# OPERATORS ON LEBESGUE SPACES
# WITH GENERAL MEASURES

By

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# OPERATORS ON LEBESGUE SPACES WITH GENERAL MEASURES

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## ABSTRACT

This thesis is concerned with the study of integral operators of the form

$$(Kf)(x) = \int_{-\infty}^{\infty} k(x,t)f(t)\,d\mu(t), \qquad x\in R,$$

between Lebesgue and "weak" Lebesgue spaces with general measures. For large classes of kernels we characterize the measures $\mu$ and $\nu$ for which the operator $K: L^p_\mu \to L^q_\nu$, or $K: L^p_\mu \to \text{weak } L^q_\nu$, $0<q<\infty$, $1\le p<\infty$ is bounded. If K is the Hardy operator our results are applied to prove a weighted Marcinkiewicz intepolation theorem and if K is the Stieltjes transform our characterization has an application to the Hilbert double series. In the case that K is the Fourier transform, we consider also the higher dimensional analogue and prove several weighted norm inequalities for near optimal weights. The one-dimensional form is applied to prove Laplace representation theorems for functions in weighted Bergman spaces.

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# CHAPTER 1

## INTRODUCTION AND PRELIMINARIES

The purpose of this dissertation is to study mapping properties of integral operators of the form

$$(Kf)(x) = \int_{-\infty}^{\infty} k(x,t)f(t)\ d\mu(t)$$

between $L^p$ spaces with general measures. In particular we characterize measures and weights for which $K: L^p_\mu \to L^q_\nu$, $0<q<\infty, 1\le p<\infty$, is bounded. The operators include the Hardy operator, $\int_0^x f(t)dt$; the Stieltjes transforms, $\int_0^\infty (x+t)^{-\lambda} f(t)dt$, $\lambda>0$; the Fourier transform, $\int_{-\infty}^{\infty} e^{-2\pi ixt} f(t)dt$, and the Laplace transform, $\int_0^\infty e^{-xt} f(t)dt$.

The study of the Hardy operator originated in 1915 with a note by G.H. Hardy [20] showing that

$$(1.1) \qquad \int_a^\infty \left((1/x)\int_a^x f(t)\ dt\right)^2 dx < \infty, \qquad a>0,$$

if and only if

$$(1.2)\qquad \int_a^\infty \int_a^\infty (x+y)^{-1} f(x)f(y)\,dx\,dy < \infty.$$

The fact that (1.2) holds whenever $\int_a^\infty f(x)^2 dx < \infty$ is a theorem due to Hilbert, and is essentially the statement that the Stieltjes transform maps $L^2$ to $L^2$. Before Hardy's work, the only known proofs of this fact were quite difficult. Over the next few years, he presented several elementary proofs [21],[22],[23] of various forms of the inequality

$$(1.3)\qquad \int_a^\infty \left((1/x)\int_a^x f(t)\,dt\right)^2 dx \le C\int_a^\infty f(x)^2\,dx,$$

thus providing a simple proof of Hilbert's result.

In 1928 Hardy [24] gave the following more general results, often referred to as Hardy's inequalities. If $p>1$, then for $r>1$ we have

$$(1.4)\qquad \int_0^\infty x^{-r}\left|\int_0^x f(t)\,dt\right|^p dx \le (p/(r-1))^p \int_0^\infty x^{-r}|xf(x)|^p\,dx,$$

and for $r<1$ we have

$$(1.5)\qquad \int_0^\infty x^{-r}\left|\int_x^\infty f(t)\,dt\right|^p dx \le (p/(1-r))^p \int_0^\infty x^{-r}|xf(x)|^p\,dx.$$

Since these inequalities provide a relationship between a function and its derivative they have applications to the theory of Sobolev spaces. Moreover, the Marcinkiewicz interpolation theorem, the Hardy-Littlewood maximal theorem and Sobolev's theorem are proved using Hardy's inequalities. It is therefore not unexpected that the following weighted form of Hardy's inequality

$$(1.6)\qquad \left(\int_0^\infty\left|\int_0^x f(t)\,dt\right|^q v(x)\,dx\right)^{1/q} \le C\left(\int_0^\infty |f(t)|^p w(t)\,dt\right)^{1/p}.$$

leads to far more general results and numerous applications (see, for example, [12],[26],[27],[40]).

In 1969 Artola [3], Talenti [51], and Tomaselli [52] characterized all those non-negative, locally integrable weight pairs v, w for which (1.6) holds with $p=q$. They showed that the condition

$$(1.7)\qquad \sup_{y>0}\left(\int_y^\infty v(x)\,dx\right)^{1/q}\left(\int_0^y w(t)^{1-p'}\,dt\right)^{1/p'} < \infty,$$

again with $p=q$, is both necessary and sufficient for (1.6). Muckenhoupt [37] provided a simple proof of their result, gave the dual inequality (which generalizes (1.5)), and extended both of these from weights to measures. Bradley

[11], Andersen and Muckenhoupt [2], and Kokilashvili [33] independently proved that, in the case $p \le q$, (1.6) holds if and only if (1.7) holds.

In the case $1 \le q < p$, Mazja and Rozin [36] showed that (1.6) holds if and only if

$$(1.8) \quad \left( \int_0^\infty \left( \int_t^\infty v(x)\, dx \right)^{r/q} \left( \int_0^t w(s)^{1-p'}\, ds \right)^{r/q'} w(t)^{1-p'}\, dt \right)^{1/r} < \infty,$$

where $1/r = 1/q - 1/p$. Sawyer [46], while studying mapping properties of the modified Hardy operators $x^{-\eta}\int_0^x f(t)dt$ on Lorentz spaces, proved that (1.6) holds in the range $0 < q < p$, $p \ge 1$ if and only if

$$\sup_{\ldots x_k < x_{k+1} \ldots} \sum_k \left( \left( \int_{x_k}^{x_{k+1}} v \right)^{1/q} \left( \int_{x_{k-1}}^{x_k} w^{1-p'} \right)^{1/p'} \right)^r < \infty$$

where $1/r = 1/q - 1/p$ and the supremum is taken over all positive increasing sequences $\{x_k\}$. But this implies that for $1 \le q < p$ this weight condition is equivalent to (1.8) so the question arises whether or not these two conditions are equivalent in the range $0 < q < 1$, $p \ge 1$. As a consequence of the work here we see that this is indeed the case.

In this dissertation we consider the more general

Hardy inequality

$$(1.9)\qquad \left(\int_{-\infty}^{\infty}\left|\int_{-\infty}^{x} f(t)\,d\mu(t)\right|^{q} d\nu(x)\right)^{1/q} \le C\left(\int_{-\infty}^{\infty}|f(t)|^{p}\,d\mu(t)\right)^{1/p},$$

which differs from (1.6) in several respects. Firstly, the weights are replaced by measures and secondly, the Lebesgue measure in the classical Hardy operator is replaced by the measure $\mu$. Further, the integrals in (1.9) all have $-\infty$ as the lower endpoint instead of 0. This relatively unimportant change is adopted in order to replace duality arguments by simple changes of variable. One advantage of giving results for general measures is that both weighted integral inequalities and series inequalities may be recognized as special cases of the same theorem. Also, this may be taken to be a more natural Hardy operator for in the right side of (1.9), $f$ is assumed to be $\mu$-measurable so it seems reasonable to integrate $f$ against $\mu$ on the left side as well. Writing the inequality in this way also simplifies the conditions which correspond to (1.7) and (1.8), and makes the extension to measures much more natural.

The characterizations (1.7) and (1.8) follow from the results of Chapter 2. Muckenhoupt considered only the classical Hardy operator in his extension from weights to measures, and so obtained results which neither imply nor are implied by those given here.

Another approach to the characterization of weight pairs satisfying the weighted Hardy inequality (1.6) is in terms of the existence of solutions to certain differential equations (see, for example, [5], [6], [10], [18], and [52]). Another approach, (cf. [34], [18], [50]) involves finding an expression for one weight in terms of the other. However, in this dissertation we concern ourselves entirely with integral conditions on weights and measures.

We mention here that a characterization of weights (in terms of integral conditions) for the two-dimensional weighted Hardy inequality

$$\left( \int_0^\infty \int_0^\infty \left| \int_0^x \int_0^y f(s,t)\, dt\, ds \right|^q v(x,y)\, dy\, dx \right)^{1/q} \le C \left( \int_0^\infty \int_0^\infty |f(x,y)|^p w(x,y)\, dy\, dx \right)^{1/p}$$

was recently given by Sawyer [47]. The characterization of weights for the higher dimensional generalization of this inequality is still open.

The Hardy operator is fundamental in the study of integral operators of the form

$$(Kf)(x) = \int_{-\infty}^{\infty} k(x,t)f(t)\, d\mu(t).$$

In Chapter 3 we characterize measures $\mu$ and $\nu$ for which $K:L^p_\mu \to L^q_\nu$, $1\le q<p<\infty$, with rather general kernels, is bounded.

The Stieltjes kernel $k(x,t)=(x+t)^{-\lambda}$, $\lambda>0$, is a special case and therefore we obtain a new characterization of measures for which the Stieltjes transform is bounded from $L^p_\mu$ to $L^q_\nu$, $1\le q<p<\infty$. Andersen [1] obtained such results in the case $1<p\le q<\infty$, however, our proofs are quite different. As in Andersen's work we are able to give a generalization of the Hilbert double series theorem.

Also in Chapter 3 we prove a weighted, weak type mapping property of operators whose kernels are monotone in one variable. This result has been given by different authors for several particular operators [1],[2],[48].

In 1966, Calderón [13] showed that a sublinear operator T is of weak type $(p_0,q_0)$ and $(p_1,q_1)$ if and only if

$$(1.10)\quad (Tf)^*(t) \le C\left(t^{-1/q_0}\int_0^{t^\alpha} s^{1/p_0-1}f^*(s)\,ds + t^{-1/q_1}\int_{t^\alpha}^{\infty} s^{1/p_1-1}f^*(s)\,ds\right)$$

where * indicates the rearrangement with respect to Lebesgue measure and $\alpha=(1/q_0-1/q_1)/(1/p_0-1/p_1)$. Since it is often easy to verify weak type conditions this characterization has wide applicability and is especially useful for proving boundedness of many classical operators, whose weak mapping properties are well known. Heinig [26], [27] applied some of the previously mentioned results on the weighted Hardy inequalities to estimate the right side of (1.10), giving conditions which imply weighted norm inequalities for very general operators. In particular this approach leads to a characterization of weighted Fourier estimates when the weights are assumed to be monotone. In Chapter 4 we follow his development using the weight conditions derived in Chapter 2. To illustrate the results obtained, specific (not necessarily monotone) weights are given for which a weighted Fourier inequality holds. Also in Chapter 4 we use weighted Hardy inequalities to prove a variant of the Marcinkiewicz interpolation theorem in which the usual weak type hypotheses are replaced by integral conditions. Chapter 4 concludes with an application of the discrete form of the general Hardy inequality. We show that the operator S defined by $(Su)_n=\alpha_n\sum_{k=n}^{\infty}u_k/\alpha_k$ is bounded on $l^p$ if and only if the sequence $\{\alpha_n\}$ satisfies a type of monotonicity condition.

The weighted Hardy space $H_w^p$, $1\le p<\infty$, consists of those functions f(z), holomorphic in the right half plane, Re(z)>0, for which

(1.11) $$\sup_{x>0}\int_{-\infty}^{\infty} w(x+iy)\,|f(x+iy)|^p\,dy < \infty$$

and f has boundary values. For example, if w=1 and p=2, Paley and Wiener [41] proved in the late 1920's that $f\in H^2$ if and only if there exists a function $F\in L^2(0,\infty)$ such that

(1.12) $$f(z) = \int_0^{\infty} e^{-zt}F(t)\,dt, \qquad \mathrm{Re}(z)>0.$$

This result was generalized by Doetsch [14] and later in a series of papers by Rooney [42],[43] involving certain power weights. More recently Benedetto, Heinig and Johnson [7] extended these results further by permitting more general weights. For example, they have shown that if u and v belong to some weight class $F_{p,q}$, $1\le p\le q<\infty$, $q>1$ with u, v radial and u and 1/v non-increasing as a function on $(0,\infty)$, and if f satisfies (1.11) then $f\in H_v^p$ and there is an $F\in L_u^p(0,\infty)$, such that f has the Laplace representation (1.12). Moreover for all $x>0$,

$$\left(\int_0^\infty e^{-xt}u(t)|F(t)|^q\,dt\right)^{1/q} \le C\left(\int_{-\infty}^\infty |f(x+iy)|^p v(x+iy)\,dy\right)^{1/p}.$$

Also, with the same weight functions u and v, if $F\in L_v^p(0,\infty)$ and f is defined by (1.12) then f is holomorphic and $f\in H_u^q$.

If instead of the weighted Hardy spaces one considers the Bergman spaces of functions f(z), holomorphic in the right half plane such that

$$\int_0^\infty\left(\int_{-\infty}^\infty w(x+iy)|f(x+iy)|^p\,dy\right)^{q/p}dx<\infty, \tag{1.13}$$

$1<p<\infty$, $1<q<\infty$, then (1.13) does not in general imply the existence of boundary values of f, not even for the weight $w=1$. Yet Rooney [42, Theorem 3] has shown that in the case $p=q=2$ with $w(x+iy)=|x|^\alpha$, $\alpha>-1$, a Laplace representation of such functions is possible. More recently Genchev [17] has given Laplace representations of functions in Bergman spaces satisfying (1.13) with $w=1$, $1<p\leq 2$ and $q=p'$.

Generalizations of the results of Genchev to quite general weights and a larger range of indices are given in Section 3 of Chapter 5. As in the Hardy space case, representation theories for functions in weighted Bergman spaces depend strongly on weighted Fourier inequalities. We establish such estimates in Section 1. For the monotone weights we consider, the $A_p$-weight class of Muckenhoupt is shown to be optimal. These results were recently proved in [8], however our proofs are different and depend on a weak type inequality which seems to be new.

Weighted Fourier norm inequalities are important in the solution of many problems of analysis and very few results are known in higher dimensions. Section 2 of Chapter 5 gives such estimates for weights which are products of one-variable, monotone weights and for weights which are radial and monotone on $(0,\infty)$. The general case seems very difficult and is unknown. However, we give an n-dimensional (one weight) generalization of a result of Kerman and Sawyer [47] for weights which satisfy a monotonicity condition in each variable separately. This in turn leads to a weighted n-dimensional form of Plancherel's theorem and consequently a weighted Hausdorff-Young theorem.

The work of Chapter 5 is in collaboration with Dr. Heinig. We particularly appreciate that the material on the Laplace representation of functions in weighted Bergman spaces is included here.

The definitions and theorems in the remainder of this chapter are well known and are included here for easy reference. For the definitions and notation of set and measure theory we follow Royden [45]. A table of symbols begins on page 20.

A subset E of $\mathbb{R}$ is <u>open</u> if for every $x\in E$ there exists a $\delta>0$ such that $(x-\delta,x+\delta)\subseteq E$.

Two sets are <u>disjoint</u> if their intersection is the

<u>empty set</u>, $\emptyset$, the unique set having no elements.

A set E is <u>countable</u> if its elements may be placed in a one to one correspondence with the elements of a subset of the integers. In particular each <u>finite</u> set (having only finitely many elements) is countable.

Every open subset of $\mathbb{R}$ may be expressed as the union of countably many disjoint open intervals.

A collection of sets $\{E_\alpha\}$ is said to <u>cover</u> a set E if $E \subseteq \cup E_\alpha$.

A $\sigma$-<u>algebra</u> of sets is a collection of sets closed under complementation and countable union.

A <u>Borel</u> set is a member of the smallest $\sigma$-algebra containing all the open sets.

A <u>measure</u> $\mu$ is a map from a $\sigma$-algebra to $\mathbb{R}^+ \cup \{\infty\}$ satisfying $\mu\emptyset=0$ and $\mu(\cup E_n)=\Sigma\mu E_n$ for any sequence $E_n$ of pairwise disjoint sets. The sets in the $\sigma$-algebra are called $\mu$-<u>measurable</u>.

A measure $\mu$ is a <u>Borel measure</u> if all Borel sets are $\mu$-measurable.

A measure $\mu$ on $\mathbb{R}$ is $\sigma$-<u>finite</u> if there exist sets $E_n$

such that $E_n \subseteq E_{n+1}$, $\cup E_n = \mathbb{R}$, and $\mu E_n < \infty$.

A measure $\mu$ is complete if every subset of a set of $\mu$-measure zero is $\mu$-measurable.

A set E has full $\mu$-measure if the complement of E has zero $\mu$-measure. In this case we say that $\mu$ is supported on E. If some property holds on a set of full $\mu$-measure we say that it holds $\mu$-almost everywhere ($\mu$-a.e.) or that it holds for $\mu$-almost every x ($\mu$-a.e. x).

The supremum of a set E is the least upper bound of E written sup E. The infimum of E is the greatest lower bound of E written inf E.

The essential supremum with respect to the measure $\mu$ of the function f over the set E is given by $\operatorname{esssup}\{f(x) : x \in E\} = \sup\{\alpha : \mu\{f(x) \geq \alpha\} > 0\}$. The essential infimum is defined similarly.

Lebesque measure on $\mathbb{R}$ is a complete, Borel measure with respect to which the measure of an interval is its length. This is the measure of "ordinary" integration. For a detailed construction see Royden [45].

A function f is called $\mu$-measurable if for all $\alpha \in \mathbb{R}$

$\{x:f(x)>\alpha\}$ is a $\mu$-measurable set.

A <u>simple function</u> is a function which takes only finitely many values. If a simple function f which takes values $a_1,\ldots,a_n$ is $\mu$-measurable then the integral of f with respect to $\mu$ is defined by $\int f d\mu = \Sigma a_k \mu\{x:f(x)=a_k\}$. The integral of an arbitrary non-negative $\mu$-measurable function f is defined to be the supremum of the integrals of simple functions which are less than or equal to f.

A function f <u>majorizes</u> a function g if $f \geq g$ at every point. We call f a <u>majorant</u> of g.

The <u>monotone convergence theorem</u>: If $f_n \to f$ pointwise as $n \to \infty$ and $f_n \leq f_{n+1}$ for $n=1,2,\ldots$ then $\int f d\mu = \lim \int f_n d\mu$.

A point x is an <u>atom</u> for a measure $\mu$ if $\mu\{x\}>0$. If the $\mu$-measure of every set E is the sum of the measures of the atoms in E then $\mu$ is an <u>atomic measure</u>. We will call an atomic measure <u>discrete</u> if all of its atoms are integers.

A measure $\mu$ is <u>absolutely continuous</u> (with respect to Lebesgue measure) if $\mu E=0$ whenever E has Lebesgue measure zero. Any such measure may be represented by a non-negative function in the sense that if $\mu$ is absolutely continuous then there exists $w \geq 0$ such that $\int f d\mu = \int f w$ for all f. We refer to w as a weight and identify w and $\mu$. The non-negative function $|x|^{\alpha}$ is an example of a commonly considered weight called a power weight.

The norm of the Lebesgue space $L^p_\mu$, $p \geq 1$, is $\|f\|_{p,\mu} = \left(\int |f|^p d\mu\right)^{1/p}$. The triangle inequality for this norm is Minkowski's inequality which states that $\|f+g\|_{p,\mu} \leq \|f\|_{p,\mu} + \|g\|_{p,\mu}$.

For any positive p we define the harmonic conjugate p' of p to be the solution to $1/p+1/p'=1$. Observe that if $0<p<1$ then $p'<0$ and if $p=1$ then $p'=\infty$.

Hölder's inequality provides the following relationship between the spaces $L^p_\mu$ and $L^{p'}_\mu$ for $1<p<\infty$:

$$\|f\|_{p,\mu} = \sup\{\int fg \, d\mu : \|g\|_{p',\mu} \leq 1\}.$$

This expresses the fact that $L^p_\mu$ and $L^{p'}_\mu$ are dual spaces.

The simple functions are dense in $L^p_\mu$ for $1 \leq p < \infty$, that is, given a non-negative function f and $\varepsilon > 0$ there exists a simple function g such that $\|f-g\|_{p,\mu} < \varepsilon$. For this reason it often suffices to establish the validity of a proposition for simple functions in order to conclude that it holds for all functions in $L^p_\mu$.

Given two measures $\mu$ and $\nu$ on $\mathbb{R}$ we define $(\mu\times\nu)(A\times B)$ to be $(\mu A)(\nu B)$ for all $\mu$-measurable A and all $\nu$-measurable B. The map $\mu\times\nu$ extends to a complete measure on $\mathbb{R}\times\mathbb{R}$ called the <u>product</u> <u>measure</u> of $\mu$ and $\nu$.

<u>Tonelli's</u> <u>theorem</u>: If $\mu$ and $\nu$ are $\sigma$-finite measures and f is non-negative and $\mu\times\nu$-measurable then f(x,y) is a $\mu$-measurable function of x for $\nu$-a.e. y, $\int f(x,y)d\nu(y)$ is a $\mu$-measurable function of x, and $\int\left(\int f d\nu\right)d\mu=\int f d(\mu\times\nu)$.

<u>Fubini's</u> <u>theorem</u>: If $\mu$ and $\nu$ are complete measures and $\int|f|d(\mu\times\nu)<\infty$ then $\int|f(x,y)|d\mu(x)<\infty$ for $\nu$-a.e. y, $\int\left|\int f(x,y)d\mu(x)\right|d\nu(y)<\infty$, and $\int\left(\int f d\nu\right)d\mu=\int f d(\mu\times\nu)$.

<u>Minkowski's</u> <u>integral</u> <u>inequality</u>: If $\mu$ and $\nu$ are $\sigma$-finite measures and f is $\mu\times\nu$-measurable then

$$\left(\int\left(\int|f|\,d\mu\right)^p d\nu\right)^{1/p}\le\int\left(\int|f|^p\,d\nu\right)^{1/p}d\mu.$$

An <u>operator</u> T (a map on a space of functions) is <u>linear</u> if $T(\alpha f+g)=\alpha Tf+Tg$ where f and g are functions and $\alpha\in\mathbb{R}$. T is <u>sublinear</u> if $|T(f+g)|\le|Tf|+|Tg|$.

The (non-increasing) rearrangement of a function f with respect to a measure $\mu$ is $f^{\circledast}(t)=\inf\{\beta:\mu\{x:|f(x)|>\beta\}\le t\}$. The rearrangement with respect to Lebesgue measure is denoted $f^*$. The symmetric rearrangement of f with respect to Lebesgue measure is denoted $f^{\circledast}$, $f^{\circledast}(t)=f^*(2|t|)$. (For properties of the rearrangement see [9]).

An operator T is of strong type $((p,\mu),(q,\nu))$ if T is bounded as a map from $L^p_\mu$ to $L^q_\nu$. That is, there exists a constant $C>0$ such that for all f $\|Tf\|_{q,\nu}\le C\|f\|_{p,\mu}$. T is of weak type $((p,\mu),(q,\nu))$ if $(\nu\{x:|(Tf)(x)|>\beta\})^{1/q}\le(C/\beta)\|f\|_{p,\mu}$ or equivalently $(Tf)^{\circledast}(t)\le t^{-1/q}\|f\|_{p,\mu}$.

The Marcinkiewicz interpolation theorem: If $p_0\ne p_1$ and T is a sublinear operator of weak type $((p_0,\mu),(q_0,\nu))$ and $((p_1,\mu),(q_1,\nu))$ then $T:L^p_\mu\to L^q_\nu$ is bounded whenever $p\le q$ where $1/p=(1-\alpha)/p_0+\alpha/p_1$, $1/q=(1-\alpha)/q_0+\alpha/q_1$, and $0<\alpha<1$.

The Riesz-Thorin theorem: If $p_0\ne p_1$, $q_0\ne q_1$ and T is a linear operator of strong type $((p_0,\mu),(q_0,\nu))$ and $((p_1,\mu),(q_1,\nu))$ then $T:L^p_\mu\to L^q_\nu$ is bounded where $1/p=(1-\alpha)/p_0+\alpha/p_1$, $1/q=(1-\alpha)/q_0+\alpha/q_1$, and $0<\alpha<1$.

The Fourier transform of f on R is defined by

$\hat{f}(x)=\int e^{-2\pi ixt}f(t)dt$. On $\mathbb{R}^n$ it takes the form $\hat{f}(x)=\int_{\mathbb{R}^n} e^{-2\pi ix\cdot t}f(t)dt$ where $x\cdot t=x_1t_1+\ldots+x_nt_n$. We will occasionally find it convenient to omit the factor $2\pi$ in the exponent.

We will adopt the following conventions: A product of the form $0.\infty$ is taken to be 0. $A\leq B$ means that if B is finite then A is finite and A is less than or equal to B. The symbol C denotes a positive constant whose value may be different at different occurrences.

# Table of Symbols

| | |
|---|---|
| $\emptyset$ | the empty set |
| $\mathbf{Z}$ | the integers |
| $\mathbf{Z}^+$ | the positive integers including zero |
| $\mathbf{Q}$ | the rational numbers |
| $\mathbf{R}$ | the real numbers |
| $\mathbf{R}^n$ | n-fold Cartesian product of $\mathbf{R}$ |
| $\mathbf{R}^+$ | the positive real numbers including zero |
| $\mathbf{R}^n_+$ | n-fold Cartesian product of $\mathbf{R}^+$ |
| $\{x \in X : P(x)\}$ | the set of x in X satisfying P(x) |
| $[a,b]$ | $\{x : a \leq x \leq b\}$, the closed interval from a to b |
| $(a,b)$ | $\{x : a < x < b\}$, the open interval from a to b |
| $A \cup B$ | the union of A and B |
| $A \cap B$ | the intersection of A and B |
| $A \subseteq B$ | A is a subset of B |
| $A \not\subseteq B$ | A is not a subset of B |
| $x \in A$ | x is an element of A |
| $x \notin B$ | x is not an element of A |
| $A \times B$ | the Cartesian product of A and B |

| | |
|---|---|
| $A \setminus B$ | $\{x \in A : x \notin B\}$, the relative complement |
| $\|x\|$ | absolute value or modulus |
| $\to$ | approaches |
| $\max(a,b)$ | the larger of a and b |
| $\min(a,b)$ | the smaller of a and b |
| $\chi_E$ | the characteristic function of the set E |
| $\mu E$ | the $\mu$-measure of the set E |
| $\int_E f d\mu$ | the integral of f over E |
| $\int_a^b f d\mu$ | the integral of f over [a,b] |
| $\mu$-a.e. | $\mu$-almost everywhere, $\mu$-almost every |
| $\mu \times \nu$ | the product measure |
| $f(x-)$ | the left limit of f at x |
| $p'$ | the harmonic complement |
| $L_\mu^p(X)$ | Lebesgue space of index p over $(X,\mu)$ |
| $L_\mu^p$ | Lebesgue space of index p over $(\mathbb{R},\mu)$ |
| $L^p$ | Lebesgue space over $\mathbb{R}$, Lebesgue measure |
| $l^p$ | Lebesgue space over $Z^+$, counting measure |
| $\\|f\\|_{p,\mu}$ | norm on $L_u^p$ |
| $\\|f\\|_p$ | norm on $L^p$ |

| | |
|---|---|
| $f \sim g$ | f is bounded above and below by g |
| $f^{\textcircled{\mu}}$ | rearrangement of f with respect to $\mu$ |
| $f^{\textcircled{\nu}}$ | with respect to $\nu$ |
| $f^{*}$ | with respect to Lebesgue measure |
| $f^{\circledast}$ | symmetric rearrangement, Lebesgue measure |
| $\equiv$ | is defined to be |
| $\hat{f}$ | the Fourier transform of f |
| $\{f_n\}_{n=0}^{\infty}$ | the sequence $f_0, f_1, \ldots$ |
| $f * g$ | the convolution of f and g |
| $\mathrm{Re}(z)$ | the real part of the complex number z |
| $\mathrm{sgn}(x)$ | 1 if $x>0$, $-1$ if $x<0$, the sign of x |
| $C_0^{\infty}(0,1)$ | functions on $(0,1)$ which vanish at 0 and 1, and have derivatives of all orders. |
| $C_0^{\infty}(\mathbb{R})$ | functions on $\mathbb{R}$ which vanish at $\infty$ and $-\infty$ and have derivatives of all orders. |

# CHAPTER 2

## THE GENERAL HARDY INEQUALITY

The purpose of this chapter is to study the general Hardy operator $(Kf)(x) = \int_{-\infty}^{x} f(t)d\mu(t)$ and its dual. Specifically, we give necessary and sufficient conditions on non-negative measures $\mu$ and $\nu$ such that

$$(2.1)\quad \left(\int_{-\infty}^{\infty}\left|\int_{-\infty}^{x} f(t)\, d\mu(t)\right|^{q} d\nu(x)\right)^{1/q} \le C\left(\int_{-\infty}^{\infty}|f(t)|^{p}\, d\mu(t)\right)^{1/p}$$

$1\le p<\infty$, $0<q<\infty$, holds for all $\mu$-measurable functions f.

For various values of the indices p and q and specific measures this problem was solved by various authors. For example, if $p=q\ge 1$ and $\mu$ and $\nu$ are absolutely continuous and defined on $\mathbb{R}^{+}$, then Artola [3], Tomaselli [52], Talenti [51] and Muckenhoupt [37] gave a complete solution to this problem. Indeed Muckenhoupt considered also general measures on $\mathbb{R}^{+}$. The case $1\le p\le q\le\infty$ with absolutely continuous measures on $\mathbb{R}^{+}$ was solved by Bradley [11] and independently by Andersen and Muckenhoupt [2] and

Kokilashvili [33]. The weight conditions given there are quite similar to the conditions we give here for measures. The case $1\le q<p<\infty$, for absolutely continuous measures on $\mathbb{R}^+$, was solved by Mazja and Rozin [36] and independently by Sawyer [46]. Sawyer's weight condition is quite different from that of Mazja and Rozin and his method also extends the result to the range $0<q<p$, $p\ge 1$.

