ON THE EXISTENCE OF ALMOST UNIFORM LINEAR SPACES

By

MONICA ANETTE ORAVAS

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AUTHOR: Monica Anette Oravas
(B.Sc., M.Sc.- McMaster University)

SUPERVISOR: Dr. Alexander Rosa

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ABSTRACT

In this thesis we investigate the existence of a particular class of linear space called an almost uniform linear space. An almost uniform linear space is a linear space in which exactly two lines (called long lines) have sizes u and w, respectively, and all other lines (called short lines) have the same size k ($k \geq 2$). We determine the necessary conditions for the existence of an almost uniform linear space, in the cases where the long lines intersect (or are disjoint) and have the same size (or distinct sizes).

Next, we are interested in establishing the sufficiency of said conditions for almost uniform linear spaces in which the short lines all have size two, three, four or five. If we assume that the short lines all have size two, this follows immediately. Also, we can show that the conditions are sufficient for almost uniform linear spaces in which the short lines have size three and

(i) the two long lines intersect and have the same size u, or

(ii) the two long lines intersect (or are disjoint) and have sizes $u \in \{5, 7, 9\}$ and $w \in \{7, 9, 13, 15\}$, where $u \neq w$.

By generalizing the conditions in (ii), we provide partial answers to the existence question for almost uniform linear spaces in which one long line has size $6t + 5$, $6t + 7$ or $6t + 9$ ($t > 0$) and the other long line has size w, $w > 6t + r$ ($r = 5,7,9$).

There are only partial solutions for the case of short lines of size four or five.

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TABLE OF CONTENTS

		PAGE
	INTRODUCTION	1
CHAPTER 1	BASIC CONCEPTS	
	§1.1 Preliminaries	5
	§1.2 Elementary Relations	9
	§1.3 Methods of Construction	14
CHAPTER 2	ALMOST UNIFORM LINEAR SPACES WITH SHORT LINES OF SIZE THREE	
	§2.1 Almost uniform linear spaces with two long lines of size u and short lines of size three	29
	§2.2 Almost uniform linear spaces with one long line of size 6t + 5, one long line of size w and short lines of size three	52
	§2.3 Almost uniform linear spaces with one long line of size 6t + 7, one long line of size w and short lines of size three	69
	§2.4 Almost uniform linear spaces with one long line of size 6t + 9, one long line of size w and short lines of size three	87
CHAPTER 3	ALMOST UNIFORM LINEAR SPACES WITH SHORT LINES OF SIZE FOUR	
	§3.1 Almost uniform linear spaces with two long lines of size u and short lines of size four	114
	§3.2 Almost uniform linear spaces with one long line of size u, one long line of size w and short lines of size four	117

TABLE OF CONTENTS(continued)

		PAGE
CHAPTER 4	ALMOST UNIFORM LINEAR SPACES WITH SHORT LINES OF SIZE FIVE	
	§4.1 Almost uniform linear spaces with two long lines of size u and short lines of size five	124
	§4.2 Almost uniform linear spaces with one long line of size u, one long line of size w and short lines of size five	126
CONCLUSION		132
BIBLIOGRAPHY		134

Introduction

A triple (**P**,**L**,I) is an incidence structure if the sets **P** and **L** are disjoint sets and I ⊂ **P** × **L** is an (incidence) relation. The elements of the set **P** are called points, and the elements of **L** are called lines. If P is a point in **P** and l is a line in **L**, and (P,l) ∈ I, we write P I l; we sometimes say that a point P is contained in line l. An incidence structure (**P**,**L**,I) in which every line contains two or more points, and every pair of points is contained on exactly one line is called a linear space. Well-known examples from geometry are affine and projective planes, and in higher dimensions, affine and projective spaces. Another example that is of special concern to us is a Steiner system S(2,k,v) which consists of a set of v points, and a set of blocks (lines), where each block has exactly k points, $k \geq 2$, and every pair of points is contained in one block (line); in particular, an S(2,3,v) is a Steiner triple system of order v, denoted by STS(v). In other words, Steiner systems S(2,k,v) are the uniform linear spaces.

There has been an extensive study of many classes of linear spaces using the traditional approach of classical synthetic geometry (cf.[B1] for a comprehensive bibliography). However, there are also strong connections between linear spaces and certain incidence systems called designs. A design is a pair (V,**B**) where V is a finite set of points and $\mathbf{B} = \{B_i : i \in I\}$ is a family of subsets of V called blocks. The order of the design (V,**B**) is $v = |V|$, the cardinality of V, and $K = \{|B_i| : i \in I\}$ is the set of block sizes of the design. Furthermore, a pairwise balanced design (PBD) with index 1 is a design in which every pair of points is contained in exactly one block. It is immediate from this definition that a linear space is essentially a PBD with index 1. From this viewpoint, we are then able to include in our considerations various results of combinatorial design theory.

Existence problems of PBDs with specific sets of block sizes are of particular interest. For instance, problems regarding the existence of PBDs with index 1 having exactly one block of size w and all other blocks of the same size k (these are sometimes called near uniform linear spaces) has been explored in more recent years. When k = 3, the necessary conditions were determined and the sufficiency of said conditions were established, in part, by Doyen and Wilson [D1] in 1973 for $w \equiv 1, 3 \pmod 6$, and by Mendelsohn and Rosa [M1] in 1983 for $w \equiv 5 \pmod 6$. The combined work of Brouwer, Lenz, Bermond, Bond, Wei, Zhu, Rees, and Stinson [R6] provided solution of the same problem for k = 4. The existence of PBDs with one block of size w and all other blocks of size $k \geq 5$ is largely an open question; Hamel, Mills, Mullin, Rees, Stinson and Yin [H2] have recently given an almost complete solution to the existence problem of a PBD with one block of size nine or thirteen, and all other blocks of size five.

A natural extension of the previous problem is to examine the existence of linear spaces with v points in which exactly two lines have designated sizes u and w, and the other lines have the same size k; these spaces will be called almost uniform linear spaces. There is a further distinction in that the two special lines (called long lines) may either intersect or be disjoint, and these two possibilities must be considered separately when determining the necessary conditions. We attempt to prove the sufficiency of these conditions, when the other lines (called short lines) all have the same size two, three, four and five, respectively. Most of our efforts will be concentrated on determining the existence of almost uniform linear spaces with short lines of size three.

Generally, to prove sufficiency for a given value of k, we must construct almost uniform linear spaces for each admissible v, u and w (the values v, u and w that satisfy the necessary conditions). We usually begin by constructing, whenever possible, the almost uniform linear space with minimum value of v (the minimum order). This is either accomplished by a direct or recursive construction. It will be evident later that, as a consequence of a very important result in combinatorial design theory, the Doyen-Wilson theorem, and Rees-Stinson theorem (cf. Theorems 1.7 and 1.41), the number of constructions to be found can be considerably reduced, provided that k = 3 or 4.

There are several kinds of recursive constructions which are employed throughout the thesis. Many of the almost uniform linear spaces can be built from existing group - divisible designs (cf. Definition 1.2) and balanced incomplete block designs (cf. Definition 1.5), possibly in conjunction with specific embeddings of particular designs into larger designs. This is most useful for almost uniform linear spaces whose order v is sufficiently large. Another recursive technique involves Skolem n-sequences and hooked Skolem n - sequences (cf. Definitions 1.26 and 1.28) which can be utilized to provide a convenient method of constructing some of the short lines of size three in almost uniform linear spaces of certain orders. For smaller values of v, direct constructions are often needed. It is remarked that we are able to completely settle the existence question for almost uniform linear spaces with short lines of size three and the two long lines of sizes u and w, where $u \in \{5, 7, 9\}$ and $w \in \{7, 9, 13, 15\}$.

An overview of the content of the thesis follows:

Chapter 1 covers many of the important definitions, terminology and theoretical results. The existence of almost uniform linear spaces with short lines of size two is established and the necessary conditions for the existence of almost uniform linear spaces with short lines of size $k > 2$ are then determined. A summary of recursive procedures is given in §1.3.

The central part of this thesis is concerned with constructions of almost uniform linear spaces with short lines of size three. First if we assume that the two long lines have the same size u, and that they intersect, the necessary conditions from §1.2 are proven to be sufficient in §2.1. Analogously, if the two long lines are disjoint, a partial solution is obtained. In the remainder of Chapter 2, there are numerous constructions of almost uniform linear spaces in which one of the long lines either has size $6t + 5$, $6t + 7$ or $6t + 9$, $t \geq 0$, and the other line has size w. The problem of existence is solved completely when $t = 0$ and $w \in \{7, 9, 13, 15\}$.

In Chapters 3 and 4, it is only possible to apply recursive techniques similar to the ones outlined in §1.3 in order to construct almost uniform linear spaces with short lines of size four or five. Since no suitable direct constructions could be found, there are far fewer constructions given, especially when we assume that the short lines have size five, since we do not even have an analogue of the Doyen-Wilson theorem.

In the conclusion some possible future research problems and other open questions are discussed, along with a prospectus of the results.

Chapter 1

Basic Concepts

§1.1 Preliminaries

Definition 1.1 An almost uniform linear space (AULS) is a linear space with v points in which two lines have sizes u and w (called long lines) and all other lines have the same size k (called short lines), denoted by $LS_i(v;\{k, u^*, w^*\})$ or $LS_d(v;\{k, u^*, w^*\})$, according to whether the long lines intersect or are disjoint. (For undefined design theoretical terms, we refer the reader to the books [B1], [B2], [H1] and [S4].)

If the two long lines have the same size u, we write $LS_i(v;\{k, u^{**}\})$ or $LS_d(v;\{k, u^{**}\})$. We define $LS_i(k,u^*,w^*) = \{v: \exists\ LS_i(v;\{k,u^*,w^*\})\}$, $LS_d(k,u^*,w^*) = \{v: \exists LS_d(v;\{k,u^*,w^*\})\}$. Furthermore, $LS(k,u^*,w^*) = LS_d(k,u^*,w^*) \cap LS_i(k,u^*,w^*)$. The sets $LS(k,u^*,w^*)$, $LS_i(k,u^*,w^*)$, $LS_d(k,u^*,w^*)$ are often referred to as the spectrum of almost uniform linear spaces (the spectrum for AULSs with intersecting long lines, and with disjoint long lines, respectively) for given k, u and w. In this paper we shall investigate the existence of AULSs with short lines of size k, where k is a positive integer such that $2 \le k \le 5$.

There are several kinds of designs which prove to be very useful in the recursive constructions given in this paper.

Definition 1.2 A group-divisible design (GDD) is a triple (X, **G**, **B**) such that
(1) **G** is a partition of X into subsets called groups,
(2) **B** is a class of subsets of X such that a group and a block contain at most one common point, and

(3) every pair of points from distinct groups occur in a unique block.

The group-type or simply type of a GDD(X, **G**, **B**) is the multiset $\{|G| : G \in \mathbf{G}\}$. To denote the type of a GDD, we use "exponential" notation; thus, a GDD of type

$g_1^{t_1} \cdots g_n^{t_n}$ is one where there are t_i groups of size g_i, $1 \le i \le n$. If K is a set of

positive integers, then we say that a GDD is a K-GDD if $|B| \in K$ for every $B \in \mathbf{B}$.

Definition 1.3 A GDD is resolvable if the block set can be partitioned into parallel classes (i.e. sets of blocks each of which partitions the point set).

Definition 1.4 A transversal design(TD(r, n)) is a GDD on rn points with r groups of size n and n^2 blocks of size r.

Definition 1.5 A balanced incomplete block design(BIBD) is a pair (V, **B**) with parameters v, k, λ which satisfy

(1) $|V| = v$,

(2) $|B| = k$ for all $B \in \mathbf{B}$,

(3) $|\{B: \{x, y\} \subset B, x \neq y, x, y \in V\}| = \lambda$, and

(4) $k < v$.

A BIBD with $\lambda = 1$ is denoted by (v, k, 1)-BIBD. In particular, a (v, 3, 1)-BIBD is called a Steiner triple system of order v (STS(v)).

The notion of embedding a Steiner triple system into a Steiner triple system of a larger order plays a crucial role in developing many of the recursive constructions of AULSs with short lines of size three.

Definition 1.6 An STS(w) (W, **A**) is embedded in an STS(v) (V, **B**) if $W \subset V$ and $\mathbf{A} \subset \mathbf{B}$.

When (W, **A**) is embedded in (V, **B**), we also say that (W, **A**) is a subsystem of (V, **B**).

A powerful theorem concerning embeddings of Steiner triple systems was proved by Doyen and Wilson in 1973 [D1]. It has direct consequences in the problems under consideration.

Theorem 1.7 Any STS(w) can be (properly) embedded in an STS(v) if and only if $v \geq 2w + 1$.

The Doyen-Wilson theorem can be interpreted in another way. We first introduce some essential terminology.

Definition 1.8 A PBD (W, **A**) is a subdesign of the PBD (V, **B**) if $W \subseteq V$ and $\mathbf{A} \subseteq \mathbf{B}$.

If $W \neq V$, then (W, **A**) is a proper subdesign.

Definition 1.9 An incomplete PBD(IPBD) is a triple (V, Y, **B**) where V is a set of points, $Y \subseteq V$ and **B** is a set of blocks satisfying:

(1) for any $B \in \mathbf{B}$, $|B \cap Y| \leq 1$, and

(2) any pair x, y such that not both x and y are in Y, is contained in exactly one block.

The set Y is called a hole. If $|B| \in K$ for every $B \in \mathbf{B}$, then we may write IPBD(v, w; K) where $v = |V|$ and $w = |Y|$. In particular, if $K = \{k\}$, we may then consider the existence problem for an IPBD with one hole Y of size w and all blocks B in **B** of size k:

Theorem 1.10 The necessary conditions for the existence of an IPBD(v, w; {k}) are $v \geq (k - 1)w + 1$, $v \equiv 1, w \pmod{k - 1}$ and $v(v - 1) \equiv w(w - 1) \pmod{k(k - 1)}$.

Proof: Consider a point P which is not in the set Y. Since P is contained in at least w blocks of size k, and each block contains k - 1 points other than P, $v \geq (k - 1)w + 1$. If P is a point which is not in the set Y, then all blocks through P have size k, and there are v - 1 points other than P. Then v - 1 points can be partitioned into (k - 1)-tuples; thus, $v \equiv 1 \pmod{k - 1}$. Similarly, v - w points can be partitioned into (k - 1)-tuples by considering a point R which is a point in the set Y. Hence, $v \equiv w \pmod{k - 1}$. Next, count the pairs of points in this IPBD. There are $v(v - 1)/2$ pairs of points, and $w(w - 1)/2$ pairs of points are from the set Y, and $tk(k - 1)/2$ pairs of points are from the t blocks of size k. Thus, $v(v - 1)/2 = w(w - 1)/2 + tk(k - 1)/2$ or $v(v - 1) \equiv w(w - 1) \pmod{k(k - 1)}$.

We note that definitions 1.6 and 1.8 imply that a Steiner triple system (V, **B**) contains Steiner triple system (W, **A**) as a subdesign since (V, **B**) is a PBD(v;{3}). By deleting all the blocks in **A**, we can view W as a hole. Thus, an equivalent way of stating Theorem 1.7 is:

Theorem 1.11 Let $v \equiv 1, 3 \pmod 6$, $w \equiv 1, 3 \pmod 6$ and $v \geq 2w + 1$. Then there exists an IPBD(v, w;{3}).

Mendelsohn and Rosa [M1] have proved an analogue of the Doyen-Wilson theorem:

Theorem 1.12 Let $v, w \equiv 5 \pmod 6$ and $v \geq 2w + 1$. Then there exists an IPBD(v, w;{3}).

The embedding given in definition 1.6 and the design constructed in Theorem 1.12 will be used as building blocks in various recursive constructions.

Thus far, we have stated some pertinent definitions and theorems which will be valuable in constructing numerous classes of AULSs that are covered in succeeding chapters. The connection of the above discussion with the problems examined in this thesis will be explored in greater detail later, when the basic methods of construction are described.

§1.2 Elementary Relations

We can easily settle the existence problem for AULSs with the short lines of size two.

Theorem 1.13 Suppose that u, w $\geq$ 2.

(1) $LS_d(2, u^*, w^*\} = \{v: v \geq u + w\}$.

(2) $LS_i(2, u^*, w^*) = \{v: v \geq u + w - 1\}$.

Proof: Clearly, the total number of points v is at least u + w, or u + w - 1, according to whether the two long lines are disjoint or intersect. On the other hand, including a line of size two joining any two points not both on the same long line yields the desired linear space.

Next we investigate such problems for AULSs with short lines of size greater than two. It requires that we distinguish not only between the possibilities that the long lines may intersect or be disjoint, but also whether the long lines have the same size or not. Initially we shall determine the necessary conditions for the existence of AULSs with long lines of size u and short lines of size k, where k > 2.

Theorem 1.14 If $v \in LS_d(k, u^{**})$ then

$v \geq ku$ (1)

$v \equiv 1 \pmod{k - 1}$ (2)

$v \equiv u \pmod{k - 1}$ (2)′

$v(v - 1) \equiv 2u(u - 1) \pmod{k(k - 1)}$ (3)

Proof: (1) Let P be a point on one of the long lines. Since there are at least u short lines passing through P, and each short line contains k - 1 points other than P, the total number of points $v \geq (k - 1)u + u = ku$.

(2) Consider a point R which is not incident with either of the long lines. All lines through R are short lines, and there are in total v - 1 points other than R. These v - 1 points can be partitioned into (k - 1)-tuples, so $v \equiv 1 \pmod{k - 1}$. Similarly, by considering a point on one of the long lines, we conclude that the v - u points not on the long line can be partitioned into (k - 1)-tuples, hence $v \equiv u \pmod{k - 1}$.

(3) There are v(v - 1)/2 pairs of points altogether in the linear space. If we let r be the number of short lines, then there are rk(k - 1)/2 pairs of points determined from these lines, along with u(u - 1) pairs of points from the long lines. Thus, in total, $v(v - 1)/2 = u(u - 1) + rk(k - 1)/2$, which can be written as
$v(v - 1) = 2u(u - 1) + rk(k - 1)$. Hence, $v(v - 1) \equiv 2u(u - 1) \pmod{k(k - 1)}$.

Corollary 1.15

(i) If $v \in LS_d(3, u^{**})$ then $v \geq 3u$, $u \equiv 1, 3 \pmod 6$ and $v \equiv 1, 3 \pmod 6$.

(ii) If $v \in LS_d(4, u^{**})$ then $v \geq 4u$, $u \equiv 1,4 \pmod{12}$ and $v \equiv 1,4 \pmod{12}$;
or $u \equiv 7,10 \pmod{12}$ and $v \equiv 1, 4 \pmod{12}$.

(iii) If $v \in LS_d(5, u^{**})$ then $v \geq 5u$, $u \equiv 1, 5 \pmod{20}$ and $v \equiv 1, 5 \pmod{20}$;
or $u \equiv 13 \pmod{20}$ and $v \equiv 9, 17 \pmod{20}$.

Proof:

(i) Statement (2) of Theorem 1.14 gives $v \equiv 1 \pmod 2$. Also, u cannot be even, for otherwise, in (2)′ $v \equiv 0 \pmod 2$, which contradicts the above. If $u \equiv 1, 3 \pmod 6$, then from (3), $v(v - 1) \equiv 0 \pmod 6$ i.e. $v \equiv 1, 3 \pmod 6$. We cannot have $u \equiv 5 \pmod 6$, since (3) then gives $v(v - 1) \equiv 4 \pmod 6$ which has no solution.

(ii) We first note that $u \not\equiv r \pmod{12}$ where $r \neq 1, 4, 7, 10$ since (2)′ then implies that $v \equiv 0, 2 \pmod 3$, contradicting (2). If $u \equiv 1, 4 \pmod{12}$ or $u \equiv 7, 10 \pmod{12}$ then $v(v - 1) \equiv 0 \pmod{12}$, hence $v \equiv 1, 4 \pmod{12}$.

(iii) Firstly, $u \not\equiv r \pmod{20}$ where $r \neq 1, 5, 9, 13$ or 17 since (2)′ gives

$v \equiv 0,2$ or $3 \pmod 4$ which contradicts (2). If $u \equiv 1, 5 \pmod{20}$ then $v(v - 1) \equiv 0 \pmod{20}$ or $v \equiv 1, 5 \pmod{20}$. If $u \equiv 13 \pmod{20}$ then $v(v - 1) \equiv 12 \pmod{20}$ i.e. $v \equiv 9, 17 \pmod{20}$. Finally $u \equiv 9, 17 \pmod{20}$ is impossible since $v(v - 1) \equiv 4 \pmod{20}$ has no solution.

Theorem 1.16 If $v \in LS_i(k, u^{**})$ then

$v \geq ku - k + 1$ (1)

$v \equiv 1 \pmod{k - 1}$ (2)

$v \equiv u \pmod{k - 1}$ (2)′

$v(v - 1) \equiv 2u(u - 1) \pmod{k(k - 1)}$ (3)

Proof: We observe that (2), (2)′ and (3) follow by applying precisely the same arguments as in Theorem 1.14. In order to prove (1), let P be a point on one of the long lines. Since there are at least $u - 1$ short lines through P, it readily follows that $v \geq (k - 1)(u - 1) + u = ku - k + 1$.

Corollary 1.17

(i) If $v \in LS_i(3, u^{**})$ then $v \geq 3u - 2$, $u \equiv 1, 3 \pmod 6$ and $v \equiv 1, 3 \pmod 6$.

(ii) If $v \in LS_i(4, u^{**})$ then $v \geq 4u - 3$, $u \equiv 1, 4 \pmod{12}$ and $v \equiv 1, 4 \pmod{12}$; or $u \equiv 7, 10 \pmod{12}$ and $v \equiv 1, 4 \pmod{12}$.

(iii) If $v \in LS_i(5, u^{**})$ then $v \geq 5u - 4$, $u \equiv 1, 5 \pmod{20}$ and $v \equiv 1, 5 \pmod{20}$; or $u \equiv 13 \pmod{20}$ and $v \equiv 9, 17 \pmod{20}$.

Proof: The arguments are precisely the same as in Corollary 1.15.

Theorem 1.18 Let $u < w$. If $v \in LS_d(k, u^*, w^*)$ then

$v \geq (k - 1)w + u$ (1)

$v \equiv 1 \pmod{k - 1}$ (2)

$v \equiv u \pmod{k - 1}$, $v \equiv w \pmod{k - 1}$ (2)′

$v(v - 1) \equiv u(u - 1) + w(w - 1) \pmod{k(k - 1)}$ (3)

Proof: (2) and (2)′ are again obtained in the same way as before. To prove (1), take a point P on the long line of size u. Since there are at least w short lines through P, clearly $v \geq (k - 1)w + u$. We know $v(v - 1)/2$ is the total number of pairs of points. On the other hand, if r is the number of short lines, there are $rk(k - 1)/2$ pairs of points determined from these lines, as well as $(u(u-1) + w(w-1))/2$ pairs of points from the long lines. Therefore, $v(v-1)/2 = (u(u-1) + w(w-1) + rk(k-1))/2$ or $v(v-1) \equiv u(u-1) + w(w-1) \pmod{k(k-1)}$.

Corollary 1.19

(i) If $v \in LS_d(3, u^*, w^*)$ then $v \geq 2w + u$, $u \equiv 1, 3 \pmod 6$, $w \equiv 1, 3 \pmod 6$ and $v \equiv 1, 3 \pmod 6$; or $u \equiv 5 \pmod 6$, $w \equiv 1, 3 \pmod 6$ and $v \equiv 5 \pmod 6$.

(ii) If $v \in LS_d(4, u^*, w^*)$ then $v \geq 3w + u$, $u \equiv 1, 4 \pmod{12}$, $w \equiv 1, 4 \pmod{12}$ and $v \equiv 1, 4 \pmod{12}$; or $u \equiv 7, 10 \pmod{12}$, $w \equiv 7, 10 \pmod{12}$ and $v \equiv 1, 4 \pmod{12}$; or $u \equiv 1, 4 \pmod{12}$, $w \equiv 7, 10 \pmod{12}$ and $v \equiv 7, 10 \pmod{12}$.

(iii) If $v \in LS_d(5, u^*, w^*)$ then $v \geq 4w + u$, $u \equiv 1, 5 \pmod{20}$, $w \equiv 1, 5 \pmod{20}$ and $v \equiv 1, 5 \pmod{20}$; or $u \equiv 1, 5 \pmod{20}$, $w \equiv 13 \pmod{20}$ and $v \equiv 13 \pmod{20}$; or $u \equiv 1, 5 \pmod{20}$, $w \equiv 9, 17 \pmod{20}$ and $v \equiv 9, 17 \pmod{20}$; or $u \equiv 13 \pmod{20}$, $w \equiv 13 \pmod{20}$ and $v \equiv 9, 17 \pmod{20}$.

Proof:

(i) We note that neither u nor w can be even. If $u \equiv 0 \pmod 2$ or $w \equiv 0 \pmod 2$, then $v \equiv 0 \pmod 2$ by (2)′ of Theorem 1.18, contradicting (2). If $u \equiv 1, 3 \pmod 6$ and $w \equiv 1, 3 \pmod 6$, then (3) implies that $v(v - 1) \equiv 0 \pmod 6$, in all cases, hence $v \equiv 1, 3 \pmod 6$. If $u \equiv 5 \pmod 6$ and $w \equiv 1, 3 \pmod 6$ then $v(v - 1) \equiv 2 \pmod 6$ or $v \equiv 5 \pmod 6$. We cannot have $u \equiv 5 \pmod 6$ and $w \equiv 5 \pmod 6$, since (3) gives $v(v - 1) \equiv 4 \pmod 6$ which has no solution.

(ii) Neither u nor w can be congruent to 0, 2, 3, 5, 6, 8, 9 or 11 (mod 12), since $v \equiv 0$ or $2 \pmod 3$ which is impossible. If $u \equiv 1, 4 \pmod{12}$ and $w \equiv 1, 4 \pmod{12}$, or $u \equiv 7, 10 \pmod{12}$ and $w \equiv 7, 10 \pmod{12}$, then $v(v - 1) \equiv 0 \pmod{12}$ i.e. $v \equiv 1, 4 \pmod{12}$. If $u \equiv 1, 4 \pmod{12}$ and $w \equiv 7, 10 \pmod{12}$ then

$v(v - 1) \equiv 6 \pmod{12}$ i.e. $v \equiv 7, 10 \pmod{12}$.

(iii) Certainly neither u nor w can be congruent to r (mod 20) where $r \neq 1, 5, 9, 13$ or 17, since we would otherwise obtain $v \equiv 0, 2$ or $3 \pmod 4$ which contradicts (2). If $u \equiv 1, 5 \pmod{20}$ and $w \equiv 1, 5 \pmod{20}$, then $v(v - 1) \equiv 0 \pmod{20}$ or $v \equiv 1, 5 \pmod{20}$. If $u \equiv 1,5 \pmod{20}$ and $w \equiv 9, 17 \pmod{20}$, or $u \equiv 13 \pmod{20}$ and $w \equiv 13 \pmod{20}$ then $v(v - 1) \equiv 12 \pmod{20}$ i.e. $v \equiv 9, 17 \pmod{20}$. If $u \equiv 1, 5 \pmod{20}$ and $w \equiv 13 \pmod{20}$ then $v(v - 1) \equiv 16 \pmod{20}$ i.e. $v \equiv 13 \pmod{20}$. We note that $u \equiv 9 \pmod{20}$ and $w \equiv 9, 17 \pmod{20}$, or $u \equiv 17 \pmod{20}$ and $w \equiv 17 \pmod{20}$ is not possible since (3) gives $v(v - 1) \equiv 4 \pmod{20}$. Also, $u \equiv 9 \pmod{20}$ and $w \equiv 13 \pmod{20}$, or $u \equiv 13 \pmod{20}$ and $w \equiv 17 \pmod{20}$ since $v(v - 1) \equiv 8 \pmod{20}$ has no solution.

Theorem 1.20: Let $u < w$. If $v \in LS_i(k, u^*, w^*)$ then

$v \geq (k - 1)w + u - k + 1$ (1)

$v \equiv 1 \pmod{k - 1}$ (2)

$v \equiv u \pmod{k - 1}$, $v \equiv w \pmod{k - 1}$ (2)′

$v(v - 1) \equiv u(u - 1) + w(w - 1) \pmod{k(k - 1)}$. (3)

Proof: (2), (2)′ and (3) follow as in Theorem 1.18. Let R be a point on one of the long lines. Since there are at least w - 1 short lines through R,

$v \geq (k - 1)(w - 1) + u = (k - 1)w + u - k + 1.$

Corollary 1.21

(i) If $v \in LS_i(3, u^*, w^*)$ then $v \geq 2w + u - 2$, $u \equiv 1, 3 \pmod 6$, $w \equiv 1, 3 \pmod 6$ and $v \equiv 1, 3 \pmod 6$; or $u \equiv 5 \pmod 6$, $w \equiv 1, 3 \pmod 6$ and $v \equiv 5 \pmod 6$.

(ii) If $v \in LS_i(4, u^*, w^*)$ then $v \geq 3w + u - 3$, $u \equiv 1, 4 \pmod{12}$, $w \equiv 1, 4 \pmod{12}$ and $v \equiv 1, 4 \pmod{12}$; or $u \equiv 7, 10 \pmod{12}$, $w \equiv 7, 10 \pmod{12}$ and $v \equiv 1, 4 \pmod{12}$; or $u \equiv 1, 4 \pmod{12}$, $w \equiv 7, 10 \pmod{12}$ and $v \equiv 7, 10 \pmod{12}$.

(iii) If $v \in LS_i(5, u^*, w^*)$ then $v \geq 4w + u - 4$, $u \equiv 1, 5 \pmod{20}$, $w \equiv 1, 5 \pmod{20}$ and $v \equiv 1, 5 \pmod{20}$; or $u \equiv 1, 5 \pmod{20}$, $w \equiv 13 \pmod{20}$ and $v \equiv 13 \pmod{20}$; or $u \equiv 1,5 \pmod{20}$, $w \equiv 9, 17 \pmod{20}$ and $v \equiv 9, 17 \pmod{20}$; or $u \equiv 13 \pmod{20}$,

$w \equiv 13 \pmod{20}$ and $v \equiv 9, 17 \pmod{20}$.

Proof: These conditions follow as in Corollary 1.19.

§ 1.3 Methods of Construction

There are essentially two types of construction that we use: direct and recursive. We will describe the recursive techniques which are applicable for constructing AULSs. In general, we partition the v points of the set P into certain subsets; for recursive constructions this is done in such a way that various design - theoretic results may be applied.

Definition 1.22 A partition π of the set P is a family of subsets P_i ($i \in I$) such that $\bigcup_{i \in I} P_i = P$

and for any P_i, P_j, either $P_i = P_j$ or $P_i \cap P_j = \emptyset$ for $i \neq j$.

The subsets P_i are called cells. In numerous constructions, we shall designate these cells by capital letters such as A, B and C. If there are n_i occurrences of cells of size v_i, we write $\Pi(v_1^{n_1},\ldots,v_k^{n_k})$. In particular, $\Pi(1^1, v_1^{n_1},\ldots,v_{k-1}^{n_{k-1}})$ is a partition in which the set $\{\infty\}$ is a cell.

The first construction makes extensive use of existing {3}-GDDs to build the required linear spaces. One particular class of {3}-GDD proved to be invaluable in many of our constructions. Colbourn, Hoffman and Rees [C2] established the necessary and sufficient conditions for the existence of a {3}-GDD of type $g^t x^1$ (the group of x points is called the long group). Their result is stated below.

Theorem 1.23 Let g, t and x be nonnegative integers. There exists a {3}-GDD of type $g^t x^1$ if and only if the following conditions are all satisfied:

(i) if $g > 0$, then $t \geq 3$, or $t = 2$ and $x = g$, or $t = 1$ and $x = 0$, or $t = 0$;

(ii) $x \leq g(t - 1)$ or $gt = 0$;

(iii) $g(t - 1) + x \equiv 0 \pmod 2$ or $gt = 0$;

(iv) $gt \equiv 0 \pmod 2$ or $x = 0$;

(v) $\frac{1}{2}g^2t(t - 1) + gtx \equiv 0 \pmod 3$.

The significance of applying Theorem 1.23 is that every pair of points from two distinct groups is therefore already contained in a short line. However, if two points are from the same group, then the pair of points is not contained in a short line. Thus, if we wish to construct an AULS from such a GDD, then each of these pairs of points must be contained either in a short line or in one of the long lines. Consequently, an appropriate additional "structure" must be imposed on each of the t groups of size g and the one group of size x. An amended procedure is necessary for an AULS in which the two long lines intersect, and we will find that Theorems 1.7 and 1.12 need to be utilized in certain constructions for v sufficiently large. The general methods of approach are herewith summarized.

Theorem 1.24 Assume g, t and x are nonnegative integers which satisfy the conditions of Theorem 1.23.

(a) If $t \equiv 0,2 \pmod 6$ and $g,x \equiv 1 \pmod 6$, or $g \equiv 1 \pmod 6$, $x \equiv 3 \pmod 6$ and $t \equiv 0,4 \pmod 6$, or $g \equiv 3 \pmod 6$, $x \equiv 1 \pmod 6$ and $t \equiv 0 \pmod 2$, or $g,x \equiv 3 \pmod 6$ and $t \equiv 0 \pmod 2$ then $gt + x \in LS_d(3, g^{**}) \cup LS_d(3,g^*,x^*)$.

Suppose that g_1 is a nonnegative integer such that $g_1 \equiv 1, 3 \pmod 6$ and

$7 \leq g_1 \leq \frac{1}{2}(g - 1)$ or $7 \leq g_1 \leq \frac{1}{2}(x - 1)$. Then $gt + x \in LS_d(3,g_1^*,g^*) \cup LS_d(3,g_1^*,x^*)$.

(b) If $g, x \equiv 0 \pmod 6$ and $t \geq 3$, or $g \equiv 0 \pmod 6$, $x \equiv 2 \pmod 6$ and $t \geq 3$, or $g \equiv 2 \pmod 6$, $x \equiv 0 \pmod 6$ and $t \equiv 0, 1, 3, 4 \pmod 6$, or $g, x \equiv 2 \pmod 6$ and

$t \equiv 0, 2, 3, 5 \pmod 6$, then $gt + x + 1 \in LS_i(3,(g+1)^{**}) \cup LS_i(3,(g+1)^*,(x+1)^*)$.

Suppose that g_1 is a nonnegative integer such that $g_1 \equiv 1, 3 \pmod 6$ and $7 \le g_1 \le \frac{1}{2}g$ or $7 \le g_1 \le \frac{1}{2}x$. Then $gt + x + 1 \in LS_d(3,g_1^*,(g+1)^*) \cup LS_d(3,g_1^*,(x+1)^*)$.

If $g_1 \equiv 0, 2 \pmod 6$ and $6 \le g_1 \le \frac{1}{2}g - 1$ or $6 \le g_1 \le \frac{1}{2}x - 1$, then

$gt + x + 1 \in LS_i(3,(g_1+1)^*,(g+1)^*) \cup LS_i(3,(g_1+1)^*,(x+1)^*)$.

(c) If $g \equiv 0 \pmod 6$, $x \equiv 4 \pmod 6$ and $t \ge 3$, or $g \equiv 2 \pmod 6$, $x \equiv 4 \pmod 6$ and $t \equiv 0,3 \pmod 6$, then $gt + x + 1 \in LS_i(3,(g + 1)^*, (x + 1)^*)$.
Suppose that g_1 is a nonnegative integer such that $g_1 \equiv 5 \pmod 6$ and $5 \le g_1 \le \frac{1}{2}x$, then $gt + x + 1 \in LS_d(3,g_1^*,(g+1)^*)$. If $g_1 \equiv 4 \pmod 6$ and $4 \le g_1 \le \frac{1}{2}x - 1$ then

$gt + x + 1 \in LS_i(3,(g_1+1)^*,(g+1)^*)$.

Proof: (a) Form a partition $\pi(g^t, x^1)$ and construct a {3}-GDD of type $g^t x^1$. Replace two of the groups of the GDD either by two lines both of size g, or by one line of size g and one line of size x. The remaining t - 1 groups of points either all have size g or t - 2 of the groups have size g and one group has size x. Since the number of points in each group is congruent to 1 or 3 (mod 6), we replace each of these groups by a Steiner triple system. Next, since $7 \le g_1 \le \frac{1}{2}(g - 1)$ or $7 \le g_1 \le \frac{1}{2}(x - 1)$ and $g_1 \equiv 1, 3 \pmod 6$, by the Doyen - Wilson theorem (Theorem 1.7), we can embed an STS(g_1) into an STS(g) or an STS(x). Replace the subdesign by a line of size g_1. If STS(g_1) is embedded in an STS(g), replace the remaining groups by copies of an

STS(g) in order to obtain an $LS_d(gt+x;\{3, g_1^\bullet, x^\bullet\})$. We must form an STS(x) in order to obtain an $LS_d(gt+x;\{3, g_1^\bullet, g^\bullet\})$. If STS(g_1) is embedded in an STS(x), we replace one group of points by a line of size g and replace each of the t - 1 remaining groups of size g by copies of an STS(g) in order to obtain an $LS_d(gt+x;\{3, g_1^\bullet, g^\bullet\})$.

(b) Define ∞ to be the intersection point of the two long lines. Form the partition $\pi(1^1, g^t, x^1)$ and construct a {3}-GDD of type $g^t x^1$. We declare the two long lines on the union of ∞ and the points of two appropriate groups, and form either an STS(g + 1) or an STS(x + 1) on ∞ and the points of a group with size g or x. Next, either construct an STS(g + 1) or an STS(x + 1) which contains ∞ and the subdesign STS(g_1), where ∞ is not a point in STS(g_1). Replace the subdesign by a line of size g_1, and place copies of an STS(g + 1) on ∞ and the points in each remaining cell of size g and, wherever appropriate, form an STS(x + 1) on ∞ and the points in the cell of size x, thereby proving that $gt + x + 1 \in LS_d(3, g_1^\bullet, (g+1)^\bullet) \cup LS_d(3, g_1^\bullet, (x+1)^\bullet)$.

Finally, if $g_1 \equiv 0, 2 \pmod 6$ and $6 \leq g_1 \leq \frac{1}{2}g - 1$ or $6 \leq g_1 \leq \frac{1}{2}x - 1$, follow the basic arguments in the previous paragraph except that an STS(g_1 + 1) is embedded in an STS(g +1) or an STS(x + 1), and the subdesign STS(g_1 + 1) also contains ∞.

(c) Form the partition $\pi(1^1, g^t, x^1)$ and construct a {3}-GDD of type $g^t x^1$. Declare two long lines on ∞ and two cells of sizes g and x respectively. Place copies of an STS(g + 1) on ∞ and the points of each remaining cell. Next, apply Theorem 1.12 to construct an IPBD(x + 1, g_1;{3}) which contains ∞, but the hole does not contain ∞. Proceed as in the second part of (b) to prove that

$gt + x + 1 \in LS_d(3, g_1^\bullet, (g+1)^\bullet)$. Finally, construct an IPBD(x + 1, g_1 + 1;{3}), where ∞ belongs to the IPBD and the hole. The rest of the arguments are the same as above, thereby giving $gt + x + 1 \in LS_i(3, (g_1+1)^*, (g+1)^*)$.

Corollary 1.25 Assume that g, t, x are nonnegative integers that satisfy Theorem 1.24. Suppose that $g_1 \equiv 1, 3 \pmod 6$ and $7 \le g_1 \le \frac{1}{2}g$ or $7 \le g_1 \le \frac{1}{2}x$. Then $gt + x + 1 \in LS(3, g_1^*, (g+1)^*) \cup LS(3, g_1^*, (x+1)^*)$. If $g_1 \equiv 5 \pmod 6$ and $5 \le g_1 \le \frac{1}{2}x$, then $gt + x + 1 \in LS(3, g_1^*, (g+1)^*)$.

Proof: In order to show that $gt + x + 1 \in LS_d(3, g_1^*, (g+1)^*) \cup LS_d(3, g_1^*, (x+1)^*)$ apply Theorem 1.24(b). To show that $gt + x + 1 \in LS_i(3, g_1^*, (g+1)^*) \cup LS_i(3, g_1^*, (x+1)^*)$, since $g_1 - 1 \equiv 0, 2 \pmod 6$ and $6 \le g_1 - 1 \le \frac{1}{2}g - 1$ or $6 \le g_1 - 1 \le \frac{1}{2}x - 1$ apply Theorem 1.24(b) where g_1 is replaced by $g_1 - 1$. Hence, we have shown that $gt + x + 1 \in LS(3, g_1^*, (g+1)^*) \cup LS(3, g_1^*, (x+1)^*)$. If $g_1 \equiv 5 \pmod 6$ and $5 \le g_1 \le \frac{1}{2}x$, follow the above procedure, applying Theorem 1.24(c).

It is not always possible to partition the points of an AULS in such a way that a {3}-GDD of the special class considered earlier may be used. Quite often transversal designs (cf. Definition 1.4) may assist us in forming the essential {3}-GDD. We are primarily interested in transversal designs TD(5, n) and TD(6, n). Typically what we do is to begin with a particular transversal design, and delete some points from two groups to obtain either a {4, 5, 6}-GDD or a {3, 4, 5}-GDD. We then recursively build the desired {3}-GDD using a form of Wilson's fundamental construction (FC) [W3]:

(FC) Let (X, G, B) be a GDD. Let **G** = $\{G_1, \dots, G_m\}$. Let each v ϵ X have an associated weight w(v). Suppose that for each block $\{v_1, \dots, v_k\}$ in **B**, there is a {3}-GDD with k groups, having sizes $w(v_1), \dots, w(v_k)$. Then there is a {3}-GDD whose groups have sizes $\sum_{v \in G_i} w(v)$ for i = 1, ... ,m.

Once the {3}-GDD has been recursively constructed, we then complete the task by applying basically the same arguments as in Theorem 1.24. Occasionally, we delete a block of a transversal design, deleting some points from two groups of the resulting GDD and apply FC again to obtain the required underlying {3}-GDD.

Although the majority of recursive constructions depend on existing {3}-GDDs, for many of the smaller values of v, there are no GDDs available. Another technique which permits us to construct AULSs of certain orders, mostly applicable when the two long lines intersect, involves the use of Skolem and hooked Skolem sequences.

Definition 1.26 A <u>Skolem n-sequence</u> is a sequence of length 2n, in which every integer 1, ... ,n appears exactly twice and the two appearances of the integer i are i apart.

It is well-known that a Skolem n-sequence is equivalent to an (A, n)-system, a set of ordered pairs $\{(a_i, b_i): i = 1, \dots, n\}$ such that $b_i - a_i = i$ for every i, and

$$\bigcup_{i=1}^{n} \{a_i, b_i\} = \{0, 1, \dots, 2n - 1\}.$$

Theorem 1.27 An (A, n)-system exists if and only if $n \equiv 0, 1 \pmod 4$.

Proof: See, e.g., [C3].

Definition 1.28 A <u>hooked Skolem n-sequence</u> is a sequence of length 2n + 1, in which every integer 1, ... ,n appears exactly twice, and the two appearances of the integer i are i apart, but the (2n)-th member of the sequence is a "hook" (or a "hole", or "blank").

A hooked Skolem n-sequence is equivalent to a (B, n)-system which is a set of ordered pairs $\{(a_i, b_i): i = 1, \ldots, n\}$ such that $b_i - a_i = i$ for every i, and $\bigcup_{i=1}^{n}\{a_i, b_i\} = \{0, 1, \ldots, 2n-2, 2n\}$.

Theorem 1.29 A (B, n)-system exists if and only if $n \equiv 2, 3 \pmod 4$.

Proof: See, e.g., [C3].

We need more definitions for our description of the second major construction.

Definition 1.30 Let $D_1 = (V_1, \mathbf{B}_1)$ and $D_2 = (V_2, \mathbf{B}_2)$ be two designs. Then a bijection
$\alpha: V_1 \to V_2$ is an <u>isomorphism</u> if the induced mapping $\alpha: \mathbf{B}_1 \to \mathbf{B}_2$ given by $\alpha(B) = \{\alpha(a): a \in B\}$ is also a bijection from $\mathbf{B}_1$ onto $\mathbf{B}_2$. We say that D_1 and D_2 are <u>isomorphic</u>; if $D_1 = D_2$, then α is an <u>automorphism</u>.

Automorphisms of a design form a group under the composition of mappings.

Definition 1.31 Let α be an automorphism of a design D with v points. Two points x and y of D are in the same orbit of points under α if $\alpha^t(x) = y$ for some $t \geq 1$. Two blocks B_m and B_n of D are in the same orbit of blocks if $\alpha^s(B_m) = B_n$ for some $s \geq 1$.

An automorphism α of D partitions the points and blocks of D into disjoint orbits since the property of being in the same orbit is an equivalence relation. An orbit of blocks (points) can be generated by any one of its members. Hence, we need only give one block called a starter block or base block, to represent each orbit of blocks in a design. Usually, for cyclic constructions, with $V = Z_v$, the natural automorphism $i \rightarrow i + 1 \pmod{v}$ is chosen. It can be proven that each of the differences between pairs of points, mod v, appears exactly once in a design with index $\lambda = 1$. To illustrate this, consider the construction of a cyclic STS(15): Let $V = Z_{15}$. We can generate the triples of our design from the starter triples 0 1 4, 0 2 8 and 0 5 10. There are two full orbits of triples, {0 1 4, 1 2 5, ... ,14 0 3} and {0 2 8, 1 3 9, ... ,14 1 6} i.e. each of these two orbits has length 15. The third orbit of triples is {0 5 10, 1 6 11, 2 7 12, 3 8 13, 4 9 14}, a short orbit of length 5, namely a ⅓- orbit. Observe that the differences are ± 1, ± 2, ± 3, ... , ± 7 and any pair of points with these differences appears exactly once.

In many of our constructions, we will find that it is convenient to represent the point - set P as the union $(\bigcup_{i=1}^{r} Z_{w_i} \times \{i\}) \cup \{\infty\}$ where $1 + \sum_{i=1}^{r} w_i = v$. Since every point of an AULS has a subscript, we shall state that pure differences arise from pairs of points of a line which have the same subscript, whilst mixed differences arise from those pairs of points of a line which have distinct subscripts. (For a general description of Bose's method of "symmetrically repeated differences", see, e.g., [H1].)

The second major recursive construction can now be presented.

Lemma 1.32 Suppose that u, v and w satisfy the necessary conditions of Corollaries 1.15, 1.17, 1.19 and 1.21, and $u < w$.

(a) If $u \equiv 3 \pmod 6$ and $w \equiv 1 \pmod 2$ or $u \equiv 5 \pmod 6$ and $w \equiv 1 \pmod 6$, then there exists an $LS_i(u + 2w - 2; \{3, u^*, w^*\})$.

(b) If there is a nonnegative integer $u_1 \equiv 0 \pmod 2$, $u_1 \geq 2u$, $w \equiv 3 \pmod 6$, $u_1 \equiv 0 \pmod 2$ and $u \equiv 1, 3$ or $5 \pmod 6$, or $w \equiv 5 \pmod 6$, $u_1 \equiv 0 \pmod 6$ and $u \equiv 1, 3 \pmod 6$, and $u_1 > w - 1$, then there exists an $LS(2u_1 + w ; \{3, u^*, w^*\})$.

Furthermore, if $u_1 \equiv 2 \pmod 6$, $w \equiv 1 \pmod 2$ and $u \equiv 1, 3 \pmod 6$, or $u_1 \equiv 4 \pmod 6$, $w \equiv 1 \pmod 6$ and $u \equiv 5 \pmod 6$, and $u_1 < w-1$, then there exists an $LS(u_1 + 2w - 1;\{3,u^*,w^*\})$.

Proof: (a) Form the partition $\pi(1^1, (u - 1)^1, (w - 1)^2)$.

<u>Case 1</u>: $u = 6t + 9$, $0 \leq t < (w - 9)/6$, where $w = 6r + s$; $s = 1, 3$ or 5, and $r \geq 1$. Cell A is the set $\{\infty\} \times \{1, ..., 6t + 8\}$. Cells B and C are given by the sets $Z_{6r+s-1} \times \{i\}$ where $i = 1, 2$. The point ∞ is the intersection point of the two long lines $\infty_1 \cdots \infty_{6t+8}$ and $0_2 1_2 \cdots (6r + s - 2)_2\infty$. We can classify the short lines to be of types ABC, BBC, ∞BB and BBB. By an easy method of counting, there are $(6t + 8)(6r + s - 1)$ lines of type ABC, $\frac{1}{2}(6r + s - 1)(6r + s - 6t - 9)$ lines of type BBC, $\frac{1}{2}(6r + s - 1)$ lines of type ∞BB and $(6r + s - 1)(t + 1)$ lines of type BBB. Construct the lines of types ∞BB and BBB, by considering the pure differences among the points of cell B,namely $\pm1, \pm2, \pm3, ..., \pm\frac{1}{2}(6r+s-1)$. The lines of type ∞BB can be generated from the starter line $\infty 0_1(\frac{1}{2}(6r + s - 1))_1$, developed mod $6r+s-1$, where $\infty + j = \infty$; $1 \leq j \leq \frac{1}{2}(6r + s - 1)$. The lines of type BBB are contained in $t + 1$ full orbits and therefore will cover $3t + 3$ pure differences. There are $\frac{1}{2}(6r - 6t + s - 9)$ remaining pure differences, denoted by $\pm j_1, ..., \pm j_{\frac{1}{2}(6r-6t+s-9)}$, which must be covered by lines of type BBC. We will construct Skolem or hooked Skolem sequences to assist us in formulating an easy way to determine these lines. If $s = 1$ and $r \equiv 2,3 \pmod 4$, or $s = 3$ and $r \equiv 0,3 \pmod 4$,or $s = 5$ and

r ≡ 0,1(mod 4), we can construct a Skolem ($\frac{1}{2}$(6r+s-3))-sequence, consisting of the pairs $(a_1, b_1), \ldots, (a_{\frac{1}{2}(6r+s-3)}, b_{\frac{1}{2}(6r+s-3)})$ such that $\bigcup_{i=1}^{\frac{1}{2}(6r+s-3)} \{a_i, b_i\} = \{0,1,\ldots,6r+s-4\}$.

Since the short lines of type BBC will cover some of the 6r+s-1 mixed differences, we can regard these mixed differences as elements of the ordered pairs in the Skolem sequence constructed above. Therefore, define the starter lines of type BBC to be $0_1(j_1)_1(b_{j_1})_2, \ldots, 0_1(j_{\frac{1}{2}(6r-6t+s-9)})_1(b_{j_{\frac{1}{2}(6r-6t+s-9)}})_2$. The pairs $(0+i)_1(a_{j_l}+i)_2$ (i = 1,...,6r+s-1) are covered since $a_{j_l} = b_{j_l} - j_l$ for any l=1,...,$\frac{1}{2}$(6r-6t+s-9). Clearly, the pairs $(0+i)_1(b_{j_l}+i)_2$ (i = 1, ... ,6r + s - 1) are also covered by lines of type BBC.

Since there are (6r + s - 1) - 2($\frac{1}{2}$(6r - 6t + s - 9)) = 6t + 8 mixed differences left, these must be covered by starter lines of type ABC. Therefore, the conclusion from the above discussion is that the starter lines of type ABC are

$$\infty_1 0_1(a_{k_1})_2, \ldots, \infty_{3t+3} 0_1(a_{k_{3t+3}})_2, \infty_{3t+4} 0_1(b_{k_1})_2, \ldots, \infty_{6t+6} 0_1(b_{k_{3t+3}})_2, \infty_{6t+7} 0_1(6r+s-3)_2,$$

$\infty_{6t+8} 0_1(6r+s-2)_2$,where $1 \le k_q \le \frac{1}{2}(6r+s-3)$ and $k_q \ne j_l$.If s=1 and r≡0,1(mod4),or s=3 and r ≡ 1,2 (mod4),or s=5 and r ≡ 2,3 (mod 4) then construct a hooked Skolem ($\frac{1}{2}$(6r+s-3))- sequence.The starter lines of types ∞BB,BBB and BBC are the same as before.The starter lines of type ABC are

$$\infty_1 0_1(a_{k_1})_2, \ldots, \infty_{3t+3} 0_1(a_{k_{3t+3}})_2, \infty_{3t+4} 0_1(b_{k_1})_2, \ldots, \infty_{6t+6} 0_1(b_{k_{3t+3}})_2, \infty_{6t+7} 0_1(6r+s-4)_2 \quad ,$$

$\infty_{6t+8}0_1(6r+s-2)_2$.

Case 2: u = 6t + 5, $0 \le t < (w - 5)/6$ and w = 6r + 1; $r \ge 1$.
Cell A is the set $\{\infty\} \times \{1, ..., 6t + 4\}$ and cells B and C are given by the sets $Z_{6r} \times \{i\}$; i = 1, 2. The point ∞ is the intersection point of the two long lines $\infty_1 \cdots \infty_{6t+4}\infty$ and $0_2 1_2 \cdots (6r - 1)_2\infty$. As in case 1, by counting, there are (6t + 4)(6r) lines of type ABC, (6r)(3r-3t-2) lines of type BBC, 3r lines of type ∞BB, and (6r)(⅓ + t) lines of type BBB. The lines of type ∞BB are generated from $\infty 0_1(3r)_1$, developed mod 6r. Since the lines of type BBB are contained in t full orbits and one ⅓ - orbit, we can generate the lines from the starter line $0_1(2r)_1(4r)_1$, developed mod 6r (the ⅓ - orbit) and t starter lines of the form $0_1 a_1(2a + 1)_1$, where $a \ge 1$. The remaining 3r - 3t - 2 differences $\pm j_1, ..., \pm j_{3r-3t-2}$ are covered by lines of type BBC. If $r \equiv 2, 3 \pmod 4$, construct a Skolem (3r - 1)-sequence such that

$\bigcup_{i=1}^{3r-1} \{a_i, b_i\} = \{0, 1, ..., 6r - 3\}$. The starter lines of type BBC are

$0_1(j_1)_1(b_{j_1})_2, ..., 0_1(j_{3r-3t-2})_1(b_{j_{3r-3t-2}})_2$. Using similar reasoning as in Case 1, the starter

lines of type ABC are $\infty_1 0_1(a_{k_1})_2, ..., \infty_{3t+1}0_1(a_{k_{3t+1}})_2, \infty_{3t+2}0_1(b_{k_1})_2, ..., \infty_{6t+2}0_1(b_{k_{3t+1}})_2$

$\infty_{6t+3}0_1(6r - 2)_2, \infty_{6t+4}0_1(6r - 1)_2$, where $1 \le k_q \le 3r - 1$, and $k_q \ne j_l$. If $r \equiv 0, 1$ (mod 4), construct a hooked Skolem (3r - 1)-sequence such that

$\bigcup_{i=1}^{3r-1} \{a_i, b_i\} = \{0, 1, ..., 6r - 4, 6r-2\}$. Lines of type BBC are generated as above. The

starter lines of type ABC are

$\infty_1 0_1 (a_{k_1})_2, \ldots, \infty_{3t+1} 0_1 (a_{k_{3t+1}})_2, \infty_{3t+2} 0_1 (b_{k_1})_2, \ldots, \infty_{6t+2} 0_1 (b_{k_{3t+1}})_2, \infty_{6t+3} 0_1 (6r - 3)_2,$

$\infty_{6t+4} 0_1 (6r - 1)_2$

(b) Form the partition $\pi(1^1, u_1^2, (w - 1)^1)$. If u ≡ 1, 3 (mod 6), by Theorem 1.7, construct an STS(u_1 + 1) one of whose points is ∞, which contains an STS(u) as a subdesign, and if u ≡ 5 (mod 6), by Theorem 1.12, construct an IPBD(u_1 + 1, u; {3}). Replace the STS(u) by a line of size u. Let cell A have size w - 1, and cells B, C each has size u_1. Apply the method of construction described in (a), constructing a Skolem (½(u_1 - 2))-sequence or a hooked Skolem (½(u_1 - 2))-sequence in order to generate the short lines of types BBC and ABC. If u_1 < w - 1, form partition $\pi(1^1, u_1^1, (w - 1)^2)$ and construct a Skolem (½(w - 3)) - sequence or a hooked Skolem (½(w - 3))-sequence following the construction in (a).

For some cases when we need to construct an AULS, most often of minimum order v, where the two disjoint long lines of sizes u and w, and u, w ≡ 1, 3 (mod 6), we use two results regarding Steiner triple systems [R7].

Theorem 1.33 Let S_1, S_2 be two Steiner triple systems of order n and 6k + 1, respectively, where k is a nonnegative integer. Then for any n ≥ 6k + 1 there exists a Steiner triple system S of order 2n + 6k + 1 containing S_1 and S_2 as disjoint subsystems.

Theorem 1.34 Let S_1,S_2 be two Steiner triple systems of order n(n ≡ 3 (mod 6)) and 6k + 3, respectively, where k is a nonnegative integer. Then for any n ≥ 6k + 3 there exists a Steiner triple system S of order 2n + 6k + 3 containing S_1 and S_2 as disjoint subsystems.

These theorems lead to the following lemmas:

Lemma 1.35 If $n \equiv 1, 3 \pmod 6$ and $6k + 1$ are integers such that $n \geq 6k + 1$, then there exists an $LS_d(2n + 6k + 1; \{3, n^*, (6k + 1)^*\})$.

Proof: In Theorem 1.33, replace the subsystems by lines of size n and $6k + 1$, respectively.

Lemma 1.36 If $n \equiv 3 \pmod 6$ and $6k + 3$ are integers such that $n \geq 6k + 3$, then there exists an $LS_d(2n + 6k + 3; \{3, n^*, (6k + 3)^*\})$.

Proof: In Theorem 1.34, replace the subsystems by lines of size n and $6k + 3$, respectively.

In order to completely determine the spectrum of AULSs with two long lines of sizes u and w, and short lines of size three, we need to construct such spaces for all values of u, v and w satisfying the conditions in Corollaries 1.15, 1.17, 1.19 and 1.21. However, since the two long lines can be replaced by two Steiner systems, we are able to exploit the Doyen- Wilson theorem to radically reduce the number of constructions to be found.

Lemma 1.37 Let $u \equiv 1, 3 \pmod 6$, $u \geq 7$ and $v_0 \in LS_d(3, u^{**})$ ($v_0 \in LS_i(3, u^{**})$). Then $v \in LS_d(3, u^{**})$ ($v \in LS_i(3, u^{**})$) for all $v \geq 2v_0 + 1$, $v \equiv 1, 3 \pmod 6$.

Proof: Replace each of the long lines of size u by an STS(u) in $LS_d(v_0;\{3,u^{**}\})(LS_i(v_0;\{3,u^{**}\}))$, obtaining an $STS(v_0)$. By Theorem 1.7, we can embed the $STS(v_0)$ into an STS(v), and reintroduce the two long lines in the subsystem $STS(v_0)$, resulting in an AULS with two long lines of size u and the short lines of size three.

Lemma 1.38 Let u, w ≡ 1, 3 (mod 6), u, w ≥ 7 and u < w and $v_0 \in LS_d(3, u^*, w^*)$ ($v_0 \in LS_i(3, u^*, w^*)$). Then $v \in LS_d(3, u^*, w^*)$ ($v \in LS_i(3, u^*, w^*)$) for all $v \geq 2v_0 + 1$; v ≡ 1, 3 (mod 6).

Proof: The arguments are essentially the same as in Lemma 1.37.

If one of the long lines has size congruent to 5 (mod 6), we can apply Theorem 1.12 to obtain the following result.

Lemma 1.39 Let u ≡ 5 (mod 6) and w ≡ 1, 3 (mod 6), u ≥ 5, w ≥ 7 and u < w, $v_0 \in LS_d(3, u^*, w^*)$ ($v_0 \in LS_i(3, u^*, w^*)$). Then $v \in LS_d(3, u^*, w^*)$ ($v \in LS_i(3, u^*, w^*)$) for all $v \geq 2v_0 + 1$, where v ≡ 5 (mod 6), $v_0 \equiv 5$ (mod 6).

Proof: Replace the long line of size u in $LS_d(v_0; \{3, u^*, w^*\})$ by a hole on u points and replace the line of size w by an STS(w). We therefore have an IPBD(v_0, u; {3}). By Theorem 1.12, there exists an IPBD(v, v_0; {3}) for all $v \geq 2v_0 + 1$, where we consider IPBD(v_0, u; {3}) to be the hole. We can always reintroduce the two long lines in the subdesign IPBD(v_0, u; {3}). Thus, we have constructed an $LS_d(v; \{3, u^*, w^*\})$ for all $v \geq 2v_0 + 1$.

The recursive techniques are far more restricted for the construction of AULSs with short lines of size four. We can build our linear spaces in some cases by using various {4} - GDDs. The necessary and sufficient conditions for the existence of a {4}-GDD all of whose groups have the same size was established by Brouwer, Hanani and Schrijver [B5].

Theorem 1.40 Suppose t > 1.There exists a {4}-GDD of type g^t if and only if t ≥ 4, g(t - 1) ≡ 0 (mod 3), $g^2t(t - 1) \equiv 0$ (mod 4) and (g, t) ≠ (2, 4) or (6, 4).

Also, several {4}-GDDs of small orders [R6] can be used to recursively construct the required {4}-GDDs by means of Wilson's fundamental construction [W3], which play a central role in building AULSs of particular orders, in an analogous way to the methods described in Theorem 1.24. On occasion we will find that the inclusion of particular embeddings of BIBDs with block size four will enable us to produce some constructions, an obvious analogue of embeddings of Steiner triple systems applied previously. The necessary conditions for such embeddings to exist have been proven to be sufficient in [R6], [W1], [W2]. In fact, in [R6], an analogue of the Doyen-Wilson theorem is proven:

Theorem 1.41 There exists an IPBD(v, w; {4}) if and only if $v \geq 3w + 1$, $v \equiv 1$ or 4 (mod 12), $w \equiv 1$ or 4 (mod 12), or $v \equiv 7, 10$ (mod 12) and $w \equiv 7$ or 10 (mod 12).

Some infinite classes of linear spaces are obtained by considering resolvable {3}-GDDS. We know that the block set is partitioned into parallel classes, and that this can be done as long as the groups all have the same size. If to each of the r parallel classes we adjoin a point at infinity, we obtain a {4}-GDD of type $g^t r^1$, which can be described as the "completion" of the resolvable {3}-GDD [R6]. We note that to apply such designs in the present context, we are necessarily restricted to group sizes which, possibly with an additional point ∞, will give us a BIBD with block size four, since our linear spaces have short lines of size four.

We can also apply the notion of completion for the construction of desired {5}-GDDs, in tandem with BIBDs having block size five whose order must be congruent to 1 or 5 (mod 20) [H3], in order to construct AULSs of particular orders with short lines of size five. The recursive techniques involving the use of specific GDDs, Wilson's fundamental construction, and certain embeddings of BIBDs can be applied, albeit in a more limited way. Since we do not have an analogue of the Doyen-Wilson theorem, the spectrum for AULSs with short lines of size five is far from complete. Therefore, we are able to obtain only what are essentially individual constructions, along with some infinite classes, by following the general procedures developed earlier.

Chapter 2

Almost uniform linear spaces with short lines of size three

§2.1 Almost uniform linear spaces with two long lines of size u and short lines of size three

We shall first assume that the two long lines intersect. In Corollary 1.17(i), the necessary conditions for the existence of these linear spaces were determined, and are now shown to be sufficient.

Theorem 2.1 Let $u \geq 7$.
$v \in LS_i(3,u^{**})$ if and only if $v \geq 3u - 2$, $u \equiv 1, 3 \pmod 6$ and $v \equiv 1, 3 \pmod 6$.

Proof: The necessity of these conditions has been established in Corollary 1.17(i). To prove sufficiency, first consider the orders v such that $3u - 2 \leq v \leq 6u - 5$. Since $v \equiv 1,3 \pmod 6$, $v = 6u - j$ where $j \equiv 5 \pmod 6$ or $v = 6u - k$ where $k \equiv 3 \pmod 6$, $k \geq 9$.

<u>Case 1</u>: $v = 6u - j$, $j \equiv 5 \pmod 6$.

Case 1a: $v = 6u - j$, $j = 5 + 6r$ and $u \geq 6r + 1$.

Form the partition $\pi(1^1, (u-1)^5, (u-6r-1)^1)$, when $u > 6r + 1$. Apply Theorem 1.24(b) where $g = u - 1$, $t = 5$ and $x = u - 6r - 1$. If $u = 6r + 1$, form the partition $\pi(1^1, (u-1)^5)$ and set $g = u - 1$, $t = 5$ and $x = 0$.

Case 1b: $v = 6u - j$ where $j = 5 + 6r$, $2r < u < 6r + 1$ and $r \geq 2$.

Form $\pi(1^1, (u-1)^3, (3u-6r-3)^1)$ and apply Theorem 1.24(b) with $g = u - 1$, $t = 3$ and $x = 3u - 6r - 3$.

<u>Case 2</u>: $v = 6u - k$, $k \equiv 3 \pmod 6$ and $k \geq 9$: $k = 9 + 6r$.

We consider $u \equiv 1 \pmod 6$ and $u \equiv 3 \pmod 6$ separately.

Case 2a: $u \equiv 1 \pmod 6$ and $u \geq 6r + 5$.
Form $\pi(1^1, (u - 1)^5, (u - 6r - 5)^1)$ and apply Theorem 1.24(b) with $g = u - 1$, $t = 5$ and $x = u - 6r - 5$.
Case 2b: $u \equiv 3 \pmod 6$ and $u \geq 3 + 3r$.
Form $\pi(1^1, (u - 1)^4, (2u - 6 - 6r)^1)$ and apply Theorem 1.24(b) with $g = u - 1$, $t = 4$ and $x = 2u - 6 - 6r$.
Case 2c: $u \equiv 1 \pmod 6$, $2r + 1 < u < 6r + 5$, $r \geq 1$.
Form $\pi(1^1, (u - 1)^3, (3u - 6r - 7)^1)$ and apply Theorem 1.24(b) with $g = u - 1$, $t = 3$ and $x = 3u - 6r - 7$.
Case 2d: $u \equiv 3 \pmod 6$, $2r+1 < u < 3+3r$ where $r \equiv 1 \pmod 2$ and $r \geq 3$, or $r \equiv 0 \pmod 2$ and $r \geq 6$.
Form $\pi(1^1, (u - 1)^3, (3u - 6r - 7)^1)$ and apply Theorem 1.24(b) with $g = u - 1$, $t = 3$ and $x = 3r - 7$. Finally, consider $v \equiv 1, 3 \pmod 6$ and $v \geq 6u - 3$. Since we have shown that $3u - 2 \in LS_i(3, u^{**})$, by Lemma 1.37, $v \in LS_i(3, u^{**})$ for all $v \geq 6u - 3$, $v \equiv 1, 3 \pmod 6$.

Next, assume that the two long lines are disjoint. We approach this problem in a similar way, however we are only able to provide a partial solution.

Theorem 2.2

(a) If $u \equiv 1, 3 \pmod 6$, then $LS_d(3u;\{3, u^{**}\})$ exists and $3u = \min\{v: \exists LS_d(v;\{3, u^{**}\})\}$.
(b) Suppose $u \equiv 1 \pmod 6$. $v \in LS_d(3,u^{**})$ if $v \equiv 1 \pmod 6$ and $4u + 3 \leq v \leq 6u - 5$, or $v \equiv 3 \pmod 6$ and $v = 6u - 3$.
(c) Suppose $u \equiv 3 \pmod 6$. $v \in LS_d(3, u^{**})$ if $v \equiv 1 \pmod 6$ and $4u + 1 \leq v \leq 6u - 5$, or $v \equiv 3 \pmod 6$ and $4u + 3 \leq v \leq 6u - 3$; $33 \in LS_d(3,9^{**})$.

Proof:

(a) $v \geq 3u$ from Corollary 1.15(i). Form a partition $\pi(u^3)$ and apply Theorem 1.24(a).

(b) If $v \equiv 1 \pmod 6$ and $4u + 3 \le v \le 6u - 5$, form $\pi(u^4, x^1)$ and apply Theorem 1.24(a) with $g = u$, $t = 4$ and $3 \le x \le 2u - 5$, $x \equiv 3 \pmod 6$. If $v \equiv 3 \pmod 6$ and $v = 6u - 3$, form $\pi(1^1, (u-1)^4, (2u)^1)$ and apply Theorem 1.24(b) where $g_1 = u$, $g = u - 1$, $x = 2u$.

(c) If $4u + 3 \le v \le 6u - 3$, form $\pi(u^4, x^1)$ and apply Theorem 1.24(a) with $g = u$, $t = 4$ and $3 \le x \le 2u - 3$, $x \equiv 3 \pmod 6$.

Next, we prove $33 \in LS_d(3, 9^{**})$. Form $\pi(6^1, 9^3)$. The three cells A,B and D are given by the sets $Z_9 \times \{i\}$; i=1,2,4. Cell C is given by $Z_6 \times \{3\}$. The short lines are of

type ABD: $0_1 i_2 (2i)_4$ (i = 0,1,...,4) $1_18_20_4$ $2_1(4-i)_2(2i+2)_4$ (i = 0,1,2,3) $3_13_23_4$
$4_14_24_4$ $5_13_25_4$ $6_16_26_4$ $7_17_27_4$ $8_18_28_4$ $1_12_23_4$ $3_14_25_4$ $4_15_26_4$ $5_16_27_4$
$6_17_28_4$ $7_15_20_4$ $8_10_21_4$ $1_11_25_4$ $3_11_27_4$ $4_16_28_4$ $5_17_20_4$ $6_18_21_4$ $7_10_22_4$
$8_15_23_4$ $1_14_27_4$ $3_16_20_4$ $4_17_21_4$ $5_18_22_4$ $6_10_23_4$ $7_11_24_4$ $8_12_25_4$ $1_13_21_4$
$2_16_21_4$ $3_17_22_4$ $4_18_23_4$ $5_10_24_4$ $6_15_25_4$ $7_14_26_4$ $8_13_27_4$

type ACD: $0_11_31_4$ $1_10_32_4$ $2_12_30_4$ $3_10_31_4$ $4_10_30_4$ $5_12_31_4$ $6_11_30_4$ $7_14_31_4$
$8_13_30_4$ $0_10_33_4$ $1_15_34_4$ $2_11_33_4$ $3_12_34_4$ $4_15_32_4$ $5_15_33_4$ $6_14_32_4$ $7_13_33_4$
$8_12_32_4$ $0_12_35_4$ $1_13_36_4$ $2_14_35_4$ $3_11_36_4$ $4_13_35_4$ $5_10_36_4$ $6_13_34_4$ $7_11_35_4$
$8_14_34_4$ $0_13_37_4$ $1_14_38_4$ $2_15_37_4$ $3_13_38_4$ $4_14_37_4$ $5_11_38_4$ $6_12_37_4$ $7_15_38_4$
$8_15_36_4$

type ABC: $0_18_24_3$ $1_16_22_3$ $2_17_23_3$ $3_15_24_3$ $4_10_22_3$ $5_12_23_3$ $6_14_20_3$ $7_13_22_3$
$8_14_21_3$ $0_16_25_3$ $1_17_21_3$ $2_15_20_3$ $3_10_25_3$ $4_11_21_3$ $5_11_24_3$ $6_13_25_3$ $7_12_20_3$
$8_11_20_3$

type ABB: $0_15_27_2$ $1_10_25_2$ $2_10_28_2$ $3_12_28_2$ $4_12_23_2$ $5_14_25_2$ $6_11_22_2$ $7_16_28_2$
$8_16_27_2$

type BCD: $7_20_34_4$ $6_20_35_4$ $8_20_37_4$ $3_20_38_4$ $6_21_34_4$ $0_21_37_4$ $5_21_32_4$ $4_22_33_4$
$8_22_36_4$ $5_22_38_4$ $3_23_32_4$ $4_23_31_4$ $2_24_30_4$ $0_24_36_4$ $7_24_33_4$ $2_25_31_4$ $1_25_30_4$
$8_25_35_4$

type BCC: $0_20_33_3$ $1_22_33_3$ $2_21_32_3$ $3_21_34_3$ $4_24_35_3$ $5_23_35_3$ $6_23_34_3$ $7_22_35_3$
$8_21_33_3$

type CCC: $0_31_35_3$ $0_32_34_3$

type BBD: $3_2 4_2 0_4$ $1_2 5_2 1_4$ $2_2 6_2 2_4$ $1_2 6_2 3_4$ $5_2 8_2 4_4$ $0_2 7_2 5_4$ $1_2 7_2 6_4$ $2_2 5_2 7_4$ $0_2 2_2 8_4$

type BBB: $0_2 1_2 3_2$ $0_2 4_2 6_2$ $1_2 4_2 8_2$ $2_2 4_2 7_2$ $3_2 5_2 6_2$ $3_2 7_2 8_2$

The long lines are $0_1 1_1 \cdots 8_1$ and $0_4 1_4 \cdots 8_4$. If $4u + 1 \leq v \leq 6u - 5$, form $\pi(u^4, x^1)$ where $1 \leq x \leq 2u - 5$, $x \equiv 1 \pmod 6$ and apply Theorem 1.24(a) with $g = u$, $t = 4$.

