MYOPIC POLICIES FOR INVENTORY CONTROL

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Abstract

In this thesis we study a typical retailer's problem characterized by a single item, periodic review of inventory levels in a multi-period setting, and stochastic demands. We consider the case of full backlogging where backorders are penalized via fixed and proportional backorder costs simultaneously. This treatment of backorder costs is a nonstandard aspect of our study.

The discussion begins with an introduction in Chapter 1. Next, a review of the relevant literature is provided in Chapter 2. In Chapter 3 we study the infinite horizon case which is of both theoretical and practical interest. From a theoretical point of view the infinite horizon solution represents the limiting behavior of the finite horizon case. Solving the infinite horizon problem has also its own practical benefits since its solution is easier to compute.

Our motivation to study the infinite horizon case in the first place is pragmatic. We prove that a myopic base-stock policy is optimal for the infinite horizon case, and this result provides a basis for our study. We show that the optimal myopic policy can be computed easily for the Erlang demand in Chapter 4; solve a disposal problem which arises under the myopic policy in Chapter 5, and also study in Chapters 6 and 7 the finite horizon problem for which a myopic policy is not optimal.

For the finite horizon problem computation of the exact policy may require a substantial effort. From a computational point of view, there is a need for developing a method that overcomes this burden. In Chapter 6 we develop a model for such a method by restricting our attention to the class of myopic base-stock policies, and call the resulting policy the 'best myopic' policy. We discuss analytical and numerical

results for the computation of the best myopic policy in Chapter 7. Finally we present a summary of our main findings in Chapter 8.

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Chapter 1

Introduction

1.1 Stochastic Dynamic Inventory Models

This thesis deals with the single item, stochastic demand, periodic review, dynamic inventory control problems within the framework of stochastic optimization and renewal theory. Our problems of study are characterized by stochastic demands, full backordering of unsatisfied demands, and negligible lead times. A typical example is a retailer's problem where inventory control of a single product or a group of independent products is of interest. Inventories are reviewed periodically, e.g., each month, and the retailer wants to decide the order quantity at the start of each month.

In general, there are two types of review processes associated with monitoring inventories; periodic review and continuous review. If a continuous review process is used, the transactions are processed immediately using a computerized system. Inventory levels are known exactly on a continuous basis, and theoretically an order can be placed any time. On the other hand, if a periodic review process is under operation inventory levels might not be known exactly at all times, probably due to the lack of a computerized system, or physical difficulties associated with counting inventories on a continuous basis. Then inventory levels are measured at specific time points, and orders (if any) are placed at these times. Traditionally, periodic review systems had been popular because of the costs and the implementation difficulty associated with computerized transactions systems. As computer technology has increased, the trend has been towards continuous review systems.

Today, almost every retail store has a computerized system. Even considering today's technology, however, there are still reasons for studying periodic review models. Most retail stores would need actual inventory counts at specific time points, as if a periodic review process was operated, for the consideration of losses due to breakage, shoplifting, mistakes, etc. In some cases, although a continuous review system is available, theoretically a periodic review model is appropriate because the supplier can only make regular deliveries once in every period so that one can schedule order receipts only at specific time points as if operating a periodic review system. Furthermore, from a practical point of view, for the dynamic problem unless a closed form solution is obtained, computing the solution of the continuous review model may require discretization, and in turn the continuous review system is essentially treated as a periodic review system.

For the periodic review system let D_n be the demand during period n. The sequence $\{D_1, D_2, \ldots\}$ represents a set of independent and identically distributed (i.i.d.) random variables. Period ends are review instants, and naturally ending inventory of period $n-1$ is equal to the beginning inventory of period n. The beginning inventory of period n is denoted by s_n. Negative inventory levels represent outstanding backorders. Thus, knowing the beginning inventory s_n of period n, the problem is to specify the size z_n of the order to be placed.

Periodic review dynamic inventory problems have been investigated since the 1950s. We provide a review of the relevant literature in Chapter 2. The traditional approach is through Dynamic Programming (DP) which is basically useful for obtaining structural results regarding the form of the optimal operating doctrines, i.e., policies. In this thesis, our terminology and methodology is different from the usual DP approach. Our notation and analysis for inventory systems are in line with Heyman and Sobel's [15] presentation in their book.

The purpose of formulating a DP model is to use it as an aid in selecting reasonable operating policies. The criterion to arrive at a policy can be profit maximization or cost minimization. Some of the well known inventory control policies are base-stock policies and (s, S) policies which are discussed in Chapter 2. There are a number of other policies (e.g., generalized (s, S) policies, multiple (s, S) policies, state dependent (s, S) policies, etc.) for which references are also provided in Chapter 2.

The costs associated with operating an inventory system play an important role in the determination of the optimal policy. Usual cost categories are procurement, holding, and stockout costs. In this thesis, we study the case of full backlogging where backorders are penalized via two types of backorder costs simultaneously. The first type is the standard backorder cost which is linear in size of the backorders, i.e., the proportional backorder cost. The second type is the one-time penalty cost that is independent of the size of the backorders, i.e., fixed backorder cost which associates a fixed charge with the stockout situation. This treatment of backorder costs is a nonstandard aspect of our study. On the other hand, we consider a unit procurement cost only, and assume that the setup cost of placing an order is negligible.

Our assumptions on negligible setup costs, immediate delivery and the single item case provide simplicity in mathematical analysis which in turn permit us to derive analytical results, and closed form solutions.

For the cost structure that we assume in the development of this thesis, base-stock policies are optimal under a set of mild assumptions on the demand density and model parameters. Optimality of base-stock policies, and the assumptions of full backordering and immediate delivery lead to a regenerative structure which in turn allow the application of renewal theory concepts in this thesis.

Once the optimality of an operating policy is established the next step is the computation. Computational difficulties associated with the finite horizon periodic review inventory models have long been established. The finite horizon solution is computed via backward DP which may demand a substantial effort. One of the traditional approaches to avoid computational difficulty is to assume an infinite horizon, and thus to approximate the finite horizon solution with an infinite horizon solution. Assuming an infinite horizon for a finite horizon problem is equivalent to Veinott's [44] well-known terminal conditions. Nevertheless, computational ease is provided at the price of a set of conditions which may not completely apply in real life. In this thesis, we are interested in other ways of avoiding computational difficulty which are superior to Veinott's terminal condition.

In the following section we present an overview of this thesis and describe its organization assuming some prior knowledge of inventory theory on the part of the reader. A review of the literature and descriptions of some of the basic concepts, e.g., inventory control policies, limiting behavior, myopia, etc., can be found in Chapter 2.

1.2 Organization and Overview

Although it is virtually impossible to present a comprehensive account of the literature on inventory theory in this thesis, the next chapter attempts to provide a review focusing on the single item, stochastic demand, periodic review inventory models.

Our mathematical analyses start with the assumption that the inventory system will operate for a long time, and thus the planning horizon is infinite. We formulate and study the infinite horizon problem in Chapter 3. The objective is to find an ordering policy that maximizes the total expected discounted *revenue net of costs*. We use the term *revenue net of costs*, rather than the term *profit*, intentionally, and provide an explanation for this choice in Section 3.1. Because of the inclusion of the fixed backorder cost, a base-stock policy is not necessarily optimal. We discuss a sufficient condition to guarantee the optimality of a myopic base-stock policy. In general, the infinite horizon solution represents the limiting behavior associated with the finite horizon solution. As we mentioned, our presentation of the problem is similar to Heyman and Sobel's [15], and this presentation is useful in the reduction of the multi-period (dynamic) problem to a forward sequence of single period (static) problems.

We continue to study the infinite horizon problem assuming Erlang demands in Chapter 4. We observe that optimality of the myopic base-stock policy is guaranteed for Erlang demand densities, and the computation of this policy is very simple. That is, the sufficient condition for a myopic optimum is not very restrictive in the sense that it holds for Erlang demand densities, and computing this solution is as easy as finding the unique, finite maximizer of a nonlinear function. Provided that the coefficient of variation of a random variable is less than one, then its distribution can be approximated by the Erlang density with a suitable choice of shape and scale parameters. Since a myopic policy is optimal for Erlang demand in general, it can be considered as an approximate solution when the exact optimal policy is complicated. Our analyses in Chapters 3 and 4 provide some theoretical results for the development in Chapters 5 and 6 where we analyze inventory disposal and finite horizon problems.

We discuss the inventory disposal decisions associated with our infinite horizon problem in Chapter 5. For the infinite horizon problem, initial inventory is a model parameter, and under the optimal policy the best we can do in monetary terms depends on the value of this parameter. In Chapter 5 we suppose that the initial inventory can be liquidated immediately under a disposal opportunity, and our objective is to choose an inventory level to start. To this end, we formulate a disposal problem and study its optimal solution. We show that for Erlang demand, finding the optimal solution reduces to computing the maximizer of a concave function.

We analyze the finite horizon case in Chapter 6 where we assume that the retailer will not be selling the product after T periods. In this case, the exact policy may require a substantial computational effort. Of course, there are some traditional methods for avoiding computational difficulties, such as computing and implementing the infinite horizon solution or assuming Veinott's [44] terminal conditions. However, there is not much work done in terms of providing a method that overcomes the computational burden associated with the finite horizon solution, but which is also superior to the traditional approaches. In order to achieve some computational ease, we restrict our attention to the class of myopic base-stock policies and optimize with respect to this set of operating doctrines for our finite horizon problem. In Chapter 6, we develop the underlying model that will be used for computing the 'best' myopic policy for our finite horizon problem.

In Chapter 7, we summarize our analytical and numerical results for computing the best myopic base-stock policy assuming Erlang demand. We are able to present analytical results if the system is not overstocked initially. If the system has excess inventory then we provide numerical results, and illustrate the dependence of the best policy on the initial inventory. We also present a comparison of the performance of the best myopic policy with the infinite horizon myopic policy.

Chapter 2

Literature Review

The history of mathematical inventory theory goes back to the beginning of this century. In 1915, Harris [14] studied the simplest problem of inventory management, and developed the economic order quantity formula (Hadley and Whitin [13]). Since then, the study of inventory models have attracted both practitioners and academics in the field of Management Science/Operations Research (MS/OR). The main purpose of the original development of inventory models was to solve practical problems. However, inventory theory had become more and more popular among academics since it contains challenging mathematical problems. As a result, the literature on inventory control is abundant and providing a comprehensive review of the literature goes beyond the objective of this chapter. In general, inventory models are classified as single item versus multi-item, single-echolon versus multi-echelon, continuous versus periodic review, deterministic versus probabilistic, and single period versus multi-period models, etc. Scarf [37] provides an account of the basic analytical techniques in inventory theory. Aggarwal [1] discusses the status and the applications of inventory theory. Porteus [33] presents a review with an emphasis on stochastic inventory theory. A comprehensive critique of the literature on single item models can be found in a recent paper by Lee and Nahmias [22]. In the following discussion, we concentrate on the class of single item, single location, periodic review, stochastic, dynamic inventory models which we find relevant to our study. Our intention is to provide a basic idea about the models and the results which are relevant to this thesis. The relationship between previous work and our analyses will be discussed in the respective chapters.

2.1 Fundamental Results

In the context of periodic review systems an inventory policy (or operating doctrine) is defined as a set of rules that specify the order quantity of each period using the information accumulated up to that time (Scarf [37]). Most of the efforts in stochastic inventory theory are devoted to the study and analysis of different policies. Historically, characterization and evaluation of optimal policies have been a center of interest. This normative approach has both theoretical and practical value.

Following the fundamental work of Arrow, Harris and Marchak [3] inventory models with stochastic demands have received a lot of attention. Arrow, Harris, and Marchak analyzed the so called 'newsboy problem' characterized by a single period, and a one time procurement opportunity for the demand of the entire period. Under the assumption of linear procurement and shortage costs the solution is simple. The optimal policy is a base-stock policy which involves a single critical number, denoted by S, and usually called the base-stock level. If initial inventory is above the base-stock level then no action is taken, otherwise the policy is to raise the stock to the base-stock level. If a setup cost component is associated with the procurement in addition to the linear procurement cost, then the optimal policy for the single period problem is specified by two numbers s and S (Scarf [36]). Specifically, this policy requires the stock to be raised to S if initially it is below s; and no order is necessary if the stock is above s. Such a policy is called a simple (s, S) policy. It is worth noting that the base-stock policy is a special case of the simple (s, S) policy where $s = S$. Thus, whenever we refer to the class of simple (s, S) policies, this class includes the base-stock policies.

Bellman, Glicksberg and Gross [5] analyzed the optimal inventory policy for the multi-period problem. They assumed i.i.d. demands in successive periods, a linear procurement cost without a setup component, linear holding and shortage costs, and proved the optimality of a base-stock policy.

For a problem with T periods, the base-stock policy involves a sequence of critical numbers $\{S^{(n)}\}_{n=1}^{T}$ with $S^{(n)}$ specifying the base-stock level of period n. In particular, under this policy, if the beginning inventory s_n of period n is below the critical level $S^{(n)}$ then an order of size $S^{(n)} - s_n$ is placed so that upon the immediate arrival of this order the inventory level can be raised to the base-stock level $S^{(n)}$. For this reason the base-stock levels are sometimes called 'order-up-to levels'.

Considering a setup cost along with the linear procurement, holding and shortage costs, Scarf [36] provided a proof of the optimality of simple (s, S) policies for the multi-period problem. In the multi-period case, the simple (s, S) policy associates two critical numbers $s^{(n)}$ and $S^{(n)}$ with each period, and in turn for a problem with T periods, the policy consists of two sequences: $\{s^{(n)}\}_{n=1}^{T}$ and $\{S^{(n)}\}_{n=1}^{T}$. An order is placed only if beginning inventory s_n is below $s^{(n)}$, and its size should be $S^{(n)} - s_n$. The multi-period base-stock policy is also a special (s, S) policy where $s^{(n)} = S^{(n)}$.

Scarf's linearity assumptions on holding and shortage costs could be replaced with a weaker assumption which requires the convexity of single period expected holding plus shortage costs. Further, his proof was easily generalized in the case of a fixed time lag in delivery. In general, our discussion in this chapter is concentrated on the case of immediate delivery. However, most of the results summarized in this chapter can be generalized to the case of a fixed lead time.

Veinott [46] also studied the multi-period problem, but he obtained a different set of conditions from Scarf [36] for the optimality of the (s, S) policy. Scarf's approach was based on geometric concepts (i.e., K-convexity) whereas Veinott's proof was analytical. Veinott's results replaced the convexity assumption on the single period expected holding plus shortage cost functions with unimodality. Nonetheless, this generality was achieved at the expense of an additional assumption which required the absolute minima of the single period expected holding plus shortage cost functions to rise over time.

So far we have summarized the structural results for the classical multi-period problem. Hereafter when we mention the 'classical' problem we refer to the multi-period problem with i.i.d. demands, linear procurement, and holding and shortage costs with or without a setup cost of procurement.

Porteus [31] and [32] also analyzed the classical problem under a more general cost structure, i.e., with a concave increasing procurement cost function with m distinct marginal costs containing a setup component, rather than a simply linear procurement cost with the setup cost component. Porteus [31] was able to show the optimality of generalized (s, S) policies which were characterized by m pairs of critical numbers. However, he imposed the condition that the demand density must be of one-sided Pólya type. In 1972, Porteus [32] presented the same result for uniform demand density which is not of Pólya type. Later Schäl [38] gave a unified proof with conditions that generalize those of Scarf [36], Veinott [46] and Porteus [31].

2.2 Dynamic Problem and the Disposal Decisions

Excess inventory disposal is a critical problem for inventory managers. Rosenfield [34] reports that although the issue of disposal of excess inventory has been a problem of interest for many years, previous treatments have not addressed the probabilistic nature of demand. The traditional formulations that we discussed in the previous section only allow a salvage value for excess inventory at the end of the horizon. Thus, according to these formulations inventory disposal is only possible at the end of the horizon, but not in the other periods. As a generalization of the classical problem Lovejoy [23] allows the decision maker to exercise the disposal option any time he/she is willing to pay the price of disposal. His problem is more general, and thus its solution is more complicated. Lovejoy does not pursue an optimal policy. Assuming linear procurement, holding and shortage costs he analyzes the problem by restricting attention to the class of inventory policies that behave myopically up to a specified stopping time.

2.3 The Limiting Behavior

Limiting behavior of the classical problem was studied by Iglehart [17]. Under Scarf's [36] assumptions, Iglehart proved that the sequences $\{s^{(n)}\}_{n=1}^{T}$ and $\{S^{(n)}\}_{n=1}^{T}$ converged to limits s and S. In fact, it was then shown that limit points s and S

characterized the optimal policy for the infinite horizon problem. Thus, in contrast to the finite horizon case, a 'stationary' (s, S) policy is optimal for the infinite horizon problem. The term 'stationary' implies that critical numbers s and S do not vary over time. The infinite horizon model is sometimes referred as the steady-state model, and its solution is called the steady-state (s, S) policy.

Time dependent and limiting distributions of the inventory level under the simple (s, S) policy have also been of interest. Karlin [19] analyzed the limiting distribution of inventory level under simple (s, S) policies in order to obtain steady-state solutions for the multi-period problem. His approach was also renewal theoretic. Greenberg [12] computed the transient solution for the probability density of the inventory level for the periodic review (s, S) model. In general, the main purpose of obtaining the time dependent and limiting distributions was the computation of (s, S) policies. We mention this issue in Section 2.8. In a recent book where periodic review was treated as a special case of continuous review, Şahin [35] provided a discussion on the use of renewal theory for the derivation of the time dependent and limiting distributions of the inventory level (pp. 14–16 in [35]).

2.4 Modeling Stockout Situations

Most inventory models assume that if a stockout occurs then unfilled demand is either lost or backordered. In some situations, however, some of the demands are lost whereas some are backordered. Scarf's [36] seminal paper deals with the case where excess demand can be fully backordered. The corresponding lost sales model of Scarf's [36] problem is discussed by Bertsekas (in [6], pp. 92–93) where optimality of the (s, S) policies is proved by S. E. Shreve. Veinott's [46] alternative proof extends the class of problems for which an (s, S) policy is optimal. This class also includes partial lost sales and partial backordering models (Şahin [35], p. 10).

Aneja and Noori [2] studied the classical problem under a different assumption describing a stockout situation. They associated a fixed penalty with stockouts and assumed that if demand could not be satisfied immediately (due to a stockout) then it would not be backordered or lost, but rather would be satisfied through some external arrangement. Since Aneja and Noori's problem was neither a pure backorders nor a pure lost sales problem, we called it the 'rush-orders' problem for the reasons given by the authors. Recently, Çetinkaya and Parlar [10] presented a criticism of a sufficient condition that was used by Aneja and Noori in order to prove the optimality of the (s, S) policy. In fact, they were able to present a counter example which proved that Aneja and Noori's proof was in error. Çetinkaya and Parlar also suggested an alternative 'pseudo-backorders' model that can handle the rush-orders case. Under a set of mild conditions (s, S) policy is optimal for Çetinkaya and Parlar's alternative model for a large class of demand densities, e.g., one-sided Pólya type densities, as presented elsewhere. Çetinkaya and Parlar also provided a general condition under which an (s, S) policy is optimal for the pseudo-backorders model.

2.5 The Cost Structure

As we mentioned, the costs incurred to operate the system have a significant role in determining the optimal policy. In the context of dynamic problems, a typical model considers procurement, holding, and stockout costs. Holding and stockout costs are usually linear whereas procurement costs may or may not offer economies of scale. Under this usual cost structure, if the procurement cost does not offer economies of scale, i.e., the setup cost of placing an order is negligible and the procurement cost is proportional to the order quantity, then the base-stock policy is optimal. If a setup cost is associated with the procurement then an (s, S) policy is optimal. Of course, many extensions[1] have been studied. For example, Porteus (see [31][2] and [32]) modeled the quantity discounts case, and considered a concave increasing procurement cost function with m distinct marginal costs containing a setup component. Parlar and Rempala's [29][3] problem with quadratic holding costs constitutes another example.

In this thesis, we study the case of full backlogging where backorders are penalized via two types of backorder costs simultaneously. As we mentioned in the previous chapter the first type is the standard proportional backorder cost, and the second type is the fixed backorder cost. The idea of associating a fixed penalty with a stockout situation had been studied by previous authors. For example, in analyzing the rush-orders case that we already discussed, Aneja and Noori [2] consider two types of shortage costs: The first type is the standard shortage cost that is linear in size of the shortage (i.e., the proportional stockout cost), and the second type is the lump-sum penalty cost (i.e., fixed stockout cost) that is independent of the size of the shortage. Aneja and Noori state that problems with lump-sum penalty costs commonly occur in practice. As an example, they cite Bell and Noori [4] who

[1] In most cases, the optimal policy has a complicated form in contrast to the class of (s, S) policies.
[2] Porteus [31] also described the generalized (s, S) policies.
[3] Parlar and Rempala [29] discussed state dependent (s, S) policies.

discuss a branch bank operation for which the treatment of stockout cost requires the proportional and lump-sum components. [Bell and Noori, in turn, cite one of their earlier papers (Noori and Bell [25]) that deals with the simpler single period version of the lump-sum cost model.] Since the unfilled demand must be satisfied using emergency ordering via rush-orders, the cost of expediting the order is represented by the lump-sum component. In particular, for the case of the branch bank that deals in foreign currency transactions, whenever the branch runs out of a particular foreign currency its staff must telephone other local branches to find those that can supply the required amount. When a sufficient amount of currency is found, it is then personally picked up by an employee. It is these activities of search and acquisition that result in the lump-sum cost for the bank's cash-management system. Aneja and Noori's rush-orders problem was in fact described in 1955 by Bellman, Glicksberg, and Gross [5] who presented the related DP equation. They used the term "red-tape cost" for Aneja and Noori's "lump-sum cost." Boylan [8] discusses numerical examples for which a fixed cost is associated with stockouts and an (s, S) policy is not optimal (Boylan's examples come from Dvoretsky, Kiefer, and Wolfowitz [11]). The fact that the rather 'innocuous' looking fixed stockout cost term may create difficulties in the solution of the problem was also observed by Bellman, Glicksberg, and Gross. In general, optimal policy may not fall in the class of (s, S) policies.

2.6 More Complicated (s, S) Policies

Many researchers have contributed to specifying the forms of optimal policies. More complicated policies have been related to variations of Scarf's [36] classical problem.

For example, Boylan (see [8] and [9]) introduced the multiple (s, S) policies defined by a sequence of triples. Parlar and Rempala [29] considered quadratic holding costs, and showed the optimality of state dependent (s, S) policies. Under state dependent (s, S) policies, for each period a pair of critical numbers is associated with each possible state of the inventory process.

All the papers that we mentioned so far assume that the demands in successive periods are i.i.d. random variables. Other generalizations include the dependent demand models, the Markovian demand models, and the controllable demand models. Karlin and Fabens [20] provided a model for the case of dependent demands where demand history was modeled as a Markov process. Iglehart and Karlin [18] studied a similar problem with negligible setup costs, and showed the optimality of base-stock policies. Waldmann [49] related the demand process to a so called 'environmental process' which may contain factors of an economic and statistical nature. Then, he discussed the optimality of 'environment-state-dependent' (s, S) policies. In the context of periodic review systems[4], Sethi and Cheng [39] assumed that demand process forms a Markov chain and proved the optimality of (s, S) policies for different types of problems with, e.g., no ordering periods, and storage and service level constraints.

Most of the above mentioned papers concentrated on the modeling of the demand process in random environments. Modeling the supply process has also received attention recently. Parlar, Wang and Gerchak [30] supposed that the supply

[4]In the context of continuous review systems, Song and Zipkin [41] also considered the effects of environmental factors. However, these factors were represented by the demand state, and demand was assumed to be a Markov-modulated Poisson process. Song and Zipkin showed the optimality of the environment-state-dependent (s, S) policies for this problem.

state was a 0-1 random variable which represented the availability. Then, under the assumption that the supply availability forms a Markov chain, they showed the optimality of 'supply-state-dependent' (s, S) policies. Özekici and Parlar [26] have modeled the effects of a randomly changing demand process, supply availability, and cost parameters. Thus, their assumptions regarding the dependency of demand distributions, supplier availability process, and cost parameters on the environmental conditions were quite general, whereas most of the previous work relates either only the demand or only the supply with the environment for which Özekici and Parlar have shown the optimality of environment-state-dependent policies.

Computational difficulty increases as the policy becomes more complicated. "A policy which is neither easy to calculate nor simple to describe is generally considered not worth reporting" (Wheeler [48], p. 601). Although the structure of (s, S) policies is simple, the actual computation turns out to be difficult. Considering the computational burden of simple (s, S) policies, complicated policies are not practical.

2.7 Myopic Policies

Formally, for a dynamic problem an optimum is said to be *myopic* if it can be deduced from an optimum of a static problem (Sobel [40]). In inventory theory, a policy is said to be myopic if it is the optimal policy for a single period model that is defined explicitly in terms of the original model parameters (Porteus [33]). Single period problems are static by nature.

Under certain assumptions Markov Decision Processes (MDP) theory allows us to decompose the multi-period problem into a sequence of single period problems. If such a decomposition is possible then the optimal policy is myopic. In turn, solving the dynamic multi-period problem is as easy as solving an equivalent static single period problem. For this reason, "very simple solutions are possible for models with a linear ordering/production cost that are not optimal otherwise" (Porteus [33], p. 629). Examples can be found in Heyman and Sobel [15], pp. 63–82.

Let us recall that, for a problem with T periods the base-stock policy involves a sequence of critical numbers $\{S^{(n)}\}_{n=1}^{T}$ with $S^{(n)}$ specifying the base-stock level of period n. This policy is said to be myopic if the sequence $\{S^{(n)}\}_{n=1}^{T}$ can be represented by a single critical number, say S where S is the base-stock level for a single period problem. Then, by definition a myopic base-stock policy is stationary, and it is specified by

$$S^{(n)} = S, \quad \forall n.$$

Veinott (see [44] and [45]) established a set of conditions for the optimality of myopic base-stock policies for different dynamic inventory problems. A review of Veinott's results is provided by Porteus [33]. In the context of inventory management, myopic policies are favored because they are simple to implement, easy to compute, and in fact optimal in certain settings.

2.8 The Computational Problem

Computation of (s, S) policies is itself another research area. In particular, efforts have been directed to computing steady-state (s, S) policies, i.e., the infinite horizon solution. Although the steady-state (s, S) policy has a simple structure, its computation turns out to be difficult. Howard's [16] 'value iteration' and 'policy iteration' are two traditional techniques for searching for an optimal steady-state policy. Another approach is known as 'stationary analysis' which require the determination of the limiting distribution of inventory level under an (s, S) policy (Şahin [35], p. 14). For example, Veinott and Wagner's [47] algorithm is based on stationary analysis. In a recent paper, Zheng and Federgruen [50] present an efficient algorithm which is superior to previously suggested algorithms in terms of its computational complexity.

In contrast to the steady-state policy for the infinite horizon case the exact policy of the finite horizon problem is not stationary, i.e., critical numbers $s^{(n)}$ and $S^{(n)}$ of period n vary over time, and thus are indexed by n. That is, computation of an exact policy for the finite horizon problem requires the use of a DP approach which may demand a substantial effort. The computational difficulty is related to the recursive calculations of the DP approach. This approach requires the calculation of optimal revenue (or cost) for a vast number of states which may not be entered. Further, for each stage of the recursion one may need to maximize (or minimize) a nonconcave (or nonconvex) function with multiple relative maxima (or minima).

Wheeler [48] concentrated on stationary (s, S) policies for a finite horizon problem. Wheeler's point of departure was the computational difficulty associated with the exact optimal policy. He provided bounds on the sets $\{S^{(n)}\}_{n=1}^{T}$ and $\{s^{(n)}\}_{n=1}^{T}$. He reported that these bounds might be used for the computation of stationary and steady-state policies.

Steady-state (s, S) policies have received a lot of interest because they are relatively easy to compute, simple to apply and they are optimal under the assumption that the system operates for a long period of time. Although they represent good approximations as the horizon length increases, for the finite horizon problems they are suboptimal. The traditional approach to avoiding the difficulty associated with computing the exact policy for a finite horizon problem is to compute the suboptimal steady-state policy. For the problems of interest in this thesis this approach is equivalent to assuming Veinott's [44] terminal condition, which imposes some restrictions on the model parameters, e.g., according to Veinott's terminal condition, the unit salvage value at the end of the horizon should be equal to the unit procurement cost. Then, under Veinott's terminal condition a steady-state policy is optimal for the finite horizon problem.

Under Veinott's terminal condition, solving the finite horizon problem is as easy as solving the infinite horizon problem. Of course, this computational ease is provided at the cost of a set of conditions which may not completely apply in retail situations. Under our profit maximization criterion, one of these conditions states that leftover items can be salvaged immediately at the rate of the unit procurement cost. The other condition requires that, if a stockout occurs during the last period then the backordered demand is satisfied free of charge (Heyman and Sobel [15], p. 79).

Hadley and Whitin ([13], p. 21) report that in some cases the task of determining the optimal policy is so difficult that it is either impossible or uneconomical, and instead one optimizes with respect to some subset of operating policies. Lovejoy [23] also emphasizes the difficulty associated with computing the optimal stocking policies for dynamic inventory models, and states that a myopic policy is easily calculated and implemented. Other than these suggestions, from a computational point of view, there is not much work done in terms of providing a method that overcomes the computational burden associated with the finite horizon solution, and which is also superior to the traditional approaches. In Chapter 6 of this thesis, we present a model that may constitute a framework for such a method. For this purpose we restrict our attention to the class of myopic base-stock policies and optimize with respect to this set of operating doctrines for our finite horizon problem.

Chapter 3

Infinite Horizon Problem

3.1 Problem Formulation

In this chapter we study the infinite horizon problem assuming that the product remains on the market for many periods which are indexed by $n = 1, 2, \ldots$. As we have already mentioned, our problem is a typical retailer's problem which requires the determination of order quantities following the periodic review of inventory levels. On one hand, we have holding costs which penalize an overstocked system, and on the other hand we have fixed and proportional backorder costs which penalize stockout situations. The infinite horizon assumption may sound quite strong because under this assumption the retailer is supposed to sell the product at all future times. In today's competitive markets most products have quite short life cycles at the end of which they are out of the market, and thus the infinite horizon assumption does not hold in general. However, there are other products which have been on the market for such a long time, e.g., Ivory Soap, that the infinite horizon assumption is not particularly violated either. As a matter of fact, studying the infinite horizon problem is of both theoretical and practical interest. The infinite horizon solution represents the limiting behavior associated with the finite horizon solution. From a theoretical point of view analyzing the infinite horizon problem may provide intuition about the finite horizon case, and may also lead to interesting results in terms of the steady-state behavior of the inventory system. Solving the infinite horizon problem has certainly its own practical benefits. For example, the optimal solution of the infinite horizon problem that we study is easy to compute as we discuss in the next chapter. Further, the infinite horizon policy presents a good approximation as the horizon length increases. Our motivation for studying the infinite horizon case in the first place is pragmatic in the sense that the results of this chapter are used extensively in the following chapters of this thesis.

We suppose that the demands, $D_n, n = 1, 2, \ldots$, in successive periods are i.i.d. random variables, an assumption which holds during the maturity years of the product life cycle and simplifies our problem a great deal. The variable s_n denotes the inventory level at the beginning of period n when an order quantity of z_n is placed. The initial inventory level, s_1, is a given model parameter. Since immediate delivery is assumed, the total number of items available to satisfy the demand at the beginning of n denoted by a_n is defined by

$$a_n \equiv s_n + z_n. \tag{3.1}$$

Excess demand in period n is backordered, so that successive periods' inventory levels are related by the following balance equations:

$$s_{n+1} = a_n - D_n, \quad n = 1, 2, \ldots. \tag{3.2}$$

Figure 3.1 illustrates a realization of the inventory system under an arbitrary ordering policy.

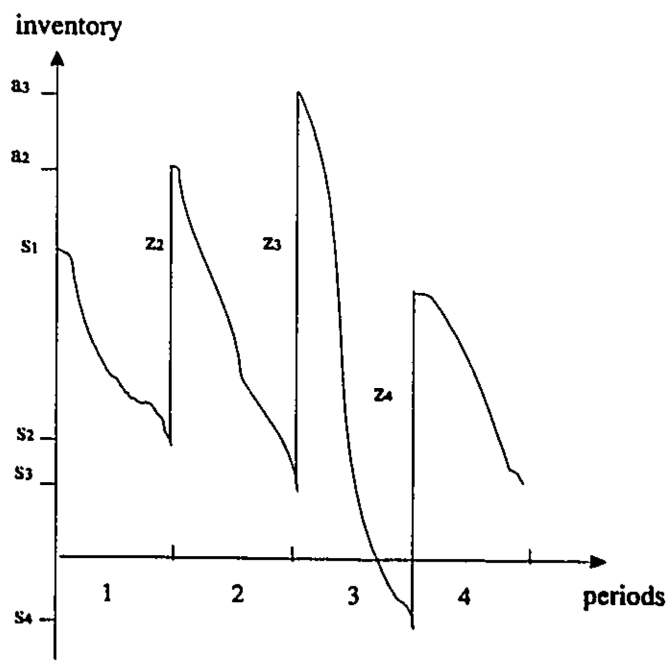

Figure 3.1: Inventory system under an arbitrary ordering policy.

At the beginning of period n, order quantity z_n is purchased. Let c denote the unit procurement cost. The total purchasing cost in period n is denoted by PC_n, and $PC_n = cz_n$. Using (3.1) and (3.2) we have

$$PC_n = c(a_n - a_{n-1} + D_{n-1}), \quad n = 2, 3, \ldots, \tag{3.3}$$

and

$$PC_1 = c(a_1 - s_1), \tag{3.4}$$

where the initial inventory s_1 is a model parameter as indicated before.

If there is excess inventory at the end of period n, i.e., $a_n \geq D_n$, then the total inventory holding cost in period n, denoted by HC_n, is given as

$$HC_n = h(a_n - D_n)^+, \quad n = 1, 2, \ldots, \tag{3.5}$$

where $(a - D)^+ \equiv \max(0, a - D)$, and h denotes unit inventory holding cost per period.

However, if demand exceeds the total number of items available at the beginning of period n, i.e., $D_n > a_n$, then the system is out of stock, in which case backorder costs are incurred. Two types of backorder costs are taken into account in order to penalize stockout situations. The first type is a fixed backorder cost which we denote by B. It corresponds to a fixed penalty associated with a stockout and it is incurred regardless of the size of the stockout, i.e., number of backordered items. Let

$$\delta(u) = \begin{cases} 1 & \text{if } u > 0, \\ 0 & \text{if } u \leq 0. \end{cases}$$

Thus, the fixed backorder cost of period n is represented by $B\delta(D_n - a_n)$. On the other hand, the second type of backorder cost that we consider is the proportional backorder cost which associates a penalty proportional to the size of backorders. The proportional backorder cost per unit is denoted by b. Hence, the total proportional backorder cost of period n is a linear function of the number of backorders $(a_n - D_n)^+$ and equals $b(D_n - a_n)^+$. Fixed backorder and total proportional backorder cost add up to total backorder cost of period n which is denoted by BC_n. Thus, BC_n is expressed as

$$BC_n = B\delta(D_n - a_n) + b(D_n - a_n)^+, \quad n = 1, 2, \ldots. \tag{3.6}$$

As we mentioned in Chapter 1, this treatment of backorder costs is a nonstandard aspect of our study. The idea of associating a fixed penalty with a stockout situation had been studied previously, and the fact that the rather 'innocuous' looking fixed stockout cost term, (3.6), may create difficulties in the solution of the problem was observed by Bellman, Glicksberg, and Gross [5]. Boylan [8] presented numerical examples for which a fixed cost was associated with stockouts and the optimal policy had a complicated form. Later in 1987 Aneja and Noori [2] attempted to analyze the so-called 'rush-orders' case by associating a fixed cost with stockouts which they call the 'lump-sum penalty.' In the context of the rush-orders case, the fixed stockout cost represents the fixed cost of placing a rush-order. As we mentioned in Sections 2.4 and 2.5, Çetinkaya and Parlar [10] have recently shown that Aneja and Noori's [2] analysis of the rush-orders case was in error, and under certain assumptions the models presented in this thesis can in fact handle the rush-orders problem. Here we note that the fixed and proportional backorder costs may also represent intangible factors such as foregone opportunities and loss of goodwill. The fixed backorder cost may correspond to a one-shot penalty, such as a negative market image, because of an occurrence of a stockout situation, whereas proportional backorder cost may correspond to a penalty for each demanded, but yet waiting to be satisfied item.

Unit revenue is a fixed known parameter denoted by r. During period n, if demand exceeds the total number of items available, i.e., $D_n > a_n$, then only a_n units are sold. Otherwise, if $a_n \geq D_n$, then D_n units are sold. That is, revenue received during n is denoted by R_n, and given by $R_n = r\min(a_n, D_n)$. One can easily show that

$$\min(a_n, D_n) = a_n - (a_n - D_n)^+ = D_n - (D_n - a_n)^+.$$

So, we can write

$$R_n = ra_n - r(a_n - D_n)^+, \quad n = 1, 2, \ldots. \tag{3.7}$$

Let ρ denote the discounting factor where $0 < \rho < 1$. The optimization criterion that we use is the expected net present value of the revenues net of costs. We use the term 'revenue net of costs' rather than the term 'profit' because of the incorporation of the backorder costs. Backorder costs may represent intangible factors such as loss of goodwill, and they are not actual costs that are shown by accounting records. Thus, sales revenue minus the total cost does not represent the actual profit, and it is called 'revenue net of costs'. Our objective is to maximize the expected value of the following infinite sum which is denoted by NPV:

$$NPV = \sum_{n=1}^{\infty} \rho^{n-1}(R_n - PC_n - HC_n - BC_n). \tag{3.8}$$

Recall that the problem is to find those values of order quantities, z_n, $n = 1, 2, \ldots$, that maximize the expected value of NPV. Thus, the objective function of our problem is denoted by $E[NPV]$, and it equals the expected value of (3.8).

