A THEORY OF OPTIMAL
WORST-CASE DESIGN

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EMBODYING

CENTERING, TOLERANCING AND TUNING, WITH CIRCUIT APPLICATIONS

By

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SCOPE AND CONTENTS

This thesis presents a unified treatment of circuit and system design methods embodying centering, tolerancing and tuning. The approach incorporates the nominal parameter values, the corresponding tolerances and tuning variables simultaneously into an optimization procedure designed to obtain the best values for all of them in an effort to reduce cost, or make an otherwise impractically toleranced design more attractive. Intuitively, the aim is to produce the best nominal point to permit the largest tolerances and the smallest tuning ranges (preferably zero) such that one can guarantee, in advance, that in the worst case, the design will meet all the constraints and specifications.

Reduced problems are formulated for digital computer implementation. Interpretations are given. Implications of biquadratic functions in the circuit tolerance problems are investigated. Practical implementation in circuit design problems in the frequency domain is treated. The thesis also includes illustrative examples and two realistic problems.

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CHAPTER 1

INTRODUCTION

With readily available and ever increasing computing power at hand, computer-aided designers are now venturing to deal with more realistic problems. Useful and important material in computer-aided circuit design may be found, for example, in the collection of reprints in COMPUTER-AIDED CIRCUIT DESIGN, edited by Director (1973), in COMPUTER-AIDED FILTER DESIGN, edited by Szentirmai (1973), in MODERN FILTER THEORY AND DESIGN, edited by Temes and Mitra (1973), in the 1971 Special Issue on Computer-Aided Circuit Design of the IEEE TRANSACTIONS ON CIRCUIT THEORY and also in the 1974 Special Issue on Computer-Oriented Microwave Practices of the IEEE TRANSACTIONS ON MICROWAVE THEORY AND TECHNIQUES.

The *tolerance problem*, which is also known as the design centering and tolerance assignment problem, has attracted deep interest among designers. Besides books by Géher (1971) and Calahan (1972) which deal briefly with this subject, some relevant papers are also contained in Szentirmai's selection. A short list of recent publications in this area is included in the Additional Bibliography to give an indication of current efforts.

The two objectives in the tolerance problem are:

(1) Some strict tolerance limits may be met by placing the nominal values of a design at a suitable 'center' (called

design centering) and distributing the corresponding tolerances (called *tolerance assignment*).

(2) A more economical design may be obtained by minimizing a function which describes the cost-tolerance relationship.

Four recent, relevant approaches have been proposed in the area of circuit design.

(1) One approach is based on the concept of *large-change sensitivity* as described by Butler (1971a, 1971b) to center a design. It involves *performance contours* and deals with pairwise parameter interaction to specify tolerances. The centering and tolerancing are separate procedures. See Butler (1971) and also Karafin (1971).

(2) A second approach is based on the concept of *statistical moments* which are parameters describing a distribution of values. It finds the maximum possible moments of each component value distribution given the constraints on the second moment of the circuit or system response. See, for example, Seth and Roe (1971) and Seth (1972).

(3) Another approach is based on a *sensitivity model*. Multivariate Taylor series approximations of the circuit responses evaluated at the nominal point are used in the formulation of constraints for a nonlinear program. It is,

essentially, an extension to the *first-order sensitivity* method. Computation may be saved by evaluating some well-chosen first- or second-order derivatives. See Pinel and Roberts (1972). By introducing extra variables which represent changes in nominal values, Pinel (1973) reported that the approach can also deal with centering and tolerancing simultaneously with some success.

(4) The last approach is based on containing the *tolerance region* (a set of all possible outcomes of a design) in a *constraint region* (a set of points in the parameter space with performance specifications and design constraints satisfied). To save some computational effort, a well-chosen set of points from the tolerance region should be used. An appropriate cost function and a set of transformed constraints are employed in the optimization. See Bandler (1972, 1974) and Bandler and Liu (1973, 1974a). Both centering and tolerancing are treated simultaneously for the benefits of increased tolerances by permitting the nominal point to move. No approximation is used by this approach. The idea of a floating and expanding polytope may give some intuitive insight into the method.

Except for the second approach, all the other three are deterministic in nature. These are commonly known as *worst-case design* methods.

In the worst-case approach, the aim is to meet the performance specifications in all possible cases, even in the "worst" cases. Thus, it is also sometimes called the *100% yield design*. For the small-change sensitivity model, the worst case always occurs at a vertex of the tolerance region indicated by signs opposite to those of the corresponding partial derivatives. This is also true if the response of the circuit or system varies monotonically with respect to the variations in the component values taken one at a time. For large-change variations, however, this is not always true. Assumptions to predict the worst points have to be made and, subsequently, these assumptions have to be tested.

Another important practical consideration in design is the *tuning problem*. A design often requires tuning or alignment as a post-manufacturing process (Pinel 1971).

The work described in this thesis provides a theory of *optimal worst-case design* embodying all the *centering*, *tolerancing* and *tuning* problems in a unified manner at the design stage. The approach incorporates the nominal design parameter values, the corresponding tolerances and tuning variables simultaneously into an optimization procedure so as to obtain the best values for all of them in an effort to reduce cost, or make an otherwise impractically toleranced design more attractive. Intuitively, the aim is to produce the best nominal point to permit the largest tolerances and the smallest tuning ranges (preferably zero) such that we can guarantee, in advance and in the worst case, the design satisfies all the constraints and meets all the performance specifications. See Bandler and Liu (1974c, 1974d), Bandler, Liu and Chen (1974a, 1974b, 1975), Bandler, Liu and Tromp (1975a, 1975b).

The formulation is general such that the worst-case purely toleranced problem and the purely tuned problem fall out as special cases. Any of the nominal values, tolerances or tuning (relative or absolute) can be fixed or varied. Solutions can be continuous or discrete. Variable specifications such as tuned circuits can be extended without any additional theoretical difficulty.

The general formulation is presented in Chapter 2. Reduced problems to simplify computation are also treated and conditions of validity are stated in appropriate theorems. A geometric interpretation using concepts of projection and slack variables is discussed. Simple examples are studied to illustrate the effects of tuning and the interdependency of tolerancing, tuning and centering.

Chapter 3 deals with constraints arising from certain circuit applications. Implications of *biquadratic functions* in the circuit tolerance problem are studied deriving some necessary conditions to have the worst case occurring at the boundary of an interval. A one-dimensional case is studied. See Bandler and Liu (1974b, 1975).

Chapter 4 suggests practical implementation which may lead to the development of user-oriented design optimization packages. Part 1 discusses topics such as *vertex selection schemes*, *design symmetry* and its implications, performance specifications and parameter constraints. Implementation of the tolerance problem is demonstrated. Part 2 deals with tuning problems. Cases with separated as well as mixed tolerancing and tuning components are treated. Part 3 presents the results for two real problems reported by industry (Karafin 1971, Pinel and Roberts 1972, Pinel 1974, and Roberts 1974).

Circuit examples throughout the thesis are confined to lumped or distributed, linear, time-invariant networks in the frequency domain. The optimization in the minimax sense of the 2-section 10:1 quarter-wave transmission-line transformer has been previously studied by Matthaei, Young and Jones (1964), Bandler and Macdonald (1969), Bandler and Charalambous (1972a), and Bandler, Srinivasan and Charalambous (1972). The study of the 5-section transmission-line filter has been reported by Brancher, Maffioli and Premoli (1970), Bandler and Charalambous (1972a), and Bandler, Srinivasan and Charalambous (1972). The adjoint network approach for evaluating the gradients of the response function with respect to network parameters was used (Director and Rohrer 1969, Bandler and Seviora 1970).

For the sake of conciseness and continuity, related material is presented in the Appendices including mathematical concepts, nonlinear (continuous and discrete) programming, a basic theorem concerning convexity and a proposal for a user-oriented tolerance optimization package.

The major contributions claimed for this thesis are:

(1) A unified approach to the theory of optimal worst-case design embodying centering, tolerancing and tuning.

(2) The statement and formulation of reduced problems adaptable to computer implementation.

(3) A geometric interpretation of tuning and tolerancing.

(4) Necessary conditions for a biquadratic function of a single variable to be pseudoconcave or pseudoconvex, and some implications of these conditions in the circuit tolerance problem.

(5) Special algorithms to exploit symmetry and monotonicity of the response functions.

CHAPTER 2

OPTIMAL WORST-CASE DESIGN

2.1 Introduction

Component tolerance assignment is now considered to be an integral part of the design process. The optimal worst-case tolerance problem with variable nominal point has benefitted in terms of increased tolerances (Bandler and Liu 1974a). Tuning, on the other hand, does not seem to have been given its proper place in the design process. This work, therefore, brings in tuning of one or more components basically to further increase tolerances to reduce cost or to make unrealistically toleranced solutions more attractive. In this chapter, the mathematical formulation of an approach which embodies centering, tolerancing and tuning in a unified manner is presented (Bandler and Liu 1974c, 1974d). Simplified problems and appropriate geometric interpretations are discussed. The worst-case purely toleranced problem and purely tuned problem fall out as special cases, as is to be expected. Numerical examples involving some simple functions illustrate the concepts.

2.2 Fundamental Concepts and Definitions

A *design* consists of design data of the *nominal point* $\underset{\sim}{\phi}^0$, the *tolerance vector* $\underset{\sim}{\varepsilon}$ and the *tuning vector* $\underset{\sim}{t}$, where

$$\underset{\sim}{\phi}^0 \triangleq \begin{bmatrix} \phi_1^0 \\ \phi_2^0 \\ \cdot \\ \cdot \\ \cdot \\ \phi_k^0 \end{bmatrix}, \quad \underset{\sim}{\varepsilon} \triangleq \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \cdot \\ \cdot \\ \cdot \\ \varepsilon_k \end{bmatrix} \text{ and } \quad \underset{\sim}{t} \triangleq \begin{bmatrix} t_1 \\ t_2 \\ \cdot \\ \cdot \\ \cdot \\ t_k \end{bmatrix}. \tag{2.1}$$

k is the number, for example, of network parameters which may be indexed by

$$I_\phi \triangleq \{1, 2, \ldots, k\}. \tag{2.2}$$

We will assume that (1) the parameters can be varied continuously, and (2) the parameters can be chosen independently. Extra conditions such as discretization and imposed parameter bounds may be treated as constraints. See Bandler, Liu and Chen (1974a, 1974b, 1975). Some of the parameters can be set to zero or held constant.

An *outcome* $\{\underset{\sim}{\phi}^0, \underset{\sim}{\varepsilon}, \underset{\sim}{\mu}\}$ of a *design* $\{\underset{\sim}{\phi}^0, \underset{\sim}{\varepsilon}, \underset{\sim}{t}\}$ implies a point in the parameter space given by

$$\underset{\sim}{\phi} = \underset{\sim}{\phi}^0 + \underset{\sim}{E}\underset{\sim}{\mu}, \tag{2.3}$$

where

$$\underset{\sim}{E} \triangleq \begin{bmatrix} \varepsilon_1 & & & & \\ & \varepsilon_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & \varepsilon_k \end{bmatrix}, \tag{2.4}$$

and $\underset{\sim}{\mu}\epsilon R_\mu$. R_μ is a set of multipliers determined from realistic situations of the tolerance spread. For example,

$$R_\mu \triangleq \{\underset{\sim}{\mu} \mid -1 \leq \mu_i \leq -a_i,\ a_i \leq \mu_i \leq 1,\ i\epsilon I_\phi\}, \tag{2.5}$$

where

$$0 \leq a_i \leq 1. \tag{2.6}$$

The most commonly used continuous range is obtained by setting a_i to zero. A commercial stock will probably have the better toleranced components taken out, thus $0 < a_i \leq 1$. Unless otherwise stated, the case

$$R_\mu \triangleq \{\underset{\sim}{\mu} \mid -1 \leq \mu_i \leq 1,\ i\epsilon I_\phi\} \tag{2.7}$$

is considered (Bandler and Liu 1974a).

The *tolerance region* R_ϵ, as described by Butler (1971) and Bandler (1972, 1974), is a set of points defined by (2.3) for all $\underset{\sim}{\mu}\epsilon R_\mu$. In the case of $-1 \leq \mu_i \leq 1$, $i\epsilon I_\phi$,

$$R_\epsilon \triangleq \{\underset{\sim}{\phi} \mid \phi_i = \phi_i^0 + \epsilon_i\mu_i,\ -1 \leq \mu_i \leq 1,\ i\epsilon I_\phi\}, \tag{2.8}$$

which is a convex *regular polytope* of k dimensions with sides of length $2\epsilon_i$, $i\epsilon I_\phi$, and centered at $\underset{\sim}{\phi}^0$. The extreme points of R_ϵ are obtained by setting $\mu_i = \pm 1$. Thus, the set of *vertices* may be defined as

$$R_v \triangleq \{\underset{\sim}{\phi} \mid \phi_i = \phi_i^0 + \varepsilon_i \mu_i,\ \mu_i \varepsilon \{-1, 1\},\ i \varepsilon I_\phi\}. \tag{2.9}$$

The number of points in R_v is 2^k. Let each of these points be indexed by $\underset{\sim}{\phi}^i$, $i \varepsilon I_v$, where

$$I_v \triangleq \{1, 2, \ldots, 2^k\}. \tag{2.10}$$

Thus, $R_v = \{\underset{\sim}{\phi}^1, \underset{\sim}{\phi}^2, \ldots, \underset{\sim}{\phi}^{2^k}\}$.

The *tuning region* $R_t(\underset{\sim}{\mu})$ is defined as the set of points (see Bandler and Liu 1974c, 1974d)

$$\underset{\sim}{\phi} = \underset{\sim}{\phi}^0 + \underset{\sim}{E}\underset{\sim}{\mu} + \underset{\sim}{T}\underset{\sim}{\rho}, \tag{2.11}$$

for all $\underset{\sim}{\rho} \varepsilon R_\rho$, where

$$\underset{\sim}{T} \triangleq \begin{bmatrix} t_1 & & & \\ & t_2 & & \\ & & \ddots & \\ & & & t_k \end{bmatrix}. \tag{2.12}$$

Some of the common examples of R_ρ are

$$R_{\rho} \triangleq \{\underset{\sim}{\rho} \mid -1 \leq \rho_i \leq 1, \; i \in I_{\phi}\}, \tag{2.13}$$

or in the case of *one-way tuning* or *irreversible trimming*,

$$R_{\rho} = \{\underset{\sim}{\rho} \mid 0 \leq \rho_i \leq 1, \; i \in I_{\phi}\}, \tag{2.14}$$

or

$$R_{\rho} = \{\underset{\sim}{\rho} \mid -1 \leq \rho_i \leq 0, \; i \in I_{\phi}\}. \tag{2.15}$$

Unless otherwise indicated, the case given by (2.13) is considered.

The *constraint region* R_c is defined as (Butler 1971, Bandler 1972, 1974);

$$R_c \triangleq \{\underset{\sim}{\phi} \mid g_i(\underset{\sim}{\phi}) \geq 0, \; i \in I_c\}, \tag{2.16}$$

where

$$I_c \triangleq \{1, 2, \ldots, m_c\} \tag{2.17}$$

is the index set for the performance specifications and parameter constraints. R_c is assumed to be not empty. Other conditions and assumptions will be imposed on R_c as the theory is developed further.

The definitions are illustrated in Fig. 2.1 by a two-dimensional example.

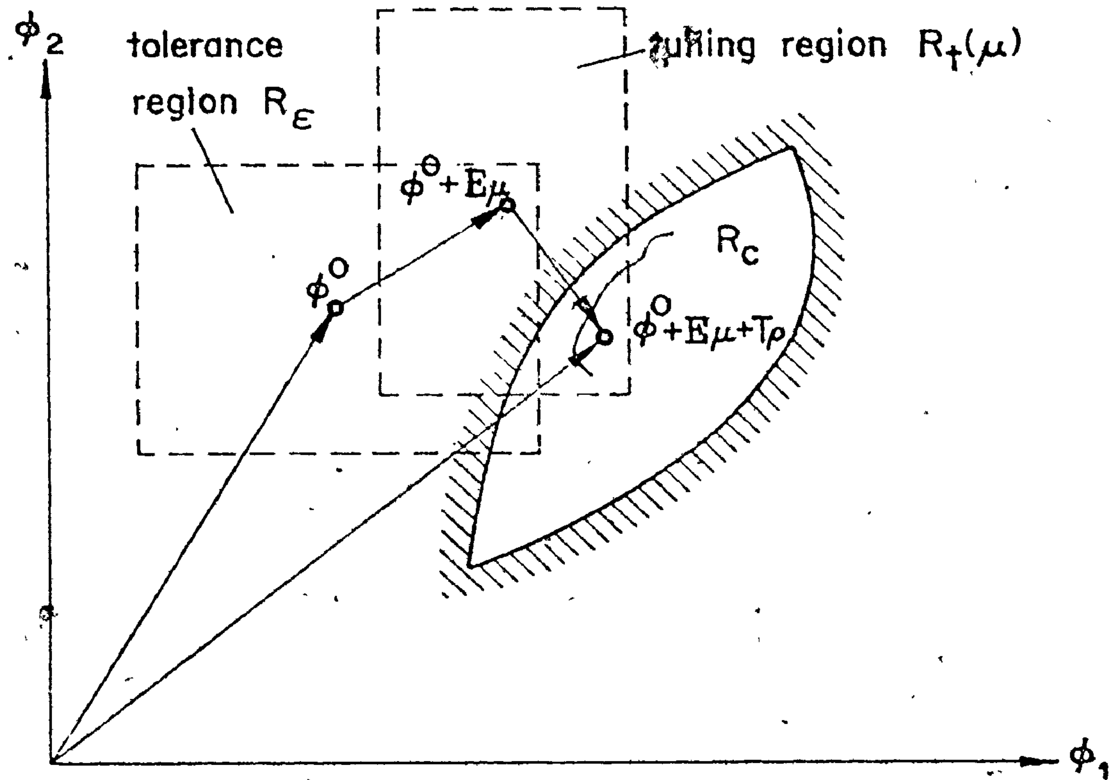


Fig. 2.1 An illustration of regions R_ε, R_t and R_c.

A *tunable constraint region* is denoted by $R_c(\underset{\sim}{\psi})$, where $\underset{\sim}{\psi}$ represents other independent variables. Figure 2.2 depicts three different regions of an example of $R_c(\underset{\sim}{\psi})$. Overlapping of these regions is allowable. The value of $\underset{\sim}{\psi}$ may be continuous or discrete. $R_c(\underset{\sim}{\psi}) = R_c$ in the ordinary sense if $\underset{\sim}{\psi}$ is a constant.

2.3 The Original Problem P_0

The problem may be stated as follows: obtain a set of optimal design values $\{\underset{\sim}{\phi}^0, \underset{\sim}{\varepsilon}, \underset{\sim}{t}\}$ such that any outcome $\{\underset{\sim}{\phi}^0, \underset{\sim}{\varepsilon}, \underset{\sim}{\mu}\}$, $\underset{\sim}{\mu} \varepsilon R_\mu$, may be tuned into R_c for some $\underset{\sim}{\rho} \varepsilon R_\rho$.

It is formulated as the nonlinear programming problem:

$$P_0 : \quad \text{minimize } C\,(\underset{\sim}{\phi}^0, \underset{\sim}{\varepsilon}, \underset{\sim}{t}),$$
$$\text{subject to } \underset{\sim}{\phi} \;\varepsilon\; R_c,$$

where

$$\underset{\sim}{\phi} = \underset{\sim}{\phi}^0 + \underset{\sim}{E}\underset{\sim}{\mu} + \underset{\sim}{T}\underset{\sim}{\rho} \tag{2.18}$$

and constraints $\underset{\sim}{\phi}^0$, $\underset{\sim}{\varepsilon}$, $\underset{\sim}{t} \geq \underset{\sim}{0}$, for all $\underset{\sim}{\mu} \varepsilon R_\mu$ and some $\underset{\sim}{\rho} \varepsilon R_\rho$. C is an appropriate function chosen to represent a reasonable approximation to known component cost data.

Stated in an abstract sense, the *worst-case solution* of the problem must satisfy

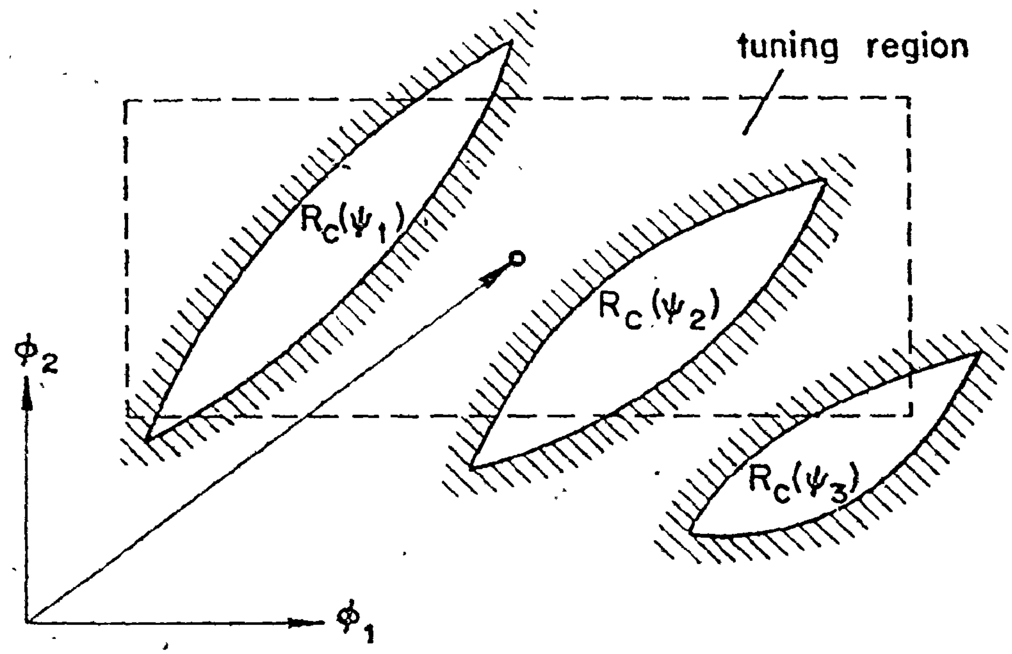


Fig. 2.2 An example of three different settings of the tunable constraint regions.

$$R_t(\underset{\sim}{\mu}) \cap R_c \neq \emptyset, \tag{2.19}$$

for all $\underset{\sim}{\mu} \varepsilon R_{\mu}$, where $\emptyset$ denotes a null set.

2.4 The Reduced Problem P_1

The original problem P_0 of the preceding section can be reduced by separating the components into *effectively tuned* and *effectively toleranced* parameters. Let

$$I_{\varepsilon} \triangleq \{i \mid \varepsilon_i > t_i,\ i \varepsilon I_{\phi}\}, \tag{2.20}$$

$$I_t \triangleq \{i \mid t_i \geq \varepsilon_i,\ i \varepsilon I_{\phi}\}, \tag{2.21}$$

$$\varepsilon_i' \triangleq \varepsilon_i - t_i,\ i \ \varepsilon\ I_{\varepsilon}, \tag{2.22}$$

and

$$t_i' = t_i - \varepsilon_i,\ i \ \varepsilon\ I_t. \tag{2.23}$$

It is obvious that I_t and I_{ε} are disjoint and $I_t \cup I_{\varepsilon} = I_{\phi}$.

Now, consider the problem

$$P_1 : \quad \text{minimize } C\,(\underset{\sim}{\phi}^0, \underset{\sim}{\varepsilon}, \underset{\sim}{t}),$$
$$\text{subject to } \underset{\sim}{\phi} \ \varepsilon\ R_c,$$

where

$$\phi_i = \phi_i^0 + \begin{cases} \varepsilon_i'\mu_i & \text{for} \quad i \in I_\varepsilon \\ t_i'\rho_i' & \text{for} \quad i \in I_t, \end{cases} \tag{2.24}$$

for all $-1 \leq \mu_i \leq 1$, $i\varepsilon I_\varepsilon$, and for some $-1 \leq \rho_i' \leq 1$, $i\varepsilon I_t$.

2.4.1 Theorem 2.1

A feasible solution to the *reduced problem* P_1 is a feasible solution to the original problem P_0.

<u>Proof</u> Given $\underset{\sim}{\phi}^0$, $\underset{\sim}{\varepsilon}$, $\underset{\sim}{t}$ we will show that

$$(1) \quad \varepsilon_i\mu_i + t_i\rho_i = \varepsilon_i'\mu_i \ , \ i \in I_\varepsilon, \tag{2.25}$$

$$(2) \quad \varepsilon_i\mu_i + t_i\rho_i = t_i'\rho_i' \ , \ i \in I_t, \tag{2.26}$$

under the restrictions on μ_i, ρ_i and ρ_i'.

(1) Since ρ_i can be freely chosen from $-1 \leq \rho_i \leq 1$, we can let $\rho_i = -\mu_i$ giving

$$(\varepsilon_i - t_i)\mu_i = \varepsilon_i'\mu_i. \tag{2.27}$$

(2) For any $-1 \leq \rho_i' \leq 1$ and all $-1 \leq \mu_i \leq 1$, we can choose

$$-1 \leq \rho_i = \frac{(t_i - \varepsilon_i)\rho_i' - \varepsilon_i \mu_i}{t_i} \leq 1 \quad , \quad t_i \neq 0. \tag{2.28}$$

Thus, any point with components represented by (2.24) of the reduced problem can be represented by (2.18) of the original problem. See Bandler and Liu (1974d).

Intuitively, this theorem states the fact that a feasible solution to a restrictive problem is also a feasible solution to an appropriate less restrictive problem. The variable transformation equations (2.22) and (2.23) may be considered as extraneous constraints to be satisfied.

2.4.2 Concept of One-Dimensional Convexity

The concept of one-dimensional convexity is important in this study. A region R is said to be *convex* if

$$\underset{\sim}{\phi}^a, \underset{\sim}{\phi}^b \in R$$

implies that

$$\underset{\sim}{\phi} = \underset{\sim}{\phi}^a + \lambda(\underset{\sim}{\phi}^b - \underset{\sim}{\phi}^a) \in R \tag{2.29}$$

for all $0 \leq \lambda \leq 1$. See Mangasarian (1969). We define a region R to be *one-dimensionally convex* (see Bandler 1972) if, for all $j \in I_\phi$,

$$\underset{\sim}{\phi}^a,\ \underset{\sim}{\phi}^{b(j)} \triangleq \underset{\sim}{\phi}^a + \alpha \underset{\sim}{e}_j \in R, \tag{2.30}$$

where α is a constant and $\underset{\sim}{e}_j$ is the jth unit vector, implies that

$$\underset{\sim}{\phi} = \underset{\sim}{\phi}^a + \lambda(\underset{\sim}{\phi}^{b(j)} - \underset{\sim}{\phi}^a) \in R, \tag{2.31}$$

for all $0 \le \lambda \le 1$. See Fig. 2.3 for some illustrations. R_1 is both convex and one-dimensionally convex whereas R_2 is one-dimensionally convex only. R_3 is neither. Since convexity implies one-dimensional convexity, the latter is less restrictive.

2.4.3 Theorem 2.2

A feasible solution to the original problem P_0 implies a feasible solution to the reduced problem P_1 if R_c is one-dimensionally convex.

<u>Proof</u> Consider the following.

