morphism $f: \Omega^{A} \longrightarrow \mathrm{Idl}(\mathcal{A})$ such that $\uparrow = m \circ f$ and

$$f \dashv m \quad ((\Omega_{\leq})^{A}, \mathrm{Idl}(\mathcal{A})_{\leq}).$$

Again by 1.4.3.7, m classifies $|U^{o} = U|$, hence by 1.4.2.3 there is a morphism $g: \Omega^{A} \longrightarrow \mathrm{Idl}(\mathcal{A})$ such that $(\;)^{o} = m\,g$ and

$$g \dashv m \quad (((\Omega_{\leq})^{A})^{o}, (\mathrm{Idl}(\mathcal{A})_{\leq})^{o}).$$

Thus, with respect to $(\Omega_{\leq})^{A}$ and $\mathrm{Idl}(\mathcal{A})_{\leq}$, we have $f \dashv m \dashv g$.

By 1.4.1.5 the partial order of $\mathrm{Idl}(\mathcal{A})_{\leq}$ induces a Heyting algebra structure $\underset{\sim}{\mathrm{Idl}}(\mathcal{A})$ with

$$b \Rightarrow d = g(mb \Rightarrow md)$$

or equivalently

$$m(b \Rightarrow d) = (mb \Rightarrow md)^{o}.$$

<u>1.4.3.9 Factoring the diagonal $\Delta: \Omega \longrightarrow \Omega^A$ through the object of ideals of a pre-ordered structure on A.</u>

From $(p \wedge (a \prec b)) \Rightarrow p$ we deduce

$$\forall a \forall b(((a \in \Delta p) \wedge (a \prec b)) \Rightarrow (b \in \Delta p))$$

thus $idl(\Delta p)$. By 1.4.3.6, $\dagger\Delta p = \Delta p$ and $(\Delta p)^o = \Delta p$. By 1.4.3.8 we have

$$m \circ f \circ \Delta = \Delta = m \circ g \circ \Delta.$$

We let $D = f \circ \Delta = g \circ \Delta$; $D: \Omega \longrightarrow Idl(\mathcal{A})$.

<u>1.4.3.10 Proposition.</u> $\exists_A \circ m \dashv D \dashv \forall_A \circ m$.

<u>Proof.</u> $(p \leq \forall_A mx) \Leftrightarrow (\Delta p \subset mx)$ since $\Delta \dashv \forall_A$

$\Leftrightarrow (f\Delta p \leq x)$ since $f \dashv m$

$\Leftrightarrow (Dp \leq x)$,

hence $D \dashv \forall_A m$.

$(x \leq Dp) \Leftrightarrow (x \leq g\Delta p)$

$\Leftrightarrow (mx \leq \Delta p)$ since $m \dashv g$

$\Leftrightarrow (\exists_A mx \leq p)$ since $\exists_A \dashv \Delta$. □

<u>1.4.3.11 Proposition.</u> D is a Heyting algebra homomorphism from $\underset{\sim}{\Omega}$ to $\underset{\sim}{Idl}(\mathcal{A})$, where $\mathcal{A}$ is a pre-ordered object.

<u>Proof.</u> Since $\exists_A \circ m \dashv D \dashv \forall_A \circ m$, D preserves 0, 1, $\wedge$ and $\vee$.

We show D preserves $\Rightarrow$:

$(Dp \Rightarrow Dq) = g(mDp \Rightarrow mDq)$ by definition of $\Rightarrow$ on $\underset{\sim}{Idl}(\mathcal{A})$

$= g(\Delta p \Rightarrow \Delta q)$ since $m \circ D = m \circ g \circ \Delta = \Delta$

$= (g \circ \Delta)(p \Rightarrow q)$ since Δ preserves $\Rightarrow$, by 0.6.17.8

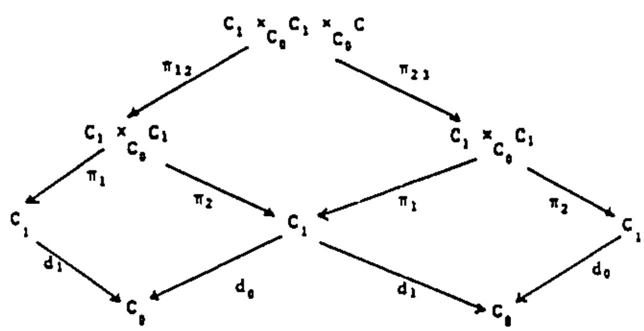
$= D(p \Rightarrow q)$.□

Section 1.5 The Topos $\mathcal{E}^{\underline{C}^{o}}$ and its Subobject Classifiier.

In this section we introduce the basic data describing $\mathcal{E}^{\underline{C}^{o}}$ and its subobject classifier, where $\underline{C}$ is an internal category in the topos $\mathcal{E}$. We rely on [J2] and [J1] for the description and we have adopted the notation of these sources. The purpose of the section is to convert the facts represented in various diagrams into the compact symbolic language $\Phi(\mathcal{E})$.

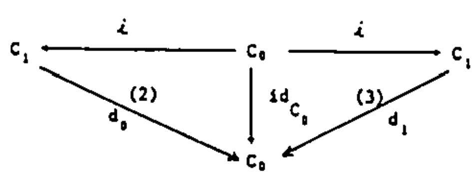
1.5.1 Definition of $\mathcal{E}^{\underline{C}^{o}}$. We record herein the diagrams involved in describing $\mathcal{E}^{\underline{C}^{o}}$ as a category.

1.5.1.1 Definition of an internal category $\underline{C}$ in $\mathcal{E}$. $\underline{C}$ is a 6-tuple $(C_0, C_1, d_0, d_1, i, m)$ where $d_0: C_1 \longrightarrow C_0$, $d_1: C_1 \longrightarrow C_0$, $i: C_0 \longrightarrow C_1$ and $m: C_1 \times_{C_0} C_1 \longrightarrow C_1$. These are such that (2), (3), (6), (7), (8), (9) and (12) commute. From d_0 and d_1 we construct pullbacks.

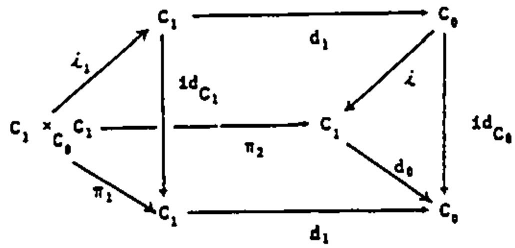


(1)

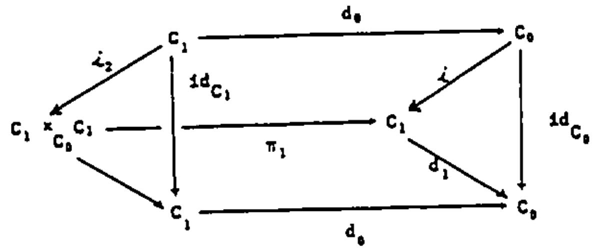
The morphism $i: C_0 \longrightarrow C_1$ is such that



commutes. The two triangles give rise to the two pullback diagrams.

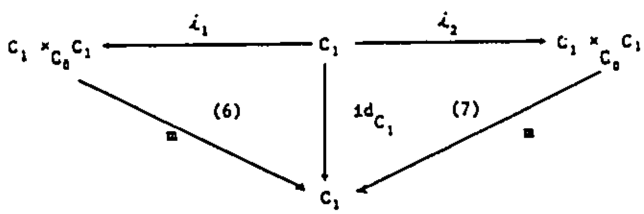


(4)

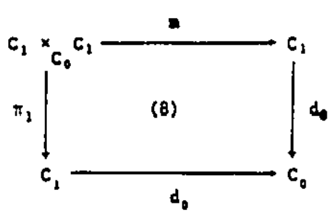


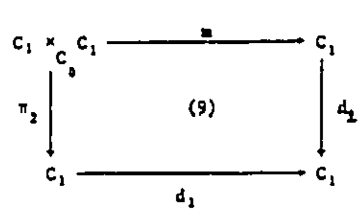
(5)

The morphism $m: C_1 \times_{C_0} C_1 \longrightarrow C_1$ is such that

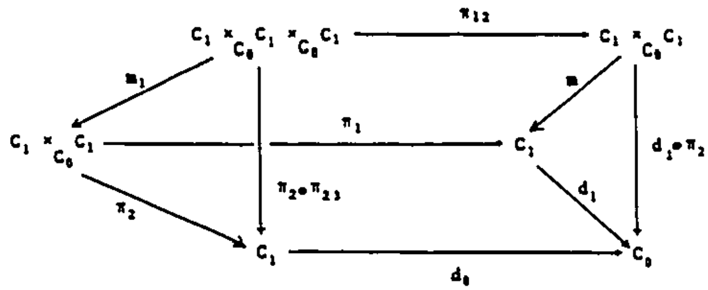


commutes, as well as the two squares

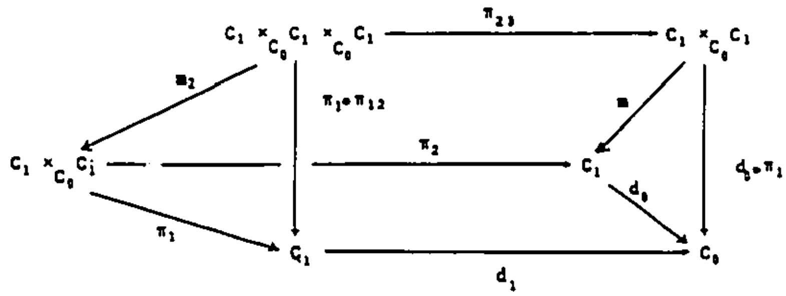




These give rise to the pullbacks (10) and (11).

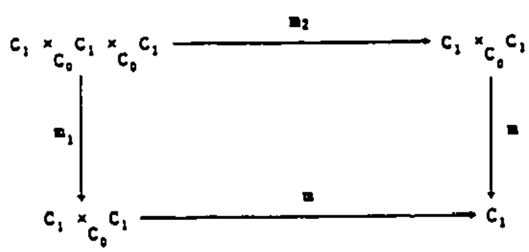


(10)



(11)

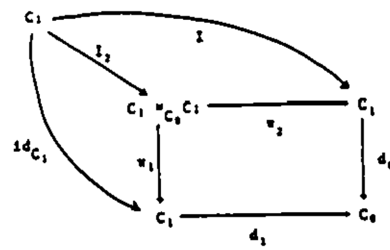
Finally we require that the square below commutes.

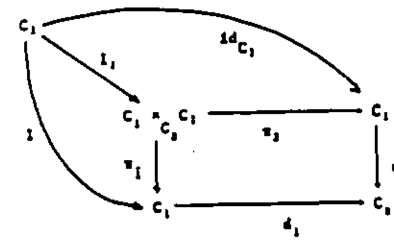


1.5.1.2 Definition. Let $\underline{C}$ be an internal category and let $I: C_1 \longrightarrow C_1$ be a morphism such that

(1) $d_0 = d_1 \circ I$ and (2) $d_1 = d_0 \circ I$.

From (1) and (2) we have induced morphisms I_1 and I_2



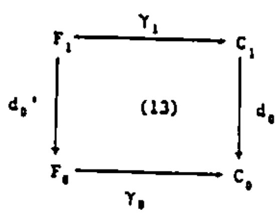


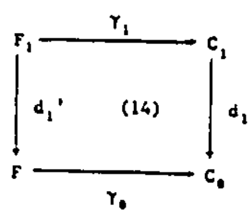
We call I an inverse on C if (1), (2),

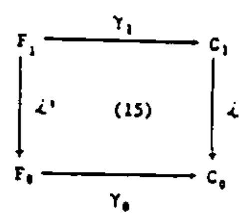
(3) $m \circ I_2 = i \circ d_0$ and (4) $m \circ I_1 = i \circ d_1$

all hold. We call C a groupoid if it has an inverse.

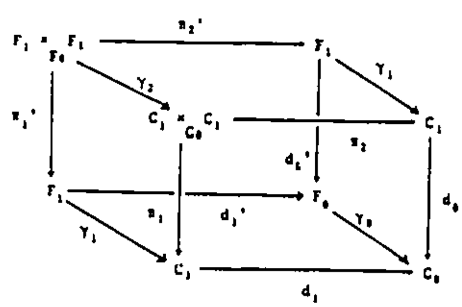
1.5.1.3 Internal functors and internal discrete fibrations. Let $\underline{F} = (F_0, F_1, d_0', d_1', i', m')$ and $\underline{C} = (C_0, C_1, d_0, d_1, i, m)$ be internal categories. An internal functor is a 4-tuple $\underline{\gamma} = (\underline{F}, \underline{C}, \gamma_0, \gamma_1)$ where $\gamma_0 \colon F_0 \longrightarrow C_0$ and $\gamma_1 \colon F_1 \longrightarrow C_1$, such that (13), (14), (15) and (17) commute.





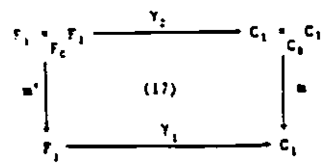


We construct γ_2 from (13) and (14).



(16)

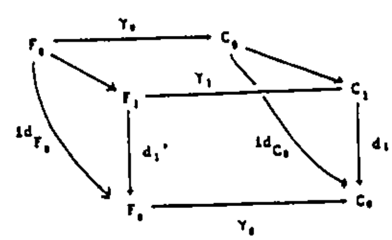
We require that (17) commute



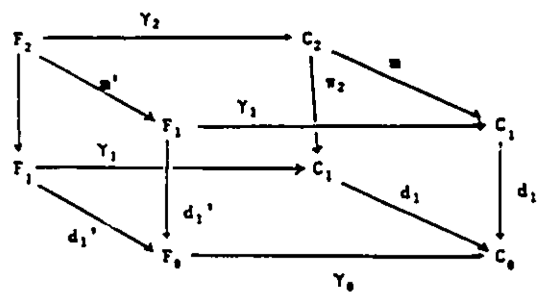
The internal categories and internal functors of a topos $\underline{\mathcal{E}}$ form a category Cat($\underline{\mathcal{E}}$).

An internal functor $\underline{\gamma}$: $\underline{F} \longrightarrow \underline{C}$ is called an internal discrete fibration over $\underline{C}$ if the square (14), above, is a pullback (we abbreviate this to "(14) pb."). By the category of discrete fibrations over $\underline{C}$ we mean the full subcategory of the comma category Cat($\underline{\mathcal{E}}$)/$\underline{C}$ whose objects are internal discrete fibrations over $\underline{C}$.

The condition (14) pb. allows us to shorten the list of objects and morphisms that we have taken as given for $\underline{F}$ and $\underline{\gamma}$. Thus F_1, γ_1 and d_1' are determined through (14) pb. by γ_0 and d_1. The morphism i' is required to satisfy (3) and (15), thus it induced by (14) pb.



In the cube (19) below, the back square is the top square of (16), the bottom square is the square (14), the right side square is (9) and the front square is (14) pb. Thus m' is the induced map making the upper square commute, which is (17), and making the left side square commute which is (8) for the category $\underline{F}$.



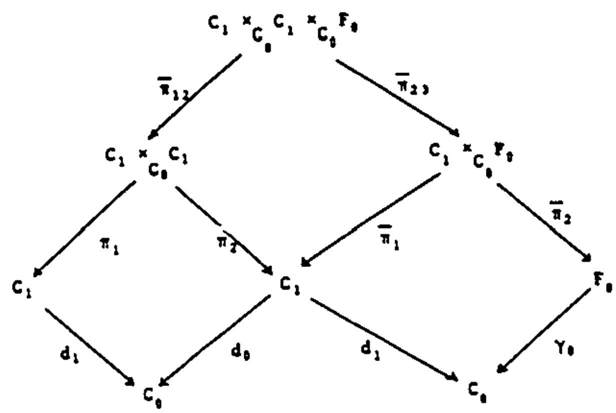
(19)

Thus a discrete fibration $\underline{\gamma}: \underline{F} \longrightarrow \underline{C}$ is uniquely determined (up to a choice of pullbacks) by $\underline{C}$, $\gamma_0: F_0 \longrightarrow C_0$, and $d_0': C_1 \times_{C_0} F_0 \longrightarrow F_0$. We now start all over again with this "short list" and collect all the conditions necessary to build a discrete fibration.

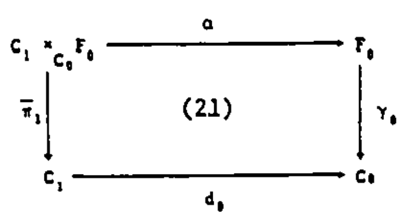
<u>1.5.1.4 Definition of an internal presheaf F on $\underline{C}$.</u> ([J1] p.6)

F is a triple $F = (F_0, \gamma_0, \alpha)$ where $\gamma_0 : F_0 \longrightarrow C_0$ and $\alpha : C_1 \times_{C_0} F_0 \longrightarrow F_0$. These are such that (21), (23) and (26) commute.

From γ_0 we construct pullbacks.

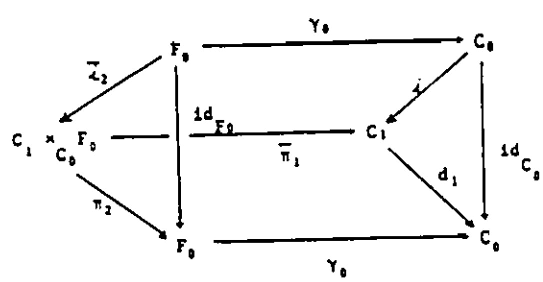


The morphism $\alpha : C_1 \times_{C_0} F_0 \longrightarrow F_0$ is such that



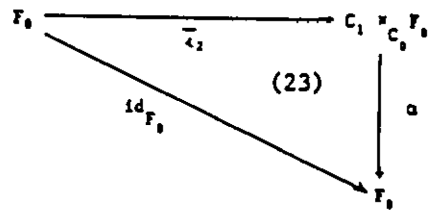
commutes.

We construct $\bar{\lambda}_2 : F_0 \longrightarrow C_1 \times_{C_0} F_0$ using (3)



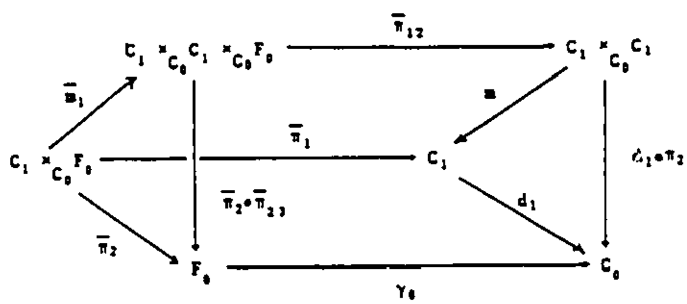
(22)

We require that



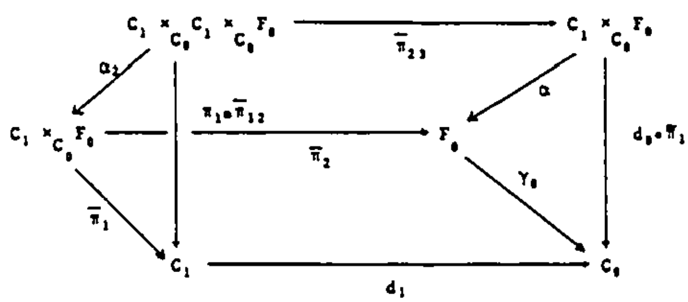
commutes.

We construct a pullback using (9) and (20).



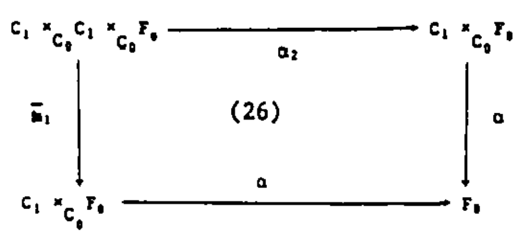
(24)

Using (11) we construct pullbacks:



(25)

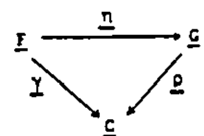
We construct α_2 by using (20) and (21).



commutes.

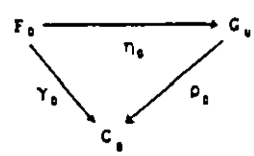
From the presheaf F on $\underline{C}$ we define the category $\underline{F} = (F_0, F_1, d_0', d_1', i', m')$ where $F_1 = C_1 \times_{C_0} F_0$, $d_0' = \alpha$, $d_1' = \overline{\pi}_2$, $i' = \overline{i}_2$ and $m' = \overline{m}$. We define a functor $\underline{\gamma} = (\underline{F}, \underline{C}, \gamma_0, \gamma_1)$ by putting $\gamma_1 = \overline{\pi}_1$. Then $\underline{\gamma}$ is a discrete fibration over $\underline{C}$.

<u>1.5.1.5 Morphisms of internal presheaves.</u> Let F and G be presheaves on $\underline{C}$, and let $\underline{\gamma}: \underline{F} \longrightarrow \underline{C}$ and $\underline{\rho}: \underline{G} \longrightarrow \underline{C}$ be the corresponding discrete fibrations. By definition a morphism in the category of discrete fibrations over $\underline{C}$, from $\underline{\gamma}$ to $\underline{\rho}$, is a functor $\underline{\eta}: \underline{F} \longrightarrow \underline{G}$ such that



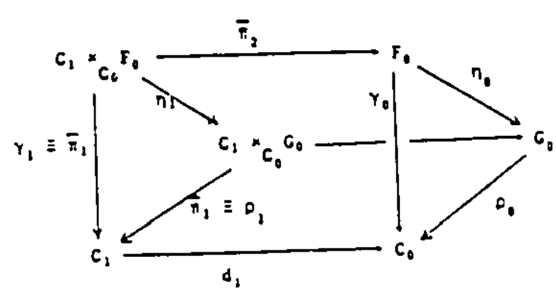
commutes.

We define a presheaf morphism $\eta: F \longrightarrow G$ in such a way that we can construct $\underline{\eta}: \underline{F} \longrightarrow \underline{G}$ from η. We put $\eta = (F, G, \eta_0)$ where η_0 is such that



(27)

commutes. We take $\eta_1: F_1 \longrightarrow G_1$ to be the induced map in (28)



(28)

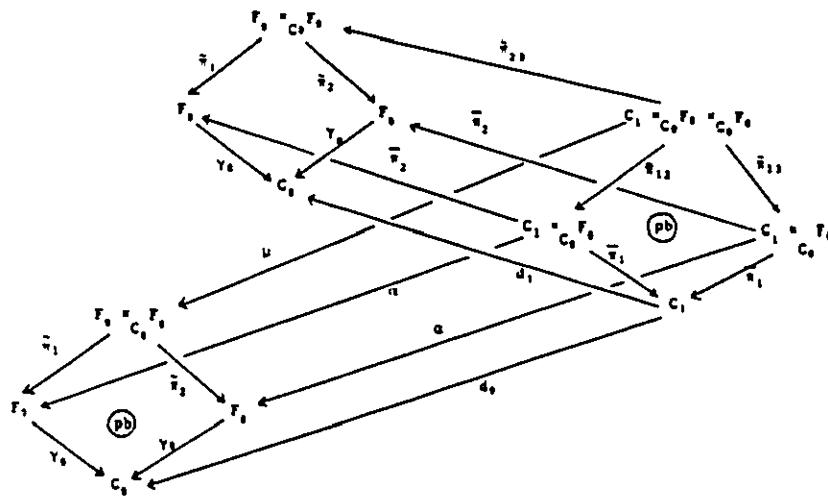
Here $G = (G_0,\rho_0,\beta)$, $\beta\colon G_1 \longrightarrow G_0$, $\rho_1\colon G_1 \longrightarrow C_1$. We require that (29) commute.

$$\begin{array}{ccc} C_1 \times_{C_0} F_0 & \xrightarrow{\ \eta_1\ } & C_1 \times_{C_0} G_0 \\ \Big\downarrow{\scriptstyle \alpha} & & \Big\downarrow{\scriptstyle \beta} \\ F_0 & \xrightarrow{\ \eta_0\ } & G_0 \end{array} \qquad (29)$$

The presheaves on a category $\underline{C}$ in $\underline{\mathcal{E}}$ together with the presheaf morphisms form a topos $\underline{\mathcal{E}}^{\underline{C}^{o}}$ which is essentially the category of discrete fibrations over $\underline{C}$. Thus we can regard $\underline{\mathcal{E}}^{\underline{C}^{o}}$ as a full subcategory of $\mathrm{Cat}(\underline{\mathcal{E}})/\underline{C}$. The functor $U\colon \underline{\mathcal{E}}^{\underline{C}^{o}} \longrightarrow \underline{\mathcal{E}}/C_0$; given on a presheaf $F = (F_0,\gamma_0,\alpha)$ by $U(F) = (F\ ,\gamma_0)$ and on a presheaf morphism $\underline{\eta} = (\underline{F},\underline{G},\eta_0,\eta_1)$, where $\underline{G} = (G_0,\rho_0,\beta)$, by $U(\underline{\eta}) = (\gamma_0,\rho_0,\eta_0)$, is monadic, comonadic, faithful and creates limits (see [J1]).

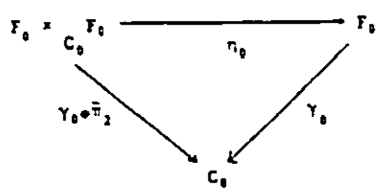
<u>1.5.1.6 Description of a morphism $\eta\colon F \times F \longrightarrow F$ in $\underline{\mathcal{E}}^{\underline{C}^{o}}$.</u> In 1.6.1 we examine the meet operation on the subobject classifier in $\underline{\mathcal{E}}^{\underline{C}^{o}}$. In the following diagrams we display the morphisms of $\underline{\mathcal{E}}$ which are involved in constructing a product $F \times F$ in $\underline{\mathcal{E}}^{\underline{C}^{o}}$ and a morphism $\eta\colon F \times F \longrightarrow F$.

Since $U\colon \underline{\mathcal{E}}^{\underline{C}^{o}} \longrightarrow \underline{\mathcal{E}}/C_0$ creates limits the product structure of $F \times F$ is induced by the product structure of $F_0 \xrightarrow{\ \gamma_0\ } C_0$ with itself in $\underline{\mathcal{E}}/\ C_0$, where $F = (F_0,\gamma_0,\alpha)$ and $\underline{C} = (C_0,C_1,d_0,d_1,i,m)$. We have $F \times F = (F_0 \times_{C_0} F_0, \gamma_0 \cdot \tilde{\pi}_2, \mu)$ where μ



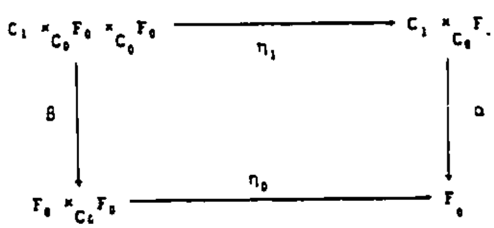
(30)

is the unique induced map to the lower pb . In order that $\eta = (F \times F, F, \eta_0)$ be a morphism of presheaves, from $F \times F$ to F we require that



(31)

and



(32)

commute.

1.5.2 The conversion of the partial operation of composition on C_1, for an internal category $\underline{C}$, to operations on subsets of $\underline{C}$ expressed in $\Phi(\mathcal{E})$. Given the attitude of Johnstone towards the language of [J2] and [J1], it is understandable that in [J1] certain expressions are introduced to describe maps, which, though their intended interpretation is clear enough, are not, strictly speaking, well-defined terms of the language. Thus for example on p. 55 of [J1], an expression appears, which, in our notation would be

$$m\langle f,\pi_1 s\rangle$$

where $m\colon C_1 \times_{C_0} C_1 \longrightarrow C_1$, f is a variable of type C_1 and $\pi_1 s$ is a certain term of type C_1. The inductive clause of our shared language only allows this expression to be a well-defined term of the language if $C_1 \times_{C_0} C_1$ is $C_1 \times C_1$. Since in our case we wish to carry out deductions and assert truths, in contrast to simply constructing useful labels for maps that have complicated constructions in $\underline{\mathcal{E}}$, we must either explain how our logic must be modified in order to derive truths about partial composition (i.e. admit $m\langle f,\pi_1 s\rangle$ as a term) or, maintain the language as described (in [J1]) and find some other way of describing partial composition. We have chosen to do the latter.

1.5.2.1 Definition. Let W, U, V be variables of type Ω^C ; w, x, y be variables of type C_1; θ, α, β be variables ot type $C_1 \times_{C_0} C$ be a variable of type $C_1 \times_{C_0} C_1 \times_{C_0} C_1$. We define a composition morphism $\Omega^{C_1} \times \Omega^{C_1} \longrightarrow \Omega^C$, using 1.5.1.1, by

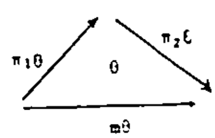
$(x \in U\bullet W) \Leftrightarrow \exists\theta((\pi_2\theta \in U) \wedge (\pi_1\theta \in W) \wedge (m\theta = x))$

and a ternary composition morphism $\Omega^{C_1} \times \Omega^{C_1} \times \Omega^{C_1} \longrightarrow \Omega^{C_1}$ by

$$(x \in U\cdot W\cdot V) \Leftrightarrow \exists\mu((\pi_1\pi_{12}\mu \in V) \wedge (\pi_2\pi_{12}\mu \in W) \wedge (\pi_2\pi_{23}\mu \in U) \wedge (mm_1\mu = x))$$

<u>1.5.2.2 Reconciliation of descriptive conventions used for $\underline{M}$-$\underline{\mathcal{E}}$ with those used in [J1] for $\underline{\mathcal{E}}^{\underline{C}^o}$.</u> In [J1] an internal monoid is defined as an internal category for which the object of objects is the terminal object. Thus it would seem natural for the internal monoid $\underline{M}$ defined in 1.3.2.1 to be construed as an internal category $\underline{C}$ by taking $C_0 = 1$, $C_1 = M$, $d_0 = d_1 = u_M$, the morphisms π_1 and $\pi_2: C_2 \longrightarrow C_1$ that appear in (1) 1.5.1.1 to be the first and second projections respectively from $M \times M$ to M, and finally the composition morphism $m: C_2 \longrightarrow C_1$ to be the multiplication morphism from $M \times M$ to M. Because, however, of the independent ways in which monoids and categories arise, it turns out that $\underline{M}$-$\underline{\mathcal{E}}$ is isomorphic to $\underline{\mathcal{E}}^{\underline{C}^o}$, and not, in general (i.e. unless $\underline{M}$ is commutative) to $\underline{\mathcal{E}}^{\underline{C}}$. To explain this apparent discrepancy we shall give a picture of what seems to lie behind Johnstone's description of internal categories.

We view a commutative triangle θ as going literally from left to right:



(1)

where π_1, π_2 and m are as in (1) 1.5.1.1. Our definition 1.5.2.1 of composition of sets of morphism, on Ω^{C_1}, leads to the equation

$$\{\pi_2\theta\}\circ\{\pi_1\theta\} = \{m\theta\}.$$

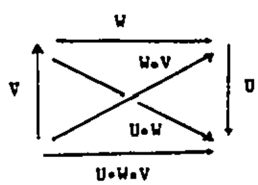
When $C_0 = 1$ we can replace θ by a pair $\langle x,y\rangle$, then $\pi_1\langle x,y\rangle = x$, $\pi_2\langle x,y\rangle = y$ and so

$$\{y\}\bullet\{x\} = \{m\langle x,y\rangle\}.$$

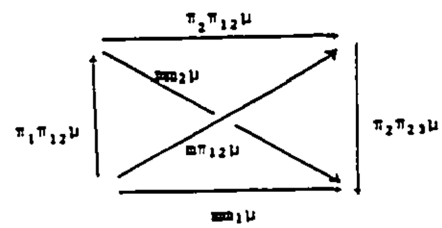
But we have already introduced an infix "•" for multiplication based on the order in which the arguments of m appear (see (1), (4) 1.3.1.1 and 1.3.2.1), thus $x\cdot y = m\langle x,y\rangle$, hence $\{y\}\bullet\{x\} = \{x\cdot y\}$.

One consequence of these different uses of an infix is that what in [Gb] is called the object of <u>left</u> ideals (U for which $s\cdot U \subset U$) is exactly the same as what is called the object of <u>right</u> ideals (U for which $U\bullet\{x\} \subset U$) in [J1].

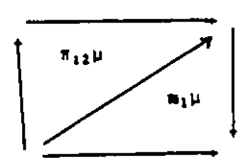
For future reference we display the morphisms involved when the ternary composition of 1.5.2.1 is interpreted externally: μ is a tetrahedron,



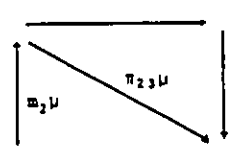
(2)



(3)



(4)



(5)

$\pi_{12}\mu$, $m_1\mu$, $\pi_{23}\mu$ and $m_2\mu$ are commutative triangles, and in (3) we have given one of several possible labellings of the edges (morphism) of μ.

<u>1.5.2.3.1 Proposition.</u> $(U \cdot W) \cdot V \subset U \cdot W \cdot V$.

<u>Proof.</u> We unravel the formula $\varphi \equiv [y \in (U \cdot W) \cdot V]$:

$\varphi \Leftrightarrow \exists\alpha((\pi_1\alpha \in V) \wedge (\pi_2\alpha \in (U \cdot W)) \wedge (m\alpha = y))$. Let $\psi \equiv (\pi_2\alpha \in (U \cdot W))$,

then $\psi \equiv \exists\beta((\pi_1\beta \in W) \wedge (\pi_2\beta \in U) \wedge (m\beta = \pi_2\alpha))$.

By (16) of 0.6.10.9 we have

$\varphi \Leftrightarrow \exists\alpha\exists\beta((\pi_1\alpha \in V) \wedge (\pi_1\beta \in W) \wedge (\pi_2\beta \in U) \wedge (m\alpha = y) \wedge (m\beta = \pi_2\alpha))$.

From the upper square of (11) of 1.5.1.1, and 0.6.16.8, we have

$(m\beta = \pi_2\alpha) \Rightarrow \exists\rho((m_2\rho = \alpha) \wedge (\pi_{23}\rho = \beta))$. Thus applying (16) 0.6.10.9

and from (11) and (12) of 1.5.1.1 we have

$$\varphi \Rightarrow \exists\alpha\exists\beta\exists\rho((\pi_1\alpha \in V) \wedge (\pi_1\beta \in W) \wedge (\pi_2\beta \in U) \wedge (m\alpha = y) \wedge (m_2\rho = \alpha) \wedge (\pi_{23}\rho = \beta))$$

$$\Rightarrow \exists\rho((\pi_1 m_2\rho \in V) \wedge (\pi_1\pi_{23}\rho \in W) \wedge (\pi_2\pi_{23}\rho \in U) \wedge (mm_2\rho = y))$$

$$\Rightarrow \exists\rho((\pi_1\pi_{12}\rho \in V) \wedge (\pi_2\pi_{12}\rho \in W) \wedge (\pi_2\pi_{23}\rho \in U) \wedge (mm_1\rho = y))$$

$$\Rightarrow (y \in U\ W\ V). \ \square$$

<u>1.5.2.3.2 Proposition.</u> $U \cdot W \cdot V \subset (U \cdot W) \cdot V$.

<u>Proof.</u> Let $\varphi \equiv (\pi_1\pi_{12}\mu \in V) \wedge (\pi_2\pi_{12}\mu \in W) \wedge (\pi_2\pi_{23}\mu \in U) \wedge (mm_1\mu = x)$.

We want to show

$$\varphi \Rightarrow (x \in (U \cdot W) \cdot V).$$

We make a normalized substitution

$$\begin{pmatrix} \mu & V & W & U & x \\ \overline{\mu} & \overline{V} & \overline{W} & \overline{U} & \overline{x} \end{pmatrix},$$

then $\pi_1\pi_{12}\mu \in \overline{V}$, $\pi_2\pi_{12}\mu \in \overline{W}$, $\pi_2\pi_{23}\mu \in \overline{U}$, and $mm_1\overline{\mu} = \overline{x}$ hence

$\pi_1\pi_{23}\overline{\mu} \in \overline{W}$ from (1) 1.5.1.1

$\pi_1 m_2\overline{\mu} \in \overline{V}$ from (11) 1.5.1.1

$mm_2\overline{\mu} = mm_1\overline{\mu}$ from (12) 1.5.1.1

$m\pi_{23}\overline{\mu} = \pi_2 m_2\overline{\mu}$ from (11) 1.5.1.1

thus

$(\pi_1(m_2\bar{\mu}) \in \bar{V}) \wedge (\pi_1(\pi_{23}\bar{\mu}) \in \bar{W}) \wedge (\pi_2(\pi_{23}\bar{\mu}) \in \bar{U})$

$$\wedge (m(m_2\bar{\mu}) = \bar{x}) \wedge (m(\pi_{23}\bar{\mu}) = \pi_2(m_2\bar{\mu}))$$

hence $\exists\theta\exists\alpha((\pi_2\alpha \in \bar{U}) \wedge (\pi_1\alpha \in \bar{W}) \wedge (m\alpha = \pi_2\theta) \wedge (\pi_1\theta \in \bar{V}) \wedge (m\theta = x))$

hence $x \in (\bar{U}\bullet\bar{W})\circ\bar{V}$. □

1.5.2.3.3 Proposition. $U\bullet(W\bullet V) = U\circ W\bullet V = (U\bullet W)\bullet V$.

Proof. $U\bullet W\bullet V = (U\bullet W)\bullet V$ by 1.5.2.3.1 and 1.5.2.3.2. For the equation $U\circ(W\bullet V) = U\bullet W\bullet V$, apply the preceding equation to the opposite category $\underline{C}^{o}$.□

1.5.2.4.1 Proposition.

(1) $(x_1 \in \{y\}\bullet\{w\}) \wedge (x_2 \in \{y\}\bullet\{w\}) \Rightarrow (x_1 = x_2)$

(2) $(x \in \{y\}\bullet\{w\}) \Leftrightarrow (\{x\} = \{y\}\bullet\{w\})$

Proof. (1) Let $\varphi \equiv (\pi_1\theta_1 = y) \wedge (\pi_2\theta_1 = w) \wedge (m\theta_1 = x_1) \wedge (\pi_1\theta_2 = y) \wedge (\pi_2\theta_2 = w) \wedge (m\theta_2 = x_2)$.

Then (1) is equivalent to $\varphi \Rightarrow (x_1 = x_2)$. We have

$(\pi_1\theta_1 = \pi_1\theta) \wedge (\pi_2\theta_1 = \pi_2\theta_2) \Rightarrow (\pi_1 \sqcap \pi_2)\theta_1 = (\pi_1 \sqcap \pi_2)\theta_2$.

But $\pi_1 \sqcap \pi_2 \colon C_1 \times_{C_0} C_1 \rightarrowtail C_1 \times C_1$ is a mono hence

$$\varphi \Rightarrow (\theta_1 = \theta_2) \wedge (m\theta_1 = x_1) \wedge (m\theta_1 = x_2)$$

$$\Rightarrow x_1 = x_2 \text{ .□}$$

(2) From (1) we have, using 0.6.16.1,

$$x \in \{y\}\bullet\{w\} \Rightarrow \forall u((u \in \{y\}\bullet\{w\}) \Rightarrow (u \in \{x\}))$$

$$\Rightarrow \{y\}\bullet\{w\} \subset \{x\}$$

$$\Rightarrow \{y\}\circ\{w\} = \{x\} \text{ . □}$$

1.5.2.4.2 Proposition.

$x \in \{y\}\bullet\{w\} \Rightarrow (d_0x = d_0w) \wedge (d_0y = d_1w) \wedge (d_1x = d_1y)$.

Proof. Let $\psi \equiv (d_0x = d_0w) \wedge (d_0y = d_1w) \wedge (d_1x = d_1y)$

$\varphi \equiv [(\pi_2\theta = y) \wedge (\pi_1\theta = w) \wedge (m\theta = x)$. We want to show $\varphi \Rightarrow \psi$. We have

$$(\pi_2\theta = y) \Rightarrow (d_0 y = d_0\pi_2\theta = d_1\pi_1\theta) \wedge (d_1 y = d_1\pi_2\theta)$$

$$(\pi_1\theta = w) \Rightarrow (d_1 w = d_1\pi_1\theta) \wedge (d_0 w = d_0\pi_1\theta)$$

$$(m\theta = x) \Rightarrow (d_0 x = d_0 m = d_0\pi_1\theta) \wedge (d_1 x = d_1 m = d_1\pi_2\theta).$$

Hence $\varphi \Rightarrow \psi$.□

<u>1.5.2.4.3 Proposition.</u>

(1) $\{id_1 y\}\circ\{y\} = \{y\}$

(2) $\{x\}\circ\{id_0 x\} = \{x\}$.

<u>Proof.</u> (1) By (2) 1.5.2.4.1 it suffices to show $y \in \{id_1 y\}\circ\{y\}$; that is $\exists\theta((\pi_2\theta = id_1 y) \wedge (\pi_1\theta = y) \wedge (m\theta = y))$.

From (4) 1.5.1.1 $\pi_2 i_1 y = id_1 y$ and $\pi_1 i_1 y = y$.

From (6) 1.5.1.1 $m i_1 y = y$, hence,

$$(\pi_2(i_1 y) = id_1 y) \wedge (\pi_1(i_1 y) = y) \wedge (m(i_1 y) = y).$$

(2) is similar -it is (1) applied to $\underline{C}^{o}$.□

<u>1.5.2.4.4 Corollary.</u>

(1) $(d_1 y = a) \Leftrightarrow (\{ia\}\circ\{y\} = \{y\})$

(2) $(d_0 x = a) \Leftrightarrow (\{x\}\circ\{ia\} = \{x\})$.

<u>Proof.</u> (1) By 1.5.2.4.3 (1), $(d_1 y = a) \Rightarrow \{ia\}\circ\{y\} = \{y\}$, and

$$(\{ia\}\circ\{y\} = \{y\}) \Rightarrow y \in \{ia\}\circ\{y\}$$
$$\Rightarrow d_1 y = d_1 ia \qquad \text{by 1.5.2.4.3}$$
$$\Rightarrow d_1 y = a.$$

(2) is similar.□

1.5.2.4.5 Proposition.

(1) $(\{v\}\bullet\{u\})\circ\{x\} = \{w\} \Rightarrow \exists z(\{z\} = \{v\}\circ\{u\})$

(2) $\{x\}\circ(\{v\}\circ\{u\}) = \{w\} \Rightarrow \exists z(\{z\} = \{v\}\circ\{u\})$.

Proof. (1) $w \in ((\{v\}\bullet\{u\})\bullet\{x\})$

$\Rightarrow \exists\theta((\pi_2\theta \in \{v\}\bullet\{u\}) \wedge (\pi_1\theta \in \{x\}) \wedge (m\theta = w))$

$\Rightarrow \exists\theta((\{\pi_2\theta\} = \{v\}\circ\{u\}) \wedge (\pi_1\theta = x) \wedge (m\theta = w))$

$\Rightarrow \exists\theta(\{\pi_2\theta\} = \{v\}\circ\{u\}$.

But $\{z\} = \{v\}\bullet\{u\} \Rightarrow \exists z(\{z\} = \{v\}\circ\{u\})$, hence

$\{\pi_2\theta\} = \{v\}\bullet\{u\} \Rightarrow \exists z(\{z\} = \{v\}\bullet\{u\})$, hence

$\exists\theta(\{\pi_2\theta\} = \{v\}\bullet\{u\}) \Rightarrow \exists z(\{z\} = \{v\}\circ\{u\})$. Hence

$w \in (\{v\}\bullet\{u\})\{x\}) \Rightarrow \exists z(\{z\} = \{v\}\circ\{u\})$.

(2) follows from (1) applied to $\underline{C}^{o}$.□

1.5.2.5 Proposition.

(1) $\forall w((w \in W) \Rightarrow (\{x\}\bullet\{w\} \subset U)) \Leftrightarrow (\{x\}\circ W \subset U)$

(2) $\forall w((w \in W) \Rightarrow (\{w\}\bullet\{x\} \subset U)) \Leftrightarrow (W\bullet\{x\} \subset U)$.

Proof. (1) $\forall w((w \in W) \Rightarrow (\{x\}\bullet\{w\} \subset U))$

$\Leftrightarrow \forall w((w \in W) \Rightarrow \forall y((y \in \{x\}\circ\{w\}) \Rightarrow (y \in U)))$

$\Leftrightarrow \forall w\forall y((\exists\theta((w \in W) \wedge (m\theta = y) \wedge (\pi_1\theta = w) \wedge (\pi_2\theta = x))) \Rightarrow (y \in U))$

$\Leftrightarrow \forall\theta\forall y[((\exists w((w \in W) \wedge (\pi_1\theta = w))) \wedge ((m\theta = y) \wedge (\pi_2\theta = x))) \Rightarrow (y \in U)]$

$\Leftrightarrow \forall\theta\forall y[((\pi_1\theta \in W) \wedge (m\theta = y) \wedge (\pi_2\theta = x)) \Rightarrow (y \in U)]$

$\Leftrightarrow \forall y(\exists\theta((\pi_1\theta \in W) \wedge (m\theta = y) \wedge (\pi_2\theta \in \{x\})) \Rightarrow (y \in U))$

$\Leftrightarrow \forall y((y \in \{x\}\ W) \Rightarrow (y \in U))$. .

(2) Apply the above to $\underline{C}^{o}$, that is, interchange π_1 and π_2.□

<u>1.5.2.6 Proposition.</u> $(U_1 \subset U_2) \Rightarrow (U_1 \bullet W \subset U_2 \bullet W)$.

<u>Proof.</u> $(U_1 \subset U_2) \wedge (m\theta = x) \wedge (\pi_1\theta \in W) \wedge (\pi_2\theta \in U_1)$

$\Rightarrow (m\theta = x) \wedge (\pi_1\theta \in W) \wedge (\pi_2\theta \in U_2)$, hence

$(U_1 \subset U_2) \wedge \exists\theta((m\theta = x) \wedge (\pi_1\theta \in W) \wedge (\pi_2\theta \in U_1))$

$\Rightarrow \exists\theta((m\theta = x) \wedge (\pi_1\theta \in W) \wedge (\pi_2\theta \in U_2))$ hence

$(U_1 \subset U_2) \Rightarrow \forall x((x \in U_1 \bullet W) \Rightarrow (x \in U_2 \bullet W))$.□

<u>1.5.2.7 Conversion of the axioms for an inverse I on $\underline{C}$.</u> Let $\underline{C}$ be an internal category and $I\colon C_1 \longrightarrow C_1$ satisfying (1) and (2) of 1.5.1.2. Call I a left inverse if (3) of 1.5.1.2 holds, and a right inverse if (4) of 1.5.1.2 holds. We shall show in 1.5.2.7.3 that both are inverses.

<u>1.5.2.7.1 Proposition.</u> Let $\underline{C}$ be an internal category and let $I\colon C_1 \longrightarrow C_1$. I is a left inverse iff

$$\{Ix\}\circ\{x\} = \{i d_0 x\} .$$

<u>Proof.</u> ($\rightarrow$) Suppose I is a right inverse, then

(1) $d_0 x = d_1 Ix$ and (2) $d_1 x = d_0 Ix$.

By definition of I .

(3) $\pi_1 I_2 x = x$ and (4) $\pi_2 I_2 x = Ix$.

From (3) of 1.5.1.2 we have

$$(5) \quad mI_2 x = i d_0 x.$$

From (3) and (4) we have

(6) $\exists\theta((\pi_1\theta = x) \wedge (\pi_2\theta = Ix) \wedge (\theta = I_2 x))$.

Now $(\theta = I_2 x) \Rightarrow (m\theta = mI_2 x)$

$\Rightarrow (m\theta = d_0 x)$ by (5).

Hence by (6)

(7) $\exists\theta((\pi_1\theta = x) \wedge (\pi_2\theta = Ix) \wedge (m\theta = id_0x))$.

Thus from 1.5.2.1

$$id_0x \in \{Ix\}\circ\{x\} \text{ . } \square$$

($\Leftarrow$) We suppose $id_0x \in \{Ix\}\circ\{x\}$, so that (7) holds. By 1.5.2.4.2, $d_0Ix = d_1x$ and $d_1Ix = d_0x$, so as in 1.5.1.2 we can construct $I_2 : C_1 \longrightarrow C_2$ such that $\pi_1I_2x = x$ and $\pi_2I\ x = Ix$. Since $\pi_1 \sqcap \pi_2 : C_2 \rightarrowtail C_1 \times C_1$ is a mono we have

$(\pi_1\theta = \pi_1I_2x) \wedge (\pi_2\theta = \pi_2I_2x) \Rightarrow (\theta = I_2x)$, so

$(\pi_1\theta = x) \wedge (\pi_2\theta = Ix) \Rightarrow (\theta = I_2x)$.

Combining this with (7) yields $\exists\theta((\theta = I_2x) \wedge (m\theta = id_0x))$, hence $mI_2x = id_0x$, so (3) holds. Thus I is a left inverse.□

By applying this proposition to $\underline{C}^{o}$ we have.

<u>1.5.2.7.2 Corollary.</u> I is a right inverse on $\underline{C}$ iff $\{x\}\circ\{Ix\} = \{id_1x\}$.□

<u>1.5.2.7.3 Proposition.</u> I is a left inverse iff I is a right inverse.

<u>Proof.</u> We show only one implication. Suppose $\{Ix\}\circ\{x\} = \{id_0x\}$, then $\{IIx\}\circ\{Ix\} = \{id_1x\}$ hence

$\{IIx\} = \{IIx\}\circ\{id_0IIx\} = \{IIx\}\circ\{id_1Ix\} = \{IIx\}\circ\{id_0x\} = \{IIx\}\circ\{Ix\}\circ\{x\}$

$\quad = \{id_1x\}\circ\{x\} = \{x\}$

hence $\{x\}\circ\{Ix\} = \{id_1x\}$.□

<u>1.5.2.7.4 Proposition.</u> (1) If I is an inverse on $\underline{C}$ then $I\circ I = id_{C_1}$.

(2) If $\underline{C}$ is a groupoid it has a uniquely determined inverse.

<u>Proof.</u> (1) $\{IIx\} = \{IIx\}\circ\{id_0 IIx\}$ by 1.5.2.4.3

$= \{IIx\}\circ\{id_0 x\}$ by (1) and (2) of 1.5.1.2

$= \{IIx\}\circ(\{Ix\}\circ\{x\})$ by (2) of 1.5.2.7

$= (\{IIx\}\circ\{Ix\})\circ\{x\}$ by 1.5.2.3.1

$= \{id_0 Ix\}\circ\{x\}$ by (2) of 1.5.7.2

$= \{id_1 x\}\circ\{x\}$ by (2) of 1.5.1.2

$= \{x\}$ by (1) of 1.5.2.4.3.□

(2) Let I and $\bar{I}$ be inverses.

$\{\bar{I}x\} = \{\bar{I}x\}\circ\{id_0 \bar{I}x\}$ by 1.5.2.4.3

$= \{\bar{I}x\}\circ\{id_1 x\}$ by (2) of 1.5.1.2

$= \{\bar{I}x\}\circ(\{x\}\circ\{Ix\})$ by (1) of 1.5.2.7

$= (\{\bar{I}x\}\circ\{x\})\circ\{Ix\}$ by 1.5.2.3.1

$= \{id_0 x\}\circ\{Ix\}$ by (2) of 1.5.1.2

$= \{id_1 Ix\}\circ\{Ix\}$ by (1) of 1.5.1.2

$= \{Ix\}$ by 1.5.2.4.3 .□

1.5.3 The pre-order of divisibility on C_1.

1.5.3.1 Definition. We define a binary relation $\prec$ on C_1 by

$$(x \prec y) \Leftrightarrow \exists w(\{x\}\bullet\{w\} = \{y\})$$

1.5.3.2 Proposition. $(x \prec y) \Leftrightarrow \exists\theta((\pi_2\theta = x) \wedge (m\theta = y))$.

Proof. $(x \prec y) \Leftrightarrow \exists w(y \in \{x\}\bullet\{w\})$

$$\Leftrightarrow \exists w(\exists\theta((\pi_1\theta = w) \wedge (\pi_2\theta = x) \wedge (m\theta = y)))$$

$$\Leftrightarrow \exists\theta((\exists w(\pi_1\theta = w)) \wedge (\pi_2\theta = x) \wedge (m\theta = y))$$

$$\Leftrightarrow \exists\theta((\pi_2\theta = x) \wedge (m\theta = y)).\square$$

1.5.3.3 Proposition. $\prec$ is a preorder; that is

(1) $x \prec x$

(2) $(x \prec y) \wedge (y \prec w) \Rightarrow (x \prec w)$.

Proof. (1) From 1.5.4.3, $\{x\}\bullet\{\iota d_0 x\} = \{x\}$, hence (1).

(2) Let $\varphi \equiv (\{x\}\circ\{u\} = \{y\}) \wedge (\{y\}\circ\{v\} = \{w\})$. Then

$\varphi \Rightarrow (\{x\}\circ\{u\})\circ\{v\} = \{w\}$

$\Rightarrow \{x\}\circ(\{u\}\circ\{v\}) = \{w\}$

$\Rightarrow \exists z(\{z\} = \{u\}\bullet\{v\})$ by 1.5.2.4.4

$\Rightarrow \exists z(\{x\}\bullet\{z\} = \{w\})$

$\Rightarrow x \prec w$.$\square$

1.5.3.4 Proposition. $(d_1 y = a) \Leftrightarrow (\iota a \prec y)$.

Proof. ($\Rightarrow$). By 1.5.2.4.4, $(d_1 y = a) \Rightarrow \{\iota a\}\bullet\{y\} = \{y\}$. Hence $(d_1 y = a) \Rightarrow \exists w(\{\iota a\}\bullet\{w\} = \{y\}).\square$

($\Leftarrow$) By 1.5.2.4.2, $(y \in \{\iota a\}\bullet\{w\}) \Rightarrow (d_1 y = d_1 \iota a)$, hence $\exists w(\{\iota a\}\bullet\{w\} = \{y\}) \Rightarrow (d_1 y = a).\square$

For the preorder of divisibility on C_1, the predicate idl gets interpreted as "right ideal", that is, closed under composition from the right; thus in [J1] the predicate is denoted by "rid".

1.5.3.5 Preposition.

(1) $idl(U) \Leftrightarrow \forall w(U \circ \{w\} \subset U)$

(2) $idl(U) \Leftrightarrow \forall x \forall w((x \in U) \Rightarrow (\{x\} \circ \{w\} \subset U))$.

