FIXED POINTS OF NONEXPANSIVE TYPE
MULTIVALUED MAPS

By

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MULTIVALUED MAPS

To my Parents and my

Uncle Mohammad Yunis Khan

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ABSTRACT

Let (X,d) be a metric space. Husain and Tarafdar call a map $J: X \to 2^X$ (nonempty subsets of X) nonexpansive (here it is called (H·T)-nonexpansive) if for all $x \in X$, $u_x \in J(x)$ there exists $v_y \in J(y)$ for all $y \in X$ such that

$$d(u_x, v_y) \le d(x,y) .$$

Clearly this notion generalizes the usual concept of nonexpansive maps and coincides with it for single-valued maps. We study fixed points for such mappings. Further, if in the above inequality we have

$$d(u_x, v_y) \le h\, d(x,y)$$

for some fixed h, $0 \le h < 1$, then we generalize the notion of contraction for single valued maps. We prove a fixed point theorem which contains the Banach fixed point theorem as a special case. We introduce and study two more classes of set-valued nonexpansive mappings: s-nonexpansive mappings and f-(H·T)-nonexpansive mappings. In general, these two types of maps are not related. But both of them contain the class of all single-valued nonexpansive maps. However, the class of all f-(H·T)-nonexpansive mappings contains the class of all (H·T)-nonexpansive mappings. It is shown that not every (H·T)-nonexpansive mapping on a nonempty closed convex and bounded subset of a Banach space has a fixed point.

Husain and Tarafdar proved that if M is a compact interval of the real line then each (H·T)-nonexpansive closed convex valued map $J: M \to 2^M$ has a fixed point. In this thesis, we extend this result which contains some well-known results due to Browder and Karlovitz. Moreover, it includes, as a special case, the result: every single-valued nonexpansive mapping on a nonempty closed convex bounded subset of a reflexive Banach space satisfying Opial's condition has a fixed point. This last result can also be derived from a result due to Kirk, in view of the fact that the Opial's condition implies normal structure. We also prove a fixed point theorem for multivalued s-nonexpansive mappings, from which it is derived that every closed convex-valued s-nonexpansive mapping on a nonempty closed convex bounded subset of a Hilbert space has a fixed point. In addition, we have a fixed point theorem for set-valued (H·T)-contractive type mappings from which a result due to Kannan can be derived as a special case. Finally, a common fixed point theorem for such mappings is proved.

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INTRODUCTION

Probably the best known and the most frequently cited fixed point theorem in mathematical literature is 'the Contraction Principle' or the 'Banach fixed point theorem' which is stated as follows: If (X,d) is a complete metric space and f a self mapping of X which satisfies

$$d(f(x),f(y)) \leq h\,d(x,y)$$

for a fixed number h, $0 \leq h < 1$ and all $x,y \in X$, then there exists a unique point $x_0 \in X$ such that $f(x_0) = x_0$, and moreover for any fixed $x \in X$, the sequence $\{f^n(x)\}$ converges to x_0. There exists a great number of generalizations of the contraction principle, for example, see Edelstein [21], [22], [23], Kasahara [37], Krasnoselskii [39], Sehgal [49], Rakotch [47], [48], Boyd and Wong [8], etc.

In the context of complete metric spaces the assumption $0 \leq h < 1$ is crucial even for the existence part of the above result, but in a more restrictive yet quite natural setting, i.e., when $h = 1$, an elaborate fixed point theory exists. Mappings of this type are called nonexpansive. On the one hand a nonexpansive self mapping of a complete metric space need not have a fixed point (consider a translation operator $x \to x + y$ with $y \neq 0$, in a Banach space) and on the other a fixed point of such a mapping need not be unique (consider $f = I$).

The study of the problem of determining those subsets of Banach spaces which have the fixed point property for nonexpansive self mappings has its origins in four papers which appeared in 1965. In the first of these [11], Browder, drawing on concepts from the theory of monotone operators, proved that bounded closed convex subsets of Hilbert spaces have this property, and subsequently in [12], he extended this result to a much larger class of all uniformly convex Banach spaces. At the same time, this result was obtained independently by Göhde [26], while in [38], Kirk, exploiting a property shared by all uniformly convex spaces, obtained the same result for an even larger class of spaces, namely reflexive Banach spaces. Karlovitz [35] obtained a similar result using a generalized notion of orthogonality. The class of nonexpansive mappings has been further generalized by a new class of mappings satisfying weaker conditions, for example, see Istrătescu [31], Browder and Petryshyn [14] etc.

Much work has been done on fixed points of multivalued functions. For instance in [33], Kakutani extended Brouwer's fixed point theorem to compact convex set-valued mappings on a compact convex subset of Euclidean spaces. The extension of Schauder's fixed point theorem was given independently by Bohnenblust and Karlin [7] and by Glicksberg [25]. An extension of Tychonoff's theorem was given by Ky Fan [41]. Further results in this direction were obtained by Browder [15] and others. Nadler [44] defined set-valued contractive mappings and proved that such mappings have fixed points in the case of complete metric spaces. These results were extended by Fraser and Nadler [24]. Smithson [50], defined contractive set-valued mappings and proved a fixed point theorem for such mappings. Husain and Tarafdar [30] have defined the concept of normal structure of bounded convex subsets of locally convex linear topological spaces and also the notion of a multivalued mapping of nonexpansive type on such spaces and proved a fixed point theorem for such mappings. In the same paper they call a map $J: X \to 2^X$ nonexpansive if for any $x \in X$, $u_x \in J(x)$ there exists $v_y \in J(y)$ for all $y \in X$ such that

$$d(u_x, v_y) \leq d(x,y) .$$

This clearly generalizes the usual notion of nonexpansive maps. In this thesis we shall call this notion (H·T)-nonexpansive. They proved a fixed point theorem for such mappings on a closed bounded interval of the real line and raised the question if it could be established for more general spaces.

In this thesis we are concerned with this question among others and prove a fixed point theorem which includes Husain and Tarafdar's result. We also introduce the notions of (H·T)-contractive and f-(H·T)-contractive mappings (which generalize the usual notion of contractiveness for single-valued maps) and prove some fixed point theorems for such mappings which contain the Banach fixed point theorem as a special case. We extend the notion of (H·T)-nonexpansive mapping to f-(H·T)-nonexpansiveness. The notion of selectively nonexpansive (denoted by s-nonexpansive) maps which we introduce and then study fixed points for such mappings. Moreover we prove a fixed point theorem for (H·T)-contractive type mappings which contains Kannan's [34] result as a special case. Finally, we prove some common fixed point theorems for a family of mappings.

Specifically, Chapter I is devoted to basic definitions, terminology and background information needed throughout the thesis. Moreover some classical results on fixed points are also presented here.

In Chapter II, we study two concepts of multivalued nonexpansive mappings, one of which was introduced by Husain and Tarafdar [30]. Both these concepts generalize the usual notion of nonexpansiveness for single-valued maps. We start out with examples and some properties of (H·T)-nonexpansive maps. In addition, we show by examples that not every (H·T)-nonexpansive mapping on a nonempty closed convex bounded subset of a Banach space has a fixed point. Here we prove our main fixed point theorem for s-nonexpansive mapping, from which it is particularly derived that each s-nonexpansive closed convex valued map $J: M \to 2^M$ has a fixed point, when M is a nonempty closed convex bounded subset of a Hilbert space X. This extends a result due to Husain and Tarafdar [30], for s-nonexpansive mappings and includes a result due to Browder [11] for single-valued maps.

Chapter III begins with the notion of (H·T)-contractiveness. Here we show that (H·T)-nonexpansive maps may not be (H·T)-contractive and prove some fixed point theorems. For instance, every closed-valued (H·T)-contractive mapping of a closed nonempty subset of a complete metric space has a fixed point. This contains the Banach fixed point theorem as a special case. We also prove a fixed point theorem for a generalized

(H·T)-contractive mapping, calling such a mapping f-(H·T)-contractive. For this, we prove a fixed point theorem by using Theorem 2.1 which extends the Husain and Tarafdar [30] result and includes results due to Browder [11] and Karlovitz [35] for the single-valued case. Moreover, it includes a result which can be derived from a result due to Kirk [38]. In addition, a fixed point theorem for generalized (H·T)-nonexpansive maps has been proved. The rest of the chapter is devoted to a study of the iterative type construction of fixed points for (H·T)-nonexpansive mappings.

In Chapter IV, we introduce the notion of (H·T)-contractive type maps and prove a fixed point theorem for such mappings, which contains Kannan's [34] result as a special case. Common fixed point theorems and a few results about general set-valued mappings are also given.

CHAPTER I

PRELIMINARIES

In this chapter, we briefly present the main background material needed throughout the thesis. Proofs are omitted, since they are available in standard books and research papers. Basic definitions and facts from the general theory of metric spaces, convex sets and Banach spaces are given in §1 to §3. For proofs and further details, see [18], [51], [29]. §4 contains various notions of geometric type which becomes interesting for the existence of fixed points for nonexpansive mappings. Some classical fixed point theorems and the definition of a multivalued nonexpansive mappings introduced by Husain and Tarafdar [30] are given in §5.

§1. Metric spaces.

Let X be a nonempty set. A function $d: X \times X \to \mathbb{R}$ is called a metric if the following properties hold:

(a) $d(x,y) \geq 0$ and $d(x,y) = 0$ iff $x = y$

(b) $d(x,y) = d(y,x)$

(c) $d(x,z) \leq d(x,y) + d(y,z)$ (triangle inequality).

The number $d(x,y)$ is called the distance between x and y and the pair (X,d) is called a metric space. For simplicity we write X and say that X is a metric space. An open ball with centre x_0 and radius r is defined by $B(x_0,r) = \{x \in X: d(x,x_0) < r\}$. A subset M of X is called open iff for every point $x \in M$ there is an open ball $B(x,r) \subseteq M$.

For any subset M of X, the diameter of M is the number defined by $\operatorname{diam} M = \sup\{d(x,y): x,y \in M\}$. M is called bounded iff $\operatorname{diam} M < \infty$. The notion of convergence in a metric space is defined as follows: a sequence $\{x_n\}$ of elements in X converges to an element x of X iff $d(x_n,x) \to 0$ as $n \to \infty$. This means: given any $\varepsilon > 0$, there is an integer N such that $d(x_n,x) \le \varepsilon$ whenever $n \ge N$. In this case we also say that x is the limit of $\{x_n\}$ and write $x_n \to x$ as $n \to \infty$ or $\lim_n x_n = x$. It is easy to see by the triangle inequality that the limit point of $\{x_n\}$, if it exists, is uniquely determined. A convergent sequence $\{x_n\}$ in X satisfies Cauchy's convergence condition $\lim_{n,m} d(x_n,x_m) = 0$. This means: given any $\varepsilon > 0$, there is an integer N such that $d(x_n,x_m) \le \varepsilon$ whenever $n,m \ge N$. Any sequence $\{x_n\}$ in a metric space X satisfying the Cauchy's convergence condition is called a Cauchy sequence. A metric space X is called complete if every Cauchy sequence in it converges to a limit point in X. A subset M of X is dense in X if for each point x in X, there is a sequence $\{x_n\}$ in M such that $x_n \to x$ as $n \to \infty$. A set M is closed iff whenever $x_n \in M$ for all n and $x_n \to x$, then $x \in M$. A subset M of a metric space X is compact iff for every sequence in M there is a convergent subsequence with its limit in M.

1.1. __Theorem__. A subset M of a complete metric space X is compact iff for any collection of closed sets $\{F_\alpha\}_{\alpha\in I}$ in M having the finite intersection property (that is, the intersection of any finite number of sets from the collection is nonempty), then $\bigcap_{\alpha\in I} F_\alpha \neq \phi$.

If (X,d_1) and (Y,d_2) are two metric spaces, $M \subset X$, $p \in M$ and f maps M into Y, then f is said to be continuous at p if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $d_2(f(x),f(p)) < \varepsilon$, for all points $x \in M$ for which $d_1(x,p) < \delta$. If f is continuous at every point of M then f is said to be continuous on M.

1.2. Theorem. If X and Y are two metric spaces, $M \subset X$ and $f: M \to Y$ is any function, then f is continuous at $x \in M$ iff for any sequence $\{x_n\}$ in M converging to x, the sequence $\{f(x_n)\}$ converges to $f(x)$.

A complete metric space (X,d) is called (metrically) convex if for each pair x,y in X with $x \neq y$ there exists $z \in X$, $x \neq z \neq y$, such that $d(x,z) + d(z,y) = d(x,y)$.

1.3. Theorem. If X is a metrically convex space, then each two points in X are the end points of at least one metric segment. (see [6], p.41)

§2. Convex sets.

Let X be a linear space over the field $\mathbb{F}$ (which is either $\mathbb{R}$ or $\mathbb{C}$). A subset M of X is called convex if for all x and y in M and $0 \le t \le 1$, $tx + (1-t)y \in M$. As the set $\{tx + (1-t)y : 0 \le t \le 1\}$ is the line segment joining x and y, a set M is convex iff for all x and $y \in M$ the entire line segment joining x and y is contained in

M. The intersection of any collection of convex sets is convex. For $M \neq \phi$, the convex hull of M is defined by

$$\operatorname{conv} M = \left\{ \sum_{i=1}^{n} \lambda_i x_i : \sum_{i=1}^{n} \lambda_i = 1, \ 0 \le \lambda_i \le 1, \ x_i \in M \right\},$$ i.e., the $\operatorname{conv} M$ is the smallest convex set containing M. If X is a topological linear space, then the closed convex hull, $\overline{\operatorname{conv}}\, M$ denotes the closure of $\operatorname{conv} M$. If M is a convex subset of a linear space X then a point x in M is an extreme point of M if there is no proper open line segment that contains x and lies entirely in M. Let $\operatorname{ext} M$ be the set of all extreme points of M. If $X = \mathbb{R}^2$ and $M = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \le 1\}$, then $\operatorname{ext} M = \{(x,y) : x^2 + y^2 = 1\}$. If $M = \{(x,y) \in \mathbb{R}^2 : x \le 0\}$, then $\operatorname{ext} M = \phi$.

2.1. Proposition. If M is a convex set in a linear space X and $x \in M$, the following statements are equivalent:

(a) $x \in \operatorname{ext} M$.

(b) If $x_1, x_2 \in X$ and $x = 1/2(x_1 + x_2)$, then either $x_1 \notin M$ or $x_2 \notin M$ or $x_1 = x_2 = x$.

(c) If $x_1, x_2, \ldots, x_n \in M$ and $x \in \operatorname{conv}\{x_1, x_2, \ldots, x_n\}$, then $x = x_i$ for some i.

(d) $M \setminus \{x\}$ is a convex set. (see [18])

A function $f: X \to \mathbb{R}$ is called convex if for every $x, y \in X$ and $0 \le t \le 1$, $f(tx + (1-t)y) \le tf(x) + (1-t)f(y)$.

§3. Banach spaces.

A seminorm on a linear space X is a function $p: X \to \mathbb{R}$ such that $p(x) \geq 0$, $p(x+y) \leq p(x)+p(y)$ and $p(\lambda x) = |\lambda| p(x)$ for all $x, y \in X$ and $\lambda \in \mathbb{C}$. If, in addition, $p(x) = 0 \Rightarrow x = 0$, p is called a norm. One writes $\|\cdot\|$ for a norm and the pair $(X, \|\cdot\|)$ is called a normed space. A normed space X becomes a metric space if we define $d(x,y) = \|x-y\|$. A Banach space is a complete normed linear space with respect to the metric defined by the norm. A linear functional on a linear space X is a map $f: X \to \mathbb{F}$ such that $f(x+y) = f(x) + f(y)$ and $f(\lambda x) = \lambda f(x)$ for all $x, y \in X$ and $\lambda \in \mathbb{F}$. Under the pointwise operations, the set of all linear functionals on X is a linear space. If X is a normed linear space, then the set of all bounded linear functionals on X together with the norm $\|f\| \equiv \sup\{|f(x)| : \|x\| \leq 1\}$ forms a Banach space, which we denote by X^* and call the dual space of X. The dual space of X^* is X^{**}. If $x \in X$, then x defines an element $\hat{x}$ of X^{**}; namely, define $\hat{x}: X^* \to \mathbb{F}$ by $\hat{x}(x^*) = x^*(x)$ for every x^* in X^*. Then $\|\hat{x}\| = \|x\|$ for all x in X. The map $x \to \hat{x}$ of X into X^{**} is called the natural map of X into its second dual. A normed space X is called reflexive if $X^{**} = \{\hat{x} : x \in X\}$. A reflexive space X is isometrically isomorphic to X^{**}, and hence must be a Banach space. It is not true, however, that a Banach space X that is isometric to X^{**} is reflexive. In fact, James [32] gives an example of a nonreflexive Banach space X that is isometric to X^{**}. Every finite dimensional normed space is reflexive. Also for $1 < p < \infty$, ℓ^p and L^p spaces are reflexive. But ℓ^1, L^1, c, and c_0 are nonreflexive Banach spaces.

The weak topology on a normed linear space X, denoted by $\sigma(X,X^*)$, is the topology defined by the family of seminorms $\{p_{x^*}: x^* \in X^*\}$, where $p_{x^*}(x) = |\langle x,x^*\rangle| = |x^*(x)|$. If X is an infinite dimensional nomred space then the norm topology is strictly stronger than the weak topology, but for finite dimensional spaces the norm topology coincides with the weak topology.

A sequence $\{x_n\}$ in a normed linear space X converges weakly to an element $x \in X$ if and only if $\lim_n x^*(x_n) = x^*(x)$ for all $x^* \in X^*$. We write $w\text{-}\lim_n x_n = x$ or, in short, $x_n \to x$ weakly. A convergent sequence is weakly convergent but not conversely. If $B \subseteq X$, then B is bounded if and only if $\sup\{\|b\|: b \in B\} < \infty$. A weakly convergent sequence is bounded. A sequence $\{x_n\}$ in X is a weakly Cauchy sequence if for every x^* in X^*, $\{x^*(x_n)\}$ is a Cauchy sequence in $\mathbf{F}$. X is said to be weakly sequentially complete if every Cauchy sequence in X converges weakly. If a set B in X has the property that every sequence in B has a subsequence that converges weakly to a point in X, then B is said to be weakly conditionally sequentially compact. If every subsequence in B has a subsequence that converges weakly to a point in B, then B is called weakly sequentially compact. If $B \subseteq X$, we say that B is closed (weakly closed) if B is closed in the original topology (weak topology) of X. And if we say that B is compact (weakly compact), we mean that it is compact in the original topology (weak topology) of X. Compact sets are weakly compact. A convex set B in X is closed iff it is weakly closed.

3.1. Theorem. Let X be a Banach space.

(a) A subset M of X is weakly compact if and only if it is weakly closed and weakly conditionally sequentially compact. [Eberlein's theorem].

(b) X is reflexive iff the closed unit ball $\{x \in X: \|x\| \leq 1\}$ is weakly compact.

(c) If M is a compact subset of X, then $\overline{\text{conv}}(M)$ is compact [Mazur's theorem].

(d) If M is a weakly compact subset of X, then $\overline{\text{conv}}\, M$ is weakly compact [Krein-Smulian's theorem].

3.2. Theorem. Let X be a reflexive Banach space.

(a) Every closed bounded convex subset of X is weakly compact.

(b) Every bounded sequence in X has a weakly convergent subsequence.

(c) If all the weakly convergent subsequences of a bounded sequence $\{x_n\}$ in X have the same limit x, then $x_n \to x$ weakly.

