RELATIVE NUMERICAL RANGES OF ELEMENTS

OF

BANACH AND L.M.C.-ALGEBRAS

By

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ABSTRACT

If A is a Banach algebra and A' its dual then the relative numerical range and relative numerical radius of an element $a \in A$ with respect to an element $b \in A$ are defined by

$$\overset{\circ}{V}_b(A,a) = \{f(ab): f \in A', \|f\| = 1, f(b) = \|b\|\},$$

and

$$\overset{\circ}{v}_b(a) = \sup\{|\lambda|: \lambda \in \overset{\circ}{V}_b(A,a)\}, \text{ respectively.}$$

We study certain properties of relative numerical range and radius. We introduce the concept of regular norm and examine the properties of relative numerical range in A with regular norm. We show by means of an example that number zero does not always belong to the relative numerical range in a non-unital Banach algebra, and discuss the relative numerical range in $L_1(R)$.

The generalized numerical range of $a \in A$ (whether A has unit or not) is defined. It is shown that in a commutative complex Banach algebra A which has regular norm, $V(A^+,x) = \overline{Co}\, V_A(x)$, for all $x \in A$, where A^+ is the unitization of A. An element $h \in A$ is called Hermitian if $V_A(h) \subseteq R$ and is called positive if $V_A(h) \subseteq R^+$. If A

has regular norm then for all $h \in H(A)$ (the set of all Hermitian elements in A), $\overline{V_A(h)}$ is connected and $\overline{V_A(h)} = Co(Sp_A(h))$. We also characterize the Hermitian and positive elements in A^+ and A respectively if A has regular norm. It is shown that the set of all positive elements K is a proper convex, closed and normal cone in $H(A)$ and that K generates $H(A)$. Next we discuss the generalized numerical ranges in special Banach algebras such as $A = \oplus_0 X_i$, $A = \oplus_\infty X_i$ and $C_0(X)$, where X is a locally compact Hausdorff space. Bonsall and Duncan have established that in a unital Banach algebra the numerical range is always a closed subset of the complex plane, we show here by an example that the generalized numerical range need not be closed in a non-unital Banach algebra.

The concept of relative and generalized numerical ranges is extended from Banach algebras to locally m-convex algebras. We prove a statement about elements with bounded relative numerical range. We also define the relative set Γ for A and obtain that if $b \in \Gamma$ then $\mathring{D}(A,P;b) = \xrightarrow{\lim} \mathring{D}(A,p_\alpha;b)$. Finally, we define regular l.m.c.-algebra and examine the properties of the generalized numerical range in this algebra. We also discuss the relationship between the spectrum and the generalized numerical range in a regular, complete, commutative l.m.c.-algebra.

To My Parents

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TABLE OF CONTENTS

INTRODUCTION

The numerical range of a linear operator on a normed space is a subset of the complex plane which is related to the algebraic and the norm structures of the operator. In this it differs from the spectrum, which is related to algebraic structure but independent of the norm. In the year 1961 Lumer [16] introduced the concept of semi-inner-product on a linear space and generalized the definition of numerical range of linear operators on Hilbert spaces to Banach spaces. This paper of Lumer was undoubtedly the most important in the development of the subject. He also introduced the numerical range of elements of normed algebras. Bonsall, Duncan [2(a), 2(b)], Bollobás [4,5,6], Sinclair [22], Stampfli, Williams [23] etc. have further contributed to the concepts of numerical range of linear operators on Banach spaces and of elements of unital Banach algebras.

In the year 1972, Husain, Giles and Koehler [13(a), 13(b)] extended the concept of numerical range of elements from unital Banach algebras to l.m.c. algebras and have characterized elements with bounded numerical ranges.

The contents of chapters of this thesis are arranged as follows:

Chapter I is introductory in nature wherein we collect some of the available results on linear spaces and algebras. Also we give some known results on spatial numerical range and the inner-product-numerical range of a linear operator. Further we gather some relevant results on the numerical range of elements of unital Banach algebras.

In Chapter II we define the relative numerical range and the relative numerical radius of an element of a commutative Banach algebra with respect to a given element. We prove that the relative numerical range of an idempotent relative to itself is the singleton $\{1\}$, and an example is given where the converse of this result is not true. It is proved that the relative numerical ranges of an element a in A related to x and $\phi(a)$ in B related to $\phi(x)$, where ϕ is an algebra homomorphism and an isometry from a Banach algebra A into a Banach algebra B, provided $\|\phi(x)\| = 1$, are equal. We show by means of examples that this result is not true in general if there is no isometry. We introduce the concept of regular norm on Banach algebras and obtain some results on relative numerical ranges of elements of Banach algebras with regular norm. In fact we have shown that if for an element $b \in A$, $\|b\| = 1$ and $\hat{V}_b(A,a) = \{\|a\|\}$ then A has regular norm. If a Banach algebra A has regular norm and $B(A)$ is the algebra of all bounded linear operators on A, then for $x, y \in A$ with $\|y\| = 1$ we have $\overset{\circ}{V}_y(A,x) = \overset{\circ}{V}_{T_y}(B(A),T_x)$. The relative numerical range of an element in $L_1(R)$ is also discussed. An example of a nonunital Banach algebra is given where the relative numerical range of an element with respect to some element does not contain the number zero.

In Chapter III we define the generalized numerical range and radius of elements of Banach algebras. We also study the concept of generalized numerical range in the light of regular norm. We prove that the closed convex hull of the generalized numerical range of an element is the same as the numerical range of the same element in the unitization of A, if A has regular norm and does not posses the identity.

If a Banach algebra has regular norm then it turns out that the closure of the generalized numerical range of a Hermitian element coincides with the convex hull of the spectrum of that element. We give a necessary and sufficient condition for an element of A^+ to be Hermitian if A has no identity, but has regular norm. We also characterize the positive elements of a Banach algebra with regular norm. It is also proved that if a Banach algebra has regular norm then the set of all positive elements K generates the space $H(A)$ of all Hermitian elements and K is a proper convex normal cone in $H(A)$.

We also discuss the ranges in $\oplus_0 X_i$, $\oplus_\infty X_i$ (X_i are commutative Banach algebras) and $C_0(X)$, where X is a locally compact Hausdorff space. Finally in this chapter we discuss the closedness of the generalized numerical range in a non-unital Banach algebra and find an example which proves that the generalized numerical range of an element in a non-unital Banach algebra need not be closed.

In Chapter IV the concept of relative and generalized numerical ranges of elements of Banach algebras is extended to l.m.c.-algebras. We prove a statement about elements with bounded relative numerical range. We also introduce the concept of relative set and prove that if an element b is in the relative set then the set $\mathring{D}(A,P;b)$ is the inductive limit of $\mathring{D}(A,p_\alpha;b)$. Using this result we prove that $\mathring{V}_b(A,p_\alpha;a) \subseteq \mathring{V}_b(A,p_\beta;a)$ and $\mathring{v}_b(A,p_\alpha;a) \leq \mathring{v}_b(A,p_\beta;a)$.

We define a regular l.m.c.-algebra and prove various results about the generalized numerical range of the elements of a regular, complete, commutative l.m.c.-algebra. We also establish the relationship between the generalized numerical range and spectrum of an element in a complete, regular, commutative l.m.c.-algebra. We close the chapter by proving a statement about the generalized numerical range of normal elements in a complete, regular, commutative l.m.c. algebra.

CHAPTER I

PRELIMINARIES

In this chapter we collect the relevant well-known defintions and results needed in the sequel. Proofs are omitted.

§1. Linear Spaces and Algebras.

In this section we have some known results from the theory of linear spaces and algebras. In particular we discuss the properties of linear spaces, topological vector spaces, ordered vector spaces and cones in linear spaces. Also we discuss topological algebras, in particular locally m-convex algebras and Banach algebras with a view to introduce the notion of numerical range of an element of unital Banach algebras as studied by Bonsall and Duncan [2(a)] among others.

I.1 Definition ([11], p.16). Let E be an additive abelian group and K a field. Suppose for each $\lambda \in K$ and $x \in E$, λx is defined and $\lambda x \in E$ such that for all $\lambda, \mu \in K$ and $x, y \in E$,

(i) $\lambda(x+y) = \lambda x+\lambda y$;

(ii) $(\lambda+\mu)x = \lambda x+\mu x$;

(iii) $\lambda(\mu x) = (\lambda\mu)x$;

(iv) $1\cdot x = x$, where 1 is the identity element of K.

Then E is called a vector or linear space over the field K.

If the field K is that of all real or complex numbers, the vector space E is called a real or complex vector space.

A mapping f of a vector space E into a vector space F (both over the same field K) is called linear if

$$f(\lambda x + \mu y) = \lambda f(x) + \mu f(y),$$

for all $\lambda, \mu \in K$ and $x, y \in E$. A (linear) mapping of a vector space over a field K into K is called a (linear) functional. The set of all linear functionals of a linear space E is called the algebraic dual of E and is denoted by E^*.

I.2 Definition ([20], p.191). A topological vector space E(J) is a vector space E equipped with a topology J for which the operations of addition and scalar multiplication in E are jointly continuous. If E(J) is a topological vector space, the mappings $\psi_{x_0}: E(J) \to E(J)$ and $\phi_{\lambda_0}: E(J) \to E(J)$ defined by $\psi_{x_0}(x) = x_0 + x$, $\phi_{\lambda_0}(x) = \lambda_0 x$, $x \in E$ are homeomorphisms for each x_0 in E and each nonzero scalar λ_0.

I.3 Definition ([7], p.65). If E is a vector space over a field K, a seminorm is a function $p: E \to [0,\infty)$ having the properties:

(i) $p(x+y) \le p(x) + p(y)$ for all x, y in E.

(ii) $p(\alpha x) = |\alpha| \cdot p(x)$ for all α in K and x in E.

It follows from (ii) that $p(0) = 0$. A norm is a seminorm p such that

(iii) $x = 0$ if $p(x) = 0$.

Usually a norm is denoted by $\|\cdot\|$.

<u>I.4 Definition ([7], p.65)</u>. A normed space is a pair $(E, \|\cdot\|)$, where E is a vector space and $\|\cdot\|$ is a norm on E. A Banach space is a normed space that is complete with respect to the norm. A linear functional f on a normed space E is called bounded if there is $M > 0$ such that $|f(x)| \le M\|x\|$, for all $x \in E$.

<u>I.5 Theorem (Hahn Banach)</u>. Let p be a seminorm defined on the real linear space E. Let f be a real linear functional on a subspace F of E with

$$f(x) \le p(x), \quad x \in F.$$

Then there is a real linear functional $\tilde{f}$ on E for which

$$\tilde{f}(x) = f(x), \quad x \in F; \quad \tilde{f}(x) \le p(x), \quad x \in E.$$

<u>Proof</u>: See [8], p.62.

<u>I.6 Theorem</u>. Let F be a subspace of a normed linear space E and F' be the dual of F (that is F' is the set of all bounded linear functionals on F). Then to every f' in F' there corresponds a g' in E' with

$$\|g'\| = \|f'\|; \quad g'(y) = f'(y), \quad y \in F.$$

<u>Proof</u>: See [8], p. 63.

<u>I.7 Lemma.</u> Let F be a subspace of a normed linear space E. Let $x \in E$ be such that $\inf_{y \in F} \|y-x\| = d > 0$. Then there is a functional $g' \in E'$ with

$$g'(x) = 1 ; \quad \|g'\| = 1/d ; \quad g'(y) = 0 , \quad y \in F .$$

<u>Proof:</u> See [8], p.64.

<u>I.8 Theorem (Alaoglu).</u> Let E be a normed space and E' its dual. Then the unit ball $B' = \{x' \in E' : \|x'\| \le 1\}$ of E' is $\sigma(E',E)$ compact and equicontinuous.

<u>Proof:</u> See [11], p.281.

<u>I.9 Definition ([17], p.116).</u> Let A be a normed space and A' its dual. An element $a' \in A'$ is said to be aligned with an element $a \in A$ if $a'(a) = \|a'\| \cdot \|a\|$.

<u>(a) Example ([17], p.117).</u> Let $A = L_p[a,b]$, $1 < p < \infty$, and $A' = L_q[a,b]$, $1/p + 1/q = 1$. The conditions for two functions $x \in L_p$, $y \in L_q$ to be aligned follow directly from the conditions for equality in the Hölder's inequality, namely,

$$\int_a^b x(t) \cdot y(t) dt = \left\{ \int_a^b |x(t)|^p dt \right\}^{1/p} \left\{ \int_a^b |y(t)|^q dt \right\}^{1/q}$$

if and only if $x(t) = K[\operatorname{sgn} y(t)] \cdot |y(t)|^{q/p}$ for some nonnegative constant

K. Note that if $p = 1$ then the above condition can be written as

$$\int_a^b x(t)\cdot y(t)dt = \int_a^b |x(t)|dt \cdot \sup_{t\in[a,b]}\left\{ \int |y(t)|dt \right\}$$

if and only if $x(t) = K(\mathrm{sgn}\,\overline{y(t)})$, $K \geq 0$.

I.10 Definition ([20], p.2). An ordered vector space is a real vector space E equipped with a transitive, reflexive, antisymmetric relation "$\leq$" satisfying the following conditions:

(i) If x,y,z are elements of E and $x \leq y$, then $x+z \leq y+z$.

(ii) If x,y are elements of E and α is a positive real number, then $x \leq y$ implies $\alpha x \leq \alpha y$.

I.11 Definition ([19], p.61). A cone K is a proper convex set if the following conditions are satisfied:

(i) For $\lambda \in R_+$, $x,y \in K$, we have $\lambda x \in K$ and $x+y \in K$.

(ii) For $x \in K$, $-x \in K$ jointly imply $x = 0$.

I.12 Definition ([20], p.4). The cone K generates an ordered vector space E if E is the linear subspace spanned by K, that is $E = K-K$. Observe that K generates E if and only if E is directed $(\leq)$. An element $e \in E$ is an order unit if for each $x \in E$ there is an $\alpha > 0$ such that $x \leq \alpha e$.

I.13 Definition ([20], p.61). Suppose that $E(J)$ is an ordered topological vector space and that K is the positive cone ([20], p.3) in

$E(J)$. K is normal for the topology J if there is a neighborhood basis of zero for J consisting of full sets ([20], p.61).

<u>I.14 Proposition</u>. If an ordered vector space E is equipped with a norm $\|\cdot\|$ and if K is a positive cone in E, the following assertions are equivalent:

(a) K is normal for the topology generated by the norm $\|\cdot\|$.

(b) There is an equivalent norm (a norm $\|\cdot\|_1$ is equivalent to a norm $\|\cdot\|$ on E if $\|\cdot\|_1$ and $\|\cdot\|$ generate the same topology on E) $\|\cdot\|_1$ on E such that $0 \le x \le y$ implies $\|x\|_1 \le \|y\|_1$.

(c) There is a constant $\nu > 0$ such that $0 \le x \le y$ implies $\nu\|x\| \le \|y\|$.

(d) There is a constant $\nu > 0$ such that $\|x+y\| \ge \nu \cdot \max\{\|x\|, \|y\|\}$ for all x, y in K.

(e) The set $\{\|x\| : 0 \le x \le y ;\ \|y\| \le 1\}$ is bounded above.

<u>Proof</u>: See [20], p.64.

<u>I.15 Definition ([7], p.74)</u>. Suppose $\{X_i : i \in I\}$ is a collection of normed spaces. Then $\Pi\{X_i : i \in I\}$ is a vector space if the linear operations are defined co-ordinatewise. Let $\|\cdot\|$ denote the norm on each X_i. For $1 \le p < \infty$, define

$$\oplus_p X_i = \left\{x \in \prod_i X_i : \|x\| = \left[\sum_i \|x(i)\|^p\right]^{1/p} < \infty\right\}$$

Define

$$\oplus_\infty X_i = \left\{x \in \prod_i X_i : \|x\| = \sup_i \|x(i)\| < \infty\right\}.$$

If $\{X_1, X_2, \ldots\}$ is a sequence of normed spaces, define

$$\oplus_0 X_n = \left\{x \in \prod_{n=1}^{\infty} X_n : \|x(n)\| \to 0\right\};$$

and give $\oplus_0 X_n$ the sup norm as it is a subspace of $\oplus_\infty X_n$.

I.16 Definition ([1]). Let R^n be endowed with the Lebesgue measure. A real valued function f defined on an interval I in R^n is called measurable on I, if there exists a sequence of step measurable functions $\{s_n\}$ on I such that

$$\lim_{n\to\infty} s_n(x) = f(x) \quad \text{a.e. on } I.$$

I.17 Theorem (Fubini). Assume f is Lebesgue-integrable on R^2. Then we have:

(a) There is a set T of measure 0 in R such that the Lebesgue integral $\int_R f(x,y)dx$ exists for all y in $R-T$.

(b) The function G defined on R by the equation

$$G(y) = \begin{cases} \int_R f(x,y)dx, & \text{if } y \in R-T, \\ 0, & \text{if } y \in T, \end{cases}$$

is Lebesgue integrable on R.

(c) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f = \int_R G(y)dy$. That is

$$\int_{R^2}\int f(x,y)\,d(x,y) = \int_R \left[\int_R f(x,y)\,dx\right] dy .$$

Note: There is a corresponding result which concludes that

$$\int_{R^2}\int f(x,y)\,d(x,y) = \int_R \left[\int_R f(x,y)\,dy\right] dx .$$

<u>Proof</u>: See [1], p.413.

<u>I.18 Definition ([12], p.1)</u>. Let E be a linear space over the complex field $\mathbb{C}$ (or the real field R). E is said to be a complex (real) algebra if for all $x,y \in E$, the product xy is defined and $xy \in E$, satisfying the following conditions:

$x(yz) = (xy)z = xyz$; $x(y+z) = xy+xz$; $(y+z)x = yx+zx$ and $(\lambda x)(\mu y) = (\lambda\mu)(xy)$, where $\lambda,\mu \in \mathbb{C}$, $z \in E$. If $xy = yx$

for all $x,y \in E$ then E is called a commutative algebra. An element $1 \in E$ is called an identity if $1\cdot x = x\cdot 1 = x$ for all $x \in E$. An algebra E with identity $1 \in E$ is called a unital algebra.

An algebra A with a Hausdorff topology is called a topological algebra if the maps $(x,y) \to x+y$; $(\lambda,x) \to \lambda x$ from $A \times A$ to A and $\mathbb{C} \times A$ to A respectively are continuous and the map $(x,y) \to xy$ from $A \times A$ to A is jointly continuous.

<u>I.19 Definition ([12], p.2).</u> A locally convex algebra A is a topological algebra whose topology is given by a family $\{p_\alpha\}_{\alpha\in I}$ of seminorms (cf. Def. I.3). Note that E is Hausdorff if and only if the following condition is satisfied:

$$p_\alpha(x) = 0 \text{ for all } \alpha \in I \text{ if and only if } x = 0 .$$

The continuity of $(x,y) \to xy$ means:

For each $p_\alpha \in \{p_\alpha\}$ there is $p_\beta \in \{p_\alpha\}$ such that

$$p_\alpha(xy) \le p_\beta(x)\cdot p_\beta(y) , \quad x,y \in A .$$

<u>I.20 Definition ([18], p.6).</u> A locally convex algebra A is called locally m-convex (LMC for short) if its topology is given by a family $\{p_\alpha\}_{\alpha\in I}$ of seminorms such that for each $p_\alpha \in \{p_\alpha\}_{\alpha\in I}$ we have $p_\alpha(xy) \le p_\alpha(x)\cdot p_\alpha(y)$, for all $x,y \in A$.