Corollary 2.3 If $u \equiv 1, 3 \pmod 6$ then $v \in LS_d(3, u^{**})$ for all $v \geq 6u + 1$, $v \equiv 1, 3 \pmod 6$.

Proof: This follows from Theorem 2.2 and Lemma 1.37.

We will be successful in completing the spectra for AULSs in which the two long lines both have size seven, nine, and thirteen, once we have provided constructions of the AULSs with orders not covered in Lemma 2.2.

Lemma 2.4 $25,27,33 \in LS_d(3,7^{**})$; $31 \in LS_d(3,9^{**})$; $43,45,49,51,57,63,69 \in LS_d(3,13^{**})$; $49,55 \in LS_d(3,15^{**})$.

Proof: To prove that $LS_d(25;\{3, 7^{**}\})$ exists, we give a direct construction. Form a partition $\pi(7^2, 11^1)$ and construct short lines of

type ABC: $0_1 i_2 i_3$ $1_1 i_2 (i+7)_3$ $2_1 i_2 (i+3)_3$ $3_1 i_2 (i+10)_3$ $4_1 i_2 (i+6)_3$ $5_1 i_2 (i+2)_3$ $6_1 i_2 (i+9)_3$ $(i = 0,1,...,6)$

type ACC: $0_1 7_3 9_3$ $0_1 8_3 10_3$ $1_1 3_3 5_3$ $1_1 4_3 6_3$ $2_1 0_3 1_3$ $2_1 2_3 10_3$ $3_1 6_3 7_3$ $3_1 8_3 9_3$ $4_1 2_3 3_3$ $4_1 4_3 5_3$ $5_1 0_3 9_3$ $5_1 1_3 10_3$ $6_1 5_3 7_3$ $6_1 6_3 8_3$

type BCC: $0_2 1_3 4_3$ $0_2 5_3 8_3$ $1_2 2_3 5_3$ $1_2 6_3 9_3$ $2_2 3_3 6_3$ $2_2 7_3 10_3$ $3_2 4_3 7_3$ $3_2 0_3 8_3$ $4_2 5_3 9_3$ $4_2 1_3 8_3$ $5_2 6_3 10_3$ $5_2 2_3 9_3$ $6_2 0_3 7_3$ $6_2 3_3 10_3$

type CCC: $0_3 2_3 6_3$ $3_3 7_3 8_3$ $4_3 9_3 10_3$ $0_3 3_3 4_3$ $2_3 4_3 8_3$ $1_3 2_3 7_3$ $0_3 5_3 10_3$ $1_3 3_3 9_3$ $1_3 5_3 6_3$

The long lines are $0_i 1_i \cdots 6_i$ $(i = 1,2)$. Similarly, to show that $27 \in LS_d(3,7^{**})$, form a partition $\pi(7^2, 13^1)$ and construct short lines of

type ABC: $0_1 i_2 i_3$ $1_1 i_2 (i+7)_3$ $2_1 i_2 (i+1)_3$ $3_1 i_2 (i+8)_3$ $4_1 i_2 (i+2)_3$ $5_1 i_2 (i+9)_3$ $6_1 i_2 (i+3)_3$ $(i = 0,1,...,6)$

type ACC: $0_1 7_3 12_3$ $0_1 8_3 11_3$ $0_1 9_3 10_3$ $1_1 1_3 6_3$ $1_1 2_3 5_3$ $1_1 3_3 4_3$ $2_1 0_3 10_3$ $2_1 8_3 12_3$

$2_1 9_3 11_3$ $3_1 2_3 7_3$ $3_1 3_3 6_3$ $3_1 4_3 5_3$ $4_1 0_3 11_3$ $4_1 1_3 10_3$ $4_1 9_3 12_3$ $5_1 3_3 8_3$
$5_1 4_3 7_3$ $5_1 5_3 6_3$ $6_1 0_3 12_3$ $6_1 1_3 11_3$ $6_1 2_3 10_3$

type BCC: $0_2 4_3 10_3$ $0_2 5_3 12_3$ $0_2 6_3 11_3$ $1_2 0_3 5_3$ $1_2 6_3 12_3$ $1_2 7_3 11_3$ $2_2 0_3 7_3$ $2_2 1_3 12_3$
$2_2 6_3 8_3$ $3_2 0_3 1_3$ $3_2 2_3 8_3$ $3_2 7_3 9_3$ $4_2 1_3 2_3$ $4_2 3_3 9_3$ $4_2 8_3 10_3$ $5_2 2_3 11_3$
$5_2 3_3 10_3$ $5_2 4_3 9_3$ $6_2 3_3 12_3$ $6_2 4_3 11_3$ $6_2 5_3 10_3$

type CCC: $0_3 8_3 9_3$ $0_3 4_3 6_3$ $0_3 2_3 3_3$ $1_3 3_3 7_3$ $1_3 4_3 8_3$ $1_3 5_3 9_3$ $2_3 6_3 9_3$ $2_3 4_3 12_3$ $3_3 5_3 11_3$
$5_3 7_3 8_3$ $6_3 7_3 10_3$ $10_3 11_3 12_3$

Next, $33 \in LS_d(3, 7^{**})$ by forming a partition $\pi(7^3, 12^1)$ and cells A,B, D are given by the sets $Z_7 \times \{i\}$ (i = 1,2,4) and cell C is the set $Z_{12} \times \{3\}$. Construct short lines of

type ABC: $0_1 0_2 4_3$ $1_1 1_2 0_3$ $2_1 2_2 6_3$ $3_1 3_2 3_3$ $4_1 4_2 2_3$ $5_1 5_2 2_3$ $6_1 6_2 11_3$ $1_1 0_2 11_3$
$3_1 1_2 7_3$ $5_1 2_2 4_3$ $0_1 3_2 5_3$ $2_1 4_2 5_3$ $4_1 5_2 11_3$ $0_1 6_2 6_3$ $2_1 0_2 7_3$ $4_1 1_2 4_3$
$6_1 2_2 1_3$ $1_1 3_2 1_3$ $5_1 4_2 9_3$ $3_1 5_2 9_3$ $1_1 6_2 10_3$ $3_1 0_2 10_3$ $2_1 1_2 10_3$ $0_1 2_2 8_3$

type ABD: $0_1 1_2 0_4$ $0_1 4_2 1_4$ $0_1 5_2 2_4$ $1_1 2_2 3_4$ $1_1 4_2 5_4$ $1_1 5_2 4_4$ $2_1 3_2 6_4$ $2_1 5_2 0_4$ $2_1 6_2 1_4$
$3_1 2_2 2_4$ $3_1 4_2 3_4$ $3_1 6_2 5_4$ $4_1 0_2 5_4$ $4_1 2_2 6_4$ $4_1 3_2 2_4$ $4_1 6_2 4_4$ $5_1 0_2 4_4$ $5_1 1_2 3_4$
$5_1 3_2 0_4$ $5_1 6_2 6_4$ $6_1 0_2 1_4$ $6_1 1_2 2_4$ $6_1 3_2 3_4$ $6_1 4_2 0_4$ $6_1 5_2 5_4$

type ACD: $0_1 11_3 3_4$ $0_1 2_3 4_4$ $0_1 0_3 5_4$ $0_1 7_3 6_4$ $1_1 7_3 0_4$ $1_1 5_3 1_4$ $1_1 9_3 2_4$ $1_1 8_3 6_4$
$2_1 4_3 2_4$ $2_1 0_3 3_4$ $2_1 8_3 4_4$ $2_1 9_3 5_4$ $3_1 11_3 0_4$ $3_1 6_3 1_4$ $3_1 1_3 4_4$ $3_1 2_3 6_4$
$4_1 9_3 0_4$ $4_1 7_3 1_4$ $4_1 5_3 3_4$ $5_1 0_3 1_4$ $5_1 10_3 2_4$ $5_1 8_3 5_4$ $6_1 6_3 4_4$ $6_1 10_3 6_4$

type ACC: $0_1 1_3 3_3$ $0_1 9_3 10_3$ $1_1 2_3 4_3$ $1_1 3_3 6_3$ $2_1 2_3 3_3$ $2_1 1_3 11_3$ $3_1 0_3 5_3$ $3_1 4_3 8_3$ $4_1 0_3 3_3$
$4_1 6_3 10_3$ $4_1 1_3 8_3$ $5_1 3_3 11_3$ $5_1 5_3 7_3$ $5_1 1_3 6_3$ $6_1 0_3 2_3$ $6_1 3_3 4_3$ $6_1 7_3 9_3$ $6_1 5_3 8_3$

type BCD: $0_2 6_3 0_4$ $0_2 0_3 2_4$ $0_2 1_3 3_4$ $0_2 3_3 6_4$ $1_2 1_3 1_4$ $1_2 9_3 4_4$ $1_2 11_3 5_4$ $1_2 6_3 6_4$
$2_2 0_3 0_4$ $2_2 10_3 1_4$ $2_2 7_3 4_4$ $2_2 5_3 5_4$ $3_2 8_3 1_4$ $3_2 4_3 4_4$ $3_2 10_3 5_4$ $4_2 3_3 2_4$
$4_2 0_3 4_4$ $4_2 1_3 6_4$ $5_2 3_3 1_4$ $5_2 4_3 3_4$ $5_2 5_3 6_4$ $6_2 4_3 0_4$ $6_2 2_3 2_4$ $6_2 3_3 3_4$

type BCC: $0_2 2_3 5_3$ $0_2 8_3 9_3$ $1_2 3_3 5_3$ $1_2 2_3 8_3$ $2_2 2_3 11_3$ $2_2 3_3 9_3$ $3_2 0_3 9_3$ $3_2 2_3 6_3$ $3_2 7_3 11_3$
$4_2 4_3 7_3$ $4_2 10_3 11_3$ $4_2 6_3 8_3$ $5_2 6_3 7_3$ $5_2 0_3 8_3$ $5_2 1_3 10_3$ $6_2 0_3 1_3$ $6_2 7_3 8_3$
$6_2 5_3 9_3$

type CCD: $2_3 10_3 0_4$ $1_3 5_3 0_4$ $3_3 8_3 0_4$ $2_3 9_3 1_4$ $4_3 11_3 1_4$ $8_3 11_3 2_4$ $5_3 6_3 2_4$ $1_3 7_3 2_4$
$6_3 9_3 3_4$ $2_3 7_3 3_4$ $8_3 10_3 3_4$ $3_3 10_3 4_4$ $5_3 11_3 4_4$ $3_3 7_3 5_4$ $4_3 6_3 5_4$ $1_3 2_3 5_4$
$0_3 4_3 6_4$ $9_3 11_3 6_4$

type CCC: $4_3 5_3 10_3$ $1_3 4_3 9_3$ $0_3 6_3 11_3$ $0_3 7_3 10_3$

type BBB: put an STS(7) on the seven points of cell B.

The long lines are $0_i 1_i \cdots 6_i$ (i = 1,4). We obtain $31 \in LS_d(3, 9^{**})$ by forming a partition $\pi(9^2, 13^1)$, cells A and C are given by the sets $Z_9 \times \{i\}$ (i = 1, 3) and cell B is the set $Z_{13} \times \{2\}$. Construct short lines of

type ABC: $0_13_20_3$ $0_11_21_3$ $0_12_22_3$ $0_10_23_3$ $0_111_24_3$ $0_15_25_3$ $0_19_26_3$ $0_110_27_3$ $0_18_28_3$ $1_1(i+1)_2i_3$ $(i = 0,1,\ldots,8)$ $2_1(i+2)_2i_3$ $(i = 0,1,\ldots,4)$ $2_112_25_3$ $2_18_26_3$ $2_19_27_3$ $2_17_28_3$ $3_10_20_3$ $3_14_21_3$ $3_112_22_3$ $3_19_23_3$ $3_110_24_3$ $3_18_25_3$ $3_16_26_3$ $3_17_27_3$ $3_111_28_3$ $4_14_20_3$ $4_17_21_3$ $4_16_22_3$ $4_110_23_3$ $4_18_24_3$ $4_19_25_3$ $4_112_26_3$ $4_111_27_3$ $4_13_28_3$ $5_1(i+5)_2i_3$ $(i = 0,1,\ldots,4)$ $5_12_25_3$ $5_111_26_3$ $5_11_27_3$ $5_10_28_3$ $6_19_20_3$ $6_110_21_3$ $6_18_22_3$ $6_16_23_3$ $6_17_24_3$ $6_11_25_3$ $6_13_26_3$ $6_10_27_3$ $6_112_28_3$ $7_110_20_3$ $7_18_21_3$ $7_19_22_3$ $7_17_23_3$ $7_11_24_3$ $7_111_25_3$ $7_10_26_3$ $7_112_27_3$ $7_12_28_3$ $8_1(i+8)_2i_3$ $(i = 0,1,2,5,6,7)$ $8_112_23_3$ $8_14_24_3$ $8_15_28_3$

type ABB: $0_16_27_2$ $0_14_212_2$ $1_10_212_2$ $1_110_211_2$ $2_10_211_2$ $2_11_210_2$ $3_11_23_2$ $3_12_25_2$ $4_10_22_2$ $4_11_25_2$ $5_14_210_2$ $5_13_212_2$ $6_12_211_2$ $6_14_25_2$ $7_15_26_2$ $7_13_24_2$ $8_17_211_2$ $8_13_26_2$

type BBC: $6_211_20_3$ $7_212_20_3$ $0_25_21_3$ $11_212_21_3$ $0_21_22_3$ $5_211_22_3$ $2_23_23_3$ $1_211_23_3$ $0_23_24_3$ $2_212_24_3$ $3_210_25_3$ $4_27_25_3$ $5_210_26_3$ $2_24_26_3$ $3_25_27_3$ $4_26_27_3$ $1_24_28_3$ $6_210_28_3$

type BBB: $0_24_29_2$ $0_27_210_2$ $0_26_28_2$ $5_28_212_2$ $1_26_212_2$ $9_210_212_2$ $2_26_29_2$ $2_28_210_2$ $3_27_28_2$ $1_28_29_2$ $4_28_211_2$ $5_27_29_2$ $3_29_211_2$ $1_22_27_2$

The long lines are $0_i 1_i \cdots 8_i$ (i = 1,3). Now, $43 \in LS_d(3, 13^{**})$ by forming a partition $\pi(3^1, 13^2, 14^1)$, cells A and D are the sets $Z_{13} \times \{i\}$ (i = 1, 4), cell B is the set $Z_{14} \times \{2\}$ and cell C is the set $Z_3 \times \{3\}$. Construct short lines of

type ACD: $0_1i_3i_4$ $1_1i_3(i+3)_4$ $2_1i_3(i+6)_4$ $3_1i_3(i+9)_4$ $4_1i_3(i+12)_4$ $5_1i_3(i+2)_4$ $6_1i_3(i+5)_4$ $7_1i_3(i+8)_4$ $8_1i_3(i+11)_4$ $9_1i_3(i+1)_4$ $10_1i_3(i+4)_4$ $11_1i_3(i+7)_4$ $12_1i_3(i+10)_4$ $(i = 0,1,2)$

type ABD: $0_10_23_4$ $0_113_24_4$ $0_11_25_4$ $0_17_26_4$ $0_14_27_4$ $0_16_28_4$ $0_112_29_4$ $0_18_210_4$ $0_15_211_4$ $0_19_212_4$ $1_110_20_4$ $1_111_21_4$ $1_113_22_4$ $1_112_26_4$ $1_10_27_4$ $1_11_28_4$ $1_17_29_4$ $1_15_210_4$ $1_14_211_4$ $1_16_212_4$ $2_14_20_4$ $2_19_21_4$ $2_16_22_4$ $2_13_23_4$

$2_{1}11_{2}4_{4}$ $2_{1}10_{2}5_{4}$ $2_{1}8_{2}9_{4}$ $2_{1}13_{2}10_{4}$ $2_{1}0_{2}11_{4}$ $2_{1}12_{2}12_{4}$ $3_{1}9_{2}0_{4}$ $3_{1}7_{2}1_{4}$
$3_{1}4_{2}2_{4}$ $3_{1}12_{2}3_{4}$ $3_{1}5_{2}4_{4}$ $3_{1}3_{2}5_{4}$ $3_{1}8_{2}6_{4}$ $3_{1}2_{2}7_{4}$ $3_{1}10_{2}8_{4}$ $3_{1}11_{2}12_{4}$
$4_{1}2_{2}2_{4}$ $4_{1}13_{2}3_{4}$ $4_{1}12_{2}4_{4}$ $4_{1}5_{2}5_{4}$ $4_{1}0_{2}6_{4}$ $4_{1}7_{2}7_{4}$ $4_{1}4_{2}8_{4}$ $4_{1}1_{2}9_{4}$
$4_{1}3_{2}10_{4}$ $4_{1}8_{2}11_{4}$ $5_{1}8_{2}0_{4}$ $5_{1}2_{2}1_{4}$ $5_{1}11_{2}5_{4}$ $5_{1}10_{2}6_{4}$ $5_{1}1_{2}7_{4}$ $5_{1}9_{2}8_{4}$
$5_{1}0_{2}9_{4}$ $5_{1}12_{2}10_{4}$ $5_{1}13_{2}11_{4}$ $5_{1}5_{2}12_{4}$ $6_{1}5_{2}0_{4}$ $6_{1}6_{2}1_{4}$ $6_{1}12_{2}2_{4}$ $6_{1}9_{2}3_{4}$
$6_{1}2_{2}4_{4}$ $6_{1}8_{2}8_{4}$ $6_{1}10_{2}9_{4}$ $6_{1}11_{2}10_{4}$ $6_{1}3_{2}11_{4}$ $6_{1}1_{2}12_{4}$ $7_{1}0_{2}0_{4}$ $7_{1}13_{2}1_{4}$
$7_{1}1_{2}2_{4}$ $7_{1}8_{2}3_{4}$ $7_{1}4_{2}4_{4}$ $7_{1}7_{2}5_{4}$ $7_{1}6_{2}6_{4}$ $7_{1}3_{2}7_{4}$ $7_{1}9_{2}11_{4}$ $7_{1}2_{2}12_{4}$
$8_{1}10_{2}1_{4}$ $8_{1}11_{2}2_{4}$ $8_{1}2_{2}3_{4}$ $8_{1}6_{2}4_{4}$ $8_{1}0_{2}5_{4}$ $8_{1}1_{2}6_{4}$ $8_{1}9_{2}7_{4}$ $8_{1}3_{2}8_{4}$ $8_{1}4_{2}9_{4}$
$8_{1}7_{2}10_{4}$ $9_{1}1_{2}0_{4}$ $9_{1}9_{2}4_{4}$ $9_{1}8_{2}5_{4}$ $9_{1}13_{2}6_{4}$ $9_{1}5_{2}7_{4}$ $9_{1}7_{2}8_{4}$ $9_{1}11_{2}9_{4}$
$9_{1}2_{2}10_{4}$ $9_{1}10_{2}11_{4}$ $9_{1}3_{2}12_{4}$ $10_{1}6_{2}0_{4}$ $10_{1}5_{2}1_{4}$ $10_{1}8_{2}2_{4}$ $10_{1}1_{2}3_{4}$
$10_{1}10_{2}7_{4}$ $10_{1}11_{2}8_{4}$ $10_{1}9_{2}9_{4}$ $10_{1}4_{2}10_{4}$ $10_{1}7_{2}11_{4}$ $10_{1}0_{2}12_{4}$ $11_{1}3_{2}0_{4}$
$11_{1}1_{2}1_{4}$ $11_{1}5_{2}2_{4}$ $11_{1}7_{2}3_{4}$ $11_{1}0_{2}4_{4}$ $11_{1}13_{2}5_{4}$ $11_{1}4_{2}6_{4}$ $11_{1}6_{2}10_{4}$
$11_{1}12_{2}11_{4}$ $11_{1}8_{2}12_{4}$ $12_{1}12_{2}0_{4}$ $12_{1}3_{2}1_{4}$ $12_{1}10_{2}2_{4}$ $12_{1}5_{2}3_{4}$ $12_{1}1_{2}4_{4}$
$12_{1}9_{2}5_{4}$ $12_{1}2_{2}6_{4}$ $12_{1}11_{2}7_{4}$ $12_{1}0_{2}8_{4}$ $12_{1}13_{2}9_{4}$

type ABB: $0_{1}2_{2}10_{2}$ $0_{1}3_{2}11_{2}$ $1_{1}3_{2}9_{2}$ $1_{1}2_{2}8_{2}$ $2_{1}1_{2}7_{2}$ $2_{1}2_{2}5_{2}$ $3_{1}0_{2}13_{2}$ $3_{1}1_{2}6_{2}$
$4_{1}6_{2}9_{2}$ $4_{1}10_{2}11_{2}$ $5_{1}4_{2}7_{2}$ $5_{1}3_{2}6_{2}$ $6_{1}0_{2}7_{2}$ $6_{1}4_{2}13_{2}$ $7_{1}10_{2}12_{2}$ $7_{1}5_{2}11_{2}$
$8_{1}5_{2}12_{2}$ $8_{1}8_{2}13_{2}$ $9_{1}0_{2}6_{2}$ $9_{1}4_{2}12_{2}$ $10_{1}3_{2}12_{2}$ $10_{1}2_{2}13_{2}$ $11_{1}2_{2}11_{2}$
$11_{1}9_{2}10_{2}$ $12_{1}4_{2}6_{2}$ $12_{1}7_{2}8_{2}$

type BBD: $11_{2}13_{2}0_{4}$ $2_{2}7_{2}0_{4}$ $0_{2}4_{2}1_{4}$ $8_{2}12_{2}1_{4}$ $0_{2}3_{2}2_{4}$ $7_{2}9_{2}2_{4}$ $6_{2}10_{2}3_{4}$ $4_{2}11_{2}3_{4}$
$3_{2}7_{2}4_{4}$ $8_{2}10_{2}4_{4}$ $2_{2}4_{2}5_{4}$ $6_{2}12_{2}5_{4}$ $3_{2}5_{2}6_{4}$ $9_{2}11_{2}6_{4}$ $6_{2}8_{2}7_{4}$ $12_{2}13_{2}7_{4}$
$5_{2}13_{2}8_{4}$ $2_{2}12_{2}8_{4}$ $5_{2}6_{2}9_{4}$ $2_{2}3_{2}9_{4}$ $0_{2}9_{2}10_{4}$ $1_{2}10_{2}10_{4}$ $1_{2}2_{2}11_{4}$ $6_{2}11_{2}11_{4}$
$7_{2}13_{2}12_{4}$ $4_{2}10_{2}12_{4}$

type BBB: $7_{2}11_{2}12_{2}$ $1_{2}5_{2}9_{2}$ $1_{2}3_{2}13_{2}$ $0_{2}5_{2}10_{2}$ $0_{2}1_{2}8_{2}$ $4_{2}8_{2}9_{2}$

type BBC: $9_{2}13_{2}0_{3}$ $1_{2}12_{2}0_{3}$ $5_{2}8_{2}0_{3}$ $0_{2}11_{2}0_{3}$ $3_{2}4_{2}0_{3}$ $2_{2}6_{2}0_{3}$ $7_{2}10_{2}0_{3}$ $0_{2}2_{2}1_{3}$
$10_{2}13_{2}1_{3}$ $3_{2}8_{2}1_{3}$ $1_{2}11_{2}1_{3}$ $4_{2}5_{2}1_{3}$ $6_{2}7_{2}1_{3}$ $9_{2}12_{2}1_{3}$ $1_{2}4_{2}2_{3}$ $5_{2}7_{2}2_{3}$
$3_{2}10_{2}2_{3}$ $8_{2}11_{2}2_{3}$ $6_{2}13_{2}2_{3}$ $2_{2}9_{2}2_{3}$ $0_{2}12_{2}2_{3}$

type CCC: $0_{3}1_{3}2_{3}$

The long lines are $0_i 1_i \cdots 12_i$ $(i = 1,4)$. Next, $45 \in LS_d(3, 13^{**})$ by forming a partition $\pi(3^1, 13^2, 16^1)$ where cells A and D are the sets $Z_{13} \times \{i\}$ $(i = 1, 4)$, cell B is the set $Z_{16} \times \{2\}$ and cell C is the set $Z_3 \times \{3\}$. Construct short lines of type ACD: same as for $LS_d(43;\{3, 13^{**}\})$.

type ABD: $0_1 15_2 3_4$ $0_1 14_2 4_4$ $0_1 9_2 5_4$ $0_1 0_2 6_4$ $0_1(i+4)_2(i+7)_4(i = 0,1,3,4)$ $0_1 3_2 9_4$
$0_1 1_2 12_4$ $1_1 14_2 0_4$ $1_1 11_2 1_4$ $1_1 4_2 2_4$ $1_1 9_2 6_4$ $1_1 10_2 7_4$ $1_1 2_2 8_4$ $1_1 5_2 9_4$
$1_1 3_2 10_4$ $1_1 13_2 11_4$ $1_1 0_2 12_4$ $2_1(i+4)_2 i_4(i = 0,2,3,4)$ $2_1 10_2 1_4$ $2_1 1_2 5_4$
$2_1 14_2 9_4$ $2_1 9_2 10_4$ $2_1 12_2 11_4$ $2_1 3_2 12_4$ $3_1 15_2 0_4$ $3_1 13_2 1_4$ $3_1 i_2(i+2)_4$
$(i = 0,3,5,6)$ $3_1 11_2 3_4$ $3_1 4_2 4_4$ $3_1 2_2 6_4$ $3_1 7_2 12_4$ $4_1(i+8)_2(i+2)_4$
$(i =0,1,3,4,5)$ $4_1 15_2 4_4$ $4_1 0_2 8_4$ $4_1 2_2 9_4$ $4_1 5_2 10_4$ $4_1 14_2 11_4$ $5_1 13_2 0_4$
$5_1 3_2 1_4$ $5_1(i+4)_2(i+5)_4(i = 0,3,4,7)$ $5_1 10_2 6_4$ $5_1 9_2 7_4$ $5_1 1_2 10_4$ $5_1 6_2 11_4$
$6_1 12_2 0_4$ $6_1 2_2 1_4$ $6_1 1_2 2_4$ $6_1 13_2 3_4$ $6_1 5_2 4_4$ $6_1 14_2 8_4$ $6_1 4_2 9_4$ $6_1 0_2 10_4$
$6_1 3_2 11_4$ $6_1 15_2 12_4$ $7_1 6_2 0_4$ $7_1 7_2 1_4$ $7_1 14_2 2_4$ $7_1 8_2 3_4$ $7_1 9_2 4_4$ $7_1 12_2 5_4$
$7_1 11_2 6_4$ $7_1 15_2 7_4$ $7_1 5_2 11_4$ $7_1 10_2 12_4$ $8_1 i_2 i_4(i = 1,6,7,8,9)$ $8_1 15_2 2_4$
$8_1 2_2 3_4$ $8_1 0_2 4_4$ $8_1 10_2 5_4$ $8_1 4_2 10_4$ $9_1 9_2 0_4$ $9_1 1_2 4_4$ $9_1 14_2 5_4$ $9_1 15_2 6_4$
$9_1 12_2 7_4$ $9_1 10_2 8_4$ $9_1 6_2 9_4$ $9_1 11_2 10_4$ $9_1 0_2 11_4$ $9_1 2_2 12_4$ $10_1 3_2 0_4$ $10_1 5_2 1_4$
$10_1 7_2 2_4$ $10_1 6_2 3_4$ $10_1 8_2 7_4$ $10_1 9_2 8_4$ $10_1 13_2 9_4$ $10_1 14_2 10_4$ $10_1 11_2 11_4$
$10_1 12_2 12_4$ $11_1 5_2 0_4$ $11_1 15_2 1_4$ $11_1 3_2 2_4$ $11_1 0_2 3_4$ $11_1 10_2 4_4$ $11_1 2_2 5_4$
$11_1 1_2 6_4$ $11_1 6_2 10_4$ $11_1 7_2 11_4$ $11_1 13_2 12_4$ $12_1 8_2 0_4$ $12_1 9_2 1_4$ $12_1 11_2 2_4$
$12_1 14_2 3_4$ $12_1 12_2 4_4$ $12_1 5_2 5_4$ $12_1 3_2 6_4$ $12_1 6_2 7_4$ $12_1 13_2 8_4$ $12_1 15_2 9_4$

type ABB: $0_1 10_2 13_2$ $0_1 2_2 11_2$ $0_1 6_2 12_2$ $1_1 1_2 8_2$ $1_1 6_2 15_2$ $1_1 7_2 12_2$ $2_1 0_2 5_2$ $2_1 11_2 15_2$
$2_1 2_2 13_2$ $3_1 8_2 10_2$ $3_1 9_2 12_2$ $3_1 1_2 14_2$ $4_1 4_2 7_2$ $4_1 3_2 10_2$ $4_1 1_2 6_2$ $5_1 5_2 12_2$
$5_1 2_2 15_2$ $5_1 0_2 14_2$ $6_1 6_2 10_2$ $6_1 9_2 11_2$ $6_1 7_2 8_2$ $7_1 0_2 13_2$ $7_1 1_2 4_2$ $7_1 2_2 3_2$
$8_1 5_2 14_2$ $8_1 11_2 12_2$ $8_1 3_2 13_2$ $9_1 3_2 4_2$ $9_1 5_2 8_2$ $9_1 7_2 13_2$ $10_1 1_2 15_2$ $10_1 2_2 4_2$
$10_1 0_2 10_2$ $11_1 8_2 11_2$ $11_1 4_2 9_2$ $11_1 12_2 14_2$ $12_1 1_2 2_2$ $12_1 0_2 4_2$ $12_1 7_2 10_2$

type BBD: $2_2 7_2 0_4$ $0_2 1_2 0_4$ $10_2 11_2 0_4$ $4_2 6_2 1_4$ $0_2 12_2 1_4$ $8_2 14_2 1_4$ $5_2 10_2 2_4$ $9_2 13_2 2_4$
$2_2 12_2 2_4$ $3_2 12_2 3_4$ $4_2 10_2 3_4$ $1_2 5_2 3_4$ $2_2 6_2 4_4$ $3_2 7_2 4_4$ $11_2 13_2 4_4$ $8_2 13_2 5_4$
$0_2 6_2 5_4$ $7_2 15_2 5_4$ $4_2 8_2 6_4$ $5_2 7_2 6_4$ $13_2 14_2 6_4$ $0_2 11_2 7_4$ $1_2 3_2 7_4$ $2_2 14_2 7_4$
$4_2 11_2 8_4$ $3_2 15_2 8_4$ $1_2 12_2 8_4$ $0_2 7_2 9_4$ $1_2 11_2 9_4$ $10_2 12_2 9_4$ $10_2 15_2 10_4$
$2_2 8_2 10_4$ $12_2 13_2 10_4$ $2_2 10_2 11_4$ $1_2 9_2 11_4$ $4_2 15_2 11_4$ $5_2 9_2 12_4$ $4_2 14_2 12_4$
$6_2 8_2 12_4$

type BBC: $2_2 5_2 0_3$ $6_2 9_2 0_3$ $0_2 15_2 0_3$ $3_2 8_2 0_3$ $7_2 11_2 0_3$ $4_2 12_2 0_3$ $1_2 13_2 0_3$ $10_2 14_2 0_3$
$6_2 13_2 1_3$ $0_2 2_2 1_3$ $3_2 11_2 1_3$ $9_2 10_2 1_3$ $1_2 7_2 1_3$ $8_2 12_2 1_3$ $14_2 15_2 1_3$ $4_2 5_2 1_3$

$0_2 8_2 2_3$ $4_2 13_2 2_3$ $3_2 5_2 2_3$ $11_2 14_2 2_3$ $12_2 15_2 2_3$ $6_2 7_2 2_3$ $1_2 10_2 2_3$ $2_2 9_2 2_3$

type BBB: $5_2 13_2 15_2$ $3_2 6_2 14_2$ $5_2 6_2 11_2$ $0_2 3_2 9_2$ $8_2 9_2 15_2$ $7_2 9_2 14_2$

type CCC: $0_3 1_3 2_3$

The long lines are $0_i 1_i \cdots 12_i$ $(i = 1,4)$. Also, $49 \in LS_d(3, 13^{**})$ by forming a partition $\pi(3^1, 13^2, 20^1)$, where cells A and D are sets $Z_{13} \times \{i\}(i = 1,4)$, cell B is set $Z_{20} \times \{2\}$ and cell C is set $Z_3 \times \{3\}$. Construct short lines of

type ACD: same as for $LS_d(43;\{3, 13^{**}\})$.

type ABD: $0_1 i_2 (i+3)_4 (i = 0,1,2,4,7,8)$ $0_1 10_2 6_4$ $0_1 17_2 8_4$ $0_1 16_2 9_4$ $0_1 6_2 12_4$ $1_1 10_2 0_4$ $1_1 11_2 1_4$ $1_1 12_2 2_4$ $1_1 19_2 6_4$ $1_1 3_2 7_4$ $1_1 14_2 8_4$ $1_1 13_2 9_4$ $1_1 15_2 10_4$ $1_1 18_2 11_4$ $1_1 17_2 12_4$ $2_1 0_2 0_4$ $2_1 5_2 1_4$ $2_1 2_2 2_4$ $2_1 1_2 3_4$ $2_1 4_2 4_4$ $2_1 6_2 5_4$ $2_1 18_2 9_4$ $2_1 3_2 10_4$ $2_1 7_2 11_4$ $2_1 8_2 12_4$ $3_1 5_2 0_4$ $3_1 15_2 1_4$ $3_1 18_2 2_4$ $3_1 3_2 3_4$ $3_1 (i+11)_2 (i+4)_4$ $(i = 0,1,2,3,8)$ $3_1 9_2 8_4$ $4_1 10_2 2_4$ $4_1 15_2 3_4$ $4_1 (i+19)_2 (i+4)_4 (i = 0,1,2,3,5)$ $4_1 8_2 8_4$ $4_1 17_2 10_4$ $4_1 5_2 11_4$ $5_1 15_2 0_4$ $5_1 9_2 1_4$ $5_1 4_2 5_4$ $5_1 11_2 6_4$ $5_1 6_2 7_4$ $5_1 16_2 8_4$ $5_1 14_2 9_4$ $5_1 5_2 10_4$ $5_1 10_2 11_4$ $5_1 7_2 12_4$ $6_1 18_2 0_4$ $6_1 12_2 1_4$ $6_1 15_2 2_4$ $6_1 6_2 3_4$ $6_1 8_2 4_4$ $6_1 4_2 8_4$ $6_1 3_2 9_4$ $6_1 2_2 10_4$ $6_1 1_2 11_4$ $6_1 0_2 12_4$ $7_1 13_2 0_4$ $7_1 7_2 1_4$ $7_1 16_2 2_4$ $7_1 18_2 3_4$ $7_1 10_2 4_4$ $7_1 11_2 5_4$ $7_1 12_2 6_4$ $7_1 19_2 7_4$ $7_1 14_2 11_4$ $7_1 3_2 12_4$ $8_1 17_2 1_4$ $8_1 9_2 2_4$ $8_1 12_2 3_4$ $8_1 0_2 4_4$ $8_1 18_2 5_4$ $8_1 2_2 6_4$ $8_1 13_2 7_4$ $8_1 1_2 8_4$ $8_1 19_2 9_4$ $8_1 4_2 10_4$ $9_1 14_2 0_4$ $9_1 13_2 4_4$ $9_1 10_2 5_4$ $9_1 (i+9)_2 (i+6)_4 (i = 0,2,3,6)$ $9_1 18_2 7_4$ $9_1 16_2 10_4$ $9_1 6_2 11_4$ $10_1 7_2 0_4$ $10_1 0_2 1_4$ $10_1 6_2 2_4$ $10_1 5_2 3_4$ $10_1 17_2 7_4$ $10_1 2_2 8_4$ $10_1 9_2 9_4$ $10_1 8_2 10_4$ $10_1 4_2 11_4$ $10_1 1_2 12_4$ $11_1 16_2 0_4$ $11_1 14_2 1_4$ $11_1 7_2 2_4$ $11_1 17_2 3_4$ $11_1 3_2 4_4$ $11_1 8_2 5_4$ $11_1 5_2 6_4$ $11_1 12_2 10_4$ $11_1 11_2 11_4$ $11_1 13_2 12_4$ $12_1 9_2 0_4$ $12_1 6_2 1_4$ $12_1 0_2 2_4$ $12_1 2_2 3_4$ $12_1 7_2 4_4$ $12_1 16_2 5_4$ $12_1 17_2 6_4$ $12_1 1_2 7_4$ $12_1 19_2 8_4$ $12_1 5_2 9_4$

type ABB: $0_1 3_2 15_2$ $0_1 11_2 19_2$ $0_1 12_2 13_2$ $0_1 5_2 14_2$ $0_1 9_2 18_2$ $1_1 0_2 8_2$ $1_1 1_2 9_2$ $1_1 2_2 16_2$ $1_1 5_2 6_2$ $1_1 4_2 7_2$ $2_1 10_2 12_2$ $2_1 11_2 15_2$ $2_1 13_2 17_2$ $2_1 14_2 16_2$ $2_1 9_2 19_2$ $3_1 0_2 10_2$ $3_1 1_2 16_2$ $3_1 2_2 17_2$ $3_1 6_2 7_2$ $3_1 4_2 8_2$ $4_1 7_2 16_2$ $4_1 14_2 18_2$ $4_1 6_2 13_2$ $4_1 3_2 11_2$ $4_1 9_2 12_2$ $5_1 0_2 3_2$ $5_1 1_2 8_2$ $5_1 2_2 13_2$ $5_1 17_2 18_2$ $5_1 12_2 19_2$ $6_1 7_2 11_2$ $6_1 5_2 16_2$ $6_1 9_2 13_2$ $6_1 10_2 14_2$ $6_1 17_2 19_2$ $7_1 0_2 6_2$ $7_1 1_2 2_2$ $7_1 8_2 9_2$ $7_1 4_2 5_2$ $7_1 15_2 17_2$ $8_1 7_2 15_2$ $8_1 5_2 11_2$ $8_1 3_2 8_2$ $8_1 10_2 16_2$ $8_1 6_2 14_2$ $9_1 0_2 19_2$ $9_1 1_2 4_2$

$9_1 5_2 8_2$ $9_1 3_2 17_2$ $9_1 2_2 7_2$ $10_1 3_2 18_2$ $10_1 10_2 19_2$ $10_1 11_2 12_2$ $10_1 13_2 14_2$
$10_1 15_2 16_2$ $11_1 0_2 1_2$ $11_1 2_2 15_2$ $11_1 9_2 10_2$ $11_1 18_2 19_2$ $11_1 4_2 6_2$ $12_1 10_2 18_2$
$12_1 12_2 15_2$ $12_1 4_2 11_2$ $12_1 3_2 14_2$ $12_1 8_2 13_2$

type BBD: $1_2 11_2 0_4$ $2_2 12_2 0_4$ $4_2 17_2 0_4$ $6_2 8_2 0_4$ $3_2 19_2 0_4$ $2_2 10_2 1_4$ $4_2 18_2 1_4$ $8_2 19_2 1_4$
$13_2 16_2 1_4$ $1_2 3_2 1_4$ $1_2 13_2 2_4$ $11_2 17_2 2_4$ $4_2 19_2 2_4$ $8_2 14_2 2_4$ $3_2 5_2 2_4$ $4_2 9_2 3_4$
$8_2 10_2 3_4$ $16_2 19_2 3_4$ $7_2 13_2 3_4$ $11_2 14_2 3_4$ $2_2 14_2 4_4$ $5_2 12_2 4_4$ $9_2 16_2 4_4$
$6_2 17_2 4_4$ $15_2 18_2 4_4$ $1_2 19_2 5_4$ $5_2 17_2 5_4$ $13_2 15_2 5_4$ $3_2 9_2 5_4$ $7_2 14_2 5_4$ $0_2 14_2 6_4$
$6_2 15_2 6_4$ $7_2 8_2 6_4$ $16_2 18_2 6_4$ $3_2 4_2 6_4$ $0_2 7_2 7_4$ $8_2 12_2 7_4$ $5_2 9_2 7_4$ $11_2 16_2 7_4$
$10_2 15_2 7_4$ $0_2 15_2 8_4$ $6_2 18_2 8_4$ $7_2 12_2 8_4$ $5_2 13_2 8_4$ $3_2 10_2 8_4$ $0_2 17_2 9_4$ $2_2 11_2 9_4$
$7_2 10_2 9_4$ $1_2 6_2 9_4$ $8_2 15_2 9_4$ $0_2 9_2 10_4$ $11_2 13_2 10_4$ $14_2 19_2 10_4$ $6_2 10_2 10_4$
$1_2 18_2 10_4$ $0_2 2_2 11_4$ $16_2 17_2 11_4$ $13_2 19_2 11_4$ $9_2 15_2 11_4$ $3_2 12_2 11_4$ $5_2 18_2 12_4$
$2_2 9_2 12_4$ $4_2 16_2 12_4$ $12_2 14_2 12_4$ $10_2 11_2 12_4$

type BBC: $0_2 5_2 0_3$ $1_2 10_2 0_3$ $8_2 16_2 0_3$ $3_2 7_2 0_3$ $14_2 15_2 0_3$ $2_2 4_2 0_3$ $6_2 19_2 0_3$ $12_2 17_2 0_3$
$9_2 11_2 0_3$ $13_2 18_2 0_3$ $0_2 13_2 1_3$ $1_2 12_2 1_3$ $10_2 17_2 1_3$ $6_2 9_2 1_3$ $15_2 19_2 1_3$ $2_2 5_2 1_3$
$7_2 18_2 1_3$ $4_2 14_2 1_3$ $8_2 11_2 1_3$ $3_2 16_2 1_3$ $0_2 16_2 2_3$ $1_2 14_2 2_3$ $12_2 18_2 2_3$ $3_2 13_2 2_3$
$2_2 19_2 2_3$ $5_2 10_2 2_3$ $4_2 15_2 2_3$ $8_2 17_2 2_3$ $6_2 11_2 2_3$ $7_2 9_2 2_3$

type BBB: $0_2 11_2 18_2$ $0_2 4_2 12_2$ $1_2 7_2 17_2$ $1_2 5_2 15_2$ $2_2 3_2 6_2$ $6_2 12_2 16_2$ $2_2 8_2 18_2$
$9_2 14_2 17_2$ $5_2 7_2 19_2$ $4_2 10_2 13_2$

type CCC: $0_3 1_3 2_3$

The long lines are $0_i 1_i \cdots 12_i$ $(i = 1,4)$. Next, $51 \in LS_d(3, 13^{**})$ by forming a partition $\pi(3^1, 13^2, 22^1)$, where cells A and D are the sets $Z_{13} \times \{i\}$ $(i = 1, 4)$, cells B and C are the sets $Z_{22} \times \{2\}$ and $Z_3 \times \{3\}$. Construct short lines of

type ACD: same as in all previous cases.

type ABD: $0_1 10_2 4_4$ $0_1 9_2 5_4$ $0_1 1_2 6_4$ $0_1 i_2 (i+3)_4 (i = 0,4,\ldots,8)$ $0_1 2_2 12_4$ $1_1 3_2 0_4$
$1_1 1_2 1_4$ $1_1 6_2 2_4$ $1_1 15_2 6_4$ $1_1 12_2 7_4$ $1_1 18_2 8_4$ $1_1 20_2 9_4$ $1_1 9_2 10_4$ $1_1 14_2 11_4$
$1_1 19_2 12_4$ $2_1 20_2 0_4$ $2_1 21_2 1_4$ $2_1 0_2 2_4$ $2_1 11_2 3_4$ $2_1 5_2 4_4$ $2_1 16_2 5_4$ $2_1 4_2 9_4$
$2_1 2_2 10_4$ $2_1 17_2 11_4$ $2_1 7_2 12_4$ $3_1 8_2 0_4$ $3_1 2_2 1_4$ $3_1 12_2 2_4$ $3_1 3_2 3_4$ $3_1 18_2 4_4$
$3_1 1_2 5_4$ $3_1 17_2 6_4$ $3_1 10_2 7_4$ $3_1 6_2 8_4$ $3_1 15_2 12_4$ $4_1 11_2 2_4$ $4_1 10_2 3_4$
$4_1 (i+20)_2 (i+4)_4 (i = 0,1,\ldots,4,6,7)$ $4_1 14_2 9_4$ $5_1 17_2 0_4$ $5_1 7_2 1_4$ $5_1 8_2 5_4$
$5_1 9_2 6_4$ $5_1 18_2 7_4$ $5_1 16_2 8_4$ $5_1 21_2 9_4$ $5_1 15_2 10_4$ $5_1 12_2 11_4$ $5_1 1_2 12_4$ $6_1 16_2 0_4$

$6_1 17_2 1_4$ $6_1 10_2 2_4$ $6_1 2_2 3_4$ $6_1 13_2 4_4$ $6_1 0_2 8_4$ $6_1 11_2 9_4$ $6_1 20_2 10_4$ $6_1 19_2 11_4$
$6_1 18_2 12_4$ $7_1 4_2 0_4$ $7_1 9_2 1_4$ $7_1 17_2 2_4$ $7_1 7_2 3_4$ $7_1 8_2 4_4$ $7_1 2_2 5_4$ $7_1 12_2 6_4$
$7_1 3_2 7_4$ $7_1 6_2 11_4$ $7_1 13_2 12_4$ $8_1 10_2 1_4$ $8_1 13_2 2_4$ $8_1 15_2 3_4$ $8_1 1_2 4_4$ $8_1 17_2 5_4$
$8_1 19_2 6_4$ $8_1 16_2 7_4$ $8_1 21_2 8_4$ $8_1 0_2 9_4$ $8_1 14_2 10_4$ $9_1 5_2 0_4$ $9_1 6_2 4_4$ $9_1 4_2 5_4$
$9_1 2_2 6_4$ $9_1 13_2 7_4$ $9_1 7_2 8_4$ $9_1 8_2 9_4$ $9_1 19_2 10_4$ $9_1 18_2 11_4$ $9_1 3_2 12_4$ $10_1 12_2 0_4$
$10_1 15_2 1_4$ $10_1 18_2 2_4$ $10_1 13_2 3_4$ $10_1 6_2 7_4$ $10_1 10_2 8_4$ $10_1 17_2 9_4$ $10_1 1_2 10_4$
$10_1 16_2 11_4$ $10_1 21_2 12_4$ $11_1 9_2 0_4$ $11_1 11_2 1_4$ $11_1 2_2 2_4$ $11_1 19_2 3_4$ $11_1 4_2 4_4$
$11_1 5_2 5_4$ $11_1 14_2 6_4$ $11_1 8_2 10_4$ $11_1 7_2 11_4$ $11_1 0_2 12_4$ $12_1 11_2 0_4$ $12_1 5_2 1_4$
$12_1 (i+3)_2 (i+2)_4 (i = 0,1,4,5,6)$ $12_1 15_2 4_4$ $12_1 14_2 5_4$ $12_1 12_2 9_4$

type ABB: $0_1 12_2 21_2$ $0_1 13_2 19_2$ $0_1 3_2 11_2$ $0_1 14_2 15_2$ $0_1 18_2 20_2$ $0_1 16_2 17_2$ $1_1 0_2 10_2$
$1_1 11_2 16_2$ $1_1 5_2 21_2$ $1_1 13_2 17_2$ $1_1 4_2 7_2$ $1_1 2_2 8_2$ $2_1 14_2 19_2$ $2_1 9_2 13_2$ $2_1 1_2 18_2$
$2_1 3_2 15_2$ $2_1 8_2 12_2$ $2_1 6_2 10_2$ $3_1 0_2 21_2$ $3_1 7_2 11_2$ $3_1 9_2 20_2$ $3_1 14_2 16_2$ $3_1 4_2 19_2$
$3_1 5_2 13_2$ $4_1 3_2 12_2$ $4_1 13_2 16_2$ $4_1 6_2 8_2$ $4_1 9_2 15_2$ $4_1 17_2 19_2$ $4_1 7_2 18_2$ $5_1 0_2 5_2$
$5_1 6_2 11_2$ $5_1 2_2 10_2$ $5_1 13_2 14_2$ $5_1 4_2 20_2$ $5_1 3_2 19_2$ $6_1 4_2 14_2$ $6_1 1_2 15_2$ $6_1 5_2 8_2$
$6_1 7_2 9_2$ $6_1 3_2 21_2$ $6_1 6_2 12_2$ $7_1 0_2 14_2$ $7_1 11_2 18_2$ $7_1 5_2 15_2$ $7_1 1_2 20_2$ $7_1 10_2 19_2$
$7_1 16_2 21_2$ $8_1 2_2 20_2$ $8_1 8_2 18_2$ $8_1 4_2 6_2$ $8_1 5_2 12_2$ $8_1 3_2 7_2$ $8_1 9_2 11_2$ $9_1 0_2 15_2$
$9_1 10_2 11_2$ $9_1 9_2 12_2$ $9_1 16_2 20_2$ $9_1 1_2 21_2$ $9_1 14_2 17_2$ $10_1 0_2 3_2$ $10_1 11_2 20_2$
$10_1 8_2 19_2$ $10_1 5_2 9_2$ $10_1 2_2 4_2$ $10_1 7_2 14_2$ $11_1 18_2 21_2$ $11_1 3_2 17_2$ $11_1 10_2 12_2$
$11_1 15_2 16_2$ $11_1 13_2 20_2$ $11_1 1_2 6_2$ $12_1 0_2 16_2$ $12_1 2_2 19_2$ $12_1 1_2 13_2$ $12_1 20_2 21_2$
$12_1 6_2 18_2$ $12_1 10_2 17_2$

type BBD: $0_2 13_2 0_4$ $1_2 10_2 0_4$ $6_2 7_2 0_4$ $2_2 15_2 0_4$ $14_2 18_2 0_4$ $19_2 21_2 0_4$ $0_2 4_2 1_4$ $3_2 13_2 1_4$
$6_2 20_2 1_4$ $12_2 18_2 1_4$ $8_2 14_2 1_4$ $16_2 19_2 1_4$ $4_2 15_2 2_4$ $5_2 19_2 2_4$ $7_2 20_2 2_4$
$1_2 16_2 2_4$ $8_2 9_2 2_4$ $14_2 21_2 2_4$ $1_2 5_2 3_4$ $6_2 17_2 3_4$ $12_2 14_2 3_4$ $16_2 18_2 3_4$
$8_2 20_2 3_4$ $9_2 21_2 3_4$ $0_2 7_2 4_4$ $9_2 19_2 4_4$ $12_2 16_2 4_4$ $3_2 14_2 4_4$ $17_2 21_2 4_4$ $2_2 11_2 4_4$
$0_2 18_2 5_4$ $3_2 10_2 5_4$ $7_2 13_2 5_4$ $11_2 19_2 5_4$ $6_2 15_2 5_4$ $12_2 20_2 5_4$ $10_2 21_2 6_4$
$5_2 20_2 6_4$ $4_2 11_2 6_4$ $3_2 8_2 6_4$ $13_2 18_2 6_4$ $6_2 16_2 6_4$ $0_2 17_2 7_4$ $5_2 11_2 7_4$ $7_2 19_2 7_4$
$9_2 14_2 7_4$ $2_2 21_2 7_4$ $15_2 20_2 7_4$ $11_2 13_2 8_4$ $4_2 12_2 8_4$ $1_2 3_2 8_4$ $8_2 17_2 8_4$
$15_2 19_2 8_4$ $14_2 20_2 8_4$ $1_2 19_2 9_4$ $3_2 5_2 9_4$ $7_2 10_2 9_4$ $2_2 16_2 9_4$ $9_2 18_2 9_4$
$13_2 15_2 9_4$ $11_2 17_2 10_4$ $13_2 21_2 10_4$ $0_2 12_2 10_4$ $5_2 18_2 10_4$ $10_2 16_2 10_4$ $3_2 6_2 10_4$
$0_2 9_2 11_4$ $10_2 15_2 11_4$ $2_2 13_2 11_4$ $3_2 20_2 11_4$ $1_2 4_2 11_4$ $11_2 21_2 11_4$ $11_2 12_2 12_4$

$17_2 20_2 12_4$ $5_2 6_2 12_4$ $4_2 8_2 12_4$ $9_2 16_2 12_4$ $10_2 14_2 12_4$

type BBC: $0_2 8_2 0_3$ $5_2 14_2 0_3$ $4_2 16_2 0_3$ $11_2 15_2 0_3$ $2_2 3_2 0_3$ $10_2 20_2 0_3$ $6_2 21_2 0_3$ $7_2 17_2 0_3$
$1_2 9_2 0_3$ $12_2 13_2 0_3$ $18_2 19_2 0_3$ $0_2 11_2 1_3$ $1_2 8_2 1_3$ $4_2 13_2 1_3$ $2_2 18_2 1_3$ $19_2 20_2 1_3$
$9_2 17_2 1_3$ $3_2 16_2 1_3$ $7_2 21_2 1_3$ $12_2 15_2 1_3$ $6_2 14_2 1_3$ $5_2 10_2 1_3$ $0_2 20_2 2_3$ $1_2 17_2 2_3$
$4_2 21_2 2_3$ $10_2 18_2 2_3$ $5_2 16_2 2_3$ $8_2 11_2 2_3$ $7_2 15_2 2_3$ $3_2 9_2 2_3$ $12_2 19_2 2_3$
$6_2 13_2 2_3$ $2_2 14_2 2_3$

type BBB: $0_2 1_2 2_2$ $0_2 6_2 19_2$ $4_2 9_2 10_2$ $3_2 4_2 18_2$ $1_2 11_2 14_2$ $2_2 5_2 7_2$ $2_2 6_2 9_2$ $8_2 10_2 13_2$
$1_2 7_2 12_2$ $4_2 5_2 17_2$ $2_2 12_2 17_2$ $8_2 15_2 21_2$ $7_2 8_2 16_2$ $15_2 17_2 18_2$

type CCC: $0_3 1_3 2_3$

The long lines are $0_i 1_i \cdots 12_i$ (i = 1,4). In order to verify that 57 ϵ $LS_d(3, 13^{**})$, form a partition $\pi(5^1, 13^4)$, where cells A, C, D and E are the sets $Z_{13}\times\{i\}$ (i = 1,3,4,5) and cell B is the set $Z_5 \times \{2\}$. Construct short lines of

type BCD: $0_2 0_3 11_4$ $0_2 5_3 0_4$ $0_2 11_3 12_4$ $0_2 4_3 4_4$ $0_2 9_3 5_4$ $0_2 1_3 10_4$ $0_2 7_3 2_4$ $0_2 12_3 9_4$
$0_2 8_3 1_4$ $0_2 10_3 6_4$ $0_2 6_3 8_4$ $1_2 1_3 11_4$ $1_2 6_3 6_4$ $1_2 12_3 0_4$ $1_2 5_3 3_4$ $1_2 7_3 8_4$ $1_2 2_3 4_4$
$1_2 4_3 1_4$ $1_2 9_3 12_4$ $1_2 11_3 7_4$ $1_2 8_3 9_4$ $1_2 10_3 10_4$ $2_2 2_3 10_4$ $2_2 11_3 8_4$ $2_2 12_3 2_4$
$2_2 1_3 5_4$ $2_2 7_3 9_4$ $2_2 3_3 1_4$ $2_2 8_3 6_4$ $2_2 10_3 7_4$ $2_2 6_3 4_4$ $2_2 9_3 0_4$ $2_2 5_3 11_4$ $3_2 3_3 8_4$
$3_2 8_3 3_4$ $3_2 1_3 2_4$ $3_2 2_3 1_4$ $3_2 5_3 12_4$ $3_2 9_3 11_4$ $3_2 11_3 0_4$ $3_2 12_3 10_4$ $3_2 0_3 6_4$
$3_2 10_3 9_4$ $3_2 7_3 4_4$ $4_2 4_3 5_4$ $4_2 9_3 9_4$ $4_2 5_3 7_4$ $4_2 3_3 6_4$ $4_2 7_3 0_4$ $4_2 12_3 3_4$
$4_2 1_3 8_4$ $4_2 8_3 4_4$ $4_2 10_3 11_4$ $4_2 0_3 10_4$ $4_2 11_3 1_4$

type BBC: $1_2 2_2 0_3$ $0_2 4_2 2_3$ $0_2 1_2 3_3$ $2_2 3_2 4_3$ $3_2 4_2 6_3$

type BBD: $1_2 4_2 2_4$ $0_2 2_2 3_4$ $1_2 3_2 5_4$ $0_2 3_2 7_4$ $2_2 4_2 12_4$

type ABE: $0_1 i_2 i_5$ $1_1 i_2 (i+5)_5$ $2_1 i_2 (i+10)_5$ $3_1 i_2 (i+2)_5$ $4_1 i_2 (i+7)_5$ $5_1 i_2 (i+12)_5$
$6_1 i_2 (i+4)_5$ $7_1 i_2 (i+9)_5$ $8_1 i_2 (i+1)_5$ $9_1 i_2 (i+6)_5$ $10_1 i_2 (i+11)_5$ $11_1 i_2 (i+3)_5$
$12_1 i_2 (i+8)_5 (i = 0,1,...,4)$

type ACE: $0_1 i_3 (i+5)_5$ $1_1 (i+5)_3 (i+10)_5 (i = 0,1,...,4)$ $2_1 10_3 2_5$ $2_1 11_3 3_5$ $2_1 12_3 4_5$
$2_1 1_3 5_5$ $2_1 9_3 6_5$ $3_1 4_3 7_5$ $3_1 5_3 8_5$ $3_1 6_3 9_5$ $3_1 3_3 10_5$ $3_1 2_3 11_5$ $4_1 9_3 12_5$
$4_1 4_3 0_5$ $4_1 12_3 1_5$ $4_1 8_3 2_5$ $4_1 7_3 3_5$ $5_1 1_3 4_5$ $5_1 2_3 5_5$ $5_1 3_3 6_5$ $5_1 0_3 7_5$ $5_1 7_3 8_5$
$6_1 7_3 9_5$ $6_1 6_3 10_5$ $6_1 8_3 11_5$ $6_1 5_3 12_5$ $6_1 10_3 0_5$ $7_1 0_3 1_5$ $7_1 7_3 2_5$ $7_1 10_3 3_5$
$7_1 11_3 4_5$ $7_1 12_3 5_5$ $8_1 5_3 6_5$ $8_1 1_3 7_5$ $8_1 2_3 8_5$ $8_1 3_3 9_5$ $8_1 4_3 10_5$ $9_1 9_3 11_5$
$9_1 11_3 12_5$ $9_1 7_3 0_5$ $9_1 8_3 1_5$ $9_1 6_3 2_5$ $10_1 1_3 3_5$ $10_1 2_3 4_5$ $10_1 11_3 5_5$ $10_1 12_3 6_5$

$10_1 3_3 7_5$ $11_1 4_3 8_5$ $11_1 5_3 9_5$ $11_1 0_3 10_5$ $11_1 10_3 11_5$ $11_1 6_3 12_5$ $12_1 11_3 0_5$
$12_1 10_3 1_5$ $12_1 12_3 2_5$ $12_1 8_3 3_5$ $12_1 9_3 4_5$

type ACC: $0_1 5_3 9_3$ $0_1 8_3 11_3$ $0_1 7_3 12_3$ $0_1 6_3 10_3$ $1_1 0_3 11_3$ $1_1 2_3 12_3$ $1_1 3_3 10_3$ $1_1 1_3 4_3$
$2_1 2_3 7_3$ $2_1 3_3 5_3$ $2_1 0_3 4_3$ $2_1 6_3 8_3$ $3_1 0_3 1_3$ $3_1 11_3 12_3$ $3_1 7_3 9_3$ $3_1 8_3 10_3$
$4_1 0_3 5_3$ $4_1 1_3 6_3$ $4_1 2_3 10_3$ $4_1 3_3 11_3$ $5_1 5_3 8_3$ $5_1 4_3 9_3$ $5_1 6_3 12_3$ $5_1 10_3 11_3$
$6_1 0_3 3_3$ $6_1 1_3 12_3$ $6_1 2_3 9_3$ $6_1 4_3 11_3$ $7_1 1_3 2_3$ $7_1 5_3 6_3$ $7_1 3_3 9_3$ $7_1 4_3 8_3$ $8_1 0_3 6_3$
$8_1 9_3 11_3$ $8_1 7_3 8_3$ $8_1 10_3 12_3$ $9_1 0_3 12_3$ $9_1 1_3 3_3$ $9_1 5_3 10_3$ $9_1 2_3 4_3$ $10_1 0_3 8_3$
$10_1 5_3 7_3$ $10_1 4_3 10_3$ $10_1 6_3 9_3$ $11_1 1_3 11_3$ $11_1 3_3 7_3$ $11_1 2_3 8_3$ $11_1 9_3 12_3$
$12_1 0_3 7_3$ $12_1 2_3 3_3$ $12_1 4_3 6_3$ $12_1 1_3 5_3$

type ADE: $0_1 4_4 10_5$ $0_1 1_4 11_5$ $0_1 2_4 12_5$ $1_1 3_4 2_5$ $1_1 0_4 3_5$ $1_1 1_4 4_5$ $2_1 6_4 7_5$ $2_1 5_4 8_5$
$2_1 11_4 9_5$ $3_1 8_4 12_5$ $3_1 9_4 0_5$ $3_1 4_4 1_5$ $4_1 10_4 4_5$ $4_1 12_4 5_5$ $4_1 1_4 6_5$ $5_1 10_4 9_5$
$5_1 11_4 10_5$ $5_1 3_4 11_5$ $6_1 8_4 1_5$ $6_1 6_4 2_5$ $6_1 2_4 3_5$ $7_1 9_4 6_5$ $7_1 2_4 7_5$ $7_1 7_4 8_5$
$8_1 4_4 11_5$ $8_1 11_4 12_5$ $8_1 12_4 0_5$ $9_1 5_4 3_5$ $9_1 0_4 4_5$ $9_1 7_4 5_5$ $10_1 0_4 8_5$ $10_1 9_4 9_5$
$10_1 6_4 10_5$ $11_1 2_4 0_5$ $11_1 7_4 1_5$ $11_1 5_4 2_5$ $12_1 11_4 5_5$ $12_1 8_4 6_5$ $12_1 10_4 7_5$

type ADD: $0_1 0_4 8_4$ $0_1 5_4 9_4$ $0_1 6_4 11_4$ $0_1 10_4 12_4$ $0_1 3_4 7_4$ $1_1 2_4 7_4$ $1_1 5_4 10_4$ $1_1 4_4 6_4$
$1_1 11_4 12_4$ $1_1 8_4 9_4$ $2_1 2_4 9_4$ $2_1 1_4 10_4$ $2_1 0_4 12_4$ $2_1 4_4 7_4$ $2_1 3_4 8_4$ $3_1 1_4 3_4$
$3_1 6_4 10_4$ $3_1 0_4 7_4$ $3_1 5_4 11_4$ $3_1 2_4 12_4$ $4_1 2_4 4_4$ $4_1 0_4 9_4$ $4_1 3_4 5_4$ $4_1 6_4 8_4$
$4_1 7_4 11_4$ $5_1 4_4 9_4$ $5_1 7_4 12_4$ $5_1 0_4 1_4$ $5_1 2_4 6_4$ $5_1 5_4 8_4$ $6_1 10_4 11_4$ $6_1 0_4 3_4$
$6_1 4_4 5_4$ $6_1 1_4 7_4$ $6_1 9_4 12_4$ $7_1 1_4 12_4$ $7_1 5_4 6_4$ $7_1 8_4 10_4$ $7_1 0_4 11_4$ $7_1 3_4 4_4$
$8_1 0_4 5_4$ $8_1 7_4 8_4$ $8_1 3_4 9_4$ $8_1 2_4 10_4$ $8_1 1_4 6_4$ $9_1 2_4 3_4$ $9_1 6_4 12_4$ $9_1 8_4 11_4$
$9_1 9_4 10_4$ $9_1 1_4 4_4$ $10_1 4_4 12_4$ $10_1 3_4 11_4$ $10_1 2_4 5_4$ $10_1 7_4 10_4$ $10_1 1_4 8_4$
$11_1 0_4 6_4$ $11_1 1_4 9_4$ $11_1 4_4 11_4$ $11_1 8_4 12_4$ $11_1 3_4 10_4$ $12_1 5_4 12_4$ $12_1 3_4 6_4$
$12_1 0_4 4_4$ $12_1 1_4 2_4$ $12_1 7_4 9_4$

type CDE: $0_3 0_4 0_5$ $0_3 9_4 2_5$ $0_3 12_4 3_5$ $0_3 5_4 4_5$ $0_3 1_4 8_5$ $0_3 2_4 9_5$ $0_3 8_4 11_5$ $0_3 3_4 12_5$
$0_3 4_4 6_5$ $1_3 4_4 0_5$ $1_3 3_4 1_5$ $1_3 7_4 2_5$ $1_3 6_4 8_5$ $1_3 0_4 9_5$ $1_3 9_4 10_5$ $1_3 12_4 11_5$
$1_3 1_4 12_5$ $2_3 3_4 0_5$ $2_3 5_4 1_5$ $2_3 11_4 2_5$ $2_3 6_4 3_5$ $2_3 12_4 6_5$ $2_3 8_4 9_5$ $2_3 0_4 10_5$
$2_3 9_4 12_5$ $3_3 10_4 0_5$ $3_3 0_4 1_5$ $3_3 4_4 2_5$ $3_3 7_4 3_5$ $3_3 9_4 4_5$ $3_3 3_4 5_5$ $3_3 11_4 11_5$
$3_3 12_4 12_5$ $4_3 6_4 1_5$ $4_3 12_4 2_5$ $4_3 10_4 3_5$ $4_3 3_4 4_5$ $4_3 8_4 5_5$ $4_3 0_4 6_5$ $4_3 9_4 11_5$
$4_3 7_4 12_5$ $5_3 6_4 0_5$ $5_3 9_4 1_5$ $5_3 1_4 2_5$ $5_3 8_4 3_5$ $5_3 2_4 4_5$ $5_3 4_4 5_5$ $5_3 5_4 7_5$
$5_3 10_4 11_5$ $6_3 11_4 0_5$ $6_3 12_4 1_5$ $6_3 9_4 3_5$ $6_3 7_4 4_5$ $6_3 1_4 5_5$ $6_3 3_4 6_5$ $6_3 0_4 7_5$

$6_3 10_4 8_5$ $7_3 1_4 1_5$ $7_3 6_4 4_5$ $7_3 5_4 5_5$ $7_3 10_4 6_5$ $7_3 12_4 7_5$ $7_3 3_4 10_5$ $7_3 7_4 11_5$
$8_3 11_4 4_5$ $8_3 10_4 5_5$ $8_3 5_4 6_5$ $8_3 8_4 7_5$ $8_3 2_4 8_5$ $8_3 12_4 9_5$ $8_3 7_4 10_5$ $8_3 0_4 12_5$
$9_3 7_4 0_5$ $9_3 3_4 3_5$ $9_3 2_4 5_5$ $9_3 1_4 7_5$ $9_3 4_4 8_5$ $9_3 6_4 9_5$ $9_3 10_4 10_5$ $9_3 8_4 2_5$
$10_3 12_4 4_5$ $10_3 0_4 5_5$ $10_3 2_4 6_5$ $10_3 3_4 7_5$ $10_3 8_4 8_5$ $10_3 4_4 9_5$ $10_3 1_4 10_5$
$10_3 5_4 12_5$ $11_3 2_4 2_5$ $11_3 11_4 6_5$ $11_3 4_4 7_5$ $11_3 9_4 8_5$ $11_3 3_4 9_5$ $11_3 5_4 10_5$
$11_3 6_4 11_5$ $11_3 10_4 1_5$ $12_3 8_4 0_5$ $12_3 4_4 3_5$ $12_3 7_4 7_5$ $12_3 11_4 8_5$ $12_3 1_4 9_5$
$12_3 12_4 10_5$ $12_3 5_4 11_5$ $12_3 6_4 12_5$

type CCD: $3_3 4_3 2_4$ $2_3 6_3 2_4$ $3_3 6_3 5_4$ $0_3 2_3 7_4$ $4_3 7_3 11_4$

type DDE: $1_4 5_4 0_5$ $2_4 11_4 1_5$ $0_4 10_4 2_5$ $1_4 11_4 3_5$ $4_4 8_4 4_5$ $6_4 9_4 5_5$ $6_4 7_4 6_5$ $9_4 11_4 7_5$
$3_4 12_4 8_5$ $5_4 7_4 9_5$ $2_4 8_4 10_5$ $0_4 2_4 11_5$ $4_4 10_4 12_5$

type CCC: $0_3 9_3 10_3$ $1_3 8_3 9_3$ $1_3 7_3 10_3$ $2_3 5_3 11_3$ $3_3 8_3 12_3$ $4_3 5_3 12_3$ $6_3 7_3 11_3$