3.3. Theorem. Let M be a nonempty closed convex bounded subset of a reflexive Banach space X. Then each continuous convex real valued function f on M has a minimum.

A complex linear space X is called an inner product space, if to each pair of elements x,y in X there is associated a complex number $\langle x,y\rangle$ (called the inner product of x and y) with the following properties:

(a) $\langle x+y,z\rangle = \langle x,z\rangle + \langle y,z\rangle$

(b) $\langle x,y\rangle = \overline{\langle y,x\rangle}$ (the bar denotes the complex conjugate)

(c) $\langle \lambda x,y\rangle = \lambda\langle x,y\rangle$

(d) $\langle x,x\rangle \geq 0$ and $\langle x,x\rangle \neq 0$ if $x \neq 0$.

An inner product space X becomes a normed space if we define $\|x\| = \langle x,x\rangle^{1/2}$. A complete inner product space is called a Hilbert space. Every Hilbert space is a reflexive Banach space.

3.4. Theorem. Let X be a Hilbert space. Then for all x,y in X we have:

(a) $|\langle x,y\rangle| \leq \langle x,x\rangle^{1/2}\langle y,y\rangle^{1/2}$ [Cauchy-Schwarz inequality].

(b) $\|x+y\|^2 = \|x\|^2 + 2\mathrm{Re}\langle x,y\rangle + \|y\|^2$ [Polar identity].

(c) $\|x+y\|^2+\|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$ [Parallelogram law].

3.5. Theorem. If X is a Hilbert space, M a closed convex nonempty subset of X and $x \in X$, then there is a unique point $x_0 \in M$ such that

$$\|x-x_0\| = \mathrm{dist}(x,M) = \inf\{\|x-y\| : y \in M\} .$$

A Banach space X is said to be uniformly convex [17], if for each $\varepsilon \in (0,2]$, there exists a $\delta(\varepsilon) > 0$ such that for all x,y in X whenever $\|x\| \leq 1$, $\|y\| \leq 1$, $\|x-y\| \geq \varepsilon$, then it follows that $\|x+y\| \leq 2\ (1 - \delta(\varepsilon))$. The function defined on $(0,2]$ by $\varepsilon \to \delta(\varepsilon)$ is said to be the modulus of convexity of X. If we define the modulus of convexity of a Banach space as the function on $[0,2]$ defined by the relation

$$\delta(\varepsilon) = 1/2\inf\{2 - \|x+y\| : \|x\| \le 1, \|y\| \le 1, \|x-y\| \ge \varepsilon\},$$

then it is clear that a Banach space X is uniformly convex if $\delta(\varepsilon) > 0$ for every $\varepsilon > 0$. Geometrically uniform convexity for X means that for any two points x,y on the boundary of the unit ball, the midpoint of the connecting line segment, i.e., $z = \frac{x+y}{2}$, lies within a ball about the origin of radius $r < 1$, where r depends on the distance $\|x-y\|$. Roughly speaking this means that the boundary of the unit ball is round. For example, we see: $X = \mathbb{R}^2$ with the euclidean norm, $\|x\| = \left(a_1^2 + a_2^2\right)^{1/2}$, $x = (a_1,a_2) \in X$ is a uniformly convex Banach space. But $X = \mathbb{R}^2$ with the maximum norm, $\|x\| = \max(|a_1|, |a_2|)$ is not a uniformly convex Banach space.

3.6. Examples. (a) Every Hilbert space is uniformly convex.

Proof: If $\|x\| \le 1$, $\|y\| \le 1$ and $\|x-y\| \ge \varepsilon$, then by the Parallelogram law it follows that $\|x+y\|^2 \le 2 - \varepsilon^2$. Let $\varepsilon \to \delta(\varepsilon) = 1 - \left(1 - (\varepsilon/2)^2\right)^{1/2}$. Thus we have $\|x+y\| \le 2(1 - \delta(\varepsilon))$.

(b) The L^p and ℓ^p spaces are uniformly convex for $p \in (1,\infty)$ (see [17] or [28]).

3.7. Theorem. Every uniformly convex Banach space is reflexive (see [43] or [51], p.127).

3.8. Proposition. Let $\{a_n\}$ and $\{b_n\}$ be two sequences in a uniformly convex Banach space X. Suppose

$$\overline{\lim_n}\|a_n\| \le a\,, \quad \overline{\lim_n}\|b_n\| \le a \quad \text{and} \quad \lim_n\|\lambda a_n + (1-\lambda)b_n\| = a$$

for fixed $\lambda \in (0,1)$ and fixed $a \ge 0$, then $\lim_n(a_n - b_n) = 0$.

3.9. Definitions. (a) A Banach space X is said to be locally uniformly convex if for any $x \in X$, $\|x\| = 1$ and $\varepsilon > 0$, there exists $\delta_x(\varepsilon) > 0$ such that for all y in X whenever $\|y\| \le 1$, $\|x-y\| \ge \varepsilon$ it follows that $\|x+y\| \le (1-\delta_x(\varepsilon))$. [42]

(b) A Banach space X is said to be strictly convex if every point x, $\|x\| = 1$ is an extreme point of the closed unit ball $\{x \in X: \|x\| \le 1\}$. ([17], [40]).

Obviously every uniformly convex space is locally uniformly convex. Lovaglia [42] shows that there exists a locally uniformly convex Banach space which is not isomorphic with a uniformly convex space.

3.10. Theorem. For a Banach space X the following are equivalent:

(a) X is strictly convex;

(b) If for x,y in X, $\|x\| = \|y\| \le 1$, $x \ne y$ then $\|x+y\| < 2$;

(c) If x,y in X and $\|x+y\| = \|x\| + \|y\|$ then $x = ay$ for some a.

The following theorem gives examples of strictly convex spaces.

3.11. Theorem. Every locally uniformly convex (in particular, uniformly convex) Banach space is strictly convex.

Lovaglia [42] shows that there exists a strictly convex Banach space which is not locally uniformly convex.

3.12. Lemma. If X is a strictly convex Banach space, M is a nonempty weakly compact convex subset of X, and $x \in X$, then there is a unique point $m_0 \in M$ such that

$$\|x-m_0\| = \inf\{\|x-m\| : m \in M\} .$$

(see [3] or [31] p.291).

§4. Duality mapping, Opial's condition, Normal structure And Approximately symmetric relations.

4.1. Definition. (Browder [10] or [13])

Let X be a Banach space with dual space X^* and let ϕ be a continuous strictly increasing real valued function on R^+ with $\phi(0) = 0$. A mapping $f: X \to X^*$ is said to be a duality mapping with gauge function ϕ if the following conditions are satisfied:

(i) For each $x \in X$, $\langle x, f(x)\rangle = \|f(x)\| \, \|x\|$

(ii) For each $x \in X$, $\|f(x)\| = \phi(\|x\|)$.

4.2. Lemma.

(a) For any Banach space X, there exist duality mappings f of X into X^*.

(b) If X is a strictly convex Banach space, there is exactly one duality mapping f for every function ϕ in Definition (4.1).

(c) If X is reflexive and strictly convex, then the duality mapping f corresponding to each ϕ is a continuous mapping of X into the weak topology of X^* (see [10], p.371).

4.3. Lemma. If $1 < p < \infty$, the Banach space ℓ^p has a weakly continuous duality mapping into its dual space ℓ^q, $q = p(p-1)^{-1}$ (see [13], p.267).

Opial [45] proved that for $1 < p < 2$ and $2 < p < \infty$, none of the spaces $L^p[0,2\pi]$ have a weakly continuous duality mapping (for $p = 4$, it has been explicitly proved in [13]).

4.4. Definition. (Opial [45])

A Banach space X satisfies Opial's condition, if for each $x \in X$ and each sequence $\{x_n\}$ weakly convergent to x,

$$\liminf_n \|x_n - y\| > \liminf_n \|x_n - x\| \tag{4.4.1}$$

holds for $y \neq x$.

The following lemma due to Opial [45] gives examples of Banach spaces which satisfy Opial's condition.

4.5. Lemma. Every Hilbert space satisfies Opial's condition.

Proof: Since every weakly convergent sequence is necessarily bounded, both limits in (4.4.1) are finite and the inequality follows from the polar identity

$$\|x_n-y\|^2 = \|x_n - x + x - y\|^2$$
$$= \|x_n-x\|^2 + \|x-y\|^2 + 2\,\mathrm{Re}\langle x_n-x,\ x-y\rangle ,$$

the last term tends to zero as n tends to infinity.

To give a partial positive answer of a natural question as to what extent is the Opial's condition valid in general Banach spaces one has the following:

4.6. Lemma. If a Banach space X admits a weakly continuous duality mapping, then X satisfies Opial's condition (see [27]).

Thus, by lemma (4.3), the spaces ℓ^p, $1 < p < \infty$ satisfy Opial's condition. However, there are Banach spaces which fail to satisfy Opial's condition, e.g., the spaces $L^p[0,2\pi]$ $(p \neq 2)$ (see [45]). The converse implication of lemma 4.6 does not hold. Because, if $1 < p \neq q < \infty$ the Hilbert product of ℓ^p and ℓ^q satisfies Opial's condition [27], but Bruck [16] showed that this does not admit a weakly continuous duality mapping.

4.7. Definition. (Brodski and Milman [9])

A convex bounded subset M of a Banach space X is said to have a normal structure if each convex subset N of M for which $\operatorname{diam} N > 0$ contains a point z such that

$$\sup\{\|x-z\| : x \in N\} < \operatorname{diam} N .$$

The following theorem gives examples of sets with normal structure.

4.8. Theorem. (a) Every compact convex subset of a Banach space X has a normal structure.

(b) In every uniformly convex Banach space, each closed convex and bounded set has a normal structure.

A Banach space X is said to have a normal structure if each closed convex bounded subset of X has a normal structure. Thus from the above result, every uniformly convex Banach space has a normal structure. However, there are Banach spaces which lack the normal structure property.

4.9. Examples.

(a) For $\beta \geq 1$, let X_β be the Hilbert space ℓ^2 renormed by taking

$$\|x\| = \max\{\|x\|_2 , \beta\|x\|_\infty\} , \qquad x \in \ell^2 .$$

The space X_β is a reflexive Banach space, but for $\beta \geq \sqrt{2}$, X_β fails to have a normal structure (see [4]).

(b) Consider the space c_0. The formula

$$\|x\| = \sup \left[\sum_{i=1}^{\infty} 2^{-2i} x^2(\alpha_i) \right]^{1/2} ,$$

where the supremum is taken over all permutations α of N, defines an equivalent norm [19] on c_0, which is known to be a locally uniformly convex space [46]. But c_0, endowed with this norm, fails to have a normal structure [27].

4.10. Lemma. A Banach space X with Opial's condition has a normal structure (see [27]).

A converse of this lemma does not hold, e.g., if $1 < p < \infty$, $p \neq 2$, $L^p[0,2\pi]$ has a normal structure, since these are uniformly convex spaces but we have remarked above that none of these spaces satisfies Opial's condition.

In a general Banach space we say (Birkhoff [5], James [32]) that w is orthogonal to v, $w \perp v$, if $\|w\| \leq \|w + \lambda v\|$ for scalar λ. In general $\perp$ is not symmetric. (Indeed, symmetry characterizes Hilbert spaces of dimensions strictly greater than 2.)

4.11. Definition. (Karlovitz [35]).

(a) The relation $\perp$ is called approximately symmetric if for each $x \in X$ and $\varepsilon > 0$, there exists a closed linear subspace $U = U(x,\varepsilon)$ such that

(i) U has finite codimension, and

(ii) $\|u\| \leq \|u + \lambda x\|$ for each $u \in U$, $\|u\| = 1$, and each λ, $\lambda \geq \varepsilon$.

To see that this is a natural generalization of the Hilbert space case, we note that if $\perp$ is symmetric then we can choose $U = \{u: f(u) = 0\}$ where f is a linear functional such that $\|f\| = 1$ and $f(x) = \|x\|$. Then $x \perp u$ for each $u \in U$; and, by symmetry, $u \perp x$ for each $u \in U$, which is stronger than (ii).

(b) The relation $\perp$ is called uniformly approximately symmetric if it is approximately symmetric and (ii) is replaced by the stronger condition:

(ii)' $\|u\| \le \|u + \lambda x\| - \delta$, for some $\delta = \delta(x, \varepsilon) > 0$, for each $u \in U$, $\|u\| = 1$, and each λ, $|\lambda| \ge \varepsilon$.

If the Banach space X has a uniformly convex ball, then approximate symmetry and uniform approximate symmetry are equivalent. In a Hilbert space, and more generally for the classical spaces ℓ^p, $1 < p < \infty$, the relation $\perp$ is uniformly approximately symmetric. On the other hand, in both spaces c_0 and L^p, $p \neq 2$, $\perp$ fails to be even approximately symmetric. (For further details see [35]).

A reflexive Banach space which lacks the normal structure property (see [4]) and the relation $\perp$ is not approximately symmetric (see [35]), is the space ℓ^2 renormed by $\|\cdot\| = \max\{1/2\|\cdot\|_2, \|\cdot\|_\infty\}$, where $\|\cdot\|_2$ denotes the ℓ^2 norm and $\|\cdot\|_\infty$ denotes the ℓ^∞ norm.

4.12. <u>Theorem</u>. Let X be a reflexive Banach space. Suppose that the relation $\perp$ is uniformly approximately symmetric. Then X satisfies Opial's condition. The converse also holds if in addition X is separable (see [35]).

§5. Classical fixed point theorems.

Let (X,d) be a metric space and $f: X \to X$ a self-mapping.

5.1. Definition. (a) An element $x \in X$ is called a fixed point of f if $f(x) = x$.

(b) f is called contractive (or contraction) if

$$d(f(x),f(y)) \leq hd(x,y)$$

for all x,y in X and for a fixed number h, $0 \leq h < 1$.

The well known Banach's fixed point theorem is:

5.2. Theorem. A contractive self-mapping of a complete metric space (X,d) has a unique fixed point.

A natural generalization of the class of contractive mappings is that of the so called nonexpansive mappings, i.e.,

5.3. Definition. A mapping $f: M \subseteq X \to X$ is called nonexpansive if

$$d(f(x),f(y)) \leq d(x,y)$$

for all x,y in M.

Simple examples show that, in general, for nonexpansive mappings the existence of fixed points is not assured. But there are nonexpansive mappings which have lots of fixed points.

Examples. (a) Let $X = \mathbb{R}$ and $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x+1$. Then, clearly $d(f(x),f(y)) = |x-y| = d(x,y)$ and f is without fixed points.

(b) Let $X = \mathbb{R}\times\mathbb{R}$ and define $f(x,y) = (x,0)$. Then clearly f is nonexpansive and every point of the form $(x,0)$ is a fixed point for this mapping.

In what follows we give some well-known results concerning the existence of fixed points for nonexpansive mappings.

5.4. Theorem (Browder [11]). Let M be a nonempty closed bounded convex subset of a Hilbert space X. Then any nonexpansive mapping f of M into itself has a fixed point.

In the case of uniformly convex Banach spaces, Theorem 5.4 was proved independently by Browder [12] and Göhde [26]. The following is an improved result due to Kirk [38].

5.5. Theorem. Let M be a nonempty closed bounded convex subset of a reflexive Banach space X, and suppose that M has a normal structure. If f is a nonexpansive mapping of M into itself, then f has a fixed point.

5.6. Corollary. Let M be a nonempty closed bounded convex subset of a reflexive Banach space X which satisfies Opial's condition.

Then each nonexpansive mapping of M into itself has a fixed point.

Proof: Since the Opial's condition implies that M has a normal structure (lemma 4.10), result follows immediately from Theorem 5.5.

In [36], Karlovitz has shown that in certain reflexive Banach spaces, the normal structure assumption of Theorem 5.5 is not essential. In [35], he also proves the following:

5.7. Theorem. Let M be a nonempty closed convex bounded subset of a reflexive Banach space X. Suppose that the relation $\perp$ is uniformly approximately symmetric. Then each nonexpansive mapping f of M into itself has a fixed point.

In [34], Kannan has investigated conditions under which two maps f_1 and f_2 each mapping a complete metric space X into itself have a unique common fixed point.

5.8. Theorem. If f_1 and f_2 are two mappings each mapping a complete metric space (X,d) into itself and if

$$d(f_1(x),f_2(y)) \leq h[d(x,f_1(x)) + d(y,f_2(y))]$$

where x,y in X and h is a fixed number with $0 < h < 1/2$. Then f_1 and f_2 have a unique common fixed point.

If $f_1 = f_2$ on X; then he obtained the following:

5.9. <u>Theorem</u>. Let f be a mapping of a complete metric space (X,d) into itself. If

$$d(f(x),f(y)) \leq h[d(x,f(x)) + d(y,f(y))]$$

for all x,y in X and for a fixed h, $0 < h < 1/2$. Then f has a unique fixed point.

5.10. <u>Definitions</u>. (a) Let (X,d_1) and (Y,d_2) be two metric spaces, by a multivalued map

$$J: M \subseteq X \to 2^Y$$

we mean a map which assigns to each point $x \in M$ a subset $J(x) \subseteq Y$. Here 2^Y denotes the collection of all nonempty subsets of Y.

Every single-valued map $f: X \to Y$ can be identified with a multivalued map by setting $J(x) = \{f(x)\}$ for all $x \in X$.

(b) A point x is called a fixed point of $J: M \to 2^M$ if

$$x \in J(x).$$

Note that $J(M)$ denotes the union of all sets $J(x)$ over $x \in M$, i.e.,

$$J(M) = \bigcup_{x \in M} J(x).$$

The following definition of a multivalued nonexpansive type map which, in the single-valued case coincides with the usual notion of nonexpansive mapping, is due to Husain and Tarafdar [30].

5.11. Definition (Husain and Tarafdar [30]).

Let M be a nonempty subset of a metric space (X,d). A multi-valued mapping $J: M \to 2^X$ is said to be nonexpansive (we will call it (H·T)-nonexpansive map in the sequal) if given $x \in M$ and $u_x \in J(x)$ there is a $u_y \in J(y)$ for each y in M such that

$$d(u_x,u_y) \le d(x,y) .$$

5.12. Example [30]. Let $\{f_\alpha: \alpha \in I\}$ be a family of single-valued nonexpansive self-mappings on a subset M of a metric space (X,d). Then the multivalued mapping $J: M \to 2^M$ defined by

$$J(x) = \{f_\alpha(x): \alpha \in I\}, \quad x \in M$$

is clearly a (H·T)-nonexpansive mapping.

In the sequal, this definition of (H·T)-nonexpansive maps will be one of our main concerns. It will be shown in Chapter II that not every (H·T)-nonexpansive map on a nonempty closed convex bounded subset M of a Banach space X has a fixed point. However, Husain and Tarafdar [30] have proved the following fixed point theorem for (H·T)-nonexpansive maps on the subsets of the real line $\mathbb{R}$.

5.13. Theorem. Let M be a nonempty closed convex bounded subset of $\mathbb{R}$. Let J be a (H·T)-nonexpansive mapping on M with closed and convex subsets of M as values (i.e. $J(x)$ is closed and convex for each $x \in M$). Then there is a point $x_0 \in M$ such that $x_0 \in J(x_0)$.

