QUADRATURE-AMPLITUDE-MODULATED SIGNALLING OVER A DISCRETE-MULTIPATH LINEAR TIME-VARIANT CHANNEL

by

AMBIGHAIRAJAH YASOTHARAN

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AUTHOR: Ambighairajah Yasotharan
B.Sc.(Eng) (University of Peradeniya, Sri Lanka)
M.Eng. (McMaster University)

SUPERVISORY COMMITTEE: Dr. D. P. Taylor (Chairman)
Dr. K. M. Wong (Co-Chairman)
Dr. J. P. Reilly
Dr. J. M. Chadam

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To My Parents and Teachers

Abstract

In this thesis, a new paradigm is proposed for designing the transmitter and receiver for quadrature-amplitude-modulated signalling over a mobile radio channel. The new paradigm is based on a discrete-multipath linear time-variant model of the mobile radio channel, and hence the title of the thesis. The time-variant input-output relationship of the discrete-multipath channel (DMC) is governed by a set of parameters which can be obtained in finite time by *probing*, that is, by transmitting a pre-assigned signal and then performing computations on the received signal. Therefore, once the parameters of the DMC's input-output relationship have been obtained in this manner, the receiver can, in principle, determine the subsequently transmitted data-carrying signal, or, the data itself, by performing computations on the received signal, which operation is referred to as *signalling*.[1]

Thus, the thesis proposes a philosophy of design based on alternate probing and signalling, and shows that when the transmitted signal is generated by quadrature amplitude modulation (QAM) the composition of QAM and DMC lends itself to this philosophy of design, even in the presence of intersymbol interference (ISI) and additive white Gaussian noise (AWGN).

As regards probing, it is shown that by transmitting a suitable quadrature-amplitude-modulated signal all parameters of the DMC, or, rather, of the composition of QAM and DMC, can be estimated in the presence of AWGN. In particular, the maximum-likelihood method of estimation is shown to have the statistical properties needed to justify the philosophy of design.

As regards signalling, based on the assumption that the parameters of the DMC, or, rather, of the composition of QAM and DMC, are known by the receiver, it is shown how the receiver may decide which data sequence was likely transmitted, taking into account

[1]The term signalling also refers in a wide sense to the combined operation of *probing* and *signalling*, as it does for example in the title of the thesis.

ISI and AWGN according to some optimal rule. Motivated by the classical receiver design principles used for quadrature-amplitude-modulated signalling over a linear time-invariant channel in the presence of ISI and AWGN, namely,

1. linear zero-forcing equalizer,
2. decision-feedback zero-forcing equalizer,
3. linear mean-square-error equalizer,
4. decision-feedback mean-square-error equalizer,
5. maximum-likelihood sequence estimator of the Forney-type,
6. maximum-likelihood sequence estimator of the Ungerboeck-type,

the thesis shows how these principles can be generalized for quadrature-amplitude-modulated signalling over a DMC in the presence of ISI and AWGN. Despite the DMC's being time-variant, these generalized receivers can be implemented with a bank of continuous-time time-invariant filters at the front.

The thesis, although mainly theoretical, illustrates some of the above methods through computer simulations. More specifically, numerical results are given for probing by maximum-likelihood method and signalling by a linear zero-forcing equalizer, under various system specifications and scenarios involving the geometry of propagation and speeds of movement.

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Chapter 1

Introduction

1.1 The Context of the Thesis

This thesis addresses a fundamental problem that arises in mobile radio communications. The primary aim of this chapter is to explain how exactly the problem arises, and thereby justify the choice of the title "Quadrature-Amplitude-Modulated Signalling over a Discrete-Multipath Linear Time-Variant Channel."

The last two decades have seen an intensification of research activity in the field of mobile radio communications. This intensification has been driven by the realization that the existing mobile communications technologies, based on analog modulations, will soon be unable to cope with the fast increasing demand for services. Although the aim of research has had many facets, the overall aim has been to provide reliable voice and data communications using as little resources – mainly, spectral occupancy and signal power – as possible. The requirements of spectral efficiency and integrability of voice and data have strongly suggested the use of digital modulations. But digital modulations when used in conjunction with the conventional receiver techniques – the techniques used in line-of-sight point-to-point digital microwave radio, for example – have failed to provide the required level of reliability.

This failure can be attributed to the rapid and seemingly random variations in the impulse response characteristics of the mobile radio communications channel that is in effect between moving transmitters and receivers. To gain some insight into the cause of failure, suppose that a purely sinusoidal signal - an unmodulated carrier, for example - is being transmitted in a mobile communications context. Owing to the rapid variations in the channel characteristics, the received signal will exhibit seemingly random fluctuations in the instantaneous frequency and amplitude so that it can hardly be considered sinusoidal. This phenomenon has serious implications for those digital modulations that depend on carrier synchronization, for carrier synchronization, at least in the traditional sense,[1] would be impossible to achieve. This is especially so when the carrier frequency and the speed of movement are high, for the rate of fluctuation in the received signal would then be comparable to the rate of signalling. In loose terms, this thesis is concerned with designing receivers for digital signalling over mobile radio communications channels that exhibit rapid variations in their impulse response characteristics.

It has widely been claimed that a mobile radio communications channel that exhibits rapid variations in its impulse response characteristics cannot be characterized in a deterministic sense, especially if the characterization must be simple enough to be useful in designing receivers. Perhaps as a result, all techniques for designing receivers for such rapidly varying channels have been based on stochastic characterizations of the channels. This thesis, however, demonstrates that there are certain types of mobile radio communications channels that can be characterized in a deterministic sense regardless of the rapidity of variations in their impulse response characteristics. This thesis further demonstrates the utility of this deterministic characterization in designing receivers and analyzing their performances. The techniques presented in this thesis have their roots in the solutions to the classical problem of "Quadrature-Amplitude-Modulated Signalling over a Linear Time-Invariant Channel." In fact, the problem addressed in this thesis, as stated in the title, can be considered as a generalization of the aforementioned classical problem.

1.1.1 The Organization of the Chapter

The rest of this chapter is organized as follows. In section 1.2, a model of the mobile radio channel is developed on the basis of certain postulates; in system-theoretic terms, the model is linear and time-variant. In section 1.3, the notion of the mobile radio channel's being random or stochastic is briefly reviewed. In section 1.4, some examples of mobile radio channels that may be considered as being deterministic are given, and a time-variant mobile radio channel termed "Discrete-Multipath Channel" is introduced, with a claim

[1] A technique proposed in chapter 2 can be considered as a generalized method of carrier synchronization.

that it can be considered as being deterministic. In section 1.5, the classical problem of "Quadrature-Amplitude-Modulated Signalling over a Linear Time-Invariant Channel" is discussed with emphasis on the notion of intersymbol interference. In section 1.6, the problem of "Quadrature-Amplitude-Modulated Signalling over a Mobile Radio Channel" is discussed with emphasis on the implications of the channel's being time-variant. In section 1.7, a new philosophy is proposed for designing the transmitter and receiver for quadrature-amplitude-modulated signalling over a mobile radio channel of a certain type, and thereby the problem addressed in the thesis, as stated in the title, is introduced. The specific contributions made by the thesis toward substantiating the proposed philosophy are outlined. In section 1.8, the organization of the rest of the thesis is presented.

1.2 The Mobile Radio Communications Channel

1.2.1 Multipath Propagation, Fading

In a mobile radio communications scenario, the electromagnetic energy radiated in an omnidirectional manner can reach surrounding locations of interest by line-of-sight propagation and/or by reflections and scattering from objects such as buildings, trees, vehicles, and mountains, and also by diffraction [32]. Thus, the propagation of electromagnetic energy can be considered as taking place along multiple paths. This characteristic mode of propagation, usually known as *multipath propagation*, is both a blessing and a curse; on the one hand, it makes communication possible, in principle, even in the absence of the line-of-sight; on the other hand, it is at the root of the failure of those digital modulations that depend on carrier synchronization.

To gain some insight into this negative aspect of multipath propagation, suppose that a pure sinusoidal signal – an unmodulated carrier, for example – is being transmitted. The receiver would see the sum of the sinusoidal signals received over the various paths. Assuming the transmitter, receiver, and all reflecting, scattering, and diffracting objects to be stationary, the component sinusoidals would be of the same frequency, as at the transmitter, but of possibly different amplitudes and phases. Since the phases of the components would be sensitively dependent on the electrical lengths of the respective paths of propagation, the amplitude and phase of the combined sinusoidal signal as seen by the receiver would be sensitively dependent on the spatial location of the receiver.[2] If the receiver is in motion, this spatial variation would be seen as a temporal fluctuation of the effective channel response, the rate of fluctuation being higher for higher speeds of movement. In the prevalance of a multitude of paths, the fluctuation would be rapid and appear to be random, and the received signal's strength would fall to undetectably low levels in an unpredictable manner. Therefore, this phenomenon of fluctuation has been dubbed *fading*.

It may seem that the undesirable phenomenon of fading can be avoided by selectively receiving only the strongest component, the line-of-sight component perhaps. But there may not be a line-of-sight, and, amongst the other components, none may be strong enough. Even if there were a strong component, an antenna with high directivity and tracking capabilities would be needed to receive it; using such an antenna is an impractical proposition for portable transceivers. Fading is therefore considered unavoidable.

The discussion so far about fading has been qualitative. Fading is but one effect of multipath propagation. To obtain a quantitative understanding of fading and any other effects of multipath propagation, one must consider multipath propagation in a general setting, where the transmitted signal may be non-sinusoidal and the transmitter, receiver, and all reflecting, scattering, and diffracting objects may be moving.

1.2.2 Postulates on Multipath Propagation

A mathematical model that would adequately reflect the effects of multipath propagation in a general setting can be developed based on the following postulates on multipath propagation:

1. The phenomena of line-of-sight propagation, reflection, scattering, and diffraction are all *linear*; in other words, each path of propagation, when considered as a channel, is *linear*.

2. In the vicinity of the receiver, the electromagnetic energy associated with each path propagates as a linearly polarized plane wave.[3]

[2] In analogy with an improperly terminated transmission line, a spatial standing wave may be said to exist.

[3] When sinusoidal signals are involved, circular and elliptic polarizations can be handled by appropriately decomposing them into orthogonal linear polarizations.

3. The receiver sees the sum of the effects of all the paths, or equivalently the additive interference of the aforementioned plane waves.

According to this last postulate, multipath propagation, *per se*, can be characterized by linear superposition as soon as propagation over a single path has been characterized.

1.2.3 Propagation over a Single-Path in a General Setting

Assume there to be only one path of propagation from the transmitter to the receiver. Also assume the transmitter, receiver, and all reflecting, scattering, and diffracting objects to be either *stationary* or moving with *constant velocity*, that is, moving *rectilinearly* with constant speed. According to the first and the second postulates, there is no loss of generality in restricting consideration to the following:

1. There is only the line-of-sight path.
2. The transmitter, and all reflecting, scattering, and diffracting objects are *stationary*.
3. The receiver is either *stationary* or moving with *constant velocity*.

An Arbitrary Signal

Assume the signal $x(t)$ to be transmitted, and consider its reception in the vicinity of a *reference* location. The received signal $y(t)$, as a function of location of the receiver and time, is given by

$$y(t) = \beta x\left((c(t-\tau)+d)/c\right), \tag{1.1}$$

where

1. c is the speed of propagation of the electromagnetic wave,
2. τ is the time of propagation up to the reference location,
3. d is the distance of the receiver measured from the reference location in the direction perpendicular to the planar wavefront and opposite to that of propagation,
4. β is the gain[4] of propagation, which is assumed to be relatively independent of d in the vicinity of the reference location.

[4]More appropriately, β is the scaling factor that represents the attenuation.

This representation is known as the *travelling wave* representation. If v is the velocity of the receiver in the direction perpendicular to the wavefront and opposite to that of propagation, then

$$d = vt. \tag{1.2}$$

Using this, the received signal $y(t)$, as a function of time alone, is given by

$$y(t) = \beta x\left((1 + v/c)t - \tau\right). \tag{1.3}$$

Thus, the received signal is a time-contracted, time-translated, and scaled version of the transmitted signal.

A Pure Sinusoidal Signal, Doppler Shift

Let $x(t) = \sin \omega_0 t$ be the transmitted signal. The received signal $y(t)$ is then given by

$$y(t) = \beta \sin\left(\omega_0\left((1 + v/c)t - \tau\right)\right). \tag{1.4}$$

The time-contraction of the signal manifests itself in the frequency shift of $\omega_0 v/c$ rad/s known as the *Doppler shift.*

A Narrowband Signal

Let $\Re\left(x(t)e^{j\omega_0 t}\right)$ be the transmitted narrowband signal with $x(t)$ being a complex-valued lowpass signal. The received signal $\Re\left(y(t)e^{j\omega_0 t}\right)$ is then given by

$$y(t)e^{j\omega_0 t} = \beta x\left((1 + v/c)t - \tau\right) e^{j\omega_0((1+v/c)t-\tau)}, \tag{1.5}$$

or, equivalently,

$$y(t) = \beta e^{-j\omega_0\tau} x\left((1 + v/c)t - \tau\right) e^{j\frac{\omega_0 v}{c}t}. \tag{1.6}$$

Since $x(t)$ is a lowpass signal, and since $1 + v/c \approx 1$ typically, the following approximation is possible:

$$y(t) \approx \beta e^{-j\omega_0\tau} x(t - \tau) e^{j\frac{\omega_0 v}{c}t}. \tag{1.7}$$

Observe that $x(t)$ may be the complex envelope [11] of the transmitted narrowband signal, in which case $y(t)$ would be the complex envelope of the received narrowband signal. Thus, when a narrowband signal propagates over a single path, its complex envelope undergoes a delay, phase change, Doppler shift, and (real) scaling. In theory, $x(t)$ can be reconstructed from $y(t)$ given β, v, and τ. But, in practice, ω_0 may be very large and τ may not be known accurately enough to determine the phase change to within 2π radians. Therefore, the phase change must be considered as being independent of the delay, and $y(t)$ must be considered as

$$y(t) \approx \beta x(t-\tau)e^{j\frac{\omega_0 v}{c}t} \tag{1.8}$$

with β being complex. Thus, when a narrowband signal propagates over a single path, its complex envelope undergoes a delay, Doppler shift, and complex scaling or gain. The transmitted signal will henceforth be assumed to be of the narrowband-type, and the received signal will be approximated as above.

1.2.4 Multipath Propagation of Narrowband Signals

From the discussions of the previous section, every path can be associated with a triple (β, ω, τ).

Discrete-Multipath Propagation

Multipath propagation where the set of pairs (ω, τ) is discrete, in the two-dimensional plane, is said to be *discrete*. Discrete-multipath propagation can be characterized by a summation. Thus, the complex envelopes $x(t)$ and $y(t)$ of the transmitted and received signals respectively are related by

$$y(t) = \sum_k \beta_k e^{j\omega_k t} x(t-\tau_k), \tag{1.9}$$

where k runs over a set of path indices; the variables β_k, ω_k, and τ_k are the gain, Doppler shift, and delay respectively of the k^{th} path.

Diffuse-Multipath Propagation

Multipath propagation where the set of pairs (ω, τ) is a continuum, in the two-dimensional plane, is said to be *diffuse*. Diffuse-multipath propagation must in general be characterized by an integral. Thus, the complex envelopes $x(t)$ and $y(t)$ of the transmitted and received signals respectively are related by

$$y(t) = \int\int \beta(\omega, \tau) e^{j\omega t} x(t-\tau) d\omega d\tau, \tag{1.10}$$

where $\beta(\omega, \tau)$ is possibly a continuous complex-valued function.

General Remarks

1. Observe that discrete-multipath propagation can also be characterized by an integral using

$$\beta(\omega, \tau) = \sum_k \beta_k \delta(\omega - \omega_k)\delta(\tau - \tau_k), \tag{1.11}$$

where $\delta(t)$ is the Dirac delta function.

2. Defining

$$\phi(t, \tau) = \int \beta(\omega, \tau) e^{j\omega t} d\omega, \tag{1.12}$$

in the case of diffuse-multipath propagation, or

$$\phi(t, \tau) = \sum_k \beta_k e^{j\omega_k t} \delta(\tau - \tau_k), \tag{1.13}$$

in the case of discrete-multipath propagation, one obtains

$$y(t) = \int \phi(t, \tau) x(t - \tau) d\tau, \tag{1.14}$$

which resembles the input-output relationship of a *linear time-variant channel.* Thus, from a system-theoretic point of view, the mobile radio communications channel can be considered a linear time-variant channel, and the phenomenon of fading can be considered the manifestation of the channel's being time-variant.

1.2.5 A Classification of Multipath Propagation

Time-Selective Multipath Propagation

Multipath propagation where all paths have the same delay τ_0 is said to be *time-selective,* for using $\beta(\omega, \tau) = \beta_0(\omega)\delta(\tau - \tau_0)$ one obtains the time-selective relationship

$$y(t) = x(t - \tau_0)\phi_0(t), \tag{1.15}$$

where

$$\phi_0(t) = \int \beta_0(\omega) e^{j\omega t} d\omega. \tag{1.16}$$

Frequency-Selective Multipath Propagation

Multipath propagation where all paths have the same Doppler shift ω_0 is said to be *frequency-selective*, for using $\beta(\omega, \tau) = \beta_0(\tau)\delta(\omega - \omega_0)$ one obtains the relationship

$$y(t) = \int \beta_0(\tau)e^{j\omega_0 t}x(t-\tau)d\tau, \tag{1.17}$$

which, when written in terms of the Fourier transform[5]

$$\hat{x}(\omega) = \int x(t)e^{-j\omega t}d\omega \tag{1.18}$$

of $x(t)$ and those of $\beta_0(\tau)$ and $y(t)$, similarly, yields the frequency-selective relationship

$$\hat{y}(\omega + \omega_0) = \hat{\beta}_0(\omega)\hat{x}(\omega). \tag{1.19}$$

Time- and Frequency-Selective Multipath Propagation

Multipath propagation that does not fall into one of the above classes is said to be *time- and frequency-selective*, for the relationship between the transmitted and received signals has the same form in both time and frequency domains.

In reality, the underlying multipath propagation of a mobile radio channel will be time-selective to some degree and frequency-selective to some degree; nevertheless, the purely time-selective and the purely frequency-selective models are useful in visualizing the boundary behaviour when $x(t)$ has small bandwidth or short duration respectively.

1.3 Stochastic Characterization of Mobile Radio Channels

It has widely been claimed that mobile radio communications channels should be considered as being *random* or *stochastic* (see [32], [19]). Such claims can be interpreted as saying that the particular channel *realization* that a mobile radio transmitter/receiver will experience, at a certain locality in space and time, cannot be known *a priori* with certainty.

One way of dealing with this situation is to characterize the *ensemble* of channels that a mobile radio receiver may experience in a region of a particular topography/terrain. This is the essence of stochastic characterization of mobile radio channels. In stochastic characterization, usually, the number of paths are assumed to be large, and, therefore, no distinction is made between discrete-multipath propagation and diffuse-multipath propagation.

[5] This convention of denoting the Fourier transform with a ^ ('hat') will henceforth be followed.

When a mobile radio channel has a large number of paths, the *central limit theorem* can be invoked to deduce, under some additional assumptions, that the random function $\beta(\omega, \tau)$, or a component thereof, has a *Gaussian* probability density.[6] This approach has been used by Clarke[6] and others (see [32] and the references therein) to develop stochastic models of time-selective mobile radio channels. It is also claimed[32] that the assumption that there is a large number of paths is not crucial, because the models are approximately valid provided there are at least six paths.[7]

1.4 Deterministic Characterization of Mobile Radio Channels

The claim that a mobile radio channel cannot be characterized in a deterministic manner was perhaps based on the belief that such a characterization should explicitly involve the *geometric positions* and the *velocities* of the transmitter, receiver, and reflecting, scattering, and diffracting objects. This claim is wrong, for instance, if a mobile radio channel is known *a priori* to be frequency-selective with $\omega_0 = 0$ (c.f. the discussion in section 1.2.5), for the time-invariant input-output relationship

$$y(t) = \int \beta_0(\tau) x(t - \tau) d\tau \tag{1.21}$$

of such a channel is characterized by the impulse response $\beta_0(t)$ which can be determined merely by probing the channel, that is, by transmitting a pre-assigned signal and observing the received signal. Another simple example of a mobile radio channel that can be characterized in a deterministic manner is the one-path channel (c.f. the opening sentence of section 1.2.4), for the input-output relationship

$$y(t) = \beta_0 e^{j\omega_0 t} x(t - \tau_0), \tag{1.22}$$

[6] A channel whose $\beta(\omega, \tau)$ has a *Gaussian* probability density is also a known as a *Rayleigh* fading channel. A channel whose $\beta(\omega, \tau)$ has the form

$$\beta(\omega, \tau) = \beta_0(\omega, \tau) + \beta_1 \delta(\omega - \omega_1) \delta(\tau - \tau_1), \tag{1.20}$$

where $\beta_0(\omega, \tau)$ has a *Gaussian* probability density, the quantities $|\beta_1|$, ω_1, and τ_1 are fixed, and $\arg(\beta_1)$ modulo-2π has an independent uniform probability density, is known as a *Rician* fading channel.

[7] When a mobile radio channel has a small number of paths, a better characterization may be obtained by making measurements of the set of triples $(\beta_k, \omega_k, \tau_k)$. This thesis demonstrates how those measurements can be made and how the measurements can be made use of in system design.

of such a channel is characterized by the numbers β_0, ω_0, and τ_0 which can be determined merely by probing the channel.[8]

A question arises as to whether there are more general mobile radio channels that can be characterized in a deterministic manner by probing them. This thesis demonstrates that there is a wider class of time- and frequency-selective multipath channels that can be characterized in a deterministic manner by probing them; this class of channels is introduced next but a discussion on probing them is deferred until chapter 2.

1.4.1 Discrete-Multipath Channel – The Channel Studied in the Thesis

Consider the discrete-multipath channel introduced in section 1.2.4. In general, there can be more than one path with the same Doppler shift, and hence the input-output relationship can be written as

$$y(t) = \sum_k e^{j\omega_k t} \sum_l \beta_{kl} x(t - \tau_{kl}), \tag{1.23}$$

$$= \sum_k e^{j\omega_k t} \int \phi_k(\tau) x(t - \tau) d\tau, \tag{1.24}$$

where the functions $\phi_k(t)$ are defined by

$$\phi_k(t) = \sum_l \beta_{kl} \delta(t - \tau_{kl}). \tag{1.25}$$

The further specialization obtained by requiring that the number of Doppler shifts be small, that is, a channel whose input-output relationship is of the form

$$y(t) = \sum_{k=1}^{K} e^{j\omega_k t} \int \phi_k(\tau) x(t - \tau) d\tau, \tag{1.26}$$

where K is small, is the channel studied in this thesis.[9] Such a channel will henceforth be referred to as a *Discrete-Multipath Channel* (DMC). The time-variant impulse response $\phi(t, \tau)$ of a DMC has the form

$$\phi(t, \tau) = \sum_{k=1}^{K} e^{j\omega_k t} \phi_k(\tau), \tag{1.29}$$

[8]This situation also arises as the combined problem of automatic gain control, carrier synchronization, and timing synchronization, when signalling through an otherwise ideal channel.

[9]The study applies equally well to a diffuse-multipath channel whose $\beta(\omega, \tau)$ has the form

$$\beta(\omega, \tau) = \sum_{k=1}^{K} \delta(\omega - \omega_k) \phi_k(\tau), \tag{1.27}$$

for the input-output relationship can then be written as

$$y(t) = \int \phi(t,\tau)x(t-\tau)d\tau. \tag{1.30}$$

Some studies of microcellular radio propagation [21] have shown that the signal strength variation with distance can be predicted by considering six interfering rays, or paths. This finding provides enough justification for studying the problem of digital signalling over a DMC.

This thesis specifically addresses the situation where the transmitted signal $x(t)$ is generated by quadrature amplitude modulation (see section 1.5.1). Since the linear time-invariant channel is a special case of the DMC, obtained by setting $K = 1$ and $\omega_1 = 0$, it is instructive to first consider the problem of quadrature-amplitude-modulated signalling over a linear time-invariant channel.

1.5 Quadrature-Amplitude-Modulated Signalling over a Linear Time-Invariant Channel

A host of techniques known for communicating digital data over linear time-invariant channels is reviewed in this section. The definitions, terms, and concepts introduced here will set the stage for reviewing, in section 1.6, the techniques known for communicating digital data over mobile radio channels, and for introducing, in section 1.7, the problem of communicating digital data over discrete-multipath channels.

The most fundamental problem in designing a digital communications system is that of designing the transmitter and the receiver. The problem of designing the transmitter is one of choosing a *coding/modulation* scheme, that is, a class of signals that can unambiguosly and robustly represent the digital data. The problem of designing a receiver is one of *hypothesis testing,*[10] that is, *guessing*, from the received signal, the digital data that was likely transmitted. A transmitter design is useful insofar as an associated receiver design is

for its input-output relationship is

$$y(t) = \sum_{k=1}^{K} e^{j\omega_k t} \int \phi_k(\tau)x(t-\tau)d\tau. \tag{1.28}$$

[10]The channel may introduce noise and other random distortions on the signal.

implementable and the combination performs well in terms of probability of correctness.[11] In this sense, quadrature amplitude modulation forms the basis of many useful designs.

1.5.1 Quadrature Amplitude Modulation

The term *Quadrature Amplitude Modulation* (QAM) means the generation of a complex-valued signal[12] $x(t)$ of the form

$$x(t) = \sum_n a(n)g(t - nT), \tag{1.31}$$

with the purpose of communicating the sequence $a(n)$ of complex numbers that represent the data; as such, the sequence $a(n)$ is known as the *data sequence*; the function $g(t)$ is complex-valued and square-integrable; the time interval T is known as the *symbol* period or *baud* period, and its reciprocal $1/T$ is known as the baud rate. The elements $a(n)$ are chosen from a finite set of complex numbers – known variously as the alphabet, *signal set*, signal constellation, etc.– some commonly used ones being the PSK-type sets[13] and the QAM-type sets (16-QAM, 64-QAM). This thesis is concerned with a *generic* signal set rather than with a specific one, and the choice of $a(n)$, for successive values of n, is considered as being done by a probabilistic information source in an independent manner.[14]

On the Terminology

In the literature, the acronym QAM is used in two different senses. In one sense, it means the generic modulation scheme discussed above. In the other sense, it means a more specific scheme whose signal set is a subset of a two-dimensional rectangular lattice, 16-QAM and 64-QAM for example. In this thesis, QAM is used in the first sense. However, it is the need in mobile communications for high-spectral-efficiency signal sets such as 16-QAM and 64-QAM that motivated the thesis. Therefore, no great harm is done by this loose terminology.

[11] The method of hypothesis testing that has the lowest probability of error is the one that chooses the data whose conditional probability Prob[data/received signal] is the largest; the receiver based on this method is known as the *maximum a posteriori probability receiver* [57].

[12] Recall that real bandpass signals can be represented in terms of their complex-envelopes [11].

[13] Binary Phase-Shift Keying or BPSK, Quaternary PSK or QPSK, M-ary PSK or MPSK, etc.

[14] Some of the techniques proposed in the thesis can be easily adapted to trellis-coded sequences.

1.5.2 Additive White Gaussian Noise Channel

In the Additive White Gaussian Noise (AWGN) channel, the complex-valued transmitted and received signals $x(t)$ and $z(t)$ respectively are related by

$$z(t) = x(t) + \eta_w(t), \tag{1.32}$$

where $\eta_w(t)$ is *white Gaussian noise*. More specifically, $\eta_w(t)$ is a *complex Gaussian random process* which is independent of $x(t)$ and has the following properties:[15] $E[\eta_w(t)] = 0$, $E[\eta_w(t)\eta_w^*(s)] = \mathcal{N}_0\delta(t-s)$, and $E[\eta_w(t)\eta_w(s)] = 0$ [57].

1.5.3 Receiving a QAM Signal over an AWGN Channel

Suppose that a quadrature-amplitude-modulated signal $x(t)$, as defined in section 1.5.1, is transmitted over an AWGN channel. Thus the received signal $z(t)$ is given by

$$z(t) = \sum_n a(n)g(t-nT) + \eta_w(t). \tag{1.33}$$

The task of the *receiver* is to decide from the received signal $z(t)$ as to which data sequence $a(n)$ was likely transmitted. In performing this task, the receiver first obtains a sequence of numbers

$$\int f_n^*(t)z(t)dt \tag{1.34}$$

defined by an appropriate sequence of square-integrable functions $f_n(t)$. If the closed subspace (of the space $\mathcal{L}^2$ of square-integrable functions) spanned by the sequence of functions $f_n(t)$ coincides with the closed subspace spanned by all possible forms of $x(t)$, then the sequence of numbers $\int f_n^*(t)z(t)dt$ is said to constitute a set of *sufficient statistics*, and all decisions can be based on these numbers without loss of optimality. Since the closed subspace spanned by all possible forms of $x(t)$ coincides with the closed subspace spanned by the sequence of functions $g(t-nT)$, an obvious set of sufficient statistics is defined by $\int g^*(t-nT)z(t)dt$ and obtained by feeding $z(t)$ into the filter matched[16] to $g(t)$ and sampling the output at the instants $t = nT$.

[15] Here $E[.]$ denotes the expectation of the random variable within the brackets, and the superscript $*$ denotes complex conjugation.

[16] The filter $f(t) = g^*(-t)$ is said to be matched to $g(t)$, and, therefore, $\int f(nT-t)z(t)dt = \int g^*(t-nT)z(t)dt$.

1.5.4 Intersymbol Interference

The case when $g(t)$ satisfies the conditions

$$\int g^*(t)g(t-nT)dt = 0 \quad \text{if} \quad n \neq 0 \tag{1.35}$$

is important. In this case, denoting the sufficient statistics as

$$b(n) = \int g^*(t-nT)z(t)dt, \tag{1.36}$$

one observes that

$$b(n) = a(n)\int |g(t)|^2\,dt + c(n), \tag{1.37}$$

where $c(n)$ given by

$$c(n) = \int g^*(t-nT)\eta_w(t)dt \tag{1.38}$$

is a complex *Gaussian random variable* with mean zero and variance $\mathcal{N}_0 \int |g(t)|^2\,dt$. Furthermore, if $n_1 \neq n_2$ then $c(n_1)$ and $c(n_2)$ are independent.[17] Therefore, assuming an uncoded data sequence, a decision on the likely $a(n)$, for any particular n, can be based without loss of optimality on $b(n)$ alone by the maximum *a posteriori* probability rule of hypothesis testing. In otherwords, there is no *Intersymbol Interference* (ISI). The condition given by equation 1.35 is necessary and sufficient for there to be no ISI.[18] Thus, the answer to the question whether there is ISI or not is inherent in the sequence of functions $g(t-nT)$ and not in the sequence of functions $f_n(t)$ that is used to obtain the set of sufficient statistics. The condition given by equation 1.35 is equivalent to the Nyquist criterion

$$\sum_k \left|\hat{g}\left(\omega + k\frac{2\pi}{T}\right)\right|^2 = \text{constant}, \tag{1.41}$$

where $\hat{g}(\omega) = \int g(t)e^{-j\omega t}dt$ is the Fourier transform of $g(t)$.

[17] For all n_1 and n_2, $E[c(n_1)c(n_2)] = 0$ and $E[c^*(n_1)c(n_2)] = \delta(n_1 - n_2)\mathcal{N}_0 \int |g(t)|^2\,dt$.

[18] The condition is equivalent to the existence of a sequence of functions $f_n(t)$ that satisfy

$$\int f_n^*(t)g(t-mT)dt = \begin{cases} 0 & \text{if} \quad n \neq m, \\ 1 & \text{if} \quad n = m, \end{cases} \tag{1.39}$$

and

$$\int f_n^*(t)f_m(t)dt = 0 \quad \text{if} \quad n \neq m. \tag{1.40}$$

1.5.5 Receiving a QAM Signal over a Linear Time-Invariant Channel in the Presence of ISI and AWGN

Suppose that a quadrature-amplitude-modulated signal $x(t)$, as defined in section 1.5.1, is transmitted over a linear time-invariant channel whose output has an additive white Gaussian noise (AWGN) component. Thus the received signal $z(t)$ takes the form

$$z(t) = \int \phi(\tau)x(t-\tau)d\tau + \eta_w(t), \tag{1.42}$$

where $\phi(t)$ is the complex impulse response of the channel. This case can be reduced to that discussed in sections 1.5.2 to 1.5.4 by observing that

$$z(t) = \sum_n a(n)h(t-nT) + \eta_w(t), \tag{1.43}$$

where

$$h(t) = \int \phi(\tau)g(t-\tau)d\tau. \tag{1.44}$$

Accordingly, given the function $\phi(t)$, it may be possible to choose $g(t)$ so as there be no intersymbol interference (ISI), but there may be both theoretical and practical reasons that preclude this choice. Therefore, ISI may be inevitable, and it would become the task of the receiver to deal with it.

1.5.6 Techniques for Dealing with ISI and AWGN

The techniques of receiving a quadrature-amplitude-modulated signal in the presence of intersymbol interference and additive white Gaussian noise, that are of interest in this thesis, fall into two main classes: equalizers and sequence estimators. The *equalizers* of interest in this thesis are:

1. linear zero-forcing equalizer [27] [3],
2. linear minimum-mean-square-error equalizer,
3. decision-feedback zero-forcing equalizer [27] [3] [35],
4. decision-feedback minimum-mean-square-error equalizer [29].

The *sequence estimators* of interest in this thesis are:

1. maximum-likelihood sequence estimator of the Forney-type [13] [14] [28].

2. maximum-likelihood sequence estimator of the Ungerboeck-type[52].

Discussions of the aforementioned techniques are deferred until later chapters of the thesis where the techniques are generalized so as to be applicable to the discrete-multipath channel whose output has an additive white Gaussian noise component.

1.5.7 QAM Signalling over an Unknown Linear Time-Invariant or a Linear Quasi-Time-Invariant Channel

Before signalling, that is, trasmitting and receiving data, can take place over an unknown linear time-invariant channel, knowledge of the impulse response $\phi(t)$ or of the function $h(t) = \int \phi(\tau)g(t-\tau)d\tau$ must be acquired by some auxiliary method. Conceptually, either $\phi(t)$ or $h(t)$ may be computed from the received signal using knowledge available of the transmitted signal which is not intended, at first, for signalling; then, using either $\phi(t)$ or $h(t)$, the parameters of the receiver may be computed, after which the signalling may commence.

In the case of the receiver techniques discussed in section 1.5.6, however, it is more feasible to compute the parameters of the receiver directly – that is, without actually computing either $\phi(t)$ or $h(t)$ – by using methods of *stochastic* approximation. Methods of stochastic approximation, as applying to quadrature-amplitude-modulated signalling, fall into two classes:

- methods that require deterministic knowledge of the transmitted signal; in this case, the transmitted signal is known as a training signal, and the stochastic approximation methods are known variously as *training*, *adaptive filtering*, or *adaptive equalization*, or even *adaptive receiving* (see [38], [15], and [16]).

- methods that require only statistical knowledge of the transmitted signal; in this case, the stochastic approximation methods are known as *blind deconvolution*, or *blind equalization* (see [18] and [16]).

When the parameters of the receiver have been sufficiently closely approximated, signalling may be commenced; in this situation, either the parameters can be *frozen* or they can be *continually adapted* under the assumption that the receiver is already operating essentially without error and, therefore, that deterministic knowledge of the transmitted signal is available; this latter mode of operation is said to be *decision-directed.* The decision-directed mode is particularly suited to quadrature-amplitude-modulated signalling over a quasi-time-invariant channel.[19]

Some applications where the receiver techniques of section 1.5.6 are used in conjunction with stochastic approximation methods are telephone line data modems and line-of-sight point-to-point digital microwave radios. Hereafter, in this thesis, these combined receiver techniques shall be referred to as *conventional techniques.*

1.6 Quadrature-Amplitude-Modulated Signalling over a Mobile Radio Channel

Recall from section 1.2.4 the representation $\phi(t,\tau) = \int \beta(\omega,\tau)e^{j\omega t}d\omega$ of the time-variant impulse response $\phi(t,\tau)$ of the mobile radio channel, in terms of the underlying multipath propagation. In the choice of strategy for designing the transmitter and receiver for quadrature-amplitude-modulated signalling over a mobile radio channel, a significant role is played by the intended baud rate as taken *relative* to the 'Doppler spread' and the 1/('delay spread'), where 'Doppler spread' and 'delay spread' are measures of the spread of the function $\beta(\omega,\tau)$ with respect to the Doppler shift ω and the delay τ respectively.[20] Some insight into this role could be gained by considering the composition of quadrature amplitude

[19] A quasi-time-invariant channel is a time-variant channel whose impulse response varies *slowly* enough with time that the channel may be considered approximately time-invariant over a *limited* interval of time. From the point of view of quadrature-amplitude-modulated signalling, a channel may be considered quasi-time-invariant if a receiver based on a time-invariant approximation of the channel would perform satisfactorily for data sequences of limited length.

[20] In terms of a stochastic characterization, the *root mean square*

$$\left(\int (\omega-\omega_0)^2\sigma(\omega)d\omega\right)^{1/2}, \tag{1.45}$$

where

$$\sigma(\omega) = \int E\left[|\beta(\omega,\tau)|^2\right]d\tau \Big/ \int E\left[|\beta(\omega,\tau)|^2\right]d\omega d\tau, \tag{1.46}$$

$$\omega_0 = \int \omega\sigma(\omega)d\omega, \tag{1.47}$$

is a popular measure of the Doppler spread. A similar measure can be defined for the delay spread.

modulation and mobile radio channel as described next.

1.6.1 The Composition of Quadrature Amplitude Modulation and Mobile Radio Channel

Suppose that a quadrature-amplitude-modulated signal $x(t)$ given by

$$x(t) = \sum_n a(n)g(t - nT), \tag{1.48}$$

as first defined in section 1.5.1, is transmitted over a mobile radio channel. The received signal $z(t)$ takes the form

$$z(t) = \int \phi(t,\tau)x(t-\tau)d\tau + \eta_w(t), \tag{1.49}$$

where $\phi(t,\tau)$ is the time-variant impulse response of the channel and $\eta_w(t)$ is white Gaussian noise. The received signal $z(t)$ can be written, owing to the linearity of the channel, as

$$z(t) = \sum_n a(n)h^n(t) + \eta_w(t), \tag{1.50}$$

where

$$h^n(t) = \int \phi(t,\tau)g(t - nT - \tau)d\tau \tag{1.51}$$

for all n. The latter two equations define the *composition* of quadrature amplitude modulation and mobile radio channel.

From the point of view of designing the transmitter and receiver, one must be concerned with the sequence of functions $h^n(t)$. Were the channel time-invariant, the functions $h^n(t + nT)$, obtained by translating the functions $h^n(t)$ to the time origin, would all be identical. Since the mobile radio channel is time-variant, however, the function $h^n(t + nT)$ varies, in general, with n; thus, fading manifests itself in the function $h^n(t + nT)$'s being dependent on n. Moreover, the functions $h^n(t)$ can physically overlap with one another, potentially causing intersymbol interference. Qualitatively speaking, the choice of a strategy for designing the transmitter and receiver is usually based on the effective rate of fading, that is, the rate at which the function $h^n(t+nT)$ varies with n, and on the potential amount of intersymbol interference, that is, the extent of overlapping among the functions $h^n(t)$. Empirical evidence suggests that, given a mobile radio channel, both the effective rate of fading and the potential amount of intersymbol interference depend on the baud rate as described next.

The Effective Rate of Fading

Given a mobile radio channel with a certain Doppler spread, the dimensionless ratio

$$\frac{\text{Doppler spread}}{\text{baud rate}} \tag{1.52}$$

is usually considered a good measure of the effective rate of fading.

The Potential Amount of Intersymbol Interference

Given a mobile radio channel with a certain delay spread, the dimensionless product

$$\text{delay spread} \times \text{baud rate} \tag{1.53}$$

is usually considered a good measure of the potential amount of intersymbol interference.

The Behaviour of the Composition of Quadrature Amplitude Modulation and Mobile Radio Channel at Various Baud Rates

Depending on the baud rate, as taken relative to the Dopplerspread and 1/(delay spread) of the underlying multipath propagation, the composition of quadrature amplitude modulation and mobile radio channel may exhibit different behaviour as follows:[21]

- at high baud rates, the rate of fading is low but the amount of intersymbol interference is high; the mobile radio channel appears to be quasi-time-invariant.
- at low baud rates, the amount of intersymbol interference is low but the rate of fading is high.
- at moderate baud rates, both the rate of fading rate and the amount of intersymbol interference are moderate.

1.6.2 Existing Strategies for Designing the Transmitter and Receiver for Quadrature-Amplitude-Modulated Signalling over a Mobile Radio Channel

The discussion of the previous section shows that if the baud rate can be considered high, then the conventional techniques, as briefly described in section 1.5.7, may be applicable.

[21] In the literature, this is usually explained in terms of the coherence time and the coherence bandwidth, which are the reciprocals of the Doppler spread and the delay spread respectively.

But, the high-baud-rate assumption is not valid in the context of some proposals for mobile radio communications systems [37] [31] [36], where it is observed that the effective rate of fading could be significant enough to degrade the performance of the conventional techniques. In an attempt to overcome this degradation, many strategies have been proposed (see [37], [7], and the references therein). These strategies, based on either *periodic training* or periodic insertion of known data symbols, are mere modifications of the conventional techniques, for the assumption that the channel is quasi-time-invariant, although recognized as being less valid, has not been entirely relaxed.[22]

When the effective rate of fading is significant, recourse must be made, in general, to a stochastic characterization of the mobile radio channel, or, rather, of the sequence of functions $\{h^n(t) : \forall n \in \mathbb{Z}\}$. If a complete stochastic characterization is available, then the receiver design may be based on either the *maximum a posteriori probability* or the *maximum likelihood* principle.[23] Unfortunately, implementations of such designs tend to be exceedingly complex in all but a few special cases.

The case where the set of functions $\{h^n(t) : \forall n \in \mathbb{Z}\}$ has a Gaussian probability density is of much interest, for the conditional probability density prob[received signal/data], the so-called *likelihood* of the conditioning data, can be expressed as

$$\text{prob[received signal/data]} \sim \det[F_{\text{data}}] \exp\left(- \int\int z^*(t) F_{\text{data}}(t,s) z(s) dt ds \right), \quad (1.54)$$

where $z(t)$ is the received signal, and $F_{\text{data}}(t,s)$ and $\det[F_{\text{data}}]$ are the *Fredholm resolvent* and the *Fredholm determinant* of the covariance kernel of $z(t)$ conditioned on the data. Even in this case, the receiver tends to be very complex owing to the complexity of the functional dependence of $F_{\text{data}}(t,s)$ and $\det[F_{\text{data}}]$ on the data. This is especially true when there is intersymbol interference. Even if there were no intersymbol interference, to keep the complexity in manageable proportions, it is often considered necessary to restrict the choice of the modulation scheme to those that give rise to a $\det[F_{\text{data}}]$ that is independent of the data; such modulation schemes have typically low spectral efficiencies.

[22]This attitude owes, perhaps, partly to the simplicity of the conventional techniques, but mainly to the lack of a feasible general technique for dealing with a time-variant mobile radio channel. The techniques presented in the thesis achieve a significant level of generality while maintaining the simplicity of the conventional techniques.

[23]Since Prob[data/received signal] = prob[received signal/data]Prob[data]/prob[received signal], the maximum a posteriori receiver is also the one that maximizes prob[received signal/data]Prob[data]; if the data are *equally likely* then only prob[received signal/data] need be maximized. A receiver that maximizes prob[received signal/data] is known as the *maximum likelihood receiver*.

1.7 The Novelty of the Thesis

The novelty of the thesis lies in the recognition that when a mobile radio channel can be modelled as a Discrete-Multipath Channel, that is, when its impulse response $\phi(t,\tau)$ has the functional form

$$\phi(t,\tau) = \sum_{k=1}^{K} e^{j\omega_k t}\phi_k(\tau), \tag{1.55}$$

as first defined in section 1.4.1, receiver design can be based on a deterministic characterization of the mobile radio channel, regardless of the effective rate of fading.[24] This recognition is based on the reasonable assumption that a discrete-multipath mobile radio channel is *quasi-static* in the sense that the number of effective wavefronts, their strengths, and orientations are *quasi-static*, that is, they remain the same over fairly long time intervals. Based on this recognition, the thesis proposes the following philosophy of design for the transmitter and receiver, to effect quadrature-amplitude-modulated signalling over a discrete-multipath mobile radio channel.

1.7.1 A Philosophy of Design for Discrete-Multipath Mobile Radio Channels Based on their Deterministic Characterization

To effect quadrature-amplitude-modulated signalling over a discrete-multipath mobile radio channel, the transmitter and the receiver together may *alternate* between the following two phases:

Probing Phase

The receiver obtains a deterministic characterization of the channel, that is, it estimates from the received signal $z(t)$ the parameters of $\phi(t,\tau)$, that is, the integer K, the set of numbers $\{\omega_k : k = 1,2,\ldots,K\}$, and the set of functions $\{\phi_k(t) : k = 1,2,\ldots,K\}$. In this, the receiver is aided by an appropriate transmitted signal. Thus, a deterministic characterization of the sequence of functions $\{h^n(t) : n \in \mathbb{Z}\}$, where

$$h^n(t) = \int \phi(t,\tau)g(t - nT - \tau)d\tau, \tag{1.56}$$

or, more specifically, of estimates thereof, becomes available to the receiver.

[24]The term 'fading' here loses its stochastic connotation.

Signalling Phase

A quadrature-amplitude-modulated signal

$$x(t) = \sum_n a(n)g(t - nT) \tag{1.57}$$

is transmitted. The receiver decides from the received signal $z(t)$, that is,

$$z(t) = \sum_n a(n)h^n(t) + \eta_w(t), \tag{1.58}$$

as to which data sequence $a(n)$ was likely transmitted. In this, the receiver assumes that the deterministic characterization of the sequence of functions $\{h^n(t) : n \in \mathbb{Z}\}$ obtained in the preceding probing phase is accurate.

1.7.2 The Contributions of the Thesis

The contribution of the thesis as a whole can be summarized as the proposing of the above philosophy of design and demonstrating, analytically, that the proposed philosophy of design leads to attractive solutions to the problems posed under the probing and the signalling phases.

The contributions of the thesis as regards the problem posed under the signalling phase can be put in proper context by considering the special case of a discrete-multipath channel obtained by setting $K = 1$ and $\omega_1 = 0$, that is, a time-invariant channel.

In quadrature-amplitude-modulated signalling over a linear time-invariant channel, there may be intersymbol interference (ISI) as discussed in sections 1.5.4 and 1.5.5. The notion of ISI given in section 1.5.4 can be generalized in a straightforward manner so as to be valid in the context of quadrature-amplitude-modulated signalling over a discrete-multipath linear time-variant channel. Thus, there is no ISI if and only if, for any n and m, $n \neq m$ implies

$$\int h^{n*}(t)h^m(t)dt = 0. \tag{1.59}$$

Accordingly, if there is no ISI, then the solution to the problem posed under the signalling phase is straightforward. Thus, for any particular n, the quantity

$$b(n) = \int h^{n*}(t)z(t)dt \tag{1.60}$$

satisfies

$$b(n) = a(n) \int |h^n(t)|^2 \, dt + c(n) \tag{1.61}$$

where $c(n)$ given by

$$c(n) = \int h^{n*}(t)\eta_w(t)dt \tag{1.62}$$

is a complex Gaussian random variable with mean zero and variance $\mathcal{N}_0 \int |h^n(t)|^2 \, dt$. Furthermore, if $n_1 \neq n_2$ then $c(n_1)$ and $c(n_2)$ are independent.[25] Provided that the data symbols were chosen in an independent manner, and $\int |h^n(t)|^2 \, dt \neq 0$, a decision on the likely $a(n)$ can be based without loss of optimality on $b(n)$ alone.

Given a time-variant discrete-multipath channel, it seems extremely unlikely that the function $g(t)$ can be chosen so as there be no ISI, unless the baud period T can be chosen large enough that the functions $h^n(t)$ do not physically overlap with one another. Choosing T large is an option only if the baud rate can be low. At moderate-to-high baud rates, ISI seems inevitable and, therefore, it becomes the task of the receiver to deal with it.

The techniques mentioned in section 1.5.6 for dealing with ISI and additive white Gaussian noise (AWGN), which are applicable in the special case of the discrete-multipath channel obtained by setting $K = 1$ and $\omega_1 = 0$, are well known for their simplicity. The thesis demonstrates, analytically, that those techniques can each be generalized so as to be applicable for quadrature-amplitude-modulated signalling over a discrete-multipath time-variant channel. The generalizations may be described as techniques for dealing with fading, ISI, and AWGN in a joint manner. The generalization is made possible by a certain representation of the sequence of functions $\{h^n(t) : n \in \mathbb{Z}\}$ induced by the functional form of $\phi(t, \tau)$. The generalization is particularly simple in the time-variant special case obtained by setting $K = 1$ while ω_1 may be non-zero.

1.7.3 Some Merits of the Proposed Philosophy of Design

As long as a mobile radio channel can be modelled as a discrete-multipath channel, the proposed philosophy calls for the continual optimization of the receiver to the actual channel in effect. In contrast, a strategy based on a stochastic characterization of the mobile radio channel calls only for the optimization of the receiver to the statistical ensemble of channels

[25] The sequence of random variables $c(n)$ has the following additional properties: $E[c^*(n_1)c(n_2)] = 0$ for all $n_1 \neq n_2$; $E[c(n_1)c(n_2)] = 0$ for all n_1 and n_2.

that the receiver may encounter in a topography of interest. Thus, providing that the overhead of the probing phase is small enough, a receiver designed under the proposed philosophy will yield better performance than a similar receiver designed under a strategy based on a stochastic characterization of the mobile radio channel.

As long as a mobile radio channel can be modelled as a discrete-multipath channel, the proposed philosophy is comprehensive in the following senses:

1. it is applicable regardless of the effective rate of fading or the potential amount of inter-symbol interference; in other words, it provides a unified framework for dealing with time-selective fading, frequency-selective fading, and time- and frequency-selective fading.

2. it is applicable regardless of the underlying statistics of fading which may be dependent on the topography.[26]

1.8 Organization of the Thesis

The rest of the thesis is organized as follows. Chapter 2 introduces the *composition* of quadrature amplitude modulation and discrete-multipath channel, and describes a method of estimating this composition; the method provides a justification for the philosophy of design proposed in Chapter 1, and for the assumption made in the five subsequent chapters that the composition is known. Chapters 3 to 6 discuss receivers termed *equalizers*; chapters 3 and 5 discuss linear equalizers of the zero-forcing type and the mean-square-error type respectively; chapters 4 and 6 discuss decision-feedback equalizers of the zero-forcing type and the mean-square-error type respectively. Chapter 7 discusses receivers termed *maximum-likelihood sequence estimators* of the Forney-type and the Ungerboeck-type. Chapter 8 presents numerical results pertaining to some specific real world scenarios. Chapter 9 summarizes the thesis, draws some conclusions, and makes suggestions for further work. Appendix A gives some of the mathematical results needed in chapters 3 to 7. Appendices B and C give the additional mathematical results needed in chapters 4 and 6 on equalizers of the mean-square-error type.

[26]When the statistics are known, however, the average performance of a receiver designed under the proposed philosophy may be evaluated.

Chapter 2

The Composition of Quadrature Amplitude Modulation and Discrete-Multipath Channel

2.1 Introduction

This chapter is aimed at substantiating the claim made in chapter 1, section 1.4 that the class of time- and frequency-selective channels termed discrete-multipath channels can be characterized in a deterministic manner by probing them. Thus, the chapter is also aimed at substantiating the probing phase of the philosophy of design proposed in section 1.7.1 of chapter 1. Under certain assumptions, the probing of the discrete-multipath channel with a quadrature-amplitude-modulated signal is shown to lead to a deterministic characterization of the discrete-multipath channel. This suggests that the composition of quadrature amplitude modulation and discrete-multipath channel – a system whose input is the data sequence and the output is the received signal – can be characterized directly by choosing the probing signal appropriately. In fact, the probing can be effected by appropriate design of the data sequence.

2.2 Channel Probing Heuristics

Consider the Discrete-Multipath Channel (DMC) to be noise-free; thus, the transmitted signal $x(t)$ and the received signal $y(t)$ are related by

$$y(t) = \sum_{k=1}^{K} e^{j\omega_k t} \int \phi_k(\tau) x(t-\tau) d\tau \tag{2.1}$$

as first defined in section 1.4.1 of chapter 1. A characterization of the noise-free DMC consists in knowing the integer K, the set of functions $\{\phi_k(t) : k = 1, 2, \ldots, K\}$, and the set of numbers $\{\omega_k : k = 1, 2, \ldots, K\}$. Under certain assumptions, such a characterization can be obtained by probing the noise-free DMC as described next.

Suppose that the transmitted signal $x(t)$ consists of a train of impulse functions, that is,

$$x(t) = \sum_n \delta(t - nT). \tag{2.2}$$

Then the received signal $y(t)$ can be written as

$$\begin{aligned} y(t) &= \sum_n \sum_{k=1}^{K} e^{j\omega_k t} \int \phi_k(\tau) \delta(t - nT - \tau) d\tau, && (2.3) \\ &= \sum_n \sum_{k=1}^{K} e^{j\omega_k t} \phi_k(t - nT), && (2.4) \\ &= \sum_n \sum_{k=1}^{K} e^{j\omega_k nT} e^{j\omega_k (t-nT)} \phi_k(t - nT), && (2.5) \\ &= \sum_n \sum_{k=1}^{K} e^{j\omega_k nT} \psi_k(t - nT), && (2.6) \end{aligned}$$

where

$$\psi_k(t) = e^{j\omega_k t} \phi_k(t) \tag{2.7}$$

for $k = 1, 2, \ldots, K$. Observe that if the integer K, the set of functions $\{\psi_k(t) : k = 1, 2, \ldots, K\}$, and the set of numbers $\{\omega_k : k = 1, 2, \ldots, K\}$ can be determined, then the set of functions $\{\phi_k(t) : k = 1, 2, \ldots, K\}$ can in turn be determined, and the DMC can thus be characterized.

Suppose that the set of functions $\{\phi_k(t) : k = 1, 2, \ldots, K\}$ is known *a priori* to be *time-limited* to an interval of length less than T, that is, the functions are zero outside this common time interval. Then the functions

$$\sum_{k=1}^{K} e^{j\omega_k nT} \psi_k(t - nT) \tag{2.8}$$

do not physically overlap with one another, and, therefore, can be *individually* observed. By translating these observations to the time-origin, a set of observations of the form

$$y_n(t) = \sum_{k=1}^{K} e^{j\omega_k nT}\psi_k(t) \tag{2.9}$$

can thus be obtained.[1] Consider the contiguous set of observations $\{y_n(t) : n = 0, 1, 2, \ldots, L\}$; observe that there is no loss of generality in choosing zero as the starting value of n, for a non-zero case can be reduced to the zero case by redefining the set of functions $\{\psi_k(t) : k = 1, 2, \ldots, K\}$. The set of observations $\{y_n(t) : n = 0, 1, 2, \ldots, L\}$ satisfies the matrix equation

$$\begin{bmatrix} y_0(t) \\ y_1(t) \\ y_2(t) \\ \vdots \\ y_L(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 & \ldots & 1 \\ e^{j\omega_1 T} & e^{j\omega_2 T} & \ldots & e^{j\omega_K T} \\ e^{j\omega_1 2T} & e^{j\omega_2 2T} & \ldots & e^{j\omega_K 2T} \\ \vdots & \vdots & & \vdots \\ e^{j\omega_1 LT} & e^{j\omega_2 LT} & \ldots & e^{j\omega_K LT} \end{bmatrix} \begin{bmatrix} \psi_1(t) \\ \psi_2(t) \\ \vdots \\ \psi_K(t) \end{bmatrix}. \tag{2.10}$$

This form of a matrix equation, where K and $\left\{\psi_k(t), e^{j\omega_k T} : k = 1, 2, \ldots, K\right\}$ are the unknowns, has been studied extensively in connection with many other problems. However, a discussion of the solution to such an equation does not, at this stage, make much engineering sense, for impulse functions are not practical realities. In section 2.3.1, a more general and practical form of channel probing is shown to lead to a matrix equation similar to equation 2.10; therefore, a discussion of the solution to such equations is deferred until section 2.4.

2.3 The Composition of QAM and the Noise-Free DMC

Suppose that the transmitted signal $x(t)$ is generated by Quadrature Amplitude Modulation (QAM), that is,

$$x(t) = \sum_n a(n)g(t - nT). \tag{2.11}$$

[1] In other words, the received signal can be written as $y(t) = \sum_n y_n(t - nT)$ with the functions $y_n(t - nT)$ not physically overlapping with one another.

Recalling the input-output relationship

$$y(t) = \sum_{k=1}^{K} e^{j\omega_k t} \int \phi_k(\tau)x(t-\tau)d\tau \tag{2.12}$$

of the noise-free DMC, the received signal $y(t)$ can be written as

$$\begin{aligned} y(t) &= \sum_n a(n) \sum_{k=1}^{K} e^{j\omega_k t} \int \phi_k(\tau)g(t-nT-\tau)d\tau, && (2.13)\\ &= \sum_n a(n) \sum_{k=1}^{K} e^{j\omega_k nT} e^{j\omega_k(t-nT)} \int \phi_k(\tau)g(t-nT-\tau)d\tau, && (2.14)\\ &= \sum_n a(n) \sum_{k=1}^{K} e^{j\omega_k nT} h_k(t-nT), && (2.15) \end{aligned}$$

where

$$h_k(t) = e^{j\omega_k t} \int \phi_k(\tau)g(t-\tau)d\tau \tag{2.16}$$

for $k = 1, 2, \ldots, K$. The relationship between the data sequence $a(n)$ and the received signal $y(t)$, given by equation 2.15, shall be known as the composition of QAM and the noise-free DMC. It is this composition that will be of direct relevance in designing a receiver for quadrature-amplitude-modulated signalling over the DMC. Observe that, according to the notation of section 1.6.1 of chapter 1,

$$h^n(t) = \sum_{k=1}^{K} e^{j\omega_k nT} h_k(t-nT). \tag{2.17}$$

This functional form is made use of in the subsequent chapters to substantiate the signalling phase of the philosophy of design proposed in section 1.7.1 of chapter 1. A characterization of the composition of QAM and the noise-free DMC consists in knowing the integer K, the set of functions $\{h_k(t) : k = 1, 2, \ldots, K\}$, and the set of numbers $\left\{e^{j\omega_k T} : k = 1, 2, \ldots, K\right\}$. Under certain assumptions, such a characterization can be obtained directly by probing the noise-free DMC as described next.

2.3.1 Probing the Noise-Free DMC with a Quadrature-Amplitude-Modulated Signal

The form of the composition of QAM and the noise-free DMC, given by equation 2.15, suggests that the noise-free DMC can be probed with a quadrature-amplitude-modulated signal $x(t)$ of the form

$$x(t) = \sum_n g(t - nT) \tag{2.18}$$

for appropriately chosen function $g(t)$ and parameter T. The received signal $y(t)$ is then obtained by setting the data sequence as $a(n) = 1$ for all n in the composition given by equation 2.15. Thus,

$$y(t) = \sum_n \sum_{k=1}^{K} e^{j\omega_k nT} h_k(t - nT), \tag{2.19}$$

where the set of functions $\{h_k(t) : k = 1, 2, \ldots, K\}$ is as defined by equation 2.16.

In the manner of the discussion in section 2.2, if the function $g(t)$ and the parameter T are such that the set of functions $\{h_k(t) : k = 1, 2, \ldots, K\}$ is *time-limited* to an interval of length less than T, then the functions

$$\sum_{k=1}^{K} e^{j\omega_k nT} h_k(t - nT) \tag{2.20}$$

can be individually observed. Denoting

$$y_n(t) = \sum_{k=1}^{K} e^{j\omega_k nT} h_k(t), \tag{2.21}$$

the contiguous set of observations $\{y_n(t) : n = 0, 1, 2, \ldots, L\}$ satisfies the matrix equation

$$\begin{bmatrix} y_0(t) \\ y_1(t) \\ y_2(t) \\ \vdots \\ y_L(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 & \ldots & 1 \\ e^{j\omega_1 T} & e^{j\omega_2 T} & \ldots & e^{j\omega_K T} \\ e^{j\omega_1 2T} & e^{j\omega_2 2T} & \ldots & e^{j\omega_K 2T} \\ \vdots & \vdots & & \vdots \\ e^{j\omega_1 LT} & e^{j\omega_2 LT} & \ldots & e^{j\omega_K LT} \end{bmatrix} \begin{bmatrix} h_1(t) \\ h_2(t) \\ \vdots \\ h_K(t) \end{bmatrix}, \tag{2.22}$$

which is similar to the matrix equation 2.10 of section 2.2. Discussion on solving equation 2.22 for K and $\left\{h_k(t), e^{j\omega_k T} : k = 1, 2, \ldots, K\right\}$ is deferred until section 2.4.

If the function $g(t)$ and the parameter T intended for use in signalling can also be used in probing, then the solution of equation 2.22 will constitute a characterization of the composition of QAM and the noise-free DMC which will be of direct relevance in designing receivers. However, the function $g(t)$ and the parameter T intended for use in signalling may not be appropriate for use in probing. For instance, the set of functions

$\{h_k(t) : k = 1, 2, \ldots, K\}$, defined by

$$h_k(t) = e^{j\omega_k t} \int \phi_k(\tau) g(t-\tau) d\tau, \tag{2.23}$$

for $k = 1, 2, \ldots, K$, may not be time-limited to an interval of length less than T. On the contrary, in striving to achieve high data-rates with band-limited signals, the functions

$$\sum_{k=1}^{K} e^{j\omega_k nT} h_k(t - nT) \tag{2.24}$$

must inevitably be allowed to overlap with one another.

If the function $g(t)$ used in probing is different from that intended for use in signalling, then the set of functions $\{\phi_k(t) : k = 1, 2, \ldots, K\}$ and the set of numbers $\{\omega_k : k = 1, 2, \ldots, K\}$ may be determined as follows:

- under the assumption that

$$-\pi < \omega_k T < \pi \tag{2.25}$$

  for $k = 1, 2, \ldots, K$, the set of numbers $\{\omega_k : k = 1, 2, \ldots, K\}$ can be determined from $\left\{e^{j\omega_k T} : k = 1, 2, \ldots, K\right\}$,

- under the further assumption that $g(t)$ is time-limited, the set of functions $\{\phi_k(t) : k = 1, 2, \ldots, K\}$ can be determined from $\{\int \phi_k(\tau) g(t-\tau) d\tau : k = 1, 2, \ldots, K\}$.

But if the function $g(t)$ intended for use in signalling could be used in probing as well, then there may be no need for determining $\{\phi_k(t), \omega_k : k = 1, 2, \ldots, K\}$.

2.3.2 Probing and Signalling with a Common $g(t)$

Suppose that the set of functions $\{h_k(t) : k = 1, 2, \ldots, K\}$ defined by equation 2.23 is time-limited to an interval of length T_p. By probing the noise-free DMC with the quadrature-amplitude-modulated signal $x(t)$ given by

$$x(t) = \sum_n g(t - nT_p), \tag{2.26}$$

one obtains a matrix equation that is identical to the matrix equation 2.22 except that T is replaced with T_p, for the parameter T (or T_p) does not enter into the definition of the set of functions $\{h_k(t) : k = 1, 2, \ldots, K\}$ (see equation 2.23). By solving the resulting matrix equation for K and $\left\{h_k(t), e^{j\omega_k T_p} : k = 1, 2, \ldots, K\right\}$, the set of numbers $\left\{e^{j\omega_k T} : k = 1, 2, \ldots, K\right\}$ can in turn be determined from $\left\{e^{j\omega_k T_p} : k = 1, 2, \ldots, K\right\}$ under the additional assumption that

$$-\pi < \omega_k T_p < \pi \tag{2.27}$$

for $k = 1, 2, \ldots, K$. Thus, a characterization of the composition of QAM and the noise-free DMC, which will be of direct relevance in designing receivers, can be obtained without determining $\{\phi_k(t), \omega_k : k = 1, 2, \ldots, K\}$.

The assumption that $-\pi < \omega_k T_p < \pi$ for $k = 1, 2, \ldots, K$ is expected to be valid in practice. In case it proves restrictive,[2] it may be obviated with added complexity as follows. Suppose T_{p_1} and T_{p_2} are two numbers such that

- $T_{p_1} \geq T_p$ and $T_{p_2} \geq T_p$,
- $T_{p_1} \neq T_{p_2}$,
- $-\pi < \omega_k(T_{p_1} - T_{p_2}) < \pi$ for $k = 1, 2, \ldots, K$.

By probing the noise-free DMC twice, first with the signal

$$x(t) = \sum_n g(t - nT_{p_1}), \tag{2.28}$$

and then with the signal

$$x(t) = \sum_n g(t - nT_{p_2}), \tag{2.29}$$

two matrix equations of the form of equation 2.22 can be obtained. By solving these matrix equations for K and $\left\{h_k(t), e^{j\omega_k T_{p_1}}, e^{j\omega_k T_{p_2}} : k = 1, 2, \ldots, K\right\}$, the set of numbers $\left\{e^{j\omega_k T} : k = 1, 2, \ldots, K\right\}$ can in turn be determined from $\left\{e^{j\omega_k (T_{p_1} - T_{p_2})} : k = 1, 2, \ldots, K\right\}$. This scenario will no longer be considered in the thesis.

The discussion so far has been on the composition of QAM and the noise-free DMC. The consideration of noise at the output of the DMC, however, may have a role in the choice of the function $g(t)$. For instance, the $g(t)$ chosen for probing must ensure accurate estimation of the set $\left\{h_k(t), e^{j\omega_k T} : k = 1, 2, \ldots, K\right\}$, but the $g(t)$ chosen for signalling must ensure correct decisions on the data sequence. The issue of choosing $g(t)$ to achieve the

[2]This question was raised by Dr. J. P. Reilly.

aforementioned ends is beyond the scope of this thesis. Nevertheless, the discussion in this section has demonstrated that choosing a common $g(t)$ for probing and signalling may be advantageous from the point of view of complexity.

2.3.3 Framing the Data to Effect Probing and Signalling with a Common $g(t)$

Suppose that the consideration of noise allows the use of the same function $g(t)$, or scaled versions thereof, in probing as well as in signalling. Suppose also that T_p is an integral multiple of the baud period T. Then the transmitted signal for alternate probing and signalling can be generated by framing the data as shown below:

$$\begin{array}{|c|c|} \text{probing} & \text{signalling} \\ 0\ldots0a0\ldots0a0\ldots\ldots0a0\ldots0a0\ldots0 & a(1)a(2)a(3)\ldots\ldots a(N-1)a(N) \end{array} \tag{2.30}$$

Thus during the probing phase, a periodic data sequence of a's interleaved with blocks of $(T_p/T)-1$ zeros is transmitted at the baud rate $1/T$.

2.3.4 On Choosing the Data a and the Lengths of the Probing and the Signalling Phases

In the presence of noise, the data a and the length of the probing phase must be large enough to ensure the 'correctness' of the estimate of K and the 'desired accuracy' of the estimates of $\left\{h_k(t), e^{j\omega_k T} : k = 1,2,\ldots,K\right\}$. The 'desired accuracy' of the estimates of $\left\{e^{j\omega_k T} : k = 1,2,\ldots,K\right\}$ is determined by the 'desired accuracy' of the estimates of $\left\{e^{j\omega_k lT} : k = 1,2,\ldots,K; 1 \le l \le \mathcal{L}\right\}$, where $\mathcal{L}$ is the length of the data frame.[3] Therefore, the data a and the length of the probing phase are determined by the length of the signalling phase.

[3]In the case where $K = 1$, in simple terms, frequency error results in accumulating phase error!

2.4 The Matrix Equation 2.22

2.4.1 Previous Encounters with Similar Matrix Equations

Matrix equations of the form of equation 2.22, where K and $\left\{h_k(t), e^{j\omega_k T} : k = 1, 2, \ldots, K\right\}$ are the unknowns, have been encountered in many other problems. To mention a few,[4]

1. Consider a superimposition of a set of *sinusoidal signals*; from the observations taken of the superimposition, determine the number of sinusoidals, their frequencies, and their amplitudes [51] [44].

2. Consider a superimposition of a set of *exponential signals*; from the observations taken of the superimposition, determine the number of exponentials, their exponents, and their initial values [4] [24].

3. Consider a set of *plane waves* of sinusoidal signals impinging on a linear uniform array of sensors; from the observations taken of the output of the sensors, determine the number of waves, their directions of arrival, and their amplitudes [48] [40] [25] [45] [20].

The noise-free formulations of the aforementioned problems are structurally identical to the problem of characterizing the composition of QAM and the noise-free DMC, as formulated in section 2.3.1. Moreover, the problem of characterizing the composition of QAM and the noisy DMC can be formulated, as shown in section 2.5.2, to conform to the formulations of the aforementioned problems that take noise into consideration.

The solution to the matrix equation 2.22 hinges on the Vandermonde structure of the coefficient matrix as discussed next.

[4]The study of this class of problems has a long history and there is an abundance of literature (see [15], [16], [33], [17]). Therefore, the citations may not necessarily be the original sources. Literature of more specific relevance to the thesis is cited in later sections of this chapter.

2.4.2 The Vandermonde Structure

Given a set of complex-valued scalars $\Lambda = \{\lambda_k : k = 1, 2, \ldots, K\}$, the $(L+1) \times K$ matrix $\mathbf{V}_L(\Lambda)$ defined as

$$\mathbf{V}_L(\Lambda) = \begin{bmatrix} 1 & 1 & \ldots & 1 \\ \lambda_1 & \lambda_2 & \ldots & \lambda_K \\ \lambda_1^2 & \lambda_2^2 & \ldots & \lambda_K^2 \\ \vdots & \vdots & & \vdots \\ \lambda_1^L & \lambda_2^L & \ldots & \lambda_K^L \end{bmatrix} \tag{2.31}$$

is said to have *Vandermonde* structure. The column vectors $\left(1, \lambda_k, \lambda_k^2 \ldots, \lambda_k^L\right)^T$, for $k = 1, 2, \ldots, K$, are known as Vandermonde vectors. Vandermonde matrices have the following important property:

Lemma 1 ([40]) *The columns of the matrix* $\mathbf{V}(\Lambda)$ *are* linearly independent *if and only if* $K \leq L+1$ *and* $\lambda_{k_1} \neq \lambda_{k_2}$ *whenever* $k_1 \neq k_2$.

When the columns of the matrix $\mathbf{V}(\Lambda)$ are linearly independent, the column span of the matrix is said to have a Vandermonde basis. Of direct relevance to solving the matrix equation 2.22 is the following consequence of the lemma:

Corollary 1 ([40]) *Given the column span of* $\mathbf{V}(\Lambda)$*, the column span has a unique Vandermonde basis, or, equivalently, is associated with a unique set of scalars* $\Lambda = \{\lambda_k : k = 1, 2, \ldots, K\}$*, if and only if* $K \leq L$.

2.4.3 On the Uniqueness of the Solution

Returning to the matrix equation 2.22, denote the vector-valued functions

$$\mathbf{Y}(t) = [y_0(t), y_1(t), y_2(t), \ldots, y_L(t)]^T, \tag{2.32}$$

$$\mathbf{H}(t) = [h_1(t), h_2(t), \ldots, h_K(t)]^T, \tag{2.33}$$

and the set of numbers

$$\Omega = \left\{e^{j\omega_k T} : k = 1, 2, \ldots, K\right\}. \tag{2.34}$$

Then, using the notation introduced in section 2.4.2 for Vandermonde matrices, one has

$$\mathbf{Y}(t) = \mathbf{V}_L(\Omega)\,\mathbf{H}(t). \tag{2.35}$$

The problem of characterizing the composition of QAM and the noise-free DMC, as formulated in section 2.3.1, is meaningful only if the observed vector $\mathbf{Y}(t)$ has a unique representation of the form given by equation 2.22, or, equivalently, equation 2.35. From the corollary of section 2.4.2, a necessary condition for such uniqueness is that $L \geq K$, for $\mathbf{Y}(t)$ belongs to the column span of $\mathbf{V}_L(\Omega)$ for every t. This condition is also sufficient if the set of functions $\{h_k(t) : k = 1, 2, \ldots, K\}$ is known *a priori* to be linearly independent. In the absence of such *a priori* knowledge about the set of functions $\{h_k(t) : k = 1, 2, \ldots, K\}$, the necessary and sufficient condition for uniqueness is that $L \geq 2K - 1$ [5] [54]. This can easily be met.[5]

2.5 The Composition of QAM and the Noisy DMC

Suppose that the quadrature-amplitude-modulated signal

$$x(t) = \sum_n a(n) g(t - nT) \tag{2.36}$$

is transmitted over the discrete-multipath channel. In the presence of Additive White Gaussian Noise (AWGN), the received signal $z(t)$ is dependent on the data sequence $a(n)$ as given by

$$z(t) = \sum_n a(n) \sum_{k=1}^{K} e^{j\omega_k nT} h_k(t - nT) + \eta_w(t), \tag{2.37}$$

where the set of functions $\{h_k(t) : k = 1, 2, \ldots, K\}$ are as defined by equation 2.16 of section 2.3, and $\eta_w(t)$ is a complex white Gaussian noise process as defined in section 1.5.2 of chapter 1.

2.5.1 Probing in the Presence of AWGN

Suppose that the quadrature-amplitude-modulated signal

$$x(t) = \sum_n g(t - nT) \tag{2.38}$$

[5] In the presence of noise, however, L may have to be much greater than $2K - 1$ to ensure the desired accuracy of the estimates of $\{e^{j\omega_k T} : k = 1, 2, \ldots, K\}$ typical of mobile radio channels. For the scenario of large L, Viberg, Ottersten, and Nehorai [53] have given a criterion which is satisfied here.

is transmitted, with T being large enough that the set of functions $\{h_k(t) : k = 1, 2, \ldots, K\}$ is time-limited to an interval of length less than T. Then the received signal is given by

$$z(t) = \sum_n y_n(t - nT) + \eta_w(t). \tag{2.39}$$

where the functions $\{y_n(t)\}$ are as defined by equation 2.21 of section 2.3.1. Denote

$$z_l(t) = \begin{cases} z(t + lT) & 0 \le t < T \\ 0 & \text{elsewhere} \end{cases}, \tag{2.40}$$

and

$$w_l(t) = \begin{cases} \eta_w(t + lT) & 0 \le t < T \\ 0 & \text{elsewhere} \end{cases}. \tag{2.41}$$

Then

$$z_l(t) = y_l(t) + w_l(t) \tag{2.42}$$

for all integers l. Denote the vector-valued functions

$$\mathbf{Z}(t) = [z_0(t), z_1(t), z_2(t), \ldots, z_L(t)]^T, \tag{2.43}$$

$$\mathbf{W}(t) = [w_0(t), w_1(t), w_2(t), \ldots, w_L(t)]^T. \tag{2.44}$$

Then

$$\mathbf{Z}(t) = \mathbf{Y}(t) + \mathbf{W}(t) = \mathbf{V}_L(\Omega)\mathbf{H}(t) + \mathbf{W}(t), \tag{2.45}$$

where $\mathbf{Y}(t)$, $\mathbf{H}(t)$, and $\mathbf{V}_L(\Omega)$ are as defined in section 2.4.3.

2.5.2 Characterizing the Composition of QAM and DMC in the Presence of AWGN

Characterizing the composition of QAM and DMC in the presence of AWGN is essentially estimating the integer K, the set of functions $\{h_k(t) : k = 1, 2, \ldots, K\}$, and the set of numbers $\left\{e^{j\omega_k T} : k = 1, 2, \ldots, K\right\}$ from the observations $\{z_l(t) : l = 0, 1, 2, \ldots, L\}$. The presence of AWGN necessitates that the observations $\{z_l(t)\}$ be projected onto some *finite* dimensional subspace, say $\mathcal{Y}$, of the space of square integrable functions that are also time-limited to the same interval that $\{h_k(t)\}$ is limited to, and the estimation be based (only) on the projections. If the subspace $\mathcal{Y}$ can be chosen such that $h_k(t) \in \mathcal{Y}$, for $k = 1, 2, \ldots, K$, then there is no loss of optimality in using (only) the projections. Suppose $\mathcal{Y}$ is such a subspace and $\{\phi_m(t) : m = 1, 2, \ldots, M\}$ is an orthonormal basis for $\mathcal{Y}$. Denote

$$z_{l,m} = \int \phi_m^*(t) z_l(t) dt, \tag{2.46}$$

$$w_{l,m} = \int \phi_m^*(t) w_l(t) dt, \tag{2.47}$$

for $l = 0, 1, 2, \ldots, L$ and $m = 1, 2, \ldots, M$, and

$$h_{k,m} = \int \phi_m^*(t) h_k(t) dt, \tag{2.48}$$

for $k = 1, 2, \ldots, K$ and $m = 1, 2, \ldots, M$. Observe that $w_{l,m}$ are zero mean complex Gaussian random variables that satisfy

$$E[w_{l_1,m_1}^* w_{l_2,m_2}] = \mathcal{N}_0 \delta_{l_1,l_2} \delta_{m_1,m_2}. \tag{2.49}$$

Denote the row vectors

$$\mathbf{z}_l = [z_{l,1}, z_{l,2}, \ldots, z_{l,M}], \tag{2.50}$$

$$\mathbf{w}_l = [w_{l,1}, w_{l,2}, \ldots, w_{l,M}], \tag{2.51}$$

for $l = 0, 1, 2, \ldots, L$, and

$$\mathbf{h}_k = [h_{k,1}, h_{k,2}, \ldots, h_{k,M}], \tag{2.52}$$

for $k = 1, 2, \ldots, K$. Denote the matrices

$$\mathbf{Z} = \begin{bmatrix} \mathbf{z}_0 \\ \mathbf{z}_1 \\ \mathbf{z}_2 \\ \vdots \\ \mathbf{z}_L \end{bmatrix}; \quad \mathbf{H} = \begin{bmatrix} \mathbf{h}_1 \\ \mathbf{h}_2 \\ \vdots \\ \mathbf{h}_K \end{bmatrix}; \quad \mathbf{W} = \begin{bmatrix} \mathbf{w}_0 \\ \mathbf{w}_1 \\ \mathbf{w}_2 \\ \vdots \\ \mathbf{w}_L \end{bmatrix}. \tag{2.53}$$

Then

$$\mathbf{Z} = \mathbf{V}_L(\Omega)\mathbf{H} + \mathbf{W}. \tag{2.54}$$

Given the matrix $\mathbf{Z}$, the task is to estimate the integer K, the set of numbers

$$\Omega = \left\{ e^{j\omega_k T} : k = 1, 2, \ldots, K \right\}, \tag{2.55}$$

and the matrix $\mathbf{H}$. In virtue of equation 2.16, the matrix $\mathbf{H}$ is dependent on Ω. But taking this dependence into account in the estimation of $\mathbf{H}$ and Ω seems difficult.[6] Disregarding the dependence makes the problem conform to the well studied class of problems mentioned in section 2.4.1. Therefore, the latter approach is taken in this thesis.

2.6 The Key Estimation Problem

For a set of mutually distinct real-valued scalars

$$\Theta = \{\theta_k : |\theta_k| < \pi; k = 1, 2, \ldots, K\}, \tag{2.56}$$

denote

$$\mathbf{V}(\Theta) = \begin{bmatrix} 1 & 1 & \ldots & 1 \\ e^{j\theta_1} & e^{j\theta_2} & \ldots & e^{j\theta_K} \\ e^{j\theta_1 2} & e^{j\theta_2 2} & \ldots & e^{j\theta_K 2} \\ \vdots & \vdots & & \vdots \\ e^{j\theta_1 L} & e^{j\theta_2 L} & \ldots & e^{j\theta_K L} \end{bmatrix}. \tag{2.57}$$

Then

$$\mathbf{Z} = \mathbf{V}(\Theta)\mathbf{H} + \mathbf{W}, \tag{2.58}$$

with the understanding that

$$\theta_k = \omega_k T, \tag{2.59}$$

for $k = 1, 2, \ldots, K$. The integer K, the set of real numbers Θ, and the complex matrix $\mathbf{H}$ must be estimated from $\mathbf{Z}$. The estimation of K is essentially a *hypothesis testing* problem, wherein a range of values of integers must be considered for K. A systematic method of doing this would involve the problem of estimating Θ and $\mathbf{H}$ for a given K. Therefore, the problem of estimating Θ and $\mathbf{H}$ assuming that K is known is discussed next, and an overview methods of estimating K is deferred until section 2.6.7.

2.6.1 An Overview of Methods of Estimating Θ and H

Equations of the form 2.58 arise also in problems 1, 2, and 3 stated in section 2.4.1. Among these, two cases can be distinguished based on the size of the matrix $\mathbf{Z}$:

[6]From the definition of the functions $\{h_k(t) : k = 1, 2, \ldots, K\}$ given by equation 2.16, any attempt at making use of the dependence between $\mathbf{H}$ and Ω is likely to introduce the functions $\{\phi_k(t) : k = 1, 2, \ldots, K\}$ into the estimation problem.

1. M is large but L is possibly small.

2. L is large but M is possibly small.

Typically, problem 3 belongs to the first case, with L being the number of *sensors* in the array and M being the so-called number of *snapshots*. However, problem 3 may belong to the second case [53]. Strictly speaking, problems 1 and 2 belong to the second case; so too does the problem of characterizing the composition of QAM and DMC as formulated in section 2.5.2.

