NONLINEAR OBSERVER FOR THE ORBITAL STATES OF AN EARTH SATELLITE

By

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ABSTRACT

The objective of satellite orbit determination is to find accurate values of a set of orbital elements which describes the orbit of the satellite, using observations of the satellite. The extended Kalman filter has been widely used for the estimation of the orbital states. The purpose of this work is to find an alternative approach that would reduce the amount of on-line computation required. A nonlinear observer is proposed for this application. Its stability problem is studied through the second method of Liapunov. The performance of the nonlinear observer is then evaluated with simulated orbits. Although many secondary effects on the dynamics of the satellite have been omitted, the simulation is indicative of results that can be obtained in real situations.

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a_{e1}, a_{e2}, a_{e3}	perturbing accelerations in orbit e-frame
Az	azimuth
C	parameter in velocity space set of orbital elements
C_{22}, S_{22}	coefficients of the cosine and sine terms of the tesseral harmonics for the gravitational potential function (m=n=2)
e_e	eccentricity of Earth
e_1, e_2, e_3	rotating polar set of unit vector defining the point-mass motion about the geocenter
$e_{01}, e_{02}, e_{03}, e_{04}$	Euler parameters defining rotation of the orbital trajectory frame about the geocenter
E, N, U	east, north and up direction or distance
E_{TOPO}	matrix for conversion from inertial to topocentric local tangent ENU frame
El	elevation
f	nonlinear dynamic model
F	Jacobian matrix of dynamic model
f_1, f_2, f_3	intermediate set of unit vectors defining co-ordinates in the instantaneous orbital plane
h	nonlinear measurement function
H	Jacobian matrix of measurement model
H_t	height of observation site above reference ellipsoid
Lo, La	longitude and lattitude
m	mass of orbital body
q	a unitary quaterion

LIST OF SYMBOLS (continued)

r	radial distance of point-mass from geocenter
R	parameter in velocity space set of orbital elements
R_e	spherical Earth's radius
t	time
α_g	Greenwich sidereal time
λ	angle of rotation from X-axis to the orbital position vector in the instantaneous orbital frame
μ	gravitational constant
ρ	range
θ	instantaneous longitude
ω_1, ω_3	angular velocity of rotating polar coordinates of the orbital body motion about the geocenter

CHAPTER ONE

INTRODUCTION

The control action required to maintain a man-made satellite in a desired orbit is based on the estimation of the orbital states from ground-based measurements. The extended Kalman filter is widely used for orbital trajectory determination and prediction [1]-[7]. However, the dynamic model of a satellite is complicated as well as nonlinear. The use of the extended Kalman filter requires re-linearization at each step for the calculation of the gain matrix. The amount of computation needed warrants a powerful computing machine.

The purpose of this work is to study the feasibility of using a Luenberger observer for the estimation of the orbital states as an alternative approach. Since the desired orbit is known beforehand, it is possible to pre-determine a set of observer gain matrices off-line. Depending upon the current states or position of the satellite, an appropriate observer gain matrix is retrieved from memory for the estimation of the next orbital states. Thus the amount of on-line computation would be very much reduced.

The basic principle of the Luenberger observer for linear systems is discussed in Chapter Two. An algorithm for the calculation of the observer gain matrix is presented.

Chapter Three discusses the extension of observer theory to nonlinear systems. The objective is to find the conditions under which the states of the nonlinear observer approach asymptotically to those of the nonlinear system.

Chapter Four is concerned with the mathematics used in orbit determination and prediction. The unified state model developed by S. Altman and the measurement model associated with it are presented. Reference coordinate frames, measurement of time and an Earth model are described.

Simulation results for the problem of satellite orbit determination and prediction using nonlinear observer are discussed in Chapter Five.

Chapter Six concludes the results of this work and suggestions for future research work are also made.

CHAPTER TWO

OBSERVER FOR LINEAR SYSTEMS

2.1 Introduction

Much control engineering is concerned with the choice of a feedback control law to achieve desired objectives, such as, optimization with respect to some performance indices, minimization of the effect of noise, reduction of the sensitivity of the system to plant parameter variations, or achieving arbitrary dynamics of the system. Many feedback control designs require the knowledge of the states of the controlled plant. But the states are not always available for measurement. One approach is to generate a suitable approximation to the state vector which is then substituted into the feedback control law. The observer proposed by D.G. Luenberger in 1964 [8] is one of the schemes that give an estimate of the state vector based upon the measurements of the system inputs and outputs.

2.2 Luenberger Observer for Linear Systems

Consider a linear nth-order system

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{2.1a}$$

$$y(t) = Cx(t) \tag{2.1b}$$

where $x(t)$ is the state vector of dimension n, $u(t)$ is the control input vector of dimension m, and $y(t)$ is the output vector of dimension p. Denote the estimate of $x(t)$ by $\hat{x}(t)$ and construct a model of the original system

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) \tag{2.2}$$

If the initial state, $x(0)$, is known, this model would provide an exact estimate of the state vector $x(t)$ for all t. But in general, this information is not available. Luenberger proposed an observer with dynamics given by

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + G[C\hat{x}(t) - y(t)] \tag{2.3}$$

where G is the observer gain matrix. The state estimate error vector

$$\tilde{x}(t) = \hat{x}(t) - x(t) \tag{2.4}$$

satisfies the differential equation

$$\dot{\tilde{x}}(t) = (A + GC)\ \tilde{x}(t) \tag{2.5}$$

The solution of this differential equation is

$$\tilde{x}(t) = \tilde{x}(0)\ e^{(A + GC)t} \tag{2.6}$$

If the observer gain matrix G is chosen in such a way that all the eigenvalues of $(A + GC)$ have negative real parts, the state estimate error vector will decay to zero eventually no matter how large the initial estimate error may be. The structure of an observer is illustrated in Figure 2.1.

2.3 Design of Luenberger Observer Gain

Corresponding to the real matrices A and C of a linear system, the set of eigenvalues of the matrix, (A + GC), can be made to correspond to the set of eigenvalues of any real matrix of the same order by suitable choice of the gain matrix G if and only if (C, A) is completely observable [9]. Although the existence of the gain matrix is guaranteed if the system is completely observable, calculation of the appropriate gain matrix to achieve given eigenvalue placement can be a difficult chore. One approach is to transform the system into the observable canonical form before proceeding further for the calculation of gain matrix.

Consider a completely observable system

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{2.7a}$$

$$y(t) = Cx(t) \tag{2.7b}$$

It can be transformed into the observable canonical form by a similarity transformation with the new state vector $\bar{x}(t)$ given by

$$\bar{x}(t) = Qx(t) \tag{2.8}$$

and

$$\bar{A} = QAQ^{-1} \tag{2.9a}$$

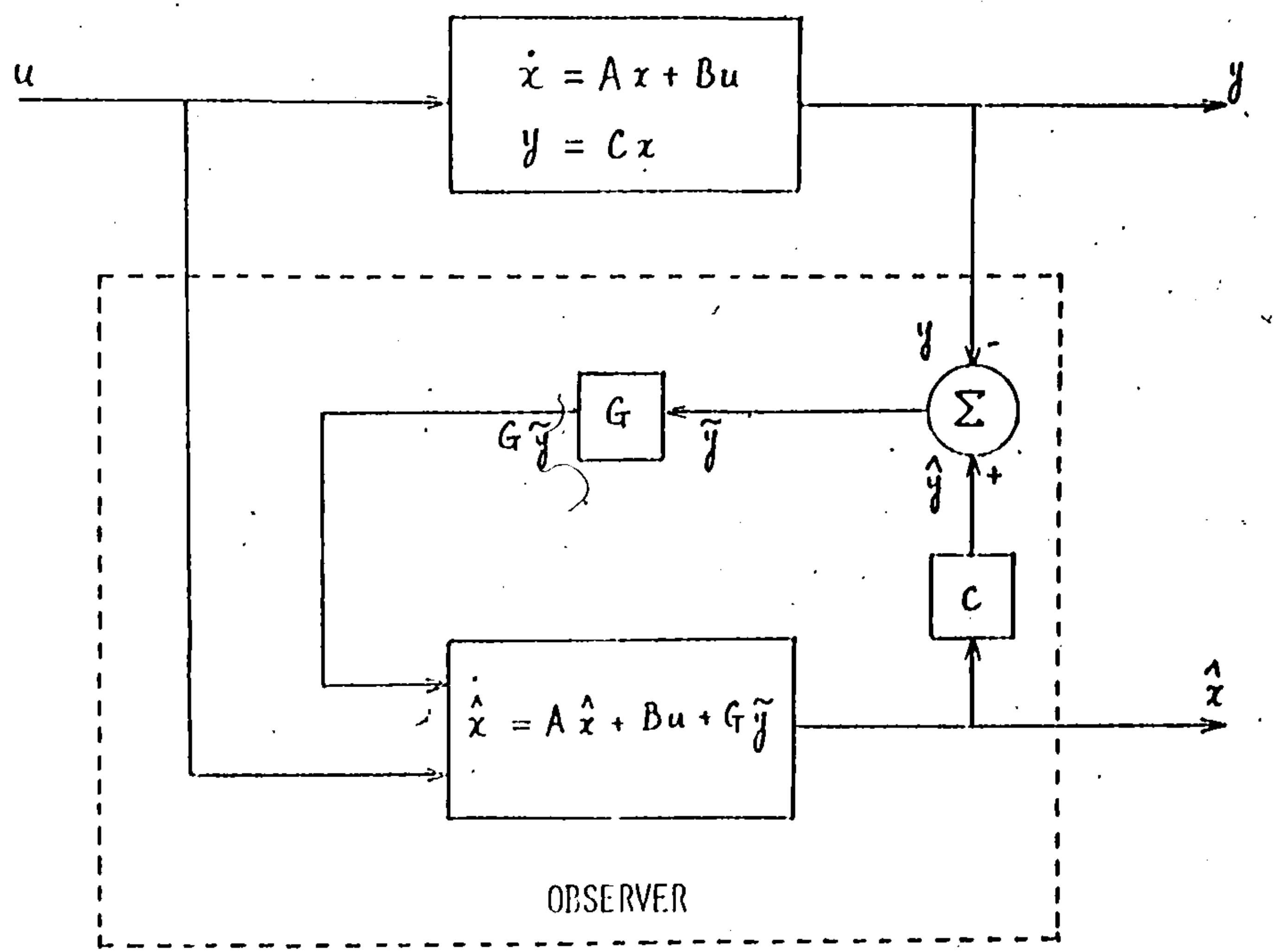


FIGURE 2.1 LUENBERGER OBSERVER FOR LINEAR SYSTEMS

$$\bar{B} = QB \tag{2.9b}$$

$$\bar{C} = CQ^{-1} \tag{2.9c}$$

In particular, the matrices $\bar{A}$ and $\bar{C}$ have the following structures

$$\bar{A} = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1p} \\ A_{21} & A_{22} & \cdots & A_{2p} \\ \vdots & \vdots & & \vdots \\ A_{p1} & A_{p2} & \cdots & A_{pp} \end{bmatrix} \tag{2.10}$$

with

$$A_{ij} = \delta_{ij} N_{n_i} - a(ij)\ e^T_{n_i} \tag{2.11}$$

and

$$\bar{C} = \begin{bmatrix} f^{n_1}_{n_1} & 0 & \cdots & 0 \\ 0 & f^{n_2}_{n_2} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & f^{n_p}_{n_p} \end{bmatrix} \tag{2.12}$$

where δ_{ij} is the Kronecker delta, N_{n_i} is a nilpotent matrix of index n_i as shown below:

$$N_{n_i} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \tag{2.13}$$

e_{n_i} is the n_ith unit vector of dimension n_i and

$$a(ij) = [a_1(ij) \quad a_2(ij) \quad \cdots \quad a_{n_i}(ij)]^T \qquad (2.14)$$

and $f_{n_i}^{n_i}$ is the n_ith unit row vector of dimension n_i and

$$\sum_{i=1}^{p} n_i = n \qquad (2.15)$$

For example,

$$A_{11} = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_1(11) \\ 1 & 0 & \cdots & 0 & -a_2(11) \\ 0 & 1 & \cdots & 0 & -a_3(11) \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{n_1}(11) \end{bmatrix} \qquad (2.16)$$

and

$$A_{12} = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_1(12) \\ 0 & 0 & \cdots & 0 & -a_2(12) \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -a_{n_1}(12) \end{bmatrix} \qquad (2.17)$$

Note that the eigenvalues of a block diagonal matrix are the eigenvalues of the blocks on the main diagonal. Now, suppose a set of eigenvalues has been chosen for the observer, we may form a block diagonal matrix D which has main diagonal blocks with the same size and structure as the corresponding blocks on the main diagonal of the $\bar{A}$ matrix.

$$D = \begin{bmatrix} D_{11} & 0 & \cdots & 0 \\ 0 & D_{22} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & D_{pp} \end{bmatrix} \tag{2.18}$$

where

$$D_{ii} = \begin{bmatrix} 0 & 0 & \cdots & -d_0 \\ 1 & 0 & \cdots & -d_1 \\ 0 & 1 & \cdots & -d_2 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & -d_{n_i-1} \end{bmatrix} \tag{2.19}$$

Note that the characteristic equation of the block D_{ii} is

$$s^{n_i} + d_{n_i-1} s^{n_i-1} + \ldots + d_2 s^2 + d_1 s + d_0 = 0 \tag{2.20}$$

Therefore, the matrix D can be formed easily once the eigenvalues have been chosen and the characteristic equation for each block has been obtained.

The observer gain matrix $\bar{G}$ for the system in the observable canonical form can be readily calculated by solving the matrix equation

$$\bar{A} + \overline{GC} = D \tag{2.21}$$

Then the observer gain matrix for the original system can be otained by the transformation

$$G = Q^{-1} \bar{G} \tag{2.22}$$

Since the eigenvalues of a matrix are invariant under similarity transformation, the eigenvalues of $(A + GC)$ are the same as those of the matrix D.

$$D = Q(A + GC)\ Q^{-1} \tag{2.23}$$

In principle, the choice of the observer eigenvalues is completely arbitrary. The eigenvalues can be moved towards minus infinity, thus, yielding extremely rapid convergence. However, to do so would require very large gains. Since the output $y(t)$ will inevitably contain at least a small amount of measurement noise, this will tend to be magnified if the observer is too fast. So in practice, the eigenvalues of the observer are selected to be somewhat more negative than those of the system to be observed so that convergence of the estimated states to the states of the system is faster than other system effects.

CHAPTER THREE

OBSERVER FOR NONLINEAR SYSTEMS

3.1 Introduction

Since the first publication of Luenberger observer theory which concentrated on purely deterministic, continuous-time, linear, and time-invariant systems, it has been extended to include time-varying systems, discrete systems and stochastic systems [12]-[14]. For nonlinear systems, some studies have also been reported [15]-[17]. However, these studies invariably involve re-linearization of the nonlinear system function at each step to calculate a new observer gain matrix. In this chapter, attempts are made to find the conditions under which the states of a nonlinear observer can be made to approach those of the nonlinear system without re-linearization at every step.

In the case of linear systems, the question of convergence of the state estimate error can be studied by examining the eigenvalues of the observer. But the concept of eigenvalues is meaningless in the case of nonlinear systems. Hence, the second method of Liapunov will be used to study the stability problem of nonlinear observers [18]-[21].

3.2 Stability of Nonlinear Systems

3.2.1 Definitions of Stability

Consider a nonlinear system

$$\dot{x}(t) = f(x,t) \tag{3.1}$$

The following stability definitions are based on the nature of the system time response that results from initial conditions in a particular region of the state space. Let $S(\alpha)$ be an open spherical region of radius $\alpha > 0$ around the origin, that is, $S(\alpha)$ consists of all the points x such that

$$||x|| \leq \alpha \tag{3.2}$$

where $||x||$ is the Euclidean norm of x given by

$$||x|| = (x_1^2 + x_2^2 + x_3^2 + \ldots + x_n^2)^{1/2} \tag{3.3}$$

Definition 1 (stability in the sense of Liapunov)

The origin of the state space of eqn. (3.1) is said to be stable in the sense of Liapunov if corresponding to each S(R), there is an S(r) such that solution of eqn. (3.1) starting in S(r) does not leave S(R) as t approaches infinity.

Definition 2 (Asymptotic Stability)

If the origin of the state space of eqn. (3.1) is stable in the sense of Liapunov, and if, in addition, every solution of eqn. (3.1) starting in S(r) not only stays within S(R) but approaches the origin as $t \to \infty$, then the system is said to be asymptotically stable.

Definition 3 (Instability)

The origin of the state space of eqn. (3.1) is said to be unstable if for some positive number R and any r, no matter how small, there is always a point in S(r) such that a trajectory of the solution starting at this point leaves S(R).

According to Definition 1, an oscillator is stable in the sense of Liapunov if the amplitude of the oscillation remains fixed with time. We shall not refer to systems that are stable in the sense of Liapunov as being stable. Rather we shall use the word stable to refer to systems which are asymptotically stable.

Note that the region S(R) is a function of S(r). But the relationship of the size of S(R) with respect to S(r) is not known. Definition 2 says nothing about the extent of the region of allowable initial conditions, other than to specify a rather vague region S(r). For a nonlinear system, this region may be very small. The second method of Liapunov may be used to determine the stability of a system as well as to give an approximate estimate of the size of the stability region.