CHAPTER II

FIXED POINTS FOR (H·T)-NONEXPANSIVE AND s-NONEXPANSIVE MAPS

In this chapter, we study two concepts of nonexpansive multivalued maps, one of which was introduced by Husain and Tarafdar [30]. Both these concepts generalize the usual notion of nonexpansiveness for single-valued maps. Examples 2.1(a), (b) show that not every (H·T)-nonexpansive multivalued map on a nonempty closed convex bounded subset of a Banach space has a fixed point. However, we prove some fixed point theorems for (H·T)-nonexpansive maps which are improved in Chapter III. Our fixed point result for s-nonexpansive map (Theorem 3.3) extends a result due to Husain and Tarafdar (Theorem I.5.13) for s-nonexpansive maps and includes the result of Browder (Theorem I.5.4) as a special case.

§1. Examples and general properties of (H·T)-nonexpansive maps.

Let M be a nonempty subset of a metric space (X,d). A multivalued mapping $J: M \to 2^X$ is said to be (H·T)-nonexpansive if given $x \in M$ and $u_x \in J(x)$ there is a $u_y \in J(y)$ for each $y \in M$, such that

$$d(u_x,u_y) \le d(x,y) .$$

We note that each single-valued map is nonexpansive iff it is (H·T)-nonexpansive.

1.1. <u>Examples</u>. (a) Let $X = \ell^2$ with the metric

$$d(x,y) = \left(\sum_{i=1}^{\infty} |x_i - y_i|^2\right)^{1/2}$$

for each $x = \{x_1, x_2, \ldots\}$ and $y = \{y_1, y_2, \ldots\}$ in ℓ^2. Consider M, the Hilbert cube, i.e.,

$$M = \{x \in \ell^2 : |x_i| \le 1/i \text{ for all } i \ge 1\}.$$

For each $n \ge 1$, we define

$$f_n(x) = \{x_1, x_2, \ldots, x_n, 0, 0, \ldots\}, \qquad (x \in M).$$

Clearly for each $n \ge 1$, f_n is a single-valued nonexpansive mapping of M into itself. Define

$$J(x) = \{f_n(x) : n \ge 1\}, \qquad (x \in M).$$

Then J is a (H·T)-nonexpansive mapping of M into 2^M. For, suppose $x \in M$, $u_x \in J(x)$ implies there exists some j such that $u_x = f_j(x)$. Let y be an arbitrary element of M, put $u_y = f_j(y)$. Then clearly $u_y \in J(y)$ and

$$\begin{aligned} d(u_x, u_y) &= d(f_j(x), f_j(y)) \\ &\le d(x,y) \qquad \text{(by nonexpansiveness of } f_j\text{)}. \end{aligned}$$

(b) Let M be the closed unit ball in R^2 with the usual euclidean metric d. Define

$$J(x) = \left\{\pm\left(\frac{x_1}{2},x_2\right), \pm\left(x_1,\frac{x_2}{2}\right)\right\}, \qquad (x = (x_1,x_2) \in M).$$

Then J is (H·T)-nonexpansive mapping of M into 2^M. For, if $x = (x_1,x_2) \in M$ and $u_x = \left(\frac{x_1}{2},x_2\right) \in J(x)$. Let $y = (y_1,y_2)$ be an arbitrary element of M. Then there is $u_y = \left(\frac{y_1}{2},y_2\right) \in J(y)$ such that

$$\begin{aligned}[d(u_x,u_y)]^2 &= \frac{1}{4}(x_1-y_1)^2 + (x_2-y_2)^2 \\ &\le (x_1-y_1)^2 + (x_2-y_2)^2 = [d(x,y)]^2\end{aligned}$$

i.e., $d(u_x,u_y) \le d(x,y)$. The other three cases work in a similar fashion.

(c) Let $M = \{x \in R: x \ge 0\}$ with the usual metric d. Then

$$J(x) = \left\{\frac{x}{1+x}, \ln(1+x)\right\}, \qquad (x \in M)$$

defines a (H·T)-nonexpansive mapping. For, if $u_x = \frac{x}{1+x} \in J(x)$ then for each element $y \in M$ there is $u_y = \frac{y}{1+y} \in J(y)$ with

$$d(u_x,u_y) = \left|\frac{x}{1+x} - \frac{y}{1+y}\right| = \frac{|x-y|}{(1+x)(1+y)} \le |x-y| = d(x,y).$$

If $u_x = \ln(1+x) \in J(x)$, then its derivative is: $u_x' = \frac{1}{1+x}$. So, by the mean value theorem there exists η with $x < \eta < y$, for all $y \in M$ and $|u_x - u_y| = |u_x'(\eta)(x-y)| < |x-y|$. Note that $u_y = \ln(1+y) \in J(y)$.

Further examples of (H·T)-nonexpansive maps will be considered later. Now we present some elementary results for (H·T)-nonexpansive mappings, some of which will be used in the sequel.

Let $J_1, J_2 \colon M \subseteq X \to 2^X$ be two multivalued mappings. Define the product by

$$J_1 J_2(x) = \bigcup_{y \in J_2(x)} J_1(y) \qquad (x \in M),$$

which is also a multivalued mapping of M into 2^X.

1.2. <u>Lemma</u>. The product of two (H·T)-nonexpansive mappings of M into 2^M is a (H·T)-nonexpansive mapping.

<u>Proof</u>: Let $J_1, J_2 \colon M \to 2^M$ be two (H·T)-nonexpansive maps. Let $x \in M$ and $u_x \in J_1 J_2(x)$, that is, $u_x \in J_1(a)$ for some $a \in J_2(x)$. Since J_2 is a (H·T)-nonexpansive mapping, there is a $b_y \in J_2(y)$ for all $y \in M$ such that

$$d(a, b_y) \le d(x, y) .$$

Also, since $u_x \in J_1(a)$, by definition of J_1 , there is a $v \in J_1(b_y)$ such that

$$d(u_x,v) \le d(a,b_y)$$
$$\le d(x,y) .$$

Clearly $v \in J_1(b_y) \subset \bigcup_{z\in J_2(y)} J_1(z) = J_1J_2(y)$ and the lemma is proved.

1.3. Lemma. Let M be a subset of a metric space (X,d) and $J: M \to 2^M$ a $(H\cdot T)$-nonexpansive mapping.

(a) If $M \subseteq N \subseteq X$ and $f: N \to M$ is a single-valued nonexpansive mapping then Jf is a $(H\cdot T)$-nonexpansive mapping of N into 2^M.

(b) If $f: M \to M$ is a single-valued nonexpansive mapping, then fJ is a $(H\cdot T)$-nonexpansive mapping of M into $2^{f(M)}$.

Proof: (a) Clearly $(Jf)(x) = Jf(x)$ maps N into $2^M \subseteq 2^N$. Let x,y be in M and $u \in Jf(x)$. Since $f(x)$ and $f(y)$ are in M, by the definition of J, there is a $v \in Jf(y) = (Jf)(y)$ such that

$$d(u,v) \le d(f(x),f(y))$$
$$\le d(x,y) . \qquad \text{(since } f \text{ is nonexpansive).}$$

(b) If $x \in M$, $(fJ)(x) = fJ(x) = \{f(z): z \in J(x) \subseteq M\} \subseteq f(M)$. Thus $(fJ)(x)$ maps M into $2^{f(M)}$. Now let $u \in (fJ)(x)$ implies $u = f(w)$, for some $w \in J(x)$. By the definition of J, there is a $z \in J(y)$ for each $y \in M$ such that

$$(1.3.1) \qquad d(w,z) \le d(x,y) .$$

Clearly $z \in M$, put $v = f(z)$. Then $v \in \{f(a): a \in J(y)\} = (fJ)(y)$ and

$$\begin{aligned} d(u,v) &= d(f(w),f(z)) \\ &\le d(w,z) \qquad \text{(by nonexpansiveness of } f\text{)} \\ &\le d(x,y) \,. \qquad \text{by (1.3.1)} \end{aligned}$$

1.4. Lemma. Let M be a nonempty subset of a normed linear space X and $J: M \to 2^X$ a (H·T)-nonexpansive mapping, we define a set-valued mapping $\tilde{J}$ by the relation

$$\tilde{J}(x) = \operatorname{conv} J(x) \qquad (x \in M) \,.$$

Then $\tilde{J}$ is a (H·T)-nonexpansive mapping.

Proof: Given $x \in M$, $u \in \tilde{J}(x) = \operatorname{conv} J(x)$. Then, by definition of $\operatorname{conv} J(x)$, $u = \sum_{i=1}^{n} \lambda_i x_i$, where $0 \le \lambda_i \le 1$, $\sum_{i=1}^{n} \lambda_i = 1$ and $x_i \in J(x)$. Since J is a (H·T)-nonexpansive, there is a $y_i \in J(y)$ for each $y \in M$ such that

$$\|x_i - y_i\| \le \|x-y\|$$

Put $v = \sum_{i=1}^{n} \lambda_i y_i$. Then, clearly $v \in \operatorname{conv} J(y) = \tilde{J}(y)$ and

$$\|u-v\| = \|\sum_{i=1}^{n}\lambda_i x_i - \sum_{i=1}^{n}\lambda_i y_i\| \le \sum_{i=1}^{n}\lambda_i\|x_i-y_i\| \le \|x-y\| ,$$

hence the lemma is proved.

1.5. Remark. If in lemma 1.4 we consider X to be a Banach space and J a compact-valued (H·T)-nonexpansive mapping, then $\tilde{J}(x) = \overline{\text{conv}}\, J(x)$ also defines a compact-valued (H·T)-nonexpansive mapping.

The following lemma gives a very useful property of a (H·T)-nonexpansive mapping. This property will be frequently used to derive fixed point results in the sequel.

1.6. Lemma. Let M be a nonempty subset of a Banach space X which satisfies Opial's condition and $J: M \to 2^M$ a compact-valued (H·T)-nonexpansive mapping. Let $\{x_n\} \subset M$ be a sequence which converges weakly to an element $x \in M$ and if $y_n \in x_n - J(x_n)$ such that $\{y_n\}$ converges to $y \in X$ then $y \in x - J(x)$.

Proof: If $y_n \in x_n - J(x_n)$, then we can write $y_n = x_n - u_n$ for some $u_n \in J(x_n)$. Since J is a (H·T)-nonexpansive map, there is a $v_n \in J(x)$ such that

$$\|u_n - v_n\| \le \|x_n - x\| ,$$

it follows that

$$(1.6.1) \qquad \liminf_n \|x_n - x\| \geq \liminf_n \|u_n - v_n\|$$

$$= \liminf_n \|x_n - y_n - v_n\| \; .$$

Since every weakly convergent sequence is necessarily bounded, both limits in (1.6.1) are finite. Now, since $\{v_n\}$ is contained in the compact set $J(x)$, there is a subsequence of $\{v_n\}$, also denoted by $\{v_n\}$, converging to $v \in J(x)$. Therefore,

$$\liminf_n \|x_n - y_n - v_n\| = \liminf_n \|x_n - y_n - v_n - (y+v) + (y+v)\|$$

$$\geq \liminf_n [\|x_n - (y+v)\| - \|(y_n + v_n) - (y+v)\|]$$

$$\geq \liminf_n \|x_n - (y+v)\| + \liminf_n (-\|y_n + v_n - y - v\|)$$

$$= \liminf_n \|x_n - (y+v)\| \; .$$

Thus we have shown:

$$\liminf_n \|x_n - x\| \geq \liminf_n \|x_n - (y+v)\| \; .$$

Since $x_n \to x$ weakly, we have, by Opial's condition, $x = y + v$, so $y = x - v \in x - J(x)$ and the lemma is proved.

1.7. Lemma. Let M be a nonempty subset of a metric space (X,d). Suppose

(i) $J\colon M \to 2^X$ is a (H·T)-nonexpansive mapping with $J(x)$ being a closed set for each $x \in M$.

(ii) If $x_0 \in M$ and $x_n \in J(x_{n-1})$, $n = 1,2,\ldots$, such that $\{x_n\}$ converges to an element $x \in M$.

Then x is a fixed point of J.

Proof: Since $x_n \in J(x_{n-1})$ and J is (H·T)-nonexpansive, there is $v_n \in J(x)$ such that

$$d(x_n,v_n) \le d(x_{n-1},x) .$$

But

$$\begin{aligned} d(x,v_n) &\le d(x,x_n) + d(x_n,v_n) \\ &\le d(x,x_n) + d(x_{n-1},x) \end{aligned}$$

implies $d(x,v_n) \to 0$ as $n \to \infty$, since $x_n \to x$ as $n \to \infty$. As $v_n \in J(x)$ and $J(x)$ is closed, we get $x \in J(x)$, proving that x is a fixed point of J.

§2. (H·T)-nonexpansive maps and fixed points.

To see that not every (H·T)-nonexpansive mapping on a nonempty closed convex bounded subset M of a Banach space X has a fixed point, we consider the following examples.

2.1. Examples. (a) Let $X = c_0$, the Banach space of all complex sequences converging to zero with the sup norm. Let M denote the closed unit ball of c_0. For each integer $n \geq 1$, let

$$f_n(x) = e_n + s_n(x) \qquad (x \in M)$$

where $e_n = \{\delta_{ni}\}_{i\geq 1}$ (the usual basis of c_0) and $s_n(x) = \{x_1, x_2, \ldots, x_{n-1}, 0, x_n, \ldots\}$ with 0 in the n^{th} position. It is easy to see that each $f_n : M \to M$ is a single-valued nonexpansive fixed-point-free map. The multivalued map

$$J(x) = \bigcup_{n\geq 1} \{f_n(x)\} \qquad (x \in M)$$

is clearly (H·T)-nonexpansive and has no fixed point. We note that M is not weakly compact or equivalently, X is not reflexive.

Similarly, if we fix $n_0 \geq 2$ for each positive integer $n \leq n_0$, let

$$f_n(x) = \{\underbrace{1,1,1,\ldots,1}_{n\text{-times}}, x_1, x_2, \ldots\} \qquad (x \in M)$$

Then for each n, f_n is a nonexpansive fixed-point-free mapping of M into itself. The compact-valued (H·T)-nonexpansive map

$$J(x) = \bigcup_{n=1}^{n_0} \{f_n(x)\}$$

has no fixed point.

(b) Let $X = C[0,1]$, the Banach space of continuous real-valued functions on $[0,1]$, which is not reflexive. Let

$$M = \{f \in X: f(0) = 0,\ f(1) = 1,\ 0 \le f(x) \le 1\}.$$

M is closed, convex and bounded. Define the mapping J as follows:

$$J(f(x)) = \{xf(x),\ x^2f(x),\ x^3f(x), \dots\}$$

where $f \in M$ and $0 \le x \le 1$. Then, it is easily seen that J is a (H·T)-nonexpansive mapping of M into 2^M, and has no fixed point.

On the other hand, Husain and Tarafdar [30] proved that if M is a compact interval of the real line, then each (H·T)-nonexpansive closed-convex valued map $J: M \to 2^M$ has a fixed point.

Now we prove a fixed point theorem for (H.T)-nonexpansive multi-valued maps on nonempty weakly compact convex subsets of a Banach space under certain conditions, since, as the above examples show, in general such maps need not have fixed points.

2.2. Theorem. Let M be a nonempty weakly compact convex subset of a Banach space X which satisfies Opial's condition (Definition I.4.4). Let $J: M \to 2^M$ be a (H.T)-nonexpansive compact-valued map. Assume the following holds:

(2.2.1) For a fixed $w \in M$ and $0 < \gamma_n < 1$, $\gamma_n \to 1$, there is $u_x \in J(x)$ for all $x \in M$ such that each single-valued self map $f_n(x) = \gamma_n u_x + (1-\gamma_n)w$ of M has a fixed point $x_n \in M$.

Then J has a fixed point.

Proof: Since M is weakly compact and $x_n \in M$, there is a weakly convergent subsequence of $\{x_n\}$. We denote the subsequence also by $\{x_n\}$ for convenience. Let $x_0 = w\text{-}\lim_n x_n$. By the weak closedness of M, $x_0 \in M$. We prove that x_0 is a fixed point of M. By the definition of f_n, there is $u_n \in J(x_n)$ such that

$$x_n = f_n(x_n) = \gamma_n u_n + (1-\gamma_n)w .$$

Hence

$$(2.2.2) \quad \|u_n - x_n\| = \|\gamma_n u_n + (1-\gamma_n)w - x_n + (1-\gamma_n)(u_n - w)\|$$
$$= \|f_n(x_n) - x_n + (1-\gamma_n)(u_n - w)\|$$
$$= (1-\gamma_n)\|(u_n - w)\| .$$

Put $y_n = x_n - u_n$. Since M is bounded, $u_n \in J(x_n) \subset M$ implies $\{\|u_n - w\|\}$ is bounded and so (2.2.2) shows that $\|y_n\| \to 0$ as $n \to \infty$. Since $\{x_n\} \subset M$, $w\text{-}\lim_n x_n = x_0 \in M$ and $y_n \in x_n - J(x_n)$ such that $\|y_n\| \to 0$, so lemma (1.6) implies that $x_0 \in J(x_0)$ and hence the theorem is proved.

Since each closed convex bounded subset of a reflexive Banach space is weakly compact, and each Hilbert space satisfies Opial's condition, we derive the following corollaries from the theorem.

2.3. Corollary. Let M be a nonempty closed convex bounded subset of a reflexive Banach space X satisfying Opial's condition. Then each compact-valued (H·T)-nonexpansive map $J: M \to 2^M$ satisfying (2.2.1), has a fixed point.

2.4. Corollary. Let M be a nonempty closed convex bounded subset of a Hilbert space X. Then each compact-valued (H·T)-nonexpansive map $J: M \to 2^M$ satisfying (2.2.1), has a fixed point.

2.5. Remark. As noted above for single-valued maps, the concept of (H·T)-nonexpansive maps coincides with that of nonexpansive maps. Moreover, for single-valued maps $J: M \to M$, where M is a nonempty closed convex bounded subset of a Banach space X, each f_n in (2.2.1) is a contraction map and so by the Banach contraction principle, each f_n has a fixed point $x_n \in M$. Thus the assumption (2.2.1) becomes redundant. Moreover, $\lim_n \|J(x) - f_n(x)\| = 0$, i.e., each such J is the pointwise limit of contraction maps.

2.6. Theorem. Let M be a nonempty weakly compact subset of a Banach space X which satisfies Opial's condition. Let $J: M \to 2^M$ be a compact-valued (H·T)-nonexpansive map. Suppose that for $n \geq 1$, there is a single-valued nonexpansive mapping f_n of M into M such that

(a) $\|f_n(x) - x\| \le n^{-1}$ $(x \in M)$,

(b) f_nJ has a fixed point.

Then there exists $x_0 \in M$ such that $x_0 \in J(x_0)$.

Proof: By lemma 1.3(b) , for each $n \ge 1$, f_nJ is a (H·T)-nonexpansive map of $f_n(M)$ into $2^{f_n(M)}$. Thus (b) implies there exists $y_n \in f_n(M)$ such that

$$y_n \in (f_nJ)(y_n) = f_nJ(y_n) = \bigcup_{\alpha_n \in J(y_n)} f_n(\alpha_n)$$

it follows that $y_n = f_n(w_n)$ for some $w_n \in J(y_n) \subset M$. Since M is weakly compact and $y_n \in M$, there is a weakly convergent subsequence of $\{y_n\}$. We denote the subsequence also by $\{y_n\}$ for convenience. Let $y_0 = w - \lim_n y_n$. Clearly $y_0 \in M$. We prove that y_0 is a fixed point of J . Since

$$\|y_n - w_n\| = \|f_n(w_n) - w_n\| \to 0 \quad \text{as} \quad n \to \infty .$$

Put $z_n = y_n - w_n$, thus $\|z_n\| \to 0$ as $n \to \infty$. As $y_0 = w - \lim_n y_n$, $z_n \in y_n - J(y_n)$ with $\{z_n\}$ converges to zero, so lemma (1.6) gives that $y_0 \in J(y_0)$. Hence the theorem.