In the first case a method known as *MUSIC* [48] and its variants are widely used under the precondition that $L \geq K$ and the matrix $\mathbf{H}$ has full row rank. In this class of methods – known collectively as *subspace* based methods – first the column span of the matrix $\mathbf{Z}$ is decomposed into two subspaces known as the (estimated) *signal subspace* and the (estimated) *noise subspace*.[7] Then the vector $\hat{\Theta}$ for which the column span of $\mathbf{V}(\hat{\Theta})$ gives the 'best' fit to the (estimated) signal subspace is considered the estimate of Θ. The matrix

$$\hat{\mathbf{H}} = \left(\hat{\mathbf{V}}^H\hat{\mathbf{V}}\right)^{-1}\hat{\mathbf{V}}^H\mathbf{Z}, \tag{2.60}$$

where $\hat{\mathbf{V}} = \mathbf{V}(\hat{\Theta})$, is then taken for the estimate of $\mathbf{H}$. For a performance analysis of MUSIC, see [50]. In the second case, the observations can be rearranged so as to satisfy the preconditions of the subspace based methods. Some possibilities[8] are forward smoothing, backward smoothing, and their combination forward-backward smoothing:

1. Forward Smoothing:- Here the subspace decomposition is done on the matrix $\mathbf{Z}_f$ given by

$$\mathbf{Z}_f = \mathbf{V}(\Theta)\mathbf{H}_f + \mathbf{W}_f, \tag{2.61}$$

[7]This is done, for instance, by finding the eigen decomposition of $\frac{1}{M}\mathbf{Z}\mathbf{Z}^H$ and choosing for the (estimated) signal subspace the span of the eigenvectors corresponding to the K largest eigen values. When $M \to \infty$ and/or $\mathcal{N}_0 \to 0$, the (estimated) signal subspace converges to the column span of $\mathbf{V}(\Theta)$, assuming $\frac{1}{M}\mathbf{H}\mathbf{H}^H$ converges to a full rank matrix. The method also gives an estimate of $\mathcal{N}_0$.

[8]In section 2.6.5, some remarks are made on the suitability of these methods in the context of the thesis.

where

$$\mathbf{Z}_f = \begin{bmatrix} \mathbf{z}_0 & \mathbf{z}_1 & \mathbf{z}_2 & \dots & \mathbf{z}_{L-\tilde{L}} \\ \mathbf{z}_1 & \mathbf{z}_2 & \mathbf{z}_3 & \dots & \mathbf{z}_{L-\tilde{L}+1} \\ \mathbf{z}_2 & \mathbf{z}_3 & \mathbf{z}_4 & \dots & \mathbf{z}_{L-\tilde{L}+2} \\ \vdots & & & & \vdots \\ \mathbf{z}_{\tilde{L}-1} & \mathbf{z}_{\tilde{L}} & \mathbf{z}_{\tilde{L}+1} & \dots & \mathbf{z}_{L-1} \\ \mathbf{z}_{\tilde{L}} & \mathbf{z}_{\tilde{L}+1} & \mathbf{z}_{\tilde{L}+2} & \dots & \mathbf{z}_L \end{bmatrix}, \tag{2.62}$$

similary for $\mathbf{W}_f$,

$$\mathbf{H}_f = \begin{bmatrix} \mathbf{h}_1 & \mathbf{h}_1 e^{j\theta_1} & \mathbf{h}_1 e^{j\theta_1 2} & \dots & \mathbf{h}_1 e^{j\theta_1 (L-\tilde{L})} \\ \mathbf{h}_2 & \mathbf{h}_2 e^{j\theta_2} & \mathbf{h}_2 e^{j\theta_2 2} & \dots & \mathbf{h}_2 e^{j\theta_2 (L-\tilde{L})} \\ \vdots & & & & \\ \mathbf{h}_K & \mathbf{h}_K e^{j\theta_K} & \mathbf{h}_K e^{j\theta_K 2} & \dots & \mathbf{h}_K e^{j\theta_K (L-\tilde{L})} \end{bmatrix}, \tag{2.63}$$

and $\mathbf{V}(\Theta)$ is the $(\tilde{L}+1) \times K$ Vandermonde matrix parameterized by Θ. Here $\tilde{L}$ is chosen according to $K \le \tilde{L} \le L - K + 1$ to satisfy the preconditions [49]. For a discussion on the tradeoffs involved in choosing $\tilde{L}$, see [50].

2. Backward Smoothing:- Here the subspace decomposition is done on the matrix $\mathbf{Z}_b$ given by

$$\mathbf{Z}_b = \mathbf{V}(\Theta)\mathbf{H}_b + \mathbf{W}_b, \tag{2.64}$$

where

$$\mathbf{Z}_b = \begin{bmatrix} \mathbf{z}^*_L & \dots & \mathbf{z}^*_{\tilde{L}+2} & \mathbf{z}^*_{\tilde{L}+1} & \mathbf{z}^*_{\tilde{L}} \\ \mathbf{z}^*_{L-1} & \dots & \mathbf{z}^*_{\tilde{L}+1} & \mathbf{z}^*_{\tilde{L}} & \mathbf{z}^*_{\tilde{L}-1} \\ \vdots & & & & \vdots \\ \mathbf{z}^*_{L-\tilde{L}+2} & \dots & \mathbf{z}^*_4 & \mathbf{z}^*_3 & \mathbf{z}^*_2 \\ \mathbf{z}^*_{L-\tilde{L}+1} & \dots & \mathbf{z}^*_3 & \mathbf{z}^*_2 & \mathbf{z}^*_1 \\ \mathbf{z}^*_{L-\tilde{L}} & \dots & \mathbf{z}^*_2 & \mathbf{z}^*_1 & \mathbf{z}^*_0 \end{bmatrix}, \tag{2.65}$$

similarly for $\mathbf{W}_b$,

$$\mathbf{H}_b = \begin{bmatrix} \mathbf{h}^*_1 e^{-j\theta_1 L} & \mathbf{h}^*_1 e^{-j\theta_1 (L-1)} & \mathbf{h}^*_1 e^{-j\theta_1 (L-2)} & \dots & \mathbf{h}^*_1 e^{-j\theta_1 \tilde{L}} \\ \mathbf{h}^*_2 e^{-j\theta_2 L} & \mathbf{h}^*_2 e^{-j\theta_2 (L-1)} & \mathbf{h}^*_2 e^{-j\theta_2 (L-2)} & \dots & \mathbf{h}^*_2 e^{-j\theta_2 \tilde{L}} \\ \vdots & & & & \\ \mathbf{h}^*_K e^{-j\theta_K L} & \mathbf{h}^*_K e^{-j\theta_K (L-1)} & \mathbf{h}^*_K e^{-j\theta_K (L-2)} & \dots & \mathbf{h}^*_K e^{-j\theta_K \tilde{L}} \end{bmatrix}, \tag{2.66}$$

and $\mathbf{V}(\Theta)$ is the $(\tilde{L}+1)\times K$ Vandermonde matrix parameterized by Θ, with $\tilde{L}$ satisfying $K \leq \tilde{L} \leq L-K+1$.

3. Forward-Backward Smoothing:- Here the subspace decomposition is done on the matrix $\mathbf{Z}_{f,b}$ given by

$$\mathbf{Z}_{f,b} = [\mathbf{Z}_f, \mathbf{Z}_b] = \mathbf{V}(\Theta)[\mathbf{H}_f, \mathbf{H}_b] + [\mathbf{W}_f, \mathbf{W}_b], \tag{2.67}$$

with $\tilde{L}$ satisfying $K \leq \tilde{L} \leq L - K/2$ [34].

A method applicable regardless of the relative magnitudes of L and M is the *maximum likelihood* method to be discussed next.[9]

2.6.2 The Maximum Likelihood Method

Given the observation matrix $\mathbf{Z}$, the conditional complex Gaussian probability density[10]

$$p(\mathbf{Z}|\Theta,\mathbf{H},\mathcal{N}_0) = \frac{1}{\pi^{(L+1)M}\mathcal{N}_0^{(L+1)M}}\exp\left(-\frac{1}{\mathcal{N}_0}\mathrm{trace}\left((\mathbf{Z}-\mathbf{VH})^H(\mathbf{Z}-\mathbf{VH})\right)\right), \tag{2.68}$$

considered as a function of Θ, $\mathbf{H}$, and $\mathcal{N}_0$, is known as the *likelihood* of Θ, $\mathbf{H}$, and $\mathcal{N}_0$. The maximum likelihood method seeks to find those values of Θ, $\mathbf{H}$, and $\mathcal{N}_0$ whose likelihood is maximum. This is conveniently done by maximizing the logarithm of the likelihood

$$\begin{aligned}\mathcal{L}(\Theta,\mathbf{H},\mathcal{N}_0) &= \log p(\mathbf{Z}|\Theta,\mathbf{H},\mathcal{N}_0), &(2.69)\\ &= -\frac{1}{\mathcal{N}_0}\mathrm{trace}\left((\mathbf{Z}-\mathbf{VH})^H(\mathbf{Z}-\mathbf{VH})\right) - (L+1)M\log\mathcal{N}_0 - (L+1)M\log\pi. &(2.70)\end{aligned}$$

Thus, the maximum likelihood estimates of Θ, $\mathbf{V}$, $\mathbf{H}$, and $\mathcal{N}_0$ are given respectively by [60]

$$\begin{aligned}\hat{\Theta} &= \underset{\Theta}{\arg\min}\ \mathrm{trace}\left(\left(\mathbf{I}-\mathbf{V}\left(\mathbf{V}^H\mathbf{V}\right)^{-1}\mathbf{V}^H\right)\mathbf{Z}\mathbf{Z}^H\right), &(2.71)\\ \hat{\mathbf{V}} &= \mathbf{V}(\hat{\Theta}), &(2.72)\\ \hat{\mathbf{H}} &= \left(\hat{\mathbf{V}}^H\hat{\mathbf{V}}\right)^{-1}\hat{\mathbf{V}}^H\mathbf{Z}, &(2.73)\end{aligned}$$

[9] In the context of problem 3, it is also known as the *deterministic* maximum likelihood method.
[10] For convenience, $\mathbf{V}(\Theta)$ is denoted simply by $\mathbf{V}$.

and

$$\hat{\mathcal{N}_0} = \frac{1}{(L+1)M}\text{trace}\left(\left(\mathbf{I} - \hat{\mathbf{V}}\left(\hat{\mathbf{V}}^H\hat{\mathbf{V}}\right)^{-1}\hat{\mathbf{V}}^H\right)\mathbf{Z}\mathbf{Z}^H\right). \tag{2.74}$$

These maximum likelihood estimates are known to have desirable asymptotic properties. Before discussing the asymptotic properties, a discussion of the Cramer-Rao bound is in order.

2.6.3 The Cramer-Rao Lower Bound

Denote by $\mathbf{p}$ the $(1+2KM+K)$-dimensional vector comprising the unknown parameters, that is,

$$\mathbf{p} = \begin{bmatrix} \mathcal{N}_0 \\ \mathbf{p_H} \\ \mathbf{p_\Theta} \end{bmatrix}, \tag{2.75}$$

where

$$\mathbf{p_\Theta} = [\theta_1, \ldots, \theta_K]^T, \tag{2.76}$$

and

$$\begin{aligned} \mathbf{p_H} \;=\; & [\Re(h_{1,1}), \ldots, \Re(h_{K,1}), \Im(h_{1,1}), \ldots, \Im(h_{K,1}), \Re(h_{1,2}), \ldots \\ & \ldots, \Im(h_{K,M-1}), \Re(h_{1,M}), \ldots, \Re(h_{K,M}), \Im(h_{1,M}), \ldots, \Im(h_{K,M})]^T \end{aligned} \tag{2.77}$$

where $\Re(.)$ and $\Im(.)$ denote the real and imaginary parts respectively. Denote by $\mathbf{q}$ the gradient of the log-likelihood function $\mathcal{L}(\mathbf{p})$ with respect to the parameter vector $\mathbf{p}$, that is,

$$\mathbf{q} = \frac{\partial \mathcal{L}}{\partial \mathbf{p}}. \tag{2.78}$$

Then the Cramer-Rao lower bound on the covariance of any *unbiased* estimator of $\mathbf{p}$ is given by the matrix

$$\mathbf{Q} = \left[E\left(\mathbf{q}\mathbf{q}^T\right)\right]^{-1}, \tag{2.79}$$

where the expectation is taken with respect to the noise $\mathbf{W}$ of equation 2.58. A formula for the inverse of $\mathbf{Q}$ has been derived by Stoica and Nehorai [50] in the context of problems 1, 2, and 3 described in section 2.4.1. Accordingly, denote by $\mathbf{U}(\Theta)$ the matrix whose k^{th} column is the derivative of the k^{th} column of the matrix $\mathbf{V}(\Theta)$ taken with repect to the parameter θ_k, for $k = 1, 2, \ldots, K$, that is,

$$\mathbf{U}(\Theta) = \begin{bmatrix} 0 & 0 & \ldots & 0 \\ je^{j\theta_1} & je^{j\theta_2} & \ldots & je^{j\theta_K} \\ j2e^{j\theta_1 2} & j2e^{j\theta_2 2} & \ldots & j2e^{j\theta_K 2} \\ \vdots & \vdots & & \vdots \\ jLe^{j\theta_1 L} & jLe^{j\theta_2 L} & \ldots & jLe^{j\theta_K L} \end{bmatrix}. \tag{2.80}$$

Denote by $\mathbf{H}_m$ the diagonal matrix comprising the elements of $(h_{1,m}, h_{2,m}, \ldots, h_{K,m})$, for $m = 1, 2, \ldots, M$, that is,

$$\mathbf{H}_m = \begin{bmatrix} h_{1,m} & 0 & \ldots & 0 & 0 \\ 0 & h_{2,m} & & & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & & & h_{K-1,m} & 0 \\ 0 & 0 & \ldots & 0 & h_{K,m} \end{bmatrix}. \tag{2.81}$$

Denote the $K \times K$ matrices[11]

$$\mathbf{E} = \frac{2}{\mathcal{N}_0}\left(\sum_{m=1}^{M} \Re\left(\mathbf{H}_m^H \mathbf{U}^H \mathbf{U} \mathbf{H}_m\right)\right), \tag{2.82}$$

$$= \frac{2}{\mathcal{N}_0}\Re\left(\left(\mathbf{U}^H\mathbf{U}\right) \odot \left(\mathbf{H}\mathbf{H}^H\right)^*\right), \tag{2.83}$$

$$\mathbf{G} = \frac{2}{\mathcal{N}_0}\mathbf{V}^H\mathbf{V}, \tag{2.84}$$

and

$$\mathbf{F}_m = \frac{2}{\mathcal{N}_0}\mathbf{V}^H\mathbf{U}\mathbf{H}_m, \tag{2.85}$$

for $m = 1, 2, \ldots, M$. Then, the matrix $\mathbf{Q}^{-1}$ has the partitioned form [50]

$$\mathbf{Q}^{-1} = \begin{bmatrix} \frac{M(L+1)}{\mathcal{N}_0^2} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_{1,1} & \mathbf{A}_{1,2} \\ \mathbf{0} & \mathbf{A}_{2,1} & \mathbf{A}_{2,2} \end{bmatrix}, \tag{2.86}$$

[11] Here $\odot$ denotes the element-wise (Hadamard) product.

where

$$\mathbf{A}_{1,1} = \begin{bmatrix} \Re(\mathbf{G}) & -\Im(\mathbf{G}) & \dots & \mathbf{0} & \mathbf{0} \\ \Im(\mathbf{G}) & \Re(\mathbf{G}) & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & & \ddots & & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \Re(\mathbf{G}) & -\Im(\mathbf{G}) \\ \mathbf{0} & \mathbf{0} & \dots & \Im(\mathbf{G}) & \Re(\mathbf{G}) \end{bmatrix}, \tag{2.87}$$

that is, a $2KM \times 2KM$ block-diagonal matrix with $2K \times 2K$ blocks along the diagonal,

$$\mathbf{A}_{2,1} = \mathbf{A}_{1,2}^T = \left[\ \Re(\mathbf{F}_1^T) \quad \Im(\mathbf{F}_1^T) \quad \dots \quad \Re(\mathbf{F}_M^T) \quad \Im(\mathbf{F}_M^T)\ \right], \tag{2.88}$$

and

$$\mathbf{A}_{2,2} = \mathbf{E}. \tag{2.89}$$

The conformal partition of $\mathbf{Q}$ has the form

$$\mathbf{Q} = \begin{bmatrix} \frac{\mathcal{N}_0^2}{M(L+1)} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_{1,1} & \mathbf{B}_{1,2} \\ \mathbf{0} & \mathbf{B}_{2,1} & \mathbf{B}_{2,2} \end{bmatrix}, \tag{2.90}$$

where

$$\mathbf{B}_{1,1} = \left[\mathbf{A}_{1,1} - \mathbf{A}_{1,2}\mathbf{A}_{2,2}^{-1}\mathbf{A}_{2,1}\right]^{-1}, \tag{2.91}$$

$$\mathbf{B}_{2,2} = \left[\mathbf{A}_{2,2} - \mathbf{A}_{2,1}\mathbf{A}_{1,1}^{-1}\mathbf{A}_{1,2}\right]^{-1}, \tag{2.92}$$

$$\mathbf{B}_{1,2} = -\mathbf{A}_{1,1}^{-1}\mathbf{A}_{1,2}\mathbf{B}_{2,2}, \tag{2.93}$$

$$\mathbf{B}_{2,1} = -\mathbf{A}_{2,2}^{-1}\mathbf{A}_{2,1}\mathbf{B}_{1,1}. \tag{2.94}$$

Stoica and Nehorai [50] have further shown that

$$\mathbf{B}_{2,2} = \frac{\mathcal{N}_0}{2}\left[\sum_{m=1}^{M} \Re\left(\mathbf{H}_m^H \mathbf{U}^H[\mathbf{I} - \mathbf{V}\left(\mathbf{V}^H\mathbf{V}\right)^{-1}\mathbf{V}^H]\mathbf{U}\mathbf{H}_m\right)\right]^{-1}, \tag{2.95}$$

which can be written more compactly as

$$\mathbf{B}_{2,2} = \frac{\mathcal{N}_0}{2}\left[\Re\left(\left(\mathbf{U}^H[\mathbf{I} - \mathbf{V}\left(\mathbf{V}^H\mathbf{V}\right)^{-1}\mathbf{V}^H]\mathbf{U}\right) \odot \left(\mathbf{H}\mathbf{H}^H\right)^*\right)\right]^{-1}, \tag{2.96}$$

where $\odot$ represents the element-wise product.

The Case $K = 1$

In this case,

$$\mathbf{B}_{2,2} = \frac{6\mathcal{N}_0}{L(L+1)(L+2)(\mathbf{H}\mathbf{H}^H)}, \tag{2.97}$$

as given by Stoica and Nehorai [50].

2.6.4 Asymptotic Properties of the Maximum Likelihood Method

The behaviour of the maximum likelihood estimate of the parameter vector $\mathbf{p}$, when L tends to infinity, has been studied in different contexts by Rao and Zhao [39] (the case where $M = 1$), and by Viberg, Ottersten, and Nehorai [53] (the case where, in general, $M \geq 1$). Rao and Zhao have shown, for the case where $M = 1$, that the maximum likelihood estimate $\hat{\mathbf{p}}$ of $\mathbf{p}$ has the following properties as L tends to infinity:

- it is *strongly consistent*, that is, $\hat{\mathbf{p}}$ converges in probability to $\mathbf{p}$,
- it is asymptotically *normal* (in other words, Gaussian),
- it is asymptotically *efficient*, that is, the covariance of $\hat{\mathbf{p}}$ approaches the Cramer-Rao lower bound.

Viberg, Ottersten, and Nehorai have shown, for the general case where $M \geq 1$, that the maximum likelihood estimates are *consistent*, asymptotically *normal*, and asymptotically *efficient*.

Convergence Rates of the Maximum Likelihood Estimates

Rao and Zhao have also derived the asymptotic rates of convergence for the estimates $\hat{\mathcal{N}}_0$, $\hat{\mathbf{H}}$, and $\hat{\Theta}$ for the case where $M = 1$. Their results can easily be extended to the case where $M \geq 1$ using the following results from [50]:

$$\lim_{L\to\infty} \frac{1}{L}\mathbf{V}^H\mathbf{V} = \mathbf{I}, \tag{2.98}$$

$$\lim_{L\to\infty} \frac{1}{L^2}\mathbf{V}^H\mathbf{U} = \frac{j}{2}\mathbf{I}, \tag{2.99}$$

$$\lim_{L\to\infty} \frac{1}{L^3}\mathbf{U}^H\mathbf{U} = \frac{1}{3}\mathbf{I}. \tag{2.100}$$

By using the above results in equations 2.87, 2.88, and 2.89, one obtains

$$\lim_{L\to\infty} \frac{1}{L}\mathbf{A}_{1,1} = \frac{2}{\mathcal{N}_0}\mathbf{I}_{2KM\times 2KM}, \tag{2.101}$$

$$\lim_{L\to\infty} \frac{1}{L^2}\mathbf{A}_{2,1} = \lim_{L\to\infty} \frac{1}{L^2}\mathbf{A}_{1,2}^T, \tag{2.102}$$

$$= \frac{1}{\mathcal{N}_0}\mathbf{X}, \tag{2.103}$$

$$\lim_{L\to\infty} \frac{1}{L^3}\mathbf{A}_{2,2} = \frac{2}{3\mathcal{N}_0}\left(\mathbf{I}_{K\times K} \odot \left(\mathbf{H}\mathbf{H}^H\right)\right), \tag{2.104}$$

$$= \frac{2}{3\mathcal{N}_0}\mathbf{X}\mathbf{X}^T, \tag{2.105}$$

where

$$\mathbf{X} = \begin{bmatrix} -\Im(\mathbf{H}_1) & \Re(\mathbf{H}_1) & \ldots & -\Im(\mathbf{H}_m) & \Re(\mathbf{H}_m) \end{bmatrix}. \tag{2.106}$$

Denoting

$$\mathbf{D} = \begin{bmatrix} L^{1/2} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & L^{1/2}\mathbf{I}_{2KM\times 2KM} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & L^{3/2}\mathbf{I}_{K\times K} \end{bmatrix}, \tag{2.107}$$

one can observe from equation 2.86 that

$$\lim_{L\to\infty} \mathbf{D}^{-1}\mathbf{Q}^{-1}\mathbf{D}^{-1} = \frac{1}{\mathcal{N}_0}\begin{bmatrix} \frac{M}{\mathcal{N}_0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 2\mathbf{I}_{2KM\times 2KM} & \mathbf{X}^T \\ \mathbf{0} & \mathbf{X} & \frac{2}{3}\mathbf{X}\mathbf{X}^T \end{bmatrix}. \tag{2.108}$$

Therefore,

$$\lim_{L\to\infty} \mathbf{DQD} = \mathcal{N}_0\begin{bmatrix} \frac{\mathcal{N}_0}{M} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_{1,1} & \mathbf{C}_{1,2} \\ \mathbf{0} & \mathbf{C}_{2,1} & \mathbf{C}_{2,2} \end{bmatrix}, \tag{2.109}$$

where

$$\mathbf{C}_{1,1} = \left[2\mathbf{I}_{2KM\times 2KM} - \frac{3}{2}\mathbf{X}^T\left(\mathbf{X}\mathbf{X}^T\right)^{-1}\mathbf{X}\right]^{-1}, \tag{2.110}$$

$$\mathbf{C}_{2,2} = \left[\frac{2}{3}\mathbf{X}\mathbf{X}^T - \frac{1}{2}\mathbf{X}\mathbf{X}^T\right]^{-1}, \tag{2.111}$$

$$= 6\left(\mathbf{X}\mathbf{X}^T\right)^{-1}, \tag{2.112}$$

$$\mathbf{C}_{1,2} = -\frac{1}{2}\mathbf{X}^T\mathbf{C}_{2,2}, \tag{2.113}$$

$$\begin{aligned} &= -6\mathbf{X}^T\left(\mathbf{X}\mathbf{X}^T\right)^{-1}, & (2.114)\\ \mathbf{C}_{2,1} &= \mathbf{C}_{1,2}^T, & (2.115)\\ &= -6\left(\mathbf{X}\mathbf{X}^T\right)^{-1}\mathbf{X}. & (2.116)\end{aligned}$$

The matrix $\mathbf{C}_{2,2}$ can also be obtained, using the result [50]

$$\lim_{L\to\infty}\frac{1}{L^3}\mathbf{U}^H[\mathbf{I}-\mathbf{V}\left(\mathbf{V}^H\mathbf{V}\right)^{-1}\mathbf{V}^H]\mathbf{U} = \frac{1}{12}\mathbf{I}_{K\times K}, \tag{2.117}$$

as

$$\begin{aligned} \mathcal{N}_0\mathbf{C}_{2,2} &= \lim_{L\to\infty} L^3\mathbf{B}_{2,2}, & (2.118)\\ &= 6\mathcal{N}_0\left(\mathbf{I}_{K\times K}\odot\mathbf{H}\mathbf{H}^H\right)^{-1}, & (2.119)\\ &= 6\mathcal{N}_0\left(\mathbf{X}\mathbf{X}^T\right)^{-1}. & (2.120)\end{aligned}$$

Since the matrix $\mathbf{Q}$ is the asymptotic covariance of the estimate $\hat{\mathbf{p}}$, the matrix $\mathbf{DQD}$ is the asymptotic covariance of $\mathbf{D}\hat{\mathbf{p}}$. This observation combined with the result that the matrix $\mathbf{DQD}$ is asymptotically constant shows that the maximum likelihood estimates of $\mathcal{N}_0$, $\mathbf{H}$, and Θ have convergence rates of $L^{-1/2}$, $L^{-1/2}$, and $L^{-3/2}$ respectively.

2.6.5 On the Necessity of Faster-than-L^{-1} Convergence Rate of the Estimate $\hat{\Theta}$ of Θ

The philosophy of design proposed in section 1.7.1 of chapter 1, based on alternate probing and signalling, can be justified only if the estimate $\hat{\Theta}$ converges to Θ at a rate faster than L^{-1}. This fact is explained below.

Suppose that the probing is done over an interval of $\mathcal{L}_{\text{probe}}$ bauds and signalling is done over an interval of $\mathcal{L}_{\text{signal}}$ bauds. Suppose further that the probing is done with the signal

$$\sum_{n=0}^{L} g(t-nT_p), \tag{2.121}$$

where T_p is chosen according to the discussion in section[12] 2.3.1 and $(L+1)$ is the largest integer smaller than $\mathcal{L}_{\text{probe}}\left(\frac{T}{T_p}\right)$ with T being the baud period for signalling. Thus one

[12]That is, such that the set of functions $\{h_k(t) : k = 1, 2, \ldots, K\}$ defined by equation 2.16 is limited to an interval of length less than T_p.

obtains equation 2.58 with the understanding that

$$\theta_k = \omega_k T_p, \quad |\theta_k| < \pi, \tag{2.122}$$

for $k = 1, 2, \ldots, K$. Denote by $\hat{\Theta}$ the estimate of $\Theta = (\theta_1, \theta_2, \ldots, \theta_K)$.

In section 2.3.4, it was observed that the receiver will need 'accurate' estimates of

$$\{\omega_k lT : k = 1, 2, \ldots, K; 1 \leq l \leq \mathcal{L}\}, \tag{2.123}$$

where $\mathcal{L}$ is the length of the data frame given by $\mathcal{L} = \mathcal{L}_{\text{probe}} + \mathcal{L}_{\text{signal}}$. Roughly speaking, the denser the signal set[13] the higher will be the 'desired accuracy.' Suppose that

$$\Theta_0 = \operatorname{diag}(\vartheta_1, \vartheta_2, \ldots, \vartheta_K) \tag{2.124}$$

is the positive definite matrix of the maximum allowable variances of the errors in the estimates of

$$\{\omega_k lT : k = 1, 2, \ldots, K; 1 \leq l \leq \mathcal{L}\}. \tag{2.125}$$

Then

$$l^2 \left(\frac{T}{T_p}\right)^2 E\left[\left(\hat{\Theta} - \Theta\right)\left(\hat{\Theta} - \Theta\right)^H\right] < \Theta_0, \tag{2.126}$$

for $1 \leq l \leq \mathcal{L}$. Particularly,

$$L^2 E\left[\left(\hat{\Theta} - \Theta\right)\left(\hat{\Theta} - \Theta\right)^H\right] < \Theta_0. \tag{2.127}$$

For arbitrarily small values of Θ_0 and signal-to-noise ratio, the above inequality can be satisfied with a sufficiently large L if and only if

$$\lim_{L \to \infty} L^2 E\left[\left(\hat{\Theta} - \Theta\right)\left(\hat{\Theta} - \Theta\right)^H\right] = 0, \tag{2.128}$$

or, in otherwords, if and only if, the estimate $\hat{\Theta}$ converges to Θ at a rate faster than L^{-1}.

The maximum likelihood estimate $\hat{\Theta}$ defined by equation 2.71 has a faster-than-L^{-1} convergence rate, as was shown in section 2.6.4. Therefore, the philosophy of design proposed in the thesis can be justified on the basis of the maximum likelihood method.

[13]Density here means the number of points per unit area obtained after normalizing the boundary; for example, 64 QAM is denser than 16 QAM.

As for the subspace-based methods – the *MUSIC* algorithm in conjunction with smoothing – described in section 2.6.1, Stoica and Nehorai [50] seem to suggest[14] that the estimate $\hat{\Theta}$ has a convergence rate of only $L^{-1/2}$. Therefore, the *MUSIC* algorithm in conjunction with smoothing cannot be used in the context of the thesis, except perhaps in finding initial values for the search algorithm that is used ultimately in finding the maximum likelihood estimate $\hat{\Theta}$.

2.6.6 A Guide-Line for Choosing $\mathcal{L}_{\text{probe}}$ under the Maximum Likelihood Method

Denote, for convenience,

$$\Xi = E\left[\left(\hat{\Theta}-\Theta\right)\left(\hat{\Theta}-\Theta\right)^H\right]. \tag{2.129}$$

From equation 2.126 of section 2.6.5,

$$\mathcal{L}^2_{\text{probe}}\left(\frac{T}{T_p}\right)^2\Xi < \left(\mathcal{L}_{\text{probe}}+\mathcal{L}_{\text{signal}}\right)^2\left(\frac{T}{T_p}\right)^2\Xi \leq \Theta_0, \tag{2.130}$$

or, equivalently,

$$\mathcal{L}^2_{\text{probe}}\mathbf{I}_{K\times K} < \left(\mathcal{L}_{\text{probe}}+\mathcal{L}_{\text{signal}}\right)^2\mathbf{I}_{K\times K} \leq \left(\frac{T_p}{T}\right)^2\Xi^{-1}\Theta_0. \tag{2.131}$$

Recall, from section 2.6.4, that for the maximum likelihood method, when $\mathcal{L}_{\text{probe}}$ is large,

$$\Xi \approx \frac{6\mathcal{N}_0\left(\mathbf{I}_{K\times K}\odot\mathbf{H}\mathbf{H}^H\right)^{-1}}{\mathcal{L}^3_{\text{probe}}\left(\frac{T}{T_p}\right)^3}. \tag{2.132}$$

Using this in equation 2.131,

$$\mathcal{L}^2_{\text{probe}}\mathbf{I}_{K\times K} < \left(\mathcal{L}_{\text{probe}}+\mathcal{L}_{\text{signal}}\right)^2\mathbf{I}_{K\times K} \leq \mathcal{L}^3_{\text{probe}}\left(\frac{T}{T_p}\right)\frac{\left(\mathbf{I}_{K\times K}\odot\mathbf{H}\mathbf{H}^H\right)\Theta_0}{6\mathcal{N}_0}. \tag{2.133}$$

Therefore, for a given $\mathcal{L}_{\text{signal}}$, the smallest allowable $\mathcal{L}_{\text{probe}}$ is given by the solution to the cubic equation

$$\left(\mathcal{L}_{\text{probe}}+\mathcal{L}_{\text{signal}}\right)^2 = \alpha\mathcal{L}^3_{\text{probe}}, \tag{2.134}$$

[14] Their proof is for the case where the elements of $\mathbf{H}$ (c.f. equation 2.58) constitute a wide sense stationary random process and where $M \to \infty$. Whether it applies to the matrices $\mathbf{H}_f$, $\mathbf{H}_b$, or $[\mathbf{H}_f, \mathbf{H}_b]$ obtained by smoothing and where $L \to \infty$ for fixed $\tilde{L}$ is not certain to the author of the thesis.

where α is the smallest element of the diagonal matrix

$$\left(\frac{T}{T_p}\right) \frac{\left(\mathbf{I}_{K\times K} \odot \mathbf{H}\mathbf{H}^H\right) \Theta_0}{6\mathcal{N}_0}. \tag{2.135}$$

2.6.7 An Overview of Methods of Estimating K

The problem of estimating the actual number of paths K is essentially one of choosing from a family of models, parametrized by the assumed number of paths, a model that provides the 'best' fit to the $(L+1) \times M$ observation matrix $\mathbf{Z}$ defined in section 2.5.2. This problem can be solved by choosing, for the estimated number of paths, the value of K that minimizes a criterion function of the form

$$-2\mathcal{L}\left(\hat{\Theta}, \hat{\mathbf{H}}, \hat{\mathcal{N}_0}\right) + p\mathcal{P}(n), \tag{2.136}$$

where $\mathcal{L}(\Theta, \mathbf{H}, \mathcal{N}_0)$ is the log-likelihood function of Θ, $\mathbf{H}$, and $\mathcal{N}_0$ for an assumed value of K, and $\hat{\Theta}$, $\hat{\mathbf{H}}$, and $\hat{\mathcal{N}_0}$, respectively, are their maximum-likelihood estimates, as discussed in section 2.6.2, p is the number of *free real scalar parameters* as determined by K, n is the number of *real scalar observations*, and $\mathcal{P}(n)$ is a positive function, whose choice determines the criterion. From section 2.6.2,

$$\mathcal{L}\left(\hat{\Theta}, \hat{\mathbf{H}}, \hat{\mathcal{N}_0}\right) = -(L+1)M\left[1 + \log\pi + \log C_K\right], \tag{2.137}$$

where

$$C_K = \min_{\Theta} \frac{1}{(L+1)M} \operatorname{trace}\left(\left(\mathbf{I} - \mathbf{V}\left(\mathbf{V}^H\mathbf{V}\right)^{-1}\mathbf{V}^H\right)\mathbf{Z}\mathbf{Z}^H\right), \tag{2.138}$$

with $\mathbf{V}$ denoting $\mathbf{V}_L(\Theta)$. Moreover, $p = K + 2KM + 1$, and $n = 2(L+1)M$. Therefore, the estimated number of paths is the value of K that minimizes

$$2(L+1)M\left[1 + \log\pi + \log C_K\right] + (K + 2KM + 1)\mathcal{P}(2(L+1)M). \tag{2.139}$$

As the assumed number of paths K increases, the maximized log-likelihood $\mathcal{L}\left(\hat{\Theta}, \hat{\mathbf{H}}, \hat{\mathcal{N}_0}\right)$ increases as well,[15] and therefore the function $\mathcal{P}(n)$ is designed to penalize the choice of

[15]If $K_1 < K_2$, then a model with an assumed number of paths K_1 is also a legitimate model with assumed number of paths K_2, but with some $(K_2 - K_1)$ paths being of strength zero. Therefore, the maximization of the likelihood with K_1 paths can be viewed as the maximization of the likelihood with K_2 paths, but with some constraints.

larger values of K. Two criterion functions of the above form that are based on information theoretic principles are the *Akaike's Information Criterion* (AIC) [1] and the *Minimum Description Length* (MDL) criterion [42]. The AIC uses $\mathcal{P}(n) = 2$ and the MDL criterion uses $\mathcal{P}(n) = \log n$. The AIC asymptotically tends to *over-estimate* the number of free parameters, that is, as n tends to infinity, although the probability of under-estimation becomes vanishingly small, the probability of over-estimation remains non-zero. The MDL criterion is asymptotically *consistent*, that is, as n tends to infinity the probability of erroneous estimation of either kind become vanishingly small.

Observe that the above method of estimating K subsumes the method of estimating Θ given in section 2.6.2. However, the above method is computationally intensive. Two computationally less intensive methods of estimating the number of real undamped/damped sinusoids in noise, due to Reddy and Biradar [41] and Fuchs [12], can be adapted for estimating the number of paths K. These methods are reviewed in the following.

The Method of Reddy and Biradar

Recall from section 2.6.1 the equation

$$\mathbf{Z}_f = \mathbf{V}_{\bar{L}}(\Theta)\mathbf{H}_f + \mathbf{W}_f \tag{2.140}$$

obtained by forward smoothing. The method of Reddy and Biradar [41] is based on the fact that if $K < \bar{L} \leq (L-K+1)$ then both $\mathbf{V}_{\bar{L}}(\Theta)\mathbf{H}_f$ and the sub-matrix of $\mathbf{V}_{\bar{L}}(\Theta)\mathbf{H}_f$ consisting of all but the last row, that is, $\mathbf{V}_{\bar{L}-1}(\Theta)\mathbf{H}_f$, have rank K. Therefore, the partition

$$\mathbf{Z}_f = \begin{bmatrix} \mathbf{Z}_{\text{matrix}} \\ \mathbf{Z}_{\text{row}} \end{bmatrix}, \tag{2.141}$$

where $\mathbf{Z}_{\text{matrix}}$ consists of all but the last row of $\mathbf{Z}_f$ (and, therefore, where $\mathbf{Z}_{\text{row}}$ is the last row of $\mathbf{Z}_f$), and the conformal partition

$$\mathbf{W}_f = \begin{bmatrix} \mathbf{W}_{\text{matrix}} \\ \mathbf{W}_{\text{row}} \end{bmatrix} \tag{2.142}$$

satisfy

$$\mathbf{Z}_{\text{matrix}} = \mathbf{V}_{\bar{L}-1}(\Theta)\mathbf{H}_f + \mathbf{W}_{\text{matrix}}, \tag{2.143}$$

$$\mathbf{Z}_{\text{row}} = \mathbf{a}^H\mathbf{V}_{\bar{L}-1}(\Theta)\mathbf{H}_f + \mathbf{W}_{\text{row}}, \tag{2.144}$$

where a is some $\tilde{L}$-dimensional column vector.[16] Then denoting

$$\mathbf{A} = \mathbf{V}_{\tilde{L}-1}(\Theta)\mathbf{H}_f \tag{2.145}$$

for convenience, $\mathbf{Z}_{\text{row}}$ has the probability density function

$$p(\mathbf{Z}_{\text{row}}\,|\mathbf{a}, \mathbf{A}, \mathcal{N}_0) = \frac{1}{(\pi\mathcal{N}_0)^{(L-\tilde{L}+1)M}} \exp\left(-\frac{1}{\mathcal{N}_0}\left(\mathbf{Z}_{\text{row}} - \mathbf{a}^H\mathbf{A}\right)\left(\mathbf{Z}_{\text{row}} - \mathbf{a}^H\mathbf{A}\right)^H\right) \tag{2.146}$$

parametrized by a, A, and $\mathcal{N}_0$. From here onwards, the development is somewhat heuristic. Thus, for an assumed number of paths K, the Singular-Value-Decomposition(SVD)-based rank-K-approximant[17] of $\mathbf{Z}_{\text{matrix}}$ can be taken as an estimate of A. A family of such estimates, obtained by varying K over a range of values, can be used to compute an *approximate* family of probability density functions for $\mathbf{Z}_{\text{row}}$, which is then considered parametrized by only the unique *minimum norm* a and $\mathcal{N}_0$. Therefore, the number of free parameters in this approximate family of models is $p = K + 1$. The estimated number of paths is the value of K which minimizes

$$2(L - \tilde{L} + 1)M\,(1 + \log\pi + \log C_K) + (K + 1)\mathcal{P}(2(L - \tilde{L} + 1)M), \tag{2.147}$$

where C_K is defined in terms of the SVD of $\mathbf{Z}_{\text{matrix}}$ as

$$C_K = \frac{1}{(L - \tilde{L} + 1)M}(\mathbf{Z}_{\text{row}}\mathbf{Y}_K)(\mathbf{Z}_{\text{row}}\mathbf{Y}_K)^H, \tag{2.148}$$

where $\mathbf{Y}_K$ contains in its columns all right singular vectors of $\mathbf{Z}_{\text{matrix}}$ except those corresponding to the K largest singular values. Here $\mathcal{P}(n)$ can be chosen as $\mathcal{P}(n) = 2$ to obtain the AIC version, or as $\mathcal{P}(n) = \log n$ to obtain the MDL version.

Using some ideas from [59], Reddy and Biradar have derived an approximate probability of correct estimation for these methods, in the context of undamped/damped sinusoidals, and verified them by computer simulations. The derived probabilities of correct estimation were optimistic for both the AIC and MDL versions but were quite close for the MDL version. Moreover, the AIC version exhibited a significant tendency to produce over-estimates while the MDL version did not exhibit any such tendency. However, no proof is given for the consistency of the MDL version.

[16] The vector a satisfies $(\mathbf{a}^H, -1)\mathbf{V}_{\tilde{L}}(\Theta) = 0$.

[17] Suppose $\mathbf{Z}_{\text{matrix}} = \sum_{k=1}^{\tilde{L}} \sigma_k \mathbf{u}_k \mathbf{v}_k^H$, where $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_{\tilde{L}}$ are the singular values of $\mathbf{Z}_{\text{matrix}}$ and $\mathbf{u}_k$, $\mathbf{v}_k$ are the left and right singular vectors, respectively, corresponding to σ_k for $k = 1, 2, \ldots, \tilde{L}$, then the rank-K-approximant of $\mathbf{Z}_{\text{matrix}}$ is $\sum_{k=1}^{K} \sigma_k \mathbf{u}_k \mathbf{v}_k^H$. This is based on the fact that $\mathbf{Z}_{\text{matrix}}$ is of full rank due to noise.

The Method of Fuchs

The method of Fuchs [12] is essentially a sequence of tests of hypotheses, wherein the question "Is the number of paths at least k?" is asked for $k = 1, 2, \ldots$ in sequence until the answer becomes "No", and the largest k for which the answer is "Yes" is taken as the estimate of the number of paths K. The method is based on the eigen-decomposition of the matrix[18] $\mathbf{Z}_f\mathbf{Z}_f^H$, where the $(\bar{L}+1) \times (L - \bar{L} + 1)M$ matrix $\mathbf{Z}_f$ satisfies

$$\mathbf{Z}_f = \mathbf{V}_{\bar{L}}(\Theta)\mathbf{H}_f + \mathbf{W}_f \tag{2.149}$$

as described in section 2.6.1. For convenience, let $\mathbf{V}_{\bar{L}}$ denote $\mathbf{V}_{\bar{L}}(\Theta)$, and for matrices $\mathbf{X}_f$, $\mathbf{Y}_f$ having N columns each, define $\mathbf{R}_{XY} = \frac{1}{N}\mathbf{X}_f\mathbf{Y}_f^H$, where $N = (L - \bar{L} + 1)M$. In terms of the above notations,

$$\mathbf{R}_{ZZ} = \mathbf{V}_{\bar{L}}\mathbf{R}_{HH}\mathbf{V}_{\bar{L}}^H + \mathcal{N}_0\mathbf{I} + \Delta_1 + \Delta_2, \tag{2.150}$$

where[19]

$$\Delta_1 = \mathbf{V}_{\bar{L}}\mathbf{R}_{HW} + \mathbf{R}_{WH}\mathbf{V}_{\bar{L}}^H, \tag{2.151}$$

$$\Delta_2 = \mathbf{R}_{WW} - \mathcal{N}_0\mathbf{I}. \tag{2.152}$$

Recall that for $K \leq \bar{L} \leq (L - K + 1)$, with K being the number of paths, the matrix $\mathbf{V}_{\bar{L}}\mathbf{H}_f$ has rank K, and hence, so does the matrix $\mathbf{V}_{\bar{L}}\mathbf{R}_{HH}\mathbf{V}_{\bar{L}}^H$. Therefore, K is given by the smallest integer k such that the $(\bar{L}+1-k)$ smallest eigen values of $\left(\mathbf{V}_{\bar{L}}\mathbf{R}_{HH}\mathbf{V}_{\bar{L}}^H + \mathcal{N}_0\mathbf{I}\right)$ are all equal. But, due to the pertubation $(\Delta_1 + \Delta_2)$, the $(\bar{L}+1-K)$ smallest eigen values of $\mathbf{R}_{ZZ}$ are all different with probability 1, and the 'gap' between the K^{th} largest eigen value and the $(K+1)^{\text{th}}$ largest eigen value is random.

Suppose, for $k = 1, 2, \ldots, (\bar{L}+1)$, the eigen vectors of $\mathbf{R}_{ZZ}$ corresponding to the $(\bar{L} + 1 - k)$ smallest eigen values are arranged in the matrix $\hat{\mathbf{U}}_k$, left to right in the order of decreasing eigen values. Then, for $k = K$, Fuchs shows that there exists a matrix $\mathbf{U}$ with orthonormal columns such that

1. $\hat{\mathbf{U}}_K = \mathbf{U} + \mathcal{O}(N^{-1/2})$,

[18] The method also applies to $\mathbf{Z}_b\mathbf{Z}_b^H$ and more generally to the centro-symmetric $[\mathbf{Z}_f, \mathbf{Z}_b][\mathbf{Z}_f, \mathbf{Z}_b]^H$.

[19] The Hermitian matrix Δ_1 contains zero-mean Gaussian random variables. The Hermitian matrix Δ_2 contains zero-mean random variables which, for large L, are approximately Gaussian. Furthermore, for large L, the variances of the elements of both Δ_1 and Δ_2 are $\mathcal{O}(N^{-1})$.

2. $\mathbf{U}^H\mathbf{V}_{\bar{L}} = \mathbf{0}$, and therefore $\mathbf{U}^H\mathbf{R}_{ZZ}\mathbf{U} = \mathcal{N}_0\mathbf{I} + \mathbf{U}^H\mathbf{\Delta}_2\mathbf{U}$.

3. the $(\bar{L}+1-K)$ smallest eigen values of $\mathbf{R}_{ZZ}$, that is, the diagonal elements of $\hat{\mathbf{U}}_K^H\mathbf{R}_{ZZ}\hat{\mathbf{U}}_K$, are obtained, to within an approximation $\mathcal{O}(N^{-1})$, by perturbing $\mathcal{N}_0$ with the eigen values of $\mathbf{U}^H\mathbf{\Delta}_2\mathbf{U}$.

Therefore, a hypothesis test for the equality of the $(\bar{L}+1-K)$ smallest eigen values of $\mathbf{R}_{ZZ}$ may be based on the probability density of the eigen values of $\mathbf{U}^H\mathbf{\Delta}_2\mathbf{U}$. However, Fuchs believes that its computation would be impossible. Therefore, Fuchs proposes to base the hypothesis test on the probability density of the diagonal elements of $\mathbf{U}^H\mathbf{\Delta}_2\mathbf{U}$, on the strength of his observation that the elements of $\mathbf{U}^H\mathbf{\Delta}_2\mathbf{U}$, for large L, are all linear combinations of some $(\bar{L}+1-K)$ zero-mean independent Gaussian random variables. Specifically, he proposes to estimate the diagonal elements of $\mathbf{U}^H\mathbf{\Delta}_2\mathbf{U}$ by the diagonal matrix $\left(\hat{\mathbf{U}}_K^H\mathbf{R}_{ZZ}\hat{\mathbf{U}}_K - \hat{\mathcal{N}_0}\mathbf{I}\right)$, where $\hat{\mathcal{N}_0} = \text{trace}\left(\hat{\mathbf{U}}_K^H\mathbf{R}_{ZZ}\hat{\mathbf{U}}_K\right)/(\bar{L}+1-K)$, and perform a χ^2 test of $2(\bar{L}+1-K)$ degrees of freedom.[20] Thus, Fuchs chooses a sequence of thresholds t_k according to

$$\text{Prob}(\mu_k \geq t_k) = P_{oe}, \tag{2.153}$$

where μ_k is a random variable with χ^2 probability density with $2(\bar{L}+1-k)$ degrees of freedom, for $k = 1, 2, \ldots, (\bar{L}+1)$, and where P_{oe} is the fixed probability of over-estimation of K. Suppose $\hat{\mathbf{q}}_k$ is the column vector containing the diagonal elements of $\left(\hat{\mathbf{U}}_k^H\mathbf{R}_{ZZ}\hat{\mathbf{U}}_k - \hat{\mathcal{N}_0}\mathbf{I}\right)$, where $\hat{\mathcal{N}_0} = \text{trace}\left(\hat{\mathbf{U}}_k^H\mathbf{R}_{ZZ}\hat{\mathbf{U}}_k\right)/(\bar{L}+1-k)$, and $\hat{\mathbf{Q}}_k$ is its covariance estimated under the assumption $k = K+1$, then the test statistic $\hat{\mu}_k = \hat{\mathbf{q}}_k^H\hat{\mathbf{Q}}_k^{-1}\hat{\mathbf{q}}_k$ will be greater than t_k with high probability[21] for $k \leq K$, but less than t_k with probability $(1-P_{oe})$ for $k = K+1$. Thus the method consists in asking whether $\hat{\mu}_k \geq t_k$ is true, in sequence for $k = 1, 2, \ldots$, until the answer is "No", at which point $k = K+1$.

2.6.8 On the Consequences of Erroneous Estimation of K

The consequences, for signalling, of erroneous estimation of K would depend on the mutual separations of the elements of $\Theta = (\theta_k : k = 1, 2, \ldots, K)$. If the mutual separations are all

[20] If $\mathbf{q}$ is the column vector containing the zero-mean diagonal elements of $\mathbf{U}^H\mathbf{\Delta}_2\mathbf{U}$ and $\mathbf{Q}$ is the covariance matrix of $\mathbf{q}$, then $\mathbf{q}^H\mathbf{Q}^{-1}\mathbf{q}$ has a χ^2 probability density with $2(\bar{L}+1-K)$ degrees of freedom.

[21] Fuchs claims this to be true, but does not provide any quantification. Thus he does not provide any means of fixing, or even finding, the probability of under-estimation. Nor does he discuss the tradeoff between the probabilities of over-estimation and under-estimation.

large, then the consequences of under-estimating K can be a severe. On the other hand, if the separation between two elements of Θ is very small, then not only will the corresponding paths be difficult to resolve, but also the consequences of not resolving them may be mild. However, this notion has yet to be precisely quantified.

The consequences of over-estimating K may be mild. Thus if the estimated number of paths $\hat{K}$ is more than the actual number of paths K, and if $\hat{\Theta}$ and $\hat{\mathbf{H}}$ are the corresponding maximum-likelihood estimates of Θ and $\mathbf{H}$, respectively, then for large L,

1. some K elements of $\hat{\Theta} = \left\{\hat{\theta}_k : k = 1, 2, \ldots, \hat{K}\right\}$ will be very close to the elements of $\Theta = \{\theta_k : k = 1, 2, \ldots, K\}$, and

2. the remaining $(\hat{K} - K)$ elements of $\hat{\Theta}$ will correspond to rows of $\hat{\mathbf{H}}$ that are close to the all-zero row vector.

In the following, a proof of the first statement is sketched based on the ideas of Bai et. al. [2]. The second statement will then follow from the first.

When K is known, the problem of estimating $\Theta = \{\theta_k : k = 1, 2, \ldots, K\}$ can be transformed into the problem of estimating the coefficients of the polynomial whose roots are $\left\{e^{j\theta_k} : k = 1, 2, \ldots, K\right\}$, or equivalently, the *unique* $(K + 1)$-dimensional column vector

$$\tilde{\mathbf{b}} = (\tilde{b}_0, \tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_{K-1}, \tilde{b}_K)^H \tag{2.154}$$

that satisfies $||\tilde{\mathbf{b}}||^2 = \tilde{\mathbf{b}}^H\tilde{\mathbf{b}} = 1$, $\tilde{b}_K > 0$, and

$$\sum_{k=0}^{K} \tilde{b}_k z^k = \tilde{b}_K \prod_{k=0}^{K} (z - e^{j\theta_k}). \tag{2.155}$$

Thus the maximum likelihood estimate $\hat{\mathbf{b}}$ of $\tilde{\mathbf{b}}$ is given by [4] [2]

$$\hat{\mathbf{b}} = \begin{array}{c}\arg\min \\ \mathbf{b} \\ ||\mathbf{b}|| = 1, b_K > 0\end{array} \quad \mathrm{trace}\left(\left(\mathbf{B}\left(\mathbf{B}^H\mathbf{B}\right)^{-1}\mathbf{B}^H\right)\mathbf{Z}\mathbf{Z}^H\right), \tag{2.156}$$

where $\mathbf{b} = (b_0, b_1, b_2, \ldots, b_{K-1}, b_K)^H$ is a $(K + 1)$-dimensional column vector that defines the $(L+1)\times(L+1-K)$ matrix $\mathbf{B}=\mathbf{B}_L(\mathbf{b})$ as

$$\mathbf{B}_L^H(\mathbf{b})=\begin{pmatrix} b_0 & b_1 & b_2 & \dots & b_{K-1} & b_K & 0 & 0 & \dots & 0\\ 0 & b_0 & b_1 & b_2 & \dots & b_{K-1} & b_K & 0 & \dots & 0\\ \vdots & \ddots & \ddots & & & & \ddots & \ddots & & \vdots\\ 0 & \dots & 0 & b_0 & b_1 & b_2 & \dots & b_{K-1} & b_K & 0\\ 0 & \dots & 0 & 0 & b_0 & b_1 & b_2 & \dots & b_{K-1} & b_K \end{pmatrix}. \tag{2.157}$$

The estimate $\hat{\mathbf{b}}$ is *strongly consistent* as shown by Bai et. al. [2]. More specifically, for every $\epsilon>0$, there exits a constant $c>0$ and an integer L_0 such that for all $L\geq L_0$

$$\text{Prob}(||\hat{\mathbf{b}}-\tilde{\mathbf{b}}||\geq\epsilon)\leq e^{-cL}. \tag{2.158}$$

Suppose K is not known exactly but is known to be less than $\hat{K}$, and

$$\hat{\mathbf{c}}=\underset{||\mathbf{c}||=1,c_{\hat{K}}>0}{\overset{\arg\min}{\mathbf{c}}}\ \text{trace}\left(\left(\mathbf{C}\left(\mathbf{C}^H\mathbf{C}\right)^{-1}\mathbf{C}^H\right)\mathbf{Z}\mathbf{Z}^H\right), \tag{2.159}$$

where $\mathbf{c}=(c_0,c_1,c_2,\dots,c_{\hat{K}-1},c_{\hat{K}})^H$ is a $(\hat{K}+1)$-dimensional vector that defines the $(L+1)\times(L+1-\hat{K})$ matrix $\mathbf{C}=\mathbf{C}_L(\mathbf{c})$ as

$$\mathbf{C}_L^H(\mathbf{c})=\begin{pmatrix} c_0 & c_1 & c_2 & \dots & c_{\hat{K}-1} & c_{\hat{K}} & 0 & 0 & \dots & 0\\ 0 & c_0 & c_1 & c_2 & \dots & c_{\hat{K}-1} & c_{\hat{K}} & 0 & \dots & 0\\ \vdots & \ddots & \ddots & & & & \ddots & \ddots & & \vdots\\ 0 & \dots & 0 & c_0 & c_1 & c_2 & \dots & c_{\hat{K}-1} & c_{\hat{K}} & 0\\ 0 & \dots & 0 & 0 & c_0 & c_1 & c_2 & \dots & c_{\hat{K}-1} & c_{\hat{K}} \end{pmatrix}. \tag{2.160}$$

In the absence of noise, $\hat{\mathbf{c}}$ is non-unique, but it belongs to the column span of the $(\hat{K}+1)\times(\hat{K}-K+1)$ matrix $\mathbf{B}_{\hat{K}}(\tilde{\mathbf{b}})$ given by

$$\mathbf{B}_{\hat{K}}^H(\tilde{\mathbf{b}})=\begin{pmatrix} \tilde{b}_0 & \tilde{b}_1 & \tilde{b}_2 & \dots & \tilde{b}_{K-1} & \tilde{b}_K & 0 & 0 & \dots & 0\\ 0 & \tilde{b}_0 & \tilde{b}_1 & \tilde{b}_2 & \dots & \tilde{b}_{K-1} & \tilde{b}_K & 0 & \dots & 0\\ \vdots & \ddots & \ddots & & & & \ddots & \ddots & & \vdots\\ 0 & \dots & 0 & \tilde{b}_0 & \tilde{b}_1 & \tilde{b}_2 & \dots & \tilde{b}_{K-1} & \tilde{b}_K & 0\\ 0 & \dots & 0 & 0 & \tilde{b}_0 & \tilde{b}_1 & \tilde{b}_2 & \dots & \tilde{b}_{K-1} & \tilde{b}_K \end{pmatrix}. \tag{2.161}$$

In the presence of noise, using the ideas of Bai et. al. [2], one can show that, for every $\epsilon > 0$, there exits a constant $c > 0$ and an integer L_0 such that for all $L \geq L_0$

$$\text{Prob}\left(\hat{\mathbf{c}}^H \left(\mathbf{I} - \mathbf{B}\left(\mathbf{B}^H\mathbf{B}\right)^{-1}\mathbf{B}^H\right)\hat{\mathbf{c}} \geq \epsilon\right) \leq e^{-cL}, \tag{2.162}$$

where $\mathbf{B} = \mathbf{B}_{\hat{K}}\left(\tilde{\mathbf{b}}\right)$, or equivalently,

$$\text{Prob}\left(\hat{\mathbf{c}}^H\mathbf{V}\left(\mathbf{V}^H\mathbf{V}\right)^{-1}\mathbf{V}^H\hat{\mathbf{c}} \geq \epsilon\right) \leq e^{-cL}, \tag{2.163}$$

where $\mathbf{V} = \mathbf{V}_{\hat{K}}(\Theta)$, the $(\hat{K}+1) \times K$ Vandermonde matrix parametrized by Θ. Therefore, $\lim_{L\to\infty} \hat{\mathbf{c}}^H\mathbf{V}_{\hat{K}}(\Theta) = \mathbf{0}$, or equivalently, some K zeros of the polynomial $\sum_{k=1}^{\hat{K}} \hat{c}_k z^k$, where $\hat{\mathbf{c}} = \left(\hat{c}_0, \hat{c}_1, \hat{c}_2, \ldots, \hat{c}_{\hat{K}-1}, \hat{c}_{\hat{K}}\right)^H$, tend to $\left\{e^{j\theta_k} : k = 1, 2, \ldots, K\right\}$.

Chapter 3

Linear Zero-Forcing Equalizers

3.1 Introduction

A class of receivers for the composition of Quadrature Amplitude Modulation (QAM) and Discrete-Multipath Channel (DMC) is derived in this chapter. Receivers of this class shall be referred to as *linear zero-forcing equalizers* in view of the fact that they are extensions of the linear zero-forcing equalizer known for the composition of QAM and linear time-invariant channel with additive white Gaussian noise (AWGN). (For information on the latter, see, for example, [27] and [3]). The stationarity of the set of sequences $\{h_k(t-nT) : k = 1, 2, \ldots, K; \forall n\}$ and the consequent isomorphism between the Hilbert spaces $\mathcal{H}$ and $\mathcal{L}^2_{\mathbf{H}}$ as discussed in appendix A, section A.5 play a central role in the derivations. To facilitate the derivations, the set of stationary sequences $\{h_k(t-nT) : k = 1, 2, \ldots, K; \forall n\}$ is assumed to satisfy the minimality condition as discussed in appendix A, section A.9. Issues of specification, implementation, and performance analysis of the receivers are addressed. Many of the concepts introduced in this chapter will be useful in the subsequent chapters on other kinds of equalizers as well.

3.1.1 Some Preliminaries

Recall from chapter 2, section 2.5 that the received signal $z(t)$ can be written in terms of the transmitted data sequence $a(n)$ as

$$z(t) = \sum_n a(n) \sum_{k=1}^{K} e^{j\omega_k nT} h_k(t-nT) + \eta_w(t). \tag{3.1}$$

For a square-integrable function function $f(t)$, the integral

$$\hat{a}_f = \int f^*(t)z(t)dt, \tag{3.2}$$

defined in the mean-square-sense with respect to both data and noise, is called the correlation of $z(t)$ with $f(t)$. Thus $\hat{a}_f$ is a complex random variable which, in view of equation 3.1, has a decomposition as the sum of a data-dependent part $\hat{b}_f$ and a noise-dependent part $\hat{c}_f$ given by

$$\hat{b}_f = \sum_n a(n) \sum_{k=1}^{K} e^{j\omega_k nT} \int f^*(t)h_k(t - nT)dt \tag{3.3}$$

and

$$\hat{c}_f = \int f^*(t)\eta_w(t)dt, \tag{3.4}$$

repectively. Thus $\hat{c}_f$ is a complex Gaussian random variable with mean zero and variance

$$E\left[|\hat{c}_f|^2\right] = \mathcal{N}_0 \int |f(t)|^2 \, dt. \tag{3.5}$$

3.2 Linear Equalizers

The task of a receiver is to decide from the received signal $z(t)$ which data sequence $a(n)$ was likely transmitted. A linear equalizer performs this task by forming a sequence of correlations

$$\hat{a}_{f_m} = \int f_m^*(t)z(t)dt, \tag{3.6}$$

for some optimum sequence of functions $f_m(t)$, and then considering the sequence of numbers

$$\hat{a}(m) = \begin{matrix}\arg\min \\ a \in \mathcal{A}\end{matrix} \; |a - \hat{a}_{f_m}| \tag{3.7}$$

as the data sequence that was likely transmitted; here $\mathcal{A}$ denotes the signal set. The latter operation, known as *quantization*, is schematically shown in figure 3.2.

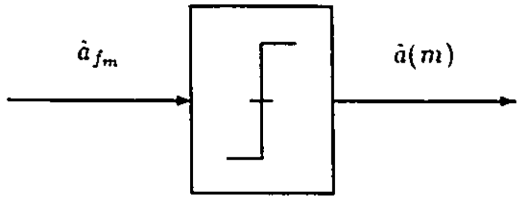

Figure 3.1: Quantization

The sequence of functions $f_m(t)$ may be optimized under various criteria that judge the closeness of $\hat{a}_{f_m}$ to $a(m)$. This thesis considers two such criteria:

1. minimum noise variance under the zero-forcing constraint.

2. minimum mean-square-error.

The first of the above criteria is considered in this chapter, and the second criterion is considered in chapter 5.

3.2.1 On Specifying a Linear Equalizer

For the composition of QAM and linear time-invariant channel, linear equalizers optimized under one of the aforementioned criteria have the property that

$$f_m(t) = f_0(t - mT) \tag{3.8}$$

for all m, where T is the baud period. Thus a linear equalizer for the composition of QAM and linear time-invariant channel is completely specified by the optimum function $f_0(t)$ and the parameter T. A linear equalizer for the composition for QAM and DMC, however, may not be specified as easily because the DMC is time-variant. In general, an entire *sequence* of optimum functions would be needed to specify a linear equalizer for the composition of QAM and DMC. The complexity involved in specifying such a sequence of functions is of major concern in this chapter and the subsequent ones on other kinds of equalizers as well. For the time being, however, a linear equalizer may be identified with an optimum sequence of functions.

3.3 The Optimality Criterion – Minimum Noise Variance under the Zero-Forcing Constraint

Suppose that, for some function $f(t)$, the correlation $\hat{a}_f = \int f^*(t)z(t)dt$ is used to obtain

$$\hat{a}(m) = \underset{a \in \mathcal{A}}{\arg\min} \; |a - \hat{a}_f| . \tag{3.9}$$

In considering $\hat{a}(m)$ as the m^{th} data symbol that was likely transmitted, both the noise and the data are potential causes of error. To guard against an error being caused by the data, the data-dependent part $\hat{b}_f$ of $\hat{a}_f$ defined by equation 3.3 must be constrained as

$$\hat{b}_f = a(m), \tag{3.10}$$

or equivalently, the function $f(t)$ must be constrained as

$$\sum_{k=1}^{K} e^{j\omega_k nT} \int f^*(t)h_k(t - nT)dt = \delta_m(n), \tag{3.11}$$

where $\delta_m(n)$ is the Kronecker delta defined as

$$\delta_m(n) = \begin{cases} 0 & \text{if} \quad n \neq m, \\ 1 & \text{if} \quad n = m. \end{cases} \tag{3.12}$$

This constraint is known as the *zero-forcing* constraint because it forces to zero the interference caused by data other than $a(m)$, the data symbol being decided upon. Assuming for the moment that there exist functions that satisfy this constraint, one obtains

$$\hat{a}_f - a(m) = \hat{c}_f, \tag{3.13}$$

where $\hat{c}_f$ is the noise-dependent part of $\hat{a}_f$ defined by equation 3.4. Since $\hat{c}_f$ is a Gaussian random variable, its variance $E\left[|\hat{c}_f|^2\right]$ must preferably be minimized. Thus the integral

$$\int |f(t)|^2 \, dt \tag{3.14}$$

must be minimized with respect to the function $f(t)$ under the zero-forcing constraint.

The derivation of a linear equalizer under the aforementioned optimality criterion – *minimum noise variance under the zero-forcing constraint* – is attempted in the next section.

It turns out there that this approach, although optimal, is not feasible from an implementation point of view. In the subsequent sections, suboptimal formulations of the problem obtained by using stronger zero-forcing constraints will be considered. Henceforth, the term *linear zero-forcing equalizer* shall describe the solution to the optimal formulation as well the solutions to the suboptimal formulations.

3.4 The Optimal Formulation of the Linear Zero-Forcing Equalizer Problem

From the discussions of sections 3.2 and 3.3, it is evident that the problem of deriving a linear zero-forcing equalizer can be cast as a family of constrained minimization problems indexed by integers, the m^{th} problem being to minimize the quantity $\int |f(t)|^2\, dt$ under the constraint C_m defined by equation 3.16. The solution to the m^{th} problem shall be denoted by $f_m(t)$, and the minimum so achieved shall be denoted by

$$\lambda_m = \int |f_m(t)|^2 dt. \tag{3.15}$$

3.4.1 Zero-Forcing Constraint C_m

Denote by C_m the following constraint on the function $f(t)$:

$$\sum_{k=1}^{K} e^{j\omega_k nT} \int f^*(t) h_k(t - nT) dt = \delta_m(n). \tag{3.16}$$

3.4.2 A Consideration in the Hilbert Space $\mathcal{L}^2$

Recall that, by assumption, the functions $h_k(t) \in \mathcal{L}^2$ for $k = 1, 2, \ldots, K$, and observe that a function $f(t) \in \mathcal{L}^2$ is being sought. Since $\mathcal{L}^2$ is a Hilbert space under the inner product

$$(f, g) = \int f^*(t) g(t) dt, \tag{3.17}$$

the problem can be restated as follows: minimize (f, f) under the constraint

$$(f, h^n) = \delta_m(n), \tag{3.18}$$

where

$$h^n(t) = \sum_{k=1}^{K} e^{j\omega_k nT} h_k(t - nT). \tag{3.19}$$

This restatement shows that the solution, if it exists, must belong to the subspace[1]

$$\mathcal{H}_\Sigma = \text{Clos.Span}\{h^n(t) : \forall n\}. \tag{3.20}$$

A proof of this simple fact is as follows: by the orthogonal projection theorem [47], an arbitrary $f(t) \in \mathcal{L}^2$ has a unique decomposition

$$f(t) = g_1(t) + g_2(t), \tag{3.21}$$

where $g_1(t) \in \mathcal{H}_\Sigma$ and $g_2(t) \in \mathcal{H}_\Sigma^\perp$ (the orthogonal complement of $\mathcal{H}_\Sigma$ in $\mathcal{L}^2$); since

$$(f, h^n) = (g_1, h^n) \tag{3.22}$$

for all n, the function $f(t)$ satisfies C_m if and only if $g_1(t)$ satisfies C_m; moreover, since $(f, f) = (g_1, g_1) + (g_2, g_2)$, if $f(t)$ is optimum then necessarily $g_2(t) = 0$, for otherwise $g_1(t)$ would be better than $f(t)$, thereby contradicting optimality.

3.4.3 Further Consideration in the Hilbert Space $\mathcal{L}^2$

To obtain further insight into the solution to the m^{th} problem, denote the subspace

$$\mathcal{H}_{\Sigma(\neq m)} = \text{Clos.Span}\{h^n(t) : n \neq m\}, \tag{3.23}$$

and consider an arbitrary $f(t) \in \mathcal{H}_\Sigma$. By the orthogonal projection theorem, there exist unique decompositions

$$h^m(t) = x(t) + y(t), \tag{3.24}$$

$$f(t) = u(t) + v(t), \tag{3.25}$$

where $x(t), u(t) \in \mathcal{H}^\perp_{\Sigma(\neq m)}$ (the orthogonal complement of $\mathcal{H}_{\Sigma(\neq m)}$ in $\mathcal{H}_\Sigma$) and $y(t), v(t) \in \mathcal{H}_{\Sigma(\neq m)}$. If $f(t)$ satisfies C_m then

$$(f, h^n) = (v, h^n) = 0 \tag{3.26}$$

for all $n \neq m$. This implies that $v(t) = 0$, and therefore

$$(f, f) = (u, u), \tag{3.27}$$

$$(f, h^m) = (u, x) = 1. \tag{3.28}$$

[1] This is the smallest closed subspace containing the set $\{h^n(t) : \forall n\}$.

This shows that a solution to the m^{th} problem exists if and only if $x(t) \neq 0$. Under this assumption, the optimum solution is unique and is given by

$$f_m(t) = x(t)/(x,x), \tag{3.29}$$

and

$$\lambda_m = 1/(x,x). \tag{3.30}$$

3.4.4 Special Cases

Finding $x(t)$ is straightforward in the trivial case where there is no intersymbol interference, that is, where $n_1 \neq n_2 \Rightarrow (h^{n_1}, h^{n_2}) = 0$ as defined in chapter 1, section 1.7.2. In this case, $y(t) = 0$ and therefore $x(t) = h^m(t)$.

In the case where $K = 1$, although the DMC is time-variant in general, the sequence of functions $\{h^n(t) : \forall n\}$ is stationary as shown in appendix A, section A.1.2. Therefore, even if there is intersymbol interference, the technique used for the composition of QAM and linear time-invariant channel can be used essentially without difference. Thus the linear zero-forcing equalizer exists if and only if

$$\int_{-\pi/T}^{\pi/T} \frac{1}{H(\omega)} d\omega < \infty, \tag{3.31}$$

where $H(\omega)$ is the spectral density function of the stationary sequence of functions $\{h_1(t - nT) : \forall n\}$ as discussed in appendix A, section A.4. Under this assumption, recalling the isomorphism between the Hilbert spaces $\mathcal{H}$ and $\mathcal{L}_H^2$ as discussed in appendix A, section A.5, the isomorph $c(\omega) \in \mathcal{L}_H^2$ of $f_m(t)$ is given by

$$c(\omega) = Te^{j(\omega_1 - \omega)mT}/H(\omega), \tag{3.32}$$

$$\lambda_m = \frac{T^2}{2\pi} \int_{-\pi/T}^{\pi/T} \frac{1}{H(\omega)} d\omega. \tag{3.33}$$

3.4.5 The General Case

In the general case where $K > 1$ and there is intersymbol interference, it does not seem a trivial task to ascertain the existence of the solution or even to find it. However, an approximate solution may be obtained as follows: since the optimum solution $f_m \in \mathcal{H}_\Sigma$, heuristically,

$$f_m(t) = \sum_n \alpha_m(n) h^n(t) \tag{3.34}$$

for some sequence $\alpha_m(n)$; this implies

$$\sum_{n'} \left(h^n, h^{n'}\right) \alpha_m(n') = \delta_m(n), \tag{3.35}$$

and

$$(f_m, f_m) = \sum_n \alpha_m(n)(f_m, h^n) = \alpha_m(m); \tag{3.36}$$

an approximation of equation 3.35 obtained by restricting both n and n' to a finite set can be solved by matrix methods. The approximate solution so obtained will not strictly satisfy the zero-forcing constraint C_m. However, it will satisfy a weaker constraint than C_m, and therefore correspond to a lower bound on λ_m. In the next section, it is shown how, for roughly the same order of complexity, a function $f(t)$ that minimizes (f, f) while satisfying a stronger constraint than C_m can be found; the solution so obtained will automatically satisfy the zero-forcing constraint C_m and correspond to an upper bound on λ_m.

3.5 A Suboptimal Formulation of the Linear Zero-Forcing Equalizer Problem

The linear zero-forcing equalizer problem can be formulated in a suboptimal manner by replacing the constraint set C_m with a stronger constraint set C^0_{mu} defined by equations 3.38 and 3.39. Thus the suboptimal formulation of the linear zero-forcing equalizer problem is a family of constrained minimization problems indexed by integers, the m^{th} problem being to minimize $\int |f(t)|^2\, dt$ under the constraint C^0_{mu}. The solution to the m^{th} problem shall be denoted by $f^0_{mu}(t)$, and the minimum so achieved shall be denoted by

$$\lambda^0_{mu} = \int |f^0_{mu}(t)|^2 dt. \tag{3.37}$$

Then λ^0_{mu} will be an upper bound[2] on λ_m. The motivation for this suboptimal formulation is that the solution is attractive from the points of view of specification and implementation.

3.5.1 Zero-Forcing Constraint C^0_{mu}

Denote by C^0_{mu} the following set of constraints on the function $f(t)$:

[2] This is the reason for the subscript 'u' in C^0_{mu} etc.. The reason for the superscript '0' will become apparent later.

- for $n \neq m$ and $k = 1, 2, \ldots, K$,

$$\int f^*(t) h_k(t - nT) dt = 0, \tag{3.38}$$

-

$$\sum_{k=1}^{K} e^{j\omega_k mT} \int f^*(t) h_k(t - mT) dt = 1. \tag{3.39}$$

Observe that C^0_{mu} is stronger than C_m in the sense that if a function $f(t)$ satisfies C^0_{mu} then it satisfies C_m.

3.5.2 Considerations in the Hilbert Space $\mathcal{L}^2$

An argument similar to that used in sections 3.4.2 and 3.4.3 gives the following conclusions: a solution to the m^{th} problem, if it exists, must belong to the subspace

$$\mathcal{H} = \text{Clos.Span}\left\{h_k(t - nT) : k = 1, 2, \ldots, K; \forall n\right\}, \tag{3.40}$$

but be orthogonal to the subspace

$$\mathcal{H}_{(\neq m)} = \text{Clos.Span}\left\{h_k(t - nT) : k = 1, 2, \ldots, K; n \neq m\right\}; \tag{3.41}$$

suppose that

$$h^m(t) = \sum_{k=1}^{K} e^{j\omega_k mT} h_k(t - mT) = x(t) + y(t) \tag{3.42}$$

is the unique decomposition of $h^m(t)$ such that $y(t) \in \mathcal{H}_{(\neq m)}$ and $x(t) \in \mathcal{H}^{\perp}_{(\neq m)}$ (the orthogonal complement of $\mathcal{H}_{(\neq m)}$ in $\mathcal{H}$); a solution exists if and only if $x(t) \neq 0$, under which assumption it is given by

$$f^0_{mu}(t) = x(t)/(x, x), \tag{3.43}$$

and $\lambda^0_{mu} = 1/(x, x)$.

3.5.3 Solution

For the special case where $K = 1$, the suboptimal formulation is equivalent to the optimal formulation, and therefore the corresponding result of section 3.4.4 is applicable.

For a general case, denote

$$\mathbf{\Omega}_m = \left[e^{j\omega_1 mT}, e^{j\omega_2 mT}, \ldots, e^{j\omega_K mT}\right]^T, \tag{3.44}$$

and recall the isomorphism between the Hilbert spaces $\mathcal{H}$ and $\mathcal{L}^2_{\mathbf{H}}$ as discussed in appendix A, section A.5. The isomorph of $\sum_{k=1}^{K} e^{j\omega_k mT} h_k(t - mT)$ is then $e^{-j\omega mT}\mathbf{\Omega}_m \in \mathcal{L}^2_{\mathbf{H}}$. Let $\mathbf{u}(\omega) \in \mathcal{L}^2_{\mathbf{H}}$ be the isomorph of $x(t)$. The isomorph of $y(t)$ is then $\left(e^{-j\omega mT}\mathbf{\Omega}_m - \mathbf{u}(\omega)\right) \in \mathcal{L}^2_{\mathbf{H}}$. To find $\mathbf{u}(\omega)$, observe that $\left(e^{-j\omega mT}\mathbf{\Omega}_m - \mathbf{u}(\omega)\right)$ is orthogonal to functions $\mathbf{v}(\omega) \in \mathcal{L}^2_{\mathbf{H}}$ that are the isomorphs of functions that belong to $\mathcal{H}^{\perp}_{(\neq m)}$. Therefore

$$\int_{-\pi/T}^{\pi/T} \mathbf{v}^H(\omega)\mathbf{H}(\omega)\left(e^{-j\omega mT}\mathbf{\Omega}_m - \mathbf{u}(\omega)\right) d\omega = 0 \tag{3.45}$$

for all functions $\mathbf{v}(\omega)$ that satisfy

$$\int_{-\pi/T}^{\pi/T} \mathbf{v}^H(\omega)\mathbf{H}(\omega)\mathbf{e}_k e^{-j\omega nT} d\omega = 0 \tag{3.46}$$

for $n \neq m$ and $k = 1, 2, \ldots, K$, where $\mathbf{e}_k$ is the k^{th} unit vector of the standard Euclidean basis; equivalently, the functions $\mathbf{v}(\omega)$ satisfy

$$\mathbf{H}(\omega)\mathbf{v}(\omega) = \mathbf{b}e^{-j\omega mT} \tag{3.47}$$

for some complex vector-valued constant $\mathbf{b}$. Also, the function $\mathbf{u}(\omega)$ satisfies

$$\mathbf{H}(\omega)\mathbf{u}(\omega) = \mathbf{a}e^{-j\omega mT} \tag{3.48}$$

for some complex vector-valued constant $\mathbf{a}$. Therefore, denoting the generalized inverse of $\mathbf{H}(\omega)$ by $\mathbf{G}(\omega)$, equation 3.45 can be written as

$$\int_{-\pi/T}^{\pi/T} \mathbf{v}^H(\omega)\mathbf{H}(\omega)\mathbf{G}(\omega)\mathbf{H}(\omega)\left(e^{-j\omega mT}\mathbf{\Omega}_m - \mathbf{u}(\omega)\right) d\omega = 0. \tag{3.49}$$

This implies

$$\int_{-\pi/T}^{\pi/T} \mathbf{b}^H\mathbf{G}(\omega)\left(\mathbf{H}(\omega)\mathbf{\Omega}_m - \mathbf{a}\right) d\omega = 0 \tag{3.50}$$

for all complex vector-valued constants $\mathbf{b}$.

Assumptions that Facilitate the Solution

The following assumptions facilitate the solution:

$$\mathbf{G}(\omega)\mathbf{H}(\omega) = \mathbf{I}. \tag{3.51}$$

$$\int_{-\pi/T}^{\pi/T} \text{Trace}\left[\mathbf{G}(\omega)\right] d\omega < \infty. \tag{3.52}$$

This pair of conditions is equivalent to the set of stationary sequences

$$\{h_k^n(t) = h_k(t - nT) : k = 1, 2, \ldots, K; \forall n\} \tag{3.53}$$

satisfying the *minimality* condition as discussed in appendix A, section A.9.

Solution

Under these assumptions,

$$\mathbf{\Omega}_m = \mathbf{G}_0\mathbf{a}, \tag{3.54}$$

where $\mathbf{G}_0$ is the positive definite matrix given by

$$\mathbf{G}_0 = \frac{T}{2\pi}\int_{-\pi/T}^{\pi/T} \mathbf{G}(\omega) d\omega. \tag{3.55}$$

Using $\mathbf{a} = \mathbf{G}_0^{-1}\mathbf{\Omega}_m$ one obtains

$$\begin{aligned} \mathbf{u}(\omega) &= e^{-j\omega mT}\mathbf{G}(\omega)\mathbf{G}_0^{-1}\mathbf{\Omega}_m, && (3.56)\\ \frac{1}{2\pi}\int_{-\pi/T}^{\pi/T} \mathbf{u}^H(\omega)\mathbf{H}(\omega)\mathbf{u}(\omega)d\omega &= \frac{1}{2\pi}\int_{-\pi/T}^{\pi/T} \mathbf{\Omega}_m^H\mathbf{G}_0^{-1}\mathbf{G}(\omega)\mathbf{G}_0^{-1}\mathbf{\Omega}_m d\omega, && (3.57)\\ &= T^{-1}\mathbf{\Omega}_m^H\mathbf{G}_0^{-1}\mathbf{\Omega}_m. && (3.58) \end{aligned}$$

Observe that $\mathbf{\Omega}_m^H\mathbf{G}_0^{-1}\mathbf{\Omega}_m > 0$ owing to the positive definiteness of $\mathbf{G}_0$. Let $\mathbf{c}(\omega) \in \mathcal{L}_{\mathbf{H}}^2$ be the isomorph of $f_{mu}^0(t)$. Then[3]

$$\begin{aligned} \mathbf{c}(\omega) &= T\left[\mathbf{\Omega}_m^H\mathbf{G}_0^{-1}\mathbf{\Omega}_m\right]^{-1} e^{-j\omega mT}\mathbf{G}(\omega)\mathbf{G}_0^{-1}\mathbf{\Omega}_m, && (3.59)\\ \lambda_{mu}^0 &= T\left[\mathbf{\Omega}_m^H\mathbf{G}_0^{-1}\mathbf{\Omega}_m\right]^{-1}. && (3.60) \end{aligned}$$

[3] Compare with results given in 3.4.4.

