STRONGLY HOMOGENEOUS SPACES

by

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STRONGLY HOMOGENEOUS SPACES

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INTRODUCTION

In recent years many authors have called topological spaces with certain properties "strongly homogeneous", and have investigated the group of homeomorphisms of such spaces. Thus far, all the definitions have one disadvantage, namely that none of them implies homogeneity. In this thesis we give a definition of strong homogeneity which, in the metric case at least, implies homogeneity. We relate this definition to the previous definitions, and show that it includes most of the spaces studied by previous authors. Moreover this definition is a local as well as a global definition; it applies to any open subset of a topological space X rather than just to X itself, as is the case with the original definitions. For some results, for example Theorem 4.5, it suffices that the space contain a strongly homogeneous subset.

This thesis is divided into four sections. Section one includes the definition of strong homogeneity and the proof that every strongly homogeneous subset of a metric space is homogeneous. We also show that strong homogeneity is preserved by homeomorphisms. In section two it is proved that every strongly homogeneous metric space is a strong local homogeneity (see Ford [6]). We prove that the unit ball in a normed linear space is strongly homogeneous and deduce that every normed linear space is strongly homogeneous. We also give an example of a space which is a strong local homogeneity, but is not strongly homogeneous. We then show that every S.H. regular perfect space is a Galois space. (see [4]) and is representable (see [5]), and deduce that every S.H. metric space is reasonable. (See [5]). In section three we show that the definition of strong homogeneity embraces that of strong local setwise homogeneity (in the sense of Brechner [2]), so that the group of homeomorphisms of a strongly homogeneous space is at least one dimensional. We give an example of a space which is strongly locally setwise homogeneous, but not strongly homogeneous. Section four deals with the group of ergodic homeomorphisms of a strongly homogeneous space, and a similar result to that of Sears [11] is proved. We also prove that the Cantor set is strongly homogeneous.

The following notation is used throughout the text. If X is any topological space, then H(X) denotes the group of all self-homeomorphisms of X. If U is an open subset of X and $f \in H(X)$ is the identity map on the complement of U, we write $f \in U'$. We denote the complement of a subset U of X by CU, and its closure by $\bar{U}$. In any metric space (X,d) we denote the d-ball of radius ε about $x \in X$ by $N(x,\varepsilon)$ and we write $\delta(U)$ for the diameter of a set U, that is

$$\delta(U) = \sup\{d(a,b) : a,b \in U\}.$$

We write $\subset$ for "is a subset of" and $\subsetneq$ for "is a proper subset of". Throughout the text R^n is the n dimensional Euclidean space with the usual topology. Finally, the numbers appearing in square brackets throughout the text refer to the bibliography which appears at the end of the thesis.

§1. A STRONG FORM OF HOMOGENEITY.

In this section we prove that a strongly homogeneous set in a metric space is homogeneous.

Definition 1.1. Let (X,τ) be a topological space and let U be an open set in X. U is said to be $\mathcal{B}$-*freely open* if there is a basis $\mathcal{B}$ for τ with the property that if $V,W \in \mathcal{B}$ are proper subsets of U with $\bar{W} \cap (\bar{U} - U) = \phi$, then for each $x \in V$ there is a homeomorphism $h \in H(X)$ such that

(i) $x \in h(W)$ and $h(\bar{W}) \subseteq V$.
(ii) $h \in U'$.

Definition 1.2. Let U be an open set in a topological space (X,τ). A basis $\mathcal{B}$ for τ is said to be a *free basis for* U if every element of $\{B \in \mathcal{B} : B \subset U\}$ is $\mathcal{B}$-freely open.

Definition 1.3. Let U be an open set in a topological space X. U is said to be *strongly homogeneous* (S.H.) if U is $\mathcal{B}$-freely open for some free basis $\mathcal{B}$ for U. The space X is S.H. if it is S.H. as a subset of itself.

Note that every element of a free basis is S.H. A simple example of a free basis is the basis of open intervals for the Euclidean topology on the space of real numbers. [Fig. I].

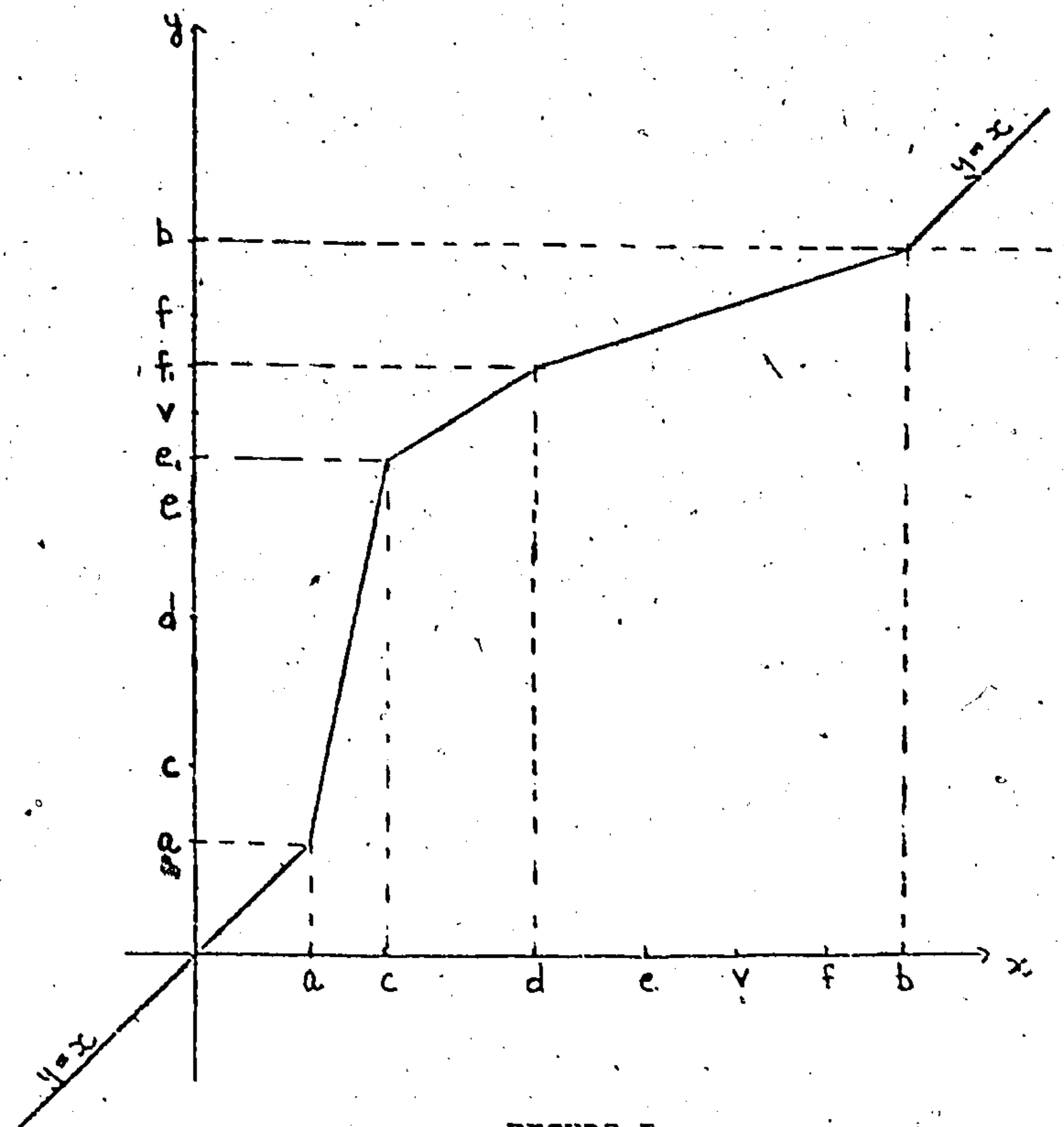

FIGURE I.

