BOUNDING METHODS FOR FACILITIES LOCATION PROBLEMS

by

Paul Douglas Dowling, B.Sc., M.Sc., M.B.A.

A Thesis

Submitted to the School of Graduate Studies

in Partial Fulfillment of the Requirements

for the Degree

Doctor of Philosophy

Management Science/Systems

McMaster University

September 1985 ©

ISBN 0-315-30558-4

BOUNDING METHODS FOR FACILITIES LOCATION PROBLEMS

DOCTOR OF PHILOSPHY (1985)
(Management Science/Systems)

MCMASTER UNIVERSITY
Hamilton, Ontario

TITLE: Bounding Methods For Facilities Location Problems

AUTHOR: Paul Douglas Dowling, B.Sc. (McMaster University)
M.Sc. (McMaster University)
M.B.A. (McMaster University)

SUPERVISOR: Professor R.F. Love

NUMBER OF PAGES: viii, 99

ABSTRACT

Several methods have been proposed and tested for calculating lower bounds on the objective function of facilities problems. These methods contribute to the efficiency of iterative solution methods by allowing the user to terminate the computation process when the objective function comes within a predetermined fraction of the optimal solution. Two of the existing bounding methods have been presented only for single facility location models with Euclidean (straight-line) distances. One of these methods uses the dual of the single facility location model to compute a lower bound. This thesis introduces a method for generating a feasible dual solution from any primal solution by means of a projection matrix. The projection matrix method is applied to single and multi-facility models. The second bounding method, which involves the solution of a rectilinear distance model to obtain a lower bound, is extended in this thesis to include a generalized distance function and the multi-facility situation. Computation results for the two new bounding methods are compared with several existing bounding methods. These results should aid practitioners in selecting an appropriate bounding method for an iterative solution method to a facilties location problem.

ACKNOWLEDGEMENTS

I wish to acknowledge the efforts of my committee members, Dr. G.O. Wesolowsky and Dr. I.Z. Chorneyko, for their advice and patience throughout the dissertation process. A special note of thanks to Dr. N.P. Archer who has always managed to find the time, above and beyond what was expected of him as Ph.D. program director, to provide valuable information and counselling throughout my attendance in the program.

It is difficult to describe the considerable demands that I have made upon my supervisor, Dr. R.F. Love, and the appreciation and admiration that I have for him. Fortunately, the effort, care and consideration exhibited by Dr. Love in the dissertation and course work phases of my program has always exceeded the demands and challenges that I have thrust upon him. And finally, a note of thanks to the many people who helped make this dissertation a reality through contributions which ranged from finding "lost" computer files on backup tapes to providing encouragement when my morale was low.

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CHAPTER 1

INTRODUCTION

1.1 Single and Multi-Facility Location Models

Facilities location problems are concerned with finding an optimal location for an object or objects which interact with other objects whose locations are known. The first known formulation of a location problem dates back to the early 17th century. Fermat posed the problem in terms of finding a fourth point in the plane such that the sum of the distances to three fixed points was a minimum. By 1640 Torricelli had solved the three point problem by means of a geometric construction. His procedure was to draw circles circumscribing the equilateral triangles constructed on the sides of the triangle formed by the three fixed points as vertices. The intersection point of the circles provided the location of the fourth point, called the Torricelli point. In 1647 Cavalieri showed that the angle formed by joining any of the two fixed points to the Torricelli point (as vertex) was 120°. In 1834, F. Heinen proved that if the triangle formed with the three fixed points as vertices had one angle greater than or equal to 120°, then the location of the fourth point was at the vertex of the greatest angle. The first generalization of the Fermat problem appeared as an exercise given by Simpson (1750), in which he asked for a minimum weighted sum of distances from three points. Weber (1909) used the three point problem to find the best location for a central facility to produce a single product for a market point while receiving raw materials from two given distinct points. Over the last three hundred years the Fermat problem has also been known as the "Steiner Problem" and "Weber Problem". A more detailed history of this problem is provided by Kuhn (1967).

In a Mathematical Appendix to Weber's book, G. Pick suggested a solution method using weights and strings. This was the Varignon Frame, a mechanical analog for determining an optimal solution to the single facility straight-line weighted-distance problem. Using a board with holes drilled in it to represent the location of existing facilities, a string is passed through each hole and a weight corresponding to the transportation cost is attached to the lower end of each string. On the top of the board, all the strings are attached at a common knot. When the knot is released it will rest at the optimal location (assuming the absence of friction).

Each weight or demand in the location model can be used to represent a traffic flow or a monetary value for converting the distance travelled between a new and existing facility into a cost. The weight must be taken per unit distance per unit time, e.g. the number of trips per week from various departments to a central storage area or the daily cost of operating a truck per kilometer. An application of the single facilty model which includes detailed calculations of trucking costs from a truck terminal to various customers is given by Love, Truscott and Walker (1985).

Weiszfeld (1937) introduced an iterative method for solving the continuous space single facility weighted Euclidean (straight-line) distance location model. The Weiszfeld method remained in relative obscurity for many years until it was re-discovered independently by Miehle (1958), Kuhn and Kuenne (1962) and Cooper (1963). About this same time the single facility location model with rectilinear distances was formulated by Bindschedler and Moore (1961) and Francis (1963). The rectilinear (rectangular) distance measure occurs when travel is restricted to routes which are parallel or orthogonal to each other. Streets which form a rectangular grid and factory floor plans with aisles along rectangular bays may give rise to rectangular or rectilinear distances. If there are one-way streets in a city or obstacles on a rectilinear floor layout, the actual distances travelled in moving from one point to another can be even greater than the rectangular distance between the two points. By having travel distance represented by a distance function rather than actual distances in the location model, the distance function can be fitted for the appropriate situation using distances ranging from Euclidean to rectangular. The multi-facility rectilinear distance problem was introduced by Francis (1964), but the proposed solution method was limited to describing a region where the new facilities could be located.

If two or more new facilities are to be located simultaneously among a set of existing facilities, then this is an example of a multi-facility problem. Each new facility can interact with the existing facilities and the other new facilities. If there is no interaction between pairs of new facilities then this is a special case of the multi-facility problem, which can be solved as a series of separate single facility models. A more complex multi-facility situation occurs when the interactions between new facilities and existing facilities are not specified. As an example of this location-allocation situation, consider the problem of locating several new warehouses to serve a set of retail outlets where the stores have not been pre-assigned to a warehouse. The idea is to simultaneously locate the new facilities and to assign the outlets to the appropriate warehouse so that the total cost of serving the warehouses is minimized.

In the past 25 years there has been considerable effort expended on solving facility location problems. If rectilinear distances are used then exact solutions can be determined for the single and multi-facility models. Linear programming was applied to solve rectilinear distance models by Cabot, Francis and Stary (1970) and Wesolowsky and Love (1971a) but the number of constraints and variables increases considerably with the problem size. Linear programming is essentially confined to solving location problems when there are linear constraints on the sites for new facilities, provided that the constrained region for locating the new sites is convex. An example of this would be locating a new machine on a plant floor where access to overhead cranes would require that certain regions be considered as unsuitable for the new location.

For single and multi-facility models there exists a corresponding dual problem which can be solved independently to obtain identical solutions to the original (primal) model. The dual is a maximization problem and the primal and dual problems are equalized at the optimal solution for either problem given there is no "duality gap". The dual for the single facility Euclidean distance model was formulated by Kuhn and Kuenne (1962) and Bellman (1965). A multi-facility dual has been developed by Francis and Cabot (1970), Love and Kraemer (1973) and Love (1974). Kuhn and Kuenne refer to the lengthy history of the single facility Euclidean distance dual, and Kuhn (1967) provides a more detailed description dating back to the 19th century. The special case for three fixed points and unit weights was published by Fasbender (1846). By constructing an equilateral triangle with maximum height which circumscribed the three fixed points, Fasbender showed that the altitude was equal to the minimum distance sum from the Torricelli point to the three fixed points. The dual models for the single and multi-facility problems do not contain any of the primal variables and can provide alternate computational possibilities in solving for the optimal locations.

1.2 User Decisions in Applying a Location Model

When a location model is used, a number of decisions must be made concerning the number of new facilities to be located, the choice of a distance function, the solution method used to solve for the optimal location or locations, and when to stop the solution procedure if an iterative computational technique is used.

If the number of new facilities has not been specified, the location and minimum cost for a single new facility can be determined. A multi-facility model can then be used to solve for the optimal locations and minimum cost for two, three, four or more new facilities. By combining the fixed cost for new facilities with their respective variable (transportation) costs, the total cost for locating one or more new facilities is obtained. The lowest total cost will then identify the appropriate number of new facilities that are required.

When calculating the transportation cost, a distance function is used to estimate the travel distance between a new and existing facility. Various functions for modelling travel distances have been suggested by Love and Morris (1972, 1979), Ward and Wendell (1980, 1982), Berens and Körling (1984), Love and Dowling (1985) and Juel and Love (1985). The usual procedure is to take a random sample of actual distances travelled between fixed points and use these distances to estimate the parameters in the distance function chosen for that particular application. The distances between any two points can then be calculated as a function of the co-ordinates of the two points and the estimated parameters.

When the number of new facilities to be located, the distance model, and the demands or weights needed to convert distance into cost have been specified, then a solution method can be considered. A closed form solution to the single facility Euclidean distance problem has not been developed, but it is possible to solve the problem using an iterative method. Iterative techniques have been proposed by Weiszfeld (1937), Miehle (1958), Kuhn and Kuenne (1962), Cooper (1963), Katz (1969), Kuhn (1973), Cordelier and Fiorat (1978), Drezner and Wesolowsky (1978b), Ostresh (1978a), Calamai and Conn (1980) and Overton (1983). In contrast to this, an exact solution can easily be found for the single facility rectilinear model. The location-allocation models require the solution of a large number of individual location problems, so that a single computer run can take considerable time.

Iterative procedures which are guaranteed to converge still leave the practitioner with the decision as to when the computation process should be stopped. One of the arbitrary procedures utilized has been to stop when the reduction in the cost at some iteration reaches a "small" value. This could mean that the solution value is proceeding on a long shallow descent, and the process could be stopped when the current location is a considerable geographical distance from an optimal one. Another technique that has been used involves the derivatives (slope) of the cost function at the current solution point. The procedure is stopped when the derivatives are "close" to zero. A method for determining a lower bound on the objective function for a stepwise location-allocation problem with Euclidean distances was given by Ostresh (1978a). Juel (1978, 1984) and Love and Yeong (1981) provided lower bounds to the optimal solution which could be used to terminate the computation procedure when the maximum percentage improvement that could be made in the current cost reached a preset value. Both of these bounds could be applied to single or multi-facility models with a generalized distance function. Drezner (1984) developed a bound for the single-facility Euclidean distance model, and Wendell and Peterson (1984) have outlined a dual approach for obtaining a lower bound to the single facility model with a generalized distance function.

1.3 The Importance of Bounding Methods for Location Models

The knowledge of how close the current location is to an optimal solution is one which is critical in stopping an iterative procedure. There are three reasons why it is advantageous to compute a lower bound on the optimal cost of a location problem; considerable computer time savings may be achieved, consistency can be obtained in comparing costs when several different problems have to be solved, and user satisfaction can be increased knowing that a solution is as close to optimality as desired.

When using an iterative technique to solve a facility location model, a bounding method allows the user to stop the procedure when the current solution is within a preset percentage error difference of an optimal solution.

Computation time can be considerable when solving a large location-allocation type problem. In some situations an exact solution can be determined. Love (1976) provides an efficient solution for the one dimensional location-allocation problem using dynamic programming where examples with 150 customers and seven new facilities can be solved in six or seven minutes on a Univac 1108. Kuenne and Soland (1972), and Love and Morris (1975b) provide solutions to the two dimensional location-allocation problems with rectangular distances. However, in many cases a heuristic algorithm is required to provide a solution for the location-allocation model, and these do not necessarily provide an optimal solution. Some of the heuristics given by Love and Juel (1982) involve a series of allocation changes, where each change requires the solution of a single facility location model for each new facility in order to determine the optimal locations with respect to a given set of allocations. Using rectangular distance measure in the single facility location model allows an exact solution to be calculated for each new facility location. Then, the total cost for the current set of allocations can be compared with the lowest total cost obtained from all previous allocation changes. If a cost reduction occurs, the best allocation and its cost are updated. A bounding method would allow the use of distance measures other than rectangular in the location model. Each facility location solution from an iterative procedure can be calculated within a preset tolerance of an optimal solution, allowing a consistent basis of comparison between total costs for two different sets of allocations.

If a single facility model with a generalized distance function is selected, the user has a choice of the Juel (1984), Love and Yeong (1981), or Wendell and Peterson (1984) bound in terminating an iterative solution technique. For the Euclidean distance model, the choice can be expanded to include the Ostresh (1978a) and Drezner (1984) bounding methods. When a multi-facility model is chosen, only the Juel, and Love and Yeong bounds are applicable. Since the Ostresh bound is a special case of the Love and Yeong bound, only the Juel, Love and Yeong, Drezner, and the Wendell and Peterson bounding methods will be considered in this thesis. Both Juel (1984) and Elzinga and Hearn (1983) have proven that the Juel bound is always as good or better than the Love and Yeong bound. These have been the only theoretical comparisons that have been published to date. The dual lower bound for the single facility Euclidean distance model has been compared with the Juel, and the Love and Yeong bounds by Wendell and Peterson (1984) but only four small test problems with special structures were given. In order to use the dual as a lower bound, a feasible dual solution must be obtained from the current primal solution and a method for obtaining this dual feasible solution has not been published for the multi-facility case. No comparison has been made between the Drezner bound and the other three bounds, and the single facility Euclidean distance Drezner bound has not been extended to encompass multi-facility models or generalized distances. Some preliminary work by the author has indicated that the Love and Yeong bound may, in most cases, provide a better bound than the dual. This is the rationale for proceeding with a comprehensive comparison of the Love and Yeong bound and the dual, since the Juel bound is always as good or better than the Love and Yeong bound.

1.4 Objectives of the Thesis

The purposes of this thesis are as follows.

1. Given a single or multi-facility location model with a generalized distance function and the current location or locations for the new facilities as determined by some iterative procedure, develop a mathematical method to calculate a feasible solution to the dual so that the value of the dual objective function can be used as a lower bound to the optimal solution.

2. Write computer programs to implement this technique and provide computational results for the single and multi-facility cases in order to provide a comprehensive comparison of the dual and Love-Yeong bounds.
3. Extend the Drezner bound for the single facility Euclidean distances model to include a generalized distance function and develop a bound for the multi-facility model with a generalized distance function.
4. Provide computational and, where possible, theoretical comparisons of the four bounding methods for single and multi-facility location models.

CHAPTER 2

SIGNIFICANT PRIOR RESEARCH

In this chapter, mathematical models for the single facility location problem will be discussed along with properties and solution methods for these models. The dual problem will be given for the single facility Euclidean distance model. Four bounding methods will then be described that can be used with single facility location methods. Mathematical models for the multi-facility location problem will be introduced, and properties and solution methods discussed. The duals for the multi-facility models will be presented and the two bounding methods that are available for use with multi-facility iterative solution techniques will be described.

2.1 Single Facility Location Models

The single facility ℓ_p distance location problem is given as:

$$\text{minimize } W_p(x) = \sum_{j=1}^{n} w_j \ell_p(x, a_j) \tag{2.1}$$

where n is the number of existing facilities (or "demand points"),

w_j converts the distance between the new facility and existing facility j into cost,

$x' = (x_1, x_2)$ is the location of the new facility on the plane,

$a_j' = (a_{j1}, a_{j2})$ is the location of existing facility j,

$\ell_p(x, a_j)$ is the distance between the new facility and existing facility j where

$$\ell_p(x, a_j) = (|x_1 - a_{j1}|^p + |x_2 - a_{j2}|^p)^{1/p},$$

and the prime denotes transpose.

We will use the notation $D(x, a_j)$ to represent a generalized distance function, for which $\ell_p(x, a_j)$ is a special case and $d_2(x, a_j)$ or $\ell_2(x, a_j)$ will be used to denote Euclidean distances. This chapter will focus on models where the new and existing facilities are treated as points in the plane, demands and costs are known, and transportation costs are assumed to be proportional to distance travelled.

If $p = 1$, the rectilinear distance problem is given by:

$$\text{minimize } W_1(x) = \sum_{j=1}^{n} w_j (|x_1 - a_{j1}| + |x_2 - a_{j2}|) \ . \tag{2.2}$$

Since $W_1(x)$ is separable, it can be written as the sum of two functions, $W_{11}(x_1)$ and $W_{12}(x_2)$ where

$$W_{1k}(x_k) = \sum_{j=1}^{n} w_j |x_k - a_{jk}|$$

for $k = 1,2$. Minimizing $W_1(x)$ is equivalent to minimizing $W_{11}(x_1)$ and $W_{12}(x_2)$ separately as:

$$\min_x W_1(x) = \min_{x_1} W_{11}(x_1) + \min_{x_2} W_{12}(x_2) \ . \tag{2.3}$$

Each problem involving

$$\min_{x_k} W_{1k}(x_k)$$

can be easily solved, as shown by Francis (1963), to yield an exact optimal solution $x^{*\prime} = (x_1^*, x_2^*)$. For $p > 1$, no such exact solution method for (2.1) has been found to date.

For $p = 2$, the Euclidean distance model becomes

$$\text{minimize } W_2(x) = \sum_{j=1}^{n} w_j [(x_1 - a_{j1})^2 + (x_2 - a_{j2})^2]^{1/2} \ . \tag{2.4}$$

As mentioned previously in the introduction, Weiszfeld (1937) proposed an iterative solution to problem (2.4) and several others rediscovered it independently in the late 1950's and early 1960's. Weiszfeld (1937), Katz (1969), Kuhn (1963), Kuhn and Keunne (1962), Ostresh (1978b) and others have discussed the convergence properties of the iterative method. While other iterative procedures exist for solving problem (2.4), the Weiszfeld technique is so well-known that a generalized version of it is used to provide solutions for all test problems used in this thesis. The iterative technique was developed by equating the two partial derivatives of $W_2(x)$ in (2.4) to zero, and isolating an x_1 or x_2 term on one side of each equation to obtain an expression that could be used in a recursive manner. At iteration k, a point $x^{k'} = (x_1^k, x_2^k)$ is generated by

$$x_t^k = \sum_{j=1}^{n} \frac{w_j\, a_{jk}}{d_2(x_{k-1}, a_j)} \Bigg/ \sum_{j=1}^{n} \frac{w_j}{d_2(x_{k-1}, a_j)}\,, \quad \text{for } t = 1, 2\,. \tag{2.5}$$

This procedure assumes that the optimal location for the new facility does not coincide with an existing location $a_j, j = 1,\ldots,n$. It has been shown by Kuhn (1962) that each existing facility location can be checked to determine if it is the optimal location for the new facility. Kuhn proved that the rth existing location (a_{r1}, a_{r2}) is optimal if and only if

$$\left(\left[\sum_{\substack{j=1 \\ j \neq r}}^{n} \frac{w_j (a_{r1} - a_{j1})}{d_2(a_r, a_j)} \right]^2 + \left[\sum_{\substack{j=1 \\ j \neq r}}^{n} \frac{w_j (a_{r2} - a_{j2})}{d_2(a_r, a_j)} \right]^2 \right)^{1/2} \leq w_r\,. \tag{2.6}$$

If none of the existing facility locations satisfies condition (2.6), then the iterative procedure can be used to determine the optimal location of the new facility.

An upper and lower bound for the Euclidean distance optimal solution was given by Pritsker and Ghare (1970, 1972), who discovered a relationship between the rectilinear and Euclidean distance solution values. If $W_2(x_s^*)$, $W_{11}(x_{1R}^*)$ and $W_{12}(x_{2R}^*)$ represent the objective function values for the optimal Euclidean distance and rectangular distance solution values, then

$$W_2(x_R^*) \geq W_2(x_s^*) \geq [(W_{11}(x_{1R}^*))^2 + (W_{12}(x_{2R}^*))^2]^{1/2}\,. \tag{2.7}$$

When the optimal location for a new facility occurs at an existing facility location, discontinuities occur in the derivatives of $W_p(x)$, as noted by Love (1967, 1968). For $1 < p < 2$, Love and Morris (1972) show that the ℓ_p distance function has convexity and derivative discontinuity properties similar to the Euclidean distance function.