<u>I.21 Definition ([18], p.9).</u> Given a locally m-convex algebra A and for $p_\alpha \in \{p_\alpha\}_{\alpha\in I}$ let N_α denote the null space of p_α, A_α denotes the quotient space A/N_α and $\|\cdot\|_\alpha$ denotes the norm on A_α defined by $\|x_\alpha\|_\alpha = \|x+N_\alpha\|_\alpha = p_\alpha(x)$, where $x_\alpha = x+N_\alpha$. Note that A_α is a normed algebra.

<u>I.22 Definition ([12], p.3).</u> If the topology of a (Hausdorff) l.m.c. algebra A is given by a single norm $p(x) = \|x\|$ (that is $\|x\| \ge 0$, $\|x\| = 0$ if and only if $x = 0$, $\|\lambda x\| = |\lambda|\cdot\|x\|$, $\|x+y\| \le \|x\| + \|y\|$ and $\|xy\| \le \|x\|\cdot\|y\|$) then A is called a normed algebra.

A complete normed algebra is called a Banach algebra. If A has identity 1, then A can be given an equivalent norm $\|\cdot\|'$ such that $\|1\|' = 1$ (see [21]).

Suppose that A is an algebra without identity and denote by A^+ the set of all pairs (x,λ), $x \in A$, $\lambda \in \mathbb{C}$. A^+ becomes an algebra with the following definitions of algebraic operations:

(i) $(x,\lambda) + (y,\mu) = (x+y, \lambda+\mu)$;

(ii) $\mu(x,\lambda) = (\mu x, \mu\lambda)$;

(iii) $(x,\lambda)\cdot(y,\mu) = (xy + \lambda y + \mu x, \lambda\mu)$ $(x,y \in A,\ \lambda,\mu \in \mathbb{C})$.

The element $e = (0,1) \in A^+$ is an identity for A^+.

A *-algebra is an algebra A equipped with a map $*: A \to A$ which is conjugate linear (that is $(x+y)^* = x^* + y^*$ and $(\lambda x)^* = \bar{\lambda} x^*$ for all $x,y \in A$ and $\lambda \in \mathbb{C}$) reverse multiplicative (that is $(xy)^* = y^* x^*$ for all $x,y \in A$) and has period at most two (that is $x^{**} = x$ for all $x \in A$). The map $*$ is called the involution on A. A Banach *-algebra A is said to be B*-algebra if $\|xx^*\| = \|x\|^2$ for all $x \in A$. A closed sub-algebra of $B(H)$ (where H is a Hilbert space) which is closed under conjugation is called a C^*-algebra (see [9], p.1).

An approximate identity in A is a net of elements $\{u_\alpha\}_{\alpha\in I}$ such that $\lim_\alpha u_\alpha x = \lim_\alpha x \cdot u_\alpha = x$ for every $x \in A$. If the set of norms $\|u_\alpha\|$ is bounded, it is called a bounded approximate identity. In the obvious way left and right approximate identities are defined, when the Banach algebra is not commutative, see [26], p.22.

<u>(a) Example.</u> Let $A = L_1(-\infty,\infty)$. We put

$$U_n(t) = \begin{cases} n/2, & \text{for } -1/n \le t \le 1/n, \\ 0 & \text{for } |t| > 1/n. \end{cases}$$

Then $\{U_n\}_{n \in I}$ is an approximate identity in $L_1(-\infty,\infty)$.

<u>I.23 Definition ([15], p.54).</u> (a) Let A be a Banach algebra and let $x \in A$. If A has an identity 1, then the spectrum of x denoted by $sp_A(x)$ is the set of all $\lambda \in \mathbb{C}$ such that $(x - \lambda \cdot 1)$ is not invertible (an element not invertible is called singular); if A is without identity, then $Sp_A(x)$ is the set of all $\lambda \in \mathbb{C}$, $\lambda \neq 0$, such that x/λ is quasi-singular, together with $\lambda = 0$. Note that if A is without identity, then $0 \in Sp_A(x)$, $x \in A$. If A has identity, then $0 \in Sp_A(x)$ if and only if x is singular.

(b) ([21]): The spectral radius is defined by

$$\rho_A(x) = \sup\{|\lambda| : \lambda \in Sp_A(x)\}.$$

<u>I.24 Theorem.</u> Let A be a Banach algebra without identity. If $x \in A$ then $Sp_A(x) = Sp_{A^+}(x)$.

<u>Proof:</u> See [15].

I.25 Theorem. Let A be a complete, locally m-convex algebra, with m-base $\{U_i\}$ and let $x \in A$. Then $Sp_A(x) = \bigcup_i Sp_{A_i}(x_i)$.

Proof: See [18], Corollary 5.3(a).

I.26 Theorem (Spectral Mapping). Let A be a commutative Banach algebra, let $x \in A$ and suppose f is a complex valued function that is defined and analytic on some open set $P \supset Sp(x)$. Then:

(i) If A has an identity, then $f(Sp(x)) = Sp(f(x))$.

(ii) If A is without identity and $f(0) = 0$, then for $x \in A$ implies $f(x) \in A$ and $f(Sp(x)) = Sp(f(x))$.

Proof: See [15], p.152.

§2. Numerical Ranges of operators and of elements of unital Banach algebras.

In this section we collect known definitions and results on spatial numerical range $V(T)$ and the inner product numerical range $W(X,T)$ of a linear operator T.

Further the concept of numerical range $V(A,a)$ of an element of a unital Banach algebra A is defined and we give some relevant known results.

I.27 Definition ([7], p.2). Let X be a complex (real) vector space. We shall say that a complex (real) inner product is defined on X if to any x,y belonging to X, there corresponds a complex (real) number $\langle x,y \rangle$ and the following properties hold:

(i) $\langle x+y, z\rangle = \langle x,z\rangle + \langle y,z\rangle$, for $x,y,z \in X$;

(ii) $\langle px,y\rangle = p\langle x,y\rangle$, for $x,y \in X$, p complex (real);

(iii) $\langle x,y\rangle = \overline{\langle y,x\rangle}$;

(iv) $\langle x,x\rangle > 0$ for $x \neq 0$;

(v) If $\langle x,x\rangle = 0$ then $x = 0$.

X, with an inner product $\langle , \rangle$ is called an I.P. space.

<u>I.28 Definition ([16])</u>. Given an I.P. space $(X,\langle,\rangle)$ and $T \in B(X)$, where $B(X)$ denotes the space of all bounded linear operators on X. The numerical range $W(X,T)$ is defined as the set of complex numbers, given by:

$$W(X,T) = \{\langle Tx,x\rangle : \langle x,x\rangle = 1\}.$$

<u>I.29 Definition ([2(a)], p.81)</u>. Let X denote a Banach space over a field of real or complex numbers, $S(X)$ its unit sphere $\{x \in X: \|x\| = 1\}$, and X' its dual space. We denote by $\Pi(X)$ the subset of the cartesian product $X \times X'$ defined by

$$\Pi(X) = \{(x,f): x \in S(X),\ f \in S(X'),\ f(x) = 1\}.$$

Let $B(X)$ denote the Banach algebra of all bounded linear operators on X with the operator norm $|\cdot|$ given by $|T| = \sup\{\|Tx\|: x \in S(X)\}$, $T \in B(X)$.

The Spatial Numerical Range $V(T)$ of an element $T \in B(X)$ is defined by

$$V(T) = \{f(Tx) : (x,f) \in \Pi(X)\} .$$

I.30 Theorem. $\Pi(X)$ is a connected subset of $X \times X'$ with the norm × weak *-topology, unless X has dimension one over R.

Proof: See [3].

I.31 Theorem. $V(T)$ is connected.

Proof: See [2(a)], p.102.

I.32 Theorem. Let X be a normed linear space, $<,>$ be an I.P. on X satisfying $\|x\| = <x,x>^{1/2}$, $x \in X$ and $T \in B(X)$. Then we have

$$W(X,T) \subset V(T) .$$

Proof: See [2(a)], p.86.

I.33 Definition ([4]). Let A be a unital Banach algebra over the field of complex numbers. Let A' be the dual space of A: For $x \in A$ the numerical range of x is defined to be:

$$V(A,x) = \{f(x) : f \in A', \|f\| = 1 = f(1)\} ,$$

and the numerical radius is

$$\nu_A(x) = \sup\{|\lambda| : \lambda \in V(A,x)\} .$$

We write $D(A,x) = \{f \in A' : \|f\| = 1,\ f(x) = \|x\| = 1\}$.

I.34 Theorem. Let A be a subalgebra of the algebra $B(X)$ of all bounded linear operators on a Banach space X containing the identity of $B(X)$. Then for each T in A we have $\overline{Co}\, V(T) = V(A,T)$.

Proof: See [2(a)], p.84.

I.35 Definition ([2(a)]). Let A be a unital Banach algebra. Given $a \in A$ we define $V(A,a,x) = \{f(ax): f \in D(A,x)\}$.

I.36 Proposition. Let A be a unital Banach algebra and $a \in A$. Then

$$V(A,a) = V(A,a,1) .$$

Proof: See [5], p.378.

I.37 Proposition. Let A be a unital Banach algebra. For each $a \in A$, we have the following:

(i) $D(A,1)$ is a convex weak *-compact subset of A' and $V(A,a)$ is a nonempty convex compact subset of $\mathbb{C}$.

(ii) $Sp_A(a)$ is a nonempty compact subset of $\mathbb{C}$.

(iii) $Sp_A(a) \subset V(A,a)$.

Proof: (i) $D(A,1)$ is a closed subset of the unit ball in A' and hence it is a weak *-closed subset of A' by the Theorem (I.8) of §1. Convexity of $D(A,1)$ is obvious. The set $V(A,a,1)$ is the image of $D(A,1)$ under the weak *-continuous linear mapping $f \to f(a)$ and so is a compact convex subset of $\mathbb{C}$. Also $D(A,1)$ is nonempty by the Theorem (I.5) of §1 and hence $V(A,a)$ is nonempty.

(ii) See [16], p.56.

(iii) See [2(a)], p.19.

I.38 Lemma. Let A,B be unital Banach algebras and let ϕ be a norm decreasing homomorphism from A to B with $\phi(1) = 1$, then $V(B,\phi(a)) \subset V(A,a)$.

Proof: See [2(b)], p.42.

I.39 Lemma. Let A_R denote the algebra A regarded as a unital Banach algebra over R. Then for $a \in A$, we have

$$V(A_R,a) = R_e V(A,a) .$$

Proof: See [2(b)], p.43.

I.40 Theorem. Let A be a unital Banach algebra. Then ~~the~~ following are equivalent:

(i) $V(A,a) = \text{Co}\, Sp_A(a) \quad (a \in A)$,

(ii) $V(A,a^2) \subset \text{Co}\{z^2 : z \in V(A,a)\}$.

Proof: See [2(b)], p.48.

I.41 Theorem. Let A be a unital Banach algebra with $\nu_A = \rho_A$. Then

$$\|a\| \le \frac{1}{2} e \rho_A(a) \quad (a \in A) .$$

Proof: See [2(b)], p.49.

I.42 Corollary. Let $\nu_A = \rho_A$ in A, let $a \in A$ with $\nu_A(a) = 1$, and let p be a polynomial that maps the unit disc into itself. Then

$$\|p(a)\| \le \frac{1}{2}e .$$

Proof: Follows from the Theorems (I.41) and (I.26).

I.43 Proposition. Let A be a unital Banach algebra and let B be a subalgebra of A containing the identity element. Then for each $b \in B$ we have the following:

(i) $V(B,b) = V(A,b)$,

(ii) $Sp_A(b) \subset Sp_B(b)$,

(iii) $\rho_B(b) = \rho_A(b)$.

Proof: (i) See [2(a)], p.16.

(ii) Since for $\lambda \in \mathbb{C}$, $(b - \lambda\cdot 1)$ is singular in A implies that $(b - \lambda\cdot 1)$ is singular in B, we have $Sp_A(b) \subset Sp_B(b)$.

(iii) Follows from [21], Theorem (1.4.1).

I.44 Theorem. Let A be a unital Banach algebra and $a \in A$. Let $f(n)$; $g(n)$ be positive real numbers such that $\nu_A(a^n) \le f(n)\cdot\nu_A^n(a)$ and $\|a^n\| \le g(n)\cdot\nu_A^n(a)$. Then $g(n) = n!\,(e/n)^n$ is the best constant satisfying the above inequality and $f(n) \le g(n)$.

Proof: See [6].

I.45 Theorem. Let A be a commutative unital Banach algebra. For all $a \in A$, $\nu_A(a)$ is a norm on A and is equivalent to the original norm on A. In fact we have

$$\|a\| \geq \nu_A(a) \geq \frac{1}{e} \cdot \|a\| .$$

Proof: See [2(a)], p.34.

I.46 Definition. An element $h \in A$ is called Hermitian if $V(A,h) \subseteq R$ and we denote the set of all Hermitian elements by $H(A)$.

I.47 Lemma. Let A be a unital Banach algebra. Given $h \in H(A)$ the following statements are equivalent:

(i) $h \in H(A)$;

(ii) $\lim_{\alpha \to 0} \frac{1}{\alpha}(\|1 + i\alpha h\| - 1) = 0$;

(iii) $\|\exp(i\alpha h)\| = 1$, $\alpha \in R$.

Proof: See [2(a)], p.46.

Remark. The condition (ii) above was Vidav's original definition of Hermitian element in A, see [25].

I.48 Proposition. Let A be a complex unital B^*-algebra. Then $h \in H(A)$ if and only if $h^* = h$

Proof: See [2(a)], p.47.

I.49 Theorem. Let A be a unital Banach algebra. Then we have the following:

(i) $H(A)$ is a real Banach space and $i(hk - kh) \in H(A)$, whenever $h,k \in H(A)$.

(ii) For $h \in H(A)$, $\max V(A,h) = \max Sp_A(h)$.

(iii) $V(A,h) = Co\, Sp_A(h) \quad (h \in H(A))$.

(iv) $\|h\| \leq e\rho_A(h) \quad (h \in H(A))$.

(v) $\rho_A(h) = \|h\| \quad (h \in H(A))$.

(vi) If $h_1, h_2 \in H(A)$ and $h_1 + ih_2 = 0$ then $h_1 = h_2 = 0$.

Proof: (i) See [2(a)], p.47.

(ii) See [22].

(iii) See [2(a)], p.53.

(iv) See [2(a)], p.54.

(v) See [22].

(vi) See [2(a)].

I.50 Definition ([2(a)], p.54). We say that $a \in A$ (unital Banach algebra) is normal if $a = h + ik$ with $h,k \in H(A)$, $hk = kh$.

I.51 Theorem. Let $a \in A$ be normal in a unital Banach algebra A. Then

$$V(A,a) = Co\, Sp_A(a).$$

Proof: See [2(a)], p.54.

CHAPTER II

RELATIVE NUMERICAL RANGES OF ELEMENTS OF BANACH ALGEBRAS

In this chapter we define the relative numerical range and the relative numerical radius of an element of a Commutative Banach algebra with respect to a given element. This definition coincides with the classical definition of numerical range of an element in a unital algebra if the reference element is the identity of the algebra. From a general result we derive that the relative numerical range of an idempotent relative to itself consists of a singleton $\{1\}$. But the converse is not true (Example (II.9)). If ϕ is an algebra homomorphism and an isometry from A into B then for $x \in A$, the relative numerical range of $\phi(a)$ with respect to $\phi(x)$ provided $\|\phi(x)\| = 1$ is equal to the relative numerical range of a with respect to x (Theorem (II.10)). Examples (II.11) and (II.12) show that Theorem (II.10) is not true in general without the isometry.

A special kind of norm has been introduced on Banach algebras which we call a regular norm. Indeed this norm coincides with the usual norm if the Banach algebra has identity or it has bounded approximate identity. We use this regular norm to advantage.

We also consider some specific examples of the relative numerical range of an element of $L_1(R)$. Example (II.25) shows that the relative numerical range of an element in a non-unital Banach algebra does not necessarily contain number zero.

§1. Relative Numerical Range And Some Properties.

In this section we introduce the concept of relative numerical range and relative numerical radius and discuss some of their properties.

II.1 Definition. Let A be a Banach algebra and A' the dual of A. Suppose $a, b \in A$. We write

$$\overset{\circ}{D}(A,b) = \{f \in A' : \|f\| = 1 , \ f(b) = \|b\|\} ,$$

and we define the relative numerical range of a with respect to b as follows:

$$\overset{\circ}{V}_b(A,a) = \{f(ab) : f \in \overset{\circ}{D}(A,b)\} .$$

The relative numerical radius of a with respect to b is defined as

$$\overset{\circ}{v}_b(a) = \sup\{|\lambda| : \lambda \in \overset{\circ}{V}_b(A,a)\} .$$

We note that the set $\overset{\circ}{D}(A,b)$ is nonempty in view of the Hahn Banach theorem. Moreover $\overset{\circ}{D}(A,b) = D(A,b)$, if $\|b\| = 1$.

Remark: If the Banach algebra A has identity 1, $\|1\| = 1$, then

$$\mathring{V}_1(A,a) = \{f(a): f \in \mathring{D}(A,1)\} = \{f(a): f \in A', \|f\| = 1, f(1) = 1\}$$
$$= V(A,a) \quad \text{(cf. Def. (I.33))} .$$

Thus our definition of relative numerical range of an element relative to identity coincides with the usual concept of numerical range in unital Banach algebras.

Note: All algebras are commutative unless otherwise stated.

II.2 Lemma. Given $a,b,x \in A$, $\alpha \in C$, then:

(i) $\mathring{V}_x(A,a+b) \subset \mathring{V}_x(A,a) + \mathring{V}_x(A,b)$,

(ii) $\mathring{V}_x(A,\alpha a) = \alpha \cdot \mathring{V}_x(A,a)$,

(iii) $\mathring{v}_x(a+b) \le \mathring{v}_x(a) + \mathring{v}_x(b)$,

(iv) $\mathring{v}_x(\alpha a) \le |\alpha| \cdot \mathring{v}_x(a)$.

Proof: (i) to (iv) are obvious by the linearity of f and the definition above.

