8-1-1975

A Study on the Problem of Optimum Economic Growth

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by

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A Thesis

Submitted to the School of Graduate Studies

in Partial Fulfilment of the Requirements

for the Degree

Master of Arts

McMaster University

August 1975

MASTER OF ARTS (1975)
(Economics)

McMASTER UNIVERSITY
Hamilton, Ontario

TITLE: A Study on the Problem of Optimum Economic Growth

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SUPERVISOR: Professor D. W. Butterfield

NUMBER OF PAGES: 61, v

ABSTRACT

The theory of optimum economic growth has centred around the 1928 paper of Ramsey and extensively developed by subsequent authors. Samuelson and Solow extended Ramsey's analysis to a world involving multiple capital goods. Following Ramsey's formulation of his problem in terms of constrained maximization of an integral over infinite time, Tinbergen, Koopmans, Cass, Weizacker and Mirrlees worked in an infinite time horizon, allowing for certain modifications. Chakravarty pointed out that the integral need not converge even if the policy proposed by Ramsey (as being optimal) were adopted. Since there is considerable difficulty in demonstrating convergence in an infinite time horizon, Chakravarty and Goodwin tackled this problem in a finite time horizon.

In our thesis, we are concerned with the problem of investigating the existence of an optimum savings programme in a finite time horizon. We provide a rigorous proof of the existence of such an optimum savings programme. We also demonstrate the uniqueness of the optimal programme. Furthermore, we have given a rigorous characterization of an optimal savings programme as being efficient. Rigorous proof of uniqueness of an optimal savings programme and the property that it is efficient, have nowhere appeared in the literature either in the context of an infinite or in that of a finite time horizon model.

ACKNOWLEDGMENTS

I would like to express my gratitude to Professor D. W. Butterfield for his comments and encouragements during the preparation of this work.

I wish to thank Professor S. Ahmed and Professor S. Mestelman for their constructive criticisms and helpful comments.

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INTRODUCTION

Recent years have seen a great deal of activity in the construction of planning models designed to aid responsible political authorities to eradicate permanent poverty which is widely prevalent among the majority of the people of almost all underdeveloped countries. It is well to recognize that economic growth is a brutal, sordid process. There are no short-cuts leading to a change in the sub-human level of living of the masses. The essence of it lies in making the labourer produce more than he is allowed to consume for his immediate needs, and to invest and re-invest the savings thus obtained. In the course of practical planning in underdeveloped countries, there arises, at some point, the question of what the rate of saving should be. There are several possible approaches to answering this question, and we will consider the answer following Ramsey's classic paper of 1928. Although this paper created the subject of optimum economic growth, it has only recently been developed to indicate the insight into the problem of planning capital investment decisions, especially for economies where the basic bottleneck relates to capital.

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[1] F. P. Ramsey, "A Mathematical Theory of Saving," Economic Journal (1928), Vol. 38, pp. 543-559.

The main conflict involved in the choice of the rate of saving is one between present consumption and future consumption, and naturally it cannot be solved without some intertemporal value judgements. Easily the most widely used method of solution of the intertemporal allocation problem is to employ a utility function, in the theory of optimum growth. The main strength of the utility maximization approach lies in the acceptability of the concept of diminishing marginal utility of increasing consumption. It is a widely observed phenomenon that we seem to care less for a marginal unit of consumption when we are rich than when we are poor, and this provides a good common-sense ground against having too high or too low a rate of saving, leading to an enormous inequality between the present and the future.

If consumption at any single instant of time is assumed to be a different commodity from consumption at any other instant then the intertemporal utility function becomes a functional, where time is treated as a continuous variable. A functional defines a real number for any given function defined over a domain. The optimum savings programme may have either a finite or an infinite time horizon and, given the utility functional, our problem in optimum economic growth is to find the path for which $\int_0^T U dt$ is a maximum, where T may be a fixed end-point or a variable end-point (i.e., $T \to \infty$).

One of the most crucial variables in the theory of planning over time is the length of the planning period. On the national plane, unless extinction is a very likely possibility, all planning models should be constructed on the basis of an infinite time horizon, because one cannot assume that the world comes to an end at a finite time period T. When the planning horizon is extended to infinity, several conceptual difficulties arise.[1] This can be seen very simply. Let $U(C_1), \ldots\ldots\ldots\ldots, U(C_n)$, be the sequence of utility levels, corresponding to the consumption stream $C_1, \ldots\ldots\ldots, C_n$. Making the (classic) assumption of additive separability, the integral $\int_0^{\infty} U\,dt$ may be an unbounded number since every periods consumption will contribute positively to aggregate welfare over the entire period. In this situation there is no possibility of introducing any order on the policy space through each mapping from the policy to the utility space. Hence, except in the sense of point-wise dominance, no functional is defined that can help us to compare alternative programmes. Point-wise dominance is a special case, and cannot be assumed on an a-priori basis.

The moment we introduce a finite time horizon, we are at once relieved of the problem of convergence; but simultaneously, we face the choice problem. Suppose that we plan only for period T but do not assume that the world comes to an end after T. Then, clearly, if some consumption is to take place beyond T, we must leave some capital at T for the sake of the future. Thus any such finite horizon model must postulate some terminal capital stock, as it is the only way in which the well-being of the generations living beyond the horizon T can be taken into account. Thus, our concern is essentially one that transcends the requirement of a single specific horizon T. The same argument would apply if, instead of

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1 S.Chakravarty, "The Existence of an Optimum Savings Programme", Econometrica. 30. (1962).

T, any other horizon T*>T were used. Hence the argument for using an infinite horizon is logically very compelling[1]. A planning model that uses a planning horizon of a finite number of years but makes provision for terminal capital stock is a surrogate for an infinite horizon model.

The finite horizon model raises important problem of defining an order over course of actions. Even when the issue of defining an order is settled, there are also important questions connected with determining whether an optimal mode of action exists. The finite horizon model approach, involving a terminal capital stock, is both conceptually and computationally simpler than working with an infinite horizon model. But the choice of a set of terminal conditions is very far from being a trivial problem. In other words the appropriate choice of a terminal stock of capital and the length of the time horizon are crucial in making the finite time horizon model a surrogate for an infinite horizon model, which is logically compelling in the theory of national planning. In this connection it is to be noted that according to Arrow,[2] any choice of a time horizon and of the terminal capital stock is bound to be arbitrary because of the impossibility of deriving a complete social ordering based on an aggregation of individual orderings.

It is a fact that, in actuality, people discriminate between earlier and later occurrences of consumption. The concept of a psychological

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1 S.Chakravarty, Capital and Development Planning, The M.I.T. Press, Cambridge, Massachusetts, 1969, pp 20.

2 K.J.Arrow, "Application of Control Theory to Economic Growth" in Mathematics of the decision Sciences, Part 2 , edited by G.B.Dantzig, and others, American Mathematical Society, 1968.

discontent of the future is of respectable antiquity. A distant object "looks" smaller, and we tend to value , it is claimed, a unit of consumption in the future less than we value the same now. If this difference occurs because of the distance in time, then the position is symmetrical. A future object looks less important now and, similarly, a present object will look less important in the future.

Time preference can be defined in a variety of ways. One of the ways is in terms of the asymmetry of the indifference curves between consumption in successive time periods, along the 45^0 line through the origin. This means that even in the absence of uncertainty and on the assumption that commodities are the same in the present and in the future, there may be an implied systematic bias against future consumption. While it is true that the decision has to be taken now there is no necessary reason why today's discount of tommorow should be used, and not tommorow's discount of today., This time bias may be called an expression of time preference .

The element of time discount might be significant for social choice: "one of the reasons for prefering a unit of present consumption to the same in the future is the uncertainty associated with the future. This might arise for resons other than the possibility of death of the present consumers. Now to a certain extent this uncertainty (say about production) is present even for the society, and if an individual discounts the future yields because he does not know whether these yields will be obtained the same argument may apply in the case of the society as well. It should, however , be added that : (a) the uncertainty facing an individual is not the same as that facing the society as a whole, and (b) the individual assessment of the uncertainty might be wrong because he does not know how other individuals are acting. Thus, this partial justification of a time discount is not the same as justifying the use of the individual's "pure" time discount in the social optimization problems." (Sen)[1]

The problem of aggregating time-preference maps of individuals for collective decisions into a single social time-preference map is a special case of the general problem of aggregating individual utility functions into a social welfare function. The more general problem has been investigated by Kenneth Arrow[2] and others, and Arrow's negative conclusion that "democratic" aggregation is impossible unless we restrict the allowable class of individual preference functions or abandon one or more intuitively appealing axioms about preferences is too familiar to require elaboration. And it does not help to recognize that in a modern state, no matter how democratic, collective decisions are taken by a relatively small number of policy-makers exercising proxies granted directly or indirectly by the community. For the proxy preference orderings of different policy-makers must nevertheless be aggregated.

In the literature of optimum growth, the extent of time preference is denoted by a single number such as a percentage rate of discount to be applied to the utility of future consumption. Assuming time preference at the

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[1] A.K.Sen, " On Optimizing the Rate of Saving ". Economic Journal, 71, (1961).

[2] K.J.Arrow, Social Choice and Individual Values, Second Edition, John Wiley and Sons, New York, 1964.

rate ρ, the total utility for the period [0,T] is $\int_0^T Ue^{-\rho t}dt$. This, of course, implies an additively separable functional form. Assuming an additively separable form, the fundamental question here is whether there is any valid ground for assuming a positive discount rate while formulating an intertemporal utility function for the society as whole. The opinions so far advanced in the literature have been classified by Chakravarty[1] into three groups. The Pigou-Ramsey[2] point of view against the assumption of a positive discount rate is that it is ethically inappropriate to discount future satisfaction just because it takes place in the future. The subsequent writers[3] contested this view and argued that a government functioning democratically should take into account the wishes of the people it represents. This represents a much more intricate problem of knowing how high this discount rate ought to be. It is to be noted that, in both cases, the nature and extent of time preference is introduced by way of an explicit postulate of behavioural patterns. In recent investigations starting with a set of postulates about utility functions, which have no explicit reference to time preference, Koopmans, Diamond, and Williamson have shown that the complete preference orderings do exibit what we have earlier defined as time preference[4]. Their basic assumption is that 'a continuous utility

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[1] S.Chakravarty, Ibid., p. 35

[2] A.C.Pigou,Economics of Welfare , London, Macmillan, 1952; F.P.Ramsey, Ibid.