(1) We note, for $i \in I_\varepsilon$, that

$$\phi_i^0 - \varepsilon_i + t_i\rho_i(-1) \le \phi_i^0 - \varepsilon_i + t_i \le \phi_i^0 + (\varepsilon_i - t_i)\mu_i$$

$$\le \phi_i^0 + \varepsilon_i - t_i \le \phi_i^0 + \varepsilon_i + t_i\rho_i(1) \tag{2.32}$$

where $\rho_i(-1)$ corresponds to $\mu_i = -1$ and $\rho_i(1)$ corresponds to

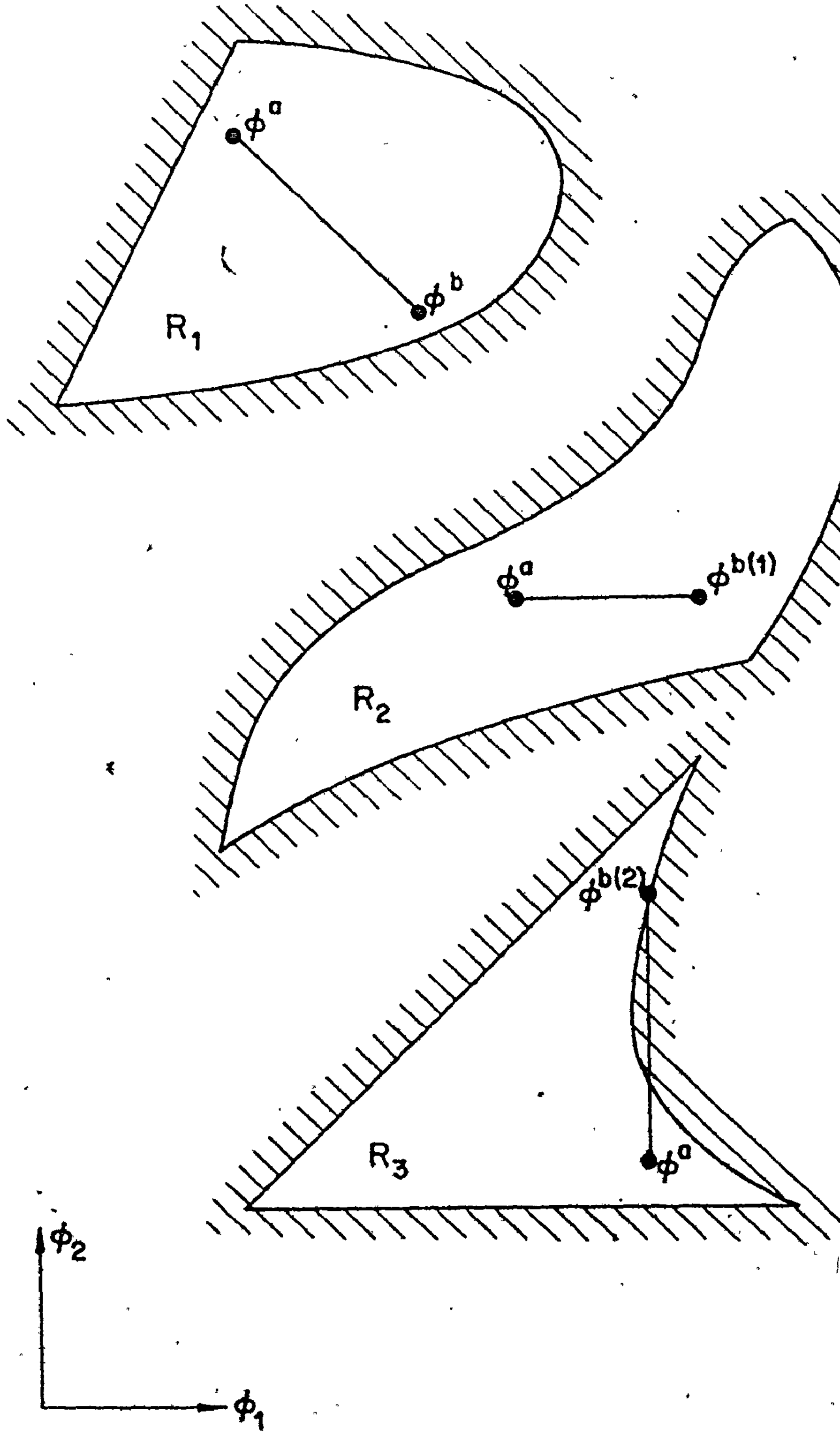


Fig. 2.3 Illustrations of convex, one-dimensionally convex and nonconvex regions.

$\mu_i = 1$. If R_c is one-dimensionally convex, the following assumption

$$\begin{bmatrix} \vdots \\ \phi_i^0 - \varepsilon_i + t_i\rho_i(-1) \\ \vdots \end{bmatrix}, \qquad \begin{bmatrix} \vdots \\ \phi_i^0 + \varepsilon_i + t_i\rho_i(1) \\ \vdots \end{bmatrix} \varepsilon R_c \tag{2.33}$$

implies that

$$\begin{bmatrix} \vdots \\ \phi_i^0 + (\varepsilon_i - t_i)\mu_i \\ \vdots \end{bmatrix} \varepsilon R_c, \tag{2.34}$$

where we consider changes in the ith component only and impose the required restrictions on μ_i and ρ_i.

(2) On the other hand, for $i\varepsilon I_t$, given feasible $\rho_i(-1)$ and $\rho_i(1)$ such that

$$\phi_i^0 - \varepsilon_i + t_i\rho_i(-1) \leq \phi_i^0 + \varepsilon_i + t_i\rho_i(1), \tag{2.35}$$

there exists a feasible ρ_i' such that

$$\phi_i^0 - \varepsilon_i + t_i\rho_i(-1) \leq \phi_i^0 + (t_i - \varepsilon_i)\rho_i' \leq \phi_i^0 + \varepsilon_i + t_i\rho_i(1). \tag{2.36}$$

This is true for $t_i = \varepsilon_i$ and can be seen for $t_i > \varepsilon_i$ by rewriting this inequality as

$$\frac{-\varepsilon_i + t_i \rho_i(-1)}{t_i - \varepsilon_i} \leq \rho_i' \leq \frac{\varepsilon_i + t_i \rho_i(1)}{t_i - \varepsilon_i}. \tag{2.37}$$

Hence, if R_c is one-dimensionally convex, the assumption implies that

$$\begin{bmatrix} \vdots \\ \phi_i^0 + (t_i - \varepsilon_i)\rho_i' \\ \vdots \end{bmatrix} \in R_c. \tag{2.38}$$

Thus, a feasible solution to the original problem can be transformed to a feasible solution of the reduced problem P_1. See Bandler and Liu (1974c, 1974d).

2.5 A Geometric Interpretation

Let us define a *projection matrix* $\underset{\sim}{P}$ as a diagonal matrix such that

$$\underset{\sim}{P} \triangleq \begin{bmatrix} p_1 & & & \\ & p_2 & & \\ & & \ddots & \\ & & & p_k \end{bmatrix}, \tag{2.39}$$

where

$$P_i = \begin{cases} 0 & \text{for} \quad i \in I_t \\ 1 & \text{for} \quad i \in I_\varepsilon . \end{cases} \tag{2.40}$$

In general, a projection operator p is defined as a linear operator such that $p^2 = p$. P obviously obeys such a property. See Finkbeiner (1960), Yale(1968) and Lancaster(1969), for some properties of a projection operator.

The projection of a point $\underset{\sim}{\phi}$ may be denoted as $\underset{\sim}{\phi}_p = \underset{\sim}{P}\underset{\sim}{\phi}$. It may be noted that the projections of two points $\underset{\sim}{\phi}^a$, $\underset{\sim}{\phi}^{b(j)} = \underset{\sim}{\phi}^a + \alpha \underset{\sim}{e}_j$, for $j \varepsilon I_t$, and some constant α, coincide. The projection concept and the introduction of slack variables provide a key to understanding the tuning concept.

Let

$$R_{\varepsilon t} \triangleq \{\underset{\sim}{\phi} | \phi_i^0 - \varepsilon_i' \leq \phi_i \leq \phi_i^0 + \varepsilon_i' \ , \ i \varepsilon I_\varepsilon\}, \tag{2.41}$$

and

$$R_{t\varepsilon} \triangleq \{\underset{\sim}{\phi} | \phi_i^0 - t_i' \leq \phi_i \leq \phi_i^0 + t_i' \ , \ i \varepsilon I_t\}, \tag{2.42}$$

denote the regions defined by the effectively toleranced and effectively tuned parameters. Then consider the following regions

$$R_{\varepsilon tp} \triangleq \{\underset{\sim}{\phi}_p | \underset{\sim}{\phi}_p = \underset{\sim}{P}\underset{\sim}{\phi} \ , \ \underset{\sim}{\phi} \varepsilon R_{\varepsilon t}\}, \tag{2.43}$$

$$R_{ct\varepsilon} \triangleq R_c \cap R_{t\varepsilon}, \tag{2.44}$$

and

$$R_{ct\varepsilon p} \triangleq \{\underset{\sim}{\phi}_p \mid \underset{\sim}{\phi}_p = \underset{\sim}{P}\underset{\sim}{\phi}\ ,\ \underset{\sim}{\phi}\varepsilon R_{ct\varepsilon}\}. \tag{2.45}$$

Figure 2.4 illustrates the definition of the regions. Any point whose components are given by (2.24) lies in the intersection of $R_{\varepsilon t}$ and $R_{t\varepsilon}$. Suppose the projection of $R_{ct\varepsilon}$ onto the subspace spanned by the effectively toleranced parameters includes the projection of that point. Then it may be tuned into $R_{ct\varepsilon}$ by adjusting the value of ρ_i', $i\varepsilon I_t$.

The reduced problem P_1 may be stated as: solve a pure tolerance problem (i.e., no tuning) in the subspace spanned by the toleranced variables with $R_{\varepsilon tp}$ as the tolerance region and $R_{ct\varepsilon p}$ as the constraint region.

In other words, the regions defined by a feasible solution must satisfy the condition that

$$R_{\varepsilon tp} \subseteq R_{ct\varepsilon p}. \tag{2.46}$$

Figure 2.5 illustrates a case where $R_{\varepsilon tp} \not\subseteq R_{ct\varepsilon p}$. An outcome at $\underset{\sim}{\phi}^0$ cannot be tuned to R_c within the effective tuning range. However, there exists a solution to the original formulation by tuning both components. R_c is not one-dimensionally convex in this case.

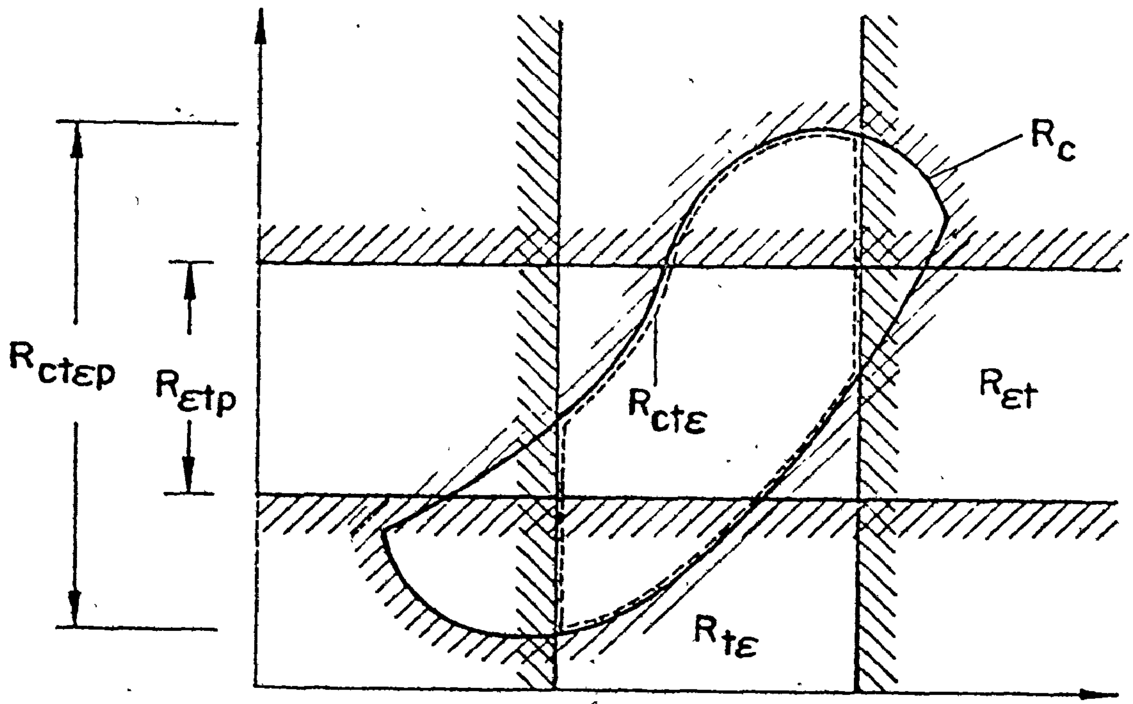


Fig. 2.4 A geometric interpretation of the reduced problem P_1.

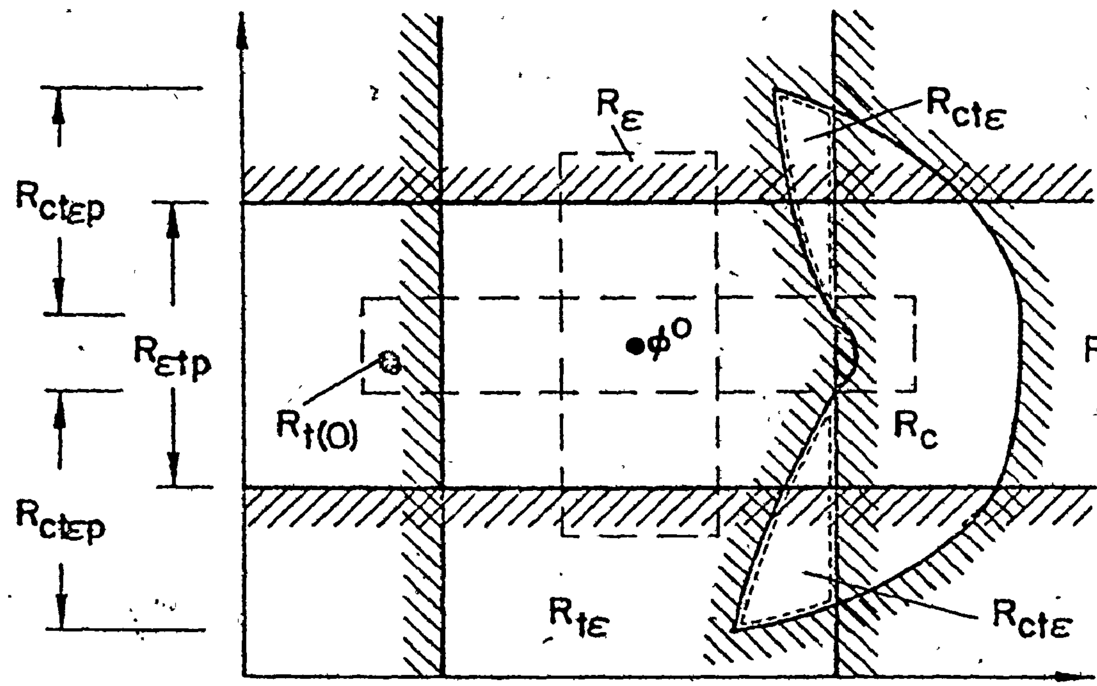


Fig. 2.5 An example of $R_{\varepsilon tp} \not\subseteq R_{ct\varepsilon p}$.

2.5.1 Special Cases

We will consider two special cases.

Case 1: $I_\varepsilon = \emptyset$, the *pure tuning problem.*

In this case, $R_{\varepsilon t}$ is the entire space and $\underset{\sim}{P}$ is a zero matrix. $R_{\varepsilon tp}$ is a single point at the origin. The problem has a solution if

$$R_{ct\varepsilon} \neq \emptyset. \tag{2.47}$$

Case 2: $I_t = \emptyset$, the *pure tolerance problem.*

In this case, $R_{t\varepsilon}$ is the entire space and $\underset{\sim}{P}$ is a unit matrix, $R_{\varepsilon tp} = R_{\varepsilon t}$ and $R_{ct\varepsilon p} = R_{ct\varepsilon} = R_c$. The problem has a solution if

$$R_{\varepsilon t} \subseteq R_c. \tag{2.48}$$

From a tolerance-tuning point of view, the first case is a trivial case theoretically. Except when there is only one single point R_c, the pure tuning problem is equivalent to an optimization of the nominal parameter values. On the other hand, the pure tolerance problem is very important from a practical point of view.

2.6 Extension of P_1 for Tunable Constraint Region

Three types of components can be identified when the constraint region is considered to be tunable. They are:

(a) Toleranced components,

(b) Components tuned by the manufacturer, and

(c) Components tunable by the customer.

In this case,

$$\underset{\sim}{\phi} \in R_c(\underset{\sim}{\psi})$$

where

$$\phi_i = \phi_i^0 + \begin{cases} \varepsilon_i' \mu_i & \text{for} \quad i \in I_\varepsilon \\ t_i' \rho_i' & \text{for} \quad i \in I_{tm} \\ t_i' \rho_i'(\underset{\sim}{\psi}) & \text{for} \quad i \in I_{tc} \end{cases} \tag{2.49}$$

where I_{tm} identifies components (b) and I_{tc} identifies components (c).

Setting the $\underset{\sim}{\psi}$ to a particular value will control the setting of ρ_i', $i \in I_{tc}$, such that $\underset{\sim}{\phi}$ will be in that particular constraint region $R_c(\underset{\sim}{\psi})$.

2.7 The Reduced Problem P_2

It is impossible to test all the points in $R_{\varepsilon tp}$ to be in $R_{c\varepsilon\varepsilon p}$. In order to make the problem tractable a number of simplifying assumptions could be made to obtain an acceptable solution to the problem with reasonable computational effort.

To this end we replace the continuous range $-1 \leq \mu_i \leq 1$ by a discrete set $\mu_i \varepsilon \{-1,1\}$, $i \varepsilon I_\varepsilon$.

Now, consider the problem

$$P_2 : \quad \text{minimize } C\,(\underset{\sim}{\phi}^0, \underset{\sim}{\varepsilon}, \underset{\sim}{t}),$$
$$\text{subject to } \underset{\sim}{\phi} \;\varepsilon\; R_c,$$

where

$$\phi_i = \phi_i^0 + \begin{cases} \varepsilon_i' \mu_i & \text{for} \quad i \;\varepsilon\; I_\varepsilon \\ t_i' \rho_i' & \text{for} \quad i \;\varepsilon\; I_t, \end{cases} \tag{2.50}$$

for all $\mu_i \varepsilon \{-1,1\}$, $i \varepsilon I_\varepsilon$, and some $-1 \leq \rho_i' \leq 1$, $i \varepsilon I_t$.

Let us define the set of *projected vertices* (or the vertices of the projected region) by

$$R_{vp} \triangleq \{\underset{\sim}{\phi}_p \,|\, \underset{\sim}{\phi}_p = \underset{\sim}{P}\underset{\sim}{\phi},\ \underset{\sim}{\phi} \;\varepsilon\; R_v\}, \tag{2.51}$$

The condition may be now stated as

$$R_{vp} \subseteq R_{c t \varepsilon p}\ .$$

2.7.1 Theorem 2.3

A feasible solution to *reduced problem* P_2 implies a feasible solution to reduced problem P_1 if $R_{ct\epsilon p}$ is one-dimensionally convex.

This is a pure tolerance problem in the subspace spanned by the effectively toleranced parameters. For a proof in the tolerance parameter space, see Appendix B which describes the proof by Bandler (1972, 1974).

2.8 The Objective Function

Several *objective functions* (or *cost functions*) have been proposed by Bandler (1972, 1974), Pinel and Roberts (1972) and Bandler and Liu (1973, 1974a). In practice, a suitable modelling problem would have to be solved to determine the cost-tolerance relationship. Here, it is assumed that the tolerances and tuning ranges (either absolute or relative) are the main variables and that the total cost of the design is the sum of the cost of the individual components.

The objective function should have the following properties,

$$
\begin{aligned}
&C(\underset{\sim}{\phi}^0,\underset{\sim}{\epsilon},\underset{\sim}{t}) \rightarrow c && \text{as} \quad \underset{\sim}{\epsilon} \rightarrow \underset{\sim}{\infty}, \\
&C(\underset{\sim}{\phi}^0,\underset{\sim}{\epsilon},\underset{\sim}{t}) \rightarrow \infty && \text{for any } \epsilon_i \rightarrow 0, \\
&C(\underset{\sim}{\phi}^0,\underset{\sim}{\epsilon},\underset{\sim}{t}) \rightarrow C(\underset{\sim}{\phi}^0,\underset{\sim}{\epsilon}) && \text{as} \quad \underset{\sim}{t} \rightarrow \underset{\sim}{0}, \\
&C(\underset{\sim}{\phi}^0,\underset{\sim}{\epsilon},\underset{\sim}{t}) \rightarrow \infty && \text{for any } t_i \rightarrow \infty.
\end{aligned}
\qquad (2.52)
$$

Suitable objective functions will be, for example, of the form

$$C = \sum_{i=1}^{k} \frac{c_i}{x_i} + \sum_{i=1}^{k} c_i' y_i, \tag{2.53}$$

where x_i and y_i denote the tolerances and tuning ranges, respectively. In the case of relative tolerances or relative tuning ranges

$$x_i = \frac{\varepsilon_i}{\phi_i^0} \times 100, \tag{2.54}$$

$$y_i = \frac{t_i}{\phi_i^0} \times 100. \tag{2.55}$$

We may set the appropriate c_i' to zero if tuning is considered either free, or fixed or is not required. c_i may be set to zero if the corresponding tolerance is fixed.

2.9 A Tolerance Example

Consider the constraints

$$\phi_2 - \phi_1 - 2 \geq 0, \tag{2.56}$$

$$-\phi_2^2 + 16\phi_1 \geq 0. \tag{2.57}$$

A convex region R_c is defined by these constraints.

We will take R_μ as an infinite set of discrete points

$\underset{\sim}{\mu}(i)$, $i = 1, 2, \ldots$, where $-1 \le \mu_1(i) \le 1$ and $-1 \le \mu_2(i) \le 1$. Thus a relevant problem may be formulated as follows. Minimize

$$C = \frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2} \tag{2.58}$$

with respect to ε_1, ε_2, ϕ_1^0 and ϕ_2^0, and subject to

$$g_1 = \varepsilon_1 \ge 0, \quad g_2 = \varepsilon_2 \ge 0, \quad g_3 = \phi_1^0 \ge 0, \quad g_4 = \phi_2^0 \ge 0, \tag{2.59}$$

$$g_5(i) = (\phi_2^0 + \varepsilon_2\mu_2(i)) - (\phi_1^0 + \varepsilon_1\mu_1(i)) - 2 \ge 0, \; i = 1, 2, \ldots \tag{2.60}$$

$$g_6(i) = -(\phi_2^0 + \varepsilon_2\mu_2(i))^2 + 16(\phi_1^0 + \varepsilon_1\mu_1(i)) \ge 0, \; i = 1, 2, \ldots \tag{2.61}$$

where $-1 \le \mu_1(i) \le 1$ and $-1 \le \mu_2(i) \le 1$.

The Kuhn-Tucker (1951) necessary conditions for a constrained minimum require that (see also Bandler 1973)

$$\begin{bmatrix} -\frac{1}{\varepsilon_1^2} \\ -\frac{1}{\varepsilon_2^2} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} + \sum_i u_5(i) \begin{bmatrix} -\mu_1(i) \\ \mu_2(i) \\ -1 \\ 1 \end{bmatrix} + \sum_i u_6(i) \begin{bmatrix} 16\mu_1(i) \\ -2\mu_2(i)(\phi_2^0+\varepsilon_2\mu_2(i)) \\ 16 \\ -2(\phi_2^0+\varepsilon_2\mu_2(i)) \end{bmatrix} \tag{2.62}$$

$$u_1 g_1 = \ldots = u_4 g_4 = u_5(i) g_5(i) = u_6(i) g_6(i) = 0, \quad i = 1, 2, \ldots \tag{2.63}$$

$$u_1, \ldots, u_4, u_5(i), u_6(i) \geq 0, \; i = 1, 2, \ldots \tag{2.64}$$

where u denotes a multiplier. To solve the above equations, assume that ε_1, ε_2, ϕ_1^0 and ϕ_2^0 are not zero, therefore, set u_1, u_2, u_3 and u_4 to zero. Minimize $g_5(i)$ of (2.60) and $g_6(i)$ of (2.61) with respect to $\underset{\sim}{\mu}(i)$. This leads, respectively, to

$$(\phi_2^0 - \varepsilon_2) - (\phi_1^0 + \varepsilon_1) - 2 \geq 0 \tag{2.65}$$

using $\underset{\sim}{\mu}(i) = [1 \;\; -1]^T$ and

$$-(\phi_2^0 + \varepsilon_2)^2 + 16(\phi_1^0 - \varepsilon_1) \geq 0, \tag{2.66}$$

using $\underset{\sim}{\mu}(i) = [-1 \;\; 1]^T$. The optimality conditions (2.62) - (2.64) are correspondingly reduced yielding the solution

$$\varepsilon_1 = 0.5, \qquad \varepsilon_2 = 0.5, \qquad \phi_1^0 = 4.5, \qquad \phi_2^0 = 7.5.$$

2.10 A Tuning Example

Consider the problem of minimizing

$$C = \frac{1}{\varepsilon_2}, \tag{2.67}$$

with respect to t_1', ε_2, ϕ_1^0, ϕ_2^0 and $\rho_1(i)$, and subject to

$$g_1 = t_1' \geq 0,\ g_2 = \varepsilon_2 \geq 0,\ g_3 = \phi_1^0 \geq 0,\ g_4 = \phi_2^0 \geq 0, \tag{2.68}$$

$$g_5 = 0.1 - \frac{t_1'}{\phi_1^0} \geq 0, \tag{2.69}$$

$$g_6(i) = (\phi_2^0+\varepsilon_2\mu_2(i))-(\phi_1^0+t_1'\rho_1'(i))-2 \geq 0,\ i = 1,\ 2,\ \ldots \tag{2.70}$$

$$g_7(i) = -(\phi_2^0+\varepsilon_2\mu_2(i))^2+16(\phi_1^0+t_1'\rho_1'(i)) \geq 0,\ i = 1,\ 2,\ \ldots \tag{2.71}$$

$$g_8(i) = 1 - \rho_1'(i) \geq 0,\ i = 1,\ 2,\ \ldots \tag{2.72}$$

$$g_9(i) = 1 + \rho_1'(i) \geq 0,\ i = 1,\ 2,\ \ldots . \tag{2.73}$$

and $-1 \leq \mu_2(i) \leq 1$.

Here, ε_1 is considered fixed at 0.5 and there is a maximum effective tuning range of 10%. Hence, the first component does not contribute to the cost. The effective tuning range $t_1' = t_1 - 0.5$ is used as a variable.

The optimality conditions require that

$$\begin{bmatrix} 0 \\ -\frac{1}{\varepsilon_2^2} \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ 0 \end{bmatrix} + u_5 \begin{bmatrix} -\frac{1}{\phi_1^0} \\ 0 \\ \frac{t_1'}{\phi_1^{02}} \\ 0 \\ 0 \end{bmatrix} + \sum_i u_6(i) \begin{bmatrix} -\rho_1'(i) \\ \mu_2(i) \\ -1 \\ 1 \\ -t_1'e_i \end{bmatrix}$$

$$+\sum_i u_7(i)\begin{bmatrix} 16\rho_1'(i) \\ -2(\phi_2^0+\varepsilon_2\mu_2(i))\mu_2(i) \\ 16 \\ -2(\phi_2^0+\varepsilon_2\mu_2(i)) \\ 16t_1'\underset{\sim}{e}_i \end{bmatrix} + \sum_i u_8(i)\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -\underset{\sim}{e}_i \end{bmatrix}$$

$$+\sum_i u_9(i)\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \underset{\sim}{e}_i \end{bmatrix}, \tag{2.74}$$

$$u_1 g_1 = \dots = u_5 g_5 = u_6(i) g_6(i) = \dots = u_9(i) g_9(i) = 0, \quad i = 1, 2, \dots \tag{2.75}$$

$$u_1, \dots, u_5, u_6(i), \dots, u_9(i) \geq 0, \; i = 1, 2, \dots \tag{2.76}$$

Minimize $g_6(i)$ of (2.70) and $g_7(i)$ of (2.71) with respect to $\mu_2(i)$. We use $\mu_2(i) = -1$ in (2.70) and $\mu_2(i) = 1$ in (2.71) for this purpose. The corresponding $\rho_1'(i) = -1$ and $\rho_1'(i) = 1$, respectively, are obtained by maximizing $g_6(i)$ and $g_7(i)$ with respect to $\rho_1'(i)$. This yields the solution

$$t_1' = 0.5432, \quad \varepsilon_2 = 1.444, \quad \phi_1^0 = 5.4321, \quad \phi_2^0 = 8.3333.$$

As expected, the inclusion of tunable elements can increase the tolerance on the components. The tolerance of the second parameter increases from $\varepsilon_2 = 0.5$ to $\varepsilon_2 = 1.444$ when the first component is allowed to have a maximum effective tuning range of 10%. This means that an actual absolute tuning of 1.0432 and a tolerance of 0.5 are designed for ϕ_1. The result can only be accomplished by allowing the nominal point to move. For example, the first component moved from 3.5 to 5.4321, a shift of 55%.

2.11 Summary

In this chapter, the problem of design centering, tolerancing and tuning has been presented in a unified manner. Definitions of constraint, tolerance and tuning regions are given. The concept of a tunable constraint region that allows variable specifications as set by the customer has also been treated. Reduced problems and conditions of validity are stated and proved in appropriate theorems. A geometric interpretation is discussed. Two simple examples have been studied to give some insight.

