HADRON MASSES AND THE NON-RELATIVISTIC QUARK MODEL

By

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TO MY WIFE
JOAN

" As far as the laws of
mathematics refer to reality,
they are not certain;
and as far as they are certain,
they do not refer to reality. "

A. Einstein

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ABSTRACT

The hadron masses are calculated in the non-relativistic quark model by solving the Schrödinger equation with a suitable potential. The three-body baryon problem is solved by the Feshbach-Rubinow method.

The Fermi-Breit potential proposed by De Rújula, Glashow and Georgi is shown to be unsuitable for non-relativistic hadron systems. Some phenomenological potentials are then examined by which light hadron masses may be fitted and remain only marginally relativistic. Moreover, such interactions indicate the necessity for long-range spin dependence and permit the validity of a perturbative approach to be tested.

An attempt is made to consistently fit charmonium together with the lighter hadrons by using a logarithmic potential and incorporating a perturbative type estimate of the relativistic kinetic energy. It is found that while the vector meson masses are well reproduced the baryonic masses are somewhat overestimated.

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CHAPTER I

INTRODUCTION

I.1 Historical Background

The search for understanding what constitutes matter is an ancient one. The availability of higher energy accelerators provides a more efficient probe into the subnuclear. Consequently, more and more data is extracted on the so-called "elementary" particles and their list grows longer.

In order to simplify the classification of these particles Gell-Mann and Zweig (c. 1964) posited that the hadrons are composites of more elementary particles called quarks which come in various "flavours" (up, down, strange, etc.). In this model mesons are regarded as a quark-antiquark pair while baryons are considered to be made up of three quarks (see appendix A.1).

The model has since been extended to incorporate quark dynamics. Weinberg and Salam (c. 1967) proposed the electro-weak interaction which is governed by gauge fields. This interaction will not be discussed in this work. In the early 1970's it was postulated that the strong interaction is also governed by gauge fields whose quanta are an octet of vector bosons known as gluons. Quantum chromodynamics (Q.C.D.), as it is now called, is becoming widely accepted as a viable theory of the strong interaction.

Both the gluon of Q.C.D. and the familiar photon of Q.E.D. (quantum electrodynamics) are massless, neutral vector bosons and one may expect some similarities between Q.C.D. and Q.E.D. The non-relativistic reduction of one photon exchange leads to the Fermi-Breit interaction, which for S-states is characterized by a Coulomb potential plus its associated short-range spin dependence, together with momentum dependent terms. Thus, as is discussed by De Rújula, Georgi and Glashow (DGG) in their pioneering work (De 75), one might also expect the quark-quark interaction to be Coulomb like. However it should be noted that there are essential differences between the two theories.

In the standard model there are four flavours of fractionally charged spin 1/2 quarks (up(u), down (d), strange (s) and charmed (c)) which, together with the gluons, carry a new quantum number known as colour, which transforms as a triplet under colour SU(3). Hence, each quark comes in one of three colours and it is postulated that coloured particles are confined. The usual assumptions that colour SU(3) is an exact symmetry and that physical states form an antisymmetric colour singlet, serve to retain the desirable standard Fermi-statistics.

Colour confinement is probably intimately linked with a property known as "infrared slavery". Unlike Q.E.D., Q.C.D.

is a non-abelian gauge theory and this leads to the gluon having a highly non-linear, strong self-coupling. In Q.C.D. the effective strong coupling constant (α) is a function of the momentum transfer and the fact that α becomes large for small momentum (or long distances) is known as infrared slavery. The converse property, asymptotic freedom, asserts that α becomes small for high momentum or short distances.

I.2 The Non-Relativistic Model

DGG (De 75) used the asymptotic freedom property to support their suggestion that the dominant short-range quark-quark interaction is Coulomb-like. An essential feature of this assumption is that the spin-dependent Fermi term is of short range. Thus the one-gluon quark-quark potential proposed by DGG is of the form

$$V_{ij} = U(r_{ij}) + k\alpha_s S_{ij}$$

and

$$S_{ij} = \frac{1}{r_{ij}} - \frac{1}{2m_i m_j}\left(\frac{\underset{\sim}{p}_i \cdot \underset{\sim}{p}_j}{r_{ij}} + \frac{\underset{\sim}{r}_{ij} \cdot (\underset{\sim}{r}_{ij} \cdot \underset{\sim}{p}_i)\underset{\sim}{p}_j}{r_{ij}^2}\right) - \frac{\pi}{2}\,\delta(\underset{\sim}{r}_{ij})\left(\frac{1}{m_i^2} + \frac{1}{m_j^2}\right.$$

$$\left. + \frac{4\underset{\sim}{\sigma}_i \cdot \underset{\sim}{\sigma}_j}{3m_i m_j}\right) + \text{spin-orbit and tensor terms} \qquad (1)$$

where m_i are the quark masses and σ_i the usual Pauli spin matrices. $U(r_{ij})$ is a universal confinement potential, proposed on arguments based on lattice guage theories, and $k = -\frac{2}{3}$ for qq and $-\frac{4}{3}$ for $q\bar{q}$. The factor of two arises from the usual attitude that the potential is generated through gluon exchange. Now since the effective quark-antiquark coupling ($g_{q\bar{q}}$) is of the order 2 g_{qq}, it is assumed that the mesonic $q\bar{q}$ potential is twice the baryonic qq potential. Likewise, if the confinement potential, $U(r_{ij})$, was also generated via the exchange of gluons (see e.g., Pu 75, Sc 76, Ph 77, Ta 79), there would be a similar factor of two difference between the mesonic and baryonic cases.

By treating interaction (1) in a perturbative spirit, DGG (De 75) were able to derive mass formulas for the lighter baryons and 1^- mesons which agree well with experiment. They also indicated that the same potential should not be expected to fit the 0^- mesons, which they suggested may be a result of the increased importance of two gluon exchange.

Since DGG (De 75) did not solve the Schrödinger equation with interaction (1) but rather parametrized matrix elements such as $\langle\psi|\delta(\underset{\sim}{r}_{ij})|\psi\rangle$, the nice fits they obtained serve to demonstrate the importance of the spin-spin coefficient $(m_i m_j)^{-1}$. However, their conclusions remain independent of the range or radial form of the potential.

If one wants to solve the Schrödinger equation with potential (1), $\delta(\underset{\sim}{r}_{ij})$ must be smoothed by a suitable short range form factor, otherwise the system collapses for an attractive δ-function. Similarly, other singularities in the potential should be regularized consistently.

In the succeeding chapter we address ourselves to the problem of examining the radial dependence of the potential. One should realize that in contradistinction to the atomic case, the hadronic hyperfine splittings may be quite large (e.g., between N and Δ) and thus a perturbative treatment is at least questionable. This point is further discussed in Chapter II. To our knowledge no one has attempted a dynamical solution to the three quark problem with interaction (1). Although Shankar and Warke (Sh 79) proposed a variational calculation, but since they retained the δ-function, the significance of their results is perhaps debatable for the reason previously given.

In this connection we solve the hadronic Schrödinger equation dynamically. Since the quark-quark potential is spin-dependent, the hadronic wavefunction ψ_h takes the form

$$\psi_h \sim \psi_c \, \psi_{sf} \, \psi_s$$

where ψ_c is the totally antisymmetric colour singlet wavefunction and ψ_{sf} the spin flavour function for which explicit forms are given in appendix A.2. The spatial wavefunction, ψ_s is obtained by solving the Schrödinger equation. The three quark Schrödinger equation for the baryonic system is solved in the Feshbach-Rubinow (F-R) approximation (Fe 55) generalized for unequal masses and force bonds (Ab 62, Bh 67). This method has been shown to be satisfactory in both atomic (Bh 76) and nuclear problems (Mc 65). In this approximation the S-state three body wavefunction ψ_0 is assumed to be a function of the single variable $R = \frac{1}{2}(r_{23} + r_{13} + \eta r_{12})$ where η is a variational asymmetry parameter. When the masses and force bonds are equal $\eta = 1$, otherwise η is varied to obtain the minimum energy. The method is tested against an exactly solvable harmonic model presented in the next chapter and the accuracy is found to be satisfactory. Further details of the method are given in appendix B.1.

In chapter II we solve the S-state baryonic problem by the above F-R method, taking a suitably regularized interaction of the form (1), in which we use a ramp as the confinement potential. The parameters (the quark masses, α_s and an overall constant) were varied to fit the masses of N(939), Δ(1232), N*(1470) and Λ(1116). A number of sets could be obtained but in each case the nucleon rms radius shrank to 0.3 fm or less and the total kinetic energy of the three quarks was > 1600 MeV. Since the up quark mass was ∿ 500 MeV, the system is clearly relativistic.

As a result of this, we show that some simple phenomenological qq potentials may be constructed that retain the essential ingredients of the underlying field theory and fit the ground state masses while remaining marginally nonrelativistic. The potentials are chosen to exhibit the characteristic confinement property and a spin-spin interaction whose coefficient is mass dependent.

Several such potentials are examined in the succeeding chapter and we demonstrate the need for a long-range spin interaction in fitting the hadron masses while remaining within the desirable, marginally non-relativistic framework. In addition, we show that as the range is decreased the kinetic energy subsequently rises and that a perturbative estimate progressively deteriorates. In this chapter, the emphasis is on the ground states of the lighter hadrons since, as a result of the increased kinetic energy in the excited states, spectroscopic calculations may be dubious. Furthermore, for charmonium, where relativistic corrections are expected to be considerably less, the potentials are unable to fit the spectrum.

In chapter III the emphasis is shifted from the lighter baryons to the heavier mesons. Here we determine the potential parameters by fitting the less relativistic mesons (e.g. ψ and ϕ). This approach is further advantageous since the two-body meson system may be solved numerically to any desired accuracy, unlike the baryons which are treated in the F-R approximation.

For this purpose we examine a simple logarithmic potential, which in a loose sense is equivalent to a Coulomb potential coupled to ramp confinement. This potential has the remarkable property that the kinetic energy is a constant that remains unchanged in the excited states. Thus there exists the possibility of performing meaningful spectroscopic calculations. This potential is not unlike one that has been proposed for charmonium (Qu 77, Ma 78, Qu 80). Having illustrated the importance of the long-range nature of the spin interaction, for simplicity we consider a constant spin-spin term together with the logarithmic potential. Furthermore, ad hoc perturbative estimates are made of the relativistic corrections and satisfactory fits are obtained for the vector mesons, including charmonium, while the baryons are reasonably reproduced.