However, if (3.3), (3.4), (3.5), (3.6), and (3.7) are substituted into (3.8) then it is observed that NPV can be expressed in terms of the variables a_n, $n = 1, 2, \ldots$. Observe that (3.1) implies

$$z_n = a_n - s_n. \tag{3.9}$$

That is, once the value of a_n is determined then the value of z_n can be specified uniquely at the beginning of n where inventory level s_n is observed. Using the decision variables a_n, $n = 1, 2, \ldots$, rather than z_n, $n = 1, 2, \ldots$, provides technical ease in finding the optimal solution. Thus, we choose to write the problem in terms of decision variables a_n, $n = 1, 2, \ldots$.

Clearly, the natural constraints of our problem are given by $z_n \geq 0$, $n = 1, 2, \ldots$. It follows from (3.9) that these constraints imply

$$a_n \geq s_n, \quad n = 1, 2, \ldots. \tag{3.10}$$

Then the infinite horizon problem that we study is called $\mathcal{P}_\infty$ and formulated as

$$\begin{aligned} \mathcal{P}_\infty: \quad & \text{Maximize} \quad E[NPV] \\ & \text{Subject to} \quad a_n \geq s_n, \quad n = 1, 2, \ldots. \end{aligned}$$

An explicit expression of the objective function, $E[NPV]$, in terms of decision variables a_n, $n = 1, 2, \ldots$, is given in Section 3.2. Here we note that $\mathcal{P}_\infty$ aims to maximize a function, $E[NPV]$, of infinitely many variables over the space defined by (3.10). This initial formulation of $\mathcal{P}_\infty$ looks quite complicated. However, once $E[NPV]$ is expressed in terms of a_n, $n = 1, 2, \ldots$, then in the following section useful observations are made that reduce the problem to a simpler form.

3.2 Reduction to a Less Complicated Formulation

Observe that (3.8) implies

$$NPV = R_1 - PC_1 - HC_1 - BC_1 + \sum_{n=2}^{\infty} \rho^{n-1}(R_n - PC_n - HC_n - BC_n).$$

Using (3.3), (3.4), (3.5), (3.6), and (3.7) we have

$$\begin{aligned} NPV &= ra_1 - r(a_1 - D_1)^+ - c(a_1 - s_1) - h(a_1 - D_1)^+ \\ &\quad - B\delta(D_1 - a_1) - b(D_1 - a_1)^+ \\ &\quad + \sum_{n=2}^{\infty} \rho^{n-1}[ra_n - r(a_n - D_n)^+ - c(a_n - a_{n-1} + D_{n-1}) \\ &\quad - h(a_n - D_n)^+ - B\delta(D_n - a_n) - b(D_n - a_n)^+]. \end{aligned}$$

Rearranging the terms in the above expression we obtain

$$\begin{aligned} NPV &= cs_1 + (r-c)a_1 - (r+h)(a_1 - D_1)^+ - B\delta(D_1 - a_1) - b(D_1 - a_1)^+ \\ &\quad + \sum_{n=2}^{\infty} \rho^{n-1}[(r-c)a_n - (r+h)(a_n - D_n)^+ - B\delta(D_n - a_n) - b(D_n - a_n)^+] \\ &\quad + \sum_{n=2}^{\infty} \rho^{n-1} c(a_{n-1} - D_{n-1}). \end{aligned}$$

It follows that,

$$\begin{aligned} NPV &= cs_1 + \sum_{n=1}^{\infty} \rho^{n-1}[(r-c)a_n - (r+h)(a_n - D_n)^+ - B\delta(D_n - a_n) \\ &\quad - b(D_n - a_n)^+)] + \sum_{n=2}^{\infty} \rho^{n-1} c(a_{n-1} - D_{n-1}). \end{aligned} \tag{3.11}$$

The last term of (3.11) can be expressed as

$$\sum_{n=2}^{\infty} \rho^{n-1} c(a_{n-1} - D_{n-1}) = \sum_{n=1}^{\infty} \rho^{n-1}[\rho c(a_n - D_n)]. \tag{3.12}$$

If we substitute (3.12) in (3.11) we obtain

$$\begin{aligned} NPV &= cs_1 + \sum_{n=1}^{\infty} \rho^{n-1}[(r-c)a_n - (r+h)(a_n - D_n)^+ \\ &\quad -B\delta(D_n - a_n) - b(D_n - a_n)^+ + \rho c(a_n - D_n)]. \end{aligned}$$

and rearranging the terms gives

$$\begin{aligned} NPV &= cs_1 + \sum_{n=1}^{\infty} \rho^{n-1}[(r-c+\rho c)a_n - \rho c D_n - (r+h)(a_n - D_n)^+ \\ &\quad -B\delta(D_n - a_n) - b(D_n - a_n)^+]. \end{aligned} \tag{3.13}$$

Let us define

$$\begin{aligned} w(a_n, D_n) &\equiv (r-c+\rho c)a_n - \rho c D_n - (r+h)(a_n - D_n)^+ \\ &\quad -B\delta(D_n - a_n) - b(D_n - a_n)^+. \end{aligned} \tag{3.14}$$

If we insert $w(a_n, D_n)$ in (3.13) and take the expected value of both sides, we obtain

$$E[NPV] = cs_1 + \sum_{n=1}^{\infty} \rho^{n-1} E[w(a_n, D_n)]. \tag{3.15}$$

Observe that expected value of $w(a_n, D_n)$ depends on the distribution of D_n as well as the distribution of a_n. Nevertheless, the following theorem states that expected value of $w(a_n, D_n)$ depends only on the distribution of a_n because of stochastic independence between a_n and D_n. This result is very useful for further analysis of the problem, and introduces simplicity for obtaining its solution.

Theorem 1

$$E[w(a_n, D_n)] = E[G(a_n)], \tag{3.16}$$

where

$$\begin{aligned} G(a) &= (r-c+\rho c)a - \rho c E(D_1) - (r+h)E[(a-D_1)^+] \\ &\quad -BE[\delta(D_1 - a)] - bE[(D_1 - a)^+], \;\; a \in (-\infty, +\infty). \end{aligned} \tag{3.17}$$

Proof The proof of this theorem is the same as the proof of a very similar theorem discussed by Heyman and Sobel (in [15], p. 65). Here we give the proof in order to preserve completeness.

Observe that a_n can only depend on the information available at the beginning of period n. That is, a_n depends on

$$s_1, a_1, D_1, s_2, a_2, D_2, \ldots, s_{n-1}, a_{n-1}, D_{n-1}, \text{ and } s_n.$$

Since a_n and D_n are stochastically independent we have

$$E[w(a_n, D_n)] = E\{E[w(a_n, D_n)|a_n]\}. \tag{3.18}$$

Note that,

$$\begin{aligned} E[w(a_n, D_n)|a_n] &= (r - c + \rho c)a_n - \rho c E[D_n] - (r + h)E[(a_n - D_n)^+] \\ &\quad - BE[\delta(D_n - a_n)] - bE[(D_n - a_n)^+]. \end{aligned} \tag{3.19}$$

By assumption, $D_n, n = 1, 2, \ldots$, are i.i.d. random variables. Therefore, we can replace D_n that appears on the right hand side of (3.19) by D_1. Furthermore, using (3.18) and (3.19) we can write

$$\begin{aligned} E[w(a_n, D_n)] &= E\{(r - c + \rho c)a_n - \rho c E(D_1) - (r + h)E[(a_n - D_1)^+] \\ &\quad - BE[\delta(D_1 - a_n)] - bE[(D_1 - a_n)^+]\}. \end{aligned} \tag{3.20}$$

Equation (3.20) implies (3.16). ■

Now, if we insert (3.16) in (3.15) we obtain

$$E[NPV] = cs_1 + \sum_{n=1}^{\infty} \rho^{n-1} E[G(a_n)]. \tag{3.21}$$

Thus, $\mathcal{P}_\infty$ is equivalent to

$$\begin{array}{llll} \mathcal{P}_\infty: & \text{Maximize} & cs_1 + \sum_{n=1}^{\infty} \rho^{n-1} E[G(a_n)] & \\ & \text{Subject to} & a_n \geq s_n, & n = 1, 2, \ldots. \end{array}$$

Given s_1, the initial inventory level, cs_1 is a constant. Thus, it can be ignored in the above formulation. That is, $\mathcal{P}_\infty$ can be considered as a maximization problem with a separable objective function where the constraint set $a_n \geq s_n$, $n = 1, 2, \ldots$, has a diagonal structure. Note that each separable term $\rho^{n-1}E[G(a_n)]$, $n = 1, 2, \ldots$, of this problem corresponds to the present value of the expected return net of costs for period n.

Function $G(\cdot)$ is called the 'single period return net of costs' function. We use the term 'single period *return* net of costs' rather than 'single period *revenue* net of costs', because if we add discounted single period revenues net of costs then we should have total discounted revenue net of costs, i.e., *NPV*. However, if we add functions $G(a_n)$ by discounting them the result is different from *NPV* by cs_1. Thus, we say $G(\cdot)$ represents single period return net of costs instead of single period revenue net of costs.

In order to find the retailer's optimal policy for a given period n, it is sufficient to maximize the net present value of the expected single period return net of costs, represented by $\rho^{n-1}E[G(a_n)]$ subject to the constraint $a_n \geq s_n$, without any consideration of the future returns, and as if current decisions have no effect on the future events, i.e., as if the level of a_n would not affect $s_{n+1}, s_{n+2}, \ldots$. Heyman and Sobel (in [15], p. 63) call such a policy *myopic*, because future consequences of current decisions can be ignored safely.

As we mentioned in Section 2.7 of Chapter 2, for a dynamic problem an optimum is said to be *myopic* if it can be deduced from an optimum of a static problem (Sobel [40]). Porteus [33] states that an inventory policy is said to be myopic if it is the optimal policy for a single period model that is defined explicitly in terms of the original model parameters. In fact, through straightforward algebraic manipulations described above we managed to reduce the multi-period dynamic inventory problem to a forward sequence of single period, static problems. Thus, our periodic review model ignores the dynamic nature of the problem and reduces to a sequence of single period problems which are solved via maximizing $G(\cdot)$. Since it is possible to decompose the multi-period problem into a sequence of single period problems we have a myopic optimal policy. In turn solving the dynamic multi-period problem is as easy as maximizing $\rho^{n-1}E[G(a_n)]$. The myopic policies are important for a couple of reasons. First of all they are simple to implement and easier to compute. Secondly, as for our problem they are *in fact* optimal in certain settings. Before discussing the myopic optimal policy for $\mathcal{P}_\infty$ we analyze the functional form $G(\cdot)$.

3.3 Analysis of the Single Period Return Net of Costs Function, $G(\cdot)$

Let $f(\cdot)$ and $F(\cdot)$ denote the demand density and corresponding distribution functions, respectively. Recall that demand is a positive random variable. Thus, $G(a)$ described by (3.17) can be stated as

$$\begin{aligned} G(a) &= (r - c + \rho c)a - \rho c E(D_1) - (r + h)\int_0^a (a - x)f(x)dx \\ &\quad - B\int_a^\infty f(x)dx - b\int_a^\infty (x - a)f(x)dx, \quad a \in (-\infty, +\infty). \end{aligned}$$

If we substitute $\int_a^\infty f(x)dx = 1 - F(a)$, in the above expression, then

$$\begin{aligned} G(a) &= (r - c + \rho c)a - \rho c E(D_1) - (r + h)\int_0^a (a - x)f(x)dx \\ &\quad - B[1 - F(a)] - b\int_a^\infty (x - a)f(x)dx, \quad a \in (-\infty, +\infty). \end{aligned} \tag{3.22}$$

Proposition 1 *For positive demand densities, the global maximizer of $G(\cdot)$ is finite and positive.*

Proof Applying Leibnitz's rule we can write

$$\frac{d}{da}\left[\int_0^a (a - x)f(x)dx\right] = F(a), \tag{3.23}$$

and

$$\frac{d}{da}\left[\int_a^\infty (x - a)f(x)dx\right] = F(a) - 1. \tag{3.24}$$

Then using (3.22), (3.23), and (3.24) the first derivative of $G(a)$ with respect to a, denoted by $G'(a)$, is stated as

$$G'(a) = (r - c + \rho c + b) - (r + h + b)F(a) + Bf(a). \tag{3.25}$$

Also note that,

$$\lim_{a \to -\infty} G'(a) = r - c + \rho c + b > 0, \tag{3.26}$$

and

$$\lim_{a \to +\infty} G'(a) = -[c(1 - \rho) + h] < 0. \tag{3.27}$$

Observe that (3.26) and (3.27) imply $G(a)$ decreases as $a \to -\infty$, and as $a \to +\infty$. Thus, the global maximizer of $G(\cdot)$ is finite.

Since demand is a positive random variable, if $a \leq 0$ then $f(a) = F(a) = 0$. Thus, (3.25) implies

$$G'(a) = r - c + \rho c + b > 0, \quad \forall a \leq 0. \tag{3.28}$$

This in turn leads to

$$G(0) \geq G(a), \quad \forall a \leq 0.$$

It follows that global maximizer of $G(\cdot)$ is positive. ■

In fact, (3.22) implies that if $a < 0$ then

$$G(a) = (r - c + \rho c + b)a - (\rho c + b)E(D_1) - B. \tag{3.29}$$

That is, $G(a)$ is a linearly increasing function for $a < 0$.

Let S denote the global maximizer of $G(\cdot)$. Proposition 1 states that $S \in (0, +\infty)$. Therefore, maximizing $G(a)$ over $(-\infty, +\infty)$ is equivalent to maximizing $G(a)$ over $(0, +\infty)$.

3.3.1 A Condition Under Which $G(\cdot)$ is Concave

Note that, it may be difficult to compute S because $G(a)$ may have many local maximizers. Proposition 2 imposes a condition on $G(a)$ to be concave over $[0, \infty)$ so that S uniquely solves $G'(a) = 0$.

Proposition 2 *$G(\cdot)$ is concave over the positive axis provided that*

$$\frac{f'(a)}{f(a)} < \frac{r + h + b}{B}, \quad a > 0. \tag{3.30}$$

If (3.30) is satisfied then S is the unique positive solution of

$$(r - c + \rho c + b) - (r + h + b)F(a) + Bf(a) = 0. \tag{3.31}$$

Proof Expression (3.25) implies that the second derivative of $G(\cdot)$ denoted by $G''(a)$ is given by

$$G''(a) = -(r + h + b)f(a) + Bf'(a), \tag{3.32}$$

where $f'(a)$ denotes the first derivative of $f(a)$.

For concavity of $G(a)$ over the positive axis we need $G''(a) \leq 0$ for $a > 0$. Thus, (3.32) implies that $G(a)$ is strictly concave over $(0, +\infty)$ if (3.30) holds. Further, under (3.30) it follows from concavity of $G(\cdot)$ and Proposition 1 that S is the unique positive solution of (3.31). ■

If the demand density is nonincreasing, e.g., uniform or exponential, then (3.30) holds immediately. Otherwise, we need model parameters r, h, b, B, and demand density $f(\cdot)$ to satisfy (3.30) in order to utilize the implications of concavity of $G(\cdot)$. Here we would like to note that concavity of $G(\cdot)$ is a sufficient condition in order to have a myopic optimal solution. We give a more detailed explanation about this in Section 3.5.2. In the next section we discuss a less restrictive sufficient condition than concavity of $G(\cdot)$.

3.3.2 A Less Restrictive Assumption than Concavity of $G(\cdot)$

Here we obtain a condition on $G(\cdot)$ under which for all a_n and a'_n such that $S \leq a_n \leq a'_n$, we have

$$G(S) \geq G(a_n) \geq G(a'_n). \tag{3.33}$$

That is, we attempt to find a condition under which $G(\cdot)$ is nonincreasing over $[S, +\infty)$.

In order for $G(\cdot)$ to be nonincreasing over $[S, +\infty)$ we need

$$G'(S+u) \leq 0, \ \ \forall u > 0,$$

or equivalently

$$G'(S) - G'(S+u) \geq 0, \ \ \forall u > 0, \tag{3.34}$$

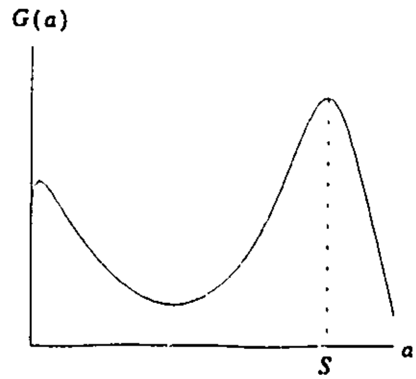

Figure 3.2: An illustration: $G(\cdot)$ satisfies (3.35).

since $G'(S) = 0$. Using (3.25) equation (3.34) can now be expressed as

$$(r + h + b)[F(S) - F(S + u)] - B[f(S) - f(S + u)] \leq 0, \ \ \forall u > 0. \tag{3.35}$$

Thus, if (3.35) holds then $G(\cdot)$ is nonincreasing over $[S, +\infty)$. Obviously, (3.35) holds immediately if $G(\cdot)$ is concave. Figure 3.2 provides an illustration for which (3.35) holds.

3.4 The Optimal Solution of $\mathcal{P}_\infty$

Now, we discuss the optimal policy for $\mathcal{P}_\infty$.

Proposition 3 *If $s_1 \leq S$ then $a_n = S$, $n = 1, 2, \ldots$, is optimal for $\mathcal{P}_\infty$.*

Proof The proof is very similar to the case discussed by Heyman and Sobel (in [15], p. 66) where backorders are not allowed. Here we present it for the sake of completeness. The proof consists of two parts. In Part (a), we show that if $s_1 \leq S$

then $a_n = S$ is a feasible solution for $\mathcal{P}_\infty$. In Part (b), the optimality of this solution is proved.

Part (a) Feasibility proof by induction: According to (3.10) we need $a_1 \geq s_1$. Then it is feasible to have $a_1 = S$ because $s_1 \leq S$. Assume that it is feasible to have $a_{n-1} = S$. Since $s_n = a_{n-1} - D_{n-1}$ and $D_{n-1} > 0$, we have $s_n = S - D_{n-1} < S$. In turn, constraint $a_n \geq s_n$ allows $a_n = S$, and this completes the feasibility proof.

Part (b) Optimality proof: Since S is the global maximizer of $G(\cdot)$ we know $G(S) \geq G(a)$, for all a. Then for every n and a_n we have $E[G(a_n)] \leq G(S)$. Therefore,

$$\sum_{n=1}^{\infty} \rho^{n-1} E[G(a_n)] \;\leq\; \sum_{n=1}^{\infty} \rho^{n-1} G(S).$$

It follows that, $a_n = S$, $n = 1, 2, \ldots$, is optimal for $\mathcal{P}_\infty$. ∎

Proposition 4 *Provided that (3.35) holds, an optimal policy for $\mathcal{P}_\infty$ is specified by*

$$a_n = \max(s_n, S), \;\; n = 1, 2, \ldots. \tag{3.36}$$

Proof First observe that, if $s_1 \leq S$ then policy (3.36) and inventory balance restrictions (3.2) imply $s_n \leq S$ for all n, and thus $a_n = S$ for all n. That is, (3.36) is equivalent to the policy stated by Proposition 3. It follows that (3.36) is optimal if $s_1 \leq S$.

Next we prove the optimality of (3.36) when $s_1 \geq S$. Let $a_1, a_2, \ldots$ denote the decisions when (3.36) is employed. Note that, s_1 is a model parameter, and let $s_2, s_3, \ldots$ denote the inventory levels observed after decisions $a_1, a_2, \ldots$ are taken.

Let N be the first period such that (3.36) causes the inventory level at the beginning of N, i.e., s_N, to be less than or equal to S. That is,

$$N = \inf\{n : s_n \leq S\}. \tag{3.37}$$

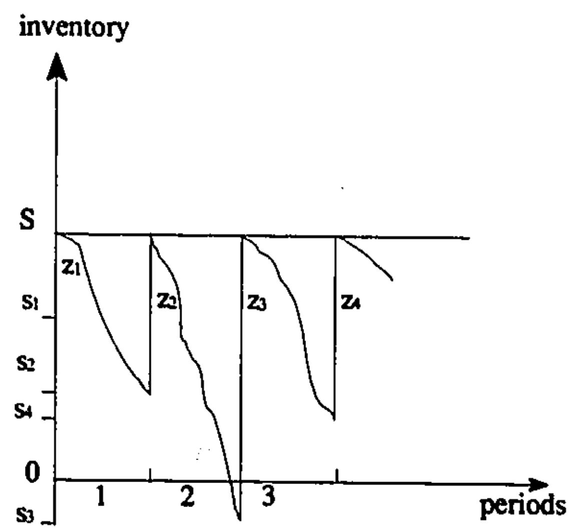

Figure 3.3: Inventory system under the optimal policy when $s_1 \leq S$.

We define N for the case $s_1 > S$, and thus minimum value that N can take is 2.

According to (3.37) until the beginning of period N the inventory level is always strictly greater than S, i.e.,

$$s_n > S, \quad n = 1, 2, \ldots, N-1. \tag{3.38}$$

Because $a_1, a_2, \ldots,$ denote the decisions under (3.36) we also have

$$a_n = \max(S, s_n), \quad n = 1, 2, \ldots, N-1. \tag{3.39}$$

Using (3.38) and (3.39) we have

$$a_n = s_n, \quad n = 1, 2, \ldots, N-1. \tag{3.40}$$

Furthermore, (3.37) implies that inventory levels $s_N, s_{N+1}, \ldots$ are all less than or equal to S, i.e.,

$$s_n \leq S, \quad n = N, N+1, \ldots. \tag{3.41}$$

Note (3.41) means that (3.36) will lead to the following solution without violating the feasibility restrictions (3.10) of $\mathcal{P}_\infty$.

$$a_n = S, \quad n = N, N+1, \ldots. \tag{3.42}$$

Let us recall equations (3.2) which relate successive periods' inventory levels and decisions. Combining (3.2) and (3.40) gives

$$s_{n+1} = s_n - D_n, \quad n = 1, 2, \ldots, N-1. \tag{3.43}$$

Equations (3.43) written explicitly imply

$$s_2 = s_1 - D_1,$$

$$\begin{aligned} s_3 &= s_2 - D_2, \\ s_4 &= s_3 - D_3, \\ &\vdots \\ s_N &= s_{N-1} - D_{N-1}. \end{aligned}$$

Summing up the above equations we obtain

$$s_n = s_1 - \sum_{i=1}^{n-1} D_i, \quad n = 2, 3, \ldots, N. \tag{3.44}$$

Let $a'_1, a'_2, \ldots$ denote the decisions under a given policy other than (3.36), and $s'_2, s'_3, \ldots$ denote the inventory levels observed after decisions $a'_1, a'_2, \ldots$ are taken. Note that, because of feasibility restrictions (3.10), and inventory balance equations (3.2) imposed by $\mathcal{P}_\infty$ we need

$$a'_n \geq s'_n, \quad n = 1, 2, \ldots, \tag{3.45}$$

and

$$s'_{n+1} = a'_n - D'_n, \quad n = 1, 2, \ldots. \tag{3.46}$$

Also note that, since initial inventory level is a given constant we have $s'_1 = s_1$. Thus, (3.45) means we need

$$a'_1 \geq s'_1 = s_1. \tag{3.47}$$

In order to complete the proof we first show that

$$G(a_n) \geq G(a'_n), \quad \forall n \geq N. \tag{3.48}$$

Next we show

$$G(a_n) \geq G(a'_n), \quad \forall n < N. \tag{3.49}$$

Then (3.48) and (3.49) imply that

$$G(a_n) \geq G(a'_n), \ \ \forall n,$$

which in turn implies

$$\sum_{n=1}^{\infty} \rho^{n-1} E[G(a_n)] \ \geq \ \sum_{n=1}^{\infty} \rho^{n-1} E[G(a'_n)],$$

and completes the proof.

If $n \geq N$ using (3.42) we have

$$G(a_n) = G(S) \geq G(a'_n), \ \ \forall n \geq N,$$

and this verifies (3.48) since S is the global maximizer of $G(\cdot)$.

On the other hand, if $n < N$, i.e., $n \leq N-1$, then (3.40) and (3.44) yield

$$a_1 = s_1, \tag{3.50}$$

$$a_n = s_1 - \sum_{i=1}^{n-1} D_i, \ \ n = 2,3,\ldots,N-1. \tag{3.51}$$

Equations (3.51) lead to

$$a_n \leq s_1 - \sum_{i=1}^{n-1} [D_i - (a'_i - s'_i)], \ \ n = 2,3,\ldots,N-1. \tag{3.52}$$

since (3.45) implies $(a'_n - s'_n) \geq 0$, $i = 1,2,\ldots,N-1$. Rearranging the terms of inequalities (3.52) and using (3.47) we obtain

$$a_n \leq -\sum_{i=2}^{n-1} s'_i + \sum_{i=1}^{n-1} (a'_i - D'_i)], \ \ n = 2,3,\ldots,N-1. \tag{3.53}$$

Recalling (3.46) we replace $(a'_i - D'_i)$ by s'_{i+1} in (3.53). Thus, inequalities (3.53) become

$$a_n \leq -\sum_{i=2}^{n-1} s'_i + \sum_{i=1}^{n-1} s'_{i+1}, \ \ n = 2,3,\ldots,N-1,$$

and they reduce to

$$a_n \le s'_n, \quad n = 2, 3, \ldots, N-1. \tag{3.54}$$

Further, using (3.47) and (3.50) we can write

$$a_1 = s_1 = s'_1 \le a'_1. \tag{3.55}$$

Combining (3.38), (3.40), (3.45), (3.54), (3.55) we have

$$S < a_n \le a'_n, \quad n = 1, 2, 3, \ldots, N-1. \tag{3.56}$$

We assume (3.35) which in turn implies (3.33) for all a_n and a'_n related by (3.56). This leads to (3.49) and completes the proof. ∎

3.5 Remarks and Technical Details

In this section we make additional remarks regarding the optimal policy, and discuss some technical details which we refer to elsewhere in the thesis.

3.5.1 Myopic Base-Stock Policy

Proposition 4 states that a base-stock policy is optimal. As we mentioned in Section 2 the value of S is called the 'critical number' or the 'base-stock level'. According to the optimal policy, at the beginning of a period an order is or is not placed depending on whether the inventory level is above or below S. If an order is placed then its amount should be sufficient to raise the inventory level up to the base-stock level S. In general, a base-stock policy involves a set of critical numbers each associated with a given period. For example, for periods 1, 2 and 3, an arbitrary base-stock policy may involve critical numbers $S^{(1)}, S^{(2)}$, and $S^{(3)}$ respectively. On the other hand, for

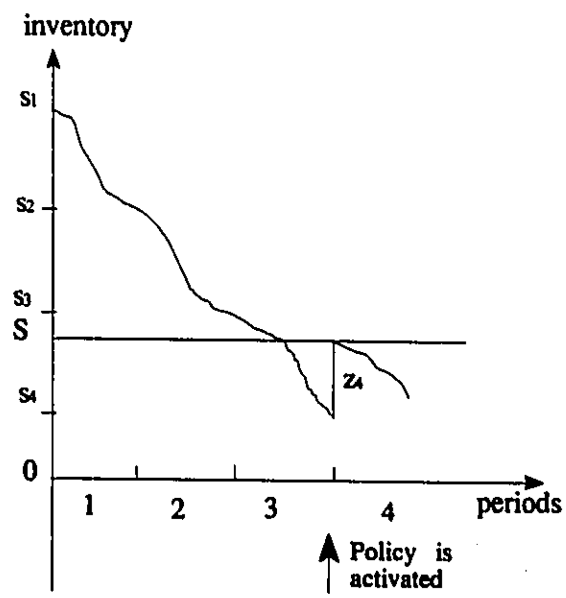

Figure 3.4: Inventory system under the optimal policy when $s_1 > S$, and $N = 4$.

our infinite horizon problem we have $S^{(1)} = S, S^{(2)} = S, S^{(3)} = S, \ldots$ Intuitively, since the system operates for a long time period the base-stock levels converge to a limit. That is, for our infinite horizon problem the same critical number S is optimal regardless of the future of the inventory system, i.e., optimal policy is also myopic. For this reason it is classified as a "myopic base-stock" policy.

3.5.2 The Sufficient Condition for the Myopic Optimum

Let us emphasize that if $s_1 \leq S$ then the policy given by (3.36) is optimal in general, i.e., no assumptions are made regarding the shape of $G(\cdot)$. In this case, optimal order quantities are given by $z_n = S - s_n$ for all n. On the other hand, if $s_1 > S$ then we need to assume (3.35) in order to prove optimality of (3.36). That is, a sufficient condition for (3.36) to be optimal in this case is to assume $G(\cdot)$ to be nonincreasing over $[S, +\infty)$. We show in Chapter 4 that this assumption holds immediately for gamma distributed demand. We consider the gamma density with an integer shape parameter which is known as the Erlang density. Let $E_{k,\lambda}$ denote the Erlang random variable with shape parameter k and scale parameter λ. The mean and variance of $E_{k,\lambda}$ is given by k/λ and k/λ^2 respectively. It follows that the coefficient of variation of $E_{k,\lambda}$ is $1/\sqrt{k} \leq 1$. Hence by a careful choice of the parameter k any random variable with coefficient of variation less than unity can be approximated by an Erlang variable. Kleinrock [21] provides a discussion of "Erlang's method of stages" to approximate a nonexponential random variable with an Erlang random variable. We suggest that condition (3.35) is not very restrictive in the sense that it holds immediately for Erlang demand which may provide good approximations for other demand densities.

3.5.3 The Optimal Order Quantities

Under the myopic base-stock policy (3.36), number of items available for sale at the beginning of periods $N, N+1, \ldots$ is always set to S by placing a sufficient order quantity. Thus, we say that ordering up to S is optimal once inventory level goes below S, i.e., period N is reached. If $s_1 \geq S$ optimal order quantities are given by

$$z_n = \begin{cases} 0 & \text{if } n < N, \\ S - s_n & \text{if } n \geq N. \end{cases}$$

If $s_1 < S$ then the optimal order quantities are given by $z_n = S - s_n, \ \forall n$.

3.5.4 Time of Placing the First Order

Under (3.35), if $s_1 > S$ then using the definition of N optimal policy defined by (3.36) can be stated as

$$a_n = \begin{cases} s_n = s_1 - \sum_{i=1}^{n-1} D_i & \text{if } n < N, \\ S & \text{if } n \geq N. \end{cases} \tag{3.57}$$

That is, if $s_1 > S$ then until the beginning of N inventory levels are related by

$$s_1 \geq s_2 \geq \ldots \geq s_{N-1} > S \geq s_N.$$

Thus, we wait until the beginning of period N which is a random variable, so that inventory level goes below S. Once period N is reached we say that 'the policy is activated', and

$$s_n \leq S, \ \forall n \geq N.$$

Using (3.37) and (3.44) we can write

$$N = \inf \left\{ n : \sum_{i=1}^{n-1} D_i \geq s_1 - S \right\}. \tag{3.58}$$

That is, at the beginning of period N we observe that cumulative demand reaches or exceeds $s_1 - S$.

Let us compute the distribution of N for future reference. Using (3.58), we conclude that

$$\{N > n\} \iff \left\{\sum_{i=1}^{n-1} D_i < s_1 - S\right\}, \quad n = 2, 3, \ldots. \tag{3.59}$$

Since N has been defined under the assumption that $s_1 > S$, by definition $P(N > 1) = 1$. Noting

$$P(N = n) = P(N > n - 1) - P(N > n),$$

it follows from relation (3.59) that

$$P(N = n) = P\left(\sum_{i=1}^{n-2} D_i < s_1 - S\right) - P\left(\sum_{i=1}^{n-1} D_i < s_1 - S\right), \quad n = 2, 3 \ldots. \tag{3.60}$$

Let $F^{(n)}(\cdot)$ denote the n-fold convolution of demand distribution $F(\cdot)$. We say $F^{(0)}(x) \equiv 1$. Then (3.60) can be stated as

$$P(N = n) = F^{(n-2)}(s_1 - S) - F^{(n-1)}(s_1 - S), \quad n = 2, 3, \ldots. \tag{3.61}$$

3.6 The Optimal Value of the Objective Function

Under the optimal policy $a_n = \max(s_n, S), n = 1, 2, \ldots$ of Proposition 4 the best we can do in monetary terms depends on s_1. For example, if $s_1 > S$ then no orders are placed until the inventory level goes below S, i.e., until the beginning of random period N. Consequently, our procurement decisions and costs are affected by the initial inventory level s_1.

In order to show the relation between s_1 and the total expected discounted revenue net of costs over an infinite horizon we substitute the optimal solution (3.36) in (3.21). That is, we compute the optimal value of the objective function $E[NPV]$. The resulting expression depends on s_1, and for this reason it is denoted by $NPV(s_1)$. The optimal myopic policy (3.36) implies that two cases are possible depending on the values of s_1 and S. Namely,

- **Case 1:** $s_1 < S$, and
- **Case 2:** $s_1 \geq S$.

In turn $NPV(s_1)$ has two expressions which are computed below.

3.6.1 Case 1: Expression for $NPV(s_1)$ if $s_1 < S$

In this simpler case, (3.36) implies $a_n = S$, $n = 1, 2, \ldots$, and if we substitute this in (3.21) we have

$$NPV(s_1) = cs_1 + \frac{G(S)}{1-\rho}, \quad s_1 < S, \tag{3.62}$$

since $E[G(S)] = G(S)$.

3.6.2 Case 2: Expression for $NPV(s_1)$ if $s_1 \geq S$

If $s_1 \geq S$ then (3.36) can also be expressed as (3.57). Substituting (3.57) in (3.21) which states

$$E[NPV] = cs_1 + \sum_{n=1}^{\infty} \rho^{n-1} E[G(a_n)],$$

we obtain

$$NPV(s_1) = cs_1 + E\left[\sum_{n=1}^{N-1} \rho^{n-1} G\left(s_1 - \sum_{i=1}^{n-1} D_i\right) + \sum_{n=N}^{\infty} \rho^{n-1} G(S)\right] \tag{3.63}$$

Using the infinite sum formula expression, (3.63) can be reduced to

$$NPV(s_1) = cs_1 + E\left[\sum_{n=1}^{N-1} \rho^{n-1} G\left(s_1 - \sum_{i=1}^{n-1} D_i\right)\right] + \frac{G(S)}{1-\rho} E\left[\rho^{N-1}\right]. \tag{3.64}$$

As we explained on page 33, function $G(a)$ represents the single period return net of costs starting with a units of inventory. Then

$$E\left[\sum_{n=1}^{N-1} \rho^{n-1} G\left(s_1 - \sum_{i=1}^{n-1} D_i\right)\right] \tag{3.65}$$

that appears on the right hand side of (3.64) is the expected discounted return net of costs throughout periods $1, 2, \ldots, N-1$, starting with s_1 units and without placing any orders. Let $C(s_1)$ denote this quantity. In other words, $C(s_1)$ is the expected discounted return net of costs until the end of random period $N-1$ during which cumulative demand reached or exceeded $s_1 - S$ units for the *first* time. Since $G(s_1)$ is the single period return net of costs starting with s_1 units then using conditioning we can write

$$C(s_1|D_1 = x) = \begin{cases} G(s_1|D_1 = x) & \text{if } x \geq s_1 - S, \\ G(s_1|D_1 = x) + \rho C(s_1 - x) & \text{if } x < s_1 - S. \end{cases} \tag{3.66}$$

Proposition 5

$$C(s_1) = G(s_1) + \int_0^{s_1 - S} G(s_1 - x) m_\rho(x) dx. \tag{3.67}$$

where

$$m_\rho(x) = \sum_{n=1}^{\infty} \rho^n f^{(n)}(x), \tag{3.68}$$

and $f^{(n)}(x)$ denotes the n-fold convolution of demand density $f(x)$.

Proof Using (3.66) in

$$C(s_1) = \int_0^{\infty} C(s_1|D_1 = x) f(x) dx,$$

we have

$$C(s_1) = G(s_1) + \rho \int_0^{s_1 - S} C(s_1 - x) f(x) dx. \tag{3.69}$$

Let $\tilde{C}(\cdot)$, $\tilde{G}(\cdot)$, and $\tilde{f}(\cdot)$ denote Laplace transforms of $C(\cdot)$, $G(\cdot)$, and $f(\cdot)$ respectively. Taking the Laplace transform of (3.69), and solving $\tilde{C}(\cdot)$ we obtain

$$\tilde{C}(\cdot) = \frac{\tilde{G}(\cdot)}{1-\rho\tilde{f}(\cdot)}.$$

However, the above formula can also be expressed as

$$\tilde{C}(\cdot) = \tilde{G}(\cdot) + \frac{\rho\tilde{G}(\cdot)\tilde{f}(\cdot)}{1-\rho\tilde{f}(\cdot)},$$

and using the infinite summation formula in this alternative expression we have

$$\tilde{C}(\cdot) = \tilde{G}(\cdot) + \tilde{G}(\cdot)\sum_{n=1}^{\infty}\rho^n[\tilde{f}(\cdot)]^n \tag{3.70}$$

Let $\tilde{m}_\rho(\cdot)$ denote the Laplace transform of $m_\rho(\cdot)$. Then (3.68) implies that

$$\tilde{m}_\rho(\cdot) = \sum_{n=1}^{\infty}\rho^n[\tilde{f}(\cdot)]^n, \tag{3.71}$$

since the Laplace transform of n-fold convolution of demand density denoted by $\tilde{f}^{(n)}(\cdot)$ satisfies

$$\tilde{f}^{(n)}(\cdot) = [\tilde{f}(\cdot)]^n.$$

Then substituting (3.71) in (3.70) leads to

$$\tilde{C}(\cdot) = \tilde{G}(\cdot) + \tilde{G}(\cdot)\tilde{m}_\rho(\cdot), \tag{3.72}$$

and taking the inverse transform of both sides gives (3.69). ■

If we use (3.67) in (3.64) we have

$$NPV(s_1) = cs_1 + G(s_1) + \int_0^{s_1-S} G(s_1-x)m_\rho(x)dx + \frac{G(S)}{1-\rho}E\left[\rho^{N-1}\right], \quad s_1 \geq S. \tag{3.73}$$

Now, we want to compute $E[\rho^{N-1}]$ that appears on the right hand side of (3.73). The well-known formula for computing expected values states that

$$E[\rho^{N-1}] = \sum_{n=1}^{\infty} \rho^n P(N-1=n). \tag{3.74}$$

Substituting (3.61) in (3.74) yields

$$E\left[\rho^{N-1}\right] = \rho \sum_{n=0}^{\infty} \rho^n F^{(n)}(s_1 - S) - \sum_{n=1}^{\infty} \rho^n F^{(n)}(s_1 - S) \tag{3.75}$$

We also define

$$M_\rho(x) = \sum_{n=1}^{\infty} \rho^n F^{(n)}(x), \tag{3.76}$$

and thus we have

$$M_\rho(x) = \int_0^x m_\rho(t)dt. \tag{3.77}$$

If we use (3.76) in (3.75) we simply obtain

$$E[\rho^{N-1}] = \rho - (1-\rho)M_\rho(s_1 - S). \tag{3.78}$$

Then substituting (3.78) in (3.73) gives

$$\begin{aligned} NPV(s_1) &= cs_1 + G(s_1) + \int_0^{s_1 - S} G(s_1 - x)m_\rho(x)dx \\ &\quad + \frac{\rho}{1-\rho}G(S) - M_\rho(s_1 - S)G(S), \quad s_1 \geq S. \end{aligned} \tag{3.79}$$

In summary, under the optimal policy the expected total discounted revenue net of costs is given by

$$NPV(s_1) = \begin{cases} (3.62) & \text{if } s_1 < S, \\ (3.79) & \text{if } s_1 \geq S. \end{cases} \tag{3.80}$$

Function $NPV(s_1)$ represents the best we can do in monetary terms as a function of initial inventory level, and it can be used to decide how much of an excess inventory to dispose. We utilize this expression of $NPV(s_1)$ in Chapter 5 where we discuss a disposal problem which is related to our infinite horizon problem $\mathcal{P}_\infty$.