The long lines are $0_i 1_i \cdots 12_i$ $(i = 1,5)$. In order to prove that $63 \in LS_d(3, 13^{**})$, form a partition $\pi(7^1, 13^2, 30^1)$, where cells A and D are the sets $Z_{13} \times \{i\}$ $(i = 1, 4)$, cell B is the set $Z_{30} \times \{2\}$, and cell C is the set $Z_7 \times \{3\}$. Construct short lines of

type ABD: $0_1 4_2 0_4$ $0_1 22_2 3_4$ $0_1 0_2 4_4$ $0_1 17_2 7_4$ $0_1 i_2 i_4 (i = 1,2,5,6,8,\dots,12)$ $1_1 (i+13)_2 i_4$
$(i = 0,1,3,5,7,10,12)$ $1_1 7_2 2_4$ $1_1 21_2 4_4$ $1_1 24_2 6_4$ $1_1 11_2 8_4$ $1_1 17_2 9_4$
$1_1 22_2 11_4$ $2_1 (i+26)_2 i_4 (i = 0,1,5,6,9,10,11)$ $2_1 19_2 2_4$ $2_1 25_2 3_4$ $2_1 4_2 4_4$
$2_1 23_2 7_4$ $2_1 0_2 8_4$ $2_1 3_2 12_4$ $3_1 (i+9)_2 i_4 (i = 0,1,\dots,5,7,\dots,11)$ $3_1 7_2 6_4$
$3_1 28_2 12_4$ $4_1 (i+22)_2 i_4 (i = 0,5,6,9,10)$ $4_1 24_2 1_4$ $4_1 23_2 2_4$ $4_1 29_2 3_4$
$4_1 20_2 4_4$ $4_1 3_2 7_4$ $4_1 4_2 8_4$ $4_1 25_2 11_4$ $4_1 0_2 12_4$ $5_1 (i+5)_2 i_4$
$(i = 0,1,3,\dots,9,11,12)$ $5_1 3_2 2_4$ $5_1 29_2 10_4$ $6_1 (i+18)_2 i_4 (i = 0,2,4,5,7,8,9)$
$6_1 21_2 1_4$ $6_1 24_2 3_4$ $6_1 19_2 6_4$ $6_1 5_2 10_4$ $6_1 3_2 11_4$ $6_1 4_2 12_4$ $7_1 15_2 2_4$ $7_1 0_2 3_4$
$7_1 3_2 6_4$ $7_1 23_2 8_4$ $7_1 (i+1)_2 i_4 (i = 0,1,4,5,7,9,\dots,12)$ $8_1 (i+14)_2 i_4$
$(i = 0,2,3,4,9)$ $8_1 7_2 1_4$ $8_1 11_2 5_4$ $8_1 0_2 6_4$ $8_1 29_2 7_4$ $8_1 24_2 8_4$ $8_1 22_2 10_4$
$8_1 9_2 11_4$ $8_1 20_2 12_4$ $9_1 (i+27)_2 i_4 (i = 0,1,5,7,8,12)$ $9_1 25_2 2_4$ $9_1 26_2 3_4$
$9_1 8_2 4_4$ $9_1 15_2 6_4$ $9_1 21_2 9_4$ $9_1 3_2 10_4$ $9_1 1_2 11_4$ $10_1 (i+10)_2 i_4$
$(i = 0,1,\dots,6,8,11,12)$ $10_1 7_2 7_4$ $10_1 24_2 9_4$ $10_1 26_2 10_4$ $11_1 25_2 0_4$ $11_1 23_2 1_4$
$11_1 24_2 2_4$ $11_1 20_2 3_4$ $11_1 15_2 4_4$ $11_1 28_2 5_4$ $11_1 29_2 6_4$ $11_1 i_2 (i+7)_4$
$(i = 0,1,2,4,5)$ $11_1 27_2 10_4$ $12_1 (i+6)_2 i_4 (i = 0,2,3,4,6,7,10,12)$ $12_1 17_2 1_4$
$12_1 21_2 5_4$ $12_1 20_2 8_4$ $12_1 7_2 9_4$ $12_1 15_2 11_4$

type ABC: $0_1 13_2 0_3$ $0_1 26_2 1_3$ $0_1 29_2 2_3$ $0_1 16_2 3_3$ $0_1 7_2 4_3$ $0_1 18_2 5_3$ $0_1 24_2 6_3$ $1_1 26_2 0_3$
$1_1 27_2 1_3$ $1_1 19_2 2_3$ $1_1 29_2 3_3$ $1_1 0_2 4_3$ $1_1 1_2 5_3$ $1_1 2_2 6_3$ $2_1 (i+9)_2 i_3$
$(i = 0,1,3,4,5)$ $2_1 21_2 2_3$ $2_1 17_2 6_3$ $3_1 (i+22)_2 i_3 (i = 0,1,\ldots,5)$ $3_1 21_2 6_3$
$4_1 5_2 0_3$ $4_1 21_2 1_3$ $4_1 (i+7)_2 (i+2)_3 (i = 0,1,\ldots,4)$ $5_1 (i+18)_2 i_3 (i = 0,2,4,5)$
$5_1 24_2 1_3$ $5_1 4_2 3_3$ $5_1 15_2 6_3$ $6_1 29_2 0_3$ $6_1 2_2 1_3$ $6_1 15_2 2_3$ $6_1 0_2 3_3$ $6_1 28_2 4_3$
$6_1 6_2 5_3$ $6_1 8_2 6_3$ $7_1 14_2 0_3$ $7_1 25_2 1_3$ $7_1 16_2 2_3$ $7_1 17_2 3_3$ $7_1 24_2 4_3$ $7_1 20_2 5_3$
$7_1 26_2 6_3$ $8_1 4_2 0_3$ $8_1 28_2 1_3$ $8_1 3_2 2_3$ $8_1 26_2 3_3$ $8_1 1_2 4_3$ $8_1 2_2 5_3$ $8_1 19_2 6_3$
$9_1 (i+10)_2 i_3 (i = 0,1,\ldots,4)$ $9_1 7_2 5_3$ $9_1 16_2 6_3$ $10_1 (i+23)_2 i_3 (i = 0,2,4,6)$
$10_1 17_2 1_3$ $10_1 20_2 3_3$ $10_1 19_2 5_3$ $11_1 3_2 1_3$ $11_1 (i+6)_2 i_3 (i = 0,2,\ldots,6)$
$12_1 28_2 0_3$ $12_1 14_2 1_3$ $12_1 11_2 2_3$ $12_1 (i+22)_2 (i+3)_3 (i = 0,1,2,3)$

type ABB: $0_1 14_2 27_2$ $0_1 20_2 21_2$ $0_1 15_2 19_2$ $0_1 3_2 23_2$ $0_1 25_2 28_2$ $1_1 3_2 28_2$ $1_1 4_2 12_2$
$1_1 5_2 9_2$ $1_1 6_2 10_2$ $1_1 8_2 15_2$ $2_1 16_2 24_2$ $2_1 15_2 29_2$ $2_1 18_2 28_2$ $2_1 8_2 22_2$
$2_1 11_2 20_2$ $3_1 0_2 29_2$ $3_1 1_2 8_2$ $3_1 2_2 15_2$ $3_1 3_2 5_2$ $3_1 4_2 6_2$ $4_1 6_2 12_2$ $4_1 13_2 26_2$
$4_1 14_2 19_2$ $4_1 15_2 18_2$ $4_1 16_2 17_2$ $5_1 0_2 21_2$ $5_1 1_2 25_2$ $5_1 2_2 28_2$ $5_1 19_2 26_2$
$5_1 7_2 27_2$ $6_1 1_2 17_2$ $6_1 9_2 16_2$ $6_1 7_2 10_2$ $6_1 11_2 14_2$ $6_1 12_2 13_2$ $7_1 4_2 21_2$
$7_1 22_2 27_2$ $7_1 9_2 28_2$ $7_1 19_2 29_2$ $7_1 7_2 18_2$ $8_1 21_2 27_2$ $8_1 5_2 12_2$ $8_1 6_2 13_2$
$8_1 10_2 15_2$ $8_1 8_2 25_2$ $9_1 17_2 20_2$ $9_1 22_2 29_2$ $9_1 18_2 19_2$ $9_1 0_2 23_2$ $9_1 6_2 24_2$
$10_1 0_2 2_2$ $10_1 1_2 28_2$ $10_1 8_2 9_2$ $10_1 3_2 6_2$ $10_1 4_2 5_2$ $11_1 13_2 19_2$ $11_1 14_2 21_2$
$11_1 7_2 26_2$ $11_1 16_2 18_2$ $11_1 17_2 22_2$ $12_1 0_2 5_2$ $12_1 1_2 26_2$ $12_1 2_2 27_2$ $12_1 3_2 29_2$
$12_1 4_2 19_2$

type BCD: $2_2 0_3 0_4$ $15_2 1_3 0_4$ $0_2 2_3 0_4$ $3_2 3_3 0_4$ $8_2 4_3 0_4$ $12_2 5_3 0_4$ $7_2 6_3 0_4$ $8_2 0_3 1_4$ $4_2 1_3 1_4$
$5_2 2_3 1_4$ $18_2 3_3 1_4$ $20_2 4_3 1_4$ $13_2 5_3 1_4$ $0_2 6_3 1_4$ $27_2 0_3 2_4$ $5_2 1_3 2_4$ $17_2 2_3 2_4$
$6_2 3_3 2_4$ $4_2 4_3 2_4$ $28_2 5_3 2_4$ $22_2 6_3 2_4$ $15_2 0_3 3_4$ $7_2 1_3 3_4$ $10_2 2_3 3_4$ $21_2 3_3 3_4$
$19_2 4_3 3_4$ $4_2 5_3 3_4$ $23_2 6_3 3_4$ $1_2 0_3 4_4$ $16_2 1_3 4_4$ $6_2 2_3 4_4$ $23_2 3_3 4_4$ $25_2 4_3 4_4$
$26_2 5_3 4_4$ $28_2 6_3 4_4$ $12_2 0_3 5_4$ $19_2 1_3 5_4$ $22_2 2_3 5_4$ $24_2 3_3 5_4$ $29_2 4_3 5_4$ $17_2 5_3 5_4$
$3_2 6_3 5_4$ $17_2 0_3 6_4$ $18_2 1_3 6_4$ $23_2 2_3 6_4$ $27_2 3_3 6_4$ $21_2 4_3 6_4$ $22_2 5_3 6_4$ $9_2 6_3 6_4$
$24_2 0_3 7_4$ $22_2 1_3 7_4$ $26_2 2_3 7_4$ $19_2 3_3 7_4$ $6_2 4_3 7_4$ $5_2 5_3 7_4$ $10_2 6_3 7_4$ $25_2 0_3 8_4$
$12_2 1_3 8_4$ $27_2 2_3 8_4$ $2_2 3_3 8_4$ $15_2 4_3 8_4$ $16_2 5_3 8_4$ $6_2 6_3 8_4$ $19_2 0_3 9_4$ $0_2 1_3 9_4$
$13_2 2_3 9_4$ $15_2 3_3 9_4$ $11_2 4_3 9_4$ $8_2 5_3 9_4$ $20_2 6_3 9_4$ $7_2 0_3 10_4$ $1_2 1_3 10_4$ $14_2 2_3 10_4$
$28_2 3_3 10_4$ $12_2 4_3 10_4$ $9_2 5_3 10_4$ $18_2 6_3 10_4$ $0_2 0_3 11_4$ $13_2 1_3 11_4$ $18_2 2_3 11_4$

$5_2 3_3 11_4$ $17_2 4_3 11_4$ $29_2 5_3 11_4$ $27_2 6_3 11_4$ $11_2 0_3 12_4$ $29_2 1_3 12_4$ $1_2 2_3 12_4$
$10_2 3_3 12_4$ $16_2 4_3 12_4$ $15_2 5_3 12_4$ $14_2 6_3 12_4$

type BBC: $3_2 21_2 0_3$ $16_2 20_2 0_3$ $8_2 20_2 1_3$ $6_2 9_2 1_3$ $4_2 28_2 2_3$ $2_2 9_2 2_3$ $1_2 11_2 3_3$ $7_2 14_2 3_3$
$2_2 3_2 4_3$ $5_2 18_2 4_3$ $0_2 3_2 5_3$ $21_2 25_2 5_3$ $1_2 4_2 6_3$ $5_2 13_2 6_3$

type BBD: $11_2 29_2 0_4$ $16_2 21_2 0_4$ $19_2 20_2 0_4$ $23_2 28_2 0_4$ $17_2 24_2 0_4$ $3_2 19_2 1_4$ $9_2 22_2 1_4$
$16_2 25_2 1_4$ $12_2 15_2 1_4$ $26_2 29_2 1_4$ $0_2 1_2 2_4$ $9_2 10_2 2_4$ $13_2 14_2 2_4$ $18_2 26_2 2_4$
$21_2 29_2 2_4$ $1_2 2_2 3_4$ $6_2 27_2 3_4$ $11_2 18_2 3_4$ $3_2 14_2 3_4$ $5_2 28_2 3_4$ $2_2 17_2 4_4$ $3_2 7_2 4_4$
$11_2 12_2 4_4$ $19_2 27_2 4_4$ $24_2 29_2 4_4$ $0_2 8_2 5_4$ $4_2 7_2 5_4$ $13_2 20_2 5_4$ $16_2 26_2 5_4$
$9_2 25_2 5_4$ $10_2 20_2 6_4$ $1_2 5_2 6_4$ $8_2 13_2 6_4$ $14_2 26_2 6_4$ $4_2 25_2 6_4$ $1_2 9_2 7_4$ $2_2 21_2 7_4$
$14_2 18_2 7_4$ $15_2 28_2 7_4$ $11_2 27_2 7_4$ $3_2 10_2 8_4$ $21_2 28_2 8_4$ $9_2 19_2 8_4$ $14_2 22_2 8_4$
$7_2 29_2 8_4$ $4_2 16_2 9_4$ $3_2 12_2 9_4$ $25_2 29_2 9_4$ $26_2 28_2 9_4$ $6_2 22_2 9_4$ $0_2 13_2 10_4$
$4_2 8_2 10_4$ $17_2 21_2 10_4$ $20_2 25_2 10_4$ $15_2 24_2 10_4$ $8_2 10_2 11_4$ $2_2 19_2 11_4$ $6_2 28_2 11_4$
$14_2 24_2 11_4$ $23_2 26_2 11_4$ $2_2 7_2 12_4$ $8_2 19_2 12_4$ $6_2 21_2 12_4$ $24_2 26_2 12_4$
$23_2 27_2 12_4$

type BBB: $0_2 6_2 7_2$ $0_2 17_2 26_2$ $0_2 12_2 16_2$ $0_2 14_2 20_2$ $0_2 9_2 15_2$ $0_2 18_2 22_2$ $4_2 13_2 23_2$
$0_2 19_2 28_2$ $0_2 25_2 27_2$ $1_2 12_2 29_2$ $1_2 6_2 14_2$ $1_2 13_2 16_2$ $1_2 15_2 27_2$ $1_2 7_2 19_2$
$1_2 3_2 18_2$ $1_2 10_2 21_2$ $1_2 20_2 22_2$ $2_2 5_2 25_2$ $2_2 18_2 23_2$ $2_2 10_2 12_2$ $9_2 13_2 17_2$
$2_2 14_2 16_2$ $2_2 20_2 29_2$ $2_2 11_2 26_2$ $3_2 13_2 25_2$ $8_2 17_2 18_2$ $3_2 9_2 20_2$ $3_2 15_2 17_2$
$3_2 16_2 27_2$ $3_2 4_2 26_2$ $8_2 11_2 23_2$ $19_2 22_2 24_2$ $15_2 23_2 25_2$ $14_2 17_2 29_2$
$18_2 21_2 24_2$ $9_2 21_2 26_2$ $4_2 20_2 27_2$ $10_2 27_2 28_2$ $5_2 15_2 26_2$ $5_2 20_2 23_2$ $5_2 19_2 21_2$
$5_2 7_2 8_2$ $5_2 16_2 22_2$ $5_2 11_2 17_2$ $8_2 14_2 28_2$ $5_2 24_2 27_2$ $5_2 6_2 29_2$ $6_2 18_2 20_2$
$2_2 6_2 8_2$ $6_2 11_2 19_2$ $6_2 25_2 26_2$ $12_2 20_2 26_2$ $6_2 15_2 16_2$ $6_2 17_2 23_2$ $4_2 18_2 29_2$
$9_2 27_2 29_2$ $4_2 14_2 15_2$ $17_2 19_2 25_2$ $13_2 15_2 22_2$ $7_2 15_2 20_2$ $7_2 16_2 23_2$ $7_2 17_2 28_2$
$8_2 12_2 21_2$ $5_2 10_2 14_2$ $8_2 16_2 29_2$ $7_2 9_2 11_2$ $4_2 9_2 24_2$ $9_2 12_2 18_2$ $10_2 11_2 13_2$
$10_2 16_2 19_2$ $4_2 10_2 17_2$ $10_2 18_2 25_2$ $10_2 22_2 26_2$ $21_2 22_2 23_2$ $0_2 4_2 11_2$
$12_2 17_2 27_2$ $13_2 28_2 29_2$ $7_2 12_2 24_2$ $12_2 14_2 25_2$ $7_2 22_2 25_2$ $11_2 15_2 21_2$
$3_2 11_2 22_2$ $8_2 26_2 27_2$ $3_2 8_2 24_2$ $9_2 14_2 23_2$ $13_2 18_2 27_2$ $11_2 16_2 28_2$ $11_2 24_2 25_2$
$20_2 24_2 28_2$ $12_2 19_2 23_2$ $12_2 22_2 28_2$ $7_2 13_2 21_2$ $0_2 10_2 24_2$ $10_2 23_2 29_2$
$2_2 13_2 24_2$ $1_2 23_2 24_2$ $2_2 4_2 22_2$

type CCC: put an STS(7) on the seven points of cell C.

The long lines are $0_i 1_i \cdots 12_i$ (i = 1,4). Next, form a partition $\pi(1^1, 12^2, 18^1, 26^1)$, embed an STS(13) into an STS(27) which contains the twenty-six points of cell D and the point ∞, and construct short lines of

type ABD: $0_1 i_2 i_4$ $\;1_1 i_2 (i+12)_4$ $\;2_1 i_2 (i+24)_4$ $\;3_1 i_2 (i+10)_4$ $\;4_1 i_2 (i+22)_4$ $\;5_1 i_2 (i+8)_4$
$6_1 i_2 (i+20)_4$ $\;7_1 i_2 (i+6)_4$ $\;8_1 i_2 (i+18)_4$ $\;9_1 i_2 (i+4)_4$ $\;10_1 i_2 (i+16)_4$
$11_1 i_2 (i+2)_4 (i = 0,1,\dots,11)$

type ACD: $0_1 i_3 (i+12)_4 (i = 0,2,5,7,8,10,11)$ $\;0_1 15_3 13_4$ $\;0_1 4_3 15_4$ $\;0_1 3_3 16_4$ $\;0_1 17_3 18_4$
$0_1 1_3 21_4$ $\;0_1 13_3 24_4$ $\;0_1 9_3 25_4$ $\;1_1 16_3 24_4$ $\;1_1 3_3 25_4$ $\;1_1 4_3 0_4$ $\;1_1 15_3 1_4$ $\;1_1 10_3 2_4$
$1_1 17_3 3_4$ $\;1_1 (i+2)_3 (i+4)_4 (i = 0,3,4,5,6,7)$ $\;1_1 11_3 5_4$ $\;1_1 12_3 6_4$ $\;2_1 16_3 10_4$
$2_1 17_3 11_4$ $\;2_1 4_3 12_4$ $\;2_1 13_3 13_4$ $\;2_1 14_3 14_4$ $\;2_1 1_3 15_4$ $\;2_1 0_3 16_4$ $\;2_1 3_3 17_4$
$2_1 10_3 18_4$ $\;2_1 15_3 19_4$ $\;2_1 2_3 20_4$ $\;2_1 9_3 21_4$ $\;2_1 12_3 22_4$ $\;2_1 5_3 23_4$ $\;3_1 5_3 22_4$
$3_1 7_3 23_4$ $\;3_1 10_3 24_4$ $\;3_1 2_3 25_4$ $\;3_1 0_3 0_4$ $\;3_1 1_3 1_4$ $\;3_1 12_3 2_4$ $\;3_1 13_3 3_4$ $\;3_1 14_3 4_4$
$3_1 17_3 5_4$ $\;3_1 8_3 6_4$ $\;3_1 6_3 7_4$ $\;3_1 16_3 8_4$ $\;3_1 15_3 9_4$ $\;4_1 4_3 8_4$ $\;4_1 9_3 9_4$ $\;4_1 3_3 10_4$
$4_1 5_3 11_4$ $\;4_1 6_3 12_4$ $\;4_1 11_3 13_4$ $\;4_1 8_3 14_4$ $\;4_1 7_3 15_4$ $\;4_1 14_3 16_4$ $\;4_1 17_3 17_4$
$4_1 2_3 18_4$ $\;4_1 1_3 19_4$ $\;4_1 12_3 20_4$ $\;4_1 13_3 21_4$ $\;5_1 16_3 20_4$ $\;5_1 4_3 21_4$
$5_1 0_3 22_4$ $\;5_1 15_3 23_4$ $\;5_1 2_3 24_4$ $\;5_1 1_3 25_4$ $\;5_1 9_3 0_4$ $\;5_1 (i+5)_3 (i+1)_4$
$(i = 0,1,3,5)$ $\;5_1 11_3 3_4$ $\;5_1 14_3 5_4$ $\;5_1 7_3 7_4$ $\;6_1 16_3 6_4$ $\;6_1 3_3 7_4$ $\;6_1 14_3 8_4$
$6_1 4_3 9_4$ $\;6_1 0_3 10_4$ $\;6_1 13_3 11_4$ $\;6_1 (i+1)_3 (i+12)_4 (i = 0,1,4,5,6,7)$ $\;6_1 11_3 14_4$
$6_1 17_3 15_4$ $\;7_1 9_3 18_4$ $\;7_1 0_3 19_4$ $\;7_1 11_3 20_4$ $\;7_1 17_3 21_4$ $\;7_1 15_3 22_4$ $\;7_1 16_3 23_4$
$7_1 4_3 24_4$ $\;7_1 12_3 25_4$ $\;7_1 14_3 0_4$ $\;7_1 10_3 1_4$ $\;7_1 1_3 2_4$ $\;7_1 3_3 3_4$ $\;7_1 13_3 4_4$ $\;7_1 2_3 5_4$
$8_1 (i+5)_3 (i+4)_4 (i = 0,1,3,5,9,11,13)$ $\;8_1 11_3 6_4$ $\;8_1 12_3 8_4$ $\;8_1 17_3 10_4$ $\;8_1 1_3 11_4$
$8_1 3_3 12_4$ $\;8_1 4_3 14_4$ $\;8_1 13_3 16_4$ $\;9_1 15_3 16_4$ $\;9_1 (i+2)_3 (i+17)_4$
$(i = 0,2,3,\dots,6,8,\dots,11)$ $\;9_1 1_3 18_4$ $\;9_1 14_3 24_4$ $\;9_1 9_3 3_4$ $\;10_1 (i+17)_3 (i+2)_4$
$(i = 0,1,2,5)$ $\;10_1 3_3 5_4$ $\;10_1 15_3 6_4$ $\;10_1 9_3 8_4$ $\;10_1 2_3 9_4$ $\;10_1 (i+5)_3 (i+10)_4$
$(i = 0,1,2,3,5)$ $\;10_1 12_3 14_4$ $\;11_1 17_3 14_4$ $\;11_1 (i+12)_3 (i+15)_4$
$(i = 0,3,6,7,8,11,12)$ $\;11_1 9_3 16_4$ $\;11_1 16_3 17_4$ $\;11_1 14_3 19_4$ $\;11_1 4_3 20_4$
$11_1 11_3 24_4$ $\;11_1 13_3 25_4$

type ACC: $0_1 12_3 14_3$ $\;0_1 6_3 16_3$ $\;1_1 0_3 1_3$ $\;1_1 13_3 14_3$ $\;2_1 6_3 7_3$ $\;2_1 8_3 11_3$ $\;3_1 3_3 11_3$ $\;3_1 4_3 9_3$
$4_1 0_3 10_3$ $\;4_1 15_3 16_3$ $\;5_1 3_3 12_3$ $\;5_1 13_3 17_3$ $\;6_1 10_3 12_3$ $\;6_1 9_3 15_3$ $\;7_1 5_3 7_3$ $\;7_1 6_3 8_3$
$8_1 2_3 15_3$ $\;8_1 7_3 9_3$ $\;9_1 0_3 17_3$ $\;9_1 3_3 16_3$ $\;10_1 11_3 14_3$ $\;10_1 13_3 16_3$ $\;11_1 3_3 7_3$

$11_1 8_3 10_3$

type BCD: $0_2 17_3 1_4$ $0_2 4_3 3_4$ $0_2 5_3 5_4$ $0_2 11_3 7_4$ $0_2 1_3 9_4$ $0_2 10_3 11_4$ $0_2 7_3 13_4$ $0_2 15_3 14_4$
$0_2 13_3 15_4$ $0_2 14_3 17_4$ $0_2 12_3 19_4$ $0_2 16_3 21_4$ $0_2 3_3 23_4$ $0_2 6_3 25_4$ $1_2 8_3 0_4$
$1_2 2_3 2_4$ $1_2 3_3 4_4$ $1_2 6_3 6_4$ $1_2 0_3 8_4$ $1_2 9_3 10_4$ $1_2 10_3 12_4$ $1_2 13_3 14_4$ $1_2 14_3 15_4$
$1_2 16_3 16_4$ $1_2 12_3 18_4$ $1_2 15_3 20_4$ $1_2 11_3 22_4$ $1_2 5_3 24_4$ $2_2 13_3 1_4$ $2_2 12_3 3_4$
$2_2 7_3 5_4$ $2_2 14_3 7_4$ $2_2 11_3 9_4$ $2_2 3_3 11_4$ $2_2 4_3 13_4$ $2_2 2_3 15_4$ $2_2 8_3 16_4$ $2_2 15_3 17_4$
$2_2 9_3 19_4$ $2_2 5_3 21_4$ $2_2 6_3 23_4$ $2_2 0_3 25_4$ $3_2 15_3 0_4$ $3_2 11_3 2_4$ $3_2 (i+6)_3 (i+4)_4$
$(i = 0,4,6,8,10)$ $3_2 1_3 6_4$ $3_2 2_3 16_4$ $3_2 (i+4)_3 (i+17)_4 (i = 0,1,3,5)$ $3_2 17_3 24_4$
$4_2 2_3 1_4$ $4_2 5_3 3_4$ $4_2 10_3 5_4$ $4_2 i_3 (i+7)_4 (i = 0,4,8,11)$ $4_2 14_3 9_4$ $4_2 16_3 13_4$
$4_2 7_3 17_4$ $4_2 13_3 19_4$ $4_2 3_3 21_4$ $4_2 1_3 23_4$ $4_2 17_3 25_4$ $5_2 (i+3)_3 i_4$
$(i = 0,2,10,12)$ $5_2 10_3 4_4$ $5_2 14_3 6_4$ $5_2 7_3 8_4$ $5_2 9_3 14_4$ $5_2 4_3 16_4$ $5_2 6_3 18_4$
$5_2 2_3 19_4$ $5_2 17_3 20_4$ $5_2 8_3 22_4$ $5_2 0_3 24_4$ $6_2 9_3 1_4$ $6_2 6_3 3_4$ $6_2 0_3 5_4$
$6_2 16_3 7_4$ $6_2 3_3 9_4$ $6_2 14_3 11_4$ $6_2 17_3 13_4$ $6_2 15_3 15_4$ $6_2 12_3 17_4$ $6_2 5_3 19_4$
$6_2 1_3 20_4$ $6_2 11_3 21_4$ $6_2 4_3 23_4$ $6_2 8_3 25_4$ $7_2 6_3 0_4$ $7_2 4_3 2_4$ $7_2 7_3 4_4$ $7_2 17_3 6_4$
$7_2 13_3 8_4$ $7_2 14_3 10_4$ $7_2 16_3 12_4$ $7_2 0_3 14_4$ $7_2 1_3 16_4$ $7_2 8_3 18_4$ $7_2 10_3 20_4$
$7_2 15_3 21_4$ $7_2 3_3 22_4$ $7_2 9_3 24_4$ $8_2 7_3 1_4$ $8_2 8_3 3_4$ $8_2 1_3 5_4$ $8_2 13_3 7_4$ $8_2 12_3 9_4$
$8_2 0_3 11_4$ $8_2 9_3 13_4$ $8_2 6_3 15_4$ $8_2 11_3 17_4$ $8_2 16_3 19_4$ $8_2 10_3 21_4$ $8_2 2_3 22_4$
$8_2 17_3 23_4$ $8_2 15_3 25_4$ $9_2 2_3 0_4$ $9_2 7_3 2_4$ $9_2 9_3 4_4$ $9_2 13_3 6_4$ $9_2 15_3 8_4$ $9_2 4_3 10_4$
$9_2 11_3 12_4$ $9_2 3_3 14_4$ $9_2 10_3 16_4$ $9_2 0_3 18_4$ $9_2 14_3 20_4$ $9_2 16_3 22_4$ $9_2 12_3 23_4$
$9_2 8_3 24_4$ $10_2 3_3 1_4$ $10_2 1_3 3_4$ $10_2 9_3 5_4$ $10_2 2_3 7_4$ $10_2 16_3 9_4$ $10_2 15_3 11_4$
$10_2 12_3 13_4$ $10_2 5_3 15_4$ $10_2 10_3 17_4$ $10_2 11_3 19_4$ $10_2 14_3 21_4$ $10_2 0_3 23_4$
$10_2 7_3 24_4$ $10_2 4_3 25_4$ $11_2 12_3 0_4$ $11_2 0_3 2_4$ $11_2 11_3 4_4$ $11_2 9_3 6_4$ $11_2 8_3 8_4$
$11_2 10_3 10_4$ $11_2 2_3 12_4$ $11_2 1_3 14_4$ $11_2 7_3 16_4$ $11_2 3_3 18_4$ $11_2 6_3 20_4$ $11_2 13_3 22_4$
$11_2 15_3 24_4$ $11_2 5_3 25_4$

type BCC: $0_2 0_3 2_3$ $0_2 8_3 9_3$ $1_2 7_3 17_3$ $1_2 1_3 4_3$ $2_2 1_3 10_3$ $2_2 16_3 17_3$ $3_2 0_3 13_3$ $3_2 3_3 8_3$
$4_2 12_3 15_3$ $4_2 6_3 9_3$ $5_2 11_3 12_3$ $5_2 1_3 16_3$ $6_2 7_3 13_3$ $6_2 2_3 10_3$ $7_2 2_3 12_3$ $7_2 5_3 11_3$
$8_2 5_3 14_3$ $8_2 3_3 4_3$ $9_2 1_3 6_3$ $9_2 5_3 17_3$ $10_2 8_3 17_3$ $10_2 6_3 13_3$ $11_2 14_3 17_3$
$11_2 4_3 16_3$

type CCD: $10_3 17_3 0_4$ $1_3 13_3 0_4$ $7_3 16_3 0_4$ $4_3 11_3 1_4$ $0_3 14_3 1_4$ $8_3 16_3 1_4$ $8_3 15_3 2_4$ $9_3 16_3 2_4$
$3_3 14_3 2_4$ $10_3 16_3 3_4$ $7_3 15_3 3_4$ $2_3 14_3 3_4$ $15_3 17_3 4_4$ $0_3 16_3 4_4$ $4_3 12_3 4_4$

$13_3 15_3 5_4$ $4_3 8_3 5_4$ $12_3 16_3 5_4$ $0_3 7_3 6_4$ $2_3 3_3 6_4$ $4_3 5_3 6_4$ $10_3 15_3 7_4$ $1_3 17_3 7_4$
$9_3 12_3 7_4$ $1_3 11_3 8_4$ $2_3 17_3 8_4$ $3_3 5_3 8_4$ $0_3 8_3 9_4$ $5_3 13_3 9_4$ $6_3 17_3 9_4$ $1_3 7_3 10_4$
$2_3 6_3 10_4$ $11_3 15_3 10_4$ $2_3 16_3 11_4$ $7_3 11_3 11_4$ $8_3 12_3 11_4$ $12_3 13_3 12_4$ $5_3 8_3 12_4$
$9_3 17_3 12_4$ $5_3 10_3 13_4$ $0_3 6_3 13_4$ $1_3 3_3 13_4$ $5_3 6_3 14_4$ $7_3 10_3 14_4$ $3_3 9_3 15_4$
$0_3 11_3 15_4$ $11_3 17_3 16_4$ $6_3 12_3 16_4$ $8_3 13_3 17_4$ $1_3 9_3 17_4$ $4_3 13_3 18_4$ $14_3 16_3 18_4$
$6_3 10_3 19_4$ $3_3 17_3 19_4$ $0_3 9_3 20_4$ $3_3 13_3 20_4$ $7_3 12_3 21_4$ $2_3 8_3 21_4$ $4_3 17_3 22_4$
$6_3 14_3 22_4$ $9_3 14_3 23_4$ $10_3 13_3 23_4$ $1_3 12_3 24_4$ $3_3 6_3 24_4$ $7_3 14_3 25_4$
$11_3 16_3 25_4$

type CCC: $1_3 8_3 14_3$ $4_3 6_3 15_3$ $0_3 3_3 15_3$ $4_3 10_3 14_3$ $2_3 4_3 7_3$ $0_3 5_3 12_3$ $1_3 5_3 15_3$ $9_3 10_3 11_3$
$2_3 11_3 13_3$ $2_3 5_3 9_3$

type ∞CC: $\infty 7_3 8_3$ $\infty 14_3 15_3$ $\infty 1_3 2_3$ $\infty 12_3 17_3$ $\infty 5_3 16_3$ $\infty 6_3 11_3$ $\infty 3_3 10_3$ $\infty 9_3 13_3$
$\infty 0_3 4_3$

types ∞BB and BBB: place an STS(13) on the twelve points of cell B and the point ∞. One long line is $0_1 1_1 \cdots 11_1 \infty$ and the other long line is obtained by replacing the subsystem STS(13). Hence, we have proven that $69 \in LS_d(3, 13^{**})$. We present our final direct constructions. Form a partition $\pi(6^1, 13^1, 15^2)$, where cells A and D are the sets $Z_{15} \times \{i\}$ $(i = 1, 4)$, cell B is the set $Z_6 \times \{2\}$ and cell C is the set $Z_{13} \times \{3\}$. Construct short lines of

type ABD: $0_1 1_2 1_4$ $0_1 3_2 3_4$ $0_1 2_2 6_4$ $0_1 0_2 7_4$ $1_1 1_2 0_4$ $1_1 4_2 4_4$ $1_1 0_2 6_4$ $1_1 5_2 13_4$
$2_1 2_2 8_4$ $2_1 5_2 9_4$ $2_1 3_2 10_4$ $2_1 4_2 11_4$ $3_1 5_2 7_4$ $3_1 0_2 12_4$ $3_1 1_2 13_4$ $3_1 2_2 14_4$
$4_1 0_2 1_4$ $4_1 4_2 3_4$ $4_1 5_2 4_4$ $4_1 3_2 6_4$ $5_1 2_2 2_4$ $5_1 1_2 5_4$ $5_1 3_2 7_4$ $5_1 4_2 8_4$ $6_1 4_2 6_4$
$6_1 0_2 9_4$ $6_1 2_2 10_4$ $6_1 3_2 11_4$ $7_1 2_2 0_4$ $7_1 3_2 1_4$ $7_1 5_2 5_4$ $7_1 1_2 14_4$ $8_1 0_2 2_4$
$8_1 1_2 3_4$ $8_1 2_2 4_4$ $8_1 4_2 13_4$ $9_1 5_2 0_4$ $9_1 3_2 2_4$ $9_1 0_2 8_4$ $9_1 1_2 9_4$ $10_1 4_2 5_4$
$10_1 0_2 10_4$ $10_1 5_2 11_4$ $10_1 2_2 12_4$ $11_1 5_2 1_4$ $11_1 4_2 2_4$ $11_1 1_2 7_4$ $11_1 3_2 14_4$
$12_1 0_2 3_4$ $12_1 1_2 4_4$ $12_1 4_2 12_4$ $12_1 3_2 13_4$ $13_1 0_2 0_4$ $13_1 3_2 8_4$ $13_1 2_2 9_4$
$13_1 5_2 10_4$ $14_1 2_2 5_4$ $14_1 1_2 11_4$ $14_1 5_2 12_4$ $14_1 4_2 14_4$

type ABB: $0_1 4_2 5_2$ $1_1 2_2 3_2$ $2_1 0_2 1_2$ $3_1 3_2 4_2$ $4_1 1_2 2_2$ $5_1 0_2 5_2$ $6_1 1_2 5_2$ $7_1 0_2 4_2$ $8_1 3_2 5_2$
$9_1 2_2 4_2$ $10_1 1_2 3_2$ $11_1 0_2 2_2$ $12_1 2_2 5_2$ $13_1 1_2 4_2$ $14_1 0_2 3_2$

type ACD: $0_1 10_3 0_4$ $0_1 6_3 2_4$ $0_1 8_3 4_4$ $0_1 2_3 5_4$ $0_1 3_3 8_4$ $0_1 5_3 9_4$ $0_1 4_3 10_4$ $0_1 9_3 11_4$
$0_1 1_3 12_4$ $0_1 7_3 13_4$ $0_1 11_3 14_4$ $1_1 9_3 1_4$ $1_1 8_3 2_4$ $1_1 4_3 3_4$ $1_1 0_3 5_4$ $1_1 2_3 7_4$

$1_16_38_4$ $1_112_39_4$ $1_110_310_4$ $1_15_311_4$ $1_17_312_4$ $1_13_314_4$ $2_111_30_4$ $2_18_31_4$
$2_11_32_4$ $2_17_33_4$ $2_19_34_4$ $2_16_35_4$ $2_112_36_4$ $2_14_37_4$ $2_13_312_4$ $2_10_313_4$
$2_15_314_4$ $3_18_30_4$ $3_11_31_4$ $3_110_32_4$ $3_112_33_4$ $3_15_34_4$ $3_14_35_4$ $3_17_36_4$
$3_19_38_4$ $3_13_39_4$ $3_111_310_4$ $3_12_311_4$ $4_13_30_4$ $4_17_32_4$ $4_19_35_4$ $4_110_37_4$
$4_112_38_4$ $4_16_39_4$ $4_11_310_4$ $4_111_311_4$ $4_15_312_4$ $4_12_313_4$ $4_10_314_4$ $5_16_30_4$
$5_111_31_4$ $5_15_33_4$ $5_112_34_4$ $5_11_36_4$ $5_18_39_4$ $5_19_310_4$ $5_17_311_4$ $5_110_312_4$
$5_14_313_4$ $5_12_314_4$ $6_10_30_4$ $6_12_31_4$ $6_111_32_4$ $6_18_33_4$ $6_16_34_4$ $6_15_35_4$
$6_17_37_4$ $6_110_38_4$ $6_14_312_4$ $6_11_313_4$ $6_112_314_4$ $7_112_32_4$ $7_110_33_4$ $7_13_34_4$
$7_111_36_4$ $7_19_37_4$ $7_11_38_4$ $7_10_39_4$ $7_15_310_4$ $7_14_311_4$ $7_16_312_4$ $7_18_313_4$
$8_17_30_4$ $8_110_31_4$ $8_11_35_4$ $8_16_36_4$ $8_10_37_4$ $8_111_38_4$ $8_14_39_4$ $8_13_310_4$
$8_112_311_4$ $8_12_312_4$ $8_18_314_4$ $9_17_31_4$ $9_12_33_4$ $9_111_34_4$ $9_13_35_4$ $9_18_36_4$
$9_16_37_4$ $9_10_310_4$ $9_11_311_4$ $9_112_312_4$ $9_15_313_4$ $9_14_314_4$ $10_12_30_4$ $10_13_31_4$
$10_19_32_4$ $10_111_33_4$ $10_110_34_4$ $10_14_36_4$ $10_18_37_4$ $10_10_38_4$ $10_17_39_4$
$10_16_313_4$ $10_11_314_4$ $11_14_30_4$ $11_16_33_4$ $11_12_34_4$ $11_110_35_4$ $11_13_36_4$
$11_17_38_4$ $11_11_39_4$ $11_112_310_4$ $11_10_311_4$ $11_18_312_4$ $11_19_313_4$ $12_11_30_4$
$12_112_31_4$ $12_10_32_4$ $12_18_35_4$ $12_15_36_4$ $12_111_37_4$ $12_14_38_4$ $12_19_39_4$
$12_12_310_4$ $12_13_311_4$ $12_16_314_4$ $13_16_31_4$ $13_14_32_4$ $13_13_33_4$ $13_11_34_4$
$13_111_35_4$ $13_12_36_4$ $13_15_37_4$ $13_110_311_4$ $13_19_312_4$ $13_112_313_4$ $13_17_314_4$
$14_112_30_4$ $14_14_31_4$ $14_12_32_4$ $14_11_33_4$ $14_17_34_4$ $14_10_36_4$ $14_13_37_4$ $14_15_38_4$
$14_111_39_4$ $14_16_310_4$ $14_110_313_4$

type ACC: $0_10_312_3$ $1_11_311_3$ $2_12_310_3$ $3_10_36_3$ $4_14_38_3$ $5_10_33_3$ $6_13_39_3$ $7_12_37_3$
$8_15_39_3$ $9_19_310_3$ $10_15_312_3$ $11_15_311_3$ $12_17_310_3$ $13_10_38_3$ $14_18_39_3$

type BCC: $0_20_31_3$ $0_25_37_3$ $0_22_36_3$ $0_29_311_3$ $3_23_34_3$ $3_28_312_3$ $3_26_39_3$ $3_21_32_3$
$1_24_310_3$ $1_21_36_3$ $1_27_312_3$ $1_23_311_3$ $2_22_33_3$ $2_21_38_3$ $2_24_37_3$ $2_20_310_3$
$4_24_35_3$ $4_211_312_3$ $4_23_36_3$ $4_28_310_3$ $5_27_311_3$ $5_25_36_3$ $5_21_34_3$ $5_22_312_3$

type CCC: $0_32_35_3$ $1_35_310_3$ $2_34_39_3$ $3_35_38_3$ $4_36_312_3$ $6_310_311_3$ $1_33_37_3$ $0_37_39_3$
$1_39_312_3$ $2_38_311_3$ $3_310_312_3$ $0_34_311_3$ $6_37_38_3$

type BCD: $0_24_34_4$ $0_212_35_4$ $0_28_311_4$ $0_23_313_4$ $0_210_314_4$ $1_25_32_4$ $1_29_36_4$ $1_22_38_4$
$1_28_310_4$ $1_20_312_4$ $2_25_31_4$ $2_29_33_4$ $2_212_37_4$ $2_26_311_4$ $2_211_313_4$ $3_25_30_4$
$3_20_34_4$ $3_27_35_4$ $3_210_39_4$ $3_211_312_4$ $4_29_30_4$ $4_20_31_4$ $4_21_37_4$ $4_22_39_4$

$4_2 7_3 10_4$ $5_2 3_3 2_4$ $5_2 0_3 3_4$ $5_2 10_3 6_4$ $5_2 8_3 8_4$ $5_2 9_3 14_4$

The long lines are $0_i 1_i \cdots 14_i$ (i = 1,4). Finally, 55 ϵ $LS_d(3, 15^{**})$ by forming a partition $\pi(15^2, 25^1)$, where cells A and C are the sets $Z_{15} \times \{i\}$ (i = 1, 3) and cell B is the set $Z_{25} \times \{2\}$. Construct short lines of

type ABC: $0_1 10_2 0_3$ $0_1 18_2 1_3$ $0_1 6_2 2_3$ $0_1 17_2 3_3$ $0_1 4_2 4_3$ $0_1 1_2 5_3$ $0_1 23_2 6_3$ $0_1 7_2 7_3$
$0_1 12_2 8_3$ $0_1 11_2 9_3$ $0_1 15_2 10_3$ $0_1 14_2 11_3$ $0_1 2_2 12_3$ $0_1 24_2 13_3$ $0_1 19_2 14_3$
$1_1 8_2 0_3$ $1_1 20_2 1_3$ $1_1 15_2 2_3$ $1_1 1_2 3_3$ $1_1 10_2 4_3$ $1_1 5_2 5_3$ $1_1 22_2 6_3$ $1_1 24_2 7_3$
$1_1 6_2 8_3$ $1_1 3_2 9_3$ $1_1 17_2 10_3$ $1_1 0_2 11_3$ $1_1 18_2 12_3$ $1_1 14_2 13_3$ $1_1 12_2 14_3$ $2_1 4_2 0_3$
$2_1 21_2 1_3$ $2_1 24_2 2_3$ $2_1 7_2 3_3$ $2_1 23_2 4_3$ $2_1 3_2 5_3$ $2_1 11_2 6_3$ $2_1 1_2 7_3$ $2_1 5_2 8_3$
$2_1 12_2 9_3$ $2_1 9_2 10_3$ $2_1 15_2 11_3$ $2_1 19_2 12_3$ $2_1 16_2 13_3$ $2_1 10_2 14_3$ $3_1 12_2 0_3$
$3_1 7_2 1_3$ $3_1 22_2 2_3$ $3_1 15_2 3_3$ $3_1 9_2 4_3$ $3_1 6_2 5_3$ $3_1 18_2 6_3$ $3_1 19_2 7_3$ $3_1 10_2 8_3$
$3_1 23_2 9_3$ $3_1 16_2 10_3$ $3_1 13_2 11_3$ $3_1 4_2 12_3$ $3_1 20_2 13_3$ $3_1 2_2 14_3$ $4_1 3_2 0_3$ $4_1 8_2 1_3$
$4_1 18_2 2_3$ $4_1 22_2 3_3$ $4_1 6_2 4_3$ $4_1 17_2 5_3$ $4_1 9_2 6_3$ $4_1 14_2 7_3$ $4_1 16_2 8_3$ $4_1 24_2 9_3$
$4_1 1_2 10_3$ $4_1 19_2 11_3$ $4_1 20_2 12_3$ $4_1 5_2 13_3$ $4_1 13_2 14_3$ $5_1 6_2 0_3$ $5_1 12_2 1_3$ $5_1 2_2 2_3$
$5_1 8_2 3_3$ $5_1 18_2 4_3$ $5_1 13_2 5_3$ $5_1 7_2 6_3$ $5_1 16_2 7_3$ $5_1 19_2 8_3$ $5_1 5_2 9_3$ $5_1 20_2 10_3$
$5_1 4_2 11_3$ $5_1 14_2 12_3$ $5_1 23_2 13_3$ $5_1 3_2 14_3$ $6_1 24_2 0_3$ $6_1 9_2 1_3$ $6_1 4_2 2_3$ $6_1 10_2 3_3$
$6_1 13_2 4_3$ $6_1 11_2 5_3$ $6_1 16_2 6_3$ $6_1 20_2 7_3$ $6_1 23_2 8_3$ $6_1 6_2 9_3$ $6_1 7_2 10_3$ $6_1 8_2 11_3$
$6_1 0_2 12_3$ $6_1 17_2 13_3$ $6_1 21_2 14_3$ $7_1 22_2 0_3$ $7_1 19_2 1_3$ $7_1 17_2 2_3$ $7_1 24_2 3_3$ $7_1 5_2 4_3$
$7_1 20_2 5_3$ $7_1 3_2 6_3$ $7_1 15_2 7_3$ $7_1 0_2 8_3$ $7_1 7_2 9_3$ $7_1 13_2 10_3$ $7_1 18_2 11_3$ $7_1 1_2 12_3$
$7_1 2_2 13_3$ $7_1 16_2 14_3$ $8_1 20_2 0_3$ $8_1 16_2 1_3$ $8_1 0_2 2_3$ $8_1 2_2 3_3$ $8_1 14_2 4_3$ $8_1 21_2 5_3$
$8_1 10_2 6_3$ $8_1 23_2 7_3$ $8_1 13_2 8_3$ $8_1 8_2 9_3$ $8_1 4_2 10_3$ $8_1 12_2 11_3$ $8_1 24_2 12_3$
$8_1 11_2 13_3$ $8_1 22_2 14_3$ $9_1 5_2 0_3$ $9_1 1_2 1_3$ $9_1 20_2 2_3$ $9_1 9_2 3_3$ $9_1 3_2 4_3$ $9_1 22_2 5_3$
$9_1 24_2 6_3$ $9_1 0_2 7_3$ $9_1 21_2 8_3$ $9_1 2_2 9_3$ $9_1 14_2 10_3$ $9_1 6_2 11_3$ $9_1 8_2 12_3$ $9_1 12_2 13_3$
$9_1 18_2 14_3$ $10_1 17_2 0_3$ $10_1 24_2 1_3$ $10_1 1_2 2_3$ $10_1 19_2 3_3$ $10_1 11_2 4_3$ $10_1 14_2 5_3$
$10_1 12_2 6_3$ $10_1 3_2 7_3$ $10_1 2_2 8_3$ $10_1 4_2 9_3$ $10_1 8_2 10_3$ $10_1 21_2 11_3$ $10_1 16_2 12_3$
$10_1 7_2 13_3$ $10_1 9_2 14_3$ $11_1 18_2 0_3$ $11_1 0_2 1_3$ $11_1 13_2 2_3$ $11_1 21_2 3_3$ $11_1 7_2 4_3$
$11_1 15_2 5_3$ $11_1 2_2 6_3$ $11_1 4_2 7_3$ $11_1 17_2 8_3$ $11_1 9_2 9_3$ $11_1 5_2 10_3$ $11_1 11_2 11_3$
$11_1 23_2 12_3$ $11_1 19_2 13_3$ $11_1 14_2 14_3$ $12_1 16_2 0_3$ $12_1 11_2 1_3$ $12_1 3_2 2_3$ $12_1 5_2 3_3$
$12_1 8_2 4_3$ $12_1 0_2 5_3$ $12_1 6_2 6_3$ $12_1 12_2 7_3$ $12_1 7_2 8_3$ $12_1 17_2 9_3$ $12_1 23_2 10_3$
$12_1 10_2 11_3$ $12_1 13_2 12_3$ $12_1 22_2 13_3$ $12_1 1_2 14_3$ $13_1 19_2 0_3$ $13_1 23_2 1_3$

$13_1 10_2 2_3$ $13_1 4_2 3_3$ $13_1 2_2 4_3$ $13_1 12_2 5_3$ $13_1 13_2 6_3$ $13_1 6_2 7_3$ $13_1 1_2 8_3$
$13_1 0_2 9_3$ $13_1 11_2 10_3$ $13_1 22_2 11_3$ $13_1 15_2 12_3$ $13_1 21_2 13_3$ $13_1 7_2 14_3$
$14_1 23_2 0_3$ $14_1 15_2 1_3$ $14_1 8_2 2_3$ $14_1 11_2 3_3$ $14_1 0_2 4_3$ $14_1 16_2 5_3$ $14_1 19_2 6_3$
$14_1 5_2 7_3$ $14_1 14_2 8_3$ $14_1 21_2 9_3$ $14_1 3_2 10_3$ $14_1 20_2 11_3$ $14_1 9_2 12_3$
$14_1 13_2 13_3$ $14_1 17_2 14_3$

type ABB: $0_1 8_2 13_2$ $0_1 0_2 22_2$ $0_1 5_2 20_2$ $0_1 16_2 21_2$ $0_1 3_2 9_2$ $1_1 2_2 9_2$ $1_1 11_2 13_2$ $1_1 7_2 19_2$
$1_1 21_2 23_2$ $1_1 4_2 16_2$ $2_1 13_2 18_2$ $2_1 14_2 20_2$ $2_1 0_2 2_2$ $2_1 6_2 8_2$ $2_1 17_2 22_2$
$3_1 0_2 11_2$ $3_1 1_2 5_2$ $3_1 3_2 8_2$ $3_1 14_2 17_2$ $3_1 21_2 24_2$ $4_1 12_2 23_2$ $4_1 2_2 7_2$ $4_1 10_2 11_2$
$4_1 4_2 15_2$ $4_1 0_2 21_2$ $5_1 15_2 21_2$ $5_1 11_2 22_2$ $5_1 10_2 17_2$ $5_1 0_2 9_2$ $5_1 1_2 24_2$
$6_1 12_2 14_2$ $6_1 1_2 22_2$ $6_1 5_2 15_2$ $6_1 18_2 19_2$ $6_1 2_2 3_2$ $7_1 4_2 9_2$ $7_1 11_2 23_2$ $7_1 8_2 10_2$
$7_1 14_2 21_2$ $7_1 6_2 12_2$ $8_1 9_2 15_2$ $8_1 6_2 7_2$ $8_1 1_2 3_2$ $8_1 17_2 18_2$ $8_1 5_2 19_2$ $9_1 7_2 23_2$
$9_1 4_2 10_2$ $9_1 13_2 15_2$ $9_1 11_2 16_2$ $9_1 17_2 19_2$ $10_1 10_2 13_2$ $10_1 0_2 23_2$ $10_1 5_2 22_2$
$10_1 15_2 20_2$ $10_1 6_2 18_2$ $11_1 6_2 10_2$ $11_1 22_2 24_2$ $11_1 1_2 20_2$ $11_1 3_2 16_2$ $11_1 8_2 12_2$
$12_1 20_2 21_2$ $12_1 9_2 18_2$ $12_1 2_2 24_2$ $12_1 15_2 19_2$ $12_1 4_2 14_2$ $13_1 18_2 24_2$
$13_1 9_2 16_2$ $13_1 3_2 14_2$ $13_1 17_2 20_2$ $13_1 5_2 8_2$ $14_1 6_2 24_2$ $14_1 4_2 18_2$ $14_1 7_2 12_2$
$14_1 1_2 10_2$ $14_1 2_2 22_2$

type BBC: $11_2 14_2 0_3$ $7_2 13_2 0_3$ $0_2 1_2 0_3$ $9_2 21_2 0_3$ $2_2 15_2 0_3$ $3_2 22_2 1_3$ $13_2 14_2 1_3$ $5_2 10_2 1_3$
$6_2 17_2 1_3$ $2_2 4_2 1_3$ $7_2 16_2 2_3$ $5_2 14_2 2_3$ $9_2 23_2 2_3$ $11_2 19_2 2_3$ $12_2 21_2 2_3$
$12_2 13_2 3_3$ $18_2 20_2 3_3$ $16_2 23_2 3_3$ $0_2 14_2 3_3$ $3_2 6_2 3_3$ $15_2 24_2 4_3$ $19_2 20_2 4_3$
$16_2 17_2 4_3$ $1_2 12_2 4_3$ $21_2 22_2 4_3$ $4_2 23_2 5_3$ $2_2 8_2 5_3$ $10_2 19_2 5_3$ $7_2 18_2 5_3$
$9_2 24_2 5_3$ $1_2 21_2 6_3$ $0_2 20_2 6_3$ $4_2 5_2 6_3$ $8_2 14_2 6_3$ $15_2 17_2 6_3$ $2_2 13_2 7_3$ $9_2 10_2 7_3$
$8_2 22_2 7_3$ $11_2 17_2 7_3$ $18_2 21_2 7_3$ $3_2 4_2 8_3$ $9_2 11_2 8_3$ $8_2 24_2 8_3$ $15_2 18_2 8_3$
$20_2 22_2 8_3$ $1_2 14_2 9_3$ $16_2 20_2 9_3$ $10_2 18_2 9_3$ $15_2 22_2 9_3$ $13_2 19_2 9_3$ $12_2 18_2 10_3$
$2_2 21_2 10_3$ $6_2 19_2 10_3$ $10_2 22_2 10_3$ $0_2 24_2 10_3$ $3_2 7_2 11_3$ $9_2 17_2 11_3$ $1_2 2_2 11_3$
$23_2 24_2 11_3$ $5_2 16_2 11_3$ $6_2 11_2 12_3$ $5_2 7_2 12_3$ $12_2 22_2 12_3 10_2 21_2 12_3$ $3_2 17_2 12_3$
$1_2 4_2 13_3$ $8_2 9_2 13_3$ $3_2 18_2 13_3$ $0_2 6_2 13_3$ $10_2 15_2 13_3$ $0_2 15_2 14_3$ $4_2 24_2 14_3$
$5_2 6_2 14_3$ $8_2 11_2 14_3$ $20_2 23_2 14_3$

type BBB: $0_2 4_2 13_2$ $1_2 7_2 9_2$ $2_2 6_2 16_2$ $14_2 15_2 16_2$ $4_2 8_2 17_2$ $5_2 18_2 23_2$ $10_2 12_2 16_2$
$3_2 11_2 21_2$ $8_2 19_2 21_2$ $13_2 22_2 23_2$ $3_2 10_2 23_2$ $7_2 11_2 15_2$ $0_2 16_2 19_2$ $1_2 13_2 17_2$
$2_2 14_2 18_2$ $3_2 12_2 15_2$ $4_2 6_2 20_2$ $5_2 17_2 21_2$ $16_2 18_2 22_2$ $9_2 14_2 19_2$ $7_2 8_2 20_2$

$2_2 5_2 11_2$ $0_2 3_2 5_2$ $1_2 6_2 15_2$ $10_2 14_2 24_2$ $0_2 7_2 10_2$ $1_2 8_2 16_2$ $2_2 17_2 23_2$
$4_2 11_2 12_2$ $9_2 12_2 20_2$ $6_2 13_2 21_2$ $7_2 14_2 22_2$ $8_2 15_2 23_2$ $5_2 12_2 24_2$ $0_2 12_2 17_2$
$1_2 11_2 18_2$ $2_2 12_2 19_2$ $3_2 13_2 20_2$ $4_2 7_2 21_2$ $6_2 9_2 22_2$ $6_2 14_2 23_2$ $7_2 17_2 24_2$
$0_2 8_2 18_2$ $1_2 19_2 23_2$ $2_2 10_2 20_2$ $11_2 20_2 24_2$ $4_2 19_2 22_2$ $5_2 9_2 13_2$ $3_2 19_2 24_2$
$13_2 16_2 24_2$

The long lines are $0_i 1_i \cdots 14_i$ (i = 1,3).

Lemma 2.5 Let u = 7, 9 and 13. Then $LS_d(3, u^{**}) = \{v: v \geq 3u, v \equiv 1, 3 \pmod 6\}$. If u = 15, $v \in LS_d(3,15^{**})$ for all $v \geq 45$, $v \equiv 1,3 \pmod 6$ except possibly v = 51,57.

Proof: This follows from Theorem 2.2, Corollary 2.3 and Lemma 2.4.

§2.2 Almost uniform linear spaces with one long line of size 6t + 5, one long line of size w and short lines of size three

From the necessary conditions in Corollary 1.19(i) and Corollary 1.21(i), we must have in such an $LS(v;\{3, (6t + 5)^*, w^*\})$, $w \equiv 1, 3 \pmod 6$ and $v \equiv 5 \pmod 6$. Furthermore, we consider that $w > 6t + 5$. If $w \equiv 1 \pmod 6$ and the two long lines intersect, we shall employ a method of construction described in §1.3 to recursively build an AULS with the minimum number of points v.

Lemma 2.6 If $w \equiv 1 \pmod 6$ and $w > 6t+5$, then there exists an $LS_i(2w+6t+3;\{3,(6t + 5)^*,w^*\})$ where $2w+6t+3 = \min\{v: \exists LS_i(v;\{3,(6t + 5)^*,w^*\})\}$.
Proof: We know that $v \geq 2w + u - 2$, by Corollary 1.21(i). Since $u = 6t + 5$, $v \geq 2w + 6t + 3$. Form the partition $\pi(1^1, (6t + 4)^1, (w - 1)^2)$ and apply Lemma 1.32(a).

Corollary 2.7 If $w \equiv 1 \pmod 6$ and $w > 6t + 5$, then $v \in LS_i(3, (6t + 5)^*, w^*)$ for all $v \geq 4w + 12t + 7$, $v \equiv 5 \pmod 6$.
Proof: This follows easily from Lemma 2.6 and Lemma 1.39.

Corollary 2.8 If $w \equiv 3 \pmod 6$ then $4w + 6t + 5 \in LS(3, (6t + 5)^*, w^*)$.
Proof: Construct an $LS_i(4w + 6t + 5;\{3, (6t + 5)^*, (2w + 1)^*\})$ by applying Lemma 2.6. Replace the line of size $2w + 1$ by an $STS(2w + 1)$. By Theorem 1.7, an $STS(2w + 1)$ contains a subdesign $STS(w)$. The subdesign is replaced by a line of size w.

There is, unfortunately, no apparent construction for an AULS of minimum order in which the two long lines intersect, and one long line has size w congruent to 3 (mod 6). However, it is possible to construct an AULS of minimum order in which the two long lines are disjoint, with one long line of size $w \equiv 1 \pmod 6$, provided that $t = 0$.

Lemma 2.9 If $w \equiv 1 \pmod 6$, then $2w + 9 \in LS_d(3, 5^*, w^*)$ and $2w + 9 = \min\{v: \exists LS_d(v;\{3, 5^*, w^*\})\}$.

Proof: If $w \geq 13$, form the partition $\pi(1^1, 10^1, (w-1)^2)$ and essentially follow the arguments in case 2 of Lemma 1.32(a) except that the line $\infty_1\infty_2 \cdots \infty_{10}\infty$ is replaced by an IPBD(11, 5;{3}) (cf. Theorem 1.12); the hole does not contain ∞. The line of size five is placed on this hole. If $w = 7$, $23 \in LS_d(3, 5^*, 7^*)$ is obtained by direct construction. Form the partition $\pi(5^2, 6^1, 7^1)$. The cells A and B are given by the sets $Z_5 \times \{i\}$ $(i = 1, 2)$, cell C is the set $Z_6 \times \{3\}$, and cell D is the set $Z_7 \times \{4\}$. The short lines are of

type ACD: $0_12_31_4$ $0_10_32_4$ $0_14_30_4$ $0_13_35_4$ $1_12_30_4$ $1_13_32_4$ $1_15_36_4$ $1_14_34_4$ $2_12_32_4$ $2_13_34_4$ $2_11_33_4$ $2_15_31_4$ $3_14_33_4$ $3_15_34_4$ $3_11_35_4$ $3_10_36_4$ $4_14_36_4$ $4_10_30_4$ $4_12_35_4$ $4_11_34_4$

type ABD: $0_10_24_4$ $0_11_23_4$ $0_12_26_4$ $1_13_21_4$ $1_14_23_4$ $1_10_25_4$ $2_11_20_4$ $2_12_25_4$ $2_13_26_4$ $3_14_20_4$ $3_10_21_4$ $3_11_22_4$ $4_14_21_4$ $4_10_22_4$ $4_12_23_4$

type ABC: $0_13_21_3$ $0_14_25_3$ $1_11_20_3$ $1_12_21_3$ $2_10_20_3$ $2_14_24_3$ $3_13_22_3$ $3_12_23_3$ $4_13_23_3$ $4_11_25_3$

type BCD: $0_21_30_4$ $0_22_36_4$ $1_21_31_4$ $1_23_36_4$ $2_20_34_4$ $2_24_31_4$ $3_24_32_4$ $3_25_35_4$ $4_21_36_4$ $4_22_34_4$

type BBC: $3_24_20_3$ $1_22_22_3$ $0_24_23_3$ $0_21_24_3$ $0_22_25_3$

type BBD: $2_23_20_4$ $2_24_22_4$ $0_23_23_4$ $1_23_24_4$ $1_24_25_4$

type CCD: $3_35_30_4$ $0_33_31_4$ $1_35_32_4$ $0_35_33_4$ $2_33_33_4$ $0_34_35_4$

type CCC: $0_31_32_3$ $1_33_34_3$ $2_34_35_3$

The long lines are $0_11_12_13_14_1$ and $0_41_4 \cdots 6_4$.

Corollary 2.10 If $w \equiv 1 \pmod 6$, then $v \in LS_d(3, 5^*, w^*)$ for all $v \geq 4w + 19$, $v \equiv 5 \pmod 6$.

Proof: This is clearly a consequence of Lemma 2.9 and Lemma 1.39.

We have no general result for proving the existence of AULSs of minimum order, where the two long lines are disjoint and one long line has size congruent to 3 (mod 6). We can do much better when one long line has size five and the other long line has size nine or fifteen.

Lemma 2.11 $23 \in LS_d(3,5^*,9^*), 35 \in LS(3,5^*,15^*)$ and $23 = \min\{v: \exists LS_d(v;\{3,5^*,9^*\})\}$ and $35 = \min\{v: \exists LS(v;\{3, 5^*, 15^*\})\}$.

Proof: Form the partition $\pi(5^1,9^2)$. The long lines are $0_1 1_1 2_1 3_1 4_1$ and $0_3 1_3 \cdots 8_3$. The short lines are of

type ABC: $0_1 i_2 i_3 (i = 0,1,...,4,7)$ $0_1 6_2 5_3$ $0_1 8_2 6_3$ $0_1 5_2 8_3$ $1_1 i_2 (i+2)_3 (i = 0,1,...,5,7)$ $1_1 8_2 8_3$ $1_1 6_2 1_3$ $2_1 i_2 (i+4)_3 (i = 0,1,...,5,7)$ $2_1 8_2 1_3$ $2_1 6_2 3_3$ $3_1 i_2 (i+6)_3$ $(i = 0,1,...,4,7)$ $3_1 6_2 2_3$ $3_1 8_2 3_3$ $3_1 5_2 5_3$ $4_1 i_2 (i+8)_3 (i = 0,1,...,5,7)$ $4_1 8_2 5_3$ $4_1 6_2 7_3$

type BBC: $2_2 6_2 0_3$ $4_2 8_2 0_3$ $0_2 3_2 1_3$ $5_2 7_2 1_3$ $1_2 4_2 2_3$ $5_2 8_2 2_3$ $0_2 7_2 3_3$ $2_2 5_2 3_3$ $1_2 3_2 4_3$ $6_2 8_2 4_3$ $0_2 4_2 5_3$ $2_2 7_2 5_3$ $1_2 5_2 6_3$ $3_2 6_2 6_3$ $0_2 8_2 7_3$ $2_2 4_2 7_3$ $1_2 6_2 8_3$ $3_2 7_2 8_3$

type BBB: $0_2 1_2 2_2$ $0_2 5_2 6_2$ $1_2 7_2 8_2$ $3_2 4_2 5_2$ $4_2 6_2 7_2$ $2_2 3_2 8_2$

Next, form $\pi(5^1, 15^2)$. The long lines are $0_1 1_1 2_1 3_1 4_1$ and $0_3 1_3 \cdots 14_3$. The short lines are of

type ABC: $0_1 i_2 i_3 (i = 0,2,6,7,9,10,11,13)$ $0_1 12_2 1_3$ $0_1 14_2 3_3$ $0_1 8_2 4_3$ $0_1 4_2 5_3$ $0_1 5_2 8_3$ $0_1 3_2 12_3$ $0_1 1_2 14_3$ $1_1 10_2 1_3$ $1_1 12_2 2_3$ $1_1 (i+2)_2 (i+3)_3 (i = 0,1,4,5,9,12)$ $1_1 8_2 5_3$ $1_1 0_2 6_3$ $1_1 5_2 9_3$ $1_1 13_2 10_3$ $1_1 9_2 11_3$ $1_1 1_2 13_3$ $1_1 4_2 14_3$ $2_1 i_2 (i+2)_3$ $(i = 0,2,6,11,12,14)$ $2_1 3_2 3_3$ $2_1 1_2 5_3$ $2_1 7_2 6_3$ $2_1 10_2 7_3$ $2_1 8_2 9_3$ $2_1 5_2 10_3$ $2_1 13_2 11_3$ $2_1 4_2 12_3$ $2_1 9_2 0_3$ $3_1 i_2 (i+3)_3 (i = 0,2,3,6,7,11,13,14)$ $3_1 12_2 4_3$ $3_1 8_2 7_3$ $3_1 9_2 8_3$ $3_1 5_2 11_3$ $3_1 10_2 12_3$ $3_1 4_2 13_3$ $3_1 1_2 0_3$ $4_1 i_2 (i+4)_3$ $(i = 0,6,7,11,13)$ $4_1 12_2 5_3$ $4_1 4_2 6_3$ $4_1 14_2 7_3$ $4_1 8_2 8_3$ $4_1 10_2 9_3$ $4_1 5_2 12_3$ $4_1 9_2 14_3$ $4_1 3_2 1_3$ $4_1 2_2 13_3$ $4_1 1_2 3_3$

type BBC: $12_2 13_2 0_3$ $2_2 5_2 0_3$ $4_2 6_2 0_3$ $7_2 8_2 0_3$ $3_2 10_2 0_3$ $2_2 6_2 1_3$ $7_2 9_2 1_3$ $0_2 11_2 1_3$

$4_2 8_2 1_3$ $1_2 5_2 1_3$ $3_2 7_2 2_3$ $1_2 9_2 2_3$ $5_2 6_2 2_3$ $4_2 11_2 2_3$ $8_2 10_2 2_3$ $10_2 13_2 3_3$ $7_2 12_2 3_3$ $6_2 9_2 3_3$ $8_2 11_2 3_3$ $4_2 5_2 3_3$ $10_2 11_2 4_3$ $6_2 13_2 4_3$ $1_2 7_2 4_3$ $5_2 9_2 4_3$ $4_2 14_2 4_3$ $0_2 7_2 5_3$ $6_2 10_2 5_3$ $9_2 11_2 5_3$ $5_2 13_2 5_3$ $3_2 14_2 5_3$ $5_2 10_2 6_3$ $1_2 8_2 6_3$ $2_2 12_2 6_3$ $11_2 14_2 6_3$ $9_2 13_2 6_3$ $0_2 2_2 7_3$ $9_2 12_2 7_3$ $1_2 4_2 7_3$ $3_2 5_2 7_3$ $11_2 13_2 7_3$ $0_2 3_2 8_3$ $11_2 12_2 8_3$ $2_2 10_2 8_3$ $4_2 13_2 8_3$ $1_2 14_2 8_3$ $0_2 13_2 9_3$ $12_2 14_2 9_3$ $2_2 11_2 9_3$ $4_2 7_2 9_3$ $1_2 3_2 9_3$ $0_2 9_2 10_3$ $1_2 12_2 10_3$ $2_2 4_2 10_3$ $3_2 11_2 10_3$ $8_2 14_2 10_3$ $0_2 4_2 11_3$ $8_2 12_2 11_3$ $6_2 14_2 11_3$ $2_2 3_2 11_3$ $1_2 10_2 11_3$ $0_2 1_2 12_3$ $6_2 12_2 12_3$ $2_2 7_2 12_3$ $8_2 13_2 12_3$ $9_2 14_2 12_3$ $0_2 8_2 13_3$ $5_2 12_2 13_3$ $10_2 14_2 13_3$ $3_2 9_2 13_3$ $6_2 7_2 13_3$ $5_2 8_2 14_3$ $0_2 6_2 14_3$ $3_2 13_2 14_3$ $2_2 14_2 14_3$ $7_2 10_2 14_3$

type BBB: $0_2 10_2 12_2$ $3_2 4_2 12_2$ $2_2 8_2 9_2$ $3_2 6_2 8_2$ $5_2 7_2 11_2$ $4_2 9_2 10_2$ $1_2 6_2 11_2$ $7_2 13_2 14_2$ $0_2 5_2 14_2$ $1_2 2_2 13_2$

Finally, 35 ϵ $LS_i(3, 5^*, 15^*)$ by forming the partition $\pi(1^1, 2^1, 4^1, 14^2)$ where cell B is set $Z_2 \times \{2\}$ and cells C, D are sets $Z_{14} \times \{i\}$ (i = 3, 4) and constructing the short lines of

type ABD: $0_1 0_2 0_4$ $0_1 1_2 1_4$ $1_1 0_2 2_4$ $1_1 1_2 3_4$ $2_1 0_2 4_4$ $2_1 1_2 5_4$ $3_1 0_2 6_4$ $3_1 1_2 7_4$

type ACD: $0_1 i_3 (i+2)_4 (i = 0,4,6,8)$ $0_1 i_3 i_4 (i = 3,5,9,13)$ $0_1 12_3 4_4$ $0_1 11_3 7_4$ $0_1 7_3 11_4$ $0_1 1_3 12_4$ $1_1 1_3 0_4$ $1_1 2_3 1_4$ $1_1 i_3 (i+4)_4 (i = 0,4,6,8)$ $1_1 10_3 5_4$ $1_1 13_3 6_4$ $1_1 5_3 7_4$ $1_1 3_3 9_4$ $1_1 12_3 11_4$ $1_1 7_3 13_4$ $2_1 12_3 0_4$ $2_1 13_3 1_4$ $2_1 11_3 2_4$ $2_1 2_3 3_4$ $2_1 0_3 6_4$ $2_1 1_3 7_4$ $2_1 10_3 8_4$ $2_1 5_3 9_4$ $2_1 4_3 10_4$ $2_1 3_3 11_4$ $2_1 6_3 12_4$ $2_1 9_3 13_4$ $3_1 8_3 0_4$ $3_1 7_3 1_4$ $3_1 3_3 2_4$ $3_1 5_3 3_4$ $3_1 11_3 4_4$ $3_1 2_3 5_4$ $3_1 0_3 8_4$ $3_1 1_3 9_4$ $3_1 12_3 10_4$ $3_1 13_3 11_4$ $3_1 4_3 12_4$ $3_1 10_3 13_4$

type ACC: $0_1 2_3 10_3$ $1_1 9_3 11_3$ $2_1 7_3 8_3$ $3_1 6_3 9_3$

type BCD: $0_2 5_3 1_4$ $0_2 10_3 3_4$ $0_2 0_3 5_4$ $0_2 7_3 7_4$ $0_2 11_3 8_4$ $0_2 13_3 9_4$ $0_2 9_3 10_4$ $0_2 1_3 11_4$ $0_2 12_3 12_4$ $0_2 4_3 13_4$ $1_2 10_3 0_4$ $1_2 9_3 2_4$ $1_2 4_3 4_4$ $1_2 1_3 6_4$ $1_2 3_3 8_4$ $1_2 6_3 9_4$ $1_2 2_3 10_4$ $1_2 8_3 11_4$ $1_2 5_3 12_4$ $1_2 11_3 13_4$

type BCC: $0_2 3_3 6_3$ $0_2 2_3 8_3$ $1_2 0_3 13_3$ $1_2 7_3 12_3$

type CCD: $5_3 9_3 0_4$ $3_3 13_3 0_4$ $4_3 6_3 0_4$ $7_3 11_3 0_4$ $0_3 2_3 0_4$ $0_3 6_3 1_4$ $1_3 9_3 1_4$ $4_3 8_3 1_4$ $3_3 11_3 1_4$ $10_3 12_3 1_4$ $1_3 6_3 2_4$ $4_3 7_3 2_4$ $5_3 10_3 2_4$ $8_3 12_3 2_4$ $2_3 13_3 2_4$ $0_3 8_3 3_4$ $1_3 7_3 3_4$ $11_3 12_3 3_4$ $4_3 9_3 3_4$ $6_3 13_3 3_4$ $1_3 5_3 4_4$ $3_3 8_3 4_4$ $7_3 9_3 4_4$ $2_3 6_3 4_4$ $10_3 13_3 4_4$ $6_3 12_3 5_4$ $4_3 13_3 5_4$ $8_3 9_3 5_4$ $1_3 11_3 5_4$ $3_3 7_3 5_4$ $2_3 11_3 6_4$ $3_3 10_3 6_4$

$5_3 8_3 6_4$ $6_3 7_3 6_4$ $9_3 12_3 6_4$ $0_3 12_3 7_4$ $2_3 4_3 7_4$ $3_3 9_3 7_4$ $8_3 13_3 7_4$ $6_3 10_3 7_4$
$1_3 8_3 8_4$ $2_3 12_3 8_4$ $9_3 13_3 8_4$ $5_3 7_3 8_4$ $0_3 10_3 9_4$ $8_3 11_3 9_4$ $4_3 12_3 9_4$ $2_3 7_3 9_4$
$0_3 7_3 10_4$ $1_3 13_3 10_4$ $3_3 5_3 10_4$ $10_3 11_3 10_4$ $0_3 4_3 11_4$ $9_3 10_3 11_4$ $6_3 11_3 11_4$
$2_3 5_3 11_4$ $0_3 9_3 12_4$ $7_3 10_3 12_4$ $11_3 13_3 12_4$ $2_3 3_3 12_4$ $0_3 5_3 13_4$ $1_3 2_3 13_4$
$6_3 8_3 13_4$ $3_3 12_3 13_4$

type CCC: $0_3 1_3 3_3$ $1_3 4_3 10_3$ $5_3 12_3 13_3$ $4_3 5_3 11_3$

type ∞CC: $\infty 0_3 11_3$ $\infty 1_3 12_3$ $\infty 8_3 10_3$ $\infty 5_3 6_3$ $\infty 7_3 13_3$ $\infty 3_3 4_3$ $\infty 2_3 9_3$

type ∞BB: $\infty 0_2 1_2$

The long lines are $0_1 1_1 2_1 3_1 \infty$ and $0_4 1_4 \cdots 13_4 \infty$.