Proof. (1) $idl(U)$

$\Leftrightarrow \forall x \forall y((x \in U) \wedge (x \prec y) \Rightarrow (y \in U))$

$\Leftrightarrow \forall x \forall y \forall w((x \in U) \wedge (y \in \{x\} \circ \{w\})) \Rightarrow (y \in U))$

$\Leftrightarrow \forall \theta \forall x \forall y \forall w((x \in U) \wedge (m\theta = y) \wedge (\pi_1 \theta = w) \wedge (\pi_2 \theta = x) \Rightarrow (y \in U))$

$\Leftrightarrow \forall \theta \forall y \forall w((\exists x((x \in U) \wedge (\pi_2 \theta = x))) \wedge (m\theta = y) \wedge (\pi_1 \theta = w) \Rightarrow (y \in U))$

$\Leftrightarrow \forall y \forall w(\exists \theta((\pi_2 \theta \in U) \wedge (m\theta = y) \wedge (\pi_1 \theta = w))) \Rightarrow (y \in U))$

$\Leftrightarrow \forall y \forall w((y \in U \quad w) \Rightarrow (y \in U))$

$\Leftrightarrow \forall w(U \circ \{w\} \subset U)$. □

(2) Substitute U for W in 1.5.2.5.1 and combine with (1).□

1.5.3.6 Proposition. Let $\underline{C}$ be an internal category and let $J: C_1 \times C_1 \longrightarrow \Omega$ be the relation defined by

(1) $J/x,y/ \Leftrightarrow (\{y\} \circ \{x\} = \{id_0 x\})$,

then (2) $J/w,x/ \wedge J/x,y/ \Rightarrow (w = y)$

(3) $J/w,x/ \wedge J/x,y/ \Rightarrow J/y,x/$

(4) $J/w,x/ \wedge J/x,y/ \Rightarrow J/x,w/$.

Proof. (2) Suppose $J/\bar{w},\bar{x}/$ and $J/\bar{x},\bar{y}/$. Then $\{\bar{x}\} \circ \{\bar{w}\} = \{id_0 \bar{w}\}$ and $\{\bar{y}\} \circ \{\bar{x}\} = \{id_0 \bar{x}\}$ hence $d_1 \bar{x} = d_0 \bar{w} = d_0 y$ and $d_1 \bar{y} = d_0 \bar{x} = d_1 \bar{w}$ hence $\{\bar{y}\} = \{\bar{y}\} \circ \{id_0 \bar{y}\} = \{\bar{y}\} \circ \{id_0 \bar{w}\} = \{\bar{y}\} \circ \{\bar{x}\} \circ \{\bar{w}\} = \{id_0 \bar{x}\} \circ \{\bar{w}\}$

$= \{id_1\overline{w}\}\bullet\{\overline{w}\} = \{\overline{w}\}$

hence $\overline{y} = \overline{w}$.□

(3) and (4) both follow directly from (2).□

1.5.3.7 Proposition. Let $\underline{C}$ be an internal groupoid. The following are equivalent

(1) $\prec$ is an equivalence relation on C_1

(2) $\forall x(x \prec id_1 x)$

(3) $\forall y \exists x\ J/x,y/$

(4) J is a functional symmetric relation on C_1

(5) $\underline{C}$ is a groupoid.

Proof. (1) → (2) We have $id_1 x \prec x$ from 1.5.3.4, and $(id_1 x \prec x) \Rightarrow (x \prec id_1 x)$ from (1), hence $x \prec id_1 x$.□

(2)↔ (3). $\exists x\ J/x,y/ \Leftrightarrow \exists x(\{y\}\bullet\{x\} = \{id_0 x\})$

$\Leftrightarrow \exists x(\{y\}\bullet\{x\} = \{id_1 y\})$

$\Leftrightarrow y \prec d_1 y$. □

(3) → (4). From (3) 1.5.3.6

$$(\exists x\ J/w,x/) \Rightarrow (J/x,y/ \Rightarrow J/y,x/).$$

Hence, by (3), $J/x,y/ \Rightarrow J/y,x/$. From (2) 1.5.3.6 we get

$$J/x,w/ \wedge J/x,y/ \Rightarrow w = y.$$

Hence J is functional and symmetric.

(4) → (5). By 0.6.16.7 there is a uniquely determined isomorphism I such that

$$J/x,y/ \Leftrightarrow (Ix = y).$$

We have $J/x,Ix/$, hence $J/Ix,x/$, hence

$$IIx = x.$$

<u>1.5.4 Construction of the subobject classifier for the topos $\mathcal{E}^{\underline{C}^o}$.</u>

The subobject classifier in $\mathcal{E}^{\underline{C}^o}$ is a morphism $\underline{\text{true}}$: $\underline{1} \longrightarrow \underline{\Omega}$ of presheaves. We have identified the terminal object of $\mathcal{E}^{\underline{C}^o}$, in 1.5.1.7, as the presheaf $\underline{1} = (C_0, id_{C_0}, d_0)$. Herein we shall present Johnstone's construction of the presheaf $\underline{\Omega} = (Sv(\underline{C}), \alpha_1, \psi)$ and the presheaf morphism $\underline{\text{true}} = (\underline{1}, \underline{\Omega}, u_1)$. We shall rely on [J1] Proposition 5.6 for a proof that this construction does in fact give the subobject classifier in $\mathcal{E}^{\underline{C}^o}$. As in previous sections of 1.5 we shall reformulate the construction in terms of $\Phi(\mathcal{E})$.

<u>1.5.4.1 Rid($\underline{C}$), the object of right ideals</u> ([J1] Definition 5.2 (ii)).

Let $m\colon C_2 \longrightarrow C_1$ and $\pi_2\colon C_2 \longrightarrow C_1$ be as in 1.5.1.1. The monomorphism $\tilde{m}\colon \text{Rid}(\underline{C}) \rightarrowtail \Omega^{C_1}$ is the equalizer of $id_{\Omega^{C_1}}$ and $\exists_m \circ \pi_2^{-1}$. We now show that the latter morphism is the closure operator $\uparrow\colon \Omega^{C_1} \longrightarrow \Omega^{C_1}$ (1.4.3.3, 1.4.3.4) defined in terms of the preorder of divisibility $\prec$ on C_1.

<u>1.5.4.2 Proposition.</u> $\uparrow U = \exists_m \pi_2^{-1} U$.

<u>Proof.</u> $(y \in \uparrow U) \leftrightarrow \exists x((x \prec y) \wedge (x \in U))$

$\leftrightarrow \exists x((\exists\theta((\pi_2\theta = x) \wedge (m\theta = y)) \wedge (x \in U))$

$\leftrightarrow \exists\theta((\exists x((\pi_2\theta = x) \wedge (x \in U))) \wedge (m\theta = y))$

$\leftrightarrow \exists\theta((\pi_2\theta \in U) \wedge (m\theta = y))$ by 0.6.10.7

$\leftrightarrow (y \in \exists_m \pi_2^{-1} U)$. □

Thus $\tilde{m}\colon \text{Rid}(\underline{C}) \longrightarrow \Omega^{C_1}$ classifies $|\uparrow U = U|$, which is the predicate $\text{idl}\colon \Omega^{C_1} \longrightarrow C_1$ (1.4.3.3, 1.4.3.7 (1)), so $\tilde{m}$ is the order embedding of the object of ideals (1.4.3.8) of the pre-ordered object $(C_1, \prec)$ into

$$Ix = y \Rightarrow (\{y\} \circ \{x\} = \{\iota d_0 x\})$$

$$\Rightarrow \iota d_0 x \in \{y\} \circ \{x\}$$

$$\Rightarrow (d_0 x = d_1 y) \wedge (d_1 x = d_0 y)$$

hence $d_0 x = d_1 Ix$ and $d_1 x = d_0 Ix$.□

(5) → (1) We want to show

$$(\{x\} \circ \{w\} = \{y\}) \Rightarrow \exists z(\{y\} \circ \{z\} = \{x\})$$

$$\{x\} \circ \{w\} = \{y\} \Rightarrow \{x\} \circ \{w\} \circ \{Iw\} = \{y\} \circ \{Iw\}$$

$$\Rightarrow \{x\} = \{y\} \circ \{Iw\}$$

$$\Rightarrow . \exists z(\{x\} = \{y\} \circ \{z\})$$

$$\Rightarrow x < y$$

Hence $(y < x) \Rightarrow (x < y)$.□

$(\Omega_{\leq})^{C_1}$. We think of $\mathrm{Rid}(\underline{C})$ as an object of $\underline{\mathcal{E}}/C_0$ in the canonical way; that is we apply the cartesian functor $C_0^*: \underline{\mathcal{E}} \longrightarrow \underline{\mathcal{E}}/C_0$:

$$C_0^*(\mathrm{Rid}(\underline{C})) = (C_0 \times \mathrm{Rid}(\underline{C}), \pi_1),$$

where $\pi_1: C_0 \times \mathrm{Rid}(\underline{C}) \longrightarrow C_0$ is the first projection.

<u>1.5.4.3 Proposition.</u> (1) $|[\iota a,\rightarrow)|: C_0 \longrightarrow \Omega^{C_1}$ is a monomorphism.

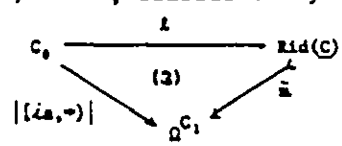
(2) There is a uniquely determined monomorphism $\ell: C_0 \rightarrowtail \mathrm{Rid}(\underline{C})$ such that $\tilde{m}\ell a = [\iota a,\rightarrow)$.

<u>Proof.</u> (1) $([\iota a,\rightarrow) = [\iota b,\rightarrow)) \Rightarrow (\iota a \prec \iota b)$

$$\Rightarrow (d_1 \iota a = d_1 \iota b)$$

$$\Rightarrow (a = b).\square$$

(2) By 1.4.3.3.1, $\mathrm{idl}[\iota a,\rightarrow)$, hence $|[\iota a,\rightarrow)|$ factors through the mono- $\tilde{m}: \mathrm{Rid}(\underline{C}) \rightarrowtail \Omega^{C_1}$, classified by idl. Thus (3) commutes.



<u>1.5.4.4. Definition.</u> Select $k: \mathrm{Fam}\ (\underline{C}) \rightarrowtail C_0 \times \Omega^{C_1}$, the "families of morphisms of $\underline{C}$" to be a mono classified by the morphism fam defined by

$$\mathrm{fam}\,[a,U] \Leftrightarrow \forall x((x \in U) \Rightarrow (d_1 x = a)).$$

<u>1.5.4.4.1 Proposition.</u>

(1) $\mathrm{fam}[a,U] \Leftrightarrow (U \subset [\iota a,\rightarrow))$

(2) $\mathrm{fam}[a,[\iota a,\rightarrow)]$.

<u>Proof.</u> (1) $\mathrm{fam}[a,U] \Leftrightarrow \forall x((x \in U) \Rightarrow (d_1 x = a))$

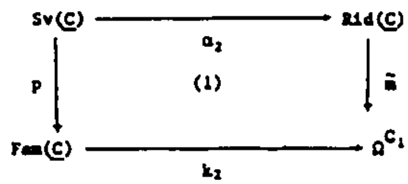
$$\Leftrightarrow \forall x((x \in U) \Rightarrow (\iota a \prec x)) \quad \text{by 1.5.3.4}$$

$$\Leftrightarrow U \subset [\iota a,\rightarrow).$$

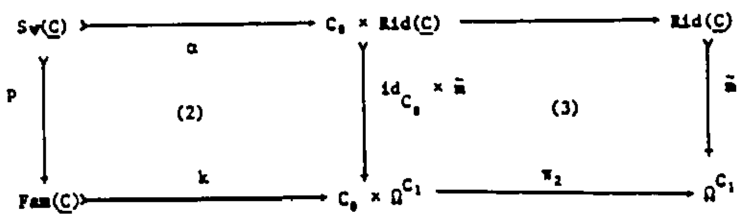
(2) Substitute $[\iota a,\rightarrow)$ for U in (1).$\square$

1.5.4.5. Sv($\underline{C}$), the object of sieves on $\underline{C}$ ([J1] Definition 5.2 (ii)).

This object is constructed by taking the pullback (1),



which can be constructed as the composite of two pullbacks:



We let α_1: Sv($\underline{C}$) $\longrightarrow$ C_0 be the composite of α with the projection $C_0 \times$ Rid($\underline{C}$) $\longrightarrow$ C_0. We define the predicate sv: $C_0 \times \Omega^{C_1} \longrightarrow \Omega$ by

$$sv\langle a,U\rangle \leftrightarrow (fam\langle a,U\rangle \wedge idl\ U)$$

so that

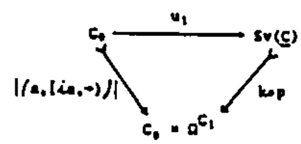
$$\|sv\langle a,U\rangle\| = \|fam\langle a,U\rangle\| \wedge [\![aU \mid idl\ U]\!]$$
$$= [\![k]\!] \wedge [\![id_{C_0} \times \tilde{m}]\!]$$
$$= [\![k \circ p]\!],$$

that is, sv classifies $k \circ p$.

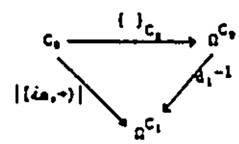
1.5.4.5.1 Proposition. $sv\langle a,[\iota a,\rightarrow)\rangle$.

Proof. $fam\langle a,[\iota a,\rightarrow)\rangle$ from 1.5.4.4.1 (2), and $idl[\iota a,\rightarrow)$ from 1.4.3.3.1, hence $sv\langle a,[\iota a,\rightarrow)\rangle$.□

1.5.4.5.2 Definition of u_1: $C_0 \rightarrowtail$ Sv($\underline{C}$). Since $sv\langle a,[\iota a,\rightarrow)\rangle$, there is a factorization



1.5.4.5.3 Proposition.



commutes.

Proof. $d_1^{-1}\{a\} = \{x\colon d_1 x = a\}$

$= \{x\colon ia \prec x\}$

$= [ia,\rightarrow)$. ◻

1.5.4.5.4 In [J1] (Definition 5.5, Proposition 5.6) the morphism true is shown to arise from a morphism from $(C_0,\ id_{C_0})$ to $(Sv(\underline{C}),\alpha_1)$ in $\underline{\mathcal{E}}/C_0$ as follows. From $C_0 \xrightarrow[\{\ \}_{C_0}]{} \Omega^{C} \xrightarrow[d_1^{-1}]{} \Omega^{C_1}$ and $C_0 \xrightarrow[\ ]{id} C_0$ is constructed the induced mono $C_0 \rightarrowtail C_0 \times \Omega^{C_1}$, which, by 1.5.4.5.3, is $|/a,[ia,\rightarrow)/|$; this mono factors through $k\circ p$ to produce $C_0 \overset{u_1}{\rightarrowtail} Sv(\underline{C})$. The morphism

$$((C_0,id_{C_0}),(Sv(\underline{C}),\alpha_1),u_1)\colon (C_0,id_{C_0}) \longrightarrow (Sv(\underline{C}),\alpha_1)$$

in $\underline{\mathcal{E}}/C_0$ is the underlying morphism of true.

<u>1.5.4.6. Proposition.</u> The following hold.

(1) $kpu_1a = \langle a,[ia,\rightarrow)\rangle$

(2) $\alpha_1u_1a = a$

(3) $\alpha_2u_1a = \ell a$

(4) $\alpha u_1a = \langle a,\ell a\rangle$

(5) $sv(kpu_1a)$.

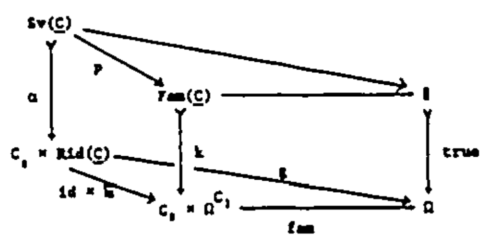
<u>Proof.</u> (1) By 1.5.4.5.2.□ (2) $\alpha_1u_1a = k_1pu_1a = a$ by (1) and (2) of 1.5.4.5.□ (3) $\tilde{m}\alpha_2u_1a = k_2pu_1a = [ia,\rightarrow) = \tilde{m}\ell a$, by (1) of 1.5.4.5, (1) above, and (2) of 1.5.4.3; since $\tilde{m}$ is a mono, $\alpha_2u_1a = \ell a$.□ (4) By (2) and (3).□ (5) From (1) and 1.5.4.5.1.□

<u>1.5.4.7 Proposition.</u> (1) $\alpha:Sv(\underline{C}) \rightarrowtail C_0 \times Rid(\underline{C})$ is classified by $\lambda ar\cdot(\tilde{m}r \subset [ia,\rightarrow)): C_0 \times Rid(\underline{C}) \longrightarrow \Omega$.

(2) α is classified by $\lambda ar\cdot(r \leq \ell a)$.

(3) α is the equalizer of $|\langle a,r\rangle|$ and $|\langle a,r \wedge \ell a\rangle|$.

<u>Proof.</u> (1) Let g classify α. Each rectangle is



a pullback and the lower triangle commutes. Thus

$g\langle a,r\rangle \leftrightarrow fam(id_{C_0} \times \tilde{m})\langle a,r\rangle \leftrightarrow fam\langle a,\tilde{m}r\rangle$

$\leftrightarrow \tilde{m}r \subset [ia,\rightarrow)$,

hence $g = \lambda ar\cdot(\tilde{m}r \subset [ia,\rightarrow))$.□

(2) $(\tilde{m}r \subset [ia,\rightarrow)) \leftrightarrow (\tilde{m}r \subset \tilde{m}\ a)$ by (2) 1.5.4.3

$\leftrightarrow (r \leq \ell a)$ since $\tilde{m}$ is an order embedding. □

(3) $(r \leq \ell a) \leftrightarrow (\langle a,r\rangle = \langle a,r \wedge \ell a\rangle)$. By (2), α is classified by

$|/a,r/ = /a,r \wedge \ell a/|$, hence, by 0.6.11.12, α is the equalizer of $|/a,r/|$ and $|/a,r \wedge \ell a/|$.□

<u>1.5.4.8 Definition.</u> $\psi_0 \colon C_1 \times \Omega^{C_1} \longrightarrow \Omega^{C_1}$ is given by
$(y \in \psi_0/x,U/) \Leftrightarrow (d_0x = d_1y) \wedge (\{x\}\bullet\{y\} \subset U)$.

<u>1.5.4.9 Proposition.</u> $fam/d_0x,\psi_0/x,U//$.

<u>Proof.</u> The assertion is equivalent to $(y \in \psi_0/x,U/) \Rightarrow (d_1y = d_0x)$ from 1.5.4.4.1.But this follows immediately from 1.5.4.8. □

<u>1.5.4.10 Proposition.</u> $(W \subset \psi_0/x,U/) \Leftrightarrow (W \subset [\iota d_0x,\to)) \wedge (\{x\}\bullet W \subset U)$.

<u>Proof.</u> $(W \subset \psi_0/x,U/)$

$\Leftrightarrow \forall y((y \in W) \Rightarrow (d_0x = d_1y) \wedge (\{x\}\bullet\{y\} \subset U))$

$\Leftrightarrow \forall y(((y \in W) \Rightarrow (\iota d_0x \prec y)) \wedge ((y \in W) \Rightarrow (\{x\}\bullet\{y\} \subset U)))$

$\Leftrightarrow (\forall y((y \in W) \Rightarrow (y \in [\iota d_0x,\to))) \wedge \forall y((y \in W) \Rightarrow (\{x\}\bullet\{y\} \subset U)))$

$\Leftrightarrow (W \subset [\iota d_0x,\to)) \wedge ((\{x\}\bullet W) \subset U)$. □

<u>1.5.4.11 Proposition.</u> (1) $idl(U) \Rightarrow idl(\psi_0/x,U/)$.

(2) $idl(U) \Rightarrow sv/d_0x,\psi_0/x,U//$

(3) $(d_1x = a) \wedge sv/a,U/ \Rightarrow sv/d_0x,\psi_0/x,U//$.

<u>Proof.</u> (1) Assume $idl(\overline{U})$. We want to show $idl(\psi_0/\overline{x},\overline{U}/)$, that is

$$(y \in \psi_0/\overline{x},\overline{U}/) \Rightarrow (\{y\}\circ\{w\} \subset \psi_0/\overline{x},\overline{U}/)$$

which is equivalent to

$(y \in [\iota d_0\overline{x},\to)) \wedge (\{\overline{x}\}\circ\{y\} \subset \overline{U}) \Rightarrow (\{y\}\bullet\{w\} \subset [\iota d_0\overline{x},\to)) \wedge (\{\overline{x}\}\circ(\{y\}\bullet\{w\}) \subset \overline{U})$.

Since $idl([\iota d_0\overline{x},\to))$ we have, by 1.5.3.5

$(y \in [\iota d_0\overline{x},\to)) \Rightarrow (\{y\}\circ\{w\} \subset [\iota d_0\overline{x},\to))$. Also

$(\{\overline{x}\}\circ\{y\} \subset \overline{U}) \Rightarrow ((\{\overline{x}\}\circ\{y\})\circ\{w\}) \subset (\overline{U}\circ\{w\}))$ by 1.5.2.6

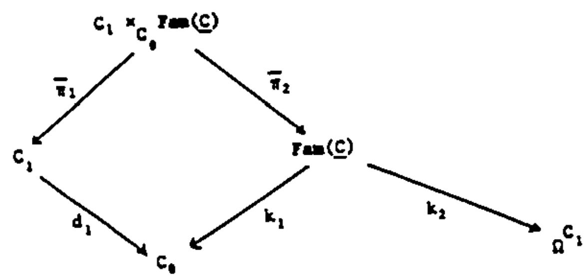
$\Rightarrow (\{\overline{x}\}\bullet(\{y\}\bullet\{w\})) \subset \overline{U}$ by 1.5.3.5 and 1.5.2.3.3.□

(2) From (1) and 1.5.4.9. □

(3) From the definition of sv in 1.5.4.5. we have $sv(a,U) \Rightarrow idl(U)$.

Hence from (2), (3) follows.□

<u>1.5.4.12 Construction of the action on $(Fam(\underline{C}),k_1)$.</u> We now follow [J1] Lemma 5.4 for the construction of an "action" ψ_2 on $(Fam(\underline{C}),k_1)$. The domain of the action we construct from the pullback



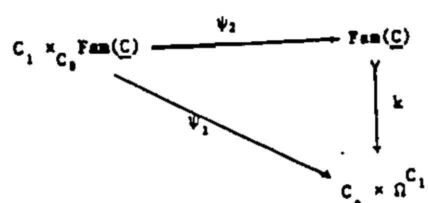
Let θ be a variable of type $C_1 \times_{C_0} Fam(\underline{C})$ then in [J] Lemma 5.4 the morphism $\psi_1: C_1 \times_{C_0} Fam(\underline{C}) \longrightarrow C_0 \times \Omega^{C_1}$ is given by

$$\psi_1\theta = (d_0\overline{\pi}_1\theta, \{x:(d_1x = d_0\overline{\pi}_1\theta) \wedge (\{\overline{\pi}_1\theta\}\bullet\{x\} \subset (k_2\overline{\pi}_2\theta))\})$$
$$= (d_0\overline{\pi}_1\theta, \psi_0(\overline{\pi}_1\theta, k_2\overline{\pi}_2\theta))$$

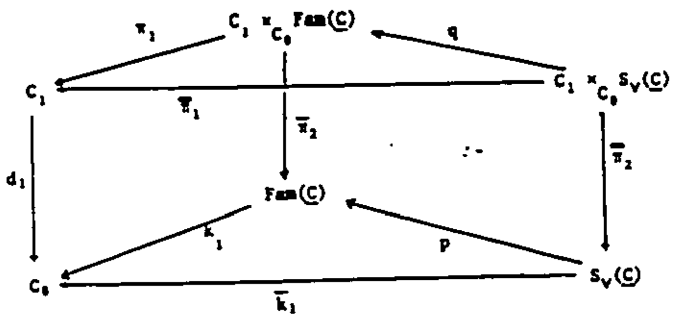
<u>1.5.4.13 Proposition.</u> $fam(\psi_1\theta)$.

<u>Proof.</u> This follows from 1.5.4.9 by substituting $\overline{\pi}_1\theta$ for x and $k_2\overline{\pi}_2\theta$ for U.□

Thus we have a factorization $k\psi_2\theta = \psi_1\theta$



We restrict ψ_2 to the object of sieves.



Let s be a variable of type $C_0 \times_{C_0} Sv(\underline{C})$. We have

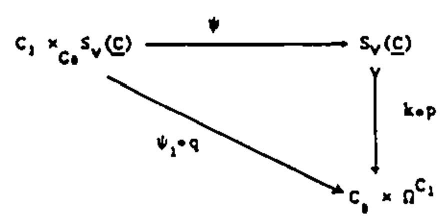
<u>1.5.4.14 Proposition.</u> $sv(\psi_1 qs)$.

<u>Proof.</u> $\psi_1 qs = (d_0\overline{\pi}_1 qs, \psi_0(\overline{\pi}_1 qs, k_2\overline{\pi}_2 qs))$

$= (d_0\overline{\pi}_1 s, \psi_0(\overline{\pi}_1 s, k_2 p\overline{\overline{\pi}}_2 s))$.

By 1.5.4.13, $fam(\psi_1 qs)$. We want to show $idl(\psi_0(\overline{\pi}_1 s, k_2\overline{\overline{\pi}}_2))$. By 1.5.4.11 (1), it suffices to show $idl(k_2 p\overline{\overline{\pi}}_2 s)$. From 1.5.4.5 (1) this is equivalent to $idl(\tilde{m}\alpha_2\overline{\overline{\pi}}_2 s)$, which is valid since $\tilde{m}$ classifies idl.□

Thus $\psi_1 q$: $C_1 \times_{C_0} Sv(\underline{C}) \longrightarrow C_0 \times \Omega^{C_1}$ factors through $k \cdot p$ (see 1.5.4.4). We let ψ be this factor:



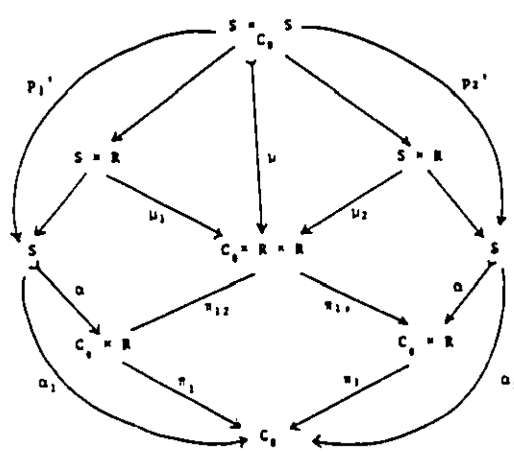
From [J1] Definition 5.5 and Proposition 5.6, $\underline{\Omega} = (Sv(\underline{C}), \alpha_1, \psi)$ is a presheaf in $\mathcal{E}^{\underline{C}^o}$; <u>true</u> $= (\underline{1}, \underline{\Omega}, u_1)$ is a presheaf morphism from $\underline{1} = (C_0, id_{C_0}, d_0)$ to $\underline{\Omega}$, and <u>true</u>: $\underline{1} \longrightarrow \underline{\Omega}$ is the subobject classifier of $\mathcal{E}^{\underline{C}^o}$.

1.6 Relationship between the Propositional Logic of $\underset{\sim}{\mathcal{E}}^{\underset{\sim}{C}^0}$ and of $\underset{\sim}{\mathcal{E}}$

1.6.1 The meet operation on $\underset{\sim}{\Omega}$ in $\underset{\sim}{\mathcal{E}}^{\underset{\sim}{C}^0}$. We put $S = Sv(\underset{\sim}{C})$ the object of sieves on $\underset{\sim}{C}$ in $\underset{\sim}{\mathcal{E}}$, $\underset{\sim}{S} = (S,\alpha_1)$ (where $\alpha_1 : S \longrightarrow C_0$) the object in the comma category $\underset{\sim}{\mathcal{E}}/C_0$ which indexes the sieves, $R = Rid(\underset{\sim}{C})$ the object of right ideals of $\underset{\sim}{C}$, and $\underset{\sim}{R} = C_0^{*}(R) = (C_0 \times R, \pi_1^{C_0,R})$.

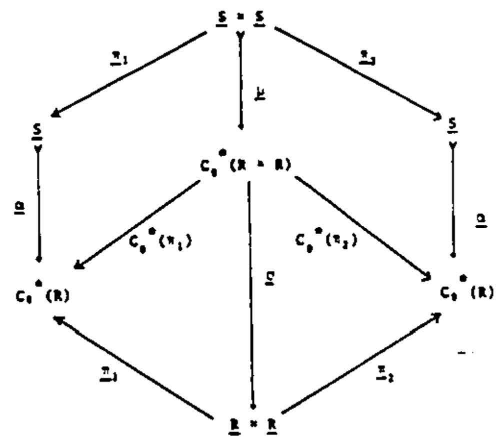
Given that $\underset{\sim}{\mathcal{E}}^{\underset{\sim}{C}^0}$ is a topos with subobject classifier $\underset{\approx}{\text{true}}: \underset{\sim}{1} \longrightarrow \underset{\sim}{\Omega}$, we know that there is a canonically defined meet operation $\underset{\approx}{\wedge}: \underset{\sim}{\Omega} \times \underset{\sim}{\Omega} \longrightarrow \underset{\sim}{\Omega}$. The functor $U: \underset{\sim}{\mathcal{E}}^{\underset{\sim}{C}^0} \longrightarrow \underset{\sim}{\mathcal{E}}/C_0$ which "forgets" the "action" of presheaves sends $\underset{\approx}{\wedge}$ to a morphism $U(\underset{\approx}{\wedge}): \underset{\sim}{S} \times \underset{\sim}{S} \longrightarrow \underset{\sim}{S}$ in $\underset{\sim}{\mathcal{E}}/C_0$; the functor $\Sigma: \underset{\sim}{\mathcal{E}}/C_0 \longrightarrow \underset{\sim}{\mathcal{E}}$ sends $U(\underset{\approx}{\wedge})$ to a morphism $\Sigma U(\underset{\approx}{\wedge}): S \times_{C_0} S \longrightarrow S$ in $\underset{\sim}{\mathcal{E}}$. In this section we shall use the meet operation $\wedge: R^2 \longrightarrow R$ on the object of ideals of $(C_1,\prec)$ to define a morphism $\dot{\lambda}: S \times_{C_0} S \longrightarrow S$ which we shall show must give rise to $\underset{\approx}{\wedge}$ so that $\Sigma U(\underset{\approx}{\wedge}) = \dot{\lambda}$.

1.6.1.1 Embedding $\underset{\sim}{S}$ in $\underset{\sim}{R}$ and $\underset{\sim}{S} \times \underset{\sim}{S}$ in $\underset{\sim}{R} \times \underset{\sim}{R}$. We let $\underset{\sim}{\alpha}: \underset{\sim}{S} \longrightarrow \underset{\sim}{R}$ be $\underset{\sim}{\alpha} = (\underset{\sim}{S},\underset{\sim}{R},\alpha)$. We form a pullback over C_0 in $\underset{\sim}{\mathcal{E}}$:



(1)

This gives rise to a diagram in $\mathcal{E}/C_0$ where $\underline{S} \times \underline{S} = (S \times_{C_0} S, \alpha_1 \circ p_1')$ in $\mathcal{E}/C_0$, $\underline{\pi}_i' = (\underline{S} \times \underline{S}, \underline{S}, p_i')$ $(i = 1,2)$ and $\underline{\mu} = (\underline{S} \times \underline{S}, C_0^*(R^2), \tau \circ \mu)$ where $\tau\langle a, r_1, r_2\rangle = \langle a, \langle r_1, r_2\rangle\rangle$.



In the lower part of the diagram, $\underline{\sigma} = (C_0^*(R^2), \underline{R}^2, \sigma)$ is a canonical isomorphism induced since C_0^* is a cartesian functor; $\pi_i\colon R^2 \longrightarrow R$ and $\underline{\pi}_i\colon \underline{R}^2 \longrightarrow \underline{R}$ $(i = 1,2)$ are the projections. By composing $\underline{\pi}_i$ with $\underline{\alpha} \times \underline{\alpha}$ and $\underline{\sigma} \circ \underline{\mu}$ for $i = 1,2$, we have

$\underline{\pi}_i \circ (\underline{\alpha} \times \underline{\alpha}) = \underline{\alpha} \circ \underline{\pi}_i' = \underline{\mu} \circ C_0^*(\pi_i) = \underline{\pi}_i \circ (\underline{\sigma} \circ \underline{\mu})$, hence $\underline{\alpha} \times \underline{\alpha} = \underline{\sigma} \circ \underline{\mu}$.

<u>1.6.1.2 Proposition.</u>

(1) $[\![\mu]\!] = [\![a r_1 r_2 \mid (r_1 \leq \ell a) \wedge (r_2 \leq \ell a)]\!]$

(2) $\mu\theta = \langle \alpha_1 p_1'\theta, \alpha_2 p_1'\theta, \alpha_2 p_2'\theta\rangle$.

<u>Proof.</u> (1) $[\![\mu]\!] = [\![\mu_1]\!] \wedge [\![\mu_2]\!]$

$$= \pi_{12}^{-1}[\![\alpha]\!] \wedge \pi_{13}^{-1}[\![\alpha]\!]$$

$$= \pi_{12}^{-1}[\![a r_1 \mid r_1 \leq \ell a]\!] \wedge \pi_{13}^{-1}[\![a r_2 \mid r_2 \leq \ell a]\!]$$

$$= [\![a r_1 r_2 \mid r_1 \leq \ell a]\!] \wedge [\![a r_1 r_2 \mid r_2 \leq \ell a]\!]$$

$$= [\![a r_1 r_2 \mid (r_1 \leq \ell a) \wedge (r_2 \leq \ell a)]\!]. \square$$

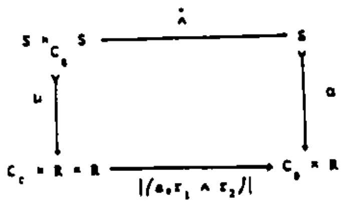
(2) Let π_1, π_2, π_{12}, π_{13} be as in diagram (1) of 1.6.1.1.

Let $\mu\theta = \langle \ell_1\theta, \ell_2\theta, \ell_3\theta \rangle$. then

$\ell_1\theta = \pi_1\pi_{12}\mu\theta = \alpha_1 p_1{}'\theta = \alpha_1 p_2{}'\theta$; $\ell_2\theta = \pi_2\pi_{12}\mu\theta = \pi_2\alpha p_1{}'\theta = \alpha_2 p_1{}'\theta$

$\ell_3\theta = \pi_2\pi_{13}\mu\theta = \pi_2\alpha p_2{}'\theta = \alpha_2 p_2{}'\theta$.□

<u>1.6.1.3 Proposition.</u> There is a factorization

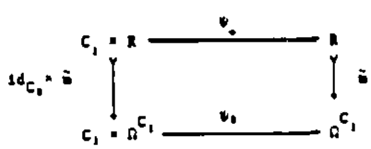


<u>Proof.</u> $(|\langle a, r_1 \wedge r_2 \rangle| \bullet \mu)\theta = |\langle a, r_1 \wedge r_2 \rangle| \langle \alpha_1 p_1{}'\theta, \alpha_2 p_1{}'\theta, \alpha_2 p_2{}'\theta \rangle$

$= \langle \alpha_1 p_1{}'\theta, (\alpha_2 p_1{}'\theta) \wedge (\alpha_2 p_2{}'\theta) \rangle$.

Since, by 1.5.4.7 (2), α classifies $\lambda ar.(r \leq \ell a)$ we have $\alpha_2 s \leq \ell\alpha_1 s$, hence $\alpha_2 p_1{}'\theta \leq \ell\alpha_1 p_1{}'\theta$, hence $\alpha_2 p_1{}'\theta \wedge \alpha_2 p_2{}'\theta \leq \ell\alpha_1 p_1{}'\theta$, hence there must be a morphism $\lambda: S \times_{C_0} S \longrightarrow S$ such that $\alpha \bullet \dot{\lambda} = |\langle a, r_1 \wedge r_2 \rangle| \mu$.□

<u>1.6.1.4 Proposition.</u> Let ψ_0 be as in 1.5.4.8. There is a factorization



<u>Proof.</u> It suffices to show $idl(\psi_0\langle x, \tilde{m}r \rangle)$. By 1.5.4.11 (1) this follows from $idl(\tilde{m}r)$.□

<u>1.6.1.5 Proposition.</u>

(1) $\psi_0\langle x, U_1 \cap U_2 \rangle = \psi_0\langle x, U_1 \rangle \cap \psi_0\langle x, U_2 \rangle$

(2) $\psi_+\langle x, r_1 \wedge r_2 \rangle = \psi_+\langle x, r_1 \rangle \wedge \psi_+\langle x, r_2 \rangle$

<u>Proof.</u> (1) $y \in \psi_0\langle x, U_1 \cap U_2 \rangle$

$\leftrightarrow ((d_0 x = d_1 y) \wedge ((\{x\} \circ \{y\}) \subset (U_1 \cap U_2)))$ by 1.5.4.8

$\leftrightarrow (d_0x = d_1y) \wedge ((\{x\}\circ\{y\}) \subset U_1) \wedge ((\{x\}\circ\{y\} \subset U_2)$

$\leftrightarrow (y \in \psi_0(x,U_1)) \wedge (y \in \psi_0(x,U_2))$

$\leftrightarrow (y \in (\psi_0(x,U_1) \cap \psi_0(x,U_2)).\square$

$$
\begin{aligned}
(2)\quad \tilde{m}\psi_+(x,r_1 \wedge r_2) &= \psi_0(x,\tilde{m}(r_1 \wedge r_2)) && \text{by } 1.6.1.4 \\
&= \psi_0(x,\tilde{m}r_1 \cap \tilde{m}r_2) && \text{by } 1.4.2.3 \text{ and } 1.4.1.3\ (2) \\
&= \psi_0(x,\tilde{m}r_1) \cap \psi_0(x,\tilde{m}r_2) && \text{by } (1) \\
&= \tilde{m}\psi_+(x,r_1) \cap \tilde{m}\psi_+(x,r_2) \\
&= \tilde{m}(\psi_+(x,r_1) \wedge \psi_+(x,r_2))
\end{aligned}
$$

hence $\psi_+(x,r_1 \wedge r_2) = \psi_+(x,r_1) \wedge \psi_+(x,r_2)$. $\square$

<u>1.6.1.6.1</u> We define the morphisms in the rectangle

$$
\begin{array}{ccc}
C_1 \times C_0 \times R \times R & \xrightarrow{\quad f_2 \quad} & C_0 \times R \times R \\
\downarrow{\scriptstyle g_2} & & \downarrow{\scriptstyle g_1} \\
C_1 \times C_0 \times R & \xrightarrow{\quad f_1 \quad} & C_0 \times R
\end{array}
$$

as follows:

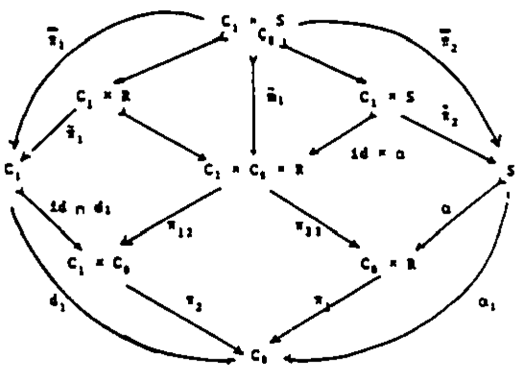
$f_1(x,a,r) = (d_0x,\psi_+(x,r))$ $\quad f_2(x,a,r_1,r_2) = (d_0x,\psi_+(x,r_1),\psi_+(x,r_2))$

$g_1(a,r_1,r_2) = (a,r_1 \wedge r_2)$ $\quad g_2(x,a,r_1,r_2) = (x,a,r_1 \wedge r_2)$.

<u>1.6.1.6.2 Proposition.</u> $g_1\circ f_2 = f_1\circ g_2$.

<u>Proof.</u>
$$
\begin{aligned}
g_1f_2(x,a,r_1,r_2) &= g_1(d_0x,\psi_+(x,r_1),\psi_+(x,r_2)) \\
&= (d_0x,\psi_+(x,r_1) \wedge \psi_+(x,r_2)) \\
&= (d_0x,\psi_+(x,r_1 \wedge r_2)) \\
&= f_1(x,a,r_1 \wedge r_2) \\
&= f_1g_2(x,a,r_1,r_2)\ .\square
\end{aligned}
$$

<u>1.6.1.7</u> The inclusion $\tilde{m}_1: C_1 \times_{C_0} S \longrightarrow C_1 \times C_0 \times R$ is defined by the pullbacks:



<u>1.6.1.8 Proposition.</u>

(1) $[\![\tilde{m}_1]\!] = [xar|(d_1x = a) \wedge (r \leq \ell a)]$

(2) $\tilde{m}_1\theta = \langle\bar{\pi}_1\theta, d_1\bar{\pi}_1\theta, \alpha_2\bar{\pi}_2\theta\rangle$

<u>Proof.</u> (1) $[\![\tilde{m}_1]\!] = \pi_{12}^{-1}[\![id_{C_1} \cap d_1]\!] \wedge \pi_{23}^{-1}[\![\alpha]\!]$

$= \pi_{12}^{-1}[\![xa|d_1x = a]\!] \wedge \pi_{23}^{-1}[\![ar|r \leq \ell a]\!]$

$= [\![xar|(d_1x = a) \wedge (r \leq \ell a)]\!]$.□

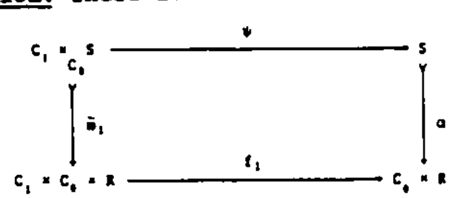
(2) Let π_i be the i-th projection from $C_1 \times C_0 \times R$ $(i = 1,2,3)$.

$\pi_1\tilde{m}_1\theta = \bar{\pi}_1\theta$

$\pi_2\tilde{m}_1\theta = d_1\bar{\pi}_1\theta$

$\pi_3\tilde{m}_1\theta = \tilde{\pi}_2\pi_{23}\tilde{m}_1\theta = \tilde{\pi}_2\alpha\bar{\pi}_2\theta = \alpha_2\bar{\pi}_2\theta$. □

<u>1.6.1.9 Proposition.</u> There is a factorization



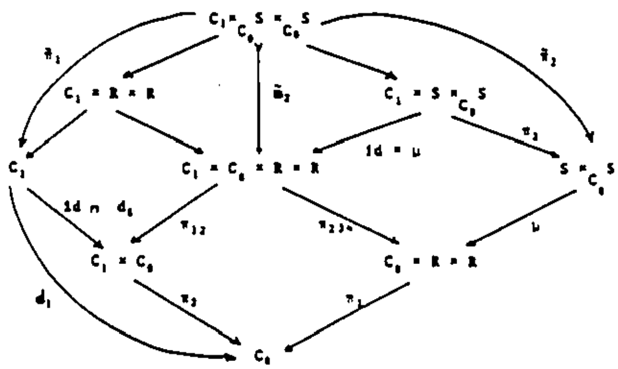
<u>Proof.</u> $f_1\tilde{m}_1\theta = f_1\langle\bar{\pi}_1\theta, d_1\bar{\pi}_2\theta, \alpha_2\bar{\pi}_2\theta\rangle$ by 1.6.1.8 (2)

$= \langle d_0\bar{\pi}_1\theta, \psi_+\langle\bar{\pi}_1\theta, \alpha_2\bar{\pi}_2\theta\rangle\rangle$ (definition of f_1: 1.6.1.6.1).

We will show $\psi_+\langle\bar{\pi}_1\theta,\alpha_2\bar{\pi}_2\theta\rangle \leq \ell d_0\bar{\pi}_1\theta$. By the definition of ψ_0 (1.5.4.8) we have $(y \in \psi_0\langle\bar{\pi}_1\theta,\tilde{m}\alpha_2\bar{\pi}_2\theta\rangle \Rightarrow (d_0\bar{\pi}_1\theta = d_1 y)$ therefore $\psi_0\langle\bar{\pi}_1\theta,\tilde{m}\alpha_2\bar{\pi}_2\theta\rangle \subset [d_0\bar{\pi}_1\theta,\rightarrow)$ hence $\tilde{m}\psi_+\langle\bar{\pi}_1\theta,\alpha_2\bar{\pi}_2\theta\rangle \subset \tilde{m}\ell d_0\bar{\pi}_1\theta$ hence $\psi_+\langle\bar{\pi}_1\theta,\alpha_2\bar{\pi}_2\theta\rangle \leq \ell d_0\bar{\pi}_1\theta$.

Since, by 1.5.4.7.(2), α is classified by $\lambda ar.(r \leq \ell a)$, and $(\lambda ar.(r \leq \ell a))\bullet(f_1\bullet\tilde{m}_1)$ factors through true, we have $f_1\bullet\tilde{m}_1$ factors through α.□

<u>1.6.1.10.</u> The inclusion $\tilde{m}_2\colon C_1 \times_{C_0} S \times_{C_0} S \rightarrowtail C_1 \times C_0 \times R \times R$ is defined via the pullbacks:



<u>1.6.1.11 Proposition.</u>

(1) $[\![\tilde{m}_2]\!] = [\![xar_1r_2 | (a = d_1x) \wedge (r_1 \leq \ell a) \wedge (r_2 \leq \ell a)]\!]$

(2) $\tilde{m}_2\theta = \langle\tilde{\pi}_1\theta, d_1\tilde{\pi}_1\theta\ \alpha_2 p_1'\tilde{\pi}_2\theta, \alpha_2 p_2'\tilde{\pi}_2\theta\rangle$

<u>Proof.</u> (1) $[\![\tilde{m}_2]\!] = \pi_{12}^{-1}[\![id \sqcap d_1]\!] \wedge \pi_{234}^{-1}[\![\mu]\!]$

$= \pi_{12}^{-1}[\![xa | a = d_1x]\!] \wedge \pi_{234}^{-1}[\![ar_1r_2 | (r_1 \leq \ell a) \wedge (r_2 \leq \ell a)]\!]$

$= [\![xar_1r_2 | (a = d_1x) \wedge (r_1 \leq \ell a) \wedge (r_2 \leq \ell a)]\!]$.□

(2) Let π_i $(i = 1,2,3,4)$ be the i-th projection from $C_1 \times C_0 \times R \times R$.

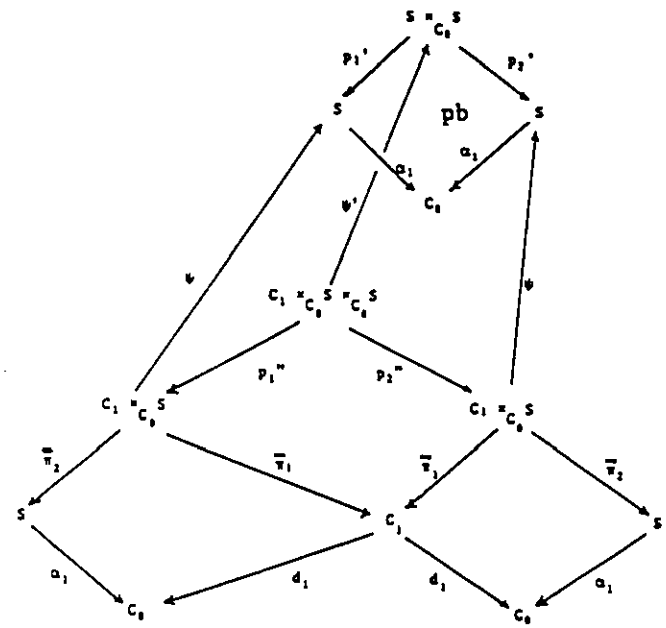
$\pi_1\tilde{m}_2\theta = \tilde{\pi}_1\theta \qquad \pi_2\tilde{m}_2\theta = d_1\tilde{\pi}_1\theta$

$\pi_3\tilde{m}_2\theta = \pi_2\pi_{234}\tilde{m}_2\theta = \pi_2\mu\tilde{\pi}_2\theta = \pi_2\pi_{12}\mu\tilde{\pi}_2\theta = \pi_2\alpha p_1{}'\tilde{\pi}_2\theta = \alpha_2 p_1{}'\tilde{\pi}_2\theta$

$\pi_4\tilde{m}_2\theta = \pi_3\pi_{234}\tilde{m}_2\theta = \pi_3\mu\tilde{\pi}_2\theta = \alpha_2 p_2{}'\tilde{\pi}_2\theta.$ □

1.6.1.12 Proposition. There is a morphism

$\psi': C_1 \times_{C_0} S \times_{C_0} S \longrightarrow S \times_{C_0} S$ making the following diagram commute.



Proof. We will show that $\alpha_1\circ(\psi\circ p_1'') = \alpha_1\circ(\psi\circ p_2'')$.

$\alpha_1\psi p_1''\theta = \pi_1 f_1\tilde{m}_1 p_1''\theta$ by 1.6.1.9

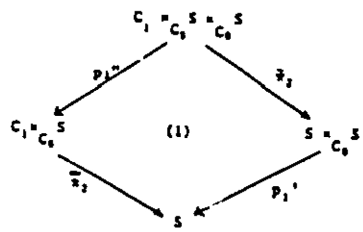
$= \pi_1 f_1(\overline{\pi}_1 p_1''\theta, d_1\overline{\pi}_1 p_1''\theta, \alpha_2\overline{\pi}_2 p_1''\theta)$

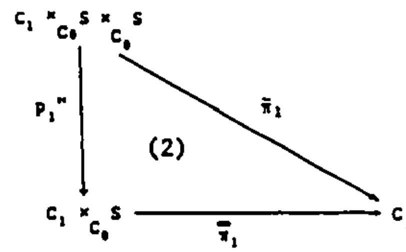
$= d_0\overline{\pi}_1 p_1''\theta = d_0\overline{\pi}_1 p_2''\theta = \alpha_2\psi p_2''\theta$.

Hence we can construct ψ' such that $p_1'\circ\psi' = \psi\circ p_1''$ and

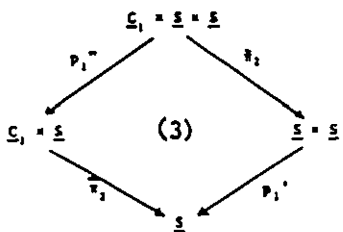
$p_2'\circ\psi' = \psi\circ p_2''$. □

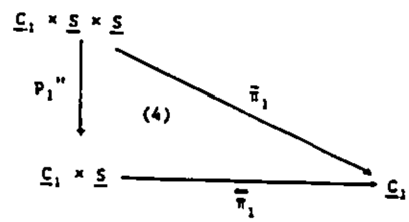
<u>1.6.1.13 Proposition.</u> The following diagrams commute.





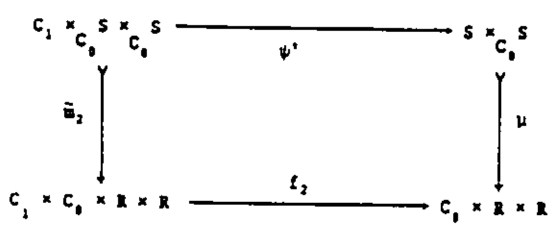
<u>Proof.</u> Define $\underline{C}_1 = (C_1, d_1)$, an object of $\underline{\mathcal{E}}/C_0$. Diagrams (3) and (4) commute since they involve canonical projections in $\underline{\mathcal{E}}/C_0$.





The morphisms $\bar{\pi}_1$ and $\bar{\pi}_2$ are defined in the pullback diagram preceding 1.5.4.14. Thus by applying $\Sigma\colon \underline{\mathcal{E}}/C_0 \longrightarrow \underline{\mathcal{E}}$ to (3) and (4) we get (1) and (2). □

<u>1.6.1.14 Proposition.</u> The following square commutes.



<u>Proof.</u> $f_2\tilde{m}_2\theta = f_2(\tilde{\pi}_1\theta, d_1\tilde{\pi}_1\theta, \alpha_2 p_1'\tilde{\pi}_2\theta, \alpha_2 p_2'\tilde{\pi}_2\theta)$

$= (d_0\tilde{\pi}_1\theta, \psi_+(\tilde{\pi}_1\theta, \alpha_2 p_1'\tilde{\pi}_2\theta), \psi_+(\tilde{\pi}_1\theta, \alpha_2 p_2'\tilde{\pi}_2\theta))$

$\mu\psi'\theta = (\alpha_1 p_1'\psi'\theta, \alpha_2 p_1'\psi'\theta, \alpha_2 p_2'\psi'\theta)$

$= (\alpha_1\psi p_1''\theta, \alpha_2\ \psi p_1''\theta, \alpha_2\psi\ {}_2''\theta)$

$$= \big(\pi_1 f_1 \tilde{m}_1 p_1''\theta, \pi_2 f_1 \tilde{m}_1 p_1''\theta, \pi_2 f_1 \tilde{m}_1 p_2''\theta\big)$$

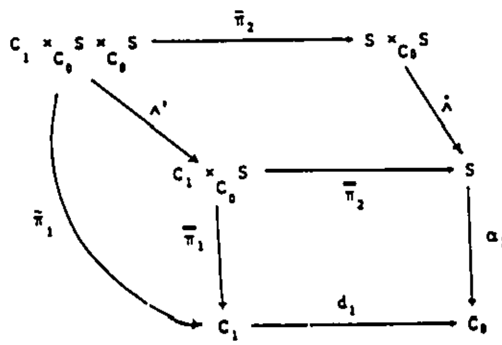
$$= \big(d_0 \bar{\pi}_1 p_1''\theta, \psi_+\big(\bar{\pi}_1 p_1''\theta, \alpha_2 \bar{\pi}_2 p_1''\theta\big), \psi_+\big(\bar{\pi}_1 p_2''\theta, \alpha_2\, \bar{\pi}_2 p_2''\theta\big)\big)$$

$$= \big(d_0 \tilde{\pi}_1 \theta, \psi_+\big(\tilde{\pi}_1\theta, \alpha_2 p_1' \tilde{\pi}_2\theta\big), \psi_+\big(\tilde{\pi}_2\theta, \alpha_2 p_2' \tilde{\pi}_2\theta\big)\big)$$

the last equation follows from 1.6.1.13 and from similar commutative diagrams formed by replacing p_1' by p_2' and p_1'' by p_2''.□

1.6.1.15 Construction of Λ': $C_1 \times_{C_0} S \times_{C_0} S \longrightarrow C_1 \times_{C_0} S$.

Given that the front rectangle is a pullback we show that $\alpha_1 \circ (\dot{\Lambda} \bullet \tilde{\pi}_2) = d_1 \bullet \tilde{\pi}_1$, from which it follows that we have a map Λ' such that $\bar{\pi}_2 \bullet \Lambda' = \dot{\Lambda} \bullet \tilde{\pi}_2$ and $\bar{\pi}_1 \bullet \Lambda' = \tilde{\pi}_1$ (see diagram (28) and (29) of 1.5.1.5, here $\eta_1 = \Lambda'$ and $\eta_0 = \dot{\Lambda}$ and $F_0 = S \times_{C_0} S$, $G_0 = S$).



$\alpha_1 \dot{\Lambda} \tilde{\pi}_2 \theta = \pi_1 g_1 \mu \tilde{\pi}_2 \theta$ by 1.6.1.3 since $g_1 = \big|\big(a, r_1 \wedge r_2\big)\big|$

$$= \pi_1 g_1 \big(\alpha_1 p_1' \tilde{\pi}_2\theta, \alpha_2 p_1' \tilde{\pi}_2\theta, \alpha_2 p_2' \tilde{\pi}_2\theta\big)$$

$$= \pi_1 \big(\alpha_1 p_1' \tilde{\pi}_2\theta, \alpha_2 p_1' \tilde{\pi}_2\theta \wedge \alpha_2 p_2' \tilde{\pi}_2\theta\big)$$

$$= \alpha_1 p_1' \tilde{\pi}_2 \theta$$

$$= \alpha_1 \bar{\pi}_2 p_1''\theta$$

$$= d_1 \bar{\pi}_1 p_1''\theta \qquad \text{from 1.6.1.7}$$

$$= d_1 \tilde{\pi}_1 \theta$$

hence $\bar{\pi}_2 \Lambda'\theta = \dot{\Lambda}\tilde{\pi}_2\theta$ and $\bar{\pi}_1 \Lambda'\theta = \tilde{\pi}_1\theta$. □

1.6.1.16 Proposition. $g_2 \circ \tilde{m}_2 = \tilde{m}_1 \circ \wedge'$

$$\begin{array}{ccc} C_1 \times_{C_0} S \times_{C_0} S & \xrightarrow{\tilde{m}_2} & C_1 \times C_0 \times R \times R \\ \downarrow{\scriptstyle \wedge'} & & \downarrow{\scriptstyle g_2} \\ C_1 \times_{C_0} S & \xrightarrow{\tilde{m}_1} & C_1 \times C_0 \times R \end{array}$$

Proof.