In order to solve the problem of discontinuities in the derivatives of $W_p(x)$ in (2.1), a uniformly convergent fitted function with continuous derivatives was developed by Love (1969). A hyperbolic approximation was suggested by Wesolowsky and Love (1972) and Eyster, White and Wierville (1973). The hyperbolic approximation to problem (2.1), formulated by Eyster, White and Wierville for $p=2$, was adapted by Love and Morris (1975a) by replacing the ℓ_p distance function with

$$\ell_{ph}(a,b) = \left[\sum_{t=1}^{N} |a_t - b_t|^p + \varepsilon \right]^{1/p},$$

where a and b are two points in N-dimension space. Verdini (1976) has shown that the hyperbolic approximation using ℓ_{ph} is not appropriate when using the Weiszfeld procedure for the general situation of $p \geq 1$. Morris and Verdini (1979) replace the ℓ_p distance function with

$$L_{pj}(x) = \left\{ \sum_{t=1}^{N} [(x_t - a_{jt})^2 + \varepsilon]^{p/2} \right\}^{1/p}, \quad p \geq 1, \varepsilon > 0$$

where x and a_j are N-dimensional vectors, and show that L_{pj} is differentiable and strictly convex. Using the hyperbolic approximation suggested by Verdini and Morris (1979), problem (2.1) can be written as

$$\text{minimize } W_{ph}(x) = \sum_{j=1}^{n} w_j \left([(x_1 - a_{j1})^2 + \varepsilon]^{p/2} + [(x_2 - a_{j2})^2 + \varepsilon]^{p/2} \right)^{1/p}. \tag{2.8}$$

The approximation function $W_{ph}(x)$ always gives values greater than the true objective function $W_p(x)$; the maximum difference is given by

$$\max[W_{ph}(x) - W_p(x)] = 2^{1/p} \varepsilon^{1/2} \left(\sum_{j=1}^{n} w_j \right). \tag{2.9}$$

This difference was given by Love and Yeong (1981) using a concept and property developed by Love (1969) in establishing the uniform convergence of his fitted function. $W_{ph}(x)$ has the properties that it is strictly convex and all orders of derivatives are continuous at all points. The practitioner can come very close to minimizing $W_p(x)$ by choosing a small value for ε when minimizing $W_{ph}(x)$.

Under certain conditions an iterative procedure is not required to solve for the optimal solution. The optimality condition (2.6) for a fixed point in the single facility Euclidean distance model has been extended to include generalized distances which include ℓ_p distances as a special case. Juel and Love (1981a) have proved that the rth existing facility $a_r = (a_{r1}, a_{r2})$ is optimal if and only if

$$[|R_{r1}|^{p/(p-1)} + |R_{r2}|^{p/(p-1)}]^{(p-1)/p} \leq w_r$$

where

$$R_{rk} = \sum_{\substack{j=1 \\ j \neq r}}^{n} \frac{w_j \operatorname{sign}(a_{rk} - a_{jk})|a_{rk} - a_{jk}|^{p-1}}{[\ell_p(a_r, a_j)]^{p-1}} \quad \text{for } k = 1, 2 \text{ and } p > 1 \,. \tag{2.10}$$

In the limiting case as $p \rightarrow 1^+$, the rectilinear distance model is minimized at the point a_r if and only if

$$\max(|R_{r1}|, |R_{r2}|) \leq w_r \,.$$

Witzgall (1965) showed that if one weight, w_t, was greater than or equal to the sum of the remaining weights, then $x^* = a_t$ was the optimal location. The Witzgall condition does not provide as strong a result as the condition given by (2.10). Juel and Love (1982) have provided conditions where an optimal location can be constructed.

In situations which do not fit the previously mentioned special cases, the practitioner is faced with using an iterative procedure to determine the optimal solution. The Weiszfeld technique given by (2.5) can be extended to include the hyperbolic approximation (2.8). By setting the partial derivatives equal to zero and isolating x_1 and x_2 on one side, Verdini (1976) and Morris and Verdini (1979) showed that a recursive relationship can be obtained for the solution at the kth iteration as $x^{k'} = (x_1{}^k, x_2{}^k)$ with

$$x_t^k = \frac{\displaystyle\sum_{j=1}^{n} \frac{a_{jt} w_j}{d'(x^{k-1}, a_j) d''(x_t^{k-1}, a_{jt})}}{\displaystyle\sum_{j=1}^{n} \frac{w_j}{d'(x^{k-1}, a_j) d''(x_t^{k-1}, a_{jt})}}$$

where

$$d'(x^{k-1}, a_j) = \left[\left((x_1^{k-1} - a_{j1})^2 + \varepsilon\right)^{p/2} + \left((x_2^{k-1} - a_{j2})^2 + \varepsilon\right)^{p/2}\right]^{(p-1)/2}$$

and

$$d''(x_t^{k-1}, a_{jt}) \doteq \left[(x_t^{k-1} - a_{jt})^2 + \varepsilon\right]^{(2-p)/2} \qquad \text{for } t = 1, 2. \tag{2.11}$$

Setting ε to a very small number such as $1.0\text{x}10^{-6}$ provides protection against the possibility of division by zero in a computer program when the optimal location converges quite closely to an existing facility. A convergence proof is given by Verdini (1976) for $p=1$ and $p=2$ and in general for $1 \leq p \leq 2$ by Morris (1981).

The dual formulation of the single facility Euclidean distance model (2.4) was given by Kuhn (1962) and Witzgall (1965) and appears implicitly in a dynamic programming procedure by Bellman (1965). Kuhn (1967) provides a history of the dual as well as a geometric and algebraic derivation of it. The dual for the single facility Euclidean distance problem (2.4) is

$$\text{maximize} - \sum_{j=1}^{n} a_j' U_j \tag{2.12}$$

$$\text{subject to} \quad \sum_{j=1}^{n} U_j = \vec{0}$$

$$|U_j| \leq w_j \qquad \text{for } j = 1, \ldots, n$$

where

$U_j' = (u_j, v_j)$ are dual variables.

Francis and Cabot (1972) have described properties for the single and multi-facility Euclidean distance duals. One of the properties of the single facility Euclidean distance dual is that the dual variables at optimality represent the direction vectors from the existing facilities to the optimal location, providing the optimal solution is not at an existing facility location. When the optimal location coincides with an existing facility, the dual variables are zero. This property can be expressed as

$$u_j^* = w_j(x_1^* - a_{j1})/d_2(x^*,a_j)$$

and $$v_j^* = w_j(x_2^* - a_{j2})/d_2(x^*,a_j) \quad \text{for } x^* \neq a_j, \text{ otherwise}$$

$$u_j^* = v_j^* = 0 \quad \text{for } x^* = a_j. \tag{2.13}$$

Bellman (1965) proposed a dual similar to (2.12) but it included the equalities $|U_j| = w_j$ for $j = 1,\ldots,n$, since the possible discontinuities in the derivatives of the primal problem were ignored. The single facility dual containing a mixture of ℓ_1 and ℓ_2 norms was given by Planchart and Hurter (1975), where the dual was solved by means of a decomposition method.

Given an optimal dual solution, the optimal primal solution can be calculated using $x^* = a_j + k_jU_j^*$ where $k_j \geq 0$. If all $|U_j^*| = w_j$, then using any two of the a_j and the corresponding U_j^*, four equations in four unknowns can be solved to obtain x^*. If $|U_t^*| < w_t$, then $x^* = a_t$.

2.2 Bounding Methods for Single Facility Location Models

The bound given by Love and Yeong (1981) for the single facility Euclidean distance model (2.4) is

$$W_2(x^*) \geq W_2(x^k) - \bar{\sigma}(x^k)|\nabla W_2(x^k)| \tag{2.14}$$

where Ω is the convex hull of the a_j,

$$\bar{\sigma}(x) = \max_{y \in \Omega} [d_2(x,y)],$$

$\nabla W_p(x)$ is the gradient of $W_p(x)$, and x^k is the value of x at the kth iteration of the solution procedure. The bound given by Juel (1984) is

$$W_2(x^*) \geq W_2(x^k) - \nabla W_2(x^k)' x^k + \min_{y \in \Omega} [\nabla W_2(x^k)' y] \tag{2.15}$$

where the prime denotes transpose. These two methods are applicable to both single and multi-facility location problems and can accommodate generalized ℓ_p distances. Juel (1984) and Elzinga and Hearn (1983) have proved that the Juel bound (2.15) must always be at least as good as the bound (2.14) given by Love and Yeong.

A third bound is given by Drezner (1984) for the single facility case with Euclidean distances. Drezner showed that

$$W_2(x^*) \geq \min_{x_1, x_2} \sum_{j=1}^{n} [w_j/d_2(a_j, x^k)][|x_1 - a_{j1}||a_{j1} - x_1^k| + |x_2 - a_{j2}||a_{j2} - x_2^k|]. \tag{2.16}$$

At each iteration of a solution process this bound is evaluated by solving the following rectilinear distance problem:

$$\min_{x_1} \sum_{j=1}^{n} w_j' |x_1 - a_{j1}| + \min_{x_2} \sum_{j=1}^{n} w_j'' |x_2 - a_{j2}| ,$$

where the "created" weights w_j' and w_j'' are defined as

$$w_j' = [w_j/d_2(a_j, x^k)] \, |a_{j1} - x_1^k| ,$$

and

$$w_j'' = [w_j/d_2(a_j, x^k)] \, |a_{j2} - x_2^k| \quad \text{for } j = 1, \ldots, n.$$

A fourth bounding method proposed by Wendell and Peterson (1984) utilized the dual of the location problem. The dual of (2.4) is given by

$$\text{maximize} - \sum_{j=1}^{n} U_j' a_j \tag{2.17a}$$

$$\text{subject to} \sum_{j=1}^{n} U_j = \vec{0} \tag{2.17b}$$

$$|U_j| \leq w_j , \qquad j = 1, \ldots, n. \tag{2.17c}$$

where the U_j are vectors of dual variables. To evaluate the dual at each iteration of the solution procedure, a dual feasible solution must be generated using the current primal solution. When the dual feasible solution has been calculated, the corresponding dual objective function can then be used as a lower bound for the optimal primal solution. Wendell and Peterson construct a vector from the current primal solution which is then projected into the subspace (2.17b) where

$$\sum_{j=1}^{n} U_j = \vec{0} ,$$

and then the resulting projected vector is shrunk to satisfy the norm constraints (2.17c). Unfortunately no description was given of the projection method used and only four small examples were included in the paper comparing the Love and Yeong, Juel, and dual bounds.

2.3 Multi-Facility Location Models

The first multi-facility location model was the rectangular distance model developed by Francis (1964) to locate several new facilities where each new facility could interact with a group of existing facilities as well as the remaining new facilities. The transportation costs or interactions between two new facilities are assumed to be proportional to the distance between them. The multi-facility ℓ_p distance model is given by

$$WM_p(x) = \sum_{i=1}^{m} \sum_{j=1}^{n} w_{1ij} \ell_p(x_i, a_j) + \sum_{i=1}^{m-1} \sum_{r=i+1}^{m} w_{2ir} \ell_p(x_i, x_r) \tag{2.18}$$

where

- m is the number of new facilities,
- n is the number of existing facilities,
- w_{1ij} is the nonnegative parameter which converts the distance between new facility i and existing facility j into cost,
- w_{2ir} is the nonnegative parameter which converts the distance between the ith and rth new facilities into cost $(i \neq r)$,
- $x_i' = (x_{i1}, x_{i2})$ are the location coordinates of new facility i,
- $a_j' = (a_{j1}, a_{j2})$ are the location coordinates of existing facility j, and p is the ℓ_p distance parameter.

Substituting p = 1 yields the rectilinear distance model:

$$\text{minimize } WM_1(x) = \sum_{i=1}^{m} \sum_{j=1}^{n} w_{1ij}(|x_{i1} - a_{j1}| + |x_{i2} - a_{j2}|)$$

$$+ \sum_{i=1}^{m-1} \sum_{r=i+1}^{m} w_{2ir}(|x_{i1} - x_{r1}| + |x_{i2} - x_{r2}|). \quad (2.19)$$

This rectilinear distance model has convexity and separability properties similar to the single facility rectangular distance model (2.2), and can be written as

$$\min WM_1(x) = \min_{x_{11}, \ldots, x_{m1}} \left[\sum_{i=1}^{m} \sum_{j=1}^{n} w_{1ij}|x_{i1} - a_{j1}| + \sum_{i=1}^{m-1} \sum_{r=i+1}^{m} w_{2ir}|x_{i1} - x_{r1}| \right]$$

$$+ \min_{x_{12}, \ldots, x_{m2}} \left[\sum_{i=1}^{m} \sum_{j=1}^{n} w_{1ij}|x_{i2} - a_{j2}| + \sum_{i=1}^{m-1} \sum_{r=i+1}^{m} w_{2ir}|x_{i2} - x_{r2}| \right]. \quad (2.20)$$

Exact solutions to the multi-facility rectilinear distance problem were obtained by Cabot, Francis and Stary (1970) and Wesolowsky and Love (1971a) using linear programming. The Wesolowsky and Love linear programming formulation introduces an equality constraint and two new variables for every absolute value term in (2.19), which makes its use impractical for large multi-facility problems. The linear programming approach does allow the introduction of additional constraints when restrictions on the locations of new facilities are required. There are a great number of discontinuities in the partial derivatives of $WM_1(x)$; this makes the use of gradient search techniques infeasible. Discontinuities occur when a new facility coincides with an existing location or with another new facility as reported by Love (1967, 1968, 1969). Several methods are available to solve large scale problems. Wesolowsky and Love (1972) used a hyperbolic approximation to the terms involving rectangular distances. Approximating $WM_1(x)$ by a non-linear convex function with continuous partial derivatives allowed an iterative gradient descent method to be used to determine an optimal solution to the approximation function. An algorithm was proposed by Juel and Love (1976) which uses a modified edge descent procedure and can solve large problems in short computational times (1 second for $m = 70$ and $n = 350$ on a Univac 1110).

The Euclidean distance multi-facility problem is

$$\min WM_2(x) = \sum_{i=1}^{m} \sum_{j=1}^{n} w_{1ij}[(x_{i1} - a_{j1})^2 + (x_{i2} - a_{j2})^2]^{1/2}$$

$$+ \sum_{i=1}^{m-1} \sum_{r=i+1}^{m} w_{2ir}[(x_{i1} - a_{j1})^2 + (x_{i2} - x_{r2})^2]^{1/2} \quad (2.21)$$

Various approaches have been employed to solve this problem. Vergin and Rogers (1967) used a heuristic which located all the new facilities in a step-wise manner, and then took each new facility in turn and located it optimally, considering all other facilities as being fixed. This procedure was continued until no further improvement could be made to the objective function value. A three-dimensional formulation of (2.21) was solved by Love (1969) using a convex programming algorithm after first using the Method of Fitted Functions to overcome the discontinuities in the partial derivatives of the objective function.

Francis and Cabot (1972) described the dual for the unconstrained Euclidean distance problem but no solution method was given. Although the primal problem (2.21) has discontinuities in the partial derivatives, the dual does not have these differentiability problems. Love and Kraemer (1973) used a nonlinear decomposition technique to solve the multi-facility Euclidean dual with linear constraints. For the unconstrained rectangular distances problem, the dual was given by Cabot, Francis and Stary (1970) and Wesolowsky and Love (1971a) provided duals for the linearly constrained and unconstrained cases. The Euclidean distance multi-facility dual for the primal problem (2.21) has dual variables defined by $U_{1ij}' = (u_{1ij}, v_{1ij})$ for $i = 1,\ldots,m$; $j = 1,\ldots,n$ and $U_{2ir}' = (u_{2ir}, v_{2ir})$ for $i = 1,\ldots,m-1$; $r = i+1,\ldots,m$.

The multi-facility dual problem is expressed as

$$\text{maximize } -\sum_{i=1}^{m}\sum_{j=1}^{n} a_j' \, U_{1ij} \tag{2.22}$$

$$\text{subject to } \quad -\sum_{r=1}^{i-1} U_{2ri} + \sum_{r=i+1}^{m} U_{2ir} + \sum_{j=1}^{n} U_{1ij} = \vec{0} \quad \text{for } i = 1, \ldots, m$$

$$|U_{1ij}| \le w_{1ij} \qquad i = 1, \ldots, m \, ; \; j = 1, \ldots, n$$

$$|U_{2ir}| \le w_{2ir} \qquad i = 1, \ldots, m-1 \, ; \; r = i+1, \ldots, m.$$

The inequalities arise in the situation where $x_i = a_j$, in which case $|U_{1ij}| = 0$ (otherwise $|U_{1ij}| = w_{1ij}$), or when $x_i = x_r$, in which case $|U_{2ir}| = 0$ (otherwise $|U_{2ir}| = w_{2ir}$). One of the properties given by Francis and Cabot (1972) for the multi-facility Euclidean distance dual provides a relationship between the dual and primal variables at optimality. The U^*_{1ij} dual variables satisfy the condition that the line through the existing facility a_j which is parallel to U^*_{1ij} passes through the optimal new facility location x^*_i, providing that the new facility does not coincide with the existing facility a_j. Similarly, if new facilities x^*_i and x^*_r do not coincide, then the line through new facility x^*_r parallel to U^*_{2ir} passes through x^*_i. If a new facility coincides with an existing facility or if two new facilities coincide, the corresponding dual variables are zero and vice versa. This property can be expressed as

$$u^*_{1ij} = w_{1ij}(x^*_{i1} - a_{j1})/[(x^*_{i1} - a_{j1})^2 + (x^*_{i2} - a_{j2})^2]^{1/2}$$

$$v^*_{1ij} = w_{1ij}(x^*_{i2} - a_{j2})/[(x^*_{i1} - a_{j1})^2 + (x^*_{i2} - a_{j2})^2]^{1/2} \qquad \text{for } x^* \ne a_j$$

$$u^*_{2ir} = w_{2ir}(x^*_{i1} - x^*_{r1})/[(x^*_{i1} - x^*_{r1})^2 + (x^*_{i2} - x^*_{r2})^2]^{1/2}, \tag{2.23}$$

$$v^*_{2ir} = w_{2ir}(x^*_{i2} - x^*_{r2})/[(x^*_{i1} - x^*_{r1})^2 + (x^*_{i2} - x^*_{r2})^2]^{1/2} \qquad \text{for } x^*_i \ne x^*_r,$$

otherwise

$$u^*_{1ij} = v^*_{1ij} = 0 \text{ if } x^*_i = a_j \text{ and } u^*_{2ir} = v^*_{2ir} = 0 \quad \text{if } x^*_i = x^*_r.$$

The multi-facility Euclidean distances optimal solution can also be bounded by condition (2.7), as shown by Pritsker and Ghare (1970, 1972). Sufficient conditions for optimal facility locations to coincide for some multi-facility problems are given by Juel and Love (1980). In some cases, by checking the interfacility weights to determine if they satisfy a few simple inequalities, it is possible to solve a multi-facility problem. Also, a multi-facility generalization of Witzgall's majority rule was given by Juel and Love, whereby if

$$w_{111} \geq \sum_{j=1}^{n} w_{1j1} + \sum_{i=2}^{m} w_{21i},$$

then $x^*_1 = a_1$ may be assumed without loss of generality.

To solve the ℓ_p distance multi-facility problem, an approximating function $WM_{ph}(x)$ can be used to overcome the problems of differentiability of (2.18), as proposed by Wesolowsky and Love (1972) and Eyster, White and Wierville (1973). The multi-facility hyperbolic approximation version of (2.8) as given by Morris and Verdini (1979) is:

$$WM_{ph}(x) = \sum_{i=1}^{m} \sum_{j=1}^{n} w_{1ij} \left\{ [(x_{i1} - a_{j1})^2 + \varepsilon]^{p/2} + [(x_{i2} - a_{j2})^2 + \varepsilon]^{p/2} \right\}^{1/p}$$

$$+ \sum_{i<r} w_{2ir} \left\{ [(x_{i1} - a_{r1})^2 + \varepsilon]^{p/2} + [(x_{i2} - x_{r2})^2 + \varepsilon]^{p/2} \right\}^{1/p}, \tag{2.24}$$

where $\varepsilon > 0$. They show that the function $WM_{ph}(x)$ is strictly convex and is differentiable to any order everywhere. It can easily be shown that $WM_{ph}(x)$ is uniformly convergent to $WM_p(x)$ as $\varepsilon \to 0$, since

$$\max[WM_{ph}(x) - WM_p(x)] = 2^{1/p} \varepsilon^{1/2} \left(\sum_{i=1}^{m} \sum_{j=1}^{n} w_{1ij} + \sum_{i<r} w_{2ir} \right),$$

as given by Love and Yeong (1981). Morris and Verdini (1979) showed that the iterative sequence given by (2.11) generalizes to the multi-facility case for $r = 1,\ldots,m$ and $s = 1,2$ as:

$$x_{rs}^{k+1} = (A + B)/(C + D) \tag{2.25}$$

where

$$A = \sum_{i=1}^{m} \frac{w_{2ri}\, x_{is}^{k}}{\{[(x_{r1}^{k} - x_{i1}^{k})^2 + \varepsilon]^{p/2} + [(x_{r2}^{k} - x_{i2}^{k})^2 + \varepsilon]^{p/2}\}^{(p-1)/p}[(x_{rs}^{k} - x_{is}^{k})^2 + \varepsilon]^{(2-p)/2}} ,$$

$$B = \sum_{j=1}^{n} \frac{w_{1rj}\, a_{js}}{\{[(x_{r1}^{k} - a_{j1})^2 + \varepsilon]^{p/2} + [(x_{r2}^{k} - a_{j2})^2 + \varepsilon]^{p/2}\}^{(p-1)/p}[(x_{rs}^{k} - a_{js})^2 + \varepsilon]^{(2-p)/2}} ,$$

$$C = \sum_{i=1}^{m} \frac{w_{2ri}}{\{[(x_{r1}^{k} - x_{i1}^{k})^2 + \varepsilon]^{p/2} + [(x_{r2}^{k} - x_{i2}^{k})^2 + \varepsilon]^{p/2}\}^{(p-1)/p}[(x_{rs}^{k} - x_{is}^{k})^2 + \varepsilon]^{(2-p)/2}} ,$$

and

$$D = \sum_{j=1}^{n} \frac{w_{1rj}}{\{[(x_{r1}^{k} - a_{j1})^2 + \varepsilon]^{p/2} + [(x_{r2}^{k} - a_{j2})^2 + \varepsilon]^{p/2}\}^{(p-1)/p}[(x_{rs}^{k} - a_{js})^2 + \varepsilon]^{(2-p)/2}} .$$

Ostresh (1978b) has proved that for $p=2$ this generalized Weiszfeld sequence is strictly decreasing and Morris (1981) has given a convergence proof for $1 \le p \le 2$.