[3] S.A.Marglin. "The Social Rate of Discount and the Optimal Rate of Investment", Quaterly Journal of Economics, 77, (1965).

[4] S.Chakravarty, Ibid., p. 37

function exists on the space of consumptions streams extending to infinity![1] Koopmans has shown that if we postulate the existance of a continuous functional displaying sensitivity with respect to changes in first period consumption, and if we also assume limited noncomplementarity and stationarity, we cannot reject time preference without involving ourselves in a logical contradiction. In the opinion of Chakravarty,'depending on the circumstances pertaining to technology, preferences or the nature of primary factor availabilities, even an incomplete ordering may do the job of isolating the optimal mode of action'.[2]

After introducing a positive rate of discount, the problem of optimum growth becomes one of maximizing an integral of discounted utilities of instantaneous consumption euther for the period $[0,T]$ or $[o,\infty]$. The solution to this problem in the context of an infinite time horizon is well established in the literature. For an infinite time horizon model, Chakravarty has made an attempt to show the existence of a solution to the problem. But a formal and rigorous proof of existence does not appear in the literature. In this study we have investigated the problem of establishing the existence of a solution to the problem of optimum growth in a rigorous manner. Subsequently, we have shown that the optimum growth path is unique and efficent. The notion of efficiency, in the context of optimum growth is completely new in the literature.

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[1] P.A.Diamond, T.C.Koopmans, and B.R.Willamson, "Stationarity, Utility and Time Perspective", Econometrica, 32, 1964.

The first part of the study reviews the literature which has appeared since 1928 on the problem in the context of a one-commodity model with no uncertainty. The second part is devoted to demonstrating the existence, uniqueness, and efficiency of an optimum programme of accumulation of capital in a finite horizon planning model.

PART 1

SURVEY OF LITERATURE

The discussion of the problem providing a theory of optimum economic growth centred around the 1928 paper of Ramsey.[1] The problem is one of determining the proportion in which net capital formation should be divided between capital goods and consumer goods in the economy. What does the word "should" mean? For what target "should" a programme be defined ? The usual answer to this question is to relate welfare to consumption (aggregate or percapita) so that optimal development is defined as maximizing an increasing function of consumption; by doing so, one can determine the optimal policy. At the behest of Keynes, F.P.Ramsey devoted himself to investigating the problem of optimal savings and taxation, and the present day problem of the theory of optimum growth appeared in the literature as the theory of optimum savings. It is to be noted that the theory of optimum savings, capital accumulation and growth investigated the same class of problem

Until 1950 (prior to Tinbergen's contribution)[2], there is no other literature in this area. Perhaps interest in the topic lapsed because of the Great Depression and the War. Although their approaches differed,

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[1] F.P.Ramsey, "A Mathematical Theory of Saving". Economic Journal, 1928.

[2] J.Tinbergen,"The Optimum Rate of Saving". Economic Journal, 66, 1956.

both Ramsey and Tinbergen investigated the problem in an infinite time horizon with a single homogeneous capital good. Since the real world involves a great variety of heterogeneous capital goods, Samuelson and Solow,[1] almost at the same time, generalized the Ramsey model to any number of capital goods. Because of the paradoxes of infinity, Tinbergen reformulated his earlier model[2] in 1960. Since there is considerable difficulty in demonstrating convergence in an infinite time horizon, Chakravarty[3] and Goodwin[4] tackled the problem in a finite time horizon. At the same time Srinivasan[5] investigated the problem in a two sector economy.

Although there is a problem of convergence in an infinite time horizon, this approach is logical because it is not possible, on the national plane, to choose a particular cut-off point and at the same time avoid being arbitrary, without explicitly introducing uncertainty. Koopmans[6] discovered that the infinite horizon formulation, contrary to some people's expectation, may really describe the immidiate future more

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[1] P.A.Samuelson, and R.M.Solow, "A Complete Capital Model Involving Heterogeneous Capital Goods". Quaterly Journal of Economics, 70, 1956.

[2] J.Tinbergen, "Optimum Savings and Utility Maximization over Time". Econometrica, 28, 1960.

[3] S.Chakravarty, "Optimal Savings with a Finite Planning Horizon". International Economic Review, 3, 1962.

[4] R.M.Goodwin, "The Optimal Growth Path for an Underdeveloped Economy". Economic Journal, 71,1961.

[5] T.N.Srinivasan, "Optimal Savings in a Two Sector Model". Econometrica. 32, 1962.

[6] T.C.Koopmans, "On the Concept of Optimal Economic Growth". In The Econometric Approach to Development, Skokie, Ill., Rand McNally & Co. 1966

than it does the infinite future. This finding can be expressed as follows: "One is guiding a ship on a long journey by keeping it lined up with a point on the horizon even though one knows that long before that point is reached the weather will change (but in an unpredictable way) and it will be necessary to pick up a new course with a new reference point, again on the horizon rather than just a short distance ahead".[1]

Thus, there was an enthusiastic revival of the problem in an infinite time horizon and it was solved by Koopmans,[2] Cass,[3] Weizsacker,[4] and Mirrlees[5] in the 1960s. In this part, we shall attempt to point out important results obtained in the literature. First, we shall consider the problem posed by Ramsey and Tinbergen[6] in infinite time horizon in some detail. Next, following the historical sequence of the literature, we shall discuss the problem in a finite time horizon considering the work of Goodwin and Chakravarty.[7] Finally, the cardinal aspect of the enquiry resulting from the revival of the problem in an infinite time horizon will be discussed.

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[1] D.Gale, "On Optimal Development in a Multisector Economy". Review of Economic Studies, 34, 1967.

[2] T.C.Koopmans, Ibid.

[3] D.Cass, "Optimum Growth in an Aggregate Model of Capital Accumulation". Review of Economic Studies, 32, 1965.

[4] C.C.von Weizsacker, "Existence of Optimal Program of Accumulation for an Infinite Time Horizon". Review of Economic Studies, 32, 1965.

[5] J.A.Mirrlees, "Optimum Growth When Technology is Changing". Review of Economic Studies, 34, 1967.

[6] F.P.Ramsey, Ibid. ; and J.Tinbergen, Ibid.

[7] R.M.Goodwin, Ibid. ; and S. Chakravarty, Ibid.

An Infinite Time Horizon: (a) Ramsey Model

Frank Ramsey considered the problem of optimum savings systematically by maximizing an integral of instantaneous utility functions for an infinite time period in an economy which produces only one commodity with homogeneous capital and labour; and where population is stationary. That is, his problem was to maximize $\int_0^\infty U(C_t)\, dt$, where C_t is consumption at time t, assumed to increase monotonically with time. Ramsey avoided the problem of convergence by assuming that the instantaneous utility functions are all bounded from above. He used the term 'Bliss', which he defined as the maximum obtainable rate of enjoyment or utility. Assuming diminishing marginal utility, he argued that as consumption increases over time, the utility associated with the level of consumption increases to a maximum point; he represented bliss by B. So he seeks to maximize

$$\int_0^\infty [U(C_t) - B]\, dt.$$

Here we must distinguish between two cases: (i) B is achieved for a finite level of consumption (as a result of production limitations, because of resourse constraints); or(ii) B is reached only when C is infinitely large (because of the assumption of the law of diminishing marginal utility).

Now consider whether $\int_0^\infty [\, U(Ct) - B]\, dt$ can define a functional at all. Note that both $\int_0^\infty U(C_t)\, dt$ and $\int_0^\infty B\, dt$ are unbounded. Does the integral of the difference between the two define a mapping, discriminating among alternative consumption programmes? Considering case (i), we assume that there exists a finite consumption level $\overline{C}$ that can be reached by following a policy that is permitted by technology and initial conditions for any length of time t*. Then $\int_{t^*}^{\infty}[U(C)-B]\,dt = 0$ and the integral $\int_0^{t^*}[U(C)-B]\,dt$ will be a finite number. Thus for the consumption programmes which reach $\bar{\bar{C}}$ for some finite value of t, $\int_0^{\infty}[U(C)-B]\,dt$ defines a finite functional. For case (ii), we must make some assumption regarding the speed of convergence to the bliss level. Assuming bliss, to get an optimum rate of saving, Ramsey's problem becomes one of minimizing

$$\int_0^{\infty}[B-U(C_t)]\,dt \,.$$

Ramsey worked with two factors, homogeneous capital and labour. Since the growth rate of labour is assumed to be zero , labour is excluded from the mainstream of the analysis. He considered the production relation which is neoclassical in nature, denoted by,

$$Y = F(K, L),$$

where K and L are the capital stock and labour respectively. In the absence of labour, the savings-investment equality for all time will ensure ordinary equilibrium in the one commodity model. Using the savings-investment equality, our problem can be written as,

$$\min.\int\{B - U(f(K, L)-\dot{K})\}\,dt, \qquad \dot{K}=dK/dt$$

where $B=\lim_{t\to\infty} U(C_t)$. This can be solved with the help of the calculus of variations. Using Euler's equation, we get,

$$\left(\frac{\partial f}{\partial K}\right)_t = -\frac{\frac{du'}{dt}}{U'},$$

which states that the rate of interest is equal to the rate at which marginal utility is diminishing. Samuelson holds the opinion that perhaps this is the most correct theory of the rate of interest in a world of homogeneous capital.[1] This is Ramsey rule 1.