(a,b) is an open interval and (c,d) and (e,f) are subintervals of (a,b) with $c \neq a$, $d \neq b$ and $v \in (e,f)$. Choose e_1, f_1 such that $e < e_1 < f_1 < f$ and $v \in (e_1,f_1)$. The required homeomorphism is shown above.

Lemma 1.4. *Let h be a homeomorphism from a topological space X onto a topological space Y. Then*

(1) *U is B-freely open in X if and only if $h(U)$ is $h(B)$-freely open in Y.*

(2) *B is a free basis for U if and only if $h(B)$ is a free basis for $h(U)$.*

(3) *U is a S.H. set if and only if $h(U)$ is a S.H. set.*

Proof. (1) Assume that U is B-freely open. Then $h(B)$ is a basis for the topology on Y. Let $V, W \in h(B)$ be proper subsets of $h(U)$ with $\bar{W} \cap [\overline{h(U)} - h(U)] = \phi$. Take $v \in V$. Then $h^{-1}(V)$ and $h^{-1}(W)$ are elements of B and are proper subsets of U. Also $\overline{h^{-1}(W)} \cap (\bar{U} - U) = \phi$ and $h^{-1}(v) \in h^{-1}(V)$. Since U is B-freely open there is a homeomorphism $g \in H(X)$ such that

(i) $h^{-1}(v) \in g(h^{-1}(W))$ and $\overline{g(h^{-1}(W))} \subsetneq h^{-1}(V)$

(ii) $g \in U'$.

Let $f = hgh^{-1}$. Then $f \in H(Y)$ and (i) ensures that $v \in f(W)$ and $f(\bar{W}) \subsetneq V$. Also if $x \notin h(U)$, then $h^{-1}(x) \notin U$, so $gh^{-1}(x) = h^{-1}(x)$, and thus $f(x) = hgh^{-1}(x) = x$. Thus $f \in (h(U))'$, and $h(U)$ is $h(B)$-freely open. Similarly if $h(U)$ is $h(B)$-freely open, then using h^{-1}, U is B-freely open.

(2) follows by a similar proof to that of (1);

(3) follows from (1) and (2).

Theorem 1.5. *If U is a S.H. set in a metric space X, then U is homogeneous.*

Proof. Let $\mathcal{B}$ be a free basis such that U is $\mathcal{B}$-freely open. Let a,b be any two elements of U. We show that there is a homeomorphism $F \in H(X)$ with $F(a) = b$. Let $W_1, Y_1 \in \mathcal{B}$ be proper subsets of U with $a \in W_1$, $b \in Y_1$, $\bar{W}_1 \cap (\bar{U} - U) = \phi$, $\delta(W_1) < 1$ and $\delta(Y_1) < 1$. Then there is a homeomorphism $h_1 \in H(X)$ such that

(i) $b \in h_1(W_1)$ and $h_1(\bar{W}_1) \subsetneq Y_1$

(ii) $h_1 \in U'$.

Let $V_1 = h_1(W_1)$, $a_1 = h_1(a)$, $\mathcal{B}_1 = h_1(\mathcal{B})$. Then by lemma 1.4, $\mathcal{B}_1$ is a free basis. Let $W_2, Y_2 \in \mathcal{B}_1$ be proper subsets of V_1 with $a_1 \in W_2$, $b \in Y_2$, $\bar{W}_2 \cap (\bar{V}_1 - V_1) = \phi$, $\delta(h_1^{-1} W_2) < \frac{1}{2}$, $\delta(Y_2) < \frac{1}{2}$. Since $V_1 \in \mathcal{B}_1$, V_1 is $\mathcal{B}_1$-freely open, so there is a homeomorphism $h_2 \in H(X)$ such that

(i) $b \in h_2(W_2)$ and $h_2(\bar{W}_2) \subsetneq Y_2$.

(ii) $h_2 \in V_1'$.

Now let $V_2 = h_2(W_2)$, $a_2 = h_2(a_1)$, $\mathcal{B}_2 = h_2(\mathcal{B}_1)$ and proceed as above.

Continuing in this way we get sequences (W_n) and (V_n) of S.H. sets, and homeomorphisms $h_n \in H(X)$ such that for all $n = 1,2,3,\dots$ we have

(a) $\overline{V}_{n+1} \subsetneq V_n$

(b) $h_n(W_n) = V_n$ and $\overline{W}_n \cap (\overline{V}_{n-1} \setminus V_{n-1}) = \phi$ $(V_o = U)$

(c) $\delta(h_1^{-1} h_2^{-1} \dots h_n^{-1} W_{n+1}) < \frac{1}{n+1}$ and $\delta(W_1) < 1$

(d) $\delta(V_n) \le \frac{1}{n}$

(e) $h_n \in V'_{n-1}$ $(V_o = U)$

(f) $a_n = h_n h_{n-1} \dots h_2 h_1(a) \in W_{n+1}$.

Let $F_n = h_n h_{n-1} \dots h_1$. Define the map F by

$$F(x) = \lim_{n \to \infty} F_n(x) \quad \text{for all } x \in X.$$

We show that F exists for all $x \in X$, that $F \in H(X)$ and that $F(a) = b$.

Take $x \in X$. If $x \notin U$, then $F(x) = x$. If $F_n(x) \notin V_n$ for some n, then $F(x) = F_n(x)$ by (a) and (c). If $F_n(x) \in V_n$ for all n, then $h_n h_{n-1} \dots h_1(x) \in h_n(W_n)$ for all n. So $F_{n-1}(x) = h_{n-1} h_{n-2} \dots h_1(x) \in W_n$ for all n, that is, $x \in F_{n-1}^{-1}(W_n)$ for all n. Thus by (c), $d(x,a) < \frac{1}{n}$

for all n, and so $x = a$. Also $F_n(a) = a_n \in V_n$ for all n, so by (d) $d(F_n(a),b) < \frac{1}{n}$ for all n, which means that $F(a) = b$.

Thus $F(x) = \lim_{n\to\infty} F_n(x)$ exists for all $x \in X$. (1)

Also by (a) and the proof of (1), F is one to one. (2)

To show that F is onto, take $x \in X$. If $x \in CU$, then $F(x) = x$. If $x \in V_n$ for all n, then by (d), $d(x,b) < \frac{1}{n}$ for all n, so $x = b$ and $F(a) = x$. Otherwise $x \notin V_n$ for some n. Say $x \notin V_N$. Now for all n, there is an element x_n of X such that $F_n(x_n) = x$. Thus in particular $F_N(x_N) \notin V_N$, and so $F(x_N) = F_N(x_N) = x$. Thus F is onto. (3)

We now show that F is continuous. Take $x \in X$. There are two cases to consider.

(i) If $x \neq a$, then there is a positive integer m such that $F_m(x) \notin V_m$, so that $F(x) = F_k(x)$ for all $k \geq m$. Now $\bar{W}_{m+1} \cap (\bar{V}_m - V_m) = \phi$, so if $F_m(x) \in \bar{W}_{m+1}$, then $F(x) = F_m(x) \notin \bar{V}_m$. If $F_m(x) \notin \bar{W}_{m+1}$, then $F_{m+1}(x) \notin h_{m+1}(\bar{W}_{m+1}) = \bar{V}_{m+1}$, so $F(x) \notin \bar{V}_{m+1}$. Thus in both cases, $F(x) = F_{m+1}(x) \notin \bar{V}_{m+1}$, which means that $x \in F_{m+1}^{-1}(X - \bar{V}_{m+1})$, which is open. Thus there is a $\delta_1(x) > 0$ such that $N(x,\delta_1) \subset F_{m+1}^{-1}(X - \bar{V}_{m+1})$. So if $d(x,y) < \delta_1$, then $F(y) = F_{m+1}(y)$. Now let $\varepsilon > 0$ be given. Then there is a $\delta_2(x,\varepsilon)$ such that if $d(x,y) < \delta_2$ then $d(F_{m+1}(y), F_{m+1}(x)) < \varepsilon$. Let $\delta = \min\{\delta_1,\delta_2\}$.