3.7 Summary

In this chapter we formulated the infinite horizon dynamic problem which we reduced to a sequence of static problems through algebraic manipulations. The stockouts are penalized via fixed and proportional backorder costs simultaneously. We provided conditions under which a myopic base-stock policy is optimal for this problem. In general, the infinite horizon solution represents the limiting behavior associated with the finite horizon solution. The presentation that we used was useful in the reduction of the multi-period dynamic problem to a forward sequence of single period, static problems. We note that each single period problem is of the classical newsboy type except for the consideration of the fixed backorder cost. Calculation of the optimal policy, i.e., computing the maximizer S of $G(\cdot)$ will be illustrated at the end of the next chapter.

We also discussed the properties and implications of the myopic policy, and introduced some basic concepts such as the time at which the policy is activated, N. The theory and results delivered in this chapter constitute a basis for the coming chapters where we study Erlang demand, disposal decisions and the finite horizon problem.

Chapter 4

The Infinite Horizon Policy for Erlang Demand

4.1 Effect of Demand Density

The optimal policy for $\mathcal{P}_\infty$ depends on the initial inventory level, s_1, and the shape of the single period return net of costs function, $G(\cdot)$. Two cases are possible:

- If initial inventory level, s_1, is less than or equal to the global maximizer S of $G(\cdot)$, then myopic policy (3.36) is optimal in general. Although optimality of (3.36) is guaranteed, in this case finding the value of S may require some computational effort depending on the shape of $G(\cdot)$.

- If the initial inventory level, s_1, is greater than the global maximizer S of $G(\cdot)$, then the optimality of (3.36) is not guaranteed. However, if (3.35) holds, that is if $G(\cdot)$ is nonincreasing over $[S, +\infty)$, then myopic policy (3.36) is optimal. The ease of computing the value of S again depends on the shape of $G(\cdot)$.

In fact, $G(\cdot)$ depends on the demand density. Therefore, the computational effort required to compute the optimal policy is related to the demand density.

If the demand density is nonincreasing it follows from Proposition 2 that $G(\cdot)$ is concave, and S can be computed easily.

For example, if demand has a uniform distribution over $[\eta_0, \eta_1]$ it can be easily shown that $G(\cdot)$ is linearly increasing over $(-\infty, \eta_0)$, strictly concave over $[\eta_0,\eta_1]$, and linearly decreasing over $(\eta_1, +\infty)$. Function $G(\cdot)$ is also continuous and differentiable in this case, and it follows that for uniform demand, $S \in [\eta_0, \eta_1]$, and

$$S = \frac{B + (r - c + \rho c + b)\eta_1 + [c(1-\rho) + h]\eta_0}{h + b + r}. \tag{4.1}$$

Observe that, according to (4.1)

- S increases as B and/or b increase,
- $S \to \eta_0$ as $h \to \infty$, and
- $S \to \eta_1$ as $r \to \infty$.

Another example of an easy-to-obtain solution is provided by exponential demand with parameter λ. In this case, $G(\cdot)$ is linearly increasing over $(-\infty,0]$, and strictly concave over $(0, +\infty)$. Again $G(\cdot)$ is continuous and differentiable, and

$$S = \frac{1}{\lambda} \ln \left[\frac{B\lambda + h + b + r}{c(1-\rho) + h} \right]. \tag{4.2}$$

Note that, for the case of exponential demand we have $D_n \in (0, +\infty)$. Using (4.2) we have

- S increases as B and/or b, and/or r increase, and
- $S \to 0$ as $h \to \infty$.

We also have

- $S \to \infty$ as $\lambda \to 0$, and
- $S \to 0$ as $\lambda \to \infty$

since the mean demand μ is given by $1/\lambda$ so that

- $\mu \to \infty$ as $\lambda \to 0$, and
- $\mu \to 0$ as $\lambda \to \infty$.

Throughout this chapter, two main issues are of interest to us:

1. Provided that condition (3.35) holds the myopic policy is optimal regardless of the value of s_1. We want to show that (3.35) is not a very restrictive assumption in the sense that it holds for the Erlang family of demand densities which is an important class of density functions in probability theory.

2. The myopic policy (3.36) may not be optimal if $s_1 > S$ and because (3.35) may not hold for some demand densities. If its coefficient of variation is less than one then the demand can be easily approximated by an Erlang variable with a suitable choice of shape and scale parameters. For this reason, the Erlang densities not only represent a quite general class of probability functions but also provide good approximations for practical purposes. Since the myopic policy (3.36) is optimal for Erlang demand, it can be considered as an approximate solution when the exact optimal policy requires substantial effort to compute. Use of an approximate myopic policy instead of the optimal policy can also be justified on the basis of its ease of implementation.

4.2 Analysis of $G(\cdot)$ for Erlang Demand

Let $\gamma(x, \lambda, k)$ denote the Erlang density with scale parameter λ and shape parameter k where $k \geq 2$ is an integer. That is,

$$\gamma(x, \lambda, k) = \frac{\lambda(\lambda x)^{k-1} \exp(-\lambda x)}{(k-1)!}, \ x > 0. \tag{4.3}$$

Also, let $\Gamma(x, \lambda, k)$ denote the corresponding distribution function. It can be shown that (see Bhat [7], p. 201)

$$\Gamma(x, \lambda, k) = 1 - \exp(-\lambda x) \sum_{j=0}^{k-1} \frac{(\lambda x)^j}{j!}, \ x > 0. \tag{4.4}$$

Throughout this section we assume that demand density and distribution functions, denoted by $f(x)$ and $F(x)$ are given by (4.3) and (4.4) respectively. Note that, if $k = 1$ then (4.3) is the exponential density with parameter λ.

4.2.1 Form of $G(\cdot)$

Here we show that if the demand density $f(x)$ is given by (4.3) then

$$G(a) = (r - c + \rho c + b)a - (\rho c + b)k/\lambda - B, \ a \leq 0, \tag{4.5}$$

and

$$\begin{aligned} G(a) &= -B \exp(-\lambda a) \sum_{j=0}^{k-1} \frac{(\lambda a)^j}{j!} - \frac{h + b + r}{\lambda} \exp(-\lambda a) \sum_{j=0}^{k-1} (k - j) \frac{(\lambda a)^j}{j!} \\ &\quad -[c(1-\rho) + h]a + (r - \rho c + h)k/\lambda, \ a > 0. \end{aligned} \tag{4.6}$$

Let μ denote the mean demand, i.e., $\mu = E[D_n], n = 1, 2, \ldots$. Then for Erlang demand we have

$$\mu = k/\lambda, \tag{4.7}$$

and thus equation (3.29) implies (4.5).

Next we want to prove (4.6). If we replace $f(x)$ with $\gamma(x,\lambda,k)$, $F(x)$ with $\Gamma(x,\lambda,k)$, and use (4.7) in (3.22) then for $a>0$ we have

$$\begin{aligned} G(a) &= (r-c+\rho c)a-\rho ck/\lambda-(r+h)a\Gamma(a,\lambda,k)+(r+h)\int_0^a x\gamma(x,\lambda,k)dx \\ &\quad -B[1-\Gamma(a,\lambda,k)]-b\int_a^\infty x\gamma(x,\lambda,k)dx+ba[1-\Gamma(a,\lambda,k)]. \end{aligned} \tag{4.8}$$

Substituting (4.4) in (4.8) gives

$$\begin{aligned} G(a) &= (r-c+\rho c)a-\rho ck/\lambda-(r+h)a+(r+h)a\exp(-\lambda a)\sum_{j=0}^{k-1}\frac{(\lambda a)^j}{j!} \\ &\quad +(r+h)\int_0^a x\gamma(x,\lambda,k)dx-B\exp(-\lambda a)\sum_{j=0}^{k-1}\frac{(\lambda a)^j}{j!} \\ &\quad -b\int_a^\infty x\gamma(x,\lambda,k)dx+ba\exp(-\lambda a)\sum_{j=0}^{k-1}\frac{(\lambda a)^j}{j!}. \end{aligned}$$

After simplifications $G(a)$ can be expressed as

$$\begin{aligned} G(a) &= -[c(1-\rho)+h]a-\rho ck/\lambda-B\exp(-\lambda a)\sum_{j=0}^{k-1}\frac{(\lambda a)^j}{j!} \\ &\quad +(h+b+r)a\exp(-\lambda a)\sum_{j=0}^{k-1}\frac{(\lambda a)^j}{j!}-b\int_a^\infty x\gamma(x,\lambda,k)dx \\ &\quad +(r+h)\int_0^a x\gamma(x,\lambda,k)dx. \end{aligned}$$

If we add and subtract

$$(r+h)\int_a^\infty x\gamma(x,\lambda,k)dx$$

in the above expression, and then substitute

$$(r+h)\int_0^\infty x\gamma(x,\lambda,k)dx=(r+h)k/\lambda,$$

we have

$$\begin{aligned} G(a) &= -[c(1-\rho)+h]a - \rho ck/\lambda - B\exp(-\lambda a)\sum_{j=0}^{k-1}\frac{(\lambda a)^j}{j!} \\ &+(h+b+r)a\exp(-\lambda a)\sum_{j=0}^{k-1}\frac{(\lambda a)^j}{j!} \\ &-(h+b+r)\int_a^\infty x\gamma(x,\lambda,k)dx + (r+h)k/\lambda. \end{aligned}$$

Rearranging the terms yields

$$\begin{aligned} G(a) &= -B\exp(-\lambda a)\sum_{j=0}^{k-1}\frac{(\lambda a)^j}{j!} - (h+b+r)\int_a^\infty x\gamma(x,\lambda,k)dx \\ &+\frac{(h+b+r)}{\lambda}\exp(-\lambda a)\sum_{j=0}^{k-1}\frac{(\lambda a)^{j+1}}{j!} \\ &-[c(1-\rho)+h]a + (r-\rho c+h)k/\lambda. \end{aligned} \tag{4.9}$$

Next we compute

$$\int_a^\infty x\gamma(x,\lambda,k)dx$$

in order to express (4.9) explicitly. Observe that (4.3) implies

$$\int_a^\infty x\gamma(x,\lambda,k)dx = \frac{k}{\lambda}\int_a^\infty \frac{\lambda(\lambda x)^k\exp(-\lambda x)}{k!}dx.$$

Then, it follows that

$$\int_a^\infty x\gamma(x,\lambda,k)dx = \frac{k}{\lambda}[1-\Gamma(a,\lambda,k+1)],$$

and using (4.4) we conclude

$$\int_a^\infty x\gamma(x,\lambda,k)dx = \frac{k}{\lambda}\exp(-\lambda a)\sum_{j=0}^{k}\frac{(\lambda a)^j}{j!}. \tag{4.10}$$

If we substitute (4.10) in (4.9) we have

$$\begin{aligned} G(a) &= -B\exp(-\lambda a)\sum_{j=0}^{k-1}\frac{(\lambda a)^j}{j!} \\ &\quad -\frac{(h+b+r)}{\lambda}\exp(-\lambda a)\left[\sum_{j=0}^{k}k\frac{(\lambda a)^j}{j!}-\sum_{j=0}^{k-1}\frac{(\lambda a)^{j+1}}{j!}\right] \\ &\quad -[c(1-\rho)+h]a+(r-\rho c+h)k/\lambda. \qquad (4.11) \end{aligned}$$

We can write

$$\sum_{j=0}^{k-1}\frac{(\lambda a)^{j+1}}{j!} = \sum_{j=1}^{k}j\frac{(\lambda a)^j}{j!}$$

in (4.11). Then (4.6) follows.

4.2.2 The First Derivative of $G(\cdot)$

Here we compute the first derivative, $G'(\cdot)$, of $G(\cdot)$. Equations (4.5) and (4.6) imply

$$G'(a) = (r-c+\rho c+b)a, \; a \le 0, \qquad (4.12)$$

and

$$\begin{aligned} G'(a) &= B\lambda\exp(-\lambda a)\frac{(\lambda a)^{k-1}}{(k-1)!} \\ &\quad +(h+b+r)\exp(-\lambda a)\sum_{j=0}^{k-1}\frac{(\lambda a)^j}{j!}-[c(1-\rho)+h], \; a>0, \qquad (4.13) \end{aligned}$$

respectively. Expression (4.12) can be verified easily. Next, we verify (4.13).

Note that, according to (4.6), if $a > 0$ then

$$\begin{aligned} G'(a) &= B\lambda\exp(-\lambda a)\sum_{j=0}^{k-1}\frac{(\lambda a)^j}{j!}-B\lambda\exp(-\lambda a)\sum_{j=1}^{k-1}\frac{(\lambda a)^{j-1}}{(j-1)!} \\ &\quad +(h+b+r)\exp(-\lambda a)\sum_{j=0}^{k-1}(k-j)\frac{(\lambda a)^j}{j!} \end{aligned}$$

$$-(h+b+r)\exp(-\lambda a)\sum_{j=1}^{k-1}(k-j)\frac{(\lambda a)^{j-1}}{(j-1)!}-[c(1-\rho)+h].$$

After simplifications, if $a > 0$ then

$$\begin{aligned} G'(a) &= B\lambda\exp(-\lambda a)\frac{(\lambda a)^{k-1}}{(k-1)!} \\ &\quad +(h+b+r)\exp(-\lambda a)\left[\sum_{j=0}^{k-1}(k-j)\frac{(\lambda a)^j}{j!}-\sum_{j=0}^{k-2}(k-j-1)\frac{(\lambda a)^j}{j!}\right] \\ &\quad -[c(1-\rho)+h]. \end{aligned}$$

Cancellations among summation terms in the above expression lead to (4.13). Later, we will use (4.13) in order to compute the value of S for Erlang demand.

4.2.3 The Second Derivative of $G(\cdot)$

It follows from (4.12) that, $G''(a) = 0$ if $a \leq 0$. We now compute $G''(a)$ for $a > 0$ using (4.13). Differentiating (4.13) results in the following:

$$\begin{aligned} G''(a) &= -B\lambda^2\exp(-\lambda a)\frac{(\lambda a)^{k-1}}{(k-1)!}+B\lambda^2\exp(-\lambda a)\frac{(\lambda a)^{k-2}}{(k-2)!} \\ &\quad -(h+b+r)\lambda\exp(-\lambda a)\sum_{j=0}^{k-1}\frac{(\lambda a)^j}{j!}+(h+b+r)\lambda\exp(-\lambda a)\sum_{j=1}^{k-1}\frac{(\lambda a)^{j-1}}{(j-1)!}. \end{aligned}$$

Thus, we write

$$\begin{aligned} G''(a) &= B\lambda^2\exp(-\lambda a)\frac{(\lambda a)^{k-2}}{(k-2)!}\left(1-\frac{\lambda a}{k-1}\right) \\ &\quad -(h+b+r)\lambda\exp(-\lambda a)\left[\sum_{j=0}^{k-1}\frac{(\lambda a)^j}{j!}-\sum_{j=0}^{k-2}\frac{(\lambda a)^j}{j!}\right], \ a>0. \end{aligned}$$

After simplifications, the above expression is reduced to

$$G''(a) = \frac{\lambda^k}{(k-1)!}[(k-1)B-(B\lambda+h+b+r)a]a^{k-2}\exp(-\lambda a), \ a > 0. \qquad (4.14)$$

The second derivative of $G(\cdot)$ helps us to define the region over which $G(\cdot)$ is concave.

4.2.4 Concavity Region Of $G(\cdot)$

Since unit revenue, r, is at least as large as the unit procurement cost, c, we already know from (4.12) that $G(\cdot)$ is a linear increasing function if $a \leq 0$. Thus, those values of $a > 0$ such that $G''(a) = 0$ give us the region of concavity for $G(\cdot)$.

If we equate (4.14) to zero and solve for $a > 0$, we observe that $G''(a)$ changes its sign at

$$\frac{(k-1)B}{B\lambda + h + b + r}.$$

That is, $G''(a) \geq 0$ over

$$\left(0, \frac{(k-1)B}{B\lambda + h + b + r}\right], \tag{4.15}$$

and $G''(a) < 0$ over

$$\left(\frac{(k-1)B}{B\lambda + h + b + r}, +\infty\right). \tag{4.16}$$

Therefore, we conclude that if demand density is Erlang with parameters λ and k then $G(\cdot)$ is linear over $(-\infty,0]$, convex over (4.15), and strictly concave over (4.16).

4.3 Computing the Value of S

Using (4.5), (4.6), (4.12) and (4.13) we can write

$$\lim_{a\to 0^-} G(a) = \lim_{a\to 0^+} G(a) = -(\rho c + b)k/\lambda - B,$$

and

$$\lim_{a\to 0^-} G'(a) = \lim_{a\to 0^+} G'(a) = r - c + \rho c + b > 0.$$

Then $G(\cdot)$ is increasing at the left-most point of its convexity region (4.15). Therefore, $G(\cdot)$ is increasing over its convexity region (4.15). It follows that $G(\cdot)$ is increasing over

$$\left(-\infty, \frac{(k-1)B}{B\lambda + h + b + r}\right],$$

since we know from (4.12) that it is linearly increasing over $(-\infty, 0]$. Then maximizer S of $G(\cdot)$ lies in the region (4.16) where $G(\cdot)$ is concave.

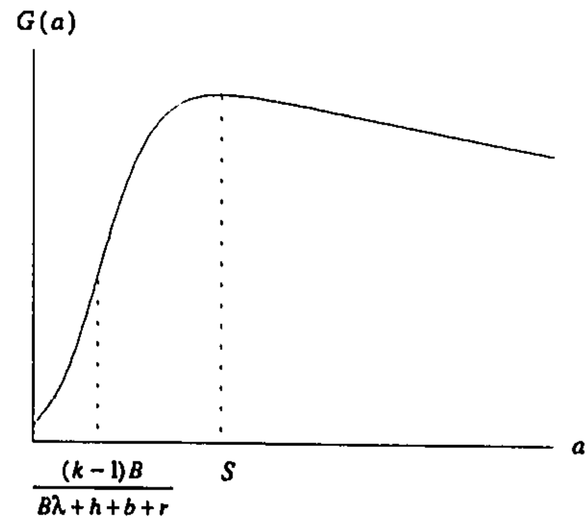

Figure 4.1: $G(\cdot)$ for Erlang demand with parameters λ and k.

Maximizer S solves $G'(a) = 0$ where $G'(a)$ is given by (4.13). That is, S solves

$$B\lambda \exp(-\lambda S)\frac{(\lambda S)^{k-1}}{(k-1)!} + (h+b+r)\exp(-\lambda S)\sum_{j=0}^{k-1}\frac{(\lambda S)^j}{j!} = c(1-\rho) + h. \quad (4.17)$$

Let $g_{\lambda,k}(\cdot)$ denote the left hand side of the above equation. Next we show that S is the unique positive solution of $g_{\lambda,k}(a) = c(1-\rho) + h$. Note that, $g_{\lambda,k}(0) = h + b + r$. Then

$$g_{\lambda,k}(0) > c(1-\rho) + h, \quad (4.18)$$

since $r > c > c(1-\rho)$. Note that, first derivative of $g_{\lambda,k}(a)$ is given by $G''(\cdot)$. Then using (4.14) we can write

$$\frac{d}{da}\left[g_{\lambda,k}(a)\right] = \frac{\lambda^k}{(k-1)!}[(k-1)B - (B\lambda + h + b + r)a]a^{k-2}\exp(-\lambda a). \qquad (4.19)$$

Expression (4.19) implies that $g_{\lambda,k}(a)$ is strictly decreasing over the concavity region of (4.16) $G(\cdot)$. Further, it is increasing over the convexity region (4.15) of $G(\cdot)$. Then inequality (4.18) imposes that any positive solution of (4.17) should lie in (4.16). Also these results regarding the shape of $g_{\lambda,k}(a)$ and inequality (4.18) implies that a positive solution of (4.17) is unique because the right hand side is a constant. We conclude that S is the unique positive solution of (4.17). Theorem 2 summarizes our results.

Theorem 2 *If the demand density is Erlang with parameters λ and k then $G(\cdot)$ is linear over $(-\infty,0]$, convex over (4.15), and strictly concave over (4.16). Moreover, maximizer S is the unique positive solution of (4.17), and S lies in the region (4.16) where $G(\cdot)$ is concave.*

4.4 Example

Based on our results summarized by Theorem 2 computing the optimal policy, i.e., the maximizer S of the single period return net of costs function $G(\cdot)$, is quite simple. In order to find the value of S we coded the well-known Newton-Raphson method, an iterative algorithm which searches for the solution of $G'(a) = 0$ or equivalently (4.17).

We use the data indicated by Table 4.1 and solve some numerical examples. Note that, when $k = 1$ the Erlang density (4.3) reduces to the exponential density

Table 4.1: Sample data for $\mathcal{P}_\infty$

ρ	r	c	h	b	B
.99	38	20	.5	30	50

Table 4.2: Examples: Value of S.

k	μ	S
1	5	24
2	10	34
3	15	43
4	20	51
5	25	59
6	30	66
7	35	73
8	40	81
9	45	88
10	50	94

with parameter λ. Varying k, and thus the mean demand $\mu = k/\lambda$, we compute the value of S, as shown in Table 4.2. As we expect, if the mean demand μ increases then the value of S also increases. Other examples regarding the computation of S are also presented in Section 7.1.3 of Chapter 7 where we discuss the computation of the best myopic policy of the finite horizon problem. In fact, the results of this chapter are used extensively in Chapter 7 in order to compare the infinite horizon myopic policy with the best myopic policy for the finite horizon problem.

4.5 Summary

Our results in Chapter 3 imply that if initial inventory level is above S in order for the myopic policy to be optimal a sufficient condition [given by inequality (3.35)] should hold. In this chapter we have shown that the sufficient condition for the optimality of a myopic policy holds immediately for Erlang demand. Thus, the myopic policy is optimal in general for Erlang demand, i.e., whether the initial inventory level is above or below the critical number S, and regardless of the values of cost parameters. We also suggest the myopic policy to be considered as an approximate solution, if the exact optimal policy is intractable. We emphasize that the myopic policy may lead to a good approximation since it is optimal for Erlang demand density which might quite well represent most densities with coefficient of variation not exceeding unity.

Chapter 5

Infinite Horizon Problem with a Disposal Opportunity

5.1 Inventory Disposal Decisions

The financial management of overstocked businesses is a critical problem. In many cases potential sales income from excess stock does not exceed the capital outlay associated with storage and holding. Under this scenario, many businesses would seek the disposal alternatives, probably by specifying a reduced price which is usually called disposal value. The question is, given the stochastic nature of the demand, how much to dispose of in an excess inventory situation.

In this chapter we consider a one-time disposal opportunity associated with $\mathcal{P}_\infty$, and in relation to this opportunity we discuss a disposal problem. For our infinite horizon problem, under the optimal policy, the inventory level is never above the base-stock level S unless initially we have an excess stock level of $s_1 > S$. If $s_1 > S$ then this level may represent an overstock situation and we may consider liquidating some portion of it. Once a disposal value (i.e., discounted price) which is denoted by v is specified we can compute an optimal level of items to keep. Naturally, the disposal value is expected to be less than r.

The retailer's disposal decision represents a one time opportunity for the customers to buy at a discounted price. We suppose that the discounted price is specified such that the retailer can dispose of any amount at this price because the discount is an opportunity that the customers won't miss. This is a simplifying assumption which leads to a simple optimal solution. In general, demand during the discount period is affected by the discounted price. Furthermore, the future demand can also be affected (reduced). We comment on these generalizations of the problem in Remark 2 on page 78. Provided that $v > c$, i.e., there is a unit profit of $v-c$ associated with disposed merchandise, then it would be optimal to dispose of the entire amount of initial stock. However, solution of the disposal problem is not always this trivial since v may be less than c in general. Therefore, we give a rigorous formulation of the problem. The dependence of total expected discounted revenue net of costs on the initial inventory level is already expressed by function $NPV(s_1)$ in Section 3.6. Since we have this function then it can be used to formulate the problem of how much of an excess initial inventory to sell, and how much to keep.

In the remaining parts of this chapter we first discuss the basics that lead to the formulation of our disposal problem which we denote by $\mathcal{P}^{\mathcal{D}}_{\infty}$. The problem formulation is given in Section 5.2 and properties of this formulation are analyzed in Section 5.3. In Section 5.4 we provide a discussion of our results associated with the optimal solution. We also discuss some computational issues, and find that solving $\mathcal{P}^{\mathcal{D}}_{\infty}$ requires computing what we call "renewal-like" functions whose analyses are presented at the end of this chapter. The problem of finding the optimal solution of $\mathcal{P}^{\mathcal{D}}_{\infty}$ is nontrivial only if $c > v$, and the initial inventory is above the critical number of the infinite horizon myopic policy S. We are able to show that the objective of $\mathcal{P}^{\mathcal{D}}_{\infty}$ is concave for Erlang demand for the nontrivial case, and thus the optimal solution is easy to compute. Otherwise, the optimal solution asserts that either the entire initial stock or none should be disposed of.

5.2 Modeling a Disposal Opportunity

Given an initial inventory level s_1, the retailer may consider disposing of some portion of s_1, by offering a special discount at the beginning of the planning horizon. This may avoid carrying excess inventory. Once a unit discounted price or disposal value which we denote by v is specified then the problem is to specify how much inventory to keep. We impose $0 < v < r$ in order for the problem to be well defined. The retailer may choose to sell $s_1 - u$ units for a total disposal income of $v(s_1 - u)$ dollars, where u denotes the number of items kept after disposal. The disposal decision involves reducing the inventory level from s_1 to u where $0 \leq u \leq s_1$, and we need to specify the optimal value of u. As we mentioned previously, if the retailer offers a discount then demand for the low price items is assumed to be unlimited, i.e., any on-hand amount can be disposed of, and disposal income is received immediately.

As we have already explained, if we start with u units of inventory, then the best we can do under (3.36) is specified by $NPV(u)$. Consequently, the best value of u maximizes

$$NPV(u) + v(s_1 - u),$$

over

$$0 \leq u \leq s_1. \tag{5.1}$$

We define

$$Q(u) \equiv NPV(u) + v(s_1 - u), \tag{5.2}$$

and it follows from equations (3.62) and (3.79) that

$$Q(u) = (c-v)u + \frac{G(S)}{1-\rho} + vs_1, \;\; u < S, \tag{5.3}$$

and

$$\begin{aligned} Q(u) \;\; = \;\; & (c-v)u + G(u) + \int_0^{u-S} G(u-x)m_\rho(x)dx \\ & + \frac{\rho}{1-\rho}G(S) - M_\rho(u-S)G(S) + vs_1, \;\; u \geq S. \end{aligned} \tag{5.4}$$

Thus, the disposal problem $\mathcal{P}_\infty^\mathcal{D}$ can be stated as

$$\begin{aligned} \mathcal{P}_\infty^\mathcal{D}: \quad & \text{Maximize} \quad && Q(u) \\ & \text{Subject to} \quad && 0 \leq u \leq s_1. \end{aligned}$$

We note that in their book Heyman and Sobel [15] provide a very brief discussion of a similar disposal model. They give the model formulation for discrete demand densities where excess demand is lost and the fixed backorder cost B is *not* considered. However, they do not analyze mathematical properties of the objective function, and they do not study the optimal solution. In his review paper Porteus [33] mentions Heyman and Sobel's model formulation and states that this sort of model arises in cash management.

5.3 Properties of the Disposal Problem

In this section, we analyze the properties of the objective function $Q(\cdot)$. We first show that $Q(\cdot)$ is continuous and differentiable over $[0, +\infty)$. Then we find a condition under which $Q(\cdot)$ is concave.

We assume that demand is a positive random variable, and thus $F(0) = 0$. Also (3.76) implies that $M_\rho(0) = 0$. Then using (5.3) and (5.4) one can easily verify that

$$\lim_{\varepsilon\to 0^-} Q(S+\varepsilon) = \lim_{\varepsilon\to 0^+} Q(S+\varepsilon) = (c-v)S + \frac{G(S)}{1-\rho} + vs_1.$$

Therefore, $Q(u)$ is continuous over $[0, +\infty)$ since both (5.3) and (5.4) describe continuous functions.

Using (5.3), (5.4), and Leibnitz's rule of differentiation under the integral sign, we have

$$\frac{dQ(u)}{du} = c - v, \ u < S. \tag{5.5}$$

$$\begin{aligned}\frac{dQ(u)}{du} &= (c-v) + \frac{dG(u)}{du} + \int_0^{u-S} \frac{dG(u-x)}{du} m_\rho(x)dx \\ &\quad + G(S)m_\rho(u-S) - \frac{dM_\rho(u-S)}{du}G(S), \ u \geq S.\end{aligned} \tag{5.6}$$

By definition [see (3.68), (3.76), and (3.77)],

$$\frac{dM_\rho(u-S)}{du} = m_\rho(u-S),$$

and substituting this in (5.6) gives

$$\frac{dQ(u)}{du} = (c-v) + \frac{dG(u)}{du} + \int_0^{u-S} \frac{dG(u-x)}{du} m_\rho(x)dx, \ u \geq S. \tag{5.7}$$

The right limit of (5.7) at S is given by

$$\lim_{\epsilon\to 0^+} \left.\frac{dQ(u)}{du}\right|_{u=S+\epsilon} = (c-v) + \lim_{\epsilon\to 0^+} \left.\frac{dG(u)}{du}\right|_{u=S+\epsilon}.$$

Since S maximizes $G(\cdot)$ we already have

$$\lim_{\epsilon\to 0^+} \left.\frac{dG(u)}{du}\right|_{u=S+\epsilon} = \left.\frac{dG(u)}{du}\right|_{u=S} = 0,$$

and thus

$$\lim_{\epsilon \to 0^+} \left. \frac{dQ(u)}{du} \right|_{u=S+\epsilon} = (c - v). \tag{5.8}$$

Expressions (5.5) and (5.8) prove that $Q(u)$ is differentiable at S. Assuming the demand density is differentiable (5.3) and (5.4) describe differentiable functions over $[0, S)$ and $[S, +\infty)$, respectively. Then $Q(u)$ is also differentiable over $[0, +\infty)$.

It is worth noting that in order for the problem to be well posed we require

$$\lim_{u \to +\infty} Q'(u) < 0.$$

That is, in order to have a finite solution to our problem the objective function should be decreasing as its argument approaches infinity. The remainder of this chapter assumes that the problem is well posed so that we have a finite solution. The following theorem imposes a condition for $Q(\cdot)$ to be concave.

Theorem 3 *Provided that $G(\cdot)$ is concave over $[S, +\infty)$ then $Q(\cdot)$ is concave over $[0, +\infty)$.*

Proof If we apply Leibnitz' rule to (5.5), (5.7), and use the fact that $G'(S) = 0$ then we can write

$$\frac{d^2Q(u)}{du^2} = \begin{cases} 0 & \text{if } u < S, \\ \frac{d^2G(u)}{du^2} + \int_0^{u-S} \frac{d^2G(u-x)}{du^2} m_\rho(x) dx & \text{if } u \geq S. \end{cases} \tag{5.9}$$

Assume that $G(\cdot)$ is concave over $[S, +\infty)$, i.e.,

$$\frac{d^2G(u)}{du^2} \leq 0, \ \forall u \geq S. \tag{5.10}$$

In fact, this assumption is not as restrictive as it seems to be. In Chapter 4 we already showed that it is true in general for the Erlang demand density. According to expression (5.9) in order to show that

$$\frac{d^2Q(u)}{du^2} \leq 0,$$

it suffices to show that

$$\frac{d^2G(u)}{du^2} + \int_0^{u-S} \frac{d^2G(u-x)}{du^2} m_\rho(x)dx \leq 0 \ \ \forall u \geq S. \tag{5.11}$$

Note that, if $0 \leq x \leq u - S$, it follows from (5.10) that

$$\frac{d^2G(u-x)}{du^2} \leq 0, \ \ \forall u \geq S.$$

Then

$$\int_0^{u-S} \frac{d^2G(u-x)}{du^2} m_\rho(x)dx \leq 0 \ \ \forall u \geq S, \tag{5.12}$$

since by definition $m_\rho(x) \geq 0$ for all $x \geq 0$. Inequalities (5.10) and (5.12) imply that (5.11) holds. This completes the proof. ■

5.4 Analysis and Solution

In this section we analyze the optimal solution of the problem using our results about the properties of $Q(\cdot)$. The optimal solution is denoted by u^*. We study two cases in which $0 \leq s_1 < S$ or $s_1 \geq S$.

5.4.1 No Excess Inventory Case: $0 \leq s_1 \leq S$

In this trivial case, $\mathcal{P}_\infty^{\mathcal{D}}$ reduces to finding the maximizer of (5.3) over $[0, s_1)$. As we already noted, (5.3) is a linear function, and it follows that u^* is either at 0 or s_1 depending on the sign of slope, $(c - v)$, of (5.3). That is, the solution is specified by

$$u^* = \begin{cases} 0 & \text{if } c < v, \\ s_1 & \text{if } c \geq v. \end{cases} \tag{5.13}$$

Figures 5.1 and 5.2 illustrate this case.

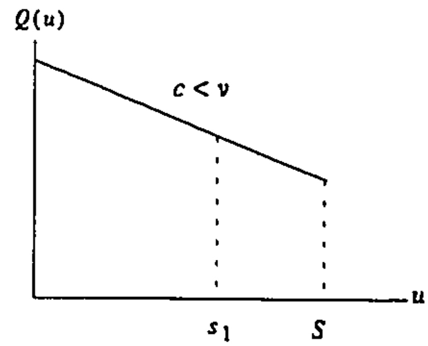

Figure 5.1: $Q(u)$ over the feasible region if $0 \leq s_1 \leq S$ and $c < v$.

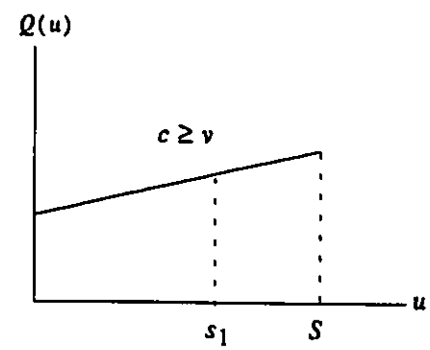

Figure 5.2: $Q(u)$ over the feasible region if $0 \leq s_1 \leq S$ and $c \geq v$.

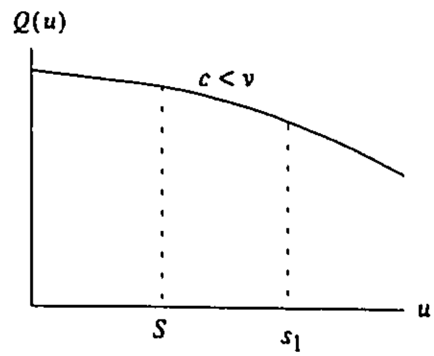

Figure 5.3: $Q(u)$ over the feasible region if $s_1 \geq S$ and $c < v$.

5.4.2 Overstock Situation: $s_1 > S$

In this case, our problem is to optimize $Q(u)$ over $[0,s_1]$ where $s_1 \geq S$. Throughout the remaining part of this chapter we assume that $G(\cdot)$ is concave over $[S, +\infty)$. Thus, Theorem 3 asserts that $Q(\cdot)$ is concave, and our problem $\mathcal{P}_\infty^\mathcal{D}$ reduces to finding the maximizer of a concave function over a finite interval. We study the following two cases.

Case 1. Value of u^* if $s_1 > S$ and $c < v$.

According to expression (5.3), if $c < v$ then $Q(u)$ is a linearly decreasing function over $[0,S)$. Since $Q(u)$ is concave, if it is decreasing at 0 then it should be decreasing over $[0, +\infty)$, and therefore if $c < v$ then

$$u^* = 0. \tag{5.14}$$

This case is illustrated by Figure 5.3.

Case 2. Value of u^* if $s_1 > S$ and $c \geq v$.

If $c \geq v$ then $Q(u)$ is an increasing (nondecreasing) linear function over $[0, S)$.

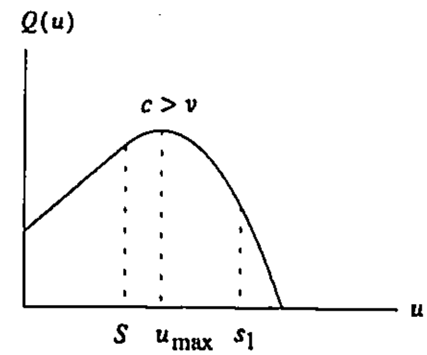

Figure 5.4: $Q(u)$ over the feasible region if $s_1 \geq S$, $c \geq v$, and $u_{max} < s_1$.