CHAPTER II

THE FERMI-BREIT INTERACTION VERSUS SOME PHENOMENOLOGICAL POTENTIALS

II.1 The Fermi-Breit Interaction

As discussed in the preceeding chapter De Rŭjula ét al (De 75) first proposed the coulombic quark-quark interaction. As was mentioned, since the three body Hamiltonian with an attractive δ-function has no lower bound, we replace it by a short range form factor $f_D(r,r_0)$ where the range r_0 is chosen, in analogy with the atomic case (Bl 65, Bl 78), to be $\sim \alpha_s/m_q$ where m_q is a typical quark mass. For simplicity we fix $r_0 = \alpha_s/m_u$. The singularities in the momentum dependent terms are regularized consistently and some details are given in appendix A.3. Since we investigate S-state baryonic masses we drop spin-orbit and tensor contributions. The two body quark-quark interaction takes the form

$$V_{ij} = -\frac{2}{3}\alpha_s S_{ij}$$

$$S_{ij} = C + \frac{1}{r} - \frac{r}{a^2}\Big(+ \frac{1}{2m_i m_j c^2}\,[f(r,r_0)\,\underset{\sim}{p}_i\cdot\underset{\sim}{p}_j + \frac{g(r,r_0)}{r^2}\,\underset{\sim}{r}\cdot(\underset{\sim}{r}\cdot\underset{\sim}{p}_i)\underset{\sim}{p}_j]$$

$$-\frac{\pi}{2}\frac{\hbar^2}{c^2}\,f_D(r,r_0)\,[\frac{1}{m_i^2} + \frac{1}{m_j^2} + \frac{4\underset{\sim}{\sigma}_i\cdot\underset{\sim}{\sigma}_j}{3m_i m_j}]$$

where

$$r = |\underset{\sim}{r}_i - \underset{\sim}{r}_j|$$

$$f_D(r,r_0) = \frac{1}{2\pi r_0^2}\left(\frac{1}{r} - \frac{1}{4r_0}\right)e^{-r/r_0}$$

$$f(r,r_0) = \frac{1}{r}\left(1-e^{-r/r_0}\right) + \frac{1}{r_0}e^{-r/r_0}$$

$$g(r,r_0) = \frac{1}{r}\left(1-e^{-r/r_0}\right) - \frac{1}{r_0}e^{-r/r_0} \qquad (3)$$

and C is an overall constant, σ_i the usual Pauli matrices and m_i the quark masses. To solve for the baryonic masses we use the Feshbach-Rubinow method (appendix B.1). This leads to a Schrödinger equation in the variable $R = \frac{1}{2}(r_{13}+r_{23}+\eta r_{12})$ of the form

$$(\hat{T} + v_{FR}(R))\phi(R) = E_{eff}\phi(R) \qquad (4)$$

where $\hat{T}$ is a kinetic energy operator, v_{FR} the F-R effective potential and E_{eff} = const E where E is the binding energy of the system. In terms of $u(R) = R^{5/2}\phi(R)$ the F-R equation becomes

$$-\frac{\hbar^2}{4m_{eff}}\left(u'' - \frac{15u}{4R^2}\right) + \frac{\hbar^2\alpha_s}{12m_1c^2}\left[\frac{d}{dR}(u'v_p) - \frac{5u}{2R}\left[\frac{3v_p}{2R} + v_p'\right]\right]$$

$$+ v_{eff}u = 2\xi(E + 2C\alpha_s)u \qquad (5)$$

where for unequal mass and force bonds, which we refer to as the asymmetric case, the quark masses $m_1 = m_2 \neq m_3$. The effective mass, m_{eff}, (appendix B.1) is given by

$$\frac{1}{m_{eff}} = [\frac{\eta^2+1}{m_1} + \frac{1}{m_3}]2\xi + [\frac{2}{m_1} + \frac{1}{m_3}]\eta\zeta$$

where

$$2\xi = \frac{8(\eta^2+5\eta+8)}{15(\eta+1)^5}$$

and

$$\zeta = \frac{8(\eta+5)}{15(\eta+1)^5} .$$

For the symmetric system where all quark masses and force bonds are equal and $\eta = 1, m_{eff} = m$, where m is the quark mass. The effective potential $V_{eff} = V_{coul}(R)+V_{spin}(R)+V_{conf}(R)$ where $V_{coul}(R)$ is the effective coulomb potential ($\sim \frac{1}{R}$) and V_{conf} the confining potential ($\sim R$). V_p derives from the momentum dependent terms in interaction (3) and the explicit forms for V_{eff} and V_p are given in appendix B.2.

Using the appropriate quark composition (appendix A.1) equation (5) may be solved numerically to give the S-state masses of the baryons. For this purpose we choose $m_u = m_d$ and restrict m_u/m_s to be ~ 0.6.

In the standard quark model the baryon magnetic moments (μ_B) may be written in terms of the quark magnetic moments (μ_q) and hence from a knowledge of μ_B one may find μ_q (see e.g., Li 78). If one then assumes the quarks to be pure Dirac particles the ratio $m_u/m_s \sim 0.7$ and $m_u \sim m_d \sim 336$ MeV. DGG (De 75) found that $m_u/m_s \sim 0.62$ which we consider to be a reasonably good guideline for our purposes.

The parameters m_u, m_s, α_s and the constant C are then varied to fit the masses of N(939), $N^*(1470)$, $\Delta(1232)$ and $\Lambda(1116)$. It was possible to obtain several parameter sets but in each case, as mentioned in the preceeding chapter, the system is relativistic. The best set was obtained with $m_u c^2$ = 475 MeV, $m_s c^2$ = 730 MeV, α_s = 258 MeV-fm, a = 1.03 fm^2 and $2C\alpha_s$ = 90 MeV. The results for this set are given in Table II.1.1 and one can clearly see that the system is relativistic. For the nucleon, the rms radius is 0.28 fm and $\langle T\rangle_N/3m_u \approx 1.4$. Even for the Ω particle $\langle T\rangle_\Omega/3m_s \approx .6$ which corresponds to $\langle v^2/c^2\rangle \approx 0.5$ for the quark. For the asymmetric baryons we have shown in appendix B.2 that

$$\frac{\langle T_3\rangle}{\langle T_1\rangle} = \frac{2m_1}{m_3} \quad \frac{2\eta^2+10\eta+8}{\eta^4+5\eta^3+11\eta^2+15\eta+8} \tag{6}$$

where T_i is the kinetic energy of the ith quark.

For $\Xi(1533)$ $m_1 = m_s$, $m_3 = m_u$ and we find $\langle T_3\rangle = 1.29\langle T_1\rangle$ and thus $\frac{\langle T_3\rangle}{m_u} \approx .94$ while $\frac{\langle T_1\rangle}{m_s} \approx 0.47$.

From these results it is evident that the Fermi-Breit interaction is unsuitable for the baryonic problem within a non-relativistic framework. From Table II.1.1 one can see that, in magnitude, the contribution to the binding energy of the Coulomb potential is of the same order as the kinetic energy. From the usual virial theorem this seems to imply that the potential is Coulomb dominated and this may be part of the difficulty.

TABLE II.1.1 Results of Fermi-Breit Potential in the Feshbach Rubinow Approximation[a]

[a] The particle masses (in Mev) in the last column are obtained by numerical solution of equation (5) and the resulting wavefunctions are used to find the expectation values (in Mev) shown, where T refers to the total kinetic energy, V_{coul} the coulomb potential, V_{conf} the ramp potential, V_{spin} the spin-spin potential and V(p) the momentum-dependent potential. In addition there is a constant potential of -90 Mev. The rms charge radius (in fm) is denoted by $\langle r^2\rangle^{\frac{1}{2}}$ and is given for the highest charged state.

TABLE II.1.1 Results of Fermi-Breit Potential[b] in the Feshbach-Rubinow Approximation

	η	Restmass	$\langle T\rangle$	$\langle V_{coul}\rangle$	$\langle V_{conf}\rangle$	$\langle V(p)\rangle$	$\langle V_{spin}\rangle$	$\langle r^2\rangle^{1/2}$	Mass
N(939)	1.	1425	1973	-1746	208	-924	93	0.28	939
N*(1470)	1.	1425	1499	-1100	392	-685	48	0.53	1489
Δ(1233)	1.	1425	1085	-1294	273	-451	286	0.37	1233
Ω(1672)	1.	2190	1257	-1728	206	-379	193	0.28	1649
Λ(1116)	1.37	1680	1923	-1802	201	-816	17	-	1113
Σ(1193)	0.74	1680	1575	-1666	216	-616	100	0.34	1201
Ξ(1318)	1.01	1935	1660	-1830	196	-574	40	0.18	1323
Σ(1385)	0.90	1680	1087	-1382	256	-399	251	0.40	1403
Ξ(1533)	1.25	1935	1140	-1523	233	-372	221	0.21	1543

b For notation see note on previous page.

We suspect that the quark-quark potential may be substantially modified as a consequence of the coupling being a function of the momentum transfer. This may be understood by first briefly reviewing the usual Q.E.D. derivation of the Fermi-Breit interaction. This is obtained by taking the Fourier transform of some $v(\underset{\sim}{q})$ arising from one photon exchange (Be 71). The leading term $\sim \frac{\alpha}{q^2}$, where α is the coupling constant, gives rise to the usual Coulomb potential, while the higher order contributions yield the remaining Fermi-Breit terms.

By analogy, the Q.C.D. quark-quark potential deriving from gluon exchange is taken to be

$$V_{qq} \sim \int d\underset{\sim}{q}\; e^{i\underset{\sim}{q}\cdot\underset{\sim}{r}}\, \frac{\alpha(q^2)}{q^2}\,(1 + \cdots\cdots) \;. \qquad (7)$$

The asymptotic freedom argument given by De Rújula et al. (De 75) is equivalent to approximating (7) by

$$V_{qq} \sim \alpha_s \int d\underset{\sim}{q}\; e^{i\underset{\sim}{q}\cdot\underset{\sim}{r}}\, \frac{1}{q^2}\,(1 + \cdots\cdot) \qquad (8)$$

which of course leads to the usual Fermi-Breit interaction since α_s is taken constant. This approximation is questionable. Recently, Levine and Tomozawa (Le 79) have made some interesting advances in finding $\alpha(q^2)$ and the leading contribution to V_{qq} which is quite different from a Coulomb potential.