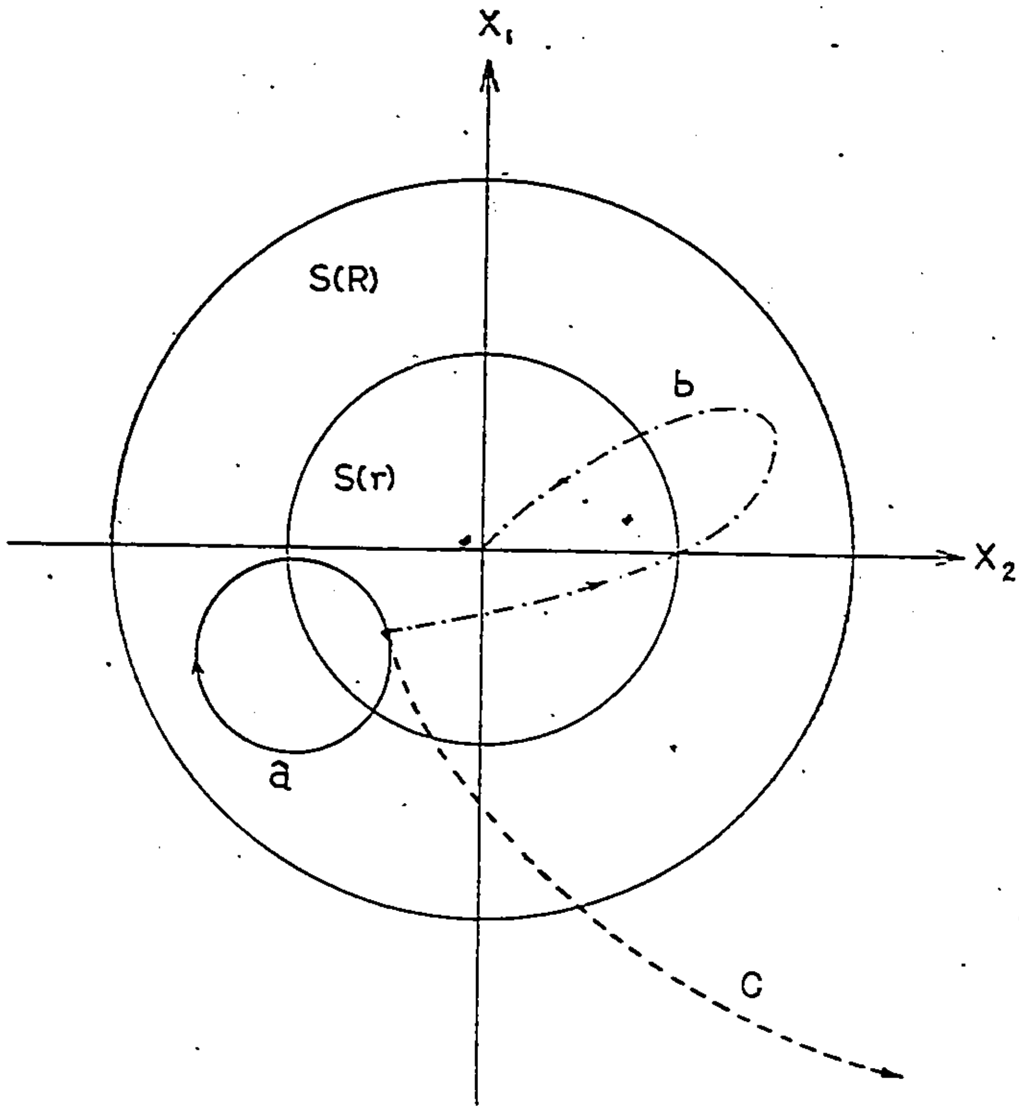


FIGURE 3.1 GRAPHICAL REPRESENTATION OF STABILITY DEFINITIONS

3.2.2 The Second Method of Liapunov

A.M. Liapunov presented two methods for the study of the stability of systems of ordinary differential equations given by

$$\dot{x} = f(x) \tag{3.4}$$

The first, or indirect, method requires a knowledge of the solution of the differential equation. The second, or direct, method requires no knowledge of the solution and provides only stability information.

Basically, the second method is a generalization of the energy concept. A system is known to be stable if its total energy is continuously decreasing. With this as the starting point, Liapunov formed a generalized energy function, known as Liapunov function. Any positive definite scalar function V(x) with continuous first partial derivatives is a possible Liapunov function.

<u>Theorem 1</u> (Local Asymptotic Stability)

If there exists a real scalar function V(x), continuous with continuous first partial derivatives, such that,

(1) $V(0) = 0$

(2) $V(x) > 0$ for $x \neq 0$

(3) $\dot{V}(x) < 0$ for $x \neq 0$

then, eqn. (3.4) is stable in the neighbourhood of the origin.

<u>Theorem 2</u> (Global Asymptotic Stability)

If there exists a real scalar function $V(x)$, continuous with continuous first partial derivatives, such that,

(1) $V(0) = 0$

(2) $V(x) > 0$ for $x \neq 0$

(3) $V(x) \to \infty$ as $||x|| \to \infty$

(4) $\dot{V}(x) \leq 0$

(5) $\dot{V}(x)$ not identically zero along any trajectory of the solution of the system other than the origin

then eqn. (3.4) is globally asymptotically stable.

The most powerful feature of the second method of Liapunov is the fact that the Liapunov function $V(x)$ is not unique. As long as one Liapunov function for a system can be shown to exist, the system is at least asymptotically stable in the neighbourhood of the origin, or equilibrium point, of the system. No longer has one to search for a single unique solution to the differential equation $\dot{x} = f(x)$, but one out of many possible Liapunov functions. The search for such a function is the main difficulty with this approach.

3.2.3 Liapunov Function

In order to show that a system is asymptotically stable, it is sufficient to show that at least one Liapunov function exists. However, it must be noted that failure to find one does not necessarily imply that the system is unstable. The process of finding one possible Liapunov function ~~for~~ a system is presented below.

(a) Linear Systems

Consider a linear system

$$\dot{x} = Ax \tag{3.5}$$

and a general quadratic function

$$V(x) = x^T Rx \tag{3.6}$$

Differentiate V(x) with respect to time,

$$\dot{V}(x) = x^T (RA + A^T R)\ x \tag{3.7}$$

Let

$$Q = -(RA + A^T R) \tag{3.8}$$

If an arbitrary positive definite matrix is selected for the matrix Q and then solve eqn. (3.8) for R, the positive definiteness of R is both necessary and sufficient for asymptotic stability of the linear system.

To determine the positive definiteness of the matrix R, we may apply Sylvester's Theorem which states that:

In order that a matrix R be positive definite, it is necessary and sufficient that each of the following

$$\det[r_{11}],\ \det\begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix},\ \ldots,\ \det(R)$$

be positive.

(b) Nonlinear Systems

Consider a nonlinear system

$$\dot{x} = f(x) \tag{3.9}$$

Expand $f(x)$ in a Taylor series about x_0,

$$f(x) = f(x_0) + \left.\frac{\partial f}{\partial x^T}\right|_{x=x_0} (x - x_0) + \text{higher order terms} \tag{3.10}$$

For simplicity, let $x_0 = 0$ be the origin and equilibrium point of the system. This is always possible by a simple linear transformation of coordinates.

$$f(x_0) = 0 \tag{3.11}$$

since x_0 is the equilibrium point of the system. Eqn. (3.10) becomes

$$\dot{x} = Fx + g(x) \tag{3.12}$$

where F is the Jacobian matrix and g(x) consists of the second and higher order terms. Consider a general quadratic function

$$V(x) = x^T R x \tag{3.13}$$

Differentiate V(x) with respect to time

$$\dot{V}(x) = -x^T Q x + 2x^T R g(x) \tag{3.14}$$

where

$$Q = -(RF + F^T R) \tag{3.15}$$

If an arbitrary positive definite matrix is selected for Q and solve eqn. (3.15) for R, the positive definiteness of R is both necessary and sufficient for asymtotic stability of the linear portion of the nonlinear system. If the linear portion is asymptotically stable, a finite region of stability for the nonlinear system is assured since the last term in eqn. (3.14) can only contain terms of third order or higher. By making x sufficiently small, the first term which contains only second-order terms predominates in the neighbourhood of the origin.

3.3 Nonlinear Observer

The objective of this section is to find the conditions under which the states of a nonlinear observer will approach those of the nonlinear system. Consider an nth-order nonlinear system

$$\dot{x} = f(x) \tag{3.16a}$$

$$y = h(x) \tag{3.16b}$$

Let x_0 be a nominal operating point and expand f(x) and h(x) about x_0 in Taylor series.

$$\dot{x} = f(x) = f(x_0) + F[x - x_0] + f_1(x - x_0) \tag{3.17}$$

$$y = h(x) = h(x_0) + H[x - x_0] + h_1(x - x_0) \tag{3.18}$$

where F and H are the Jacobian matrices consisting of first partial derivative of f(x) and h(x) evaluated at $x = x_0$ and $f_1(x - x_0)$ and $h_1(x - x_0)$ consist of second and higher order terms.

Construct a nonlinear observer

$$\dot{\hat{x}} = f(\hat{x}) + G[h(\hat{x}) - y] \tag{3.19}$$

where G is the observer gain matrix such that all the eigenvalues of (F + GH) have negative real parts. The state estimate error vector

$$\tilde{x} = \hat{x} - x \tag{3.20}$$

is governed by the differential equation

$$\dot{\tilde{x}} = (F + GH)\tilde{x} + [f_1(x - x_0 + \tilde{x}) - f_1(x - x_0)] + G[h_1(x - x_0 + \tilde{x}) - h_1(x - x_0)] \tag{3.21}$$

Assume $f_1(x - x_0)$ satisfies a Lipschitz condition for all $x - x_0$ in a certain region S(a) containing the origin $\tilde{x} = 0$

$$||f_1(x - x_0 + \tilde{x}) - f_1(x - x_0)|| \leq \alpha \, ||\tilde{x}|| \tag{3.22}$$

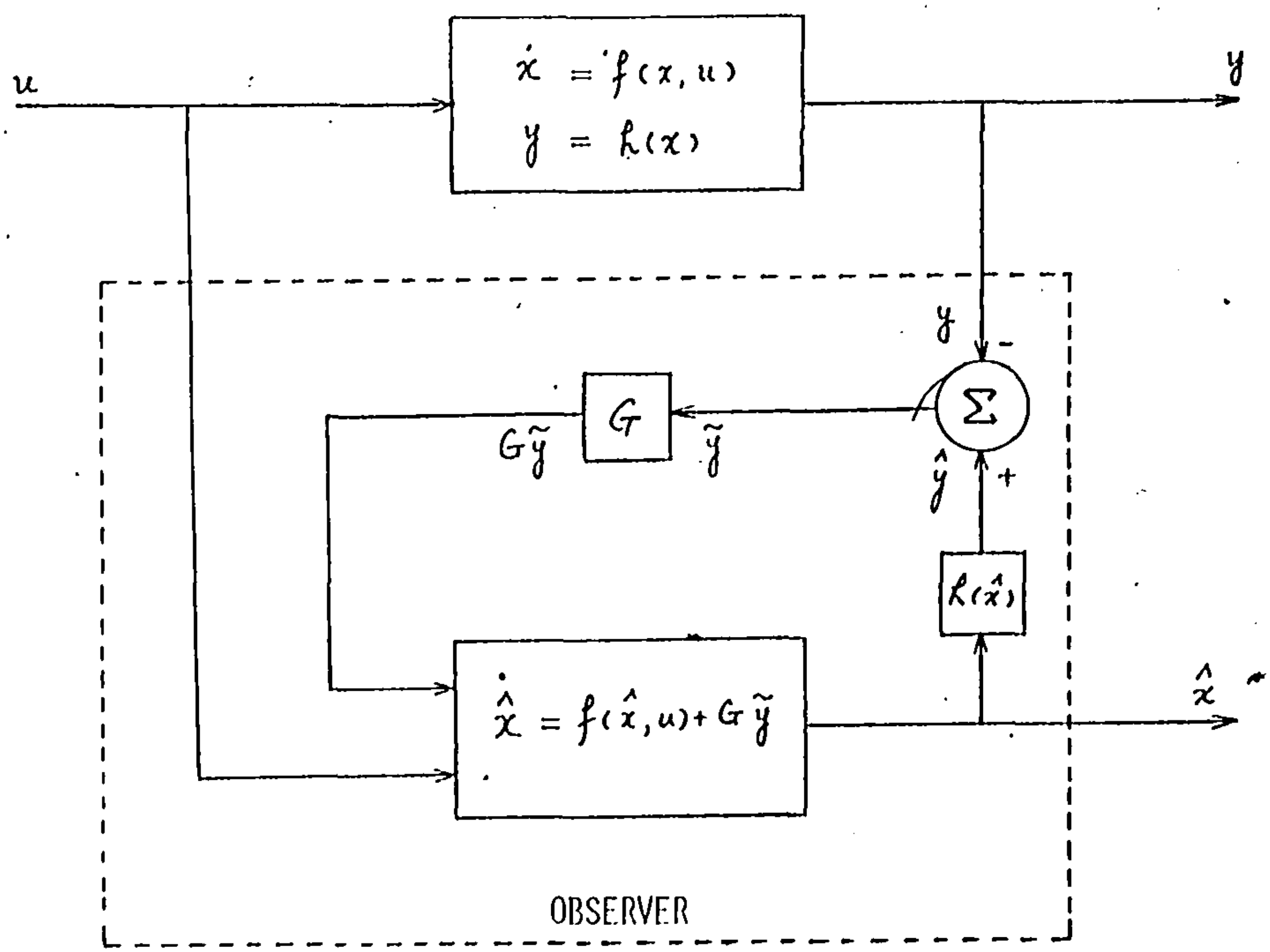


FIGURE 3.2 NONLINEAR OBSERVER

where α is a positive constant. Assume also that $Gh_1(x-x_0)$ satisfies a Lipschitz condition for all $x - x_0$ in a certain region S(b) containing the origin $\bar{x} = 0$

$$||Gh_1(x - x_0 + \bar{x}) - Gh_1(x - x_0)|| \leq \beta \; ||\bar{x}|| \qquad (3.23)$$

where β is a positive constant.

Consider a quadratic function

$$V(\bar{x}) = \bar{x}^T R\bar{x} \qquad (3.24)$$

Evaluate the derivative of $V(\bar{x})$ along the trajectory of eqn. (3.21).

$$\begin{aligned}\dot{V}(\bar{x}) &= \bar{x}^T [R(F + GH) + (F + GH)^T R]\bar{x} \\ &\quad + 2\bar{x}^T R[f_1(x - x_0 + \bar{x}) - f_1(x - x_0)] \\ &\quad + 2\bar{x}^T R\{G[h_1(x - x_0 + \bar{x}) - h_1(x - x_0)]\} \qquad (3.25)\end{aligned}$$

Let

$$-Q = R(F + GH) + (F + GH)^T R \qquad (3.26)$$

Then

$$\dot{V}(\bar{x}) \leq -||\bar{x}^T Q\bar{x}|| + 2||\bar{x}^T R|| \; (\alpha||\bar{x}||) + 2||\bar{x}^T R|| \; (\beta||\bar{x}||) \qquad (3.27)$$

where $||\bar{x}||$, $||Q||$, $||R||$ are the Euclidean norms given by

$$||\bar{x}|| = (\sum_{i=1}^{n} \bar{x}_i^2)^{1/2} \qquad (3.28)$$

$$||Q|| = (\sum_{i=1}^{n} \sum_{j=1}^{n} Q_{ij}^2)^{1/2} \qquad (3.29)$$

$$||R|| = \left(\sum_{i=1}^{n} \sum_{j=1}^{n} R_{ij}^2\right)^{1/2} \tag{3.30}$$

By Schwarz's inequality

$$\dot{V}(\tilde{x}) \le -||Q||\cdot||\tilde{x}||^2 + 2\alpha||R||\cdot||\tilde{x}||^2 + 2\beta||R||\cdot||\tilde{x}||^2 \tag{3.31}$$

If

$$||Q|| = q \tag{3.32}$$

$$||R|| = r \tag{3.33}$$

where q and r are positive numbers, then

$$\dot{V}(\tilde{x}) \le -(q - 2\alpha r - 2\beta r)\ ||\tilde{x}||^2 \tag{3.34}$$

Thus, if

$$q - 2r(\alpha + \beta) \ge 0 \tag{3.35}$$

then $\dot{V}(\tilde{x}) \le 0$, and eqn. (3.21) is locally asymptotically stable.

The above result implies that the state estimate error, $\tilde{x}$, will eventually approach zero if

(1) (F, H) is a completely observable pair,

(2) the nonlinear terms in the system to be observed satisfy a Lipschitz condition in the region $\min[S(a),S(b)]$ containing the origin $\tilde{x} = 0$,

(3) the initial state estimate error, $\tilde{x}(0)$, is sufficiently small that it is within the stability region, and

(4) (3.35) is satisfied.