Corollary 2.12 $v \in LS_d(3, 5^*, 9^*)$ for all $v \geq 47$ and $v \in LS(3, 5^*, 15^*)$ for all $v \geq 71$.
Proof: This follows from Lemma 2.11 and Lemma 1.39.

Some individual recursive constructions for AULSs where the two long lines are either disjoint or intersecting are now provided.

Lemma 2.13 If $w \equiv 1 \pmod 6$, then $4w+12t+7, 4w+12t+13 \in LS_d(3, (6t+5)^*, w^*)$.
Proof: Start with a partition $\pi(1^1, (w-1)^4, (12t+10)^1)$ or $\pi(1^1, (w-1)^4, (12t+16)^1)$ and apply Corollary 1.25.

Lemma 2.14 If $w \equiv 9 \pmod{12}$ and $0 < t < (w-5)/6$, then $4w + 12t + 5 \in LS_d(3, (6t+5)^*, w^*)$.
Proof: Form the partition $\pi(1^1, (w-1)^1, ((3w+12t+5)/2)^2)$ and apply Lemma 1.32(b).

Lemma 2.15 If $w \equiv 1 \pmod 6$ and $0 \leq t \leq (w-5)/12$, or j is a nonnegative integer such that $j = 5, 11$ or 17, $0 \leq t \leq (w+j-4)/12$ and $w \geq j+8$, then $4w + 12t + 1, 4w + 12t - j \in LS(3, (6t+5)^*, w^*)$.
Proof: Start with a partition $\pi(1^1, (w-1)^3, (w+12t+3)^1)$ or $\pi(1^1, (w-1)^3, (w+12t-j+2)^1)$ and apply Corollary 1.25.

Corollary 2.16 17 $\in LS_i(3, 5^*, 7^*)$, 23 $\in LS(3, 5^*, 7^*)$; 35, 41 $\in LS(3, 5^*, 13^*)$ and 59 $\in LS(3, 5^*, 19^*)$.

Proof: 17 $\in LS_i(3, 5^*, 7^*)$ by Lemma 2.6. We have 23 $\in LS_i(3, 5^*, 7^*)$ since there exists a {3}-GDD of type $6^3 4^1$[C2], and therefore we can apply Theorem 1.24(c). By Lemma 2.9, 23 $\in LS_d(3, 5^*, 7^*)$. In order to prove that 35 $\in LS(3, 5^*, 13^*)$, we give a direct construction. Form $\pi(1^1, 6^2, 10^1, 12^1)$, where cell A is $Z_{10} \times \{1\}$, cells B, C are the sets $Z_6 \times \{i\}$ (i = 2, 3) and cell D is the set $Z_{12} \times \{4\}$. Construct an IPBD(11, 5;{3}) which contains the point ∞. The short lines are of

type ABD: $0_1 i_2 (2i)_4 (i = 0,1...5)$ $1_1 i_2 (2i+1)_4 (i = 0,1...,5)$ $2_1 0_2 10_4$ $2_1 1_2 0_4$ $2_1 2_2 6_4$ $2_1 3_2 2_4$ $2_1 4_2 4_4$ $2_1 5_2 8_4$ $3_1 i_2 (2i+3)_4 (i = 0,1,...,4)$ $3_1 5_2 1_4$ $4_1 0_2 2_4$ $4_1 1_2 6_4$ $4_1 2_2 0_4$ $4_1 3_2 8_4$ $4_1 4_2 10_4$ $4_1 5_2 4_4$ $5_1 0_2 9_4$ $5_1 1_2 11_4$ $5_1 2_2 3_4$ $5_1 3_2 5_4$ $5_1 4_2 1_4$ $5_1 5_2 7_4$ $6_1 0_2 6_4$ $6_1 1_2 8_4$ $6_1 2_2 10_4$ $6_1 3_2 4_4$ $6_1 4_2 0_4$ $6_1 5_2 2_4$ $7_1 0_2 7_4$ $7_1 1_2 9_4$ $7_1 2_2 1_4$ $7_1 3_2 11_4$ $7_1 4_2 5_4$ $7_1 5_2 3_4$ $8_1 0_2 4_4$ $8_1 1_2 10_4$ $8_1 2_2 8_4$ $8_1 3_2 0_4$ $8_1 4_2 2_4$ $8_1 5_2 6_4$ $9_1 0_2 5_4$ $9_1 1_2 1_4$ $9_1 2_2 11_4$ $9_1 3_2 3_4$ $9_1 4_2 7_4$ $9_1 5_2 9_4$

type ACD: $0_1 0_3 1_4$ $0_1 4_3 3_4$ $0_1 2_3 5_4$ $0_1 5_3 7_4$ $0_1 3_3 9_4$ $0_1 1_3 11_4$ $1_1 2_3 0_4$ $1_1 0_3 2_4$ $1_1 4_3 4_4$ $1_1 3_3 6_4$ $1_1 5_3 8_4$ $1_1 1_3 10_4$ $2_1 5_3 1_4$ $2_1 0_3 3_4$ $2_1 3_3 5_4$ $2_1 2_3 7_4$ $2_1 1_3 9_4$ $2_1 4_3 11_4$ $3_1 0_3 0_4$ $3_1 1_3 2_4$ $3_1 3_3 4_4$ $3_1 5_3 6_4$ $3_1 2_3 8_4$ $3_1 4_3 10_4$ $4_1 2_3 1_4$ $4_1 5_3 3_4$ $4_1 1_3 5_4$ $4_1 0_3 7_4$ $4_1 4_3 9_4$ $4_1 3_3 11_4$ $5_1 4_3 0_4$ $5_1 2_3 2_4$ $5_1 5_3 4_4$ $5_1 1_3 6_4$ $5_1 3_3 8_4$ $5_1 0_3 10_4$ $6_1 3_3 1_4$ $6_1 2_3 3_4$ $6_1 4_3 5_4$ $6_1 1_3 7_4$ $6_1 5_3 9_4$ $6_1 0_3 11_4$ $7_1 5_3 0_4$ $7_1 4_3 2_4$ $7_1 2_3 4_4$ $7_1 0_3 6_4$ $7_1 1_3 8_4$ $7_1 3_3 10_4$ $8_1 1_3 1_4$ $8_1 3_3 3_4$ $8_1 0_3 5_4$ $8_1 4_3 7_4$ $8_1 2_3 9_4$ $8_1 5_3 11_4$ $9_1 1_3 0_4$ $9_1 3_3 2_4$ $9_1 0_3 4_4$ $9_1 2_3 6_4$ $9_1 4_3 8_4$ $9_1 5_3 10_4$

type BCD: $0_2 0_3 8_4$ $0_2 2_3 11_4$ $1_2 1_3 4_4$ $1_2 3_3 7_4$ $2_2 5_3 2_4$ $2_2 0_3 9_4$ $3_2 4_3 1_4$ $3_2 2_3 10_4$ $4_2 1_3 3_4$ $4_2 4_3 6_4$ $5_2 5_3 5_4$ $5_2 3_3 0_4$

type BCC: $0_2 1_3 3_3$ $0_2 4_3 5_3$ $1_2 0_3 4_3$ $1_2 2_3 5_3$ $2_2 1_3 2_3$ $2_2 3_3 4_3$ $3_2 0_3 1_3$ $3_2 3_3 5_3$ $4_2 0_3 5_3$ $4_2 2_3 3_3$ $5_2 0_3 2_3$ $5_2 1_3 4_3$

type ∞BB: $\infty 0_2 5_2$ $\infty 1_2 4_2$ $\infty 2_2 3_2$

type ∞CC: $\infty 0_3 3_3$ $\infty 1_3 5_3$ $\infty 2_3 4_3$

type BBB: $0_2 1_2 2_2$ $0_2 3_2 4_2$ $1_2 3_2 5_2$ $2_2 4_2 5_2$

One long line is $0_4 1_4 \cdots 11_4 \infty$ and the other long line is formed on the hole of

IPBD(11, 5;{3}). Thus, 35 ϵ $LS_d(3, 5^*, 13^*)$ if the hole in the IPBD(11, 5;{3}) does not contain ∞, and 35 ϵ $LS_i(3, 5^*, 13^*)$ if the hole does contain ∞. Hence,
35 ϵ $LS(3, 5^*, 13^*)$. Next, 41 ϵ $LS_i(3, 5^*, 13^*)$ since there exists a {3}-GDD of type $12^3 4^1$[C2] and we thereby can apply Theorem 1.24(c). We need a direct construction to prove that 41 ϵ $LS_d(3, 5^*, 13^*)$.
Form $\pi(5^1, 10^1, 13^2)$. The short lines are of

type BCD: $0_21_32_4$ $0_23_36_4$ $0_25_310_4$ $0_24_34_4$ $0_29_31_4$ $0_210_312_4$ $1_22_33_4$ $1_27_38_4$
$1_28_35_4$ $1_212_310_4$ $1_211_31_4$ $1_25_37_4$ $2_23_32_4$ $2_22_34_4$ $2_24_35_4$ $2_25_38_4$
$2_26_310_4$ $2_27_312_4$ $3_28_33_4$ $3_211_310_4$ $3_24_36_4$ $3_26_39_4$ $3_27_311_4$ $3_23_30_4$
$4_20_30_4$ $4_25_36_4$ $4_23_34_4$ $4_27_310_4$ $4_24_38_4$ $4_22_312_4$ $5_25_35_4$ $5_212_311_4$
$5_23_39_4$ $5_27_32_4$ $5_24_30_4$ $5_21_36_4$ $6_210_37_4$ $6_211_312_4$ $6_26_32_4$ $6_29_33_4$
$6_28_39_4$ $6_20_34_4$ $7_21_31_4$ $7_210_38_4$ $7_212_30_4$ $7_26_311_4$ $7_20_35_4$ $7_28_34_4$
$8_26_38_4$ $8_25_39_4$ $8_24_311_4$ $8_211_32_4$ $8_29_37_4$ $8_210_31_4$ $9_26_36_4$ $9_29_39_4$
$9_211_30_4$ $9_212_33_4$ $9_27_37_4$ $9_28_311_4$

type ABC: $0_14_26_3$ $0_18_28_3$ $0_12_20_3$ $0_13_21_3$ $1_17_27_3$ $1_12_211_3$ $1_14_210_3$ $1_15_29_3$
$2_16_27_3$ $2_13_29_3$ $2_11_23_3$ $2_15_28_3$ $3_14_28_3$ $3_17_25_3$ $3_19_22_3$ $3_11_21_3$ $4_16_21_3$
$4_11_26_3$ $4_13_25_3$ $4_15_210_3$

type BBC: $1_25_20_3$ $3_28_20_3$ $0_29_20_3$ $2_29_21_3$ $4_28_21_3$ $3_27_22_3$ $0_26_22_3$ $5_28_22_3$
$8_29_23_3$ $6_27_23_3$ $1_26_24_3$ $7_29_24_3$ $6_29_25_3$ $0_25_26_3$ $0_28_27_3$ $0_22_28_3$ $1_24_29_3$
$2_27_29_3$ $3_29_210_3$ $1_22_210_3$ $4_25_211_3$ $0_27_211_3$ $0_24_212_3$ $6_28_212_3$ $2_23_212_3$

type ABD: $0_10_20_4$ $0_19_28_4$ $0_11_211_4$ $0_16_26_4$ $0_17_212_4$ $0_15_210_4$ $1_19_21_4$ $1_11_20_4$
$1_13_212_4$ $1_18_25_4$ $1_10_28_4$ $1_16_210_4$ $2_10_23_4$ $2_12_211_4$ $2_14_22_4$ $2_19_212_4$
$2_17_27_4$ $2_18_20_4$ $3_10_25_4$ $3_16_28_4$ $3_12_21_4$ $3_13_22_4$ $3_15_212_4$ $3_18_23_4$ $4_14_27_4$
$4_17_22_4$ $4_12_29_4$ $4_19_210_4$ $4_10_211_4$ $4_18_24_4$

type BBD: $2_26_20_4$ $3_24_21_4$ $5_26_21_4$ $1_29_22_4$ $2_24_23_4$ $5_27_23_4$ $1_23_24_4$ $5_29_24_4$
$3_26_25_4$ $4_29_25_4$ $1_27_26_4$ $2_28_26_4$ $2_25_27_4$ $0_23_27_4$ $3_25_28_4$ $0_21_29_4$ $4_27_29_4$
$7_28_210_4$ $4_26_211_4$ $1_28_212_4$

type ACD: $0_12_31_4$ $0_14_32_4$ $0_15_33_4$ $0_19_34_4$ $0_110_35_4$ $0_111_37_4$ $0_112_39_4$ $1_10_32_4$
$1_16_33_4$ $1_112_34_4$ $1_18_36_4$ $1_11_37_4$ $1_12_39_4$ $1_13_311_4$ $2_15_31_4$ $2_11_34_4$
$2_112_35_4$ $2_110_36_4$ $2_12_38_4$ $2_14_39_4$ $2_10_310_4$ $3_19_30_4$ $3_16_34_4$ $3_111_36_4$

$3_13_37_4$ $3_10_39_4$ $3_14_310_4$ $3_110_311_4$ $4_12_30_4$ $4_14_31_4$ $4_17_33_4$ $4_111_35_4$
$4_19_36_4$ $4_112_38_4$ $4_18_312_4$

type ACC: $0_13_37_3$ $1_14_35_3$ $2_16_311_3$ $3_17_312_3$ $4_10_33_3$

type CCD: $1_37_30_4$ $5_38_30_4$ $6_310_30_4$ $0_37_31_4$ $8_312_31_4$ $3_36_31_4$ $8_39_32_4$ $5_310_32_4$
$2_312_32_4$ $0_31_33_4$ $4_310_33_4$ $3_311_33_4$ $5_37_34_4$ $10_311_34_4$ $1_33_35_4$ $2_36_35_4$
$7_39_35_4$ $0_312_36_4$ $2_37_36_4$ $0_32_37_4$ $6_38_37_4$ $4_312_37_4$ $0_311_38_4$ $1_38_38_4$
$3_39_38_4$ $1_310_39_4$ $7_311_39_4$ $1_39_310_4$ $3_38_310_4$ $2_310_310_4$ $0_35_311_4$ $1_311_311_4$
$2_39_311_4$ $0_36_312_4$ $3_34_312_4$ $1_35_312_4$ $9_312_312_4$

type CCC: $0_34_38_3$ $0_39_310_3$ $4_39_311_3$ $1_32_34_3$ $1_36_312_3$ $3_310_312_3$ $2_33_35_3$ $7_38_310_3$
$5_311_312_3$ $4_36_37_3$ $5_36_39_3$ $2_38_311_3$

The long lines are $0_11_12_13_14_1$ and $0_41_4 \cdots 12_4$. Finally, a direct construction is necessary to show that $59 \in LS(3, 5^*, 19^*)$. Form $\pi(1^1, 10^1, 12^1, 18^2)$ and construct an IPBD(11, 5;{3}). The short lines are of

type ABD: $0_17_22_4$ $0_15_24_4$ $0_14_25_4$ $0_12_27_4$ $0_1i_2i_4(i = 0,1,3,6,8,...,11)$ $1_17_23_4$ $1_15_25_4$
$1_14_26_4$ $1_12_28_4$ $1_1i_2(i+1)_4(i = 0,1,3,6,8,...,11)$ $2_17_24_4$ $2_15_26_4$ $2_14_27_4$
$2_12_29_4$ $2_1i_2(i+2)_4(i = 0,1,3,6,8,...,11)$ $3_17_26_4$ $3_15_27_4$ $3_14_28_4$ $3_13_210_4$
$3_1i_2(i+3)_4(i = 0,1,2,6,8,...,11)$ $4_1i_2(i+4)_4(i = 0,1,2,3,6,...,11)$ $4_15_28_4$
$4_14_29_4$ $5_17_27_4$ $5_15_29_4$ $5_14_210_4$ $5_12_212_4$ $5_1i_2(i+5)_4(i = 0,1,3,6,8,...,11)$
$6_17_28_4$ $6_15_210_4$ $6_14_211_4$ $6_12_213_4$ $6_1i_2(i+6)_4(i = 0,1,3,6,8,...,11)$
$7_17_29_4$ $7_12_210_4$ $7_15_211_4$ $7_14_212_4$ $7_13_214_4$ $7_1i_2(i+7)_4(i = 0,1,6,8,...,11)$
$8_17_210_4$ $8_15_212_4$ $8_14_213_4$ $8_12_215_4$ $8_1i_2(i+8)_4(i = 0,1,3,6,8,...,11)$
$9_1i_2(i+9)_4(i = 0,1,2,3,6,...,11)$ $9_15_213_4$ $9_14_214_4$

type ACD: $0_1i_3(i+12)_4(i = 0,2,4,5)$ $0_114_313_4$ $0_116_315_4$ $1_16_30_4$ $1_1(i+7)_3(i+13)_4$
$(i = 0,1,...,4)$ $2_14_30_4$ $2_113_31_4$ $2_114_314_4$ $2_115_315_4$ $2_12_316_4$ $2_117_317_4$
$3_1i_2(i+15)_4(i = 0,1,3,4)$ $3_18_317_4$ $3_112_32_4$ $4_1(i+6)_3(i+16)_4$
$(i = 0,1,...,5)$ $5_1(i+12)_3(i+17)_4(i = 0,1,2,3,5)$ $5_10_33_4$ $6_1i_3i_4$
$(i = 0,1,...,5)$ $7_1(i+6)_3(i+1)_4(i = 0,1,...,5)$ $8_15_32_4$ $8_1(i+13)_3(i+3)_4$
$(i = 0,1,...,4)$ $9_116_33_4$ $9_11_34_4$ $9_16_35_4$ $9_13_36_4$ $9_14_37_4$ $9_112_38_4$

type ACC: $0_16_37_3$ $0_19_317_3$ $0_110_311_3$ $0_11_313_3$ $0_112_315_3$ $0_13_38_3$ $1_10_31_3$ $1_12_33_3$
$1_14_314_3$ $1_15_313_3$ $1_115_316_3$ $1_112_317_3$ $2_10_312_3$ $2_11_310_3$ $2_15_36_3$ $2_13_37_3$

$2_{1}8_{3}11_{3}$ $2_{1}9_{3}16_{3}$ $3_{1}6_{3}14_{3}$ $3_{1}7_{3}15_{3}$ $3_{1}16_{3}17_{3}$ $3_{1}9_{3}13_{3}$ $3_{1}5_{3}10_{3}$ $3_{1}2_{3}11_{3}$
$4_{1}0_{3}2_{3}$ $4_{1}1_{3}3_{3}$ $4_{1}12_{3}13_{3}$ $4_{1}4_{3}15_{3}$ $4_{1}5_{3}17_{3}$ $4_{1}14_{3}16_{3}$ $5_{1}8_{3}16_{3}$ $5_{1}1_{3}9_{3}$
$5_{1}2_{3}10_{3}$ $5_{1}3_{3}6_{3}$ $5_{1}5_{3}11_{3}$ $5_{1}4_{3}7_{3}$ $6_{1}6_{3}10_{3}$ $6_{1}7_{3}11_{3}$ $6_{1}8_{3}12_{3}$ $6_{1}9_{3}14_{3}$
$6_{1}13_{3}16_{3}$ $6_{1}15_{3}17_{3}$ $7_{1}0_{3}5_{3}$ $7_{1}1_{3}16_{3}$ $7_{1}2_{3}12_{3}$ $7_{1}3_{3}15_{3}$ $7_{1}4_{3}13_{3}$
$7_{1}14_{3}17_{3}$ $8_{1}0_{3}6_{3}$ $8_{1}1_{3}7_{3}$ $8_{1}2_{3}8_{3}$ $8_{1}3_{3}9_{3}$ $8_{1}4_{3}10_{3}$ $8_{1}11_{3}12_{3}$ $9_{1}2_{3}5_{3}$
$9_{1}7_{3}10_{3}$ $9_{1}8_{3}15_{3}$ $9_{1}0_{3}9_{3}$ $9_{1}13_{3}14_{3}$ $9_{1}11_{3}17_{3}$

type BCD: $0_{2}i_{3}(i+10)_{4}(i = 0,1,2,3)$ $0_{2}15_{3}14_{4}$ $0_{2}5_{3}15_{4}$ $0_{2}7_{3}16_{4}$ $0_{2}6_{3}17_{4}$ $1_{2}9_{3}0_{4}$
$1_{2}17_{3}11_{4}$ $1_{2}(i+10)_{3}(i+12)_{4}(i = 0,1,...,5)$ $2_{2}16_{3}0_{4}$ $2_{2}17_{3}1_{4}$ $2_{2}3_{3}14_{4}$
$2_{2}11_{3}16_{4}$ $2_{2}8_{3}2_{4}$ $2_{2}5_{3}3_{4}$ $2_{2}6_{3}4_{4}$ $2_{2}9_{3}17_{4}$ $3_{2}7_{3}0_{4}$ $3_{2}10_{3}1_{4}$ $3_{2}9_{3}13_{4}$
$3_{2}12_{3}15_{4}$ $3_{2}13_{3}16_{4}$ $3_{2}16_{3}17_{4}$ $3_{2}4_{3}2_{4}$ $3_{2}8_{3}6_{4}$ $4_{2}15_{3}0_{4}$ $4_{2}2_{3}1_{4}$ $4_{2}17_{3}2_{4}$
$4_{2}1_{3}3_{4}$ $4_{2}7_{3}15_{4}$ $4_{2}16_{3}16_{4}$ $4_{2}3_{3}17_{4}$ $4_{2}8_{3}4_{4}$ $5_{2}5_{3}0_{4}$ $5_{2}7_{3}1_{4}$ $5_{2}6_{3}2_{4}$
$5_{2}9_{3}3_{4}$ $5_{2}0_{3}14_{4}$ $5_{2}11_{3}15_{4}$ $5_{2}12_{3}16_{4}$ $5_{2}10_{3}17_{4}$ $6_{2}12_{3}0_{4}$ $6_{2}15_{3}1_{4}$
$6_{2}13_{3}2_{4}$ $6_{2}17_{3}3_{4}$ $6_{2}2_{3}4_{4}$ $6_{2}1_{3}5_{4}$ $6_{2}9_{3}16_{4}$ $6_{2}14_{3}17_{4}$ $7_{2}10_{3}0_{4}$ $7_{2}3_{3}1_{4}$
$7_{2}7_{3}5_{4}$ $7_{2}1_{3}12_{4}$ $7_{2}0_{3}13_{4}$ $7_{2}11_{3}14_{4}$ $7_{2}2_{3}15_{4}$ $7_{2}13_{3}17_{4}$ $8_{2}11_{3}0_{4}$ $8_{2}12_{3}1_{4}$
$8_{2}(i+14)_{3}(i+2)_{4}(i = 0,1,3,4)$ $8_{2}13_{3}4_{4}$ $8_{2}5_{3}7_{4}$ $9_{2}16_{3}1_{4}$ $9_{2}(i+1)_{3}(i+2)_{4}$
$(i = 0,2,4,6)$ $9_{2}4_{3}3_{4}$ $9_{2}2_{3}5_{4}$ $9_{2}8_{3}7_{4}$ $10_{2}(i+9)_{3}(i+2)_{4}(i = 0,1,2,4,6,7)$
$10_{2}14_{3}5_{4}$ $10_{2}12_{3}7_{4}$ $11_{2}6_{3}3_{4}$ $11_{2}5_{3}4_{4}$ $11_{2}16_{3}5_{4}$ $11_{2}2_{3}6_{4}$ $11_{2}1_{3}7_{4}$
$11_{2}3_{3}8_{4}$ $11_{2}10_{3}9_{4}$ $11_{2}17_{3}10_{4}$

type CCD: $1_{3}17_{3}0_{4}$ $2_{3}14_{3}0_{4}$ $0_{3}11_{3}1_{4}$ $5_{3}8_{3}1_{4}$ $0_{3}16_{3}2_{4}$ $3_{3}11_{3}2_{4}$ $2_{3}7_{3}3_{4}$ $12_{3}14_{3}3_{4}$
$0_{3}7_{3}4_{4}$ $10_{3}15_{3}4_{4}$ $12_{3}16_{3}4_{4}$ $3_{3}13_{3}5_{4}$ $0_{3}8_{3}5_{4}$ $9_{3}11_{3}5_{4}$ $4_{3}12_{3}5_{4}$ $1_{3}4_{3}6_{4}$
$6_{3}9_{3}6_{4}$ $7_{3}12_{3}6_{4}$ $10_{3}17_{3}6_{4}$ $14_{3}15_{3}6_{4}$ $0_{3}3_{3}7_{4}$ $2_{3}9_{3}7_{4}$ $6_{3}16_{3}7_{4}$ $7_{3}14_{3}7_{4}$
$10_{3}13_{3}7_{4}$ $11_{3}15_{3}7_{4}$ $0_{3}10_{3}8_{4}$ $1_{3}14_{3}8_{4}$ $11_{3}16_{3}8_{4}$ $8_{3}13_{3}8_{4}$ $5_{3}9_{3}8_{4}$
$4_{3}17_{3}8_{4}$ $2_{3}6_{3}8_{4}$ $0_{3}13_{3}9_{4}$ $1_{3}12_{3}9_{4}$ $7_{3}17_{3}9_{4}$ $3_{3}4_{3}9_{4}$ $6_{3}11_{3}9_{4}$ $8_{3}9_{3}9_{4}$
$2_{3}15_{3}9_{4}$ $5_{3}14_{3}9_{4}$ $1_{3}5_{3}10_{4}$ $2_{3}4_{3}10_{4}$ $3_{3}12_{3}10_{4}$ $13_{3}15_{3}10_{4}$ $6_{3}8_{3}10_{4}$
$7_{3}9_{3}10_{4}$ $10_{3}16_{3}10_{4}$ $11_{3}14_{3}10_{4}$ $0_{3}14_{3}11_{4}$ $2_{3}16_{3}11_{4}$ $3_{3}10_{3}11_{4}$ $5_{3}12_{3}11_{4}$
$7_{3}8_{3}11_{4}$ $6_{3}15_{3}11_{4}$ $11_{3}13_{3}11_{4}$ $4_{3}9_{3}11_{4}$ $9_{3}12_{3}12_{4}$ $8_{3}14_{3}12_{4}$ $6_{3}17_{3}12_{4}$
$3_{3}16_{3}12_{4}$ $4_{3}11_{3}12_{4}$ $7_{3}13_{3}12_{4}$ $5_{3}15_{3}12_{4}$ $5_{3}16_{3}13_{4}$ $10_{3}12_{3}13_{4}$ $4_{3}8_{3}13_{4}$
$6_{3}13_{3}13_{4}$ $2_{3}17_{3}13_{4}$ $1_{3}15_{3}13_{4}$ $13_{3}17_{3}14_{4}$ $4_{3}16_{3}14_{4}$ $1_{3}6_{3}14_{4}$ $9_{3}10_{3}14_{4}$
$5_{3}7_{3}14_{4}$ $1_{3}8_{3}15_{4}$ $4_{3}6_{3}15_{4}$ $10_{3}14_{3}15_{4}$ $3_{3}17_{3}15_{4}$ $3_{3}5_{3}16_{4}$ $0_{3}15_{3}16_{4}$
$8_{3}17_{3}16_{4}$ $0_{3}4_{3}17_{4}$ $1_{3}2_{3}17_{4}$

type ∞CC: $\infty 0_3 17_3$ $\infty 2_3 13_3$ $\infty 3_3 14_3$ $\infty 6_3 12_3$ $\infty 7_3 16_3$ $\infty 8_3 10_3$ $\infty 1_3 11_3$ $\infty 9_3 15_3$ $\infty 4_3 5_3$

type BBC: $1_2 3_2 0_3$ $2_2 4_2 0_3$ $6_2 9_2 0_3$ $10_2 11_2 0_3$ $1_2 5_2 1_3$ $2_2 10_2 1_3$ $3_2 8_2 1_3$ $1_2 10_2 2_3$
$3_2 5_2 2_3$ $2_2 8_2 2_3$ $1_2 8_2 3_3$ $3_2 10_2 3_3$ $5_2 6_2 3_3$ $0_2 6_2 4_3$ $1_2 4_2 4_3$ $7_2 8_2 4_3$ $5_2 10_2 4_3$
$2_2 11_2 4_3$ $1_2 6_2 5_3$ $4_2 10_2 5_3$ $3_2 7_2 5_3$ $1_2 9_2 6_3$ $7_2 10_2 6_3$ $3_2 6_2 6_3$ $4_2 8_2 6_3$
$1_2 11_2 7_3$ $2_2 6_2 7_3$ $8_2 10_2 7_3$ $0_2 5_2 8_3$ $1_2 7_2 8_3$ $8_2 11_2 8_3$ $6_2 10_2 8_3$ $0_2 8_2 9_3$
$4_2 7_2 9_3$ $9_2 11_2 9_3$ $0_2 2_2 10_3$ $4_2 6_2 10_3$ $8_2 9_2 10_3$ $0_2 9_2 11_3$ $3_2 4_2 11_3$ $6_2 11_2 11_3$
$0_2 7_2 12_3$ $2_2 9_2 12_3$ $4_2 11_2 12_3$ $0_2 11_2 13_3$ $2_2 5_2 13_3$ $4_2 9_2 13_3$ $0_2 4_2 14_3$
$2_2 7_2 14_3$ $3_2 9_2 14_3$ $5_2 11_2 14_3$ $7_2 11_2 15_3$ $5_2 9_2 15_3$ $2_2 3_2 15_3$ $0_2 1_2 16_3$
$6_2 7_2 16_3$ $5_2 8_2 16_3$ $0_2 3_2 17_3$ $5_2 7_2 17_3$ $9_2 10_2 17_3$

type ∞BB: $\infty 0_2 10_2$ $\infty 1_2 2_2$ $\infty 3_2 11_2$ $\infty 4_2 5_2$ $\infty 6_2 8_2$ $\infty 7_2 9_2$

One long line is $0_4 1_4 \cdots 17_4 \infty$ and the other long line is formed on the hole of IPBD(11, 5:{3}). We complete the construction in precisely the same way as in the case of an LS(35;{3, 5^*, 13^*}).

Lemma 2.17 Let $w \equiv 3 \pmod 6$. If $j = 1, 7, 13$ or 19, $0 \le t \le (w + j - 4)/12$, $w \ge 8 + j$, then $4w + 12t - j \in LS(3, (6t + 5)^*, w^*)$.

Proof: Form $\pi(1^1, (w-1)^3, (w + 12t - j + 2)^1)$ and apply Corollary 1.25.

Corollary 2.18 $23, 29 \in LS(3, 5^*, 9^*)$; $41, 47 \in LS(3, 5^*, 15^*)$; $65 \in LS(3, 5^*, 21^*)$.

Proof: It is proven in Lemma 2.11 that $23 \in LS_d(3, 5^*, 9^*)$. A direct construction is essential to show that $23 \in LS_i(3, 5^*, 9^*)$. Form $\pi(1^1, 2^1, 4^1, 8^2)$, where cell A is the set $Z_4 \times \{1\}$, cell B is the set $Z_2 \times \{2\}$ and cells C, D are the sets $Z_8 \times \{i\}$ $(i = 3,4)$. The short lines are of

type ABD: $0_1 0_2 0_4$ $0_1 1_2 4_4$ $1_1 0_2 1_4$ $1_1 1_2 5_4$ $2_1 0_2 2_4$ $2_1 1_2 6_4$ $3_1 0_2 3_4$ $3_1 1_2 7_4$

type ACD: $0_1 0_3 1_4$ $0_1 6_3 2_4$ $0_1 4_3 3_4$ $0_1 3_3 5_4$ $0_1 2_3 6_4$ $0_1 7_3 7_4$ $1_1 6_3 0_4$ $1_1 5_3 2_4$
$1_1 i_3 (i+3)_4 (i = 0,1,3,4)$ $2_1 1_3 0_4$ $2_1 5_3 1_4$ $2_1 6_3 3_4$ $2_1 2_3 4_4$ $2_1 7_3 5_4$ $2_1 0_3 7_4$
$3_1 4_3 0_4$ $3_1 3_3 1_4$ $3_1 2_3 2_4$ $3_1 5_3 4_4$ $3_1 1_3 5_4$ $3_1 7_3 6_4$

type ACC: $0_1 1_3 5_3$ $1_1 2_3 7_3$ $2_1 3_3 4_3$ $3_1 0_3 6_3$

type BCD: $0_20_34_4$ $0_22_35_4$ $0_26_36_4$ $0_23_37_4$ $1_25_30_4$ $1_24_31_4$ $1_21_32_4$ $1_27_33_4$

type BCC: $0_21_34_3$ $0_25_37_3$ $1_20_32_3$ $1_23_36_3$

type CCD: $0_37_30_4$ $2_33_30_4$ $1_37_31_4$ $2_36_31_4$ $0_33_32_4$ $4_37_32_4$ $1_32_33_4$ $3_35_33_4$
$3_37_34_4$ $4_36_34_4$ $0_34_35_4$ $5_36_35_4$ $0_31_36_4$ $4_35_36_4$ $1_36_37_4$ $2_35_37_4$

type ∞BB: $\infty 0_21_2$

type ∞CC: $\infty 0_35_3$ $\infty 1_33_3$ $\infty 2_34_3$ $\infty 6_37_3$

The long lines are $0_11_12_13_1\infty$ and $0_41_4 \cdots 7_4\infty$. We prove that $41 \in LS(3, 5^*, 15^*)$ by forming the partition $\pi(1^1, 2^1, 10^1, 14^2)$, where cell A is the set $Z_{10} \times \{1\}$, cell B is the set $Z_2 \times \{2\}$, and cells C, D are the sets $Z_{14} \times \{i\}$ ($i = 3, 4$). Construct an IPBD(11, 5;{3}) which contains ∞. The short lines are of

type ABD: $0_10_21_4$ $0_11_22_4$ $1_10_23_4$ $1_11_24_4$ $2_10_25_4$ $2_11_26_4$ $3_10_27_4$ $3_11_28_4$
$4_10_29_4$ $4_11_210_4$ $5_10_211_4$ $5_11_212_4$ $6_10_213_4$ $6_11_20_4$ $7_10_22_4$ $7_11_23_4$
$8_10_26_4$ $8_11_27_4$ $9_10_28_4$ $9_11_29_4$

type ACD: $0_14_30_4$ $0_12_33_4$ $0_15_34_4$ $0_110_35_4$ $0_11_36_4$ $0_16_37_4$ $0_1i_3i_4 (i = 8,9,11,12,13)$
$0_13_310_4$ $1_12_30_4$ $1_13_31_4$ $1_11_32_4$ $1_15_35_4$ $1_19_36_4$ $1_110_37_4$ $1_14_38_4$
$1_18_39_4$ $1_112_310_4$ $1_17_311_4$ $1_16_312_4$ $1_10_313_4$ $2_11_30_4$ $2_10_31_4$ $2_15_32_4$
$2_13_33_4$ $2_110_34_4$ $2_112_37_4$ $2_12_38_4$ $2_14_39_4$ $2_17_310_4$ $2_18_311_4$ $2_111_312_4$
$2_16_313_4$ $3_15_30_4$ $3_18_31_4$ $3_13_32_4$ $3_14_33_4$ $3_113_34_4$ $3_12_35_4$ $3_110_36_4$ $3_17_39_4$
$3_16_310_4$ $3_112_311_4$ $3_19_312_4$ $3_111_313_4$ $4_16_30_4$ $4_19_31_4$ $4_10_32_4$ $4_15_33_4$
$4_11_34_4$ $4_113_35_4$ $4_14_36_4$ $4_111_37_4$ $4_112_38_4$ $4_13_311_4$ $4_12_312_4$ $4_17_313_4$
$5_17_30_4$ $5_16_31_4$ $5_12_32_4$ $5_10_33_4$ $5_14_34_4$ $5_19_35_4$ $5_18_36_4$ $5_11_37_4$ $5_111_38_4$
$5_110_39_4$ $5_113_310_4$ $5_15_313_4$ $6_111_31_4$ $6_19_32_4$ $6_17_33_4$ $6_112_34_4$ $6_13_35_4$
$6_10_36_4$ $6_12_37_4$ $6_110_38_4$ $6_11_39_4$ $6_18_310_4$ $6_15_311_4$ $6_113_312_4$ $7_113_30_4$
$7_110_31_4$ $7_16_34_4$ $7_111_35_4$ $7_13_36_4$ $7_15_37_4$ $7_19_38_4$ $7_10_39_4$ $7_11_310_4$
$7_14_311_4$ $7_18_312_4$ $7_112_313_4$ $8_110_30_4$ $8_14_31_4$ $8_18_32_4$ $8_11_33_4$ $8_111_34_4$
$8_17_35_4$ $8_15_38_4$ $8_113_39_4$ $8_19_310_4$ $8_10_311_4$ $8_13_312_4$ $8_12_313_4$ $9_112_30_4$
$9_11_31_4$ $9_17_32_4$ $9_110_33_4$ $9_12_34_4$ $9_14_35_4$ $9_111_36_4$ $9_19_37_4$ $9_15_310_4$
$9_16_311_4$ $9_10_312_4$ $9_13_313_4$

type ACC: $0_10_37_3$ $1_111_313_3$ $2_19_313_3$ $3_10_31_3$ $4_18_310_3$ $5_13_312_3$ $6_14_36_3$ $7_12_37_3$
$8_16_312_3$ $9_18_313_3$

type BCD: $0_29_30_4$ $0_20_34_4$ $0_210_310_4$ $0_25_312_4$ $1_22_31_4$ $1_28_35_4$ $1_21_311_4$ $1_29_313_4$

type BCC: $0_22_34_3$ $0_21_33_3$ $0_26_38_3$ $0_211_312_3$ $0_27_313_3$ $1_23_35_3$ $1_24_312_3$ $1_26_37_3$
$1_20_311_3$ $1_210_313_3$

type CCD: $8_311_30_4$ $0_33_30_4$ $5_37_31_4$ $12_313_31_4$ $6_311_32_4$ $4_313_32_4$ $10_312_32_4$ $6_313_33_4$
$9_311_33_4$ $8_312_33_4$ $7_39_34_4$ $3_38_34_4$ $1_36_35_4$ $0_312_35_4$ $2_36_36_4$ $7_312_36_4$
$5_313_36_4$ $0_34_37_4$ $7_38_37_4$ $3_313_37_4$ $3_37_38_4$ $1_313_38_4$ $0_36_38_4$ $3_311_39_4$
$5_36_39_4$ $2_312_39_4$ $0_32_310_4$ $4_311_310_4$ $9_310_311_4$ $2_313_311_4$ $1_310_312_4$
$4_37_312_4$ $1_38_313_4$ $4_310_313_4$

type CCC: $1_32_35_3$ $0_38_39_3$ $3_36_310_3$ $2_33_39_3$ $1_34_39_3$ $2_310_311_3$ $4_35_38_3$ $5_39_312_3$
$0_35_310_3$ $1_37_311_3$

type ∞BB: $\infty 0_21_2$

type ∞CC: $\infty 6_39_3$ $\infty 7_310_3$ $\infty 2_38_3$ $\infty 1_312_3$ $\infty 0_313_3$ $\infty 3_34_3$ $\infty 5_311_3$

One long line is $0_41_4 \cdots 13_4\infty$ and the other long line is formed on the hole of IPBD(11, 5;{3}). The rest of the arguments parallel the previous construction. We note here that, by Lemma 1.39, $v \in LS_i(3, 5^*, 9^*)$ for all $v \geq 47$, $v \equiv 5 \pmod 6$. We approach the problem of establishing that $47 \in LS(3, 5^*, 15^*)$ similarly, working with the partition $\pi(1^1, 8^2, 14^1, 16^1)$ where cell A is $Z_{16} \times \{1\}$, cells B, C are the sets $Z_8 \times \{i\}$ ($i = 2, 3$) and cell D is the set $Z_{14} \times \{4\}$, and constructing an IPBD(17, 5;{3}). The short lines are of

type ABD: $i_10_2i_4$ $(i+14)_11_2i_4$ $(i+12)_12_2i_4$ $(i+10)_13_2i_4$ $(i+8)_14_2i_4$ $(i+6)_15_2i_4$
$(i+4)_16_2i_4$ $(i+2)_17_2i_4 (i = 0,1,...,13)$

type ACD: $(i+1)_10_3i_4$ $(i+3)_11_3i_4$ $(i+5)_12_3i_4$ $(i+7)_13_3i_4$ $(i+9)_14_3i_4$ $(i+11)_15_3i_4$
$(i+13)_16_3i_4$ $(i+15)_17_3i_4 (i = 0,1,...,13)$

type ABC: $0_17_20_3$ $1_17_21_3$ $2_16_21_3$ $3_16_22_3$ $4_15_22_3$ $5_15_23_3$ $6_14_23_3$ $7_14_24_3$
$8_13_24_3$ $9_13_25_3$ $10_12_25_3$ $11_12_26_3$ $12_11_26_3$ $13_11_27_3$ $14_10_27_3$ $15_10_20_3$

type BCC: $0_21_36_3$ $0_22_35_3$ $0_23_34_3$ $1_20_35_3$ $1_22_34_3$ $1_21_33_3$ $2_20_32_3$ $2_21_34_3$
$2_23_37_3$ $3_20_33_3$ $3_21_32_3$ $3_26_37_3$ $4_20_36_3$ $4_21_35_3$ $4_22_37_3$ $5_20_31_3$ $5_25_37_3$
$5_24_36_3$ $6_20_37_3$ $6_23_36_3$ $6_24_35_3$ $7_22_36_3$ $7_24_37_3$ $7_23_35_3$

type ∞BB: $\infty 0_24_2$ $\infty 1_27_2$ $\infty 2_23_2$ $\infty 5_26_2$

type ∞CC: $\infty 0_34_3$ $\infty 1_37_3$ $\infty 2_33_3$ $\infty 5_36_3$

type BBB: $0_2 1_2 2_2$ $0_2 3_2 5_2$ $0_2 6_2 7_2$ $1_2 3_2 6_2$ $1_2 4_2 5_2$ $2_2 4_2 6_2$ $2_2 5_2 7_2$ $3_2 4_2 7_2$

One long line is $0_4 1_4 \cdots 13_4 \infty$ and the other long line is formed on the hole of IPBD(17, 5;{3}). We are able to give a recursive construction to prove that $65 \in LS(3, 5^*, 21^*)$. Form $\pi(1^1, 20^1, 22^2)$ and apply Lemma 1.32(b).

Lemma 2.19 Let $w \equiv 1 \pmod 6$ and $w \geq 19$. Then $4w - 23 \in LS(3, 5^*, w^*)$.

Proof: If $w \geq 31$, form $\pi(1^1, (w-21)^1, (w-1)^3)$, construct an IPBD(w - 20, 5;{3}) and apply Corollary 1.25. When w = 19, form $\pi(1^1, 16^1, 18^2)$ and apply Lemma 1.32(b). For w = 25, form $\pi(1^1, 6^1, 22^1, 24^2)$ where cell A is the set $Z_{22} \times \{1\}$, cell B is the set $Z_6 \times \{2\}$, cells C and D are the sets $Z_{24} \times \{i\}$ (i = 3, 4). Construct an IPBD(23, 5;{3}) and construct the short lines of

type ABD: $i_1 0_2 i_4$ $i_1 1_2 (i+22)_4$ $i_1 2_2 (i+20)_4$ $i_1 3_2 (i+18)_4$ $i_1 4_2 (i+16)_4$ $i_1 5_2 (i+14)_4 (i=0,1,\ldots,21)$

type ACD: $0_1 21_3 1_4$ $0_1 12_3 2_4$ $0_1 3_3 3_4$ $0_1 4_3 4_4$ $0_1 18_3 5_4$ $0_1 17_3 6_4$ $0_1 6_3 7_4$ $0_1 9_3 8_4$ $0_1 0_3 9_4$ $0_1 7_3 10_4$ $0_1 1_3 11_4$ $0_1 14_3 12_4$ $0_1 5_3 13_4$ $0_1 11_3 15_4$ $0_1 2_3 17_4$ $0_1 13_3 19_4$ $0_1 8_3 21_4$ $0_1 10_3 23_4$ $1_1 11_3 0_4$ $1_1 (i+19)_3 (i+2)_4$ $(i = 0,3,4,6,7,10,20)$ $1_1 17_3 3_4$ $1_1 7_3 4_4$ $1_1 14_3 7_4$ $1_1 16_3 10_4$ $1_1 8_3 11_4$ $1_1 0_3 13_4$ $1_1 20_3 14_4$ $1_1 6_3 16_4$ $1_1 18_3 18_4$ $1_1 4_3 20_4$ $2_1 14_3 1_4$ $2_1 13_3 3_4$ $2_1 8_3 4_4$ $2_1 11_3 5_4$ $2_1 2_3 6_4$ $2_1 16_3 7_4$ $2_1 18_3 8_4$ $2_1 19_3 9_4$ $2_1 15_3 10_4$ $2_1 (i+21)_3 (i+11)_4$ $(i = 0,1,\ldots,4)$ $2_1 10_3 17_4$ $2_1 7_3 19_4$ $2_1 3_3 21_4$ $2_1 6_3 23_4$ $3_1 2_3 0_4$ $3_1 17_3 2_4$ $3_1 6_3 4_4$ $3_1 9_3 5_4$ $3_1 20_3 6_4$ $3_1 23_3 7_4$ $3_1 5_3 8_4$ $3_1 13_3 9_4$ $3_1 4_3 10_4$ $3_1 16_3 11_4$ $3_1 12_3 12_4$ $3_1 7_3 13_4$ $3_1 11_3 14_4$ $3_1 19_3 15_4$ $3_1 14_3 16_4$ $3_1 3_3 18_4$ $3_1 22_3 20_4$ $3_1 10_3 22_4$ $4_1 17_3 1_4$ $4_1 (i+1)_3 (i+3)_4 (i = 0,3,11,12)$ $4_1 14_3 5_4$ $4_1 2_3 7_4$ $4_1 10_3 8_4$ $4_1 18_3 9_4$ $4_1 21_3 10_4$ $4_1 6_3 11_4$ $4_1 9_3 12_4$ $4_1 22_3 13_4$ $4_1 5_3 16_4$ $4_1 8_3 17_4$ $4_1 11_3 19_4$ $4_1 20_3 21_4$ $4_1 7_3 23_4$ $5_1 19_3 0_4$ $5_1 15_3 2_4$ $5_1 20_3 4_4$ $5_1 21_3 6_4$ $5_1 7_3 7_4$ $5_1 23_3 8_4$ $5_1 14_3 9_4$ $5_1 1_3 10_4$ $5_1 2_3 11_4$ $5_1 16_3 12_4$ $5_1 4_3 13_4$ $5_1 6_3 14_4$ $5_1 17_3 15_4$ $5_1 10_3 16_4$ $5_1 0_3 17_4$ $5_1 11_3 18_4$ $5_1 5_3 20_4$ $5_1 22_3 22_4$ $6_1 22_3 1_4$ $6_1 12_3 3_4$ $6_1 4_3 5_4$ $6_1 10_3 7_4$ $6_1 7_3 8_4$ $6_1 1_3 9_4$ $6_1 23_3 10_4$ $6_1 13_3 11_4$ $6_1 2_3 12_4$ $6_1 8_3 13_4$ $6_1 19_3 14_4$ $6_1 5_3 15_4$ $6_1 17_3 16_4$ $6_1 3_3 17_4$ $6_1 16_3 18_4$

$6_{1}21_{3}19_{4}$ $6_{1}15_{3}21_{4}$ $6_{1}11_{3}23_{4}$ $7_{1}16_{3}0_{4}$ $7_{1}10_{3}2_{4}$ $7_{1}18_{3}4_{4}$ $7_{1}6_{3}6_{4}$ $7_{1}20_{3}8_{4}$
$7_{1}9_{3}9_{4}$ $7_{1}13_{3}10_{4}$ $7_{1}4_{3}11_{4}$ $7_{1}7_{3}12_{4}$ $7_{1}11_{3}13_{4}$ $7_{1}14_{3}14_{4}$ $7_{1}21_{3}15_{4}$
$7_{1}19_{3}16_{4}$ $7_{1}23_{3}17_{4}$ $7_{1}22_{3}18_{4}$ $7_{1}8_{3}19_{4}$ $7_{1}15_{3}20_{4}$ $7_{1}17_{3}22_{4}$ $8_{1}20_{3}1_{4}$
$8_{1}7_{3}3_{4}$ $8_{1}1_{3}5_{4}$ $8_{1}21_{3}7_{4}$ $8_{1}15_{3}9_{4}$ $8_{1}0_{3}10_{4}$ $8_{1}18_{3}11_{4}$ $8_{1}10_{3}12_{4}$
$8_{1}6_{3}13_{4}$ $8_{1}16_{3}14_{4}$ $8_{1}2_{3}15_{4}$ $8_{1}4_{3}16_{4}$ $8_{1}13_{3}17_{4}$ $8_{1}12_{3}18_{4}$ $8_{1}23_{3}19_{4}$
$8_{1}17_{3}20_{4}$ $8_{1}11_{3}21_{4}$ $8_{1}9_{3}23_{4}$ $9_{1}7_{3}0_{4}$ $9_{1}8_{3}2_{4}$ $9_{1}19_{3}4_{4}$ $9_{1}10_{3}6_{4}$ $9_{1}16_{3}8_{4}$
$9_{1}22_{3}10_{4}$ $9_{1}15_{3}11_{4}$ $9_{1}0_{3}12_{4}$ $9_{1}1_{3}13_{4}$ $9_{1}3_{3}14_{4}$ $9_{1}18_{3}15_{4}$ $9_{1}2_{3}16_{4}$
$9_{1}17_{3}17_{4}$ $9_{1}23_{3}18_{4}$ $9_{1}14_{3}19_{4}$ $9_{1}6_{3}20_{4}$ $9_{1}5_{3}21_{4}$ $9_{1}11_{3}22_{4}$
$10_{1}(i+2)_{3}(i+1)_{4}(i = 0,2,10,13,14,22)$ $10_{1}13_{3}5_{4}$ $10_{1}9_{3}7_{4}$ $10_{1}11_{3}9_{4}$
$10_{1}20_{3}12_{4}$ $10_{1}18_{3}13_{4}$ $10_{1}22_{3}16_{4}$ $10_{1}19_{3}17_{4}$ $10_{1}17_{3}18_{4}$ $10_{1}5_{3}19_{4}$
$10_{1}1_{3}20_{4}$ $10_{1}14_{3}21_{4}$ $10_{1}3_{3}22_{4}$ $11_{1}8_{3}0_{4}$ $11_{1}2_{3}2_{4}$ $11_{1}10_{3}4_{4}$ $11_{1}3_{3}6_{4}$
$11_{1}17_{3}8_{4}$ $11_{1}20_{3}10_{4}$ $11_{1}13_{3}12_{4}$ $11_{1}14_{3}13_{4}$ $11_{1}23_{3}14_{4}$ $11_{1}4_{3}15_{4}$
$11_{1}11_{3}16_{4}$ $11_{1}15_{3}17_{4}$ $11_{1}0_{3}18_{4}$ $11_{1}16_{3}19_{4}$ $11_{1}12_{3}20_{4}$ $11_{1}19_{3}21_{4}$
$11_{1}18_{3}22_{4}$ $11_{1}21_{3}23_{4}$ $12_{1}14_{3}0_{4}$ $12_{1}8_{3}1_{4}$ $12_{1}9_{3}3_{4}$ $12_{1}0_{3}5_{4}$ $12_{1}1_{3}7_{4}$
$12_{1}6_{3}9_{4}$ $12_{1}20_{3}11_{4}$ $12_{1}15_{3}13_{4}$ $12_{1}5_{3}14_{4}$ $12_{1}22_{3}15_{4}$ $12_{1}18_{3}16_{4}$
$12_{1}21_{3}17_{4}$ $12_{1}2_{3}18_{4}$ $12_{1}3_{3}19_{4}$ $12_{1}13_{3}20_{4}$ $12_{1}4_{3}21_{4}$ $12_{1}12_{3}22_{4}$
$12_{1}23_{3}23_{4}$ $13_{1}20_{3}0_{4}$ $13_{1}10_{3}1_{4}$ $13_{1}23_{3}2_{4}$ $13_{1}2_{3}4_{4}$ $13_{1}19_{3}6_{4}$ $13_{1}21_{3}8_{4}$
$13_{1}3_{3}10_{4}$ $13_{1}18_{3}12_{4}$ $13_{1}1_{3}14_{4}$ $13_{1}6_{3}15_{4}$ $13_{1}15_{3}16_{4}$ $13_{1}16_{3}17_{4}$
$13_{1}8_{3}18_{4}$ $13_{1}0_{3}19_{4}$ $13_{1}7_{3}20_{4}$ $13_{1}17_{3}21_{4}$ $13_{1}9_{3}22_{4}$ $13_{1}5_{3}23_{4}$ $14_{1}22_{3}0_{4}$
$14_{1}13_{3}1_{4}$ $14_{1}14_{3}2_{4}$ $14_{1}5_{3}3_{4}$ $14_{1}10_{3}5_{4}$ $14_{1}17_{3}7_{4}$ $14_{1}16_{3}9_{4}$ $14_{1}7_{3}11_{4}$
$14_{1}19_{3}13_{4}$ $14_{1}20_{3}15_{4}$ $14_{1}23_{3}16_{4}$ $14_{1}18_{3}17_{4}$ $14_{1}9_{3}18_{4}$ $14_{1}15_{3}19_{4}$
$14_{1}8_{3}20_{4}$ $14_{1}1_{3}21_{4}$ $14_{1}2_{3}22_{4}$ $14_{1}4_{3}23_{4}$ $15_{1}23_{3}0_{4}$ $15_{1}5_{3}1_{4}$ $15_{1}9_{3}2_{4}$
$15_{1}6_{3}3_{4}$ $15_{1}22_{3}4_{4}$ $15_{1}16_{3}6_{4}$ $15_{1}0_{3}8_{4}$ $15_{1}2_{3}10_{4}$ $15_{1}3_{3}12_{4}$ $15_{1}13_{3}14_{4}$
$15_{1}12_{3}16_{4}$ $15_{1}7_{3}17_{4}$ $15_{1}10_{3}18_{4}$ $15_{1}4_{3}19_{4}$ $15_{1}11_{3}20_{4}$ $15_{1}21_{3}21_{4}$
$15_{1}19_{3}22_{4}$ $15_{1}20_{3}23_{4}$ $16_{1}3_{3}0_{4}$ $16_{1}11_{3}1_{4}$ $16_{1}6_{3}2_{4}$ $16_{1}2_{3}3_{4}$ $16_{1}1_{3}4_{4}$
$16_{1}12_{3}5_{4}$ $16_{1}8_{3}7_{4}$ $16_{1}10_{3}9_{4}$ $16_{1}9_{3}11_{4}$ $16_{1}13_{3}13_{4}$ $16_{1}15_{3}15_{4}$ $16_{1}4_{3}17_{4}$
$16_{1}5_{3}18_{4}$ $16_{1}17_{3}19_{4}$ $16_{1}14_{3}20_{4}$ $16_{1}7_{3}21_{4}$ $16_{1}16_{3}22_{4}$ $16_{1}22_{3}23_{4}$
$17_{1}5_{3}0_{4}$ $17_{1}15_{3}1_{4}$ $17_{1}20_{3}2_{4}$ $17_{1}23_{3}3_{4}$ $17_{1}9_{3}4_{4}$ $17_{1}19_{3}5_{4}$ $17_{1}11_{3}6_{4}$
$17_{1}14_{3}8_{4}$ $17_{1}10_{3}10_{4}$ $17_{1}21_{3}12_{4}$ $17_{1}22_{3}14_{4}$ $17_{1}3_{3}16_{4}$ $17_{1}1_{3}18_{4}$
$17_{1}18_{3}19_{4}$ $17_{1}16_{3}20_{4}$ $17_{1}2_{3}21_{4}$ $17_{1}6_{3}22_{4}$ $17_{1}8_{3}23_{4}$ $18_{1}18_{3}0_{4}$ $18_{1}4_{3}1_{4}$

$18_1 13_3 2_4$ $18_1 20_3 3_4$ $18_1 11_3 4_4$ $18_1 8_3 5_4$ $18_1 15_3 6_4$ $18_1 0_3 7_4$ $18_1 17_3 9_4$
$18_1 23_3 11_4$ $18_1 2_3 13_4$ $18_1 12_3 15_4$ $18_1 9_3 17_4$ $18_1 19_3 19_4$ $18_1 10_3 20_4$
$18_1 16_3 21_4$ $18_1 21_3 22_4$ $18_1 1_3 23_4$ $19_1 10_3 0_4$ $19_1 12_3 1_4$ $19_1 3_3 2_4$
$19_1 8_3 3_4$ $19_1 15_3 4_4$ $19_1 6_3 5_4$ $19_1 5_3 6_4$ $19_1 18_3 7_4$ $19_1 4_3 8_4$ $19_1 11_3 10_4$
$19_1 23_3 12_4$ $19_1 9_3 14_4$ $19_1 13_3 16_4$ $19_1 21_3 18_4$ $19_1 20_3 20_4$ $19_1 0_3 21_4$
$19_1 7_3 22_4$ $19_1 19_3 23_4$ $20_1 21_3 0_4$ $20_1 16_3 1_4$ $20_1 22_3 2_4$ $20_1 14_3 3_4$ $20_1 0_3 4_4$
$20_1 5_3 5_4$ $20_1 1_3 6_4$ $20_1 12_3 7_4$ $20_1 15_3 8_4$ $20_1 8_3 9_4$ $20_1 3_3 11_4$ $20_1 9_3 13_4$
$20_1 10_3 15_4$ $20_1 6_3 17_4$ $20_1 20_3 19_4$ $20_1 23_3 21_4$ $20_1 4_3 22_4$ $20_1 13_3 23_4$
$21_1 0_3 0_4$ $21_1 23_3 1_4$ $21_1 5_3 2_4$ $21_1 11_3 3_4$ $21_1 17_3 4_4$ $21_1 7_3 5_4$ $21_1 22_3 6_4$
$21_1 19_3 7_4$ $21_1 3_3 8_4$ $21_1 20_3 9_4$ $21_1 18_3 10_4$ $21_1 4_3 12_4$ $21_1 21_3 14_4$ $21_1 8_3 16_4$
$21_1 6_3 18_4$ $21_1 2_3 20_4$ $21_1 1_3 22_4$ $21_1 12_3 23_4$

type BCD: $0_2 5_3 22_4$ $0_2 2_3 23_4$ $1_2 0_3 20_4$ $1_2 6_3 21_4$ $2_2 4_3 18_4$ $2_2 22_3 19_4$ $3_2 1_3 16_4$
$3_2 12_3 17_4$ $4_2 17_3 14_4$ $4_2 14_3 15_4$ $5_2 17_3 12_4$ $5_2 21_3 13_4$

type ACC: $0_1 20_3 23_3$ $0_1 19_3 22_3$ $0_1 15_3 16_3$ $1_1 9_3 12_3$ $1_1 3_3 10_3$ $1_1 13_3 21_3$ $2_1 5_3 12_3$
$2_1 9_3 17_3$ $2_1 4_3 20_3$ $3_1 8_3 21_3$ $3_1 15_3 18_3$ $3_1 0_3 1_3$ $4_1 3_3 15_3$ $4_1 19_3 23_3$ $4_1 0_3 16_3$
$5_1 3_3 9_3$ $5_1 13_3 18_3$ $5_1 8_3 12_3$ $6_1 6_3 18_3$ $6_1 14_3 20_3$ $6_1 0_3 9_3$ $7_1 0_3 5_3$ $7_1 1_3 3_3$
$7_1 2_3 12_3$ $8_1 3_3 8_3$ $8_1 5_3 19_3$ $8_1 14_3 22_3$ $9_1 13_3 20_3$ $9_1 4_3 9_3$ $9_1 12_3 21_3$
$10_1 7_3 10_3$ $10_1 21_3 23_3$ $10_1 6_3 8_3$ $11_1 6_3 7_3$ $11_1 1_3 5_3$ $11_1 9_3 22_3$ $12_1 7_3 16_3$
$12_1 10_3 17_3$ $12_1 11_3 19_3$ $13_1 12_3 14_3$ $13_1 11_3 22_3$ $13_1 4_3 13_3$ $14_1 11_3 12_3$
$14_1 0_3 3_3$ $14_1 6_3 21_3$ $15_1 17_3 18_3$ $15_1 1_3 15_3$ $15_1 8_3 14_3$ $16_1 0_3 21_3$ $16_1 18_3 23_3$
$16_1 19_3 20_3$ $17_1 12_3 13_3$ $17_1 7_3 17_3$ $17_1 0_3 4_3$ $18_1 6_3 14_3$ $18_1 5_3 7_3$ $18_1 3_3 22_3$
$19_1 1_3 2_3$ $19_1 14_3 17_3$ $19_1 16_3 22_3$ $20_1 11_3 17_3$ $20_1 18_3 19_3$ $20_1 2_3 7_3$
$21_1 13_3 16_3$ $21_1 10_3 14_3$ $21_1 9_3 15_3$

type CCD: $4_3 17_3 0_4$ $1_3 12_3 0_4$ $6_3 9_3 0_4$ $13_3 15_3 0_4$ $1_3 9_3 1_4$ $3_3 6_3 1_4$ $7_3 19_3 1_4$ $0_3 18_3 1_4$
$11_3 16_3 2_4$ $1_3 18_3 2_4$ $0_3 7_3 2_4$ $4_3 21_3 2_4$ $0_3 22_3 3_4$ $16_3 19_3 3_4$ $10_3 18_3 3_4$
$15_3 21_3 3_4$ $3_3 12_3 4_4$ $14_3 16_3 4_4$ $5_3 21_3 4_4$ $13_3 23_3 4_4$ $3_3 21_3 5_4$ $15_3 23_3 5_4$
$2_3 17_3 5_4$ $16_3 20_3 5_4$ $8_3 18_3 6_4$ $9_3 14_3 6_4$ $0_3 12_3 6_4$ $7_3 13_3 6_4$ $3_3 20_3 7_4$ $5_3 15_3 7_4$
$4_3 22_3 7_4$ $11_3 13_3 7_4$ $12_3 19_3 8_4$ $6_3 22_3 8_4$ $2_3 13_3 8_4$ $8_3 11_3 8_4$ $3_3 5_3 9_4$ $7_3 21_3 9_4$
$12_3 22_3 9_4$ $4_3 23_3 9_4$ $5_3 6_3 10_4$ $8_3 9_3 10_4$ $12_3 17_3 10_4$ $14_3 19_3 10_4$ $0_3 17_3 11_4$
$10_3 19_3 11_4$ $5_3 22_3 11_4$ $11_3 14_3 11_4$ $1_3 8_3 12_4$ $6_3 19_3 12_4$ $11_3 15_3 12_4$

$12_3 20_3 13_4$ $10_3 16_3 13_4$ $3_3 17_3 13_4$ $4_3 7_3 14_4$ $8_3 10_3 14_4$ $2_3 18_3 14_4$ $0_3 8_3 15_4$
$3_3 7_3 15_4$ $9_3 23_3 15_4$ $16_3 21_3 16_4$ $0_3 20_3 16_4$ $7_3 9_3 16_4$ $5_3 14_3 17_4$ $1_3 11_3 17_4$
$20_3 22_3 17_4$ $7_3 14_3 18_4$ $13_3 19_3 18_4$ $15_3 20_3 18_4$ $1_3 10_3 19_4$ $2_3 9_3 19_4$ $6_3 12_3 19_4$
$18_3 21_3 20_4$ $9_3 19_3 20_4$ $3_3 23_3 20_4$ $10_3 12_3 21_4$ $18_3 22_3 21_4$ $9_3 13_3 21_4$
$0_3 13_3 22_4$ $14_3 23_3 22_4$ $8_3 20_3 22_4$ $16_3 17_3 23_4$ $14_3 15_3 23_4$ $3_3 18_3 23_4$

type BCC: $0_2 1_3 23_3$ $0_2 9_3 20_3$ $0_2 3_3 16_3$ $0_2 0_3 10_3$ $0_2 6_3 11_3$ $0_2 13_3 17_3$ $0_2 7_3 8_3$ $0_2 4_3 12_3$
$0_2 15_3 22_3$ $0_2 19_3 21_3$ $0_2 14_3 18_3$ $1_2 10_3 20_3$ $1_2 1_3 22_3$ $1_2 2_3 19_3$ $1_2 8_3 23_3$
$1_2 4_3 15_3$ $1_2 3_3 11_3$ $1_2 7_3 18_3$ $1_2 13_3 14_3$ $1_2 12_3 16_3$ $1_2 5_3 9_3$ $1_2 17_3 21_3$
$2_2 9_3 10_3$ $2_2 1_3 14_3$ $2_2 2_3 3_3$ $2_2 6_3 16_3$ $2_2 12_3 18_3$ $2_2 17_3 19_3$ $2_2 5_3 20_3$ $2_2 0_3 15_3$
$2_2 11_3 21_3$ $2_2 7_3 23_3$ $2_2 8_3 13_3$ $3_2 8_3 19_3$ $3_2 16_3 18_3$ $3_2 3_3 13_3$ $3_2 17_3 23_3$
$3_2 0_3 6_3$ $3_2 4_3 14_3$ $3_2 7_3 22_3$ $3_2 9_3 11_3$ $3_2 20_3 21_3$ $3_2 10_3 15_3$ $3_2 2_3 5_3$ $4_2 5_3 11_3$
$4_2 1_3 16_3$ $4_2 10_3 13_3$ $4_2 7_3 12_3$ $4_2 3_3 4_3$ $4_2 0_3 23_3$ $4_2 2_3 6_3$ $4_2 9_3 21_3$ $4_2 18_3 20_3$
$4_2 15_3 19_3$ $4_2 8_3 22_3$ $5_2 4_3 6_3$ $5_2 1_3 7_3$ $5_2 5_3 10_3$ $5_2 8_3 16_3$ $5_2 13_3 22_3$ $5_2 0_3 19_3$
$5_2 3_3 14_3$ $5_2 9_3 18_3$ $5_2 12_3 15_3$ $5_2 2_3 20_3$ $5_2 11_3 23_3$

type CCC: $6_3 10_3 23_3$ $1_3 6_3 13_3$ $4_3 5_3 18_3$ $5_3 16_3 23_3$ $5_3 8_3 17_3$ $2_3 4_3 16_3$ $10_3 21_3 22_3$
$1_3 4_3 19_3$ $1_3 17_3 20_3$ $0_3 2_3 11_3$ $2_3 14_3 21_3$ $4_3 10_3 11_3$ $6_3 15_3 17_3$ $7_3 11_3 20_3$
$2_3 22_3 23_3$ $2_3 8_3 15_3$

type ∞CC: $\infty 5_3 13_3$ $\infty 11_3 18_3$ $\infty 3_3 19_3$ $\infty 7_3 15_3$ $\infty 12_3 23_3$ $\infty 17_3 22_3$ $\infty 4_3 8_3$ $\infty 1_3 21_3$
$\infty 9_3 16_3$ $\infty 6_3 20_3$ $\infty 2_3 10_3$ $\infty 0_3 14_3$

types ∞BB and BBB: Form an STS(7) on ∞ and the six points from cell B.
One long line is $0_4 1_4 \cdots 23_4 \infty$ and the other long line is formed on the hole of IPBD(23, 5;{3}).

Thus the spectrum for AULSs in which one long line has size five and the other long line has size seven, nine, thirteen or fifteen has thereby been completely determined.

Lemma 2.20 $LS_d(3,5^*,7^*) = \{v: v \geq 23, v \equiv 5 \pmod 6\}$;
$LS_i(3,5^*,7^*) = \{v: v \geq 17, v \equiv 5 \pmod 6\}$; $LS(3,5^*,9^*) = \{v: v \geq 23, v \equiv 5 \pmod 6\}$;
$LS_d(3,5^*,13^*) = \{v: v \geq 35, v \equiv 5 \pmod 6\}$; $LS_i(3,5^*,13^*) = \{v: v \geq 29, v \equiv 5 \pmod 6\}$;
$LS(3,5^*,15^*) = \{v: v \geq 35, v \equiv 5 \pmod 6\}$.

Proof: We obtain $v \in LS_d(3,5^*,7^*)$ for all $v \geq 23$ and $v \equiv 5 \pmod 6$, from Lemma 2.9, Corollary 2.10, Lemma 2.13 and Lemma 2.15. Also, $v \in LS_i(3,5^*,7^*)$ for all $v \geq 17$, $v \equiv 5 \pmod 6$, by Lemma 2.6, Corollary 2.7, Lemma 2.15 and Corollary 2.16. Next, $v \in LS_d(3,5^*,9^*)$ for all $v \geq 23$, $v \equiv 5 \pmod 6$, from Lemma 2.11, Corollary 2.12, Corollary 2.8, Lemma 2.17 and Corollary 2.18. In order to prove that $v \in LS_i(3,5^*,9^*)$ for all $v \geq 23$, $v \equiv 5 \pmod 6$, apply Corollary 2.8, Lemma 2.17 and Corollary 2.18. Hence, by definition, $v \in LS(3,5^*,9^*)$ for all $v \geq 23$, $v \equiv 5 \pmod 6$. By Lemma 2.9, Corollary 2.10, Lemmas 2.13 and 2.15, and Corollary 2.16, $v \in LS_d(3,5^*,13^*)$ for all $v \geq 35$, $v \equiv 5 \pmod 6$. By Lemma 2.6, Corollary 2.7, Lemma 2.15 and Corollary 2.16, $v \in LS_i(3,5^*,13^*)$ for all $v \geq 29$, $v \equiv 5 \pmod 6$. Finally, Lemma 2.11, Corollary 2.12, Corollary 2.8, Lemma 2.17, and Corollary 2.18 yield $v \in LS_d(3,5^*,15^*)$ for all $v \geq 35$, $v \equiv 5 \pmod 6$. Similarly, by applying Corollary 2.8, Lemma 2.11, Corollary 2.12, Lemma 2.17 and Corollary 2.18, we obtain $v \in LS_i(3,5^*,15^*)$ for all $v \geq 35$, $v \equiv 5 \pmod 6$. Thus, $v \in LS(3,5^*,15^*)$ for all $v \geq 35$, $v \equiv 5 \pmod 6$.

§2.3 Almost uniform linear spaces with one long line of size 6t + 7, one long line of size w and short lines of size three

From Corollaries 1.19(i) and 1.21(i), if an AULS has one long line of size $6t + 7$, then the other long line must either have size $w \equiv 1,3 \pmod 6$ and $v \equiv 1,3 \pmod 6$, or $w \equiv 5 \pmod 6$ and $v \equiv 5 \pmod 6$. We shall also assume that $w > 6t + 7$. We begin by recursively constructing an AULS where one long line is of size $6t + 7$ and one long line is of size $w \equiv 1,3 \pmod 6$ and has minimum order $v = 2w + 6t + 7$.