$g_2\tilde{m}_2\theta = g_2(\tilde{\pi}_1\theta, d_1\tilde{\pi}_1\theta, \alpha_2 p_1'\tilde{\pi}_2\theta, \alpha_2 p_2'\tilde{\pi}_2\theta)$ by 1.6.1.11 (2)

$= (\tilde{\pi}_1\theta, d_1\tilde{\pi}_1\theta, \alpha_2 p_1'\tilde{\pi}_2\theta \wedge \alpha_2 p_2'\tilde{\pi}_2\theta)$ by definition of g_2 (1.6.1.6.1)

$\tilde{m}_1\wedge'\theta = (\overline{\pi}_1\wedge'\theta, d_1\overline{\pi}_1\wedge'\theta, \alpha_2\overline{\pi}_2\wedge'\theta)$ by 1.6.1.8 (2)

$= (\tilde{\pi}_1\theta, d_1\tilde{\pi}_1\theta, \alpha_2\dot{\wedge}\tilde{\pi}_2\theta)$ by 1.6.1.15

$\alpha_2\dot{\wedge}\tilde{\pi}_2\theta = \pi_2\alpha\dot{\wedge}\tilde{\pi}_2\theta$

$= \pi_2 g_1\mu\tilde{\pi}_2\theta$ by 1.6.1.3

$= \pi_2 g_1(\alpha_1 p_1'\tilde{\pi}_2\theta, \alpha_2 p_1'\tilde{\pi}_2\theta, \alpha_2 p_2'\tilde{\pi}_2\theta)$ by 1.6.1.2 (2)

$= \alpha_2 p_1'\tilde{\pi}_2\theta \wedge \alpha_2 p_2'\tilde{\pi}_2\theta$.□

1.6.1.17 Proposition. $\dot{\wedge}\circ\psi' = \psi\circ\wedge'$

$$\begin{array}{ccc} C_1 \times_{C_0} S \times_{C_0} S & \xrightarrow{\psi'} & S \times_{C_0} S \\ \downarrow{\scriptstyle \wedge'} & & \downarrow{\scriptstyle \wedge} \\ C_1 \times_{C_0} S & \xrightarrow{\psi} & S \end{array}$$

Proof. $\alpha\dot{\wedge}\psi'\theta = g_1\mu\psi'\theta$ by 1.6.1.3

$= g_1 f_2\tilde{m}_2\theta$ by 1.6.1.14

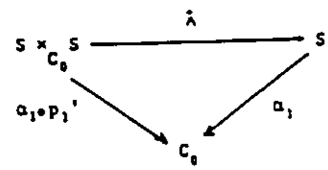
$= f_1 g_2\tilde{m}_2\theta$ by 1.6.1.6.2

$= f_1\tilde{m}_1\wedge'\theta$ by 1.6.1.16

$= \alpha\psi\wedge'\theta$ by 1.6.1.9

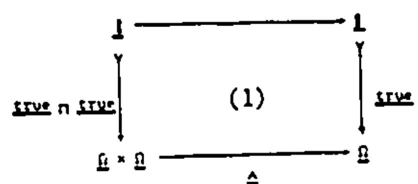
Since α is a mono, $\dot{\wedge}\psi'\theta = \psi\wedge'\theta$.□

<u>1.6.1.18 Definition of $\underline{\wedge}$: $\underline{\Omega}^2 \longrightarrow \underline{\Omega}$.</u> We define a presheaf morphism $\underline{\wedge}$ but do not establish that this is the naturally defined meet operation on $\underline{\Omega}$, until the end of 1.6.1. From 1.6.1.1 $\underline{S} \times \underline{S} = (S \times_{C_0} S, \alpha_1 \circ p_1')$ is a product in $\mathcal{E}/C_0$ with projections $\underline{\pi}_i'$: $(\underline{S} \times \underline{S}, \underline{S}, p_i')$ for $i = 1,2$. By the construction of $\dot{\wedge}$ in 1.6.1.3 we have

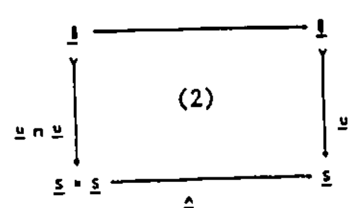


commuting, thus $\underline{\wedge} = (\underline{S} \times \underline{S}, \underline{S}, \dot{\wedge})$ is a morphism of $\mathcal{E}/C_0$. By [J1] Proposition 5.6 $\underline{\Omega} = (S, \alpha_1, \psi)$ is the codomain of the subobject classifier in $\mathcal{E}^{\underline{C}^{o}}$, hence by 1.6.1.17, 1.6.1.16, 1.6.1.15, 1.5.1.6 (31) and (32), $\underline{\wedge}$ is the underlying morphism of a morphism $\underline{\wedge}$: $\underline{\Omega}^2 \longrightarrow \underline{\Omega}$ in $\mathcal{E}^{\underline{C}^{o}}$.

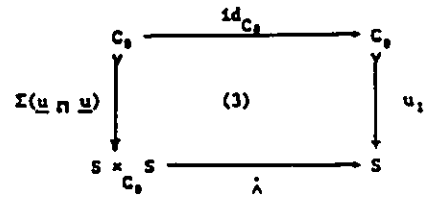
<u>1.6.1.19 A condition equivalent to $\underline{\wedge}$ being the meet operation on $\underline{\Omega}$ in $\mathcal{E}^{\underline{C}^{o}}$.</u> The morphism $\underline{\wedge}$ is the meet operation on $\underline{\Omega}$ iff



is a pullback in $\mathcal{E}^{\underline{C}^{o}}$. Since U: $\mathcal{E}^{\underline{C}^{o}} \longrightarrow \mathcal{E}/C_0$ is monadic ([J1] pg. 50 Proposition 2.2.1), this is equivalent to (2) being a pullback in $\mathcal{E}/C_0$



where $\underline{u} = (\underline{1},\underline{S},u_1)$ and u_1 is as defined in 1.5.4.5.4. In $\underline{\mathcal{E}}$ this means that (3)



is a pullback in $\underline{\mathcal{E}}$.

<u>1.6.1.20 Proposition.</u> There is a morphism $u_2 \colon C_0 \rightarrowtail S \times_{C_0} S$ for which the following hold:

(1) $\mu u_2 a = \lceil a, \ell a, \ell a \rfloor$,

(2) $\alpha_1 p_1' u_2 a = \alpha_1 p_2' u_2 a = a$,

(3) $\alpha_2 p_1' u_2 a = \alpha_2 p_2' u_2 a = \ell a$.

<u>Proof</u>. Let $P \colon C_0 \times R \times R \longrightarrow \Omega$ the classifying morphism for μ (see 1.6.1.2 (1)) so that

$$P\lceil a, r_1, r_2 \rfloor \leftrightarrow (r_1 \leq \ell a) \wedge (r_2 \leq \ell a)$$

then $P(a, \ell a, \ell a)$ so $P \bullet |\lceil a, \ell a, \ell a \rfloor| = true_{C_0}$ thus there exists a morphism u_2 such that

$$\mu \bullet u_2 = |\lceil a, \ell a, \ell a \rfloor| \quad \text{by } 0.6.2.3,$$

therefore (1) holds. Combining (1) with 1.6.1.2 (1) and (2), (2) and (3) (above) hold.□

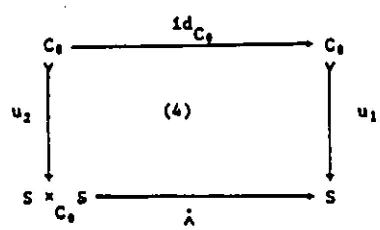
<u>1.6.1.21 Proposition.</u> $u_2 = \Sigma(\underline{u} \sqcap \underline{u})$.

<u>Proof.</u> From 1.6.1.20 (2), there is a map $\underline{u}_2 \colon \underline{S} \times \underline{S} \longrightarrow \underline{S}$ such that $\Sigma\underline{u}_2 = u_2$. We have $\underline{u}_2 = \underline{u} \sqcap \underline{u}$ iff $\underline{\pi}_i \bullet \underline{u}_2 = \underline{u}$ for $\pi_i \colon \underline{S} \times \underline{S} \longrightarrow \underline{S}$ projections $(i = 1,2)$ iff $(\Sigma\underline{\pi}_i)\bullet(\Sigma\underline{u}_2) = \Sigma\underline{u}$ for $i = 1,2$ iff $p_i' \bullet u_2 = u_1$ for $i = 1,2$. But since $\alpha = \alpha_1 \sqcap \alpha_2$ is a mono, this last statement is equivalent to

$$\alpha_1 p_i' u_2 a = \alpha_1 u_1 a \quad \text{and} \quad \alpha_2 p_i' u_2 a = \alpha_2 u_1 a \quad \text{for } i = 1,2,$$

which follows from 1.6.1.20 (2) and (3).□

<u>1.6.1.22 Proposition.</u> The square (4) is a pullback diagram in $\underline{\mathcal{E}}$.



<u>Proof.</u> (4) is a pullback

iff $(\dot{\wedge}\theta = u_1 a) \leftrightarrow \exists! b((id_{C_0} b = a) \wedge (u_2 b = \theta))$

iff $(\dot{\wedge}\theta = u_1 a) \leftrightarrow (u_2 a = \theta)$

iff $(\alpha\dot{\wedge}\theta = \alpha u_1 a) \leftrightarrow (\mu u_2 a = \mu\theta)$ (since α and μ are monos)

iff $(\alpha\dot{\wedge}\theta = \langle a, \ell a\rangle) \leftrightarrow (\mu\theta = \langle a, \ell a, \ell a\rangle)$

iff $(g_1\mu\theta = \langle a, \ell a\rangle) \leftrightarrow (\mu\theta = \langle a, \ell a, \ell a\rangle)$ by 1.6.1.3.

Now we prove the internal double implication.

$(\leftarrow)$: $(\mu\theta = \langle a, \ell a, \ell a\rangle) \rightarrow (g_1\mu\theta = \langle a, \ell a \wedge \ell a\rangle)$ (see 1.6.1.6.1)

$\rightarrow (g_1\mu\theta = \langle a, \ell a\rangle)$.

$(\rightarrow)$: We shall prove $(g_1\mu\theta = \langle a, \ell a\rangle) \rightarrow (\mu\theta = \langle a, \ell a, \ell a\rangle)$. By 1.6.1.2 (2) the implication holds iff

$(\langle\alpha_1p_1'\theta, \alpha_2p_1'\theta \wedge \alpha_2p_2'\theta\rangle = \langle a, \ell a\rangle)$

$\Rightarrow (\langle\alpha_1p_1'\theta, \alpha_2p_1'\theta, \alpha_2p_2'\theta\rangle = \langle a, \ell a, \ell a\rangle)$

iff $((\alpha_1p_1'\theta = a) \wedge ((\alpha_2p_1'\theta \wedge \alpha_2p_2'\theta) = \ell a))$

$\Rightarrow ((\alpha_2p_1'\theta = \ell a) \wedge (\alpha_2p_2'\theta = \ell a))$.

If we let $s \in V_S$, then $\alpha s = \langle\alpha_1 s, \alpha_2 s\rangle$, so by 1.5.4.7, $\alpha_2 s \leq \ell\alpha_1 s$

therefore $\alpha_2p_i'\theta \leq \ell\alpha_1p_i'\theta$ for $i = 1,2$, hence

$(\alpha_1p_i'\theta = a) \Rightarrow (\alpha_2p_i'\theta \leq \ell a)$ for $i = 1,2$.

But $\alpha_1p_1'\theta = \alpha_1p_2'\theta$, (see 1.6.1.1 (1)), hence

$(\alpha_1p_1'\theta = a) \Rightarrow ((\alpha_2p_1'\theta \leq \ell a) \wedge (\alpha_2p_2'\theta \leq \ell a))$.

But $((\alpha_2p_1'\theta \wedge \alpha_2p_2'\theta) = \ell a) \Rightarrow (\ell a \leq \alpha_2p_i'\theta)$ for $i = 1,2$.

Hence

$((\alpha_1p_1'\theta = a) \wedge ((\alpha_2p_1'\theta \wedge \alpha_2p_2'\theta) = \ell a)) \Rightarrow ((\ell a = \alpha_2p_1'\theta) \wedge (\ell a = \alpha_2p_2'\theta))$.□

<u>1.6.1.23 Proposition.</u> $\underline{\wedge}$ is the meet operation on $\underline{\Omega}$ in $\mathcal{E}^{C^{o}}$.

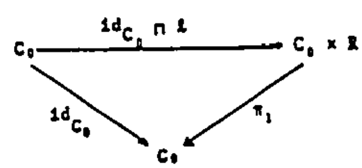
<u>Proof.</u> By 1.6.1.22, (4) is a pullback. By 1.6.1.21, $u_2 = \Sigma(\underline{u} \sqcap \underline{u})$.

Hence (3) in 1.6.1.19 is a pullback therefore (1) in 1.6.1.19 is a pullback.□

1.6.2 The equivalence of the proposition logic of $\underline{\mathcal{E}}^{\underline{C}^0}$ and the propositional logic associated with the internal Heyting algebra $\mathcal{R}$ of right ideals of $\underline{C}$ in $\underline{\mathcal{E}}$.

By 1.4.3.8 and 1.5.4.1, R , the object of right ideals of $\underline{C}$, has the structure of an internal Heyting algebra, $\mathcal{R}$. By 0.6.12.10, since $C_0^*: \underline{\mathcal{E}} \longrightarrow \underline{\mathcal{E}}/C_0$ is cartesian, $C_0^*(\mathcal{R})$ is a Heyting algebra in $\underline{\mathcal{E}}/C_0$. More generally, for any $t \in \mathrm{Poly(H)}$, if $\mathcal{R} \models t = 1$ then $C_0^*(\mathcal{R}) \models t = 1$. Even if C_0^* is not a faithful functor we can prove the converse. Moreover we can prove it from the weaker looking hypothesis that t, interpreted in $C_0^*(\mathcal{R})$, is bounded below by a certain constant $\underline{\ell}$ (which only coincides with the top element of $C_0^*(\mathcal{R})$ when $u_{C_0} : C_0 \longrightarrow \mathbb{I}$ is a mono).

We construct $\underline{\ell}$ from the morphism $\ell: C_0 \rightarrowtail R$ of 1.5.4.3. In $\underline{\mathcal{E}}$ we have the commutative triangle



Thus $\underline{\ell}' = ((C_0, id_{C_0}), (C_0 \times R, \pi_1), id_{C_0} \sqcap \ell)$ is a morphism from $\underline{\mathbb{I}}$ to $C_0^*(R)$ in $\underline{\mathcal{E}}/C_0$. We put $\underline{\ell} = \underline{\ell}'^*$. Let $\underline{r}$, a, x be variables of types $\underline{R} = C_0^*(R)$, C_0 and R respectively.

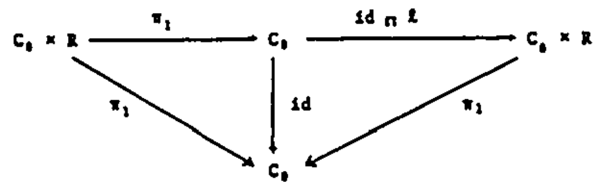
1.6.2.1 Proposition. $\Sigma|\underline{r} \wedge \underline{\ell}| = |/a, x \wedge \ell a/|$.

Proof. $|\underline{r} \wedge \underline{\ell}| = \underline{\wedge}\circ(id_{\underline{R}} \sqcap \underline{\ell}' \circ u_{\underline{R}})$ where $u_{\underline{R}} = \lambda \underline{r}.* : \underline{R} \longrightarrow \underline{\mathbb{I}}$.

Construction of $\underline{\ell}' \circ u_{\underline{R}}$:

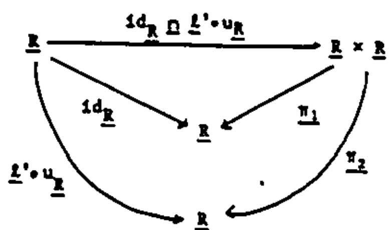
$$\underline{R} \xrightarrow{\ u_{\underline{R}}\ } \underline{\Omega} \xrightarrow{\ \underline{\ell}'\ } \underline{R} \qquad \text{in } \underline{\mathcal{E}}/C_0$$

has underlying morphisms in $\underline{\mathcal{E}}$

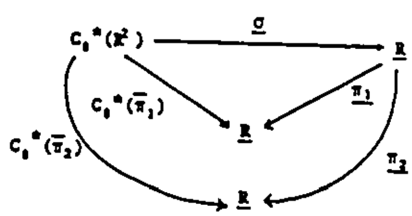


thus $\Sigma\underline{\ell}'\circ u_{\underline{R}} = (id_{C_0} \sqcap \ell)\circ\pi_1 = \lambda ax.(a,\ell a)$.

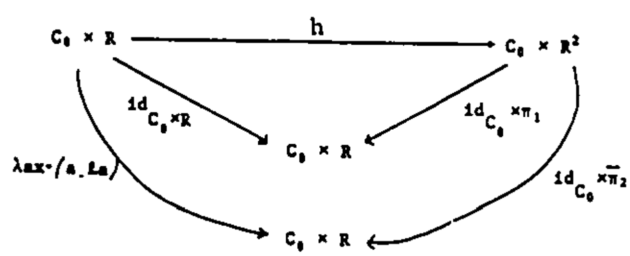
Construction of $id_{\underline{R}} \,\underline{\sqcap}\, \underline{\ell}'\circ u_{\underline{R}}$:



To take the underlying diagram we introduce the canonical isomorphism $\underline{\sigma}$:



We put $h = \Sigma(\underline{\sigma}^{-1}\circ(id_{\underline{R}} \,\underline{\sqcap}\, \underline{\ell}'\circ u_{\underline{R}}))$, then, in $\underline{\mathcal{E}}$, we have the underlying diagram.



Thus $(id_{C_0} \times \overline{\pi}_1)\, h = id_{C_0 \times R}$ and $(id_{C_0} \times \overline{\pi}_2)\circ h = \lambda ax.(a,\ell a)$.

Let $h = h_1 \sqcap (h_2 \sqcap h_3)$ so that $h(a,x) = (h_1(a,x),(h_2(a,x),h_3(a,x)))$,

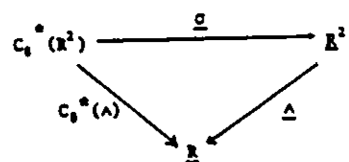
then the equations become

$(h_1(a,x),h_2(a,x)) = (a,x)$ and $(h_1(a,x),h_3(a,x)) = (a,\ell a)$

hence $h_1(a,x) = a$, $h_2(a,x) = x$ and $h_3(a,x) = \ell a$

so $h(a,x) = (a,(x,\ell a))$, hence $h = |(a,(x,\ell a))|$.

By definition of $\underline{\wedge}$ on $\underline{R}$ we have



commuting.

Thus $\Sigma|\underline{r} \wedge \underline{\ell}| = \Sigma(\underline{\wedge}\circ\underline{\sigma})\ \Sigma(\underline{\sigma}^{-1}\circ(id_{\underline{R}} \sqcap \underline{\ell}'\circ u_{\underline{R}}))$

$= (id_{C_0} \times \wedge) \circ |(a,(x,\ell a))| = |(a,x \wedge \ell a)|$.□

<u>1.6.2.2 Proposition.</u> Let $\overline{t}$ and $\overline{\overline{t}}$ be the interpretations in $\mathcal{R}$ and $C_0{}^*(\mathcal{R})$ respectively, of the H algebra polynomial t. The following are equivalent:

(1) $\underline{\ell} \leq \overline{\overline{t}}$

(2) $\overline{t} = \underline{1}$

(3) $\overline{\overline{t}} = \overline{\overline{1}}$.

<u>Proof.</u> Let $\vec{\underline{r}} = var(\overline{\overline{t}})$, $\vec{x} = var(\overline{t})$, $\tau_0(\underline{s}) = C_0{}^*(R)$, $\tau_0(y) = R$, $\tau_0(a) = C_0$, $n = \ell(\vec{x}) = \ell(\vec{\underline{r}})$, $\tau_0(u) = C_1$.

Let $\underline{\sigma}_n$: $(C_0{}^*(R))^n \longrightarrow C_0{}^*(R^n)$ be the canonical isomorphism of 0.4.4.2 for $n \geq 2$, $\underline{\sigma}_1 = id_{\underline{R}}$ and $\underline{\sigma}_0$: $\underline{\mathbb{1}} \longrightarrow C_0{}^*(\mathbb{1})$ the uniquely determined isomorphism, then

$$\sigma_n = \Sigma(\underline{\sigma}_n): \Sigma((C_0{}^*(R))^n) \longrightarrow C_0 \times R^n$$

is an isomorphism, $\Sigma(\underline{\sigma}_1) = id_{C_0 \times R}$, and $\Sigma(\underline{\sigma}_0) = \sigma_0: C_0 \longrightarrow C_0 \times 1$.

(1) → (2): Since $\underline{\ell} = \underline{\ell} \wedge \overline{t}$ we have

$\lambda\vec{\underline{r}}.\underline{\ell} = \lambda\vec{\underline{r}}.(\underline{\ell} \wedge \overline{t}) = (\lambda\underline{s}.(\underline{s} \wedge \underline{\ell}))\circ(\lambda\vec{\underline{r}}.\overline{t})$.

By 0.6.12.3, $\lambda\vec{\underline{r}}.\overline{t} = (C_0{}^*(\lambda\vec{x}.\overline{t}))\circ\underline{\sigma}_n$, and

$\lambda\vec{\underline{r}}.\underline{\ell} = \underline{\ell}'\circ(\lambda\vec{\underline{r}}.*) = \underline{\ell}'\circ\underline{\sigma}_0{}^{-1}\circ(C_0{}^*(\lambda\vec{x}.*))\circ\underline{\sigma}_n$ therefore

$\underline{\ell}'\circ\underline{\sigma}_0{}^{-1}\circ(C_0{}^*(\lambda\vec{x}.*)) = \lambda\underline{s}.(\underline{s} \wedge \underline{\ell})\circ(C_0{}^*(\lambda\vec{x}.\overline{t}))$.

We now apply the functor $\Sigma: \underline{\mathcal{E}}/C_0 \longrightarrow \underline{\mathcal{E}}$ to get

(4) $\Sigma(\underline{\ell}')\circ\Sigma(\underline{\sigma}_0{}^{-1})\circ\Sigma(C_0{}^*(\lambda\vec{x}.*)) = \Sigma(\lambda s.(\underline{s} \wedge \underline{\ell}))\circ\Sigma(C_0{}^*(\lambda\vec{x}.\overline{t}))$, where

$\Sigma(\underline{\ell}') = \lambda a.\langle a,\ell a\rangle$, $\Sigma(\underline{\sigma}_0{}^{-1}) = \sigma_0{}^{-1}$, and $\Sigma(C_0{}^*(\lambda\vec{x}.*)) = id_{C_0} \times(\lambda\vec{x}.*)$.

Let $f: \tau\, a\vec{x} \longrightarrow \tau(a) \times \tau(\vec{x})$ be the isomorphism

$$f = \lambda a\vec{x}.\langle a,\pi(\vec{x})\rangle$$

then $\sigma_0{}^{-1}\circ(id_{C_0} \times (\lambda\vec{x}.*))\circ f = \lambda a\vec{x}.a$, thus the left side of (4) composed with f is: $(\lambda a.\langle a,\ell a\rangle)\circ(\lambda a\vec{x}.a) = \lambda a\vec{x}.\langle a,\ell a\rangle$.

$\Sigma(C_0{}^*(\lambda\vec{x}.\overline{t}))\circ f = (id_{C_0} \times \lambda x.\overline{t}) \circ (\lambda a\vec{x}.\langle a,\pi(\vec{x})\rangle) = \lambda a\vec{x}.\langle a,\overline{t}\rangle$

and $\Sigma(\lambda\underline{s}.\langle\underline{s} \wedge \underline{\ell}\rangle) = \lambda ay.\langle a,y \wedge \ell a\rangle$ by 1.6.2.1, thus the right side of (4) composed with f is:

$(\lambda ay.\langle a,y \wedge \ell a\rangle)\circ(\lambda a\vec{x}.\langle a,\overline{t}\rangle) = \lambda a\vec{x}.\langle a,\overline{t} \wedge \ell a\rangle$.

Thus from (4) we derive

$$\lambda a\vec{x}.\langle a,\ell a\rangle = \lambda a\vec{x}.\langle a,t \wedge \ell a\rangle$$

therefore $\ell a = \overline{t} \wedge \ell a$ and we conclude that $\ell a \leq \overline{t}$.

Applying the order embedding $\tilde{m}: R \rightarrowtail \Omega^{C_1}$ we have: $\tilde{m}\ell a \subset \tilde{m}\overline{t}$

therefore $[\iota a,\rightarrow) \subset \tilde{m}\overline{t}$ by 1.5.4.3

therefore $\mathit{i}a \in \overline{\tilde{m}t}$

therefore $\mathit{i}d_1u \in \overline{\tilde{m}t}$

therefore $u \in \overline{\tilde{m}t}$ since $\overline{\tilde{m}t}$ is an ideal (1.5.4)

therefore $\overline{\tilde{m}t} = \{u:\top\} = \overline{\tilde{m}\underline{1}}$

therefore $\overline{t} = \overline{\underline{1}}$.□

(2) (3): Since C_0^* is cartesian.□

(3) (1): Since $\underline{\ell} \leq \overline{\underline{1}}$.□

In 1.6.2.3 we shall show that the monomorphism $\underline{\alpha}: \underline{S} \rightarrowtail \underline{R}$, defined in 1.6.1.1, is classified by $\lambda\underline{r}:(\underline{r} \leq \underline{\ell})$. It will follow (1.6.2.4) that for the Heyting algebra structure $\underline{\mathcal{S}}$ induced by $\underline{\alpha}$ and $\underline{R}$ we have $\underline{\mathcal{S}} \models t = \underline{1}$ iff $\underline{\ell} \leq \overline{t}$.

<u>1.6.2.3 Proposition.</u> $\underline{\alpha}: \underline{S} \longrightarrow \underline{R}$ is classified by $|\underline{r} \leq \underline{\ell}|$.

<u>Proof.</u> $\alpha: S \rightarrowtail C_0 \times R$ is classified by $\lambda ar.(r \leq \ell a) = \lambda ar.(\langle a,r\rangle = \langle a,r \wedge \ell a\rangle)$, hence by 1.5.4.7, α is the equalizer of $id_{C_0 \times R}$ and $|\langle a,r \wedge \ell a\rangle|$. Since $\Sigma\underline{\alpha} = \alpha$, $\Sigma id_{\underline{R}} = id_{C_0 \times R}$ and $\Sigma|\underline{r} \wedge \underline{\ell}| = |\langle a,r \wedge \ell a\rangle|$, $\underline{\alpha}$ is the equalizer of $id_{\underline{R}}$ and $|\underline{r} \wedge \underline{\ell}|$, hence, by 0.6.11.12, $\underline{\alpha}$ is classified by $|\underline{r} = \underline{r} \wedge \underline{\ell}| = |\underline{r} \leq \underline{\ell}|$.□

<u>1.6.2.4 Proposition.</u> Let $t \in \mathrm{Poly}(H)$, $\underline{\mathcal{R}} = C_0^*(\mathcal{R})$ the Heyting algebra object in $\underline{\mathcal{E}}/C_0$ which is the image under $C_0^*: \underline{\mathcal{E}} \longrightarrow \underline{\mathcal{E}}/C_0$ of the object of right ideals of $\underline{C}$, $\overline{t}$ the interpretation of t in $\underline{\mathcal{R}}$, $\tilde{t}$ the interpretation of t in $\underline{\mathcal{S}}$, then

$$\underline{\mathcal{R}} \models t = \underline{1} \quad \text{iff} \quad \underline{\mathcal{S}} \models t = \underline{1}.$$

Proof. By 1.4.2.4 (1), the embedding $\underline{S} \rightarrowtail^{\alpha} \underline{R}$ has a right adjoint $\underline{e}$ with respect to the order relation of $\underline{\mathcal{R}}$ and the induced order on $\underline{S}$, by (3) $\underline{e}$ is a Heyting algebra homomorphism, and by (4)

$$\underline{e}\ \underline{r} = \overline{\underline{1}} \Leftrightarrow \underline{\ell} \leq \underline{r} \qquad \text{therefore} \qquad \underline{e}\ \overline{\overline{t}} = \overline{\underline{1}} \text{ iff } \underline{\ell} \leq \overline{\overline{t}} .$$

Since $\underline{e}$ is a homomorphism, by 0.6.17.3 and 0.6.17.6,

$$\underline{e}\ \overline{\overline{t}} = S(\alpha_e)(\tilde{t}).$$

(←) From $\underline{\mathcal{S}} \models t = \underline{1}$ we have $\tilde{t} = \underline{\tilde{1}}$, hence $S(\alpha_{\underline{e}})(\tilde{t}) = S(\alpha_{\underline{e}})\underline{\tilde{1}}$, so $\underline{\ell}\ \overline{\overline{t}} = \underline{\tilde{1}}$ and this implies that $\underline{\ell} \leq \overline{\overline{t}}$. By 1.6.2.2, $\overline{\overline{t}} = \overline{\underline{1}}$, and therefore $\underline{\mathcal{R}} \models t = \underline{1}$.□

(→) Since $\underline{e}$ is a retraction it is an epi and a homomorphism therefore, by 0.6.17.11, $\overline{\overline{t}} = \overline{\underline{1}}$ implies $\tilde{t} = \underline{\tilde{1}}$.□

<u>1.6.2.5 Proposition.</u> For $t \in \mathrm{Poly}(H)$, if the formula $t = \underline{1}$ is valid on any one of the internal Heyting algebras: $\underline{\Omega}$, $\underline{\mathcal{S}}$, $C_0^{\,*}(\mathcal{R})$, $\mathcal{R}$ then it is valid on all of them.

Proof. Since the functor $U: \mathcal{E}^{C^0} \longrightarrow \underline{\mathcal{E}}/C_0$ is cartesian and faithful we have, by 0.6.12.10 and 0.6.12.11, for any $t \in \mathrm{Poly}(H)$,

$$\underline{\Omega} \models t = \underline{1} \text{ iff } U(\underline{\Omega}) \models t = \underline{1}.$$

It follows that $U(\underline{\Omega})$ is a Heyting algebra object in $\underline{\mathcal{E}}/C_0$ which, by our construction (1.6.1.18) has meet operation $\underline{\wedge}: \underline{S}^2 \longrightarrow \underline{S}$ for which, by 1.6.1.3, we have

$$\underline{\alpha}(\underline{\theta}_1 \wedge \underline{\theta}_2) = \underline{\alpha}(\underline{\theta}_1) \wedge \underline{\alpha}(\underline{\theta}_2).$$

But by 1.4.2.4 (2) the Heyting algebra structure $\underline{\mathcal{S}}$ has a meet operation $\underline{\wedge}''$ for which we also have

$$\underline{\alpha}(\underline{\theta}_1 \wedge'' \underline{\theta}_2) = \underline{\alpha}(\underline{\theta}_1) \wedge \underline{\alpha}(\underline{\theta}_2).$$

Since $\underline{\alpha}$ is a monomorphism we must have $\underline{\wedge}'' = \underline{\wedge}$. Now, by 0.6.7.16, since the Heyting algebra structures $\underline{\mathcal{S}}$ and $U(\underset{\sim}{\Omega})$ coincide for the meet operation we must have

$$\underline{\mathcal{S}} = U(\underset{\sim}{\Omega}).$$

Thus $\underset{\sim}{\Omega} \models t = 1$ iff $\underline{\mathcal{S}} \models t = 1$ iff $C_0^*(\mathcal{R}) \models t = 1$ iff $\mathcal{R} \models t = 1$. □

<u>1.6.2.6.</u> We can now show that $\underline{\mathcal{E}}/C_0$ inherits the propositional logic of $\underline{\mathcal{E}}^{C_0}$. Note that $\underline{\Omega} = C_0^*(\Omega)$ is the codomain of the subobject classifier of $\underline{\mathcal{E}}/C_0$. The morphism $\underline{\alpha}$: $\underline{S} \rightarrowtail \underline{R}$ has, with respect to the partial orders of $\underline{\mathcal{S}}$ and $\underline{\mathcal{R}}$, the right adjoint $\underline{e}$ by 1.4.2.4.(1) (as in the proof of 1.6.2.4). We define

$$\underline{g}\colon \underline{\Omega} \longrightarrow \underline{S}$$

by $$\underline{g} = \underline{e} \circ C_0^*(D).$$

We shall show that $\underline{g}$ has a retraction $\underline{f}$.

<u>1.6.2.7 Definition of the morphism $\underline{f}$.</u>

We define $f'\colon C_0 \times R \longrightarrow C_0 \times \Omega$ by

$$f'/a,r/ = /a, ia \in \tilde{m}r/$$

where $\tilde{m}\colon R \rightarrowtail \Omega^{C_1}$ is the embedding of 1.5.4.1. Then we have a morphism $\underline{f}'\colon \underline{R} \longrightarrow \underline{\Omega}$ for which $f' = \Sigma(\underline{f}')$. We define $\underline{f}\colon \underline{S} \longrightarrow \underline{\Omega}$ to be the composite

$$\underline{f} = \underline{f}' \circ \underline{\alpha}\ .$$

<u>1.6.2.8 Proposition.</u> $\underline{f}$ is a retraction of $\underline{g}$.

<u>Proof.</u> $\underline{f} \circ \underline{g} = id_{\underline{\Omega}}$

iff $\underline{f}' \circ \underline{\alpha} \circ \underline{e} \circ C_0^*(D) = id_{\underline{\Omega}}$

iff $f' \circ |/a, r \wedge \ell a/| \circ (id_{C_0} \times D) = id_{C_0} \times \Omega$

iff $f'/a, Dp \wedge \ell a/ = /a,p/$

iff $\quad [\![a, \mathit{ia} \in \tilde{m}(Dp \wedge \ell a)]\!] = [\![a,p]\!]$.

$\tilde{m}[\![Dp \wedge \ell a]\!] = \tilde{m}Dp \cap \tilde{m}\ell a = \{x:p\} \cap [\mathit{ia},\rightarrow)$ hence

$(\mathit{ia} \in \tilde{m}(Dp \wedge \ell a)) \leftrightarrow ((\mathit{ia} \in \{x:p\}) \wedge (\mathit{ia} \in [\mathit{ia},\rightarrow)))$

$\leftrightarrow p$. □

<u>1.6.2.9 Proposition.</u> Let $t \in \mathrm{Poly}(\mathbb{H})$.

(1) If $\underline{\underline{\Omega}} \models t = \bot$ then $\underset{=}{\Omega} \models t = \bot$,

(2) If C_0 is inhabited and $\underline{\underline{\Omega}} \models t = \bot$ then $\underset{\sim}{\Omega} \models t = \bot$, where $\underline{\underline{\Omega}}$, $\underset{=}{\Omega}$ and $\underset{\sim}{\Omega}$ are the natural $\mathbb{H}$-algebra structures on the subobject classifiers of $\underline{\mathcal{E}}^{C_0}$, $\underline{\mathcal{E}}/C$ and $\underline{\mathcal{E}}$ respectively.

<u>Proof.</u> (1): By 1.4.3.11, D is an $\mathbb{H}$-homomorphism from $\underset{\sim}{\Omega}$ to $\mathcal{R}$ in $\underline{\mathcal{E}}$; and since C_0^* is cartesian, by 0.6.17.12, $C_0^*(D)$ is an $\mathbb{H}$-homomorphism from $\underset{=}{\Omega}$ to $\underline{\mathcal{R}}$ in $\underline{\mathcal{E}}/C_0$. By 1.4.2.4 (3), $\underline{e}$ is an $\mathbb{H}$-homomorphism from $\underline{\mathcal{R}}$ to $\underline{\mathcal{S}}$, so, by 0.6.17.13, the composite $\underline{g} = \underline{e} \circ C_0^*(D)$ is an $\mathbb{H}$-homomorphism from $\underset{=}{\Omega}$ to $\underline{\mathcal{S}}$. By 1.6.2.8, $\underline{g}$ is also a monomorphism. By 1.6.2.5, from $\underline{\underline{\Omega}} \models t = \bot$ we have $\underline{\mathcal{S}} \models t = \bot$, and by 0.6.17.7, we get $\underset{=}{\Omega} \models t = \bot$. □

(2): We first prove that if C_0 is inhabited then C_0^* is faithful. Suppose $id_{C_0} \times f = id_{C_0} \times g$ for $f: A \longrightarrow B$, $g: A \longrightarrow B$, then $[\![x, fa]\!] = [\![x, ga]\!]$ therefore $(x = x) \rightarrow (fa = ga)$, hence $\exists x(x = x) \rightarrow fa = ga$; since C_0 is inhabited $fa = ga$, hence $f = g$. Thus C_0^* is faithful. If $\underline{\underline{\Omega}} \models t = \bot$, then, by (1) above, $\underset{=}{\Omega} \models t = \bot$, hence, by 0.6.12.11, $\underset{\sim}{\Omega} \models t = \bot$.□

1.6.3 A characterization of $\underline{S}$ in $\mathcal{E}/C_0$ when $\underline{C}$ is a groupoid in $\mathcal{E}$.

We shall prove that $\underline{C}$ is a groupoid in $\mathcal{E}$ iff the embedding $\underline{g}: \underline{\Omega} \longrightarrow \underline{S}$ is an isomorphism iff $\underline{S}$ is isomorphic to $\underline{\Omega}(\underline{S} \approx \underline{\Omega})$. The last equivalence depends on a result of Higgs; it states that for the subobject classifier $1 \longrightarrow \Omega$ of a topos, every monomorphism $m: \Omega \rightarrowtail \Omega$ is an involution. A proof of this result appears in [J3].

1.6.3.1 Proposition. $\underline{S} \approx \underline{\Omega}$ iff $\underline{g} \cdot \underline{f} = id_{\underline{S}}$.

Proof. ($\leftarrow$): By 1.6.3.3, $\underline{f} \cdot \underline{g} = id_{\underline{\Omega}}$. Hence if $\underline{g} \cdot \underline{f} = id_{\underline{S}}$ then $\underline{S} \approx \underline{\Omega}$.□

($\rightarrow$): Suppose $\underline{S} \approx \underline{\Omega}$. Let $\underline{\sigma}: \underline{S} \longrightarrow \underline{\Omega}$ be an isomorphism. Since $\underline{g}: \underline{\Omega} \rightarrowtail \underline{S}$ is a mono, $\underline{\sigma} \cdot \underline{g}: \underline{\Omega} \rightarrowtail \underline{\Omega}$ is a mono

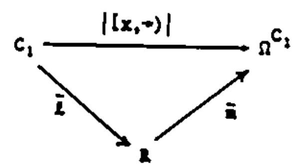
hence $\underline{\sigma} \cdot \underline{g} \cdot \underline{\sigma} \cdot \underline{g} = id_{\underline{\Omega}}$,

hence $\underline{g} \cdot (\underline{\sigma} \cdot \underline{g} \cdot \underline{\sigma}) = id_{\underline{S}}$,

hence $\underline{g}$ is an isomorphism with inverse

$$\underline{f} = \underline{\sigma} \cdot \underline{g} \cdot \underline{\sigma} \; . \; \square$$

1.6.3.2 Construction of $\tilde{\ell}: C_1 \longrightarrow R$. By 1.4.3.3.1 $[x,\rightarrow)$ is an ideal (i.e. $idl[x,\rightarrow)$), hence $|[x,\rightarrow)| : C_1 \longrightarrow \Omega^{C_1}$ factors through the monomorphism $\tilde{m}: R \rightarrowtail \Omega^{C_1}$ which is classified by $idl: \Omega^{C_1} \longrightarrow \Omega$. We let $\tilde{\ell}: C_1 \longrightarrow R$ be the morphism such that



commutes. In terms of $\Phi(\mathcal{E})$ we have $\tilde{m}\tilde{\ell}x = [x,\rightarrow)$.

We wish to prove that $\underline{g}\circ\underline{f} = id_{\underline{S}}$ iff $\underline{C}$ is a groupoid. Since $\underline{\alpha}$ is a monomorphism and $\underline{e}$ is an epimorphism, $\underline{g}\circ\underline{f} = id_{\underline{S}}$ is equivalent to $\underline{\alpha}\circ\underline{g}\circ\underline{f}\circ\underline{e} = \underline{\alpha}\circ\underline{e}$. We shall first calculate the morphism $\Sigma(\underline{\alpha}\circ\underline{g}\circ\underline{f}\circ\underline{e})$ in terms of $\Phi(\mathcal{E})$.

<u>1.6.3.3 Proposition.</u> $\Sigma(\underline{\alpha}\circ\underline{g}\circ\underline{f}\circ\underline{e}) = |/a, D(ia \in \tilde{m}r) \wedge \ell a/|$.

<u>Proof.</u> $\Sigma(\underline{\alpha}\circ\underline{g}\circ\underline{f}\circ\underline{e}) = (\Sigma(\underline{\alpha}\circ\underline{e}))\circ(\Sigma C_0{}^*(D))\circ(\Sigma\underline{f}')\circ(\Sigma(\underline{\alpha}\ \underline{e}))$

$$= |/a', r' \wedge \ell a'/|\circ(id_{C_0} \times D)\circ|f'/a, r \wedge \ell a/|$$

$$= |/a', r' \wedge \ell a'/|\circ|(id_{C_0} \times D)/a, ia \in \tilde{m}r/|$$

$$= |/a', r' \wedge \ell a'/|\circ|/a, D(ia \in \tilde{m}r)/|$$

$$= |/a, D(ia \in \tilde{m}r) \wedge \ell a/| .\square$$

<u>1.6.3.4 Proposition.</u> $\underline{S}$ is isomorphic to $\underline{\Omega}$ iff $\underline{C}$ is a groupoid.

<u>Proof.</u> $\underline{S} \approx \underline{\Omega}$

iff $\underline{\alpha}\circ\underline{g}\circ\underline{f}\circ\underline{e} = \underline{\alpha}\circ\underline{e}$, since $\underline{\alpha}$ is mono and $\underline{e}$ is epi,

iff $\Sigma(\underline{\alpha}\circ\underline{g}\circ\underline{f}\circ\underline{e}) = \Sigma(\underline{\alpha}\circ\underline{e})$, since Σ is faithful,

iff $D(ia \in \tilde{m}r) \wedge \ell a = r \wedge \ell a$, by 1.6.3.6,

iff $\tilde{m}(D(ia \in \tilde{m}r) \wedge \ell a) = \tilde{m}(r \wedge \ell a)$, since $\tilde{m}$ is mono,

iff $\Delta(ia \in \tilde{m}r) \cap [ia,\rightarrow) = \tilde{m}r \cap [ia,\rightarrow)$,

iff $((ia \in \tilde{m}r) \wedge (ia \prec x)) \leftrightarrow ((x \in \tilde{m}r) \wedge (ia \prec x))$.

Since $\tilde{m}: R \rightarrowtail \Omega^{C_1}$ is classified by idl: $\Omega^{C_1} \longrightarrow \Omega$,

we have

$(\iota a \in \tilde{m}r) \wedge (\iota a \prec x) \Rightarrow (x \in \tilde{m}r) \wedge (\iota a \prec x)$.

Thus we have: $\underline{S} \approx \underline{\Omega}$ iff

(1) $(\iota a \prec x) \wedge (x \in \tilde{m}r) \Rightarrow (\iota a \in \tilde{m}r)$.

We now prove that $\underline{S} \approx \underline{\Omega}$ implies $\underline{C}$ is a groupoid.

Substituting $\tilde{\ell}x$ for r and d_1x for a in (1) yields

$$((\iota d_1 x \prec x) \wedge (x \in [x,\rightarrow))) \Rightarrow (\iota d_1 x \in [x,\rightarrow)).$$

By 1.5.3.4, $\iota d_1 x \prec x$, hence

(3) $x \prec \iota d_1 x$.

Hence, by 1.5.3.7, $\underline{C}$ is a groupoid. □

Conversely, suppose $\underline{C}$ is a groupoid, so that (2) holds, then, by 1.5.3.4

$(\iota a \prec x) \wedge (x \in \tilde{m}r) \Rightarrow (a = d_1 x) \wedge (\iota d_1 x \in \tilde{m}r)$

$\Rightarrow (\iota a \in \tilde{m}r)$.□

1.6.4 Conditions on the propositional logic of $\underline{\mathcal{E}}$ and $\underline{\mathcal{E}}^{\underline{C}^0}$ that force $\underline{C}$ to be a groupoid.

We first prove the direct generalization to toposes. of the statement that, for $\underline{C}$ a small category, if $\mathcal{S}^{\underline{C}^0}$ is Boolean then $\underline{C}$ is a groupoid. Although it also follows from 1.6.4.11 we wish to present a proof at this stage to show that it is unnecessary to use the morphism N of 1.6.4.4 to prove this basic case of 1.6.4.11.

__1.6.4.1 Proposition.__ If $\underline{\mathcal{E}}^{\underline{C}^0} \models p \vee \neg p$ then $\underline{C}$ is a groupoid.

__Proof.__ Let $\tau_0(r) = R = Rid(\underline{C})$, $\tau_0(x) = C_1 = \tau_0(y)$.

Suppose $\underline{\mathcal{E}}^{\underline{C}^0} \models p \vee \neg p$. By 1.6.2.5, $r \vee \neg r = 1$, hence $\tilde{m}r \cup \tilde{m} \neg r = \tilde{m}1$.

By 1.4.3.8, $\tilde{m} \neg r = (\neg \tilde{m}r)^0$, hence $\tilde{m}r \cup (\neg \tilde{m}r)^0 = \{x:T\}$, therefore

$$(x \in \tilde{m}r) \vee ([x,\rightarrow) \cap \tilde{m}r = \phi)$$

Substitute $\tilde{\ell}y$ for r and id_1y for x and replace $\tilde{m}\ell y$ by $[y,\rightarrow)$ (see 1.6.3.2):

$$(id_1y \in [y,\rightarrow)) \vee ([id_1y,\rightarrow) \cap [y,\rightarrow) = \phi).$$

By 1.5.4.3, $id_1y \prec y$, hence

$$([id_1y,\rightarrow) \cup [y,\rightarrow) = \phi) \Rightarrow (y \in \phi)$$
$$\Rightarrow \bot$$

and therefore $y \prec id_1y$. By 1.5.3.7, $\underline{C}$ is a groupoid.□

To deal with the more complex conditions which also are sufficient for $\underline{C}$ to be a groupoid we introduce a new morphism

$$N: \Omega \longrightarrow R.$$

__1.6.4.2 Definition.__ $N_1: \Omega \longrightarrow \Omega^{C_1}$ is given by

$$N_1p = \{x: (x \prec id_1x) \Rightarrow p\}.$$

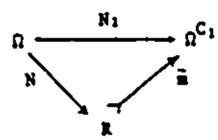
<u>1.6.4.3 Proposition.</u> $\mathrm{idl}(N_1 p)$.

<u>Proof.</u> $((x \prec y) \wedge (y \prec \iota d_1 y)) \Rightarrow (x \prec \iota d_1 x)$ by 1.5.3.4 and transitivity of $\prec$, hence

$$(x \prec y) \Rightarrow ((y \prec \iota d_1 y) \Rightarrow (x \prec \iota d_1 x))$$
$$\Rightarrow (((x \prec \iota d_1 x) \Rightarrow p) \Rightarrow ((y \prec \iota d_1 y) \Rightarrow p))$$
$$\Rightarrow ((x \in N_1 p) \Rightarrow (y \in N_1 p))$$

hence $((x \prec y) \wedge (x \in N_1 p)) \Rightarrow (y \in N_1 p)$.□

<u>1.6.4.4 Definition.</u> By 1.6.4.3 N_1 factors through $\tilde{m}$, since $\tilde{m}$ is classified by idl. We let N be the factor;



hence $\tilde{m}Np = N_1 p$.

<u>1.6.4.5 Proposition.</u> $|\exists a(\iota a \in \tilde{m}r)| \dashv N \quad (\Omega_{\leq}, R_{\leq})$, where $R_{\leq}$ is the partial order induced on R by $\mathcal{R}$.

<u>Proof.</u> $(r \leq Np) \Leftrightarrow (\tilde{m}r \subset N_1 p)$

$$\Leftrightarrow \forall x((x \in \tilde{m}r) \Rightarrow ((x \prec \iota d_1 x) \Rightarrow p))$$
$$\Leftrightarrow \forall x(((x \in \tilde{m}r) \wedge (x \prec \iota d_1 x)) \Rightarrow p)$$
$$\Leftrightarrow ((\exists x((x \in \tilde{m}r) \wedge (x \prec \iota d_1 x))) \Rightarrow p).$$

This establishes that N does indeed have a left adjoint, namely $\lambda r.\varphi$, where

$$\varphi \equiv \exists x((x \in \tilde{m}r) \wedge (x \prec \iota d_1 x)).$$

We now show that $\varphi \Leftrightarrow \exists a(\iota a \in \tilde{m}r)$.

($\Rightarrow$): $((x \in \tilde{m}r) \wedge (x \prec \iota d_1 x)) \Rightarrow (\iota d_1 x \in \tilde{m}r)$ since idl $\tilde{m}r$,

$(\iota d_1 x \in \tilde{m}r) \Rightarrow \exists a(\iota a \in \tilde{m}r)$ by 0.6.10.2,

hence $((x \in \tilde{m}r) \wedge (x \prec \iota d_1 x)) \Rightarrow \exists a(\iota a \in \tilde{m}r)$, hence $\varphi \Rightarrow \exists a(\iota a \in \tilde{m}r)$ by 0.6.10.4.

($\Leftarrow$): $((x \in \tilde{m}r) \wedge (x \prec \iota d_1 x)) \Rightarrow \varphi$ by 0.6.10.2.

Substitute ιa for x

$$(\iota a \in \tilde{m}r) \wedge (\iota a \prec \iota d_1 \iota a) \Rightarrow \varphi.$$

Since $\iota a = \iota d_1 \iota a$, we have

$$\exists a(\iota a \in \tilde{m}r) \Rightarrow \varphi.\square$$

<u>1.6.4.6 Proposition.</u> $Dp \leq Np$.

<u>Proof.</u> $p \Rightarrow ((x \prec \iota d_1 x) \Rightarrow p)$

hence $(x \in \Delta p) \Rightarrow (x \in N_1 p)$

hence $\tilde{m}Dp \subset \tilde{m}Np$

hence $Dp \leq Np.\square$

<u>1.6.4.7 Proposition.</u> The following are equivalent:

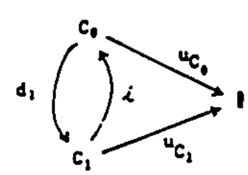
(1) C_0 is inhabited

(2) C_1 is inhabited

(3) D is an order embedding

(4) N is an order embedding.

<u>Proof.</u> (1) $\leftrightarrow$ (2): $u_{C_1} = u_{C_0} \circ d_1$ and $u_{C_0} = u_{C_1} \circ \iota$,



hence u_{C_1} is an epi iff u_{C_0} is an epi. The equivalence of (1) and (2) follows from the proof of 0.6.17.9.$\square$

(2) $\rightarrow$ (3): By 0.6.17.9, Δ is a mono. Since $\Delta = \tilde{m} \circ D$, D is a mono. By 1.4.1.2 and 1.4.3.10 D is an order embedding.$\square$

(3) $\rightarrow$ (2): Since D is a mono its left adjoint, given in 1.4.3.10, is, by 1.4.1.2, a retraction:

$$\exists_{C_1} \circ \tilde{m} \circ D = id_\Omega$$

hence $p \Leftrightarrow \exists_{C_1} \Delta p \Leftrightarrow \exists x(x \in \{y:p\}) \Leftrightarrow \exists x((x = x) \wedge p)$.

Substituting $\top$ for p we get

$$\exists x(x = x).\square$$

(1) $\rightarrow$ (4): By 1.4.1.2 and 1.6.4.5 is suffices to show that N is a mono.

$(Np = Nq) \Rightarrow (((x \prec id_1 x) \Rightarrow p) \Leftrightarrow ((x \prec id_1 x) \Rightarrow q)$.

Substitute ia for x and use the fact that $ia = id_1 ia$ to get

$$\exists a(a = a) \Rightarrow ((Np = Nq) \Rightarrow (p = q)),$$

hence $\quad (Np = Nq) \Rightarrow (p = q).\square$

(4) $\rightarrow$ (1). If N is a mono its left adjoint is a retraction thus

$$p \Leftrightarrow \exists a(ia \in \tilde{m}Np).$$

Since $\tilde{m}N\top = \{x: (x \prec id_1 x) \Rightarrow \top\} = \{x:\top\}$, we have, after substituting $\top$ for p:

$$\exists a(a = a).\square$$

Using N we can formulate a new condition equivalent to $\underline{C}$ being a groupoid.

1.6.4.8 Proposition. $\underline{C}$ is a groupoid iff $N_1 p \subset \Delta p$.

Proof. ($\rightarrow$): Since $\underline{C}$ is a groupoid, $x \prec id_1 x$, hence $((x \prec id_1 x) \Rightarrow p) \Rightarrow p$ hence $N_1 p \subset \Delta p$ $\square$

($\leftarrow$): $N_1 p \subset \Delta p$ is equivalent to $((x \prec id_1 x) \Rightarrow p) \Rightarrow p$, hence, substituting $(x \prec id_1 x)$ for p:

$$((x \prec id_1 x) \Rightarrow (x \prec id_1 x)) \Rightarrow (x \prec id_1 x)$$

hence $x \prec id_1 x$, so $\underline{C}$ is a groupoid.□

The above formulation will be the most convenient for us, but we note also that $N_1 p \subset \Delta p$ is equivalent to: $N_1 p = \Delta p$, $N_1 \equiv \Delta$, $Np \leq Dp$, $Np = Dp$, and $N \equiv D$.

Once we have established that, for a formula φ, we have the inclusion

$$N_1\varphi \subset \Delta\varphi$$

we can derive other inclusions:

<u>1.6.4.9 Proposition.</u> If $N_1\varphi \subset \Delta\varphi$ then

(1) $N_1(p \Rightarrow \varphi) \subset \Delta(p \Rightarrow \varphi)$, and

(2) $(((p \Rightarrow \varphi) \Rightarrow \varphi) \Rightarrow p) \Rightarrow (N_1 p \subset \Delta p)$.

<u>Proof</u>. (1) $(\Delta p) \cap (N_1(p \Rightarrow \varphi)) \subset (N_1 p \cap N(p \Rightarrow \varphi))$ by 1.6.4.6

$\subset N_1(p \wedge (p \Rightarrow \varphi))$ by 1.4.1.3 (2.)