In this chapter we generalize and unify these results. Section 1 contains a number of estimates involving general measures, required in subsequent sections. These results are summarized in Proposition 2.5. The content of Theorems 2.6, 2.7 and 2.8 of Section 2 is the characterization of the measures $\mu$ and $\nu$ for which (2.1) holds in the cases $1<p\le q<\infty$; $0<q<p$, $p=1$; and $1\le q<p<\infty$. In order to prove the characterization in the case $0<q<1$, $1<p<\infty$, we need to construct a function $g^\circ$ for every $\mu$-measurable bounded $g$ such that $g^\circ$ is non-increasing and satisfies

$$\int_{-\infty}^{x} g(t)\, d\mu(t) \le \int_{-\infty}^{x} g^\circ(t)\, d\mu(t)$$

and $\|g^\circ\|_{p,\mu} \le \|g\|_{p,\mu}$, $p\ge 1$. This construction, we think, is of independent interest and is contained in Section 3. Halperin [19] defined functions with similar properties in the case of absolutely continuous measures $\mu$, but our results resemble the simplifications of Halperin's work due to Lorentz [35]. Section 4 concludes our work on the general Hardy inequality by proving the characterization in the index range $0<q<1$, $1<p<\infty$. In addition, a number of specific cases are given to illustrate our results. Particularly noteworthy is Theorem 2.19 in which a Hardy inequality for $0<p=q<1$ is given provided f satisfies a monotonicity condition.

## 1. INTEGRAL ESTIMATES WITH MEASURES

For a non-negative, Lebesgue measurable function f and an $x>0$ integration yields

$$p\int_0^x\left(\int_0^t f(y)\,dy\right)^{p-1} f(t)\,dt = \left(\int_0^x f(t)\,dt\right)^p$$

for all $p>0$, and

$$|p|\int_x^\infty\left(\int_0^t f(y)\,dy\right)^{p-1} f(t)\,dt \le \left(\int_0^x f(t)\,dt\right)^p$$

for all $p<0$. These inequalities play an important role in generalizing the classical Hardy inequality. In order to prove our results we require a number of estimates of this type in which Lebesgue measure on the half line is replaced by a measure $\mu$ on $\mathbb{R}$.

These estimates are given in the next four lemmas.

LEMMA 2.1. *If* $\mu(-\infty,x]<\infty$ *and* $0<\alpha\le 1$, *then there is a* $z\in(-\infty,x]$, *depending on* $\alpha$, *such that*

i) $\mu(-\infty,z]\ge\alpha\mu(-\infty,x]$;

ii) $\mu[z,x]\ge(1-\alpha)\mu(-\infty,x]$.

*Proof*. Let $m=\mu(-\infty,x]$. The lemma holds trivially if $m=0$ so we may assume $m>0$. Set $\lambda(s)=(1/m)\mu(-\infty,s]$ for $s\in\mathbb{R}$.

The set $E=\{s:\lambda(s)\ge\alpha\}$ is non-empty and bounded below since $x\in E$ and $\lim_{s\to\infty}\lambda(s)=0<\alpha$. Let $z=\inf E$ then by the right continuity of $\lambda$, $\lambda(z)\ge\alpha$. Therefore $\mu(-\infty,z]=\lambda(z)m\ge\alpha m$ and i) follows.

Also $\mu[z,x]=\lim_{\varepsilon\to 0^+}\mu(z-\varepsilon,x]=m-m\lim_{\varepsilon\to 0^+}\lambda(z-\varepsilon)\ge(1-\alpha)m$ which proves ii).

LEMMA 2.2. *For any measure* $\mu$ *on* $\mathbb{R}$ *and any* $p \geq 1$, *there exists a constant* C *such that*

$$C\left(\int_{-\infty}^{x} d\mu\right)^{p} \leq \int_{-\infty}^{x}\left(\int_{-\infty}^{t} d\mu\right)^{p-1} d\mu(t) \leq \left(\int_{-\infty}^{x} d\mu\right)^{p}$$

*for all* $x \in \mathbb{R}$.

*Proof*. Since the conclusion of the lemma is trivial in the case $p=1$, we proceed to the case $p>1$.

The inequality on the right follows by extending the range of the inner integral from $(-\infty,t]$ to $(-\infty,x]$.

To prove the other inequality, we choose $z \in (-\infty,x]$ as follows:

i) If $\int_{-\infty}^{x} d\mu = \infty$, choose z so that both $\int_{-\infty}^{z} d\mu$ and $\int_{z}^{x} d\mu$ are non-zero. Observe that at least one of the integrals must be infinite.

ii) If $\int_{-\infty}^{x} d\mu < \infty$, let z be as in Lemma 2.1 with $\alpha = 1/p'$, then both inequalities

$$\int_{-\infty}^{z} d\mu \geq (1/p') \int_{-\infty}^{x} d\mu \quad \text{and} \quad \int_{z}^{x} d\mu \geq (1/p) \int_{-\infty}^{x} d\mu$$

hold.

In either case

$$\left(\int_{-\infty}^{x} d\mu\right)^{p} \le \left(p\int_{z}^{x} d\mu\right)\left(p'\int_{-\infty}^{z} d\mu\right)^{p-1} = C\int_{z}^{x}\left(\int_{-\infty}^{z} d\mu\right)^{p-1} d\mu(t)$$
$$\le C\int_{z}^{x}\left(\int_{-\infty}^{t} d\mu\right)^{p-1} d\mu(t) \le C\int_{-\infty}^{x}\left(\int_{-\infty}^{t} d\mu\right)^{p-1} d\mu(t).$$

This completes the proof.

LEMMA 2.3. _For any measure_ $\mu$ _on_ $\mathbb{R}$ _and any_ $p\ge 1$, _there is a constant_ $C$ _such that_

(2.2) $$\left(\int_{-\infty}^{x} d\mu\right)^{1/p} \le \int_{-\infty}^{x}\left(\int_{-\infty}^{t} d\mu\right)^{1/p-1} d\mu(t) \le C\left(\int_{-\infty}^{x} d\mu\right)^{1/p}$$

_for all_ $x\in R$.

_Proof_. Since (2.2) holds trivially for $p=1$, assume $p>1$.

The left inequality follows by extending the range of the inner integral in (2.2) from $(-\infty,t]$ to $(-\infty,x]$.

To prove the other inequality in (2.2), we assume that $0<\int_{-\infty}^{x} d\mu<\infty$, for otherwise the conclusion is trivial. Fix $x\in R$ and $a>1$ and let $E_n=\{t\le x: \int_{-\infty}^{t} d\mu \le a^{-n}\int_{-\infty}^{x} d\mu\}$, $n=0,1,2,\dots,$

then $E_{n+1} \subseteq E_n$ for all n, and $E_0 = (-\infty, x]$. Since $\int_{-\infty}^{t} d\mu$ is non-decreasing as a function of t, there are numbers $t_n$, such that $E_n$ is $(-\infty, t_n)$ or $(-\infty, t_n]$. If $E_n = (-\infty, t_n)$ then

$$\int_{E_n} d\mu = \lim_{\varepsilon \to 0+} \int_{-\infty}^{t_n-\varepsilon} d\mu \le a^{-n} \int_{-\infty}^{x} d\mu$$

and otherwise

$$\int_{E_n} d\mu = \int_{-\infty}^{t_n} d\mu \le a^{-n} \int_{-\infty}^{x} d\mu.$$

Using this and the fact that for $t \notin E_{n+1}$

$$\int_{-\infty}^{t} d\mu > a^{-(n+1)} \int_{-\infty}^{x} d\mu$$

we get (since $1/p-1<0$)

$$\int_{-\infty}^{x} \left( \int_{-\infty}^{t} d\mu \right)^{1/p-1} d\mu(t) = \sum_{n=0}^{\infty} \int_{E_n \setminus E_{n+1}} \left( \int_{-\infty}^{t} d\mu \right)^{1/p-1} d\mu(t)$$

$$\le \sum_{n=0}^{\infty} \int_{E_n \setminus E_{n+1}} \left( a^{-(n+1)} \int_{-\infty}^{x} d\mu \right)^{1/p-1} d\mu(t) \le \left( \int_{-\infty}^{x} d\mu \right)^{1/p-1} \sum_{n=0}^{\infty} a^{(n+1)/p'} \int_{E_n} d\mu$$

$$\le \left( \int_{-\infty}^{x} d\mu \right)^{1/p-1} \sum_{n=0}^{\infty} a^{(n+1)/p'} a^{-n} \int_{-\infty}^{x} d\mu = \left( \int_{-\infty}^{x} d\mu \right)^{1/p} a^{1/p'} \sum_{n=0}^{\infty} a^{-n/p} = C \left( \int_{-\infty}^{x} d\mu \right)^{1/p},$$

where $C = a/(a^{1/p}-1)$.

REMARK. Minimizing $a/(a^{1/p}-1)$, $a>1$, we get

$p/(p')^{p/p'}$. This contrasts with the constant p of the inequality

$$\int_t^\infty (\mu[x,\infty))^{-1/p'}\, d\mu(x) \le p(\mu[t,\infty))^{1/p}$$

given without proof in [37].

LEMMA 2.4. _For any measure_ $\mu$ _on_ $\mathbb{R}$ _and_ $p>1$, _there exists a constant_ C _such that_

$$\int_x^\infty \left(\int_{-\infty}^t d\mu\right)^{-p} d\mu(t) \le C\left(\int_{-\infty}^x d\mu\right)^{1-p}$$

_for all_ $x\in\mathbb{R}$.

_Proof_. If $\int_{-\infty}^x d\mu=0$, the result is trivial. If $\int_{-\infty}^x d\mu=\infty$, then $\left(\int_{-\infty}^t d\mu\right)^{-p}=0$ for all $t\ge x$, so again the conclusion holds.

We now consider the case $0<\int_{-\infty}^x d\mu<\infty$. Fix $x\in\mathbb{R}$ and $a>1$ and set $F_n=\{t\in\mathbb{R}: \int_{-\infty}^t d\mu\le a^n \int_{-\infty}^x d\mu\}$, $n=0,1,2,\ldots$, and $F_{-1}=(-\infty,x)$. Then $F_n\subseteq F_{n+1}$, for all n and $\bigcup_{n\ge -1} F_n=\mathbb{R}$. Arguing as for the $E_n$'s in Lemma 2.3 we derive

$$\int_{F_n} d\mu \le a^n \int_{-\infty}^x d\mu.$$

Using this and the fact that $t \notin E_{n-1}$ implies

$$\int_{-\infty}^{t} d\mu > a^{n-1} \int_{-\infty}^{x} d\mu$$

we get

$$\int_x^\infty \left( \int_{-\infty}^{t} d\mu \right)^{-p} d\mu(t) = \sum_{n=0}^{\infty} \int_{F_n \setminus F_{n-1}} \left( \int_{-\infty}^{t} d\mu \right)^{-p} d\mu(t)$$

$$\leq \sum_{n=0}^{\infty} \int_{F_n \setminus F_{n-1}} \left( a^{n-1} \int_{-\infty}^{x} d\mu \right)^{-p} d\mu(t) \leq \left( \int_{-\infty}^{x} d\mu \right)^{-p} \sum_{n=0}^{\infty} a^{(n-1)(-p)} \int_{F_n} d\mu$$

$$\leq \left( \int_{-\infty}^{x} d\mu \right)^{-p} a^p \sum_{n=0}^{\infty} a^{-np} a^n \int_{-\infty}^{x} d\mu \leq \left( \int_{-\infty}^{x} d\mu \right)^{1-p} a^p \sum_{n=0}^{\infty} a^{n(1-p)} = C \left( \int_{-\infty}^{x} d\mu \right)^{1-p}$$

where $C = a^p/(1-a^{1-p})$. Minimizing this expression we obtain $(p'+1)^{p'+1}/p'^{p'}$.

PROPOSITION 2.5. Let $p>1$ and $f$ be non-negative and $\mu$-measurable. The following estimates hold with constants depending only on $p$:

i) $$\left( \int_{-\infty}^{x} f(t)\, d\mu(t) \right)^{p} \sim \int_{-\infty}^{x} \left( \int_{-\infty}^{t} f(s)\, d\mu(s) \right)^{p-1} f(t)\, d\mu(t);$$

ii) $$\left( \int_{x}^{\infty} f(t)\, d\mu(t) \right)^{p} \sim \int_{x}^{\infty} \left( \int_{t}^{\infty} f(s)\, d\mu(s) \right)^{p-1} f(t)\, d\mu(t);$$

iii) $$\left( \int_{-\infty}^{x} f(t)\, d\mu(t) \right)^{1/p} \sim \int_{-\infty}^{x} \left( \int_{-\infty}^{t} f(s)\, d\mu(s) \right)^{-1/p'} f(t)\, d\mu(t);$$

iv) $\left(\int_x^\infty f(t)\, d\mu(t)\right)^{1/p} \sim \int_x^\infty \left(\int_t^\infty f(s)\, d\mu(s)\right)^{-1/p'} f(t)\, d\mu(t);$

v) $\int_x^\infty \left(\int_{-\infty}^t f(s)\, d\mu(s)\right)^{-p} f(t)\, d\mu(t) \le C\left(\int_{-\infty}^x f(t)\, d\mu(t)\right)^{1-p};$

vi) $\int_{-\infty}^x \left(\int_t^\infty f(s)\, d\mu(s)\right)^{-p} f(t)\, d\mu(t) \le C\left(\int_x^\infty f(t)\, d\mu(t)\right)^{1-p}.$

*Proof*. Statements i), iii), and v) are immediate consequences of Lemmas 2.2, 2.3, and 2.4 respectively. The only change is that we apply the lemmas to the measure $fd\mu$. Statements ii), iv), and vi) may be obtained from i), iii), and v) by making the change of variables $t\to -t$, $s\to -s$, and $x\to -x$.

## 2. HARDY'S INEQUALITY I

In order to prove the general Hardy inequality (2.1) it clearly suffices to consider only non-negative $f$ on $R$. We shall assume therefore that $f\ge 0$ throughout.

Let $\mu$ and $\nu$ be $\sigma$-finite, Borel measures on $R$ and define $M$ and $N$ by

$$M(y) = \int_{-\infty}^{y} d\mu, \quad N(y) = \int_{y}^{\infty} d\nu. \tag{2.3}$$

Our first main result is

THEOREM 2.6. Suppose $1<p\le q<\infty$, then there exists a constant $C>0$ such that

$$\left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{x} f(t)\,d\mu(t)\right)^{q} d\nu(x)\right)^{1/q} \le C\left(\int_{-\infty}^{\infty} f(t)^{p}\,d\mu(t)\right)^{1/p} \tag{2.4}$$

holds for all $f\ge 0$ if and only if

$$\sup_{y\in R} N(y)^{1/q}\, M(y)^{1/p'} < \infty. \tag{2.5}$$

Proof. Note that $M(y)>0$ $\mu$-a.e. $y$ and $N(y)>0$ $\nu$-a.e. $y$. Fix $x\in R$. If $M(t)=\infty$ for some $t\le x$, then $M(x)=\infty$; so in view of (2.5), $N(x)=0$. Consequently $M(t)<\infty$ for all $t\le x$ $\mu$-a.e. $x$. Hence we may write

$$\int_{-\infty}^{x} f(t)\,d\mu(t) = \int_{-\infty}^{x} f(t)M(t)^{1/pp'}M(t)^{-1/pp'}\,d\mu(t)$$

$\nu$-a.e. $x$.

Now by Hölder's inequality and Minkowski's integral inequality

$$\left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{x} f(t)\, d\mu(t)\right)^{q} d\nu(x)\right)^{1/q}$$

$$= \left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{x} f(t)M(t)^{1/pp'}M(t)^{-1/pp'}\, d\mu(t)\right)^{q} d\nu(x)\right)^{1/q}$$

$$\leq \left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{x} f(t)^{p} M(t)^{1/p'}\, d\mu(t)\right)^{q/p}\left(\int_{-\infty}^{x} M(t)^{-1/p}\, d\mu(t)\right)^{q/p'} d\nu(x)\right)^{1/q}$$

$$= \left(\left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{x} f(t)^{p} M(t)^{1/p'}\left(\int_{-\infty}^{x} M(s)^{-1/p}\, d\mu(s)\right)^{p/p'} d\mu(t)\right)^{q/p} d\nu(x)\right)^{p/q}\right)^{1/p}$$

$$\leq \left(\int_{-\infty}^{\infty}\left(\int_{t}^{\infty}\left(f(t)^{p} M(t)^{1/p'}\left(\int_{-\infty}^{x} M(s)^{-1/p}\, d\mu(s)\right)^{p/p'}\right)^{q/p} d\nu(x)\right)^{p/q} d\mu(t)\right)^{1/p}$$

$$(2.6)\quad = \left(\int_{-\infty}^{\infty} f(t)^{p} M(t)^{1/p'}\left(\int_{t}^{\infty}\left(\int_{-\infty}^{x} M(s)^{-1/p}\, d\mu(s)\right)^{q/p'} d\nu(x)\right)^{p/q} d\mu(t)\right)^{1/p}.$$

By Lemma 2.3 and assumption (2.5) the inner integral satisfies $\int_{-\infty}^{x} M(s)^{-1/p} d\mu(s) \leq CM(x)^{1/p'} \leq CN(x)^{-1/q}$, so that

$$\int_{t}^{\infty}\left(\int_{-\infty}^{x} M(s)^{-1/p}\, d\mu(s)\right)^{q/p'} d\nu(x)$$

$$\leq C\int_{t}^{\infty} N(x)^{-1/p'}\, d\nu(x) \leq CN(t)^{1/p} = C(N(t)^{1/q})^{q/p} \leq CM(t)^{-q/pp'},$$

where the last two inequalities follow from Proposition 2.5 (iv) with $\mu=\nu$ and $f=1$ and from assumption (2.5) respectively. Substituting this into (2.6), it follows that (2.6) is dominated by

$$C\left(\int_{-\infty}^{\infty} f(t)^p M(t)^{1/p'}(M(t)^{-q/pp'})^{p/q}\, d\mu(t)\right)^{1/p} = C\left(\int_{-\infty}^{\infty} f(t)^p\, d\mu(t)\right)^{1/p}$$

which proves that (2.5) implies (2.4).

To prove the converse note that since $\mu$ is $\sigma$-finite, there are sets $E_n \subseteq \mathbb{R}$ such that $E_n \subseteq E_{n+1}$, $\bigcup_n E_n = \mathbb{R}$, and $\mu E_n < \infty$. Fix $y \in \mathbb{R}$ and let $f_n(t) = \chi_{E_n}(t)\chi_{(-\infty,y]}(t)$ for each n. Now by (2.4)

$$\left(\int_{y}^{\infty} d\nu(x)\right)^{1/q}\left(\int_{-\infty}^{y} f_n(t)\, d\mu(t)\right)$$

$$= \left(\int_{y}^{\infty}\left(\int_{-\infty}^{y} f_n(t)\, d\mu(t)\right)^q d\nu(x)\right)^{1/q} \le \left(\int_{y}^{\infty}\left(\int_{-\infty}^{x} f_n(t)\, d\mu(t)\right)^q d\nu(x)\right)^{1/q}$$

$$\le \left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{x} f_n(t)\, d\mu(t)\right)^q d\nu(y)\right)^{1/q} \le C\left(\int_{-\infty}^{\infty} f_n(t)^p\, d\mu(t)\right)^{1/p}$$

$$= C\left(\int_{-\infty}^{y} f_n(t)\, d\mu(t)\right)^{1/p}.$$

Dividing both sides by $\left(\int_{-\infty}^{y} f_n(t)d\mu(t)\right)^{1/p}$, we have for each n

$$\left(\int_{y}^{\infty} d\nu(y)\right)^{1/q}\left(\int_{-\infty}^{y} f_n(t)\, d\mu(t)\right)^{1/p'} \le C$$

As $n\to\infty$, $N(y)^{1/q} M(y)^{1/p'} \le C$ for all $y \in \mathbb{R}$.

Unlike the known results for weights, the case p=1 for measures is much easier to state and prove separately.

THEOREM 2.7. *Suppose that* $0<q<\infty$, *and let* $E=\{x:M(x)\neq 0\}$. *Then*

$$\left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{x} f(t)\,d\mu(t)\right)^{q} d\nu(x)\right)^{1/q} \le C\int_{-\infty}^{\infty} f(t)\,d\mu(t) \tag{2.7}$$

*if and only if* $\int_E d\nu<\infty$.

*Proof*. Since M is a non-decreasing function, the set E is an interval of the form $E=(z,\infty)$ or $E=[z,\infty)$. If $x\notin E$ we have $M(x)=0$, and so $\int_{-\infty}^{x} f(t)d\mu(t)=0$ for any f. Therefore

$$\left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{x} f(t)\,d\mu(t)\right)^{q} d\nu(x)\right)^{1/q}$$

$$= \left(\int_{E}\left(\int_{-\infty}^{x} f(t)\,d\mu(t)\right)^{q} d\nu(x)\right)^{1/q} \le \left(\int_{E} d\nu(x)\right)^{1/q}\int_{-\infty}^{\infty} f(t)\,d\mu(t).$$

The proof of necessity is given in two parts. Firstly, suppose that z in an atom for $\mu$. Set $f(t)=(1/\mu\{z\})\chi_{\{z\}}(t)$. Since $z\in(-\infty,x]$ for every $x\in E$ we have by (2.7)

$$\left(\int_E d\nu\right)^{1/q} = \left(\int_E\left(\int_{-\infty}^{x} f(t)\, d\mu(t)\right)^q d\nu(x)\right)^{1/q}$$

$$\le \left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{x} f(t)\, d\mu(t)\right)^q d\nu(x)\right)^{1/q} \le C\int_{-\infty}^{\infty} f(t)\, d\mu(t) = C < \infty.$$

Secondly, suppose that $z$ is not an atom for $\mu$. Let $\varepsilon>0$, and $f(t)=[1/\mu(z,z+\varepsilon)]\chi_{(z,z+\varepsilon)}(t)$. Thus by (2.6)

$$\left(\int_{z+\varepsilon}^{\infty} d\nu\right)^{1/q} = \left(\int_{z+\varepsilon}^{\infty}\left(\int_{-\infty}^{x} f(t)\, d\mu(t)\right)^q d\nu(x)\right)^{1/q} \le C\int_{-\infty}^{\infty} f(t)\, d\mu(t) = C.$$

As $\varepsilon\to 0^+$, we have $\left(\int_E d\nu\right)^{1/q} \le C<\infty$.

We now study the general Hardy inequality in the case $1\le q<p<\infty$.

THEOREM 2.8. <u>Let</u> M <u>and</u> N <u>be given by</u> (2.3). <u>Suppose</u> $1\le q<p<\infty$, $1/r=1/q-1/p$, <u>then there exists a constant</u> $C>0$, <u>such that</u> (2.4) <u>holds if and only if</u>

(2.8) $$\left(\int_{-\infty}^{\infty} N(t)^{r/q} M(t)^{r/q'}\, d\mu(t)\right)^{1/r} < \infty.$$

<u>Proof</u>. We consider only the case $q>1$. The proof for $q=1$ follows by the same argument with obvious simplifica-

tions. To prove sufficiency we first show that

$$(2.9)\quad \left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{t} f(s)\,d\mu(s)\right)^{p} M(t)^{-p}\,d\mu(t)\right)^{1/p} \le C\left(\int_{-\infty}^{\infty} f(t)^{p}\,d\mu(t)\right)^{1/p}$$

holds. By Theorem 2.6, this follows provided

$$\sup_{x\in\mathbb{R}}\left(\int_{x}^{\infty} M(t)^{-p}\,d\mu(t)\right)^{1/p} M(x)^{1/p'} < \infty.$$

But by Proposition 2.5 (v)

$$\left(\int_{x}^{\infty} M(t)^{-p}\,d\mu(t)\right)^{1/p} M(x)^{1/p'} \le CM(x)^{(1-p)/p}\, M(x)^{1/p'} = C < \infty,$$

which proves (2.9).

To estimate the left side of (2.4), we argue as in [28]. Utilizing Proposition 2.5 (i), and then interchanging the order of integration we have

$$\left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{x} f(t)\,d\mu(t)\right)^{q} d\nu(x)\right)^{1/q}$$

$$\le C\left(\int_{-\infty}^{\infty}\int_{-\infty}^{x}\left(\int_{-\infty}^{t} f(s)\,d\mu(s)\right)^{q-1} f(t)\,d\mu(t)\,d\nu(x)\right)^{1/q}$$

$$= C\left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{t} f(s)\,d\mu(s)\right)^{q-1} f(t)\int_{t}^{\infty} d\nu(x)\,d\mu(t)\right)^{1/q}.$$

We multiply by $M(t)^{1-q}M(t)^{q-1}$ (which is justified in much the same way as in the proof of Theorem 2.6) and apply

Hölder's inequality with indices p, p/(q-1), and r/q to obtain

$$\left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{x} f(t)\, d\mu(t)\right)^{q} d\nu(x)\right)^{1/q}$$

$$\le C\left(\int_{-\infty}^{\infty} f(t)\left(\int_{-\infty}^{t} f(s)\, d\mu(s)\right)^{q-1} M(t)^{1-q}N(t)M(t)^{q-1}\, d\mu(t)\right)^{1/q}$$

$$\le C\left(\left(\int_{-\infty}^{\infty} f(t)^{p}\, d\mu(t)\right)^{1/p}\left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{t} f(s)\, d\mu(s)\right)^{p} M(t)^{-p}\, d\mu(t)\right)^{(q-1)/p} \times\right.$$

$$\left.\left(\int_{-\infty}^{\infty} N(t)^{r/q}M(t)^{r/q'}\, d\mu(t)\right)^{q/r}\right)^{1/q}.$$

By (2.9) the second factor in the above expression is bounded by (a multiple of) the first. The third factor is finite by hypothesis and hence the sufficiency part of the theorem follows.

Since $\mu$ is $\sigma$-finite, there are sets $E_n \subseteq \mathbb{R}$ such that $E_n \subseteq E_{n+1}$, $\bigcup_n E_n = \mathbb{R}$ and $\mu E_n < \infty$. Set $M_n(t) = \int_{-\infty}^{t} \chi_{E_n}(s)\, d\mu(s)$, and $f_n(t) = M_n(t)^{r/pq'} \min(n, N(t)^{r/pq})\chi_{E_n}(t)$ for each n.

We first show that

$$f_n(t)M_n(t) \le C\int_{-\infty}^{t} f_n(s)\, d\mu(s). \tag{2.10}$$

N(t) is non-increasing, so for $s \le t$, $\min(n, N(t)^{r/pq}) \le \min(n, N(s)^{r/pq})$, hence for $s \in E_n \cap (-\infty, t]$, the definition of

$f_n$ implies that $f_n(t)M_n(t)^{-r/pq'} \le f_n(s)M_n(s)^{-r/pq'}$. By Proposition 2.5 (i) (with f replaced by $\chi_{E_n}$) and this argument

$$\begin{aligned} f_n(t)M_n(t) &= f_n(t)M_n(t)^{-r/pq'}M_n(t)^{r/pq'+1} \\ &\le Cf_n(t)M_n(t)^{-r/pq'}\int_{-\infty}^{t} M_n(s)^{r/pq'}\chi_{E_n}(s)\, d\mu(s) \\ &\le C\int_{-\infty}^{t} f_n(s)M_n(s)^{-r/pq'}M_n(s)^{r/pq'}\chi_{E_n}(s)\, d\mu(s) = C\int_{-\infty}^{t} f_n(s)\, d\mu(s), \end{aligned}$$

which proves (2.10).

Next, by the definition of $f_n$, $f_n(t)^{p-q} \le M_n(t)^{q-1}N(t)$ for each $t \in E_n$. Applying this inequality and (2.10) yields

$$\begin{aligned} \left(\int_{-\infty}^{\infty} f_n(t)^p\, d\mu(t)\right)^{1/q} &= \left(\int_{-\infty}^{\infty} f_n(t)f_n(t)^{q-1}f_n(t)^{p-q}\, d\mu(t)\right)^{1/q} \\ &\le \left(\int_{-\infty}^{\infty} f_n(t)(f_n(t)M_n(t))^{q-1}N(t)\, d\mu(t)\right)^{1/q} \\ &\le C\left(\int_{-\infty}^{\infty} f_n(t)\left(\int_{-\infty}^{t} f_n(s)\, d\mu(s)\right)^{q-1}N(t)\, d\mu(t)\right)^{1/q}. \end{aligned}$$

But $N(t)=\int_t^{\infty} d\nu(x)$, so an interchange of the order of integration and Proposition 2.5 (i) show that the last integral takes the form

$$= C\left(\int_{-\infty}^{\infty}\int_{-\infty}^{x} f_n(t)\left(\int_{-\infty}^{t} f_n(s)\, d\mu(s)\right)^{q-1} d\mu(t)\, d\nu(x)\right)^{1/q}$$

$$\le C\left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{x} f_n(t)\, d\mu(t)\right)^q d\nu(x)\right)^{1/q} \le C\left(\int_{-\infty}^{\infty} f_n(t)^p\, d\mu(t)\right)^{1/p},$$

where the last inequality follows from (2.4). Since $f_n$ is supported on the $\sigma$-finite set $E_n$ and is bounded above (by $n(\mu E_n)^{r/pq'}$), we may divide by (the finite quantity) $\left(\int_{-\infty}^{\infty} f_n(t)^p d\mu(t)\right)^{1/p}$, to obtain $\left(\int_{-\infty}^{\infty} f_n(t)^p d\mu(t)\right)^{1/r} \le C$. Since $f_n(t) \to M(t)^{r/pq'} N(t)^{r/pq}$ as $n \to \infty$, the conclusion follows from the monotone convergence theorem.