CHAPTER 3

SOME IMPLICATIONS OF BIQUADRATIC FUNCTIONS

3.1 Introduction

It has been stated in Chapter 2 that the constraint region R_c may be defined by a set of constraint functions. However, Chapter 2 is primarily concerned with the region itself rather than the functions. Conditions for the worst cases to occur at the vertices of the tolerance region will be studied in this chapter. In practice, two kinds of constraint functions may be identified. The first kind which determines the *feasibility* of a design is denoted as $\underset{\sim}{g}_f(\underset{\sim}{\phi})$. The second kind which determines the *acceptability* of a design is denoted as $\underset{\sim}{g}_a(\underset{\sim}{\phi})$. $\underset{\sim}{g}_f(\underset{\sim}{\phi})$ is usually derived from physical considerations such as nonnegativity of parameter values, component bounds or any other physical restrictions in manufacturing. $\underset{\sim}{g}_a(\underset{\sim}{\phi})$, on the other hand, is derived from performance specifications. We shall be concerned mainly with the latter kind of constraint functions. In particular, this chapter is motivated by those electrical circuit responses which can be expressed as biquadratic functions of the parameter of interest. A one-dimensional case is presented. See Fidler and Nightingale (1972) for some biquadratic relationships; Parker, Peskin and Chirlian (1965) and Géher (1971) for some circuit properties; Mangasarian (1969) and Zangwill (1969) for a discussion of functions more general than concave and convex functions. See also Bandler and Liu (1974b, 1975).

We elaborate in this chapter on an underlying assumption made in a theorem proposed by Bandler (1972, 1974). See Appendix B.

3.2 The Biquadratic Functions

3.2.1 General Properties

Consider the *biquadratic function*

$$F(\phi) = \frac{N(\phi)}{M(\phi)} = \frac{c\phi^2 + 2d\phi + e}{\phi^2 + 2a\phi + b}. \tag{3.1}$$

The first derivative of $F(\phi)$ is

$$F'(\phi) = 2\,\frac{(c\phi+d)M(\phi) - (\phi+a)N(\phi)}{M^2(\phi)}. \tag{3.2}$$

It may be noted that the numerator of (3.2) is a quadratic function of ϕ which implies that the derivative has at most two changes of sign for finite values of ϕ. Furthermore, the function value approaches the value of c as $\phi \to \pm\infty$.

Take any two points ϕ^r and ϕ^s and let $\Delta\phi = \phi^s - \phi^r$. $F(\phi^s)$ may be expressed in terms of ϕ^r, $\Delta\phi$ and the coefficients of $N(\phi)$ and $M(\phi)$ as follows:

$$F(\phi^s) = \frac{N(\phi^s)}{M(\phi^s)} = \frac{N(\phi^r) + 2\Delta\phi(c\phi^r+d) + c\Delta\phi^2}{M(\phi^r) + 2\Delta\phi(\phi^r+a) + \Delta\phi^2}. \tag{3.3}$$

The *large change sensitivity* is

$$\frac{\Delta F}{\Delta\phi} \triangleq \frac{F(\phi^s) - F(\phi^r)}{\phi^s - \phi^r} \tag{3.4}$$

may be related to the first differential sensitivity $F'(\phi^r)$. We have

$$F(\phi^s) - F(\phi^r) = \frac{2\Delta\phi\{(c\phi^r+d)M(\phi^r)-(\phi^r+a)N(\phi^r)\}-\Delta\phi^2\{N(\phi^r)-cM(\phi^r)\}}{M(\phi^r)M(\phi^s)}$$

$$= \Delta\phi F'(\phi^r)\,\frac{M(\phi^r)}{M(\phi^s)} - \Delta\phi^2\,\frac{(F(\phi^r) - c)}{M(\phi^s)} ,$$

therefore,

$$M(\phi^s)\,\frac{\Delta F}{\Delta\phi} = F'(\phi^r)M(\phi^r) - \Delta\phi(F(\phi^r) - c). \tag{3.5}$$

Given a fixed value ϕ^r, we can find uniquely one other point ϕ^s such that $F(\phi^s) = F(\phi^r)$, except when the function $F(\phi^r) = c$, $F'(\phi^r) = 0$, or $M(\phi^r) = 0$. The point ϕ^s is given, using (3.5) with $\Delta F = 0$, by

$$\phi^s = \phi^r + \frac{F'(\phi^r)M(\phi^r)}{F(\phi^r) - c} . \tag{3.6}$$

For the case $F'(\phi^r) = 0$, the point ϕ^r is either at the maximum or at the minimum of the function. There is only one finite point ϕ^c such that $F(\phi^c) = c$. The other points with the same value can only be at infinity. See, for example, Fig. 3.1.

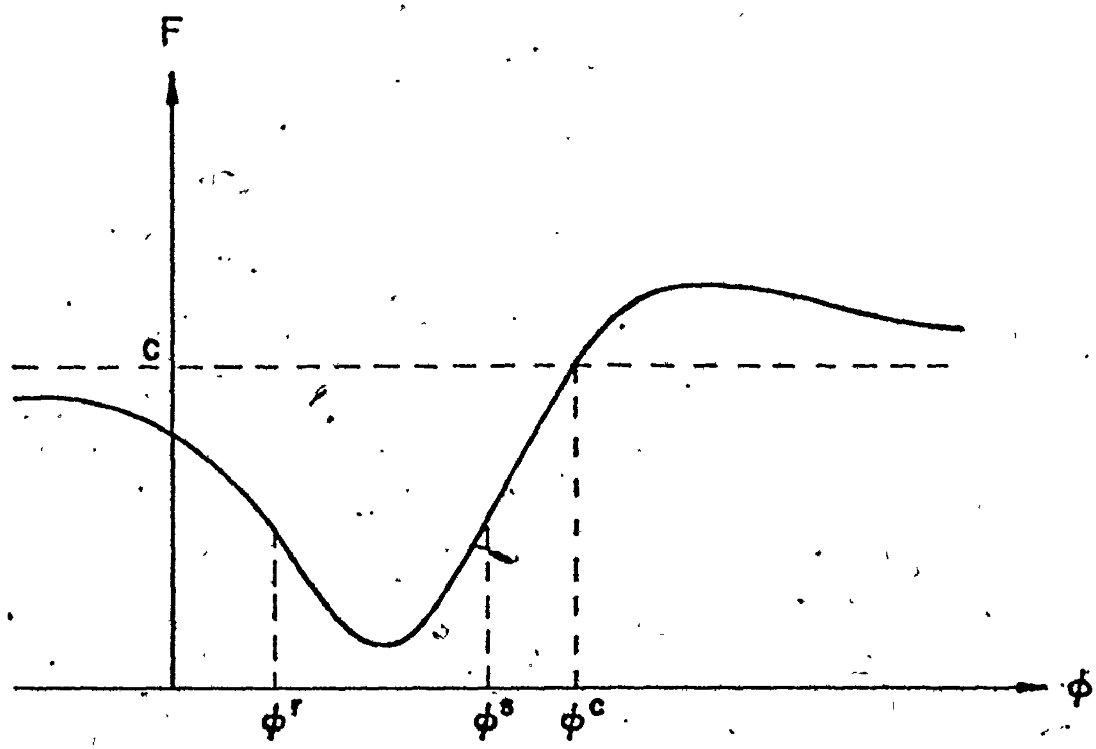


Fig. 3.1 A general biquadratic function.

3.2.2 Assumptions

In the following discussion, we shall assume that $M(\phi)$ does not change sign on $[\phi^r, \phi^s]$. We shall also exclude points where $M(\phi) = 0$ since the derivative of $F(\phi)$ is not defined at such points.

3.3 Some Lemmas and Theorems

3.3.1 Lemma 3.1

$F(\phi^r + \lambda(\phi^s - \phi^r)) > \min[F(\phi^r), F(\phi^s)]$ for all λ satisfying $0 < \lambda < 1$ provided that

$$\frac{\Delta F}{\Delta \phi} \cdot \left.\frac{dF}{d\phi}\right|_{\phi=\check{\phi}} > 0, \tag{3.7}$$

where $\frac{\Delta F}{\Delta \phi}$ is given in (3.4), $\check{\phi}$ is ϕ^r or ϕ^s whichever corresponds to the lower function value.

Figure 3.2 illustrates this lemma.

<u>Proof</u> The case $F(\phi^s) > F(\phi^r)$ will be considered first. From (3.5), we have

$$M(\phi) \frac{F(\phi) - F(\phi^r)}{\lambda \Delta \phi} = F'(\phi^r)M(\phi^r) - \lambda \Delta \phi (F(\phi^r) - c), \tag{3.8}$$

where

$$\phi = \phi^r + \lambda(\phi^s - \phi^r), \quad 0 < \lambda < 1. \tag{3.9}$$

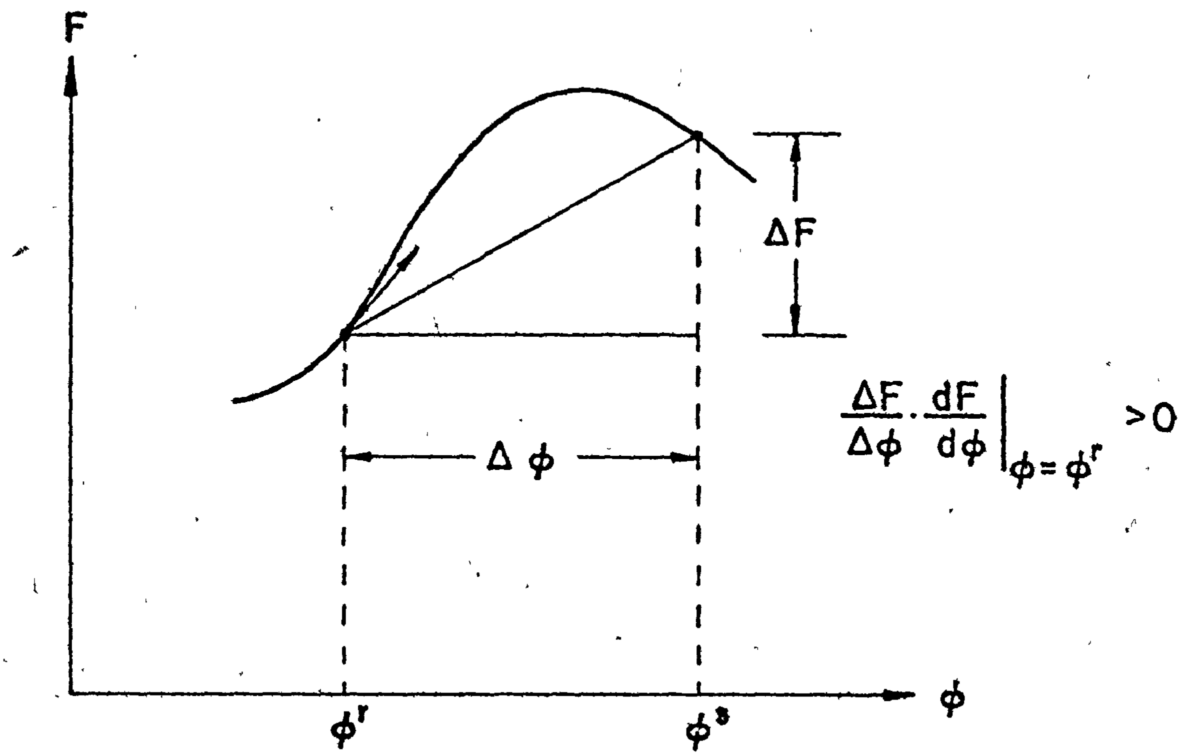


Fig. 3.2 Illustration of pseudoconcavity on an interval.

If condition (3.7) is satisfied, $F'(\phi^r) = \left.\frac{dF}{d\phi}\right|_{\phi=\phi^r} > 0$, then

$$\frac{1}{M(\phi^s)}\left[F'(\phi^r)M(\phi^r) - \Delta\phi(F(\phi^r) - c)\right] > 0 \qquad (3.10)$$

implies, since $M(\phi)$ must not change sign, that

$$\frac{1}{M(\phi)}\left[F'(\phi^r)M(\phi^r) - \lambda\Delta\phi(F(\phi^r) - c)\right] > 0. \qquad (3.11)$$

Therefore,

$$F(\phi) - F(\phi^r) > 0. \qquad (3.12)$$

Similarly, for the case when $F(\phi^r) > F(\phi^s)$, it is required from (3.7) that $F'(\phi^s) = \left.\frac{dF}{d\phi}\right|_{\phi=\phi^s} < 0$. The equations corresponding to (3.5) and (3.8) are, respectively,

$$M(\phi^r)\,\frac{F(\phi^s) - F(\phi^r)}{\Delta\phi} = F'(\phi^s)M(\phi^s) + \Delta\phi(F(\phi^s) - c) \qquad (3.13)$$

and

$$M(\phi)\,\frac{F(\phi^s) - F(\phi)}{(1-\lambda)\Delta\phi} = F'(\phi^s)M(\phi^s) + (1-\lambda)\Delta\phi(F(\phi^s) - c). \qquad (3.14)$$

Since $\frac{\Delta F}{\Delta\phi} < 0$,

$$\frac{1}{M(\phi^r)}\left[F'(\phi^s)M(\phi^s) + \Delta\phi(F(\phi^s) - c)\right] < 0 \qquad (3.15)$$

implies, since $M(\phi)$ must not change sign, that

$$\frac{1}{M(\phi)}\left[F'(\phi^s)M(\phi^s) + (1-\lambda)\Delta\phi(F(\phi^s) - c)\right] < 0. \tag{3.16}$$

and hence that

$$F(\phi) - F(\phi^s) > 0. \tag{3.17}$$

Inequalities (3.12) and (3.17) are true for all $0 < \lambda < 1$, hence the lemma is proved.

Corollary: $F(\phi^r + \lambda(\phi^s - \phi^r)) < \max[F(\phi^r), F(\phi^s)]$, where $0 < \lambda < 1$ provided that

$$\frac{\Delta F}{\Delta\phi} \cdot \frac{dF}{d\phi}\bigg|_{\phi=\hat{\phi}} > 0, \tag{3.18}$$

where $\hat{\phi}$ is ϕ^r or ϕ^s whichever corresponds to the higher function value.

The corollary may be proved by defining a new function $G(\phi) = - F(\phi)$ and applying Lemma 3.1. See Fig. 3.3 for an illustration. Figure 3.4 shows an example where both the lemma and its corollary apply.

3.3.2 Lemma 3.2

The function $F(\phi)$ is *pseudoconcave* (see Appendix A) on the

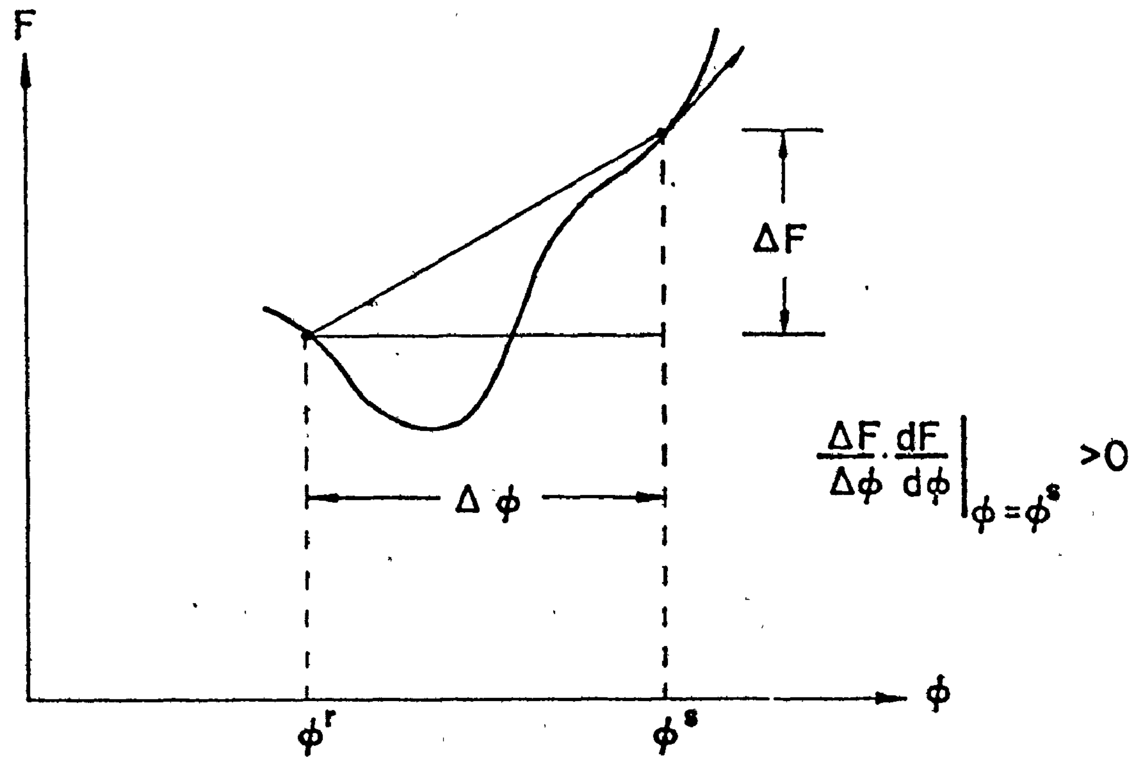


Fig. 3.3 Illustration of pseudoconvexity on an interval.

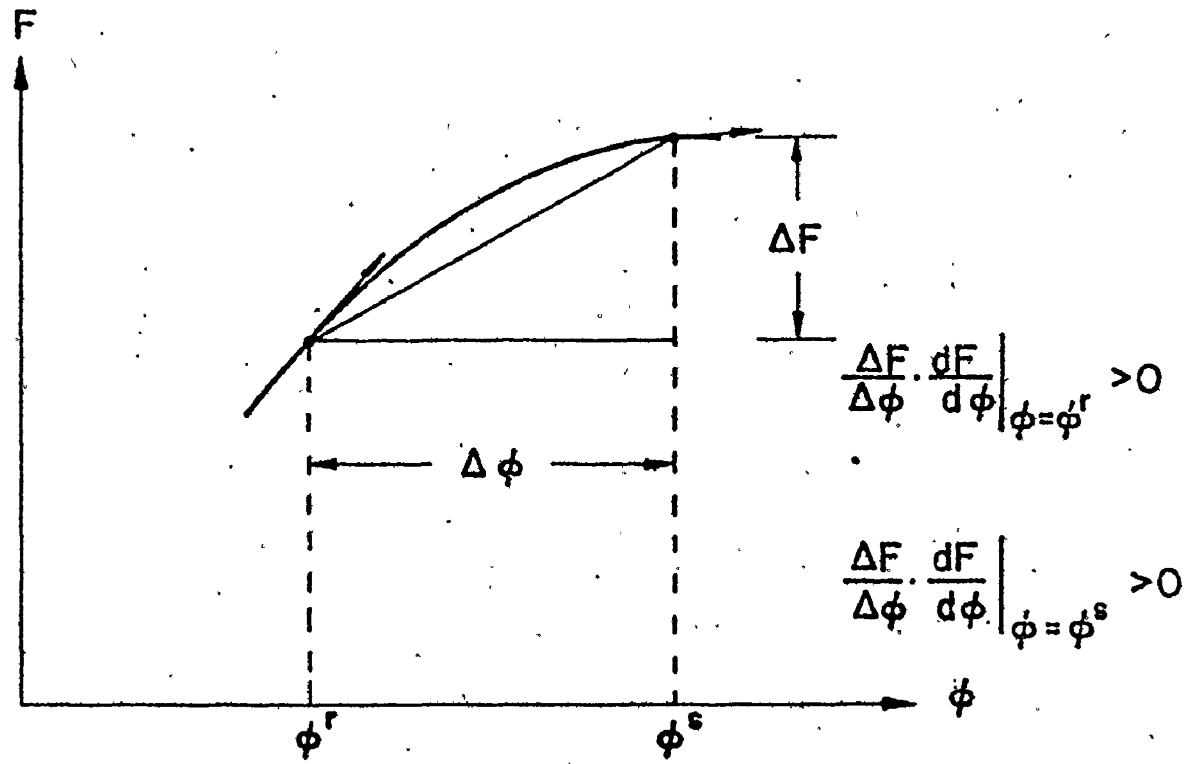


Fig. 3.4 Illustration of monotonicity on an interval.

interval $[\phi^r,\phi^s]$ except where $M(\phi) = 0$ if the conditions of Lemma 3.1 are satisfied.

<u>Proof</u> Consider the case $F(\phi^s) > F(\phi^r)$. The other case follows a similar argument. Let us assume that the function has more than one turning point in the interval. By the nature of the biquadratic function, there are at most two turning points. If we assume that there are two turning points on $[\phi^r,\phi^s]$, there exist two points $\phi^\alpha = \phi^r + \alpha\Delta\phi$ and $\phi^\beta = \phi^r + \beta\Delta\phi$, where $0 < \alpha < \beta < 1$ such that the following inequalities hold:

$$F(\phi^\alpha) > F(\phi^\beta) \tag{3.19}$$

and

$$F'(\phi^\beta) > 0. \tag{3.20}$$

As a direct consequence of Lemma 3.1 and inequality (3.20), the following inequalities can be made to hold:

$$F(\phi^s) > F(\phi^\beta) \tag{3.21}$$

and

$$F(\phi^\beta) > F(\phi^r). \tag{3.22}$$

Rewriting the function values in terms of $F'(\phi^\beta)$, $F(\phi^\beta)$ and $M(\phi^\beta)$ as in (3.5), we obtain

$$\frac{1}{M(\phi^{\alpha})}\left[F'(\phi^{\beta})M(\phi^{\beta}) + (\beta-\alpha)\Delta\phi(F(\phi^{\beta}) - c)\right] < 0, \tag{3.23}$$

$$\frac{1}{M(\phi^{r})}\left[F'(\phi^{\beta})M(\phi^{\beta}) + \beta\Delta\phi(F(\phi^{\beta}) - c)\right] > 0, \tag{3.24}$$

and

$$\frac{1}{M(\phi^{s})}\left[F'(\phi^{\beta})M(\phi^{\beta}) - (1-\beta)\Delta\phi(F(\phi^{\beta}) - c)\right] > 0. \tag{3.25}$$

Multiply (3.23) by $M(\phi^{\alpha})$, (3.24) by $M(\phi^{r})$ and (3.25) by $M(\phi^{s})$. Subtracting appropriately, we have

$$\alpha\Delta\phi(F(\phi^{\beta}) - c)\begin{cases} > 0 & \text{for} \quad M > 0 \\ < 0 & \text{for} \quad M < 0, \end{cases} \tag{3.26}$$

and

$$-(1-\alpha)\Delta\phi(F(\phi^{\beta}) - c)\begin{cases} > 0 & \text{for} \quad M > 0 \\ < 0 & \text{for} \quad M < 0. \end{cases} \tag{3.27}$$

The last two pairs of inequalities are inconsistent, therefore, the assumption that there are two turning points on the interval is false. $F(\phi)$, $\phi\epsilon[\phi^{r},\phi^{s}]$, is unimodal with a positive derivative at ϕ^{r}.

Given any two points ϕ^{a} and ϕ^{b}, such that $F(\phi^{b}) > F(\phi^{a})$, we will consider the following:

(1) $F'(\phi^{a}) > 0$, then $\phi^{b} > \phi^{a}$ because F is an increasing function between ϕ^{r} and ϕ^{a}.

(2) $F'(\phi^{a}) < 0$, then $\phi^{b} < \phi^{a}$ because F is a decreasing function between ϕ^{a} and ϕ^{s}.

Therefore, in both cases $F(\phi^{b}) > F(\phi^{a})$ implies $F'(\phi^{a})(\phi^{b}-\phi^{a}) > 0$, which proves the lemma.

Corollary: The function $F(\phi)$ is *pseudoconvex* (see Appendix A) on the interval $[\phi^r, \phi^s]$ except where $M(\phi) = 0$ if the conditions of the corollary to Lemma 3.1 are satisfied.

3.3.3 Theorem 3.1

The $\begin{smallmatrix}\text{minimum}\\ \text{maximum}\end{smallmatrix}$ of $F(\phi)$, $\phi\varepsilon[\phi^r, \phi^s]$, lies on the boundary of the interval if one of the following conditions is satisfied.

$$F'(\phi^r) \gtrless 0 \quad \text{and} \quad F'(\phi^s) \lessgtr 0 \tag{3.28a, 3.28b}$$

$$F'(\phi^r) > 0, \qquad F'(\phi^s) > 0 \quad \text{and} \quad F(\phi^r) < F(\phi^s) \tag{3.29}$$

or

$$F'(\phi^r) < 0, \qquad F'(\phi^s) < 0 \quad \text{and} \quad F(\phi^r) > F(\phi^s). \tag{3.30}$$

See, for example, Figs. 3.2 - 3.4.

<u>Proof</u> We will prove the case for the minimum of $F(\phi)$ to be on the boundary of an interval for the conditions of (3.28a), (3.29) and (3.30).

(1) Take $\check{\phi} = \phi^r$, then $F(\phi^s) > F(\phi^r)$ and $\frac{\Delta F}{\Delta \phi} > 0$. Using Lemma 3.1, $F(\phi^r + \lambda(\phi^s - \phi^r)) > \min[F(\phi^r), F(\phi^s)]$, $0 < \lambda < 1$, will hold if $F'(\phi^r) > 0$. This is satisfied in (3.28a) and (3.29).

(2) Take $\check{\phi} = \phi^s$, then $F(\phi^r) > F(\phi^s)$ and $\frac{\Delta F}{\Delta \phi} < 0$. Using Lemma 3.1 again, the requirement that $F'(\phi^s) < 0$

will be met in (3.28a) and (3.30).

(3) Suppose $F(\phi^r) = F(\phi^s)$ and hence $\frac{\Delta F}{\Delta \phi} = 0$. We can find one point ϕ^a such that $F(\phi^a) > F(\phi^r) = F(\phi^s)$. Two subintervals are thus obtained, each of which may be considered under cases (1) and (2) above.

It should be noted that, from Lemma 3.2, (3.28a), (3.29) and (3.30) imply pseudoconcavity. From its corollary, (3.28b), (3.29) and (3.30) imply pseudoconvexity.

3.3.4 Theorem 3.2

An *acceptable interval* denoted by I_a as

$$I_a \triangleq \{\phi | S_{ui} - F_i(\phi) \geq 0,\ i \epsilon I_u,\ F_j(\phi) - S_{\ell j} \geq 0,\ j \epsilon I_\ell\}, \tag{3.31}$$

where S_{ui}, $i \epsilon I_u$, and $S_{\ell i}$, $i \epsilon I_\ell$, are the upper and lower specifications, respectively, and where I_u and I_ℓ are disjoint index sets, is convex if the condition (3.28a), (3.29) or (3.30) is satisfied by $F_i(\phi)$, for all $i \epsilon I_\ell$, and condition (3.28b), (3.29) or (3.30) is satisfied by $F_i(\phi)$, for all $i \epsilon I_u$.

<u>Proof</u> Consider the case $i \epsilon I_\ell$ and let

$$I_i \triangleq \{\phi | F_i(\phi) - S_{\ell i} \geq 0\},\ i \ \epsilon\ I_\ell. \tag{3.32}$$

Take two different points ϕ^r, ϕ^s εI_i. If the condition (3.28a), (3.29) or (3.30) is satisfied, then from Theorem 3.1

$$F_i(\phi^\lambda) = F_i(\phi^r + \lambda(\phi^s - \phi^r)) > \min[F_i(\phi^r), F_i(\phi^s)], \tag{3.33}$$

$$0 < \lambda < 1,$$

thus

$$F_i(\phi^\lambda) - S_{\ell i} > \min[F_i(\phi^r) - S_{\ell i}, F_i(\phi^s) - S_{\ell i}], \tag{3.34}$$

$$0 < \lambda < 1.$$

Since

$$\phi^r, \phi^s \ \varepsilon\ I_i,$$

$$F_i(\phi^\lambda) - S_{\ell i} > 0. \tag{3.35}$$

Therefore,

$$\phi^\lambda = \phi^r + \lambda(\phi^s - \phi^r) \ \varepsilon\ I_i. \tag{3.36}$$

Hence I_i, $i\varepsilon I_\ell$, is a convex interval by definition of a convex set. Similarly, for the case $i\varepsilon I_u$, if the condition (3.28b), (3.29) or (3.30) is satisfied, using Theorem 3.1, we may prove that I_i, $i\varepsilon I_u$, is convex.

The intersection of convex sets is convex, and since by

~~definition~~

$$I_a = \bigcap_{\substack{i \in I_\ell \\ i \in I_u}} I_i, \tag{3.37}$$

I_a is convex.

If any $F(\phi)$ has both upper and lower specifications, the required conditions for a convex acceptable interval are restricted to (3.29) and (3.30).

3.4 The Network Tolerance Problem

We consider a *bilinear network function* of the form $(A + \phi B)/(C + \phi D)$ where A, B, C, and D are, in general, complex and frequency dependent. For a discussion on bilinear network functions, see Parker, Peskin and Chirlian (1965) and Géher (1971). Thus, we assume a function of the form

$$F(\phi) = \left| \frac{A + \phi B}{C + \phi D} \right|^2 = \frac{N(\phi)}{M(\phi)}. \tag{3.38}$$

In this case $N, M \geq 0$. The coefficients of (3.1) are related to the bilinear function as follows:

$$\left.\begin{aligned} a &= \frac{C_r D_r + C_i D_i}{|D|^2}, & b &= \frac{|C|^2}{|D|^2}, \\ c &= \frac{|B|^2}{|D|^2}, & d &= \frac{A_r B_r + A_i B_i}{|D|^2}, \\ e &= \frac{|A|^2}{|D|^2}, \end{aligned}\right\} \tag{3.39}$$

where the subscripts i and r denote the imaginary and real parts of the complex number.