Now to get Ramsey rule 2, we will employ the second Euler equation. Since U is dependent on t, we know that the Euler equation is always integrable in the form:

$$U - \dot{K} U_{\dot{K}} = \mu,$$

where μ is any arbitrary constant. Furthermore,[2] the necessary boundary condition at infinity is that:

$$\text{as } t \to \infty, \quad U - \dot{K} U_{\dot{K}} \to 0.$$

Since μ is an arbitrary constant, the boundary condition at infinity implies that $\mu = 0$. From the above equation, we obtain the basic relationship:

$$\dot{K} = \frac{B - U}{U'} \quad \text{for all } t .$$

This is the principle of optimum savings due to Ramsey. "The rate of saving multiplied by the marginal utility of money should always be equal to the amount by which the net rate of enjoyment falls short of the maximum possible rate of enjoyment".[3] The optimum amount of savings is given by the excess of the level of utility enjoyed at bliss over the utility of current consumption, divided by the marginal utility of consumption. The assumption of diminishing marginal utility satisfies

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[1] P.A.Samuelson, and R.M.Solow, Ibid.

[2] S,Chakravarty, Capital and Development Planning, M.I.T. Press. Cambridge, Mass., 1969, p. 81.

[3] F.P.Ramsey, Ibid.

the second order condition which is known as Legendre's condition.

The Ramsey model holds good if we accept the notion of bliss. The relevant question is whether the level B is attainable for a finite level of consumption or whether there is merely an asymptotic approach towards B. In the first case, the functional $\int_0^\infty [B - U(C_t)]\, dt$ defines a meaningful order and we may work out the optimal savings programme in the light of the order. But in the second situation, where bliss is approached asymptotically, we may still run into difficulties because the functional may not define a non-discriminating mapping from the policy to the utility space.[1] On the whole, the assumption of bliss, although mathematically helpful, is not economically meaningful, because non-availability of resources and non-appearence of new commodities, which are the underlying assumptions of production bliss and utility bliss respectively cannot be accepted.

Considering Ramsey's problem as a variable end-point problem in the (t, K) plane. Samuelson and Solow[2] proved Ramsey principle in a different way. Suppose $\frac{dK}{dt} > 0$ for all t; then the integral:

$$\int_0^\infty U[F(K, \bar{L}) - \dot{K}]\, dt,$$

can be replaced by the equivalent integral:

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1 S. Chakravarty, "The Existence of an Optimum Savings Programme". Econometrica, 30, 1962.

2 P.A.Samuelson, and R.M.Solow, Ibid.

$$\int_{K(0)}^{K^*} \frac{U[F(K, \bar{L}) - \dot{K}]}{K}\, dK .$$

Here, the upper limit of integration is now fixed at K^*, where $\dot{K} = 0$. It is to be noted that , due to the choice of units, we now have $B = 0$. Defining $F(K, \bar{L}) = f(k)$, we have the equivalent integral:

$$\int_{K(0)}^{K^*} \left(\frac{U(C)}{f(k) - C} \right) dK .$$

Using the Euler equation, we obtain,

$$\frac{\partial}{\partial C} \left(\frac{U}{f(k) - C} \right) = 0$$

This would give us the Ramsey rule again. In this connection we can make a note of the following important implication of their analysis: "Even though there is no such thing as a single abstract capital substance that transmutes itself from one machine form to another like a restless reincarnating soul, the rigorous investigation of a heterogeneous capital-goods model shows that over extended periods of time an economic society can in a perfectly straightforward way reconstruct the composition of the diverse capital-goods so that there may remain great heuristic value in the simpler J. B. Clark - Ramsey models of abstract capital substance".[1]

[1] P.A.Samuelson, and R.M.Solow, "A Complete Capital Model Involving Heterogeneous Capital Goods". Quaterly Journal of Economics, 70, 1956.

AN INFINITE TIME HORIZON: (b) TINBERGEN - I

Tinbergen [40] does not adopt the specific Ramsey assumption of "finite bliss", defined as a maximum conceivable state of satisfaction, but is essentially concerned with discovering the policy implications of a one-commodity capital model by using economically tested utility and production functions. His first paper [40] was somewhat restricted in scope. He was concerned with finding a savings ratio which would be optimal for all future years, given the utility and production functions, and the initial endowment of capital. He also assumed a subjective rate of time preference, independent of diminishing utility or uncertainty. In his second paper [41], his problem was one of maximising the integral of discounted utility over time with respect to the savings ratio. This is a very restricted problem as it considers only programs with fixed savings ratios. But, what is optimal among this class of programs would not be optimal in the sense of maximising an integral of discounted utilities over time. Properly formulated, this problem is one in the variational calculus. This is what Tinbergen does in his second article.

In the second article, he starts with the marginal utility function of the form

$$U'_t = (C_t - \bar{C})^{-\nu},$$

where U'_t is the marginal utility in period t, C_t is the consumption in t, $\bar{C}$ is the subsistence consumption, and ν is the elasticity of marginal utility with respect to surplus consumption $(C_t - \bar{C})$. This can be shown as follows:

$$\frac{\dfrac{dU'_t}{d(C_t-\bar{C})}}{\dfrac{U'_t}{(C_t-\bar{C})}} = \frac{-\nu(C_t - \bar{C})^{-\nu-1}}{(C_t - \bar{C})^{-\nu-1}} = -\nu.$$

From statistical estimation, Tinbergen observed that $\nu < 1$, which is of crucial importance in Tinbergen's model. From this marginal utility function, we can find the total utility function:

$$U_t = \frac{(C_t - \bar{C})^{1-\nu}}{1 - \nu}.$$

An important point to be noted here is that Ramsey did not introduce a precise mathematical utility function. Tinbergen assumes a production function of the Harrod-Domar type:

$$Y_t = \frac{K_t}{\alpha}.$$

This implies either that factors other than capital are freely available or that their influence is already reflected in the coefficient α. Because savings equals investment, we have:

$$I_t = S_t,$$

or

$$\dot{K}_t = \frac{K_t}{\alpha} - C_t,$$

or

$$C_t = \frac{K_t}{\alpha} - \dot{K}_t.$$

With these in mind, our problem is to find:

$$\max\int_0^\infty \frac{(C_t - \bar{C})^{1-\nu}}{1 - \nu}\, dt,$$

subject to the constraint,

$$C_t = \frac{K_t}{\alpha} - \dot{K}_t .$$

Using Euler's first order condition, we obtain:

$$\frac{dC}{dt} = \frac{1}{\alpha\nu} C_t - \frac{1}{\alpha\nu}\bar{C},$$

and the solution of this non-homogeneous first order differential equation is:

$$C_t = A\ e^{\frac{1}{\alpha\nu}t} + \bar{C} .$$

This is the optimum consumption path. Since there is a one-to-one correspondence between savings and consumption programs corresponding to this optimum consumption path, we have an optimum savings program. The optimum path of capital accumulation can be obtained by putting the value of consumption in

$$\dot{K}_t = \frac{K_t}{\alpha} - C_t ,$$

and we get:

$$\dot{K}_t = \frac{K_t}{\alpha} - A\ e^{\frac{1}{\alpha\nu}t} - \bar{C} .$$

The solution of this first order non-homogeneous differential equation is:

$$K_t = B_1\ e^{\frac{1}{\alpha}t} + B_2\ e^{\frac{1}{\alpha\nu}t} + \alpha\bar{C} ,$$

where

$$A = \frac{B_2}{\alpha}(1 - \frac{1}{\nu}) ,$$

and B_1 and B_2 are arbitrary constants. With this relation, our consumption path is:

$$C_t = \frac{B_2}{\alpha}(1 - \frac{1}{\nu}) \cdot e^{\frac{1}{\alpha\nu}t} + \bar{C} .$$

From the capital path we can easily find the optimum savings path since $S_t = \dot{K}_t$.

Tinbergen introduces some boundary conditions for economic meaningfulness: (a) capital stock must be non-negative in any period, i.e., $K_t \geqslant 0$; and (b) consumption in any period must be greater than or equal to the subsistence consumption, i.e., $C_t \geqslant \bar{C}$. In the complete solution of the capital path, $\alpha\bar{C}$ is constant and the other two parts are exponentially increasing. So, $\alpha\bar{C}$ is negligible. The question now is which one of the exponential parts is greater or which one will dominate? Since the first exponent involves t/α and the second one involves $t/\alpha\nu$, the second one will dominate the capital path. Thus, for non-negative K_t,

$$B_2 \geqslant 0$$

as $e^{t/\alpha\nu}$ is non-negative.

Again, from the consumption path,

$$C_t = \frac{B_2}{\alpha} (1 - \frac{1}{\nu}) e^{t/\alpha\nu} + \bar{C} .$$

For non-negative C_t, we must have

$$\frac{B_2}{\alpha} (1 - \frac{1}{\nu}) e^{t/\alpha\nu} \geqslant 0 .$$

Since $\nu < 1$, and thus $1 - \frac{1}{\nu} < 0$, therefore

$$B_2 \lessgtr 0.$$

The two boundary conditions imply that $B_2 = 0$. When B_2 becomes zero, the two expressions will be:

$$K_t = B_1 e^{t/\alpha} + \alpha\bar{C} ,$$

and $C_t = \overline{C}$.

Hence $\frac{\dot{K}_t}{K_t} \simeq \frac{1}{\alpha}$. Savings and accumulation would come up only after making the subsistence consumption, i.e., only after meeting the subsistence consumption requirements the rest of the output can be saved. The optimal policy is to save everything greater than that needed for subsistence consumption in each period for ever.

The above paradox emerges in Tinbergen's model due to the following three assumptions:

(i) We are trying to maximize some total of discounted utility and have no time preference at all. This zero time preference gives the same weight to the present as well as the future;

(ii) The constant returns to scale; and

(iii) $\nu < 1$ or $1 - \nu > 0$. Utility can be increased by increasing consumption and, since future utility is the same as that of the present, by postponing present consumption, future utility can be increased.