This seems to imply that not only the Coulomb potential but also the short-range spin term may be quite modified as a result. We do not pursue this approach further but rather, adopt the attitude that the choice of V_{qq} should not be restricted to the Fermi-Breit potential. We thus examine some simple ad hoc potentials, $V_{q\dot{q}}$, and as we show in the next section, the short-range nature of the Fermi spin-spin interaction seems also to be unsuitable in the context of a non-relativistic model.

II.2 Some Simple Potentials

In this section we show that some simple phenomenological potentials, that contain a mass dependent spin term and exhibit the confinement property, may be used to fit the ground state masses without the kinetic energy being unreasonably high. As an example, we consider the harmonic interaction

$$V_{ij} = \underset{\sim}{F}_i \cdot \underset{\sim}{F}_j \{- \frac{\kappa}{2}[1+(m_i m_j)^n \lambda \underset{\sim}{\sigma}_i \cdot \underset{\sim}{\sigma}_j] r^2 + C\} \qquad (9)$$

where $\langle \underset{\sim}{F}_i \cdot \underset{\sim}{F}_j \rangle = -\frac{2}{3}$ for qq, $-\frac{4}{3}$ for $q\bar{q}$ and κ, λ and C are parameters adjusted to fit the masses of N and Δ and the rms radius of the nucleon which we restrict to be in the range 0.6 to 0.7 fm. Our choice is also influenced by the $N^*(1470)$ mass but as we see from the results, this viewpoint is questionable due to the larger relativistic corrections of the excited states.

Equation (9) offers the advantage of being exactly solvable for both mesonic and baryonic systems. The value $n = -\frac{1}{2}$ was found to give the best mass splittings while we chose $\kappa = 437.5$ MeV fm^{-2}, $\lambda = 101.6$ MeV and C = 433.5 MeV. We fixed the quark masses $m_u = m_d = 336$ MeV and a good overall fit for the strange baryons was obtained by setting $m_s = 595$ MeV.

For the three quark system the Hamiltonian with interaction (9) may be written

$$H = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \frac{p_3^2}{2m'} + \frac{1}{2} b_0 (r_{13}^2 + r_{23}^2 + (1+\varepsilon) r_{12}^2) - 2C \quad (10)$$

where

$$b_0 = \frac{2\kappa}{3} \{1 + (mm')^n \lambda \langle \underset{\sim}{\sigma}_1 \cdot \underset{\sim}{\sigma}_3 \rangle\}$$

$$b_0 \varepsilon = \frac{2\kappa}{3} (mm')^n \lambda \langle \underset{\sim}{\sigma}_1 \cdot \underset{\sim}{\sigma}_2 - \underset{\sim}{\sigma}_1 \cdot \underset{\sim}{\sigma}_3 \rangle \;.$$

This may be rewritten in terms of the Jacobi co-ordinates $\underset{\sim}{R} = [m(\underset{\sim}{r}_{13} + \underset{\sim}{r}_{23}) + m'\underset{\sim}{r}_{12}]/M$, $\underset{\sim}{\rho} = (\underset{\sim}{r}_{23} - \underset{\sim}{r}_{13})/\sqrt{2}$ and $\underset{\sim}{\mu} = (\underset{\sim}{r}_{23} + \underset{\sim}{r}_{13} - 2\underset{\sim}{r}_{12})/\sqrt{6}$ where M is the total quark mass. After removing the center of mass motion (10) reduces to two uncoupled oscillators with

$$H_\rho = \frac{p_\rho^2}{2m} + b_0 \rho^2 (\frac{3}{2} + \varepsilon)$$

$$H_\mu = \frac{p_\mu^2}{2m_\mu} + \frac{3}{2} b_0 \mu^2 \quad (11)$$

where $m_\mu = \frac{3mm'}{M}$.

The baryon masses calculated in this way are presented in the second column of tables II.2.1 and II.2.2. For these parameters we find for the nucleon $\langle T \rangle_N / 3m_u \approx 0.40$ which corresponds to $\langle v^2/c^2 \rangle \approx 0.44$ which we regard as marginally relativistic. For the $\Xi(1533)$ we now have $\langle T_1 \rangle / m_s \approx 0.19$ and $\langle T_3 \rangle / m_u \approx 0.64$ which is a considerable improvement over the value of 0.94 found in the last section. To determine these values we have used

$$\frac{\langle T_3 \rangle}{\langle T_1 \rangle} = \frac{4m_1}{3m_3} \sqrt{\frac{3m_3}{M(2\varepsilon/3+1)}} \qquad (12)$$

which may easily be derived from equations (11).

By solving interaction (9) in the Feshbach-Rubinow approximation, we are able to test the accuracy of the three-body method. For this potential, the F-R equation (5) given previously has V_p set to zero and α_s replaced by $\kappa/2$. The effective potential becomes (see appendix B.1)

$$V_{eff}(R) = [A_x(2,\eta) + \frac{1}{2}(1+\varepsilon)A_z(2,\eta)]b_0R^2 \qquad (13)$$

where $A_i(n,\eta)$ are given in the appendix.

For the symmetric case this reduces to $6b_0R^2/35$. Solving, we find the nucleon mass to be overestimated by about 19.7 MeV and the constant C in equation (9) is readjusted accordingly to 443.4 MeV. The results obtained by the F-R method are found to be satisfactory, as shown in tables II.2.1 and II.2.2. By way of comparison, we find $\langle T\rangle_N/3m_u \approx 0.41$ and using equation (6) from the previous section, that $\langle T_1\rangle/m_s \approx 0.21$ and $\langle T_3\rangle/m_u \approx 0.59$ for $\Xi(1533)$.

The S-state masses of the 1^- mesons were also calculated using the same interaction (9) with $\langle \underset{\sim}{F}_i \cdot \underset{\sim}{F}_j\rangle = -\frac{4}{3}$. The results, displayed in table II.2.3 are found to be reasonable. As discussed by DGG (De 75), we do not expect the 0^- mesons to be reproduced by the same interaction. We find them to be greatly overestimated.

An even simpler form of harmonic interaction, proposed for the non-strange baryons by Liberman (Li 77) may also be solved analytically. Here we use

$$V_{ij} = \underset{\sim}{F}_i \cdot \underset{\sim}{F}_j \{-\frac{\kappa r^2}{2} - (m_i m_j)^{-1}\lambda \underset{\sim}{\sigma}_i \cdot \underset{\sim}{\sigma}_j + C\} . \qquad (14)$$

This may be solved in the same way as interaction (9) and we find a comparable fit with $m_u = m_d = 336$ MeV, $m_s = 574$ MeV, $\kappa = 305.3$ MeV fm^{-2}, $\lambda/m_u^2 = 73.3$ MeV and C = 359.5 MeV. From the results, presented in tables II.2.1 and II.2.2, we

RESULTS FOR THE SIMPLE MODELS[a]

[a]The notation used in the following tables is the same as that given in table II.1.1. For the symmetric baryons (N, Δ and Ω) the value of $\eta = 1$.

TABLE II.2.1 <u>Results of Simple Model Calculations for Octet Baryons</u>

Model	Eq. (9)		Eq. (14)	Eq. (15)	
				Set (a)	Set (b)
Method	Exact	F-R	Exact	F-R	F-R
N(939)	939	939	939	939	939
$\langle T\rangle_N$	399	409	325	468	503
$\langle r^2\rangle^{1/2}_N$	0.66	0.67		0.63	0.61
$N^*(1471)$	1471	1484	1470	1470	1475
$\langle T\rangle_{N^*}$	665	681	591	645	662
$\langle r^2\rangle^{1/2}_{N^*}$		0.86		0.89	0.88
$\Lambda(1116)$	1103	1106	1118	1113	1112
$\langle T\rangle_\Lambda$	352	363	296	439	485
η_Λ		0.84		0.66	1.04
$\Sigma(1193)$	1187	1187	1199	1177	1194
$\langle T\rangle_\Sigma$	394	403	336	460	453
$\langle r^2\rangle^{1/2}_\Sigma$		0.74		0.70	0.72
η_Σ		1.15		0.97	0.68
$\Xi(1318)$	1337	1337	1342	1320	1336
$\langle T\rangle_\Xi$	340	349	289	422	438
$\langle r^2\rangle^{1/2}_\Xi$		0.40		0.41	0.60
η_Ξ		1.53		1.41	1.10

TABLE II.2.2 Results of Simple Model Calculations for Decuplet Baryons

Model	Eq. (9)		Eq. (14)	Eq. (15)	
				Set (a)	Set (b)
Method	Exact	F-R	Exact	F-R	F-R
$\Delta(1232)$	1231	1239	1232	1245	1234
$\langle T\rangle_\Delta$	545	559	472	569	404
$\langle r^2\rangle_\Delta^{1/2}$		0.59		0.57	0.67
$\Sigma(1385)$	1385	1391	1370	1390	1378
$\langle T\rangle_\Sigma$	493	505	422	532	391
$\langle r^2\rangle_\Sigma^{1/2}$		0.67		0.66	0.76
η_Σ		0.86		0.83	0.80
$\Xi(1533)$	1540	1542	1513	1533	1522
$\langle T\rangle_\Xi$	441	451	374	493	375
$\langle r^2\rangle_\Xi^{1/2}$		0.52		0.48	0.53
η_Ξ		1.19		1.24	1.25
$\Omega(1672)$	1695	1694	1663	1673	1668
$\langle T\rangle_\Omega$	389	398	330	453	354
$\langle r^2\rangle_\Omega^{1/2}$		0.51		0.48	0.54

TABLE II.2.3 Results of Simple Model Calculations for Mesons[a]

Model	Eq. (9)	Eq. (14)	Eq. (15) Set (a)	Eq. (15) Set (b)
$J^P = 1^-$				
$\rho(773)$	723	750	714	735
$\langle T\rangle_\rho$	315	279	342	237
$K^*(892)$	893	893	872	887
$\langle T\rangle_{K^*}$	270	231	308	228
$\phi(1020)$	1060	1054	1022	1039
$\langle T\rangle_\phi$	224	193	272	210
$J^P = 0^-$				
$\pi(138)$	261	360	325	269
$\langle T\rangle_\pi$	84	84	212	427
$K(495)$	628	670	608	605
$\langle T\rangle_K$	138	120	220	335

a The calculations for equations (9) and (14) were analytical while equation (15) was treated numerically. The results for the scaler mesons are given for completeness but cannot be fitted by the same parameters (see text).

find $\langle T\rangle_N\ 3m_u \gtrsim 0.32$ which corresponds to $\langle v^2/c^2\rangle \gtrsim 0.39$ and for $\Xi(1533)$, $\langle T_1\rangle/m_s \gtrsim 0.17$ and $\langle T_3\rangle/m_u \gtrsim 0.54$ which may be calculated from Eq. (12) with $\varepsilon = 0$. The meson masses calculated from equation (14) are displayed in table II.2.3.