Lemma 2.21 If $w \equiv 1,3 \pmod 6$, then $2w + 6t + 7 \in LS_d(3, (6t + 7)^*, w^*)$, and $2w + 6t + 7 = \min\{v: \exists LS_d(v;\{3, (6t + 7)^*, w^*\})\}$.

Proof: By Corollary 1.19(i), $v \geq 2w + 6t + 7$. The result follows by Lemma 1.35.

Corollary 2.22 If $w \equiv 1,3 \pmod 6$ and $w > 6t + 7$, then $v \in LS_d(3, (6t + 7)^*, w^*)$ for all $v \geq 4w + 12t + 15$.

Proof: Apply Lemma 1.38.

If we consider an AULS with two intersecting lines of sizes $6t + 7$ and $w \equiv 1 \pmod 6$, there is an analogous result, although there are more restrictions which must be imposed on w when $t > 0$.

Lemma 2.23 If $w \equiv 1 \pmod{6t + 6}$ then $2w + 6t + 5 \in LS_i(3, (6t + 7)^*, w^*)$ and $2w + 6t + 5 = \min\{v: \exists LS_i(v;\{3, (6t + 7)^*, w^*\})\}$.

Proof: From Corollary 1.21(i), we get $v \geq 2w + 6t + 5$. Since $w \equiv 1 \pmod{6t + 6}$ and $w > 6t + 7$, there exists an integer $r \geq 2$ such that $w - 1 = r(6t + 6)$. Form the partition $\pi(1^1, (6t + 6)^{r+1}, (w - 1)^1)$ and apply Theorem 1.24(b).

Corollary 2.24 If $w \equiv 1 \pmod{6t + 6}$, then $v \in LS_i(3, (6t + 7)^*, w^*)$ for all $v \geq 4w + 12t + 11$.

Proof: Apply Lemma 1.38.

There is no obvious general recursive construction when the AULS of minimum order has two long lines which intersect, one of which has size $w \equiv 3 \pmod 6$. However, if we assume that $t = 0$, $u = 7$ and $w = 9$ or 15, we can provide direct constructions.

Lemma 2.25 There exist AULSs $LS_i(25;\{3, 7^*, 9^*\})$ and $LS_i(37;\{3, 7^*, 15^*\})$.

Proof: Form the partition $\pi(1^1, 6^1, 8^1, 10^1)$ and construct the short lines of

type ABC: $i_1 i_2 0_3 (i = 0,1,2,4,5) \quad 0_1 1_2 5_3 \quad 0_1 2_2 4_3 \quad 0_1 3_2 6_3 \quad 0_1 4_2 8_3 \quad 0_1 5_2 1_3 \quad 0_1 6_2 3_3$
$0_1 7_2 7_3 \quad 1_1 0_2 8_3 \quad 1_1 2_2 9_3 \quad 1_1 3_2 4_3 \quad 1_1 4_2 3_3 \quad 1_1 5_2 7_3 \quad 1_1 6_2 5_3 \quad 1_1 7_2 6_3 \quad 2_1 0_2 2_3$
$2_1 1_2 1_3 \quad 2_1 3_2 9_3 \quad 2_1 4_2 4_3 \quad 2_1 5_2 6_3 \quad 2_1 6_2 7_3 \quad 2_1 7_2 3_3 \quad 3_1 0_2 3_3 \quad 3_1 1_2 2_3 \quad 3_1 2_2 1_3$
$3_1 3_2 7_3 \quad 3_1 4_2 5_3 \quad 3_1 5_2 9_3 \quad 3_1 6_2 8_3 \quad 3_1 7_2 0_3 \quad 4_1 0_2 4_3 \quad 4_1 1_2 9_3 \quad 4_1 2_2 2_3 \quad 4_1 3_2 8_3$
$4_1 5_2 5_3 \quad 4_1 6_2 6_3 \quad 4_1 7_2 1_3 \quad 5_1 0_2 5_3 \quad 5_1 1_2 4_3 \quad 5_1 2_2 3_3 \quad 5_1 3_2 2_3 \quad 5_1 4_2 6_3 \quad 5_1 6_2 9_3$
$5_1 7_2 8_3$

type ACC: $0_1 2_3 9_3 \quad 1_1 1_3 2_3 \quad 2_1 5_3 8_3 \quad 3_1 4_3 6_3 \quad 4_1 3_3 7_3 \quad 5_1 1_3 7_3$

type BCC: $0_2 1_3 6_3 \quad 0_2 7_3 9_3 \quad 1_2 3_3 6_3 \quad 1_2 7_3 8_3 \quad 2_2 5_3 7_3 \quad 2_2 6_3 8_3 \quad 3_2 3_3 5_3 \quad 3_2 0_3 1_3$
$4_2 2_3 7_3 \quad 4_2 1_3 9_3 \quad 5_2 2_3 8_3 \quad 5_2 3_3 4_3 \quad 6_2 0_3 2_3 \quad 6_2 1_3 4_3 \quad 7_2 5_3 9_3 \quad 7_2 2_3 4_3$

type CCC: $0_3 3_3 9_3 \quad 0_3 4_3 5_3 \quad 0_3 6_3 7_3 \quad 1_3 3_3 8_3 \quad 2_3 5_3 6_3 \quad 4_3 8_3 9_3$

type ∞CC: $\infty 0_3 8_3 \quad \infty 1_3 5_3 \quad \infty 2_3 3_3 \quad \infty 6_3 9_3 \quad \infty 4_3 7_3$

The long lines are $0_1 1_1 2_1 3_1 4_1 5_1 \infty$ and $0_2 1_2 2_2 3_2 4_2 5_2 6_2 7_2 \infty$. Therefore, we have obtained, by direct construction, the space $LS_i(25;\{3, 7^*, 9^*\})$. In order to prove the existence of $LS_i(37;\{3, 7^*, 15^*\})$, we give another direct construction. Form the partition $\pi(1^1, 6^1, 8^2, 14^1)$ and the short lines are of

type ABD: $0_1 0_2 0_4 \quad 0_1 2_2 1_4 \quad 0_1 4_2 2_4 \quad 0_1 6_2 9_4 \quad 0_1 1_2 4_4 \quad 0_1 3_2 12_4 \quad 1_1 1_2 2_4 \quad 1_1 3_2 1_4$
$1_1 5_2 4_4 \quad 1_1 7_2 5_4 \quad 1_1 2_2 6_4 \quad 1_1 4_2 7_4 \quad 2_1 2_2 4_4 \quad 2_1 4_2 12_4 \quad 2_1 6_2 6_4 \quad 2_1 0_2 7_4 \quad 2_1 3_2 8_4$
$2_1 5_2 9_4 \quad 3_1 3_2 6_4 \quad 3_1 5_2 7_4 \quad 3_1 7_2 8_4 \quad 3_1 1_2 5_4 \quad 3_1 4_2 10_4 \quad 3_1 6_2 11_4 \quad 4_1 4_2 8_4 \quad 4_1 6_2 3_4$
$4_1 0_2 10_4 \quad 4_1 2_2 11_4 \quad 4_1 5_2 5_4 \quad 4_1 7_2 13_4 \quad 5_1 5_2 10_4 \quad 5_1 7_2 11_4 \quad 5_1 1_2 3_4 \quad 5_1 3_2 13_4$
$5_1 6_2 0_4 \quad 5_1 0_2 9_4$

type ABB: $0_1 5_2 7_2$ $1_1 0_2 6_2$ $2_1 1_2 7_2$ $3_1 0_2 2_2$ $4_1 1_2 3_2$ $5_1 2_2 4_2$

type ACD: $0_1 i_3 (i+6)_4 (i = 0,1,2,4,5,7)$ $0_1 3_3 4_4$ $0_1 6_3 5_4$ $1_1 1_3 0_4$ $1_1 2_3 3_4$ $1_1 0_3 8_4$
$1_1 4_3 9_4$ $1_1 3_3 10_4$ $1_1 7_3 11_4$ $1_1 5_3 12_4$ $1_1 6_3 13_4$ $2_1 3_3 0_4$ $2_1 5_3 1_4$ $2_1 1_3 2_4$
$2_1 0_3 3_4$ $2_1 2_3 10_4$ $2_1 6_3 11_4$ $2_1 7_3 5_4$ $2_1 4_3 13_4$ $3_1 6_3 0_4$ $3_1 7_3 1_4$ $3_1 5_3 3_4$ $3_1 4_3 2_4$
$3_1 0_3 4_4$ $3_1 1_3 12_4$ $3_1 2_3 9_4$ $3_1 3_3 13_4$ $4_1 0_3 0_4$ $4_1 3_3 1_4$ $4_1 6_3 2_4$ $4_1 7_3 4_4$ $4_1 4_3 12_4$
$4_1 5_3 9_4$ $4_1 1_3 6_4$ $4_1 2_3 7_4$ $5_1 3_3 2_4$ $5_1 2_3 4_4$ $5_1 1_3 1_4$ $5_1 6_3 12_4$ $5_1 4_3 6_4$ $5_1 5_3 7_4$
$5_1 7_3 8_4$ $5_1 0_3 5_4$

type BCD: $0_2 0_3 2_4$ $0_2 1_3 4_4$ $0_2 2_3 5_4$ $0_2 3_3 11_4$ $0_2 4_3 1_4$ $0_2 5_3 13_4$ $0_2 6_3 8_4$ $0_2 7_3 6_4$ $1_2 0_3 1_4$
$1_2 1_3 8_4$ $1_2 2_3 0_4$ $1_2 3_3 9_4$ $1_2 4_3 11_4$ $1_2 5_3 6_4$ $1_2 6_3 10_4$ $1_2 7_3 7_4$ $2_2 0_3 9_4$ $2_2 1_3 5_4$
$2_2 2_3 12_4$ $2_2 3_3 8_4$ $2_2 4_3 7_4$ $2_2 5_3 0_4$ $2_2 6_3 3_4$ $2_2 7_3 10_4$ $3_2 0_3 7_4$ $3_2 1_3 3_4$
$3_2 2_3 11_4$ $3_2 3_3 4_4$ $3_2 4_3 5_4$ $3_2 5_3 10_4$ $3_2 6_3 9_4$ $3_2 7_3 2_4$ $4_2 0_3 11_4$ $4_2 1_3 9_4$
$4_2 2_3 13_4$ $4_2 3_3 6_4$ $4_2 4_3 0_4$ $4_2 5_3 5_4$ $4_2 6_3 4_4$ $4_2 7_3 3_4$ $5_2 0_3 13_4$ $5_2 1_3 11_4$
$5_2 2_3 2_4$ $5_2 3_3 12_4$ $5_2 4_3 3_4$ $5_2 5_3 8_4$ $5_2 6_3 6_4$ $5_2 7_3 0_4$ $6_2 0_3 10_4$ $6_2 1_3 13_4$
$6_2 2_3 1_4$ $6_2 3_3 5_4$ $6_2 4_3 8_4$ $6_2 5_3 4_4$ $6_2 6_3 7_4$ $6_2 7_3 12_4$ $7_2 0_3 12_4$ $7_2 1_3 10_4$
$7_2 2_3 6_4$ $7_2 3_3 7_4$ $7_2 4_3 4_4$ $7_2 5_3 2_4$ $7_2 6_3 1_4$ $7_2 7_3 9_4$

type BBD: $3_2 7_2 0_4$ $4_2 5_2 1_4$ $2_2 6_2 2_4$ $0_2 7_2 3_4$ $0_2 1_2 12_4$ $1_2 2_2 13_4$

type ∞BB: $\infty 0_2 5_2$ $\infty 1_2 4_2$ $\infty 2_2 7_2$ $\infty 3_2 6_2$

type BBB: $0_2 3_2 4_2$ $2_2 3_2 5_2$ $4_2 6_2 7_2$ $1_2 5_2 6_2$

type ∞CC: $\infty 0_3 1_3$ $\infty 2_3 4_3$ $\infty 3_3 7_3$ $\infty 5_3 6_3$

type CCC: $0_3 2_3 3_3$ $0_3 4_3 6_3$ $0_3 5_3 7_3$ $1_3 2_3 5_3$ $1_3 3_3 6_3$ $1_3 4_3 7_3$ $3_3 4_3 5_3$ $2_3 6_3 7_3$

The long lines are $0_1 1_1 2_1 3_1 4_1 5_1 \infty$ and $0_4 1_4 \cdots 13_4 \infty$.

Corollary 2.26 $v \in LS_i(3, 7^*, 9^*)$ for all $v \geq 51$, and $v \in LS_i(3, 7^*, 15^*)$ for all $v \geq 75$, where $v \equiv 1,3 \pmod 6$.

Proof: Lemma 2.25 and Lemma 1.38.

The following lemmas provide various recursive and direct constructions to prove that AULSs of certain orders exist.

Lemma 2.27 If $0 \le t < (w-7)/6$, $t \equiv 0 \pmod 2$ and $w \equiv 3t + 3 \pmod{6t + 6}$, then $4w + 12t + 13 \in LS(3, (6t + 7)^*, w^*)$.

Proof: Form the partition $\pi(1^1, (6t + 6)^{(w+6t+6)/(3t+3)}, (2w)^1)$. We note that $(w + 6t + 6)/(3t + 3)$ is an integer since it is assumed that $w \equiv 0 \pmod{3t + 3}$. Use Corollary 1.25 with $g_1 = w$, $g = 6t + 6$ and $x = 2w$ to obtain the desired result.

Corollary 2.28 If $0 < t < (w-7)/6$ and $w \equiv 3 \pmod{3t + 3}$, or $t \equiv 1 \pmod 6$ and $w \equiv 9 \pmod{12}$, then $4w + 12t + 13 \in LS(3, (6t + 7)^*, w^*)$.

Proof: If $w \equiv 3 \pmod{3t + 3}$, form the partition $\pi(1^1, (6t + 6)^{(w+6t+3)/(3t+3)}, (2w + 6)^1)$ and apply Corollary 1.25 with $g_1 = w$, $g = 6t + 6$ and $x = 2w + 6$. If $t \equiv 1 \pmod 6$ and $w \equiv 9 \pmod{12}$, form the partition $\pi(1^1, (w-1)^1, ((3w + 12t + 13)/2)^2)$ and apply Lemma 1.32(b) where $u_1 = (3w + 12t + 13)/2$.

Lemma 2.29 If $w \equiv 1 \pmod 6$ then $4w + 12t + 11 \in LS_d(3, (6t + 7)^*, w^*)$.

Proof: Form the partition $\pi(1^1, (w-1)^4, (12t + 14)^1)$ and apply Theorem 1.24(b) with $g_1 = 6t + 7$, $g = w - 1$ and $x = 12t + 14$.

Lemma 2.30

(a) If $t \equiv 0 \pmod 2$ and $w \equiv 1$ or $3 \pmod 6$ then $4w + 12t + 9 \in LS_d(3,(6t + 7)^*,w^*)$.

(b) If $t \equiv 0 \pmod 2$, $t \ne 2$ or $t \equiv 1 \pmod 2$ and $w \equiv 1 \pmod{6t + 6}$, or $t = 0$ and $w \equiv 1,3 \pmod 6$, or $t = 2$ and $w \equiv 1 \pmod 6$, or $t \equiv 1,4$ or $7 \pmod 9$ and $w \equiv 4t+5 \pmod{6t+6}$, or $t \equiv 3,6 \pmod 9$ and $w \equiv 2t+3 \pmod{6t+6}$, or $t \equiv 9 \pmod{45}$ and $w \equiv 2t+3 \pmod{6t+6}$, then $4w + 12t + 9 \in LS_i(3, (6t + 7)^*, w^*)$.

Proof:

(a) First, construct an STS$(4w + 12t + 9)$ which contains disjoint subsystems STS$(6t + 7)$ and STS$(2w + 3t + 1)$, by using Theorem 1.33. By Theorem 1.7, embed an STS(w) into STS$(2w + 3t + 1)$, replacing STS(w) by a line of size w. Finally, replace STS(6t + 7) by a line of size 6t + 7.

(b) Form the partition $\pi(1^1, (6t + 6)^{(w+4t+3)/(2t+2)}, (w - 1)^1)$ and apply Theorem 1.24(b).

Corollary 2.31 If t ≡ 1,2,4 or 5(mod 6) and w ≡ 1(mod6), or t ≡ 0,2,3 or 5(mod 6) and w ≡ 3(mod 6); w ≥ 8t + 11, then $4w + 12t + 9 \in LS(3, (6t + 7)^*, w^*)$.

Proof: Form a partition $\pi(1^1,(w + 4t+ 3)^3,(w - 1)^1)$ and apply Corollary 1.25 with $g_1 = 6t + 7$, $g = w + 4t + 3$ and $x = w - 1$.

Lemma 2.32 If w ≡ 3 (mod 3t + 3) and w > 6t + 7, then $4w + 12t + 7 \in LS(3, (6t + 7)^*, w^*)$.

Proof: Form $\pi(1^1, (6t + 6)^{(w+6t+3)/(3t+3)}, (2w)^1)$ and apply Corollary 1.25 with $g_1 = w$, $g = 6t + 6$ and $x = 2w$.

Lemma 2.33

(a) If j is an odd nonnegative integer, 1 ≤ j ≤ 5, t is a nonnegative integer such that 0 ≤ t ≤ (w-j-4)/12, and w ≥ 12-j, w ≡ 1,3(mod 6),then $4w+12t+j \in LS(3,(6t+7)^*,w^*)$.

(b) If j is an odd nonnegative integer, 1 ≤ j ≤ 23;0 ≤ t ≤ (w+j-4)/12 and w ≥ 12+ j, then $4w + 12t - j \in LS(3, (6t + 7)^*, w^*)$.

Proof: Form a partition $\pi(1^1, (w - 1)^3, (w + 12t \pm j + 2)^1)$ and apply Corollary 1.25 with $g_1 = 6t + 7$, $g = w - 1$ and $x = w + 12t + j + 2$ (for (a)), or $x = w + 12t - j + 2$ (for (b)).

Corollary 2.34

(a) 25, 27, 31, 37 $\in LS(3, 7^*, 9^*)$.

(b) 37, 39, 43 $\in LS(3, 7^*, 13^*)$.

(c) 39, 43, 45, 49, 55 $\in LS(3, 7^*, 15^*)$.

(d) 55, 57, 63, 67 $\in LS(3, 7^*, 19^*)$.

(e) If $w \geq 19$ and $w \equiv 1 \pmod 6$, $t \equiv 0,1,3$ or $4 \pmod 6$, and $w \geq 8t + 19$, then $4w + 12t - 15 \in LS(3, (6t+7)^*, w^*)$.

(f) $61, 67, 69, 73 \in LS(3, 7^*, 21^*)$.

(g) If $w \geq 21$, $w \equiv 1 \pmod 6$ and $t \equiv 0,2,3,5 \pmod 6$, or $w \equiv 3 \pmod 6$ and $t \equiv 0,1,3$ or $4 \pmod 6$, $w \geq 8t + 21$, then $4w + 12t - 21 \in LS(3, (6t+7)^*, w^*)$.

Proof:

(a) It follows from Lemma 2.21 that $25 \in LS_d(3, 7^*, 9^*)$ and $25 \in LS_i(3, 7^*, 9^*)$ by Lemma 2.25. Hence, $25 \in LS(3, 7^*, 9^*)$. Next, $27 \in LS_i(3, 7^*, 9^*)$ by forming partition $\pi(1^1, 6^3, 8^1)$ and applying Theorem 1.24(b). We obtain $27 \in LS_d(3, 7^*, 9^*)$ by direct construction. Form the partition $\pi(7^1, 9^1, 11^1)$ where cell A is $Z_{11} \times \{1\}$, cell B is $Z_7 \times \{2\}$ and cell C is $Z_9 \times \{3\}$. Construct short lines of

type ABC: $i_1 0_2 i_3$ $(i+9)_1 1_2 i_3$ $(i+7)_1 2_2 i_3$ $(i+5)_1 3_2 i_3$ $(i+3)_1 4_2 i_3$ $(i+1)_1 5_2 i_3$ $(i+10)_1 6_2 i_3$ $(i = 0,1,\ldots,8)$

type AAB: $9_1 10_1 0_2$ $7_1 8_1 1_2$ $5_1 6_1 2_2$ $3_1 4_1 3_2$ $1_1 2_1 4_2$ $0_1 10_1 5_2$ $8_1 9_1 6_2$

type AAC: $2_1 4_1 0_3$ $6_1 8_1 0_3$ $3_1 7_1 1_3$ $5_1 9_1 1_3$ $4_1 6_1 2_3$ $8_1 10_1 2_3$ $5_1 7_1 3_3$ $0_1 9_1 3_3$ $6_1 10_1 4_3$ $1_1 8_1 4_3$ $0_1 7_1 5_3$ $2_1 9_1 5_3$ $1_1 10_1 6_3$ $3_1 8_1 6_3$ $0_1 2_1 7_3$ $4_1 9_1 7_3$ $5_1 10_1 8_3$ $1_1 3_1 8_3$

type AAA: $0_1 3_1 5_1$ $0_1 4_1 8_1$ $0_1 1_1 6_1$ $1_1 4_1 5_1$ $1_1 7_1 9_1$ $2_1 3_1 10_1$ $2_1 5_1 8_1$ $2_1 6_1 7_1$ $3_1 6_1 9_1$ $4_1 7_1 10_1$

The long lines are $0_2 1_2 \cdots 6_2$ and $0_3 1_3 \cdots 8_3$. Forming the partition $\pi(1^1, 6^1, 8^3)$ and applying Theorem 1.24(b) yields $31 \in LS_i(3, 7^*, 9^*)$. A direct construction is necessary to show that $31 \in LS_d(3, 7^*, 9^*)$. Form the partition $\pi(7^2, 8^1, 9^1)$ and construct short lines of

type ABC: $0_1 0_2 0_3$ $1_1 1_2 1_3$ $2_1 2_2 3_3$ $3_1 3_2 1_3$ $4_1 4_2 2_3$ $5_1 5_2 7_3$ $6_1 6_2 5_3$ $0_1 2_2 2_3$ $1_1 4_2 6_3$ $2_1 6_2 4_3$ $3_1 1_2 5_3$ $4_1 3_2 7_3$ $5_1 0_2 2_3$ $6_1 5_2 3_3$

type ABD: $0_1 1_2 0_4$ $0_1 3_2 2_4$ $0_1 4_2 4_4$ $0_1 5_2 6_4$ $0_1 6_2 8_4$ $1_1 0_2 1_4$ $1_1 2_2 3_4$ $1_1 3_2 5_4$ $1_1 5_2 7_4$ $1_1 6_2 0_4$ $2_1 0_2 0_4$ $2_1 1_2 2_4$ $2_1 5_2 4_4$ $2_1 3_2 6_4$ $2_1 4_2 8_4$ $3_1 0_2 3_4$ $3_1 2_2 5_4$ $3_1 4_2 7_4$ $3_1 5_2 1_4$ $3_1 6_2 2_4$ $4_1 0_2 4_4$ $4_1 1_2 6_4$ $4_1 2_2 8_4$ $4_1 5_2 5_4$ $4_1 6_2 1_4$ $5_1 1_2 5_4$ $5_1 2_2 7_4$ $5_1 3_2 3_4$ $5_1 4_2 1_4$ $5_1 6_2 6_4$ $6_1 0_2 8_4$ $6_1 1_2 4_4$ $6_1 2_2 2_4$ $6_1 3_2 0_4$ $6_1 4_2 3_4$

type ACD: $0_1 3_3 1_4$ $0_1 1_3 3_4$ $0_1 5_3 5_4$ $0_1 7_3 7_4$ $1_1 0_3 2_4$ $1_1 5_3 4_4$ $1_1 4_3 6_4$ $1_1 2_3 8_4$

$2_1 1_3 1_4$ $2_1 5_3 3_4$ $2_1 7_3 5_4$ $2_1 0_3 7_4$ $3_1 4_3 0_4$ $3_1 0_3 4_4$ $3_1 2_3 6_4$ $3_1 3_3 8_4$ $4_1 3_3 0_4$
$4_1 4_3 2_4$ $4_1 6_3 3_4$ $4_1 1_3 7_4$ $5_1 5_3 0_4$ $5_1 1_3 2_4$ $5_1 3_3 4_4$ $5_1 4_3 8_4$ $6_1 2_3 1_4$ $6_1 0_3 5_4$
$6_1 6_3 6_4$ $6_1 4_3 7_4$

type ACC: $0_1 4_3 6_3$ $1_1 3_3 7_3$ $2_1 2_3 6_3$ $3_1 6_3 7_3$ $4_1 0_3 5_3$ $5_1 0_3 6_3$ $6_1 1_3 7_3$

type BCD: $0_2 3_3 2_4$ $0_2 4_3 5_4$ $0_2 1_3 6_4$ $0_2 6_3 7_4$ $1_2 6_3 1_4$ $1_2 3_3 3_4$ $1_2 2_3 7_4$ $1_2 7_3 8_4$ $2_2 0_3 0_4$
$2_2 5_3 1_4$ $2_2 6_3 4_4$ $2_2 7_3 6_4$ $3_2 4_3 1_4$ $3_2 2_3 4_4$ $3_2 5_3 7_4$ $3_2 0_3 8_4$ $4_2 7_3 0_4$ $4_2 5_3 2_4$
$4_2 1_3 5_4$ $4_2 0_3 6_4$ $5_2 2_3 0_4$ $5_2 6_3 2_4$ $5_2 0_3 3_4$ $5_2 1_3 8_4$ $6_2 7_3 3_4$ $6_2 1_3 4_4$ $6_2 6_3 5_4$
$6_2 3_3 7_4$

type BCC: $0_2 5_3 7_3$ $1_2 0_3 4_3$ $2_2 1_3 4_3$ $3_2 3_3 6_3$ $4_2 3_3 4_3$ $5_2 4_3 5_3$ $6_2 0_3 2_3$

type CCD: $1_3 6_3 0_4$ $0_3 7_3 1_4$ $2_3 7_3 2_4$ $2_3 4_3 3_4$ $4_3 7_3 4_4$ $2_3 3_3 5_4$ $3_3 5_3 6_4$ $5_3 6_3 8_4$

type CCC: $0_3 1_3 3_3$ $1_3 2_3 5_3$

type BBB: put an STS(7) on the points of cell B.

The long lines are $0_1 1_1 \cdots 6_1$ and $0_4 1_4 \cdots 8_4$.

Form the partition $\pi(1^1, 8^1, 14^2)$ and apply Lemma 1.32(b) with $u_1 = 14$, $w = 9$ and $u = 7$ to prove that $37 \in LS_i(3, 7^*, 9^*)$. For $37 \in LS_d(3, 7^*, 9^*)$, form the partition $\pi(7^4, 9^1)$ and apply Theorem 1.24(a).

(b) Form the partition $\pi(1^1, 6^4, 12^1)$ and apply Theorem 1.24(b) to obtain $37 \in LS_i(3, 7^*, 13^*)$. It is demonstrated that $37 \in LS_d(3, 7^*, 13^*)$ by direct construction. Form the partition $\pi(7^2, 10^1, 13^1)$ and construct short lines of

type ABC: $0_1 0_2 8_3$ $i_1 i_2 i_3 (i = 1,2,\ldots,6)$

type ABD: $0_1 (i+1)_2 (2i)_4 (i = 0,1,\ldots,5)$ $1_1 0_2 1_4$ $1_1 (i+2)_2 (2i+3)_4 (i = 0,1,\ldots,4)$
$2_1 0_2 2_4$ $2_1 1_2 4_4$ $2_1 i_2 (2i)_4 (i = 3,\ldots,6)$ $3_1 0_2 3_4$ $3_1 1_2 5_4$ $3_1 2_2 7_4$ $3_1 4_2 9_4$
$3_1 5_2 11_4$ $3_1 6_2 1_4$ $4_1 i_2 (2i+5)_4 (i = 0,1,2,3)$ $4_1 5_2 1_4$ $4_1 6_2 3_4$ $5_1 i_2 (2i+4)_4$
$(i = 0,1,\ldots,4)$ $5_1 6_2 0_4$ $6_1 0_2 0_4$ $6_1 1_2 2_4$ $6_1 2_2 5_4$ $6_1 3_2 12_4$ $6_1 4_2 1_4$ $6_1 5_2 3_4$

type ACD: $0_1 i_3 i_4 (i = 1,3,5,7,9)$ $0_1 2_3 11_4$ $0_1 4_3 12_4$ $1_1 i_3 i_4 (i = 0,2,6,8)$ $1_1 3_3 4_4$
$1_1 4_3 10_4$ $1_1 5_3 12_4$ $2_1 5_3 0_4$ $2_1 3_3 1_4$ $2_1 0_3 3_4$ $2_1 1_3 5_4$ $2_1 9_3 7_4$ $2_1 4_3 9_4$
$2_1 6_3 11_4$ $3_1 4_3 0_4$ $3_1 5_3 2_4$ $3_1 6_3 4_4$ $3_1 8_3 6_4$ $3_1 7_3 8_4$ $3_1 0_3 10_4$ $3_1 2_3 12_4$
$4_1 7_3 0_4$ $4_1 6_3 2_4$ $4_1 8_3 4_4$ $4_1 1_3 6_4$ $4_1 3_3 8_4$ $4_1 5_3 10_4$ $4_1 0_3 12_4$ $5_1 0_3 1_4$ $5_1 9_3 2_4$
$5_1 1_3 3_4$ $5_1 4_3 5_4$ $5_1 6_3 7_4$ $5_1 7_3 9_4$ $5_1 8_3 11_4$ $6_1 7_3 4_4$ $6_1 0_3 6_4$ $6_1 2_3 7_4$ $6_1 5_3 8_4$
$6_1 8_3 9_4$ $6_1 9_3 10_4$ $6_1 1_3 11_4$

type ACC: $0_1 0_3 6_3 \quad 1_1 7_3 9_3 \quad 2_1 7_3 8_3 \quad 3_1 1_3 9_3 \quad 4_1 2_3 9_3 \quad 5_1 2_3 3_3 \quad 6_1 3_3 4_3$

type BCD: $0_2 3_3 7_4 \quad 0_2 7_3 6_4 \quad 0_2 5_3 9_4 \quad 0_2 6_3 8_4 \quad 0_2 9_3 11_4 \quad 0_2 2_3 10_4 \quad 0_2 1_3 12_4 \quad 1_2 7_3 3_4$
$1_2 2_3 1_4 \quad 1_2 0_3 9_4 \quad 1_2 4_3 8_4 \quad 1_2 6_3 10_4 \quad 1_2 3_3 11_4 \quad 1_2 8_3 12_4 \quad 2_2 1_3 0_4 \quad 2_2 3_3 12_4$
$2_2 0_3 11_4 \quad 2_2 5_3 4_4 \quad 2_2 7_3 10_4 \quad 2_2 8_3 1_4 \quad 2_2 9_3 6_4 \quad 3_2 0_3 2_4 \quad 3_2 2_3 0_4 \quad 3_2 4_3 1_4 \quad 3_2 5_3 7_4$
$3_2 6_3 9_4 \quad 3_2 8_3 3_4 \quad 3_2 9_3 8_4 \quad 4_2 0_3 5_4 \quad 4_2 9_3 4_4 \quad 4_2 3_3 0_4 \quad 4_2 5_3 11_4 \quad 4_2 6_3 3_4 \quad 4_2 7_3 2_4$
$4_2 1_3 10_4 \quad 5_2 0_3 4_4 \quad 5_2 1_3 2_4 \quad 5_2 9_3 12_4 \quad 5_2 4_3 7_4 \quad 5_2 6_3 0_4 \quad 5_2 3_3 6_4 \quad 5_2 8_3 5_4 \quad 6_2 1_3 8_4$
$6_2 4_3 4_4 \quad 6_2 3_3 2_4 \quad 6_2 5_3 6_4 \quad 6_2 2_3 9_4 \quad 6_2 8_3 7_4 \quad 6_2 9_3 5_4$

type BCC: $0_2 0_3 4_3 \quad 1_2 5_3 9_3 \quad 2_2 4_3 6_3 \quad 3_2 1_3 7_3 \quad 4_2 2_3 8_3 \quad 5_2 2_3 7_3 \quad 6_2 0_3 7_3$

type CCD: $8_3 9_3 0_4 \quad 5_3 7_3 1_4 \quad 6_3 9_3 1_4 \quad 4_3 8_3 2_4 \quad 2_3 5_3 3_4 \quad 4_3 9_3 3_4 \quad 1_3 2_3 4_4 \quad 3_3 7_3 5_4 \quad 2_3 6_3 5_4$
$2_3 4_3 6_4 \quad 0_3 1_3 7_4 \quad 0_3 2_3 8_4 \quad 1_3 3_3 9_4 \quad 3_3 8_3 10_4 \quad 4_3 7_3 11_4 \quad 6_3 7_3 12_4$

type CCC: $0_3 3_3 9_3 \quad 1_3 6_3 8_3 \quad 0_3 5_3 8_3 \quad 3_3 5_3 6_3 \quad 1_3 4_3 5_3$

type BBB: put an STS(7) on the points of cell B.

The long lines are $0_1 1_1 \cdots 6_1$ and $0_4 1_4 \cdots 12_4$. In order to show that $39 \in LS_i(3,7^*,13^*)$ form the partition $\pi(1^1, 6^1, 8^1, 12^2)$ and construct the short lines of

type ABC: $0_1 0_2 8_3 \quad 0_1 3_2 2_3 \quad 0_1 5_2 9_3 \quad 0_1 7_2 6_3 \quad 1_1 1_2 0_3 \quad 1_1 4_2 3_3 \quad 1_1 6_2 5_3 \quad 1_1 0_2 1_3 \quad 2_1 2_2 5_3$
$2_1 5_2 4_3 \quad 2_1 7_2 8_3 \quad 2_1 1_2 11_3 \quad 3_1 3_2 1_3 \quad 3_1 6_2 0_3 \quad 3_1 0_2 3_3 \quad 3_1 2_2 8_3 \quad 4_1 4_2 4_3 \quad 4_1 7_2 10_3$
$4_1 1_2 7_3 \quad 4_1 3_2 9_3 \quad 5_1 5_2 1_3 \quad 5_1 0_2 5_3 \quad 5_1 2_2 4_3 \quad 5_1 4_2 7_3$

type ABD: $0_1 1_2 0_4 \quad 0_1 2_2 2_4 \quad 0_1 4_2 4_4 \quad 0_1 6_2 6_4 \quad 1_1 2_2 1_4 \quad 1_1 3_2 3_4 \quad 1_1 5_2 5_4 \quad 1_1 7_2 7_4 \quad 2_1 0_2 4_4$
$2_1 3_2 6_4 \quad 2_1 4_2 8_4 \quad 2_1 6_2 10_4 \quad 3_1 1_2 5_4 \quad 3_1 4_2 7_4 \quad 3_1 5_2 9_4 \quad 3_1 7_2 11_4 \quad 4_1 0_2 0_4 \quad 4_1 2_2 8_4$
$4_1 5_2 10_4 \quad 4_1 6_2 2_4 \quad 5_1 1_2 1_4 \quad 5_1 3_2 9_4 \quad 5_1 6_2 11_4 \quad 5_1 7_2 3_4$

type BCD: $0_2 9_3 11_4 \quad 0_2 2_3 1_4 \quad 0_2 4_3 3_4 \quad 0_2 7_3 7_4 \quad 1_2 1_3 2_4 \quad 1_2 2_3 6_4 \quad 1_2 9_3 8_4 \quad 2_2 11_3 4_4 \quad 2_2 7_3 6_4$
$2_2 0_3 10_4 \quad 3_2 8_3 0_4 \quad 3_2 6_3 5_4 \quad 3_2 3_3 7_4 \quad 4_2 6_3 9_4 \quad 4_2 10_3 1_4 \quad 4_2 11_3 11_4 \quad 5_2 0_3 0_4$
$5_2 5_3 2_4 \quad 5_2 10_3 3_4 \quad 6_2 1_3 8_4 \quad 6_2 8_3 9_4 \quad 7_2 3_3 10_4 \quad 7_2 5_3 4_4 \quad 7_2 4_3 5_4$

type ACD: $0_1 11_3 1_4 \quad 0_1 5_3 3_4 \quad 0_1 1_3 7_4 \quad 0_1 3_3 5_4 \quad 0_1 7_3 9_4 \quad 0_1 0_3 11_4 \quad 0_1 10_3 8_4 \quad 0_1 4_3 10_4$
$1_1 4_3 4_4 \quad 1_1 2_3 0_4 \quad 1_1 11_3 2_4 \quad 1_1 10_3 9_4 \quad 1_1 8_3 8_4 \quad 1_1 7_3 10_4 \quad 1_1 6_3 11_4 \quad 1_1 9_3 6_4$
$2_1 0_3 1_4 \quad 2_1 9_3 2_4 \quad 2_1 3_3 0_4 \quad 2_1 2_3 3_4 \quad 2_1 10_3 11_4 \quad 2_1 7_3 5_4 \quad 2_1 6_3 7_4 \quad 2_1 1_3 9_4 \quad 3_1 9_3 4_4$
$3_1 2_3 2_4 \quad 3_1 4_3 1_4 \quad 3_1 10_3 6_4 \quad 3_1 7_3 8_4 \quad 3_1 5_3 0_4 \quad 3_1 6_3 10_4 \quad 3_1 11_3 3_4 \quad 4_1 8_3 3_4$
$4_1 2_3 4_4 \quad 4_1 5_3 5_4 \quad 4_1 0_3 7_4 \quad 4_1 3_3 11_4 \quad 4_1 1_3 1_4 \quad 4_1 6_3 6_4 \quad 4_1 11_3 9_4 \quad 5_1 8_3 5_4 \quad 5_1 0_3 6_4$
$5_1 3_3 2_4 \quad 5_1 2_3 8_4 \quad 5_1 6_3 4_4 \quad 5_1 11_3 7_4 \quad 5_1 9_3 10_4 \quad 5_1 10_3 0_4$

type BBD: $2_2 4_2 0_4 \quad 6_2 7_2 0_4 \quad 3_2 6_2 1_4 \quad 5_2 7_2 1_4 \quad 0_2 3_2 2_4 \quad 4_2 7_2 2_4 \quad 1_2 2_2 3_4 \quad 4_2 6_2 3_4 \quad 1_2 3_2 4_4$

$5_26_24_4 \quad 0_24_25_4 \quad 2_26_25_4 \quad 0_27_26_4 \quad 4_25_26_4 \quad 1_26_27_4 \quad 2_25_27_4 \quad 0_25_28_4 \quad 3_27_28_4$
$0_22_29_4 \quad 1_27_29_4 \quad 0_21_210_4 \quad 3_24_210_4 \quad 1_25_211_4 \quad 2_23_211_4$

type ∞BB: $\infty 0_26_2 \quad \infty 1_24_2 \quad \infty 2_27_2 \quad \infty 3_25_2$

type BCC: $0_26_311_3 \quad 0_20_310_3 \quad 1_24_35_3 \quad 1_23_36_3 \quad 1_28_310_3 \quad 2_21_39_3 \quad 2_23_310_3 \quad 2_22_36_3$
$3_20_34_3 \quad 3_25_37_3 \quad 3_210_311_3 \quad 4_20_31_3 \quad 4_25_39_3 \quad 4_22_38_3 \quad 5_22_37_3 \quad 5_26_38_3$
$5_23_311_3 \quad 6_23_39_3 \quad 6_22_311_3 \quad 6_26_310_3 \quad 6_24_37_3 \quad 7_20_37_3 \quad 7_21_32_3 \quad 7_29_311_3$

type CCD: $1_34_30_4 \quad 7_311_30_4 \quad 6_39_30_4 \quad 5_36_31_4 \quad 3_37_31_4 \quad 8_39_31_4 \quad 0_36_32_4 \quad 4_310_32_4 \quad 7_38_32_4$
$7_39_33_4 \quad 1_36_33_4 \quad 0_33_33_4 \quad 0_38_34_4 \quad 7_310_34_4 \quad 1_33_34_4 \quad 0_39_35_4 \quad 2_310_35_4$
$1_311_35_4 \quad 3_34_36_4 \quad 5_311_36_4 \quad 1_38_36_4 \quad 2_39_37_4 \quad 5_310_37_4 \quad 4_38_37_4 \quad 3_35_38_4$
$4_36_38_4 \quad 0_311_38_4 \quad 0_35_39_4 \quad 2_33_39_4 \quad 4_39_39_4 \quad 8_311_310_4 \quad 2_35_310_4 \quad 1_310_310_4$
$2_34_311_4 \quad 5_38_311_4 \quad 1_37_311_4$

type ∞CC: $\infty 3_38_3 \quad \infty 0_32_3 \quad \infty 1_35_3 \quad \infty 4_311_3 \quad \infty 6_37_3 \quad \infty 9_310_3$

The long lines are $0_11_1 \cdots 5_1\infty$ and $0_41_4 \cdots 11_4\infty$. Next, by direct construction, we show that $39 \in LS_d(3,7^*,13^*)$, by forming a partition $\pi(7^2,12^1,13^1)$ and constructing short lines of

type ACD: $0_10_30_4 \quad 0_19_32_4 \quad 0_12_34_4 \quad 0_13_36_4 \quad 0_110_38_4 \quad 0_15_310_4 \quad 0_16_312_4 \quad 0_17_31_4$
$0_14_33_4 \quad 0_18_35_4 \quad 1_11_31_4 \quad 1_12_33_4 \quad 1_17_35_4 \quad 1_110_37_4 \quad 1_15_39_4 \quad 1_16_311_4 \quad 1_13_30_4$
$1_18_32_4 \quad 1_19_34_4 \quad 1_14_36_4 \quad 2_12_32_4 \quad 2_17_34_4 \quad 2_110_36_4 \quad 2_15_38_4 \quad 2_16_310_4$
$2_13_312_4 \quad 2_14_31_4 \quad 2_18_33_4 \quad 2_11_35_4 \quad 2_111_37_4 \quad 3_13_33_4 \quad 3_110_35_4 \quad 3_15_37_4$
$3_16_39_4 \quad 3_17_311_4 \quad 3_18_30_4 \quad 3_10_32_4 \quad 3_14_34_4 \quad 3_111_36_4 \quad 3_19_38_4 \quad 4_110_34_4$
$4_15_36_4 \quad 4_16_38_4 \quad 4_17_310_4 \quad 4_14_312_4 \quad 4_18_31_4 \quad 4_19_33_4 \quad 4_111_32_4 \quad 4_10_37_4$
$4_11_39_4 \quad 5_15_35_4 \quad 5_16_37_4 \quad 5_17_39_4 \quad 5_18_311_4 \quad 5_19_30_4 \quad 5_14_32_4 \quad 5_111_34_4 \quad 5_10_36_4$
$5_11_38_4 \quad 5_12_310_4 \quad 6_16_36_4 \quad 6_17_38_4 \quad 6_14_310_4 \quad 6_18_312_4 \quad 6_19_31_4 \quad 6_111_33_4$
$6_10_35_4 \quad 6_110_32_4 \quad 6_12_39_4 \quad 6_13_311_4$

type ACC: $0_11_311_3 \quad 1_10_311_3 \quad 2_10_39_3 \quad 3_11_32_3 \quad 4_12_33_3 \quad 5_13_310_3 \quad 6_11_35_3$

type ABD: $0_16_27_4 \quad 0_12_29_4 \quad 0_11_211_4 \quad 1_16_28_4 \quad 1_13_210_4 \quad 1_10_212_4 \quad 2_13_20_4 \quad 2_15_29_4$
$2_12_211_4 \quad 3_13_21_4 \quad 3_14_210_4 \quad 3_11_212_4 \quad 4_16_20_4 \quad 4_13_25_4 \quad 4_14_211_4 \quad 5_10_21_4$
$5_12_23_4 \quad 5_15_212_4 \quad 6_10_20_4 \quad 6_11_27_4 \quad 6_15_24_4$

type ABB: $0_10_24_2 \quad 0_13_25_2 \quad 1_11_25_2 \quad 1_12_24_2 \quad 2_10_26_2 \quad 2_11_24_2 \quad 3_10_22_2 \quad 3_15_26_2 \quad 4_12_25_2$
$4_10_21_2 \quad 5_13_24_2 \quad 5_11_26_2 \quad 6_14_26_2 \quad 6_12_23_2$

type BCD: $0_24_39_4$ $0_27_37_4$ $0_29_35_4$ $0_22_311_4$ $0_26_34_4$ $0_21_32_4$ $0_211_310_4$ $0_20_33_4$
$1_22_31_4$ $1_23_39_4$ $1_210_310_4$ $1_27_33_4$ $1_21_30_4$ $1_28_34_4$ $1_211_35_4$ $1_29_36_4$
$2_22_35_4$ $2_27_32_4$ $2_210_30_4$ $2_25_31_4$ $2_23_34_4$ $2_28_310_4$ $2_24_37_4$ $2_21_36_4$ $3_21_33_4$
$3_210_311_4$ $3_25_312_4$ $3_20_34_4$ $3_22_36_4$ $3_28_37_4$ $4_21_37_4$ $4_26_32_4$ $4_25_34_4$
$4_27_312_4$ $4_23_35_4$ $4_210_33_4$ $4_211_38_4$ $4_28_36_4$ $4_20_39_4$ $4_24_30_4$ $5_21_310_4$
$5_28_38_4$ $5_25_33_4$ $5_29_311_4$ $5_23_32_4$ $5_22_37_4$ $5_211_30_4$ $5_26_35_4$ $6_23_31_4$ $6_25_32_4$
$6_27_36_4$ $6_29_310_4$ $6_26_33_4$ $6_24_35_4$ $6_21_34_4$ $6_211_311_4$

type BBD: $4_25_21_4$ $1_23_22_4$ $0_25_26_4$ $0_23_28_4$ $1_22_28_4$ $3_26_29_4$ $2_26_212_4$

type BCC: $0_23_38_3$ $0_25_310_3$ $1_20_34_3$ $1_25_36_3$ $2_20_36_3$ $2_29_311_3$ $3_26_39_3$ $3_23_37_3$
$3_24_311_3$ $4_22_39_3$ $5_20_37_3$ $5_24_310_3$ $6_20_38_3$ $6_22_310_3$

type CCD: $2_36_30_4$ $5_37_30_4$ $0_310_31_4$ $6_311_31_4$ $3_39_37_4$ $0_32_38_4$ $3_34_38_4$ $8_39_39_4$
$10_311_39_4$ $0_33_310_4$ $0_35_311_4$ $1_34_311_4$ $0_31_312_4$ $9_310_312_4$ $2_311_312_4$

type CCC: $2_35_38_3$ $2_34_37_3$ $3_35_311_3$ $1_33_36_3$ $1_37_39_3$ $4_35_39_3$ $4_36_38_3$ $6_37_310_3$
$1_38_310_3$ $7_38_311_3$

The long lines are $0_11_1 \cdots 6_1$ and $0_41_4 \cdots 12_4$. Finally, $43 \in LS_i(3, 7^*, 13^*)$ since we can form a partition $\pi(1^1, 6^5, 12^1)$ and apply Theorem 1.24(b). Now $43 \in LS_d(3, 7^*, 13^*)$ by first applying Theorem 1.33 to construct an STS(43) which contains disjoint subsystems STS(15) and STS(13). By Theorem 1.7, embed an STS(7) into an STS(15), and replace the subsystem STS(7) by a line of size seven. Replace the STS(13) by a line of size thirteen.

(c) Form the partition $\pi(1^1, 6^4, 14^1)$ and apply Theorem 1.24(b) to show that $39 \in LS_i(3, 7^*, 15^*)$. A direct construction is needed to demonstrate that $39 \in LS_d(3, 7^*, 15^*)$. Form a partition $\pi(7^2, 10^1, 15^1)$ and construct the short lines of

type ABD: $0_10_20_4$ $0_1(i+1)_2(2i+3)_4(i = 0,1,...,5)$ $1_1(i+2)_2(2i+4)_4(i = 0,1,...,5)$
$2_1(i+3)_2(2i+5)_4(i = 0,1,...,5)$ $3_14_26_4$ $3_1(i+5)_2(2i+8)_4(i = 0,1,2,3)$
$3_12_21_4$ $4_1(i+5)_2(2i+7)_4(i = 0,1,...,4)$ $4_13_22_4$ $5_1(i+6)_2(2i+8)_4$
$(i = 0,1,2,3)$ $5_13_21_4$ $5_14_23_4$ $6_10_29_4$ $6_11_211_4$ $6_12_213_4$ $6_13_20_4$ $6_14_22_4$
$6_15_24_4$ $i_1i_2i_4(i = 1,2,...,6)$

type ACD: $0_18_31_4$ $0_10_32_4$ $0_13_34_4$ $0_14_36_4$ $0_19_38_4$ $0_11_310_4$ $0_17_312_4$ $0_16_314_4$
$1_16_30_4$ $1_13_32_4$ $1_11_33_4$ $1_19_35_4$ $1_10_37_4$ $1_17_39_4$ $1_14_311_4$ $1_12_313_4$ $2_17_31_4$

$2_{1}2_{3}3_{4}$ $2_{1}0_{3}4_{4}$ $2_{1}3_{3}6_{4}$ $2_{1}6_{3}8_{4}$ $2_{1}4_{3}10_{4}$ $2_{1}8_{3}12_{4}$ $2_{1}5_{3}14_{4}$ $3_{1}2_{3}0_{4}$
$3_{1}1_{3}2_{4}$ $3_{1}5_{3}4_{4}$ $3_{1}0_{3}5_{4}$ $3_{1}7_{3}7_{4}$ $3_{1}9_{3}9_{4}$ $3_{1}6_{3}11_{4}$ $3_{1}3_{3}13_{4}$ $4_{1}9_{3}1_{4}$ $4_{1}3_{3}3_{4}$
$4_{1}7_{3}5_{4}$ $4_{1}8_{3}6_{4}$ $4_{1}5_{3}8_{4}$ $4_{1}0_{3}10_{4}$ $4_{1}6_{3}12_{4}$ $4_{1}4_{3}14_{4}$ $5_{1}9_{3}0_{4}$ $5_{1}5_{3}2_{4}$
$5_{1}6_{3}4_{4}$ $5_{1}0_{3}6_{4}$ $5_{1}8_{3}7_{4}$ $5_{1}3_{3}9_{4}$ $5_{1}2_{3}11_{4}$ $5_{1}1_{3}13_{4}$ $6_{1}5_{3}1_{4}$ $6_{1}0_{3}3_{4}$ $6_{1}3_{3}5_{4}$
$6_{1}1_{3}7_{4}$ $6_{1}7_{3}8_{4}$ $6_{1}6_{3}10_{4}$ $6_{1}4_{3}12_{4}$ $6_{1}2_{3}14_{4}$

type ACC: $0_{1}2_{3}5_{3}$ $1_{1}5_{3}8_{3}$ $2_{1}1_{3}9_{3}$ $3_{1}4_{3}8_{3}$ $4_{1}1_{3}2_{3}$ $5_{1}4_{3}7_{3}$ $6_{1}8_{3}9_{3}$

type BCD: $0_{2}2_{3}1_{4}$ $0_{2}9_{3}2_{4}$ $0_{2}7_{3}3_{4}$ $0_{2}4_{3}4_{4}$ $0_{2}8_{3}5_{4}$ $0_{2}1_{3}6_{4}$ $0_{2}5_{3}7_{4}$ $0_{2}0_{3}8_{4}$ $1_{2}4_{3}2_{4}$
$1_{2}2_{3}4_{4}$ $1_{2}1_{3}5_{4}$ $1_{2}5_{3}6_{4}$ $1_{2}3_{3}7_{4}$ $1_{2}8_{3}8_{4}$ $1_{2}0_{3}9_{4}$ $1_{2}9_{3}10_{4}$ $2_{2}4_{3}3_{4}$ $2_{2}6_{3}6_{4}$
$2_{2}9_{3}7_{4}$ $2_{2}1_{3}8_{4}$ $2_{2}5_{3}9_{4}$ $2_{2}8_{3}10_{4}$ $2_{2}7_{3}11_{4}$ $2_{2}3_{3}12_{4}$ $3_{2}1_{3}4_{4}$ $3_{2}4_{3}8_{4}$
$3_{2}6_{3}9_{4}$ $3_{2}2_{3}10_{4}$ $3_{2}0_{3}11_{4}$ $3_{2}9_{3}12_{4}$ $3_{2}8_{3}13_{4}$ $3_{2}7_{3}14_{4}$ $4_{2}8_{3}0_{4}$ $4_{2}0_{3}1_{4}$
$4_{2}6_{3}5_{4}$ $4_{2}3_{3}10_{4}$ $4_{2}5_{3}11_{4}$ $4_{2}1_{3}12_{4}$ $4_{2}7_{3}13_{4}$ $4_{2}9_{3}14_{4}$ $5_{2}5_{3}0_{4}$ $5_{2}4_{3}1_{4}$
$5_{2}8_{3}2_{4}$ $5_{2}6_{3}3_{4}$ $5_{2}7_{3}6_{4}$ $5_{2}2_{3}12_{4}$ $5_{2}9_{3}13_{4}$ $5_{2}1_{3}14_{4}$ $6_{2}7_{3}0_{4}$ $6_{2}3_{3}1_{4}$
$6_{2}6_{3}2_{4}$ $6_{2}8_{3}3_{4}$ $6_{2}9_{3}4_{4}$ $6_{2}2_{3}5_{4}$ $6_{2}4_{3}7_{4}$ $6_{2}0_{3}14_{4}$

type BCC: $0_{2}3_{3}6_{3}$ $1_{2}6_{3}7_{3}$ $2_{2}0_{3}2_{3}$ $3_{2}3_{3}5_{3}$ $4_{2}2_{3}4_{3}$ $5_{2}0_{3}3_{3}$ $6_{2}1_{3}5_{3}$

type CCD: $0_{3}1_{3}0_{4}$ $3_{3}4_{3}0_{4}$ $1_{3}6_{3}1_{4}$ $2_{3}7_{3}2_{4}$ $5_{3}9_{3}3_{4}$ $7_{3}8_{3}4_{4}$ $4_{3}5_{3}5_{4}$ $2_{3}9_{3}6_{4}$ $2_{3}6_{3}7_{4}$
$2_{3}3_{3}8_{4}$ $1_{3}4_{3}9_{4}$ $2_{3}8_{3}9_{4}$ $5_{3}7_{3}10_{4}$ $3_{3}9_{3}11_{4}$ $1_{3}8_{3}11_{4}$ $0_{3}5_{3}12_{4}$ $0_{3}4_{3}13_{4}$
$5_{3}6_{3}13_{4}$ $3_{3}8_{3}14_{4}$

type CCC: $0_{3}7_{3}9_{3}$ $0_{3}6_{3}8_{3}$ $1_{3}3_{3}7_{3}$ $4_{3}6_{3}9_{3}$

type BBB: put an STS(7) on the points of cell B.

The long lines are $0_1 1_1 \cdots 6_1$ and $0_4 1_4 \cdots 14_4$. Form the partition $\pi(1^1, 14^3)$ and apply Corollary 1.25 where $g_1 = 7$, $g = 14$, $t = 3$ and $x = 0$ to show that $43 \in LS(3, 7^*, 15^*)$. Next, $45 \in LS(3, 7^*, 15^*)$ by forming the partition $\pi(1^1, 2^1, 14^3)$ and applying Corollary 1.25 where $g_1 = 7$, $g = 14$, $t = 3$ and $x = 2$. Form the partition $\pi(1^1, 6^1, 14^3)$ and apply Corollary 1.25 where $g_1 = 7$, $g = 14$, $t = 3$ and $x = 6$ to prove that $49 \in LS(3, 7^*, 15^*)$. Finally, $55 \in LS(3, 7^*, 15^*)$ by forming the partition $\pi(1^1, 12^1, 14^3)$ and applying Corollary 1.25 where $g_1 = 7$, $g = 14$, $t = 3$ and $x = 12$.

(d) Form the partition $\pi(1^1, x^1, 18^3)$ and apply Corollary 1.25 where $g_1 = 7$, $g = 18$, $t = 3$ and $x = 0, 2, 8$ or 12.

(e) Form the partition $\pi(1^1, (w + 4t - 5)^3, (w - 1)^1)$ and apply Corollary 1.25 where

$g_1 = 6t + 7$, $g = w + 4t - 5$, $t = 3$ and $x = w - 1$.

(f) Form the partition $\pi(1^1, x^1, 20^3)$ and apply Corollary 1.25 where $g_1 = 6t + 7$, $g = 20$, $t = 3$ and $x = 0, 6, 8$ or 12.

(g) Form the partition $\pi(1^1, (w + 4t - 7)^3, (w - 1)^1)$ and apply Corollary 1.25 where $g_1 = 6t + 7$, $g = w + 4t - 7$, $t = 3$ and $x = w - 1$.

The following lemma outlines the construction of AULSs where the two long lines intersect, and one has size $w \equiv 5 \pmod 6$.

Lemma 2.35 If $w = 5 + 6r$ ($r > 0$) and $w > 6t + 7$, where $2w + 6t + 7 \leq v < 4w + 12t + 15$ and $k \equiv (3 + 3r) \pmod{t + 1}$, $1 \leq k \leq 2 + 3r$, then $4w + 12t + 15 - 6k \in LS_i(3, (6t + 7)^*, w^*)$.

Proof: Form the partition $\pi(1^1, (6t + 6)^{(w+4t+5-2k)/(2t+2)}, (w - 1)^1)$ and apply Theorem 1.24(b).

Corollary 2.36 For $v \equiv 5 \pmod 6$ and $w \equiv 5 \pmod 6$, $v \in LS_i(3, 7^*, w^*)$ for all $v \geq 4w + 15$.

Proof: This follows from Lemma 2.35 and Lemma 1.39.

The next two results give constructions of AULSs with two disjoint long lines, one having size $w \equiv 5 \pmod 6$.

Lemma 2.37 Let $w \equiv 5 \pmod 6$.

(a)If $0 \leq t \leq (w-13)/8$,and $t \equiv 0,2,3$ or $5 \pmod 6$,then $4w+12t+3 \in LS_d(3,(6t+7)^*,w^*)$.

(b)If $0 \leq t \leq (w-11)/8$ and $t \equiv 0,1,3$ or $4 \pmod 6$,then $4w+12t+9 \in LS_d(3,(6t+7)^*,w^*)$.

Proof:

(a) Form the partition $\pi(1^1, (w - 1)^1, (w + 4t + 1)^3)$ and apply Theorem 1.24(b) with $g_1 = 6t + 7$, $g = w + 4t + 1$, $t = 3$ and $x = w - 1$.

(b) The arguments are as in (a), with partition $\pi(1^1, (w - 1)^1, (w + 4t + 3)^3)$.

Lemma 2.38 Let $w \equiv 5 \pmod 6$.
(a) If $0 \le t \le (w-17)/8$ and $t \equiv 0,1,3$ or $4 \pmod 6$ then $4w+12t-9 \in LS_d(3,(6t+7)^*,w^*)$.
(b) If $0 \le t \le (w-19)/8$ and $t \equiv 0,2,3$ or $5 \pmod 6$ then $4w+12t-15 \in LS_d(3,(6t+7)^*,w^*)$.

Proof: For (a) and (b), form either the partition $\pi(1^1, (w+4t-3)^3, (w-1)^1)$ or $\pi(1^1, (w+4t-5)^3, (w-1)^1)$ and follow the same arguments as in Lemma 2.33.

The final results regard the construction of AULSs whose long lines are both of sizes congruent to 1 (mod 6) and intersect. We utilize Wilson's fundamental construction [W3] which was described generally in §1.3.

Lemma 2.39 Let $m \ge 5$ and t be a nonnegative integer such that $1 \le t < m$ and $6m \not\equiv 0 (\mathrm{mod}\ 6m - 6t)$. Then $36m - 12t + 1 \in LS_i(3, (6m-6t+1)^*, (6m+1)^*)$.

Proof: There are two parts to this proof.
Suppose that TD(6, m) exists, then delete t points from two of its groups to obtain a {4,5,6}-GDD of type $m^4(m-t)^2$. Put a weight of six on every point, apply FC (cf. §1.3), giving a {3}-GDD of type $(6m)^4(6m-6t)^2$. Add a point ∞, and then apply Theorem 1.24(b).
Otherwise, for $m \in \{6,10,14,18,22,26,30,34,42\}$, delete a block from TD(6, m + 1) to obtain a {5,6}-GDD of type m^6. Delete t points from two groups of the {5,6}-GDD to obtain a {3,4,5,6}-GDD of type $m^4(m-t)^2$. Put a weight of six on every point of the GDD, apply FC, resulting in a {3}-GDD of type $(6m)^4(6m-6t)^2$. Add a point ∞, and apply Theorem 1.24(b).

Corollary 2.40 $61 \in LS_i(3, 7^*, 13^*)$, $109 \in LS_i(3, 7^*, 25^*)$.
Proof: Form the partition $\pi(1^1, 6^8, 12^1)$ and apply Theorem 1.24(b) to prove that $61 \in LS_i(3, 7^*, 13^*)$. Similarly, $109 \in LS_i(3, 7^*, 25^*)$ by forming the partition $\pi(1^1, 6^{14}, 24^1)$ and applying Theorem 1.24(b).

Lemma 2.41 Let $m \geq 4$ ($m \neq 6, 10$) and t be a nonnegative integer such that $1 \leq t < m$, where $6m \neq 0 (\text{mod } 6m\text{-}6t)$. Then $30m\text{-}12t + 1 \in LS_i(3,(6m\text{-}6t+1)^*,(6m+1)^*)$.

Proof: Since TD(5, m) exists, delete t points from two of its groups to obtain a {3,4,5}-GDD of type $m^3(m - t)^2$. Put a weight of six on every point, apply FC, to obtain a {3}-GDD of type $(6m)^3(6m - 6t)^2$. Add a point ∞ and apply Theorem 1.24(b).

As in §2.2, our goal is to complete the spectrum for AULSs with one long line of size seven, and the other long line of size nine, thirteen or fifteen. First we will fill in any gaps by giving constructions of AULSs with orders that were not covered by any of the previous lemmas or corollaries.

Lemma 2.42 $33 \in LS(3,7^*,9^*)$; $45,49 \in LS(3,7^*,13^*)$; $33 \in LS_i(3,7^*,13^*)$; $51 \in LS(3,7^*,15^*)$.

Proof: First, form the partition $\pi(7^2, 9^1, 10^1)$ where cells A, B are the sets $Z_7 \times \{i\}$ ($i = 1,2$), cell C is the set $Z_{10} \times \{3\}$, and cell D is the set $Z_9 \times \{4\}$ and construct short lines of

type ABC: $0_10_21_3$ $0_11_23_3$ $1_11_20_3$ $1_12_29_3$ $2_12_22_3$ $2_13_24_3$ $3_13_25_3$ $3_14_27_3$ $4_14_24_3$
$4_15_27_3$ $5_15_25_3$ $5_16_28_3$ $6_16_23_3$ $6_10_24_3$

type ABD: $0_12_20_4$ $0_13_22_4$ $0_14_26_4$ $0_15_28_4$ $0_16_21_4$ $1_10_21_4$ $1_13_23_4$ $1_14_25_4$ $1_15_27_4$
$1_16_20_4$ $2_10_24_4$ $2_11_26_4$ $2_14_28_4$ $2_15_20_4$ $2_16_22_4$ $3_10_25_4$ $3_11_27_4$ $3_12_23_4$
$3_15_21_4$ $3_16_24_4$ $4_10_28_4$ $4_11_20_4$ $4_12_22_4$ $4_13_24_4$ $4_16_26_4$ $5_10_27_4$ $5_11_23_4$
$5_12_25_4$ $5_13_21_4$ $5_14_22_4$ $6_11_24_4$ $6_12_28_4$ $6_13_26_4$ $6_14_20_4$ $6_15_22_4$

type ACD: $0_10_33_4$ $0_19_34_4$ $0_17_35_4$ $0_15_37_4$ $1_17_32_4$ $1_18_34_4$ $1_16_36_4$ $1_12_38_4$ $2_16_31_4$
$2_13_33_4$ $2_15_35_4$ $2_17_37_4$ $3_13_30_4$ $3_10_32_4$ $3_18_36_4$ $3_14_38_4$ $4_12_31_4$ $4_11_33_4$
$4_10_35_4$ $4_16_37_4$ $5_17_30_4$ $5_13_34_4$ $5_14_36_4$ $5_11_38_4$ $6_18_31_4$ $6_15_33_4$ $6_12_35_4$
$6_11_37_4$

type ACC: $0_12_36_3$ $0_14_38_3$ $1_13_34_3$ $1_11_35_3$ $2_10_38_3$ $2_11_39_3$ $3_11_36_3$ $3_12_39_3$ $4_18_39_3$

$4_1 3_3 5_3$ $5_1 0_3 2_3$ $5_1 6_3 9_3$ $6_1 7_3 9_3$ $6_1 0_3 6_3$

type BCD: $0_2 5_3 0_4$ $0_2 9_3 2_4$ $0_2 6_3 3_4$ $0_2 0_3 6_4$ $1_2 9_3 1_4$ $1_2 4_3 2_4$ $1_2 1_3 5_4$ $1_2 6_3 8_4$
$2_2 1_3 1_4$ $2_2 0_3 4_4$ $2_2 5_3 6_4$ $2_2 4_3 7_4$ $3_2 2_3 0_4$ $3_2 9_3 7_4$ $3_2 3_3 5_4$ $3_2 8_3 8_4$ $4_2 3_3 1_4$
$4_2 8_3 3_4$ $4_2 6_3 4_4$ $4_2 0_3 7_4$ $5_2 9_3 3_4$ $5_2 1_3 4_4$ $5_2 8_3 5_4$ $5_2 2_3 6_4$ $6_2 7_3 3_4$ $6_2 6_3 5_4$
$6_2 2_3 7_4$ $6_2 9_3 8_4$

type BCC: $0_2 2_3 3_3$ $0_2 7_3 8_3$ $1_2 2_3 7_3$ $1_2 5_3 8_3$ $2_2 3_3 7_3$ $2_2 6_3 8_3$ $3_2 0_3 1_3$ $3_2 6_3 7_3$
$4_2 1_3 2_3$ $4_2 5_3 9_3$ $5_2 0_3 4_3$ $5_2 3_3 6_3$ $6_2 0_3 5_3$ $6_2 1_3 4_3$

type CCD: $0_3 9_3 0_4$ $4_3 6_3 0_4$ $1_3 8_3 0_4$ $0_3 7_3 1_4$ $4_3 5_3 1_4$ $2_3 8_3 2_4$ $5_3 6_3 2_4$ $1_3 3_3 2_4$
$2_3 4_3 3_4$ $2_3 5_3 4_4$ $4_3 7_3 4_4$ $4_3 9_3 5_4$ $3_3 9_3 6_4$ $1_3 7_3 6_4$ $3_3 8_3 7_4$ $5_3 7_3 8_4$ $0_3 3_3 8_4$

type BBB: put an STS(7) on the points of cell B.