## 3. THE LEVEL FUNCTION

Given a finite measure $\lambda$ and a bounded $\lambda$-measurable function $g$, we construct a function $g^\circ$ with the following properties:

1. $g^\circ$ is non-increasing on $\mathbb{R}$;
2. $\int_{-\infty}^{x} g(t)d\lambda(t) \le \int_{-\infty}^{x} g^\circ(t)d\lambda(t)$;
3. $\|g^\circ\|_{p,\lambda} \le \|g\|_{p,\lambda}$ for all $p \ge 1$.

Since we deal with integrals of the form $\int_{-\infty}^{x} d\lambda$ for every $x$, we must assume that $\lambda$ is a Borel measure.

DEFINITION 2.9. <u>A function</u> $G$ <u>on</u> $\mathbb{R}$ <u>is</u> $\lambda$-<u>concave if</u>

*for all* $a \le x \le b$

(2.11) $$(\Lambda(b)-\Lambda(x))(G(x)-G(a)) \ge (G(b)-G(x))(\Lambda(x)-\Lambda(a))$$

*where* $\Lambda(t) = \int_{-\infty}^{t} d\lambda$.

*We may also write* (2.11) *in the form*

(2.12) $$G(x)(\Lambda(b)-\Lambda(a)) \ge G(a)(\Lambda(b)-\Lambda(x))+G(b)(\Lambda(x)-\Lambda(a)).$$

Throughout this section we assume g to be non-negative and bounded on $\mathbb{R}$ and

$$G(x) = \int_{-\infty}^{x} g(t)\, d\lambda(t).$$

THEOREM 2.10. *There exist non-negative functions* $G^\circ$ *and* $g^\circ$ *on* $\mathbb{R}$ *such that:*

i) $G^\circ$ *is the least* $\lambda$-*concave majorant of* $G$;

ii) $\lim_{x \to -\infty} G^\circ(x) = 0$;

iii) $G^\circ$ *is non-decreasing*;

iv) $G^\circ$ *is right continuous*;

v) $G^\circ(x) = \int_{-\infty}^{x} g^\circ(t)\, d\lambda(t)$;

vi) $g^\circ$ *is non-increasing* $\lambda$-a.e..

<u>Proof</u>. Let B be a bound for g. Then $B\Lambda(x)$ is $\lambda$-concave and dominates G. Also the constant function $\lim_{x\to\infty} G(x)$ (which is finite) is a $\lambda$-concave majorant of G. Set $G^\circ(x)=\inf\{\bar{G}(x):\bar{G}$ is a $\lambda$-concave majorant of $G\}$.

i) If we show that $G^\circ$ is $\lambda$-concave then it is clearly the least $\lambda$-concave majorant of G. To show that $G^\circ$ has this property, fix $a\le x\le b$. If $\Lambda(a)=\Lambda(b)$, then $\Lambda(a)=\Lambda(x)=\Lambda(b)$, so (2.11) holds trivially for $G^\circ$. If $\Lambda(a)<\Lambda(b)$, fix $\varepsilon>0$ then by the definition of $G^\circ$ choose a $\lambda$-concave majorant $\bar{G}$ of G such that $\bar{G}(x)-G^\circ(x)\le\varepsilon/(\Lambda(b)-\Lambda(a))$. We have

$$G^\circ(x)(\Lambda(b)-\Lambda(a))+\varepsilon \ge \bar{G}(x)(\Lambda(b)-\Lambda(a))$$

$$\ge \bar{G}(a)(\Lambda(b)-\Lambda(x))+\bar{G}(b)(\Lambda(x)-\Lambda(a)) \ge G^\circ(a)(\Lambda(b)-\Lambda(x))+G^\circ(b)(\Lambda(x)-\Lambda(a))$$

and since $\varepsilon$ is arbitrary,

$$G^\circ(x)(\Lambda(b)-\Lambda(a)) \ge G^\circ(a)(\Lambda(b)-\Lambda(x))+G^\circ(b)(\Lambda(x)-\Lambda(a)).$$

Therefore $G^\circ$ is $\lambda$-concave.

ii) $B\Lambda(x)$ is a $\lambda$-concave majorant of $G(x)$ so $0 \le G^{\circ}(x) \le B\Lambda(x)$, but $\lim_{x\to-\infty} B\Lambda(x)=0$. Therefore $\lim_{x\to-\infty} G^{\circ}(x)=0$.

iii) Suppose to the contrary that $G^{\circ}$ is not non-decreasing, then there are numbers a and b such that $a<b$ and $G^{\circ}(a)>G^{\circ}(b)$. Since $\lim_{x\to\infty} G(x)$ is a $\lambda$-concave majorant of G, we have $\lim_{x\to\infty} G(x) \ge G^{\circ}(a) > G^{\circ}(b)$, so there exists a $y>b$, such that $G(y)>G^{\circ}(b)$. It follows that $G^{\circ}(y)>G^{\circ}(b)$. But by i) $G^{\circ}$ is $\lambda$-concave, that is

$$(2.13)\quad (\Lambda(y)-\Lambda(b))(G^{\circ}(b)-G^{\circ}(a)) \ge (G^{\circ}(y)-G^{\circ}(b))(\Lambda(b)-\Lambda(a)).$$

Since $G^{\circ}(y)-G^{\circ}(b)>0$ and $\Lambda(b)-\Lambda(a)\ge 0$, the left side of (2.13) is non-negative. Moreover the factors satisfy $G^{\circ}(b)-G^{\circ}(a)<0$ and $\Lambda(y)-\Lambda(b)\ge 0$. This can only occur if $\Lambda(y)-\Lambda(b)=0$. Hence

$$0 = B\int_{(b,y]} d\lambda(t) \ge \int_{(b,y]} g(t)\, d\lambda(t) = G(y)-G(b) \ge G(y)-G^{\circ}(b).$$

This contradicts the choice of y. Therefore $G^{\circ}$ is non-decreasing.

iv) Fix $x\in\mathbb{R}$ and $y>x$. If $\Lambda(x)=0$, then by iii)

$$0 \le G^\circ(y)-G^\circ(x) \le G^\circ(y) \le B\Lambda(y) = B(\Lambda(y)-\Lambda(x))$$

and since $\Lambda$ is right continuous

$$\Leftarrow \quad 0 \le \lim_{y\to x^+} G^\circ(y)-G^\circ(x) \le \lim_{y\to x^+} B(\Lambda(y)-\Lambda(x)) = 0.$$

If $\Lambda(x)>0$, and $a<x<y$, then by (2.11)

$$(\Lambda(x)-\Lambda(a))(G^\circ(y)-G^\circ(x)) \le (G^\circ(x)-G^\circ(a))(\Lambda(y)-\Lambda(x)).$$

Let $a\to-\infty$, then by ii) $\Lambda(x)(G^\circ(y)-G^\circ(x))\le G^\circ(x)(\Lambda(y)-\Lambda(x))$. Now as in the previous case, the right continuity of $\Lambda$ implies

$$0 = \lim_{y\to x^+} G^\circ(y)-G^\circ(x) \le (G^\circ(x)/\Lambda(x))\lim_{y\to x^+}\Lambda(y)-\Lambda(x) = 0.$$

Therefore $G^\circ$ is right continuous at x.

v) Let $\lambda_B$ be the restriction of $\lambda$ to the $\sigma$-algebra of Borel sets. Since $G^\circ$ is non-decreasing and right continuous, [45, p. 262, Prop. 12] asserts the existence of a Borel measure $\nu$, such that for all $a<b$ $\nu(a,b]=G^\circ(b)-G^\circ(a)$.

Claim: $\nu$ is absolutely continuous with respect to $\lambda_B$.

For this it suffices to show that $\nu(a,b) \le B\lambda_B(a,b)$ for all $a<b$.

Fix a and b and let $\delta>0$, such that, $a<b-\delta$. First consider the case $\lambda_B(-\infty,b)=0$. Since

$$\nu(a,b) = \lim_{\delta\to 0^+} \nu(a,b-\delta] = \lim_{\delta\to 0^+} G^\circ(b-\delta)-G^\circ(a) \le \lim_{\delta\to 0^+} G^\circ(b-\delta)$$

$$\le \lim_{\delta\to 0^+} B\Lambda(b-\delta) = B\lambda_B(-\infty,b) = 0,$$

$\nu(a,b) \le B\lambda_B(a,b)$. In case $\lambda_B(-\infty,b)>0$, choose $\delta$ so small that $\Lambda(b-\delta)>\Lambda(x)$ as $x\to-\infty$. So by $\lambda$-concavity

$$\nu(a,b-\delta] = G^\circ(b-\delta)-G^\circ(a)$$

$$\le [(\Lambda(b-\delta)-\Lambda(a))/(\Lambda(b-\delta)-\Lambda(x))](G^\circ(b-\delta)-G^\circ(x)).$$

Now let $x\to-\infty$, then $\Lambda(x)\to 0$ and by ii) $G^\circ(x)\to 0$. Therefore

$$\nu(a,b) = \lim_{\delta\to 0^+} \nu(a,b-\delta] = \lim_{\delta\to 0^+}[(\Lambda(b-\delta)-\Lambda(a))/\Lambda(b-\delta)]G^\circ(b-\delta)$$

$$\le B \lim_{\delta\to 0^+}(\Lambda(b-\delta)-\Lambda(a)) = B\lambda_B(a,b),$$

where as before B is the bound for g. Thus $\nu$ is absolutely continuous with respect to $\lambda_B$. By the Radon-Nikodym theorem [45, p. 238, Theorem 23] there exists a non-negative function $g^\circ$ such that $\nu(E)=\int_E g^\circ(t)d\lambda(t)$ for all Borel sets E. In particular,

$$G^\circ(x) = \lim_{a\to-\infty} \nu(a,x] = \lim_{a\to-\infty} \int_{(a,x]} g^\circ(t)\, d\lambda(t) = \int_{-\infty}^{x} g^\circ(t)\, d\lambda(t).$$

This completes the proof of part v). Specifically, we have shown that

$$\int_{(a,b)} g^\circ(t)\, d\lambda(t) = \nu(a,b) \le B\lambda_B(a,b) = B\lambda(a,b)$$

for all a<b, where as before B is the bound for g. Consequently

$$\int_E g^\circ(t)\, d\lambda(t) \le B\lambda E, \tag{2.14}$$

E a Borel set.

vi) For each x let $\alpha_x = \operatorname{essinf}_{t<x} g^\circ(t)$; $\beta_x = \operatorname{esssup}_{x<t} g^\circ(t)$ where the essential infimum and essential supremum are with respect to the measure $\lambda$. We wish to show first that $\alpha_x \ge \beta_x$.

Suppose to the contrary that $\alpha_x<\beta_x$ for some x. Choose $\alpha$ and $\beta$ such that $\alpha_x<\alpha<\beta<\beta_x$ and let $A=\{t<x:g^\circ(t)<\alpha\}$ and $B=\{t>x:g^\circ(t)>\beta\}$. Clearly $\lambda A\ne0$ and $\lambda B\ne0$. Fix $\varepsilon>0$ and choose open covers $\{A_i\}$ and $\{B_i\}$ of A and B respectively with $A_i \subseteq (-\infty, x)$ and $B_i \subseteq (x, \infty)$ such that $\sum_i \lambda A_i < (1+\varepsilon)\lambda A$ and $\sum_i \lambda B_i < (1+\varepsilon)\lambda B$. This is possible since $\lambda$ coincides with its outer measure on A and B (see [45, Ch. 12, Sect. 2]). It is easily shown that there must exist an A' among the $A_i$, and a B' among the $B_i$ such that

$$(2.15) \qquad 0 < \lambda A' < (1+\varepsilon)\lambda(A \cap A'),$$

and

$$(2.16) \qquad 0 < \lambda B' < (1+\varepsilon)\lambda(B \cap B').$$

Let $A'=(a_0,a_1)$ and $B'=(b_0,b_1)$. Since $a_0<a_1<x<b_0<b_1$, the $\lambda$-concavity of $G^\circ$ implies

$$[G^\circ(a_1-\delta)-G^\circ(a_0)]/[\Lambda(a_1-\delta)-\Lambda(a_0)]$$

$$\geq [G^\circ(b_0)-G^\circ(a_0)]/[\Lambda(b_0)-\Lambda(a_0)] \geq [G^\circ(b_1-\delta)-G^\circ(b_0)]/[\Lambda(b_1-\delta)-\Lambda(b_0)]$$

for all sufficiently small $\delta>0$. Hence

$$(2.17) \quad \left(\int_{A'} g^\circ(t)\, d\lambda(t)\right)\left(\int_{B'} d\lambda(t)\right)$$

$$= \lim_{\delta\to 0}(G^\circ(a_1-\delta)-G^\circ(a_0))(\Lambda(b_1-\delta)-\Lambda(b_0))$$

$$\geq \lim_{\delta\to 0}(G^\circ(b_1-\delta)-G^\circ(b_0))(\Lambda(a_1-\delta)-\Lambda(a_0)) = \left(\int_{B'} g^\circ(t)\, d\lambda(t)\right)\left(\int_{A'} d\lambda(t)\right).$$

We now apply (2.14) and the fact that $g^\circ(t) \le \alpha$ on A, then

$$\int_{A'} g^\circ(t)\,d\lambda(t) = \int_{A'\cap A} g^\circ(t)\,d\lambda(t) + \int_{A'\setminus A} g^\circ(t)\,d\lambda(t)$$

$$\le \alpha\lambda(A\cap A') + B\lambda(A'\setminus A) \le \alpha\lambda A' + B(\lambda A' - \lambda(A'\cap A))$$

$$\le \alpha\lambda A' + B\varepsilon\lambda(A\cap A') \le (\alpha + B\varepsilon)\lambda A',$$

where the second last inequality follows from (2.15). Also, by (2.16)

$$\int_{B'} g^\circ(t)\,d\lambda(t) \ge \int_{B'\cap B} g^\circ(t)\,d\lambda(t) \ge \beta\lambda(B'\cap B) \ge \beta\lambda B'/(1+\varepsilon).$$

Therefore by (2.17)

$$(\alpha+B\varepsilon)\lambda A'\lambda B' \ge \left(\int_{A'} g^\circ(t)\,d\lambda(t)\right)\lambda B'$$

$$\ge \left(\int_{B'} g^\circ(t)\,d\lambda(t)\right)\lambda A' \ge [(\beta\lambda B')/(1+\varepsilon)]\lambda A'$$

and hence $\alpha + \varepsilon B \ge \beta/(1+\varepsilon)$. As $\varepsilon \to 0$ we obtain $\alpha \ge \beta$, a contradiction.

Having established that $\alpha_x \ge \beta_x$ for each x, we construct a set S of λ-measure zero such that $g^\circ$ is non-increasing on $\mathbb{R}\setminus S$, for once this is done, $g^\circ$ is non-increasing $\lambda$-a.e..

For each rational $q$, let $S^{+}_{q,n}=\{x>q:\beta_q<g^\circ(x)-1/n\}$ and $S^{-}_{q,n}=\{x<q:\alpha_q>g^\circ(x)+1/n\}$, where $\beta_q=\underset{q<t}{\text{esssup}}\ g^\circ(t)$ and $\alpha_q=\underset{q>t}{\text{essinf}}\ g^\circ(t)$. Clearly, the sets $S^{+}_{q,n}$ and $S^{-}_{q,n}$ have $\lambda$-measure zero. Therefore $S^{+}_{q}=\bigcup_{n=1}^{\infty} S^{+}_{q,n}=\{x>q:\beta_q<g^\circ(x)\}$, $S^{-}_{q}=\bigcup_{n=1}^{\infty} S^{-}_{q,n}=\{x<q:\alpha_q>g^\circ(x)\}$, and $S=\bigcup_{q\in\mathbf{Q}}(S^{+}_{q}\cup S^{-}_{q})$ have $\lambda$-measure zero. To show $g^\circ$ is non-increasing, suppose $x<y$ with $x,y\notin S$ and choose a rational $q$ such that $x<q<y$. $g^\circ(x)\geq\alpha_q\geq\beta_q\geq g^\circ(y)$. This completes the proof of the theorem.

Since $G^\circ$ is a majorant of $G$, the second of the three properties stated at the beginning of this section follows immediately. The first, of course, is part iv) of Theorem 2.10. In order to prove Property 3, that is, $\|g^\circ\|_{p,\lambda}\leq\|g\|_{p,\lambda}$, we must examine more closely the structure of the function $g^\circ$.

LEMMA 2.11. _Let_ $U=\{x:G^\circ(x)>G(x)$ _and_ $G^\circ(x-)>G(x-)\}$ _where_ $G^\circ$ _is the majorant of_ $G$ _of Theorem_ 2.10.

i) _If_ $G^\circ(x)>G(x)$ _then there exists_ $b>x$ _such that_ $(x,b)\subseteq U$.

ii) *If* $G^\circ(x-)>G(x-)$ *then there exists* $a<x$ *such that* $(a,x)\subseteq U$.

*Proof*. i) $G$ is right continuous, so we may choose $b>x$ so that $G^\circ(x)>G(b)$. If $y\in(x,b)$ then $y\in U$ since $G^\circ(y)\geq G^\circ(y-)\geq G^\circ(x)>G(b)\geq G(y)\geq G(y-)$.

ii) Choose $a<x$ so that $G^\circ(a)>G(x-)$. If $y\in(a,x)$, then $y\in U$ since $G^\circ(y)\geq G^\circ(y-)\geq G^\circ(a)>G(x-)\geq G(y)\geq G(y-)$.

COROLLARY 2.12. $U$ *is open*.

We can write $U=\bigcup_{i=1}^{\infty}(a_i,b_i)$, where $(a_i,b_i)$, $i=1,2,\ldots$, are disjoint.

DEFINITION 2.13. *Let* $I_i$ *be one of* $(a_i,b_i)$, $[a_i,b_i)$, $(a_i,b_i]$, $[a_i,b_i]$ *where* $a_i\in I_i$ *whenever* $G^\circ(a_i)>G(a_i)$ *and* $b_i\in I_i$ *whenever* $G^\circ(b_i-)>G(b_i-)$.

Note that $I_i$, $i=1,2,\ldots$, are disjoint, for if $b_i=a_j=c$ for some $i$, $j$ and $b_i\in I_i$ and $a_j\in I_j$, then by Definition 2.13 $G^\circ(c)>G(c)$ and $G^\circ(c-)>G(c-)$ and therefore $c\in U$ which is impossible.

THEOREM 2.14. *The function* $g^\circ$ *constructed in Theorem* 2.10 *satisfies*

$$g^\circ(t) = (1/\lambda I_i)\int_{I_i} g(s)\, d\lambda(s)$$

for $\lambda$-a.e. $t \in I_i$ $i=1,2,\ldots.$

*Proof*. Let $I=I_i$, $a=a_i$, $b=b_i$, and for $\varepsilon>0$

$$a_\varepsilon = \begin{cases} a & a \notin I \\ a-\varepsilon & a \in I \end{cases} \quad \text{and} \quad b_\varepsilon = \begin{cases} b & b \in I \\ b-\varepsilon & b \notin I \end{cases}.$$

We assume that $\lambda I>0$ since otherwise the theorem is trivial. Now, for $\varepsilon$ sufficiently small and $G^\circ$ as in Theorem 2.10, the denominator of $m_\varepsilon=(G^\circ(b_\varepsilon)-G^\circ(a_\varepsilon))/(\Lambda(b_\varepsilon)-\Lambda(a_\varepsilon))$ is not zero. Let $C_I=\lim_{\varepsilon\to 0}(G^\circ(b_\varepsilon)-m_\varepsilon\Lambda(b_\varepsilon))$ then an easy calculation shows that $C_I=\lim_{\varepsilon\to 0}(G^\circ(a_\varepsilon)-m_\varepsilon\Lambda(a_\varepsilon))$. If we let $m=\lim_{\varepsilon\to 0} m_\varepsilon$ and $C_m=\sup_{x\in R}(G(x)-m\Lambda(x))$ then $m\Lambda(x)+C_m$ is a $\lambda$-concave majorant of $G(x)$ so that

$$G^\circ(x) \le m\Lambda(x)+C_m \tag{2.18}$$

holds for each $x$ and in particular $C_I \le C_m$.

To show that $C_m \le C_I$ we take a sequence $\{d_n\}$ of real numbers such that $\lim_{n\to\infty}(G(d_n)-m\Lambda(d_n))=C_m$. For each n either $d_n>b$, $d_n<a$, or $a\le d_n\le b$ and at least one of these conditions must hold for infinitely many n. We distinguish three cases based on this observation.

First suppose that $d_n>b$ for infinitely many n. Since $G^\circ$ is $\lambda$-concave we have

$$(\Lambda(d_n)-\Lambda(b_\varepsilon))(G^\circ(b_\varepsilon)-G^\circ(a_\varepsilon)) \ge (G^\circ(d_n)-G^\circ(b_\varepsilon))(\Lambda(b_\varepsilon)-\Lambda(a_\varepsilon))$$

or equivalently $G^\circ(b_\varepsilon)-m_\varepsilon\Lambda(b_\varepsilon) \ge G^\circ(d_n)-m_\varepsilon\Lambda(d_n)$. Allowing $\varepsilon\to 0$, we obtain $C_I\ge G^\circ(d_n)-m\Lambda(d_n)\ge G(d_n)-m\Lambda(d_n)$ for infinitely many n. This implies that $C_I\ge C_m$.

Next suppose $d_n<a$ for infinitely many n. By $\lambda$-concavity of $G^\circ$ we have

$$(\Lambda(b_\varepsilon)-\Lambda(a_\varepsilon))(G^\circ(a_\varepsilon)-G^\circ(d_n)) \ge (G^\circ(b_\varepsilon)-G^\circ(a_\varepsilon))(\Lambda(a_\varepsilon)-\Lambda(d_n)),$$

for all sufficiently small $\varepsilon$, that is $G^\circ(a_\varepsilon)-m_\varepsilon\Lambda(a_\varepsilon) \ge G^\circ(d_n)-m_\varepsilon\Lambda(d_n)$. Allowing $\varepsilon\to 0$, we have $C_I \ge G^\circ(d_n)-m\Lambda(d_n) \ge G(d_n)-m\Lambda(d_n)$ for infinitely many n so that $C_I\ge C_m$.

In the remaining case, $a\le d_n\le b$ for infinitely many

n. Since the sequence $\{d_n\}$ has a limit point, say, $d\in[a,b]$, (2.18) implies that if d is a right limit point, then $C_m=G(d)-m\Lambda(d)\le G^\circ(d)-m\Lambda(d)\le C_m$ and if d is a left limit point then $C_m=G(d-)-m\Lambda(d-)\le G^\circ(d-)-m\Lambda(d-)\le C_m$. Clearly, then, $G(d)=G^\circ(d)$ or $G(d-)=G^\circ(d-)$ so $d\notin U$. (see Lemma 2.11). We are left with two possibilities, either d=a or d=b. If d=a then d must be a right limit point so $G(a)=G^\circ(a)$ which implies that $a\notin I$ and hence $C_I=G(a)-m\Lambda(a)=C_m$. Similarly, if d=b, d is a left limit point and therefore $G(b-)=G^\circ(b-)$, $b\in I$, and $C_I=G(b-)-m\Lambda(b-)=C_m$.

Now we use the fact that $C_m=C_I$ to show that $g^\circ$ is constant $\lambda$-a.e. on I. Let $x\in I$, then for all $\varepsilon$ sufficiently small, $a_\varepsilon\le x\le b_\varepsilon$ and since $G^\circ$ is $\lambda$-concave

$$(\Lambda(b_\varepsilon)-\Lambda(x))(G^\circ(x)-G^\circ(a_\varepsilon)) \ge (G^\circ(b_\varepsilon)-G^\circ(x))(\Lambda(x)-\Lambda(a_\varepsilon))$$

which can be written in the form $G^\circ(x)-m_\varepsilon\Lambda(x) \ge G^\circ(b_\varepsilon)-m_\varepsilon\Lambda(b_\varepsilon)$. Let $\varepsilon\to 0$ in this inequality, then applying (2.18) we obtain

$$G^\circ(x)-m\Lambda(x) \ge C_I = C_m \ge G^\circ(x)-m\Lambda(x)$$

for all $x\in I$. This implies that $\int_{-\infty}^{x}(g^\circ(t)-m)d\lambda(t)$ is constant on I and hence $g^\circ(t)=m$ $\lambda$-a.e. on I, except possibly at $x=a$. If $a\in I$ and $\lambda\{a\}>0$ then

$$(g^\circ(a)-m)\lambda\{a\} = \int_{-\infty}^{a}(g^\circ(t)-m)\, d\lambda(t)-\lim_{\varepsilon\to 0}\int_{-\infty}^{a-\varepsilon}(g^\circ(t)-m)\, d\lambda(t) = C_I-C_I = 0$$

so $g^\circ(a)=m$ as well.

To complete the proof it remains to show that $m=(1/\lambda I)\int_I g(s)d\lambda(s)$. If $a\in I$ then by Definition 2.13 $G^\circ(a)>G(a)$, but $a\notin U$ so $G^\circ(a-)=G(a-)$, hence $\lim_{\varepsilon\to 0} G^\circ(a_\varepsilon)$ $=G^\circ(a-)=G(a-)=\lim_{\varepsilon\to 0} G(a_\varepsilon)$. If $a\notin I$ then $G^\circ(a)=G(a)$, so $\lim_{\varepsilon\to 0} G^\circ(a_\varepsilon)=G^\circ(a)=G(a)=\lim_{\varepsilon\to 0} G(a_\varepsilon)$. In the same way we show that $\lim_{\varepsilon\to 0} G^\circ(b_\varepsilon)=\lim_{\varepsilon\to 0} G(b_\varepsilon)$. Since $m_\varepsilon=(G^\circ(b_\varepsilon)-G^\circ(a_\varepsilon))/(\Lambda(b_\varepsilon)-\Lambda(a_\varepsilon))$, $\lim_{\varepsilon\to 0}(G^\circ(b_\varepsilon)-G^\circ(a_\varepsilon))$ $=\lim_{\varepsilon\to 0}(G(b_\varepsilon)-G(a_\varepsilon))=\int_I g(t)d\lambda(t)$, and $\lim_{\varepsilon\to 0} \Lambda(b_\varepsilon)-\Lambda(a_\varepsilon)=\lambda(I)$ we obtain $m=\lim_{\varepsilon\to 0} m_\varepsilon=1/\lambda(I)\int_I g(t)d\lambda(t)$.

THEOREM 2.15. $g^\circ(t)=g(t)$ $\lambda$-a.e. <u>for</u> $t\notin \bigcup_{i=1}^{\infty} I_i$.

<u>Proof</u>. Let $F=\mathbb{R}\setminus\bigcup_{i=1}^{\infty} I_i$, $h(t)=g(t)\chi_F(t)+g^\circ(t)\chi_{\mathbb{R}\setminus F}(t)$, and $H(\lambda)=\int_{-\infty}^{x}h(t)d\lambda(t)$. It is enough to show that $H(x)=G^\circ(x)$ for all $x\in\mathbb{R}$ for then $g^\circ=h$ $\lambda$-a.e. and hence

$g°=g$ $\lambda$-a.e. on $F$.

Suppose $x \in F$. If $G°(x)>G(x)$ then by Lemma 2.11 there exists $b>x$ such that $(x,b) \subseteq U$. But $x \in F$ implies $x \notin U$ so $x$ is the left endpoint of some interval $I_i$ and by Definition 2.13 $x \in I_i$ which is a contradiction. Therefore $G°(x)=G(x)$. Now $(-\infty,x]\backslash F=\bigcup_{i\in J} I_i$ where $J=\{i:I_i \subseteq (-\infty,x]\}$, so by Theorem 2.14

$$H(x) = \int_{-\infty}^{x} h(t)\, d\lambda(t) = \sum_{i\in J}\int_{I_i} g°(t)\, d\lambda(t)+ \int_{(-\infty,x]\cap F} g(t)\, d\lambda(t)$$

$$=\sum_{i\in J}\int_{I_i} g(t)\, d\lambda(t)+ \int_{(-\infty,x]\cap F} g(t)\, d\lambda(t) = \int_{-\infty}^{x} g(t)\, d\lambda(t) = G(x) = G°(x).$$

Similar arguments show that $H(a_i-)=G°(a_i-)$ whenever $a_i \in I_i$ and $H(a_i)=G°(a_i)$ whenever $a_i \notin I_i$. (Recall that $a_i$ and $b_i$ are the endpoints of the interval $I_i$.)

Suppose $x \notin F$, then $x \in I_i$ for some $i$. If $a_i \in I_i$ then $[a_i,x] \subseteq I_i$ so

$$H(x) = H(a_i-)+\int_{a_i}^{x} h(t)\, d\lambda(t) = G°(a_i-)+\int_{a_i}^{x} g°(t)\, d\lambda(t) = G°(x).$$

If $a_i \notin I_i$ then

$$H(x) = H(a_i)+ \int_{(a_i,x]} h(t)\, d\lambda(t) = G°(a_i)+ \int_{(a_i,x]} g°(t)\, d\lambda(t) = G°(x).$$

This completes the proof.