We say that a nonempty subset M of a normed space X has a (H·T)-fixed point property (for short, (H·T)-f.p.p) if each (H·T)-nonexpansive multivalued mapping $J: M \to 2^M$ has a fixed point.

2.7. Theorem. Let M and N be nonempty subsets of a normed space X such that $M \subset N$ and $f: N \to M$ a single-valued nonexpansive map such that $f = I$ on M, where I is the identity map. If N has the (H·T)-f.p.p., then M also has the (H.T)-f.p.p.

Proof: Let $J: M \to 2^M$ be a (H·T)-nonexpansive mapping. Define $\Phi: N \to 2^N$ by $\Phi(x) = (Jf)(x) = Jf(x)$.

Then, by lemma 1.3(a) Φ is a (H·T)-nonexpansive map. Since N has a (H·T)-f.p.p., so there exists $x_0 \in N$ such that

$$x_0 \in \Phi(x_0) = Jf(x_0) \subset M .$$

So by definition of f, $f(x_0) = x_0$ and hence $x_0 \in Jf(x_0) = J(x_0)$.

The following lemma gives examples of a function f considered in Theorem 2.7.

2.8. Lemma. Let M be a nonempty closed convex subset of a Hilbert space H. Then there exists a single-valued nonexpansive mapping f of H into M such that $f = I$ on M, where I is the identity mapping of M.

Proof: Define $f: H \to M$ by

$$x \leadsto \tilde{x} \text{ with } \|x-\tilde{x}\| = \inf\{\|x-z\| : z \in M\} .$$

Clearly from Theorem I.3.5, f is well-defined and $f = I$ on M. Let $x,y \in H$ and denote $f(x) = \tilde{x}$, $f(y) = \tilde{y}$ with

$$\|x-\tilde{x}\| = \inf\{\|x-z\| : z \in M\} \quad \text{and} \quad \|y-\tilde{y}\| = \inf\{\|y-z\| : z \in M\} .$$

For nonexpansiveness of f, it is enough to show that

$$\|\tilde{x}-\tilde{y}\| \le \|x-y\| .$$

As $\tilde{x} \in M$ and M is convex, for any $z \in M$ and $t > 0$ implies

$$tz + (1-t)\tilde{x} \in M .$$

We have

$$\|x-\tilde{x}\|^2 \le \|x-(tz+(1-t)\tilde{x})\|^2$$

implies

$$\langle x-tz-\tilde{x}+t\tilde{x},\ x-tz-\tilde{x}+t\tilde{x}\rangle \ge \langle x-\tilde{x},\ x-\tilde{x}\rangle .$$

After expanding we have

$$\langle x-\tilde{x},\ \tilde{x}-z\rangle \ge -t/2\|z-\tilde{x}\|^2 \to 0 \quad \text{as} \quad t \to 0^+$$

and so $\langle x-\tilde{x},\ \tilde{x}-z\rangle \ge 0$.

Now for $z = \tilde{y}$ we obtain

$$\langle x-\tilde{x},\ \tilde{x}-\tilde{y}\rangle \geq 0$$

and thus

$$(2.8.1) \qquad \langle \tilde{x},\tilde{x}-\tilde{y}\rangle \leq \langle x,\tilde{x}-\tilde{y}\rangle .$$

Similarly, from

$$\|y-\tilde{y}\|^2 \leq \|y-(tz+(1-t)\tilde{y})\|^2$$

we have

$$\langle y-\tilde{y},\ \tilde{y}-z\rangle \geq 0 .$$

Now for $z = \tilde{x}$, we get

$$\langle y-\tilde{y},\ \tilde{y}-\tilde{x}\rangle \geq 0$$

which implies

$$(2.8.2) \qquad -\langle \tilde{y},\tilde{x}-\tilde{y}\rangle \leq \langle -y,\tilde{x}-\tilde{y}\rangle$$

Adding (2.8.1) and (2.8.2), we have

$$\|\tilde{x}-\tilde{y}\|^2 \le \langle x-y, \tilde{x}-\tilde{y}\rangle$$
$$\le \|x-y\| \, \|\tilde{x}-\tilde{y}\| \qquad \text{(by I.3.4(a))} .$$

Hence

$$\|\tilde{x}-\tilde{y}\| \le \|x-y\| .$$

2.9. <u>Theorem</u>. Let M and N be nonempty subsets of a normed space X. Suppose M has the (H·T)-f.p.p. and M is homeomorphic to N under a linear mapping f with $\|f^{-1}\| \, \|f\| \le 1$. Then N also has the (H·T)-f.p.p..

<u>Proof</u>: Let $J: N \to 2^N$ be a (H·T)-nonexpansive mapping and $f: M \to N$ a linear homeomorphism with $\|f^{-1}\| \, \|f\| \le 1$. Define

$$S(x) = f^{-1}(Jf(x)) \qquad (x \in M)$$
$$= \{f^{-1}(z) : z \in Jf(x)\} \subset M$$

Clearly S is a mapping of M into 2^M. Now we prove that S is a (H·T)-nonexpansive mapping. Let $x \in M$, $u \in S(x)$, then from the definition of S, we can write $u = f^{-1}(w)$ for some $w \in Jf(x)$. From the (H·T)-nonexpansiveness of J, there is a $z \in Jf(y)$ for each y in M such that

$$\text{(2.9.1)} \qquad \|w-z\| \le \|f(x)-f(y)\| .$$

As $z \in Jf(y) \subset N$, put $u_y = f^{-1}(z)$, then

$$u_y = f^{-1}(z) \in \{f^{-1}(\alpha): \alpha \in Jf(y)\} = f^{-1}(Jf(y)) = S(y),$$

and

$$\begin{aligned} \|u-u_y\| = \|f^{-1}(w)-f^{-1}(z)\| = \|f^{-1}(w-z)\| &\le \|f^{-1}\| \, \|w-z\| \le \\ &\le \|f^{-1}\| \, \|f(x)-f(y)\| \qquad \text{(by 2.9.1)} \\ &\le \|f^{-1}\| \, \|f(x-y)\| \le \|f^{-1}\| \, \|f\| \, \|x-y\| \\ &\le \|x-y\| \qquad (\text{since } \|f^{-1}\| \, \|f\| \le 1). \end{aligned}$$

Thus S is (H·T)-nonexpansive. Now since M has a (H·T)-f.p.p., there exists $x_0 \in M$ such that

$$x_0 \in S(x_0) = f^{-1}(Jf(x_0))$$

$\Rightarrow f(x_0) \in Jf(x_0)$, i.e., J has a fixed point $f(x_0) \in N$ and hence the theorem is proved.

§3. <u>Fixed point theorems for s-nonexpansive maps</u>.

In this section we show that under certain natural conditions the (2.2.1) condition of theorem 2.2 is dispensible. For this we introduce another notion of nonexpansiveness.

3.1. <u>Definition</u>. Let M be a nonempty subset of a normed space X. A multivalued map $J: M \to 2^X$ is said to be selectively nonexpansive (for short, s-nonexpansive) if given $x \in M$ and $u_x \in J(x)$ with

$$\|x-u_x\| = \inf\{\|x-z\|: z \in J(x)\}$$

there is a $u_y \in J(y)$ with

$$\|y-u_y\| = \inf\{\|y-w\|: w \in J(y)\} \text{ for each } y \in M.$$

such that

$$\|u_x-u_y\| \le \|x-y\| .$$

We note that if J is single-valued, then $u_x = J(x)$, $u_y = J(y)$ and we have the usual notion of nonexpansive single-valued maps.

As an example of a s-nonexpansive map, we consider the following:

3.2. <u>Example</u>. Let $M = \{(a,b) \in \mathbb{R}^2: a^2+b^2 \le 1\}$. For each $x = (a,b)$, we define $J(x) = D_x$, where D_x is the unique diameter perpendicular to the straight line joining the points (a,b) and $(0,0)$. Clearly $u_x = (0,0)$ (as defined in Definition 3.1) for all $x \in M$ and so

$$0 = \|u_x-u_y\| \le \|x-y\|$$

showing that J is a s-nonexpansive multivalued map.

When X is a Hilbert space we can dispense with the condition (2.2.1) of Theorem 2.2 as well as Opial's condition and we have an improved fixed point theorem for s-nonexpansive maps.

3.3. Theorem. Let M be a nonempty closed, convex and bounded subset of a Hilbert space X. Then each closed convex-valued s-nonexpansive map $J: M \to 2^M$ has a fixed point.

Proof: Since $J(x)$ is a nonempty closed convex subset of a Hilbert space, each u_x in the Definition 3.1 is unique (I.3.5) and by the definition of s-nonexpansiveness there is a unique $u_y \in J(y)$ such that

$$\|u_x - u_y\| \le \|x-y\| .$$

Let $w \in M$ be fixed and consider a sequence of positive numbers $\{\gamma_n\}$ converging to 1 and $\gamma_n < 1$. For each $n \ge 1$, we consider the single-valued mappings $f_n: M \to M$ defined by

$$f_n(x) = \gamma_n u_x + (1-\gamma_n)w .$$

For all x,y in M, we have

$$\|f_n(x) - f_n(y)\| = \|\gamma_n u_x + (1-\gamma_n)w - \gamma_n u_y - (1-\gamma_n)w\|$$
$$= \gamma_n\|u_x - u_y\| \le \gamma_n\|x-y\| .$$

Thus for each $n \ge 1$, f_n is a contraction map and so by Theorem I.5.2, has a fixed point $x_n \in M$ for each $n \ge 1$. Since M is a closed, convex and bounded subset of a Hilbert space, it is (Theorem I.3.2(a)) weakly compact and so there is a weakly convergent subsequence of $\{x_n\}$. We can suppose, w.o.l.g. that $\{x_n\}$ has just this property i.e., $\{x_n\}$ is weakly convergent. Let $x_0 = w - \lim_n x_n$. Clearly $x_0 \in M$, because C is weakly closed. We prove that x_0 is a fixed point of J. Let x be an arbitrary point of X, and thus we have

$$\|x_n - x\|^2 = \|(x_n - x_0) + (x_0 - x)\|^2$$
$$= \|x_n - x_0\|^2 + \|x_0 - x\|^2 + 2\mathrm{Re}\langle x_n - x_0, x_0 - x\rangle$$

We know that $2\mathrm{Re}\langle x_n - x_0, x_0 - x\rangle \to 0$, because $x_0 = w - \lim_n x_n$. Hence

(3.3.1) $$\lim_n [\|x_n - x\|^2 - \|x_n - x_0\|^2] = \|x_0 - x\|^2$$

Since $J(x_0)$ and $J(x_n)$ are nonempty closed convex subsets of X, there exists a unique $u_0 \in J(x_0)$ such that

$$\|x_0 - u_0\| = \inf\{\|x_0 - z\| : z \in J(x_0)\}$$

and a unique $v_n \in J(x_n)$ such that

$$\|x_n - v_n\| = \inf\{\|x_n - t_n\| : t_n \in J(x_n)\} .$$

Thus it follows from the definition of J that

$$\|v_n - u_0\| \le \|x_n - x_0\| .$$

But then

$$\begin{aligned}\|x_n - u_0\| &\le \|x_n - v_n\| + \|v_n - u_0\| \\ &\le \|x_n - v_n\| + \|x_n - x_0\|\end{aligned}$$

implies that

$$(3.3.2) \qquad \overline{\lim_n}[\|x_n - u_0\| - \|x_n - x_0\|] \le \overline{\lim_n}\|x_n - v_n\| .$$

But since

$$\begin{aligned}v_n - x_n &= \gamma_n v_n + (1-\gamma_n)w - x_n + (1-\gamma_n)(v_n - w) \\ &= f_n(x_n) - x_n + (1-\gamma_n)(v_n - w) = (1-\gamma_n)(v_n - w) .\end{aligned}$$

We have

$$(3.3.3) \qquad \|v_n - x_n\| = (1-\gamma_n)\ \|(v_n - w)\| .$$

Since M is bounded, $v_n \in J(x_n) \subset M$ implies $\{\|v_n - w\|\}$ is bounded, and so (3.3.3) shows that $\|v_n - x_n\| \to 0$ as $n \to \infty$. Thus from (3.3.2), we have

$$\overline{\lim_n}[\|x_n - u_0\| - \|x_n - x_0\|] \le 0 .$$

From the boundedness of M there exists a $k > 0$ such that for all $n \ge 1$,

$$\|x_n - u_0\| + \|x_n - x_0\| \le k .$$

But

$$\|x_n - u_0\|^2 - \|x_n - x_0\|^2 = (\|x_n - u_0\| - \|x_n - x_0\|)(\|x_n - u_0\| + \|x_n - x_0\|)$$
$$\le k\cdot(\|x_n - u_0\| - \|x_n - x_0\|)$$

implies

$$\overline{\lim_n}[\|x_n - u_0\|^2 - \|x_n - x_0\|^2] \le k\cdot\overline{\lim_n}[\|x_n - u_0\| - \|x_n - x_0\|] \le k\cdot 0 = 0 .$$

$$\Rightarrow \quad \overline{\lim_n}[\|x_n - u_0\|^2 - \|x_n - x_0\|^2] \le 0 .$$

But from (3.3.1),

$$\overline{\lim_n}[\|x_n - u_0\|^2 - \|x_n - x_0\|^2] = \|x_0 - u_0\|^2 \le 0$$

$\Rightarrow$ $\|x_0 - u_0\| = 0$ implies $x_0 = u_0 \in J(x_0)$, proving that x_0 is a fixed point of J and hence the theorem is proved.

This extends the result of Husain and Tarafdar (I.5.13) for s-nonexpansive maps and includes the result of Browder (I.5.4) as a special case.

Since each closed convex bounded subset of a reflexive Banach space is weakly compact and each Hilbert space is strictly convex, the following fixed point theorem for s-nonexpansive maps extends Theorem 3.3.

3.4. <u>Theorem</u>. Let M be a nonempty weakly compact convex subset of a strictly convex Banach space (in particular (I.3.11), locally uniformly convex Banach space) X which satisfies the Opial's condition. Then each closed convex-valued s-nonexpansive mapping $J: M \to 2^M$ has a fixed point.

<u>Proof</u>: Since $J(x)$ is a nonempty closed convex subset of a weakly compact set M, $J(x)$ itself is weakly compact. As $J(x)$ is a weakly compact convex subset of a strictly convex space X, each u_x in the definition 3.1 (I.3.12) is unique and by the definition of s-nonexpansiveness there is a unique $u_y \in J(y)$ such that

$$\|u_x - u_y\| \le \|x-y\| .$$

Thus, as in the proof of Theorem 3.3, we can find a weakly convergent subsequence $\{x_n\}$ in M such that $f_n(x_n) = x_n$ for all $n \geq 1$. Let $x_0 = w-\lim_n x_n$. Clearly $x_0 \in M$. Now we complete the proof similarly as that of Theorem 2.2.

CHAPTER III

(H·T)-CONTRACTIVE MAPS AND THEIR GENERALIZATIONS

In this chapter we introduce the notion of (H·T)-contractive multivalued mapping and give examples of this notion. It is shown that in general an (H·T)-nonexpansive mapping is not (H·T)-contractive (§1). In §2 we prove fixed point theorems for (H·T)-contractive maps, one of which contains the classical Banach fixed point theorem as a special case. §3 consists of a fixed point theorem for a new class of mappings, namely, f-(H·T)-contractive maps. In §4 we establish fixed point theorems for (H·T)-nonexpansive maps, one of which extends Husain and Tarafdar's result (Theorem I.5.13) and includes the results of Browder (Theorem I.5.4) and Karlovitz (Theorem I.5.7) for single-valued nonexpansive maps. Also in the same section we study a larger class of multivalued maps which contains in particular (H·T)-nonexpansive maps. In §5 we concern with the iterative type construction of fixed points.

§1. Examples of (H·T)-contractive maps.

1.1. Definition. Let M be a nonempty subset of a metric space (X,d). A multivalued mapping $J\colon M \to 2^X$ is said to be (H·T)-contractive if there exists a real number h with $0 \le h < 1$ and for any $x \in M$, $u_x \in J(x)$ there is a $u_y \in J(y)$ for all y in M such that

(1.1.1) $$d(u_x,u_y) \le h\,d(x,y)\,.$$

Clearly, each (H·T)-contractive mapping is a (H·T)-nonexpansive map. Moreover, each single-valued map is contractive iff it is a (H·T)-contractive map. That is, the concept of (H·T)-contractiveness generalizes the usual notion of contractiveness for single-valued maps.

As examples of (H·T)-contractive maps, we consider the following:

1.2. Examples. (a) Let $\{f_\alpha: \alpha \in I\}$ be a family of single-valued contractive mappings on a non-empty subset M of a metric space (X,d) into X, with the same Lipschitz constant $h = h_\alpha$ [i.e. for each $\alpha \in I$, $f_\alpha: M \to X$ and $d(f_\alpha(x),f_\alpha(y) \le h\,d(x,y)$ for all $x,y \in M$, $0 \le h < 1$]. Then the multivalued mapping

$$J(x) = \{f_\alpha(x): \alpha \in I\} \qquad (x \in M)\,,$$

defines a (H·T)-contractive map of M into 2^X, with the Lipschitz constant h. Indeed:

Let $u \in J(x)$, this implies $u = f_j(x)$ for some $j \in I$. Put $u_y = f_j(y)$ for an arbitrary $y \in M$, then $u_y \in J(y)$ and

$$d(u,u_y) = d(f_j(x),f_j(y)) \le h\,d(x,y)\,.$$

(b) Let $X = \mathbb{R}^2$ with the usual metric d, i.e., $d(x,y) = \left(\sum_{i=1}^{2} |x_i - y_i|^2\right)^{1/2}$ for $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in $\mathbb{R}^2$.

Let $M = \{x = (x_1, x_2) \in X: x_i \geq 0, \quad i = 1,2\}$ and define

$$J(x) = \left\{\pm\left(\frac{x_1}{2}, \frac{x_2}{2}\right), \pm\left(\frac{x_1}{3}, \frac{x_2}{3}\right), \ldots\right\} \in 2^X \qquad (x \in M).$$

Then J is a (H·T)-contractive mapping with $h = \frac{1}{2}$. For if $u_x = \left(\frac{x_1}{n}, \frac{x_2}{n}\right) \in J(x)$, where n is any positive integer greater than or equal to 2. For any $y \in M$, put $u_y = \left(\frac{y_1}{n}, \frac{y_2}{n}\right) \in J(y)$. Then clearly

$$d(u_x, u_y) = \sqrt{\frac{1}{n^2}\left[(x_1 - y_1)^2 + (x_2 - y_2)^2\right]}$$

$$\leq \frac{1}{2} d(x,y).$$

This inequality is also true for any negative integer n less than or equal to -2.

(c) Let $X = \mathbb{R}$ with the usual metric d and $M = [0,1]$. Consider

$$J(x) = \left[\frac{x}{3}, \frac{x}{2}\right] \qquad (x \in M),$$

then it defines an (H·T)-contractive mapping of M into 2^X with $h = \frac{1}{2}$. More generally, the multivalued mapping

$$J(x) = \left[\frac{x}{n+1}, \frac{x}{n}\right] \qquad (x \in M)$$

is a (H·T)-contractive mapping of M into 2^X with $h = \frac{1}{n}$ for any positive integer $n \geq 2$.