3.5.4 An Alternative Approach

The isomorph $\mathbf{c}(\omega) \in \mathcal{L}^2_{\mathbf{H}}$ of $f^0_{mu}(t)$ is that which minimizes the quantity

$$\frac{1}{2\pi}\int_{-\pi/T}^{\pi/T} \mathbf{c}^H(\omega)\mathbf{H}(\omega)\mathbf{c}(\omega)d\omega \tag{3.61}$$

under the constraints

$$\frac{1}{2\pi}\int_{-\pi/T}^{\pi/T} \mathbf{c}^H(\omega)\mathbf{H}(\omega)\mathbf{\Omega}_m e^{-j\omega mT}d\omega = 1, \tag{3.62}$$

$$\mathbf{H}(\omega)\mathbf{c}(\omega) = \mathbf{a}e^{-j\omega mT} \tag{3.63}$$

for some complex vector-valued constant $\mathbf{a}$. Equivalently, the complex vector-valued constant $\mathbf{a}$ is that which minimizes

$$\begin{aligned} \frac{1}{2\pi}\int_{-\pi/T}^{\pi/T} \mathbf{c}^H(\omega)\mathbf{H}(\omega)\mathbf{c}(\omega)d\omega &= \frac{1}{2\pi}\int_{-\pi/T}^{\pi/T} \mathbf{a}^H\mathbf{G}(\omega)\mathbf{a}d\omega, && (3.64)\\ &= T^{-1}\mathbf{a}^H\mathbf{G}_0\mathbf{a} && (3.65)\end{aligned}$$

under the constraint

$$\mathbf{a}^H\mathbf{\Omega}_m = T. \tag{3.66}$$

Denoting $\mathbf{b} = \mathbf{G}_0^{1/2}\mathbf{a}$, the complex vector-valued constant $\mathbf{b}$ is that which minimizes

$$\mathbf{a}^H\mathbf{G}_0\mathbf{a} = \mathbf{b}^H\mathbf{b} \tag{3.67}$$

under the constraint

$$\mathbf{b}^H\mathbf{G}_0^{-1/2}\mathbf{\Omega}_m = T; \tag{3.68}$$

here $\mathbf{G}_0^{1/2}$ and $\mathbf{G}_0^{-1/2}$ are the positive square roots of $\mathbf{G}_0$ and $\mathbf{G}_0^{-1}$ respectively. Since an arbitrary $\mathbf{b}$ has the unique orthogonal decomposition

$$\mathbf{b} = \alpha\mathbf{G}_0^{-1/2}\mathbf{\Omega}_m + \mathbf{f}, \tag{3.69}$$

where α is a scalar and $\mathbf{f}$ is orthogonal to $\mathbf{G}_0^{-1/2}\mathbf{\Omega}_m$, the complex vector-valued constant $\mathbf{b}$ which minimizes

$$\mathbf{b}^H\mathbf{b} = |\alpha|^2\mathbf{\Omega}_m^H\mathbf{G}_0^{-1}\mathbf{\Omega}_m + \mathbf{f}^H\mathbf{f} \tag{3.70}$$

under the constraint

$$\alpha^*\mathbf{\Omega}_m^H\mathbf{G}_0^{-1}\mathbf{\Omega}_m = T \tag{3.71}$$

is given by $\mathbf{f} = 0$ and $\alpha = T\left[\mathbf{\Omega}_m^H\mathbf{G}_0^{-1}\mathbf{\Omega}_m\right]^{-1}$. Thus $\mathbf{a} = T\left[\mathbf{\Omega}_m^H\mathbf{G}_0^{-1}\mathbf{\Omega}_m\right]^{-1}\mathbf{G}_0^{-1}\mathbf{\Omega}_m$ and $T^{-1}\mathbf{a}^H\mathbf{G}_0\mathbf{a} = T\left[\mathbf{\Omega}_m^H\mathbf{G}_0^{-1}\mathbf{\Omega}_m\right]^{-1}$.

3.5.5 On Specifying and Implementing the Solution

The linear equalizer $\{f^0_{mu}(t) : \forall m\}$ is specified by the matrix $\mathbf{G}(\omega)$, the vector $\mathbf{\Omega}_1$, and the parameter T. More specifically, it is specified by the isomorphs of the columns of the matrix

$$\mathbf{G}(\omega)\mathbf{G}_0^{-1}, \tag{3.72}$$

the matrix $\mathbf{G}_0^{-1}$, the vector $\mathbf{\Omega}_1$, and the parameter T. The linear equalizer can be implemented as shown in figure 3.2. Thus the received signal $z(t)$ is fed into the bank of K continuous-time time-invariant filters matched to the isomorphs of the columns of the matrix $\mathbf{G}(\omega)\mathbf{G}_0^{-1}$. The outputs of these filters are sampled once every T seconds and the samples are linearly combined according to the weight vector $\mathbf{\Omega}^*_m$. The combined output is scaled by $T\left[\mathbf{\Omega}^H_m\mathbf{G}_0^{-1}\mathbf{\Omega}_m\right]^{-1}$, and then quantized to the signal set to obtain $\hat{a}(n)$.

3.6 Generalized Suboptimal Formulation of the Linear Zero-Forcing Equalizer Problem

The suboptimal formulation of the linear zero-forcing equalizer problem of section 3.5 can be generalized by replacing the constraint set C^0_{mu} with a generalized constraint set C^q_{mu} defined by equations 3.74, 3.75, and 3.76. Thus a generalized suboptimal formulation of the linear zero-forcing equalizer problem is a family of constrained minimization problems indexed by integers, the m^{th} problem being to minimize $\int |f(t)|^2 dt$ under the constraint set C^q_{mu}. The solution to the m^{th} problem shall be denoted by $f^q_{mu}(t)$ and the minimum so achieved shall be denoted by

$$\lambda^q_{mu} = \int |f^q_{mu}(t)|^2\, dt. \tag{3.73}$$

3.6.1 Zero-Forcing Constraint C^q_{mu}

Let q be a non-negative integer. Denote by C^q_{mu} the following set of constraints on the function $f(t)$:

- for $|n - m| > q$ and $k = 1, 2, \ldots, K$,

$$\int f^*(t) h_k(t - nT)dt = 0, \tag{3.74}$$

- for $0 < |n - m| \le q$,

$$\sum_{k=1}^{K} e^{j\omega_k nT} \int f^*(t) h_k(t - nT) dt = 0, \tag{3.75}$$

- $$\sum_{k=1}^{K} e^{j\omega_k mT} \int f^*(t) h_k(t - mT) dt = 1. \tag{3.76}$$

Observe that C_{mu}^q is stronger than C_m in the sense that if a function $f(t)$ satisfies C_{mu}^q then it satisfies C_m. In the same sense, C_{mu}^0 is stronger than C_{mu}^q, and if $q_1 < q_2$ then $C_{mu}^{q_1}$ is stronger than $C_{mu}^{q_2}$. Therefore,

$$\lambda_m \le \lambda_{mu}^{q_2} \le \lambda_{mu}^{q_1} \le \lambda_{mu}^0. \tag{3.77}$$

In otherwords, for increasing q, the quantities λ_{mu}^q constitute a hierarchy of tighter upper bounds on λ_m. This is the reason for the subscript 'u' and the superscript 'q.' One may call q the degree of optimality.

3.6.2 Solution

In the manner of the discussion of section 3.5.4, the isomorph $\mathbf{c}(\omega) \in \mathcal{L}_{\mathbf{H}}^2$ of $f_{mu}^q(t)$ is that which minimizes the quantity

$$\frac{1}{2\pi} \int_{-\pi/T}^{\pi/T} \mathbf{c}^H(\omega) \mathbf{H}(\omega) \mathbf{c}(\omega) d\omega \tag{3.78}$$

under the following set of constraints:

- for $|n - m| > q$ and $k = 1, 2, \ldots, K$

$$\int_{-\pi/T}^{\pi/T} \mathbf{c}^H(\omega) \mathbf{H}(\omega) \mathbf{e}_k e^{-j\omega nT} d\omega = 0, \tag{3.79}$$

- for $0 < |n - m| \le q$,

$$\frac{1}{2\pi} \int_{-\pi/T}^{\pi/T} \mathbf{c}^H(\omega) \mathbf{H}(\omega) \mathbf{\Omega}_n e^{-j\omega nT} d\omega = 0, \tag{3.80}$$

- $$\frac{1}{2\pi} \int_{-\pi/T}^{\pi/T} \mathbf{c}^H(\omega) \mathbf{H}(\omega) \mathbf{\Omega}_m e^{-j\omega mT} d\omega = 1, \tag{3.81}$$

where $\mathbf{e}_k$ is the k^{th} unit vector of the standard Euclidean basis, and

$$\boldsymbol{\Omega}_m = \left[e^{j\omega_1 mT}, e^{j\omega_2 mT}, \ldots, e^{j\omega_K mT}\right]^T. \tag{3.82}$$

The first set of constraints say that

$$\mathbf{H}(\omega)\mathbf{c}(\omega) = e^{-j\omega mT} \sum_{|p|\leq q} \mathbf{a}_p e^{-j\omega pT} \tag{3.83}$$

for some set of complex vector-valued constants $\{\mathbf{a}_p : |p| \leq q\}$. The set $\{\mathbf{a}_p : |p| \leq q\}$ is that which minimizes the quantity

$$\begin{aligned} \frac{1}{2\pi}\int_{-\pi/T}^{\pi/T} \mathbf{c}^H(\omega)\mathbf{H}(\omega)\mathbf{c}(\omega)d\omega &= \sum_{|p_1|\leq q}\sum_{|p_2|\leq q} \mathbf{a}_{p_1}^H \left(\frac{1}{2\pi}\int_{-\pi/T}^{\pi/T} \mathbf{G}(\omega)e^{j\omega(p_1-p_2)T}d\omega\right)\mathbf{a}_{p_2}, & (3.84) \\ &= T^{-1}\sum_{|p_1|\leq q}\sum_{|p_2|\leq q} \mathbf{a}_{p_1}^H \mathbf{G}_{p_1-p_2}\mathbf{a}_{p_2}, & (3.85) \end{aligned}$$

under the constraints

$$\sum_{|p|\leq q} \mathbf{a}_p^H \boldsymbol{\Omega}_n \frac{1}{2\pi}\int_{-\pi/T}^{\pi/T} e^{j\omega(m+p-n)T}d\omega = \begin{cases} 0 & \text{if} \quad 0 < |n-m| \leq q, \\ 1 & \text{if} \quad n-m=0, \end{cases} \tag{3.86}$$

or equivalently,

$$\mathbf{a}_p^H \boldsymbol{\Omega}_{p+m} = \begin{cases} 0 & \text{if} \quad 0 < |p| \leq q, \\ T & \text{if} \quad p=0. \end{cases} \tag{3.87}$$

In the above, the denotion

$$\mathbf{G}_n = \frac{T}{2\pi}\int_{-\pi/T}^{\pi/T} \mathbf{G}(\omega)e^{j\omega nT}d\omega \tag{3.88}$$

has been used.

The Case $q = 1$

The set $\{\mathbf{a}_p : |p| \leq 1\}$ is that which minimizes the quantity

$$\sum_{|p_1|\leq 1}\sum_{|p_2|\leq 1} \mathbf{a}_{p_1}^H \mathbf{G}_{p_1-p_2}\mathbf{a}_{p_2} = \left[\mathbf{a}_1^H, \mathbf{a}_0^H, \mathbf{a}_{-1}^H\right] \begin{bmatrix} \mathbf{G}_0 & \mathbf{G}_1 & \mathbf{G}_2 \\ \mathbf{G}_{-1} & \mathbf{G}_0 & \mathbf{G}_1 \\ \mathbf{G}_{-2} & \mathbf{G}_{-1} & \mathbf{G}_0 \end{bmatrix} \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_0 \\ \mathbf{a}_{-1} \end{bmatrix}, \tag{3.89}$$

under the constraint

$$\begin{bmatrix} \Omega_{m+1}^H & \mathbf{0}^H & \mathbf{0}^H \\ \mathbf{0}^H & \Omega_m^H & \mathbf{0}^H \\ \mathbf{0}^H & \mathbf{0}^H & \Omega_{m-1}^H \end{bmatrix} \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_0 \\ \mathbf{a}_{-1} \end{bmatrix} = \begin{bmatrix} 0 \\ T \\ 0 \end{bmatrix}, \tag{3.90}$$

where $\mathbf{0}$ denotes the K-dimensional column vector of zeros, and hence $\mathbf{0}^H$ denotes the K-dimensional row vector of zeros. Denote

$$\mathbf{G} = \begin{bmatrix} \mathbf{G}_0 & \mathbf{G}_1 & \mathbf{G}_2 \\ \mathbf{G}_{-1} & \mathbf{G}_0 & \mathbf{G}_1 \\ \mathbf{G}_{-2} & \mathbf{G}_{-1} & \mathbf{G}_0 \end{bmatrix}. \tag{3.91}$$

Denote by $\mathbf{G}^{1/2}$ and $\mathbf{G}^{-1/2}$ the positive square roots of $\mathbf{G}$ and $\mathbf{G}^{-1}$ respectively. Then the set $\{\mathbf{a}_p : |p| \leq 1\}$ is that which minimizes the quantity

$$\sum_{|p_1|\leq 1} \sum_{|p_2|\leq 1} \mathbf{a}_{p_1}^H \mathbf{G}_{p_1-p_2} \mathbf{a}_{p_2} = \left[\mathbf{a}_1^H, \mathbf{a}_0^H, \mathbf{a}_{-1}^H\right] \mathbf{G}^{1/2}\mathbf{G}^{1/2} \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_0 \\ \mathbf{a}_{-1} \end{bmatrix}, \tag{3.92}$$

under the constraint

$$\begin{bmatrix} \Omega_{m+1}^H & \mathbf{0}^H & \mathbf{0}^H \\ \mathbf{0}^H & \Omega_m^H & \mathbf{0}^H \\ \mathbf{0}^H & \mathbf{0}^H & \Omega_{m-1}^H \end{bmatrix} \mathbf{G}^{-1/2}\mathbf{G}^{1/2} \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_0 \\ \mathbf{a}_{-1} \end{bmatrix} = \begin{bmatrix} 0 \\ T \\ 0 \end{bmatrix}. \tag{3.93}$$

For an arbitrary set $\{\mathbf{a}_p : |p| \leq 1\}$, there exists a unique decomposition

$$\mathbf{G}^{1/2} \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_0 \\ \mathbf{a}_{-1} \end{bmatrix} = \mathbf{G}^{-1/2} \begin{bmatrix} \Omega_{m+1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \Omega_m & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \Omega_{m-1} \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_0 \\ \gamma_{-1} \end{bmatrix} + \mathbf{f}, \tag{3.94}$$

where γ_1, γ_0, γ_{-1} are scalars and $\mathbf{f}$ is a $3K$-dimensional vector orthogonal to the span of the columns of

$$\mathbf{G}^{-1/2} \begin{bmatrix} \Omega_{m+1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \Omega_m & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \Omega_{m-1} \end{bmatrix}. \tag{3.95}$$

The set $\{\mathbf{a}_p : |p| \leq 1\}$ which minimizes

$$\sum_{|p_1|\leq 1} \sum_{|p_2|\leq 1} \mathbf{a}_{p_1}^H \mathbf{G}_{p_1-p_2} \mathbf{a}_{p_2} = \mathbf{f}^H \mathbf{f} + \tag{3.96}$$

$$[\gamma_1^*, \gamma_0^*, \gamma_{-1}^*] \begin{bmatrix} \Omega_{m+1}^H & \mathbf{0}^H & \mathbf{0}^H \\ \mathbf{0}^H & \Omega_m^H & \mathbf{0}^H \\ \mathbf{0}^H & \mathbf{0}^H & \Omega_{m-1}^H \end{bmatrix} \mathbf{G}^{-1} \begin{bmatrix} \Omega_{m+1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \Omega_m & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \Omega_{m-1} \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_0 \\ \gamma_{-1} \end{bmatrix} \tag{3.97}$$

under the constraint

$$\begin{bmatrix} \Omega_{m+1}^H & \mathbf{0}^H & \mathbf{0}^H \\ \mathbf{0}^H & \Omega_m^H & \mathbf{0}^H \\ \mathbf{0}^H & \mathbf{0}^H & \Omega_{m-1}^H \end{bmatrix} \mathbf{G}^{-1} \begin{bmatrix} \Omega_{m+1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \Omega_m & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \Omega_{m-1} \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_0 \\ \gamma_{-1} \end{bmatrix} = \begin{bmatrix} 0 \\ T \\ 0 \end{bmatrix} \tag{3.98}$$

is given by $\mathbf{f} = \mathbf{0}$ and

$$\begin{bmatrix} \gamma_1 \\ \gamma_0 \\ \gamma_{-1} \end{bmatrix} = T \left[\begin{bmatrix} \Omega_{m+1}^H & \mathbf{0}^H & \mathbf{0}^H \\ \mathbf{0}^H & \Omega_m^H & \mathbf{0}^H \\ \mathbf{0}^H & \mathbf{0}^H & \Omega_{m-1}^H \end{bmatrix} \mathbf{G}^{-1} \begin{bmatrix} \Omega_{m+1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \Omega_m & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \Omega_{m-1} \end{bmatrix} \right]^{-1} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}. \tag{3.99}$$

The solution can now be given in terms of the scalars $\gamma_1, \gamma_0, \gamma_{-1}$ as follows:

$$\begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_0 \\ \mathbf{a}_{-1} \end{bmatrix} = \mathbf{G}^{-1} \begin{bmatrix} \Omega_{m+1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \Omega_m & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \Omega_{m-1} \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_0 \\ \gamma_{-1} \end{bmatrix}, \tag{3.100}$$

$$\mathbf{c}(\omega) = \mathbf{G}(\omega) \left[e^{-j\omega(m+1)T}\mathbf{I}, e^{-j\omega mT}\mathbf{I}, e^{-j\omega(m-1)T}\mathbf{I} \right] \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_0 \\ \mathbf{a}_{-1} \end{bmatrix}, \tag{3.101}$$

$$\lambda_{mu}^1 = \frac{1}{2\pi} \int_{-\pi/T}^{\pi/T} \mathbf{c}^H(\omega)\mathbf{H}(\omega)\mathbf{c}(\omega) d\omega = T^{-1} \sum_{|p_1|\leq 1} \sum_{|p_2|\leq 1} \mathbf{a}_{p_1}^H \mathbf{G}_{p_1-p_2} \mathbf{a}_{p_2} = \gamma_0; \tag{3.102}$$

here $\mathbf{I}$ denotes the $K \times K$ identity matrix.

3.6.3 On Specifying and Implementing the Solution for $q = 1$

The linear equalizer $\{f_{mu}^1(t) : \forall m\}$ is thus specified by the isomorphs of the columns of the matrix $\mathbf{G}(\omega)$, the $3K \times 3K$ matrix $\mathbf{G}^{-1}$, the vector Ω_1, and the parameter T.

The linear equalizer can be implemented as shown in figure 3.3. Thus the received signal $z(t)$ is fed into the bank of K continuous-time time-invariant filters matched to the isomorphs of the columns of the matrix $\mathbf{G}(\omega)$. The outputs of these filters are sampled once every T seconds and the samples are fed into K parallel tapped delay lines each with 3 taps. The values at these $3K$ taps are arranged into a column vector which is then multiplied by the $3K \times 3K$ matrix $\mathbf{G}^{-1}$. The resulting column vector is used to form a scalar product with the vector $\left[\gamma_1^* \Omega^H_{(m+1)}, \gamma_0^* \Omega^H_m, \gamma_{-1}^* \Omega^H_{(m-1)}\right]^T$: observe that the scalars $\gamma_1, \gamma_0, \gamma_{-1}$ are dependent on m. The resulting scalar is then quantized to the signal set to obtain $\hat{a}(n)$.

In view of this implementation and that given in section 3.5.5, the matrix $\mathbf{G}(\omega)$ may be said to define a canonical front end for a class of linear zero-forcing equalizers.

3.7 A Lower Bound on λ_m

When $K > 1$ and there is intersymbol interference, the linear zero-forcing equalizer derived in section 3.5 is, in general, suboptimal in the sense that $\lambda_m < \lambda^0_{mu}$. An accurate comparison of λ_m with λ^0_{mu} cannot be made without knowing λ_m. However, an approximate comparison can be made if a lower bound on λ_m is known. A lower bound can be found by replacing the constraint set C_m of the optimum formulation of the linear zero-forcing equalizer problem of section 3.4 with the weaker constraint set C^0_{ml} defined by equation 3.104. Thus solutions are sought for a family of constrained minimization problems indexed by integers, the m^{th} problem being to minimize $\int |f(t)|^2\, dt$ under the constraint C^0_{ml}. The solution to the m^{th} problem shall be denoted by $f^0_{ml}(t)$, and the minimum so achieved shall be denoted by

$$\lambda^0_{ml} = \int |f^0_{ml}(t)|^2 dt. \tag{3.103}$$

Then λ^0_{ml} will be a lower bound[4] on λ_m. It turns out that the expression for λ^0_{ml} is similar to that of λ^0_{mu} obtained in section 3.5.3.

3.7.1 Constraint C^0_{ml}

Denote by C^0_{ml} the following constraint on the function $f(t)$:

$$\sum_{k=1}^{K} e^{j\omega_k mT} \int f^*(t) h_k(t - mT)dt = 1. \tag{3.104}$$

[4]This is the reason for the subscript 'l' in C^0_{ml} etc.. The reason for the superscript '0' will become apparent later.

Observe that C^0_{ml} is weaker than C_m in the sense that if a function $f(t)$ satisfies C_m then it satisfies C^0_{ml}. Also, observe that C^0_{ml} is *not*, in general, a zero-forcing constraint.

3.7.2 Solution

By an argument similar to that used in section 3.4.2, the solution $f^0_{ml}(t)$ takes the form

$$\alpha \sum_{k=1}^{K} e^{j\omega_k mT} h_k(t - mT) \tag{3.105}$$

for some scalar α. Therefore, in terms of its isomorph $\mathbf{c}(\omega) \in \mathcal{L}^2_{\mathbf{H}}$ given by

$$\mathbf{c}(\omega) = \alpha e^{-j\omega mT} \Omega_m, \tag{3.106}$$

the quantity

$$\frac{1}{2\pi}\int_{-\pi/T}^{\pi/T} \mathbf{c}^H(\omega)\mathbf{H}(\omega)\mathbf{c}(\omega)d\omega = \frac{|\alpha|^2}{2\pi}\int_{-\pi/T}^{\pi/T} \Omega_m^H(\omega)\mathbf{H}(\omega)\Omega_m d\omega = T^{-1}|\alpha|^2 \Omega_m^H \mathbf{H}_0 \Omega_m \tag{3.107}$$

must be minimized under the constraint

$$\frac{1}{2\pi}\int_{-\pi/T}^{\pi/T} \mathbf{c}^H(\omega)\mathbf{H}(\omega)\Omega_m e^{-j\omega mT} d\omega = \frac{\alpha^*}{2\pi}\int_{-\pi/T}^{\pi/T} \Omega_m^H \mathbf{H}(\omega)\Omega_m d\omega = T^{-1}\alpha^* \Omega_m^H \mathbf{H}_0 \Omega_m = 1, \tag{3.108}$$

where

$$\mathbf{H}_0 = \frac{T}{2\pi}\int_{-\pi/T}^{\pi/T} \mathbf{H}(\omega)d\omega. \tag{3.109}$$

Therefore, a solution exists if and only if $\Omega_m^H \mathbf{H}_0 \Omega_m \neq 0$. Observe that this condition is implied by minimality.[5] Under this assumption,

$$\alpha = T\left[\Omega_m^H \mathbf{H}_0 \Omega_m\right]^{-1}, \tag{3.110}$$

$$\mathbf{c}(\omega) = T\left[\Omega_m^H \mathbf{H}_0 \Omega_m\right]^{-1} e^{-j\omega mT}\Omega_m, \tag{3.111}$$

$$f^0_{ml}(t) = T\left[\Omega_m^H \mathbf{H}_0 \Omega_m\right]^{-1} \sum_{k=1}^{K} e^{j\omega_k mT} h_k(t - mT), \tag{3.112}$$

$$\lambda^0_{ml} = T\left[\Omega_m^H \mathbf{H}_0 \Omega_m\right]^{-1}. \tag{3.113}$$

Observe that the expression for λ^0_{ml} has the same form as the expression for λ^0_{mu} given by equation 3.60. The expression for the isomorph of $f^0_{ml}(t)$ is also of the same form as the expression for the isomorph of $f^0_{mu}(t)$ given by equation 3.59.

[5] Under minimality, $0 < \mathbf{G}_0^{-1} \leq \mathbf{H}_0$.

3.8 A Generalized Lower Bound on λ_m

The lower bound on λ_m, derived in section 3.7, can be generalized by replacing the constraint C^0_{ml} with a generalized constraint set C^q_{ml} defined by equations 3.115 and 3.116. Thus solutions are sought for a family of constrained minimization problems indexed by integers m, the m^{th} problem being to minimize $\int |f(t)|^2\,dt$ under the constraint set C^q_{ml}. The solution to the m^{th} problem shall be denoted by $f^q_{ml}(t)$, and the minimum so achieved shall be denoted by

$$\lambda^q_{ml} = \int |f^q_{ml}(t)|^2 dt. \tag{3.114}$$

The functions $f^q_{ml}(t)$ are the approximate solutions suggested, in section 3.4.5, to the optimal formulation of the linear zero-forcing equalizer problem.

3.8.1 Constraint C^q_{ml}

Let q be a non-negative integer. Denote by C^q_{ml} the following set of constraints on the function $f(t)$:

- for $0 < |n - m| \leq q$,

$$\sum_{k=1}^{K} e^{j\omega_k nT} \int f^*(t) h_k(t - nT) dt = 0, \tag{3.115}$$

-

$$\sum_{k=1}^{K} e^{j\omega_k mT} \int f^*(t) h_k(t - mT) dt = 1. \tag{3.116}$$

Observe that C^q_{ml} is weaker than C_m in the sense that if a function $f(t)$ satisfies C_m then it satisfies C^q_{ml}. In the same sense, C^0_{ml} is weaker than C^q_{ml}, and if $q_1 < q_2$ then $C^{q_1}_{ml}$ is weaker than $C^{q_2}_{ml}$. Therefore,

$$\lambda^0_{ml} \leq \lambda^{q_1}_{ml} \leq \lambda^{q_2}_{ml} \leq \lambda_m. \tag{3.117}$$

In other words, for increasing q, the quantities λ^q_{ml} constitute a hierarchy of tighter lower bounds on λ_m. This is the reason for the subscript 'l' and the superscript 'q.'

3.8.2 Solution

By an argument similar to that used in section 3.4.2, the solution takes the form

$$f_{ml}^{?}(t) = \sum_{|p|\le q} \alpha_p \sum_{k=1}^{K} e^{j\omega_k(m+p)T} h_k(t-(m+p)T). \tag{3.118}$$

for some set of scalars $\{\alpha_p : |p| \le q\}$. Therefore, in terms of its isomorph $\mathbf{c}(\omega) \in \mathcal{L}^2_{\mathbf{H}}$ given by

$$\mathbf{c}(\omega) = \sum_{|p|\le q} \alpha_p \sum_{k=1}^{K} e^{-j\omega(m+p)T} \mathbf{\Omega}_{m+p}. \tag{3.119}$$

the quantity

$$\frac{1}{2\pi}\int_{-\pi/T}^{\pi/T} \mathbf{c}^H(\omega)\mathbf{H}(\omega)\mathbf{c}(\omega)d\omega \tag{3.120}$$

$$\begin{aligned} &= \sum_{|p_1|\le q}\sum_{|p_2|\le q} \alpha^*_{p_1}\alpha_{p_2}\mathbf{\Omega}^H_{m+p_1}\left(\frac{1}{2\pi}\int_{-\pi/T}^{\pi/T}\mathbf{H}(\omega)e^{j\omega(p_1-p_2)T}d\omega\right)\mathbf{\Omega}_{m+p_2}. \\ &= T^{-1}\sum_{|p_1|\le q}\sum_{|p_2|\le q} \alpha^*_{p_1}\alpha_{p_2}\mathbf{\Omega}^H_{m+p_1}\mathbf{H}_{p_1-p_2}\mathbf{\Omega}_{m+p_2} \end{aligned} \tag{3.121}$$

must be minimized under the constraint

$$\frac{1}{2\pi}\int_{-\pi/T}^{\pi/T} \mathbf{c}^H(\omega)\mathbf{H}(\omega)\mathbf{\Omega}_{m+p_2}e^{-j\omega(m+p_2)T}d\omega \tag{3.122}$$

$$\begin{aligned} &= \sum_{|p_1|\le q} \alpha^*_{p_1}\mathbf{\Omega}^H_{m+p_1}\left(\frac{1}{2\pi}\int_{-\pi/T}^{\pi/T}\mathbf{H}(\omega)e^{j\omega(p_1-p_2)T}d\omega\right)\mathbf{\Omega}_{m+p_2}, \\ &= T^{-1}\sum_{|p_1|\le q} \alpha^*_{p_1}\mathbf{\Omega}^H_{m+p_1}\mathbf{H}_{p_1-p_2}\mathbf{\Omega}_{m+p_2}, \\ &= \begin{cases} 0 & \text{if } \; 0<|p_2|\le q, \\ 1 & \text{if } \; p_2 = 0, \end{cases} \end{aligned} \tag{3.123}$$

where

$$\mathbf{H}_n = \frac{T}{2\pi}\int_{-\pi/T}^{\pi/T}\mathbf{H}(\omega)e^{j\omega nT}d\omega. \tag{3.124}$$

The Case $q = 1$

The set of scalars $\{\alpha_p : |p| \leq 1\}$ is that which minimizes

$$[\alpha_1^*, \alpha_0^*, \alpha_{-1}^*] \begin{bmatrix} \Omega_{m+1}^H & \mathbf{0}^H & \mathbf{0}^H \\ \mathbf{0}^H & \Omega_m^H & \mathbf{0}^H \\ \mathbf{0}^H & \mathbf{0}^H & \Omega_{m-1}^H \end{bmatrix} \mathbf{H} \begin{bmatrix} \Omega_{m+1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \Omega_m & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \Omega_{m-1} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_0 \\ \alpha_{-1} \end{bmatrix} \tag{3.125}$$

under the constraint

$$\begin{bmatrix} \Omega_{m+1}^H & \mathbf{0}^H & \mathbf{0}^H \\ \mathbf{0}^H & \Omega_m^H & \mathbf{0}^H \\ \mathbf{0}^H & \mathbf{0}^H & \Omega_{m-1}^H \end{bmatrix} \mathbf{H} \begin{bmatrix} \Omega_{m+1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \Omega_m & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \Omega_{m-1} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_0 \\ \alpha_{-1} \end{bmatrix} = \begin{bmatrix} 0 \\ T \\ 0 \end{bmatrix}. \tag{3.126}$$

where

$$\mathbf{H} = \begin{bmatrix} \mathbf{H}_0 & \mathbf{H}_1 & \mathbf{H}_2 \\ \mathbf{H}_{-1} & \mathbf{H}_0 & \mathbf{H}_1 \\ \mathbf{H}_{-2} & \mathbf{H}_{-1} & \mathbf{H}_0 \end{bmatrix}. \tag{3.127}$$

In terms of the scalar α_0, the solution is given by

$$\lambda_{ml}^1 = \alpha_0. \tag{3.128}$$

Observe that the expression for λ_{ml}^1 has the same form as the expression for λ_{mu}^1 given by equation 3.102. The expression for the isomorph of $f_{ml}^1(t)$ is also of the same form as the expression for the isomorph of $f_{mu}^1(t)$ given by equation 3.101.

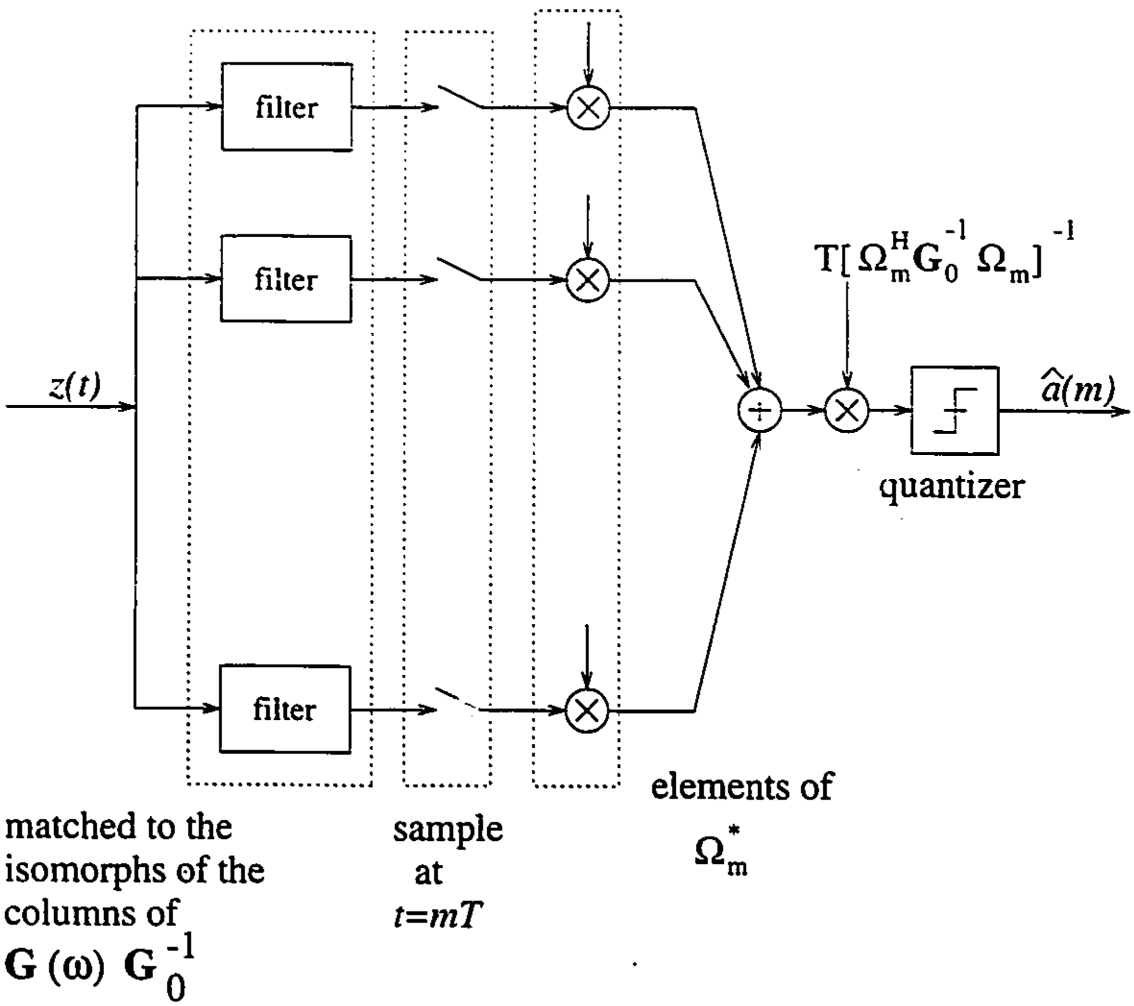

Figure 3.2: Schematic Diagram of the Linear Zero-Forcing Equalizer for $q = 0$

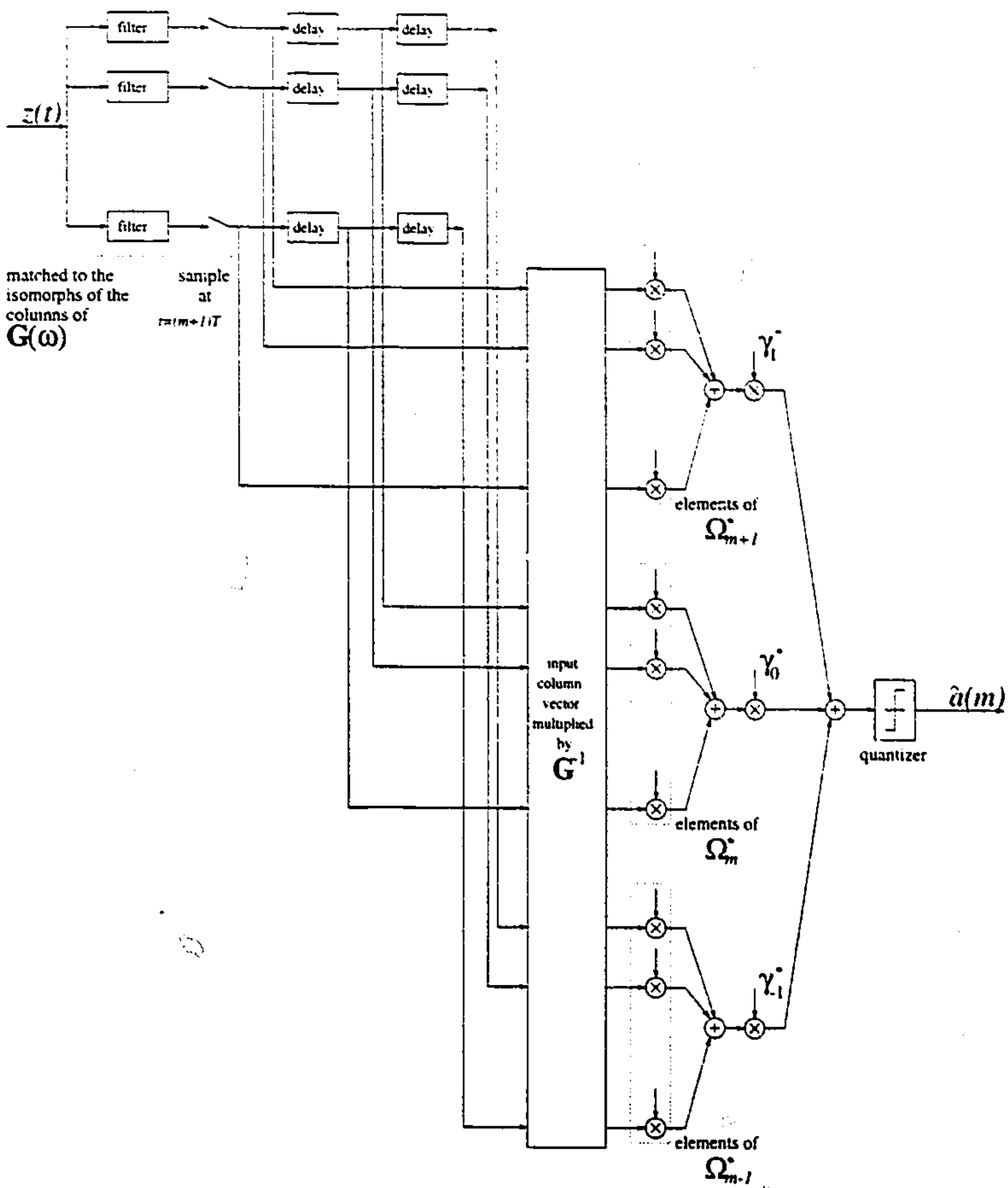

Figure 3.3: Schematic Diagram of the Linear Zero-Forcing Equalizer for $q = 1$

Chapter 4

Decision-Feedback Zero-Forcing Equalizers

4.1 Introduction

A class of receivers for the composition of Quadrature Amplitude Modulation (QAM) and Discrete-Multipath Channel (DMC) is derived in this chapter. Receivers of this class shall be referred to as *decision-feedback zero-forcing equalizers* in view of the fact that they are extensions of the decision-feedback zero-forcing equalizer known for the composition of QAM and linear time-invariant channel with additive white Gaussian noise (AWGN). (For information on the latter, see, for example [27] and [3].) The stationarity of the set of sequences $\{h_k(t - nT) : k = 1, 2, \ldots, K; \forall n\}$ and the consequent isomorphism between the Hilbert spaces $\mathcal{H}$ and $\mathcal{L}^2_{\mathbf{H}}$ as discussed in appendix A, section A.5 play a central role in the derivations. Most of the concepts introduced in chapter 3 on linear zero-forcing equalizers will be useful in this chapter. However, to facilitate the derivations, the set of stationary sequences $\{h_k(t - nT) : k = 1, 2, \ldots, K; \forall n\}$ is assumed to satisfy the *regularity* condition as discussed in appendix A, section A.8. Issues of specification, implementation, and performance analysis of the receivers are addressed.

4.1.1 Some Preliminaries

The preliminaries given in chapter 3, section 3.1.1 are needed here as well.

4.2 Decision-Feedback Equalizers

The task of a receiver is to decide from the received signal $z(t)$ which data sequence $a(n)$ was likely transmitted. A decision-feedback equalizer performs this task in a recursive manner as follows. Suppose $\hat{a}(n) \in \mathcal{A}$ is the n^{th} data symbol that was likely transmitted for $n < m$; here $\mathcal{A}$ denotes the signal set. A random variable $\hat{a}_{f_m,\beta_m}$ defined as

$$\hat{a}_{f_m,\beta_m} = \int f_m^*(t)z(t)dt - \sum_{n<m} \beta_m(n)\hat{a}(n), \tag{4.1}$$

for some optimum pair of functions $(f_m(t), \beta_m(n))$, is formed, and then the number

$$\hat{a}(m) = \underset{a \in \mathcal{A}}{\arg\min} \; |a - \hat{a}_{f_m,\beta_m}| \tag{4.2}$$

is considered as the m^{th} data symbol that was likely transmitted. The sequence of pairs of functions $(f_m(t), \beta_m(n))$ may be optimized under various criteria that judge the closeness of $\hat{a}_{f_m,\beta_m}$ to $a(m)$. This thesis considers two such criteria:

1. minimum noise variance under the zero-forcing constraint.

2. minimum mean-square-error.

The first of the above criteria is considered in this chapter, and the second criterion is considered in chapter 6.

4.2.1 A Simplifying Assumption

Observe that the $\hat{a}(n)$ are random variables dependent on both the data and the noise and, therefore, their probabilistic/statistical characterization may be exceedingly difficult. Therefore, in optimizing the pair of functions $(f_m(t), \beta_m(n))$, it is necessary to assume that $\hat{a}(n) = a(n)$ for $n < m$; in practice, this assumption will hold with high probability if the signal-to-noise ratio $\mathcal{S}_0/\mathcal{N}_0$ is high. Under this simplifying assumption,

$$\hat{a}_{f,\beta} = \int f^*(t)z(t)dt - \sum_{n<m} \beta(n)a(n), \tag{4.3}$$

where $\int f^*(t)z(t)dt$ is the correlation defined in chapter 3, section 3.1.1, and where the summation $\sum_{n<m} \beta(n)a(n)$ is defined in the mean-square-sense with respect to the data.

In view of equation 3.1 of chapter 3, the random variable $\hat{a}_{f,\beta}$ has a decomposition as the sum of a data-dependent part $\hat{b}_{f,\beta}$ given by

$$\hat{b}_{f,\beta} = \sum_{n\geq m} a(n) \sum_{k=1}^{K} e^{j\omega_k nT} \int f^*(t) h_k(t-nT)dt$$
$$- \sum_{n<m} a(n) \left[\sum_{k=1}^{K} e^{j\omega_k nT} \int f^*(t) h_k(t-nT)dt - \beta(n) \right] \qquad (4.4)$$

and a noise-dependent part $\hat{c}_f$ given by

$$\hat{c}_f = \int f^*(t) \eta_w(t) dt. \qquad (4.5)$$

For later use, observe that the random variable

$$\hat{a}_\beta = \sum_{n<m} \beta(n) a(n) \qquad (4.6)$$

has mean zero and variance

$$E\left[|\hat{a}_\beta|^2 \right] = \mathcal{S}_0 \sum_{n<m} |\beta(n)|^2. \qquad (4.7)$$

Therefore, from an implementation point of view, the condition

$$\sum_{n<m} |\beta(n)|^2 < \infty \qquad (4.8)$$

is desired.

4.2.2 On Specifying a Decision-Feedback Equalizer

For the composition of QAM and linear time-invariant channel, decision-feedback equalizers optimized under one of the aforementioned criteria have the property that

$$f_m(t) = f_0(t - mT), \qquad (4.9)$$
$$\beta_m(n) = \beta_0(n - m) \qquad (4.10)$$

for all m, where T is the baud period. Thus a decision-feedback equalizer for the composition of QAM and linear time-invariant channel is completely specified by the optimum pair of functions $(f_0(t), \beta_0(n))$ and the parameter T. A decision-feedback equalizer for the composition of QAM and DMC, however, may not be specified as easily because the DMC is time-variant. In general, an entire sequence of optimum pairs of functions would be needed to specify a decision-feedback equalizer for the composition of QAM and DMC. The complexity involved in specifying such a sequence of pairs of functions is of major concern in this chapter and chapter 6. For the time being, however, a decision-feedback equalizer may be identified with an optimum sequence of pairs of functions $(f_m(t), \beta_m(n))$.

4.3 The Optimality Criterion – Minimum Noise Variance under the Zero-Forcing Constraint

Suppose that, for some pair of functions $(f(t), \beta(n))$, the random variable

$$\hat{a}_{f,\beta} = \int f^*(t)z(t)dt - \sum_{n<m} \beta(n)a(n) \tag{4.11}$$

is used to obtain

$$\hat{a}(m) = \begin{array}{c}\arg\min \\ a \in \mathcal{A}\end{array} |a - \hat{a}_{f,\beta}| . \tag{4.12}$$

In considering $\hat{a}(m)$ as the m^{th} data symbol that was likely transmitted, both the data and the noise are potential causes of error. To guard against an error being caused by the data, the data-dependent part $\hat{b}_{f,\beta}$ of $\hat{a}_{f,\beta}$ defined by equation 4.4 must be constrained as

$$\hat{b}_{f,\beta} = a(m), \tag{4.13}$$

or equivalently the pair of functions $(f(t), \beta(n))$ must be constrained as

$$\sum_{k=1}^{K} e^{j\omega_k nT} \int f^*(t)h_k(t-nT)dt = \begin{cases} 0 & \text{if} \quad n > m, \\ 1 & \text{if} \quad n = m, \end{cases} \tag{4.14}$$

and[1]

$$\beta(n) = \sum_{k=1}^{K} e^{j\omega_k nT} \int f^*(t)h_k(t-nT)dt \tag{4.15}$$

for $n < m$. This set of constraints is known as the *zero-forcing* constraint because it forces to zero the interference caused by data other than $a(m)$, the data symbol being decided upon. Assuming for the moment that there exist functions that satisfy this constraint, one obtains

$$\hat{a}_{f,\beta} - a(m) = \hat{c}_f, \tag{4.16}$$

[1] For later use, observe that by Cauchy inequality $|\beta(n)|^2 \le K \sum_{k=1}^{K} \left|\int f^*(t)h_k(t-nT)dt\right|^2$.

where $\hat{c}_f$ is the noise-dependent part of $\hat{a}_{f,\beta}$ defined by equation 4.5. Since $\hat{c}_f$ is a Gaussian random variable, its variance $E\left[|\hat{c}_f|^2\right]$ must preferably be minimized. Thus the integral

$$\int |f(t)|^2\, dt \tag{4.17}$$

must be minimized with respect to the function $f(t)$ under the first part of the zero-forcing constraint given by equation 4.14. The function $\beta(n)$ is then straightforwardly determined by the second part of the zero-forcing constraint given by equation 4.15.

The derivation of a decision-feedback equalizer under the aforementioned optimality criterion – *minimum noise variance under the zero-forcing constraint* – will be attempted in the next section. It turns out there that this approach, although optimal, is not feasible from an implementation point of view. In the subsequent sections, suboptimal formulations of the problem obtained by using stronger zero-forcing constraints will be considered. Henceforth, the term *decision-feedback zero-forcing equalizer* shall describe the solution to the optimal formulation as well the solution to a suboptimal formulation.

4.4 The Optimal Formulation of the Decision-Feedback Zero-Forcing Equalizer Problem

From the discussions of sections 4.2 and 4.3, it is evident that the problem of deriving a decision-feedback zero-forcing equalizer can be cast as a family of constrained minimization problems indexed by integers, the m^{th} problem being to minimize the quantity $\int |f(t)|^2\, dt$ under the constraint $\mathcal{D}_m$ defined by equation 4.19. The solution to the m^{th} problem shall be denoted by $f_m(t)$, and the minimum so achieved shall be denoted by

$$\lambda_m = \int |f_m(t)|^2 dt. \tag{4.18}$$

4.4.1 Zero-Forcing Constraint $\mathcal{D}_m$

Denote by $\mathcal{D}_m$ the following constraint on the function $f(t)$:

$$\sum_{k=1}^{K} e^{j\omega_k nT} \int f^*(t) h_k(t-nT) dt = \begin{cases} 0 & \text{if} \quad n > m, \\ 1 & \text{if} \quad n = m. \end{cases} \tag{4.19}$$

4.4.2 Some Considerations in the Hilbert Space $\mathcal{L}^2$

Following the discussions of chapter 3, section 3.4.2, the problem can be restated as follows: minimize (f, f) under the constraint

$$(f, h^n) = \begin{cases} 0 & \text{if} \quad n > m, \\ 1 & \text{if} \quad n = m. \end{cases} \tag{4.20}$$

where

$$h^n(t) = \sum_{k=1}^{K} e^{j\omega_k nT} h_k(t - nT). \tag{4.21}$$

This restatement shows that the solution, if it exists, must belong to the subspace[2]

$$\mathcal{H}_{\Sigma(\geq m)} = \text{Clos.Span}\left\{h^n(t) : n \geq m\right\}. \tag{4.22}$$

In the manner of the discussion of chapter 3, section 3.4.3, denote the subspace

$$\mathcal{H}_{\Sigma(>m)} = \text{Clos.Span}\left\{h^n(t) : n > m\right\}. \tag{4.23}$$

Suppose that

$$h^m(t) = x(t) + y(t), \tag{4.24}$$

is the unique decomposition of $h^m(t)$ such that $x(t) \in \mathcal{H}^{\perp}_{\Sigma(>m)}$ (the orthogonal complement of $\mathcal{H}_{\Sigma(>m)}$ in $\mathcal{H}_{\Sigma(\geq m)}$) and $y(t) \in \mathcal{H}_{\Sigma(>m)}$. A solution to the m^{th} problem exists if and only if $x(t) \not\equiv 0$. Under this assumption, the optimum solution is unique and is given by

$$f_m(t) = x(t)/(x, x), \tag{4.25}$$

and

$$\lambda_m = 1/(x, x). \tag{4.26}$$

4.4.3 Special Cases

In the trivial case where there is no intersymbol interference, the decision-feedback zero-forcing equalizer is the same as the linear zero-forcing equalizer. Thus $x(t) = h^m(t)$ as observed in chapter 3, section 3.4.4, and $\beta_m(n) = 0$.

[2]This is the smallest closed subspace containing the set $\{h^n(t) : n \geq m\}$.

In the case where $K = 1$, although the DMC is time-variant in general, the sequence of functions $\{h^n(t) : \forall n\}$ is stationary as shown in appendix A, section A.1.2. Therefore, even if there is intersymbol interference, the technique used for the composition of QAM and linear time-invariant channel can be used essentially without difference. Thus the decision-feedback zero-forcing equalizer exists if and only if

$$\int_{-\pi/T}^{\pi/T} \log H(\omega) d\omega > -\infty, \tag{4.27}$$

where $H(\omega)$ is the spectral density function of the stationary sequence of functions $\{h_1(t - nT) : \forall n\}$. Under this assumption, there exist causal functions $C(\omega)$ that satisfy

$$H(\omega) = |C(\omega)|^2, \tag{4.28}$$

as observed in appendix A, section A.8.1. In terms of a maximal causal function $C(\omega)$, the isomorph $c(\omega) \in \mathcal{L}_H^2$ of $f_m(t)$ is given by

$$c(\omega) = Te^{j(\omega_1 - \omega)mT}/(C_0^* C(\omega)), \tag{4.29}$$

$$\lambda_m = T|C_0|^{-2}, \tag{4.30}$$

and

$$\beta_m(n) = e^{j\omega_1(n-m)T} C_{m-n}/C_0, \tag{4.31}$$

where

$$C_n = \frac{T}{2\pi} \int_{-\pi/T}^{\pi/T} C(\omega) e^{j\omega nT} d\omega. \tag{4.32}$$

Observe that, since $C(\omega) \in \mathcal{L}^2[-\pi/T, \pi/T]$, the condition $\sum_{n<m} |\beta(n)|^2 < \infty$ is satisfied.

4.4.4 The General Case

In the general case where $K > 1$ and there is intersymbol interference, it does not seem a trivial task to ascertain the existence of the solution or even to find it. However, an approximate solution may be obtained as follows: since the optimum solution $f_m \in \mathcal{H}_{\Sigma(\geq m)}$, heuristically,

$$f_m(t) = \sum_{n \geq m} \alpha_m(n) h^n(t) \tag{4.33}$$

for some one-sided sequence $a_m(n)$: this implies

$$\sum_{n' \geq m} \left(h^n, h^{n'}\right) a_m(n') = \begin{cases} 0 & \text{if} \quad n > m, \\ 1 & \text{if} \quad n = m, \end{cases} \tag{4.34}$$

and

$$(f_m, f_m) = \sum_{n \geq m} a_m(n)\,(f_m, h^n) = a_m(m); \tag{4.35}$$

an approximation of equation 4.34 obtained by letting both n and n' vary over a finite set can be solved by matrix methods. The approximate solution so obtained will not strictly satisfy the zero-forcing constraint $\mathcal{D}_m$. But it will satisfy a weaker constraint than $\mathcal{D}_m$, and therefore correspond to a lower bound on λ_m. In the next section, it is shown how, for roughly the same order of complexity, a function $f(t)$ that minimizes (f, f) while satisfying a stronger constraint than $\mathcal{D}_m$ can be found: the solution so obtained will automatically satisfy the zero-forcing constraint $\mathcal{D}_m$ and correspond to an upper bound on λ_m.

4.5 A Suboptimal Formulation of the Decision-Feedback Zero-Forcing Equalizer Problem

The decision-feedback zero-forcing equalizer problem can be formulated in a suboptimal manner by replacing the constraint set $\mathcal{D}_m$ with a stronger constraint set $\mathcal{D}^0_{mu}$ defined by equations 4.37 and 4.38. Thus the suboptimal formulation of the decision-feedback zero-forcing equalizer problem is a family of constrained minimization problems indexed by integers, the m^{th} problem being to minimize $\int |f(t)|^2\, dt$ under the constraint $\mathcal{D}^0_{mu}$. The solution to the m^{th} problem shall be denoted by $f^0_{mu}(t)$, and the minimum so achieved shall be denoted by

$$\lambda^0_{mu} = \int |f^0_{mu}(t)|^2 dt. \tag{4.36}$$

Then λ^0_{mu} will be an upper bound[3] on λ_m. The motivation for this suboptimal formulation is that the solution is attractive from the points of view of specification and implementation.

4.5.1 Zero-Forcing Constraint $\mathcal{D}^0_{mu}$

Denote by $\mathcal{D}^0_{mu}$ the following set of constraints on the function $f(t)$:

[3]This is the reason for the subscript 'u' in $\mathcal{D}^0_{mu}$ etc.. The reason for the superscript '0' will become apparent later.

- for $n > m$ and $k = 1, 2, \ldots, K$,

$$\int f^*(t) h_k(t - nT) dt = 0. \tag{4.37}$$

-

$$\sum_{k=1}^{K} e^{j\omega_k mT} \int f^*(t) h_k(t - mT) dt = 1. \tag{4.38}$$

Observe that $\mathcal{D}^0_{mu}$ is stronger than $\mathcal{D}_m$ in the sense that if a function $f(t)$ satisfies $\mathcal{D}^0_{mu}$ then it satisfies $\mathcal{D}_m$.

4.5.2 Considerations in the Hilbert Space $\mathcal{L}^2$

An argument similar to that given in chapter 3, sections 3.4.2 and 3.4.3 gives the following conclusions: the solution, if it exists, must belong to the subspace

$$\mathcal{H}_{(\geq m)} = \text{Clos.Span}\left\{h_k(t - nT) : k = 1, 2, \ldots, K; n \geq m\right\}, \tag{4.39}$$

but be orthogonal to the subspace

$$\mathcal{H}_{(\geq m+1)} = \text{Clos.Span}\left\{h_k(t - nT) : k = 1, 2, \ldots, K; n \geq m + 1\right\}; \tag{4.40}$$

suppose that

$$h^m(t) = \sum_{k=1}^{K} e^{j\omega_k mT} h_k(t - mT) = x(t) + y(t) \tag{4.41}$$

is the unique decomposition of $h^m(t) \in \mathcal{H}_{(\geq m)}$ such that $y(t) \in \mathcal{H}_{(\geq m+1)}$ and $x(t) \in \mathcal{H}^{\perp}_{(\geq m+1)}$ (the orthogonal complement of $\mathcal{H}_{(\geq m+1)}$ in $\mathcal{H}$); a solution exists if and only if $x(t) \neq 0$, under which assumption it is given by

$$f^0_{mu}(t) = x(t)/(x, x), \tag{4.42}$$

and $\lambda^0_{mu} = 1/(x, x)$.

4.5.3 Solution

For the special case where $K = 1$, the suboptimal formulation is equivalent to the optimal formulation, and therefore the corresponding result of section 4.4.3 is applicable.

In the general case, a necessary condition for $x(t) \neq 0$ is that the set of stationary sequences

$$\{h_k(t-nT) : k = 1, 2, \ldots, K; \forall n\} \tag{4.43}$$

be *non singular*. However, a stronger condition which also facilitates the solution is that the set of stationary sequences be *regular*, or, equivalently, that it have a *Wold* decomposition as discussed in appendix A, section A.8.

Assumptions that Facilitate the Solution

The assumption of regularity is equivalent to the following: the spectral density matrix $\mathbf{H}(\omega)$, of the set of stationary sequences

$$\{h_k(t-nT) : k = 1, 2, \ldots, K; \forall n\}, \tag{4.44}$$

has constant rank L almost everywhere, and admits a factorization. Let $\mathbf{C}(\omega)$ be an $L \times K$ maximal causal matrix satisfying

$$\mathbf{H}(\omega) = \mathbf{C}^H(\omega)\mathbf{C}(\omega), \tag{4.45}$$

and let the $K \times L$ matrix $\mathbf{A}(\omega) = [\mathbf{a}_1(\omega), \mathbf{a}_2(\omega), \ldots, \mathbf{a}_L(\omega)]$ satisfy

$$\mathbf{C}(\omega)\mathbf{A}(\omega) = \mathbf{I}_{L\times L}. \tag{4.46}$$

Then the set of functions

$$\left\{\sqrt{T}\mathbf{a}_l(\omega)e^{-j\omega nT} : l = 1, 2, \ldots, L; \forall n\right\} \tag{4.47}$$

constitutes a Wold decomposition of $\mathcal{L}^2_{\mathbf{H}}$, the Hilbert space isomorphic to $\mathcal{H}$ as discussed in appendix A, section A.5.

Solution

Denote

$$\Omega_m = \left[e^{j\omega_1 mT}, e^{j\omega_2 mT}, \ldots, e^{j\omega_K mT}\right]^T. \tag{4.48}$$

The isomorph of $h^m(t)$ is then $\Omega_m e^{-j\omega mT} \in \mathcal{L}^2_{\mathbf{H}}$. The isomorph $\mathbf{u}(\omega) \in \mathcal{L}^2_{\mathbf{H}}$ of $x(t)$ is obtained using the Wold decomposition as follows:

$$\begin{aligned}
\mathbf{u}(\omega) &= \sum_{l=1}^{L}\left(\frac{1}{2\pi}\int_{-\pi/T}^{\pi/T}\sqrt{T}\mathbf{a}_l^H(\omega)\mathbf{H}(\omega)\Omega_m d\omega\right)\sqrt{T}\mathbf{a}_l(\omega)e^{-j\omega mT}. &(4.49)\\
&= e^{-j\omega mT}\mathbf{A}(\omega)\sum_{l=1}^{L}\mathbf{e}_l\left(\frac{T}{2\pi}\int_{-\pi/T}^{\pi/T}\mathbf{e}_l^H\mathbf{A}^H(\omega)\mathbf{H}(\omega)\Omega_m d\omega\right), &(4.50)\\
&= e^{-j\omega mT}\mathbf{A}(\omega)\sum_{l=1}^{L}\mathbf{e}_l\left(\frac{T}{2\pi}\int_{-\pi/T}^{\pi/T}\mathbf{e}_l^H\mathbf{C}(\omega)\Omega_m d\omega\right), &(4.51)\\
&= e^{-j\omega mT}\mathbf{A}(\omega)\left(\sum_{l=1}^{L}\mathbf{e}_l\mathbf{e}_l^H\right)\mathbf{C}_0\Omega_m, &(4.52)\\
&= e^{-j\omega mT}\mathbf{A}(\omega)\mathbf{C}_0\Omega_m, &(4.53)
\end{aligned}$$

where[4]

$$\mathbf{C}_0 = \frac{T}{2\pi}\int_{-\pi/T}^{\pi/T}\mathbf{C}(\omega)d\omega. \quad (4.54)$$

Therefore

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\pi/T}^{\pi/T}\mathbf{u}^H(\omega)\mathbf{H}(\omega)\mathbf{u}(\omega)d\omega &= \Omega_m^H\mathbf{C}_0^H\left(\frac{1}{2\pi}\int_{-\pi/T}^{\pi/T}\mathbf{A}^H(\omega)\mathbf{H}(\omega)\mathbf{A}(\omega)d\omega\right)\mathbf{C}_0\Omega_m, \\
& &(4.55)\\
&= T^{-1}\Omega_m^H\mathbf{C}_0^H\mathbf{C}_0\Omega_m. &(4.56)
\end{aligned}$$

Threfore a solution exists if and only if Ω_m does not belong to the null space of $\mathbf{C}_0$. Under this assumption, the isomorph $\mathbf{c}(\omega)$ of $f^0_{mu}(t)$ is given by[5]

$$\begin{aligned}
\mathbf{c}(\omega) &= T\left[\Omega_m^H\mathbf{C}_0^H\mathbf{C}_0\Omega_m\right]^{-1}e^{-j\omega mT}\mathbf{A}(\omega)\mathbf{C}_0\Omega_m, &(4.57)\\
\lambda^0_{mu} &= T\left[\Omega_m^H\mathbf{C}_0^H\mathbf{C}_0\Omega_m\right]^{-1}. &(4.58)
\end{aligned}$$

If $L = K$ then $\mathbf{C}_0$ has rank K, and therefore a solution exists.

4.5.4 The Optimum Function $\beta_m(n)$

The optimum function $\beta_m(n)$, given by

$$\beta_m(n) = \sum_{k=1}^{K} e^{j\omega_k nT}\int f^{0\,*}_{mu}(t)h_k(t-nT)dt \quad (4.59)$$

[4]The elements of $\mathbf{C}(\omega)$ are in $\mathcal{L}^2[-\pi/T,\pi/T]$.

[5]Compare with results given in 4.4.3.

for $n < m$, can be obtained using the isomorphism between $\mathcal{H}$ and $\mathcal{L}^2_{\mathbf{H}}$ as follows:

$$\begin{aligned}\beta_m(n) &= \frac{1}{2\pi}\int_{-\pi/T}^{\pi/T} \mathbf{c}^H(\omega)\mathbf{H}(\omega)\Omega_n e^{-j\omega nT}d\omega. && (4.60)\\ &= T\left[\Omega_m^H \mathbf{C}_0^H \mathbf{C}_0 \Omega_m\right]^{-1}\Omega_m^H \mathbf{C}_0^H\left(\frac{1}{2\pi}\int_{-\pi/T}^{\pi/T}\mathbf{A}^H(\omega)\mathbf{H}(\omega)\Omega_n e^{j\omega(m-n)T}d\omega\right). && (4.61)\\ &= \left[\Omega_m^H \mathbf{C}_0^H \mathbf{C}_0 \Omega_m\right]^{-1}\Omega_m^H \mathbf{C}_0^H\left(\frac{T}{2\pi}\int_{-\pi/T}^{\pi/T}\mathbf{C}(\omega)e^{j\omega(m-n)T}d\omega\right)\Omega_n. && (4.62)\\ &= \left[\Omega_m^H \mathbf{C}_0^H \mathbf{C}_0 \Omega_m\right]^{-1}\Omega_m^H \mathbf{C}_0^H \mathbf{C}_{m-n}\Omega_n, && (4.63)\end{aligned}$$

where[6]

$$\mathbf{C}_n = \frac{T}{2\pi}\int_{-\pi/T}^{\pi/T}\mathbf{C}(\omega)e^{j\omega nT}d\omega. \qquad (4.64)$$

That the condition $\sum_{n<m}|\beta_m(n)|^2 < \infty$ is satisfied is evident from the equation

$$\int f_{mu}^{0\,*}(t)h_k(t-nT)dt = \left[\Omega_m^H \mathbf{C}_0^H \mathbf{C}_0 \Omega_m\right]^{-1}\Omega_m^H \mathbf{C}_0^H \mathbf{C}_{m-n}\mathbf{e}_k. \qquad (4.65)$$

the fact that the elements of $\mathbf{C}(\omega)$ are in $\mathcal{L}^2[-\pi/T,\pi/T]$, and the inequality $|\beta_m(n)|^2 \leq K\sum_{k=1}^{K}\left|\int f_{mu}^{0\,*}(t)h_k(t-nT)dt\right|^2$.

4.5.5 An Alternative Approach

Since $f_{mu}^0(t)$ is a function that belongs to $\mathcal{H}_{(\geq m)}$ but is orthogonal to $\mathcal{H}_{(\geq m+1)}$, its isomorph $\mathbf{c}(\omega) \in \mathcal{L}^2_{\mathbf{H}}$ has the form

$$\mathbf{c}(\omega) = \mathbf{A}(\omega)\mathbf{b}e^{-j\omega mT} \qquad (4.66)$$

for some complex L-dimensional vector-valued constant $\mathbf{b}$. Using this form, one obtains

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi/T}^{\pi/T}\mathbf{c}^H(\omega)\mathbf{H}(\omega)\mathbf{c}(\omega)d\omega &= \frac{1}{2\pi}\int_{-\pi/T}^{\pi/T}\mathbf{b}^H\mathbf{A}^H(\omega)\mathbf{H}(\omega)\mathbf{A}(\omega)\mathbf{b}d\omega, && (4.67)\\ &= T^{-1}\mathbf{b}^H\mathbf{b}, && (4.68)\end{aligned}$$

and

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi/T}^{\pi/T}\mathbf{c}^H(\omega)\mathbf{H}(\omega)\Omega_m(\omega)d\omega &= \frac{1}{2\pi}\int_{-\pi/T}^{\pi/T}\mathbf{b}^H\mathbf{A}^H(\omega)\mathbf{H}(\omega)\Omega_m d\omega, && (4.69)\\ &= T^{-1}\mathbf{b}^H\mathbf{C}_0\Omega_m. && (4.70)\end{aligned}$$

[6]The elements of $\mathbf{C}(\omega)$ are in $\mathcal{L}^2[-\pi/T,\pi/T]$.

The complex vector-valued constant $\mathbf{b}$ that minimizes $T^{-1}\mathbf{b}^H\mathbf{b}$ under the constraint $T^{-1}\mathbf{b}^H\mathbf{C}_0\mathbf{\Omega}_m = 1$ is then given by

$$\mathbf{b} = T\left[\mathbf{\Omega}_m^H\mathbf{C}_0^H\mathbf{C}_0\mathbf{\Omega}_m\right]^{-1}\mathbf{C}_0\mathbf{\Omega}_m, \tag{4.71}$$

for which $T^{-1}\mathbf{b}^H\mathbf{b} = T\left[\mathbf{\Omega}_m^H\mathbf{C}_0^H\mathbf{C}_0\mathbf{\Omega}_m\right]^{-1}$.

4.5.6 On Specifying and Implementing the Solution

The decision-feedback equalizer $\{f_{mu}^0(t), \beta_m(n) : \forall m\}$ is specified by the isomorphs of the columns of the matrix

$$\mathbf{A}(\omega)\mathbf{C}_0, \tag{4.72}$$

the sequence of matrices $\mathbf{C}_0^H\mathbf{C}_n$ for $n \geq 0$, the vector $\mathbf{\Omega}_1$, and the parameter T. Observe that, although $\mathbf{C}(\omega)$ is non-unique, $\mathbf{A}(\omega)\mathbf{C}_0$ and $\mathbf{C}_0^H\mathbf{C}_n$, for all n, are unique.

For the purpose of implementation, the decision-feedback equalizer can be considered as having the following four functional blocks:

1. *forward filter* – a bank of K continuous-time time-invariant filters matched to the isomorphs of the columns of the matrix $T\mathbf{A}(\omega)\mathbf{C}_0$.

2. *backward filter* – a K-input K-output discrete-time time-invariant filter with impulse response

$$\mathbf{B}_n = \begin{cases} \mathbf{C}_0^H\mathbf{C}_n & \text{if } n > 0, \\ \mathbf{0} & \text{if } n \leq 0. \end{cases} \tag{4.73}$$

3. *subtractor* – a bank of K elementary subtractors, that is devices each with two inputs and an output, the output being the signed difference of the inputs.

4. *combiner* – a device that linearly combines K values according to a set of weights.

The implementation in terms of these blocks is shown in figure 4.1.

Thus the received signal $z(t)$ is used as input to the forward filter. The output of the forward filter is sampled once every T seconds and the samples are used as the positive input to the subtractor. The sequence[7] $\mathbf{\Omega}_n\hat{a}(n)$ is used as input to the backward filter. The output of the backward filter is used as the negative input to the subtractor. The output of the subtractor is combined according to the weight vector $\mathbf{\Omega}_m^*$. The combined output is then scaled by $\left[\mathbf{\Omega}_m^H\mathbf{C}_0^H\mathbf{C}_0\mathbf{\Omega}_m\right]^{-1}$.

[7] Recall from section 4.2 that $\hat{a}(n)$ is the decision as to which data sequence was likely transmitted.

4.6 Generalized Suboptimal Formulation of the Decision-Feedback Zero-Forcing Equalizer Problem

The suboptimal formulation of the decision-feedback zero-forcing equalizer problem of section 4.5 can be generalized by replacing the constraint set $\mathcal{D}^0_{mu}$ with a generalized constraint set $\mathcal{D}^q_{mu}$ defined by equations 4.75, 4.76, and 4.77. Thus a generalized suboptimal formulation of the decision-feedback zero-forcing equalizer problem is a family of constrained minimization problems indexed by integers, the m^{th} problem being to minimize $\int |f(t)|^2 dt$ under the constraint set $\mathcal{D}^q_{mu}$. The solution to the m^{th} problem shall be denoted by $f^q_{mu}(t)$ and the minimum so achieved shall be denoted by

$$\lambda^q_{mu} = \int |f^q_{mu}(t)|^2\, dt. \tag{4.74}$$

4.6.1 Zero-Forcing Constraint $\mathcal{D}^q_{mu}$

Let q be a non-negative integer. Denote by $\mathcal{D}^q_{mu}$ the following set of constraints on the function $f(t)$:

- for $n - m > q$ and $k = 1, 2, \ldots, K$,

$$\int f^*(t) h_k(t - nT) dt = 0, \tag{4.75}$$

- for $0 < n - m \leq q$,

$$\sum_{k=1}^{K} e^{j\omega_k nT} \int f^*(t) h_k(t - nT) dt = 0, \tag{4.76}$$

-

$$\sum_{k=1}^{K} e^{j\omega_k mT} \int f^*(t) h_k(t - mT) dt = 1. \tag{4.77}$$

Observe that $\mathcal{D}^q_{mu}$ is stronger than $\mathcal{D}_m$ in the sense that if a function $f(t)$ satisfies $\mathcal{D}^q_{mu}$ then it satisfies $\mathcal{D}_m$. In the same sense, $\mathcal{D}^0_{mu}$ is stronger than $\mathcal{D}^q_{mu}$, and if $q_1 < q_2$ then $\mathcal{D}^{q_1}_{mu}$ is stronger than $\mathcal{D}^{q_2}_{mu}$. Therefore,

$$\lambda_m \leq \lambda^{q_2}_{mu} \leq \lambda^{q_1}_{mu} \leq \lambda^0_{mu}. \tag{4.78}$$

In other words, for increasing q, the quantities λ^q_{mu} constitute a hierarchy of tighter upper bounds on λ_m. This is the reason for the subscript 'u' and the superscript 'q.' One may call q the degree of optimality.

4.6.2 Solution

An argument similar to that given in chapter 3, sections 3.4.2 and 3.4.3 gives the following conclusions: the solution, if it exists, must belong to the subspace

$$\mathcal{H}_{(\geq m)} = \text{Clos.Span}\left\{h_k(t-nT) : k = 1,2,\ldots,K; n \geq m\right\}. \tag{4.79}$$

but be orthogonal to the subspace

$$\mathcal{H}_{(\geq m+q+1)} = \text{Clos.Span}\left\{h_k(t-nT) : k = 1,2,\ldots,K; n \geq m+q+1\right\}. \tag{4.80}$$

Under the assumptions stated in 4.5.3, the isomorph $\mathbf{c}(\omega) \in \mathcal{L}^2_{\mathbf{H}}$ of $f^q_{mu}(t)$ has the form

$$\mathbf{c}(\omega) = \mathbf{A}(\omega)e^{-j\omega mT}\sum_{p=0}^{q}\mathbf{b}_p e^{-j\omega pT} \tag{4.81}$$

for some set of L-dimensional complex vector-valued constants $\{\mathbf{b}_p : p = 0,1,\ldots,q\}$. Using this form, one obtains

$$\frac{1}{2\pi}\int_{-\pi/T}^{\pi/T}\mathbf{c}^H(\omega)\mathbf{H}(\omega)\mathbf{c}(\omega)d\omega = \sum_{p_1=0}^{q}\sum_{p_2=0}^{q}\frac{1}{2\pi}\int_{-\pi/T}^{\pi/T}\mathbf{b}_{p_1}^H\mathbf{A}^H(\omega)\mathbf{H}(\omega)\mathbf{A}(\omega)\mathbf{b}_{p_2}e^{j\omega(p_1-p_2)T}d\omega, \tag{4.82}$$

$$= T^{-1}\sum_{p=0}^{q}\mathbf{b}_p^H\mathbf{b}_p, \tag{4.83}$$

and

$$\frac{1}{2\pi}\int_{-\pi/T}^{\pi/T}\mathbf{c}^H(\omega)\mathbf{H}(\omega)\mathbf{\Omega}_n e^{-j\omega nT}d\omega = \sum_{p=0}^{q}\mathbf{b}_p^H\frac{1}{2\pi}\int_{-\pi/T}^{\pi/T}\mathbf{A}^H(\omega)\mathbf{H}(\omega)\mathbf{\Omega}_n e^{-j\omega(n-m-p)T}d\omega, \tag{4.84}$$

$$= T^{-1}\sum_{p=0}^{q}\mathbf{b}_p^H\mathbf{C}_{p-(n-m)}\mathbf{\Omega}_n. \tag{4.85}$$

The set of complex vector-valued constants $\{\mathbf{b}_p : p = 0,1,\ldots,q\}$ is that which minimizes $T^{-1}\sum_{p=0}^{q}\mathbf{b}_p^H\mathbf{b}_p$ under the constraint

$$T^{-1}\sum_{p=n-m}^{q}\mathbf{b}_p^H\mathbf{C}_{p-(n-m)}\mathbf{\Omega}_n = \begin{cases} 0 & \text{if} \quad 0 < n-m \leq q, \\ 1 & \text{if} \quad n-m = 0, \end{cases} \tag{4.86}$$

or equivalently, by a change of variable,

$$T^{-1}\sum_{p=n}^{q}\mathbf{b}_p^H\mathbf{C}_{p-n}\mathbf{\Omega}_{n+m} = \begin{cases} 0 & \text{if} \quad 0 < n \leq q, \\ 1 & \text{if} \quad n = 0. \end{cases} \tag{4.87}$$

The Case $q = 1$

The set of complex vector-valued constants $\{\mathbf{b}_p : p = 0, 1\}$ is that which minimizes

$$\left[\mathbf{b}_1^H, \mathbf{b}_0^H\right] \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_0 \end{bmatrix} \tag{4.88}$$

under the constraint

$$\left[\mathbf{b}_1^H, \mathbf{b}_0^H\right] \begin{bmatrix} \mathbf{C}_0 & \mathbf{C}_1 \\ \mathbf{0} & \mathbf{C}_0 \end{bmatrix} \begin{bmatrix} \boldsymbol{\Omega}_{m+1} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Omega}_m \end{bmatrix} = [0, T]. \tag{4.89}$$

An arbitrary set of complex vector-valued constants $\{\mathbf{b}_p : p = 0, 1\}$ has a unique decomposition

$$\begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_0 \end{bmatrix} = \begin{bmatrix} \mathbf{C}_0 & \mathbf{C}_1 \\ \mathbf{0} & \mathbf{C}_0 \end{bmatrix} \begin{bmatrix} \boldsymbol{\Omega}_{m+1} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Omega}_m \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_0 \end{bmatrix} + \mathbf{f}, \tag{4.90}$$

where γ_0 and γ_1 are scalars and $\mathbf{f}$ is a $2L$-dimensional vector that is orthogonal to the column span of

$$\begin{bmatrix} \mathbf{C}_0 & \mathbf{C}_1 \\ \mathbf{0} & \mathbf{C}_0 \end{bmatrix} \begin{bmatrix} \boldsymbol{\Omega}_{m+1} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Omega}_m \end{bmatrix}. \tag{4.91}$$

The set of complex vector-valued constants $\{\mathbf{b}_p : p = 0, 1\}$ which minimizes

$$\left[\mathbf{b}_1^H, \mathbf{b}_0^H\right] \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_0 \end{bmatrix} = \mathbf{f}^H\mathbf{f} + \tag{4.92}$$

$$[\gamma_1^*, \gamma_0^*] \begin{bmatrix} \boldsymbol{\Omega}_{m+1}^H & \mathbf{0}^H \\ \mathbf{0}^H & \boldsymbol{\Omega}_m^H \end{bmatrix} \begin{bmatrix} \mathbf{C}_0^H & \mathbf{0} \\ \mathbf{C}_1^H & \mathbf{C}_0^H \end{bmatrix} \begin{bmatrix} \mathbf{C}_0 & \mathbf{C}_1 \\ \mathbf{0} & \mathbf{C}_0 \end{bmatrix} \begin{bmatrix} \boldsymbol{\Omega}_{m+1} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Omega}_m \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_0 \end{bmatrix} \tag{4.93}$$

under the constraint 4.89 is given by $\mathbf{f} = \mathbf{0}$ and scalars γ_0 and γ_1 that satisfy the constraint

$$\begin{bmatrix} \boldsymbol{\Omega}_{m+1}^H & \mathbf{0}^H \\ \mathbf{0}^H & \boldsymbol{\Omega}_m^H \end{bmatrix} \begin{bmatrix} \mathbf{C}_0^H & \mathbf{0} \\ \mathbf{C}_1^H & \mathbf{C}_0^H \end{bmatrix} \begin{bmatrix} \mathbf{C}_0 & \mathbf{C}_1 \\ \mathbf{0} & \mathbf{C}_0 \end{bmatrix} \begin{bmatrix} \boldsymbol{\Omega}_{m+1} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Omega}_m \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_0 \end{bmatrix} = \begin{bmatrix} 0 \\ T \end{bmatrix}. \tag{4.94}$$

Such scalars γ_0 and γ_1 exist if and only if the vector

$$\mathbf{e} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{4.95}$$

belongs to the range space of the coefficient matrix

$$\tilde{\mathbf{C}} = \begin{bmatrix} \Omega_{m+1}^H & \mathbf{0}^H \\ \mathbf{0}^H & \Omega_m^H \end{bmatrix} \begin{bmatrix} \mathbf{C}_0^H & \mathbf{0} \\ \mathbf{C}_1^H & \mathbf{C}_0^H \end{bmatrix} \begin{bmatrix} \mathbf{C}_0 & \mathbf{C}_1 \\ \mathbf{0} & \mathbf{C}_0 \end{bmatrix} \begin{bmatrix} \Omega_{m+1} & \mathbf{0} \\ \mathbf{0} & \Omega_m \end{bmatrix}. \tag{4.96}$$

or, equivalently, if and only if the vector $\mathbf{e}$ does not belong to the null space of the matrix $\tilde{\mathbf{C}}$. This implies that such scalars γ_0 and γ_1 exist if and only if the vector

$$\begin{bmatrix} \mathbf{0} \\ \Omega_m \end{bmatrix} \tag{4.97}$$

does not belong to the null space of the matrix

$$\begin{bmatrix} \mathbf{C}_0 & \mathbf{C}_1 \\ \mathbf{0} & \mathbf{C}_0 \end{bmatrix}. \tag{4.98}$$

or, equivalently, if and only if the vector Ω_m does not simultaneously belong to the null spaces of $\mathbf{C}_0$ and $\mathbf{C}_1$. Under this assumption, the solution can be given in terms of the scalars γ_0 and γ_1 as follows:

$$\begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_0 \end{bmatrix} = \begin{bmatrix} \mathbf{C}_0 & \mathbf{C}_1 \\ \mathbf{0} & \mathbf{C}_0 \end{bmatrix} \begin{bmatrix} \Omega_{m+1} & \mathbf{0} \\ \mathbf{0} & \Omega_m \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_0 \end{bmatrix}, \tag{4.99}$$

$$\mathbf{c}(\omega) = \mathbf{A}(\omega)\left[e^{-j\omega(m+1)T}\mathbf{I}, e^{-j\omega mT}\mathbf{I}\right] \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_0 \end{bmatrix}, \tag{4.100}$$

$$\lambda_{mu}^1 = \frac{1}{2\pi}\int_{-\pi/T}^{\pi/T} \mathbf{c}^H(\omega)\mathbf{H}(\omega)\mathbf{c}(\omega)d\omega = T^{-1}\sum_{p=0}^{1} \mathbf{b}_p^H \mathbf{b}_p = \gamma_0; \tag{4.101}$$

here $\mathbf{I}$ denotes the $L \times L$ identity matrix.