3.4 Example

Consider a second-order nonlinear system

$$\dot{x}_1 = -x_1^2 - 3x_2$$
$$\dot{x}_2 = x_1 - x_2 + u \tag{3.36a}$$

and

$$y = x_1^2 + x_2 \tag{3.36b}$$

where u is the input and y is the output of the system. The Jacobian matrices, F and H, are given by

$$F = \begin{bmatrix} -2x_1 & -3 \\ 1 & -1 \end{bmatrix}\Bigg|_{x=x_0} \tag{3.37}$$

and

$$H = [2x_1 \quad 1]\Big|_{x=x_0} \tag{3.38}$$

If the initial state esitmate is $\hat{x}_1(0) = 0$ and $\hat{x}_2(0) = 0$

$$F = \begin{bmatrix} 0 & -3 \\ 1 & -1 \end{bmatrix} \tag{3.39}$$

and

$$H = [0 \quad 1] \tag{3.40}$$

The eigenvalues of F are at $s = -0.5 \pm j(\sqrt{11}/2)$. Note that (F, H) is already in observable canonical form. If the eigenvalues of the linearized portion of the observer are chosen to be at $s = -2$, then

$$G = \begin{bmatrix} -1 \\ -3 \end{bmatrix} \tag{3.41}$$

The nonlinear observer is given by

$$\begin{aligned} \dot{\hat{x}}_1 &= -\hat{x}_1^2 - 3\hat{x}_2 - \bar{y} \\ \dot{\hat{x}}_2 &= \hat{x}_1 - \hat{x}_2 + u - 3\bar{y} \end{aligned} \tag{3.42}$$

where

$$\bar{y} = y - \hat{x}_1^2 - \hat{x}_2 \tag{3.43}$$

If the control input $u = -2$, the steady state solution of the system is

$$x_{1(ss)} = 1.372$$

$$x_{2(ss)} = -0.628$$

Figures 3.3(a) and 3.3(b) indicate the time responses of the system and the observer for initial states, $x_1(0) = 0.5$ and $x_2(0) = 0.5$. The states of the observer approach those of the system in spite of the initial estimate errors.

However, if the process to be observed starts at the initial states, $x_1(0) = -0.55$ and $x_2(0) = 1$, the nonlinear observer fails to give close estimates of the states because the initial estimate error vector is so large that it falls outside the stability region of the nonlinear observer. Figures 3.4(a) and 3.4(b) show the trajectories of the system and the observer.

As is evident from eqn. (3.23), the size of the stability region is affected by the choice of observer gain matrix G. If the eigenvalues of the linear portion of the observer are chosen to be at s = -3 instead of -2, the new observer gain matrix becomes

$$G = \begin{bmatrix} -6 \\ -5 \end{bmatrix} \tag{3.44}$$

and the nonlinear observer is given by

$$\begin{aligned} \dot{\hat{x}}_1 &= -\hat{x}_1^2 - 3\hat{x}_2 - 6\tilde{y} \\ \dot{\hat{x}}_2 &= \hat{x}_1 - \hat{x}_2 + u - 5\tilde{y} \end{aligned} \tag{3.45}$$

The change of the observer gain matrix alters the stability region of the observer. With the initial states of the system, $x_1(0) = -0.55$ and $x_2(0) = 1$, the nonlinear observer given by eqn. (3.45) is able to give a correct estimate of the states of the system. The motions of the system (3.26) and the nonlinear observer (3.45) are shown in Figures 3.5(a) and 3.5(b).

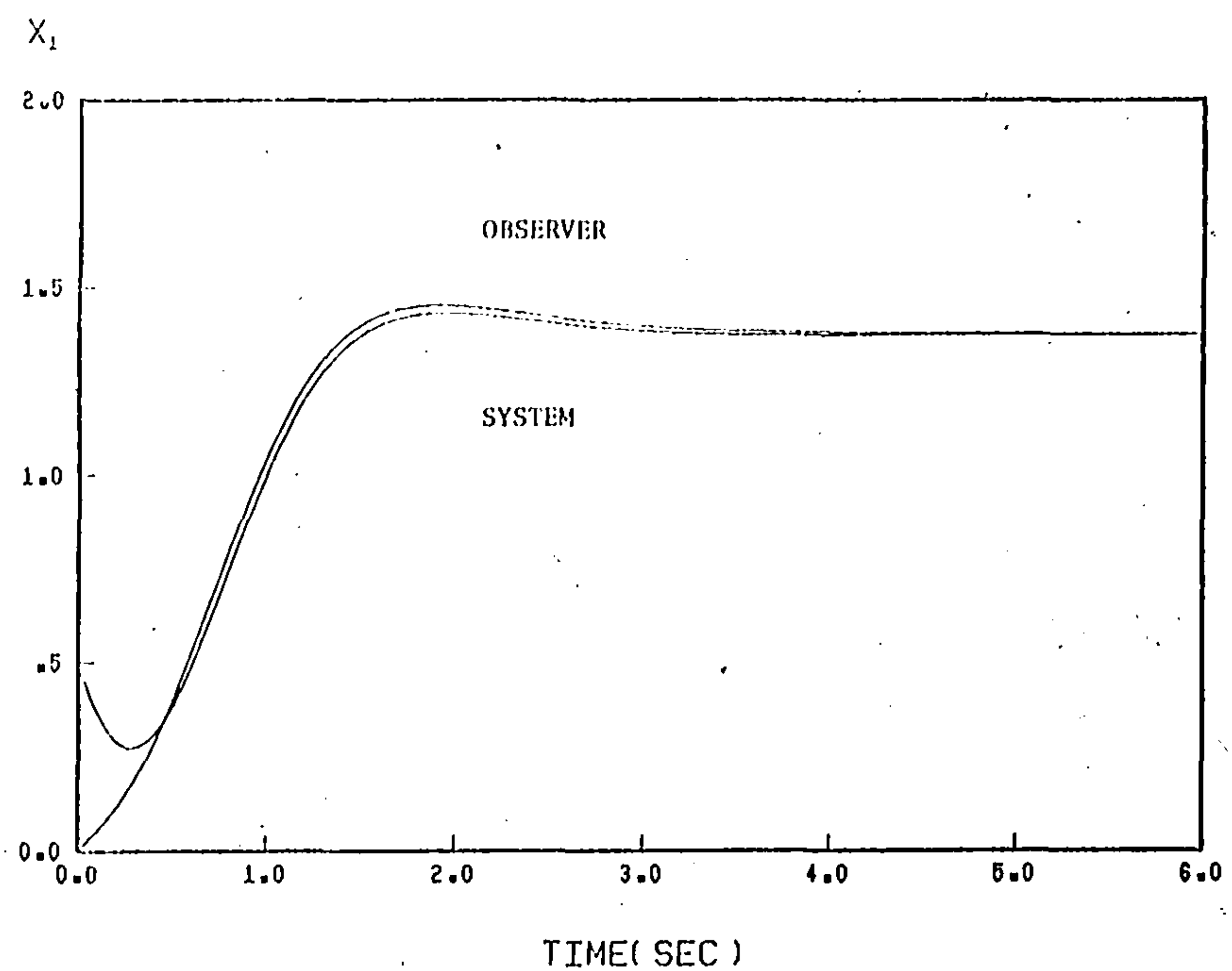


FIGURE 3.3 TRAJECTORIES OF DYNAMIC SYSTEM AND OBSERVER (CASE 1)

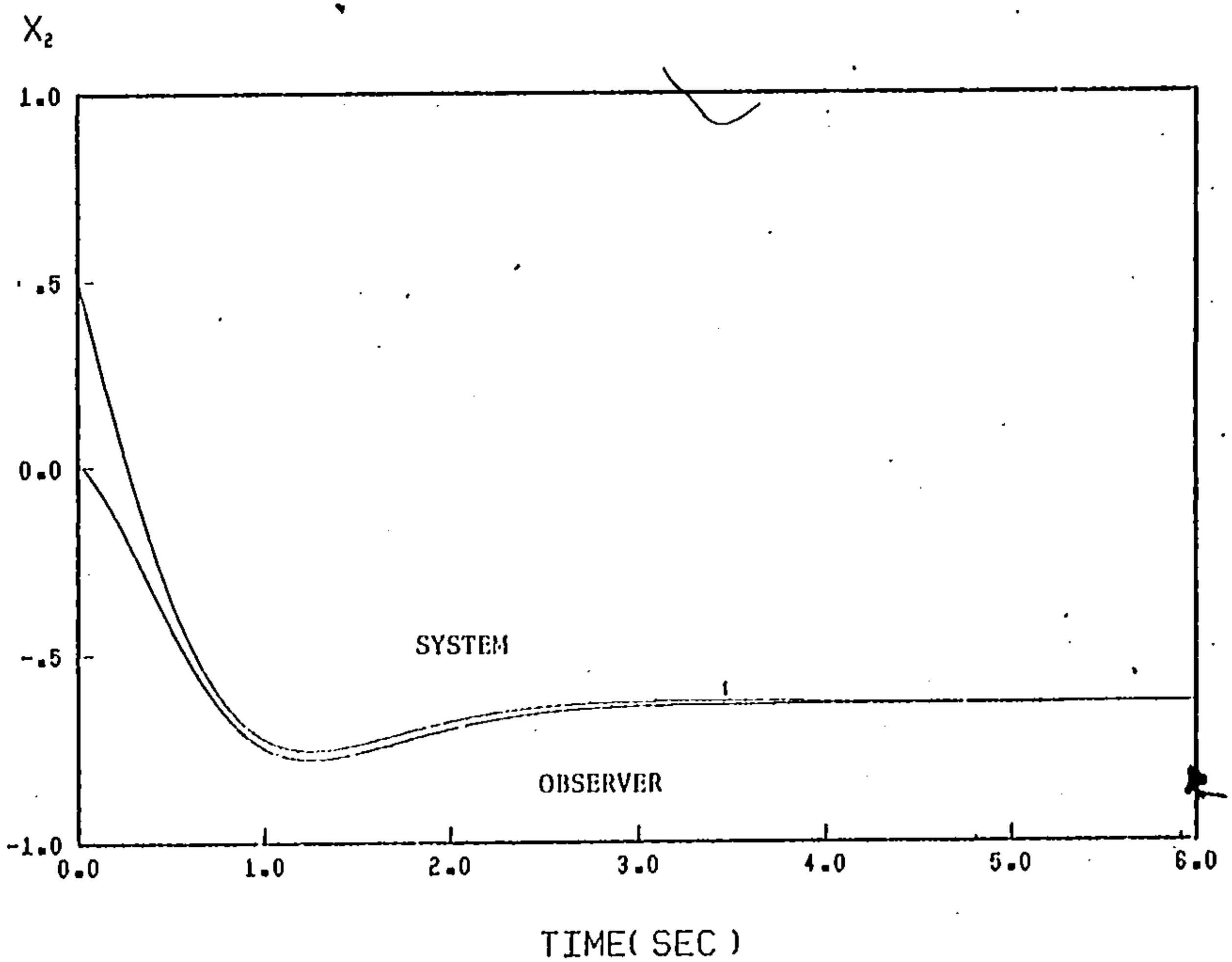


FIGURE 3.3 TRAJECTORIES OF DYNAMIC SYSTEM AND OBSERVER (CASE 1)

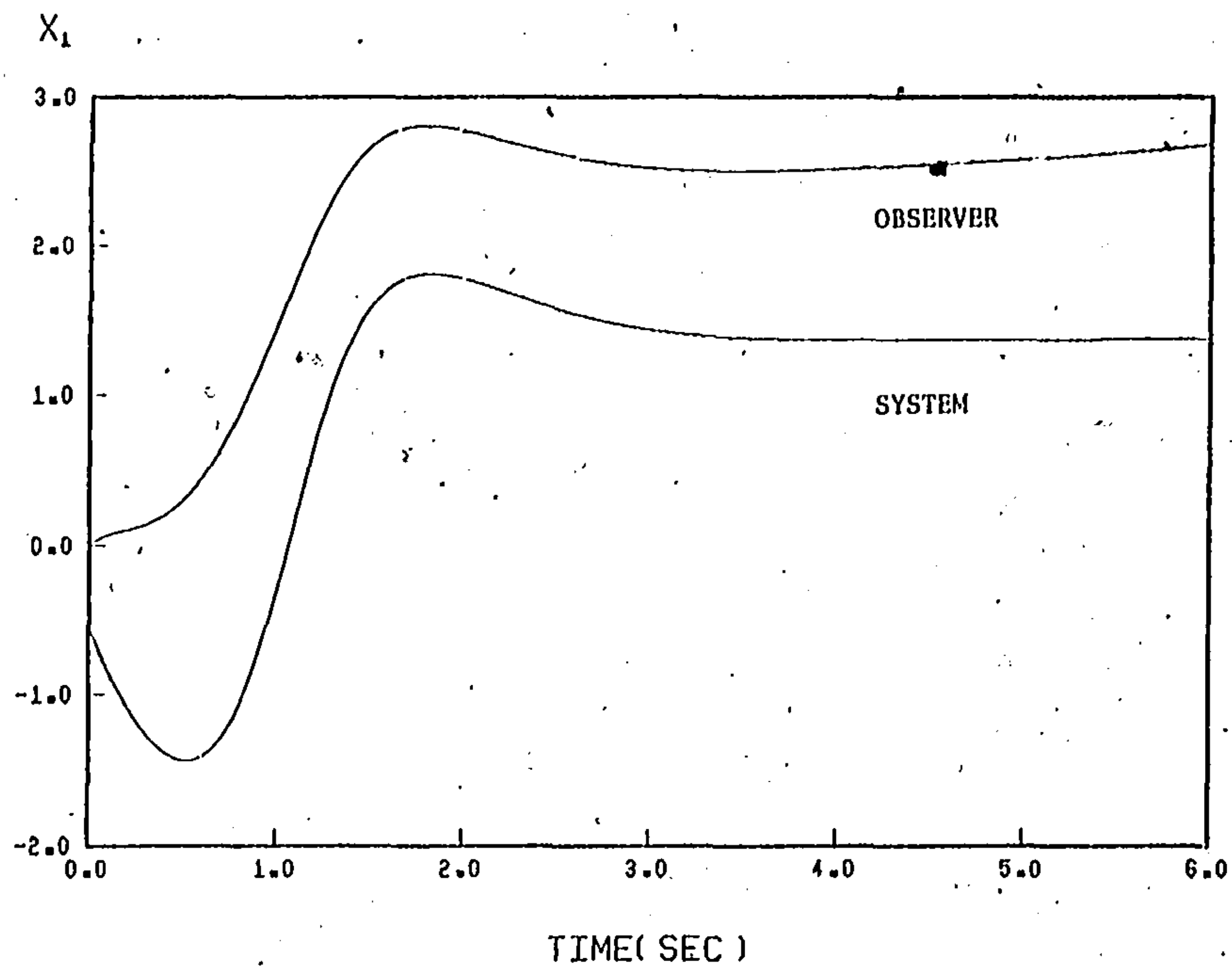


FIGURE 3.4 TRAJECTORIES OF DYNAMIC SYSTEM AND OBSERVER (CASE 2)

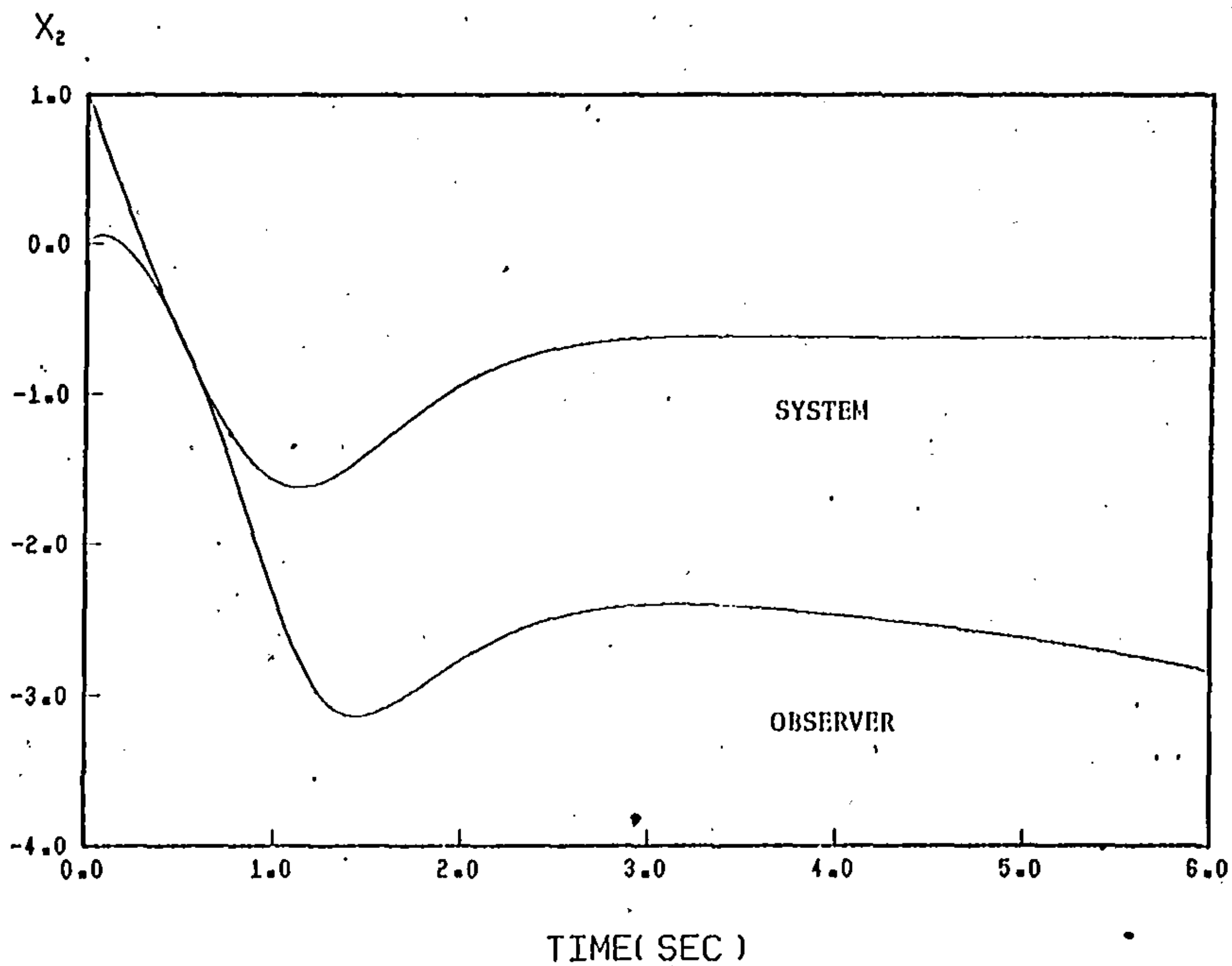


FIGURE 3.4 TRAJECTORIES OF DYNAMIC SYSTEM AND OBSERVER (CASE 2)