II.3 Example. In general, the equality does not hold in (i). For an example, consider the two dimensional space $A = (R^2, \|\cdot\|_\infty)$ and $e_1 = (1,0)$, $e_2 = (0,1)$. Then for $x = (1,1) \in (R^2, \|\cdot\|_\infty)$ and $f = (\lambda, 1-\lambda) \in (R^2)'$, $\lambda \in [0,1]$, arbitrary

$$\mathring{V}_x(R^2,e_1) + \mathring{V}_x(R^2,e_2) = [0;2] ,$$

On the other hand $\mathring{V}_x(R^2, e_1+e_2) = \{1\}$, which shows that

$$\mathring{V}_x(R^2, e_1+e_2) = \{1\} \neq \mathring{V}_x(R^2, e_1) + \mathring{V}_x(R^2, e_2) = [0,2] .$$

<u>II.4 Proposition</u>. (i) For all $a \in A$, $\mathring{V}_b(A,a)$ is bounded.
(ii) If $\|b\| = 1$, then $\mathring{V}_b(A,a)$ is a convex subset of the complex plane.

<u>Proof</u>: (i) If $\lambda \in \mathring{V}_b(A,a)$ then for some $f \in \mathring{D}(A,b)$, $|\lambda| = |f(ab)| \le \|ab\|$ and so $\mathring{V}_b(A,a)$ is bounded.

(ii) For the convexity, suppose $0 \le \zeta \le 1$ and $\lambda, \mu \in \mathring{V}_b(A,a)$. Then for some $f, g \in A'$ with $\|f\| = 1 = \|g\|$, $f(b) = 1 = g(b)$, we have

$$\lambda = f(ab) \quad \text{and} \quad \mu = g(ab) .$$

Define a functional h on A as follows: for all $x \in A$, $h(x) = (1-\zeta)f(x) + \zeta g(x)$, $0 \le \zeta \le 1$. Then clearly h is linear and $\|h\| \le 1$. Also for $b \in A$, $\|b\| = 1$, $h(b) = 1$, so $\|h\| = 1$. Thus

$$\begin{aligned} h(ab) &= (1-\zeta)f(ab) + \zeta g(ab) \\ &= (1-\zeta)\lambda + \zeta \cdot \mu \in \mathring{V}_b(A,a) , \end{aligned}$$

proving that $\mathring{V}_b(A,a)$ is convex.

<u>II.5 Proposition</u>. For $a, b \in A$, $\|ab\| \ge \mathring{v}_b(a)$.

<u>Proof</u>: Suppose that $\lambda \in \mathring{V}_b(A,a)$. Then for some $f \in A'$ such that $\|f\| = 1$, $f(b) = \|b\|$ we have

$$\lambda = f(ab).$$

Therefore, $|\lambda| \le \|f\| \cdot \|ab\| \le \|ab\|$,
hence $\mathring{v}_b(a) = \sup\{|\lambda| : \lambda \in \mathring{V}_b(A,a)\} \le \|ab\|$.

<u>II.6 Theorem</u>. Let A and B be Banach algebras. Suppose ϕ is a homomorphism from A to B such that $\|\phi(z)\| \le \|z\|$, for all $z \in A$. If $x \in A$ with $\|x\| = 1$ such that $\|\phi(x)\| = 1$, then

$$\mathring{V}_{\phi(x)}(B,\phi(a)) \subset \mathring{V}_x(A,a), \quad \text{for all } a \in A.$$

<u>Proof</u>: Let $\lambda \in \mathring{V}_{\phi(x)}(B,\phi(a))$ then for some $g \in B'$, $\|g\| = 1$, $g(\phi(x)) = \|\phi(x)\| = 1$, it follows that $\lambda = g(\phi(ax))$ because ϕ is a homomorphism.

Now define f on A by, $f(y) = g(\phi(y))$, $y \in A$. Then clearly $\|f\| \le 1$ and since $g(\phi(x)) = f(x) = 1$ for $\|x\| = 1$, we have $\|f\| = 1$. Hence $\lambda = f(ax) \in \mathring{V}_x(A,a)$, which proves the theorem.

<u>II.7 Theorem</u>. Assume $a,b \in A$ and $\|b\| = 1$. Then

(i) If $\mathring{V}_b(A,a) = \{1\}$ then either $ab = b$ or $0 < \operatorname{dist}(b,\mathbb{C}ab) < 1$ where $\mathbb{C}ab$ is the span of ab.

(ii) If $ab = b$ then $\mathring{V}_b(A,a) = \{1\}$.

<u>Proof</u>: (i) Suppose that $\mathring{V}_b(A,a) = \{1\}$ and we claim either $ab = b$ or $0 < \operatorname{dist}(b, \mathbb{C}\, ab) < 1$.

Assume to the contrary that $ab \neq b$ and that $\operatorname{dist}(b, \mathbb{C}\, ab) = 1$. If $b = \lambda ab$, $\lambda \neq 1$, then $f(b) = f(\lambda ab) = \lambda f(ab) = \lambda$, for any $f \in A'$ for which $\|f\| = f(b) = 1$ which is a contradiction because $f(b) = 1 = \|b\|$. Hence $ab = b$.

Now if b is not in the span of ab and $\operatorname{dist}(b, \mathbb{C}\, ab) = 1$ then by Lemma (I.7) there exists $f \in A'$, $\|f\| = 1$, $f(b) = \|b\| = 1$ with $f(ab) = 0$, which is a contradiction. Hence $0 < \operatorname{dist}(b, \mathbb{C}\, ab) < 1$.

(ii) Suppose that $ab = b$ then

$$\begin{aligned}\mathring{V}_b(A,a) &= \{f(ab) = f(b) : f \in A',\ \|f\| = 1,\ f(b) = \|b\| = 1\} \\ &= \{1\}.\end{aligned}$$

<u>II.8 Corollary</u>. Let $a \in A$ be such that $\|a\| = 1$ and $a^2 = a$, then $\mathring{V}_a(A,a) = \{1\}$.

<u>Proof</u>: Follows from Theorem (II.7).

<u>II.9 Example</u>. The statement "if $\mathring{V}_a(A,a) = \{1\}$ then $a^2 = a$" is false.

Suppose $A = \{f \in C[0,1] : f(0) = 0\}$, then A is a Banach algebra with supremum norm and the multiplication is defined as follows:

$$(fg)(t) = f(t) \cdot g(t), \quad \text{for all } t \in [0,1].$$

This algebra has no unit.

Suppose $f \in A$ is such that $f(x) = x$, for all $x \in [0,1]$. Since A is a closed subspace of $C[0,1]$, we have $A' \cong C'[0,1]/A^{\perp}$, where $A^{\perp}$ is the annihilator of A. Suppose $\mu \in A'$, then $\mu \in C'[0,1]$ such that $\mu(g) = 0$ for all $g \in A$ implies that $\mu(\{0\}) = 0$.

Now consider

$$\mathring{V}_f(A,f) = \{\mu(f\cdot f) : \|\mu\| = 1,\ \mu \in A',\ \mu(f) = \|f\|,\ \mu(\{0\}) = 0\}$$

$$= \left\{ \int_0^1 x^2 d\mu : \|\mu\| = 1 = \int_0^1 x d\mu,\ \mu(\{0\}) = 0 \right\} .$$

In particular let E be a Borel set in $[0,1]$ and define

$$\mu_0(E) = \begin{cases} 1 , & 1 \in E , \\ 0 , & 1 \notin E . \end{cases}$$

μ_0 is the point measure given by $\mu_0 = \delta_1$, where for any $s \in [0,1]$

$$\delta_s(g) = \int g d\delta_s = g(s) , \quad \text{for all } g \in C[0,1] ,$$

and hence

$$\mu_0(f) = \delta_1(f) = \int f d\delta_1 = f(1) = 1 ,$$

such that $\mu_0(\{0\}) = 0$; $\|\mu_0\| = 1$ and $\int x d\mu_0 = 1$. Because $f(x) = x$ has exactly one maximizing point $x = 1$ and $\mu_0(f) = \|\mu_0\| \cdot \|f\|$ is supported on points which maximize $|f|$, that is there are no δ_s except δ_1 in the set

$$\left\{\mu : \mu(\{0\}) = 0, \|\mu\| = 1, \int x d\mu = 1\right\},$$

so that

$$\int x d\delta_s = s < 1 \quad \text{unless} \quad s = 1 .$$

Thus

$$\int_0^1 x^2 d\mu_0 = \mu_0(f^2) = f^2(1) = 1 .$$

Hence $\mathring{V}_f(A,f) = \{1\}$, though $f^2 \neq f$.

<u>II.10 Theorem</u>. Let A and B be Banach algebras. Suppose ϕ is an algebra homomorphism and an isometry from A into B . If $x \in A$ such that $\|x\| = 1$, then

$$\mathring{V}_{\phi(x)}(B,\phi(a)) = \mathring{V}_x(A,a) , \quad (a \in A) .$$

<u>Proof</u>: Since $\|\phi(x)\| = \|x\| = 1$, the inclusion $\mathring{V}_{\phi(x)}(B,\phi(a)) \subseteq \mathring{V}_x(A,a)$ follows from Theorem (II.6).

Conversely suppose that $\mu \in \mathring{V}_x(A,a)$ then for some $f \in A'$, $\|f\| = 1$, $f(x) = \|x\|$, we have $\mu = f(ax)$.

Define g on $\phi(A) = \{\phi(x): x \in A\}$ by $g(\phi(y)) = f(y)$, for all $y \in A$. Since ϕ is an isometry so g is linear and bounded. Also

$$\|g\| = \sup_{\|\phi(y)\| \le 1} |g(\phi(y))|$$

$$= \sup_{\|y\| \le 1} |f(y)| = \|f\| = 1 .$$

Hence by Theorem (I.6) g can be extended to h on B such that $\|g\| = 1 = \|h\|$ and $g(\phi(x)) = h(\phi(x)) = f(x)$. Also

$$g(\phi(ax)) = h(\phi(ax)) = f(ax) = \mu .$$

This implies that $\mu \in \mathring{V}_{\phi(x)}(B,\phi(a))$, proving that $\mathring{V}_x(A,a) \subseteq \mathring{V}_{\phi(x)}(B,\phi(a))$. Combining the two inequalities we have the result.

<u>II.11 Example</u>. The converse of the Theorem (II.10), that is if ϕ is a homomorphism from A to B such that for every $x \in A$ with $\|x\| = 1$ and for each $a \in A$ we have

$$\mathring{V}_{\phi(x)}(B,\phi(a)) = \mathring{V}_x(A,a)$$

then ϕ still need not be an isometry.

Consider the vector space X of dimension greater than or equal to 1 , suppose

$$Y = \begin{bmatrix} X \\ \oplus \\ \mathbb{C} \end{bmatrix} = \left\{ \begin{bmatrix} x \\ \lambda \end{bmatrix} : x \in X, \quad \lambda \in \mathbb{C} \right\}$$

then Y is an algebra. Let $B(Y)$ be the algebra of all bounded linear operators on Y . Then

$$B(Y) \cong \left\{ \begin{bmatrix} A & w \\ f & \alpha \end{bmatrix} : A \in B(X), \quad f \in X', \quad w \in X, \quad \alpha \in \mathbb{C} \right\}.$$

Consider $B_1 = \left\{ \begin{bmatrix} 0 & w \\ 0 & 0 \end{bmatrix} : w \in X \right\}$. Clearly B_1 is a subalgebra of $B(Y)$ and for $a,x \in B_1$ the sets $\mathring{V}_x(B_1,a) = \mathring{V}_{T_x}(B(Y),T_a) = \{0\}$, where T_z is the left regular representation on $B(Y)$ that is $T_z u = zu$ for all $u \in Y$. But the mapping $T|_{B_1} : B_1 \to B(Y)$ is not an isometry because for $x_1, x_2 \in X$,

$$\begin{bmatrix} 0 & x_1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & x_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

though it is a homomorphism.

Remark: The above example is equally applicable if we take the unitization B_1^+ of B_1, that is

$$B_1^+ = \left\{ \begin{bmatrix} \alpha I & w \\ 0 & \alpha \cdot 1 \end{bmatrix} : \begin{matrix} w \in X, \\ \alpha \in C \end{matrix} \right\},$$

in place of $B(Y)$. Note that B_1^+ is a commutative algebra.

II.12 Example. Isometry in Theorem (II.10) is vital because if we drop this condition from the theorem then the conclusion fails, as shown in the following example.

Let T be the unit circle. Suppose $L_1(T)$ denotes the commutative Banach algebra of all measurable functions on T with $\int |f(t)| dt < \infty$. It has no unit. The multiplication is defined by convolution:

$$f*g(u) = \frac{1}{2\pi} \int_0^{2\pi} f(t) \cdot g(u-t) dt, \quad f, g \in L_1(T),$$

and the norm is given by $\|f\| = \int_0^{2\pi} |f(t)| \, dt/2\pi$.

Define $\phi: L_1(T) \to L_1(T)$ by $\phi(f) = f*D_1$, where $D_1(x) = 1 + 2\cos x$ is the Dirichlet kernel. Now we claim that $D_1 * D_1 = D_1$. For,

$$D_1 * D_1(u) = \frac{1}{2\pi}\int_0^{2\pi} D_1(t)\cdot D_1(u-t)\,dt$$

$$= \frac{1}{2\pi}\int_0^{2\pi} (1+2\cos t)(1+2\cos(u-t))\,dt$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \left(1+e^{it}+e^{-it}\right)\left(1+e^{i(u-t)}+e^{-i(u-t)}\right)dt$$

$$= \frac{1}{2\pi}\left(2\pi + e^{iu}\cdot 2\pi + e^{-iu}\cdot 2\pi\right)$$

$$= 1 + e^{iu} + e^{-iu} = 1+2\cos u = D_1(u) ,$$

proving that D_1 is an idempotent.

Linearity of ϕ is obvious. Now, since D_1 is an idempotent we have

$$\phi(f*g) = (f*D_1)*(g*D_1) = \phi(f)*\phi(g) ,$$

which shows that ϕ is an algebra homomorphism.

We claim that ϕ is not an isometry for $f = e^{2it}$, and $D_1(x) = 1+2\cos x$. For,

$$\phi(f) = f*D_1(u) = e^{2it}*D_1(u)$$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{2it}(1+2\cos(u-t))dt$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\left(e^{2it}+e^{it}\cdot e^{iu}+e^{-iu}\cdot e^{3it}\right)dt$$

$$= 0 .$$

So $\phi(f) = f*D_1 = 0$, proving that ϕ is not an isometry. Thus we see that ϕ is an algebra homomorphism but not an isometry. Now

$$\mathring{V}_f(L_1(T),g) = \left\{\psi(g*f) : \psi \in [L_1(T)]' = L_\infty ,\ \|\psi\|_\infty = 1 ,\quad \psi(f) = \|f\|\right\}$$

The only possible choices of ψ are

$$\psi(x) = \begin{cases} \overline{\text{sgn}}\, f(x) , & x \in (\text{support of } f) , \\ \alpha(x) \text{ measurable with } |\alpha(x)| \le 1 & \text{for } x \notin (\text{support of } f) , \end{cases}$$

see Definition (I.9).

Denote

$$X = \left\{ \psi(f*g) = \int_{(\text{support of } f)} (\overline{\text{sgn } f})\cdot(f*g) + \int_{(\text{support of } f)^c} \alpha(g*f) : \|\alpha\|_\infty \le 1 \right\} .$$

In particular, if $f = e^{2it}$ then

$$\psi(f*g) = \int_0^{2\pi} e^{-2iu}\left[\frac{1}{2\pi}\int_0^{2\pi} e^{2it}\; g(u-t)dt\right] du$$

$$= \int_0^{2\pi} e^{-2iu}\left[\frac{1}{2\pi}\int_0^{2\pi} g(t)\cdot e^{2i(u-t)}dt\right] du$$

$$= \int_0^{2\pi} \left[\frac{1}{2\pi}\int_0^{2\pi} g(t)\cdot e^{-2it}dt\right] du$$

$$= \int_0^{2\pi} g(t)\cdot e^{-2it}dt \; .$$

Therefore

$$\mathring{V}_{f=e^{2it}}(L_1(T),g) = \left\{\int_0^{2\pi} g(t)\cdot e^{-2it}dt\right\} .$$

In contrast, the set

$$\mathring{V}_{\phi(f)}(L_1(T),\phi(g)) = \{\psi(\phi(g*f)) : \psi \in L_\infty ,\ \|\psi\|_\infty = 1 ,\ \psi(\phi(f)) = \|\phi(f)\|\}$$

$$= \{0\} ,$$

because for $f = e^{2it}$, $\phi(f) = 0$, and hence

$$\mathring{V}_{f=e^{2it}}(L_1(T),g) \neq \mathring{V}_{\phi(e^{2it})}(L_1(T),\phi(g)) .$$

<u>II.13 Corollary</u>. Suppose B is a Banach algebra and A is a sub-algebra of B then for $x,a \in A$, $\|x\| = 1$, $\mathring{V}_x(A,a) = \mathring{V}_x(B,a)$.

<u>Proof</u>: Put $\phi = i$ (identity map) in Theorem (II.10) .

We apply Theorem (II.10) to obtain some examples.

<u>II.14 Examples</u>. (a) Suppose A is a commutative complex Banach algebra with identity. Assume for all $x \in A$, $\|x^2\| = \|x\|^2$. Let $\Gamma: A \to C(X_A,\mathbb{C})$ be the Gelfand transform.

Where X_A is the set of all characters of A (X_A is a compact Hausdorff space) and $C(X_A,\mathbb{C})$ is the algebra of all continuous complex valued functions on X_A. Then Γ is an isometry (see [9], p.29), and an algebra homomorphism. So all the conditions of Theorem (II.10) are satisfied and hence for any $x,y \in A$ such that $\|\hat{x}\|_\infty = 1 = \|x\|$

$$(*) \qquad \mathring{V}_{\Gamma(x)}(C(X_A,\mathbb{C}) , \Gamma(y)) = \mathring{V}_x(A,y) ,$$

or

$$\mathring{V}_{\hat{x}}(C(X_A,\mathbb{C}) , \hat{y}) = \mathring{V}_x(A,y) ,$$

where $\Gamma(x) = \hat{x}$ defined by $\hat{x}(f) = f(x)$, $f \in C(X_A,\mathbb{C})$.

(b) If the algebra A is a commutative complex B^*-algebra with identity then (*) above holds. For, since A is a B^*-algebra, every $x \in A$ is normal and so $\|x\|^2 = \|x^2\|$, see [24], p.427.

(c) Let A and B be complex C^*-algebras with identity (Definition (I.22)) and let $\phi: A \to B$ be a complex $|\cdot|$, *-algebra map. Then ϕ is an isometry, see [9], p.33. Thus the conclusion (*) of (a) follows.

(d) Suppose A is a commutative Banach algebra with a continuous involution (Definition (I.22)), but without identity such that for every $x \in A$, $\hat{x}^* = \bar{\hat{x}}$ and $\|x^2\| = \|x\|^2$.

We know that each $\hat{x}$ is a continuous function on the maximal ideal space m of A, which vanishes at infinity.

Put $\hat{X} = \{\hat{x}: x \in A\} \subset C(m)$. Suppose $\psi: X \to \hat{X}$, then the mapping is an isometry, see [10], p.269. Now the conditions of Theorem (II.10) are satisfied. Thus for $x \in A$ such that $\|x\| = \|\psi(x)\| = 1$

$$\mathring{V}_{\psi(x)}(\hat{X},\psi(y)) = \mathring{V}_x(A,y), \quad \text{for all } y \in A .$$

§2. Regular norm and Relative Numerical Range.