4.6.3 The Optimum Function $\beta_m(n)$

The optimum function $\beta_m(n)$, defined by

$$\beta_m(n) = \sum_{k=1}^{K} e^{j\omega_k nT} \int f_{mu}^{1\,*}(t) h_k(t - nT)dt \tag{4.102}$$

for $n < m$, can be obtained using the isomorphism between $\mathcal{H}$ and $\mathcal{L}_{\mathbf{H}}^2$ as follows:

$$\beta_m(n) = \frac{1}{2\pi}\int_{-\pi/T}^{\pi/T} \mathbf{c}^H(\omega)\mathbf{H}(\omega)\Omega_n e^{-j\omega nT} d\omega, \tag{4.103}$$

$$= T^{-1}\sum_{p=0}^{q} \mathbf{b}_p^H \mathbf{C}_{p-(n-m)}\Omega_n. \tag{4.104}$$

The Case $q = 1$

$$\beta_m(n) = T^{-1} \begin{bmatrix} \mathbf{b}_1^H . \mathbf{b}_0^H \end{bmatrix} \begin{bmatrix} \mathbf{C}_{m+1-n} \\ \mathbf{C}_{m-n} \end{bmatrix} \mathbf{\Omega}_n. \tag{4.105}$$

That the condition $\sum_{n<m} |\beta_m(n)|^2 < \infty$ is satisfied is evident from the equation

$$\int f_{mu}^{1}{}^{*}(t)h_k(t-nT)dt = T^{-1} \begin{bmatrix} \mathbf{b}_1^H . \mathbf{b}_0^H \end{bmatrix} \begin{bmatrix} \mathbf{C}_{m+1-n} \\ \mathbf{C}_{m-n} \end{bmatrix} \mathbf{e}_k. \tag{4.106}$$

the fact that the elements of $\mathbf{C}(\omega)$ are in $\mathcal{L}^2[-\pi/T, \pi/T]$, and the inequality $|\beta_m(n)|^2 \leq K\sum_{k=1}^{K} |\int f_{mu}^{1}{}^{*}(t)h_k(t-nT)dt|^2$.

4.6.4 On Specifying and Implementing the Solution for $q = 1$

The decision-feedback equalizer is specified by the isomorphs of the columns of the matrix

$$\mathbf{A}(\omega), \tag{4.107}$$

the sequence of matrices $\mathbf{C}_n$ for $n \geq 0$, the vector $\mathbf{\Omega}_1$, and the parameter T.

For the purpose of implementation, the decision-feedback equalizer can be considered as having the following seven functional blocks:

1. *forward filter* – a bank of L continuous-time time-invariant filters matched to the isomorphs of the columns of the matrix $\mathbf{A}(\omega)$.

2. *backward filter (main)* – a K-input L-output discrete-time time-invariant filter with impulse response
$$\mathbf{B}_n = \begin{cases} \mathbf{C}_{n+1} & \text{if } n > 0, \\ \mathbf{0} & \text{if } n \leq 0. \end{cases} \tag{4.108}$$

3. *backward filter (auxiliary)* – a K-input L-output discrete-time time-invariant filter with impulse response
$$\mathbf{D}_n = \begin{cases} \mathbf{C}_1 & \text{if } n = 1, \\ \mathbf{0} & \text{if } n \neq 1. \end{cases} \tag{4.109}$$

4. *subtractor (main)* – a bank of L elementary subtractors, that is devices each with two inputs and an output, the output being the signed difference of the inputs.

5. *subtractor (auxiliary)* – same as above.

6. *tapped delay line* – a bank of L elementary delay lines each with two taps.

7. *combiner* – a device that linearly combines $2L$ values according to a set of weights.

The definition of the main and auxiliary backward filters is motivated by the following observation which simplifies the implementation of $\sum_{n<m} \beta_m(n)\hat{a}(n)$:

$$\sum_{n<m} \mathbf{C}_{m+1-n}\Omega_n\hat{a}(n) = \sum_n \mathbf{B}_{m-n}\Omega_n\hat{a}(n). \tag{4.110}$$

$$\sum_{n<m} \mathbf{C}_{m-n}\Omega_n\hat{a}(n) = \sum_{n<m-1} \mathbf{C}_{m-n}\Omega_n\hat{a}(n) + \mathbf{C}_1\Omega_{m-1}\hat{a}(m-1), \tag{4.111}$$

$$= \sum_n \mathbf{B}_{m-1-n}\Omega_n\hat{a}(n) + \sum_n \mathbf{D}_{m-n}\Omega_n\hat{a}(n). \tag{4.112}$$

The implementation in terms of these blocks is shown in figure 4.2.

Thus the received signal $z(t)$ is used as input to the forward filter. The output of the forward filter is sampled once every T seconds and the samples used as the positive input to the subtractor (main). The sequence $T^{-1}\Omega_n\hat{a}(n)$ is used as input to both backward filter (main) and backward filter (auxiliary). The output of the backward filter (main) is used as the negative input to the subtractor (main). The output of the subtractor (main) is used as input to the tapped delay line. The values at the delayed tap is used as the positive input to the subtractor (auxiliary). The output of the backward filter (auxiliary) is used as the negative input to the subtractor (auxiliary). The values at the non-delayed tap and the values at the output of the subtractor (auxiliary) are arranged into a column vector which is then multiplied by the $2K \times 2L$ matrix

$$\begin{bmatrix} \mathbf{C}_0 & \mathbf{C}_1 \\ \mathbf{0} & \mathbf{C}_0 \end{bmatrix}^H. \tag{4.113}$$

The resulting column vector is used to form a scalar product with the vector $\left[\gamma_1^*\Omega_{(m+1)}^H, \gamma_0^*\Omega_m^H\right]^T$; observe that the scalars γ_1, γ_0 are dependent on m. The resulting scalar is then quantized to the signal set to obtain $\hat{a}(n)$.

4.7 A Lower Bound on λ_m

When $K > 1$ and there is intersymbol interference, the decision-feedback zero-forcing equalizer derived in section 4.5 is, in general, suboptimal in the sense that $\lambda_m < \lambda_{mu}^0$. An accurate comparison of λ_m with λ^0_{mu} cannot be made without knowing λ_m. However, an approximate comparison can be made if a lower bound on λ_m is known. A lower bound can be found by replacing the constraint set $\mathcal{D}_m$ of the optimum formulation of the decision-feedback zero-forcing equalizer problem of section 4.4 with the weaker constraint set $\mathcal{D}^0_{ml}$ defined by equation 4.115. Thus solutions are sought for a family of constrained minimization problems indexed by integers, the m^{th} problem being to minimize $\int |f(t)|^2\,dt$ under the constraint $\mathcal{D}^0_{ml}$. The solution to the m^{th} problem shall be denoted by $f^0_{ml}(t)$, and the minimum so achieved shall be denoted by

$$\lambda^0_{ml} = \int |f^0_{ml}(t)|^2 dt. \tag{4.114}$$

Then λ^0_{ml} will be a lower bound[8] on λ_m. It turns out that the expression for λ^0_{ml} is similar to that of λ^0_{mu} obtained in section 4.5.3.

4.7.1 Constraint $\mathcal{D}^0_{ml}$

Denote by $\mathcal{D}^0_{ml}$ the following constraint on the function $f(t)$:

$$\sum_{k=1}^{K} e^{j\omega_k mT} \int f^*(t) h_k(t - mT)dt = 1. \tag{4.115}$$

Observe that $\mathcal{D}^0_{ml}$ is weaker than $\mathcal{D}_m$ in the sense that if a function $f(t)$ satisfies $\mathcal{D}_m$ then it satisfies $\mathcal{D}^0_{ml}$. Also, observe that $\mathcal{D}^0_{ml}$ is *not*, in general, a zero-forcing constraint.

4.7.2 Solution

An identical problem is solved in section 3.7. Thus the isomorph $\mathbf{c}(\omega) \in \mathcal{L}^2_{\mathbf{H}}$ of $f^0_{ml}(t)$ is given by

$$\mathbf{c}(\omega) = T\left[\Omega^H_m \mathbf{H}_0 \Omega_m\right]^{-1} e^{-j\omega mT}\Omega_m, \tag{4.116}$$

and

$$f^0_{ml}(t) = T\left[\Omega^H_m \mathbf{H}_0 \Omega_m\right]^{-1} \sum_{k=1}^{K} e^{j\omega_k mT} h_k(t - mT), \tag{4.117}$$

$$\lambda^0_{ml} = T\left[\Omega^H_m \mathbf{H}_0 \Omega_m\right]^{-1}. \tag{4.118}$$

[8]This is the reason for the subscript 'l' in $\mathcal{D}^0_{ml}$ etc.. The reason for the superscript '0' will become apparent later.

Observe that the expression for λ^0_{ml} has the same form as the expression for λ^0_{mu} given by equation 4.58. The expression for the isomorph of $f^0_{ml}(t)$ is also of the same form as the expression for the isomorph of $f^0_{mu}(t)$ given by equation 4.57.

4.8 A Generalized Lower Bound on λ_m

The lower bound on λ_m, derived in section 4.7, can be generalized by replacing the constraint $\mathcal{D}^0_{ml}$ with a generalized constraint set $\mathcal{D}^q_{ml}$ defined by equations 4.120 and 4.121. Thus solutions are sought for a family of constrained minimization problems indexed by integers, the m^{th} problem being to minimize $\int |f(t)|^2\,dt$ under the constraint set $\mathcal{D}^q_{ml}$. The solution to the m^{th} problem shall be denoted by $f^q_{ml}(t)$, and the minimum so achieved shall be denoted by

$$\lambda^q_{ml} = \int |f^q_{ml}(t)|^2 dt. \tag{4.119}$$

The functions $f^q_{ml}(t)$ are the approximate solutions suggested, in section 4.4.4, to the optimal formulation of the decision-feedback zero-forcing equalizer problem.

4.8.1 Constraint $\mathcal{D}^q_{ml}$

Let q be a non-negative integer. Denote by $\mathcal{D}^q_{ml}$ the following set of constraints on the function $f(t)$:

- for $0 < n - m \leq q$,

$$\sum_{k=1}^{K} e^{j\omega_k nT} \int f^*(t) h_k(t - nT)dt = 0, \tag{4.120}$$

-

$$\sum_{k=1}^{K} e^{j\omega_k mT} \int f^*(t) h_k(t - mT)dt = 1. \tag{4.121}$$

Observe that $\mathcal{D}^q_{ml}$ is weaker than $\mathcal{D}_m$ in the sense that if a function $f(t)$ satisfies $\mathcal{D}_m$ then it satisfies $\mathcal{D}^q_{ml}$. In the same sense, $\mathcal{D}^0_{ml}$ is weaker than $\mathcal{D}^q_{ml}$, and if $q_1 < q_2$ then $\mathcal{D}^{q_1}_{ml}$ is weaker than $\mathcal{D}^{q_2}_{ml}$. Therefore,

$$\lambda^0_{ml} \leq \lambda^{q_1}_{ml} \leq \lambda^{q_2}_{ml} \leq \lambda_m. \tag{4.122}$$

In other words, for increasing q, the quantities λ^q_{ml} constitute a hierarchy of tighter lower bounds on λ_m. This is the reason for the subscript 'l' and the superscript 'q.'

4.8.2 Solution

Observe that the problem is similar to that discussed in section 3.8. Thus the solution takes the form

$$f_{ml}^{q}(t) = \sum_{p=0}^{q} \alpha_p \sum_{k=1}^{K} e^{j\omega_k(m+p)T} h_k(t-(m+p)T). \tag{4.123}$$

where $\{\alpha_p : 0 \leq p \leq q\}$ is a set of scalars. In terms of its isomorph $\mathbf{c}(\omega) \in \mathcal{L}_{\mathbf{H}}^2$ given by

$$\mathbf{c}(\omega) = \sum_{p=0}^{q} \alpha_p \sum_{k=1}^{K} e^{-j\omega(m+p)T} \mathbf{\Omega}_{m+p}, \tag{4.124}$$

the quantity

$$\frac{1}{2\pi}\int_{-\pi/T}^{\pi/T} \mathbf{c}^H(\omega)\mathbf{H}(\omega)\mathbf{c}(\omega)d\omega = T^{-1}\sum_{p_1=0}^{q}\sum_{p_2=0}^{q} \alpha_{p_1}^{*}\alpha_{p_2}\mathbf{\Omega}_{m+p_1}^H \mathbf{H}_{p_1-p_2}\mathbf{\Omega}_{m+p_2} \tag{4.125}$$

must be minimized under the constraint

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\pi/T}^{\pi/T} \mathbf{c}^H(\omega)\mathbf{H}(\omega)\mathbf{\Omega}_{m+p_2}e^{-j\omega(m+p_2)T}d\omega &= T^{-1}\sum_{p_1=0}^{q}\alpha_{p_1}^{*}\mathbf{\Omega}_{m+p_1}^H\mathbf{H}_{p_1-p_2}\mathbf{\Omega}_{m+p_2}, \\
&= \begin{cases} 0 & \text{if} \quad 0 < p_2 \leq q, \\ 1 & \text{if} \quad p_2 = 0, \end{cases}
\end{aligned} \tag{4.126}$$

where

$$\mathbf{H}_n = \frac{T}{2\pi}\int_{-\pi/T}^{\pi/T} \mathbf{H}(\omega)e^{j\omega nT}d\omega. \tag{4.127}$$

The Case $q = 1$

The set of scalars $\{\alpha_p : p = 0, 1\}$ is that which minimizes

$$[\alpha_1^{*}, \alpha_0^{*}] \begin{bmatrix} \mathbf{\Omega}_{m+1}^H & \mathbf{0}^H \\ \mathbf{0}^H & \mathbf{\Omega}_m^H \end{bmatrix} \mathbf{H} \begin{bmatrix} \mathbf{\Omega}_{m+1} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Omega}_m \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_0 \end{bmatrix} \tag{4.128}$$

under the constraint

$$\begin{bmatrix} \mathbf{\Omega}_{m+1}^H & \mathbf{0}^H \\ \mathbf{0}^H & \mathbf{\Omega}_m^H \end{bmatrix} \mathbf{H} \begin{bmatrix} \mathbf{\Omega}_{m+1} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Omega}_m \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_0 \end{bmatrix} = \begin{bmatrix} 0 \\ T \end{bmatrix}, \tag{4.129}$$

where

$$\mathbf{H} = \begin{bmatrix} \mathbf{H}_0 & \mathbf{H}_1 \\ \mathbf{H}_{-1} & \mathbf{H}_0 \end{bmatrix}. \tag{4.130}$$

In terms of the set of scalars $\{\alpha_p : p = 0, 1\}$, the solution is given by

$$\lambda^1_{ml} = \alpha_0. \tag{4.131}$$

Observe that the expression for λ^1_{ml} has the same form as the expression for λ^1_{mu} given by equation 4.101. The expression for the isomorph of $f^1_{ml}(t)$ is also of the same form as the expression for the isomorph of $f^1_{mu}(t)$ given by equation 4.100.

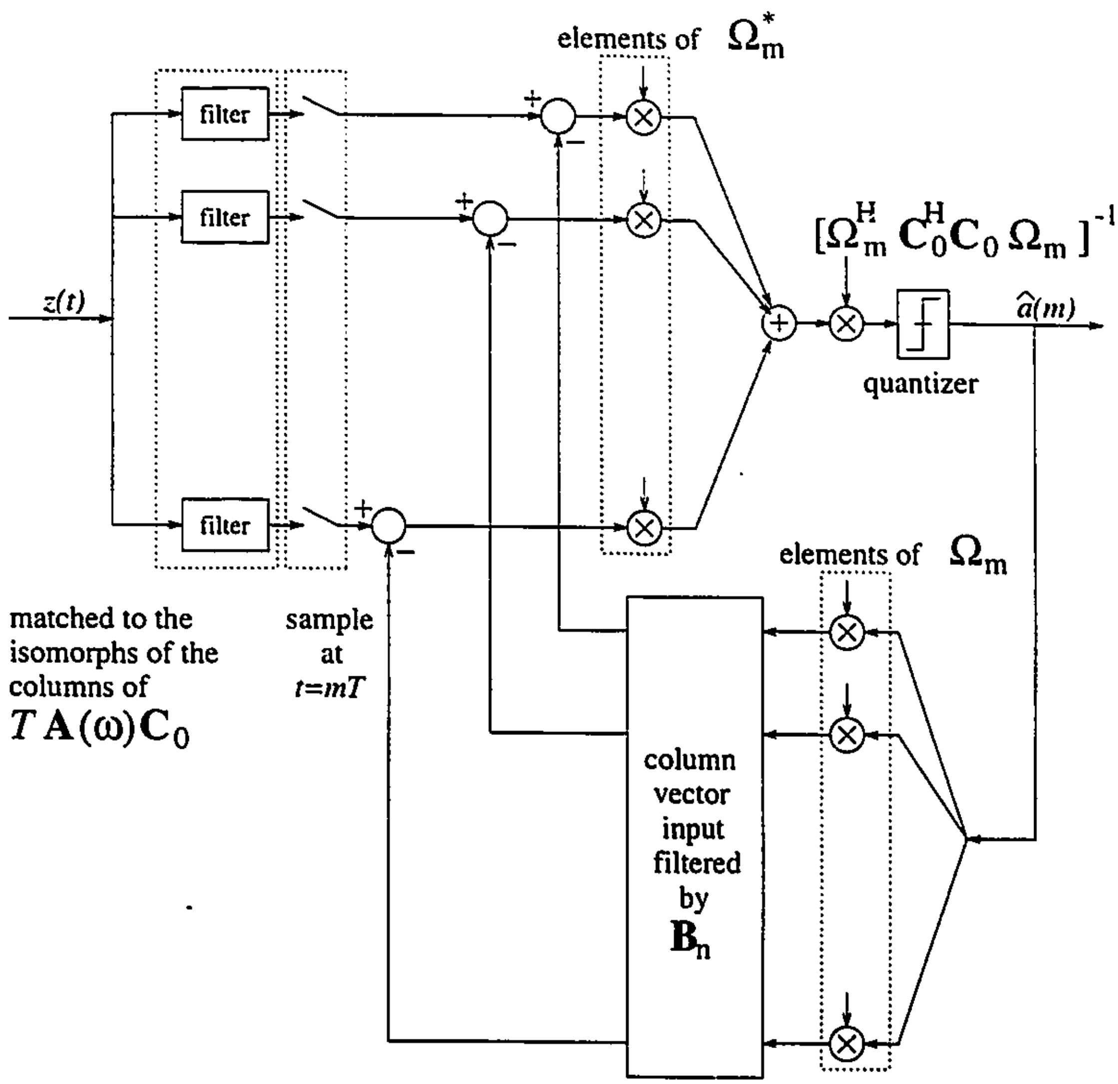

Figure 4.1: Schematic Diagram of the Decision-Feedback Zero-Forcing Equalizer for $q = 0$

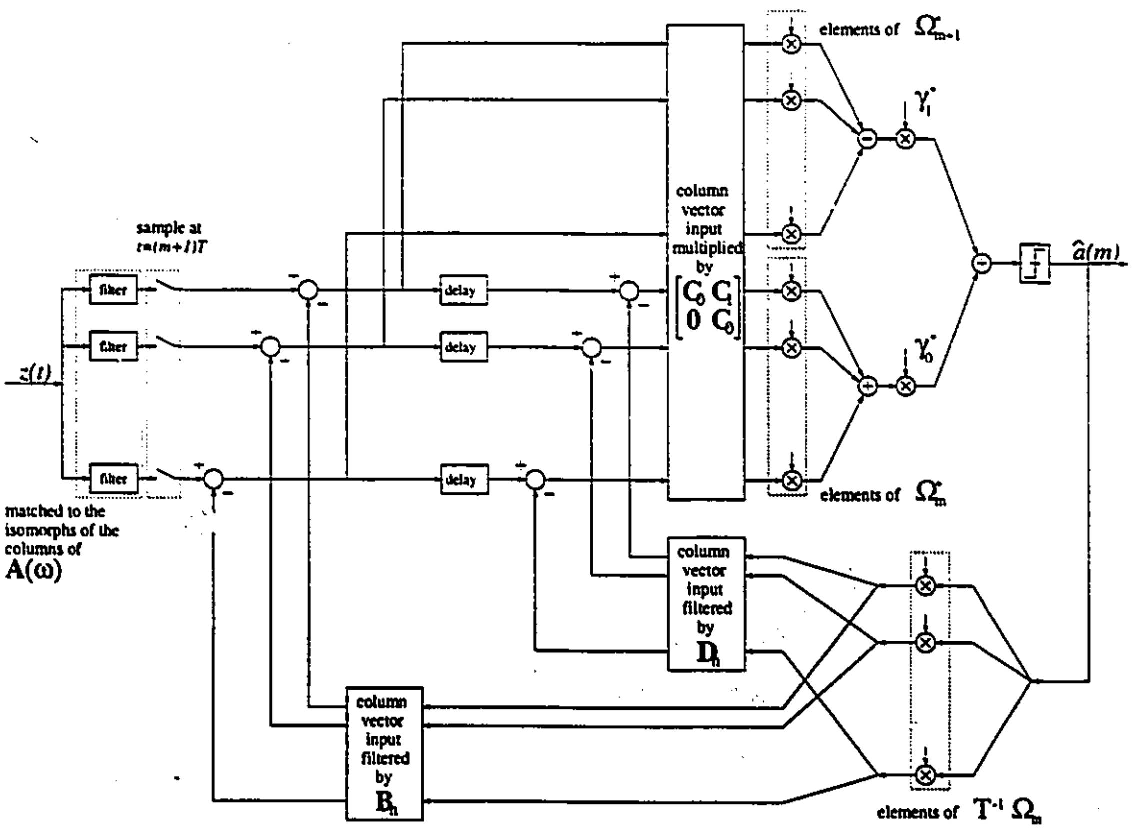

Figure 4.2: Schematic Diagram of the Decision-Feedback Zero-Forcing Equalizer for $q = 1$

Chapter 5

Linear Mean-Square-Error Equalizers

5.1 Introduction

A class of receivers for the composition of Quadrature Amplitude Modulation (QAM) and Discrete-Multipath Channel (DMC) is derived in this chapter. Receivers of this class shall be referred to as *linear mean-square-error equalizers* in view of the fact that they are extensions of the linear mean-square-error equalizer known for the composition of QAM and linear time-invariant channel with additive white Gaussian noise (AWGN). (For information on the latter, see, for example, [3]). The stationarity of the set of sequences $\{h_k(t-nT) : k = 1, 2, \ldots, K; \forall n\}$ and the consequent isomorphism between the Hilbert spaces $\mathcal{H}$ and $\mathcal{L}^2_{\mathbf{H}}$ as discussed in appendix A, section A.5 play a central role in the derivations. However, no further assumptions are made on the set of stationary sequences $\{h_k(t-nT) : k = 1, 2, \ldots, K; \forall n\}$. Issues of specification, implementation, and performance analysis of the receivers are addressed. Many of the concepts introduced in this chapter will be useful in chapter 6 on decision-feedback mean-square-error equalizers as well.

5.1.1 Some Preliminaries

The preliminaries given in chapter 3, section 3.1.1 are needed here as well. In addition to those, the data sequence $a(n)$ is considered a wide-sense-stationary uncorrelated random process with mean zero and variance $E\left[|a(n)|^2\right] = \mathcal{S}_0$.

5.1.2 Linear Equalizers – Their Definition and Specification

The definition of a linear equalizer given in chapter 3, section 3.2 is followed in this chapter. The discussion on specifying a linear equalizer given in chapter 3, section 3.2.1 is valid in this chapter as well. The criterion of optimality is, however, minimum mean-square-error as defined in section 5.2.

5.2 The Optimality Criterion – Minimum Mean-Square-Error

Suppose that, for some function $f(t)$, the correlation $\hat{a}_f = \int f^*(t)z(t)dt$ is used to obtain

$$\hat{a}(m) = \underset{a \in \mathcal{A}}{\arg\min} \; |a - \hat{a}_f| . \tag{5.1}$$

In considering $\hat{a}(m)$ as the m^{th} data symbol that was likely transmitted, both the noise and the data are potential causes of error. The probability of error may be reduced by choosing $f(t)$ so as to minimize the *mean-square-error*

$$\Lambda_m[f] = E\left[|\hat{a}_f - a(m)|^2\right], \tag{5.2}$$

where $E[.]$ denotes the expectation with respect to both data and noise. To obtain an expression for $\Lambda_m[f]$, observe that $(\hat{a}_f - a(m))$ can be decomposed as the sum of a data-dependent part and a noise-dependent part in the manner that $\hat{a}_f$ was decomposed as $\hat{a}_f = \hat{b}_f + \hat{c}_f$ in chapter 3, section 3.1.1. Thus the data-dependent part of $(\hat{a}_f - a(m))$ is a complex random variable given by

$$\hat{b}_f - a(m) = \sum_n a(n) \left(\sum_{k=1}^{K} e^{j\omega_k nT} \int f^*(t) h_k(t - nT)dt - \delta_m(n) \right), \tag{5.3}$$

where $\delta_m(n)$ is the Kronecker delta defined as

$$\delta_m(n) = \begin{cases} 0 & \text{if} \quad n \neq m, \\ 1 & \text{if} \quad n = m. \end{cases} \tag{5.4}$$

The data-dependent part has mean zero and variance

$$E\left[\left|\hat{b}_f - a(m)\right|^2\right] = \mathcal{S}_0 \sum_n \left| \sum_{k=1}^{K} e^{j\omega_k nT} \int f^*(t) h_k(t - nT)dt - \delta_m(n) \right|^2 . \tag{5.5}$$

The noise-dependent part of $(\hat{a}_f - a(m))$ is $\hat{c}(f) = \int f^*(t)\eta_w(t)dt$ which has mean zero and variance

$$E\left[|\hat{c}_f|^2\right] = \mathcal{N}_0 \int |f(t)|^2\, dt. \tag{5.6}$$

Since the data-dependent part and the noise-dependent part are independent, the mean-square-error is given by

$$\Lambda_m[f] = \mathcal{S}_0 \sum_n \left| \sum_{k=1}^{K} e^{j\omega_k nT} \int f^*(t) h_k(t - nT)dt - \delta_m(n) \right|^2 + \mathcal{N}_0 \int |f(t)|^2\, dt. \tag{5.7}$$

The derivation of a linear equalizer under the aforementioned optimality criterion – *minimum mean-square-error* – is attempted in the next section. It turns out there that this approach, although optimal, is not feasible from an implementation point of view. In the subsequent sections, suboptimal formulations of the problem, obtained by using upper bounds on the mean-square-error, will be considered. Henceforth, the term *linear mean-square-error equalizer* shall describe the solution to the optimal formulation as well the solution to a suboptimal formulation.

5.3 The Optimal Formulation of the Linear Mean-Square-Error Equalizer Problem

From the discussions of section 5.2 and chapter 3, section 3.2.1, it is evident that the problem of deriving a linear mean-square-error equalizer can be cast as a family of unconstrained minimization problems indexed by integers, the m^{th} problem being to minimize the cost $\Lambda_m[f]$ defined by equation 5.10. The solution to the m^{th} problem shall be denoted by $f_m(t)$, and the minimum so achieved shall be denoted by λ_m. Thus

$$f_m(t) = \underset{f}{\arg\min}\ \Lambda_m[f], \tag{5.8}$$

$$\lambda_m = \min_f\ \Lambda_m[f]. \tag{5.9}$$

5.3.1 Cost $\Lambda_m[f]$

Denote by $\Lambda_m[f]$ the following cost on the function $f(t)$:

$$\Lambda_m[f] = \mathcal{S}_0 \sum_n \left| \sum_{k=1}^{K} e^{j\omega_k nT} \int f^*(t) h_k(t-nT)dt - \delta_m(n) \right|^2 + \mathcal{N}_0 \int |f(t)|^2 \, dt. \tag{5.10}$$

5.3.2 A Consideration in the Hilbert Space $\mathcal{L}^2$

Recall that, by assumption, the functions $h_k(t) \in \mathcal{L}^2$ for $k = 1, 2, \ldots, K$, and observe that a function $f(t) \in \mathcal{L}^2$ is being sought. Since $\mathcal{L}^2$ is a Hilbert space under the inner product

$$(f, g) = \int f^*(t) g(t) dt, \tag{5.11}$$

the problem can be restated as follows: minimize

$$\Lambda_m[f] = \mathcal{S}_0 \sum_n |(f, h^n) - \delta_m(n)|^2 + \mathcal{N}_0 (f, f), \tag{5.12}$$

where

$$h^n(t) = \sum_{k=1}^{K} e^{j\omega_k nT} h_k(t-nT). \tag{5.13}$$

This restatement shows that the solution must belong to the subspace[1]

$$\mathcal{H}_\Sigma = \text{Clos.Span} \{h^n(t) : \forall n\}. \tag{5.14}$$

A proof of this simple fact is as follows: by the orthogonal projection theorem [47], an arbitrary $f(t) \in \mathcal{L}^2$ has a unique decomposition

$$f(t) = g_1(t) + g_2(t), \tag{5.15}$$

where $g_1(t) \in \mathcal{H}_\Sigma$ and $g_2(t) \in \mathcal{H}_\Sigma^\perp$ (the orthogonal complement of $\mathcal{H}_\Sigma$ in $\mathcal{L}^2$); since

$$(f, h^n) = (g_1, h^n) \tag{5.16}$$

for all n, and $(f, f) = (g_1, g_1) + (g_2, g_2)$, one obtains

$$\Lambda_m[f] = \Lambda_m[g_1] + \mathcal{N}_0 (g_2, g_2). \tag{5.17}$$

Therefore, if $f(t)$ is optimum then necessarily $g_2(t) = 0$, for otherwise $g_1(t)$ would be better than $f(t)$, thereby contradicting optimality.

[1] This is the smallest closed subspace containing the set $\{h^n(t) : \forall n\}$.

5.3.3 A Consideration in the Calculus of Variations

The conclusion of the previous section can be stated, heuristically, as follows: the optimum $f_m(t)$ has a series representation

$$f_m(t) = \sum_n \alpha_m(n) h^n(t). \tag{5.18}$$

This representation can also be directly inferred by using the calculus of variations to minimize $\Lambda_m[f]$; implicit definitions for the optimum sequence $\alpha_m(n)$ and the minimum λ_m can also be obtained as a result. Thus, for an arbitrary $g(t)$ and a real number ϵ, one has

$$\lim_{\epsilon\to 0} \frac{\partial\Lambda[f+\epsilon g]}{\partial\epsilon} = \mathcal{S}_0 \sum_n 2\Re\left[(f, h^n) - \delta_m(n)\right]^* (g, h^n) + \mathcal{N}_0 2\Re(g, f), \tag{5.19}$$

where $\Re(.)$ denotes the real part of the quantity within the parentheses. Replacing $g(t)$ with $jg(t)$, one obtains

$$\lim_{\epsilon\to 0} \frac{\partial\Lambda[f+j\epsilon g]}{\partial\epsilon} = \mathcal{S}_0 \sum_n 2\Im\left[(f, h^n) - \delta_m(n)\right]^* (g, h^n) + \mathcal{N}_0 2\Im(g, f), \tag{5.20}$$

where $\Im(.)$ denotes the imaginary part of the quantity within the parentheses. The optimum function $f_m(t)$ is that for which both of the above quantities are zero for every $g(t)$. Thus

$$\mathcal{S}_0 \sum_n \left[(f_m, h^n) - \delta_m(n)\right]^* h^n(t) + \mathcal{N}_0 f_m(t) = 0, \tag{5.21}$$

which is a series representation for $f_m(t)$ of the form given by equation 5.18. The optimum sequence $\alpha_m(n)$ must satisfy

$$\mathcal{S}_0 \left[\sum_{n'} \left(h^n, h^{n'}\right) \alpha_m(n') - \delta_m(n)\right] + \mathcal{N}_0 \alpha_m(n) = 0, \tag{5.22}$$

$$\sum_{n'} \left[\left(h^n, h^{n'}\right) + \frac{\mathcal{N}_0}{\mathcal{S}_0}\delta_n(n')\right] \alpha_m(n') = \delta_m(n), \tag{5.23}$$

where

$$\left(h^n, h^{n'}\right) = \sum_{k=1}^{K} \sum_{k'=1}^{K} e^{-j\omega_k nT} e^{j\omega_{k'} n'T} \int h_k^*(t-nT) h_{k'}(t-n'T)\,dt. \tag{5.24}$$

To obtain λ_m, observe from 5.21 that

$$\mathcal{S}_0 \sum_n \left[(f_m, h^n) - \delta_m(n)\right]^* (f_m, h^n) + \mathcal{N}_0 (f_m, f_m) = 0, \tag{5.25}$$

which, when compared with equation 5.12, gives

$$\begin{aligned}\lambda_m &= -\mathcal{S}_0 \sum_n [(f_m, h^n) - \delta_m(n)]^* \delta_m(n), &(5.26)\\ &= -\mathcal{S}_0 [(f_m, h^m) - 1]^*, &(5.27)\\ &= \mathcal{N}_0 \alpha_m(m). &(5.28)\end{aligned}$$

5.3.4 Special Cases

Finding $f_m(t)$ is straightforward in the trivial case where there is no intersymbol interference, that is, when $n_1 \neq n_2 \Rightarrow (h^{n_1}, h^{n_2}) = 0$, as defined in chapter 1, section 1.7.2. In this case,

$$\begin{aligned}f_m(t) &= h^m(t) / \left((h^m, h^m) + \frac{\mathcal{N}_0}{\mathcal{S}_0} \right), &(5.29)\\ \lambda_m &= \mathcal{N}_0 / \left((h^m, h^m) + \frac{\mathcal{N}_0}{\mathcal{S}_0} \right). &(5.30)\end{aligned}$$

In the case where $K = 1$, although the DMC is time-variant in general, the sequence of functions $\{h^n(t) : \forall n\}$ is stationary as shown in appendix A, section A.1.2. Therefore, even if there is intersymbol interference, the technique used for the composition of QAM and linear time-invariant channel can be used essentially without difference. Thus the isomorph $c(\omega) \in \mathcal{L}_H^2$ of $f_m(t)$ is given by

$$\begin{aligned}c(\omega) &= \frac{T e^{j(\omega_1 - \omega) m T}}{\left(H(\omega) + \frac{\mathcal{N}_0 T}{\mathcal{S}_0} \right)}, &(5.31)\\ \lambda_m &= \frac{T^2}{2\pi} \int_{-\pi/T}^{+\pi/T} \frac{\mathcal{N}_0}{\left(H(\omega) + \frac{\mathcal{N}_0 T}{\mathcal{S}_0} \right)} d\omega, &(5.32)\end{aligned}$$

where $H(\omega)$ is the spectral density function of the stationary sequence of functions $\{h_1(t - nT) : \forall n\}$.

5.3.5 The General Case

In the general case where $K > 1$ and there is intersymbol interference, it does not seem a trivial task to find the solution. An approximation of equation 5.23 obtained by restricting both n and n' to a finite set can be solved by matrix methods. The approximate solution so obtained will correspond to a lower bound on λ_m. In the next section, it is shown how, for roughly the same order of complexity, a function $f(t)$ that minimizes an upper bound on $\Lambda_m[f]$ can be found; the solution so obtained will correspond to an upper bound on λ_m.

5.4 A Suboptimal Formulation of the Linear Mean-Square-Error Equalizer Problem

The linear mean-square-error equalizer problem can be formulated in a suboptimal manner by replacing the cost $\Lambda_m[f]$ with its upperbound $\Lambda^0_{mu}[f,\alpha]$ defined by equations 5.36 and 5.37. Thus the suboptimal formulation of the linear mean-square-error equalizer problem is a family of constrained minimization problems indexed by integers, the m^{th} problem being to minimize $\Lambda^0_{mu}[f,\alpha]$ under the constraint given by equation 5.36. The solution to the m^{th} problem shall be denoted by $f^0_{mu}(t)$, and the minimum so achieved shall be denoted by λ^0_{mu}. Thus

$$f^0_{mu}(t) = \underset{f}{\arg\min} \; \min_{\alpha} \; \Lambda^0_{mu}[f,\alpha], \tag{5.33}$$

$$\lambda^0_{mu} = \min_{f,\alpha} \Lambda^0_{mu}[f,\alpha]. \tag{5.34}$$

Then λ^0_{mu} will be an upper bound[2] on both λ_m and $\Lambda_m[f^0_{mu}]$. More specifically,

$$\lambda_m \leq \Lambda_m[f^0_{mu}] \leq \lambda^0_{mu}. \tag{5.35}$$

The motivation for this suboptimal formulation is that the solution is attractive from the points of view of specification and implementation.

5.4.1 Cost $\Lambda^0_{mu}[f,\alpha]$

Let the set of scalars $\{\alpha_k : k = 1,2,\ldots,K\}$ satisfy

$$\sum_{k=1}^{K} \alpha_k = 1. \tag{5.36}$$

Denote by $\Lambda^0_{mu}[f,\alpha]$ the following cost on the function $f(t)$:

$$\Lambda^0_{mu}[f,\alpha] = KS_0 \sum_n \sum_{k=1}^{K} \left| \int f^*(t)h_k(t-nT)dt - \alpha_k e^{-j\omega_k mT}\delta_m(n)\right|^2 + \mathcal{N}_0 \int |f(t)|^2 dt. \tag{5.37}$$

[2]This is the reason for the subscript 'u' in Λ^0_{mu} etc.. The reason for the superscript '0' will become apparent later.

Then

$$\Lambda_m[f] \leq \Lambda^0_{mu}[f, \alpha]. \tag{5.38}$$

To prove this inequality, observe first that

$$\Lambda_m[f] = S_0 \sum_n \left| \sum_{k=1}^{K} \left(e^{j\omega_k nT} \int f^*(t) h_k(t - nT)dt - \alpha_k \delta_m(n) \right) \right|^2 + N_0 \int |f(t)|^2 \, dt, \tag{5.39}$$

and then by the Cauchy inequality that

$$\left| \sum_{k=1}^{K} \left(e^{j\omega_k nT} \int f^*(t) h_k(t - nT)dt - \alpha_k \delta_m(n) \right) \right|^2 \tag{5.40}$$

$$\leq K \sum_{k=1}^{K} \left| \int f^*(t) h_k(t - nT)dt - \alpha_k e^{-j\omega_k mT} \delta_m(n) \right|^2 \tag{5.41}$$

for all n. The conclusion is now obvious.

5.4.2 Solution

For the special case $K = 1$, the suboptimal formulation is equivalent to the optimal formulation, and therefore the corresponding result of section 5.3.4 is applicable.

In the general case, the problem is solved in two stages as follows. In the first stage, the cost $\Lambda^0_{mu}[f, \alpha]$ is minimized with respect to $f(t)$ for a fixed set of scalars $\{\alpha_k : k = 1, 2, \ldots, K\}$. Denote

$$f_\alpha(t) = \underset{f}{\arg\min} \; \Lambda^0_{mu}[f, \alpha], \tag{5.42}$$

$$\lambda_\alpha = \min_f \; \Lambda^0_{mu}[f, \alpha]. \tag{5.43}$$

In the second stage, λ_α is minimized under the constraint $\sum_{k=1}^{K} \alpha_k = 1$. Thus

$$\lambda^0_{mu} = \min_\alpha \; \lambda_\alpha, \tag{5.44}$$

$$f^0_{mu}(t) = f_\beta(t), \tag{5.45}$$

where the set of scalars $\{\beta_k : k = 1, 2, \ldots, K\}$ is given by

$$\beta = \underset{\alpha}{\arg\min} \; \lambda_\alpha. \tag{5.46}$$

An argument similar to that used in section 5.3.2 gives the following conclusion: the solution to the first stage must belong to the subspace

$$\mathcal{H} = \text{Clos.Span}\left\{h_k(t-nT) : k = 1,2,\ldots,K;\forall n\right\}. \tag{5.47}$$

The first stage is a special case of the generic problem treated in appendix B, where use is made of the isomorphism between the Hilbert spaces $\mathcal{H}$ and $\mathcal{L}^2_{\mathbf{H}}$ as discussed in appendix A, section A.5. In accordance with the discussion given in appendix B, denote

$$\mathbf{a} = \left[\alpha_1^* e^{j\omega_1 mT}, \alpha_2^* e^{j\omega_2 mT}, \ldots, \alpha_K^* e^{j\omega_K mT}\right]^T, \tag{5.48}$$

$$\mathbf{b}(\omega) = T\mathbf{a}e^{-j\omega mT}, \tag{5.49}$$

$$\mathbf{G}(\omega) = \left(\mathbf{H}(\omega) + \frac{\mathcal{N}_0 T}{K S_0}\mathbf{I}\right)^{-1}, \tag{5.50}$$

and

$$\mathbf{G}_0 = \frac{T}{2\pi}\int_{-\pi/T}^{\pi/T} \mathbf{G}(\omega)d\omega. \tag{5.51}$$

Then the isomorph $\mathbf{c}_\alpha(\omega) \in \mathcal{L}^2_{\mathbf{H}}$ of $f_\alpha(t)$ is given by

$$\mathbf{c}_\alpha(\omega) = Te^{-j\omega mT}\mathbf{G}(\omega)\mathbf{a}, \tag{5.52}$$

and

$$\lambda_\alpha = \frac{\mathcal{N}_0}{2\pi}\int_{-\pi/T}^{\pi/T} \mathbf{b}^H(\omega)\left(\mathbf{H}(\omega) + \frac{\mathcal{N}_0 T}{K S_0}\mathbf{I}\right)^{-1}\mathbf{b}(\omega)d\omega, \tag{5.53}$$

$$= \mathcal{N}_0 T\mathbf{a}^H\mathbf{G}_0\mathbf{a}. \tag{5.54}$$

The first stage is now solved. To solve the second stage, observe that the constraint $\sum_{k=1}^{K}\alpha_k = 1$ can be written as

$$\mathbf{a}^H\mathbf{\Omega}_m = 1, \tag{5.55}$$

where

$$\mathbf{\Omega}_m = \left[e^{j\omega_1 mT}, e^{j\omega_2 mT}, \ldots, e^{j\omega_K mT}\right]^T. \tag{5.56}$$

The solution to the second stage can be obtained using the method discussed in chapter 3, section 3.5.4. Thus the optimum a is given by[3]

$$\mathbf{a} = \left[\mathbf{\Omega}_m^H\mathbf{G}_0^{-1}\mathbf{\Omega}_m\right]^{-1}\mathbf{G}_0^{-1}\mathbf{\Omega}_m, \tag{5.57}$$

[3]The matrix $\mathbf{G}_0$ is positive definite

the isomorph $\mathbf{c}(\omega) \in \mathcal{L}^2_{\mathbf{H}}$ of $f^0_{mu}(t)$ is given by

$$\mathbf{c}(\omega) = T\left[\Omega_m^H \mathbf{G}_0^{-1}\Omega_m\right]^{-1} e^{-j\omega mT}\mathbf{G}(\omega)\mathbf{G}_0^{-1}\Omega_m, \tag{5.58}$$

and the minimized $\Lambda^0_{mu}[f,\alpha]$ is given by

$$\lambda^0_{mu} = \mathcal{N}_0 T\left[\Omega_m^H \mathbf{G}_0^{-1}\Omega_m\right]^{-1}. \tag{5.59}$$

Observe that, except for the constant $\mathcal{N}_0$, the above expression is similar to that given by equation 3.60 of chapter 3.

5.4.3 On Specifying and Implementing the Solution

The discussion given in chapter 3, section 3.5.5 is valid here with the appropriate definitions of $\mathbf{G}(\omega)$ and $\mathbf{G}_0$. Thus figure 3.2 of chapter 3 is also a schematic diagram of the implementation of the linear mean square error equalizer for $q = 0$.

5.5 Generalized Suboptimal Formulation of the Linear Mean-Square-Error Equalizer Problem

The suboptimal formulation of the linear mean-square-error equalizer problem of section 5.4 can be generalized by replacing the cost $\Lambda^0_{mu}[f,\alpha]$ with a generalized cost $\Lambda^q_{mu}[f,\alpha]$ defined by equations 5.63 and 5.64. Thus a generalized suboptimal formulation of the linear mean-square-error equalizer problem is a family of constrained minimization problems indexed by integers, the m^{th} problem being to minimize $\Lambda^q_{mu}[f,\alpha]$ under the constraint given by equation 5.63. The solution to the m^{th} problem shall be denoted by $f^q_{mu}(t)$, and the minimum so achieved shall be denoted by λ^q_{mu}. Thus

$$f^q_{mu}(t) \;=\; \underset{f}{\arg\min}\;\min_{\alpha}\;\Lambda^q_{mu}[f,\alpha], \tag{5.60}$$

$$\lambda^q_{mu} \;:=\; \min_{f,\alpha}\;\Lambda^q_{mu}[f,\alpha]. \tag{5.61}$$

Then λ^q_{mu} will be an upper bound on both λ_m and $\Lambda_m[f^q_{mu}]$. More specifically,

$$\lambda_m \le \Lambda_m[f^q_{mu}] \le \lambda^q_{mu}. \tag{5.62}$$

5.5.1 Cost $\Lambda^q_{mu}[f,\alpha]$

Let q be a non-negative integer and let the set of scalars $\{\alpha_{pk} : k = 1, 2, \ldots, K; |p| \leq q\}$ satisfy

$$\sum_{k=1}^{K} \alpha_{pk} = \begin{cases} 0 & \text{if} \quad 0 < |p| \leq q, \\ 1 & \text{if} \quad p = 0. \end{cases} \tag{5.63}$$

Denote by $\Lambda^q_{mu}[f,\alpha]$ the following cost on the function $f(t)$:

$$\Lambda^q_{mu}[f,\alpha] = K\mathcal{S}_0 \sum_n \sum_{k=1}^{K} \left| \int f^*(t) h_k(t-nT)dt - \sum_{|p|\leq q} \alpha_{pk} e^{-j\omega_k(m+p)T} \delta_{m+p}(n) \right|^2 + \mathcal{N}_0 \int |f(t)|^2 \, dt. \tag{5.64}$$

Then

$$\Lambda_m[f] \leq \Lambda^q_{mu}[f,\alpha]. \tag{5.65}$$

To prove this inequality, observe first that

$$\sum_{k=1}^{K} \sum_{|p|\leq q} \alpha_{pk} \delta_{m+p}(n) = \sum_{|p|\leq q} \delta_0(p) \delta_{m+p}(n) = \delta_m(n), \tag{5.66}$$

and therefore that

$$\Lambda_m[f] = \mathcal{S}_0 \sum_n \left| \sum_{k=1}^{K} \left(e^{j\omega_k nT} \int f^*(t) h_k(t-nT)dt - \sum_{|p|\leq q} \alpha_{pk} \delta_{m+p}(n) \right) \right|^2 + \mathcal{N}_0 \int |f(t)|^2 \, dt, \tag{5.67}$$

and by the Cauchy inequality that,

$$\left| \sum_{k=1}^{K} \left(e^{j\omega_k nT} \int f^*(t) h_k(t-nT)dt - \sum_{|p|\leq q} \alpha_{pk} \delta_{m+p}(n) \right) \right|^2 \tag{5.68}$$

$$\leq K \sum_{k=1}^{K} \left| \int f^*(t) h_k(t-nT)dt - e^{-j\omega_k nT} \sum_{|p|\leq q} \alpha_{pk} \delta_{m+p}(n) \right|^2, \tag{5.69}$$

$$= K \sum_{k=1}^{K} \left| \int f^*(t) h_k(t-nT)dt - \sum_{|p|\leq q} \alpha_{pk} e^{-j\omega_k(m+p)T} \delta_{m+p}(n) \right|^2 \tag{5.70}$$

for all n. The conclusion is now obvious.

Observe that if q_1 and q_2 are non-negative integers satisfying $q_1 < q_2$ then

$$\lambda_m \leq \lambda^{q_2}_{mu} \leq \lambda^{q_1}_{mu} \leq \lambda^0_{mu}. \tag{5.71}$$

In other words, for increasing q, the quantities λ^q_{mu} constitute a hierarchy of tighter upper bounds on λ_m. This is the reason for the subscript 'u' and the superscript 'q.' One may call q the degree of optimality.

5.5.2 Solution

In the manner of the discussion of section 5.4.2, the problem is solved in two stages as follows. In the first stage, $\Lambda^q_{mu}[f, \alpha]$ is minimized with respect to $f(t)$ for a fixed set of scalars $\{\alpha_{pk} : k = 1, 2, \ldots, K; |p| \leq q\}$. Denote

$$f_\alpha(t) = \underset{f}{\arg\min}\ \Lambda^q_{mu}[f, \alpha], \tag{5.72}$$

$$\lambda_\alpha = \min_f\ \Lambda^q_{mu}[f, \alpha]. \tag{5.73}$$

In the second stage, λ_α is minimized under the constraint given by equation 5.63. Thus

$$\lambda^q_{mu} = \min_\alpha\ \lambda_\alpha, \tag{5.74}$$

$$f^q_{mu}(t) = f_\beta(t), \tag{5.75}$$

where the set of scalars $\{\beta_{pk} : k = 1, 2, \ldots, K; |p| \leq q\}$ is given by

$$\beta = \underset{\alpha}{\arg\min}\ \lambda_\alpha. \tag{5.76}$$

An argument similar to that used in section 5.3.2 gives the following conclusion: the solution to the first stage must belong to the subspace

$$\mathcal{H} = \text{Clos.Span}\left\{h_k(t - nT) : k = 1, 2, \ldots, K; \forall n\right\}. \tag{5.77}$$

The first stage is a special case of the generic problem treated in appendix B, where use is made of the isomorphism between the Hilbert spaces $\mathcal{H}$ and $\mathcal{L}^2_{\mathbf{H}}$ as discussed in appendix A, section A.5. In accordance with the discussion given in appendix B, denote

$$\mathbf{a}_p = \left[\alpha^*_{p1} e^{j\omega_1 (m+p)T}, \alpha^*_{p2} e^{j\omega_2 (m+p)T}, \ldots, \alpha^*_{pK} e^{j\omega_K (m+p)T}\right]^T \tag{5.78}$$

for $|p| \leq q$,

$$\mathbf{b}(\omega) = Te^{-j\omega mT} \sum_{|p|\leq q} \mathbf{a}_p e^{-j\omega pT}, \tag{5.79}$$

$$\mathbf{G}(\omega) = \left(\mathbf{H}(\omega) + \frac{\mathcal{N}_0 T}{K\mathcal{S}_0}\mathbf{I}\right)^{-1}, \tag{5.80}$$

and

$$\mathbf{G}_n = \frac{T}{2\pi}\int_{-\pi/T}^{\pi/T} \mathbf{G}(\omega)e^{j\omega nT}d\omega. \tag{5.81}$$

Then the isomorph $\mathbf{c}_\alpha(\omega) \in \mathcal{L}^2_{\mathbf{H}}$ of $f_\alpha(t)$ is given by

$$\mathbf{c}_\alpha(\omega) = Te^{-j\omega mT}\mathbf{G}(\omega) \sum_{|p|\leq q} \mathbf{a}_p e^{-j\omega pT}, \tag{5.82}$$

and

$$\begin{aligned}
\lambda_\alpha &= \frac{\mathcal{N}_0}{2\pi}\int_{-\pi/T}^{\pi/T} \mathbf{b}^H(\omega)\left(\mathbf{H}(\omega) + \frac{\mathcal{N}_0 T}{K\mathcal{S}_0}\mathbf{I}\right)^{-1}\mathbf{b}(\omega)d\omega, && (5.83)\\
&= \sum_{|p_1|\leq q}\sum_{|p_2|\leq q} \frac{\mathcal{N}_0 T^2}{2\pi}\int_{-\pi/T}^{\pi/T} \mathbf{a}_{p_1}^H \mathbf{G}(\omega)\mathbf{a}_{p_2} e^{j\omega(p_1-p_2)T}d\omega, && (5.84)\\
&= \mathcal{N}_0 T \sum_{|p_1|\leq q}\sum_{|p_2|\leq q} \mathbf{a}_{p_1}^H \mathbf{G}_{p_1-p_2}\mathbf{a}_{p_2}. && (5.85)
\end{aligned}$$

The first stage is now solved. To solve the second stage, observe that the constraint given by equation 5.63 can be written as

$$\mathbf{a}_p^H \Omega_{m+p} = \begin{cases} 0 & \text{if } \; 0 < |p| \leq q, \\ 1 & \text{if } \; p = 0, \end{cases} \tag{5.86}$$

where

$$\Omega_{m+p} = \left[e^{j\omega_1(m+p)T}, e^{j\omega_2(m+p)T}, \ldots, e^{j\omega_K(m+p)T}\right]^T \tag{5.87}$$

for $|p| \leq q$.

The Case $q = 1$

Following the discussion given in chapter 3, section 3.6.2, denote

$$\mathbf{G} = \begin{bmatrix} \mathbf{G}_0 & \mathbf{G}_1 & \mathbf{G}_2 \\ \mathbf{G}_{-1} & \mathbf{G}_0 & \mathbf{G}_1 \\ \mathbf{G}_{-2} & \mathbf{G}_{-1} & \mathbf{G}_0 \end{bmatrix}, \tag{5.88}$$

and define[4]

$$\begin{bmatrix} \gamma_1 \\ \gamma_0 \\ \gamma_{-1} \end{bmatrix} = \left[\begin{bmatrix} \Omega_{m+1}^H & 0^H & 0^H \\ 0^H & \Omega_m^H & 0^H \\ 0^H & 0^H & \Omega_{m-1}^H \end{bmatrix} \mathbf{G}^{-1} \begin{bmatrix} \Omega_{m+1} & 0 & 0 \\ 0 & \Omega_m & 0 \\ 0 & 0 & \Omega_{m-1} \end{bmatrix} \right]^{-1} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}. \tag{5.89}$$

Thus in terms of the scalars $\gamma_1, \gamma_0, \gamma_{-1}$, the optimum set of vectors $\{\mathbf{a}_p : |p| \leq 1\}$ is given by

$$\begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_0 \\ \mathbf{a}_{-1} \end{bmatrix} = \mathbf{G}^{-1} \begin{bmatrix} \Omega_{m+1} & 0 & 0 \\ 0 & \Omega_m & 0 \\ 0 & 0 & \Omega_{m-1} \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_0 \\ \gamma_{-1} \end{bmatrix}, \tag{5.90}$$

the isomorph $\mathbf{c}(\omega) \in \mathcal{L}^2_{\mathbf{H}}$ of $f^1_{mu}(t)$ is given by

$$\mathbf{c}(\omega) = T\mathbf{G}(\omega) \left[e^{-j\omega(m+1)T}\mathbf{I}, e^{-j\omega mT}\mathbf{I}, e^{-j\omega(m-1)T}\mathbf{I} \right] \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_0 \\ \mathbf{a}_{-1} \end{bmatrix}, \tag{5.91}$$

where $\mathbf{I}$ denotes the $K \times K$ identity matrix, and the minimum cost is given by

$$\lambda^1_{mu} = \mathcal{N}_0 T \gamma_0. \tag{5.92}$$

5.5.3 On Specifying and Implementing the Solution

The discussion given in chapter 3, section 3.6.3 is valid here with the appropriate definitions of $\mathbf{G}(\omega)$ and $\mathbf{G}$ and the scalars $\gamma_1, \gamma_0, \gamma_{-1}$; the parameter T is being introduced separately here as opposed to being incorporated in the scalars $\gamma_1, \gamma_0, \gamma_{-1}$ there. Thus figure 3.3 of chapter 3 is also a schematic diagram of the implementation of the linear mean square error equalizer for $q = 1$.

5.6 A Lower Bound on λ_m

When $K > 1$ and there is intersymbol interference, the linear mean-square-error equalizer derived in section 5.4 is, in general, suboptimal in the sense that $\lambda_m < \lambda^0_{mu}$. An accurate comparison of λ_m with λ^0_{mu} cannot be made without knowing λ_m. However, an approximate

[4] The matrix $\mathbf{G}$ is positive definite.

comparison can be made if a lower bound on λ_m is known. A lower bound can be found by replacing the cost $\Lambda_m[f]$ of the optimum formulation of the linear mean-square-error equalizer problem of section 5.3 with its lower bound $\Lambda^0_{ml}[f]$ defined by equation 5.96. Thus solutions are sought for a family of minimization problems indexed by integers, the m^{th} problem being to minimize the cost $\Lambda^0_{ml}[f]$. The solution to the m^{th} problem shall be denoted by $f^0_{ml}(t)$, and the minimum so achieved shall be denoted by λ^0_{ml}. Thus

$$f^0_{ml}(t) = \underset{f}{\arg\min}\ \Lambda^0_{ml}[f], \tag{5.93}$$

$$\lambda^0_{ml} = \min_{f}\ \Lambda^0_{ml}[f]. \tag{5.94}$$

Then λ^0_{ml} will be a lower bound[5] on both λ_m and $\Lambda_m[f^0_{ml}]$. More specifically,

$$\lambda^0_{ml} \le \lambda_m \le \Lambda_m[f^0_{ml}]. \tag{5.95}$$

It turns out that the expression for λ^0_{ml} is similar to that of λ^0_{mu} obtained in section 5.4.2.

5.6.1 Cost $\Lambda^0_{ml}[f]$

Denote by $\Lambda^0_{ml}[f]$ the following cost on the function $f(t)$:

$$\Lambda^0_{ml}[f] = \mathcal{S}_0 \left| \sum_{k=1}^{K} e^{j\omega_k mT} \int f^*(t) h_k(t - mT)dt - 1 \right|^2 + \mathcal{N}_0 \int |f(t)|^2\, dt. \tag{5.96}$$

It is obvious that

$$\Lambda^0_{ml}[f] \le \Lambda_m[f]. \tag{5.97}$$

5.6.2 Solution

By an argument similar to that used in section 5.3.2, the solution $f^0_{ml}(t)$ takes the form

$$\alpha \sum_{k=1}^{K} e^{j\omega_k mT} h_k(t - mT) \tag{5.98}$$

[5]This is the reason for the subscript 'l' in Λ^0_{ml} etc.. The reason for the superscript '0' will become apparent later.

for some scalar α. Therefore, in terms of its isomorph $\mathbf{c}(\omega) \in \mathcal{L}^2_{\mathbf{H}}$ given by

$$\mathbf{c}(\omega) = \alpha e^{-j\omega mT}\boldsymbol{\Omega}_m, \tag{5.99}$$

the quantity

$$\Lambda^0_{ml}[f] = \mathcal{S}_0 \left|\alpha^* T^{-1}\beta - 1\right|^2 + \mathcal{N}_0 |\alpha|^2 T^{-1}\beta \tag{5.100}$$

must be minimized with respect to α; here the following denotions have been used:

$$\begin{aligned} \boldsymbol{\Omega}_m &= \left[e^{j\omega_1 mT}, e^{j\omega_2 mT}, \ldots, e^{j\omega_K mT}\right]^T, & (5.101)\\ \mathbf{H}_0 &= \frac{T}{2\pi}\int_{-\pi/T}^{\pi/T} \mathbf{H}(\omega)d\omega, & (5.102)\\ \beta &= \boldsymbol{\Omega}_m^H \mathbf{H}_0 \boldsymbol{\Omega}_m. & (5.103) \end{aligned}$$

Observing that $\Lambda^0_{ml}[f]$ is a quadratic function of α, completing the square gives

$$\Lambda^0_{ml}[f] = \mathcal{S}_0 \left[\left|\alpha\left(T^{-1}\beta\right)^{1/2}\left(T^{-1}\beta + \frac{\mathcal{N}_0}{\mathcal{S}_0}\right)^{1/2} - \left(T^{-1}\beta\right)^{1/2}\left(T^{-1}\beta + \frac{\mathcal{N}_0}{\mathcal{S}_0}\right)^{-1/2}\right|^2 \right. \tag{5.104}$$

$$\left. + \left[1 - \left(T^{-1}\beta\right)\left(T^{-1}\beta + \frac{\mathcal{N}_0}{\mathcal{S}_0}\right)^{-1}\right]\right]. \tag{5.105}$$

Therefore, the optimum α is given by

$$\alpha = \left(T^{-1}\beta + \frac{\mathcal{N}_0}{\mathcal{S}_0}\right)^{-1}, \tag{5.106}$$

and the minimum cost λ^0_{ml} is given by

$$\begin{aligned} \lambda^0_{ml} &= \mathcal{S}_0\left[1 - \left(T^{-1}\beta\right)\left(T^{-1}\beta + \frac{\mathcal{N}_0}{\mathcal{S}_0}\right)^{-1}\right], & (5.107)\\ &= \mathcal{N}_0\left(T^{-1}\beta + \frac{\mathcal{N}_0}{\mathcal{S}_0}\right)^{-1}. & (5.108) \end{aligned}$$

Observe that

$$\left(T^{-1}\beta + \frac{\mathcal{N}_0}{\mathcal{S}_0}\right) = T^{-1}\boldsymbol{\Omega}_m^H \mathbf{H}^0 \boldsymbol{\Omega}_m, \tag{5.109}$$

where

$$\mathbf{H}^0 = \mathbf{H}_0 + \frac{\mathcal{N}_0 T}{K\mathcal{S}_0}\mathbf{I}. \tag{5.110}$$

In terms of the matrix $\mathbf{H}^0$, the solutions are given by

$$\begin{aligned} \mathbf{c}(\omega) &= T\left[\Omega_m^H \mathbf{H}^0 \Omega_m\right]^{-1} e^{-j\omega m T}\Omega_m, & (5.111)\\ f_{ml}^0(t) &= T\left[\Omega_m^H \mathbf{H}^0 \Omega_m\right]^{-1} \sum_{k=1}^{K} e^{j\omega_k m T} h_k(t-mT), & (5.112)\\ \lambda_{ml}^0 &= \mathcal{N}_0 T\left[\Omega_m^H \mathbf{H}^0 \Omega_m\right]^{-1}. & (5.113)\end{aligned}$$

Observe that the expression for λ_{ml}^0 has the same form as the expression for λ_{mu}^0 given by equation 5.59. The expression for the isomorph of $f_{ml}^0(t)$ is also of the same form as the expression for the isomorph of $f_{mu}^0(t)$ given by equation 5.58.

5.7 A Generalized Lower Bound on λ_m

The lower bound on λ_m, derived in section 5.6, can be generalized by replacing the cost $\Lambda_{ml}^0[f]$ with a generalized cost $\Lambda_{ml}^q[f]$ defined by equation 5.116. Thus solutions are sought for a family of minimization problems indexed by integers, the m^{th} problem being to minimize the cost $\Lambda_{ml}^q[f]$. The solution to the m^{th} problem shall be denoted by $f_{ml}^q(t)$, and the minimum so achieved shall be denoted by λ_{ml}^q. Thus

$$f_{ml}^q(t) = \arg\min_f \Lambda_{ml}^q[f], \quad (5.114)$$

$$\lambda_{ml}^q = \min_f \Lambda_{ml}^q[f]. \quad (5.115)$$

The functions $f_{ml}^q(t)$ are the approximate solutions suggested in section 5.3.5 to the optimal formulation of the linear mean-square-error equalizer problem.

5.7.1 Cost $\Lambda_{ml}^q[f]$

Let q be a non-negative integer. Denote by $\Lambda_{ml}^q[f]$ the following cost on the function $f(t)$:

$$\Lambda_{ml}^q[f] = \mathcal{S}_0 \sum_{|n-m|\le q} \left|\sum_{k=1}^{K} e^{j\omega_k n T}\int f^*(t)h_k(t-nT)dt - \delta_m(n)\right|^2 + \mathcal{N}_0 \int |f(t)|^2\, dt. \quad (5.116)$$

The following are obvious:

- $$\Lambda^0_{ml}[f] \leq \Lambda^q_{ml}[f] \leq \Lambda_m[f]. \tag{5.117}$$

- if $q_1 < q_2$ then
$$\Lambda^{q_1}_{ml}[f] \leq \Lambda^{q_2}_{ml}[f]. \tag{5.118}$$

Therefore,

$$\lambda^0_{ml} \leq \lambda^{q_1}_{ml} \leq \lambda^{q_2}_{ml} \leq \lambda_m. \tag{5.119}$$

In other words, for increasing q, the quantities λ^q_{ml} constitute a hierarchy of tighter lower bounds on λ_m. This is the reason for the subscript 'l' and the superscript 'q.'

5.7.2 Solution

By an argument similar to that used in section 5.3.2, the solution $f^q_{ml}(t)$ takes the form

$$\sum_{|p|\leq q} \alpha_p \sum_{k=1}^{K} e^{j\omega_k(m+p)T} h_k(t-(m+p)T), \tag{5.120}$$

where $\{\alpha_p : |p| \leq q\}$ is a set of scalars. Therefore, in terms of its isomorph $\mathbf{c}(\omega) \in \mathcal{L}^2_{\mathbf{H}}$ given by

$$\mathbf{c}(\omega) = \sum_{|p|\leq q} \alpha_p e^{-j\omega(m+p)T} \boldsymbol{\Omega}_{m+p}, \tag{5.121}$$

the quantity

$$\Lambda^q_{ml}[f] = \mathcal{S}_0 \sum_{|p|\leq q} \left| T^{-1} \sum_{|p_1|\leq q} \alpha^*_{p_1} \beta_{p_1,p} - \delta_0(p) \right|^2 + \mathcal{N}_0 T^{-1} \sum_{|p_1|\leq q} \sum_{|p_2|\leq q} \alpha^*_{p_1} \alpha_{p_2} \beta_{p_1,p_2} \tag{5.122}$$

$$\doteq \mathcal{S}_0 \left[T^{-2} \sum_{|p|\leq q} \sum_{|p_1|\leq q} \sum_{|p_2|\leq q} \alpha^*_{p_1} \beta_{p_1,p} \beta^*_{p_2,p} \alpha_{p_2} - T^{-1} 2\Re \sum_{|p_1|\leq q} \alpha^*_{p_1} \beta_{p_1,0} + 1 \right] + \mathcal{N}_0 T^{-1} \sum_{|p_1|\leq q} \sum_{|p_2|\leq q} \alpha^*_{p_1} \alpha_{p_2} \beta_{p_1,p_2} \tag{5.123}$$

must be minimized with respect to the set of scalars $\{\alpha_p : |p| \leq q\}$; here the following denotions have been used:

$$\boldsymbol{\Omega}_{m+p} = \left[e^{j\omega_1(m+p)T}, e^{j\omega_2(m+p)T}, \ldots, e^{j\omega_K(m+p)T} \right]^T, \tag{5.124}$$

$$\mathbf{H}_p = \frac{T}{2\pi} \int_{-\pi/T}^{\pi/T} \mathbf{H}(\omega) e^{j\omega pT} d\omega, \tag{5.125}$$

$$\tag{5.126}$$

for $|p| \leq q$, and

$$\beta_{p_1,p_2} = \Omega^H_{m+p_1} \mathbf{H}_{p_1-p_2} \Omega_{m+p_2}, \tag{5.127}$$

for $|p_1|, |p_2| \leq q$.

The Case $q = 1$

Denoting

$$\begin{aligned} \mathbf{a} &= [a_1, a_0, a_{-1}]^T, & (5.128)\\ \mathbf{e} &= [0,1,0]^T, & (5.129)\\ \mathbf{B} &= \begin{bmatrix} \beta_{1,1} & \beta_{1,0} & \beta_{1,-1} \\ \beta_{0,1} & \beta_{0,0} & \beta_{0,-1} \\ \beta_{-1,1} & \beta_{-1,0} & \beta_{-1,-1} \end{bmatrix}, & (5.130) \end{aligned}$$

the quantity $\Lambda^1_{ml}[f]$ can be rewritten as

$$\begin{aligned} \Lambda^1_{ml}[f] &= \mathcal{S}_0 \left[T^{-2}\mathbf{a}^H\mathbf{B}\mathbf{B}\mathbf{a} - T^{-1}2\Re\mathbf{a}^H\mathbf{B}\mathbf{e} + 1\right] + \mathcal{N}_0 T^{-1}\mathbf{a}^H\mathbf{B}\mathbf{a}, & (5.131)\\ &= \mathcal{S}_0 \left[T^{-2}\mathbf{a}^H\left(\mathbf{B}\mathbf{B} + \frac{\mathcal{N}_0 T}{\mathcal{S}_0}\mathbf{B}\right)\mathbf{a} - T^{-1}2\Re\mathbf{a}^H\mathbf{B}\mathbf{e} + 1\right], & (5.132)\\ &= \mathcal{S}_0 \left[T^{-2}\mathbf{a}^H\mathbf{B}^{1/2}\left(\mathbf{B} + \frac{\mathcal{N}_0 T}{\mathcal{S}_0}\mathbf{I}\right)\mathbf{B}^{1/2}\mathbf{a} - T^{-1}2\Re\mathbf{a}^H\mathbf{B}\mathbf{e} + \mathbf{e}^H\mathbf{e}\right], & (5.133) \end{aligned}$$

where $\mathbf{B}^{1/2}$ is the positive square root of the Hermitian positive semidefinite matrix $\mathbf{B}$. Observing that $\Lambda^1_{ml}[f]$ is a quadratic function of $\mathbf{a}$, completing the square gives

$$\begin{aligned} \Lambda^1_{ml}[f] = \mathcal{S}_0 &\left[T^{-1}\left(\mathbf{B} + \frac{\mathcal{N}_0 T}{\mathcal{S}_0}\mathbf{I}\right)^{1/2}\mathbf{B}^{1/2}\mathbf{a} - \left(\mathbf{B} + \frac{\mathcal{N}_0 T}{\mathcal{S}_0}\mathbf{I}\right)^{-1/2}\mathbf{B}^{1/2}\mathbf{e}\right]^H \\ &\left[T^{-1}\left(\mathbf{B} + \frac{\mathcal{N}_0 T}{\mathcal{S}_0}\mathbf{I}\right)^{1/2}\mathbf{B}^{1/2}\mathbf{a} - \left(\mathbf{B} + \frac{\mathcal{N}_0 T}{\mathcal{S}_0}\mathbf{I}\right)^{-1/2}\mathbf{B}^{1/2}\mathbf{e}\right], \\ &+ \mathcal{S}_0\left[\mathbf{e}^H\mathbf{e} - \mathbf{e}^H\mathbf{B}^{1/2}\left(\mathbf{B} + \frac{\mathcal{N}_0 T}{\mathcal{S}_0}\mathbf{I}\right)^{-1}\mathbf{B}^{1/2}\mathbf{e}\right], \end{aligned} \tag{5.134}$$

where $\left(\mathbf{B} + \frac{\mathcal{N}_0 T}{\mathcal{S}_0}\mathbf{I}\right)^{1/2}$ and $\left(\mathbf{B} + \frac{\mathcal{N}_0 T}{\mathcal{S}_0}\mathbf{I}\right)^{-1/2}$ are the positive square roots of $\left(\mathbf{B} + \frac{\mathcal{N}_0 T}{\mathcal{S}_0}\mathbf{I}\right)$ and $\left(\mathbf{B} + \frac{\mathcal{N}_0 T}{\mathcal{S}_0}\mathbf{I}\right)^{-1}$ respectively. Therefore, the optimum $\mathbf{a}$ is given by

$$\begin{aligned} T^{-1}\left(\mathbf{B} + \frac{\mathcal{N}_0 T}{\mathcal{S}_0}\mathbf{I}\right)^{1/2}\mathbf{B}^{1/2}\mathbf{a} &= \left(\mathbf{B} + \frac{\mathcal{N}_0 T}{\mathcal{S}_0}\mathbf{I}\right)^{-1/2}\mathbf{B}^{1/2}\mathbf{e}, & (5.135)\\ T^{-1}\mathbf{B}\mathbf{a} &= \mathbf{B}^{1/2}\left(\mathbf{B} + \frac{\mathcal{N}_0 T}{\mathcal{S}_0}\mathbf{I}\right)^{-1}\mathbf{B}^{1/2}\mathbf{e}, & (5.136) \end{aligned}$$

and the minimum cost λ^1_{ml} is given by

$$\lambda^1_{ml} = \mathcal{S}_0 \mathbf{e}^H \left[\mathbf{I} - \mathbf{B}^{1/2}\left(\mathbf{B} + \frac{\mathcal{N}_0 T}{\mathcal{S}_0}\mathbf{I}\right)^{-1}\mathbf{B}^{1/2}\right]\mathbf{e}. \tag{5.137}$$

These results can be simplified using the matrix identities

$$\mathbf{B}^{1/2}\left(\mathbf{B} + \frac{\mathcal{N}_0 T}{\mathcal{S}_0}\mathbf{I}\right)^{-1}\mathbf{B}^{1/2} = \mathbf{B}\left(\mathbf{B} + \frac{\mathcal{N}_0 T}{\mathcal{S}_0}\mathbf{I}\right)^{-1}, \tag{5.138}$$

$$\mathbf{I} - \mathbf{B}^{1/2}\left(\mathbf{B} + \frac{\mathcal{N}_0 T}{\mathcal{S}_0}\mathbf{I}\right)^{-1}\mathbf{B}^{1/2} = \frac{\mathcal{N}_0 T}{\mathcal{S}_0}\left(\mathbf{B} + \frac{\mathcal{N}_0 T}{\mathcal{S}_0}\mathbf{I}\right)^{-1}. \tag{5.139}$$

Thus, the optimum $\mathbf{a}$ is given by

$$\mathbf{a} = T\left(\mathbf{B} + \frac{\mathcal{N}_0 T}{\mathcal{S}_0}\mathbf{I}\right)^{-1}\mathbf{e}, \tag{5.140}$$

and the minimum cost λ^1_{ml} is given by

$$\lambda^1_{ml} = \mathcal{N}_0 T \mathbf{e}^H \left(\mathbf{B} + \frac{\mathcal{N}_0 T}{\mathcal{S}_0}\mathbf{I}\right)^{-1}\mathbf{e}. \tag{5.141}$$

Observe that

$$\mathbf{B} + \frac{\mathcal{N}_0 T}{\mathcal{S}_0}\mathbf{I} = \begin{bmatrix} \Omega^H_{m+1} & \mathbf{0}^H & \mathbf{0}^H \\ \mathbf{0}^H & \Omega^H_m & \mathbf{0}^H \\ \mathbf{0}^H & \mathbf{0}^H & \Omega^H_{m-1} \end{bmatrix} \mathbf{H} \begin{bmatrix} \Omega_{m+1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \Omega_m & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \Omega_{m-1} \end{bmatrix}, \tag{5.142}$$

where $\mathbf{H}$ denotes the matrix

$$\mathbf{H} = \begin{bmatrix} \mathbf{H}^0 & \mathbf{H}_1 & \mathbf{H}_2 \\ \mathbf{H}_{-1} & \mathbf{H}^0 & \mathbf{H}_1 \\ \mathbf{H}_{-2} & \mathbf{H}_{-1} & \mathbf{H}^0 \end{bmatrix}, \tag{5.143}$$

where

$$\mathbf{H}^0 = \mathbf{H}_0 + \frac{\mathcal{N}_0 T}{K\mathcal{S}_0}\mathbf{I}, \tag{5.144}$$

and

$$\mathbf{H}_p = \frac{T}{2\pi}\int_{-\pi/T}^{\pi/T} \mathbf{H}(\omega) e^{j\omega p T} d\omega. \tag{5.145}$$

Therefore,

$$\begin{bmatrix} \alpha_1 \\ \alpha_0 \\ \alpha_{-1} \end{bmatrix} = T\left[\begin{bmatrix} \Omega^H_{m+1} & \mathbf{0}^H & \mathbf{0}^H \\ \mathbf{0}^H & \Omega^H_m & \mathbf{0}^H \\ \mathbf{0}^H & \mathbf{0}^H & \Omega^H_{m-1} \end{bmatrix} \mathbf{H} \begin{bmatrix} \Omega_{m+1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \Omega_m & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \Omega_{m-1} \end{bmatrix}\right]^{-1} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \tag{5.146}$$

and

$$\lambda^1_{ml} = \lambda_0 \alpha_0. \tag{5.147}$$

Observe that the expression for λ^1_{ml} has the same form as the expression for λ^1_{mu} given by equation 5.92. The expression for the isomorph of $f^1_{ml}(t)$ is also of the same form as the expression for the isomorph of $f^1_{mu}(t)$ given by equation 5.91.

Chapter 6

Decision-Feedback Mean-Square-Error Equalizers

6.1 Introduction

A class of receivers for the composition of Quadrature Amplitude Modulation (QAM) and Discrete-Multipath Channel (DMC) is derived in this chapter. Receivers of this class shall be referred to as *decision-feedback mean-square-error equalizers* in view of the fact that they are extensions of the decision-feedback mean-square-error equalizer known for the composition of QAM and linear time-invariant channel with additive white Gaussian noise (AWGN). (For information on the latter, see, for example, [3] and [29]). The stationarity of the set of sequences $\{h_k(t-nT) : k = 1, 2, \ldots, K; \forall n\}$ and the consequent isomorphism between the Hilbert spaces $\mathcal{H}$ and $\mathcal{L}^2_{\mathbf{H}}$ as discussed in appendix A, section A.5 play a central role in the derivations. However, no further assumptions are made on the set of stationary sequences $\{h_k(t-nT) : k = 1, 2, \ldots, K; \forall n\}$. Many of the concepts introduced in chapter 5 on linear mean-square-error equalizers will be useful in this chapter as well. Issues of specification, implementation, and performance analysis of the receivers are addressed.

6.1.1 Some Preliminaries

The preliminaries given in chapter 3, section 3.1.1 are needed here as well. In addition to those, the data sequence $a(n)$ is considered a wide-sense-stationary uncorrelated random process with mean zero and variance $E\left[|a(n)|^2\right] = \mathcal{S}_0$.

6.1.2 Decision-Feedback Equalizers – Their Definition and Specification

The definition of a decision-feedback equalizer given in chapter 4, section 4.2 is followed in this chapter. The criterion of optimality is, however, minimum mean-square-error as defined in section 6.2. The simplifying assumption made in chapter 4, section 4.2.1, and the discussion on specifying a decision-feedback equalizer given in chapter 4, section 4.2.2 are valid in this chapter as well.

6.2 The Optimality Criterion – Minimum Mean-Square-Error

Suppose that, for some pair of functions $(f(t), \beta(n))$, the random variable

$$\hat{a}_{f,\beta} = \int f^*(t)z(t)dt - \sum_{n<m} \beta(n)a(n) \tag{6.1}$$

is used to obtain

$$\hat{a}(m) = \underset{a \in \mathcal{A}}{\arg\min} \; |a - \hat{a}_{f,\beta}| \,. \tag{6.2}$$

In considering $\hat{a}(m)$ as the m^{th} data symbol that was likely transmitted, both the noise and the data are potential causes of error. The probability of error may be reduced by choosing $f(t)$ and $\beta(n)$ so as to minimize the *mean-square-error*

$$E\left[|\hat{a}_{f,\beta} - a(m)|^2\right], \tag{6.3}$$

where $E[.]$ denotes the expectation with respect to both data and noise. To obtain an expression for the mean-square-error, observe that $(\hat{a}_{f,\beta} - a(m))$ can be decomposed as the sum of a data-dependent part and a noise-dependent part in the manner that $\hat{a}_{f,\beta}$ was decomposed as $\hat{a}_{f,\beta} = \hat{b}_{f,\beta} + \hat{c}_f$ in chapter 4, section 4.2.1. Thus the data-dependent part of $(\hat{a}_{f,\beta} - a(m))$ is a complex random variable given by

$$\begin{aligned}
\hat{b}_{f,\beta} - a(m) &= \sum_{n\geq m} a(n)\left(\sum_{k=1}^{K} e^{j\omega_k nT}\int f^*(t)h_k(t-nT)dt - \delta_m(n)\right) \\
&\quad - \sum_{n<m} a(n)\left[\sum_{k=1}^{K} e^{j\omega_k nT}\int f^*(t)h_k(t-nT)dt - \beta(n)\right],
\end{aligned} \tag{6.4}$$

where $\delta_m(n)$ is the Kronecker delta defined as

$$\delta_m(n) = \begin{cases} 0 & \text{if} \quad n \neq m, \\ 1 & \text{if} \quad n = m. \end{cases} \tag{6.5}$$

The data-dependent part has mean zero and variance

$$\begin{aligned} E\left[\left|\hat{b}_{f,\beta} - a(m)\right|^2\right] &= \mathcal{S}_0 \sum_{n \geq m} \left| \sum_{k=1}^{K} e^{j\omega_k nT} \int f^*(t) h_k(t - nT) dt - \delta_m(n) \right|^2 \\ &\quad + \mathcal{S}_0 \sum_{n<m} \left| \sum_{k=1}^{K} e^{j\omega_k nT} \int f^*(t) h_k(t - nT) dt - \beta(n) \right|^2 . \end{aligned} \tag{6.6}$$

The noise-dependent part of $(\hat{a}_{f,\beta} - a(m))$ is $\hat{c}(f) = \int f^*(t)\eta_w(t)dt$ which has mean zero and variance

$$E\left[|\hat{c}_f|^2\right] = \mathcal{N}_0 \int |f(t)|^2 \, dt. \tag{6.7}$$

Since the data-dependent part and the noise-dependent part are independent, the mean-square-error is given by

$$\begin{aligned} E\left[|\hat{a}_{f,\beta} - a(m)|^2\right] &= \mathcal{S}_0 \sum_{n \geq m} \left| \sum_{k=1}^{K} e^{j\omega_k nT} \int f^*(t) h_k(t - nT) dt - \delta_m(n) \right|^2 \\ &\quad + \mathcal{S}_0 \sum_{n<m} \left| \sum_{k=1}^{K} e^{j\omega_k nT} \int f^*(t) h_k(t - nT) dt - \beta(n) \right|^2 \\ &\quad + \mathcal{N}_0 \int |f(t)|^2 \, dt. \end{aligned} \tag{6.8}$$

Observe that, for a fixed function $f(t)$, the mean-square-error is minimized by the function

$$\beta(n) = \sum_{k=1}^{K} e^{j\omega_k nT} \int f^*(t) h_k(t - nT) dt \tag{6.9}$$

for $n < m$, and the minimum so achieved is given by

$$\Gamma_m[f] = \mathcal{S}_0 \sum_{n \geq m} \left| \sum_{k=1}^{K} e^{j\omega_k nT} \int f^*(t) h_k(t - nT) dt - \delta_m(n) \right|^2 + \mathcal{N}_0 \int |f(t)|^2 \, dt. \tag{6.10}$$

The derivation of a decision-feedback equalizer under the aforementioned optimality criterion – *minimum mean-square-error* – is attempted in the next section. It turns out there that this approach, although optimal, is not feasible from an implementation point of view. In the subsequent sections, suboptimal formulations of the problem, obtained by using upper bounds on the mean-square-error, will be considered. Henceforth, the term *decision-feedback mean-square-error equalizer* shall describe the solution to the optimal formulation as well the solution to a suboptimal formulation.