Then if $d(x,y) < \delta$, $F(y) = F_{m+1}(y)$ and so $d(F(x),F(y)) < \varepsilon$. Thus F is continuous at x for $x \neq a$.

(ii) Suppose $x = a$. Let $\varepsilon > 0$ be given and let N be a positive integer with $0 < \frac{1}{N} < \varepsilon$. Since F_{N-1} is continuous and $a \in F_{N-1}^{-1}(W_N)$ by (f), there is a $\delta(a) > 0$ such that $N(a,\delta) \subset F_{N-1}^{-1}(W_N)$. So if $d(y,a) < \delta$, then $y \in F_{N-1}^{-1}(W_N)$, and so $F_N(y) \in h_N(W_N) = V_N$, which means that $F(y) \in V_N$. Thus $d(F(y),F(a)) = d(F(y),b) < \frac{1}{N} < \varepsilon$. So F is continuous at $x = a$.

Thus F is continuous. (4)

It remains to show that F^{-1} is continuous. Take $x \in X$. There are again two cases to consider.

(i) If $x \neq b$, then there is an element z of X such that $F(z) = x$ and $z \neq a$. Now $F(z) = F_N(z) \notin \bar{V}_N$ for some positive integer N so that $F^{-1}(x) = F_N^{-1}(x)$. Thus $x \in X - \bar{V}_N$ which is open. Let $\delta_3(x) > 0$ be such that $N(x,\delta_3) \subset X - \bar{V}_N$. Thus if $y \in N(x,\delta_3)$, then $F^{-1}(y) = F_N^{-1}(y)$. Now let $\varepsilon > 0$ be given. Since F_N^{-1} is continuous, there is a $\delta_4(x,\varepsilon) > 0$ such that if $d(x,y) < \delta_4$, then $d(F_N^{-1}(x), F_N^{-1}(y)) < \varepsilon$. Let $\delta = \min\{\delta_3,\delta_4\}$. Then if $d(x,y) < \delta$, then $d(F^{-1}(x),F^{-1}(y)) = d(F_N^{-1}(x),F_N^{-1}(y)) < \varepsilon$. So F^{-1} is continuous at x for $x \neq b$.

(ii) Suppose $x = b$, so that $F^{-1}(x) = a$. Let $\varepsilon > 0$ be given and let N be a positive integer with $0 < \frac{1}{N} < \varepsilon$. Choose $\delta(b)$ such that $N(b,\delta) \subset V_N$. Then if $y \in N(b,\delta)$, then $F^{-1}(y) \in F^{-1}(V_N)$, so $F^{-1}(y) \in F_N^{-1}(V_N)$ since $F^{-1}(V_N) \subset F_N^{-1}(V_N)$. But $\delta(F_N^{-1}(V_N)) = \delta(F_N^{-1}(h_N(W_N))) = \delta(F_{N-1}^{-1}(W_N)) < \frac{1}{N}$ by (c). So if $d(b,y) < \delta$, then $d(F^{-1}(y),a) < \frac{1}{N} < \varepsilon$. Thus if $d(b,y) < \delta$ then $d(F^{-1}(b),F^{-1}(y)) < \varepsilon$. So F^{-1} is continuous at $x = b$.

Thus F^{-1} is continuous. (5)

Thus by equations (1) to (5), $F \in H(X)$ and $F(a) = b$. Thus U is a homogeneous set.

The following example, used by Ford [6], is an example of a compact metric space which is homogeneous but not S.H. Let C be the Cantor set and let S' be the one dimensional circle. Let X be the Cartesian product of C and S' with the usual product topology. Since both C and S' are compact, metric, homogeneous spaces, X is compact, metric and homogeneous. We shall therefore assume that X is embedded on the surface of a right circular cylinder in Euclidean 3-space with the metric topology. The space is evidently not S.H. since in order to move a point $x \in X$ to a neighbouring point on a different circle, we have to move the whole circle. We show in section four that the Cantor set is S.H. Since S^1 is S.H., the above example shows that the product of two S.H. spaces need not be S.H.

§2. A STRONGLY HOMOGENEOUS SPACE IS A STRONG LOCAL HOMOGENEITY AND A GALOIS SPACE.

The following definitions are due to Ford [6].

Definition 2.1. Let X be a completely regular Hausdorff space, and let G be any transitive subgroup of H(X). G is said to have a *reasonable topology over X* if G is a topological group under this topology, and the coset space G/C_x (left cosets) is homeomorphic to X under the map η: $G/C_x \to X$ defined by $\eta(gC_x) = g(x)$. [Where $C_x = \{g \in G: g(x) = x\}$].

Definition 2.2. Let X be a Hausdorff space. X is a *strong local homogeneity* (S.L.H.) if for any neighbourhood of any point x, there exists a subneighbourhood U(x) such that for each $z \in U(x)$ there is a homeomorphism g with

(i) $g(x) = z$.

(ii) $g \in (U(x))'$.

Ford [6] claimed to have proved that if X is a completely regular, Hausdorff S.L.H., and H(X) is a transitive group, then H(X) is reasonable over X under any topology induced by a uniform structure on X.

However Mostert [9] showed that Ford had in fact not proved this. Mostert proved the following:

Theorem 2.3 *Let X be a homogeneous, locally compact, Hausdorff S.L.H. Then*

(i) *the uniform structure on X obtained from its one point compactification induces on H(X) a reasonable topology over X,*

(ii) *the uniform structure on X obtained from its Stone-Čech compactification induces on H(X) a reasonable topology over X,*

(iii) *If X is connected, then for any uniformity U on X, the group of homeomorphisms uniformly continuous relative to U is reasonable over X.*

The result used in the proof of this theorem is that if G is a group of uniformly continuous homeomorphisms on a uniform space X, then G is a topological transformation group on X when given the uniform topology. (See [6])

Now suppose that X is any S.H. metric space. Then by Theorem 1.5, X is a S.L.H. and X is homogeneous. Thus from Theorem 2.3, we have:

Theorem 2.4. *Let X be a S.H. locally compact metric space. Then*

(i) *the uniform structure on X obtained from its one point compactification induces on $H(X)$ a reasonable topology over X,*

(ii) *the uniform structure on X obtained from its Stone-Čech compactification induces on $H(X)$ a reasonable topology over X,*

(iii) *if X is connected, then the group of uniformly continuous homeomorphisms (that is, homeomorphisms g such that g and g^{-1} are uniformly continuous) is reasonable over X.*

Ford proved that every normed linear space is a S.L.H. We extend this result by proving that every normed linear space is S.H. (all spaces are taken over the field of real numbers).

Theorem 2.5. *Let X be a normed linear space and let $S = \{x : ||x|| < 1\}$. Then S is a S.H. set.*

Proof. Define $f : X \to S$ by

$$f(x) = \frac{x}{1 + ||x||}$$

f is a homeomorphism from X onto S with inverse f^{-1}

given by

$$f^{-1}(x) = \frac{x}{1 - \|x\|}.$$

Let B be the usual basis for the norm topology on X.
Let V, $W \epsilon B$ be proper subsets of S with $\bar{W} \cap (\bar{S} - S) = \emptyset$ and take $v \epsilon V$. Say $W = N(b, \delta_1)$ and $U = N(v, \delta_2)$ where $\bar{U} \subsetneq V$.
We shall show that $W \subset f(W_1)$ and $U \supset f(U_1)$ where $W_1 = N(f^{-1}(b), \epsilon_1)$ and $U_1 = N(f^{-1}(v), \epsilon_2)$ for some $\epsilon_1, \epsilon_2 > 0$. (1)
Assuming this is true, let $a = f^{-1}(b)$ and $v_1 = f^{-1}(v)$
Define T: $X \to X$ by

$$T(x) = \frac{\epsilon_2}{\epsilon_1}(x-a) + v_1.$$

Then $T \epsilon H(X)$, $T(a) = v_1$ and $T(W_1) = U_1$ since $\|x-a\| < \epsilon_1$ if and only if $\|T(x) - v_1\| < \epsilon_2$. Now define $R: X \to X$ by

$$R(x) = (fTf^{-1})(x) \quad \text{if } \|x\| < 1$$
$$= x \quad \text{if } \|x\| \geq 1$$

We show that $R \epsilon H(X)$ and that $R(\bar{W}) \subsetneq V$. To show that $R \epsilon H(X)$, we show that $\lim_{\|x\| \to 1^-} R(x) = x$.