An Infinite Time Horizon: (c) Tinbergen - II

In order to have a meaningful solution in the infinite time horizon planning model, we must ensure a convergence condition. In Ramsey's case, it was ensured by the consumption of "bliss" and in Tinbergen's [40] model, convergence arises[1] when $C_t = \bar{C}$. But, as we have seen, there are certain valid objections to both conditions; these are unacceptable. What condition will give a non-trivial solution? A way out has been provided by introducing a discount factor, i.e., by introducing time preference in favour of present consumption over the future. Our problem is to examine the case of boundedness and to solve Tinbergen's case by introducing a discount factor.

Ramsey solved his optimum problem assuming bliss. Such bliss can be attained in two ways: (i) by putting a restriction on consumption, or (ii) by putting a constraint on production (assuming diminishing returns to scale due to a limitation of a primary factor; thus production will fall). So, consumption cannot increase after a certain point. Neither the assumption of bliss nor diminishing returns are meaningful in the context of planning. In planning analysis, underdeveloped countries come into consideration where the assumption of a sub-additive scale or increasing returns to scale are more important and pragmatic. The above type of restraints, therefore, are discarded.

Now we introduce new boundedness conditions. There is no restriction on consumption and utility but there is one on the discount factor. Strotz [39] pointed out that we are to introduce

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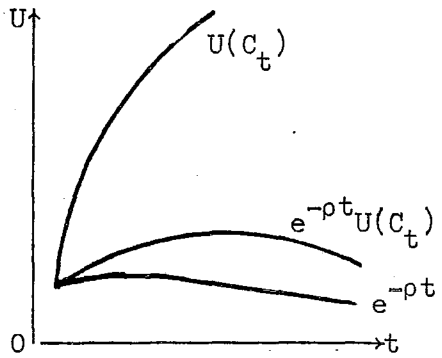
1 - Later it will be shown that the boundary conditions which are necessary for convergence lead to a situation, $C_t = \bar{C}$.

a discount factor which takes the form of an exponential function $e^{-\rho t}$. Thus, with the introduction of time preference, our problem becomes:

$$\max \int_0^\infty e^{-\rho t} U(C_t)dt, \text{ where } \rho > 0, \text{ constant.}$$

The idea is that C_t grows over time but $e^{-\rho t}$ will be so great that it will pull down $U(C_t)$. This is illustrated diagrammatically:



Put another way, the upward pull in the consumption is caused by the marginal productivity of capital and by the elasticity of utility, but ρ will neutralise the upward movement. Like the previous case, we solve the problem of maximisation of a total of discounted utility:

$$\max \int_0^\infty e^{-\rho t} U(C_t)dt ,$$

$$\text{where } U(C_t) = \frac{(C_t - \bar{C})^{1-\nu}}{1 - \nu} .$$

Using this specific form of a utility function, our problem can be written as:

$$\max \int_0^\infty e^{-\rho t}\frac{(C_t - \bar{C})^{1-\nu}}{1 - \nu}dt \ .$$

Again, considering the savings-investment equality and the production condition, our constraint becomes:

$$C_t = \frac{K_t}{\alpha} - \dot{K}_t \ ,$$

where the accumulation of capital (i.e., change of capital stock) is the same as investment.

From Euler's first order condition, we get:

$$\frac{\delta e^{-\rho t}U}{\delta K} = \frac{d}{dt}\left(\frac{\delta e^{-\rho t}U}{\delta \dot{K}}\right) \ .$$

Solving this, we obtain:

$$\frac{dC}{dt} = \frac{1}{\nu}\left(\frac{1}{\alpha} - \rho\right)C_t - \frac{1}{\nu}\left(\frac{1}{\alpha} - \rho\right)\bar{C} \ ,$$

a first order non-homogeneous differential equation, and its solution can be written as:

$$C_t = Ae^{\frac{1}{\nu}\left(\frac{1}{\alpha} - \rho\right)t} + \bar{C} \ ,$$

where A is any arbitrary constant. This is the optimum consumption profile.

Now, to get the optimum capital path, let us start with

$$\frac{dK}{dt} = \frac{K_t}{\alpha} - Ae^{\frac{1}{\nu}\left(\frac{1}{\alpha} - \rho\right)t} - \bar{C} \ ,$$

and the solution of this first order non-homogeneous differential equation is

$$K_t = B_1e^{\frac{1}{\alpha}t} + B_2e^{\frac{1}{\nu}\left(\frac{1}{\alpha} - \rho\right)t} + \alpha\bar{C} \ ,$$

with a restriction on A such that

$$A = \frac{B_2}{\alpha}\left[1 - (1 - \frac{\alpha\rho}{\nu})\right] ,$$

where B_1 and B_2 are any arbitrary constants.

Now, for the sake of convergence, define

$$Z = e^{-\rho t}\frac{(C_t - \bar{C})^{1-\nu}}{1 - \nu} ;$$

and set the condition

$$\frac{dZ/dt}{Z} < 0 .$$

This is the convergence condition which will make the solution a meaningful one; when this inequality holds, it is possible to show that the consumption path is well behaved, since $1/\alpha(1 - \nu) < \rho$ implies $\frac{1}{\nu}\left(\frac{1}{\alpha} - \rho\right) < \frac{1}{\alpha}$. This convergence condition can be written as:

$$\frac{1}{\nu}\left(\frac{1}{\alpha} - \rho\right) < \frac{1}{\alpha} .$$

Thus, the first part of the capital path will dominate while in the original case the second part was dominant. There was no way to make the expression free from domination of the second term and the entire problem arose from this. Using this convergent condition, it can be shown that consumption is not equal to subsistence consumption. It will be increasing over time. This is one advantage of introducing a convergence condition. Another advantage is when $B_2 \neq 0$; it makes the case for assuming that capital stock can be divided into two parts, one for subsistence consumption and the other for growing consumption. Lastly, with the help of the convergence condition, we get the well-behaved path for consumption and capital.

But this is not the end of the story. When the convergence condition is introduced, the entire solution hinges on the condition $\frac{1}{\alpha}(1 - \nu) < \rho$, where ν, α and ρ are given from outside. Here we face the same Harrodian type long run problem of $\frac{s}{v} = n$, where different factors are to be solved by different agents. As there was no reason why such equality should hold good, here also there is no reason why the inequality would hold. There is another fundamental objection which is ethical in nature. The individual always prefers present consumption to future consumption. Society also has to make a choice but it need not necessarily follow that social preference will be the algebraic summation of individual preferences. Again, society can never die, so there is no reason to prefer present consumption.

To conclude, in the infinite time horizon planning model, boundary conditions are essential to get a meaningful solution. Tinbergen aims low, preferring to maintain constant utility for an indefinite period. His model gives a trivial solution both mathematically and economically.

In order to have a meaningful solution in the infinite time horizon planning model, we must ensure the convergence condition. In Ramsey's case, it was ensured by the assumption of "bliss" and in Tinbergen's model [40] convergence is assured when $C_t = \bar{C}$. But, as we have seen, both the conditions are unacceptable; so, what condition will give a non-trivial solution? With the introduction of time preference, Tinbergen, in his second paper [41], provides a way out.

Problem in Finite Time Horizon: (a) Goodwin Model

If a nation's utility (whether discounted or not) at any point of time is solely a function of its instantaneous rate of consumption, and if this nation is assumed to live forever, then the proper planning horizon for the optimization of its rate of saving is infinity. This is the horizon which is mostly found in the literature on optimum savings. The problem with an infinite time horizon is that unless we introduce some crucial boundedness assumption (explicitly or implicitly) at some stage of the argument, infinite programs give rise to conceptual difficulties which may be briefly described as *paradoxes of infinity* [8].

Both Goodwin [15] and Chakravarty [9] propose to sidetrack the difficulties of an infinite program by considering plans extending over only a finite number of time periods. Chakravarty pointed out that from a logical point of view, this is not a satisfactory approach unless we can show that the program optimal for finite time tends in the limit to an optimal program on some relevant definition of order. Despite this logical shortcoming, arising from the fact that such a limit does not exist, for practical purposes a finite horizon, sufficiently large, is deemed adequate, provided the future is taken care of by leaving some capital at the end of the period under consideration [9]. If, in the absence of uncertainty, there is no natural cut-off point in time, the correspondence between logic and practice requires that we should be careful to pose the planning problem in such a way that we are, in fact, assured of the existence of an optimal solution for an infinite future. Failing any or all of these, in practical planning, we have to fall back on a finite horizon model.

Goodwin [15] was concerned with the optimal growth path. Optimal growth was defined as the maximum of welfare over a finite period, welfare being taken as per capita consumption valued in some manner. The planners consider only consumption per head, thus ignoring any inequalities as well as, by implication, holding growth in numbers to be by itself, no gain. There is no time preference, i.e., all consumption is equally desirable regardless of when it occurs. Thus, his problem is to maximize the integral

$$\int_o^T u(x_t)dt.$$

The labour force is growing at a given and constant rate. For realism, he assumed that it seems best to start with a considerable excess of labour, though this is not essential, since initial unemployment increases in realistic conditions. Thus labour is included as an element of per capita income (consumption), but not as a factor in the production function; consequently, capital goods are the only scarce factor.

Assuming a fixed coefficient production function, in equilibrium, we obtain

$$\dot{K}_t = \frac{K_t}{\alpha} - C_t ,$$

where the production function is

$$Y_t = \frac{K_t}{\alpha} .$$

In terms of per capita consumption, this equilibrium relation gives us

$$x_t = \frac{\frac{K_t}{\alpha} - \dot{K}_t}{L_0 e^{nt}} ,$$

where labour force at time period t is given by

$$L_t = L_0 e^{nt} ,$$

so that

$$\frac{\dot{L}}{L} = n .$$

Assuming the lag between investment and capacity, Goodwin adopted the technique of finite difference, treating time as a discrete variable in the first part of his article. In order to use the classical technique of calculus of variations, Goodwin drops the lag between investment and capacity. Now the planners must seek, amongst all possible x(t)'s, the one which will maximize

$$\int_0^T U(x_t)dt ,$$

subject to

$$x_t = \frac{\frac{K_t}{\alpha} - \dot{K}_t}{L_0 e^{nt}} = \Psi(t, K_{(t)}, \dot{K}_{(t)}) .$$

So, the problem reduces to

$$\max_{K_{(t)} \mid t \in [0,T]} J = \int_0^T U[\Psi(t, K_{(t)}, \dot{K}_{(t)})]dt .$$

The necessary and sufficient condition for the stationarity of this expression is given by Euler's equation:

$$U'(x_t)\frac{\delta\Psi}{\delta K} = \frac{d}{dt}U'(x_t)\frac{\delta\Psi}{\delta K} ,$$

which in this case reduces to:

$$\frac{dU'(x_t)/dt}{U'(x_t)} = -\left(\frac{1}{\alpha} - n\right) .$$

This gives us an optimum rule of savings, which is slightly different from the one we have called Ramsey's Rule.