THEOREM 2.16. $\int_{-\infty}^{\infty} g^\circ(t)^\alpha g(t)\,d\lambda(t) = \int_{-\infty}^{\infty} g^\circ(t)^{\alpha+1} d\lambda(t)$ *for* $\alpha\in\mathbb{R}$.

*Proof*. Let $m_i=(1/\lambda I)\int_{I_i} g(t)\,d\lambda(t)$, and $F=\mathbb{R}\setminus\bigcup_{i=1}^{\infty} I_i$ as above. Since $g^\circ=m_i$ $\lambda$-a.e. on $I_i$ and $g=g^\circ$, $\lambda$-a.e. on F

$$\int_{-\infty}^{\infty} g^\circ(t)^\alpha g(t)\,d\lambda(t) = \sum_i \int_{I_i} g^\circ(t)^\alpha g(t)\,d\lambda(t) + \int_F g^\circ(t)^\alpha g(t)\,d\lambda(t)$$

$$= \sum_i m_i^\alpha \int_{I_i} g(t)\,d\lambda(t) + \int_F g^\circ(t)^{\alpha+1}\,d\lambda(t) = \sum_i m_i^{\alpha+1}\lambda I_i + \int_F g^\circ(t)^{\alpha+1}\,d\lambda(t)$$

$$= \sum_i \int_{I_i} g^\circ(t)^{\alpha+1}\,d\lambda(t) + \int_F g^\circ(t)^{\alpha+1}\,d\lambda(t) = \int_{-\infty}^{\infty} g^\circ(t)^{\alpha+1}\,d\lambda(t).$$

Property 3 of the level function is given by

COROLLARY 2.17. $\|g^\circ\|_{p,\lambda}\le\|g\|_{p,\lambda}$ *for all* $p\ge 1$.

*Proof*. From Theorem 2.16 with $\alpha=p-1$ and Hölder's inequality we obtain

$$\int_{-\infty}^{\infty} g^\circ(t)^p\,d\lambda(t) = \int_{-\infty}^{\infty} g^\circ(t)^{p-1} g(t)\,d\lambda(t)$$

$$\le \left(\int_{-\infty}^{\infty} g^\circ(t)^{(p-1)p'}\, d\lambda(t)\right)^{1/p'} \left(\int_{-\infty}^{\infty} g(t)^p\, d\lambda(t)\right)^{1/p}$$

$$= \left(\int_{-\infty}^{\infty} g^\circ(t)^p\, d\lambda(t)\right)^{1/p'} \left(\int_{-\infty}^{\infty} g(t)^p\, d\lambda(t)\right)^{1/p}.$$

Since $\lambda$ is finite and $g^\circ$ is bounded we may divide by the first factor to obtain

$$\left(\int_{-\infty}^{\infty} g^\circ(t)^p\, d\lambda(t)\right)^{1/p} \le \left(\int_{-\infty}^{\infty} g(t)^p\, d\lambda(t)\right)^{1/p}.$$

## 4. HARDY'S INEQUALITY II

In this section we return to the general Hardy inequality

$$\text{(2.19)} \qquad \left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{x} f(t)\, d\mu(t)\right)^q d\nu(x)\right)^{1/q} \le C\left(\int_{-\infty}^{\infty} f(t)^p\, d\mu(t)\right)^{1/p}.$$

Our main result is to extend the characterization of Theorem 2.8 to the range of indices $0<q<1$, $p>1$. In addition we characterize the measures for which (2.19) holds where $0<p=q<1$, provided f is non-decreasing. This inequality arises naturally in the theory of interpolation [9, p. 20]. To complete the chapter we summarize the results for general measures, weights, and discrete measures together with their dual results.

Recall that M and N are defined by (2.3) as follows

$$M(y) = \int_{-\infty}^{y} d\mu, \quad N(y) = \int_{y}^{\infty} d\nu .$$

THEOREM 2.18. *Suppose* $0<q<1$ *and* $1<p<\infty$, *then there exists a constant* $C>0$ *such that* (2.19) *holds for all* $f\geq 0$, *if and only if*

$$(2.20) \qquad \left(\int_{-\infty}^{\infty} N(t)^{r/q} M(t)^{r/q'} \, d\mu(t)\right)^{1/r} = B < \infty$$

*holds*.

*Proof*. (Sufficiency) Since $\mu$ is $\sigma$-finite, there are sets $E_n$ such that $E_n \subseteq E_{n+1}$, $\mu E_n < \infty$ and $\bigcup_n E_n = R$. With M and N as above and B as in (2.20) define $\mu_n$, $M_n$ and $B_n$ by $d\mu_n(t) = \chi_{E_n}(t) d\mu(t)$; $M_n(t) = \int_{-\infty}^{t} d\mu_n$; $B_n = \left(\int_{-\infty}^{\infty} N(t)^{r/q} M_n(t)^{r/q'} d\mu_n(t)\right)^{1/r}$. Proposition 2.5 ii) and iii) yield

$$B_n = \left(\int_{-\infty}^{\infty} N(t)^{r/q} M_n(t)^{r/q'} \, d\mu_n(t)\right)^{1/r} .$$

$$\sim \left(\int_{-\infty}^{\infty}\left(\int_{t}^{\infty}N(x)^{r/p}\,d\nu(x)\right)M_n(t)^{r/q'}\,d\mu_n(t)\right)^{1/r}$$

$$= \left(\int_{-\infty}^{\infty}N(x)^{r/p}\left(\int_{-\infty}^{x}M_n(t)^{r/q'}\,d\mu_n(t)\right)\,d\nu(x)\right)^{1/r}$$

$$\sim \left(\int_{-\infty}^{\infty}N(x)^{r/p}\,M_n(x)^{r/p'}\,d\nu(x)\right)^{1/r}.$$

By the monotone convergence theorem, this last expression tends to $\left(\int_{-\infty}^{\infty}N(x)^{r/p}M(x)^{r/p'}d\nu(x)\right)^{1/r}$ as $n\to\infty$. Repeating the above calculation with $\mu_n$ replaced by $\mu$ shows that $\left(\int_{-\infty}^{\infty}N(x)^{r/p}M(x)^{r/p'}d\nu(x)\right)^{1/r}\sim B$. We conclude that $B_n\le CB$ for sufficiently large n.

Fix $f\ge 0$ and let $f_n(t)=\min(f(t),n)$, $n=1,2,\ldots$. The results of Section 3 imply the existence of a non-negative, non-increasing function $f_n^\circ=(f_n)^\circ$ such that $\int_{-\infty}^{x}f_n(t)d\mu_n(t)$ $\le \int_{-\infty}^{x}f_n^\circ(t)d\mu_n(t)$ and $\|f_n^\circ\|_{p,\mu_n}\le\|f_n\|_{p,\mu_n}$. Using this fact we proceed as in Theorem 2.8. Applying Proposition 2.5 iii) and interchanging the order of integration we have

$$\left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{x}f_n(t)\,d\mu_n(t)\right)^{q}d\nu(x)\right)^{1/q} \le \left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{x}f_n^\circ(t)\,d\mu_n(t)\right)^{q}d\nu(x)\right)^{1/q}$$

$$\le C\left(\int_{-\infty}^{\infty}\int_{-\infty}^{x}\left(\int_{-\infty}^{t}f_n^\circ(s)\,d\mu_n(s)\right)^{q-1}f_n^\circ(t)\,d\mu_n(t)\,d\nu(x)\right)^{1/q}$$

$$= C\left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{t}f_n^\circ(s)\,d\mu_n(s)\right)^{q-1}f_n^\circ(t)\int_{t}^{\infty}d\nu(x)\,d\mu_n(t)\right)^{1/q}.$$

Since M is never infinite and is positive $\mu_n$-a.e., we multiply and divide by $M^{q-1}$ and apply Hölder's inequality with indices p/q and p/(p-q) to obtain

$$\left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{x} f_n(t)\, d\mu_n(t)\right)^q d\nu(x)\right)^{1/q}$$

$$\le C\left(\int_{-\infty}^{\infty} f_n^\circ(t)\left(\int_{-\infty}^{t} f_n^\circ(s)\, d\mu_n(s)\right)^{q-1} M_n(t)^{1-q}\, N(t) M_n(t)^{q-1}\, d\mu_n(t)\right)^{1/q}$$

$$\le C\left(\int_{-\infty}^{\infty} f_n^\circ(t)^{p/q}\left(\int_{-\infty}^{t} f_n^\circ(s)\, d\mu_n(s)\right)^{p/q'} M_n(t)^{-p/q'}\, d\mu_n(t)\right)^{1/p} \times$$

$$\left(\int_{-\infty}^{\infty} N(t)^{r/q} M_n(t)^{r/q'}\, d\mu_n(t)\right)^{1/r}$$

$$= CB_n\left(\int_{-\infty}^{\infty} f_n^\circ(t)^{p/q}\left(\int_{-\infty}^{t} f_n^\circ(s)\, d\mu_n(s)\right)^{p/q'} M_n(t)^{-p/q'}\, d\mu_n(t)\right)^{1/p}.$$

Since $f_n^\circ$ is non-increasing, $f_n^\circ(t) \le M_n(t)^{-1}\int_{-\infty}^{t} f_n^\circ(s)\,d\mu_n(s)$ so by (2.9) with $\mu=\mu_n$, $f=f_n^\circ$ the last expression is dominated by

$$CB_n\left(\int_{-\infty}^{\infty} M_n(t)^{-p/q}\left(\int_{-\infty}^{t} f_n^\circ(s)\, d\mu_n(s)\right)^{p/q}\left(\int_{-\infty}^{t} f_n^\circ(s)\, d\mu_n(s)\right)^{p/q'} M_n(t)^{-p/q'}\, d\mu_n(t)\right)^{1/p}$$

$$= CB_n\left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{t} f_n^\circ(s)\, d\mu_n(s)\right)^{p} M_n(t)^{-p}\, d\mu_n(t)\right)^{1/p}$$

$$\le CB_n\left(\int_{-\infty}^{\infty} f_n^\circ(t)^p\, d\mu_n(t)\right)^{1/p} \le CB_n\left(\int_{-\infty}^{\infty} f_n(t)^p\, d\mu_n(t)\right)^{1/p}.$$

As $n\to\infty$ we obtain

$$\left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{x} f(t)\, d\mu(t)\right)^{q} d\nu(x)\right)^{1/q} \le CB\left(\int_{-\infty}^{\infty} f(t)^{p}\, d\mu(t)\right)^{1/p}.$$

To prove that (2.20) is also necessary for (2.19), we require the inequality

$$(2.21)\quad \int_{-\infty}^{\infty}\left(\int_{-\infty}^{t} f(s)M(s)^{r/q'}\, d\mu(s)\right)^{p/q} M(t)^{-p}\, d\mu(t) \le C\int_{-\infty}^{\infty} f(t)^{p/q} M(t)^{r/q'}\, d\mu(t).$$

But Theorem 2.6 shows that (2.21) holds whenever

$$\sup_{x\in\mathbb{R}}\left(\int_{x}^{\infty} M(t)^{-p}\, d\mu(t)\right)^{q/p}\left(\int_{-\infty}^{x} M(t)^{r/q'}\, d\mu(t)\right)^{q/r} < \infty.$$

Applying Proposition 2.5 v) and iii) the left side is less than or equal to $\sup_{x\in\mathbb{R}} M(x)^{[(1-p)q/p]}M(x)^{[(r/q'+1)q/r]}$, which is finite since $(1-p)q/p+(r/q'+1)q/r=0$. Therefore (2.21) holds.

Since $\mu$ is $\sigma$-finite, there are sets $E_n$ such that $E_n \subseteq E_{n+1}$, $\mu E_n<\infty$ and $\bigcup_n E_n=\mathbb{R}$. Set $g(t)=M(t)^{q-1}N(t)$, $g_n(t)=\min(n,g(t))\chi_{E_n}(t)$, and $h_n(t)=M(t)^{q-1}g_n(t)^{r/p}$ then

$$\left(\int_{-\infty}^{\infty} g_n(t)^{r/q}\, d\mu(t)\right)^{1/q} \le \left(\int_{-\infty}^{\infty} g_n(t)^{r/p} g(t)\, d\mu(t)\right)^{1/q}$$

$$= \left(\int_{-\infty}^{\infty} h_n(t)N(t)\, d\mu(t)\right)^{1/q} = \left(\int_{-\infty}^{\infty} h_n(t)\int_{t}^{\infty} d\nu(x)\, d\mu(t)\right)^{1/q}.$$

After interchanging the order of integration, we apply

Proposition 2.5 i) and use the hypothesis (2.19) with $f(t)=\left(\int_{-\infty}^{t} h_n(s)\,d\mu(s)\right)^{-1/q'} h_n(t)$ to obtain

$$\left(\int_{-\infty}^{\infty} g_n(t)^{r/q}\,d\mu(t)\right)^{1/q} = \left(\int_{-\infty}^{\infty}\int_{-\infty}^{x} h_n(t)\,d\mu(t)\,d\nu(x)\right)^{1/q}$$

$$\le C\left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{x}\left(\int_{-\infty}^{t} h_n(s)\,d\mu(s)\right)^{-1/q'} h_n(t)\,d\mu(t)\right)^{q} d\nu(x)\right)^{1/q}$$

$$\le C\left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{t} h_n(s)\,d\mu(s)\right)^{-p/q'} h_n(t)^p\,d\mu(t)\right)^{1/p}$$

$$= C\left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{t} h_n(s)\,d\mu(s)\right)^{-p/q'} M(t)^{[p(q-1)]} g_n(t)^r\,d\mu(t)\right)^{1/p}.$$

Now Hölder's inequality with indices $1/(1-q)$ and $1/q$, followed by (2.21) with $f(s)=h_n(s)M(s)^{-r/q'}$ yield

$$\left(\int_{-\infty}^{\infty} g_n(t)^{r/q}\,d\mu(t)\right)^{1/q}$$

$$\le C\left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{t} h_n(s)\,d\mu(s)\right)^{p/q} M(t)^{-p}\,d\mu(t)\right)^{(1-q)/p}\left(\int_{-\infty}^{\infty} g_n(t)^{r/q}\,d\mu(t)\right)^{q/p}$$

$$= C\left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{t} h_n(s)M(s)^{-r/q'}M(s)^{r/q'}\,d\mu(s)\right)^{p/q} M(t)^{-p}\,d\mu(t)\right)^{(1-q)/p} \times$$

$$\left(\int_{-\infty}^{\infty} g_n(t)^{r/q}\,d\mu(t)\right)^{q/p}$$

$$\le C\left(\int_{-\infty}^{\infty}(h_n(t)M(t)^{-r/q'})^{p/q} M(t)^{r/q'}\,d\mu(t)\right)^{(1-q)/p}\left(\int_{-\infty}^{\infty} g_n(t)^{r/q}\,d\mu(t)\right)^{q/p}$$

$$= C\left(\int_{-\infty}^{\infty} g_n(t)^{r/q}\,d\mu(t)\right)^{1/p},$$

where multiplying and dividing by $M(s)^{r/q'}$ is justified since $M>0$ $\mu$-a.e., and if $M(s)=\infty$ then $M(t)^{-p}=0$ for all $t\geq s$. Dividing by $\left(\int_{-\infty}^{\infty} g_n(t)^{r/q} d\mu(t)\right)^{1/p}$, we have $\left(\int_{-\infty}^{\infty} g_n(t)^{r/q} d\mu(t)\right)^{1/r} \leq C$. As $n\to\infty$, the monotone convergence theorem implies that $\left(\int_{-\infty}^{\infty} N(t)^{r/q} M(t)^{r/q'} d\mu(t)\right)^{1/r} <\infty$ which completes the theorem.

If $f\in L^p_\mu$, $0<p<1$, it is still possible to characterize the measures for which the general Hardy operator is bounded on $L^p_\nu$, provided f is non-decreasing.

THEOREM 2.19. <u>Suppose that</u> $0<p<1$, <u>then</u>

$$\left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{x} f(t)\, d\mu(t)\right)^{p} d\nu(x)\right)^{1/p} \leq C\left(\int_{-\infty}^{\infty} f(t)^{p}\, d\mu(t)\right)^{1/p} \tag{2.22}$$

<u>holds for all non-decreasing functions</u> $f\geq 0$, <u>if and only if</u>

$$\int_{y}^{\infty}\left(\int_{y}^{x} d\mu\right)^{p} d\nu(x) \leq C\int_{y}^{\infty} d\mu \tag{2.23}$$

<u>holds for all</u> $y\in\mathbb{R}$. <u>Also</u>

$$\left(\int_{-\infty}^{\infty}\left(\int_{x}^{\infty} f(t)\, d\mu(t)\right)^{p} d\nu(x)\right)^{1/p} \leq C\left(\int_{-\infty}^{\infty} f(t)^{p}\, d\mu(t)\right)^{1/p}$$

*holds for all non-increasing functions* $f\geq 0$ *if and only if*

$$\int_{-\infty}^{y}\left(\int_{x}^{y} d\mu\right)^{p} d\nu(x) \leq C\int_{-\infty}^{y} d\mu$$

*holds for all* $y\in\mathbb{R}$.

*Proof*. It clearly suffices to prove only the first equivalence since the second follows from a change of variable.

Necessity follows if we note that (2.23) is just (2.22) with $f(t)=\chi_{[y,\infty]}(t)$.

We proceed with the proof of sufficiency. For each $n\in\mathbb{Z}$, let $E_n=\{x:2^n<f(x)\}$. Since f is non-decreasing, $E_n$ is an interval of the form $(y_n,\infty)$ or $[y_n,\infty)$. If $E_n$ is empty, let $y_n=\infty$.

Since $\sum_{n\in\mathbb{Z}}\chi_{E_n\setminus E_{n+1}}(t)=1$ whenever $f(t)>0$ and $f(t)\leq 2^{n+1}$ if $t\notin E_{n+1}$ we obtain

$$\int_{-\infty}^{\infty}\left(\int_{-\infty}^{x} f(t)\, d\mu(t)\right)^{p} d\nu(x) = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{x}\sum_{n\in\mathbb{Z}}\chi_{E_n\setminus E_{n+1}}(t)f(t)\, d\mu(t)\right)^{p} d\nu(x)$$

$$= \int_{-\infty}^{\infty}\left(\sum_{n}\int_{-\infty}^{x}\chi_{E_n\setminus E_{n+1}}(t)f(t)\, d\mu(t)\right)^{p} d\nu(x)$$

$$\leq \int_{-\infty}^{\infty}\left(\sum_{n}2^{n+1}\int_{-\infty}^{x}\chi_{E_n\setminus E_{n+1}}(t)\, d\mu(t)\right)^{p} d\nu(x)$$

$$\le \int_{-\infty}^{\infty}\left(\sum_n 2^{n+1}\int_{-\infty}^{x}\chi_{E_n}(t)\,d\mu(t)\right)^p d\nu(x) \le \int_{-\infty}^{\infty}\sum_n 2^{(n+1)p}\left(\int_{-\infty}^{x}\chi_{E_n}(t)\,d\mu(t)\right)^p d\nu(x)$$

$$= C\sum_n 2^{np}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{x}\chi_{E_n}(t)\,d\mu(t)\right)^p d\nu(x),$$

where the last inequality relies on the fact that $0<p<1$.

If $y_n \in E_n$ then

$$\int_{-\infty}^{\infty}\left(\int_{-\infty}^{x}\chi_{E_n}(t)\,d\mu(t)\right)^p d\nu(x) = \int_{y_n}^{\infty}\left(\int_{y_n}^{x}d\mu(t)\right)^p d\nu(x) \le C\int_{y_n}^{\infty}d\mu(t) = C\int_{E_n}d\mu(t)$$

by (2.23) If $y_n \notin E_n$, a limiting argument gives the same estimate. Substituting we get

$$\int_{-\infty}^{\infty}\left(\int_{-\infty}^{x}f(t)\,d\mu(t)\right)^p d\nu(x) \le C\sum_n 2^{np}\int_{E_n}d\mu(t)$$

$$= C\sum_{n\in Z}2^{np}\sum_{k\ge n}\int_{E_k\setminus E_{k+1}}d\mu(t) = C\sum_{k\in Z}\int_{E_k\setminus E_{k+1}}\left(\sum_{n\le k}2^{np}\right)d\mu(t)$$

$$= C\sum_{k\in Z}\int_{E_k\setminus E_{k+1}}2^{kp}\,d\mu(t) \le C\sum_{k\in Z}\int_{E_k\setminus E_{k+1}}f(t)^p\,d\mu(t) = C\int_{-\infty}^{\infty}f(t)^p\,d\mu(t).$$

We now summarize our results for the Hardy operator and its dual.

THEOREM 2.20. *Suppose* $\mu$ *and* $\nu$ *are* $\sigma$*-finite measures on* $\mathbb{R}$*. Then there is a constant* $C>0$ *such that*

$$(2.24)\quad \left(\int_{-\infty}^{\infty} \left|\int_{-\infty}^{x} f(t)\, d\mu(t)\right|^{q} d\nu(x)\right)^{1/q} \le C\left(\int_{-\infty}^{\infty} |f(t)|^{p}\, d\mu(t)\right)^{1/p}$$

*for all* $\mu$*-measurable functions* $f$, *if and only if*

i) *for* $1<p\le q<\infty$

$$\sup_{y\in\mathbb{R}}\left(\int_{y}^{\infty} d\nu\right)^{1/q}\left(\int_{-\infty}^{y} d\mu\right)^{1/p'} < \infty;$$

ii) *for* $0<q<p$, $1<p<\infty$, *and* $1/r=1/q-1/p$

$$\left(\int_{-\infty}^{\infty}\left(\int_{t}^{\infty} d\nu\right)^{r/q}\left(\int_{-\infty}^{t} d\mu\right)^{r/q'} d\mu(t)\right)^{1/r} < \infty;$$

iii) *for* $p=1$, $0<q<\infty$

$$\int_{E} d\nu < \infty \text{ where } E = \{x\in\mathbb{R}: \int_{-\infty}^{x} d\mu>0\}.$$

If we apply Theorem 2.20 to the measures $\bar{\mu}$, $\bar{\nu}$ where $\bar{\mu}E=\mu\{-x:x\in E\}$ and $\bar{\nu}E=\nu\{-x:x\in E\}$ we obtain dual results.

THEOREM 2.21. *Suppose* $\mu$, *and* $\nu$ *are as in Theorem* 2.20. *Then there is a constant* $C$ *such that*

$$\left(\int_{-\infty}^{\infty}\left|\int_{x}^{\infty} f(t)\, d\mu(t)\right|^{q} d\nu(x)\right)^{1/q} \le C\left(\int_{-\infty}^{\infty} |f(t)|^{p}\, d\mu(t)\right)^{1/p}$$

*holds for all* $\mu$*-measurable functions* $f$, *if and only if*

i) *for* $1<p\le q<\infty$

$$\sup_{y\in R}\left(\int_{-\infty}^{y}d\nu\right)^{1/q}\left(\int_{y}^{\infty}d\mu\right)^{1/p'}<\infty;$$

ii) *for* $0<q<p$, $1<p<\infty$, *and* $1/r=1/q-1/p$

$$\left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{t}d\nu\right)^{r/q}\left(\int_{t}^{\infty}d\mu\right)^{r/q'}d\mu(t)\right)^{1/r}<\infty;$$

iii) *for* $p=1$, $0<q<\infty$

$$\int_{E}d\nu<\infty \text{ where } E=\{x\in R:\int_{x}^{\infty}d\mu>0\}.$$

Weighted inequalities for the classical Hardy operator are obtained by considering absolutely continuous measures supported on $(0,\infty)$. Specifically, if in Theorems 2.20 and 2.21 $d\mu(t)=u(t)^{1-p}dt$, $d\nu(x)=v(x)dx$ and $g(t)=f(t)u(t)^{1-p'}$ we obtain:

COROLLARY 2.22. *Suppose* $u>0$ *and* $v\ge 0$ *are weight functions defined on* $(0,\infty)$. *Then there is a constant* C *such that*

$$\left(\int_{0}^{\infty}\left|\int_{0}^{x}g(t)\,dt\right|^{q}v(x)\,dx\right)^{1/q}\le C\left(\int_{0}^{\infty}|g(t)|^{p}u(t)\,dt\right)^{1/p}$$

*holds for all measurable functions g, if and only if*

i) *for* $1<p\le q<\infty$

$$\sup_{y>0}\left(\int_y^\infty v(x)\,dx\right)^{1/q}\left(\int_0^y u(t)^{1-p'}\,dt\right)^{1/p'}<\infty;$$

ii) *for* $0<q<p$, $1<p<\infty$, *and* $1/r=1/q-1/p$

$$\left(\int_0^\infty\left(\int_t^\infty v(x)\,dx\right)^{r/q}\left(\int_0^t u(s)^{1-p'}\,ds\right)^{r/q'}u(t)^{1-p'}\,dt\right)^{1/r}<\infty;$$

iii) *for* $p=1$, $0<q<\infty$

$$\int_E v(x)\,dx<\infty \text{ where } E=\{x:\int_0^x u(t)^{1-p'}dt>0\}.$$

COROLLARY 2.23. *Suppose u, and v are as in Corollary* 2.22. *Then there is a constant* C *such that*

$$\left(\int_0^\infty\left|\int_x^\infty g(t)\,dt\right|^q v(x)\,dx\right)^{1/q}\le C\left(\int_0^\infty|g(t)|^p u(t)\,dt\right)^{1/p}$$

*holds for all measurable functions g, if and only if*

i) *for* $1<p\le q<\infty$

$$\sup_{y>0}\left(\int_0^y v(x)\,dx\right)^{1/q}\left(\int_y^\infty u(t)^{1-p'}\,dt\right)^{1/p'}<\infty;$$

ii) *for* $0<q<p$, $1<p<\infty$, *and* $1/r=1/q-1/p$

$$\left(\int_0^\infty\left(\int_0^t v(x)\,dx\right)^{r/q}\left(\int_t^\infty u(s)^{1-p'}\,ds\right)^{r/q'}u(t)^{1-p'}\,dt\right)^{1/r}<\infty;$$

iii) *for* $p=1$, $0<q<\infty$

$$\int_E v(x)\,dx<\infty \text{ where } E=\{x: \int_x^\infty u(t)^{1-p'}dt>0\}.$$

Even in the classical setting the results for $0<q<1$, $p>1$ seem to be new.

The corresponding series inequalities are obtained by letting $\mu S=\sum_{k\in S}u_k^{1-p'}$, $\nu S=\sum_{n\in S}v_n$ and $a_k=f(k)u_k^{1-p'}$ in Theorems 2.20 and 2.21.

COROLLARY 2.24. *Suppose* $u_n>0$ *and* $v_n\geq 0$ *for* $n=0,1,2,\ldots$. *Then there is a constant* $C$ *such that*

$$\left(\sum_{n=0}^\infty\left|\sum_{k=0}^n a_k\right|^q v_n\right)^{1/q}\leq C\left(\sum_{k=0}^\infty|a_k|^p u_k\right)^{1/p}$$

*holds for all sequences* $\{a_k\}$ *if and only if*

i) *for* $1<p\leq q<\infty$

$$\sup_{m\in Z^+}\left(\sum_{n=m}^{\infty} v_n\right)^{1/q}\left(\sum_{k=0}^{m} u_k^{1-p'}\right)^{1/p'} < \infty;$$

ii) for $0<q<p$, $1<p<\infty$, and $1/r=1/q-1/p$

$$\left(\sum_{k=0}^{\infty}\left(\sum_{n=k}^{\infty} v_n\right)^{r/q}\left(\sum_{j=0}^{k} u_j^{1-p'}\right)^{r/q'} u_k^{1-p'}\right)^{1/r} < \infty;$$

iii) for $p=1$, $0<q<\infty$

$$\sum_{n\in E} v_n < \infty \text{ where } E = \{n: \sum_{k=0}^{n} u_k^{1-p'} > 0\}.$$

COROLLARY 2.25. *Suppose* $u_n$, *and* $v_n$ *are as in Corollary* 2.24. *Then there is a constant* C *such that*

$$\left(\sum_{n=0}^{\infty}\left|\sum_{k=n}^{\infty} a_k\right|^q v_n\right)^{1/q} \le C\left(\sum_{k=0}^{\infty} |a_k|^p u_k\right)^{1/p}$$

*holds for all sequences* $\{a_k\}$ *if and only if*

i) for $1<p\le q<\infty$

$$\sup_{m\in Z^+}\left(\sum_{n=0}^{m} v_n\right)^{1/q}\left(\sum_{k=m}^{\infty} u_k^{1-p'}\right)^{1/p'} < \infty;$$

ii) for $0<q<p$, $1<p<\infty$, and $1/r=1/q-1/p$

$$\left(\sum_{k=0}^{\infty}\left(\sum_{n=0}^{k} v_n\right)^{r/q}\left(\sum_{j=k}^{\infty} u_j^{1-p'}\right)^{r/q'} u_k^{1-p'}\right)^{1/r} < \infty;$$

iii) <u>for</u> $p=1$, $0<q<\infty$

$$\sum_{n\in E} v_n < \infty \text{ where } E = \{n: \sum_{k=n}^{\infty} u_k^{1-p'} > 0\}.$$

# CHAPTER 3

## MAPPING PROPERTIES OF INTEGRAL OPERATORS WITH POSITIVE KERNELS

Let K be the operator defined by

$$(Kf)(x) = \int_{-\infty}^{\infty} k(x,t)f(t)\, d\mu(t)$$

where k(x,t) belongs to some class of positive kernels. In this chapter we study the boundedness of this operator and specifically answer the following question: For which σ-finite, Borel measures $\mu$ and $\nu$ does the inequality

$$\left(\int_{-\infty}^{\infty} |(Kf)(x)|^q\, d\nu(x)\right)^{1/q} \le C\left(\int_{-\infty}^{\infty} |f(t)|^p\, d\mu(t)\right)^{1/p},$$

$1<q<p<\infty$, hold? In Section 1 we answer this question by showing that the condition

$$\left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} k(x,t)^q\, d\nu(x)\right)^{r/q}\left(\int_{-\infty}^{\infty} (k(t,s)/k(t,t))^{p'}\, d\mu(s)\right)^{r/q'} d\mu(t)\right)^{1/r} < \infty,$$

where $1/r=1/q-1/p$, is necessary and sufficient for boundedness of $K:L^p_\mu \to L^q_\nu$. The class of kernels that we consider is defined below, but we note that the Stieltjes kernels

$k(x,t)=(x+t)^{-\alpha}$ are members of this class, so that we also obtain new mapping properties for the Stieltjes transform and the Hilbert double series. These results are given in Section 2. We point out that a characterization of weights for which the Stieltjes transform is bounded from $L^p_\mu$ to $L^q_\nu$ for $1<p\le q<\infty$ was recently given by Andersen [1] and his work has motivated our study here. Our results complement Andersen's work even though the proofs are quite different in the index range $1<q<p<\infty$.