In this section we will define regular norm on a Banach algebra A and give some of its properties and study the relative numerical range of an element in A when A has a regular norm.

<u>II.15 Definition</u>. A Banach algebra A is said to have regular norm if for all $x \in A$, $\|x\| = \sup_{\|y\| \le 1} \|xy\|$.

If the above equality holds for some $x \in A$, then x is said to have regular norm.

<u>II.16 Proposition</u>. Each Banach algebra A with bounded approximate identity $\{u_\alpha\}_{\alpha \in I}$, $\|u_\alpha\| = 1$ (Definition (I.22)) has regular norm.

<u>Proof</u>: Since $\{u_\alpha\}_{\alpha \in I}$ is an approximate identity,

$$\lim_\alpha \|au_\alpha - a\| = \lim_\alpha \|u_\alpha a - a\| = 0, \quad (a \in A).$$

Which implies that

$$\|a\| \le \sup_{\|u_\alpha\| \le 1} \|au_\alpha\|$$

$$\le \sup_{\|b\| \le 1} \|ab\| \le \|a\|$$

and so $\|a\| = \sup_{\|b\| \le 1} \|ab\|$.

<u>II.17 Corollary</u>. The above proposition is also true if A has identity 1 of norm one.

Now we show by an example that a Banach algebra with regular norm need not have a bounded approximate identity.

<u>II.18 Example</u>. Let $\Delta = \{\lambda: \lambda \in C, |\lambda| \le 1\}$ be the closed unit disc in C. Suppose $A(\Delta)$ denotes the set of all $f \in C(\Delta)$ such that f is analytic on $\text{int}\cdot(\Delta) = \{\lambda \in C: |\lambda| < 1\}$, with the usual pointwise operations and the sup norm,

$$\|f\|_\infty = \sup_{\lambda \in \Delta} |f(\lambda)| \qquad (f \in A(\Delta)) .$$

$A(\Delta)$ is a commutative Banach algebra.

Consider $A_0 = \{f \in A(\Delta): f(0) = 0\}$ and let $g \in A_0$ such that $g(\lambda) \equiv \lambda$. Then for $f \in A_0$,

$$\|f\| = \sup_{\Delta} |f(\lambda)\cdot\lambda| = \sup_{\Delta} |f(\lambda)\cdot g(\lambda)| = \|fg\| .$$

Suppose $\{\delta_n\}$ is a bounded approximate identity of A_0. Then $\delta_n(0) = 0$. Define a linear functional ϕ as follows:

$$\phi(f) = f'(0) , \quad \text{for all } f \in A_0 ,$$

where $f'(0)$ is the derivative of f at 0. Then clearly $\phi \in A_0'$, and

$$\delta_n(\lambda)\cdot g(\lambda) \longrightarrow g(\lambda) , \quad (\lambda \in \Delta) ,$$

and by taking the derivatives on both sides we get,

$$\delta_n'(\lambda)\cdot g(\lambda) + g'(\lambda)\cdot\delta_n(\lambda) \longrightarrow g'(\lambda) = 1 ,$$

hence $\phi(\delta_n g) = \delta_n'(0)\cdot g(0) + g'(0)\cdot\delta_n(0) \longrightarrow g'(0) = 1$.

But the left hand side of above expression is zero, because $\delta_n(0) = 0 = g(0)$, while the right hand side is equal to one, this leads to an absurdity: $0 = 1$. This proves that A_0 cannot have a bounded approximate identity.

<u>II.19 Theorem.</u> Let A be a Banach algebra and suppose there is $b \in A$ such that $\|b\| = 1$ and $\mathring{V}_b(A,a) = \{\|a\|\}$. Then a has regular norm.

<u>Proof</u>: We prove that $\|a\| = \sup_{\|x\|\le 1} \|ax\|$. Clearly $\sup_{\|x\|\le 1} \|ax\| \le \|a\|$.

On the other hand, since $\mathring{V}_b(A,a) = \{\|a\|\}$ then for all $f \in A'$ with $\|f\| = 1$, $f(b) = 1 = \|b\|$, we have $f(ab) = \|a\|$, and hence

$$\|a\| = |f(ab)| \le \|ab\| \le \sup_{\|x\|\le 1} \|ax\| ,$$

proving that $\|a\| = \sup_{\|x\|\le 1} \|ax\|$.

<u>II.20 Theorem.</u> Let A be a Banach algebra with regular norm. Suppose $B(A)$ is the algebra of all bounded linear operators on A . Then for $x,y \in A$, $\|y\| = 1$,

$$\mathring{V}_y(A,x) = \mathring{V}_{T_y}(B(A),T_x) ,$$

where T_z is the left regular representation operator on A .

Proof: Suppose A^+ is the unitization of A. Define an operator $T_{(a,\lambda)}$ on A by $T_{(a,\lambda)}(b) = ab + \lambda b$, $(b \in A,\ \lambda \in \mathbb{C})$. Then clearly this operator is linear and bounded. Also $A^+ \to B(A)$ is an algebra homomorphism. In fact $(a,\lambda) \to T_{(a,\lambda)}$ is a monomorphism.

Define the norm $\|\cdot\|_+$ on A^+ by $\|(a,\lambda)\|_+ = \|T_{(a,\lambda)}\|$. Then

$$\|(a,0)\|_+ = \|T_{(a,0)}\| = \sup_{\|b\|\le 1} \|T_{(a,0)}(b)\| = \sup_{\|b\|\le 1} \|ab + 0\cdot b\|$$

$$= \sup_{\|b\|\le 1} \|ab\| = \|a\| \quad \text{(by hypothesis).}$$

In this way we have the extension of the original norm on A to the norm on A^+. Therefore we have an algebra homomorphism from A to $B(A)$ and an isometry from A into $B(A)$, (because $A \subseteq A^+ \subseteq B(A)$). Hence the theorem follows from Theorem (II.10).

II.21 Theorem. Let A be a unital Banach algebra and $B(A)$ the algebra of all bounded linear operators on A, such that the norm on A is induced by that of $B(A)$. Then for $y \in A$, $\|y\| = 1$, we have $\mathring{V}_y(A,x) = \mathring{V}_{T_y}(B(A),T_x)$, $(x \in A)$.

Proof: First of all, note that if we endow A with the norm $\|x\|' = \|T_x\|$, then $\|\cdot\|'$ is equivalent to the original norm $\|\cdot\|$. We, thus assume that A is endowed with this norm which we also denote by $\|\cdot\|$. Thus the embedding of A into $B(A)$ via: $x \to T_x$ is an isometry and so we can apply Theorem (II.10). Explicitly, we may show it as follows:

Suppose $\alpha \in \mathring{V}_y(A,x)$ then for some $f \in A'$, $\|f\| = 1$, $f(y) = \|y\|'$, we have $\alpha = f(xy)$. Consider $T_{(A)} = \{S \in B(A) : \exists z \in A$ such that $S = T_z\}$. Then $T_{(A)}$ is a subalgebra of $B(A)$. Now define a functional ϕ on $T_{(A)}$ by

$$\phi(T_a) = f(a)\,, \quad a \in A\,.$$

Clearly ϕ is linear and $|\phi(T_a)| = |f(a)| \le \|f\|\cdot\|a\|' \le \|T_a\|$, which implies that $\|\phi\| \le 1$. Also $\phi(T_y) = f(y) = \|y\|' = \|T_y\| = 1$. Therefore $\|\phi\| = 1$. Since $T_{(A)}$ is a subalgebra of $B(A)$, by Theorem (I.6) ϕ can be extended to $\hat{\phi}$ on $B(A)$ such that $\|\phi\| = \|\hat{\phi}\| = 1$, and

$$\begin{aligned} \phi(T_x \cdot T_y) &= \phi(T_{xy}) \\ &= \hat{\phi}(T_{xy}) \\ &= f(xy) = \alpha\,. \end{aligned}$$

Which shows that $\alpha \in \mathring{V}_{T_y}(B(A), T_x)$.

On the other hand, suppose that $\beta \in \mathring{V}_{T_y}(B(A), T_x)$. Then for some $\psi \in [B(A)]'$, $\|\psi\| = 1$, $\psi(T_y) = \|T_y\| = 1$ implies that $\|f\| = 1$.

Thus $f(xy) = \psi(T_{xy}) = \beta \in \mathring{V}_y(A,x)$, and the proof is complete.

Now we introduce another notion of relative numerical range which includes the previously introduced relative numerical range.

II.22 Definition. Let A be Banach algebra. Then for $a,b \in A$ we define

$$\mathring{V}(A,a;b) = \{f(ab) : f \in A', \ \|f\| \le 1, \ f(b) = \|b\|\}.$$

In this definition we have "$\|f\| \le 1$" instead of "$\|f\| = 1$" and so $\mathring{V}_b(A,a) \subset \mathring{V}(A,a;b)$.

II.23 Theorem. Let A be commutative Banach algebra. Then for all $a,b \in A$ we have the following:

(i) $\mathring{V}(A,a;b) = \mathring{V}(A,a;-b)$,

(ii) if $b = b^2$, $b \neq 0$ then $\mathring{V}(A,ab;b) = \mathring{V}(A,a;b)$,

(iii) if $0 \neq b = b^2$, then $\mathring{V}(A,a;b) = \mathring{V}(A_b,ab;b)$. Where A_b is the closed subalgebra generated by b.

Proof:

(i)
$$\begin{aligned}\mathring{V}(A,a;-b) &= \{-f(ab) : f \in A', \ \|f\| \le 1, \ f(-b) = \|-b\|\} \\ &= \{g(ab) : -g \in A', \ \|g\| \le 1, \ g(b) = \|b\|\}, \text{ where } -f = g \\ &= \{g(ab) : g \in -A' = A', \ \|g\| \le 1, \ g(b) = \|b\|\} \\ &= \mathring{V}(A,a;b).\end{aligned}$$

(ii)
$$\begin{aligned}\mathring{V}(A,ab;b) &= \{f(ab^2) : f \in A', \ \|f\| \le 1, \ f(b) = \|b\|\} \\ &= \{f(ab) : f \in A', \ \|f\| \le 1, \ f(b) = \|b\|\} \quad (\text{because } b = b^2) \\ &= \mathring{V}(A,a;b).\end{aligned}$$

(iii) Let $\alpha \in \mathring{V}(A,a;b)$ then for some $f \in A'$, $\|f\| \le 1$, $f(b) = \|b\|$, we have $\alpha = f(ab)$. Define $f\big|_{A_b} = g$. Then $g(b) = \|b\|$ and $\|g\| \le 1$. Hence, using $b = b^2$, $\alpha = g(ab^2)$, proving that $\alpha \in \mathring{V}(A_b, ab; b)$.

The reverse inclusion follows from Theorem (I.6), since A_b is a subalgebra of A.

§3. Relative Numerical Range Of An Element In $L_1(R)$.

In this section we are going to consider the Banach algebra $L_1(R)$ and describe the relative numerical range of its elements relative to a special element of $L_1(R)$. We note that $L_1(R)$ is a commutative Banach algebra without identity. Multiplication in $L_1(R)$ is given by convolution:

$$f*g(t) = \int_R f(s)\cdot g(t-s)ds .$$

II.24 Theorem. For $f,g \in L_1(R)$, $f \neq 0$, $\mathring{V}_f(L_1(R),g) = \{\langle g, \phi * Ff\rangle : \phi \in \mathring{D}(L_1(R),f)\}$, where $(\phi*Ff)(t) = \int_R \phi(s)\, f(s-t)ds$, and Ff = Flip of f, that is, $Ff(t) = f(-t)$, for all t.

In particular for $f(\sigma) = \begin{cases} e^{-\sigma}, & \sigma \ge 0 \\ 0 & \sigma < 0 \end{cases}$ in $L_1(R)$, the set $\mathring{V}_f(L_1(R),g)$ is a translated disc (disc not necessarily centred at zero) for any g in $L_1(R)$ with radius less than or equal to the norm of g.

Moreover, if $g > 0$ then that translated disc has a specific radius which is given by $\int_{-\infty}^{0}(1-e^{t})\cdot g(t)\cdot dt$.

<u>Proof</u>: For given $f,g \in L_1(R)$, by Definition (II.1) we have

$$\mathring{V}_f(L_1(R),g) = \{\langle f*g,\phi\rangle : \phi \in L_\infty(R)\ ,\ \|\phi\|_\infty = 1 \ \underline{\text{and}}\ \langle f,\phi\rangle = \|f\|\}\ .$$

Consider all functions $\phi \in L_\infty(R)$ with

$$\phi(t) = \begin{cases} \dfrac{|f(t)|}{f(t)}\ , & \text{if } f(t) \neq 0\ , \\ \text{measurable,}\ |\phi(t)| \leq 1 & \text{on } f^{-1}(0)\ . \end{cases} \qquad \text{(I)}$$

These functions are identified as exactly those for which $\|\phi\|_\infty = 1$ and $\langle f,\phi\rangle = \|f\|_1$.

Now $\lambda \in \mathring{V}_f(L_1(R),g)$ if and only if there is a $\phi \in L_1'(R) = L_\infty(R)$ such that

$$\lambda = \phi(f*g) = \int_R \phi(t)\cdot\left\{\int_R g(s)\cdot f(t-s)ds\right\}dt$$

$$= \int_{R^2} \phi(t)\cdot g(s)\cdot f(t-s)dt\cdot ds$$

$$= \langle g,\phi*Ff\rangle\ .$$

This proves the first part of the theorem.

For the second part now suppose

$$f(\sigma) = \begin{cases} e^{-\sigma} , & \sigma \geq 0 , \\ 0 , & \sigma < 0 . \end{cases} \tag{II}$$

Then clearly $f \in L_1(R)$ and $\|f\|_1 = 1$. Define ϕ as follows:

$$\phi(t) = \begin{cases} 1 , & t \geq 0 , \\ \in L_\infty(-\infty,0) \text{ with norm } \leq 1 & \text{for } t < 0 . \end{cases}$$

Then $\lambda \in \mathring{V}_f(L_1(R),g)$ if and only if $\lambda = \langle g,\phi*Ff\rangle$ for ϕ as above, $\|\phi\| = 1$ and $\phi(f) = \|f\| = 1$. Denote

$$w(t) = \int_t^\infty \phi(s)\cdot f(s-t)ds .$$

Case 1). When $t \geq 0$, then

$$w(t) = \int_t^\infty f(s-t)ds , \quad (\text{because } \phi(s) = 1 , \; s > 0) ,$$

$$= \int_0^\infty f(\sigma)d\sigma , \quad \text{where} \quad s-t = \sigma$$

$$= 1 ,$$

that is $w(t) = 1$ for all $t \geq 0$.

Case 2). When $t < 0$, then

$$w(t) = \int_t^0 \phi(s)\cdot f(s-t)ds + \int_0^\infty f(s-t)ds$$

$$= \int_0^{-t} \phi(t+\sigma)\cdot f(\sigma)d\sigma + \int_{-t}^\infty f(\sigma)d\sigma \text{, where } s = t+\sigma \text{.}$$

If $f(\sigma)$ is in (II) then we calculate

$$w(t) = \begin{cases} 1, & t \geq 0, \\ \int_0^{-t} \phi(t+\sigma)e^{-\sigma}d\sigma + e^t, & t < 0. \end{cases} \qquad \text{(III)}.$$

Note that $\int_{-t}^\infty f(\sigma)d\sigma = \int_{-t}^\infty e^{-\sigma}d\sigma = e^t$.

Now

$$\lambda = \langle g,\phi * Ff\rangle = \int_{-\infty}^\infty g(t)\cdot w(t)dt$$

$$= \int_{-\infty}^0 g(t)\left[\int_0^{-t} \phi(t+\sigma)\cdot e^{-\sigma}d\sigma + e^t\right]dt + \int_0^\infty g(t)dt ,$$

from (II) and (III) above. That is

$$\lambda = \left(\int_{-\infty}^{0} e^{t}\cdot g(t)dt + \int_{0}^{\infty} g(t)dt\right) + \int_{-\infty}^{0} g(t)\left\{\int_{0}^{-t}\phi(t+\sigma)e^{-\sigma}d\sigma\right\}dt \qquad \text{(IV)} .$$

Let $M_\phi = \int_{-\infty}^{0} g(t)\left\{\int_{0}^{-t}\phi(t+\sigma)e^{-\sigma}d\sigma\right\}dt$. Then by Theorem (I.17) it follows that $M_\phi = \int_{0}^{\infty}\left\{\int_{-\infty}^{-\sigma} g(t)\cdot\phi(t+\sigma)dt\right\}e^{-\sigma}d\sigma$, because $-\infty < t < 0$ and $0 < \sigma < -t$ are equivalent to $-\infty < t < -\sigma$ and $0 < \sigma < \infty$. If $t = -\zeta$ then we have

$$M_\phi = \int_{0}^{\infty}\left\{\int_{\sigma}^{\infty} g(-\zeta)\cdot\phi(\sigma-\zeta)d\zeta\right\}e^{-\sigma}d\sigma .$$

Put $\zeta = \sigma + \alpha$ then

$$M_\phi = \int_{0}^{\infty}\int_{0}^{\infty} g(-\sigma-\alpha)\cdot\phi(-\alpha)e^{-\sigma}d\alpha d\sigma$$

$$= \int_{0}^{\infty}\left\{\int_{0}^{\infty} e^{-\sigma}\cdot g(-\sigma-\alpha)d\sigma\right\}\phi(-\alpha)d\alpha .$$

Let $P = \{M_\phi : \phi \in \mathring{D}(L_1(R),f)\}$. Then by choosing ϕ as a constant = 1 on $(-\infty,0)$ we make $P \supseteq$ Disc with the radius $\left|\int_0^\infty \int_0^\infty e^{-\sigma} g(-\sigma-\alpha)d\sigma \cdot d\alpha\right|$. Denote

$$r = \left|\int_0^\infty \left\{\int_0^\infty g(-\sigma-\alpha)d\alpha\right\} e^{-\sigma}d\sigma\right| .$$

Put $-\sigma-\alpha = \eta$, then

$$r = \left|\int_0^\infty \left\{\int_{-\infty}^{-\sigma} g(\eta)d\eta\right\} e^{-\sigma}d\sigma\right|$$

$$\leq \int_0^\infty \left\{\int_{-\infty}^{-\sigma} |g(\eta)|d\eta\right\} e^{-\sigma}d\sigma$$

$$\leq \|g\|_1 .$$

So we have proved that for an arbitrary $g \in L_1(R)$, and f as in (II) the relative numerical range $\mathring{V}_f(L_1(R),g)$ is a translated disc with centre $C = \int_{-\infty}^0 e^t \cdot g(t)dt + \int_0^\infty g(t)dt$, (see (IV)), and radius less than or equal to the norm of g.

Further if $g > 0$ then $\int_0^\infty \left\{ \int_{-\infty}^{-\sigma} g(\eta) \cdot d\eta \right\} e^{-\sigma} d\sigma \in P$.