We consider two further simple potentials of the form

$$V_{ij} = \underset{\sim}{F}_i \cdot \underset{\sim}{F}_j [-\alpha_s \{r^{m_1} + \frac{\lambda}{(m_i m_j)^n} \underset{\sim}{\sigma}_i \cdot \underset{\sim}{\sigma}_j\, r^{m_2} - C\}] \ . \quad (15)$$

The F-R equation now takes the form (5) with V_p' set to zero and V_{eff} given by (see appendix B.1)

$$V_{eff}(R) = \frac{2}{3}\ \alpha_s [\{2A_x(m_1,\eta)+A_z(m_1,\eta)\}R^{m_1} + \frac{\lambda}{(m_i m_j)^n} \{$$

$$2A_x(m_2,\eta)\langle \underset{\sim}{\sigma}_1 \cdot \underset{\sim}{\sigma}_3\rangle + A_z(m_2,\eta)\langle \underset{\sim}{\sigma}_1 \cdot \underset{\sim}{\sigma}_2\rangle\}R^{m_2}] \quad (16)$$

We fit the baryons by the same procedure as used previously and for set (a) we take $m_1 = m_2 = 1$, $n = 1/2$, $m_u = m_d = 336$ MeV, $m_s = 595$ MeV, $\alpha_s = 555.6$ MeV fm^{-1}, $\lambda = 49.33$ MeV and $C = 1.324$ fm. For set (b) we use the values $m_1 = 1$, $m_2 = -1$, $n = 1$, $m_u = m_d = 336$ MeV, $m_s = 585$ MeV, $\alpha_s = 445.5$ MeV fm^{-1}, $\lambda = 0.3625\ (c\hbar)^2$ and $C = 1.418$ fm. The results for these two cases of equation (15) for the baryons and mesons are set out in tables II.2.1, II.2.2 and II.2.3. Here again we see that the calculated masses are quite reasonable. Set (b), which in the nucleon has $\langle v^2/c^2\rangle \gtrsim 0.50$ for the quark, is of particular interest since 1/r-type spin-dependence arises naturally if the ramp confinement is generated by Lorentz vector exchange (Sc 76, Kh 78, Ta 79).

In order to demonstrate the importance of a long-range spin-spin interaction and investigate the accuracy of a perturbation estimate we consider the potential

$$V_{ij} = \underset{\sim}{F}_i \cdot \underset{\sim}{F}_j [- \frac{\kappa}{2} r^2 - (m_i m_j)^{-1} \lambda f_D(r, r_0) \underset{\sim}{\sigma}_i \cdot \underset{\sim}{\sigma}_j + C] \qquad (17)$$

where $f_D(r,r_0)$ is defined as in equation (3). We fix $m_u = m_d$ = 336 MeV and κ = 241.5 MeV fm^{-2} and then for a series of values of r_0, interaction (17) is solved by the Feshbach-Rubinow method and λ and C adjusted to fit the masses of N(939) and Δ(1232). Using oscillator wavefunctions for a perturbative estimate, the nucleon kinetic energy is found to be 355 MeV while the rms radius 0.7 fm. These values, together with the estimated Δ-N splittings, may be compared with the results of the dynamical calculation shown in table II.2.4. It can be seen that for larger ranges the perturbation estimate is adequate but as $r_0 \lesssim 1$ fm the nucleon kinetic energy rises $\gtrsim$ 1700 MeV. We should point out that the dramatic effect illustrated here is suppressed somewhat for larger values of κ.

Similar trends are exhibited by the $q\bar{q}$ system, which we demonstrate with the K-meson. The parameters κ and C are fixed as above and m_s is taken to be 500 MeV, then λ is adjusted to fit the K mass obtained by solving the Schrödinger equation with interaction (17). For r_0 = 3 fm the kinetic

TABLE II.2.4 Investigation of the Range of the Spin-Spin Potential[a]

Method			F-R		Perturbative Estimate
r_0	$\lambda/(\hbar c)^3$	C	$\langle r^2\rangle_N^{1/2}$	$\langle T\rangle_N$	$M_\Delta - M_N$
5	177.5	321.3	0.66	415	283
2	37.8	319.7	0.65	430	279
1	14.5	316.6	0.64	453	270
0.5	7.13	311.2	0.61	501	248
0.1	2.60	288.8	0.49	1711	70

a The results tabulated are obtained from Eq. (17). Perturbation theory estimates the nucleon rms radius to be 0.7 fm and the kinetic energy to be 355 MeV (see text). The last column gives the estimated Δ-N mass splitting in MeV.

energy is 300 MeV whereas for $r_0 \lesssim 0.2$ fm it rises above 950 MeV.

From these calculations we conclude that for an attractive spin-spin interaction of range << 1 fm the kinetic energies of the quarks become very large and subsequently relativistic corrections play a role of increasing importance. We have also shown that perturbative estimates can be misleading for such short-range potentials. Furthermore, we have demonstrated that it is possible to construct simple phenomenological interactions that fit the masses of the baryons and vector mesons within a reasonably non-relativistic framework. Such potentials are characterized by confinement and a long-range spin-spin interaction.

We should also mention that the potentials we have used for the lighter hadrons are unable to fit the charmonium spectrum although for a suitable choice of the charmed quark mass (m_c) the ground state may always be fitted. In the succeeding chapter we attempt to remedy this situation by considering a logarithmic potential.

CHAPTER III

THE LOGARITHMIC POTENTIAL

III.1 The Logarithmic Model

As we have shown in the last section, it is possible to fit the baryon masses with some simple potentials such that the system is marginally relativistic. We now attempt to fit the hadrons in such a way as to minimize the effects of the relativistic corrections. To this end, we fit the masses of the least relativistic mesons in order to determine the parameters. This approach has the added advantage that we do not introduce uncertainties in the solutions since the two-body Schrödinger equation is numerically solvable.

We now consider a two-body interaction which takes the form

$$V_{ij} = \underset{\sim}{F}_i \cdot \underset{\sim}{F}_j [-\alpha_s \{\ln(\Lambda r_{ij}) + \frac{\beta}{m_i m_j} \underset{\sim}{\sigma}_i \cdot \underset{\sim}{\sigma}_j\}] \qquad (18)$$

where, as previously, $\underset{\sim}{\sigma}_i$ are the Pauli matrices and $\langle \underset{\sim}{F}_i \cdot \underset{\sim}{F}_j \rangle =$ $-2/3$ for qq and $-4/3$ for $q\bar{q}$. As remarked in the introductory chapter, potential (18) has the remarkable property that, in any state, the kinetic energy $\langle T \rangle$ (appendix C.1) is

$$\langle T \rangle = -\frac{\alpha_s}{2} \sum_{i>j} \langle \underset{\sim}{F}_i \cdot \underset{\sim}{F}_j \rangle . \qquad (19)$$

Thus for baryons with three qq bonds the kinetic energy is simply α_s, while for mesons it is $2\alpha_s/3$.

For the purpose of investigating the mesonic systems, we transform to the Jacobi coordinates $\underset{\sim}{R} = (m_1\underset{\sim}{r}_1+m_2\underset{\sim}{r}_2)/(m_1+m_2)$ and $\underset{\sim}{\rho} = \underset{\sim}{r}_1-\underset{\sim}{r}_2$, then after removing the center of mass motion, equation (17) leads to the Hamiltonian

$$H = m_1+m_2 + \frac{p_\rho^2}{2\mu} - \underset{\sim}{F}_i\cdot\underset{\sim}{F}_j \; (\alpha_s\{\ell n(\Lambda\rho) + \frac{\beta}{m_1 m_2}\underset{\sim}{\sigma}_1\cdot\underset{\sim}{\sigma}_2\}) \quad (20)$$

where μ is the usual reduced mass. In order to calculate the charmonium spectrum we add to equation (20) the usual centrifugal potential

$$v = \ell(\ell+1)\frac{\hbar^2}{2\mu\rho^2} \quad (21)$$

We find that the 1P-1S splitting for the logarithmic potential is a function only of the parameter α_s and thus we may determine its value from the 1P-1S splitting of 427 MeV in charmonium. A variational calculation using Coulomb wavefunctions predicts this splitting to be $7\alpha_s/9$ which suggests the value $\alpha_s = 549$ MeV, however from the numerical solution we find we need to take $\alpha_s = 537$ MeV. Hence, using equation (18) we see that for mesons the kinetic energy is 358 MeV while for baryons it is 537 MeV.

Since we have chosen a constant spin term in the potential, spin splittings between baryons of identical mass quarks may be simply determined and we find

$$\Delta(1232)-N(939) = \frac{4\alpha_s\beta}{m_u^2}$$

$$\Sigma(1193)-\Lambda(1116) = \frac{8\alpha_s\beta}{3m_u^2}\left(1 - \frac{m_u}{m_s}\right) . \qquad (22)$$

Now, since we insist that the above splittings be fitted, we take $m_u/m_s = 0.6$ and $\beta/m_u^2 = 0.1364$.

In order to find the baryonic masses we solve the interaction (18) by the F-R method. As we have already seen, this leads to an effective one body equation

$$-\frac{\hbar^2}{4m_{eff}}\left(u'' - \frac{15u}{4R^2}\right) + V_L u = 2\xi(E-V_s)u \qquad (23)$$

where we use the same notation as in the preceding chapter and

$$V_s = \frac{2\alpha_s\beta}{3} \sum_{i>j} \frac{\langle \underset{\sim}{\sigma}_i \cdot \underset{\sim}{\sigma}_j \rangle}{m_i m_j}$$

$$V_L = \frac{2\alpha_s}{3}\left[6\xi \ln\Lambda R + G(\eta)\right]$$

where $G(\eta)$ is given in appendix B.3 and for the symmetric case reduces to -199/600.

Now as we have already indicated α_s is chosen to be 537 Mev to reproduce the $\psi_{1p}-\psi_{1s}$ while the Δ-N and Σ-n splittings fix $m_u/m_s = 0.6$ and $\beta/m_u^2 = 0.1364$.