$\subset N_1\varphi$ since N_1 is order preserving (by 1.4.1.2 (3.))

$\subset \Delta\varphi$ by 1.6.4.6,

hence $N_1(p \Rightarrow \varphi) \subset (\Delta p \Rightarrow \Delta\varphi)$

$\subset \Delta(p \Rightarrow \varphi)$ by 0.6.17.8. □

(2) Substitute $(p \Rightarrow \varphi)$ for p:

$$N_1((p \Rightarrow \varphi) \Rightarrow \varphi) \subset \Delta((p \Rightarrow \varphi) \Rightarrow \varphi)$$

then $(((p \Rightarrow \varphi) \Rightarrow \varphi) = q) \Rightarrow (N_1 q \subset \Delta q)$ where $q \notin s_F(p\varphi)$.

Since $p \Rightarrow ((p \Rightarrow \varphi) \Rightarrow \varphi)$, after substituting p for q, we get (2).□

<u>1.6.4.10 Corollary.</u> If $N_1\bot \subset \Delta\bot$ then

(1) $N_1(\neg p) \subset \Delta(\neg p)$, and

(2) $(\neg\neg p \Rightarrow p) \Rightarrow (N_1 p \subset \Delta p)$.

Proof. Let $\varphi = \top$ in 1.6.4.9.□

In the next proposition we shall let $t \in \text{Poly}(H)$, $R = \text{Rid}(\underset{\sim}{C})$, $\bar{t}$ and $\bar{\bar{t}}$ be the interpretations of t in $\mathcal{R}$ and $\underset{\sim}{\Omega}$ respectively, $r \in (V_R - s_F(\bar{t}))$, $s_F(\vec{q}) \subset s_F(\bar{\bar{t}})$, $a \in V_{C_0}$.

1.6.4.11 Proposition. If (1) $r \vee (r \Rightarrow \bar{t}) = \top$ then
(2) $N_1(\forall\vec{q}\ \bar{\bar{t}}) \subset \Delta(\forall\vec{q}\ \bar{\bar{t}})$.

Proof. Applying the order embedding $\tilde{m}: R \longrightarrow \Omega^{C_1}$ to (1) yields:

$$\tilde{m}r \cup (\tilde{m}r \Rightarrow \tilde{m}\bar{t})^0 = \{x:\top\}$$

hence (1') $(ia \in \tilde{m}r) \vee (([ia,\rightarrow) \cap \tilde{m}r) \subset \tilde{m}\bar{t})$.

Since $D: \underset{\sim}{\Omega} \longrightarrow \mathcal{R}$ is an H-algebra homomorphism we have

$$S(\alpha_D)(\bar{t}) = D\bar{\bar{t}}$$

by 0.6.17.3 and 0.6.17.6.

Making the substitution $S(\alpha_D)$ in (1)' yields

$$(ia \in \tilde{m}r) \vee (([ia,\rightarrow) \cap \tilde{m}r) \subset \Delta\bar{\bar{t}})$$

since $\tilde{m}D\bar{\bar{t}} = \Delta\bar{\bar{t}}$.

Let $\varphi = \forall\vec{q}\ \bar{\bar{t}}$ and substitute $N\varphi$ for r:

(3) $(ia \in \tilde{m}N\varphi) \vee (([ia,\rightarrow) \cap \tilde{m}N\varphi) \subset \Delta\bar{\bar{t}})$.

The first component of this formula reduces:

$(ia \in \tilde{m}N\varphi) \Leftrightarrow ((ia < id_1 ia) \Rightarrow \varphi)$

$\Leftrightarrow \varphi$

We will show (4) $\varphi \Rightarrow (([ia,\rightarrow) \cap N_1\varphi) \subset \Delta\bar{\bar{t}})$.

$\forall\vec{q}\bar{\bar{t}} \Rightarrow \bar{\bar{t}}$, i.e. $\varphi \Rightarrow \bar{\bar{t}}$, hence $\varphi \wedge (x \in ([ia,\rightarrow) \cap N_1\varphi)) \Rightarrow \bar{\bar{t}}$ so (4) holds.

Hence (3) reduces to

(5) $([ia,\rightarrow) \cap N_1\varphi) \subset \Delta\bar{\bar{t}}$.

Hence $(x \in N_1\varphi) \Rightarrow (x \in ([id_1 x, \vec{\ }) \cap N_1\varphi))$

$\Rightarrow x \in \Delta\bar{t}$ by (5)

$\Rightarrow \bar{t}$

$\Rightarrow \forall \vec{q}\,\bar{t}$

$\Rightarrow (x \in \Delta\varphi)$

therefore $N_1\varphi \subset \Delta\varphi$.□

<u>1.6.4.12</u> Before treating our main example we shall show how the Boolean case, 1.6.4.1, can be deduced from 1.6.4.11. Take t (in 1.6.4.11) to be a variable whose interpretations in R and $\underline{\Omega}$ are s and q respectively, then (1) becomes

$$r \vee (r \Rightarrow s) = \top$$

which is equivalent to

$$r \vee \neg r = \top .$$

Take $\vec{q}$ to be the empty string, then (2) reduces to $N_1 q \subset \Delta q$ which is equivalent to $\underline{C}$ being a groupoid, by 1.6.4.8.

<u>1.6.4.13.</u> We now begin the investigation of our main example of a non-Boolean propositional axiom: $\Delta^2(\underline{0}) \equiv p \vee (p \Rightarrow (q \vee \neg q))$. We first establish two tautologies and then a consequence of $\Delta^2(\underline{0})$.

<u>1.6.4.14 Proposition.</u> In a topos (1) and (2) hold.

(1) $(q = (q \vee \neg q)) \Leftrightarrow \neg\neg q$

(2) $(\neg\neg q \Rightarrow (q \vee \neg q)) \Rightarrow (\neg\neg q \Rightarrow q)$

(3) If $p \vee (p \Rightarrow (q \vee \neg q))$ holds then so does $\neg\neg q \vee (\neg\neg q \Rightarrow q)$.

<u>Proof.</u> (1) $(\Rightarrow)$:

$(q = (q \vee \neg q) \Rightarrow (\neg\neg q = \neg\neg(q \vee \neg q))$

$\Rightarrow (\neg\neg q = \top)$

$\Rightarrow \neg\neg q.$

(1) ($\Leftarrow$): $\neg\neg q \Rightarrow (\neg q = \bot)$

$\Rightarrow ((q \vee \neg q) = (q \vee \bot))$

$(q = (q \vee \neg q)).\square$

(2): $(\neg\neg q \Rightarrow (q \vee \neg q)) \Rightarrow (\neg\neg q \Rightarrow ((q = (q \vee \neg q)) \wedge (q \vee \neg q)))$

$\Rightarrow (\neg\neg q \Rightarrow q).\square$

(3) Substitute $\neg\neg q$ for p to get

$$\neg\neg q \vee (\neg\neg q \Rightarrow (q \vee \neg q)).$$

By (2) $\neg\neg q \vee (\neg\neg q \Rightarrow q).\square$

<u>1.6.4.15.</u> Although the condition $N_1\bot \subset \Delta\bot$ of 1.6.4.10 is not in general strong enough to imply $\underline{C}$ is a groupoid, in the Boolean case it is, since

$$(N_1\bot \subset \Delta\bot) \Leftrightarrow \forall x(\neg\neg(x < id_1 x)).$$

When $\mathcal{E}$ is not Boolean the condition appears to "almost" imply $\underline{C}$ is a groupoid in that it says "for every morphism x ($\neg\neg$ (x is a retraction))". In fact, in the presence of $\Delta^2(\underline{0})$, it does imply $\underline{C}$ is a groupoid.

<u>1.6.4.16 Proposition.</u> If $\mathcal{E}^{\underline{C}^0} \models p \vee (p \Rightarrow (q \vee \neg q))$ and $N_1(\bot) = \phi$ then $\underline{C}$ is a groupoid.

<u>Proof.</u> Taking $\vec{q} = \phi$ in (2) of 1.6.4.11 we have $N_1(p \vee \neg p) \subset \Delta(p \vee \neg p)$ hence $(p = (p \vee \neg p)) \Rightarrow (N_1 p \subset \Delta p)$, so by (1) of 1.6.4.14,

(a) $\neg\neg p \Rightarrow (N_1 p \subset \Delta p)$.

By hypothesis $N_1\bot \subset \Delta\bot$, hence, by (2) of 1.6.4.10,

(b) $(\neg\neg p \Rightarrow p) \Rightarrow (N_1 p \subset \Delta p)$.

Combining (a) and (b) we get

$$(\neg\neg p \vee (\neg\neg p \Rightarrow p)) \Rightarrow (N_1 p \subset \Delta p)$$

Hence, by (3) of 1.6.4.14, $N_1 p \subset \Delta p$, so by 1.6.4.8, $\underline{C}$ is a groupoid.□

The class of presheaf categories introduced in I of 1.2.3.4.1 -that included $\dot{\mathcal{D}}gph$ - satisfies $\Delta^2(\underline{0})$ as well as the "negative" condition: $\neg\forall p(p \vee \neg p)$. For toposes satisfying these axioms we have the following theorem.

<u>1.6.4.17 Theorem.</u> If $\mathcal{E}^{\underline{C}^0} \models p \vee (p \Rightarrow (q \vee \neg q)$ and $\mathcal{E} \models \neg\forall p(p \vee \neg p)$ then $\underline{C}$ is a groupoid.

<u>Proof.</u> From $\mathcal{E}^{\underline{C}^0} \models p \vee (p \Rightarrow (q \vee \neg q))$ we have by 1.6.2.5 $\mathcal{R} \models r \vee (r \Rightarrow (s \vee \neg s)) = \top$ where $\mathcal{R}$ is the Heyting algebra of right ideals of $\underline{C}$. Put $\vec{q} = q$, then by 1.6.4.11 (2)

$$N_1(\forall q(q \vee \neg q)) \subset \Delta(\forall q(q \vee \neg q)).$$

Since $(\forall p(p \vee \neg p)) = \bot$ we have

$$N_1(\bot) \subset \Delta(\bot).$$

Hence by 1.6.4.16, $\underline{C}$ is a groupoid.□

The second class of presheaf categories satisfying $\Delta^2(\underline{0})$ - that which included $\mathcal{D}gph$ - satisfied $\neg\neg\forall p(p \vee \neg p)$ as well as the rule

$$\frac{\forall p(p \vee \neg p) \vee \psi}{\psi} \; s_F(\psi) = \phi,$$

which is inconsistent with the logic of $\underline{\mathcal{S}}$.

As we can see from the example 1.2.5.7.6.1 we cannot prove a theorem of the generality of 1.6.4.17. We do however have a theorem that includes the case of $\underline{C}$ a monoid, as well as the case of $\underline{C}$ a finite external category. The latter can be represented as an internal category by taking C_0 and C_1 to be finite coproducts of $\mathbf{1}$. Our generalization of the two cases is an internal category $\underline{C}$ for which there exist a finite number of constants k_j $(1 \le j \le n)$ of type C_0 such that

$$\bigvee_{j=1}^{n} (a = k_j)$$

where a is a variable of type C_0. We first prove that if k is a constant of type C_0 (a "constant object") then, under the "logical conditions" (1), (2) and (3), $N_1\perp$ is "disjoint from the set of morphisms of $\underline{C}$ with codomain k".

<u>1.6.4.18 Proposition.</u> If for a topos $\underline{\mathcal{E}}$ and an internal category $\underline{C}$ we have

(1) $\underline{\mathcal{E}}^{\underline{C}^0} \models p \vee (p \Rightarrow (q \vee \neg q))$, and in the language of $\underline{\mathcal{E}}$,

(2) $\neg\neg \forall p(p \vee \neg p)$ and

(3) $\dfrac{\forall p(p \vee \neg p) \vee \psi}{\psi}$. $s_F(\psi) = \phi$

then

(4) $([ik,\rightarrow) \cap (N_1\perp)) = \phi$.

<u>Proof.</u> Put $e = \forall p(p \vee \neg p)$. By 1.6.2.5 and (1) (above), we have

$$(x \in \tilde{m}r) \vee (([x,\rightarrow) \cap \tilde{m}r) \subset \tilde{m}(s \vee \neg s))$$

where x is of type Ω^{C_1}, $\tilde{m}:R \rightarrowtail \Omega^{C_1}$ is the embedding of right ideals of $\underline{C}$ into Ω^{C_1}, and both r and s are of type R. Substitute De for r, $N\perp$ for s, and ik for x:

$$e \vee (([ik,\rightarrow) \cap \Delta e) \subset \tilde{m}(N\perp \vee \neg N\perp).$$

By (3), and since $\tilde{m}$ preserves ,

$$([ik,\rightarrow) \cap \Delta e) \subset (\tilde{m}N\perp \cup (\neg \tilde{m}N\perp)^0),$$

hence $e \Rightarrow ((ik \in N_1\perp) \vee (ik \in (\neg N_1\perp)^0)$

$\Rightarrow \neg (id_1 ik < ik) \vee ([ik,\rightarrow) \subset \neg N_1\perp)$

$\Rightarrow \perp \vee (([ik,\rightarrow) \cap (N_1\perp)) = \phi)$

$\Rightarrow \neg\, \exists x(x \in ([\iota k,\to) \cap N_1\bot))$,

hence $\neg\neg e \Rightarrow \neg\, \exists x(x \in ([\iota k,\to) \cap N_1\bot))$.

Now, by (2), $([\iota k,\to) \cap N_1\bot) = \phi$.□

<u>1.6.4.19 Theorem.</u> If for a topos $\underline{\mathcal{E}}$ and an internal category $\underline{C}$ we have

(1) $\underline{\mathcal{E}}^{\underline{C}^0} \models p \vee (p \Rightarrow (q \vee \neg q))$,

and in the language of $\underline{\mathcal{E}}$,

(2) $\bigvee_{j=1}^{n} (a = k_j)$, where $\tau_0(a) = C_0$,

(3) $\neg\neg\, \forall p(p \vee \neg p)$,

as well as the rule

(4) $\dfrac{\forall p(p \vee \neg p) \vee \psi}{\psi}$ $s_F(\psi) = \phi$,

then $\underline{C}$ is a groupoid.

<u>Proof.</u> Let $\tau_0(x) = C_1$. From (2) we have

(5) $\bigvee_{j=1}^{n} ([\iota d_1 x,\to) = [\iota k_j,\to))$.

By 1.6.4.18, for each j,

$$([\iota k_j,\to) \cap N_1\bot) = \phi$$

hence, for each j,

$$([\iota d_1 x,\to) = \iota k_j,\to)) \Rightarrow ([\iota d_1 x,\to) \cap N_1\bot = \phi),$$

hence

$$\bigvee_{j=1}^{n} \Big(([\iota d_1 x,\to) = [\iota k_j,\to))\Big) \Rightarrow (([\iota d_1 x,\to) \cap N_1\bot) = \phi).$$

By (5), $([\iota d_1 x,\to) \cap N_1\bot) = \phi$,

hence $\neg\,(x \in N_1\bot)$, hence $N_1\bot = \phi$, therefore, by (1) and 1.6.4.16, $\underline{C}$ is a groupoid.□

Section 1.7 Internalizing the condition: $\varphi \in \Gamma(\mathcal{E})$

In this section we shall show that for each $\varphi \in \mathcal{C}^{\rightarrow}_{\dashv\vdash}$ and topos $\underline{\mathcal{E}}$ we can construct a formula $\varphi^{\#}$ of the language of $\underline{\mathcal{E}}$ such that

$$\varphi \in \Gamma(\underline{\mathcal{E}}) \text{ iff } \underline{\mathcal{E}} \models \varphi^{\#}.$$

In our endeavours to replace the subobject classifier of $\underline{\mathcal{E}}^{\underline{C}^0}$, in its capacity as the determining structure of the propositional logic of $\underline{\mathcal{E}}^{\underline{C}^0}$, by a structure which was, in terms of the language of $\underline{\mathcal{E}}$, more manageable, we introduced preorders and ideals of preorders. We are now going to reformulate the converse logic $\Gamma(\underline{\mathcal{E}})$ in terms of preorders and their ideals. This reformulation is the first stage in a series of "relativizations" of this converse logic to subclasses $\mathcal{D}_i$ ($i = -1,0,1,2,3,4$) of the class PreOrd of all preordered objects of $\underline{\mathcal{E}}$.

We shall call subclasses of the class PreOrd simply "classes", within this introduction. We identify the classes we work with for $i = -1,0,1$, in terms of structures we have already studied and for $i = 2,3,4$, by references which locate the new construction.

$\mathcal{D}_{-1}$ consists of all $\mathcal{A}$ for which there exists an internal category $\underline{C}$ such that $\mathcal{A} = (C_1,\prec)$ where $\prec$ is the preorder of divisibility of the object of morphisms C_1 of $\underline{C}$.

$\mathcal{D}_0$ = PreOrd

$\mathcal{D}_1$ is the class of partially ordered objects.

$\mathcal{D}_2$ is the class of all $\tilde{\mathcal{A}}$ where $\mathcal{A}$ varies over $\mathcal{D}_1$ and $\tilde{\mathcal{A}}$ is the "simplification" of $\mathcal{A}$ defined in 1.7.3.2.

$\mathcal{D}_3$ is the class of all $\hat{\mathcal{A}}$ where $\mathcal{A}$ varies over $\mathcal{D}_1$ and $\hat{\mathcal{A}}$ is the "bounded" partially ordered object defined in 1.7.4.1.

$\mathcal{D}_4$ is the class of all $\mathcal{D}(f)$, order embedded in $(\Omega_{\leq})^c \times\Omega$, and indexed by the set of all $f \in [\Omega,\Omega]$; $\mathcal{D}(f)$ is defined in 1.7.4.1 also.

To give the definitions of the converse logics $\Gamma_i(\underline{\mathcal{E}})$, relativized by the class $\mathcal{D}_i$, we introduce further classes:

Eqv is the class of all $\mathcal{A} \equiv (A,\prec)$ where $\prec$ is an equivalence relation; members of Eqv will simply be called "equivalences".

For each polynomial φ,

PreOrd(φ) is the class of all $\mathcal{A}$ such that $\underset{\sim}{\text{Idl}}\, \mathcal{A} \models \varphi = \underline{1}$.

Although membership of $\mathcal{A}$ in PreOrd(φ) is not determined by equations given over $|\mathcal{A}|$, nevertheless $\mathcal{A}$ is determined by an equation; thus PreOrd(φ) is at least analogous to a "variety generated by φ".

For each class $\mathcal{D}_i$, we define the corresponding converse logic $\Gamma_i(\underline{\mathcal{E}})$ by

$$\Gamma_i(\underline{\mathcal{E}}) \equiv \{\varphi \mid \mathcal{D}_i \cap \text{PreOrd}(\varphi) \subset \text{Eqv}\}.$$

We shall call a function from one class to another a <u>class map</u>. We shall introduce properties of class maps $\delta\colon \mathcal{D}_j \longrightarrow \mathcal{D}_i$ which will allow us to relate the converse logic of $\mathcal{D}_j$ to that of $\mathcal{D}_i$.

We say $\delta\colon \mathcal{D}_j \longrightarrow \mathcal{D}_i$ <u>reflects equivalences</u> if, for all $\mathcal{A}$, $\delta(\mathcal{A}) \in \mathcal{D}_i \cap \text{Eqv}$ implies $\mathcal{A} \in \mathcal{D}_j \cap \text{Eqv}$.

Let $\varphi \in$ Poly H. We say $\delta\colon \mathcal{D}_j \longrightarrow \mathcal{D}_i$ <u>preserves φ</u> if, for all $\mathcal{A}$, $\mathcal{A} \in \mathcal{D}_j \cap \text{PreOrd}(\varphi)$ implies $\delta(\mathcal{A}) \in \mathcal{D}_i \cap \text{PreOrd}(\varphi)$.

Let $\Sigma \subset$ Poly H. We say $\delta\colon \mathcal{D}_j \longrightarrow \mathcal{D}_i$ <u>preserves Σ</u> if δ preserves φ for all $\varphi \in \Sigma$.

We say $\delta\colon \mathcal{D}_j \longrightarrow \mathcal{D}_i$ <u>preserves polynomials</u> if δ preserves Poly H.

We call $\delta\colon \mathcal{D}_j \longrightarrow \mathcal{D}$, <u>$\varphi$-representative</u>, <u>$\Sigma$-representative</u>, or <u>representative</u>, if δ reflects equivalences and δ, respectively, preserves φ, preserves Σ

preserves polynomials.

By simply unravelling the definitions involved we can

<u>Claim</u>: If $\varphi \in \Gamma_i(\underline{\mathcal{E}})$ and if $\delta: \mathcal{D}_j \longrightarrow \mathcal{D}_i$ is φ-representative then $\varphi \in \Gamma_j(\underline{\mathcal{E}})$.

<u>Proof of claim</u>: To show $\varphi \in \Gamma_j(\underline{\mathcal{E}})$ we suppose $\mathcal{A} \in \mathcal{D}_j \cap \text{PreOrd}(\varphi)$. Since δ preserves φ, $\delta(\mathcal{A}) \in \mathcal{D}_i \cap \text{PreOrd}(\varphi)$. Since $\varphi \in \Gamma_i(\underline{\mathcal{E}})$, $\delta(\mathcal{A}) \in$ Eqv. Since δ reflects equivalences, $\mathcal{A} \in$ Eqv.□

<u>Corollaries to claim</u>: (a) If $\delta: \mathcal{D}_j \longrightarrow \mathcal{D}_i$ is Σ-representative then $\Gamma_i(\underline{\mathcal{E}}) \cap \Sigma \subset \Gamma_j(\underline{\mathcal{E}}) \cap \Sigma$.□

(b) If $\delta: \mathcal{D}_j \longrightarrow \mathcal{D}_i$ is representative then $\Gamma_i(\underline{\mathcal{E}}) \subset \Gamma_j(\underline{\mathcal{E}})$.□

(c) If $\mathcal{D}_j \subset \mathcal{D}_i$ then $\Gamma_i(\underline{\mathcal{E}}) \subset \Gamma_j(\underline{\mathcal{E}})$.□

We shall now analyse the condition "$\delta: \mathcal{D}_j \longrightarrow \mathcal{D}_i$ preserves polynomials", for classes of preordered <u>sets</u>. In $\underline{\mathcal{S}}$ we have

$\delta: \mathcal{D}_j \longrightarrow \mathcal{D}_i$ preserves polynomials iff

for all $\mathcal{A}$, $\{\varphi \mid \underset{\sim}{\text{Idl}}\, \mathcal{A} \models \varphi = \underline{1}\} \subset \{\varphi \mid \underset{\sim}{\text{Idl}}\, \delta(\mathcal{A}) \models \varphi = \underline{1}\}$ iff

for all $\mathcal{A}$, $\underset{\sim}{\text{Idl}}\, \delta(\mathcal{A}) \in \text{HSP}\, \{\underset{\sim}{\text{Idl}}\, \mathcal{A}\}$.

Products from {Idl $\mathcal{A}$} are just powers. Furthermore, even internally, we can show $(\underset{\sim}{\text{Idl}}\, \mathcal{A})^I \approx \underset{\sim}{\text{Idl}}(\mathcal{A} \times I)$, where, in its appearance is the right hand factor of the product, I is considered to have equality as its ordering (the "discrete" ordering). Thus, in $\underline{\mathcal{S}}$, $\delta: \mathcal{D}_j \longrightarrow \mathcal{D}_i$ preserves polynomials iff, for each $\mathcal{A}$, there exists, a set I, a mono m, and an epi e, such that both m and e are H-homomorphisms connecting $\mathcal{A}$ and $\delta(\mathcal{A})$ as in the diagram

$$\underset{\sim}{\text{Idl}}\, \delta(\mathcal{A}) \overset{e}{\twoheadleftarrow} \underset{\sim}{?} \overset{m}{\rightarrowtail} \underset{\sim}{\text{Idl}}(\mathcal{A} \times I).$$

By our work on internal algebras in 0.6, it is clear that we have just given at least a sufficient condition in a topos $\underline{\mathcal{E}}$ for $\delta: \mathcal{D}_j \longrightarrow \mathcal{D}_i$ to preserve polynomials. We shall restrict ourselves to attempting to define <u>either</u> a mono <u>or</u> an epi (in $\underline{\mathcal{E}}$):

(1) $\quad \underset{\sim}{\text{Idl}}\ \delta(\mathcal{A}) \overset{m}{\rightarrowtail} \underset{\sim}{\text{Idl}}\ (\mathcal{A} \times I)$

(2) $\quad \underset{\sim}{\text{Idl}}\ (\mathcal{A} \times I) \overset{e}{\twoheadrightarrow} \underset{\sim}{\text{Idl}}(\delta(\mathcal{A}))$.

All our constructions take place within the class of preorders and in all cases come equipped with maps of preorders thus our natural immediate goal will be to construct maps of preorders which induce (1) and (2).

In 1.7.1 we shall study relevant properties or preorder preserving maps $f: \mathcal{A} \longrightarrow \mathcal{B}$; we refer to them as "maps", unqualified , within this introduction. We extend the construction of $\text{Idl}_{\leq}\mathcal{A}$ from preordered objects $\mathcal{A}$, given in 1.4 to maps, to produce a map $f^*: \text{Idl}_{\leq}\mathcal{B} \longrightarrow \text{Idl}_{\leq}\mathcal{A}$ from a map $f: \mathcal{A} \longrightarrow \mathcal{B}$. We will show that if f is epi, respectively mono, then f* is mono, respectively epi; and that if f is an h-map then f* is an H-homomorphism;(in the case where $\mathcal{B}$ is partially ordered the converse of the last property also holds). In practice we will need to establish either that a given epi μ is an h-map, or that a given mono η is an h-map.

(1)' $\quad \mathcal{A} \times I \overset{\mu}{\twoheadrightarrow} \delta(\mathcal{A})$

(2)' $\quad \delta(\mathcal{A}) \overset{\eta}{\rightarrowtail} \mathcal{A} \times I$

The morphisms of (1)' and (2)' will give rise to (1) and (2); that is, we take $m = \mu^*$ and $e = \eta^*$.

What we have presented so far is a possible technique for reformulating $\Gamma(\underline{\mathcal{E}})$, or at least part of it. The representative class maps which are to be introduced we exhibit below in (3) and (4); the class map of (5) is $\mathcal{C}^{\rightarrow}_{\dashv\vdash}$ representative.

(3) $\mathcal{D}_{-1} \subset \mathcal{D}_0 \xrightarrow{\overline{(\ )}} \mathcal{D}_1$

(4) $\mathcal{D}_{-1} \xleftarrow[\widetilde{(\ )}]{} \mathcal{D}_0 \supset \mathcal{D}_1 \supset \mathcal{D}_4 \supset \mathcal{D}_3 \xleftarrow{\widehat{(s(\ ))}} \mathcal{D}_2$

(5) $\mathcal{D}_1 \dashrightarrow \mathcal{D}_2$

The existence of the representative class maps of (3) and (4) implies

(6) $\Gamma(\underline{\mathcal{E}}) = \Gamma_0(\underline{\mathcal{E}}) = \Gamma_1(\underline{\mathcal{E}}) \subset \Gamma_4(\underline{\mathcal{E}}) \subset \Gamma_3(\underline{\mathcal{E}}) \subset \Gamma_2(\underline{\mathcal{E}})$.

Combining (6) with the existance of (5), yields

$$\Gamma_2(\underline{\mathcal{E}}) \cap \mathcal{C}^{\rightarrow}_{\dashv\vdash} = \Gamma_1(\underline{\mathcal{E}}) \cap \mathcal{C}^{\rightarrow}_{\dashv\vdash}$$

and hence

(7) $\Gamma_4(\underline{\mathcal{E}}) \cap \mathcal{C}^{\rightarrow}_{\dashv\vdash} = \Gamma(\underline{\mathcal{E}}) \cap \mathcal{C}^{\rightarrow}_{\dashv\vdash}$.

We shall now discuss each of the class maps and the mediating h-maps, in the case of (3) and (4). The class map $\mathcal{D}_0 \longrightarrow \mathcal{D}_{-1}$ associates with each $\mathcal{A}$ the preorder of divisibility, $\mathcal{C}$, of the natural internal category associated with $\mathcal{A}$; this construction comes equipped with an h-map, e, which is an

$$\begin{array}{ccc} \mathcal{C} & \rightarrowtail & \mathcal{A} \times A \\ {\scriptstyle e}\downarrow & \swarrow {\scriptstyle \pi_1} & \\ \mathcal{A} & & \end{array}$$

epi; hence $\mathcal{D}_0 \longrightarrow \mathcal{D}_{-1}$ preserves polynomials. The reflection of equivalences is, as in all the other cases less problematic; in fact we begin each construction by requiring that it reflect equivalences. The class map $\mathcal{D}_0 \xrightarrow{\overline{(\ )}} \mathcal{D}_1$ associates with each $\mathcal{A}$ the quotient partial order $\bar{\mathcal{A}}$, and comes equipped with a quotient map

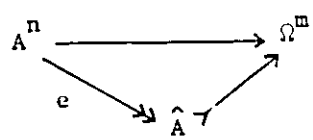
$$\mathcal{A} \overset{e}{\twoheadrightarrow} \bar{\mathcal{A}}$$

The next stages, $\mathcal{D}_2$, $\mathcal{D}_3$ and $\mathcal{D}_4$, arise out of an attempt to find a "bounded" class which maintains links with $\mathcal{D}_1$. By a bounded class we mean one all of whose members can be order embedded in some fixed preordered object - the "bounding" object. The first such class we encounter will be $\mathcal{D}_3$. The class $\mathcal{D}_4$ will share with $\mathcal{D}_3$ the same bounding object with the additional feature that its defining clause will contain no reference to a class of preordered objects, but only to the set of morphisms from Ω to Ω. The only non trivial "given" object (not 0 or 1) is the basic object Ω, thus we should expect that a bounding object will be built up from Ω using products and exponentiation.

The bounding object will be $(\Omega_{\leq})^{0} \times \Omega$.

There is a natural class of choices for a morphism $A^n \longrightarrow \Omega^m$. Any m-tuple of relations defined by the partial order and equality of the same arity, n, will correspond to an m-tuple of characteristic morphisms to Ω, and thus a single morphism $A^n \longrightarrow \Omega^m$.

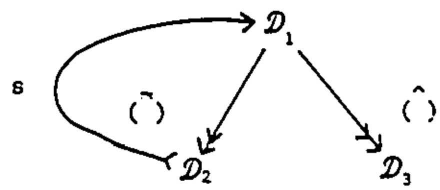
It is from among the images of such morphisms



that we have sought a bounded class. Our experiments with constructions of this kind have met with only partial success. For one natural example:

$A^2 \xrightarrow{|(x \leq y) \Rightarrow (x = y)|} \Omega$, the epi e is a map from $\mathcal{A} \times A$ to $\hat{\mathcal{A}}$ where the order of $\hat{\mathcal{A}}$ is induced by $\Omega_{\leq}$, and the class map reflects equivalences, however we are unable to verify that e is an h-map.

Only by simplifying the partial order used as the domain of the constructed morphism were we able to produce a morphism satisfying all the conditions required. But this meant that we had to cut down $\mathcal{D}_1$ to a subclass $\mathcal{D}_2$. For this "simplified" class and for the bounded class $\mathcal{D}_3$ we can establish the existence of a representative class map from $\mathcal{D}_2$ to $\mathcal{D}_3$. Both these classes arise as the image of class maps with domain $\mathcal{D}_1$:



Thus, to define a class map from $\mathcal{D}_2$ to $\mathcal{D}_3$ we take a section s of $\tilde{(\ )}$ and compose with $\hat{(\ )}$. The composite $\hat{(\ )}\circ s\colon \mathcal{D}_2 \longrightarrow \mathcal{D}_3$ preserves φ iff for all $\mathcal{A}$ in $\mathcal{D}_1$

$\tilde{\mathcal{A}} \in \mathcal{D}_2 \cap \text{PreOrd}(\varphi)$ implies

$\hat{\mathcal{A}} \in \mathcal{D}_3 \cap \text{PreOrd}(\varphi)$,

(since $\hat{\mathcal{A}} \cong (s(\tilde{\mathcal{A}}))^{\wedge}$). This is proven in 1.7.4 by producing an epi which is an h-map $k\colon \tilde{\mathcal{A}} \twoheadrightarrow \hat{\mathcal{A}}$.

Because $\mathcal{D}_2$ arose from an attempt to compensate for the apparent failure of a map $\mathcal{A} \times I \twoheadrightarrow \hat{\mathcal{A}}$ to be an h-map we should not expect that the gap between $\mathcal{D}_1$ and $\mathcal{D}_2$ can be bridged by a representative class map; nevertheless there are features of our analysis which can be used to establish that the class map $\tilde{(\ )}\colon \mathcal{D}_1 \longrightarrow \mathcal{D}_2$ is Σ-representative for a substantial subset $\Sigma \subset$ Poly H. The class map $\tilde{(\ )}\colon \mathcal{D}_1 \longrightarrow \mathcal{D}_2$ comes equipped with an order embedding

$$\tilde{\mathcal{A}} \overset{\alpha}{\rightarrowtail} \mathcal{A} \times A^2$$

for which we at least have a lattice homomorphism α^* which is an epi

$$\mathrm{Idl}_{\leq}(\mathcal{A}^2 \times A^2) \xrightarrow{\alpha^*} \mathrm{Idl}_{\leq}(\tilde{\mathcal{A}}).$$

The map α^* also has both adjoints (we have not been explicit about this in the sequel). Since α^* is an epi, its left adjoint σ is a section, that is

$$(*) \qquad \forall k(\alpha^*(\sigma(k)) = k).$$

Moreover σ preserves the binary join operation,a condition equivalent (when σ has a right adjoint α^*) to

$$(**) \quad \forall k \forall K(\alpha^*(\sigma k \Rightarrow K) = (k \Rightarrow \alpha^* K)).$$

Thus although α is <u>not</u> in general an h-map α^* preserves $\Rightarrow$ if the left argument of $\Rightarrow$ is suitable restricted (since $\alpha^*\sigma k = k$). When, in fact, we examine particular cases, we find that (*) and (**) can be used to establish that $\tilde{(\ )}: \mathcal{D}_1 \longrightarrow \mathcal{D}_2$ is $\{v \vee \neg v, R_2, \Delta^2(\underline{0})\}$-representative. If we attempt to generalize to arbitrary φ, the proofs for these three cases, we find that in the inductive clause involving $\Rightarrow$ we must restrict the first argument to variables in order to be successful . This will establish that the class map $\tilde{(\ )}: \mathcal{D}_1 \longrightarrow \mathcal{D}_2$ is $\mathcal{C}$-representative. A further argument - one not used for the three given cases -establishes that this class map is $\mathcal{C}^{\Rightarrow}$-representative.

The class $\mathcal{D}_3$, though bounded, still makes reference to the unbounded class $\mathcal{D}_1$ in its definition . We shall replace $\mathcal{D}_3$ by a (perhaps larger) class $\mathcal{D}_4$ consisting of partial orders $\mathcal{D}(f)$ order embedded in $(\Omega_{\leq})^0 \times \Omega$ and indexed by the set of all $f \in [\Omega,\Omega]$. We proceed to the internalization of $\varphi \in \Gamma_4(\mathcal{E})$ as follows. We will show that for each φ, the class $\mathcal{D}_4 \cap \mathrm{PreOrd}(\varphi)$ has a largest member $\mathcal{D}(f_\varphi)$. Thus for each $\mathcal{A}$ in $\mathcal{D}_4$,

$$\underset{\sim}{\mathrm{Idl}}\, \mathcal{A} \models \varphi = \underline{1} \ \text{ iff } \mathcal{A} \subset \mathcal{D}(f_\varphi).$$

Hence $\varphi \in \Gamma_4(\underline{\mathcal{E}})$ iff $\mathcal{D}(f_\varphi) \in$ Eqv.

A translation into the internal language of the statement that $\mathcal{D}(f_\varphi)$ is discrete leads us to an equivalence:

$$\varphi \in \Gamma_4(\underline{\mathcal{E}}) \text{ iff } \forall p((f_\varphi p) \Rightarrow p).$$

In the sense that the morphism $f_\varphi: \Omega \longrightarrow \Omega$ arises out of several constructions on Ω, the formula $f_\varphi p$ is excessively complicated. The goal of 1.7.6 is to convert $f_\varphi p$ to a purer form: one which uses only propositional variables and the standard logical operations. We could attempt to discover such a form immediately by trying to construct inductively a "purer" formula equivalent to $f_\varphi p$. That is: replace φ by $\underline{0}$, $\underline{1}$, and $v(v \in V)$ and find the simplest equivalents of $f_{\underline{0}}p$, $f_{\underline{1}}p$ and $f_v p$; then find a simple way of expressing $f_{\varphi \otimes \psi}p$ in terms of $f_\varphi p$ and $f_\varphi q$ where $\otimes$ is one of the infixes: $\wedge$, $\vee$ or $\Rightarrow$. This plan will eventually be carried through;however in its present form, "$f_\varphi p$", it is difficult to see how the above connections are to be made, particularly for the relation between $f_{\varphi \Rightarrow \psi}p$ and the two formulas $f_\varphi p$ and $f_\psi p$. Thus we shall prepare the way by re-expressing $f_\varphi p$ in terms of the more basic data from which it has been built up. Once this has been accomplished it is not difficult to "discover" the correct inductive clauses. We have presented the resulting inductive definition at the beginning of 1.7.6; at the end of the subsection, in 1.7.6.10 shall verify its equivalence with $f_\varphi p$.

The last subsection, 1.7.7,is a further fine-tuning of the internalization of $\varphi \in \Gamma(\underline{\mathcal{E}})$ for certain special cases, which include $v \vee \neg v$, R_2, $\Delta^2(\underline{0})$ and $\neg v \vee \neg\neg v$.

1.7.1 Heyting algebra homomorphisms between objects of ideals.

In this section we continue our development of preorders, and the object of ideals constructed from them, begun in 1.4.3.7. Let $\mathcal{A}$ and $\mathcal{B}$ be pre-ordered objects with carriers A and B, and let $f: A \longrightarrow B$. We shall investigate criteria under which f induces an H-homomorphism from $\underset{\sim}{\text{Idl}}\,\mathcal{B}$ to $\underset{\sim}{\text{Idl}}\,\mathcal{A}$, and apply these criteria to several examples; other examples will appear in future sections. In 1.7.1.18 - 1.7.1.22 we shall show that $\underset{\sim}{\text{Idl}}(\mathcal{A} \times B)$ and $\underset{\sim}{\text{Idl}}(\mathcal{A})^B$ are H-isomorphic, where B is considered as the pre-order having equality as its pre-ordering.

Throughout we will follow the conventions of 1.4.1.

1.7.1.1 Proposition. The following are equivalent.

(1) $\exists_f[a,\rightarrow) \subset [fa,\rightarrow)$

(2) $(a \prec a') \Rightarrow (fa \prec fa')$

(3) $\text{idl } U \Rightarrow \text{idl } f^{-1}U$

(4) There is a bounded lattice homomorphism f^* such that

$$\begin{array}{ccc} \text{Idl}\mathcal{B} & \xrightarrow{\quad f^* \quad} & \text{Idl}\mathcal{A} \\ {\scriptstyle m_2}\big\downarrow & & \big\downarrow{\scriptstyle m_1} \\ \Omega^B & \xrightarrow{\quad f^{-1} \quad} & \Omega^A \end{array}$$

commutes.

Proof. (1) $\leftrightarrow$ (2). By 1.4.1.7, $\exists_f \dashv f^{-1}$, hence

$(\exists_f[a,\rightarrow) \subset [fa,\rightarrow)) \Leftrightarrow ([a,\rightarrow) \subset f^{-1}[fa,\rightarrow))$

$\Leftrightarrow \forall a'((a \prec a') \Rightarrow (fa \prec fa')).\square$

(2) → (3). idl U ∧ (a ≺ a') ∧ (fa ∈ U)

⇒ idl U ∧ (fa ≺ fa') ∧ (fa ∈ U)

⇒ fa' ∈ U

⇒ a' ∈ $f^{-1}U$

hence idl U ⇒ ∀aa'[((a ≺ a') ∧ (a ∈ $f^{-1}U$)) ⇒ (a' ∈ $f^{-1}U$)].□

(3) → (4). We construct m_1 and m_2 as in 1.4.3.8. By (3) there is a factorization $m_1 \cdot f^* \equiv f^{-1} \cdot m_2$. By 1.4.1.7, f^{-1} has a left and right adjoint; by 1.4.3.8, m_2 has a left and right adjoint, hence both f^{-1} and m_2 preserve 0, 1, ∨ and ∨, by 1.4.1.3. Hence their composite and $m_1 \cdot f^*$ preserve 0, 1, ∨ and ∧. Since m_1 itself preserves 0, 1, ∨ and ∧ we have by 0.6.17.13 that

$m_1 f^* 0 = 0 = m_1 0$; $m_1 f^* 1 = 1 = m_1 1$;

$m_1 f^*(K_1 \vee K_2) = K_1 \vee K_2 = m_1 f^* K_1 \vee m_1 f^* K_2 = m_1(f^* K_1 \vee f^* K_2)$

$m_1 f^*(K_1 \vee K_2) = K_1 \vee K_2 = (m_1 f^* K_1 \vee m_1 f^* K_2) = m_1(f^* K_1 \vee f^* K_2)$.

Since m_1 is a monomorphism, f^* preserves 0, 1, ∨ and ∧ .□

(4) → (3) To prove (3) it suffices to show that idl $f^{-1}(m_2 K)$ (0.6.5.1, (4) → (2)). By (4) this is equivalent to idl $m_1(f^* K)$.□

(3) → (2) By 1.4.3.3.1, idl[fa,→), hence by (3), idl f^{-1}[fa,→). Thus (a ≺ a') ∧ (a ∈ f^{-1}[fa,→)) ⇒ (a' ∈ f^{-1}[fa,→)), hence (a ≺ a') ∧ (fa ≺ fa) ⇒ (fa ≺ fa'), hence (a ≺ a') ∧ (fa ≺ fa'). □

<u>1.7.1.2 Proposition.</u> If f is order preserving (i.e. satisfies (2) of 1.7.1.1) then

$$f^*(K_1 \Rightarrow K_2) \leq (f^* K_1 \Rightarrow f^* K_2).$$

<u>Proof.</u> $K_1 \wedge (K_1 \Rightarrow K_2) \leq K_2$, thus since f^* preserves ∧ and is itself order preserving $(f^* K_1 \wedge f^*(K_1 \Rightarrow K_2)) \leq f^* K_2$, hence $f^*(K_1 \Rightarrow K_2) \leq (f^* K_1 \Rightarrow f^* K_2)$. □

The pair $(A, \mathrm{Idl}\,\mathcal{A} \rightarrowtail \Omega^A)$ can be regarded as a topological space in $\underline{\mathcal{E}}$, with ideals being interpreted as open sets. From this point of view f order preserving is equivalent to f continuous: the inverse image of open sets are open; and f is open iff the image morphism $\exists_f$ takes open sets to open sets.

We call $\mathcal{A}$ discrete if $\mathrm{Idl}\,\mathcal{A} \overset{m}{\rightarrowtail} \Omega^A$ is an isomorphism.

<u>1.7.1.2.1 Proposition.</u> $\mathcal{A}$ is discrete iff $(x \prec y) \Rightarrow (x = y)$.

<u>Proof.</u> $(\rightarrow)$ We have $(x \in mk) \wedge (x \prec y) \Rightarrow (y \in mk)$.

Since $m: \mathrm{Idl} \rightarrowtail \Omega^A$ is an epi, $(x \in U) \wedge (x \prec y) \Rightarrow (y \in U)$, in particular $(x \in \{x\}) \wedge (x \prec y) \Rightarrow (y \in \{x\})$ hence $(x \prec y) \Rightarrow (y = x)$.□

$(\leftarrow)$ $m: \mathrm{Idl}\,\mathcal{A} \rightarrowtail \Omega^A$ is classified by $\mathrm{idl}: \Omega^A \longrightarrow \Omega$ where $\mathrm{idl}\ U \Leftrightarrow \forall xy(((x \in U) \wedge (x \prec y)) \Rightarrow (y \in U))$.

Since $(x \prec y) \Rightarrow (x = y)$ by hypothesis, and $x \prec x$, we have $(x \prec y) \Leftrightarrow (x = y)$, hence $\mathrm{idl}\ U \Leftrightarrow \forall xy(((x \in U) \wedge (x = y)) \Rightarrow (y \in U))$, hence $\mathrm{idl} \equiv \mathrm{true}_{\Omega^A}$, hence m is an iso.□

The next proposition characterizes open maps.

<u>1.7.1.3 Proposition.</u> The following are equivalent

(1) $[fa,\rightarrow) \subset \exists_f[a,\rightarrow)$

(2) $(fa < b) \Rightarrow \exists a'((a < a') \wedge (fa' = b))$

(3) $\mathrm{idl}\ W \Rightarrow \mathrm{idl}\ \exists_f\ W$

(4) There is a $\vee$ and 0 preserving morphism $f_!$ such that

$$\begin{array}{ccc} \mathrm{Idl}\,\mathcal{A} & \xrightarrow{\ f_!\ } & \mathrm{Idl}\,\mathcal{B} \\ {\scriptstyle m_1}\big\downarrow & & \big\downarrow{\scriptstyle m_2} \\ \Omega^A & \xrightarrow{\ \exists_f\ } & \Omega^B \end{array}$$

commutes.

Proof. (1) $\leftrightarrow$ (2). Clear.□

We compute an internal equivalence of idl $\exists_f W$:

$\mathrm{idl}\ \exists_f W \leftrightarrow \forall bb'[((b \prec b') \wedge (b \in \exists_f W)) \to (b' \in \exists_f W)]$

$\leftrightarrow \forall bb'[\exists a((a \in W) \wedge (fa = b)) \to ((b \prec b') \to (b' \in \exists_f W))$

$\leftrightarrow \forall b'a\forall b[(fa = b) \to (((a \in W) \wedge (b \prec b')) \to (b' \in \exists_f W))]$

$\leftrightarrow \forall b'a[((a \in W) \wedge (fa \prec b')) \to (b' \in \exists_f W)]$.

Hence $(\mathrm{idl}\ W \to \mathrm{idl}\ \exists_f W) \leftrightarrow \forall ba[[\mathrm{idl}\ W \wedge (a \in W) \wedge (fa \prec b)) \to (b \in \exists_f W)]$.

(2) $\to$ (3) We suppose idl $\overline{W}$, $\overline{a} \in \overline{W}$ and $f\overline{a} \prec \overline{b}$.

By (2), $\exists a'((\overline{a} \prec a') \wedge (fa' = b))$, also $(a \prec a') \to (a' \in W)$, since idl W and $a \in W$, hence $\exists a'((a'\ W) \wedge (fa' = b))$, hence $b \in \exists_f W$.□

(3) $\to$ (1) We have $a \in [a,\to)$ and idl $[a,\to)$. Thus from (3), idl $\exists_f[a,\to)$; also $fa \in \exists_f[a,\to)$. Hence $[fa,\to) \subset \exists_f[a,\to)$.□

(3) $\leftrightarrow$ (4). (3) is equivalent to the existence of a morphism $f!$ making the square of (4) commute. We show such a morphism must preserve 0 and v. $m_2 f!0 = \exists_f m_1 0 = \exists_f \phi = \phi = m_2 0$, hence $f!0 = 0$.

$m_2 f!(J \vee J') = \exists_f m_1(J \vee J') = \exists_f(m_1 J \cup m_1 J') = (\exists_f m_1 J \cup \exists_f m_1 J')$

$= (m_2 f! J \cup m_2 f! J') = m_2(f! J \vee f! J')$,

hence $f!(J \vee J') = f!J \vee f!J'$.□

1.7.1.3.1 Corollary. f is both continuous and open from $\mathcal{A}$ to $\mathcal{B}$ iff $\forall a([fa,\to) = \exists_f[a,\to))$.□

1.7.1.4 Proposition. If f is both continuous and open with respect to the ideal topologies then $f! \dashv f^*$.

Proof. Since $\exists_f \dashv f^{-1}$ we have $(\exists_f m_1 J \subset m_2 K) \leftrightarrow (m_1 J \subset f^{-1} m_2 K)$ hence $m_2 f! J \subset m_2 K \leftrightarrow (m_1 J \subset m_1 f^* K)$ since m_1 and m_2 are order embeddings $(f!J \leq K) \leftrightarrow (J \leq f^* K)$.□

The condition on an order preserving morphism f that f^* be an $\mathbb{H}$-homomorphism is given as follows.

1.7.1.5 Proposition. Let f be continuous. The following are equivalent.

(1) f^* is an $\mathbb{H}$-homomorphism

(2) $f^*K_1 \Rightarrow f^*K_2 \leq f^*(K_1 \Rightarrow K_2)$

(3) $(\uparrow\exists_f[a,\rightarrow)) \cap m_2K \subset \uparrow\exists_f([a,\rightarrow) \cap f^{-1}m_2K)$

(4) $(fa \prec b) \Rightarrow \exists a'((fa' \prec b) \wedge (b \prec fa') \wedge (a \prec a'))$.

Proof. (1) $\rightarrow$ (2) Clear.□ (2) $\rightarrow$ (1) follows from 1.7.1.1 (3) and 1.7.1.2.□

(2) $\leftrightarrow$ (3) We let $f_1 \dashv m_1$ and $f_2 \dashv m_2$, then $m_1f_1U = \uparrow U$ and $m_2f_2W = \uparrow W$.

We have (2.i) $\leftrightarrow$ 2.(i + 1)) for $1 \leq i \leq 5$, (2) $\leftrightarrow$ (2.1), and (2.5) $\leftrightarrow$ (3), for the reasons stated:

(2.1) $\quad m_1(f^*K_1 \Rightarrow f^*K_2) \subset m_1(f^*(K_1 \Rightarrow K_2))$,

since m_1 is an order embedding. We have $(a \in m_1J) \Leftrightarrow \{a\} \subset m_1J$

$\Leftrightarrow f_1\{a\} \leq J$.

(2.2) $((f_1\{a\} \wedge f^*K_1) \leq f^*K_2) \Rightarrow (f_1\{a\} \leq f^*(K_1 \Rightarrow K_2))$.

(2.3) $(([a,\rightarrow) \cap f^{-1}m_2K_1) \subset f^{-1}m_2K_1) \Rightarrow ([a,\rightarrow) \subset f^{-1}m_2(K_1 \Rightarrow K_2))$,

since m_1 is an order embedding and $f^{-1}\circ m_2 = m_1 \circ f^*$.

(2.4) $((f_2\exists_f([a,\rightarrow) \cap f^{-1}m_2K_1)) \leq K_2) \Rightarrow ((f_2\exists_f[a,\rightarrow)) \wedge K_1) \leq K_2$

since $\exists_f \dashv f^{-1}$ and $f_2 \dashv m_2$.

(2.5) $((f_2\exists_f[a,\rightarrow)) \wedge K) \leq f_2\exists_f([a,\rightarrow) \cap f^{-1}m_2K)$,

by making the appropriate substitution for K_2 and, for the converse, using the transitivity of $\leq$.

(3) $((\uparrow(\exists_f[a,\rightarrow))) \cap m_2K) \subset \uparrow(\exists_f([a,\rightarrow) \cap f^{-1}m_2K))$

since m_2 is an order embedding.□

(3) $\leftrightarrow$ (4) We first show (5) $(b \in \uparrow\exists_f U) \Leftrightarrow \exists a((fa \prec b) \wedge (a \in U))$.

$$(b \in \uparrow\exists_f U) \Leftrightarrow \exists b'((b' \prec b) \wedge (b' \in \exists_f U))$$

$$\Leftrightarrow \exists b'((b' \prec b) \wedge \exists a((b' = fa) \wedge (a \in U)))$$

$$\Leftrightarrow \exists a \exists b'((b' = fa) \wedge (fa \prec b) \wedge (a \in U))$$

$$\Leftrightarrow \exists a((\exists b'(b' = fa)) \wedge (fa \prec b) \wedge (a \in U))$$

$$\Leftrightarrow \exists a((fa \prec b) \wedge (a \in U)).$$

(3) $\rightarrow$ (4) Substitute $f_2\{b\}$ for K to get

(3.1) $(\uparrow\exists_f[a,\rightarrow)) \cap [b,\rightarrow) \subset \uparrow\exists_f([a,\rightarrow) \cap f^{-1}[b,\rightarrow))$, $a \in [a,\rightarrow)$, hence

$fa \in \exists_f[a,\rightarrow)$, hence $fa \prec b \Rightarrow b \in ((\uparrow\exists_f[a,\rightarrow)) \cap [b,\rightarrow))$

$$\Rightarrow b \in \uparrow\exists_f([a,\rightarrow) \cap f^{-1}[b,\rightarrow)) \quad \text{by (3.1)}$$

$$\Rightarrow \exists a'((fa' \prec b) \wedge (a' \in ([a,\rightarrow) \cap f^{-1}[b,\rightarrow)))) \text{ by (5)}$$

$$\Rightarrow \exists a'((fa' \prec b) \wedge (b \prec fa') \wedge (a \prec a')). \ \square$$

(4) $\rightarrow$ (3) Suppose $b \in ((\uparrow\exists_f[a,\rightarrow)) \cap m_2K)$, then

$(\exists a'((fa' \prec b) \wedge (a \prec a'))) \wedge (b \in m_2K)$ hence by (4)

(3.2) $\exists a'(\exists a''((fa'' \prec b) \wedge (b \prec fa'') \wedge (a \prec a''))) \wedge (a \prec a') \wedge (b \in m_2K))$

We want to show $b \in \uparrow\exists_f([a,\rightarrow) \cap f^{-1}m_2K)$ which by (5) is equivalent to

(3.3) $\exists a''((fa'' \prec b) \wedge (a \prec a'') \wedge (fa'' \in m_2K))$.

Since $(a \prec a') \wedge (a' \prec a'') \Rightarrow (a \prec a'')$ and

$(b \in m_2K) \wedge (b \prec fa'') \Rightarrow (fa'' \in m_2K)$, we have, from (3.2), (3.3).$\square$

<u>1.7.1.6 Corollary.</u> If f is continuous and open, then f^* is an H-homomorphism.

<u>Proof.</u> $(fa' = b) \Rightarrow ((fa' \prec b) \wedge (b \prec fa'))$. Since f is open 1.7.1.3 (1) holds, hence 1.7.1.5 (4) holds.$\square$

<u>1.7.1.7 The equivalence relation determined by $\prec$.</u>

Define $|a \sim a'| : A^2 \longrightarrow \Omega$ by

$$(a \sim a') \Leftrightarrow ((a \prec a') \wedge (a' \prec a)).$$

By the metatheory, $|a \sim a'|$ is an equivalence relation on A^2, i.e. it is a symmetric preorder.

Define $\theta: A \longrightarrow \Omega^A$ by $\theta a = \{b: b \sim a\}$.

<u>1.7.1.8 Properties connecting $\prec$, $\sim$ and θ.</u>

(1) $(a \prec b) \Rightarrow (\leftarrow,a] \subset (\leftarrow,b]$

(2) $(a \prec b) \Rightarrow [b,\rightarrow) \subset [a,\rightarrow)$

(3) $(a \in \theta a)$

(4) $(\theta a \subset \theta b) \Rightarrow (a \sim b)$

(5) $\theta a = (\leftarrow,a] \cap [a,\rightarrow)$

(6) $(a \sim b) \Rightarrow (\theta a = \theta b)$

(7) $(a \sim b) \Leftrightarrow (\theta a = \theta b)$

(8) $(c \in (\theta a \cap \theta b)) \Rightarrow (\theta a = \theta b)$.