Since $0 < \sigma < \infty$ and $-\infty < \eta < -\sigma$ are equivalent to $-\infty < \eta < 0$ and $0 < \sigma < -\eta$ then it follows by Theorem (I.17) that

$$\int_{-\infty}^{0} \left\{ \int_0^{-\eta} e^{-\sigma} d\sigma \right\} g(\eta) d\eta \in P \quad ,$$

which implies that

$$\int_{-\infty}^{0} g(\eta) d\eta - \int_{-\infty}^{0} e^{\eta} g(\eta) d\eta \in P \quad .$$

Recall that λ is in the disc with centre c , which is given by

$$c = \int_{-\infty}^{0} e^{t} \cdot g(t) dt + \int_0^\infty g(t) dt \quad ,$$

$$\left(\text{because from (IV)} \quad \bar{\lambda} - \left\{ \int_{-\infty}^{0} e^{t} g(t) dt + \int_0^\infty g(t) dt \right\} \in P \right) ,$$

and the radius is given by

$$r = \int_{-\infty}^{0} (1-e^t)g(t)dt \ .$$

which proves the theorem.

Remark (1): From the above Theorem (II.24) it is clear that the left most point on the disc is given by

$$(c-r) = \int_{0}^{\infty} g(t)dt + \int_{-\infty}^{0} (2e^t-1)g(t)dt \ ,$$

while the right most point is given by $\|g\|_1$ that is $(c+r) = \|g\|_1$.

Remark (2): Note that for $g < 0$, Theorem (II.24) holds good and the proof goes on the same lines as for $g > 0$.

The following example shows that if a Banach algebra has no identity then it is not true that the number zero will belong to the relative numerical range:

II.25 Example. Let $A = \{f \in C[0,1]: f(0) = 0\}$. Clearly A has no unit.

Consider $f \in A$ such that $f(x) = x$, for all $x \in [0,1]$. Now we claim that the number zero does not belong to $\mathring{V}_g(A,f)$ for some $g \in A$, with $\|g\| = 1$.

Suppose on the contrary that $0 \in \mathring{V}_g(A,f)$. Then there exists a $\phi \in A'$ such that $\phi(g) = 1$, $\|g\| = 1$, $\|\phi\| = 1$, with $0 = \phi(fg)$.

Let $\Omega_g = \{t \in [0,1]: |g(t)| = 1\}$. Consequently, $\inf. \Omega_g > 0$, since Ω_g is closed and does not contain zero, ϕ is a measure with support $\subseteq \Omega_g$ and $|\phi|$ is a probability measure.

Now

$$0 = \phi(fg) = \int_{\Omega_g} f(t) \cdot g(t) \cdot d\phi(t)$$

$$= \int_{\Omega_g} t \cdot g(t) \cdot d\phi(t) .$$

It is clear by the above alignment of ϕ and g (see Definition (I.9)) that $g(t) \cdot d\phi(t)$ is a measure such that $g(t) \cdot d\phi(t) \geq 0$.

Therefore

$$0 = \phi(fg) \geq (\inf. \Omega_g) \cdot \int_{\Omega_g} g(t) \cdot d\phi(t)$$

$$= (\inf. \Omega_g) \cdot 1 > 0 \quad - \quad (I) ,$$

because $t \geq \inf. \Omega_g$. Which is a contradiction and hence $0 \notin \mathring{V}_g(A,f)$.

CHAPTER III

GENERALIZED NUMERICAL RANGES OF ELEMENTS OF BANACH ALGEBRAS

In this chapter we examine the properties of the generalized numerical range of elements of Banach algebra with regular norm. Theorem (III.3) establishes that the closed convex hull of the generalized numerical range of an element in A with regular norm coincides with the numerical range of the same element in A^+. We also characterize the spectral state of A (A with regular norm) in terms of the generalized numerical range of the normal elements (cf. Def. (I.50)). The generalized numerical ranges of Hermitian and positive elements of A with regular norm are discussed here. It is shown that the closure of the generalized numerical range of $h \in H(A)$ is exactly the convex hull of the spectrum of h. If A has regular norm and has no identity then Theorem (III.18) serves as a characterization of Hermitian elements in A^+. Also there is a relation between K and $H(A)$ and in fact we have Theorem (III.20). The generalized numerical ranges of elements in $\oplus_0 X_i$ and $\oplus_\infty X_i$ (X_i are commutative Banach algebras) as well as in $C(X)$, where X is a compact or locally compact Hausdorff space are discussed in this chapter. An example (III.25) shows that in a nonunital Banach algebra the generalized numerical range of an element need not be closed, while the numerical range is always closed in a unital Banach algebra, according to Proposition (I.37).

All algebras are over the complex field unless otherwise stated.

§1. Generalized Numerical Ranges of elements of Banach algebras with regular norm.

In this section we will define the generalized numerical range and generalized numerical radius of elements of a Banach algebra. In particular we will establish a relationship between the generalized numerical range of an element of an algebra A with regular norm and the numerical range of the same element in A^+.

III.1 Definition. Given a Banach algebra A, let $S(A)$ and A' denote respectively the unit sphere $\{x: \|x\| = 1\}$ of A and the dual space of A. For each $x \in A$ we denote by $D_A(x)$ the following set

$$D_A(x) = \{f \in A': \|f\| = 1 = f(x)\},$$

and we write

$$V_x(A,a) = \{f(ax): f \in D_A(x)\}.$$

Note that if $\|x\| = 1$, then $D_A(x) = \mathring{D}(A,x) = D(A,x)$.

Now given an element a of a Banach algebra A, the generalized numerical range $V_A(a)$ is defined by

$$V_A(a) = \{f(ax): f \in A', \|f\| = 1 = f(x) = \|x\|\},$$
$$= \cup\{V_x(A,a): x \in S(A)\}.$$

For each $a \in A$ the generalized numerical radius $v_A(a)$ is defined by

$$v_A(a) = \sup\{|\lambda| : \lambda \in V_A(a)\}.$$

Note that $V_A(a)$ is a bounded subset of the complex plane, bounded by $\|a\|$.

Remark: If A is a Banach algebra and $B(A)$ is the algebra of all bounded linear operators on A then for each $a \in A$ and T_a in $B(A)$ (where T_z denotes the left regular representation of A that is for each x in A $T_z x = zx$),

$$V_A(a) = V(T_a),$$

where $V(T_a)$ is the spatial numerical range of T_a (cf. Def. (I.29)).

Proof: From the Definition (I.29) we have

$$\begin{aligned} V(T_a) &= \{f(T_a x) : f \in A', \|f\| = 1 = f(x) = \|x\|\}, \\ &= \{f(ax) : f \in A', \|f\| = 1 = f(x) = \|x\|\}, \\ &= V_A(a) \quad \text{(by Definition (III.1) above).} \end{aligned}$$

III.2 Lemma. Given $a, b \in A$ and $\alpha \in \mathbb{C}$, then

(i) $V_A(a+b) \subset V_A(a) + V_A(b)$,

(ii) $V_A(\alpha a) = \alpha \cdot V_A(a)$,

(iii) $v_A(a+b) \le v_A(a) + v_A(b)$,

(iv) $v_A(\alpha a) = |\alpha| \cdot v_A(a)$.

Proof: Same as in Lemma (II.2).

III.3 Theorem. Let A be a commutative complex Banach algebra without identity but with regular norm (Definition II.15)). Suppose that A^+ is the unitization of A then for each $x \in A$ we have:

(i) $\bar{C}o\, V_A(x) = V(A^+,x)$ (where $\bar{C}o\, X$ denotes the closed convex hull of the set X that is the intersection of all closed convex sets containing X),

(ii) $v_A(\cdot)$ is a norm on A which is equivalent to the given norm on A and for all $x \in A$, $\|x\| \geq v_A(x) \geq \frac{\|x\|}{e}$, where e is the constant $\exp(1)$,

(iii) $Sp_A(x) \subseteq \bar{C}o\, V_A(x)$.

Proof: (i) Since A^+ is the unitization of A, A^+ is also a Banach algebra with unit $(1,0)$ and the norm $\|(\lambda,x)\| = \|x\| + |\lambda|$. Thus

$$\phi: A \longrightarrow A^+$$
$$x \rightsquigarrow (0,x)$$

is an isometry.

We note that the regular multiplication operator on A^+ is given by:

$$T^+_{(\lambda,x)}(\mu,a) = (\mu,a)(\lambda,x) = (\lambda\mu, \mu x + \lambda a + ax).$$

For each $x \in A$, $\lambda \in \mathbb{C}$, define

$$T_{(\lambda,x)}: A \to A \text{ by } T_{(\lambda,x)}(a) = \lambda a + xa, \quad (a \in A).$$

Each $T^{+}_{(\lambda,x)}$ is a bounded linear operator on A^{+} with

$$\|T^{+}_{(\lambda,x)}(\mu,a)\| \le \|(\mu,a)\| \cdot \|(\lambda,x)\| .$$

In particular, $T_{(\lambda,x)}$ is linear, bounded and

$$\begin{aligned}\|T_{(\lambda,x)}(a)\| &= \|\lambda a + xa\| , \\ &\le \|\lambda a\| + \|xa\| , \\ &\le (|\lambda| + \|x\|) \cdot \|a\| = \|(\lambda,x)\| \cdot \|a\| .\end{aligned}$$

Suppose $B(A)$ denotes the algebra of all bounded linear operators on A. Define $\psi: A^{+} \to B(A)$ by

$$\psi((\lambda,x)) = T_{(\lambda,x)} ,$$

which is the restriction of $T^{+}_{(\lambda,x)}$ on A. Clearly ψ is an algebra homomorphism. To prove ψ is a monomorphism, suppose $\psi(\lambda,x) = 0$ then for all $a \in A$ we have

$$T_{(\lambda,x)}(a) = \lambda a + xa = 0 .$$

Now if $\lambda = 0$ then $xa = 0$ for all $a \in A$, and so $\|xa\| = 0$. Since the norm on A is regular by hypothesis, we have,

$$\sup_{\|a\| \le 1} \|xa\| = \|x\| = 0 ,$$

which shows that $x = 0$, hence $(\lambda,x) = (0,0)$. But if $\lambda \neq 0$, then we have,

$$a + \frac{xa}{\lambda} = 0 ,$$

that is $a = (-x/\lambda)a$, for all $a \in A$, proving that $(-x/\lambda)$ is the identity in A. But this is contrary to our assumption and hence $(\lambda,x) = (0,0)$.

Now consider the norm $\|\cdot\|_+$ on A^+ defined by $\|(\lambda,x)\|_+ = \|T_{(\lambda,x)}\|$. Clearly $\|(\lambda,x)\|_+$ is submultiplicative. Also for each $x \in A$,

$$\begin{aligned}\|x\|_+ = \|(0,x)\|_+ &= \|T_{(0,x)}\| \\ &= \sup_{\|a\|\le 1} \|T_{(0,x)}(a)\| \\ &= \sup_{\|a\|\le 1} \|xa\| \\ &= \|x\| , \text{ (by hypothesis).}\end{aligned}$$

This shows that the new norm coincides with the initial norm on A. Therefore we have the isometry of A (with $\|\cdot\|_+$) into $B(A)$:

$$A \hookrightarrow A^+ \hookrightarrow B(A) .$$

Since $(1,0)$ is the unit of A^+, we have $\psi((1,0)) = T_{(1,0)} = I$, the identity in $B(A)$ that is $T_{(1,0)}(a) = a$ for all $a \in A$. Since A^+

is a subalgebra of $B(A)$, therefore by Theorem (I.34) we have

$$V(A^+,(\lambda,x)) = \bar{C}o\, V(T_{(\lambda,x)}) .$$

But by the remark after Definition (III.1) it follows in particular for all $x \in A$,

$$V(A^+,x) = \bar{C}o\, V_A(x) ,$$

proving (i).

(ii) To prove this part, first we prove that

$$v_A(x) = v_{A^+}(x) , \quad \text{for all } x \in A .$$

For this it is sufficient to prove that if $D(\gamma)$ is the closed disc of radius γ then the following two subsets D_1 and D_2 of R are equal:

$$D_1 = \{\gamma : V(A^+,x) \subseteq D(\gamma)\} ,$$

$$D_2 = \{\gamma : V_A(x) \subseteq D(\gamma)\} .$$

Suppose $\gamma \in D_1$ then $D(\gamma) \supseteq V(A^+,x)$, hence by (i) above it follows that $D(\gamma) \supseteq V_A(x)$, which implies that $\gamma \in D_2$ and hence $D_1 \subseteq D_2$.

Conversely suppose that $s \in D_2$ then $D(s) \supseteq V_A(x)$. But $D(s)$ is closed and convex, therefore

$$D(s) \supseteq \bar{C}o\, V_A(x) = V(A^+,x) \;, \quad \text{(by (i) above).}$$

This shows that,

$$D_2 \subseteq D_1 \;,$$

proving that $D_1 = D_2$.

Since $\inf \cdot D_1 = \inf \cdot D_2$ it follows that $v_A(x) = v_{A^+}(x)$, for all $x \in A$. But A^+ is a Banach algebra with unit, so by Theorem (I.45) we have

$$\|(0,x)\|_+ \geq v_{A^+}(x) \geq \frac{\|(0,x)\|}{e} \;,$$

which implies that $\|x\| \geq v_A(x) \geq \frac{\|x\|}{e}$ and (ii) follows.

(iii) By Theorem (I.24) we have for all $x \in A$,

$$Sp_A(x) = Sp_{A^+}(0,x) \;.$$

But by Theorem (I.37), we have

$$Sp_{A^+}(0,x) \subseteq V(A^+,(0,x)) \;.$$

Therefore from (i) it follows that

$$Sp_A(x) \subseteq \bar{C}o\, V_A(x) \;.$$

III.4 Corollary. Let A be a commutative complex Banach algebra with a bounded approximate identity $\{u_\alpha\}$, $\|u_\alpha\| \le 1$. Then the conclusion of Theorem (III.3) holds.

Proof: This is obvious from Proposition (II.16).

III.5 Corollary. Let A be a Banach algebra with bounded approximate identity and $v_A(x) = \rho_A(x)$, for all $x \in A$. Then for $a \in A$,

$$\|a\| \le \frac{1}{2} e \rho_A(a) .$$

Proof: In view of Corollary (III.4), $\bar{C}o\, V_A(x) = V(A^+, x)$, for all $x \in A$. This implies that $\rho_{A^+}(a) = v_{A^+}(a) = v_A(a) = \rho_A(a)$, for all $a \in A$.

Hence the corollary follows from Theorem (I.41) applied to A^+.

III.6 Corollary. Let A be as in Corollary (III.5). Suppose $a \in A$ with $v_A(a) = 1$ and p is a polynomial that maps the unit disc into itself. Then

$$\|p(a)\| \le \frac{1}{2} e .$$

Proof: By Theorem (I.26), we have,

$$Sp_{A^+}(p(a)) = p(Sp_{A^+}(a)) ,$$

and then it follows from Theorem (I.24),

$$Sp_A(p(a)) = p(Sp_A(a)) \; .$$

Hence by Corollary (III.5) we get

$$\|p(a)\| \leq \frac{1}{2} e \; .$$

III.7 Corollary. Let A_R denote a Banach algebra A over R with bounded approximate identity. Then for every $a \in A$,

$$\bar{C}o\, V_{A_R}(a) = R_e V(A^+, a) \; .$$

Proof: By Lemma (I.39), $\operatorname{Re} V(A^+, a) = V(A_R^+, a)$ and by Corollary (III.4)

$$\bar{C}o\, V_{A_R}(a) = V(A_R^+, a) \; .$$

Hence

$$\bar{C}o\, V_{A_R}(a) = R_e V(A^+, a) \; .$$

Now we define the spectral states of A and characterize them in terms of the generalized numerical range of normal elements in A (cf. Def. (I.50)).

III.8 Definition. Let A be a complex Banach algebra, we say that $f \in A'$ is a spectral state of A if

$$f(a) \in Co(Sp_A(a)) \; , \quad \text{for every } a \in A \; .$$

We write $\Omega(A)$ for the set of all spectral states of A.

III.9 Theorem. Let A be a complex Banach algebra with a regular norm. Then for all normal elements $a \in A$,

$$f \in \Omega(A) \text{ if and only if } f(a) \in \overline{Co}\, V_A(a) .$$

Proof: $f \in \Omega(A)$ if and only if $f(a) \in Co(Sp_A(a))$, $a \in A$, from Definition (III.8). Thus $f \in \Omega(A)$ if and only if $f(a) \in Co(Sp_{A^+}(a))$, for all $a \in A$, by Theorem (I.24).

Hence from Theorem (I.51) it follows that

$$f \in \Omega(A) \text{ if and only if } f(a) \in V(A^+,a) , \text{ for all } a \in A ,$$

and the theorem follows from the Theorem (III.3).

III.10 Theorem. For a Banach algebra A with a regular norm the following two conditions are equivalent:

(i) For all $a \in A$, $\overline{Co}(Sp_A(a)) = \overline{Co}\, V_A(a)$,

(ii) for all $a \in A$, $Co\{\lambda^2 : \lambda \in V(A^+,a)\} \supseteq V(A^+,a^2)$.

Proof: By Theorem (III.3(i)), we have

$$V(A^+,a) = \overline{Co}\, V_A(a) , \text{ for all } a \in A ,$$

and by Theorem (I.24) $Sp_A(a) = Sp_{A^+}(a)$, for all $a \in A$. Hence by

Theorem (I.40),

$$\bar{C}o(Sp_A(a)) = \bar{C}o(Sp_{A^+}(a)) = V(A^+,a) = \bar{C}o\, V_A(a) \ ,$$

is equivalent to (ii).

§2. Generalized Numerical Range of Hermitian and positive elements.

In this section we will discuss the generalized numerical ranges of Hermitian and positive elements in a Banach algebra which has regular norm. If the algebra has a unit then we have for all $h \in H(A)$, $V(A,h) = Co(Sp_A(h))$, see Theorem (I.49(iii)) but if the algebra is without identity and has regular norm it turns out that the generalized numerical range of $h \in H(A)$ is connected and hence the closure of $V_A(h)$ is exactly the convex hull of $Sp_A(h)$.

We define a positive element in terms of its generalized numerical range and it will be shown that an element k is positive in A (when A has regular norm) if k is Hermitian and has non-negative spectrum. Moreover, if an element has real generalized numerical range and has positive spectrum then that element is positive.

III.11 Definition. Let A be a Banach algebra, then an element $a \in A$ is called Hermitian if $V_A(a) \subseteq R$ and is called positive if $V_A(a) \subseteq R^+$. Let $H(A)$ denote the set of all Hermitian elements and K the set of all positive elements.

III.12 Theorem. Let A be a commutative Banach algebra with a regular norm. Then

$$H(A) = A \cap H(A^+) .$$

Proof: Let $x \in H(A)$, then $x \in A$ and $V_A(x) \subseteq R$ (Definition (III.11)),

if and only if $x \in A$ and $V(A^+, x) \subseteq R$ (by Theorem (III.3(i))),

if and only if $x \in A$ and $x \in H(A^+)$.