The following examples show that (H·T)-nonexpansive maps may not be (H·T)-contractive.

1.3. Examples. (a) Let $M = X = R$ with the usual metric d. Define

$$J(x) = \left\{x-\tan^{-1}(x), \frac{x}{2} - \tan^{-1}(x)\right\}, \qquad (x \in X).$$

Then J is a (H.T)-nonexpansive mapping, but not (H.T)-contractive. For, if $u_x = x-\tan^{-1}(x) \in J(x)$, for the derivative $u'_x = 1 - \frac{1}{1+x^2}$, used in the mean value theorem we have,

$$d(u_x,u_y) = |u_x-u_y| \leq \left|1 - \frac{1}{1+\eta^2}\right| |x-y| < |x-y| = d(x,y).$$

where $x < \eta < y$. Clearly $u_y = y-\tan^{-1}(y) \in J(y)$. Similarly, if $u_x = \frac{x}{2} - \tan^{-1}(x) \in J(x)$, we have

$$d(u_x,u_y) \leq \left|\frac{1}{2} - \frac{1}{1+\nu^2}\right| |x-y| < d(x,y).$$

Note that η and ν both depend on x and y, so J cannot be a (H·T)-contractive mapping.

(b) Let $X = \mathbb{R}$ with the usual metric d and $M = \mathbb{R}^+$. Define

$$J(x) = \left\{1+\left(x^2+1\right)^{1/2}, \frac{x}{2}+2\right\} \qquad (x \in M).$$

Then J is a (H·T)-nonexpansive mapping of M into 2^M, but not a (H·T)-contractive mapping.

§2. Fixed points for (H·T)-contractive maps.

2.1. Theorem. Let M be a nonempty closed subset of a complete metric space (X,d) and $J\colon M \to 2^M$ a multivalued (H·T)-contractive mapping with closed subsets of M as values. Then there is a point $x \in M$ such that $x \in J(x)$.

Proof: Let x_0 be an arbitrary but fixed element of M and choose an $x_1 \in J(x_0)$ with $d(x_0,x_1) > 0$. If there is no such x_1, then x_0 is already a fixed point of J. Since $x_1 \in J(x_0)$ and J is a (H·T)-contractive map, there is a $x_2 \in J(x_1)$ such that

$$d(x_1,x_2) \le h\, d(x_0,x_1) \qquad 0 \le h < 1.$$

Similarly, since $x_2 \in J(x_1)$, by the definition of J, we can find a point $x_3 \in J(x_2)$ such that

$$d(x_2,x_3) \le h\, d(x_1,x_2).$$

In general, there is $x_{n+1} \in J(x_n)$ such that

$$d(x_n, x_{n+1}) \le h\, d(x_{n-1}, x_n) \qquad (\forall n \ge 1).$$

Thus

$$(2.1.1) \qquad d(x_n, x_{n+1}) \le h\, d(x_{n-1}, x_n) \le h^2 d(x_{n-2}, x_{n-1}) \le \ldots \le \le h^n d(x_0, x_1).$$

For m and n positive integers $(m > n)$, repeated application of the triangle inequality and finally the sum formula for a geometric series yields:

$$(2.1.2) \qquad d(x_n, x_m) \le d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{m-1}, x_m)$$
$$\le (h^n + h^{n+1} + \ldots + h^{m-1})\, d(x_0, x_1) \qquad \text{by (2.1.1)}$$
$$\le \sum_{i=n}^{\infty} h^i d(x_0, x_1) = \frac{h^n}{1-h} d(x_0, x_1).$$

Since $0 \le h < 1$, we have $d(x_n, x_m) \to 0$ as $m, n \to \infty$. Thus $\{x_n\}$ is a Cauchy sequence in the complete metric space X. There exists an element $p \in X$ such that, $\lim_n x_n = p$. Since $J(M) = \bigcup_{x \in M} J(x) \subset M$ and $x_0 \in M$, we have $x_n \in M$ for all n. Since the sequence $\{x_n\}$ is in the closed set M and $\lim_n x_n = p$, it follows that $p \in M$. Since each (H·T)-contractive map is (H·T)-nonexpansive, by lemma II.1.7 p is a fixed point of J.

2.2. Remark. From the proof of Theorem 2.1 we conclude the following:

(a) The sequence $\{x_n\}_{n\geq 1}$ converges to a fixed point of J, for an arbitrary choice of initial point x_0 in M.

(b) For all $n = 0,1,2,\ldots,$ we have

$$d(x_n,p) \leq h^n(1-h)^{-1} d(x_0,x_1) .$$

For, since $x_n \to p$ for sufficiently large n, so this inequality follows from (2.1.2) by letting $m \to \infty$.

(c) For all $n = 0,1,2,\ldots,$ we have

$$d(x_{n+1},p) \leq h(1-h)^{-1} d(x_n,x_{n+1}) .$$

For, if m and n are positive integers with $m > n$, we have

$$\begin{aligned} d(x_{n+1},x_{m+1}) &\leq d(x_{n+1},x_{n+2}) + d(x_{n+2},x_{n+3}) + \ldots + d(x_m,x_{m+1}) \\ &\leq h\,d(x_n,x_{n+1}) + h^2 d(x_n,x_{n+1}) + \ldots + h^{m-n} d(x_n,x_{n+1}) \quad \text{(by(2.1.1))} \\ &= (h + h^2 + \ldots + h^{m-n}) d(x_n,x_{n+1}) \\ &\leq \sum_{i=1}^{\infty} h^i d(x_n,x_{n+1}) = h(1-h)^{-1} d(x_n,x_{n+1}) . \end{aligned}$$

Letting $m \to \infty$ in the last inequality and using the fact that $x_n \to p$ for sufficiently large n, we get the required inequality.

(d) For any $x \in M$, there is a $u \in J(x)$ such that

$$(2.2.1) \qquad d(x,p) \le (1-h)^{-1} d(x,u) .$$

For, since $p \in J(p)$, by definition of J there is a $v \in J(x)$ such that

$$d(v,p) \le h\, d(x,p) .$$

But

$$\begin{aligned} d(x,p) &\le d(x,v) + d(v,p) \\ &\le d(x,v) + h\, d(x,p) \\ \Rightarrow d(x,p) &\le (1-h)^{-1} d(x,v) \qquad (\text{since } (1-h) > 0) . \end{aligned}$$

A point v does the required job. Note that (2.2.1) gives a neighbourhood of x in which a fixed point of J must lie.

If we take $M = X$ in Theorem 2.1, then we have the following:

2.3. <u>Corollary</u>. Each (H·T)-contractive mapping $J: X \to 2^X$ with closed subsets of X as values, has a fixed point.

This contains the Banach fixed point theorem as a special case. Now, we want to show that the closedness of M in Theorem 2.1 is essential:

2.4. Counterexamples. (a) Let $X = \mathbb{R}$ with the usual metric d and $M = \mathbb{R}\setminus\{0\}$, note that M is not closed. Define

$$J(x) = \left\{\frac{x}{2}, \frac{x}{3}, \ldots, \frac{x}{n}\right\} \subset M$$

for each $x \in M$. Then J is a (H·T)-contractive mapping of M with $J(x)$ as a closed subset of M and $h = \frac{1}{2}$. Clearly J is a fixed-point-free mapping.

(b) Let $X = \mathbb{R}$ with the usual metric and $M = (0,1]$. Then, the (H·T)-contractive closed-valued mapping J defined in example 1.2(c) has no fixed point.

Now we prove a fixed point theorem for (H·T)-contractive mappings in a complete metric space which is metrically convex.

We need the following:

From Theorem I.1.3 it follows that, if M is a closed subset of a complete convex metric space X, $x \in M$ and $y \notin M$, then there exists a point w in the boundary of M such that

(A) $$d(x,w) + d(w,y) = d(x,y) .$$

We use the symbol ∂M to denote the boundary of M and $\partial_N M$ denotes the boundary of M relative to $N \subset X$ i.e.,

$$\partial_N M = \{x \in M\colon B(x,\gamma) \cap N \backslash M \neq \phi \text{ for each } \gamma > 0\}$$

where $B(x,\gamma)$ is an open ball in X.

2.5. Theorem. Let M be a nonempty closed subset of a complete convex metric space (X,d). Suppose that:

(a) $J\colon M \to 2^X$ is a (H·T)-contractive closed-valued mapping and;

(b) $J(x) \subset M$ for each $x \in \partial M$.

Then J has a fixed point.

Proof: Let y_0 be an arbitrary but fixed element of M and choose an $x_1 \in J(y_0)$ with $d(y_0,x_1) > 0$, otherwise we are done. Now if $x_1 \in M$, let $x_1 = y_1$.

If $x_1 \notin M$, then by (A), there exists a point $y_1 \in \partial M$ such that

$$d(y_0,y_1) + d(y_1,x_1) = d(y_0,x_1).$$

Since $x_1 \in J(y_0)$, $y_1 \in M$ and J is a (H·T)-contractive, there is a $x_2 \in J(y_1)$ such that

$$d(x_1,x_2) \le h\, d(y_0,y_1).$$

Similarly, if $x_2 \in M$, let $x_2 = y_2$.

If $x_2 \notin M$, there exists a point $y_2 \in \partial M$ such that

$$d(y_1,y_2) + d(y_2,x_2) = d(y_1,x_2) .$$

Since $x_2 \in J(y_1)$, $y_2 \in M$, so by definition of J, there is a $x_3 \in J(y_2)$ such that

$$d(x_2,x_3) \le h\, d(y_1,y_2) .$$

In general, we may obtain sequences $\{x_n\}$, $\{y_n\}$ such that for all $n \ge 1$

(i) $x_{n+1} \in J(y_n)$

(ii) $d(x_n,x_{n+1}) \le h\, d(y_{n-1},y_n)$

(iii) $x_{n+1} = y_{n+1}$ if $x_{n+1} \in M$

(iv) If $x_{n+1} \notin M$, there is $y_{n+1} \in \partial M$ such that

$$d(y_n,y_{n+1}) + d(y_{n+1},x_{n+1}) = d(y_n,x_{n+1}) .$$

Now let, $A = \{y_k \in \{y_n\}: y_k = x_k, \quad k = 1,2,\ldots\}$

and $B = \{y_k \in \{y_n\}: y_k \neq x_k, \quad k = 1,2,\ldots\}$.

It is clear from the construction and (b) that if $y_j \in B$ for some j, then $y_{j+1} \in A$. [For instance if $y_1 \neq x_1$, that is $x_1 \notin M$, then by construction $y_1 \in \partial M$ and $x_2 \in J(y_1)$ but from hypothesis (b), $J(y_1) \subset M$. Thus $x_2 \in M$, so by construction $x_2 = y_2$, that is $y_2 \in A$].

Claim I. For $n \ge 2$

$$d(y_n, y_{n+1}) \le \begin{cases} h\, d(y_{n-1}, y_n) \\ \qquad\qquad \text{OR} \\ h\, d(y_{n-1}, y_{n-2}) \end{cases} . \tag{2.5.1}$$

Here we discuss the three possibilities which can occur. If $y_n \in A$ and $y_{n+1} \in B$, then

$$\begin{aligned} d(y_n, y_{n+1}) &\le d(y_n, x_{n+1}) && \text{by (iv)} \\ &= d(x_n, x_{n+1}) && (\text{since } y_n \in A) \\ &\le h\, d(y_{n-1}, y_n) && \text{by (ii)}. \end{aligned}$$

If $y_n \in A$, $y_{n+1} \in A$, then

$$\begin{aligned} d(y_n, y_{n+1}) &= d(x_n, x_{n+1}) \\ &\le h\, d(y_{n-1}, y_n) && \text{by (ii)}. \end{aligned}$$

If $y_n \in B$ and $y_{n+1} \in A$. Since $y_n \in B$, by the above observation $y_{n-1} \in A$, that is $y_{n-1} = x_{n-1}$.

$$\begin{aligned} d(y_n, y_{n+1}) &\le d(y_n, x_n) + d(x_n, y_{n+1}) \\ &= d(y_n, x_n) + d(x_n, x_{n+1}) \\ &\le d(y_n, x_n) + h\, d(y_{n-1}, y_n) && \text{by (ii)} \\ &\le d(y_{n-1}, x_n) && \text{by (iv)} \\ &= d(x_{n-1}, x_n) \le h\, d(y_{n-2}, y_{n-1}). \end{aligned}$$

By the above observation, the fourth possibility, viz., $y_n \in B$, $y_{n+1} \in B$ cannot occur. Now let

(2.5.2) $$\lambda = (\sqrt{h})^{-1} \max\{d(y_0,y_1),\ d(y_1,y_2)\}$$

Claim II. For $n \geq 1$

(2.5.3) $$d(y_n,y_{n+1}) \leq (\sqrt{h})^n \lambda .$$

For $n = 1$

$$d(y_1,y_2) \leq \sqrt{h}\,\lambda \qquad \text{by (2.5.2)} .$$

If $n = 2$

$$\begin{aligned} d(y_2,y_3) &\leq h\, d(y_1,y_2) && \text{by (2.5.1)} \\ &\leq h\sqrt{h}\,\lambda && \text{by (2.5.2)} \\ &\leq h\lambda . \end{aligned}$$

Again
$$\begin{aligned} d(y_2,y_3) &\leq h\, d(y_0,y_1) \\ &\leq h\sqrt{h}\,\lambda \leq h\lambda . \end{aligned}$$

Assuming that (2.5.3) is true for $1 \leq n \leq k$, we will show it is true for $n = k+1$.

$$d(y_{k+1},y_{k+2}) \le h\,d(y_k,y_{k+1}) \qquad \text{by (2.5.1)}$$

$$\le h(\sqrt{h})^k\lambda = (\sqrt{h})^{k+2}\lambda$$

$$\le (\sqrt{h})^{k+1}\lambda .$$

Again $$d(y_{k+1},y_{k+2}) \le h\,d(y_{k-1},y_k)$$

$$\le h(\sqrt{h})^{k-1}\lambda = (\sqrt{h})^{k+1}\lambda .$$

Thus the induction process is completed, hence the claim. Now for m and n positive integers with $(m > n \ge 1)$, by the repeated application of the triangle inequality we have:

$$d(y_n,y_m) \le d(y_n,y_{n+1}) + d(y_{n+1},y_{n+2}) + \dots + d(y_{m-1},y_m)$$

$$\le \left(h^{\frac{n}{2}} + h^{\frac{n+1}{2}} + \dots + h^{\frac{m-1}{2}}\right)\lambda \qquad \text{by (2.5.3)}$$

$$\le \frac{h^{\frac{n}{2}}}{1-\sqrt{h}}\lambda .$$

Since $0 \le h < 1$, we have $d(y_n,y_m) \to 0$ as $m,n \to \infty$. Thus since $\{y_n\}$ is a Cauchy sequence in the complete metric space X, there exists an element p in X such that $\lim_n y_n = p$. Since the sequence $\{y_n\}$ is in M and M is closed, so $p \in M$. By hypothesis (b) and the construction of the sequence $\{y_n\}$ we observe that there exists a convenient subsequence $\{y_m\}$ of $\{y_n\}$ such that $y_m \in A$ that is, $y_m = x_m$, $x_m \in M$. Moreover $\lim_n y_m = p$. Since J is a (H·T)-contractive closed-valued mapping of M into 2^X and $p, y_m \in M$ with $y_m = x_m \in J(x_{m-1})$, $y_m \to p$ as $m \to \infty$, by lemma (II.1.7) we get $p \in J(p)$.

2.6. Corollary. Let M be a nonempty closed convex subset of a Banach space X and N a nonempty closed subset of M. Suppose that:

(a) $J: N \to 2^M$ is a (H·T)-contractive closed-valued mapping;

(b) $J(x) \subset N$ if $x \in \partial_M N$.

Then there exists $x_0 \in N$ such that $x_0 \in J(x_0)$.

Proof: We observe that $\partial_M N = \partial N$. Since each closed subset of a complete space is complete, the corollary follows from Theorem 2.5.

§3. A class of mappings related to (H·T)-contractive mappings.

As we know there exists a vast literature which is concerned with various extensions of contraction mappings. It is natural to ask about the classes of multivalued mappings containing as a particular case the multivalued (H·T)-contractive mappings. As we have seen one such class is that of (H·T)-nonexpansive mappings and another class is defined below:

3.1. Definition. Let M be a nonempty subset of a metric space X and f a single-valued mapping of M into X. A multivalued mapping $J: M \to 2^X$ is said to be f-(H·T)-contractive if there exists a real number h with $0 \le h < 1$ and for any $x \in M$, $u_x \in J(x)$ there is a $u_y \in J(y)$ for all $y \in M$ such that

$$d(u_x,u_y) \le h\,d(f(x),f(y)).$$

In particular, if f is an identity map on M, then a multi-valued map is f-(H·T)-contractive iff it is (H·T)-contractive.

Now we consider an example of a f-(H·T)-contractive map which is not a (H·T)-contractive map.

3.2. Example. Let $X = \mathbb{R}^2$ with the usual metric d. Let $M = \{x = (x_1,x_2) \in \mathbb{R}^2 : x_i > 0,\ i = 1,2\}$ and for $x \in M$, define

$$J(x) = \left\{(0,0),\ \left(7x_1,\ \frac{x_2}{3}+\alpha\right)\right\}$$

$$f(x) = \left(11x_1,\ \frac{x_2}{2}+\beta\right),$$

where α and β are arbitrary but fixed numbers in $\mathbb{R}$ with $\beta > 0$. Then J is a fixed-point-free f-(H·T)-contractive mapping of M into 2^X with the Lipschitz constant $h = \frac{2}{3}$, but J is not a (H·T)-contractive mapping. For, it is clear that f maps M into $M \subset X$ and J maps M into 2^X. Let $x = (x_1,x_2)$ and $y = (y_1,y_2)$ be two points in M. If $u_x = (0,0) \in J(x)$, then there is a $u_y = (0,0) \in J(y)$ such that

$$d(u_x,u_y) = 0 \le \frac{2}{3} d(f(x),f(y)).$$

If $u_x = \left(7x_1,\ \frac{x_2}{3}+\alpha\right) \in J(x)$, then

$$[d(f(x),f(y))]^2 = \left[d\left(\left(11x_1, \frac{x_2}{2}+\beta\right), \left(11y_1, \frac{y_2}{2}+\beta\right)\right)\right]^2$$

$$= (11)^2(x_1-y_1)^2 + \frac{1}{2^2}\left(x_2-y_2\right)^2$$

$$\geq \left(\frac{3}{2}\right)^2 (7)^2(x_1-y_1)^2 + \frac{1}{2^2}\left(x_2-y_2\right)^2$$

$$= \left(\frac{3}{2}\right)^2 \left[(7)^2 (x_1-y_1)^2 + \frac{1}{3^2}\left(x_2-y_2\right)^2\right]$$

$$= \left(\frac{3}{2}\right)^2 \left[d\left(\left(7x_1, \frac{x_2}{3}+\alpha\right), \left(7y_1, \frac{y_2}{3}+\alpha\right)\right)\right]^2$$

$$= \left(\frac{3}{2}\right)^2 \left[d(u_x,u_y)\right]^2 \quad \text{where} \quad u_y = \left(7y_1, \frac{y_2}{3}+\alpha\right) \in J(y)$$

$$\Rightarrow \quad d(u_x,u_y) \leq \frac{2}{3} d(f(x),f(y)) .$$

Thus J is a f-(H·T)-contractive mapping and clearly J has no fixed point. Now, if we take $x = (x_1,x_2)$, $y = (y_1,x_2)$ in M and $u_x = \left(7x_1, \frac{x_2}{3}+\alpha\right) \in J(x)$, then there exists no $u_y \in J(y)$ with

$$d(u_x,u_y) \leq h\, d(x,y)$$

suppose $u_y = \left(7y_1, \frac{x_2}{3}+\alpha\right) \in J(y)$, then

$$d(u_x,u_y) = 7|(x_1-y_1)| = 7d(x,y) \nleq h\,d(x,y) \quad \text{for any} \quad 0 \le h < 1 .$$

As we have seen in this example that on a general nonempty subset of a complete metric space X, a f-(H·T)-contractive mapping fails to have a fixed point. To obtain a positive result for the existence of fixed points of f-(H·T)-contractive mappings, we need certain additional conditions.