Take $x \in S$.

$$R(x) = (fTf^{-1})(x)$$

$$= fT\left(\frac{x}{1-||x||}\right)$$

$$= f\left(\alpha\left(\frac{x}{1-||x||} - a\right) + v_1\right) \quad \text{where} \quad \alpha = \frac{\varepsilon_2}{\varepsilon_1}$$

$$= \frac{\alpha\left(\frac{x}{1-||x||} - a\right) + v_1}{1 + \left|\left|\alpha\left(\frac{x}{1-||x||} - a\right) + v_1\right|\right|}$$

$$= \frac{\alpha x + (1-||x||)(v_1 - \alpha a)}{1 - ||x|| + ||\alpha x + (1-||x||)(v_1 - \alpha a)||}$$

So

$$||Rx-x|| = \left|\left|\frac{\alpha x+(1-||x||)(v_1-\alpha a)-x(1-||x||)-x||\alpha x+(1-||x||)(v_1-\alpha a)||}{1-||x|| + ||\alpha x + (1-||x||)(v_1-\alpha a)||}\right|\right|$$

Therefore $\lim\limits_{||x||\to 1^-} ||R(x) - x|| = \lim\limits_{||x||\to 1^-}\left|\left|\frac{\alpha x - x||\alpha x||}{||\alpha x||}\right|\right|$

$$= \lim_{||x||\to 1^-} \frac{\alpha||x||}{\alpha||x||}(1-||x||)$$

$$= 0.$$

Similarly $\lim\limits_{||x||\to 1^-} ||R^{-1}(x) - x|| = \lim\limits_{||x||\to 1^-}||fT^{-1}f^{-1}(x) - x|| = 0.$

Thus $R \in H(X)$.

Also $R(W) = (fTf^{-1})(W) \subset (fT)(W_1) = f(U_1) \subset U$.

So $R(\bar{W}) \subset \bar{U} \subsetneq V$.

Finally $R(b) = (fTf^{-1})\ (b) = (fT)(a) = f(v_1) = v$. So $v \in R(W)$.

It therefore remains to prove (1). To show that $f^{-1}(W) \subset N(f^{-1}(b), \varepsilon_1)$ for some $\varepsilon_1 > 0$, note that there is a number $K>0$ such that $\|w\| < 1 - K$ for all $w \in W$.

Now take $x \in f^{-1}(W)$. Then $x = \dfrac{w}{1 - \|w\|}$ where $\|w-b\| < \delta_1$

$$\text{Then } \left\|x - f^{-1}(b)\right\| = \left\| \frac{w}{1-\|w\|} - \frac{b}{1-\|b\|} \right\|$$

$$= \frac{\|w - b - \|b\|w + \|w\|b\|}{(1-\|w\|)(1-\|b\|)}$$

$$\leq \frac{1}{K(1-\|b\|)} (\|w-b\| + 2\|w\|\,\|b\|)$$

$$\leq \frac{1}{K(1-\|b\|)} (\delta_1 + 2)$$

Thus taking $\varepsilon_1 = \dfrac{\delta_1 + 2}{K(1-\|b\|)}$ we have that $f^{-1}(W) \subset N(f^{-1}(v), \varepsilon_1)$.

Now let $\varepsilon_2 = \delta_2 - 2$, and $U_1 = N(f^{-1}(v), \varepsilon_2)$. Take $y \in f(U_1)$.

Then $y = f(x)$ where $\left\| x - \dfrac{v}{1-\|v\|} \right\| < \delta_2 - 2$

So $x = \dfrac{y}{1-\|y\|}$ and $\left\| \dfrac{y}{1-\|y\|} - \dfrac{v}{1-\|v\|} \right\| < \delta_2 - 2$ (2)

But $\left\| \frac{y}{1-\|y\|} - \frac{v}{1-\|v\|} \right\| = \frac{\|y - v - \|v\|y + \|y\|\,v\|}{(1-\|y\|)\ (1-\|v\|)}$

$$\geq \|y - v\| - 2\,\|y\|\,\|v\|$$

$$\geq \|y - v\| - 2$$

From (2) $\|y - v\| < \delta_2$. Thus $y \epsilon U$ and $f(U_1) \subset U$.

Thus S is *B* freely open. Now any set of the form $N(x,\varepsilon)$ is homeomorphic to S, so *B* is a free basis for S and S is a S. H. set.

Corollary 2.6. *Every normed linear space is S.H.*

Corollary 2.7. *For every positive integer* n, *any open interval in* R^n *is a S.H. set, and so* R^n *is a S.H. set.*

Proof. The result follows from Theorem 2.5. and Lemma 1.4., since every open interval in R^n is homeomorphic to $\{x \in R^n : \|x\| \leq 1\}$.

Corollary 2.8. *Let* X *be a topological space. If* U *is an open subset of* X *which is homeomorphic to* R^n *for some positive integer* n, *then* U *is a S.H. set.*

We can, in fact, prove a much stronger result than Corollary 2.7.

Definition 2.9. Let U be an open subset of a topological space X. U is said to be *strongly setwise homogeneous* (S.S.H.) if each pair of proper open subset V,W of U with $\bar{W} \cap (\bar{U} - U) = \phi$ has the property that for each $x \in V$ there is a homeomorphism $h \in H(X)$ such that

(i) $x \in h(W)$ and $h(\bar{W}) \subsetneq V$

(ii) $h \in U'$.

It is obvious that if U is S.S.H., then U is S.H.

Theorem 2.10. *Any open interval in R^1 is a S.S.H. set.*

Proof. Let $J = (0,1)$. By Lemma 1.4., it suffices to show that J is S.S.H. Let V,W be proper open subsets of J with $\bar{W} \cap (\bar{J} - J) = \phi$. Choose $v \in V$. Then there is an open interval $K = (v-\varepsilon, v+\varepsilon)$ with $\bar{K} \subsetneq V$ and there is an open interval $L = (a,b)$ with $W \subset L$ and $\bar{L} \subset J$. Choose $\delta < \varepsilon$ and let $M = (v-\delta, v+\delta)$. Choose $N = (c,d)$ such that $a < c < d < b$ and $N \subset W$. Then we can define a homeomorphism $h \in H(R^1)$ such that

(i) $h(x) = x$ for all $x \in R-J$

(ii) $h(N) = M$

(iii) $h((a,c]) = (v-\varepsilon, v+\delta]$

(iv) $h([d,b)) = [v+\delta, v-\varepsilon)$.

Obviously there is such a homeomorphism h [Fig. II] .