Dual formulations for the unconstrained and linearly constrained ℓ_p distance location model and the hyperbolic approximating function model were provided by Love (1974). The dual for the multi-facility location model was extended by Juel and Love (1981) to include generalized distances and linear constraints. For the ℓ_p distance primal (2.18), the corresponding dual, as given by Love (1974), is

$$\text{maximize } D_p(U) = \text{maximize} - \sum_{i=1}^{m} \sum_{j=1}^{n} a_j{}' \, U_{1ij} \qquad (2.26)$$

$$\text{subject to} \quad -\sum_{r=1}^{i-1} U_{2ri} + \sum_{r=i+1}^{m} U_{2ir} + \sum_{j=1}^{n} U_{1ij} = \vec{0} \quad i = 1, \ldots, m$$

$$|U_{1ij}|_q = w_{1ij} \qquad i = 1, \ldots, m \,;\; j = 1, \ldots, n$$

$$|U_{2ir}|_q = w_{2ir} \qquad i = 1, \ldots, m-1 \,;\; r = i+1, \ldots, m.$$

given that the derivatives $\partial WM(x)/\partial x_{ki}$ $k=1,2$ and $i=1,\ldots,m$ exist and that $1/p+1/q=1$, $|\cdot|_q$ is the ℓ_q norm, and $U_{1ij}{}' = (u_{1ij}, v_{1ij})$ and $U_{2ir}{}' = (u_{2ir}, v_{2ir})$ are dual variables.

Love (1974) also showed that the hyperbolic approximating function

$$\min WMH_p(x) = \sum_{i=1}^{m}\sum_{j=1}^{n} w_{1ij}[|x_{i1} - a_{j1}|^p + |x_{i2} - a_{j2}|^p + \varepsilon^p]^{1/p}$$

$$+ \sum_{i=1}^{m-1}\sum_{r=i+1}^{m} w_{2ir}[|x_{i1} - x_{r1}|^p + |x_{i2} - x_{r2}|^p + \varepsilon^p]^{1/p} . \qquad (2.27)$$

has a corresponding dual

$$\max D_{ph}(U, Z) = \max - \sum_{i=1}^{m}\sum_{j=1}^{n} a_j U_{1ij} - \sum_{i=1}^{m}\sum_{j=1}^{n} \varepsilon Z_{1ij} \qquad (2.28)$$

subject to $-\sum_{r=1}^{i-1} U_{2ri} + \sum_{r=i+1}^{m} U_{2ir} + \sum_{j=1}^{n} U_{1ij} = \vec{0} \quad i = 1, ..., m$

$[(\|U_{1ij}\|_q)^q + |Z_{1ij}|^q]^{1/p} = w_{1ij}^{1/(p-1)} \quad i = 1, ..., m ; j = 1, ..., n$

$[(\|U_{2ir}\|_q)^q + |Z_{1ir}|^q]^{1/p} = w_{2ir}^{1/(p-1)} \quad i = 1, ..., m-1 ; r = i+1, ..., m.$

where $Z' = (Z_{111}, \ldots Z_{11n}, \ldots, Z_{1mn})$ is a vector of additional variables and $\|\cdot\|_q$ is the ℓ_q norm.

Love (1974) has shown that

$$\lim_{\varepsilon \to 0} D_{ph}(U, Z) = D_p(U),$$

so that the dual of the limiting case when $\varepsilon \to 0$ is given by

$$\max \; -\sum \sum a_j' U_{1ij} \qquad (2.29)$$

subject to $-\sum_{r=1}^{i-1} U_{2ri} + \sum_{r=i+1}^{m} U_{2ir} + \sum_{j=1}^{n} U_{1ij} = \vec{0} \quad i = 1, ..., m$

$|U_{1ij}|_q = w_{1ij}$ or 0, $\quad i = 1, ..., m ; j = 1, ..., n$

$|U_{2ir}|_q = w_{2ir}$ or 0 $\quad i = 1, ..., m-1 ; r = i+1, ..., m.$

If the solution $U_{2ir} = 0$ is obtained, then $x_i = x_r$ in the optimal primal solution, and if $U_{1ij} = 0$ then $x_i = a_j$. The converse is also true.

2.4 Bounding Methods for Multi-Facility Location Models

Two of the bounding methods for single facility location models have been applied to multi-facility ℓ_p distance location models. The extension of the Love-Yeong bound (2.14) and Juel bound (2.15) to include the multi-facility case (2.18) was given by Love and Yeong (1981) as

$$WM_p(x^*) \geq WM_p(x^k) - \bar{\bar{\sigma}}(x^k)|\nabla WM_p(x^k)|, \tag{2.30}$$

and

$$WM_p(x^*) \geq WM_p(x^k) - \nabla WM_p(x^k)' x^k + \min_{y \in \bar{\Omega}} \{\nabla WM_p(x^k)' y\}, \tag{2.31}$$

where

$\bar{\Omega} = \{s = (s_1, \ldots, s_m) | s_i \in \Omega, i = 1, \ldots, m\}$, Ω is the convex hull of the a_j, $j = 1, \ldots, n$

$\bar{\bar{\sigma}}(x) = \max\{d(x, y) | y \in \bar{\Omega}\}$

and $x^{k'} = (x_{11}{}^k, x_{12}{}^k, \ldots, x_{m1}{}^k, x_{m2}{}^k)$ is a point generated by a computational procedure at the kth iteration. When an approximation function $WM_{ph}(x)$, as in (2.24), is utilized then the bounds can be calculated using

$$WM_p(x^*) \geq WM_{ph}(x^k) - \bar{\bar{\sigma}}|\nabla WM_{ph}(x^k)| - 2^{1/p}\varepsilon^{1/2}\left(\sum_{i=1}^{m}\sum_{j=1}^{n} w_{1ij} + \sum_{i=1}^{m-1}\sum_{r=i+1}^{m} w_{2ir}\right), \tag{2.32}$$

and

$$WM_p(x^*) \geq WM_{ph}(x^k) + \min_{y \in \bar{\Omega}}\{\nabla WM_{ph}(x^k)'[y - x^k]\} - 2^{1/p}\varepsilon^{1/2}\left(\sum_{i=1}^{m}\sum_{j=1}^{n} w_{1ij} + \sum_{i=1}^{m-1}\sum_{r=i+1}^{m} w_{2ir}\right). \tag{2.33}$$

No generalization of the single facility Drezner bound (2.16) to include an ℓ_p distance function and/or the multi-facility problem has been published to date. Also, the dual has not been used to bound the multi-facility primal since a method for obtaining a feasible dual solution from the current primal solution is required.

CHAPTER 3

A COMPARISON OF BOUNDING METHODS FOR SINGLE FACILITY LOCATION MODELS

In this chapter, the dual and Drezner bounds will be developed for the single facility location model using ℓ_p distances. A projection matrix technique will be used to generate dual-feasible solutions from a given primal solution, so that the dual can be used as a lower bound to the primal problem. Results will be given for a computational comparison of the Juel, Love and Yeong, Drezner and dual bounds, followed by conclusions regarding the use of these bounding methods.

3.1 A Lower Bound Obtained from the Dual

Recall that the single facility location problem with ℓ_p distances is given by

$$\text{minimize} \sum_{j=1}^{n} w_j \ell_p(x, a_j) \tag{3.1}$$

and the corresponding dual by

$$\text{maximize} \; -\sum_{j=1}^{n} a_j' U_j \tag{3.2a}$$

$$\text{subject to} \quad \sum_{j=1}^{n} U_j = \vec{0} \tag{3.2b}$$

$$|U_j|_q \le w_j \tag{3.2c}$$

for $j = 1, \ldots, n$ and $1/p + 1/q = 1$.

A lower bound can be determined by obtaining a feasible solution to the dual in the following manner. Given an iterative computation procedure and the current solution $x^{k'} = (x_1^k, x_2^k)$ at iteration k, the direction vectors U_j are estimated using

$U_j^{k'} = (u_j^k, v_j^k) = (x_1^k - a_{j1}, x_2^k - a_{j2})$ for $j = 1,\ldots,n$. Condition (2.13), which gives the relationships between the dual and primal variables at optimality for the single facility Euclidean distance model, provided the motivation for this method of obtaining an initial estimate of the dual variables. The non-normalized U_j^k direction vectors are used to obtain vector $U^{k'} = (u_1^k, \ldots, u_n^k, v_1^k, \ldots, v_n^k)$. This initial dual solution need not satisfy constraints (3.2b) and (3.2c). The U_j^k vectors are adjusted so that constraints (3.2b) and (3.2c) are satisfied, thus generating a dual feasible solution at the kth iteration. Each U_j^k is adjusted so that the equality holds true in (3.2c), using $c_j = w_j/|U_j^k|_q$ to obtain the adjusted vectors $\overline{U}_j^k = c_j U_j^k$, for $j = 1,2,\ldots,n$. Although the norm constraint is satisfied, the adjusted U_j^k vectors may not satisfy the linear equality constraints. In order to satisfy (3.2b), the vector $\overline{U}^{k'} = (c_1 u_1^k, \ldots, c_n v_n^k)$ is projected into the intersection of the two planes determined by (3.2b), using the projection matrix P as given by Rosen (1960).

P is defined as $P = I - A_g'(A_g A_g')^{-1} A_g$, where A_g is the matrix whose rows are the coefficients of the variables in the dual using constraints (3.2b). The constraints in (3.2b) can be expressed as

$$A_g U = \vec{0} \tag{3.3}$$

where A_g is a $2 \times 2n$ matrix given by

$$A_g = \begin{bmatrix} 1\,1 & \ldots & 1 & 0\,0 & \ldots & 0 \\ 0\,0 & \ldots & 0 & 1\,1 & \ldots & 1 \end{bmatrix}.$$

In general, P will be of the form

$$P = \begin{bmatrix} R & O \\ O & R \end{bmatrix},$$

where submatrix R is an $n \times n$ matrix with elements

$$r_{ij} = \begin{cases} (n-1)/n & i = j \\ -1/n & i \neq j \end{cases} \quad i,j = 1, 2, \ldots, n.$$

The derivation of this result can be found in Appendix A.

The projected vector $\bar{\bar{U}}^k = P \cdot \bar{U}^k$ now satisfies (3.2b). However, some of the $\bar{\bar{U}}_j^k$ vectors may no longer satisfy (3.2c), or all of the $\bar{\bar{U}}_j^k$ may satisfy the strict inequality $|\bar{\bar{U}}_j^k|_q < w_j$, for $j = 1,\ldots,n$. A final adjustment can be made to $\bar{\bar{U}}^k$ so that at least one equality condition in (3.2c) holds true. Calculate $\bar{c}_j = w_j / |\bar{\bar{U}}_j^k|$ and $c = \min_j \{\bar{c}_j\}$; $c\bar{\bar{U}}^k$ will then provide a feasible solution to the dual at iteration k. For the case where all $|\bar{\bar{U}}_j^k|_q < w_j$, for $j = 1,\ldots,n$, $\bar{\bar{U}}^k$ will provide a feasible solution but $c\bar{\bar{U}}^k$ provides a better objective function value since each $\bar{c}_j > 1$, for $j = 1,\ldots,n$, and hence $c > 1$. A lower bound to the primal objective function can be calculated using the dual objective function value

$$-\sum_{j=1}^{n} c\, a_j' \, \bar{\bar{U}}_j^k .$$

It would require less computation time to use the projection matrix on the 2n components of U^k to obtain $\hat{U}^k = PU^k$ and then adjust the resulting vector by a constant

$$c = \min_j \{w_j / |\hat{U}_j^k|_q\}.$$

However, computational experience has shown that the solutions obtained are extremely poor. This occurs because any adjustment must be applied simultaneously to each of the 2n components of PU^k so that constraint (3.2b) is still satisfied. Furthermore, the adjustment made is based on the worst violation of (3.2c).

3.2 The Drezner Bound for Single Facility ℓ_p Distance Problems

Drezner (1984) has shown that for Euclidean distance problems

$$W_2(x^*) \geq \min_{x_1,x_2} \sum_{j=1}^{n} [w_j / d_2(a_j, x^k)] \cdot [|x_1 - a_{j1}||a_{j1} - x_1^k| + |x_2 - a_{j2}||a_{j2} - x_2^k|],$$

where $x^{k\prime} = (x_1^k, x_2^k)$. At each iteration of a solution process this bound is evaluated by solving a rectilinear distance problem. The Drezner bound is obtained by solving the following problem, given by (2.16):

$$\min_{x_1} \sum_{j=1}^{n} w_j'|x_1 - a_{j1}| + \min_{x_2} \sum_{j=1}^{n} w_j''|x_2 - a_{j2}|$$

where the "created" weights w_j' and w_j'' are defined as

$$w_j' = [w_j / d_2(a_j, x^k)] \cdot |a_{j1} - x_1^k| ,$$

and

$$w_j'' = [w_j / d_2(a_j, x^k)] \cdot |a_{j2} - x_2^k| \quad \text{for } j = 1, \ldots, n.$$

The single facility Drezner bound can be generalized to ℓ_p distances in the following manner, using the Hölder inequality which is given by:

$$\left| \sum_{i=1}^{N} b_i \cdot c_i \right| \le \left(\sum_{i=1}^{N} |b_i|^p \right)^{1/p} \left(\sum_{i=1}^{N} |c_i|^q \right)^{1/q} ,$$

where $\{b_N\}$ and $\{c_N\}$ are real sequences, $p>1$ and $1/p + 1/q = 1$. Let $b_i = |x_i - a_{ji}|$ and $c_i = |a_{ji} - x_i^k|$ for $i = 1,2$ and $j = 1,\ldots,n$. Then

$$\sum_{i=1}^{2} |x_i - a_{ji}||a_{ji} - x_i^k| \le \{[|x_1 - a_{j1}|^p + |x_2 - a_{j2}|^p]^{1/p}\}\{[|a_{j1} - x_1^k|^q + |a_{j2} - x_2^k|^q]^{1/q}\} .$$

This can be written as

$$[|x_1 - a_{j1}|^p + |x_2 - a_{j2}|^p]^{1/p} \ge \frac{|x_1 - a_{j1}||a_{j1} - x_1^k| + |x_2 - a_{j2}||a_{j2} - x_2^k|}{[|a_{j1} - x_1^k|^q + |a_{j2} - x_2^k|^q]^{1/q}} ,$$

or

$$W_p(x) = \sum_{j=1}^{n} w_j \ell_p(x, a_j) \ge \sum_{j=1}^{n} w_j'|x_1 - a_{j1}| + \sum_{j=1}^{n} w_j''|x_2 - a_{j2}| ,$$

where

$$w_j' = w_j |a_{j1} - x_1^k| \,/\, [|a_{j1} - x_1^k|^q + |a_{j2} - x_2^k|^q]^{1/q}$$

and

$$w_j'' = w_j |a_{j2} - x_2^k| \,/\, [|a_{j1} - x_1^k|^q + |a_{j2} - x_2^k|^q]^{1/q} .$$

Since

$$W_p(x) \ge \sum_{j=1}^{n} w_j'|x_1 - a_{j1}| + \sum_{j=1}^{n} w_j''|x_2 - a_{j2}| ,$$

then

$$W_p(x^*) = \min_x W_p(x) \geq \min_{x_1,x_2} \left[\sum_{j=1}^{n} w_j'|x_1 - a_{j1}| + \sum_{j=1}^{n} w_j''|x_2 - a_{j2}| \right],$$

or

$$W_p(x^*) \geq \min_{x_1} \sum_{j=1}^{n} w_j'|x_1 - a_{j1}| + \min_{x_2} \sum_{j=1}^{n} w_j''|x_2 - a_{j2}| .$$

This result can be used to generate the rectangular bound for the single facility ℓ_p distance model. At each iteration of the solution process, a single facility rectilinear problem

$$\text{minimize } R(x) = \min_{x_1} \sum_{j=1}^{n} w_j'|x_1 - a_{j1}| + \min_{x_2} \sum_{j=1}^{n} w_j''|x_2 - a_{j2}|$$

is constructed, using the fixed facilities a_j, weights w_j and current solution x^k to calculate w_j' and w_j'' for $j = 1,\ldots,n$. The two optimization problems can be solved independently and an optimal solution x_R^* can be used to calculate $R(x_R^*)$, which is the lower bound on $W_p(x^*)$ at the $k^{\underline{th}}$ iteration.

While it may appear that adding another optimization problem and solving it has increased the work required to find a lower bound, this procedure has several advantages. The rectilinear problem is separable and each part can be solved rapidly. Also, it is not necessary to find the hull points which are used in both the Love-Yeong and Juel bounds.

In order to test the effectiveness and efficiency of the four bounding methods, several single facility test problems were randomly generated. Comparisons and observations are presented in section 3.3 for these test runs.

3.3 Bound Comparisons for the Single Facility ℓ_p Distance Model

Four programs were written to incorporate the generalized Weiszfeld procedure with each of the lower bound methods. At each iteration of the solution procedure the bound was calculated and tested against the current solution. By entering a proportionate error difference e, a stopping rule calculated as

$$|(\text{bound value} - \text{objective function value})|/(\text{objective function value}) \leq e$$

was used to terminate the process. In all sample runs e = 0.01, and the initial starting solution used in the Weiszfeld procedure was (0,0). Samples of size n = 6, 10, 15, 20 existing facilities locations were randomly generated. In the first set of runs a unit value was assigned to the w_j weights, and ℓ_p distances were calculated for p = 2, 1.8, 1.6, 1.4, 1.2.

For a given value of n and p, a series of test runs was made using each of the four programs. For each bounding method the iterations were terminated using the stopping rule with e = 0.01. The number of iterations required, the objective function value, the value of the bound, and the CPU compilation and execution times were recorded in each case. The bound values are displayed in Table 3.1, where B1, B2, B3, B4 refer to the Love-Yeong, Juel, Drezner and dual bounds, respectively. The average computation times for various sample runs are in Tables 3.3 and 3.4.

From Table 3.1, it is quite evident that for p = 2 the Drezner bound provided superior results. However, it is also quite evident that for p<2 the Drezner and dual bounds may not converge. For example, with n = 6 and p = 1.6 the Drezner bound did not reach the 1% error difference in 25 iterations. The closest it came was at iteration 9 when the error difference was 1.06%. At successive iterations after the ninth, the percentage error difference increased in value. To further study this phenomenon, a second set of test samples was created using weights randomly selected from the range [1,10]. For each n and p combination a series of four runs was made and the data were recorded. Then a new set of weights was generated for the next n and p combination. The results for these test runs are shown in Table 3.2.

The second series of test runs provided data that supported the earlier observations. The instability of the Drezner bound makes its use impractical except for models with p equal to two. The apparent convergence of the Drezner bound for p=1.8 was due to the magnitude of the error difference value (e = 0.01). If a smaller value had been used for e, say e = 0.00001, the Drezner bound would diverge before that error difference value could be attained. However, the test results show that for the Euclidean distance model the Drezner bound was always superior to the Juel bound. Also, the Drezner bound is computationally more efficient than the other three bounds. Average compilation and execution times for Euclidean distances are shown in Tables 3.3 and 3.4 for a CDC Cyber 170/730.

n		6			10			15			20		
p		No. of Iter	Obj Funct	Bound	No. of Iter	Obj Funct	Bound	No. of Iter	Obj Funct	Bound	No. of Iter	Obj Funct	Bound
2	B1	12	117.72	116.81	6	187.34	185.53	7	290.37	288.60	8	389.25	386.03
	B2	7	117.74	116.86	6	187.34	186.11	7	290.37	288.83	8	389.25	386.64
	B3	4	117.83	117.01	4	187.39	186.40	4	290.55	288.54	4	389.97	386.07
	B4	6	117.76	116.93	6	187.34	186.34	6	290.38	287.96	7	389.27	385.50
1.8	B1	11	120.94	119.91	7	193.48	191.59	8	297.86	295.31	9	399.11	395.58
	B2	5	120.97	119.80	7	193.48	192.09	8	297.86	295.55	9	399.11	396.11
	B3	4	121.00	120.02	4	193.59	191.80	6	297.92	295.62	6	399.32	395.68
	B4	6	120.96	119.95	7	193.48	191.74	11	297.85	295.11	25*	399.10	393.18
1.6	B1	7	125.36	124.35	9	201.69	200.03	10	308.02	305.27	11	412.40	409.10
	B2	4	125.38	124.35	8	201.69	199.73	10	308.02	305.49	11	412.40	409.52
	B3	25*	125.36	124.03	9	201.69	199.68	25*	308.01	304.36	25*	412.39	407.53
	B4	25*	125.36	123.87	25*	201.69	198.14	25*	308.01	300.23	25*	412.39	397.89
1.4	B1	19	131.64	130.41	12	213.10	211.29	13	322.26	319.61	16	431.13	427.22
	B2	18	131.64	130.34	11	213.10	211.36	12	322.26	319.05	16	431.13	427.87
	B3	25*	131.64	127.72	25*	213.09	207.90	25*	322.25	311.91	25*	431.12	419.71
	B4	25*	131.64	127.21	25*	213.09	204.42	25*	322.25	303.07	25*	431.12	407.17
1.2	B1	25*	141.01	139.50	20	229.78	227.74	13	343.08	340.89	25	459.07	454.99
	B2	25	141.01	139.61	18	229.79	227.50	13	343.08	341.11	24	459.08	455.00
	B3	25*	141.01	130.54	25*	229.77	216.18	25*	343.08	322.47	25*	459.07	431.07
	B4	25*	141.01	128.30	25*	229.77	208.34	25*	343.08	306.76	25*	459.07	414.89

*did not converge to within 1% error difference in 25 iterations.