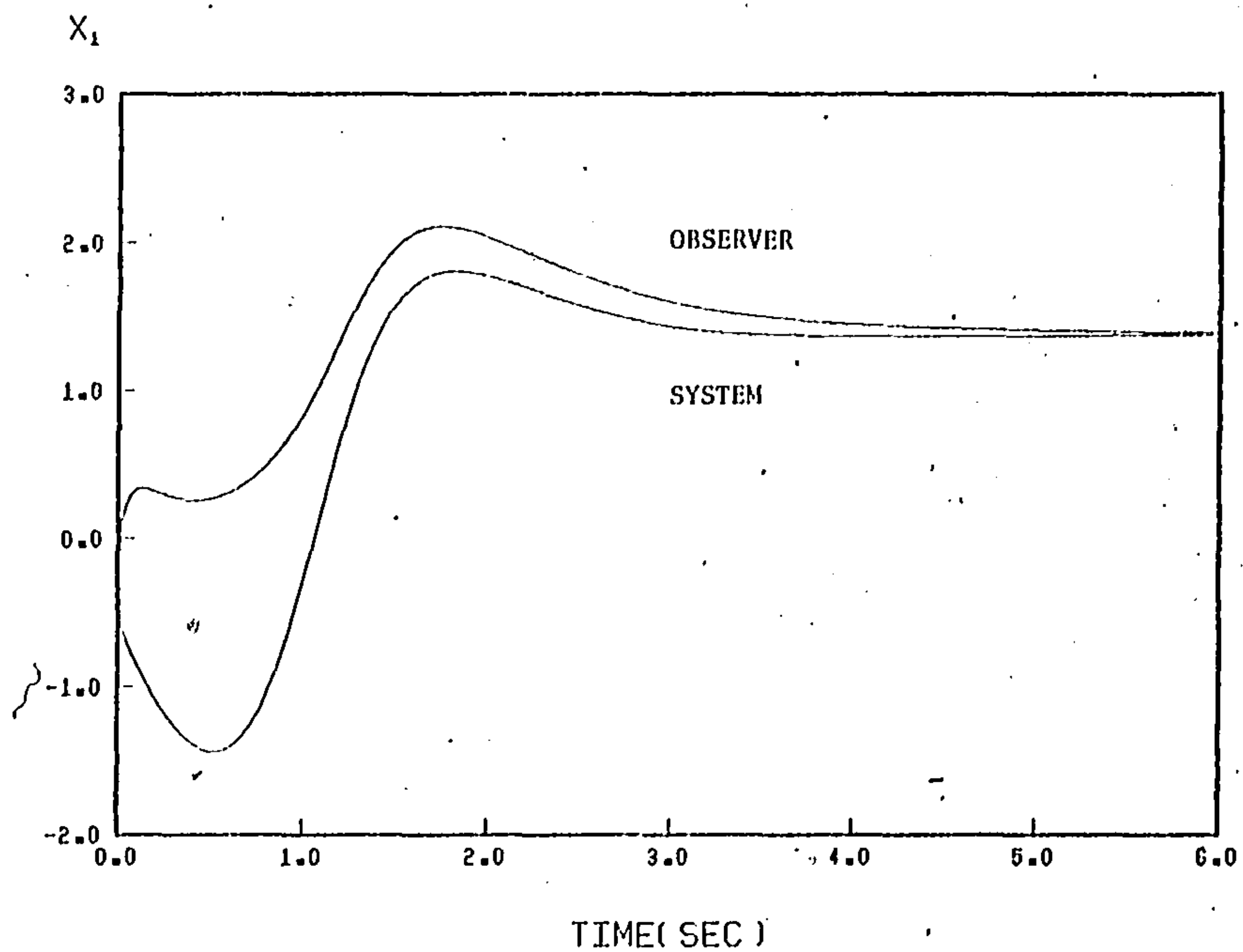


FIGURE 3.5 TRAJECTORIES OF DYNAMIC SYSTEM AND OBSERVER (CASE 3)

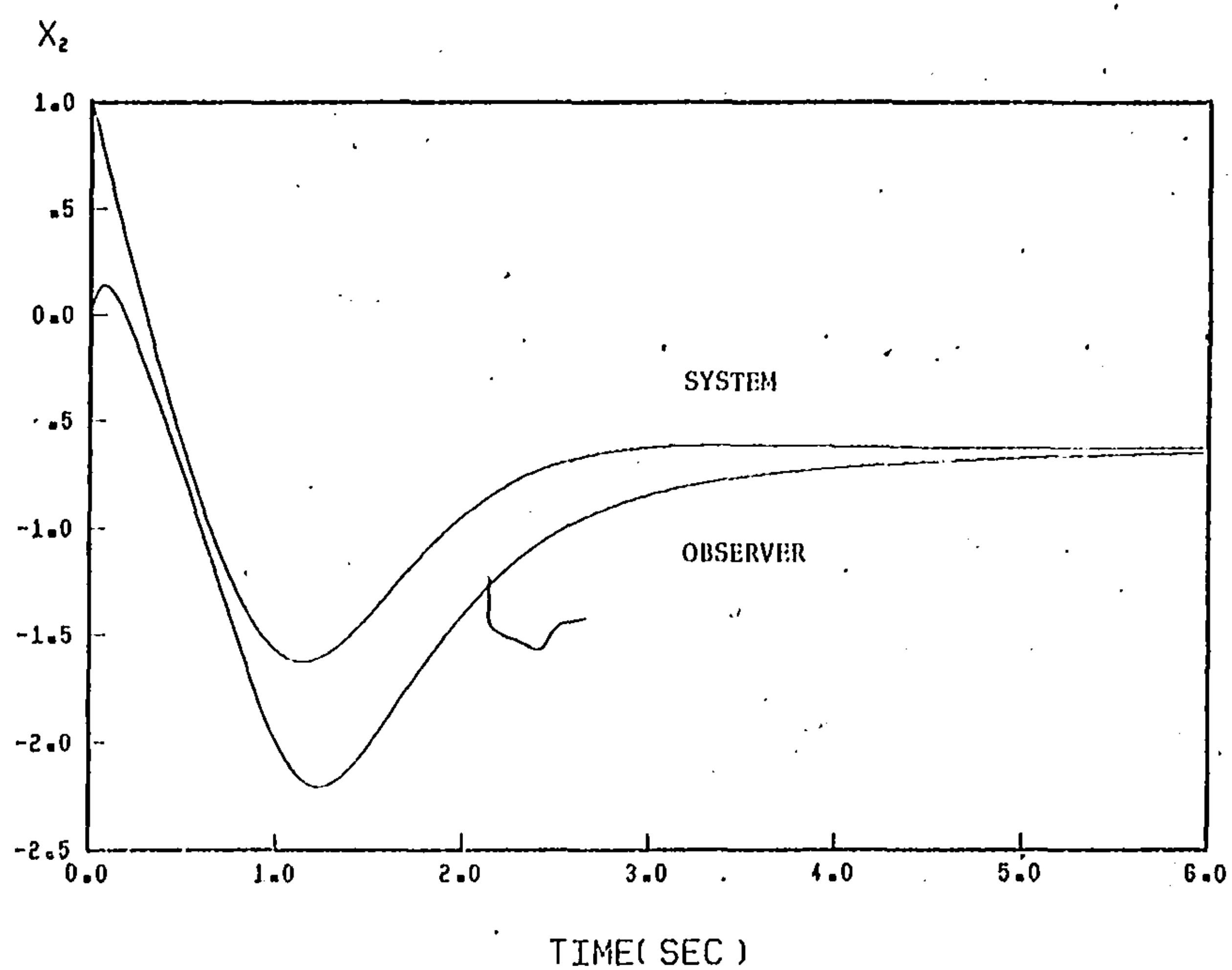


FIGURE 3.5 TRAJECTORIES OF DYNAMIC SYSTEM AND OBSERVER (CASE 3)

CHAPTER 4

ORBITAL DYNAMICS

4.1 Introduction

The orbit of a satellite is defined by a set of orbital elements at a specified time. The objective of orbit determination is to produce an accurate set of orbital elements and use them to predict the satellite position and velocity at any desired time.

Many sets of dynamical variables are available for selection as orbital elements and use in aerospace system and trajectory formulation for design and operational computation. Although no one set can be universally most efficient for all mission requirements, significant differences exist in their utility as the result of diverse functional forms of the consequent constraint equations. The dynamic model of a satellite used in this work is the unified state model developed by S. Altman [27], as further modified by P. Choda [28].

In this chapter, the mathematics used in satellite orbit determination are described. These include the state model of a satellite, the co-ordinate sets, and the measurement model which relates the observation measurements to the orbital elements of the state model.

4.2 Reference Frames

The motion and the position of a satellite are best described in a coordinate frame that is fixed both in space and in time. For this purpose, we choose an inertial frame and fix it at a certain epoch. However, the measurements of the position of a satellite from an observation site on the surface of Earth are made in a different coordinate frame, known as the observation frame. The dynamic model of the satellite uses yet another set of coordinate frames so that it is in a simple and compact form. Several other coordinate frames which serve as links will be described below along with these three coordinate frames.

4.2.1 Inertial Reference Frame

Origin	-	at Earth center of gravity and coordinate axes
X	-	direction of the vernal equinox at epoch
Y	-	direction forms right-handed system with X and Z axes
Z	-	directed North and normal to equator

4.2.2 Earth Fixed Cartesian

Origin	-	at Earth center of gravity and coordinate axes
X	-	direction of prime meridian intersection with equator
Y	-	on equator forming right-handed system with X and Z axes

Z - directed North and normal to the equator

4.2.3 Earth Fixed Spherical

Origin - at Earth center of gravity and coordinate

r - radial distance to the point being measured

La - latitude, positive north of equator

Lo - longitude, east of prime meridian

4.2.4 Topocentric Local Tangent

Origin - at observation site and coordinate axes

E - projection of the East direction in the local horizon plane

N - projection of the North direction in the local horizon plane

U - local vertical

4.2.5 Observation Frame

Origin - at observation site and coordinates

El - elevation, angular measurement from the horizon to the range vector

Az - azimuth, an angle measured in the plane of the local horizon. It is measured from the projection of the North direction eastward to the projection of the range vector

ρ - radial distance to the point being measured

4.2.6 Orbit Frame (f_1 f_2 f_3)

Origin - at Earth center of gravity and axes

f_1 - in the orbit plane such that the angle between f_1 and the ascending node is equal to the longitude of the ascending node in inertial coordinates

f_2 - in the orbit plane and normal to f_1 such that the satellite moves from f_1 to f_2

f_3 - forms a right-handed coordinate system with f_1 and f_2 and is in the direction of the orbit angular momentum

4.2.7 Orbit Frame (e_1 e_2 e_3)

Origin - at Earth center of gravity and axes

e_1 - directed towards the satellite

e_2 - in the orbit plane and normal to e_1 such that the satellite moves from e_1 to e_2

e_3 - normal to the orbit plane and forms right-handed coordinate system with e_1 and e_2

4.3 Transformation between Coordinate Frames

4.3.1 Transformation from Orbit Frame f_1 f_2 f_3 to Orbit Frame e_1 e_2 e_3

This is simply a rotation by an angle λ about the f_3 axis to bring the f_1 axis in line with the direction of the satellite.

$$\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} \cos\lambda & \sin\lambda & 0 \\ -\sin\lambda & \cos\lambda & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} \qquad (4.1)$$