The long lines are $0_1 1_1 \cdots 6_1$ and $0_4 1_4 \cdots 8_4$. Since we can form a partition $\pi(1^1, 6^4, 8^1)$ and apply Theorem 1.24(b), $33 \in LS_i(3, 7^*, 9^*)$. We can prove that $45 \in LS(3, 7^*, 13^*)$ by forming the partition $\pi(1^1, 6^1, 12^2, 14^1)$, where cell A is the set $Z_{14} \times \{1\}$, cell B is the set $Z_6 \times \{2\}$, cells C and D are the sets $Z_{12} \times \{i\}$ $(i = 3, 4)$, and embedding an STS(7) into an STS(15) which contains ∞ and the fourteen points of cell A. Construct short lines of

type ACD: $i_1 i_3 i_4 (i = 0,1,2,4,5,6,8,10)$ $3_1 2_3 3_4$ $7_1 10_3 7_4$ $12_1 9_3 9_4$ $12_1 11_3 11_4$ $2_1 0_3 1_4$
$3_1 1_3 2_4$ $4_1 3_3 3_4$ $5_1 2_3 4_4$ $6_1 4_3 5_4$ $7_1 5_3 6_4$ $13_1 7_3 7_4$ $9_1 7_3 8_4$ $10_1 8_3 9_4$
$13_1 9_3 10_4$ $11_1 7_3 11_4$ $8_1 11_3 0_4$ $4_1 0_3 2_4$ $5_1 1_3 3_4$ $6_1 3_3 4_4$ $7_1 2_3 5_4$ $13_1 4_3 6_4$
$12_1 5_3 7_4$ $10_1 6_3 8_4$ $8_1 7_3 9_4$ $11_1 8_3 10_4$ $9_1 9_3 11_4$ $1_1 7_3 0_4$ $3_1 11_3 1_4$ $6_1 0_3 3_4$
$7_1 1_3 4_4$ $8_1 3_3 5_4$ $8_1 2_3 6_4$ $10_1 4_3 7_4$ $13_1 5_3 8_4$ $9_1 10_3 9_4$ $12_1 7_3 10_4$ $0_1 8_3 11_4$
$2_1 9_3 0_4$ $4_1 7_3 1_4$ $5_1 11_3 2_4$ $8_1 0_3 4_4$ $9_1 1_3 5_4$ $10_1 3_3 6_4$ $11_1 3_3 7_4$ $11_1 4_3 8_4$
$11_1 5_3 9_4$ $0_1 6_3 10_4$ $1_1 6_3 11_4$ $3_1 8_3 0_4$ $5_1 9_3 1_4$ $6_1 10_3 2_4$ $7_1 11_3 3_4$ $10_1 0_3 5_4$
$11_1 1_3 6_4$ $9_1 2_3 7_4$ $12_1 3_3 8_4$ $0_1 4_3 9_4$ $1_1 5_3 10_4$ $2_1 10_3 11_4$ $4_1 6_3 0_4$ $6_1 8_3 1_4$
$7_1 9_3 2_4$ $8_1 10_3 3_4$ $9_1 11_3 4_4$ $12_1 0_3 6_4$ $8_1 1_3 7_4$ $0_1 2_3 8_4$ $1_1 2_3 9_4$ $2_1 4_3 10_4$
$3_1 5_3 11_4$ $5_1 10_3 0_4$ $13_1 10_3 1_4$ $12_1 8_3 2_4$ $10_1 9_3 3_4$ $12_1 6_3 4_4$ $11_1 11_3 5_4$ $1_1 0_3 7_4$
$2_1 1_3 8_4$ $3_1 3_3 9_4$ $4_1 2_3 10_4$ $5_1 4_3 11_4$ $6_1 5_3 0_4$ $7_1 6_3 1_4$ $10_1 7_3 2_4$ $9_1 8_3 3_4$
$11_1 9_3 4_4$ $13_1 6_3 5_4$ $1_1 11_3 6_4$ $3_1 0_3 8_4$ $4_1 1_3 9_4$ $5_1 3_3 10_4$ $6_1 2_3 11_4$ $7_1 4_3 0_4$
$10_1 5_3 1_4$ $8_1 6_3 2_4$ $12_1 4_3 3_4$ $13_1 8_3 4_4$ $0_1 9_3 5_4$ $2_1 7_3 6_4$ $4_1 11_3 7_4$ $5_1 0_3 9_4$
$6_1 1_3 10_4$ $7_1 3_3 11_4$ $13_1 3_3 0_4$ $9_1 4_3 1_4$ $9_1 5_3 2_4$ $11_1 6_3 3_4$ $0_1 7_3 4_4$ $1_1 8_3 5_4$

$3_1 9_3 6_4$ $5_1 6_3 7_4$ $6_1 11_3 8_4$

type ABD: $0_1 1_2 1_4$ $0_1 4_2 2_4$ $0_1 3_2 3_4$ $0_1 2_2 6_4$ $0_1 5_2 7_4$ $1_1 3_2 2_4$ $1_1 4_2 3_4$ $1_1 1_2 4_4$ $1_1 0_2 8_4$
$2_1 2_2 3_4$ $2_1 5_2 4_4$ $2_1 3_2 5_4$ $2_1 4_2 7_4$ $2_1 0_2 9_4$ $3_1 4_2 4_4$ $3_1 2_2 5_4$ $3_1 0_2 7_4$ $3_1 3_2 10_4$
$4_1 1_2 5_4$ $4_1 3_2 6_4$ $4_1 2_2 8_4$ $4_1 5_2 11_4$ $5_1 0_2 6_4$ $5_1 5_2 8_4$ $6_1 1_2 7_4$ $6_1 5_2 9_4$ $7_1 3_2 8_4$
$7_1 4_2 9_4$ $7_1 1_2 10_4$ $8_1 4_2 10_4$ $8_1 2_2 11_4$ $8_1 0_2 1_4$ $9_1 0_2 0_4$ $9_1 2_2 10_4$ $9_1 4_2 6_4$
$10_1 3_2 0_4$ $10_1 2_2 4_4$ $10_1 0_2 11_4$ $11_1 5_2 0_4$ $11_1 4_2 1_4$ $11_1 2_2 2_4$ $12_1 1_2 0_4$ $12_1 4_2 5_4$
$12_1 5_2 1_4$ $13_1 5_2 2_4$ $13_1 3_2 9_4$ $13_1 1_2 3_4$ $13_1 4_2 11_4$

type BCD: $0_2 3_3 2_4$ $0_2 7_3 3_4$ $0_2 5_3 4_4$ $0_2 10_3 5_4$ $0_2 0_3 10_4$ $1_2 10_3 6_4$ $1_2 11_3 9_4$ $1_2 4_3 2_4$
$1_2 9_3 8_4$ $1_2 1_3 11_4$ $2_2 1_3 0_4$ $2_2 8_3 7_4$ $2_2 6_3 9_4$ $2_2 2_3 1_4$ $3_2 10_3 4_4$ $3_2 3_3 1_4$ $3_2 9_3 7_4$
$3_2 0_3 11_4$ $4_2 10_3 8_4$ $4_2 2_3 0_4$ $5_2 11_3 10_4$ $5_2 5_3 3_4$ $5_2 7_3 5_4$ $5_2 8_3 6_4$

type ABC: $13_1 0_2 1_3$ $4_1 0_2 8_3$ $6_1 0_2 9_3$ $0_1 0_2 11_3$ $12_1 0_2 2_3$ $5_1 1_2 8_3$ $11_1 1_2 0_3$ $8_1 1_2 5_3$
$3_1 1_2 7_3$ $2_1 1_2 3_3$ $1_1 2_2 3_3$ $12_1 2_2 10_3$ $6_1 2_2 7_3$ $13_1 2_2 11_3$ $12_1 3_2 1_3$ $8_1 3_2 4_3$
$4_1 4_2 5_3$ $5_1 4_2 7_3$ $7_1 5_2 0_3$ $10_1 5_2 2_3$ $1_1 5_2 10_3$ $9_1 5_2 6_3$ $8_1 5_2 9_3$ $3_1 5_2 4_3$

type ABB: $5_1 2_2 3_2$ $6_1 3_2 4_2$ $7_1 0_2 2_2$ $9_1 1_2 3_2$ $10_1 1_2 4_2$ $11_1 0_2 3_2$

type ∞BB: $\infty 0_2 1_2$ $\infty 2_2 4_2$ $\infty 3_2 5_2$

type BBB: $0_2 4_2 5_2$ $1_2 2_2 5_2$

type ACC: $0_1 1_3 5_3$ $0_1 3_3 10_3$ $1_1 4_3 9_3$ $2_1 5_3 6_3$ $2_1 8_3 11_3$ $3_1 6_3 10_3$ $4_1 9_3 10_3$ $7_1 7_3 8_3$
$9_1 0_3 3_3$ $10_1 1_3 11_3$ $11_1 2_3 10_3$ $13_1 0_3 2_3$

type BCC: $0_2 4_3 6_3$ $1_2 2_3 6_3$ $2_2 0_3 4_3$ $2_2 5_3 9_3$ $3_2 2_3 5_3$ $3_2 6_3 8_3$ $3_2 7_3 11_3$ $4_2 0_3 8_3$ $4_2 3_3 6_3$
$4_2 1_3 4_3$ $4_2 9_3 11_3$ $5_2 1_3 3_3$

type ∞CC: $\infty 0_3 10_3$ $\infty 1_3 6_3$ $\infty 8_3 9_3$ $\infty 2_3 11_3$ $\infty 4_3 5_3$ $\infty 3_3 7_3$

type CCC: $0_3 5_3 7_3$ $0_3 1_3 9_3$ $0_3 6_3 11_3$ $6_3 7_3 9_3$ $1_3 2_3 7_3$ $2_3 3_3 9_3$ $3_3 4_3 11_3$ $2_3 4_3 8_3$ $3_3 5_3 8_3$
$1_3 8_3 10_3$ $4_3 7_3 10_3$ $5_3 10_3 11_3$

One long line is $0_4 1_4 \cdots 11_4 \infty$ and the other long line is formed by replacing the subsystem of STS(15). Next, $49 \in LS(3, 7^*, 13^*)$ by forming a partition $\pi(1^1, 6^1, 12^2, 18^1)$, where cell A is the set $Z_{18} \times \{1\}$, cell B is the set $Z_6 \times \{2\}$, cells C and D are the sets $Z_{12} \times \{i\}$ $(i = 3, 4)$, and embedding an STS(7) into an STS(19) which contains ∞ and the eighteen points of the cell A. Construct short lines of

type ABD: $i_1 0_2 i_4$ $(i+12)_1 1_2 i_4$ $(i+6)_1 2_2 i_4$ $(11\text{-}i)_1 3_2 i_4 (i = 0,1,\ldots,11)$ $(5\text{-}i)_1 4_2 i_4$
$(i = 0,1,\ldots,5)$ $(17\text{-}i)_1 4_2 (i+6)_4 (i = 0,1,\ldots,5)$ $(17\text{-}i)_1 5_2 i_4 (i = 0,1,\ldots,11)$

type ABC: $0_1 2_2 2_3$ $0_1 5_2 1_3$ $1_1 2_2 8_3$ $1_1 5_2 6_3$ $2_1 2_2 4_3$ $2_1 5_2 9_3$ $3_1 2_2 7_3$ $3_1 5_2 10_3$
$4_1 2_2 0_3$ $4_1 5_2 5_3$ $5_1 2_2 5_3$ $5_1 5_2 11_3$ $6_1 1_2 8_3$ $6_1 4_2 7_3$ $7_1 1_2 0_3$ $7_1 4_2 6_3$ $8_1 1_2 6_3$
$8_1 4_2 5_3$ $9_1 1_2 4_3$ $9_1 4_2 1_3$ $10_1 1_2 9_3$ $10_1 4_2 11_3$ $11_1 1_2 1_3$ $11_1 4_2 10_3$ $12_1 0_2 2_3$
$12_1 3_2 10_3$ $13_1 0_2 8_3$ $13_1 3_2 3_3$ $14_1 0_2 3_3$ $14_1 3_2 5_3$ $15_1 0_2 0_3$ $15_1 3_2 4_3$
$16_1 0_2 10_3$ $16_1 3_2 7_3$ $17_1 0_2 9_3$ $17_1 3_2 8_3$

type BCC: $0_2 6_3 11_3$ $0_2 1_3 7_3$ $0_2 4_3 5_3$ $1_2 10_3 11_3$ $1_2 3_3 5_3$ $1_2 2_3 7_3$ $2_2 9_3 11_3$ $2_2 3_3 6_3$
$2_2 1_3 10_3$ $3_2 2_3 11_3$ $3_2 0_3 6_3$ $3_2 1_3 9_3$ $4_2 2_3 8_3$ $4_2 0_3 4_3$ $4_2 3_3 9_3$ $5_2 2_3 3_3$ $5_2 0_3 8_3$
$5_2 4_3 7_3$

type ACD: $0_1 4_3 1_4$ $0_1 0_3 2_4$ $0_1 11_3 3_4$ $0_1 10_3 4_4$ $0_1 9_3 7_4$ $0_1 7_3 8_4$ $0_1 6_3 9_4$ $0_1 5_3 10_4$
$1_1 7_3 0_4$ $1_1 1_3 2_4$ $1_1 9_3 3_4$ $1_1 10_3 5_4$ $1_1 5_3 6_4$ $1_1 11_3 8_4$ $1_1 3_3 9_4$ $1_1 4_3 11_4$
$2_1 3_3 0_4$ $2_1 8_3 1_4$ $2_1 1_3 4_4$ $2_1 0_3 5_4$ $2_1 10_3 6_4$ $2_1 11_3 7_4$ $2_1 7_3 10_4$ $2_1 5_3 11_4$
$3_1 8_3 0_4$ $3_1 3_3 1_4$ $3_1 2_3 4_4$ $3_1 5_3 5_4$ $3_1 6_3 6_4$ $3_1 0_3 7_4$ $3_1 11_3 10_4$ $3_1 1_3 11_4$
$4_1 1_3 0_4$ $4_1 7_3 2_4$ $4_1 10_3 3_4$ $4_1 11_3 5_4$ $4_1 2_3 6_4$ $4_1 3_3 8_4$ $4_1 4_3 9_4$ $4_1 8_3 11_4$
$5_1 2_3 1_4$ $5_1 8_3 2_4$ $5_1 7_3 3_4$ $5_1 0_3 4_4$ $5_1 4_3 7_4$ $5_1 6_3 8_4$ $5_1 10_3 9_4$ $5_1 9_3 10_4$ $6_1 1_3 1_4$
$6_1 3_3 2_4$ $6_1 0_3 3_4$ $6_1 4_3 4_4$ $6_1 6_3 7_4$ $6_1 9_3 8_4$ $6_1 11_3 9_4$ $6_1 10_3 10_4$ $7_1 4_3 0_4$
$7_1 5_3 2_4$ $7_1 8_3 3_4$ $7_1 2_3 5_4$ $7_1 9_3 6_4$ $7_1 10_3 8_4$ $7_1 1_3 9_4$ $7_1 3_3 11_4$ $8_1 2_3 0_4$ $8_1 7_3 1_4$
$8_1 11_3 4_4$ $8_1 8_3 5_4$ $8_1 0_3 6_4$ $8_1 3_3 7_4$ $8_1 1_3 10_4$ $8_1 9_3 11_4$ $9_1 9_3 0_4$ $9_1 10_3 1_4$
$9_1 8_3 4_4$ $9_1 3_3 5_4$ $9_1 11_3 6_4$ $9_1 2_3 7_4$ $9_1 6_3 10_4$ $9_1 7_3 11_4$ $10_1 6_3 0_4$ $10_1 4_3 2_4$
$10_1 2_3 3_4$ $10_1 7_3 5_4$ $10_1 3_3 6_4$ $10_1 8_3 8_4$ $10_1 5_3 9_4$ $10_1 10_3 11_4$ $11_1 9_3 1_4$
$11_1 11_3 2_4$ $11_1 4_3 3_4$ $11_1 6_3 4_4$ $11_1 5_3 7_4$ $11_1 0_3 8_4$ $11_1 8_3 9_4$ $11_1 2_3 10_4$
$12_1 5_3 1_4$ $12_1 9_3 2_4$ $12_1 6_3 3_4$ $12_1 7_3 4_4$ $12_1 8_3 7_4$ $12_1 1_3 8_4$ $12_1 0_3 9_4$
$12_1 4_3 10_4$ $13_1 0_3 0_4$ $13_1 10_3 2_4$ $13_1 5_3 3_4$ $13_1 4_3 5_4$ $13_1 1_3 6_4$ $13_1 2_3 8_4$
$13_1 9_3 9_4$ $13_1 11_3 11_4$ $14_1 11_3 0_4$ $14_1 0_3 1_4$ $14_1 9_3 4_4$ $14_1 1_3 5_4$ $14_1 4_3 6_4$
$14_1 7_3 7_4$ $14_1 8_3 10_4$ $14_1 6_3 11_4$ $15_1 10_3 0_4$ $15_1 11_3 1_4$ $15_1 5_3 4_4$ $15_1 6_3 5_4$
$15_1 7_3 6_4$ $15_1 1_3 7_4$ $15_1 3_3 10_4$ $15_1 2_3 11_4$ $16_1 5_3 0_4$ $16_1 6_3 2_4$ $16_1 3_3 3_4$
$16_1 9_3 5_4$ $16_1 8_3 6_4$ $16_1 4_3 8_4$ $16_1 2_3 9_4$ $16_1 0_3 11_4$ $17_1 6_3 1_4$ $17_1 2_3 2_4$ $17_1 1_3 3_4$
$17_1 3_3 4_4$ $17_1 10_3 7_4$ $17_1 5_3 8_4$ $17_1 7_3 9_4$ $17_1 0_3 10_4$

type ACC: $0_1 3_3 8_3$ $1_1 0_3 2_3$ $2_1 2_3 6_3$ $3_1 4_3 9_3$ $4_1 6_3 9_3$ $5_1 1_3 3_3$ $6_1 2_3 5_3$ $7_1 7_3 11_3$
$8_1 4_3 10_3$ $9_1 0_3 5_3$ $10_1 0_3 1_3$ $11_1 3_3 7_3$ $12_1 3_3 11_3$ $13_1 6_3 7_3$ $14_1 2_3 10_3$ $15_1 8_3 9_3$
$16_1 1_3 11_3$ $17_1 4_3 11_3$

type CCC: $5_3 8_3 11_3$ $0_3 7_3 9_3$ $1_3 2_3 4_3$ $5_3 9_3 10_3$ $0_3 3_3 10_3$ $4_3 6_3 8_3$ $7_3 8_3 10_3$ $1_3 5_3 6_3$

type ∞CC: $\infty 2_3 9_3$ $\infty 1_3 8_3$ $\infty 3_3 4_3$ $\infty 6_3 10_3$ $\infty 0_3 11_3$ $\infty 5_3 7_3$

type ∞BB: $\infty 0_2 5_2$ $\infty 1_2 4_2$ $\infty 2_2 3_2$

type BBB: $0_2 1_2 2_2$ $0_2 3_2 4_2$ $1_2 3_2 5_2$ $2_2 4_2 5_2$

One long line is $0_4 1_4 \cdots 11_4 \infty$ and the other long line is formed by replacing the subsystem of STS(19). Next, $33 \in LS_i(3, 7^*, 13^*)$ by forming a partition $\pi(1^1, 6^2, 8^1, 12^1)$ and constructing short lines of

type ACD: $0_1 i_3 (i+1)_4$ $1_1 i_3 (i+3)_4$ $2_1 i_3 (i+5)_4$ $3_1 i_3 (i+7)_4$ $4_1 i_3 (i+9)_4$ $5_1 i_3 (i+11)_4$ $(i = 0,1,...,7)$

type BCD: $0_2 i_3 (i+2)_4$ $1_2 i_3 (i+4)_4$ $2_2 i_3 (i+6)_4$ $3_2 i_3 (i+8)_4$ $4_2 i_3 (i+10)_4$ $5_2 i_3 i_4$ $(i = 0,1,...,7)$

type ABD: $(i+5)_1 0_2 (i+10)_4$ $i_1 1_2 i_4$ $(i+1)_1 2_2 (i+2)_4$ $(i+2)_1 3_2 (i+4)_4$ $(i+3)_1 4_2 (i+6)_4$ $(i+4)_1 5_2 (i+8)_4 (i = 0,1,2,3)$

type ABB: $0_1 2_2 3_2$ $1_1 3_2 4_2$ $2_1 4_2 5_2$ $3_1 0_2 5_2$ $4_1 0_2 1_2$ $5_1 1_2 2_2$

type ∞BB: $\infty 0_2 3_2$ $\infty 1_2 4_2$ $\infty 2_2 5_2$

type BBB: $0_2 2_2 4_2$ $1_2 3_2 5_2$

type ∞CC and type CCC: form an STS(9) on ∞ and the eight points of cell C. The long lines are $0_1 1_1 \cdots 5_1 \infty$ and $0_4 1_4 \cdots 11_4 \infty$. Form the partition $\pi(1^1, 8^1, 14^3)$ and apply Corollary 1.25 with $g_1 = 7$, $g = 14$, $t = 3$ and $x = 8$ to prove that $51 \in LS(3, 7^*, 15^*)$.

We now are able to summarize all of the above results in the following theorem.

Theorem 2.43

$LS(3,7^*,9^*) = \{v: v \geq 25, v \equiv 1,3 (\text{mod } 6)\}$; $LS_d(3,7^*,13^*) = \{v: v \geq 33, v \equiv 1,3 (\text{mod} 6)\}$; $LS_i(3, 7^*, 13^*) = \{v: v \geq 31, v \equiv 1,3 (\text{mod } 6)\}$; $LS(3, 7^*, 15^*) = \{v: v \geq 37, v \equiv 1,3 (\text{mod } 6)\}$.

Proof: By Lemma 2.21, Corollary 2.22 and Lemmas 2.27, 2.30, 2.32, 2.33 and 2.42, and Corollary 2.34, $v \in LS_d(3, 7^*, 9^*)$ for all $v \geq 25$, $v \equiv 1,3$ (mod 6). Similarly, $v \in LS_i(3, 7^*, 9^*)$ for all $v \geq 25$, $v \equiv 1,3$ (mod 6) from Lemma 2.25, Corollary 2.26 and

Lemmas 2.27, 2.30, 2.32 2.33 and 2.42, and Corollary 2.34. Next, $v \in LS_d(3, 7^*, 13^*)$ for all $v \geq 33$, $v \equiv 1,3 \pmod 6$, from Lemma 2.21, Corollary 2.22 and Lemmas 2.29, 2.30, 2.33 and 2.42, and Corollary 2.34. Now, $v \in LS_i(3, 7^*, 13^*)$ for all $v \geq 31$, $v \equiv 1,3 \pmod 6$ from Lemmas 2.23, 2.30, 2.33 and 2.42, and Corollaries 2.24 and 2.34. We have $v \in LS_d(3, 7^*, 15^*)$ for all $v \geq 37$, $v \equiv 1,3 \pmod 6$ from Lemmas 2.21, 2.27, 2.30, 2.32, 2.33 and 2.42, and Corollaries 2.22 and 2.34. Finally, $v \in LS_i(3, 7^*, 15^*)$ for all $v \geq 37$, $v \equiv 1,3 \pmod 6$, from Lemmas 2.25, 2.27, 2.32, 2.33 and 2.42, and Corollaries 2.26, 2.31 and 2.34.

§2.4 Almost uniform linear spaces with one long line of size 6t + 9, one long line of size w and short lines of size three

From Corollaries 1.19 and 1.21 if an AULS has one long line of size $6t + 9$ ($t \geq 0$) then the other long line has size $w \equiv 1,3 \pmod 6$ and $v \equiv 1,3 \pmod 6$ or size $w \equiv 5 \pmod 6$ and $v \equiv 5 \pmod 6$; also, $w > 6t + 9$. Firstly assume that the two long lines intersect and start by constructing an AULS of minimum order.

Lemma 2.44 Let $w \equiv 1 \pmod 2$, $w \geq 11$.
Then $2w+6t+7 \in LS_i(3,(6t+9)^*,w^*)$ and $2w+6t+7 = \min\{v: \exists LS_i(v;\{3,(6t+9)^*,w^*\})\}$.
Proof: As a consequence of Corollary 1.21, $v \geq 2w + 6t + 7$. Apply Lemma 1.32(a) to prove that $LS_i(2w + 6t + 7; \{3, (6t + 9)^*, w^*\})$ exists.

Corollary 2.45 If $w \equiv 1 \pmod 2$, $w \geq 11$, then $v \in LS_i(3, (6t + 9)^*, w^*)$ for all $v \geq 4w + 12t + 15$, where $v \equiv 1,3$ or $5 \pmod 6$.

Proof: Apply Lemmas 2.44, 1.38 and 1.39.

If the two long lines are disjoint, we can prove an analogous result when

$w \equiv 3 \pmod 6$.

Lemma 2.46 If $w \equiv 3 \pmod 6$, $0 \le t < (w-9)/6$, then $2w+6t+9 \in LS_d(3,(6t+9)^*, w^*)$ and $2w + 6t + 9 = \min\{v: \exists LS_d(v; \{3, (6t+9)^*, w^*\})\}$.

Proof: By Corollary 1.19(i), $v \ge 2w+6t+9$. There exists an $LS_d(2w+6t+9;\{3,(6t+9)^*,w^*\})$ by Lemma 1.36.

Corollary 2.47 If $w \equiv 3 \pmod 6$, then $v \in LS_d(3,(6t+9)^*,w^*)$ for all $v \ge 4w+12t+19$; $v \equiv 1,3 \pmod 6$.

Proof: Lemmas 2.46 and 1.38.

There are no apparent recursive constructions if $w \equiv 1$ or $5 \pmod 6$. However, by setting $t = 0$, $u = 9$ and $w = 13$, we can provide a direct construction to verify that $LS_d(37;\{3, 9^*, 13^*\})$ exists.

Lemma 2.48 There exists an $LS_d(37;\{3, 9^*, 13^*\})$.

Proof: Form the partition $\pi(6^1, 9^2, 13^1)$, where cells A,B are the sets $Z_9 \times \{1\}$ $(i = 1,2)$, cell C is the set $Z_6 \times \{i\}$ and cell D is the set $Z_{13} \times \{4\}$, and construct short lines of

type ABD: $0_1 i_2 i_4 (i = 0,1,2,4,6,7,8)$ $0_1 5_2 3_4$ $0_1 3_2 5_4$ $1_1 0_2 9_4$ $1_1 1_2 10_4$ $1_1 2_2 11_4$
$1_1 5_2 12_4$ $1_1 4_2 0_4$ $1_1 3_2 1_4$ $1_1 6_2 2_4$ $1_1 8_2 3_4$ $1_1 7_2 4_4$ $2_1 i_2 (i+5)_4$
$(i = 0,1,2,4,6,7,8)$ $2_1 5_2 8_4$ $2_1 3_2 10_4$ $3_1 i_2 (i+1)_4 (i = 0,1,...,8)$
$4_1 i_2 (i+10)_4 (i = 0,1,...,6)$ $4_1 8_2 4_4$ $4_1 7_2 5_4$ $5_1 i_2 (i+6)_4 (i = 0,1,...,8)$
$6_1 i_2 (i+2)_4 (i = 0,1,2,4,6,7,8)$ $6_1 5_2 5_4$ $6_1 3_2 7_4$ $7_1 i_2 (i+11)_4 (i = 0,1,2,4,6)$
$7_1 5_2 1_4$ $7_1 3_2 3_4$ $7_1 8_2 5_4$ $7_1 7_2 6_4$ $8_1 i_2 (i+7)_4 (i = 0,1,2,4,6,7,8)$ $8_1 5_2 10_4$
$8_1 3_2 12_4$

type ACD: $0_1 0_3 9_4$ $0_1 1_3 10_4$ $0_1 3_3 11_4$ $0_1 5_3 12_4$ $1_1 0_3 5_4$ $1_1 1_3 6_4$ $1_1 4_3 7_4$ $1_1 5_3 8_4$
$2_1 2_3 1_4$ $2_1 3_3 2_4$ $2_1 0_3 3_4$ $2_1 5_3 4_4$ $3_1 0_3 0_4$ $3_1 3_3 10_4$ $3_1 2_3 11_4$ $3_1 4_3 12_4$
$4_1 0_3 6_4$ $4_1 3_3 7_4$ $4_1 4_3 8_4$ $4_1 5_3 9_4$ $5_1 5_3 2_4$ $5_1 2_3 3_4$ $5_1 1_3 4_4$ $5_1 3_3 5_4$ $6_1 1_3 0_4$

$6_1 3_3 1_4$ $6_1 4_3 11_4$ $6_1 0_3 12_4$ $7_1 0_3 7_4$ $7_1 2_3 8_4$ $7_1 1_3 9_4$ $7_1 4_3 10_4$ $8_1 1_3 3_4$
$8_1 0_3 4_4$ $8_1 2_3 5_4$ $8_1 3_3 6_4$

type ACC: $0_1 2_3 4_3$ $1_1 2_3 3_3$ $2_1 1_3 4_3$ $3_1 1_3 5_3$ $4_1 1_3 2_3$ $5_1 0_3 4_3$ $6_1 2_3 5_3$ $7_1 3_3 5_3$
$8_1 4_3 5_3$

type BCD: $0_2 5_3 3_4$ $0_2 2_3 4_4$ $0_2 1_3 8_4$ $0_2 3_3 12_4$ $1_2 2_3 0_4$ $1_2 4_3 4_4$ $1_2 5_3 5_4$ $1_2 3_3 9_4$
$2_2 4_3 1_4$ $2_2 1_3 5_4$ $2_2 5_3 6_4$ $2_2 0_3 10_4$ $3_2 4_3 2_4$ $3_2 2_3 6_4$ $3_2 1_3 11_4$ $3_2 0_3 8_4$ $4_2 4_3 3_4$
$4_2 5_3 7_4$ $4_2 3_3 8_4$ $4_2 2_3 12_4$ $5_2 1_3 7_4$ $5_2 5_3 0_4$ $5_2 3_3 4_4$ $5_2 4_3 9_4$ $6_2 1_3 1_4$ $6_2 4_3 5_4$
$6_2 2_3 9_4$ $6_2 5_3 10_4$ $7_2 1_3 2_4$ $7_2 2_3 10_4$ $7_2 0_3 11_4$ $7_2 3_3 3_4$ $8_2 4_3 6_4$ $8_2 2_3 7_4$
$8_2 5_3 11_4$ $8_2 1_3 12_4$

type CCD: $3_3 4_3 0_4$ $0_3 5_3 1_4$ $0_3 2_3 2_4$

type BBC: $0_2 1_2 0_3$ $5_2 6_2 0_3$ $4_2 8_2 0_3$ $1_2 4_2 1_3$ $2_2 5_2 2_3$ $2_2 6_2 3_3$ $3_2 8_2 3_3$ $0_2 7_2 4_3$ $3_2 7_2 5_3$

type BBB: $0_2 2_2 8_2$ $0_2 3_2 5_2$ $0_2 4_2 6_2$ $2_2 3_2 4_2$ $1_2 2_2 7_2$ $1_2 3_2 6_2$ $6_2 7_2 8_2$ $4_2 5_2 7_2$ $1_2 5_2 8_2$

type CCC: $0_3 1_3 3_3$

The long lines are $0_1 1_1 \cdots 8_1$ and $0_4 1_4 \cdots 12_4$.

Corollary 2.49 $v \in LS_d(3, 9^*, 13^*)$ for all $v \geq 75$; $v \equiv 1,3 \pmod 6$.

We will now present constructions of AULSs in which the two long lines may either intersect or be disjoint, provided that certain conditions on t and w are met.

Lemma 2.50 If $w \equiv 6t + 7 \pmod{12t + 18}$ and $0 \leq t \leq (w - 16)/12$, then $2w + 6t + 11 \in LS_d(3, (6t + 9)^*, w^*)$.

Proof: Form the partition $\pi((6t + 9)^{(w+6t+11)/(6t+9)}, w^1)$ and apply Theorem 1.24(a).

Lemma 2.51 If $w \equiv 6t + 5 \pmod{12t + 18}$ and $0 \leq t \leq (w - 14)/12$, then $2w + 6t + 13 \in LS_d(3, (6t + 9)^*, w^*)$.

Proof: This readily follows, as in Lemma 2.50, by forming the partition $\pi((6t + 9)^{(w+6t+13)/(6t+9)}, w^1)$.

Lemma 2.52

(a) If $w \equiv 1,3 (\mathrm{mod}\ 6)$ and $0 \leq t < (w-9)/6$, then $4w+12t+15 \in LS_d(3,(6t+9)^*,w^*)$.

(b) If $w \equiv 1 (\mathrm{mod}\ 6)$ and $0 \leq t < (w-9)/6$,

then $4w + 12t + 17$, $4w + 12t + 21 \in LS_d(3,(6t+9)^*,w^*)$.

Proof:

(a) Form the partition $\pi(1^1, (w-1)^4, (12t+18)^1)$ and apply Theorem 1.24(b) with $g_1 = 6t + 9$, $g = w - 1$, $t = 4$ and $x = 12t + 18$.

(b) Form the partition $\pi(1^1, (w-1)^4, (12t+r)^1)$ where $r = 20$ or 24, and apply Theorem 1.24(b) where $g_1 = 6t + 9$, $g = w - 1$, $t = 4$ and $x = 12t + r$.

Lemma 2.53

(a) If $j \in \{1,3,5,7,9,13\}$, $0 \leq t \leq (w-j-4)/12$ and $w \geq 16-j$,

then $4w+12t+j \in LS(3,(6t+9)^*,w^*)$.

(b) If j is odd and $1 \leq j \leq 19$, $0 \leq t \leq (w+j-4)/12$ and $w \geq 16+j$,

then $4w+12t-j \in LS(3,(6t+9)^*,w^*)$.

Proof: Form the partition $\pi(1^1,(w-1)^3,(w+12t \pm j+2)^1)$ and apply Corollary 1.25 where $g_1 = 6t + 9$, $g = w - 1$ and choose $x = w + 12t + j + 2$ for (a), $x = w + 12t - j + 2$ for (b).

Corollary 2.54

(a) $37, 39, 43, 45, 49 \in LS(3, 9^*, 13^*)$.

(b) $43, 45, 49, 51, 55, 57, 73 \in LS(3, 9^*, 15^*)$.

Proof:

(a) By Lemma 2.48, $37 \in LS_d(3, 9^*, 13^*)$. We have $37 \in LS_i(3, 9^*, 13^*)$ since we can form the partition $\pi(1^1, 8^3, 12^1)$ and can apply Theorem 1.24(b). A direct construction is necessary to show that $39 \in LS_d(3, 9^*, 13^*)$. Form the partition $\pi(8^1, 9^2, 13^1)$, where cells A, B are the sets $Z_9 \times \{i\}$ $(i = 1, 2)$, cell C is the set $Z_8 \times \{3\}$ and cell D is the set $Z_{13} \times \{4\}$, and construct short lines of

type ABC: $0_1 0_2 3_3$ $1_1 1_2 5_3$ $2_1 2_2 7_3$ $3_1 3_2 4_3$ $4_1 4_2 0_3$ $5_1 5_2 0_3$ $6_1 6_2 6_3$ $7_1 7_2 7_3$ $8_1 8_2 4_3$

type ABD: $0_1 (i+1)_2 i_4 (i = 0,1,\ldots,7)$ $1_1 0_2 8_4$ $1_1 (i+2)_2 (i+9)_4 (i = 0,1,\ldots,6)$ $2_1 0_2 3_4$

$2_1 1_2 4_4$ $2_1(i+3)_2(i+5)_4$ $(i = 0,1,\dots,5)$ $3_1 0_2 11_4$ $3_1 1_2 12_4$ $3_1 2_2 0_4$ $3_1(i+4)_2(i+1)_4 (i=0,1,\dots,4)$ $4_1 i_2(i+6)_4 (i = 0,1,2,3)$ $4_1(i+5)_2(i+10)_4$ $(i = 0,1,2,3)$ $5_1 i_2(i+1)_4 (i = 0,\dots,4)$ $5_1 6_2 6_4$ $5_1 7_2 7_4$ $5_1 8_2 8_4$ $6_1 i_2(i+9)_4$ $(i = 0,1,\dots,5)$ $6_1 7_2 2_4$ $6_1 8_2 3_4$ $7_1 i_2(i+4)_4 (i = 0,2,3,4,6)$ $7_1 1_2 9_4$ $7_1 5_2 5_4$ $7_1 8_2 11_4$ $8_1 0_2 12_4$ $8_1 1_2 5_4$ $8_1 2_2 2_4$ $8_1 3_2 6_4$ $8_1 4_2 4_4$ $8_1 5_2 3_4$ $8_1 6_2 1_4$ $8_1 7_2 0_4$

type ACD: $0_1 4_3 8_4$ $0_1 2_3 9_4$ $0_1 1_3 10_4$ $0_1 6_3 11_4$ $0_1 0_3 12_4$ $1_1 6_3 3_4$ $1_1 2_3 4_4$ $1_1 4_3 5_4$ $1_1 0_3 6_4$ $1_1 3_3 7_4$ $2_1 4_3 0_4$ $2_1 3_3 1_4$ $2_1 2_3 2_4$ $2_1 5_3 11_4$ $2_1 1_3 12_4$ $3_1 2_3 6_4$ $3_1 5_3 7_4$ $3_1 7_3 8_4$ $3_1 0_3 9_4$ $3_1 6_3 10_4$ $4_1 6_3 1_4$ $4_1 3_3 2_4$ $4_1 7_3 3_4$ $4_1 4_3 4_4$ $4_1 5_3 5_4$ $5_1 6_3 0_4$ $5_1 1_3 9_4$ $5_1 4_3 10_4$ $5_1 3_3 11_4$ $5_1 7_3 12_4$ $6_1 7_3 4_4$ $6_1 0_3 5_4$ $6_1 1_3 6_4$ $6_1 4_3 7_4$ $6_1 5_3 8_4$ $7_1 1_3 0_4$ $7_1 5_3 1_4$ $7_1 6_3 2_4$ $7_1 2_3 3_4$ $7_1 4_3 12_4$ $8_1 2_3 7_4$ $8_1 6_3 8_4$ $8_1 7_3 9_4$ $8_1 0_3 10_4$ $8_1 1_3 11_4$

type ACC: $0_1 5_3 7_3$ $1_1 1_3 7_3$ $2_1 0_3 6_3$ $3_1 1_3 3_3$ $4_1 1_3 2_3$ $5_1 2_3 5_3$ $6_1 2_3 3_3$ $7_1 0_3 3_3$ $8_1 3_3 5_3$

type BCD: $0_2 2_3 0_4$ $0_2 5_3 2_4$ $0_2 1_3 5_4$ $0_2 0_3 7_4$ $0_2 7_3 10_4$ $1_2 1_3 1_4$ $1_2 3_3 3_4$ $1_2 6_3 6_4$ $1_2 0_3 8_4$ $1_2 4_3 11_4$ $2_2 5_3 4_4$ $2_2 2_3 5_4$ $2_2 1_3 7_4$ $2_2 3_3 10_4$ $2_2 6_3 12_4$ $3_2 3_3 0_4$ $3_2 0_3 1_4$ $3_2 1_3 3_4$ $3_2 2_3 8_4$ $3_2 7_3 11_4$ $4_2 7_3 2_4$ $4_2 6_3 7_4$ $4_2 5_3 9_4$ $4_2 2_3 10_4$ $4_2 3_3 12_4$ $5_2 5_3 0_4$ $5_2 7_3 6_4$ $5_2 3_3 8_4$ $5_2 4_3 9_4$ $5_2 2_3 11_4$ $6_2 4_3 2_4$ $6_2 1_3 4_4$ $6_2 7_3 7_4$ $6_2 3_3 9_4$ $6_2 5_3 12_4$ $7_2 4_3 3_4$ $7_2 3_3 5_4$ $7_2 1_3 8_4$ $7_2 5_3 10_4$ $7_2 0_3 11_4$ $8_2 7_3 1_4$ $8_2 0_3 4_4$ $8_2 3_3 6_4$ $8_2 6_3 9_4$ $8_2 2_3 12_4$

type BCC: $0_2 4_3 6_3$ $1_2 2_3 7_3$ $2_2 0_3 4_3$ $3_2 5_3 6_3$ $4_2 1_3 4_3$ $5_2 1_3 6_3$ $6_2 0_3 2_3$ $7_2 2_3 6_3$ $8_2 1_3 5_3$

type CCD: $0_3 7_3 0_4$ $2_3 4_3 1_4$ $0_3 1_3 2_4$ $0_3 5_3 3_4$ $3_3 6_3 4_4$ $6_3 7_3 5_4$ $4_3 5_3 6_4$

type CCC: $3_3 4_3 7_3$

type BBB: form an STS(9) on the points of cell B.

The long lines are $0_1 1_1 \cdots 8_1$ and $0_4 1_4 \cdots 12_4$. In order to prove that $39 \in LS_i(3, 9^*, 13^*)$, a direct construction is required. Form the partition $\pi(1^1, 6^1, 8^1, 12^2)$, where cell A is the set $Z_8 \times \{1\}$, cell B is the set $Z_6 \times \{2\}$, cells C and D are the sets $Z_{12} \times \{i\}$ $(i = 3, 4)$, and construct short lines of

type BCD: $0_2 i_3 i_4 (i = 0,1,2,5,6,7)$ $0_2 11_3 3_4$ $0_2 9_3 4_4$ $1_2 8_3 8_4$ $1_2 4_3 9_4$ $1_2(i+9)_3(i+10)_4$ $(i = 0,3,4,5)$ $1_2 3_3 11_4$ $1_2 11_3 4_4$ $2_2 7_3 5_4$ $2_2(i+5)_3(i+6)_4 (i = 0,1,5,6)$ $2_2 9_3 8_4$ $2_2 3_3 9_4$ $2_2 4_3 10_4$ $3_2 0_3 4_4$ $3_2 1_3 5_4$ $3_2 2_3 6_4$ $3_2 11_3 7_4$ $3_2 7_3 8_4$ $3_2 5_3 3_4$ $3_2 6_3 2_4$ $3_2 10_3 1_4$ $4_2 3_3 0_4$ $4_2 7_3 1_4$ $4_2 9_3 2_4$ $4_2 4_3 3_4$ $4_2 8_3 11_4$ $4_2 1_3 4_4$

$4_2 2_3 10_4$ $4_2 11_3 9_4$ $5_2 10_3 9_4$ $5_2 5_3 8_4$ $5_2 6_3 10_4$ $5_2 7_3 0_4$ $5_2 4_3 11_4$ $5_2 8_3 7_4$
$5_2 9_3 6_4$ $5_2 3_3 5_4$

type BCC: $0_2 4_3 8_3$ $0_2 3_3 10_3$ $1_2 7_3 10_3$ $1_2 5_3 6_3$ $2_2 0_3 1_3$ $2_2 2_3 8_3$ $3_2 3_3 8_3$ $3_2 4_3 9_3$
$4_2 0_3 5_3$ $4_2 6_3 10_3$ $5_2 0_3 2_3$ $5_2 1_3 11_3$

type ABD: $0_1 0_2 8_4$ $1_1 0_2 9_4$ $5_1 0_2 10_4$ $4_1 0_2 11_4$ $3_1 1_2 0_4$ $5_1 1_2 5_4$ $6_1 1_2 6_4$ $7_1 1_2 7_4$
$0_1 2_2 1_4$ $6_1 2_2 2_4$ $2_1 2_2 3_4$ $3_1 2_2 4_4$ $4_1 3_2 0_4$ $5_1 3_2 9_4$ $6_1 3_2 10_4$ $7_1 3_2 11_4$ $4_1 4_2 5_4$
$1_1 4_2 6_4$ $2_1 4_2 7_4$ $7_1 4_2 8_4$ $(i+4)_1 5_2 (i+1)_4 (i = 0,1,2,3)$

type ABB: $0_1 1_2 5_2$ $0_1 3_2 4_2$ $1_1 1_2 3_2$ $1_1 2_2 5_2$ $2_1 0_2 1_2$ $2_1 3_2 5_2$ $3_1 0_2 3_2$ $3_1 4_2 5_2$ $4_1 1_2 2_2$
$5_1 2_2 4_2$ $6_1 0_2 4_2$ $7_1 0_2 2_2$

type ∞BB: $\infty 0_2 5_2$ $\infty 1_2 4_2$ $\infty 2_2 3_2$

type ACD: $0_1 2_3 0_4$ $0_1 7_3 2_4$ $0_1 8_3 3_4$ $0_1 4_3 4_4$ $0_1 9_3 5_4$ $0_1 1_3 6_4$ $0_1 0_3 7_4$ $0_1 6_3 9_4$ $0_1 3_3 10_4$
$0_1 5_3 11_4$ $1_1 5_3 0_4$ $1_1 3_3 1_4$ $1_1 8_3 2_4$ $1_1 1_3 3_4$ $1_1 10_3 4_4$ $1_1 11_3 5_4$ $1_1 4_3 7_4$
$1_1 2_3 8_4$ $1_1 0_3 10_4$ $1_1 9_3 11_4$ $2_1 9_3 0_4$ $2_1 4_3 1_4$ $2_1 11_3 2_4$ $2_1 7_3 4_4$ $2_1 6_3 5_4$
$2_1 3_3 6_4$ $2_1 10_3 8_4$ $2_1 2_3 9_4$ $2_1 5_3 10_4$ $2_1 0_3 11_4$ $3_1 9_3 1_4$ $3_1 5_3 2_4$ $3_1 6_3 3_4$
$3_1 2_3 5_4$ $3_1 4_3 6_4$ $3_1 3_3 7_4$ $3_1 0_3 8_4$ $3_1 1_3 9_4$ $3_1 10_3 10_4$ $3_1 11_3 11_4$ $4_1 3_3 2_4$
$4_1 9_3 3_4$ $4_1 6_3 4_4$ $4_1 7_3 6_4$ $4_1 2_3 7_4$ $4_1 11_3 8_4$ $4_1 5_3 9_4$ $4_1 8_3 10_4$ $5_1 10_3 0_4$
$5_1 8_3 1_4$ $5_1 3_3 3_4$ $5_1 2_3 4_4$ $5_1 0_3 6_4$ $5_1 1_3 7_4$ $5_1 6_3 8_4$ $5_1 7_3 11_4$ $6_1 1_3 0_4$ $6_1 5_3 1_4$
$6_1 8_3 4_4$ $6_1 10_3 5_4$ $6_1 9_3 7_4$ $6_1 4_3 8_4$ $6_1 7_3 9_4$ $6_1 2_3 11_4$ $7_1 4_3 0_4$ $7_1 6_3 1_4$ $7_1 0_3 2_4$
$7_1 10_3 3_4$ $7_1 8_3 5_4$ $7_1 11_3 6_4$ $7_1 9_3 9_4$ $7_1 1_3 10_4$

type ACC: $0_1 10_3 11_3$ $1_1 6_3 7_3$ $2_1 1_3 8_3$ $3_1 7_3 8_3$ $4_1 1_3 4_3$ $4_1 0_3 10_3$ $5_1 5_3 9_3$ $5_1 4_3 11_3$
$6_1 0_3 11_3$ $6_1 3_3 6_3$ $7_1 2_3 3_3$ $7_1 5_3 7_3$

type CCD: $6_3 8_3 0_4$ $2_3 11_3 1_4$ $4_3 10_3 2_4$ $0_3 7_3 3_4$ $3_3 5_3 4_4$ $0_3 4_3 5_4$ $8_3 10_3 6_4$ $5_3 10_3 7_4$
$1_3 3_3 8_4$ $0_3 8_3 9_4$ $7_3 11_3 10_4$ $1_3 6_3 11_4$

type CCC: $0_3 6_3 9_3$ $1_3 2_3 5_3$ $1_3 7_3 9_3$ $2_3 4_3 6_3$ $5_3 8_3 11_3$ $2_3 9_3 10_3$ $3_3 4_3 7_3$ $3_3 9_3 11_3$

type ∞CC: $\infty 0_3 3_3$ $\infty 1_3 10_3$ $\infty 2_3 7_3$ $\infty 6_3 11_3$ $\infty 4_3 5_3$ $\infty 8_3 9_3$

The long lines are $0_1 1_1 \cdots 7_1 \infty$ and $0_4 1_4 \cdots 11_4 \infty$. Next, to show that $43 \in LS_d(3, 9^*, 13^*)$, form the partition $\pi(9^2, 12^1, 13^1)$ and construct short lines of

type ABC: $0_1 0_2 4_3$ $0_1 1_2 2_3$ $1_1 2_2 3_3$ $2_1 3_2 4_3$ $3_1 3_2 2_3$ $3_1 4_2 5_3$ $4_1 5_2 10_3$ $5_1 6_2 7_3$ $6_1 6_2 10_3$
$6_1 7_2 6_3$ $7_1 8_2 1_3$ $8_1 8_2 6_3$ $8_1 0_2 11_3$ $i_1 i_2 i_3 (i = 1,2,4,5,7)$

type ABD: $0_1 (i+2)_2 i_4 (i = 0,1,\ldots,6)$ $1_1 0_2 7_4$ $1_1 (i+3)_2 (i+8)_4 (i = 0,1,\ldots,5)$ $2_1 0_2 1_4$

$2_1 1_2 2_4$ $2_1(i+4)_2(i+3)_4$ $(i = 0,1,\ldots,4)$ $3_1 0_2 8_4$ $3_1 1_2 9_4$ $3_1 2_2 10_4$ $3_1(i+5)_2(i+11)_4 (i = 0,1,2,3)$ $4_1 i_2 (i+2)_4 (i = 0,1,2,3)$ $4_1 6_2 6_4$ $4_1 7_2 7_4$ $4_1 8_2 8_4$ $5_1 i_2 (i+9)_4 (i = 0,1,\ldots,4)$ $5_1 7_2 1_4$ $5_1 8_2 2_4$ $6_1 i_2 (i+3)_4 (i = 0,1,\ldots,5)$ $6_1 8_2 9_4$ $7_1 i_2 (i+10)_4 (i = 0,1,\ldots,6)$ $8_1 7_2 4_4$ $8_1(i+1)_2(i+5)_4 (i = 0,1,\ldots,5)$

type ACD: $0_1 1_3 7_4$ $0_1 3_3 8_4$ $0_1 5_3 9_4$ $0_1 10_3 10_4$ $0_1 8_3 11_4$ $0_1 9_3 12_4$ $1_1 0_3 1_4$ $1_1 11_3 2_4$ $1_1 8_3 3_4$ $1_1 6_3 4_4$ $1_1 10_3 5_4$ $1_1 9_3 6_4$ $2_1 6_3 0_4$ $2_1 5_3 8_4$ $2_1 10_3 9_4$ $2_1 9_3 10_4$ $2_1 11_3 11_4$ $2_1 1_3 12_4$ $3_1 9_3 2_4$ $3_1 7_3 3_4$ $3_1 3_3 4_4$ $3_1 0_3 5_4$ $3_1 1_3 6_4$ $3_1 10_3 7_4$ $4_1 5_3 0_4$ $4_1 9_3 1_4$ $4_1 6_3 9_4$ $4_1 11_3 10_4$ $4_1 2_3 11_4$ $4_1 7_3 12_4$ $5_1 1_3 3_4$ $5_1 2_3 4_4$ $5_1 4_3 5_4$ $5_1 3_3 6_4$ $5_1 6_3 7_4$ $5_1 8_3 8_4$ $6_1 11_3 0_4$ $6_1 5_3 1_4$ $6_1 7_3 2_4$ $6_1 1_3 10_4$ $6_1 4_3 11_4$ $6_1 8_3 12_4$ $7_1 4_3 4_4$ $7_1 3_3 5_4$ $7_1 8_3 6_4$ $7_1 5_3 7_4$ $7_1 6_3 8_4$ $7_1 0_3 9_4$ $8_1 8_3 0_4$ $8_1 7_3 1_4$ $8_1 4_3 2_4$ $8_1 3_3 3_4$ $8_1 9_3 11_4$ $8_1 10_3 12_4$

type ACC: $0_1 6_3 7_3$ $0_1 0_3 11_3$ $1_1 5_3 7_3$ $1_1 2_3 4_3$ $2_1 3_3 7_3$ $2_1 0_3 8_3$ $3_1 6_3 11_3$ $3_1 4_3 8_3$ $4_1 0_3 1_3$ $4_1 3_3 8_3$ $5_1 0_3 9_3$ $5_1 10_3 11_3$ $6_1 2_3 9_3$ $6_1 0_3 3_3$ $7_1 2_3 11_3$ $7_1 9_3 10_3$ $8_1 1_3 5_3$ $8_1 0_3 2_3$

type BCD: $0_2 2_3 0_4$ $0_2 5_3 4_4$ $0_2 8_3 5_4$ $0_2 7_3 6_4$ $0_2 3_3 11_4$ $0_2 0_3 12_4$ $1_2 7_3 0_4$ $1_2 4_3 1_4$ $1_2 10_3 6_4$ $1_2 11_3 7_4$ $1_2 0_3 8_4$ $1_2 6_3 12_4$ $2_2 11_3 1_4$ $2_2 6_3 2_4$ $2_2 5_3 3_4$ $2_2 0_3 7_4$ $2_2 9_3 8_4$ $2_2 4_3 9_4$ $3_2 1_3 2_4$ $3_2 0_3 3_4$ $3_2 10_3 4_4$ $3_2 3_3 9_4$ $3_2 7_3 10_4$ $3_2 5_3 11_4$ $4_2 9_3 4_4$ $4_2 6_3 5_4$ $4_2 2_3 6_4$ $4_2 0_3 10_4$ $4_2 1_3 11_4$ $4_2 3_3 12_4$ $5_2 9_3 0_4$ $5_2 8_3 1_4$ $5_2 1_3 5_4$ $5_2 11_3 6_4$ $5_2 3_3 7_4$ $5_2 4_3 12_4$ $6_2 0_3 0_4$ $6_2 6_3 1_4$ $6_2 3_3 2_4$ $6_2 8_3 7_4$ $6_2 11_3 8_4$ $6_2 2_3 9_4$ $7_2 0_3 2_4$ $7_2 2_3 3_4$ $7_2 1_3 8_4$ $7_2 9_3 9_4$ $7_2 4_3 10_4$ $7_2 10_3 11_4$ $8_2 10_3 3_4$ $8_2 0_3 4_4$ $8_2 11_3 5_4$ $8_2 8_3 10_4$ $8_2 7_3 11_4$ $8_2 2_3 12_4$

type BCC: $0_2 6_3 10_3$ $0_2 1_3 9_3$ $1_2 3_3 5_3$ $1_2 8_3 9_3$ $2_2 7_3 10_3$ $2_2 1_3 8_3$ $3_2 9_3 11_3$ $3_2 6_3 8_3$ $4_2 8_3 10_3$ $4_2 7_3 11_3$ $5_2 2_3 6_3$ $5_2 0_3 7_3$ $6_2 1_3 4_3$ $6_2 5_3 9_3$ $7_2 3_3 11_3$ $7_2 5_3 8_3$ $8_2 4_3 5_3$ $8_2 3_3 9_3$

type CCD: $1_3 10_3 0_4$ $3_3 4_3 0_4$ $1_3 2_3 1_4$ $3_3 10_3 1_4$ $2_3 8_3 2_4$ $5_3 10_3 2_4$ $6_3 9_3 3_4$ $4_3 11_3 3_4$ $1_3 7_3 4_4$ $8_3 11_3 4_4$ $2_3 5_3 5_4$ $7_3 9_3 5_4$ $4_3 6_3 6_4$ $0_3 5_3 6_4$ $4_3 9_3 7_4$ $2_3 7_3 7_4$ $2_3 10_3 8_4$ $4_3 7_3 8_4$ $1_3 11_3 9_4$ $7_3 8_3 9_4$ $5_3 6_3 10_4$ $2_3 3_3 10_4$ $0_3 6_3 11_4$ $5_3 11_3 12_4$

type CCC: $0_3 4_3 10_3$ $1_3 3_3 6_3$

type BBB: Place an STS(9) on the points of cell B.

The long lines are $0_1 1_1 \cdots 8_1$ and $0_4 1_4 \cdots 12_4$. We can show that

$43 \in LS_i(3, 9^*, 13^*)$ by forming the partition $\pi(1^1, 8^2, 14^1, 12^1)$ and constructing short lines of

type ABD: $0_16_20_4$ $0_15_22_4$ $0_12_24_4$ $0_10_26_4$ $0_14_28_4$ $0_13_23_4$ $1_13_21_4$ $1_17_23_4$
$1_16_25_4$ $1_14_27_4$ $1_15_29_4$ $1_11_24_4$ $2_17_22_4$ $2_10_24_4$ $2_14_26_4$ $2_15_28_4$ $2_13_210_4$
$2_12_25_4$ $3_15_23_4$ $3_14_25_4$ $3_13_27_4$ $3_11_29_4$ $3_17_211_4$ $3_16_26_4$ $4_14_24_4$ $4_13_26_4$
$4_11_28_4$ $4_17_210_4$ $4_10_21_4$ $4_15_27_4$ $5_13_25_4$ $5_11_27_4$ $5_17_29_4$ $5_10_211_4$ $5_16_22_4$
$5_12_28_4$ $6_15_26_4$ $6_17_28_4$ $6_10_210_4$ $6_11_20_4$ $6_12_23_4$ $6_16_29_4$ $7_12_27_4$ $7_10_29_4$
$7_13_211_4$ $7_17_21_4$ $7_15_24_4$ $7_14_210_4$

type ABC: $0_11_212_3$ $0_17_213_3$ $1_10_213_3$ $1_12_24_3$ $2_16_24_3$ $2_11_22_3$ $3_10_26_3$ $3_12_27_3$
$4_12_28_3$ $4_16_26_3$ $5_15_210_3$ $5_14_211_3$ $6_14_213_3$ $6_13_24_3$ $7_11_210_3$ $7_16_21_3$

type ACD: $0_12_31_4$ $0_14_35_4$ $0_16_37_4$ $0_111_39_4$ $0_110_310_4$ $0_11_311_4$ $1_11_30_4$ $1_111_32_4$
$1_18_36_4$ $1_17_38_4$ $1_15_310_4$ $1_12_311_4$ $2_112_30_4$ $2_13_31_4$ $2_16_33_4$ $2_18_37_4$
$2_19_39_4$ $2_110_311_4$ $3_12_30_4$ $3_11_31_4$ $3_113_32_4$ $3_19_34_4$ $3_13_38_4$ $3_112_310_4$
$4_110_30_4$ $4_13_32_4$ $4_12_33_4$ $4_19_35_4$ $4_112_39_4$ $4_14_311_4$ $5_113_30_4$ $5_18_31_4$
$5_17_33_4$ $5_15_34_4$ $5_112_36_4$ $5_19_310_4$ $6_110_31_4$ 6_1[illegible]$_32_4$ $6_112_34_4$ $6_10_35_4$
$6_12_37_4$ $6_111_311_4$ $7_111_30_4$ $7_12_32_4$ $7_19_33_4$ $7_17_35_4$ $7_15_36_4$ $7_16_38_4$

type ACC: $0_10_37_3$ $0_18_39_3$ $0_13_35_3$ $1_10_36_3$ $1_13_312_3$ $1_19_310_3$ $2_10_35_3$ $2_17_311_3$
$2_11_313_3$ $3_10_38_3$ $3_14_35_3$ $3_110_311_3$ $4_10_31_3$ $4_15_311_3$ $4_17_313_3$ $5_11_33_3$
$5_14_36_3$ $5_10_32_3$ $6_11_39_3$ $6_13_36_3$ $6_15_37_3$ $7_10_313_3$ $7_13_38_3$ $7_14_312_3$

type BCD: $0_27_37_4$ $0_25_32_4$ $0_210_33_4$ $0_24_30_4$ $0_28_38_4$ $0_23_35_4$ $1_21_32_4$ $1_26_35_4$ $1_211_31_4$
$1_213_311_4$ $1_23_310_4$ $1_24_36_4$ $1_20_33_4$ $2_213_39_4$ $2_211_310_4$ $2_20_30_4$ $2_25_31_4$
$2_26_32_4$ $2_21_36_4$ $2_29_311_4$ $3_26_30_4$ $3_20_32_4$ $3_22_34_4$ $3_212_38_4$ $3_23_39_4$ $4_23_30_4$
$4_24_31_4$ $4_29_32_4$ $4_21_33_4$ $4_27_39_4$ $4_20_311_4$ $5_20_310_4$ $5_25_30_4$ $5_27_31_4$
$5_212_311_4$ $5_28_35_4$ $6_23_33_4$ $6_210_34_4$ $6_211_38_4$ $6_213_31_4$ $6_20_37_4$ $6_25_311_4$
$6_28_310_4$ $7_29_30_4$ $7_25_35_4$ $7_20_36_4$ $7_21_37_4$ $7_27_34_4$

type BCC: $0_22_312_3$ $0_20_39_3$ $0_21_311_3$ $1_25_38_3$ $1_27_39_3$ $2_22_33_3$ $2_210_312_3$ $3_28_310_3$
$3_21_37_3$ $3_25_39_3$ $3_211_313_3$ $4_22_35_3$ $4_28_312_3$ $4_26_310_3$ $5_21_36_3$ $5_23_313_3$
$5_29_311_3$ $5_22_34_3$ $6_22_37_3$ $6_29_312_3$ $7_23_34_3$ $7_22_310_3$ $7_26_312_3$ $7_28_311_3$

type CCD: $7_38_30_4$ $0_312_31_4$ $6_39_31_4$ $4_310_32_4$ $7_312_32_4$ $4_38_33_4$ $11_312_33_4$ $5_313_33_4$
$0_33_34_4$ $6_311_34_4$ $8_313_34_4$ $1_34_34_4$ $2_311_35_4$ $12_313_35_4$ $1_310_35_4$ $7_310_36_4$

$3_311_36_4$ $2_39_36_4$ $6_313_36_4$ $3_310_37_4$ $4_311_37_4$ $5_312_37_4$ $9_313_37_4$ $4_39_38_4$
$1_35_38_4$ $2_313_38_4$ $0_310_38_4$ $1_38_39_4$ $0_34_39_4$ $2_36_39_4$ $5_310_39_4$ $1_32_310_4$
$6_37_310_4$ $4_313_310_4$ $3_37_311_4$ $6_38_311_4$

type ∞BB: $\infty 0_27_2$ $\infty 1_26_2$ $\infty 2_25_2$ $\infty 3_24_2$

type ∞CC: $\infty 1_312_3$ $\infty 5_36_3$ $\infty 2_38_3$ $\infty 4_37_3$ $\infty 3_39_3$ $\infty 10_313_3$ $\infty 0_311_3$

type BBB: $0_21_22_2$ $0_23_25_2$ $0_24_26_2$ $1_23_27_2$ $1_24_25_2$ $2_23_26_2$ $2_24_27_2$ $5_26_27_2$

The long lines are $0_11_1 \cdots 7_1\infty$ and $0_41_4 \cdots 11_4\infty$. We need a direct construction to show that $45 \in LS_d(3, 9^*, 13^*)$. Form the partition $\pi(9^2, 13^1, 14^1)$, where cells A, B are the sets $Z_9 \times \{i\}$ $(i = 1, 2)$, cell C is the set $Z_{14} \times \{3\}$, and cell D is the set $Z_{13} \times \{4\}$, and construct short lines of

type ABD: $0_11_29_4$ $0_14_25_4$ $0_12_21_4$ $0_13_210_4$ $1_12_210_4$ $1_15_26_4$ $1_16_22_4$ $1_14_211_4$
$2_15_211_4$ $2_13_27_4$ $2_14_23_4$ $2_16_212_4$ $3_15_23_4$ $3_13_212_4$ $3_14_28_4$ $3_16_24_4$
$3_17_20_4$ $4_13_24_4$ $4_15_20_4$ $4_16_29_4$ $4_17_25_4$ $4_18_21_4$ $5_16_21_4$ $5_17_210_4$ $5_18_26_4$
$5_10_22_4$ $6_13_26_4$ $6_17_22_4$ $6_18_211_4$ $6_11_27_4$ $6_10_23_4$ $7_18_23_4$ $7_11_212_4$ $7_10_28_4$
$7_12_24_4$ $8_10_24_4$ $8_11_20_4$ $8_14_29_4$ $8_12_25_4$ $i_1i_2i_4 (i = 0,1,2,5,7,8)$

type ABC: $0_1(i+5)_2i_3 (i = 0,1,2,3)$ $1_10_21_3$ $1_13_20_3$ $1_17_24_3$ $1_18_27_3$ $2_10_210_3$ $2_11_25_3$
$2_17_28_3$ $2_18_212_3$ $3_10_29_3$ $3_11_213_3$ $3_12_20_3$ $3_18_26_3$ $4_10_211_3$ $4_11_23_3$
$4_12_29_3$ $4_14_21_3$ $5_11_212_3$ $5_12_24_3$ $5_14_29_3$ $5_13_210_3$ $6_14_27_3$ $6_12_211_3$
$6_16_25_3$ $6_15_23_3$ $7_14_25_3$ $7_13_21_3$ $7_15_22_3$ $7_16_23_3$ $8_16_211_3$ $8_15_29_3$ $8_13_25_3$
$8_17_27_3$

type ACD: $0_14_32_4$ $0_19_33_4$ $0_16_34_4$ $0_17_36_4$ $0_15_37_4$ $0_110_38_4$ $0_112_311_4$ $0_111_312_4$
$1_19_30_4$ $1_113_33_4$ $1_111_34_4$ $1_15_35_4$ $1_112_37_4$ $1_13_38_4$ $1_18_39_4$ $1_110_312_4$
$2_10_30_4$ $2_16_31_4$ $2_12_34_4$ $2_13_35_4$ $2_111_36_4$ $2_19_38_4$ $2_11_39_4$ $2_113_310_4$
$3_18_31_4$ $3_110_32_4$ $3_17_35_4$ $3_112_36_4$ $3_13_37_4$ $3_15_39_4$ $3_111_310_4$ $3_14_311_4$
$4_18_32_4$ $4_14_33_4$ $4_110_36_4$ $4_17_37_4$ $4_12_38_4$ $4_15_310_4$ $4_113_311_4$ $4_10_312_4$
$5_12_30_4$ $5_13_33_4$ $5_15_34_4$ $5_16_37_4$ $5_113_38_4$ $5_111_39_4$ $5_18_311_4$ $5_11_312_4$
$6_113_30_4$ $6_11_31_4$ $6_10_34_4$ $6_18_35_4$ $6_14_38_4$ $6_112_39_4$ $6_19_310_4$ $6_12_312_4$
$7_112_30_4$ $7_14_31_4$ $7_16_32_4$ $7_19_35_4$ $7_113_36_4$ $7_110_39_4$ $7_17_310_4$ $7_10_311_4$
$8_10_31_4$ $8_11_32_4$ $8_12_33_4$ $8_18_36_4$ $8_113_37_4$ $8_112_310_4$ $8_110_311_4$ $8_13_312_4$

type ACC: $0_18_313_3$ $1_12_36_3$ $2_14_37_3$ $3_11_32_3$ $4_16_312_3$ $5_10_37_3$ $6_16_310_3$ $7_18_311_3$

$8_1 4_3 6_3$

type BCD: $0_2 3_3 1_4$ $0_2 6_3 5_4$ $0_2 2_3 7_4$ $0_2 8_3 12_4$ $1_2 8_3 3_4$ $1_2 0_3 2_4$ $1_2 1_3 4_4$ $1_2 2_3 6_4$
$2_2 6_3 0_4$ $2_2 2_3 11_4$ $2_2 7_3 9_4$ $2_2 5_3 8_4$ $3_2 12_3 1_4$ $3_2 2_3 9_4$ $3_2 4_3 0_4$ $3_2 11_3 2_4$
$4_2 3_3 0_4$ $4_2 13_3 4_4$ $4_2 4_3 6_4$ $4_2 10_3 7_4$ $5_2 6_3 12_4$ $5_2 13_3 2_4$ $5_2 7_3 4_4$ $5_2 11_3 7_4$
$6_2 0_3 3_4$ $6_2 6_3 11_4$ $6_2 12_3 5_4$ $6_2 8_3 8_4$ $7_2 9_3 1_4$ $7_2 10_3 3_4$ $7_2 13_3 9_4$ $7_2 12_3 12_4$
$8_2 1_3 5_4$ $8_2 9_3 7_4$ $8_2 0_3 9_4$ $8_2 4_3 12_4$

type BCC: $0_2 4_3 5_3$ $0_2 0_3 12_3$ $0_2 7_3 13_3$ $1_2 6_3 7_3$ $1_2 4_3 11_3$ $1_2 9_3 10_3$ $2_2 3_3 10_3$ $2_2 1_3 8_3$
$2_2 12_3 13_3$ $3_2 3_3 7_3$ $3_2 6_3 13_3$ $3_2 8_3 9_3$ $4_2 0_3 2_3$ $5_2 8_3 12_3$ $5_2 4_3 10_3$ $4_2 6_3 8_3$
$5_2 1_3 5_3$ $4_2 11_3 12_3$ $6_2 4_3 13_3$ $6_2 2_3 9_3$ $6_2 7_3 10_3$ $7_2 0_3 5_3$ $7_2 6_3 11_3$ $7_2 1_3 3_3$
$8_2 2_3 11_3$ $8_2 5_3 8_3$ $8_2 10_3 13_3$

type CCD: $1_3 10_3 0_4$ $5_3 11_3 0_4$ $7_3 8_3 0_4$ $2_3 7_3 1_4$ $11_3 13_3 1_4$ $5_3 10_3 1_4$ $2_3 3_3 2_4$ $5_3 7_3 2_4$
$9_3 12_3 2_4$ $5_3 6_3 3_4$ $1_3 12_3 3_4$ $7_3 11_3 3_4$ $3_3 9_3 4_4$ $10_3 12_3 4_4$ $4_3 8_3 4_4$ $2_3 13_3 5_4$
$10_3 11_3 5_4$ $0_3 4_3 5_4$ $0_3 1_3 6_4$ $6_3 9_3 6_4$ $3_3 5_3 6_4$ $0_3 8_3 7_4$ $1_3 4_3 7_4$ $0_3 6_3 8_4$
$7_3 12_3 8_4$ $1_3 11_3 8_4$ $3_3 6_3 9_4$ $4_3 9_3 9_4$ $0_3 10_3 10_4$ $1_3 6_3 10_4$ $2_3 4_3 10_4$ $3_3 8_3 10_4$
$5_3 9_3 11_4$ $1_3 7_3 11_4$ $3_3 11_3 11_4$ $5_3 13_3 12_4$ $7_3 9_3 12_4$

type CCC: $2_3 5_3 12_3$ $0_3 3_3 13_3$ $1_3 9_3 13_3$ $2_3 8_3 10_3$ $3_3 4_3 12_3$ $0_3 9_3 11_3$

type BBB: $0_2 4_2 7_2$ $1_2 2_2 8_2$ $1_2 5_2 6_2$ $2_2 5_2 7_2$ $3_2 4_2 6_2$ $0_2 3_2 8_2$

type BBD: $6_2 8_2 0_4$ $4_2 5_2 1_4$ $4_2 8_2 2_4$ $2_2 3_2 3_4$ $7_2 8_2 4_4$ $1_2 3_2 5_4$ $0_2 2_2 6_4$ $6_2 7_2 6_4$
$2_2 6_2 7_4$ $3_2 5_2 8_4$ $1_2 7_2 8_4$ $0_2 5_2 9_4$ $0_2 6_2 10_4$ $1_2 4_2 10_4$ $5_2 8_2 10_4$ $0_2 1_2 11_4$
$3_2 7_2 11_4$ $2_2 4_2 12_4$

Clearly, $45 \in LS_i(3, 9^*, 13^*)$ since a partition $\pi(1^1, 8^1, 12^3)$ can be formed and Theorem 1.24(b) is applicable. A direct construction is given to verify that $49 \in LS(3, 9^*, 13^*)$. Form the partition $\pi(1^1, 6^1, 12^2, 18^1)$, where cell A is the set $Z_{18} \times \{1\}$, cell B is the set $Z_6 \times \{2\}$, cells C and D are the sets $Z_{12} \times \{i\}$ (i = 3, 4), and embed an STS(9) into an STS(19) which contains ∞ and the eighteen points of cell A. Construct short lines of

type ACD: $i_1 0_3 i_4 (i = 0,1,...,11)$ $(i+12)_1 1_3 i_4 (i = 0,1,...,11)$ $(i+6)_1 2_3 i_4$
$(i = 0,1,...,11)$ $5_1 3_3 0_4$ $(10-i)_1 3_3 (i+1)_4 (i = 0,1,...,4)$ $17_1 3_3 6_4$ $4_1 3_3 7_4$
$12_1 3_3 8_4$ $2_1 3_3 9_4$ $14_1 3_3 10_4$ $0_1 3_3 11_4$ $11_1 4_3 0_4$ $(4-i)_1 4_3 (i+1)_4 (i = 0,1,...,4)$
$5_1 4_3 6_4$ $(16-i)_1 4_3 (i+7)_4 (i = 0,1,...,4)$ $17_1 5_3 0_4$ $16_1 5_3 1_4$ $0_1 5_3 2_4$ $1_1 5_3 3_4$

$13_{1}5_{3}4_{4}$ $3_{1}5_{3}5_{4}$ $11_{1}5_{3}6_{4}$ $(10-i)_{1}5_{3}(i+7)_{4}(i = 0,1,\ldots,4)$ $1_{1}6_{3}0_{4}$ $0_{1}6_{3}1_{4}$
$(11-i)_{1}6_{3}(i+2)_{4}(i = 0,1,\ldots,9)$ $13_{1}7_{3}0_{4}$ $12_{1}7_{3}1_{4}$ $(5-i)_{1}7_{3}(i+2)_{4}$
$(i = 0,1,\ldots,5)$ $(17-i)_{1}7_{3}(i+8)_{4}$ $(i = 0,1,2,3)$ $16_{1}8_{3}0_{4}$ $17_{1}8_{3}1_{4}$
$(6+i)_{1}8_{3}(i+2)_{4}$ $(i = 0,1,\ldots,5)$ $3_{1}8_{3}8_{4}$ $13_{1}8_{3}9_{4}$ $1_{1}8_{3}10_{4}$ $15_{1}8_{3}11_{4}$
$10_{1}9_{3}0_{4}$ $11_{1}9_{3}1_{4}$ $15_{1}9_{3}2_{4}$ $14_{1}9_{3}3_{4}$ $2_{1}9_{3}4_{4}$ $12_{1}9_{3}5_{4}$ $(4+i)_{1}9_{3}(i+6)_{4}$
$(i = 0,1,\ldots,5)$ $4_{1}10_{3}0_{4}$ $5_{1}10_{3}1_{4}$ $12_{1}10_{3}2_{4}$ $(13+i)_{1}10_{3}(i+3)_{4}$
$(i = 0,1,\ldots,4)$ $i_{1}10_{3}(i+8)_{4}(i = 0,1,2,3)$ $(9-i)_{1}11_{3}i_{4}(i = 0,1,2,3)$
$(17-i)_{1}11_{3}(i+4)_{4}(i = 0,1,\ldots,7)$

type ABD: $0_{1}0_{2}3_{4}$ $0_{1}2_{2}4_{4}$ $0_{1}4_{2}9_{4}$ $0_{1}1_{2}10_{4}$ $1_{1}0_{2}2_{4}$ $1_{1}1_{2}5_{4}$ $1_{1}5_{2}8_{4}$ $1_{1}3_{2}11_{4}$
$2_{1}1_{2}0_{4}$ $2_{1}2_{2}1_{4}$ $2_{1}4_{2}6_{4}$ $2_{1}0_{2}7_{4}$ $3_{1}2_{2}0_{4}$ $3_{1}3_{2}1_{4}$ $3_{1}5_{2}6_{4}$ $3_{1}1_{2}7_{4}$ $4_{1}2_{2}2_{4}$
$4_{1}3_{2}5_{4}$ $4_{1}4_{2}8_{4}$ $4_{1}5_{2}11_{4}$ $5_{1}1_{2}3_{4}$ $5_{1}0_{2}4_{4}$ $5_{1}2_{2}9_{4}$ $5_{1}3_{2}10_{4}$ $6_{1}1_{2}1_{4}$ $6_{1}4_{2}4_{4}$
$6_{1}0_{2}9_{4}$ $6_{1}5_{2}10_{4}$ $7_{1}3_{2}0_{4}$ $7_{1}5_{2}5_{4}$ $7_{1}1_{2}8_{4}$ $7_{1}4_{2}11_{4}$ $8_{1}4_{2}0_{4}$ $8_{1}0_{2}6_{4}$ $8_{1}3_{2}7_{4}$
$8_{1}2_{2}11_{4}$ $9_{1}4_{2}1_{4}$ $9_{1}1_{2}6_{4}$ $9_{1}5_{2}7_{4}$ $9_{1}2_{2}10_{4}$ $10_{1}5_{2}2_{4}$ $10_{1}0_{2}5_{4}$ $10_{1}2_{2}8_{4}$
$10_{1}3_{2}9_{4}$ $11_{1}2_{2}3_{4}$ $11_{1}1_{2}4_{4}$ $11_{1}0_{2}8_{4}$ $11_{1}5_{2}9_{4}$ $12_{1}3_{2}3_{4}$ $12_{1}5_{2}4_{4}$ $12_{1}2_{2}7_{4}$
$12_{1}4_{2}10_{4}$ $13_{1}4_{2}2_{4}$ $13_{1}2_{2}5_{4}$ $13_{1}3_{2}6_{4}$ $13_{1}1_{2}11_{4}$ $14_{1}0_{2}0_{4}$ $14_{1}5_{2}1_{4}$ $14_{1}4_{2}5_{4}$
$14_{1}2_{2}6_{4}$ $15_{1}5_{2}0_{4}$ $15_{1}0_{2}1_{4}$ $15_{1}3_{2}4_{4}$ $15_{1}4_{2}7_{4}$ $16_{1}1_{2}2_{4}$ $16_{1}4_{2}3_{4}$ $16_{1}3_{2}8_{4}$
$16_{1}0_{2}11_{4}$ $17_{1}3_{2}2_{4}$ $17_{1}5_{2}3_{4}$ $17_{1}1_{2}9_{4}$ $17_{1}0_{2}10_{4}$

type ABC: $0_{1}3_{2}2_{3}$ $0_{1}5_{2}8_{3}$ $1_{1}4_{2}9_{3}$ $1_{1}2_{2}11_{3}$ $2_{1}3_{2}11_{3}$ $2_{1}5_{2}2_{3}$ $3_{1}0_{2}11_{3}$ $3_{1}4_{2}2_{3}$
$4_{1}0_{2}8_{3}$ $4_{1}1_{2}11_{3}$ $5_{1}4_{2}8_{3}$ $5_{1}5_{2}5_{3}$ $6_{1}2_{2}1_{3}$ $6_{1}3_{2}4_{3}$ $7_{1}0_{2}7_{3}$ $7_{1}2_{2}10_{3}$ $8_{1}1_{2}1_{3}$
$8_{1}5_{2}7_{3}$ $9_{1}0_{2}4_{3}$ $9_{1}3_{2}7_{3}$ $10_{1}1_{2}10_{3}$ $10_{1}4_{2}1_{3}$ $11_{1}3_{2}3_{3}$ $11_{1}4_{2}10_{3}$ $12_{1}0_{2}0_{3}$
$12_{1}1_{2}8_{3}$ $13_{1}0_{2}6_{3}$ $13_{1}5_{2}9_{3}$ $14_{1}1_{2}0_{3}$ $14_{1}3_{2}5_{3}$ $15_{1}1_{2}5_{3}$ $15_{1}2_{2}3_{3}$ $16_{1}2_{2}6_{3}$
$16_{1}5_{2}3_{3}$ $17_{1}2_{2}0_{3}$ $17_{1}4_{2}6_{3}$

type ACC: $0_{1}9_{3}11_{3}$ $1_{1}2_{3}3_{3}$ $2_{1}5_{3}8_{3}$ $3_{1}3_{3}9_{3}$ $4_{1}2_{3}5_{3}$ $5_{1}2_{3}11_{3}$ $6_{1}7_{3}10_{3}$ $7_{1}1_{3}4_{3}$
$8_{1}4_{3}10_{3}$ $9_{1}1_{3}10_{3}$ $10_{1}4_{3}7_{3}$ $11_{1}1_{3}7_{3}$ $12_{1}5_{3}6_{3}$ $13_{1}0_{3}3_{3}$ $14_{1}6_{3}8_{3}$ $15_{1}0_{3}6_{3}$
$16_{1}0_{3}9_{3}$ $17_{1}4_{3}9_{3}$

type BCC: $0_{2}1_{3}3_{3}$ $0_{2}2_{3}9_{3}$ $0_{2}5_{3}10_{3}$ $1_{2}2_{3}4_{3}$ $1_{2}7_{3}9_{3}$ $1_{2}3_{3}6_{3}$ $2_{2}2_{3}7_{3}$ $2_{2}4_{3}5_{3}$ $2_{2}8_{3}9_{3}$
$3_{2}0_{3}1_{3}$ $3_{2}6_{3}9_{3}$ $3_{2}8_{3}10_{3}$ $4_{2}0_{3}4_{3}$ $4_{2}3_{3}7_{3}$ $4_{2}5_{3}11_{3}$ $5_{2}0_{3}10_{3}$ $5_{2}1_{3}6_{3}$
$5_{2}4_{3}11_{3}$

type CCC: $0_{3}5_{3}7_{3}$ $0_{3}2_{3}8_{3}$ $1_{3}5_{3}9_{3}$ $1_{3}8_{3}11_{3}$ $2_{3}6_{3}10_{3}$ $3_{3}4_{3}8_{3}$ $3_{3}10_{3}11_{3}$ $6_{3}7_{3}11_{3}$

type ∞CC: $\infty 0_{3}11_{3}$ $\infty 3_{3}5_{3}$ $\infty 1_{3}2_{3}$ $\infty 4_{3}6_{3}$ $\infty 7_{3}8_{3}$ $\infty 9_{3}10_{3}$

type BBB: $0_21_22_2$ $0_23_24_2$ $1_23_25_2$ $2_24_25_2$

type ∞BB: $\infty 0_25_2$ $\infty 1_24_2$ $\infty 2_23_2$

One long line is $0_41_4 \cdots 11_4\infty$ and the other long line is formed by replacing the subsystem of STS(19).