III.13 Corollary: Let A be a commutative Banach algebra with a bounded approximate identity. Suppose $h_1, h_2 \in H(A)$. Then we have the following:

(i) If $h_1 + ih_2 = 0$ then $h_1 = h_2 = 0$,

(ii) $i(h_1h_2 - h_2h_1) \in H(A)$,

(iii) for all $h \in H(A)$, $\rho_A(h) = \|h\|$.

Proof: By Theorem (III.12) and Corollary (III.4) we have

$$h_1, h_2 \in A \cap H(A^+) .$$

This implies that $h_1, h_2 \in H(A^+)$ and the corollary follows from Theorem (I.49) applied to A^+.

In the following results A is commutative and has a bounded approximate identity.

III.14 Corollary. Let A be a B^*-algebra. Then $h \in H(A)$ if and only if $h^* = h$.

Proof: Suppose that $h \in H(A)$. Then by Corollary (III.4) and Theorem (III.12) we have

$$h \in H(A^+) \cap A .$$

This implies that $h \in H(A^+)$ and $h^* = h$ by Proposition (I.48).

Conversely suppose that $h \in A$ and $h^* = h$. Since $A \subseteq A^+$, we have $h \in A^+$ so that $h^* = h$. Thus $h \in H(A^+)$ by Proposition (I.48). That is, $V(A^+,h) \subseteq R$, and by Theorem (III.3(i)) we have

$$V_A(h) \subseteq R ,$$

proving that $h \in H(A)$.

III.15 Proposition. $H(A)$ is a real Banach space and for $h \in H(A)$,

$$V_A(h) \subseteq Co(Sp_A(h)) \subseteq \bar{C}o(Sp_A(h)) .$$

Proof: If A has identity then $H(A)$ is a real Banach space, see Theorem (I.49).

Now by Theorem (III.12) it is obvious that $H(A)$ is a real Banach space. Also by Corollary (III.4) and by Theorem (I.49) we have

$$V_A(h) \subseteq Co(Sp_{A^+}(h)) ,$$

and the proposition follows from Theorem (I.24).

<u>III.16 Corollary</u>. For $h \in H(A)$, $\bar{C}o\,V_A(h) = \bar{C}o(Sp_A(h))$.

<u>Proof</u>: From Proposition (III.15) it follows that

$$\bar{C}o\,V_A(h) \subseteq \bar{C}o(Sp_A(h)),$$

and from Corollary (III.4) we have

$$\bar{C}o(Sp_A(h) \subseteq \bar{C}o\,V_A(h).$$

Hence the Corollary.

<u>Remark</u>: Let A be a Commutative Banach algebra. Then the following are equivalent:

(i) If $a \in H(A)$ then $a^2 \in H(A)$,

(ii) if $b,c \in H(A)$ then $bc \in H(A)$.

<u>Proof</u>: Suppose that (i) is true. Then $b,c \in H(A)$ implies that $(b+c)^2 \in H(A)$. Now

$$bc = \frac{1}{2}\left\{(b+c)^2 - b^2 - c^2\right\} \in H(A),$$

so (ii) holds.

Conversely (ii) implies (i) by putting $a = b = c$ in (ii).

<u>III.17 Theorem</u>. For all $h \in H(A)$, $\overline{V_A(h)} = Co(Sp_A(h))$.

<u>Proof</u>: First we show that $V_A(h)$ is connected. By the remark after Definition (III.1) we have for each h,

$$V_A(h) = V(T_h) .$$

But by Theorem (I.31), $V(T_h)$ is connected and hence it follows that $V_A(h)$ is connected.

Since $V_A(h)$ is connected and $V_A(h) \subseteq R$ so $V_A(h)$ is convex. Therefore by Corollary (III.4) we have

$$\overline{Co}\, V_A(h) = V(A^+,h) = \overline{V_A(h)} .$$

Thus by Theorem (I.49(iii)) it follows that

$$\overline{V_A(h)} = V(A^+,h) = Co(Sp_{A^+}(h)) .$$

Hence the theorem follows from Theorem (I.24).

<u>III.18 Theorem</u>. Let A be a commutative Banach algebra without identity and with regular norm. Then $(\lambda,h) \in H(A^+)$ if and only if $h \in H(A)$ and $\lambda \in R$.

<u>Proof</u>: Suppose that $(\lambda,h) \in H(A^+)$ then

$$V(A^+,(\lambda,h)) \subseteq R .$$

Therefore for all $f \in (A^+)'$ such that $\|f\| = 1$, $f((1,0)) = 1$, it follows that

$$f((\lambda,h)) \in R .$$

Since A has regular norm so T_a is an isometry and since A^+ is complete so $T_A = \{T_a : a \in A\}$ is closed in A^+ and $I \notin T_A$. Now we claim that

$$\text{dist}(I, T_A) = \inf\{\|I - T_a\| : a \in A\} = 1 .$$

Since $\|I-0\| = 1$, we have $\inf_{a \in A} \|I - T_a\| \le 1$. Now suppose that $\text{dist}(I, T_A) < 1$, then there exists a such that

$$\|I - T_a\| < 1 .$$

So T_a is invertible in A^+, that is $(T_a)^{-1} \in A^+$.

Define $p = (T_a)^{-1} a \in A$. Then p acts on A as identity, for, $x \in A$,

$$px = (T_a)^{-1} ax = (T_a)^{-1}(T_a x) = ((T_a)^{-1}(T_a))x ,$$

$$= I \cdot x = x .$$

This is absurd because A does not have a unit. Therefore

$$\text{dist}(I, T_A) = 1 .$$

Consequently by Lemma (I.7) there exists $f_1 \in (A^+)'$ such that

$$f_1((1,0)) = 1 \text{ and } f_1((0,h)) = 0 .$$

Now $(\lambda, h) = (0,h) + \lambda(1,0)$. (I)

Therefore

$$f_1((\lambda,h)) = f_1[(0,h) + \lambda(1,0)] ,$$
$$= f_1((0,h)) + \lambda f_1((1,0)) ,$$
$$= 0+\lambda = \lambda ,$$

and hence $\lambda \in R$.

Also $(0,h) = (\lambda,h) - \lambda(1,0)$ and hence $h \in H(A)$, because $H(A)$ is a real Banach space.

Conversely suppose that $h \in H(A)$ and $\lambda \in R$ then from (I) above it follows that $(\lambda,h) \in H(A^+)$, proving the theorem.

<u>III.19 Theorem.</u> Let A be a commutative Banach algebra with regular norm. An element $k \in A$ is positive if and only if $k \in H(A)$ and $Sp_A(k) \subseteq R^+$.

<u>Proof</u>: Suppose k is positive then obviously k is Hermitian. And by Theorem (III.3) it follows that $Sp_A(k) \subseteq R^+$.

Conversely suppose that k is Hermitian and $Sp_A(k) \subseteq R^+$. Then by Theorem (III.17) we conclude that k is positive.

§3. <u>Cones And Generalized Numerical Range In $\oplus_0 X_i$ And $\oplus_\infty X_i$</u>.

In this section we obtain a relation between K and $H(A)$ for a commutative Banach algebra. In fact it will be shown that the set of all positive elements in a Banach algebra A is a proper convex, closed and normal cone in $H(A)$. If A has regular norm, it turns out that for such an A, K generates the whole space $H(A)$.

<u>III.20 Theorem</u>. Let A be a Banach algebra with regular norm. Then K is a proper convex, closed and normal cone in $H(A)$. Further K generates $H(A)$.

<u>Proof</u>: <u>K is proper convex</u>: K will be proper convex if it has the following geometric properties (cf. Def. (I.11)):

(C1) $K+K \subset K$,

(C2) $\alpha K \subset K$ for each positive real number α,

(C3) $K \cap (-K) = \{0\}$, where 0 denotes the zero element in $H(A)$.

The Conditions (C1) and (C2) are clear from the definition of positive elements and Lemma (III.2). To prove (C3), let $a \in K \cap (-K)$. Then by Lemma (III.2),

$$V_A(a) \subseteq R^+ \text{ and } -V_A(a) = V_A(-a) \subseteq R^+ .$$

This implies that $V_A(a) \subseteq R^+ \cap R^-$, and hence $V_A(a) = \{0\}$. Therefore the numerical radius $v_A(a) = 0$ and since $v_A(a)$ is a norm on A (Theorem III.3(ii)) so it follows that $a = 0$, thus we have

$$K \cap (-K) = \{0\} .$$

<u>K is closed</u>: Let $\{a_n\}$ be a sequence of positive elements of A such that $a_n \to a$. Suppose $f \in D_A(b)$, then

$$f(a_n b) \longrightarrow f(ab) .$$

But $f(a_n b) \geq 0 \Rightarrow f(ab) \geq 0$ and so $a \in K$. This proves that K is closed.

K is normal: Let $a,b \in K$, then we claim that $v_A(a) \leq v_A(a+b)$. Suppose for $c \in A$, $\|c\| = 1$, $f \in D_A(c)$. Then

$$\begin{aligned} v_A(a) &= \sup\{|\lambda| : \lambda \in V_A(a)\} \\ &= \sup\{|f(ac)| : f(ac) \in V_c(A,a)\} \qquad (I) . \end{aligned}$$

But since $a,b \in K$ implies that

$$\begin{aligned} V_A(a) &= \cup\{V_c(A,a) : \|c\| = 1\} \subset R^+ \quad \text{and} \\ V_A(b) &= \cup\{V_c(A,b) : \|c\| = 1\} \subset R^+ , \end{aligned}$$

$\Rightarrow f(ac) \geq 0$, $f(bc) \geq 0$ and so

$$f(ac) + f(bc) \geq f(ac) .$$

Therefore from (I) above we have

$$\begin{aligned} v_A(a) &\leq \sup\{f(ac) + f(bc) : f(ac) + f(bc) \in V_c(A,a+b)\} , \\ &\leq \sup\{|\mu| : \mu \in V_A(a+b)\} , \\ &= v_A(a+b) \quad \text{(by Definition (III.1))}. \end{aligned}$$

So we have proved that $v_A(a) \le v_A(a+b)$, for $a,b \in K$. Hence by Proposition (I.14d) it follows that K is normal, because as noted above $v_A(a)$ is a norm on A.

<u>K generates H(A)</u>: Suppose that A has unit e with $\|e\| = 1$. We claim that $H(A) = K - K$. It is obvious that $K - K \subseteq H(A)$. Since $V_A(a)$ is bounded, so for $h \in A$,

$V_A(h)$ is bounded then there exists $\lambda \ge 0$ such that $h + \lambda e \in K$. Clearly $\lambda e \in K$, $h \in H(A)$. Now $h = (h+\lambda e) - \lambda e \in K-K$ and so $H(A) = K-K$.

If A has no identity then suppose that A^+ is the unitization of A, where A has regular norm. Let $h \in H(A)$, then

$$(0,h) = (\lambda_1,h_1) - (\lambda_2,h_2) ,$$

$\lambda_1 > 0$, $\lambda_2 > 0$; $h_1,h_2 \in K$. This implies that $\lambda_1 - \lambda_2 = 0$ and $h_1 - h_2 = h$.

$\Rightarrow \lambda_1 = \lambda_2$ and $h_1 - h_2 = h$, which proves that $H = K - K$ and hence the theorem.

The following theorem gives the generalized numerical range of elements of the direct c_0 sum of a family of Commutative Banach algebras.

III.21 Theorem. Suppose $\{X_i : i \in I\}$, where I is a discrete set, is an indexed family of Commutative Banach algebras. Let $X = \oplus_0 X_i$ be the Banach algebra consisting of $x = (x_i)_{i \in I}$ with norm $\|x\| = \sup_i \|x(i)\| < \infty$. Then for all $x \in X$, $x = (x_i)$ with $x_i \in X_i$, we have

$$\mathrm{Co}\left[\bigcup_i V_{X_i}(x_i)\right] = V_X(x) .$$

Proof: Suppose that $a \in \mathrm{Co}\left[\bigcup_i V_{X_i}(x_i)\right]$. Then a can be written as a finite linear combination of elements of $\bigcup_i V_{X_i}(x_i)$, that is

$$a = \sum_{j=1}^{n} \lambda_j a_j ; \quad \sum_{j=1}^{n} \lambda_j = 1 , \quad \lambda_j \geq 0 , \quad a_j \in \bigcup_i V_{X_i}(x_i) \qquad \text{(I)} .$$

Since $a_j \in \bigcup_i V_{X_i}(x_i)$, then for some $i \in I$, $y_i \in X_i$, we have

$$a_j \in V_{y_{i(j)}}(X_{i(j)} ; x_{i(j)}) .$$

Therefore for some $\psi_j \in (X_{i(j)})'$ such that $\|\psi_j\| = 1$, $\|y_{i(j)}\| = 1$, $\psi_j(y_{i(j)}) = 1$, we have

$$a_j = \psi_j(x_{i(j)} \cdot y_{i(j)}) .$$

Now for all $z \in X$, define a functional $\phi \in X'$ as follows:

$$\phi(z) = \sum_{j=1}^{n} \lambda_j \psi_j\left(\eta_{i(j)}(z)\right),$$

where $\eta_{i(j)}: X \to X_{i(j)}$ is the projection with $\eta_{i(j)}(x) = x_{i(j)}$ and $\|\eta_{i(j)}(x)\| \le \|x\|$.

Clearly ϕ is linear and $\|\phi\| \le 1$. Also for $y \in X$, $y = (y_{i(j)})$ where $y_i = 0$ for $i(j) \neq 0$, $\|y\| = 1$, we have

$$\phi(y) = \sum_{j=1}^{n} \lambda_j \psi_j\left(\eta_{i(j)}(y)\right)$$

$$= \sum_{j=1}^{n} \lambda_j \psi_j\left(y_{i(j)}\right)$$

$$= 1.$$

Thus $\|\phi\| = 1$. Finally

$$\phi(xy) = \sum_{j=1}^{n} \lambda_j \psi_j\left(\eta_{i(j)}(xy)\right)$$

$$= \sum_{j=1}^{n} \lambda_j \psi_j\left(x_{i(j)} \cdot y_{i(j)}\right)$$

$$= \sum_{j=1}^{n} \lambda_j a_j \quad \text{(because } a_j = \psi_j(x_{i(j)} \cdot y_{i(j)})\text{)}$$

$$= a \quad \text{(from (I), above).}$$

That is $a \in V_y(X,x)$ and hence $a \in \cup\{V_y(X,x) : \|y\| = 1\}$. Thus we have

$$\mathrm{Co}\left[\bigcup_i V_{X_i}(x_i)\right] \subseteq V_X(x) \qquad \text{(II)} .$$

Conversely suppose that $\lambda \in V_X(x)$. Then for $\phi \in X'$, $\|\phi\| = 1$ and $y \in X$, $\|y\| = 1 = \sup_{i \in I}\|y_i\|$, we get

$$\lambda = \phi(xy) .$$

Since $\phi \in X'$ it means that $\phi \in \oplus_1 X_i'$ that is $\phi = \sum_{i \in I} \phi_i$ and $y \in X$ means that $y = \left\{y_i\right\}_{i \in I}$. Therefore

$$\phi(xy) = \lambda = \sum_{i \in I} \phi_i(x_i y_i) ; \quad \phi(y) = \sum_{i \in I} \phi_i(y_i) = 1 ,$$

where

$$\sup_{i \in I}\|y_i\| = 1 \text{ and } \sum_{i \in I}\|\phi_i\| = \|\phi\| = 1 .$$

Now we claim that for all i, $\|\phi_i\| = 0$ if $\|y_i\| < 1$. Suppose on contrary that $\|y_s\| < 1$, $\|\phi_s\| > 0$ for some $s \in I$. Then

$$1 = \phi(y) = R_e\,\phi_s(y_s) + R_e \sum_{i\in I\setminus\{s\}} \phi_i(y_i)$$

$$\le \|\phi_s\|\cdot\|y_s\| + \sum_{i\in I\setminus\{s\}} \|\phi_i\|\cdot\|y_i\|$$

$$\le \|\phi_s\|\cdot\|y_s\| + \sum_{i\in I\setminus\{s\}} \|\phi_i\|$$

$$= \|\phi_s\| - (1-\|y_s\|)\cdot\|\phi_s\| + \sum_{i\in I\setminus\{s\}} \|\phi_i\|$$

$$= 1 - (1-\|y_s\|)\cdot\|\phi_s\|$$

$$= 1 - \|\phi_s\| + \|y_s\|\cdot\|\phi_s\| < 1\ ,$$

which is a contradiction, hence proving that $\|y_i\| < 1$ implies $\phi_i = 0$.

Let $\Omega_y = \{i : \|y_i\| = 1\}$. Then Ω_y is finite, nonempty since $\|y_i\| \to 0$, and support of $\phi \subseteq \Omega_y$. Now

$$\lambda = \phi(xy) = \sum_{i\in(\text{support of }\phi)} \phi_i(x_i y_i)$$

$$= \sum_{i\in(\text{support of }\phi)} \|\phi_i\| \cdot \frac{\phi_i(x_i y_i)}{\|\phi_i\|} .$$

Choose weights $\mu_i = \|\phi_i\|$ then $\sum_i \mu_i = 1$ (because $\|\phi\| = \sum \|\phi_i\| = \sum \mu_i = 1$), and

$$\lambda = \phi(xy) = \sum_{i \in (\text{support of } \phi)} \mu_i \frac{\phi_i(x_i y_i)}{\mu_i} ,$$

which proves that $\lambda \in \mathrm{Co}\left[\bigcup_i V_{X_i}(x_i)\right]$ and hence

$$V_X(x) \subseteq \mathrm{Co}\left[\bigcup_i V_{X_i}(x_i)\right] \qquad \text{(III)} .$$

Thus theorem follows by combining (II) and (III) above.

In Theorem (III.21) we have discussed the generalized numerical range in $\oplus_0 X_i$. But what will happen to the generalized numerical range in $X = \oplus_\infty X_i$? It turns out that in that case the equality does not hold and this can be seen in the example below. First we have the following proposition:

III.22 Proposition: Suppose $\{X_i : i \in I\}$ is a collection of commutative Banach algebras. Let $X = \oplus_\infty X_i = \{x \in \prod_i X_i : \|x\| = \sup_i \|x(i)\| < \infty\}$. Then for all $x \in X$, $x_i \in X_i$, we have

$$\mathrm{Co}\left[\bigcup_i V_{X_i}(x_i)\right] \subseteq V_X(x) .$$

Proof: Same as in Theorem (III.21).

III.23 Example. The reverse inclusion in the above proposition is not true. For this we have this following example:

Suppose $\mathbb{C}$ denotes the Banach algebra of complex numbers. Let $x = (x_n)$, $n = 1,2,\ldots$ be the sequence defined by

$$x_n = (-1)^n (1 - 1/n) .$$

Then by Proposition (I.37) it is obvious that $V_X(x)$ is the closed interval $[-1,1]$. On the other hand,

$$\mathrm{Co}\left\{(-1)^n(1-1/n)\right\}_{n=1}^{\infty} = \mathrm{Co}\left(\left\{-1+\frac{1}{2n-1}\right\}_{n=1}^{\infty} \cup \left\{1-\frac{1}{2n}\right\}_{n=1}^{\infty}\right)$$

$$= \mathrm{Co} \bigcup_n \left(-1+\frac{1}{2n-1}\,,\ 1+\frac{1}{2n}\right)$$

$$= (-1,1) ,$$

which is an open set and hence we see that

$$V_X(x) = [-1,1] \subsetneq (-1,1) = \mathrm{Co}\left[\bigcup_i V_{X_i}(x_i)\right] .$$

§4. Applications.