<u>Proof.</u> (1) $(a \prec b) \wedge (x \prec a) \Rightarrow (x \prec b)$, hence

$(a \prec b) \Rightarrow \forall x((x \in (\leftarrow,a]) \Rightarrow (x \in (\leftarrow,b]))$.□

(2) Similar to (1).□ (3) Clear.□

(4) $(\theta a \subset \theta b) \Rightarrow ((a \in \theta a) \Rightarrow (a \in \theta b)) \Rightarrow (a \sim b)$.□

(5) $(x \in \theta a) \Leftrightarrow (x \sim a) \Leftrightarrow ((x \prec a) \wedge (a \prec x)) \Leftrightarrow (x \in ((\leftarrow,a] \cap [a,\rightarrow)))$.□

(6) $(a \sim b) \Rightarrow (a \prec b) \wedge (b \prec a)$

$\Rightarrow ((\leftarrow,a] \subset (\leftarrow,b]) \wedge ([a,\rightarrow) \subset [b,\rightarrow))$ by (2)

$\Rightarrow (((\leftarrow,a] \cap [a,\rightarrow)) \subset ((\leftarrow,b] \cap [b,\rightarrow))) \Rightarrow (\theta a \subset \theta b)$

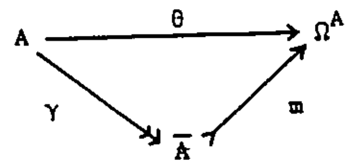
thus also $(a \sim b) \Rightarrow (b \sim a) \Rightarrow (\theta b \subset \theta a)$ hence $(a \sim b) \Rightarrow (\theta a = \theta b)$.□

(7) By (6) and (4).□

(8) $(c \in (\theta a \cap \theta b)) \Rightarrow ((c \sim a) \wedge (c \sim b)) \Rightarrow ((\theta c = \theta a) \wedge (\theta c = \theta b))$

$\Rightarrow (\theta a = \theta b)$.□

<u>1.7.1.9 The quotient of A by $\sim$.</u> We form an epi-mono factorization of θ and call γ the



quotient map for $\sim$.

We define a relation $|x \leq y| : \bar{A}^2 \longrightarrow \Omega$ by

$$(x \leq y) \Leftrightarrow \exists a \exists b((a \prec b) \wedge (\gamma a = x) \wedge (\gamma b = y)).$$

<u>1.7.1.10 Proposition.</u> The relation $|x \leq y|$ is a partial order.

<u>Proof.</u> Reflexivity:

$((a \prec b) \wedge (\gamma a = x) \wedge (\gamma b = x)) \Rightarrow \exists ab((a \prec b) \wedge (\gamma a = x) \wedge (\gamma b = x).$

Substituting a for b, using the reflexivity of $\prec$, and introducing $\exists a$, we get $\exists a(\gamma a = x) \Rightarrow (x \leq x)$.

Since γ is an epi, $x \leq x$.□

Transitivity: We want to show $(x \leq y) \wedge (y \leq z) \Rightarrow (x \leq z)$, i.e.

$[(a \prec b) \wedge (\gamma a = x) \wedge (\gamma b = y) \wedge (c \prec d) \wedge (\gamma c = y) \wedge (\gamma c = y) \wedge (\gamma d = z)]$

$\Rightarrow (x \leq z)$.

We suppose $\bar{a} \prec \bar{b}$, $\gamma\bar{a} = \bar{x}$, $\gamma\bar{b} = \bar{y}$, $\bar{c} \prec \bar{d}$, $\gamma\bar{c} = \bar{y}$ and $\gamma\bar{d} = \bar{z}$. Then $\gamma\bar{c} = \gamma\bar{b}$, hence $\theta\bar{c} = \theta\bar{b}$, hence by 1.7.1.8 (7) $\bar{c} \sim \bar{b}$, hence $\bar{b} \prec \bar{c}$, hence by the transitivity of $\prec$, $\bar{a} \prec \bar{d}$, thus $(\bar{a} \prec \bar{d}) \wedge (\gamma\bar{a} = \bar{x}) \wedge (\gamma\bar{d} = \bar{z})$, hence $\exists a \exists d((a \prec d) \wedge (\gamma a = \bar{x}) \wedge (\gamma d = \bar{z})$, hence $\bar{x} \leq \bar{z}$.□

Antisymmetry. We want to show $(x \leq y) \wedge (y \leq x) \Rightarrow (x = y)$.

We suppose $\bar{a} \prec \bar{b}$, $\gamma\bar{a} = \bar{x}$, $\gamma\bar{b} = \bar{y}$, $\bar{c} \prec \bar{d}$, $\gamma\bar{c} = \bar{y}$ and $\gamma\bar{d} = \bar{x}$ then $\gamma\bar{b} = \gamma\bar{c}$ and $\gamma\bar{a} = \gamma\bar{d}$, hence $\bar{b} \sim \bar{c}$ and $\bar{a} \sim \bar{d}$, hence $\bar{b} \prec \bar{c}$ and $\bar{d} \prec \bar{a}$. From $\bar{b} \prec \bar{c}$, $\bar{c} \prec \bar{d}$ and $\bar{d} \prec \bar{a}$ we deduce $\bar{b} \prec \bar{a}$. Hence $\bar{a} \sim \bar{b}$, hence $\bar{x} = \gamma\bar{a} = \gamma\bar{b} = \bar{y}$. Thus $(\bar{x} \leq \bar{y})$ and $(\bar{y} \leq \bar{x})$ imply $(\bar{x} = \bar{y})$.□

<u>1.7.1.10.1</u> We let $\bar{A}$ be the partially ordered object with carrier $\bar{A}$ and the above ordering, derived from the preordered object A.

<u>1.7.1.11 Proposition.</u> $(a \prec b) \Leftrightarrow (\gamma a \leq \gamma b)$.

<u>Proof.</u> ($\Rightarrow$) We have

$[(c \prec d) \wedge (\gamma c = \gamma a) \wedge (\gamma d = \gamma b) \Rightarrow \exists cd\ (c \prec d) \wedge (\gamma c = \gamma a) \wedge (\gamma d = \gamma b)]$

Substitute a for c and b for d to get $(a \prec b) \Rightarrow (\gamma a \leq \gamma b)$.□

($\Leftarrow$) $[(c \prec d) \wedge (\gamma c = \gamma a) \wedge (\gamma d = \gamma b)] \Rightarrow [(c \prec d) \wedge (c \sim a) \wedge (d \sim b)]$

$$\Rightarrow a \prec b.$$

Now introduce $\exists cd$ to get $(\gamma a \leq \gamma b) \Rightarrow (a \prec b)$.□

The condition on f in our next proposition is a generalization of the order embeddingbetween partially ordered objects as given in 1.4.2.2 and the property of γ given in 1.7.1.11. For $f: A \longrightarrow B$ we define

$$\text{Im } f = \{b: \exists a\ (fa = b)\}.$$

<u>1.7.1.12 Proposition.</u> Let $\mathcal{A}$ and $\mathcal{B}$ be preordered objects, $A \equiv |\mathcal{A}|$, $B \equiv |\mathcal{B}|$ and $f: A \longrightarrow B$, satisfying $(a \prec a') \Leftrightarrow (fa \prec fa')$.

The following are equivalent.

(1) idl(Im f)

(2) f is open.

<u>Proof.</u> (1) holds iff $((b \in \text{Im } f) \wedge (b \prec b')) \Rightarrow (b' \in \text{Im } f)$

iff $(fa = b) \wedge (b \prec b') \Rightarrow \exists a'(fa' = b')$

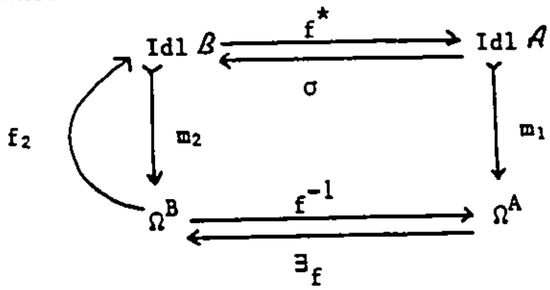
iff $\exists b(fa = b) \Rightarrow ((fa \prec b') \Rightarrow \exists a'(fa' = b'))$

iff $(fa \prec b') \Rightarrow \exists a'((fa' = b') \wedge (fa \prec fa'))$

iff $(fa \prec b') \Rightarrow \exists a'((fa' = b') \wedge (a \prec a'))$

iff (2).□

<u>1.7.1.12.1</u> From the order preserving morphism $f: \mathcal{A} \longrightarrow \mathcal{B}$ we construct f^* and σ where



(1) $\sigma \equiv f_2 \circ \exists_f \circ m_1$, and $m_2 \dashv f_2$. We have

(2) $m_2 \circ f_2 \equiv \uparrow$ where $\uparrow: \Omega^B \longrightarrow \Omega^B$ is the closure operation induced by $\mathcal{B}$, and

(3) $f^{-1} \circ m_2 \equiv m_1 \circ f^*$.

<u>1.7.1.12.2 Proposition.</u> If f satisfies $(a \prec a') \leftrightarrow (fa \prec fa')$ then

(1) $f^* \sigma J = J$ and

(2) $([fa,\rightarrow) \cap \exists_f m_1 J) = \exists_f([a,\rightarrow) \cap m_1 J)$.

<u>Proof.</u> (1) $m_1 f^* \sigma J = f^{-1} m_2 f_2 \exists_f m_1 J = f^{-1} \uparrow \exists_f m_1 J$.

We shall prove $f^{-1} \uparrow \exists_f m_1 J = m_1 J$.

$$(a \in f^{-1} \uparrow \exists_f m_1 J) \leftrightarrow (fa \in (\uparrow \exists_f m_1 J))$$
$$\leftrightarrow \exists b((b \prec fa) \wedge (\exists a'((fa' = b) \wedge (a' \in m_1 J)))$$
$$\leftrightarrow \exists a'((\exists b(fa' = b)) \wedge (fa' \prec fa) \wedge (a' \in m_1 J))$$
$$\leftrightarrow (\exists a'((a' \prec a) \wedge (a' \in m_1 J))) \leftrightarrow (a \in m_1 J).$$

Hence $m_1 f^* \sigma J = m_1 J$, hence $f^* \sigma J = J$.□

(2)
$$b \in ([fa,\rightarrow) \cap (\exists_f m_1 J)) \leftrightarrow ((fa \prec b) \wedge \exists a'((fa' = b) \wedge (a' \in m_1 J)))$$
$$\leftrightarrow \exists a'((fa \prec fa') \wedge (fa' = b) \wedge (a' \in m_1 J))$$
$$\leftrightarrow \exists a'((fa' = b) \wedge (a' \in ([a,\rightarrow) \cap m_1 J)))$$
$$\leftrightarrow (b \in \exists_f([a,\rightarrow) \cap m_1 J)).□$$

<u>1.7.1.13 Corollary.</u> Let $\mathcal{B}$ be a partially ordered object and let $P\colon B \longrightarrow \Omega$ satisfy $Pb \wedge (b \leq b') \Rightarrow Pb'$.

If $A \rightarrowtail^{f} B$ is classified by P and $\mathcal{A}$ is the partially ordered structure induced by f and $\mathcal{B}$ then f is continuous and open.

<u>Proof.</u> Since $(a \leq a') \Leftrightarrow (fa \leq fa')$, f is continuous.

$\mathrm{Im}\ f = \{b\colon \exists a(fa = b)\} = \{b\colon Pb\}$, hence

$$\mathrm{Idl}(\mathrm{Im}\ f) \Leftrightarrow \forall bb'(((b \in \mathrm{Im}\ f) \wedge (b \leq b')) \Rightarrow (b' \in \mathrm{Im}\ f))$$

$$\Leftrightarrow (\forall bb'((Pb \wedge (b \leq b')) \Rightarrow Pb')) \Leftrightarrow \top . \square$$

By 1.7.1.12, f is open. $\square$

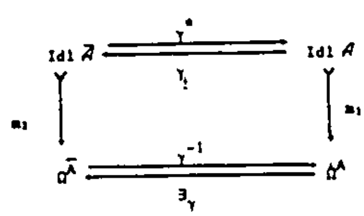
<u>1.7.1.14 Corollary.</u> Let $\mathcal{A}$ be a preordered object with $\gamma\colon A \longrightarrow \bar{A}$ its quotient map, then γ is continuous and open from $\mathcal{A}$ to $\bar{\mathcal{A}}$.

<u>Proof.</u> By 1.7.1.11, γ is order preserving and hence continuous with respect to the ideal-topologies. Since γ is an epi

$\mathrm{Im}\ \gamma = \{x\colon \exists a(fa = x)\} = \{x\colon\top\} = \Delta\top$, hence $\mathrm{idl}\ \mathrm{Im}\ \gamma$, hence by 1.7.1.12, γ is open with respect to the ideal topologies. $\square$

<u>1.7.1.15 Proposition.</u> $\mathrm{Idl}\ \mathcal{A}$ is isomorphic to $\mathrm{Idl}\ \bar{\mathcal{A}}$ as Heyting algebras. More precisely, γ^{*} has inverse $\gamma_{!}$ and both are H-homomorphisms.

<u>Proof.</u> By 1.7.1.14 we have $\gamma_{!} \dashv \gamma^{*}$ and γ^{*} is an H-homomorphism. Since $\gamma\colon A \twoheadrightarrow \bar{A}$ is an epi, $\gamma^{-1}\colon \Omega^{\bar{A}} \rightarrowtail \Omega^{A}$ is a mono, hence $\gamma^{*}\colon \mathrm{Idl}\ \bar{\mathcal{A}} \longrightarrow \mathrm{Idl}\ \mathcal{A}$ is a mono. By 1.4.1.2 (10) $\gamma_{!}\gamma^{*}J = J$.



We want to show $\gamma^{*}\gamma_{!}K = K$. Since $\gamma_{!} \dashv \gamma^{*}$ we have, by 1.4.1.2 (2.), $K \leq \gamma^{*}\gamma_{!}K$. Thus we must show $\gamma^{*}\gamma_{!}K \leq K$. Since m_1 is an order embedding this holds iff $m_1\gamma^{*}\gamma_{!}K \subset m_1K$

iff $\gamma^{-1}m_2\gamma_{!}K \subset m_1K$

iff $\gamma^{-1}\exists_{\gamma}m_1K \subset m_1K$.

$$(a \in \gamma^{-1}\exists_{\gamma}m_1K) \Rightarrow (\gamma a \in \exists_{\gamma}m_1K) \Rightarrow (\exists b((\gamma a = \gamma b) \wedge (b \in m_1K)))$$

$$\Rightarrow \exists b((b \prec a) \wedge (b \in m_1K))$$

$$\Rightarrow (a \in m_1K).$$

Thus $\gamma_{!}$ is the inverse of γ^{*}. Since γ^{*} is an H-homomorphism, by 0.6.17.14.1 $\gamma_{!}$ is also an H-homomorphism. Thus $\underset{\sim}{\text{Idl}}\, \mathcal{A}$ and $\underset{\sim}{\text{Idl}}\, \bar{\mathcal{A}}$ are H-isomorphic.□

<u>1.7.1.16 Definition of $\mathcal{A} \times B$.</u> Let $\mathcal{A}$ be a preordered object with $A \equiv |\mathcal{A}|$ and let B be an object. We define a relation $\prec$ on $A \times B$ by

$$\langle a,x\rangle \prec \langle b,y\rangle \Leftrightarrow (a \prec b) \wedge (x = y).$$

We let $\mathcal{A} \times B$ be the structure with $|\mathcal{A} \times B| \equiv A \times B$, the single binary relation given above, and equality.

<u>1.7.1.17 Proposition.</u> $\mathcal{A} \times B$ is a preordered object, and if, in addition, $\mathcal{A}$ is partially ordered, then $\mathcal{A} \times B$ is partially ordered.

<u>Proof.</u> Reflexivity: $(a \prec a) \wedge (x = x)$ hence $\langle a,x\rangle \prec \langle a,x\rangle$.

Transitivity: $(\langle a,x\rangle \prec \langle b,y\rangle) \wedge (\langle b,y\rangle \prec \langle c,z\rangle)$

$$\Rightarrow (a \prec b) \wedge (x = y) \wedge (b \prec c) \wedge (y = z)$$

$$\Rightarrow ((a \prec c) \wedge (x = z)) \Rightarrow (\langle a,x\rangle \prec \langle c,z\rangle).\square$$

We assume the pre-order of $\mathcal{A}$ is antisymmetric.

Antisymmetry: $(\langle a,x\rangle \prec \langle b,y\rangle) \wedge (\langle b,y\rangle \prec \langle a,x\rangle)$

$$\Rightarrow (a \prec b) \wedge (x = y) \wedge (b \prec a) \wedge (y = x)$$

$$\Rightarrow ((a = b) \wedge (x = y)) \Rightarrow (\langle a,x\rangle = \langle b,y\rangle).\square$$

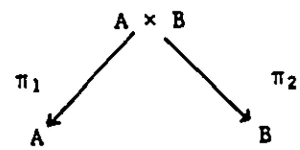
<u>1.7.1.17.1 Proposition.</u> In $\mathcal{A} \times B$ we have

$$(\langle a,b \rangle \in \uparrow U) \Leftrightarrow \exists a'((a' \prec a) \wedge (\langle a',b \rangle \in U)).$$

<u>Proof.</u> We have

$$(\langle a,b \rangle \in \uparrow U) \Leftrightarrow \exists \theta((\pi_1\theta \prec a) \wedge (\pi_2\theta = b) \wedge (\langle \pi_1\theta,b \rangle \in U))$$

where



are projections.

$(\Rightarrow)$: $[(a' \prec a) \wedge (\langle a',b \rangle \in U)] \Rightarrow \exists a'[\langle a' \prec a \rangle \wedge (\langle a',b \rangle \in U)]$

Substitute $\pi_1\theta$ for a'

$[(\pi_1\theta \prec a) \wedge (\langle \pi_1\theta,b \rangle \in U)] \Rightarrow \exists a'[\langle a' \prec a \rangle \wedge (\langle a',b \rangle \in U)]$

Hence $[(\pi_1\theta \prec a) \wedge (\pi_2\theta = b) \wedge (\langle \pi_1\theta,b \rangle \in U)] \Rightarrow \exists a'[(a' \prec a) \wedge (\langle a',b \rangle \in U)]$

hence $(\langle a,b \rangle \in \uparrow U) \Rightarrow \exists a'[(a' \prec a) \wedge (\langle a',b \rangle \in U)]$.

$(\Leftarrow)$ We have

$[(\pi_1\theta \prec a) \wedge (\pi_2\theta = b) \wedge (\langle \pi_1\theta,b \rangle \in U)] \Rightarrow (\langle a,b \rangle \in \uparrow U)$.

Substitute $\langle a',b \rangle$ for θ:

$[(a' \prec a) \wedge (\langle a',b \rangle \in U)] \Rightarrow (\langle a,b \rangle \in \uparrow U)$ hence

$\exists a'[(a' \prec a) \wedge (\langle a',b \rangle \in U)] \Rightarrow (\langle a,b \rangle \in \uparrow U)$.□

<u>1.7.1.18 Proposition.</u> Let $\mathcal{A}$ be a preordered object and let $\pi: A \times B \longrightarrow A$ be the first projection, then π is both continuous and open with respect to the ideal topologies of $\mathcal{A} \times B$ an $\mathcal{A}$.

<u>Proof.</u> $(\langle a,x \rangle \prec \langle b,y \rangle) \Rightarrow (a \prec b) \Rightarrow (\pi\langle a,x \rangle \prec \pi\langle b,y \rangle)$, hence π is continuous.□

To show π is open we must prove $(\pi\theta_1 \prec b) \Rightarrow \exists\theta_2((\pi\theta_2 = b) \wedge (\theta_1 \prec \theta_2))$

which is equivalent to

$(\pi\langle a,x \rangle \prec b) \Rightarrow \exists cz((\pi\langle c,z \rangle = b) \wedge (\langle a,x \rangle \prec \langle c,z \rangle))$ i.e.

$(a \prec b) \Rightarrow \exists cz((c = b) \wedge (a \prec c) \wedge (x = z))$ i.e.

$(a \prec b) \Rightarrow \exists c \exists z(((c = b) \wedge (x = z)) \wedge (a \prec b))$ i.e.

$(a \prec b) \Rightarrow (\exists c(c = b) \wedge \exists z(x = z) \wedge (a \prec b))$.□

1.7.1.19 The natural isomorphism $Idl(\mathcal{A} \times B) \simeq Idl(\mathcal{A})^B$.

By an exercise in [J2] (p. 72 exercises 3, 4), $Cat(\underline{\mathcal{E}})$, the category of internal categories in $\underline{\mathcal{E}}$ and internal functors between them, is cartesian closed. Thus there is a natural isomorphism

$$\underline{C}^{(\underline{A} \times \underline{B})} \approx (\underline{C}^{\underline{A}})^{\underline{B}}$$

where $\underline{A}$, $\underline{B}$ and $\underline{C}$ are internal categories.
As we shall show in 1.7.2.1 there is an obvious way of associating an internal category with a preordered object, and the full subcategory of $Cat(\underline{\mathcal{E}})$ determined by such internal categories is itself cartesian closed, thus the natural isomorphism above is between preordered objects when $\underline{A}$, $\underline{B}$ and $\underline{C}$ are preordered objects. For the preordered object $\underline{\Omega_{\leq}}$ order preserving internal functions from $\underline{A}$ to $\underline{\Omega_{\leq}}$ correspond to internal ideals of A:
$\forall ab(((a \prec b) \wedge (a \in U)) \Rightarrow (b \in U)) \Leftrightarrow \forall ab((a \prec b) \Rightarrow (U[a] \leq U[b]))$.
thus when $\underline{C}$ is replaced by $\underline{\Omega_{\leq}}$ we have $Idl(\underline{A} \times \underline{B}) \approx (Idl(\underline{A}))^{\underline{B}}$.
Since the isomorphism is an order isomorphism one can infer based on 0.6.8.1 that it is an H-isomorphism. We shall establish this in the special case that $\underline{B}$ is discrete.

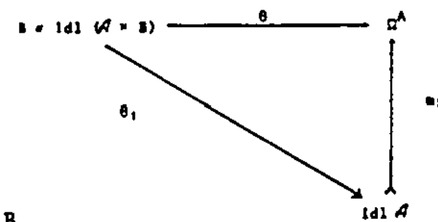
Now we consider $\mathcal{A}$ and B as in 1.7.1.16. We let $m_1\colon Idl\,\mathcal{A} \rightarrowtail \Omega^A$ and $m_2\colon Idl(\mathcal{A} \times B) \rightarrowtail \Omega^{A \times B}$ be the natural inclusions of ideals in subsets. Define
$\theta\colon B \times Idl(\mathcal{A} \times B) \longrightarrow \Omega^A$ by $(a \in \theta\langle x,J\rangle) \Leftrightarrow (\langle a,x\rangle \in m_2 J)$.

<u>1.7.1.20 Proposition.</u> $\mathrm{idl}\,\theta/x,J/$.

<u>Proof.</u> $((a \in \theta/x,J/) \wedge (a \prec b)) \Rightarrow ((/a,x/ \in m_2 J) \wedge (/a,x/ \prec /b,x/))$

$\Rightarrow (/b,x/ \in m_2 J) \Rightarrow (b \in \theta/x,J/)$.□

As a consequence we have a factorization:



We define $\overline{\theta}$: $\mathrm{Idl}(\mathcal{A} \times B) \longrightarrow (\mathrm{Idl}\,\mathcal{A})^B$ by

$$(\overline{\theta}J)[x] = \theta_1/x,J/.$$

The reverse morphism is slightly simpler.

Define K: $(\mathrm{Idl}\,\mathcal{A})^B \longrightarrow \Omega^{A \times B}$ by

$$(/a,x/ \in K\mathcal{f}) \Leftrightarrow (a \in (m_1(\mathcal{f}[x]))).$$

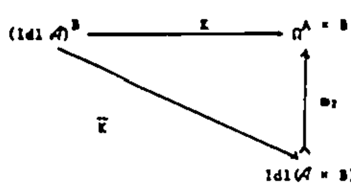
<u>1.7.1.21 Proposition.</u> $\mathrm{idl}(K\mathcal{f})$.

<u>Proof.</u> $(/a,x/ \in K\mathcal{f}) \wedge (/a,x/ \prec /b,y/)$

$\Rightarrow (a \in m_1(\mathcal{f}[x])) \wedge (a \prec b) \wedge (x = y)$

$\Rightarrow (b \in m_1(\mathcal{f}[y])) \Rightarrow (/b,y/ \in K\mathcal{f})$. □

We have the factorization



<u>1.7.1.22 Proposition.</u> (1): $\overline{K}\,\overline{\theta}J = J$ and (2): $\overline{\theta}\overline{K}\mathcal{f} = \mathcal{f}$.

<u>Proof.</u> (1) $(/a,x/ \in \overline{K}\overline{\theta}J) \Leftrightarrow (a \in (m_1((\overline{\theta}J)[x]))) \Leftrightarrow (a \in (m_1\theta_1/x,J/))$

$\Leftrightarrow (a \in \theta/x,J/) \Leftrightarrow (/a,x/ \in m_2 J)$

hence $m_2\overline{K}\,\overline{\theta}J = m_2 J$ hence $\overline{K}\,\overline{\theta}J = J$. □

(2) $(a \in m_1((\overline{\theta}\,\overline{K}f)[x])) \leftrightarrow (a \in m_1\theta_1\langle x,\overline{K}f\rangle) \leftrightarrow (a \in \theta\langle x,\overline{K}f\rangle)$

$\leftrightarrow (\langle a,x\rangle \in m_2\overline{K}f) \leftrightarrow (\langle a,x\rangle \in Kf)$

$\leftrightarrow (a \in m_1(f[x]))$

Hence $m((\overline{\theta}\,\overline{K}f)[x]) = m_1(f[x])$, hence $(\overline{\theta}\,\overline{K}f)[x] = f[x]$, hence

$\overline{\theta}\,\overline{K}f = f$. □

1.7.1.23 Proposition. Both $\overline{K}$ and $\overline{\theta}$ are order embeddings.

Proof. $(\overline{K}f_1 \leq \overline{K}f_2) \leftrightarrow Kf_1 \subset Kf_2$

$\leftrightarrow \forall ax((\langle a,x\rangle \in Kf_1) \rightarrow (\langle a,x\rangle \in Kf_2))$

$\leftrightarrow \forall ax((a \in m_1(f_1[x])) \rightarrow (a \in m_1(f_2[x]))$

$\leftrightarrow \forall x(m_1(f\ [x]) \subset m_1(f_2[x]))$

$\leftrightarrow \forall x(f_1[x] \leq f_2[x]) \leftrightarrow (f_1 \leq f_2)$.

Since $\overline{K}$ is an order embedding,

$(\overline{\theta}J_1 \leq \overline{\theta}J_2) \leftrightarrow (\overline{K}\,\overline{\theta}J_2 \leq \overline{K}\,\overline{\theta}J_2) \leftrightarrow (J_1 \leq J_2)$. □

We will show that if two internal Heyting algebras are order isomorphic then they are isomorphic as Heyting algebras.

1.7.1.24 Proposition. Let $\mathcal{C}$ and $\mathcal{D}$ be internal Heyting algebras with object parts C and D respectively, and let $C_{\leq}$ and $D_{\leq}$ be the partially ordered objects compatible with $\mathcal{C}$ and $\mathcal{D}$ respectively. If $f\colon C \longrightarrow D$ is an isomorphism and an order embedding from $C_{\leq}$ to $D_{\leq}$ then $f\colon\mathcal{C} \longrightarrow \mathcal{D}$ is an H-isomorphism.

Proof. By 0.6.17.15.1 there is an induced H-algebra structure $\mathcal{D}'$ on D such that $f\colon\mathcal{C} \longrightarrow \mathcal{D}$ is an H-homomorphism. By 0.6.17.11 $\mathcal{D}'$ is a Heyting algebra. Let $D_{\leq'}$ be the induced partially ordered object corresponding to $\mathcal{D}'$. We have

$(x \leq' y) \leftrightarrow ((x \Rightarrow' y) = 1')$

$\leftrightarrow (((f^{-1}x) \Rightarrow (f^{-1}y)) = 1)$ since $f^{-1}\colon\mathcal{D}' \longrightarrow \mathcal{C}$

$\leftrightarrow ((f^{-1}x) \leq (f^{-1}y)) \leftrightarrow (x \leq y)$.

Since $D_{\leq} \equiv D_{\leq'}$, we have $\mathcal{D} \equiv \mathcal{D}'$ (see 0.6.6.8.1)□

1.7.1.25 Corollary. Both $\overline{K}$ and $\overline{\theta}$ are H-isomorphisms.□

1.7.2 Replacing internal categories by internal partially ordered objects.

Define $\Gamma_1(\underline{\mathcal{E}})$ to be the set of all $\varphi \in \text{Poly}(H)$ such that for all partially ordered objects $\mathcal{A}$ in $\underline{\mathcal{E}}$, if $\underline{\text{Idl}}\,\mathcal{A} \models \varphi = \underline{1}$ then $\mathcal{A}$ is discrete.

In this section we shall show that $\Gamma_1(\underline{\mathcal{E}}) \equiv \Gamma(\underline{\mathcal{E}})$.

1.7.2.1 Associating an internal category with a preordered object.

We let $\mathcal{A}$ be a preordered object with carrier A. No binary predicate sign has explicitly been introduced to the formula elphabet for $\mathcal{A}$ (see 1.5.3.3). For partial orders we do however have the sign $\textcircled{\leq}$ of 0.6.8 and the convention: if $\bar{\textcircled{\leq}}$ is the interpretation of $\textcircled{\leq}$ in some internal partially ordered structure with, say, object part B, and if t and s are terms of type B then

$$(t \leq s) \equiv \bar{\textcircled{\leq}}/t,s/.$$

Thus we would naturally want any sign $\textcircled{\prec}$ used for the theory of preorders to be interpreted so that $(t \prec s) \equiv \bar{\textcircled{\prec}}/t,s/$.

When we come to interpret $\mathcal{A}$ as an internal category our choice of orientation of arrows (in an internal sense) will be governed by a comparison we make between the preorder of divisibility of the associated internal category,and the given preorder. It turns out that this choice has, in an internal sense, an arrow going from a to b whenever $b \prec a$, thus the natural morphism $C_1 \rightarrowtail C_0 \times C_0$ will be classified by $\lambda ab.(b \prec a)$, and not by $\bar{\textcircled{\prec}} \equiv |b \prec a|$.

We let $\ell \equiv \lambda ab.(b \prec a)$ so that $\ell/a,b/ \leftrightarrow (b \prec a)$; $C_0 \equiv A$; $\rho\colon C_1 \rightarrowtail C_0 \times C_0$ is the mono classified by ℓ, so that

$$(b \prec a) \leftrightarrow \exists y(\rho y = /a,b/).$$

We let $\pi'_i\colon C_0 \times C_0 \longrightarrow C_0$, $(i \equiv 1,2)$ be the projections:

$$\pi'_1\langle a,b\rangle = a \qquad \text{and} \qquad \pi'_2\langle a,b\rangle = b.$$

We let $d_0 \equiv \pi'_1 \bullet \rho$ and $d_1 \equiv \pi'_2 \bullet \rho$, so that $\rho\langle a,b\rangle = \langle d_0 a, d_1 b\rangle$,

$$(b \prec a) \Leftrightarrow \exists y((d_0 y = a) \wedge (d_1 y = b))$$

and $d_1 y \prec d_0 y$ (y of type C_1).

We can now construct diagram (1) of 1.5.1.1. For y, z of type C_1 and θ of type $C_2 \equiv C_1 \times_{C_0} C_1$ we have from this diagram

$$(d_1 z = d_0 y) \Leftrightarrow \exists\theta((\pi_1\theta = z) \wedge (\pi_2\theta = y)).$$

<u>1.7.2.1.1 Proposition.</u> There is exactly one morphism $i\colon C_0 \rightarrowtail C_1$ such that $d_0 \bullet i \equiv id_A \equiv d_1 \bullet i$.

<u>Proof.</u> We have, since ρ is a mono, $(b \prec a) \Leftrightarrow \exists! y(\rho y = \langle a,b\rangle)$.

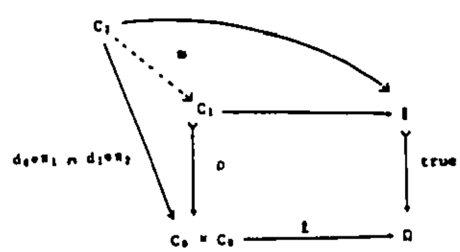
Since $a \prec a$, it follows that $\forall a \exists! y(\rho y = \langle a,a\rangle)$.

Hence by 0.6.16.6 there is exactly one morphism $i\colon C_0 \rightarrowtail C_1$ such that $\rho i a = \langle a,a\rangle$, or equivalently, $d_0 i a = a$ and $d_1 i a = a$. □

Diagrams (1), (2) and (3) of 1.5.1.1 give rise to (4) and (5).

<u>1.7.2.1.2 Proposition.</u> There is exactly one morphism $m\colon C_2 \longrightarrow C_1$ such that diagrams (8) and (9) of 1.5.1.1 commute.

<u>Proof.</u> We shall show that the outer figure below commutes and hence conclude that



there is exactly one morphism $m: C_2 \longrightarrow C_1$ such that

$\rho \bullet m \equiv d_0 \bullet \pi_1 \sqcap d_1 \bullet \pi_2$.

We want to show that $\ell \bullet (d_0 \bullet \pi_1 \sqcap d_1 \bullet \pi_2) \equiv true_{C_2}$. By definition of $\rho \equiv d_0 \sqcap d_1$, $d_1\pi_2\theta \prec d_0\pi_2\theta$ and $d_1\pi_1\theta \prec d_0\pi_1\theta$ and from (1) 1.5.1.1, $d_0\pi_2\theta = d_1\pi_1\theta$ hence, by transitivity, $d_1\pi_2\theta \prec d_0\pi_1\theta$, thus

$$\ell \bullet (d_0 \bullet \pi_1 \sqcap d_1 \bullet \pi_2) \equiv true_{C_2}.$$

The equation which determines m is equivalent, by composing with the projections π'_i, to $d_0 \bullet m \equiv d_0 \bullet \pi_1$ and $d_1 \bullet m \equiv d_1 \bullet \pi_2$.

<u>1.7.2.1.3 Proposition.</u> The morphism $m: C_2 \longrightarrow C_1$ is such that diagrams (6), (7) and (12) of 1.5.1.1, commute.

<u>Proof.</u> (6): $mi_1x = x$.

From (4) 1.5.1.1, $\pi_1 i_1 x = x$, hence $d_0 m i_1 x = d_0 \pi_1 i_1 x$ by (2) 1.5.1.1

$= d_0 x$

also $d_1 m i_1 x = d_1 \pi_2 i_1 x$ by (9) of 1.5.1.1

$= d_1 i d_1 x$ by (4) of 1.5.1.1

$= d_1 x$ by (5) of 1.5.1.1

hence $(d_0 \sqcap d_1) m i_1 x = (d_0 \sqcap d_1) x$, hence $m i_1 x = x$.□

(7): $m i_2 x = x$. Similar to (6).□

(12): $m m_1 \gamma = m m_2 \gamma$ (where γ is of type C_3).

The equation is equivalent to the conjuction of (12.1) $d_0 m m_1 \gamma = d_0 m m_2 \gamma$ and (12.2) $d_1 m m_1 \gamma = d_1 m m_2 \gamma$. We show (12.1) holds:

$d_0 m m_1 \gamma = d_0 \pi_1 m_1 \gamma$ by (8) of 1.5.1.1

$= d_0 m \pi_{12} \gamma$ by (10) of 1.5.1.1

$= d_0 \pi_1 \pi_{12} \gamma$ by (8) of 1.5.1.1

$= d_0 \pi_1 m_2 \gamma$ by (11) of 1.5.1.1

(12.2) is similar.□

This completes our verification that the structure $\underline{C} \equiv (C_0, C_1, d_0, d_1, i, m)$, which we have associated with the preordered object $\mathcal{A}$, is an internal category.

<u>1.7.2.2 Comparison of the preorder of $\mathcal{A}$ with the preorder of divisibility of the internal category $\underline{C}$ associated with $\mathcal{A}$.</u>

Let $\mathcal{C}$ be the preordered object having carrier C_1 and preorder the preorder of divisibility. We shall connect $\mathcal{C}$ to $\mathcal{A} \times C_0$ by the morphism $\rho: C_1 \longrightarrow C_0 \times C_0$. We have:

<u>1.7.2.2.1 Proposition.</u> $(x \prec y) \leftrightarrow (\rho x \prec \rho y)$ where the preorders are those of $\mathcal{C}$ and $\mathcal{A} \times C_0$.

<u>Proof.</u> By definition the preorder of divisibility is given by

$$(x \prec y) \leftrightarrow \exists z(\{x\} \circ \{z\} = \{y\})$$

which by 1.5.3.2 is equivalent to

$$(x \prec y) \leftrightarrow \exists \theta((\pi_2 \theta = x) \wedge (m\theta = y)).$$

Using the same diagram of 1.5.1.1 as we did in 1.7.2.1, we have,

$$(\pi_2\theta = x) \leftrightarrow (d_0\pi_2\theta = d_0x) \wedge (d_1\pi_2\theta = d_1x)$$

$$\leftrightarrow (d_1\pi_1\theta = d_0x) \wedge (d_1\pi_2\theta = d_1x) \quad \text{and}$$

$$(m\theta = y) \leftrightarrow (d_0m\theta = d_0y) \wedge (d_1m\theta = d_1y)$$

$$\leftrightarrow (d_0\pi_1\theta = d_0y) \wedge (d_1\pi_2\theta = d_1y).$$

We first show $(x \prec y) \rightarrow (\rho x \prec \rho y)$. We have

$$((\pi_2\theta = x) \wedge (m\theta = y)) \rightarrow (d_1\pi_1\theta = d_0y) \wedge (d_0\pi_1\theta = d_0x)$$

$$\rightarrow \exists z((d_1z = d_0y) \wedge (d_0z = d_0x))$$

$$\rightarrow (d_0y \prec d_0x)$$

and $((\pi_2\theta = x) \wedge (m\theta = y) \rightarrow ((d_1\pi_2\theta = d_1x) \wedge (d_1\pi_2\theta = d_1y))$

$$\rightarrow (d_1y = d_1x)$$

hence $\exists\theta((\pi_2\theta = x) \wedge (m\theta = y)) \Rightarrow ((d_0y \prec d_0x) \wedge (d_1y = d_1x))$

that is $(x \prec y) \Rightarrow (\rho x \prec \rho y)$.□

We show $(\rho x \prec \rho y) \Rightarrow (x \prec y)$. Let $\psi_1 \equiv (d_1\pi_1\theta = d_0x) \wedge (d_1\pi_2\theta = d_1x)$

and $\psi_2 \equiv (d_0\pi_1\theta = d_0y) \wedge (d_1\pi_2\theta = d_1y)$. We want to show

$$(\rho x \prec \rho y) \Rightarrow \exists\theta(\psi_1 \wedge \psi_2)$$

We have $(\rho x \prec \rho y) \Leftrightarrow ((d_0x \prec d_0y) \wedge (d_1x = d_1y))$

$$\Leftrightarrow (\exists z((d_0z = d_0y) \wedge (d_1z = d_0x)) \wedge (d_1x = d_1y)).$$

Thus we must show

$((d_0z = d_0y) \wedge (d_1z = d_0x) \wedge (d_1x = d_1y)) \Rightarrow \exists\theta(\psi_1 \wedge \psi_2)$.

By (1) 1.5.1.1, $(d_1z = d_0x) \Leftrightarrow \exists\theta((\pi_1\theta = z) \wedge (\pi_2\theta = x))$. Let

$\psi \equiv (d_0z = d_0y) \wedge (\pi_1\theta = z) \wedge (\pi_2\theta = x) \wedge (d_1z = d_1y)$.

We will show $\psi \Rightarrow (\psi_1 \wedge \psi_2)$.

We assume $d_0z = d_0y$, $\pi_1\theta = z$, $\pi_2\theta = x$ and $d_1x = d_1y$.

We shall show $d_1\pi_1\theta = d_0x$, $d_1\pi_2\theta = d_1x$, $d_0\pi_1\theta = d_0y$ and $d_1\pi_2\theta = d_1y$.

$d_0x = d_0\pi_2\theta = d_1\pi_1\theta$ by (1) 1.5.1.1; $d_1x = d_1\pi_2\theta$

$d_0y = d_0z = d_0\pi_1\theta$; $d_1y = d_1x = d_1\pi_2\theta$.

The latter three follow directly from the assumptions. Thus

$\psi \Rightarrow (\psi_1 \wedge \psi_2)$, and hence $\exists\theta\psi \Rightarrow \exists\theta(\psi_1 \wedge \psi_2)$ □

<u>1.7.2.2.2 Proposition.</u> idl(Im(ρ)).

<u>Proof.</u> $(\langle a,b\rangle \in \mathrm{Im}\,\rho) \Leftrightarrow \exists x(\rho x = \langle a,b\rangle) \Leftrightarrow (b \prec a)$.

We have $(\langle a,b\rangle \in \mathrm{Im}\,\rho) \wedge (\langle a,b\rangle \prec \langle c,d\rangle)$

$$\Rightarrow ((b \prec a) \wedge (a \prec c) \wedge (b = d)) \Rightarrow (d \prec c) \Rightarrow (\langle c,d\rangle \in \mathrm{Im}\,\rho) \square$$

<u>1.7.2.2.3 Corollary.</u> ρ is a continuous and open map from $\mathcal{C}$ to $\mathcal{A} \times C_0$.

<u>Proof.</u> By 1.7.2.2.2 and 1.7.1.12 ρ is open. □

1.7.2.2.4 Corollary. d_0 is a continuous and open map from $\mathcal{C}$ to $\mathcal{A}$.

Proof. $d_0 \equiv \pi_1' \circ \rho$ and both π_1' and ρ are open and continuous -the former morphism by 1.7.1.18, thus

$$\begin{aligned} \exists_{d_0}[x,\rightarrow) &= \exists_{\pi_1'}(\exists\rho[x,\rightarrow)) \\ &= \exists_{\pi_1'}[\rho x,\rightarrow) \\ &= \exists_{\pi_1'}[/d_0x,d_1x/,\rightarrow) \\ &= [d_0x,\rightarrow) \end{aligned}$$

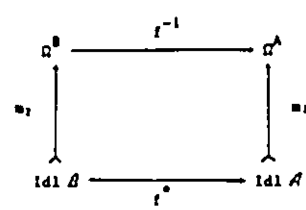
so d_0 is a continuous and open map.□

1.7.2.2.5 Proposition. If $f: A \longrightarrow B$ is an epi then $f^{-1}: \Omega^B \longrightarrow \Omega^A$ is a mono.

Proof. The functor from $\mathcal{E}^0$ to $\mathcal{E}$ that takes any $A \xrightarrow{g} B$ in $\mathcal{E}^0$ to $C^A \xrightarrow{C^g} C^B$, has as its left adjoint the functor from $\mathcal{E}$ to $\mathcal{E}^0$ that takes $B \xrightarrow{g} A$ to $C^B \xrightarrow{C^g} C^A$.
Thus the functor from $\mathcal{E}^0$ to $\mathcal{E}$ takes monos in $\mathcal{E}^0$ to monos in $\mathcal{E}$, i.e. epis in $\mathcal{E}$ to monos in $\mathcal{E}$.□

1.7.2.2.6 Corollary. If $f: A \longrightarrow B$ is an epi and f is continuous from $\mathcal{A}$ to $\mathcal{B}$ then $f^*: \text{Idl}\,\mathcal{B} \longrightarrow \text{Idl}\,\mathcal{A}$ is a mono.

Proof. From the diagram



we have (J_1 and J_2 are of type Idl $\mathcal{B}$)

$$\begin{aligned} (f^*J_1 = f^*J_2) &\Rightarrow (m_1f^*J_1 = m_1f^*J_2) \\ &\Rightarrow (f^{-1}m_2J_1 = f^{-1}m_2J_2) \\ &\Rightarrow (m_2J_1 = m_2J_2) \text{ since } f^{-1} \text{ is a mono by} \\ &\Rightarrow J_1 = J_2.\square \end{aligned}$$

<u>1.7.2.2.7 Proposition.</u> If $f: A \longrightarrow B$ is a mono then $f^{-1}\exists_f U = U$.

<u>Proof.</u> Since $\exists_f \dashv f^{-1}$ we have, by 1.4.12 (2), $U \subset f^{-1}\exists_f U$.

$(a \in f^{-1}\exists_f U) \Rightarrow (fa \in \exists_f U) \Rightarrow (\exists a_1((a_1 \in U) \wedge (fa = fa_1)))$

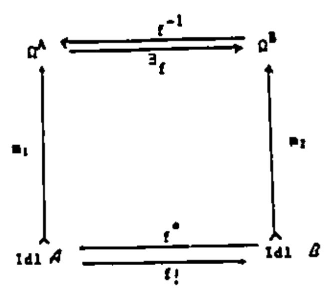
$\Rightarrow (\exists a_1((a_1 \in U) \wedge (a = a_1)))$ since f is a mono

$\Rightarrow (a \in U)$,

hence $f^{-1}\exists_f U \subset U$. □

<u>1.7.2.2.8 Corollary.</u> If $f: A \longrightarrow B$ is a mono and is open and continuous from $\mathcal{A}$ to $\mathcal{B}$ then $f^* f_! J = J$.

<u>Proof.</u> We have



$m_1 f^* f_! J = f^{-1} m_2 f_! J = f^{-1}\exists_f m_1 J = m_1 J$ by 1.7.2.2 7

hence $f^* f_! J = J$.□

<u>1.7.2.2.9 Proposition.</u> Let $\mathcal{A}$ be a preordered object with carrier C_0, let $\underline{C}$ be the associated internal category, and let $\mathcal{C}$ be the preorder of divisibility of $\underline{C}$. For $\varphi \in$ Poly H we have

$$\underline{\text{Idl}}\,\mathcal{A} \vdash \varphi = \underline{1} \quad \text{iff} \quad \underline{\text{Idl}}\,\mathcal{C} \vdash \varphi = \underline{1}\ .$$

<u>Proof.</u> ($\rightarrow$) We suppose $\underline{\text{Idl}}\,\mathcal{A} \models \varphi = \underline{1}$. Since $(\ )^{C_0}$ is cartesian we have, by 0.6.12.10 (see 0.6.14) $(\underline{\text{Idl}}\mathcal{A})^{C_0} \models \varphi = \underline{1}$. Since, by 1.7.1.24, $(\underline{\text{Idl}}\mathcal{A})^{C_0}$ is H-isomorphic to $\underline{\text{Idl}}(\mathcal{A} \times C_0)$, we have $\underline{\text{Idl}}(\mathcal{A} \times C_0) \models \varphi = \underline{1}$. The morphism $\rho: C_1 \longrightarrow C_0 \times C_0$ is a mono and is both continuous and open from $\mathcal{C}$ to $\mathcal{A}$ C_0, by 1.7.2.2.3, hence we have maps $\rho_! \dashv \rho^*$ with

$\rho^{*}\rho_{!}J = J$ (for J of type Idl $\mathcal{C}$). Thus ρ^{*} is an epi and an H-homomorphism from $\underset{\sim}{\text{Idl}}(\mathcal{A} \times C_0)$ to $\underset{\sim}{\text{Idl}}\,\mathcal{C}$ by 1.7.1.6. Thus, by 0.6.17.11, $\underset{\sim}{\text{Idl}}\,\mathcal{C} \models \varphi = \underline{1}$.□

(←) We suppose $\underset{\sim}{\text{Idl}}\,\mathcal{C} \models \varphi = \underline{1}$. By 1.7.2.1.1 and 1.7.2.2.4 $d_1\colon C_1 \twoheadrightarrow C_0$ is an epi and is open and continuous from $\mathcal{C}$ to $\mathcal{A}$. Hence by 1.7.1.6, $d_1^{*}\colon \text{Idl}\,\mathcal{A} \longrightarrow \text{Idl}\,\mathcal{C}$, is an H-homomorphism from $\underset{\sim}{\text{Idl}}\,\mathcal{A}$ to $\underset{\sim}{\text{Idl}}\,\mathcal{C}$. By 1.7.2.2.6, d_1^{*} is a mono. Hence by 0.6.17.7 (2), $\underset{\sim}{\text{Idl}}\,\mathcal{A} \models \varphi = \underline{1}$.□

__1.7.2.2.10 Proposition.__ With the same hypothesis as for 1.7.2.2.9, we have: the preorder of $\mathcal{A}$ is an equivalence relation iff the preorder of $\mathcal{C}$ is an equivalence relation.

__Proof.__ (←) Suppose the preorder of $\mathcal{C}$ is an equivalence relation. We will show that for $\mathcal{A}$ we have $(a \prec b) \Rightarrow (b \prec a)$, $(a \prec b) \Rightarrow \exists y((d_1y = a) \wedge (d_0y = b))$, thus it suffices to show

$$((d_1y = a) \wedge (d_0y = b)) \Rightarrow (b \prec a).$$

Since $a \prec a$, we have $\exists z((d_1z = a) \wedge (d_0z = a))$. Thus it suffices to show

$$((d_1y = a) \wedge (d_1z = a) \wedge (d_0z = a) \wedge (d_0y = b)) \Rightarrow (b \prec a).$$

We assume $d_1\bar{y} = \bar{a}$, $d_1\bar{z} = \bar{a}$, $d_0\bar{z} = \bar{a}$ and $d_0\bar{y} = \bar{b}$, then $\bar{a} \prec \bar{b}$ and hence $d_0\bar{z} \prec d_0\bar{y}$. Also $d_1\bar{z} = \bar{a} = d_1\bar{y}$, hence in $\mathcal{A} \times C_0$, $\langle d_0\bar{z}, d_1\bar{z}\rangle \prec \langle d_0\bar{y}, d_1\bar{y}\rangle$, hence $\bar{z} \prec \bar{y}$. But the preorder of $\mathcal{C}$ is an equivalence, hence $\bar{y} \prec \bar{z}$, hence $d_0\bar{y} \prec d_0\bar{z}$, hence $\bar{b} \prec \bar{a}$.□

(→) Suppose the pre-order of $\mathcal{A}$ is an equivalence

$x \prec y \Rightarrow ((d_0x \prec d_0y) \wedge (d_1x = d_1y)) \Rightarrow ((d_0y \prec d_0x) \wedge (d_1y = d_1x))$

$\Rightarrow y \prec x$.□

<u>1.7.2.2.11 Proposition.</u> Let $\mathcal{A}$ be a preordered object and let $\overline{\mathcal{A}}$ be the partially ordered object formed by taking the equivalence determined by the preorder, then $\mathcal{A}$ is an equivalence iff $\overline{\mathcal{A}}$ is discrete.

<u>Proof.</u> ($\rightarrow$) Assume $\mathcal{A}$ is an equivalence, i.e. $(a \prec b) \Rightarrow (b \prec a)$. We have

$$(x \leq y) \Rightarrow \exists a \exists b((a \prec b) \wedge (\gamma a = x) \wedge (\gamma b = y))$$
$$\Rightarrow \exists b \exists a((b \prec a) \wedge (\gamma b = y) \wedge (\gamma a = x))$$
$$\Rightarrow ((y \leq x) \wedge (x \leq y)) \Rightarrow (x = y). \square$$

($\leftarrow$) Assume $(x \leq y) \Rightarrow (x = y)$. We have, by 1.7.1.11,

$$(a \prec b) \Rightarrow (\gamma a \leq \gamma b) \Rightarrow (\gamma b \leq \gamma a) \Rightarrow (b \prec a). \square$$

<u>1.7.2.3 Theorem.</u> $\Gamma(\mathcal{E}) \equiv \Gamma_1(\mathcal{E})$.

<u>Proof.</u> $\Gamma(\mathcal{E}) \subset \Gamma_1(\mathcal{E})$: We assume that for all internal categories $\underline{C}$, if $\varphi \in L(\mathcal{E}^{\underline{C}^0})$ then $\underline{C}$ is a groupoid. Let $\mathcal{A}$ be an internal partially ordered object for which $\underline{\text{Idl}}\,\mathcal{A} \models \varphi = \underline{1}$. Let $\underline{C}$ be the internal category associated with $\mathcal{A}$. By 1.7.2.2.9 we have $\underline{\text{Idl}}\,\mathcal{C} \models \varphi = \underline{1}$, where $\mathcal{C}$ is the preorder of divisibility for $\underline{C}$. By 1.6.2.5 $\varphi \in L(\mathcal{E}^{\underline{C}^0})$, hence, since $\varphi \in \Gamma(\mathcal{E})$, $\underline{C}$ is a groupoid, hence by 1.5.3.7, $\prec$ is an equivalence relation on C_1. By 1.7.2.2.10, the preorder on $\mathcal{A}$ is an equivalence relation. But $\mathcal{A}$ is a partial order, hence $\mathcal{A}$ is discrete. This establishes $\varphi \in \Gamma_1(\mathcal{E})$.

$\Gamma_1(\mathcal{E}) \subset \Gamma(\mathcal{E})$: We assume that for all partially ordered objects $\mathcal{A}$, if $\underline{\text{Idl}}\,\mathcal{A} \models \varphi = \underline{1}$ then $\mathcal{A}$ is discrete. Let $\underline{C}$ be an internal category for which $\varphi \in L(\mathcal{E}^{\underline{C}^0})$. Let $\mathcal{C}$ be C_1 equipped with the preorder of divisibility, then by 1.6.2.5, $\underline{\text{Idl}}\,\mathcal{C} \models \varphi = \underline{1}$. By 1.7.1.15 $\underline{\text{Idl}}\,\mathcal{C}$ is isomorphic as a Heyting algebra to $\underline{\text{Idl}}\,\overline{\mathcal{C}}$, where $\overline{\mathcal{C}}$ is the quotient defined in 1.7.1.9. Hence $\underline{\text{Idl}}\,\overline{\mathcal{C}} \models \varphi = \underline{1}$. But $\overline{\mathcal{C}}$ is a partially ordered object, by 1.7.1.10, hence by $\varphi \in \Gamma_1(\mathcal{E})$, $\overline{\mathcal{C}}$ is discrete. By 1.7.2.2.11, $\mathcal{C}$ is an equivalence, hence by 1.5.3.7, $\underline{C}$ is an internal groupoid.$\square$

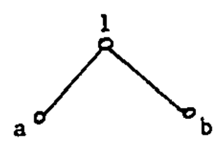
1.7.3 Simplifying a partially ordered object and restricting to polynomials in $\mathcal{C}^{\bullet}$.