6.3 The Optimal Formulation of the Decision-Feedback Mean-Square-Error Equalizer Problem

From the discussions of section 6.2 and chapter 4, section 4.2, it is evident that the problem of deriving a decision-feedback mean-square-error equalizer can be cast as a family of unconstrained minimization problems indexed by integers, the m^{th} problem being to minimize the cost $\Gamma_m[f]$ defined by equation 6.13. The solution to the m^{th} problem shall be denoted by $f_m(t)$, and the minimum so achieved shall be denoted by λ_m. Thus

$$f_m(t) = \underset{f}{\arg\min}\ \Gamma_m[f], \tag{6.11}$$

$$\lambda_m = \underset{f}{\min}\ \Gamma_m[f]. \tag{6.12}$$

6.3.1 Cost $\Gamma_m[f]$

Denote by $\Gamma_m[f]$ the following cost on the function $f(t)$:

$$\Gamma_m[f] = \mathcal{S}_0 \sum_{n\geq m} \left| \sum_{k=1}^{K} e^{j\omega_k nT} \int f^*(t) h_k(t-nT)dt - \delta_m(n) \right|^2 + \mathcal{N}_0 \int |f(t)|^2\, dt. \tag{6.13}$$

6.3.2 A Consideration in the Hilbert Space $\mathcal{L}^2$

By an argument similar to that used in section 5.3.2, the solution $f_m(t)$ must belong to the subspace[1]

$$\mathcal{H}_\Sigma = \text{Clos.Span}\left\{h^n(t) : n \geq m\right\}. \tag{6.14}$$

Therefore, heuristically,

$$f_m(t) = \sum_{n\geq m} \alpha_m(n) h^n(t), \tag{6.15}$$

[1]This is the smallest closed subspace containing the set $\{h^n(t) : n \geq m\}$.

for some sequence $\alpha_m(n)$.

6.3.3 A Consideration in the Calculus of Variations

In the manner of the discussion of chapter 5, section 5.3.3, the optimum solution $f_m(t)$ must satisfy

$$\mathcal{S}_0 \sum_{n \geq m} \left[(f_m, h^n) - \delta_m(n) \right]^* h^n(t) + \mathcal{N}_0 f_m(t) = 0. \tag{6.16}$$

Therefore the sequence $\alpha_m(n)$ must satisfy

$$\sum_{n' \geq m} \left[\left(h^n, h^{n'} \right) + \frac{\mathcal{N}_0}{\mathcal{S}_0} \delta_n(n') \right] \alpha_m(n') = \delta_m(n), \tag{6.17}$$

for $n \geq m$, where

$$\left(h^n, h^{n'} \right) = \sum_{k=1}^{K} \sum_{k'=1}^{K} e^{-j\omega_k nT} e^{j\omega_{k'} n' T} \int h_k^*(t - nT) h_{k'}(t - n') dt. \tag{6.18}$$

Also

$$\lambda_m = \mathcal{N}_0 \alpha_m(m). \tag{6.19}$$

6.3.4 Special Cases

In the trivial case where there is no intersymbol interference, the decision-feedback mean-square-error equalizer is the same as the linear mean-square-error equalizer. Thus $f_m(t) = h^m(t) / \left((h^m, h^m) + \mathcal{N}_0/\mathcal{S}_0 \right)$ and $\lambda_m = \mathcal{N}_0 / \left((h^m, h^m) + \mathcal{N}_0/\mathcal{S}_0 \right)$ as observed in chapter 5, section 5.3.4, and $\beta_m(n) = 0$ for $n \geq m$.

In the case where $K = 1$, although the DMC is time-variant in general, the sequence of functions $\{h^n(t) : \forall n\}$ is stationary as shown in appendix A, section A.1.2. Therefore, even if there is intersymbol interference, the technique used for the composition of QAM and linear time-invariant channel can be used essentially without difference. Thus the decision-feedback mean-square-error equalizer always exists, for there exist, as observed in appendix A, section A.8.1, causal functions $C(\omega)$ that satisfy

$$H(\omega) + \mathcal{N}_0 T / \mathcal{S}_0 = |C(\omega)|^2, \tag{6.20}$$

where $H(\omega)$ is the spectral density function of the stationary sequence of functions $\{h_1(t-nT) : \forall n\}$. In terms of a maximal causal function $C(\omega)$, the isomorph $c(\omega) \in \mathcal{L}^2_H$ of $f_m(t)$ is given by

$$c(\omega) = Te^{j(\omega_1-\omega)mT}/(C_0^* C(\omega)), \tag{6.21}$$

$$\lambda_m = T\mathcal{N}_0 |C_0|^{-2}, \tag{6.22}$$

and

$$\beta_m(n) = e^{j\omega_1(n-m)T} C_{m-n}/C_0, \tag{6.23}$$

where

$$C_n = \frac{T}{2\pi}\int_{-\pi/T}^{\pi/T} C(\omega)e^{j\omega nT}d\omega. \tag{6.24}$$

Observe that, since $C(\omega) \in \mathcal{L}^2[-\pi/T, \pi/T]$, the condition $\sum_{n<m} |\beta(n)|^2 < \infty$ is satisfied.

6.3.5 The General Case

In the general case where $K > 1$ and there is intersymbol interference, it does not seem a trivial task to find the solution. An approximation of equation 6.17 obtained by restricting both n and n' to a finite set can be solved by matrix methods. The approximate solution so obtained will correspond to a lower bound on λ_m. In the next section, it is shown how, for roughly the same order of complexity, a function $f(t)$ that minimizes an upper bound on $\Gamma_m[f]$ can be found; the solution so obtained will correspond to an upper bound on λ_m.

6.4 A Suboptimal Formulation of the Decision-Feedback Mean-Square-Error Equalizer Problem

The decision-feedback mean-square-error equalizer problem can be formulated in a suboptimal manner by replacing the cost $\Gamma_m[f]$ with its upperbound $\Gamma^0_{mu}[f,\alpha]$ defined by equations 6.28 and 6.29. Thus the suboptimal formulation of the decision-feedback mean-square-error equalizer problem is a family of constrained minimization problems indexed by integers, the m^{th} problem being to minimize $\Gamma^0_{mu}[f,\alpha]$ under the constraint given by equation 6.28. The solution to the m^{th} problem shall be denoted by $f^0_{mu}(t)$, and the minimum so achieved shall be denoted by λ^0_{mu}. Thus

$$f^0_{mu}(t) = \underset{f}{\arg\min}\ \min_{\alpha}\ \Gamma^0_{mu}[f,\alpha], \tag{6.25}$$

$$\lambda^0_{mu} = \min_{f,\alpha} \Gamma^0_{mu}[f,\alpha]. \tag{6.26}$$

Then λ^0_{mu} will be an upper bound[2] on both λ_m and $\Gamma_m[f^0_{mu}]$. More specifically,

$$\lambda_m \le \Gamma_m[f^0_{mu}] \le \lambda^0_{mu}. \tag{6.27}$$

The motivation for this suboptimal formulation is that the solution is attractive from the points of view of specification and implementation.

6.4.1 Cost $\Gamma^0_{mu}[f,\alpha]$

Let the set of scalars $\{\alpha_k : k = 1,2,\ldots,K\}$ satisfy

$$\sum_{k=1}^{K} \alpha_k = 1. \tag{6.28}$$

Denote by $\Gamma^0_{mu}[f,\alpha]$ the following cost on the function $f(t)$:

$$\Gamma^0_{mu}[f,\alpha] = KS_0 \sum_{n\ge m} \sum_{k=1}^{K} \left| \int f^*(t) h_k(t-nT)dt - \alpha_k e^{-j\omega_k mT} \delta_m(n) \right|^2 + \mathcal{N}_0 \int |f(t)|^2\, dt. \tag{6.29}$$

Then

$$\Gamma_m[f] \le \Gamma^0_{mu}[f,\alpha]. \tag{6.30}$$

The proof is similar to that given in section 5.4.1 for $\Lambda_m[f] \le \Lambda^0_{mu}[f,\alpha]$.

6.4.2 Solution

For the special case $K = 1$, the suboptimal formulation is equivalent to the optimal formulation, and therefore the corresponding result of section 6.3.4 is applicable.

In the general case, the problem is solved in two stages as follows. In the first stage, the cost $\Gamma^0_{mu}[f,\alpha]$ is minimized with respect to $f(t)$ for a fixed set of scalars $\{\alpha_k : k = 1,2,\ldots,K\}$. Denote

$$f_\alpha(t) = \arg\min_{f} \Gamma^0_{mu}[f,\alpha], \tag{6.31}$$

[2]This is the reason for the subscript 'u' in Γ^0_{mu} etc.. The reason for the superscript '0' will become apparent later.

$$\lambda_\alpha = \min_f \Gamma^0_{mu}[f,\alpha]. \quad (6.32)$$

In the second stage, λ_α is minimized under the constraint $\sum_{k=1}^{K} \alpha_k = 1$. Thus

$$\lambda^0_{mu} = \min_\alpha \lambda_\alpha, \quad (6.33)$$

$$f^0_{mu}(t) = f_\beta(t), \quad (6.34)$$

where the set of scalars $\{\beta_k : k = 1, 2, \ldots, K\}$ is given by

$$\beta = \arg\min_\alpha \lambda_\alpha. \quad (6.35)$$

An argument similar to that used in chapter 5, section 5.3.2 gives the following conclusion: the solution to the first stage must belong to the subspace

$$\mathcal{H} = \text{Clos.Span}\{h_k(t - nT) : k = 1, 2, \ldots, K; \forall n\}. \quad (6.36)$$

The first stage is a special case of the generic problem treated in appendix C, where use is made of the isomorphism between the Hilbert spaces $\mathcal{H}$ and $\mathcal{L}^2_{\mathbf{H}}$ as discussed in appendix A, section A.5. In accordance with the discussion given in appendix C, denote

$$\mathbf{b} = T\left[\alpha_1^* e^{j\omega_1 mT}, \alpha_2^* e^{j\omega_2 mT}, \ldots, \alpha_K^* e^{j\omega_K mT}\right]^T. \quad (6.37)$$

Let $\mathbf{C}(\omega)$ be a maximal causal matrix satisfying

$$\left(\mathbf{H}(\omega) + \frac{N_0 T}{K S_0}\mathbf{I}\right) = \mathbf{C}^H(\omega)\mathbf{C}(\omega), \quad (6.38)$$

and let

$$\mathbf{C}(\omega)\mathbf{A}(\omega) = \mathbf{I}. \quad (6.39)$$

Then the isomorph $\mathbf{c}_\alpha(\omega) \in \mathcal{L}^2_{\mathbf{H}}$ of $f_\alpha(t)$ is given by

$$\mathbf{c}_\alpha(\omega) = e^{-j\omega mT}\mathbf{A}(\omega)\mathbf{d}, \quad (6.40)$$

and

$$\lambda_\alpha = N_0 T^{-1}\mathbf{d}^H\mathbf{d}, \quad (6.41)$$

where

$$\mathbf{d} = \mathbf{A}_0^H \mathbf{b}, \tag{6.42}$$

or, equivalently,

$$\mathbf{b} = \mathbf{C}_0^H \mathbf{d}, \tag{6.43}$$

where

$$\mathbf{C}_0 = \frac{T}{2\pi} \int_{-\pi/T}^{\pi/T} \mathbf{C}(\omega) d\omega, \tag{6.44}$$

$$\mathbf{A}_0 = \frac{T}{2\pi} \int_{-\pi/T}^{\pi/T} \mathbf{A}(\omega) d\omega. \tag{6.45}$$

The first stage is now solved. To solve the second stage, observe that the constraint $\sum_{k=1}^{K} \alpha_k = 1$ can be written as

$$\mathbf{b}^H \mathbf{\Omega}_m = T, \tag{6.46}$$

where

$$\mathbf{\Omega}_m = \left[e^{j\omega_1 mT}, e^{j\omega_2 mT}, \ldots, e^{j\omega_K mT}\right]^T, \tag{6.47}$$

or, equivalently, as

$$\mathbf{d}^H \mathbf{C}_0 \mathbf{\Omega}_m = T. \tag{6.48}$$

The solution to the second stage can be obtained using the method discussed in chapter 3, section 3.5.4. Thus

$$\begin{aligned} \mathbf{d} &= T\left[\mathbf{\Omega}_m^H \mathbf{C}_0^H \mathbf{C}_0 \mathbf{\Omega}_m\right]^{-1} \mathbf{C}_0 \mathbf{\Omega}_m, & (6.49)\\ \mathbf{c}(\omega) &= T\left[\mathbf{\Omega}_m^H \mathbf{C}_0^H \mathbf{C}_0 \mathbf{\Omega}_m\right]^{-1} e^{-j\omega mT} \mathbf{A}(\omega) \mathbf{C}_0 \mathbf{\Omega}_m, & (6.50)\\ \lambda_{mu}^0 &= \mathcal{N}_0 T\left[\mathbf{\Omega}_m^H \mathbf{C}_0^H \mathbf{C}_0 \mathbf{\Omega}_m\right]^{-1}. & (6.51) \end{aligned}$$

6.4.3 The Optimum Function $\beta_m(n)$

The optimum function $\beta_m(n)$, defined by

$$\beta_m(n) = \sum_{k=1}^{K} e^{j\omega_k nT} \int f_{mu}^{0\,*}(t) h_k(t - nT) dt, \tag{6.52}$$

for $n < m$, can be derived as was done in chapter 4, section 4.5.4. Thus

$$\beta_m(n) = \left[\mathbf{\Omega}_m^H \mathbf{C}_0^H \mathbf{C}_0 \mathbf{\Omega}_m\right]^{-1} \mathbf{\Omega}_m^H \mathbf{C}_0^H \mathbf{C}_{m-n} \mathbf{\Omega}_n. \tag{6.53}$$

The argument given in chapter 4, section 4.5.4 for $\sum_{n<m} |\beta_m(n)|^2 < \infty$ is valid here as well.

6.4.4 On Specifying and Implementing the Solution

The discussion given in section 4.5.6 is valid here with the appropriate definitions of $\mathbf{C}(\omega)$ and $\mathbf{A}(\omega)$. Thus figure 4.1 of chapter 4 is also a schematic diagram of the implementation of the decision-feedback mean-square-error equalizer for $q = 0$.

6.5 Generalized Suboptimal Formulation of the Decision-Feedback Mean-Square-Error Equalizer Problem

The suboptimal formulation of the decision-feedback mean-square-error equalizer problem of section 6.4 can be generalized by replacing the cost $\Gamma^0_{mu}[f,\alpha]$ with a generalized cost $\Gamma^q_{mu}[f,\alpha]$ defined by equations 6.57 and 6.58. Thus a generalized suboptimal formulation of the decision-feedback mean-square-error equalizer problem is a family of constrained minimization problems indexed by integers, the m^{th} problem being to minimize $\Gamma^q_{mu}[f,\alpha]$ under the constraint given by equation 6.57. The solution to the m^{th} problem shall be denoted by $f^q_{mu}(t)$, and the minimum so achieved shall be denoted by λ^q_{mu}. Thus

$$f^q_{mu}(t) = \underset{f}{\arg\min} \; \underset{\alpha}{\min} \; \Gamma^q_{mu}[f,\alpha], \tag{6.54}$$

$$\lambda^q_{mu} = \underset{f,\alpha}{\min} \; \Gamma^q_{mu}[f,\alpha]. \tag{6.55}$$

Then λ^q_{mu} will be an upper bound on both λ_m and $\Gamma_m[f^q_{mu}]$. More specifically,

$$\lambda_m \le \Gamma_m[f^q_{mu}] \le \lambda^q_{mu}. \tag{6.56}$$

6.5.1 Cost $\Gamma^q_{mu}[f,\alpha]$

Let q be a non-negative integer and let the set of scalars $\{\alpha_{pk} : k = 1,2,\ldots,K; 0 \le p \le q\}$ satisfy

$$\sum_{k=1}^{K} \alpha_{pk} = \begin{cases} 0 & \text{if } \; 0 < p \le q, \\ 1 & \text{if } \; p = 0. \end{cases} \tag{6.57}$$

Denote by $\Gamma^q_{mu}[f,\alpha]$ the following cost on the function $f(t)$:

$$\Gamma^q_{mu}[f,\alpha] = KS_0 \sum_{n\geq m} \sum_{k=1}^{K} \left| \int f^*(t) h_k(t-nT)dt - \sum_{p=0}^{q} \alpha_{pk} e^{-j\omega_k(m+p)T} \delta_{m+p}(n) \right|^2 + N_0 \int |f(t)|^2 dt. \tag{6.58}$$

Then

$$\Gamma_m[f] \leq \Gamma^q_{mu}[f,\alpha]. \tag{6.59}$$

The proof is the same as that given in chapter 5, section 5.5.1 for $\Lambda_m[f] \leq \Lambda^q_{mu}[f,\alpha]$.

Observe that if q_1 and q_2 are non-negative integers satisfying $q_1 < q_2$ then

$$\lambda_m \leq \lambda^{q_2}_{mu} \leq \lambda^{q_1}_{mu} \leq \lambda^0_{mu}. \tag{6.60}$$

In other words, for increasing q, the quantities λ^q_{mu} constitute a hierarchy of tighter upper bounds on λ_m. This is the reason for the subscript 'u' and the superscript 'q.' One may call q the degree of optimality.

6.5.2 Solution

In the manner of the discussion of section 6.4.2, the problem is solved in two stages as follows. In the first stage, $\Gamma^q_{mu}[f,\alpha]$ is minimized with respect to $f(t)$ for a fixed set of scalars $\{\alpha_{pk} : k = 1,2,\ldots,K; 0 \leq p \leq q\}$. Denote

$$f_\alpha(t) = \underset{f}{\arg\min}\; \Gamma^q_{mu}[f,\alpha], \tag{6.61}$$

$$\lambda_\alpha = \min_f \Gamma^q_{mu}[f,\alpha]. \tag{6.62}$$

In the second stage, λ_α is minimized under the constraint given by equation 6.57. Thus

$$\lambda^q_{mu} = \min_\alpha \lambda_\alpha, \tag{6.63}$$

$$f^q_{mu}(t) = f_\beta(t), \tag{6.64}$$

where the set of scalars $\{\beta_{pk} : k = 1,2,\ldots,K; 0 \leq p \leq q\}$ is given by

$$\beta = \underset{\alpha}{\arg\min}\; \lambda_\alpha. \tag{6.65}$$

An argument similar to that used in chapter 5, section 5.3.2 gives the following conclusion: the solution to the first stage must belong to the subspace

$$\mathcal{H} = \text{Clos.Span}\left\{h_k(t - nT) : k = 1, 2, \ldots, K; \forall n\right\}. \tag{6.66}$$

The first stage is a special case of the generic problem treated in appendix C, where use is made of the isomorphism between the Hilbert spaces $\mathcal{H}$ and $\mathcal{L}^2_{\mathbf{H}}$ as discussed in appendix A, section A.5. In accordance with the discussion given in appendix C, denote

$$\mathbf{b}_p = T\left[\alpha^*_{p1}e^{j\omega_1(m+p)T}, \alpha^*_{p2}e^{j\omega_2(m+p)T}, \ldots, \alpha^*_{pK}e^{j\omega_K(m+p)T}\right]^T \tag{6.67}$$

for $0 \leq p \leq q$, and

$$\mathbf{b}(\omega) = \sum_{p=0}^{q} \mathbf{b}_p e^{-j\omega pT}. \tag{6.68}$$

Let $\mathbf{C}(\omega)$ be a maximal causal matrix satisfying

$$\left(\mathbf{H}(\omega) + \frac{\mathcal{N}_0 T}{K\mathcal{S}_0}\mathbf{I}\right) = \mathbf{C}^H(\omega)\mathbf{C}(\omega), \tag{6.69}$$

and let

$$\mathbf{C}(\omega)\mathbf{A}(\omega) = \mathbf{I}. \tag{6.70}$$

Then the isomorph $\mathbf{c}_\alpha(\omega) \in \mathcal{L}^2_{\mathbf{H}}$ of $f_\alpha(t)$ is given by

$$\mathbf{c}_\alpha(\omega) = e^{-j\omega mT}\mathbf{A}(\omega)\sum_{p=0}^{q} \mathbf{d}_p e^{-j\omega pT}, \tag{6.71}$$

and

$$\lambda_\alpha = \mathcal{N}_0 T^{-1}\sum_{p=0}^{q} \mathbf{d}_p^H \mathbf{d}_p, \tag{6.72}$$

where

$$\mathbf{d}_p = \sum_{p'=p}^{q} \mathbf{A}^H_{p'-p}\mathbf{b}_{p'}, \tag{6.73}$$

or, equivalently,

$$\mathbf{b}_p = \sum_{p'=p}^{q} \mathbf{C}^H_{p'-p}\mathbf{d}_{p'}, \tag{6.74}$$

where

$$\mathbf{C}_n = \frac{T}{2\pi}\int_{-\pi/T}^{\pi/T} \mathbf{C}(\omega)e^{j\omega nT}d\omega, \tag{6.75}$$

$$\mathbf{A}_n = \frac{T}{2\pi}\int_{-\pi/T}^{\pi/T} \mathbf{A}(\omega)e^{j\omega nT}d\omega. \tag{6.76}$$

The first stage is now solved. To solve the second stage, observe that the constraint given by equation 6.57 can be written as

$$\mathbf{b}_p^H \boldsymbol{\Omega}_{m+p} = \begin{cases} 0 & \text{if} \quad 0 < p \le q, \\ T & \text{if} \quad p = 0, \end{cases} \tag{6.77}$$

or, equivalently, as

$$\sum_{p'=p}^{q} \mathbf{d}_{p'}^H \mathbf{C}_{p'-p} \boldsymbol{\Omega}_{m+p} = \begin{cases} 0 & \text{if} \quad 0 < p \le q, \\ T & \text{if} \quad p = 0, \end{cases} \tag{6.78}$$

where

$$\boldsymbol{\Omega}_{m+p} = \left[e^{j\omega_1 (m+p)T}, e^{j\omega_2 (m+p)T}, \ldots, e^{j\omega_K (m+p)T}\right]^T. \tag{6.79}$$

The Case $q = 1$

The set of complex vector-valued constants $\{\mathbf{b}_p : p = 0, 1\}$ is that which minimizes

$$\left[\mathbf{d}_1^H, \mathbf{d}_0^H\right] \begin{bmatrix} \mathbf{d}_1 \\ \mathbf{d}_0 \end{bmatrix} \tag{6.80}$$

under the constraint

$$\left[\mathbf{d}_1^H, \mathbf{d}_0^H\right] \begin{bmatrix} \mathbf{C}_0 & \mathbf{C}_1 \\ \mathbf{0} & \mathbf{C}_0 \end{bmatrix} \begin{bmatrix} \boldsymbol{\Omega}_{m+1} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Omega}_m \end{bmatrix} = [0, T]. \tag{6.81}$$

Following the discussion in chapter 4, section 4.6.2, denote

$$\begin{bmatrix} \gamma_1 \\ \gamma_0 \end{bmatrix} = T \left[\begin{bmatrix} \boldsymbol{\Omega}_{m+1}^H & \mathbf{0}^H \\ \mathbf{0}^H & \boldsymbol{\Omega}_m^H \end{bmatrix} \begin{bmatrix} \mathbf{C}_0^H & \mathbf{0} \\ \mathbf{C}_1^H & \mathbf{C}_0^H \end{bmatrix} \begin{bmatrix} \mathbf{C}_0 & \mathbf{C}_1 \\ \mathbf{0} & \mathbf{C}_0 \end{bmatrix} \begin{bmatrix} \boldsymbol{\Omega}_{m+1} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Omega}_m \end{bmatrix} \right]^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{6.82}$$

The solution can now be given in terms of the scalars γ_0 and γ_1 as follows.

$$\begin{bmatrix} \mathbf{d}_1 \\ \mathbf{d}_0 \end{bmatrix} = \begin{bmatrix} \mathbf{C}_0 & \mathbf{C}_1 \\ \mathbf{0} & \mathbf{C}_0 \end{bmatrix} \begin{bmatrix} \boldsymbol{\Omega}_{m+1} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Omega}_m \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_0 \end{bmatrix}, \tag{6.83}$$

$$\mathbf{c}(\omega) = \mathbf{A}(\omega) \left[e^{-j\omega(m+1)T}\mathbf{I}, e^{-j\omega mT}\mathbf{I}\right] \begin{bmatrix} \mathbf{d}_1 \\ \mathbf{d}_0 \end{bmatrix}, \tag{6.84}$$

$$\lambda_{mu}^1 = \mathcal{N}_0 \gamma_0. \tag{6.85}$$

6.5.3 The Optimum Function $\beta_m(n)$

The optimum function $\beta_m(n)$, defined by

$$\beta_m(n) = \sum_{k=1}^{K} e^{j\omega_k nT} \int f_{mu}^{1*}(t) h_k(t - nT) dt, \tag{6.86}$$

for $n < m$, can be derived as was done in chapter 4, sections 4.5.3 and 4.6.3. Thus

$$\beta_m(n) = T^{-1} \sum_{p=0}^{q} \mathbf{d}_p^H \mathbf{C}_{p-(n-m)} \mathbf{\Omega}_n. \tag{6.87}$$

The Case $q = 1$

$$\beta_m(n) \quad = \quad T^{-1} \quad \left[\mathbf{d}_1^H, \mathbf{d}_0^H\right] \quad \begin{bmatrix} \mathbf{C}_{m+1-n}\mathbf{\Omega}_n \\ \mathbf{C}_{m-n}\mathbf{\Omega}_n \end{bmatrix}. \tag{6.88}$$

6.5.4 On Specifying and Implementing the Solution for $q = 1$

The discussion given in section 4.6.4 is valid here with the appropriate definitions of $\mathbf{C}(\omega)$ and $\mathbf{A}(\omega)$, and with L being equal to K. Thus figure 4.2 of chapter 4 is also a schematic diagram of the implementation of the decision-feedback mean-square-error equalizer for $q = 1$.

6.6 A Lower Bound on λ_m

When $K > 1$ and there is intersymbol interference, the decision-feedback mean-square-error equalizer derived in section 6.4 is, in general, suboptimal in the sense that $\lambda_m < \lambda_{mu}^0$. An accurate comparison of λ_m with λ_{mu}^0 cannot be made without knowing λ_m. However, an approximate comparison can be made if a lower bound on λ_m is known. A lower bound can be found by replacing the cost $\Gamma_m[f]$ of the optimum formulation of the decision-feedback mean-square-error equalizer problem of section 6.3 with its lower bound $\Gamma_{ml}^0[f]$ defined by equation 6.92. Thus solutions are sought for a family of minimization problems indexed by integers, the m^{th} problem being to minimize the cost $\Gamma_{ml}^0[f]$. The solution to the m^{th} problem shall be denoted by $f_{ml}^0(t)$, and the minimum so achieved shall be denoted by λ_{ml}^0. Thus

$$f_{ml}^0(t) \quad = \quad \underset{f}{\arg\min} \; \Gamma_{ml}^0[f], \tag{6.89}$$

$$\lambda^0_{ml} = \min_f \Gamma^0_{ml}[f]. \tag{6.90}$$

Then λ^0_{ml} will be a lower bound[3] on both λ_m and $\Gamma_m[f^0_{ml}]$. More specifically,

$$\lambda^0_{ml} \leq \lambda_m \leq \Gamma_m[f^0_{ml}]. \tag{6.91}$$

It turns out that the expression for λ^0_{ml} is similar to that of λ^0_{mu} obtained in section 6.4.2.

6.6.1 Cost $\Gamma^0_{ml}[f]$

Denote by $\Gamma^0_{ml}[f]$ the following cost on the function $f(t)$:

$$\Gamma^0_{ml}[f] = \mathcal{S}_0 \left| \sum_{k=1}^{K} e^{j\omega_k mT} \int f^*(t) h_k(t - mT)dt - 1 \right|^2 + \mathcal{N}_0 \int |f(t)|^2 \, dt. \tag{6.92}$$

It is obvious that

$$\Gamma^0_{ml}[f] \leq \Gamma_m[f]. \tag{6.93}$$

6.6.2 Solution

An identical problem is solved in chapter 5, section 5.6. Thus the isomorph $\mathbf{c}(\omega) \in \mathcal{L}^2_{\mathbf{H}}$ of $f^0_{ml}(t)$ is given by

$$\mathbf{c}(\omega) = T \left[\Omega^H_m \mathbf{H}^0 \Omega_m \right]^{-1} e^{-j\omega mT} \Omega_m, \tag{6.94}$$

and

$$f^0_{ml}(t) = T \left[\Omega^H_m \mathbf{H}^0 \Omega_m \right]^{-1} \sum_{k=1}^{K} e^{j\omega_k mT} h_k(t - mT), \tag{6.95}$$

$$\lambda^0_{ml} = \mathcal{N}_0 T \left[\Omega^H_m \mathbf{H}^0 \Omega_m \right]^{-1}, \tag{6.96}$$

where

$$\mathbf{H}^0 = \mathbf{H}_0 + \frac{\mathcal{N}_0 T}{K \mathcal{S}_0} \mathbf{I}. \tag{6.97}$$

Observe that the expression for λ^0_{ml} has the same form as the expression for λ^0_{mu} given by equation 6.51. The expression for the isomorph of $f^0_{ml}(t)$ is also of the same form as the expression for the isomorph of $f^0_{mu}(t)$ given by equation 6.50.

[3] This is the reason for the subscript 'l' in Γ^0_{ml} etc.. The reason for the superscript '0' will become apparent later.

6.7 A Generalized Lower Bound on λ_m

The lower bound on λ_m, derived in section 6.6, can be generalized by replacing the cost $\Gamma^0_{ml}[f]$ with a generalized cost $\Gamma^q_{ml}[f]$ defined by equation 6.100. Thus solutions are sought for a family of minimization problems indexed by integers, the m^{th} problem being to minimize the cost $\Gamma^q_{ml}[f]$. The solution to the m^{th} problem shall be denoted by $f^q_{ml}(t)$, and the minimum so achieved shall be denoted by λ^q_{ml}. Thus

$$f^q_{ml}(t) = \underset{f}{\arg\min}\; \Gamma^q_{ml}[f]. \tag{6.98}$$

$$\lambda^q_{ml} = \min_f \Gamma^q_{ml}[f]. \tag{6.99}$$

The functions $f^q_{ml}(t)$ are the approximate solutions suggested in section 6.3.5 to the optimal formulation of the decision-feedback mean-square-error equalizer problem.

6.7.1 Cost $\Gamma^q_{ml}[f]$

Let q be a non-negative integer. Denote by $\Gamma^q_{ml}[f]$ the following cost on the function $f(t)$:

$$\Gamma^q_{ml}[f] = S_0 \sum_{(n-m)=0}^{q} \left| \sum_{k=1}^{K} e^{j\omega_k nT} \int f^*(t) h_k(t-nT)\,dt - \delta_m(n) \right|^2 + \mathcal{N}_0 \int |f(t)|^2\,dt. \tag{6.100}$$

The following are obvious:

- $$\Gamma^0_{ml}[f] \le \Gamma^q_{ml}[f] \le \Gamma_m[f]. \tag{6.101}$$

- if $q_1 < q_2$ then
$$\Gamma^{q_1}_{ml}[f] \le \Gamma^{q_2}_{ml}[f]. \tag{6.102}$$

Therefore,

$$\lambda^0_{ml} \le \lambda^{q_1}_{ml} \le \lambda^{q_2}_{ml} \le \lambda_m. \tag{6.103}$$

In other words, for increasing q, the quantities λ^q_{ml} constitute a hierarchy of tighter lower bounds on λ_m. This is the reason for the subscript 'l' and the superscript 'q.'

6.7.2 Solution

Observe that the problem is similar to that discussed in chapter 5, section 5.7. Thus, the solution $f^q_{ml}(t)$ takes the form

$$\sum_{p=0}^{q} \alpha_p \sum_{k=1}^{K} e^{j\omega_k (m+p)T} h_k(t-(m+p)T), \tag{6.104}$$

where $\{\alpha_p : 0 \le p \le q\}$ is a set of scalars. In terms of its isomorph $\mathbf{c}(\omega) \in \mathcal{L}^2_{\mathbf{H}}$ given by

$$\mathbf{c}(\omega) = \sum_{p=0}^{q} \alpha_p e^{-j\omega(m+p)T} \boldsymbol{\Omega}_{m+p}, \tag{6.105}$$

the quantity

$$\Gamma^q_{ml}[f] = \mathcal{S}_0 \sum_{p=0}^{q} \left| T^{-1} \sum_{p_1=0}^{q} \alpha^*_{p_1} \beta_{p_1,p} - \delta_0(p) \right|^2 + \mathcal{N}_0 T^{-1} \sum_{p_1=0}^{q} \sum_{p_2=0}^{q} \alpha^*_{p_1} \alpha_{p_2} \beta_{p_1,p_2} \tag{6.106}$$

must be minimized with respect to the set of scalars $\{\alpha_p : 0 \le p \le q\}$; here the following denotions have been used:

$$\boldsymbol{\Omega}_{m+p} = \left[e^{j\omega_1(m+p)T}, e^{j\omega_2(m+p)T}, \ldots, e^{j\omega_K(m+p)T} \right]^T, \tag{6.107}$$

for $0 \le p \le q$

$$\beta_{p_1,p_2} = \boldsymbol{\Omega}^H_{m+p_1} \mathbf{H}_{p_1-p_2} \boldsymbol{\Omega}_{m+p_2}, \tag{6.108}$$

for $0 \le p_1, p_2 \le q$, and

$$\mathbf{H}_p = \frac{T}{2\pi} \int_{-\pi/T}^{\pi/T} \mathbf{H}(\omega) e^{j\omega pT} d\omega, \tag{6.109}$$

for $|p| \le q$.

The Case $q = 1$

Denoting

$$\mathbf{a} = [\alpha_1, \alpha_0]^T, \tag{6.110}$$

$$\mathbf{e} = [0, 1]^T, \tag{6.111}$$

$$\mathbf{B} = \begin{bmatrix} \beta_{1,1} & \beta_{1,0} \\ \beta_{0,1} & \beta_{0,0} \end{bmatrix}, \tag{6.112}$$

the quantity $\Gamma^1_{ml}[f]$ can be rewritten as

$$\Gamma^1_{ml}[f] = \mathcal{S}_0\left[T^{-2}\mathbf{a}^H\mathbf{B}\mathbf{B}\mathbf{a} - T^{-1}2\Re\mathbf{a}^H\mathbf{B}\mathbf{e} + 1\right] + \mathcal{N}_0 T^{-1}\mathbf{a}^H\mathbf{B}\mathbf{a}. \tag{6.113}$$

Thus, the optimum $\mathbf{a}$ is given by

$$\mathbf{a} = T\left(\mathbf{B} + \frac{\mathcal{N}_0 T}{\mathcal{S}_0}\mathbf{I}\right)^{-1}\mathbf{e}, \tag{6.114}$$

and the minimum cost λ^1_{ml} is given by

$$\lambda^1_{ml} = \mathcal{N}_0 T\mathbf{e}^H\left(\mathbf{B} + \frac{\mathcal{N}_0 T}{\mathcal{S}_0}\mathbf{I}\right)^{-1}\mathbf{e}. \tag{6.115}$$

Observe that

$$\mathbf{B} + \frac{\mathcal{N}_0 T}{\mathcal{S}_0}\mathbf{I} = \begin{bmatrix} \Omega^H_{m+1} & \mathbf{0}^H \\ \mathbf{0}^H & \Omega^H_m \end{bmatrix} \mathbf{H} \begin{bmatrix} \Omega_{m+1} & \mathbf{0} \\ \mathbf{0} & \Omega_m \end{bmatrix}. \tag{6.116}$$

where $\mathbf{H}$ denotes the matrix

$$\mathbf{H} = \begin{bmatrix} \mathbf{H}^0 & \mathbf{H}_1 \\ \mathbf{H}_{-1} & \mathbf{H}^0 \end{bmatrix}, \tag{6.117}$$

where

$$\mathbf{H}^0 = \mathbf{H}_0 + \frac{\mathcal{N}_0 T}{K\mathcal{S}_0}\mathbf{I}, \tag{6.118}$$

and

$$\mathbf{H}_p = \frac{T}{2\pi}\int_{-\pi/T}^{\pi/T}\mathbf{H}(\omega)e^{j\omega pT}d\omega. \tag{6.119}$$

Therefore,

$$\begin{bmatrix} \alpha_1 \\ \alpha_0 \end{bmatrix} = T\left[\begin{bmatrix} \Omega^H_{m+1} & \mathbf{0}^H \\ \mathbf{0}^H & \Omega^H_m \end{bmatrix} \mathbf{H} \begin{bmatrix} \Omega_{m+1} & \mathbf{0} \\ \mathbf{0} & \Omega_m \end{bmatrix}\right]^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{6.120}$$

and

$$\lambda^1_{ml} = \mathcal{N}_0\alpha_0. \tag{6.121}$$

Observe that the expression for λ^1_{ml} has the same form as the expression for λ^1_{mu} given by equation 6.85. The expression for the isomorph of $f^1_{ml}(t)$ is also of the same form as the expression for the isomorph of $f^1_{mu}(t)$ given by equation 6.84.

Chapter 7

Maximum Likelihood Sequence Estimators

7.1 Introduction

Two receivers for the composition of Quadrature Amplitude Modulation (QAM) and Discrete-Multipath Channel (DMC), based on the *maximum likelihood sequence* criterion, are derived in this chapter. These receivers shall be referred to as the Maximum Likelihood Sequence Estimator (MLSE) of the Forney type and the Maximum Likelihood Sequence Estimator of the Ungerboeck type in view of the fact that they are extensions of their namesakes known for the composition of QAM and linear time-invariant channel with Additive White Gaussian Noise (AWGN). (For information on the latter, see [13], [14], and [52]). The stationarity of the set of sequences $\{h_k(t-nT) : k = 1, 2, \ldots, K; \forall n\}$ and the consequent isomorphism between the Hilbert spaces $\mathcal{H}$ and $\mathcal{L}^2_{\mathbf{H}}$ as discussed in appendix A, section A.5 play a central role in the derivation of the MLSE of the Forney type and a marginal role in the derivation of the MLSE of the Ungerboeck type. The receivers exist under the condition that the functions $\{h_k(t) : k = 1, 2, \ldots, K\}$ are time-limited. Some issues of implementation are addressed.

7.2 The Optimality Criterion – Maximum Likelihood Sequence

The task of the receiver is to decide from the received signal $z(t)$ which data sequence $a(n)$ was likely transmitted. A maximum likelihood sequence estimator performs this task by choosing the data sequence $a(n)$ for which the following likelihood ratio is maximum:

$$\mathrm{LR} = \frac{\text{Likelihood of the hypothesis that } z(t) \text{ is due to that data sequence and noise}}{\text{Likelihood of the hypothesis that } z(t) \text{ is due to noise alone}}. \tag{7.1}$$

Recalling the composition of QAM and DMC from chapter 2, section 2.5, the maximizing of the likelihood ratio is equivalent to finding the data sequence $a(n)$ for which the following quantity is maximum:[1]

$$\Lambda[a] = -\int |z(t) - y(t)|^2 dt + \int |z(t)|^2 dt, \tag{7.3}$$

$$= 2\Re \int z(t) y^*(t) dt - \int |y(t)|^2 dt, \tag{7.4}$$

where

$$y(t) = \sum_n a(n) \sum_{k=1}^{K} e^{j\omega_k nT} h_k(t - nT), \tag{7.5}$$

and $\Re(.)$ denotes the real part.

Recall that the elements of the data sequence belong to a discrete set $\mathcal{A}$ known as the alphabet (see chapter 1, section 1.5.1), and therefore the task of the maximum likelihood sequence estimator is to perform a combinatorial optimization. The computational cost of the brute force approach of computing $\Lambda[a]$ for all possible data sequences and then finding the largest increases *exponentially* with the length of the data sequence. In the case where $K = 1$ and $\omega_1 = 0$, that is, the composition of QAM and linear time-invariant channel with AWGN, Forney and Ungerboeck have shown that, under some mild conditions on the stationary sequence $\{h_1(t - nT) : \forall n\}$, the combinatorial optimization can be performed with a computational cost that increases *linearly* with the length of the data sequence. The methods of Forney and Ungerboeck can be extended to the case where $K = 1$ and $\omega_1 \neq 0$

[1]The likelihood ratio

$$\mathrm{LR} = \frac{\exp\left(-\frac{1}{\mathcal{N}_0} \int |z(t) - y(t)|^2 dt\right)}{\exp\left(-\frac{1}{\mathcal{N}_0} \int |z(t)|^2 dt\right)}. \tag{7.2}$$

Then $\Lambda[a] = \mathcal{N}_0 \log \mathrm{LR}$. The equivalence follows from the monotonicity of Λ with respect to LR.

and indeed to the case where $K > 1$, under some mild conditions on the set of stationary sequences $\{h_k(t - nT) : k = 1, 2, \ldots, K; \forall n\}$.

7.3 The Common Thread of Forney's and Ungerboeck's Methods

The common thread of Forney's [13] [14] and Ungerboeck's [52] methods is being able, under some conditions, to write

$$\Lambda[a] = \sum_n \lambda_n[\text{state}(a, n-1), a(n)] + C, \tag{7.6}$$

where

$$\text{state}(a, n-1) = (a(n-n_0), a(n-n_0+1), \ldots, a(n-2), a(n-1)) \tag{7.7}$$

for some n_0, the $\lambda_n[.]$'s are some functions of the data sequence, and C is a constant independent of the data sequence. Although $\lambda_n[.]$ is a function of the vector

$$(a(n-n_0), a(n-n_0+1), \ldots, a(n-2), a(n-1), a(n)) \tag{7.8}$$

obtained by appending the data symbol $a(n)$ to the state$(a, n-1)$, the notion of the *state* has conceptual consequences in the maximization of $\Lambda[a]$ as recapitulated below.[2]

Denote by segment(a, n) the *initial segment* of the data sequence a up to and including the n^{th} data symbol, that is,

$$\text{segment}(a, n) = (\ldots\ldots, a(n-2), a(n-1), a(n)). \tag{7.9}$$

Suppose that the sequence $\hat{a}$ is the solution to the maximization problem, that is,

$$\hat{a} = \underset{a}{\arg\max}\ \Lambda[a]. \tag{7.10}$$

Then

$$\text{segment}(\hat{a}, m) = \underset{\substack{\text{segment}(b, m) : \\ \text{state}(b, m) = \text{state}(\hat{a}, m)}}{\arg\max} \sum_{n \leq m} \lambda_n[\text{state}(b, n-1), b(n)]. \tag{7.11}$$

[2] The argument is also known to have roots in the subject of *Dynamic Programming* founded by R. Bellman, as pointed out to me by Dr. D. P. Taylor.

In other words, of all the data sequences whose state at time m concides with the state of $\hat{a}$ at time m, the partial sum $\sum_{n\leq m} \lambda_n[\text{state}(b, n-1), b(n)]$ is maximum for those data sequences whose initial segment up to time m concides with the initial segment up to time m of $\hat{a}$. To see this, write

$$\Lambda[\hat{a}] = \sum_{n\leq m} \lambda_n[\text{state}(\hat{a}, n-1), \hat{a}(n)] + \sum_{n>m} \lambda_n[\text{state}(\hat{a}, n-1), \hat{a}(n)], \qquad (7.12)$$

and observe that if the result of equation 7.11 were false then there exists a sequence b whose state at time m concides with the state at time m of $\hat{a}$ such that

$$\sum_{n\leq m} \lambda_n[\text{state}(b, n-1), b(n)] > \sum_{n\leq m} \lambda_n[\text{state}(\hat{a}, n-1), \hat{a}(n)]. \qquad (7.13)$$

More specifically, the sequence b can be chosen such that $b(n) = \hat{a}(n)$ for $n > m$ also. In other words, the sequence b can be obtained by replacing the initial segment up to time m of $\hat{a}$ with the solution to the problem

$$\underset{\substack{\text{segment}(b,m):\\ \text{state}(b,m)=\text{state}(\hat{a},m)}}{\arg\max} \sum_{n\leq m} \lambda_n[\text{state}(b, n-1), b(n)]. \qquad (7.14)$$

For such a sequence b,

$$\sum_{n>m} \lambda_n[\text{state}(b, n-1), b(n)] = \sum_{n>m} \lambda_n[\text{state}(\hat{a}, n-1), \hat{a}(n)], \qquad (7.15)$$

and therefore

$$\Lambda[b] > \Lambda[\hat{a}], \qquad (7.16)$$

which contradicts the definition of $\hat{a}$ given by equation 7.10. This proves the result of equation 7.11.

Since the elements of the data sequence belong to the discrete and finite set $\mathcal{A}$, the values taken by state(a, n) belong to a *discrete* and *finite* set,[3] say $\mathcal{S}$. Consider the solutions to the problems

$$\underset{\substack{\text{segment}(b,m):\\ \text{state}(b,m)=s}}{\arg\max} \sum_{n\leq m} \lambda_n[\text{state}(b, n-1), b(n)], \quad \forall s \in \mathcal{S}. \qquad (7.17)$$

[3]In mathematical terms, $\mathcal{S}$ is the Cartesian product $\mathcal{A} \times \mathcal{A} \times \ldots \times \mathcal{A}$ (n_0 times).

The result of equation 7.11 says that one of these solutions must concide with segment$(\hat{a}, m)$. By an argument similar to that used above, one can prove the following: the initial segment up to time m of the solutions to the problems

$$\begin{array}{l}\arg\max \\ \text{segment}(b, m+1): \\ \text{state}(b, m+1) = s\end{array} \sum_{n \leq m+1} \lambda_n[\text{state}(b, n-1), b(n)], \quad \forall s \in \mathcal{S} \tag{7.18}$$

must be among the solutions to the problems of equation 7.17. Therefore, in order to solve the problem of equation 7.18, the maximization need be done only over the *extensions* of the solutions to the problem of equation 7.17. In other words, as m increases, the problems of equation 7.17 can be solved *recursively*, storing at each stage the maximizing initial segments and the maxima so achieved. For data sequences that terminate at a known time with a known state, the final step of the recursion will yield $\hat{a}$. The optimization procedure described above is easily visualized in terms of a directed graph known as the trellis diagram [13], [14], [52].

For the above method of maximizing $\Lambda[a]$, it is clear that the number of computations increases only *linearly* with the length of the data sequence.

7.4 Maximum Likelihood Sequence Estimator of the Ungerboeck Type

To derive the MLSE of the Ungerboeck type, denote

$$h^n(t) = \sum_{k=1}^{K} e^{j\omega_k nT} h_k(t - nT), \tag{7.19}$$

$$c(n_1, n_2) = \int h^{n_1 *}(t) h^{n_2}(t) dt, \tag{7.20}$$

$$b(n) = \int z(t) h^{n*}(t) dt. \tag{7.21}$$

Then

$$\int z(t) y^*(t) dt = \sum_n a^*(n) b(n), \tag{7.22}$$

as easily observed, and

$$\int |y(t)|^2 dt = 2\Re \sum_n a^*(n) \left[\frac{1}{2} c(n, n) a(n) + \sum_{m<n} c(n, m) a(m)\right], \tag{7.23}$$

as shown in section 7.4.2.

Under the assumption that the functions $\{h_k(t) : k = 1, 2, \ldots, K\}$ are time-limited, there exists a finite integer n_0 such that

$$c(n_1, n_2) = 0 \qquad \text{for} \quad |n_1 - n_2| > n_0. \tag{7.24}$$

Combining equations 7.4, 7.22, 7.23, and 7.24, one obtains

$$\Lambda[a] = 2\Re \sum_n a^*(n)\left[b(n) - \left(\frac{1}{2}c(n,n)a(n) + \sum_{m=n-n_0}^{n-1} c(n,m)a(m)\right)\right], \tag{7.25}$$

which can be identified with equation 7.6 by defining

$$\lambda_n[\text{state}(a, n-1), a(n)] = 2\Re\left(a^*(n)\left[b(n) - \left(\frac{1}{2}c(n,n)a(n) + \sum_{m=n-n_0}^{n-1} c(n,m)a(m)\right)\right]\right), \tag{7.26}$$

$$C = 0. \tag{7.27}$$

The expression of equation 7.25 is clearly an extension of that given by Ungerboeck [52] for the composition of QAM and linear time-invariant channel with AWGN where $c(n, m)$ takes the form $\tilde{c}(n - m)$.

7.4.1 On the Implementation

Although the stationarity of the set of sequences $\{h_k(t - nT) : k = 1, 2, \ldots, K; \forall n\}$ has played no role in the derivation of the MLSE of the Ungerboeck type, it has a role in the implementation of the MLSE.

First, it facilitates the computation of $c(n_1, n_2)$ as[4]

$$\begin{aligned} c(n_1, n_2) &= \frac{1}{2\pi}\int_{-\pi/T}^{\pi/T} e^{j\omega n_1 T}\Omega_{n_1}^H \mathbf{H}(\omega)\Omega_{n_2} e^{-j\omega n_2 T} d\omega, & (7.28)\\ &= T^{-1}\Omega_{n_1}^H \mathbf{H}_{(n_1-n_2)}\Omega_{n_2}, & (7.29)\end{aligned}$$

where

$$\begin{aligned} \Omega_n &= \left[e^{j\omega_1 nT}, e^{j\omega_2 nT}, \ldots, e^{j\omega_K nT}\right]^T, & (7.30)\\ \mathbf{H}_n &= \frac{T}{2\pi}\int_{-\pi/T}^{\pi/T} \mathbf{H}(\omega) e^{j\omega nT} d\omega. & (7.31)\end{aligned}$$

[4]Here $\mathbf{H}(\omega)$ is the spectral density matrix of the set of stationary sequences $\{h_k(t - nT) : k = 1, 2, \ldots, K; \forall n\}$.

Thus only the set of matrices $\{\mathbf{H}_n : n = 0, 1, \ldots, n_0\}$ and the set of real numbers $\{\omega_k : k = 1, 2, \ldots, K\}$ need be known.

Second, it facilitates the computation of $b(n)$ as

$$b(n) = \sum_{k=1}^{K} e^{-j\omega_k nT} b_k(n), \tag{7.32}$$

where

$$b_k(n) = \int z(t) h_k^*(t - nT)dt. \tag{7.33}$$

The MLSE can be implemented as schematically shown in figure 7.1. Thus the received signal $z(t)$ is fed into the bank of filters matched to $\{h_k(t) : k = 1, 2, \ldots, K\}$ and the outputs are sampled once every T seconds to obtain the numbers $\{b_k(n) : k = 1, 2, \ldots, K\}$. These numbers are then linearly combined according to the weight vector Ω_n^* to obtain $b(n)$.

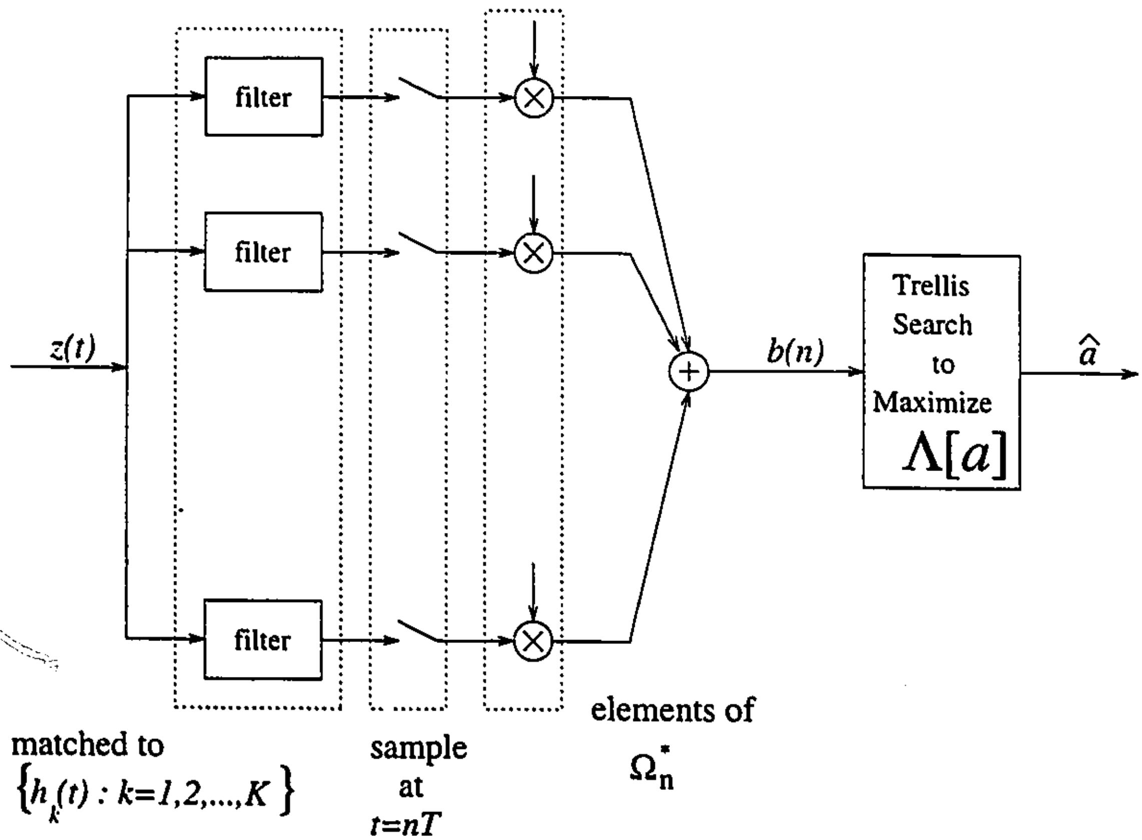

Figure 7.1: Schematic Diagram of the Implementation of the Ungerboeck type MLSE

7.4.2 Proof of Equation 7.23

To prove equation 7.23, write

$$\begin{aligned}\int |y(t)|^2 dt &= \sum_{n_1}\sum_{n_2} a^*(n_1)a(n_2)c(n_1,n_2), && (7.34)\\ &= \sum_{n_1}\sum_{n_2<n_1} a^*(n_1)a(n_2)c(n_1,n_2) && (7.35)\\ &\quad + \sum_{n_1} a^*(n_1)a(n_1)c(n_1,n_1) && (7.36)\\ &\quad + \sum_{n_1}\sum_{n_2>n_1} a^*(n_1)a(n_2)c(n_1,n_2), && (7.37)\end{aligned}$$

and observe that the last double sum is the complex conjugate of the first double sum. To see this, interchange the variables n_1 and n_2, and also interchange the order of summation. Thus

$$\begin{aligned}\sum_{n_1}\sum_{n_2>n_1} a^*(n_1)a(n_2)c(n_1,n_2) &= \sum_{n_2}\sum_{n_1>n_2} a^*(n_2)a(n_1)c(n_2,n_1), && (7.38)\\ &= \sum_{n_1}\sum_{n_2<n_1} a(n_1)a^*(n_2)c^*(n_1,n_2), && (7.39)\end{aligned}$$

where, in the last step, the Hermitian property

$$c(n_2,n_1) = c^*(n_1,n_2) \qquad (7.40)$$

is used. Therefore,

$$\begin{aligned}\int |y(t)|^2 dt &= \sum_{n_1} a^*(n_1) \sum_{n_2<n_1} c(n_1,n_2)a(n_2) && (7.41)\\ &\quad + \sum_{n_1} a^*(n_1)c(n_1,n_1)a(n_1) && (7.42)\\ &\quad + \sum_{n_1} a(n_1)\left(\sum_{n_2<n_1} c(n_1,n_2)a(n_2)\right)^*, && (7.43)\end{aligned}$$

which can be written as in equation 7.23.

7.5 Maximum Likelihood Sequence Estimator of the Forney Type

To derive the MLSE of the Forney type, denote

$$\mathcal{H} = \text{Clos.Span}\left\{h_k(t-nT) : k = 1,2,\ldots,K;\forall n\right\}, \qquad (7.44)$$

and observe from equation 7.5 that $y(t) \in \mathcal{H}$ for every data sequence $a(n)$. Under the assumption that the set of functions $\{h_k(t) : k = 1, 2, \ldots, K\}$ is time-limited, the spectral density matrix $\mathbf{H}(\omega)$ of the set of stationary $\{h_k(t - nT) : k = 1, 2, \ldots, K; \forall n\}$ is a polynomial in $e^{-j\omega T}$, that is, it has the form

$$\mathbf{H}(\omega) = \sum_{n=-n_0}^{n_0} \mathbf{H}_n e^{-j\omega n T} \tag{7.45}$$

for some finite n_0. Therefore, as discussed in appendix A, section A.7, there exists a set of orthonormal stationary sequences,[5] say,

$$\{g_l(t - nT) : l = 1, 2, \ldots, L; \forall n\}, \tag{7.46}$$

that constitutes an orthonormal basis for $\mathcal{H}$, and the functions $\{h_k(t) : k = 1, 2, \ldots, K\}$ can be represented in the form[6]

$$h_k(t) = \sum_{n=0}^{n_0} \sum_{l=1}^{L} c_{l,k}(n) g_l(t - nT). \tag{7.49}$$

In terms of this representation,

$$\begin{aligned} y(t) &= \sum_n a(n) \sum_{k=1}^{K} e^{j\omega_k n T} \sum_{m=0}^{n_0} \sum_{l=1}^{L} c_{l,k}(m) g_l(t - (n+m)T), && (7.50) \\ &= \sum_n \sum_{l=1}^{L} g_l(t - nT) \sum_{m=0}^{n_0} \sum_{k=1}^{K} c_{l,k}(m) e^{j\omega_k (n-m)T} a(n - m), && (7.51) \end{aligned}$$

and therefore,

$$\int |y(t)|^2 dt = \sum_n \sum_{l=1}^{L} \left| \sum_{m=0}^{n_0} \sum_{k=1}^{K} c_{l,k}(m) e^{j\omega_k (n-m)T} a(n - m) \right|^2. \tag{7.52}$$

Denote

$$b_l(n) = \int z(t) g_l^*(t - nT) dt. \tag{7.53}$$

[5] A Wold decomposition also exists as discussed in appendix A, section A.8.1.

[6] Here

$$c_{l,k}(n) = \frac{\sqrt{T}}{2\pi} \int_{-\pi/T}^{\pi/T} \mathbf{e}_l^H \mathbf{C}(\omega) \mathbf{e}_k e^{j\omega n T} d\omega, \tag{7.47}$$

where $\mathbf{C}(\omega)$ is a causal polynomial in $e^{-j\omega T}$, of degree n_0, satisfying $\mathbf{H}(\omega) = \mathbf{C}^H(\omega)\mathbf{C}(\omega)$, and the functions $g_l(t) \in \mathcal{H}$ are the isomorphs of the functions $\sqrt{T}\mathbf{a}_l(\omega) \in \mathcal{L}^2_{\mathbf{H}}$, where

$$\mathbf{C}(\omega) [\mathbf{a}_1(\omega), \mathbf{a}_2(\omega), \ldots, \mathbf{a}_L(\omega)] = \mathbf{I}_{L \times L}. \tag{7.48}$$

Therefore

$$\int z(t)y^*(t)dt = \sum_n \sum_{l=1}^{L} b_l(n) \left(\sum_{m=0}^{n_0} \sum_{k=1}^{K} c_{l,k}(m) e^{j\omega_k (n-m)T} a(n-m) \right)^* . \tag{7.54}$$

Combining equations 7.4, 7.52, and 7.54, one obtains

$$\Lambda[a] = 2\Re \sum_n \sum_{l=1}^{L} b_l(n) \left(\sum_{m=0}^{n_0} \sum_{k=1}^{K} c_{l,k}(m) e^{j\omega_k (n-m)T} a(n-m) \right)^* \tag{7.55}$$

$$- \sum_n \sum_{l=1}^{L} \left| \sum_{m=0}^{n_0} \sum_{k=1}^{K} c_{l,k}(m) e^{j\omega_k (n-m)T} a(n-m) \right|^2 . \tag{7.56}$$

By completing the square,

$$\Lambda[a] = \sum_n \sum_{l=1}^{L} |b_l(n)|^2 - \sum_n \sum_{l=1}^{L} \left| b_l(n) - \sum_{m=0}^{n_0} \sum_{k=1}^{K} c_{l,k}(m) e^{j\omega_k (n-m)T} a(n-m) \right|^2 , \tag{7.57}$$

which can be identified with equation 7.6 by defining

$$\lambda_n[\text{state}(a, n-1), a(n)] = - \sum_{l=1}^{L} \left| b_l(n) - \sum_{m=0}^{n_0} \sum_{k=1}^{K} c_{l,k}(m) e^{j\omega_k (n-m)T} a(n-m) \right|^2 , \tag{7.58}$$

$$C = \sum_n \sum_{l=1}^{L} |b_l(n)|^2 . \tag{7.59}$$

Disregarding the constant term, the optimization problem can be posed as follows:

minimize

$$\Gamma[a] = \sum_n \sum_{l=1}^{L} \left| b_l(n) - \sum_{m=0}^{n_0} \sum_{k=1}^{K} c_{l,k}(m) e^{j\omega_k (n-m)T} a(n-m) \right|^2 . \tag{7.60}$$

The expression of equation 7.60 is clearly an extension of that (squared Euclidean distance) given by Forney [13] for the composition of QAM and linear time-invariant channel with AWGN.

7.5.1 On the Implementation

The MLSE can be implemented as schematically shown in figure 7.2. Thus the received signal $z(t)$ is fed into the bank of filters matched to $\{g_l(t) : l = 1, 2, \ldots, L\}$. The outputs are sampled once every T seconds to obtain the numbers $b_l(n)$. The front end filter bank is an extension of the *whitened matched filter* derived by Forney [13].

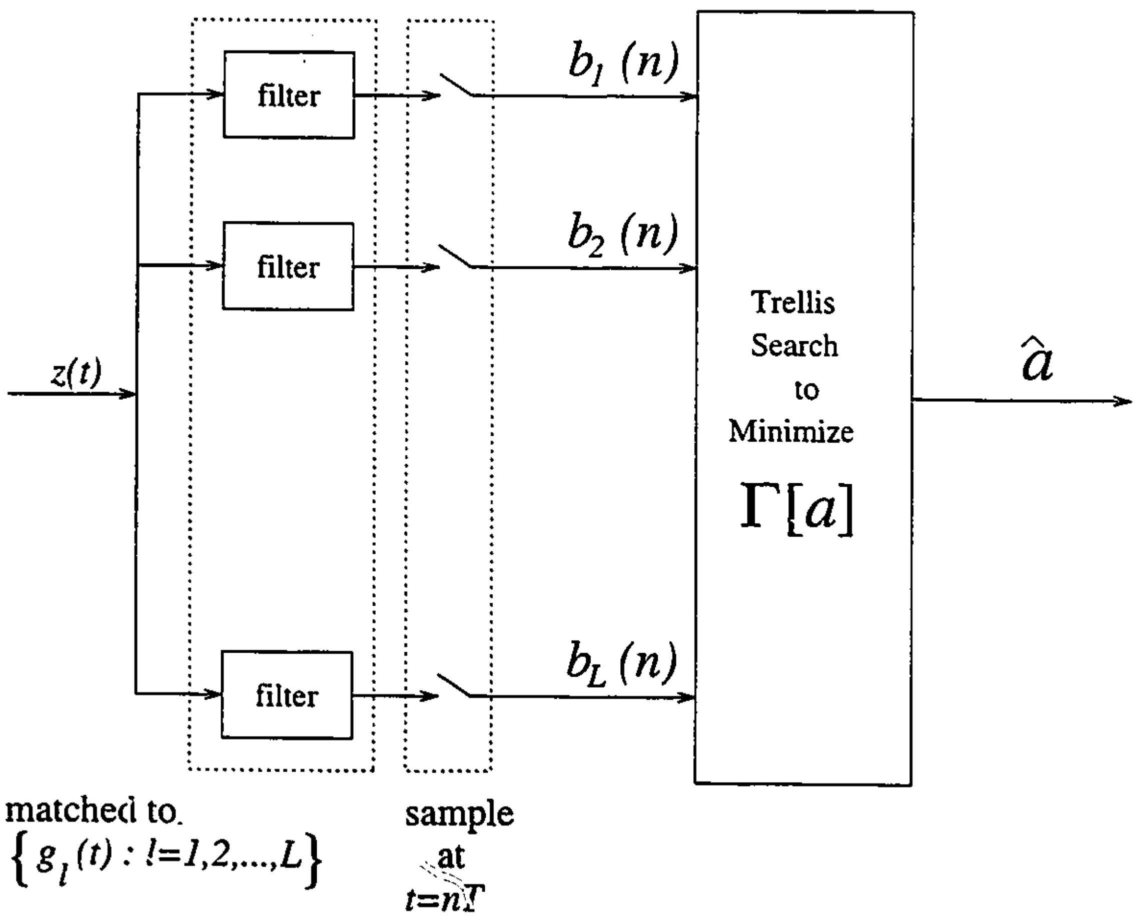

Figure 7.2: Schematic Diagram of the Implementation of the Forney type MLSE

Chapter 8

Results of Computer Simulations

8.1 Introduction

In this chapter, results of computer simulations are presented for the following:

- maximum-likelihood method of estimating the parameters of the *composition* of quadrature amplitude modulation (QAM) and discrete-multipath channel (DMC) proposed in chapter 2, section 2.6.2,
- linear zero-forcing equalizer proposed in chapter 3, section 3.5, based on the *actual* parameters of the composition of QAM and DMC,
- linear zero-forcing equalizer proposed in chapter 3, section 3.5, based on the *estimated* parameters of the composition of QAM and DMC.

Wherever possible, results of computer simulations are compared to their theoretical counterparts. All computations were performed by using MATLAB, an interactive high level programming environment.

8.2 The QAM-based Transmitter

8.2.1 Signal Set

The 16 QAM signal set shown in figure 8.1 was considered, with an *equiprobable* choice of signals. This signal set, when scaled to have a minimum distance of 2 between signals, has a mean square value of 10.

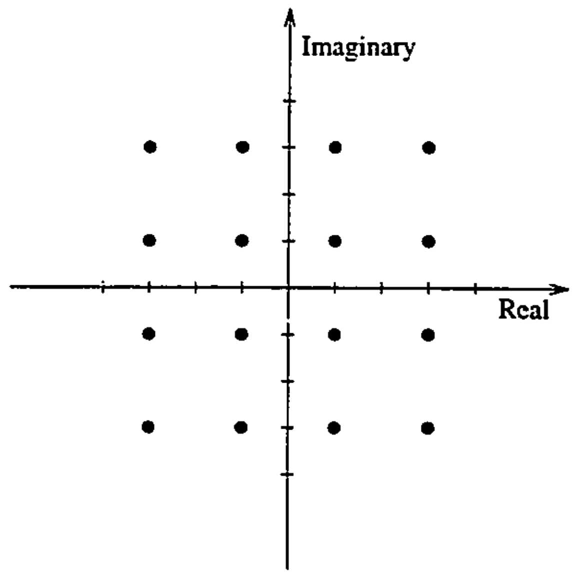

Figure 8.1: The 16 QAM Signal Set

8.2.2 Transmitter Pulses

The following two pulses were considered:

- A pulse of a rectangular shape in the time domain, one baud duration, and unit energy, that is,

$$g(t) = \begin{cases} \frac{1}{\sqrt{T}} & \text{for} \quad 0 \le t \le T, \\ 0 & \text{elsewhere.} \end{cases} \tag{8.1}$$

- A pulse of a square root raised cosine shape in the frequency domain with roll-off factor 1.0, and unit energy, that is,[1]

$$\hat{g}(\omega) = \begin{cases} \sqrt{\frac{T}{2}} \left(1 + \cos \frac{\omega T}{2}\right)^{1/2} & \text{for} \quad -\frac{2\pi}{T} \le \omega \le \frac{2\pi}{T}, \\ 0 & \text{elsewhere.} \end{cases} \tag{8.2}$$

$$= \begin{cases} \sqrt{T} \cos \frac{\omega T}{4} & \text{for} \quad -\frac{2\pi}{T} \le \omega \le \frac{2\pi}{T}, \\ 0 & \text{elsewhere.} \end{cases} \tag{8.3}$$

[1]In the time domain, $g(t) = \frac{1}{2\pi} \int \hat{g}(\omega) e^{j\omega t} d\omega$, the inverse Fourier transform of $\hat{g}$; since $g(t)$ is not time-limited only a truncated version can be implemented.

8.2.3 Carrier Frequency, Baud Rate

The following combination of carrier frequency and baud rate was considered:[2]

1 GHz, 20 kbauds/s,

8.2.4 Signal-to-Noise Ratio

Unless specified otherwise, signal-to-noise ratio (SNR) will mean the following:

$$\mathrm{SNR} = \frac{\text{The average } \textit{transmitted} \text{ signal energy per baud period}}{\text{The noise power spectral density}}. \tag{8.4}$$

8.3 The Discrete-Multipath Channel (DMC)

Discrete-Multipath Channels having up to three paths were considered. Compositions of QAM and DMC with Doppler-shifts of up to $\pi/200$ radians/baud and relative delays of up to 0.2 bauds were considered. Some scenarios where the values of Doppler-shifts and delays considered will arise are given below. In all these scenarios, without loss of generality, the transmitter and the reflector/scatterer are assumed to be stationary and the receiver is assumed to be moving. Moreover, the transmitter and the reflector/scatterer are assumed to be far from the receiver such that the assumptions made in chapter 1, section 1.2.2 are valid.

Recall from chapter 1, section 1.2.3, that the Doppler-shift ω_k of the k^{th} path is

$$\omega_k = \frac{\omega_0 v_k}{c}, \tag{8.5}$$

where v_k is the velocity of the receiver taken relative to the k^{th} path,[3] ω_0 is the carrier frequency, and c is the speed of propagation of the electromagnetic wave.

8.3.1 Scenario 0

In scenario 0, the DMC is assumed to have one path, and the receiver is assumed to be moving such that $v_1 = 0$. Therefore, the Doppler-shift is zero.

[2]The results of this chapter are, however, valid for any combination that gives rise to the same *normalized* values of Doppler shifts and delays, the normalization being with respect to the baud rate and baud period respectively.

[3]More specifically, the velocity in the direction perpendicular to the planar wave front associated with the k^{th} path and opposite to that of propagation.

The frame length is assumed to be 400 bauds, with the lengths of the probing and the signalling phases being 200 bauds each. Observe that the composition of QAM and DMC is time-invariant.

8.3.2 Scenario 1

In scenario 1, the DMC is assumed to have two paths, and the receiver is assumed to be moving such that $v_1 = 0$ and $v_2 = 15$ m/s (15 m/s is equivalent to 54 km/h). The relative delay is assumed to be zero.

At the carrier frequency of $f_0 = 1$ GHz, the Doppler-shifts are given by,

$$\begin{aligned} \omega_1 &= 0, && (8.6) \\ \omega_2 &= 2\pi \times 10^9 \times 15/(3 \times 10^8), && (8.7) \\ &= 100\pi \text{ radians/s.} && (8.8) \end{aligned}$$

At the baud rate of 20 kbauds/s, the baud period is $T = 50$ μs. Therefore, $\omega_1 T = 0$ and $\omega_2 T = 100\pi \times 50 \times 10^{-6} = \pi/200$ radians/baud.

The frame length is assumed to be 400 bauds, with the lengths of the probing and the signalling phases being 200 bauds each. Observe that, with the differential Doppler-shift being $\pi/200$ radians/baud, the composition of QAM and DMC can be considered as periodically time-variant with a period of 400 bauds.

8.3.3 Scenario 2

Scenario 2 is identical to scenario 1, except for the delay. In scenario 2, a relative delay of[4] 10 μs, or, equivalently, 0.2 bauds, is assumed.

8.3.4 Scenario 3

In Scenario 3, the DMC is assumed to have three paths, and the receiver is assumed to be moving such that $v_1 = -10$ m/s, $v_2 = 0$, and $v_3 = 10$ m/s (10 m/s is equivalent to 36 km/h). The relative delays are assumed to be zero.

[4] According to [37], relative delays can be as much as 100 μs. According to [8], the ETSI/GSM specification assumes relative delays of up to 16 μs.

At the carrier frequency of $f_0 = 1$ GHz, the Doppler-shifts are given by,

$$\omega_1 = -2\pi \times 10^9 \times 10/(3 \times 10^8), \tag{8.9}$$
$$= -200\pi/3 \text{ radians/s}, \tag{8.10}$$
$$\omega_2 = 0, \tag{8.11}$$
$$\omega_3 = 200\pi/3 \text{ radians/s}. \tag{8.12}$$

At the baud rate of 20 kbauds/s, the baud period is $T = 50$ μs. Therefore, $\omega_1 T = -200\pi \times 50 \times 10^{-6}/3 = -\pi/300$ radians/baud, $\omega_2 T = 0$, and $\omega_3 T = \pi/300$ radians/baud.

The frame length is assumed to be 600 bauds, with the lengths of the probing and the signalling phases being 300 bauds each. Observe that, with the differential Doppler-shifts being integer multiples of $\pi/300$ radians/baud, the composition of QAM and DMC can be considered as periodically time-variant with a period of 600 bauds.

8.3.5 Scenario 4

Scenario 4 is identical to scenario 3, except for the delays. In scenario 4, a relative delay of 10 μs, or, equivalently, 0.2 bauds, is assumed.

8.3.6 Scenario 5

In Scenario 5, the DMC is assumed to have three paths, and the receiver is assumed to be moving such that $v_1 = -15$ m/s, $v_2 = 0$, and $v_3 = 15$ m/s (15 m/s is equivalent to 54 km/h). The relative delays are assumed to be zero.

At the carrier frequency of $f_0 = 1$ GHz, the Doppler-shifts are given by,

$$\omega_1 = -2\pi \times 10^9 \times 15/(3 \times 10^8), \tag{8.13}$$
$$= -100\pi \text{ radians/s}, \tag{8.14}$$
$$\omega_2 = 0, \tag{8.15}$$
$$\omega_3 = 100\pi \text{ radians/s}. \tag{8.16}$$

At the baud rate of 20 kbauds/s, the baud period is $T = 50$ μs. Therefore, $\omega_1 T = -100\pi \times 50 \times 10^{-6} = -\pi/200$ radians/baud, $\omega_2 T = 0$, and $\omega_3 T = \pi/200$ radians/baud.

The frame length is assumed to be 400 bauds, with the lengths of the probing and the signalling phases being 200 bauds each. Observe that, with the differential Doppler-shifts being integer multiples of $\pi/200$ radians/baud, the composition of QAM and DMC can be considered as periodically time-variant with a period of 400 bauds.

8.3.7 Scenario 6

Scenario 6 is identical to scenario 5, except for the delays. In scenario 6, a relative delay of 10 μs, or, equivalently, 0.2 bauds, is assumed.

8.4 Errors in the Estimated Doppler-Shifts

The probing of the DMC with a QAM signal and the maximum-likelihood method of estimating the parameters of the composition of QAM and DMC were simulated[5] for the scenarios described in section 8.3. The SNR-dependence of the simulated errors in the maximum-likelihood estimates of the Doppler-shifts are shown in figure 8.2 through figure 8.16. These figures also show the SNR-dependence of the Cramer-Rao bounds given by equation 2.96 of chapter 2. These figures, therefore, substantiate the result quoted in chapter 2, section 2.6.4 that the maximum-likelihood estimates of the Doppler-shifts are asymptotically efficient.

8.5 Post-Equalizer Signal-to-Noise Ratios

Recalling the generic structure of the linear equalizer from chapter 3, section 3.2, the post-equalizer signal-to-noise ratio (Post-Equalizer SNR) is defined as follows:

$$\text{Post-Equalizer SNR} = \frac{\text{The mean square value of the signal at the input to the quantizer}}{\text{The mean square value of the noise at the input to the quantizer}}. \tag{8.17}$$

Observe that the defining quantities are, in general, functions of discrete-time (measured in bauds), for the DMC is time-variant.

For the linear zero-forcing equalizers of chapter 3, one can define a *normalized* Post-Equalizer SNR as follows:

$$\text{Normalized Post-Equalizer SNR} = \frac{\text{Post-Equalizer SNR}}{\text{SNR}} = \frac{1}{\lambda^q_{mu}}, \tag{8.18}$$

[5]The MATLAB function *fmins* was used to find the estimate $\hat{\Theta}$ defined by equation 2.71 of chapter 2, using the actual value Θ as the initial point, with the resolution parameter being set at 10^{-7}.

where SNR is as defined in section 8.2.4, and the λ^q_{mu}'s are as derived in chapter 3 (sections 3.5 and 3.6). The time-dependence of the hierarchy of these normalized Post-Equalizer SNR's corresponding to $q = 0, 1, 2$ is shown in figure 8.17 through figure 8.32 for the various scenarios described in section 8.3. These figures also show the time-dependence of the hierarchy of upper bounds[6] $\frac{1}{\lambda^q_{ml}}$ corresponding to $q = 0, 1, 2$. In these figures, the time interval shown is that of a frame.

The figures also show that the common understanding of *fading*, gained by transmitting a sinusoidal signal and observing the received signal, may be too simplistic when it comes to quadrature-amplitude-modulated signalling in the presence of intersymbol interference.

8.6 Symbol-Error Rates

The symbol-error rate over a particular time interval is defined as follows:

$$\text{Symbol-Error Rate} = \frac{\text{The expected number of symbol errors over the time interval}}{\text{The length of the time interval}}. \tag{8.19}$$

Observe that, since the DMC is time-variant, the symbol-error rate is specific to the time-interval.

Figure 8.33 through figure 8.47 show the SNR-dependence of the simulated symbol-error rates of a linear zero-forcing equalizer, with $q = 0$, over the signalling phase,[7] for the various scenarios decribed in section 8.3. The simulated symbol-error rates were found by simply counting the symbol errors over the time interval, dividing the count by the length of the time interval, and then averaging the result over the trials. The figures also show the theoretical SNR-dependence of the symbol-error rates of the hierarchy of linear zero-forcing equalizers corresonding to $q = 0, 1, 2$, calculated according to the discussion of section 8.6.1 by using λ^q_{mu}. These figures also show the SNR-dependence of the hierarchy of lower bounds corresponding to $q = 0, 1, 2$, calculated according to the discussion of section 8.6.1, but by using λ^q_{ml}.

---

[6]Recall that $\lambda^0_{ml} \le \lambda^1_{ml} \le \lambda^2_{ml} \le \lambda^2_{mu} \le \lambda^1_{mu} \le \lambda^0_{mu}$.

[7]The time interval was approximately that of the signalling phase, with an allowance having been made for the effects of the probing phase to die down.

8.6.1 Theoretical Calculation of Symbol-Error Rate

To calculate the symbol-error rate for a linear equalizer, consider the string X of binary random variables

$$X = (X_1, X_2, \ldots, X_N), \tag{8.20}$$

where each binary random variable X_n takes value 0 or 1 with the joint probability distribution

$$\text{Prob.}[X_1 = x_1, X_2 = x_2, \ldots, X_N = x_N] = p(x_1, x_2, \ldots, x_N), \tag{8.21}$$

and the marginal probability distributions

$$\text{Prob.}[X_n = x] = p_n(x), \tag{8.22}$$

for $n = 1, 2, \ldots, N$. Denote by N_X the expected number of 1's in the string X. Then

$$\begin{aligned} N_X &= \sum_{\text{all } x=(x_1,x_2,\ldots,x_N)} (x_1 + x_2 + \ldots + x_N) p(x_1, x_2, \ldots, x_N), && (8.23) \\ &= \sum_{n=1}^{N} \left(\sum_{\text{all } x \text{ with} x_n=1} p(x_1, x_2, \ldots, x_N) \right), && (8.24) \\ &= \sum_{n=1}^{N} p_n(1). && (8.25) \end{aligned}$$

By identifying the event that $X_n = 1$ with the event that the n^{th} symbol is in *error*, one has, for the 16 QAM signal set shown in figure 8.1,

$$p_n(1) = 3Q_n - 2.25Q_n^2, \tag{8.26}$$

where

$$Q_n = \frac{1}{\sqrt{2\pi}} \int_{\alpha_n}^{\infty} e^{-x^2/2} dx, \tag{8.27}$$

and

$$\frac{\alpha_n^2}{2} = \frac{\text{Post-Equalizer SNR}}{10} = \frac{\text{SNR}}{10\lambda_{nu}^q}. \tag{8.28}$$

Therefore, the symbol-error rate over the time-interval 1 to N is given by

$$\text{Symbol-Error Rate} = \frac{1}{N} \sum_{n=1}^{N} p_n(1), \tag{8.29}$$

where $p_n(1)$ is given by equations 8.26.