The value of m_s is determined by fitting $\phi(1019)$ and m_c from $\psi(3095)$ while the value of Λ is chosen that gives the best overall fit to the hadrons. With $\Lambda = 0.8\ \text{fm}^{-1}$, we find $m_s = 639$ Mev and $m_c = 1887$ Mev and we thus use

$m_u = 383.4$ Mev. The results of the hadron mass calculations based on equation (18) for these parameters are presented in table III.1.1.

As can be seen, the vector mesons are reproduced reasonably well as is the charmonium spectrum. The scaler mesons are overestimated as was the case for the other simple models, while the baryons are ~ 50 Mev too high over and above the underbinding introduced by the R-R approximation, which we expect to be ~ 20 Mev. Yet, the calculated Ω mass is only 26 Mev above the experimental value and we therefore suspect that the overestimated masses may arise as a result of neglected relativistic effects. For the $\langle T\rangle/3m_u \approx 0.47$ which corresponds to a quark velocity $\langle v^2/c^2\rangle \approx 0.48$, whereas for Ω, $\langle T\rangle/3m_s \approx 0.28$ while $\langle T_3\rangle/m_u \approx 0.47$ and $\langle T_1\rangle/m_s \approx 0.28$ for the Ξ particle. For comparison $\langle T\rangle/2m_c \approx 0.09$ in charmonium, which is equivalent to $\langle v^2/c^2\rangle \approx 0.16$ while $\langle T\rangle/2m_u \approx 0.47$ for the ρ particle. Thus, although we may consider charmonium to be non-relativistic we expect relativistic corrections to be of importance for the light hadrons. In the next section we attempt to improve the model by incorporating an estimate of the relativistic correction to the kinetic energy.

TABLE III.1.1 HADRON MASSES CALCULATED USING INTERACTION (18)[a]

MESONS	MASS	BARYONS	MASS	η
ψ_{1s}(3095)	3096	N(939)	1007	1.
ψ_{1p}(3522)	3524	N*(1470)	1470	1.
ψ_{2s}(3684)	3671	Δ (1232)	1300	1.
ψ_{1d}(3772)	3790	Ω (1672)	1698	1.
ψ_{3s}(4028)	3988	Λ (1116)	1181	0.804
ψ_{2d}(4160)	4058	Σ (1193)	1259	0.804
ψ_{4s}(4414)	4207	Ξ (1318)	1392	1.27
		Σ (1385)	1435	0.804
ϕ(1019)	1019	Ξ (1533)	1568	1.27
ρ(770)	753	Λ_c(2285)	2322	0.58
K*(892)	890			
D*(2010)	1997			
π(138)	363			
K(492)	655			
D(1867)	1918			

[a]Baryon masses are calculated by the F-R method. The meson kinetic energy is 358 Mev, while for baryons it is 537 Mev.

III.2 The Model with Ad Hoc Relativistic Corrections

In attempting a potential model calculation that includes both the heavy and light hadrons we feel that relativistic effects are not unimportant. We therefore attempt an estimation of the correction to the kinetic energy. In order to get some idea as to the order of magnitude we first make a crude estimate in the following way.

The two body interaction (18) leads to an effective one body Hamiltonian (20) for the meson system. To a first approximation then the kinetic energy operator, $p^2/2\mu$, may be considered as deriving from E_μ with

$$E_\mu \approx \sqrt{p^2+\mu^2} \quad (\approx \mu + \frac{p^2}{2\mu} - \frac{p^3}{8\mu^3} + \cdots) \ . \tag{24}$$

Now in the spirit of a closure approximation we consider

$$E_\mu \approx \sqrt{2\mu\langle T\rangle+\mu^2} \tag{24a}$$

and since $\langle T\rangle = 2\alpha_s/3$ for mesons we take the correction ΔT to be of the order

$$\Delta T \approx \sqrt{\frac{4\mu\alpha_s}{3} + \mu^2} - (\mu + \frac{2\alpha_s}{3}) \ . \tag{25}$$

In this approximation we consider in essence a particle of mass μ moving in a one-body field.

The same estimate may be applied to the F-R equation (23) and we then find for the baryons that the correction to the kinetic energy becomes

$$\Delta T \gtrsim \sqrt{2\mu_B\alpha_s+\mu_B^2} - (\mu_B+\alpha_s) \qquad (26)$$

where

$$\mu_B = 4\,\xi\, m_{eff}.$$

With this correction included the masses are refitted as before and we now take $m_s = 543$ Mev, $m_u = 326$ Mev, $m_c = 1790$ Mev and $\Lambda = 1.1\ fm^{-1}$. The results, which still suffer from overestimating the baryons, are not markedly different from those in the previous set. We list the masses together with the estimated correction in Table III.2.1.

We now try an improved method of estimating ΔT by a perturbative calculation. To accomplish this we use Gaussian wavefunctions to find the expectation value of the Hamiltonian

$$h = \sum_i (p_i^2+m_i^2)^{\frac{1}{2}} \qquad (27)$$

where the sum runs over the number of quarks. This may be readily done in momentum ($\vec{p}$) space.

For the mesons in the center of mass frame $\underset{\sim}{p}_1 = -\underset{\sim}{p}_2 \equiv \underset{\sim}{p}$ and we use the $\vec{p}$-space wavefunction

$$\psi(p) = \left[\frac{b}{\sqrt{\pi}}\right]^{3/2} e^{-\frac{1}{2}b^2p^2}$$

where the constant b is determined by the condition that $\langle p^2\rangle/2\mu = 2\alpha_s/3$. We then find

TABLE III.2.1 RESULTS WITH ΔT BASED ON EFFECTIVE ONE BODY CALCULATION[a]

MESONS			BARYONS			
$J^P = 1^-$						
	MASS	ΔT		MASS	ΔT	η
$\psi_{1s}(3095)$	3096	-52	N(939)	1006	-257	1.
$\psi_{1p}(3522)$	3524	-52	N*(1470)	1470	-257	1.
$\psi_{2s}(3684)$	3671	-52	Δ(1232)	1299	-257	1.
$\psi_{1d}(3772)$	3790	-52	Ω(1672)	1629	-210	1.
$\psi_{3s}(4028)$	3988	-52				
$\psi_{2d}(4160)$	4058	-52	Λ(1116)	1157	-256	1.18
$\psi_{4s}(4414)$	4207	-52	Σ(1192)	1235	-256	1.18
			Σ*(1385)	1411	-256	1.18
D*(2010)	2010	-111	Ξ(1318)	1319	-264	1.70
φ(1019)	1002	-112	Ξ*(1533)	1495	-264	1.70
K*(892)	894	-129				
ρ(770)	782	-143	$\Lambda_c(2286)$	2328	-221	0.90
$J^P = 0^-$						
D(1867)	1939	-111				
K(492)	659	-129				
π(138)	391	-143				

[a]ΔT is calculated using equations (25) and (26).

$$<(p^2+m_i^2)^{\frac{1}{2}}> = \frac{1}{b}\, J_1(b^2 m_i^2) \qquad (28)$$

with $b^2 = \frac{9}{8\mu\alpha_s}$

and

$$J_1(x) = \frac{4}{\sqrt{\pi}} \int_0^\infty dy\; y^2 (y^2+x)^{\frac{1}{2}} e^{-y^2}$$

$$= \frac{1}{\sqrt{\pi}}\, x\, e^{x/2} K_1(x/2) \qquad (29)$$

where $K_1(x/2)$ is the modified bessel function. The kinetic energy correction for mesons in this approximation then becomes

$$\Delta T \approx \sum_{i=1}^{2} [<(p^2+m_i^2)^{\frac{1}{2}}> - m_i] - 2\alpha_s/3 \; . \qquad (30)$$

Due to the judicious choice of Gaussian wavefunction, the three body system may be treated in a similar fashion. We have shown in the preceding chapter that the three body harmonic oscillator may be reduced to two uncoupled oscillators (equations (10) and (11)). Since the only asymmetry in interaction (18) arises as a result of mass differences we take $\varepsilon = 0$ in equations (11) and the solutions in $\vec{p}$-space, after removing center of mass motion, may be written

$$\psi(p_1, p_3) = \left(\frac{a^2\sqrt{3\kappa}}{\pi}\right)^{3/2} e^{-\frac{1}{2}a^2[2p_1^2 + \frac{(\kappa+1)}{2} p_3^2 + 2\underset{\sim}{p}_i \cdot \underset{\sim}{p}_2]}$$

where we have used $\underset{\sim}{p}_1 = \underset{\sim}{p}_\rho/\sqrt{2} + \underset{\sim}{p}_\mu/\sqrt{6}$, $\underset{\sim}{p}_\lambda = -\underset{\sim}{p}_\rho/\sqrt{2} + \underset{\sim}{p}_\mu/\sqrt{6}$, $\underset{\sim}{p}_3 = -2\underset{\sim}{p}_\mu/\sqrt{6}$ and $\kappa = (M/3m')^{\frac{1}{2}}$ is a mass symmetry constant.

As was done above for the mesons, we fix the parameter a by insisting that

$$\frac{\langle p_1^2\rangle}{2m} + \frac{\langle p_2^2\rangle}{2m} + \frac{\langle p_3^2\rangle}{2m'} = \alpha_s \qquad (32)$$

from which we find

$$a^2 = \frac{1}{4\alpha_s m}\ [3 + \frac{2m/m'+1}{\kappa}]\ . \qquad (33)$$

The baryonic kinetic energy correction is then given by

$$\Delta T \approx 2\langle(p_1^2+m^2)^{\frac{1}{2}}\rangle + \langle(p_3^2+m'^2)^{\frac{1}{2}}\rangle - (\alpha_s+2m+m') \qquad (34)$$

where

$$\langle(p_3^2+m'^2)^{\frac{1}{2}}\rangle = \frac{1}{a}\sqrt{\frac{2}{3\kappa}}\ J_1[\frac{3\kappa}{2}\ (m'a)^2]$$

$$\langle(p_1^2+m^2)^{\frac{1}{2}}\rangle = \frac{1}{a}\sqrt{\frac{3\kappa+1}{6\kappa}}\ J_1[\frac{6\kappa}{3\kappa+1}\ (ma)^2]$$

and $J_1(x)$ is given in equation (29). One might note that for equal mass sytems $\kappa = 1$ and $\langle(p_1^2+m^2)^{\frac{1}{2}}\rangle = \langle(p_3^2+m^2)^{\frac{1}{2}}\rangle$.