Table 3.1: Lower Bound Data for Single Facility Samples, $w_j = 1$

n		6			10			15			20		
p		No. of Iter	Obj Funct	Bound	No. of Iter	Obj Funct	Bound	No. of Iter	Obj Funct	Bound	No. of Iter	Obj Funct	Bound
2	B1	18	728.52	721.48	9	1095.51	1086.70	9	1591.44	1579.82	10	2089.37	2071.77
	B2	17	728.55	721.63	8	1095.57	1084.87	8	1591.51	1581.93	9	2089.47	2074.56
	B3	6	730.19	722.95	6	1096.05	1086.91	7	1591.73	1582.03	6	2089.06	2078.33
	B4	25*	728.44	718.42	12	1095.47	1087.42	11	1591.41	1579.65	12	2089.31	2069.41
1.8	B1	21	745.67	738.86	10	1128.55	1117.71	10	1632.40	1616.50	12	2146.56	2130.53
	B2	20	745.70	739.18	10	1128.55	1119.78	9	1632.50	1620.85	10	2146.79	2128.17
	B3	10	747.01	739.60	13	1128.48	1117.31	8	1632.76	1622.70	7	2149.51	2133.78
	B4	25*	745.61	698.18	25*	1128.46	1073.78	10	1632.40	1619.47	12	2146.56	2130.01
1.6	B1	25	768.80	761.66	12	1171.53	1160.11	13	1687.84	1674.63	14	2223.67	2203.53
	B2	24	768.84	762.15	12	1171.53	1162.09	10	1688.21	1671.76	12	2223.93	2203.68
	B3	25*	768.80	757.98	25*	1171.40	1137.90	10	1688.21	1672.04	10	2224.91	2208.01
	B4	25*	768.80	695.85	25*	1171.40	1043.85	25*	1687.79	1664.51	25*	2223.58	2196.76
1.4	B1	25*	803.25	769.26	16	1228.62	1216.82	17	1766.00	1749.97	19	2331.54	2311.34
	B2	25*	803.25	774.02	15	1228.70	1216.64	13	1766.50	1750.05	17	2331.72	2313.36
	B3	25*	803.25	773.99	25*	1228.43	1151.36	25*	1765.90	1716.65	25*	2331.49	2278.06
	B4	25*	803.25	657.98	25*	1228.43	1028.75	25*	1765.90	1702.23	25*	2331.49	2273.57
1.2	B1	24	848.14	842.44	25	1308.03	1295.75	25*	1880.68	1855.33	24	2490.78	2468.99
	B2	24	848.14	843.12	25	1308.03	1295.81	19	1881.73	1864.06	23	2490.81	2467.26
	B3	25*	848.14	781.65	25*	1308.03	1187.83	25*	1180.68	1734.62	25*	2490.78	2357.87
	B4	25*	848.14	718.70	25*	1308.03	1140.52	25*	1880.68	1710.66	25*	2490.78	2353.82

*did not converge to within 1% error difference in 25 iterations.

Table 3.2: Lower Bound Data for Single Facility Samples, $w_j \in [1,10]$

Love-Yeong	Juel	Drezner
1.961	2.028	1.992

Table 3.3: Average Compilation Time (secs) for Program and Bound, p = 2

n	6	10	15	20
Love-Yeong	0.545	0.699	0.587	0.524
Juel	0.542	0.487	0.628	0.544
Drezner	0.358	0.421	0.415	0.414

Table 3.4: Average Execution Time (secs) for Solution and Bound, p = 2

From Table 3.1 where all weights have a unit value, the Juel and Love-Yeong bounds provided better bound values than the dual as p decreased in value when n was fixed. Also, for p fixed, the Juel and Love-Yeong bounds provided better bound values than the dual as n increased, except for p = 2. In Table 3.2 where the weights are from the interval [1,10], the Juel and Love-Yeong bounding methods provided better bound values than the dual for 19 out of 20 n and p combinations.

In the following section, it will be proven that the Drezner bound is superior to the Juel bound for the single facility Euclidean distance model.

3.4 Comparison of the Drezner and Juel Bounds for Euclidean Distances

The bound values in Tables 3.1 and 3.2 confirm the theoretical results given by Juel (1984) and Elizinga and Hearn (1984) which establish that the Juel bound gives as good or better results than the Love and Yeong bound. A comparison of the Drezner and Juel bounds will now be made for the Euclidean distances location model.

The Juel bound at iteration k is given by

$$J(x^k) = W_2(x^k) - \nabla W_2(x^k)' \, x^k + \min_{y \in \Omega} [\nabla W_2(x^k)' \, y] .$$

For Euclidean distances,

$$W_2(x^k) = \sum_{j=1}^{n} w_j d_2(x^k, a_j), \text{ where } d_2(x^k, a_j) = \left[(x_1^k - a_{j1})^2 + (x_2^k - a_{j2})^2 \right]^{1/2}$$

and

$$\nabla W_2(x^k)' = \left(\sum_{j=1}^{n} w_j (x_1^k - a_{j1}) / d_2(x^k, a_j), \ \sum_{j=1}^{n} w_j (x_2^k - a_{j2}) / d_2(x^k, a_j) \right) .$$

Substituting in the Juel bound gives

$$J(x^k) = \sum_{j=1}^{n} w_j d_2(x^k, a_j) - \sum_{j=1}^{n} w_j (x_1^k - a_{j1}) x_1^k / d_2(x^k, a_j)$$

$$- \sum_{j=1}^{n} w_j (x_2^k - a_{j2}) x_2^k / d_2(x^k, a_j) + \min_{(y_1, y_2) \in \Omega} \left[\sum_{j=1}^{n} w_j (x_1^k - a_{j1}) y_1 / d_2(x^k, a_j) \right.$$

$$\left. + \sum_{j=1}^{n} w_j (x_2^k - a_{j2}) y_2 / d_2(x^k, a_j) \right] .$$

The Drezner bound at the k^{th} iteration is specified by minimizing

$$R(x) = \sum_{j=1}^{n} w_j' |x_1 - a_{j1}| + \sum_{j=1}^{n} w_j'' |x_2 - a_{j2}|$$

where

$$w_j' = w_j |a_{j1} - x_1^k| / d_2(x^k, a_j) \text{ and } w_j'' = w_j |a_{j2} - x_2^k| / d_2(x^k, a_j) .$$

Let $x^*{}_R{}^k = (x^*{}_{R1}{}^k, x^*{}_{R2}{}^k)$ represent an optimal solution obtained by minimizing R(x). Define $S_1 = \{a_{11}, a_{21}, \ldots, a_{n1}\}$ and $S_2 = \{a_{12}, a_{22}, \ldots, a_{n2}\}$ as the sets of first and second coordinates respectively from the existing facility locations a_j, for $j=1,\ldots,n$. An optimal solution to the rectilinear distance problem must be an element from the set $S_1 \times S_2$. Wendell and Hurter (1973) have shown that $x^*{}_R{}^k \in (S_1 \times S_2) \cap \Omega$ for at least one optimal solution.

It will now be shown that at iteration k the Drezner bound is at least as good as the Juel bound.

<u>THEOREM 1</u>

For $p = 2$, $R(x^*{}_R{}^k) \geq J(x^k)$.

Proof:

$$R(x^*{}_R^k) = \sum_{t=1}^{2}\sum_{j=1}^{n} w_j |a_{jt} - x_t^k| \, |x^*{}_{Rt}^k - a_{jt}| / d_2(x^k, a_j) \, ,$$

$$= \sum_{t=1}^{2}\sum_{j=1}^{n} w_j |x_t^k - a_{jt}| \, |(x_t^k - a_{jt}) - (x_t^k - x_{Rt}^{*k})| / d_2(x^k, a_j) \, .$$

$$\geq \sum_{t=1}^{2}\sum_{j=1}^{n} w_j (x_t^k - a_{jt})[(x_t^k - a_{jt}) - (x_t^k - x_{Rt}^{*k})] / d_2(x^k, a_j)$$

$$= \sum_{t=1}^{2}\sum_{j=1}^{n} w_j (x_t^k - a_{jt})^2 / d_2(x^k, a_j) - \sum_{t=1}^{2}\sum_{j=1}^{n} w_j (x_t^k - a_{jt}) x_t^k / d_2(x^k, a_j)$$

$$+ \sum_{t=1}^{2}\sum_{j=1}^{n} w_j (x_t^k - a_{jt}) x_{Rt}^{*k} / d_2(x^k, a_j)$$

$$= W_2(x^k) - \nabla W_2(x^k)' x^k + \nabla W_2(x^k)' x_R^{*k}$$

Thus $R(x^*{}_R{}^k) \geq J(x^k)$ if

$$\sum_{t=1}^{2}\sum_{j=1}^{n} w_j (x_t^k - a_{jt}) x_{Rt}^{*k} / d_2(x^k, a_j) \geq \min_{y \in \Omega} \left[\sum_{t=1}^{2}\sum_{j=1}^{n} w_j (x_t^k - a_{jt}) y_t / d_2(x^k, a_j) \right] .$$

For $x^*_R{}^k$ lying in Ω, the convex (Euclidean) hull, then $R(x^*_R{}^k) \geq J(x^k)$ with equality holding only when $x^*_R{}^k = y^* \in \Omega$. If multiple optimal solutions exist, at least one optimal solution must lie in Ω. Let $x^*_R{}^k$ and $z^*_R{}^k$ represent optimal solutions where $x^*_R{}^k \in \Omega$ and $z^*_R{}^k \notin \Omega$. Then, $R(z^*_R{}^k) = R(x^*_R{}^k) \geq J(x^k)$. Thus, therorem 1 holds true for $p=2$ and any optimal solution.

This establishes that the rectangular bound can be used without any trepidation about its convergence with Euclidean distances since $R(x^*_R{}^k)$ will converge to $W_2(x^*)$. Considerable computation time can be saved using this bound, as it required fewer iterations to reach the same level of percentage error difference as the other two bounds.

Theorem 1 in conjunction with the results of Juel (1984) and Elzinga and Hearn (1984) establishes that $B3 \geq B2 \geq B1$ for the Euclidean distance single facility model. In section 3.3, Tables 3.1 and 3.2 revealed that the Juel and Love-Yeong bounding methods provided better results than the dual in a majority of the examples. For $p = 2$, the dual outperformed the Love-Yeong bound only when the weights had unit values. For $p < 2$, the Love-Yeong bound provided superior results to the dual in 29 out of 32 test problems. The nature of the dual makes it difficult to provide a theoretical comparison with the other three bounds. In the next section, the dual and the Love and Yeong bound will be compared using numerous examples for $p = 2$.

3.5 Comparison of the Dual (B4) and Love-Yeong Bound (B1)

Several hundred test examples were constructed and run on a CDC Cyber 170/730 computer in order to investigate the performance of bounds B1 and B4 with respect to the number of existing facilities, the locations of the existing facilities and the values for the positive weights. Computational experience has revealed that there are three factors which influence a comparison of bounds B1 and B4 for Euclidean distances: the number of existing facilities, an outlier among the existing facilities, and the range of values for the weights. It is the difference between the maximum and minimum values of the weights rather than the actual values of the weights which affects the relative performance of bounds B1 and B4. For the numerical examples, the positive weights w_j were chosen from a range of values $[r_1, r_2]$ where $r_1 \leq r_2$. The interval $[r_1, r_2]$ could be mapped onto [1,w] where $w = r_2/r_1$. When w was unity, or very close to unity, the dual provided a better bound than B1 at each iteration. As the number of facilities increased, the dual's performance was diminished, and as w increased, the dual no longer provided a better bound than B1. Figure 3.1 illustrates the relationship between n and w that was observed from running sample problems with n=5, 10, 20, 30, 40, 50, 75 and 100 for various values of w. For each value of n, a value of w was selected and the w_j weights were randomly generated over the interval [1,w]. Twenty different values for w were used for each value of n, and the bounds B1 and B4 were compared for 25 iterations of the test problem. In most cases the dual provided a better bound for the first 3 or 4 iterations, then B1 provided a better bound. More than 160 test problems were used to plot the graph in Figure 3.1. Some test problems were run several times with the same weights and existing facility coordinates but with the weights assigned to different facilities on each run.

The effect of outliers was tested on many examples where each facility coordinate, except for the outlier, was generated randomly from the interval [0,50]. The weights were randomly generated from the interval [1,20]. Each example was run 9 times, using 3 outliers and 3 different outlier weights. The results are displayed in Table 3.5, which shows the better bound obtained at each iteration.

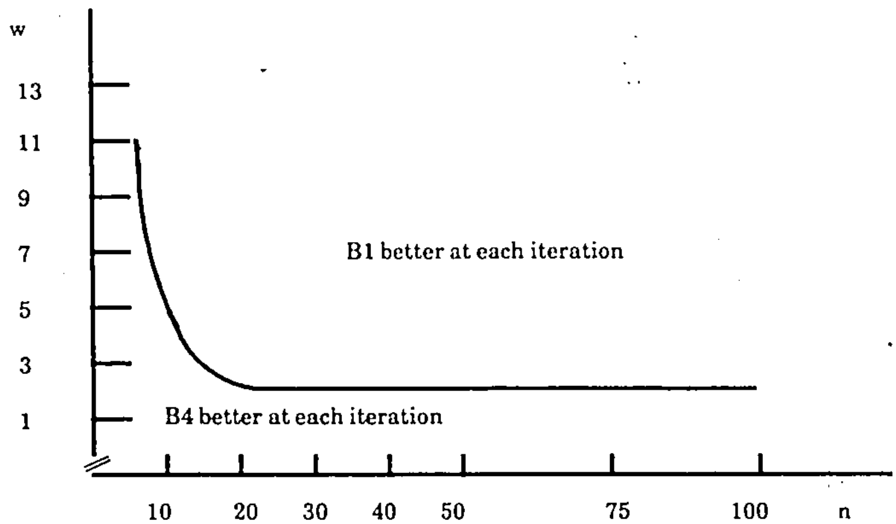

Figure 3.1: Dual (B4) and Love-Yeong (B1) Comparison

n	Outlier	Better Bound Obtained Outlier weight 3	11	19
10	(78, 93)	B4	B4	B4
	(126, 112)	B4	B4	B4
	(163, 186)	B4	B4	B4
20	(78, 93)	B4	B4	B4
	(126, 112)	B4	B4	B4
	(163, 186)	B4	B4	B4
30	(78, 93)	B1	B1	B1
	(126, 112)	B4	B1	B1
	(163, 186)	B4	B1	B1

Table 3.5: B1 and B4 Comparison with 1 Outlier.

For a small number of facilities, when there is a single outlier with a small weight attached to it, the dual will usually provide a better bound than B1. As the weight attached to the outlier increases and/or the number of facilities increases, the effect of the outlier is minimized. An outlier has the effect on B1 of increasing $\bar{\sigma}$ which has a detrimental effect on the bound value. The dual receives preferential treatment since the direction vector from the outlier will have either two large negative components or a large negative component and a smaller positive component which increases the value of the dual objective function. As the number of facilities increases, the effect on the dual is diluted. Increasing the weight associated with the outlier has the tendency to shift the solution towards the outlier, decreasing $\bar{\sigma}$ and the negative components of the outlier direction vector.

While the dual may provide a better bound than B1 in some special cases, this is attained at considerable cost in terms of CPU execution time. Table 3.6 gives the average CPU compile and execution times in seconds based on three runs for each example using weights randomly generated from the intervals [1,1], [1,2] and [1,10]. As the number of facilities increases, the proportion of CPU execution time required by the dual as compared with B1 increases. For n=30, 50 and 100 the ratio of CPU execution time is 2.9, 3.6 and 4.3 respectively.

	Weiszfeld Solution		Weiszfeld Solution and Bound B1		Weiszfeld Solution and Bound B4	
n	Compile	Execution	Compile	Execution	Compile	Execution
10	0.184	0.280	0.202	0.313	0.293	0.599
30	0.192	0.843	0.223	0.933	0.315	2.726
50	0.208	1.678	0.237	1.787	0.338	6.387

Table 3.6: CPU Timings (seconds) for 20 Iterations on CDC CYBER 170/730

If the dual could give a lower bound in fewer iterations than B1, then it may be possible to overcome the difference in CPU execution times. In order to compare the timing and number of iterations required to obtain equivalent solutions, the program was run until the percentage difference between the total cost function and the lower bound was less than a prescribed constant. The weights selected were unity so that the examples were biased in favor of the dual bound. The CPU timings are displayed in Table 3.7. The execution time needed to obtain the same percentage difference is much less for B1 than B4. Usually, if the dual does provide a better bound at each iteration, B1 is never lagging more than 2 or 3 iterations behind it.

n		% difference	No. of Iterations	Objective Function	Bound Value	Execution Time (secs)
50	B1	0.1	11	900.906	900.247	1.37
	B4	0.1	10	900.906	900.387	3.66
	B1	0.0001	23	900.906	900.906	1.93
	B4	0.0001	21	900.906	900.905	6.69
100	B1	0.001	13	12132.6	12132.5	4.43
	B4	0.001	12	12132.6	12132.5	14.59

Table 3.7: CPU Timings to Reach Equivalent Solutions

Both the dual and B1 can provide equivalent bounds for an iterative solution procedure. The examples in Table 3.8, one favouring B4 and the other favouring B1, illustrate that a wide discrepancy in bound values may exist over the first five iterations but very little difference is observable by the 25th iteration. However, when computation time is taken into consideration, B1 is generally superior to the dual bound.

n	Weights	Iteration	Objective Function	Bound B1	Bound B4
20	[1, 20] outlier with weight 3	1	15958.8	0	10517.0
		5	15619.3	13694.6	15045.3
		10	15617.6	15301.2	15551.9
		15	15617.5	15550.4	15603.3
		20	15617.5	15602.9	15614.4
		25	15617.5	15614.2	15616.8
30	[1, 80]	1	25364.9	20988.9	6144.0
		5	24941.0	24190.2	20856.5
		10	24932.4	24867.7	24524.7
		15	24932.3	24926.9	24897.2
		20	24932.3	24931.9	24929.4
		25	24932.3	24932.3	24932.2

Table 3.8: Bound Comparisons at Selected Iterations

3.6 Conclusions

For p=2, the Drezner method provides a better bound at each iteration than the other three methods. In some cases, the Juel, Love and Yeong, and dual bounds required twice as many iterations to reach the same value as the Drezner bound. The computational savings achieved by using the Drezner bound with an iterative solution technique could be considerable.

Another advantage of the Drezner bound was observed when a series of test problems with 30, 40 and 50 existing facilities were randomly generated with coordinates from the interval [1,50], an outlier at (30,78), and weights equal to 1. The bound results for B1, B2, B3 and B4 are displayed in Table 3.9. For the situation in Table 3.9 with 40 existing facilities, the optimal solution was very close to an existing facility. This slowness of convergence of bounds B1, B2 and B4 has always been observed in test problems where the optimal solution was very close to an existing facility. The Drezner bound has never been affected by this situation.

n		B1	B2	B3	B4
	Iter. No.	10	10	5	8
30	Bound	595.6	595.6	595.4	595.0
	Obj. Fn.	600.0	600.0	600.6	600.1
	Iter. No.	25*	25*	4	25*
40	Bound	771.5	784.8	795.9	787.4
	Obj. Fn.	797.1	797.1	799.0	797.1
	Iter. No.	8	7	4	8
50	Bound	978.2	977.9	981.6	978.2
	Obj. Fn.	984.8	984.8	988.0	984.8

* Did not converge to within e = 1% in 25 iterations

Table 3.9: Comparison of Bounds for Weights = 1

A final comparison of Drezner's bound (B3) and the dual bound (B4) using the four examples from Wendell and Peterson (1984) is given in Table 3.10. The solution procedure was terminated when e reached 1% or less. The Wendell and Peterson examples are of the type that favour the performance of the dual over B1 since they have small numbers of fixed points, an outlier, and uniform weights. However, as shown in Table 3.10, the Drezner bound is clearly superior to the dual bound in each case.

When $1 < p < 2$, the Juel bound provides the best bound from among the four methods. Both the Drezner and dual bounds may experience convergence problems when $p<2$. The Juel bound is always as good or better than the Love and Yeong bound, but computational experience reveals that in most instances the Love and Yeong bound is never lagging any more than 1 or 2 iterations behind the Juel bound.

Example	Drezner Bound B3 Number of Iterations	Dual Bound B4 Number of Iterations
1	3	7
2	1	6
3	4	10
4	1	25*

* Did not converge to within e = 1% in 25 iterations.