In this section we give some applications of previous results. It has been shown by Stampfli and Williams [23] that if $A = C(X)$ where X is a compact space, the numerical range of any $f \in C(X)$ is the closed convex hull of $f(X)$. We show here that if X is a locally compact Hausdorff space and if $C_0(X)$ denotes the algebra of all continuous complex valued functions on X, vanishing at infinity, then for all $g \in C_0(X)$, we have

$$Co(R(g)) \subseteq V_{C_0(X)}(g) \subseteq Co(\overline{R(g)}) .$$

Bonsall and Duncan [2(a)], p.16 have proved that the numerical range of an element in a unital Banach algebra is a closed subset of C, but what happens if the algebra has no identity? The answer to this question is given by Example (III.25).

III.24 Theorem. Let X be a locally compact Hausdorff space. Suppose $C_0(X)$ denotes the algebra of all continuous complex valued functions on X, vanishing at infinity. Then for all $g \in C_0(X)$,

$$Co(R(g)) \subseteq V_{C_0(X)}(g) \subseteq Co(\overline{R(g)}) ,$$

where $R(g)$ denotes the range of g.

Proof: Let $y \in Co(R(g))$, then y can be written as the convex combination of finitely many points from $R(g)$, that is

$$y = \sum_{j=1}^{n} \lambda_j y_j \qquad \text{(I)} .$$

Where $y_j \in R(g)$, $\lambda_j \geq 0$ and $\sum_{j=1}^{n} \lambda_j = 1$.

Since $y_j \in R(g)$, for some $x_j \in X$, $y_j = g(x_j)$. Define the functional $\phi \in C_0'(X)$ as follows:

$$\text{for all } h \in C_0(X) , \quad \phi(h) = \sum_{j=1}^{n} \lambda_j h(x_j) .$$

Clearly if $h \geq 0$, $\phi(h) \geq 0$ and ϕ is linear so ϕ is a measure with total variation 1 . Hence $\|\phi\|$ = total variation of $\phi = 1$.

Now complete regularity of X allows us to choose $f \in C_0(X)$ as follows:

$$f: X \longrightarrow [0,1] \text{ such that } f(x_i) = 1 , \quad i = 1,2,\ldots,n, \ \|f\| = 1 .$$

Then $\phi(f) = \sum_{j=1}^{n} \lambda_j f(x_j) = 1$ and

$$\phi(fg) = \sum_{j=1}^{n} \lambda_j (fg)(x_j)$$

$$= \sum_{j=1}^{n} \lambda_j f(x_j) \cdot g(x_j)$$

$$= \sum_{j=1}^{n} \lambda_j g(x_j) \quad \text{(because } f(x_j) = 1\text{)}$$

$$= \sum_{j=1}^{n} \lambda_j y_j$$

$$= y \quad \text{(from (I) above)} .$$

Hence for $g \in C_0(X)$, $Co(R(g)) \subseteq V_{C_0(X)}(g)$ (II) .

Further suppose that $\lambda \in V_{C_0(X)}(g)$ then there exists $f \in C_0(X)$, $\|f\| = 1$ such that $\lambda \in V_f(C_0(X),g)$. Then for some $\phi \in C_0'(X)$, $\|\phi\| = 1$, $\phi(f) = 1$,

$$\lambda = \phi(fg) .$$

Define the set E as follows: $E = \{x \in X: |f(x)| = 1\}$. Clearly E is compact (because $\{x \in X: |f(x)| = 1\} = \{x \in X: |f(x)| \geq 1\}$ is compact since f vanishes at ∞).

Since $\phi(f) = 1$ and $\|\phi\| = 1$, ϕ is a measure supported on E. Now define a function w on E as follows:

$$w(x) = \begin{cases} \dfrac{\overline{f(x)}}{|f(x)|}, & x \in E, \\ \text{Continuously extended to } X \text{ with norm } \leq 1 . \end{cases}$$

Let M_w be the pointwise multiplication by w on $C_0(X)$ and M^* be the adjoint of M acting on measures $\psi \in C_0'(X)$.

Denote by $\bar{w}$ the conjugate of w. Now define a measure ψ as follows:

$$\psi = M_{\bar{w}}^* \phi \text{ , since } w\bar{w} = 1 \text{ on } E, \quad M_w^* \psi = M_w^* M_{\bar{w}}^* \phi = \phi .$$

that is

$$\phi = M_w^* \psi .$$

Now

$$\begin{aligned} \lambda = \phi(fg) &= M_w^* \psi(fg) \\ &= M_w^* \psi(M_f g) \\ &= \psi(M_w M_f g) \\ &= \psi(M_{wf} g) \\ &= \psi(M_{|f|} g) \\ &= \psi(|f| g) = \psi(g) . \end{aligned}$$

Since $\psi \geq 0$ and $\|\psi\| = 1$ so ψ is a probability measure and therefore we have

$$\lambda = \psi(g) = \lim_{\max\cdot[\psi(E_j)]\to 0} \sum_j \psi(E_j)\, g(x_j) \;, \quad x_j \in E \;.$$

Hence $\lambda \in Co(\overline{R(g)})$. Thus we have

$$V_{C_0(X)}(g) \subseteq Co(\overline{R(g)}) \qquad \text{(III)} \;.$$

And the theorem follows from (II) and (III) above.

III.25 Example. In this example we show that if a Banach algebra does not have identity then the generalized numerical range of each element need not be closed:

Let $A = \{f \in C[0,1]: f(0) = 0\}$. Clearly A is a Banach algebra without identity. Consider $f \in A$ such that $f(x) = x$ for all $x \in [0,1]$. Now we claim that $V_A(f) = (0,1]$.

We have already proved that $0 \notin V_g(A,f)$, for $g \in A$ with $\|g\| = 1$ in Example (II.25) and hence by Definition (III.1) we get $0 \notin V_A(f)$.

Now we claim that $(0,1] \subseteq V_A(f)$. Given $t \in (0,1]$, we construct a continuous function $g(x)$ as follows:

$$g_t(x) = \begin{cases} 0 \ , & \text{if } x = 0 \ , \\ 1 \ , & \text{if } x = t \ , \\ \in [-1,1] \ , & \text{elsewhere} \ . \end{cases}$$

We pick $\phi = \delta_t$ so that $\phi(g_t) = 1$, $\|\phi\| = 1$ and $\phi(fg) = t$, this shows that

$$(0,1] \subseteq V_A(f) \ .$$

On the other hand let $0 \neq \lambda \in V_A(f)$ then there exists $g \in A$, $\|g\| = 1$ such that $\lambda \in V_g(A,f)$ and hence for a $\phi \in A'$, $\|\phi\| = 1 = \phi(g)$,

$$\lambda = \phi(fg) \ .$$

Now

$$\lambda = \phi(fg) = \int_0^1 t \cdot g(t) \cdot d\phi(t) \in (0,1] \ ,$$

because of the property of the integral and convexity of $(0,1]$. But we have shown above that $0 \notin V_A(f)$. So we have

$$V_A(f) = (0,1] \ ,$$

which is not closed.

CHAPTER IV

RELATIVE RANGES AND GENERALIZED NUMERICAL RANGES IN LMC-ALGEBRAS

Husain, Giles and Koehler [13] have defined the numerical range of $a \in A$ as

$$V(A,\{p_\alpha\};a) = \{f(a) : f \in D(A,\{p_\alpha\};1)\}$$

Where $(A,\{p_\alpha\})$ is a unital lmc-algebra and the sets $D_\alpha(A,p_\alpha;1)$ and $D(A,\{p_\alpha\};1)$ are subsets of A'.

They have shown that for $\alpha, \beta \in I$, $\alpha \leq \beta$, $V(A_\alpha,\|\cdot\|_\alpha;a_\alpha) \subseteq V(A_\beta,\|\cdot\|_\beta;a_\beta)$ and that if $(A,\{p_\alpha\})$ is complete and for each $x \in A$, $\alpha \in I$, $p(x) = \sup_\alpha p_\alpha(x) < \infty$, then

$$V(A,\{p_\alpha\};a) = V(A,p;a) .$$

In this chapter we introduce the definition of relative numerical range of an element $a \in (A,P)$ which reduces to the familiar definition of the numerical range of $a \in A$ when A has unit.

Husain and Giles have shown that if $V(A,\{p_\alpha\};a)$ is bounded for $a \in A$, then $\sup_\alpha p_\alpha(a) < \infty$. Here we show that if for $a,b \in A$, $V_b(A,p;a)$ is bounded by a positive constant M then $\sup_\alpha p_\alpha(ab) \leq eM$.

where e is the constant exp. (1). Conversely, if $\sup_{\alpha} p_{\alpha}(ab) = M < \infty$, then it turns out that $\mathring{V}_b(A,p;a)$ is bounded (Theorem (IV.4)).

Further we define a relative set which we denote by Γ. If $b \in \Gamma$ then the set $D(A,p;b)$ is the inductive limit of $D(A;p_{\alpha};b)$ (Theorem (IV.9)) and for $\alpha \leq \beta$, $\mathring{V}_b(A,p_{\alpha};a) \leq \mathring{V}_b(A,p_{\beta};a)$; $\mathring{v}_b(A,p_{\alpha};a) \geq \mathring{v}_b(A,p_{\beta};a)$.

If A is a commutative lmc-algebra and suppose $b \in A$ such that $ab^2 = ab$ for all $a \in A$ and denote B the subalgebra generated by b. Then

$$\mathring{V}_b(A,p;a) = V_b(B,p;ab) .$$

The relationship between the spectrum and generalized numerical range is also investigated.

§1. Relative Numerical Range Of Elements In LMC-Algebras.

IV.1 Definition. Let A be a lmc-algebra and suppose that

$\rho(A) = \{P:$ P is an indexed family of submultiplicative semi-norms on A and P is an upper semilattice.$\}$

We assume that any P in $\rho(A)$ defines an equivalent topology of A, and let I be the indexing set for the elements in a particular family P.

We write (A,P) as the algebra A with a particular family of seminorms $P \in \mathcal{P}(A)$. Given A and $p_\alpha \in P$, $\alpha \in I$, we define the set

$$\overset{\circ}{D}(A,p_\alpha;b) = \{f \in A' : f(b) = p_\alpha(b) \text{ and for all } x \in A,\ |f(x)| \le p_\alpha(x)\},$$

and write

$$\overset{\circ}{D}(A,P;b) = \bigcup_\alpha \{\overset{\circ}{D}(A,p_\alpha;b) : p_\alpha \in P\}.$$

Also for each $a \in A$, let

$$\overset{\circ}{V}_b(A,p_\alpha;a) = \{f(ab) : f \in \overset{\circ}{D}(A,p_\alpha;b)\}.$$

We define the relative numerical range and relative numerical radius of a with respect to b as follows:

$$\begin{aligned}\overset{\circ}{V}_b(A,P;a) &= \{f(ab) : f \in \overset{\circ}{D}(A,P;b)\}\\ &= \{f(ab) : f \in \bigcup_\alpha \overset{\circ}{D}(A,p_\alpha;b)\}\\ &= \bigcup_\alpha \{f(ab) : f \in \overset{\circ}{D}(A,p_\alpha;b)\},\end{aligned}$$

and

$$\overset{\circ}{v}_b(A,P;a) = \sup\{|\lambda| : \lambda \in \overset{\circ}{V}_b(A,P;a)\}.$$

IV.2 Theorem. Let A be a l.m.c.-algebra and $a,b(\neq 0) \in A$. If $a_\alpha, b_\alpha \in A_\alpha$ (cf. Def. I.21) for each α, then we have the following

(i) $\mathring{V}_b(A,p_\alpha;a) = \mathring{V}_{b_\alpha}(A_\alpha,\|\cdot\|_\alpha;a_\alpha)$,

(ii) $\mathring{V}_b(A,P;a) = \bigcup_\alpha \mathring{V}_{b_\alpha}(A_\alpha,\|\cdot\|_\alpha;a_\alpha)$,

(iii) $\mathring{v}_b(A,p_\alpha;a) \le p_\alpha(ab)$,

(iv) $\mathring{v}_b(A,P;a) = \sup_\alpha \mathring{v}_b(A,p_\alpha;a) = \sup_\alpha \mathring{v}_{b_\alpha}(A_\alpha,\|\cdot\|_\alpha;a_\alpha)$.

Proof: For $p_\alpha \in P$ let N_α denote the null space of p_α (cf. Def. (I.21)). We observe that to each linear functional f on A which annihilates N_α, we can define the linear functional F on $(A_\alpha,\|\cdot\|_\alpha)$ by

$$F(x_\alpha) = f(x),$$

and to each linear functional F on (A,p_α) we can define the linear functional f on A by

$$f(x) = F(x_\alpha) .$$

(i) Let $f \in \mathring{D}(A,p_\alpha;b)$. Then for $b \in A$, we have

$$f(b) = F(b_\alpha) = p_\alpha(b) = \|b_\alpha\|_\alpha \quad \text{(cf. Def. (I.21))} .$$

Since $f \in \mathring{D}(A,p_\alpha;b)$, we have $|f(x)| \le p_\alpha(x)$, for all $x \in A$ and so $|F(x_\alpha)| \le \|x_\alpha\|_\alpha$, which implies that $\|F\|_\alpha \le 1$.

Further $|F(b_\alpha)| = \|b_\alpha\|_\alpha \leq \|F\|_\alpha \cdot \|b_\alpha\|_\alpha$, implies that $1 \geq \|F\|_\alpha$, provided $\|b_\alpha\|_\alpha \neq 0$. Hence $\|F\|_\alpha = 1$. So $\mathring{D}(A,p_\alpha;b)$ can be identified with $\mathring{D}(A_\alpha,\|\cdot\|_\alpha;b_\alpha)$ and for $a \in A$,

$$\mathring{V}_b(A,p_\alpha;a) = \mathring{V}_{b_\alpha}(A_\alpha,\|\cdot\|_\alpha;a_\alpha), \text{ proving (i)}.$$

(ii) From (i) above it follows that

$$\bigcup_\alpha\{\mathring{V}_b(A,p_\alpha;a): p_\alpha \in P\} = \bigcup_\alpha \mathring{V}_{b_\alpha}(A_\alpha,\|\cdot\|_\alpha;a_\alpha),$$

hence by Definition (IV.1) we have

$$\mathring{V}_b(A,P;a) = \bigcup_\alpha \mathring{V}_{b_\alpha}(A_\alpha,\|\cdot\|_\alpha;a_\alpha).$$

(iii) Suppose $\lambda \in \mathring{V}_b(A,p_\alpha;a)$ then for some $f \in \mathring{D}(A,p_\alpha;b)$, $\lambda = f(ab)$, implies that

$$|\lambda| \leq p_\alpha(ab),$$

and therefore

$$\sup\{|\lambda|: \lambda \in \mathring{V}_b(A,p_\alpha;a)\} \leq p_\alpha(ab),$$

that is

$$\mathring{v}_b(A,p_\alpha;a) \le p_\alpha(ab) .$$

(iv) $$\mathring{v}_b(A,P;a) = \sup\{|\lambda|: \lambda \in \mathring{V}_b(A,P;a)\}$$

$$= \sup\{|\lambda|: \lambda \in \bigcup_\alpha \mathring{V}_b(A,p_\alpha;a)\} \quad \text{(by (ii) above)}$$

$$= \sup_\alpha\{\sup\{|\lambda|: \lambda \in \mathring{V}_b(A,p_\alpha;a)\}\}$$

$$= \sup_\alpha \mathring{v}_b(A,p_\alpha;a)$$

$$= \sup_\alpha \mathring{v}_{b_\alpha}(A_\alpha,\|\cdot\|_\alpha;a_\alpha) \quad \text{(from (i) above).}$$

<u>IV.3 Corollary.</u> (i) $\mathring{v}_b(A,P;a) \le \sup_\alpha(p_\alpha(a)\cdot p_\alpha(b)) = \sup_\alpha(\|a_\alpha\|_\alpha \cdot \|b_\alpha\|_\alpha)$,

(ii) $\mathring{v}_b(A,p_\alpha;a) \le \mathring{v}_b(A,P;a)$.

<u>Proof:</u> (i) By Proposition (II.5) and Theorem (IV.2(iv)) we have

$$\sup_\alpha \mathring{v}_{b_\alpha}(A_\alpha,\|\cdot\|_\alpha;a_\alpha) = \mathring{v}_b(A,P;a) \le \sup_\alpha\|a_\alpha b_\alpha\|_\alpha \le \sup_\alpha(\|a_\alpha\|_\alpha \cdot \|b_\alpha\|_\alpha)$$

(ii) It follows clearly from Theorem (IV.2(iv)).

If the algebra has identity then it has been proved by Husain, Giles and Koehler [13] that boundedness of the numerical range of $a \in A$ implies that $\sup_\alpha p_\alpha(a) < \infty$. But we will establish the relationship between the boundedness of the relative numerical range of an element $a \in A$ and $\sup_\alpha p_\alpha(ab)$ in the following theorem:

IV.4 Theorem. Given A and $a,b \in A^*$ we have the following:

(i) If $\sup_{\alpha\in I} p_\alpha(ab) = M < \infty$, then $\overset{\circ}{V}_b(A,P;a)$ is bounded by M,

(ii) if $\overset{\circ}{V}_b(A,P;a)$ is bounded by a Constant M (>0) then

$\sup_{\alpha\in I} p_\alpha(ab) \le eM$, where e is the Constant exp.(1).

Proof: (i) Suppose $\sup_{\alpha\in I} p_\alpha(ab) = M < \infty$. Then by Theorem (IV.2(iii)) we have

$$\overset{\circ}{v}_b(A,P;a) \le \sup_{\alpha\in I} p_\alpha(ab) = M .$$

Hence $\overset{\circ}{V}_b(A,P;a)$ is a bounded set.

(ii) By Theorem (IV.2(ii)) we have

$$|\lambda| \le M ,$$

for all $\lambda \in \overset{\circ}{V}_{b_\alpha}(A_\alpha,\|\cdot\|_\alpha;a_\alpha)$.

This implies that $\overset{\circ}{v}_{b_\alpha}(a_\alpha) = \overset{\circ}{v}_{b_\alpha}(A_\alpha,\|\cdot\|_\alpha;a_\alpha) \le M$.