For another class of kernels $k(x,t)$, we characterize the measures $\mu$ and $\nu$ for which the weak type inequality

$$(\nu\{x:|(Kf)(x)|>\lambda\})^{1/q} \le (A/\lambda)\|f\|_{p,\nu}$$

holds in the range $q>0$, $p\ge1$. This is done in Section 3.

## 1. STRONG TYPE INEQUALITIES FOR $1<q<p<\infty$

Suppose the kernel $k$ of the integral operator $K$ is a product of the form $k(x,t)=a(t)\bar{a}(x)$ where $a$ and $\bar{a}$ are non-negative functions. A duality argument shows that for $q>1$, $p>1$, the boundedness of $K:L^p_\mu\to L^q_\nu$ is equivalent to

$$\left(\int_{-\infty}^{\infty} a(t)^{p'}\, d\mu(t)\right)^{1/p'} \left(\int_{-\infty}^{\infty} \bar{a}(x)^{q}\, d\nu(x)\right)^{1/q} < \infty$$

which, in the case $1<q<p<\infty$, may be written as

$$(3.1)\quad \left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} k(x,t)^{q}\, d\nu(x)\right)^{r/q}\left(\int_{-\infty}^{\infty}(k(t,s)/k(t,t))^{p'}\, d\mu(s)\right)^{r/q'} d\mu(t)\right)^{1/r} < \infty$$

without explicit dependence on the functions $a$ and $\bar{a}$. This suggests that (3.1) may characterize the boundedness of $K: L_\mu^p \to L_\nu^q$ for a larger class of kernels.

Suppose now that there exist positive functions $a$, $\bar{a}$ and $b$, $\bar{b}$ such that

$$k(x,t) \sim a(t)\bar{a}(x) \qquad t \le x$$

and

$$k(x,t) \sim b(t)\bar{b}(x) \qquad t \ge x.$$

If we impose these conditions, then clearly

$$(3.2)\quad \begin{aligned} k(x,t)/k(x,x) &\sim a(t)/a(x) \qquad && t \le x; \\ k(x,t)/k(x,x) &\sim b(t)/b(x) \qquad && t \ge x. \end{aligned}$$

DEFINITION 3.1. <u>A non-negative kernel</u> $k(x,t)$ <u>is said to belong to class</u> $\mathcal{K}$ <u>if there exist positive functions</u>

*a and b such that (3.2) holds.*

With these preliminaries we give our first result:

THEOREM 3.2. *If k is a kernel of class* $\mathcal{K}$ *then* (3.1) *with* $1<q<p<\infty$ *implies* $K:L^p_\mu \to L^q_\nu$ *is bounded.*

*Proof.* By Minkowski's inequality and (3.2)

$$\left(\int_{-\infty}^{\infty} |(Kf)(x)|^q \, d\nu(x)\right)^{1/q}$$

$$\le \left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{x} k(x,t)f(t)\,d\mu(t)\right)^q d\nu(x)\right)^{1/q} + \left(\int_{-\infty}^{\infty}\left(\int_{x}^{\infty} k(x,t)f(t)\,d\mu(t)\right)^q d\nu(x)\right)^{1/q}$$

$$\le C\left[\left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{x} a(t)f(t)\,d\mu(t)\right)^q [k(x,x)/a(x)]^q \, d\nu(x)\right)^{1/q}\right.$$

$$\left. + \left(\int_{-\infty}^{\infty}\left(\int_{x}^{\infty} b(t)f(t)\,d\mu(t)\right)^q [k(x,x)/b(x)]^q \, d\nu(x)\right)^{1/q}\right] = C[J_1+J_2],$$

respectively.

We apply Theorem 2.8 with $f(t)$ replaced by $f(t)a(t)^{1-p'}$, $d\mu(t)$ replaced by $a(t)^{p'}d\mu(t)$, and $d\nu(x)$ replaced by $[k(x,x)/a(x)]^q d\nu(x)$ to obtain

$$J_1 \le C\left(\int_{-\infty}^{\infty} (f(t)a(t)^{1-p'})^p a(t)^{p'} \, d\mu(t)\right)^{1/p} = C\left(\int_{-\infty}^{\infty} f(t)^p \, d\mu(t)\right)^{1/p}$$

provided

$$\left(\int_{-\infty}^{\infty}\left(\int_{t}^{\infty}(k(x,x)/a(x))^{q}\,d\nu(x)\right)^{r/q}\left(\int_{-\infty}^{t}a(s)^{p'}\,d\mu(s)\right)^{r/q'}a(t)^{p'}\,d\mu(t)\right)^{1/r}<\infty.$$

But $k(x,x)\le Ck(x,t)a(x)/a(t)$ for $t\le x$, and $a(s)\le Ca(t)k(t,s)/k(t,t)$ for $s\le t$ so the last integral is dominated by

$$\left(\int_{-\infty}^{\infty}\left(\int_{t}^{\infty}k(x,t)^{q}\,d\nu(x)\right)^{r/q}\left(\int_{-\infty}^{t}(k(t,s)/k(t,t))^{p'}\,d\mu(s)\right)^{r/q'}a(t)^{p'-r+p'r/q'}\,d\mu(t)\right)^{1/r}$$

and since $p'-r+p'r/q'=0$ this is finite by hypothesis (3.1).

To estimate $J_2$ replace $f(t)$ by $f(t)b(t)^{1-p'}$, $d\mu(t)$ by $b(t)^{p'}d\mu(t)$ and $d\nu(x)$ by $[k(x,x)/b(x)]^{q}d\nu(x)$ in Theorem 2.21, then the same argument shows that

$$J_2\le C\left(\int_{-\infty}^{\infty}f(t)^{p}\,d\mu(t)\right)^{1/p}.$$

In order to give a converse to Theorem 3.2 we restrict ourselves to the class of kernels defined below.

DEFINITION 3.3. <u>A non-negative kernel is said to be of class</u> $\mathcal{K}$ <u>provided there exist positive functions</u> a <u>and</u> b <u>such that</u>

i) $k(x,t)/k(x,x) \sim a(t)/a(x)$ <u>for all</u> $t \le x$;

ii) $k(x,t)/k(x,x) \sim b(t)/b(x)$ <u>for all</u> $t \ge x$;

iii) $k(x,t)/a(t)$ <u>is non-increasing in</u> t <u>for each</u> x;

iv) $k(x,t)/b(t)$ <u>is non-decreasing in</u> t <u>for each</u> x.

We remark that conditions i) and ii) above ensure that $\bar{\mathcal{K}} \subseteq \mathcal{K}$.

THEOREM 3.4. <u>If</u> k <u>is a kernel of class</u> $\bar{\mathcal{K}}$, $1<q<p<\infty$, <u>and</u> $K: L^p_\mu \to L^q_\nu$ <u>is bounded then</u>

$$\left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} k(x,t)^q \, d\nu(x)\right)^{r/q}\left(\int_{-\infty}^{\infty}(k(t,s)/k(t,t))^{p'} \, d\mu(s)\right)^{r/q'} d\mu(t)\right)^{1/r} < \infty.$$

<u>Proof</u>. The proof is in two parts. First we construct two sequences of sets $E_n^*$ and $F_n^*$ such that:

i) $E_n^* \subseteq E_{n+1}^*$, $\mu E_n^* < \infty$, $\mu(\mathbb{R} \setminus \bigcup_n E_n^*) = 0$;

ii) $F_n^* \subseteq F_{n+1}^*$, $\mu F_n^* < \infty$, $\mu(\mathbb{R} \setminus \bigcup_n F_n^*) = 0$;

iii) $\left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} k(x,t)^q \, d\nu_n(x)\right)^{r/q}\left(\int_{-\infty}^{\infty}(k(t,s)/k(t,t))^{p'} \, d\mu_n(s)\right)^{r/q'} d\mu_n(t)\right)^{1/r} < \infty$

where $d\mu_n(t) = \chi_{E_n^*}(t)\,d\mu(t)$ and $d\nu_n(t) = \chi_{F_n^*}(t)\,d\nu(t)$. In the second part we show that

$$(3.3)\quad \left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}(k(x,t))^q\,d\nu_n(x)\right)^{r/q}\left(\int_{-\infty}^{\infty}(k(t,s)/k(t,t))^{p'}\,d\mu_n(s)\right)^{r/q'}d\mu_n(t)\right)^{1/r}\le C$$

with C independent of n. By i) and ii) above and the monotone convergence theorem this will prove the result.

We now construct the sequences $\{E_n^*\}$ and $\{F_n^*\}$. Since $\mu$ is $\sigma$-finite there exist sets $E_n$ such that $E_n \subseteq E_{n+1}$, $\mu E_n < \infty$, and $\bigcup_n E_n = R$. Let

$$E_n^* = E_n \cap \{x: 1/n < a(x) < n\} \cap \{x: 1/n < b(x) < n\} \qquad n=1,2,\ldots.$$

Clearly $E_n^* \subseteq E_{n+1}^*$ and $\mu E_n^* < \infty$. Since a and b are positive functions, finite $\mu$-a.e., $\bigcup_n E_n^*$ has full $\mu$-measure, completing i).

We may certainly assume that $\mu$ is not the zero measure so that $\bigcup_n E_n^* \neq \emptyset$. Since k is measurable with respect to the product measure $\nu \times \mu$, Tonelli's Theorem implies the existence of $z \in \bigcup_n E_n^*$ such that $k(x,z) < \infty$ for $\nu$-a.e. $x \in R$. Without loss of generality we assume that $z \in E_n^*$ for all n. This z will be used in the definition of $F_n^*$ below.

The measure $\nu$ is $\sigma$-finite so there are sets $F_n$ such that $F_n \subseteq F_{n+1}$, $\nu F_n < \infty$, and $\bigcup_n F_n = \mathbb{R}$. Define

$$F_n^* = F_n \cap \{x: 1/n < a(x) < n\} \cap \{x: 1/n < b(x) < n\} \cap \{x: k(x,z) < n\}.$$

It is clear that $F_n^* \subseteq F_{n+1}^*$ and $\nu F_n^* < \infty$. As before we have $0 < a(x) < \infty$, $0 < b(x) < \infty$, and $k(x,z) < \infty$ for $\nu$-a.e. $x \in \mathbb{R}$ so $\nu(\mathbb{R} \setminus \bigcup_n F_n^*) = 0$ and ii) is complete.

To prove iii) it is enough to show that $\int_{-\infty}^{\infty} k(x,t)^q d\nu_n(x)$ and $\int_{-\infty}^{\infty} (k(t,s)/k(t,t))^{p'} d\mu_n(s)$ are bounded above for $t \in E_n^*$. The first integral we split at z and apply the monotonicity of $k(x,t)/a(t)$ and $k(x,t)/b(t)$.

$$\int_{-\infty}^{\infty} k(x,t)^q \, d\nu_n(x) \le \int_z^{\infty} k(x,t)^q \, d\nu_n(x) + \int_{-\infty}^{z} k(x,t)^q \, d\nu_n(x)$$

$$= \int_z^{\infty} a(t)^q (k(x,t)/a(t))^q \, d\nu_n(x) + \int_{-\infty}^{z} b(t)^q (k(x,t)/b(t))^q \, d\nu_n(x)$$

$$\le \int_z^{\infty} a(t)^q (k(x,z)/a(z))^q \, d\nu_n(x) + \int_{-\infty}^{z} b(t)^q (k(x,z)/b(z))^q \, d\nu_n(x)$$

$$\le \int_z^{\infty} n^q (n/(1/n))^q \, d\nu_n(x) + \int_{-\infty}^{z} n^q (n/(1/n))^q \, d\nu_n(x) \le 2n^{3q} \, \nu F_n^* < \infty.$$

To bound the second integral we split the range at t and use $k(t,s)/k(t,t) \sim a(s)/a(t)$ on the first summand and $k(t,s)/k(t,t) \sim b(s)/b(t)$ on the second, so that

$$\int_{-\infty}^{\infty}(k(t,s)/k(t,t))^{p'}\,d\mu_n(s) \le \int_{-\infty}^{t}(k(t,s)/k(t,t))^{p'}\,d\mu_n(s)+\int_{t}^{\infty}(k(t,s)/k(t,t))^{p'}\,d\mu_n(s)$$

$$\le C\left(\int_{-\infty}^{t}(a(s)/a(t))^{p'}\,d\mu_n(s)+\int_{t}^{\infty}(b(s)/b(t))^{p'}\,d\mu_n(s)\right)$$

$$\le C\left(\int_{-\infty}^{t}(n/(1/n))^{p'}\,d\mu_n(s)+\int_{t}^{\infty}(n/(1/n))^{p'}\,d\mu_n(s)\right) \le 2Cn^{2p'}\mu E_n^{*} < \infty.$$

For the second part of the proof we fix n and define $\bar{f}$, $\bar{f}_1$ and $\bar{f}_2$ by

$$\bar{f}(t)^p = \left(\int_{-\infty}^{\infty}k(x,t)^q\,d\nu_n(x)\right)^{r/q}\left(\int_{-\infty}^{\infty}(k(t,s)/k(t,t))^{p'}\,d\mu_n(s)\right)^{r/q'},$$

$$\bar{f}_1(t)^p = \left(\int_{-\infty}^{\infty}k(x,t)^q\,d\nu_n(x)\right)^{r/q}\left(\int_{-\infty}^{t}(a(s)/a(t))^{p'}\,d\mu_n(s)\right)^{r/q'},$$

$$\bar{f}_2(t)^p = \left(\int_{-\infty}^{\infty}k(x,t)^q\,d\nu_n(x)\right)^{r/q}\left(\int_{t}^{\infty}(b(s)/b(t))^{p'}\,d\mu_n(s)\right)^{r/q'}.$$

Note that we have shown in Part 1 that $\int_{-\infty}^{\infty}\bar{f}(t)^p d\mu_n(t)<\infty$ and now to prove (3.3) we must show that $\left(\int_{-\infty}^{\infty}\bar{f}(t)^p d\mu_n(t)\right)^{1/r}\le C$ with C independent of n.

Since

$$\int_{-\infty}^{\infty}(k(t,s)/k(t,t))^{p'}\,d\mu_n(s) \le C\left(\int_{-\infty}^{t}(a(s)/a(t))^{p'}\,d\mu_n(s)+\int_{t}^{\infty}(b(s)/b(t))^{p'}\,d\mu_n(s)\right)$$

we have $\bar{f}(t)\le C(\bar{f}_1(t)+\bar{f}_2(t))$ and therefore, by Minkowski's inequality

$$\left(\int_{-\infty}^{\infty}\bar{f}(t)^p\, d\mu_n(t)\right)^{1/p} \le C\left(\left(\int_{-\infty}^{\infty}\bar{f}_1(t)\, d\mu_n(t)\right)^{1/p}+\left(\int_{-\infty}^{\infty}\bar{f}_2(t)^p\, d\mu_n(t)\right)^{1/p}\right).$$

Using the definition of $\bar{f}_1$ and interchanging the order of integration we have

$$\int_{-\infty}^{\infty}\bar{f}_1(t)^p\, d\mu_n(t) = \int_{-\infty}^{\infty}\bar{f}_1(t)^q\, \bar{f}_1(t)^{p-q}\, d\mu_n(t)$$

$$= \int_{-\infty}^{\infty}\bar{f}_1(t)^q\left(\left(\int_{-\infty}^{t}(a(s)/a(t))^{p'}\, d\mu_n(s)\right)^{r/q'}\left(\int_{-\infty}^{\infty}k(x,t)^q\, d\nu_n(x)\right)^{r/q}\right)^{p-q} d\mu_n(t)$$

$$= \int_{-\infty}^{\infty}\bar{f}_1(t)\left(\bar{f}_1(t)\int_{-\infty}^{t}(a(s)/a(t))^{p'}\, d\mu_n(s)\right)^{q-1}\left(\int_{-\infty}^{\infty}k(x,t)^q\, d\nu_n(x)\right) d\mu_n(t)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}k(x,t)\bar{f}_1(t)\left(k(x,t)\bar{f}_1(t)\int_{-\infty}^{t}(a(s)/a(t))^{p'}\, d\mu_n(s)\right)^{q-1} d\mu_n(t)\, d\nu_n(x).$$

If we show that

$$(3.4)\quad k(x,t)\bar{f}_1(t)\int_{-\infty}^{t}(a(s)/a(t))^{p'}\, d\mu_n(s) \le C\int_{-\infty}^{t}k(x,s)\bar{f}_1(s)\, d\mu_n(s),$$

then we may apply Proposition 2.5 i) and the boundedness of $K:L^p_\mu \to L^q_\nu$ to obtain that

$$\int_{-\infty}^{\infty}\bar{f}_1(t)^p\, d\mu_n(t) \le C\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}k(x,t)\bar{f}_1(t)\, d\mu_n(t)\right)^q d\nu_n(x)$$

$$\le C\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}k(x,t)[\bar{f}_1(t)\chi_{E_n^*}(t)]\, d\mu(t)\right)^q d\nu(x)$$

$$\le C\Big(\int_{-\infty}^{\infty}(\bar f_1(t)\chi_{E_n^*}(t))^p\,d\mu(t)\Big)^{q/p} = C\Big(\int_{-\infty}^{\infty}\bar f_1(t)^p\,d\mu_n(t)\Big)^{q/p}.$$

By a completely analogous argument

$$\int_{-\infty}^{\infty}\bar f_2(t)^p\,d\mu_n(t) \le C\Big(\int_{-\infty}^{\infty}\bar f_2(t)^p\,d\mu_n(t)\Big)^{q/p}.$$

Therefore

$$(3.5)\quad \Big(\int_{-\infty}^{\infty}\bar f(t)\,d\mu_n(t)\Big) \le C\Big(\Big(\int_{-\infty}^{\infty}\bar f_1(t)^p\,d\mu_n(t)\Big)^{q/p}+\Big(\int_{-\infty}^{\infty}\bar f_2(t)^p\,d\mu_n(t)\Big)^{q/p}\Big)$$

where C is independent of n. But since by hypothesis

$$\int_{-\infty}^{t}(a(s)/a(t))^{p'}\,d\mu_n(s) \le C\int_{-\infty}^{\infty}(k(t,s)/k(t,t))^{p'}\,d\mu_n(s)$$

and

$$\int_{t}^{\infty}(b(s)/b(t))^{p'}\,d\mu_n(s) \le C\int_{-\infty}^{\infty}(k(t,s)/k(t,t))^{p'}\,d\mu_n(s)$$

we have $\bar f_1(t)\le C\bar f(t)$ and $\bar f_2(t)\le C\bar f(t)$. Therefore by (3.5)

$$\Big(\int_{-\infty}^{\infty}\bar f(t)^p\,d\mu_n(t)\Big)^{1/q} \le C\Big(\int_{-\infty}^{\infty}\bar f(t)^p\,d\mu_n(t)\Big)^{1/p}$$

so that $\Big(\int_{-\infty}^{\infty}\bar f(t)^p d\mu_n(t)\Big)^{1/r}\le C$ as required.

It remains to show that (3.4) holds. Observe that $a(t)^{p'(r/pq'+1)}=a(t)^{-(1+r/q)}$ then by Proposition 2.5 i)

$$k(x,t)\bar{f}_1(t)\int_{-\infty}^{t}(a(s)/a(t))^{p'}\,d\mu_n(s)$$

$$= k(x,t)\left(\int_{-\infty}^{\infty}k(y,t)^q\,d\nu_n(y)\right)^{r/pq}\left(\int_{-\infty}^{t}(a(s)/a(t))^{p'}\,d\mu_n(s)\right)^{r/pq'+1}$$

$$= C[k(x,t)/a(t)]\left(\int_{-\infty}^{\infty}[k(y,t)/a(t)]^q\,d\nu_n(y)\right)^{r/pq}\left(\int_{-\infty}^{t}a(s)^{p'}\,d\mu_n(s)\right)^{r/pq'+1}$$

$$\le C[k(x,t)/a(t)]\left(\int_{-\infty}^{\infty}[k(y,t)/a(t)]^q\,d\nu_n(y)\right)^{r/pq}\times$$

$$\int_{-\infty}^{t}\left(\int_{-\infty}^{s}a(y)^{p'}\,d\mu_n(y)\right)^{r/pq'}a(s)^{p'}\,d\mu_n(s)$$

$$\le C\int_{-\infty}^{t}[k(x,s)/a(s)]\left(\int_{-\infty}^{\infty}[k(y,s)/a(s)]^q\,d\nu_n(y)\right)^{r/pq}\times$$

$$\left(\int_{-\infty}^{s}a(y)^{p'}\,d\mu_n(y)\right)^{r/pq'}a(s)^{p'}\,d\mu_n(s)$$

$$= C\int_{-\infty}^{t}k(x,s)\left(\int_{-\infty}^{\infty}k(y,s)^q\,d\nu_n(y)\right)^{r/pq}\left(\int_{-\infty}^{s}(a(y)/a(s))^{p'}\,d\mu_n(y)\right)^{r/pq'}d\mu_n(s)$$

$$= C\int_{-\infty}^{t}k(x,s)\bar{f}_1(s)\,d\mu_n(s).$$

The last inequality is a consequence of the monotonicity of $k(x,t)/a(t)$. This proves (3.4) and hence the theorem.

## 2. THE STIELTJES TRANSFORM AND HILBERT'S DOUBLE SERIES

The class of kernels $\bar{\mathcal{K}}$ considered in the previous section includes kernels of homogeneous type.

LEMMA 3.5. *Suppose that for some* $\alpha<0$ *the kernel* $k(x,t)$ *satisfies*:

i) $k(x,t)\geq 0$ *is defined on* $(0,\infty)\times(0,\infty)$;

ii) $k(xs,ts) = s^{\alpha}k(x,t)$, $\alpha<0$, $s>0$;

iii) $k(x,t)$ *is non-increasing in both* $x$ *and* $t$;

iv) $k(0,1)$, $k(1,0)$, $k(1,1)$ *are positive and finite*;

*then* $k$ *belongs to the class* $\bar{\mathcal{K}}$.

*Proof*. Let $a(t)=1$, $b(t)=t^{\alpha}$ for all $t\geq 0$. First we show that if $t\leq x$, $a(x)k(x,t)\sim a(t)k(x,x)$. This follows at once since $a(x)k(x,t)=k(x,t)\leq k(x,0)=x^{\alpha}k(1,0)=Cx^{\alpha}k(1,1)$ $=Ck(x,x)=Ca(t)k(x,x)$, where $C=k(1,0)/k(1,1)$, and $a(x)k(x,t)$ $=k(x,t)\geq k(x,x)=a(t)k(x,x)$.

Next, if $t\geq x$, then $b(x)k(x,t)=x^{\alpha}k(x,t)\leq x^{\alpha}k(0,t)$ $=t^{\alpha}x^{\alpha}k(0,1)=Ct^{\alpha}x^{\alpha}k(1,1)=Ct^{\alpha}k(x,x)=Cb(t)k(x,x)$, where $C=k(0,1)/k(1,1)$, and $b(x)k(x,t)=x^{\alpha}k(x,t)\geq x^{\alpha}k(t,t)$ $=t^{\alpha}x^{\alpha}k(1,1)=b(t)k(x,x)$, so that $b(x)k(x,t)\sim b(t)k(x,x)$. Therefore $k\in\mathcal{K}$.

$k\in\bar{\mathcal{K}}$ since $k(x,t)/a(t)=k(x,t)$ is non-increasing in $t$

for each x, and $k(x,t)/b(t)=t^{-\alpha}k(x,t)=k(x/t,1)$ is non-decreasing in t for each x.

Let $(S_\lambda f)(x)=\int_0^\infty (x+t)^{-\lambda}f(t)\,d\mu(t)$, $\lambda>0$, denote the Stieltjes transform with measure $\mu$. In the next theorem we characterize the ($\sigma$-finite, Borel) measures $\mu$ and $\nu$ for which $S_\lambda:L^p_\mu\to L^q_\nu$ is bounded with $1<q<p<\infty$.

THEOREM 3.6. _Suppose_ $1<q<p<\infty$, $1/r=1/q-1/p$, _and_ $\lambda>0$. _Then_,

$$\left(\int_0^\infty |(S_\lambda f)(x)|^q\,d\nu(x)\right)^{1/q}\le C\left(\int_0^\infty |f(t)|^p\,d\mu(t)\right)^{1/p}$$

_holds if and only if_

$$\left(\int_0^\infty\left(\int_0^\infty (x+t)^{-q\lambda}\,d\nu(x)\right)^{r/q}\left(\int_0^\infty (t/(t+s))^{p'\lambda}\,d\mu(s)\right)^{r/q'}d\mu(t)\right)^{1/r}<\infty.$$

_Proof_. $k(x,t)=(x+t)^{-\lambda}$ satisfies the conditions of Lemma 3.5 so, taking $\mu$ and $\nu$ to be supported on $(0,\infty)$, Theorems 3.2 and 3.4 yield the result.

For the Stieltjes transform with Lebesgue measure

$$(S_\lambda f)(x) = \int_0^\infty [f(t)/(x+t)^\lambda]\,dt$$

we deduce the following weighted norm inequality.

COROLLARY 3.7. *Suppose* p, q, r, *and* $\lambda$ *are as above and* $u>0$ *and* $v\geq 0$ *are weights*. *Then*,

$$\left(\int_0^\infty |(S_\lambda g)(x)|^q\, v(x)\,dx\right)^{1/q} \leq C\left(\int_0^\infty |g(t)|^p\, u(t)\,dt\right)^{1/p}$$

*if and only if*

$$\left(\int_0^\infty\left(\int_0^\infty (x+t)^{-q\lambda} v(x)\,dx\right)^{r/q}\left(\int_0^\infty (t/(t+s))^{p'\lambda} u(s)^{1-p'}\,ds\right)^{r/q'} u(t)^{1-p'}\,dt\right)^{1/r}.$$

*Proof*. Let $d\nu(x)=v(x)dx$ and $d\mu(t)=u(t)^{1-p'}dt$ in Theorem 3.6, and make the substitution $g(t)=f(t)u(t)^{1-p'}$.

Andersen [1] obtained such a characterization in the range $1<p\leq q<\infty$.

We also obtain the following extension of the Hilbert double series theorem:

COROLLARY 3.8. *Suppose* p, q, r *and* $\lambda$ *are as in Theorem* 3.6, *and* $\{u_n\}$ *and* $\{v_n\}$ *are positive sequences. Then*

$$\left|\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(n+k)^{-\lambda}a_k b_n\right| \le C\left(\sum_{k=1}^{\infty}|a_k|^p u_k\right)^{1/p}\left(\sum_{n=1}^{\infty}|b_n|^{q'}v_n\right)^{1/q'}$$

*if and only if*

$$(3.6)\ \sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}(n+k)^{-\lambda q}v_n^{1-q}\right)^{r/q}\left(\sum_{j=1}^{\infty}(k/(k+j))^{p'\lambda}u_j^{1-p'}\right)^{r/q'}u_k^{1-p'} < \infty.$$

*Proof*. Define $\mu$ and $\nu$ by $\mu E=\sum_{n\in E}u_n^{1-p'}$ and $\nu E=\sum_{n\in E}v_n^{1-q}$. Applying Theorem 3.6 and making the substitution $a_k=f(k)u_k^{1-p'}$, we obtain that

$$\left(\sum_{n=1}^{\infty}\left|\sum_{k=1}^{\infty}(n+k)^{-\lambda}a_k\right|^q v_n^{1-q}\right)^{1/q} \le C\left(\sum_{k=1}^{\infty}|a_k|^p u_k\right)^{1/p}$$

holds if and only if (3.6) holds. The conclusion follows by Hölder's inequality.

3. A WEAK TYPE RESULT

In this section we give a weak type result for the operator $(Kf)(x)=\int_{-\infty}^{\infty}k(x,t)f(t)d\mu(t)$, $x\in R$, assuming only that k(x,t) is non-negative and monotone in the first variable.