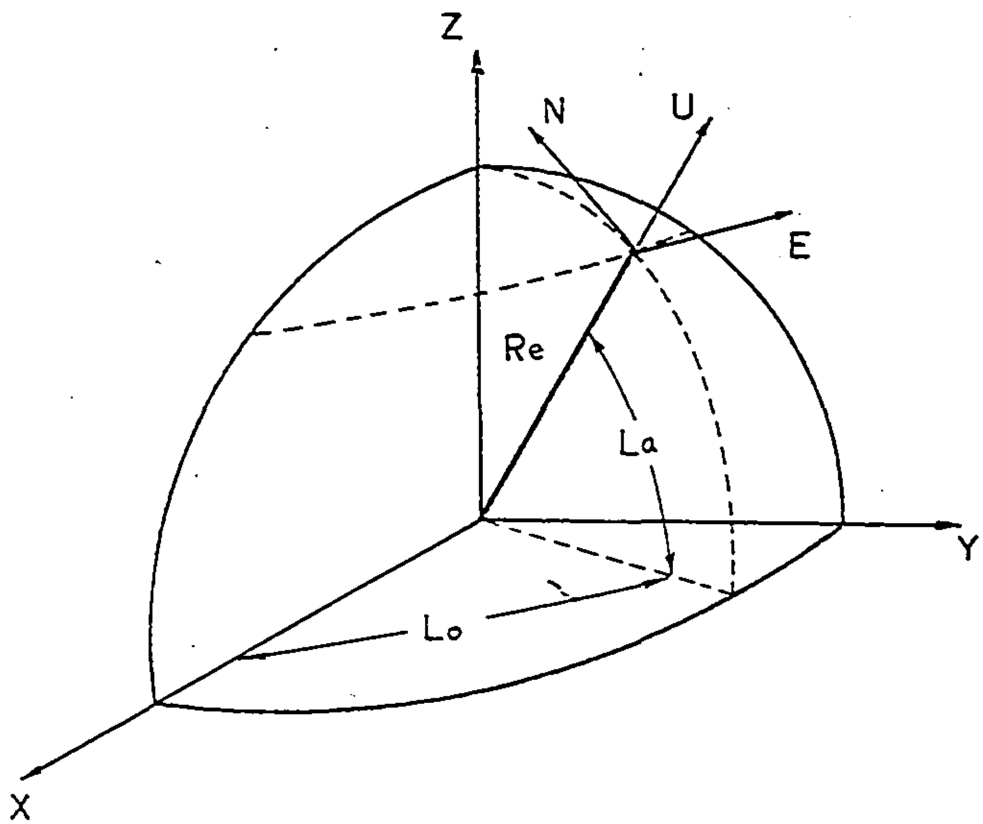


FIGURE 4.1 CO-ORDINATE SET FOR A GROUND-BASED SITE

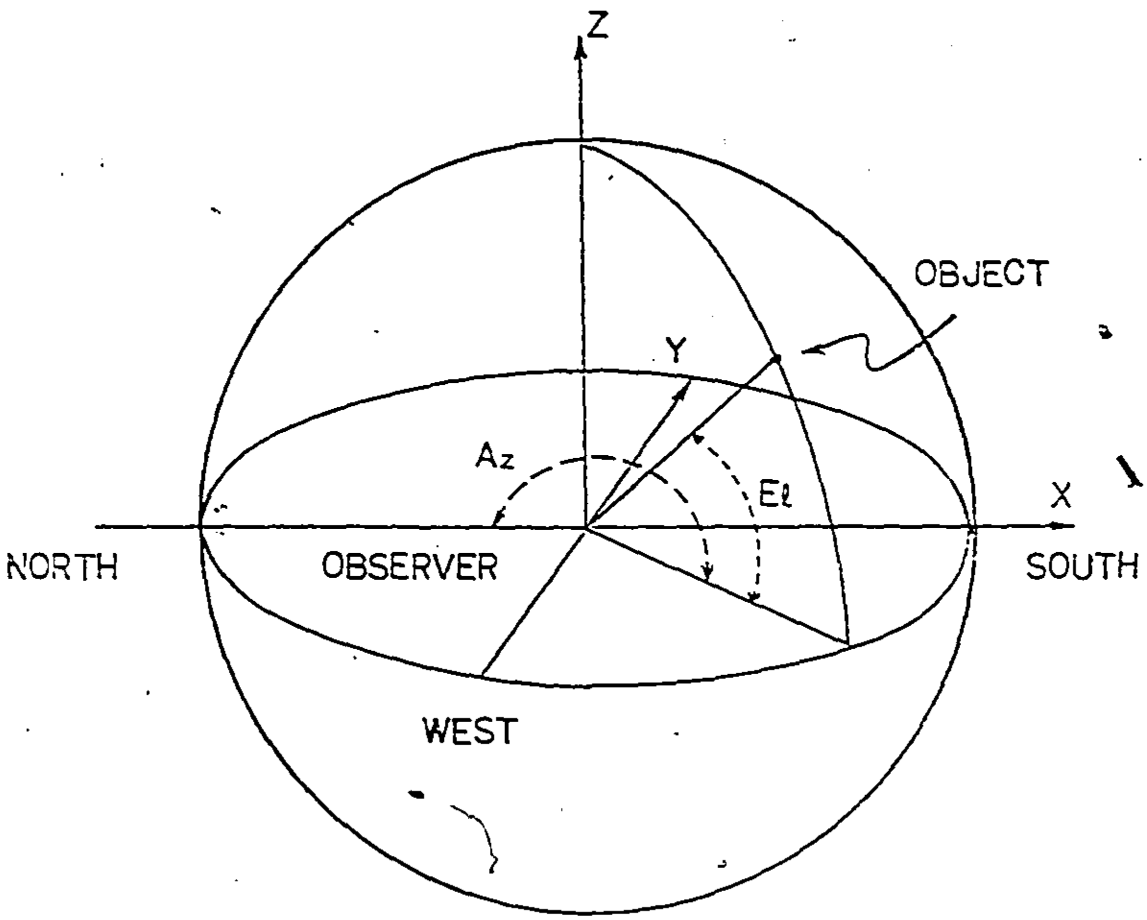


FIGURE 4.2 OBSERVATION FRAME

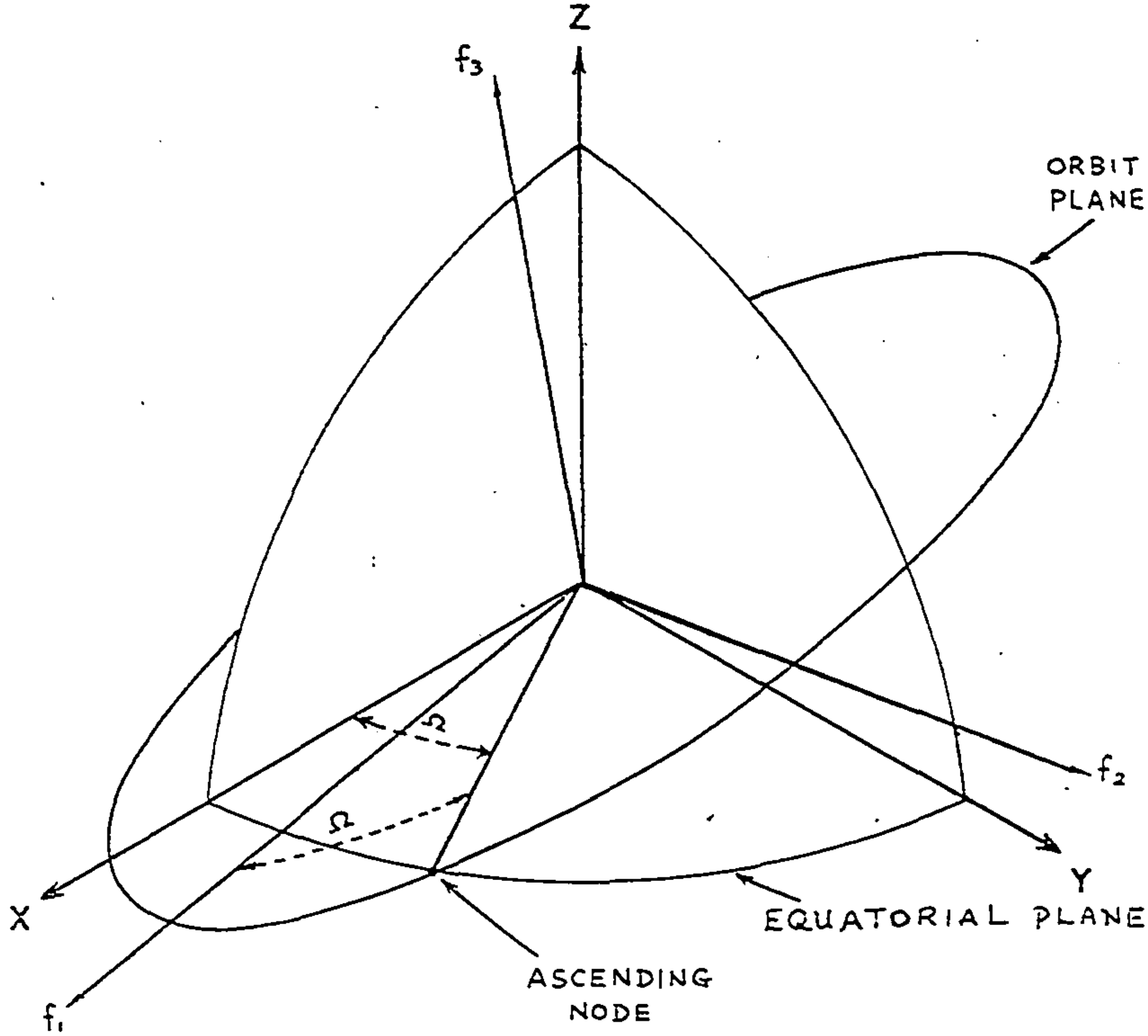


FIGURE 4.3 DEFINITION OF f_1 f_2 f_3 CO-ORDINATES

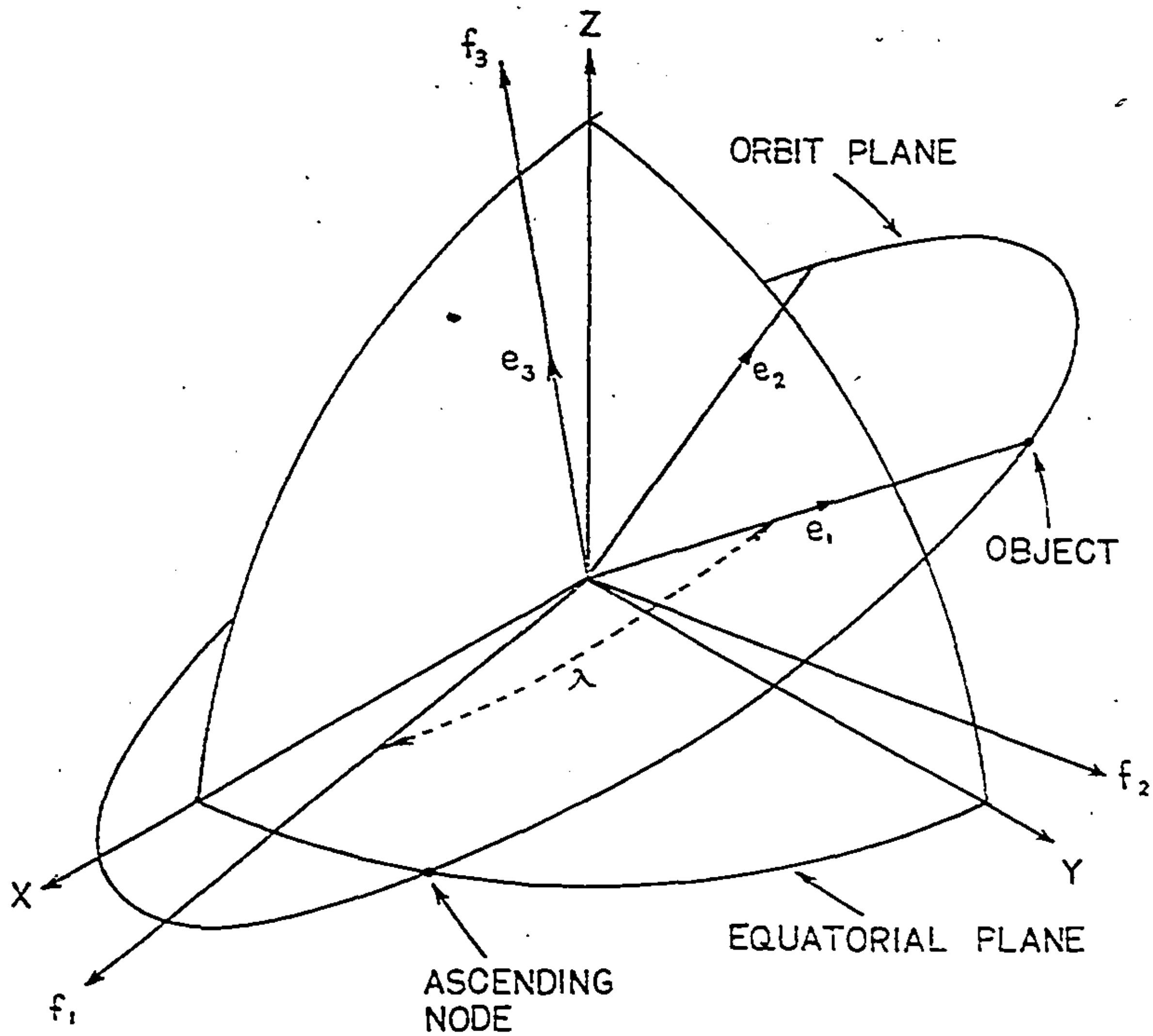


FIGURE 4.4 DEFINITION OF e_1 e_2 e_3 CO-ORDINATES

4.3.2 Transformation from Orbit Frame e_1 e_2 e_3 to Intertial Frame

The rotation matrix from e-frame to inertial frame is the inverse of the E-matrix defined in Section 4.5. Because the E-matrix is a unitary matrix, the rotation matrix is simply its transpose.

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = E^T \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} \tag{4.2}$$

4.3.3 Transformation from Inertial Frame to Topocentric Local Tangent

The origin of the inertial frame is translated to the observation site from the Earth's center of gravity. Rotate about the new Z axis by inertial longitude θ.

$$\theta = Lo + \alpha_g \tag{4.3}$$

where Lo is longitude of observation site and α_g is Greenwich Sidereal Time. Then rotate about the new Y-axis by the negative of latitude of observation site.

$$\begin{bmatrix} E \\ N \\ U \end{bmatrix} = E_{TOPO} \left[\begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_{satellite} - \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_{site} \right] \tag{4.4}$$

where

$$E_{TOPO} = \begin{bmatrix} \sin\theta & \cos\theta & 0 \\ -\sin La \cos\theta & -\sin La \sin\theta & \cos La \\ \cos La \cos\theta & \cos La \sin\theta & \sin La \end{bmatrix} \quad (4.5)$$

4.3.4 Transformation from Topocentric Local Tangent to Observation Frame

$$El = \sin^{-1} \frac{U}{(E^2 + N^2 + U^2)^{1/2}} \quad (4.6a)$$

$$Ax = \tan^{-1} (E/N) \quad (4.6b)$$

$$\rho = (E^2 + N^2 + U^2)^{1/2} \quad (4.6c)$$

4.4 The Measurement of Time

The intent of this section is to relate the necessary definitions and equations for the determination of time.

4.4.1 Universal Time

Astronomical observations are customarily reported in Universal Time, given in units of mean solar days, hours, minutes, and seconds. The mean solar time at the Greenwich Observatory is called Universal Time (U.T.), or Greenwich Mean Time (G.M.T.). One mean solar day is the elapsed time between successive passages of the meridian of the observer past the mean Sun. The mean Sun is a fictitious Sun that moves along the celetial equator at the average speed with which the true Sun apparently moves along the ecliptic throughout the year.

For the sake of convenience, the Earth is divided into twenty-four time zones. The Greenwich meridian is exactly in the middle of time zone number 0. If an observer is z time zones west of the Greenwich meridian and the local mean time is x hours, then the Universal Time can be obtained by

$$U.T. = x + z \tag{4.7}$$

4.4.2 The Julian Date

The Julian Date, denoted by J.D., is an arbitrary benchmark which is a continuous count of each day elapsed since a particular epoch. That epoch was selected to be January 1, 4713 B.C. Each Julian Date is measured from noon to noon, and hence, is an exact integer twelve hours after midnight. The Inertial Frame in this work is fixed at the mean equator and equinox of 1950.0. The Julian Date for this epoch is 2433282.5.

4.4.3 Sidereal Time

If the Earth's axis of rotation coincides with the Z-axis of the inertial reference frame, there is a unique angle between the Greenwich prime meridian and the inertial X-axis. The angle is denoted by α_g and is defined as the sidereal time of the Greenwich prime meridian. The Greenwich sidereal time can be calculated by means of the following formula [29]:

$$\alpha_g = 100.075542 + 360.985647346\ (t - 2433282.5)$$

$$+ 0.29 \times 10^{-12}\ (t - 2433282.5)^2 \qquad (4.8)$$

where t is time measured in Julian date. The instantaneous longitude of an observation site which is on east longitude Lo is given by

$$\theta(t) = \alpha_g(t) + Lo \qquad (4.9)$$

This equation provides the basic link between the rotating reference frames and the inertial reference frame.

4.4.4 Ephemeris Time

Both mean solar time and sidereal time are based on the rotation of the Earth about its axis. It has been found that the Earth suffers from periodic and secular variations in its rotational rate. In order to have a more uniform time, Ephemeris Time was developed. Emphemeris Time, denoted by E.T. is defined as

$$E.T. = U.T. + \Delta T \qquad (4.10)$$

where ΔT is an annual increment tabulated in the American Ephemeris and Nautical Almanac. It must be noted that ΔT cannot be calculated in advance and is always an estimated quantity. Ephemeris Time must be used if the time period under consideration is long and high accuracy is required.

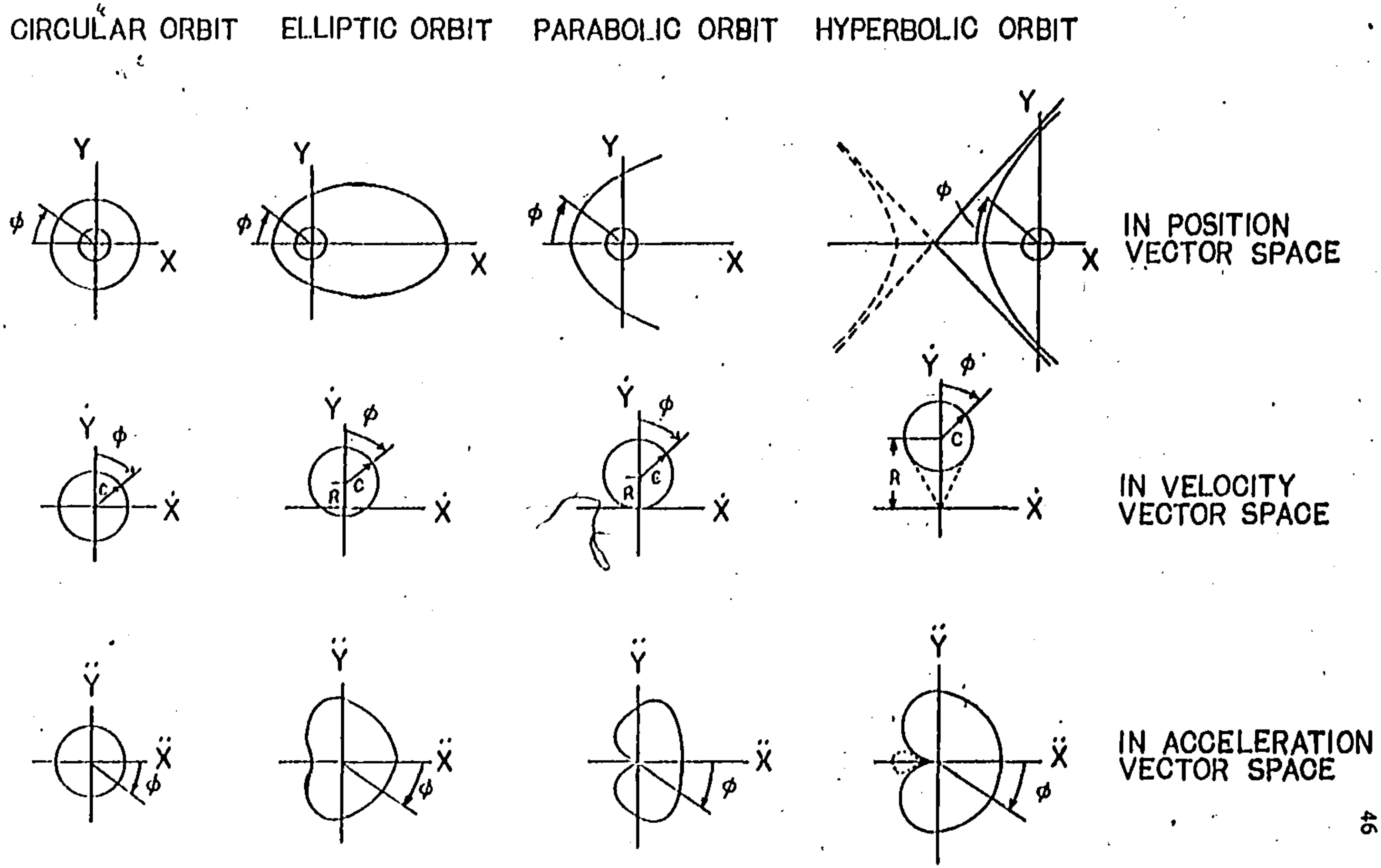
4.5 The Unified State Model

A unified state model has been developed by S.P. Altman to define the trajectory and attitude dynamics of an orbital spacecraft. Only the portion concerning orbit dynamics will be described in this section. In the unified state model, the state variables are momenta and the coordinate variables are the Euler parameters [27].

An unperturbed orbital trajectory, occuring in the presence of the simple spherical harmonic function of gravity field due to one celestial body, is represented by cyclic figures lying in an orbital plane in position, velocity, and acceleration vector spaces as shown in Fig. 4.5. The orbital figure is a conic in position space, a circle in velocity space, and a limacon-like figure in acceleration space. As the orbital energy level changes, only the velocity map remains invariant in geometric figure. As a result, a differential formulation of the orbital trajectory dynamics will be free from singularities in the state variables. Since the velocity space map corresponds to the position space map point by point, the position state can be obtained directly from the velocity state by means of algebraic transform equations.