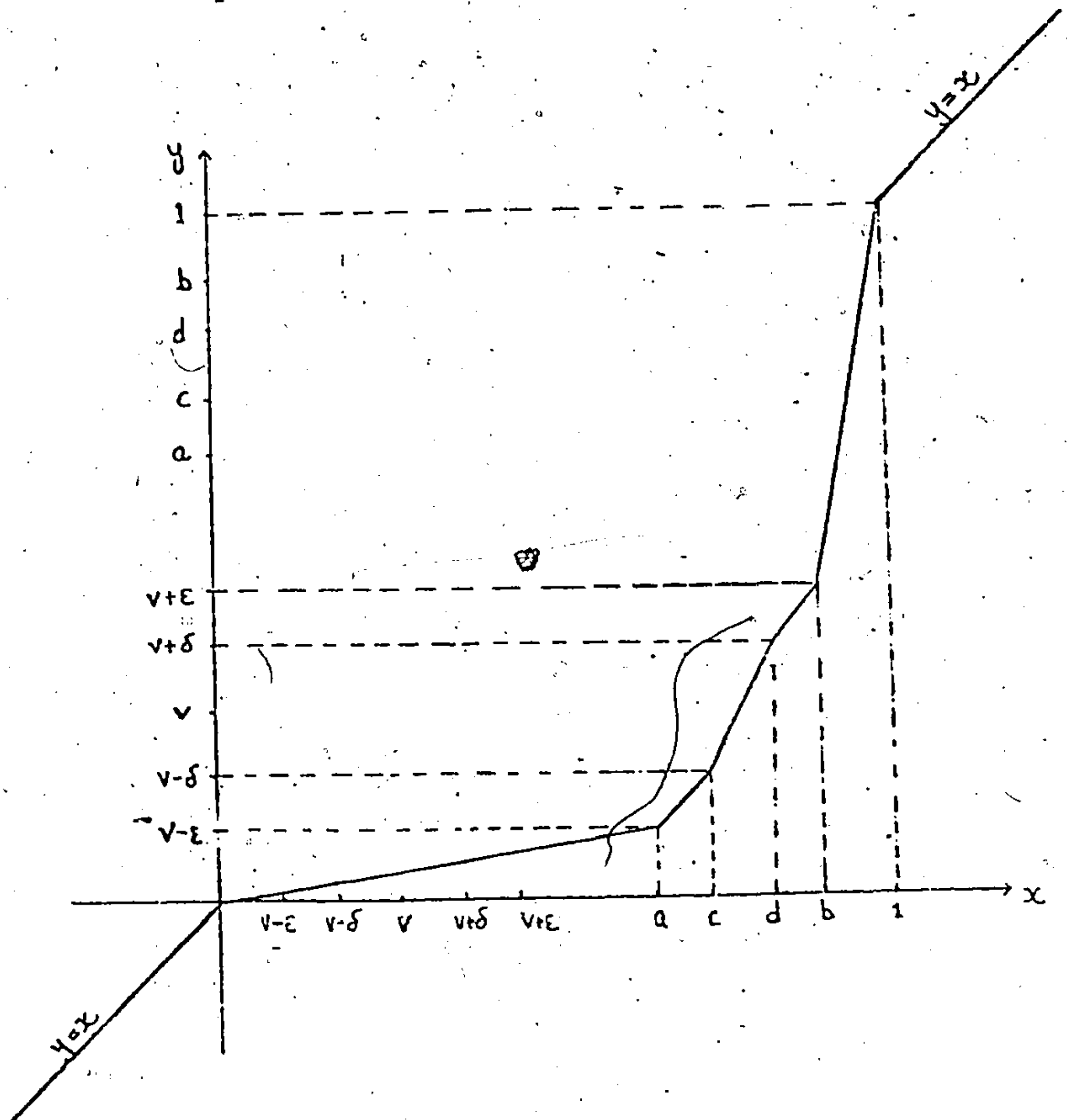

FIGURE II.

Now $h(\bar{W}) \subset h(\bar{L}) = \bar{K} \subsetneq V$, and $v \in h(N) \subset h(W)$. Also $h \in \mathcal{J}$. Thus J is S.S.H.

Corollary 2.11. *For every positive integer n, any open interval in R^n is S.S.H., and so R^n is S.S.H.*

Proof. Let $J = (0,1)$. It suffices to show that J^n is S.S.H. in R^n. Let V,W be proper open subsets of J^n with $\bar{W} \cap (\overline{J^n} - J^n) = \phi$. Choose $v = (v_k)_{1 \le k \le n} \in V$. Then there are proper open subsets V_k, W_k of J such that $\prod_{k=1}^{n} V_k \subset V$, $W \subsetneq \prod_{k=1}^{n} W_k$ and $\bar{W}_k \cap (\bar{J} - J) = \phi$, $v \in \prod_{k=1}^{n} V_k$ $(1 \le k \le n)$. Now $v_k \in V_k$ for all k, $1 \le k \le n$, so there are homeomorphisms $h_k \in H(R')$ with

(i) $v_k \in h_k(W_k)$ and $h_k(\bar{W}_k) \subsetneq V_k$

(ii) $h_k \in J'$.

Let $f = \prod_{k=1}^{n} h_k$. Then $f \in H(R^n)$ with $f \in J'$, $v \in f(W)$ and $f(W) \subsetneq V$. Thus J^n is S.S.H. in R^n.

There are many examples of spaces which are S.L.H. but not S.H. (in the sense of Ford [6]). For example, let C be the Cantor set in [0,1]. Let $Y = C \cup (2,3)$. Y is obviously not S.H. since it is not homogeneous. However Y is S.L.H.

The following definition is due to Fletcher and Snider [4].

Definition 2.12. A space X is a *Galois space* if for each closed set F and each $x \in CF$ there is a homeomorphism $h \in H(X)$ such that h is the identity map on F and $h(x) \neq x$.

Every S.H. regular perfect space is a Galois space; in fact every S.L.H. without isolated points is a Galois space. The following example, given by Fletcher and Snider [4], is an example of a homogeneous space with no isolated points which is not a Galois space. Let X be the plane. For each point $p = (a,b)$ let

$$D_{(p,\varepsilon)} = \{p\} \cup \{(x,y) : y = b \text{ and } \sqrt{(x-a)^2+(y-b)^2} < \varepsilon\}.$$

Let $\mathcal{B} = \{D_{(p,\varepsilon)} : p \in X, \varepsilon > 0\}$. Then $\mathcal{B}$ is a basis for a topology τ and with this topology X is a homogeneous space with no isolated points. However X is not a Galois space (see [4]).

Definition 2.13. A topological space X is said to be *representable* provided that if F is a closed set and $x \notin F$ then $\{h(x): h = \text{identity map on } F,\ h \in H(X)\}$ contains an open set about x.

It is obvious that every S.H. metric space is a representable space; in fact every S.L.H. is a representable space (see [5]). In [5] Fletcher and McCoy prove that every homogeneous completely regular representable space X is a reasonable space i.e. there is a topology on H(X) such that H(X) is a topological group and the map $\gamma: H(X)/C_x \to X$ defined by $\gamma(gC_x) = g(x)$ is a homeomorphism where $H(X)/C_x$ has the quotient topology and $C_x = \{f \in H(X): f(x) = x\}$. Thus we have

Theorem 2.14. *Every S.H. metric space is reasonable.*

§3. A STRONGLY HOMOGENEOUS SPACE IS STRONGLY LOCALLY SETWISE HOMOGENEOUS.

The following definitions are due to Brechner [2]. All spaces are taken to be separable metric. A continuum, is a compact, connected Hausdorff space.

Definition 3.1. Let X be a locally connected continuum, and let G be a subgroup of $H(X)$. X is called *locally setwise homogeneous under* G if and only if there exist both a basis $\mathcal{B}$ of connected open sets of X and a dense subset A of X such that for any $B \in \mathcal{B}$ and $a,b \in A \cap B$, there is a homeomorphism $g \in G$ such that $g \in B'$ and $g(a) = b$. $\{X,A,\mathcal{B},G\}$ is called a *locally setwise homogeneous structure* for X. If X is locally setwise homogeneous under $H(X)$, then X is called *locally setwise homogeneous*, denoted l-s-h.

Definition 3.2. A locally connected continuum X is called *strongly locally setwise homogeneous* (s-l-s-h) if and only if there exists a locally setwise homogeneous structure $\{X,A,\mathcal{B},G\}$ for X such that for each $B \in \mathcal{B}$ and $x \in A \cap B$ there is a neighbourhood U of x with $\bar{U} \subset B$, satisfying the following property:
For each open subset V of B there exists a homeomorphism $h \in G$ such that

(1) $h \in B'$.

(2) $h(\bar{U}) \subset V$.

Definition 3.3. Let $\{X,A,B,H\}$ be a l-s-h structure and let $B \in B$, $B \neq \phi$. Let $\mathcal{H} = \{h(B) : h \in H\}$. $\mathcal{H}$ is called a *near basis* for X if and only if every open set U of X contains the closure of an element of $\mathcal{H}$.