3.4.1 Filter Example

We have studied the behaviour of $|\rho|^2$, the modulus squared of the reflection coefficient ρ, for the LC lowpass filter (Fig. 3.5) with respect to the variations of L, C_2 and C_3, respectively. Figure 3.6 shows some of the curves for different values of frequency. The three vertical lines on each drawing represent the nominal values and the extreme values of ±25% relative tolerance. The nominal values for L, C_2 and C_3 are 2, .125 and 1, respectively. $C_1 = C_3$ for reasons of symmetry.

The curves for L and C_2 have two turning points each. For example, at $\omega = 1$, (ω denotes frequency in rad/sec.)

$$|\rho(L)|^2 = \frac{81L^2 - 144L + 64}{82L^2 - 160L + 128} . \tag{3.40}$$

The turning points are at L = .889 and L = 8.0 corresponding to the minimum of $|\rho|^2 = 0$ and the maximum of $|\rho|^2 = 1$, respectively. Setting $|\rho|^2 = \frac{81}{82} = c$, we can solve for one unique point L = 4.44 at which the curve is divided into two parts: $|\rho|^2 \geq .988$ for $L \geq 4.44$ and $|\rho|^2 \leq .988$ for $L \leq 4.44$. The maximum and minimum function values occur separately in the two parts. The derivatives at the boundary of the tolerance region are both positive, indicating that the curve is monotonic in the region (both

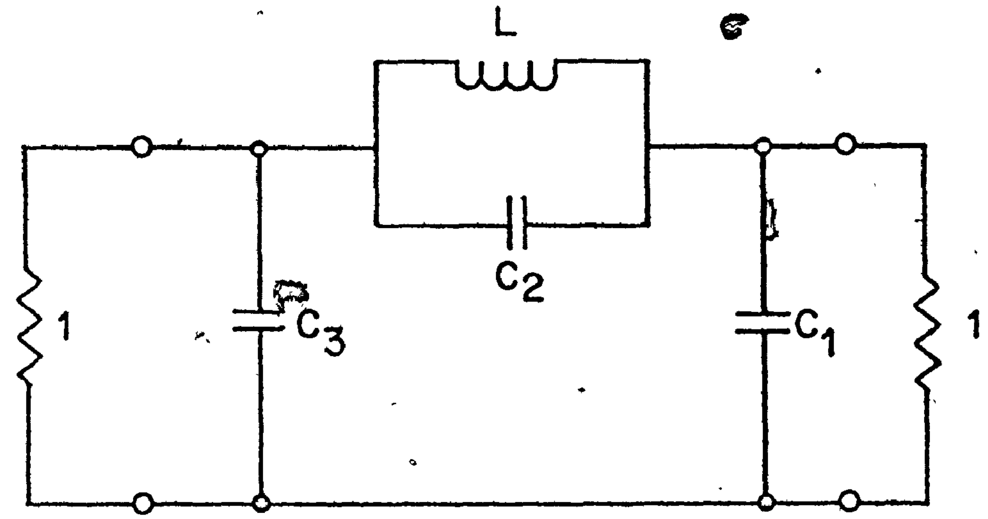


Fig. 3.5 An LC elliptic lowpass filter example.

pseudoconvex and pseudoconcave).

For parameter C_2 at $\omega = 1$

$$|\rho(C_2)|^2 = \frac{4C_2^2 + 4C_2 + 1}{8C_2^2 + 2} . \tag{3.41}$$

The maximum and minimum occur at values of .5 and −.5. At $C_2 = 0$, the curve is again divided into two parts for $|\rho|^2 \geq .5$ and $|\rho|^2 \leq .5$ for positive or negative C_2 values, respectively.

The curves for C_3 have only one turning point. The biquadratic function is of the form

$$|\rho(C_3)|^2 = \frac{C_3^2 + 2aC_3 + e}{C_3^2 + 2aC_3 + b} . \tag{3.42}$$

The minimum occurs at $C_3 = -a$. The curves are pseudoconvex on $(-\infty,\infty)$ for frequencies in both the passband $(0 \leq \omega \leq 1)$ and stopband $(\omega \geq 2)$. For the worst case at stopband frequencies to occur at the boundary of an interval, it is required that the curves corresponding to these frequencies also be pseudoconcave on the interval, i.e., the curves should be monotonic on the interval.

A situation which violates the conditions may be found, for example, by studying the $\omega = 2.0$ curve of Fig. 3.6(a) for L_1 between 0 and 1.

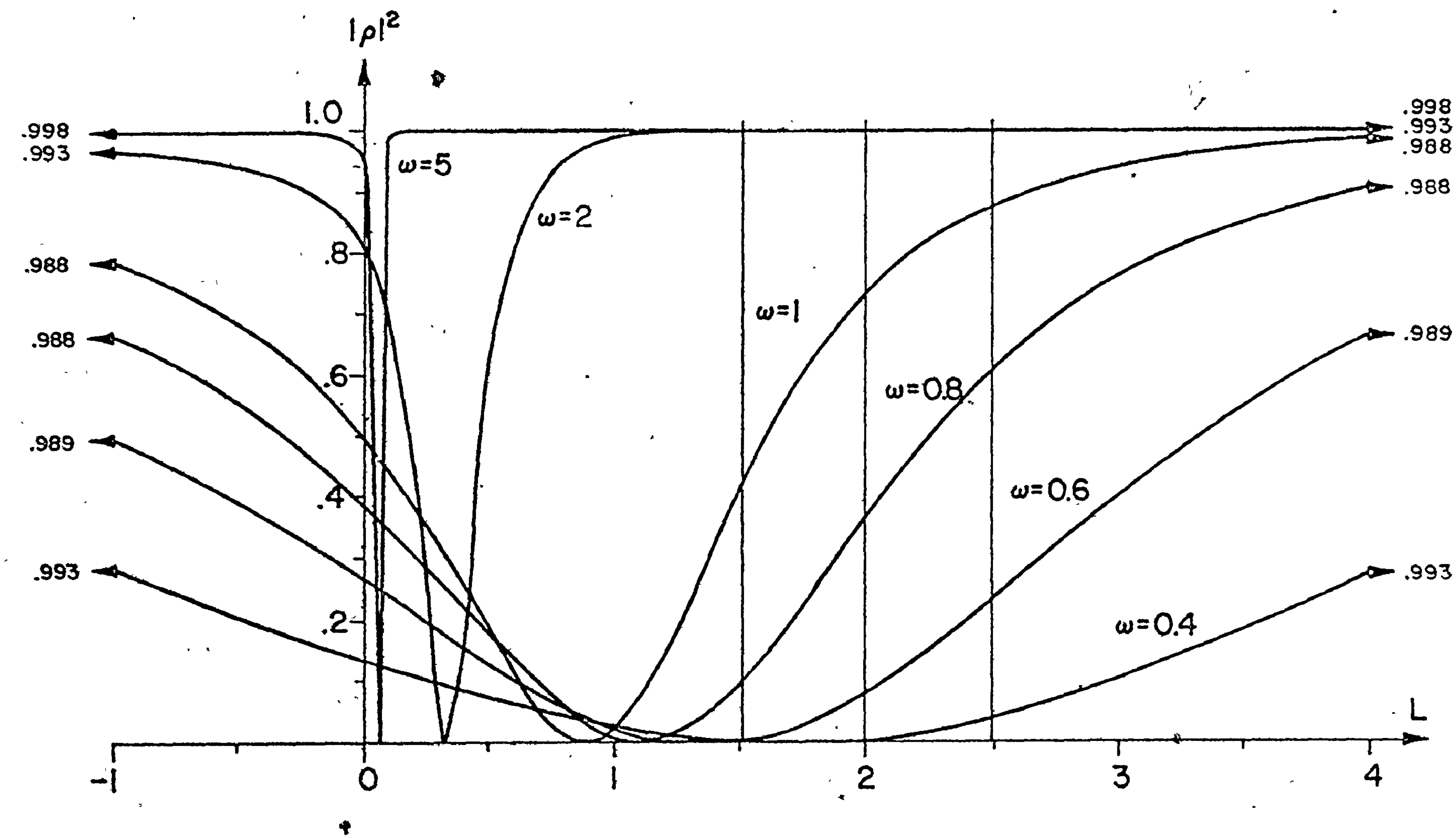


Fig. 3.6(a) $|\rho|^2$ versus L for the elliptic filter example.

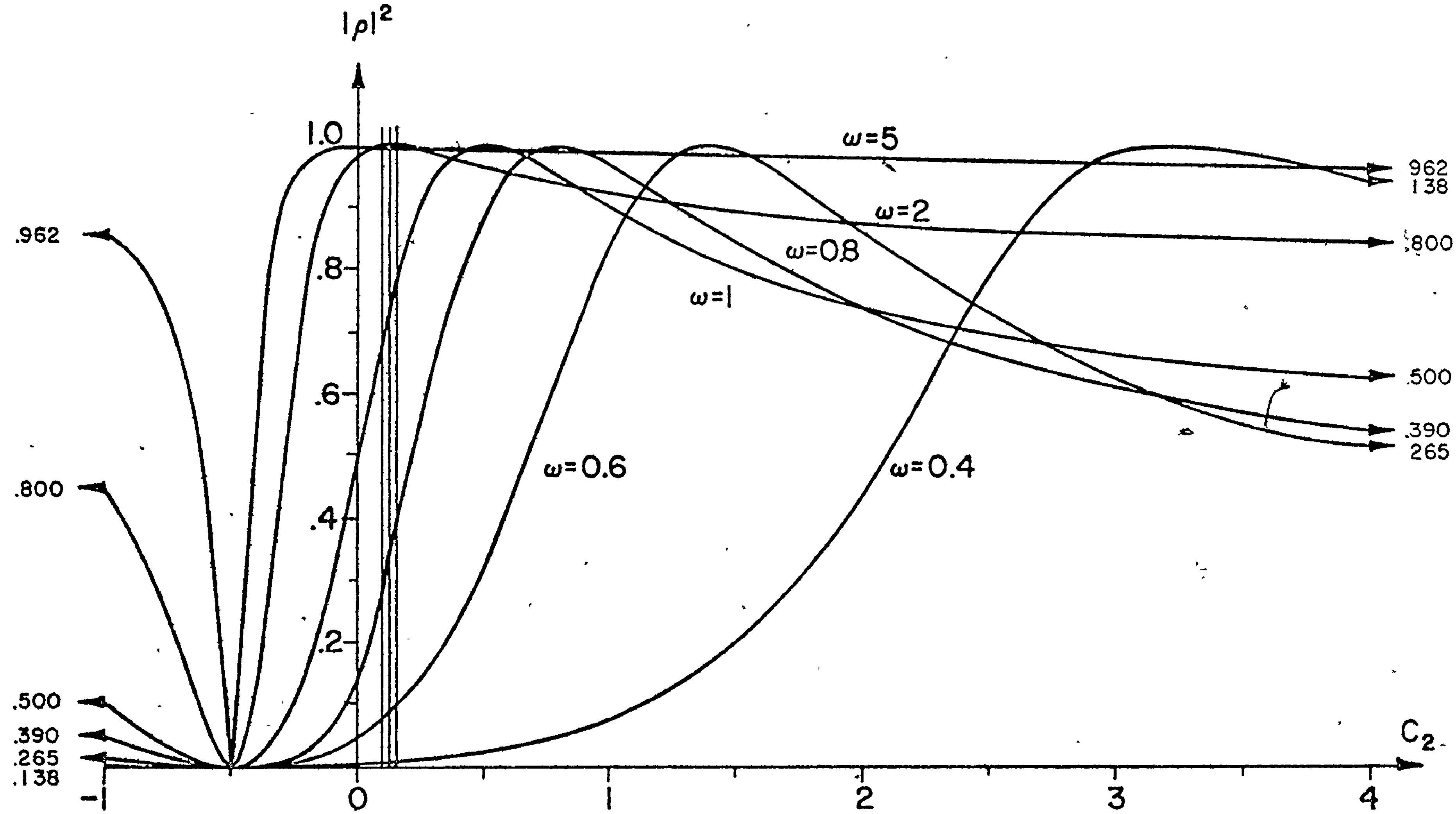


Fig. 3.6(b) $|\rho|^2$ versus C_2 for the elliptic filter example.

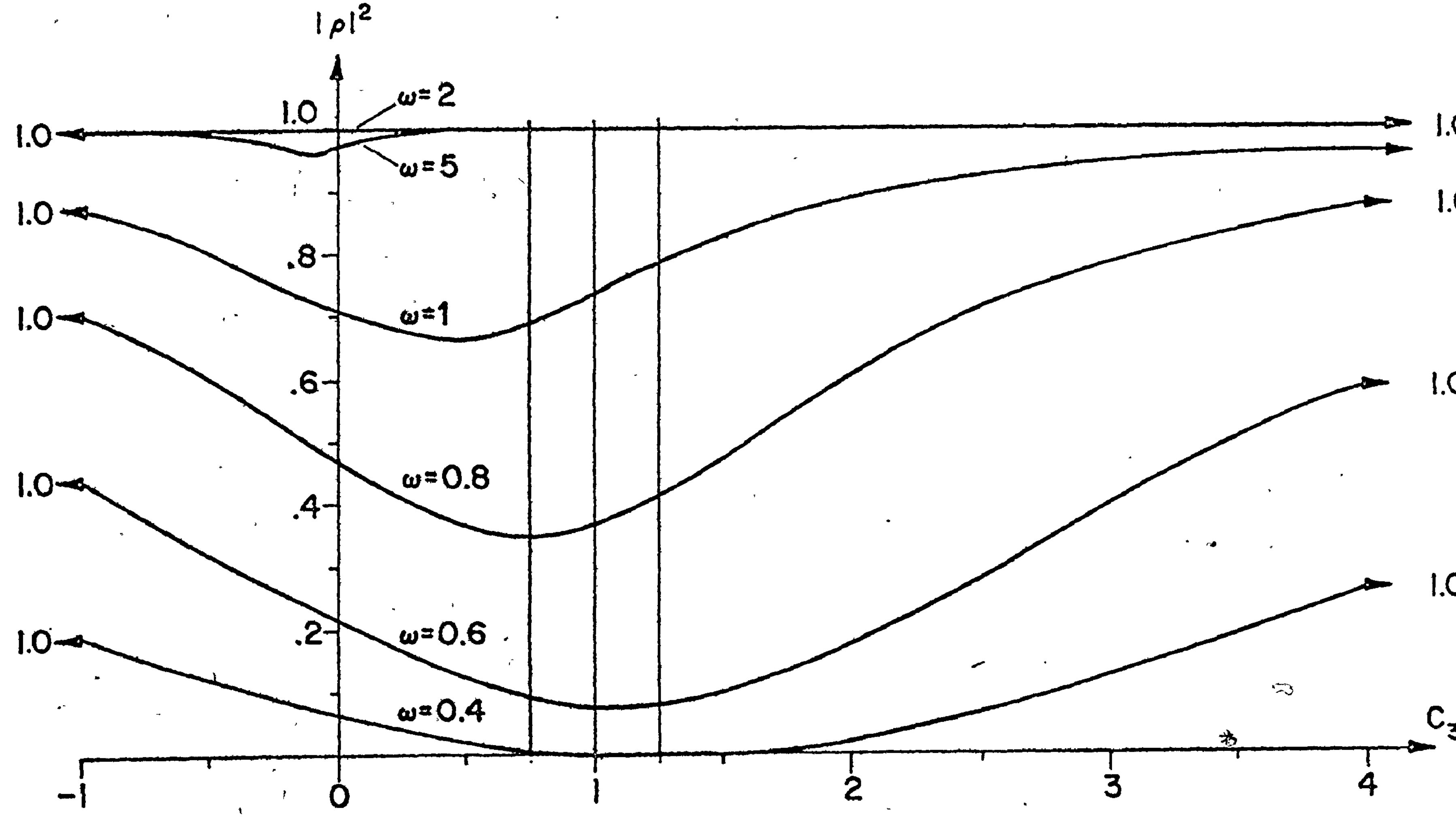


Fig. 3.6(c) $|\rho|^2$ versus C_3 for the elliptic filter example.

3.5 Conclusions

Conditions for the worst case to occur at the boundary of an interval have been presented. The conditions may be used at least to partially justify the usual assumption that the worst case occurs at a vertex of the tolerance region. The present chapter deals with a one-dimensional case. Bandler (1972, 1974) has already related a one-dimensional convexity assumption for the acceptable interval to that of the k-dimensional case. Thus, Theorem 3.1 involves necessary conditions for the vertices of a k-dimensional region.

CHAPTER 4

IMPLEMENTATION IN NETWORK DESIGN

4.1 Introduction

In this chapter, it is shown how to implement the ideas of Chapters 2 and 3 on a digital computer. Objective functions, performance specifications and parameter constraints are handled in a manner such that any of the nominal values, tolerances or tuning parameters can be fixed or varied. Time-saving techniques for choosing constraints (vertices selection) are discussed in detail. Schemes based on the assumptions of generalized convexity and monotonicity properties of the constraint functions are proposed. The schemes also check the conditions listed in Chapter 3 and perform a worst-case analysis. The schemes suggest the development of a general user-oriented computer program package called TOLOPT (TOLerance OPTimization) described in Appendix D. See also Bandler, Liu and Chen (1974b, 1975).

This chapter contains a brief discussion of network symmetry and how it may be implemented to further reduce the number of constraints.

The optimal worst-case tolerance problem which is very important in its own right is treated in Part 1. Part 2 brings in the tuning of one or more circuit components basically in order to further increase tolerances on all the components. The implementation of tolerance-tuning problems is similar to the implementation of the tolerance problem. See Bandler, Liu and Tromp (1975a, 1975b).

The nonlinear programming problem takes the general form:

$$\text{minimize} \quad f(\underset{\sim}{x})$$
$$\text{subject to} \quad g_i(\underset{\sim}{x}) \geq 0, \qquad i = 1, 2, \ldots, m.$$

f is the chosen objective function. The vector $\underset{\sim}{x}$ represents a set of design variables which include the nominal values, the relative and/or absolute tolerances or tuning variables of the network components and all the slack variables associated with each distinct outcome. The constraint functions $g_1(\underset{\sim}{x})$, $g_2(\underset{\sim}{x})$, ..., $g_m(\underset{\sim}{x})$, comprise the selected response specifications, component bounds, slack variable bounds and any other constraints. The constraints are numbered from 1 to m for simplicity.

Unless otherwise indicated, the examples in this chapter are solved by the following methods. The nonlinear programming problem is transformed into an unconstrained minimax problem by the Bandler-Charalambous technique (1972a, 1974). The solution of the resulting minimax problem is found by least pth approximation algorithms also proposed by Bandler and Charalambous (1972b, 1972c). Fletcher's minimization methods (1970, 1972) are used to minimize the transformed unconstrained function. The solution of discrete problems in this thesis are obtained by the branch and bound approach (Dakin 1966, Garfinkel and Nemhauser 1972). These methods are featured in a user-oriented computer program called DISOPT (see Bandler and Chen 1974, Chen 1974) which is described in Appendix C so as not to interrupt the flow of the chapter.

Part 3 deals with two realistic circuit design problems. The bandpass filter was studied by Butler (1971), Karafin (1971) and Pinel and Roberts (1972). Substantial improvement is obtained by our method. The highpass filter was suggested by Pinel (1974) and Roberts (1974). They did not exploit the advantages of tuning. We have, however, explored the effects of tuning in this example.

4.2 Numbering Scheme for Vertices

The set of vertices of a tolerance region R_v is given by (2.9). We will label each vertex by an integer from the index set I_v such that

$$\underset{\sim}{\phi}^r \triangleq \underset{\sim}{\phi}^0 + \underset{\sim}{E}\, \underset{\sim}{\mu}^r \tag{4.1}$$

where $\mu_j^r \in \{-1, 1\}$ and satisfies the relation

$$r = 1 + \sum_{j=1}^{k} \left[\frac{\mu_j^r + 1}{2}\right] 2^{j-1}. \tag{4.2}$$

Thus,

$$\underset{\sim}{\mu}^1 \triangleq \begin{bmatrix} -1 \\ -1 \\ \cdot \\ \cdot \\ \cdot \\ -1 \end{bmatrix}, \quad \underset{\sim}{\mu}^2 \triangleq \begin{bmatrix} +1 \\ -1 \\ \cdot \\ \cdot \\ \cdot \\ -1 \end{bmatrix}, \quad \underset{\sim}{\mu}^3 \triangleq \begin{bmatrix} -1 \\ +1 \\ \cdot \\ \cdot \\ \cdot \\ -1 \end{bmatrix}, \ldots, \underset{\sim}{\mu}^{2^k} \triangleq \begin{bmatrix} +1 \\ +1 \\ \cdot \\ \cdot \\ \cdot \\ +1 \end{bmatrix}. \tag{4.3}$$

The set of vertices may now be identified as

$$R_v = \{\underset{\sim}{\phi}^1, \underset{\sim}{\phi}^2, \ldots, \underset{\sim}{\phi}^{2^k}\}. \tag{4.4}$$

This notation will be used throughout this chapter unless otherwise indicated.

4.3 One-Dimensional Quasiconcave Functions

A function $g(\underset{\sim}{\phi})$ is said to be *quasiconcave* in a region if, for all $\underset{\sim}{\phi}^a$, $\underset{\sim}{\phi}^b$ in the region,

$$g(\underset{\sim}{\phi}^a + \lambda(\underset{\sim}{\phi}^b-\underset{\sim}{\phi}^a)) \geq \min[g(\underset{\sim}{\phi}^a),\ g(\underset{\sim}{\phi}^b)], \tag{4.5}$$

for all $0 \leq \lambda \leq 1$. See Mangasarian (1969) and Appendix A for some other definitions and some properties of the function. An immediate consequence of (4.5) is that the region defined as $\{\underset{\sim}{\phi}|g(\underset{\sim}{\phi}) \geq 0\}$ is convex. It can be proved that the intersection of convex regions is also convex. Now, the convexity condition implies the one-dimensional convexity condition necessary for Theorem 2.2 and Theorem 2.3. We have given the term *one-dimensional quasiconcave function* to a function which satisfies (4.5) when $\underset{\sim}{\phi}^b$ is given by

$$\underset{\sim}{\phi}^b = \underset{\sim}{\phi}^{b(j)} \triangleq \underset{\sim}{\phi}^a + \alpha \underset{\sim}{e}_j, \tag{4.6}$$

for some constant α. The region defined by such functions is called a *one-dimensional convex region*. Pseudoconcavity implies quasiconcavity. The conditions for concavity and monotonicity with respect to each variable discussed in Chapter 3 certainly apply to the case of one-dimensional quasiconcave functions.

4.4 Conditions for Monotonicity

Given a differentiable one-dimensional quasiconcave function $g(\phi)$ (here we consider one variable denoted by ϕ for convenience), the function is *monotonic* with respect to ϕ on an interval $[\phi^a,\phi^b]$ if $\mathrm{sgn}(g'(\phi^a)) = \mathrm{sgn}(g'(\phi^b))$, where g' is the first derivative of g with respect to ϕ, and $\mathrm{sgn}(\cdot)$ denotes the sign of the function. Furthermore, the minimum of $g(\phi)$ is at

$$\phi = \frac{1}{2}\,[\phi^a + \phi^b - \mathrm{sgn}(g'(\phi^a))(\phi^b-\phi^a)]. \tag{4.7}$$

This may be proved as follows.

Consider the case $\mathrm{sgn}(g'(\phi^a)) = \mathrm{sgn}(g'(\phi^b)) > 0$. Suppose $g(\phi)$ is not monotonic. Then there exist two points

$$\phi^1,\ \phi^2 \ \varepsilon\ (\phi^a,\ \phi^b), \tag{4.8}$$

where

$$\phi^2 > \phi^1, \tag{4.9}$$

such that $g'(\phi^1) < 0$ and

$$g(\phi^2) > g(\phi^1). \tag{4.10}$$

Thus, for some $0 < \lambda < 1$

$$g(\phi^1 + \lambda(\phi^2 - \phi^1)) < g(\phi^1), \qquad (4.11)$$

which contradicts the definition of quasiconcavity. The assumption that $g(\phi)$ is not monotonic is wrong, hence, $g(\phi)$ is monotonic. Furthermore, it is nondecreasing on $[\phi^a, \phi^b]$. The minimum is at

$$\phi^a = \frac{1}{2}\left[\phi^a + \phi^b - \mathrm{sgn}(g'(\phi^a))(\phi^b - \phi^a)\right] \qquad (4.12)$$

in this case.

Similarly, it may be proved that the case $\mathrm{sgn}(g'(\phi^a)) = \mathrm{sgn}(g'(\phi^b)) < 0$ implies monotonicity with $g(\phi)$ nonincreasing on $[\phi^a, \phi^b]$. The minimum is at ϕ^b.

4.5 Implications of Monotonicity

Suppose g_i is monotonic in the same direction with respect to ϕ_j throughout R_ε. Then the minimum of g_i is on the hyperplane $\phi_j = \phi_j^0 - \varepsilon_j \,\mathrm{sgn}(\frac{\partial g_i}{\partial \phi_j})$. Hence, only those vertices which lie on that hyperplane need to be constrained. In general, if there are ℓ variables with respect to which the function g_i is monotonic in this way, the $2^{k-\ell}$ vertices lying on the intersection of the hyperplanes are constrained. In the case where $\ell = k$, the vertex of minimum g occurs at $\underset{\sim}{\phi}^r$, where

$$\phi_j^r = \phi_j^0 - \varepsilon_j \,\mathrm{sgn}(\frac{\partial g_i}{\partial \phi_j}), \qquad \text{for all } j \in I_\phi. \qquad (4.13)$$

4.6 The Vertices Elimination Scheme

Various schemes may be developed to identify or to predict the most critical vertices that are likely to give rise to active constraints. Any scheme proposed should eliminate all but one vertex for each constraint function in the most favourable conditions. When monotonicity assumptions are not sufficient to describe the function behaviour, the schemes should increase the number of vertices until, at worst, all the 2^k vertices are included.

In principle, our schemes may be stated as follows:

Step (1): Systematic generation, for $\alpha \geq 0$, sets of points

$$\underset{\sim}{\phi}^a, \ \underset{\sim}{\phi}^{b(j)} = \underset{\sim}{\phi}^a + \alpha \underset{\sim}{e}_j. \tag{4.14}$$

Step (2): Evaluation of the function values and the partial derivatives at these points.

Step (3): If $\operatorname{sgn}\left(\frac{\partial g_i}{\partial \phi_j}\bigg|_{\underset{\sim}{\phi}=\underset{\sim}{\phi}^a}\right) = \operatorname{sgn}\left(\frac{\partial g_i}{\partial \phi_j}\bigg|_{\underset{\sim}{\phi}=\underset{\sim}{\phi}^{b(j)}}\right)$,

eliminate the vertices $\underset{\sim}{\phi}^r \in R_v$ on the hyperplane

$$\phi_j = \phi_j^0 + \varepsilon_j \operatorname{sgn}\left(\frac{\partial g_i}{\partial \phi_j}\right). \tag{4.15}$$

If $\operatorname{sgn}\left(\frac{\partial g_i}{\partial \phi_j}\bigg|_{\underset{\sim}{\phi}=\underset{\sim}{\phi}^a}\right) < 0$ and $\operatorname{sgn}\left(\frac{\partial g_i}{\partial \phi_j}\bigg|_{\underset{\sim}{\phi}=\underset{\sim}{\phi}^{b(j)}}\right) > 0$, note that the quasiconcavity assumption is violated.

The different schemes depend on the different ways of implementing Step (1). Three methods of increasing complexity can be described as follows:

(a) $\underset{\sim}{\phi}^a = \underset{\sim}{\phi}^b = \underset{\sim}{\phi}^0$,

(b) $\underset{\sim}{\phi}^a = \underset{\sim}{\phi}^0 - \varepsilon_j \underset{\sim}{e}_j$ and $\underset{\sim}{\phi}^b = \underset{\sim}{\phi}^0 + \varepsilon_j \underset{\sim}{e}_j$, for all $j \in I_\phi$,

(c) the vertices of R_ε.

Method (a) is a special case for which the first part of (3) is applicable. For method (c), a worst-case check can be accomplished as a by-product of the vertices elimination scheme since function values are computed at each vertex.

It is possible to further eliminate some vertices by ranking the values of $g(\underset{\sim}{\phi}^r)$, where $\underset{\sim}{\phi}^r$ are the selected vertices, in ascending order and rejecting those having sufficiently large values.

Since the schemes are based on local information, the vertices chosen should be updated at suitable intervals.

4.7 Symmetry Considerations

A designer should exploit symmetry to reduce computation time. The following is an example of how it may be done in the tolerance problem.

A function is said to be *symmetrical* with respect to $\underset{\sim}{S}$ in a region if

$$g(\underset{\sim}{S}\,\underset{\sim}{\phi}) = g(\underset{\sim}{\phi}), \tag{4.16}$$

where $\underset{\sim}{S}$ is a matrix obtained by interchanging suitable rows of a unit matrix. It has exactly one entry of 1 in each row and in each column, all other entries being 0.