The velocity space parameters, C and R, are functions of radial momentum ($p_r = mv_{e1}$) and angular momentum ($p_\lambda = mv_{e2}$) as defined by

FIGURE 4.5 STATE MAP OF AN ORBIT



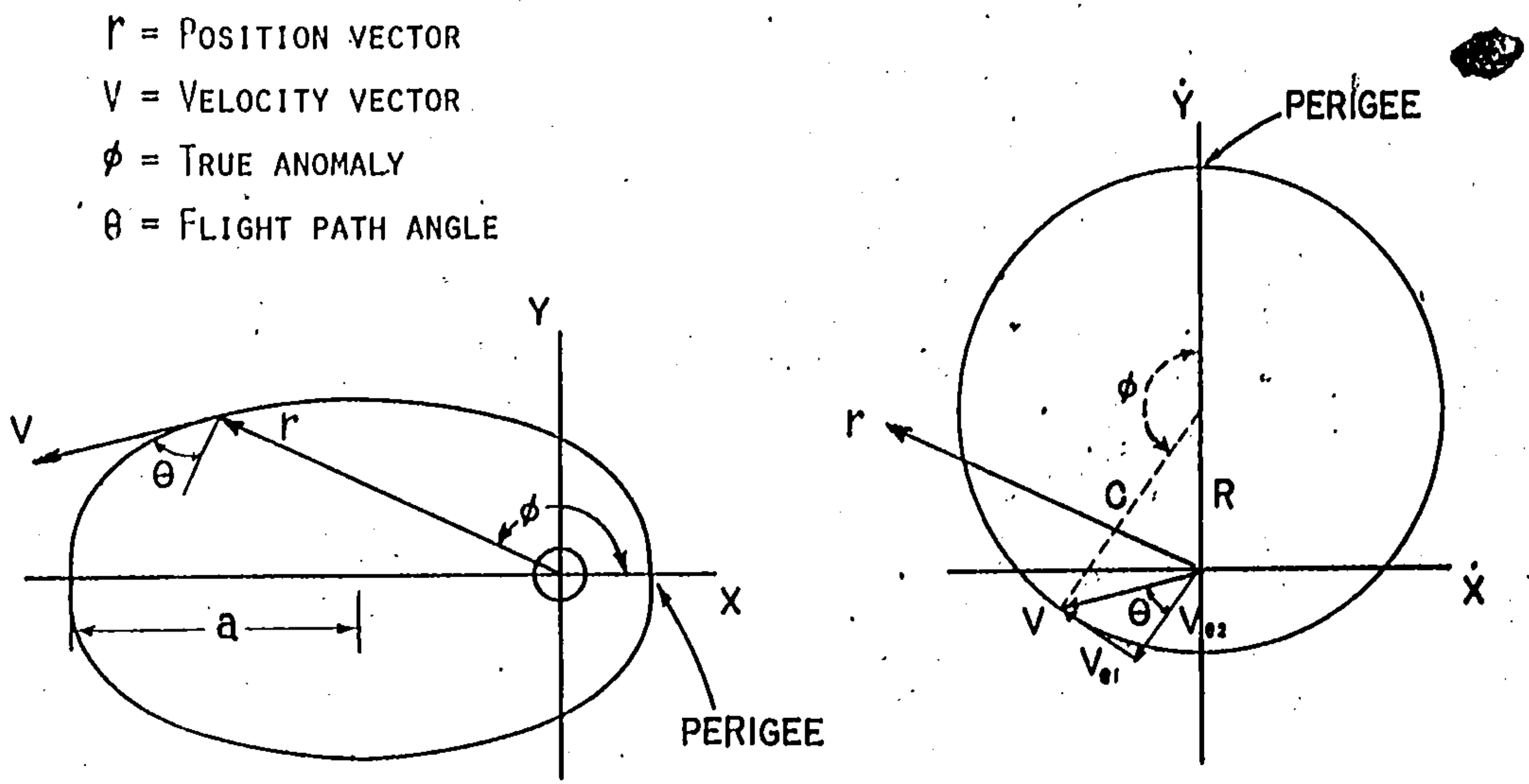


FIGURE 4.6 VELOCITY MAPPING OF AN ORBIT

$$C = \mu m/p_\lambda \tag{4.11}$$

$$R = [2E + C^2]^{1/2} \tag{4.12}$$

or

$$R = [(p_r/m)^2 + (p_\lambda/mr - m/p_\lambda)^2]^{1/2} \tag{4.13}$$

where m is the mass of the spacecraft and E is the orbital energy per unit mass. The state parameters, C and R, are therefore implicit forms of orbital momenta.

The orbital state variables of the unified state model are the three parameters, C, R_{f1}, and R_{f2}, the velocity state variable R being defined in two components. Together with the Euler parameters as coordinate variables, the elements of the unified state model parameter set are

$$x^T = (C,\ R_{f1},\ R_{f2},\ e_{01},\ e_{02},\ e_{03},\ e_{04}) \tag{4.14}$$

The Euler parameters are defined in Appendix A.

The orbital dynamic equations for the unified state model are

$$\begin{bmatrix} \dot{C} \\ \dot{R}_{f1} \\ \dot{R}_{f2} \end{bmatrix} = \begin{bmatrix} 0 & -p & 0 \\ \cos\lambda & -(1+p)\sin\lambda & -\gamma R_{f2}/V_{e2} \\ \sin\lambda & (1+p)\cos\lambda & \gamma R_{f1}/V_{e2} \end{bmatrix} \begin{bmatrix} a_{e1} \\ a_{e2} \\ a_{e3} \end{bmatrix} \tag{4.15}$$

$$\begin{bmatrix} \dot{e}_{01} \\ \dot{e}_{02} \\ \dot{e}_{03} \\ \dot{e}_{04} \end{bmatrix} = 1/2 \begin{bmatrix} 0 & \omega_3 & 0 & \omega_1 \\ -\omega_3 & 0 & \omega_1 & 0 \\ 0 & -\omega_1 & 0 & \omega_3 \\ -\omega_1 & 0 & -\omega_3 & 0 \end{bmatrix} \begin{bmatrix} e_{01} \\ e_{02} \\ e_{03} \\ e_{04} \end{bmatrix}$$

where

$$p = \frac{C}{V_{e2}} \tag{4.16}$$

$$\gamma = \frac{e_{01}e_{03} - e_{02}e_{04}}{e_{03}^2 + e_{04}^2} \tag{4.17}$$

$$\omega_1 = \frac{a_{e3}}{V_{e2}} \tag{4.18}$$

$$\omega_3 = \frac{CV_{e2}^2}{\mu} \tag{4.19}$$

$$\sin \lambda = \frac{2e_{03}e_{04}}{e_{03}^2 + e_{04}^2} \tag{4.20}$$

$$\cos \lambda = \frac{e_{04}^2 - e_{03}^2}{e_{03}^2 + e_{04}^2} \tag{4.21}$$

and μ is the planetary gravitational constant with numerical value 0.3986×10^6 km^3/sec^2. a_{e1}, a_{e2}, a_{e3} denote the e-frame components of the total perturbing accelerations. The components of velocity in e-frame are

$$V_{e1} = R_{f1} \cos \lambda + R_{f2} \sin \lambda \tag{4.22}$$

$$V_{e2} = C - R_{f1} \sin \lambda + R_{f2} \cos \lambda \tag{4.23}$$

The perturbating accelerations due to the zonal harmonics (J_2, J_3, J_4) and the tesseral harmonic ($n = 2$, $m = 2$) are defined for the state model in Appendix B.

Note that the unified state model parameter set contains one more element than the minimum of six required to describe an orbit because it uses four Euler parameters while the description of a rotation requires only three independent parameters. So the Euler parameters must satisfy a contraint equation

$$e_{01}^2 + e_{02}^2 + e_{03}^2 + e_{04}^2 = 1 \tag{4.24}$$

The rotation matrix from inertial frame to e-frame, denoted by E, is given in terms of the Euler parameters by

$$E = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix} \tag{4.25}$$

where

$$\varepsilon_{11} = 1 - 2\ (e_{02}^2 + e_{03}^2)$$

$$\varepsilon_{12} = 2\ (e_{01}\ e_{02} + e_{03}\ e_{04})$$

$$\varepsilon_{13} = 2\ (e_{01}\ e_{03} - e_{02}\ e_{04})$$

$$\varepsilon_{21} = 2\ (e_{01}\ e_{02} - e_{03}\ e_{04})$$

$$\varepsilon_{22} = 1 - 2\ (e_{01}^2 + e_{03}^2)$$

$$\varepsilon_{23} = 2\ (e_{02}\ e_{03} + e_{01}\ e_{04})$$

$$\epsilon_{31} = 2\ (e_{01}\ e_{03} + e_{02}\ e_{04})$$

$$\epsilon_{32} = 2\ (e_{02}\ e_{03} - e_{01}\ e_{04})$$

$$\epsilon_{33} = 1 - 2\ (e_{01}^2 + e_{02}^2)$$

4.6 The Measurement Model

The observables of a satellite are elevation, azimuth and range. The measurement model maps estimated orbital states into predicted observables.

$$y = h(x) \tag{4.26}$$

where

$$x^T = (C,\ R_{f1},\ R_{f2},\ e_{01},\ e_{02},\ e_{03},\ e_{04})$$

and

$$y^T = (El,\ Az,\ \rho) \tag{4.27}$$

The first step in calculating the observables is to find the inertial position of the observation site. The Earth's shape will be taken to be an ellipsoid of revolution. The coordinates of a site are measured with respect to this reference ellipsoid. La is the geodetic lattitude. Lo is the geographic longitude of the site. H_t is the height of the site above the reference ellipsoid, measured along the normal of the ellipsoid. Figure 4.7, taken as a cross-section of the Earth in the plane of the meridian of the site, depicts the definition of La and H_t. The cross-section of the reference ellipsoid is an ellipse having semi-major axis R_e and eccentricity e_e.

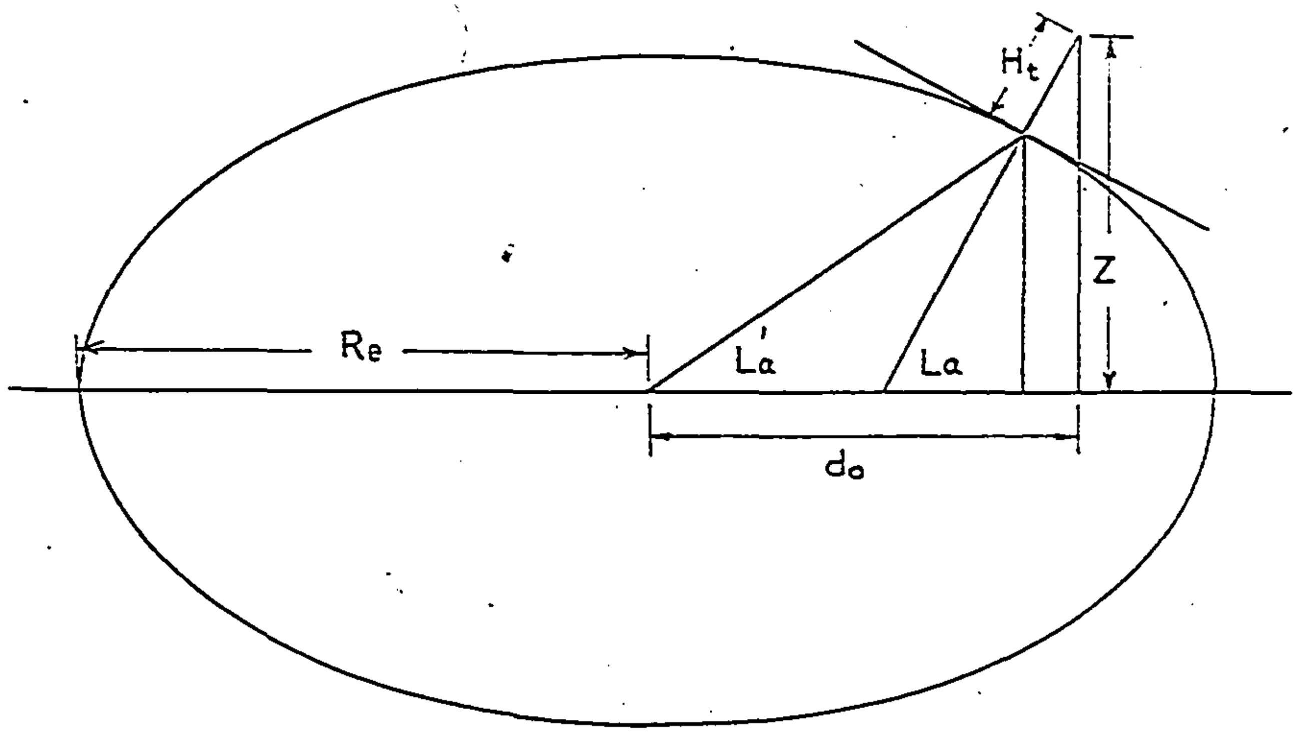


FIGURE 4.7 CROSS-SECTION OF EARTH MODEL

$$R_e = 6378.166 \text{ km}$$

$$1/f = 298.3$$

where f is the flattening of the ellipse.

The rectangular coordinates of the site in the cross-section plane are given by [29]:

$$\begin{aligned} d_o &= \left[\frac{R_e}{\sqrt{1 - e_e^2 \sin^2 La}} + H_t\right] \cos La \\ z &= \left[\frac{R_e (1 - e_e^2)}{\sqrt{1 - e_e^2 \sin^2 La}} + H_t\right] \sin La \end{aligned} \tag{4.28}$$

where e_e is the eccentricity of Earth. The inertial longitude of the site has been defined by eqn. (4.9). So the inertial frame components of the observation are

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_{SITE} = \begin{bmatrix} d_o \cos \theta \\ d_o \sin \theta \\ z \end{bmatrix} \tag{4.29}$$

Having calculated the instantaneous inertial position of the site, the procedure in transforming the orbital states to observables is described as follows.

The co-ordinates of the satellite in orbit frame (e_1, e_2, e_3) are first calculated from the orbital elements.

$$r = r_{e1} = \mu/(CV_{e2}) \tag{4.30}$$

They are then transformed into co-ordinates in inertial frame by

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_{satellite} = E^T \begin{bmatrix} r \\ 0 \\ 0 \end{bmatrix} \tag{4.31}$$

The position is then expressed in terms of the ENU-frame components

$$\begin{bmatrix} E \\ N \\ U \end{bmatrix} = E_{TOPO} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_{satellite} - \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_{site} \tag{4.32}$$

and finally the observables are obtained by

$$El = \sin^{-1} \frac{U}{(E^2 + N^2 + U^2)^{1/2}} \tag{4.33}$$

$$Az = \tan^{-1} (E/N) \tag{4.34}$$

$$\rho = (E^2 + N^2 + U^2)^{1/2} \tag{4.35}$$

CHAPTER FIVE

SATELLITE ORBIT DETERMINATION AND PREDICTION

5.1 Introduction

It has been shown in Chapter Three that a nonlinear observer is at least locally asymptotically stable if the nonlinear system to be observed satisfies certain conditions. In this chapter, attempts are made to evaluate the feasibility of applying the nonlinear observer to the problem of earth satellite orbit determination and prediction.

The seven-parameter set unified state model described in the last chapter is used as the dynamic model of a satellite. First, it will be shown that this nonlinear system satisfies the conditions listed at the end of Chapter Three for local asymptotic stability. Then the performance of nonlinear observer in the application of satellite orbit determination and prediction is evaluated with simulated noise-free observations, i.e., a true orbit is assumed and observations are computed.

5.2 Applicability of the Nonlinear Observer

Let the nonlinear dynamic model of the satellite given by eqn. (4.15) be represented by

$$\dot{x} = f(x) \tag{5.1}$$

where $x^T = (C, R_{f1}, R_{f2}, e_{01}, e_{02}, e_{03}, e_{04})$ and the measurement model be represented by

$$y = h(x) \tag{5.2}$$

where $y^T = (E\ell, Az, \rho)$.

Expanding f(x) and h(x) in Taylor series about the initial state estimate, x_o, we have

$$\dot{x} = f(x) = f(x_o) + F[x-x_o] + f_1(x-x_o) \tag{5.3}$$

$$y = h(x) = h(x_o) + H[x-x_o] + h_1(x-x_o) \tag{5.4}$$

where F and H are the Jacobian matrices consisting of first partial derivatives of f(x) and h(x) with respect to x and evaluated at $x = x_o$, and $f_1(x-x_o)$ and $h_1(x-x_o)$ consist of second and higher order terms.

Let the nonlinear observer of the satellite be represented by

$$\dot{\hat{x}} = f(\hat{x}) + G\,[h(\hat{x}) - y] \tag{5.5}$$

where G is the observer gain matrix. The state estimate error

$$\bar{x} = \hat{x} - x \tag{5.6}$$

will decay to zero eventually if at least one Liapunov function can be shown to exist.

If all the eigenvalues of (F+GH) have negative real parts, then the existence of a positive definite matrix, R, is assured for an arbitrarily chosen positive definite matrix, Q, such that

$$Q = -[R(F+GH) + (F+GH)^T R] \tag{5.7}$$

In order that all the eigenvalues of (F+GH) can be placed arbitrarily in the left-half of the complex plane, (F,H) must be a completely observable pair. So the first test for applicability of nonlinear observer is to check the observability of (F,H).

There are seven states in the dynamic model, one more than the absolute minimum of six to describe the motion of a satellite. The unified state model is not a minimal representation of the system. Thus, the linearized system (F,H) is not necessarily completely observable. To test its observability, an observability matrix, V, is formed

$$V = \begin{bmatrix} H \\ H\ F \\ \vdots \\ HF^6 \end{bmatrix} \tag{5.8}$$

If the rank of V is seven, (F,H) is completely observable. But if the rank of V is less than seven, another model for the satellite has to be chosen.