In order to ascertain the degree of variance in the kinetic energy correction, we compare, in table III.2.2, the two above methods with a W.K.B. approximation and a perturbative estimate using coulomb wavefunctions. For this purpose we use m_c = 1790 Mev, m_s = 543 Mev, m_u = 326 Mev and α_s = 358 Mev as before. Although the coulomb wavefunction is probably a more realistic wavefunction for a logarithmic

TABLE III.2.2 COMPARISON OF THE VALUES OF ΔT FOR MESONS BY DIFFERENT METHODS[a]

METHOD	EFFECTIVE 3-BODY Eq. (25)	WKB	COULOMB WFN	GAUSS WFN Eq. (30)
ψ	-52	-38	-41	-25
ϕ	-112	-84	-83	-61
K	-129	-102	-100	-78
ρ	-143	-109	-107	-84
D	-111	-119	-115	-96

[a]The values of ΔT are given for the parameters quoted in the text. No details of the WKB method or the method using coulomb wavefunctions is provided.

potential. The three body calculation is non-trivial. Thus we feel that using Gaussian wavefunctions tends to underestimate ΔT, and this effect may be more important for the baryons and we therefore concede that such estimates are not entirely satisfactory, however the Gaussion wavefunction method may be used for both mesonic and baryonic systems and the results are considered to serve as an ad hoc estimate.

Using equations (30) and (34) based on the Gaussion wavefunction, we refit the masses by assuming that the corrections in the excited charmonium states are the same as that in the ground state and thus we retain the value $\alpha_s = 537$ Mev. In tables III.2.3 and III.2.4 where we display the results, we have used $m_c = 1831$ Mev, $m_s = 592$ Mev, $m_u = 355.2$ Mev and $\Lambda = 0.95\ fm^{-1}$.

The baryon masses are still somewhat overestimated although the Ω -particle, which seems to be less relativistic (in the sense that ΔT is smaller), falls the expected 29 Mev or so above the experimental value. We therefore feel that within the context of this simple model the calculated masses are reasonably satisfactory and although the model tends to underestimate the baryon binding energies the mass splittings are well reproduced.

The mesons are fairly well reproduced by this model with the exception of an extra S-state at 4.21 Gev in the charmonium spectrum. As is suggested by Kang (Ka 79) this may be a fault inherent in the simple logarithmic potential.

TABLES III.2.3 AND III.2.4

HADRON MASSES CALCULATED FROM THE LOGARITHMIC POTENTIAL WITH A GAUSSION ESTIMATE FOR ΔT[a]

[a]The calculations are based on interaction (18) with kinetic energy corrections, ΔT, found from Eq. (30) for mesons and Eq. (34) for the baryons.

TABLE III.2.3 MESON MASSES[b]

	MASS (MeV)	ΔT
$\psi_{1s}(3095)$	3094	-24
$\psi_{1p}(3522)$	3511	-24
$\psi_{2s}(3684)$	3669	-24
$\psi_{1d}(3772)$	3788	-24
$\psi_{2d}(\ ?\ )$	3886	-24
$\psi_{3s}(4028)$	3985	-24
$\psi_{2d}(4160)$	4055	-24
$\psi_{3p}(\ ?\ )$	4129	-24
$\psi_{3d}(\ ?\ )$	4253	-24
$\psi_{4s}(4414)$	4205	-24
$\psi_{5s}(\ ?\ )$	4386	-24
$\phi(1019)$	1018	-58
F*(∿2140)	2083	-60
K*(892)	891	-74
$\rho(770)$	767	-80
D*(2010)	1969	-90
F(∿2020)	2030	-60
K(492)	656	-74
$\pi(138)$	377	-80
D(1867)	1893	-90

	MASS (GeV)	ΔT
$\Upsilon_{1s}(9.46)$	9.46	-0.01
$\Upsilon_{1p}(\ ?\ )$	9.89	-0.01
$\Upsilon_{2s}(10.01)$	10.03	-0.01
$\Upsilon_{1d}(\ ?\ )$	10.15	-0.01
$\Upsilon_{2p}(\ ?\ )$	10.25	-0.01
$\Upsilon_{3s}(10.41)$	10.35	-0.01
$\Upsilon_{2d}(\ ?\ )$	10.42	-0.01
$\Upsilon_{3p}(\ ?\ )$	10.50	-0.01
$\Upsilon_{4s}(\ ?\ )$	10.57	-0.01
PREDICTIONS		
$M(u\bar{b})$	5.27	-0.11
$M(s\bar{b})$	5.36	-0.08
$M(c\bar{b})$	6.31	-0.03

[b]The value of m_b = 5.195 Gev was chosen to fit the ground state of $\Upsilon(b\bar{b})$. This value is used in masses of the predicted "bottomed" 1^- mesons given above and baryons given overleaf.

TABLE III.2.4 BARYON MASSES[c]

	MASS (MeV)	ΔT	η		MASS (MeV)	ΔT	η
N(939)	1028	-120	1.	Δ(1232)	1321	-120	1.
N*(1470)	1491	-120	1.	Σ*(1385)	1444	-114	.804
Λ(1116)	1190	-114	.804	Ξ*(1533)	1569	-103	1.27
Σ(1192)	1269	-114	.804	Ω(1672)	1697	- 86	1.
Ξ(1318)	1393	-103	1.27				
Λ_c(2273)	2306	-127	.56				
Σ_c(∿2430)	2463	-127	.56				

PREDICTIONS

	MASS (GeV)	ΔT	η		MASS (GeV)	ΔT	η
Ξ_{cc}(?)	3.61	-0.11	2.23	Ξ^*_{cc}(?)	3.67	-0.11	2.23
				Σ^*_c (?)	2.52	-0.13	0.56
Ω_c(?)	2.73	-.09	.64	Ω^*_c (?)	2.76	-0.09	0.64
Ω_{cc}(?)	3.76	-.07	1.73	Ω^*_{cc}(?)	3.79	-0.07	1.73
				Ω_{ccc}(?)	4.81	-0.04	1.
				Ω_{bbb}	14.27	-0.02	1.

[c]In this calculation we find the nucleon rms charge radius to be 0.58 fm. The notation used for the charmed bayons is the same as used by Lichtenberg (Li 78).

The bottomonium spectrum is found using $m_b = 5.195$ GeV to fit the ground state and the three known states come reasonably well. Using the same parameters we also calculate some as yet unobserved hadrons in the charmed and "bottomed" sector.

Thus we have been able to fit the hadron masses reasonably well by including an ad hoc relativistic correction to the energy, however, we feel that our approach may be underestimating the corrections and this may be more severe for the more relativistic baryons. In addition we have retained the simple logarithmic potential and if one is to be consistent, relativistic corrections to the potential should also be included.

CHAPTER IV

CONCLUSIONS

IV.1 Summary

In this work we calculated hadron masses by dynamically solving the Schrödinger equation where the three-body baryonic system was treated in the Feshbach-Rubinow approximation. Initially, we investigated a suitably regularized Fermi-Breit interaction of the form suggested by De Rújula et al (De 75) and we found that, although we could fit the baryon masses, the kinetic energies were very large, which led us to believe that such a potential was unsuitable in a non-relativistic model. We felt that this was in part due not only to the fact that it was Coulomb dominated but also as a result of the inherent short-range nature of the spin potential.

Consequently, we examined some phenomenological interactions that retain the characteristic confining property and a mass-dependent spin term. We illustrated that it seems necessary for such a spin-spin potential to have long range and furthermore, that as the range is reduced, not only does the kinetic energy rise but that a perturbative estimate of the spin potential progressively deteriorates. In addition, we showed that by using such simple potentials, it was possible not only to fit the light baryons but also the low-lying vector mesons. Unfortunately, the charmonium spectrum was unable to be reproduced using the same parameters. Since the light hadrons were found to be marginally relativistic while charmonium is expected to be reasonably non-relativistic this is perhaps not surprising.

We therefore adopted the approach of determining the potential parameters by fitting the least relativistic hadrons and attempting to include a perturbative-type estimate of the relativistic kinetic energy corrections. This was done for a simple logarithmic potential which has the remarkable property that the kinetic energy is a constant. We found that although the vector mesons could be fitted well the baryons were overestimated. This is due in part to the variational character of the F-R approximation and more importantly to the probable underestimated corrections. Although this point deserves further study, the inadequacy of the estimate is supported by the fact that for the Ω-particle, where the correction is not too large, the mass is reasonably well reproduced. One may therefore expect the less relativistic baryon masses, such as the charmed omega, to be predicted fairly well. We therefore feel that relativistic effects play an essential role if one tries to use an effective potential to consistently fit the hadron masses and that this deserves further attention.

IV.2 Suggestions for Future Work

In the context of the potential model, the kinetic energy corrections need to be calculated more reliably. One simple approach would be to do this numerically using the unperturbed wavefunctions found by solving the Schrödinger equation. In addition the relativistic corrections to the potential should also be included.

This may perhaps best be accomplished by starting with a momentum space Q.C.D. potential, such as that proposed by Levine and Tomozawa (Le 79). One could then add the appropriate relativistic corrections and solve the integral Schrödinger equation in momentum space. This offers the advantage that kinetic corrections may be readily found using the wavefunctions so obtained.

It is interesting that one may use a potential approach in a non-relativistic model to determine the masses of some hadrons, but it would be more satisfying to be able to consistently calculate the masses of both the baryons and the mesons. This work is a step in that direction but it is felt that the subject merits further investigation.

APPENDIX A

APPENDIX A.1 QUARK COMPOSITION OF HADRONS

No details of the model are given here but an excellent review of the quark model is given by Lipkin (Li 73). We list below the quark compositions, but in this work we ignore electromagnetic splittings and thus for example we consider the n and p to be degenerate.

BARYONS

$J^P = 1/2^+$ PARTICLE	COMPOSITION	$J^P = 3/2^+$ PARTICLE	COMPOSITION
p	uud	Δ^{++}	uuu
n	udd	Δ^{+}	uud
Λ^{o}	uds	Δ^{o}	udd
Σ^{+}	uus	Δ^{-}	udd
Σ^{o}	uds	Σ^{**}	uus
Σ^{-}	dds	Σ^{*o}	uds
Ξ^{o}	uss	Σ^{*-}	dds
Ξ^{-}	dss	Ξ^{*o}	uss
		Ξ^{*-}	dss
		Ω^{*-}	sss

For the $J^P = 1^-$ mesons we use the following compositions; $\rho = u\bar{u}$, $K^+ - u\bar{s}$ and $\phi = s\bar{s}$ while for the $J^P = 0^-$ mesons we take $\pi = u\bar{u}$ and $K = u\bar{s}$.