THEOREM 3.9. *Suppose that* $0<q<\infty$, $1<p<\infty$, *and* $k(x,t)\geq 0$ *is non-decreasing in* x *for each* t. *Then*

$$(3.7)\qquad (\nu\{x:|Kf(x)|>\alpha\})^{1/q} \leq (A/\alpha)\left(\int_{-\infty}^{\infty}|f(t)|^p\,d\mu(t)\right)^{1/p}$$

*if and only if*

$$(3.8)\qquad \left(\int_{y}^{\infty} d\nu\right)^{1/q}\left(\int_{-\infty}^{\infty}k(y,t)^{p'}\,d\mu(t)\right)^{1/p'} \leq A,$$

*for each* $y\in R$.

*Proof*. Since $\mu$ is $\sigma$-finite, there exist sets $E_n$ such that $E_n\subseteq E_{n+1}$, $\mu E_n<\infty$, and $\bigcup_n E_n=R$.

Suppose that (3.7) holds and $y\in R$ is fixed. If $\int_{-\infty}^{\infty}k(y,t)^{p'}d\mu(t)=0$ there is nothing to prove. Otherwise, let $\bar{f}_n(t)=\min(n,k(y,t)^{1/(p-1)})\chi_{E_n}(t)$, and choose $\alpha_n>0$ such that $\alpha_n<\int_{-\infty}^{\infty}\bar{f}_n(t)^p d\mu(t)$ for sufficiently large n. Since for such n

$$\alpha_n < \int_{-\infty}^{\infty}\bar{f}_n(t)\bar{f}_n(t)^{p-1}\,d\mu(t) \leq \int_{-\infty}^{\infty}\bar{f}_n(t)k(y,t)\,d\mu(t),$$

the monotonicity of k implies that for $x \geq y$, $\alpha_n < (K\bar{f}_n)(x)$ and hence $[y,\infty) \subseteq \{x: (K\bar{f}_n)(x) > \alpha_n\}$. Therefore by (3.7)

$$\left(\int_y^\infty d\nu\right)^{1/q} \leq (\nu\{x:(K\bar{f}_n)(x)>\alpha_n\})^{1/q} \leq (A/\alpha_n)\left(\int_{-\infty}^\infty \bar{f}_n(t)^p \, d\mu(t)\right)^{1/p}.$$

The only restriction on $\alpha_n$ is that $\alpha_n < \int_{-\infty}^\infty \bar{f}_n(t)^p d\mu(t)$. If we allow $\alpha_n \to \int_{-\infty}^\infty \bar{f}_n(t)^p d\mu(t)$ and divide by $\left(\int_{-\infty}^\infty \bar{f}_n(t)^p d\mu(t)\right)^{1/p-1}$ then (3.8) follows from the monotone convergence theorem.

Now suppose (3.8) holds. $(Kf)(x)$ is non-decreasing so $\{x:(Kf)(x)>\alpha\}$ is an interval either of the form $[y,\infty)$ or $(y,\infty)$. In the first case, since $(Kf)(y)>\alpha$

$$(\nu\{x:(Kf)(x)>\alpha\})^{1/q} = \left(\int_y^\infty d\nu\right)^{1/q} \leq \left(\int_y^\infty d\nu\right)^{1/q}((Kf)(y)/\alpha)$$

$$\leq (1/\alpha)\left(\int_y^\infty d\nu\right)^{1/q}\left(\int_{-\infty}^\infty k(y,t)^{p'} \, d\mu(t)\right)^{1/p'}\left(\int_{-\infty}^\infty |f(t)|^p \, d\mu(t)\right)^{1/p}$$

$$\leq (A/\alpha)\left(\int_{-\infty}^\infty |f(t)|^p \, d\mu(t)\right)^{1/p},$$

where the second inequality follows from Hölder's inequality.

In the second case, the argument is similar except that we must compensate for the fact that $y \notin \{x:(Kf)(x)>\alpha\}$:

$$(\nu\{x:(Kf)(x)>\alpha\})^{1/q} = \lim_{\varepsilon \to 0+}\left(\int_{y+\varepsilon}^\infty d\nu\right)^{1/q}((Kf)(y+\varepsilon)/\alpha)$$

$$\le (1/\alpha)\left(\int_{-\infty}^{\infty}|f(t)|^p\,d\mu(t)\right)^{1/p}\lim_{\varepsilon\to 0}\left(\int_{y+\varepsilon}^{\infty}d\nu\right)^{1/q}\left(\int_{-\infty}^{\infty}k(y+\varepsilon,t)^{p'}\,d\mu(t)\right)^{1/p'}$$

$$\le (A/\alpha)\left(\int_{-\infty}^{\infty}|f(t)|^p\,d\mu(t)\right)^{1/p}.$$

In Theorem 3.9 we assumed that $k(x,t)$ was non-decreasing in $x$. The result when $k(x,t)$ is non-increasing in $x$ is:

THEOREM 3.10. *Suppose that* $0<q<\infty$, $1<p<\infty$, $k\ge 0$ *and* $k(x,t)$ *is non-increasing in* $x$ *for each* $t$. *Then,* (3.7) *holds if and only if*

$$\left(\int_{-\infty}^{y}d\nu\right)^{1/q}\left(\int_{-\infty}^{\infty}k(y,t)^{p'}\,d\mu(t)\right)^{1/p'}\le A,$$

*for all* $y\in R$.

*Proof*. Apply Theorem 3.9 to the kernel $k(-x,t)$. ∎

# CHAPTER 4

## APPLICATIONS OF HARDY'S INEQUALITY

In this chapter we give applications of the general Hardy inequality developed in Chapter 2.

Heinig [26],[27] showed that estimates for operators on weighted $L^p$ may be obtained from unweighted weak type conditions. This extension of the Marcinkiewicz interpolation theorem provides sufficient conditions on weight functions u and v for an inequality of the form

$$(4.1) \qquad \left(\int_0^\infty (Tf)^*(x)^q\, v(x)\, dx\right)^{1/q} \le C\left(\int_0^\infty f^*(x)^p\, w(x)\, dx\right)^{1/p},$$

for a large class of operators T, where * indicates the non-increasing rearrangement with respect to Lebesgue measure. In the case $0<q<p$, the weight conditions given by Heinig are unwieldy, and difficult to verify for specific weights. The conditions given here are in integral form and consequently easier to compute.

In the first section of this chapter we state and prove a weighted Marcinkiewicz interpolation theorem with weight conditions in integral form. In addition we give examples of weights u, v for which (4.1) holds.

In Section 2 a special case of the Marcinkiewicz interpolation theorem is proved, where the hypotheses of weak boundedness are replaced by different (and possibly weaker) conditions. The result is given to illustrate the role of the Hardy operator in interpolation theory.

The final section contains an application of the result on the characterization of Hardy's inequality for discrete measures. This example (Theorem 4.5) is interesting because it proves the equivalence of an $l^p$-boundedness property and a type of monotonicity condition. Moreover, the corresponding result for absolutely continuous measures is false.

## 1. A WEIGHTED INTERPOLATION THEOREM

Recall that a sublinear operator T is of weak type (p,q) if $x^{1/q}(Tf)^*(x) \le C\|f\|_p$.

THEOREM 4.1. *Suppose* T *is a sublinear operator defined on simple functions such that* T *is of weak type* $(p_0,q_0)$ *and* $(p_1,q_1)$ *with* $1\le p_0<p_1<\infty$, $1\le q_0\le\infty$, $1\le q_1\le\infty$, *and* $q_0\ne q_1$. *Let* $\alpha=(1/q_0-1/q_1)/(1/p_0-1/p_1)$, $I(t)=(t^\alpha,\infty)$ *when* $\alpha>0$, $I(t)=(0,t^\alpha)$ *when* $\alpha<0$, *and* $J(t)=(0,\infty)\setminus I(t)$. *If* $0<q<p$, $1\le p<\infty$, $1/r=1/q-1/p$, *and* v *and* w *are non-negative weights such that*

$$(4.2)\quad \left(\int_0^\infty\left(\int_{I(t)} v(x)x^{-q/q_0}\,dx\right)^{r/q}\times \left(\int_0^t w(x)^{1-p'}x^{-p'/p_0'}\,dx\right)^{r/q'} w(t)^{1-p'}t^{-p'/p_0'}\,dt\right)^{1/r}<\infty,$$

*and*

$$(4.3)\quad \left(\int_0^\infty\left(\int_{J(t)} v(x)x^{-q/q_1}\,dx\right)^{r/q}\times \left(\int_t^\infty w(x)^{1-p'}x^{-p'/p_1'}\,dx\right)^{r/q'} w(t)^{1-p'}t^{-p'/p_1'}\,dt\right)^{1/r}<\infty,$$

*then* (4.1) *holds for all simple* f.

*Proof.* Calderón [13] has shown that the weak type conditions on T imply

$$(Tf)^*(x)\le C\left(x^{-1/q_0}\int_0^{x^\alpha} t^{1/p_0-1}f^*(t)\,dt + x^{-1/q_1}\int_{x^\alpha}^\infty t^{1/p_1-1}f^*(t)\,dt\right).$$

Now by Minkowski's inequality when q>1 and directly when q<1, we have

$$(4.4)\qquad \left(\int_0^\infty (Tf)^*(x)^q v(x)\,dx\right)^{1/q}$$

$$\le C\left(\int_0^\infty \left(x^{-1/q_0}\int_0^{x^\alpha} t^{1/p_0-1}f^*(t)\,dt\right)^q v(x)\,dx\right)^{1/q}$$

$$+\,C\left(\int_0^\infty \left(x^{-1/q_1}\int_{x^\alpha}^\infty t^{1/p_1-1}f^*(t)\,dt\right)^q v(x)\,dx\right)^{1/q}.$$

We complete the proof by showing that both summands in (4.4) are bounded by $\left(\int_0^\infty f^*(x)^p w(x)dx\right)^{1/p}$. The change of variable $y=x^\alpha$ in the first summand yields

$$\left(\int_0^\infty \left(y^{-1/\alpha q_0}\int_0^y t^{1/p_0-1}f^*(t)\,dt\right)^q v(y^{1/\alpha})(1/|\alpha|)y^{1/\alpha-1}\,dy\right)^{1/q}$$

$$\le \left(\int_0^\infty (x^{1/p_0-1}f^*(x))^p w(x)x^{(1-1/p_0)p}\,dx\right)^{1/p} = \left(\int_0^\infty f^*(x)^p w(x)\,dx\right)^{1/p}$$

where the above inequality holds by Proposition 2.22, provided the condition

$$\left(\int_0^\infty\left(\int_t^\infty (1/|\alpha|)v(y^{1/\alpha})y^{-(1/\alpha)(q/q_0-1)-1}\,dy\right)^{r/q}\times\right.$$

$$\left.\left(\int_0^t w(x)^{-p'/p}x^{-p'(1-1/p_0)}\,dx\right)^{r/q'} w(t)^{-p'/p}t^{-p'(1-1/p_0)}\,dt\right)^{1/r} < \infty$$

holds. A change of variable in the first inner integral shows this to be (4.2).

The bound on the second summand of (4.4) follows from condition (4.3) in exactly the same way. We omit the details.

The following important special case of Theorem 4.1 was proved in [7] and [15] for the range $1<q<p<\infty$:

COROLLARY 4.2. *If T is of weak type* $(1,\infty)$ *and* $(2,2)$, *then for all simple functions* $f$

$$\left(\int_0^\infty (Tf)^*(x)^q v(x)\,dx\right)^{1/q} \le C\left(\int_0^\infty f^*(x)^p w(x)\,dx\right)^{1/p}$$

$0<q<p$, $p>1$, *provided*

$$\left(\int_0^\infty\left(\int_0^{1/t} v(x)\,dx\right)^{r/q}\left(\int_0^t u(x)\,dx\right)^{r/q'} u(t)\,dt\right)^{1/r} < \infty$$

(4.5)

$$\left(\int_0^\infty\left(\int_{1/t}^\infty v(x)x^{-q/2}\,dx\right)^{r/q}\left(\int_t^\infty u(x)x^{-p'/2}\,dx\right)^{r/q'} u(t)t^{-p'/2}\,dt\right)^{1/r} < \infty,$$

*where* $u(x)=w(x)^{1-p'}$ *and* $1/r=1/q-1/p$.

Since the Fourier transform is of type $(1,\infty)$ and

(2,2), (hence of weak type), Corollary 4.2 establishes weighted norm inequalities for the Fourier transform in the index range $0<q<1$, $p>1$ which are new with the integral conditions (4.5). In addition the argument given in [15] to prove norm inequalities involving the Hankel-, K- and Y-transforms for the range $1<q<\infty$, $1<p<\infty$ extends to the range $0<q<1$, $p>1$ by applying Corollary 4.2.

Finally we note that the weights

$$v(x) = Ax^{a-1}\chi_{(0,1)}(x)+Bx^{q/2-b-1}\chi_{(1,\infty)}(x);$$

$$u(x) = Cx^{c-1}\chi_{(0,1)}(x)+Dx^{p'/2-d-1}\chi_{(1,\infty)}(x)$$

satisfy condition (4.5) provided A, B, C, D, a, b, c, d are positive, $a/q+d/p'>1/2$, and $b/q+c/p'>1/2$. Specifically, Corollary 4.2 holds for these weights with $u=w^{1-p'}$.

## 2. A VARIANT OF THE MARCINKIEWICZ INTERPOLATION THEOREM

Let $f^{\Theta}$ and $f^{\Theta}$ denote the rearrangements of $f$ with regard to $\mu$ and $\nu$ respectively.

THEOREM 4.3. *Suppose* $\mu$ *and* $\nu$ *are* $\sigma$-*finite, Borel measures and* $\mu$ *is non-atomic. Let* $1<p_0<p<p_1<\infty$, *and define* $\Phi_j$ *by*

$$\Phi_j(t)=\sup\{(Tf)^{\Theta}(t):\|f\|_{p_j,\mu}\le 1\},\qquad j=0,1.$$

*If*

$$(4.6)\qquad \begin{aligned} &x^{(p-p_0)/p_0}\int_x^\infty \Phi_{p_0}(t)^p\,dt;\\ &x^{-1}\int_0^x (x-t)^{p/p_1}\,\Phi_{p_1}(t)^p\,dt \end{aligned}$$

*are bounded for all* $x>0$, *then* $\|Tf\|_{p,\nu}\le C\|f\|_{p,\mu}$.

*Proof*. Let $t>0$ be a point of continuity of $f^{\Theta}$ and define $E=\{x:|f(x)|>f^{\Theta}(t)\}$. With $f_0(x)=f(x)\chi_E(x)$ and $f_1(x)=f(x)-f_0(x)$. It is not difficult to show that $f_0^{\Theta}(s)=f^{\Theta}(s)\chi_{(0,t)}(s)$ and $f_1^{\Theta}(s)=f^{\Theta}(s+t)$, so that

$$\|f_0\|_{p_0,\mu}=\left(\int_0^t f^{\Theta}(s)^{p_0}\,ds\right)^{1/p_0}\ \text{and}\ \|f_1\|_{p_1,\mu}=\left(\int_t^\infty f^{\Theta}(s)^{p_1}\,ds\right)^{1/p_1}.$$

Since $(Tf)^{\Theta}(2t)\le(Tf_0)^{\Theta}(t)+(Tf_1)^{\Theta}(t)$, Minkowski's inequality shows that

$$\left(\int_{-\infty}^{\infty}|Tf(x)|^p\,d\nu(x)\right)^{1/p} \le C\left(\left(\int_0^\infty (Tf_0)^\theta(t)^p\,dt\right)^{1/p}+\left(\int_0^\infty (Tf_1)^\theta(t)^p\,dt\right)^{1/p}\right)$$

$$\le C\left(\left(\int_0^\infty \|f_0\|_{p_0}^p\phi_{p_0}(t)^p\,dt\right)^{1/p}+\left(\int_0^\infty \|f_1\|_{p_1}^p\phi_{p_1}(t)^p\,dt\right)^{1/p}\right)$$

$$= C\left(\left(\int_0^\infty\left(\int_0^t f^\theta(s)^{p_0}\,ds\right)^{p/p_0}\phi_{p_0}(t)^p\,dt\right)^{1/p}\right.$$

$$\left.+\left(\int_0^\infty\left(\int_t^\infty f^\theta(s)^{p_1}\,ds\right)^{p/p_1}\phi_{p_1}(t)^p\,dt\right)^{1/p}\right).$$

Hence the proof of the boundedness of $T:L_\mu^p\to L_\nu^q$ reduces to obtaining bounds for the Hardy operator and its dual. Therefore by Theorems 2.6 and 2.19

$$\left(\int_0^\infty\left(\int_0^t f^\theta(s)^{p_0}\,ds\right)^{p/p_0}\phi_{p_0}(t)^p\,dt\right)^{p_0/p} \le C\left(\int_0^\infty f^\theta(s)^p\,ds\right)^{p_0/p}$$

and

$$\int_0^\infty\left(\int_t^\infty f^\theta(s)^{p_1}\,ds\right)^{p/p_1}\phi_{p_1}(t)^p\,dt \le C\int_0^\infty f^\theta(s)^p\,ds$$

provided

$$\left(\int_x^\infty \phi_{p_0}(t)^p\,dt\right)^{p_0/p}\left(\int_0^x ds\right)^{1-p_0/p} \le C;$$

$$\int_0^x\left(\int_t^x ds\right)^{p/p_1}\phi_{p_1}(t)^p\,dt \le C\int_0^x dt$$

respectively, for all $x>0$. But this is precisely (4.6).

Since $\left(\int_0^\infty f^{o}(s)^p ds\right)^{1/p} = \|f\|_{p,\mu}$ the result follows.

## 3. AN EXAMPLE FOR SERIES

In this section we answer the following question: For what positive sequences $\{a_n\}$ is the operator S defined by $(Su)_n = a_n \sum_{k=0}^{n} u_k/a_k$ bounded on $l^p$?

DEFINITION 4.4. *A sequence $\{a_n\}$ of positive real numbers is almost strongly decreasing (a.s.d.) provided there are non-negative constants A and N such that*

i) $a_k \le Aa_n$ *whenever* $k \ge n$;

ii) $2a_k \le a_n$ *whenever* $k \ge n+N$.

THEOREM 4.5. *Given any positive sequence $\{a_n\}$, define S by $(Su)_n = a_n \sum_{k=0}^{n} u_k/a_k$. If $1<p<\infty$ then $S: l^p \to l^p$ is bounded if and only if $\{a_n\}$ is* a.s.d..

*Proof*. Since S is bounded on $l^p$,

$$\left(\sum_{n=0}^{\infty} |(Su)_n|^p\right)^{1/p} \le C\left(\sum_{k=0}^{\infty} |u_k|^p\right)^{1/p}$$

for all sequences $\{u_k\}$. If $u_k = v_k a_k^{1-p'}$ this becomes

$$\left(\sum_{n=0}^{\infty}\left|\sum_{k=0}^{n} v_k a_k^{-p'}\right| a_n^p\right)^{1/p} \le C\left(\sum_{n=0}^{\infty}|v_k|^p a_k^{-p'}\right)^{1/p}$$

for all sequences $\{v_k\}$, and by Theorem 2.26 this is equivalent to

$$\sup_n\left(\sum_{k=n}^{\infty} a_k^p\right)^{1/p}\left(\sum_{k=0}^{n} a_k^{-p'}\right)^{1/p'} < \infty. \tag{4.7}$$

It remains to show that $\{a_n\}$ is almost strongly decreasing if and only if (4.7) holds. Suppose that $\{a_n\}$ is a.s.d. then by Definition 4.4 there exist constants A and N satisfying $a_k \le A a_n$ for $k \ge n$, and $2a_k \le a_n$ whenever $k \ge n+N$. Repeated application of this last property implies $a_{n+jN} \le 2^{-j} a_n$ for $j=0,1,2,\ldots$ so that

$$\sum_{k=n}^{\infty} a_k^p = \sum_{j=0}^{\infty}\left(\sum_{k=jN+n}^{jN+n+(N-1)} a_k^p\right) \le \sum_{j=0}^{\infty} 2^{-jp}\left(\sum_{k=n}^{n+N-1} a_k^p\right) = C\sum_{k=n}^{n+N-1} a_k^p \le CNA^p a_n^p$$

since $k \ge n$.

If $n \ge N$

$$\sum_{k=0}^{n} a_k^{-p'} = \sum_{j=0}^{\infty}\left(\sum_{k\in E_j} a_k^{-p'}\right) \le \sum_{j=0}^{\infty} 2^{-jp'}\left(\sum_{k=n-N+1}^{n} a_k^{-p'}\right) = C\sum_{k=n-N+1}^{n} a_k^{-p'} \le CNA^{p'} a_n^{-p'},$$

where $E_j = \{k \ge 0 : n-(j+1)N+1 \le k \le n-jN\}$. These estimates imply

$$\sup_{n\geq N}\left(\sum_{k=n}^{\infty} a_k^p\right)^{1/p}\left(\sum_{k=0}^{n} a_n^{-p'}\right)^{1/p'} \leq CNA^2 < \infty$$

which in turn implies (4.7).

To prove that $(a_n)$ is a.s.d. suppose

$$\sup_{n}\left(\sum_{k=n}^{\infty} a_k^p\right)^{1/p}\left(\sum_{k=0}^{n} a_k^{-p'}\right)^{1/p'} = B < \infty,$$

then for any $k\geq m$, $a_k a_m^{-1} \leq B$ so i) of Definition 4.4 holds with $A=B$. If ii) of Definition 4.4 fails, that is, if for any $N$ there exist $n$ and $k$ such that $k\geq n+N$ and $2a_k > a_n$ we have

$$B \geq \left(\sum_{j=k}^{\infty} a_j^p\right)^{1/p}\left(\sum_{j=0}^{k} a_j^{-p'}\right)^{1/p'} \geq a_k\left(\sum_{j=n+1}^{n+N} a_j^{-p'}\right)^{1/p'}$$

$$\geq (1/2)a_n(N(Ba_n)^{-p'})^{1/p'} = (1/2B)N^{1/p'}.$$

Since $N$ was arbitrary we have a contradiction.

## CHAPTER 5

## LAPLACE REPRESENTATIONS OF FUNCTIONS IN WEIGHTED BERGMAN SPACES

The Fourier transform $\hat{f}$ of $f$ is defined by

$$\hat{f}(x) = \int_{R^n} e^{-2\pi i x\cdot y} f(y)\, dy, \qquad x \in R^n,$$

whenever the integral converges. Here $x=(x_1,x_2,\ldots,x_n)$, $dy=dy_1dy_2\ldots dy_n$ and $x\cdot y=x_1y_1+\ldots+x_ny_n$.

The weight class $A_p$ $1<p<\infty$, consists of all non-negative functions $w$, locally integrable, such that,

$$(5.1) \qquad \left(1/|Q|\int_Q w(x)\, dx\right)^{1/p}\left(1/|Q|\int_Q w(x)^{1-p'}\, dx\right)^{1/p'} \le C$$

for all cubes $Q \subseteq R^n$, with sides parallel to the coordinate axes. If $n=1$, cubes $Q$ in (5.1) are of course replaced by intervals $I$ with Lebesgue measure $|I|$.

In Section 1 we give a weak type Fourier estimate (Lemma 5.1) which is used to prove weighted Fourier inequalities for monotone weights in $A_p$. These inequalities were also obtained in [8], but the proofs given here

(Theorems 5.4 and 5.7) are quite different. In Section 2, several weighted inequalities are obtained for the Fourier transform in higher dimensions including a new Hausdorff-Young inequality which is particularly noteworthy.

The existence of Laplace representations for functions in weighted Bergman spaces depends on weighted Fourier inequalities. We derive such representations in Section 3.

## 1. WEIGHTED FOURIER INEQUALITIES ON $\mathbb{R}$

We now establish a number of weighted Fourier inequalities for functions defined on $\mathbb{R}$. The first result constitutes a weighted weak type estimate for a Fourier operator.

We assume in Sections 1 and 2 that f is simple. The extension to functions in weighted $L^p$ follows standard arguments (see [7]) and is therefore omitted.

LEMMA 5.1. *Suppose v is a non-negative even function and a is defined by*

$$a(x) = \operatorname{sgn}(x)\int_0^x v(1/t)\,dt,$$

*then for any* $\lambda>0$,

(5.2) $$\mu\{x\in R: |a(x)\hat{g}(x)|>\lambda\} \le 2\|g\|_1/\lambda,$$

*where*

(5.3) $$\mu E = \int_E a(x)^{-2}v(1/x)\,dx, \qquad E\subseteq R\setminus\{0\}.$$

*Proof*. Since $|\hat{g}(x)|\le\|g\|_1$,

$$\mu\{x\in R: |a(x)\hat{g}(x)|>\lambda\} \le \mu\{x\in R: a(x)\|g\|_1>\lambda\}$$

$$\le 2\int_{a^{-1}(\lambda/\|g\|_1)}^{\infty} a(x)^{-2}v(1/x)\,dx = 2\|g\|_1/\lambda,$$

and (5.2) follows. Here we interpret $a^{-1}(y)$ to be $\inf\{x:a(x)>y\}$ so that $a(a^{-1}(y))=y$.

LEMMA 5.2. *If* $v$ *is a non-negative, even function non-decreasing on* $(0,\infty)$ *and* $a$ *and* $\mu$ *are as in Lemma* 5.1, *then*

(5.4) $$\int_{-\infty}^{\infty} |a(x)\hat{g}(x)|^2 \, d\mu(x) \le C \int_{-\infty}^{\infty} |g(x)|^2 v(x) \, dx$$

*if and only if* $v \in A_2$.

*Proof*. Since (5.4) can be written in the form

$$\int_{-\infty}^{\infty} |\hat{g}(x)|^2 v(1/x) \, dx \le C \int_{-\infty}^{\infty} |g(x)|^2 v(x) \, dx$$

the result follows from [8, Theorem 3.1] with $p=2$.

THEOREM 5.3. *Suppose* $v$ *is a non-negative, even function, non-decreasing on* $(0,\infty)$ *and let* $a$ *be defined as in Lemma* 5.1. *If* $1<p\le 2$, *then*

(5.5) $$\int_{-\infty}^{\infty} |\hat{f}(x)|^p a(x)^{p-2} v(1/x) \, dx \le C \int_{-\infty}^{\infty} |f(x)|^p v(x)^{p-1} \, dx$$

*if and only if* $v \in A_2$.

*Proof*. Let $g=f/v$ in (5.2) and (5.4), then

$$\mu\{x \in \mathbb{R} : |a(x)(f/v)^{\hat{}}(x)| > \lambda\} \le (2/\lambda) \|f\|_{1,1/v}$$

and

$$\left(\int_{-\infty}^{\infty} |a(x)(f/v)^{\wedge}(x)|^2\, d\mu(x)\right)^{1/2} \le C\|f\|_{2,1/v}.$$

By the Marcinkiewicz interpolation theorem [9]

$$\int_{-\infty}^{\infty} |a(x)(f/v)^{\wedge}(x)|^p\, d\mu(x) \le C\int_{-\infty}^{\infty} |f(x)|^p v(x)^{-1}\, dx$$

for $1<p<2$ and replacing $f/v$ by $f$, (5.5) follows.

Conversely, let $f=\chi_{(0,s)}/v$, $s>0$, with $v>0$, in (5.5), then since a is an even function we obtain

$$\int_0^{\infty} a(x)^{p-2}v(1/x)\left(\int_0^s |\cos(xy)|v(y)^{-1}\, dy\right)^p dx \le C\int_0^s v(x)^{-1}\, dx.$$

Reducing the range of integration on the left from $(0,\infty)$ to $(0,1/s)$ so that $\cos(xy)>\cos(1)$ we have, for all $s>0$

$$\left(\int_0^{1/s} a(x)^{p-2}v(1/x)\, dx\right)\left(\int_0^s v(y)^{-1}\, dy\right)^{p-1} \le C.$$

Now integration and a change of variable yield

$$\int_0^{1/s} a(x)^{p-2}v(1/x)\, dx = (1/(p-1))\left(\int_s^{\infty} y^{-2}v(y)\, dy\right)^{p-1}$$

so that

$$\left(\int_s^{\infty} y^{-2}v(y)\, dy\right)\left(\int_0^s v(y)^{-1}\, dy\right) \le C.$$

But by [8, Corollary 3.2] this is equivalent to $v \in A_2$.

As a consequence we establish the weighted Hardy-Littlewood inequality given by Benedetto, Heinig and Johnson [8].

THEOREM 5.4. *With* v *and* a *as in Theorem* 5.3,

$$(5.6)\quad \int_{-\infty}^{\infty} |\hat{f}(x)|^p |x|^{p-2} v(1/x)^{p-1}\, dx \le C \int_{-\infty}^{\infty} |f(x)|^p v(x)^{p-1}\, dx, \qquad 1<p\le 2$$

*if and only if* $v^{p-1} \in A_p$.

*Proof*. Let $v\in A_2$, then by [29, Lemma 1], for each $x>0$

$$a(x) = \int_0^x v(1/t)\, dt = \int_{1/x}^{\infty} t^{-2} v(t)\, dt \le Cx^2 \int_0^{1/x} v(t)\, dt \le Cxv(1/x).$$

Since $p-2\le 0$, (5.6) follows from (5.5). If (5.6) holds then arguing as in the proof of Theorem 5.3, $v^{p-1}\in A_p$.

It remains to show that $v\in A_2$ whenever $v^{p-1}\in A_p$ with $1<p<2$. This is contained in the next lemma.

LEMMA 5.5. *Let v be a non-negative, even function non-decreasing on $(0,\infty)$, then $v\in A_2$ if and only if $v^{p-1}\in A_p$ with $1<p\le 2$.*

*Proof*. Let $1<p<2$ and $v\in A_2$, then for each interval $I\subseteq\mathbb{R}$, Hölder's inequality with index $1/(p-1)>1$ yields

$$\left((1/|I|)\int_I v(x)^{p-1}\,dx\right)^{1/p}\left((1/|I|)\int_I (v(x)^{p-1})^{1-p'}\,dx\right)^{1/p'}$$

$$\le \left((1/|I|)\int_I v(x)\,dx\right)^{1/p'}\left((1/|I|)\int_I v(x)^{-1}\,dx\right)^{1/p'} \le C$$

which shows that $v^{p-1}\in A_p$.