In this case, because $Q(u)$ is differentiable, using (5.8) we can write

$$\left.\frac{dQ(u)}{du}\right|_{u=S} = c - v. \tag{5.15}$$

Thus, if $c \geq v$ then $Q(u)$ is increasing (nondecreasing) not only over $[0, S)$, but also at S. By assumption $Q(u)$ is concave, and the problem is well posed, i.e., the unconstrained maximizer of $Q(u)$ is finite. Let u_{max} denote the unconstrained maximizer of $Q(\cdot)$ under the assumption that $c \geq v$. Then u_{max} solves

$$\frac{dQ(u)}{du} = 0,$$

and $S \leq u_{max} < +\infty$. Two cases are possible. Either $S < u^* < s_1$ or $u^* \geq s_1$. These cases are illustrated by Figures 5.4 and Figure 5.5. It follows that if $c > v$ then the optimal solution of $\mathcal{P}^{\mathcal{D}}_{\infty}$ is given by

$$u^* = \begin{cases} u_{max} & \text{if } u^* < s_1, \\ s_1 & \text{if } u^* \geq s_1. \end{cases} \tag{5.16}$$

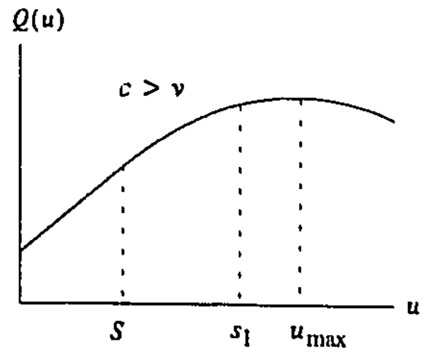

Figure 5.5: $Q(u)$ over the feasible region if $s_1 \geq S$, $c \geq v$, and $u_{max} \geq s_1$.

5.4.3 The Optimal Solution

Based on (5.13), (5.14), (5.16) and our analysis throughout this section we conclude that if $G(\cdot)$ is concave over $[S, +\infty)$ then the optimal solution is provided by Table 5.1. Let us note that the case $c < v$ for which $u^* = 0$ represents the trivial case. Apparently, if the procurement cost is less than the discounted price so that there is a profit of $v - c$ dollars per unit, then the optimal decision is to dispose of the entire initial stock since the demand during the discount period is unlimited, but otherwise it is random. On the other hand, if $c < v$ the disposal decision may or may not be taken depending on the value of u_{max} at which $Q(u)$ is maximized. If disposal is a profitable alternative then $u_{max} < s_1$ so that the optimal beginning inventory is given by $u^* = u_{max}$. Otherwise, inventory disposal is not a profitable choice so that the optimal beginning inventory is s_1.

Remark 1 If $s_1 \leq S$ and $c = v$ then according to (5.3) function $Q(u)$ is constant over the feasible region of the problem so that

$$u^* = \{u : 0 \leq u \leq s_1\}. \tag{5.17}$$

A similar result can also be obtained for the case $s_1 > S$. Observe that if $s_1 > S$ and $c = v$ then according to (5.3) function $Q(u)$ is constant over $[0,S)$, and it follows that $Q(u)$ *should be* nonincreasing over $[S, +\infty)$ since by assumption $Q(u)$ is concave. This case is illustrated by Figure 5.6. Further, if $c = v$ then (5.7) leads to

$$\left.\frac{dQ(u)}{du}\right|_{u=S} = 0,$$

so that S is a stationary point, and thus it maximizes $Q(u)$. As a matter of fact, in this case all points on $[0,S]$ maximize $Q(u)$, i.e., we have multiple optima. This result is very intuitive and trivial. Naturally, if $c = v$ then we are indifferent between 'disposing all we have and buying back S units' and 'disposing $s_1 - S$ units only'. Therefore, we have

$$u^* = \{u : 0 \leq u \leq S\}. \tag{5.18}$$

■

Remark 2 For the model under consideration we assume that the discounted price is specified such that the retailer can dispose of any amount at this price. This is a strong but simplifying assumption, and leads to an easy solution. In most cases, the amount that can be disposed is affected by the discounted price. As an example,

Table 5.1: Optimal solution of $\mathcal{P}^{\mathcal{D}}_{\infty}$.

	$s_1 < S$	$s_1 \geq S$
$c < v$	$u^* = 0$	$u^* = 0$
$c \geq v$	$u^* = s_1$	$u_{max} < s_1 \Longrightarrow u^* = u_{max}$, $u_{max} \geq s_1 \Longrightarrow u^* = s_1$

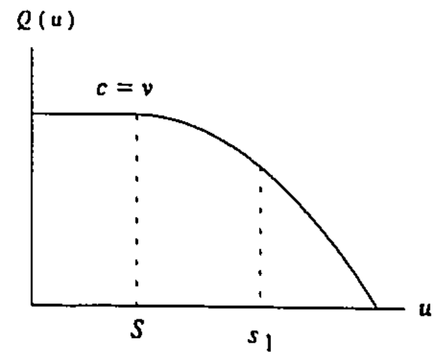

Figure 5.6: $Q(u)$ over the feasible region if $s_1 \geq S$ and $c = v$.

suppose that the amount that can be disposed, i.e., $s_1 - u$, is a linear decreasing function of v. That is,

$$s_1 - u = \alpha - \beta v, \tag{5.19}$$

where α and β are nonnegative constants. Note that, (5.19) implies

$$v = \frac{u - s_1 + \alpha}{\beta}. \tag{5.20}$$

If we substitute (5.20) in (5.3) and (5.4) then we obtain the expression of the objective function $Q(u)$ for this generalization of the problem. It is easy to observe that, although the expression of $Q(u)$ changes, the optimal solution is obtained by similar analysis applied in this chapter. A more general research problem assumes that future demand is also affected by the discounted price. This problem remains as an area for future investigation. ■

5.5 Computational Issues

5.5.1 Computing the Value of u_{max}

We need to compute u_{max} only for the nontrivial case $c > v$ and $s_1 \geq S$, because the optimal solution of $\mathcal{P}_\infty^{\mathcal{D}}$ is obtained regardless of the value of u_{max} for other cases. In this case, we know that $Q(u)$ is defined by (5.4), and its maximizer u_{max} lies on $(S, +\infty)$. It follows from the concavity of $Q(u)$ that u_{max} solves

$$\frac{dQ(u)}{du} = 0 \tag{5.21}$$

where $dQ(u)/du$ is defined by (5.7). Nevertheless, obtaining an explicit expression of $Q(u)$ may be difficult since we need to substitute the expressions for $m_\rho(\cdot)$ and $M_\rho(\cdot)$ in (5.4). In Section 5.5.2 we show that for uniform and exponential demand densities functions $m_\rho(\cdot)$ and $M_\rho(\cdot)$ have closed form expressions. However, for some demand densities, e.g., Erlang density, we cannot express $m_\rho(\cdot)$ and $M_\rho(\cdot)$ explicitly. Therefore, we seek numerical expressions of these functions over $(0, s_1 - S)$ to obtain (5.4) and (5.7), and to solve (5.21).

Next, in Section 5.5.2 we discuss how $m_\rho(\cdot)$ and $M_\rho(\cdot)$ can be obtained using Laplace transforms. In Section 5.5.4 we give integral equations in order to compute $m_\rho(\cdot)$ and $M_\rho(\cdot)$ numerically.

5.5.2 Use of Laplace Transforms

Computation of $m_\rho(\cdot)$ and $M_\rho(\cdot)$ may require some effort. Laplace transform is a powerful technique that can be used for this purpose. We use tilde ($\tilde{}$) to denote the Laplace transform of a function, e.g., for function $f(\cdot)$ we define $\tilde{f}(s)$ as follows

$$\tilde{f}(s) = \int_0^\infty f(x) \exp(-sx) dx, \ 0 \leq s < +\infty.$$

Then taking the Laplace transform of both sides of (3.68) we can write

$$\tilde{m}_\rho(s) = \sum_{n=1}^{\infty} \rho^n [\tilde{f}(s)]^n,$$

and it follows that

$$\tilde{m}_\rho(s) = \sum_{n=0}^{\infty} \rho^n [\tilde{f}(s)]^n - 1.$$

If we use the infinite summation formula the above expression reduces to

$$\tilde{m}_\rho(s) = \frac{\rho \tilde{f}(s)}{1 - \rho \tilde{f}(s)} \tag{5.22}$$

We can utilize (5.22) to obtain a closed form expression of $m_\rho(\cdot)$ for uniform demand density. If demand is uniformly distributed over $[\eta_0, \eta_1]$, then the Laplace transform of demand density is given by

$$\frac{1}{(\eta_1 - \eta_0)s}.$$

If we substitute the above expression for $\tilde{f}(s)$ in (5.22) and take the inverse transform of both sides, we obtain $m_\rho(x)$ for uniform density as

$$\frac{\rho}{\eta_1 - \eta_0} \exp\left(\frac{\rho x}{\eta_1 - \eta_0}\right). \tag{5.23}$$

Using (5.23) in (3.77) we obtain $M_\rho(x)$ for uniform density, and it is given by

$$\exp\left(\frac{\rho x}{\eta_1 - \eta_0}\right) - 1. \tag{5.24}$$

Another density for which we have closed form expressions $m_\rho(x)$ and $M_\rho(x)$ is the exponential density. The Laplace transform of exponential density with parameter λ is given by

$$\frac{\lambda}{\lambda + s}.$$

Substituting the above formula for $\bar{f}(s)$ in (5.22) and taking the inverse transform of both sides, we can write $m_\rho(x)$ for exponential density as

$$\rho\lambda \exp\left[-\lambda(1-\rho)x\right]. \tag{5.25}$$

Using (3.77) and integrating (5.25), $M_\rho(x)$ for exponential density with parameter λ is obtained as

$$\frac{\rho}{1-\rho}\{1-\exp\left[-\lambda(1-\rho)x\right]\}. \tag{5.26}$$

Next, we consider the Erlang density function with parameters λ and k, denoted by $\gamma(x, \lambda, k)$, and defined by (4.3). In this case,

$$\bar{\gamma}(s, \lambda, k) = \left(\frac{\lambda}{\lambda+s}\right)^k.$$

If we insert the above expression in (5.22), we obtain

$$\bar{m}_\rho(s) = \frac{\rho\lambda^k}{-\rho\lambda^k + (\lambda+s)^k}.$$

In order to find the inverse transform of $\bar{m}_\rho(s)$ for the Erlang demand density we need the factorial expansion of the polynomial that appears in its denominator. This polynomial has complex roots for odd values of k. In general, we cannot obtain a convenient closed form for $m_\rho(\cdot)$ for the Erlang demand density using (5.22). Since the numerical inversion of Laplace transforms can also be cumbersome, we are interested in an alternative method to compute $m_\rho(\cdot)$. In Section 5.5.4 we present integral equations that can be used to compute $M_\rho(\cdot)$ and $m_\rho(\cdot)$ numerically. However, in order to prepare the necessary background for our development in Section 5.5.4, we first discuss the underlying renewal process associated with the inventory system under consideration.

5.5.3 Use of the Underlying Renewal Process

Let $\mathcal{N}(t)$ denote the number of periods that the inventory system has completed by the time cumulative demand reaches t. We say that the first review is made right after the end of the first period, i.e., at the beginning of the second period, so that until the end of nth period $n-1$ reviews are made. Then $\{\mathcal{N}(t) : t \geq 0\}$ is a renewal process which registers successive reviews. Successive occurrence times can be represented by demand during successive periods. Thus, D_i is interpreted as the time between $(i-1)$th and the ith transition of $\{\mathcal{N}(t) : t \geq 0\}$. Let us recall that $D_1, D_2, \ldots$ are positive random variables signifying that

$$P(D_i \leq 0) = F(0) = 0, \; i = 1, 2, \ldots$$

which is a basic stipulation for a renewal process (Taylor and Karlin [42], p. 274). Cumulative demand until the end of nth period or just before the nth review is denoted by W_n. That is,

$$W_n = D_1 + D_2 + \cdots + D_n, \; n \geq 1,$$

and corresponds to the waiting time until the nth review within the usual renewal process terminology. By convention $W_0 = 0$. Clearly, W_n represents the total accumulated demand until the end of period n. Therefore, its distribution is given by the n-fold convolution distribution of demand, i.e.,

$$P(W_n \leq x) = F^{(n)}(x).$$

It follows that

$$\mathcal{N}(t) \geq n \iff W_n \leq t. \tag{5.27}$$

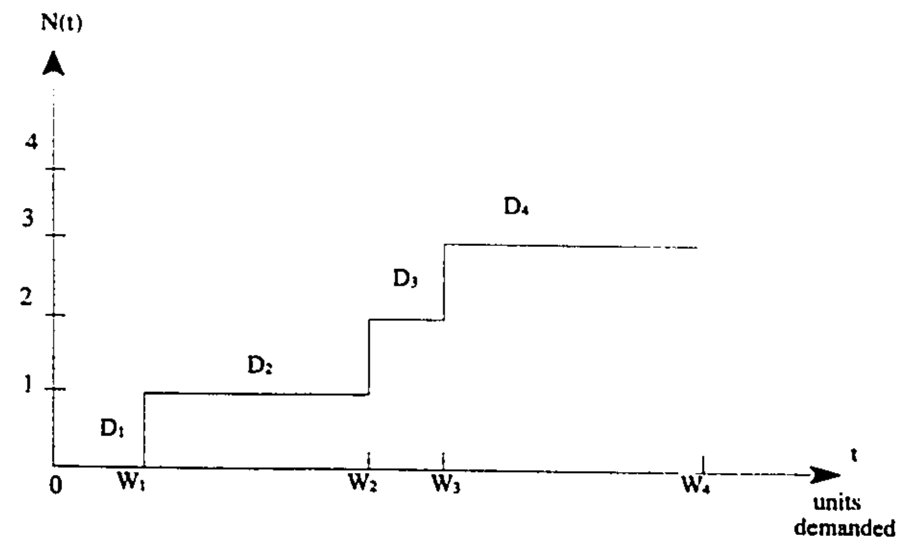

Figure 5.7: Underlying renewal process.

That is, the number of reviews until the cumulative demand reaches t is at least n if and only if the cumulative demand at the time of nth review, i.e., W_n, did not exceed t. Figure 5.7 illustrates the relations between the demand in successive periods (interoccurence times), the cumulative demand (waiting time), and the counting process $\mathcal{N}(t)$. Expression (5.27) asserts that

$$P\left(\mathcal{N}(t) \geq n\right) = P\left(W_n \leq t\right) = F^{(n)}(t), \;\; t \geq 0, n = 1, 2, ... \tag{5.28}$$

Consequently,

$$P\left(\mathcal{N}(t) = n\right) = P\left(\mathcal{N}(t) \geq n\right) - P\left(\mathcal{N}(t) \geq n+1\right),$$

leads to

$$P\left(\mathcal{N}(t) = n\right) = F^{(n)}(t) - F^{(n+1)}(t). \tag{5.29}$$

Obviously, the renewal process $\mathcal{N}(t)$ and the random variable N are related. It follows from the definition (3.37) of N that, $N = n$ if the cumulative inventory reaches or exceeds $s_1 - S$ during period $n-1$. In this case, we say the policy is activated at the beginning of n. Observe that if the cumulative inventory level reaches or exceeds $s_1 - S$ during period $n-1$, then the number of reviews made until the inventory level reaches or exceeds $s_1 - S$, i.e., $\mathcal{N}(s_1 - S)$, equals $n-2$. It follows that,

$$N = n \iff \mathcal{N}(s_1 - S) = n - 2.$$

In fact, the correspondence between equations (3.61) and (5.29) rely on this relation.

In the following section we use (5.29) and present integral equations in order to compute $M_\rho(\cdot)$ and $m_\rho(\cdot)$ numerically.

5.5.4 The Integral Equations

Since

$$E\left[\rho^{\mathcal{N}(t)}\right] = \sum_{n=1}^{\infty} \rho^n P\left[\mathcal{N}(t) = n\right],$$

using (5.29) we have

$$E\left[\rho^{\mathcal{N}(t)}\right] = \sum_{n=1}^{\infty} \rho^n F^{(n)}(x) - \sum_{n=1}^{\infty} \rho^n F^{(n+1)}(x).$$

Using the definition of $M_\rho(\cdot)$ given by (3.76) the above expression can be written as

$$E\left[\rho^{\mathcal{N}(t)}\right] = M_\rho(t) - \frac{M_\rho(t) - \rho F(t)}{\rho}.$$

If we solve this expression for $M_\rho(t)$ we obtain

$$M_\rho(t) = \frac{\rho F(t)}{1-\rho} - \frac{\rho E\left[\rho^{\mathcal{N}(t)}\right]}{1-\rho}. \tag{5.30}$$

In order to compute $M_\rho(t)$ we need to know $E\left[\rho^{\mathcal{N}(t)}\right]$. Using conditioning on the first demand we compute $E\left[\rho^{\mathcal{N}(t)}\right]$ as follows:

$$\begin{aligned}
E\left[\rho^{\mathcal{N}(t)}\right] &= \int_0^\infty E\left[\rho^{\mathcal{N}(t)}|D_1 = x\right] f(x)dx \\
&= \int_0^t E\left[\rho^{\mathcal{N}(t)}|D_1 = x\right] f(x)dx + \int_t^\infty E\left[\rho^{\mathcal{N}(t)}|D_1 = x\right] f(x)dx \\
&= \int_0^t E\left[\rho^{1+\mathcal{N}(t-x)}\right] f(x)dx + \int_t^\infty E\left[\rho^{1+\mathcal{N}(t-x)}\right] f(x)dx \\
&= \rho\int_0^t E\left[\rho^{\mathcal{N}(t-x)}\right] f(x)dx + \rho\int_t^\infty E\left[\rho^{\mathcal{N}(t-x)}\right] f(x)dx \\
&= \rho\int_0^t E\left[\rho^{\mathcal{N}(t-x)}\right] f(x)dx + \rho\int_t^\infty E[\rho^0] f(x)dx \\
&= \rho\int_0^t E\left[\rho^{\mathcal{N}(t-x)}\right] f(x)dx + \rho\int_t^\infty f(x)dx.
\end{aligned}$$

Thus, we can write

$$E\left[\rho^{\mathcal{N}(t)}\right] = \rho[1 - F(t)] + \rho\int_0^t E\left[\rho^{\mathcal{N}(t-x)}\right] f(x)dx. \tag{5.31}$$

Integral equation (5.31) and relation (5.30) can be utilized to compute $M_\rho(\cdot)$ numerically.

Alternatively, we obtain another integral equation from which we can compute $m_\rho(\cdot)$ directly. Expression (5.22) implies that

$$\tilde{m}_\rho(s) = \rho\tilde{f}(s) + \rho\tilde{m}_\rho(s)\tilde{f}(s).$$

If we take the inverse transform of both sides we have

$$m_\rho(t) = \rho f(t) + \rho m_\rho(t) \odot f(t),$$

where $\odot$ represents the convolution operator. It follows from the definition of a convolution that

$$m_\rho(t) = \rho f(t) + \rho\int_0^t m_\rho(t - x) f(x)dx. \tag{5.32}$$

Expressions (3.77) and (5.32) lead to

$$M_\rho(t) = \rho F(t) + \rho \int_0^t M_\rho(t-x) f(x) dx. \tag{5.33}$$

Observe that if we substitute $\rho = 1$ in (5.33), it reduces to the so called renewal equation, whereas $M_\rho(t)$ reduces to the so called renewal function, usually denoted by $M(t)$, and given by

$$M(t) = E[\mathcal{N}(t)] = \sum_{n=1}^{\infty} P\left(\mathcal{N}(t) \geq n\right) = \sum_{n=1}^{\infty} F^{(n)}(t).$$

For this reason we $M_\rho(\cdot)$ is called 'discounted renewal function'. Functions $m_\rho(\cdot)$ and $M_\rho(\cdot)$ are also referred as 'renewal-like' functions.

5.6 Numerical Example

In this section we present a numerical example, and illustrate the computation of the optimal solution u^*. We suppose that the data presented in Table 5.2 are available,

Table 5.2: Sample data for $\mathcal{P}_\infty^\mathcal{D}$

s_1	ρ	r	c	v	h	b	B
200	.99	40	25	20	5	30	50

and the demand density is exponential with parameter $\lambda = .05$. Then the demand density and distribution are given by

$$f(x) = .05 \exp(-.05x), \quad x > 0, \tag{5.34}$$

$$F(x) = 1 - \exp(-.05x), \quad x > 0, \tag{5.35}$$

and the mean demand is $\mu = 1/\lambda = 20$.

The optimal solution can be found by using Table 5.1. However, first we need to compute the value of S so that we can use the results stated in Table 5.1. By definition, S is the maximizer of the single period return net of costs function $G(\cdot)$. Recalling this expression of $G(\cdot)$ we can write

$$\begin{aligned} G(a) &= (r - c + \rho c)a - \rho c E(D_1) - (r + h)\int_0^a (a - x)f(x)dx \\ &\quad - B[1 - F(a)] - b\int_a^{\infty} (x - a)f(x)dx, \quad a \in (-\infty, +\infty). \end{aligned}$$

If we use the data in Table 5.2, the exponential density (5.34) and the exponential distribution (5.35) in the above expression then we have

$$G(a) = -1550\exp(-.05a) - .75(7a - 540), \quad a > 0. \tag{5.36}$$

Proposition 1 of Chapter 3 states that for positive demand densities, the global maximizer of $G(\cdot)$ is finite and positive. We also have Proposition 2 which asserts that $G(a)$ is concave for exponential demand. Therefore, we conclude that for our example S solves $G'(a) = 0$ where $G'(a)$ is computed by using (5.36) so that we have $S = 53.841$. It is worth noting that since $G(\cdot)$ is concave then the requirement of Theorem 3 holds immediately, and thus $Q(\cdot)$ is also concave.

Now, we can use Table 5.1 which suggests that we need to compute u_{max} in order to find the value of u^* since $s_1 = 200$ and $S = 53.841$ whereas $c = 25$ and $v = 20$. As mentioned previously, u_{max} denotes the unconstrained maximizer of $Q(\cdot)$ under the assumption that $c \geq v$. Function $Q(\cdot)$ has two parts given by expressions (5.3) and (5.4). However, we do not need to use (5.3) which defines $Q(\cdot)$ over $(-\infty, S)$, because (as we explained in Section 5.4.2) if $c \geq v$ and $s_1 \geq S$ then $u_{max} \geq S$. Let us recall (5.4) which gives

$$\begin{aligned} Q(u) &= (c-v)u + G(u) + \int_0^{u-S} G(u-x)m_\rho(x)dx \\ &\quad + \frac{\rho}{1-\rho}G(S) - M_\rho(u-S)G(S) + vs_1, \quad u \geq S. \end{aligned}$$

Let us also recall (5.26) which provides the expression of $M_\rho(\cdot)$ for the case of exponential demand. For our example, (5.26) implies that

$$M_\rho(x) = \frac{.99}{1-.99}\{1 - \exp[-.05(1-.99)x]\}. \tag{5.37}$$

If we substitute (5.36) and (5.37) in (5.37) we obtain that

$$Q(u) = -1078650\exp(-.0005u) - 40(13u - 27100). \tag{5.38}$$

Then using (5.38), computing its first derivative, and solving

$$\frac{dQ(u)}{du} = 0,$$

leads to $u^* = 72.981 \approx 73$. It follows that, since $s_1 = 200$ the optimal decision is to dispose of approximately 127 units of the initial stock so that the starting inventory is 73 units.

5.7 Summary

In this chapter, we formulated a disposal problem which was related to the infinite horizon problem that we studied in Chapter 3. This related formulation could in fact be considered as a generalization of the infinite horizon problem in the sense that the initial inventory was assumed to be a model parameter for the infinite horizon problem while it was treated as a decision variable under a disposal opportunity in this chapter. We provided a condition (not so restrictive in the sense that it holds for Erlang demand) for the objective function to be concave. Thus, we analyzed the optimal solution using well known nonlinear optimization arguments. Depending on the parameter values and the base-stock value S, different cases were possible which were presented in Table 5.1.

We noted that there was only one nontrivial case ($c \geq v$ and $s_1 \geq S$) for which obtaining the optimal solution required computation. We also discussed the computational issues and argued that we need the expressions of $m_\rho(\cdot)$ and $M_\rho(\cdot)$. We noted that one approach was to use the Laplace transform technique, and for uniform and exponential demand densities this method gave closed form expressions for $m_\rho(\cdot)$ and $M_\rho(\cdot)$. However, we also stated that for some demand densities, e.g., the Erlang density, it might not be possible to obtain closed form expressions of $m_\rho(\cdot)$ and $M_\rho(\cdot)$. Therefore, we sought numerical expressions of these functions. We introduced the underlying renewal process $\{\mathcal{N}(t) : t \geq 0\}$ which registers successive reviews associated with the inventory system under consideration. Finally, using the distribution of $\mathcal{N}(t)$ we obtained integral equations whose solutions would give numerical expressions of $M_\rho(\cdot)$ and $m_\rho(\cdot)$.

Chapter 6

Finite Horizon Problem

6.1 Finite Horizon Case

In this chapter, we study the finite horizon case. For this purpose we suppose that the retailer stops selling the product after T periods in which case there are a finite number of periods of operation. Consequently, we have a finite number of decision variables, and our problem is to find those values of $a_1, a_2, ..., a_T$ that maximize the expected value of discounted revenues net of costs throughout the horizon of T periods. One of our motivations for studying the finite horizon case is that in the volatile markets of the 1990s products have shorter life cycles due to competition, marketing strategies, continuous product development efforts, etc. Additionally, the finite horizon case is mainly of interest because of the seasonal nature of some products. A characteristic example of a seasonal product is ski equipment whose sales are not continued during the summer months.

From a theoretical point of view the finite horizon problem is more complicated than the infinite horizon problem in terms of the computation of the optimal policy. In Section 6.2 we give the problem formulation for the finite horizon case and observe that this formulation is slightly different than the infinite horizon problem. We provide a comparison of the infinite and finite horizon problems at the end of Section 6.2. In Section 6.3 we study some mathematical details following which we discuss the conditions that lead to a myopic solution for the finite horizon case. However, unless the conditions of Section 6.4 hold, the exact policy may require substantial computational effort. After we make some refinements to handle the finite horizon case in Section 6.5 we deliver the mathematical expressions that can be used for the computation of the 'best myopic' policy. A justification for computing the best myopic policy is presented in section 6.6. The theory developed in this chapter provides a foundation for the next chapter where we present analytical and numerical results on the computation of the best myopic policy.

6.2 Formulation of the Finite Horizon Problem

We consider the same cost structure which was explained in Section 3.1 and considered for the infinite horizon problem. For period n revenue net of costs is given by

$$R_n - PC_n - HC_n - BC_n, \ n = 1, 2, ..., T-1, \tag{6.1}$$

where R_n, PC_n, HC_n, and BC_n are described by (3.7), (3.3), (3.5), and (3.6) respectively, and

$$PC_1 = c(a_1 - s_1). \tag{6.2}$$

In this formulation, period T is the last period. We suppose that excess inventory at the end of T can be sold immediately for a unit salvage value of l dollars per unit. If $l < 0$ then we say that l is the unit disposal cost. We also impose the condition that if there is backordered demand at the end of T then it is satisfied immediately by ordering at a unit cost of c_T dollars, and selling at a unit price of r_T dollars. Then revenue net of costs for period T is given by

$$R_T - PC_T - HC_T - BC_T + \rho[l(a_T - D_T)^+ - c_T(D_T - a_T)^+ + r_T(D_T - a_T)^+]. \quad (6.3)$$

It is reasonable to assume that the unit salvage income l is less than unit procurement cost c. We distinguish c_T from c, and r_T from r for the sake of generality. One can assume $c_T = c$ and $r_T = r$. In fact, another assumption is to say that backordered demand during T is lost, and this case can be represented by $c_T = r_T = 0$. Apparently, we have $c_T \leq r_T$. Parameters l, c_T, and r_T are called 'end of horizon cost parameters' since they are associated with the end of horizon retail activities.

We denote the net present value of revenue net of costs throughout T periods by NPV_T . Then (6.1) and (6.3) lead to

$$\begin{aligned} NPV_T &= \sum_{n=1}^{T} \rho^{n-1}(R_n - PC_n - HC_n - BC_n) \\ &\quad + \rho^T \left[l(a_T - D_T)^+ - c_T(D_T - a_T)^+ + r_T(D_T - a_T)^+\right]. \end{aligned} \quad (6.4)$$

We substitute (3.3), (3.5), (3.6), (3.7), and (6.2) in expression (6.4), and perform similar algebraic manipulations as we did in Section 3.2. We obtain that

$$\begin{aligned} NPV_T &= cs_1 + \sum_{n=1}^{T} \rho^{n-1}[(r - c + \rho c)a_n - \rho c D_n \\ &\quad -(r + h)(a_n - D_n)^+ - B\delta(D_n - a_n) - b(D_n - a_n)^+)] \\ &\quad + \rho^T \left[l(a_T - D_T)^+ - c(a_T - D_T) + (r_T - c_T)(D_T - a_T)^+\right]. \end{aligned} \quad (6.5)$$

We define

$$\phi(a_T, D_T) \equiv l(a_T - D_T)^+ - c(a_T - D_T) + (r_T - c_T)(D_T - a_T)^+, \quad (6.6)$$

and

$$Y(a) \equiv lE[(a-D_1)^+] - cE[a-D_1] + (r_T - c_T)E[(D_1 - a)^+], \; a \in (-\infty, +\infty). \tag{6.7}$$

We call $Y(\cdot)$ 'end of horizon costs function' because it is associated with the end of horizon cost parameters l, c_T, and r_T.

It can easily be shown that

$$E[\phi(a_T, D_T)] = E[Y(a_T)]. \tag{6.8}$$

The proof is based on stochastic independence of a_T and D_T, and is similar to the proof of Theorem 1, on page 31, which states that $E[w(a_n, D_n)] = E[G(a_n)]$.

Taking the expected values of both sides of (6.5), and using (3.14), (3.16), (6.6), and (6.8) the expected discounted revenue net of costs throughout the T periods is expressed as

$$E[NPV_T] = cs_1 + \sum_{n=1}^{T} \rho^{n-1} E[G(a_n)] + \rho^T E[Y(a_T)]. \tag{6.9}$$

Our problem is to compute $a_1, a_2, \ldots, a_T$ that maximize $E[NPV_T]$ subject to the natural constraints $a_n \geq s_n$ for $n = 1, 2, \ldots, T$. We denote this problem as $\mathcal{P}_T$, and state it as follows:

$$\mathcal{P}_T: \quad \text{Maximize} \quad E[NPV_T]$$
$$\text{Subject to} \quad a_n \geq s_n, \quad n = 1, 2, \ldots, T.$$

The objective functions $E[NPV_T]$ of $\mathcal{P}_T$, and $E[NPV]$ of $\mathcal{P}_\infty$ given by expression (3.21) on page 32 are related. The function $E[NPV_T]$ involves the additional term $\rho^T E[Y(a_T)]$. Nevertheless, as $T \to \infty$ we have $E[NPV_T] \to E[NPV]$

since $0 < \rho < 1$. Although it vanishes for large T, the existence of the extra term $\rho^T E[Y(a_T)]$ may complicate the exact solution of $\mathcal{P}_T$. With this additional term the shape of $E[NPV_T]$ depends on both $G(\cdot)$ and $Y(\cdot)$. Later in Section 6.4 we discuss the conditions under which $E[Y(a_T)] = 0$, so that obtaining the optimal solution of $\mathcal{P}_T$ is as easy as obtaining the optimal solution of $\mathcal{P}_\infty$. Next, we continue with analyzing the properties of $Y(\cdot)$.

6.3 Analysis of the End of Horizon Costs Function, $Y(\cdot)$

Let us recall (6.7) which provides the expression of $Y(a)$. Taking the expected values that appear on the right hand side of (6.7), and noting that μ denotes the mean demand, i.e., $\mu = E[D_n], n = 1, 2, \ldots$, we can write

$$Y(a) = l\int_0^a (a-x)f(x)dx - c(a-\mu) + (r_T - c_T)\int_a^\infty (x-a)f(x)dx, \quad a \in (-\infty, +\infty).$$

After performing a few algebraic manipulations, the above expression of $Y(a)$ can be reduced to

$$\begin{aligned} Y(a) &= (c-l)(\mu - a) + (r_T - c_T + l)\int_a^\infty xf(x)dx \\ &\quad -(r_T - c_T + l)[1 - F(a)]a, \quad a \in (-\infty, +\infty). \end{aligned} \tag{6.10}$$

Since we assume that demand is a positive random variable, if $a \leq 0$ then $f(a) = F(a) = 0$. Then (6.10) implies that

$$Y(a) = (r_T - c_T + c)\mu - (r_T - c_T + c)a, \quad a \leq 0. \tag{6.11}$$

We have $-(r_T - c_T + c) < 0$ since $r_T \geq c_T$, and $c > 0$. Consequently, (6.11) indicates that $Y(\cdot)$ is a linear decreasing function over $(-\infty, 0]$ regardless of which positive demand density is applicable.

Using (6.10) and applying Leibnitz's rule the first derivative, $Y'(a)$, of $Y(a)$ can be expressed as

$$Y'(a) = (r_T - c_T + l)F(a) - (r_T - c_T + c), \quad a \in (-\infty, +\infty). \tag{6.12}$$

Then, for $a \leq 0$ the first derivative $Y'(a)$ reduces to

$$Y'(\cdot) = -(r_T - c_T + c), \quad a \leq 0,$$

since $F(a) = 0$ over this region. It follows that the end of horizon costs function $Y(\cdot)$ is not a differentiable function in general, because there is no guarantee that

$$\lim_{a \to 0^+} Y'(a) = \lim_{a \to 0^-} Y'(a).$$

If we solve $Y'(a) = 0$ using (6.12), we obtain

$$F(a) = \frac{r_T - c_T + c}{r_T - c_T + l} > 1,$$

since $l < c$. This in turn suggests that, $Y(\cdot)$ does not have a stationary point since distribution function $F(a) \leq 1$ for all a. We already know that $Y(.)$ is decreasing over $(-\infty, 0]$. Therefore, we can conclude that $Y(\cdot)$ is decreasing in general. Furthermore, (6.12) implies that

$$\lim_{a \to -\infty} Y'(a) = -(r_T - c_T + c) < 0, \tag{6.13}$$

$$\lim_{a \to +\infty} Y'(a) = l - c < 0. \tag{6.14}$$

According to (6.14) function $Y(a)$ is decreasing as $a \to +\infty$, and thus $Y(\cdot)$ should have a finite maximizer over $(0, +\infty)$. On the other hand, limit (6.13) indicates that $Y(a)$ is increasing as $a \to -\infty$. Then $Y(\cdot)$ may not have a finite maximizer over $(-\infty, 0]$, and thus we say that $Y(.)$ may not be well behaved considering our maximization problem $\mathcal{P}_T$. Nevertheless, this result does not concern us since we will be interested in the behavior of $Y(\cdot)$ over the positive axis because of reasons which will become apparent in the next chapter.

To summarize, we know that in general $Y(\cdot)$ is a decreasing function. In particular, we also know that it is decreasing linearly over the negative axis. Finally, we concentrate on the shape of $Y(\cdot)$ over the positive axis. Taking the derivative of (6.12) we obtain the second derivative, $Y''(a)$, of $Y(a)$ as

$$Y''(a) = (r_T - c_T + l)f(a). \tag{6.15}$$

If $l \geq 0$ then $Y(\cdot)$ may be convex over $(0, +\infty)$ since right hand side of (6.15) is nonnegative. That is, if there is a unit salvage income l associated with excess inventory at the end of the planning horizon then $Y(\cdot)$ may be convex over $(0, +\infty)$. However, it is possible to have $l < 0$ in which case l represents the unit disposal cost. If $l < 0$ then depending on the values of r_T and c_T, function $Y(\cdot)$ may turn out to be concave.

In the subsequent sections of this chapter as well as in Chapter 7, we study $\mathcal{P}_T$ without imposing restrictions on the shape of $Y(\cdot)$. Our analysis in this section will lead to further analytical and numerical results in Chapter 7.

6.4 Conditions for a Myopic Optima

Unlike the infinite horizon problem $\mathcal{P}_\infty$ the finite horizon problem $\mathcal{P}_T$ may require a substantial amount of computational effort. In general, the finite horizon problem does not have an easy-to-compute myopic optimum because of the additional term $\rho^T E[Y(a_T)]$ that appears on the right hand side of (6.9). However, if we assume that

- excess inventory $(a_T - D_T)$ at the end of T, i.e., when the retailer stops selling the product, can be salvaged at a unit salvage value of c dollars per unit, and
- backordered demand $(D_T - a_T)$ of period T, if any, should be satisfied with a concomitant unit purchase cost c,

then $E[Y(a_T)]$ disappears (Heyman and Sobel [15], p. 79). This is because the first assumption suggests that $l = c$, and the second assumption imposes that $c_T = c$, and $r_T = 0$, and it follows from (6.7) that $E[Y(a_T)] = 0$.

Assuming $l = c$, $c_T = c$, and $r_T = 0$ is equivalent to Veinott's well known terminal condition (see Veinott [44]). Proposition 6 states that if $s_1 \leq S$ and Veinott's terminal condition holds, then the myopic policy with critical number S which is optimal for the infinite horizon problem $\mathcal{P}_\infty$ is also optimal for the finite horizon problem $\mathcal{P}_T$.

Proposition 6 *If $s_1 \leq S$ and Veinott's terminal condition holds then $a_n = S$, $n = 1, 2, ..., T$, is optimal for $\mathcal{P}_T$.*

Proof Since $E[Y(a_T)] = 0$ under Veinott's terminal condition, the objective function (6.9) simplifies to

$$E[NPV_T] = cs_1 + \sum_{n=1}^{T} \rho^{n-1} E[G(a_n)].$$

We do not give the entire proof to avoid repetitions. Noting that $n = T$ is the final period, the proof would be the same as the proof of Proposition 3 on page 38. ■

Even if Veinott's terminal condition holds, if $s_1 > S$ then a myopic policy is not necessarily optimal. A sufficient condition for a myopic optimum in this case is the same as (3.35).