Table 3.10: Comparison of Drezner and Dual Bounds.

CHAPTER 4

A COMPARISON OF BOUNDING METHODS FOR MULTI-FACILITY LOCATION MODELS

The format for this chapter will be similar to the one used in Chapter 3. The dual and Drezner bounds will be developed for the multi-facility ℓ_p distance models. Dual feasible solutions will be constructed from a given primal solution using a projection matrix technique. The proof that the Drezner bound is as good or better than the Juel bound for the single facility Euclidean distance problem will be extended to include the multi-facility Euclidean distance model. A computational comparison of the dual, and the Love and Yeong bounds will be given as well as conclusions regarding the usage of these bounding methods.

4.1 A Lower Bound Obtained from the Dual

The multi-facility ℓ_p distance location problem is given as

$$\text{minimize } W M_p(x) = \sum_{i=1}^{m} \sum_{j=1}^{n} w_{1ij} [|x_{i1} - a_{j1}|^p + |x_{i2} - a_{j2}|^p]^{1/p} + \sum_{i<r} w_{2ir} [|x_{i1} - x_{r1}|^p + |x_{i2} - x_{r2}|^p]^{1/p}, \tag{3.1}$$

and the corresponding dual is given by

$$\text{maximize} \quad -\sum_{i=1}^{m}\sum_{j=1}^{n} a_j' U_{1ij} \tag{4.2a}$$

$$\text{subject to} \quad -\sum_{r=1}^{i-1} U_{2ri} + \sum_{r=i+1}^{m} U_{2ir} + \sum_{j=1}^{n} U_{1ij} = \vec{0}, \quad i=1,\ldots,m \tag{4.2b}$$

$$|U_{1ij}|_q \le w_{1ij} \quad i=1,\ldots,m\,;\, j=1,\ldots,n \tag{4.2c}$$

$$|U_{2ir}|_q \le w_{2ir} \quad i=1,\ldots,m-1\,;\, r=i+1,\ldots,m \tag{4.2d}$$

where $1/p + 1/q = 1$, $|U|_q = [|u|^q + |v|^q]^{1/q}$ for $U' = (u,v)$, and U_{1ij} and U_{2ir} are vectors of dual variables. At optimality, $U_{1ij}' = (u_{1ij}, v_{1ij})$ is the non-normalized direction vector from the jth existing facility to the ith new facility and $U_{2ir}' = (u_{2ir}, v_{2ir})$ is the non-normalized direction vector from the rth new to the ith new facility.

Given a primal iterative computation procedure and the current solution x_i^k for $i=1,\ldots,m$ at iteration k, vectors U_{1ij} and U_{2ir} are estimated by $U^k{}_{1ij}' = (x_{i1}^k - a_{i1}, x_{i2}^k - a_{j2})$ and $U^k{}_{2ir}' = (x_{i1}^k - x_{r1}^k, x_{i2}^k - x_{r2}^k)$. Each U_{1ij}^k and U_{2ir}^k vector is adjusted so that the equality condition holds true in (4.2c) and (4.2d), using $c_{1ij} = w_{1ij}/|U_{1ij}^k|_q$ for $i=1,\ldots,m;$ $j=1,\ldots,n$ and $c_{2ir} = w_{2ir}/|U_{2ir}^k|_q$ for $i=1,\ldots,m-1$ and $r=i+1,\ldots,m$. The adjusted vectors are $\bar{U}_{1ij}^k = c_{1ij}U_{1ij}^k$ for $i=1,\ldots,m; j=1,\ldots,n$ and $\bar{U}_{2ir}^k = c_{2ir}U_{2ir}^k$ for $i=1,2,\ldots,m-1;$ $r=i+1,\ldots,m$. Although the norm constraints in (4.2c) and (4.2d) are satisfied, the adjusted vectors may not satisfy the linear equality constraints in (4.2b). The constraints in (4.2b) can be written with the u_{2ir} and u_{1ij} variables in the first m rows and the v_{2ir} and v_{1ij} variables in the second set of m rows.

Let $\quad U^{k'} = (u^k_{212}, \ldots, u^k_{2(m-1)m}, u^k_{111}, \ldots, u^k_{1mn}, v^k_{212}, \ldots, v^k_{2(m-1)m}, v^k_{111}, \ldots, v^k_{1mn})$,

a vector with $2mn + m(m-1)$ components. Constraint (4.2b) can be expressed as $A_g U^k = \vec{0}$ where A_g is a $2m \times [m(m-1) + 2mn]$ matrix and

$$A_g = \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix}.$$

B is an $m \times [m(m-1)/2 + mn]$ matrix, defined as $B = [B_1\ B_2 \ldots B_{m-1}\ C_1\ C_2 \ldots C_m]$ where B_t is an $m \times (m-t)$ matrix with elements

$$b_{ij} = \begin{cases} 1 & i=t \\ -1 & i=j+t \\ 0 & \text{otherwise} \end{cases} \quad \text{for } t=1, \ldots, m-1$$

and C_r is an $m \times n$ matrix with elements

$$c_{ij} = \begin{cases} 1 & i=r \\ 0 & \text{otherwise} \end{cases} \quad \text{for } r = 1, \ldots, m.$$

The vector $\overline{U}^k$ is projected into the intersection of the 2m planes determined by (4.2b), using the projection matrix P as given by Rosen (1960). P is defined as $P = I - A_g'(A_g A_g')^{-1} A_g$. In order to develop the form of the projection matrix for the given matrix A_g, the first step is the calculation of $A_g A_g'$.

$$A_g A_g' = \begin{bmatrix} BB' & 0 \\ 0 & BB' \end{bmatrix}$$

$$= \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}$$

where A is an $m \times m$ symmetric matrix with elements

$$a_{ij} = \begin{cases} m+n-1 & i=j \\ -1 & i \neq j \end{cases}$$

The derivation of this result can be found in Appendix B.

In Appendix C, it is shown that the $2m \times 2m$ symmetrix matrix $(A_g A_g')^{-1}$ can be calculated using

$$\left(A_g A_g'\right)^{-1} = \begin{bmatrix} A^{-1} & 0 \\ 0 & A^{-1} \end{bmatrix}$$

where A^{-1} is an $m \times m$ symmetric matrix with elements

$$a_{ij}^{-1} = \begin{cases} (n+1)/n(n+m) & i=j \\ 1/n(n+m) & i \neq j \end{cases}$$

The final form of the projection matrix P is

$$P_{\gamma} = I - A_g'(A_g A_g')^{-1} A_g$$

$$= 1/(n+m) \begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix}$$

where R is an $[m(m-1)/2+mn] \times [m(m-1)/2+mn]$ matrix given by

$$R = \left[\begin{array}{cccc|ccccc} F_1 & G_{12} & \cdots & G_{1(m-1)} & H_1 & K_{12} & \cdots & K_{1(m-1)} & K_{1m} \\ G_{12}' & F_2 & \cdots & G_{2(m-1)} & 0 & H_2 & \cdots & K_{2(m-1)} & K_{2m} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ G_{1(m-1)}' & G_{2(m-1)}' & \cdots & F_{m-1} & 0 & 0 & \cdots & H_{m-1} & K_{(m-1)m} \\ \hline H_1' & 0 & \cdots & 0 & L & N & & \cdots & N \\ K_{12}' & H_2' & \cdots & 0 & N & L & & \cdots & N \\ \vdots & & & \vdots & \vdots & & & & \vdots \\ K_{1m}' & & \cdots & K_{(m-1)m}' & N & & & \cdots & L \end{array}\right]$$

and F_t, G_{ts}, H_t, K_{tr}, L and N are defined as follows:

F_t is an $(m-t) \times (m-t)$ matrix with elements $f_{ij} = \begin{cases} n+m-2 & i=j \\ -1 & i \neq j \end{cases}$ for $t = 1,\ldots,m-1$;

G_{ts} is an $(m-t) \times (m-s)$ matrix with elements $g_{ij} = \begin{cases} 1 & i=s-t \\ -1 & i=j+s-t \\ 0 & \text{otherwise} \end{cases}$ for $t<s = 2,\ldots,m-1$;

H_t is an $(m-t) \times n$ constant matrix with $h_{ij} = -1$ for $t = 1, \ldots, m-1$;

K_{tr} is an $(m-t) \times n$ matrix with elements $k_{ij} = \begin{cases} 1 & i=r-t \\ 0 & \text{otherwise} \end{cases}$ for $t<r = 2,\ldots,m$;

L is an $n \times n$ matrix with elements $\ell_{ij} = \begin{cases} [n(n+m)-(n+1)]/n & i=j \\ -(n+1)/n & i \neq j \end{cases}$

and N is an $n \times n$ constant matrix with elements $n_{ij} = -1/n$.

This result is obtained in Appendix D. Applying the projection matrix P to $\bar{U}^k$ yields the vector $\bar{\bar{U}}^k = P\,\bar{U}^k$, which satisfies the linear constraints in (4.2b). It is possible that some of the $\bar{\bar{U}}_{2ir}{}^k$ or $\bar{\bar{U}}_{1ij}{}^k$ vectors may now violate (4.2d) or (4.2e) respectively or each vector may satisfy the strict inequality. $\bar{\bar{U}}^k$ can be adjusted by a factor c so that at least one vector satisfies the equality in (4.2d) or (4.2e) and all remaining $\bar{\bar{U}}_{2ir}{}^k$ and $\bar{\bar{U}}_{1ij}{}^k$ vectors satisfy the strict inequality. Calculate $\bar{c}_{2ir} = w_{2ir} / |\bar{\bar{U}}_{2ir}{}^k|_q$ for $1 \le i < r \le m$; $\bar{c}_{1ij} = w_{1ij} / |\bar{\bar{U}}_{1ij}{}^k|_q$ for $i = 1,\ldots,m$ and $j = 1,\ldots,n$; and

$$c = \min\{ \min_{1 \le i < r \le m} \bar{c}_{2ir},\ \min_{\substack{i=1,\ldots,m \\ j=1,\ldots,n}} \bar{c}_{1ij} \}.$$

Vector $c\bar{\bar{U}}^k$ will provide a feasible solution to the dual problem at iteration k and the dual objective function value

$$-c \sum_{i=1}^{m} \sum_{j=1}^{n} a_j{}' \,\bar{\bar{U}}_{1ij}^{k}$$

can be used as a lower bound for the primal objective function.

While the projection matrix P appears to be quite complicated, it is not necessary to actually calculate and store it in a computer program. Any component of the projected vector $P\bar{U}^k$ can be expressed in terms of the elements of $\bar{U}^k$ and the submatrix elements from P, where the non-zero elements of P only take on the values 1, -1, n+m-2, -(n+1)/n or [n(n+m)-(n+1)]/n.

4.2 The Drezner Bound for the Multi-Facility ℓ_p Distance Problem

The multi-facility ℓ_p distance location problem is to minimize

$$WM_p(x) = \sum_{i=1}^{m} \sum_{j=1}^{n} w_{1ij} [\,|x_{i1} - a_{j1}|^p + |x_{i2} - a_{j2}|^p\,]^{1/p}$$

$$+ \sum_{i=1}^{m-1} \sum_{r=i+1}^{m} w_{2ir} [\,|x_{i1} - x_{r1}|^p + |x_{i2} - x_{r2}|^p\,]^{1/p}$$

where

$x_i' = (x_{i1}, x_{i2})$ are the location coordinates of new facility i,

$a_j' = (a_{j1}, a_{j2})$ are the location coordinates of existing facility j,

and $x_i^{k\prime} = (x^k_{i1}, x^k_{i2})$ represents the current solution at the k^{th} iteration of a solution procedure for the ith new facility location. Using the Hõlder inequality,

$$\left| \sum_{t=1}^{N} b_t c_t \right| \leq \left(\sum_{t=1}^{N} |b_t|^p \right)^{1/p} \left(\sum_{t=1}^{N} |c_t|^q \right)^{1/q} \quad \text{where } p>1 \text{ and } 1/p + 1/q = 1,$$

and substituting

$$b_t = |x_{it} - a_{jt}| \quad \text{and} \quad c_t = |a_{jt} - x^k_{it}|$$

for t = 1,2; then

$$|x_{i1}-a_{j1}||a_{j1}-x^k_{i1}| + |x_{i2}-a_{j2}||a_{j2}-x^k_{i2}| \leq [|x_{i1}-a_{j1}|^p + |x_{i2}-a_{j2}|^p]^{1/p}$$

$$\cdot [|a_{j1}-x^k_{i1}|^q + |a_{j2}-x^k_{i2}|^q]^{1/q} .$$

This can be rewritten as

$$[|x_{i1}-a_{j1}|^p + |x_{i2}-a_{j2}|^p]^{1/p} \geq \frac{|x_{i1}-a_{j1}||a_{j1}-x^k_{i1}| + |x_{i2}-a_{j2}||a_{j2}-x^k_{i2}|}{[|a_{j1}-x^k_{i1}|^q + |a_{j2}-x^k_{i2}|^q]^{1/q}} .$$

By multiplying both sides of the inequality by the nonnegative weights w_{1ij} and summing, then

$$\sum_{i=1}^{m} \sum_{j=1}^{n} w_{1ij} [|x_{i1}-a_{j1}|^p + |x_{i2}-a_{j2}|^p]^{1/p}$$

$$\geq \sum_{i=1}^{m} \sum_{j=1}^{n} \frac{w_{1ij}|x_{i1}-a_{j1}||a_{j1}-x^k_{i1}|}{[|a_{j1}-x^k_{i1}|^q + |a_{j2}-x^k_{i2}|^q]^{1/q}}$$

$$+ \sum_{i=1}^{m} \sum_{j=1}^{n} \frac{w_{1ij}|x_{i2}-a_{j2}||a_{j2}-x^k_{i2}|}{[|a_{j1}-x^k_{i1}|^q + |a_{j2}-x^k_{i2}|^q]^{1/q}} .$$

Let

$$w_{1ij}' = (w_{1ij}|a_{j1}-x^k_{i1}|)/[|a_{j1}-x^k_{i1}|^q + |a_{j2}-x^k_{i2}|^q]^{1/q}$$

and

$$w_{1ij}'' = (w_{1ij}|a_{j2}-x_{i2}^k|)/[|a_{j1}-x_{i1}^k|^q + |a_{j2}-x_{i2}^k|^q]^{1/q} ,$$

then

$$\min_{x_i} \sum_{i=1}^{m}\sum_{j=1}^{n} w_{1ij}\ell_p(x_i, a_j) \geq \min_{x_{i1}} \sum_{i=1}^{m}\sum_{j=1}^{n} w_{1ij}' |x_{i1}-a_{j1}|$$

$$+ \min_{x_{i2}} \sum_{i=1}^{m}\sum_{j=1}^{n} w_{1ij}'' |x_{i2}-a_{j2}| .$$

For the terms representing the weighted distances between pairs of new facilities, the Hölder inequality can be used in the same manner as before.

Substituting

$$b_t = |x_{it} - x_{rt}| \quad \text{and} \quad c_t = |x_{rt}^k - x_{it}^k|$$

for $t=1,2$ in the Hölder inequality, then

$$[|x_{i1} - x_{r1}|^p + |x_{i2} - x_{r2}|^p]^{1/p} \geq \frac{|x_{i1} - x_{r1}||x_{r1}^k - x_{i1}^k| + |x_{i2} - x_{r2}||x_{r2}^k - x_{i2}^k|}{[|x_{i1} - x_{r1}|^q + |x_{i2} - x_{r2}|^q]^{1/q}}$$

By multiplying both sides of the inequality by the non-negative weights w_{2ir} and summing, then

$$\sum_{i=1}^{m-1}\sum_{r=i+1}^{m} w_{2ir}[|x_{i1}-x_{r1}|^p + |x_{i2}-x_{r2}|^p]^{1/p} \geq \sum_{i=1}^{m-1}\sum_{r=i+1}^{m} w_{2ir}'|x_{i1}-x_{r1}|$$

$$+ \sum_{i=1}^{m-1}\sum_{r=i+1}^{m} w_{2ir}''|x_{i2}-x_{r2}| ,$$

where

$$w_{2ir}' = (w_{2ir}|x_{r1}^k-x_{i1}^k|)/[|x_{r1}^k-x_{i1}^k|^q + |x_{r2}^k-x_{i2}^k|^q]^{1/q}$$

and

$$w_{2ir}'' = (w_{2ir}|x_{r2}^k-x_{i2}^k|)/[|x_{r1}^k-x_{i1}^k|^q + |x_{r2}^k-x_{i2}^k|^q]^{1/q}.$$

Combining these two results gives

$$WM_p(x) \geq \sum_{i=1}^{m} \sum_{j=1}^{n} w_{1ij}'|x_{i1}-a_{j1}| + \sum_{i=1}^{m} \sum_{j=1}^{n} w_{1ij}''|x_{i2}-a_{j2}|$$

$$+ \sum_{i=1}^{m-1} \sum_{r=i+1}^{m} w_{2ir}'|x_{i1}-x_{r1}| + \sum_{i=1}^{m-1} \sum_{r=i+1}^{m} w_{2ir}''|x_{i2}-x_{r2}|$$

or

$$WM_p(x) \geq \sum_{i=1}^{m} \sum_{j=1}^{n} w_{1ij}'|x_{i1}-a_{j1}| + \sum_{i=1}^{m-1} \sum_{r=i+1}^{m} w_{2ir}'|x_{i1}-x_{r1}|$$

$$+ \sum_{i=1}^{m} \sum_{j=1}^{n} w_{1ij}''|x_{i2}-a_{j2}| + \sum_{i=1}^{m-1} \sum_{r=i+1}^{m} w_{2ir}''|x_{i2}-x_{r2}| = RM(x) .$$

The solution at each iteration of a computation procedure for the multi-facility problem is used to construct RM(x), a multi-facility rectangular distance model. An optimal solution x^*_R, which minimizes RM(x), is used to calculate RM(x^*_R) which is the lower bound.

4.3 Bound Comparisons for the Multi-Facility ℓ_p Distance Model

Programs were written to incorporate the multi-facility hyperbolic approximation version of the Weiszfeld procedure (2.23) with the Love and Yeong, Juel and dual bounding methods. At each iteration of the solution procedure the bound was calculated and tested against the current solution. By entering a proportionate error difference, e, a stopping rule calculated as

$$|(\text{bound value} - \text{objective function value})|/(\text{objective function value}) \leq e$$

was used to terminate the process. In all sample runs e = 0.01 and the initial starting solution used for each of the m new facilities in the Weizfeld procedure was (0,0). Samples of size n=5 and 10 existing facilities were randomly generated for m = 2 and 3 new facilities. The w_{1ij} weights were randomly generated from the interval [1,3], the interfacility weights w_{2ir} were 1 and ℓ_p distances were calculated for p = 2 and 1.8. An interactive computer program was written to solve the multi-facility rectilinear distance problem. When the current solution from the Weiszfeld procedure was entered, the new weights for the rectangular distance model used in the Drezner bound were calculated and an optimal solution x^*_R and lower bound value $RM(x^*_R)$ were obtained. Then $RM(x^*_R)$ was adjusted by subtracting the value of

$$2^{1/p}\varepsilon^{1/2}\left(\sum_{i=1}^{m}\sum_{j=1}^{n} w_{1ij} + \sum_{i=1}^{m-1}\sum_{r=i+1}^{m} w_{2ir}\right)$$

since a hyperbolic approximation function, (2.24) and (2.32), was used to solve the multi-facility ℓ_p distance problem. In all sample runs, $\varepsilon = 0.001$ was used in the approximating function.

For each bounding method, the number of iterations required to obtain $e \leq 0.01$, the hyperbolic and true objective function values and the value of the bound were recorded. These results are displayed for 2 and 3 new facilities in Tables 4.1 and 4.2, where B1, B2, B3, B4 refer to the Love-Yeong, Juel, Drezner and dual bounds respectively.

n		5				10			
p		No. of Iter.	Hyperbolic Objective Function	True Objective Function	Bound	No. of Iter.	Hyperbolic Objective Function	True Objective Function	Bound
2	B1	13	495.34	495.34	490.61	12	679.20	679.11	673.12
	B2	12	495.37	495.37	490.73	11	679.20	679.11	673.14
	B3	10	495.48	495.48	491.18	6	679.63	679.60	673.79
	B4	13	495.34	495.34	490.72	25*	679.19	679.10	636.18
1.8	B1	16	509.66	509.66	505.09	16	699.35	699.25	693.36
	B2	14	509.72	509.72	505.83	15	699.35	699.25	693.48
	B3	25*	509.63	509.63	495.45	8	699.88	699.79	692.92
	B4	25*	509.62	509.63	496.79	25*	699.34	699.24	653.05

* did not converge to within 1% error difference in 25 iterations.

Table 4.1: Lower Bound Data for Multi-Facility Samples, m = 2.

p		5				10			
	n	No. of Iter.	Hyperbolic Objective Function	True Objective Function	Bound	No. of Iter.	Hyperbolic Objective Function	True Objective Function	Bound
2	B1	22	691.66	691.66	685.72	15	1024.72	1024.60	1015.79
	B2	20	691.68	691.67	685.91	14	1024.73	1024.61	1015.87
	B3	12	692.99	692.98	687.54	7	1025.88	1025.82	1015.63
	B4	22	691.66	691.66	686.32	25*	1024.71	1024.59	943.14
1.8	B1	25	705.62	705.61	699.36	18	1053.09	1052.95	1043.74
	B2	23	705.63	705.63	700.23	17	1053.09	1052.95	1043.89
	B3	15	706.46	706.45	701.73	11	1053.44	1053.30	1043.00
	B4	25*	705.62	705.61	686.37	25*	1053.08	1052.94	960.67

* did not converge to within 1% error difference in 25 iterations.