3.3. <u>Theorem</u>. Let M be a nonempty closed subset of a complete metric space X, f a continuous mapping of M into itself. Suppose $J: M \to 2^M$ is a f-(H·T)-contractive closed-valued mapping such that J commutes with f and $J(M) \subseteq f(M)$.

Moreover, we assume that one of the following holds:

either (a) $f(x) \neq f^2(x)$ implies $f(x) \notin J(x)$

or (b) $f(x) \in J(x)$ implies $\lim_n f^n(x)$ exists.

Then J has a fixed point in M (which is also a fixed point of f).

<u>Proof</u>: Let x_0 be an arbitrary but fixed element of M and choose $u_1 \in J(x_0)$. Using the fact that $J(M) \subseteq f(M)$, we can choose $x_1 \in M$ such that $f(x_1) = u_1$. Since $f(x_1) \in J(x_0)$, by the definition of J there is $u_2 \in J(x_1)$ such that

(3.3.1) $$d(f(x_1),u_2) \le h\,d(f(x_0),f(x_1)) .$$

Using the fact that $J(M) \subseteq f(M)$, we may choose $x_2 \in M$ such that $u_2 = f(x_2) \in J(x_1)$; thus (3.3.1) can be written as

$$d(f(x_1),f(x_2)) \leq h\,d(f(x_1),f(x_0)) .$$

Similarly since $f(x_2) \in J(x_1)$, again by the definition of f-(H·T)-contractive and $J(M) \subseteq f(M)$, we may choose $x_3 \in M$ such that $f(x_3) \in J(x_2)$ and

$$d(f(x_3),f(x_2)) \leq h\,d(f(x_2),f(x_1)) .$$

By continuing this process, we obtain a sequence $\{x_n\}$ of points in M such that $f(x_n) \in J(x_{n-1})$ and

$$\begin{aligned} d(f(x_n),f(x_{n-1})) &\leq h\,d(f(x_{n-1}),f(x_{n-2})) \\ &\leq h^2 d(f(x_{n-2}),f(x_{n-3})) \\ &\;\vdots \\ &\leq h^{n-1} d(f(x_1),f(x_0)) . \end{aligned} \tag{3.3.2}$$

For m and n positive integers $(m > n)$, repeated application of the triangle inequality and finally the sum formula for a geometric series yields:

$$d(f(x_m),f(x_n)) \le d(f(x_n),f(x_{n+1})) + d(f(x_{n+1}),f(x_{n+2})) + \ldots + d(f(x_{m-1}),f(x_m))$$

$$\le (h^n + h^{n+1} + \ldots + h^{m-1})\, d(f(x_1),f(x_0)) \qquad \text{by (3.3.2)}$$

$$\le \frac{h^n}{1-h}\, d(f(x_1),f(x_0)) \;,$$

since $0 \le h < 1$, we have $d(f(x_m),f(x_n)) \to 0$ as $m,n \to \infty$. Thus the sequence $\{f(x_n)\}$ is a Cauchy sequence. By the completeness of X, let $\lim_n f(x_n) = p$. Since the sequence $\{f(x_n)\}$ is in M and M is closed, we get $p \in M$. Let $u_n = f(x_n)$ for all $n \ge 1$, since $\lim_n u_n = p$ and f is continuous, so by Theorem I.1.2

(3.3.3) $$\lim_n f(u_n) = f(p) \,.$$

Since $u_n \in J(x_{n-1})$, by the commutativity of J and f, we have

$$f(u_n) \in fJ(x_{n-1}) = Jf(x_{n-1}) = J(u_{n-1}) \,,$$

so then by definition of J there is a $v_n \in J(p)$ such that

$$d(f(u_n),v_n) \le h\, d(f(u_{n-1}),f(p)) \,.$$

But

$$d(v_n,f(p)) \le d(v_n,f(u_n)) + d(f(u_n),f(p))$$

$$\le h\, d(f(u_{n-1}),f(p)) + d(f(u_n),f(p)) \;,$$

and so by using (3.3.3) we get $d(v_n, f(p)) \to 0$ as $n \to \infty$. Since $J(p)$ is closed and $v_n \in J(p)$ for all $n \geq 1$, we have

$$(3.3.4) \qquad f(p) \in J(p) .$$

Hence, if (a) holds we get

$$f(p) = f^2(p) = f(f(p)) \in fJ(p) = Jf(p) ,$$

that is, $f(p)$ is a required fixed point of J.

Now if (b) holds, let

$$(3.3.5) \qquad \lim_n f^n(p) = \lim_n w_n = q .$$

Clearly $q \in M$, by the closedness of M. By continuity of f we get,

$$(3.3.6) \qquad \lim_n f(w_n) = f(q) .$$

Using (3.3.4) and the fact $Jf = fJ$, we get

$$w_n = f^n(p) = f^{n-1}(f(p)) \in f^{n-1}J(p) = Jf^{n-1}(p) = J(w_{n-1}) .$$

Since $w_n \in J(w_{n-1})$, by the definition of J, there is a $\ell_n \in J(q)$ such that

$$d(w_n, \ell_n) \le h\, d(f(w_{n-1}), f(q)) .$$

But,

$$d(\ell_n, q) \le d(\ell_n, w_n) + d(w_n, q)$$

$$\le h\, d(f(w_{n-1}), f(q)) + d(w_n, q) .$$

By using (3.3.5) and (3.3.6), we get

$$d(\ell_n, q) \to 0 \quad \text{as} \quad n \to \infty .$$

Thus the closedness of $J(q)$ implies $q \in J(q)$.

We want to show that the closedness of M and the commutativity of J with f in the hypothesis of Theorem 3.3 are essential.

3.4. Counterexamples. (a) Let $M = X = \mathbb{R}^2$ with the usual metric d. Suppose J and f are same as in the Example 3.2 with α, β $(\neq 0)$ in $\mathbb{R}$. Then easily seen is that all the hypotheses of Theorem 3.3 are satisfied except $Jf = fJ$. Clearly no fixed point exists. (In particular, if $\alpha = 0$, then J has a unique fixed point $(0,0)$, which is not a fixed point of f).

(b) Let $M = X = \mathbb{R}^2$ with the usual metric d. For $x = (x_1, x_2) \in M$, define $J(x) = \left\{\left(7x_1, \frac{x_2}{3} + \alpha\right)\right\}$, where α is a nonzero fixed real number and f is the same as in Example 3.2 with $\beta = 0$. Then all the conditions of Theorem 3.3 are satisfied except $Jf = fJ$. Clearly f has a unique fixed point $(0,0)$, which is not a fixed point of J.

(c) Let $X = \mathbb{R}$ with the usual metric d and $M = (0, \frac{1}{2})$. For $x \in M$, define $J(x) = \{x^4\}$, $f(x) = x^2$. Since

$$|x^4-y^4| = |x^2-y^2|\ |x^2+y^2| \leq \frac{1}{2}|x^2-y^2| .$$

Thus J is a f-(H·T)-contractive mapping. All the requirements of Theorem 3.3 are satisfied except that M is closed. Clearly J has no fixed point.

We conclude with an example of a multivalued mapping which is not (H·T)-contractive, which nevertheless satisfies the hypotheses of Theorem 3.3 and hence has a fixed point.

3.5. Example. Let $M = X = \mathbb{R}^2$ with the usual metric d. Suppose J and f are the same as in Example 3.2 with $\alpha = \beta = 0$. Then these maps do the job.

§4. Fixed point theorems for (H·T)-nonexpansive maps.

In this section we study a class of (H·T)-nonexpansive multivalued mappings which contains the class of all (H·T)-contractive maps. We wish to investigate the ways in which the fixed point theorems in §2 and §3 may be extended to (H·T)-nonexpansive mappings and f-(H·T)-nonexpansive mappings.

We have seen in Examples II.2.1 that not every (H·T)-nonexpansive mapping on a nonempty closed bounded convex subset of an arbitrary Banach space has a fixed point. Here we prove an improved fixed point theorem for (H·T)-nonexpansive maps on a nonempty convex weakly compact subset of a Banach space X under certain conditions, where the (2.2.1) condition of Theorem II.2.2 can be dropped.

Our basic approach to the (H·T)-nonexpansive mapping $J: M \subset X \to 2^M$ is through (H·T)-contractive mappings

$$J_n: M \to 2^M$$

with Lipschitz constants $h_n < 1$ for all $n \geq 1$ and $h_n \to 1$ as $n \to \infty$. By Theorem 2.1 J_n will have a fixed point x_n for all $n \geq 1$. The Lemma II.1.6 is then used to show the weak convergence of the sequence $\{x_n\}$ to a fixed point of J.

We need the following:

4.1. <u>Proposition</u>. Let M be a nonempty closed bounded convex subset of a Banach space X. Suppose $J: M \to 2^M$ is a (H·T)-nonexpansive compact-valued mapping. Then there exists a sequence $\{x_n\}$ in M and $u_n \in J(x_n)$ such that

$$\|x_n - u_n\| \to 0 \text{ as } n \to \infty .$$

<u>Proof</u>: Consider a sequence of positive numbers $\{h_n\}$ converging to 1 and $0 < h_n < 1$ for all $n \geq 1$ (for instance $h_n = (1-n^{-1})$).

For a given point x_0 of M we define the mapping J_n of M into 2^M by setting

(4.1.1) $$J_n(x) = h_n J(x) + (1-h_n)x_0 = \{h_n u + (1-h_n)x_0 : u \in J(x)\}$$

The mapping J_n does carry M into 2^M, since for each $x \in M$, $J_n(x)$ is the set of elements which are the convex linear combinations of the points x_0 and $u \in J(x) \subset M$, and M is convex, we have $J_n(M) \subseteq M$. Now we show that for each $n \geq 1$, J_n is a (H·T)-contractive mapping such that $J_n(x)$ is a closed subset of M for each $x \in M$.

Clearly, for each $x \in M$, $J_m(x)$ is nonempty. Now let $u_x \in J_m(x)$ then by (4.1.1), we get

$$u_x = h_m v_x + (1-h_m)x_0 , \quad \text{for some } v_x \in J(x) .$$

Since $v_x \in J(x)$ and J is (H·T)-nonexpansive, there is a $v_y \in J(y)$ for all $y \in M$ such that

(4.1.2) $$\|v_x - v_y\| \leq \|x-y\| .$$

Put $u_y = h_m v_y + (1-h_m)x_0$. Clearly, by definition of $J_m(y)$ we get $u_y \in J_m(y)$ and

$$\|u_x - u_y\| = \|h_m(v_x - v_y)\|$$
$$\leq h_m\|x-y\| \quad \text{by (4.1.2),}$$

which proves the (H·T)-contractiveness of J_m. Now we show that $J_m(x)$ is closed. Put $J_m(x) = K$ for fixed x and m. Suppose $x_k \in K$, $k = 1,2,\ldots$ and $x_k \to y_0 \in \bar{K} \subset M$. Since $x_k \in K = J_m(x)$, by (4.1.1) we get

$$x_k = h_m u_k + (1-h_m)x_0 , \qquad u_k \in J(x) . \tag{4.1.3}$$

Since $J(x)$ is compact, for a convenient subsequence still denoted by $\{u_k\}$ we have $u_k \to u \in J(x)$.

Taking the limit as $k \to \infty$ in (4.1.3), we get

$$y_0 = h_m u + (1-h_m)x_0 .$$

Thus, by the definition of $J_m(x)$ we have $y_0 \in J_m(x)$. But m and x are arbitrary, hence for each $n \geq 1$ and for each $x \in M$, $J_n(x)$ is a closed set. The Theorem 2.1 guarantees that for each $n \geq 1$, J_n has a fixed point in M, say,

$$x_n \in J_n(x_n) \subset M , \qquad n = 1,2,\ldots .$$

From the definition of $J_n(x_n)$, there is a $u_n \in J(x_n)$ such that

$$x_n = h_n u_n + (1-h_n)x_0 .$$

Thus

$$\|x_n-u_n\| = \|h_n u_n + (1-h_n)x_0 - u_n\| = (1-h_n)\|x_0-u_n\| .$$

Since M is bounded, $u_n \in J(x_n) \subset M$ implies $\{\|u_n-x_0\|\}$ is bounded and so by the fact that $h_n \to 1$ as $n \to \infty$, we have $\|x_n-u_n\| \to 0$ as $n \to \infty$.

4.2. <u>Theorem</u>. Let M be a nonempty weakly compact convex subset of a Banach space X which satisfies Opial's condition. Suppose $J: M \to 2^M$ is a (H·T)-nonexpansive compact-valued map. Then J has a fixed point.

<u>Proof</u>: Since M is a closed convex and bounded subset of X. By Proposition 4.1 there exists a sequence $\{x_n\}$ in M such that

(4.2.1) $$\|y_n\| = \|x_n-u_n\| \to 0 \text{ as } n \to \infty , \quad u_n \in J(x_n) .$$

M being a weakly compact, we can find a weakly convergent subsequence $\{x_m\}$ of $\{x_n\}$. Let $x_0 = w-\lim_m x_m$. Clearly $x_0 \in M$. From (4.2.1) we can write

$$\|y_m\| = \|x_m-u_m\|$$

where $u_m \in J(x_m)$ and $y_m \to 0$, so by Lemma II.1.6 there exists a fixed point $x_0 \in J(x_0)$ and the theorem is proved.

For the mapping $J(x) = \{x\}$ on a Banach space, each point is a fixed point of J. Thus one cannot expect any unique conclusion in Theorem 4.2.

We derive the following corollaries from Theorem 4.2.

4.3. Corollary. Let M be a nonempty weakly compact convex subset of a Banach space X having a weakly continuous duality mapping. Then each (H·T)-nonexpansive compact-valued mapping J of M into 2^M has a fixed point.

Proof: Since each Banach space with a weakly continuous duality mapping satisfies Opial's condition [I.4.6] so the corollary follows from Theorem 4.2.

4.4. Corollary. Let M be a nonempty closed bounded convex subset of a reflexive Banach space (in particular, uniformly convex space) X which satisfies Opial's condition. Then each (H·T)-nonexpansive compact-valued mapping J of M into 2^M has a fixed point.

Proof: Since each closed convex bounded subset of a reflexive Banach space is weakly compact (I.3.2(a)), the corollary follows from Theorem 4.2.

4.5. Corollary. Let M be a nonempty closed convex bounded subset of a reflexive Banach space X. Suppose the relation $\perp$ is uniformly approximately symmetric, then each (H·T)-nonexpansive compact-valued mapping J of M into 2^M has a fixed point.

Proof: Since each reflexive Banach space with relation $\perp$ uniformly approximately symmetric satisfies Opial's condition (I.4.12), the result follows from Corollary 4.4.

4.6. Corollary. Let M be a nonempty closed convex bounded subset of a Hilbert space X. Then each (H·T)-nonexpansive compact-valued mapping J of M into 2^M has a fixed point.

Proof: Since each Hilbert space is reflexive and satisfies Opial's condition (I.4.5), the corollary follows from Corollary 4.4.

4.7. Remark. Corollary 4.6 extends a result due to Husain and Tarafdar (Theorem I.5.13) and contains a result due to Browder (Theorem I.5.4) as a special case. Corollary 4.5 generalizes Karlovitz's result (Theorem I.5.7). Corollary 4.4 includes a result (I.5.6) as a special case.

Now we give an example which shows that the (H·T)-nonexpansiveness of J in the Theorem 4.2 is essential. The following example shows that the result is not true if the Lipschitz constant is greater than 1.

4.8. Example. Let $X = \ell^2$, the space of all infinite sequences $x = \{x_i\}_{i\geq 1}$ which are absolutely square-sumable, that is,

$$\sum_{i=1}^{\infty} |x_i|^2 = \|x\|^2 < \infty .$$

Let $M = \{x \in X: \|x\| \le 1\}$, which is obviously a closed bounded and convex subset of the reflexive Banach space ℓ^2, then it follows that M is a weakly compact subset of X.

For a real number $\lambda > 1$, we define

$$J(x) = \{h(1-\|x\|)e_i + s_i(x): i = 1,2,\ldots,n\}$$

for all $x \in M$, where $h < 1$, $0 < h \le (\lambda^2-1)^{1/2}$, $e_i = \left\{\delta_{ij}\right\}_{j\ge 1}$ and $s_i(x) = \{x_1,x_2,\ldots,x_{i-1},0,x_i,0,0,\ldots\}$ with 0 in the i^{th} position.

Clearly J is a compact-valued map of M into 2^M. Moreover J is a (H·T)-nonexpansive mapping with the Lipschitz constant $\lambda > 1$. For, let $u_x = h(1-\|x\|)e_k + s_k(x) \in J(x)$ for some k. Put $u_y = h(1-\|y\|)e_k + s_k(y)$. Clearly $u_y \in J(y)$ and

$$\|u_x-u_y\|^2 = \|\{x_1-y_1,\ x_2-y_2,\ldots,\ x_{k-1}-y_{k-1},\ h(\|y\|-\|x\|),\ x_k-y_k,\ldots\}\|^2$$

$$= \sum_{j=1}^{\infty} |x_j-y_j|^2 + h^2\left|\,\|y\|-\|x\|\,\right|^2$$

$$\le \|x-y\|^2 + h^2\|y-x\|^2 = (1+h^2)\|x-y\|^2$$

$$\Rightarrow \quad \|u_x-u_y\| \le \sqrt{1+h^2}\,\|x-y\| \le \lambda\|x-y\| .$$

Clearly J is a fixed-point-free mapping:

4.9. Theorem. Let X be a Banach space which satisfies Opial's condition, D a closed convex subset of X and M a nonempty weakly compact convex subset of D. Suppose $J: M \to 2^D$ is a (H·T)-nonexpansive mapping such that for each $x \in M$, $J(x)$ is a compact subset of D. If $J(x) \subset M$ whenever $x \in \partial_D M$, then J has a fixed point.

Proof: Let $x_0 \in M$ be fixed and define for $0 < h_n < 1$, $h_n \to 1$

$$J_n(x) = h_n J(x) + (1-h_n)x_0 \qquad (x \in M).$$

Then, for each $n \geq 1$, J_n is a (H·T)-contractive closed-valued mapping of M into 2^D (see proof of Proposition 4.1). Now if $x \in \partial_D M$ implies $J(x) \subset M$ and hence by the convexity of M, we get $J_n(x) \subset M$.