Proposition 7 *If Veinott's terminal condition and (3.35) holds, an optimal policy for $\mathcal{P}_T$ is specified by*

$$a_n = \max(s_n, S), \; n = 1, 2, ..., T.$$

Proof The proof is very similar to the proof of Proposition 4 where $n = T$ is the final period. We do not give its details in order to avoid repetitions. We note that, if $s_1 > S$ then the optimal policy again implies (3.57), i.e., no orders are placed until the inventory level goes below S. According to our terminology this in turn means that no orders are placed until the policy is activated at the beginning of random period N. To carry out the proof all we need is to refine the definition of N, previously defined by (3.37), so that the finite horizon case can be handled. The following discussion explains how this can be done. ∎

6.5 A Refinement to Handle the Finite Horizon Case

If we have s_1 units of initial inventory, and if this represents an overstock situation then it is possible that the inventory level may not go below the critical number S of the myopic policy throughout the entire horizon. Under these conditions, the random period N at the beginning of which the policy is activated is never reached during the horizon. That is, policy may not activate with probability $P(N > T)$. In order to handle this technical detail that arises for the finite horizon problem, we need to update the set of possible values of the random variable N, previously defined by (3.37). Using the new set of possible values for the finite horizon case we also need to update the distribution of N, previously given by (3.61) under the infinite horizon assumption.

We assume that a myopic policy with critical number S will be employed for the horizon of T periods. Under this scenario, we say that if $s_1 > S$ then policy is activated at a random time N defined by

$$N = \inf\{n : s_n \leq S, n = 1, 2, ..., T\}, \tag{6.16}$$

and if the infimum is not an element of the set defined by $\{n : s_n \leq S, n = 1, 2, ..., T\}$, then we say $N = T + 1$. We again define N for the case $s_1 > S$, but this time the possible values of N are $2, 3, ..., T, T + 1$. If it turned out that $N = T + 1$ then the inventory levels $s_1, s_2, \ldots, s_T$ were never below S. This means if $N = T + 1$ then no orders were placed, and the policy did not activate.

An important relation that we frequently used in Chapters 3 and 5 still holds with the updated definition (6.16) of N. Namely, under the myopic policy with critical number S, relation (3.57) which states that

$$a_n = \begin{cases} s_n = s_1 - \sum_{i=1}^{n-1} D_i & \text{if } n < N, \\ S & \text{if } n \geq N, \end{cases} \tag{6.17}$$

is still valid where N is defined by (6.16). In contrast, (3.59) which states that

$$\{N > n\} \iff \left\{\sum_{i=1}^{n-1} D_i < s_1 - S\right\},$$

holds only for $n = 2, ..., T$ with the updated definition (6.16) of N because $N > T + 1$ is not allowed. Then we can write

$$P(N > 1) = 1, \tag{6.18}$$

$$P(N > n) = P\left(\sum_{i=1}^{n-1} D_i < s_1 - S\right), \; n = 2, 3, ..., T, \quad (6.19)$$

$$P(N > T + 1) = 0. \quad (6.20)$$

Since we defined $F^{(0)}(x) \equiv 1$, $x > 0$, expression (6.19) implies that

$$P(N = n) = F^{(n-2)}(s_1 - S) - F^{(n-1)}(s_1 - S), \; n = 2, 3, ..., T. \quad (6.21)$$

Further, (6.18) to (6.20) suggest that

$$P(N = T + 1) = 1 - \sum_{n=2}^{T} P(N = n).$$

Using (6.21) in the above expression leads to

$$P(N = T + 1) = F^{(T-1)}(s_1 - S). \quad (6.22)$$

Combining (6.21) and (6.22) we conclude that if a myopic policy with critical number S is considered for the finite horizon problem $\mathcal{P}_T$ then the distribution of N is given by

$$P(N = n) = \begin{cases} F^{(n-2)}(s_1 - S) - F^{(n-1)}(s_1 - S), & n = 2, 3, ..., T \\ F^{(T-1)}(s_1 - S), & n = T + 1. \end{cases} \quad (6.23)$$

We also compute $E[\rho^{N-1}]$ for future reference. Since N is the first period at the beginning of which an order is placed, then $N - 1$ is the period during which the cumulative demand exceeds $s_1 - S$. In fact, $E[\rho^{N-1}]$ is the probability generating function of random variable $N - 1$ since $0 < \rho < 1$. If we use (6.23) in

$$E[\rho^{N-1}] = \sum_{n=1}^{\infty} \rho^n P(N - 1 = n),$$

then we have

$$E\left(\rho^{N-1}\right) = \rho - (1 - \rho) \sum_{n=1}^{T-1} \rho^n F^{(n)}(s_1 - S). \quad (6.24)$$

So far we have updated the set of possible values for the random variable N in order to handle some technical details required by the finite horizon case. We also presented its distribution for future reference. We conclude this section by mentioning the relation between the distribution (6.23) of N over a finite horizon of T periods and the distribution (3.61) of N over an infinite horizon. If we take the limit of the right hand side of (3.61) as $T \to \infty$, then (3.61) reduces to (6.37) as we expect. Similarly, as $T \to \infty$, (6.24) reduces to $E\left(\rho^{N-1}\right)$ of the infinite horizon problem $\mathcal{P}_\infty$ [see (3.75)].

6.6 The Best Myopic Policy

For the finite horizon problem under consideration, published results, e.g., Porteus [31] reports that 'under appropriate conditions' a base-stock policy may be shown to be optimal. However, a myopic optimal policy has not been ensured. As we discussed on page 44, when a base-stock policy is optimal for a problem over T periods we generally have a sequence of critical numbers $\left\{S^{(n)}\right\}_{n=1}^{T}$ with $S^{(n)}$ specifying the order-up-to level of period n.

The real life application of such a policy may have drawbacks. On one hand, we have the computational problem. One approach to computation is to use dynamic programming (DP) which may demand a substantial effort. In relation to this computational difficulty Wheeler [48] mentions that the DP approach requires the calculation of the optimal revenue for a vast number of states which will not be entered, and then for each stage of the recursion one must optimize probably a nonconcave function with some relative maximizers. On the other hand, we have the implementation problem. In contrast to myopic ones, if these policies are used for many products over a long but finite horizon, then we may need to keep track of a large number of parameters.

Many researchers contributed to specifying the forms of optimal policies for stochastic inventory control problems. There is an abundant literature (Scarf [36], Veinott [46], Schal [38], etc.) on simple (s, S) policies under which two different critical numbers are associated with each period. Porteus [31] has studied generalized (s, S) policies which are characterized by more than one pair of critical numbers for each period. Boylan's work (references [8] and [9]) on multiple (s, S) policies should be mentioned among these efforts. Boylan's multiple (s, S) policies also associate more than a pair of critical numbers for each period. Waldmann [49] discusses optimal 'state dependent' (s, S) policies where for each period a pair of critical numbers is associated with each possible state of the 'environmental process'. Parlar and Rempala [29] also present an optimal state dependent (s, S) policy under quadratic holding costs. More computational effort would be required as the structure of the policy gets more complicated.

The traditional approach to avoid computational difficulty associated with the finite horizon problem is to assume Veinott's terminal condition. As we already argued in Section 6.4, under Veinott's terminal condition solving the finite horizon problem is as easy as solving the infinite horizon problem. Of course, this computational ease is provided at a price of a set of conditions which may not completely apply in the retail industry. One of these assumptions is $l = c$ where in most retail environments we would expect $l < c$. Another assumption is $r_T = 0$ which implies that if some portion of demand is backordered during the last period then the retailer gives away $(D_T - a_T)$ units although (s)he pays $c_T = c > 0$ dollars per unit to purchase these. In general, we are interested in other ways of avoiding computational difficulty which are superior to Veinott's terminal condition since we seek computational ease without ignoring reality.

In summary, some policies are structured, and thus can be characterized but may be difficult to compute or implement, or worse, some policies are not structured at all. A structured optimal policy has theoretical importance whereas an easy-to-compute policy with a simple structure has practical value. Nevertheless, easy to compute simple policies, such as myopic base-stock policies, are not always optimal. However, they can still be favored on the basis of their ease of computation and implementation. In our search for an easy to compute but suboptimal policy (without Veinott's terminal condition) we restrict our attention to the class of myopic base-stock policies, i.e., base-stock policies for which the critical number associated with different periods is the same.

For this purpose let us assume that we have convinced the retailer to implement the myopic policy:

$$a_n = \max(s_n, \bar{S}), \ n = 1, 2, ..., T, \tag{6.25}$$

The initial inventory level s_1 is a given positive constant, and obviously two cases are possible depending on the value of s_1 and the critical number $\bar{S}$. These are:

- **Case (1):** $s_1 \leq \bar{S}$. In this case, myopic policy (6.25) implies that successive periods' order-up-to levels are related by

$$a_n = \bar{S}, \ n = 1, 2, ..., T. \tag{6.26}$$

- **Case (2):** $s_1 > \bar{S}$. Referring to our discussion in Section (6.5), we conclude that if $s_1 > \bar{S}$ myopic policy (6.25) implies

$$a_n = \begin{cases} s_n = s_1 - \sum_{i=1}^{n-1} D_i & \text{if } n < \bar{N}, \\ \bar{S} & \text{if } n \geq \bar{N}. \end{cases} \tag{6.27}$$

where $\bar{N}$ is the time at the beginning of which the policy is activated for our finite horizon problem under a myopic policy with critical number $\bar{S}$.

By analogy to (6.16), random variable $\bar{N}$ is defined as

$$\bar{N} = \inf\left\{n : s_n \leq \bar{S}, n = 1, 2, ..., T\right\}, \tag{6.28}$$

Again if the infimum is not an element of the set defined by

$$\left\{n : s_n \leq \bar{S}, n = 1, 2, ..., T\right\},$$

then we say $\bar{N} = T + 1$, and thus the possible values of $\bar{N}$ are $2, 3, ..., T, T + 1$.

By assumption we consider only those policies that have the form of (6.25). We eventually seek the best policy of this form. Thus, we aim to compute the value of $\bar{S}$ which maximizes the total expected discounted revenue net of costs over the entire horizon. The objective function $E[NPV_T]$ given by (6.9) corresponds to the total expected discounted revenue net of costs over the horizon of T periods. Then dependence of the total expected discounted return net of costs on the critical number $\bar{S}$ is simply computed by substituting (6.25) in (6.9). The result gives the total expected discounted return net of costs over T periods as a function of $\bar{S}$. We denote this function by $NPV_T(\bar{S})$. Once we obtain an expression of $NPV_T(\bar{S})$ then we can use it to compute the best myopic policy. Since all demands must be satisfied, the critical number of the myopic policy is not allowed to be negative. Thus, the best myopic policy should maximize $NPV_T(\bar{S})$ over $\bar{S} \geq 0$.

As we have already discussed, the myopic policy (6.25) implies (6.26) and (6.27). We observe that over $\bar{S} \leq s_1$ an expression of $NPV_T(\bar{S})$ is obtained by substituting (6.26) in (6.9). Similarly, over $\bar{S} < s_1$ an expression for $NPV_T(\bar{S})$ is computed by substituting (6.27) in (6.25). We continue with the computation of $NPV_T(\bar{S})$ in the next section.

6.7 Expression of $NPV_T(\cdot)$

In general, $NPV_T(\bar{S})$ has two parts depending on the value of s_1. Now, we analyze the two cases that lead to the expressions of $NPV_T(\bar{S})$ over $[s_1, \infty)$ and over $(-\infty, s_1)$.

Case (1): Over $[s_1, \infty)$

Over this region we have $s_1 \leq \bar{S}$, and thus the myopic policy under consideration asserts that order-up-to levels are related by (6.26). Substituting (6.26) in (6.9) leads to

$$NPV_T(\bar{S}) = cs_1 + \left(\frac{1-\rho^T}{1-\rho}\right) G(\bar{S}) + \rho^T Y(\bar{S}). \qquad (6.29)$$

Case (2): Over $(-\infty, s_1)$

Analysis of this case requires more effort, and the resulting expression of $NPV_T(\bar{S})$ is quite complicated. Over the region defined by $(-\infty, s_1)$ we have $s_1 > \bar{S}$. Then according to the myopic policy under consideration order-up-to levels are related by (6.27).

Initially, let us note that expression (6.9) can be stated as

$$E[NPV_T] = cs_1 + E\left[\sum_{n=1}^{T} \rho^{n-1} G(a_n)\right] + \rho^T E[Y(a_T)], \qquad (6.30)$$

We first compute

$$E\left[\sum_{n=1}^{T}\rho^{n-1}G(a_n)\right],$$

which appears on the right hand side of (6.30). Using conditional expectations we write

$$E\left[\sum_{n=1}^{T}\rho^{n-1}G(a_n)\right] = E\left[E\left(\sum_{n=1}^{T}\rho^{n-1}G(a_n)|\bar{N}\right)\right], \tag{6.31}$$

where $\bar{N}$ is defined by (6.28). If we use (6.27) in (6.31) then we have

$$E\left[E\left(\sum_{n=1}^{T}\rho^{n-1}G(a_n)|\bar{N}\right)\right] = E\left[E\left(\sum_{n=1}^{\bar{N}-1}\rho^{n-1}G\left(s_1-\sum_{i=1}^{n-1}D_i\right)+\sum_{n=\bar{N}}^{T}\rho^{n-1}G(\bar{S})\right)\right]. \tag{6.32}$$

Combining expressions (6.31), (6.32) and using the finite sum formula we obtain

$$E\left[\sum_{n=1}^{T}\rho^{n-1}G(a_n)\right] = E\left\{E\left[\sum_{n=1}^{\bar{N}-1}\rho^{n-1}G\left(s_1-\sum_{i=1}^{n-1}D_i\right)\right]\right\}+G(\bar{S})E\left(\frac{\rho^{\bar{N}-1}-\rho^{T}}{1-\rho}\right). \tag{6.33}$$

We interpret the first term

$$E\left\{E\left[\sum_{n=1}^{\bar{N}-1}\rho^{n-1}G\left(s_1-\sum_{i=1}^{n-1}D_i\right)\right]\right\},$$

that appears on the right hand side of (6.33) as the expected discounted total return net of costs until the inventory level goes below $\bar{S}$ because it is the expected net present value of the single return net of costs functions until the policy is activated at the beginning of $\bar{N}$. In fact, this quantity is the same as $C(s_1)$ given by equation (3.67) if S that appears in the expression of $C(s_1)$ is replaced by $\bar{S}$. Note that here we have the double expectation for the sake of notation. It follows that

$$E\left\{E\left[\sum_{n=1}^{\bar{N}-1}\rho^{n-1}G\left(s_1-\sum_{i=1}^{n-1}D_i\right)\right]\right\} = G(s_1)+\int_0^{s_1-\bar{S}}G(s_1-x)m_\rho(x)dx. \tag{6.34}$$

Using (6.34) in (6.33) we can write

$$E\left[\sum_{n=1}^{T}\rho^{n-1}G(a_n)\right] = G(s_1) + \int_0^{s_1-\bar{S}} G(s_1-x)m_\rho(x)dx + \frac{G(\bar{S})}{1-\rho}E\left(\rho^{\tilde{N}-1}\right) - \frac{\rho^T G(\bar{S})}{1-\rho}. \tag{6.35}$$

The probability generating function of $\tilde{N}$ under myopic policy (6.25) is simply obtained by replacing S with $\bar{S}$ in the right hand side of (6.24) giving

$$E\left(\rho^{\tilde{N}-1}\right) = \rho - (1-\rho)\sum_{n=1}^{T-1}\rho^n F^{(n)}(s_1-\bar{S}). \tag{6.36}$$

If we use (6.36) in (6.35) then we obtain

$$\begin{aligned} E\left[\sum_{n=1}^{T}\rho^{n-1}G(a_n)\right] &= G(s_1) + \int_0^{s_1-\bar{S}} G(s_1-x)m_\rho(x)dx + \frac{\rho G(\bar{S})}{1-\rho} \\ &\quad -G(\bar{S})\sum_{n=1}^{T-1}\rho^n F^{(n)}(s_1-\bar{S}) - \frac{\rho^T G(\bar{S})}{1-\rho}. \end{aligned} \tag{6.37}$$

We continue with the computation of the value of $E\left[Y(a_T)\right]$ under the myopic base-stock policy given by (6.27). Using conditional expectation we can write

$$E\left[Y(a_T)\right] = \sum_{n=2}^{T+1} E\left(Y(a_T)|\tilde{N}=n\right)P(\tilde{N}=n). \tag{6.38}$$

If $\tilde{N} \le T$ then (6.27) implies that $a_T = \bar{S}$, and using this result in (6.38) leads to

$$E\left[Y(a_T)\right] = \sum_{n=2}^{T} E\left[Y(\bar{S})|\tilde{N}=n\right]P(\tilde{N}=n) + E\left[Y(a_T)|\tilde{N}=T+1\right]P(\tilde{N}=T+1). \tag{6.39}$$

We simply have

$$E\left(Y(\bar{S})|\tilde{N}=n\right) = Y(\bar{S}).$$

Then (6.39) reduces to

$$E\left[Y(a_T)\right] = Y(\bar{S})P(\tilde{N}\le T) + E\left[Y(a_T)|\tilde{N}=T+1\right]P(\tilde{N}=T+1). \tag{6.40}$$

Based on our discussion in Section 6.5 we have $\bar{N} = T+1$ if the inventory level did not go below the critical number $\bar{S}$ of the myopic policy at any of the review instants of the planning horizon. Thus, $\bar{N} = T+1$ means

$$a_T = s_T = s_1 - \sum_{i=1}^{T-1} D_i,$$

so that we can write

$$E\left[Y(a_T)|\bar{N} = T+1\right] = E\left[Y\left(s_1 - \sum_{i=1}^{T-1} D_i\right)\right]. \tag{6.41}$$

If we insert (6.41) in (6.40) we have

$$E\left[Y(a_T)\right] = Y(\bar{S})P(\bar{N} \leq T) + E\left[Y\left(s_1 - \sum_{i=1}^{T-1} D_i\right)\right] P(\bar{N} = T+1). \tag{6.42}$$

The probability density of $\bar{N}$ is simply obtained by replacing S that appears on the right hand side of (6.23) with $\bar{S}$. Then using this density we can express (6.42) as follows:

$$E\left[Y(a_T)\right] = Y(\bar{S}) - Y(\bar{S})F^{(T-1)}(s_1 - \bar{S}) + E\left[Y\left(s_1 - \sum_{i=1}^{T-1} D_i\right)\right] F^{(T-1)}(s_1 - \bar{S}). \tag{6.43}$$

We want to show that the expression

$$E\left[Y\left(s_1 - \sum_{i=1}^{T-1} D_i\right)\right]$$

that appears on the right hand side of (6.43) does not depend on $\bar{S}$. According to the definition of $Y(\cdot)$ given by (6.7) we have

$$\begin{aligned} E\left[Y\left(s_1 - \sum_{i=1}^{T-1} D_i\right)\right] &= E\left\{lE\left[(s_1 - \sum_{i=1}^{T} D_i)^+\right] - cE\left(s_1 - \sum_{i=1}^{T} D_i\right)\right. \\ &\quad \left. +(r_T - c_T)E\left[\left(\sum_{i=1}^{T} D_i - s_1\right)^+\right]\right\}. \end{aligned}$$

Taking the expected values in the above expression we have

$$\begin{aligned} E\left[Y\left(s_1-\sum_{i=1}^{T-1}D_i\right)\right] &= l\int_0^{s_1}(s_1-x)f^{(T)}(x)dx - cs_1 + cTE(D_1) \\ &\quad +(r_T-c_T)\int_{s_1}^{\infty}(x-s_1)f^{(T)}(x)dx. \end{aligned}$$

It can be easily shown that the above expression implies

$$\begin{aligned} E\left[Y\left(s_1-\sum_{i=1}^{T-1}D_i\right)\right] &= (c_T-r_T-c)s_1 + cTE(D_1) + (l+r_T-c_T)s_1F^{(T)}(s_1) \\ &\quad -l\int_0^{s_1}xf^{(T)}(x)dx + (r_T-c_T)\int_{s_1}^{\infty}xf^{(T)}(x)dx. \quad (6.44) \end{aligned}$$

The right hand side of (6.44) does not depend on $\bar{S}$. Knowing this we substitute (6.37) and (6.43) in (6.30). It follows that if $\bar{S} < s_1$ then

$$\begin{aligned} NPV_T(\bar{S}) &= cs_1 + G(s_1) + \int_0^{s_1-\bar{S}} G(s_1-x)m_\rho(x)dx + \frac{\rho G(\bar{S})}{1-\rho} \\ &\quad -G(\bar{S})\sum_{n=1}^{T-1}\rho^n F^{(n)}(s_1-\bar{S}) - \frac{\rho^T G(\bar{S})}{1-\rho} + \rho^T Y(\bar{S}) \\ &\quad -\rho^T Y(\bar{S})F^{(T-1)}(s_1-\bar{S}) \\ &\quad +\rho^T E\left[Y\left(s_1-\sum_{i=1}^{T-1}D_i\right)\right]F^{(T-1)}(s_1-\bar{S}). \quad (6.45) \end{aligned}$$

6.8 Summary

In this chapter we formulated and analyzed the finite horizon problem. At the formulation stage we referred to Chapter 3 where we had studied the infinite horizon problem, and required some refinements in order to handle the finite horizon case. The finite horizon case was considered not only because it was of theoretical interest but also the horizon length was in fact finite for most real life problems due to short life cycles and the seasonal nature of some products. Unlike the infinite horizon problem the optimal policy was not myopic. However, as we argued in Section 6.4 under certain conditions the finite horizon solution reduces to the infinite horizon solution (at the price of a set of assumptions which might not exactly represent reality). On the other hand, the exact policy was complicated, and one reasonable approach was to compute the best myopic policy which could be justified on the basis of computation and implementation ease.

In the previous section we obtained the expression for $NPV_T(\bar{S})$. Function $NPV_T(\bar{S})$ represents the value of the objective function of our finite horizon problem $\mathcal{P}_T$ if a myopic policy with critical number $\bar{S}$ is employed. That is, $NPV_T(\bar{S})$ gives the expected discounted returns net of costs over T periods under the myopic policy characterized by $\bar{S}$. According to our analysis function $NPV_T(\bar{S})$ is specified by two expressions. Expressions (6.45) and (6.29) define $NPV_T(\bar{S})$ over $(-\infty, s_1)$ and $[s_1, +\infty)$ respectively. Then we have

$$NPV_T(\bar{S}) = \begin{cases} (6.45) & \text{if } \bar{S} < s_1, \\ (6.29) & \text{if } \bar{S} \geq s_1. \end{cases} \tag{6.46}$$

Different uses can be made of (6.46). Expression (6.46) itself represents the relation between the expected discounted returns over a finite horizon of T periods and the critical number $\bar{S}$. Thus, it provides a measure of the revenue generation power of any given myopic policy. It also displays the dependence of the expected discounted returns on the initial inventory level s_1 if a myopic policy is employed during a finite horizon. Further, (6.46) can be optimized in order to compute the maximizing value of $\bar{S}$ so that "the best finite horizon myopic policy" can be specified. We primarily make use of (6.46) for optimization purposes and discuss various analytical and computational results for Erlang demand in the next chapter. However, we also present numerical results which display the dependence of the best myopic policy on the initial inventory level.

Chapter 7

Computing the Best Myopic Policy For the Finite Horizon Problem

In this chapter we present our results on the computation of the best myopic policy using the approach developed in Sections 6.6 and 6.7 of the previous chapter. We suppose that the demand density is Erlangian, and $s_1 \geq 0$, i.e., there are no outstanding backorders initially. Once the initial inventory level, unit revenue, costs, and demand parameters are given then the best myopic policy can be computed numerically. The computational procedure involves optimizing $NPV_T(\cdot)$ given by (6.46). Since all demands must be satisfied, the best myopic policy cannot offer a negative base-stock level.

According to (6.46) the function $NPV_T(\bar{S})$ has two parts which are specified by the value of s_1. In searching for the global maximizer of $NPV_T(\bar{S})$ our approach is to compute the maximizer of each part over the relevant regions, and then arrive at the global maximizer by comparison. That is, we find the maximizers of (6.45) and (6.29) over the regions $[0, s_1)$ and $[s_1, +\infty)$, respectively, and then choose the one which leads to a higher value of $NPV_T(\cdot)$.

In contrast to the optimal myopic policy of the infinite horizon case, the best myopic policy for the finite horizon depends on s_1. [Note that, for the case of Erlang demand, the critical number S of the infinite horizon policy solves equation (4.17), on page 63, which does not depend on s_1.] This dependence complicates the problem of computing the best myopic policy. Let us recall (6.46) which defines $NPV_T(\bar{S})$. As we already mentioned, optimization of $NPV_T(\bar{S})$ involves computing the maximizers of (6.45) and (6.29) over the regions $[0, s_1)$ and $[s_1, +\infty)$, respectively. Nonetheless, expression (6.45) is quite messy, and its explicit evaluation requires the computation of $m_\rho(\cdot)$ either using Laplace transforms or solving integral equations. [Computational issues regarding $m_\rho(\cdot)$ were discussed in Sections 5.5.2 and 5.5.4.] It is worth noting that if $s_1 = 0$ then computing the positive maximizer of $NPV_T(\bar{S})$ requires the optimization of (6.29) only.

For any practical problem the initial inventory is known, and the best myopic policy can be computed using our approach. We present various numerical examples and illustrations by varying the initial inventory level as well as the other model parameters. To obtain general results, however, it may help to specify an initial inventory level by an assumption which is true for at least a class of problems. For this purpose, it is reasonable to assume zero as the initial inventory level since most practical problems do start with no on-hand inventory. In opposition to the frequent use of this assumption in inventory theory, Wheeler [48] states that in some cases periodic review models cannot be applied until the inventory process has been observed for a few periods, so that good assumptions regarding costs and demand can be made. He adopts the convention of assuming that the initial inventory level is $\bar{S}$, and thus he sets s_1 to the variable with respect to which the optimization is performed. That is, Wheeler suggests to assume $s_1 = \bar{S}$ instead of $s_1 = 0$. Although we assume that $s_1 = 0$ in order derive some explicit results, this assumption can be easily replaced with $s_1 = \bar{S}$, and only minor changes will be required for the extension of our results.

The rest of this chapter is organized as follows. In Section 7.1, we first study the computation of the best myopic policy assuming $s_1 = 0$. In fact, the purpose of this assumption is two-fold; on one hand it is a reasonable assumption that leads to quite general results, and on the other hand it helps to avoid manipulating (6.45), and eases computation. Thus, in Section 7.1 we assume $s_1 = 0$, and we discuss the computation of the best myopic policy for the exponential demand because of its simplicity, and then continue with the Erlang demand. We show that if $s_1 = 0$ then computing the best myopic policy via using our analytical results is easy for the case of Erlang demand. This is similar to the infinite horizon case for which computing the optimal myopic policy is as easy as computing the unique positive maximizer of a well behaved unimodal function (Theorem 2, p. 64). We present many examples and graphs in order to illustrate the sensitivity of the myopic policy to the horizon length, discounting factor, and mean demand. In Section 7.2 we discuss our computational experimentation for the more difficult case of $s_1 > 0$. We illustrate the dependence of the best myopic policy on s_1 under different scenarios of the horizon length and demand parameters. We also compare the performance of the best myopic policy of the finite horizon case with the infinite horizon myopic policy when applied to the finite horizon.

It is worth noting that, since the feasible space of our problem is the positive axis, we are interested in the expression of $NPV_T(\bar{S})$ for $\bar{S} \geq 0$. Therefore, in the following discussion whenever we provide an expression for a function of $\bar{S}$, e.g., single period revenue net of costs function $G(\bar{S})$, end of horizon costs function $Y(\bar{S})$, or $NPV_T(\bar{S})$, we refer to the case $\bar{S} \geq 0$. Similarly, unless otherwise is stated whenever we mention concavity (or convexity) we refer to the region $[0, +\infty)$.

7.1 Zero Initial Inventory

As already noted, throughout this section we assume that the initial inventory level is zero, so that there are no on-hand inventories or outstanding backorders at the start of the horizon. Under this assumption, function $NPV_T(\bar{S})$, defined by (6.46), can be expressed by inserting $s_1 = 0$ in (6.29) giving

$$NPV_T(\bar{S}) = \left(\frac{1-\rho^T}{1-\rho}\right) G(\bar{S}) + \rho^T Y(\bar{S}). \tag{7.1}$$

Then our problem reduces to

$$\max_{\bar{S}\geq 0} \left\{ \left(\frac{1-\rho^T}{1-\rho}\right) G(\bar{S}) + \rho^T Y(\bar{S}) \right\}. \tag{7.2}$$

7.1.1 Exponential Demand

In order to find the solution of our problem which is stated by (7.2), we need explicit expressions for the single period return net of costs function, $G(\cdot)$, and the end of horizon costs function, $Y(\cdot)$. The properties of $G(\cdot)$ were discussed in Section 3.3 of Chapter 3. Similarly, the function $Y(\cdot)$ was analyzed in Section 6.3 of Chapter 6. We now make use of our results presented in previous chapters, and obtain explicit expressions for $G(\cdot)$ and $Y(\cdot)$ for the case of exponential demand.

We first note that density and distribution functions of exponential demand with parameter λ are given by

$$f(x) = \lambda \exp(-\lambda x), \quad x > 0, \tag{7.3}$$

and

$$F(x) = 1 - \exp(-\lambda x), \quad x > 0, \tag{7.4}$$

where the mean demand is

$$\mu = 1/\lambda. \tag{7.5}$$

Let us recall (3.22) which states

$$\begin{aligned} G(a) &= (r - c + \rho c)a - \rho c E[D_1] - (r+h)\int_0^a (a-x)f(x)dx \\ &\quad -B[1-F(a)] - b\int_a^\infty (x-a)f(x)dx, \quad a \in (-\infty, +\infty). \end{aligned}$$

If we substitute (7.3) and (7.4) in the above expression, and perform some algebraic manipulations then we find

$$G(\bar{S}) = -\frac{B\lambda + h + b + r}{\lambda}\exp(-\lambda\bar{S}) - [c(1-\rho)+h]\bar{S} + \frac{r - \rho c + h}{\lambda}. \tag{7.6}$$

Taking the first and second derivatives of (7.6) leads to

$$G'(\bar{S}) = (B\lambda + h + b + r)\exp(-\lambda\bar{S}) - [c(1-\rho)+h], \tag{7.7}$$

$$G''(\bar{S}) = \lambda(B\lambda + h + b + r)\exp(-\lambda\bar{S}) < 0. \tag{7.8}$$

Observe that the right hand side of (7.8) is negative, and therefore $G(\cdot)$ is strictly concave over the positive axis in this case.

In order to provide an expression for the end of horizon costs function $Y(\cdot)$ for exponential demand, we substitute (7.3) and (7.5) in expression (6.10) which gives

$$\begin{aligned} Y(a) &= (c-l)(\mu-a) + (r_T - c_T + l)\int_a^{\infty} x f(x)dx \\ &\quad -(r_T - c_T + l)[1-F(a)]a, \quad a \in (-\infty, +\infty). \end{aligned}$$

As a result we obtain

$$Y(\bar{S}) = (c-l)\left(\frac{1}{\lambda} - \bar{S}\right) + \frac{r_T - c_T + l}{\lambda}\exp(-\lambda\bar{S}). \tag{7.9}$$

Taking the first and second derivatives of (7.9) results that

$$Y'(\bar{S}) = -(r_T - c_T + l)\exp(-\lambda\bar{S}) - (c-l) < 0, \tag{7.10}$$

and

$$Y''(\bar{S}) = \lambda(r_T - c_T + l)\exp(-\lambda\bar{S}). \tag{7.11}$$

As we discussed in Section 6.3, function $Y(\cdot)$ is always decreasing since the right hand side of (7.10) is negative. Provided that $l \geq 0$, the end of horizon costs function $Y(\cdot)$ is strictly convex since $r_T > c_T$ so that the right hand side of (7.11) is positive. Otherwise, $Y(\cdot)$ is concave. Figure 7.1 illustrates how the functions $G(\cdot)$ and $Y(\cdot)$ may look like under the assumption of exponential demand.

According to (7.1), function $NPV_T(\cdot)$ is a combination of function $G(\cdot)$ which is concave and function $Y(\cdot)$ which is convex unless $l < 0$. Figures 7.2 and 7.3 illustrate how the function $NPV_T(\cdot)$ may look like under the assumption that demand is exponential, and $s_1 = 0$. For the example that is used to plot Figure 7.3 function $NPV_T(\cdot)$ is concave. However, we need to make sure that it really is a well-behaved concave function for exponential demand for most parameter values, and thus we analyze its properties further. Expression (7.1) implies that

$$NPV_T'(\bar{S}) = \left(\frac{1-\rho^T}{1-\rho}\right)G'(\bar{S}) + \rho^T Y'(\bar{S}), \tag{7.12}$$

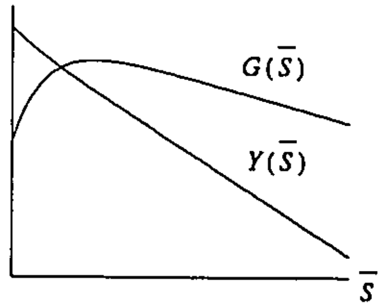

Figure 7.1: $G(\cdot)$ and $Y(\cdot)$ for exponential demand.

and

$$NPV_T''(\bar{S}) = \left(\frac{1-\rho^T}{1-\rho}\right) G''(\bar{S}) + \rho^T Y''(\bar{S}). \tag{7.13}$$

If we insert (7.7) and (7.10) in (7.12) then we have

$$\begin{aligned} NPV_T'(\bar{S}) &= \left[\left(\frac{1-\rho^T}{1-\rho}\right)(B\lambda + h + b + r) - \rho^T(r_T - c_T + l)\right]\exp(-\lambda\bar{S}) \\ &\quad - \left(\frac{1-\rho^T}{1-\rho}\right)[c(1-\rho) + h] - \rho^T(c - l). \end{aligned} \tag{7.14}$$

It follows that,

$$\lim_{\bar{S}\to+\infty} NPV_T'(\bar{S}) = -\left(\frac{1-\rho^T}{1-\rho}\right)[c(1-\rho) + h] - \rho^T(c - l), \tag{7.15}$$

and

$$\lim_{\bar{S}\to 0^+} NPV_T'(\bar{S}) = \left(\frac{1-\rho^T}{1-\rho}\right)[B\lambda + b + r - c(1-\rho)] - \rho^T(r_T - c_T + c). \tag{7.16}$$

Thus, we have the following propositions.

Proposition 8 *If the demand density is exponential with parameter* λ, *and* $s_1 = 0$, *then* $NPV_T(\cdot)$ *has a finite maximizer over the positive axis.*

Proof The right hand side of the limit (7.15) is negative since $\rho < 1$ and $c < l$. That is, $NPV_T(\bar{S})$ is decreasing as $\bar{S} \to +\infty$ so that it should have a finite maximizer over the positive axis. ■

Proposition 9 *Suppose that the demand density is exponential with parameter* λ *and* $s_1 = 0$. *Then,* $NPV_T(\cdot)$ *is increasing at zero provided that*

$$(r_T - c_T + l) \leq \frac{1-\rho^T}{\rho^T(1-\rho)}[B\lambda + b + r - c(1-\rho)]. \tag{7.17}$$

Proof In order for $NPV_T(\cdot)$ to be increasing at zero we need $\lim_{\bar{S}\to 0^+} NPV_T'(\bar{S}) \geq 0$. Thus, it follows from (7.16) that, if

$$\left(\frac{1-\rho^T}{1-\rho}\right)[B\lambda + b + r - c(1-\rho)] - \rho^T(r_T - c_T + c) \geq 0$$

holds then $NPV_T(\cdot)$ is increasing at zero. Rearranging the terms of the above inequality leads to (7.17), and this completes the proof. ■

Using (7.8) and (7.11) the second derivative of $NPV_T(\bar{S})$, provided by (7.13), can be expressed as

$$NPV_T''(\bar{S}) = \left[-\left(\frac{1-\rho^T}{1-\rho}\right)\lambda(B\lambda + h + b + r) + \rho^T\lambda(r_T - c_T + l)\right]\exp(-\lambda\bar{S}). \tag{7.18}$$

In order for $NPV_T(\cdot)$ to be concave, we need $NPV_T''(\bar{S}) \leq 0, \forall \bar{S} \geq 0$. Then, expression (7.18) implies that if

$$-\left(\frac{1-\rho^T}{1-\rho}\right)\lambda(B\lambda + h + b + r) + \rho^T\lambda(r_T - c_T + l) \leq 0,$$

holds then $NPV_T(\cdot)$ is concave. After rearranging its terms, the above requirement can be rewritten as

$$(r_T - c_T + l) \leq \frac{1-\rho^T}{\rho^T(1-\rho)}(B\lambda + h + b + r). \tag{7.19}$$

This in turn implies that, if the model parameters do not satisfy (7.19), i.e., if

$$(r_T - c_T + l) > \frac{1-\rho^T}{\rho^T(1-\rho)}(B\lambda + h + b + r), \tag{7.20}$$

then $NPV_T(\cdot)$ is convex. Observe that if (7.20) holds then (7.17) is violated because the right hand side of (7.17) is smaller than the right hand side of (7.20). Therefore, Proposition 9 and requirement (7.20) imply that if $NPV_T(\cdot)$ is convex then it is decreasing at zero. On the other hand, if function $NPV_T(\cdot)$ is concave, i.e., (7.19) holds, then it may be increasing or decreasing at zero since (7.17) may or may not be satisfied.

In summary, we have the following possibilities and consequences:

- Requirement (7.19) is not satisfied. Therefore, $NPV_T(\cdot)$ is *convex*. Since (7.19) does not hold the model parameters should satisfy (7.20) and they should violate (7.17). It follows that $NPV_T(\cdot)$ is decreasing at zero, and thus Proposition 8 asserts that $NPV_T(\cdot)$ is convex decreasing over $[0, +\infty)$ so that it does not have a stationary point. Under this scenario, the solution of our problem is at zero, i.e., $\bar{S} = 0$.