A common physical symmetry configuration is a *mirror-image symmetry* with respect to a center line. The $\underset{\sim}{S}$ matrix in this case is

$$\underset{\sim}{S} = \begin{bmatrix} 0 & & & 1 \\ & & 1 & \\ & \cdot & & \\ 1 & & & 0 \end{bmatrix}. \tag{4.17}$$

Postmultiplication of a matrix $\underset{\sim}{A}$ by any $\underset{\sim}{S}$ simply permutes the columns of $\underset{\sim}{A}$ and premultiplication of $\underset{\sim}{A}$ permutes the rows of $\underset{\sim}{A}$. $\underset{\sim}{S}\underset{\sim}{S}^T = \underset{\sim}{1}$ and $\underset{\sim}{S}^T\underset{\sim}{D}\underset{\sim}{S} = \underset{\sim}{D}_s$, where $\underset{\sim}{D}$ is a diagonal matrix and $\underset{\sim}{D}_s$ is also a diagonal matrix with diagonal entries permuted.

Consider symmetrical $\underset{\sim}{S}$, $\underset{\sim}{\phi}^0$ and $\underset{\sim}{\varepsilon}$. By this we imply

$$\underset{\sim}{S}(\underset{\sim}{S}\,\underset{\sim}{A}) = \underset{\sim}{A}, \tag{4.18}$$

$$\underset{\sim}{S}\,\underset{\sim}{\phi}^0 = \underset{\sim}{\phi}^0 \tag{4.19}$$

and

$$\underset{\sim}{S}^T \underset{\sim}{E} \underset{\sim}{S} = \underset{\sim}{E}. \tag{4.20}$$

Let us premultiply the rth vertex $\underset{\sim}{\phi}^r$ by $\underset{\sim}{S}$, giving, from (4.1)

$$\begin{aligned} \underset{\sim}{S}\,\underset{\sim}{\phi}^r &= \underset{\sim}{S}\,\underset{\sim}{\phi}^0 + \underset{\sim}{S}(\underset{\sim}{E}\,\underset{\sim}{\mu}^r), \qquad r \in I_V \\ &= \phi^0 + \underset{\sim}{S}(\underset{\sim}{S}^T \underset{\sim}{E}\,\underset{\sim}{S}\,\underset{\sim}{\mu}^r) \\ &= \underset{\sim}{\phi}^0 + \underset{\sim}{E}\,\underset{\sim}{S}\,\underset{\sim}{\mu}^r. \end{aligned} \tag{4.21}$$

Now, $\underset{\sim}{S}\,\underset{\sim}{\mu}^r$ is another vector with +1 and -1 entries. Let it be denoted by $\underset{\sim}{\mu}^s$, $s \epsilon I_V$. In some cases $\underset{\sim}{\mu}^r$ is identical to $\underset{\sim}{\mu}^s$, if the vector is symmetrical. In other cases, $\underset{\sim}{\mu}^r \neq \underset{\sim}{\mu}^s$. In all cases,

$$\underset{\sim}{S}\underset{\sim}{\phi}^r = \underset{\sim}{\phi}^s. \tag{4.22}$$

Making use of the symmetrical property of g,

$$g(\underset{\sim}{S}\underset{\sim}{\phi}^r) = g(\underset{\sim}{\phi}^r) = g(\underset{\sim}{\phi}^s). \tag{4.23}$$

Let the number of symmetrical vectors $\underset{\sim}{\mu}^r$ and the number of pairs of nonsymmetrical $\underset{\sim}{\mu}^r$ and $\underset{\sim}{\mu}^s$ be denoted by N(r=s) and N(r≠s), respectively. Then

$$N(r=s) = 2^{k-k_s}, \qquad 2k_s \leq k, \tag{4.24}$$

and

$$N(r\neq s) = (2^k - 2^{k-k_s})/2, \qquad 2k_s \leq k, \tag{4.25}$$

where k_s is the number of pairs of symmetrical components. There are, therefore, $N(r=s) + N(r\neq s)$ effective vertices as compared to 2^k topological vertices. Take, for example, $k = 6$ and $k_s = 3$. Only 36 function evaluations are required for all the 64 vertices. For more details about symmetry, see, for example, Yale (1968).

The above discussion and results may be used to reduce computation time. In general, however, it is not certain that a nominal design without tolerances yielding a symmetrical solution will imply a symmetrical optimal solution with tolerances; either in the continuous or in the discrete cases.

4.8 Formulation of Constraints

After eliminating potentially inactive vertices, each chosen vertex is associated with a *data vector* $\underset{\sim}{a}^i$, which has the form

$$\underset{\sim}{a}^i \triangleq \begin{bmatrix} r \\ \underset{\sim}{\mu} \\ \psi \\ S \\ w \end{bmatrix}, \qquad i = 1, 2, \ldots, m_a, \tag{4.26}$$

where ψ is an independent parameter denoting frequency or any number to identify a particular function for which the vertex $\underset{\sim}{\phi}^r$ is chosen, $\underset{\sim}{\mu}$ is the vector associated with $\underset{\sim}{\phi}^r$. m_a is the total number of distinct vectors $\underset{\sim}{a}^i$. S is a specification and w a weighting factor associated with each ψ. In our present work,

$$w = \begin{cases} +1 & \text{if S is an upper specification} \\ -1 & \text{if S is a lower specification.} \end{cases} \tag{4.27}$$

The *performance constraints* may now be formulated in the form of

$$g = w(S - F) \geq 0, \tag{4.28}$$

with appropriate subscripts. F is the circuit response function evaluated at the appropriate vertex and ψ.

The *parameter constraints* that define the feasibility of a design are

$$\phi_j^0 - \varepsilon_j - \phi_{\ell j} \geq 0 \tag{4.29}$$

and

$$\phi_{uj} - \phi_j^0 - \varepsilon_j \geq 0, \tag{4.30}$$

where ϕ_{uj} and $\phi_{\ell j}$, $j \in I_\phi$, are the user supplied upper and lower bounds, respectively. Let m be the total number of constraints, including both the specifications and the parameter bounds.

4.9 Examples

4.9.1 Two-Section 10:1 Quarter-Wave Transformer

To illustrate the basic ideas of different cost functions, variable nominal point, continuous and discrete solutions, a two-section 10:1 quarter-wave transformer is considered. See Bandler and Macdonald (1969), Bandler and Liu (1973, 1974a). Table 4.1 shows the specification of the design and the result of a minimax solution without tolerances. Figure 4.1 shows the contours of $\max_i |\rho_i|$ with respect to the characteristic impedances Z_1 and Z_2. ρ_i denotes the reflection coefficient at the ith sample point. The unshaded region is R_c which satisfies all the assumptions of convexity.

Two cases are presented here.

Case 1: Optimization of relative tolerances

The cost function is of the form

$$C_1 = \frac{1}{x_1} + \frac{1}{x_2}, \tag{4.31}$$

where

TABLE 4.1

SPECIFICATIONS FOR THE

TWO-SECTION 10:1 QUARTER-WAVE TRANSFORMER

Relative Bandwidth	Sample Points (GHz)	Reflection Coefficient Specification	Type
100%	0.5, 0.6, ..., 1.5	0.55	upper

Minimax solution (no tolerances) $|\rho| = 0.4286$

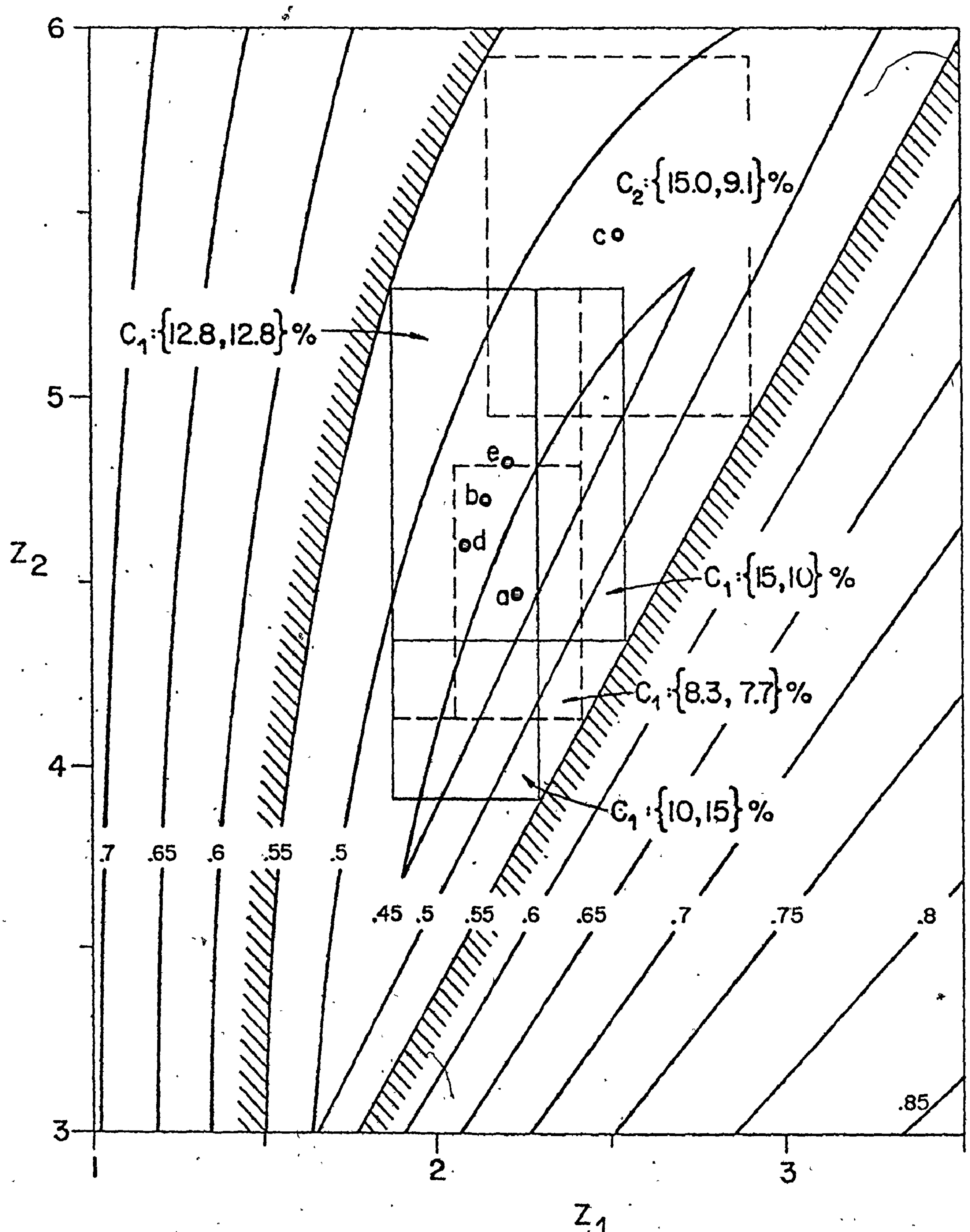


Fig. 4.1 Contours of max $|\rho_i|$ with respect to Z_1 and Z_2 for the 2- section transformer example, indicating a number of relevant solution points (see text).

$$x_1 = \frac{\varepsilon_1}{\phi_1^0} \times 100$$

$$x_2 = \frac{\varepsilon_2}{\phi_2^0} \times 100 \qquad (4.32)$$

$$x_3 = \phi_1^0 = Z_1^0$$

$$x_4 = \phi_2^0 = Z_2^0 .$$

The optimal solution of C_1 with respect to variables x_1 and x_2 and a fixed nominal point at $\underset{\sim}{a}$ yields a continuous tolerance set of {8.3, 7.7}%. For the same function with a variable nominal point, the set is {12.8, 12.8}% with optimal nominal solution at $\underset{\sim}{b}$. $\underset{\sim}{d}$ and $\underset{\sim}{e}$ correspond to the two discrete solutions with tolerances 10% and 15%. The allowable discrete tolerance set is {1, 2, 5, 10, 15, 20}%.

Case 2: Optimization of absolute tolerances

The cost function is of the form

$$C_2 = \frac{1}{x_1} + \frac{1}{x_2}, \qquad (4.33)$$

where

$$x_1 = \varepsilon_1$$

$$x_2 = \varepsilon_2$$

$$x_3 = \phi_1^0 = Z_1^0 \qquad (4.34)$$

$$x_4 = \phi_2^0 = Z_2^0 .$$

The optimal solution of C_2 with respect to x_1, x_2, x_3 and x_4 yields a tolerance set of {15.0, 9.1}% with nominal solution at $\underset{\sim}{c}$.

It may be noted from this example that an optimal discrete solution cannot always be obtained by rounding or truncating the continuous tolerances to the discrete values. The nominal points must be allowed to move.

4.9.2 Three-Component LC Lowpass Filter

A three-component LC lowpass filter is studied to illustrate some discrete solutions. The circuit is shown in Fig. 4.2. Table 4.2 summarizes the specifications. The objective function is

$$C = \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}, \qquad (4.35)$$

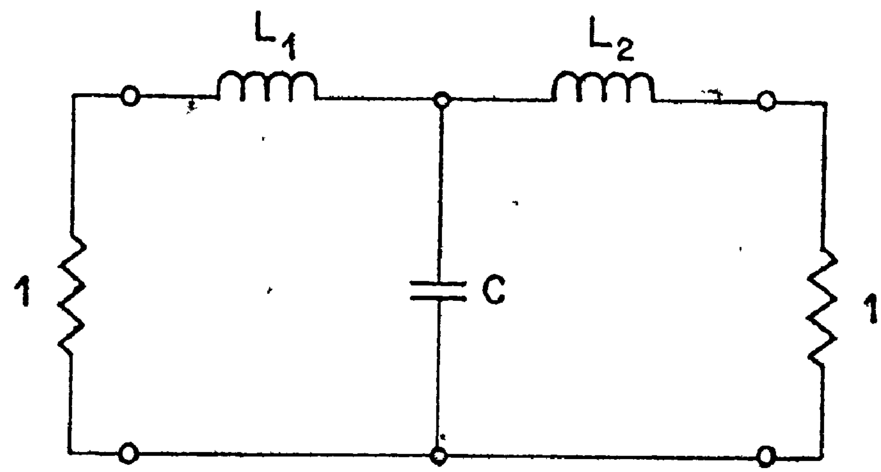


Fig. 4.2 The circuit for the LC lowpass filter example.

TABLE 4.2

SPECIFICATIONS FOR THE

LC LOWPASS FILTER

Frequency Range (rad/s)	Sample Points (rad/s)	Insertion Loss Specification (dB)	Type
0 - 1	0.5, 0.55, 0.6, 1.0	1.5	upper (passband)
2.5	2.5	25	lower (stopband)

Minimax solution (no tolerances)

passband 0.53 dB

stopband 26 dB

where

$$x_1 = \frac{\varepsilon_1}{\phi_1^0} \times 100$$

$$x_2 = \frac{\varepsilon_2}{\phi_2^0} \times 100$$

$$x_3 = \frac{\varepsilon_3}{\phi_3^0} \times 100 \tag{4.36}$$

$$x_4 = \phi_1^0 = C^0$$

$$x_5 = \phi_2^0 = L_1^0$$

$$x_6 = \phi_3^0 = L_2^0 \ .$$

Table 4.3 lists the results for both the continuous and discrete solutions. It may be noted that one of the discrete solutions as well as the continuous solution yield symmetrical results although symmetry is not assumed in the formulation of the problem.

4.9.3 Five-Section Cascaded Transmission-Line Lowpass Filter

Consider a five-section cascaded lossless transmission-line lowpass filter with characteristic impedances fixed at the values

$$Z_1^0 = Z_3^0 = Z_5^0 = 0.2, \tag{4.37}$$

$$Z_2^0 = Z_4^0 = 5.0$$

TABLE 4.3

RESULTS FOR THE LC LOWPASS FILTER

(TOLERANCE OPTIMIZATION)

Parameters	Continuous Solution Fixed Nominal	Continuous Solution Variable Nominal	Discrete Solution From {1,2,5,10,15}% 1	2	3
$100\ \varepsilon_1/L_1^0$	3.5 %	9.9 %	5 %	10 %	10 %
$100\ \varepsilon_2/C^0$	3.2 %	7.6 %	10 %	5 %	10 %
$100\ \varepsilon_3/L_2^0$	3.5 %	9.9 %	10 %	10 %	5 %
L_1^0	1.628		1.999		
C^0	1.090		0.906		
L_2^0	1.628		1.999		

and terminated in unity resistances. See Bandler and Charalambous (1972c) for a minimax solution without tolerance considerations and see Table 4.4 for the specifications. The length units are normalized with respect to $\ell_q = c/4f_0$, where $f_0 = 1$ GHz.

Two problems are presented here.

Problem 1: Optimization of length tolerances

A uniform 1% relative tolerance is allowed for each impedance. Maximize the absolute tolerances on the section lengths and find the corresponding nominal lengths. Let the cost function be

$$C = \sum_{i=1}^{5} \frac{1}{x_i}, \tag{4.38}$$

where

$$\begin{aligned} x_i &= \varepsilon_{\ell_i}, \quad i = 1, 2, \ldots, 5, \\ x_{i+5} &= \ell_i^0, \quad i = 1, 2, \ldots, 5. \end{aligned} \tag{4.39}$$

Problem 2: Optimization of impedance tolerances

A uniform absolute length tolerance of .001 is given. Maximize the relative tolerances on the impedances and obtain the corresponding nominal lengths. Let the cost function be

TABLE 4.4

SPECIFICATIONS FOR THE

FIVE-SECTION TRANSMISSION-LINE LOWPASS FILTER

Frequency Range (GHz)	Sample Points (GHz)	Insertion Loss Specification (dB)	Type
0 - 1	.35,.4,.45,.75,.8,.85,1.0	.02	upper (passband)
2.5 - 10	2.5, 10	25	lower (stopband)

$$C = \sum_{i=1}^{5} \frac{1}{x_i}, \tag{4.40}$$

where

$$\begin{aligned} x_i &= \frac{\varepsilon_{Z_i}}{Z_i^0} \times 100, \quad i = 1, 2, \ldots, 5, \\ x_{i+5} &= \ell_i^0, \quad\quad i = 1, 2, \ldots, 5. \end{aligned} \tag{4.41}$$

The filter has 10 circuit parameters which may be arranged in the order $Z_1, Z_2, \ldots, Z_5, \ell_1, \ell_2, \ldots, \ell_5$. To simplify the problem, symmetry with respect to a center line through the circuit is assumed. The matrix $\underset{\sim}{S}$ is given by

$$\underset{\sim}{S} = \begin{bmatrix} & & & & 1 & & \underset{\sim}{0} & & & \\ & & & 1 & & & & & & \\ & & 1 & & & & & & & \\ & 1 & & & & & & & & \\ 1 & & & & & & & & & \\ & & & & & & & & & 1 \\ & & & & & & & & 1 & \\ & & & & & & & 1 & & \\ & & & & & & 1 & & & \\ & \underset{\sim}{0} & & & & 1 & & & & \end{bmatrix} \tag{4.42}$$

which also implies that $\ell_1^0 = \ell_5^0$ and $\ell_2^0 = \ell_4^0$. The same kind of equalities are applied to the tolerances.

The second vertices elimination scheme is applied with values at the optimal nominal values without tolerances and the relative impedance tolerance and the absolute length tolerances at 2% and .002, respectively. A total of 46 vertices corresponding to all the frequency points were selected from a possible set of 9×2^{10}. 14 were further eliminated by symmetry. A final total of 15 constraints were chosen after comparing relative magnitudes. These 15 constraints were used throughout the optimization. The continuous and discrete solutions to the two problems are shown in Tables 4.5 and 4.6.

4.10 Discussion

The schemes discussed could be started, theoretically, from any arbitrary initial acceptable or unacceptable design to obtain continuous and/or discrete optimal nominal parameter values and tolerances simultaneously. Optimization of nominal values without tolerances should, however, preferably be done first to obtain a suitable starting point. The effort is small compared with the complete tolerance problem when a small value of p greater than unity, e.g., $p = 2$, is used in the least pth optimization. An exact minimax solution is not needed. See Charalambous (1974). This also serves as a feasibility check. If R_c is indicated to be empty, the designer has to relax some specifications or change his circuit. The solution process may also provide valuable information to the designer, e.g., parameter or frequency symmetry.

With a reasonable starting point, a prediction of the critical vertices could be more accurately done. The last example presented is a large problem from the tolerance optimization point of view. Out of a possible 9216 constraints, only 15 were chosen. The ability to choose the minimal number of constraints is very important for the branch and bound discrete optimization since each branching step involves a complete continuous optimization.

TABLE 4.5

RESULTS FOR THE

FIVE-SECTION TRANSMISSION-LINE LOWPASS FILTER

(TOLERANCE OPTIMIZATION, PROBLEM 1)

Parameters	Continuous Solution	Discrete Solution .0005 Step Size
$\varepsilon_{\ell_1} = \varepsilon_{\ell_5}$	0.0033	0.0030
$\varepsilon_{\ell_2} = \varepsilon_{\ell_4}$	0.0028	0.0030
ε_{ℓ_3}	0.0027	0.0025
$\ell_1^0 = \ell_5^0$	0.0788	
$\ell_2^0 = \ell_4^0$	0.1414	
ℓ_3^0	0.1738	

$Z_1^0 = Z_3^0 = Z_5^0 = 0.2;\ Z_2^0 = Z_4^0 = 5$

$100\ \varepsilon_{Z_i}/Z_i^0 = 1\%,\ i = 1, 2, \ldots, 5$

TABLE 4.6

RESULTS FOR THE

FIVE-SECTION TRANSMISSION-LINE LOWPASS FILTER

(TOLERANCE OPTIMIZATION, PROBLEM 2)

Parameters	Continuous Solution	Discrete Solution From {.5, 1, 1.5, 2, 3, 5}%
$100(\varepsilon_{Z_1}/Z_1^0 = \varepsilon_{Z_5}/Z_5^0)$	3.56 %	3 %
$100(\varepsilon_{Z_2}/Z_2^0 = \varepsilon_{Z_4}/Z_4^0)$	2.27 %	2 %
$100(\varepsilon_{Z_3}/Z_3^0)$	1.98 %	2 %
$\ell_1^0 = \ell_5^0$	0.0786	
$\ell_2^0 = \ell_4^0$	0.1415	
ℓ_3^0	0.1736	

$$Z_1^0 = Z_3^0 = Z_5^0 = 0.2,\ Z_2^0 = Z_4^0 = 5$$

$$\varepsilon_{\ell_i} = 0.001,\ i = 1, 2, \ldots, 5$$

Several properties of the centering and tolerance assignment process were demonstrated by the examples. In particular,

(1) Any circuit parameter can be fixed or varied, toleranced or otherwise, continuous or discrete.

(2) An optimal nominal point without tolerances may not be optimal when the components are toleranced. By allowing it to vary, tolerances may be enhanced.

(3) The best discrete solution cannot always be obtained by rounding or truncating the optimal continuous solution.

(4) A symmetrical continuous solution does not necessarily imply a symmetrical discrete solution.

PART 2

TOLERANCE-TUNING OPTIMIZATION

4.11 Formulation of Constraints

Consider the constraints of the form

$$g = w(S - F) \geq 0, \tag{4.43}$$

with appropriate subscripts. F is the circuit response function evaluated at sample point ψ and point $\underset{\sim}{\phi}$ which is given by

$$\underset{\sim}{\phi} = \underset{\sim}{P}\underset{\sim}{\phi}^r + \sum_{j\in I_t} (\phi_j^0 + t_j' \rho_j'(r))\underset{\sim}{e}_j. \tag{4.44}$$

Information required for (4.44) is contained in the vectors

$$\underset{\sim}{a}^i \triangleq \begin{bmatrix} r \\ \underset{\sim}{\mu} \\ \psi \\ S \\ w \end{bmatrix}, \quad i = 1, 2, \ldots, m_a. \tag{4.45}$$

The projection matrix $\underset{\sim}{P}$ and the index sets I_t and I_ε are fixed for a particular problem. They are determined before optimization takes place.

The vector of variables $\underset{\sim}{x}$ consists of the variable nominal values, tolerances, tuning variables and all the appropriate slack variables $\rho_j'(r)$, $j \in I_t$, $r \in I_v$.

Each of the slack variables is associated with two extra parameter constraints,

$$1 - \rho_j'(r) \geq 0 \tag{4.46}$$

and

$$1 + \rho_j'(r) \geq 0, \tag{4.47}$$

for appropriate j and r. These two constraints, however, may be combined to form

$$1 - (\rho_j'(r))^2 \geq 0. \tag{4.48}$$

Let m be the total number of constraints which include the performance specifications given by (4.43), slack variable bounds given by (4.46) and (4.47), parameter bounds given by (4.29) and (4.30), and any other extra constraints not considered above. In general, for linear network design in the frequency domain

$$n = k_0 + k_\varepsilon + k_t(1 + n_v) \tag{4.49}$$

and

$$m = \left[\sum_{i=1}^{n_\psi} n_v(i) \right] + 2k_t n_v + \ldots \qquad (4.50)$$

where k_0, k_ε and k_t are the number of variable nominal parameters, toleranced and tuned parameters, respectively; $n_v \leq 2^{k_\varepsilon}$ is the number of distinct vertices chosen; n_ψ is the number of frequency points considered; $n_v(i)$ is the number of vertices chosen at the ith frequency point and $2k_t n_v$ is the number of slack variable bounds.

4.12 Three-Component LC Lowpass Filter Examples

The LC lowpass filter presented in Section 4.9.2 is considered. For each frequency sample point $2^3 = 8$ vertices for the tolerance region can be obtained. The critical vertices selected are $\underset{\sim}{\phi}^6$ at $\psi = \psi_1, \psi_2, \psi_3$; $\underset{\sim}{\phi}^8$ at $\psi = \psi_4$ and $\underset{\sim}{\phi}^1$ at $\psi = \psi_5$, where

$$\underset{\sim}{\phi} = \begin{bmatrix} L_1 \\ C \\ L_2 \end{bmatrix}. \qquad (4.51)$$

For this problem, the vectors $\underset{\sim}{a}^i$, $i = 1, 2, \ldots, 5$, are

$$\underset{\sim}{a}^1 = \begin{bmatrix} 6 \\ +1 \\ -1 \\ +1 \\ 0.45 \\ 1.5 \\ 1 \end{bmatrix}, \quad \underset{\sim}{a}^2 = \begin{bmatrix} 6 \\ +1 \\ -1 \\ +1 \\ 0.50 \\ 1.5 \\ 1 \end{bmatrix}, \quad \underset{\sim}{a}^3 = \begin{bmatrix} 6 \\ +1 \\ -1 \\ +1 \\ 0.55 \\ 1.5 \\ 1 \end{bmatrix},$$

$$\underset{\sim}{a}^4 = \begin{bmatrix} 8 \\ +1 \\ +1 \\ +1 \\ 1.0 \\ 1.5 \\ 1 \end{bmatrix}, \quad \underset{\sim}{a}^5 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \\ 2.5 \\ 25 \\ -1 \end{bmatrix}. \tag{4.52}$$

Three problems are presented here. See Bandler, Liu and Tromp (1975a).

4.12.1 Effective Tuning for One Component

Case 1: L_1 tuned, C and L_2 toleranced.

We consider an objective function based on the relative tolerances of C and L_2 in the form

$$C = \frac{x_2}{x_5^2} + \frac{x_3}{x_6^2}, \tag{4.53}$$

where, assuming $t_C = t_{L_2} = 0$, and some fixed value of ε_{L_1},

$$x_1 = \phi_1^0 = L_1^0$$

$$x_2 = \phi_2^0 = C^0$$

$$x_3 = \phi_3^0 = L_2^0 .$$

$$\text{(4.54)}$$

$$x_4^2 = t_1' = t_{L_1} - \varepsilon_{L_1}$$

$$x_5^2 = \varepsilon_2 = \varepsilon_C$$

$$x_6^2 = \varepsilon_3 = \varepsilon_{L_2}.$$

The cost of element L_1 is assumed fixed. It, therefore, is not included in (4.53).