It is at this stage in our replacement of structures by simpler structures that we will restrict ourselves to the set $\mathcal{C}^{\bullet} \subset \mathrm{Poly}(\mathbb{H})$ defined in 1.1.1.3.1. We will introduce $\Gamma_2(\underline{\mathcal{E}}) \subset \mathrm{Poly}(\mathbb{H})$ and show that

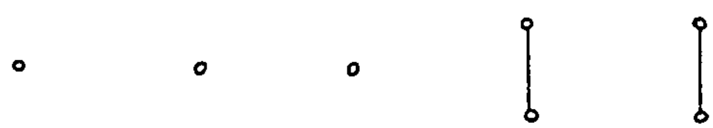
$$\mathcal{C}^{\bullet} \cap \Gamma_1(\underline{\mathcal{E}}) \equiv \mathcal{C}^{\bullet} \cap \Gamma_2(\underline{\mathcal{E}}).$$

1.7.3.1 The simplified partially ordered object associated with a partially ordered object.

If we perform the construction of this section on a partially ordered set $\mathcal{A}$ in "the" category of sets, we will get a partially ordered set $\tilde{\mathcal{A}}$ which is a disjoint union of partially ordered sets $\mathcal{A}_{(x,y)}$ where $|\mathcal{A}_{(x,y)}| \approx \{x,y\}$ and $x \leq y$. For example if $\mathcal{A}$ has Hasse diagram



then $\tilde{\mathcal{A}}$ has Hasse diagram



$\mathcal{A}_{(a,a)}$ $\mathcal{A}_{(b,b)}$ $\mathcal{A}_{(1,1)}$ $\mathcal{A}_{(a,1)}$ $\mathcal{A}_{(b,1)}$

Clearly, within the category of sets, $\mathcal{A}$ is discrete iff $\tilde{\mathcal{A}}$ is discrete.

1.7.3.2 The construction of $\tilde{\mathcal{A}}$ from $\mathcal{A}$ in a topos.

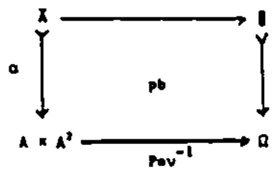
Throughout $\mathcal{A}$ will be a partially ordered object with $|\mathcal{A}| \equiv A$. We define a ternary predicate $P\colon A^3 \longrightarrow \Omega$ by

$$P/x,y,z/ \leftrightarrow (x \in \{y,z\}) \wedge (y \leq z).$$

We let $\nu: A^3 \xrightarrow{\sim} A \times A^2$ be the canonical isomorphism given by

$$\nu(x,y,z) = (x,(y,z))$$

We take a pullback



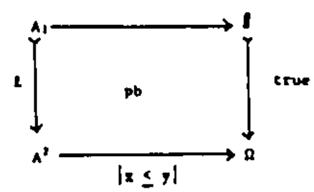
so that $P\circ\nu^{-1}$ classifies α, and P classifies $\nu^{-1}\circ\alpha$. We let $\tilde{\mathcal{A}}$ be the partially ordered object with $|\tilde{\mathcal{A}}| \equiv \tilde{A}$ and order induced by $\mathcal{A} \times A^2$ through α thus:

$$(\theta_1 \leq \theta_2) \Leftrightarrow (\alpha\theta_1 \leq \alpha\theta_2).$$

<u>1.7.3.3 The set $\Gamma_2(\mathcal{E}) \subset \mathrm{Poly}(\mathbb{H})$.</u> $\Gamma_2(\mathcal{E})$ is the set of all $\varphi \in \mathrm{Poly}(\mathbb{H})$ such that for all partially ordered objects $\mathcal{A}$ in $\mathcal{E}$, if $\underline{\mathrm{Idl}}\,\tilde{\mathcal{A}} \models \varphi = \underline{1}$ then $\tilde{\mathcal{A}}$ is discrete. It is clear that $\Gamma_1(\mathcal{E}) \subset \Gamma_2(\mathcal{E})$.

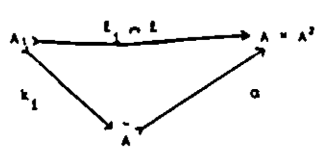
<u>1.7.3.4 The equivalence of the discreteness of $\mathcal{A}$ and $\tilde{\mathcal{A}}$.</u>

We construct a pullback:



and let $\ell \equiv \ell_1 \sqcap \ell_2$ so that $\ell_1\rho \leq \ell_2\rho$ for ρ of type A_1.

<u>1.7.3.4.1 Proposition.</u> $\ell_i \sqcap \ell$ factors through α for $i \equiv 1,2$:



Proof. We have

$\nu^{-1}(\ell_i \cap \ell)\rho = \nu^{-1}\langle \ell_i\rho, \ell\rho\rangle = \nu^{-1}\langle \ell_i\rho, \langle \ell_1\rho, \ell_2\rho\rangle\rangle$

$= \langle \ell_i\rho, \ell_1\rho, \ell_2\rho\rangle$

and $(P\circ\nu^{-1}\circ(\ell_i \cap \ell))\rho = P\langle \ell_i\rho, \ell_1\rho, \ell_2\rho\rangle$

$= (\ell_i\pi \in \{\ell_1\rho, \ell_2\rho\}) \wedge (\ell_1\rho \leq \ell_2\rho) = T$

hence $(P\circ\nu^{-1})\circ(\ell_i \cap \ell) \equiv true_A$, hence $\ell_i \cap \ell$ factors through α.□

We have monomorphisms $k_i\colon A_1 \rightarrowtail \tilde{A}$ such that $\alpha\circ k_i \equiv \ell_i \cap \ell$ for $i \equiv 1,2$.

1.7.3.4.2 Proposition. $\tilde{A}$ is discrete iff A is discrete.

Proof. (←) Suppose A is discrete then for $A \times A^2$ we have

$(\langle w,\langle x_1,x_2\rangle\rangle \leq \langle y,\langle z_1,z_2\rangle\rangle) \Rightarrow (w \leq y) \wedge (x_1 = z_1) \wedge (x_2 = z_2)$

$\Rightarrow (w = y) \wedge (x_1 = z_1) \wedge (x_2 = z_2)$

$\Rightarrow \langle w,\langle x_1,x_2\rangle\rangle = \langle y,\langle z_1,z_2\rangle\rangle$.

Thus for $\tilde{A}$ we have

$(\theta_1 \leq \theta_2) \Rightarrow (\alpha\theta_1 \leq \alpha\theta_2)$

$\Rightarrow (\alpha\theta_1 = \alpha\theta_2)$

$\Rightarrow (\theta_1 = \theta_2)$ □

(→) Suppose $\tilde{A}$ is discrete, then $(\theta_1 \leq \theta_2) \Rightarrow (\theta_1 = \theta_2)$.

Since α is an order embedding this is equivalent to

$(\alpha\theta_1 \leq \alpha\theta_2) \Rightarrow (\alpha\theta_1 = \alpha\theta_2)$.

Substitute $k_i\rho$ for θ_i $(i \equiv 1,2)$ to get $(\alpha k_1\rho \leq \alpha k_2\rho) \Rightarrow (\alpha k_1\rho = \alpha k_2\rho)$.

By 1.7.3.4.1 this is equivalent to

$(\langle \ell_1\rho, \ell\rho\rangle \leq \langle \ell_2\rho, \ell\rho\rangle) \Rightarrow (\langle \ell_1\rho, \ell\rho\rangle = \langle \ell_2\rho, \ell\rho\rangle)$

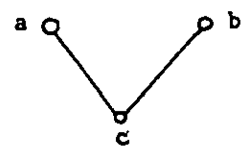
that is $(\ell_1\rho \leq \ell_2\rho) \Rightarrow (\ell_1\rho = \ell_2\rho)$ hence $\ell_1\rho = \ell_2\rho$ Hence

$(x \leq y) \Rightarrow (x = y)$.□

1.7.3.5 Properties of the order embedding $\alpha: \tilde{\mathcal{A}} \rightarrowtail \mathcal{A} \times A^2$.

1.7.3.5.1 α is not in general open from $\tilde{\mathcal{A}}$ to $\mathcal{A} \times A^2$.

We consider the classical situation (in the category of sets). Take $\mathcal{A}$ to have Hasse diagram:

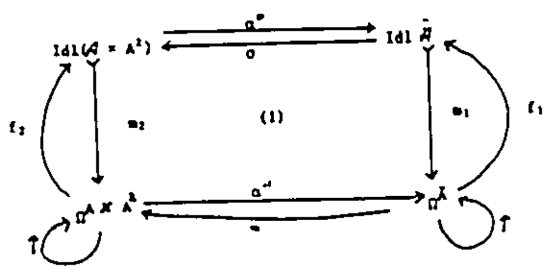


We treat α informally as an inclusion: $\tilde{\mathcal{A}} \subset A \times A^2$. Let $\theta = \langle c,\langle c,b\rangle\rangle$, then $\theta \in \tilde{A}$, and, in $\tilde{\mathcal{A}}$, $[\theta,\rightarrow) = \{\langle c,\langle c,b\rangle\rangle,\langle b,\langle c,b\rangle\rangle\}$. In $\mathcal{A} \times A^2$, $[\alpha\theta,\rightarrow) = \{\langle c,\langle c,b\rangle\rangle,\langle b,\langle c,b\rangle\rangle,\langle a,\langle c,b\rangle\rangle\}$. Since $[\alpha\theta,\rightarrow) \not\subset \exists\alpha[\theta,\rightarrow)$, by 1.7.1.3, α is not open.

Thus, although there is an epi, α^*, from $\mathrm{Idl}\,\mathcal{A} \times A^2$ to $\mathrm{Idl}\,\tilde{\mathcal{A}}$ which preserves $\underline{0}$, $\underline{1}$, $\underline{\vee}$ and $\underline{\wedge}$, it does not in general preserve $\underline{\Rightarrow}$.

1.7.3.5.2 Morphisms determined by α, $\tilde{\mathcal{A}}$ and $\mathcal{A} \times A^2$.

From the diagram of 1.7.1.12.1 we have morphisms:



We introduce variables for the types involved in this diagram:

type	A	$\tilde{A}$	$\Omega^{A \times A^2}$	Ω^{A}	$\mathrm{Idl}(\mathcal{A} \times A^2)$	$\mathrm{Idl}\,\tilde{\mathcal{A}}$
variable	x,y,z,u,v,w,u',u"	θ	W	U	K, K_i	k, k_i,
term			t_1, t_2			

There are three pairs of adjoints as well as a composite adjoint:

$m_1 \dashv f_1$, $m_2 \dashv f_2$, $\exists_\alpha \dashv \alpha^{-1}$ and $\exists_\alpha \bullet m_1 \dashv f_1 \bullet \alpha^{-1}$.

Since each of the left adjoints is a mono we have

$f_1 m_1 k = k$, $f_2 m_2 K = K$, $\alpha^{-1} \exists_\alpha U = U$ and $f_1 \alpha^{-1} \exists_\alpha m_1 k = k$.

The closure operators on $\Omega^{\tilde{A}}$ and $\Omega^{A \times A^2}$ are respectively:

$$\uparrow U = m_1 f_1 U \quad \text{and} \quad \uparrow W = m_2 f_2 W.$$

From the definition of α^* we have

$$\alpha^{-1} m_2 K = m_1 \alpha^* K.$$

The morphism σ satisfies (by 1.7.1.12.2 (1))

$$\sigma k = f_2 \exists_\alpha m_1 k \quad \text{and} \quad \alpha^* \sigma k = k.$$

1.7.3.5.3 In the next proposition we shall show that α^* preserves the operation $\Rightarrow$ when the first argument is restricted to the image of σ; that is $\alpha^*(\sigma k \Rightarrow K) = (k \Rightarrow \alpha^* K) = \alpha^*(\sigma k) \Rightarrow \alpha^* K$.

We refer to 1.7.3.5.2 for notation and basic properties of morphisms used in 1.7.3.6.

1.7.3.6 Proposition.

(1) $([\alpha\theta, \rightarrow) \cap \uparrow\exists_\alpha m_1 k) \subset \uparrow\exists_\alpha([\theta, \rightarrow) \cap m_1 k)$

(2) $(k \Rightarrow \alpha^* K) = \alpha^*((\sigma k) \Rightarrow K)$.

Proof. (1): From 1.7.1.12.2 (2), (1) is equivalent to (1.2):

$([\alpha\theta, \rightarrow) \cap \uparrow\exists_\alpha m_1 k) \subset \uparrow([\alpha\theta, \rightarrow) \cap \exists_\alpha m_1 k)$. Let $t_1 \equiv ([\langle x, \langle y, z\rangle\rangle, \rightarrow) \cap (\uparrow\exists_\alpha m_1 k))$ and $t_2 \equiv \uparrow([\langle x, \langle y, z\rangle\rangle, \rightarrow) \cap (\exists_\alpha m_1 k))$, then (1.2) holds iff

(1.3) $\quad \exists\theta((\alpha\theta = \langle x, \langle y, z\rangle\rangle) \Rightarrow (t_1 \subset t_2)$.

Since $\alpha: \tilde{A} \rightarrowtail A \times A^2$ is classified by $P: A \times A^2 \longrightarrow \Omega$, (1.3) holds iff $P\langle x, y, z\rangle \Rightarrow (t_1 \subset t_2)$ iff

(1.4) $(P\langle x,y,z\rangle \wedge (\langle u,\langle v,w\rangle\rangle \in t_1)) \rightarrow (\langle u,\langle v,w\rangle\rangle \in t_2)$.

We have $\langle u,\langle v,w\rangle\rangle \in t_1$

$\leftrightarrow ((x \leq u) \wedge (\langle y,z\rangle = \langle v,w\rangle) \wedge (\langle u,\langle y,z\rangle\rangle \in \uparrow\exists_\alpha m_1 k))$ and

$(\langle u,\langle y,z\rangle\rangle \in \uparrow\exists_\alpha m_1 k) \leftrightarrow \exists u'((u' \leq u) \wedge (\langle u',\langle y,z\rangle\rangle \in \exists_\alpha m_1 k))$ by 1.7.1.17.1

hence $\langle u,\langle y,z\rangle\rangle \in t_1$

$\leftrightarrow ((x \leq u) \wedge (\langle u,\langle y,z\rangle\rangle \in \uparrow\exists_\alpha m_1 k)$

$\leftrightarrow \exists u'((x \leq u) \wedge (u' \leq u) \wedge (\langle u',\langle y,z\rangle\rangle \in \exists_\alpha m_1 k))$.

Also $\langle u,\langle v,w\rangle\rangle \in t_2$

$\leftrightarrow \exists u'((u' \leq u) \wedge (\langle y,z\rangle = \langle v,w\rangle) \wedge (\langle u',\langle y,z\rangle\rangle \in (\exists_\alpha m_1 k)) \wedge (x \leq u')$

hence $\langle u,\langle y,z\rangle\rangle \in t_2$

$\leftrightarrow \exists u'((u' \leq u) \wedge (\langle u',\langle y,z\rangle\rangle \in (\exists_\alpha m_1 k))) \wedge (x \leq u')$

Hence for $i \equiv 1,2$

$$(\langle u,\langle v,w\rangle\rangle \in t_i) \leftrightarrow ((\langle y,z\rangle = \langle v,w\rangle) \wedge (\langle u,\langle y,z\rangle\rangle \in t_i)).$$

Thus we have (1.4) iff

$\left[(\langle y,z\rangle = \langle v,w\rangle) \wedge P\langle x,y,z\rangle \wedge (\langle u,\langle y,z\rangle\rangle \in t_1)\right] \rightarrow \left[\langle u,\langle y,z\rangle\rangle \in t_2\right]$ iff

(1.5) $\left[P\langle x,y,z\rangle \wedge (\langle u,\langle y,z\rangle\rangle \in t_1)\right] \rightarrow \left[\langle u,\langle y,z\rangle\rangle \in t_2\right]$

We will show that $(\langle u,\langle y,z\rangle\rangle \in t_1) \rightarrow (y \leq z)$ and hence that we can simplify (1.5) to the equivalent

(1.6) $\left[(x \in \{y,z\}) \wedge (\langle u,\langle y,z\rangle\rangle \in t_1)\right] \rightarrow \left[\langle u,\langle y,z\rangle \in t_2\right]$.

$\langle u,\langle y,z\rangle\rangle \in t_1$

$\rightarrow \exists u'(\langle u',\langle y,z\rangle\rangle \in \exists_\alpha m_1 k)$

$\rightarrow \exists u'\exists\theta(\alpha\theta = \langle u'\langle y,z\rangle\rangle)$

$\rightarrow (y \leq z)$.

Thus (1.5) is equivalent to (1.6). We can break (1.6) up into the conjunction of (1.7) and (1.8).

(2.1) holds

iff $m_1(k \Rightarrow (\alpha^* K)) \subset m_1\alpha^*((\sigma k \Rightarrow K)$

iff $((m_1 k) \Rightarrow (m_1\alpha^* K))^0 \subset \alpha^{-1} m_2((\sigma k \Rightarrow K)$

iff $((m_1 k) \Rightarrow (\alpha^{-1} m_2 K))^0 \subset \alpha^{-1}((m_2\sigma k \Rightarrow (m_2 K))^0$

iff $(([\theta,\to) \cap (m_1 k)) \subset (\alpha^{-1} m K)) \Rightarrow (([\alpha\theta,\to) \cap \uparrow\exists_\alpha m_1 k) \subset m_2 K)$

iff $((\exists_\alpha([\theta,\to) \cap (m_1 k)) \subset (m_2 K)) \Rightarrow ((([\alpha\theta,\to) \cap \uparrow\exists_\alpha m_1 k) \subset m_2 K)$

The last statement follows from (1).□

<u>1.7.3.6.1 Interpreting H-polynomials in $\underset{\sim}{\mathrm{Idl}}\,\tilde{\mathcal{A}}$ and $\underset{\sim}{\mathrm{Idl}}(\mathcal{A} \times A^2)$.</u>

For each $\varphi \in \mathrm{Poly}(H)$ we let $\tilde{\varphi}$ and $\bar{\varphi}$ be its interpretations in $\underset{\sim}{\mathrm{Idl}}\,\tilde{\mathcal{A}}$ and $\underset{\sim}{\mathrm{Idl}}(\mathcal{A} \times A^2)$ respectively.

Define a substitution μ: Vbls ⟶ Terms by $\mu(x) \equiv x$ for x not of type $\mathrm{Idl}(\mathcal{A} \times A^2)$ and $\mu(K_i) \equiv \sigma k_i$ where K_i and k_i are the i-th variable of type $\mathrm{Idl}(\mathcal{A} \times A^2)$ and $\mathrm{Idl}\,\tilde{\mathcal{A}}$ respectively.

<u>1.7.3.7. Proposition.</u> $\alpha^*(S(\mu)(\bar{\varphi})) = \tilde{\varphi}$ for all $\varphi \in \mathcal{C}$.

<u>Proof.</u> We shall show $\mathcal{D} \in \Phi$ where Φ is defined in 1.1.1.3.1 and $\mathcal{D} \equiv \{\varphi \in \mathrm{Poly}(H) \mid \alpha^*(S(\mu)(\bar{\varphi})) = \tilde{\varphi}\}$, by verifying the four clauses of 1.7.3.1.1.

(1) $\underline{0} \in \mathcal{D}$: $\alpha^*(S(\mu)(\bar{\underline{0}})) \equiv \alpha^*(\bar{\underline{0}}) = \tilde{\underline{0}}$.

$\underline{1} \in \mathcal{D}$: $\alpha^*(S(\mu)(\bar{\underline{1}})) \equiv \alpha^*(\bar{\underline{1}}) = \tilde{\underline{1}}$.

(2) $v_i \in \mathcal{D}$, where v_i is the i-th variable of V:

$\alpha^*(S(\mu)(\bar{v}_i)) \equiv \alpha^*(\sigma k_i) \equiv k_i \equiv \tilde{v}_i$ by 1.7.1.12.2 (1)

(1.7) $\left[(x = y) \wedge (\langle u,\langle y,z\rangle\rangle \in t_1)\right] \Rightarrow \left[\langle u,\langle y,z\rangle\rangle \in t_2\right]$ and

(1.8) $\left[(x = z) \wedge (\langle u,\langle y,z\rangle\rangle \in t_1)\right] \Rightarrow \left[\langle u,\langle y,z\rangle\rangle \in t_2\right]$.

We first show (1.7) holds.

$(x = y) \wedge (\langle u,\langle y,z\rangle\rangle \in t_1)$

$\Rightarrow \exists u'((u' \leq u) \wedge (x \leq u') \wedge (\langle u',\langle y,z\rangle\rangle \in \exists_\alpha m_1 k))$

$\Rightarrow \langle u,\langle y,z\rangle\rangle \in t_2$.

For (1.8) we have $\langle u',\langle y,z\rangle\rangle \in \exists_\alpha m_1 k$

$\Leftrightarrow \exists\theta'((\alpha\theta' = \langle u',\langle y,z\rangle\rangle) \wedge (\theta' \in m_1 k))$.

We have $(\alpha\theta' = \langle u',\langle y,z\rangle\rangle) \wedge (\theta' \in m_1 k)$

$\Leftrightarrow (\alpha\theta' = \langle u',\langle y,z\rangle\rangle) \wedge (y \leq z) \wedge (\theta' \in m_1 k)$, but

$(y \leq z) \Leftrightarrow ((z \in \{y,z\}) \wedge (y \leq z))$

$\Leftrightarrow \exists\theta(\alpha\theta = \langle z,\langle y,z\rangle\rangle)$, hence

$(\alpha\theta' = \langle u',\langle y,z\rangle\rangle) \wedge (\theta' \in m_1 k)$

$\Leftrightarrow \exists\theta((\alpha\theta = \langle z,\langle y,z\rangle\rangle) \wedge (\alpha\theta' = \langle u',\langle y,z\rangle\rangle \wedge (\theta' \in m_1 k))$

$\Rightarrow \exists\theta((\alpha\theta = \langle z,\langle y,z\rangle\rangle) \wedge (\alpha\theta' = \langle u',\langle y,z\rangle\rangle) \wedge (\theta' \leq \theta) \wedge (\theta' \in m_1 k))$

$\Rightarrow \exists\theta((\alpha\theta = \langle z,\langle y,z\rangle\rangle) \wedge (\theta \in m_1 k))$

$\Rightarrow (\langle z,\langle y,z\rangle\rangle \in \exists_\alpha m_1 k)$

hence $(\langle u',\langle y,z\rangle\rangle \in \exists_\alpha m_1 k) \Rightarrow (\langle z,\langle y,z\rangle\rangle \in \exists_\alpha m_1 k)$.

$(x = z) \wedge (\langle u,\langle y,z\rangle\rangle \in t_1)$

$\Leftrightarrow \left[(x = z) \wedge (x \leq u) \wedge \exists u''((u'' \leq u) \wedge (\langle u'',\langle y,z\rangle\rangle \in (\exists_\alpha m_1 k)))\right]$

$\Rightarrow \left[(x \leq z) \wedge (z \leq u) \wedge (\langle z,\langle y,z\rangle\rangle \in (\exists_\alpha m_1 k))\right]$

$\Rightarrow \exists u'\left[(x \leq u') \wedge (u' \leq u) \wedge (\langle u',\langle y,z\rangle\rangle \in (\exists_\alpha m_1 k))\right]$

$\Rightarrow (\langle u,\langle y,z\rangle\rangle \in t_2)$. Thus (1.8) holds, and hence (1) holds.□

(2): By 1.7.1.2 and 1.7.1.12.2 (1) we have

$$\alpha^*((\sigma k) \Rightarrow K) \leq (k \Rightarrow (\alpha^* K)).$$

(3) If φ $\varepsilon\,\mathcal{D}$ and $\varphi_2\ \varepsilon\,\mathcal{D}$ then

$$\alpha^*(S(\mu)(\overline{\varphi_1 \wedge \varphi_2})) = (\alpha^*(S(\mu)(\overline{\varphi}_1))) \wedge (\alpha^*(S(\mu)(\overline{\varphi}_2))) = \tilde{\varphi}_1 \wedge \tilde{\varphi}_2$$
$$= \widetilde{\varphi_1 \wedge \varphi_2}$$

hence $(\varphi_1 \wedge \varphi_2)\ \varepsilon\,\mathcal{D}$, and similarly $(\varphi_1 \vee \varphi_2)\ \varepsilon\,\mathcal{D}$. □

(6) Suppose $v\ \varepsilon\ V$ and $\varphi\ \varepsilon\,\mathcal{D}$. Let $K \equiv \overline{v}$ and $k \equiv \tilde{v}$.

$$\alpha^*(S(\mu)(\overline{v \Rightarrow \varphi})) = \alpha^*((\sigma k \Rightarrow S(\mu)(\overline{\varphi}))$$
$$= k \Rightarrow \alpha^*(S(\mu)(\overline{\varphi})) \quad \text{, by 1.7.3.6 (2)}$$
$$= k \Rightarrow \tilde{\varphi}$$
$$= \widetilde{(v \Rightarrow \varphi)}$$

hence $(v \Rightarrow \varphi)\ \varepsilon\,\mathcal{D}$.

It follows that $\mathcal{D}\ \varepsilon\ \Phi$; hence $\mathcal{C} \subset \mathcal{D}$, i.e. $\varphi\ \varepsilon\ \mathcal{C}$ implies $\alpha^*(S(\mu)(\overline{\varphi})) = \tilde{\varphi}$.□

<u>1.7.3.8 Proposition.</u> Let $\psi\ \varepsilon\ \mathcal{C}^{\Rightarrow}$ and suppose $\underset{\sim}{\mathrm{Idl}}\,\mathcal{A} \models \psi = \underline{1}$, then $\underset{\sim}{\mathrm{Idl}}\,\tilde{\mathcal{A}} \models \psi = \underline{1}$.

<u>Proof.</u> Let $\psi \equiv (\varphi_1 \Rightarrow \varphi_2)$ where $\varphi_i\ \varepsilon\ \mathcal{C}$ $(i \equiv 1.2)$. Suppose $\underset{\sim}{\mathrm{Idl}}\,\mathcal{A} \models ((\varphi_1 \Rightarrow \varphi_2) = \underline{1})$, that is $\overline{\varphi}_1 \leq \overline{\varphi}_2$ hence $S(\mu)(\overline{\varphi}_1) \leq S(\mu)(\overline{\varphi}_2)$ hence $\alpha^*(S(\mu)(\overline{\varphi}_1)) \leq \alpha^*(S(\mu)(\overline{\varphi}_2))$ hence $\tilde{\varphi}_1 \leq \tilde{\varphi}_2$ by 1.7.3.7 hence $\underset{\sim}{\mathrm{Idl}}\,\tilde{\mathcal{A}} \models ((\varphi_1 \Rightarrow \varphi_2) = \underline{1})$.□

<u>1.7.3.9 Theorem.</u> $\mathcal{C}^{\Rightarrow} \cap \Gamma_1(\underline{\mathcal{E}}) \equiv \mathcal{C}^{\Rightarrow} \cap \Gamma_2(\underline{\mathcal{E}})$.

<u>Proof.</u> $(\subset)$ Since $\Gamma_1(\underline{\mathcal{E}}) \subset \Gamma_2(\underline{\mathcal{E}})$. □

$(\supset)$: Let $\varphi\ \varepsilon\ \mathcal{C}^{\Rightarrow} \cap \Gamma_2(\underline{\mathcal{E}})$. We must show $\varphi\ \varepsilon\ \Gamma_1(\underline{\mathcal{E}})$. Let $\mathcal{A}$ be a partially ordered object in $\underline{\mathcal{A}}$ such that $\underset{\sim}{\mathrm{Idl}}\,\mathcal{A} \models \varphi = \underline{1}$. By 1.7.3.8, $\underset{\sim}{\mathrm{Idl}}\,\mathcal{A} \models \varphi = \underline{1}$, hence, since $\varphi\ \varepsilon\ \Gamma_2(\underline{\mathcal{E}})$, $\tilde{\mathcal{A}}$ is discrete. Hence, by 1.7.3.4.2, $\mathcal{A}$ is discrete.□

1.7.4 Reducing a partially ordered object to a subobject of Ω^2.

We continue the process of replacing an internal structure by a simpler structure which shares, for our purposes, important similarities with the original structure. This will be the last such reduction. From a given partially ordered structure $\mathcal{A}$ we constructed, in 1.7.3.2.1, the structure $\tilde{\mathcal{A}}$; now we shall construct a partially ordered object $\hat{\mathcal{A}}$ which is the image under a continuous open map of $\tilde{\mathcal{A}}$ and is order embedded in $(\Omega_{\underline{<}})^0 \times \Omega$ (see 1.4.1 and 1.7.1.16).

1.7.4.1 The construction of $\hat{\mathcal{A}}$ from $\mathcal{A}$.

We break our construction of an order embedding $\hat{\alpha}: \hat{\mathcal{A}} \rightarrowtail (\Omega_{\underline{<}})^0 \times \Omega$ into two constructions, I and II.

I: For each partially ordered structure $\mathcal{A}$ we define an endomorphism

$f_{\mathcal{A}} : \Omega \longrightarrow \Omega$ by

$$f_{\mathcal{A}}\, q = \exists yz((y \leq z) \wedge (q \Leftrightarrow (y = z)))$$

II. For each endomorphism $f: \Omega \longrightarrow \Omega$ we define $\dot{f}: \Omega^2 \longrightarrow \Omega$ by

$$\dot{f}\langle p,q\rangle \Leftrightarrow ((p \in \{q,\top\}) \wedge fq)$$

and select a monomorphism

$$m_f: D(f) \rightarrowtail \Omega^2.$$

which is classified by $\dot{f}$. We let $\mathcal{D}(f)$ be the partially ordered structure induced on $D(f)$ via m_f by $(\Omega_{\underline{<}})^0 \times \Omega$. Thus m_f is an order embedding, with the partial order, $\prec$, on Ω^2 given by

$$(\langle p,q\rangle \prec \langle p',q'\rangle) \Leftrightarrow ((p' \leq p) \wedge (q = q')).$$

We compose the two constructions and apply the composite constructions to $\mathcal{A}$; this gives us: $\hat{A} \equiv D(f_{\mathcal{A}})$, $\hat{\mathcal{A}} \equiv \mathcal{D}(f_{\mathcal{A}})$ and $\hat{\alpha} \equiv m_{f_{\mathcal{A}}}$

$$\hat{\alpha}: \hat{\mathcal{A}} \rightarrowtail (\Omega_{\underline{<}})^0 \times \Omega.$$

1.7.4.1.1. The set $\Gamma_3(\mathcal{E}) \subset \mathrm{Poly}(H)$. $\Gamma_3(\mathcal{E})$ is the set of all $\varphi \in \mathrm{Poly}(H)$ such that for all partially ordered structures $\mathcal{A}$,

$$\text{if } \ \mathrm{Idl}\,\hat{\mathcal{A}} \models \varphi \ \text{ then } \ \hat{\mathcal{A}} \ \text{ discrete.}$$

It is clear that $\Gamma_1(\mathcal{E}) \subset \Gamma_3(\mathcal{E})$.

1.7.4.2 Define $h: A \times A^2 \longrightarrow \Omega^2$ by $h\langle x,\langle y,z\rangle\rangle = \langle x = y, y = z\rangle$ and $h_i: A \times A^2 \longrightarrow \Omega$ $(i \equiv 1,2)$ by $h_1\langle x,\langle y,z\rangle\rangle = (x = y)$ and $h_2\langle x,\langle y,z\rangle\rangle = (y = z)$ so that $h \equiv h_1 \sqcap h_2$.

In the next four propositions we shall show that there is a continuous open epi, k, from $\tilde{\mathcal{A}}$ to $\hat{\mathcal{A}}$, which is the restriction to $\tilde{\mathcal{A}}$ and corestriction to $\hat{\mathcal{A}}$ of the morphism h, so that

$$\begin{array}{ccc} \tilde{A} & \xrightarrow{\ k\ } & \hat{A} \\ {\scriptstyle \alpha}\big\downarrow & & \big\downarrow{\scriptstyle \hat{\alpha}} \\ A \times A^2 & \xrightarrow[\ h\ ]{} & \Omega^2 \end{array}$$

commutes, where α is the mono introduced in 1.7.3.2.1 and $\hat{\alpha}$ is the mono introduced in 1.7.4.1.

Define $\alpha_i: \tilde{A} \longrightarrow A$ $(i \equiv 1,2,3)$ by composing $\alpha: \tilde{A} \rightarrowtail A \times A^2$ with projections: $\alpha \equiv \alpha_1 \sqcap (\alpha_2 \sqcap \alpha_3)$ and for $\theta \in V_{\tilde{A}}$,
$\alpha\theta = \langle \alpha_1\theta, \langle \alpha_2\theta, \alpha_3\theta\rangle\rangle$.

1.7.4.3 Proposition. There exists a morphism $k: \tilde{A} \longrightarrow \hat{A}$ such that $\hat{\alpha} \circ k \equiv h \circ \alpha$, so that $\hat{\alpha}k\theta = \langle \alpha_1\theta = \alpha_2\theta, \alpha_2\theta = \alpha_3\theta\rangle$.

Proof. We must show that $\dot{f}_{\mathcal{A}} \circ h \circ \alpha \equiv \mathrm{true}_{\tilde{A}}$. Let $\bar{x} \equiv \alpha_1\theta$, $\bar{y} \equiv \alpha_2\theta$ and $\bar{z} \equiv \alpha_3\theta$.

$$(\dot{f}_{\mathcal{A}} \circ h \circ \alpha)\theta = \dot{f}_{\mathcal{A}}(h\langle \bar{x},\langle \bar{y},\bar{z}\rangle\rangle) = \dot{f}_{\mathcal{A}}\langle \bar{x} = \bar{y}, \bar{y} = \bar{z}\rangle$$
$$= ((\bar{x} = \bar{y}) \in \{\bar{y} = \bar{z}, \top\}) \wedge f_{\mathcal{A}}(\bar{y} = \bar{z})$$

$$= (((\bar{x} = \bar{y}) \leftrightarrow (\bar{y} = \bar{z})) \wedge f_{\mathcal{A}}(\bar{y} = \bar{z})) \vee ((\bar{x} = \bar{y}) \wedge f_{\mathcal{A}}(\bar{y} = \bar{z})) \quad .$$

From the definition of α in 1.7.3.2

(1) $(\bar{x} = \bar{y}) \vee (\bar{x} = \bar{z})$ and (2) $\bar{y} \leq \bar{z}$.

$f_{\mathcal{A}}(\bar{y} = \bar{z}) \leftrightarrow \exists yz((y \leq z) \wedge ((\bar{y} = \bar{z}) \leftrightarrow (y = z)))$

$\leftrightarrow \top$ by (2)

hence $(\dot{f}_{\mathcal{A}} \circ h \bullet \alpha)\theta = ((\bar{x} = \bar{y}) \leftrightarrow (\bar{y} = \bar{z})) \vee (\bar{x} = \bar{y})$

$(\bar{x} = \bar{z}) \Rightarrow ((\bar{x} = \bar{y}) \leftrightarrow (\bar{y} = \bar{z}))$ hence by (1), $\dot{f}_{\mathcal{A}} \bullet h \bullet \alpha \equiv true_{\tilde{A}}$.

Thus, since $\dot{f}_{\mathcal{A}}$ classifies $\hat{\alpha}$, there exists a morphism $k: \tilde{A} \longrightarrow \hat{A}$

such that $\hat{\alpha} \bullet k \equiv h \bullet \alpha$. Hence

$\hat{\alpha}k\theta = h\alpha\theta = h\langle\alpha_1\theta, \langle\alpha_2\theta, \alpha_3\theta\rangle\rangle = \langle\alpha_1\theta = \alpha_2\theta, \alpha_2\theta = \alpha_3\theta\rangle$.□

<u>1.7.4.4 Proposition.</u> $k: \tilde{\mathcal{A}} \longrightarrow \hat{\mathcal{A}}$ is order preserving.

<u>Proof.</u> We want to show $(\theta \leq \theta') \Rightarrow (k\theta \leq k\theta')$.

Since α and $\hat{\alpha}$ are order embeddings, this is equivalent to

$(\alpha\theta \leq \alpha\theta) \Rightarrow (\alpha k\theta \prec \alpha k\theta')$.

We suppose $\alpha\bar{\theta} \leq \alpha\bar{\theta}'$ and show $h\alpha\bar{\theta} \prec h\alpha\bar{\theta}'$. Let $\bar{x} \equiv \alpha_1\bar{\theta}$, $\bar{y} \equiv \alpha_2\bar{\theta}$, $\bar{z} \equiv \alpha_3\bar{\theta}$, $\bar{x}_1 \equiv \alpha_1\bar{\theta}'$, $\bar{y}_1 \equiv \alpha_2\bar{\theta}'$, $\bar{z}_1 \equiv \alpha_3\bar{\theta}'$. Our supposition then is equivalent to

(1) $\bar{x} \leq \bar{x}_1$ (2) $\bar{y} = \bar{y}_1$ and (3) $\bar{z} = \bar{z}_1$.

From the definition of α we have

(4) $(\bar{x} = \bar{y}) \vee (\bar{x} = \bar{z})$ (5) $\bar{y} \leq \bar{z}$

(6) $(\bar{x}_1 = \bar{y}_1) \vee (\bar{x}_1 = \bar{z}_1)$ (7) $\bar{y}_1 \leq \bar{z}_1$

By (2) and (3), (7) is equivalent to (5), and (6) to

(6)' $(\bar{x}_1 = \bar{y}) \vee (\bar{x}_1 = \bar{z})$.

We have $h\alpha\bar{\theta} \prec h\alpha\bar{\theta}'$

iff $h\langle\bar{x}, \langle\bar{y}, \bar{z}\rangle\rangle \prec h\langle\bar{x}_1, \langle\bar{y}_1, \bar{z}_1\rangle\rangle$, iff $h\langle \bar{x},\langle \bar{y},\bar{z}\rangle\rangle \prec h\langle \bar{x}_1,\langle \bar{y},\bar{z}\rangle\rangle$, by (2) and (3)

iff $\langle \bar{x} = \bar{y}, \bar{y} = \bar{z}\rangle \prec \langle \bar{x}_1 = \bar{y}, \bar{y} = \bar{z}\rangle$

iff (8) $(\bar{x}_1 = \bar{y}) \Rightarrow (\bar{x} = \bar{y})$. By (1) we have

(9) $(\bar{x}_1 = \bar{y}) \Rightarrow (\bar{x} \leq \bar{y})$, hence it will suffice to show

(10) $\bar{y} \leq \bar{x}$.

We have (11) $(\bar{x} = \bar{y}) \Rightarrow (\bar{y} \leq \bar{x})$, and

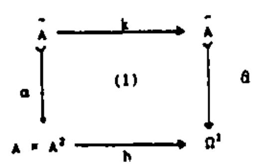
(12) $(\bar{x} = \bar{z}) \Rightarrow (\bar{y} \leq \bar{x})$ by (5)

From (11), (12) and (4) we have (10) $\bar{y} \leq \bar{x}$. Combining (9) and (10) we have (8) $(\bar{x}_1 = \bar{y}) \Rightarrow (\bar{x} = \bar{y})$

which means that $h\alpha\bar{\theta} \prec h\alpha\bar{\theta}'$. □

<u>1.7.4.5 Proposition.</u> $k\colon \tilde{A} \longrightarrow \hat{A}$ is an epi.

<u>Proof.</u> We have



commuting. Let $\theta \in V_{\tilde{A}}$, $\rho \in V_{\hat{A}}$, $\hat{\alpha}_i\colon \hat{A} \longrightarrow \Omega$ $(i \equiv 1,2)$ be the composites of $\hat{\alpha}$ with the projections so that $\hat{\alpha}\rho = \langle \hat{\alpha}_1\rho, \hat{\alpha}_2\rho\rangle$.

We want to show that $\forall\rho\exists\theta(k\theta = \rho)$. We have $\exists\theta(k\theta = \rho)$

iff $\exists\theta(h\alpha\theta = \hat{\alpha}\rho)$ from (1)

iff $\exists\theta(\exists xyz(\alpha\theta = \langle x,\langle y,z\rangle\rangle) \wedge (h\alpha\theta = \hat{\alpha}\rho))$

iff $\exists xyz(\exists\theta(\alpha\theta = \langle x,\langle y,z\rangle\rangle) \wedge (h\langle x,\langle y,z\rangle\rangle = \hat{\alpha}\rho))$

iff $\exists yz\exists x((x \in \{y,z\}) \wedge (y \leq z) \wedge (h\langle x,\langle y,z\rangle\rangle = \hat{\alpha}\rho))$

iff $\exists yz((y \leq z) \wedge [\exists x((x = y) \wedge (h\langle y,\langle y,z\rangle\rangle = \hat{\alpha}\rho))$

$\vee\ \exists x((x = z) \wedge (h\langle z,\langle y,z\rangle\rangle = \hat{\alpha}\rho))]\ )$

iff $\exists yz((y \leq z) \wedge ((h/y,/y,z// = \hat{\alpha}\rho) \vee (h/z,/y,z// = \hat{\alpha}\rho)))$.

$(h/z,/y,z// = \hat{\alpha}\rho) \leftrightarrow [((y = z) \leftrightarrow \hat{\alpha}_1\rho) \wedge ((y = z) \leftrightarrow \hat{\alpha}_2\rho)]$

$\leftrightarrow [(\hat{\alpha}_1\rho = \hat{\alpha}_2\rho) \wedge ((y = z) \leftrightarrow \hat{\alpha}_2\rho)]$

and $(h/y,/y,z// = \hat{\alpha}\rho) \leftrightarrow [((y = y) \leftrightarrow \hat{\alpha}_1\rho) \wedge ((y = z) \leftrightarrow \hat{\alpha}_2\rho]$

$\leftrightarrow [(\hat{\alpha}_1\rho = T) \wedge ((y = z) \leftrightarrow \hat{\alpha}_2\rho)]$

hence $\exists\theta(k\theta = \rho)$

iff $\exists yz((y \leq z) \wedge ((y = z) \leftrightarrow \hat{\alpha}_2\rho) \wedge (\hat{\alpha}_1\rho \in \{\hat{\alpha}_2\rho, T\}$

iff $\dot{f}_{\mathcal{A}}/\hat{\alpha}_1\rho, \hat{\alpha}_2\rho/$ iff $\dot{f}_{\mathcal{A}} \bullet \hat{\alpha} \equiv true_{\hat{A}}$, which follows from the definition of $\hat{\alpha}$ in 1.7.4.2.□

<u>1.7.4.6 Proposition.</u> k is open from $\tilde{\mathcal{A}}$ to $\hat{\mathcal{A}}$.

<u>Proof.</u> We want to show

(1) $(k\theta \leq \rho) \rightarrow \exists\theta_1((\theta \leq \theta_1) \wedge (\rho = k\theta_1))$.

Since k is an epi $\exists\theta_2(k\theta_2 = \rho)$, hence (1) holds iff

(2) $(k\theta \leq k\theta_2) \rightarrow \exists\theta_1((\theta \leq \theta_1) \wedge (k\theta_2 = k\theta_1))$ iff

(3) $(h\alpha\theta \prec h\alpha\theta_2) \rightarrow \exists\theta_1((\alpha\theta \leq \alpha\theta_1) \wedge (h\alpha\theta_2 = h\alpha\theta_1))$.

We shall suppose

(4) $h\alpha\bar{\theta} \prec h\alpha\bar{\theta}_2$

and show

(5) $\exists\theta_1((\alpha\bar{\theta} \leq \alpha\theta_1) \wedge (h\alpha\bar{\theta}_2 = h\alpha\theta_1))$.

We let $\bar{x} \equiv \alpha_1\bar{\theta}$, $\bar{y} \equiv \alpha_2\bar{\theta}$, $\bar{z} \equiv \alpha_3\bar{\theta}$, $\bar{x}_2 \equiv \alpha_1\bar{\theta}_2$, $\bar{y}_2 \equiv \alpha_2\bar{\theta}_2$, and $\bar{z}_2 \equiv \alpha_3\bar{\theta}_2$. By the definition of α we have

(6) $\bar{y} \leq \bar{z}$ (7) $(\bar{x} = \bar{y}) \vee (\bar{x} = \bar{z})$ (8) $\bar{y}_2 \leq \bar{z}_2$

(9) $(\bar{x}_2 = \bar{y}_2) \vee (\bar{x}_2 = \bar{z}_2)$.

From (4) we have $/\bar{x} = \bar{y}, \bar{y} = \bar{z}/ \prec (\bar{x}_2 = \bar{y}_2, \bar{y}_2 = \bar{z}_2)$ hence

(10) $(\bar{x}_2 = \bar{y}_2) \rightarrow (\bar{x} = \bar{y})$ and (11) $(\bar{y} = \bar{z}) \leftrightarrow (\bar{y}_2 = \bar{z}_2)$.

We shall replace (5) by an equivalent formula, (15), with a single bound variable of type A. When we combine (5) with $\exists x_1 y_1 z_1(\alpha\theta_1 = \langle x_1,\langle y_1,z_1\rangle\rangle)$ we have (5) iff

(12) $\exists x_1 y_1 z_1((\alpha\bar{\theta} \leq \langle x_1,\langle y_1,z_1\rangle\rangle) \wedge (h\alpha\bar{\theta}_2 = h\langle x_1,\langle y_1,z_1\rangle\rangle))$.

We have $(\alpha\bar{\theta} \leq \langle x_1,\langle y_1,z_1\rangle\rangle) \leftrightarrow ((\bar{x} \leq x_1) \wedge (\bar{y} = y_1) \wedge (\bar{z} = z_1))$

hence (12) iff

(13) $\exists x_1((\bar{x} \leq x_1) \wedge (h\alpha\bar{\theta}_2 = h\langle x_1,\langle \bar{y},\bar{z}\rangle\rangle))$ iff

(14) $\exists x_1((\bar{x} \leq x\ ) \wedge ((\bar{x}_2 = \bar{y}_2) \leftrightarrow (x_1 = \bar{y})) \wedge ((\bar{y}_2 = \bar{z}_2) \leftrightarrow (\bar{y} = \bar{z})))$.

By (11) we have (14) iff

(15) $\exists x_1((\bar{x} \leq x_1) \wedge (\bar{x}_2 = \bar{y}_2) \leftrightarrow (\bar{x}_1 = \bar{y})))$.

Now let $\psi \equiv (\bar{x} \leq x_1) \wedge ((\bar{x}_2 = \bar{y}_2) \leftrightarrow (x_1 = \bar{y}))$,

$\varphi_1 \equiv (\bar{x} = \bar{y}) \wedge (\bar{x}_2 = \bar{y}_2)$, $\varphi_2 \equiv (\bar{x} = \bar{y}) \wedge (\bar{x}_2 = \bar{z}_2)$,

$\varphi_3 \equiv (\bar{x} = \bar{z}) \wedge (\bar{x}_2 = \bar{y}_2)$, $\varphi_4 \equiv (\bar{x} = \bar{z}) \wedge (\bar{x}_2 = \bar{z}_2)$.

By taking the conjunction of (7) and (9) we get (16) $\bigvee_{i=1}^{4} \varphi_i$.

We shall show

(17) $\varphi_i \rightarrow \exists x_1\psi$ for $1 \leq i \leq 4$, by first proving

(18) $\varphi_1 \rightarrow \psi[x_1|\bar{y}]$

(19) $\varphi_2 \rightarrow \psi[x_1|\bar{z}]$

(20) $\varphi_3 \rightarrow \psi[x_1|\bar{y}]$

(21) $\varphi_4 \rightarrow \psi[x_1|\bar{z}]$

We have $\psi[x_1|\bar{y}] \leftrightarrow (\bar{x} \leq \bar{y}) \wedge (\bar{x}_2 = \bar{y}_2)$

$\leftrightarrow (\bar{x}_2 = \bar{y}_2)$ by (10)

and $\psi[x_1|\bar{z}] \leftrightarrow (\bar{x} \leq \bar{z}) \wedge ((\bar{x}_2 = \bar{y}_2) \leftrightarrow (\bar{z} = \bar{y}))$

$\leftrightarrow (\bar{x} \leq \bar{z}) \wedge ((\bar{x}_2 = \bar{y}_2) \leftrightarrow (\bar{z}_2 = \bar{y}_2))$ by (11).

We now establish (18), (19), (20) and (21).

(18): $[(\bar{x} = \bar{y}) \wedge (\bar{x}_2 = \bar{y}_2) \Rightarrow (\bar{x}_2 = \bar{y}_2)$.

(19) $(\bar{x} = \bar{y}) \Rightarrow (\bar{x} \leq \bar{y})$ by (6), and $(\bar{x}_2 = \bar{z}_2) \Rightarrow ((\bar{x}_2 = \bar{y}_2) \Leftrightarrow (\bar{z}_2 = \bar{y}_2))$

hence $((\bar{x} = \bar{y}) \vee (\bar{x} = \bar{z})) \Rightarrow ((\bar{x} \leq \bar{y}) \vee ((\bar{x}_2 = \bar{y}_2) \Leftrightarrow (\bar{z}_2 = \bar{y}_2)))$.

(20) $[(\bar{x} = \bar{z}) \wedge (\bar{x}_2 = \bar{y}_2)] \Rightarrow (\bar{x}_2 = \bar{y}_2)$.

(21) $(\bar{x} = \bar{z}) \Rightarrow (\bar{x} \leq \bar{z})$

$(\bar{x}_2 = \bar{z}_2) \Rightarrow ((\bar{x}_2 = \bar{y}_2) \Leftrightarrow (\bar{z}_2 = \bar{y}_2))$.

Now $\psi \Rightarrow \exists x_1\psi$, hence $\psi[x_1|\bar{z}] \Rightarrow \exists x_1\psi$ and $\psi[x_1|\bar{y}] \Rightarrow \exists x_1\psi$ hence

$\bigvee_{i=1}^{4} \varphi_i \Rightarrow \exists x_1\psi$, hence $\exists x_1\psi$.□

1.7.4.7 Proposition. For each partially ordered object A and each $\varphi \in \mathrm{Poly}(H)$, if $\underset{\sim}{\mathrm{Idl}}\, \tilde{A} \models \varphi = \underline{1}$ then $\underset{\sim}{\mathrm{Idl}}\, \hat{A} \models \varphi = \underline{1}$.

Proof. By 1.7.4.4, 1.7.4.6 and 1.7.1.6, k^* is an H-homomorphism from $\underset{\sim}{\mathrm{Idl}}\, \hat{A}$ to $\underset{\sim}{\mathrm{Idl}}\, \tilde{A}$. Since, by 1.7.4.5, k is an epi, by 1.7.2.2.6, f^* is a mono. Hence by 0.6.17.7, the result follows.□

1.7.4.8 Proposition. If $\tilde{A}$ is discrete then $\hat{A}$ is discrete.

Proof. $(\rho_1 \leq \rho_2) \Leftrightarrow [(\exists\theta_1(k\theta_1 = \rho_1)) \wedge (\rho_1 \leq \rho_2)$ since k is an epi

$\Leftrightarrow \exists\theta_1((k\theta_1 = \rho_1) \wedge (k\theta_1 \leq \rho_2))$

$\Leftrightarrow \exists\theta_1((k\theta_1 = \rho_1) \wedge \exists\theta_2((\theta_1 \leq \theta_2) \wedge (k\theta_2 = \rho_2)))$,

since k is continuous and open, hence

$(\rho_1 \leq \rho_2) \Leftrightarrow \exists\theta_1\theta_2((k\theta_1 = \rho_1) \wedge (k\theta_2 = \rho_2) \wedge (\theta_1 \leq \theta_2))$.

Suppose $\tilde{A}$ is discrete. We have $(\theta_1 \leq \theta_2) \Rightarrow (\theta_1 = \theta_2)$ hence

$(k\theta_1 = \rho_1) \wedge (k\theta_2 = \rho_2) \wedge (\theta_1 \leq \theta_2) \Rightarrow (\rho_1 = \rho_2)$ hence

$(\rho_1 \leq \rho_2) \Rightarrow (\rho_1 = \rho_2)$.□

1.7.4.9 Corollary. $\Gamma_3(\mathcal{E}) \subset \Gamma_2(\mathcal{E})$.

Proof. Suppose $\varphi \in \Gamma_3(\mathcal{E})$. Let A be a partially ordered structure for which $\underset{\sim}{\mathrm{Idl}}\, \tilde{A} \models \varphi = \underline{1}$. By 1.7.4.7, $\underset{\sim}{\mathrm{Idl}}\, \hat{A} \models \varphi = \underline{1}$. Hence $\hat{A}$ is discrete. By 1.7.4.8, $\tilde{A}$ is discrete.□

<u>1.7.4.10 Theorem.</u> $\mathcal{C}^{\to} \cap \Gamma_1(\underline{\mathcal{E}}) \equiv \mathcal{C}^{\to} \cap \Gamma_3(\underline{\mathcal{E}})$.

<u>Proof.</u> We have, from the definition of $\Gamma_3(\underline{\mathcal{E}})$ (1.7.4.1.1), $\mathcal{C}^{\to} \cap \Gamma_1(\underline{\mathcal{E}}) \subset \mathcal{C}^{\to} \cap \Gamma_3(\underline{\mathcal{E}})$. Combining 1.7.3.9 and 1.7.4.9 we get $\mathcal{C}^{\to} \cap \Gamma_3(\underline{\mathcal{E}}) \subset \mathcal{C}^{\to} \cap \Gamma_1(\underline{\mathcal{E}})$.□

<u>1.7.4.11 The set $\Gamma_4(\underline{\mathcal{E}}) \subset \text{Poly}(H)$.</u>

The definitions of $\Gamma_i(\underline{\mathcal{E}})$ ($i \equiv 1,2,3$) each contain a reference to an arbitrary partially ordered structure $\mathcal{A}$. In the case of $\Gamma_2(\underline{\mathcal{E}})$ and $\Gamma_3(\underline{\mathcal{E}})$ the structures $\tilde{\mathcal{A}}$ and $\hat{\mathcal{A}}$ are special kinds of partial orders but we have not yet described a smaller class into which either falls which is free of any reference to structures which may be "arbitrarily large". We now define such a class based in the construction, II, of 1.7.4.1. We take $\Gamma_4(\underline{\mathcal{E}})$ to be the set of all $\varphi \in \text{Poly}(H)$ such that for all endomorphisms $f: \Omega \longrightarrow \Omega$, if $\underset{\sim}{\text{Idl}}\, \mathcal{D}(f) \models \varphi = \underline{1}$ then $\mathcal{D}(f)$ is discrete.

<u>1.7.4.12 Theorem.</u> $\mathcal{C}^{\to} \cap \Gamma_1(\underline{\mathcal{E}}) \equiv \mathcal{C}^{\to} \cap \Gamma_4(\underline{\mathcal{E}})$.

<u>Proof.</u> ($\subset$): It is clear that $\Gamma_1(\underline{\mathcal{E}}) \subset \Gamma_4(\underline{\mathcal{E}})$, and hence $\mathcal{C}^{\to} \cap \Gamma_1(\underline{\mathcal{E}}) \subset \mathcal{C}^{\to} \cap \Gamma_4(\underline{\mathcal{E}})$.□

($\supset$): Let $\varphi \in \Gamma_4(\underline{\mathcal{E}})$ and let $\mathcal{A}$ be a partially ordered structure such that $\underset{\sim}{\text{Idl}}\, \mathcal{A} \models \varphi = \underline{1}$. Since $\mathcal{D}(f_{\mathcal{A}}) \equiv \hat{\mathcal{A}}$, $\hat{\mathcal{A}}$ is discrete. Hence $\varphi \in \Gamma_3(\underline{\mathcal{E}})$. Thus $\Gamma_4(\underline{\mathcal{E}}) \subset \Gamma_3(\underline{\mathcal{E}})$, and hence $\mathcal{C}^{\to} \cap \Gamma_4(\underline{\mathcal{E}}) \subset \mathcal{C}^{\to} \cap \Gamma_3(\underline{\mathcal{E}}) \equiv \mathcal{C}^{\to} \cap \Gamma_1(\underline{\mathcal{E}})$.□

1.7.5 Internalizing the definition of the set $\Gamma_4(\underline{\mathcal{E}})$.

In this section we shall show that the property that an H-polynomial φ, is required to have in order that it belong to $\Gamma_4(\underline{\mathcal{E}})$ is equivalent to the validity in $\underline{\mathcal{E}}$ of a formula $f_\varphi p \Rightarrow p$, where $f_\varphi \colon \Omega \longrightarrow \Omega$.