8.7 Graphs

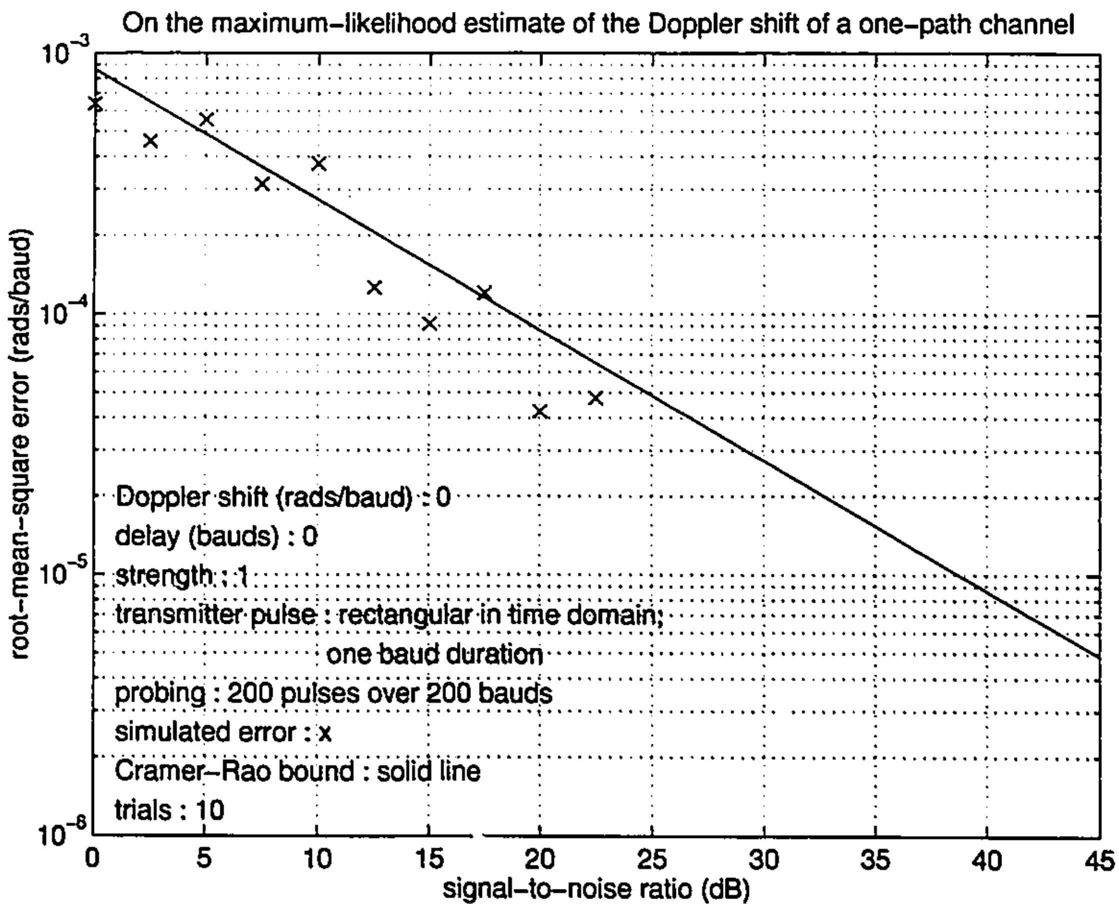

Figure 8.2: An Example in Scenario 0

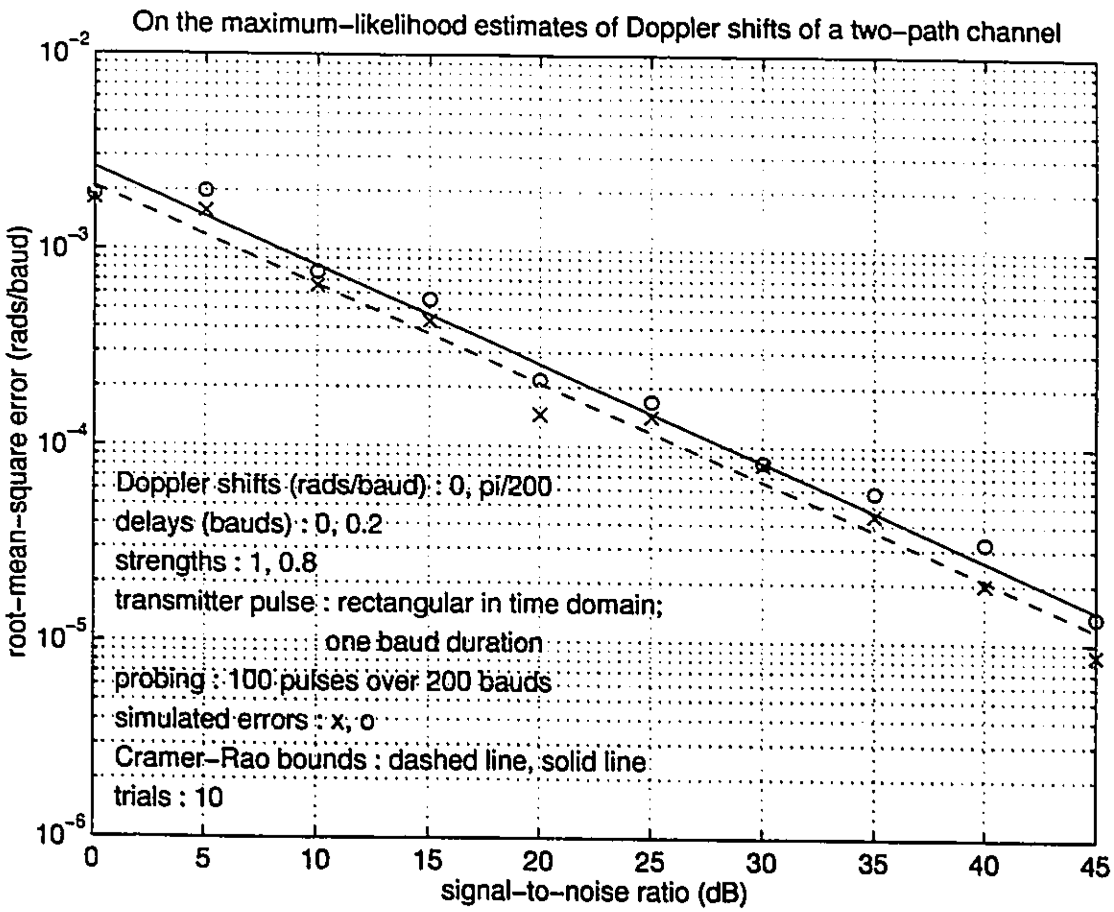

Figure 8.3: An Example in Scenario 2

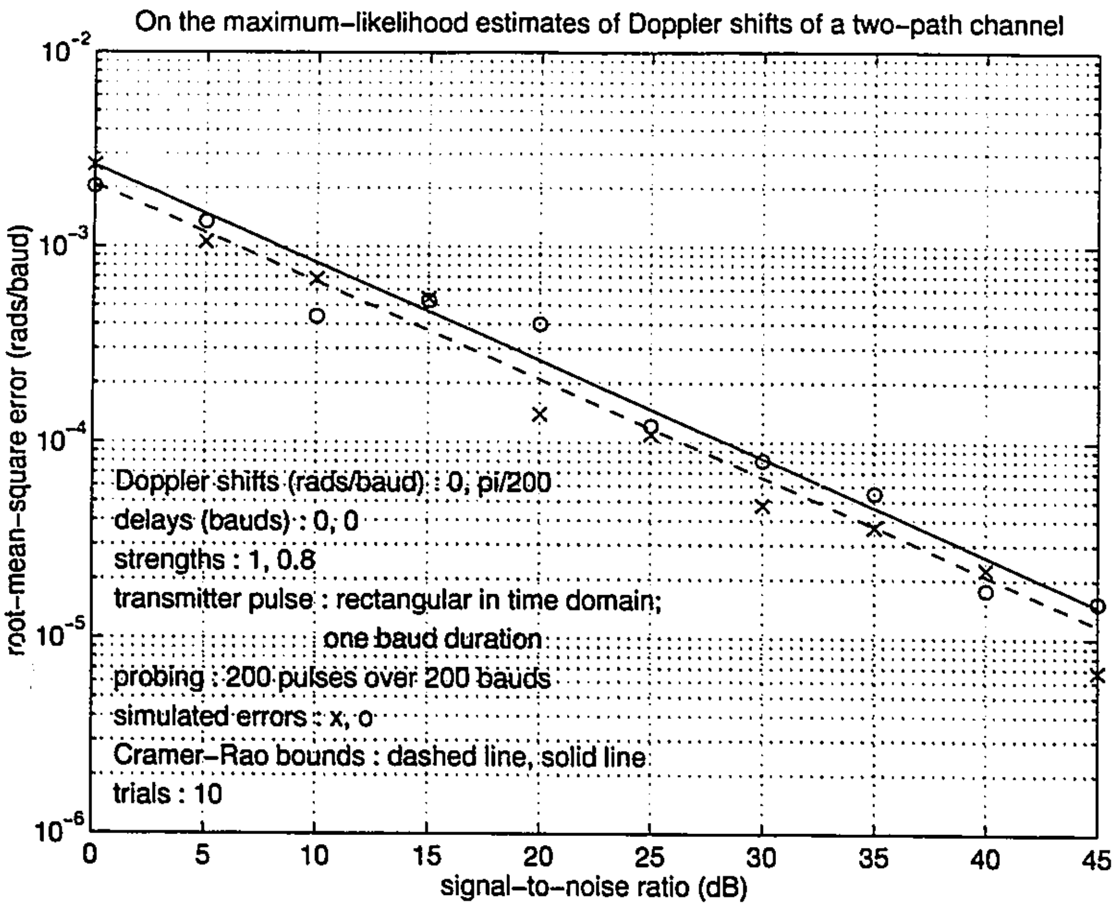

Figure 8.4: An Example in Scenario 1

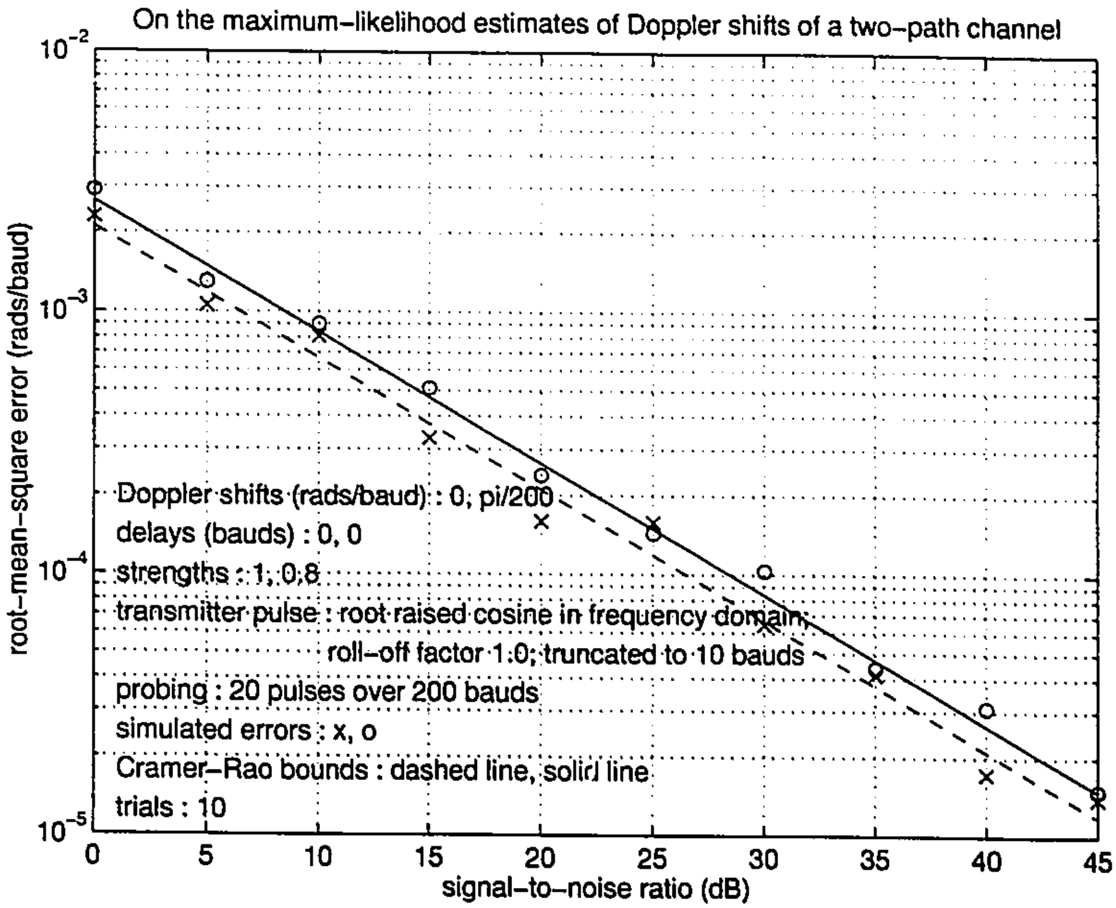

Figure 8.5: An Example in Scenario 1

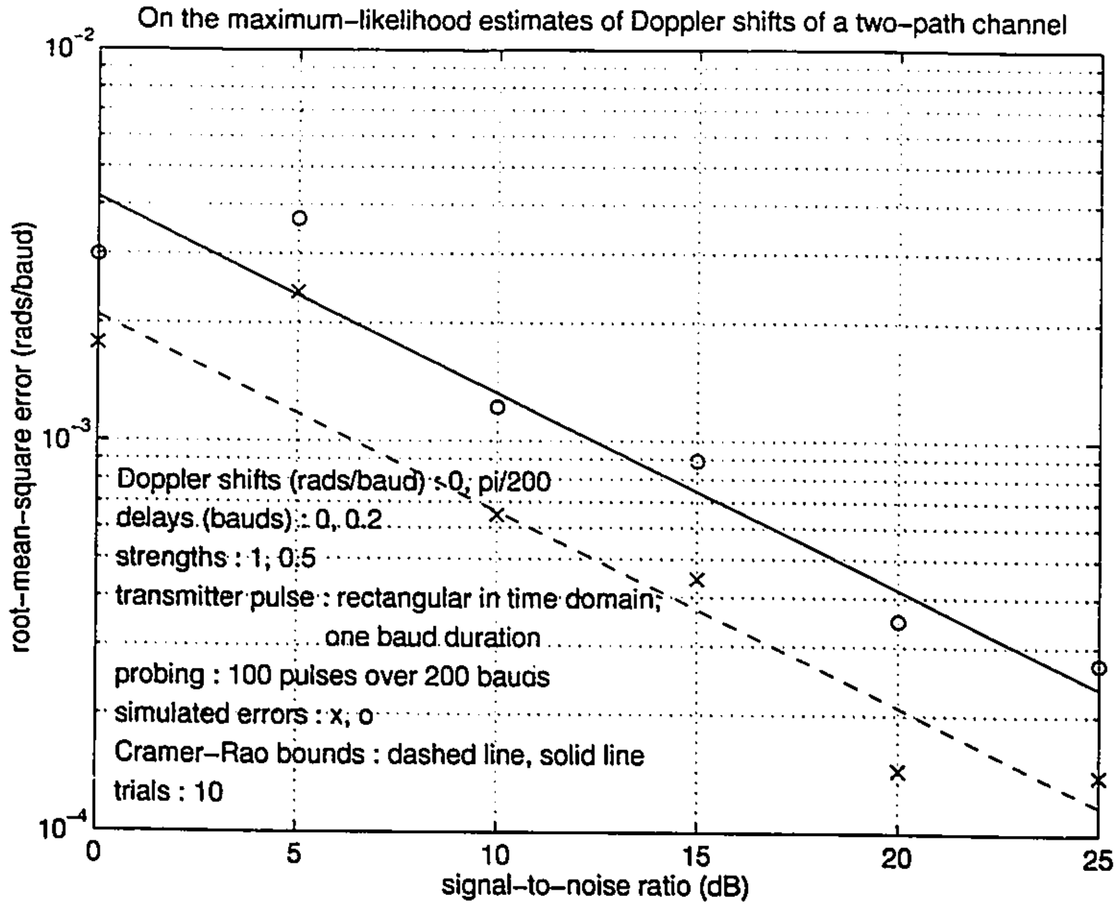

Figure 8.6: An Example in Scenario 2

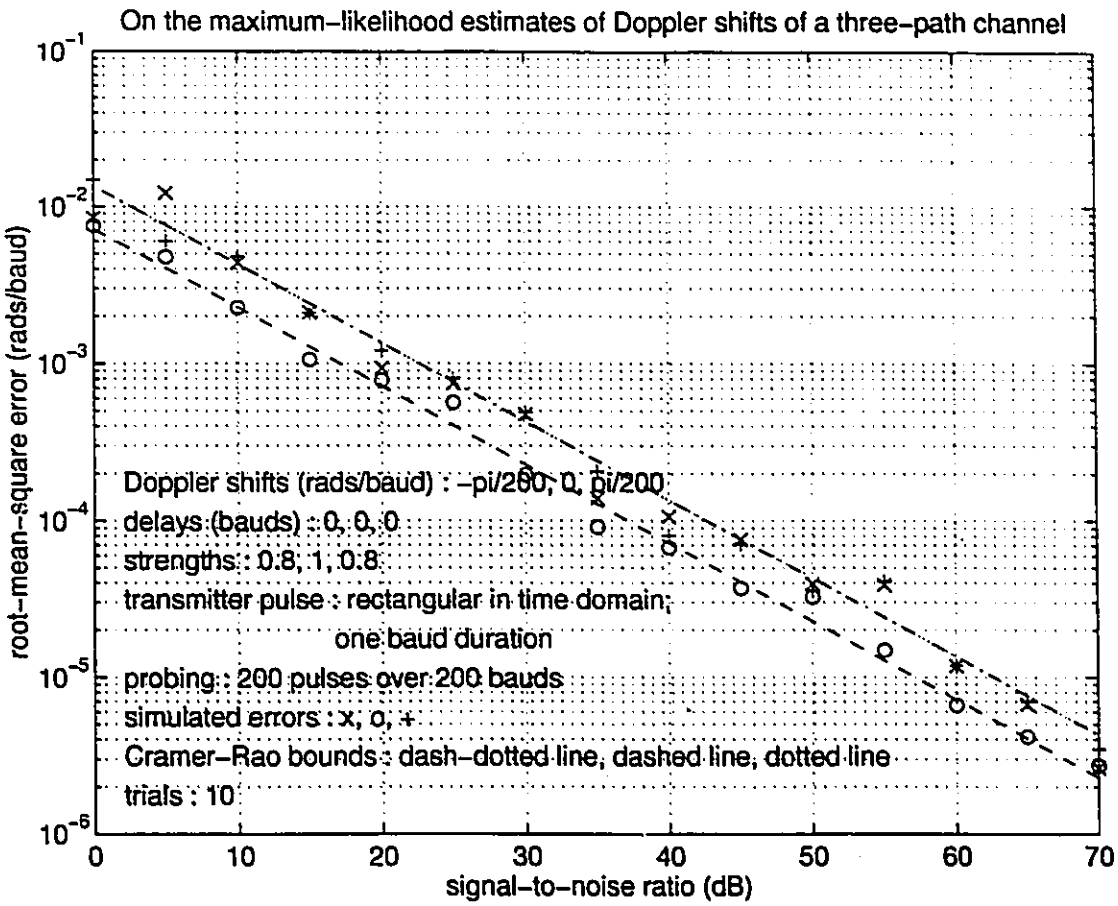

Figure 8.7: An Example in Scenario 5

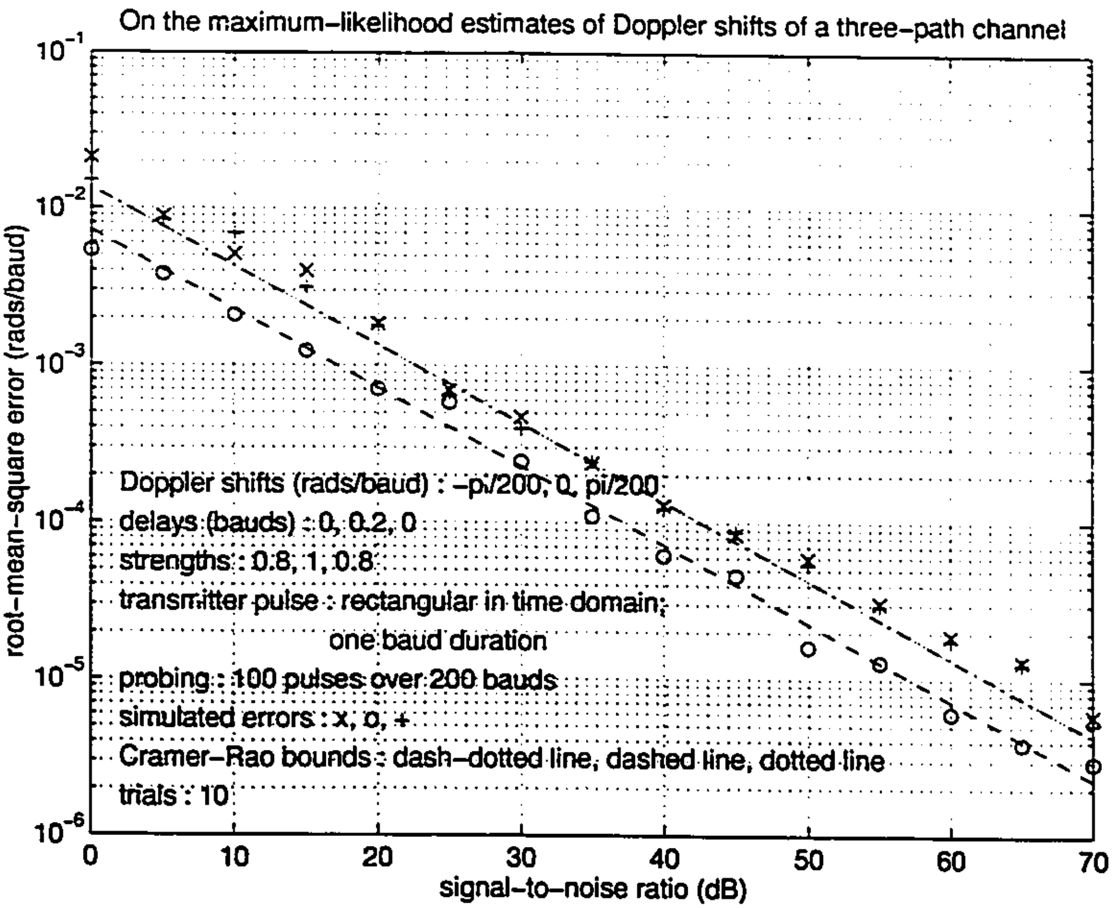

Figure 8.8: An Example in Scenario 6

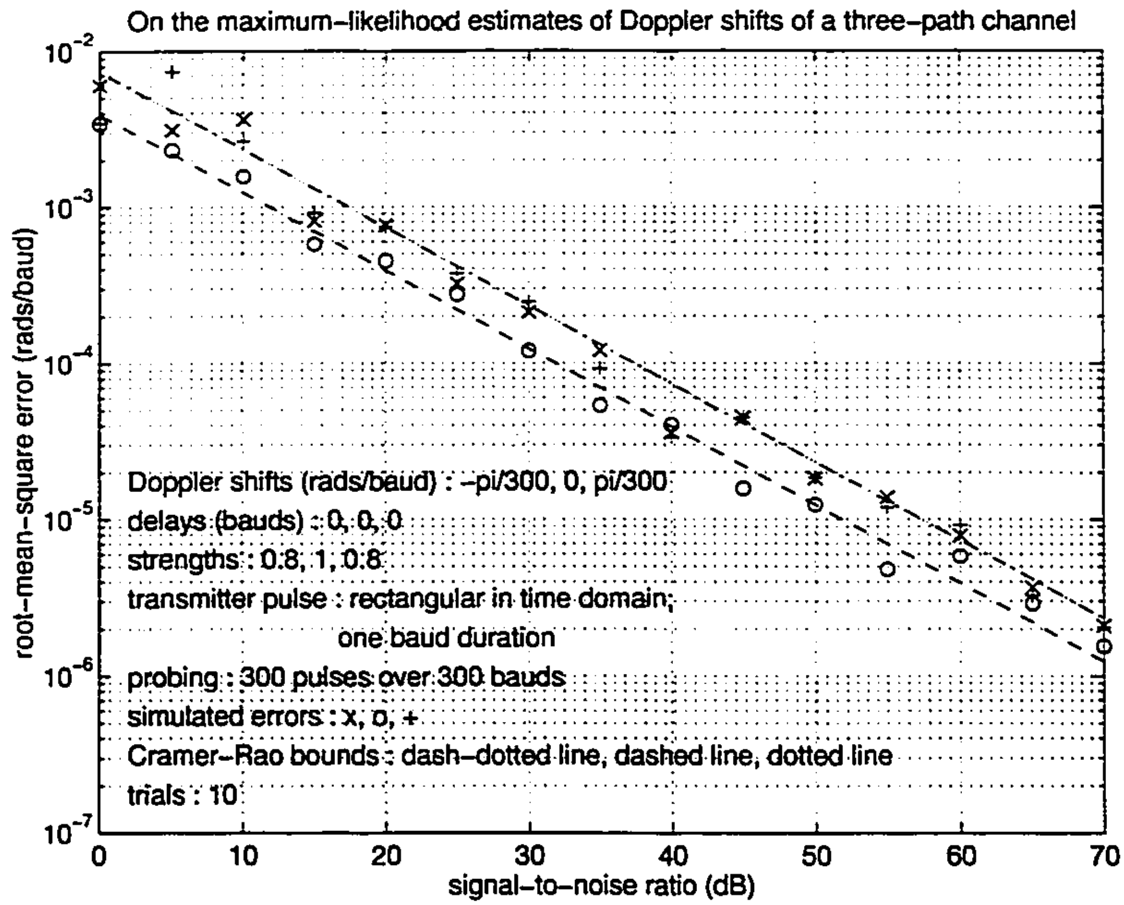

Figure 8.9: An Example in Scenario 3

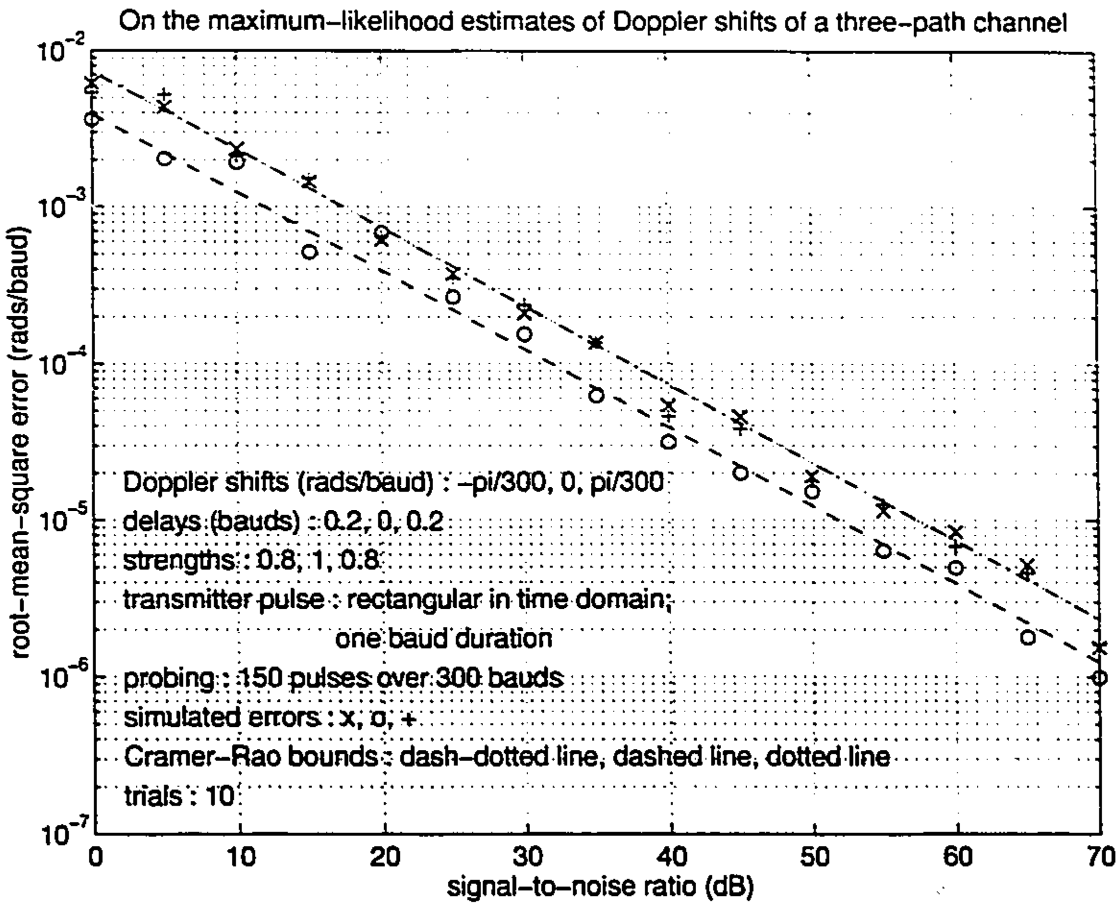

Figure 8.10: An Example in Scenario 4

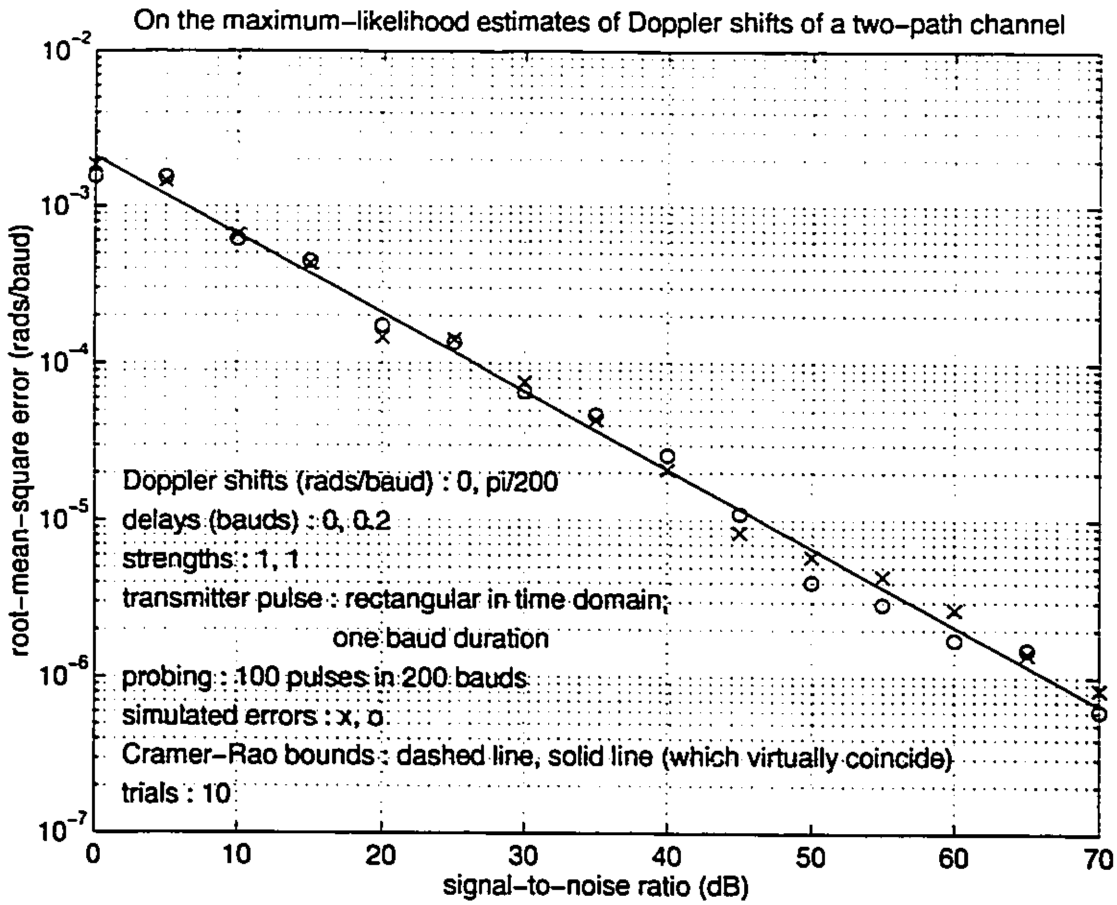

Figure 8.11: An Example in Scenario 2

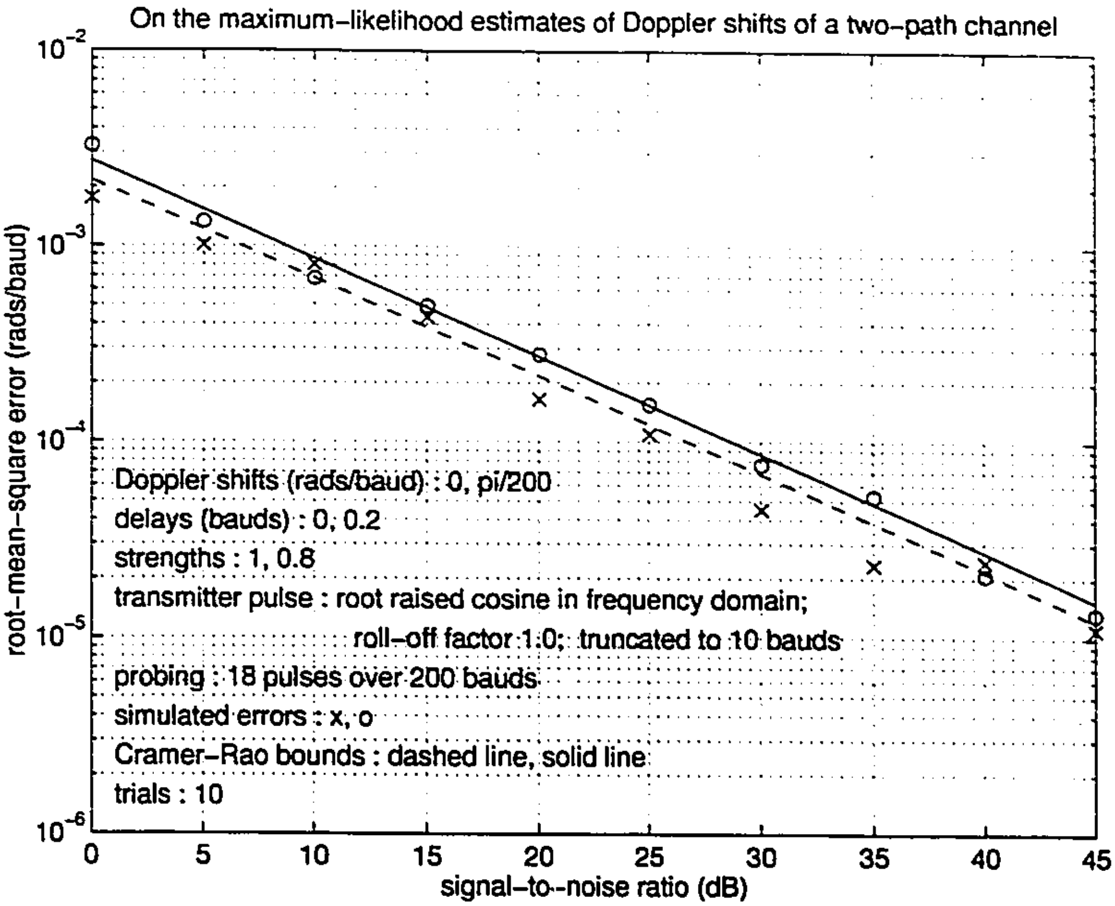

Figure 8.12: An Example in Scenario 2

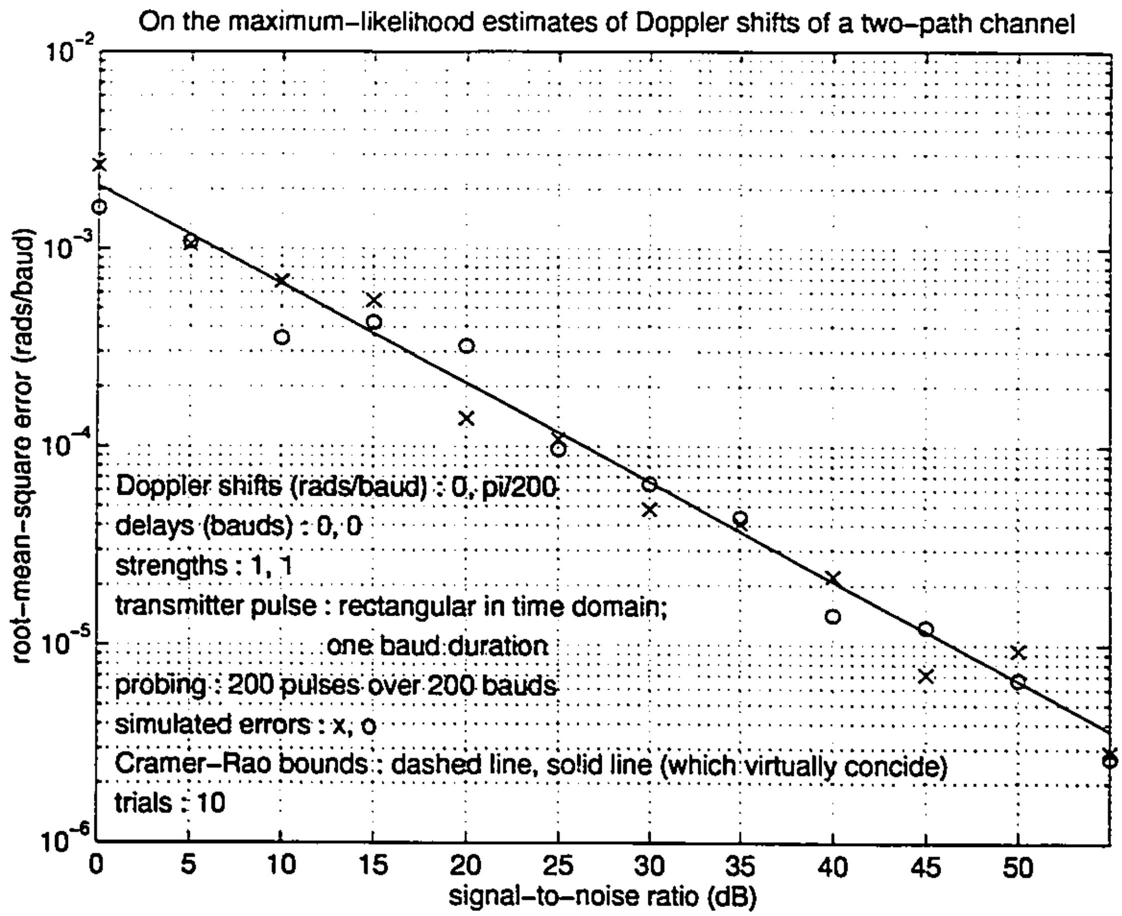

Figure 8.13: An Example in Scenario 1

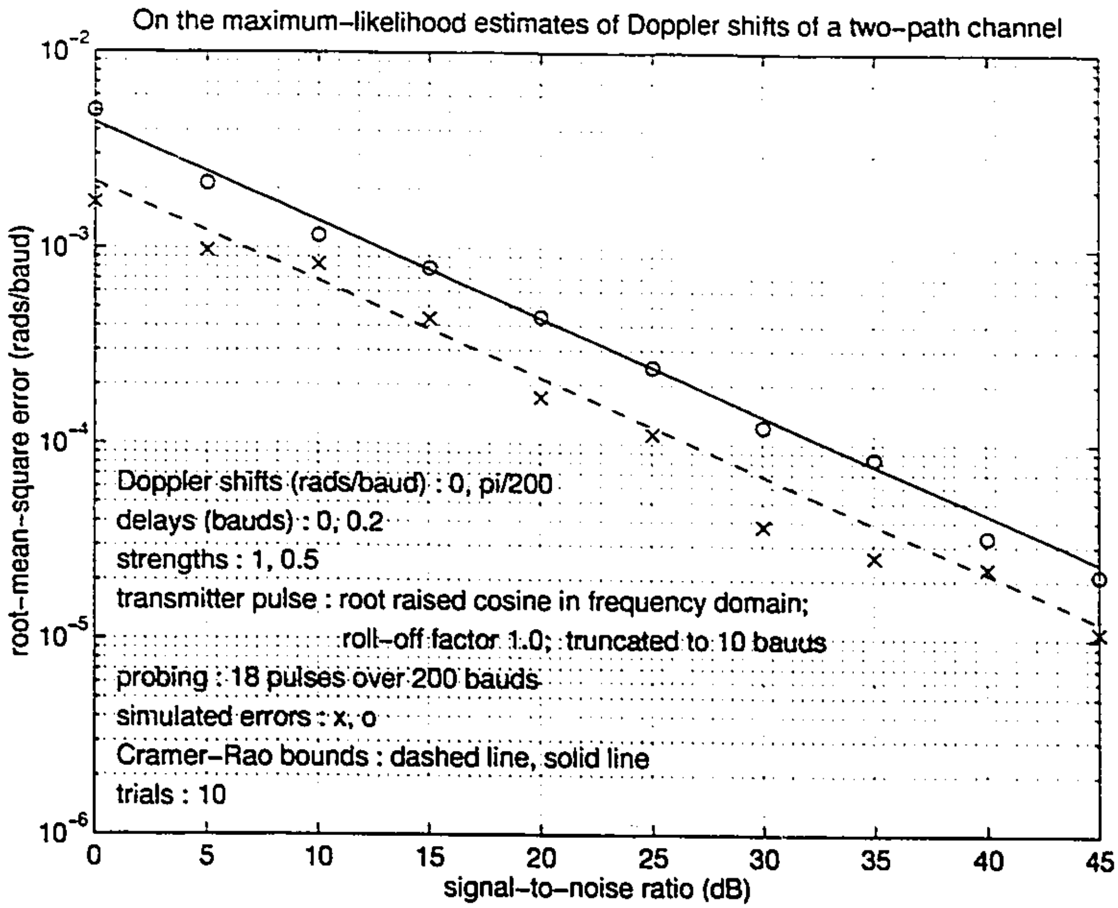

Figure 8.14: An Example in Scenario 2

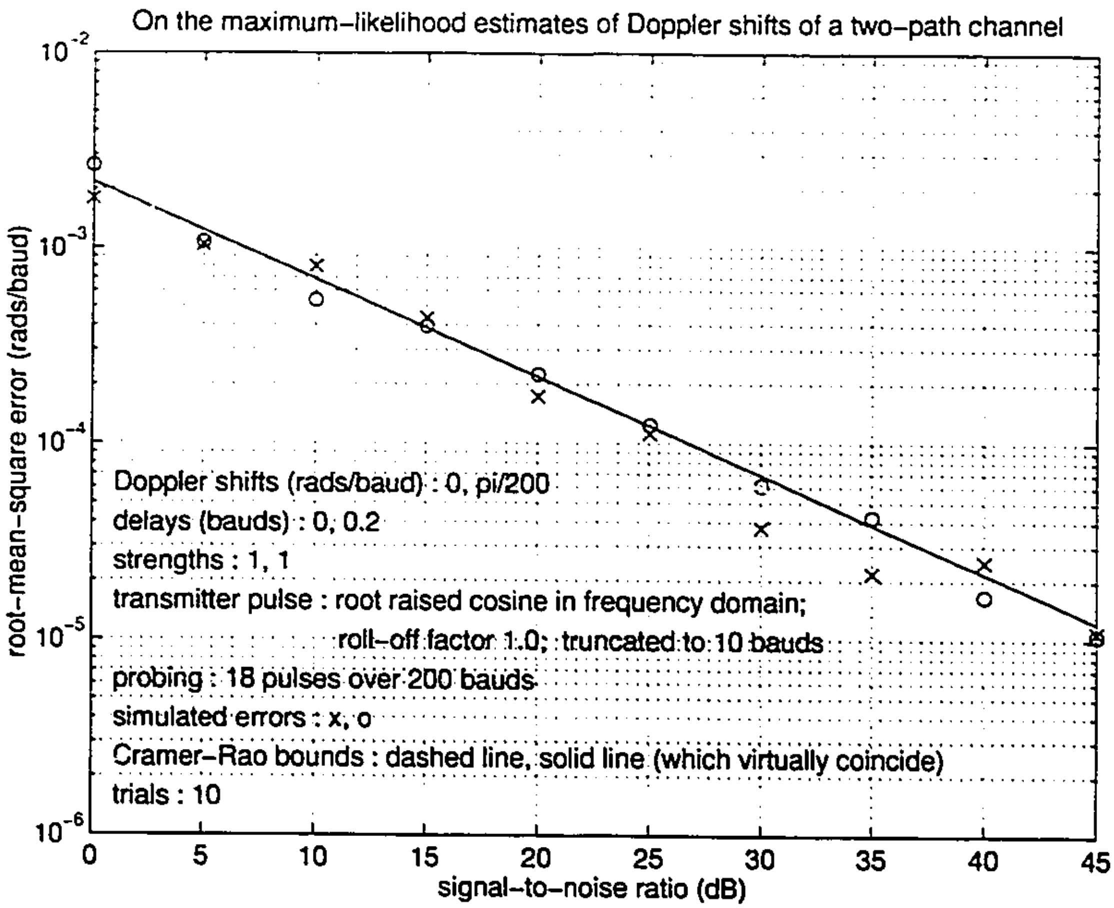

Figure 8.15: An Example in Scenario 2

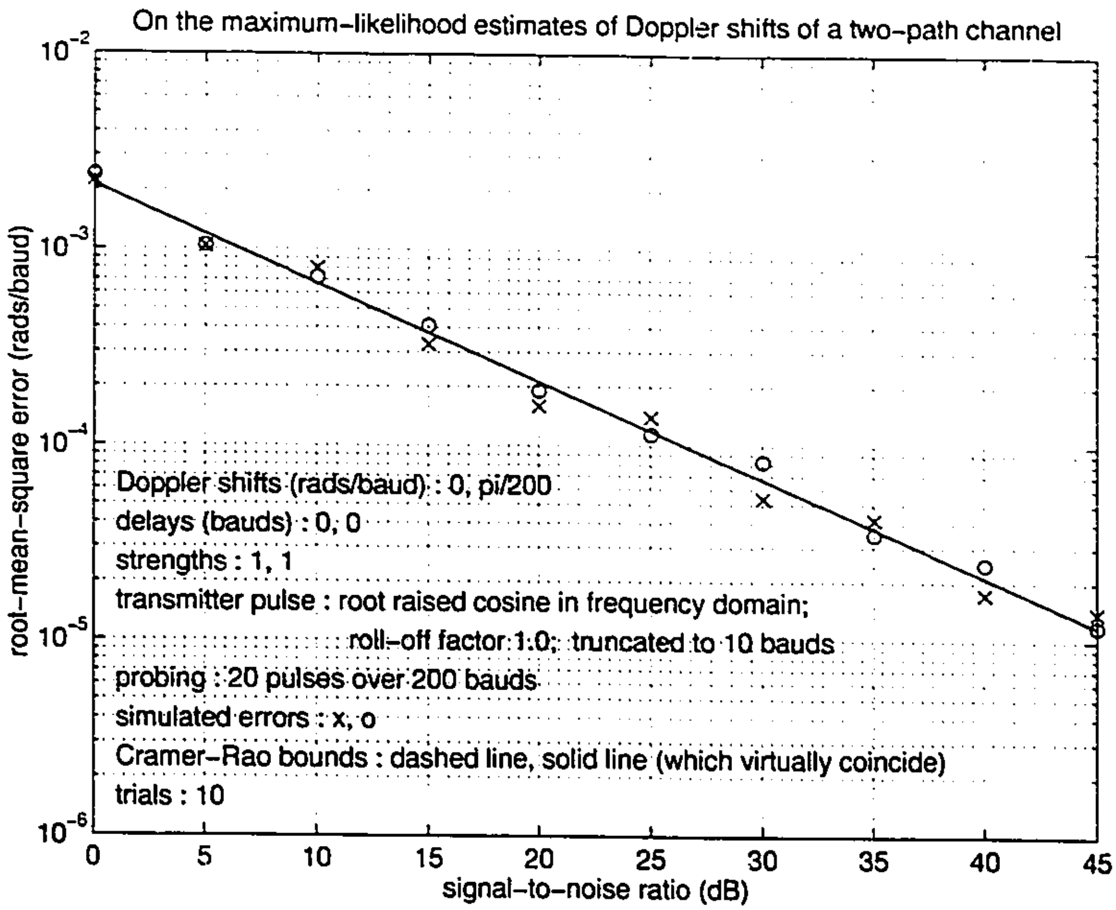

Figure 8.16: An Example in Scenario 1

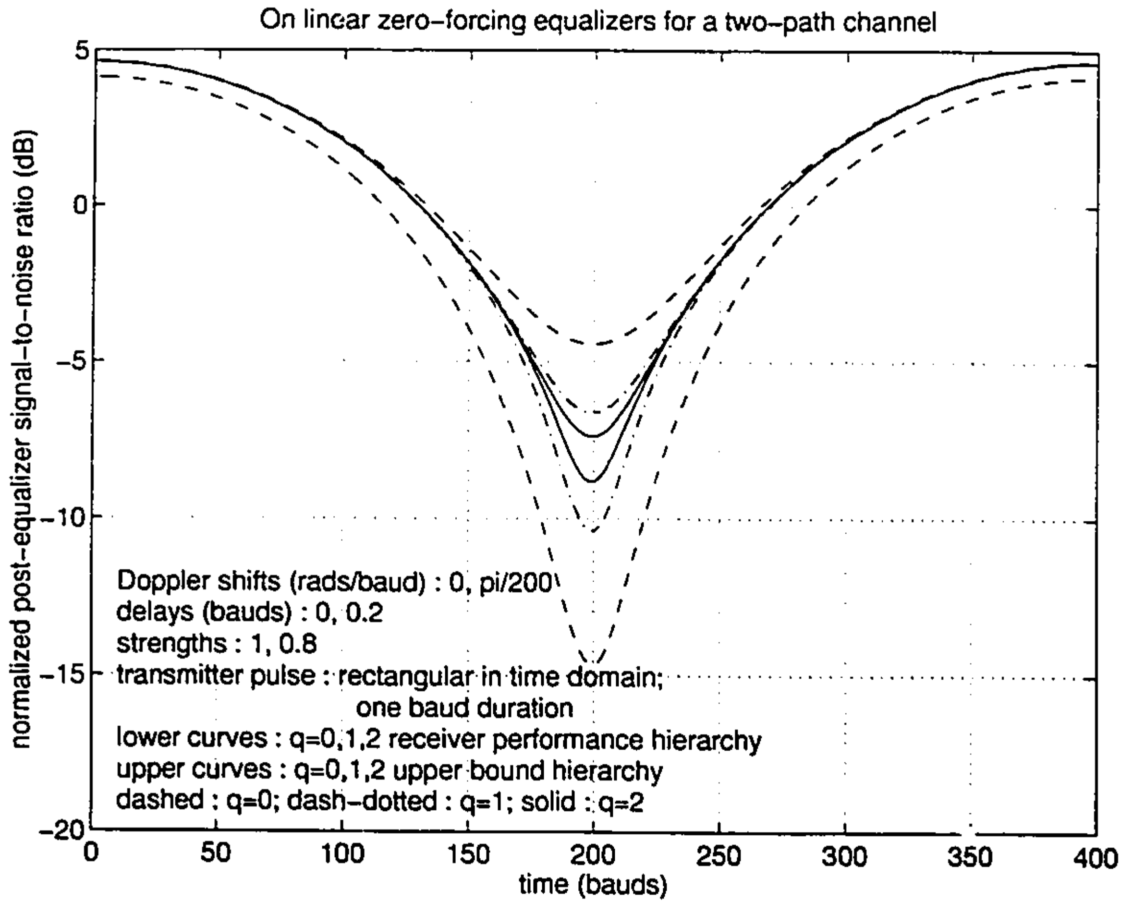

Figure 8.17: An Example in Scenario 2

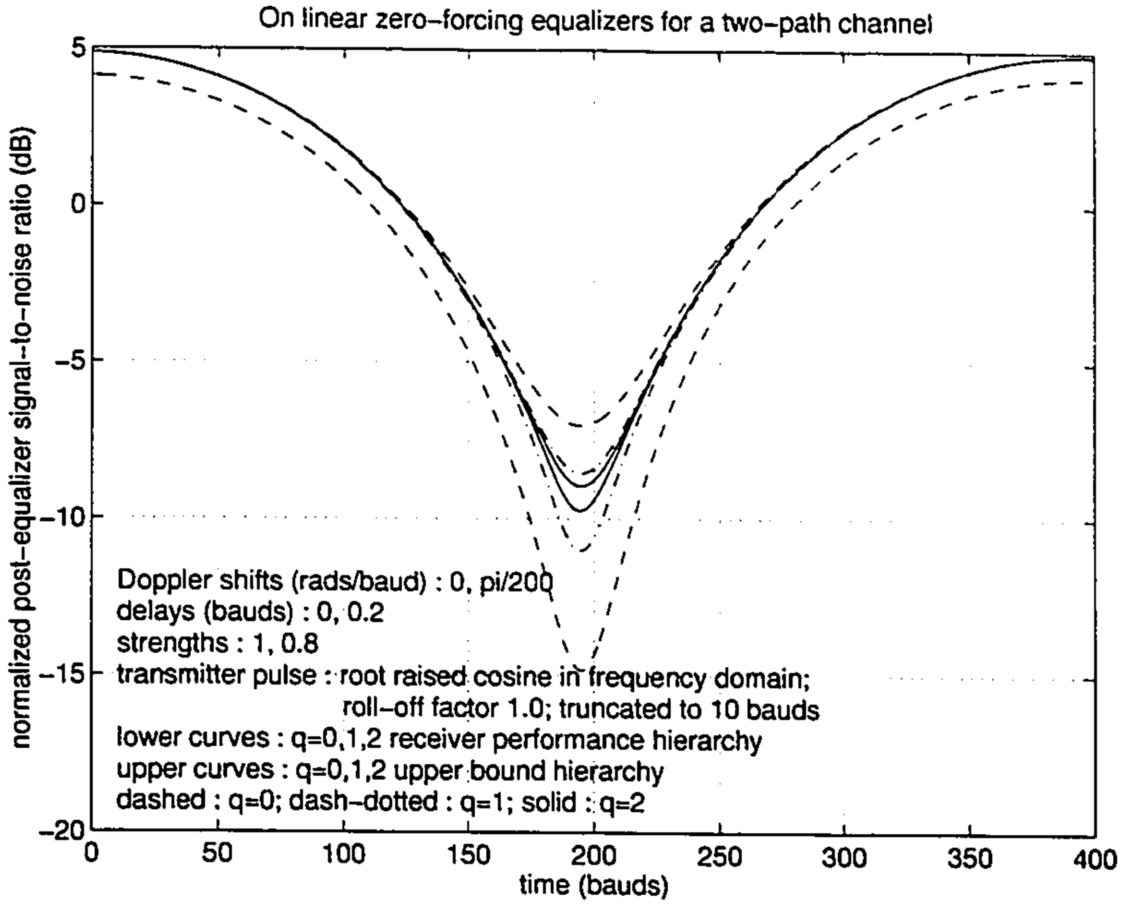

Figure 8.18: An Example in Scenario 2

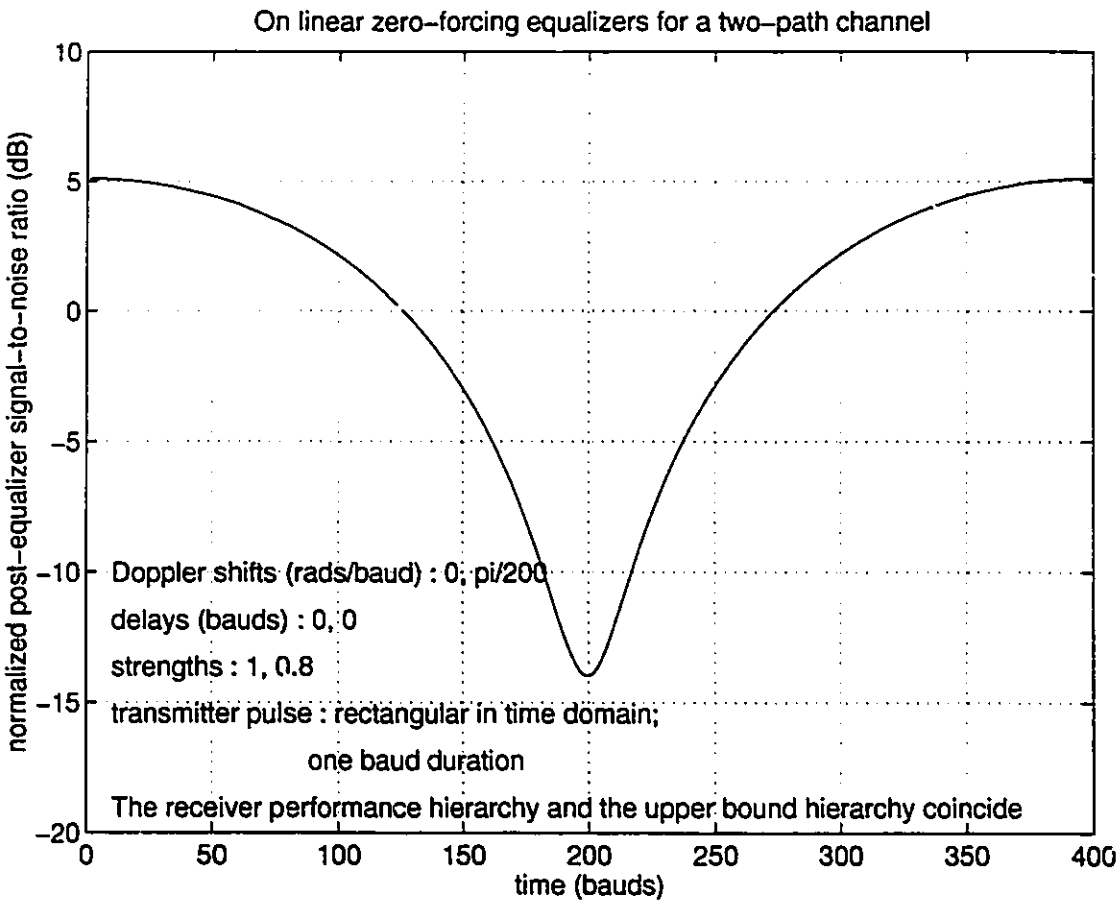

Figure 8.19: An Example in Scenario 1

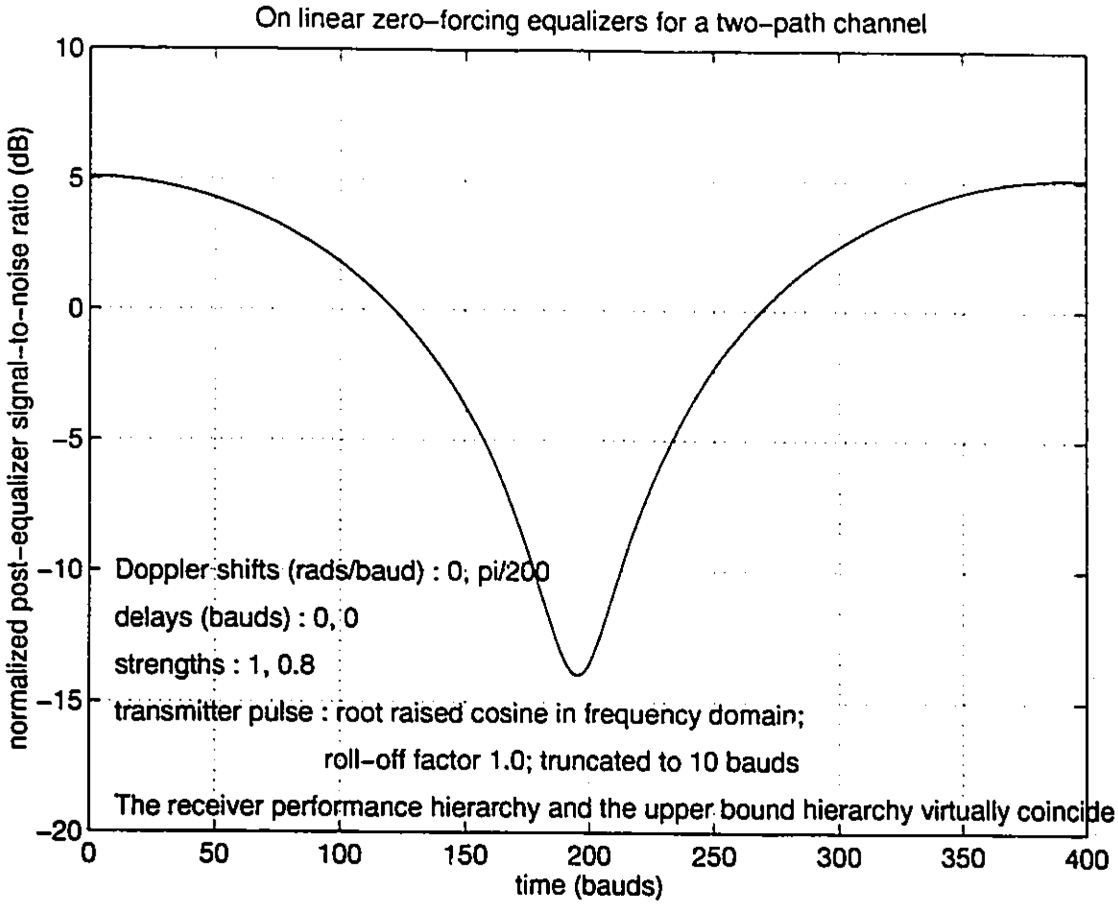

Figure 8.20: An Example in Scenario 1

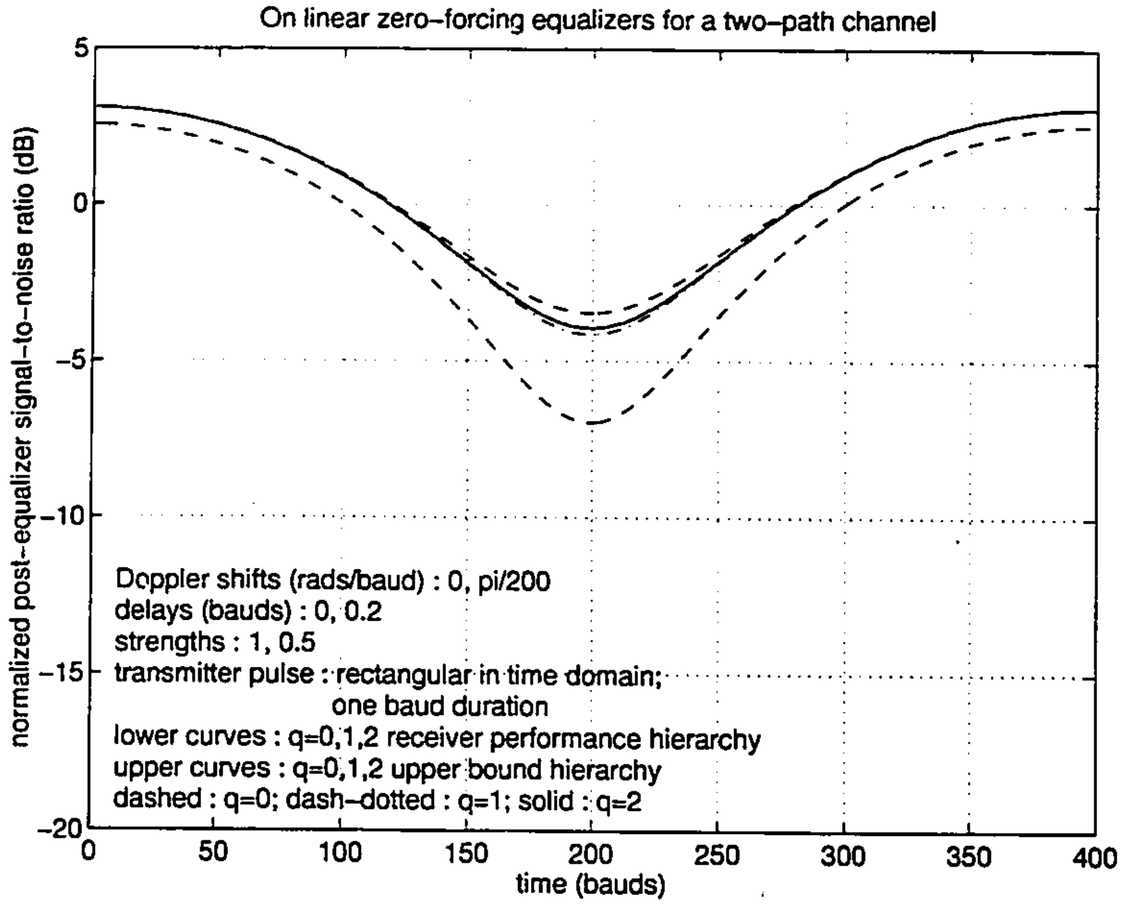

Figure 8.21: An Example in Scenario 2

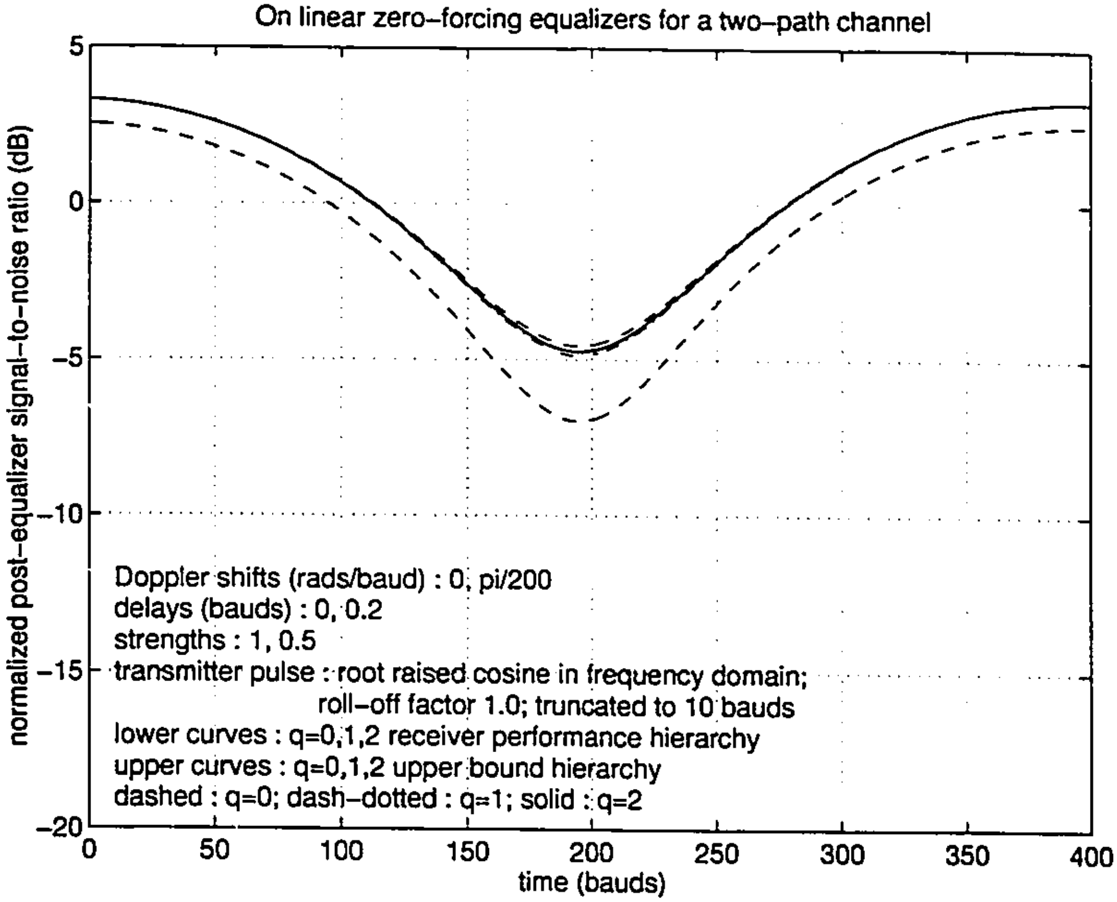

Figure 8.22: An Example in Scenario 1

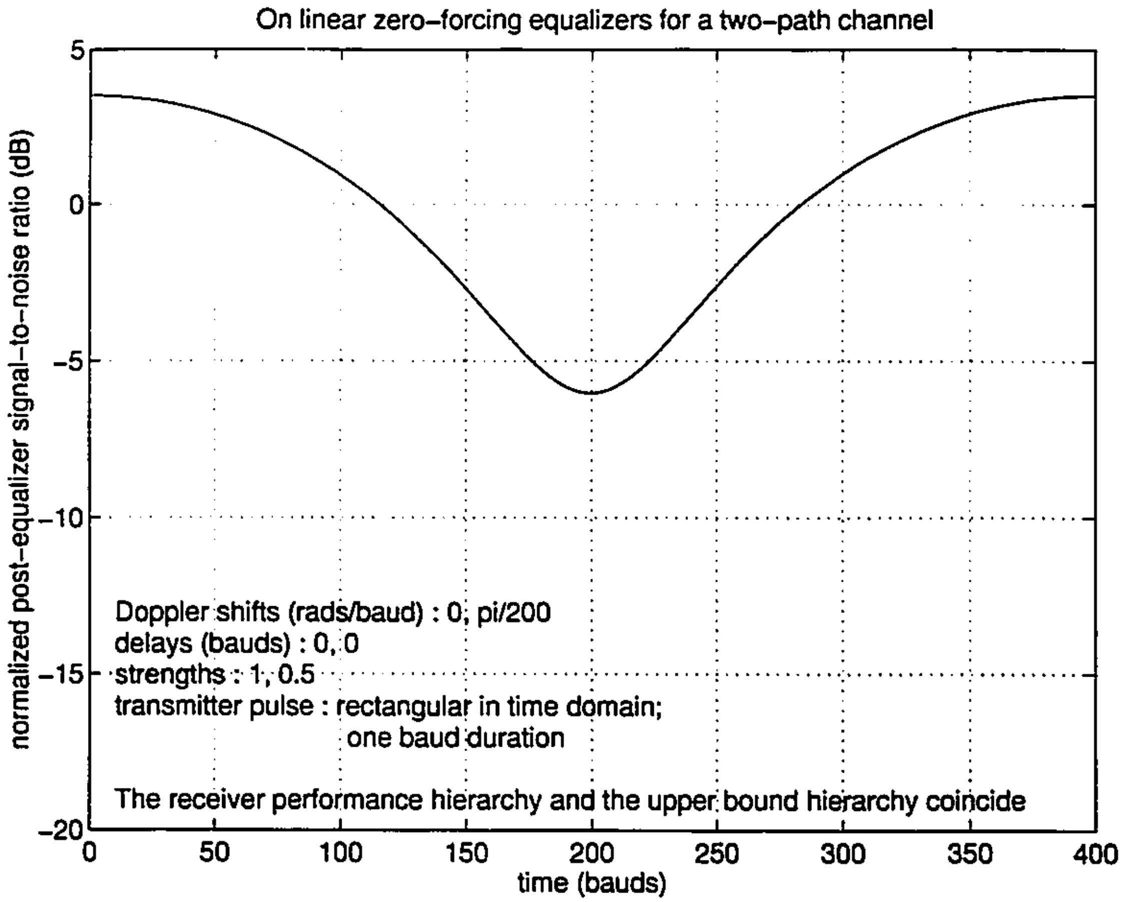

Figure 8.23: An Example in Scenario 1

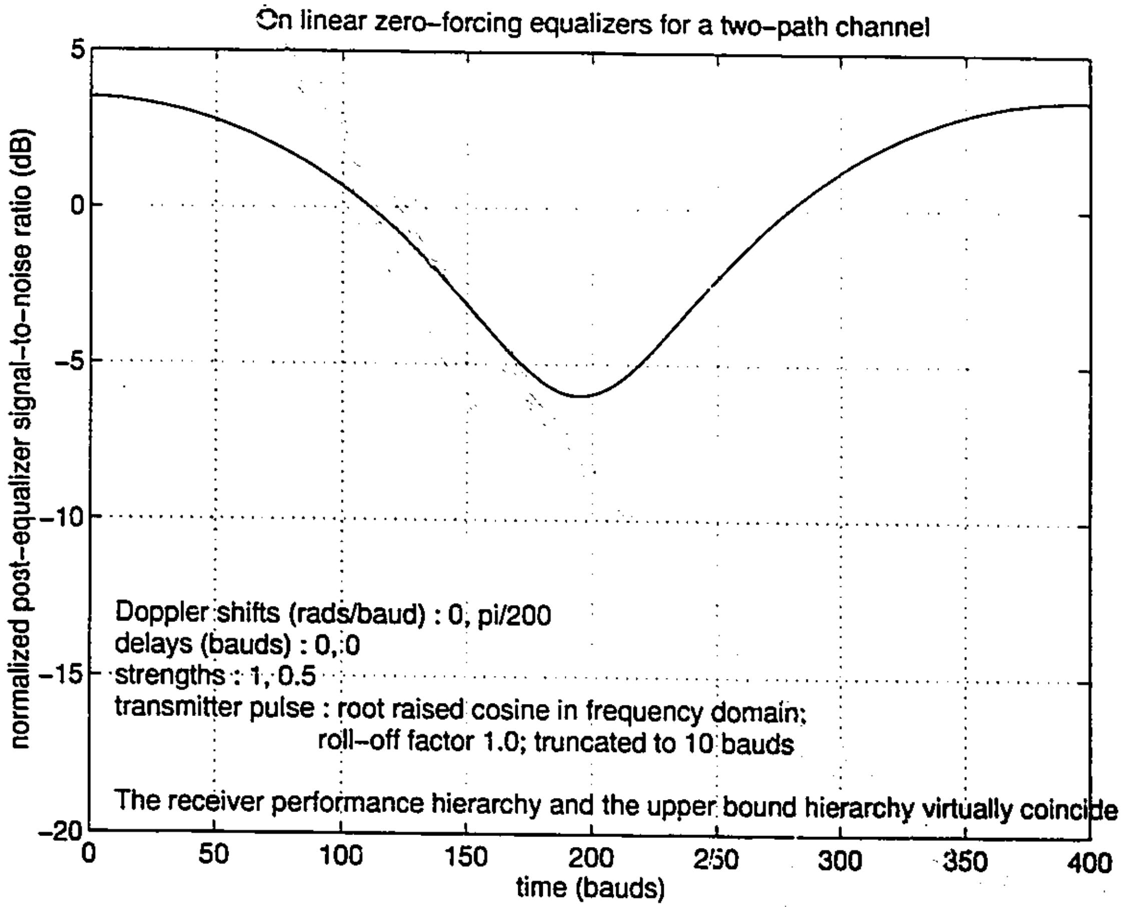

Figure 8.24: An Example in Scenario 1

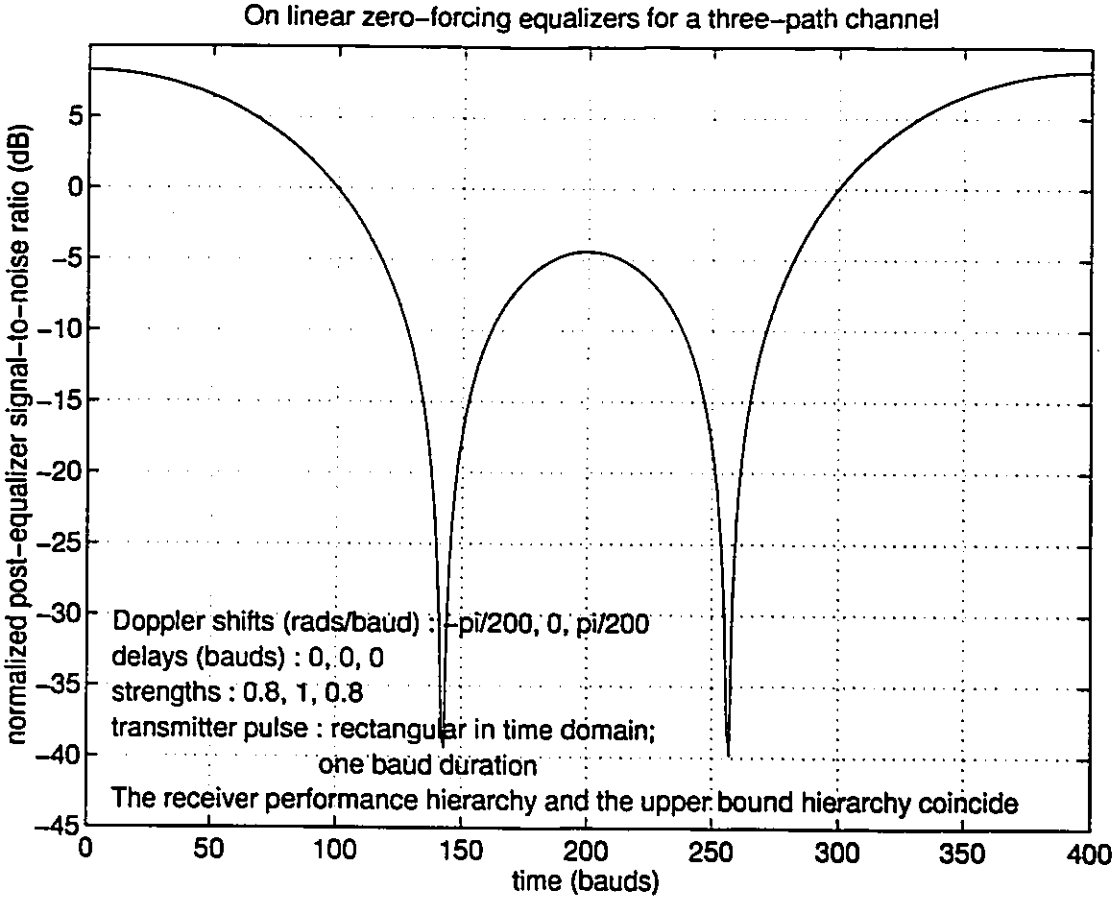

Figure 8.25: An Example in Scenario 5

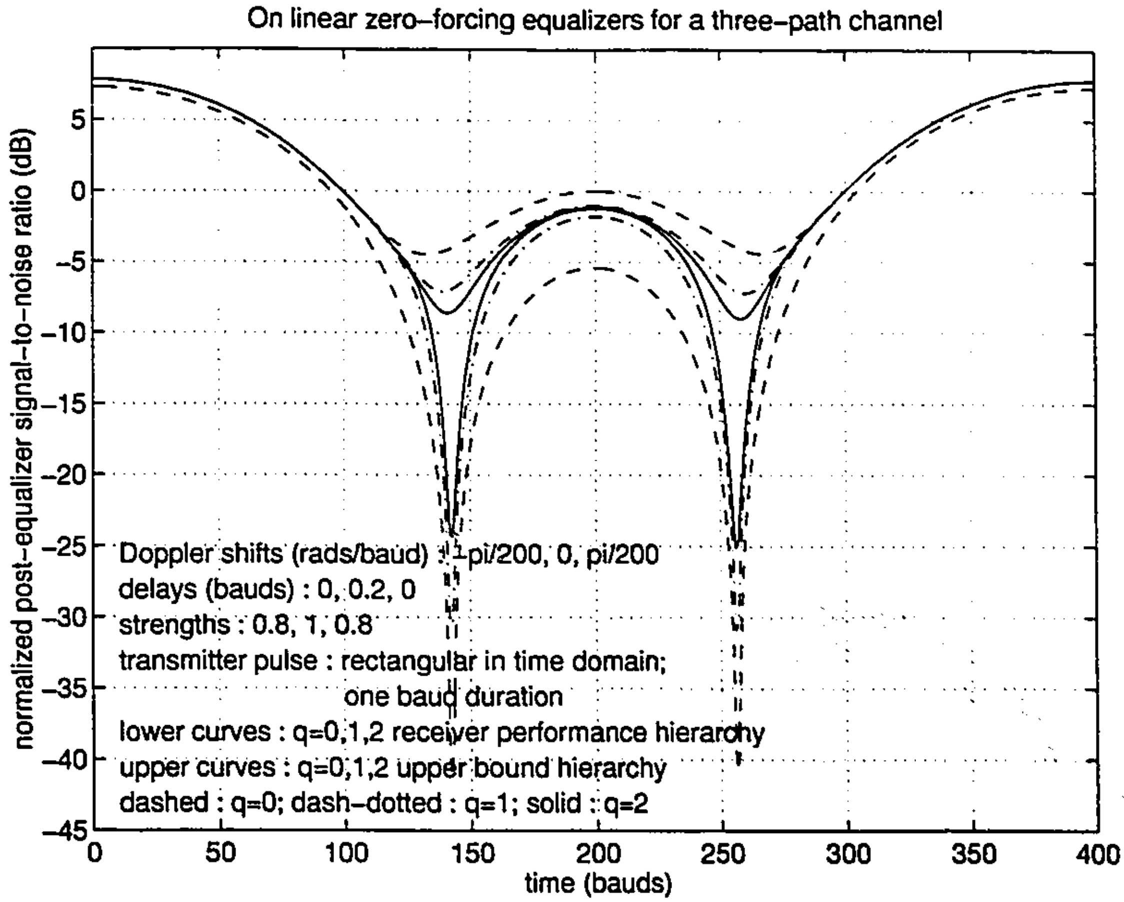

Figure 8.26: An Example in Scenario 6

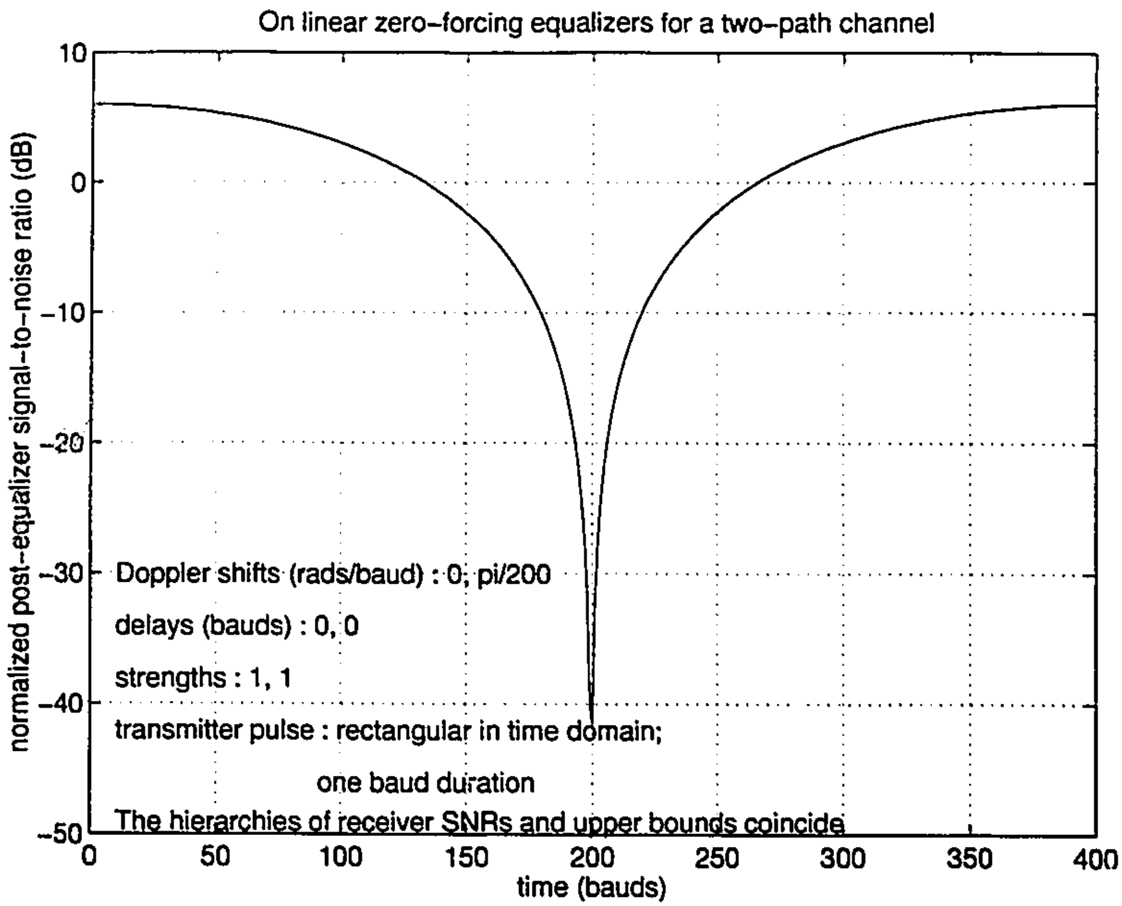

Figure 8.27: An Example in Scenario 1

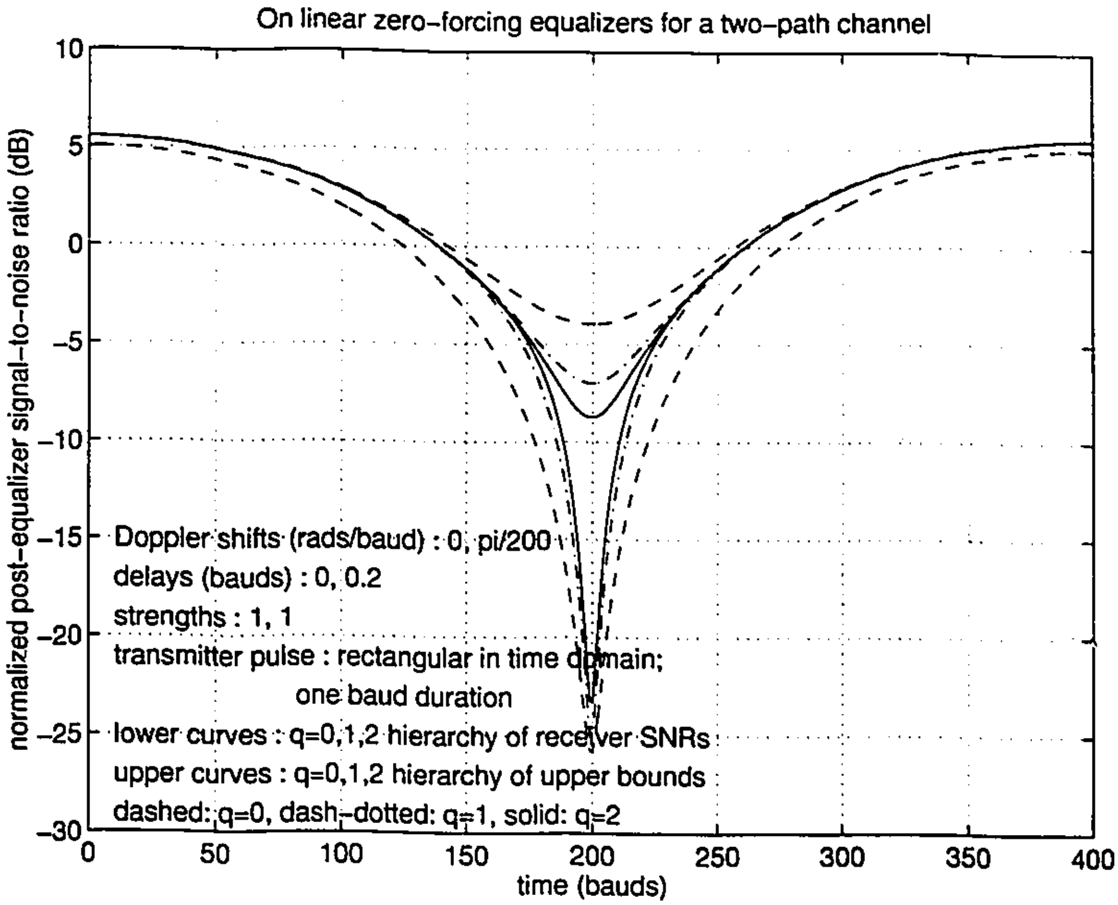

Figure 8.28: An Example in Scenario 2

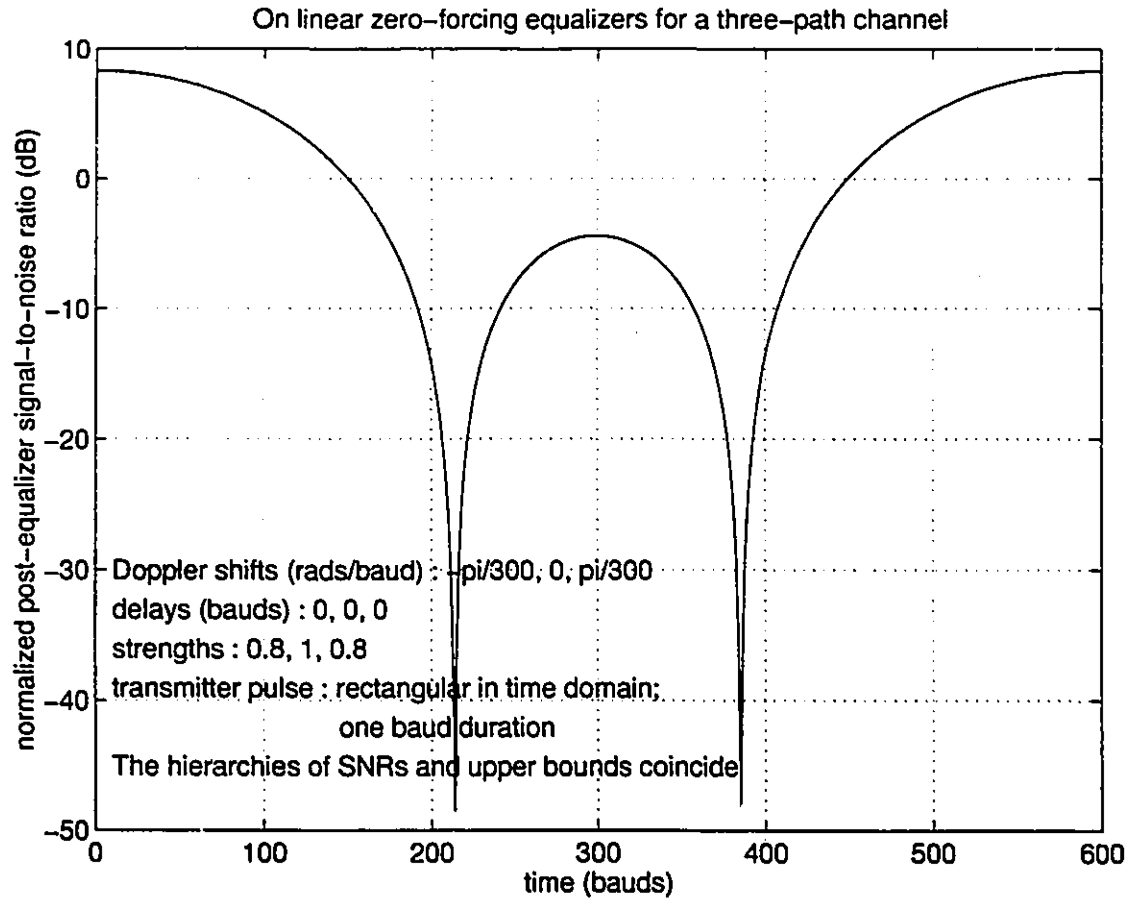

Figure 8.29: An Example in Scenario 3

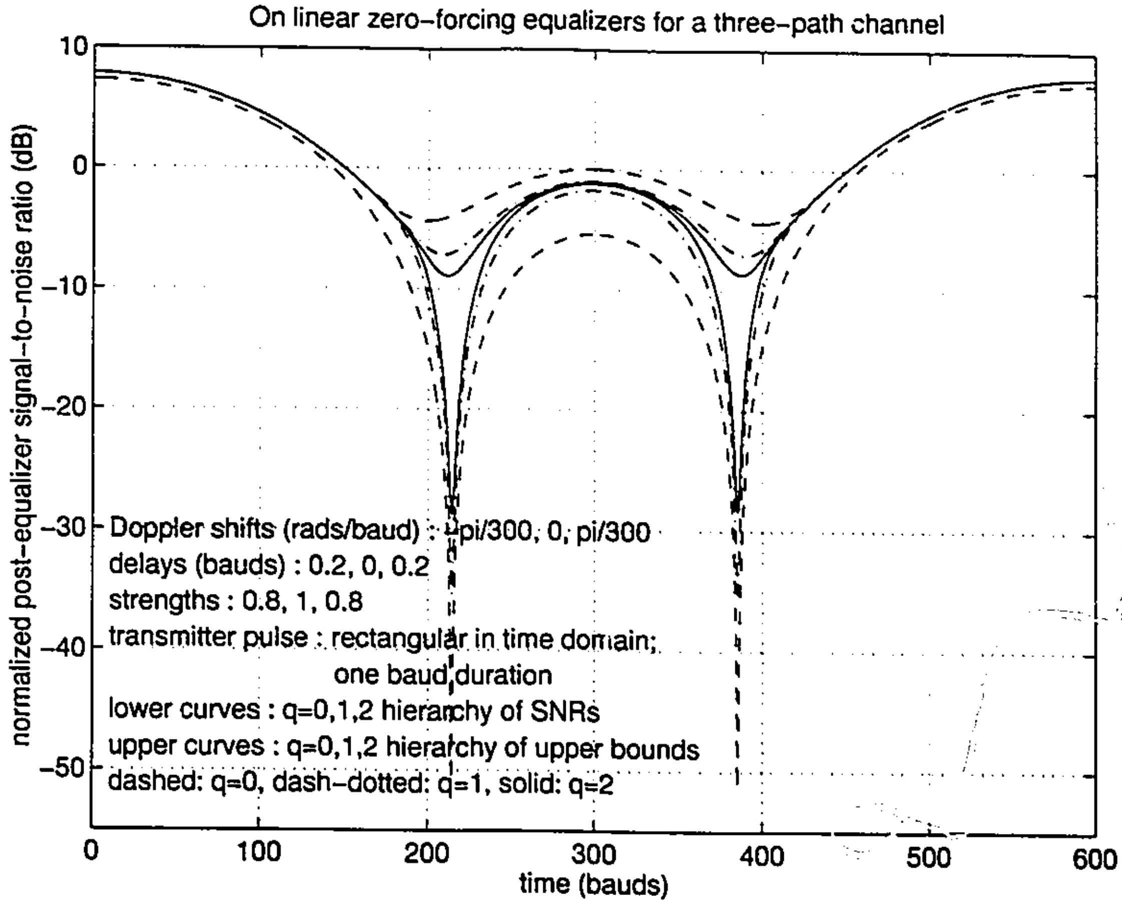

Figure 8.30: An Example in Scenario 3

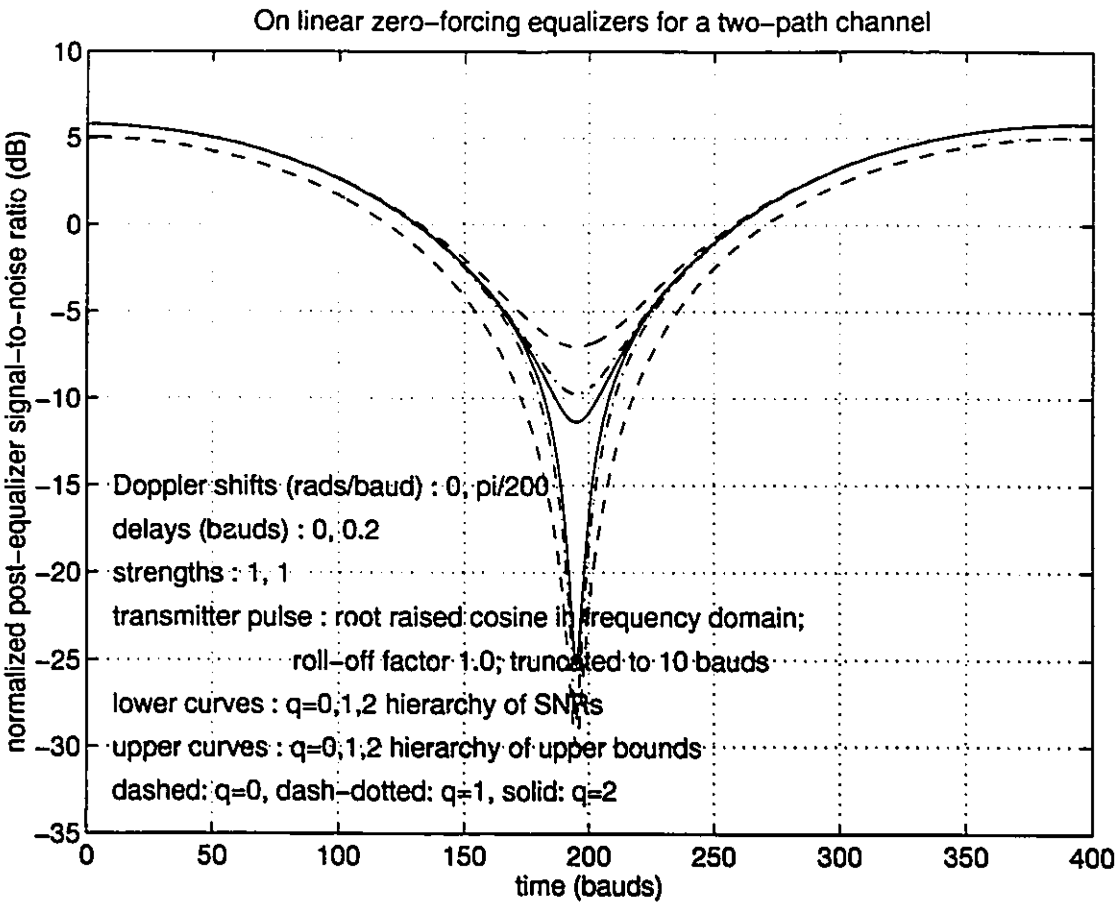

Figure 8.31: An Example in Scenario 2

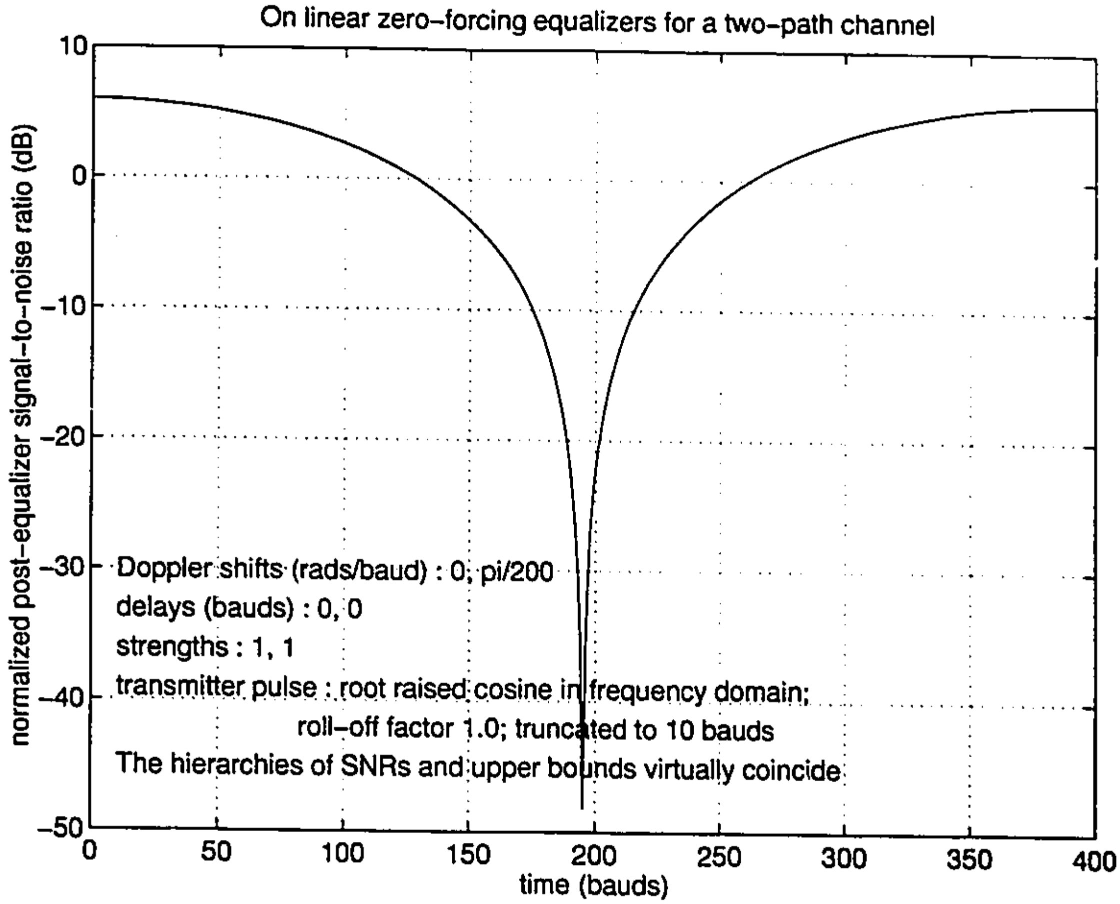

Figure 8.32: An Example in Scenario 1

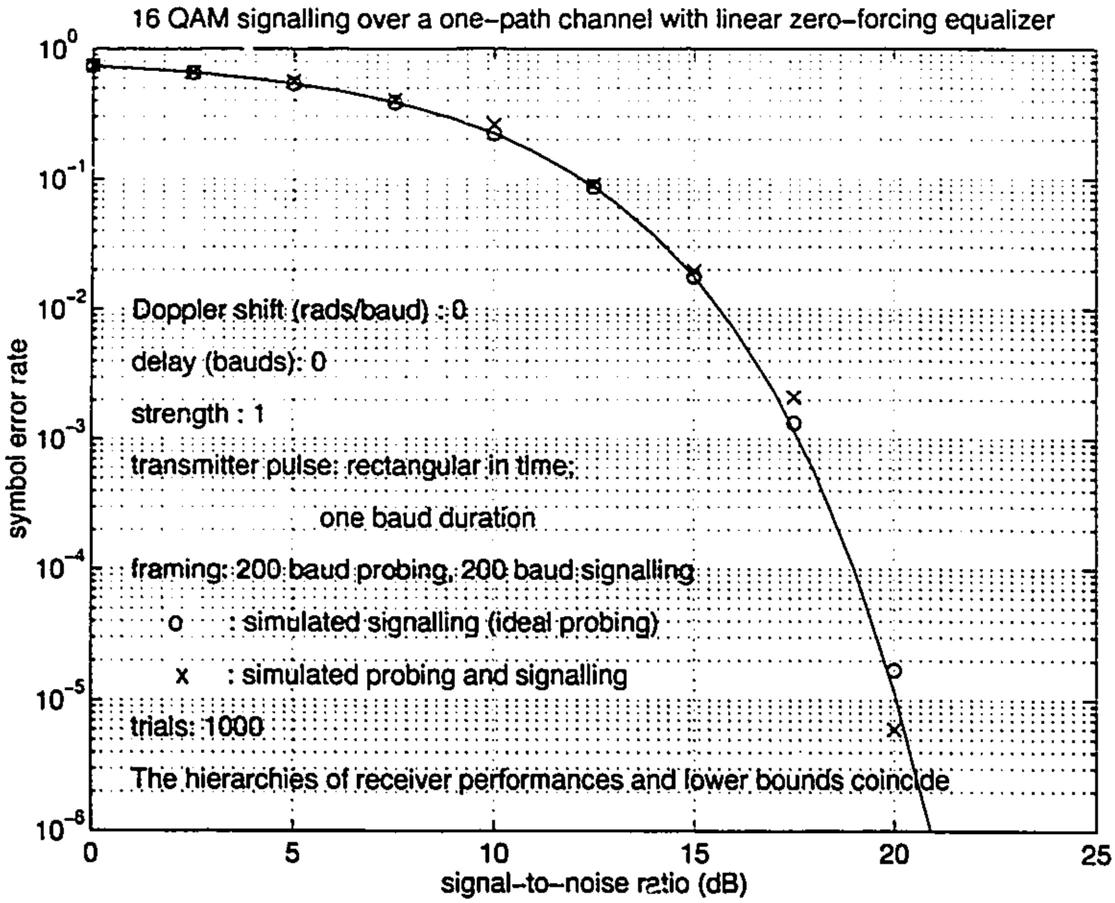

Figure 8.33: An Example in Scenario 0

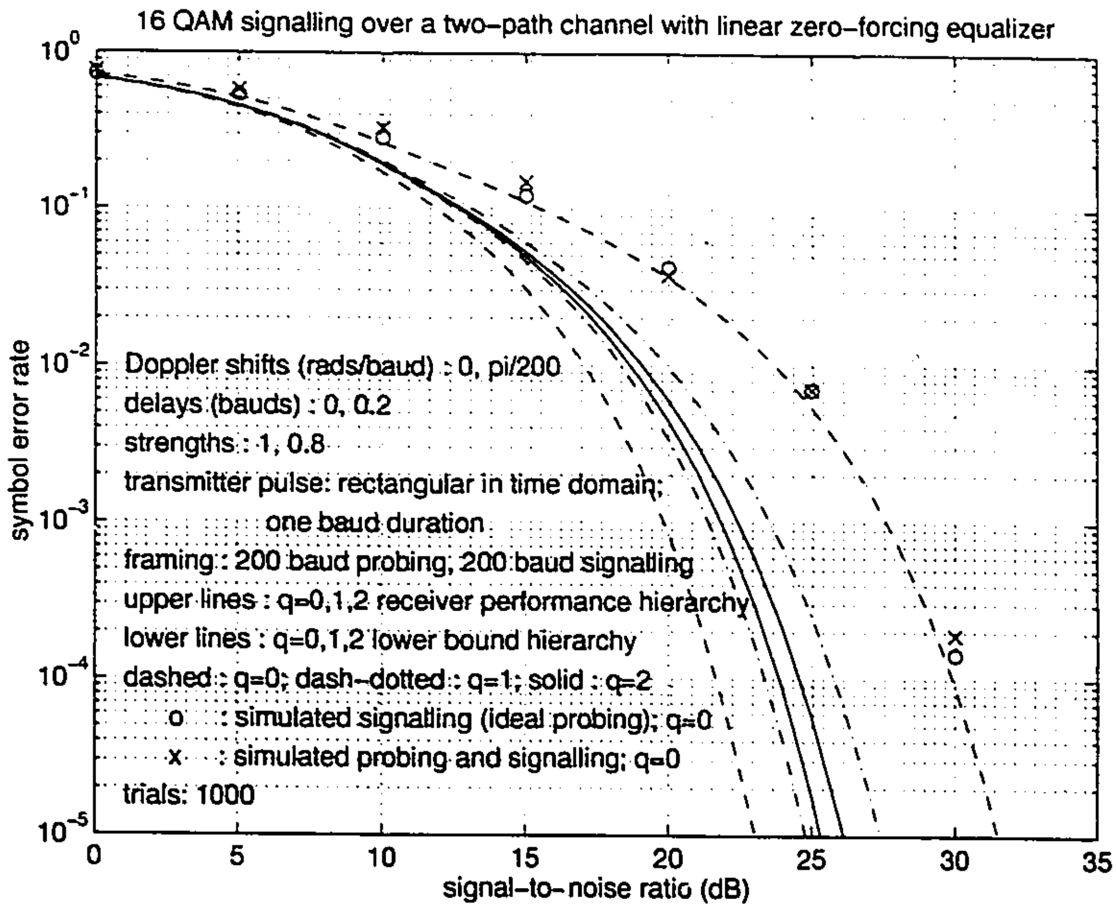

Figure 8.34: An Example in Scenario 2

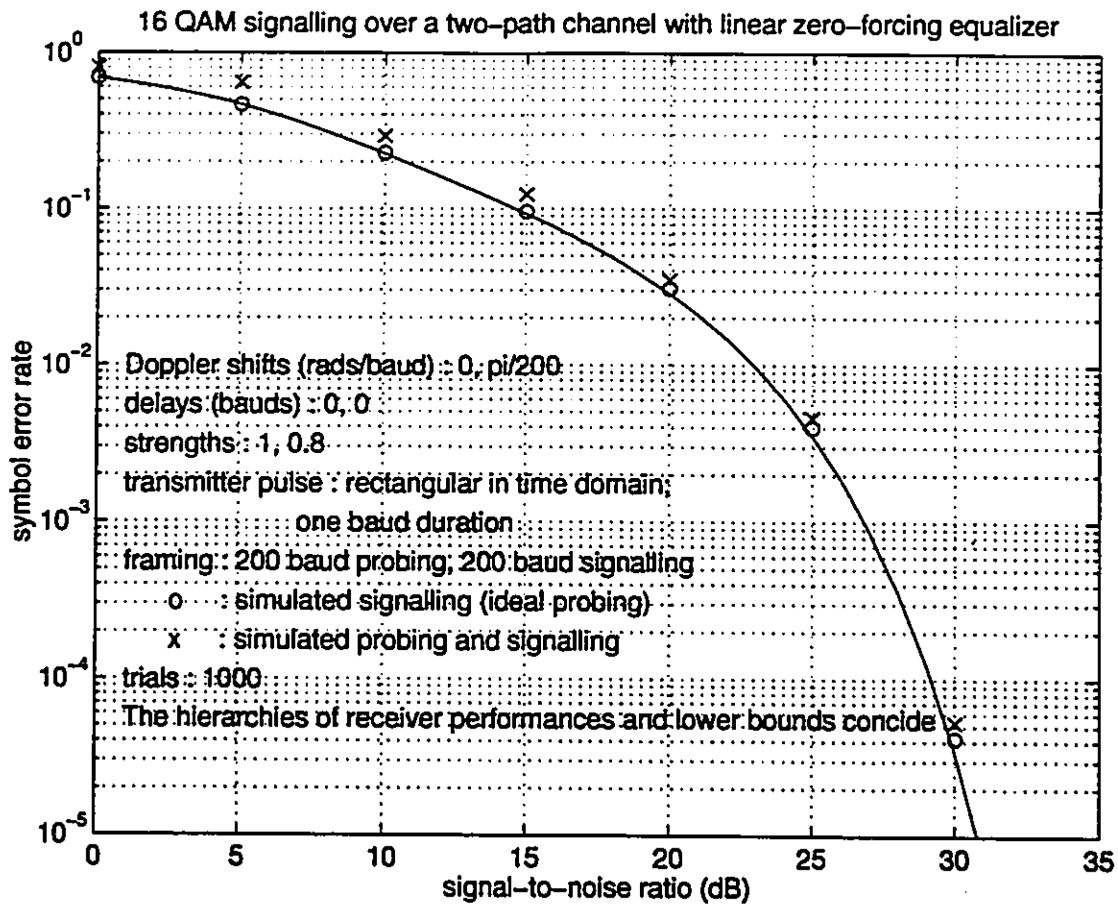

Figure 8.35: An Example in Scenario 1

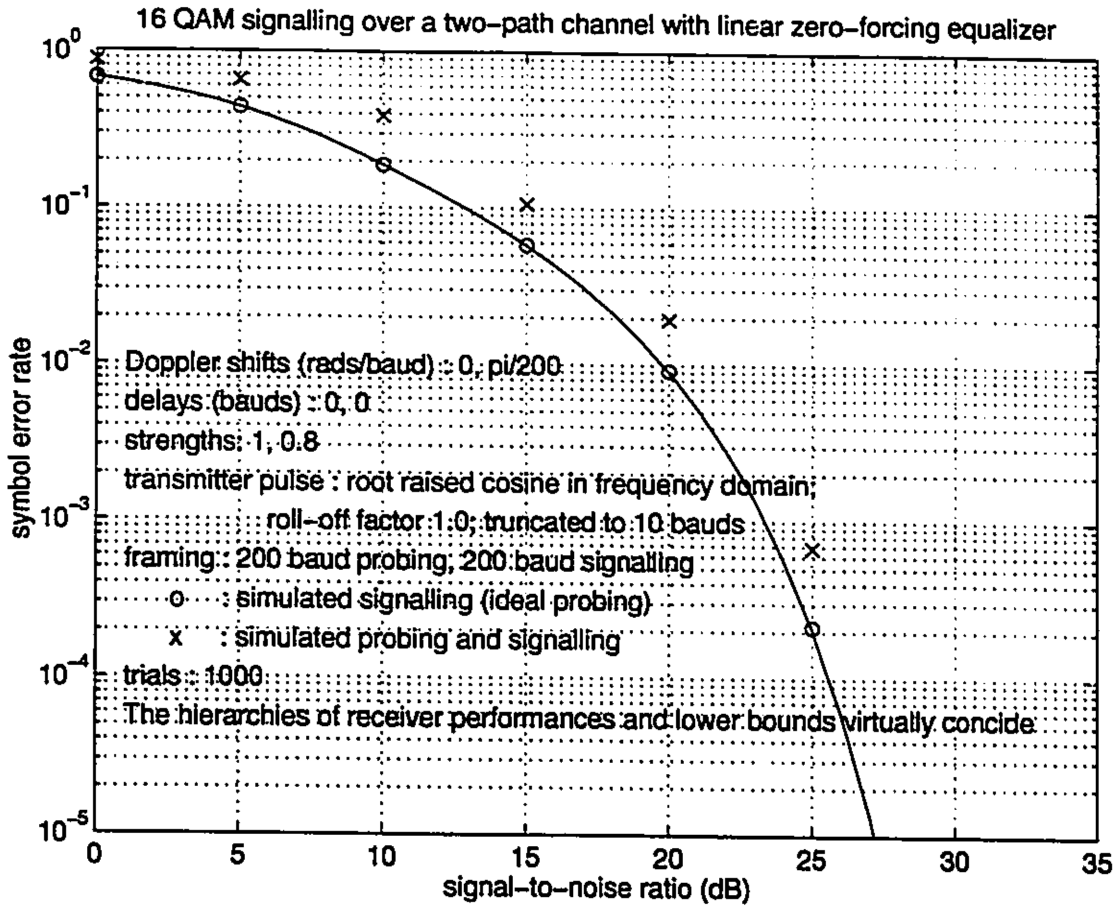

Figure 8.36: An Example in Scenario 1

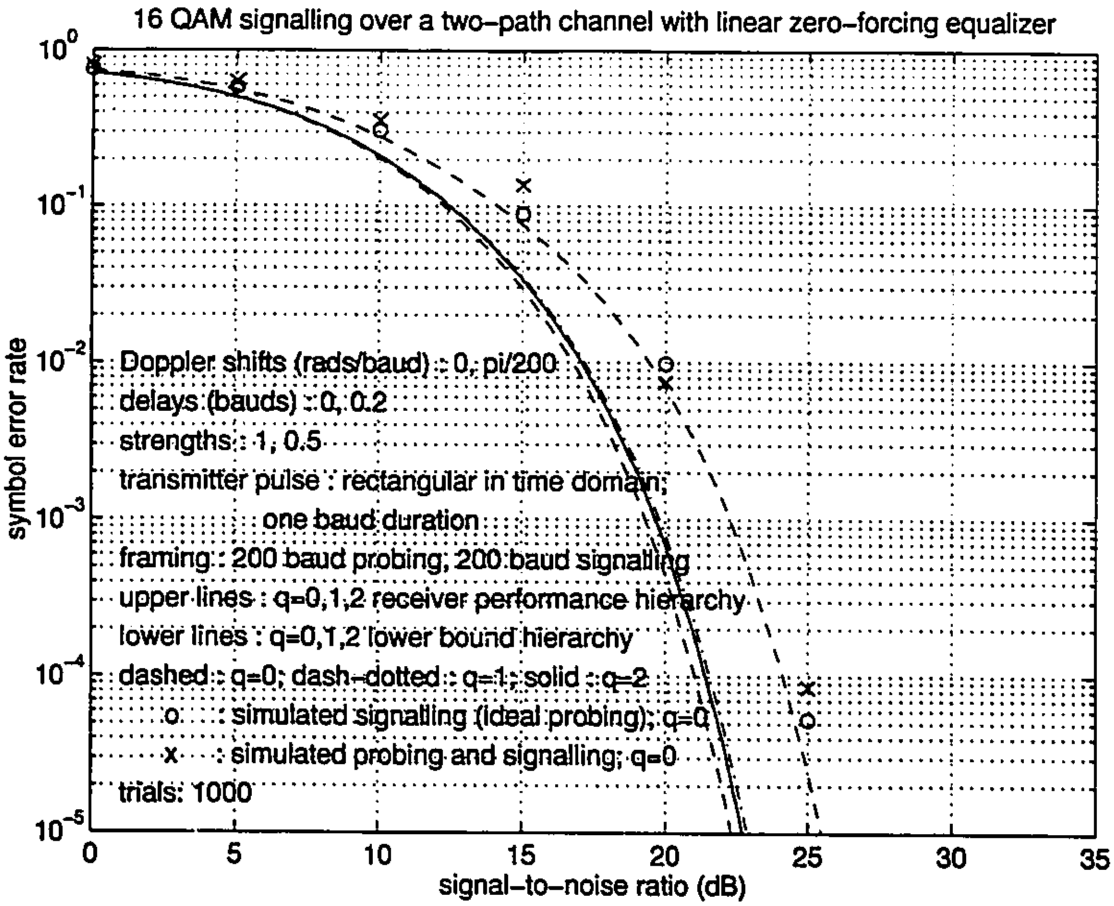

Figure 8.37: An Example in Scenario 2

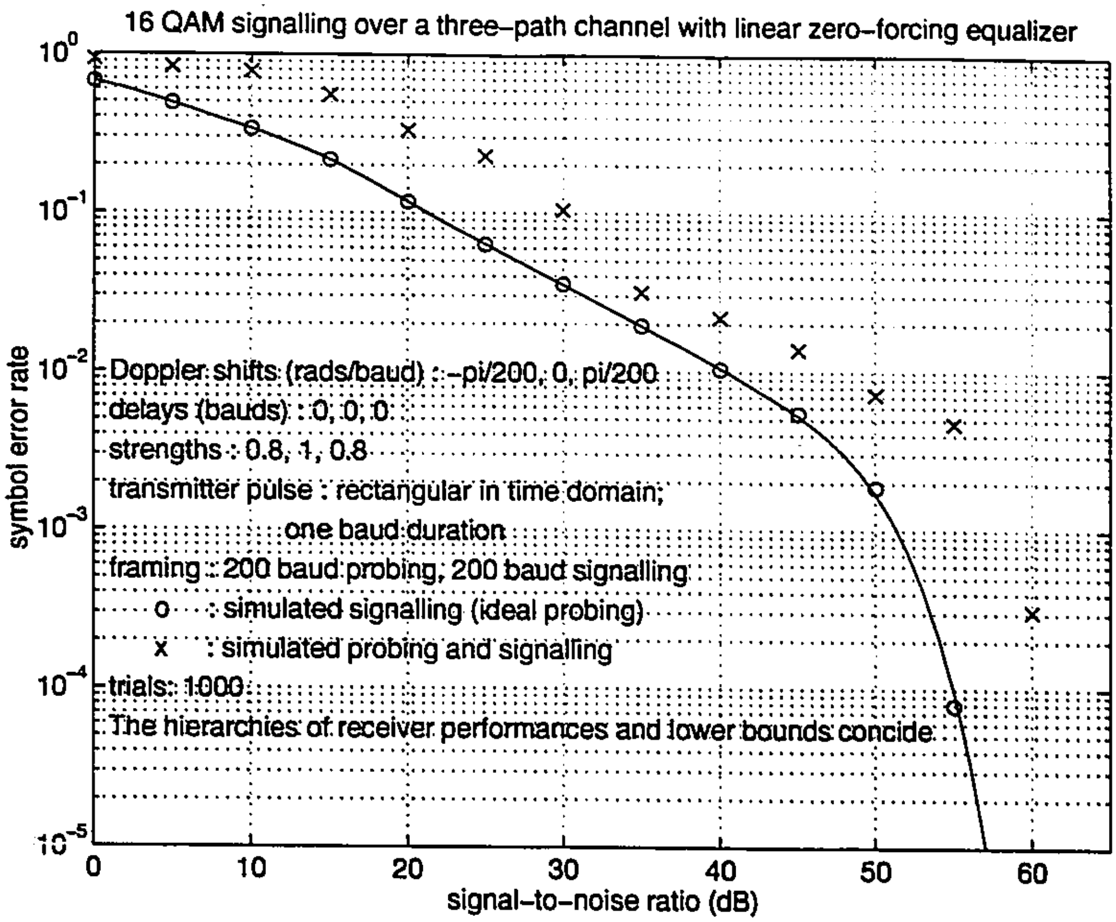

Figure 8.38: An Example in Scenario 5

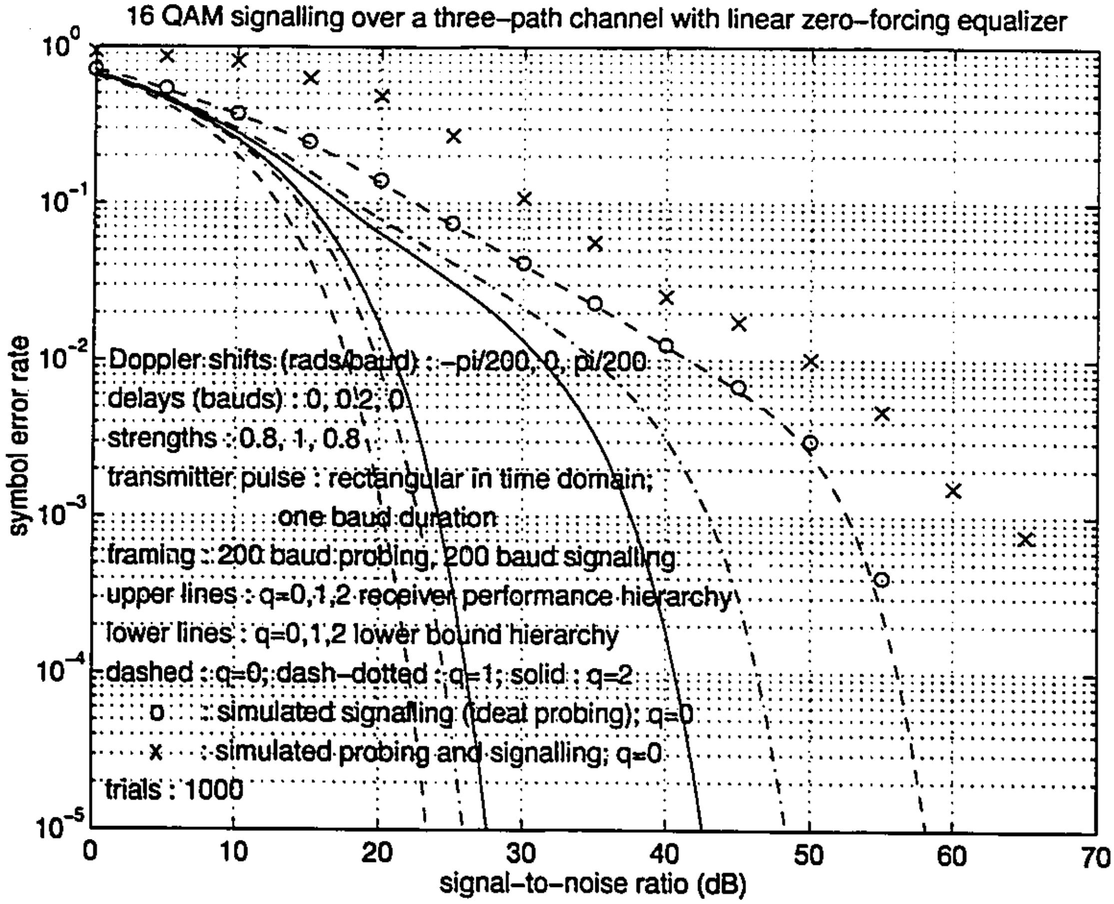

Figure 8.39: An Example in Scenario 6

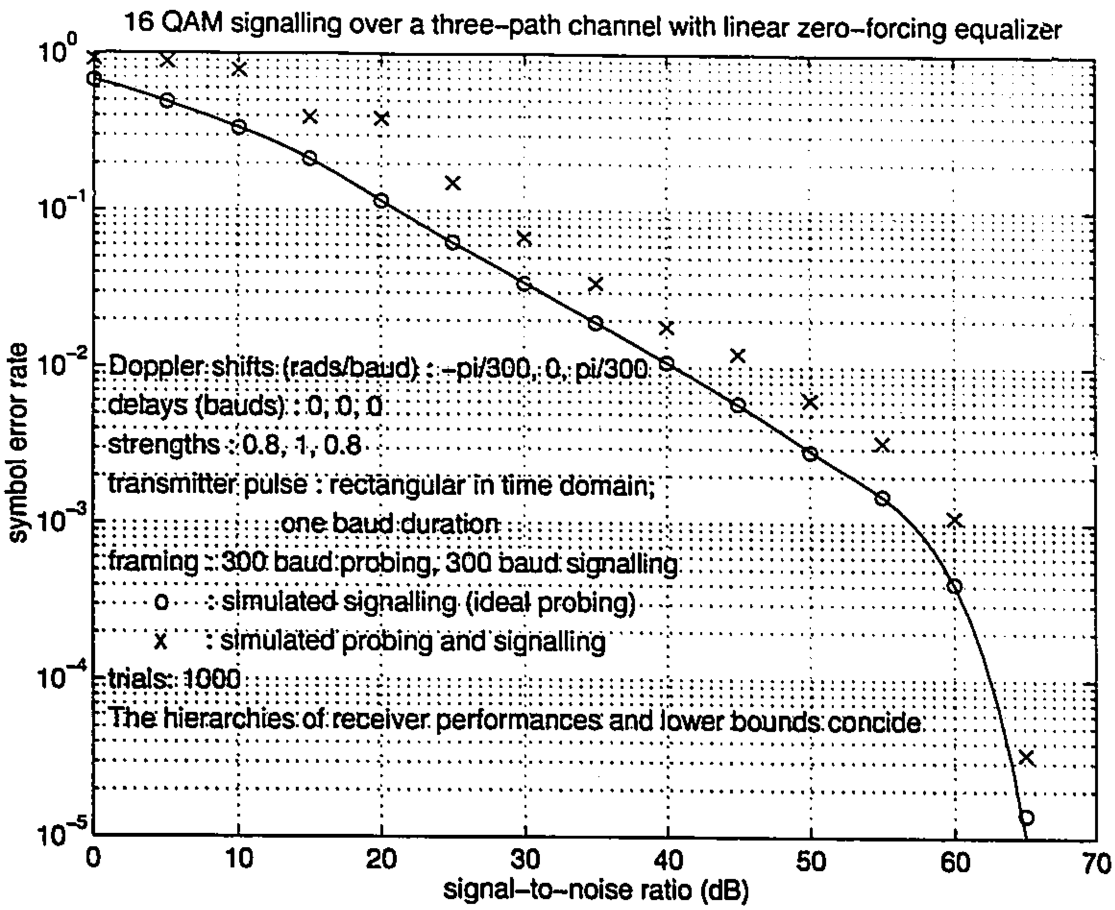

Figure 8.40: An Example in Scenario 3

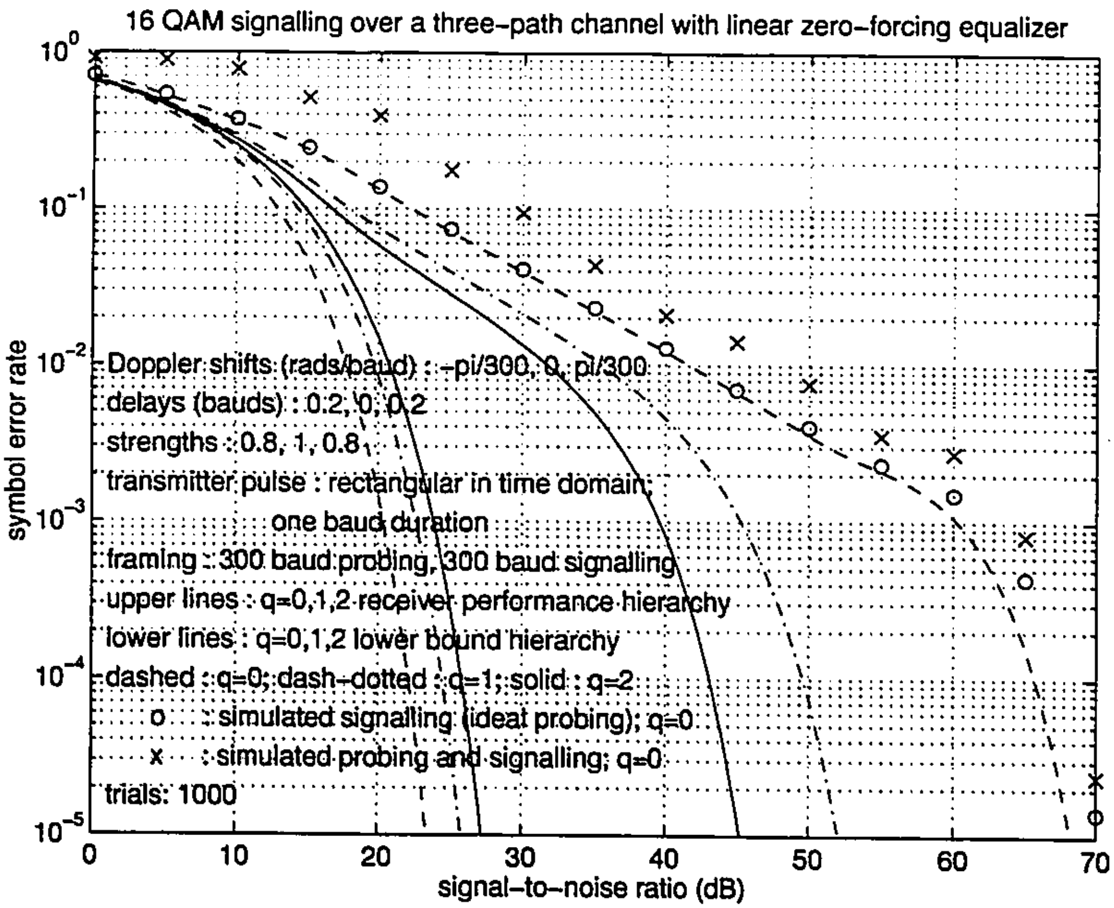

Figure 8.41: An Example in Scenario 4

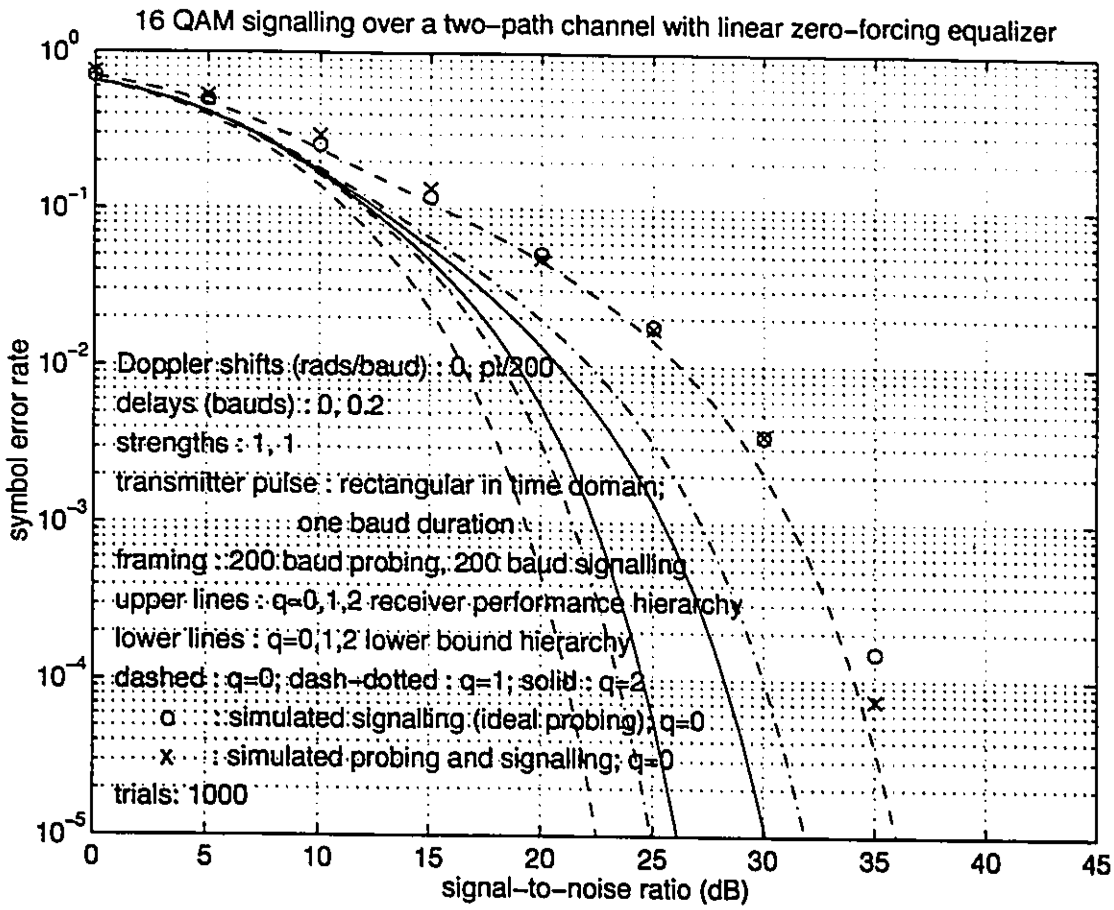

Figure 8.42: An Example in Scenario 2

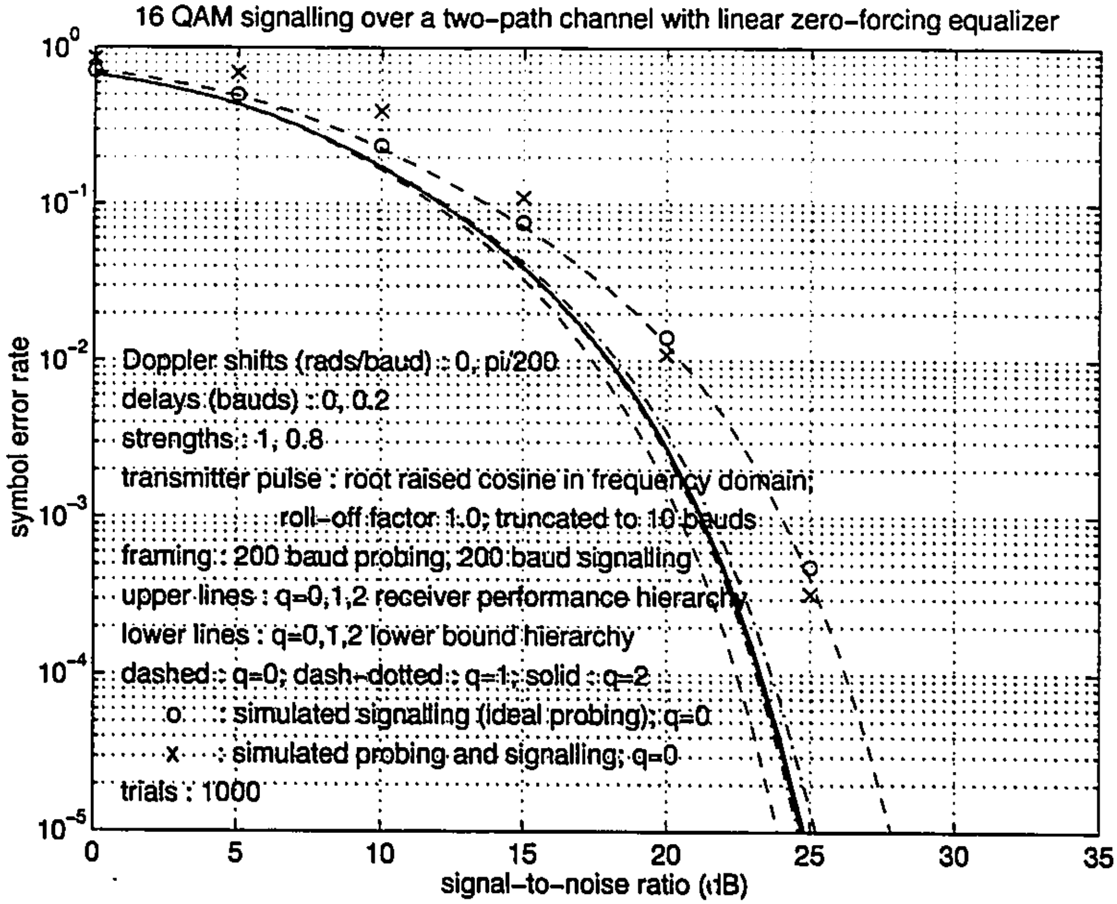

Figure 8.43: An Example in Scenario 2

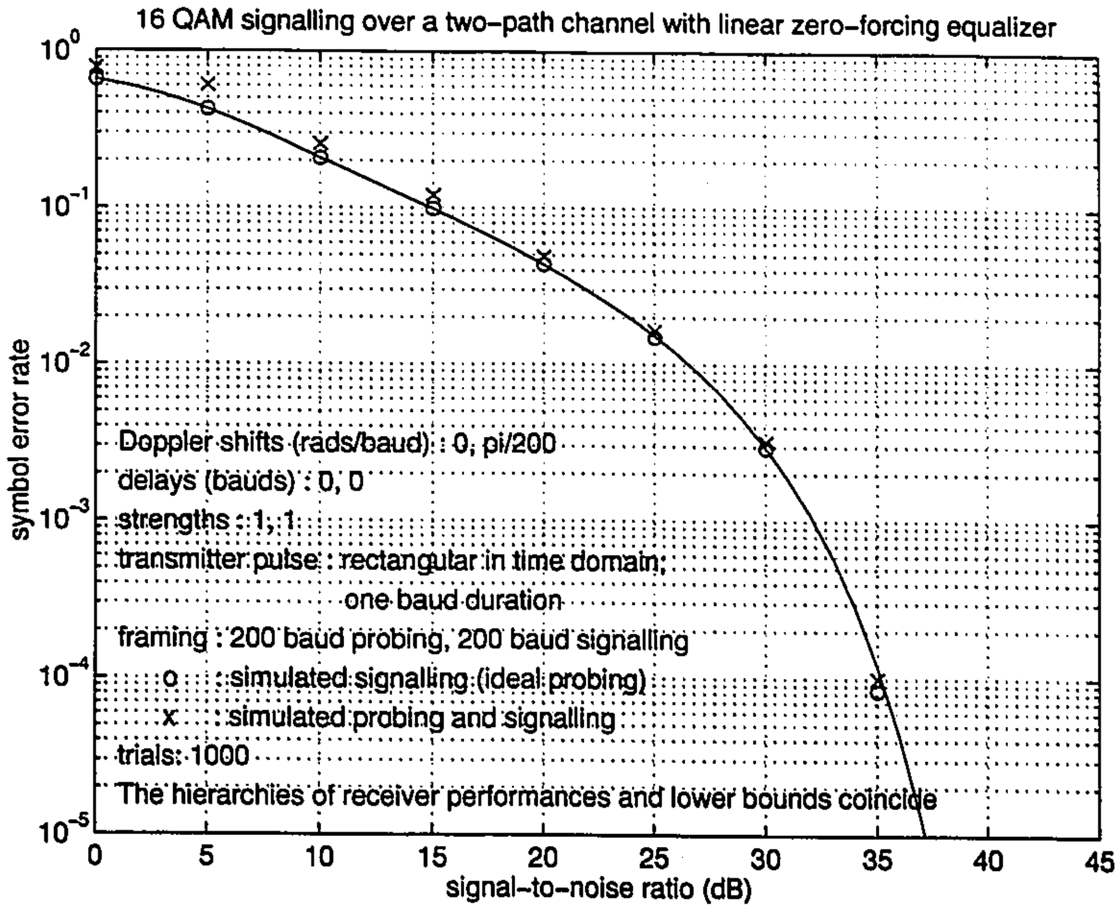

Figure 8.44: An Example in Scenario 1

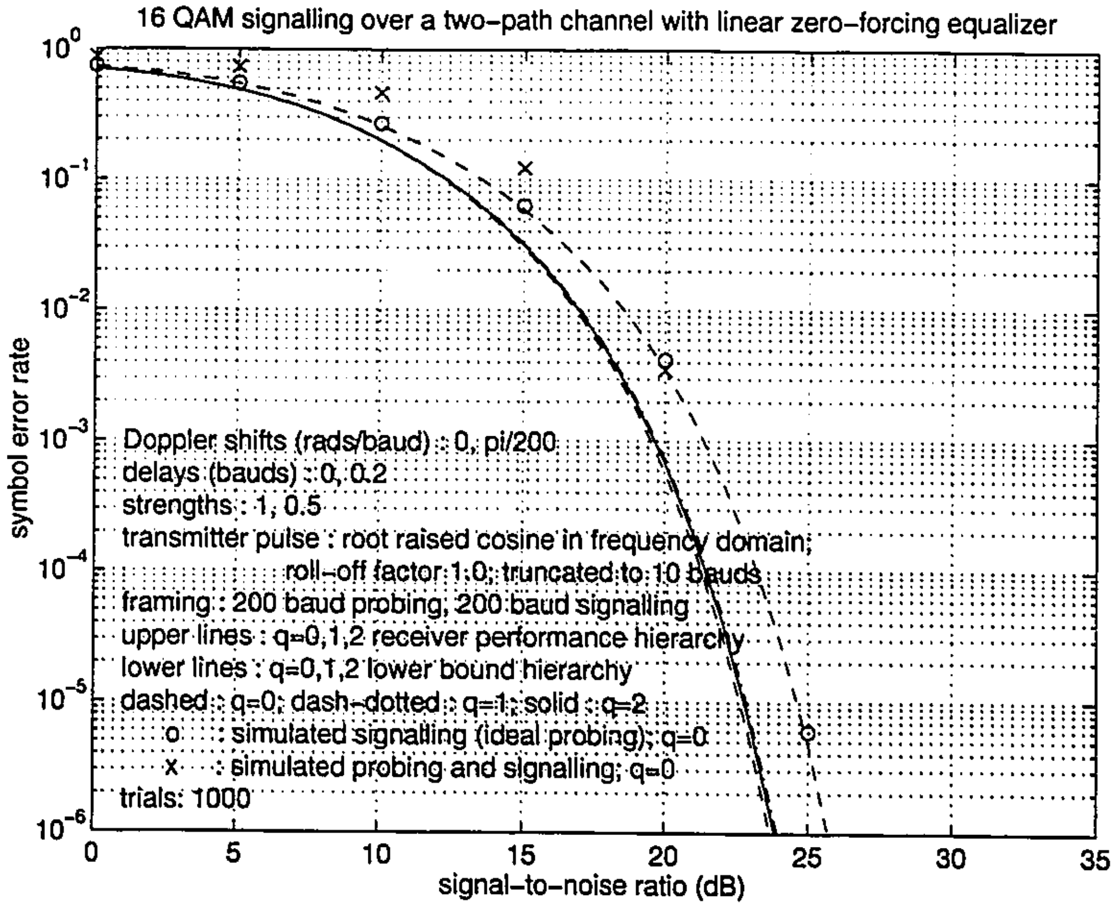

Figure 8.45: An Example in Scenario 2

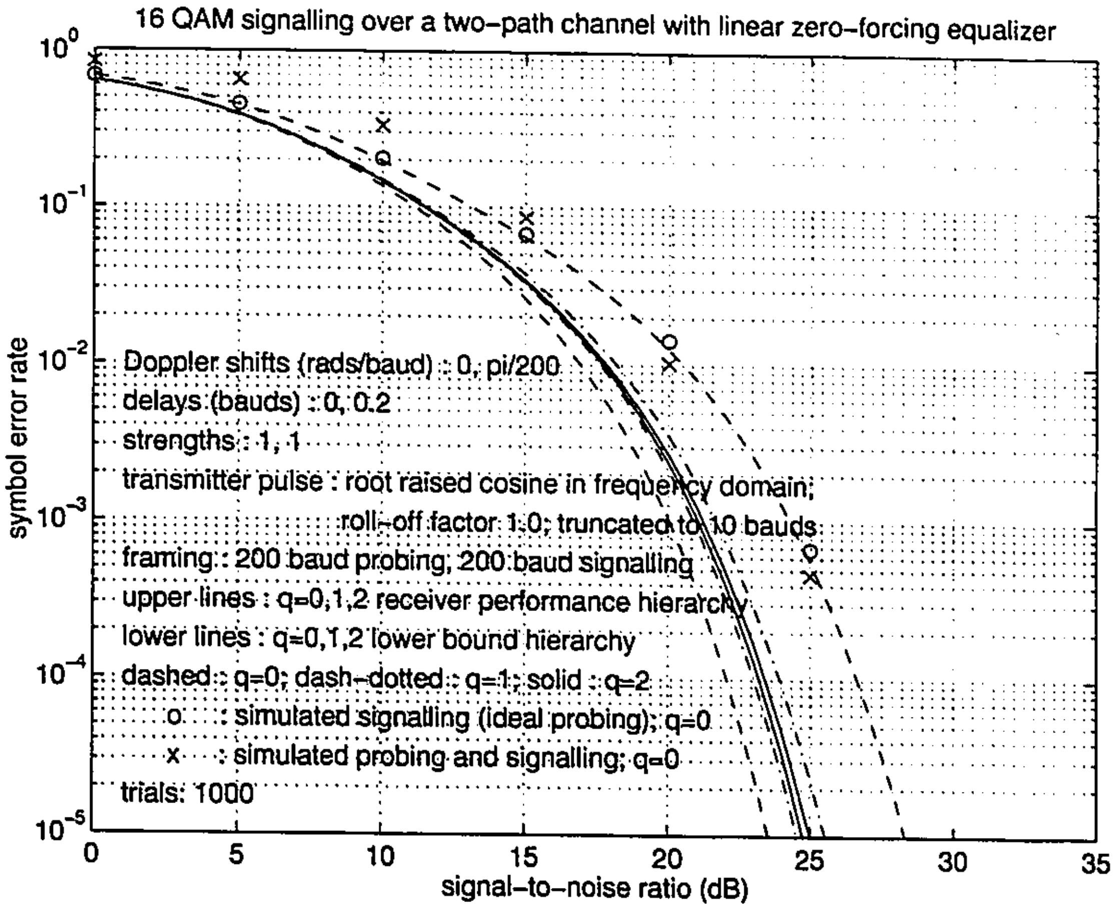

Figure 8.46: An Example in Scenario 2

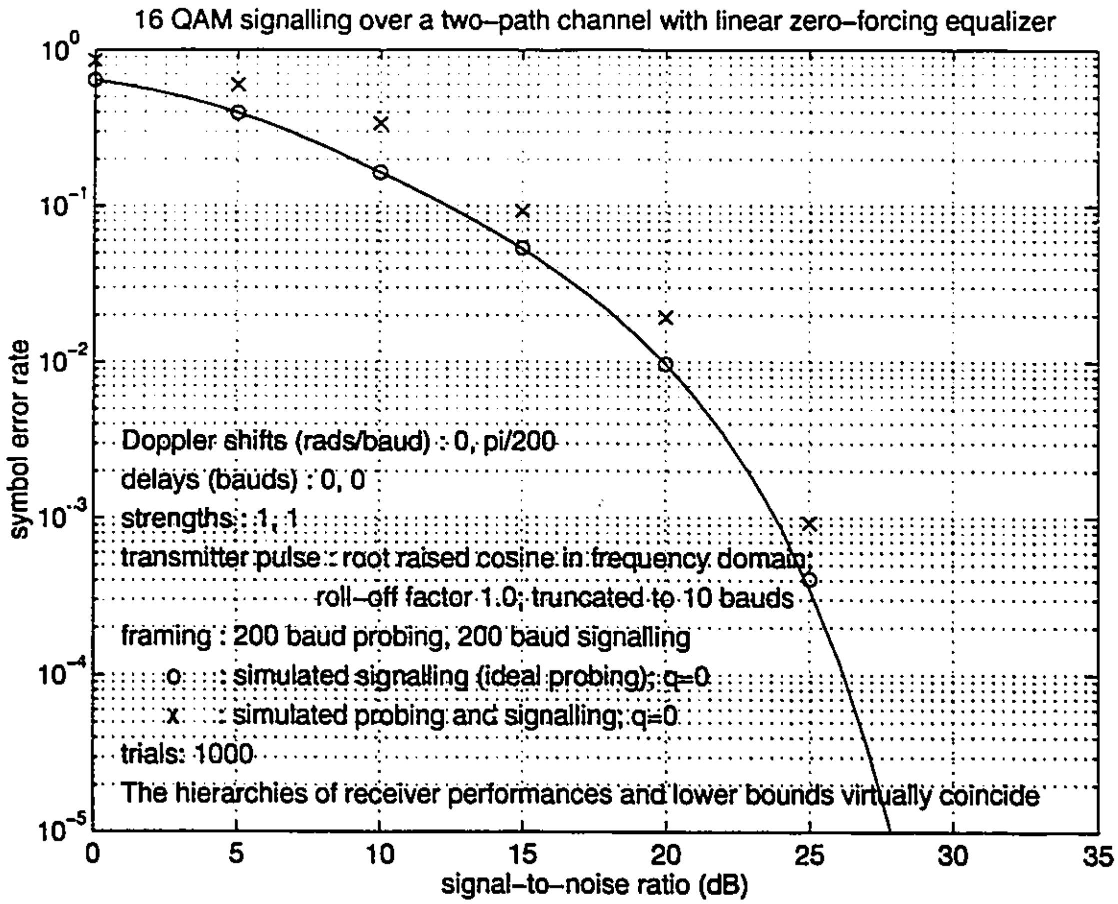

Figure 8.47: An Example in Scenario 1

Chapter 9

Summary, Conclusions, and Suggestions for Further Work

9.1 Summary of the Thesis

In this thesis, a new paradigm has been proposed for designing the transmitter and receiver for quadrature-amplitude-modulated signalling over a mobile radio channel. The new paradigm is based on a discrete-multipath linear time-variant model of the mobile radio channel, and hence the title of the thesis. The following is a chapter-wise summary of the thesis.

9.1.1 Chapter 1 – Introduction

1. The Discrete-Multipath Channel (DMC) was introduced as a model for the mobile radio channel. The DMC is a linear time-variant channel whose input $x(t)$ and output $y(t)$ are related, in the absence of noise, as

$$y(t) = \sum_{k=1}^{K} e^{j\omega_k t} \int \phi_k(t-\tau)x(\tau)d\tau. \tag{9.1}$$

2. A new philosophy, based on alternate phases of probing and signalling, was proposed for designing the transmitter and receiver for quadrature-amplitude-modulated signalling over a DMC.

9.1.2 Chapter 2 – The Composition of QAM and DMC

1. The composition of quadrature amplitude modulation (QAM) and DMC was shown to have the input-output description
$$z(t) = \sum_n a(n) \sum_{k=1}^{K} e^{j\omega_k nT} h_k(t - nT) + \eta_w(t), \tag{9.2}$$
as per following denotions:
   - $a(n)$: transmitted data sequence, which is the input to the composition,
   - T : baud period,
   - $z(t)$: complex envelope of the received signal, which is the output of the composition,
   - $\eta_w(t)$: complex white Gaussian noise,
   - K, $\{\omega_k, h_k(t) : k = 1, 2, \ldots, K\}$: parameters of the composition; here the ω_k's are real and the $h_k(t)$'s are complex-valued square-integrable functions.
2. The probing of the DMC with a suitable quadrature-amplitude-modulated signal was shown to lead to a characterization of the composition of QAM and DMC, that is, to estimates of K and $\{\omega_k, h_k(t) : k = 1, 2, \ldots, K\}$.
3. Under the assumption that K is known, the maximum likelihood estimates of $\{\omega_k, h_k(t) : k = 1, 2, \ldots, K\}$ were discussed with emphasis on their asymptotic properties of strong consistency, normality, and efficiency when the duration of the probing phase tends to infinity.
4. The result that
$$\lim_{L\to\infty} L^3 E\left[\left(\hat{\Theta} - \Theta\right)\left(\hat{\Theta} - \Theta\right)^T\right] < \infty, \tag{9.3}$$
was used to justify the philosophy of design proposed in chapter 1. Here $\hat{\Theta}$ is the maximum-likelihood estimate of $\Theta = T(\omega_1, \omega_2, \ldots, \omega_K)^T$, and L is the probing duration.
5. An overview of methods of estimating K was given, and the possible consequences of erroneous estimation of K was discussed.

9.1.3 Chapter 3 – Linear Zero-Forcing Equalizers

1. The concept of a linear equalizer for the composition of QAM and DMC was introduced; the linear equalizer was identified with an optimum sequence of functions $f_m(t)$.

2. The optimal zero-forcing criterion – minimum noise variance under the zero-forcing constraint – for designing a linear equalizer was stated.

3. The use of the optimal zero-forcing criterion was shown to be feasible in the special cases where either there is no intersymbol interference or $K = 1$, but was seen to be difficult in the general case where $K > 1$ and there is intersymbol interference.

4. A hierarchy of suboptimal zero-forcing criteria, based on stronger versions of the zero-forcing constraint, was used for designing linear equalizers in the general case; the output noise variances of the suboptimal equalizers so designed form a hierarchy of upper bounds on the output noise variance of the optimal equalizer.

5. Despite the DMC's being time-variant, the sequence of functions $f^q_{mu}(t)$ defining a suboptimal equalizer was easily specified. Moreover, the suboptimal equalizers were shown to be implementable using a bank of continuous-time linear time-invariant filters at the front end; the baud-rate samples of the outputs of these filters have to be combined in a linear time-variant manner and fed into a quantizer.

6. To assess the performance degradation of the suboptimal equalizers relative to the optimal equalizer, a hierarchy of lower bounds on the output noise variance of the optimal equalizer was derived.

9.1.4 Chapter 4 – Decision-Feedback Zero-Forcing Equalizers

1. The concept of a decision-feedback equalizer for the composition of QAM and DMC was introduced; the decision-feedback equalizer was identified with an optimum sequence of pairs of functions $(f_m(t), \beta_m(n))$.

2. The optimal zero-forcing criterion – minimum noise variance under the zero-forcing constraint – for designing a decision-feedback equalizer was stated.

3. The use of the optimal zero-forcing criterion was shown to be feasible in the special cases where either there is no intersymbol interference or $K = 1$, but was seen to be difficult in the general case where $K > 1$ and there is intersymbol interference.

4. A hierarchy of suboptimal zero-forcing criteria, based on stronger versions of the zero-forcing constraint, was used for designing decision-feedback equalizers in the general case; the output noise variances of the suboptimal equalizers so designed form a hierarchy of upper bounds on the output noise variance of the optimal equalizer.

5. Despite the DMC's being time-variant, the sequence of pairs of functions $(f^q_{mu}(t), \beta_m(n))$ defining a suboptimal equalizer was easily specified. Moreover, the suboptimal equalizers were shown to be implementable using a bank of L $(\leq K)$ continuous-time linear time-invariant filters at the front end; the baud-rate samples of the outputs of these filters and the decisions on the past data have to be combined in a linear time-variant manner and fed into a quantizer; a discrete-time linear time-invariant filter plays a central role in combining the past decisions.

6. To assess the performance degradation of the suboptimal equalizers relative to the optimal equalizer, a hierarchy of lower bounds on the output noise variance of the optimal equalizer was derived.

9.1.5 Chapter 5 – Linear Mean-Square-Error Equalizers

1. The concept of a linear equalizer for the composition of QAM and DMC was recalled from chapter 3.

2. The optimal mean-square-error criterion – minimum mean-square-error – for designing a linear equalizer was stated.

3. The use of the optimal mean-square-error criterion was shown to be feasible in the special cases where either there is no intersymbol interference or $K = 1$, but was seen to be difficult in the general case where $K > 1$ and there is intersymbol interference.

4. A hierarchy of suboptimal mean-square-error criteria, based on upper bounds on the mean-square-error, was used for designing linear equalizers in the general case; the

mean-square-error of the suboptimal equalizers so designed form a hierarchy of upper bounds on the mean-square-error of the optimal equalizer.