The last three transformations are chosen to avoid changes of sign. There are three distinct projected vertices: $\underset{\sim}{\phi}_p^6$, $\underset{\sim}{\phi}_p^8$ and $\underset{\sim}{\phi}_p^1$. The projection matrix in this case is

$$\underset{\sim}{P} = \begin{bmatrix} 0 & & \\ & 1 & \\ & & 1 \end{bmatrix}. \qquad (4.55)$$

Therefore, the other variables may be identified as

$$x_7 = \rho_1'(6), \quad x_8 = \rho_1'(8), \quad x_9 = \rho_1'(1). \tag{4.56}$$

Substituting the numerical values from (4.52) into (4.44) we have the following.

$$\underset{\sim}{a}^1, \underset{\sim}{a}^2, \underset{\sim}{a}^3 \Rightarrow \underset{\sim}{\phi} = \underset{\approx}{P}\underset{\sim}{\phi}^6 + (\phi_1^0 + t_1'\rho_1'(6))\underset{\sim}{e}_1$$

$$= \begin{bmatrix} x_1 + x_4^2 x_7 \\ x_2 - x_5^2 \\ x_3 + x_6^2 \end{bmatrix}, \tag{4.57}$$

$$\underset{\sim}{a}^4 \Rightarrow \underset{\sim}{\phi} = \underset{\approx}{P}\underset{\sim}{\phi}^8 + (\phi_1^0 + t_1'\rho_1'(8))\underset{\sim}{e}_1$$

$$= \begin{bmatrix} x_1 + x_4^2 x_8 \\ x_2 + x_5^2 \\ x_3 + x_6^2 \end{bmatrix}, \tag{4.58}$$

$$\underset{\sim}{a}^5 \Rightarrow \underset{\sim}{\phi} = \underset{\approx}{P}\underset{\sim}{\phi}^1 + (\phi_1^0 + t_1'\rho_1'(1))\underset{\sim}{e}_1$$

$$= \begin{bmatrix} x_1 + x_4^2 x_9 \\ x_2 - x_5^2 \\ x_3 - x_6^2 \end{bmatrix}. \tag{4.59}$$

The performance specifications g_i, $i = 1, 2, \ldots, 5$, may now be formed. Additional constraints are given by

$$\left.\begin{aligned} g_{5+2i-1} &= 1 + x_{6+i} \\ g_{5+2i} &= 1 - x_{6+i} \end{aligned}\right\} \quad i = 1, 2, 3, \tag{4.60}$$

$$g_{12} = t_r - x_4^2/x_1.$$

The last constraint g_{12} is designed to limit the tuning range to t_r. Table 4.7 shows results for three values of t_r. The same results are obtained replacing the term $x_1 + x_4^2 x_i$ by $x_1(1 + t_r x_i)$, $i = 7, 8, 9$, allowing g_{12} to be removed, and reducing the number of variables by one, since g_{12} is active.

Case 2: C tuned, L_1 and L_2 toleranced

We consider an objective function based on the relative tolerances of L_1 and L_2 in the form

$$C' = \frac{x_1}{x_4^2} + \frac{x_3}{x_6^2}, \tag{4.61}$$

where x_1, x_2, x_3 and x_6 are as before but where,

$$\begin{aligned} x_4^2 &= \varepsilon_1 = \varepsilon_{L_1}, \\ x_5^2 &= t_2' = t_C - \varepsilon_C, \end{aligned} \tag{4.62}$$

with $t_1 = t_3 = 0$, and some fixed ε_C. In this case

TABLE 4.7

RESULTS FOR THE LC LOWPASS FILTER

(L_1 TUNED, C AND L_2 TOLERANCED)

Parameters	$t_r = 0.2$	$t_r = 0.1$	$t_r = 0.05$
L_1^0	2.0932	2.2442	2.1953
C^0	0.9360	0.9059	0.9062
L_2^0	1.7718	1.7569	1.7920
$100\ t_1'/L_1^0$	20.00 %	10.00 %	5.00 %
$100\ \varepsilon_2/C^0$	15.99 %	14.23 %	12.60 %
$100\ \varepsilon_3/L_2^0$	21.62 %	18.41 %	16.23 %
$\rho_1'(6)$		-1.0000	
$\rho_1'(8)$		-1.0000	
$\rho_1'(1)$		1.0000	

$n = 9 \qquad m = 12$

$$\underset{\sim}{P} = \begin{bmatrix} 1 & & \\ & 0 & \\ & & 1 \end{bmatrix}, \tag{4.63}$$

and there are only two distinct projected vertices $\underset{\sim}{\phi}_p^6 = \underset{\sim}{\phi}_p^8$ and $\underset{\sim}{\phi}_p^1$. The slack variables are

$$x_7 = \rho_2'(6), \qquad x_8 = \rho_2'(2). \tag{4.64}$$

We have now,

$$\underset{\sim}{a}^1, \underset{\sim}{a}^2, \underset{\sim}{a}^3, \underset{\sim}{a}^4 \Rightarrow \underset{\sim}{\phi} = \underset{\sim}{P}\underset{\sim}{\phi}^6 + (\phi_2^0 + t_2'\rho_2'(6))\underset{\sim}{e}_2$$

$$= \begin{bmatrix} x_1 + x_4^2 \\ x_2 + x_5^2 x_7 \\ x_3 + x_6^2 \end{bmatrix}, \tag{4.65}$$

$$\underset{\sim}{a}^5 \Rightarrow \underset{\sim}{\phi} = \underset{\sim}{P}\underset{\sim}{\phi}^1 + (\phi_2^0 + t_2'\rho_2'(1))\underset{\sim}{e}_2$$

$$= \begin{bmatrix} x_1 - x_4^2 \\ x_2 + x_5^2 x_8 \\ x_3 - x_6^2 \end{bmatrix}. \tag{4.66}$$

Additional constraints are given by

$$\left.\begin{aligned} g_{5+2i-1} &= 1 + x_{6+i} \\ g_{5+2i} &= 1 - x_{6+i} \end{aligned}\right\} \quad i = 1,\ 2, \tag{4.67}$$

$$g_{10} = t_r - x_5^2/x_2.$$

Table 4.8 shows results for three values of t_r. The same results are obtained replacing the term $x_2 + x_5^2 x_i$ by $x_2(1 + t_r x_i)$, $i = 7, 8$, removing constraint g_{10} and reducing the number of variables by one. We note that larger tolerances are obtained than before for corresponding tuning ranges.

4.12.2 Tolerancing and Tuning for One Component

We consider C to be both toleranced and tuned and minimize

$$C = \frac{x_1}{x_4^2} + \frac{x_2}{x_5^2} + \frac{x_3}{x_6^2}, \tag{4.68}$$

where x_1, x_2 and x_3 are as before but where

$$\begin{aligned} x_4^2 &= \varepsilon_1 = \varepsilon_{L_1} \\ x_5^2 &= \varepsilon_2 = \varepsilon_C \\ x_6^2 &= \varepsilon_3 = \varepsilon_{L_2}, \end{aligned} \tag{4.69}$$

TABLE 4.8

RESULTS FOR THE LC LOWPASS FILTER

(C TUNED, L_1 AND L_2 TOLERANCED)

Parameters	$t_r = 0.2$	$t_r = 0.1$	$t_r = 0.05$
L_1^0	1.8664	1.9536	2.0002
C^0	1.1336	1.0077	0.9546
L_2^0	1.8664	1.9536	2.0002
$100\ \varepsilon_1/L_1^0$	27.54 %	21.84 %	19.00 %
$100\ t_2'/C^0$	20.00 %	10.00 %	5.00 %
$100\ \varepsilon_3/L_2^0$	27.54 %	21.84 %	19.00 %
$\rho_2'(6)$		-1.0000	
$\rho_2'(1)$		1.0000	

$n = 8 \qquad m = 10$

and $t_1 = t_2 = 0$. The cost of tuning is assumed fixed. It is, therefore, not included in (4.68). The slack variables are

$$x_7 = \rho_2(6), \qquad x_8 = \rho_2(8), \qquad x_9 = \rho_2(1). \tag{4.70}$$

Here,

$$\underset{\sim}{a}^1,\ \underset{\sim}{a}^2,\ \underset{\sim}{a}^3 \Rightarrow \underset{\sim}{\phi} = \underset{\sim}{\phi}^6 + t_2\rho_2(6)\underset{\sim}{e}_2$$

$$= \begin{bmatrix} x_1 + x_4^2 \\ x_2 - x_5^2 + t_r x_2 x_7 \\ x_3 + x_6^2 \end{bmatrix}, \tag{4.71}$$

$$\underset{\sim}{a}^4 \Rightarrow \underset{\sim}{\phi} = \underset{\sim}{\phi}^8 + t_2\rho_2(8)\underset{\sim}{e}_2$$

$$= \begin{bmatrix} x_1 + x_4^2 \\ x_2 + x_5^2 + t_r x_2 x_8 \\ x_3 + x_6^2 \end{bmatrix}, \tag{4.72}$$

$$\underset{\sim}{a}^5 \Rightarrow \underset{\sim}{\phi} = \underset{\sim}{\phi}^1 + t_2\rho_2(1)\underset{\sim}{e}_2$$

$$= \begin{bmatrix} x_1 - x_4^2 \\ x_2 - x_5^2 + t_r x_2 x_9 \\ x_5 - x_6^2 \end{bmatrix}, \tag{4.73}$$

with $t_2 = t_r C^0$. Constraints g_6 to g_{11} are as in (4.60).

The results are shown in Table 4.9 where we note that for 5% and 10% tuning we have an effective tolerance problem, whereas for 20% tuning we have an effective tuning problem. Rerunning the same problem with $t_r = 0.05$ and $x_7 = 1$, $x_8 = -1$, $x_9 = 1$, which imply effective tolerances, the same solution as for the 5% tuning range is obtained.

4.12.3 Optimal Tuning

In this example we include the tuning range in the objective function. Two cases are presented.

Case 1: Tolerancing and tuning for one component

We take a similar formulation to the last example except that

$$C = \frac{x_1}{x_4^2} + \frac{x_2}{x_5^2} + \frac{x_3}{x_6^2} + c\,\frac{x_7^2}{x_2}, \tag{4.74}$$

where c is a weighting factor and the term $t_r x_2$ is replaced by x_7^2, x_i by x_{i+1}, $i = 7, 8, 9$. This implies that $t_2 = x_7^2$. The constraints remain the same except for g_6 to g_{11} with x_i updated by x_{i+1}.

Table 4.10 shows results for different values of c. Note that a threshold value of c seems to occur somewhere between 10 and 20. Below that threshold, the solution in terms of an effective tuning and tolerance problem is unaffected. Note

TABLE 4.9

RESULTS FOR THE LC LOWPASS FILTER

(TOLERANCING AND TUNING FOR C. L_1 AND L_2 TOLERANCED)

Parameters	$t_r = 0.2$	$t_r = 0.1$	$t_r = 0.05$
L_1^0	2.0178	2.0380	2.0209
C^0	0.9366	0.9061	0.9040
L_2^0	2.0178	2.0380	2.0209
$100\ \varepsilon_1/L_1^0$	17.96 %	14.81 %	12.41 %
$100\ \varepsilon_2/C^0$	16.83 %	11.66 %	9.64 %
$100\ \varepsilon_3/L_2^0$	17.96 %	14.81 %	12.41 %
$100\ t_2/C^0$	20.00 %	10.00 %	5.00 %
$\rho_2'(8)$		1.0000	
$\rho_2'(6)$		-1.0000	
$\rho_2'(1)$		1.0000	
$100/C^0 x$	$t_2' = 3.17$ %	$\varepsilon_2' = 1.66$ %	$\varepsilon_2' = 4.64$ %
	$n = 9$	$m = 11$	

TABLE 4.10

RESULTS FOR THE LC LOWPASS FILTER

(OPTIMAL TUNING, CASE 1)

Parameters	$c = 1$	$c = 10$	$c = 20$	$c = 50$	$c = 100$	$c = 1000$
L_1^0	1.8440	1.8440	1.9221	2.0492	2.0227	1.9990
C^0	1.1730	1.1730	1.0486	0.9069	0.9043	0.9056
L_2^0	1.8440	1.8440	1.9221	2.0492	2.0227	1.9990
$100\ \varepsilon_1/L_1^0$	29.08 %	29.08 %	23.84 %	16.15 %	12.69 %	9.89 %
$100\ \varepsilon_2/C^0$	100.00 %	31.62 %	22.36 %	14.14 %	10.00 %	7.60 %
$100\ \varepsilon_3/L_2^0$	29.08 %	29.08 %	23.84 %	16.15 %	12.69 %	9.89 %
$100\ t_2/C^0$	122.69 %	54.31 %	35.88 %	14.14 %	5.71 %	0.00 %
$\rho_2(6)$			1.0000			
$\rho_2(8)$			-1.0000			
$\rho_2(1)$			1.0000			
$100/C^0x$	t_2'=22.69%	t_2'=22.69%	t_2'=13.52%	t_2'=0.00%	ε_2'=4.29%	ε_2'=7.60%

$n = 10 \qquad m = 11$

also the transition for c = 50 from effective tuning to effective tolerancing. When c is very large we obtain the tolerance solution presented in Table 4.3.

Case 2: Tolerancing and tuning for three components

The objective function considered is of the form

$$C = \sum_{i=1}^{3} \left(\frac{\phi_i^0}{\varepsilon_i} + c\,\frac{t_i}{\phi_i^0} \right). \tag{4.75}$$

We consider one additional vertex $\underset{\sim}{\phi}^3$ in order to bound the solution during optimization.

We omit details of the constraints, and summarize the final results in Table 4.11 for different c. The results are the same as in Table 4.10, but the computational effort has substantially increased. This formulation, however, has verified that $\phi_2 = C$ should be effectively tuned for c less than 50, and the other parameters effectively toleranced. The values of $\underset{\sim}{\rho}(6)$, $\underset{\sim}{\rho}(8)$, $\underset{\sim}{\rho}(1)$ and $\underset{\sim}{\rho}(3)$ confirm these observations.

4.13 Discussion

The formulation of the constraints for the tolerance-tuning problem has been treated. By its very nature the problem is a large one, even for designs with a relatively small number of parameters. Practical implementation depends heavily on one's ability to select

TABLE 4.11

RESULTS FOR THE LC LOWPASS FILTER

(OPTIMAL TUNING, CASE 2)

Parameters	c = 10	c = 20	c = 50
$L_1^0 = L_2^0$	1.8440	1.9221	2.0492
C^0	1.1730	1.0486	0.9069
$100\ \varepsilon_1/L_1^0 = 100\ \varepsilon_3/L_2^0$	31.62 %	23.84 %	16.15 %
$100\ \varepsilon_2/C^0$	31.62 %	22.36 %	14.14 %
$100\ t_1/L_1^0 = 100\ t_3/L_2^0$	2.54 %	0.00 %	0.00 %
$100\ t_2/C^0$	54.31 %	35.89 %	14.14 %
$\rho_1(6)$	-1.0000	-0.7165	0.9743
$\rho_2(6)$	0.1645	0.2466	1.0000
$\rho_3(6)$	-1.0000	-0.9992	-0.9846
$\rho_1(8)$	-1.0000	-1.0000	-0.8813
$\rho_2(8)$	-1.0000	-1.0000	-1.0000
$\rho_3(8)$	-1.0000	-1.0000	-0.9876
$\rho_1(1)$	1.0000	0.9887	0.9933
$\rho_2(1)$	1.0000	1.0000	1.0000
$\rho_3(1)$	1.0000	0.9989	0.9029
$\rho_1(3)$	1.0000	0.8433	-0.6051
$\rho_2(3)$	-0.1645	-0.1468	0.6434
$\rho_3(3)$	1.0000	0.8944	0.6441
$100\ \varepsilon_1'/L_1^0 = 100\ \varepsilon_3'/L_2^0$	29.08 %	23.84 %	14.14 %
$100\ t_2'/C^0$	22.69 %	13.53 %	0.00 %

$n = 21 \qquad m = 36$

a sufficiently small number of relevant vertices or critical points and constraints likely to be active, as well as meaningful variables.

Several properties of the centering, tolerancing and tuning process that have been noted previously were very much in evidence in the examples studied. In particular,

(1) Tuning one or more components enhances the overall tolerances significantly. The results presented could not have been obtained without considering centering, tolerancing and tuning in an integrated manner.

(2) Tuning of C conserves the symmetrical properties of the filter and a set of larger tolerances is obtained than by tuning L_1.

(3) When the tuning range does not appear in the objective function, a bound is needed.

(4) The results of the investigation seem to justify the reduction of the general tuning problem into one containing effectively toleranced and effectively tuned components, where appropriate. If the separation of the components is not decided in advance, the general problem as demonstrated in Section 4.12.3 with the cost function reflecting both tolerances and tuning ranges is appropriate, since an optimization program requires an explicit number of variables and constraints in advance. Compare the results of Tables 4.10 and 4.11.

(5) Zero tuning is indicated when the cost becomes too high.

(6) Except for the last problem considered, all the slack variables assume either the value of 1 or -1. This observation may indicate ways of simplifying constraints and eliminating some slack variables.

PART 3

REALISTIC DESIGN PROBLEMS

4.14 Introduction

Two realistic circuit design problems are now studied. The circuits under investigation have been reported to be in production in the telephone industry. The first circuit is a bandpass filter which is subjected to tolerance optimization. It has been studied by Butler (1971), Karafin (1971) and later by Pinel and Roberts (1972). The other circuit is a highpass filter for a digital receiver. It was suggested by Pinel (1974) and Roberts (1974). We have investigated it as a tolerance-tuning problem.

4.15 Tolerance Optimization of a Bandpass Filter

The circuit schematic is shown in Fig. 4.3. Specifications of insertion loss are shown in Table 4.12 and a frequency response at the nominal values obtained from Karafin's result is shown in Fig. 4.4. The reference frequency is at 420 Hz. Six frequency points are taken, two for the passband. A constant Q is assumed for the four inductors and, therefore, the four corresponding resistances are dependent variables. Parameter values are scaled by normalizing with respect to the central frequency and the load resistance such that the inductors and capacitors will have the same order of magnitude to avoid ill-conditioning during optimization.

We have considered three different objective functions

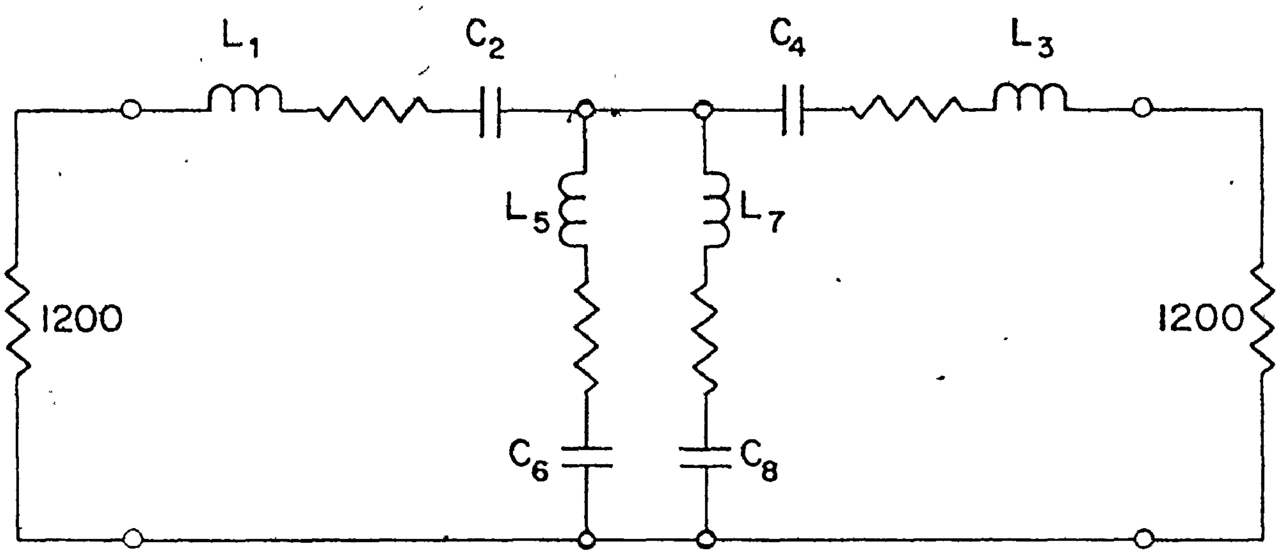


Fig. 4.3 The circuit for Karafin's bandpass filter.

TABLE 4.12

SPECIFICATIONS FOR KARAFIN'S BANDPASS FILTER

Frequency Range (Hz)	Sample Points (Hz)	Relative Insertion Loss Specification (dB)	Type
0 - 240	170, 240	35	lower (stopband)
360 - 490	360, 490	3	upper (passband)
700 - 1000	700, 1000	35	lower (stopband)

Reference Frequency: 420 Hz

A constant Q is assumed for the inductors.

$$C_1 = \sum_{i=1}^{8} \frac{\phi_i^0}{\varepsilon_i}, \tag{4.76}$$

$$C_2 = \sum_{i=1}^{8} \frac{1}{\varepsilon_i}, \tag{4.77}$$

and

$$C_3 = \sum_{i=1}^{8} \log_e \frac{\phi_i^0}{\varepsilon_i}, \tag{4.78}$$

where

$$\underset{\sim}{\phi}^0 = \begin{bmatrix} L_1^0 \\ C_2^0 \\ L_3^0 \\ C_4^0 \\ L_5^0 \\ C_6^0 \\ L_7^0 \\ C_8^0 \end{bmatrix}, \qquad \underset{\sim}{\varepsilon} = \begin{bmatrix} \varepsilon_{L_1} \\ \varepsilon_{C_2} \\ \varepsilon_{L_3} \\ \varepsilon_{C_4} \\ \varepsilon_{L_5} \\ \varepsilon_{C_6} \\ \varepsilon_{L_7} \\ \varepsilon_{C_8} \end{bmatrix}. \tag{4.79}$$

Initially, components L_3 and C_4 are assumed equal to L_1 and C_2, respectively, reducing the number of variables to 6 and the number of vertices to 2^6. Because of some violations, symmetry is not assumed for the objective function C_1.

The SUMT method (Fiacco and McCormick 1968) is used for this particular problem with starting nominal values used by Pinel and

Roberts and a ½% tolerance for each component. The penalty parameter r (see Appendix C) is set to 1 and is made successively smaller by a factor of 10. Table 4.13 shows some results and Fig. 4.4 shows the optimized nominal response using C_1. Note that the cost listed in Table 4.13 is $\sum_{i=1}^{8} \frac{\phi_i^0}{\varepsilon_i} \times .01$. There are no violations observed for both the Monte Carlo and worst-case analysis at the specified frequencies assuming 2^8 vertices. The relative insertion loss, however, becomes negative in some instances at other uncontrolled frequencies in the passband.

4.16 Tolerance-Tuning Optimization of a Highpass Filter

The circuit diagram is shown in Fig. 4.5 and the basic specifications for the design are listed in Table 4.14. The insertion loss relative to the loss at 990 Hz is to be constrained as indicated with resistances R_5 and R_7 related to L_5^0 and L_7^0 with constant Q. The terminations are fixed, the designable parameters being C_1, C_2, C_3, C_4, L_5, C_6 and L_7.

The objective function throughout was taken as

$$C = \sum_{i=1}^{7} \frac{\phi_i^0}{\varepsilon_i}, \tag{4.80}$$

where

TABLE 4.13

RESULTS FOR KARAFIN'S BANDPASS FILTER

(TOLERANCE OPTIMIZATION)

Parameters	Karafin, Pinel and Roberts	C_1	C_2	C_3
ϕ_1^0	1.824×10^{0}	3.0142×10^{0}	2.3206×10^{0}	2.7682×10^{0}
ϕ_2^0	7.870×10^{-8}	4.9750×10^{-8}	6.3694×10^{-8}	5.2611×10^{-8}
ϕ_3^0	1.824×10^{0}	2.9020×10^{0}	2.3206×10^{0}	2.7682×10^{0}
ϕ_4^0	7.870×10^{-8}	5.0729×10^{-8}	6.3694×10^{-8}	5.2611×10^{-8}
ϕ_5^0	4.272×10^{-1}	8.2836×10^{-1}	6.0517×10^{-1}	7.7895×10^{-1}
ϕ_6^0	9.880×10^{-7}	5.5531×10^{-7}	7.7708×10^{-7}	5.8726×10^{-7}
ϕ_7^0	1.437×10^{-1}	3.0319×10^{-1}	2.1677×10^{-1}	2.5438×10^{-1}
ϕ_8^0	3.400×10^{-7}	1.6377×10^{-7}	2.2630×10^{-7}	1.8981×10^{-7}
$100\ \epsilon_1/\phi_1^0$	3 , 3.32	6.99	2.29	7.67
$100\ \epsilon_2/\phi_2^0$	5 , 2.41	6.52	11.26	6.53
$100\ \epsilon_3/\phi_3^0$	5 , 3.30	6.97	2.29	7.67
$100\ \epsilon_4/\phi_4^0$	3 , 2.41	6.55	11.26	6.53
$100\ \epsilon_5/\phi_5^0$	2 , 1.14	4.36	3.30	4.33
$100\ \epsilon_6/\phi_6^0$	2 , 1.89	5.69	3.02	8.10
$100\ \epsilon_7/\phi_7^0$	3 , 7.80	6.80	6.61	5.85
$100\ \epsilon_8/\phi_8^0$	5 , 2.07	5.25	4.40	2.71
Cost	2.60, 3.45	1.34	2.06	1.46

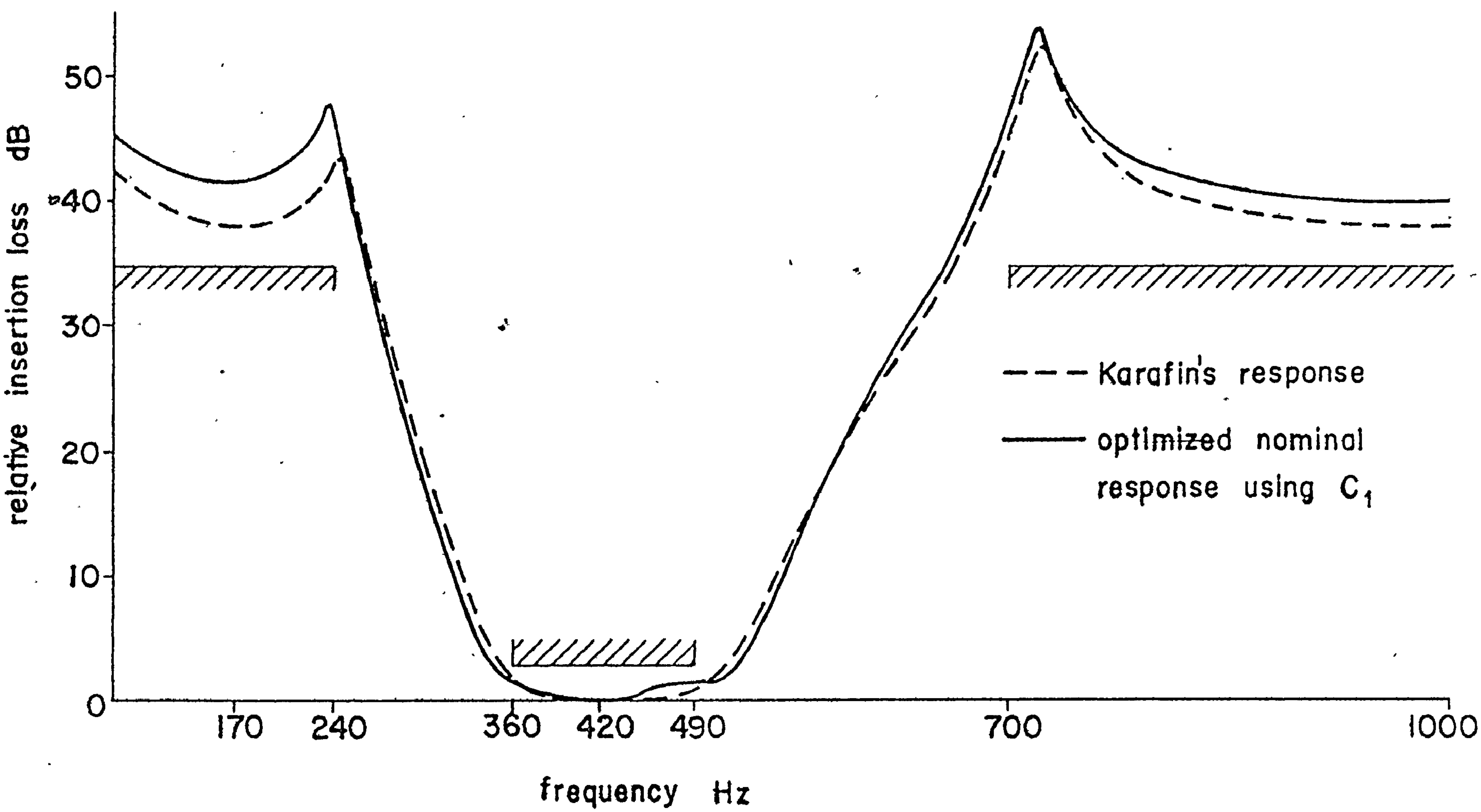


Fig. 4.4 Optimized response of Karafin's bandpass filter.