We return to the construction, II, of 1.7.4.1. Define $\partial \colon [\Omega,\Omega] \longrightarrow [\Omega^2,\Omega]$ by $\partial(f) \equiv \dot{f}$ for each $f \in [\Omega,\Omega]$.

<u>1.7.5.1 Proposition.</u> $\partial \colon [\Omega,\Omega_{\underline{\leq}}] \longrightarrow [\Omega^2,\Omega_{\underline{\leq}}]$ is an order embedding.

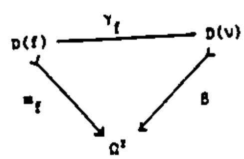
<u>Proof.</u> $\dot{f} \leq \dot{g}$

iff $((q \in \{p,\top\}) \wedge fp) \Rightarrow ((q \in \{p,\top\}) \wedge gp)$

iff $\exists q(q \in \{p,\top\}) \Rightarrow (fp \leq gp)$

iff $f \leq g$.□

<u>1.7.5.1.1</u> We put $\nu \equiv \mathrm{true}_\Omega$, the top element of $[\Omega,\Omega_{\underline{\leq}}]$, and $\beta \equiv m_\nu$. For any $f \colon \Omega \longrightarrow \Omega$ we have $\dot{f} \leq \dot{\nu}$ hence $[\![m_f]\!] \leq$ [illegible]$]\!]$, hence there is a uniquely determined morphism, γ_f, such that



commutes.

<u>1.7.5.2 Proposition.</u> γ_f is an order embedding from $\mathcal{D}(f)$ to $\mathcal{D}(\nu)$.

<u>Proof.</u> $(\gamma_f\theta \leq \gamma_f\theta') \Leftrightarrow (\beta\gamma_f\theta \prec \beta\gamma_f\theta')$

$\Leftrightarrow (m_f\theta \prec m_f\theta')$

$\Leftrightarrow (\theta \leq \theta')$. □

We let $k_f \colon D(\nu) \longrightarrow \Omega$ be the characteristic morphism of $\gamma_f \colon D(f) \rightarrowtail D(\nu)$.

1.7.5.3 Proposition. (1) $(k_f\rho \wedge (\rho \leq \rho')) \Rightarrow k_f\rho'$

Proof. $k_f\rho \Leftrightarrow \exists\theta(\gamma_f\theta = \rho) \Leftrightarrow \exists\theta(m_f\theta = \beta\rho)$

$\Leftrightarrow \exists\theta(m_f\theta = \langle\beta_1\rho,\beta_2\rho\rangle)$

$\Leftrightarrow (\beta_1\rho \in \{\beta_2\rho,\top\}) \wedge f\beta_2\rho$

$\Leftrightarrow f\beta_2\rho$ since $\beta_1\rho \in \{\beta_2\rho,\top\}$

and $(\rho \leq \rho') \Leftrightarrow (\beta\rho \prec \beta\rho')$

$\Leftrightarrow \langle\beta_1\rho,\beta_2\rho\rangle \prec \langle\beta_1\rho',\beta_2\rho'\rangle$

$\Leftrightarrow (\beta_1\rho' \leq \beta_1\rho) \wedge (\beta_2\rho = \beta_2\rho')$

hence $(k_f\rho \wedge (\rho \leq \rho')) \Rightarrow f\beta_2\rho \wedge (\beta_2\rho = \beta_2\rho')$

$\Rightarrow f\beta_2\rho'$

$\Rightarrow k_f\rho'$. □

1.7.5.4 Corollary. γ_f is continuous and open from $\mathcal{D}(f)$ to $\mathcal{D}(\nu)$ for each $f\colon \Omega \longrightarrow \Omega$.

Proof. By 1.7.5.3 and 1.7.1.13.□

1.7.5.5 Definition of the set $\Gamma_5(\mathcal{E}) \subset \mathrm{Poly}(H)$.

The morphism idl: $\Omega^{D(\nu)} \longrightarrow \Omega$, determined by $\mathcal{D}(\nu)$, classifies a monomorphism

$$\eta\colon \ \mathrm{Idl}\,\mathcal{D}(\nu) \rightarrowtail \Omega^{D(\nu)}.$$

We define

$$\mu\colon (\mathrm{Idl}\,\mathcal{D}(\nu)) \times \Omega \longrightarrow \Omega$$

by $\mu\langle K,p\rangle = (\beta^{-1}(\{p,\top\} \times \{p\}) \subset \eta K)$.

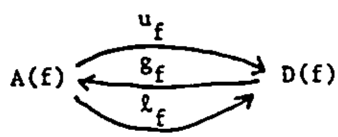
Let $\vec{v} \equiv \mathrm{var}\,\varphi$ where $\varphi \in \mathrm{Poly}\ H$. Let $\tilde{\varphi}$ be its interpretation in Idl $\mathcal{D}(\nu)$ and let $\vec{\vec{v}}$ be the interpretation of the string of distinct variables, $\vec{v}$, as a string of distinct variables of type Idl $\mathcal{D}(\nu)$, so that $\vec{\vec{v}} \equiv \mathrm{var}\,\tilde{\varphi}$.

Define $f_\varphi: \Omega \longrightarrow \Omega$ by $f_\varphi p = \forall\vec{v}\mu(\vec{\varphi},p)$; $\Gamma s(\mathcal{E})$ is the set of all $\varphi \in \mathrm{Poly}(H)$ such that $(f_\varphi p) \Rightarrow p$ is valid in $\mathcal{E}$.

<u>1.7.5.6</u> The function ∂ corresponds to an order embedding

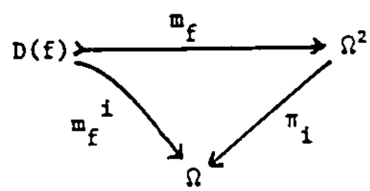
$$\partial': \mathrm{Sub}(\Omega) \longrightarrow \mathrm{Sub}(\Omega^2).$$

Let $f: \Omega \longrightarrow \Omega$ and let $A(f) \overset{\alpha_f}{\rightarrowtail} \Omega$ be any mono classified by f. We will define morphisms



such that if $A(f)$ is given the discrete order then, with respect to $\mathcal{D}(f)$, we have $\ell_f \dashv g_f \dashv u_f$.

<u>1.7.5.7 Definition of g_f.</u> Define $m_f^{\,i}$ $(i \equiv 1,2)$ via the projections

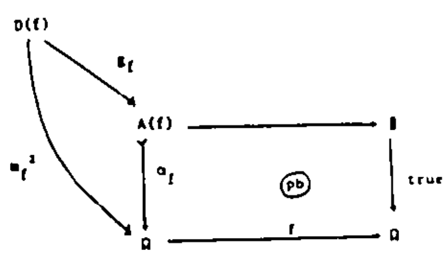


Since $\dot{f}$ classifies m_f we have $\mathrm{true}_{D(f)} \equiv \dot{f} \bullet m_f$ hence

$= \dot{f}(m_f^{\,1}\theta, m_f^{\,2}\theta) = (m_f^{\,1}\theta \in \{m_f^{\,2}\theta, T\}) \wedge (f m_f^{\,2}\theta)$

hence (1) $m_f^{\,1}\theta \in \{m_f^{\,2}\theta, T\}$ and (2) $f \bullet m_f^{\,2} \equiv \mathrm{true}_{D(f)}$.

Since f classifies α_f there exists exactly one morphism g_f such that (3)' commutes



hence (3) $\alpha_f g_f \theta = m_f{}^2\theta$.

<u>1.7.5.8 Definition of u_f and ℓ_f.</u> Since f classifies α_f we have

(4) $f\alpha_f x$.

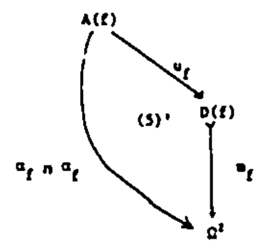
By definition of $\dot{f}$ we have $\dot{f}\langle q,\alpha_f x\rangle \leftrightarrow (q \in \{\alpha_f x, \top\}) \wedge f\alpha_f x$

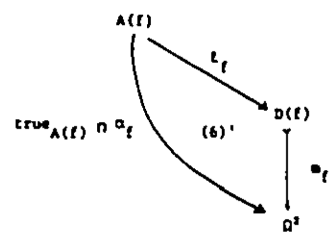
$$\leftrightarrow (q \in \{\alpha_f x, \top\})$$

hence $\dot{f}\langle \alpha_f x, \alpha_f x\rangle \leftrightarrow \top \leftrightarrow \dot{f}\langle \top, \alpha_f x\rangle$.

hence $\dot{f}\bullet(\alpha_f \cap \alpha_f) \equiv true_{A(f)} \equiv \dot{f}\bullet(true_{A(f)} \cap \alpha_f)$

hence there exist morphisms u_f and ℓ_f such that (5)' and (6)' commute





hence (5) $m_f u_f x = \langle \alpha_f x, \alpha_f x\rangle$ and (6) $m_f \ell_f x = \langle \top, \alpha_f x\rangle$.

<u>1.7.5.9 Proposition.</u> $\ell_f \dashv g_f \dashv u_f$ with respect to the discrete ordering on $A(f)$ and the partially ordered structure $\mathcal{D}(f)$; that is

$$(\ell_f x \leq \theta) \leftrightarrow (x = g_f\theta) \leftrightarrow (\theta \leq u_f x).$$

<u>Proof.</u> $(\ell_f x \leq \theta) \leftrightarrow (m_f \ell_f x \prec m_f\theta)$

$$\leftrightarrow (m_f{}^1\theta \leq m_f{}^1\ell_f x) \wedge (m_f{}^2\theta = m_f{}^2\ell_f x)$$

$$\leftrightarrow (m_f{}^2\theta = \alpha_f x) \qquad \text{by (6)}$$

$$\leftrightarrow (\alpha_f g_f\theta = \alpha_f x) \qquad \text{by (3)}$$

$$\leftrightarrow (x = g_f\theta)$$

hence $\ell_f \dashv g_f$.

$(\theta \leq u_f x) \Leftrightarrow (m_f\theta \prec m_f u_f x)$

$\Leftrightarrow (m_f{}^1 u_f x \leq m_f{}^1\theta) \wedge (m_f{}^2 u_f x = m_f{}^2\theta)$

$\Leftrightarrow (\alpha_f x \leq m_f{}^1\theta) \wedge (\alpha_f x = m_f{}^2\theta)$ by (5)

$\Leftrightarrow (m_f{}^2\theta \leq m_f{}^1\theta) \wedge (\alpha_f x = \alpha_f g_f \theta)$ by (3)

$\Leftrightarrow (g_f\theta = x)$ by (1)

hence $g_f \dashv \ell_f$.□

1.7.5.10 Corollary.

(1) $g_f \ell_f x = x = g_f u_f x$

(2) $(\theta \leq \ell_f x) \Leftrightarrow (\theta = \ell_f x)$

(3) $(u_f x \leq \theta) \Leftrightarrow (\theta = u_f x)$.

Proofs. (1) by substituting $\ell_f x$ for θ and $u_f x$ for θ. □

(2) $(\theta \leq \ell_f x) \Rightarrow g_f\theta = g_f \quad = g_f\ell_f x$, since g_f is order preserving

$\Rightarrow g_f\theta = x$ by (1)

$\Rightarrow \ell_f x \leq \theta$

$\Rightarrow (\theta = \ell_f x)$.□

(3) similar to (2).□

1.7.5.11 Proposition. Given $f:\Omega \longrightarrow \Omega$ we have

(1) $(W \subset (\{m_f{}^2\theta, T\} \times \{m_f{}^2\theta\})) \Rightarrow (\exists_{m_f} m_f{}^{-1}(W) = W)$.

Proof. Let $m \equiv m_f$ and $m_2 \equiv m_f{}^2$. Since $\exists_m \dashv m^{-1}$ we have

$\exists_m m^{-1}(W) \subset W$.

$W \subset \exists_m m^{-1}(W)) \Leftrightarrow \forall pq\ (\langle p,q\rangle \in W) \Rightarrow \exists\theta'((m\theta' = \langle p,q\rangle) \wedge (m\theta' \in W))$

$\Leftrightarrow \forall pq\ (\langle p,q\rangle \in W) \Rightarrow \dot{f}\langle p,q\rangle$.

Hence (1) is equivalent to

(2) $(W \subset (\{m_2\theta, T\} \times \{m_2\theta\})) \wedge (\langle p,q\rangle \in W) \Rightarrow \dot{f}\langle p,q\rangle$.

We have $\dot{f}\langle m_2\theta, m_2\theta\rangle$ and $\dot{f}\langle \top, m_2\theta\rangle$, hence

$\langle p,q\rangle \in (\{m_2\theta\ , \top\} \times \{m_2\theta\})$

$\Rightarrow (\langle p,q\rangle = \langle m_2\theta, m_2\theta\rangle) \vee (\langle p,q\rangle = \langle \top, m_2\theta\rangle)$

$\Rightarrow \dot{f}\langle p,q\rangle$, hence (2) holds.□

<u>1.7.5.12 Corollary.</u> For any $f\colon \Omega \longrightarrow \Omega$ we have

(1) $\exists_{m_f} m_f{}^{-1}(\{m_f{}^2\theta, \top\} \times \{m_f{}^2\theta\}) = \{m_f{}^2\theta, \top\} \times \{m_f{}^2\theta\}$

(2) $(W \subset (\{\alpha_f x, \top\} \times \{\alpha_f x\})) \Rightarrow (\exists_{m_f} m_f{}^{-1}(W) = W)$.

By taking $f \equiv true_\Omega$ we have $\alpha_f \equiv id_\Omega$ and $m_f \equiv \beta$; this gives us

(3) $(W \subset (\{\beta_2\theta, \top\} \times \{\beta_2\theta\})) \Rightarrow (\exists_\beta \beta^{-1}(W) = W)$

(4) $\exists_\beta \beta^{-1}(\{\beta_2\theta, \top\} \times \{\beta_2\theta\}) = \{\beta_2\theta, \top\} \times \{\beta_2\theta\}$

(5) $(W \subset (\{p, \top\} \times \{p\})) \Rightarrow (\exists_\beta \beta^{-1}(W) = W)$

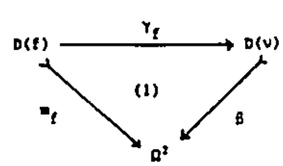
(6) $\exists_\beta \beta^{-1}(\{p, \top\} \times \{p\}) = \{p, \top\} \times \{p\}$

<u>Proof.</u> (1) From 1.7.5.11 (1).□

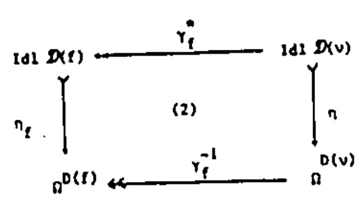
(2) Substitute $u_f x$ for θ in 1.7.5.12 (1) and use 1.7.5.8 (5).□

The remaining statements arise from 1.7.5.11 (1), and (1) and (2) above, by specializing to $f \equiv true_\Omega$.□

<u>1.7.5.13</u> Let $f \in [\Omega, \Omega]$. We have commutative diagrams



and



we put $\gamma \equiv \gamma_f$, $m \equiv m_f$, $m_2 \equiv m_f^{\,2}$, $m_1 \equiv m_f^{\,1}$.

Let $\varphi \in \mathrm{Poly}(H)$ and let $\tilde{\tilde{\varphi}}$ and $\tilde{\varphi}$ be the interpretations of φ in Idl $\mathcal{D}(f)$ and Idl $\mathcal{D}(\nu)$ respectively. Let $\vec{v} \equiv \mathrm{var}\,\varphi$; let $\vec{K}$ and $\vec{A}$ be the interpretations of $\vec{v}$ as strings of variables of types Idl $\mathcal{D}(f)$ and Idl $\mathcal{D}(\nu)$ respectively,and let K_i and A_i be the i-th variable of $\vec{K}$ and $\vec{A}$ respectively., with this notation we have:

1.7.5.14 Proposition. The following are equivalent

(1) Idl $\mathcal{D}(f) \models \varphi = \underline{1}$

(2) $\theta \in \gamma_f^{-1}\eta\,\varphi$

(3) $\{m_f^{\,2}\theta, T\} \times \{m_f^{\,2}\theta\} \subset \exists_\beta\, \eta\tilde{\varphi}$

(4) $f \leq f_\varphi$.

Proof. Abbreviate γ_f to γ , m_f to m, $m_f^{\,1}$ to m_1 and $m_f^{\,2}$ to m_2.

(1) → (2): $\tilde{\tilde{\varphi}} = \tilde{\tilde{1}}$, hence

$$\left(\mathop{\mathsf{S}}_i \begin{pmatrix} K_i \\ \gamma^* A_i \end{pmatrix}\right)(\tilde{\varphi}) = \tilde{\tilde{\underline{1}}}$$

since γ^* is a homomorphism from Idl $\mathcal{D}(\nu)$ to Idl $\mathcal{D}(f)$,

$$\gamma^*\tilde{\varphi} = \left(\mathop{\mathsf{S}}_i \begin{pmatrix} i \\ \gamma^* A_i \end{pmatrix}\right)(\tilde{\varphi}) \quad \text{hence } \gamma^*\tilde{\varphi} = \tilde{\tilde{\underline{1}}} \text{ , hence } \gamma^{-1}\eta\tilde{\varphi} = \eta_f\gamma^*\tilde{\varphi} = \eta_f\tilde{\tilde{\underline{1}}} \text{ ;}$$

since $\eta_f\tilde{\tilde{\underline{1}}} = \{\theta : T\}$, $\theta \in \gamma^{-1}\eta\tilde{\varphi}$.□

(2) → (1): $\theta \in \gamma^{-1}\eta\tilde{\varphi}$, hence $\eta_f\gamma^*\tilde{\varphi} = \gamma^{-1}\eta\tilde{\varphi} = \eta_f\tilde{\tilde{1}}$, hence $\gamma^*\tilde{\varphi} = \tilde{\tilde{1}}$,

hence $\left(\mathop{\mathsf{S}}_i \begin{pmatrix} K_i \\ \gamma^* A_i \end{pmatrix}\right)(\tilde{\varphi}) = \tilde{\tilde{1}}$, hence $\bigwedge_i (K_i = \gamma^* A_i)) \Rightarrow (\tilde{\varphi} = \tilde{\tilde{1}})$.

Since γ^* is an epi $\exists A_i(K_i = \gamma^* A_i)$, hence $\exists\vec{A}(\bigwedge_i (K_i = \gamma^* A_i))$, hence $\tilde{\tilde{\varphi}} = \tilde{\tilde{1}}$.□

(2) → (3). $\theta \in \gamma^{-1}\eta^{\sim}$, hence $m^{-1}(\{m_2\theta ,\top\} \times \{m_2\theta\}) \subset \gamma^{-1}\eta\tilde{\varphi}$

hence $\exists_m m^{-1}(\{m_2\theta,\top\} \times \{m_2\theta\}) \subset \exists_m \gamma^{-1}\eta\tilde{\varphi}$, hence, by 1.7.5.12 (1),

$(\{m_2\theta,\top\} \times \{m_2\theta\}) \subset \exists_m \gamma^{-1}\eta\tilde{\varphi}$.

Since $\exists_m \gamma^{-1}\eta\tilde{\varphi} = \exists_\beta(\exists_\gamma \gamma^{-1}(\eta\tilde{\varphi})) \subset \exists_\beta(\eta\tilde{\varphi})$, we have

$(\{m_2\theta,\top\} \times \{m_2\theta\}) \subset \exists_\beta \eta\tilde{\varphi}$

(3) → (2). We have $\{m_2\theta,\top\} \times \{m_2\theta\} \subset \exists_\beta \eta\tilde{\varphi}$. Since $m_1\theta \in \{m_2\theta,\top\}$,

$m\theta \in (\{m_2\theta,\top\} \times \{m_2\theta\})$. Hence $m\theta \in \exists_\beta \eta\tilde{\varphi}$, hence $\theta \in m^{-1}\exists_\beta \eta\tilde{\varphi}$.

Since $m \equiv \beta\circ\gamma$, $\exists_m \equiv \exists_\beta \circ \exists_\gamma$, hence

$\exists_\beta(\exists_\gamma(m^{-1}\exists_\beta \eta^{\sim})) = \exists_m m^{-1}(\exists_\beta \eta\tilde{\varphi}) \subset \exists_\beta(\eta\tilde{\varphi})$. Since $\exists_\beta$ is an order embedding

$\exists_\gamma(m^{-1}\exists_\beta \eta\tilde{\varphi}) \subset \eta\tilde{\varphi}$, hence $m^{-1}\exists_\beta \eta\tilde{\varphi} \subset \gamma^{-1}\eta\tilde{\varphi}$, hence $\theta \in \gamma^{-1}\eta\tilde{\varphi}$.□

(4) ↔ (3) $f \leq f_\varphi$

iff $(\dot{f}p) \Rightarrow (\dot{f}_\varphi p)$ since ∂ is an order embedding

iff $\dot{f}_\varphi\, m\theta$ since $\dot{f}$ classifies m,

iff $(m_1\theta \in \{m_2\theta,\top\}) \wedge (f_\varphi\, m_2\theta)$

iff $\mu\langle\tilde{\varphi},m_2\theta\rangle$

iff $\beta^{-1}(\{m_2\theta,\top\} \times \{m_2\theta\}) \subset \eta\tilde{\varphi}$

iff $\exists_\beta(\beta^{-1}(\{m_2\theta,\top\} \times \{m_2\theta\})) \subset \exists_\beta \eta\tilde{\varphi}$ since $\exists_\beta$ is an order embedding

iff $(\{m_2\theta,\top\} \times \{m_2\theta\}) \subset \exists_\beta \eta\tilde{\varphi}$ by 1.7.5.12 (6).□

<u>1.7.5.15 Proposition.</u> $\mathcal{D}(f)$ is discrete iff $f \leq id_\Omega$.

<u>Proof.</u> (←): Suppose $fp \leq p$. Let $m_f^{\,i}: D(f) \longrightarrow \Omega$ $(i \equiv 1,2)$ be

such that $m_f\theta = \langle m_f^{\,1}\theta, m_f^{\,2}\theta\rangle$, put $m \equiv m_f$, $m_i \equiv m_f^{\,i}$.

$(\theta \leq \theta') \leftrightarrow (\langle m_1\theta,m_2\theta\rangle \prec \langle m_1\theta',m_2\theta'\rangle)$

$\leftrightarrow ((m_1\theta' \leq m_1\theta) \wedge (m_2\theta = m_2\theta'))$

Since $\dot{f}\circ m \equiv true_{D(f)}$, we have $m_1\theta \in \{m_2\theta,\top\}$ and $fm_2\theta$. Since

$fm_2\theta \Rightarrow m_2\theta$ we have $m_2\theta = \top$, hence $m_1\theta = \top$, hence

$(m_1\theta' \leq m_1\theta) \Rightarrow (m_1\theta' = m_1\theta)$, hence $(\theta \leq \theta') \Rightarrow (\theta = \theta')$.□

(⇒) Suppose $\mathcal{D}(f)$ is discrete. We have $\dot{f}\ulcorner\top,p\urcorner \Leftrightarrow fp$ and $\dot{f}\ulcorner p,p\urcorner \Rightarrow fp$

hence $fp \Rightarrow (\dot{f}\ulcorner p,p\urcorner \wedge \dot{f}\ulcorner\top,p\urcorner)$

$\Rightarrow \exists\theta\theta'((m = \ulcorner p,p\urcorner) \wedge (m\theta' = \ulcorner\top,p\urcorner))$

$\Leftrightarrow \exists\theta\theta'((m = \ulcorner p,p\urcorner) \wedge (m\theta' = \ulcorner\top,p\urcorner) \wedge (m\theta' \prec m\theta))$

but $(m\theta' \prec m\theta) \Rightarrow (\theta' \leq \theta)$

$\Rightarrow (\theta' = \theta)$

$\Rightarrow (m\theta' = m\theta)$

hence $fp \Rightarrow \exists\theta\theta'((m\theta = \ulcorner p,p\urcorner) \wedge (m\theta' = \ulcorner\top,p\urcorner) \wedge p)$

hence $fp \Rightarrow p$, hence $f \leq id_\Omega$.□

<u>1.7.5.16 Theorem.</u> $\Gamma_4(\underline{\mathcal{E}}) \equiv \Gamma_5(\underline{\mathcal{E}})$.

<u>Proof.</u> By 1.7.5.14 and 1.7.5.15, $\varphi \in \Gamma_4(\underline{\mathcal{E}})$ iff $\varphi \in$ Poly(H) and for all $f: \Omega \longrightarrow \Omega$, if $f \leq f_\varphi$ then $f \leq id_\Omega$. Thus

$\varphi \in \Gamma_4(\underline{\mathcal{E}})$ iff $\varphi \in$ Poly(H) and $f_\varphi \leq id_\Omega$

iff $\varphi \in \Gamma_5(\underline{\mathcal{E}})$.□

1.7.6 Simplification of the formula $(f_\varphi p) \Rightarrow p$.

1.7.6.1 We shall introduce another equivalent formula of the language of the topos $\underline{\mathcal{E}}$, which contains only propositional variables and is built up from them using the logical connectives $\top$, $\bot$, $\wedge$, $\vee$, and $\Rightarrow$, and the quantifier $\forall$.

1.7.6.1.1 We fix disjoint infinite subsets V_1 and V_2 of V, a bijection from V_1 to V_2, and a variable $v \in V - (V_1 \cup V_2)$. For each $u \in V_1$ we let $\ddot{u} \in V_2$ be the corresponding variable in V_2. For each $\varphi \in \text{Poly}\, V_1$ we define $\varphi' \in \text{Poly}(V_1 \cup V_2 \cup \{v\})$ inductively:

$u' \equiv (\ddot{u} \vee v) \wedge u$

$\underline{0}' \equiv \underline{0}$

$\underline{1}' \equiv \underline{1}$

$(\varphi \wedge \psi)' \equiv \varphi' \wedge \psi'$

$(\varphi \vee \psi)' \equiv \varphi' \vee \psi'$

$(\varphi \Rightarrow \psi)' \equiv (\varphi' \Rightarrow \psi') \wedge (\varphi \Rightarrow \psi)$

For any string $\vec{u}$ of distinct variable from V_1, we let $\ddot{\vec{u}}$ be the corresponding string of variables from V_2, so that if u_i is the i-th variable of $\vec{u}$ then $\ddot{u}_i$ is the i-th variable of $\ddot{\vec{u}}$.

1.7.6.1.2 We interpret polynomials and strings of distinct variables from V in a topos $\underline{\mathcal{E}}$. For each $\psi \in \text{Poly}\, H$. We let $\tilde{\psi}$ and $\bar{\psi}$ be the interpretations of ψ in the algebras $\underset{\sim}{\text{Idl}}\, \mathcal{D}(\nu)$ and $\underset{\sim}{\Omega}$ respectively, and for each string of distinct variables from $\vec{v}$, we let $\tilde{\vec{v}}$ and $\bar{\vec{v}}$ be the interpretations of $\vec{v}$ as strings of variables of types $\text{Idl}\, \mathcal{D}(\nu)$ and Ω respectively.

1.7.6.1.3 We can now identify the "propositional" formula equivalent to $(f_\varphi p) \Rightarrow p$. Given $\varphi \in \text{Poly } V_1$, let $\vec{v} \equiv \text{var}(\varphi)$, $\vec{r} \equiv \vec{\bar{v}}$, $\vec{s} \equiv \vec{\bar{\bar{v}}}$ and $p \equiv \bar{v}$. We shall show

$$f_\varphi p \Leftrightarrow \forall \vec{r}\ \vec{s}\ \overline{\varphi'}$$

and hence $((f_\varphi p) \Rightarrow p) \Leftrightarrow ((\forall \vec{r}\ \vec{s}\ \overline{\varphi'}) \Rightarrow p)$.

1.7.6.2 Notation for morphisms associated with $\mathcal{D}(\nu)$.

The identity morphism $id_\Omega : \Omega \longrightarrow \Omega$ is classified by $\nu \equiv true_\Omega : \Omega \longrightarrow \Omega$, so we may take $id_\Omega \equiv \alpha_\nu$ and $\Omega \equiv A(\nu)$ in 1.7.5.7. We have put $\beta \equiv m_\nu$ in 1.7.5.1.1; we put, $\beta_1 \equiv m_\nu{}^1 \equiv \pi_1 \circ \beta$ and $\beta_2 \equiv m_\nu{}^2 \equiv \pi_2 \circ \beta$ (from 1.7.5.7), thus:

$$\beta_2 \equiv g_\nu .$$

We drop subscripts in 1.7.5.8:

$$u \equiv u_\nu \qquad \text{and} \qquad \ell \equiv \ell_\nu$$

then by 1.7.5.8, (5) and (6), we have

$$\beta u p = \langle p,p \rangle \quad \text{and} \quad \beta \ell p = \langle \top,p \rangle .$$

Proposition 1.7.5.9 becomes: $\ell \dashv \beta_2 \dashv u$, between the partially ordered structures $\mathcal{D}(\nu)$, and Ω with the discrete ordering, thus we have

$$(\ell p \leq \theta) \Leftrightarrow (p = \beta_2 \theta) \Leftrightarrow (\theta \leq up).$$

Corollary 1.7.5.10 becomes

$$\beta_2 \ell p = p = \beta_2 up$$

$$(\theta \leq \ell p) \Leftrightarrow (\theta = \ell p)$$

$$(up \leq \theta) \Leftrightarrow (\theta = up).$$

Finally we note that it is implicit in the diagram of 1.7.5.13 (2) that $\eta \equiv \eta_\nu$.

<u>1.7.6.3 Ideals of $\mathcal{D}(\nu)$.</u> We give a characterization of subsets of $D(\nu)$ which are ideals of $\mathcal{D}(\nu)$.

<u>1.7.6.3.1 Proposition.</u> $\mathrm{idl}\ W \leftrightarrow \forall\theta((\langle\top,\beta_2\theta\rangle \in \exists_\beta W) \rightarrow (\langle\beta_2\theta,\beta_2\theta\rangle \in \exists_\beta W))$.

<u>Proof.</u> ($\rightarrow$): $[(\mathrm{idl}\ W) \wedge (\theta' \le \theta) \wedge (\theta' \in W)] \rightarrow (\theta \in W)$.

Substitute $\ell\beta_2\theta$ for θ' and $u\beta_2\theta$ for θ:

$[\mathrm{idl}\ W \wedge (\ell\beta_2\theta \le u\beta_2\theta) \wedge (\ell\beta_2\theta \in W)] \rightarrow (u\beta_2\theta \in W)$.

Since $\beta\colon D(\nu) \rightarrowtail \Omega^2$ is an order embedding from $\mathcal{D}(\nu)$ to $(\Omega_{\le})^0 \times \Omega$ we have:

$[\mathrm{idl}\ W \wedge (\beta\ell\beta_2\theta \prec \beta u\beta_2\theta) \wedge (\beta\ell\beta_2\theta \in \exists_\beta W)] \rightarrow ((\beta u\beta_2\theta \in \exists_\beta W)$.

Since $\beta u\beta_2\theta = \langle\beta_2\theta,\beta_2\theta\rangle$ and $\beta\ell\beta_2\theta = \langle\top,\beta_2\theta\rangle$,

we have $[\mathrm{idl}\ W \wedge (\langle\top,\beta_2\theta\rangle \in \exists_\beta W)] \rightarrow (\langle\beta_2\theta,\beta_2\theta\rangle \in \exists_\beta W)$, hence

$\mathrm{idl}\ W \rightarrow \forall\theta[(\langle\top,\beta_2\theta\rangle \in \exists_\beta W) \rightarrow (\langle\beta_2\theta,\beta_2\theta\rangle \in \exists_\beta W)]$ □

($\leftarrow$): $\mathrm{idl}\ W$

$\leftrightarrow \forall\theta\theta'[((\theta \in W) \wedge (\theta \le \theta')) \rightarrow (\theta' \in W)]$

$\leftrightarrow \forall\theta\theta'[((\langle\beta_1\theta,\beta_2\theta\rangle \in \exists_\beta W) \wedge (\beta_1\theta' \le \beta_1\theta) \wedge (\beta_2\theta = \beta_2\theta')) \rightarrow (\langle\beta_1\theta',\beta_2\theta\rangle \in W)]$.

Define $Q\colon \Omega \times \Omega \times D(\nu) \times \Omega^{D(\nu)} \longrightarrow \Omega$ by

$Q\langle p,p',\theta,W\rangle \leftrightarrow [((\langle p,\beta_2\theta\rangle \in \exists_\beta W) \wedge (p' \le p) \wedge (\beta_2\theta = \beta_2\theta')) \rightarrow (\langle p',\beta_2\theta\rangle \in W)]$.

then $\mathrm{idl}\ W \leftrightarrow \forall\theta\theta' Q\langle\beta_1\theta,\beta_1\theta',\theta,W\rangle$.

We have $\beta_1\theta \in \{\beta_2\theta,\top\}$ and $\beta_1\theta' \in \{\beta_2\theta',\top\}$.

We let $\varphi_1 \equiv (\beta_1\theta = \beta_2\theta) \wedge (\beta_1\theta' = \beta_2\theta')$, $\varphi_2 \equiv (\beta_1\theta = \top) \wedge (\beta_1\theta' = \top)$,

$\varphi_3 \equiv (\beta_1\theta = \beta_2\theta) \wedge (\beta_1\theta' = \top)$, and $\varphi_4 \equiv (\beta_1\theta = \top) \wedge (\beta_1\theta' = \beta_2\theta')$,

then $(\beta_1\theta \in \{\beta_2\theta,\top\}) \wedge (\beta_1\theta \in \{\beta_2\theta,\top\}) \leftrightarrow \bigvee_{i=1}^{4} \varphi_i$ hence $\bigvee_{i=1}^{4} \varphi_i$. We have

$\mathrm{idl}\ W \leftrightarrow \forall\theta\theta'((\bigvee_{i=1}^{4} \varphi_i) \rightarrow Q\langle\beta_1\theta,\beta_1\theta',\theta,W\rangle)$

$\leftrightarrow \forall\theta\theta'(\bigwedge_{i=1}^{4} (\varphi_i \rightarrow Q\langle\beta_1\theta,\beta_1\theta',\theta,W\rangle))$

Let $Q_1 \equiv Q\ulcorner\beta_2\theta,\beta_2\theta',\theta,W\urcorner$, $\quad Q_2 \equiv Q\ulcorner T,T,\theta,W\urcorner$,

$Q_3 \equiv Q\ulcorner\beta_2\theta,T,\theta,W\urcorner$, $\quad Q_4 \equiv Q\ulcorner T,\beta_2\theta',\theta,W\urcorner$,

and $Q' \equiv Q\ulcorner\beta_1\theta,\beta_1\theta',\theta,W\urcorner$.

We have, for each i, $(\varphi_i \to Q') \leftrightarrow (\varphi_i \to (Q' \wedge \varphi_i))$

$\leftrightarrow (\varphi_i \to Q_i)$, so

$\text{idl } W \leftrightarrow \forall\theta\theta'(\bigwedge_{i=1}^{4} (\varphi_i \to Q_i))$.

For $i \equiv 1,2,3$, Q_i is valid, hence $\text{idl } W \to \forall\theta\theta'(\varphi_4 \to Q_4)$.

$(\varphi_4 \to Q_4)$

$\leftrightarrow [((\beta_1\theta = T) \wedge (\beta_1\theta' = \beta_2\theta')) \to (((\ulcorner T,\beta_2\theta\urcorner \in \exists_\beta W) \wedge (\beta_2\theta = \beta_2\theta'))$

$\to (\ulcorner\beta_2\theta',\beta_2\theta\urcorner \in \exists_\beta W))]$

$\leftrightarrow [((\beta\theta = \ulcorner T,\beta_2\theta'\urcorner) \wedge (\ulcorner T,\beta_2\theta'\urcorner \in \exists_\beta W) \wedge (\beta_1\theta' = \beta_2\theta')) \to (\ulcorner\beta_2\theta',\beta_2\theta'\urcorner \in \exists_\beta W)]$

hence idl W

$\leftrightarrow \forall\theta'[(\exists\theta(\theta = \ulcorner T,\beta_2\theta'\urcorner) \wedge (\ulcorner T,\beta_2\theta'\urcorner \in \exists_\beta W) \wedge (\beta_1\theta' = \beta_2\theta'))$

$\to (\ulcorner\beta_2\theta',\beta_2\theta'\urcorner \in \exists_\beta W)]$

but $(\ulcorner T,\beta_2\theta'\urcorner \in \exists_\beta W) \to \exists\theta(\beta\theta = \ulcorner T,\beta_2\theta'\urcorner)$ hence

$\text{idl } W \leftrightarrow \forall\theta[((\beta_1\theta = \beta_2\theta) \wedge (\ulcorner T,\beta_2\theta\urcorner \in \exists_\beta W)) \to (\ulcorner\beta_2\theta\beta_2\theta\urcorner \in \exists_\beta W)]$ hence

$\forall\theta((\ulcorner T,\beta_2\theta\urcorner \in \exists_\beta W) \to (\ulcorner\beta_2\theta,\beta_2\theta\urcorner \in \exists_\beta W)) \to \text{idl } W$. □

<u>1.7.6.3.2 Proposition.</u> $\text{idl } W \leftrightarrow \forall p((\ulcorner T,p\urcorner \in \exists_\beta W) \to (\ulcorner p,p\urcorner \in \exists_\beta W))$.

<u>Proof.</u> (→) By 1.7.6.3.1, $\text{idl } W \to ((\ulcorner T,\beta_2\theta\urcorner \in \exists_\beta W) \to (\ulcorner\beta_2\theta,\beta_2\theta\urcorner \in \exists_\beta W))$.

Substitute up for θ and use the fact that $\beta_2 up = p$ to get

$\text{idl } W \to ((\ulcorner T,p\urcorner \in \exists_\beta W) \to (\ulcorner p,p\urcorner \in \exists_\beta W))$, hence

$\text{idl } W \to \forall p((\ulcorner T,p\urcorner \in \exists_\beta W) \to (\ulcorner p,p\urcorner \in \exists_\beta W))$.□

(←) Conversely $\forall p((\ulcorner T,p\urcorner \in \exists_\beta W) \to (\ulcorner p,p\urcorner \in \exists_\beta W))$

$\to ((\ulcorner T,\beta_2\theta\urcorner \in \exists_\beta W) \to (\ulcorner\beta_2\theta,\beta_2\theta\urcorner \in \exists_\beta W))$, hence

$\forall p((\ulcorner T,p\urcorner \in \exists_\beta W) \to (\ulcorner p,p\urcorner \in \exists_\beta W)) \to \text{idl } W$.□

<u>1.7.6.4 Proposition.</u> $(U \subset \{p,\tau\}) \Rightarrow [\mathrm{idl}(\beta^{-1}(U \times \{p\})) \Leftrightarrow ((\tau \in U) \Rightarrow (p \in U))]$

<u>Proof.</u> By 1.7.6.3.2 we have

$$\mathrm{idl}(\beta^{-1}(U \times \{p\})) \Leftrightarrow \forall q((\ulcorner\tau,q\urcorner \in \exists_\beta(\beta^{-1}(U \times \{p\}))) \Rightarrow (\ulcorner q,q\urcorner \in \exists_\beta(\beta^{-1}(U \times \{p\})))$$

By 1.7.5.14 (5), $(U \subset \{p,\tau\}) \Rightarrow (\exists_\beta(\beta^{-1}(U \times \{p\})) = U \times \{p\})$.

Suppose $\bar{U} \subset \{\bar{p},\tau\}$, then

$$\begin{aligned}\mathrm{idl}(\beta^{-1}(\bar{U} \times \{\bar{p}\})) &\Leftrightarrow \forall q((\ulcorner\tau,q\urcorner \in (\bar{U} \times \{\bar{p}\})) \Rightarrow (\ulcorner q,q\urcorner \in (\bar{U} \times \{\bar{p}\}))) \\ &\Leftrightarrow \forall q(((\tau \in \bar{U}) \wedge (q = \bar{p})) \Rightarrow (q \in \bar{U}) \wedge (q = \bar{p})) \\ &\Leftrightarrow \forall q((q = \bar{p}) \Rightarrow ((\tau \in \bar{U}) \Rightarrow (\bar{p} \in \bar{U}))) \\ &\Leftrightarrow ((\tau \in \bar{U}) \Rightarrow (\bar{p} \in \bar{U})).\square\end{aligned}$$

<u>1.7.6.5 Parametrizing subsets of $\{p,\tau\}$.</u>

We define $P: \Omega^3 \longrightarrow \Omega^\Omega$ by

$$P\ulcorner p,r,s\urcorner = (\{q:r\} \cap \{\tau\}) \cup (\{q:s\} \cap \{p\})$$

As immediate consequences of the definition we have:

(1) $P\ulcorner p,r,s\urcorner \subset \{p,\tau\}$

(2) $(\tau \in P\ulcorner p,r,s\urcorner) \Leftrightarrow (r \vee (s \wedge p))$

(3) $(p \in P\ulcorner p,r,s\urcorner) \Leftrightarrow ((r \wedge p) \vee s)$.

We will show that $(U \subset \{p,\tau\}) \Leftrightarrow \exists rs(U = P\ulcorner p,r,s\urcorner)$. Thus in an internal sense Ω^2 is being used to parametrize the subsets of $\{p,\tau\}$, and we can think of r and s as parameters.

<u>1.7.6.5.1 Proposition.</u>

(a) $U \cap \{\psi\} = \{q: \psi \in U\} \cap \{\psi\}$ for any formula ψ

(b) $P\ulcorner p,\tau \in U,p \in U\urcorner = U \cap \{p,\tau\}$

(c) $(U \subset \{p,\tau\}) \Leftrightarrow \exists rs(U = P\ulcorner p,r,s\urcorner)$.

Proof. (a) $(U \cap \{\psi\}) = \{q: (q \in U) \wedge (q = \psi)\}$

$= \{q: (\psi \in U) \wedge (q = \psi)\}$

$= \{q: \psi \in U\} \cap \{\psi\}$.□

(b) $P/p, \top \in U, p \in U/ = (\{q: \top \in U\} \cap \{\top\}) \cup (\{q: p \in U\} \cap \{p\})$

$= (U \cap \{\top\}) \cup (U \cap \{p\})$ by (a)

$= U \cap \{p, \top\}$. □

(c) ($\Rightarrow$) $(U \subset \{p, \top\}) \Rightarrow (U = P/p, \top \in U, p \in U/)$ and

$(U = P/p, \top \in U, p \in U/) \Rightarrow \exists rs(U = P/p, r, s/)$ hence

$(U \subset \{p, \top\}) \Rightarrow \exists rs(U = P/p, r, s/)$.

($\Leftarrow$) By 1.7.6.5 (1) $(U = P/p, r, s/) \Rightarrow (U \subset \{p, \top\})$, hence

$\exists rs(U = P/p, r, s/) \Rightarrow (U \subset \{p, \top\})$.□

1.7.6.6 Parametrizing subsets of $\{p, \top\}$ which give rise to ideals of $\mathcal{D}(\nu)$.

We seek a condition on the parameters r and s, which is equivalent to

$$\text{idl } \beta^{-1}((P/p, r, s/) \times \{p\}).$$

1.7.6.61 Proposition. $\text{idl } \beta^{-1}((P/p, r, s/) \times \{p\}) \Leftrightarrow (r \leq (p \vee s))$.

Proof. By 1.7.6.4 and 1.7.6.5 (1), (2) and (3):

$\text{idl}\beta^{-1}((P/p, r, s/) \times \{p\}) \Leftrightarrow ((\top \in P/p, r, s/) \Rightarrow (p \in P/p, r, s/)$

$\Leftrightarrow ((r \vee (s \wedge p)) \Rightarrow ((r \wedge p) \vee s))$

$\Leftrightarrow (r \Rightarrow ((r \wedge p) \vee s)$

$\Leftrightarrow (r \leq (p \vee s))$.□

1.7.6.6.2 We define $I: \Omega^3 \longrightarrow \Omega^\Omega$ by

$$I/p, r, s/ = P/p, r \wedge s, s/.$$

By 1.7.6.6.1 we have $\text{idl } \beta^{-1}(I/p, r, s/ \times \{p\})$

hence there exists a morphism $J: \Omega^3 \longrightarrow \text{Idl } \mathcal{D}(\nu)$ such that

$\eta J/p, r, s/ = \beta^{-1}(I/p, r, s/ \times \{p\})$. We put $\gamma \equiv \exists_\beta \circ \eta$, and apply $\exists_\beta$ to get

$$\gamma J/p,r,s/ = I/p,r,s/ \times \{p\},$$

by 1.7.5.6 (5) and 1.7.6.5 (1).

<u>1.7.6.6.3 Proposition.</u> $((U \subset \{p,\top\}) \wedge \mathrm{idl}(\beta^{-1}(U \times \{p\}))) \Leftrightarrow \exists rs(U = I/p,r,s/)$.

<u>Proof.</u> ($\Leftarrow$) Clear.□

($\Rightarrow$) $(U \subset \{p,\top\}) \wedge \mathrm{idl}(\beta^{-1}(U \times \{p\}))$

$\Rightarrow$ $(U \subset \{p,\top\}) \wedge ((\top \in U) \Rightarrow (p \in U))$ by 1.7.6.4

$\Rightarrow$ $(U = P/p,\top \in U,p \in U/) \wedge ((\top \in U) \Rightarrow (p \in U))$

$\Rightarrow$ $(U = P/p,(\top \in U) \wedge (p \in U),p \in U/)$

$\Rightarrow$ $U = I/p,\top \in U,p \in U/$

$\Rightarrow$ $\exists rs(U = I/p,r,s/)$. □

<u>1.7.6.6.4</u> We define $h\colon \Omega \longrightarrow \mathrm{Idl}\,\mathcal{D}(\nu)$ by $hp = J/p,\top,\top/$, then

$\gamma hp = I/p,\top,\top/ \times \{p\} = P/p,\top,\top/ \times \{p\}$

$= ((\{q:\top\} \cap \{\top\}) \cup (\{q:\top\} \cap \{p\})) \times \{p\} = \{p,\top\} \times \{p\}$.

We abbreviate $\hat{p} \equiv hp$.

<u>1.7.6.6.5</u> We define $\mathrm{inv}\colon \Omega^{\Omega \times \Omega} \times \Omega \longrightarrow \Omega$ by

$$\mathrm{inv}/W,p/ = \{q: /q,p/ \in W\} .$$

For any terms $\overline{W}$, ψ of types $\Omega^{\Omega \times \Omega}$ and Ω we put

$$\overline{W}^{-1}\{\psi\} \equiv \mathrm{inv}/\overline{W},\psi/ ,$$

so that $(q \in W^{-1}\{p\}) \Leftrightarrow (/q,p/ \in W)$.

<u>1.7.6.6.6 Proposition.</u> (1) $(W \subset U \times \{p\}) \Rightarrow (W^{-1}\{p\} \subset U)$

(2) $W^{-1}\{p\} \times \{p\} \subset W$

(3) $(W \subset \{q:\top\} \times \{p\}) \Rightarrow (W = (W^{-1}\{p\} \times \{p\}))$.

<u>Proof.</u> (1) $(W \subset U \times \{p\}) \Rightarrow \forall q((/q,p/ \in W) \Rightarrow (q \in U))$

$\Rightarrow (W^{-1}\{p\} \subset U)$.□

(2): $\langle q,q' \rangle \in (W^{-1}\{p\} \times \{p\}) \Rightarrow (\langle q,p \rangle \in W) \wedge (q' = p)$

$\Rightarrow (\langle q,q' \rangle \in W).\square$

(3): This follows from (2) and

$(W \subset \{q:T\} \times \{p\}) \wedge (\langle q,q' \rangle \in W) \Rightarrow ((\langle q,p \rangle \in W) \wedge (q' = p))).\square$

<u>1.7.6.6.7 Proposition.</u> $(K \leq \hat{p}) \Leftrightarrow \exists rs(K = J\langle p,r,s \rangle)$.

<u>Proof.</u> $(\Leftarrow)$ $K = J\langle p,r,s \rangle \Rightarrow (\gamma K = (I\langle p,r,s \rangle \times \{p\}))$

$\Rightarrow (\gamma K \subset (\{p,T\} \times \{p\}))$

$\Rightarrow (\gamma K \subset \gamma\hat{p})$

$\Rightarrow (K \leq \hat{p})$

hence $\exists rs(K = J\langle p,r,s \rangle) \Rightarrow (K \leq \hat{p}).\square$

$(\Rightarrow)$ $(K \leq \hat{p}) \Rightarrow (\gamma K) \subset (\{p,T\} \times \{p\})$

$\Rightarrow (\gamma K) = (((\gamma K)^{-1}\{p\}) \times \{p\})$ by 1.7.6.6.(3)

$\Rightarrow (\beta^{-1}\exists_\beta \eta K = \beta^{-1}(((\gamma K)^{-1}\{p\}) \times \{p\}))$

$\Rightarrow \eta K = \beta^{-1}(((\gamma K)^{-1}\{p\}) \times \{p\})$ by 1.7.2.2.7

$\Rightarrow$ idl $\beta^{-1}(((\gamma K)^{-1}\{p\}) \times \{p\})$, since idl ηK

$\Rightarrow \exists rs((\gamma K)^{-1}\{p\} = I\langle p,r,s \rangle)$, by 1.7.6.6.3

$\Rightarrow \exists rs(((\ K)^{-1}\{p\}) \times \{p\} = I\langle p,r,s \rangle \times \{p\})$

$\Rightarrow \exists rs(\gamma K = \gamma J\langle p,r,s \rangle)$, by 1.7.6.6.6 (3) and 1.7.6.6.2

$\Rightarrow \exists rs(K = J\langle p,r,s \rangle).\ \square$

<u>1.7.6.6.8 Proposition.</u> $(\{p,T\} \times \{p\} \subset \gamma K) \Leftrightarrow (\langle T,p \rangle \in \gamma K)$.

<u>Proof.</u> $(\Rightarrow)$: Clear.$\square$

$(\Leftarrow)$: $\gamma K = \exists_\beta \eta K$ and idl ηK hence by 1.7.6.3.2,

$(\langle T,p \rangle \in \gamma K) \Rightarrow (\langle p,p \rangle \in \gamma K)$

hence $(\langle T,p \rangle \in \gamma K) \Rightarrow (\langle p,p \rangle \in \gamma K) \wedge (\langle T,p \rangle \in \gamma K)$

$\Rightarrow (\{\langle p,p \rangle, \langle T,p \rangle\} \subset \gamma K)$

$\Rightarrow (\{p,T\} \times \{p\} \subset \gamma K).\square$

1.7.6.7 Some local properties of the interpretation of H-algebra polynomials in an internal Heyting algebra $\mathcal{A}$.

Let $\varphi \in$ Poly(H), let x and y be distinct variables of type $\ddot{A}$, where $\ddot{A}$ is the carrier of $\mathcal{A}$, and let $\tilde{\varphi}$ be the interpretation of φ in $\mathcal{A}$.

1.7.6.7.1 Proposition. $(\tilde{\varphi}[x|x \wedge y]) \wedge y = \tilde{\varphi} \wedge y$.

Proof. We proceed by induction on the length of φ. For φ a constant or a variable u such that $\tilde{u} \not\equiv x$ we have $\tilde{\varphi}[x|x \wedge y] \equiv \tilde{\varphi}$.
For $\tilde{u} \equiv x$ we have $(\tilde{u}[x|x \wedge y]) \wedge y \equiv (x \wedge y) \wedge y = x \wedge y \equiv \tilde{u} \wedge y$.

Induction steps:

$$
\begin{aligned}
((\varphi_1 \wedge \varphi_2)^{\sim}[x|x \wedge y]) \wedge y &= ((\tilde{\varphi}_1[x|x \wedge y]) \wedge y) \wedge ((\tilde{\varphi}_2[x|x \wedge y]) \wedge y) \\
&= (\tilde{\varphi}_1 \wedge y) \wedge (\tilde{\varphi}_2 \wedge y) \quad \text{, by induction} \\
&= (\varphi_1 \wedge \varphi_2)^{\sim} \wedge y
\end{aligned}
$$

$$
\begin{aligned}
((\varphi_1 \vee \varphi_2)^{\sim}[x|x \wedge y]) \wedge y &= ((\tilde{\varphi}_1[x|x \wedge y]) \wedge y) \vee ((\tilde{\varphi}_2[x|x \wedge y]) \wedge y) \\
&= (\tilde{\varphi}_1 \wedge y) \vee (\tilde{\varphi}_2 \wedge y) \text{ , by induction} \\
&= (\varphi_1 \vee \varphi_2)^{\sim} \wedge y
\end{aligned}
$$

$$
\begin{aligned}
((\varphi_1 \Rightarrow \varphi_2)^{\sim}[x|x \wedge y]) \wedge y &= [\tilde{\varphi}_1[x|x \wedge y]) \Rightarrow (\tilde{\varphi}_2[x|x \wedge y]) \wedge y \\
&= ((\tilde{\varphi}_1[x|x \wedge y]) \wedge y) \Rightarrow ((\tilde{\varphi}_2[x|x \wedge y]) \wedge y)] \wedge y \\
&= (\tilde{\varphi}_1 \wedge y) \Rightarrow (\tilde{\varphi}_2 \wedge y)] \wedge y \quad \text{, by induction} \\
&= (\varphi_1 \Rightarrow \varphi_2)^{\sim} \wedge y. \ \square
\end{aligned}
$$

The steps immediately preceding and following the induction step (for $\Rightarrow$) is justified by the identity $(x_1 \Rightarrow x_2) \wedge y = ((x_1 \wedge y) \Rightarrow (x_2 \wedge y)) \wedge y$ which is proven in [BD] (Theorem 3 (vii), p.174); it is also used in the proof of 1.1.2.1.

We extend this result to several variables. Let $s_F(\tilde{\varphi}) \subset s_F(\dot{x})$, where $\vec{x}$ is a string of distinct variables of type A of length $n + 1$, with x_i $(0 \le i \le n)$ the i-th variable of $\vec{x}$; let $y \notin s_F(\vec{x})$.

1.7.6.7.2 Proposition. $\forall\vec{x}(y \le \tilde{\varphi}) \Leftrightarrow \forall\vec{x}((\bigwedge_{i=0}^{n} (x_i \le y)) \Rightarrow (y \le \tilde{\varphi}))$.