Conversely, suppose $v^{p-1}\in A_p$ and let I be the interval $[a,b]$. Since v is even we may assume without loss of generality that $0\le|a|<b$. Hence

$$\left((b-a)^{-1}\int_a^b v(x)\,dx\right)^{p-1} \le v(b)^{p-1} = v(b)^{p-1}\int_b^\infty ((b-a)/2)(x-(b+a)/2)^{-2}\,dx$$

$$\le (1/2)\int_b^\infty (b-a)(x-(b+a)/2)^{-2}v(x)^{p-1}\,dx$$

by the monotonicity of v. Now $v^{p-1}\in A_p$ implies $v^{p-1}\in A_2$ so by [29, Lemma 1] the last expression is bounded by

$$C(b-a)^{-1}\int_a^b v(x)^{p-1}\,dx.$$

Since $v^{p-1} \in A_p$ we have

$$\left((b-a)^{-1}\int_a^b v(x)\,dx\right)^{1/2}\left((b-a)^{-1}\int_a^b v(x)^{-1}\,dx\right)^{1/2}$$

$$\le C\left((b-a)^{-1}\int_a^b v(x)^{p-1}\,dx\right)^{1/2(p-1)}\left((b-a)^{-1}\int_a^b (v(x)^{p-1})^{1-p'}\,dx\right)^{1/2} \le C$$

so $v \in A_2$. This proves the lemma.

The dual of Theorem 5.4 is the following.

THEOREM 5.6. <u>Let</u> $v$ <u>be a non-negative, even function non-decreasing on</u> $(0,\infty)$, <u>then</u>

$$\left(\int_{-\infty}^{\infty} |\hat{f}(x)|^{p'} v(x)^{1-p'}\,dx\right)^{1/p'} \le C\left(\int_{-\infty}^{\infty} |f(x)|^{p'} |x|^{p'-2} v(1/x)^{1-p'}\,dx\right)^{1/p'},$$

$1<p\le 2$, <u>if and only if</u> $v^{1-p'} \in A_{p'}$.

REMARK. It is easily seen that $v^{1-p'} \in A_{p'}$ if and only if $v \in A_p$. But by Lemma 5.5, $v \in A_p$ if and only if $v^{1/(p-1)} \in A_2$ and since $1/(p-1)=p'-1$, $v^{p'-1} \in A_2$. But this holds if and only if $v^{1-p'} \in A_2$. Hence with $u=v^{1-p'}$ we have shown that <u>if</u> $u$ <u>is even and non-increasing then</u> $u \in A_{p'}$ <u>with</u> $2 \le p < \infty$ <u>if and only if</u> $u \in A_2$.

Our next result is a weighted Hausdorff-Young inequality.

THEOREM 5.7. _Let_ w _be an even, non-decreasing function on_ $(0,\infty)$, _then_

$$(5.7)\quad \left(\int_{-\infty}^{\infty} |\hat{f}(x)|^{p'} w(1/x)^{p'-1}\, dx\right)^{1/p'} \le C\left(\int_{-\infty}^{\infty} |f(x)|^p w(x)\, dx\right)^{1/p},$$

$1<p\le 2$, _if and only if_ $w\in A_p$.

_Proof_. Consider $(f/v)^\wedge$, where v is even and non-decreasing on $(0,\infty)$. Since $\|(f/v)^\wedge\|_\infty \le \|f\|_{1,1/v}$, (5.4) and the Riesz-Thorin interpolation theorem [9] show that

$$\left(\int_{-\infty}^{\infty} |(f/v)^\wedge(x)|^{p'} v(1/x)\, dx\right)^{1/p'} \le C\left(\int_{-\infty}^{\infty} |f(x)|^p v(x)^{-1}\, dx\right)^{1/p},$$

$1<p\le 2$, whenever $v\in A_2$. Now replacing f/v by f and $v^{p-1}$ by w we obtain (5.7) whenever $w^{1/(p-1)}\in A_2$. But Lemma 5.5 shows that this holds whenever $w\in A_p$.

The sufficiency part is standard and hence omitted.

We conclude this section by quoting two results of [8] which may be derived from Theorems 5.4 and 5.7 (cf. [44]).

THEOREM 5.8. *Suppose* $w$ *is even and non-decreasing on* $(0,\infty)$ *and* $1<p\le q\le p'<\infty$ *then*

a) $$\left(\int_{-\infty}^{\infty}|\hat{f}(x)|^q|x|^{q/p'-1}w(1/x)^{q/p}\,dx\right)^{1/q}\le C\left(\int_{-\infty}^{\infty}|f(x)|^p w(x)\,dx\right)^{1/p}$$

*if and only if* $w^{q/p}\in A_{1+q/p'}$;

b) $$\left(\int_{-\infty}^{\infty}|\hat{f}(x)|^{p'}w(x)^{1-p'}\,dx\right)^{1/p'}\le C\left(\int_{-\infty}^{\infty}|f(x)|^{q'}|x|^{q'/p-1}w(1/x)^{-q'/p}\,dx\right)^{1/q'}$$

*if and only if* $w^{1-p'}\in A_{1+p'/q}$.

2. HIGHER DIMENSIONAL CASES

We now study Fourier inequalities for functions on $R^n$. We shall denote a point $x\in R^n$ by $x=(x_1,\dots,x_n)$, $|x|=(x_1^2+\dots+x_n^2)^{1/2}$ and $dx=dx_1\dots dx_n$ is the n-dimensional Lebesgue measure. Also $1/x=(1/x_1,1/x_2,\dots,1/x_n)$.

Since the n-dimensional Fourier transform is a product operator we obtain the following n-dimensional generalization of Theorems 5.7 and 5.4:

THEOREM 5.9. Suppose $w(x)=\prod_{i=1}^{n} w_i(x_i)$, where $w_i(x_i)$ $i=1,2,\ldots,n$ are non-negative, even functions, non-decreasing on $(0,\infty)$.

Let $1<p\le 2$, then

$$(5.8)\quad \left(\int_{\mathbb{R}^n} |\hat{f}(x)|^{p'} w(1/x)^{p'-1}\,dx\right)^{1/p'} \le C\left(\int_{\mathbb{R}^n} |f(x)|^p\, w(x)\,dx\right)^{1/p}$$

and

$$(5.9)\quad \left(\int_{\mathbb{R}^n} |\hat{f}(x)|^p\, w(1/x) \prod_{i=1}^{n} |x_i|^{p-2}\,dx\right)^{1/p} \le C\left(\int_{\mathbb{R}^n} |f(x)|^p\, w(x)\,dx\right)^{1/p}$$

if and only if $w_i \in A_p$, for $i=1,2,\ldots,n$.

Proof. We consider only inequality (5.8). The proof of (5.9) is similar and hence omitted. Moreover we only prove the result for n=2 since then the general case follows by induction.

Following the argument of [4, Lemma 2] we obtain

$$\int_{\mathbb{R}^2} w(1/x)^{p'-1} \left| \int_{-\infty}^{\infty} e^{-2\pi i x_1 y_1} \int_{-\infty}^{\infty} e^{-2\pi i x_2 y_2}\, f(y)\,dy \right|^{p'} dx$$

$$= \int_{-\infty}^{\infty} w_2(1/x_2)^{p'-1} \int_{-\infty}^{\infty} w_1(1/x_1)^{p'-1} \left| \int_{-\infty}^{\infty} e^{-2\pi i x_1 y_1} (F_2 f)(y_1, x_2)\, dy_1 \right|^{p'} dx$$

where

$$(F_2 f)(y_1, x_2) = \int_{-\infty}^{\infty} e^{-2\pi i x_2 y_2} f(y_1, y_2)\, dy_2 .$$

Now by (5.7) of Theorem 2.8 the triple integral above is not larger than

$$C_1 \int_{-\infty}^{\infty} w_2(1/x_2)^{p'-1} \left( \int_{-\infty}^{\infty} w_1(y_1) |(F_2 f)(y_1, x_2)|^p\, dy_1 \right)^{p'/p} dx_2$$

$$\le C_1 \left( \int_{-\infty}^{\infty} w_1(y_1) \left( \int_{-\infty}^{\infty} w_2(1/x_2)^{p'-1} \left| \int_{-\infty}^{\infty} e^{-2\pi i x_2 y_2} f(y)\, dy_2 \right|^{p'} dx_2 \right)^{p/p'} dy_1 \right)^{p'/p}$$

$$\le C_1 C_2 \left( \int_{-\infty}^{\infty} w_1(y_1) \left( \int_{-\infty}^{\infty} w_2(y_2) |f(y)|^p\, dy_2 \right) dy_1 \right)^{p'/p}$$

$$= C_1 C_2 \left( \int_{\mathbb{R}^2} w(y) |f(y)|^p\, dy \right)^{p'/p} .$$

Here we applied Minkowski's integral inequality and then (5.7) again.

Conversely, suppose (5.8) holds (with n=2). Define f by $f(y)=w(y)^{1-p'} \prod_{i=1}^{2} \chi_{(0,s_i)}(|y_i|)$, where $0<s_i<\infty$, then f is even in each variable separately, so that (5.8) takes the form

$$\left( \int_0^{\infty} w_2(1/x_2)^{p'-1} \int_0^{\infty} w_1(1/x_1)^{p'-1} \left| \int_0^{s_1} \cos(x_1 y_1) w(y_1)^{1-p'}\, dy_1 \times \right.\right.$$

$$\int_0^{s_2} \cos(x_2y_2)w(y_2)^{1-p'}\,dy_2\Big|^{p'}\,dx_1\,dx_2\Big)^{1/p'} \le C\left(\int_0^{s_1}\int_0^{s_2} w(y)^{1-p'}\,dy\right)^{1/p}.$$

Reducing the range of integration on the left side from $(0,\infty)$ to $(0,1/s_i)$, $i=1,2$, then, since $\cos(x_iy_i)\ge\cos(1)$, $i=1,2$,

$$\left(\int_0^{1/s_2}\int_0^{1/s_1} w(1/x)^{p'-1}\,dx\right)^{1/p'}\left(\int_0^{s_1}\int_0^{s_2} w(y)^{1-p'}\,dy\right)^{1/p'} \le C$$

for all $s_1,s_2>0$. Fixing $s_2$ and then $s_1$ it follows that

$$\left(\int_0^{1/s_i} w_i(1/x_i)^{p'-1}\,dx_i\right)^{1/p'}\left(\int_0^{s_i} w_i(y_i)^{1-p'}\,dy_i\right)^{1/p'} \le C_i$$

$i=1,2$. Making the change of variable $x_i\to 1/y_i$ in the first integral [8, Corollary 3.2] shows that $w_i^{p'-1}\in A_2$, $i=1,2$. But then Lemma 5.5 shows that $w_i\in A_p$ for $i=1,2$.

If the weight functions are radial the previous argument does not apply. To obtain weighted inequalities in this case we recall the following result of Muckenhoupt [38],[39] and Jurkat and Sampson [31]:

Suppose $u^*$ and $(1/v)^*$ denote the decreasing rearrangement of $u$ and $1/v$ respectively (recall that $g^*(t)=\inf\{y>0:|\{x:g(x)>y\}|\le t\}$ ). If

$$\sup_{s>0}\left(\int_0^{1/s} u^*(t)\,dt\right)\left(\int_0^{s}(1/v)^*(t)\,dt\right) < \infty$$

then

$$\int_{\mathbb{R}^n} |\hat{f}(x)|^2 u(x)\,dx \le C\int_{\mathbb{R}^n} |f(x)|^2 v(x)\,dx.$$

Let $\theta^n$ denote the volume of the unit sphere in $\mathbb{R}^n$, then the radial rearrangement of a function g is defined by $g^{\circledast}(t)=g^*(\theta^n|t|^n)$. The above integral condition then takes the form

$$\sup_{s>0}\left(\int_0^{1/(\theta s^{1/n})} u^{\circledast}(x)x^{n-1}\,dx\right)\left(\int_0^{(s^{1/n})/\theta}(1/v)^{\circledast}(x)x^{n-1}\,dx\right) < \infty.$$

Now let $v(x)$ be radial on $\mathbb{R}^n$ and with $|x|=t$, $v(t)$ is non-decreasing. If $u(|x|)=v(1/|x|)$ then $u^{\circledast}=u$ and $(1/v)^{\circledast}=1/v$. Therefore it follows that

$$\sup_{s>0}\left(\int_0^{1/\theta s} v(1/t)t^{n-1}\,dt\right)\left(\int_0^{s/\theta} v(t)^{-1}t^{n-1}\,dt\right) < \infty \tag{5.10}$$

implies

$$\int_{\mathbb{R}^n} |\hat{f}(x)|^2 v(1/|x|)\,dx \le C\int_{\mathbb{R}^n} |f(x)|^2 v(|x|)\,dx. \tag{5.11}$$

The result corresponding to Lemma 5.2 is now the following:

LEMMA 5.10. *If v is a non-negative, radial function defined on* $\mathbb{R}^n$, *such that* $t^{1-n}v(t)$ $(t=|x|)$ *is non-decreasing on* $(0,\infty)$, *then* (5.11) *holds if and only if* $t^{1-n}v(t)\in A_2$.

*Proof*. If $\theta$ is (as before) the $n^{th}$ root of the volume of the unit n-sphere in $\mathbb{R}^n$ and $t^{1-n}v(t)\in A_2$ then for $s>0$,

$$C \geq (1/\theta s)\left(\int_0^{\theta s} t^{1-n}v(t)\,dt\right)^{1/2}\left(\int_0^{\theta s}(t^{1-n}v(t))^{-1}\,dt\right)^{1/2}$$

$$\geq (1/\theta s)\left(\int_0^{s\theta} t^{1-n}v(t)\,dt\right)^{1/2}\left(\int_0^{s/\theta} t^{n-1}v(t)^{-1}\,dt\right)^{1/2}$$

since $\theta>1$. But by [29, Lemma 1]

$$\left(\int_{\theta s}^{\infty} t^{1-n}v(t)/t^2\,dt\right)^{1/2} \leq (C/\theta s)\left(\int_0^{\theta s} t^{1-n}v(t)\,dt\right)^{1/2}.$$

Substituting this into the above estimate yields

$$C \geq \left(\int_{\theta s}^{\infty} t^{-1-n}v(t)\,dt\right)^{1/2}\left(\int_0^{s/\theta} t^{n-1}v(t)^{-1}\,dt\right)^{1/2}.$$

But this is clearly equivalent to (5.10). Therefore (5.11) holds.

Conversely, suppose (5.11) holds. If f is radial, so is $\hat{f}$ and (5.11) takes the form ([49])

$$\int_0^\infty t^{n-1}v(1/t)\left(t^{1-n/2}\int_0^\infty y^{n/2}J_{(n-2)/2}(ty)|f(y)|\,dy\right)^2 dt \le C\int_0^\infty t^{n-1}v(t)|f(t)|^2\,dt$$

where $J_{(n-2)/2}$ is the Bessel function of order $(n-2)/2$. Now let $f(t)=v(t)^{-1}\chi_{(0,s)}(t)$, $s>0$, and reducing the range of integration on the left to $(0,1/s)$ we obtain

$$\int_0^{1/s} v(1/t)t\left(\int_0^s y^{n/2}v(y)^{-1}J_{(n-2)/2}(ty)\,dy\right)^2 dt \le C\int_0^s t^{n-1}v(t)^{-1}\,dt.$$

Since $J_{(n-2)/2}(ty)\ge C(ty)^{(n-2)/2}$ for $0<ty<1$ this yields

$$\left(\int_0^{1/s} t^{n-1}v(1/t)\,dt\right)\left(\int_0^s y^{n-1}v(y)^{-1}\,dy\right) \le C.$$

But the change of variable $t\to 1/y$ in the first integral and the fact that $y^{1-n}v(y)$ is non-decreasing show that $y^{1-n}v(y)\in A_2$ by [8, Corollary 3.2].

The n-dimensional weak type inequality corresponding to Lemma 5.1 is given next.

LEMMA 5.11. *Let* v *be a non-negative, radial* *function defined on* $\mathbb{R}^n$ *and let* a *be given by*

$$a(x) = \int_0^{|x|} v(1/t)t^{n-1}\,dt.$$

*If*

$$\mu E = \int_E a(x)^{-2}v(1/|x|)\,dx \qquad E\subseteq\mathbb{R}^n\backslash\{0\}$$

*then*

$$\mu\{x\in\mathbb{R}^n:|a(x)\hat{f}(x)|>\lambda\} \le (C/\lambda)\|f\|_1.$$

*Proof*. Let $\Omega$ be the surface of the unit n-sphere, then

$$\mu\{x\in\mathbb{R}^n:|a(x)\hat{f}(x)|>\lambda\} \le \mu\{x\in\mathbb{R}^n:a(x)\|f\|_1>\lambda\}$$

$$= \int_{\{x\in\mathbb{R}^n:a(x)>\lambda/\|f\|_1\}} a(x)^{-2}v(1/|x|)\,dx = \int_\Omega d\sigma \int_{\{t>0:a(t)>\lambda/\|f\|_1\}} a(t)^{-2}t^{n-1}v(1/t)\,dt$$

$$= C\int_{a^{-1}(\lambda/\|f\|_1)} a(t)^{-2}\,da(t) = (C/\lambda)\|f\|_1.$$

Here we interpret $a^{-1}(y)$ to be $\inf\{x>0:a(x)>y\}$ so that $a(a^{-1}(y))=y$.

As in the one dimensional case of Theorem 5.3, these two results yield a higher dimensional extension of the Hardy-Littlewood theorem.

THEOREM 5.12. *Let* $v$ *be a non-negative, radial function on* $\mathbb{R}^n$, *such that* $t^{1-n}v(t)$ *(with* $t=|x|$*) is non-decreasing. If* $a$ *is defined as above and* $1<p\le 2$, *then*

$$\int_{\mathbb{R}^n} |\hat{f}(x)|^p a(x)^{p-2} v(1/|x|)\, dx \le C \int_{-\infty}^{\infty} |f(x)|^p v(x)^{p-1}\, dx,$$

*if and only if* $t^{1-n}v(t)\in A_2$.

Now on application of [29, Lemma 1] together with the fact that $t^{1-n}v(t)$, $(t=|x|)$ is non-decreasing we obtain as in the proof of Theorem 5.4

$$a(x) \le |x|^n\, v(1/|x|).$$

Therefore by Theorem 5.12 we have for such $v$ and $1<p\le 2$

$$\int_{\mathbb{R}^n} |\hat{f}(x)|^p |x|^{n(p-2)} v(1/|x|)^{p-1}\, dx \le C \int_{\mathbb{R}^n} |f(x)|^p v(x)^{p-1}\, dx$$

whenever $t^{1-n}v(t)\in A_2$.

As observed in the introduction, the

characterization of weights for which weighted Fourier inequalities such as those above hold is still open. (For a discussion of these questions see also [7] and [8].) Kerman and Sawyer [47], [32] however, proved that if w and v are weights on $\mathbb{R}^2$ which are decreasing, respectively, increasing in each variable separately and satisfy certain additional conditions then

$$\int_{\mathbb{R}^2} |\hat{f}(x)|^2 w(x)\, dx \le C \int_{\mathbb{R}^2} |f(x)|^2 v(x)\, dx.$$

Basic to this estimate is the two dimensional extension of an inequality of Jodeit and Torchinsky [30], namely

$$\int_0^s \int_0^t |\hat{f}(x,y)|^2\, dx\, dy \le C \int_0^s \int_0^t \left( \int_0^{1/x} \int_0^{1/y} \tilde{f}(\xi,\eta)\, d\xi\, d\eta \right)^2 dx\, dy$$

where $\tilde{f}(\xi,\eta)$ is the rearrangement of f first with respect to the first variable and then with respect to the second.

We now carry out the n-dimensional generalization of these inequalities and give a Hausdorff-Young inequality.

As before we denote $x \in \mathbb{R}^n$ by $x=(x_1,x_2,\ldots,x_n)$, $1/x=(1/x_1,\ldots,1/x_n)$ and

$$\int_0^x g(y)\,dy = \int_0^{x_1} \cdots \int_0^{x_n} g(y_1,y_2,\ldots,y_n)\,dy_n\ldots dy_1.$$

Recall that if $g^*(t)$ is the rearrangement of $|g(x)|$, $x\in\mathbb{R}$, then the symmetric rearrangement of $|g(x)|$, $x\in\mathbb{R}$, is $g^{\circledast}(t)=g^*(2t)$, $t>0$, extended as an even function.

DEFINITION 5.13. _The symmetric rearrangement_ $R_j$ _with respect to the_ $j^{th}$ _variable of a function_ $f(x)$, $x\in\mathbb{R}^n$ _is defined by_

$$(R_jf)(x) = [f(x_1,\ldots,x_{j-1},\cdot,x_{j+1},\ldots,x_n)]^{\circledast}(x_j), \qquad 1\le j\le n.$$

_Then_ $\tilde{f}$, _the rearrangement of_ $f$ _with respect to each variable separately, is defined by_

$$\tilde{f}(x) = (R_nR_{n-1}\ldots R_1f)(x).$$

Note that in $\mathbb{R}^2$, $\tilde{f}(x,y)=(R_2R_1f)(x,y)=R_2[(R_1f)(x,y)]=R_2(f_y)^{\circledast}(x)=(R_2F_x)(y)=F_x^{\circledast}(y)$, where $F_x(y)=(f_y)^{\circledast}(x)$. To simplify the notation we write the Fourier transform of a function $g$ on $\mathbb{R}^n$ by

$$\hat{g}(x) = \int_{\mathbb{R}^n} e^{-ix\cdot y}g(y)\,dy, \qquad x\in\mathbb{R}^n,$$

whenever the integral exists.

LEMMA 5.14. *Let* $D_s(x)=\prod_{j=1}^{n}\chi_{(-s_j,s_j)}(x_j)/(2s_j)$, *then there is an* $\alpha>0$, *such that* $|\hat{D}_{1/s}(x)|>1/2$ *for all* $x\in R^n$ *with* $x_j/s_j\in(-\alpha,\alpha)$, $j=1,2,\dots,n$.

*Proof*. Since $D_1=D_{(1,1,\dots,1)}$ is bounded and

$$\hat{D}_1(0) = \int_{\mathbb{R}^n} D_1(t)\,dt = \prod_{j=1}^{n}(1/2s_j)\int_{-s_j}^{s_j} dt_j = 1$$

the continuity of $\hat{D}_1$ implies that there exists $\alpha>0$, such that $|\hat{D}_1(x)|>1/2$, whenever $-\alpha<x_j<\alpha$, $j=1,2,\dots,n$. Now

$$\hat{D}_{1/s}(x) = \prod_{j=1}^{n}\int_{\mathbb{R}} e^{-ix_jt_j}[(s_j/2)\chi_{(-1/s_j,1/s_j)}(t_j)]\,dt_j$$

$$= (1/2^n)\prod_{j=1}^{n}\int_{-1}^{1} e^{-ix_jy_j/s_j}\,dy_j \qquad (y_j = s_jt_j)$$

$$= \int_{\mathbb{R}^n} e^{-i(x_1/s_1,\dots,x_n/s_n)\cdot y}\,D_1(y)\,dy = \hat{D}_1(x_1/s_1,\dots,x_n/s_n)$$

so that $|\hat{D}_{1/s}(x)|>1/2$ whenever $-\alpha<x_j/s_j<\alpha$, $j=1,2,\dots,n$.

LEMMA 5.15. *Suppose* a, b *and* c *are non-negative functions on* $R^n$, *then*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} a(x-y)b(x)c(y)\,dx\,dy \le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \tilde{a}(x-y)\tilde{b}(x)\tilde{c}(y)\,dx\,dy,$$

where $\tilde{f}$ denotes the rearrangement of Definition 5.13.

Proof. If we show that

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} a(x-y)b(x)c(y)\,dx\,dy \le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (R_ja)(x-y)(R_jb)(x)(R_jc)(y)\,dx\,dy$$

for any $j=1,2,\ldots,n$, then the result follows by induction. Now by F. Riesz [25, Theorem 379]

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} a(x-y)b(x)c(y)\,dx\,dy$$

$$= \int_{\mathbb{R}^2}\cdots\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2} a(x-y)b(x)c(y)\,dx_j\,dy_j\right)(dx\,dy)^{(j)}$$

$$\le \int_{\mathbb{R}^2}\cdots\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2} (R_ja)(x-y)(R_jb)(x)(R_jc)(y)\,dx_j\,dy_j\right)(dx\,dy)^{(j)}$$

$$= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (R_ja)(x-y)(R_jb)(x)(R_jc)(y)\,dx\,dy$$

where $(dx\,dy)^{(j)}=dx_1dy_1\ldots dx_{j-1}dy_{j-1}dx_{j+1}dy_{j+1}\ldots dx_ndy_n$, so the result follows.

LEMMA 5.16. For any integer k, $0\le k\le n$

$$(5.12)\quad \int_0^{s_1}\cdots\int_0^{s_k}\int_{s_{k+1}}^{\infty}\cdots\int_{s_n}^{\infty}(D_s*\tilde{f})(x)^2\,dx \le C\int_0^{1/s}\left(\int_0^{1/x}\tilde{f}(u)\,du\right)^2 dx.$$

(If k=0 or k=n the integral on the left of (5.12) is not "mixed" and taken to be of the same form.)

Proof. First observe that

$$(D_s*\tilde{f})(x) = \int_{\mathbb{R}^n} D_s(u)\tilde{f}(x-u)\,du$$

$$= [C/(s_1s_2\ldots s_n)]\int_{-s_1}^{s_1}\cdots\int_{-s_n}^{s_n}\tilde{f}(x-u)\,du = [C/(s_1s_2\ldots s_n)]\int_{x-s}^{x+s}\tilde{f}(u)\,du.$$

Since $\tilde{f}$ is even and decreasing on $(0,\infty)$ in each variable separately

$$(1/2s_j)\int_{x_j-s_j}^{x_j+s_j}\tilde{f}(u)\,du_j \le (1/2s_j)\int_{-s_j}^{s_j}\tilde{f}(u)\,du_j = (1/s_j)\int_0^{s_j}\tilde{f}(u)\,du_j.$$

Also averages of decreasing functions decrease, so that for $x_j>s_j$

$$(1/2s_j)\int_{x_j-s_j}^{x_j+s_j}\tilde{f}(u)\,du_j \le (1/x_j)\int_0^{x_j}\tilde{f}(u)\,du_j.$$

Fix k, then using the first inequality if $j\le k$ and the second if $j>k$, we have

$$\int_0^{s_1}\cdots\int_0^{s_k}\int_{s_{k+1}}^{\infty}\cdots\int_{s_n}^{\infty}(D_s*\tilde{f})(x)^2\,dx$$

$$\le C\int_0^{s_1}\cdots\int_0^{s_k}\int_{s_{k+1}}^{\infty}\cdots\int_{s_n}^{\infty}\Big((\prod_{j=1}^{k}s_j)^{-1}(\prod_{j=k+1}^{n}x_j)^{-1}\times$$

$$\int_0^{s_1}\cdots\int_0^{s_k}\int_0^{x_{k+1}}\cdots\int_0^{x_n}\tilde{f}(u)\,du\Big)^2\,dx.$$

Replacing $x_j$ by $1/x_j$ for $j=k+1,\ldots,n$ this becomes

$$C\int_0^{s_1}\cdots\int_0^{s_k}\int_0^{1/s_{k+1}}\cdots\int_0^{1/s_n}\Big((\prod_{j=1}^{k}s_j)^{-1}\int_0^{s_1}\cdots\int_0^{s_k}\int_0^{1/x_{k+1}}\cdots\int_0^{1/x_k}\tilde{f}(u)\,du\Big)^2\,dx$$

$$= C\int_0^{1/s_{k+1}}\cdots\int_0^{1/s_n}(\prod_{j=1}^{k}s_j)^{-1}\Big(\int_0^{1/x_{k+1}}\cdots\int_0^{1/x_n}\Big(\int_0^{s_1}\cdots\times$$

$$\int_0^{s_k}\tilde{f}(u)\,du_k\ldots du_1\Big)\,du_n\ldots du_{k+1}\Big)^2\,dx_n\ldots dx_{k+1}$$

$$= C\int_0^{1/s}\Big(\int_0^{1/x_{k+1}}\cdots\int_0^{1/x_n}\Big(\int_0^{s_1}\cdots\int_0^{s_k}\tilde{f}(u)\,du_k\ldots du_1\Big)\,du_n\ldots du_{k+1}\Big)^2\,dx$$

$$\le C\int_0^{1/s}\Big(\int_0^{1/x}\tilde{f}(u)\,du\Big)^2\,dx$$

since for $j\le k$, $s_j\le 1/x_j$. Note that we used the facts

$$\int_0^{1/s_1}\cdots\int_0^{1/s_k}dx_k\ldots dx_1 = (\prod_{j=1}^{k}s_j)^{-1}\text{ and }\int_0^{s_1}\cdots\int_0^{s_k}dx_k\ldots dx_1 = \prod_{j=1}^{k}s_j.$$

PROPOSITION 5.17. _If_ $s=(s_1,s_2,\ldots,s_n)$, $s_j>0$, $j=1,2,\ldots,n$ _and_ $f\in L^1(\mathbb{R}^n)$, _then_

$$\int_0^{s}|\hat{f}(x)|^2\,dx \le C\int_0^{s}\Big(\int_0^{1/y}\tilde{f}(u)\,du\Big)^2\,dy$$

*where $\tilde{f}$ is the rearrangement of f with respect to each variable separately.*

*Proof*. We proceed as in the corresponding two dimensional case given by Kerman and Sawyer [47]. Let $\alpha$ be chosen as in Lemma 5.14, then the well known convolution property of the Fourier transform, Plancherel's theorem and Lemma 5.15 yield

$$\int_0^{\alpha s} |\hat{f}(x)|^2\, dx \le C\int_0^{\alpha s} |\hat{D}_{1/s}(x)\hat{f}(x)|^2\, dx \le C\int_{\mathbb{R}^n} |(D_{1/s}*f)\hat{\ }(x)|^2\, dx$$

$$= C\int_{\mathbb{R}^n} |(D_{1/s}*f)(x)|^2\, dx = C\int_{\mathbb{R}^n}\left|\int_{\mathbb{R}^n} D_{1/s}(x-u)f(u)\, du\right|^2 dx$$

$$\le C\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} D_{1/s}(x-u)D_{1/s}(x-v)\,|f(u)|\,|f(v)|\, du\, dv\, dx$$

$$= C\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} D_{1/s}(x-u)D_{1/s}(x-v)\, dx\right)|f(u)|\,|f(v)|\, du\, dv$$

$$= C\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} L_s(u-v)\,|f(u)|\,|f(v)|\, du\, dv \le C\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \tilde{L}_s(u-v)\tilde{f}(u)\tilde{f}(v)\, du\, dv$$

where

$$L_s(u-v) = \int_{\mathbb{R}^n} D_{1/s}(y-(u-v))D_{1/s}(y)\, dy.$$

It is clear that $L_s$ is even and decreasing on $(0,\infty)$ in each variable separately so that $L_s = \tilde{L}_s$. Since all functions are positive we can reverse the argument so that

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} L_s(u-v)\tilde{f}(u)\tilde{f}(v)\,du\,dv = \int_{\mathbb{R}^n} |(D_{1/s}*\tilde{f})(x)|^2\,dx$$

$$= C\int_0^\infty \cdots \int_0^\infty |(D_{1/s}*\tilde{f})(x)|^2\,dx.$$

But this integral can be written as a sum of $2^n$ integrals each of the form

$$\int_0^{1/s_1} \cdots \int_0^{1/s_k} \int_{1/s_{k+1}}^\infty \cdots \int_{1/s_n}^\infty (D_{1/s}*\tilde{f})(x)^2\,dx$$

$k=0,1,2,\ldots n$, (again when $k=0$ or $k=n$ the integral is of the form $\int_{1/s}^\infty$ respectively $\int_0^{1/s}$, $1/s=(1/s_1,1/s_2,\ldots,1/s_n)$).
Therefore by Lemma 5.16

$$\int_0^{\alpha s} |\hat{f}(x)|^2\,dx \le C\int_0^s \left(\int_0^{1/x} \tilde{f}(u)\,du\right)^2 dx$$

and if $\alpha \ge 1$ the result follows. If $\alpha < 1$ then (with s replaced by $s/\alpha$) we have

$$\int_0^s |\hat{f}(x)|^2\,dx \le C\int_0^{s/\alpha} \left(\int_0^{1/x} \tilde{f}(u)\,du\right)^2 dx = C\int_0^s \left(\int_0^{\alpha/y} \tilde{f}(u)\,du\right)^2 dy \le C\int_0^s \left(\int_0^{1/y} \tilde{f}(u)\,du\right)^2 dy$$

where $y=\alpha x$ and $\alpha/y=\alpha(1/y_1,\ldots,1/y_n)$.