Table 4.2: Lower Bound Data for Multi-Facility Samples, m = 3.

From Tables 4.1 and 4.2, it is quite evident that for p = 2 the Drezner bound provided superior results. As before, when p decreases in value, the Drezner and dual bounds may not converge. The Juel and Love-Yeong bounding methods provided better bound values than the dual in most of the examples. In the following section it will be proven that the Drezner bound is superior to the Juel bound for the multi-facility Euclidean distance model.

4.4 Comparison of Drezner and Juel Bounds for the Multi-Facility Euclidean Distance Model

The lower bound for the multi-facility ℓ_p model by Love and Yeong (1981) is given by

$$WM_p(x^*) \geq WM_p(x^k) - \bar{\bar{\sigma}}(x^k)\|\nabla WM_p(x^k)\|,$$

where

$$\bar{\Omega} = \{s = (s_1, s_2, \ldots, s_m) \mid s_i \in \Omega, i = 1, \ldots, m\}$$

$$\bar{\bar{\sigma}}(x) = \max\{d(x, y) \mid y \in \bar{\Omega}\},$$

and $x^{k\prime} = (x^k_{11}, x^k_{12}, \ldots, x^k_{m1}, x^k_{m2})$ is a point generated by any procedure at the kth iteration.

For the same model, the lower bound by Juel (1984) is

$$WM_p(x^*) \geq WM_p(x^k) - \nabla WM_p(x^k)' x^k + \min_{y \in \bar{\Omega}} \{\nabla WM_p(x^k)' y\}.$$

For p = 2, the gradient $\nabla WM_2(x)$ has components $\partial WM_2(x)/\partial x_{it}$, where

$$\frac{\partial WM_2(x)}{\partial x_{it}} = \sum_{j=1}^{n} w_{1ij}(x_{it} - a_{jt})/d_2(x_i, a_j) + \sum_{\substack{r=1 \\ r \neq i}}^{m} w_{2ir}(x_{it} - x_{rt})/d_2(x_i, x_r),$$

$$w_{2ir} = \begin{cases} w_{2ri} & r \neq i, r = 1, 2, \ldots, m \\ 0 & r = i \end{cases}$$

for t = 1,2, and i = 1,...,m.

By substituting for the gradient, the Juel bound for p = 2 can be expressed as

$$J(x^k) = WM_2(x^k) - \sum_{t=1}^{2} \Bigg[\sum_{i=1}^{m} \sum_{j=1}^{n} w_{1ij}(x^k_{it} - a_{jt})x^k_{it}/d_2(x^k_i, a_j)$$

$$+ \sum_{i=1}^{m} \sum_{\substack{r=1 \\ r \neq i}}^{m} w_{2ir}(x^k_{it} - x^k_{rt})x^k_{it}/d_2(x^k_i, x^k_r) \Bigg] + \min_{y \in \bar{\Omega}} \{\nabla WM_2(x^k)' y\}$$

Since the Juel bound is as good or better than the Love-Yeong bound at each iteration, only the Juel bound need be compared to the multi-facility rectangular bound. It will now be shown that the multi-facility rectangular bound will be at least as good as the Juel bound. Before proceeding with this proof, the following Lemma is required.

Lemma 1

$$\sum_{i=1}^{m-1} \sum_{r=i+1}^{m} w_{2ir}(x_i - x_r)(x_i - x_r) = \sum_{i=1}^{m} \sum_{\substack{r=1 \\ r \neq i}}^{m} w_{2ir}(x_i - x_r)\,x_i$$

Proof:

$$-\sum_{i=1}^{m-1} \sum_{r=i+1}^{m} w_{2ir}(x_i - x_r)x_r = \sum_{i=1}^{m-1} \sum_{r=i+1}^{m} w_{2ir}(x_r - x_i)x_r$$

$$= \sum_{r=2}^{m} w_{21r}(x_r - x_1)x_r + \sum_{r=3}^{m} w_{22r}(x_r - x_2)x_r$$

$$+ \ldots + \sum_{r=m}^{m} w_{2(m-1)r}(x_r - x_{m-1})x_r .$$

By grouping the ( )x_2, ( )x_3, ..., ( )x_m terms together, and using the fact that $w_{2ir} = w_{2ri}$ for i, r = 1,...,m and $i \neq r$, then

$$-\sum_{i=1}^{m-1} \sum_{r=i+1}^{m} w_{2ir}(x_i - x_r)x_r = \sum_{r=2}^{m} \sum_{i=1}^{r-1} w_{2ri}(x_r - x_i)x_r$$

$$= \sum_{i=2}^{m} \sum_{r=1}^{i-1} w_{2ir}(x_i - x_r)x_i$$

$$= \sum_{i=1}^{m} \sum_{\substack{r=1 \\ r \neq i}}^{i-1} w_{2ir}(x_i - x_r)x_i .$$

Therefore

$$\text{LHS} = \sum_{i=1}^{m-1} \sum_{r=i+1}^{m} w_{2ir}(x_i - x_r)x_i + \sum_{i=1}^{m} \sum_{\substack{r=1 \\ r\neq i}}^{i-1} w_{2ir}(x_i - x_r)x_i$$

$$= \sum_{i=1}^{m} \sum_{\substack{r=i+1 \\ r\neq i}}^{m} w_{2ir}(x_i - x_r)x_i + \sum_{i=1}^{m} \sum_{\substack{r=1 \\ r\neq i}}^{i-1} w_{2ir}(x_i - x_r)x_i$$

$$= \sum_{i=1}^{m} \sum_{\substack{r=1 \\ r\neq i}}^{m} w_{2ir}(x_i - x_r)x_i .$$

This establishes that

$$\sum_{i=1}^{m-1} \sum_{r=i+1}^{m} w_{2ir}(x_i - x_r)(x_i - x_r) = \sum_{i=1}^{m} \sum_{\substack{r=1 \\ r\neq i}}^{m} w_{2ir}(x_i - x_r)x_i .$$

It also follows that

$$\sum_{i=1}^{m-1} \sum_{r=i+1}^{m} w_{2ir}(x_i - x_r)(x_i^* - x_r^*) = \sum_{i=1}^{m} \sum_{\substack{r=1 \\ r\neq i}}^{m} w_{2ir}(x_i - x_r)x_i^* .$$

Let $x^*{}_R{}^{k'} = (x^*{}_{R11}{}^k, x^*{}_{R12}{}^k, \ldots, x^*{}_{Rm1}{}^k, x^*{}_{Rm2}{}^k)$, represent an optimal solution at the kth iteration obtained by solving the multi-facility rectangular distance model RM(x), where $x^*{}_{Ri}{}^k$ is the optimal location of the ith new facility. Hansen, Perreur and Thisse (1980), Theorem 2, have proven that $x^*{}_{Ri}{}^k \in (S_1 \times S_2) \cap \Omega$ for at least one optimal solution. Using this result for $x^*{}_{Ri}{}^k$ and the definition of $\overline{\Omega}$, it follows that $x^*{}_R{}^k \in \overline{\Omega}$ for at least one optimal solution.

Theorem 2

For p=2, $$RM(x^*{}_R{}^k) \geq J(x^k)$$

Proof:

By substituting for w'_{1ij}, w''_{1ij}, w'_{2ir} and w''_{2ir},

$$RM(x_R^{*k}) = \sum_{t=1}^{2}\Big[\sum_{i=1}^{m}\sum_{j=1}^{n} w_{1ij}|a_{jt}-x_{it}^k||x_{Rit}^{*k}-a_{jt}|/d_2(x_i^k,a_j)$$

$$+\sum_{i=1}^{m-1}\sum_{r=i+1}^{m} w_{2ir}|x_{it}^k-x_{rt}^k||x_{Rit}^{*k}-x_{Rrt}^{*k}|/d_2(x_i^k,x_r^k)\Big]$$

$$=\sum_{t=1}^{2}\Big[\sum_{i=1}^{m}\sum_{j=1}^{n} w_{1ij}|x_{it}^k-a_{jt}||x_{it}^k-x_{it}^k+x_{Rit}^{*k}-a_{jt}|/d_2(x_i^k,a_j)$$

$$+\sum_{i=1}^{m-1}\sum_{r=i+1}^{m} w_{2ir}|(x_{it}^k-x_{rt}^k)||(x_{it}^k-x_{rt}^k)-(x_{it}^k-x_{rt}^k)+x_{Rit}^{*k}-x_{Rrt}^{*k}|/d_2(x_i^k,x_r^k)\Big]$$

$$\geq\sum_{t=1}^{2}\Big[\sum_{i=1}^{m}\sum_{j=1}^{n} w_{1ij}(x_{it}^k-a_{jt})[(x_{it}^k-a_{jt})-(x_{it}^k-x_{Rit}^{*k})]/d_2(x_i^k,a_j)$$

$$+\sum_{i=1}^{m-1}\sum_{r=i+1}^{m} w_{2ir}(x_{it}^k-x_{rt}^k)\Big[(x_{it}^k-x_{rt}^k)-(x_{it}^k-x_{rt}^k)+(x_{Rit}^{*k}-x_{Rrt}^{*k})\Big]/d_2(x_i^k,x_r^k)\Big]$$

$$=\sum_{t=1}^{2}\Big[\sum_{i=1}^{m}\sum_{j=1}^{n} w_{1ij}(x_{it}^k-a_{jt})^2/d_2(x_i^k,a_j)+\sum_{i=1}^{m-1}\sum_{r=i+1}^{m} w_{2ir}(x_{it}^k-x_{rt}^k)^2/d_2(x_i^k,x_r^k)\Big]$$

$$-\sum_{t=1}^{2}\Big[\sum_{i=1}^{m}\sum_{j=1}^{n} w_{1ij}(x_{it}^k-a_{jt})x_{it}^k/d_2(x_i^k,a_j)+\sum_{i=1}^{m-1}\sum_{r=i+1}^{m} w_{2ir}(x_{it}^k-x_{rt}^k)(x_{it}^k-x_{rt}^k)/d_2(x_i^k,x_r^k)\Big]$$

$$+\sum_{t=1}^{2}\Big[\sum_{i=1}^{m}\sum_{j=1}^{n} w_{1ij}(x_{it}^k-a_{jt})x_{Rit}^{*k}/d_2(x_i^k,a_j)+\sum_{i=1}^{m-1}\sum_{r=i+1}^{m} w_{2ir}(x_{it}^k-x_{rt}^k)(x_{Rit}^{*k}-x_{Rrt}^{*k})/d_2(x_i^k,x_r^k)\Big]$$

Applying Lemma 1 to

$$\sum_{i=1}^{m-1}\sum_{r=i+1}^{m} w_{2ir}(x_{it}^k-x_{rt}^k)(x_{it}^k-x_{rt}^k)$$

and

$$\sum_{i=1}^{m-1}\sum_{r=i+1}^{m} w_{2ir}(x_{it}^k-x_{rt}^k)(x_{Rit}^{*k}-x_{Rrt}^{*k}),$$

then

$$RM(x_R^{*k}) \geq WM_2(x^k) - \sum_{t=1}^{2} \Bigg[\sum_{i=1}^{m} \sum_{j=1}^{n} w_{1ij}(x_{it}^k - a_{jt})\, x_{it}^k / d_2(x_i^k, a_j)$$

$$+ \sum_{i=1}^{m} \sum_{\substack{r=1 \\ r\neq i}}^{m} w_{2ir}(x_{it}^k - x_{rt}^k)\, x_{it}^k / d_2(x_i^k, x_r^k) \Bigg]$$

$$+ \sum_{t=1}^{2} \Bigg[\sum_{i=1}^{m} \sum_{j=1}^{n} w_{1ij}(x_{it}^k - a_{jt})\, x_{Rit}^{*k} / d_2(x_i^k, a_j)$$

$$+ \sum_{i=1}^{m} \sum_{\substack{r=1 \\ r\neq i}}^{m} w_{2ir}(x_{it}^k - x_{rt}^k)\, x_{Rit}^{*k} / d_2(x_i^k, x_r^k) \Bigg]$$

$$= WM_2(x^k) - \sum_{t=1}^{2} \sum_{i=1}^{m} \frac{\partial WM_2(x)}{\partial x_{it}} x_{it}^k + \sum_{t=1}^{2} \sum_{i=1}^{m} \frac{\partial WM_2(x)}{\partial x_{it}} x_{Rit}^{*k}$$

Therefore,

$$RM(x_R^{*k}) \geq WM_2(x^k) - \nabla WM_2(x^k)'\, x^k + \nabla WM_2(x^k)'\, x_R^{*k}$$

This expression can be compared to the Juel bound,

$$J(x^k) = WM_2(x^k) - \nabla WM_2(x^k)'\, x^k + \min_{y\in\overline{\Omega}} \{\nabla WM_2(x^k)'\, y\} .$$

By choosing $x^*{}_R{}^k \in \overline{\Omega}$, then

$$\min_{y\in\overline{\Omega}} \{\nabla WM_p(x^k)'\, y\} \leq \nabla WM_p(x^k)'\, x_R^{*k} ,$$

which means

$$RM(x_R^{*k}) \geq J(x^k) \qquad \text{for } x_R^{*k} \in \overline{\Omega} .$$

Multiple optimal solutions can occur when solving the multi-facility rectangular distance problem RM(x) at iteration k. An optimal solution may or may not be an element of $\overline{\Omega}$, but Theorem 2 from Hansen, Perreur and Thisse (1980) guarantees that at least one optimal solution is an element of $\overline{\Omega}$. Let $x^*{}_R{}^k$ and $z^*{}_R{}^k$ represent optimal solutions to RM(x) where $x^*{}_R{}^k \in \overline{\Omega}$ and $z^*{}_R{}^k \notin \overline{\Omega}$. Then, $RM(z^*{}_R{}^k) = RM(x^*{}_R{}^k) \geq J(x^k)$. Thus, for p=2, Theorem 2 holds for any optimal solution to the multi-facility rectangular distance problem at iteration k.

This establishes that the rectangular bound can be used without any doubts about its convergence with Euclidean distances. Considerable computation time can be saved using this bound, as it required fewer iterations to reach the same level of percentage error difference as the other two bounds.

Theorem 2 in conjunction with the results of Juel (1984) and Elzinga and Hearn (1984) establishes that B3 ≥ B2 ≥ B1 for the Euclidean distance multi-facility model. The nature of the dual makes it difficult to provide a theoretical comparison with the other three bounds. In Tables 4.1 and 4.2 the Love-Yeong bounding method provided better bound values than the dual in a majority of the examples. The dual outperformed the Love-Yeong bound in only 2 of the 8 test problems, where the n and p values were 5 and 2 respectively. In the next section, the dual and the Love and Yeong bound will be compared using numerous examples for $p = 2$.

4.5 Comparison of the Dual (B4) and Love-Yeong Bound (B1)

Several test examples were constructed for the multi-facility hyperbolic approximation model with Euclidean distances (2.22) and solved using the generalized Weiszfeld iterative technique (2.23). At each iteration, the Love-Yeong and dual bounds were recorded in order to investigate the performance of B1 and B4 with respect to the number of existing facilities, the number of new facilities, the locations of the existing facilities and the values for the non-negative weights.

Test problems were constructed with $m = 2$ and 3 new facilities and $n = 5, 10, 20$ and 30 existing facilities. For each value of m and n, a value of $w = 1, 2, 3, 4, 7, 8$ and 10 was selected and the w_{1ij} weights were randomly generated over the interval [1,w]. For each of these test problems with n, m and w fixed, the bounds B1 and B4 were compared with the true objective function value for 25 iterations with $w_{2ir} = 1$. Then each problem was run again with the w_{2ir} weights randomly generated over the interval [1,w]. The bound values for these sample runs are displayed in Tables 4.3 to 4.10.

When m is fixed and n increases, this situation usually favours the Love and Yeong bound. For example, in Tables 4.3 and 4.7 for m = 2 and 3 respectively, when w is small and n is small the dual provided a better bound over the first 25 iterations. However, as n increases as in Tables 4.6 and 4.10 with w small, the dual usually provided a better bound over the first few iterations and then the Love and Yeong bound was better at each iteration. When n and m are fixed and w increases in value, this usually has the tendency to favour the Love and Yeong bound. Also, when the w_{1ij} weights were selected from the interval [1,w] and the w_{2ir} weights were changed from a unit value to a value from the range [1,w], then this situation usually favoured the Love and Yeong bound. No definite patterns have really emerged for the Love-Yeong and dual bounds involving n, m and w. In Table 4.5 where m = 2, n = 20, $w_{1ij} \in [1,4]$ and $w_{2ir} \in [1,1]$, the dual provided a better bound over iterations 1-3 and 15-25 and the Love-Yeong bound was better over iterations 4-14. It is doubtful if any pattern occurs involving n, m and w where either bound can be declared to be superior to the other for all situations.

The effect of outliers was tested for m = 2 and 3 new facilities where each existing facility coordinate, except for the single outlier, was randomly generated from the interval [0,50]. The w_{1ij} weights were randomly generated from the interval [1,4] and the w_{2ir} weights were assigned a value of 1, 4 or 8. Each multi-facility Euclidean distance model was solved 6 times, using 2 outlier values and 3 different outlier weights. The B1 and B4 bound values and true objective function are shown in Table 4.11 for sample runs with 2 and 3 new facilities and 5 and 10 existing facilities. In each test run, the solution procedure was terminated when an error difference of $e \leq 0.01$ was reached or when 25 iterations were completed.

Interval for w_{1ij} weights	Interval for w_{2ir} weights	Iteration 25			Iteration Range	
		Obj. Fn.	B1	B4	B1 > B4	B4 > B1
[1,1]	[1,1]	216.68	178.16	173.19	14-25	1-13
[1,2]	[1,1]	391.73	353.90	357.44		1-25
	[1,2]	397.21	307.96	312.70		1-25
[1,3]	[1,1]	405.11	403.41	404.76	3-4	1-2
						5-25
	[1,3]	415.80	237.40	236.59	25	1-24
[1,4]	[1,1]	462.62	460.39	461.27	6-18	1-5
						19-25
	[1,4]	477.11	400.08	408.46		1-25
[1,7]	[1,1]	1097.18	1094.97	1096.92	6-17	1-5
						18-25
	[1,7]	1154.77	731.86	839.23		1-25
[1,8]	[1,1]	879.82	877.35	878.40	4-21	1-3
						22-25
	[1,8]	1017.97	547.61	630.58		1-25
[1,10]	[1,1]	1212.54	1203.24	1185.38	3-25	1-2
	[1,10]	1269.16	843.11	866.77		1-25

Table 4.3: Love-Yeong and Dual Comparison for 2 New and 5 Existing Facilities

Interval for w_{1ij} weights	Interval for w_{2ir} weights	Obj. Fn.	Iteration 25 B1	B4	Iteration Range B1 > B4	B4 > B1
[1,1]	[1,1]	309.29	306.11	277.13	22-25	1-21
[1,2]	[1,1]	489.30	425.78	438.77		1-25
	[1,2]	513.77	373.48	394.93		1-25
[1,3]	[1,1]	571.12	568.90	547.64	19-25	1-18
	[1,3]	600.89	518.89	544.57		1-25
[1,4]	[1,1]	576.32	494.50	494.94	18-25	1-17
	[1,4]	611.31	443.98	437.75	21-25	1-20
[1-7]	[1,1]	934.20	930.41	727.76	12-25	1-11
	[1,7]	1326.88	1193.57	1151.63	3-25	1-2
[1-8]	[1,1]	1479.95	1469.58	1421.50	15-25	1-14
	[1,8]	1532.61	1495.15	1472.98	22-25	1-21
[1,10]	[1,1]	2041.33	2036.10	1609.61	2-25	1
	[1,10]	2140.41	2036.66	1876.98	2-25	1

Table 4.4: Love-Yeong and Dual Comparison for 2 New and 10 Existing Facilities

Interval for w_{1ij} weights	Interval for w_{2ir} weights	Iteration 25 Obj. Fn.	B1	B4	Iteration Range B1 > B4	B4 > B1
[1,1]	[1,1]	776.77	740.81	727.27	24-25	1-23
[1,2]	[1,1]	1188.34	1184.39	1187.24	6-9	1-5
						10-25
	[1,2]	1189.99	1185.15	1187.22	10-13	1-9
						14-25
[1,3]	[1,1]	1473.61	1470.10	1472.51		1-25
	[1,3]	1476.15	1414.44	1387.50	6-25	1-5
[1,4]	[1,1]	2056.15	2051.66	2051.99	4-14	1-3
						15-25
	[1,4]	2057.17	1967.25	1826.09	3-25	1-2
[1,7]	[1,1]	2609.87	2602.19	2599.96	2-25	1
	[1,7]	2622.17	2600.72	2562.38	3-25	1-2
[1,8]	[1,1]	3580.08	3571.33	3572.95	2-12	1
						13-25
	[1,8]	3590.35	3410.88	3352.38	8-25	1-7
[1,10]	[1,1]	5020.56	5006.36	5007.63	2-10	1
						11-25
	[1,10]	5040.93	4879.46	4538.16	3-25	1-2