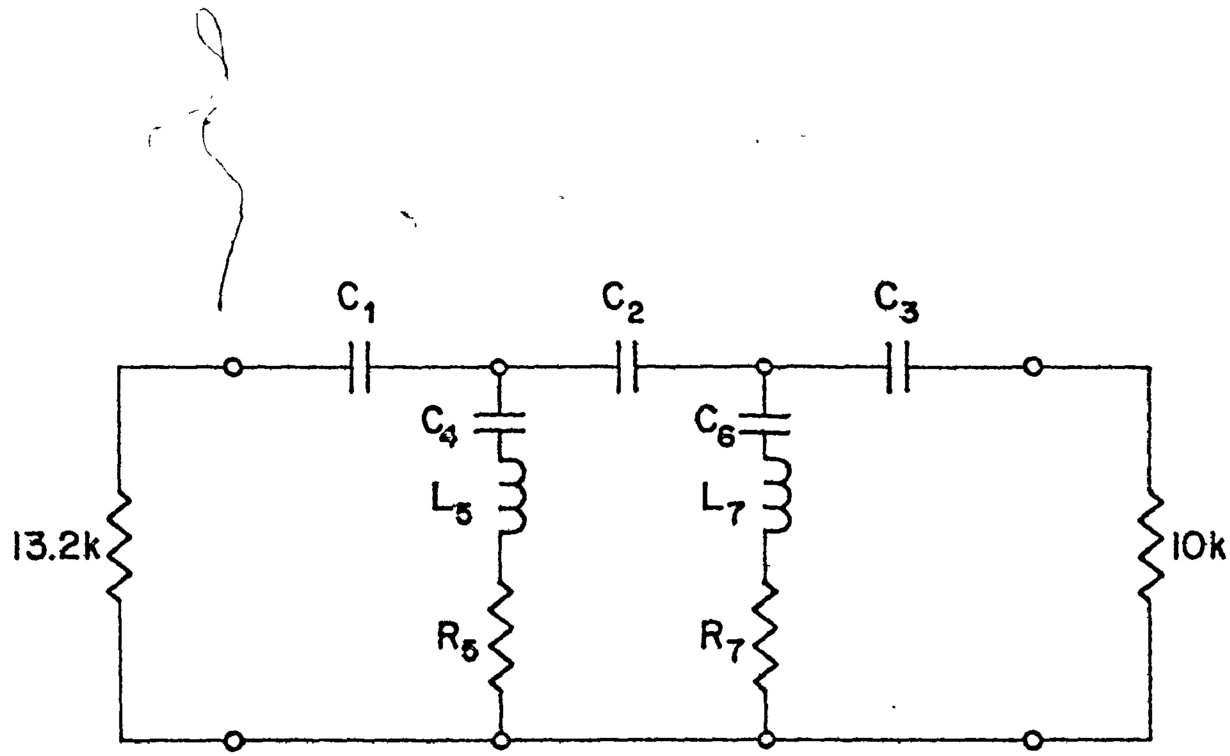


Fig. 4.5 The circuit for the highpass filter example.

TABLE 4.14

SPECIFICATIONS FOR THE HIGHPASS FILTER

Frequency Range (Hz)	Basic Sample Points (Hz)	Relative Insertion Loss (dB)	Weight w
170	170	45.	-1
360	360	49.	-1
440	440	42.	-1
630 - 680	630	4.	+1
680 - 1800	680 710 725 740	1.75	+1
630 - 1800	630 650 680 860 910 930 1050	-0.05	-1

Reference Frequency: 990 Hz

R_5, R_7 related to L_5^0 and L_7^0 through $Q = \dfrac{2\pi 990 L_5^0}{R_5} = \dfrac{2\pi 990 L_7^0}{R_7} = 1456.$

$$\underset{\sim}{\phi}^0 = \begin{bmatrix} C_1^0 \\ C_2^0 \\ C_3^0 \\ C_4^0 \\ L_5^0 \\ C_6^0 \\ L_7^0 \end{bmatrix} \qquad \underset{\sim}{\varepsilon} = \begin{bmatrix} \varepsilon_{C_1} \\ \varepsilon_{C_2} \\ \varepsilon_{C_3} \\ \varepsilon_{C_4} \\ \varepsilon_{L_5} \\ \varepsilon_{C_6} \\ \varepsilon_{L_7} \end{bmatrix}. \tag{4.81}$$

Verification of the designs to be described was carried out using all 2^7 vertices plus the nominal point at 170, 360, 440, 630-680 and 680-1800 Hz. Forty-two logarithmically spaced points were taken for the latter interval, and eight for the former interval.

Four cases are presented here.

Case 1: No tuning

Table 4.15 summarizes the particular frequencies, specifications and the particular vertex number employed to obtain the final tolerances listed in Table 4.16. The total number of variables and constraints are indicated in Table 4.15. Table 4.16 also lists the shifts in nominal parameter values with respect to those of an uncentered design by Pinel and Roberts.

Case 2: 3% tuning for L_5

Results corresponding to the ones for Case 1 are tabulated in Tables 4.15 and 4.16. Note that all the tolerances have

TABLE 4.15

DATA FOR CONSTRAINTS

OF THE HIGHPASS FILTER EXAMPLE

Frequency (Hz)	S (dB)	w	Vertex Number			
			Case 1 No Tuning	Case 2 L_5 Tuned	Case 3 L_5 and L_7 Tuned	Case 4 L_7 Tuned
170	45	-1	8	8	8	8
360	49	-1	48	48	48	48
440	42	-1	128	128	128	128
630	4	+1	1	1	1	1
630	-0.05	-1	60,100,104, 108,120,126	58,60,100, 104,108,120 126	60,108,120	60,87,95 100,104,108, 120,126
637	-0.05	-1	-	-	-	87
640	-0.05	-1	-	58	108	52,58,60
643	-0.05	-1	-	-	-	85,93,117
650	-0.05	-1	nominal,12, 50,58,102	nominal,12, 34,42,50,58, 102,106,126	nominal,12,34, 42,44,58,106, 126	nominal,12, 36,42,50,58, 85,93,94, 102,106,126

to be continued

TABLE 4.15 - continued

Frequency (Hz)	S (dB)	w	Vertex Number: Case 1 No Tuning	Case 2 L_5 Tuned	Case 3 L_5 and L_7 Tuned	Case 4 L_7 Tuned
658	-0.05	-1	-	-	42	58,69,85
665	-0.05	-1	-	-	34,42	34,58
670	-0.05	-1	-	-	-	2
680	1.75	+1	123	123	123	123
680	-0.05	-1	2,6	2,6	2,6	2,6
710	1.75	+1	43,83	43,83	43,83,123	43,83
725	1.75	+1	43,83	43,83	43,83	43,83
730	1.75	+1	-	-	43,83	43
740	1.75	+1	43,83	43,83	43,83	43,83
860	-0.05	-1	118,126	118,126	118,126	118,126
910	-0.05	-1	118,126	118,126	118,126	118,126
930	-0.05	-1	118,126	118,126	118,126	118,126
1040	-0.05	-1	-	-	-	3
1050	-0.05	-1	3	3	3	3

to be continued

TABLE 4.15 - continued

Number of Constraints and Variables	Case 1 No Tuning	Case 2 L_5 Tuned	Case 3 L_5 and L_7 Tuned	Case 4 L_7 Tuned
Number of Response Constraints	31	37	37	55
Total Number of Constraints m	45	51	51	69
Number of Variables n	14	14	14	14

TABLE 4.16

RESULTS FOR THE HIGHPASS FILTER

Parameters		Case 1 No Tuning	Case 2 L_5 Tuned	Case 3 L_5 and L_7 Tuned	Case 4 L_7 Tuned
C_1	tolerance (%)	5.71	6.77	7.90	6.63
	nom. shift(%)	+18.1	+17.8	+18.3	+17.6
C_2	tolerance (%)	4.33	4.97	5.32	4.77
	nom. shift(%)	+16.2	+15.2	+14.4	+15.3
C_3	tolerance (%)	4.72	5.81	7.23	5.83
	nom. shift(%)	+16.6	+18.0	+18.8	+17.8
C_4	tolerance (%)	4.54	5.03	5.15	4.78
	nom. shift(%)	- 3.8	- 2.2	- 1.2	- 3.1
L_5	tolerance (%)	3.29	3.95	4.44	3.82
	nom. shift(%)	- 3.0	- 3.0	- 4.3	- 4.1
C_6	tolerance (%)	6.32	7.05	7.27	6.66
	nom. shift(%)	- 7.3	- 5.1	- 3.6	- 6.0
L_7	tolerance (%)	3.64	4.34	5.04	4.32
	nom. shift(%)	- 6.4	- 7.9	- 7.9	- 6.3
Cost		157	135	121	138*

*Violation of specifications. Relative Loss = -0.052 dB at 658 Hz.

increased over the results of Case 1. Figure 4.6 shows the nominal response as well as the worst upper and lower outcomes based on all 2^7 vertices.

A more detailed verification of the results was made. Sixty logarithmically spaced points were taken from the critical region 630-680 Hz as well as forty from 600-630 Hz. All the vertices were checked plus the nominal point, followed by 4000 Monte Carlo simulations uniformly distributed in the effective tolerance region. No violations were detected, and the upper and lower limits of response given by the vertices bounded the results from the Monte Carlo analysis except at 638.2 Hz, where the lowest relative loss obtained from the vertices was -0.0243 dB, whereas the Monte Carlo analysis yielded -0.0246 dB.

As a further check on the optimality of these results, L_5 was allowed to be both toleranced and tuned as distinct from being effectively toleranced from the point of view of optimization. The same vertices, an additional 25 ρ variables and 50 additional constraints on the ρ variables were used without any significant improvement in the results. The values of the ρ variables confirmed the assumption that L_5 should be effectively toleranced for 3% tuning.

Case 3: 3% tuning for L_5 and L_7

As indicated by Table 4.16, a further improvement in all tolerances has been obtained.

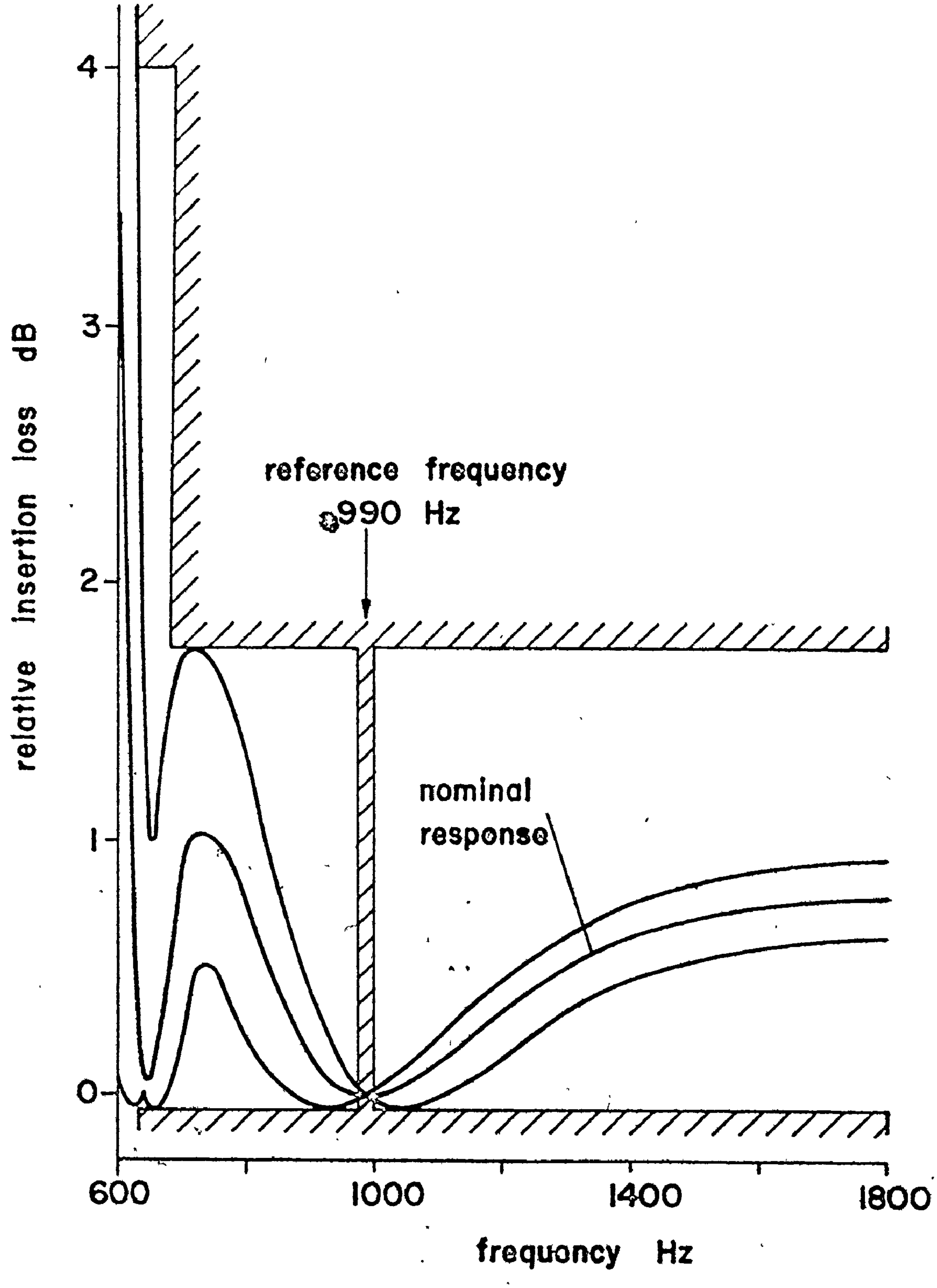


Fig. 4.6 Passband details of the optimized highpass filter (Case 2).

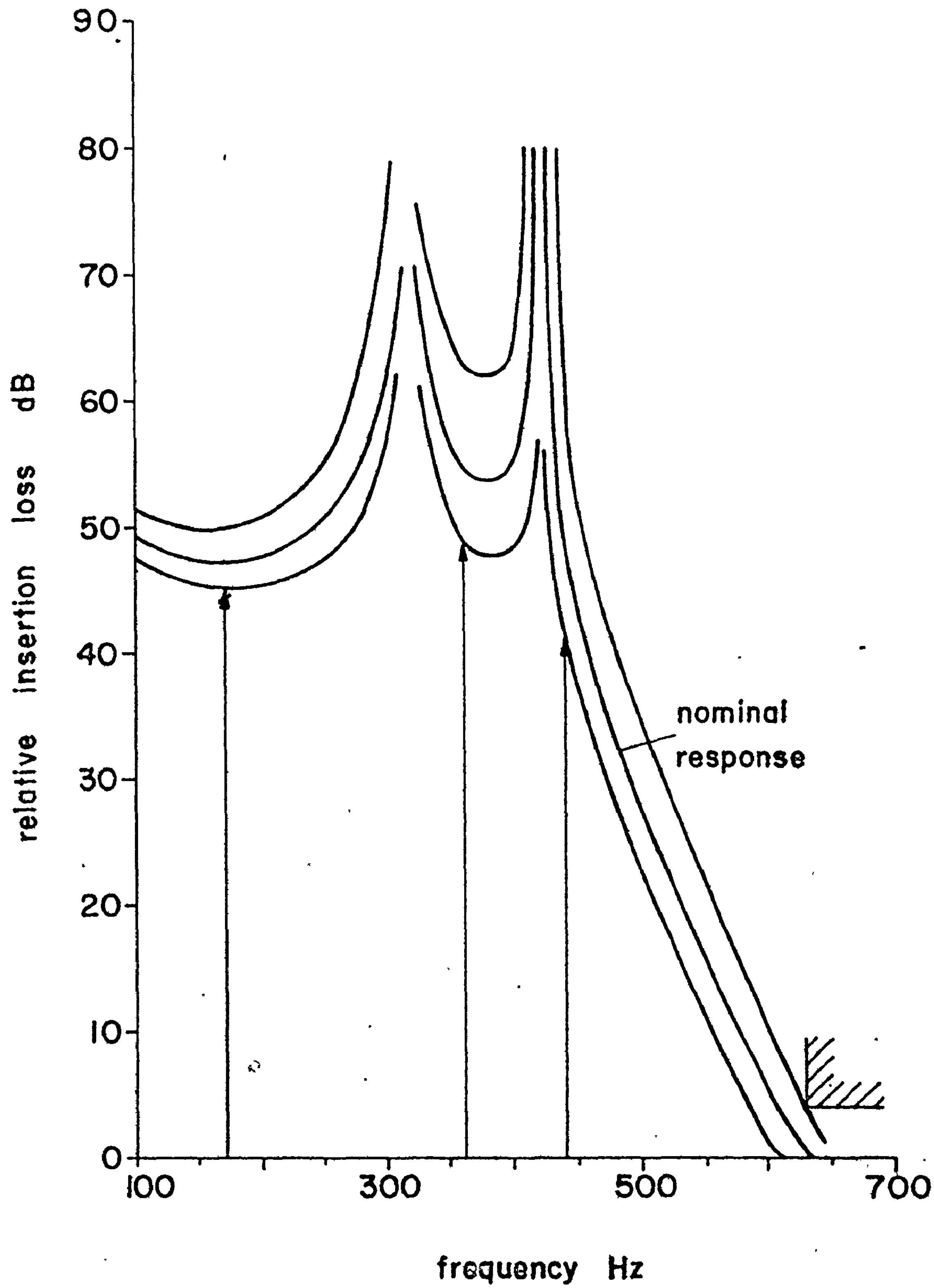


Fig. 4.7 Stopband details of the optimized highpass filter (2).

Case 4: 3% tuning for L_7

The results for this case are, as shown by Table 4.16, slightly worse than those for Case 2. A slight violation of the specifications at 658 Hz was detected. We conclude that if only one inductor is to be tuned, L_5 should be chosen.

4.17 Discussion

The problems studied are large from a computational point of view. The following comments regarding them can be made.

(1) Sometimes several preliminary runs are required to establish a reasonable choice of relevant vertices and constraints before a full optimization is attempted.
(2) Both problems demonstrate that the choice of sampling frequency points is very important in practical cases. Violations may occur at uncontrolled frequencies. This ill-conditioning property may be due to the formulation of relative insertion loss in the passband, noting that it is the difference of two responses of similar magnitudes.
(3) The Monte Carlo technique may be employed to test the assumptions of convexity after the final optimization.

Besides the comments made above, other pertinent remarks on advantages and observations presented in Part 1 and Part 2 also apply. For some more results and illustrations not included in this thesis, see Bandler, Liu and Tromp (1975b).

4.18 Conclusions

The advantages of the integrated approach to circuit design embodying centering, tolerancing and tuning have been shown and the sucessful implementations have been demonstrated by numerous examples. The introduction of tuning variables and allowing the nominal point to move have enhanced tolerances and subsequently reduced the cost of eventual fabrication. Time-saving techniques including vertices selection strategies and symmetry considerations have been presented and shown to be indispensible for an efficient automated algorithm. Two realistic problems have been studied. Typically, less than 2 minutes of CDC 6400 computer time is sufficient to optimize small problems and 5 to 10 minutes is sufficient for larger problems.

CHAPTER 5

CONCLUSIONS

In this thesis we have considered the problem of design centering, tolerancing and tuning in a unified manner. The concept of a tunable constraint region that allows variable specifications as set by the customer has also been incorporated. This may find application, for example, in tunable filters. Reduced problems adaptable for computer implementation have been treated. The purely toleranced and purely tuned problems turn out to be special cases. The examples we have studied seem to justify the reduction of the general tolerance-tuning problem into one containing effectively toleranced and effectively tuned components, where appropriate. If the separation of the components is not decided in advance, the general problems as in Section 4.12 with a cost function reflecting both tolerances and tuning ranges is appropriate, since an optimization program requires an explicit number of variables and constraints in advance.

A cost function tending to maximize tolerances and minimize tuning has been implemented successfully in this context. Zero tuning ranges were indicated when the cost became too high.

As far as the author is aware, this formulation seems to be the most general to date dealing with the centering, tolerancing and tuning problem at the design stage. Tuning uncertainties can also be taken care of in the formulation by associating tolerances with the tuning.

On the computational side the concept of one-dimensional convexity is essential. The application of this generalized convexity enables us to reduce an infinite number of constraints and variables to a manageable number. A class of functions that, under certain conditions, will give rise to such a region, in particular, the class of one-dimensional biquadratic functions, was investigated. These functions include the frequency response magnitudes of common linear, lumped, time-invariant circuits. Further reduction has been demonstrated by exploiting monotonicity and symmetrical properties of the network functions.

Reduction of computation time remains a challenging hurdle to overcome, particularly for discrete problems.

This work has revealed promising directions conceptually and algorithmically for future investigation.

(1) Extension of the formulation to correlated parameters. The deviation from the nominal of one component is often a function of another. This tracking problem is common in integrated circuit fabrication.

(2) A two-dimensional equivalent of (3.1) is

$$F(\phi_1,\phi_2) = \frac{N(\phi_1,\phi_2)}{M(\phi_1,\phi_2)} = \frac{\underset{\sim}{X}^T \underset{\sim}{A} \underset{\sim}{Y}}{\underset{\sim}{X}^T \underset{\sim}{B} \underset{\sim}{Y}}$$

and

$$\frac{\partial F}{\partial \phi_1} = \frac{\underset{\sim}{Y}'^T \underset{\sim}{B}^T \underset{\sim}{\Phi}_1 \underset{\sim}{A} \underset{\sim}{Y}'}{M^2}$$

where

$$\underset{\sim}{X} \triangleq \begin{bmatrix} 1 \\ \phi_1 \\ \phi_1^2 \end{bmatrix}, \quad \underset{\sim}{Y} \triangleq \begin{bmatrix} 1 \\ \phi_2 \\ \phi_2^2 \end{bmatrix}, \quad \underset{\sim}{Y}' \triangleq \begin{bmatrix} 0 \\ 1 \\ 2\phi_2 \end{bmatrix}, \quad \underset{\sim}{\Phi}_1 \triangleq \begin{bmatrix} 0 & 1 & 2\phi_1 \\ -1 & 0 & \phi_1^2 \\ -2\phi_1 & -\phi_1^2 & 0 \end{bmatrix},$$

and $\underset{\sim}{A}$ and $\underset{\sim}{B}$ are 3×3 matrices of the coefficients of N and M, respectively. Conditions for the worst case to occur at one of the vertices of the tolerance region can be investigated.

(3) Instead of considering exact 100% yield problems, bounds on the magnitudes of the constraint function may be obtained, say, from a multi-dimensional extension of equations (3.5) and (3.6) to predict the yield of a given design without a Monte Carlo simulation.

(4) Practical applications of tolerance-tuning ideas to optimize circuits subjected to parasitic loss effects (Temes 1962), stray elements and uncertainties in modelling. See, for example, some efforts by Bandler, Liu and Tromp (1975b).

(5) A special purpose optimization method which will choose and update constraints in the optimization process. A preliminary thought is as follows. Piecewise linearize all

the constraints, out of which choose the active ones and solve the subproblems in an iterative manner.

(6) The idea of generalized concave functions and the implications of signs of derivatives over a region could be applied to speed up some statistical methods that require repeated evaluation of function values.

APPENDIX A

GENERALIZATION OF CONCAVE/CONVEX FUNCTIONS

There is a vast volume of literature on generalized concave/convex functions. See, for example, relevant papers by Ponstein (1967), Greenberg and Pierskalla (1971) and books by Mangasarian (1969), Zangwill (1969), and by Roberts and Varbarg (1973). Unless otherwise indicated, we will follow definitions used by Zangwill.

Definition A.1 : A set $R \subseteq E^n$ is *convex* if $\underset{\sim}{\phi}^a, \underset{\sim}{\phi}^b \in R$ implies

$$\underset{\sim}{\phi}^a + \lambda(\underset{\sim}{\phi}^b - \underset{\sim}{\phi}^a) \in R \tag{A.1}$$

for any $0 \leq \lambda \leq 1$.

Lemma A.1: Let R_i, $i = 1, \ldots, m$, be convex sets. Then the set

$$R \triangleq \bigcap_{i=1}^{m} R_i \tag{A.2}$$

is also convex.

Definition A.2 : A function g on a convex set R is a *concave function* if $\underset{\sim}{\phi}^a, \underset{\sim}{\phi}^b \in R$ implies

$$g(\underset{\sim}{\phi}^a + \lambda(\underset{\sim}{\phi}^b-\underset{\sim}{\phi}^a)) \geq g(\underset{\sim}{\phi}^a) + \lambda(g(\underset{\sim}{\phi}^b) - g(\underset{\sim}{\phi}^a)) \qquad (A.3)$$

for any $0 \leq \lambda \leq 1$.

Definition A.3 : A function g on a convex set R is a *convex function* if $-g$ is concave.

Lemma A.2: Let g_i, $i = 1, 2, \ldots, m$, each be concave on a convex set R. If $a_i \geq 0$, $i = 1, \ldots, m$, the function

$$g(\underset{\sim}{\phi}) \triangleq \sum_{i=1}^{m} a_i g_i(\underset{\sim}{\phi}) \qquad (A.4)$$

is concave on R.

Lemma A.3: Let g be differentiable on a convex open set R. Then g is concave if and only if

$$g(\underset{\sim}{\phi}^b) \leq g(\underset{\sim}{\phi}^a) + \underset{\sim}{\nabla} g^T(\underset{\sim}{\phi}^a)(\underset{\sim}{\phi}^b-\underset{\sim}{\phi}^a), \qquad (A.5)$$

for any $\underset{\sim}{\phi}^a, \underset{\sim}{\phi}^b \in R$.

Lemma A.4: Let g be a concave function on a convex set R. Then for any fixed scalar γ the set

$$H_\gamma \triangleq \{\underset{\sim}{\phi} | g(\underset{\sim}{\phi}) \geq \gamma\} \qquad (A.6)$$

is convex.

Definition A.4 : A differentiable function $g : E^n \to E^1$ is *pseudoconcave* on a convex set R if for all $\underset{\sim}{\phi}^a, \underset{\sim}{\phi}^b \in R$,

$$\underset{\sim}{\nabla} g(\underset{\sim}{\phi}^a)^T(\underset{\sim}{\phi}^b - \underset{\sim}{\phi}^a) \leq 0 \tag{A.7}$$

implies

$$g(\underset{\sim}{\phi}^b) \leq g(\underset{\sim}{\phi}^a). \tag{A.8}$$

Definition A.5 : A function g is *pseudoconvex* if -g is pseudoconcave.

Definition A.6 : A function $g : E^n \to E^1$ is called *quasiconcave* on a convex set R if given $\underset{\sim}{\phi}^a, \underset{\sim}{\phi}^b \in R$

$$g(\underset{\sim}{\phi}^a + \lambda(\underset{\sim}{\phi}^b - \underset{\sim}{\phi}^a)) \geq \min[g(\underset{\sim}{\phi}^a), g(\underset{\sim}{\phi}^b)], \tag{A.9}$$

for any $0 \leq \lambda \leq 1$.

Definition A.7 : A function g is *quasiconvex* if -g is quasiconcave.

Lemma A.5: A function g is quasiconcave if and only if the set

$$H_\gamma \triangleq \{\underset{\sim}{\phi} | g(\underset{\sim}{\phi}) \geq \gamma\} \tag{A.10}$$

is convex for any scalar γ.

Definition A.8 : A set $R \subseteq E^n$ is *one-dimensional convex* if given any $\underset{\sim}{\phi}^a, \underset{\sim}{\phi}^{b(j)} \in R$, $j = 1, 2, \ldots, n$, where

$$\underset{\sim}{\phi}^{b(j)} \triangleq \underset{\sim}{\phi}^a + \alpha \underset{\sim}{e}_j, \tag{A.11}$$

for some scalar α, implies

$$\underset{\sim}{\phi}^a + \lambda(\underset{\sim}{\phi}^{b(j)} - \underset{\sim}{\phi}^a) \in R \tag{A.12}$$

for all $0 \leq \lambda \leq 1$.

Definition A.9 : A differentiable function $g : E^n \to E^1$ is *one-dimensional pseudoconcave* on a convex set R if given any $\underset{\sim}{\phi}^a, \underset{\sim}{\phi}^{b(j)} \in R$, $j = 1, 2, \ldots, n$, $\underset{\sim}{\phi}^{b(j)}$ as in (A.11), for some α

$$\frac{\partial g}{\partial \phi_j}(\underset{\sim}{\phi}^a) \cdot \alpha \leq 0 \tag{A.13}$$

implies

$$g(\underset{\sim}{\phi}^{b(j)}) \leq g(\underset{\sim}{\phi}^a). \tag{A.14}$$

The logical equivalent statement of (A.13) and (A.14) is as follows:

$$g(\underset{\sim}{\phi}^{b(j)}) > g(\underset{\sim}{\phi}^{a}) \tag{A.15}$$

implies

$$\frac{\partial g}{\partial \phi_j}(\underset{\sim}{\phi}^{a}) \cdot \alpha > 0. \tag{A.16}$$

Definition A.10: A function g is one-dimensional pseudoconvex if $-g$ is one-dimensional pseudoconcave.

Definition A.11: A function $g : E^n \to E^1$ is *one-dimensional quasiconcave* on a convex set R if for some α and for all $j = 1, 2, \ldots, n$, $\underset{\sim}{\phi}^{a}$, $\underset{\sim}{\phi}^{b(j)} \in R$,

$$g(\underset{\sim}{\phi}^{a} + \lambda(\underset{\sim}{\phi}^{b(j)} - \underset{\sim}{\phi}^{a})) \geq \min[g(\underset{\sim}{\phi}^{a}), g(\underset{\sim}{\phi}^{b(j)})], \tag{A.17}$$

for any $0 \leq \lambda \leq 1$.