In [2], Brechner proved that every s-l-s-h continuum has a near basis.

To prove that every locally connected, S.H. continuum is s-l-s-h, we need the following:

Definition 3.4. A simple chain connecting two points a,b of a space X is a sequence $U_1, U_2, \ldots U_n$ of open sets of X such that $a \in U_1$ only, $b \in U_n$ only, and $U_i \cap U_j \neq \phi$ if and only if $|i - j| \leq 1$.

Lemma 3.5. *If X is connected, and $\mathcal{U}$ is any open cover of X, then any two points a,b of X can be connected by a simple chain of elements of $\mathcal{U}$.*

Proof. See [12], page 195.

Corollary 3.6. *Let $\mathcal{B}$ be any basis for the topology of a space X and let U be any connected open subset of X. Then any two points a,b of U can be connected by a chain $B_1,B_2,\ldots B_n$ of elements of $\mathcal{B}$ such that $B_i \subset U$ for all $i = 1,2,\ldots n$.*

Proof. Let $\mathcal{B}_1 = \{B \in \mathcal{B} : B \subset U\}$. Then since U is open, $\mathcal{B}_1$ is a basis for the subspace topology on the connected subspace U. The result follows from lemma 3.5.

Theorem 3.7. *If X is a locally connected, S.H. Continuum, then X is s-l-s-h.*

Proof. Let $\mathcal{D}$ be a free basis for X such that X is $\mathcal{D}$-freely open. We construct a locally setwise homogeneous structure $\{X,A,\mathcal{B},G\}$ by taking $\mathcal{B}$ to be any basis of connected open sets, $A = X$, and $G = H(X)$. Take $B \in \mathcal{B}$ and let $a,b \in B$. Let $D_1,D_2,\ldots D_n$ be a chain of elements of $\mathcal{D}$ connecting a and b, with $D_i \subset B$ for all $i = 1,2,\ldots n$. Let $d_i \in D_i \cap D_{i+1}$ for all $i = 1,2,\ldots n-1$. Now D_1 is S.H. and $a,d_1 \in D_1$. So by Theorem 1.5 there is a homeomorphism $h_1 \in H(X)$ such that $h_1(a) = d_1$. Also, by the proof of Theorem 1.5, $h_1 \in D_1'$. Now D_2 is S.H. and $d_1,d_2 \in D_2$. So there is a homeomorphism $h_2 \in H(X)$ such that $h_2(d_1) = d_2$ and $h_2 \in D_2'$. Continuing in this way we get homeomorphisms $h_1, h_2, \ldots h_n \in H(X)$ such that

(i) $h_1(a) = d_1$, $h_{i+1}(d_i) = d_{i+1}$ for all $i = 1,\ldots n-2$,

$$h_n(d_{n-1}) = b.$$

(ii) $h \in D_i'$ for all $i = 1,\ldots n-1$.

Let $f = h_n h_{n-1} \ldots h_1$. Then $f \in H(X)$, $f \in B'$, and $f(a) = b$. Thus X is l-s-h.

Now take $B \in \mathcal{B}$ and $x \in B$. Choose $D \in \mathcal{D}$ with $x \in D \subset B$, and let $U \in \mathcal{D}$ with $x \in U \subset D$, $\bar{U} \cap (\bar{D}-D) = \phi$. Let V be any open subset of B, and choose $v \in V$. Let $\mathcal{D}_1$ be the free basis for X consisting of all the elements of $\mathcal{D}$ with the restriction that the basis sets containing x must be contained in U. Now there is a chain $A_1, A_2, \ldots A_n$ of elements of $\mathcal{D}_1$ connecting x and v. Since $A_1 \in \mathcal{D}_1$, $A_1 \cap U \neq \phi$. If there is an integer m such that $U \cap A_m \neq \phi$ and $U \cap A_{m+1} = \phi$, consider the chain $U, A_m, A_{m+1}, \ldots A_n$ which connects x and v. If $U \cap A_m \neq \phi$ for all $m = 1,2,\ldots n$, consider the chain U, A_n. Let $A^*_m, A^*_{m+1}, \ldots A^*_{n-1}$ be elements of $\mathcal{D}_1$ such that $A^*_m \subset U \cap A_m$, $A^*_m \cap (\bar{A}_m - A_m) = \phi$ and $A^*_i \subset A_{i-1} \cap A_i$ with $A^*_i \cap (\bar{A}_i - A_i) = \phi$ for all

$i = m+1, m+2, \ldots n-1$. Also choose $D_1 \in \mathcal{D}_1$ with $v \in D_1 \subset V \cap A_n$.

Now $U, A_m^* \in \mathcal{D}_1$ are contained in the $\mathcal{D}_1$-freely open set D, and $\bar{U} \cap (\bar{D} - D) = \phi$. So here is a homeomorphism $h_m \in H(X)$ such that $h_m \in D'$ and $h_m(\bar{U}) \subsetneqq A_m^*$. The elements A_m^*, A_{m+1}^* of $\mathcal{D}_1$ are contained in the $\mathcal{D}_1$ freely open set A_m and $A_m^* \cap (\bar{A}_m - A_m) = \phi$. So there is a homeomorphism $h_{m+1} \in H(X)$ such that $h_{m+1} \in A_m'$ and $h_{m+1}(A_m^*) \subsetneqq A_{m+1}^*$. Continuing in this way we get homeomorphisms $h_m, h_{m+1} \ldots h_n$ in $H(X)$ such that

(i) $h_m(\bar{U}) \subsetneqq A_m^*$, $h_i(A_{i-1}^*) \subsetneqq \bar{A}_i^*$ for all $i = m+1, m+2, \ldots n-1$, $h_n(\overline{A_{n-1}^*}) \subsetneqq D_1$.

(ii) $h_m \in D'$, $h_i \in A_{i-1}'$ for all $i = m+1, m+2, \ldots n-1$.

Let $f = h_n h_{n-1} \ldots h_{m+1} h_m$. Then $f \in H(X)$, $f \in B'$, and $f(\bar{U}) \subsetneqq D_1 \subset V$.

Therefore U is a neighbourhood of x with $\bar{U} \subset B$ such that for each open subset V of B there exists a homeomorphism $f \in H(X)$ such that $f \in B'$ and $f(\bar{U}) \subset V$. Thus X is s.l.s.h.

Note that every S.H. locally connected continuum has a near basis.

The following definition, essentially due to Menger, can be found in [7].

Definition 3.9. Let X be a separable metric space. The empty set, and only the empty set, has *dimension* -1. X has *dimension* $\leq n$ (where n is a non-negative integer) *at a point* p if p has arbitrarily small neighbourhoods whose bounderies have dimension $\leq n-1$.
X has *dimension* $\leq n$ if X has dimension $\leq n$ at each of its points.
X has *dimension* n *at a point* p if X has dimension $\leq n$ at p but X does not have dimension $\leq n-1$ at p.
X has *dimension* n if X has dimension $\leq n$ but does not have dimension $\leq n-1$.
X has *dimension* ∞ if the dimension of $X \nleq n$ for each n.

It is known that if X is a 1-s-h continuum, then $H(X)$ is at least one dimensional (see [1]).

Definition 3.10. Let X be any topological space and let $x \in X$. Then *the orbit of* x, denoted $O(x)$, is defined by

$$O(x) = \{h(x) : h \in H(X)\}.$$

It is known that if X is 1-s-h then for all $x \in X$, if $O(x)$ is not dense, it is nowhere dense. (See [2]). We therefore have:

Theorem 3.11. Let X be a locally connected, S.H. continuum

(i) $H(X)$ is at least one dimensional.

(ii) If $O(x)$ is not dense in X for some $x \in X$, then $O(x)$ is nowhere dense. (If X is metric then $O(x) = X$, so this is obvious).