Since the numerical range of a_α is unaltered by replacing A_α by its completion, see [2(a)], p.17, hence by Theorem (I.45) it follows that

$$\overset{\circ}{v}_{b_\alpha}(a_\alpha) \ge \frac{\|a_\alpha b_\alpha\|_\alpha}{e} ,$$

whence by Definition (I.21), we have

$$p_\alpha(ab) = \|a_\alpha b_\alpha\|_\alpha \leq e\cdot M \quad .$$

This proves that

$$\sup_{\alpha\in I} p_\alpha(ab) \leq e\cdot M \quad .$$

IV.5 Remark. It is clear that the relative numerical range and relative numerical radius have the following properties:

For $a,b,x \in A$, $\lambda \in C$,

(i) $\mathring{V}_b(A,P;a+x) \subseteq \mathring{V}_b(A,P;a) + \mathring{V}_b(A,P;x)$,

(ii) $\mathring{v}_b(A,P;a+x) \leq \mathring{v}_b(A,P;a) + \mathring{v}_b(A,P;x)$,

(iii) $\mathring{v}_b(A,P;\lambda a) = |\lambda|\cdot\mathring{v}_b(A,P;a)$.

IV.6 Definition. Let A be a l.m.c.-algebra. Then the following set Γ is called the relative set for A, and is defined by

$$\Gamma = \{b\in A: \exists\alpha_0 , \text{ for all } p_\alpha \geq p_{\alpha_0} , \text{ for all } p_\beta \geq p_{\alpha_0} \Rightarrow p_\alpha(b) = p_\beta(b)\} \; .$$

IV.7 Examples of the relative set Γ for A.

(a) Consider the algebra $C^\infty[0,1]$ which is topologized by

$$p_m(f) = \sum_{j=0}^{m} \frac{\sup|f^{(j)}(x)|}{j!}$$

Then the relative set Γ for $C^\infty[0,1]$ is the set $P[0,1]$ of polynomials; which is dense in A.

(b) If we take $A = P[0,1]$ then $\Gamma = A$.

(c) Consider the algebra $A = \mathrm{span}\{e^{\alpha x} : \alpha \in \mathbb{C}\}$ which is a subalgebra of $C^\infty[0,1]$, then in this case,

$$\Gamma = \mathrm{span}\{1\}, \text{ because } A \cap P[0,1] = \mathrm{span}\{1\} .$$

(d) Let us consider the algebra $C_0^\infty[0,1]$, which has no identity and suppose that $A = \mathrm{span}\{e^{\alpha x} - e^{\beta x} : \alpha, \beta \in C\}$. Then the relative set Γ for A will be given by $\{0\}$, (this follows from (c) above).

(e) Let $A = C(X)$, where X is not a real compact space, see [11], p. 324, the seminorms are given by

$$p_K(f) = \sup_{x \in K} |f(x)| ,$$

where K is a compact subset of X, then $b \in \Gamma$ if and only if $|b|$ has a maximum in X.

<u>IV.8 Definition [14]</u>. Let $\{E_\alpha\}_{\alpha \in I}$ be a family of topological spaces, with I a directed set. Let $\{f_{\alpha\beta}\}$ be a family of mappings such that $f_{\alpha\beta} : E_\alpha \to E_\beta$ is continuous whenever $\alpha \le \beta$ and the following conditions hold:

(i) For $\alpha \le \beta \le \nu$, $f_{\alpha\nu} = f_{\beta\nu} \circ f_{\alpha\beta}$;

(ii) for each $\alpha \in I$, $f_{\alpha \cdot \alpha}$ is the identity. Then $(E_\alpha, I, f_{\beta\alpha})$ is called an inductive system.

If $(E_\alpha, I, f_{\beta\alpha})$ is an inductive system and let G be the subset of ΠE_α such that for all $\alpha \le \beta$, $f_\alpha(x_\beta) = x_\alpha$. Let G be endowed with the finest locally convex topology making each $f_\alpha : E_\alpha \to E$ continuous. Then G is called the inductive limit of $(E_\alpha, I, f_{\beta\alpha})$. The inductive limit E of $(E_\alpha, I, f_{\beta\alpha})$ is denoted by $E = \varinjlim E_\alpha$. We recall that each complete commutative l.m.c.-algebra is a projective limit of Banach algebras, see [18], p.20.

Now by means of the relative set Γ for a l.m.c.-algebra A, we are in a position to establish the following:

<u>IV.9 Theorem.</u> Let A be a l.m.c.-algebra and $b \in \Gamma$. Then we have

$$\mathring{D}(A,P;b) = \varinjlim \mathring{D}(A,p_\alpha;b) .$$

<u>Proof:</u> First we show that for $\alpha \le \beta$, $p_\alpha, p_\beta \in P$ such that $p_\alpha(b) = p_\beta(b)$ (because $b \in \Gamma$), we have

$$\mathring{D}(A,p_\alpha;b) \subseteq \mathring{D}(A,p_\beta;b) \qquad \text{(I)} .$$

Suppose that $f \in \mathring{D}(A,p_\alpha;b)$, then $f \in A'$ such that $f(b) = p_\alpha(b)$ and for all $x \in A$, $|f(x)| \le p_\alpha(x)$. Since for $\alpha \le \beta$ means $p_\alpha(x) \le p_\beta(x)$, for all $x \in A$ and $p_\alpha(b) = p_\beta(b)$ we have $f \in \mathring{D}(A,p_\beta;b)$.

Now $\{\mathring{D}(A,p_\alpha;b)\}_I$ is a family of subsets of A' indexed by I and $b \in \Gamma$ and (I) holds. Also for each $\alpha \in I$ we have $\mathring{D}(A,p_\alpha;b)$ is isomorphic to $\mathring{D}(A_\alpha, \|\cdot\|_\alpha; b_\alpha)$ (II), by Theorem (IV.2(i)), where A_α as in Definition (I.21).

Define for $\alpha \leq \beta$, $S_{\beta\alpha}: (A_\beta, \|\cdot\|_\beta) \longrightarrow (A_\alpha, \|\cdot\|_\alpha)$, by $S_{\beta\alpha}(a+N_\beta) = a+N_\alpha$, for all $a \in A$.

<u>$S_{\beta\alpha}$ is well-defined</u>: For $a_1, a_2 \in A$, if $a_1 + N_\beta = a_2 + N_\beta$, then

$$(a_1 + N_\alpha) - (a_2 + N_\alpha) = (a_1 - a_2) + N_\alpha \subseteq N_\alpha ,$$

because $a_1 - a_2 \in N_\beta$ and $p_\alpha \leq p_\beta$ implies that $N_\beta \subseteq N_\alpha$.

<u>$S_{\beta\alpha}$ is surjective</u>: obvious.

Now, since $S_{\beta\alpha}(a+N_\beta) = a+N_\alpha$, for all $a \in A$, we have

$$\begin{aligned} |S_{\beta\alpha}(a+N_\beta)| &= \|a+N_\alpha\|_\alpha \\ &= p_\alpha(a) , \quad \text{for all } a \in A , \\ &\leq p_\beta(a) = \|a+N_\beta\|_\beta , \end{aligned}$$

that is $\|S_{\beta\alpha}\| \leq 1$.

Suppose $\phi_\alpha \in \overset{\circ}{D}(A_\alpha, \|\cdot\|_\alpha; b_\alpha)$ then $\phi_\alpha(b_\alpha) = \|b_\alpha\|_\alpha = p_\alpha(b)$ and $|\phi_\alpha(x+N_\alpha)| \leq p_\alpha(x)$, (from (II) above), for all $x \in A$.

Now for $\alpha \leq \beta$, $\phi_\beta(x+N_\beta) = \phi_\alpha(x+N_\alpha)$, and

$$\begin{aligned} \phi_\beta(x+N_\beta) &= \phi_\alpha[S_{\beta\alpha}(x+N_\beta)] \\ &= (\phi_\alpha \cdot S_{\beta\alpha})(x+N_\beta) , \end{aligned}$$

so $\phi_\beta = \phi_\alpha \cdot S_{\beta\alpha} = (S_{\beta\alpha})^* \cdot \phi_\alpha$ (III) .

Now take $\mathring{i}_{\alpha\beta}$ to be the canonical injection:

$$\mathring{D}(A_\alpha, \|\cdot\|_\alpha; b_\alpha) \longrightarrow \mathring{D}(A_\beta, \|\cdot\|_\beta; b_\beta) .$$

Then from (III) it follows that

$$\phi_\beta = (S_{\beta\alpha})^* \cdot \phi_\alpha = \mathring{i}_{\alpha\beta}(\phi_\alpha) .$$

Hence $\xrightarrow{\lim} \mathring{D}(A, p_\alpha; b)$ is identified canonically with

$$\cup\{\mathring{D}(A, p_\alpha; b) : p_\alpha \in P\} ,$$

and the theorem follows from Definition (IV.1).

<u>IV.10 Corollary</u>. Let A be a l.m.c.-algebra and $b \in \Gamma$. Suppose $P, Q \in \mathcal{P}(A)$ such that $P \leq Q$. Then for $\alpha \leq \beta$ and $p_\alpha, p_\beta \in P$, we have the following:

(i) $\mathring{V}_b(A, p_\alpha; a) \subseteq \mathring{V}_b(A, p_\beta; a)$,

(ii) $\mathring{v}_b(A, p_\alpha; a) \leq \mathring{v}_b(A, p_\beta; a)$,

(iii) $\mathring{V}_b(A, P; a) \subseteq \mathring{V}_b(A, Q; a)$,

(iv) $\mathring{v}_b(A, P; a) \leq \mathring{v}_b(A, Q; a)$.

<u>Proof</u>: (i) It follows from (I) in the Theorem (IV.9) above and (ii) is a consequence of (i).

From (i) it follows that $\bigcup_{\alpha} \mathring{V}_b(A,p_\alpha;a) \subseteq \bigcup_{\beta} \mathring{V}_b(A,p_\beta;a)$;
And (iii) follows from Definition (IV.1) and by the hypothesis $P \leq Q$.
(iv) This follows clearly from (iii).

IV.11 Corollary. Let A be a l.m.c.-algebra such that for each $x \in A$ and each $\alpha \in I$, $p_\alpha(x) < \infty$. Suppose $p(x) = \sup_\alpha p_\alpha(x) < \infty$. Then for $a \in A$ and $b \in \Gamma$,

$$\mathring{V}_b(A,P;a) \subseteq \mathring{V}_b(A,p;a) \ .$$

Proof: The Corollary follows by Definition (IV.1) and Corollary (IV.10(i)) above.

IV.12 Theorem. Suppose A is a commutative l.m.c.-algebra. Let $b \in A$ such that $ab^2 = ab$, for all $a \in A$ and denote by B the subalgebra generated by b. Then

$$\mathring{V}_b(A,P;a) = \mathring{V}_b(B,P;ab) \ .$$

Proof: Let $\lambda \in \mathring{V}_b(A,P;a)$ then for some $f \in \mathring{D}(A,P;b)$, we have

$$\lambda = f(ab) \ .$$

But $f \in \mathring{D}(A,P;b)$ implies that there is α such that $f(b) = p_\alpha(b)$ and for all $x \in A$, $|f(x)| \leq p_\alpha(x)$. Put $g = f|_B$ then $g(ab) = f(ab) = p_\alpha(ab)$ and $|g(y)| = |f(y)| \leq p_\alpha(y)$, for all $y \in B$.

Hence, $g(ab^2) = f(ab^2) = f(ab) = \lambda$, so $\lambda \in V_b(B,P;ab)$.

The reverse inclusion follows from Theorem (I.6), since B is a subalgebra of A .

§2. Generalized Numerical range of elements of LMC-algebras.

In this section we define the generalized numerical range of an element of a l.m.c.-algebra and study some of its properties. The results of this section are true whether or not the algebra has unit.

IV.13 Definition. Let A be a l.m.c.-algebra. Given (A,P) , $P \in \rho(A)$, we define the set

$$D_{(A,p_\alpha)}(b) = \{f \in A' : f(b) = 1 = p_\alpha(b) \text{ and } |f(x)| \le p_\alpha(x) \text{ for all } x \in A\} ,$$

and we write,

$$D_{(A,P)}(b) = \bigcup_\alpha \left\{ D_{(A,p_\alpha)}(b) : p_\alpha \in P \right\} .$$

For each $a \in A$, we write

$$V_b(A,p_\alpha;a) = \left\{ f(ab) : f \in D_{(A,p_\alpha)}(b) \right\} ,$$

and define the generalized numerical range of a as the set,

$$V_{(A,P)}(a) = \bigcup_\alpha \left\{ V_b(A,p_\alpha;a) : p_\alpha \in P \right\} .$$

For each $a \in A$ the generalized numerical radius of $a \in A$ is defined as the number

$$v_{(A,P)}(a) = \sup\left\{|\lambda| : \lambda \in V_{(A,P)}(a)\right\} .$$

IV.14 Theorem. For each $a \in A$ we have the following:

(i) $V_{(A,P)}(a) = \bigcup_{\alpha}\left\{V_{(A_\alpha,\|\cdot\|_\alpha)}(a_\alpha)\right\}$.

(ii) $v_{(A,P)}(a) = \sup_{\alpha} v_{(A_\alpha,\|\cdot\|_\alpha)}(a_\alpha)$.

Proof: Same as in Theorem (IV.2).

IV.15 Definition. (a) A submultiplicative seminorm p on a locally m-convex algebra is called regular if $p(a) = \sup_{p(x)\le 1} p(ax)$.

(b) A l.m.c.-algebra A is called a regular l.m.c.-algebra if there exists a family $P \in \mathcal{P}(A)$ of submultiplicative regular seminorms defining the topology of A .

An approximate identity $\left\{e_\beta\right\}_{\beta\in I}$ in a l.m.c.-algebra is said to be uniformly bounded if $\sup_{\alpha} \sup_{\beta} p_\alpha(e_\beta) \le 1$, $p_\alpha \in P$.

It is easy to see that if a l.m.c.-algebra (A,P) has a uniformly bounded approximate identity then each seminorm p_α is regular. Also if A has identity e such that $\sup_{\alpha} p_\alpha(e) \le 1$, then each $p_\alpha \in P$ is a regular seminorm.

If the algebra A is a regular l.m.c.-algebra then we can establish a relationship between the numerical range of an element $a \in A^+$ and the generalized numerical range of the same element a in A . This relationship can be seen in the following theorem:

<u>IV.16 Theorem</u>. Let A be a regular complete commutative l.m.c.-algebra. Then for all $a \in A$, $V(A^+,P;a) \subseteq \bar{C}o\, V_{(A,P)}(a)$.

<u>Proof</u>: From Theorem (IV.14(i)) it follows that

$$V(A^+,P;a) = \bigcup_\alpha V(A_\alpha^+, \|\cdot\|_\alpha; a_\alpha) .$$

Also by Theorem (III.3) we have

$$\overline{V(A_\alpha^+, \|\cdot\|_\alpha; a_\alpha)} = \bar{C}o\, V_{(A_\alpha, \|\cdot\|_\alpha)}(a_\alpha) .$$

Hence

$$\begin{aligned} V(A^+,P;a) &= \bigcup_\alpha \left[\bar{C}o\, V_{(A_\alpha,\|\cdot\|_\alpha)}(a_\alpha)\right] \\ &\subseteq \bar{C}o \left[\bigcup_\alpha V_{(A_\alpha,\|\cdot\|_\alpha)}(a_\alpha)\right] \\ &= \bar{C}o\, V_{(A,P)}(a) , \end{aligned}$$

by Theorem (IV.14(i)).

<u>IV.17 Theorem</u>. Let A be a regular complete commutative l.m.c.-algebra. Then for all $a \in A$, we have the following:

(i) $v_{(A,P)}(a) \leq \sup_\alpha p_\alpha(a)$,

(ii) $Sp_A(a) \subseteq \bar{C}o\, V_{(A,P)}(a)$,

(iii) $\bigcup_{\alpha} Sp_{A_\alpha}(a_\alpha) \subseteq \overline{Co}\, V_{(A,P)}(a)$.

<u>Proof</u>: (i) Since $v_{A_\alpha}(a_\alpha) \leq \|a_\alpha\|_\alpha$, (by Theorem (I.45)), therefore by Definition (I.21) we have

$$v_{A_\alpha}(a_\alpha) \leq p_\alpha(a) .$$

Hence

$$\sup_\alpha v_{A_\alpha}(a_\alpha) \leq \sup_\alpha p_\alpha(a) ,$$

and by Theorem (IV.14(ii)) it follows that

$$v_{(A,P)}(a) \leq \sup_\alpha p_\alpha(a) ,$$

which proves (i).

(ii) From Theorem (I.24) we have

$$Sp_A(a) = Sp_{A^+}(a) ,$$

and by [13] we know that $Sp_{A^+}(a) \subseteq V(A^+,P;a)$. Hence by Theorem (IV.16) it follows that

$$Sp_A(a) \subseteq \overline{Co}\, V_{(A,P)}(a) ,$$

proving (ii).

(iii) From Theorems (I.24) and (I.25) we have

$$Sp_A(a) = \bigcup_\alpha Sp_{A_\alpha^+}(a_\alpha) ,$$

and hence from (ii) above we have,

$$\bigcup_\alpha Sp_{A_\alpha^+}(a_\alpha) \subseteq \bar{C}o\, V_{(A,P)}(a) .$$

Thus (iii) follows from Theorem (I.24).

<u>IV.18 Theorem</u>. Let A be as in Theorem (IV.17). Then for each normal element (cf. Def. (I.50)) a_α in A_α, $\alpha \in I$, we have the following:

(i) $\bigcup_\alpha V(A_\alpha^+, \|\cdot\|_\alpha; a_\alpha^2) \subset Co\{\lambda^2 : \lambda \in \bigcup_\alpha V(A_\alpha^+, \|\cdot\|_\alpha; a_\alpha)\}$,

(ii) $V(A^+, P; a^2) = \bar{C}o\{\lambda^2 : \lambda \in \bar{C}o\, V_{(A,P)}(a)\}$.

<u>Proof</u>: (i) Since we know by [13]

$$Sp_{A^+}(a) \subseteq V(A^+, P; a) ,$$

which implies that

$$Co(Sp_{A^+}(a)) \subseteq V(A^+, P; a) .$$

Therefore by Theorem (IV.14(i)) it follows that

$$\mathrm{Co}(\mathrm{Sp}_{A^+}(a)) \subseteq \bigcup_\alpha V(A_\alpha^+, \|\cdot\|_\alpha; a_\alpha) \ .$$

Since each a_α is normal then by Theorem (I.51) we get.

$$\bigcup_\alpha \mathrm{Co}\left(\mathrm{Sp}_{A_\alpha^+}(a_\alpha)\right) = \bigcup_\alpha V(A_\alpha^+, \|\cdot\|_\alpha; a_\alpha) \ ,$$

and consequently by Theorem (I.40) we have

$$\begin{aligned} \bigcup_\alpha V(A_\alpha^+, \|\cdot\|_\alpha; a^2) &\subset \bigcup_\alpha \mathrm{Co}\{\lambda^2 : \lambda \in V(A_\alpha^+, \|\cdot\|_\alpha; a_\alpha)\} \\ &\subset \mathrm{Co}\{\lambda^2 : \lambda \in \bigcup_\alpha V(A_\alpha^+, \|\cdot\|_\alpha; a_\alpha)\} \ , \end{aligned}$$

proving (i).

(ii) follows from Theorems (IV.14(i)) and (IV.16).

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