With F and H evaluated at the initial state estimate, it has been found that the rank of V is seven. Thus (F,H) is completely observable. The eigenvalues of (F+GH) can, therefore, be placed at any desired locations by suitable choice of the gain matrix G.

Secondly, the existence of Liapunov functions depends on whether $f_1(x-x_o)$ and $h_1(x-x_o)$ satisfy the Lipschitz condition at least in the neighbourhood of $(x=x_o)$. It may be noted that a continuous function with bounded continuous first partial derivatives satisfies the Lipschitz condition.

f(x) is continuous with bounded and continuous partial derivatives except for two classes of orbits. The first class of these orbits are the ones for which

$$e_{03}^2 + e_{04}^2 = 0$$

In such case, the angle λ and thus the orbit frame (f_1 f_2 f_3) are not defined. Since the f-frame is required for the definition of the states R_{f1} and R_{f2}, the orbit is not representable. This happens when the orbit has an inclination of 180^o. The second class of orbits are the rectilinear orbits. For these orbits, the angular momentum of the satellite is zero and thus the orbital state, C, is not defined.

For the measurement model h(x), the problem of discontinuity occurs when the satellite is directly above the observation site. In this case, the azimuth angle is undefined.

Finally, the existence of Liapunov functions depends on whether the following condition is satisfied

$$-q + 2r(\alpha+\beta) \leq 0$$

where

$$q = ||Q||$$

$$r = ||R||$$

$$||f_1(x-x_o+\bar{x}) - f_1(x-x_o)|| \leq \alpha \; ||\bar{x}||$$

$$||Gh_1(x-x_o+\bar{x}) - Gh_1(x-x_o)|| \leq \beta \; ||\bar{x}||$$

This condition is always satisified by making α and β sufficiently small. In order to get a feel of the order of magnitude of $||x||$, an arbitrary positive definite matrix Q was chosen and its corresponding matrix R was obtained. It was found that

$$q/2r \approx 10^{-7}$$

So,

$$\alpha + \beta \leq 10^{-7}$$

and $||\bar{x}||$ is in the order of 10^{-1}. However, it must be noted that this value of $||x||$ does not provide any information about the largest possible estimate error for each individual state. In the simulation, it has been found that estimate error of this magnitude is tolerable for the states C, R_{f1} and R_{f2} but not for the Euler parameters.

5.3 Simulation Results

The dynamical modelling problem is an extremely difficult one in earth satellite orbit determination and prediction. The model dynamics are two-body whereas the real dynamics are not. Earth oblateness causes secular as well as periodic variation in the orbital elements. Additional innumerable accelerations act on the earth satellite. These include the gravitational accelerations of the sun, moon and planets, solar radiation pressure, and drag caused by the earth's atmosphere. Nevertheless, the simulation is indicative of the results that can be obtained in real situations.

5.3.1 Numerical Solution of Ordinary Differential Equation

The predictor-corrector algorithm was used to solve eqn. (4.15). The predictor step is

$$x_{n+1}^{(0)} = x_n + \frac{T}{12} [23 f(x_n) - 16 f(x_{n-1}) + 5 f(x_{n-2})] \qquad (5.9)$$

and the corrector step is

$$x_{n+1}^{(j+1)} = x_n + \frac{T}{24} [9 f(x_{n+1}^{(j)}) + 19 f(x_n) - 5 f(x_{n-1}) + f(x_{n-2})] \qquad (5.10)$$

where T is the step size which was taken as one second in this study.

5.3.2 Transformation to Observable Canonical Form

To simplify the calculation of the observer gain matrix G, the linearized system (F,H) is transformed to the observable canonical form $(\bar{F},\bar{H})$. The algorithm used in this study was proposed by Hickin and Sinha [36] and is described below.

To start, the observability matrix is obtained in the following way

$$V = \begin{bmatrix} H \\ H\ F \\ \vdots \\ HF^6 \end{bmatrix} \tag{5.11}$$

Beginning with the first row of V, a non-zero pivot is selected, say v_{1i}, and then proceed as below

$$\bar{v}_1 = v_1 - e^i \tag{5.12}$$

$$V^{(1)} = V - (v_{1i})^{-1}\ v_i\ \bar{v}_1 \tag{5.13}$$

where e^i is the ith unit row vector of dimension n which is 7 in this case, v_i is the i^{th} column of V and v_1 is the first row of V.

The process is repeated for rows 2, 3 etc. to get different row unit vectors in each case, and skip a row if it is a linear combination of those above it. Since (F,H) is observable, the rank of V is seven. So after seven steps, $V^{(7)}$ will contain seven unit row vectors in addition to other rows which are linear combinations of the unit row vectors above them. Permutation of the columns so that the unit row vectors appear in their natural order gives the Hermite Normal form of the matrix. The first three rows define $\bar{H}$ and the next seven rows define $\bar{F}$, where $(\bar{F},\bar{H})$ is in output identifiable form.

From $\bar{F}$ and $\bar{H}$, select three unit row vectors

$$r^i = \bar{h}^i \bar{F}^{n_i-1} \qquad i = 1, 2, 3 \tag{5.14}$$

where n_i is the largest possible integer for which the right-hand side is a unit vector. Defining the transpose of r^i as the unit column vector r_i, one then generates the matrix

$$\begin{aligned} R = [r_1\ \bar{F}r_1\ \ldots\ \bar{F}^{n_1-1} r_1\ r_2\ \ldots\ \bar{F}^{n_2-1} r_2\ r_3 \\ \ldots\ \bar{F}^{n_2-1} r_3\ \bar{F}^{n_1} r_1\ \bar{F}^{n_2} r_2\ \ \bar{F}^{n_3} r_3] \end{aligned} \tag{5.15}$$

Transformation of R into Hermite normal form gives

$$\bar{R} = [I_n \mid \beta_1\ \beta_2\ \beta_3] \tag{5.16}$$

Hence, the observable canonical form is obtained as

$$\begin{aligned} \bar{F} = [e_2\ e_3\ \ldots\ e_{n_1}\ \beta_1\ e_{n_1+2}\ \ldots\ e_{n_1+n_2}\ \beta_2 \\ e_{n_1+n_2+2}\ \ldots\ e_{n_1+n_2+n_3}\ \beta_3] \end{aligned} \tag{5.17}$$

and

$$\bar{H} = R_{pn} \tag{5.18}$$

where R_{pn} consists of the first three rows and seven columns of R.

In the process of transforming (F,H) to observable canonical form, numerical problem was encountered. This was probably due to the fact that the numbers in F are very small while those in H are large. This disparity in magnitude introduced significantly large rounding errors. In order to improve the numerical results, the time scale was changed from second to microsecond, i.e., F was scaled up by a factor of 10^6. At the same time, the last row of H, $\partial\rho/\partial x$, was scaled down by a factor of 10^{-3}. In addition, while reducing V to the Hermite normal form, attention must be paid to Eq. (5.13) with regard to the order of arithmetic operation being performed. Tremendous improvement in results is obtained if division is deferred until the end of operation.

5.3.3 Performance of Nonlinear Observer

Both the dynamic system of the satellite given by eqn. (4.15) and the nonlinear observer given by eqn. (5.5) are simulated on a CDC 6400 computer. The initial states of the dynamic system are assumed to be

$$C = 3.066694035 \text{ km/sec.}$$

$$R_{f1} = 1.198952484 \times 10^{-2} \text{ km/sec.}$$

$$R_{f2} = -1.425617897 \times 10^{-4} \text{ km/sec.}$$

$$e_{01} = -8.431775050 \times 10^{-4}$$
$$e_{02} = -7.622086208 \times 10^{-4}$$
$$e_{03} = -6.616178898 \times 10^{-1}$$
$$e_{04} = 7.498403003 \times 10^{-1}$$

This is the orbit of a stationary satellite. The time taken for it to complete one revolution is the same as that of the Earth rotating about the Z-axis. The satellite stays in full view of the observation station all the time.

In order to choose the eigenvalues for the linearized portion of the nonlinear observer, it is necessary to know the order of magnitude of the eigenvalues of the dynamic system. It has been found that the real part of the largest eigenvalues of the system is in the order of -1×10^{-9}. This indicates that the system has very large time constants. The effect of any perturbation on the system will take a very long time to die out. This presents some difficulties in the choice of observer eigenvalues. To choose a set of observer eigenvalues such that the state estimate error will decay to a significantly low level within a reasonably short period of time would require tremendously large gains. This results in not only that the measurement noise are greatly magnified but also that the observer may fall outside the stability region. On the other hand, if the eigenvalues chosen are too small, it would take ages before the estimated states approach fairly close to the actual orbital states. After trying with different values, it has been decided that the eigenvalues of the linearized portion of the observer should be placed at $s = -1 \times 10^{-7}$.

The nonlinear system is relinearized at every sixty steps at the latest state estimate and a new observer gain is used.

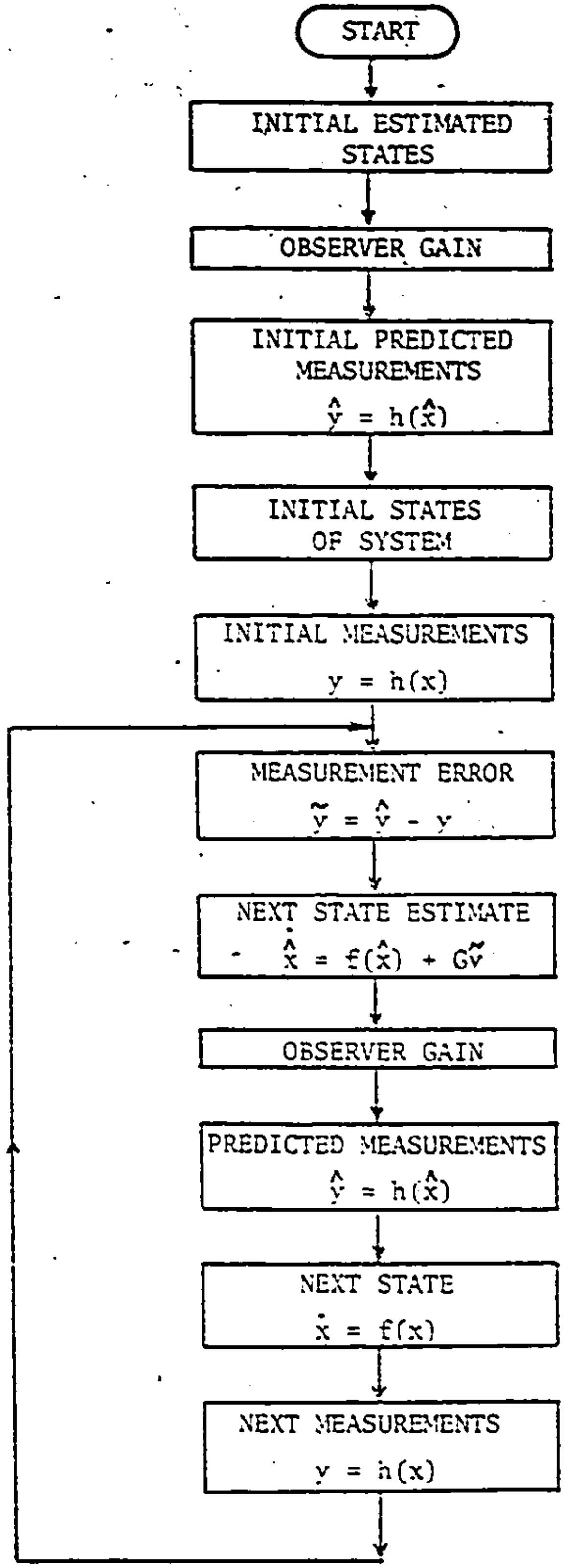
Figure 5.1 is a flow chart of the simulation program. The initial estimated orbital states are

$$C = 3.066694 \text{ km/sec.}$$
$$R_{f1} = 2.53382 \times 10^{-2} \text{ km/sec.}$$
$$R_{f2} = -1.335137 \times 10^{-1} \text{ km/sec.}$$
$$e_{01} = -8.152435 \times 10^{-4}$$
$$e_{02} = -7.937526 \times 10^{-4}$$
$$e_{03} = -6.616458 \times 10^{-1}$$
$$e_{04} = 7.497742 \times 10^{-1}$$

The simulation covers a period of four hours with step size of one second. Figure 5.2 is a plot of $||\tilde{x}||$ versus time on semilog scale. Figure 5.3 is a plot of the position estimation error $(\hat{r}-r)$.

5.4 Discussion

At the start of this work, it was hoped that the Luenberger observer may be used as an alternative approach to the extended Kalman filter in the application of satellite orbit determination and prediction. It has been


START
INITIAL ESTIMATED STATES
OBSERVER GAIN
INITIAL PREDICTED MEASUREMENTS
$\hat{y} = h(\hat{x})$
INITIAL STATES OF SYSTEM
INITIAL MEASUREMENTS
$y = h(x)$
MEASUREMENT ERROR
$\tilde{y} = \hat{y} - y$
NEXT STATE ESTIMATE
$\dot{\hat{x}} = f(\hat{x}) + G\tilde{y}$
OBSERVER GAIN
PREDICTED MEASUREMENTS
$\hat{y} = h(\hat{x})$
NEXT STATE
$\dot{x} = f(x)$
NEXT MEASUREMENTS
$y = h(x)$

shown in this work that this application is theoretically feasible. However, simulation results indicate that the use of Luenberger observer in this application is not particularly promising, at least with the dynamic model used in this work.

For one thing, the time needed for the estimate to converge to the orbital state is too long and it is not possible to use a larger gain to enable faster convergence. But the most serious problem involves the size of initial estimate error. To make a sufficiently close initial state estimate is not very likely in some cases. If the initial state estimate errors are not small enough, then the nonlinear observer may become unstable. However, once the observer has reached the states, it can track the states fairly accurately with much less computation than is necessary with other approaches. Moreover, it is only necessary to relinearize the dynamic model after as many as 60 observations to update the observer parameters. This may be contrasted with the extended Kalman filter where relinearization of the dynamic model is necessary after each iteration in order to maintain similar degree of accuracy.

A listing of the computer simulation program in Fortran will be published as an internal report by the Group on Simulation, Optimization and Control, McMaster University, Hamilton, Canada.