(b) We demonstrate that $43 \in LS(3, 9^*, 15^*)$ by giving two direct constructions. First, form the partition $\pi(9^2, 10^1, 15^1)$ and construct short lines of

type ABD: $0_1i_2i_4$ $0_17_22_4$ $0_18_26_4$ $0_12_27_4$ $0_16_28_4$ $1_1i_2(i+2)_4$
$1_17_24_4$ $1_18_28_4$ $1_12_29_4$ $1_16_210_4$ $2_1i_2(i+4)_4$
$2_17_26_4$ $2_18_210_4$ $2_12_211_4$ $2_16_212_4$ $3_1i_2(i+6)_4$
$3_17_28_4$ $3_18_212_4$ $3_12_213_4$ $3_16_214_4$ $4_1i_2(i+8)_4$
$4_17_210_4$ $4_18_214_4$ $4_12_20_4$ $4_16_21_4$ $5_1i_2(i+10)_4$
$5_17_212_4$ $5_18_21_4$ $5_12_22_4$ $5_16_23_4$ $6_1i_2(i+12)_4$
$6_17_214_4$ $6_18_23_4$ $6_12_24_4$ $6_16_25_4$ $7_1i_2(i+14)_4$
$7_17_21_4$ $7_18_25_4$ $7_12_26_4$ $7_16_27_4$ $8_1i_2(i+1)_4$ $(i = 0,1,3,4,5)$
$8_17_23_4$ $8_18_27_4$ $8_12_28_4$ $8_16_29_4$

type ACD: $0_11_39_4$ $0_10_310_4$ $0_12_311_4$ $0_19_312_4$ $0_14_313_4$ $0_16_314_4$ $1_11_30_4$ $1_10_31_4$
$1_19_311_4$ $1_14_312_4$ $1_16_313_4$ $1_18_314_4$ $2_12_30_4$ $2_19_31_4$ $2_14_32_4$ $2_16_33_4$
$2_18_313_4$ $2_17_314_4$ $3_19_30_4$ $3_14_31_4$ $3_16_32_4$ $3_18_33_4$ $3_17_34_4$ $3_15_35_4$ $4_10_32_4$
$4_17_33_4$ $4_16_34_4$ $4_13_35_4$ $4_15_36_4$ $4_11_37_4$ $5_14_34_4$ $5_16_35_4$ $5_10_36_4$ $5_17_37_4$
$5_13_38_4$ $5_15_39_4$ $6_18_36_4$ $6_1(i+5)_3(i+7)_4$ $(i = 0,1,4,5)$ $6_13_39_4$ $7_15_38_4$
$7_10_39_4$ $7_14_310_4$ $7_11_311_4$ $7_12_312_4$ $7_13_313_4$ $8_18_30_4$ $8_17_310_4$ $8_13_311_4$
$8_10_312_4$ $8_11_313_4$ $8_12_314_4$

type ACC: $0_13_35_3$ $0_17_38_3$ $1_12_35_3$ $1_13_37_3$ $2_10_35_3$ $2_11_33_3$ $3_10_31_3$ $3_12_33_3$ $4_12_34_3$
$4_18_39_3$ $5_12_38_3$ $5_11_39_3$ $6_11_34_3$ $6_12_36_3$ $7_17_39_3$ $7_16_38_3$ $8_16_39_3$ $8_14_35_3$

type BCD: $0_23_33_4$ $0_21_35_4$ $0_22_37_4$ $0_29_39_4$ $0_24_311_4$ $0_27_313_4$ $1_25_34_4$ $1_26_36_4$ $1_28_38_4$
$1_22_310_4$ $1_23_312_4$ $1_21_314_4$ $2_23_31_4$ $2_22_33_4$ $2_29_35_4$ $2_28_310_4$ $2_27_312_4$
$2_25_314_4$ $3_27_31_4$ $3_23_36_4$ $3_22_38_4$ $3_26_310_4$ $3_21_312_4$ $3_29_314_4$ $4_24_30_4$
$4_27_32_4$ $4_26_37_4$ $4_28_39_4$ $4_25_311_4$ $4_29_313_4$ $5_22_31_4$ $5_21_33_4$ $5_20_38_4$
$5_23_310_4$ $5_26_312_4$ $5_24_314_4$ $6_23_30_4$ $6_25_32_4$ $6_21_34_4$ $6_24_36_4$ $6_26_311_4$
$6_20_313_4$ $7_26_30_4$ $7_22_35_4$ $7_23_37_4$ $7_24_39_4$ $7_28_311_4$ $7_25_313_4$ $8_20_30_4$

$8_2 8_3 2_4$ $8_2 9_3 4_4$ $8_2 6_3 9_4$ $8_2 7_3 11_4$ $8_2 2_3 13_4$

type BBC: $0_2 7_2 0_3$ $2_2 4_2 0_3$ $1_2 3_2 0_3$ $4_2 7_2 1_3$ $2_2 8_2 1_3$ $4_2 6_2 2_3$ $4_2 8_2 3_3$ $1_2 2_2 4_3$ $3_2 8_2 4_3$ $0_2 3_2 5_3$ $5_2 8_2 5_3$ $0_2 2_2 6_3$ $1_2 5_2 7_3$ $6_2 7_2 7_3$ $0_2 5_2 8_3$ $3_2 6_2 8_3$ $1_2 6_2 9_3$ $5_2 7_2 9_3$

type BBB: $0_2 1_2 4_2$ $0_2 6_2 8_2$ $1_2 7_2 8_2$ $2_2 3_2 7_2$ $3_2 4_2 5_2$ $2_2 5_2 6_2$

type CCD: $5_3 7_3 0_4$ $1_3 8_3 1_4$ $5_3 6_3 1_4$ $1_3 2_3 2_4$ $3_3 9_3 2_4$ $0_3 4_3 3_4$ $5_3 9_3 3_4$ $3_3 8_3 4_4$ $0_3 2_3 4_4$ $0_3 8_3 5_4$ $4_3 7_3 5_4$ $1_3 7_3 6_4$ $2_3 9_3 6_4$ $0_3 9_3 7_4$ $4_3 8_3 7_4$ $1_3 6_3 8_4$ $4_3 9_3 8_4$ $2_3 7_3 9_4$ $1_3 5_3 10_4$ $5_3 8_3 12_4$ $0_3 3_3 14_4$

type CCC: $0_3 6_3 7_3$ $3_3 4_3 6_3$

The long lines are $0_1 1_1 \cdots 8_1$ and $0_4 1_4 \cdots 14_4$. Second, form the partition $\pi(1^1, 8^2, 12^1, 14^1)$ and construct short lines of

type ABD: $0_1 i_2 i_4$ $1_1 i_2 (i+2)_4$ $2_1 i_2 (i+4)_4$ $3_1 i_2 (i+6)_4$ $4_1 i_2 (i+8)_4$ $5_1 i_2 (i+10)_4$ $6_1 i_2 (i+12)_4$ $7_1 i_2 (i+1)_4 (i = 0,1,...,7)$

type ACD: $0_1 i_3 (i+8)_4 (i = 0,1,...,4)$ $0_1 8_3 13_4$ $1_1 11_3 10_4$ $1_1 6_3 11_4$ $1_1 0_3 12_4$ $1_1 9_3 13_4$ $1_1 4_3 0_4$ $1_1 7_3 1_4$ $2_1 1_3 12_4$ $2_1 5_3 13_4$ $2_1 2_3 0_4$ $2_1 3_3 1_4$ $2_1 4_3 2_4$ $2_1 7_3 3_4$ $3_1 6_3 0_4$ $3_1 11_3 1_4$ $3_1 5_3 2_4$ $3_1 9_3 3_4$ $3_1 10_3 4_4$ $3_1 0_3 5_4$ $4_1 i_3 (i+2)_4 (i = 0,1,4,5)$ $4_1 3_3 4_4$ $4_1 2_3 5_4$ $5_1 6_3 4_4$ $5_1 11_3 5_4$ $5_1 8_3 6_4$ $5_1 9_3 7_4$ $5_1 10_3 8_4$ $5_1 7_3 9_4$ $6_1 i_3 (i+6)_4$ $(i = 0,1,...,5)$ $7_1 (i+6)_3 (i+9)_4 (i = 0,1,2,3,5)$ $7_1 4_3 13_4$

type ACC: $0_1 6_3 11_3$ $0_1 7_3 10_3$ $0_1 5_3 9_3$ $1_1 5_3 8_3$ $1_1 1_3 10_3$ $1_1 2_3 3_3$ $2_1 10_3 11_3$ $2_1 0_3 8_3$ $2_1 6_3 9_3$ $3_1 4_3 7_3$ $3_1 1_3 3_3$ $3_1 2_3 8_3$ $4_1 7_3 11_3$ $4_1 6_3 8_3$ $4_1 9_3 10_3$ $5_1 0_3 1_3$ $5_1 2_3 5_3$ $5_1 3_3 4_3$ $6_1 9_3 11_3$ $6_1 6_3 7_3$ $6_1 8_3 10_3$ $7_1 0_3 3_3$ $7_1 1_3 2_3$ $7_1 5_3 10_3$

type BCD: $0_2 8_3 3_4$ $0_2 1_3 5_4$ $0_2 3_3 7_4$ $0_2 10_3 9_4$ $0_2 11_3 11_4$ $0_2 2_3 13_4$ $1_2 5_3 0_4$ $1_2 7_3 4_4$ $1_2 9_3 6_4$ $1_2 8_3 8_4$ $1_2 10_3 10_4$ $1_2 6_3 12_4$ $2_2 5_3 1_4$ $2_2 3_3 5_4$ $2_2 2_3 7_4$ $2_2 8_3 9_4$ $2_2 1_3 11_4$ $2_2 10_3 13_4$ $3_2 7_3 0_4$ $3_2 11_3 2_4$ $3_2 10_3 6_4$ $3_2 9_3 8_4$ $3_2 1_3 10_4$ $3_2 8_3 12_4$ $4_2 2_3 1_4$ $4_2 0_3 3_4$ $4_2 7_3 7_4$ $4_2 11_3 9_4$ $4_2 10_3 11_4$ $4_2 6_3 13_4$ $5_2 3_3 0_4$ $5_2 8_3 2_4$ $5_2 5_3 4_4$ $5_2 6_3 8_4$ $5_2 9_3 10_4$ $5_2 10_3 12_4$ $6_2 1_3 1_4$ $6_2 2_3 3_4$ $6_2 10_3 5_4$ $6_2 5_3 9_4$ $6_2 9_3 11_4$ $6_2 0_3 13_4$ $7_2 10_3 0_4$ $7_2 3_3 2_4$ $7_2 8_3 4_4$ $7_2 11_3 6_4$ $7_2 6_3 10_4$ $7_2 7_3 12_4$

type BBC: $0_2 2_2 0_3$ $1_2 5_2 0_3$ $3_2 7_2 0_3$ $1_2 7_2 1_3$ $4_2 5_2 1_3$ $1_2 3_2 2_3$ $5_2 7_2 2_3$ $1_2 4_2 3_3$ $3_2 6_2 3_3$ $1_2 6_2 4_3$ $3_2 5_2 4_3$ $0_2 4_2 4_3$ $2_2 7_2 4_3$ $0_2 3_2 5_3$ $4_2 7_2 5_3$ $0_2 6_2 6_3$ $2_2 3_2 6_3$ $0_2 5_2 7_3$ $2_2 6_2 7_3$ $4_2 6_2 8_3$ $0_2 7_2 9_3$ $2_2 4_2 9_3$ $1_2 2_2 11_3$ $5_2 6_2 11_3$

type ∞BB: $\infty 0_2 1_2$ $\infty 2_2 5_2$ $\infty 3_2 4_2$ $\infty 6_2 7_2$

type CCD: $0_3 9_3 0_4$ $1_3 8_3 0_4$ $4_3 10_3 1_4$ $8_3 9_3 1_4$ $0_3 6_3 1_4$ $1_3 6_3 2_4$ $2_3 10_3 2_4$ $7_3 9_3 2_4$ $3_3 10_3 3_4$ $4_3 6_3 3_4$ $5_3 11_3 3_4$ $0_3 2_3 4_4$ $1_3 9_3 4_4$ $4_3 11_3 4_4$ $7_3 8_3 5_4$ $4_3 9_3 5_4$ $5_3 6_3 5_4$ $1_3 5_3 6_4$ $3_3 6_3 6_4$ $2_3 7_3 6_4$ $4_3 8_3 7_4$ $6_3 10_3 7_4$ $0_3 11_3 7_4$ $1_3 7_3 8_4$ $3_3 11_3 8_4$ $4_3 5_3 8_4$ $0_3 4_3 9_4$ $2_3 9_3 9_4$ $0_3 5_3 10_4$ $3_3 8_3 10_4$ $0_3 7_3 11_4$ $2_3 4_3 11_4$ $2_3 11_3 12_4$ $3_3 5_3 12_4$ $3_3 7_3 13_4$ $1_3 11_3 13_4$

type ∞CC: $\infty 2_3 6_3$ $\infty 0_3 10_3$ $\infty 5_3 7_3$ $\infty 1_3 4_3$ $\infty 8_3 11_3$ $\infty 3_3 9_3$

The long lines are $0_1 1_1 \cdots 7_1 \infty$ and $0_4 1_4 \cdots 13_4 \infty$. Similarly, direct constructions are provided to show that $45 \in LS(3, 9^*, 15^*)$. Form the partition $\pi(9^2, 12^1, 15^1)$ and construct short lines of

type ABD: $0_1 i_2 i_4 (i = 0,2,4,6)$ $0_1 7_2 1_4$ $0_1 5_2 3_4$ $0_1 8_2 5_4$ $0_1 3_2 7_4$ $0_1 1_2 8_4$ $1_1 0_2 2_4$ $1_1 (i+1)_2 (i+3)_4 (i = 0,1,\ldots,5)$ $1_1 8_2 9_4$ $1_1 7_2 10_4$ $2_1 i_2 (i+4)_4 (i = 0,2,4,6,7,8)$ $2_1 5_2 5_4$ $2_1 1_2 7_4$ $2_1 3_2 9_4$ $3_1 7_2 5_4$ $3_1 i_2 (i+6)_4 (i = 0,2,4,6,8)$ $3_1 5_2 9_4$ $3_1 3_2 11_4$ $3_1 1_2 13_4$ $4_1 i_2 (i+8)_4 (i = 0,2,6,7,8)$ $4_1 4_2 9_4$ $4_1 5_2 11_4$ $4_1 1_2 12_4$ $4_1 3_2 13_4$ $5_1 0_2 10_4$ $5_1 4_2 11_4$ $5_1 2_2 12_4$ $5_1 5_2 13_4$ $5_1 1_2 14_4$ $5_1 3_2 0_4$ $5_1 6_2 1_4$ $5_1 7_2 2_4$ $5_1 8_2 3_4$ $6_1 0_2 12_4$ $6_1 7_2 13_4$ $6_1 2_2 14_4$ $6_1 5_2 0_4$ $6_1 4_2 1_4$ $6_1 3_2 2_4$ $6_1 6_2 3_4$ $6_1 1_2 4_4$ $6_1 8_2 7_4$ $7_1 i_2 (i+14)_4 (i = 0,1,2,4,6)$ $7_1 5_2 2_4$ $7_1 3_2 4_4$ $7_1 8_2 6_4$ $7_1 7_2 7_4$ $8_1 i_2 (i+1)_4 (i = 0,1,2,4,6)$ $8_1 5_2 4_4$ $8_1 3_2 6_4$ $8_1 8_2 8_4$ $8_1 7_2 9_4$

type ACD: $0_1 5_3 9_4$ $0_1 11_3 10_4$ $0_1 2_3 11_4$ $0_1 0_3 12_4$ $0_1 7_3 13_4$ $0_1 4_3 14_4$ $1_1 (i+6)_3 (i+11)_4$ $(i = 0,2,3,4)$ $1_1 3_3 12_4$ $1_1 1_3 1_4$ $2_1 1_3 13_4$ $2_1 0_3 14_4$ $2_1 2_3 0_4$ $2_1 7_3 1_4$ $2_1 4_3 2_4$ $2_1 3_3 3_4$ $3_1 0_3 0_4$ $3_1 6_3 1_4$ $3_1 8_3 2_4$ $3_1 9_3 3_4$ $3_1 10_3 4_4$ $3_1 11_3 7_4$ $4_1 5_3 2_4$ $4_1 1_3 3_4$ $4_1 2_3 4_4$ $4_1 10_3 5_4$ $4_1 4_3 6_4$ $4_1 8_3 7_4$ $5_1 6_3 4_4$ $5_1 7_3 5_4$ $5_1 8_3 6_4$ $5_1 4_3 7_4$ $5_1 0_3 8_4$ $5_1 1_3 9_4$ $6_1 3_3 5_4$ $6_1 10_3 6_4$ $6_1 2_3 8_4$ $6_1 9_3 9_4$ $6_1 4_3 10_4$ $6_1 7_3 11_4$ $7_1 (i+6)_3 (i+8)_4 (i = 0,1,\ldots,5)$ $8_1 0_3 10_4$ $8_1 3_3 11_4$ $8_1 11_3 12_4$ $8_1 9_3 13_4$ $8_1 2_3 14_4$ $8_1 7_3 0_4$

type ACC: $0_1 1_3 6_3$ $0_1 8_3 10_3$ $0_1 3_3 9_3$ $1_1 0_3 5_3$ $1_1 2_3 4_3$ $1_1 7_3 11_3$ $2_1 6_3 8_3$ $2_1 9_3 11_3$ $2_1 5_3 10_3$ $3_1 4_3 7_3$ $3_1 1_3 5_3$ $3_1 2_3 3_3$ $4_1 6_3 9_3$ $4_1 3_3 11_3$ $4_1 0_3 7_3$ $5_1 3_3 10_3$ $5_1 2_3 11_3$ $5_1 5_3 9_3$ $6_1 5_3 6_3$ $6_1 8_3 11_3$ $6_1 0_3 1_3$ $7_1 0_3 2_3$ $7_1 1_3 4_3$ $7_1 3_3 5_3$ $8_1 4_3 6_3$ $8_1 5_3 8_3$ $8_1 1_3 10_3$

type BCD: $0_2 7_3 3_4$ $0_2 5_3 5_4$ $0_2 9_3 7_4$ $0_2 11_3 9_4$ $0_2 0_3 11_4$ $0_2 2_3 13_4$ $1_2 11_3 1_4$ $1_2 5_3 6_4$

$1_2 6_3 5_4 \quad 1_2 3_3 9_4 \quad 1_2 9_3 10_4 \quad 1_2 4_3 11_4 \quad 2_2 5_3 0_4 \quad 2_2 2_3 5_4 \quad 2_2 3_3 7_4 \quad 2_2 4_3 9_4$
$2_2 1_3 11_4 \quad 2_2 6_3 13_4 \quad 3_2 8_3 1_4 \quad 3_2 0_3 3_4 \quad 3_2 5_3 8_4 \quad 3_2 1_3 10_4 \quad 3_2 9_3 12_4 \quad 3_2 7_3 14_4$
$4_2 6_3 0_4 \quad 4_2 2_3 2_4 \quad 4_2 1_3 7_4 \quad 4_2 8_3 12_4 \quad 4_2 5_3 13_4 \quad 4_2 10_3 14_4 \quad 5_2 5_3 1_4 \quad 5_2 3_3 6_4$
$5_2 11_3 8_4 \quad 5_2 10_3 10_4 \quad 5_2 7_3 12_4 \quad 5_2 8_3 14_4 \quad 6_2 11_3 0_4 \quad 6_2 10_3 2_4 \quad 6_2 3_3 4_4 \quad 6_2 2_3 9_4$
$6_2 5_3 11_4 \quad 6_2 4_3 13_4 \quad 7_2 6_3 3_4 \quad 7_2 0_3 4_4 \quad 7_2 11_3 6_4 \quad 7_2 8_3 8_4 \quad 7_2 1_3 12_4 \quad 7_2 5_3 14_4$
$8_2 1_3 0_4 \quad 8_2 6_3 2_4 \quad 8_2 5_3 4_4 \quad 8_2 2_3 10_4 \quad 8_2 8_3 11_4 \quad 8_2 3_3 13_4$

type BBC: $5_2 6_2 0_3 \quad 1_2 2_2 0_3 \quad 4_2 8_2 0_3 \quad 0_2 6_2 1_3 \quad 1_2 5_2 1_3 \quad 1_2 3_2 2_3 \quad 5_2 7_2 2_3 \quad 3_2 4_2 3_3$
$0_2 7_2 3_3 \quad 3_2 7_2 4_3 \quad 0_2 4_2 4_3 \quad 5_2 8_2 4_3 \quad 0_2 5_2 6_3 \quad 3_2 6_2 6_3 \quad 1_2 4_2 7_3 \quad 2_2 6_2 7_3 \quad 7_2 8_2 7_3$
$0_2 2_2 8_3 \quad 1_2 6_2 8_3 \quad 4_2 5_2 9_3 \quad 6_2 8_2 9_3 \quad 2_2 7_2 9_3 \quad 0_2 3_2 10_3 \quad 2_2 8_2 10_3 \quad i_2 7_2 10_3$
$3_2 8_2 11_3 \quad 2_2 4_2 11_3$

type BBB: $0_2 1_2 8_2 \quad 2_2 3_2 5_2 \quad 4_2 6_2 7_2$

type CCD: $3_3 8_3 0_4 \quad 4_3 9_3 0_4 \quad 0_3 3_3 1_4 \quad 2_3 9_3 1_4 \quad 4_3 10_3 1_4 \quad 1_3 9_3 2_4 \quad 0_3 11_3 2_4 \quad 3_3 7_3 2_4$
$4_3 8_3 3_4 \quad 5_3 11_3 3_4 \quad 2_3 10_3 3_4 \quad 1_3 8_3 4_4 \quad 4_3 11_3 4_4 \quad 7_3 9_3 4_4 \quad 1_3 11_3 5_4 \quad 8_3 9_3 5_4$
$0_3 4_3 5_4 \quad 1_3 2_3 6_4 \quad 6_3 7_3 6_4 \quad 0_3 9_3 6_4 \quad 0_3 6_3 7_4 \quad 2_3 5_3 7_4 \quad 7_3 10_3 7_4 \quad 9_3 10_3 8_4 \quad 3_3 4_3 8_4$
$1_3 7_3 8_4 \quad 0_3 8_3 9_4 \quad 6_3 10_3 9_4 \quad 3_3 6_3 10_4 \quad 5_3 7_3 10_4 \quad 10_3 11_3 11_4 \quad 2_3 6_3 12_4 \quad 4_3 5_3 12_4$
$0_3 10_3 13_4 \quad 6_3 11_3 14_4 \quad 1_3 3_3 14_4$

type CCC: $2_3 7_3 8_3$

The long lines are $0_1 1_1 \cdots 8_1$ and $0_4 1_4 \cdots 14_4$. Form the partition $\pi(1^1, 2^1, 8^1, 14^1, 20^1)$, where cell A is the set $Z_8 \times \{1\}$, cell B is the set $Z_2 \times \{2\}$, cell C is the set $Z_{20} \times \{3\}$ and cell D is the set $Z_{14} \times \{4\}$, and construct short lines of

type ABD: $0_1 0_2 0_4 \quad 0_1 1_2 1_4 \quad 1_1 0_2 2_4 \quad 1_1 1_2 3_4 \quad 2_1 0_2 4_4 \quad 2_1 1_2 5_4 \quad 3_1 0_2 6_4 \quad 3_1 1_2 7_4 \quad 4_1 0_2 8_4$
$4_1 1_2 9_4 \quad 5_1 0_2 10_4 \quad 5_1 1_2 11_4 \quad 6_1 0_2 12_4 \quad 6_1 1_2 13_4 \quad 7_1 0_2 1_4 \quad 7_1 1_2 2_4$

type ACD: $0_1 i_3 (i+2)_4 (i = 0,1,3,5,6) \quad 0_1 12_3 4_4 \quad 0_1 19_3 6_4 \quad 0_1 9_3 9_4 \quad 0_1 15_3 10_4 \quad 0_1 7_3 11_4$
$0_1 14_3 12_4 \quad 0_1 16_3 13_4 \quad 1_1 11_3 0_4 \quad 1_1 13_3 1_4 \quad 1_1 8_3 4_4 \quad 1_1 19_3 5_4 \quad 1_1 15_3 6_4$
$1_1 (i+17)_3 (i+7)_4 (i = 0,1,3,4,6) \quad 1_1 2_3 9_4 \quad 1_1 12_3 12_4 \quad 2_1 3_3 0_4 \quad 2_1 7_3 1_4$
$2_1 (i+4)_3 (i+2)_4 (i = 0,1,4,5,6,9,11) \quad 2_1 16_3 9_4 \quad 2_1 11_3 10_4 \quad 2_1 19_3 12_4 \quad 3_1 13_3 0_4$
$3_1 17_3 1_4 \quad 3_1 18_3 2_4 \quad 3_1 10_3 3_4 \quad 3_1 0_3 4_4 \quad 3_1 1_3 5_4 \quad 3_1 (i+2)_3 (i+8)_4 (i = 0,1,3,4)$
$3_1 16_3 10_4 \quad 3_1 9_3 13_4 \quad 4_1 8_3 0_4 \quad 4_1 9_3 1_4 \quad 4_1 15_3 2_4 \quad 4_1 3_3 3_4 \quad 4_1 7_3 4_4 \quad 4_1 13_3 5_4$
$4_1 4_3 6_4 \quad 4_1 14_3 7_4 \quad 4_1 10_3 10_4 \quad 4_1 17_3 11_4 \quad 4_1 18_3 12_4 \quad 4_1 19_3 13_4 \quad 5_1 i_3 i_4$
$(i = 0,1,2,4,5,7,8) \quad 5_1 19_3 3_4 \quad 5_1 3_3 6_4 \quad 5_1 10_3 9_4 \quad 5_1 11_3 12_4 \quad 5_1 14_3 13_4 \quad 6_1 9_3 0_4$

$6_1 10_3 1_4$ $6_1 19_3 2_4$ $6_1 14_3 3_4$ $6_1 2_3 4_4$ $6_1 17_3 5_4$ $6_1 18_3 6_4$ $6_1 15_3 7_4$ $6_1 0_3 8_4$
$6_1 1_3 9_4$ $6_1 6_3 10_4$ $6_1 3_3 11_4$ $7_1 11_3 3_4$ $7_1 5_3 4_4$ $7_1 12_3 5_4$ $7_1 6_3 6_4$ $7_1 8_3 7_4$
$7_1 13_3 8_4$ $7_1 14_3 9_4$ $7_1 2_3 10_4$ $7_1 16_3 11_4$ $7_1 9_3 12_4$ $7_1 4_3 13_4$ $7_1 10_3 0_4$

type ACC: $0_1 10_3 18_3$ $0_1 4_3 13_3$ $0_1 2_3 8_3$ $0_1 11_3 17_3$ $1_1 7_3 9_3$ $1_1 5_3 10_3$ $1_1 6_3 16_3$ $1_1 4_3 14_3$
$2_1 0_3 12_3$ $2_1 1_3 2_3$ $2_1 14_3 17_3$ $2_1 6_3 18_3$ $3_1 4_3 15_3$ $3_1 8_3 11_3$ $3_1 12_3 14_3$ $3_1 7_3 19_3$
$4_1 0_3 16_3$ $4_1 1_3 6_3$ $4_1 2_3 5_3$ $4_1 11_3 12_3$ $5_1 9_3 12_3$ $5_1 13_3 18_3$ $5_1 16_3 17_3$ $5_1 6_3 15_3$
$6_1 8_3 16_3$ $6_1 4_3 5_3$ $6_1 11_3 13_3$ $6_1 7_3 12_3$ $7_1 0_3 15_3$ $7_1 1_3 18_3$ $7_1 17_3 19_3$ $7_1 3_3 7_3$

type BCD: $0_2 0_3 3_4$ $0_2 2_3 5_4$ $0_2 6_3 7_4$ $0_2 12_3 9_4$ $0_2 11_3 11_4$ $0_2 7_3 13_4$ $1_2 1_3 0_4$ $1_2 3_3 4_4$
$1_2 5_3 6_4$ $1_2 4_3 8_4$ $1_2 12_3 10_4$ $1_2 0_3 12_4$

type BCC: $0_2 1_3 5_3$ $0_2 3_3 4_3$ $0_2 10_3 19_3$ $0_2 8_3 14_3$ $0_2 13_3 15_3$ $0_2 17_3 18_3$ $0_2 9_3 16_3$ $1_2 2_3 15_3$
$1_2 6_3 11_3$ $1_2 7_3 17_3$ $1_2 8_3 9_3$ $1_2 10_3 13_3$ $1_2 16_3 19_3$ $1_2 14_3 18_3$

type CCD: $2_3 14_3 0_4$ $15_3 19_3 0_4$ $5_3 17_3 0_4$ $4_3 6_3 0_4$ $7_3 18_3 0_4$ $12_3 16_3 0_4$ $0_3 6_3 1_4$ $2_3 19_3 1_4$
$4_3 12_3 1_4$ $16_3 18_3 1_4$ $3_3 15_3 1_4$ $5_3 8_3 1_4$ $11_3 14_3 1_4$ $1_3 7_3 2_4$ $3_3 13_3 2_4$ $11_3 16_3 2_4$
$5_3 12_3 2_4$ $8_3 10_3 2_4$ $6_3 17_3 2_4$ $9_3 14_3 2_4$ $2_3 16_3 3_4$ $4_3 7_3 3_4$ $13_3 17_3 3_4$ $6_3 9_3 3_4$
$8_3 18_3 3_4$ $12_3 15_3 3_4$ $10_3 11_3 4_4$ $6_3 14_3 4_4$ $1_3 16_3 4_4$ $18_3 19_3 4_4$ $9_3 13_3 4_4$
$15_3 17_3 4_4$ $0_3 10_3 5_4$ $7_3 11_3 5_4$ $6_3 8_3 5_4$ $15_3 18_3 5_4$ $4_3 9_3 5_4$ $14_3 16_3 5_4$ $0_3 13_3 6_4$
$1_3 14_3 6_4$ $7_3 16_3 6_4$ $2_3 12_3 6_4$ $10_3 17_3 6_4$ $9_3 11_3 6_4$ $0_3 1_3 7_4$ $2_3 10_3 7_4$ $3_3 19_3 7_4$
$4_3 16_3 7_4$ $11_3 18_3 7_4$ $12_3 13_3 7_4$ $1_3 9_3 8_4$ $12_3 17_3 8_4$ $11_3 15_3 8_4$ $5_3 19_3 8_4$
$7_3 14_3 8_4$ $3_3 16_3 8_4$ $0_3 11_3 9_4$ $7_3 15_3 9_4$ $4_3 17_3 9_4$ $5_3 18_3 9_4$ $6_3 19_3 9_4$ $8_3 13_3 9_4$
$1_3 17_3 10_4$ $13_3 14_3 10_4$ $5_3 7_3 10_4$ $8_3 19_3 10_4$ $3_3 9_3 10_4$ $4_3 18_3 10_4$ $0_3 4_3 11_4$
$8_3 15_3 11_4$ $10_3 14_3 11_4$ $12_3 19_3 11_4$ $2_3 6_3 11_4$ $9_3 18_3 11_4$ $1_3 15_3 12_4$ $10_3 16_3 12_4$
$5_3 13_3 12_4$ $7_3 8_3 12_4$ $2_3 4_3 12_4$ $3_3 17_3 12_4$ $0_3 18_3 13_4$ $10_3 12_3 13_4$ $2_3 13_3 13_4$
$1_3 11_3 13_4$ $5_3 6_3 13_4$ $8_3 17_3 13_4$

type CCC: $0_3 5_3 9_3$ $0_3 3_3 8_3$ $1_3 8_3 12_3$ $1_3 13_3 19_3$ $2_3 3_3 11_3$ $2_3 9_3 17_3$ $3_3 12_3 18_3$ $3_3 5_3 14_3$
$6_3 7_3 13_3$ $5_3 15_3 16_3$ $3_3 6_3 10_3$ $4_3 11_3 19_3$ $0_3 2_3 7_3$ $0_3 14_3 19_3$ $1_3 4_3 10_3$
$9_3 10_3 15_3$

type ∞CC: $\infty 5_3 11_3$ $\infty 14_3 15_3$ $\infty 2_3 18_3$ $\infty 0_3 17_3$ $\infty 7_3 10_3$ $\infty 1_3 3_3$ $\infty 6_3 12_3$ $\infty 13_3 16_3$
$\infty 4_3 8_3$ $\infty 9_3 19_3$

type ∞BB: $\infty 0_2 1_2$

The long lines are $0_1 1_1 \cdots 7_1 \infty$ and $0_4 1_4 \cdots 13_4 \infty$. In order to show that

$49 \in LS_d(3, 9^*, 15^*)$, form the partition $\pi(9^1, 10^1, 15^2)$ and construct short lines of

type ABD: $0_1 i_2 i_4\,(i = 0,1,4,5,8,9)$ $0_1 3_2 2_4$ $0_1 2_2 3_4$ $0_1 7_2 6_4$ $0_1 6_2 7_4$ $1_1 2_2 1_4$
$1_1 (i+1)_2 (i+2)_4 (i = 0,2,3,4,7,8)$ $1_1 0_2 3_4$ $1_1 7_2 7_4$ $1_1 6_2 8_4$ $2_1 i_2 (i+2)_4$
$(i = 0,1,\ldots,5,8,9)$ $2_1 7_2 8_4$ $2_1 6_2 9_4$ $3_1 3_2 3_4$ $3_1 (i+1)_2 (i+4)_4 (i = 0,3,4,7,8)$
$3_1 0_2 5_4$ $3_1 2_2 6_4$ $3_1 7_2 9_4$ $3_1 6_2 10_4$ $4_1 i_2 (i+4)_4 (i = 0,1,4,5,8,9)$ $4_1 3_2 6_4$
$4_1 2_2 7_4$ $4_1 7_2 10_4$ $4_1 6_2 11_4$ $5_1 2_2 5_4$ $5_1 (i+1)_2 (i+6)_4 (i = 0,2,3,4,7,8)$
$5_1 0_2 7_4$ $5_1 7_2 11_4$ $5_1 6_2 12_4$ $6_1 i_2 (i+6)_4 (i = 0,1,\ldots,5,8,9)$ $6_1 7_2 12_4$ $6_1 6_2 13_4$
$7_1 3_2 7_4$ $7_1 (i+1)_2 (i+8)_4 (i = 0,3,4,7,8)$ $7_1 0_2 9_4$ $7_1 2_2 10_4$ $7_1 7_2 13_4$ $7_1 6_2 14_4$
$8_1 i_2 (i+8)_4 (i = 0,1,4,5)$ $8_1 3_2 10_4$ $8_1 2_2 11_4$ $8_1 7_2 14_4$ $8_1 6_2 0_4$ $8_1 8_2 1_4$
$8_1 9_2 2_4$

type ACD: $0_1 i_3 (i+10)_4 (i = 0,1,2,4)$ $0_1 8_3 13_4$ $1_1 12_3 11_4$ $1_1 3_3 12_4$ $1_1 7_3 13_4$ $1_1 8_3 14_4$
$1_1 9_3 0_4$ $2_1 (i+10)_3 (i+12)_4 (i = 0,1,\ldots,4)$ $3_1 0_3 13_4$ $3_1 i_3 (i+13)_4$
$(i = 0,1,\ldots,4)$ $4_1 (i+5)_3 (i+14)_4 (i = 0,1,\ldots,4)$ $5_1 (i+10)_3 i_4 (i = 0,1,\ldots,4)$
$6_1 i_3 (i+1)_4 (i = 0,1,\ldots,3)$ $6_1 12_3 5_4$ $7_1 (i+5)_3 (i+2)_4 (i = 0,1,2,4)$ $7_1 4_3 5_4$
$8_1 10_3 3_4$ $8_1 13_3 4_4$ $8_1 6_3 5_4$ $8_1 11_3 6_4$ $8_1 14_3 7_4$

type ACC: $0_1 11_3 12_3$ $0_1 3_3 9_3$ $0_1 7_3 10_3$ $0_1 5_3 13_3$ $0_1 6_3 14_3$ $1_1 0_3 4_3$ $1_1 1_3 10_3$ $1_1 2_3 11_3$
$1_1 5_3 6_3$ $1_1 13_3 14_3$ $2_1 0_3 9_3$ $2_1 1_3 4_3$ $2_1 2_3 7_3$ $2_1 3_3 6_3$ $2_1 5_3 8_3$ $3_1 12_3 13_3$
$3_1 8_3 10_3$ $3_1 6_3 7_3$ $3_1 9_3 11_3$ $3_1 5_3 14_3$ $4_1 0_3 12_3$ $4_1 1_3 11_3$ $4_1 2_3 13_3$ $4_1 3_3 14_3$
$4_1 4_3 10_3$ $5_1 0_3 6_3$ $5_1 1_3 7_3$ $5_1 2_3 8_3$ $5_1 3_3 4_3$ $5_1 5_3 9_3$ $6_1 7_3 9_3$ $6_1 6_3 8_3$ $6_1 4_3 5_3$
$6_1 10_3 13_3$ $6_1 11_3 14_3$ $7_1 0_3 10_3$ $7_1 1_3 14_3$ $7_1 2_3 12_3$ $7_1 3_3 13_3$ $7_1 8_3 11_3$ $8_1 0_3 7_3$
$8_1 1_3 9_3$ $8_1 2_3 3_3$ $8_1 5_3 12_3$ $8_1 4_3 8_3$

type BCD: $0_2 2_3 1_4$ $0_2 1_3 10_4$ $0_2 5_3 11_4$ $0_2 6_3 12_4$ $0_2 14_3 13_4$ $0_2 0_3 14_4$ $1_2 8_3 10_4$
$1_2 (i+7)_3 (i+11)_4 (i = 0,2,3,4)$ $1_2 5_3 12_4$ $2_2 7_3 0_4$ $2_2 9_3 2_4$ $2_2 0_3 9_4$ $2_2 13_3 12_4$
$2_2 4_3 13_4$ $2_2 6_3 14_4$ $3_2 1_3 0_4$ $3_2 4_3 1_4$ $3_2 2_3 11_4$ $3_2 8_3 12_4$ $3_2 5_3 13_4$ $3_2 3_3 14_4$
$4_2 (i+10)_3 (i+13)_4 (i = 0,1,3,4,5)$ $4_2 5_3 0_4$ $5_2 2_3 14_4$ $5_2 3_3 0_4$ $5_2 6_3 1_4$ $5_2 7_3 2_4$
$5_2 1_3 3_4$ $5_2 4_3 4_4$ $6_2 12_3 1_4$ $6_2 3_3 2_4$ $6_2 8_3 3_4$ $6_2 0_3 4_4$ $6_2 11_3 5_4$ $6_2 2_3 6_4$ $7_2 4_3 0_4$
$7_2 9_3 1_4$ $7_2 13_3 2_4$ $7_2 5_3 3_4$ $7_2 10_3 4_4$ $7_2 1_3 5_4$ $8_2 6_3 2_4$ $8_2 14_3 3_4$ $8_2 5_3 4_4$
$8_2 13_3 5_4$ $8_2 7_3 6_4$ $8_2 8_3 7_4$ $9_2 11_3 3_4$ $9_2 9_3 4_4$ $9_2 10_3 5_4$ $9_2 14_3 6_4$ $9_2 12_3 7_4$
$9_2 3_3 8_4$

type BBC: $1_2 8_2 0_3$ $3_2 9_2 0_3$ $5_2 7_2 0_3$ $1_2 6_2 1_3$ $2_2 8_2 1_3$ $4_2 9_2 1_3$ $1_2 2_2 2_3$ $4_2 7_2 2_3$ $8_2 9_2 2_3$

$1_2 4_2 3_3$ $2_2 7_2 3_3$ $0_2 8_2 3_3$ $0_2 1_2 4_3$ $4_2 8_2 4_3$ $6_2 9_2 4_3$ $2_2 9_2 5_3$ $5_2 6_2 5_3$ $1_2 9_2 6_3$
$3_2 4_2 6_3$ $6_2 7_2 6_3$ $0_2 3_2 7_3$ $4_2 6_2 7_3$ $7_2 9_2 7_3$ $0_2 7_2 8_3$ $5_2 9_2 8_3$ $2_2 4_2 8_3$ $0_2 6_2 9_3$
$3_2 8_2 9_3$ $4_2 5_2 9_3$ $0_2 5_2 10_3$ $2_2 3_2 10_3$ $6_2 8_2 10_3$ $0_2 2_2 11_3$ $3_2 7_2 11_3$ $5_2 8_2 11_3$
$0_2 4_2 12_3$ $1_2 3_2 12_3$ $2_2 5_2 12_3$ $7_2 8_2 12_3$ $0_2 9_2 13_3$ $1_2 5_2 13_3$ $3_2 6_2 13_3$ $3_2 5_2 14_3$
$2_2 6_2 14_3$ $1_2 7_2 14_3$

type CCD: $0_3 8_3 0_4$ $12_3 14_3 0_4$ $1_3 8_3 1_4$ $5_3 10_3 1_4$ $0_3 2_3 2_4$ $10_3 11_3 2_4$ $3_3 7_3 3_4$ $4_3 12_3 3_4$
$1_3 2_3 4_4$ $6_3 11_3 4_4$ $8_3 12_3 4_4$ $0_3 3_3 5_4$ $2_3 9_3 5_4$ $5_3 7_3 5_4$ $8_3 14_3 5_4$ $0_3 5_3 6_4$
$1_3 3_3 6_4$ $4_3 6_3 6_4$ $8_3 13_3 6_4$ $10_3 12_3 6_4$ $0_3 13_3 7_4$ $1_3 6_3 7_4$ $2_3 10_3 7_4$ $3_3 5_3 7_4$
$7_3 11_3 7_4$ $4_3 9_3 7_4$ $0_3 14_3 8_4$ $1_3 5_3 8_4$ $2_3 4_3 8_4$ $8_3 9_3 8_4$ $11_3 13_3 8_4$ $7_3 12_3 8_4$
$6_3 10_3 8_4$ $1_3 12_3 9_4$ $2_3 14_3 9_4$ $3_3 10_3 9_4$ $4_3 13_3 9_4$ $7_3 8_3 9_4$ $6_3 9_3 9_4$ $5_3 11_3 9_4$
$2_3 5_3 10_4$ $6_3 12_3 10_4$ $4_3 7_3 10_4$ $3_3 11_3 10_4$ $10_3 14_3 10_4$ $9_3 13_3 10_4$ $0_3 11_3 11_4$
$6_3 13_3 11_4$ $3_3 8_3 11_4$ $4_3 14_3 11_4$ $9_3 10_3 11_4$ $0_3 1_3 12_4$ $4_3 11_3 12_4$ $7_3 14_3 12_4$
$9_3 12_3 12_4$ $1_3 13_3 13_4$ $2_3 6_3 13_4$ $3_3 12_3 13_4$ $7_3 13_3 14_4$ $9_3 14_3 14_4$

The long lines are $0_1 1_1 \cdots 8_1$ and $0_4 1_4 \cdots 14_4$.

We can prove that $LS_i(49;\{3, 9^*, 15^*)$ exists by forming the partition $\pi(1^1, 8^1, 12^1, 14^2)$ and constructing short lines of

type ABD: $0_1 2_2 0_4$ $0_1 1_2 2_4$ $0_1 0_2 4_4$ $0_1 (i+3)_2 (2i+6)_4 (i = 0,1,2,3)$ $0_1 7_2 1_4$ $0_1 8_2 3_4$
$0_1 9_2 5_4$ $1_1 7_2 3_4$ $1_1 (i+3)_2 (2i+5)_4 (i = 0,1,2,3)$ $1_1 2_2 13_4$ $1_1 8_2 2_4$ $1_1 9_2 4_4$
$1_1 10_2 6_4$ $2_1 0_2 2_4$ $2_1 3_2 4_4$ $2_1 (i+4)_2 (2i+6)_4 (i = 0,1,\ldots,4)$ $2_1 9_2 3_4$ $2_1 10_2 5_4$
$2_1 11_2 7_4$ $3_1 (i+4)_2 (2i+5)_4 (i = 0,1,\ldots,5)$ $3_1 10_2 4_4$ $3_1 11_2 6_4$ $3_1 2_2 8_4$
$4_1 (i+5)_2 (2i+6)_4 (i = 0,1,\ldots,5)$ $4_1 11_2 5_4$ $4_1 2_2 7_4$ $4_1 1_2 9_4$ $5_1 (i+6)_2 (2i+7)_4$
$(i = 0,1,\ldots,5)$ $5_1 2_2 6_4$ $5_1 1_2 8_4$ $5_1 0_2 10_4$ $6_1 7_2 8_4$ $6_1 8_2 10_4$ $6_1 9_2 12_4$
$6_1 10_2 0_4$ $6_1 11_2 2_4$ $6_1 2_2 4_4$ $6_1 1_2 7_4$ $6_1 0_2 9_4$ $6_1 3_2 11_4$ $7_1 8_2 9_4$ $7_1 9_2 11_4$
$7_1 10_2 13_4$ $7_1 11_2 1_4$ $7_1 2_2 3_4$ $7_1 1_2 5_4$ $7_1 0_2 8_4$ $7_1 3_2 10_4$ $7_1 4_2 12_4$ $i_1 i_2 i_4$
$(i = 1,3,4,5,6,7)$

type ACD: $0_1 0_3 7_4$ $0_1 1_3 9_4$ $0_1 12_3 11_4$ $0_1 7_3 13_4$ $1_1 4_3 0_4$ $1_1 11_3 8_4$ $1_1 7_3 10_4$ $1_1 13_3 12_4$
$2_1 9_3 1_4$ $2_1 8_3 9_4$ $2_1 10_3 11_4$ $2_1 3_3 13_4$ $3_1 2_3 0_4$ $3_1 3_3 2_4$ $3_1 11_3 10_4$ $3_1 1_3 12_4$
$4_1 12_3 1_4$ $4_1 7_3 3_4$ $4_1 4_3 11_4$ $4_1 2_3 13_4$ $5_1 6_3 0_4$ $5_1 11_3 2_4$ $5_1 10_3 4_4$ $5_1 8_3 12_4$
$6_1 10_3 1_4$ $6_1 11_3 3_4$ $6_1 2_3 5_4$ $6_1 13_3 13_4$ $7_1 0_3 0_4$ $7_1 1_3 2_4$ $7_1 2_3 4_4$ $7_1 6_3 6_4$

type ABC: $0_1 10_2 4_3$ $0_1 11_2 9_3$ $1_1 11_2 5_3$ $1_1 0_2 9_3$ $2_1 2_2 12_3$ $2_1 1_2 13_3$ $3_1 1_2 12_3$ $3_1 0_2 7_3$

$4_1 0_2 6_3$ $4_1 3_2 11_3$ $5_1 3_2 9_3$ $5_1 4_2 3_3$ $6_1 4_2 0_3$ $6_1 5_2 1_3$ $7_1 5_2 4_3$ $7_1 6_2 3_3$

type ACC: $0_1 5_3 6_3$ $0_1 11_3 13_3$ $0_1 3_3 10_3$ $0_1 2_3 8_3$ $1_1 0_3 10_3$ $1_1 1_3 8_3$ $1_1 2_3 12_3$ $1_1 3_3 6_3$
$2_1 0_3 11_3$ $2_1 1_3 6_3$ $2_1 2_3 5_3$ $2_1 4_3 7_3$ $3_1 4_3 10_3$ $3_1 5_3 13_3$ $3_1 6_3 8_3$ $3_1 0_3 9_3$
$4_1 0_3 5_3$ $4_1 1_3 10_3$ $4_1 3_3 8_3$ $4_1 9_3 13_3$ $5_1 0_3 7_3$ $5_1 2_3 13_3$ $5_1 4_3 5_3$ $5_1 1_3 12_3$
$6_1 6_3 12_3$ $6_1 3_3 7_3$ $6_1 4_3 9_3$ $6_1 5_3 8_3$ $7_1 5_3 7_3$ $7_1 10_3 11_3$ $7_1 8_3 9_3$ $7_1 12_3 13_3$

type BCD: $0_2 12_3 0_4$ $0_2 4_3 1_4$ $0_2 1_3 5_4$ $0_2 3_3 6_4$ $0_2 10_3 7_4$ $0_2 2_3 11_4$ $0_2 5_3 3_4$ $1_2 1_3 0_4$
$1_2 4_3 3_4$ $1_2 5_3 4_4$ $1_2 7_3 6_4$ $1_2 3_3 10_4$ $1_2 9_3 11_4$ $2_2 0_3 1_4$ $2_2 2_3 2_4$ $2_2 7_3 5_4$ $2_2 4_3 9_4$
$2_2 5_3 10_4$ $2_2 3_3 11_4$ $3_2 7_3 0_4$ $3_2 3_3 1_4$ $3_2 6_3 2_4$ $3_2 8_3 7_4$ $3_2 5_3 8_4$ $3_2 0_3 9_4$ $4_2 5_3 0_4$
$4_2 13_3 1_4$ $4_2 8_3 2_4$ $4_2 9_3 3_4$ $4_2 10_3 9_4$ $4_2 1_3 10_4$ $5_2 13_3 0_4$ $5_2 8_3 1_4$ $5_2 10_3 2_4$
$5_2 0_3 3_4$ $5_2 6_3 4_4$ $5_2 5_3 11_4$ $6_2 8_3 0_4$ $6_2 5_3 1_4$ $6_2 0_3 2_4$ $6_2 1_3 3_4$ $6_2 4_3 4_4$ $6_2 13_3 5_4$
$7_2 9_3 0_4$ $7_2 13_3 2_4$ $7_2 3_3 4_4$ $7_2 11_3 5_4$ $7_2 8_3 6_4$ $8_2 1_3 1_4$ $8_2 0_3 4_4$ $8_2 12_3 5_4$
$8_2 4_3 6_4$ $8_2 5_3 7_4$ $8_2 6_3 8_4$ $9_2 4_3 2_4$ $9_2 5_3 6_4$ $9_2 13_3 7_4$ $9_2 3_3 8_4$ $9_2 9_3 9_4$ $9_2 8_3 10_4$
$10_2 6_3 3_4$ $10_2 11_3 7_4$ $10_2 9_3 8_4$ $10_2 13_3 9_4$ $10_2 12_3 10_4$ $10_2 7_3 12_4$ $11_2 10_3 0_4$
$11_2 7_3 4_4$ $11_2 13_3 8_4$ $11_2 12_3 9_4$ $11_2 6_3 10_4$ $11_2 11_3 11_4$

type BBC: $1_2 11_2 0_3$ $0_2 10_2 0_3$ $7_2 9_2 0_3$ $9_2 10_2 1_3$ $3_2 11_2 1_3$ $2_2 7_2 1_3$ $1_2 10_2 2_3$ $3_2 9_2 2_3$
$4_2 5_2 2_3$ $6_2 11_2 2_3$ $7_2 8_2 2_3$ $5_2 11_2 3_3$ $8_2 10_2 3_3$ $3_2 4_2 4_3$ $7_2 11_2 4_3$ $7_2 10_2 5_3$
$1_2 9_2 6_3$ $6_2 7_2 6_3$ $2_2 4_2 6_3$ $6_2 9_2 7_3$ $5_2 7_2 7_3$ $4_2 8_2 7_3$ $0_2 2_2 8_3$ $1_2 8_2 8_3$ $10_2 11_2 8_3$
$5_2 8_2 9_3$ $2_2 6_2 9_3$ $2_2 3_2 10_3$ $1_2 7_2 10_3$ $6_2 10_2 10_3$ $8_2 9_2 10_3$ $4_2 6_2 11_3$ $0_2 8_2 11_3$
$1_2 2_2 11_3$ $5_2 9_2 11_3$ $3_2 7_2 12_3$ $4_2 9_2 12_3$ $5_2 6_2 12_3$ $2_2 8_2 13_3$ $0_2 3_2 13_3$

type BBD: $4_2 10_2 11_4$ $2_2 11_2 12_4$ $1_2 3_2 12_4$ $0_2 5_2 12_4$ $0_2 7_2 13_4$ $1_2 6_2 13_4$ $4_2 11_2 13_4$
$3_2 5_2 13_4$

type ∞BB: $\infty 4_2 7_2$ $\infty 1_2 5_2$ $\infty 8_2 11_2$ $\infty 3_2 10_2$ $\infty 2_2 9_2$ $\infty 0_2 6_2$

type BBB: $0_2 1_2 4_2$ $0_2 9_2 11_2$ $3_2 6_2 8_2$ $2_2 5_2 10_2$

type CCD: $3_3 11_3 0_4$ $2_3 7_3 1_4$ $6_3 11_3 1_4$ $7_3 9_3 2_4$ $5_3 12_3 2_4$ $2_3 10_3 3_4$ $3_3 12_3 3_4$ $8_3 13_3 3_4$
$1_3 13_3 4_4$ $9_3 12_3 4_4$ $8_3 11_3 4_4$ $0_3 3_3 5_4$ $4_3 6_3 5_4$ $5_3 9_3 5_4$ $8_3 10_3 5_4$ $0_3 13_3 6_4$
$1_3 9_3 6_4$ $10_3 12_3 6_4$ $2_3 11_3 6_4$ $1_3 4_3 7_4$ $7_3 12_3 7_4$ $3_3 9_3 7_4$ $2_3 6_3 7_4$ $0_3 8_3 8_4$
$1_3 2_3 8_4$ $7_3 10_3 8_4$ $4_3 12_3 8_4$ $6_3 7_3 9_4$ $5_3 11_3 9_4$ $2_3 3_3 9_4$ $10_3 13_3 10_4$ $2_3 9_3 10_4$
$0_3 4_3 10_4$ $0_3 1_3 11_4$ $6_3 13_3 11_4$ $7_3 8_3 11_4$ $0_3 6_3 12_4$ $11_3 12_3 12_4$ $2_3 4_3 12_4$
$3_3 5_3 12_4$ $9_3 10_3 12_4$ $0_3 12_3 13_4$ $1_3 5_3 13_4$ $4_3 8_3 13_4$ $6_3 10_3 13_4$ $9_3 11_3 13_4$

type CCC: $3_3 4_3 13_3$ $1_3 7_3 11_3$

type ∞CC: $\infty 7_3 13_3$ $\infty 5_3 10_3$ $\infty 8_3 12_3$ $\infty 4_3 11_3$ $\infty 1_3 3_3$ $\infty 6_3 9_3$ $\infty 0_3 2_3$

The long lines are $0_1 1_1 \cdots 7_1 \infty$ and $0_4 1_4 \cdots 13_4 \infty$.

Use Corollary 1.32(b) with $u_1 = 18$, $u = 9$ and $w = 15$ or $u_1 = 20$, $u = 9$ and $w = 15$, to show that $51, 55 \in LS(3, 9^*, 15^*)$. Now, $57 \in LS_d(3, 9^*, 15^*)$ follows since Lemma 1.36 can be applied to construct an $LS_d(57;\{3, 15^*, 21^*\})$. Replace the line of size twenty-one by an STS(21), which contains a subdesign STS(9); such an STS(21) exists by Theorem 1.7. This subdesign is then replaced by a line of size nine. A direct construction yields $57 \in LS_i(3, 9^*, 15^*)$. Form the partition $\pi(1^1, 2^1, 14^1, 20^2)$,where cell A is the set $Z_{14} \times \{1\}$, cell B is the set $Z_2 \times \{2\}$,cells C and D are the sets $Z_{20} \times \{i\}(i = 3,4)$,embed an STS(9) into an STS(21) which contains ∞ and the twenty points of cell D and construct short lines of

type ABD: $0_1 0_2 0_4$ $0_1 1_2 1_4$ $1_1 0_2 2_4$ $1_1 1_2 3_4$ $2_1 0_2 4_4$ $2_1 1_2 5_4$ $3_1 0_2 6_4$ $3_1 1_2 7_4$ $4_1 0_2 8_4$ $4_1 1_2 9_4$ $5_1 0_2 10_4$ $5_1 1_2 11_4$ $6_1 0_2 12_4$ $6_1 1_2 13_4$ $7_1 0_2 14_4$ $7_1 1_2 15_4$ $8_1 0_2 16_4$ $8_1 1_2 17_4$ $9_1 0_2 18_4$ $9_1 1_2 19_4$ $10_1 0_2 1_4$ $10_1 1_2 2_4$ $11_1 0_2 3_4$ $11_1 1_2 4_4$ $12_1 0_2 5_4$ $12_1 1_2 6_4$ $13_1 0_2 7_4$ $13_1 1_2 8_4$

type ACD: $0_1 i_3 (i+2)_4 (i = 0,1,5,6,8,10,11,12,15,17)$ $0_1 7_3 4_4$ $0_1 13_3 5_4$ $0_1 19_3 6_4$ $0_1 16_3 9_4$ $0_1 18_3 11_4$ $0_1 3_3 15_4$ $0_1 2_3 16_4$ $0_1 14_3 18_4$ $1_1 18_3 4_4$ $1_1 2_3 5_4$ $1_1 0_3 6_4$ $1_1 1_3 7_4$ $1_1 19_3 8_4$ $1_1 (i+3)_3 (i+9)_4 (i = 0,2,3,5,...,12)$ $1_1 7_3 10_4$ $1_1 16_3 13_4$ $2_1 (i+16)_3 (i+6)_4 (i = 0,1,2,4,5,7,9,...,17)$ $2_1 2_3 9_4$ $2_1 19_3 12_4$ $2_1 14_3 14_4$ $3_1 4_3 8_4$ $3_1 (i+15)_3 (i+9)_4 (i = 0,1,2,5,6,10,11,15,16)$ $3_1 9_3 12_4$ $3_1 2_3 13_4$ $3_1 19_3 16_4$ $3_1 18_3 17_4$ $3_1 7_3 18_4$ $3_1 8_3 1_4$ $3_1 14_3 2_4$ $3_1 3_3 3_4$ $4_1 11_3 10_4$ $4_1 7_3 11_4$ $4_1 8_3 12_4$ $4_1 15_3 13_4$ $4_1 18_3 14_4$ $4_1 17_3 15_4$ $4_1 16_3 16_4$ $4_1 2_3 17_4$ $4_1 i_3 (i+18)_4$ $(i = 0,1,5,6)$ $4_1 4_3 0_4$ $4_1 14_3 1_4$ $4_1 3_3 2_4$ $4_1 19_3 5_4$ $4_1 13_3 6_4$ $4_1 9_3 7_4$ $5_1 (i+11)_3 (i+12)_4 (i = 0,2,4,6,10,14)$ $5_1 10_3 13_4$ $5_1 12_3 15_4$ $5_1 3_3 17_4$ $5_1 19_3 19_4$ $5_1 2_3 0_4$ $5_1 9_3 1_4$ $5_1 0_3 3_4$ $5_1 14_3 4_4$ $5_1 7_3 5_4$ $5_1 4_3 7_4$ $5_1 16_3 8_4$ $5_1 6_3 9_4$ $6_1 (i+9)_3 (i+14)_4 (i = 0,2,6,8,12,16)$ $6_1 18_3 15_4$ $6_1 10_3 17_4$ $6_1 3_3 18_4$ $6_1 12_3 19_4$ $6_1 4_3 1_4$ $6_1 14_3 3_4$ $6_1 2_3 4_4$ $6_1 16_3 5_4$ $6_1 0_3 7_4$ $6_1 13_3 8_4$ $6_1 7_3 9_4$ $6_1 19_3 11_4$ $7_1 3_3 16_4$ $7_1 6_3 17_4$ $7_1 (i+9)_3 (i+18)_4 (i = 0,2,4,6,8,12)$ $7_1 7_3 19_4$ $7_1 10_3 1_4$ $7_1 12_3 3_4$ $7_1 8_3 5_4$ $7_1 16_3 7_4$ $7_1 2_3 8_4$ $7_1 18_3 9_4$ $7_1 0_3 11_4$ $7_1 14_3 12_4$ $7_1 19_3 13_4$ $8_1 5_3 18_4$ $8_1 3_3 19_4$ $8_1 19_3 0_4$ $8_1 6_3 1_4$ $8_1 9_3 2_4$ $8_1 7_3 3_4$

$8_1 11_3 4_4$ $8_1 10_3 5_4$ $8_1 2_3 6_4$ $8_1 15_3 7_4$ $8_1 8_3 8_4$ $8_1 4_3 9_4$ $8_1 17_3 10_4$ $8_1 16_3 11_4$
$8_1 12_3 12_4$ $8_1 18_3 13_4$ $8_1 1_3 14_4$ $8_1 0_3 15_4$ $9_1 13_3 0_4$ $9_1 19_3 1_4$ $9_1 (i+5)_3 (i+2)_4$
$(i = 0,4,10,12)$ $9_1 2_3 3_4$ $9_1 4_3 4_4$ $9_1 (i+6)_3 (i+5)_4 (i = 0,4,6,8,10)$ $9_1 7_3 7_4$
$9_1 11_3 8_4$ $9_1 3_3 10_4$ $9_1 18_3 16_4$ $9_1 8_3 17_4$ $10_1 19_3 3_4$ $10_1 5_3 4_4$ $10_1 0_3 5_4$
$10_1 4_3 6_4$ $10_1 (i+6)_3 (i+7)_4$ $(i = 0,1,6,10,12)$ $10_1 9_3 9_4$ $10_1 14_3 10_4$
$10_1 8_3 11_4$ $10_1 3_3 12_4$ $10_1 10_3 14_4$ $10_1 11_3 15_4$ $10_1 13_3 16_4$ $10_1 2_3 18_4$
$10_1 17_3 0_4$ $11_1 4_3 5_4$ $11_1 (i+18)_3 (i+6)_4 (i = 0,1,2,3)$ $11_1 6_3 10_4$ $11_1 13_3 11_4$
$11_1 5_3 12_4$ $11_1 17_3 13_4$ $11_1 2_3 14_4$ $11_1 (i+10)_3 (i+15)_4$ $(i = 0,2,4,6)$
$11_1 9_3 16_4$ $11_1 11_3 18_4$ $11_1 7_3 0_4$ $11_1 15_3 2_4$ $12_1 12_3 7_4$ $12_1 15_3 8_4$ $12_1 19_3 9_4$
$12_1 9_3 10_4$ $12_1 3_3 11_4$ $12_1 2_3 12_4$ $12_1 13_3 13_4$ $12_1 11_3 14_4$ $12_1 14_3 15_4$
$12_1 8_3 16_4$ $12_1 17_3 17_4$ $12_1 10_3 18_4$ $12_1 16_3 19_4$ $12_1 0_3 0_4$ $12_1 7_3 1_4$ $12_1 4_3 2_4$
$12_1 18_3 3_4$ $12_1 1_3 4_4$ $13_1 i_3 (i+9)_4 (i = 0,4,5,10)$ $13_1 13_3 10_4$ $13_1 6_3 11_4$
$13_1 1_3 12_4$ $13_1 2_3 15_4$ $13_1 17_3 16_4$ $13_1 9_3 17_4$ $13_1 15_3 18_4$ $13_1 12_3 0_4$ $13_1 3_3 1_4$
$13_1 11_3 2_4$ $13_1 8_3 3_4$ $13_1 16_3 4_4$ $13_1 18_3 5_4$ $13_1 7_3 6_4$

type ACC: $0_1 4_3 9_3$ $1_1 4_3 17_3$ $2_1 4_3 15_3$ $3_1 12_3 13_3$ $4_1 10_3 12_3$ $5_1 8_3 18_3$ $6_1 6_3 8_3$ $7_1 4_3 5_3$
$8_1 13_3 14_3$ $9_1 0_3 1_3$ $10_1 1_3 15_3$ $11_1 3_3 8_3$ $12_1 5_3 6_3$ $13_1 14_3 19_3$

type BCD: $0_2 12_3 9_4$ $0_2 4_3 11_4$ $0_2 5_3 13_4$ $0_2 19_3 15_4$ $0_2 13_3 17_4$ $0_2 8_3 19_4$ $1_2 1_3 0_4$
$1_2 10_3 10_4$ $1_2 18_3 12_4$ $1_2 7_3 14_4$ $1_2 12_3 16_4$ $1_2 13_3 18_4$

type BCC: $0_2 1_3 9_3$ $0_2 0_3 10_3$ $0_2 3_3 11_3$ $0_2 2_3 14_3$ $0_2 7_3 15_3$ $0_2 16_3 18_3$ $0_2 6_3 17_3$ $1_2 0_3 11_3$
$1_2 4_3 19_3$ $1_2 5_3 8_3$ $1_2 3_3 6_3$ $1_2 14_3 16_3$ $1_2 2_3 15_3$ $1_2 9_3 17_3$

type CCD: $3_3 16_3 0_4$ $5_3 18_3 0_4$ $8_3 9_3 0_4$ $0_3 5_3 1_4$ $1_3 17_3 1_4$ $12_3 18_3 1_4$ $2_3 13_3 1_4$ $2_3 8_3 2_4$
$6_3 10_3 2_4$ $7_3 18_3 2_4$ $16_3 19_3 2_4$ $16_3 17_3 3_4$ $9_3 10_3 3_4$ $4_3 11_3 3_4$ $6_3 15_3 3_4$ $0_3 3_3 4_4$
$17_3 19_3 4_4$ $9_3 13_3 4_4$ $8_3 12_3 4_4$ $1_3 3_3 5_4$ $5_3 12_3 5_4$ $9_3 14_3 5_4$ $15_3 17_3 5_4$ $8_3 15_3 6_4$
$6_3 14_3 6_4$ $3_3 12_3 6_4$ $10_3 11_3 6_4$ $14_3 18_3 7_4$ $2_3 3_3 7_4$ $10_3 13_3 7_4$ $8_3 11_3 7_4$
$1_3 10_3 8_4$ $3_3 5_3 8_4$ $9_3 12_3 8_4$ $14_3 17_3 8_4$ $8_3 17_3 9_4$ $5_3 14_3 9_4$ $11_3 13_3 9_4$ $4_3 12_3 10_4$
$2_3 19_3 10_4$ $15_3 18_3 10_4$ $10_3 15_3 11_4$ $11_3 14_3 11_4$ $2_3 9_3 11_4$ $0_3 17_3 12_4$ $4_3 13_3 12_4$
$7_3 16_3 12_4$ $0_3 7_3 13_4$ $1_3 8_3 13_4$ $6_3 9_3 13_4$ $3_3 4_3 14_4$ $6_3 16_3 14_4$ $15_3 19_3 14_4$
$13_3 15_3 15_4$ $4_3 6_3 15_4$ $7_3 8_3 15_4$ $0_3 14_3 16_4$ $1_3 4_3 16_4$ $5_3 7_3 16_4$ $1_3 14_3 17_4$
$5_3 19_3 17_4$ $0_3 4_3 17_4$ $1_3 18_3 18_4$ $6_3 19_3 18_4$ $4_3 16_3 18_4$ $0_3 15_3 19_4$ $6_3 11_3 19_4$
$2_3 4_3 19_4$

type CCC: $0_3 9_3 19_3$ $0_3 6_3 18_3$ $0_3 12_3 16_3$ $0_3 8_3 13_3$ $1_3 13_3 16_3$ $1_3 2_3 5_3$ $1_3 6_3 12_3$ $1_3 7_3 19_3$ $4_3 8_3 14_3$ $13_3 18_3 19_3$ $3_3 14_3 15_3$ $2_3 11_3 16_3$ $2_3 10_3 18_3$ $3_3 9_3 18_3$ $8_3 10_3 19_3$ $11_3 12_3 19_3$ $5_3 10_3 16_3$ $5_3 11_3 15_3$ $2_3 6_3 7_3$ $4_3 7_3 10_3$ $2_3 12_3 17_3$ $5_3 13_3 17_3$ $9_3 15_3 16_3$ $7_3 9_3 11_3$ $11_3 17_3 18_3$ $3_3 10_3 17_3$ $7_3 12_3 14_3$ $3_3 7_3 13_3$

type ∞CC: $\infty 0_3 2_3$ $\infty 1_3 11_3$ $\infty 3_3 19_3$ $\infty 7_3 17_3$ $\infty 4_3 18_3$ $\infty 5_3 9_3$ $\infty 6_3 13_3$ $\infty 10_3 14_3$ $\infty 8_3 16_3$ $\infty 12_3 15_3$

type ∞BB: $\infty 0_2 1_2$

One long line is $0_1 1_1 \cdots 13_1 \infty$ and the other long line is formed by replacing the subsystem of STS(21). Finally, $73 \in LS(3, 9^*, 15^*)$ by forming a partition $\pi(1^1, 8^5, 32^1)$ and applying Corollary 1.25.