Since M is a closed subset of D, from the Corollary (2.6) it follows that for each $n \geq 1$, J_n has a fixed point in M, i.e.,

$$x_n \in J_n(x_n) \cap M \qquad n = 1,2,\ldots .$$

We complete the proof similarly as that of Theorem 4.2.

Now we define a class of multivalued mappings containing as a particular case, the multivalued (H·T)-nonexpansive maps and also contains a class of f-(H·T)-contractive multivalued maps.

4.10. Definition. Let M be a nonempty subset of a metric space (X,d) and f a mapping of M into X. A multivalued mapping $J: M \to 2^X$ is said to be f-(H·T)-nonexpansive if for any $x \in M$, $u_x \in J(x)$ there is a $u_y \in J(y)$ for all $y \in M$ such that

$$d(u_x, u_y) \leq d(f(x), f(y)) .$$

Obviously each f-(H·T)-contractive map is a f-(H·T)-nonexpansive map. If $f = I$ on M, then a multivalued map is f-(H·T)-nonexpansive iff it is (H·T)-nonexpansive.

Here we prove a fixed point theorem for f-(H·T)-nonexpansive maps which contains Theorem 4.2 as a special case.

4.11. Theorem. Let M be a nonempty compact convex subset of a Banach space X which satisfies Opial's condition. Suppose f is a continuous affine mapping of M into itself and $J: M \to 2^M$ a f-(H·T)-nonexpansive mapping. Suppose:

(a) for each $x \in M$, $J(x)$ is a compact subset of M;

(b) f commutes with J and $J(M) \subset f(M)$;

(c) $f(x) \neq f^2(x)$ implies $\lambda f(x) \notin J(x)$, $\lambda \geq 1$.

Then J has a fixed point (which is also a fixed point of f).

Proof: We may assume without loss of generality, that $0 \in M$. For each h_n with $0 < h_n < 1$ and $h_n \to 1$, we define

$$J_n(x) = \{h_n u: u \in J(x)\} \qquad (x \in M) .$$

Then for each $n \geq 1$, J_n is a closed-valued mapping of M into 2^M. Now we show that for all $n \geq 1$, J_n is a f-(H·T)-contractive mapping such that J_n commutes with f and $J_n(M) \subset f(M)$.

Let $u_x \in J_n(x)$, then $u_x = h_n v_x$, for some $v_x \in J(x)$. Since J is f-(H·T)-nonexpansive, there is a $v_y \in J(y)$ for all $y \in M$ such that

$$\|v_x - v_y\| \leq \|f(x)-f(y)\| .$$

Put $u_y = h_n v_y$, then $u_y \in J_n(y)$ and

$$\|u_x - u_y\| = h_n\|v_x - v_y\| \leq h_n\|f(x)-f(y)\| .$$

This proves the f-(H·T)-contractiveness of J_n. Now for $x \in M$

$$J_n f(x) = \{h_n u: u \in J(f(x)) = fJ(x) = \bigcup_{z \in J(x)} f(z)\}$$

$$= \{h_n f(z): z \in J(x)\} = \{f(h_n z): z \in J(x)\}$$

$$= \{f(v): v \in h_n J(x) = J_n(x)\} = fJ_n(x) .$$

Now let $w \in J_n(x)$ which implies $w = h_n u$ for some $u \in J(x) \subset f(M)$. Since $u \in f(M)$ implies $u = f(z)$ for some $z \in M$, the convexity of M implies $h_n z \in M$. Thus

$$w = h_n f(z) = f(h_n z) \in f(M) .$$

But since x and w are arbitrary, we have $J_n f = f J_n$ and $J_n(M) \subset f(M)$. Finally we show that if $f(x) \neq f^2(x)$, then $f(x) \notin J_n(x)$ for all $n \geq 1$. Suppose $f(x) \in J_n(x) = h_n J(x)$ implies $f(x) = h_n u$ for some $u \in J(x)$ which implies $(h_n)^{-1} f(x) \in J(x)$ and this contradicts hypothesis (c). Since all the conditions of Theorem 3.3 are satisfied for the f-(H·T)-contractive mapping J_n, thus for each $n \geq 1$, J_n has a fixed point which is also a fixed point of f. Say,

$$x_n \in M , \quad x_n \in J_n(x_n) \quad \text{and} \quad f(x_n) = x_n , \qquad n = 1,2,\ldots .$$

M being compact, for a convenient subsequence $\{x_m\}$ of $\{x_n\}$, we have $x_m \to x_0 \in M$, so continuity of f and $f(x_m) = x_m$ imply

$$x_0 = \lim_m x_m = \lim_m f(x_m) = f(x_0) \in M .$$

Since $x_m \in J_m(x_m)$, this implies $x_m = h_m u_m$ for some $u_m \in J(x_m)$. By f-(H·T)-nonexpansiveness of J, there is a $v_m \in J(x_0)$ such that

$$\|u_m - v_m\| \leq \|f(x_m) - f(x_0)\| = \|x_m - x_0\| .$$

But

$$\|x_m - u_m\| = (1-h_m)\|u_m\| .$$

Since M is bounded, $u_m \in J(x_m) \subset M$ implies $\{\|u_m\|\}$ is bounded and so by the fact that $h_m \to 1$, we have

$$\|y_m\| = \|x_m - u_m\| \to 0 \qquad \text{as} \quad m \to \infty .$$

Since each convergent sequence is weakly convergent, so that $x_0 = w - \lim_m x_m$. Now we complete the proof similarly as that of Lemma II.1.6.

§5. Iterative sequences converging to fixed points of (H·T)-nonexpansive mappings.

In this section we present a result of iterative sequences on a Hilbert space X which converges to fixed points. Consider,

(5.1) $$x_{n+1} = \lambda u_n + (1-\lambda)x_n , \quad x_0 \in M \subset X, \quad u_n \in J(x_n) , \quad n \geq 0 .$$

with fixed $\lambda \in (0,1)$.

First we prove the following:

5.2. Lemma. Let M be a nonempty closed convex subset of a reflexive Banach space X which satisfies Opial's condition. Suppose $J: M \to 2^M$ is a (H·T)-nonexpansive mapping such that the following conditions hold:

(a) For each $x \in M$, $J(x)$ is compact;

(b) Fix J is a convex set (where Fix J denotes the set of all fixed points of J).

Then there exists a continuous convex functional which has a minimum on Fix J.

Proof: By Corollary 4.4, Fix J $\neq \phi$. Moreover Fix J is a closed subset of X. For, let $y_n \in$ Fix J, $n = 1,2,...$ and $y_n \to y \in \overline{\text{Fix } J} \subset M$. We show that $y \in$ Fix J.

Since $y_n \in$ Fix J implies $y_n \in J(y_n)$ for each $n \geq 1$, by (H·T)–nonexpansiveness of J there is $z_n \in J(y)$ such that

(5.2.1) $$\|y_n - z_n\| \leq \|y_n - y\| .$$

J(y) being compact, for a convenient subsequence $\{z_m\}$ of $\{z_n\}$, we have $z_m \to z \in J(y)$.

Now

$$\|z-y\| \leq \|z-z_m\| + \|z_m-y_m\| + \|y_m-y\|$$

$$\leq \|z-z_m\| + 2\|y_m-y\| \qquad \text{by (5.2.1).}$$

Since the right hand side of this inequality converges to zero as $m \to \infty$, we have $y = z \in J(y)$, i.e., $y \in$ Fix J.

Since M is convex, the sequence $\{x_n\}$ defined in (5.1) lies in M. If $p \in J(p)$, then by the definition of J there is $u_n \in J(x_n)$ such that

$$\|p-u_n\| \le \|x_n-p\| .$$

But

$$\begin{aligned}\|x_{n+1}-p\| &= \|\lambda(u_n-p) + (1-\lambda)(x_n-p)\| \\ &\le \lambda\|u_n-p\| + (1-\lambda)\|x_n-p\| \\ &\le \lambda\|x_n-p\| + (1-\lambda)\|x_n-p\| \\ &= \|x_n-p\| .\end{aligned}$$

Since the sequence of numbers $\{\|x_n-p\|\}$ is monotonically decreasing and bounded, hence convergent. Now define

$$f(y) = \lim_n \|x_n-y\| , \quad \text{for all } y \in \text{Fix } J .$$

We show that f is the required functional. Let y and y' be any two elements of Fix J, then

$$\begin{aligned}|f(y)-f(y')| &\le \left|\lim_n(\|x_n-y\| - \|x_n-y'\|)\right| \\ &\le \|y-y'\| ,\end{aligned}$$

which implies that f is continuous. For the convexity of f, we let $0 \le \eta \le 1$ and put $z = \eta y + (1-\eta)y'$, then

$$\|x_n - z\| = \|x_n - \eta x_n + \eta x_n - z\|$$
$$= \|\eta(x_n - y) + (1-\eta)x_n - (1-\eta)y'\|$$
$$\leq (1-\eta)\|x_n - y'\| + \eta\|x_n - y\| .$$

Taking the limit as $n \to \infty$, we get

$$f(z) = f(\eta y + (1-\eta)y') \leq \eta f(y) + (1-\eta)f(y') ,$$

that is, f is a convex functional on Fix J. By (I.3.3) the functional f has a minimum value on the nonempty closed bounded convex set Fix J.

5.3. Theorem. Let M be a nonempty closed convex bounded subset of a Hilbert space X. Suppose $J: M \to 2^M$ is a $(H \cdot T)$-nonexpansive mapping satisfying (a) and (b) of Lemma 5.2. Then for each fixed $\lambda \in (0,1)$, the sequence $\{x_n\}$ constructed above in (5.1) converges weakly to a fixed point of J.

Proof: By Lemma 5.2, there exists a continuous convex functional

(5.3.1) $$f(y) = \overline{\lim_n}\|x_n - y\| \qquad \forall y \in \text{Fix } J ,$$

which has its minimum value on Fix J. We choose a fixed minimal point $y_0 \in$ Fix J, then

(5.3.2) $$f(y) \geq f(y_0) \qquad \text{for all } y \in \text{Fix } J .$$

M being a closed bounded subset of a Hilbert space is weakly compact (I.3.2(a)), for a convenient subsequence $\{x_m\}$ of $\{x_n\}$, we have $x_0 = w-\lim_m x_m$. Clearly $x_0 \in M$. We show that x_0 is a fixed point of J.

Since $y_0 \in J(y_0)$ and $x_m \in M$, by the (H·T)-nonexpansiveness of J, there is $u_m \in J(x_m)$ such that

$$\|u_m - y_0\| \leq \|x_m - y_0\| .$$

Put $u_m - y_0 = v_m$ and $x_m - y_0 = v'_m$, then it follows that

$$\overline{\lim_m}\|v_m\| \leq \overline{\lim_m}\|v'_m\| = \lim_m\|x_m - y_0\| = f(y_0), \quad \text{by} \quad (5.3.1) .$$

Now it follows from (5.1),

$$\|\lambda v_m + (1-\lambda)v'_m\| = \|\lambda(u_m - y_0) + (1-\lambda)(x_m - y_0)\|$$

$$= \|\lambda u_m + (1-\lambda)x_m - y_0\| = \|x_{m+1} - y_0\| .$$

Since $y_0 \in J(y_0)$, we get

$$\lim_m\|\lambda v_m + (1-\lambda)v'_m\| = \lim_m\|x_{m+1} - y_0\| = f(y_0) .$$

Since each Hilbert space is a uniformly convex space, it follows from (I.3.8) that

$$y_m = v'_m - v_m = x_m - u_m \to 0 \quad \text{as} \quad m \to \infty .$$

Since $x_0 = w - \lim_m x_m$, $y_m \in x_m - J(x_m)$, $y_m \to 0$, and each Hilbert space satisfies Opial's condition, it follows from Lemma II.1.6 that $x_0 \in J(x_0)$.

Now we show that $x_0 = y_0$. Let <,> be the scalar product on X. From the identity

$$(5.3.3) \qquad \|x_m - y_0\|^2 = \|x_m - x_0\|^2 + \|x_0 - y_0\|^2 + 2\,\mathrm{Re}\langle x_m - x_0,\ x_0 - y_0\rangle ,$$

(since $x_0 = w - \lim_m x_m$), (5.3.3) and (5.3.1) imply that, as $m \to \infty$

$$\begin{aligned}[f(y_0)]^2 &= [f(x_0)]^2 + \|x_0 - y_0\|^2 \\ &\geq [f(y_0)]^2 + \|x_0 - y_0\|^2 \qquad \text{by (5.3.2)} .\end{aligned}$$

So that $x_0 = y_0$. Thus all the weakly convergent subsequences of a bounded sequence $\{x_n\}$ have the same limit y_0. By (I.3.2(c)), the original sequence $\{x_n\}$ converges weakly to $y_0 \in \mathrm{Fix}\, J$, and hence the theorem.

Now we give an improved result for a Banach space with uniformly convex balls.

5.4. Theorem. Let M be a nonempty closed convex bounded subset of a uniformly convex Banach space which satisfies Opial's condition.

Suppose $J: M \to 2^M$ is a (H·T)-nonexpansive mapping satisfying (a) and (b) of Lemma 5.2. Then for each $\lambda \in (0,1)$, the sequence $\{x_n\}$ constructed above in (5.1) converges weakly to a fixed point of J.

Proof: We proceed exactly as in the proof of Theorem 5.3 and show $x_0 \in J(x_0)$, where $x_0 = w - \lim_m x_m$. If $y_0 \neq x_0$, then by Opial's condition and (5.3.1) we get

$$\liminf_m \|x_m - x_0\| \le \liminf_m \|x_m - y_0\|$$

$$\Rightarrow \quad \lim_m \|x_m - x_0\| \le \lim_m \|x_m - y_0\|$$

$$\Rightarrow \quad f(x_0) \le f(y_0).$$

But it is impossible by (5.3.2) unless $x_0 = y_0$, and therefore the whole sequence $\{x_n\}$ converges weakly to $y_0 \in \text{Fix } J$.

CHAPTER IV

COMMON FIXED POINTS FOR MULTIVALUED MAPS

In §1, a fixed point theorem for generalized K-contractive type mappings is proved which contains Kannan's result (Theorem I.5.9) as a special case. In §2 we generalize (Theorem 2.1) a result due to Kannan (Theorem I.5.8). Our Theorem 2.2 includes Theorem III.2.1 as a special case. §3 deals with fixed points for usual multivalued mappings.

§1. Fixed points for (H·T)-contractive type maps.

In [34] Kannan has proved a fixed point theorem for single-valued mappings satisfying the following condition:

Let (X,d) be a metric space. A single-valued mapping $f: X \to X$ is K-contractive type, if

$$d(f(x),f(y)) \le h[d(x,f(x)) + d(y,f(y))]$$

for all x,y in X and for a fixed number h, $0 \le h < \frac{1}{2}$.

Here we define a general notion of K-contractive type multivalued maps as follows:

1.1. Definition. Let M be a nonempty subset of a metric space (X,d). A multivalued mapping $J: M \to 2^X$ is said to be (H·T)-contractive type if there exists a nonnegative number $h < \frac{1}{2}$ and for any $x \in M$, $u_x \in J(x)$ there is a $u_y \in J(y)$ for all $y \in M$ such that

$$d(u_x,u_y) \le h[d(x,u_x) + d(y,u_y)] .$$

Clearly each single-valued map is K-contractive type iff it is a (H·T)-contractive type.

1.2. Example. Let $X = \mathbb{R}$ with the usual metric d and $M = [0,1]$. Let $f: M \to M$ be given by

$$f(x) = \begin{cases} \frac{x}{4} ; & 0 \le x < \frac{1}{2} \\ \frac{x}{5} ; & \frac{1}{2} \le x \le 1 . \end{cases}$$

Define

$$J(x) = \{w\} \cup \{f(x)\} \qquad (x \in M) ,$$

where w is an arbitrary fixed number in $[0,1]$. Then J maps M into 2^M and is a (H·T)-contractive mapping. For, if $x,y \in M$, and $u_x = w \in J(x)$, then there is $u_y = w \in J(y)$ such that

$$d(w,w) = 0 \le h[d(x,w) + d(y,w)] .$$

But if $u_x = f(x) \in J(x)$, then four cases arise:

Case (i) $0 \le x,y < \frac{1}{2}$, then there is $u_y = \frac{y}{4} \in J(y)$ such that

$$d\left(\frac{x}{4}, \frac{y}{4}\right) = \left|\frac{x}{4} - \frac{y}{4}\right| \le \frac{1}{3}|x+y| = \frac{4}{9}\left[\frac{3(x+y)}{4}\right].$$

Hence

$$d\left(\frac{x}{4}, \frac{y}{4}\right) \le \frac{4}{9}\left[d\left(x, \frac{x}{4}\right) + d\left(y, \frac{y}{4}\right)\right].$$

Case (ii) $\frac{1}{2} \le x, y \le 1$, then there is $u_y = \frac{y}{5} \in J(y)$ such that

$$d\left(\frac{x}{5}, \frac{y}{5}\right) = \frac{1}{5}|x-y| \le \frac{1}{4} \cdot \frac{4}{5}|x+y| \le \frac{4}{9}\left[\frac{4}{5}(x+y)\right]$$

so

$$d\left(\frac{x}{5}, \frac{y}{5}\right) \le \frac{4}{9}\left[d\left(x, \frac{x}{5}\right) + d\left(y, \frac{y}{5}\right)\right].$$

Case (iii) $0 \le x < \frac{1}{2}$, $\frac{1}{2} \le y \le 1$, then there is $u_y = \frac{y}{5} \in J(y)$ such that

$$20d\left(\frac{x}{4}, \frac{y}{5}\right) = |5x-4y| \le |5x+4y| \le \frac{4}{9}(15x+16y).$$

Hence

$$d\left(\frac{x}{4}, \frac{y}{5}\right) \le \frac{4}{9}\left[d\left(x, \frac{x}{4}\right) + d\left(y, \frac{y}{5}\right)\right].$$

Case (iv) $\frac{1}{2} \le x \le 1$, $0 \le y < \frac{1}{2}$, we take $u_y = \frac{y}{4} \in J(y)$, then

$$d\left(\frac{x}{5}, \frac{y}{4}\right) \le \frac{4}{9}\left[d\left(x, \frac{x}{5}\right) + d\left(y, \frac{y}{4}\right)\right].$$

Therefore J is a (H·T)-contractive type mapping with $h = \frac{4}{9}$.

Now we prove a fixed point theorem for (H·T)-contractive type multivalued mappings which contains the Kannan's result [34] (see I.5.9) as a special case.

1.3. Theorem. Let M be a nonempty closed subset of a complete metric space (X,d). Then each (H·T)-contractive type closed-valued mapping J of M into 2^M has a fixed point.

Proof: Let x_0 be an arbitrary element of M and choose an $x_1 \in J(x_0)$, then by definition of J there is an $x_2 \in J(x_1)$ such that

$$d(x_1,x_2) \le h[d(x_0,x_1) + d(x_1,x_2)]$$

$$\le \frac{h}{1-h} d(x_0,x_1) .$$

Similarly, there is an $x_3 \in J(x_2)$ such that

$$d(x_2,x_3) \le \frac{h}{1-h} d(x_1,x_2) \le \left[\frac{h}{1-h}\right]^2 d(x_0,x_1) .$$

In general, there is an $x_{n+1} \in J(x_n)$ such that

(1.3.1) $$d(x_n,x_{n+1}) \le \left[\frac{h}{1-h}\right]^n d(x_0,x_1) , \quad \text{for all } n \ge 1 .$$

Put $\lambda = \frac{h}{1-h}$, then $0 \le \lambda < 1$, since $0 \le h < \frac{1}{2}$.