Problem In Finite Time Horizon: (b) Chakravarty's Case

Chakravarty [9] started with the assumption of (i) a zero rate of time preference, (ii) a constant marginal productivity of capital, equal to average productivity, and (iii) a constant elasticity marginal utility schedule like that used by Tinbergen [41]. The coefficient of elasticity can be any non-negative number. He then relaxed these initial assumptions to take into account non-zero time preference and a variable productivity of capital. Here we will consider his model 2: optimal savings with discount factor. We consider model 2 because it can be shown that model 1 is a special case of model 2, by setting the discount factor $\rho = 0$. With nonlinear production functions, the optimal path of capital accumulation is no longer a linear combination of exponential growth paths. He considered a specific nonlinear case, setting β of the Cobb-Douglas production function equal to one-half; it turns out to have a parabola-type behaviour.

Consider an economy where terminal capital stock is growing at the rate g in the time interval 0 to T such that:

$$K(T) = K_0 e^{gT} \ ,$$

where K_0 is the initial stock of capital received from one's predecessors. Our problem is to find out the optimum savings path which

$$\max J = \int_0^T \frac{C_t^{\ 1-\nu}}{1-\nu} e^{-\rho t} dt \ ,$$

$$\text{subject to } C_t = \frac{K_t}{\alpha} - \dot{K}_t \ ,$$

from the equilibrium condition, where $U(C_t) = \dfrac{C_t^{\ 1-\nu}}{1-\nu}$.

The problem is to:

$$\underset{K_t \,|\, t \in [0,T]}{\max} J = \int_0^T \frac{1}{1-\nu} \left[\frac{1}{\alpha} K_t - \dot{K}_t\right]^{1-\nu} e^{-\rho t} dt \ .$$

Note that for $\nu = 0$ and $\rho = 0$, the integrand is a linear expression and, therefore, the problem may be solved by methods of dynamic programming [41].

Using Euler's first order condition, we get

$$\bar{C} = \frac{1}{\nu}\left(\frac{1}{\ } - \rho\right)^{c} \ ,$$

a homogeneous first order differential equation; and its solution is:

$$C_{(t)} = A \ e^{\frac{1}{\nu}\left(\frac{1}{\alpha} - \rho\right)t} \ ,$$

where A is any constant determined by the initial condition. Substituting this value of C_t in the equilibrium condition, and solving this non-homogeneous differential equation in K, we get:

$$K_{(t)} = B_1 e^{\frac{t}{\alpha}} + B_2 e^{\frac{1}{\nu}\left(\frac{1}{\alpha} - \rho\right)t} \ ,$$

where B_1 and B_2 are arbitrary constants such that

$$A = B_2\left[\frac{1}{\alpha} - \frac{1}{\nu}\left(\frac{1}{\alpha} - \rho\right)\right] .$$

One constant is determined by the initial capital stock $K(0)$, and the other constant is to be determined by choosing a terminal condition. We have for $t = T$, $K(T) = K_0 e^{gT}$, where the terminal condition is expressed with two parameters: g which indicates the provision for the future ($g < 1/\alpha$), and T which stands for the period over which the provision is made. From a sensitivity test, Chakravarty finds that (a) the best consumption profiles are, in general, insensitive to changes in terminal capital stock within a wide range, and (b) these profiles are sensitive to changes in the time horizon in all cases where time discount is not admitted. With the introduction of time discount, we may get an invariance with respect to T provided a certain inequality is satisfied between time preference, productivity and the coefficient of elasticity of marginal utility.

Maneschi [24] independently made a sensitivity test and observed that "with appropriate initial and terminal conditions ... optimal consumption paths [are obtained] which are 'quite insensitive' to changes in terminal equipment, for any fixed choice of horizon."[24] This will be valid only if we grant a rather inappropriate terminal condition, namely that g should be so low as to produce a drastic decumulation of capital towards the end of the plan. If g is restricted to the values for which investment at the end of the plan is non-negative, the consumption profile is no longer insensitive to changes in g. Again, when $1/\alpha - (1 - \nu) \leq \rho$, the problem is well posed because "the result is independent of such 'irrelevant details', e.g., whether we put T = 20 or 30." [24] With regard to this contention, it was shown that $1/\alpha(1 - \nu) < \rho$ is a necessary condition for invariance of the consumption profile with respect to T. The necessary and sufficient condition for this is that

$$g = \frac{1}{\nu}\left(\frac{1}{\alpha} - \rho\right) .$$

Maneschi further concluded that both types of insensitivity with respect to g and to T cannot occur at the same time. If g can assume any value in the interval $0 \leqslant g \leqslant 1/\alpha$, g needs to be at the lower end of this range if insensitivity of the consumption profile with respect to g is to obtain. On the other hand, if insensitivity with respect to T is to hold, g must be at the upper end of the range since consumption, investment and income all grow at the rate $\frac{1}{\nu}\left(\frac{1}{\alpha} - \rho\right)$ so that investment at time T is necessarily positive.

Maneschi concluded that because insensitivity of the consumption profile with respect to T is indispensable if inconsistency in decision-making is to be avoided, and since such insensitivity is in general not found in finite horizon plans, the optimization of the rate of saving is justifiable only over a infinite horizon.

This controversy between Professor Chakravarty and Maneschi produces many interesting findings, but we cannot comment on any as being correct or incorrect, because any statement of sensitivity usually presupposes a certain range of variations for the parameters in question.

Problem in Infinite Time Horizon: Revisited

Generalizations of Ramsey's study were made, independently and more or less simultaneously, in several papers by Cass [7], Koopman [21], Malinvaud [23], and Weizsäcker [42] (= CKMW) respectively, with considerable overlap in the results. In the amalgam of their models to be discussed here, a discount factor $e^{-\rho t}$ is introduced, without precluding the possibility that the discount rate ρ is zero. In their model, a social intertemporal preference structure is specified exogenously in the form of a social welfare functional, and that takes the form:

$$J = \int_0^\infty e^{-\rho t} U(x_t)dt \ ,$$

where x_t denotes per capita consumption at time t, $u(x_t)$ is a strictly increasing and strictly concave function giving the utility flow arising from a consumption flow x. ρ is a non-negative continuous time discount rate applied to utility, and $e^{-\rho t}$ the corresponding discount factor at time t. In their models, the utility and production functions are independent of time and satisfy the following assumptions:

(a) $u'(x) > 0 \qquad u''(x) < 0 \qquad$ for $x > 0$;

(b) $f'(k) > 0 \qquad f''(k) < 0 \qquad$ for all $k \geq 0$;

(c) $f'(0) = \infty \qquad f'(\infty) = 0 \qquad$ and $f(0) = 0$;

and Koopmans also assumed

(d) $\lim u(x) \to \infty$ as $x \to 0$ with $x \geq 0$.

He explained that "this means a strong incentive to avoid periods of very low consumption as much as is feasible."|21| If $x = 0$ for very a very small time interval, then by assumption (d) the objective integral diverges to $-\infty$. In essence, this assumption guarantees an interior solution.

The other departure from Ramsey consists in the introduction of exogenous exponential population growth,

$$L_t = L_0 e^{nt} , \qquad n > 0$$

equating population with labour force. This new assumption immediately raises a new ethical question: whether one should maximize, as in (i) an integral over discounted per capita utility $u(x_t)$; or (ii) an integral over a discounted sum $L_t u(x_t)$ of individual utilities, as in Mirrless' |26| and Phelps' |29| (=MP) models. While this is an important question of principle, there are no essential mathematical differences in the models as long as both population growth and the discounting formula are exponential. The only difference then is one of interpretation of the parameters. If we write $\delta = \rho - n$, then δ is the discounted rate applied to per capita utilities in the models of (CKMW) and ρ is that applied to individual utilities in model (MP).

The outcome of model (CKMW) is the existence of a unique optimal path for both consumption and the capital stock, for any non-negative δ, regardless of whether or not there can be saturation with consumption, or with capital. The reason is that now mere maintenance of any given level of per capita consumption requires continual net investment in order to maintain a constant ratio of the capital stock to the growing labour force. As a consequence, consumption per head cannot indefinitely remain at (or above) a level exceeding a highest sustainable level.

The golden rule path has been developed independently by Phelps, Srinivasan and Weizäcker, and presupposes an initial per worker capital stock $k_0 = \bar{k}$ that just allows the highest sustainable level of consumption per head to be attained at all times. With the nonlinear specification of the production function, f, Koopmans showed that there exists a unique optimal attainable path for the infinite horizon problem, which approaches the modified golden rule path with $\delta > 0$.

Assume that output is at all time to be optimally divided between consumption and net investment; then for a positive labour force, the production relation becomes:

$$x_t = f(k) - nk - \dot{k} .$$

Euler's condition solves the problem of maximizing J subject to the production relation, and gives us:

$$\dot{x}_t = - \frac{u'}{u''}\left[f(k) - (n + \delta)\right]$$

Ignoring the possibility of a corner solution (which may arise due to the non-negativity conditions $k_t \geq 0$, $x_t \geq 0$), then the feasible Euler path is the one that satisfies the equations

$$x_t = f(k) - nk - \dot{k}$$

and $$\dot{x}_t = - \frac{u'}{u''}\left[f(k) - (n + \rho)\right]$$

simultaneously.