We now consider the $A_p$-class of weights associated with n-dimensional rectangles.

DEFINITION 5.18. a) *Let* J *denote the family of bounded n-dimensional intervals* $I=[a_1,b_1]\times\ldots\times[a_n,b_n]$. *The strong maximal function is defined by*

$$(M_s f)(x) = \sup_{x\in I\in J}(1/|I|)\int_I |f(y)|\,dy.$$

b) *A non-negative function* w *on* $\mathbb{R}^n$ *is said to belong to* $A_p^*$ *if*

$$\left((1/|I|)\int_I w(x)\,dx\right)^{1/p}\left((1/|I|)\int_I w(x)^{1-p'}\,dx\right)^{1/p'} \le C$$

*for all* $I\in J$.

We also need the following known result [16, Theorems 6.2, 6.5; pp. 453-6]:

THEOREM 5.19. a) $\|M_s f\|_{p,w} \le C\|f\|_{p,w}$ *if and only if* $w\in A_p^*$ $1<p<\infty$.

b) *If* $w\in A_p^*$ $p>1$ *then there exist constants* C *and* $\delta>0$ *such that the "reversed Hölder's inequality"*

$$(1/|I|)\int_I w(x)^{1+\delta}\,dx \le \left((C/|I|)\int_I w(x)\,dx\right)^{1+\delta}$$

*holds for all* $I \in J$.

Now since $u(x)=w(x)^{1-p'}$ satisfies $A_{p'}^*$, if $w \in A_p^*$ then applying the "reverse Hölder's inequality" to u we see that w satisfies $A_{p-\varepsilon}^*$ with $\varepsilon=(p-1)\delta/(1+\delta)$. Using this fact we can prove:

LEMMA 5.20. *If* $1<p<\infty$ *and* $w \in A_p^*$ *then there is a constant* $C_p$ *such that*

$$\int_{h_1}^{\infty}\cdots\int_{h_n}^{\infty}(x_1x_2\ldots x_n)^{-p}w(x)\,dx \le C_p(h_1h_2\ldots h_n)^{-p}\int_0^{h_1}\cdots\int_0^{h_n}w(x)\,dx.$$

*Proof*. The argument is the same as the proof of [29, Lemma 1].

We are now able to give the n-dimensional generalization of the Kerman-Sawyer result [47],[32] (and also Lemma 5.2).

THEOREM 5.21. *Suppose* $v$, *defined on* $\mathbb{R}^n$, *is even and non-decreasing in each variable separately. Let* $w(x)=v(1/x)$ *and* $(-1)^n(\partial^n w)/(\partial x_1 \ldots \partial x_n)$ *be non-negative and locally integrable. Then for all* $f \in L^1$

$$\int_{\mathbb{R}^n} |\hat{f}(x)|^2 v(1/x)\, dx \le C \int_{\mathbb{R}^n} |f(x)|^2 v(x)\, dx \tag{5.13}$$

*if and only if* $v \in A_2^*$.

*Proof*. We may assume $w(x)$ is compactly supported, for otherwise let $\theta \in C_0^\infty(\mathbb{R})$ such that $\theta(0)=1$, $0 \le \theta(t) \le 1$, and $\theta'(t) \le 0$ for $t \ge 0$. Let

$$w_\varepsilon(x_1, x_2, \ldots, x_n) = \theta(\varepsilon|x_1|)\ldots\theta(\varepsilon|x_n|)w(x_1, \ldots, x_n),$$

then $w_\varepsilon$ satisfies the conditions of the theorem and increases to $w$ as $\varepsilon \to 0$. Therefore if (5.13) holds with $w_\varepsilon$ then Fatou's lemma and the Lebesgue dominated convergence theorem give the result. We may further assume that $(\text{supp } f) \subseteq \mathbb{R}^n_+$, then integration by parts and Proposition 5.17 yield

$$\int_{R_+^n} |\hat{f}(x)|^2 v(1/x)\, dx = \int_{R_+^n} |\hat{f}(x)|^2 w(x)\, dx$$

$$= (-1)^n \int_{R_+^n} |\hat{f}(x)|^2 \left( \int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} (\partial^n w(t))/(\partial t_1 \ldots \partial t_n)\, dt \right) dx$$

$$= (-1)^n \int_{R_+^n} [(\partial^n w(t))/(\partial t_1 \ldots \partial t_n)] \left( \int_0^{t1} \cdots \int_0^{tn} |\hat{f}(x)|^2\, dx \right) dt$$

$$\leq (-1)^n c \int_{R_+^n} [(\partial^n w(t))/(\partial t_1 \ldots \partial t_n)] \int_0^t \left( \int_0^{1/y} \tilde{f}(u)\, du \right)^2 dy\, dt$$

$$= (-1)^n c \int_{R_+^n} \left( \int_0^{1/y} \tilde{f}(u)\, du \right)^2 \int_y^{\infty} [(\partial^n w(t))/(\partial t_1 \ldots \partial t_n)]\, dt$$

$$= c \int_{R_+^n} w(y) \left( \int_0^{1/y} \tilde{f}(u)\, du \right)^2 dy \qquad (y=1/x)$$

$$= c \int_{R_+^n} v(x) \left( (1/x_1 \ldots x_n) \int_0^x \tilde{f}(u)\, du \right)^2 dx.$$

But $v \in A_2^*$ and $(1/x_1 \ldots x_n) \int_0^x \tilde{f}(u)du \leq (M_s \tilde{f})(x)$ where $M_s$ is the strong maximal function of Definition 5.18. Therefore by Theorem 5.19 a)

$$\int_{R_+^n} v(x) \left( (1/x_1 \ldots x_n) \int_0^x \tilde{f}(u)\, du \right)^2 dx \leq C \int_{R_+^n} v(x) \tilde{f}(x)^2\, dx$$

which proves the sufficiency since the monotonicity of v implies $\int_{R_+^n} v(x)\tilde{f}(x)^2 dx \leq \int_{R_+^n} v(x)|f(x)|^2 dx$.

Conversely, suppose (5.13) holds and let $f(x) = v(x) \prod_{i=1}^n \chi_{(0,s_i)}(|x_i|)$ then f is even in each variable separately so that (5.13) takes the form (after reducing the range of integration on the left side)

$$\int_0^{1/s} v(1/x)\left(\int_0^s \left(\prod_{i=1}^n \cos(x_i y_i)\right) v(y)^{-1}\, dy\right)^2 dx \le C\int_0^s v(x)^{-1}\, dx.$$

Since $0<x_i<1/s_i$ and $0<y_i\le s_i$, we have $0<x_i y_i\le 1$ and $\prod_{i=1}^n \cos(x_i y_i)\ge(\cos(1))^n$. Therefore for $s=(s_1,\dots s_n)$, $s_i>0$,

$$\left(\int_0^{1/s} v(1/x)\, dx\right)\left(\int_0^s v(y)^{-1}\, dy\right) \le C.$$

The change of variable $x=1/y$ in the first integral yields

$$\left(\int_s^\infty v(y)\left(\prod_{i=1}^n y_i\right)^{-2} dy\right)\left(\int_0^s v(y)^{-1}\, dy\right) \le C.$$

Now we argue as in [8, Theorem 3.1] n times to obtain that $v\in A_2^*$.

It now is easy to deduce the Hausdorff-Young inequality corresponding to Theorem 5.7.

THEOREM 5.22. Let $v$ satisfy the conditions of Theorem 5.21 then

$$\left(\int_{\mathbb{R}^n} |\hat{f}(x)|^{p'} v(1/x)\, dx\right)^{1/p'} \le C\left(\int_{\mathbb{R}^n} |f(x)|^p v(x)^{p-1}\, dx\right)^{1/p}, \qquad 1<p\le 2,$$

*if and only if* $v \in A_2^*$.

*Proof*. Let f be simple, then $\|(f/v)^{\hat{}}\|_\infty \le \|f\|_{1,1/v}$ and by (5.13)

$$\int_{\mathbb{R}^n} |(f/v)^{\hat{}}(x)|^2 v(1/x)\,dx \le C\int_{\mathbb{R}^n} |f(x)|^2 v(x)^{-1}\,dx.$$

By the Riesz Thorin interpolation theorem we have

$$\left(\int_{\mathbb{R}^n} |(f/v)^{\hat{}}(x)|^{p'} v(1/x)\,dx\right)^{1/p'} \le C\left(\int_{\mathbb{R}^n} |f(x)|^p v(x)^{-1}\,dx\right)^{1/p}$$

$1<p\le 2$. Now replace f/v by f and the result holds for simple functions. The general case follows from the density of simple functions in $L_v^p$.

The converse follows along well established lines and is omitted.

## 3. REPRESENTATIONS IN BERGMAN SPACES

DEFINITION 5.23. *Let* w *be a non-negative, locally integrable function. We say* f *belongs to the weighted Bergman space* $L^q(H_w^p)$ *if* f(z) *is holomorphic in the* right half plane and

$$\int_0^\infty \left( \int_{-\infty}^\infty w(x+iy)\,|f(x+iy)|^p\,dy \right)^{q/p} dx < \infty \tag{5.14}$$

where $1<p<\infty$, $1<q<\infty$.

In order to prove the Laplace representation theory for functions in $L^q(H_w^p)$ we need the following results:

LEMMA 5.24. ([8, Proposition 3.3]) Let $w$ be even, non-decreasing on $(0,\infty)$. Then, for $1<p<\infty$, $w\in A_p$ if and only if $|x|^{p'-2}w(1/x)^{1-p'}\in A_{p'}$.

Recently Johnson and Neugebauer (personal communication) have shown that this result holds also without the monotonicity condition imposed on $w$.

LEMMA 5.25. ([17]) Let $u$ be a non-negative function in $C_0^\infty(0,1)$, such that

$$\int_0^1 u(x)\,dx = 1.$$

*If*

(5.15) $$U(z) = \int_0^1 e^{-zt} u(t)\, dt$$

*then* U *is an entire function and* $|U(z)| \le C_n(1+|z|)^{-n}$, $\mathrm{Re}(z) \ge 0$, $n=0,1,2,\ldots$; ($C_n$ *a constant*).

THEOREM 5.26. *If* $f \in L^q(H^p_w)$, $1<p\le q\le p'<\infty$ *and* $w^{q/p} \in A_{1+p/q'}$ *is continuous, non-decreasing on* $(0,\infty)$, *and* $w(z)=w(|z|)$, *then there exists a function* F, *such that*

$$f(z) = \int_0^\infty e^{-zt} F(t)\, dt, \qquad \mathrm{Re}(z)>0,$$

*with*

$$\int_0^\infty |F(t)|^q\, t^{q/p'-2}\, w(1/t)^{q/p}\, dt < \infty.$$

*Proof*. For $x \ge 0$ let

$$I_p(x) = \int_{-\infty}^{\infty} |f(x+iy)|^p\, w(x+iy)\, dy,$$

then $f \in L^q(H^p_w)$ implies that (5.14) holds and therefore $I_p(x)$ is finite a.e.. Moreover, there exists a sequence

$\{x_j\}_{j=1}^{\infty}$ with $x_j \to \infty$, such that $I_p(x_j) \to 0$ as $j \to \infty$. Now let

$$\Omega = \{x \in R^+ : I_p(x) < \infty\}$$

and define $f_\delta(z) = f(z+\delta)$ where $\delta \in \Omega$ and $\mathrm{Re}(z) \geq 0$. Let

$$F_\delta(t) = (1/2\pi i) \int_{-i\infty}^{i\infty} e^{tz} f_\delta(z) U(\delta z)\, dz,$$

where U is given by (5.15) and the integral is taken over the line $\mathrm{Re}(z)=0$. Clearly the integral exists, for by Hölder's inequality and Lemma 5.25

$$\left| \int_{-\infty}^{\infty} e^{ity} f_\delta(iy) w(\delta+iy)^{1/p-1/p}\, U(i\delta y)\, dy \right|$$

$$\leq C\left(\int_{-\infty}^{\infty} |f(\delta+iy)|^p w(\delta+iy)\, dy\right)^{1/p} \left(\int_{-\infty}^{\infty} w(\delta+iy)^{1-p'} (1+|\delta y|)^{-np'}\, dy\right)^{1/p'}$$

$$\leq C I_p(\delta) w(\delta)^{-1/p} \left(\int_{-\infty}^{\infty} (1+|\delta y|)^{-np'}\, dy\right)^{1/p'} < \infty.$$

We now claim that for $x \in R^+$ and $x+\delta \in \Omega$

(5.16) $$F_\delta(t) = (1/2\pi i) \int_{x-i\infty}^{x+i\infty} e^{tz} f_\delta(z) U(\delta z)\, dz.$$

To prove this, note that by Fubini's theorem and Hölder's inequality with $w_\delta(z) = w(\delta+z)$

$$\int_{-\infty}^{\infty} \int_{0}^{x} |f_\delta(\xi+iy)|^p w_\delta(\xi+iy)\, d\xi\, dy = \int_{0}^{x} \int_{-\infty}^{\infty} |f_\delta(\xi+iy)|^p w_\delta(\xi+iy)\, dy\, d\xi$$

$$\leq \left( \int_0^x \left( \int_{-\infty}^{\infty} |f_\delta(\xi+iy)|^p w_\delta(\xi+iy)\ dy \right)^{q/p} d\xi \right)^{p/q} x^{1-p/q} = J(x) < \infty.$$

Fubini's theorem, Hölder's inequality twice and Lemma 5.25 yield

$$\int_{-\infty}^{\infty} \int_0^x |f_\delta(\xi+iy)||U(\delta(\xi+iy))|\ d\xi\ dy$$

$$\leq \int_{-\infty}^{\infty} \left( \int_0^x |f_\delta(\xi+iy)|^p w_\delta(\xi+iy)\ d\xi \right)^{1/p} \times$$

$$\left( \int_0^x w_\delta(\xi+iy)^{1-p'} (1+|\delta(\xi+iy)|)^{-np'}\ d\xi \right)^{1/p'} dy$$

$$\leq C \left( \int_{-\infty}^{\infty} \int_0^x |f_\delta(\xi+iy)|^p\ w_\delta(\xi+iy)\ d\xi\ dy \right)^{1/p} \times$$

$$\left( \int_{-\infty}^{\infty} \int_0^x w_\delta(\xi+iy)^{1-p'} (1+|\delta(\xi+iy)|)^{-np'}\ d\xi\ dy \right)^{1/p'}$$

$$\leq CJ(x)^{1/p} \left( \left( \int_0^x w(\xi)^{1-p'}\ d\xi \right) \left( \int_{-\infty}^{\infty} (1+|\delta y|)^{-np'}\ dy \right) \right)^{1/p'} < \infty,$$

where we used the fact that $w^{q/p} \epsilon A_{1+q/p'}$. But now there exist sequences $\{a_k\}_{k=0}^{\infty}$, $\{b_k\}_{k=0}^{\infty}$ with $a_k \to -\infty$, $b_k \to +\infty$ as $k \to \infty$ such that

$$\lim_{k\to\infty} \int_0^x |f_\delta(\xi+ia_k)||U(\delta(\xi+ia_k))|\ d\xi = 0$$

and

$$\lim_{k\to\infty} \int_0^x |f_\delta(\xi+ib_k)||U(\delta(\xi+ib_k))|\ d\xi = 0.$$

Consider the rectangle $T: 0 \le \xi \le x$, $a_k \le y \le b_k$ and integrate the holomorphic function $z \to e^{tz} f_\delta(z) U(\delta z)$, $z = x+iy$ along the boundary of T. Then

$$0 = (1/2\pi i)\int_T e^{tz} f_\delta(z) U(\delta z)\, dz$$

$$= (1/2\pi i)\int_0^x e^{t(\xi+ia_k)} f_\delta(\xi+ia_k) U(\delta(\xi+ia_k))\, d\xi$$

$$+ (1/2\pi)\int_{a_k}^{b_k} e^{t(x+iy)} f_\delta(x+iy) U(\delta(x+iy))\, dy$$

$$+ (1/2\pi i)\int_x^0 e^{t(\xi+ib_k)} f_\delta(\xi+ib_k) U(\delta(\xi+ib_k))\, d\xi$$

$$+ (1/2\pi)\int_{b_k}^{a_k} e^{ity} f_\delta(iy) U(\delta(iy))\, dy.$$

As $k \to \infty$, the first and third summands converge to zero so that

$$F_\delta(t) = (1/2\pi)\int_{-\infty}^{\infty} e^{t(x+iy)} f_\delta(x+iy) U(\delta(x+iy))\, dy$$

which is (5.16).

Now by Hölder's inequality and Lemma 5.25

$$|F_\delta(t)| \le Ce^{tx}\left(\int_{-\infty}^{\infty} |f_\delta(x+iy)|^p w_\delta(x+iy)\, dy\right)^{1/p} \times$$

$$\left(\int_{-\infty}^{\infty} w_\delta(x+iy)^{1-p'} (1+|\delta(x+iy)|)^{-np'}\, dy\right)^{1/p'}$$

$$\le Ce^{tx} I_p(x+\delta)^{1/p}\left(\int_0^1 w(y)^{1-p'}\,dy + w(1)^{1-p'}\int_1^\infty (1+|\delta y|)^{-np'}\,dy\right)^{1/p'}.$$

Let $x=x_j$ such that $I_p(x_j+\delta)\to 0$ as $j\to\infty$, then $|F_\delta(t)|\le 0$ whenever $t\le 0$, i.e. $F_\delta(t)=0$ for $t\le 0$.

Now

$$(5.17)\qquad F_\delta(t)e^{-tx} = (1/2\pi)\int_{-\infty}^{\infty} e^{ity} f_\delta(x+iy)U(\delta(x+iy))\,dy,$$

so by Theorem 5.8 a)

$$\left(\int_0^\infty |F_\delta(t)|^q e^{-txq}|t|^{q/p'-1} w(1/t)^{q/p}\,dt\right)^{1/q}$$

$$\le C\left(\int_{-\infty}^{\infty} |f_\delta(x+iy)|^p |U(\delta(x+iy))|^p w(y)\,dy\right)^{1/p} \le C\left(\int_{-\infty}^{\infty} |f_\delta(x+iy)|^p w_\delta(x+iy)\,dy\right)^{1/p}.$$

Raising this integral to the $q^{th}$-power and integrating yields

$$\int_0^\infty |F_\delta(t)|^q |t|^{q/p'-2} w(1/t)^{q/p}\,dt \le C\int_0^\infty \left(\int_{-\infty}^{\infty} |f(x+iy)|^p w(x+iy)\,dy\right)^{q/p} dx < \infty.$$

Write $H_\delta(t)=t^{1/p'-2/q} w(1/t)^{1/p} F_\delta(t)$, then this shows that $H_\delta\in L^q(0,\infty)$ so by weak compactness there is a subsequence $\{\delta_k\}_{k=1}^\infty$ such that $\delta_k\to 0$ as $k\to\infty$, and for every $G\in L^{q'}(0,\infty)$

$$(5.18)\quad \lim_{k\to\infty}\int_0^\infty H_{\delta_k}(t)G(t)\,dt = \int_0^\infty H(t)G(t)\,dt = \int_0^\infty G(t)t^{1/p'-2/q} w(1/t)^{1/p} F(t)\,dt.$$

Now inverting (5.17) and replacing $\delta$ by $\delta_k$ we obtain

$$f_{\delta_k}(x+iy)U(\delta_k(x+iy)) = \int_0^\infty e^{-t(x+iy)}F_{\delta_k}(t)\,dt = \int_0^\infty e^{-t(x+iy)}H_{\delta_k}(t)t^{2/q-1/p'}w(1/t)^{-1/p}\,dt$$

Let $e^{-t(x+iy)}t^{2/q-1/p'}w(1/t)^{-1/p}=G(t)$, then

$$f(x+iy) = \lim_{k\to\infty} f_{\delta_k}(x+iy)U(\delta_k(x+iy)) = \lim_{k\to\infty}\int_0^\infty H_{\delta_k}(t)G(t)\,dt = \int_0^\infty e^{-t(x+iy)}F(t)\,dt$$

which proves the theorem, provided we justify the above Fourier inversion of (5.17) and show $G\in L^{q'}(0,\infty)$.

To justify the Fourier inversion we must show that $F_\delta(t)e^{-tx}\in L^1(0,\infty)$. By Hölder's inequality

$$\int_0^\infty |F_\delta(t)|e^{-tx}\,dt$$

$$\leq \left(\int_0^\infty |F_\delta(t)|^q t^{q/p'-2}w(1/t)^{q/p}\,dt\right)^{1/q}\left(\int_0^\infty e^{-xtq'}t^{2q'/q-q'/p'}w(1/t)^{-q'/p}\,dt\right)^{1/q'}.$$

The first integral, as we observed above, is finite and the second is written in the form

$$(5.19)\qquad \left(\int_0^1+\int_1^\infty\right)e^{-xtq'}t^{2q'/q-q'/p'}w(1/t)^{-q'/p}\,dt = J_1 + J_2$$

respectively. Since $q\leq p'$

$$J_1 \leq w(1)^{-q'/p}\int_0^1 t^{2q'/q-q'/p'}\,dt < \infty.$$

and Lemma 5.24 shows that $|x|^{q'/p-1}w(1/x)^{-qq'/pp}\epsilon A_{1+q'/p}$.

Therefore

$$J_2 = \int_1^\infty e^{-xtq'}w(1/t)^{(q/p-1)q'/p}t^{2q'/q-q'/p'+2}[(w(1/t)^{-qq'/pp}t^{q'/p-1})/t^{q'/p+1}]\,dt$$

$$\leq w(1)^{(q/p-1)q'/p}C\int_1^\infty (w(1/t)^{-qq'/pp}t^{q'/p-1})/t^{q'/p+1}\,dt$$

$$\leq C\int_0^1 w(1/t)^{-qq'/p^2}t^{q'/p-1}\,dt < \infty$$

by [29, Lemma 1].

This shows that $F_\delta(t)e^{-tx}\epsilon L^1(0,\infty)$ and also $G\epsilon L^{q'}(0,\infty)$. The converse result is the following:

THEOREM 5.27. _Suppose_

$$f(z) = \int_0^\infty e^{-zt}F(t)\,dt, \qquad \mathrm{Re}(z)>0,$$

_where_

$$(5.20) \qquad \int_0^\infty |F(t)|^{q'}|t|^{q'/p-2}w(1/t)^{-q'/p}\,dt < \infty$$

$1<p\leq q\leq p'<\infty$, w _non-decreasing on_ $(0,\infty)$, $w(z)=w(|z|)$, _and_ $w^{1-p'}\epsilon A_{1+p'/q}$. _Then_ $f\epsilon L^{q'}(H^{p'}_{w^{1-p'}})$.

Proof. If f has the Laplace transform representation, then by Theorem 5.8 b)

$$\left(\int_{-\infty}^{\infty}|f(x+iy)|^{p'}w(x+iy)^{1-p'}\,dy\right)^{q'/p'} \le C\int_0^{\infty}e^{-xtq'}|F(t)|^{q'}t^{q'/p-1}w(1/t)^{-q'/p}\,dt$$

and the result follows on integrating with respect to x.

It only remains to show that f is well defined if (5.20) holds. By Hölder's inequality

$$\int_0^{\infty}e^{-tx}|F(t)|\,dt$$

$$\le \left(\int_0^{\infty}|F(t)|^{q'}t^{q'/p-2}w(1/t)^{-q'/p}\,dt\right)^{1/q'}\left(\int_0^{\infty}e^{-xtq}t^{2q/q'-q/p}w(1/t)^{q/p}\,dt\right)^{1/q}$$

$$= C\left(\left(\int_0^1+\int_1^{\infty}\right)e^{-xtq}t^{2q/q'-q/p}w(1/t)^{q/p}\,dt\right)^{1/q} = C[J_1+J_2]^{1/q}$$

respectively. Clearly

$$J_2 \le w(1)^{q/p}\int_1^{\infty}e^{-txq}t^{2q/q'-q/p}\,dt < \infty$$

and

$$J_1 \le \int_0^1 t^{2(q-1)-q/p}w(1/t)^{q/p}\,dt.$$

Now $w^{1-p'}\in A_{1+p'/q}$ if and only if $w^{q/p}\in A_{1+q/p'}$ and by Lemma 5.24, $w^{q/p}\in A_{1+q/p'}$ if and only if $w(1/x)^{q/p}|x|^{q/p'-1}\in A_{1+q/p'}$.

Therefore

$$J_1 \le \int_0^1 t^{2q-2-q/p+1-q/p'} t^{q/p'-1} w(1/t)^{q/p}\,dt$$

$$= \int_0^1 t^{q-1} t^{q/p'-1} w(1/t)^{q/p}\,dt \le \int_0^1 t^{q/p'-1} w(1/t)^{q/p}\,dt < \infty.$$

REMARK. If $q=p'$ and $w=1$ in Theorem 5.26 we obtain a result of Genchev [17]. If $q=p$ the theorem takes the following form:

COROLLARY 5.28. *If* $f \in L^p(H^p_w)$, $1<p\le 2$ *and* $w \in A_p$ *is continuous, non-decreasing on* $(0,\infty)$, *and* $w(z)=w(|z|)$, *then there exists* $F$ *with*

$$\int_0^\infty |F(t)|^p t^{p-3} w(1/t)\,dt < \infty$$

*such that*

$$f(z) = \int_0^\infty e^{-zt} F(t)\,dt, \qquad \mathrm{Re}(z)>0.$$

For $p=q=2$ Theorems 5.26 and 5.27 yield the following characterization:

COROLLARY 5.29. *Let* $w$ *be continuous, non-decreasing on* $(0,\infty)$, $w(z)=w(|z|)$, *and* $w\in A_2$. *A function* $f\in L^2(H_w^2)$ *if and only if*

$$f(z) = \int_0^\infty e^{-zt}F(t)\,dt$$

*with* $F$ *satisfying*

$$\int_0^\infty |F(t)|^2 w(1/t)\,dt/t < \infty.$$

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