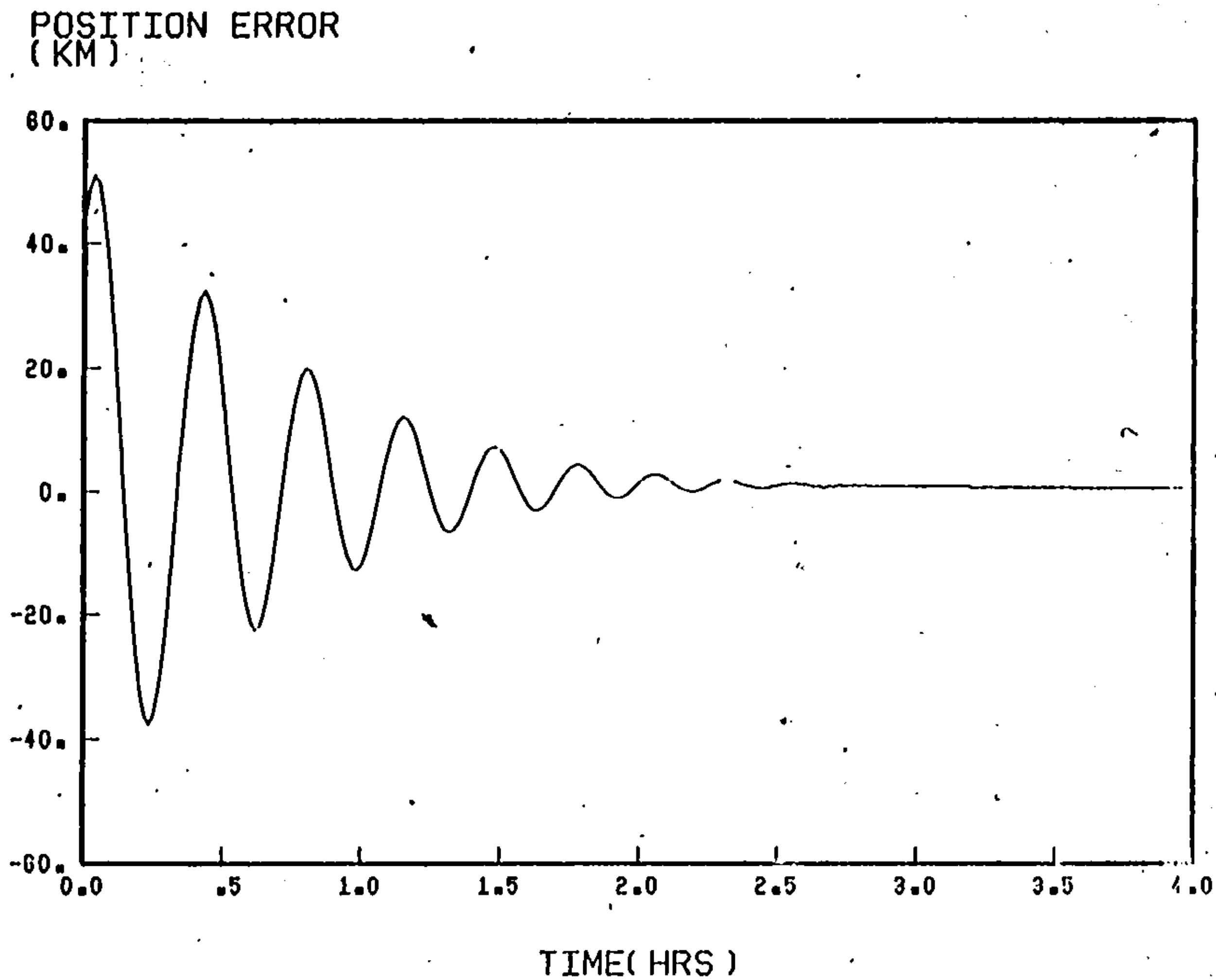


FIGURE 5.2 POSITION ERROR

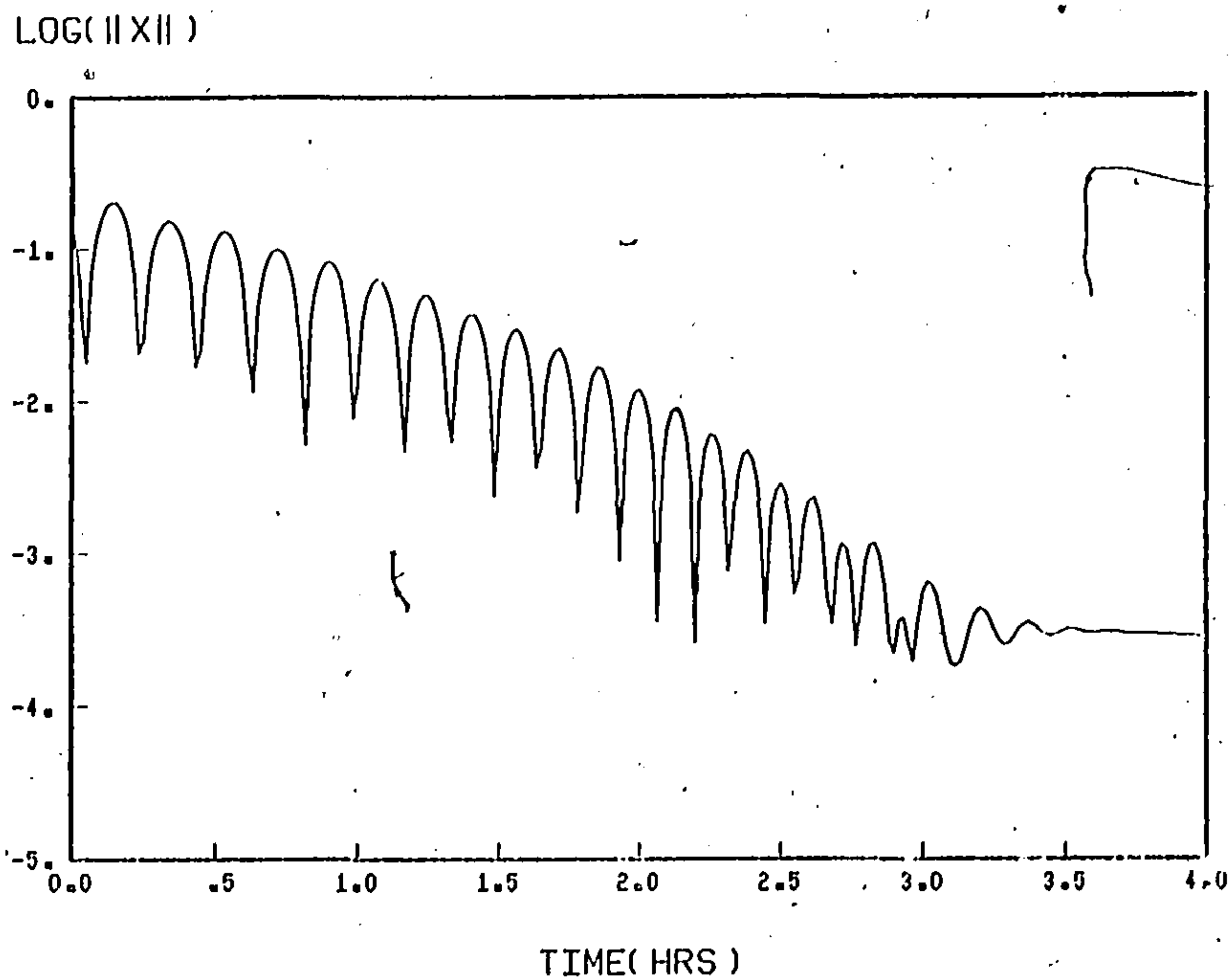


FIGURE 5.3 STATE ESTIMATION ERROR

CHAPTER SIX

CONCLUSIONS

The feasibility of applying a nonlinear observer in the problem of satellite orbit determination and prediction has been studied in this thesis. The proposed algorithm involves calculating the observer gains off-line and storing them in memory. Depending upon the current estimated orbital states of the satellite, an appropriate observer gain matrix is retrieved from memory for the estimation of the next orbital states. This scheme would save a tremendous amount of on-line computation as would be required by other algorithms, such as the extended Kalman filter since it does not require re-linearization of the dynamic model and calculation of the gain matrix at every iteration step.

Following a brief review of the basic principle of the Luenberger observer for linear systems in Chapter Two, the conditions for local asymptotic stability of nonlinear observers are established in Chapter Three. The stability problem is studied through the second method of Liapunov. It has been found that nonlinear observers may be used in applications where the linearized portion of the system is completely observable and the nonlinear portion satisfies the Lipschitz condition. Since these conditions are not overly restrictive, the nonlinear observer is applicable in many fields. However, there is one drawback in the nonlinear observer. The initial state estimate error must not be too large, otherwise the observer would become an unstable system and would not be able to recover itself.

The orbital dynamics are described in Chapter Four. The unified state model proposed by S. Altman is used as the dynamic model of a satellite. In the aerospace industry, people tend to be more comfortable with the classical model which uses semi-major axis, eccentricity, inclination, longitude of ascending node, argument of periapsis, and true anomaly of the orbit as orbital elements. With these elements, they are able to get a direct feel of the orbit of the satellite. The unified state model, on the other hand, uses a set of more abstract parameters as orbital elements. However, this abstractness is more than compensated for by a much simpler and compact form of the equations describing the motion of a satellite.

Simulation results are discussed in Chapter Five. It has been shown that the nonlinear observer is theoretically feasible in problems of orbit determination and prediction with the unified state model as the dynamic model of the satellite. The Jacobian matrices form a completely observable pair and the nonlinear portions of the dynamic and measurement models satisfy the Lipschitz condition. However, the simulation results seem to suggest that its practicability is not quite favourable. The time taken for the estimated states to converge to the actual states is long. But the more serious problem is that the initial state estimate error has to be small. This criterion turns out to be a little difficult to meet. It is comparatively easy to make a fairly accurate estimate of the orbital state C since it is related to the altitude of the satellite. But to make close estimates of the Euler parameters is not an easy matter. Yet a sufficiently accurate estimate of these states is crucial for the stabiliy of the nonlinear observer. On the other hand, the nonlinear observer is able to give a correct estimate of the orbital states once it starts tracking. This may not be attainable by other schemes which involve linearized models.

The poor performance of the nonlinear observer in the simulation does not necessarily indicate that it is not suitable for use in the problem of orbit determination and prediction. The size of the stability region is a function of the orbital states chosen. It is possible that the stability region could be considerably larger if the dynamics of the satellite are described in some other state spaces. This possibility remains to be investigated.

In this study, the measurements of the system output are assumed to be noise free. This is not true in reality. Therefore, some filtering may be needed before the output measurements are applied to the observer. The stability problem of the coupled system of a filter and a nonlinear observer has also to be studied.

APPENDIX A

DEFINITION OF THE EULER PARAMETERS

The rotation of a triad set can be generated by a scalar rotation about a directed line from the origin of the inertial space. This rotation is defined by the unitary quaternion

$$q = e_{04} + (g_1 e_{01} + g_2 e_{02} + g_3 e_{03})$$

where the set of four Euler parameters

$$\begin{bmatrix} e_{01} \\ e_{02} \\ e_{03} \\ e_{04} \end{bmatrix} = \begin{bmatrix} \cos \alpha \sin u/2 \\ \cos \beta \sin u/2 \\ \cos \gamma \sin u/2 \\ \cos u/2 \end{bmatrix}$$

consists of real scalars such that

$$e_{01}^2 + e_{02}^2 + e_{03}^2 + e_{04}^2 = 1$$

The spherical angles, α, β, γ, are shown in Fig. A1. As an alternative, the Euler parameters may be defined in terms of the Euler angles, Ω, ℓ, $u\ell$ as shown in Fig. A2 in accordance with

$$\begin{bmatrix} e_{01} \\ e_{02} \\ e_{03} \\ e_{04} \end{bmatrix} = \begin{bmatrix} \sin \ell/2 \cos (\Omega-u\ell)/2 \\ \sin \ell/2 \sin (\Omega-u\ell)/2 \\ \cos \ell/2 \sin (\Omega+u\ell)/2 \\ \cos \ell/2 \cos (\Omega+u\ell)/2 \end{bmatrix}$$

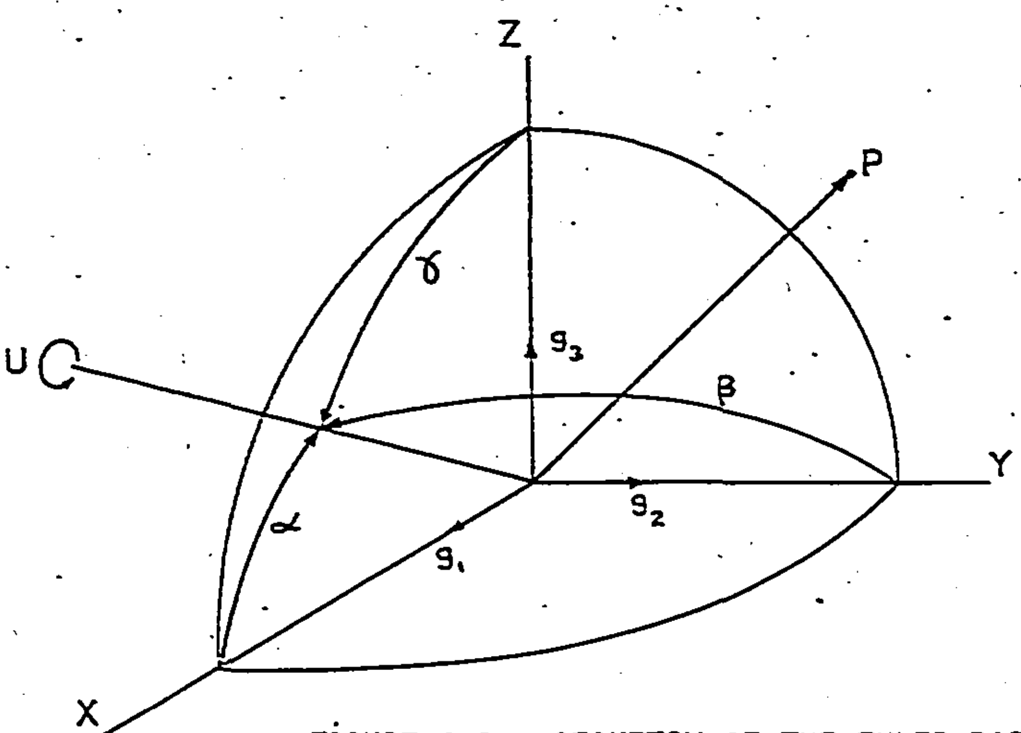


FIGURE A.1 GEOMETRY OF THE EULER PARAMETERS

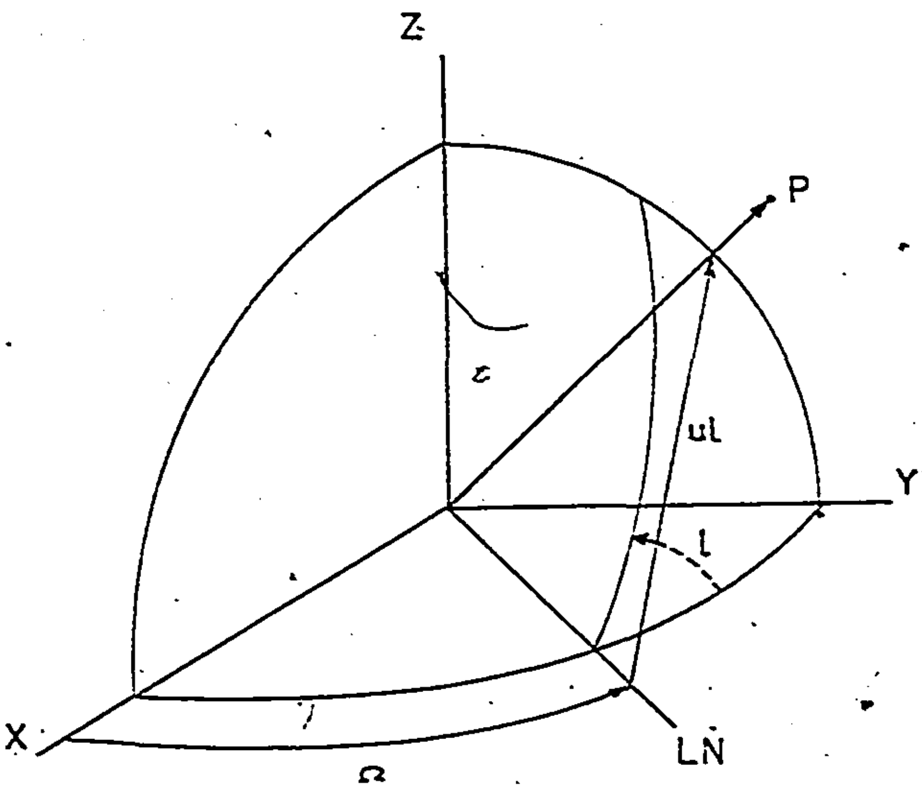


FIGURE A.2 THE EULER ANGLES

APPENDIX B

GRAVITY HARMONICS

B.1 Zonal Harmonics (J_2, J_3, J_4)

$$\begin{bmatrix} a_{e1} \\ a_{e2} \\ a_{e3} \end{bmatrix}_{J_2} = -\left(\frac{3}{2}\right) \mu R_e^2 J_2 \left(\frac{CV_{e2}}{\mu}\right)^4 \begin{bmatrix} 1 - 3\varepsilon_{13}^2 \\ 2\varepsilon_{13}\varepsilon_{23} \\ 2\varepsilon_{13}\varepsilon_{33} \end{bmatrix}$$

$$\begin{bmatrix} a_{e1} \\ a_{e2} \\ a_{e3} \end{bmatrix}_{J_3} = -\left(\frac{1}{2}\right) \mu R_e^3 J_3 \left(\frac{CV_{e2}}{\mu}\right)^5 \begin{bmatrix} 4\varepsilon_{13}(5\varepsilon_{13}^2 - 3) \\ 3\varepsilon_{23}(1 - 5\cdot\varepsilon_{13}^2) \\ 3\varepsilon_{33}(1 - 5\varepsilon_{13}^2) \end{bmatrix}$$

$$\begin{bmatrix} a_{e1} \\ a_{e2} \\ a_{e3} \end{bmatrix}_{J_4} = -\left(\frac{5}{8}\right) \mu R_e^4 J_4 \left(\frac{CV_{e2}}{\mu}\right)^6 \begin{bmatrix} 35\varepsilon_{13}^4 - 30\varepsilon_{13}^2 + 3 \\ -4\varepsilon_{13}\varepsilon_{23}(7\varepsilon_{13}^2 - 3) \\ -4\varepsilon_{13}\varepsilon_{33}(7\varepsilon_{13}^2 - 3) \end{bmatrix}$$

$$J_2 = 0.108265 \times 10^{-2}$$

$$J_3 = -0.254503 \times 10^{-5}$$

$$J_4 = -0.167150 \times 10^{-5}$$

B.2 Tesseral Harmonics (n = 2, m = 2)

$$\begin{bmatrix} a_{e1} \\ a_{e2} \\ a_{e3} \end{bmatrix} = 3 \mu R_e^2 \left(\frac{CV_{e2}}{\mu}\right)^4 \begin{bmatrix} T_{22/e1} \\ T_{22/e2} \\ T_{22/e3} \end{bmatrix}$$

$$T_{22/e1} = - (A_1 \epsilon_{11} + A_2 \epsilon_{12})$$

$$T_{22/e2} = \frac{2}{1+\epsilon} \{A_1 [(\epsilon_{23}^2 - \epsilon_{13}^2) \sin\lambda - 2\epsilon_{13} \epsilon_{23} \cos\lambda]$$

$$- A [(\epsilon_{23}^2 - \epsilon_{13}^2) \cos\lambda + 2 \epsilon_{13} \epsilon_{23} \sin\lambda]\}$$

$$T_{22/e3} = 2[A_1 (\epsilon_{23} \sin\lambda - \epsilon_{13} \cos\lambda) - A_2 (\epsilon_{13} \sin\lambda$$

$$+ \epsilon_{23} \cos\lambda)]$$

$$A_1 = C_{22} \epsilon_{11} + S_{22} \epsilon_{12}$$

$$A_2 = S_{22} \epsilon_{11} - C_{22} \epsilon_{12}$$

$$C_{22} = 0.1566511 \times 10^{-5}$$

$$S_{22} = -0.8869932 \times 10^{-6}$$

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