Now define $k^*(\rho)$ as the value of k which satisfies the equation

$$f'(k) = n + \rho$$

such that $0 \leqslant (\rho) \lesssim \bar{k}$ due to assumptions (b), (c), and (d); and also define $x^*(\rho)$ as

$$x^*(\rho) = f[k^*(\rho)] - nk^*(\rho) \ .$$

This gives us feasible Euler paths of three different types: (i) $k_t > k^*(\rho) \ \forall \ t > \bar{t}$ (for some $\bar{t} > 0$); (ii) $k_t \rightarrow k^*(\rho)$ and $x_t \rightarrow x^*(\rho)$ as $t \rightarrow \infty$; and (iii) $k_t < k^*(\rho)$, for all $t > \bar{t}$ (for some $\bar{t} > 0$). It is clear that the class of paths satisfied by (i) and (iii) are not eligible for the infinite horizon problem. For $\rho \geqslant 0$, the integral J is convergent along the second type of that path, and this path monotonically approaches $[k^*(\rho), x^*(\rho)]$ as time extends without limit. Since the problem of divergence arises when $\rho = 0$, we redefine our target function as:

$$J^* = \int_0^\infty [u(x_t) - u(x^*)]dt$$

where $x^* = x^*(0)$, and then along the second type of path J^* is convergent. The $u(x^*)$ stands for the utility of consumption associated with the golden rule. This is a modified Ramseyian device. It is to be noted that the golden rule path, derived by Phelps, and Joan Robinson can be criticised on the ground that it neglects the historically given stock of capital and labour, and that its choice set is restricted to the golden age paths. This means that even if the historically given value of the capital labour ratio happens to be on the golden rule path, it only maximizes per capita consumption in the choice set which is limited to the set of the golden age paths. Koopmans-Cass derived that the path which maximizes J^* converges to the golden rule path regardless of the initial value of k_0, as long as it satisfies eligibility conditions. Here the choice set is not limited to the golden age paths. Koopmans discovered that all the eligible paths in his formulation closely approach some fixed balanced growth path; hence they all look the same for a sufficiently large time horizon.

The model of Mirrless [26] differs from the CKMW model only in assuming that the exogenous exponential technological progress is of the labour augmenting type. He adopts the interpretation of the optimality criterion as an integral over a sum of individual utilities. He also finds that the consumption and capital stock, both taken "per augmented worker", approach finite asymptotic levels $\hat{x}$, $\hat{k}$, respectively.

In all the optimality criteria considered, the discount rate, whether zero or positive, is always a constant. A criterion defined recursively, and in which the discount factor $\phi(c)$ itself depends on the prospective consumption level C, was developed by Koopmans [21] in a model using a discrete time variable. Beals and Koopmans experimented with the maximization of this objective function in a constant technology with constant returns to capital alone. It was found that an optimal path approaching finite and positive asymptotic levels of consumption and capital can exist only if the discount rate $[1 - \phi(c)]/\phi(c)$ either increases with increasing consumption ($\phi'(c) < 0$), or, if constant, just happens to equal the constant rate of return on capital. Many economists feel, however, that if the discount rate is to be at all variable, it is more plausible that it should decrease when consumption levels increase.

Problem of Existence of Optimum Accumulation Path

Until now, we have attempted to point out the important results obtained to characterize an optimum accumulation of capital path, both in a finite time horizon and in an infinite time horizon. But we are still left with the problem of the existence of such a path. What Ramsey did was to maximize an integral over the whole non-negative part of the time axis. Chakravarty has pointed out [8] that this integral need not converge even if the Ramsey policy proposed (as being optimal) is adopted. The question is whether the divergence of the integral indicates that there exists no optimal policy. The divergence certainly compels us to find other methods than just maximizing an integral subject to certain constraints. Koopmans [21] and Weizsäcker [42], in independent attempts, have shown that the divergence of the utility integral does not necessarily mean that no optimal program of accumulation exists.

Koopmans' paper, although masterly and very penetrating, is long, and his proofs are sometimes difficult. This difficulty is partly the result of his thorough and important examination of the conditions of the existence of an optimum growth path in an infinite time horizon. Koopmans considers an economy where a single homogeneous output is produced with the use of two homogeneous factors, labour and capital, with the production function subject to constant return to scale, and are diminishing returns to individual factors. His problem is one of maximizing

$$\int_0^\infty u(x_t)e^{-\rho t}dt \;,$$

x_t being the consumption of a representative individual. He assumed positive and diminishing marginal rates of substitution.

Weizsäcker incorporates the growing labour force into his study and generalizes Koopmans' problem. He defined x(t) as total consumption of an economy C(t) as a function of time. The amount of "utility" that is produced at time t is a function, U(C, t), of the amount of consumption available at time t and of t itself. "The dependence of U on t may have different reasons. The two most important ones are: (i) the same amount of consumption produced at a more distant time may be enjoyed less than if it were produced to-day or tomorrow, because of what is generally called time preference; and (ii) the same amount of consumption today and tomorrow may have to meet the needs of populations of different sizes and hence imply different degrees of satisfaction." [42] Let Ω be the class of all feasible consumption programs C(t). Then Weizsäcker defined $C^*(t) \in \Omega$ as optimal if for each program $C(t) \leq \Omega$ there exists a T_0 such that for $T \geq T_0$

$$\int_0^T U(C^*_t)dt \geq \int_0^T U(C_t)dt \;,$$

This definition of optimality avoids the problem of divergence of utility integrals of the form

$$\int_0^\infty U(C, t)dt \ .$$

By this approach, known as partial ordering, one can say that the utility stream $U^*(t)$ represents a development of the economy better than that represented by the utility stream $U^0(t)$ if, for all sufficiently large T,

$$\int_0^T U^*(t)dt > \int_0^T U^0(t)dt \ .$$

Mirrlees simplified the proof provided by Weizsäcker by saying that "it is usually much easier to prove that there exists a policy that cannot be bettered than to prove that it is better than any other." [26] With these considerations in mind, following Mirrlees [26], one can rely on the theorem stated below to solve the problem of selecting C(t) which maximizes

$$\int_0^\infty L_t U(C_t/L_t)\ e^{-\rho t}dt \ .$$

<u>Theorem</u>: The development of the economy is optimal if, writing $s = e^{-\rho t}U(x_t)$, $\frac{ds}{dt} + s \cdot f'(k^*) = 0 \ \forall\ t > 0$; $sk^* \to \beta, < \infty$, as $t \to \infty$; and either (i) $\rho = 0$, or (ii) k^*_t, x^*_t are bounded, and bounded away from zero, where C^*_t, K^*_t, x^*_t, k^*_t are the optimal values of the respective variable.[1]

Now, this problem of existence in the context of an infinite time horizon is well established in the literature. So far as the

1 - For proof, follow Mirrlees [26].

existence of an optimum accumulation path in a finite time horizon is concerned, Chakravarty made an attempt to show the required condition, but did not provide any vigorous proof. In the next part, we shall provide a vigorous proof of the existence of such a path. We shall also attempt to show that this path is unique and efficient among all other paths.

PART II

THE EXISTENCE OF OPTIMUM PATH IN A FINITE TIME HORIZON

This part of the study is to be devoted to the problem of the existence of an optimum accumulation path in a finite time horizon. This problem has been interesting since 1962, when Chakravarty pointed out [8] that an integral over the whole non-negative part of the time axis need not converge even if the optimal policy proposed by Ramsey -- that of maximizing an integral -- is adopted. The rigorous proof to the solution of the above problem has not yet appeared in the literature. This study makes an attempt to provide a rigorous proof of the existence of the optimum accumulation of capital in a finite time horizon. We have also shown that this path is unique. In this study, we also introduce the notion of 'efficiency' in optimum growth literature and have shown that the optimum accumulation of capital path is uniquely efficient.

In section I, we shall describe the model and in section II, we shall show the existence, uniqueness and also efficiency of an optimum accumulation path in a finite time horizon model.

I

We have an economy which produces one commodity as its "national product" (Y). Some part of output is consumed; the remainder is saved and, hence, invested, at every point of time. Output is produced with the help of two homogeneous factors of production, capital (K) and labour (L). These two inputs are smoothly substitutable for each other in the production process. We are assuming a constant returns to scale production function which displays diminishing returns to individual factors. In this economy, our problem is to find out an optimum savings programme which maximises a certain stipulated functional in utilities, subject to certain restrictions on the class of admissible utility and production functions.

We consider a single utility functional which is to be maximised. This is best interpreted as the functional of the central planning authority. Such a functional defines a mapping from the space of functions defining alternative continuous consumption profiles to the real line. We consider a functional because time is being treated here as a continuous variable. Following Koopmans[1], we assume that this functional obeys the following postulates. (a) Consumption is sensitive to the magnitude of the variable in the initial period in that it has an important role on optimum savings. (b) Intertemporal inter-generational stationarity of preferences - this postulate demands that the preference maps of future consumption programmes be unaffected by the passage of time.

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[1] Koopmans, T.C., "Stationary Ordinal Utility and Impatience", Econometrica, 28, April 1960.

There are two distinct components in this postulate: intertemporal intra-generational stationarity requires that once the preference maps of individuals are drawn, they will not change during the lifetime of the individuals concerned; intertemporal inter-generational stationarity assumes that the age structure of the society is unchanged, and the preference maps of individuals in each age-group will remain unchanged across generations. Stationarity is, of course, a rather unlikely property. (c) Periodwise independence referring to three periods - the marginal rate of substitution between consumption in any two periods is independent of consumption in any third period. The implications of this assumption have been explored by Debreu. He has shown that it makes utility measurable up to a linear transformation and, indeed, expressible as a sum of functions, each of which depends only on the assumption of a single period:

$$U(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} U_i \, (x_i).$$

The cross-effects of consumption in one period on the marginal utility of consumption in another are assumed away. This is an assumption sanctified by tradition, viz, the work of Ramsey [33], Tinbergen [40, 41], Samuelson and Solow [34], etc. Chakravarty [8, 9] asserts that, although Debreu's theorem is applicable only if time is handled discretely, there is no logical difficulty in using this theorem when time is continuous. These assumptions, together with the basic theorem of Debreu, imply that such a functional can be represented by an integral of instantaneous utilities over the space of continuous real-valued functions. Thus, these postulates jointly justify the use of a decision procedure such as maximising $\int_0^T U[c(t)]dt$, where U refers to instantaneous utility.