- Requirement (7.19) is satisfied, and thus $NPV_T(\cdot)$ is *concave*. In this case, if (7.17) does not hold then $NPV_T(\cdot)$ is decreasing at zero. Then, it follows from Proposition 8 that $NPV_T(\cdot)$ is decreasing over $[0, +\infty)$, and the solution of our problem is at zero. Otherwise, i.e., if (7.17) is satisfied so that $NPV_T(\cdot)$ is increasing at zero, the solution of our problem is obtained by solving $NPV_T'(\bar{S}) = 0$.

Next, we show that the requirement (7.19) which assures the concavity of $NPV_T(\cdot)$

is not very restrictive in general, and then we provide a more formal presentation of our observations summarized above.

Let us first take a moment to analyze

$$\nu(\rho, T) \equiv \frac{1-\rho^T}{\rho^T(1-\rho)}$$

which appears on the right hand side of inequality (7.19). For any given value of $\rho \in (0,1)$, the term $\nu(\rho, T)$ is a concave increasing function of T since

$$\begin{aligned} \frac{\partial\nu(\rho,T)}{\partial T} &= -\frac{\ln\rho}{\rho^T(1-\rho)} > 0, \\ \frac{\partial^2\nu(\rho,T)}{\partial T^2} &= -\frac{(\ln\rho)^2}{\rho^T(1-\rho)} < 0. \end{aligned}$$

For the multi-period problem under consideration we have $2 \leq T < +\infty$. It follows that, for any given ρ, we have

$$\nu(\rho,T) \geq \nu(\rho,2) = \frac{1-\rho^2}{\rho^2(1-\rho)}, \quad \forall T \geq 2.$$

On the other hand, $\nu(\rho, 2)$ is a convex decreasing function of ρ since

$$\begin{aligned} \nu'(\rho,2) &= -\frac{\rho+2}{\rho^3} < 0, \\ \nu''(\rho,2) &= \frac{2(\rho+3)}{\rho^4} > 0. \end{aligned}$$

Combining our results, we conclude that a lower bound for $\nu(\rho, T)$ is provided by $\lim_{\rho\to1^-} \nu(\rho, 2)$, i.e.,

$$\nu(\rho,T) \geq \lim_{\rho\to1^-} \nu(\rho,2) = 2 \;\; \forall\, T \geq 2. \tag{7.21}$$

If we use (7.21) in the concavity requirement for $NPV_T(\cdot)$, given by (7.19), then we can write

$$(r_T - c_T + l) \leq 2(B\lambda + h + b + r). \tag{7.22}$$

Note that we naturally have $l \leq c_T$, because otherwise, i.e., when the wholesale price c_T at the end of T is less than the salvage value l, the retailer cannot expect to dispose of the excess inventory immediately at the time that she/he stops selling the product. Under this scenario, requirement (7.22) is less restrictive than

$$r_T \leq 2r + 2(B\lambda + h + b).$$

which requires that the end of horizon unit revenue r_T to be less than two times r plus a positive constant. Then, (7.22) should not impose a constraint unless r_T is unreasonably large. Consequently, because the requirement (7.19) is itself less restrictive than (7.22), it should not be constraining either for a large set of reasonable parameter values. Thus, $NPV_T(\cdot)$ is concave for most practical problems.

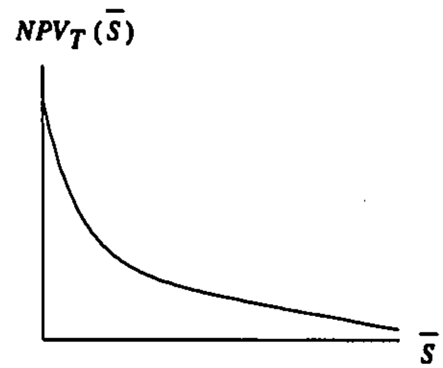

Figure 7.2: An illustration: Exponential demand, $s_1 = 0$, and $NPV_T(\cdot)$ is convex.

Let $\bar{S}_{best}$ denote the optimal solution of our problem which is stated by (7.2) and let $\bar{S}_{max}$ represent the solution of $NPV_T'(\bar{S}) = 0$. The following discussion formalizes the results that we have obtained so far.

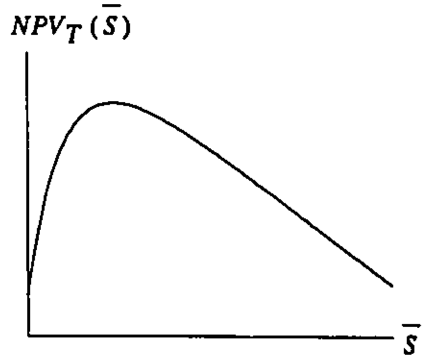

Figure 7.3: Another illustration: Exponential demand, $s_1 = 0$, and $NPV_T(\cdot)$ is concave.

Corollary 1 *Under the assumption that demand is exponential and $s_1 = 0$, if requirement (7.19) does not hold, i.e., if r_T is unreasonably large, then*

$$\bar{S}_{best} = 0. \tag{7.23}$$

Proof If (7.19) is not satisfied, i.e., r_T is unreasonably large, then $NPV_T(\bar{S})$ is convex. In this case, expression (7.15) implies that $NPV_T(\bar{S})$ is decreasing as $\bar{S} \to +\infty$. If it is convex and decreasing as $\bar{S} \to +\infty$, it should be decreasing over $[0, +\infty)$. Under these conditions, over the feasible space of our problem, function $NPV_T(\cdot)$ is maximized at zero, and this completes the proof. In other words, for the case of exponential demand with parameter λ, if $s_1 = 0$ and r_T is extremely large compared to r then the best myopic policy offers to backorder in all periods. ■

Proposition 10 *Under the assumption that demand is exponential and $s_1 = 0$, if inequality (7.17) holds then*

$$\bar{S}_{best} = \frac{1}{\lambda} \ln \left[\frac{(B\lambda + h + b + r)(1 - \rho^T) - (r_T - c_T + l)(1 - \rho)}{h(1 - \rho^T) + (c - \rho^T l)(1 - \rho)} \right]. \tag{7.24}$$

Proof Recall that if (7.17) holds, then $NPV_T(\cdot)$ is increasing at zero. Since the concavity requirement (7.19) is weaker than (7.17), $NPV_T(\cdot)$ is also concave. We already know from (7.15) that $NPV_T(\cdot)$ is decreasing at zero. Then, under the assumptions of this proposition, the function $NPV_T(\cdot)$ is not only concave but also decreasing at zero and infinity. For this reason, the critical number of the best myopic policy is given by $\bar{S}_{max}$ which solves $NPV_T'(\bar{S}) = 0$. Equating the right hand side of expression (7.14) to zero and solving leads to (7.24). ∎

Corollary 2 *Under the assumption that demand is exponential and* $s_1 = 0$*, if the revenue and procurement cost parameters are stationary, i.e.,* $c_T = c$ *and* $r_T = r$*, then the critical number of the best myopic policy is given by*

$$\bar{S}_{best} = \frac{1}{\lambda} \ln \left[\frac{(B\lambda + h + b + r)(1-\rho^T) - (r - c + l)(1-\rho)}{h(1-\rho^T) + (c - \rho^T l)(1-\rho)} \right]. \tag{7.25}$$

Proof If $r_T = r$ and $c_T = c$ then (7.22) holds immediately. This in turn implies that the concavity requirement (7.19) is satisfied, so that the critical number of the best myopic policy is given by (7.24). If we set $r_T = r$ and $c_T = c$ in (7.24) then we obtain (7.25). ∎

Theorem 4 *If demand density is exponential and* $s_1 = 0$ *then the critical number of the best myopic policy is specified by*

$$\bar{S}_{best} = \begin{cases} \bar{S}_{max}, & \text{if } (r_T - c_T + l) \leq \frac{1-\rho^T}{\rho^T(1-\rho)}[B\lambda + b + r - c(1-\rho)], \\ 0, & \text{otherwise.} \end{cases} \tag{7.26}$$

where

$$\bar{S}_{max} = \frac{1}{\lambda} \ln \left[\frac{(B\lambda + h + b + r)(1-\rho^T) - (r_T - c_T + l)(1-\rho)}{h(1-\rho^T) + (c - \rho^T l)(1-\rho)} \right]. \tag{7.27}$$

Proof The proof follows from Propositions 8, 9, 10 and Corollary 1. ∎

7.1.2 Erlang Demand

In this section we suppose that the demand density is Erlang with parameters λ and $k \geq 2$. The demand density and distribution functions are provided by

$$f(x) = \gamma(x, \lambda, k) = \frac{\lambda(\lambda x)^{k-1} \exp(-\lambda x)}{(k-1)!}, \quad x > 0, \tag{7.28}$$

and

$$F(x) = \Gamma(x, \lambda, k) = 1 - \exp(-\lambda x) \sum_{j=0}^{k-1} \frac{(\lambda x)^j}{j!}, \quad x > 0, \tag{7.29}$$

respectively. We also have

$$\mu = k/\lambda \tag{7.30}$$

as the mean demand. As in the case of exponential demand, we again need the explicit expressions of $G(\cdot)$ and $Y(\cdot)$ in order to find the solution of our problem which is stated by (7.2). Under the assumption of Erlang demand, we have already discussed the properties of the single period return net of costs function $G(\cdot)$ in Chapter 4 where we also provided explicit expressions for $G(\cdot)$, $G'(\cdot)$ and $G''(\cdot)$. Throughout this section we make extensive use of our results presented in Chapter 4. Now, we obtain an explicit expression for the end of horizon costs function, $Y(\cdot)$, for the case of Erlang demand. For this purpose, let us recall expression (6.10) which appears on page 95. If we use expressions (7.28), (7.29), (7.30), and the result stated by (4.10) which suggests that

$$\int_a^\infty x\gamma(x, \lambda, k)dx = \frac{k}{\lambda} \exp(-\lambda a) \sum_{j=0}^{k} \frac{(\lambda a)^j}{j!},$$

in (6.10), then the end of horizon costs function can be expressed as

$$Y(\bar{S}) = (c - l)\left(\frac{k}{\lambda} - \bar{S}\right) + (r_T - c_T + l)\frac{k}{\lambda} \exp(-\lambda \bar{S}) \sum_{j=0}^{k} \frac{(\lambda \bar{S})^j}{j!}$$

$$-(r_T - c_T + l)\bar{S}\exp(-\lambda\bar{S})\sum_{j=0}^{k-1}\frac{(\lambda\bar{S})^j}{j!}.$$

After applying some algebraic manipulations, the above expression for $Y(\bar{S})$ simplifies to

$$\begin{aligned} Y(\bar{S}) &= (c-l)\left(\frac{k}{\lambda}-\bar{S}\right)+(r_T-c_T+l)\left(\frac{k}{\lambda}-\bar{S}\right)\exp\left(-\lambda\bar{S}\right)\sum_{j=0}^{k}\frac{(\lambda\bar{S})^j}{j!} \\ &+(r_T-c_T+l)\bar{S}\exp(-\lambda\bar{S})\frac{(\lambda\bar{S})^k}{k!}. \end{aligned} \tag{7.31}$$

Using (7.29) instead of $F(\cdot)$ in the first derivative of $Y(\cdot)$, given by (6.12) on page 96, yields

$$Y'(\bar{S}) = (r_T - c_T + l) - (r_T - c_T + c)\exp\left(-\lambda\bar{S}\right)\sum_{j=0}^{k-1}\frac{(\lambda\bar{S})^j}{j!} - (r_T - c_T + c),$$

and after cancellations among the terms, the above expression reduces to

$$Y'(\bar{S}) = -(c-l) - (r_T - c_T + l)\exp\left(-\lambda\bar{S}\right)\sum_{j=0}^{k-1}\frac{(\lambda\bar{S})^j}{j!}. \tag{7.32}$$

In a similar fashion, if we substitute (7.28) instead of $f(\cdot)$ in the second derivative of $Y(\cdot)$, given by (6.15) on page 97, then we have

$$Y''(\bar{S}) = (r_T - c_T + l)\frac{\lambda(\lambda\bar{S})^{k-1}\exp\left(-\lambda\bar{S}\right)}{(k-1)!},$$

and this expression can be rewritten as

$$Y''(\bar{S}) = \frac{\lambda^k}{(k-1)!}[(r_T - c_T + l)\bar{S}]\bar{S}^{k-2}\exp\left(-\lambda\bar{S}\right). \tag{7.33}$$

Since we are searching for the maximizer of $NPV_T(\cdot)$, we are essentially interested in the properties of $NPV_T(\cdot)$. However, as we mentioned earlier, (7.1) asserts that $NPV_T(\cdot)$ is a combination of single period return net of costs function

$G(\cdot)$ and the end of horizon costs function $Y(\cdot)$, and in fact this is the sole reason for dealing with some of the properties of these two functions. For example, assuming Erlang demand, if we could guarantee that both $G(\cdot)$ and $Y(\cdot)$ are concave then the solution of our problem would be easier. Unfortunately, this not the case, i.e., $Y(\cdot)$ may be convex or concave, depending on the sign and magnitude of l, whereas $G(\cdot)$ is neither convex nor concave in general. [For the case of Erlang demand the properties of $G(\cdot)$ are summarized by Theorem 2 on page 64.] Figure 7.4 illustrates how the functions $G(\cdot)$ and $Y(\cdot)$ may look like under the assumption that demand is Erlang. Figure 4.1, on page 63, also illustrates the shape of $G(\cdot)$ for Erlang demand.

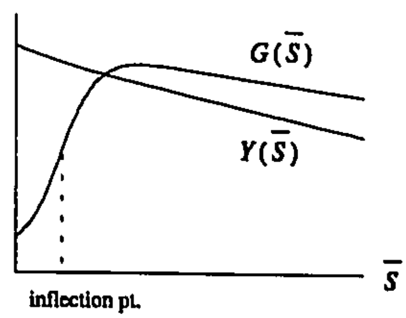

Figure 7.4: $G(\cdot)$ and $Y(\cdot)$ for Erlang demand.

Next we proceed with studying the properties of $NPV_T(\cdot)$ which finally leads us to the solution of our problem stated by (7.2). Expression (7.12) implies that

$$\lim_{\bar{S}\to+\infty} NPV_T'(\bar{S}) = \left(\frac{1-\rho^T}{1-\rho}\right) \lim_{\bar{S}\to+\infty} G'(\bar{S}) + \rho^T \lim_{\bar{S}\to+\infty} Y'(\bar{S}). \qquad (7.34)$$

For the case of Erlang demand, the first derivative of $G(\cdot)$ is given by expression

(4.13) which states

$$\begin{aligned} G'(\bar{S}) &= B\lambda \exp(-\lambda \bar{S}) \frac{(\lambda \bar{S})^{k-1}}{(k-1)!} \\ &\quad + (h+b+r)\exp(-\lambda \bar{S}) \sum_{j=0}^{k-1} \frac{(\lambda \bar{S})^j}{j!} - [c(1-\rho)+h]. \end{aligned} \tag{7.35}$$

Taking the limit of the above expression as $\bar{S} \to +\infty$ leads to

$$\lim_{\bar{S} \to +\infty} G'(\bar{S}) = -[c(1-\rho)+h] < 0$$

Similarly, if we compute the limit of (7.32) as $\bar{S} \to +\infty$ then the result is

$$\lim_{\bar{S} \to +\infty} Y'(\bar{S}) = -(c-l) < 0.$$

Substituting these limits of $G'(\bar{S})$ and $Y'(\bar{S})$ in (7.34) we can write

$$\lim_{\bar{S} \to +\infty} NPV_T'(\bar{S}) = -\left(\frac{1-\rho^T}{1-\rho}\right)[c(1-\rho)+h] - \rho^T(c-l) < 0. \tag{7.36}$$

Since the right hand side of (7.36) is negative, function $NPV_T(\bar{S})$ is strictly decreasing as $\bar{S} \to +\infty$. Thus, $NPV_T(\cdot)$ should have a finite maximizer over the positive axis, and we have the following proposition.

Proposition 11 *If the demand density is Erlang with parameters λ and k, and $s_1 = 0$ then $NPV_T(\cdot)$ has a finite maximizer over the positive axis.*

The first derivatives $G'(\cdot)$ and $Y'(\cdot)$ given by expressions (7.35) and (7.32) also imply that

$$\begin{aligned} \lim_{\bar{S} \to 0^+} G'(\cdot) &= b + r - c(1-\rho) > 0, \\ \lim_{\bar{S} \to 0^+} Y'(\cdot) &= -(r_T - c_T + c) < 0. \end{aligned}$$

Utilizing the above two expressions in

$$\lim_{\bar{S}\to 0^+} NPV'_T(\bar{S}) = \left(\frac{1-\rho^T}{1-\rho}\right) \lim_{\bar{S}\to 0^+} G'(\bar{S}) + \rho^T \lim_{\bar{S}\to 0^+} Y'(\bar{S})$$

yields

$$\lim_{\bar{S}\to 0^+} NPV'_T(\bar{S}) = \left(\frac{1-\rho^T}{1-\rho}\right)[b+r-c(1-\rho)] - (r_T - c_T + c)\rho^T. \tag{7.37}$$

The following proposition can now be easily proved.

Proposition 12 *If demand density is Erlang with parameters λ and k, and $s_1 = 0$ then $NPV_T(\cdot)$ is decreasing at zero if*

$$(r_T - c_T + c) \geq \frac{1-\rho^T}{\rho^T(1-\rho)}[b+r-(1-\rho)c], \tag{7.38}$$

It follows that $NPV_T(\cdot)$ is increasing at zero provided that

$$(r_T - c_T + c) < \frac{1-\rho^T}{\rho^T(1-\rho)}[b+r-(1-\rho)c]. \tag{7.39}$$

Proof If the model parameters satisfy inequality (7.38) then the right hand side of (7.37) is negative. It follows that, under inequality (7.38) function $NPV_T(\cdot)$ is decreasing (nonincreasing) at zero. Otherwise, [i.e., the model parameters satisfy (7.39)] the right hand side of (7.37) is positive, and thus $NPV_T(\cdot)$ is increasing at zero. ■

Let us recall (4.14), on page 61, which provides the expression of the second derivative of the single period return net of costs function, $G(\cdot)$, for Erlang demand, and states that

$$G''(\bar{S}) = \frac{\lambda^k}{(k-1)!}[(k-1)B - (B\lambda + h + b + r)\bar{S}]\bar{S}^{k-2}\exp(-\lambda\bar{S}).$$

Let us also recall expression (7.33) which gives the second derivative of the end of horizon costs function, $Y(\cdot)$, for Erlang demand, and suggests that

$$Y''(\bar{S}) = \frac{\lambda^k}{(k-1)!}[(r_T - c_T + l)\bar{S}]\bar{S}^{k-2}\exp(-\lambda\bar{S}).$$

If we insert the above two expressions in (7.13) then we have

$$\begin{aligned} NPV_T''(\bar{S}) \;=\; & \frac{\lambda^k}{(k-1)!}\left(\frac{1-\rho^T}{1-\rho}\right)(k-1)B\bar{S}^{k-2}\exp(-\lambda\bar{S}) \\ & -\frac{\lambda^k}{(k-1)!}\left(\frac{1-\rho^T}{1-\rho}\right)(B\lambda + h + b + r)\bar{S}^{k-1}\exp(-\lambda\bar{S}) \\ & +\frac{\lambda^k}{(k-1)!}\rho^T(r_T - c_T + c)\bar{S}^{k-1}\exp(-\lambda\bar{S}), \end{aligned} \tag{7.40}$$

for the case of Erlang demand. Observe that, according to (7.40), if

$$(r_T - c_T + l) > \frac{1-\rho^T}{\rho^T(1-\rho)}(B\lambda + h + b + r), \tag{7.41}$$

then

$$NPV_T''(\bar{S}) \geq 0, \;\; \forall\, \bar{S} \geq 0.$$

That is, if (7.41) holds [or equivalently r_T is unreasonably large so that (7.19) is *not* satisfied] then $NPV_T(\cdot)$ is convex over $[0, +\infty)$, and we have Proposition 13 that we discuss below. On the other hand, if (7.41) does not hold [or equivalently (7.19) is satisfied] then we have Lemma 1 which we present later.

Proposition 13 *Under the assumption that demand is Erlang and $s_1 = 0$, if the model parameters satisfy (7.41) then $NPV_T(\cdot)$ is convex over $[0, +\infty)$ in which case $\bar{S}_{best} = 0$.*

Proof As we have already explained, under the assumptions given by this proposition, $NPV_T(\cdot)$ is convex over $[0, +\infty)$. We also have expression (7.36) which implies that $NPV_T(\cdot)$ is decreasing as $\bar{S} \to +\infty$. If a convex function is decreasing at $+\infty$ then it should be decreasing (nonincreasing) elsewhere. This case is illustrated by Figure 7.5. ■

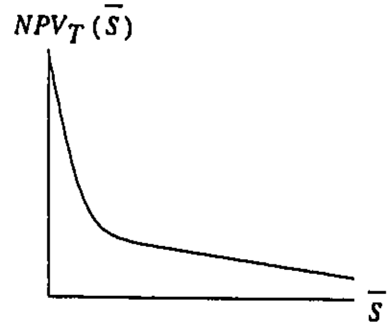

Figure 7.5: An illustration for Proposition 13.

Now, let us consider the other possibility and suppose that (7.41) is violated [and thus (7.19) holds] so that the model parameters satisfy

$$(r_T - c_T + l) \leq \frac{1-\rho^T}{\rho^T(1-\rho)}(B\lambda + h + b + r). \tag{7.42}$$

In this case, $NPV_T(\cdot)$ has an inflection point at which its second derivative changes its sign, and therefore $NPV_T(\cdot)$ is neither convex nor concave as we argue in Lemma 1 next.

Lemma 1 *Under the assumption that demand is Erlang and $s_1 = 0$, if the model parameters satisfy (7.42) then $NPV_T(\cdot)$ is convex over*

$$\left[0, \bar{S}_{inflect}\right], \tag{7.43}$$

and strictly concave over

$$\left(\bar{S}_{inflect}, +\infty\right), \tag{7.44}$$

where

$$\bar{S}_{inflect} = \frac{(1-\rho^T)(k-1)B}{(1-\rho^T)(B\lambda+h+b+r)-\rho^T(1-\rho)(r_T-c_T+l)}. \tag{7.45}$$

Proof According to the assumption of this lemma on the model parameters, we have (7.42). Using (7.40) and solving

$$NPV_T''(\bar{S}) = 0$$

for $\bar{S}$, under the assumption that (7.42) holds, we observe that $NPV_T''(\cdot)$ changes its sign at 0 and at the point

$$\frac{(1-\rho^T)(k-1)B}{(1-\rho^T)(B\lambda+h+b+r)-\rho^T(1-\rho)(r_T-c_T+l)}$$

which we denote by $\bar{S}_{inflect}$. We also observe that, if $\bar{S} \leq \bar{S}_{inflect}$ then $NPV_T''(\bar{S}) \geq 0$. Thus, $NPV_T(\cdot)$ is convex over (7.43). Similarly, if $\bar{S} > \bar{S}_{inflect}$ then $NPV_T''(\cdot) < 0$, so that $NPV_T(\cdot)$ is strictly concave over (7.44). ∎

Remark 3 *Let us recall that $c \geq l$, $0 < \rho < 1$, and except for the salvage value l all the model parameters are nonnegative. For these reasons we have*

$$\begin{aligned} r_T - c_T + l &\leq r_T - c_T + c \\ b + r - (1-\rho)c &< B\lambda + h + b + r. \end{aligned}$$

The above relations imply that if condition (7.41) of Proposition 13 holds so that $NPV_T(\cdot)$ is convex over the region $(0, +\infty]$ then (7.39) is never satisfied, i.e., $NPV_T(\cdot)$ cannot be increasing at zero.

Remark 4 *If function $NPV_T(\cdot)$ is increasing at zero then it should be increasing over (7.43) because zero is the left-most point of region (7.43) over which $NPV_T(\cdot)$ is convex. That is, if (7.39) holds so that $NPV_T(\cdot)$ is increasing at zero then it should be increasing over (7.43), and thus the maximizer of $NPV_T(\cdot)$ lies over (7.44) for which $NPV_T(\cdot)$ is strictly concave. Let $\bar{S}_{max}$ denote the stationary point of $NPV_T(\cdot)$ over (7.44). Then, if (7.39) holds then $\bar{S}_{max} > \bar{S}_{inflect}$ is the unique solution of $NPV_T''(\bar{S}) = 0$.*

Before going any further, let us summarize the results derived so far that will be used in the following development.

- Proposition 11 asserts that our problem, given by (7.2), has a finite solution.
- Proposition 12 provides conditions that can be used to check if $NPV_T(\cdot)$ is decreasing or increasing at zero.
- Remark 4 explains that if $NPV_T(\cdot)$ is increasing at zero then $\bar{S}_{best} = \bar{S}_{max}$.
- We have two possible scenarios regarding the values of the model parameters.
  - Under the first scenario the model parameters satisfy (7.41) so that we have Proposition 13 which states $NPV_T(\cdot)$ is a convex decreasing function, and $\bar{S}_{best} = 0$.
  - The other scenario requires the model parameters to satisfy (7.42) under which $NPV_T(\cdot)$ is convex over (7.43) and concave over (7.44). If this is the case then we need to use Proposition 12 in order to find the solution of our problem. In fact, the proof of the concluding theorem of this chapter that we discuss below is based on this observation.

Theorem 5 *Under the assumption that demand density is Erlang with parameters λ and k, and $s_1 = 0$, only one of the following three cases is true:*

- **Case 1.** *If the model parameters satisfy*
$$(r_T - c_T + c) < \frac{1-\rho^T}{\rho^T(1-\rho)}[b + r - (1-\rho)c],$$
then $\bar{S}_{best} = \bar{S}_{max}$.

- **Case 2.** *If the model parameters satisfy*
$$\frac{1-\rho^T}{\rho^T(1-\rho)}(B\lambda + h + b + r) \geq (r_T - c_T + c) \geq \frac{1-\rho^T}{\rho^T(1-\rho)}[b + r - (1-\rho)c],$$
and $NPV_T(\bar{S}_{max}) \geq NPV_T(0)$ then $\bar{S}_{best} = \bar{S}_{max}$.

- **Case 3.** *If neither Case 1 nor Case 2 then $\bar{S}_{best} = 0$.*

Proof

- **Case 1.** In this case, (7.39) is satisfied, and thus the result follows from Remark 4. Figure 7.6 provides an illustration for this case.

- **Case 2.** Note that both (7.42) and (7.38) are satisfied in this case. Then Proposition 12 and Lemma 1 imply that we have three possibilities which are illustrated by Figures 7.7, 7.8, 7.1.2. The maximum of $NPV_T(\cdot)$ is either at zero or at $\bar{S}_{max}$. Consequently, provided that $NPV_T(\bar{S}_{max}) \geq NPV_T(0)$ we have $\bar{S}_{best} = \bar{S}_{max}$.

- **Case 3.** Directly follows from Proposition 13 and the proof of the Case 2 that we discussed above. This case is illustrated by Figure 7.5 for which $NPV_T(\cdot)$ is convex, and Figures 7.8 and 7.1.2 for which $NPV_T(\cdot)$ is neither convex nor concave. ■

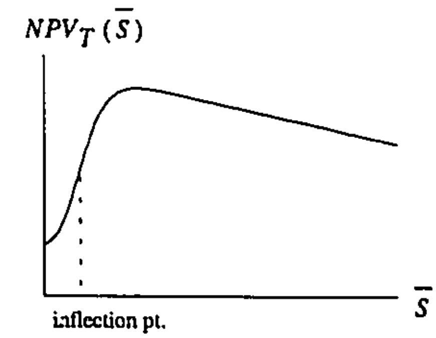

Figure 7.6: An illustration for Theorem 5, Case 1.

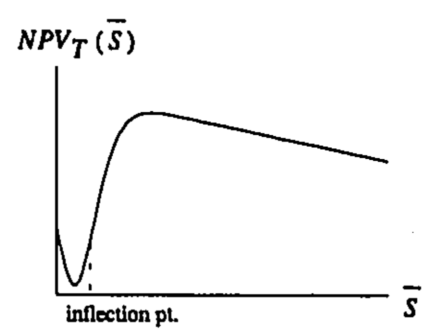

Figure 7.7: An illustration for Theorem 5, Case 2.

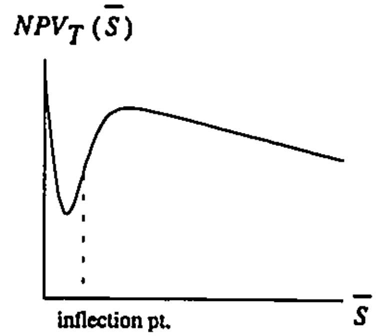

Figure 7.8: An illustration for Theorem 5, Cases 2 and 3.

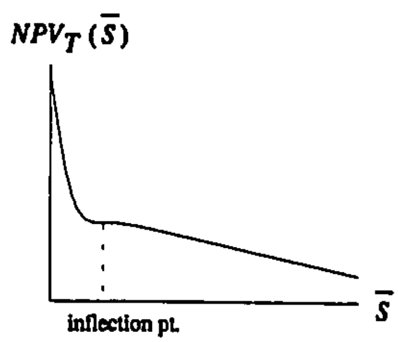

Figure 7.9: Another illustration for Theorem 5, Cases 2 and 3.

Table 7.1: Base parameter values for $\bar{S}_{best}$ computations.

k	λ	ρ	r	c	h	b	B	r_T	c_T	l	T
5	.2	.99	38	20	.5	30	50	30	25	4	15

7.1.3 Numerical Illustrations

In this section we present some numerical illustrations regarding the computation of the best myopic policy under the assumption that there is no initial inventory. Using the analytical results presented in the previous sections and summarized by Theorems 4 and 5, computing the best myopic policy for exponential and Erlang demand densities is quite simple. Theorem 5 suggests that (for the case of Erlang demand) in the most complicated case $\bar{S}_{best} = \bar{S}_{max}$ where $\bar{S}_{max} > \bar{S}_{inflect}$ as we explained in Remark 4. Therefore, the most complicated problem with Erlang demand requires searching for the unique solution of $NPV_T'(\bar{S}) = 0$ over $(\bar{S}_{inflect}, +\infty)$ for which $NPV_T(\cdot)$ is strictly concave (a result due to Lemma 1). For computational purposes, we coded the well-known Newton-Raphson method, and computed $\bar{S}_{max}$ for a number of numerical examples assuming exponential and Erlang demands. The basic parameter values are as indicated in Table 7.1, and unless otherwise stated computations are performed using these values. It is worth noting that the parameter values given in Table 7.1 satisfy condition (7.19) [which is equivalent to (7.42)] and requirement (7.39) so that Theorems 4 and 5 suggest that $\bar{S}_{best} = \bar{S}_{max} > 0$ for the examples presented below.

In Table 7.2 we tabulate the value of $\bar{S}_{best}$, i.e., the critical number of the best myopic policy, for different horizon lengths. For $T = \infty$ the best myopic policy

Table 7.2: The convergence of $\bar{S}_{best}$ to S as T increases.

S_{best} Values								
k	μ	$T = 5$	$T = 10$	$T = 15$	$T = 20$	$T = 25$	$T = 30$	$T = \infty$
1	5	15	18	19	20	21	21	24
2	10	24	27	29	30	30	31	34
3	15	31	35	37	38	39	39	43
4	20	38	43	45	46	47	47	51
5	25	45	50	52	53	54	55	59
6	30	52	57	59	61	61	62	66
7	35	59	64	66	68	69	69	73
8	40	65	70	73	74	75	76	81
9	45	72	77	80	81	82	83	88
10	50	78	84	86	88	89	90	94

is of course the optimal infinite horizon policy which is myopic itself as we discussed in complete detail in Chapters 3 and 4. That is, if $T = \infty$ then $\bar{S}_{best} = S$. Recall that, for the case of exponential demand, i.e., if $k = 1$, the value of S is computed using the formula (4.2) on page 55 of Chapter 4. On the other hand, for the case of Erlang demand, i.e., if $k \geq 2$, S is the unique positive solution of (4.17), a result which is due to Theorem 2 on page 64 of Chapter 4. As the horizon length increases the value of $\bar{S}_{best}$ also increases, and converges to S. For example, the fourth row of Table 7.2 indicates that if $k = 2$ then $\mu = 10$ (since $\mu = k/\lambda$, and the base value of λ is taken as .2), and as the horizon length T increases from 5 periods to 30 periods the critical number of the best myopic policy increases from 24 to 31 whereas the infinite horizon policy suggests a base-stock level of 34 units.

In Tables 7.3, 7.4, and 7.5 we provide a comparison of the resulting revenue net of costs using the best myopic policy instead of the infinite horizon policy.

Table 7.3: Comparison of the resulting revenues net of costs for a problem with 10 periods.

$T = 10$						
k	μ	S_{best}	$NPV_T(S_{best})$	S	$NPV_T(S)$	% Increase
1	5	18	484	24	434	11.58
2	10	27	1239	34	1181	4.95
3	15	35	2022	43	1957	3.31
4	20	43	2817	51	2747	2.55
5	25	50	3621	59	3546	2.11
6	30	57	4430	66	4351	1.81
7	35	64	5244	73	5161	1.61
8	40	70	6061	81	5974	1.45
9	45	77	6880	88	6790	1.32
10	50	84	7702	94	7609	1.22

In these tables, $NPV_T(\bar{S}_{best})$ represents the expected revenue net of costs if the best myopic policy with critical number $\bar{S}_{best}$ is employed. Recall that, since we assume $s_1 = 0$, function $NPV_T(\cdot)$ is given by (7.1), and $NPV_T(\bar{S}_{best})$ is computed by inserting the value $\bar{S}_{best}$ in (7.1). On the other hand, $NPV_T(S)$ represents the resulting expected revenue net of costs if the infinite horizon myopic policy with critical number S is employed for a problem with $T < \infty$ periods.

As we argued in Section 2.8 of Chapter 2 the traditional approach to avoid computational difficulty associated with computing the exact policy for the finite horizon problem is to compute the suboptimal steady-state policy, i.e., the infinite horizon myopic policy for our problem. In fact, for our problem this approach is equivalent to assuming Veinott's terminal condition as we explained in Sections 6.4 and 6.6 of Chapter 6. We also indicated that, from a computational point of view, there is not much work done in terms of providing a method that overcomes the

Table 7.4: Comparison of the resulting revenues net of costs for a problem with 15 periods.

$T = 15$						
k	μ	$\bar{S}_{best}$	$NPV_T(\bar{S}_{best})$	S	$NPV_T(S)$	% Increase
1	5	19	808	24	773	4.53
2	10	29	1945	34	1904	2.13
3	15	37	3114	43	3069	1.46
4	20	45	4297	51	4249	1.14
5	25	52	5491	59	5439	0.95
6	30	59	6690	66	6636	0.82
7	35	66	7895	73	7838	0.73
8	40	73	9104	81	9044	0.66
9	45	80	10315	88	10253	0.61
10	50	86	11530	94	11465	0.56

computational burden associated with the finite horizon solution, but which is also superior to the traditional approaches, and we suggested that the best myopic policy be computed. Using the results presented in Tables 7.3, 7.4, and 7.5, we observe that the best myopic policy is always superior to the infinite horizon myopic policy since $NPV_T(\bar{S}_{best}) > NPV_T(S)$ in all cases. Each of these tables includes a column in which we display the percentage increase of $NPV_T(\cdot)$ by using the best myopic policy instead of the infinite horizon myopic policy.

It is not surprising to see that the expected revenue net of costs, $NPV_T(\cdot)$, can always be improved by using the best myopic policy instead of the infinite horizon myopic policy, because $\bar{S}_{best}$ is computed by optimizing $NPV_T(\cdot)$ over *the set of all possible myopic policies.* However, for any given T, the percentage increase declines as the value of k increases. This result is due to the fact that as the value of the shape parameter k gets larger both the mean μ and the variance σ^2 of the Erlang random

Table 7.5: Comparison of the resulting revenues net of costs for a problem with 20 periods.

$T = 20$						
k	μ	S_{best}	$NPV_T(S_{best})$	S	$NPV_T(S)$	% Increase
1	5	20	1121	24	1095	2.39
2	10	30	2623	34	2592	1.17
3	15	38	4159	43	4126	0.81
4	20	46	5713	51	5677	0.64
5	25	53	7277	59	7239	0.53
6	30	61	8849	66	8808	0.46
7	35	68	10426	73	10383	0.41
8	40	74	12007	81	11963	0.37
9	45	81	13593	88	13546	0.34

variable becomes larger since $\mu = k/\lambda$ and $\sigma^2 = k/\lambda^2$. In this case, the percentage improvement by using the best myopic policy gets smaller. One conclusion that may be drawn is that one should use the optimal policy for high demand items whose demand fluctuates dramatically.

The infinite horizon solution represents the limiting behavior associated with the finite horizon solution. Therefore, as the horizon length increases the value of $\bar{S}_{best}$ approaches S, and in turn the percentage improvement from using the best myopic policy decreases. For example, Tables 7.3, 7.4, and 7.5 indicate that when $k = 8$, giving $\mu = 40$, the percentage increase of expected revenue net of costs is 1.45% for $T = 10$ and 0.66% for $T = 15$ whereas it goes down to 0.37% for $T = 20$. The percentage improvements displayed in this section may not look substantial to the reader since most of the reported savings vary between 0.5% to 5%. Nonetheless, these savings may add up to substantial amounts if the inventory control of many independent items is of interest as in the case of retail businesses. As a matter of fact, the burden of computing the exact optimal policy arises when the recursive calculations must be performed for many items. Thus, our approach not only provides a simple method of computation but also leads to potential savings for many practical problems.