We now give an example of a space which is s-1-s-h, but not S.H. The *universal plane curve* (or *Sierpinski curve*) is a continuum whose standard construction is the following: Let C be a square plus its interior in the plane. Divide C into nine equal squares, and remove the interior of the middle ninth. Break each of the remaining eight squares into nine equal squares, and remove the interiors of their middle ninths. Continue the process inductively. The set M which remains is the desired continuum. It was proved in [1] that M has a basis E of connected open sets such that the closure of each element of E is homeomorphic to M, and M was shown to be 1-s-h with locally setwise homogeneous structure $\{X, A, E, H(X)\}$ where A is the interior of M.

It was proved in [2] that M is s-l-h. However Mazurkiewicz showed that M is not homogeneous (see [8]), and so by Theorem 1.5 the universal plane curve is not S.H.

§4. ERGODIC HOMEOMORPHISMS AS A SUBSET OF ALL SELF-HOMEOMORPHISMS.

Definition 4.1. Let X be a Hausdorff space and let $T \in H(X)$. T is called *ergodic* if every closed proper subset A of X with $T(A) = A$ is nowhere dense in X. The subset of $H(X)$ consisting of the ergodic homeomorphisms is denoted $E(X)$.

It is well known that if X is a second countable Baire space and $T \in H(X)$, then T is ergodic if and only if there is an element x of X such that $\{T^n(x) : n \text{ is an integer}\}$ is dense in X.

Sears [11] found a sufficient condition for $E(X)$ to be nowhere dense in $H(X)$, where (X,d) is a compact, perfect metric space and $H(X)$ has the uniform topology generated by the metric

$$d^+(S,T) = \sup_{x \in X} d(S(x),T(x)), \qquad \text{for } S,T \in H(X).$$

To do this, Sears made a number of definitions.

Definition 4.2. Let X be a compact, perfect metric space and let $T \in H(X)$. A closed non-empty subset F of X is called a *sink invariant set under* T if $T(F) = F$ and there is an open set U with $F \subset U$ such

that $T(\bar{U}) \subsetneq U$ and $\bigcap_{i=1}^{\infty} T^i(U) = F$. F is called a *sink periodic set under* T if F is a sink invariant set under T^n for some $n > 0$.
The subset of $H(X)$ consisting of the homeomorphisms which have sink periodic sets is denoted $\Gamma(X)$.

Theorem 4.3. *Let X be a compact, perfect, metric space. Then*

(i) *If $T \in H(X)$ is such that $T^n(\bar{U}) \subsetneq U$ for some open subset U of X, then T has a sink periodic set.*

(ii) $\Gamma(X) \cap E(X) = \phi$

(iii) *$\Gamma(X)$ is open in $H(X)$.*

Proof. See [11].

Definition 4.4. Let X be a compact metric space. X is called *strongly homogeneous* at a point x_o if there is a basis of neighbourhoods $\{U_n$: n positive integer$\}$ at x_o such that every member of the basis satisfies the following condition:

(*) For every two open subsets V and W of U_n with $\bar{W} \cap (\bar{U}_n - U_n) = \phi$, there is a homeomorphism h from $\bar{U}_n$ onto itself satisfying

(i) $h(x) = x$ for $x \in \bar{U}_n - U_n$.

(ii) $h(\bar{W}) \subseteq V$.

If U is a neighbourhood of x_o satisfying condition (*), we call U a *strongly homogeneous neighbourhood* of x_o.

Sears related the concepts of strong homogeneity and homogeneity by proving that if X is strongly homogeneous at a point x_o and U is a strongly homogeneous neighbourhood of x_o, then there is a dense subset A of U with $x_o \in A$ such that X is strongly homogeneous at every point of A and A is a homogeneous set of X. (See [11]). Notice that if X has a basis $\mathcal{B}$ and U is an open subset of X such that every point of U is strongly homogeneous with respect to the basis of neighbourhoods of $\mathcal{B}$, then $\mathcal{B}$ is a free basis for U.

It was proved in [11] that if X is a compact metric space with a strongly homogeneous point, then $E(X) \subset \overline{\Gamma(X)}$. From this it was deduced that if X is a compact metric perfect space with a strongly homogeneous point then $E(X)$ is nowhere dense in $G(X)$. We prove a similar result. The method of proof follows closely that used by

Sears in [11]. Throughout we take X to be a compact metric space.

Theorem 4.5. *If X contains an open set U which has a free basis, then $E(X) \subseteq \overline{\Gamma(X)}$.*

Proof. Let $\varepsilon > 0$. Let $\mathcal{B}$ be a free basis for U and choose any $B \in \mathcal{B}$ with $\delta(B) < \varepsilon$. Take $T \in \Xi(X)$. Then some point has dense orbit under T, so there is an $x \in X$ and positive integers i such that $x \in B$ and $T^{-i}(x) \in \bar{B}$. Let n be the smallest positive integer with this property. Now $T^{-1}(\bar{B})$ does not contain any of the points $T^{-2}(x), T^{-3}(x), \ldots T^{-n}(x)$. Thus there is a neighbourhood $B_1 \in \mathcal{B}$ of x such that

(i) $T^{-n}(B_1), T^{-n+1}(B_1), \ldots T^{-2}(B_1)$ are all disjoint from $T^{-1}(B)$.

(ii) $B_1 \subset B$.

(iii) $\bar{B}_1 \cap (\bar{B} - B) = \phi$.

$T^{-n}(B_1) \cap B$ is open and nonempty since $T^{-n}(x) \in \bar{B}$. So there is a $B_2 \in \mathcal{B}$ with $B_2 \subset T^{-n}(B_1) \cap B$. Thus, since B is $\mathcal{B}$-freely open there is a homeomorphism $h \in H(X)$ such that $h(\bar{B}_1) \subsetneq B_2 \subset T^{-n}(B_1) \cap B$ and $h \in B'$. Define $S : X \to X$ by

$$S(y) = (h_o T)(y) \qquad \text{for all } y \in X.$$

Then $S \in H(X)$. We shall show that $S \in \Gamma(X)$. Now $S^n(T^{-n}(B_1)) = S(S^{n-1}(T^{-n}(B_1))) = S(T^{n-1}(T^{-n}(B_1)))$ since $h \in B'$ and the sets $T^{-n}(B_1), \ldots T^{-2}(B_1)$ are all disjoint from $T^{-1}(B)$. Thus

$S^n(T^{-n}(B_1)) = S(T^{-1}(B_1)) = h(T(T^{-1}(B_1))) = h(B_1)$. So

$S^n(T^{-n}(\bar{B}_1)) = h(\bar{B}_1) \subsetneqq T^{-n}(B_1)$. Therefore by Theorem 4.3 (i), $S \in \Gamma(X)$. Finally

$$d^+(T,S) = \sup_{x \in X} d(T(x), h(T(x))) = \sup_{x \in T^{-1}(B)} d(T(x), h(T(x))).$$

So $d^+(T,S) = \sup_{b \in B} d(b, h(b)) < \varepsilon$ since $\delta(B) < \varepsilon$.

Thus $E(X) \subset \overline{\Gamma(X)}$.

Corollary 4.6. *If X contains an open subset with a free basis, then $E(X)$ is nowhere dense in $H(X)$.*

Proof. This follows directly from Theorem 4.3 (ii) and (iii) and from the above Theorem.

Thus if X is any compact metric perfect space which has a S.H. subset, then $E(X)$ is nowhere dense in $H(X)$.

Sears proved in [11] that every point of the Cantor set is a strongly homogeneous point. Similarly by looking at the Cantor subintervals of rank r $(r = 1,2,3,\ldots)$ (see [10]) we see that the Cantor set is S.H.

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ADDENDUM.