We will be successful in determining the spectra for AULSs with one long line of size nine, and the other long line of size thirteen or fifteen, when we provide constructions for those orders which were not included in the previous lemmas.

Lemma 2.55 $51,63 \in LS(3,9^*,13^*); 39 \in LS_i(3,9^*,15^*)$.

Proof: In order to show that $LS(51;\{3, 9^*, 13^*)$ exists, form a partition $\pi(1^1, 8^1, 12^2, 18^1)$, where cell A is the set $Z_{18} \times \{1\}$, cell B is the set $Z_8 \times \{2\}$, cells C and D are the sets $Z_{12} \times \{i\}$ $(i = 3, 4)$, and embed an STS(9) into an STS(19) which contains ∞ and the eighteen points of cell A (STS(9) is replaced by a line of size nine), and construct short lines of

type ABD: $i_1 0_2 i_4$ $(i+12)_1 1_2 i_4$ $(i+6)_1 2_2 i_4$ $(11\text{-}i)_1 3_2 i_4 (i = 0,1,...,11)$ $(5\text{-}i)_1 4_2 i_4$ $(17\text{-}i)_1 4_2 (i+6)_4$ $(i = 0,1,...,5)$ $(17\text{-}i)_1 5_2 i_4 (i = 0,1,...,11)$ $(2+i)_1 6_2 i_4$ $(i = 0,1,...,9)$ $0_1 6_2 10_4$ $1_1 6_2 11_4$ $13_1 7_2 0_4$ $12_1 7_2 1_4$ $0_1 7_2 2_4$ $1_1 7_2 3_4$ $14_1 7_2 4_4$ $15_1 7_2 5_4$ $16_1 7_2 6_4$ $17_1 7_2 7_4$ $5_1 7_2 8_4$ $4_1 7_2 9_4$ $3_1 7_2 10_4$ $2_1 7_2 11_4$

type ABC: $0_1 2_2 2_3$ $0_1 5_2 9_3$ $1_1 2_2 0_3$ $1_1 5_2 11_3$ $2_1 2_2 1_3$ $2_1 5_2 4_3$ $3_1 2_2 6_3$ $3_1 5_2 7_3$ $4_1 2_2 8_3$ $4_1 5_2 5_3$ $5_1 2_2 10_3$ $5_1 5_2 1_3$ $6_1 1_2 11_3$ $6_1 4_2 0_3$ $6_1 7_2 7_3$ $7_1 1_2 0_3$ $7_1 4_2 2_3$ $7_1 7_2 3_3$ $8_1 1_2 1_3$ $8_1 4_2 4_3$ $8_1 7_2 5_3$ $9_1 1_2 5_3$ $9_1 4_2 10_3$ $9_1 7_2 11_3$ $10_1 1_2 2_3$ $10_1 4_2 8_3$ $10_1 7_2 4_3$ $11_1 1_2 8_3$ $11_1 4_2 6_3$ $11_1 7_2 10_3$ $12_1 0_2 0_3$ $12_1 3_2 3_3$ $12_1 6_2 6_3$

$13_1 0_2 1_3$ $13_1 3_2 4_3$ $13_1 6_2 5_3$ $14_1 0_2 2_3$ $14_1 3_2 6_3$ $14_1 6_2 9_3$ $15_1 0_2 8_3$ $15_1 3_2 10_3$
$15_1 6_2 11_3$ $16_1 0_2 3_3$ $16_1 3_2 5_3$ $16_1 6_2 7_3$ $17_1 0_2 4_3$ $17_1 3_2 7_3$ $17_1 6_2 10_3$

type BCC: $0_2 5_3 9_3$ $0_2 6_3 11_3$ $0_2 7_3 10_3$ $1_2 3_3 7_3$ $1_2 6_3 10_3$ $1_2 4_3 9_3$ $2_2 3_3 5_3$ $2_2 4_3 7_3$
$2_2 9_3 11_3$ $3_2 0_3 9_3$ $3_2 2_3 11_3$ $3_2 1_3 8_3$ $4_2 1_3 7_3$ $4_2 3_3 9_3$ $4_2 5_3 11_3$ $5_2 0_3 6_3$
$5_2 2_3 10_3$ $5_2 3_3 8_3$ $6_2 0_3 1_3$ $6_2 4_3 8_3$ $7_2 0_3 2_3$ $7_2 1_3 6_3$ $7_2 8_3 9_3$ $6_2 2_3 3_3$

type ACD: $0_1 10_3 1_4$ $0_1 4_3 3_4$ $0_1 5_3 4_4$ $0_1 7_3 7_4$ $0_1 6_3 8_4$ $0_1 0_3 9_4$ $1_1 10_3 0_4$ $1_1 9_3 2_4$
$1_1 1_3 5_4$ $1_1 6_3 6_4$ $1_1 2_3 8_4$ $1_1 8_3 9_4$ $2_1 8_3 1_4$ $2_1 3_3 4_4$ $2_1 6_3 5_4$ $2_1 0_3 6_4$ $2_1 5_3 7_4$
$2_1 2_3 10_4$ $3_1 4_3 0_4$ $3_1 10_3 4_4$ $3_1 5_3 5_4$ $3_1 3_3 6_4$ $3_1 0_3 7_4$ $3_1 8_3 11_4$ $4_1 7_3 0_4$
$4_1 1_3 3_4$ $4_1 0_3 5_4$ $4_1 2_3 6_4$ $4_1 3_3 8_4$ $4_1 11_3 11_4$ $5_1 6_3 1_4$ $5_1 0_3 2_4$ $5_1 4_3 4_4$ $5_1 9_3 7_4$
$5_1 5_3 9_4$ $5_1 7_3 10_4$ $6_1 9_3 1_4$ $6_1 10_3 2_4$ $6_1 5_3 3_4$ $6_1 8_3 7_4$ $6_1 1_3 8_4$ $6_1 2_3 9_4$
$6_1 4_3 10_4$ $7_1 1_3 0_4$ $7_1 11_3 2_4$ $7_1 6_3 3_4$ $7_1 5_3 6_4$ $7_1 4_3 8_4$ $7_1 10_3 9_4$ $7_1 9_3 11_4$
$8_1 2_3 0_4$ $8_1 7_3 1_4$ $8_1 11_3 4_4$ $8_1 9_3 5_4$ $8_1 6_3 7_4$ $8_1 8_3 10_4$ $8_1 3_3 11_4$ $9_1 9_3 0_4$
$9_1 3_3 1_4$ $9_1 6_3 4_4$ $9_1 4_3 5_4$ $9_1 8_3 6_4$ $9_1 1_3 10_4$ $9_1 2_3 11_4$ $10_1 11_3 0_4$ $10_1 6_3 2_4$
$10_1 0_3 3_4$ $10_1 7_3 5_4$ $10_1 9_3 6_4$ $10_1 3_3 9_4$ $10_1 10_3 11_4$ $11_1 11_3 1_4$ $11_1 7_3 2_4$
$11_1 3_3 3_4$ $11_1 0_3 4_4$ $11_1 4_3 7_4$ $11_1 5_3 8_4$ $11_1 9_3 10_4$ $12_1 1_3 2_4$ $12_1 7_3 3_4$ $12_1 8_3 4_4$
$12_1 2_3 7_4$ $12_1 9_3 8_4$ $12_1 4_3 9_4$ $12_1 11_3 10_4$ $13_1 3_3 2_4$ $13_1 11_3 3_4$ $13_1 2_3 5_4$
$13_1 10_3 6_4$ $13_1 7_3 8_4$ $13_1 9_3 9_4$ $13_1 6_3 11_4$ $14_1 3_3 0_4$ $14_1 5_3 1_4$ $14_1 8_3 5_4$
$14_1 1_3 6_4$ $14_1 11_3 7_4$ $14_1 10_3 10_4$ $14_1 7_3 11_4$ $15_1 6_3 0_4$ $15_1 2_3 1_4$ $15_1 9_3 4_4$
$15_1 7_3 6_4$ $15_1 3_3 7_4$ $15_1 0_3 10_4$ $15_1 1_3 11_4$ $16_1 0_3 0_4$ $16_1 8_3 2_4$ $16_1 9_3 3_4$
$16_1 11_3 5_4$ $16_1 10_3 8_4$ $16_1 1_3 9_4$ $16_1 4_3 11_4$ $17_1 0_3 1_4$ $17_1 5_3 2_4$ $17_1 2_3 3_4$
$17_1 1_3 4_4$ $17_1 8_3 8_4$ $17_1 11_3 9_4$ $17_1 3_3 10_4$

type ACC: $0_1 8_3 11_3$ $0_1 1_3 3_3$ $1_1 5_3 7_3$ $1_1 3_3 4_3$ $2_1 7_3 9_3$ $2_1 10_3 11_3$ $3_1 1_3 11_3$ $3_1 2_3 9_3$
$4_1 4_3 6_3$ $4_1 9_3 10_3$ $5_1 3_3 11_3$ $5_1 2_3 8_3$ $6_1 3_3 6_3$ $7_1 7_3 8_3$ $8_1 0_3 10_3$ $9_1 0_3 7_3$
$10_1 1_3 5_3$ $11_1 1_3 2_3$ $12_1 5_3 10_3$ $13_1 0_3 8_3$ $14_1 0_3 4_3$ $15_1 4_3 5_3$ $16_1 2_3 6_3$ $17_1 6_3 9_3$

type CCD: $5_3 8_3 0_4$ $1_3 4_3 1_4$ $2_3 4_3 2_4$ $8_3 10_3 3_4$ $2_3 7_3 4_4$ $3_3 10_3 5_4$ $4_3 11_3 6_4$ $1_3 10_3 7_4$
$0_3 11_3 8_4$ $6_3 7_3 9_4$ $5_3 6_3 10_4$ $0_3 5_3 11_4$

type ∞BB: $\infty 0_2 7_2$ $\infty 1_2 6_2$ $\infty 2_2 5_2$ $\infty 3_2 4_2$

type BBB: $0_2 1_2 2_2$ $0_2 3_2 5_2$ $0_2 4_2 6_2$ $1_2 3_2 7_2$ $1_2 4_2 5_2$ $2_2 3_2 6_2$ $2_2 4_2 7_2$ $5_2 6_2 7_2$

type ∞CC: $\infty 0_3 3_3$ $\infty 1_3 9_3$ $\infty 2_3 5_3$ $\infty 4_3 10_3$ $\infty 6_3 8_3$ $\infty 7_3 11_3$

The long line of size thirteen is $0_4 1_4 \cdots 11_4 \infty$. Similarly, $63 \in LS(3, 9^*, 13^*)$ by forming a partition $\pi(1^1, 12^1, 14^1, 18^2)$, where cells A, B are the sets $Z_{18} \times \{i\}$ ($i = 1, 2$), cell C is the set $Z_{14} \times \{3\}$, and cell D is the set $Z_{12} \times \{4\}$ and constructing short lines of

type ACD: $0_1 0_3 10_4$ $0_1 1_3 1_4$ $0_1 2_3 0_4$ $0_1 3_3 2_4$ $0_1 4_3 11_4$ $0_1 5_3 6_4$ $0_1 6_3 9_4$ $0_1 7_3 5_4$
$1_1 1_3 2_4$ $1_1 2_3 3_4$ $1_1 3_3 7_4$ $1_1 4_3 0_4$ $1_1 5_3 11_4$ $1_1 11_3 4_4$ $1_1 7_3 8_4$ $1_1 13_3 10_4$
$2_1 2_3 4_4$ $2_1 3_3 5_4$ $2_1 4_3 6_4$ $2_1 5_3 0_4$ $2_1 6_3 7_4$ $2_1 7_3 10_4$ $2_1 8_3 9_4$ $2_1 9_3 8_4$ $3_1 3_3 6_4$
$3_1 4_3 8_4$ $3_1 5_3 10_4$ $3_1 6_3 4_4$ $3_1 7_3 9_4$ $3_1 8_3 7_4$ $3_1 9_3 1_4$ $3_1 10_3 11_4$ $4_1 4_3 7_4$ $4_1 5_3 9_4$
$4_1 6_3 0_4$ $4_1 7_3 4_4$ $4_1 8_3 1_4$ $4_1 9_3 2_4$ $4_1 13_3 5_4$ $4_1 11_3 3_4$ $5_1 5_3 4_4$ $5_1 6_3 11_4$
$5_1 7_3 2_4$ $5_1 8_3 6_4$ $5_1 9_3 5_4$ $5_1 10_3 0_4$ $5_1 11_3 10_4$ $5_1 12_3 3_4$ $6_1 6_3 2_4$ $6_1 7_3 0_4$
$6_1 8_3 11_4$ $6_1 9_3 3_4$ $6_1 10_3 10_4$ $6_1 11_3 7_4$ $6_1 12_3 1_4$ $6_1 13_3 9_4$ $7_1 7_3 1_4$ $7_1 8_3 2_4$
$7_1 9_3 9_4$ $7_1 10_3 3_4$ $7_1 11_3 0_4$ $7_1 12_3 8_4$ $7_1 13_3 7_4$ $7_1 0_3 5_4$ $8_1 8_3 5_4$ $8_1 9_3 4_4$
$8_1 10_3 7_4$ $8_1 11_3 9_4$ $8_1 12_3 6_4$ $8_1 6_3 10_4$ $8_1 0_3 8_4$ $8_1 1_3 11_4$ $9_1 9_3 11_4$ $9_1 6_3 6_4$
$9_1 11_3 1_4$ $9_1 12_3 9_4$ $9_1 13_3 4_4$ $9_1 0_3 2_4$ $9_1 1_3 0_4$ $9_1 2_3 8_4$ $10_1 10_3 1_4$ $10_1 11_3 8_4$
$10_1 12_3 5_4$ $10_1 13_3 11_4$ $10_1 0_3 0_4$ $10_1 1_3 3_4$ $10_1 2_3 2_4$ $10_1 3_3 9_4$ $11_1 11_3 11_4$
$11_1 12_3 10_4$ $11_1 13_3 3_4$ $11_1 0_3 7_4$ $11_1 1_3 8_4$ $11_1 2_3 9_4$ $11_1 3_3 4_4$ $11_1 4_3 1_4$
$12_1 12_3 0_4$ $12_1 13_3 6_4$ $12_1 0_3 11_4$ $12_1 1_3 4_4$ $12_1 2_3 5_4$ $12_1 3_3 1_4$ $12_1 6_3 3_4$
$12_1 5_3 8_4$ $13_1 13_3 2_4$ $13_1 0_3 3_4$ $13_1 1_3 9_4$ $13_1 2_3 6_4$ $13_1 3_3 8_4$ $13_1 4_3 4_4$ $13_1 5_3 5_4$
$13_1 8_3 10_4$ $14_1 0_3 4_4$ $14_1 1_3 5_4$ $14_1 2_3 1_4$ $14_1 3_3 0_4$ $14_1 4_3 2_4$ $14_1 5_3 3_4$
$14_1 10_3 6_4$ $14_1 7_3 7_4$ $15_1 1_3 6_4$ $15_1 2_3 7_4$ $15_1 3_3 3_4$ $15_1 4_3 10_4$ $15_1 5_3 1_4$
$15_1 10_3 5_4$ $15_1 7_3 11_4$ $15_1 8_3 0_4$ $16_1 2_3 10_4$ $16_1 3_3 11_4$ $16_1 4_3 5_4$ $16_1 5_3 2_4$
$16_1 6_3 1_4$ $16_1 7_3 6_4$ $16_1 8_3 8_4$ $16_1 9_3 7_4$ $17_1 3_3 10_4$ $17_1 4_3 9_4$ $17_1 5_3 7_4$
$17_1 13_3 8_4$ $17_1 7_3 3_4$ $17_1 8_3 4_4$ $17_1 9_3 6_4$ $17_1 10_3 2_4$

type CCD: $9_3 13_3 0_4$ $0_3 13_3 1_4$ $11_3 12_3 2_4$ $4_3 8_3 3_4$ $10_3 12_3 4_4$ $6_3 11_3 5_4$ $0_3 11_3 6_4$ $1_3 12_3 7_4$
$6_3 10_3 8_4$ $0_3 10_3 9_4$ $1_3 9_3 10_4$ $2_3 12_3 11_4$

type ABC: $0_1 0_2 8_3$ $0_1 1_2 9_3$ $0_1 2_2 10_3$ $0_1 16_2 11_3$ $0_1 13_2 12_3$ $0_1 3_2 13_3$ $1_1 2_2 0_3$ $1_1 1_2 8_3$
$1_1 7_2 9_3$ $1_1 5_2 10_3$ $1_1 10_2 12_3$ $1_1 12_2 6_3$ $2_1 4_2 0_3$ $2_1 0_2 1_3$ $2_1 7_2 10_3$ $2_1 9_2 11_3$
$2_1 12_2 12_3$ $2_1 11_2 13_3$ $3_1 8_2 0_3$ $3_1 10_2 1_3$ $3_1 1_2 2_3$ $3_1 2_2 11_3$ $3_1 4_2 12_3$ $3_1 14_2 13_3$
$4_1 10_2 0_3$ $4_1 5_2 1_3$ $4_1 8_2 2_3$ $4_1 0_2 3_3$ $4_1 6_2 12_3$ $4_1 15_2 10_3$ $5_1 11_2 0_3$ $5_1 9_2 1_3$
$5_1 2_2 2_3$ $5_1 16_2 3_3$ $5_1 1_2 4_3$ $5_1 17_2 13_3$ $6_1 6_2 0_3$ $6_1 16_2 1_3$ $6_1 11_2 2_3$ $6_1 13_2 3_3$
$6_1 15_2 4_3$ $6_1 17_2 5_3$ $7_1 14_2 1_3$ $7_1 6_2 2_3$ $7_1 2_2 3_3$ $7_1 10_2 4_3$ $7_1 0_2 5_3$ $7_1 17_2 6_3$

$8_1 9_2 2_3$ $8_1 6_2 3_3$ $8_1 2_2 4_3$ $8_1 14_2 5_3$ $8_1 7_2 13_3$ $8_1 1_2 7_3$ $9_1 8_2 3_3$ $9_1 12_2 4_3$
$9_1 5_2 5_3$ $9_1 11_2 8_3$ $9_1 14_2 7_3$ $9_1 1_2 10_3$ $10_1 16_2 4_3$ $10_1 4_2 5_3$ $10_1 5_2 8_3$ $10_1 7_2 7_3$
$10_1 6_2 6_3$ $10_1 3_2 9_3$ $11_1 13_2 5_3$ $11_1 9_2 10_3$ $11_1 10_2 7_3$ $11_1 12_2 8_3$ $11_1 11_2 9_3$
$11_1 16_2 6_3$ $12_1 4_2 4_3$ $12_1 15_2 7_3$ $12_1 3_2 8_3$ $12_1 14_2 9_3$ $12_1 6_2 10_3$ $12_1 17_2 11_3$
$13_1 3_2 7_3$ $13_1 8_2 6_3$ $13_1 13_2 9_3$ $13_1 16_2 10_3$ $13_1 4_2 11_3$ $13_1 7_2 12_3$ $14_1 13_2 8_3$
$14_1 0_2 9_3$ $14_1 9_2 13_3$ $14_1 11_2 11_3$ $14_1 3_2 12_3$ $14_1 4_2 6_3$ $15_1 5_2 0_3$ $15_1 17_2 9_3$
$15_1 15_2 13_3$ $15_1 14_2 11_3$ $15_1 8_2 12_3$ $15_1 9_2 6_3$ $16_1 13_2 0_3$ $16_1 12_2 1_3$ $16_1 8_2 10_3$
$16_1 15_2 11_3$ $16_1 5_2 12_3$ $16_1 0_2 13_3$ $17_1 12_2 0_3$ $17_1 3_2 1_3$ $17_1 17_2 2_3$ $17_1 10_2 11_3$
$17_1 15_2 12_3$ $17_1 7_2 6_3$

type BCC: $0_2 0_3 2_3$ $0_2 4_3 6_3$ $0_2 10_3 11_3$ $0_2 7_3 12_3$ $1_2 0_3 12_3$ $1_2 3_3 13_3$ $1_2 1_3 6_3$ $1_2 5_3 11_3$
$2_2 1_3 5_3$ $2_2 6_3 8_3$ $2_2 7_3 13_3$ $2_2 9_3 12_3$ $3_2 0_3 4_3$ $3_2 3_3 11_3$ $3_2 5_3 6_3$ $3_2 2_3 10_3$
$4_2 2_3 13_3$ $4_2 1_3 10_3$ $4_2 7_3 9_3$ $4_2 3_3 8_3$ $5_2 2_3 3_3$ $5_2 7_3 11_3$ $5_2 4_3 13_3$ $5_2 6_3 9_3$
$6_2 1_3 7_3$ $6_2 4_3 5_3$ $6_2 8_3 13_3$ $6_2 9_3 11_3$ $7_2 0_3 5_3$ $7_2 2_3 8_3$ $7_2 3_3 4_3$ $7_2 1_3 11_3$
$8_2 1_3 13_3$ $8_2 4_3 9_3$ $8_2 8_3 11_3$ $8_2 5_3 7_3$ $9_2 0_3 8_3$ $9_2 4_3 12_3$ $9_2 3_3 7_3$ $9_2 5_3 9_3$
$10_2 2_3 5_3$ $10_2 3_3 6_3$ $10_2 10_3 13_3$ $10_2 8_3 9_3$ $11_2 1_3 3_3$ $11_2 4_3 10_3$ $11_2 5_3 12_3$
$11_2 6_3 7_3$ $12_2 2_3 7_3$ $12_2 3_3 9_3$ $12_2 5_3 10_3$ $12_2 11_3 13_3$ $13_2 7_3 10_3$ $13_2 1_3 4_3$
$13_2 6_3 13_3$ $13_2 2_3 11_3$ $14_2 0_3 6_3$ $14_2 2_3 4_3$ $14_2 3_3 10_3$ $14_2 8_3 12_3$ $15_2 0_3 9_3$
$15_2 1_3 8_3$ $15_2 2_3 6_3$ $15_2 3_3 5_3$ $16_2 0_3 7_3$ $16_2 2_3 9_3$ $16_2 5_3 8_3$ $16_2 12_3 13_3$
$17_2 8_3 10_3$ $17_2 3_3 12_3$ $17_2 0_3 1_3$ $17_2 4_3 7_3$

type ∞CC: $\infty 0_3 3_3$ $\infty 1_3 2_3$ $\infty 4_3 11_3$ $\infty 5_3 13_3$ $\infty 6_3 12_3$ $\infty 7_3 8_3$ $\infty 9_3 10_3$

type ABD: $0_1 4_2 3_4$ $0_1 6_2 4_4$ $0_1 7_2 7_4$ $0_1 5_2 8_4$ $1_1 9_2 1_4$ $1_1 17_2 5_4$ $1_1 16_2 6_4$ $1_1 0_2 9_4$ $2_1 8_2 1_4$
$2_1 17_2 2_4$ $2_1 15_2 3_4$ $2_1 10_2 11_4$ $3_1 3_2 0_4$ $3_1 16_2 2_4$ $3_1 0_2 3_4$ $3_1 11_2 5_4$ $4_1 4_2 6_4$
$4_1 14_2 8_4$ $4_1 17_2 10_4$ $4_1 7_2 11_4$ $5_1 14_2 1_4$ $5_1 10_2 7_4$ $5_1 13_2 8_4$ $5_1 15_2 9_4$ $6_1 0_2 4_4$
$6_1 7_2 5_4$ $6_1 12_2 6_4$ $6_1 2_2 8_4$ $7_1 1_2 4_4$ $7_1 8_2 6_4$ $7_1 13_2 10_4$ $7_1 9_2 11_4$ $8_1 8_2 0_4$
$8_1 10_2 1_4$ $8_1 11_2 2_4$ $8_1 12_2 3_4$ $9_1 17_2 3_4$ $9_1 3_2 5_4$ $9_1 4_2 7_4$ $9_1 7_2 10_4$ $10_1 10_2 4_4$
$10_1 2_2 6_4$ $10_1 13_2 7_4$ $10_1 11_2 10_4$ $11_1 15_2 0_4$ $11_1 14_2 2_4$ $11_1 1_2 5_4$ $11_1 3_2 6_4$
$12_1 5_2 2_4$ $12_1 11_2 7_4$ $12_1 13_2 9_4$ $12_1 8_2 10_4$ $13_1 6_2 0_4$ $13_1 5_2 1_4$ $13_1 9_2 7_4$
$13_1 15_2 11_4$ $14_1 6_2 8_4$ $14_1 1_2 9_4$ $14_1 12_2 10_4$ $14_1 5_2 11_4$ $15_1 0_2 2_4$ $15_1 12_2 4_4$
$15_1 3_2 8_4$ $15_1 16_2 9_4$ $16_1 14_2 0_4$ $16_1 6_2 3_4$ $16_1 2_2 4_4$ $16_1 9_2 9_4$ $17_1 1_2 0_4$
$17_1 4_2 1_4$ $17_1 2_2 5_4$ $17_1 16_2 11_4$

type ABB: $0_1 9_2 11_2$ $0_1 10_2 15_2$ $0_1 12_2 14_2$ $0_1 8_2 17_2$ $1_1 3_2 6_2$ $1_1 8_2 11_2$ $1_1 4_2 14_2$
$1_1 13_2 15_2$ $2_1 1_2 5_2$ $2_1 2_2 16_2$ $2_1 3_2 14_2$ $2_1 6_2 13_2$ $3_1 6_2 9_2$ $3_1 7_2 13_2$ $3_1 15_2 17_2$
$3_1 5_2 12_2$ $4_1 1_2 16_2$ $4_1 2_2 12_2$ $4_1 9_2 13_2$ $4_1 3_2 11_2$ $5_1 0_2 3_2$ $5_1 8_2 12_2$ $5_1 4_2 6_2$
$5_1 5_2 7_2$ $6_1 1_2 8_2$ $6_1 3_2 9_2$ $6_1 4_2 5_2$ $6_1 10_2 14_2$ $7_1 4_2 16_2$ $7_1 11_2 12_2$ $7_1 3_2 7_2$
$7_1 5_2 15_2$ $8_1 3_2 15_2$ $8_1 0_2 5_2$ $8_1 4_2 17_2$ $8_1 13_2 16_2$ $9_1 0_2 9_2$ $9_1 2_2 6_2$ $9_1 10_2 13_2$
$9_1 15_2 16_2$ $10_1 0_2 12_2$ $10_1 1_2 14_2$ $10_1 9_2 17_2$ $10_1 8_2 15_2$ $11_1 0_2 4_2$ $11_1 2_2 5_2$
$11_1 6_2 8_2$ $11_1 7_2 17_2$ $12_1 2_2 9_2$ $12_1 1_2 10_2$ $12_1 0_2 16_2$ $12_1 7_2 12_2$ $13_1 0_2 10_2$
$13_1 1_2 12_2$ $13_1 2_2 14_2$ $13_1 11_2 17_2$ $14_1 2_2 8_2$ $14_1 10_2 17_2$ $14_1 7_2 15_2$ $14_1 14_2 16_2$
$15_1 7_2 11_2$ $15_1 2_2 13_2$ $15_1 1_2 4_2$ $15_1 6_2 10_2$ $16_1 1_2 11_2$ $16_1 4_2 7_2$ $16_1 10_2 16_2$
$16_1 3_2 17_2$ $17_1 11_2 14_2$ $17_1 8_2 9_2$ $17_1 0_2 13_2$ $17_1 5_2 6_2$

type BBD: $0_2 7_2 0_4$ $2_2 4_2 0_4$ $11_2 13_2 0_4$ $10_2 12_2 0_4$ $9_2 16_2 0_4$ $5_2 17_2 0_4$ $0_2 1_2 1_4$ $2_2 17_2 1_4$
$3_2 13_2 1_4$ $6_2 11_2 1_4$ $7_2 16_2 1_4$ $12_2 15_2 1_4$ $12_2 13_2 2_4$ $1_2 9_2 2_4$ $7_2 8_2 2_4$ $2_2 10_2 2_4$
$6_2 15_2 2_4$ $3_2 4_2 2_4$ $5_2 9_2 3_4$ $1_2 13_2 3_4$ $2_2 3_2 3_4$ $8_2 10_2 3_4$ $11_2 16_2 3_4$ $7_2 14_2 3_4$
$3_2 8_2 4_4$ $5_2 14_2 4_4$ $7_2 9_2 4_4$ $4_2 13_2 4_4$ $11_2 15_2 4_4$ $16_2 17_2 4_4$ $0_2 8_2 5_4$ $9_2 15_2 5_4$
$4_2 10_2 5_4$ $6_2 12_2 5_4$ $5_2 16_2 5_4$ $13_2 14_2 5_4$ $9_2 14_2 6_4$ $5_2 11_2 6_4$ $1_2 15_2 6_4$ $0_2 6_2 6_4$
$7_2 10_2 6_4$ $13_2 17_2 6_4$ $0_2 17_2 7_4$ $1_2 6_2 7_4$ $3_2 5_2 7_4$ $12_2 16_2 7_4$ $2_2 15_2 7_4$ $8_2 14_2 7_4$
$0_2 11_2 8_4$ $4_2 15_2 8_4$ $8_2 16_2 8_4$ $9_2 10_2 8_4$ $12_2 17_2 8_4$ $1_2 7_2 8_4$ $6_2 7_2 9_4$ $2_2 11_2 9_4$
$3_2 10_2 9_4$ $14_2 17_2 9_4$ $4_2 12_2 9_4$ $5_2 8_2 9_4$ $4_2 9_2 10_4$ $1_2 3_2 10_4$ $0_2 2_2 10_4$ $6_2 16_2 10_4$
$5_2 10_2 10_4$ $14_2 15_2 10_4$ $8_2 13_2 11_4$ $1_2 2_2 11_4$ $0_2 14_2 11_4$ $4_2 11_2 11_4$ $6_2 17_2 11_4$
$3_2 12_2 11_4$

type ∞BB: $\infty 6_2 14_2$ $\infty 1_2 17_2$ $\infty 0_2 15_2$ $\infty 9_2 12_2$ $\infty 5_2 13_2$ $\infty 10_2 11_2$ $\infty 3_2 16_2$ $\infty 4_2 8_2$
$\infty 2_2 7_2$

One long line is $0_1 1_1 \cdots 11_1 \infty$ and the other long line is formed by replacing the subsystem of STS(19). Form the partition $\pi(1^1, 8^3, 14^1)$ and apply Theorem 1.24(b) to show that $39 \in LS_i(3,9^*,15^*)$.

Lemma 2.56

$LS_d(3,9^*,13^*) = \{v: v \geq 37, v \equiv 1,3 (\text{mod } 6)\}$; $LS_i(3,9^*,13^*) = \{v: v \geq 33, v \equiv 1,3 (\text{mod } 6)\}$;
$LS(3,9^*,15^*) = \{v: v \geq 39, v \equiv 1,3 (\text{mod } 6)\}$.

Proof: The first statement follows from Lemma 2.48, Corollary 2.49, Lemmas 2.52, 2.53 and 2.55, and Corollary 2.54. Secondly, Lemma 2.44, Corollary 2.45, Lemmas 2.53, 2.55 and Corollary 2.54 leads us to conclude that $v \in LS_i(3,9^*,13^*)$ for all $v \geq 33$, $v \equiv 1,3 (\text{mod } 6)$. As a consequence of Lemma 2.46, Corollaries 2.47, 2.54 and Lemmas 2.52 and 2.53, $v \in LS_d(3,9^*,15^*)$ for all $v \geq 39$, $v \equiv 1,3 (\text{mod } 6)$. Finally, Lemma 2.44, Corollaries 2.45, 2.54 and Lemmas 2.53 and 2.55 lead to $v \in LS_i(3,9^*,15^*)$ for all $v \geq 39$, $v \equiv 1,3 (\text{mod } 6)$.

Chapter 3

Almost uniform linear spaces with short lines of size four

§3.1 Almost uniform linear spaces with two long lines of size u and short lines of size four

We begin by demonstrating the existence of an AULS whose order v is the minimum, and the two long lines intersect. Our method is essentially analogous to the one outlined in Theorem 1.24(b).

Theorem 3.1 Let $u \equiv 1,4 \pmod{12}$. Then $4u - 3 \in LS_i(4, u^{**})$ and $4u - 3 = \min\{v: \exists LS_i(v;\{4, u^{**}\})\}$.

Proof: According to Corollary 1.17, $v \geq 4u - 3$. Form the partition $\pi(1^1, (u - 1)^4)$. By Theorem 1.40, there exists a $\{4\}$-GDD of type $(u - 1)^4$. Since $u \equiv 1,4 \pmod{12}$, we can place two copies of a (u, 4, 1)-BIBD on ∞ and u - 1 points of a cell in the partition. Place two copies of a line of size u on ∞ and the u - 1 points of a cell.

Corollary 3.2 $v \in LS_i(4, u^{**})$ for all $v \geq 12u - 8$.
Proof: This is a consequence of Theorem 1.41.

We cannot employ a recursive construction for all admissible $u \equiv 7,10 \pmod{12}$. It is possible to give individual results when u = 7, 10 or 19.

Lemma 3.3 $25 \in LS_i(4, 7^{**})$; $37 \in LS_i(4, 10^{**})$; $73 \in LS_i(4, 19^{**})$;
$25=\min\{v:\exists LS_i(v;\{4,7^{**}\})\}$, $37=\min\{v: \exists LS_i(v;\{4,10^{**}\})\}$, $73=\min\{v: \exists LS_i(v;\{4,19^{**}\})\}$.

Proof: The orders 25, 37 and 73 are minimum by Corollary 1.17(ii). Form partitions $\pi(1^1, 3^4, 6^2)$, $\pi(1^1, 3^6, 9^2)$ and $\pi(1^1, 3^{12}, 18^2)$. There exist {4}-GDDs of types $3^4 6^2$, $3^6 9^2$ and $3^{12} 18^2$[R6]. The construction is as in Theorem 3.1.

Corollary 3.4 $v \in LS_i(4, 7^{**})$ for all $v \geq 76$; $v \in LS_i(4, 10^{**})$ for all $v \geq 112$; $v \in LS_i(4, 19^{**})$ for all $v \geq 220$, where $v \equiv 1,4 \pmod{12}$.
Proof: Apply Theorem 1.41.

Next we consider an AULS with two long lines of size u which are disjoint. We can prove an analogous result provided that $u \equiv 1,4 \pmod{12}$.

Lemma 3.5 If $u \equiv 1,4 \pmod{12}$ then $4u \in LS_d(4, u^{**})$ and $4u = \min\{v: \exists LS_d(v;\{4, u^{**}\})\}$.
Proof: Form the partition $\pi(u^4)$ and construct a {4}-GDD of type u^4, by Theorem 1.40. Complete the construction as in Theorem 1.24(a).

Corollary 3.6 $\{v: v = 4u \text{ or } v \geq 12u + 1, v \equiv 1,4 \pmod{12}\} \subseteq LS_d(4, u^{**})$.
Proof: We conclude this from an analogue of Lemma 1.37.

The rest of the results in this section concern AULSs whose two long lines intersect. The scope of recursive constructions is narrower since there are fewer appropriate {4}-GDDs.

Lemma 3.7 $4u \in LS_i(4, u^{**})$ for all $u \equiv 1,4 \pmod{12}$.
Proof: Form a partition $\pi(1^1, 3^1, (u-1)^4)$ and construct a {4}-GDD of type $(u-1)^4 3^1$: delete $(u-4)/3$ points from a group of a TD(5, $(u-1)/3$) to obtain a {4,5}-GDD of type $((u-1)/3)^4 1^1$ and putting a weight of three on every point, apply FC.

Lemma 3.8 Let t be a nonnegative integer such that $4 < t \leq 12$. If $u \equiv 1 \pmod{12}$, or $u \equiv 4 \pmod{12}$ and $t \equiv 0,1 \pmod 4$, then $tu - t + 1 \in LS_i(4, u^{**})$.
Proof: Form a partition $\pi(1^1, (u-1)^t)$. We can construct a {4}-GDD of type $(u-1)^t$ by Theorem 1.40. The rest of the proof is similar to Theorem 3.1.

Lemma 3.9 $73 \in LS_i(4, 16^{**})$.
Proof: Form a partition $\pi(1^1, 12^1, 15^4)$ and form a {4}-GDD of type $15^4 12^1$, by deleting a point from a group of a TD(5, 5) to obtain a {4,5}-GDD of type $5^4 4^1$, putting a weight of three on every point and applying FC.

The next lemma basically summarizes what orders belong to the spectrum of AULS with two long lines of size u which intersect, for any admissible u which is congruent to 1 or 4 (mod 12).

Lemma 3.10
(a) If $u \equiv 1 \pmod{12}$, then $\{4u-3, 4u, 5u-4, 6u-5, \ldots, 12u-11\} \cup \{v: v \geq 12u-8\} \subseteq LS_i(4, u^{**})$.
(b) If $u \equiv 4 \pmod{12}$, then $\{4u-3, 4u, 5u-4, 8u-7, 9u-8, 12u-11\} \cup \{v: v \geq 12u-8\} \subseteq LS_i(4, u^{**})$.
Proof: The claims in (a) and (b) both follow from Theorem 3.1, Corollary 3.2, Lemmas 3.7 and 3.8.

We now state precisely which orders we have proved belong to the spectrum of AULSs with two long lines either of size thirteen or sixteen. Unlike the analogous problem in the previous chapter, we were unsuccessful in completing the spectrum even in these special cases. Since recursive techniques are more restricted, and the direct methods utilized previously do not seem to extend to this situation, we are left with some gaps.

Lemma 3.11
$\{49, 52, 61, 73, 85, 97, 109, 121, 133, 145\} \cup \{v: v \geq 148, v \equiv 1,4 \pmod{12}\} \subseteq LS_i(4, 13^{**})$.
$\{61, 64, 73, 76, 121, 136, 181\} \cup \{v: v \geq 184, v \equiv 1,4 \pmod{12}\} \subseteq LS_i(4, 16^{**})$.

Proof: The first claim follows from Lemma 3.10. The second claim follows from Lemmas 3.9 and 3.10.

It should be remarked that no similar result was determined if it was assumed that the two long lines had either size seven or ten. It appears that another sort of direct method must be developed in order to handle such cases. Therefore, a large part of the spectrum still must be determined. Finally, the spectrum of AULSs with two long lines of size u which are disjoint, for any admissible u, remains largely undetermined.

§3.2 Almost uniform linear spaces with one long line of size u, one long line of size w and short lines of size four

We are able to construct some infinite classes of such AULSs by employing the method of completion of resolvable {3}-GDDs.

Lemma 3.12 Let $u \equiv 1,4 \pmod{12}$ and r is an integer such that $r \geq 2$. Then
$(3r + 1)u - 3r \in LS_i(4,u^*,(r(u-1) + 1)^*)$
and $(3r + 1)u - 3r = \min\{v : \exists LS_i(v;\{4,u^*,(r(u-1) + 1)^*\})\}$.

Proof: Clearly, $v \geq (3r + 1)u - 3r$ by Corollary 1.21(ii). Form a partition $\pi(1^1, (u-1)^{2r+1},(r(u-1))^1)$. We note that we can construct a {4}-GDD of type $(u-1)^{2r+1}(r(u-1))^1$ by adjoining $r(u-1)$ points at infinity to $r(u-1)$ parallel classes of a resolvable {3}-GDD of type $(u-1)^{2r+1}$.Hence, the long line of size u is formed on the point ∞ and the u - 1 points of one of the cells, and the long line of size $r(u-1) + 1$ contains ∞ and the $r(u-1)$ points of the penultimate cell. Form a (u, 4, 1)-BIBD on ∞ and the u - 1 points of each remaining cell.

Corollary 3.13 $v \in LS_i(4, u^*, (r(u-1) + 1)^*)$ for all $v \geq (9r + 3)u - 9r + 1$, $v \equiv 1,4 \pmod{12}$.

Proof: This follows from Theorem 1.41 and an analogue of Lemma 1.39.

Lemma 3.14 Let $u \equiv 1 \pmod{12}$ and r is an integer such that $r \geq 2$. Then $((6r - 1)u - 6r + 3)/2 \in LS_i(4, u^*, (((2r - 1)u - 2r + 3)/2)^*)$ and $((6r - 1)u - 6r + 3)/2 = \min\{v: \exists LS_i(v;\{4, u^*, (((2r - 1)u - 2r + 3)/2)^*\})\}$.

Proof: $v \geq ((6r - 1)u - 6r + 3)/2$ by Corollary 1.21(ii). The arguments parallel those in Lemma 3.13; form a partition $\pi(1, (u - 1)^{2r}, (((u - 1)(2r - 1))/2)^1)$ and construct a {4}-GDD from a resolvable {3}-GDD of type $(u - 1)^{2r}$ by adjoining $((u - 1)(2r - 1))/2$ points at infinity to the $((u - 1)(2r - 1))/2$ parallel classes of a resolvable {3}-GDD of type $(u - 1)^{2r}$.

Corollary 3.15 $v \in LS_i(4,u^*, (((2r - 1)u - 2r + 3)/2)^*)$ for all $v \geq ((18r - 3)u - 18r + 11)/2$, $v \equiv 1,4 \pmod{12}$.

Proof: This follows from Theorem 1.41 and an analogue of Lemma 1.39.

We previously discovered that Wilson's fundamental construction could be applied in various situations to build the necessary underlying GDDs. The following lemmas make use of this technique in order to construct particular AULSs.

Lemma 3.16 Let t be a nonnegative integer such that $1 \leq t \leq 3$. Then $181 - 9t \in LS_i(4, (37 - 9t)^*, 37^*)$; $241 - 12t \in LS_i(4, (49 - 12t)^*, 49^*)$.

Proof: Form the following partitions: $\pi(1^1, (36 - 9t)^1, 36^4)$, $\pi(1^1, (48 - 12t)^1, 48^4)$. We can construct {4}-GDDs of types $36^4(36 - 9t)^1$, $48^4(48 - 12t)^1$, by first deleting t points from a TD(5, 4) to obtain a {4,5}-GDD of type $4^4(4 - t)^1$, and then either putting a weight of nine or twelve on every point of this GDD, applying FC, in order to obtain the desired GDDs. The rest of this proof is similar to that of Lemma 3.12.

Some other classes of AULSs may be obtained by applying results of Rees and Stinson [R6]:

Lemma 3.17 Suppose there is a TD(5, m) and $0 \le u \le m$. Then there is a {4}-GDD of type $(3m)^4(3u)^1$.

Lemma 3.18 Suppose there is a TD(6, m) and $m \le u \le 2m$. Then there is a {4}-GDD of type $(3m)^4(6m)^1(3u)^1$.

Lemma 3.19 Suppose there is a TD(6, m), and $0 \le u \le m$. Then there is a {4}-GDD of type $(3m)^5(6u)^1$.

From Lemma 3.17, we have the following immediate consequence.

Lemma 3.20 Suppose $m \equiv 0,1 \pmod 4$, $m \ge 4$, and u is a nonnegative integer such that $0 \le u \le m$. Then $12m + 3u + 1 \in LS_i(4, (3u + 1)^*, (3m + 1)^*)$.

Proof: Form a partition $\pi(1^1, (3u)^1, (3m)^4)$ and apply Lemma 3.17, as well as the procedure in Lemma 3.12.

Analogously, from Lemma 3.18, we can state two subsequent conclusions.

Lemma 3.21 Let $m \equiv 0,4,8 \pmod{12}$, $m \ge 8$ and $m \ne 14$; u is a nonnegative integer such that $m \le u \le 2m$. Then $18m + 3u + 1 \in LS_i(4, (3m + 1)^*, (3u + 1)^*)$.

Lemma 3.22 Let $m \equiv 0,4,8 \pmod{12}$, $m \ge 8$ and $m \ne 14$; u is a nonnegative integer such that $m \le u \le 2m$ and $u \equiv 0,1 \pmod 4$. Then $18m + 3u + 1 \in LS_i(4, (3m + 1)^*, (6m + 1)^*)$.

Lemma 3.19 leads to the next lemma.

Lemma 3.23 Suppose $m \equiv 0,1 \pmod 4$, $m \ge 4$ or $m \ge 9$, respectively, and u is a nonnegative integer such that $0 \le u \le m$. Then $15m + 6u + 1 \in LS_i(4,(3m+1)^*,(6u+1)^*)$.

We have thus far restricted our considerations to AULSs with two long lines that intersect. However, Lemma 3.17 can be linked with an embedding of a (m, 4, 1)-BIBD into a (3m + 1, 4, 1)-BIBD to obtain the following result.

Lemma 3.24 Let $m \equiv 1,4 \pmod{12}$, $m \geq 13$, and u is an integer such that $0 \leq u \leq m$. Then $12m + 3u + 1 \in LS(4, m^*, (3u + 1)^*)$.

Proof: Form a partition $\pi(1^1, (3u)^1, (3m)^4)$ and embed a (m, 4, 1)-BIBD into a (3m + 1, 4, 1)-BIBD, replacing the sub-BIBD by a line of size m. Thereafter, use the same reasoning as in Lemma 3.12.

In a similar manner, Lemma 3.18 may be used in conjunction with such an embedding to produce the next result.

Lemma 3.25 Let $m \equiv 0,4,8 \pmod{12}$, $m \geq 8$, $m \neq 14$ and u is a nonnegative integer such that $m \leq u \leq 2m$ and $u \equiv 0,1 \pmod 4$. Then $18m + 3u + 1 \in LS(4,u^*,(3m + 1)^*)$.

Proof: Form a partition $\pi(1^1, (3m)^4, (3u)^1, (6m)^1)$ and embed a (u, 4, 1)-BIBD into a (3u + 1, 4, 1)-BIBD, replacing the sub-BIBD with a line of size u. Otherwise, see Lemma 3.24 for the general method of approach.

It was possible to complete the spectrum for AULSs with two long lines of specified sizes, and short lines of size three. By contrast, we shall consider only one special case here, assuming that one long line has size thirteen and the other long line has size sixteen, and will list the orders that we can demonstrate belong to the spectrum. If the long lines have sizes congruent to 1 or 4 (mod 12), recursive techniques can be applied more readily.

Lemma 3.26

(a) $\{73,88,157,172,193,196,205,208,217\} \cup \{v: v \geq 220, v \equiv 1,4 \pmod{12}\} \subseteq LS_i(4,13^*,16^*)$.

(b) $\{136,157,169,172,193,196,205,208,217\} \cup \{v: v \geq 220, v \equiv 1,4 \pmod{12}\} \subseteq LS_d(4,13^*,16^*)$.

Proof:

(a) Form the partition $\pi(1^1, 12^1, 15^4)$, and apply Lemma 3.20 to show that $73 \in LS_i(4, 13^*, 16^*)$. Similarly, $88 \in LS_i(4, 13^*, 16^*)$ by Lemma 3.23. In order to prove that $157 \in LS_i(4, 13^*, 16^*)$, form a partition $\pi(1^1, 12^9, 48^1)$ and embed a (16, 4, 1)- BIBD into a (49, 4, 1)-BIBD which contains ∞ and the forty-eight points of the tenth cell, replacing the (16, 4, 1)-BIBD by a line of size sixteen. Construct a {4}-GDD of type $12^9 48^1$ by completing a resolvable {3}-GDD of type 12^9. Next, $172 \in LS_i(4, 13^*, 16^*)$ by applying Lemma 3.24. We have $193 \in LS_i(4, 13^*, 16^*)$ by forming a partition $\pi(1^1, 48^4)$, embedding a (13, 4, 1)-BIBD into a (49, 4, 1)-BIBD which contains ∞ and the forty-eight points of the first cell, and embedding a (16, 4, 1)-BIBD into a (49, 4, 1)-BIBD which contains ∞ and the forty-eight points of the second cell, both sub-BIBDs being replaced by lines of sizes thirteen and sixteen respectively. The rest of the proof is standard. By forming the partitions $\pi(1^1, r^1, 48^4)$, where $r = 3,15$ and 24, and the two embeddings mentioned in the previous case, we can show that $196, 208, 217 \in LS_i(4, 13^*, 16^*)$. We can construct a {4}-GDD of type $48^4 r^1$ by deleting $(48 - r)/3$ points from a group of a TD(5, 16) to obtain a {4,5}-GDD of type $16^4(r/3)^1$, putting a weight of three on every point and applying FC. Next, $205 \in LS_i(4, 13^*, 16^*)$ by forming a partition $\pi(1^1, 51^4)$, embedding a (13, 4, 1)-BIBD ((16, 4, 1)-BIBD) into a (52, 4, 1)-BIBD which contains ∞ and the fifty-one points of the first (second) cell, and replacing both sub-BIBDs by lines of sizes thirteen and sixteen respectively. Finally, since we have shown that $73 \in LS_i(4, 13^*, 16^*)$, by Theorem 1.41, we conclude that for all $v \geq 3 \cdot 73 + 1 = 220$, $v \equiv 1,4 \pmod{12}$, $v \in LS_i(4, 13^*, 16^*)$.

(b)It should be remarked at the outset that, by the recursive constructions developed in (a), $157,172,193,196,205,208,217 \in LS_d(4,13^*,16^*)$.Now, $136 \in LS_d(4, 13^*, 16^*)$ by forming a partition $\pi(16^6, 40^1)$, embedding a (13, 4, 1)-BIBD into a (40, 4, 1)-BIBD, and replacing the sub-BIBD by a line of size thirteen. The line of size sixteen is placed on the points of the first cell. On the points of each remaining cell, form a (16, 4, 1)-BIBD. We thereby can directly conclude from Theorem 1.41 that

$v \geq 3 \cdot 136 + 1 = 409$, $v \equiv 1,4 \pmod{12}$. Next, $169 \in LS_d(4, 13^*, 16^*)$: form a partition $\pi(13^9, 52^1)$ and construct a $\{4\}$-GDD of type $13^9 52^1$ by completion of a resolvable $\{3\}$-GDD of type 13^9, and embed a (16, 4, 1)-BIBD into a (52, 4, 1)-BIBD formed on the fifty-two points of the last cell. What is left to consider are $v \equiv 1,4 \pmod{12}$ and $220 \leq v \leq 400$. In order to prove that $220,232,241 \in LS_d(4, 13^*, 16^*)$, form the partitions $\pi(1^1, r^1, 48^4)$ where $r = 27, 39$ and 48, and embed a (13, 4, 1)-BIBD ((16, 4, 1)-BIBD) into a (49, 4, 1)-BIBD as before. If $r = 27$ or 39, delete $(48 - r)/3$ points from a group of a TD(5, 16) to obtain a $\{4,5\}$-GDD of type $16^4(r/3)^1$, put a weight of three on every point and apply FC, to construct a $\{4\}$-GDD of type 48^4r^1. For $r = 48$, proceed as for $193 \in LS_i(4, 13^*, 16^*)$ in (a), working with partition $\pi(1^1, 48^5)$. For $244 \leq v \leq 301$, form partitions $\pi(1^1, r^1, 60^4)$ where $r \equiv 0,3 \pmod{12}$, $3 \leq r \leq 60$; embed a (13, 4, 1)-BIBD and a (16, 4, 1)-BIBD into a (61, 4, 1)-BIBD, replacing the sub-BIBDs with the required long lines. Constuct a $\{4\}$-GDD of type 60^4r^1, for $3 \leq r \leq 51$, by deleting $(60 - r)/3$ points from a group of a TD(5, 20), to obtain a $\{4,5\}$-GDD of type $20^4(r/3)^1$, and putting a weight of three on every point and then applying the FC. If $r = 60$, the partition is $\pi(1^1, 60^5)$ and we proceed in the usual way. Next, $229, 337, 373 \in LS_d(4, 13^*, 16^*)$ by forming partitions $\pi(1^1, 12^r, (6r - 6)^1)$ where $r = 13$, 19 and 21; embed a (16, 4, 1)-BIBD into a $(6r - 5, 4, 1)$-BIBD which contains ∞ and the $6r - 6$ points of the $(r + 1)$st cell. Construct a $\{4\}$-GDD of type $12^r(6r - 6)^1$ by completing a $\{3\}$-GDD of type 12^r. To show that $316 \in LS_d(4, 13^*, 16^*)$, form a partition $\pi(1^1, 63^5)$, and embed a (13, 4, 1)-BIBD ((16, 4, 1)-BIBD) into a (64, 4, 1)-BIBD which contains ∞ and the sixty-three points of a cell. There exists a $\{4\}$-GDD of type 63^5. We have $397 \in LS_d(4, 13^*, 16^*)$ by forming a partition $\pi(1^1, 72^4, 108^1)$, embedding a (13, 4, 1)-BIBD ((16, 4, 1)-BIBD) into a (73, 4, 1)-BIBD which contains ∞ and the seventy-two points of a cell. There exists a $\{4\}$-GDD of type 72^4108^1 by completion of a $\{3\}$-GDD of type 72^4. Form partitions $\pi(1^1, r^1, 75^4)$, $r \equiv 0,3 \pmod{12}$, $3 \leq r \leq 75$, $r \neq 15, 72$ thereby proving that $304,313,325,328,340,349,352,361,364,376 \in LS_d(4,13^*,16^*)$. Embed a (13, 4, 1)-BIBD ((16, 4, 1)-BIBD) into a (76, 4, 1)-BIBD. Construct a $\{4\}$-GDD of type 75^4r^1, if $3 \leq r \leq 63$, by deleting $(75 - r)/3$ points from a group of a TD(5,25) to obtain a {4,5}-GDD of type $25^4(r/3)^1$, put a weight of three on every point and apply FC. If r = 75, form partition $\pi(1^1, 75^5)$ and proceed as usual. Finally, 385, 388, 400 $\in$ $LS_d(4, 13^*, 16^*)$ by forming partitions $\pi(1, r^1, 84^4)$, where r = 48, 51 and 63; embed a (13, 4, 1)-BIBD ((16, 4, 1)-BIBD) into an (85, 4, 1)-BIBD which contains ∞ and the eighty-four points of a cell. Construct a {4}-GDD of type 84^4r^1 by deleting (84 - r)/3 points from a TD(5, 28) to obtain a {4,5}-GDD of type $28^4(r/3)^1$, putting a weight of three on every point and apply FC.

Chapter 4

Almost uniform linear spaces with short lines of size five

§4.1 Almost uniform linear spaces with two long lines of size u and short lines of size five

Initially, we will construct an AULS with two long lines of size u, $u \equiv 1,5 \pmod{20}$ and of the minimum order v, in the case where the two long lines intersect or are disjoint.

Theorem 4.1 Let $u \equiv 1,5 \pmod{20}$.
(a) $5u - 4 \in LS_i(5, u^{**})$ and $5u - 4 = \min\{v: \exists LS_i(v;\{5, u^{**}\})\}$.
(b) $5u \in LS_d(5, u^{**})$ and $5u = \min\{v: \exists LS_d(v;\{5, u^{**}\})\}$.
Proof:
(a) By Corollary 1.17(iii), $v \geq 5u - 4$. Form a partition $\pi(1^1, (u - 1)^5)$, construct a TD(5, u - 1), and define each of the long lines on the point ∞ and the u - 1 points in a cell of the partition. Finally, form a (u, 5, 1)-BIBD on ∞ and the u - 1 points of each remaining cell.
(b) By Corollary 1.15(iii), $v \geq 5u$. Form a partition $\pi(u^5)$, define two long lines on the points of two cells, and form a (u, 5, 1)-BIBD on the points of each remaining cell.

Corollary 4.2 $20u - 15 \in LS_i(5, u^{**})$ and $20u + 1 \in LS_d(5, u^{**})$.

From Corollary 1.17(iii), the two long lines may also have size $u \equiv 13 \pmod{20}$, however there appear to be no recursive constructions which are appropriate. The remainder of the constructions pertain to AULSs with long lines that intersect.

Lemma 4.3 If $u \equiv 1,5 \pmod{20}$, $u \neq 25,41,105,121$, then $5u \in LS_i(5, u^{**})$.

Proof: Form a partition $\pi(1^1, 4^1, (u-1)^5)$ and construct a {5}-GDD of type $(u-1)^5 4^1$: delete three points from a group of a TD(6, (u - 1)/4) to obtain a {5,6}-GDD of type $((u-1)/4)^5 1^1$, put a weight of four on every point, and apply FC. The rest of the arguments are similar to those in Theorem 4.1.

Lemma 4.4 If $u \equiv 1,5 \pmod{20}$, $u \neq 25,41$ and t is a nonnegative integer such that $t > 5$, $t \equiv 0,1 \pmod 5$, then $tu - t + 1 \in LS_i(5, u^{**})$.

Proof: We note that there exists a {5}-GDD of type 4^t (delete a point from a (4t + 1, 5, 1)-BIBD. Put a weight of (u - 1)/4 on every point, apply FC, using a {5}-GDD of type $((u-1)/4)^5$, to obtain a {5}-GDD of type $(u-1)^t$. Hence, form a partition $\pi(1^1, (u-1)^t)$, construct a {5}-GDD of type $(u-1)^t$ and proceed as in Theorem 4.1.

Corollary 4.5 $241 \in LS_i(5, 41^{**})$.

Proof: Form a partition $\pi(1^1, 40^6)$. Construct a {5}-GDD of type 8^6 ([B4]), put a weight of five on every point, and apply FC.

Lemma 4.6 If $u \equiv 1 \pmod{20}$ and $t \equiv 1 \pmod 3$, $t \geq 13$, or $u \equiv 25 \pmod{60}$ and $t \equiv 1,4 \pmod{10}$, $t \geq 14$, then $((4t-1)(u-1)+3)/3 \in LS_i(5, u^{**})$.

Proof: Form a partition $\pi(1^1, (u-1)^t, (((t-1)(u-1))/3)^1)$. There is a resolvable {4}-GDD of type 4^t for all $t \equiv 1 \pmod 3$ ([H4]). Put a weight of (u - 1)/4 on every point, and apply FC. Hence, we have constructed a {4}-GDD of type $(u-1)^t$. The completion of this {4}-GDD is a {5}-GDD of type $(u-1)^t(((t-1)(u-1))/3)^1$. We note that when $u \equiv 25 \pmod{60}$, $t \equiv 1,4 \pmod{10}$, in order that a $((t - 1)(u - 1) + 3)/3, 5, 1)$-BIBD exists.

§4.2 Almost uniform linear spaces with one long line of size u, one long line of size w and short lines of size five

In the previous chapter, the completion of resolvable {3}-GDDs to {4}-GDDS, led us to form certain infinite classes of AULSs. The technique can also be applied here.

Lemma 4.7 Let $u \equiv 1,5 \pmod{20}$.
Then $9u - 8 \in LS_i(5, u^*, (2u - 1)^*)$ and $9u - 8 = \min\{v: \exists LS_i(v;\{5, u^*, (2u - 1)^*\})\}$.

Proof: By Corollary 1.21(iii), $v \geq 9u - 8$. Form a partition $\pi(1^1, (u - 1)^7, (2u - 2)^1)$ and construct a {5}-GDD of type $(u - 1)^7(2u - 2)^1$, by completing a resolvable {4}-GDD of type $(u - 1)^7$. The long line of size u contains ∞ and the $u - 1$ points of one of the cells, and the long line of size $2u - 1$ contains ∞ and the $2u - 2$ points of the cell in the partition. Form a $(u, 5, 1)$-BIBD on ∞ and the $u - 1$ points in each remaining cell.

Lemma 4.8 Let $u \equiv 1 \pmod{20}$.
Then $13u - 12 \in LS_i(5,u^*,(3u - 2)^*)$ and $13u - 12 = \min\{v: \exists LS_i(v;\{5,u^*,(2u - 1)^*\})\}$.

Proof: By Corollary 1.21(iii), $v \geq 13u - 12$. The proof is similar to that of Lemma 4.7; form a partition $\pi(1^1, (u - 1)^{10}, (3u - 3)^1)$ and complete a resolvable {4}-GDD of type $(u - 1)^{10}$, thereby constructing a {5}-GDD of type $(u - 1)^{10}(3u - 3)^1$.

Lemma 4.9 Let $u \equiv 25 \pmod{60}$.
Then $(23u-20)/3 \in LS_i(5,u^*,((5u-2)/3)^*)$;
$(23u-20)/3 = \min\{v:\exists LS_i(v;\{5,u^*,((5u-2)/3)^*\})\}$ and
$(35u-32)/3 \in LS_i(5,u^*,((8u-5)/3)^*)$;$(35u-32)/3 = \min\{v:\exists LS_i(v;\{5,u^*,((8u-5)/3)^*\})\}$.

Proof: It follows from Corollary 1.20(iii) that $v \geq (23u - 20)/3$ and $v \geq (35u - 32)/3$. Form partitions $\pi(1^1, (u-1)^6, ((5(u-1))/3)^1)$ and $\pi(1^1, (u-1)^9, ((8(u-1))/3)^1)$. Construct {5}-GDDS of types $(u-1)^6((5(u-1))/3)^1$ and $(u-1)^9((8(u-1))/3)^1$ by completion of the resolvable {4}-GDDS of types $(u-1)^6$ and $(u-1)^9$. The rest of the proof is similar to Lemma 4.7.

Another method is to use transversal designs and the fundamental constructions of Wilson. There are numerous results which arise from these considerations.

Lemma 4.10 Suppose $m = 4^n$ for any $n \geq 1$. There exists a {5}-GDD of type $(5m)^5(3m)^1$.
Proof: Delete two points from a groups of a TD(6, 5) to obtain a {5,6}-GDD of type $5^5 3^1$. Give every point a weight of m, apply FC, using {5}-GDDs of type m^5 or m^6 (the first {5}-GDD exists since there is a TD(5, m) for all m; the second exists by repeatedly applying FC to a {5}-GDD of type 4^6 with all points receiving a weight of four), to build a {5}-GDD of type $(5m)^5(3m)^1$.

Lemma 4.11 For $m = 4^n$, $n \geq 1$, $28m + 1 \in LS_i(5, (3m+1)^*, (5m+1)^*)$.

Proof: Form a partition $\pi(1^1, (3m)^1, (5m)^5)$ and, by Lemma 4.10, construct a {5}-GDD of type $(5m)^5(3m)^1$. Then proceed in the usual way (cf. previous lemmas).

Lemma 4.12 If TD(5, m) exists then there are {5}-GDDs of types $(4m)^5$ and $(4m)^6$.
Proof: A TD(5, m) exists for all $m \neq 2,3,6$ or 10. Give a weight of m to every point

of a $\{5\}$-GDD of type 4^5 or 4^6, and apply FC in order to obtain a $\{5\}$-GDD of type $(4m)^5$ or $(4m)^6$.

Lemma 4.13 If TD(5, m) exists then there is a $\{5\}$-GDD of type $(20m)^5(12m)^1$.
Proof: Delete two points from a group of a TD(6, 5) to obtain a $\{5,6\}$-GDD of type $5^5 3^1$. Give every point a weight of 4m, and apply FC, using $\{5\}$-GDDs of type $(4m)^5$ or $(4m)^6$, to build a $\{5\}$-GDD of type $(20m)^5(12m)^1$.

Lemma 4.14 If TD(5, m) exists, then $112m + 1 \in LS_i(5, (12m + 1)^*, (20m + 1)^*)$.
Proof: Form a partition $\pi(1^1, (12m)^1, (20m)^5)$ and construct, by applying Lemma 4.13, a $\{5\}$-GDD of type $(20m)^5(12m)^1$.

Corollary 4.15 $225 \in LS_i(5, 25^*, 41^*)$.
Proof: Form a partition $\pi(1^1, 24^1, 40^5)$ and construct a $\{5\}$-GDD of type $40^5 24^1$: delete two points from a group of a TD(6, 5) to obtain a $\{5,6\}$-GDD of type $5^5 3^1$, give every point a weight of eight, use FC and $\{5\}$-GDDs of types 8^5 or 8^6, to build a $\{5\}$-GDD of type $40^5 24^1$.

Lemma 4.16 If TD(5, m) exists, then there is a $\{5\}$-GDD of type $(20m)^5(16m)^1$.
Proof: Delete one point from a group of a TD(6, 5) to obtain a $\{5,6\}$-GDD of type $5^5 4^1$, put a weight of 4m to every point, and apply FC, to obtain a $\{5\}$-GDD of type $(20m)^5(16m)^1$.

Lemma 4.17 If TD(5, m) exists, then $116m + 1 \in LS_i(5, (16m + 1)^*, (20m + 1)^*)$.

Proof: Form a partition $\pi(1^1, (16m)^1, (20m)^5)$ and, by Lemma 4.16, construct a $\{5\}$-GDD of type $(20m)^5(16m)^1$.

Corollary 4.18 $233 \in LS_i(5, 33^*, 41^*)$.

Proof: Form a partition $\pi(1^1, 32^1, 40^5)$, and construct a {5}-GDD of type $40^5 32^1$: delete one point from a group of a TD(6, 5), yielding a {5,6}-GDD of type $5^5 4^1$, give every point a weight of eight, and apply FC.

Lemma 4.19 Let $1 \leq t \leq 9$. Then $265 - 4t \in LS_i(5, (45 - 4t)^*, 45^*)$.
Proof: Form a partition $\pi(1^1, (44 - 4t)^1, 44^5)$ and construct a {5}-GDD of type $44^5(44 - 4t)^1$, by deleting t points from a TD(6, 11) to obtain a {5,6}-GDD of type $11^5(11 - t)^1$, giving each point a weight of four and applying FC.

Lemma 4.20 Let $1 \leq t \leq 7$. Then $385 - 8t \in LS_i(5, (65 - 8t)^*, 65^*)$.
Proof: Form a partition $\pi(1^1, (64 - 8t)^1, 64^5)$ and construct a {5}-GDD of type $64^5(64 - 8t)^1$, by deleting t points from a group of a TD(6, 8), to obtain a {5,6}-GDD of type $8^5(8 - t)^1$. Give each point a weight of eight and apply FC.

It should be remarked that in the constructions given, various possibilities for the sizes of the long lines that arise from Corollary 1.21(iii) have been considered. For example, in Lemma 4.19, for all choices of t, $u \equiv 1,5,9,13$ or $17 \pmod{20}$, $w \equiv 5 \pmod{20}$ and $v \equiv 1,5,9,13$ or $17 \pmod{20}$. Next, we shall utilize certain embeddings of BIBDs with block size five into larger ones in order to construct almost uniform linear spaces where the long line of size u is disjoint from the long line of size w.

Lemma 4.21 If $1 \leq t \leq 18$, then there is a {5}-GDD of type $(100)^5(100 - 4t)^1$.
Proof: Delete t points from a group of a TD(6, 25) to obtain a {5,6}-GDD of type $25^5(25 - t)^1$. Give a weight of four to every point and apply FC.

Lemma 4.22 If $1 \leq t \leq 18$ then $601 - 4t \in LS(5, 25^*, (100 - 4t + 1)^*)$.
Proof: Form a partition $\pi(1^1, (100 - 4t)^1, (100)^5)$, and by Lemma 4.21, construct a {5}-GDD of type $(100)^5(100 - 4t)^1$. The line of size twenty-five is obtained by replacing a (25, 5, 1)-BIBD which is embedded in a (101, 5, 1)-BIBD containing ∞

and the one hundred points in a cell of the partition. Otherwise, proceed as previously.

Lemma 4.23 If $1 \leq t \leq 9$ and $r = 105$, or $1 \leq t \leq 11$ and $r = 121$, then $6r - 5 - 8t \in LS(5,25^*,(r - 8t)^*)$.

Proof: Form a partition $\pi(1^1, (r - 1 - 8t)^1, (r - 1)^5)$ and construct a {5}-GDD of type $(r - 1)^5(r - 1 - 8t)^1$ by deleting t points from a TD(6, (r - 1)/8) to yield a {5,6}-GDD of type $((r - 1)/8)^5((r - 8t - 1)/8)^1$, giving every point a weight of eight, and applying FC. The long line of size twenty-five replaces a (25, 5, 1)-BIBD which can be embedded into a (r, 5, 1)-BIBD which contains ∞ and the r - 1 points in a cell of the partition.

Lemma 4.24 If $1 \leq t \leq 15$, then $505 - 4t \in LS(5, 21^*, (85 - 4t)^*)$.
Proof: Form a partition $\pi(1^1, (84 - 4t)^1, 84^5)$ and construct a {5}-GDD of type $84^5(84 - 4t)^1$, by deleting t points from a TD(6, 21), yielding a {5,6}-GDD of type $21^5(21 - t)^1$, giving every point a weight of four and applying FC. The long line of size twenty-one replaces a (21, 5, 1)-BIBD which can be embedded into an (85, 5, 1)-BIBD which contains ∞ and the eighty-four points in a cell of the partition.

Finally, we look at the problem of determining the spectra of AULSs with one long line of size twenty-five and one long line of size fifty-seven. There are only a small number of orders that we can prove belong to the spectra in question. We are able to demonstrate that corresponding AULS of minimum order exists if the long lines intersect.

Lemma 4.25
(a) $\{249,557,577,657\} \subseteq LS_i(5, 25^*, 57^*)$.
(b) $\{557,577,657\} \subseteq LS_d(5, 25^*, 57^*)$.

Proof: We note that 249 is the minimum possible order of an $LS_i(v;\{5, 25^*, 57^*\})$ by Corollary 1.21(iii). Form a partition $\pi(1^1, 24^8, 56^1)$ and construct a $\{5\}$-GDD of type $24^8 56^1$: firstly, there exists a resolvable $\{4\}$-GDD of type 3^8 [L1]. By the method of completion, we obtain a $\{5\}$-GDD of type $3^8 7^1$. Put a weight of eight on every point, and apply FC. The proof for the other values in (a) is similar. Also, $557, 577, 657 \in LS_d(5, 25^*, 57^*)$ by applying Lemma 4.22 with $t = 11$, Lemma 4.23 with $t = 6$ and $r = 105$, as well as $t = 8$ and $r = 121$.

Conclusion

In this thesis, the existence problems for special type of linear spaces called almost uniform linear spaces were examined. It was an easy matter to complete the spectrum $LS(2, u^*, w^*)$. The main body of work involved the investigation of almost uniform linear spaces with short lines of size three. Recursive techniques were applied in order to yield the entire spectrum $LS_i(3, u^{**})$, and a partial answer was obtained for $LS_d(3, u^{**})$. There were also partial solutions given for the spectra $LS(3, (6t + 5)^*, w^*)$, $LS(3, (6t + 7)^*, w^*)$ and $LS(3, (6t + 9)^*, w^*)$. We were able to provide a full solution to the above problems if we assumed that $t = 0$ and $w \equiv 1,3 \pmod 6$, where $7 \leq w \leq 15$, $9 \leq w \leq 15$ and $w = 13,15$, respectively. Some of the recursive methods developed to handle the case of almost uniform linear spaces with short lines of size three had natural extensions enabling us to construct almost uniform linear spaces of particular orders with short lines of size four or five. The majority of the constructions in these cases were achieved by using the notion of completion of resolvable {3}-GDDs and {4}-GDDs, as well as Wilson's fundamental theorem. It was not possible to complete any of the spectra $LS(4, u^{**})$ or $LS(5, u^{**})$, and considerably less could be done when we assumed that the two long lines were disjoint. There were far fewer results if the long lines had different sizes, especially when the short lines were of size five, since not having an analogue of the Doyen-Wilson theorem available meant that we could not automatically conclude that all orders v which were sufficiently large belonged to the spectrum. Another reason was that we could not develop any direct method akin to the one used in Chapter 2, and this appears to be crucial for constructions of almost uniform linear spaces of small v.

It would be instructive to complete the spectra LS(3, $(6t + 5)^*$, w^*), LS(3, $(6t + 7)^*$, w^*) and LS(3, $(6t + 9)^*$, w^*), for any t, and admissible w, since these incidence structures can be viewed as IPBDs with two holes, designs which more recently have been studied by others.

It is probable that other techniques beyond the present work must be developed in order to complete the spectra LS(4, u^{**}) and LS(5, u^{**}). There are many existence problems for almost uniform linear spaces with long lines of different sizes and short lines of size k, $k \geq 4$, which are not settled. Nevertheless, we feel that the following holds:

Conjecture: If u, v and w are nonnegative integers, $u \leq w$, which satisfy the conditions in Theorems 1.14, 1.16, 1.18 or 1.20, then there exists an LS(v;{k, u^*, w^*}), $k \in \{3,4,5\}$.

One could explore similar existence questions by increasing the number of long lines in the linear space, or studying the existence of PBDs with index λ, $\lambda \geq 1$, in which there are precisely m special blocks, $m \geq 1$. There is very little in the literature in this direction for PBDs with index $\lambda > 1$.

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