Table 4.5: Love-Yeong and Dual Comparison for 2 New and 20 Existing Facilities

Interval for w_{1ij} weights	Interval for w_{2ir} weights	Iteration 25 Obj. Fn.	B1	B4	Iteration Range B1 > B4	B4 > B1
[1,1]	[1,1]	1122.52	1102.42	1080.34	11-25	1-10
[1,2]	[1,1]	1692.81	1681.63	1669.66	7-25	1-6
	[1,2]	1692.97	1646.66	1605.33	9-25	1-8
[1,3]	[1,1]	2138.26	2125.50	2126.39	3-21	1-2
						22-25
	[1,3]	2139.07	2047.53	1938.76	3-25	1-2
[1,4]	[1,1]	3206.94	3201.18	3202.79	4-11	1-3
						12-25
	[1,4]	3213.29	3064.33	2954.72	3-25	1-2
[1,7]	[1,1]	3591.73	3583.10	3585.39	1-12	1
						13-25
	[1,7]	3602.53	3582.50	3574.40	3-25	1-2
[1,8]	[1,1]	5046.98	5035.21	5033.80	3-25	1-2
	[1,8]	5072.44	5054.96	5057.89	3-23	1-2
						24-25
[1,10]	[1,1]	6902.27	6881.76	6872.66	2-25	1
	[1,10]	6903.02	6764.80	6624.59	2-25	1

Table 4.6: Love-Yeong and Dual Comparison for 2 New and 30 Existing Facilities

Interval for w_{1ij} weights	Interval for w_{2ir} weights	Iteration 25 Obj. Fn.	B1	B4	Iteration Range B1 > B4	B4 > B1
[1,1]	[1,1]	353.01	160.22	194.95		1-25
[1,2]	[1,1]	512.17	363.38	383.86		1-25
	[1,2]	539.24	261.38	333.48		1-25
[1,3]	[1,1]	599.28	435.24	497.44		1-25
	[1,3]	687.34	223.69	343.99		1-25
[1,4]	[1,1]	613.31	505.04	488.99	4-25	1-3
	[1,4]	626.73	626.81	324.23	17-25	1-16
[1,7]	[1,1]	1313.56	1309.01	1311.39	2-18	1
						19-25
	[1,7]	1357.12	846.70	718.92	3-25	1-2
[1,8]	[1,1]	1350.70	1346.22	1346.11	3-25	1-2
	[1,8]	1384.78	1194.00	947.12	5-25	1-4
[1,10]	[1,1]	1740.62	1733.44	1725.36	3-25	1-2
	[1,10]	1774.73	1260.71	1376.29		1-25

Table 4.7: Love-Yeong and Dual Comparison for 3 New and 5 Existing Facilities

Interval for w_{1ij} weights	Interval for w_{2ir} weights	Iteration 25 Obj. Fn.	Iteration 25 B1	Iteration 25 B4	Iteration Range B1 > B4	Iteration Range B4 > B1
[1,1]	[1,1]	491.46	302.19	363.38		1-25
[1,2]	[1,1]	778.99	600.45	626.60		1-25
	[1,2]	792.34	447.86	565.37		1-25
[1,3]	[1,1]	933.60	930.91	880.84	15-25	1-14
	[1,3]	947.91	523.18	704.22		1-25
[1,4]	[1,1]	1030.59	1023.30	917.90	8-25	1-7
	[1,4]	1046.84	662.28	629.30	16-25	1-15
[1,7]	[1,1]	1937.70	1904.03	1467.34	2-25	1
	[1,7]	1940.93	1583.47	1474.89	4-25	1-3
[1,8]	[1,1]	2018.00	1982.73	1881.98	8-25	1-7
	[1,8]	2046.70	1440.66	1558.69		1-25
[1,10]	[1,1]	2823.75	2816.38	2288.97	2-25	1
	[1,10]	2896.86	2210.08	2307.16	3-5	1-2
						6-25

Table 4.8: Love-Yeong and Dual Comparison for 3 New and 10 Existing Facilities

Interval for w_{1ij} weights	Interval for w_{2ir} weights	Iteration 25 Obj. Fn.	B1	B4	Iteration Range B1 > B4	B4 > B1
[1,1]	[1,1]	1167.35	1039.14	1017.73	17-25	1-16
[1,2]	[1,1]	1807.27	1802.64	1806.88	5-12	1-4
						13-25
	[1,2]	1811.23	1667.74	1626.62	6-25	1-5
[1,3]	[1,1]	2155.69	2128.11	2097.12	5-25	1-4
	[1,3]	2162.26	1921.21	1832.69	3-25	1-2
[1,4]	[1,1]	2952.19	2951.71	2951.70	4-25	1-3
	[1,4]	2973.58	2391.74	2077.12	4-25	1-3
[1,7]	[1,1]	4118.07	4108.43	4177.99	2-11	1
						12-25
	[1,7]	4131.37	3997.00	3867.86	4-25	1-3
[1,8]	[1,1]	5114.04	5102.10	5114.04	3-10	1-2
						11-25
	[1,8]	5126.90	5113.95	5125.82	3-11	1-2
						12-25
[1,10]	[1,1]	7107.97	7091.62	7107.59	2-14	1
						15-25
	[1,10]	7131.53	6594.70	4798.83	2-25	1

Table 4.9: Love-Yeong and Dual Comparison for 3 New and 20 Existing Facilities

Interval for w_{1ij} weights	Interval for w_{2ir} weights	Iteration 25 Obj. Fn.	B1	B4	Iteration Range B1 > B4	B4 > B1
[1,1]	[1,1]	1684.63	1581.42	1540.40	14-25	1-13
[1,2]	[1,1]	2537.95	2475.43	2441.66	9-25	1-8
	[1,2]	2538.43	2433.85	2403.78	8-25	1-7
[1,3]	[1,1]	3123.25	3058.57	3039.69	3-25	1-2
	[1,3]	3124.73	2938.04	2723.76	3-25	1-2
[1,4]	[1,1]	4423.13	4412.43	4422.96	4-12	1-3
						13-25
	[1,4]	4434.12	4247.04	4040.40	4-25	1-3
[1,7]	[1,1]	6536.95	6517.34	6475.04	2-25	1
	[1,7]	6547.70	6418.68	6211.72	2-25	1
[1,8]	[1,1]	7410.06	7492.02	7509.80	3-13	1-2
						14-25
	[1,8]	7519.32	7499.81	7515.89	3-18	1-2
						19-25
[1,10]	[1,1]	9924.45	9900.98	9920.50	2-16	1
						17-25
	[1,10]	9955.71	9776.32	8030.32	2-25	1

Table 4.10: Love-Yeong and Dual Comparison for 3 New and 30 Existing Facilities

m			2				3			
n	Outlier	Outlier Weight	No. Iter.	Obj. Fn.	B1	B4	No. Iter.	Obj. Fn.	B1	B4
5	(78,93)	1	25*	540.3	524.7	534.8	25*	894.8	372.5	651.1
		4	9	1002.5	993.5	996.0	25*	1468.5	1253.8	1338.3
		8	25*	1441.8	1366.5	1344.3	25*	2199.4	1980.9	1773.1
	(126,112)	1	16	614.1	604.4	611.5	25*	987.8	562.4	855.0
		4	25*	1323.0	1187.4	1160.4	25*	1955.3	1369.3	1445.3
		8	25*	2112.2	2023.8	1923.4	25*	3259.7	3029.8	2648.8
10	(78,93)	1	25*	712.4	667.8	687.5	25*	1194.9	846.1	971.0
		4	24	1258.3	1247.9	1228.0	25*	2009.0	1879.0	1896.2
		8	14	1973.8	1957.7	1953.6	25	3079.7	3048.9	3031.2
	(126,112)	1	25*	826.5	815.2	718.2	25*	1302.5	793.5	1045.2
		4	14	1635.1	1626.1	1564.0	23	2435.4	2413.1	2356.0
		8	12	2727.4	2710.6	2553.4	12	3933.0	3899.2	3616.0

* did not converge to within 1% error difference in 25 iterations.

Table 4.11: Outlier Data for 2 and 3 New Facilities with $w_{2ir} = 1$.

As in the single facility situation, the presence of an outlier among the existing facilities favours the dual. However, as the outlier weight is increased, the dual loses its advantage and the Love-Yeong bound is better. As the number of existing facilities increases, this also favours the Love and Yeong bound.

4.6 Conclusions

For p = 2, the Drezner bound is superior to the Love and Yeong, Juel, and dual bounds. In some cases the Juel, Love and Yeong, and dual bounds required twice as many iterations to reach the same value as the Drezner bound. Considerable computational savings can be achieved by using the Drezner bound with an iterative solution technique. The dual has, in most instances, been the poorest bound. In situations where the dual has been better than the Love-Yeong bound, this has been achieved by sacrificing computation time. The computation time required when calculating the Juel or Love-Yeong bound is increased, on average, by a factor of 3.2 when the dual bound is used.

When $1 < p < 2$, the Juel method provides the best bound from among the four methods. As in the single facility case, the multi-facility Drezner and dual bounds may experience convergence problems when $p < 2$.

CHAPTER 5

CONCLUSION

5.1 User Criteria for Selecting a Bounding Method

This thesis has shown that the practitioner, when considering a bounding method to terminate an iterative computational procedure for a single or multi-facility ℓ_p distance location model, can make a choice based upon the value of the parameter p.

For $1 < p < 2$, the best bound value was obtained by using the Juel method. The Juel bound is computationally efficient and has been proven to be as good or better than the Love and Yeong bound. Since the Drezner and dual bounds may experience convergence problems for $1 < p < 2$, the useage of these two bounds should be confined to models with Euclidean distances.

When $p = 2$, the best bound was obtained by using Drezner's method which requires the solution of a location model with rectangular distances. The single facility rectilinear model can easily be solved, which gives the Drezner bound a computational advantage over the other three bounds. This thesis has established the effectiveness of the single and multi-facility Drezner bounds over the Love-Yeong and Juel bounds by proving that the Drezner bound is always as good or better than the Juel bound for $p = 2$. The superiority of the Drezner bound over the dual has been shown using many examples. Since an interactive program was used to solve the multi-facility rectilinear problem to obtain the Drezner bound value, no computation times are available for the multi-facility bound. Procedures for solving the multi-facility rectilinear model have been given by Juel and Love (1976), and Drezner and Wesolowsky (1978b). One of these techniques for solving the multi-facility rectilinear model could be incorporated in the Drezner bounding method and then computation times could be obtained to compare all four methods. Even if the Drezner bound required double or triple the computation time of the Juel bound, this disadvantage could be offset by calculating the Drezner bound after every second or third iteration.

The statement made by Wendell and Peterson (1984) that the dual is often much better than the Juel or Love-Yeong bound for the single facility Euclidean distance model has been shown to have little validity. The dual requires more computation time than the other three methods and, in general, yields bound values which do not compare favourably with the other three methods.

5.2 Future Research

This thesis has extended the Drezner and dual bounds to include the multi-facility model with ℓ_p distances, and shown the superiority of the Drezner bound for the Euclidean distance situations. One problem which has emerged is that for $p < 2$ both the dual and Drezner bounds may experience convergence problems. An area for future research would entail investigating the reasons for this lack of convergence.

APPENDIX A

PROJECTION MATRIX FOR THE SINGLE FACILITY ℓ_p DISTANCE MODEL

Given the $2 \times 2n$ matrix

$$A_g = \begin{pmatrix} 1 \ldots 1 & 0 \ldots 0 \\ 0 \ldots 0 & 1 \ldots 1 \end{pmatrix}$$

the projection matrix $P = I - A_g'(A_gA_g')^{-1}A_g$ can be derived as follows. A_gA_g' is a 2×2 matrix where any element $x_{ij} \in A_gA_g'$ can be written as

$$x_{ij} = \sum_{k=1}^{2n} a_{ik}a'_{kj}$$

$$= \sum_{k=1}^{2n} a_{ik}a_{jk}$$

$$= \sum_{k=1}^{n} a_{ik}a_{jk} + \sum_{k=n+1}^{2n} a_{ik}a_{jk} \quad \text{for } i,j = 1, 2.$$

Since $a_{1k} = a_{2(n+k)} = 1$ for $k = 1, \ldots, n$ and $a_{1k} = a_{2(k-n)} = 0$ for $k = n+1, \ldots, 2n$;

$$\text{for } i = j, \quad x_{ii} = \sum_{k=1}^{n} a_{ik}a_{ik} + \sum_{k=n+1}^{2n} a_{ik}a_{ik} = n$$

$$\text{and for } i \neq j, \quad x_{ij} = \sum_{k=1}^{n} a_{ik}a_{jk} + \sum_{k=n+1}^{2n} a_{ik}a_{jk} = 0.$$

$$A_gA_g' = \begin{pmatrix} n & 0 \\ 0 & n \end{pmatrix} \quad \text{and} \quad (A_gA_g')^{-1} = \begin{pmatrix} 1/n & 0 \\ 0 & 1/n \end{pmatrix},$$

so that $A_g'(A_gA_g')^{-1} = (1/n)A_g'$ and $A_g'(A_gA_g')^{-1}A_g = (1/n)A_g'A_g$. Now, $(1/n)A_g'A_g$ is a $2n \times 2n$ matrix where any element $x_{ij} \in (1/n)A_g'A_g$ can be written as

$$x_{ij} = (1/n)\sum_{k=1}^{2} a'_{ik}a_{kj}$$

$$= (1/n)\sum_{k=1}^{2} a_{ki}a_{kj}$$

$$= (1/n)[a_{1i}a_{1j} + a_{2i}a_{2j}].$$

Therefore

$$x_{ij} = \begin{cases} (1/n)[(1)(1)+0] & i,j = 1, \ldots, n \\ (1/n)[0+(1)(1)] & i,j = n+1, \ldots, 2n \\ 0 & \text{otherwise} \end{cases}$$

or

$$x_{ij} = \begin{cases} 1/n & i,j = 1, \ldots, n;\ i,j = n+1, \ldots, 2n \\ 0 & \text{otherwise} \end{cases}$$

The $2n \times 2n$ projection matrix P can be written as

$$P = I - A'_g(A_gA'_g)^{-1}A_g$$

$$= \begin{bmatrix} R & O \\ O & R \end{bmatrix}$$

where R is an $n \times n$ matrix with elements

$$r_{ij} = \begin{cases} 1 - 1/n & i=j \\ -1/n & i \neq j \end{cases}$$

APPENDIX B

CALCULATION OF $A_g A_g'$ FOR THE MULTI-FACILITY PROJECTION MATRIX

Given the $2m \times [m(m-1) + 2mn]$ matrix

$$A_g = \begin{bmatrix} B & O \\ O & B \end{bmatrix}$$

$A_g A_g'$ can be determined using

$$A_g A_g' = \begin{bmatrix} BB' & O \\ O & BB' \end{bmatrix}$$

The elements in matrix B only take on the values 1, 0 or −1, so that row i can be described in terms of row index sets containing the column indices of the elements in row i which have values of 1 or −1. Then, using properties of these index sets, matrix BB′ can be calculated.

The $m \times [m(m-1)/2 + mn]$ matrix B has elements q_{ij} where

$$q_{ij} = \begin{cases} 1 & j \in I_1(i) \cup I_2(i) \\ -1 & j \in I_3(i) \\ 0 & \text{otherwise} \end{cases}$$

with

$$I_1(i) = \left\{ \ell + \sum_{\substack{k=1 \\ i>1}}^{i-1} (m-k) \;\middle|\; \ell = 1, \ldots, m-i; \quad i < m \right\}$$

$$I_2(i) = \left\{ [M + n(i-1)] + 1, \; \ldots \;, [M + n(i-1)] + n \right\}$$

and

$$I_3(i) = \left\{ i - \ell + \sum_{\substack{k=1 \\ \ell>1}}^{\ell-1} (m-k) \;\middle|\; \ell = 1, \ldots, i-1; \quad i \geq 2 \right\} \quad \text{for } M = m(m-1)/2.$$

The following properties are evident from the definitions of q_{ij} and the index sets.

Property 1. Let n(S) denote the number of elements in set S.

(i) $n(I_1(i)) = m-i$

(ii) $n(I_2(i)) = n$

(iii) $n(I_3(i)) = i-1$

Property 2

(i) For $r \in I_1(i)$, $t \in I_1(j)$ if $i < j$ then $r < t$.

(ii) For $r \in I_1(i)$, $t \in I_3(i)$ then $t < r$.

Property 3

(i) $I_1(i) \cap I_2(i) = \phi$

(ii) $I_1(i) \cap I_3(i) = \phi$

(iii) $I_2(i) \cap I_3(i) = \phi$

Property 4

For $i \neq j$

(i) $I_1(i) \cap I_1(j) = \phi$

(ii) $I_1(i) \cap I_2(j) = \phi$

(iii) $I_1(i) \cap I_3(j) \neq \phi$ for $i < j$, so that $n(I_1(i) \cap I_3(j)) = 1$

(iv) $I_1(i) \cap I_3(j) = \phi$ for $i > j$

(v) $I_2(i) \cap I_3(j) = \phi$

(vi) $I_2(i) \cap I_2(j) = \phi$

(vii) $I_3(i) \cap I_3(j) = \phi$

The preceding properties can be used to determine the elements x_{ij} of the $m \times m$ matrix BB', where

$$x_{ij} = \sum_{k=1}^{M+mn} q_{ik} q_{kj}$$

$$= \sum_{k=1}^{M+mn} q_{ik} q_{jk}$$

$$= \sum_{k \in I_1(i)} q_{ik} q_{jk} + \sum_{k \in I_2(i)} q_{ik} q_{jk} + \sum_{k \in I_3(i)} q_{ik} q_{jk}$$

For $i = j$,

$$x_{ii} = \sum_{k \in I_1(i)} q_{ik} q_{ik} + \sum_{k \in I_2(i)} q_{ik} q_{ik} + \sum_{k \in I_3(i)} q_{ik} q_{ik}$$

$$= n(I_1(i))[(1)(1)] + n(I_2(i))[(1)(1)] + n(I_3(i))[(-1)(-1)]$$

$$= m - i + n + i - 1$$

$$= m + n - 1.$$

For $i \neq j$,

$$x_{ij} = \sum_{\substack{k \in I_1(i) \\ k \notin I_3(j)}} q_{ik} q_{jk} + \sum_{\substack{k \in I_1(i) \\ k \in I_3(j)}} q_{ik} q_{jk} + \sum_{\substack{k \in I_2(i) \\ k \notin I_2(j)}} q_{ik} q_{jk} + \sum_{\substack{k \in I_2(i) \\ k \in I_2(j)}} q_{ik} q_{jk}$$

$$+ \sum_{\substack{k \in I_3(i) \\ k \notin I_1(j)}} q_{ik} q_{jk} + \sum_{\substack{k \in I_3(i) \\ k \in I_1(j)}} q_{ik} q_{jk}$$

$$= \sum_{\substack{k \in I_1(i) \\ k \in I_3(j)}} q_{ik} q_{jk} + \sum_{\substack{k \in I_3(i) \\ k \in I_1(j)}} q_{ik} q_{jk}$$

For $i < j$,

$$x_{ij} = \sum_{\substack{k \in I_1(i) \\ k \in I_3(i)}} q_{ik} q_{jk} = 1(-1) = -1$$

For $i > j$,

$$x_{ij} = \sum_{\substack{k \in I_3(i) \\ k \in I_1(j)}} q_{ik} q_{jk} = (-1)(1) = -1$$

Therefore

$$x_{ij} = \begin{cases} m + n - 1 & i = j \\ -1 & \text{otherwise} \qquad i, j = 1, \ldots, m. \end{cases}$$

Thus,

$$A_g A_g^{'} = \begin{bmatrix} BB' & O \\ O & BB' \end{bmatrix}$$

$$= \begin{bmatrix} A & O \\ O & A \end{bmatrix}$$

where A is an $m \times m$ symmetric matrix with elements

$$a_{ij} = \begin{cases} m + n - 1 & i = j \\ -1 & i \neq j. \end{cases}$$

APPENDIX C

CALCULATION OF $(A_g A_g')^{-1}$ FOR THE MULTI-FACILITY PROJECTION MATRIX

From Appendix B,

$$A_g A_g' = \begin{bmatrix} A & O \\ O & A \end{bmatrix}$$

where A is an $n \times n$ symmetric matrix with elements

$$a_{ij} = \begin{cases} m + n - 1 & i = j \\ -1 & i \neq j. \end{cases}$$

The $2m \times 2m$ symmetric matrix $(A_g \cdot A_g')^{-1}$ can be calculated using

$$\left(A_g A_g'\right)^{-1} = \begin{bmatrix} A^{-1} & O \\ O & A^{-1} \end{bmatrix}$$

where A^{-1} is an $m \times m$ symmetric matrix with elements

$$a_{ij}^{-1} = \begin{cases} c & i = j \\ d & i \neq j. \end{cases}$$

Since $A \cdot A^{-1} = 1$, take $x_{ij} \in I$, then

$$x_{ij} = \sum_{k=1}^{m} a_{ik} a_{kj}^{-1} \qquad i, j = 1, \ldots, n .$$

For $i = j$, $x_{ii} = 1$ and

$$1 = \sum_{k=1}^{m} a_{ik} a_{ki}^{-1}$$

$$= a_{ii} a_{ii}^{-1} + \sum_{\substack{k=1 \\ k \neq i}}^{m} a_{ik} a_{ki}^{-1}$$

$$= (n + m - 1)c + (m - 1)(-1)d .$$

This provides one of the equations required to solve for c and d,

$$(n + m - 1)c - (m - 1)d = 1 . \qquad (1)$$

For $i \neq j$, $x_{ij} = 0$ and

$$0 = \sum_{k=1}^{m} a_{ik} a_{kj}^{-1}$$

$$= a_{ii} a_{ij}^{-1} + a_{ij} a_{jj}^{-1} + \sum_{\substack{k=1 \\ k \neq i\,j}}^{m} a_{ik} a_{kj}^{-1}$$

$$= (n + m - 1)d + (-1)c + (m - 2)(-1)d\,.$$

This provides a second equation,

$$(n+1)d - c = 0\,. \qquad (2)$$

By substituting $c = (n+1)d$ into equation (1) and solving for d, the elements of A^{-1} are obtained as $d = 1/n(n+m)$ and $c = (n+1)/n(n+m)$.