Since completing this thesis, the author has weakened the metric condition in Theorem 1.5, which now becomes

Theorem 1.5′. *If U is a S.H. set in a first countable, regular, T_1 space X, then U is homogeneous.*

Proof. Let B be a free basis for U such that U is B-freely open. Let a, b be any two elements of U. We show that there is a homeomorphism F in H(X) with F(a) = b. Let C_n and D_n (n = 1,2,...) be elements of B such that $\bigcap_{n=1}^{\infty} C_n = \{a\}$ and $\bigcap_{n=1}^{\infty} D_n = \{b\}$. Now let $W_1, Y_1 \in B$ be proper subsets of U with $a \in W_1 \subset C_1$, $b \in Y_1 \subset D_1$, and $\bar{W}_1 \cap (\bar{U} - U) = \phi$. Then there is a homeomorphism $h_1 \in H(X)$ such that

(i) $b \in h_1(W_1)$ and $h_1(\bar{W}_1) \subsetneqq Y_1$.

(ii) $h_1 \in U'$.

Let $V_1 = h_1(W_1)$, $a_1 = h_1(a)$, $B_1 = h_1(B)$. Let $W_2, Y_2 \in B_1$ be proper subsets of V_1 with $a_1 \in W_2$, $b \in Y_2 \subset D_2$, $h_1^{-1}(W_2) \subset C_2$, $\bar{W}_2 \cap (\bar{V}_1 - V_1) = \phi$. Since V_1 is B_1-freely open, there is a homeomorphism $h_2 \in H(X)$ such that

(i) $b \in h_2(W_2)$ and $h_2(\bar{W}_2) \subsetneqq Y_2$.

(ii) $h_2 \in V_1'$

Now let $V_2 = h_2(W_2)$, $a_2 = h_2(a_1)$, $B_2 = h_2(B_1)$ and proceed as before.

Continuing in this way we get sequences (W_n) and (V_n) of S.H. sets and homeomorphisms $h_n \varepsilon H(X)$ such that for all $n = 1,2,\ldots$ we have:

(a) $\bar{V}_{n+1} \subsetneqq V_n$

(b) $h_n(W_n) = V_n$ and $\bar{W}_n \cap (\bar{V}_n - V_n) = \emptyset$.

(c) $(h_1^{-1}h_2^{-1}\ldots h_n^{-1})(W_{n+1}) \subset C_{n+1}$ and $W_1 \subset C_1$

(d) $V_n \subset D_n$

(e) $h_n \varepsilon V'_{n-1}$ $(V_0 = U)$

(f) $a_n = h_n h_{n-1}\ldots h_1(a) \varepsilon W_{n+1}$

Let $F_n = h_n h_{n-1}\ldots h_1$. Define the map F by

$$F(x) = \lim_{n\to\infty} F_n(x) \qquad \text{for all } x\varepsilon X$$

We show that F exists for all $x\varepsilon X$, that $F\varepsilon H(X)$ and that $F(a) = b$.

Take $x\varepsilon X$. If $x\notin U$, then $F(x) = x$. If $F_n(x)\notin V_n$ for some n, then by (a) and (e), $F(x) = F_n(x)$. If $F_n(x)\varepsilon V_n$ for all n, then $h_n h_{n-1}\ldots h_1(x)\varepsilon h_n(W_n)$ for all n so that $x\varepsilon F_{n-1}^{-1}(W_n)\subset C_n$ for all n. Thus $x = a$. Also $F_n(a) = a_n \varepsilon V_n \subset D_n$ for all n, so $F(a) = b$. Thus $F(x)$ exists for all $x\varepsilon X$ and F is one to one. To show that F is onto, take $x\varepsilon X$. If $x\notin U$, then $F(x) = x$. If $x\varepsilon V_n$ for all n, then $x = b$ and $F(a) = x$. Otherwise $x\notin V_n$ for some n. Say $x\notin V_N$. Now there is an element y of X such that $F_N(y) = x$ (because F_N is onto). Then $F_N(y)\notin V_N$ and so $F(y) = F_N(y) = x$. Thus F is onto.

We now show that F is continuous. Take $x\varepsilon X$. There are two cases to consider.

(i) If $x \neq a$, then there is a positive integer m such that $F_m(x) \notin V_m$ and $F(x) = F_k(x)$ for all $k \geq m$. Now $\bar{W}_{m+1} \cap (\bar{V}_m - V_m) = \emptyset$, so if $F_m(x) \in \bar{W}_{m+1}$, then $F_m(x) \notin \bar{V}_m$. If $F_m(x) \notin \bar{W}_{m+1}$, then $F_{m+1}(x) \notin h_{m+1}(\bar{W}_{m+1})$, so $F(x) = F_{m+1}(x) \notin \bar{V}_{m+1}$.

Thus in both cases $F(x) \notin \bar{V}_{m+1}$, which means that $x \in F_{m+1}^{-1}(X - \bar{V}_{m+1})$ which is open. So there is a neighbourhood A of x such that $A \subset F_{m+1}^{-1}(X - \bar{V}_{m+1})$. Thus if $y \in A$, then $F(y) = F_{m+1}(y)$. Now let P be any neighbourhood of $F_{m+1}(x)$. Then there is a neighbourhood P_1 of x such that $F_{m+1}(P_1) \subset P$. Let $P_2 = A \cap P_1$. Then $F(P_2) = F_{m+1}(P_2) \subset P$. So F is continuous at x for $x \neq a$.

(ii) Suppose $x = a$ and let Q be any neighbourhood of $F(x) = b$. Then there is a positive integer N such that $D_N \subset Q$. Since F_{N-1} is continuous and $a \in F_{N-1}^{-1}(W_N)$, there is a neighbourhood B of a such that $B \subset F_{N-1}^{-1}(W_N)$. So if $y \in B$, then $F_{N-1}(y) \in W_N$, so $F_N(y) \in V_N$. Thus $F(y) \in V_N \subset D_N \subset Q$. Therefore $F(B) \subset Q$ and F is continuous at $x = a$.

Thus F is continuous.

It remains to show that F^{-1} is continuous. Take $x \in X$. There are again two cases to consider

(i) If $x \neq b$, then there is an element z of X, $z \neq a$, with $F(z) = x$. Now $F(z) = F_N(z) \notin \bar{V}_N$ for some positive integer N, so that $x \in X - \bar{V}_N$ which is open. Thus there is a neighbourhood S of x such that $S \subset X - \bar{V}_N$. So if $y \in S$, then $y \notin \bar{V}_N$ and $F^{-1}(y) = F_N^{-1}(y)$. Now let E be any neighbourhood of $F^{-1}(x)$. Since F_N^{-1} is continuous, there is a neighbourhood E_1 of x such that $F_N^{-1}(E_1) \subset E$. Let $E_2 = E_1 \cap S$. Then $F^{-1}(E_2) = F_N^{-1}(E_2) \subset E$. So F^{-1} is continuous at x for $x \neq b$.

(ii) If $x = b$, let G be any neighbourhood of $F^{-1}(x) = a$. Then there is a positive integer M such that $C_M \subset G$. Now V_M is a neighbourhood of b and if $y \in V_M$, then $F^{-1}(y) \in F^{-1}(V_M) \subset F_M^{-1}(V_M)$, so $F^{-1}(y) \in F_{M-1}^{-1}(W_M)$. Thus from (c), $F^{-1}(y) \in C_M \subset G$. Therefore $F^{-1}(V_M) \subset G$ and so F^{-1} is continuous at $x = b$.

Thus $F \in H(X)$ and $F(a) = b$. Thus U is a homogeneous set.

As a result of this improvement, Theorem 2.4 and Theorem 2.14 can be strengthened respectively to:

Theorem 2.4$'$. *Let X be a S.H. locally compact first countable space. Then*

(i) *the uniform structure on X obtained from its one point compactification induces on H(X) a reasonalbe topology.*

(ii) *the uniform structure on X obtained from its Stone-Čech compactification induces on H(X) a reasonable topology over X.*

(iii) *if X is connected, then the group of uniformly continuous homeomorphisms (that is, homeomorphisms g such that g and g^{-1} are uniformly continuous) is reasonable over X.*

Theorem 2.14$'$. *Every S.H. Tychonoff first countable space is reasonable.*