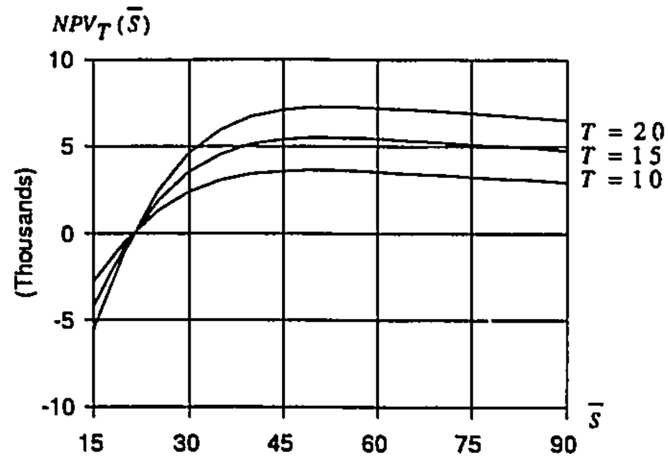

Figure 7.10: Plots of $NPV_T(\cdot)$ for different horizon lengths.

Figure 7.10 provides the plots of $NPV_T(\cdot)$ for $T = 10$, $T = 15$, and $T = 20$. According to this figure functions $NPV_{10}(\cdot)$, $NPV_{15}(\cdot)$, and $NPV_{20}(\cdot)$ look quite flat around their maximizers. However, Figure 7.10 may be misleading because of the scaling that is used, and as an illustration we also provide the individual plot for $NPV_{10}(\cdot)$ by Figure 7.11 which shows that function $NPV_{10}(\cdot)$ reaches its maximum around 50 whereas $NPV_{10}(50) = 3621$ as indicated in Table 7.3.

Varying the values of k and λ we obtain the results presented in Table 7.6. These results indicate that for constant k, when the value of λ increases, and in turn the value of μ decreases, the percentage improvement remains almost constant. That

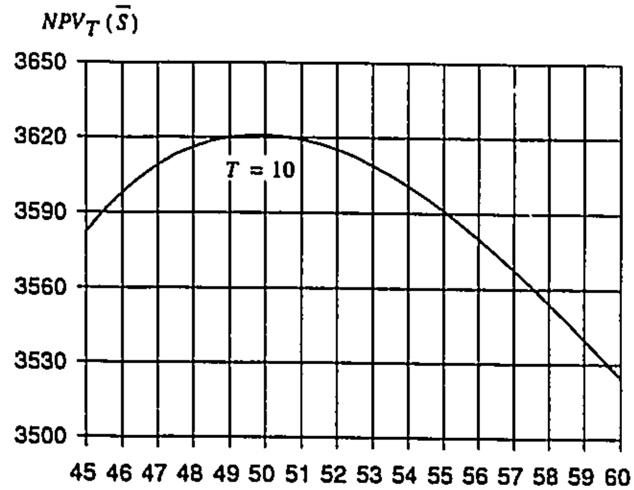

Figure 7.11: $NPV_{10}(\cdot)$ is not so flat.

is, the savings are not sensitive to the value of λ, and thus the mean demand μ, for our examples.

Figure 7.12 displays the plots of $NPV_T(\cdot)$ for different values of λ when $k = 5$. If the scale parameter k is constant then the mean demand $\mu = k/\lambda$ gets smaller while the shape parameter λ increases. Consequently, as illustrated by Figure 7.12 the value of $\bar{S}_{best}$ as well the value of $NPV_T(\bar{S}_{best})$ decreases as the shape parameter λ becomes larger.

We also have Figure 7.13 which illustrates the sensitivity of $\bar{S}_{best}$ to the value of ρ. Table 7.7 tabulates our calculations, and indicates that for smaller values of ρ the value of $\bar{S}_{best}$ is close to S. However, this observation is not surprising because as we explained on page 95 objective functions $E[NPV_T]$ of $\mathcal{P}_T$, and $E[NPV]$ of $\mathcal{P}_\infty$ are related. Function $E[NPV_T]$ involves the additional term $\rho^T E[Y(a_T)]$ which vanishes as $\rho \to 0$. Thus, as $\rho \to 0$ we have $E[NPV_T] \to E[NPV]$ so that the exact

Table 7.6: Sensitivity of savings to λ.

$k = 3$						
λ	μ	S_{best}	$NPV_T(S_{best})$	S	$NPV_T(S)$	% Increase
0.025	120	290	25023	336	24663	1.460
0.05	60	145	12503	168	12323	1.461
0.1	30	73	6243	84	6153	1.462
0.2	15	37	3114	43	3069	1.464
0.4	7.5	19	1549	22	1527	1.469
0.8	3.75	10	768	11	757	1.475
$k = 5$						
λ	μ	S_{best}	$NPV_T(S_{best})$	S	$NPV_T(S)$	% Increase
0.025	200	411	44040	464	43625	0.951
0.05	100	206	22011	232	21804	0.951
0.1	50	103	10997	116	10894	0.951
0.2	25	52	5491	59	5439	0.952
0.4	12.5	26	2738	30	2712	0.952
0.8	6.25	13	1362	15	1349	0.953
$k = 7$						
λ	μ	S_{best}	$NPV_T(S_{best})$	S	$NPV_T(S)$	% Increase
0.025	280	524	63276	582	62817	0.731
0.05	140	262	31630	291	31400	0.731
0.1	70	131	15806	146	15692	0.731
0.2	35	66	7895	73	7838	0.731
0.4	17.5	33	3940	37	3911	0.731
0.8	8.750	17	1963	19	1949	0.731

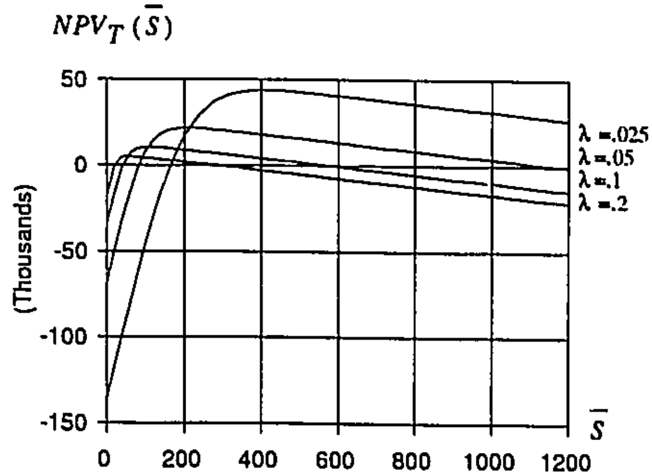

Figure 7.12: Sensitivity of $\bar{S}_{best}$ and $NPV_T(\bar{S}_{best})$ to λ when $k = 5$.

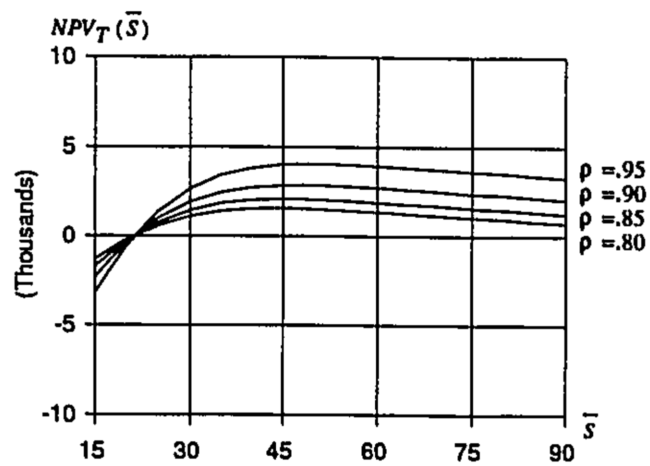

Figure 7.13: Sensitivity of $\bar{S}_{best}$ and $NPV_T(\bar{S}_{best})$ to ρ when $k = 5$.

solution of the finite horizon problem converges to the infinite horizon myopic policy which is simply given by the maximizer S of $G(\cdot)$. Furthermore, as $\rho \to 0$ we also have $\bar{S}_{best} \to S$ because in this case (7.1) asserts that the maximum of $NPV_T(\cdot)$ is also achieved at the maximizer S of $G(\cdot)$. Therefore, for smaller values of ρ the percentage increase of the expected revenue net of costs due to using the best myopic policy instead of the infinite horizon myopic policy is negligible. Nevertheless, if ρ is small then the exact policy can be safely approximated by the infinite horizon policy so that the best myopic policy is still a good solution.

In general, optimal finite horizon policies are quite different near the end of the horizon than they are at the beginning of the horizon. Near the beginning of a finite horizon of moderate duration (e.g., 10 periods) the base-stock level of the optimal policy is closer to the base-stock level of the infinite horizon policy. However, in the last periods the optimal policy may differ (decrease) dramatically from period to period. Implementing the best myopic policy is a compromise because the base-stock level for the entire horizon is constant. Although the best myopic policy performs better than the infinite horizon policy, it may not perform well in an the absolute sense. For this reason, next we compare the expected discounted revenue net of costs for the best myopic policy with other pragmatic non-stationary policies.

For a problem with T periods, suppose that the infinite horizon optimal policy is used for periods $1, \ldots, T-1$, and the single period optimal policy at the end. This policy is called $S\&S^{(1)}$ policy, and the base-stock level for the single period optimal policy is denoted by $S^{(1)}$. The expected discounted revenue net of costs of $S\&S^{(1)}$ policy is denoted by $NPV_T\left(S\&S^{(1)}\right)$. For a problem with T periods, another non-stationary policy is to implement the best myopic policy for periods $1, \ldots, T-1$

Table 7.7: Sensitivity of $\bar{S}_{best}$ to ρ.

$k=3$						
ρ	μ	S_{best}	$NPV_T(S_{best})$	S	$NPV_T(S)$	% Increase
0.75	15	29	616	29	616	0.001
0.80	15	30	818	30	818	0.007
0.85	15	32	1118	32	1117	0.029
0.90	15	33	1572	34	1570	0.116
0.95	15	35	2274	38	2264	0.444
$k=5$						
ρ	μ	S_{best}	$NPV_T(S_{best})$	S	$NPV_T(S)$	% Increase
0.75	25	42	1221	43	1221	0.001
0.80	25	44	1575	44	1575	0.004
0.85	25	46	2094	46	2094	0.018
0.90	25	48	2874	49	2872	0.074
0.95	25	50	4070	53	4058	0.288
$k=7$						
ρ	μ	S_{best}	$NPV_T(S_{best})$	S	$NPV_T(S)$	% Increase
0.75	35	55	1847	56	1847	0.001
0.80	35	57	2354	57	2354	0.003
0.85	35	59	3093	60	3093	0.014
0.90	35	61	4201	63	4198	0.057
0.95	35	64	5893	67	5880	0.222

and the single period optimal policy in period T. This alternative policy is called $\bar{S}_{best}\&S^{(1)}$ policy, and the expected discounted revenue net of costs for using $\bar{S}_{best}\&S^{(1)}$ policy is denoted by $NPV_T\left(\bar{S}_{best}\&S^{(1)}\right)$.

Table 7.8: Explanation for Tables 7.9, 7.10, and 7.11.

Column No.	Explanation
(1)	μ
(2)	$\bar{S}_{best}$
(3)	$NPV_T(\bar{S}_{best})$
(4)	S
(5)	$NPV_T(S)$
(6)	$\left[NPV_T(\bar{S}_{best}) - NPV_T(S)\right]/NPV_T(S)$
(7)	$S^{(1)}$
(8)	$NPV_T\left(S\&S^{(1)}\right)$
(9)	$\left[NPV_T\left(S\&S^{(1)}\right) - NPV_T(S)\right]/NPV_T(S)$
(10)	$NPV_T\left(\bar{S}_{best}\&S^{(1)}\right)$
(11)	$\left[NPV_T\left(\bar{S}_{best}\&S^{(1)}\right) - NPV_T(S)\right]/NPV_T(S)$

We compute the infinite horizon policy and compare its performance with the best myopic policy, $S\&S^{(1)}$ policy, and $\bar{S}_{best}\&S^{(1)}$ policy. Tables 7.9, 7.10 and 7.11 display our results. Column labels used in Tables 7.9, 7.10, and 7.11 are explained in Table 7.8. Expected percentage increase in the discounted revenue net of costs as a result of using the best myopic policy instead of the infinite horizon policy is displayed in column (6), whereas columns (9) and (11) represent the expected percentage increase by using $S\&S^{(1)}$ policy and $\bar{S}_{best}\&S^{(1)}$ policy instead of the infinite horizon policy. We observe that the best myopic policy is not better than the $S\&S^{(1)}$ policy in an absolute sense. However, $\bar{S}_{best}\&S^{(1)}$ policy is better than all the others. That is, when the best myopic policy is combined with the single period optimal policy than the expected discounted revenue net of costs can be improved significantly.

Table 7.9: Comparison of alternative policies on the basis of resulting revenues net of costs for $k = 1$ and $T = 5, 6, 7, 8, 9$.

					$T = 5$					
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
40	115	1378	184	709	% 94.48	52	1291	% 82.23	1842	% 159.99
50	144	1726	230	889	% 94.13	65	1618	% 81.92	2306	% 159.39
60	173	2074	276	1070	% 93.90	78	1944	% 81.72	2770	% 158.99
70	201	2422	322	1250	% 93.73	91	2270	% 81.57	3234	% 158.71
					$T = 6$					
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
40	122	1885	184	1293	% 45.83	52	1870	% 44.63	2365	% 82.97
50	152	2360	230	1619	% 45.74	65	2340	% 44.53	2960	% 82.80
60	182	2835	276	1946	% 45.68	78	2811	% 44.47	3555	% 82.68
					$T = 7$					
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
40	127	2402	184	1871	% 28.39	52	2442	% 30.53	2892	% 54.56
50	158	3006	230	2342	% 28.35	65	3056	% 30.48	3618	% 54.48
60	189	3610	276	2813	% 28.32	78	3670	% 30.45	4345	% 54.42
					$T = 8$					
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
40	131	2924	184	2443	% 19.68	52	3009	% 23.14	3420	% 39.95
50	163	3659	230	3058	% 19.66	65	3765	% 23.11	4278	% 39.91
60	196	4394	276	3672	% 19.64	78	4520	% 23.10	5137	% 39.87
					$T = 9$					
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
40	134	3449	184	3010	% 14.57	52	3570	% 18.60	3947	% 31.14
50	168	4315	230	3766	% 14.56	65	4466	% 18.58	4938	% 31.11
60	201	5181	276	4523	% 14.55	78	5362	% 18.57	5929	% 31.09

Table 7.10: Comparison of alternative policies on the basis of resulting revenues net of costs for $k = 1$ and $T = 10, 11, 12, 13, 14, 15$.

$T = 10$										
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
40	138	3974	184	3571	% 11.28	52	4125	% 15.52	4474	% 25.28
50	172	4971	230	4468	% 11.27	65	5160	% 15.50	5596	% 25.26
60	206	5969	276	5364	% 11.27	78	6196	% 15.50	6719	% 25.25
$T = 11$										
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
40	140	4498	184	4127	% 9.01	52	4675	% 13.30	4999	% 21.13
50	175	5627	230	5162	% 9.00	65	5848	% 13.29	6252	% 21.12
60	210	6756	276	6198	% 9.00	78	7021	% 13.28	7506	% 21.11
$T = 12$										
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
40	143	5021	184	4676	% 7.37	52	5220	% 11.61	5521	% 18.05
50	178	6281	230	5850	% 7.37	65	6529	% 11.61	6905	% 18.04
60	214	7540	276	7023	% 7.37	78	7837	% 11.60	8289	% 18.03
$T = 13$										
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
40	145	5542	184	5221	% 6.15	52	5759	% 10.30	6040	% 15.69
50	181	6931	230	6530	% 6.14	65	7202	% 10.29	7554	% 15.68
60	217	8321	276	7839	% 6.14	78	8646	% 10.29	9068	% 15.67
$T = 14$										
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
40	147	6060	184	5760	% 5.20	52	6292	% 9.24	6556	% 13.82
50	183	7579	230	7204	% 5.20	65	7869	% 9.24	8199	% 13.81
60	220	9098	276	8648	% 5.20	78	9447	% 9.23	9842	% 13.80
$T = 15$										
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
40	149	6574	184	6293	% 4.46	52	6820	% 8.37	7068	% 12.31
50	186	8222	230	7871	% 4.46	65	8530	% 8.37	8839	% 12.30
60	223	9870	276	9449	% 4.46	78	10239	% 8.37	10610	% 12.29

Table 7.11: Comparison of alternative policies on the basis of resulting revenues net of costs for $k = 1$ and $T = 16, 17, 18, 19, 20$.

$T = 16$										
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
40	150	7085	184	6822	% 3.87	52	7343	% 7.65	7576	% 11.06
50	188	8861	230	8531	% 3.86	65	9184	% 7.64	9475	% 11.06
60	225	10637	276	10241	% 3.86	78	11024	% 7.64	11373	% 11.05
$T = 17$										
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
40	152	7593	184	7345	% 3.38	52	7861	% 7.03	8081	% 10.02
50	190	9495	230	9185	% 3.38	65	9831	% 7.03	10105	% 10.02
60	227	11398	276	11026	% 3.38	78	11800	% 7.03	12130	% 10.02
$T = 18$										
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
40	153	8096	184	7862	% 2.98	52	8374	% 6.50	8581	% 9.14
50	191	10125	230	9832	% 2.98	65	10472	% 6.50	10731	% 9.14
60	229	12154	276	11802	% 2.98	78	12569	% 6.50	12881	% 9.14
$T = 19$										
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
40	154	8596	184	8375	% 2.64	52	8881	% 6.04	9078	% 8.39
50	193	10750	230	10473	% 2.64	65	11106	% 6.04	11352	% 8.39
60	231	12903	276	12571	% 2.64	78	13331	% 6.04	13626	% 8.39
$T = 20$										
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
40	156	9092	184	8882	% 2.36	52	9383	% 5.64	9570	% 7.74
50	194	11369	230	11107	% 2.36	65	11734	% 5.64	11967	% 7.74
60	233	13647	276	13333	% 2.36	78	14084	% 5.64	14364	% 7.74

7.2 Positive Initial Inventory

In this section we provide numerical examples for the computation of the best myopic policy when $s_1 > 0$. Under this assumption, function $NPV_T(\bar{S})$, defined by (6.46), has two parts with lengthy expressions which are specified by the value of s_1. Our problem is still one of finding the positive maximizer of $NPV_T(\cdot)$ whose form is no longer as simple as (7.1). As we noted at the beginning of this chapter, in searching for the positive global maximizer of $NPV_T(\bar{S})$ our approach is to compute the maximizers for each of its parts over the relevant regions, and then arrive at the global maximizer by comparison. We compute the maximizers of (6.45) and (6.29) over the regions $[0, s_1)$ and $[s_1, +\infty)$, respectively using simulated annealing[1] (Van Laarhoven and Aarts [43]), and then choose the one which gives a higher value of $NPV_T(\cdot)$. We use the data displayed by Table 7.1 with the exceptions that $k = 1$ and $\lambda = .02$. Unlike the case $s_1 = 0$ computations are more complicated, and we do not have analytical results. However, using our approach numerical computations still do not require much effort. We again denote the critical number of the best myopic policy by $\bar{S}_{best}$, and present our calculations.

In order to illustrate that if $s_1 > 0$ then maximizing $NPV_T(\cdot)$ makes it necessary to deal with lengthy expressions, we present the explicit expression of $NPV_T(\cdot)$ in Appendix B for the case of exponential demand. Here we mention that $m_\rho(\cdot)$ may not have a closed form expression for some demand densities. Referring to (6.45) we observe that writing $NPV_T(\cdot)$ explicitly requires the computation of the renewal-like function $m_\rho(\cdot)$. Nonetheless, as we discussed in Section 5.5.2 of Chapter 5

[1]For the examples presented in this section simulated annealing finds the truly optimal $\bar{S}_{best}$.

for some demand densities, such as Erlang, $m_\rho(\cdot)$ does not have a closed for expression, and thus it is computed numerically via solving integral equations, a method that we developed in Section 5.5.4. In turn, $NPV_T(\cdot)$ may not have a closed form expression but it can also be computed numerically. On the other hand, in the case of exponential demand, according to (5.25) we have $m_\rho(x) = \rho\lambda \exp\left[-\lambda(1-\rho)x\right]$. Therefore, we are able to present an explicit expression for $NPV_T(\cdot)$ for this case in Appendix B.

We expect that the best myopic policy for the finite horizon depends on the initial inventory s_1. In order to understand the nature of this dependence we compute $\bar{S}_{best}$ for different values of s_1. We realize that the relation between s_1 and $\bar{S}_{best}$ is complicated but has a structure. Table 7.12 shows our results when $T = 15$. Since base values of k and λ are given by 1 and .02, it follows that $\mu = 50$. In order to provide a comprehensive illustration, in Table 7.13 we also present similar calculations for $\mu = 40$ and $\mu = 60$. Since we assume that $T = 15$, in Tables 7.12 and 7.13 the value of s_1 varies from zero to 15μ. That is, for a given value of μ (e.g., 40, 50, or 60) we repeat our calculations when the initial inventory is zero, and when the initial inventory covers the expected demand for one period, two periods, etc. The last rows of Tables 7.12 and 7.13 show the calculations when the initial inventory is sufficient to satisfy the expected demand of the entire planning horizon. For example, in Table 7.12 we have $\mu = 50$ so that the last row displays our results when the initial inventory level $s_1 = T\mu = (15)(50) = 750$. If the initial inventory is above $T\mu$ then the system is extremely overstocked in which case the procurement problem that we deal in this chapter is not significant.

The calculations presented in Tables 7.12 and 7.13 are also performed while varying μ from 10 to 100 with a step size of 10. We tabulate the results in

Table 7.12: Dependence of $\bar{S}_{best}$ on s_1, and savings when $\mu = 50$.

$\mu = 50$						
s_1	s_1/μ	S_{best}	$NPV_T(S_{best})$	S	$NPV_T(S)$	% Increase
0	0	186	8222	230	7871	4.459
50	1	186	9222	230	8871	3.956
100	2	186	10222	230	9871	3.556
150	3	186	11222	230	10871	3.228
200	4	183	12237	230	11871	3.081
250	5	180	13263	230	12868	3.072
300	6	177	14258	230	13837	3.046
350	7	173	15223	230	14771	3.059
400	8	169	16160	230	15671	3.124
450	9	163	17085	230	16539	3.301
500	10	154	18027	230	17381	3.718
550	11	142	19032	230	18214	4.490
600	12	129	20115	230	19060	5.537
650	13	118	21230	230	19933	6.510
700	14	110	22290	230	20825	7.036
750	15	104	23223	230	21708	6.983

Table 7.13: Dependence of $\bar{S}_{best}$ on s_1, and savings for $\mu = 40$ and 60

$\mu = 40$						
s_1	s_1/μ	$\bar{S}_{best}$	$NPV_T(\bar{S}_{best})$	S	$NPV_T(S)$	% Increase
0	0	149	6574	184	6293	4.461
40	1	149	7374	184	7093	3.958
80	2	149	8174	184	7893	3.557
120	3	149	8974	184	8693	3.230
160	4	146	9786	184	9493	3.081
200	5	144	10607	184	10291	3.072
240	6	142	11403	184	11066	3.046
280	7	139	12175	184	11813	3.059
320	8	135	12925	184	12534	3.123
360	9	131	13664	184	13228	3.300
400	10	123	14419	184	13902	3.716
440	11	114	15222	184	14569	4.486
480	12	103	16088	184	15245	5.532
520	13	94	16980	184	15943	6.507
560	14	88	17829	184	16657	7.034
600	15	83	18575	184	17363	6.983
$\mu = 60$						
0	0	223	9870	276	9449	4.458
60	1	223	11070	276	10649	3.955
120	2	223	12270	276	11849	3.555
180	3	223	13470	276	13049	3.228
240	4	219	14688	276	14249	3.082
300	5	216	15920	276	15445	3.071
360	6	212	17113	276	16607	3.046
420	7	208	18270	276	17728	3.059
480	8	203	19396	276	18808	3.124
540	9	196	20505	276	19849	3.302
600	10	185	21636	276	20860	3.720
660	11	170	22842	276	21860	4.492
720	12	154	24142	276	22875	5.540
780	13	141	25480	276	23922	6.513
840	14	131	26752	276	24993	7.038
900	15	125	27872	276	26052	6.983

Appendix C. Figure 7.14 provides a plot of s_1/μ versus $\bar{S}_{best}$ for different values of μ. It is worth noting that s_1/μ represents the number of periods of expected demand that the initial inventory is sufficient to satisfy. According to Figure 7.14, as the mean demand increases the value of $\bar{S}_{best}$ increases. However, regardless of the value of μ, the critical number $\bar{S}_{best}$ is constant as long as $s_1/\mu \leq 3$ for this example. Once the initial inventory level is above the expected demand for three periods the value of $\bar{S}_{best}$ starts to decrease. That is, the value of $\bar{S}_{best}$ remains constant for tolerable levels of initial stock, but otherwise the best myopic policy offers to keep $\bar{S}_{best}$ smaller in order to avoid the end of horizon costs associated with excess inventory disposal.

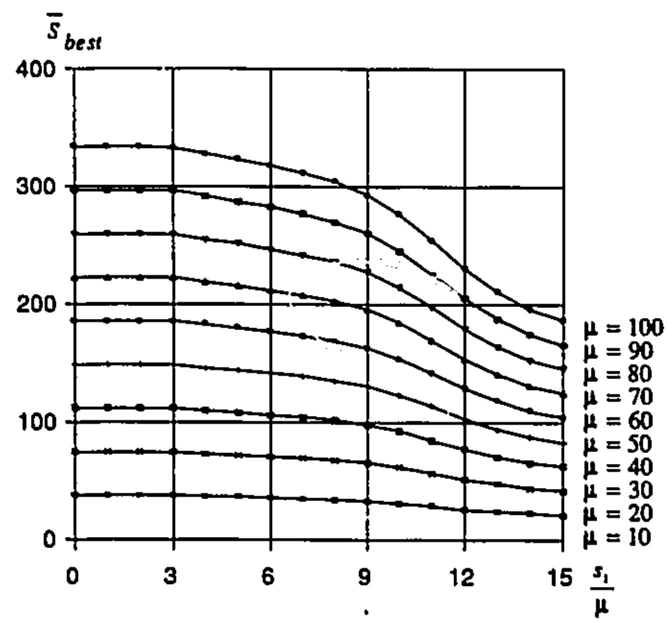

Figure 7.14: Dependence of $\bar{S}_{best}$ on s_1 for $T = 15$ and different μ values.

In order to strengthen our observations about the dependence of $\bar{S}_{best}$ on

s_1, we performed additional calculations. Assuming $\mu = 50$, we calculated $\bar{S}_{best}$ for $T = 5, 10$, and 20 while varying the value of s_1 from 0 to $T\mu$. Using these calculations we present Figure 7.15 which displays the plots of s_1/μ versus $\bar{S}_{best}$ for different values of T. Also note that, as we have already explained, our problem is meaningful when $s_1 \leq T\mu$, and therefore for any given T the value of s_1/μ varies between 0 and T. For example, if $T = 5$ we make our calculations while s_1 varies from 0 to 250, since $\mu = 50$, and we display these results by plotting s_1/μ versus $\bar{S}_{best}$ while s_1/μ varies from 0 to 5.

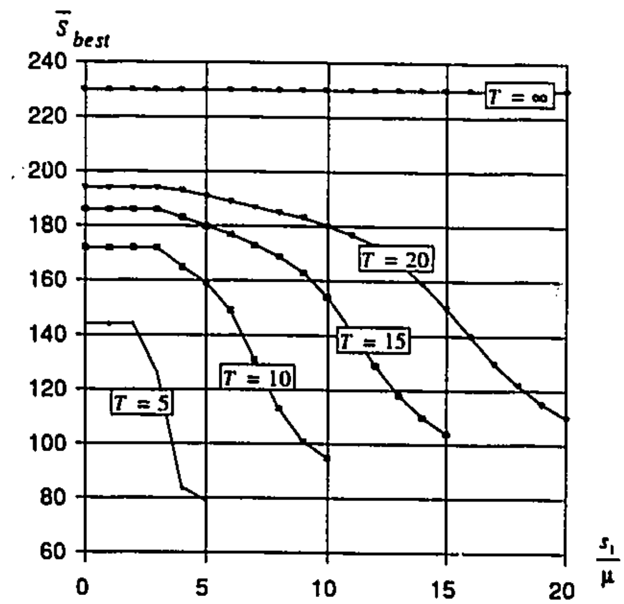

Figure 7.15: Dependence of $\bar{S}_{best}$ on s_1 for $\mu = 50$ and different T values.

According to Figure 7.15 the value $\bar{S}_{best}$ remains constant for tolerable levels of initial stock, and as we expected the upper bound on these levels depends on the horizon length. If $T = 5$ the maximum tolerable level is 100 so that $\bar{S}_{best}$ is constant for $s_1/\mu \leq 2$, whereas for $T = 15$ and $T = 20$ the tolerable level is 150 so that $\bar{S}_{best}$ is constant for $s_1/\mu \leq 3$. The infinite horizon myopic policy does not depend on the initial inventory level as shown by Figure 7.15. [Recall that, in case of Erlang demand S solves equation (4.17), on page 63, which does not depend on s_1.] In fact, since S is optimal under the assumption that there are infinitely many periods then any initial inventory level is tolerable because the possible end of horizon costs associated with excess disposal are quite far away in the future, and for this reason their discounted values are negligible.

Finally, we note that the best myopic policy is always superior to the infinite horizon myopic policy. Tables 7.12 and 7.13 include a column in which we display the possible percentage increase of $NPV_T(\cdot)$ by using the best myopic policy instead of the infinite horizon myopic policy. Figure 7.16 displays the percentage increase in the resulting expected revenue net of costs as a function of s_1/μ for $T = 10, 15$, and 20 where $\mu = 50$. Percentage improvements are significant especially for smaller values of T. Furthermore, if the system has initial overstock then again the best myopic policy performs better. For $T = 5$ and $\mu = 50$, the percentage increase in the resulting expected revenue net of costs as a function of s_1/μ is displayed by Figure 7.17. In this case, the savings are dramatic and can be as high as 94%.

7.3 Summary

In this chapter we presented original results on the computation of the best myopic policy for the cases of exponential and Erlang demand. Using the approach developed

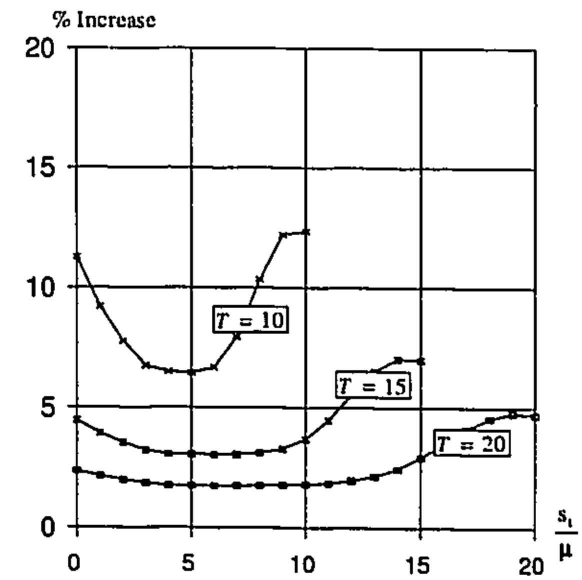

Figure 7.16: Percentage improvements for $\mu = 50$ and different T values.

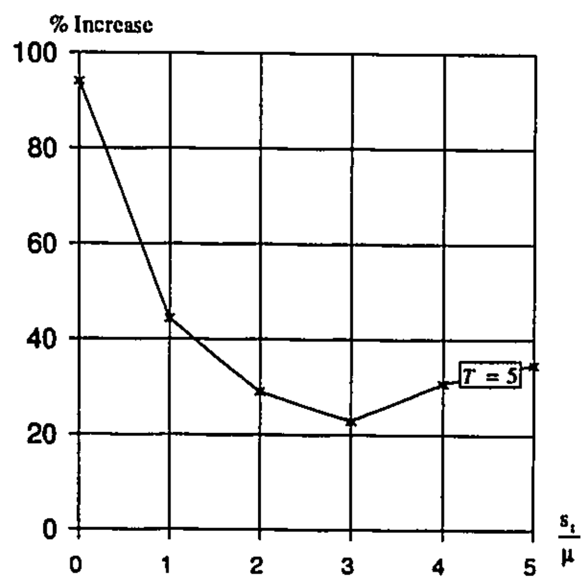

Figure 7.17: Substantial improvements for $\mu = 50$ and $T = 5$.

in Chapter 6 the critical number of the best myopic policy was computed via maximizing $NPV_T(\cdot)$, given by (6.46) on page 111. The feasible space of maximization was $[0, +\infty]$ since all demands must have been satisfied by assumption. If $s_1 = 0$ then function $NPV_T(\cdot)$ reduced to (7.1) so that we were able to analyze the properties of $NPV_T(\cdot)$. As we mentioned at the beginning of this chapter, the purpose of assuming $s_1 = 0$ was two fold: it was a convincing assumption that resulted in quite general results, and it simplified the maximization problem. Assuming $s_1 = 0$, $NPV_T(\cdot)$ was concave for exponential demand unless r_T was unreasonably large, and the critical number of the best myopic policy could be computed easily using Theorem 4. For the case of Erlang demand, if $s_1 = 0$ then the shape of $NPV_T(\cdot)$ was more complicated (Lemma 1) compared to the simpler case of exponential demand. However, the critical number of the best myopic was still easy to compute using Theorem 5.

The results summarized by Theorems 4 and 5 greatly simplify the problem of finding the best myopic policy. Further, since the Erlang density provides a good approximation for most real life densities with coefficient of variation less than unity, our approach may be considered as an easy to compute alternative to the exact policy.

As we mentioned many times throughout this thesis, the traditional approach for avoiding the computational difficulty associated with the finite horizon case was to compute the steady state policy (the infinite horizon myopic policy in the context of this study). Nonetheless, by definition, the best myopic policy was always superior to the steady state policy. Therefore, we argued that computing the best myopic policy was not only easy, but it was also a better way of avoiding the computational difficulty. We presented numerical results in order illustrate the potential savings offered by employing the best myopic policy. As we noted, computing the exact optimal policy was really cumbersome when the recursive calculations should be performed for many items. Thus, the percentage improvements achieved by employing the best myopic policy might quite well add up to a substantial amount if the inventory control of many independent items was of interest. We proposed to combine the best myopic policy with the single period optimal policy, i.e., employ the $\bar{S}_{best}\&S^{(1)}$ policy. We observed that the expected discounted revenue net of costs could be improved significantly using $\bar{S}_{best}\&S^{(1)}$ policy.

We also discussed many observations based on our computational experimentation. For example, the best myopic policy converged to the infinite horizon myopic policy as the number of periods increased. In order to present some additional examples, we also provided numerical results about the sensitivity of the best myopic policy to the mean demand, and discounting factor, etc.

If $s_1 > 0$ then the analysis was perplexing compared to the case $s_1 = 0$. We computed the best myopic policy using simulated annealing. For this case, we did not have analytical results regarding the critical number of the best myopic policy since writing the explicit expression of $NPV_T(\cdot)$ was complicated if not impossible (see Appendix B for an illustration of this claim). Using numerical examples, we observed that the dependence of the best myopic policy on the initial inventory was somehow structured (an intuitive and original result). We argued that for tolerable levels of initial inventory the critical number of the best myopic policy did not depend on s_1, however after a threshold point (the maximum tolerable level) of initial overstock the critical number declined.

Chapter 8

Summary of the Thesis and Concluding Comments

In this thesis we studied the following variations of a typical retailer's problem which required the determination of order quantities following the review of inventory levels periodically:

1. The infinite horizon case (Chapters 3 and 4).
2. The infinite horizon case with a one-time disposal opportunity (Chapter 5).
3. The finite horizon case (Chapters 6 and 7).

The basic problem was characterized by a single item, a multi-period setting, and i.i.d. demands in successive periods. We studied the case of full backlogging. Backorders were penalized via two types of backorder costs simultaneously. The first type was the standard proportional backorder cost which was linear in size of the backorders. The second type was the fixed backorder cost that was independent of the size of the backorders but rather associated a one-time penalty with the stockout situation. This treatment of backorder costs, i.e., consideration of proportional and fixed backorder costs simultaneously, was a nonstandard aspect of our study. We also considered a unit purchase cost associated with procurement decisions, and a unit inventory holding cost per period. On one hand, we had holding costs which penalized the overstocked system, and on the other hand we had fixed and proportional backorder costs which penalized the stockout situations. The objective was to find the order quantities which maximized the expected discounted revenues net of costs throughout the horizon.

Our discussion started with an introduction. Some of the basic concepts associated with the periodic review inventory systems and the motivations of this study were described in Chapter 1. A review of the relevant literature was provided in Chapter 2 where the focus was on the class of single item, single location, periodic review, stochastic, dynamic inventory models which were relevant to this study.

The mathematical development began in Chapter 3 where we studied the infinite horizon problem assuming that the product remains in the market indefinitely. We argued that the infinite horizon assumption might seem strong because in the competitive markets of the 1990s most products had quite short life cycles at the end of which they were off the market. However, there were still other products which had been in the market for a relatively long time, and thus the infinite horizon assumption was not particularly violated either. The results associated with the infinite horizon solution were of particular interest in this thesis because they provided the basis of our study. We provided the conditions under which a myopic base-stock policy was optimal for the infinite horizon problem. This result was important because of its practical benefits. In the context of inventory management myopic policies were favored since they were simple to implement, easier to compute, and optimal under certain settings. Further, the infinite horizon policy presented a good approximation as the horizon length increases, and thus the computational burden associated with the finite horizon case could be resolved by implementing the infinite horizon myopic policy for longer horizon lengths. The formulation of the infinite horizon problem that we presented in Chapter 3 was useful in the reduction of the multi-period dynamic problem to a forward sequence of single period, static problems so that it was possible to prove the optimality of a myopic policy.