Thus A^{-1} is an $m \times m$ symmetric matrix with elements

$$a_{ij}^{-1} = \begin{cases} (n+1)/n(n+m) & i = j \\ 1/n(n+m) & i \neq j. \end{cases}$$

APPENDIX D

PROJECTION MATRIX FOR THE MULTI-FACILITY ℓ_p DISTANCE MODEL

The calculation of the projection matrix $P = I - A_g'(A_gA_g')^{-1} A_g$ will be accomplished in three stages, beginning with the calculation of $A_g'(A_gA_g')^{-1}$, then $A_g'(A_gA_g')^{-1} A_g$ and ending with the final form for P.

1. Calculation of $A_g'(A_gA_g')^{-1}$

Recall that $A_g'(A_gA_g')^{-1} = \begin{bmatrix} B' & 0 \\ 0 & B' \end{bmatrix} \begin{bmatrix} A^{-1} & 0 \\ 0 & A^{-1} \end{bmatrix}$ with A^{-1} defined

as in Appendix B. A^{-1} can be written as $A^{-1} = (1/n(n+m))\, \bar{A}$ where $\bar{A}$ is an $m \times m$ symmetric matrix with elements

$$a_{ij} = \begin{cases} (n+1) & i=j \\ 1 & i \neq j \end{cases}$$

Then,

$$A_g'(A_gA_g')^{-1} = 1/n(n+m) \begin{bmatrix} B'\bar{A} & 0 \\ 0 & B'\bar{A} \end{bmatrix}.$$

Using $B = [B_1 \ldots B_{m-1}\, C_1 \ldots C_m]$, $B'\bar{A}$ will contain the submatrices $B_1'\bar{A}, \ldots, B_{m-1}'\bar{A}$, $C_1'\bar{A}, \ldots, C_m'$. From the definitions of B_t and C_r, the elements $b_{ij}' \in B_t'$ and $c_{ij}' \in C_r'$ can be expressed as

$$b_{ij}' = \begin{cases} 1 & j=t \\ -1 & j=i+t \\ 0 & \text{otherwise} \end{cases} \quad i = 1, \ldots, m-t; j = 1, \ldots, m; t = 1, \ldots, m-1$$

and $$c_{ij}' = \begin{cases} 1 & j=r \\ 0 & \text{otherwise} \end{cases} \quad i = 1, \ldots, n; j = 1, \ldots, m; r = 1, \ldots, m.$$

Take any element $x_{ij} \in B_t'\bar{A}$ for $t = 1, \ldots, m-1$;

$$x_{ij} = \sum_{k=1}^{m} b_{ij}' \, \bar{a}_{kj}$$

$$= \sum_{\substack{k=1 \\ k \neq t, i+t}}^{m} b_{ik}' \, \bar{a}_{kj} + b_{it}' \, \bar{a}_{tj} + b_{i(i+t)}' \, \bar{a}_{(i+t)j}$$

$$= \bar{a}_{tj} - \bar{a}_{(i+t)j} \quad \text{for } i = 1,\ldots, m-t \text{ and } j = 1, \ldots, m.$$

For $j=t$, $x_{it} = a_{tt} - a_{(i+t)t} = n+1-1 = n.$

For $j=i+t$, $x_{i(i+t)} = a_{t(i+t)} - a_{(i+t)(i+t)} = 1-(n+1) = -n.$

For $j \neq t, i+t$; $x_{ij} = 1-1 = 0.$

Thus, $D_t = B_t' \, A$ is an $(m-t) \times m$ matrix with elements

$$d_{ij} = \begin{cases} n & j=t \\ -n & j=i+t \\ 0 & \text{otherwise} \end{cases} \quad \text{for } t = 1,\ldots, m-1 .$$

Take any element x_{ij} from the $n \times m$ matrix product $C_r' \, A$,

$$x_{ij} = \sum_{k=1}^{m} c_{ij}' \, \bar{a}_{kj}$$

$$= \sum_{\substack{k=1 \\ k \neq r}}^{m} c_{ik}' \, \bar{a}_{kj} + c_{ir}' \, \bar{a}_{rj}$$

$$= c_{ir}' \, \bar{a}_{rj} \quad \text{for } r = 1,\ldots m.$$

For $j=r$, $x_{ir} = c_{ir}' \, a_{rr} = 1\,(n+1) = n+1.$

For $j \neq r$, $x_{ij} = c_{ir}' \, a_{rj} = 1\,(1) = 1.$

Thus $E_r = C_r' A$ is an $n \times m$ matrix with elements

$$e_{ij} = \begin{cases} n+1 & j=r \\ 1 & \text{otherwise} \end{cases} \quad \text{for } r = 1,\ldots, m.$$

2. **Calculation of $A_g'(A_gA_g')^{-1}A_g$**

$$A_g'(A_gA_g')^{-1}A_g = 1/n(n+m)\begin{bmatrix} B'\bar{A} & 0 \\ 0 & B'\bar{A} \end{bmatrix}\begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix}$$

$$= 1/n(n+m)\begin{bmatrix} B'\bar{A}B & 0 \\ 0 & B'\bar{A}B \end{bmatrix},$$

where

$$B'\bar{A}B = \begin{bmatrix} D_1 \\ \vdots \\ D_{m-1} \\ E_1 \\ \vdots \\ E_m \end{bmatrix}[B_1 \cdots B_{m-1} C_1 \cdots C_m]$$

$$= \left[\begin{array}{ccc|ccc} D_1B_1 & \cdots & D_1B_{m-1} & D_1C_1 & \cdots & D_1C_m \\ \vdots & & \vdots & \vdots & & \vdots \\ D_{m-1}B_1 & \cdots & D_{m-1}B_{m-1} & D_{m-1}C_1 & \cdots & D_{m-1}C_m \\ \hline E_1B_1 & \cdots & E_1B_{m-1} & E_1C_1 & \cdots & E_1C_m \\ \vdots & & \vdots & \vdots & & \vdots \\ E_mB_1 & \cdots & E_mB_{m-1} & E_mC_1 & \cdots & E_mC_m \end{array}\right].$$

It is necessary now to derive the elements from the various matrix products in B'AB.

2(a) **Calculation of $D_t\, B_s$**

For x_{ij} from the $(m-t)\times(m-s)$ matrix $D_t\, B_s$ and $s, t = 1, \ldots, m-1$,

$$x_{ij} = \sum_{k=1}^{m} d_{ik}\, b_{kj}$$

If $k \neq t,\ i+t,\ s$ or $s+j$ then either d_{ik} or b_{kj} is zero.

By writing x_{ij} in terms of these values for k and imposing restriction on the indices to prevent duplication of terms, then

$$x_{ij} = \begin{cases} d_{is}b_{sj} + d_{i(j+s)}b_{(j+s)j} & s=t \\ d_{i(i+t)}b_{(i+t)j} + d_{i(j+s)}b_{(j+s)j} + d_{it}b_{tj} & s\neq t,\ i+t=s,\ j+s\neq t,\ i+t\neq j+s \\ d_{it}b_{tj} + d_{i(i+t)}b_{(i+t)j} + d_{is}b_{sj} & s\neq t,\ i+t\neq s,\ j+s\neq t,\ i+t=j+s \\ d_{is}b_{sj} + d_{i(i+t)}b_{(i+t)j} + d_{i(j+s)}b_{(j+s)j} & s\neq t,\ i+t\neq s,\ j+s=t,\ i+t\neq j+s \\ d_{is}b_{sj} + d_{i(i+t)}b_{(i+t)j} + d_{it}b_{tj} + d_{i(j+s)}b_{(j+s)j} & s\neq t,\ i+t\neq s,\ j+s\neq t,\ i+t\neq j+s. \end{cases}$$

<u>Case 1</u> $s=t$ and $x_{ij} = d_{it}(1) + d_{i(j+t)}(-1) = n - d_{i(j+t)}$.

For $i = j$, $x_{ii} = n - d_{i(i+t)} = n-(-n) = 2n$

For $i\neq j$, $x_{ij} = n - 0 = n$

D_tB_t is an $(m-t)\times(m-t)$ matrix with elements

$$x_{ij} = \begin{cases} 2n & i=j \\ n & \text{otherwise} \end{cases} \quad \text{for } t=1,\ldots,m-1.$$

<u>Case 2</u> $s<t$. If $i+t=s$ then $i=s-t$ which is impossible since $s<t$. Thus $i\neq s-t$

For $i+t=j+s$, $x_{ij} = d_{it}b_{tj} + d_{i(i+t)}b_{(j+s)j} + d_{is}b_{sj}$

$$= nb_{t(i+t-s)} + (-n)(-1) + d_{is}(1)$$

Since $s<t$, then $s<t+i$ and hence $d_{is}=0$

Since $s\neq t$ and $t\neq(i+t-s)+s$, then $b_{t(i+t-s)} = 0$, and

$$x_{ij} = n \qquad \text{for } j = i+t-s$$

For $j+s=t$, $x_{ij} = d_{is}b_{sj} + d_{i(i+t)}b_{(i+t)j} + d_{it}b_{t(t-s)}$

$$= d_{is}(1) + (-n)b_{(i+t)j} + n(-1)$$

Since $s<t$, then $s<i+t$ and $d_{is}=0$. Also $i+t\neq j+s$ and so $b_{(i+t)j} = 0$ and $x_{ij} = -n$ for $j=t-s$.

For $j+s\neq t$, $i+t\neq j+s$ and $i+t\neq s$, then $d_{i(j+s)}=0$ and $b_{(i+t)j} = 0$.

Then $x_{ij} = d_{is}b_{sj} + d_{it}b_{tj}$

$$= d_{is} + nb_{tj}.$$

Since $s<t$ and $s\neq t+i$ then $d_{is}=0$. Since $j+t\neq s+j$, then $b_{tj} = 0$ and $x_{ij} = 0$ for $s<t$ and $j\neq t-s$ or $j\neq i+t-s$.

Therefore D_tB_s is an $(m-t)\times(m-s)$ matrix for $s<t=2,\ldots,m-1$:

$$\text{with elements } x_{ij} = \begin{cases} -n & j=t-s \\ n & j=i+t-s \\ 0 & \text{otherwise} \end{cases}$$

<u>Case 3</u> $s>t$. If $j+s=t$ then $j=t-s$ which contradicts $s>t$.

For $i+t=s$, $x_j = d_{is}b_{sj} + d_{i(j+s)}b_{(j+s)j} + d_{it}\, b_{tj}$

$$= d_{(t-s)s}(1) + d_{i(j+s)}(-1) + n\ b_{tj}$$

$$= -n - d_{i(j+s)} + n\ b_{tj}$$

Since $j+s \neq t$ and $j+s \neq i+t$, then $d_{i(j+s)} = 0$

Since $s>t$ and $j+s>t$, then $b_{tj}=0$ and $x_{ij} = -n$ for $i = s-t$.

For $i+t=j+s$, $x_{ij} = d_{it}\, b_{tj} + d_{i(i+t)}b_{(j+s)} + d_{ir}\, b_{rj}$

$$= n\, b_{tj} + (-n)(-1) + d_{ir}.$$

Since $t \neq j+s$ and $t<s$, then $b_{tj}=0$. Since $s>t$ and $s \neq i+t$ then $d_{is}=0$ and $x_{ij} = n$ for $i = j+s-t$.

For $i+t \neq s$, $i+t \neq j+s$ amd $j+s \neq t$, then $d_{i(i+t)} = 0$, $b_{(j+s)j} = 0$, $d_{is} = 0$, $b_{tj} = 0$, and $x_{ij} = 0$.

Thus, $D_t B_s$ is an $(m-t)\times(m-s)$ matris for $t<s=2,\ldots,m-1$.

$$\text{with elements } x_{ij} = \begin{cases} -n & i=s-t \\ n & i=j+s-t \\ 0 & \text{otherwise} \end{cases}$$

D_tB_s is the transpose of $D_s B_t$.

2(b) Calculation of D_tC_u

The calculation of D_tC_u proceeds as follows; for $x_{ij} \in D_tC_u$, $t=1,\ldots,m-1$ and $u=1,\ldots,m$

$$x_{ij} = \sum_{k=1}^{m} d_{ik} c_{kj}$$

$$= \left[\sum_{\substack{k=1 \\ k \neq t, i+t}}^{m} d_{ik} c_{kj} \right] + d_{it} c_{tj} + d_{i(i+t)} c_{(i+t)j}$$

$$= n c_{tj} - n c_{(i+t)j} \qquad i = 1,\ldots,m-t;\; j = 1,\ldots,n.$$

Case 1: $t=u$

Then $x_{ij} = nc_{uj} - nc_{(i+u)j} = n-0=n$ and D_tC_t is an $(m-t)\times n$ constant matrix for $t=1,\ldots,m-1$ with elements $x_{ij}=n$.

Case 2: $t<u$

For $i+t=u$, $x_{ij} = nc_{(u-i)j} - nc_{uj} = n(0) - n(1) = -n$.

For $i+t\neq u$, $x_{ij} = n(0) - n(0) = 0$, and $D_t C_u$ is an $(m-t)\times n$ matrix with elements

$$x_{ij} = \begin{cases} -n & i=u-t \\ 0 & \text{otherwise} \end{cases}$$

for $t<u = 2,\ldots,m$.

Case 3: $t>u$

If $i+t=u$, then $i=u-t$ which is impossible since $t>u$. If $i+t\neq u$, then $x_{ij}=0$. Thus $D_tC_u = 0$ for $u<t=2, \ldots, m-1$.

2(c) Calculation of E_rB_s

For x_{ij} from the $n\times(m-s)$ matrix $E_r B_s$ and $s = 1, \ldots, m-1;\ r = 1, \ldots, m$,

$$x_{ij} = \sum_{k=1}^{m} e_{ik} b_{kj}$$

$$= \sum_{\substack{k=1 \\ k \neq s, j+s}}^{m} e_{ik} b_{kj} + e_{is} b_{sj} + e_{i(j+s)} b_{(j+s)j}$$

$$= e_{is} - e_{i(j+s)}$$

Case 1: $s = r$

Then $x_{ij} = e_{ir} - e_{i(j+r)} = n + 1 - 1 = n$.

$E_r B_r$ is an $n \times (m-r)$ matrix for $r = 1,\ldots,m-1$ with elements $x_{ij} = n$.

Case 2: $r < s$.

Since $r < s$, then $s \neq r$, $s+j \neq r$ and $x_{ij} = 1 - 1 = 0$.

$E_r B_s = 0$ for $r < s = 2, \ldots, m-1$

Case 3: $r > s$

If $j+s = r$, $x_{ij} = e_{is} - e_{ir} = 1 - (n+1) = -n$.

If $j+s \neq r$, $x_{ij} = 1 - 1 = 0$.

$E_r B_s$ is an $n \times (m-s)$ matrix for $s < r = 2,\ldots,m$, with elements

$$x_{ij} = \begin{cases} -n & j = r-s \\ 0 & \text{otherwise} \end{cases}$$

2(d) Calculation of $E_r C_u$

For x_{ij} from the $n \times n$ matrix $E_r C_u$ and $u, r = 1, \ldots, m$;

$$x_{ij} = \sum_{k=1}^{m} e_{ik} c_{kj}$$

$$= \sum_{\substack{k=1 \\ k \neq u}}^{m} e_{ik} c_{kj} + e_{iu} c_{uj}$$

$$= e_{iu}$$

Case 1: $u = r$

Then $x_{ij} = e_{ir} = n + 1$.

$E_r C_r$ is an $n \times n$ matrix with elements $x_{ij} = n+1$ for $r = 1, \ldots, m$.

Case 2: $r \neq u$.

Then $x_{ij} = e_{iu} = 1$.

$E_r C_u$ is an $n \times n$ matrix for $r, u = 1, \ldots, m$ with elements $x_{ij} = 1$ for $r \neq u$.

3. Projection Matrix

Using the properties that

$D_t B_s$ is the transpose of $D_s B_t$ and $E_r B_s$ is the transpose of $D_s C_r$,

$E_r C_r$ is an $n \times n$ constant matrix with elements $x_{ij} = n+1$ for $r = 1, \ldots, m$ and

$E_r C_u$ is an $n \times n$ unit matrix with elements $x_{ij} = 1$ for $r, u = 1, \ldots, m$ and $r \neq u$, the final form for $(-1/n(n+m))B'AB$ can be written as

$$1/n(n+m)\left[\begin{array}{cccc|ccccc}
-D_1B_1 & -D_1B_2 & \cdots & -D_1B_{m-1} & -D_1C_1 & -D_1C_2 & \cdots & -D_1C_{m-1} & -D_1C_m \\
(-D_1B_2)' & -D_2B_2 & \cdots & -D_2B_{m-1} & 0 & -D_2C_2 & \cdots & -D_2C_{m-1} & -D_2C_m \\
\vdots & & & \vdots & \vdots & & & & \vdots \\
(-D_1B_{m-1})' & \cdots & & -D_{m-1}B_{m-1} & 0 & 0 & \cdots & -D_{m-1}C_{n-1} & -D_{m-1}C_m \\
\hline
(-D_1C_1)' & 0 & \cdots & 0 & -E_1C_1 & -E_1C_2 & \cdots & & -E_1C_2 \\
(-D_1C_2)' & (-D_2C_2)' & \cdots & 0 & -E_1C_2 & -E_1C_1 & \cdots & & -E_1C_2 \\
\vdots & & & \vdots & \vdots & \vdots & & & \vdots \\
(-D_1C_m)' & & \cdots & (-D_{m-1}C_m)' & -E_1C_2 & -E_1C_2 & \cdots & & -E_1C_1
\end{array}\right]$$

$$P = I - A_g(A_gA_g')^{-1}A_g$$

$$= \begin{bmatrix} I-(1/n(n+m))B\bar{A}B & 0 \\ 0 & I-(1/n(n+m))B\bar{A}B \end{bmatrix}$$

$$= 1/(n+m)\begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix}$$

where R is an $[(m(m-1)/2+mn]\times[m(m-1)/2+mn]$ matrix given by

$$R = \left[\begin{array}{cccc|ccccc} F_1 & G_{12} & \cdots & G_{1(m-1)} & H_1 & K_{12} & \cdots & K_{1(m-1)} & K_{1m} \\ G_{12}' & F_2 & \cdots & G_{2(m-1)} & 0 & H_2 & \cdots & K_{2(m-1)} & K_{2m} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ G_{1(m-1)}' & G_{2(m-1)}' & \cdots & F_{m-1} & 0 & 0 & \cdots & H_{m-1} & K_{(m-1)m} \\ \hline H_1' & 0 & \cdots & 0 & L & N & & \cdots & N \\ K_{12}' & H_2' & \cdots & 0 & N & L & & \cdots & N \\ \vdots & & & \vdots & \vdots & & & & \vdots \\ K_{1m}' & & \cdots & K_{(m-1)m}' & N & & & \cdots & L \end{array}\right]$$

and F_t, G_{ts}, H_t, K_{tr}, L and N are defined as follows:

F_t is an $(m-t) \times (m-t)$ matrix with elements $f_{ij} = \begin{cases} n+m-2 & i=j \\ -1 & i \neq j \end{cases}$ for $t = 1,\ldots,m-1$;

G_{ts} is an $(m-t) \times (m-s)$ matrix with elements $g_{ij} = \begin{cases} 1 & i=s-t \\ -1 & i=j+s-t \\ 0 & \text{otherwise} \end{cases}$ for $t<s = 2,\ldots,m-1$;

H_t is an $(m-t) \times n$ constant matrix with $h_{ij} = -1$ for $t = 1, \ldots, m-1$;

K_{tr} is an $(m-t) \times n$ matrix with elements $k_{ij} = \begin{cases} 1 & i=r-t \\ 0 & \text{otherwise} \end{cases}$ for $t<r = 2,\ldots,m$;

L is an $n \times n$ matrix with elements $\ell_{ij} = \begin{cases} [n(n+m)-(n+1)]/n & i=j \\ -(n+1)/n & i \neq j \end{cases}$,

and N is an $n \times n$ constant matrix with elements $n_{ij} = -1/n$.

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