APPENDIX A.2 HADRONIC WAVEFUNCTIONS

The wavefunction ψ_{sf} needs to be symmetric. Thus for the octet baryons ($J^P = 1/2^+$) we follow Schiff (Sc 64) and consider:

$$x_1 = \frac{1}{\sqrt{3}} \alpha_1 \frac{1}{\sqrt{2}} (\alpha_2\beta_3+\beta_2\alpha_3) - \sqrt{\frac{2}{3}} \beta_1\alpha_2\alpha_3 \qquad \text{(symmetric)}$$

$$x_2 = \alpha_1 \frac{1}{\sqrt{2}} (\alpha_1\beta_3-\beta_2\alpha_3) \qquad \text{(antisymmetric)}$$

where the indices refer to quarks, α to spin up and β to spin down. For the flavour part we consider η_1 and η_2 defined as above with $\alpha \to u$ and $\beta \to d$. The s quark is treated as distinguishable from u and d quarks because of the large mass difference. Thus we use:

$$\psi_{sf}(p) = \frac{1}{\sqrt{2}} (x_1\eta_1+x_2\eta_2)$$

$$\psi_{sf}(\Sigma^o) = x_1 s_1 \frac{1}{\sqrt{2}} (u_2d_3+d_2u_3)$$

$$\psi_{sf}(\Lambda) = x_2 s_1 \frac{1}{\sqrt{2}} (u_2d_3-d_2u_3)$$

$$\psi_{sf}(\Xi^o) = x_1 u_1 s_2 s_3$$

For the decouplet ($J^P = 3/2^+$) baryons the spin part may be taken as $\alpha_1\alpha_2\alpha_3$.

We need $\langle \underset{\sim}{\sigma}_i \cdot \underset{\sim}{\sigma}_j \rangle$ for the potentials considered, these may be obtained using the given wavefunctions. The results are given below where we abbreviate ψ_{sf} by its particle.

$$\langle p|\underset{\sim}{\sigma}_i\cdot\underset{\sim}{\sigma}_j|p\rangle = -1$$

$$\langle\Sigma^0|\underset{\sim}{\sigma}_1\cdot\underset{\sim}{\sigma}_2|\Sigma^0\rangle = \langle\Sigma^0|\underset{\sim}{\sigma}_1\cdot\underset{\sim}{\sigma}_3|\Sigma^0\rangle = -2$$

$$\langle\Sigma^0|\underset{\sim}{\sigma}_2\underset{\sim}{\sigma}_3|\Sigma^0\rangle = 1$$

$$\langle\Lambda|\underset{\sim}{\sigma}_1\cdot\underset{\sim}{\sigma}_2|\Lambda\rangle = \langle\Lambda|\underset{\sim}{\sigma}_1\cdot\underset{\sim}{\sigma}_3|\Lambda\rangle = 0$$

$$\langle\Lambda|\underset{\sim}{\sigma}_2\cdot\underset{\sim}{\sigma}_3|\Lambda\rangle = -3$$

$$\langle\Xi^0|\underset{\sim}{\sigma}_1\cdot\underset{\sim}{\sigma}_2|\Xi\rangle = \langle\Xi^0|\underset{\sim}{\sigma}_1\cdot\underset{\sim}{\sigma}_2|\Xi^0\rangle = -2$$

$$\langle\Xi^0|\underset{\sim}{\sigma}_2\cdot\underset{\sim}{\sigma}_3|\Xi^0\rangle = 1$$

For the decuplet baryons

$$\langle\underset{\sim}{\sigma}_i\cdot\underset{\sim}{\sigma}_j\rangle = 1.$$

Note that for simplicity the nucleon, for example, is treated as being symmetric between any two pairs.

APPENDIX A.3 REGULARIZING THE FERMI-BREIT POTENTIAL

The potential v(r) may be found by taking the Fourier transform of $v(q^2)$ (Be 71) where for example the $\delta(\underset{\sim}{r})$ derives from $v(q^2) = 1$. To regularize this we replace this by a form factor $f(q^2) = \frac{2\Lambda^2}{(q^2+\Lambda^2)} - \frac{\Lambda^4}{(q^2+\Lambda^2)^2}$, where $r_0 = \Lambda^{-1}$ $\sim$ compton wavelength of the quark. This form was chosen because it leads to comparatively simple modifications in the momentum dependent potential. The δ-function is then replaced by

$$f_D(r,\Lambda) = \frac{1}{(2\pi)^3} \int d\underset{\sim}{q}\; e^{i\underset{\sim}{q}\cdot\underset{\sim}{r}}\, F(q^2)$$

$$= \frac{\Lambda}{2\pi} \left(\frac{1}{r} - \frac{\Lambda}{4}\right) e^{-\Lambda r} = \frac{1}{2\pi r_0} \left(\frac{1}{r} - \frac{1}{4r_0}\right) e^{-r/r_0}$$

The $\bar{p}$-dependent potential now takes the form

$$V_p = \frac{4}{(2\pi)^3} \int d\underset{\sim}{q}\; \left\{\underset{\sim}{p}_i\cdot\underset{\sim}{p}_j + \frac{(\underset{\sim}{p}_i\cdot\underset{\sim}{q})(\underset{\sim}{p}_j\cdot\underset{\sim}{q})}{q^2}\right\} \frac{F(q^2)}{q^2}\, e^{i\underset{\sim}{q}\cdot\underset{\sim}{r}}$$

$$= \frac{1}{2} \left\{\underset{\sim}{p}_i\cdot\underset{\sim}{p}_j\, f(rr_0) + \frac{(\underset{\sim}{p}_i\cdot r)(\underset{\sim}{p}_j\cdot r)}{r^2}\, g(rr_0)\right\}$$

where

$$f(rr_0) = \frac{1}{r}\left(1-e^{-r/r_0}\right) + \frac{1}{r_0}\, e^{-r/r_0}$$

$$g(rr_0) = \frac{1}{r}\left(1-e^{-r/r_0}\right) - \frac{1}{r_0}\, e^{-r/r_0} .$$

APPENDIX B

APPENDIX B.1 FESHBACH-RUBINOW METHOD

This method is useful for S-state three body systems for which, in the center of mass coordinates, the wavefunction, $\psi(\underset{\sim}{r}_1,\underset{\sim}{r}_2,\underset{\sim}{r}_3)$, is a function only of three variables, $\psi(r_1,r_2,r_3)$. We treat the case where at least two masses and force bonds are equal. These bonds are given the indices 1 and 2 and thus $\underset{\sim}{r}_{23} = \underset{\sim}{r}_1$, $\underset{\sim}{r}_{13} = \underset{\sim}{r}_2$ and $\underset{\sim}{r}_{12} = \underset{\sim}{r}_3$. The idea, as discussed in the text, is to assume that the wavefunction is a function of the single variable $R = (r_1+r_2+\eta r_3)/2$, where η is an asymmetry variational parameter. The three body Schrödinger equation then reduces to an effective one body Schrödinger equation for $\phi(R)$ which may be obtained in the following way.

The kinetic energy operator takes the form $T = T_1+T_2+T_3$ with

$$\langle T_1\rangle = \langle T_2\rangle = \frac{\hbar^2}{4m_1}[(\eta^2+1) + \eta\zeta]\int_0^\infty dR\, R^5\left(\frac{d\phi(R)}{dR}\right)^2$$

and

$$\langle T_3\rangle = \frac{\hbar^2}{4m_3}[2\xi + \eta\zeta]\int_0^\infty dR\, R^5\left(\frac{d\phi(R)}{dR}\right)^2$$

where

$$\xi = \frac{4(\eta^2 + 5\eta + 8)}{15(\eta + 1)^5}$$

$$\zeta = \frac{8(\eta + 5)}{15(\eta + 1)^5}$$

Thus we have

$$\langle T\rangle = \frac{\hbar^2}{4m_{eff}} \int_0^\infty dR\, R^5 \left(\frac{d\phi(R)}{dR}\right)^2$$

with

$$\frac{1}{m_{eff}} = 2\xi\left[\frac{\eta^2+1}{m} + \frac{1}{m_3}\right] + \eta\zeta\left[\frac{2}{m} + \frac{1}{m_3}\right]$$

in which we use $m_1 = m_2 = m$.

One may thus find the ratio

$$\frac{\langle T_3\rangle}{\langle T_1\rangle} = \frac{2m}{m_3}\,\frac{2\eta^2 + 10\eta + 8}{\eta^4 + 5\eta^3 + 11\eta^2 + 15\eta + 8}$$

To find the effective potential, in general one considers the transformation

$$R = (r_1+r_2+\eta r_3)/2$$

$$R_2 = r_2$$

$$R_3 = r_3$$

and hence (Fe 55, Bh 67)

$$\int_0^\infty \int_0^\infty \int_{|r_1-r_2|}^{r_1+r_2} dr_1 dr_2 dr_3 \; r_1 r_2 r_3$$

$$= 2 \int_0^\infty dR \; \{ \int_0^R dR_2 \int_{\frac{2R-2R_2}{\eta+1}}^{\frac{2R}{\eta+1}} dR_3 \; + \int_R^{\frac{2R}{\eta+1}} dR_2 \int_{\frac{2R-2R_2}{\eta-1}}^{\frac{2R}{\eta+1}} dR_3 \; \} [$$

$$R_2 R_3 (2R - R_2 - \eta R_3)]$$

which gives

$$\langle V(r_1, r_2, r_3) \rangle = \int_0^\infty dR \; R^5 \; V_{eff}(R) \; \phi^2(R)$$

and thus

$$\langle H-E \rangle = \int_0^\infty dR \; R^5 \; F$$

where $\quad F = \frac{\hbar^2}{4m_{eff}} \; \phi'^2(R) + V_{eff}(R)\phi^2(R)$

From the variation $\delta\langle H-E\rangle = 0$ we obtain the Euler-Lagrange equation

$$-\frac{d}{dR}[\frac{\partial}{\partial\phi'}(R^5 F)] + R^5 \frac{\partial}{\partial R}(R^5 F) = 0$$

which is an effective one body Schrödinger equation that may be readily solved on a computer for $u(R) = R^{5/2}\phi(R)$. When momentum-dependent forces are present F takes the form

$$F = \frac{\hbar^2}{4m_{eff}} \phi'^2(R) + V_p(R)\phi'^2(R) + V_{eff}(R)\phi^2(R)$$

The effective potential for r^n type potentials is given below for $n > -2$.

$$\langle r_1^n \rangle = \langle r_2^n \rangle = A_x(n,\eta) \int_0^\infty dR\ R^5\ \phi^2(R)\ R^n$$

$$\langle r_3^n \rangle = A_z(n,\eta) \int_0^\infty dR\ R^5\ \phi^2(R)\ R^n$$

where $V_{eff}(R) = [2A_x(n,\eta) + A_z(n,\eta)]\ R^n$, with

$$A_x(n,\eta) = \frac{32\ \Gamma(n+2)}{(\eta^2-1)^2\Gamma(n+6)} \{(n+2)[\eta+(\frac{2}{\eta+1})^{n+3}] +$$

$$\frac{8\eta}{(\eta^2-1)} [(\frac{2}{\eta+1})^{n+2}-1]\}$$

and $$A_z(n,\eta) = \frac{\Gamma(n+3)}{3\Gamma(n+6)} (\frac{2}{\eta+1})^{n+5}[3\eta^2+3(n+5)\eta+(n+4)(n+6)]$$

For $\eta = 1$, $A_x(n,1) = A_z(n,1) = \frac{(n+6)(n+7)}{3(n+3)(n+4)(n+5)}$, while for $n = 0$, $A_x(0,\eta) = A_z(0,\eta) = 2\xi$.