5. Despite the DMC's being time-variant, the sequence of functions $f^{q}_{mu}(t)$ defining a suboptimal equalizer was easily specified. Moreover, the suboptimal equalizers were shown to be implementable using a bank of continuous-time linear time-invariant filters at the front end; the baud-rate samples of the outputs of these filters have to be combined in a linear time-variant manner and fed into a quantizer.

6. To assess the performance degradation of the suboptimal equalizers relative to the optimal equalizer, a hierarchy of lower bounds on the mean-square-error of the optimal equalizer was derived.

9.1.6 Chapter 6 – Decision-Feedback Mean-Square-Error Equalizers

1. The concept of a decision-feedback equalizer for the composition of QAM and DMC was recalled from chapter 4.

2. The optimal mean-square-error criterion – minimum mean-square-error – for designing a decision-feedback equalizer was stated.

3. The use of the optimal mean-square-error criterion was shown to be feasible in the special cases where either there is no intersymbol interference or $K = 1$, but was seen to be difficult in the general case where $K > 1$ and there is intersymbol interference.

4. A hierarchy of suboptimal mean-square-error criteria, based on upper bounds on the mean-square-error, was used for designing decision-feedback equalizers in the general case; the mean-square-error of the suboptimal equalizers so designed form a hierarchy of upper bounds on the mean-square-error of the optimal equalizer.

5. Despite the DMC's being time-variant, the sequence of pairs of functions $(f^{q}_{mu}(t), \beta_m(n))$ defining a suboptimal equalizer was easily specified. Moreover, the suboptimal equalizers were shown to be implementable using a bank of L ($\leq K$) continuous-time linear time-invariant filters at the front end; the baud-rate samples of the outputs of these filters and the decisions on the past data have to be combined in a linear time-variant

manner and fed into a quantizer; a discrete-time linear time-invariant filter plays a central role in combining the past decisions.

6. To assess the performance degradation of the suboptimal equalizers relative to the optimal equalizer, a hierarchy of lower bounds on the mean-square-error of the optimal equalizer was derived.

9.1.7 Chapter 7 – Maximum-Likelihood Sequence Estimators

1. The definition of the maximum-likelihood sequence estimator (MLSE) was stated in the context of the composition of QAM and DMC.

2. The common thread of Forney's and Ungerboeck's methods that made the implementation of the MLSE feasible in the case where $K = 1$ and $\omega_1 = 0$, that is, the composition of QAM and linear time-invariant channel, was recapitulated.

3. The Ungerboeck-type MLSE was derived for the composition of QAM and DMC. The Ungerboeck-type MLSE was shown to be implementable using a bank of K continuous-time linear time-invariant filters at the front-end; the baud-rate samples of the outputs of these filters have to be combined in a linear, time-variant, and memoryless manner and then fed into a trellis search algorithm.

4. The Forney-type MLSE was derived for the composition of QAM and DMC. The Forney-type MLSE was shown to be implementable using a bank of $L(< K)$ continuous-time linear time-invariant filters at the front-end; the baud-rate samples of the outputs of these filters have to be fed into a trellis search algorithm.

9.1.8 Chapter 8 – Results of Computer Simulations

1. Various scenarios of 16 QAM signalling over DMC's, some aspects of which were simulated, were decribed.

2. Simulated results were presented for probing using the maximum-likelihood method and signalling using the linear zero-forcing $q = 0$ equalizer.

3. Graphs depicting the signal-to-noise ratio dependence of the simulated errors in the estimated Doppler-shifts and of their Cramer-Rao bounds were presented.

4. Graphs depicting the time-dependence of the normalized post-equalizer signal-to-noise ratio's of the linear zero-forcing equalizers for $q = 0, 1, 2$ were presented.

5. Graphs depicting the signal-to-noise ratio dependence of the simulated symbol-error rate of the $q = 0$ equalizer and of the theoretical symbol-error rates of the $q = 0, 1, 2$ equalizers were presented.

9.2 Conclusions

From a theoretical standpoint, the conclusion of the thesis as a whole is, that the philosophy of design based on alternate probing and signalling leads to attractive solutions to the problem of quadrature-amplitude-modulated signalling over a discrete-multipath linear time-variant channel, even in the presence of intersymbol interference (ISI) and additive white Gaussian noise (AWGN). Probing may be effected by transmitting a pre-assigned data sequence and then applying the maximum-likelihood method of estimation. Signalling may be effected by using one of the many equalizers and sequence estimators obtained by generalizing some receiver designs known for quadrature-amplitude-modulated signalling over a linear time-invariant channel.

From a practical standpoint, the discrete-multipath channel (DMC) being an adequate model for real mobile radio channels, generic[1] quadrature-amplitude-modulated signalling over a mobile radio channel can be effected by applying the philosophy. In applying this philosophy of design, no attention need be given to whether the mobile radio channel is time-selective, frequency-selective, or time- and frequency-selective, for fading and ISI (and also AWGN) are jointly dealt with. Moreover, no attention need be given to the underlying statistics of the mobile radio channel.

9.3 Suggestions for Further Work

Much more work has to be done before the concepts and methods presented in the thesis can be implemented in a cost-effective manner, with a fair amount of work being aimed at improving the probing, for probing accuracies are crucial to the success of the signalling.

[1] the signal set being arbitrary

9.3.1 Possible Approaches to Improving the Probing

- The maximum-likelihood method of chapter 2, section 2.6.2, although has the best possible asymptotic properties, is expensive to implement, due to its high storage and computational requirements. While the computational requirements may be reduced by using a fast[2] algorithm to perform the minimization defined by equation 2.71 of chapter 2, it is extremely unlikely that the storage requirements can be reduced within the framework of the maximum-likelihood method. Therefore, it seems, that an iterative method must be devised for solving the key estimation problem of chapter 2, section 2.6, that is, a method of producing a sequence of estimates $\hat{\Theta}_L$, of Θ, as L increases, satisfying the following requirements:

  - the method needs a finite and fixed amount of storage at every step of the iteration,
  - the method needs a finite and fixed amount of computation at every step of the iteration,
  - the method has a faster-than-L^{-1} convergence rate.[3]

  Such an iterative method would account for any slow changes in the underlying geometry of multipath progation of the mobile radio channel, by de-emphasizing the remote past, in the manner that some adaptive algorithms do by incorporating a forgetting factor[15].

- As explained in chapter 2, section 2.3.4, for a given length of the probing phase, the total allowable length of the frame is limited by the accuracy of the estimate $\hat{\Theta}_L$ achievable at the end of the probing phase. In order to remove this limitation, a method of iteratively refining the estimate of Θ even through the signalling phase is needed. Such an iterative method must have a faster-than-L^{-1} convergence rate.[4] The method may be decision-directed, that is, it may use previous decisions on the

[2]The MATLAB *fmins* function proved to be too time consuming in the computer simulations.
[3]That is, the sequence of estimates $\hat{\Theta}_L$ of Θ must satisfy

$$\lim_{L\to\infty} L^{(2+\epsilon)} E\left[(\hat{\Theta}_L - \Theta)(\hat{\Theta}_L - \Theta)^T\right] < \infty, \tag{9.4}$$

where $0 < \epsilon \leq 1$. The need for such a convergence rate was explained in chapter 2, section 2.6.5.
[4]In the case where $K = 1$ and $\omega_1 = 0$, this is implicitly achieved by a carrier-phase tracking method.

data, assuming them to be correct. Such an iterative method of refining the estimate of Θ will allow the length of the probing phase to be chosen independently of the length of the signalling phase, and once the length of the probing phase has been chosen, will allow a virtually unlimited length of the signalling phase.

- When there are two Doppler-shifts that are nearly the same, they may be considered approximately as one Doppler-shift, within a limited length of the signalling phase. A precise quantification of this notion will be useful.

9.3.2 Possible Approaches to Improving Signalling

- When an equalizer is intended for use, an analytical method of choosing the value of q to achieve a certain degree of optimality will be useful.

- An analytical or a simulation study of the effects of previous decision errors on the performance of the decision-feedback equalizer will be useful.

Appendix A

Some Results from the Theory of Stationary Sequences in Hilbert Space

A.1 Introduction

A.1.1 Historical Background

A sequence[1] $\{x_n : n \in \mathbb{Z}\}$ of elements in a complex Hilbert space[2] is said to be *stationary* if the inner product (x_{n_1}, x_{n_2}) depends only on the difference $(n_1 - n_2)$ of the indices; this definition was introduced by A. N. Kolmogorov in the seminal paper [23]. Having shown that a weakly-stationary discrete-time stochastic process can be considered as a *stationary* sequence in a Hilbert space, Kolmogorov used the results of [23] to solve the problems of linear extrapolation and interpolation of weakly-stationary discrete-time stochastic processes in [22].

In [23], Kolmogorov also introduced a definition concerning pairs of stationary sequences; accordingly, stationary sequences $\{x_n : n \in \mathbb{Z}\}$ and $\{y_n : n \in \mathbb{Z}\}$ are said to be *jointly stationary* if the inner product (x_{n_1}, y_{n_2}) depends only on the difference $(n_1 - n_2)$ of the indices. Although [23] contained some significant results about a finite set of pairwise jointly stationary sequences, its main concern was a single stationary sequence. The results found in [23], concerning a single stationary sequence, were subsequently extended to a finite set of pairwise jointly stationary sequences by many researchers, notably by Rozanov [46] and

[1] Here $\mathbb{Z}$ denotes the set of integers.