In our system, utility depends on per capita consumption. This is the consumption of the "representative" individual in the society.

Define $x_t = \frac{C_t}{L_t}$, as per capita consumption at period t; we assume that

[Assumption 1] $U'(x) > 0$, $U''(x) < 0$, for all $x > 0$,

positive but diminishing marginal rate of substitution between any two generations, as well as $\lim_{x \to 0} U'(x) = \infty$. This is because of the necessity of avoiding extremely low levels of consumption per capita. This condition ensures that an optimum path will never specify a zero level of consumption per capita. Also we assume that

[Assumption 2] $x_t = x(t)$ is a non-decreasing function of time, $t \in [0,T]$.

He introduces this assumption as a political constraint which is relevant to the planner of the developing economy.

The labour force and population both grow exogenously at the positive (constant) rate n. Therefore, quantities measured in terms of the labour force are equivalent, but for a scale factor, γ, to quantities per capita. As the planning body has the authority to require all able-bodied persons to work, by the following assumption

[Assumption 3] $L(t) = L(0)\ e^{nt}$,

the whole labour force will always be productively employed. When labour is growing, one part of capital formation would be of widening variety and would have to be reckoned with in maximising indefinitely substainable consumption per head. This is because any choice of the optimal capital labour ratio must now reckon with the investment that will be needed to equip a larger labour force with the same capital labour ratio. Occasionally, the optimum capital labour ratio has been interpreted as giving us the most preferred course of action out of a set of alternative possibilities. This is incorrect since, as we shall see, the optimum refers to the best steady state situation defined in terms of intensive magnitudes but does not indicate the immediate course of action, given an arbitrary initial situation,

Given a rate of consumption c, and a rise of population $\frac{L}{\gamma}$, the total utility for the period [0,T] will be

$$J = \int_0^T \frac{L(t)}{\gamma} U(x)\ e^{-\rho t}\ dt,$$

with the assumption of the possibility of time preference, i.e., discounting at a rate ρ. This utility, U, is a continuously increasing, strictly concave, twice differentiable function.

Let our production function be

$$Y_t = F(K_t, L_t)$$

and, $\quad F_K > 0, \quad F_{KK} < 0,$

$\quad F_L > 0, \quad F_{LL} < 0.$

It is to be noted that there is no time subscript in F, which indicates that there is no possibility of changing technology over time.[1] More clearly, we are assuming that technology is given during the planning period. Since we have already assumed that what is saved is invested at each and every point of time, future consumption will be given by the relationship:

$$c_t = F(K_t, L_t) - \mu K_t - \dot{K}_t,$$

where μ denotes the rate of depreciation and the subscript t denotes the time period. We assume that the constraint on the capital stock is

[1]We discussed this problem with changing technology elsewhere. Bandyopadhyay, T., Working Paper, 1975, Department of Economics, McMaster University.

as follows:

$$K_o > 0 \text{ and given, and } K_T > 0^1$$

Since we have assumed that the production function obeys constant returns to scale, therefore,

$$F(K_t, L_t) = L_t f(k_t),$$

where $k_t = \frac{K_t}{L_t}$, the capital-labour ratio. We assume that f is a continuous, increasing, strictly concave, twice differentiable function such that

[Assumption 4] $f(k) > 0$, $f'(k) > 0$, $f''(k) < 0$, for all $k > 0$,

and $\lim_{k\to 0} f(k) = 0$, $\lim_{k\to\infty} f(k) = \infty$;

[Assumption 5] $f'(k) \geq \text{Sup } (\mu+n)$ for $t \in [0,T]$ and $k > 0$.

Again we assume that

[Assumption 6] $k = k(t)$ is a continuous real valued function.

We have

$$k = \frac{K}{L},$$

and thus, taking logarithm and differentiating with respect to t, we obtain,

$$\frac{\dot{k}}{k} = \frac{\dot{K}}{K} - n.$$

Substituting this relation in

$$c(t) = L_t f[k(t)] - \mu L(t) k(t) - \dot{K}(t),$$

future consumption can be obtained by the alternative relationship:

$$x(t) = f[k(t)] - (\mu+n) k(t) - \dot{k}(t)$$

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[1]The results are unaffected with $K_T \geq 0$, but $K_T > 0$ is more meaningful in a finite planning model. For a discussion of this point, see T. Bandyopadhyay, Working Paper 1975, Department of Economics, McMaster University.

Along with the growth of population, the planning authority recognizes that, due to time preference, tomorrow's consumption is not the same as the consumption of today; also taking intergenerational equity into account, they give some weightage to the consumption of future generations. This view is implemented by discounting future welfare at a positive rate ρ which is lower than the population growth rate n. Thus, defining $\delta = n - \rho$ and assuming $n > \rho$, social welfare associated with any particular feasible path is given by the functional representing total welfare

$$J = \frac{1}{\gamma} \int_0^T L_0 \cdot U[x(t)] e^{\delta t} \, dt.$$

The problem confronting the central planning authority is to choose a particular feasible capital path (corresponding to which we get a savings path) which

$$\underset{\{k(t) \mid t \varepsilon [0,T]\}}{\text{Max } J} = \frac{1}{\gamma} \int_0^T L_0 \cdot U[x(t)] e^{\delta t} \, dt,$$

subject to

$$x(t) = f[k(t)] - (\mu + n)\, k(t) - \dot{k}(t),$$

$k(t) > 0$ for all $t \,\varepsilon\, [0,T]$ and $k(0)$ given.

We have assumed that $x(t)$ is a non-decreasing function in the in the interval $[0,T]$. Now, since L_0 is given n and μ are constants we can rewrite the problem as

$$\underset{\{k(t) \mid t \varepsilon [0,T]\}}{\text{Maximize }} J[k] = \int_0^T \phi[t, k(t), \dot{k}(t)] \, dt$$

subject to $k(t) > 0$ for all $t \,\varepsilon\, [0,T]$,

where $k(0)$ is given and positive. Clearly, the value of the integral depends on the function $k(t)$. By changing the function $k(t)$, we can get different values of J. Suppose we have a certain class of functions Ω

(for example, the set of all differentiable functions defined on the closed interval [0,T]). Then we can consider the problem of choosing a function $k(t)$ from the class of functions Ω so as to maximize the integral J, subject to the condition that $k(0)$ is given and positive and that $k(T) > 0$. This is the type of problem which can be solved with the calculus of variations.

II

In this section we will prove the existence and the uniqueness of an optimum savings programme. To do so we need a few lemmas and theorems.

Theorem 1:[1] If $k = k^*(t) \in \Omega$ is an extremal for the functional

$$J[k] = \int_0^T \phi[t, k(t), \dot{k}(t)]\, dt,$$

and if at some point $t = t_o \in [0, T]$

$$(1) \qquad \phi^*_{\dot{k}\dot{k}} \neq 0$$

then in some neighbourhood $N_\varepsilon(t)$ of the point $t_o \in [0, T]$, the function $k^*(t) \in \Omega$ has a continuous second derivative.

Proof: On the basis of the integral equation

$$(2) \qquad \phi_{\dot{k}}[t, k^*(t), \dot{k}^*(t)] = \int_0^t \phi_k[a, k^*(a), \dot{k}^*(a)]\, da + c$$

the system

$$\phi_{\dot{k}}[t, k^*(t), \ell] = \int_0^t \phi_k[a, k^*(a), \dot{k}^*(a)]\, da + c$$

can be solved for ℓ for every t. Condition (1) expresses the fact that at $t = t_o \in [0,T]$, $\phi^*_{\dot{k}\dot{k}}$ is different from zero. According to a theorem

[1]This Theorem is a special case of Hilberts Theorem. Sagan, H. Introduction to the Calculus of Variations, McGraw-Hill Book Co., (1963), pp. 98.

of implicit functions, this system and the condition $\ell(t_o) = \dot{k}^*(t_o)$ define the functions in some neighbourhood $N_\varepsilon(t)$ for the point $t = t_o \varepsilon [0, T]$ and provide that they are continuously differentiable in the $N_\varepsilon(t)$. Thus, $\ell(t) = \dot{k}^*(t)$ and the derivative $\ddot{k}^*(t)$ exist and are continuous in some neighbourhood $N_\varepsilon(t)$ of the point $t = t_o \varepsilon [0, T]$.

[Q.E.D.]

Theorem 2:[1] If ϕ is continuous for the functional

$$J[k] = \int_0^T \phi[t, k(t), \dot{k}(t)]\, dt,$$

then (a) there exists a smooth function $k = k(t)$ that is defined in $\{N_\varepsilon(t_o) | t = t_o \varepsilon [0, T]\}$, and which satisfies the equation

$$(3) \qquad \frac{d\phi_{\dot{k}}}{dt} - \phi_k = 0;$$

and (b) this is unique.