For m and n positive integers $(n > m)$, repeated application of the triangle inequality, using (1.3.1) and finally the sum formula for a geometric series yields:

$$d(x_m,x_n) < \frac{\lambda^m}{1-\lambda} d(x_0,x_1) .$$

Since $\lambda < 1$, $d(x_m,x_n) \to 0$ as $m,n \to \infty$, that is, $\{x_n\}$ is a Cauchy sequence in a complete metric space X. Let $p = \lim_n x_n \in X$. Since $J(M) \subset M$ and $x_0 \in M$, we have $x_n \in M$ for all n. M being closed we have $p \in M$. Since $x_n \in J(x_{n-1})$, by definition of J there is $u_n \in J(p)$ such that

$$d(x_n,u_n) \le h[d(x_{n-1},x_n) + d(p,u_n)] .$$

But

$$\begin{aligned} d(p,u_n) &\le d(p,x_n) + d(x_n,u_n) \\ &\le d(p,x_n) + h[d(x_{n-1},x_n) + d(p,u_n)] \\ &\le \frac{1}{1-h} [d(p,x_n) + h\, d(x_{n-1},x_n)] . \end{aligned}$$

Letting $n \to \infty$, we get $d(p,u_n) \to 0$. Since $J(p)$ is closed and the sequence $\{u_n\}$ is in $J(p)$, we have $p \in J(p)$.

1.4. <u>Remark</u>. If $M = X$ and J is single-valued mapping then J has a unique fixed point (Kannan's result I.5.9).

§2. Common fixed points for multivalued maps.

In this section we study common fixed points for multivalued mappings of a nonempty closed subset of a complete metric space. Theorem (2.1) contains Kannan's result (see [34] or I.5.8) as a special case.

2.1. Theorem. Let M be a nonempty closed subset of a complete metric space (X,d) and $\{J_n\}$ a sequence of closed-valued mappings of M into 2^M. Suppose there exists a constant h with $0 \le h < \frac{1}{2}$ and

(2.1.1) For any two maps J_i, J_j and for any $x \in M$, $u_x \in J_i(x)$ there is a $u_y \in J_j(y)$ for all y in M with

$$d(u_x,u_y) \le h[d(x,u_x) + d(y,u_y)] .$$

Then $\{J_n\}$ has a common fixed point.

Proof: Let x_0 be an arbitrary element of M and let $x_1 \in J_1(x_0)$, then by (2.1.1) there is an $x_2 \in J_2(x_1)$ such that

$$d(x_1,x_2) \le \frac{h}{1-h} d(x_0,x_1) .$$

We proceed as in the proof of Theorem 1.3, to show that there is an $x_{n+1} \in J_{n+1}(x_n)$ such that

$$d(x_n,x_{n+1}) \le \left[\frac{h}{1-h}\right]^n d(x_0,x_1) , \quad \text{for all } n \ge 1$$

and $\{x_n\}$ is a Cauchy sequence in a complete metric space X. Let $p = \lim_n x_n \in M$. Now we show that p is a common fixed point of $\{J_n\}$.

Let J_m be an arbitrary member of $\{J_n\}$. Since $x_n \in J_n(x_{n-1})$, by (2.1.1) there is a $u_n \in J_m(p)$ such that

$$d(x_n,u_n) \le h[d(x_{n-1},x_n) + d(p,u_n)] .$$

Again we proceed as in the proof of Theorem 1.3, to show that $d(p,u_n) \to 0$ as $n \to \infty$ and finally we get $p \in J_m(p)$, but J_m is an arbitrary, hence $p \in \bigcap_{n\ge 1} J_n(p)$.

2.2. Remark. (a) If $M = X$ and $\{J_n\}_{n=1,2}$ in Theorem 2.1 such that each J_n, $n = 1,2$ is a single-valued map, then J_1 and J_2 have a unique common fixed point (Kannan's result (Theorem I.5.8)).

(b) If we take $J_j = J_1$ for all $j \ge 1$ in Theorem 2.1, then we obtain Theorem 1.3.

Now we prove a theorem which contains Theorem III.2.1 as a special case.

2.3. Theorem. Let M be a nonempty closed subset of a complete metric space (X,d) and $\{J_n\}$ a sequence of closed-valued maps of M into 2^M. Suppose that there exists a constant h with $0 \le h < 1$ such that

(2.3.1) For any two maps J_i, J_j and for any $x \in M$, $u_x \in J_i(x)$ there is a $u_y \in J_j(y)$ for all y in M with

$$d(u_x,u_y) \le h\, d(x,y) .$$

Then $\{J_n\}$ has a common fixed point.

Proof: Same proof as that of Theorem 2.1, by using the proof of Theorem III.2.1 instead of Theorem 1.3.

2.4. Remark. If we take $J_j = J_1$ for all $j \ge 1$ in Theorem 2.3, then again we obtain Theorem III.2.1.

The following first two examples show that there are sequences of mappings which do not satisfy (2.3.1) and have no common fixed points. That the condition (2.3.1) is not necessary can be seen from Example 2.5(c).

2.5. Examples. (a) Let $X = \mathbb{R}$ with the usual metric d and let $M = [-1,2]$. For $x \in M$, we define

$$J_1(x) = \left\{\frac{2}{3}-\frac{x}{2}\right\}, \quad J_2(x) = \left\{\frac{3}{4}-\frac{3x}{4}\right\}, \quad J_i = J_{i-2} \text{ for } i = 3,4,\dots .$$

Now let us consider J_1 and J_2. Then we have

$$d(u_x,u_y) = \left|\frac{3y}{4} - \frac{x}{2} - \frac{1}{12}\right| .$$

If $x = -\frac{1}{6}$, $y = -\frac{2}{3}$, then

$$d(u_x, u_y) = \frac{1}{2} = d(x,y)$$

Thus all the conditions except (2.3.1) of Theorem 2.3 are satisfied. Moreover the sequence $\{J_n\}_{n\geq 1}$ has no common fixed point. For, suppose there exists a point $x_0 \in M$ such that $x_0 \in \bigcap_{n\geq 1} J_n(x_0)$. Then

$$x_0 = \frac{2}{3} - \frac{x_0}{2} \quad \text{and} \quad x_0 = \frac{3}{4} - \frac{3x}{4}$$

$$\Rightarrow \quad x_0 = \frac{4}{9} = \frac{3}{7} \quad \text{which is impossible.}$$

(b) Let $X = M$ consist of two distinct elements x_1 and x_2 and let (X,d) be a metric space. Define

$$J_1(x_1) = \{x_2\} \quad \text{and} \quad J_1(x_2) = \{x_1\}$$

$$J_2(x_i) = \{x_i\}, \quad J_i = J_{i-2} \quad \text{for} \quad i = 3,4,\ldots .$$

Let us consider J_1 and J_3 and suppose condition (2.3.1) is true; then

$$d(x_2,x_1) \leq h\, d(x_1,x_2)$$

which is impossible, because $h < 1$. Clearly $\bigcap_{n\geq 1} J_n(x) = \phi$.

(c) Let $X = \mathbb{R}$ with the usual metric d and $M = [0,1]$. Define for $x \in M$,

$$J_1(x) = \{x\}, \quad J_2(x) = \left\{\frac{x}{2}+\frac{1}{4}\right\}, \quad J_i = J_{i-2}, \quad i = 3,4,\ldots .$$

Clearly $J_n\left(\frac{1}{2}\right) = \left\{\frac{1}{2}\right\}$ for all $n \geq 1$, and $\frac{1}{2}$ is the only fixed point. Now let us consider J_1 and J_2, then we have

$$d(u_x,u_y) = \left|x-\frac{y}{2}-\frac{1}{4}\right| .$$

If $x = \frac{1}{4}$, $y = \frac{1}{2}$, then

$$d(u_x,u_y) = \frac{1}{4} = d(x,y) .$$

Thus the condition (2.3.1) of Theorem 2.3 is not necessary.

Now we prove a general theorem about common fixed points which extends the previous results of this section.

2.6. Theorem. Let M be a nonempty closed subset of a complete metric space (X,d) and $\{J_n\}$ a sequence of closed-valued mappings of M into 2^M. Suppose that there are nonnegative real numbers h_1, h_2, h_3 with $2h_1 + 2h_2 + h_3 < 1$ and

(2.6.1) For any two maps J_i, J_j and for any $x \in M$, $u_x \in J_i(x)$ there is a $u_y \in J_j(y)$ for all $y \in M$ such that

$$d(u_x,u_y) \le h_1[d(x,u_x)+d(y,u_y)] + h_2[d(x,u_y)+d(y,u_x)] + h_3 d(x,y) .$$

Then the sequence of mappings $\{J_n\}$, has a common fixed point.

Proof: Let x_0 be an arbitrary element of M and let $x_1 \in J_1(x_0)$, then by (2.6.1) there is $x_2 \in J_2(x_1)$ such that

$$d(x_1,x_2) \le h_1[d(x_0,x_1)+d(x_1,x_2)] + h_2[d(x_0,x_2)+d(x_1,x_1)] + h_3 d(x_0,x_1)$$

$$(1-h_1)d(x_1,x_2) \le (h_1+h_3)d(x_0,x_1) + h_2 d(x_0,x_1) + h_2 d(x_1,x_2)$$

$$\Rightarrow \quad d(x_1,x_2) \le \left[\frac{h_1+h_2+h_3}{1-h_1-h_2}\right] d(x_0,x_1) .$$

Similarly since $x_2 \in J_2(x_1)$, there is an $x_3 \in J_3(x_2)$ such that

$$d(x_2,x_3) \le h_1[d(x_1,x_2)+d(x_2,x_3)] + h_2[d(x_1,x_3)+d(x_2,x_2)] + h_3 d(x_1,x_2) .$$

Therefore we have

$$d(x_2,x_3) \le \left[\frac{h_1+h_2+h_3}{1-h_1-h_2}\right] d(x_1,x_2) \le \left[\frac{h_1+h_2+h_3}{1-h_1-h_2}\right]^2 d(x_0,x_1) .$$

In general we have $x_{n+1} \in J_{n+1}(x_n)$ such that

$$d(x_n,x_{n+1}) \le \left[\frac{h_1+h_2+h_3}{1-h_1-h_2}\right]^n d(x_0,x_1).$$

Clearly since $2h_1+2h_2+h_3 < 1$, we have $\frac{h_1+h_2+h_3}{1-h_1-h_2} < 1$.

So by proceeding as in the proof of Theorem 1.3, we can show that $\{x_n\}$ is a Cauchy sequence in a complete metric space X. Let $p = \lim_n x_n$. Clearly $p \in M$. Now we show that p is a common fixed point of $\{J_n\}$.

Let J_k be an arbitrary member of $\{J_n\}$. Since by our choice $x_n \in J_n(x_{n-1})$ so (2.6.1) implies that there is an $u_n \in J_k(p)$ such that

(2.6.2) $$d(x_n,u_n) \le h_1[d(x_{n-1},x_n)+d(p,u_n)] + h_2[d(x_{n-1},u_n) + d(p,x_n)] + h_3 d(x_{n-1},p).$$

But

(2.6.3) $$d(p,u_n) \le d(p,x_n) + d(x_n,u_n).$$

From (2.6.2) and (2.6.3), we have

$$(1-h_1)d(p,u_n) \le d(p,x_n) + h_1 d(x_{n-1},x_n) + h_2 d(x_{n-1},u_n) + h_2 d(p,x_n) + h_3 d(x_{n-1},p).$$

Hence,

$$(1-h_1-h_2)d(p,u_n) \le (1+h_2)d(p,x_n) + h_1 d(x_{n-1},x_n)$$
$$+ (h_2+h_3)d(x_{n-1},p) .$$

Letting $n \to \infty$, we have $d(p,u_n) \to 0$. By the closedness of $J_k(p)$, we have $p \in J_k(p)$. But J_k is arbitrary, hence $p \in \bigcap_{n\ge 1} J_n(p)$.

2.7. Remark. If we take $h_2 = h_3 = 0$ in Theorem 2.6, then we obtain Theorem 2.1. If $h_1 = h_2 = 0$, then we obtain Theorem 2.3.

§3. Fixed points for usual multivalued maps.

3.1. Theorem. Let M be a weakly compact convex subset of a Banach space X and suppose that M has a normal structure. Let $J: M \to 2^M$ be a set-valued mapping such that

(3.1.1) Each nonempty closed convex subset of a closed convex J-invariant subset of M is J-invariant.

Then there exists a point $x_0 \in M$ such that $J(x_0) = \{x_0\}$.

Proof: Define

$$\Phi = \{C: \ C \text{ is a nonempty closed convex subset of } M \text{ with } J(C) \subset C\}$$

Clearly Φ is nonempty, since M is a member of Φ. If we order Φ by defining $C_1 \le C_2$ when $C_1 \subset C_2$, then Φ becomes a partially ordered set. Moreover it is inductive [i.e., every linearly ordered subfamily of Φ has a lower bound]. For, let Φ' be a linearly ordered subfamily of Φ. Now the intersection of C_α in Φ' is also a closed convex subset M' of M, which is invariant under J. Since each C_α is a weakly closed subset of a weakly compact set M, so M' is nonempty. Hence $\bigcap_\alpha C_\alpha = M' \in \Phi$ and M' is a lower bound of Φ'. By Zorn's lemma Φ has a minimal element K, say. Now if K consists of a single point then this is a fixed point and we are done. The rest of the proof is devoted to showing a contradiction if K consists of more than one point.

Suppose K consists of more than one point and we let $\operatorname{diam} K = d > 0$. By a normal structure there exists $\gamma \in (0,d)$, such that

$$\mathcal{L} = \{k \in K: \|k-x\| \le \gamma \quad \text{for all} \quad x \in K\} \neq \phi .$$

First we show that $\mathcal{L}$ is closed and convex. Suppose $k_i \in \mathcal{L}$, $i = 1,2,\ldots,$ and $k_i \to \ell \in \bar{\mathcal{L}} \subset K$. Since $k_i \in \mathcal{L}$ we have

$$\|k_i - x\| \le \gamma \qquad \text{for each} \quad x \in K .$$

Now for each $x \in K$

$$\|\ell - x\| \le \|\ell - k_i\| + \|k_i - x\| \le \|\ell - k_i\| + \gamma .$$

Letting $i \to \infty$, we have

$$\|\ell - x\| \le \gamma \qquad (\forall x \in K)$$

implies $\ell \in \mathcal{L}$ and $\mathcal{L}$ is therefore closed. For convexity, suppose $\lambda_i \ge 0$, $i = 1,2,\ldots,n$ with $\sum_{i=1}^{n} \lambda_i = 1$. By the convexity of K $\sum_{i=1}^{n} \lambda_i k_i \in K$. Also for all $x \in K$

$$\|k_i - x\| \le \gamma \qquad i = 1,2,\ldots,n .$$

Hence

$$\left\| \sum_{i=1}^{n} \lambda_i k_i - x \right\| = \left\| \sum_{i=1}^{n} \lambda_i k_i - \sum_{i=1}^{n} \lambda_i x \right\|$$

$$\le \sum_{i=1}^{n} \lambda_i \|k_i - x\| \le \sum_{i=1}^{n} \lambda_i \gamma = \gamma .$$

$\mathcal{L}$ is therefore convex. Thus by (3.1.1) and the minimality of K, we have $\mathcal{L} = K$.

But

$$\operatorname{diam} \mathcal{L} \le \gamma < d = \operatorname{diam} K = \operatorname{diam} \mathcal{L}$$

a contradiction, hence K is a singleton.

3.2. <u>Theorem</u>. Let M and X be same as that in Theorem 3.1. Let $J: M \to 2^M$ be a multivalued mapping such that

(3.2.1) $$J(K) \subset \{k \in K: K \subset B(k;\gamma)\} \neq \phi$$

where K is a nonempty subset of M with $\operatorname{diam} K > \gamma > 0$.

Then there exists a point $x_0 \in M$ such that $J(x_0) = \{x_0\}$.

<u>Proof</u>: By using the weak compactness of M and Zorn's lemma we find a minimal nonempty closed convex subset C of M such that $J(C) \subset C$. Let $d = \operatorname{diam} C$. If $d = 0$, then $C = \{x\}$. Suppose $d > 0$, by a normal structure there exists $\gamma \in (0,d)$ for which

$$\mathcal{L} = \{c \in C: C \subset B(c;\gamma)\} \neq \phi .$$

Since $d = \operatorname{diam} C > \gamma > 0$, so by hypothesis (3.2.1) we have $J(C) \subset \mathcal{L}$. Since $J(C) \subset C$ and C is a closed convex subset of a weakly compact subset M, it follows

$$\overline{\operatorname{conv}}\, J(C) \subset C .$$

Hence

$$J[\overline{\operatorname{conv}}\, J(C)] \subset J(C) \subset \overline{\operatorname{conv}}\, J(C) .$$

Since $\overline{\operatorname{conv}}\, J(C)$ is J-invariant and $\overline{\operatorname{conv}}\, J(C) \subset C$, by the minimality of

C, $C = \overline{\text{conv}}\, J(C)$. Since $\mathcal{L}$ is closed and convex such that $J(C) \subset \mathcal{L}$, we have

$$\mathcal{L} \subset C = \overline{\text{conv}}\, J(C) \subset \mathcal{L},$$

proving that $C = \mathcal{L}$. Now we proceed as in the proof of Theorem 3.1 to show that C is a singleton.

3.3. <u>Theorem</u>. Let M be a nonempty weakly compact convex subset of a Banach space X which satisfies Opial's condition. Let $\{J_n\}$ be a sequence of set-valued mappings of M into 2^M with a fixed point x_n for $n = 1,2,\ldots$ and let $J_0: M \to 2^M$ be a compact-valued mapping such that

(3.3.1) For any J_i, $x \in M$ and $u_x \in J_i(x)$ there is a $u_y \in J_0(y)$ for all $y \in M$ such that

$$\|u_x - u_y\| \le \|x-y\|.$$

Then there is a subsequence of $\{x_n\}$ which converges weakly to a fixed point of J_0.

<u>Proof</u>: Since $x_n \in J_n(x_n) \subset M$ and M is a weakly compact, we can find a weakly convergent subsequence $\{x_m\}$ of $\{x_n\}$ such that $x_0 = w-\lim_m x_m$. Clearly $x_0 \in M$. We prove that x_0 is a fixed point of J_0.

As $x_m \in J_m(x_m)$, so by (3.3.1) there is $u_m \in J_0(x_0)$ such that

$$\|x_m - u_m\| \leq \|x_m - x_0\| .$$

$J_0(x_0)$ being compact, for a convenient subsequence still denoted by u_m we have $u_m \to u \in J_0(x_0)$. Now then

$$\|x_m - u\| \leq \|x_m - u_m\| + \|u_m - u\| ,$$

and so

$$\|x_m - u\| - \|u_m - u\| \leq \|x_m - u_m\| \leq \|x_m - x_0\| .$$

Therefore we have

$$\liminf_m \|x_m - x_0\| \geq \liminf_m [\|x_m - u\| - \|u_m - u\|]$$

$$\geq \liminf_m \|x_m - u\| + \liminf_m (-\|u_m - u\|)$$

$$\Rightarrow \qquad \liminf_m \|x_m - x_0\| \geq \liminf_m \|x_m - u\| .$$

Since $x_0 = w - \lim_m x_m$, Opial's condition implies that $x_0 = u \in J_0(x_0)$.

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