[2] For a discussion of Hilbert spaces, see [47].

Wiener and Masani [55] [56] [26] (see [46] for more references).

A.1.2 On the Relevance of the Theory of Stationary Sequences to Quadrature-Amplitude-Modulated Signalling

Signalling Over a Linear Time-Invariant Channel

In studying quadrature-amplitude-modulated signalling over a linear time-invariant channel in the presence of intersymbol interference and additive white Gaussian noise, one is concerned with linear combinations of elements of the sequence $\{h(t-nT) : n \in \mathbb{Z}\}$ generated by a complex-valued square-integrable function $h(t)$ and a real-valued parameter T (see section 1.5.5 of chapter 1). In mathematical terms, $h(t) \in \mathcal{L}^2$, where $\mathcal{L}^2$ denotes the set of all complex-valued Lebesgue-measurable functions $f(t)$ of the real variable t that satisfy

$$\int_{-\infty}^{\infty} |f(t)|^2 dt < \infty. \tag{A.1}$$

Since $\mathcal{L}^2$ is a vector space, all finite linear combinations of functions of the sequence $\{h(t-nT) : n \in \mathbb{Z}\}$ belong to $\mathcal{L}^2$. The most important property of the sequence $\{h(t-nT) : n \in \mathbb{Z}\}$, however, stems from the fact that $\mathcal{L}^2$ is a Hilbert space under the inner product [47]

$$(f,g) = \int_{-\infty}^{\infty} f^*(t)g(t)dt. \tag{A.2}$$

Denote $h^n(t) = h(t-nT)$ for all integers n, and observe that

$$(h^{n_1}, h^{n_2}) = (h^{n_1-n_2}, h^0) \tag{A.3}$$

for all integers n_1 and n_2. Thus, the inner product (h^{n_1}, h^{n_2}) depends only on the difference $(n_1 - n_2)$ of the indices; in other words, the sequence of functions $\{h(t-nT) : n \in \mathbb{Z}\}$ constitutes a *stationary sequence* in the Hilbert space $\mathcal{L}^2$.

Questions concerning the existence of various kinds of receivers for quadrature-amplitude-modulated signalling over a linear time-invariant channel can be answered by applying Kolmogorov's results of [23] to the stationary sequence $\{h(t-nT) : n \in \mathbb{Z}\}$. In fact, a connection between pulse-amplitude-modulated signalling and weakly-stationary discrete-time stochastic processes was discovered by D. G. Messerschmitt [27]. Using this connection, Messerschmitt was able to interpret some of the results on linear extrapolation and interpolation of a stochastic process as results on decision-feedback equalization and linear equalization, respectively, of pulse-amplitude-modulated signalling. However, Messerschmitt did not seem to be aware of Kolmogorov's work.

Signalling Over a Discrete-Multipath Linear Time-Variant Channel

In studying quadrature-amplitude-modulated signalling over a discrete-multipath linear time-variant channel in the presence of intersymbol interference and additive white Gaussian noise, one is concerned with linear combinations of elements of the sequence

$$\left\{\sum_{k=1}^{K} e^{j\omega_k nT} h_k(t-nT) : n \in \mathbb{Z}\right\} \tag{A.4}$$

generated by a finite set of complex-valued square-integrable functions $\{h_k(t) \in \mathcal{L}^2 : k = 1,2,\ldots,K\}$, a set of mutually distinct real-valued scalars $\{\omega_k : k = 1,2,\ldots,K\}$, and a real-valued parameter T (see sections 2.3 and 2.5 of chapter 2). Denote

$$h^n(t) = \sum_{k=1}^{K} e^{j\omega_k nT} h_k(t-nT). \tag{A.5}$$

Then $h^n(t) \in \mathcal{L}^2$. However, in general,

$$(h^{n_1}, h^{n_2}) \neq (h^{n_1-n_2}, h^0). \tag{A.6}$$

The case where $K = 1$ is an exception, for

$$\begin{aligned}
(h^{n_1}, h^{n_2}) &= \int_{-\infty}^{\infty} e^{-j\omega_1 n_1 T} h_1^*(t-n_1T) h_1(t-n_2T) e^{j\omega_1 n_2 T} dt, &\text{(A.7)}\\
&= \int_{-\infty}^{\infty} e^{-j\omega_1 (n_1-n_2) T} h_1^*(t-(n_1-n_2)T) h_1(t) dt, &\text{(A.8)}\\
&= (h^{n_1-n_2}, h^0). &\text{(A.9)}
\end{aligned}$$

Thus, when $K > 1$, the sequence $\{h^n(t) : n \in \mathbb{Z}\}$ is not stationary in general.

Observe, however, that finite linear combinations of functions of the sequence $\{h^n(t) : n \in \mathbb{Z}\}$ are also finite linear combinations of functions of the set of sequences

$$\{h_k(t-nT) : k = 1,2,\ldots,K; n \in \mathbb{Z}\}. \tag{A.10}$$

Denote

$$h_k^n(t) = h_k(t - nT) \tag{A.11}$$

for $k = 1, 2, \ldots, K$ and all integers n. Observe that

$$(h_{k_1}^{n_1}, h_{k_2}^{n_2}) = (h_{k_1}^{n_1 - n_2}, h_{k_2}^{0}) \tag{A.12}$$

for all integers n_1, n_2; in other words, for fixed k_1 and k_2 the inner product $(h_{k_1}^{n_1}, h_{k_2}^{n_2})$ depends only on the difference $(n_1 - n_2)$ of the indices. Thus, the sequences of functions $\{h_k(t-nT) : k = 1, 2, \ldots, K; n \in \mathbb{Z}\}$ constitute a set of pairwise *jointly stationary sequences* in the Hilbert space $\mathcal{L}^2$.

The theory of jointly stationary sequences in Hilbert space provides a suitable framework within which to pose and solve receiver design problems as arising in quadrature-amplitude-modulated signalling over a discrete-multipath linear time-variant channel. Henceforth, the term 'stationary' will mean 'pairwise jointly stationary' as well.

A.1.3 The Aim of the Appendix

The primary aim of this appendix is to state from the theory of stationary sequences those results that are relevant to this thesis. A secondary aim is to sketch the proofs of the central theorems of the theory, as specialized to the concrete situation $\{h_k(t - nT) : k = 1, 2, \ldots, K; n \in \mathbb{Z}\}$ in the Hilbert space $\mathcal{L}^2$. The discussion is based on [46].

A.2 The Hilbert Space $\mathcal{H}$

Denote by $\mathcal{H}$ the closed subspace (of the Hilbert space $\mathcal{L}^2$) spanned by the sequences of functions $\{h_k(t - nT) : k = 1, 2, \ldots, K; n \in \mathbb{Z}\}$. Thus

$$\mathcal{H} = \text{Clos.Span}\{h_k(t - nT) : k = 1, 2, \ldots, K; n \in \mathbb{Z}\}. \tag{A.13}$$

Denote by $\mathcal{H}'$ the vector space spanned by $\{h_k(t - nT) : k = 1, 2, \ldots, K; n \in \mathbb{Z}\}$. Thus

$$\mathcal{H}' = \text{Span}\{h_k(t - nT) : k = 1, 2, \ldots, K; n \in \mathbb{Z}\}. \tag{A.14}$$

By definition,

$$\mathcal{H} = \text{Clos.}\mathcal{H}'. \tag{A.15}$$

The definitions of the terms used above are found in [47]. The Span and Clos. (abbreviation for Closure) conventions are followed in the thesis.

The theory of stationary sequences aims to characterize the elements of the Hilbert space $\mathcal{H}$ and of some subspaces of $\mathcal{H}$. This characterization is easily done in a certain Hilbert space that is isomorphic to $\mathcal{H}$. The following few sections are devoted to defining this Hilbert space.

A.3 The Hilbert Space $\mathcal{L}^2_{\mathbf{H}}$

The following theorem plays an important role in the theory of stationary sequences.

Theorem 1 *Let* $\mathbf{H}(\omega)$ *be a* $K \times K$ *positive semi-definite Hermitian matrix-valued function defined on* $[-\pi/T, \pi/T]$. *Denote by* $\mathcal{L}^2_{\mathbf{H}}$ *the set of all complex vector-valued functions* $\mathbf{a}(\omega) = [\hat{a}_1(\omega), \hat{a}_2(\omega), \ldots, \hat{a}_K(\omega)]^T$ *that satisfy*

$$\int_{-\pi/T}^{\pi/T} \mathbf{a}^H(\omega)\mathbf{H}(\omega)\mathbf{a}(\omega)d\omega < \infty. \tag{A.16}$$

Then the set $\mathcal{L}^2_{\mathbf{H}}$ *is a Hilbert space under the inner product*

$$(\mathbf{a}, \mathbf{b}) = \frac{1}{2\pi}\int_{-\pi/T}^{\pi/T} \mathbf{a}^H(\omega)\mathbf{H}(\omega)\mathbf{b}(\omega)d\omega. \tag{A.17}$$

For a proof of the above theorem and properties of $\mathcal{L}^2_{\mathbf{H}}$, see [46] or [43].

A study of stationary sequences in the Hilbert space $\mathcal{L}^2_{\mathbf{H}}$ can be done in its own right. In fact, results from such a study are needed in this thesis. But the main reason why this space is introduced here is that there exists an $\mathbf{H}(\omega)$ such that $\mathcal{L}^2_{\mathbf{H}}$ is isomorphic[3] to $\mathcal{H}$; the appropriate function $\mathbf{H}(\omega)$ is the so-called spectral density matrix of the set of stationary sequences $\{h_k(t - nT) : k = 1, 2, \ldots, K; n \in \mathbb{Z}\}$.

A.4 The Spectral Density Matrix $\mathbf{H}(\omega)$

A Hilbert space isomorphism known as the Fourier-Plancherel Transform and a certain result from integration theory are needed for the definition of $\mathbf{H}(\omega)$ and the proof of the

[3]Two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ are isomorphic if there is a one-to-one linear mapping Λ of $\mathcal{H}_1$ onto $\mathcal{H}_2$ which also preserves inner products, that is,

$$(x, y)_{\mathcal{H}_1} = (\Lambda x, \Lambda y)_{\mathcal{H}_2} \tag{A.18}$$

for all $x, y \in \mathcal{H}_1$, where the subscripts denote the space in which the inner product is defined [47].

isomorphism between $\mathcal{H}$ and $\mathcal{L}^2_{\mathbf{H}}$.

A.4.1 The Fourier-Plancherel Transform

The Fourier-Plancherel transform,[4] defined by

$$\hat{f}(\omega) = \lim(\text{in } \mathcal{L}^2)_{A\to\infty} \int_{-A}^{A} f(t)e^{-j\omega t}dt, \tag{A.19}$$

is an isomorphism of $\mathcal{L}^2$ onto $\mathcal{L}^2_{FP}$, where the latter space is the same as the former space except for a scaling factor in the inner product. Thus, for $f(t), g(t) \in \mathcal{L}^2$,

$$(f,g)_{\mathcal{L}^2} = \int_{-\infty}^{\infty} f^*(t)g(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}^*(\omega)\hat{g}(\omega)d\omega = (\hat{f},\hat{g})_{\mathcal{L}^2_{FP}}. \tag{A.20}$$

A useful property of the Fourier-Plancherel transform is that $g(t) = f(t-\tau)$ is equivalent to $\hat{g}(\omega) = e^{-j\omega\tau}\hat{f}(\omega)$. The inverse transform is defined by

$$f(t) = \lim(\text{in } \mathcal{L}^2)_{A\to\infty} \frac{1}{2\pi}\int_{-A}^{A} \hat{f}(\omega)e^{j\omega t}d\omega. \tag{A.21}$$

See [47] for a discussion of the Fourier-Plancherel transform.

A.4.2 A Result from Integration Theory

Lemma 2 *If $f(t), g(t) \in \mathcal{L}^2$, then the series*

$$H(\omega) = \sum_{k=-\infty}^{\infty} \hat{f}^*(\omega + k2\pi/T)\hat{g}(\omega + k2\pi/T) \tag{A.22}$$

converges almost everywhere, $H(\omega) \in \mathcal{L}^1[-\pi/T, \pi/T]$, and

$$\int_{-\infty}^{\infty} \hat{f}^*(\omega)\hat{g}(\omega)d\omega = \sum_{k=-\infty}^{\infty}\int_{-\pi/T}^{\pi/T} \hat{f}^*(\omega + k2\pi/T)\hat{g}(\omega + k2\pi/T)d\omega = \int_{-\pi/T}^{\pi/T} H(\omega)d\omega. \tag{A.23}$$

Proof

Observe that by Schwarz inequality

$$\begin{aligned} \int_{-\infty}^{\infty} \left|\hat{f}(\omega)\hat{g}(\omega)\right| d\omega &\leq \left(\int_{-\infty}^{\infty} \left|\hat{f}(\omega)\right|^2 d\omega\right)^{1/2} \left(\int_{-\infty}^{\infty} |\hat{g}(\omega)|^2 d\omega\right)^{1/2}, & \text{(A.24)} \\ &< \infty. & \text{(A.25)} \end{aligned}$$

[4]The Fourier-Plancherel transform will henceforth be referred to simply as the Fourier transform and denoted by the hat convention as shown.

The left hand side is equivalent to the series

$$\sum_{k=-\infty}^{\infty} \int_{-\pi/T}^{\pi/T} \left|\hat{f}(\omega + k2\pi/T)\hat{g}(\omega + k2\pi/T)\right| d\omega. \tag{A.26}$$

which converges because its partial sums form a monotone and bounded sequence. The conclusions of the lemma now follow from Theorem 1.38 of [47].

A.4.3 An Application of the Lemma – Definition of $\mathbf{H}(\omega)$

The lemma applies particularly to pairs of elements of $\{h_k(t) \in \mathcal{L}^2 : k = 1, 2, \ldots, K\}$. Thus, for $k_1, k_2 = 1, 2, \ldots, K$, the series

$$H_{k_1,k_2}(\omega) = \sum_{l=-\infty}^{\infty} \hat{h}^*_{k_1}(\omega + l2\pi/T)\hat{h}_{k_2}(\omega + l2\pi/T), \tag{A.27}$$

converge almost everywhere and $H_{k_1,k_2}(\omega) \in \mathcal{L}^1[-\pi/T, \pi/T]$. Denote by $\mathbf{H}(\omega)$ the $K \times K$ complex matrix-valued function

$$\mathbf{H}(\omega) = [H_{k_1,k_2}(\omega)] \tag{A.28}$$

defined on $[-\pi/T, \pi/T]$. This is called the *spectral density matrix* of the set of stationary sequences $\{h_k(t - nT) : k = 1, 2, \ldots, K; n \in \mathbb{Z}\}$. Clearly, the spectral density matrix is Hermitian. It is also positive semi-definite almost everywhere, for

$$\sum_{k_1=1}^{K} \sum_{k_2=1}^{K} \alpha^*_{k_1}\alpha_{k_2} H_{k_1,k_2}(\omega) = \sum_{l=-\infty}^{\infty} \sum_{k_1=1}^{K} \sum_{k_2=1}^{K} \alpha^*_{k_1}\alpha_{k_2}\hat{h}^*_{k_1}(\omega + 2\pi l/T)\hat{h}_{k_2}(\omega + 2\pi l/T), \tag{A.29}$$

$$= \sum_{l=-\infty}^{\infty} \left|\sum_{k=1}^{K} \alpha_k \hat{h}_k(\omega + 2\pi l/T)\right|^2, \tag{A.30}$$

$$\geq 0, \tag{A.31}$$

where $\{\alpha_k : k = 1, 2, \ldots, K\}$ is an arbitrary set of complex-valued scalars. Also, $\mathbf{H}(\omega) \in \mathcal{L}^1[-\pi/T, \pi/T]$, meaning $H_{k_1,k_2}(\omega) \in \mathcal{L}^1[-\pi/T, \pi/T]$ for $k_1, k_2 = 1, 2, \ldots, K$.

A.5 The Isomorphism Between $\mathcal{H}$ and $\mathcal{L}^2_{\mathbf{H}}$

The Hilbert space $\mathcal{L}^2_{\mathbf{H}}$ defined by the spectral density matrix $\mathbf{H}(\omega)$ of the set of stationary sequences $\{h_k(t - nT) : k = 1, 2, \ldots, K; n \in \mathbb{Z}\}$ is isomorphic to the Hilbert space $\mathcal{H}$.

This isomorphism is obtained by extending, to $\mathcal{H}$, a certain mapping from $\mathcal{H}'$ into $\mathcal{L}^2_{\mathbf{H}}$ as described in the next two sections.

A.5.1 A Linear Inner-Product-Preserving Mapping Φ from $\mathcal{H}'$ into $\mathcal{L}^2_{\mathbf{H}}$

An arbitrary function $f(t) \in \mathcal{H}'$ has the form

$$f(t) = \sum_{k=1}^{K} \sum_{n=N_{1k}}^{N_{2k}} a_k(n) h_k(t - nT) \tag{A.32}$$

for some finite set of scalars $\{a_k(n) : -\infty < N_{1k} \le n \le N_{2k} < \infty; k = 1, 2, \ldots, K\}$. Taking Fourier transforms on both sides, one obtains

$$\hat{f}(\omega) = \sum_{k=1}^{K} \sum_{n=N_{1k}}^{N_{2k}} a_k(n) e^{-j\omega nT} \hat{h}_k(\omega), \tag{A.33}$$

$$= \sum_{k=1}^{K} \hat{a}_k(\omega) \hat{h}_k(\omega), \tag{A.34}$$

where

$$\hat{a}_k(\omega) = \sum_{n=N_{1k}}^{N_{2k}} a_k(n) e^{-j\omega nT} \tag{A.35}$$

for $k = 1, 2, \ldots, K$. Denote the complex vector-valued function

$$\mathbf{a}(\omega) = [\hat{a}_1(\omega), \hat{a}_2(\omega), \ldots, \hat{a}_K(\omega)]^T. \tag{A.36}$$

Then the above defined association between $f(t)$ and $\mathbf{a}(\omega)$, written as

$$\Phi(f(t)) = \mathbf{a}(\omega), \tag{A.37}$$

is a linear one-to-one inner-product-preserving mapping from $\mathcal{H}'$ into $\mathcal{L}^2_{\mathbf{H}}$.

Towards proving these assertions, denote $\hat{f}_k(\omega) = \hat{a}_k(\omega)\hat{h}_k(\omega)$ for $k = 1, 2, \ldots, K$. Then[5]

$$\left(\hat{f}, \hat{f}\right)_{\mathcal{L}^2_{FP}} = \sum_{k_1=1}^{K} \sum_{k_2=1}^{K} \left(\hat{f}_{k_1}, \hat{f}_{k_2}\right)_{\mathcal{L}^2_{FP}}. \tag{A.38}$$

[5] Recall that $\hat{f}(\omega)$ and $\hat{f}_k(\omega)$ for $k = 1, 2, \ldots, K$ are Fourier transforms of functions in $\mathcal{L}^2$.

By the lemma and owing to the periodicity of the functions $\hat{a}_k(\omega)$,

$$\sum_{l=-\infty}^{\infty} \hat{f}^*_{k_1}(\omega + l2\pi/T)\hat{f}_{k_2}(\omega + l2\pi/T) \tag{A.39}$$

$$= \hat{a}^*_{k_1}(\omega)\hat{a}_{k_2}(\omega) \sum_{l=-\infty}^{\infty} \hat{h}^*_{k_1}(\omega + l2\pi/T)\hat{h}_{k_2}(\omega + l2\pi/T), \tag{A.40}$$

$$= \hat{a}^*_{k_1}(\omega)\hat{a}_{k_2}(\omega)H_{k_1k_2}(\omega). \tag{A.41}$$

Moreover, integrating with the use of the lemma,

$$2\pi\left(\hat{f}_{k_1}, \hat{f}_{k_2}\right)_{\mathcal{L}^2_{FP}} = \int_{-\infty}^{\infty} \hat{f}^*_{k_1}(\omega)\hat{f}_{k_2}(\omega)d\omega = \int_{-\pi/T}^{\pi/T} \hat{a}^*_{k_1}(\omega)\hat{a}_{k_2}(\omega)H_{k_1k_2}(\omega)d\omega. \tag{A.42}$$

By summing over $k_1, k_2 = 1, 2, \ldots, K$,

$$\left(\hat{f}, \hat{f}\right)_{\mathcal{L}^2_{FP}} = \frac{1}{2\pi}\sum_{k_1=1}^{K}\sum_{k_2=1}^{K}\int_{-\pi/T}^{\pi/T} \hat{a}^*_{k_1}(\omega)\hat{a}_{k_2}(\omega)H_{k_1k_2}(\omega)d\omega, \tag{A.43}$$

$$= \frac{1}{2\pi}\int_{-\pi/T}^{\pi/T} \mathbf{a}^H(\omega)\mathbf{H}(\omega)\mathbf{a}(\omega)d\omega. \tag{A.44}$$

Therefore $\mathbf{a}(\omega) \in \mathcal{L}^2_{\mathbf{H}}$. Conversely, for any complex vector-valued function $\mathbf{a}(\omega) \in \mathcal{L}^2_{\mathbf{H}}$ whose entries $\hat{a}_k(\omega)$ for $k = 1, 2, \ldots, K$ are polynomials in $e^{-j\omega T}$, there exists a function $f(t) \in \mathcal{H}'$ such that

$$\Phi\left(f(t)\right) = \mathbf{a}(\omega). \tag{A.45}$$

The linearity of $\Phi(.)$ is obvious. So is the fact that $\Phi(.)$ is one-to-one under the inner product space notions of equality in $\mathcal{H}'$ and $\mathcal{L}^2_{\mathbf{H}}$. Suppose that for a function $g(t) \in \mathcal{H}'$ and complex vector-valued function $\mathbf{b}(\omega) \in \mathcal{L}^2_{\mathbf{H}}$, whose entries $\hat{b}_k(\omega)$ for $k = 1, 2, \ldots, K$ are polynomials in $e^{-j\omega T}$, one has

$$\Phi\left(g(t)\right) = \mathbf{b}(\omega), \tag{A.46}$$

that is,

$$\hat{g}(\omega) = \sum_{k=1}^{K} \hat{b}_k(\omega)\hat{h}_k(\omega). \tag{A.47}$$

Then by the same method that was used to show that $\left(\hat{f}, \hat{f}\right)_{\mathcal{L}^2_{FP}} = (\mathbf{a}, \mathbf{a})_{\mathcal{L}^2_{\mathbf{H}}}$, it can also be shown that $\left(\hat{f}, \hat{g}\right)_{\mathcal{L}^2_{FP}} = (\mathbf{a}, \mathbf{b})_{\mathcal{L}^2_{\mathbf{H}}}$, that is, $\Phi(.)$ preserves inner products. Also, observe that $\Phi(.)$ has the property that if $\Phi(f(t)) = \mathbf{a}(\omega)$ then $\Phi(f(t - nT)) = \mathbf{a}(\omega)e^{-j\omega nT}$. In particular,[6] $\Phi(h_k(t - nT)) = \mathbf{e}_k e^{-j\omega nT}$ for $k = 1, 2, \ldots, K$ and all integers n.

[6]Here and henceforth, $\mathbf{e}_k$ will denote the standard k^{th} unit vector of the Euclidean space.

A.5.2 The Extension of Φ as an Isomorphism of $\mathcal{H}$ onto $\mathcal{L}^2_{\mathbf{H}}$

The linear one-to-one inner-product-preserving mapping $\Phi\,(.)$ defined in the previous section can be extended as an isomorphism from $\mathcal{H}$ onto $\mathcal{L}^2_{\mathbf{H}}$ by using the following fact:

$$\mathcal{L}^2_{\mathbf{H}} = \text{Clos.Span}\left\{\mathbf{e}_k e^{-j\omega nT} : k = 1, 2, \ldots, K; n \in \mathbb{Z}\right\}, \tag{A.48}$$

that is, the set of complex vector-valued functions whose entries are polynomials in $e^{-j\omega T}$ is dense in $\mathcal{L}^2_{\mathbf{H}}$. An equivalent statement is that if $\mathbf{a}(\omega) \in \mathcal{L}^2_{\mathbf{H}}$ is orthogonal to

$$\text{Span}\left\{\mathbf{e}_k e^{-j\omega nT} : k = 1, 2, \ldots, K; n \in \mathbb{Z}\right\}, \tag{A.49}$$

then $\mathbf{a}(\omega) = 0$. To prove this, observe that

$$\int_{-\pi/T}^{\pi/T} e^{j\omega nT} \mathbf{e}_k^H \mathbf{H}(\omega)\mathbf{a}(\omega) d\omega = 0 \tag{A.50}$$

for $k = 1, 2, \ldots, K$ and all integers n, implies[7] that $\mathbf{e}_k^H \mathbf{H}(\omega)\mathbf{a}(\omega) = 0$ for $k = 1, 2, \ldots, K$, which in turn implies that $\mathbf{H}(\omega)\mathbf{a}(\omega) = \mathbf{0}$ or, equivalently,

$$\int_{-\pi/T}^{\pi/T} \mathbf{a}^H(\omega)\mathbf{H}(\omega)\mathbf{a}(\omega) d\omega = 0. \tag{A.51}$$

The extension of $\Phi\,(.)$ is accomplished in a standard manner. Briefly, for any $f(t) \in \mathcal{H}$, choose a sequence of elements $f_n(t) \in \mathcal{H}'$ that converges to $f(t)$ in the norm of the Hilbert space $\mathcal{L}^2$. Denote $\Phi\,(f_n(t)) = \mathbf{a}_n(\omega)$. Then the sequence of elements $\mathbf{a}_n(\omega)$ is a Cauchy sequence in $\mathcal{L}^2_{\mathbf{H}}$ which converges to a unique $\mathbf{a}(\omega) \in \mathcal{L}^2_{\mathbf{H}}$ whatever sequence $f_n(t) \in \mathcal{H}'$ was initially chosen to approximate $f(t)$. Conversely, with an $\mathbf{a}(\omega) \in \mathcal{L}^2_{\mathbf{H}}$, one can associate, in a similar fashion, a unique $f(t) \in \mathcal{H}$; if $\mathbf{H}(\omega)$ is of full rank almost everywhere, and $\mathbf{H}^{-1}(\omega) \in \mathcal{L}^1[-\pi/T, \pi/T]$, then the sequence of partial Fourier series of $\mathbf{a}(\omega)$ converges to $\mathbf{a}(\omega)$; for a proof see [46]. The extension of $\Phi(.)$ defined by $\Phi\,(f(t)) = \mathbf{a}(\omega)$ can be shown to be an isomorphism. The extension satisfies

$$\hat{f}(\omega) = \sum_{k=1}^{K} \hat{a}_k(\omega)\hat{h}_k(\omega), \tag{A.52}$$

where $\mathbf{a}(\omega) = [\hat{a}_1(\omega), \hat{a}_2(\omega), \ldots, \hat{a}_K(\omega)]^T$, and has the property that if $\Phi\,(f(t)) = \mathbf{a}(\omega)$ then $\Phi\,(f(t - nT)) = \mathbf{a}(\omega)e^{-j\omega nT}$.

[7] A property of $\mathcal{L}^2_{\mathbf{H}}$ is that if $\mathbf{a}(\omega), \mathbf{b}(\omega) \in \mathcal{L}^2_{\mathbf{H}}$ then $\mathbf{a}^H(\omega)\mathbf{H}(\omega)\mathbf{b}(\omega) \in \mathcal{L}^1[-\pi/T, \pi/T]$.

Owing to this isomophism between $\mathcal{H}$ and $\mathcal{L}^2_{\mathbf{H}}$, the subspace structure of $\mathcal{H}$ can be inferred from the corresponding subspace structure of $\mathcal{L}^2_{\mathbf{H}}$, which in turn can be inferred from the analytic properties of $\mathbf{H}(\omega)$.

A.6 Stationary Sequences in $\mathcal{L}^2_{\mathbf{H}}$

Henceforth, unless mentioned otherwise, $\mathbf{H}(\omega)$ will be an arbitrary positive semi-definite Hermitian matrix-valued function defined on $[-\pi/T, \pi/T]$.

In the manner that stationary sequences in $\mathcal{L}^2$ were considered, stationary sequences in $\mathcal{L}^2_{\mathbf{H}}$ may also be considered. Thus, for any set of functions $\{\mathbf{a}_l(\omega) : l = 1, 2, \ldots, L\}$ in $\mathcal{L}^2_{\mathbf{H}}$, the set of functions

$$\left\{\mathbf{a}_l(\omega)e^{-j\omega nT} : l = 1, 2, \ldots, L; n \in \mathbb{Z}\right\} \tag{A.53}$$

constitutes a set of stationary sequences in $\mathcal{L}^2_{\mathbf{H}}$. One can associate with this set the $L \times L$ positive semi-definite Hermitian matrix-valued function

$$\mathbf{G}(\omega) = \mathbf{A}^H(\omega)\mathbf{H}(\omega)\mathbf{A}(\omega) \tag{A.54}$$

defined on $[-\pi/T, \pi/T]$, where $\mathbf{A}(\omega)$ denotes the $K \times L$ matrix-valued function

$$\mathbf{A}(\omega) = [\mathbf{a}_1(\omega), \mathbf{a}_2(\omega), \ldots, \mathbf{a}_L(\omega)]. \tag{A.55}$$

In the manner that $\mathbf{H}(\omega)$ defines the Hilbert space $\mathcal{L}^2_{\mathbf{H}}$, $\mathbf{G}(\omega)$ defines the Hilbert space $\mathcal{L}^2_{\mathbf{G}}$ of complex vector-valued functions $\mathbf{b}(\omega) = [\hat{b}_1(\omega), \hat{b}_2(\omega), \ldots, \hat{b}_L(\omega)]^2$ that satisfy

$$\int_{-\pi/T}^{\pi/T} \mathbf{b}^H(\omega)\mathbf{G}(\omega)\mathbf{b}(\omega)d\omega < \infty. \tag{A.56}$$

In the manner that $\mathcal{H}$ was shown to be isomorphic to $\mathcal{L}^2_{\mathbf{H}}$, the subspace

$$\text{Clos.Span}\left\{\mathbf{a}_l(\omega)e^{-j\omega nT} : l = 1, 2, \ldots, L; n \in \mathbb{Z}\right\} \tag{A.57}$$

of $\mathcal{L}^2_{\mathbf{H}}$ can be shown to be isomorphic to $\mathcal{L}^2_{\mathbf{G}}$. Thus, denoting

$$\mathcal{A} = \text{Span}\left\{\mathbf{a}_l(\omega)e^{-j\omega nT} : l = 1, 2, \ldots, L; n \in \mathbb{Z}\right\} \subset \mathcal{L}^2_{\mathbf{H}}, \tag{A.58}$$

$$\mathcal{B} = \text{Span}\left\{\mathbf{e}_l e^{-j\omega nT} : l = 1, 2, \ldots, L; n \in \mathbb{Z}\right\} \subset \mathcal{L}^2_{\mathbf{G}}, \tag{A.59}$$

for every $\mathbf{x}(\omega) \in \mathcal{A}$ there exists a $\mathbf{y}(\omega) \in \mathcal{B}$ such that $\mathbf{x}(\omega) = \mathbf{A}(\omega)\mathbf{y}(\omega)$ and vice versa. This linear one-to-one mapping preserves inner products. Since $\text{Clos.}\mathcal{B} = \mathcal{L}^2_{\mathbf{G}}$, the mapping can be extended as an isomorphism between $\text{Clos.}\mathcal{A}$ and $\mathcal{L}^2_{\mathbf{G}}$. Therefore, $\mathbf{G}(\omega)$ may be called the spectral density matrix of the set of stationary sequences given by equation A.53.

A.7 Orthonormal Stationary Sequences in $\mathcal{L}^2_{\mathbf{H}}$

In this section, it is assumed that $\mathbf{H}(\omega) \in \mathcal{L}^1[-\pi/T, \pi/T]$. Thus, $\mathbf{e}_k e^{-j\omega nT} \in \mathcal{L}^2_{\mathbf{H}}$ for $k = 1, 2, \ldots, K$ and all integers n.

The set of stationary sequences given by equation A.53 is said to be *orthonormal* if the following conditions are satisfied:

$$\frac{1}{2\pi} \int_{-\pi/T}^{\pi/T} \mathbf{a}_{l_1}^H(\omega)\mathbf{H}(\omega)\mathbf{a}_{l_2}(\omega)e^{j(n_1-n_2)\omega T} d\omega = \begin{cases} 0 & \text{if} \quad l_1 \neq l_2 \quad \text{or} \quad n_1 \neq n_2, \\ 1 & \text{if} \quad l_1 = l_2 \quad \text{and} \quad n_1 = n_2, \end{cases} \tag{A.60}$$

for $l_1, l_2 = 1, 2, \ldots, L$ and all integers n_1, n_2, or equivalently, if

$$\mathbf{G}(\omega) = \mathbf{A}^H(\omega)\mathbf{H}(\omega)\mathbf{A}(\omega) = T\mathbf{I}_{L\times L}, \tag{A.61}$$

where $\mathbf{I}_{L\times L}$ denotes the $L \times L$ identity matrix and $\mathbf{A}(\omega)$ is as defined in equation A.55, that is,

$$\mathbf{A}(\omega) = [\mathbf{a}_1(\omega), \mathbf{a}_2(\omega), \ldots, \mathbf{a}_L(\omega)]. \tag{A.62}$$

The question – under what condition does an orthonormal set of stationary sequences of the form given by equation A.53 exist such that

$$\text{Clos.Span}\left\{\mathbf{a}_l(\omega)e^{-j\omega nT} : l = 1, 2, \ldots, L; n \in \mathbb{Z}\right\} = \mathcal{L}^2_{\mathbf{H}} \tag{A.63}$$

– is important, for such an orthonormal set of stationary sequences would then constitute an orthonormal basis for $\mathcal{L}^2_{\mathbf{H}}$; the necessary and sufficient condition is that $\text{rank}\mathbf{H}(\omega) = L$ almost everywhere. For a proof, see [46].

Under this assumption, $\mathcal{L}^2_{\mathbf{H}}$ is isomorphic to $\mathcal{L}^2_{\mathbf{G}}$. Moreover, denoting by $\mathbf{c}_k(\omega) \in \mathcal{L}^2_{\mathbf{G}}$ the isomorph of $\mathbf{e}_k \in \mathcal{L}^2_{\mathbf{H}}$ for $k = 1, 2, \ldots, K$ respectively, one has

$$\mathbf{e}_{k_1}^H \mathbf{H}(\omega)\mathbf{e}_{k_2} = T\mathbf{c}_{k_1}^H(\omega)\mathbf{c}_{k_2}(\omega) \tag{A.64}$$

for $k_1, k_2 = 1, 2, \ldots, K$, and

$$\mathbf{H}(\omega)\left(\mathbf{A}(\omega)\mathbf{c}_k(\omega) - \mathbf{e}_k\right) = \mathbf{0} \tag{A.65}$$

for $k = 1, 2, \ldots, K$. The first of the above conditions is equivalent to

$$\mathbf{H}(\omega) = T\mathbf{C}^H(\omega)\mathbf{C}(\omega), \tag{A.66}$$

where

$$\mathbf{C}(\omega) = [\mathbf{c}_1(\omega), \mathbf{c}_2(\omega), \ldots, \mathbf{c}_K(\omega)], \tag{A.67}$$

an $L \times K$ matrix of rank L almost everywhere. The second condition is equivalent to

$$\begin{aligned} \mathbf{H}(\omega)\left(\mathbf{A}(\omega)\mathbf{C}(\omega) - \mathbf{I}_{K\times K}\right) &= \mathbf{0}, && \text{(A.68)} \\ \mathbf{C}^H(\omega)\left(\mathbf{C}(\omega)\mathbf{A}(\omega) - \mathbf{I}_{L\times L}\right)\mathbf{C}(\omega) &= \mathbf{0}, && \text{(A.69)} \end{aligned}$$

which implies

$$\mathbf{C}(\omega)\mathbf{A}(\omega) = \mathbf{I}_{L\times L}. \tag{A.70}$$

Conversely, for any $L \times K$ matrix $\mathbf{C}(\omega)$ that satisfies equation A.66, there exists an $\mathbf{A}(\omega)$ that satisfies equation A.70; the columns of $\mathbf{A}(\omega)$ generate, according to equation A.53, an orthonormal set of stationary sequences that constitutes an orthonormal basis for $\mathcal{L}^2_{\mathbf{H}}$.

A.8 Regularity, Wold Decomposition

In this section, it is assumed that $\mathbf{H}(\omega) \in \mathcal{L}^1[-\pi/T, \pi/T]$ and $\operatorname{rank}\mathbf{H}(\omega) = L$ almost everywhere. By the first assumption, $\mathbf{e}_k e^{-j\omega nT} \in \mathcal{L}^2_{\mathbf{H}}$ for $k = 1, 2, \ldots, K$ and all integers n, and

$$\text{Clos.Span}\left\{\mathbf{e}_k e^{-j\omega nT} : k = 1, 2, \ldots, K; n \in \mathbb{Z}\right\} = \mathcal{L}^2_{\mathbf{H}}. \tag{A.71}$$

By the second assumption, there exists an orthonormal set of stationary sequences

$$\left\{\mathbf{a}_l(\omega) e^{-j\omega nT} : l = 1, 2, \ldots, L; n \in \mathbb{Z}\right\} \tag{A.72}$$

in $\mathcal{L}^2_{\mathbf{H}}$ such that

$$\text{Clos.Span}\left\{\mathbf{a}_l(\omega) e^{-j\omega nT} : l = 1, 2, \ldots, L; n \in \mathbb{Z}\right\} = \mathcal{L}^2_{\mathbf{H}}. \tag{A.73}$$

Denote

$$\begin{aligned} \mathcal{X}_m &= \text{Clos.Span}\left\{\mathbf{e}_k e^{-j\omega nT} : k = 1, 2, \ldots, K; n \geq m\right\}, && \text{(A.74)} \\ \mathcal{Y}_m &= \text{Clos.Span}\left\{\mathbf{a}_l(\omega) e^{-j\omega nT} : l = 1, 2, \ldots, L; n \geq m\right\} && \text{(A.75)} \end{aligned}$$

for all integers m.

The question – under what condition does

$$\mathcal{X}_m = \mathcal{Y}_m \tag{A.76}$$

hold for all integers m – is important, for the set of functions

$$\left\{\mathbf{a}_l(\omega)e^{-j\omega nT} : l = 1, 2, \ldots, L; n \geq m\right\} \tag{A.77}$$

would then constitute an orthonormal basis for $\mathcal{X}_m$; the necessary and sufficient condition is that $\mathbf{H}(\omega)$ *admit a factorization* of the form

$$\mathbf{H}(\omega) = T\mathbf{C}^H(\omega)\mathbf{C}(\omega) \tag{A.78}$$

for some $L \times K$ matrix $\mathbf{C}(\omega)$ that satisfies[8]

$$\int_{-\pi/T}^{\pi/T} \mathbf{C}(\omega)e^{j\omega nT}d\omega = \mathbf{0} \tag{A.79}$$

for $n < 0$. For a proof, see [46]. The property defined in equation A.76 is referred to as *regularity* of the set of stationary sequences

$$\left\{\mathbf{e}_k e^{-j\omega nT} : k = 1, 2, \ldots, K; n \in \mathbb{Z}\right\}, \tag{A.80}$$

and the decomposition of $\mathcal{L}^2_{\mathbf{H}}$ into the family of mutually orthogonal subspaces

$$\mathbf{W}_m = \text{Span}\left\{\mathbf{a}_l(\omega)e^{-j\omega mT} : l = 1, 2, \ldots, L\right\} \tag{A.81}$$

for $-\infty \leq m \leq \infty$ is known as a *Wold decomposition.*

If there exists a matrix $\mathbf{C}(\omega)$ satisfying equations A.78 and A.79, then there exists a class of such matrices $\mathbf{C}(\omega)$. Of these matrices, there is a subclass of matrices whose corresponding set of functions

$$\left\{\mathbf{a}_l(\omega)e^{-j\omega nT} : l = 1, 2, \ldots, L; n \geq m\right\} \tag{A.82}$$

satisfies $\mathcal{X}_m = \mathcal{Y}_m$ for all m. Matrices of this subclass are known as *maximal*[9] or optimal matrices. A matrix $\tilde{\mathbf{C}}(\omega)$ that satisfies equations A.78 and A.79 is maximal if and only if

$$\tilde{\mathbf{C}}_0^H\tilde{\mathbf{C}}_0 \geq \mathbf{C}_0^H\mathbf{C}_0 \tag{A.83}$$

for all matrices $\mathbf{C}(\omega)$ that satisfy equations A.78 and A.79; here

$$\tilde{\mathbf{C}}_0 = \frac{T}{2\pi}\int_{-\pi/T}^{\pi/T} \tilde{\mathbf{C}}(\omega)d\omega, \tag{A.84}$$

$$\mathbf{C}_0 = \frac{T}{2\pi}\int_{-\pi/T}^{\pi/T} \mathbf{C}(\omega)d\omega. \tag{A.85}$$

[8]The matrix $\mathbf{C}(\omega)$ is *causal* in system-theoretic terminology.

[9]In engineering literature, the term *minimum phase* is often used, especially when $K = 1$.

A maximal matrix $\tilde{\mathbf{C}}(\omega)$ is unique to within premultiplication by an $L \times L$ constant unitary matrix. The proofs of the aforestated results are found in [46].

A.8.1 Special Cases of Regular Sequences

Stationary sequences whose spectral density matrix $\mathbf{H}(\omega)$ is a *rational* function of $e^{-j\omega T}$ are regular. For a proof see [46].

Stationary sequences whose spectral density matrix $\mathbf{H}(\omega)$ is of full rank almost everywhere and satisfies

$$\int_{-\pi/T}^{\pi/T} \log \det \mathbf{H}(\omega) d\omega > \infty \tag{A.86}$$

are regular. This condition was proven by Wiener and Masani [55]. Its special case for $K = 1$ was proven by Kolmogorov [23].

See [46] for more general cases.

A.8.2 Spectral Factorization

Given a set of regular stationary sequences, or its spectral density matrix $\mathbf{H}(\omega)$, methods of finding a maximal matrix $\tilde{\mathbf{C}}(\omega)$ are known as *spectral factorization*. When $K = 1$ and $\mathbf{H}(\omega)$ satisfies the condition of equation A.86, a closed form expression is available for the maximal $\tilde{\mathbf{C}}(\omega)$ [46]. When $\mathbf{H}(\omega)$ is a rational function of $e^{-j\omega T}$, algebraic methods are available for finding the maximal $\tilde{\mathbf{C}}(\omega)$ [46], [9], [30]. When $\mathbf{H}(\omega)$ satisfies the condition of equation A.86, iterative methods are available for finding the maximal $\tilde{\mathbf{C}}(\omega)$ [10], [58].

A.9 Minimality

In this section, it is assumed that $\mathbf{H}(\omega) \in \mathcal{L}^1[-\pi/T, \pi/T]$, and therefore that $\mathbf{e}_k e^{-j\omega nT} \in \mathcal{L}^2_{\mathbf{H}}$ for $k = 1, 2, \ldots, K$ and all integers n, and

$$\text{Clos.Span}\left\{\mathbf{e}_k e^{-j\omega nT} : k = 1, 2, \ldots, K; n \in \mathbb{Z}\right\} = \mathcal{L}^2_{\mathbf{H}}. \tag{A.87}$$

Denote by Ξ_k the closed span of all elements of $\left\{\mathbf{e}_k e^{-j\omega nT} : k = 1, 2, \ldots, K; n \in \mathbb{Z}\right\}$ except $\mathbf{e}_k$. Denote by $\mathbf{a}_k(\omega)$ the orthogonal projection of $\mathbf{e}_k$ onto Ξ_k. The set of sequences $\left\{\mathbf{e}_k e^{-j\omega nT} : k = 1, 2, \ldots, K; n \in \mathbb{Z}\right\}$ is said to be *minimal* if $\mathbf{e}_k \neq \mathbf{a}_k(\omega)$, for $k = 1, 2, \ldots, K$.

A necessary and sufficient condition for minimality is that $\mathbf{H}(\omega)$ is of full rank almost everywhere and

$$\int_{-\pi/T}^{\pi/T} \text{Trace}\left[\mathbf{H}^{-1}(\omega)\right] d\omega < \infty. \tag{A.88}$$

Appendix B

A Generic Minimization Problem that arises in Linear Mean-Square-Error Equalizer Problems

The suboptimal formulations of the linear mean-square-error equalizer problem of chapter 5 can all be viewed as two-stage minimization problems. The first stages of these problems being similar, a generic problem is stated and solved in this appendix. The result obtained here is also used in appendix C.

B.1 Statement of Problem

Minimize the quantity

$$\Lambda[f] = K\mathcal{S}_0 \sum_n \sum_{k=1}^{K} \left| \int_{-\infty}^{\infty} f^*(t) h_k(t - nT) dt - b_k(n) \right|^2 + \mathcal{N}_0 \int_{-\infty}^{\infty} |f(t)|^2 \, dt \tag{B.1}$$

with respect to the function $f(t)$, given that $h_k(t) \in \mathcal{L}^2$ for $k = 1, 2, \ldots, K$,

$$\sum_n \sum_{k=1}^{K} |b_k(n)|^2 < \infty, \tag{B.2}$$

and $\mathcal{S}_0$ and $\mathcal{N}_0$ are real positive quantities.

B.2 Preliminaries

By an argument similar to that used in chapter 5, section 5.3.2, the solution must belong to the Hilbert space

$$\mathcal{H} = \text{Clos.Span}\left\{h_k(t - nT) : k = 1, 2, \ldots, K; n \in \mathbb{Z}\right\}. \tag{B.3}$$

Recall the isomorphism between the Hilbert spaces $\mathcal{H}$ and $\mathcal{L}^2_{\mathbf{H}}$ discussed in appendix A, section A.5, and denote by $\mathbf{c}(\omega) \in \mathcal{L}^2_{\mathbf{H}}$ the isomorph of the solution.

Denote by $\mathbf{b}(\omega)$ the complex vector-valued function

$$\mathbf{b}(\omega) = [\hat{b}_1(\omega), \hat{b}_2(\omega), \ldots, \hat{b}_K(\omega)]^T \tag{B.4}$$

defined on $[-\pi/T, \pi/T]$, where

$$\hat{b}_k(\omega) = T \sum_n b_k^*(n) e^{-j\omega nT} \tag{B.5}$$

for $k = 1, 2, \ldots, K$. Then[1]

$$\frac{1}{2\pi} \int_{-\pi/T}^{\pi/T} \mathbf{b}^H(\omega) \mathbf{e}_k e^{-j\omega nT} d\omega = b_k(n) \tag{B.6}$$

for $k = 1, 2, \ldots, K$ and all integers n, and

$$\frac{1}{2\pi T} \int_{-\pi/T}^{\pi/T} \mathbf{b}^H(\omega) \mathbf{b}(\omega) d\omega = \sum_n \sum_{k=1}^{K} |b_k(n)|^2 < \infty. \tag{B.7}$$

The above are consequences of the Riesz-Fischer theorem and the Parseval Formula [47].

The quantity $\Lambda[f]$ can now be written in terms of $\mathbf{c}(\omega)$ and $\mathbf{b}(\omega)$ as[2]

$$\begin{aligned}
\Lambda[\mathbf{c}] &= K\mathcal{S}_0 \sum_n \sum_{k=1}^{K} \left| \frac{1}{2\pi} \int_{-\pi/T}^{\pi/T} \mathbf{c}^H(\omega) \mathbf{H}(\omega) \mathbf{e}_k e^{-j\omega nT} d\omega - \frac{1}{2\pi} \int_{-\pi/T}^{\pi/T} \mathbf{b}^H(\omega) \mathbf{e}_k e^{-j\omega nT} d\omega \right|^2 \\
&\qquad + \mathcal{N}_0 \frac{1}{2\pi} \int_{-\pi/T}^{\pi/T} \mathbf{c}^H(\omega) \mathbf{H}(\omega) \mathbf{c}(\omega) d\omega, \qquad \text{(B.8)} \\
&= K\mathcal{S}_0 \sum_n \sum_{k=1}^{K} \left| \frac{1}{2\pi} \int_{-\pi/T}^{\pi/T} \left[\mathbf{c}^H(\omega) \mathbf{H}(\omega) - \mathbf{b}^H(\omega) \right] \mathbf{e}_k e^{-j\omega nT} d\omega \right|^2 \\
&\qquad + \mathcal{N}_0 \frac{1}{2\pi} \int_{-\pi/T}^{\pi/T} \mathbf{c}^H(\omega) \mathbf{H}(\omega) \mathbf{c}(\omega) d\omega, \qquad \text{(B.9)}
\end{aligned}$$

[1] Here $\mathbf{e}_k$ denotes the standard k^{th} unit vector of the K-dimensional Euclidean space.
[2] Here Λ is retained for convenience.

where $\mathbf{H}(\omega)$ is the spectral density matrix of the set of sequences

$$\{h_k(t - nT) : k = 1, 2, \ldots, K; n \in \mathbb{Z}\}, \tag{B.10}$$

which defines $\mathcal{L}^2_{\mathbf{H}}$ as discussed in appendix A, section A.5.

To see that there exist non-zero $\mathbf{c}(\omega) \in \mathcal{L}^2_{\mathbf{H}}$ such that $\Lambda[\mathbf{c}] < \infty$, try[3]

$$\mathbf{c}(\omega) = (\mathbf{H}(\omega) + \alpha\mathbf{I})^{-1}\,\mathbf{b}(\omega) \tag{B.11}$$

for some real positive α; observe that since $\alpha\mathbf{I} \le (\mathbf{H}(\omega) + \alpha\mathbf{I})$ the inverse $(\mathbf{H}(\omega) + \alpha\mathbf{I})^{-1}$ exists. Then

$$\begin{aligned} \mathbf{c}^H(\omega)\mathbf{H}(\omega)\mathbf{c}(\omega) &\le \mathbf{c}^H(\omega)\,(\mathbf{H}(\omega) + \alpha\mathbf{I})\,\mathbf{c}(\omega), && \text{(B.12)}\\ &= \mathbf{b}^H(\omega)\,(\mathbf{H}(\omega) + \alpha\mathbf{I})^{-1}\,\mathbf{b}(\omega), && \text{(B.13)}\\ &\le \frac{1}{\alpha}\mathbf{b}^H(\omega)\mathbf{b}(\omega), && \text{(B.14)} \end{aligned}$$

for

$$0 < (\mathbf{H}(\omega) + \alpha\mathbf{I})^{-1} \le \frac{1}{\alpha}\mathbf{I}, \tag{B.15}$$

and therefore,

$$\int_{-\pi/T}^{\pi/T} \mathbf{c}^H(\omega)\mathbf{H}(\omega)\mathbf{c}(\omega)d\omega \le \frac{1}{\alpha}\int_{-\pi/T}^{\pi/T} \mathbf{b}^H(\omega)\mathbf{b}(\omega)d\omega < \infty, \tag{B.16}$$

that is, $\mathbf{c}(\omega) \in \mathcal{L}^2_{\mathbf{H}}$. Futhermore,

$$\begin{aligned} \mathbf{c}^H(\omega)\mathbf{H}(\omega) - \mathbf{b}^H(\omega) &= \mathbf{b}^H(\omega)\left[(\mathbf{H}(\omega) + \alpha\mathbf{I})^{-1}\,\mathbf{H}(\omega) - \mathbf{I}\right], && \text{(B.17)}\\ &= -\alpha\mathbf{b}^H(\omega)\,(\mathbf{H}(\omega) + \alpha\mathbf{I})^{-1}, && \text{(B.18)} \end{aligned}$$

$$\begin{aligned} \left[\mathbf{c}^H(\omega)\mathbf{H}(\omega) - \mathbf{b}^H(\omega)\right][\mathbf{H}(\omega)\mathbf{c}(\omega) - \mathbf{b}(\omega)] &= \alpha^2\mathbf{b}^H(\omega)\,(\mathbf{H}(\omega) + \alpha\mathbf{I})^{-2}\,\mathbf{b}(\omega), && \text{(B.19)}\\ &\le \mathbf{b}^H(\omega)\mathbf{b}(\omega), && \text{(B.20)} \end{aligned}$$

for

$$0 < (\mathbf{H}(\omega) + \alpha\mathbf{I})^{-2} \le \frac{1}{\alpha^2}\mathbf{I}, \tag{B.21}$$

[3]Here $\mathbf{I}$ denotes the $K \times K$ identity matrix.

and therefore,

$$\int_{-\pi/T}^{\pi/T}\left[\mathbf{c}^H(\omega)\mathbf{H}(\omega)-\mathbf{b}^H(\omega)\right]\left[\mathbf{H}(\omega)\mathbf{c}(\omega)-\mathbf{b}(\omega)\right]d\omega \leq \int_{-\pi/T}^{\pi/T}\mathbf{b}^H(\omega)\mathbf{b}(\omega)d\omega < \infty. \tag{B.22}$$

By the Parseval formula

$$\sum_n\sum_{k=1}^{K}\left|\frac{1}{2\pi}\int_{-\pi/T}^{\pi/T}\left[\mathbf{c}^H(\omega)\mathbf{H}(\omega)-\mathbf{b}^H(\omega)\right]\mathbf{e}_k e^{-j\omega nT}d\omega\right|^2 \tag{B.23}$$
$$=\frac{1}{2\pi T}\int_{-\pi/T}^{\pi/T}\left[\mathbf{c}^H(\omega)\mathbf{H}(\omega)-\mathbf{b}^H(\omega)\right]\left[\mathbf{H}(\omega)\mathbf{c}(\omega)-\mathbf{b}(\omega)\right]d\omega.$$

and hence $\Lambda[\mathbf{c}] < \infty$.

B.3 The Solution

For any $\mathbf{c}(\omega) \in \mathcal{L}^2_{\mathbf{H}}$ such that $\Lambda[\mathbf{c}] < \infty$, by the Riesz-Fischer theorem and the Parseval formula,

$$\Lambda[\mathbf{c}] = K\mathcal{S}_0\frac{1}{2\pi T}\int_{-\pi/T}^{\pi/T}\left[\mathbf{c}^H(\omega)\mathbf{H}(\omega)-\mathbf{b}^H(\omega)\right]\left[\mathbf{H}(\omega)\mathbf{c}(\omega)-\mathbf{b}(\omega)\right]d\omega$$
$$+\mathcal{N}_0\frac{1}{2\pi}\int_{-\pi/T}^{\pi/T}\mathbf{c}^H(\omega)\mathbf{H}(\omega)\mathbf{c}(\omega)d\omega. \tag{B.24}$$

Since both integrands in the above expression are non-negative, the quantity $\Lambda[\mathbf{c}]$ can be minimized by minimizing the sum of the integrands almost everywhere on $[-\pi/T, \pi/T]$. Moreover, since the integrand is quadratic in $\mathbf{c}(\omega)$, the minimization can be done by completing the square. Thus, by collecting together the quadratic terms in $\mathbf{c}(\omega)$, one obtains

$$\Lambda[\mathbf{c}] = \frac{K\mathcal{S}_0}{2\pi T}\int_{-\pi/T}^{\pi/T}\left[\mathbf{c}^H(\omega)\left(\mathbf{H}(\omega)\mathbf{H}(\omega)+\frac{\mathcal{N}_0 T}{K\mathcal{S}_0}\mathbf{H}(\omega)\right)\mathbf{c}(\omega)\right.$$
$$\left.-\mathbf{c}^H(\omega)\mathbf{H}(\omega)\mathbf{b}(\omega)-\mathbf{b}^H(\omega)\mathbf{H}(\omega)\mathbf{c}(\omega)+\mathbf{b}^H(\omega)\mathbf{b}(\omega)\right]d\omega, \tag{B.25}$$

$$\Lambda[\mathbf{c}] = \frac{K\mathcal{S}_0}{2\pi T}\int_{-\pi/T}^{\pi/T}\left[\mathbf{c}^H(\omega)\mathbf{H}^{1/2}(\omega)\left(\mathbf{H}(\omega)+\frac{\mathcal{N}_0 T}{K\mathcal{S}_0}\mathbf{I}\right)\mathbf{H}^{1/2}(\omega)\mathbf{c}(\omega)\right.$$
$$\left.-\mathbf{c}^H(\omega)\mathbf{H}(\omega)\mathbf{b}(\omega)-\mathbf{b}^H(\omega)\mathbf{H}(\omega)\mathbf{c}(\omega)+\mathbf{b}^H(\omega)\mathbf{b}(\omega)\right]d\omega, \tag{B.26}$$

where $\mathbf{H}^{1/2}(\omega)$ is the positive square root of $\mathbf{H}(\omega)$, and by completing the square, one obtains

$$\Lambda[\mathbf{c}] = \frac{KS_0}{2\pi T}\int_{-\pi/T}^{\pi/T}\mathbf{b}^H(\omega)\left[\mathbf{I} - \mathbf{H}^{1/2}(\omega)\left(\mathbf{H}(\omega)+\frac{\mathcal{N}_0 T}{KS_0}\mathbf{I}\right)^{-1}\mathbf{H}^{1/2}(\omega)\right]\mathbf{b}(\omega)d\omega +$$
$$\int_{-\pi/T}^{\pi/T}\left[\mathbf{c}^H(\omega)\mathbf{H}^{1/2}(\omega)\left(\mathbf{H}(\omega)+\frac{\mathcal{N}_0 T}{KS_0}\mathbf{I}\right)^{1/2} - \mathbf{b}^H(\omega)\mathbf{H}^{1/2}(\omega)\left(\mathbf{H}(\omega)+\frac{\mathcal{N}_0 T}{KS_0}\mathbf{I}\right)^{-1/2}\right]$$
$$\left[\left(\mathbf{H}(\omega)+\frac{\mathcal{N}_0 T}{KS_0}\mathbf{I}\right)^{1/2}\mathbf{H}^{1/2}(\omega)\mathbf{c}(\omega) - \left(\mathbf{H}(\omega)+\frac{\mathcal{N}_0 T}{KS_0}\mathbf{I}\right)^{-1/2}\mathbf{H}^{1/2}(\omega)\mathbf{b}(\omega)\right]d\omega, \quad \text{(B.27)}$$

where $\left(\mathbf{H}(\omega)+\frac{\mathcal{N}_0 T}{KS_0}\mathbf{I}\right)^{1/2}$ and $\left(\mathbf{H}(\omega)+\frac{\mathcal{N}_0 T}{KS_0}\mathbf{I}\right)^{-1/2}$ are the positive square roots of $\left(\mathbf{H}(\omega)+\frac{\mathcal{N}_0 T}{KS_0}\mathbf{I}\right)$ and $\left(\mathbf{H}(\omega)+\frac{\mathcal{N}_0 T}{KS_0}\mathbf{I}\right)^{-1}$ respectively.

Therefore, the optimum $\mathbf{c}(\omega)$ satisfies

$$\left(\mathbf{H}(\omega)+\frac{\mathcal{N}_0 T}{KS_0}\mathbf{I}\right)^{1/2}\mathbf{H}^{1/2}(\omega)\mathbf{c}(\omega) = \left(\mathbf{H}(\omega)+\frac{\mathcal{N}_0 T}{KS_0}\mathbf{I}\right)^{-1/2}\mathbf{H}^{1/2}(\omega)\mathbf{b}(\omega), \quad \text{(B.28)}$$

$$\begin{aligned}
\mathbf{H}^{1/2}(\omega)\mathbf{c}(\omega) &= \left(\mathbf{H}(\omega)+\frac{\mathcal{N}_0 T}{KS_0}\mathbf{I}\right)^{-1}\mathbf{H}^{1/2}(\omega)\mathbf{b}(\omega), && \text{(B.29)}\\
&= \mathbf{H}^{1/2}(\omega)\left(\mathbf{H}(\omega)+\frac{\mathcal{N}_0 T}{KS_0}\mathbf{I}\right)^{-1}\mathbf{b}(\omega), && \text{(B.30)}\\
\mathbf{c}(\omega) &= \left(\mathbf{H}(\omega)+\frac{\mathcal{N}_0 T}{KS_0}\mathbf{I}\right)^{-1}\mathbf{b}(\omega), && \text{(B.31)}
\end{aligned}$$

and the minimum so achieved is given by

$$\begin{aligned}
\Lambda[\mathbf{c}] &= \frac{KS_0}{2\pi T}\int_{-\pi/T}^{\pi/T}\mathbf{b}^H(\omega)\left[\mathbf{I} - \mathbf{H}^{1/2}(\omega)\left(\mathbf{H}(\omega)+\frac{\mathcal{N}_0 T}{KS_0}\mathbf{I}\right)^{-1}\mathbf{H}^{1/2}(\omega)\right]\mathbf{b}(\omega)d\omega, && \text{(B.32)}\\
&= \frac{\mathcal{N}_0}{2\pi}\int_{-\pi/T}^{\pi/T}\mathbf{b}^H(\omega)\left(\mathbf{H}(\omega)+\frac{\mathcal{N}_0 T}{KS_0}\mathbf{I}\right)^{-1}\mathbf{b}(\omega)d\omega, && \text{(B.33)}
\end{aligned}$$

where the last equality is a consequence of the matrix identity

$$\left[\mathbf{I} - \mathbf{H}^{1/2}(\omega)\left(\mathbf{H}(\omega)+\frac{\mathcal{N}_0 T}{KS_0}\mathbf{I}\right)^{-1}\mathbf{H}^{1/2}(\omega)\right] = \frac{\mathcal{N}_0 T}{KS_0}\left(\mathbf{H}(\omega)+\frac{\mathcal{N}_0 T}{KS_0}\mathbf{I}\right)^{-1}, \quad \text{(B.34)}$$

which is easily verified. The commutativity of matrices $\left(\mathbf{H}(\omega)+\frac{\mathcal{N}_0 T}{KS_0}\mathbf{I}\right)^{-1}$ and $\mathbf{H}^{1/2}(\omega)$, tacitly used above, is a consequence of the matrices' having the same eigenvectors.

B.4 Conclusion

The isomorph $\mathbf{c}(\omega) \in \mathcal{L}^2_{\mathbf{H}}$ of the solution $f(t)$ to the problem posed in section B.1 and the minimum $\Lambda[f] = \Lambda[\mathbf{c}]$ so achieved are given by equations B.31 and B.33 respectively.

Appendix C

A Generic Minimization Problem that arises in Decision-Feedback Mean-Square-Error Equalizer Problems

The suboptimal formulations of the decision-feedback mean-square-error equalizer problem of chapter 6 can all be viewed as two-stage minimization problems. The first stages of these problems being similar, a generic problem is stated and solved in this appendix. The result obtained in appendix B is used here.

C.1 Statement of Problem

Minimize the quantity

$$\Gamma[f] = K\mathcal{S}_0 \sum_{n\geq 0} \sum_{k=1}^{K} \left| \int_{-\infty}^{\infty} f^*(t) h_k(t-nT)\,dt - b_k(n) \right|^2 + \mathcal{N}_0 \int_{-\infty}^{\infty} |f(t)|^2\,dt \tag{C.1}$$

with respect to the function $f(t)$ under the constraint

$$\sum_{n<0} \sum_{k=1}^{K} \left| \int_{-\infty}^{\infty} f^*(t) h_k(t-nT)\,dt \right|^2 < \infty, \tag{C.2}$$

given that $h_k(t) \in \mathcal{L}^2$ for $k = 1, 2, \ldots, K$, the number of nonzero $b_k(n)$'s is finite for $n > 0$, $k = 1, 2, \ldots, K$, and $\mathcal{S}_0$ and $\mathcal{N}_0$ are real positive quantities. Condition C.2 ensures the implementability of the feedback filter of a decision-feedback equalizer.

C.2 An Equivalent Problem

The problem stated in the previous section is equivalent to the following problem:

minimize the quantity

$$\Lambda[f,b] = K\mathcal{S}_0 \sum_n \sum_{k=1}^{K} \left| \int_{-\infty}^{\infty} f^*(t) h_k(t-nT)dt - b_k(n) \right|^2 + \mathcal{N}_0 \int_{-\infty}^{\infty} |f(t)|^2 \, dt \qquad \text{(C.3)}$$

with respect to the function $f(t)$ and the set of one-sided sequences $\{b_k(n) : k = 1,2,\ldots,K; n < 0\}$ under the constraint

$$\sum_{n<0} \sum_{k=1}^{K} |b_k(n)|^2 < \infty, \qquad \text{(C.4)}$$

where $h_k(t)$ for $k = 1,2,\ldots,K$, $b_k(n)$ for $n > 0$, $k = 1,2,\ldots,K$, and $\mathcal{S}_0, \mathcal{N}_0$ are the same as those in the previous section; the argument b of $\Lambda[f,b]$ denotes the set of one-sided sequences $\{b_k(n) : k = 1,2,\ldots,K; n < 0\}$.

To prove the equivalence, first observe that, if for some function $f(t)$ and some set of sequences $\{b_k(n) : k = 1,2,\ldots,K; n \in \mathbb{Z}\}$ that satisfies

$$\sum_{n<0} \sum_{k=1}^{K} |b_k(n)|^2 < \infty, \qquad \text{(C.5)}$$

one has $\Lambda[f,b] < \infty$, then, by the triangle inequality in the metric space of square summable sequences, condition C.2 is satisfied. Furthermore, since the optimum function $f(t)$ and set of one-sided sequences $\{b_k(n) : k = 1,2,\ldots,K; n < 0\}$ must satisfy

$$\int_{-\infty}^{\infty} f^*(t) h_k(t-nT)dt = b_k(n) \qquad \text{(C.6)}$$

for $k = 1,2,\ldots,K$ and $n < 0$, one has

$$\arg\min_f \min_b \Lambda[f,b] = \arg\min_f \Gamma[f], \qquad \text{(C.7)}$$

$$\min_{f,b} \Lambda[f,b] = \min_f \Gamma[f]. \qquad \text{(C.8)}$$

Thus the equivalence is proved.

C.3 Solution

The equivalent problem stated in the previous section can be solved in two stages as follows. In the first stage, the quantity $\Lambda[f,b]$ is minimized with respect to the function $f(t)$ for a fixed set of sequences $\{b_k(n) : k = 1,2,\ldots,K; n \in \mathbb{Z}\}$ that satifies

$$\sum_n \sum_{k=1}^{K} |b_k(n)|^2 < \infty. \tag{C.9}$$

The optimum function shall be denoted by $f_b(t)$ and the minimum so achieved shall be denoted by $\Xi[b]$. Thus

$$f_b(t) = \underset{f}{\arg\min}\ \Lambda[f,b], \tag{C.10}$$

$$\Xi[b] = \min_{f} \Lambda[f,b]. \tag{C.11}$$

In the second stage, the quantity $\Xi[b]$ is minimized with respect to the set of one-sided sequences $\{b_k(n) : k = 1,2,\ldots,K; n < 0\}$.

The first stage is the generic problem treated in appendix B. Thus to write the solution to the first stage, while setting the argument b for the second stage, denote

$$\mathbf{b}^+(\omega) = [\hat{b}_1^+(\omega), \hat{b}_2^+(\omega), \ldots, \hat{b}_K^+(\omega)]^T, \tag{C.12}$$

$$\mathbf{b}^-(\omega) = [\hat{b}_1^-(\omega), \hat{b}_2^-(\omega), \ldots, \hat{b}_K^-(\omega)]^T, \tag{C.13}$$

where

$$\hat{b}_k^+(\omega) = T \sum_{n \geq 0} b_k^*(n) e^{-j\omega nT}, \tag{C.14}$$

$$\hat{b}_k^-(\omega) = T \sum_{n < 0} b_k^*(n) e^{-j\omega nT}, \tag{C.15}$$

$$\tag{C.16}$$

for $k = 1,2,\ldots,K$. Accordingly, the isomorph $\mathbf{c}_b(\omega) \in \mathcal{L}_{\mathbf{H}}^2$ of $f_b(t)$ and the quantity $\Xi[b]$ are given by

$$\mathbf{c}_b(\omega) = \left(\mathbf{H}(\omega) + \frac{\mathcal{N}_0 T}{K S_0}\mathbf{I}\right)^{-1} (\mathbf{b}^+(\omega) + \mathbf{b}^-(\omega)) \tag{C.17}$$

and

$$\Xi[b] = \frac{\mathcal{N}_0}{2\pi}\int_{-\pi/T}^{\pi/T} (\mathbf{b}^+(\omega)+\mathbf{b}^-(\omega))^H \left(\mathbf{H}(\omega)+\frac{\mathcal{N}_0 T}{K\mathcal{S}_0}\mathbf{I}\right)^{-1} (\mathbf{b}^+(\omega)+\mathbf{b}^-(\omega))\,d\omega \quad \text{(C.18)}$$

respectively.

To solve the second stage, denote by $\mathcal{Y}$ the set of complex vector-valued functions $\mathbf{y}(\omega) = [\hat{y}_1(\omega), \hat{y}_2(\omega), \ldots, \hat{y}_K(\omega)]^T$ defined on $[-\pi/T, \pi/T]$ that satisfy

$$\int_{-\pi/T}^{\pi/T} \mathbf{y}^H(\omega)\left(\mathbf{H}(\omega)+\frac{\mathcal{N}_0 T}{K\mathcal{S}_0}\mathbf{I}\right)\mathbf{y}(\omega)d\omega < \infty. \quad \text{(C.19)}$$

From the discussion of appendix A, section A.5, the set $\mathcal{Y}$ is a Hilbert space under the inner product

$$(\mathbf{x},\mathbf{y}) = \frac{1}{2\pi}\int_{-\pi/T}^{\pi/T} \mathbf{x}^H(\omega)\left(\mathbf{H}(\omega)+\frac{\mathcal{N}_0 T}{K\mathcal{S}_0}\mathbf{I}\right)\mathbf{y}(\omega)d\omega. \quad \text{(C.20)}$$

The functions[1] $\mathbf{e}_k e^{-j\omega nT} \in \mathcal{Y}$ for $k = 1, 2, \ldots, K$ and all integers n, and

$$\mathcal{Y} = \text{Clos.Span}\left\{\mathbf{e}_k e^{-j\omega nT} : k = 1, 2, \ldots, K; n \in \mathbb{Z}\right\}. \quad \text{(C.21)}$$

Denote by $\mathcal{Y}^+$ the subspace

$$\mathcal{Y}^+ = \text{Clos.Span}\left\{\mathbf{e}_k e^{-j\omega nT} : k = 1, 2, \ldots, K; n \geq 0\right\} \quad \text{(C.22)}$$

of $\mathcal{Y}$. Denote

$$\mathbf{x}(\omega) = \left(\mathbf{H}(\omega)+\frac{\mathcal{N}_0 T}{K\mathcal{S}_0}\mathbf{I}\right)^{-1}\mathbf{b}^+(\omega), \quad \text{(C.23)}$$

$$\mathbf{y}(\omega) = \left(\mathbf{H}(\omega)+\frac{\mathcal{N}_0 T}{K\mathcal{S}_0}\mathbf{I}\right)^{-1}\mathbf{b}^-(\omega). \quad \text{(C.24)}$$

Then both $\mathbf{x}(\omega)$ and $\mathbf{y}(\omega)$ belong to $\mathcal{Y}$, and therefore

$$\Xi[b] = \mathcal{N}_0\,(\mathbf{x}+\mathbf{y}, \mathbf{x}+\mathbf{y}), \quad \text{(C.25)}$$

for

$$\begin{aligned}
\int_{-\pi/T}^{\pi/T} \mathbf{x}^H(\omega)\left(\mathbf{H}(\omega)+\frac{\mathcal{N}_0 T}{K\mathcal{S}_0}\mathbf{I}\right)\mathbf{x}(\omega)d\omega &= \int_{-\pi/T}^{\pi/T} \mathbf{b}^{+H}(\omega)\left(\mathbf{H}(\omega)+\frac{\mathcal{N}_0 T}{K\mathcal{S}_0}\mathbf{I}\right)^{-1}\mathbf{b}^+(\omega)d\omega, && \text{(C.26)}\\
&\leq \frac{K\mathcal{S}_0}{\mathcal{N}_0 T}\int_{-\pi/T}^{\pi/T} \mathbf{b}^{+H}(\omega)\mathbf{b}^+(\omega)d\omega, && \text{(C.27)}\\
&< \infty, && \text{(C.28)}
\end{aligned}$$

[1]Here $\mathbf{e}_k$ denotes the standard k^{th} unit vector of the K-dimensional Euclidean space.

and similarly for $\mathbf{y}(\omega)$. Furthermore, $\mathbf{y}(\omega)$ belongs to the orthogonal complement of $\mathcal{Y}^+$ in $\mathcal{Y}$, for

$$\begin{aligned}
\int_{-\pi/T}^{\pi/T} e^{j\omega nT}\mathbf{e}_k^H\left(\mathbf{H}(\omega)+\frac{\mathcal{N}_0 T}{KS_0}\mathbf{I}\right)\mathbf{y}(\omega)d\omega &= \int_{-\pi/T}^{\pi/T} e^{j\omega nT}\mathbf{e}_k^H\mathbf{b}^-(\omega)d\omega, && \text{(C.29)}\\
&= \int_{-\pi/T}^{\pi/T} \hat{b}_k^-(\omega)e^{j\omega nT}d\omega, && \text{(C.30)}\\
&= 0 && \text{(C.31)}
\end{aligned}$$

for $k = 1, 2, \ldots, K$ and $n \geq 0$.

The second stage of the problem can now be posed in the Hilbert space $\mathcal{Y}$ as follows. Minimize

$$\mathcal{N}_0\,(\mathbf{x}+\mathbf{y}, \mathbf{x}+\mathbf{y}) \tag{C.32}$$

with respect to

$$\mathbf{y}(\omega) \in \mathcal{Y}^{+\perp} \tag{C.33}$$

under the constraint

$$\int_{-\pi/T}^{\pi/T} \mathbf{b}^{-H}(\omega)\mathbf{b}^-(\omega)d\omega < \infty. \tag{C.34}$$

In the following, the minimization is first carried out regardless of the constraint C.34, and the optimum solution is then shown to satisfy the constraint.

Suppose

$$\mathbf{x}(\omega) = \mathbf{u}(\omega) + \mathbf{v}(\omega) \tag{C.35}$$

is the unique decomposition of $\mathbf{x}(\omega)$ such that $\mathbf{u}(\omega) \in \mathcal{Y}^+$ and $\mathbf{v}(\omega) \in \mathcal{Y}^{+\perp}$. Then

$$(\mathbf{x}+\mathbf{y}, \mathbf{x}+\mathbf{y}) = (\mathbf{u}, \mathbf{u}) + (\mathbf{v}+\mathbf{y}, \mathbf{v}+\mathbf{y}). \tag{C.36}$$

This implies that the optimum $\mathbf{y}(\omega)$ is given by $\mathbf{y}(\omega) = -\mathbf{v}(\omega)$ and the minimum so achieved is given by

$$(\mathbf{x}+\mathbf{y}, \mathbf{x}+\mathbf{y}) = (\mathbf{u}, \mathbf{u}). \tag{C.37}$$

Thus, in terms of $\mathbf{u}(\omega)$, the isomorph $\mathbf{c}(\omega) \in \mathcal{L}^2_{\mathbf{H}}$ of the function $f(t)$ that minimizes $\Gamma[f]$ is given by

$$\mathbf{c}(\omega) = \mathbf{u}(\omega), \tag{C.38}$$

and the minimum so achieved is given by

$$\min_f \Gamma[f] = \mathcal{N}_0\,(\mathbf{u},\mathbf{u})\,. \tag{C.39}$$

It remains to find $\mathbf{u}(\omega)$ and then show that the constraint C.34 is satisfied.

Since the matrix $\left(\mathbf{H}(\omega) + \frac{\mathcal{N}_0 T}{K\mathcal{S}_0}\mathbf{I}\right)$ is positive definite, by a theorem of Wiener and Masani [55], the set of stationary sequences $\left\{\mathbf{e}_k e^{-j\omega nT} : k = 1,2,\ldots,K; n \in \mathbb{Z}\right\}$ in $\mathcal{Y}$ is regular. Let $\mathbf{C}(\omega)$ be the *maximal causal* matrix satisfying

$$\left(\mathbf{H}(\omega) + \frac{\mathcal{N}_0 T}{K\mathcal{S}_0}\mathbf{I}\right) = \mathbf{C}^H(\omega)\mathbf{C}(\omega), \tag{C.40}$$

and let $\mathbf{A}(\omega)$ be its inverse, that is, $\mathbf{C}(\omega)\mathbf{A}(\omega) = \mathbf{I}$. Then the set of stationary sequences $\{\sqrt{T}\mathbf{a}_k(\omega)e^{-j\omega nT} : k = 1,2,\ldots,K; n \in \mathbb{Z}\}$ is orthonormal in $\mathcal{Y}$ and

$$\mathcal{Y}^+ = \text{Clos.Span}\left\{\mathbf{a}_k(\omega)e^{-j\omega nT} : k = 1,2,\ldots,K; n \geq 0\right\}, \tag{C.41}$$

$$\mathcal{Y}^{+\perp} = \text{Clos.Span}\left\{\mathbf{a}_k(\omega)e^{-j\omega nT} : k = 1,2,\ldots,K; n < 0\right\}, \tag{C.42}$$

where

$$\mathbf{A}(\omega) = [\mathbf{a}_1(\omega), \mathbf{a}_2(\omega), \ldots, \mathbf{a}_K(\omega)]^T. \tag{C.43}$$

Therefore,

$$\mathbf{u}(\omega) = \sum_{n\geq 0}\sum_{k=1}^{K}\left(\frac{T}{2\pi}\int_{-\pi/T}^{\pi/T} e^{j\omega nT}\mathbf{a}_k^H(\omega)\left(\mathbf{H}(\omega) + \frac{\mathcal{N}_0 T}{K\mathcal{S}_0}\mathbf{I}\right)\mathbf{x}(\omega)d\omega\right)\mathbf{a}_k(\omega)e^{-j\omega nT}, \tag{C.44}$$

$$= \sum_{n\geq 0}\sum_{k=1}^{K}\left(\frac{T}{2\pi}\int_{-\pi/T}^{\pi/T}\mathbf{a}_k^H(\omega)\mathbf{b}^+(\omega)e^{j\omega nT}d\omega\right)\mathbf{a}_k(\omega)e^{-j\omega nT}, \tag{C.45}$$

$$= \mathbf{A}(\omega)\sum_{n\geq 0}\left(\frac{T}{2\pi}\int_{-\pi/T}^{\pi/T}\mathbf{A}^H(\omega)\mathbf{b}^+(\omega)e^{j\omega nT}d\omega\right)e^{-j\omega nT}, \tag{C.46}$$

and

$$(\mathbf{u},\mathbf{u}) = \sum_{n\geq 0}\sum_{k=1}^{K}\left|\frac{\sqrt{T}}{2\pi}\int_{-\pi/T}^{\pi/T}\mathbf{a}_k^H(\omega)\mathbf{b}^+(\omega)e^{j\omega nT}d\omega\right|^2, \tag{C.47}$$

$$= \sum_{n\geq 0}\left(\frac{\sqrt{T}}{2\pi}\int_{-\pi/T}^{\pi/T}\mathbf{A}^H(\omega)\mathbf{b}^+(\omega)e^{j\omega nT}d\omega\right)^H\left(\frac{\sqrt{T}}{2\pi}\int_{-\pi/T}^{\pi/T}\mathbf{A}^H(\omega)\mathbf{b}^+(\omega)e^{j\omega nT}d\omega\right). \tag{C.48}$$

Observe that, for this optimum $\mathbf{u}(\omega)$,

$$\mathbf{b}^{+}(\omega)+\mathbf{b}^{-}(\omega) = \left(\mathbf{H}(\omega)+\frac{\mathcal{N}_0 T}{K S_0}\mathbf{I}\right)(\mathbf{x}(\omega)+\mathbf{y}(\omega)), \tag{C.49}$$

$$= \mathbf{C}^{H}(\omega)\mathbf{C}(\omega)\mathbf{u}(\omega), \tag{C.50}$$

$$= \mathbf{C}^{H}\omega)\sum_{n\geq 0}\left(\frac{T}{2\pi}\int_{-\pi/T}^{\pi/T}\mathbf{A}^{H}(\omega)\mathbf{b}^{+}(\omega)e^{j\omega nT}d\omega\right)e^{-j\omega nT}. \tag{C.51}$$

Since $\mathbf{b}^{+}(\omega)$ is a polynomial in $e^{-j\omega T}$ and $\mathbf{A}^{H}(\omega)$ is anticausal, the summation

$$\sum_{n\geq 0}\left(\frac{T}{2\pi}\int_{-\pi/T}^{\pi/T}\mathbf{A}^{H}(\omega)\mathbf{b}^{+}(\omega)e^{j\omega nT}d\omega\right)e^{-j\omega nT} \tag{C.52}$$

is itself a polynomial. This, when combined with the fact that $\mathbf{C}^{H}(\omega)\in\mathcal{L}^2[-\pi/T,\pi/T]$, shows that $(\mathbf{b}^{+}(\omega)+\mathbf{b}^{-}(\omega))\in\mathcal{L}^2[-\pi/T,\pi/T]$, which immediately implies that $\mathbf{b}^{-}(\omega)\in\mathcal{L}^2[-\pi/T,\pi/T]$, that is, the constraint C.34.

C.4 Conclusion

The isomorph $\mathbf{c}(\omega)\in\mathcal{L}^2_{\mathbf{H}}$ of the solution $f(t)$ to the problem posed in section C.1 is given by

$$\mathbf{c}(\omega)=\mathbf{u}(\omega), \tag{C.53}$$

and the minimum so achieved is given by

$$\Gamma[f]=\mathcal{N}_0\,(\mathbf{u},\mathbf{u}), \tag{C.54}$$

where $\mathbf{u}(\omega)$ and $(\mathbf{u},\mathbf{u})$ are given by equations C.46 and C.48 respectively.

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