Proof: Let us assume that $\phi^*_{\dot{k}\dot{k}} \neq 0$ holds at the point $t = t_o \varepsilon [0, T]$. Differentiation of the equation, $\phi_{\dot{k}}[t, k(t), \dot{k}(t)] = \int_0^T \phi_k[a, k(a), \dot{k}(a)]da+c$ in $\{N_\varepsilon (t_o) | \ t_o \varepsilon [0,T]\}$ yields:

$$\frac{d\phi_{\dot{k}}}{dt} + \phi_{\dot{k}k}\, \dot{k}(t) + \phi_{\dot{k}\dot{k}}\, \ddot{k}(t) = \phi_k ,$$

and thus

$$(4) \qquad \ddot{k}(t) = \frac{\phi_k - \frac{d}{dt}\phi_{\dot{k}} - \phi_{\dot{k}k}\, \dot{k}(t)}{\phi_{\dot{k}\dot{k}}}$$

In some neighbourhood of $[t, k(t), \dot{k}(t)]$, the right hand side of equation (4) is continuous; in this neighbourhood it satisfies a Cauchy-

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[1]Bliss, G.C., Lectures on the Calculus of Variations, University of Chicago Press, 1956.

Lipschutz condition with respect to the arguments $k(t)$, $\dot{k}(t)$. By an existence theorem for ordinary differential equations, system (4) will have only one solution $k = k(t)$ in $\{N_\varepsilon(t_o) | t_o \in [0, T]\}$; this solution will also satisfy equation (3).

[Q.E.D.]

This establishes the existence of a unique local extremal.

Theorem 3: If a unique local extremal exists then it is global.

Proof: From Theorem 1, every smooth solution of the system (3) that satisfies $k = k(t)$ in $\{N_\varepsilon(t_o) | t_o \in [0, T]\}$ has a second derivative and, therefore, satisfies the system (4) which has only one solution.

[Q.E.D.]

Now we use the following lemmas to show that this extremal is a maximum.

Lemma 1: ϕ is a monotonic increasing function with respect to all its arguments.

Proof: Consider $\phi = \frac{1}{\gamma} L_o \, e^{\delta t} \, U[x(t)]$;

differentiating with respect to t, we obtain:

$$\phi_t = \frac{1}{\gamma} \; L_o \; e^{\delta t} \; U' \; \frac{dx_t}{dt} + \frac{1}{\gamma} \; \delta L_o \; e^{\delta t} \; U$$

By assumption (2) x_t is a non-decreasing function, and since we assume $\delta > 0$, therefore, for $t \in [0, T]$,

$$\phi_t \geq 0 \; .$$

Substituting the production relation,

$$x(t) = f[k(t)] - (\mu+n) \; k(t) - \dot{k}(t) \; .$$

Consider again

$$\phi = \frac{1}{\gamma} L_o \; e^{\delta t} \; U[f[k(t)] - (\mu+n)\; k(t) - \dot{k}(t)].$$

Differentiating with respect to k, $\dot{k}$ and remembering that

$U' > 0$ and $f'(k) \geq \sup_{0\leq t\leq T} (\mu+n)$

the proof is immediate.

[Q.E.D.]

Lemma 2: ϕ is a strictly concave function in its second and third arguments.

Proof: For strict concavity, we have to show

$$\frac{\partial^2\phi}{\partial k^2} \cdot \frac{\partial^2\phi}{\partial \dot{k}^2} - \left(\frac{\partial^2\phi}{\partial k \cdot \partial \dot{k}}\right)^2 > 0.$$

Partial differentiation of second order with respect to k and $\dot{k}$, yields

$$\frac{\partial^2\phi}{\partial k^2} = \frac{1}{\gamma} L_o \; e^{\delta t}\{u''[f'(k) - (\mu+n)]^2 + u'f''(k)\}$$

and

$$\frac{\partial^2\phi}{\partial \dot{k}^2} = \frac{1}{\gamma} L_o \; e^{\delta t} \; U'';$$

and the cross-partial derivative is:

$$\frac{\partial^2\phi}{\partial k \partial \dot{k}} = -\frac{1}{\gamma} L_o \; e^{\delta t} \; U''[f'(k) - (\mu+n)].$$

Now

$$\frac{\partial^2\phi}{\partial k^2} \cdot \frac{\partial^2\phi}{\partial \dot{k}^2} - \left(\frac{\partial^2\phi}{\partial k \partial \dot{k}}\right)^2 = \frac{1}{\gamma} L_o \; e^{\delta t} \; U''\cdot U'\cdot f''(k);$$

Since $U' > 0$, $U'' < 0$, $f'' < 0$, this completes the proof.

[Q.E.D.]

Lemma 3: ϕ is a bounded function in the interval $[0, T]$.

Proof: Since $f'(0) = \infty$ and $f'(\infty) = 0$, and, also, $k(t) > 0$ for all $t \in [0, T]$, therefore, for economic meaningfulness, we can assume that there exist numbers m and M such that

$$0 < m \leq k \leq M < \infty$$

and also numbers p and P such that

$$0 < p \leq \dot{k} \leq P < \infty.$$

Since $k(t)$ is twice differentiable in the closed interval $[0, T]$, $k(0) = \alpha$ and $k(T) = \beta$, where α and β are fixed end points, we can say that ϕ is defined on a compact set. Again, since ϕ is continuous on a compact set, therefore, it is bounded.

[Q.E.D.]

Theorem 4: Given assumptions (1) to (6), there exists a savings programme for the economy which is optimal, i.e., there exists $k(t)$ which maximizes

$$J = \frac{1}{\gamma} \int_0^T L_o\, e^{\delta t}\, U(x_t)\, dt,$$

for $t \in [0, T]$, subject to:

$$x(t) = f[k(t)] - (\mu+n)\, k(t) - \dot{k}(t)$$

$$k(t) > 0 \text{ for all } t \in [0, T] \text{ and } k(0) \text{ given.}$$

Proof: From theorems (1) and (2) there exists an extremal, and from lemmas (1), (2) and (3), the extremal is a maximum.

[Q.E.D.]

Theorem 5: If an optimal savings programme exists, then it is unique.

Proof: The proof is immediate from theorems (3) and (4).

[Q.E.D.]

We would expect that an optimal savings programme would not waste capital. We characterize this properly by the notion of 'efficiency'.

Definition: A savings programme, corresponding to a k(t), will be said to be efficient if and only if there does not exist another savings programme, to which there corresponds a $\underline{k}(t)$ such that:

(a) $\underline{k}(t) = k(0)$ for $t = 0$
$k(T)$ for $t = T$

(b) $x = f[\underline{k}(t)] - (\mu+n)\, \underline{k}(t) - \dot{\underline{k}}(t)$,
$\underline{k}(t) > 0$ for all $t \in [0, T]$,
$\underline{k}(0) > 0$ and given ;

(c) $\underline{k}(t) \leqq k(t)$ for all $t \in [0, T]$
and $\underline{k}(t) < k(t)$ for some $t \in [0, T]$
such that $\frac{1}{\gamma}\int_0^T L_o\, U[\underline{x}(t)]\, e^{\delta t}\, dt \geqq \frac{1}{\gamma}\int_0^T L_o\, U[x(t)] e^{\delta t}\, dt.$

So, we consider a savings programme to be efficient if and only if there does not exist any other savings programme (a) which starts from the same initial condition and has the same terminal capital stock, (b) is feasible, and (c) has a lower capital-labour ratio in at least one period in the planning horizon and does not have a higher capital-labour ratio in any other period; and yet achieves at least as much social welfare over the planning period.

Theorem 6: An optimal savings programme is efficient.

Proof: Suppose not. Then k*(t), corresponding to the optimal savings programme is inefficient. Let there exist a k**(t), corresponding to another savings programme for which $k^{**}(t) \leqq k^*(t)$ for all $t \in [0, T]$ and $k^{**}(t) < k^*(t)$ for some $t \in [0, T]$, such that

$$\frac{1}{\gamma} \int_0^T L_o\, U[x^{**}(t)] \quad e^{\delta t}\, dt \geqq \frac{1}{\gamma} \int_0^T L_o\, u[x^*(1)] e^{\delta t}\, dt.$$

By theorem 4, there does not exist a $k^{**}(t) \neq k^*(t)$ such that

$$\frac{1}{\gamma} \int_0^T L_o\, U[x^{**}(t)]\, e^{\delta t}\, dt > \frac{1}{\gamma} \int_0^T L_o\, U[x^*(t)] e^{\delta t}\, dt;$$

and by theorem 5, there does not exist a $k^{**}(t) \neq k^*(t)$ such that

$$\frac{1}{\gamma} \int_0^T L_o\, U[x^{**}(t)] e^{\delta t}\, dt = \frac{1}{\gamma} \int_0^T L_o\, U[x^*(t)] e^{\delta t}\, dt.$$

So, for any $k^{**}(t) \neq k^*(t)$,

$$\frac{1}{\gamma} \int_0^T L_o\, U[x^{**}(t)] e^{\delta t}\, dt < \frac{1}{\gamma} \int_0^T L_o\, U[x^*(t)] e^{\delta t}\, dt,$$

which is a contradiction.

[Q.E.D.]

III

In this paper, we have been concerned with the problem of investigating the existence of an optimal savings programme for an economy which is planning within a finite time horizon. As far as we are aware, no formal proof of the existence of an optimum savings programme for a finite time horizon model appears in the literature. Here we provide a rigorous proof of existence of such a programme.

We also demonstrate that it is unique. Furthermore, we have given a rigorous characterization of an optimal savings programme as being efficient. Rigorous proofs of uniqueness of an optimal savings programme and the property that it is efficient, have nowhere appeared in the literature either in the context of an infinite or in that of a finite time horizon model.

Since our problem has been posed in the context of development planning, we have two options with respect to justifying the sign of δ. D. Cass [7] considers the situation of planners' pessimism where the rate of discount of future welfare is greater in absolute value than the rate of growth of the population, rendering $\delta < 0$.[1] His rationalization for doing so is that the planners would be more interested in getting an optimal programme for the current and immediately succeeding generations as he is more concerned with the present; in fact, this implies that a very small weightage is given to generations beyond the immediate present, raising severe problems of social justice. We adopt the alternative approach of planners' optimism, i.e., $\delta > 0$, where planning is undertaken with a view to making an attempt at ensuring inter-generational equity as well.

[1] In the unlikely, but mathematically possible, case of $n = \rho$, the existence proof is unaffected; however, the problem becomes uninteresting.

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