Definition A.12: A function g is *one-dimensional quasiconvex* if $-g$ is one-dimensional quasiconcave.

Lemma A.6: A function g is one-dimensional quasiconcave if and only if the set

$$H_\gamma \triangleq \{\underset{\sim}{\phi} | g(\underset{\sim}{\phi}) \geq \gamma\} \tag{A.18}$$

is one-dimensional convex for any scalar γ.

Mangasarian and Ponstein have related quasiconvex functions to pseudoconvex functions and convex functions, with the conclusion that the class of quasiconvex functions is the largest class considered and the strictly convex class is the smallest. Similar statements can be made for quasiconcave functions, etc.

With the introduction of one-dimensional generalized concave/convex functions, a larger class of functions is added to the list. The class of one-dimensional generalized functions is less restrictive than the multi-dimensional counter-part. This can be demonstrated by the function

$$g(\underset{\sim}{\phi}) = \phi_1\phi_2,$$

which is convex over ϕ_1 (for any fixed ϕ_2) and over ϕ_2 (for any fixed ϕ_1) but fails the defining inequality (A.3) of convexity for

$$\underset{\sim}{\phi}^1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \underset{\sim}{\phi}^2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \lambda = \frac{1}{2}.$$

APPENDIX B

A BASIC THEOREM (Bandler 1972, 1974)

<u>Theorem</u>

If the vertices of R_ε are in R_c, then $R_\varepsilon \subseteq R_c$ if, for all $j = 1, 2, \ldots, k$

$$\underset{\sim}{\phi}^a,\ \underset{\sim}{\phi}^{b(j)} = \underset{\sim}{\phi}^a + \alpha \underset{\sim}{e}_j \in R_c \tag{B.1}$$

where α is a scalar and $\underset{\sim}{\varepsilon}$ is the jth unit vector, implies that

$$\underset{\sim}{\phi} = \underset{\sim}{\phi}^a + \lambda(\underset{\sim}{\phi}^{b(j)} - \underset{\sim}{\phi}^a) \in R_c \tag{B.2}$$

for all λ satisfying $0 \le \lambda \le 1$.

<u>Proof</u>

Let $\underset{\sim}{\phi}_\ell$ denote some point, in general, in an ℓ-dimensional linear manifold generated by the first 2^ℓ vertices as

$$\underset{\sim}{\phi}_\ell = \underset{\sim}{\phi}^0 - \underset{\sim}{\varepsilon} + 2\sum_{i=1}^{2^\ell}(p_i \sum_{j=1}^{\ell} \mu_j'(i)\varepsilon_j \underset{\sim}{e}_j) \tag{B.3}$$

with p_i satisfying

$$\sum_{i=1}^{2^{\ell}} p_i = 1, \qquad p_i \geq 0, \quad i = 1, 2, \ldots, 2^{\ell} \tag{B.4}$$

where $\mu_j'(i)\varepsilon\{0, 1\}$, $j = 1, 2, \ldots, \ell$ and $\varepsilon_j \geq 0$ is the tolerance of the jth component. The index i denotes the vertex number and must satisfy

$$i = 1 + \sum_{j=1}^{k} \mu_j'(i)2^{j-1}. \tag{B.5}$$

Assume that $\underset{\sim}{\phi}_{\ell} \in R_c$ for all $\underset{\sim}{\phi}^i \in R_c$. Now consider

$$\underset{\sim}{\phi}_{\ell+1} = \underset{\sim}{\phi}^0 - \underset{\sim}{\varepsilon} + 2\sum_{i=1}^{2^{\ell+1}} (q_i \sum_{j=1}^{\ell+1} \mu_j'(i)\varepsilon_j \underset{\sim}{e}_j) \tag{B.6}$$

with q_i satisfying

$$\sum_{i=1}^{2^{\ell+1}} q_i = 1, \qquad q_i \geq 0, \quad i = 1, 2, \ldots, 2^{\ell+1}. \tag{B.7}$$

After some manipulation, we find that

$$\underset{\sim}{\phi}_{\ell+1} = \underset{\sim}{\phi}^0 - \underset{\sim}{\varepsilon} + 2\sum_{i=1}^{2^{\ell}} [(q_i + q_{2^{\ell}+i}) \sum_{j=1}^{\ell} \mu_j'(i)\varepsilon_j \underset{\sim}{e}_j]$$

$$+ 2(\sum_{i=2^{\ell}+1}^{2^{\ell+1}} q_i)\varepsilon_{\ell+1}\underset{\sim}{e}_{\ell+1}. \tag{B.8}$$

Let

$$\lambda = \sum_{i=2^{\ell}+1}^{2^{\ell+1}} q_i \tag{B.9}$$

and

$$p_i = q_i + q_{2^\ell+1}, \quad i = 1, 2, \ldots, 2^\ell. \tag{B.10}$$

Hence (B.8) becomes

$$\underset{\sim}{\phi}_{\ell+1} = \underset{\sim}{\phi}_\ell + 2\lambda\varepsilon_{\ell+1}\underset{\sim}{e}_{\ell+1}. \tag{B.11}$$

With $\lambda=0$, $\underset{\sim}{\phi}_{\ell+1} = \underset{\sim}{\phi}_\ell \in R_c$ by assumption. If $\lambda=1$, $\underset{\sim}{\phi}_{\ell+1} = \underset{\sim}{\phi}_\ell + 2\varepsilon_{\ell+1}\underset{\sim}{e}_{\ell+1}$, which represents a translation of the ℓ-dimensional manifold. Thus, $\underset{\sim}{\phi}_{\ell+1} \in R_c$. For $0 < \lambda < 1$ we note $\underset{\sim}{\phi}_{\ell+1} \in R_c$ if (B.1) and (B.2) hold for $j = \ell+1$.

It is easy to verify that $\underset{\sim}{\phi}_1 \in R_c$ and, furthermore, that $\underset{\sim}{\phi}_2 \in R_c$ if (B.1) and (B.2) hold for $j=1$ and $j=2$, respectively. It follows by the foregoing inductive reasoning that $\underset{\sim}{\phi}_k = \underset{\sim}{\phi}$, as defined by

$$\underset{\sim}{\phi} = \underset{\sim}{\phi}^0 - \underset{\sim}{\varepsilon} + 2\sum_{i=1}^{2^k}\left(p_i \sum_{j=1}^{k} \mu_j'(i)\varepsilon_j\underset{\sim}{e}_j\right), \tag{B.12}$$

where

$$\sum_{i=1}^{2^k} p_i = 1, \quad p_i \geq 0, \quad i = 1, 2, \ldots, 2^k, \tag{B.13}$$

is in R_c under the conditions of the theorem.

APPENDIX C

OPTIMIZATION METHODS

A brief review of the techniques used for this work is presented here. Most of the algorithms described in this appendix have been incorporated in a user-oriented computer program called DISOPT. See Bandler and Chen (1974), and Chen (1974).

C.1 The Nonlinear Program

The nonlinear programming problem can be stated as

$$\text{minimize } f(\underset{\sim}{x}) \tag{C.1}$$

subject to

$$g_i(\underset{\sim}{x}) \geq 0, \qquad i = 1, 2, \ldots, m, \tag{C.2}$$

where f is the general nonlinear objective function of n parameters $\underset{\sim}{x}$, and $g_1(\underset{\sim}{x})$, $g_2(\underset{\sim}{x})$, ..., $g_m(\underset{\sim}{x})$ are, in general, nonlinear functions of the parameters. We will assume that all the functions are continuous with continuous partial derivatives.

The nonlinear program can be solved by methods such as the barrier-function method of Fiacco and McCormick (1968). We define, for example, the unconstrained function

$$B(\underset{\sim}{x},r) = f(\underset{\sim}{x}) + \sum_{i=1}^{m} \frac{r}{g_i(\underset{\sim}{x})}, \tag{C.3}$$

and minimize it with respect to $\underset{\sim}{x}$ for appropriately decreasing values of the parameter r.

Recently, Bandler and Charalambous (1972a, 1974) proposed a minimax approach which involves minimizing

$$V(\underset{\sim}{x},\alpha) = \max_{1\le i\le m} \left[f(\underset{\sim}{x}),\ f(\underset{\sim}{x}) - \alpha g_i(\underset{\sim}{x})\right], \tag{C.4}$$

where

$$\alpha > 0.$$

A sufficiently large value of α must be chosen to satisfy the inequality

$$\frac{1}{\alpha} \sum_{i=1}^{m} u_i < 1, \tag{C.5}$$

where the u_i's are the Kuhn-Tucker multipliers at the optimum.

C.2 Least pth Optimization

Several least pth optimization algorithms are available for obtaining minimax or near minimax solutions. The unconstrained function to be minimized, in the present context, can be of the form

$$U(\underset{\sim}{x}) \leftarrow (M(\underset{\sim}{x}) - \varepsilon) \left(\sum_{j \in J} \left(\frac{e_j(\underset{\sim}{x}) - \varepsilon}{M(\underset{\sim}{x}) - \varepsilon} \right)^q \right)^{\frac{1}{q}}, \tag{C.6}$$

where

$$\varepsilon \leftarrow \begin{cases} 0 \text{ for } M(\underset{\sim}{x}) \neq 0 \\ \text{small positive number for } M(\underset{\sim}{x}) = 0 \end{cases} \tag{C.7}$$

$$q \leftarrow p \ \mathrm{sgn}(M(\underset{\sim}{x}) - \varepsilon)$$

$$p > 1,$$

and

$$\text{if } M(\underset{\sim}{x}) \begin{cases} > 0, \ J \leftarrow \{j \mid e_j(\underset{\sim}{x}) > 0, \ j = 1, 2, \ldots, m+1\} \\ < 0, \ J \leftarrow \{1, 2, \ldots, m+1\}. \end{cases} \tag{C.8}$$

The definition of the e_j's, the appropriate value(s) of p and the convergence features of suitable algorithms are summarized in Table C.1. For the algorithm with large value of p, see Bandler and

TABLE C.1

FEATURES OF SOME LEAST pTH FORMULATIONS

Algorithm	Definition of e_i	Convergence feature	Value(s) of p	Number of optimizations
1	$e_i \leftarrow \begin{cases} f - \alpha g_i', & i = 1, 2, \ldots, m \\ f, & i = m+1 \end{cases}$		Large	1
2	where $\alpha > 0$	Increment of p	Increasing	Implied by the sequence but superceded by the stopping quantity
3		Extrapolation	Geometrically increasing	
4	$e_i \leftarrow \begin{cases} f - \alpha g_i - \xi^r, & i = 1, 2, \ldots, m \\ f - \xi^r, & i = m+1 \end{cases}$ where $\alpha > 0$ $\xi^r \leftarrow \begin{cases} \min[0, M^0 + \gamma], & r = 1 \\ \check{M}^{r-1} + \gamma, & r > 1 \end{cases}$ r indicates the optimization number γ is a small positive quantity	Updating of ξ^r	Finite	Depend on the stopping quantity

Charalambous (1972c), and Charalambous and Bandler (1973) for the description of Algorithm 4. See Chu (1974) for extrapolation technique used in Algorithm 3.

C.3 Existence of a Feasible Solution

The existence of a feasible solution can be detected by minimizing (C.6) when

$$e_j \leftarrow \begin{cases} -g_j, & j = 1, 2, \ldots, m \\ f - \bar{f}, & j = m + 1, \end{cases} \tag{C.9}$$

where $\bar{f}$ is an upper bound on f. A nonpositive value of M at the minimum or even before the minimum is reached indicates that a feasible solution exists. Otherwise, no feasible solution satisfying the current set of constraints and the upper bound on the objective function value is perceivable. Only one single optimization with a small value of p greater than unity is required.

C.4 Unconstrained Minimization Method

Gradient unconstrained minimization methods have become very popular because of their reported efficiency. One such program is the Fortran subroutine, which utilizes first derivatives, implemented by Fletcher (1972). The method used belongs to the class of quasi-Newton methods. The direction of search $\underset{\sim}{s}^j$ at the jth iteration is calculated by solving the set of equations

$$\underset{\sim}{B}^j \underset{\sim}{s}^j = - \underset{\sim}{\nabla} U(\underset{\sim}{x}^j), \tag{C.10}$$

where $\underset{\sim}{B}^j$ is an approximation to the Hessian matrix $\underset{\sim}{G}$ of U, $\underset{\sim}{\nabla}U$ is the gradient vector and $\underset{\sim}{x}^j$ is the estimate of the solution at the jth iteration.

As proposed by Gill and Murray (1972), the matrix $\underset{\sim}{B}^j$ is factorized as

$$\underset{\sim}{B}^j = \underset{\sim}{L}^j \underset{\sim}{D}^j \underset{\sim}{L}^{j^T}, \tag{C.11}$$

where $\underset{\sim}{L}$ is a lower unit triangular matrix and $\underset{\sim}{D}$ a diagonal matrix. It is important that $\underset{\sim}{B}^j$ must always be kept positive definite and, with the above factorization, it is easy to guarantee this by ensuring $d_{ii} > 0$ for all i.

A modification of the earlier switching strategy of Fletcher (1970) is used to determine the choice of the correction formula for the approximate Hessian matrix. The Davidon-Fletcher-Powell (DFP) formula is used if

$$\underset{\sim}{\delta}^T \underset{\sim}{L} \underset{\sim}{D} \underset{\sim}{L}^T \underset{\sim}{\delta} < \underset{\sim}{\delta}^T(\underset{\sim}{\nabla}U(\underset{\sim}{x}^{j+1}) - \underset{\sim}{\nabla}U(\underset{\sim}{x}^j)), \tag{C.12}$$

where

$$\underset{\sim}{\delta} = \underset{\sim}{x}^{j+1} - \underset{\sim}{x}^j. \tag{C.13}$$

Otherwise, the complementary DFP formula is used.

The minimization will be terminated when $|x_i^{j+1} - x_i^j|$ is less than a prescribed small quantity, for all i.

C.5 Discrete Optimization

A general strategy for solving a nonlinear discrete programming problem due to Dakin (1966) is described as follows.

Dakin's integer tree-search algorithm first finds a solution to the continuous problem. If this solution happens to be integral, the integer problem is solved. If it is not, then at least one of the integer variables, e.g., x_i, is non-integral and assumes a value x_i^*, say, in this solution. The range

$$[x_i^*] < x_i < [x_i^*] + 1, \tag{C.14}$$

where $[x_i^*]$ is the largest integer value included in x_i^*, is inadmissible and consequently we may divide all solutions to the given problem into two non-overlapping groups, namely,

(1) solutions in which

$$x_i \leq [x_i^*], \text{ and}$$

(2) solutions in which

$$x_i \geq [x_i^*] + 1.$$

Each of the constraints is added to the continuous problem sequentially and the corresponding augmented problems are solved. The procedure is repeated for each of the two solutions so obtained. Each resulting nonlinear programming problem thus constitutes a node and from each node two branches may emanate. A node will be fathomed if the following happens:

(1) the solution is integral,

(2) no feasible solution for the current set of constraints is achievable, and

(3) the current optimum solution is worse than the best integer solution obtained so far.

The search stops when all the nodes are fathomed.

It seems, then, that the most efficient way of searching would be to branch, at each stage, from the node with the lowest f(x) value. This would minimize the searching of unlikely subtrees. To do this, all information about a node has to be retained for comparison and this may require cumbersome housekeeping and excessive storage for computer implementation. One way of compromising is to search the tree in an orderly manner; each branch is followed until it is fathomed.

The tree is not, in general, unique for a given problem. The tree structure depends on the order of partitioning on the integer variables used. The amount of computation may be vastly different for different trees.

For the case of discrete programming problems subject to uniform quantization step sizes, the Dakin algorithm is modified as follows. Let x_i be the discrete variable which assumes a non-discrete solution x_i^*, and q_i be the corresponding quantization step, then the two variable constraints added sequentially after each node become

$$x_i \geq [x_i^*/q_i]q_i + q_i \tag{C.15}$$

and

$$x_i \leq [x_i^*/q_i]q_i. \tag{C.16}$$

The integer problem is thus a special case of the discrete problem with $q_i = 1$, $i = 1, 2, \ldots, n$, where n is the number of discrete variables.

If, however, a finite set of discrete values given by

$$D_i = \{d_{i1}, d_{i2}, \ldots, d_{ij}, d_{i(j+1)}, \ldots, d_{iu}\}, \ i = 1, 2, \ldots, n \tag{C.17}$$

is imposed upon each of the discrete variables, the variable constraints are then added according to the following rules:

(1) if $d_{ij} < x_i^* < d_{i(j+1)}$, then add the two constraints

$$x_i \leq d_{ij} \tag{C.18}$$

and

$$x_i \geq d_{i(j+1)} \tag{C.19}$$

sequentially,

(2) if $x_i^* < d_{il}$, only add the constraint

$$x_i \geq d_{il} \tag{C.20}$$

(3) if $x_i^* > d_{iu}$, only add the constraint

$$x_i \leq d_{iu}. \tag{C.21}$$

The resulting nonlinear programming problem at each node is solved by one of the algorithms described earlier. The feasibility check is particularly useful here since the additional variable constraints may conflict with the original constraints on the continuous problem. An upper bound, $\bar{f}$, on $f(x)$, if not specified, may be taken as the current best discrete solution. For a discrete problem, the best solution among all the discrete solutions given by letting variables assume combinations of the nearest upper and lower discrete values (if they exist) may be taken as the initial upper bound on $f(x)$.

The new variable constraint added at each node excludes the preceding optimum point from the current solution space and the constraint is therefore active if the function is locally unimodal. Thus the value of the variable under the new constraint may be optionally fixed on the constraint boundary during the next optimization. See Fig. C.1 for illustrations of the search procedure and a tree structure.

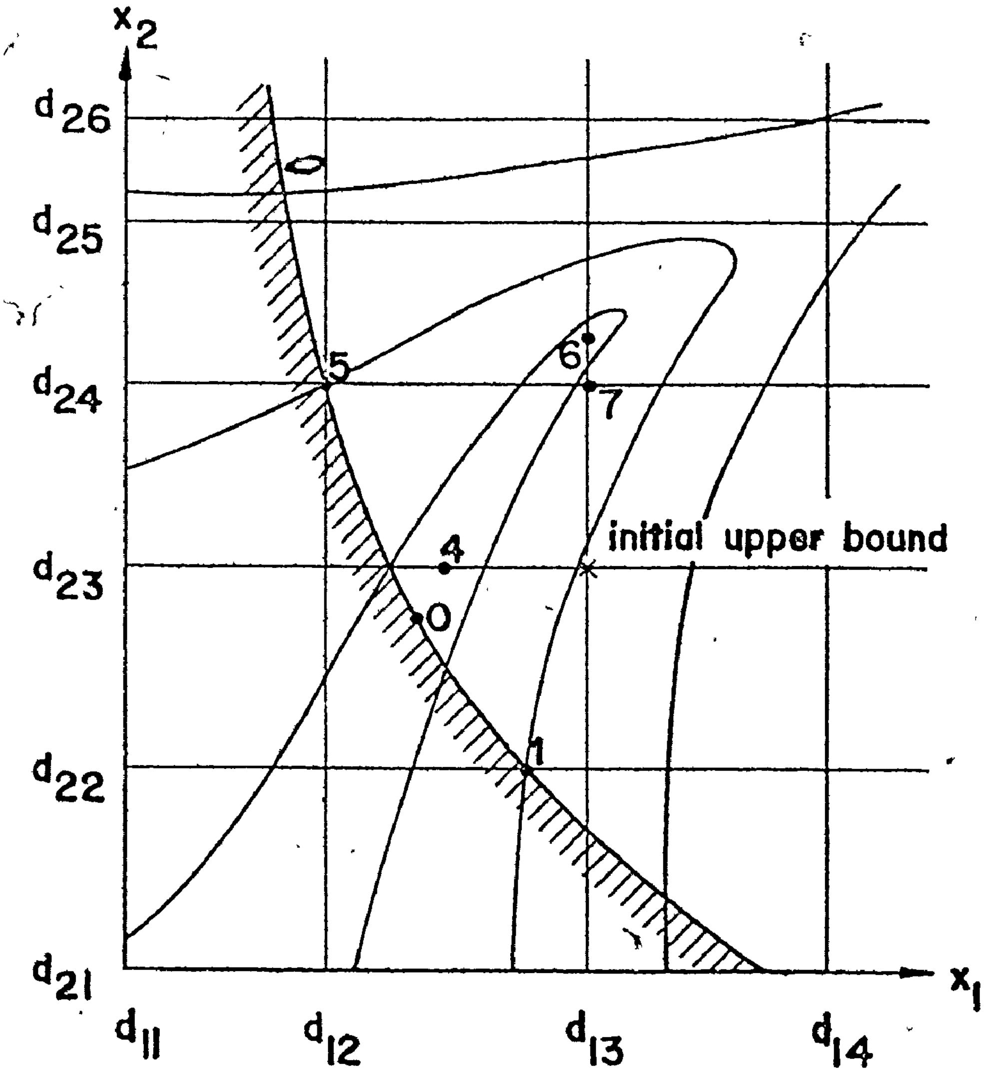
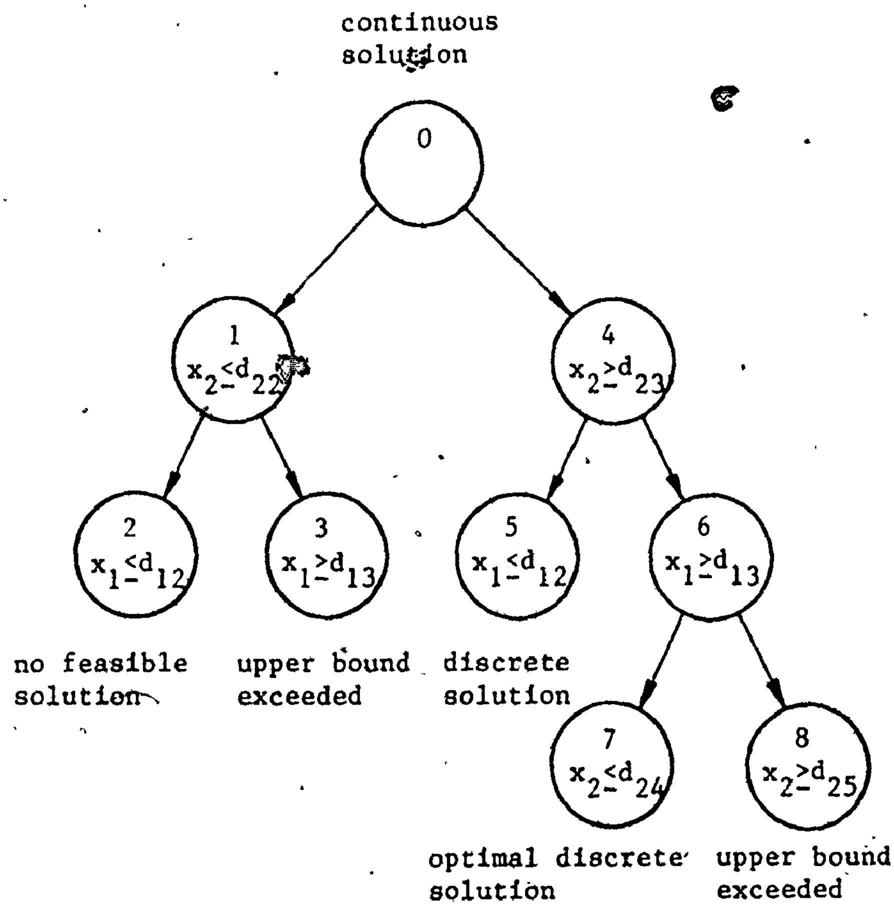


Fig. C.1 An illustration of the search for discrete solutions.
(a) Contours of a function of two variables with grid and intermediate solutions.



(b) The tree structure.

APPENDIX D

PROPOSED STRUCTURE OF A

TOLERANCE OPTIMIZATION PROGRAM

A proposal based on the techniques described in Appendix C for a TOLerance OPTimization program called TOLOPT is given here. Figure D.1 displays a block diagram of the principal subprograms comprising the program. TOLOPT is the subroutine called by the user. It organizes input data and coordinates other subprograms. Subroutine DISOP2 is a general program for continuous and discrete nonlinear programming problems. See Appendix C. Subroutine VERTST eliminates the inactive vertices of the tolerance region. Subroutine CONSTR sets up the constraint functions based on the response specifications, component bounds and other constraints supplied in the user subroutine USERCN. Subroutine COSTFN computes the cost function. The user supplied subroutine NETWRK returns the network responses and the partial derivatives.

Table D.1 is a summary of the features and options which may be incorporated into TOLOPT. See Bandler, Liu and Chen (1975).

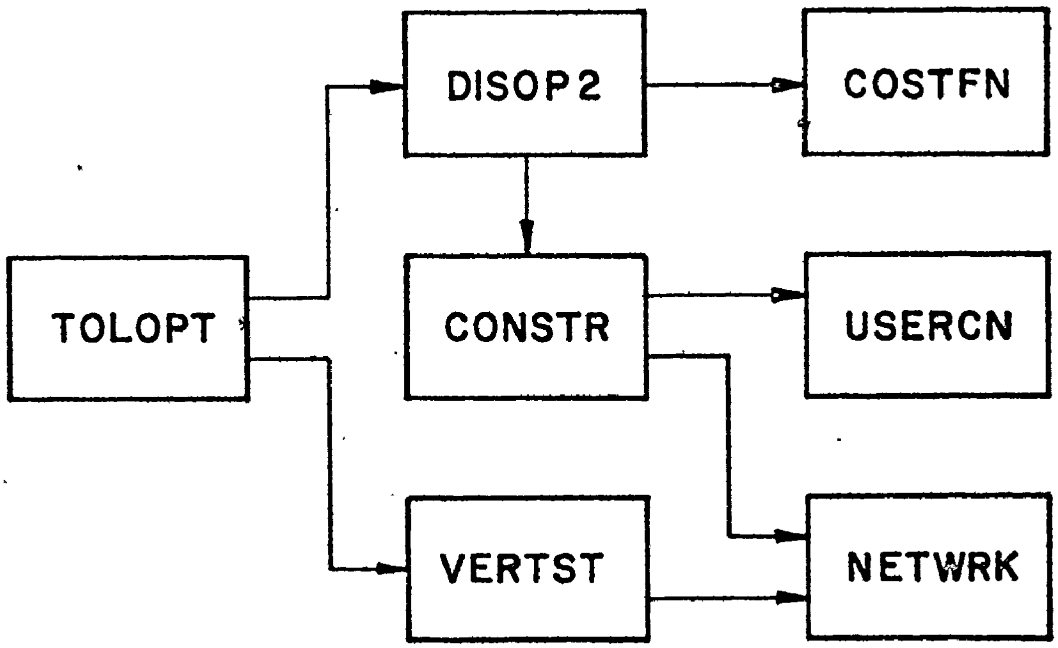


Fig. D.1 The overall structure of proposed TOLOPT. The user will be responsible for NETWRK and USERCN.

TABLE D.1

SUMMARY OF FEATURES, OPTIONS, PARAMETERS AND SUBROUTINES OF TOLOPT

Features	Type	Options	Parameters[+]/Subroutines
Design parameters	Nominal and tolerance	Variable or fixed Relative or absolute tolerances	Number of parameters Starting values Indication for fixed or variable parameters and relative or absolute tolerances
Objective function	Cost	Reciprocal of relative and/or absolute tolerances	Weighting factors
		Other	Subroutine to define the objective function and its partial derivatives
Vertices selection*	Gradient direction strategy		Maximum allowable number of calls of the vertices selection subroutine
Constraints	Specifications on functions of network parameters	Upper and/or lower	Sample points (e.g., frequency) Specifications Subroutine to calculate, for example, the network response and its partial derivatives (NETWRK)
	Network parameter bounds		Upper and lower bounds
	Other constraints	As many as required	Subroutine to define the constraint functions and their partial derivatives (USERCN)

to be continued

TABLE D.1 - continued

Features	Type	Options	Parameters† /Subroutines
Nonlinear programming	Bandler-Charalambous minimax (or suitable alternative)	Least pth optimization algorithms See, for example, Appendix C, Table C.1	Controlling parameter Value(s) of p Test quantities for termination
Solution feasibility check*	Least pth	Discrete problem Continuous and discrete problem	Constraint violation tolerance Value of p
Unconstrained minimization method	Quasi-Newton	Gradient checking at starting point by numerical perturbation	Number of function evaluations allowed Estimate of lower bound on least pth objective Test quantities for termination
Discrete optimization*	Dakin tree-search	Reduction of dimensionality User supplied or program determined initial upper bound on objective function Single or multiple optimum discrete solution *Uniform or nonuniform quantization step sizes	Upper bound on objective function Maximum permissible number of nodes Discrete values on step sizes Number of discrete variables Discrete value tolerance Order of partitioning Indication for discrete variables

† Parameters associated with the options are not explicitly listed.

* These features are optional and may be bypassed.

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