Proof. ($\Rightarrow$) Clear.□

($\Leftarrow$) Let $\psi \equiv ((\bigwedge_{i=0}^{n} (x_i \le y)) \Rightarrow (y \le \tilde{\varphi}))$, and let $\sigma_k \equiv \left(\bigcup_{i=0}^{n}\left(\begin{smallmatrix} x_i \\ x_i \wedge y\end{smallmatrix}\right)\right)$, the function which substitutes into every term, the term $x_i \wedge y$ for the variable x_i for $0 \le i \le k$ where $k \le n$. We must show

(1) $$(\forall\vec{x}\psi) \Rightarrow (y \le \tilde{\varphi}).$$

We have $\sigma_n(\psi) \equiv ((\bigwedge_{i=0}^{n}((x_i \wedge y) \le y)) \Rightarrow (\sigma_n(y \le \tilde{\varphi}))) = \sigma_n(y \le \tilde{\varphi})$, hence

(2) $$(\forall\vec{x}\psi) \Rightarrow \sigma_n(y \le \tilde{\varphi}).$$

We now proceed by induction to show

(3) $$\sigma_n(y \le \tilde{\varphi}) \Leftrightarrow (y \le \tilde{\varphi}).$$

$$\begin{aligned}\sigma_0(y \le \tilde{\varphi}) &= (y \le \tilde{\varphi}[x_0 | x_0 \wedge y]) \\ &= (y = ((\tilde{\varphi}[x_0 | x_0 \wedge y]) \wedge y)) \\ &= (y = (\tilde{\varphi} \wedge y)) \quad \text{by 1.7.6.7.1} \\ &= (y \le \tilde{\varphi}).\end{aligned}$$

Induction step:

$$\begin{aligned}\sigma_{j+1}(y \le \tilde{\varphi}) &= (\sigma_j(y \le \tilde{\varphi}))[x_{j+1} | x_{j+1} \wedge y] \\ &= (y \le \tilde{\varphi})[x_{j+1} | x_{j+1} \wedge y] \text{ , by induction,} \\ &= (y = ((\tilde{\varphi}[x_{j+1} | x_{j+1} \wedge y]) \wedge y)) \\ &= (y = (\tilde{\varphi} \wedge y)) \quad \text{by 1.7.6.7.1} \\ &= (y \le \tilde{\varphi}).\end{aligned}$$

This proves (3). Combining (3) and (2) we get (1).□

Now we take $\underset{\sim}{\text{Idl}}\,\mathcal{D}(\nu)$ for $\mathcal{A}$, we let $\vec{K}$ be a string of variables of type Idl $\mathcal{D}(\nu)$ for which $s_F(\tilde{\varphi}) \subset s_F(\vec{K})$ and we let $\hat{p} \equiv hp$, the term of type Idl $\mathcal{D}(\nu)$ defined in 1.7.6.6.4.

<u>1.7.6.7.3 Corollary.</u> $\forall\vec{K}(\hat{p} \leq \tilde{\varphi}) \Leftrightarrow \forall((\bigwedge_{i \equiv 0}^{n} (K_i \leq \hat{p})) \Rightarrow (\hat{p} \leq \tilde{\varphi})).\square$

<u>1.7.6.8</u> We now pick up the conventions of 1.7.6.1.1 and 1.7.6.1.2. For $\vec{v}$ a string of distinct variables from V_1, we let $\vec{K} \equiv \vec{\tilde{v}}$, $\vec{r} \equiv \vec{\bar{v}}$, $\vec{s} \equiv \vec{\bar{\bar{v}}}$ and $p \equiv \bar{v}$, and let K_i, r_i and s_i be the i^{th} variable in the strings $\vec{K}$, $\vec{r}$ and $\vec{s}$ respectively; $0 \leq i \leq n$ where $n+1$ is the length of $\vec{v}$. We want $\vec{v}$ to remain free of any association with a particular formula; that is we are <u>not</u> taking $\vec{v}$, as in 1.7.6.1.3, to be $\text{var}(\varphi)$; the condition $\vec{v} \equiv \text{var}\ \varphi$ is one we shall impose when necessary.

We let $J_i \equiv J\langle p, r_i, s_i\rangle$, where $J\colon \Omega^3 \longrightarrow \text{Idl}\,\mathcal{D}(\nu)$ is the morphism defined in 1.7.6.6.2. We let σ_k be the function of terms which substitutes the term J_i for the variable K_i, for $0 \leq i \leq k \leq n$, i.e.

$$\sigma_k \equiv \mathop{S}_{i \equiv 0}^{k} \begin{pmatrix} K_i \\ J_i \end{pmatrix}$$

as defined in 0.2.6.2. The mono $\gamma\colon \text{Idl}\,\mathcal{D}(\nu) \rightarrowtail \Omega^{\Omega^2}$ is defined in 1.7.6.6.2.

In the next proposition we shall replace $f_\varphi p$ by an equivalent formula whose only bound variables are the propositional variables of the string $\vec{r}\,\vec{s}$ and whose free variable is p, where $p \equiv \bar{v}$.

<u>1.7.6.8.1 Proposition.</u> Let $\vec{v} \equiv \text{var}(\varphi)$, then

$$f_\varphi p \Leftrightarrow \forall\vec{r}\,\vec{s}(\langle \tau, p\rangle \in \gamma(\sigma_n(\tilde{\varphi}))).$$

<u>Proof.</u> $f_\varphi p = \forall\vec{K}\mu\langle\tilde{\varphi}, p\rangle$

$= \forall\vec{K}(\beta^{-1}(\{p,\tau\} \times \{p\}) \subset \eta\tilde{\varphi})$ by 1.7.5.5

$= \forall\vec{K}(\exists_\beta \beta^{-1}(\{p,\tau\} \times \{p\}) \subset \exists_\beta \eta\tilde{\varphi})$

$= \forall\vec{K}((\{p,\tau\} \times \{p\}) \subset \gamma\tilde{\varphi})$ by 1.7.5.14 (6)

$= \forall\vec{K}(\hat{p} \leq \tilde{\varphi})$

$= \forall\vec{K}((\bigwedge_{i \equiv 0}^{n} (K_i \leq \hat{p})) \rightarrow (\hat{p} \leq \tilde{\varphi}))$

$= \forall\vec{K}((\bigwedge_{i \equiv 0}^{n} (\exists r_i s_i (K_i = J/p, r_i, s_i/))) \rightarrow (p \leq \tilde{\varphi}))$ by

$= \forall\vec{r}\ \vec{s}(\forall\vec{K}((\bigwedge_{i \equiv 0}^{n} (K_i = J_i)) \rightarrow (\hat{p} \leq \tilde{\varphi}))).$

We proceed by induction to modify the inner, quantifier-free, formula to the equivalent formula

$$\left(\bigwedge_{i \equiv 0}^{n} (K_i = J_i)\right) \rightarrow (\hat{p} \leq \sigma_n(\tilde{\varphi})).$$

We note that the effect of this substitution is that, whereas $s_F(\vec{K}) \subset s_F(\hat{p} \leq \tilde{\varphi})$, now, $(s_F(\vec{K}) \cap s_F(\hat{p} \leq \sigma_n(\tilde{\varphi}))) \equiv \phi$.

$(K_0 = J_0) \rightarrow (\hat{p} \leq \tilde{\varphi})$

$= (K_0 = J_0) \rightarrow (\hat{p} \leq \sigma_0(\tilde{\varphi}))$ by 0.6.9.14 and 0.6.9.15 (2)

$\bigwedge_{i \equiv 0}^{j+1} (K_i = J_i) \rightarrow (\hat{p} \leq \tilde{\varphi})$

$= (K_{j+1} = J_{j+1}) \rightarrow \left((\bigwedge_{i \equiv 0}^{j} (K_i = J_i)) \rightarrow (\hat{p} \leq \tilde{\varphi})\right)$

$= ((K_{j+1} = J_{j+1}) \rightarrow \left((\bigwedge_{i \equiv 0}^{j} (K_i = J_i)) \rightarrow (\hat{p} \leq \sigma_j(\tilde{\varphi}))\right))$

$= (\bigwedge_{i \equiv 0}^{j+1} (K_i = J_i) \rightarrow (\hat{p} \leq \sigma_{j+1}(\tilde{\varphi})))$.

We have

$\forall\vec{K}((\bigwedge_{i \equiv 0}^{n} (K_i = J_i)) \rightarrow (\hat{p} \leq \tilde{\varphi})$

$= \forall\vec{K}((\bigwedge_{i \equiv 0}^{n} (K_i = J_i)) \rightarrow (\hat{p} \leq \sigma_n(\tilde{\varphi})))$

$= (\exists\vec{K}(\bigwedge_{i=0}^{n} (K_i = J_i))) \Rightarrow (\hat{p} \leq \sigma_n(\tilde{\varphi}))$

$= \left(\bigwedge_{i=0}^{n} (\exists K_i(K_i = J_i))\right) \Rightarrow (\hat{p} \leq \sigma_n(\tilde{\varphi})$

$= \hat{p} \leq \sigma_n(\tilde{\varphi})$.

Hence $f_\varphi p = \forall\vec{r}\ \vec{s}(\hat{p} \leq \sigma_n(\tilde{\varphi}))$

$= \forall\vec{r}\ \vec{s}((\{p,T\} \times \{p\}) \subset \gamma(\sigma_n(\tilde{\varphi})))$

$= \forall\vec{r}\ \vec{s}(\langle T,p\rangle \in \gamma(\sigma_n(\tilde{\varphi}))).\square$

1.7.6.8.2 Assume only $s_F(\varphi) \subset s_F(\vec{v})$; then the formula

(1) $\langle T,p\rangle \in \gamma(\sigma_n(\tilde{\varphi}))$

is still well-defined. In order to calculate a formula of propositional logic equivalent to (1) we shall find it necessary to first calculate a formula equivalent to

(2) $\langle p,p\rangle \in \gamma(\sigma_n(\tilde{\varphi}))$.

For fixed $\vec{v}$, our "calculation" will be an inductive on the complexity of $\varphi \in$ Poly V_1. The crucial step is the unravelling of (2) and (1) when φ is of the form $\varphi_1 \Rightarrow \varphi_2$. In the next two propositions we prepare for this step.

1.7.6.9.1 Proposition.

$[\langle p,p\rangle \in \gamma(K_1 \Rightarrow K_2)] \Leftrightarrow [(\langle p,p\rangle \in \gamma K_1) \Rightarrow (\langle p,p\rangle \in \gamma K_2)]$.

Proof. $\langle p,p\rangle \in \gamma(K_1 \Rightarrow K_2)$

$\Leftrightarrow (\beta up \in \exists_\beta \eta(K_1 \Rightarrow K_2))$

$\Leftrightarrow (up \in \eta(K_1 \Rightarrow K_2))$

$\Leftrightarrow ([up,\rightarrow) \cap \eta K_1) \subset \eta K_2$

$\Leftrightarrow \forall\theta(((up \leq \theta) \wedge (\theta \in \eta K_1)) \Rightarrow (\theta \in \eta K_2))$

$\Leftrightarrow \forall\theta(((\beta_1\theta \leq p) \wedge (\beta_2\theta = p) \wedge (\beta\theta \in \gamma K_1)) \Rightarrow \beta\theta \in \gamma K_2))$

Since $\beta_1\theta \in \{\beta_2\theta, \top\}$ we have

$\langle p,p\rangle \in \gamma(K_1 \Rightarrow K_2)$

$\Leftrightarrow \forall\theta(((\beta\theta = \langle p,p\rangle) \Rightarrow ((\langle p,p\rangle \in \gamma K_1) \Rightarrow (\langle p,p\rangle \in \gamma K_2)))$

$\Leftrightarrow (\exists\theta(\beta\theta = \langle p,p\rangle) \Rightarrow ((\langle p,p\rangle \in \gamma K_1) \Rightarrow (\langle p,p\rangle \in \gamma K_2)))$

Since $\beta up = \langle p,p\rangle$ we have

$(\langle p,p\rangle \in \gamma(K_1 \Rightarrow K_2)) \Leftrightarrow ((\langle p,p\rangle \in \gamma K_1) \Rightarrow (\langle p,p\rangle \in \gamma K_2)).\square$

<u>1.7.6.9.2 Proposition.</u> $[\langle\top,p\rangle \in \gamma(K_1 \Rightarrow K_2)] \Leftrightarrow$

$[((\langle\top,p\rangle \in \gamma K_1) \Rightarrow (\langle\top,p\rangle \in \gamma K_2))] \wedge [(\langle p,p\rangle \in \gamma K_1) \Rightarrow (\langle p,p\rangle \in \gamma K_2)]$

<u>Proof.</u> $\langle\top,p\rangle \in \gamma(K_1 \Rightarrow K_2)$

$\Leftrightarrow (\beta\ell p \in \exists_\beta\eta(K_1 \Rightarrow K_2))$

$\Leftrightarrow (\ell p \in \eta(K_1 \Rightarrow K_2))$

$\Leftrightarrow (([\ell p,\rightarrow) \cap \eta K_1) \subset \eta K_2)$

$\Leftrightarrow \forall\theta(((\ell p \leq \theta) \wedge (\theta \in \eta K_1)) \Rightarrow (\theta \in \eta K_2))$

$\Leftrightarrow \forall\theta(((\beta_1\theta \leq \top) \wedge (\beta_2\theta = p) \wedge (\beta\theta \in \gamma K_1)) \Rightarrow (\beta\theta \in \gamma K_2)).$

Since $\beta_1\theta \leq \top$ and $(\beta_1\theta = \beta_2\theta) \vee (\beta_1\theta = \top)$ we have

$\langle\top,p\rangle \in \gamma(K_1 \Rightarrow K_2)$

$\Leftrightarrow \forall\theta(((\beta_1\theta = \beta_2\theta) \vee (\beta_1\theta = \top)) \Rightarrow [((\beta_2\theta = p) \wedge (\beta\theta \in \gamma K_1)) \Rightarrow (\beta\theta \in \gamma K_2)])$

The scope of $\forall\theta$ breaks down into a conjunction. For the first component of this conjunction we have

$(\beta_1\theta = \beta_2\theta) \Rightarrow [((\beta_2\theta = p) \wedge (\beta\theta \in \gamma K_1)) \Rightarrow (\beta\theta \in \gamma K_2)]$

$\Leftrightarrow [(\beta\theta = \langle p,p\rangle) \Rightarrow (\langle p,p\rangle \in \gamma K_1) \Rightarrow (\langle p,p\rangle \in \gamma K_2)]$

and for the second:

$(\beta_1\theta = \top) \to [((\beta_2\theta = p) \wedge (\beta\theta \in \gamma K_1)) \to (\beta\theta \in \gamma K_2)]$

$\leftrightarrow [(\beta\theta = /\top,p/) \to [(/\top,p/ \in \gamma K_1) \to (/\top,p/ \in \gamma K_2)]]$.

Since $\exists\theta(\beta\theta = /p,p/)$ and $\exists\theta(\beta\theta = /\top,p/)$, we have

$/\top,p/ \in \gamma(K_1 \to K_2)$

$\leftrightarrow [(/p,p/ \in \gamma K_1) \to (/p,p/ \in \gamma K_2)] \wedge [(/\top,p/ \in \gamma K_1) \to (/\top,p/ \in \gamma K_2)]$.□

1.7.6.9.3 Proposition. For each $\varphi \in \text{Poly } H$ such that $s_F(\varphi) \subset s_F(\vec{v})$, we have

$$(/p,p/ \in \gamma(\sigma_n(\tilde{\varphi}))) \leftrightarrow \overline{\varphi}$$

Proof. By induction on the complexity of φ such that $s_F(\varphi) \subset s_F(\vec{v})$.

$\underline{0}$: $(/p,p/ \in \gamma(\sigma_n(\tilde{\underline{0}}))) \equiv (/p,p/ \in \gamma\tilde{\underline{0}})$

$= (/p,p/ \in \phi)$

$= \bot \equiv \overline{\underline{0}}$.

$\underline{1}$: $(/p,p/ \in \gamma(\sigma_n(\tilde{\underline{1}}))) = (/p,p/ \in \gamma\tilde{\underline{1}})$

$= (/p,p/ \in \{q:\top\})$

$= \top \equiv \overline{\underline{1}}$.

v_i: $(/p,p/ \in \gamma(\sigma_n(\tilde{v}_i))) \equiv (/p,p/ \in (\gamma J/p,r_i,s_i/))$

$= (p \in I/p,r_i,s_i/)$

$= (p \in ((\{q:r_i \wedge s_i\} \cap \{\top\}) \cup (\{q:s_i\} \cap \{p\})))$

$= ((r_i \wedge s_i \wedge p) \vee s_i) = s_i$.

Δ: $\gamma(\sigma_n(\varphi_1 \wedge \varphi_2)^{\sim}) = \gamma(\sigma_n(\tilde{\varphi}_1 \wedge \tilde{\varphi}_2)) = \gamma((\sigma_n(\tilde{\varphi}_1) \wedge \sigma_n(\tilde{\varphi}_2))$

$= (\gamma(\sigma_n(\tilde{\varphi}_1)) \cap \gamma(\sigma_n(\tilde{\varphi}_2)))$

$/p,p/ \in \gamma(\sigma_n(\varphi_1 \wedge \varphi_2)^{\sim}) \leftrightarrow (/p,p/ \in \gamma(\sigma_n(\tilde{\varphi}_1))) \wedge (/p,p/ \in \gamma(\sigma_n(\tilde{\varphi}_2)))$

$\leftrightarrow (\overline{\varphi}_1 \wedge \overline{\varphi}_2)$ by induction

$\leftrightarrow \overline{\varphi_1 \wedge \varphi_2}$.

$\underline{\vee}$: The verification for $\varphi_1 \vee \varphi_2$ is essentially the same as for $\varphi_1 \wedge \varphi_1$.

$\underline{\Rightarrow}$: $\gamma(\sigma_n(\varphi_1 \Rightarrow \varphi_2)^{\sim}) = \gamma((\sigma_n(\tilde{\varphi}_1)) \Rightarrow (\sigma_n(\tilde{\varphi}_2)))$

$\langle p,p\rangle \in \gamma(\sigma_n(\varphi_1 \Rightarrow \varphi_2)^{\sim})$

$\Leftrightarrow ((\langle p,p\rangle \in \gamma(\sigma_n(\tilde{\varphi}_1))) \Rightarrow (\langle p,p\rangle \in \gamma(\sigma_n(\tilde{\varphi}_2))))$ by 1.7.6.9.1

$\Leftrightarrow (\overline{\varphi}_1 \Rightarrow \overline{\varphi}_2)$ by induction

$\Leftrightarrow \overline{\varphi_1 \Rightarrow \varphi_2}$.

<u>1.7.6.9.4 Proposition.</u> For each $\varphi \in$ Poly H such that $s_F(\varphi) \subset s_F(\vec{v})$, we have

$$(\langle \top,p\rangle \in \gamma(\sigma_n(\tilde{\varphi}))) \Leftrightarrow \overline{\varphi'} .$$

<u>Proof.</u> By induction on the complexity of φ such that $s_F(\varphi) \subset s_F(\vec{v})$.

$\underline{0}$: $(\langle \top,p\rangle \in \gamma(\sigma_n(\underline{\tilde{0}}))) \equiv (\langle \top,p\rangle \in \gamma\underline{\tilde{U}})$

$= \bot \equiv \overline{\underline{0}} \equiv \overline{\underline{0}'}$.

$\underline{1}$: $(\langle \top,p\rangle \in \gamma(\sigma_n(\underline{\tilde{I}}))) \equiv (\langle \top,p\rangle \in \gamma\underline{\tilde{1}})$

$= \top \equiv \overline{\underline{1}} \equiv \overline{\underline{1}'}$.

v_i: $(\langle \top,p\rangle \in \gamma(\sigma_n(\tilde{v}_i))) \equiv (\langle \top,p\rangle \in (\gamma J\langle p,r_i,s_i\rangle))$

$= (\top \in I\langle p,r_i,s_i\rangle)$

$= (\top \in ((\{q{:}r_i \wedge s_i\} \cap \{\top\}) \cup (\{q{:}s_i\} \cap \{p\})))$

$= (r_i \wedge s_i) \vee (s_i \wedge p)$

$= (r_i \vee p) \wedge s_i$

$\equiv (\overline{\ddot{v}}_i \vee \overline{v}) \wedge \overline{v}_i$

$\equiv \overline{(\ddot{v}_i \vee v) \wedge v_i}$

$\equiv \overline{v_i{}'}$.

$\underline{\wedge}$: $\gamma(\sigma_n(\varphi_1 \wedge \varphi_2)^{\sim}) = \gamma(\sigma_n(\tilde{\varphi}_1) \wedge \sigma_n(\tilde{\varphi}_2))$

$= \gamma(\sigma_n(\tilde{\varphi}_1)) \cap \gamma(\sigma_n(\tilde{\varphi}_2))$

$$/\top,p/ \in \gamma(\sigma_n(\varphi_1 \wedge \varphi_2)^{\sim}) = ((/\top,p/ \in \gamma(\sigma_n(\tilde{\varphi}_1))) \wedge (/\top,p/ \in \gamma(\sigma_n(\tilde{\varphi}_2))))$$

$$= (\overline{\varphi_1{'}} \wedge \overline{\varphi_2{'}}) \qquad \text{by induction}$$

$$= \overline{(\varphi_1 \wedge \varphi_2)}{'}$$

$\underline{\vee}$: The verification for $\varphi_1 \vee \varphi_2$ is similar.

$\underline{\Rightarrow}$: $\gamma(\sigma_n(\varphi_1 \Rightarrow \varphi_2)^{\sim}) = \gamma(\sigma_n(\tilde{\varphi}_1) \Rightarrow \sigma_n(\tilde{\varphi}_2))$

$[/\top,p/ \in \gamma(\sigma_n(\varphi_1 \Rightarrow \varphi_2)^{\sim}) \Leftrightarrow$

$[((/\top,p/ \in \gamma(\sigma_n(\tilde{\varphi}_1))) \Rightarrow (/\top,p/ \in \gamma(\sigma_n(\tilde{\varphi}_2)))) \wedge ((/p,p/ \in \gamma(\sigma_n(\tilde{\varphi}_1)))$

$\Rightarrow (/p,p/ \in \gamma(\sigma_n(\tilde{\varphi}_2))))$ by 1.7.6.9.2

$\Leftrightarrow (\overline{\varphi_1{'}} \Rightarrow \overline{\varphi_2{'}}) \wedge (\overline{\varphi_1} \Rightarrow \overline{\varphi_2})$ by induction and 1.7.6.9.3

$\Leftrightarrow (\varphi_1{'} \Rightarrow \varphi_2{'}) \wedge (\varphi_1 \Rightarrow \varphi_2)$

$\Leftrightarrow (\varphi_1 \Rightarrow \varphi_2)'$. □

<u>1.7.6.10 Theorem.</u> Let $\varphi \in$ Poly $\mathbb{H}$ and $\vec{v} \equiv \mathrm{var}(\varphi)$.

(1) $f_{\varphi} p \Leftrightarrow (\forall \vec{r}\ \vec{s}\ \overline{\varphi'})$ and hence

(2) $((f_{\varphi} p) \Rightarrow p) \Leftrightarrow ((\forall \vec{r}\ \vec{s}\ \overline{\varphi'}) \Rightarrow p)$

where $\vec{r} \equiv \bar{\vec{v}}$ and $\vec{s} \equiv \bar{\bar{\vec{v}}}$.

<u>Proof.</u> Combine 1.7.6.9. and 1.7.6.9.4.□

1.7.7 Examples of the internalization of $\varphi \in \Gamma(\underline{\mathcal{E}})$ for $\varphi \in (\{\Delta^1(\underline{0}),\Delta^2(\underline{0}),R_2\} \cup \{C_n \mid n \in N^+\})$.

1.7.7.1 Eliminating the free variable in the internalized formula.

Let $\varphi \in \mathcal{C}^{*}$ and let φ^* be the formula of the language of $\underline{\mathcal{E}}$:

(1) $(\forall\vec{r}\ \vec{s}\ \overline{\varphi'}) \to p$

defined in 1.7.6.1.3. The formula $\varphi^*[p|\bot]$, that is ,

(2) $\neg(\forall\vec{r}\ \vec{s}(\varphi'[v|\underline{0}]))$

is a consequence of (1) and thus a necessary condition for $\varphi \in \Gamma(\underline{\mathcal{E}})$.

It turns out (1.7.7.2.2 and 1.7.7.1.3) that for $\varphi \in (\{\Delta^2(\underline{0}),R_2\} \cup \{C_n \mid n \in \mathbb{N}^+\})$ (2) is also a sufficient condition for $\varphi \in \Gamma(\underline{\mathcal{E}})$. In general, this is not the case; in particular for $\varphi \equiv Z \equiv ((u_1 \to u_2) \vee (u_2 \to u_1))$, (2) is strictly weaker than (1) (1.7.7.3.2).

The form (2) has a syntactical advantage over (1): if we know that $\varphi'[v|\underline{0}] \dashv\vdash \psi$ then for $\vec{q} \equiv \mathrm{var}(\overline{\psi})$ we have

$$\forall\vec{r}\ \vec{s}(\varphi'[v|\underline{0}]) \leftrightarrow (\forall\vec{q}\overline{\psi})$$

hence $\varphi^*[p|\bot] \leftrightarrow \neg(\forall\vec{q}\overline{\psi})$.

1.7.7.1.1. Proposition. Let $\varphi, \psi \in \mathrm{Poly}\,\mathbb{H}$ and suppose $\varphi \vdash \psi$. If we interpret in the internal algebra $\underline{\Omega}$ in a topos we have

$$(\forall\vec{r}\ \overline{\varphi}) \to \overline{\psi}$$

where $\vec{r} \equiv \mathrm{var}\ \overline{\varphi}$.

Proof. We expand the similarity type $\mathbb{H}$ of 0.6.4.1 to $\mathbb{H}^{\underline{e}}$ by adding a nullary operation sign $\underline{e}$. We let $\langle i,\langle\mathbb{H},\mathrm{Eql}\rangle,\langle\mathbb{H}^{\underline{e}},\mathrm{Eql}\rangle\rangle$ be the natural inclusion of similarity types. We call an $\langle\mathbb{H}^{\underline{e}},\mathrm{Eql}\rangle$-structure, $\mathcal{A}$, an $\underline{e}$-Heyting algebra structure, if $\mathcal{A}^i$ is a Heyting algebra structure (see 0.6.7.1 for definition of $\mathcal{A}^i$). We claim that if $\mathcal{A}$ is an $\underline{e}$-Heyting structure and $\mathcal{A} \models \underline{e} \wedge \varphi = \underline{e}$ then $\models \underline{e} \wedge \psi = \underline{e}$. We suppose $\mathcal{A} \models \underline{e} \wedge \varphi = \underline{e}$ where $\mathcal{A}$ is an $\underline{e}$-Heyting algebra structure. We have an H-homomorphism from $\mathcal{A}^i$ onto $(\mathcal{A}^i)_e$, $f: \mathcal{A}^i \longrightarrow (\mathcal{A}^i)_e$, where e is the interpretation of $\underline{e}$ in $\mathcal{A}$. Now let e be the interpretation of $\underline{e}$ in $(\mathcal{A}^i)_e$ so that we have an $\langle \mathbb{H}^e, \text{Eql} \rangle$-structure $\mathcal{A}_e$. Since f is an H^e-homomorphism from $\mathcal{A}$ onto $\mathcal{A}_e$ we have $\mathcal{A}_e \models \underline{e} \wedge \varphi = \underline{e}$. Since $\mathcal{A}_e \models \underline{e} = \underline{1}$, we have $\mathcal{A}_e \models \varphi = \underline{1}$, hence $(\mathcal{A}_e)^i \models \varphi = \underline{1}$, by 0.6.7.6, hence $(\mathcal{A}_e)^i \models \psi = \underline{1}$, by hypothesis, hence $\mathcal{A}_e \models \psi = \underline{1}$, again by 0.6.7.6. We claim $\mathcal{A} \models \underline{e} = \underline{e} \wedge \psi$. Let $\beta: V \longrightarrow |\mathcal{A}|$, then we have an $\text{H}^{\underline{e}}$ homomorphism $\tilde{\beta}^{\underline{e}}: \text{Poly}(\mathbb{H}^e) \longrightarrow \mathcal{A}$ extending β. We want to show $\tilde{\beta}^{\underline{e}}(\underline{e}) \equiv \tilde{\beta}^{\underline{e}}(\underline{e} \wedge \psi)$, i.e. $\tilde{\beta}^e(\psi) \wedge e \equiv e$. The valuation $f \circ \beta: V \longrightarrow |\mathcal{A}_e|$ extends to the $\text{H}^{\underline{e}}$-homomorphism $\widetilde{(f \circ \beta)}^e: \text{Poly}(\mathbb{H}^e) \longrightarrow \mathcal{A}_e$; since $f \circ \tilde{\beta}^e \equiv \widetilde{(f \circ \beta)}^e$, we have $e \equiv \widetilde{(f \circ \beta)}^e(\underline{1}) \equiv f(\tilde{\beta}^e(\psi)) \equiv \tilde{\beta}^e(\psi) \wedge e$. This proves $\mathcal{A} \models \underline{e} = \underline{e} \wedge \psi$.

Now we interpret in a topos; $\overline{\varphi}$ and $\overline{\psi}$ are the interpretations of φ and ψ in the internal structure $\underset{\sim}{\Omega}$. Let $\vec{r} \equiv \text{var}\,\overline{\varphi}$ and let $e \equiv \forall \vec{r}\, \overline{\varphi}$. We have an internal $\langle \mathbb{H}^{\underline{e}}, \text{Eql} \rangle$ structure $\Omega^{\underline{e}}$ which has $\underline{e}$ interpreted as e. Since $(\forall \vec{r}\, \overline{\varphi}) \Rightarrow \overline{\varphi}$ we have $\underset{\sim}{\Omega}^{\underline{e}} \models \underline{e} \wedge \varphi = \underline{e}$ where $\underset{\sim}{\Omega}^{\underline{e}}$ is an $\underline{e}$-Heyting algebra. Hence by the metatheory $\underset{\sim}{\Omega}^{\underline{e}} \models \underline{e} \wedge \psi = \underline{e}$, thus $e \Rightarrow \overline{\psi}$, i.e. $(\forall \vec{r}\, \overline{\varphi}) \Rightarrow \overline{\psi}$. □

<u>1.7.7.1.2 Corollary.</u> Let $\varphi, \psi \in \text{Poly}\,\mathbb{H}$ such that $\varphi \dashv\vdash \psi$; let $\overline{\varphi}$, $\overline{\psi}$ be their interpretations in $\underset{\sim}{\Omega}$ and let $\vec{r} \equiv \text{var}\,\overline{\varphi}$, $\vec{s} \equiv \text{var}\,\overline{\psi}$, then

$$(\forall \vec{r}\, \overline{\varphi}) \Leftrightarrow (\forall \vec{s}\, \overline{\psi}).$$

<u>Proof.</u> By 1.7.7.1.1 we have $(\forall \vec{r}\, \overline{\varphi}) \Rightarrow \overline{\psi}$ and $(\forall \vec{s}\, \overline{\psi}) \Rightarrow \overline{\varphi}$, hence $(\forall \vec{r}\, \overline{\varphi}) \Rightarrow (\forall \vec{s}\, \overline{\psi})$ and $(\forall \vec{s}\, \overline{\psi}) \Rightarrow (\forall \vec{r}\, \overline{\varphi})$. □

<u>1.7.7.2 Calculation of φ' for $\varphi \in (\mathcal{C}^{\bullet} \cap \text{Poly } V_1)$</u>. Calculations are based on the inductive definition of φ' given in 1.7.6.1.1. We put $\dot{u} \equiv \ddot{u}$.

$(\Delta(\underline{0}))$: $\varphi \equiv u \vee \neg u$

$$\varphi' \equiv (u' \vee (\neg u)') \equiv u' \vee (\neg u' \wedge \neg u)$$

$$\equiv ((\dot{u} \vee v) \wedge u) \vee (\neg ((\dot{u} \vee v) \wedge u) \wedge \neg u)$$

$$= ((\dot{u} \vee v) \wedge u) \vee \neg u$$

$$\varphi' \leq (\dot{u} \vee v) \vee \neg u.$$

Let $s \equiv \bar{\bar{u}}$, $r \equiv \bar{u}$ and $p \equiv \bar{v}$. $\forall rs\overline{\varphi'} \Rightarrow \forall rs((s \vee p) \vee \neg r)$ but

$$\forall rs((s \vee p) \vee \neg r) \Rightarrow (((s \vee p) \vee \neg r)[s|\bot][r|\top])$$

$$\Rightarrow p$$

hence $(\forall rs\overline{\varphi'}) \Rightarrow p$. Thus $u \vee \neg u \in \Gamma(\underline{\mathcal{E}})$ for all toposes.

We have $u' \equiv (\dot{u} \vee v) \wedge u$

and $u'[v|\underline{0}] \equiv \dot{u} \wedge u$

$$(\neg u)'[v|0] \equiv (\neg(u') \wedge \neg u)[v|0] \equiv \neg(\dot{u} \wedge u) \wedge \neg u = \neg u.$$

(R_2): $\varphi \equiv u_1 \vee \neg u_1 \vee (u_1 \Rightarrow u_2) \equiv R_2$, then

$\varphi' \equiv u_1' \vee (\neg u_2' \wedge \neg u_2) \vee ((u_1' \Rightarrow u_2') \wedge (u_1 \Rightarrow u_2))$, and

$\models \varphi'[v|\underline{0}] = (\dot{u}_1 \wedge u_1) \vee \neg u_2 \vee (((\dot{u}_1 \wedge u_1) \Rightarrow (\dot{u}_2 \wedge u_2)) \wedge (u_1 \Rightarrow u_2))$

$(\Delta^2(0))$: $\varphi \equiv u_1 \vee (u_1 \Rightarrow (u_2 \vee \neg u_2)) \equiv \Delta^2(\underline{0})$. then

$\varphi' \equiv u_1' \vee [(u_1' \Rightarrow (u_2' \vee (\neg u_2)')) \wedge (u_1 \Rightarrow (u_2 \vee \neg u_2))$, and

$\models \varphi'[v|\underline{0}] = (\dot{u}_1 \wedge u_1) \vee [((\dot{u}_1 \wedge u_1) \Rightarrow ((\dot{u}_2 \wedge u_2) \vee \neg u_2))$

$\wedge (u_1 \Rightarrow (u_2 \vee \neg u_2))]$.

(Z): $\varphi \equiv (u_1 \Rightarrow u_2) \vee (u_2 \Rightarrow u_1) \equiv Z$, then

$\varphi' \equiv ((u_1' \Rightarrow u_2') \wedge (u_1 \Rightarrow u_2)) \vee ((u_2' \Rightarrow u_1') \wedge (u_2 \Rightarrow u_1))$, and

$\varphi'[v|0] \equiv [((\dot{u}_1 \wedge u_1) \Rightarrow (\dot{u}_2 \wedge u_2)) \wedge (u_1 \Rightarrow u_2)]$

$\vee ((\dot{u}_2 \wedge u_2) \Rightarrow (\dot{u}_1 \wedge u_1)) \wedge (u_2 \Rightarrow u_1)]$.

We will study $(\Delta^2(0))$ and (R_2) together.

1.7.7.2.1 Proposition. For $\varphi \in \{R_2, \Delta^2(\underline{0})\}$, $\varphi'[v|\underline{0}] \dashv\vdash u \vee \neg u$.

Proof. ($\vdash$) Suppose $\varphi'[v|\underline{0}] \nvdash u \vee \neg u$. Then there exists an $\mathcal{S}i$ - algebra $\mathcal{A}$ such that $\nvDash u \vee \neg u$ and $\mathcal{A} \vDash \varphi'[v|0]$. We must have $\mathbf{3}$ as a subalgebra of $\mathcal{A}$ with $|\mathbf{3}| \equiv \{0, e, 1\}$. Let $U \equiv \{u_1, u_2, \dot{u}_1, \dot{u}_2\}$.

Let $\alpha: V \longrightarrow |\mathcal{A}|$ be such that

$$\alpha|_U \equiv \begin{pmatrix} u_1 & u_2 & \dot{u}_1 & \dot{u}_2 \\ 1 & e & e & 1 \end{pmatrix}.$$

We will show that $\tilde{\alpha}(\varphi'[v|0]) \equiv e$. For R_2 we have

$$\tilde{\alpha}(R_2'[v|0]) \equiv (e \wedge 1) \vee \neg e \vee ((e \wedge 1) \Rightarrow (1 \wedge e)) \wedge (1 \Rightarrow e))$$
$$\equiv e.$$

For $\Delta^2(\underline{0})$ we have

$$\tilde{\alpha}((\Delta^2(\underline{0}))'[v|0]) \equiv (e \wedge 1) \vee [((e \wedge 1) \Rightarrow ((e \wedge e) \vee \neg e)) \wedge (1 \Rightarrow (e \vee \neg e))$$
$$\equiv e.$$

Hence for $\varphi \in \{R_2, \Delta^2(\underline{0})\}$, $\mathcal{A} \nvDash \varphi'[v|0]$ -a contradiction.□

($\dashv$) We will show $\mathbf{2} \vDash \varphi'[v|0]$. Suppose $\mathbf{2} \nvDash \varphi'[v|\underline{0}]$ then for some a_1, a_2, b_1, b_2, there is an α such that

$$\alpha|_U \equiv \begin{pmatrix} u_1 & u_2 & \dot{u}_1 & \dot{u}_2 \\ a_1 & a_2 & b_1 & b_2 \end{pmatrix}$$

and $\tilde{\alpha}(\varphi'[v|0]) \equiv 0$ in $\mathbf{2}$. For R_2 we have

$(b_1 \wedge a_1) \vee \neg a_2 \vee (((b_1 \wedge a_1) \Rightarrow (b_2 \wedge a_2)) \wedge (a_1 \Rightarrow a_2)) \equiv 0$;

this implies $b_1 \wedge a_1 \equiv 0$ and $a_2 \equiv 1$ and hence also

$(((0 \Rightarrow (b_2 \wedge 1)) \wedge (a_1 \Rightarrow 1)) \equiv 0$ -a contradiction. Hence $\mathbf{2} \vDash (R_2'[v|0])$.

For $\Delta^2(\underline{0})$ we have

$(b_1 \wedge a_1) \vee [((b_1 \wedge a_1) \Rightarrow ((b_2 \wedge a_2) \vee \neg a_2)) \wedge (a_1 \Rightarrow (a_2 \vee \neg a_2))] \equiv 0$;

this implies $b_1 \wedge a_1 \equiv \underset{\cdot}{0}$ and hence also

$(0 \Rightarrow ((b_2 \wedge a_2) \vee \neg a_2)) \wedge (a_1 \Rightarrow (a_2 \vee \neg a_2) \equiv 0$.

But on $\mathbf{2}$, $a_2 \vee \neg a_2 \equiv 1$, hence we have a contradiction. Hence

$\mathbf{2} \vdash ((\Delta^2(\underline{0}))'[v|0])$. □

<u>1.7.7.2.2 Corollary.</u> For $\varphi \in \{R_2, \Delta^2(\underline{0})\}$, if $\varphi \in \Gamma(\underline{\mathcal{E}})$ then

$\underline{\mathcal{E}} \models \neg \forall q(q \vee \neg q)$.

<u>Proof.</u> Suppose $\varphi \in \Gamma(\underline{\mathcal{E}})$. Let $r_i \equiv \bar{u}_i$, $s_i \equiv \bar{\bar{u}}_i$, $q \equiv \bar{u}$. By 1.7.7.2.1

$\neg \forall r_1 r_2 s_1 s_2 (\overline{\varphi'[v|0]}) \Leftrightarrow \neg \forall q(q \vee \neg q)$. Hence by 1.7.6.10,

$\underline{\mathcal{E}} \models \neg \forall q(q \vee \neg q)$.□

<u>1.7.7.2.3 Theorem.</u> The following are equivalent

(1) $\Delta^2(\underline{0}) \in \Gamma(\underline{\mathcal{E}})$

(2) $(\Delta^2(\underline{0}) \wedge E_n) \in \Gamma(\underline{\mathcal{E}})$ for some $n \geq 1$

(3) $(\Delta^2(\underline{0}) \wedge E_n) \in \Gamma(\underline{\mathcal{E}})$ for all $n \geq 1$

(4) $R_2 \in \Gamma(\underline{\mathcal{E}})$

(5) $\underline{\mathcal{E}} \models \neg \forall q(q \vee \neg q)$.

<u>Proof.</u> (1) → (2), (2) → (3) and (3) → (4) are clear .□

(4) → (5) by 1.7.7.2.2.□ (5) → (1) By 1.6.4.17. □

Using the topos $\dot{\underline{\mathcal{E}}}_1$ of 1.2.3.4.1 we get a stronger statement.

<u>1.7.7.2.4 Theorem</u>. The following are equivalent for $\varphi \in \mathrm{Poly}(H)$.

(1) For all toposes $\underline{\mathcal{E}}$

if $\underline{\mathcal{E}} \models \neg \forall q(q \vee \neg q)$ then $\varphi \in \Gamma(\underline{\mathcal{E}})$.

(2) $\varphi \vdash \Delta^2(\underline{0})$.

<u>Proof.</u> (1) → (2). Consider in particular the topos $\dot{\underline{\mathcal{E}}}_1 = \underline{\mathcal{S}}^{\underline{M}_1{}^0}$. We have

$\Gamma(\dot{\underline{\mathcal{E}}}_1) \equiv \{\psi | \psi \vdash \Delta^2(\underline{0})\}$ and $\dot{\underline{\mathcal{E}}}_1 \models \neg \forall q(q \vee \neg q)$. Hence if (1) holds for φ it must hold for (2).

(2) $\rightarrow$ (1). Suppose $\underline{\mathcal{E}} \models \neg \forall q(q \vee \neg q)$ and $\varphi \vdash \Delta^2(\underline{0})$. By 1.6.4.17, $\Delta^2(\underline{0}) \in \Gamma(\underline{\mathcal{E}})$ and hence $\varphi \in \Gamma(\underline{\mathcal{E}})$.□

1.7.7.2.5 From 1.7.7.2.3 we have, if $\underline{\mathcal{E}} \models \neg \forall q(q \vee \neg q)$ then $\{\varphi | \varphi \vdash \Delta^2(\underline{0})\} \subset \Gamma(\underline{\mathcal{E}})$ and in particular $\Gamma(\dot{\underline{\mathcal{E}}}_1) \equiv \{\varphi | \varphi \vdash \Delta^2(\underline{0})\}$. However the condition $\underline{\mathcal{E}} \models \neg \forall q(q \vee \neg q)$ does not determine the set $\Gamma(\underline{\mathcal{E}})$. For any finite monoid $\underline{M}$, which is not a group we will have $\underline{M}$-Sets $\models \neg \forall q(q \vee \neg q)$; and as we have seen, for monoids satisfying the conditions of 1.2.5.6.7 $\Gamma(\underline{M}\text{-Sets}) \equiv \{\varphi | \varphi \vdash X_\alpha\}$ where $\alpha \equiv \mathcal{L}(\underline{M}) \oplus 1$. By 1.1.4.7, if $\beta \rightarrowtail \alpha$ is proper, $X_\alpha \not\vdash X_\beta$, thus for example, all the sets $\Gamma(\mathcal{G}_i)$ are distinct from each other and, in particular, from $\Gamma(\mathcal{G}_1) \equiv \Gamma(\dot{\underline{\mathcal{E}}}_1)$.

1.7.7.3 Example of a formula φ for which there is a topos in which $\varphi^*[p|\bot]$ is valid and φ^* is not valid.

1.7.7.3.1 Proposition. For $\varphi \equiv (u_1 \Rightarrow u_2) \vee (u_2 \Rightarrow u_1)$

$$\varphi'[v|0] \dashv\vdash u \vee \neg u$$

and hence in a topos $\underline{\mathcal{E}}$,

$$(\varphi^*[p|\bot]) \Leftrightarrow \neg \forall q(q \vee \neg q).$$

Proof. ($\vdash$) Suppose $\varphi'[v|\underline{0}] \not\vdash u \vee \neg u$, then there exists an $\mathcal{S}\ell$ algebra A such that $A \vdash \varphi'[v|\underline{0}]$ but $A \not\vdash u \vee \neg u$. Hence $\mathbf{3}$ is a subalgebra of A, hence $\mathbf{3} \vdash \varphi'[v|\underline{0}]$. Let $|\mathbf{3}| \equiv \{0,e,1\}$ and let $\alpha: V \longrightarrow |\mathbf{3}|$ be such that for $U \equiv s_F(\varphi'[v|\underline{0}])$, $\alpha|_U \equiv \begin{pmatrix} u_1 & u_1 & u_2 & u_2 \\ e & e & 1 & 0 \end{pmatrix}$ then

$\tilde{\alpha}(\varphi'[v|\underline{0}]) \equiv ([(e \Rightarrow 0) \wedge (e \Rightarrow 1)] \vee [(0 \Rightarrow e) \wedge (1 \Rightarrow e)]) \equiv e$

-a contradiction.□

($\dashv$) Suppose $u \vee \neg u \not\vdash \varphi'[v|\underline{0}]$ then $\mathbf{2} \not\vdash \varphi'[v|\underline{0}]$ hence there exists α such that $\alpha|_U \equiv \begin{pmatrix} u_1 & u_1 & u_2 & u_2 \\ a_1 & b_1 & a_2 & b_2 \end{pmatrix}$ where $U \equiv s_F(\varphi'[v|0])$ and

and $\tilde{\alpha}(\varphi'[v|\underline{0}]) \equiv 0$, hence both $[((b_1 \wedge a_1) \Rightarrow (b_2 \wedge a_2)) \wedge (a_1 \Rightarrow a_2)] \equiv 0$ and $[((b_1 \wedge a_1) \Rightarrow (b_2 \wedge a_2)) \wedge (a_1 \Rightarrow a_2)] \equiv 0$; neither $a_1 = 0$ nor $a_2 = 0$ is possible, hence both $(b_1 \Rightarrow b_2) \equiv 0$ and $(b_2 \Rightarrow b_1) \equiv 0$; but this is not possible -a contradiction.□

1.7.7.3.2 Proposition.

(1) $\dot{\mathcal{E}}_1 \models Z^*[p|\bot]$ and (2) $\dot{\mathcal{E}}_1 \not\models Z^*$.

Proof. (1). $\dot{\mathcal{E}}_1 \models \neg \forall q(q \vee \neg q)$, hence $\dot{\mathcal{E}}_1 \models Z^*[p|\bot]$, by 1.7.7.3.1.□

(2) $Z \not\vdash \Delta^2(0)$, since $\Delta^2(\underline{C}) \dashv\vdash X_{\mathfrak{A}}$, by 1.1.4.6.7 and $\mathfrak{A} \vdash X_{\mathfrak{A}}$, by 1.1.4.2 whereas $\mathfrak{A} \vdash Z$. Since $\Gamma(\dot{\mathcal{E}}_1) \equiv \{\varphi|\ \varphi \vdash \Delta^2(\underline{0})\}$, we have $Z \notin \Gamma(\dot{\mathcal{E}}_1)$, hence $\dot{\mathcal{E}}_1 \not\models Z^*$, by 1.7.6.10.□

1.7.7.4 The degeneracy of the condition $C_n \in \Gamma(\mathcal{E})$.

The polynomials C_n were defined in 1.1.3.4 where it was established that $C_n \in \mathcal{T}^*$ and $C_n \dashv\vdash E_n$.

1.7.7.4.1 Proposition. $\models C_n'[v|\underline{0}] = C_n$.

Proof. We first show $\models\left(\left(\overset{n}{\underset{i \equiv 1}{\Rrightarrow}} u_i\right)(\underline{0})\right)'[v|0] = \left(\overset{n}{\underset{i \equiv 1}{\Rrightarrow}} u_i\right)(\underline{0})$,

by induction:

$$\begin{aligned}\models \left(\left(\overset{1}{\underset{i \equiv 1}{\Rrightarrow}} u_i\right)(\underline{0})\right)'[v|\underline{0}] &= (u_1 \Rightarrow 0)'[v|\underline{0}]\\ &= ((((\dot{u}_1 \vee v) \wedge u_1) \Rightarrow \underline{0}) \wedge (u_1 \Rightarrow \underline{0}))[v|\underline{0}]\\ &= \neg(\dot{u}_1 \wedge u_1) \wedge \neg u_1\\ &= \neg u_1\\ &= \left(\overset{1}{\underset{i \equiv 1}{\Rrightarrow}} u_i\right)(\underline{0}).\end{aligned}$$

$$\models \left(\left(\overset{k+1}{\underset{i\,\equiv\,1}{\Longrightarrow}} u_i\right)(\underline{\Omega})\right)'[v|\underline{\Omega}]$$

$$= \left(u_{k+1} \Rightarrow \left(\overset{k}{\underset{i\,\equiv\,1}{\Longrightarrow}} u_i\right)(\underline{\Omega})\right)'[v|\underline{\Omega}]$$

$$= \left[\left(u'_{k+1} \Rightarrow \left(\left(\overset{k}{\underset{i\,\equiv\,1}{\Longrightarrow}} u_i\right)(\underline{\Omega})\right)'\right) \wedge \left(u_{k+1} \Rightarrow \left(\overset{k}{\underset{i\,\equiv\,1}{\Longrightarrow}} u_i\right)(\underline{\Omega})\right)\right][v|\underline{\Omega}]$$

$$= \left((u_{k+1} \wedge u_{k+1}) \Rightarrow \left(\overset{k}{\underset{i\,\equiv\,1}{\Longrightarrow}} u_i\right)(\underline{\Omega})\right) \wedge \left(u_{k+1} \Rightarrow \left(\overset{k}{\underset{i\,\equiv\,1}{\Longrightarrow}} u_i\right)(\underline{\Omega})\right) \quad \text{by induction}$$

$$= \left(\overset{k+1}{\underset{i\,\equiv\,1}{\Longrightarrow}} u_i\right)(\underline{\Omega}).$$

Hence

$$\models C_n'[v|\underline{\Omega}] = \left(\left(\overset{n+1}{\underset{i\,\equiv\,1}{\Longrightarrow}} u_i\right)(\underline{\Omega})\right)'[v|\underline{\Omega}] \Rightarrow \left(\overset{n+1}{\underset{i\,\equiv\,1}{\bigvee}}\left(\left(\left(\overset{n+1}{\underset{j\,\equiv\,1,j\,\not\equiv\,i}{\Longrightarrow}} u_j\right)(\underline{\Omega})\right)'[v|\underline{\Omega}]\right)\right)$$

$$\wedge \left(\left(\overset{n+1}{\underset{i\,\equiv\,1}{\Longrightarrow}} u_i\right)(\underline{\Omega}) \Rightarrow \overset{n}{\underset{i\,\equiv\,1}{\bigvee}} \left(\overset{n+1}{\underset{j\,\equiv\,1,j\,\not\equiv\,i}{\Longrightarrow}} u_j\right)(\underline{\Omega})\right)$$

$$= \left(\overset{n+1}{\underset{i\,\equiv\,1}{\Longrightarrow}} u_i\right)(\underline{\Omega}) \Rightarrow \left(\overset{n+1}{\underset{i\,\equiv\,1}{\bigvee}}\left(\left(\overset{n+1}{\underset{j\,\equiv\,1,j\,\not\equiv\,i}{\Longrightarrow}} u_j\right)(\underline{\Omega})\right)\right) \wedge C_n$$

$$= C_n. \ \square$$

<u>1.7.7.4.2</u> For the purposes of this particular class of polynomials we use the converse logic $\Gamma_{max}(\underline{\mathcal{E}})$ defined in 1.2.5.3.2 (<u>3</u>). Recall that since initial categories are groupoids, we have

$$\Gamma_{max}(\underline{\mathcal{E}}) \subset \Gamma(\underline{\mathcal{E}}) .$$

We choose a particular initial category $\underline{0}$ in $\underline{\mathcal{E}}$.

<u>1.7.7.4.3 Proposition</u>. The following are equivalent for a topos $\underline{\mathcal{E}}$.

(1) $C_n \in \Gamma(\underline{\mathcal{E}})$

(2) $C_n \in \Gamma_{max}(\underline{\mathcal{E}})$

(3) $\underline{\mathcal{E}} \models \neg \forall \vec{r}\ \overline{C}_n$ where $\vec{r} \equiv var(\overline{C}_n)$.

<u>Proof.</u> (2) → (1) Clear.□ (1) → (3) By 1.7.6.10 and 1.7.7.4.1.□

(3) → (2). Suppose $\underline{\mathcal{E}} \models \neg \forall \vec{r}\, \overline{C}_n$ and $\underline{D}$ is an internal category for which $\underline{\mathcal{E}}^{\underline{D}^0} \models \overline{C}_n$. Since the pullback functor $D_0^*: \underline{\mathcal{E}} \longrightarrow \underline{\mathcal{E}}/D_0$ is logical, we have $\underline{\mathcal{E}}/D_0 \models \neg \forall \vec{r}\, \overline{C}_n$ where now $\vec{r}$ and $\overline{C}_n$ are interpreted in $\underline{\mathcal{E}}/D_0$. By 1.6.2.9, $\underline{\mathcal{E}}/D_0 \models \overline{C}_n$, hence $\underline{\mathcal{E}}/D_0 \models \forall \vec{r}\, \overline{C}_n$. Combining our conclusions we have $\underline{\mathcal{E}}/D_0 \models \bot$, hence $\underline{D}$ must be isomorphic to $\underline{\mathbf{0}}$ in $\underline{\mathcal{E}}$. □

<u>1.7.7.4.4 Corollary.</u> The following are equivalent

(1) $\underline{\mathcal{E}} \models \overline{C}_n$ and $C_n \in \Gamma(\underline{\mathcal{E}})$

(2) $\underline{\mathcal{E}}$ is degenerate.

<u>Proof.</u> (2) → (1). Clear.□ (1) → (2) Take $\underline{\mathbb{1}}$ to be the terminal category, then $\underline{\mathcal{E}} \approx \underline{\mathcal{E}}^{\underline{\mathbb{1}}}$. Suppose $\underline{\mathcal{E}} \models \overline{C}_n$ and $C_n \in \Gamma(\underline{\mathcal{E}})$, then $\underline{\mathcal{E}}^{\underline{\mathbb{1}}} \models \overline{C}_n$ and $C_n \in \Gamma_{max}(\underline{\mathcal{E}})$hence $\underline{\mathbb{1}} \approx \underline{\mathbf{0}}$, hence $\underline{\mathcal{E}}$ is degenerate.□

<u>1.7.7.4.5 A final graphic illustration.</u>

For the topos $\underline{\dot{\mathcal{G}}\mathit{ph}}$ we have $\underline{\dot{\mathcal{G}}\mathit{ph}} \models \overline{E}_2$. Since $E_2 \dashv\vdash C_2$ we must have $\underline{\dot{\mathcal{G}}\mathit{ph}} \not\models \neg \forall \vec{r}\, \overline{C}_2$, where $\vec{r} \equiv var(\overline{C}_2)$, hence $E_2 \notin \Gamma(\underline{\dot{\mathcal{G}}\mathit{ph}})$. Thus there must exist an internal category $\underline{D}$ in $\underline{\dot{\mathcal{G}}\mathit{ph}}$ which is <u>not</u> a groupoid and for which $\underline{\dot{\mathcal{G}}\mathit{ph}}^{\underline{D}^0} \models \overline{E}_2$.

The failure of E_2 to belong to $\Gamma(\underline{\dot{\mathcal{G}}\mathit{ph}})$ has, in the light of 1.7.7.4.4 less to do with any special characteristic of $\underline{\dot{\mathcal{G}}\mathit{ph}}$ and more to do with E_2. That is for <u>any</u> non-degenerate topos $\underline{\mathcal{E}}$ for which $\underline{\mathcal{E}} \models \overline{E}_2$ there must always exist an internal category $\underline{D}$ which is not a groupoid and for which $\underline{\mathcal{E}}^{\underline{D}^0} \models \overline{E}_2$.

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