The above expressions may be used to calculate the rms radius. For example

$$\text{rms charge radius} = \left[\frac{\Sigma q_i \langle x_i^2 \rangle}{\Sigma q_i} \right]^{1/2}$$

where q_i and x_i are the charge and distance from the center of mass of the i th quark and

$$\langle x_3^2 \rangle = \left[\frac{M-m_3}{M}\right]^2 (\langle r_1^2 \rangle - \langle r_3^2 \rangle/4)$$

$$\langle x_1^2 \rangle = \langle x_2^2 \rangle = \left[\frac{m_3}{M}\right]^2 \langle r_1^2 \rangle + \langle r_3^2 \rangle (1 - \left[\frac{m_3}{M}\right]^2)$$

APPENDIX B.2 FERMI-BREIT F-R POTENTIAL

Using the methods outlined in Appendix B.1 it is found that for the Fermi-Breit interaction (5) the F-R effective potentials are given by

$$V_{eff}(R) = V_{coul}(R) + V_{spin}(R) + V_{conf}(R)$$

where

$$V_{conf}(R) = \frac{2}{3}\frac{\alpha_s}{a^2}[2A_x(1,\eta)+A_z(1,\eta)]R$$

$$V_{coul}(R) = -\frac{2}{3}\alpha_s[2A_x(-1,\eta)+A_z(-1,\eta)]\frac{1}{R}$$

$$V_{spin}(R) = \frac{2}{3}\alpha_s\left(\frac{\hbar}{c}\right)^2\{(\frac{1}{m_1^2}+\frac{1}{m_3^2}+\frac{4\langle\underset{\sim}{\sigma}_2\cdot\underset{\sim}{\sigma}_3\rangle}{3m_1m_3})W_1(R) + (1+2\langle\underset{\sim}{\sigma}_1\cdot\underset{\sim}{\sigma}_2\rangle)\frac{W_3(R)}{m_1^2}\}$$

$$W_1(R) = \frac{4}{(\eta+1)^2R^4}\{\frac{1}{(\eta+1)}(R-r_0)+e^{-r/r_0}[\frac{2\eta(\eta^2+3)r_0}{(\eta+1)(\eta-1)^3}+\frac{\eta R}{(\eta-1)^2}] + e^{-\frac{2R}{(\eta+1)r_0}}[\frac{R}{(\eta-1)^2}-\frac{(\eta+1)^2r_0}{(\eta-1)^3}]\}$$

$$W_3(R) = \frac{1}{4R^4}\{2R-2\eta r_0+e^{-\frac{2R}{(\eta+1)r_0}}[2\eta r_o+\frac{(\eta-1)2R}{(\eta+1)} - \frac{4R^2}{(\eta+1)^2r_0}+\frac{8R^3}{(\eta+1)^4r_0^2}]\}$$

The momentum dependent part, v_p, takes the form

$$v_p(R) = R^{-5}\{\frac{1}{m_1} X + \frac{2}{m_3} Y\}$$

where

$$X = \frac{4\eta R^4}{9(1+\eta)^3} (3\eta^2+9\eta+8) - \frac{1}{3} \{4R^2(3\eta^2-2)I_1-4R(3\eta^3-4\eta-R/r_0)I_2$$

$$+ [(1-\eta^2)(4-3\eta^2)-4\eta R/r_0]I_3-(1-\eta^2)I_4/r_0\}$$

$$Y = \frac{32R^4}{45(\eta+1)^7} [-48\eta^3+166\eta^2+67\eta+15] - \frac{4}{15(\eta+1)^3} \{40R^2(3-2\eta)I_1$$

$$+40R[\eta R/r_0-4\eta-3]I_2+2[(\eta-1)(4\eta^2+16\eta-15)-20\eta R/r_0]I_3$$

$$+2\eta(\eta+4)(1-\eta)I_4/r_0\} - \frac{64\eta R^4}{(15(1-\eta^2)^3} \{10\ell n(\frac{2}{\eta+1})$$

$$+ \frac{(\eta-1)}{12(\eta+1)^4} (15\eta^5+171\eta^4-344\eta^3+896\eta^2+161\eta+61)$$

$$+ 4RJ_{-2}+2(5+2R/r_0)J_{-1}+5[(\eta^2-1)/R-2/r_0]J_6$$

$$+ 5[(1-\eta^2)R/r_0-11-9\eta^2]J_1/R^2+5[15\eta^2+13+(3\eta^2+1)R/r_0]J_2/R^3$$

$$- [19+35\eta^2+5(3\eta^2+1)R/r_0]J_3/R^4+(5\eta^2+1)J_4/(R^5r_0)\}$$

where the notation

$$I_n = \int_0^{2R/\eta+1} y^n e^{-y/r_0} dy$$

$$J_n = \int_R^{2R/\eta+1} y^n e^{-y/r_0} dy$$

For the symmetric case ($\eta = 1$) these expressions simplify to the following

$$V_{spin}(R) = \frac{\hbar^2 \alpha_s}{3m^2c^2} (3+2\langle \underset{\sim}{\sigma}_1 \cdot \underset{\sim}{\sigma}_2 \rangle \; [\frac{1}{R^3} - \frac{1}{R^4} + e^{-R/r_0} \{\frac{r_0}{R^4} - \frac{1}{2r_0R^2}$$

$$+ \frac{1}{12Rr_0^2}\}]$$

$$V_p(R) = \frac{10}{3R} - \frac{12r_0^2}{R^3} + \frac{16r_0^3}{R^4} + e^{-R/r_0} (\frac{4r_0}{R^2} - \frac{4r_0^2}{R^3} - \frac{16r_0^3}{R^4})$$

APPENDIX B.3 LOGARITHMIC F-R POTENTIAL

To find the effective logarithmic F-R potential one may use the results of Appendix B.1. Consider

$$I(n) = \int (dr) r_1 r_2 r_3 (\Lambda r_i)^n = C(n) \int_0^\infty dR \; R^5 \; (\Lambda R)^n \phi^2(R)$$

Thus

$$\left.\frac{dI(n)}{dn}\right|_{n=0} = \int (dr) r_1 r_2 r_3 \ln(\Lambda r_i) ,$$

$$= \left.\frac{dC(n)}{dn}\right|_{n=0} \int_0^\infty dR \; R^5 \phi^2(R) + C(0) \int_0^\infty dR \; R^5 \ln(\Lambda R) \phi^2(R)$$

Thus

$$v_{eff}(R) = (2A_x(0,\eta) + A_z(0,\eta) \ln \Lambda R$$

$$+ \left.\frac{d}{dn} \{2A_x(n,\eta) + A_z(n,\eta)\}\right|_{n=0} .$$

When this is evaluated the expression for $v_{eff}(r)$ takes the form

$$V_{eff}(R) = [6\xi \ln \Lambda R + G(\eta)]$$

where

$$G(\eta) = \frac{2}{15(\eta+1)^5} \{4[\eta^2 + 5\eta + 8 + \frac{16(3-1)}{(\eta-1)^3}] \ln(\frac{2}{\eta+1})$$

$$- \frac{1}{15} [201\eta^2 + 145\eta + 1408] + \frac{4}{(\eta-1)^2} [\eta(\eta+1)^3 + 8]\}$$

for the symmetric case

$$G(1) = -199/600.$$

APPENDIX C.1 VIRIAL THEOREM FOR LOGARITHMIC POTENTIAL

This may be readily demonstrated using scaling. Consider

$$H|\underset{\sim}{r}> = E|\underset{\sim}{r}> .$$

Then let

$$E(\beta) = <\beta\underset{\sim}{r}|H(r)|\beta\underset{\sim}{r}> . \quad (1)$$

Now, since $|\underset{\sim}{r}>$ is an eigenvector of H, then the variation

$$\frac{\partial E(\beta)}{\partial \beta} = 0$$

holds at $\beta = 1$. Normalization of the wavefunction demonstrates:

$$\begin{aligned} <\beta\underset{\sim}{r}|\beta\underset{\sim}{r}> &= 1 \\ &= \int d\underset{\sim}{r}|\psi(\beta\underset{\sim}{r})|^2 \\ &= \frac{1}{\beta^{3n}}<\underset{\sim}{r}|\underset{\sim}{r}> \\ &\Rightarrow \beta^{3n} = 1, \end{aligned}$$

where n is the number of degrees of freedom.

Now (1) may be written

$$E(\beta) = <\underset{\sim}{r}|H(r/\beta)|\underset{\sim}{r}> .$$

Since the kinetic energy operator $\sim 1/r^2$ and

$$V = - \alpha_s \sum_{i>j} \langle \underset{\sim}{F}_i \cdot \underset{\sim}{F}_j \rangle \ell n(\Lambda r_{ij}) \ .$$

Then

$$\frac{\partial E(\beta)}{\partial \beta} = \langle \underset{\sim}{r} | 2\beta \hat{T} + \alpha_s \sum_{i>j} \langle F_i \cdot F_j \rangle \frac{1}{\beta} | \underset{\sim}{r} \rangle \ .$$

Thus

$$\langle \hat{T} \rangle = - \frac{\alpha_s}{2} \sum_{i>j} \langle \underset{\sim}{F}_i \cdot \underset{\sim}{F}_j \rangle \ .$$

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