The infinite horizon myopic policy was characterized by a single critical number denoted by S, and stated that the number of items available for sale at the beginning of a period should be brought up to S whenever it was below S. We showed in Chapter 3 that if the initial inventory level was below S, then this policy would be optimal in general, i.e., it would be optimal regardless of the demand density and parameter values. On the other hand, if the initial inventory level was above S, in order for the myopic policy to be optimal a sufficient condition was required to hold. In Chapter 4 we showed that this sufficient condition was not a serious restriction in the sense that it held immediately for the Erlang demand so that the myopic policy would be optimal in general for the case of Erlang demand. We also showed that computing the value of S was quite easy for the case of Erlang demand. These results were important since the Erlang demand density might quite well represent most densities with coefficient of variation not exceeding unity.

In Chapter 5 we argued that the financial management of overstocked businesses was a critical problem, and many businesses could seek disposal alternatives, probably by specifying a reduced price. Under this scenario the problem was how much to dispose in an excess inventory situation. In order to provide an answer to this question we considered a one-time disposal opportunity associated with the infinite horizon problem of Chapter 3 so that we were able to formulate a related disposal problem. According to our assumptions, the retailer's disposal decision represented a one-time opportunity for the customers to buy at a discounted price. We formulated the problem using our results in Chapter 3, and analyzed the mathematical properties of the objective function. We showed that the objective function of the disposal problem was concave for Erlang demand, and then we presented the optimal solution. We also discussed some computational issues. Computing optimal solution of our disposal problem might require the computation of what we called renewal-like functions, namely $m_\rho(\cdot)$ and $M_\rho(\cdot)$. We proposed two alternative techniques for the computation of these functions, i.e., use of Laplace transforms or integral equations. The disposal problem of Chapter 5 could in fact be considered as a generalization of the infinite horizon problem of Chapter 3.

In Chapter 6 we studied the finite horizon case for which the myopic policy is not optimal in general. The finite horizon case was mainly of interest because of the seasonal nature of some products for which the sales were discontinued after a certain number of periods. Another motivation for studying the finite horizon case was that most products had short life cycles at the end of which their production was discontinued so that they were out of the market. For the finite horizon problem computation of the exact policy required the use of the dynamic programming (DP) approach which may demand a substantial effort. This difficulty was related to the recursive calculations of DP which involved computation of optimal revenue for a vast number of states which probably would not be entered, and the optimization of a nonconcave function at each stage of the recursion. We argued that there were two traditional approaches to avoiding this computational difficulty. One approach was to compute the infinite horizon myopic policy whereas the other approach was to assume Veinott's [44] terminal condition which imposed a set of assumptions regarding the costs incurred at the end of the horizon. We showed that these two approaches were equivalent, and they provided computational ease at a price of a set of conditions which might not completely apply in retail industry. We argued that from a computational point of view, there was a need for providing a method that overcomes the computational burden associated with the finite horizon solution, but which was also superior to the traditional approaches. In Chapter 6 of this thesis, we presented an original model that constituted a framework for our numerical study in Chapter 7. The exact policy was difficult to compute and the infinite horizon myopic policy was suboptimal for the finite horizon case. However, myopic policies were favored since they were simple to implement, easier to compute, optimal under certain settings, and presented good approximations as the horizon length increased; thus, we restricted our attention to the class of myopic base-stock policies. We optimized the expected discounted revenues net of costs with respect to the set of myopic operating doctrines for our finite horizon problem, and named the resulting policy the 'best myopic' policy. Chapter 2.8 was aimed at presenting the basic idea and developing the underlying model to be used for computation of the best myopic policy in Chapter 7.

In Chapter 7 we presented original results on the computation of the best myopic policy for the cases of exponential and Erlang demands. Under the assumption that there was no initial stock we were able to present analytical results regarding the computation of the best myopic policy. The results presented in Chapter 7 greatly simplified the problem of finding the best myopic policy if there was no initial stock.

We again argued that since the Erlang density provided a good approximation for most real life densities with coefficient of variation less than unity, the best myopic policy might be considered as an easy to compute alternative to the exact policy. By definition, the best myopic policy was always superior to the steady-state policy. Therefore, we not only developed an easy method but also provided a better way of avoiding the computational difficulty.

Based on our numerical results in Chapter 6, we observed that the best myopic policy depended on the initial inventory level, and its computation was more complex when there was initial stock compared to the case of no initial inventory. Varying the value of initial stock we presented some numerical examples in Chapter 7, and illustrated the computation of the best myopic policy. We also analyzed the dependence of the best myopic policy on the initial stock. Our examples indicated that this dependence had a structure. The best myopic policy was robust, i.e., the critical number of the best myopic policy remained constant, up to a maximum tolerable level of initial stock, and decreased for higher levels of initial stock. However, this observation was made based on our numerical examples, and its further investigation on analytical grounds remains as an area for future research.

Since the best myopic policy depends on the initial stock, along with the best myopic policy we may employ a rolling horizon schedule. In this case, the best myopic policy would be recalculated as further information becomes available regarding the inventory levels. Checking the efficiency of the best myopic policy when combined with a rolling horizon schedule also remains as an area for future investigation.

Appendix A

List of Symbols

Here we present a list of frequently used symbols.

Problems of interest:

$\mathcal{P}_\infty$	Infinite horizon problem.
$\mathcal{P}_\infty^{\mathcal{D}}$	Infinite horizon problem with a disposal opportunity.
$\mathcal{P}_T$	Finite horizon problem.

Notation related to the periods:

n	Period index.
T	Number of periods.
N	Time of placing the first order (infinite horizon case).
$\bar{N}$	Time of placing the first order (finite horizon case).

Notation related to the variables:

s_n	Inventory level at the beginning of n.
z_n	Order quantity for period n.
a_n	Order-up-to level for period n.
D_n	Demand during period n.

Notation related to the costs:

ρ	Discounting factor.
c	Unit procurement cost.
PC_n	Total procurement cost of period n.
h	Unit holding cost per period.
HC_n	Total holding cost of period n.

b	Proportional backorder cost.
B	Fixed backorder cost.
BC_n	Total backorder cost of period n.
r	Unit revenue.
R_n	Total revenue of period n.
v	Discounted price.
r_T	End of horizon revenue.
c_T	End of horizon procurement cost.
l	Salvage value.
Notation related to various functions:	
$f(\cdot)$	Demand density.
$f^{(n)}(\cdot)$	n-fold convolution of $f(\cdot)$.
$F(\cdot)$	Demand distribution.
$F^{(n)}(\cdot)$	n-fold convolution of $F(\cdot)$.
μ	Mean demand.
σ^2	Variance of demand.
NPV	Total discounted revevue net of costs for $\mathcal{P}_\infty$.
$E[NPV]$	Objective function of $\mathcal{P}_\infty$.
$G(\cdot)$	Single period return net of costs function.
S	Global maximizer of $G(\cdot)$.
$NPV(s_1)$	Optimal value of $E[NPV]$ as a function of s_1.
$C(s_1)$	Expected discounted return net of costs until the end of $N-1$.
$(\tilde{\ })$	Denotes the Laplace transform of a function.
$M_\rho(\cdot)$	Discounted renewal function.
$m_\rho(\cdot)$	First derivative of $M_\rho(\cdot)$.
λ	Scale parameter of Erlang variable, or parameter of exponential density.
k	Shape parameter of Erlang variable.
$\gamma(\cdot, \lambda, k)$	Erlang density with parameters λ and k.
$\Gamma(\cdot, \lambda, k)$	Erlang distribution parameters λ and parameter k.
$Q(\cdot)$	Objective function of $\mathcal{P}_\infty^{\mathcal{D}}$.
u^*	Optimal solution of $\mathcal{P}_\infty^{\mathcal{D}}$.
u_{max}	Global maximizer of $Q(\cdot)$.
$\mathcal{N}(t)$	Number of periods until cumulative demand reaches t.
NPV_T	Total discounted revevue net of costs for $\mathcal{P}_T$.
$E[NPV_T]$	Objective function of $\mathcal{P}_T$.
$NPV_T(\cdot)$	Objective function for computing the best myopic policy.
$\bar{S}_{best}$	Critical number of the best myopic policy.
$\bar{S}_{max}$	Positive maximizer of $NPV_T(\cdot)$.
$\bar{S}_{inflect}$	Inflection point of $NPV_T(\cdot)$.

Appendix B

Derivation of $NPV_T(\cdot)$

B.1 Expression of $NPV_T(\cdot)$ for Exponential Demand

Under the assumption that $s_1 = 0$ we studied the properties of $NPV_T(\cdot)$ for exponential and Erlang demand in complete detail (see Section 7.1 of Chapter 7). On the other hand, as we discussed in Section 7.2, if $s_1 > 0$ then the analysis is complicated and obtaining the exlicit expression of $NPV_T(\cdot)$ is a cumbersome task. Now we derive the explicit expression of function $NPV_T(\cdot)$ when $s_1 > 0$ for the case of exponential demand. Our purpose is to illustrate the the messy expressions that we encountered for the relatively simple exponential demand case.

According to our results in Section 6.7 of Chapter 6 we have expression (6.46) which gives

$$NPV_T(\bar{S}) = \begin{cases} (6.45) & \text{if } \bar{S} < s_1, \\ (6.29) & \text{if } \bar{S} \geq s_1. \end{cases}$$

Then in order to write $NPV_T(\cdot)$ explicitly we need to use expressions (6.45) and (6.29). First let us recall expression (6.45) which define $NPV_T(\bar{S})$ over $(-\infty, s_1)$. According to (6.45) if $\bar{S} < s_1$ then

$$NPV_T(\bar{S}) \;\; = \;\; cs_1 + G(s_1) + \int_0^{s_1 - \bar{S}} G(s_1 - x) m_\rho(x) dx + \frac{\rho G(\bar{S})}{1-\rho}$$

$$
\begin{aligned}
&-G(\bar{S})\sum_{n=1}^{T-1}\rho^n F^{(n)}(s_1-\bar{S}) - \frac{\rho^T G(\bar{S})}{1-\rho} + \rho^T Y(\bar{S}) \\
&-\rho^T Y(\bar{S}) F^{(T-1)}(s_1-\bar{S}) \\
&+\rho^T E\left[Y(s_1-\sum_{i=1}^{T-1} D_i)\right] F^{(T-1)}(s_1-\bar{S}). \qquad \text{(B.1)}
\end{aligned}
$$

Assuming that demand has exponential distribution with parameter λ, we already have the following results:

- Expression (7.6), on page 117, requires that

$$
\begin{aligned}
G(\bar{S}) &= -\frac{B\lambda+h+b+r}{\lambda}\exp(-\lambda\bar{S}) - [c(1-\rho)+h]\bar{S} \qquad \text{(B.2)} \\
&+\frac{r-\rho c+h}{\lambda}.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
G(s_1) &= -\frac{B\lambda+h+b+r}{\lambda}\exp(-\lambda s_1) - [c(1-\rho)+h]s_1 \qquad \text{(B.3)} \\
&+\frac{r-\rho c+h}{\lambda},
\end{aligned}
$$

and

$$
\begin{aligned}
G(s_1-\bar{S}) &= -\frac{B\lambda+h+b+r}{\lambda}\exp[-\lambda(s_1-\bar{S})] \qquad \text{(B.4)} \\
&-[c(1-\rho)+h](s_1-\bar{S}) + \frac{r-\rho c+h}{\lambda}.
\end{aligned}
$$

- Expression (5.25), on page 82, gives

$$
m_\rho(x) = \rho\lambda\exp[-\lambda(1-\rho)x]. \qquad \text{(B.5)}
$$

- If demand density is exponential with parameter λ then the n-fold convolution distribution is Erlang with scale parameter λ and shape parameter n. It follows that $F^{(n)}(s_1 - \bar{S}) = \Gamma(s_1 - \bar{S}, \lambda, k)$ so that

$$F^{(n)}(s_1 - \bar{S}) \quad = \quad = 1 - \exp\left[-\lambda(s_1 - \bar{S})\right] \sum_{j=0}^{k-1} \frac{[\lambda(s_1 - \bar{S})]^j}{j!}. \tag{B.6}$$

Also note that

$$f^{(n)}(x) \quad = \quad \gamma(x, \lambda, k) = \frac{\lambda(\lambda x)^{k-1} \exp(-\lambda x)}{(k-1)!}. \tag{B.7}$$

- Expression (7.9), on page 118, asserts that

$$Y(\bar{S}) = (c - l)\left(\frac{1}{\lambda} - \bar{S}\right) + \frac{r_T - c_T + l}{\lambda} \exp(-\lambda \bar{S}). \tag{B.8}$$

As we showed in Chapter 6,

$$E\left[Y\left(s_1 - \sum_{i=1}^{T-1} D_i\right)\right]$$

provided by (6.44) does not depend on $\bar{S}$ so that it is a constant. Recall that (6.44) implies

$$\begin{aligned} E\left[Y\left(s_1 - \sum_{i=1}^{T-1} D_i\right)\right] \quad = \quad & (c_T - r_T - c)s_1 \\ & + cTE[D_1] + (l + r_T - c_T)s_1 F^{(T)}(s_1) \\ & - l\int_0^{s_1} x f^{(T)}(x)dx + (r_T - c_T)\int_{s_1}^{\infty} x f^{(T)}(x)dx. \end{aligned} \tag{B.9}$$

Substituting (B.6), (B.7), and $E[D_1] = \mu = 1/\lambda$ in (B.9) then we can write

$$\begin{aligned} E\left[Y\left(s_1 - \sum_{i=1}^{T-1} D_i\right)\right] \quad = \quad & (l - c)\left(s_1 - \frac{T}{\lambda}\right) \\ & + (l + r_T - c_T)\exp(-\lambda s_1)\left(\frac{T}{\lambda} - s_1\right)\sum_{j=0}^{T-1} \frac{(\lambda s1)^j}{j!} \\ & + (l + r_T - c_T)\exp(-\lambda s_1)\frac{\lambda^{T-1} s_1^T}{(T-1)!}. \end{aligned} \tag{B.10}$$

We also need to compute the convolution integral

$$\int_0^{s_1-\bar{S}} G(s_1-x)m_\rho(x)dx.$$

Using (B.4) and (B.5), and taking the integral the above quantity is given by

$$\frac{\rho\exp[\lambda(1-\rho)(\bar{S}-s_1)][\bar{S}(r-c+\rho c)-s_1(h+r)]}{\rho-1}$$
$$-\frac{\rho\exp[\lambda(1-\rho)(\bar{S}-s_1)][(\rho-1)^2c-2(\rho-1)h-(2\rho-1)r}{\lambda(rho-1)^2}$$
$$-\exp(\lambda[\rho(s_1-\bar{S})-s_1])\left[2(h+b+r)\bar{S}+B+\left(\frac{2\rho+2}{\lambda\rho}-s_1\right)(h+b+r)\right]$$
$$+\exp(-\lambda s_1)\left[B+\left(\frac{2\rho+2}{\lambda\rho}-s_1\right)(h+b+r)\right]$$
$$-\frac{\rho}{\rho-1}\left[s_1(c\rho-c-h)-\frac{(\rho-1)^2c-2(\rho-1)h-(2\rho-1)r}{\lambda(\rho-1)}\right].$$

If we substitute the above lengthy expression as well as expressions (B.3), (B.4), (B.8), (B.10), and (B.6) in (B.1) then we have the explicit expression of $NPV_T(\bar{S})$ when $\bar{S}\in[0,s_1)$. Since this expression is very long and messy, it is not displayed here but rather given in Appendix B.2.

Now, we recall expression (6.29) which define $NPV_T(\bar{S})$ over $[s_1,+\infty)$. According to (6.29) if $\bar{S}\geq s_1$ then

$$NPV_T(\bar{S}) = cs_1+\left(\frac{1-\rho^T}{1-\rho}\right)G(\bar{S})+\rho^T Y(\bar{S}). \tag{B.11}$$

Inserting (B.2) and (B.8) in (B.11) leads to the explicit expression of $NPV_T(\cdot)$ if $\bar{S}\geq s_1$. That is, if $\bar{S}\geq s_1$ then

$$\begin{aligned} NPV_T(\bar{S}) &= cs_1+\left(\frac{1-\rho^T}{1-\rho}\right)\left[-\frac{B\lambda+h+b+r}{\lambda}\exp(-\lambda\bar{S})\right] \\ &\quad \left(\frac{1-\rho^T}{1-\rho}\right)-\left[\{c(1-\rho)+h\}\bar{S}-\frac{r-\rho c+h}{\lambda}\right] \\ &\quad +\rho^T\left[(c-l)\left(\frac{1}{\lambda}-\bar{S}\right)+\frac{r_T-c_T+l}{\lambda}\exp(-\lambda\bar{S})\right]. \end{aligned}$$

B.2 $NPV_T(\cdot)$ over $[0, s_1)$ for Exponential Demand

Following our discussion in Appendix B.1, if $\bar{S} \in [0, s_1)$ then $NPV_T(\bar{S})$ is given by

$$
\begin{aligned}
& cs_1 - \frac{B\lambda + h + b + r}{\lambda} \exp(-\lambda s_1) - [c(1-\rho) + h]s_1 + \frac{r - \rho c + h}{\lambda} \\
& \frac{\rho \exp[\lambda(1-\rho)(\bar{S} - s_1)][\bar{S}(r - c + \rho c) - s_1(h + r)]}{\rho - 1} \\
& - \frac{\rho \exp[\lambda(1-\rho)(\bar{S} - s_1)][(\rho - 1)^2 c - 2(\rho - 1)h - (2\rho - 1)r}{\lambda(rho - 1)^2} \\
& - \exp(\lambda[\rho(s_1 - \bar{S}) - s_1]) \left[2(h + b + r)\bar{S} + B + \left(\frac{2\rho + 2}{\lambda\rho} - s_1\right)(h + b + r)\right] \\
& + \exp(-\lambda s_1) \left[B + \left(\frac{2\rho + 2}{\lambda\rho} - s_1\right)(h + b + r)\right] \\
& - \frac{\rho}{\rho - 1}\left[s_1(c\rho - c - h) - \frac{(\rho - 1)^2 c - 2(\rho - 1)h - (2\rho - 1)r}{\lambda(\rho - 1)}\right] \\
& - \frac{\rho - \rho^T}{1 - \rho}\left(\frac{B\lambda + h + b + r}{\lambda} \exp(-\lambda\bar{S}) + [c(1-\rho) + h]\bar{S} - \frac{r - \rho c + h}{\lambda}\right) \\
& + \left(\frac{B\lambda + h + b + r}{\lambda} \exp(-\lambda\bar{S}) + [c(1-\rho) + h]\bar{S} - \frac{r - \rho c + h}{\lambda}\right) \\
& \times \sum_{n=1}^{T-1} \rho^n \left(1 - \exp[-\lambda(s_1 - \bar{S})] \sum_{j=0}^{k-1} \frac{[\lambda(s_1 - \bar{S})]^j}{j!}\right) \\
& + \rho^T \left[(c - l)\left(\frac{1}{\lambda} - \bar{S}\right) + \frac{r_T - c_T + l}{\lambda} \exp(-\lambda\bar{S})\right] \exp[-\lambda(s_1 - \bar{S})] \sum_{j=0}^{k-1} \frac{[\lambda(s_1 - \bar{S})]^j}{j!} \\
& + \rho^T \left[(l - c)\left(s_1 - \frac{T}{\lambda}\right) + (l + r_T - c_T)\exp(-\lambda s_1)\left(\frac{T}{\lambda} - s_1\right) \sum_{j=0}^{T-1} \frac{(\lambda s_1)^j}{j!}\right. \\
& \left. + (l + r_T - c_T)\exp(-\lambda s_1)\frac{\lambda^{T-1} s_1^T}{(T-1)!}\right] \left(1 - \exp[-\lambda(s_1 - \bar{S})] \sum_{j=0}^{k-1} \frac{[\lambda(s_1 - \bar{S})]^j}{j!}\right).
\end{aligned}
$$

Appendix C

Additional Computations

Here we present additional computations that support our discussion in Section 7.2 of Chapter 7. The results tabulated below are utilized to obtain Figures 7.14, 7.15, 7.16, and 7.17.

$T = 5$							
μ	s_1	s_1/μ	S_{best}	$NPV_T(S_{best})$	S	$NPV_T(S)$	% Increase
10	0	0	29	335	47	167	99.897
10	10	1	29	535	47	367	45.509
10	20	2	29	735	47	567	29.466
10	30	3	26	942	47	767	22.721
10	40	4	18	1258	47	967	30.082
10	50	5	16	1571	47	1167	34.625
20	0	0	58	682	92	348	96.247
20	20	1	58	1082	92	748	44.755
20	40	2	58	1482	92	1148	29.156
20	60	3	51	1900	92	1548	22.774
20	80	4	34	2541	92	1948	30.480
20	100	5	32	3163	92	2347	34.726

$T = 5$							
μ	s_1	s_1/μ	S_{best}	$NPV_T(S_{best})$	S	$NPV_T(S)$	% Increase
30	0	0	87	1030	138	528	95.064
30	30	1	87	1630	138	1128	44.504
30	60	2	87	2230	138	1728	29.052
30	90	3	76	2859	138	2328	22.796
30	120	4	51	3825	138	2928	30.614
30	150	5	48	4754	138	3528	34.758
40	0	0	115	1378	184	709	94.479
40	40	1	115	2178	184	1509	44.379
40	80	2	115	2978	184	2309	29.000
40	120	3	101	3818	184	3109	22.807
40	160	4	67	5108	184	3909	30.682
40	200	5	64	6346	184	4708	34.774
50	0	0	144	1726	230	889	94.129
50	50	1	144	2726	230	1889	44.303
50	100	2	144	3726	230	2889	28.969
50	150	3	126	4776	230	3889	22.815
50	200	4	84	6391	230	4889	30.723
50	250	5	79	7937	230	5889	34.783
60	0	0	173	2074	276	1070	93.897
60	60	1	173	3274	276	2270	44.253
60	120	2	173	4474	276	3470	28.948
60	180	3	151	5735	276	4670	22.820
60	240	4	100	7675	276	5870	30.750
60	300	5	95	9529	276	7069	34.789

$T = 5$							
μ	s_1	s_1/μ	S_{best}	$NPV_T(S_{best})$	S	$NPV_T(S)$	% Increase
70	0	0	201	2422	322	1250	93.732
70	70	1	201	3822	322	2650	44.217
70	140	2	201	5222	322	4050	28.933
70	210	3	176	6694	322	5450	22.823
70	280	4	117	8958	322	6850	30.769
70	350	5	111	11120	322	8250	34.793
80	0	0	230	2770	367	1431	93.608
80	80	1	230	4370	367	3031	44.190
80	160	2	230	5970	367	4631	28.922
80	240	3	201	7653	367	6231	22.826
80	320	4	133	10241	367	7831	30.784
80	400	5	126	12712	367	9430	34.797
90	0	0	259	3118	413	1611	93.512
90	90	1	259	4918	413	3411	44.169
90	180	2	259	6718	413	5211	28.913
90	270	3	226	8612	413	7011	22.828
90	360	4	150	11525	413	8811	30.795
90	450	5	142	14303	413	10611	34.799
100	0	0	287	3466	459	1792	93.435
100	100	1	287	5466	459	3792	44.153
100	200	2	287	7466	459	5792	28.906
100	300	3	251	9571	459	7792	22.830
100	400	4	166	12808	459	9792	30.804
100	500	5	158	15895	459	11791	34.801

$T = 10$							
μ	s_1	s_1/μ	S_{best}	$NPV_T(S_{best})$	S	$NPV_T(S)$	% Increase
10	0	0	35	982	47	882	11.412
10	10	1	35	1182	47	1082	9.302
10	20	2	35	1382	47	1282	7.850
10	30	3	35	1582	47	1482	6.791
10	40	4	34	1791	47	1682	6.488
10	50	5	32	2002	47	1881	6.439
10	60	6	31	2213	47	2075	6.643
10	70	7	27	2441	47	2263	7.840
10	80	8	23	2700	47	2451	10.172
10	90	9	21	2967	47	2647	12.086
10	100	10	20	3205	47	2853	12.338
20	0	0	69	1979	92	1778	11.324
20	20	1	69	2379	92	2178	9.245
20	40	2	69	2779	92	2578	7.810
20	60	3	69	3179	92	2978	6.761
20	80	4	66	3597	92	3378	6.493
20	100	5	64	4020	92	3777	6.441
20	120	6	60	4442	92	4165	6.667
20	140	7	53	4900	92	4541	7.916
20	160	8	46	5421	92	4916	10.274
20	180	9	41	5953	92	5309	12.148
20	200	10	39	6427	92	5721	12.341

$T = 10$							
μ	s_1	s_1/μ	S_{best}	$NPV_T(S_{best})$	S	$NPV_T(S)$	% Increase
30	0	0	103	2977	138	2675	11.295
30	30	1	103	3577	138	3275	9.225
30	60	2	103	4177	138	3875	7.797
30	90	3	103	4777	138	4475	6.751
30	120	4	99	5404	138	5075	6.494
30	150	5	96	6038	138	5673	6.442
30	180	6	90	6672	138	6254	6.676
30	210	7	79	7360	138	6818	7.942
30	240	8	68	8142	138	7381	10.308
30	270	9	61	8940	138	7970	12.169
30	300	10	57	9650	138	8590	12.341
40	0	0	138	3974	184	3571	11.280
40	40	1	138	4774	184	4371	9.216
40	80	2	138	5574	184	5171	7.790
40	120	3	138	6374	184	5971	6.746
40	160	4	132	7211	184	6771	6.495
40	200	5	127	8057	184	7569	6.442
40	240	6	119	8901	184	8344	6.681
40	280	7	105	9819	184	9096	7.956
40	320	8	90	10863	184	9847	10.326
40	360	9	81	11927	184	10632	12.180
40	400	10	76	12873	184	11458	12.342

$T = 10$							
μ	s_1	s_1/μ	S_{best}	$NPV_T(S_{best})$	S	$NPV_T(S)$	% Increase
50	0	0	172	4971	230	4468	11.271
50	50	1	172	5971	230	5468	9.210
50	100	2	172	6971	230	6468	7.786
50	150	3	172	7971	230	7468	6.743
50	200	4	165	9018	230	8468	6.495
50	250	5	159	10075	230	9465	6.443
50	300	6	149	11131	230	10434	6.683
50	350	7	131	12279	230	11373	7.964
50	400	8	113	13585	230	12312	10.337
50	450	9	101	14914	230	13294	12.186
50	500	10	95	16095	230	14327	12.342
60	0	0	206	5969	276	5364	11.265
60	60	1	206	7169	276	6564	9.206
60	120	2	206	8369	276	7764	7.783
60	180	3	206	9569	276	8964	6.741
60	240	4	197	10825	276	10164	6.496
60	300	5	190	12093	276	11361	6.443
60	360	6	179	13360	276	12523	6.685
60	420	7	158	14738	276	13650	7.969
60	480	8	135	16306	276	14777	10.344
60	540	9	121	17902	276	15956	12.190
60	600	10	114	19318	276	17196	12.342

$T = 10$							
μ	s_1	s_1/μ	S_{best}	$NPV_T(S_{best})$	S	$NPV_T(S)$	% Increase
70	0	0	240	6966	322	6261	11.261
70	70	1	240	8366	322	7661	9.203
70	140	2	240	9766	322	9061	7.781
70	210	3	240	11166	322	10461	6.740
70	280	4	230	12631	322	11861	6.496
70	350	5	222	14111	322	13257	6.443
70	420	6	208	15590	322	14613	6.686
70	490	7	184	17198	322	15928	7.973
70	560	8	158	19027	322	17243	10.349
70	630	9	141	20889	322	18618	12.193
70	700	10	133	22541	322	20064	12.342
80	0	0	274	7963	367	7158	11.258
80	80	1	274	9563	367	8758	9.201
80	160	2	274	11163	367	10358	7.780
80	240	3	274	12763	367	11958	6.739
80	320	4	263	14438	367	13558	6.496
80	400	5	254	16129	367	15153	6.443
80	480	6	238	17820	367	16703	6.687
80	560	7	210	19658	367	18205	7.976
80	640	8	180	21748	367	19708	10.353
80	720	9	161	23876	367	21280	12.196
80	800	10	152	25763	367	22933	12.342

$T = 10$							
μ	s_1	s_1/μ	S_{best}	$NPV_T(S_{best})$	S	$NPV_T(S)$	% Increase
90	0	0	309	8961	413	8054	11.255
90	90	1	308	10761	413	9854	9.199
90	180	2	308	12561	413	11654	7.779
90	270	3	309	14361	413	13454	6.738
90	360	4	296	16245	413	15254	6.496
90	450	5	285	18147	413	17049	6.443
90	540	6	268	20049	413	18792	6.688
90	630	7	236	22117	413	20483	7.978
90	720	8	202	24469	413	22173	10.356
90	810	9	181	26863	413	23942	12.197
90	900	10	171	28986	413	25802	12.342
100	0	0	343	9958	459	8951	11.253
100	100	1	343	11958	459	10951	9.198
100	200	2	343	13958	459	12951	7.778
100	300	3	343	15958	459	14951	6.737
100	400	4	329	18052	459	16951	6.496
100	500	5	317	20166	459	18945	6.443
100	600	6	297	22279	459	20882	6.689
100	700	7	262	24577	459	22760	7.980
100	800	8	225	27191	459	24639	10.358
100	900	9	201	29850	459	26604	12.199
100	1000	10	189	32209	459	28670	12.342

$T = 15$							
μ	s_1	s_1/μ	S_{best}	$NPV_T(S_{best})$	S	$NPV_T(S)$	% Increase
10	0	0	38	1631	47	1561	4.493
10	10	1	38	1831	47	1761	3.983
10	20	2	38	2031	47	1961	3.576
10	30	3	38	2231	47	2161	3.245
10	40	4	37	2434	47	2361	3.075
10	50	5	37	2639	47	2561	3.074
10	60	6	36	2839	47	2755	3.046
10	70	7	35	3032	47	2942	3.056
10	80	8	34	3220	47	3122	3.117
10	90	9	33	3404	47	3296	3.285
10	100	10	31	3592	47	3465	3.683
10	110	11	29	3792	47	3631	4.431
10	120	12	26	4008	47	3800	5.465
10	130	13	24	4231	47	3974	6.450
10	140	14	23	4444	47	4153	7.004
10	150	15	21	4632	47	4329	6.980
20	0	0	75	3279	92	3138	4.472
20	20	1	75	3679	92	3538	3.966
20	40	2	75	4079	92	3938	3.563
20	60	3	75	4479	92	4338	3.235
20	80	4	73	4884	92	4738	3.079
20	100	5	72	5295	92	5137	3.072
20	120	6	71	5693	92	5525	3.046
20	140	7	70	6079	92	5899	3.058
20	160	8	68	6455	92	6259	3.121
20	180	9	66	6824	92	6607	3.295
20	200	10	62	7201	92	6944	3.704
20	220	11	57	7602	92	7277	4.467
20	240	12	52	8035	92	7615	5.509
20	260	13	48	8480	92	7964	6.487
20	280	14	44	8905	92	8321	7.024
20	300	15	42	9279	92	8674	6.982

$T = 15$							
μ	s_1	s_1/μ	S_{best}	$NPV_T(S_{best})$	S	$NPV_T(S)$	% Increase
30	0	0	112	4926	138	4716	4.465
30	30	1	112	5526	138	5316	3.961
30	60	2	112	6126	138	5916	3.559
30	90	3	112	6726	138	6516	3.231
30	120	4	110	7335	138	7116	3.080
30	150	5	108	7951	138	7714	3.072
30	180	6	106	8548	138	8296	3.046
30	210	7	104	9127	138	8856	3.059
30	240	8	102	9690	138	9396	3.122
30	270	9	98	10244	138	9917	3.298
30	300	10	93	10810	138	10423	3.712
30	330	11	85	11412	138	10923	4.479
30	360	12	78	12062	138	11430	5.525
30	390	13	71	12730	138	11953	6.500
30	420	14	66	13367	138	12489	7.031
30	450	15	63	13927	138	13018	6.982
40	0	0	149	6574	184	6293	4.461
40	40	1	149	7374	184	7093	3.958
40	80	2	149	8174	184	7893	3.557
40	120	3	149	8974	184	8693	3.230
40	160	4	146	9786	184	9493	3.081
40	200	5	144	10607	184	10291	3.072
40	240	6	142	11403	184	11066	3.046
40	280	7	139	12175	184	11813	3.059
40	320	8	135	12925	184	12534	3.123
40	360	9	131	13664	184	13228	3.300
40	400	10	123	14419	184	13902	3.716
40	440	11	114	15222	184	14569	4.486
40	480	12	103	16088	184	15245	5.532
40	520	13	94	16980	184	15943	6.507
40	560	14	88	17829	184	16657	7.034
40	600	15	83	18575	184	17363	6.983

$T = 15$							
μ	s_1	s_1/μ	S_{best}	$NPV_T(S_{best})$	S	$NPV_T(S)$	% Increase
50	0	0	186	8222	230	7871	4.459
50	50	1	186	9222	230	8871	3.956
50	100	2	186	10222	230	9871	3.556
50	150	3	186	11222	230	10871	3.228
50	200	4	183	12237	230	11871	3.081
50	250	5	180	13263	230	12868	3.072
50	300	6	177	14258	230	13837	3.046
50	350	7	173	15223	230	14771	3.059
50	400	8	169	16160	230	15671	3.124
50	450	9	163	17085	230	16539	3.301
50	500	10	154	18027	230	17381	3.718
50	550	11	142	19032	230	18214	4.490
50	600	12	129	20115	230	19060	5.537
50	650	13	118	21230	230	19933	6.510
50	700	14	110	22290	230	20825	7.036
50	750	15	104	23223	230	21708	6.983
60	0	0	223	9870	276	9449	4.458
60	60	1	223	11070	276	10649	3.955
60	120	2	223	12270	276	11849	3.555
60	180	3	223	13470	276	13049	3.228
60	240	4	219	14688	276	14249	3.082
60	300	5	216	15920	276	15445	3.071
60	360	6	212	17113	276	16607	3.046
60	420	7	208	18270	276	17728	3.059
60	480	8	203	19396	276	18808	3.124
60	540	9	196	20505	276	19849	3.302
60	600	10	185	21636	276	20860	3.720
60	660	11	170	22842	276	21860	4.492
60	720	12	154	24142	276	22875	5.540
60	780	13	141	25480	276	23922	6.513
60	840	14	131	26752	276	24993	7.038
60	900	15	125	27872	276	26052	6.983

$T = 15$							
μ	s_1	s_1/μ	S_{best}	$NPV_T(S_{best})$	S	$NPV_T(S)$	% Increase
70	0	0	260	11518	322	11026	4.457
70	70	1	260	12918	322	12426	3.954
70	140	2	260	14318	322	13826	3.554
70	210	3	260	15718	322	15226	3.227
70	280	4	255	17139	322	16626	3.082
70	350	5	252	18576	322	18022	3.071
70	420	6	247	19968	322	19378	3.046
70	490	7	242	21318	322	20685	3.059
70	560	8	237	22631	322	21945	3.124
70	630	9	228	23925	322	23160	3.303
70	700	10	215	25245	322	24339	3.721
70	770	11	198	26652	322	25506	4.494
70	840	12	180	28169	322	26690	5.542
70	910	13	164	29730	322	27912	6.515
70	980	14	153	31214	322	29161	7.039
70	1050	15	146	32520	322	30397	6.983
80	0	0	297	13165	367	12604	4.456
80	80	1	297	14765	367	14204	3.954
80	160	2	297	16365	367	15804	3.554
80	240	3	297	17965	367	17404	3.227
80	320	4	292	19589	367	19004	3.082
80	400	5	287	21232	367	20599	3.071
80	480	6	283	22823	367	22149	3.046
80	560	7	277	24366	367	23643	3.059
80	640	8	270	25866	367	25083	3.124
80	720	9	261	27345	367	26471	3.303
80	800	10	246	28854	367	27819	3.722
80	880	11	226	30462	367	29152	4.495
80	960	12	206	32196	367	30505	5.544
80	1040	13	188	33980	367	31901	6.516
80	1120	14	175	35676	367	33330	7.039
80	1200	15	166	37168	367	34742	6.983

$T = 15$							
μ	s_1	s_1/μ	S_{best}	$NPV_T(S_{best})$	S	$NPV_T(S)$	% Increase
90	0	0	334	14813	413	14181	4.455
90	90	1	334	16613	413	15981	3.953
90	180	2	334	18413	413	17781	3.553
90	270	3	333	20213	413	19581	3.227
90	360	4	328	22040	413	21381	3.082
90	450	5	323	23888	413	23176	3.071
90	540	6	318	25678	413	24919	3.046
90	630	7	312	27414	413	26600	3.059
90	720	8	304	29102	413	28220	3.124
90	810	9	293	30765	413	29782	3.303
90	900	10	277	32463	413	31298	3.722
90	990	11	255	34272	413	32798	4.496
90	1080	12	231	36223	413	34320	5.545
90	1170	13	211	38230	413	35891	6.517
90	1260	14	196	40138	413	37498	7.040
90	1350	15	187	41816	413	39086	6.983
100	0	0	371	16461	459	15759	4.455
100	100	1	371	18461	459	17759	3.953
100	200	2	371	20461	459	19759	3.553
100	300	3	371	22461	459	21759	3.226
100	400	4	364	24491	459	23759	3.082
100	500	5	359	26544	459	25753	3.071
100	600	6	353	28533	459	27690	3.046
100	700	7	346	30462	459	29557	3.059
100	800	8	338	32337	459	31357	3.124
100	900	9	325	34185	459	33092	3.304
100	1000	10	307	36072	459	34777	3.723
100	1100	11	283	38082	459	36443	4.497
100	1200	12	257	40250	459	38135	5.546
100	1300	13	234	42480	459	39881	6.518
100	1400	14	218	44599	459	41666	7.040
100	1500	15	208	46464	459	43431	6.983

$T = 20$							
μ	s_1	s_1/μ	S_{best}	$NPV_T(S_{best})$	S	$NPV_T(S)$	% Increase
10	0	0	39	2259	47	2207	2.372
10	10	1	39	2459	47	2407	2.175
10	20	2	39	2659	47	2607	2.008
10	30	3	39	2859	47	2807	1.865
10	40	4	39	3060	47	3007	1.758
10	50	5	39	3264	47	3207	1.777
10	60	6	38	3461	47	3401	1.760
10	70	7	38	3651	47	3588	1.756
10	80	8	38	3835	47	3768	1.764
10	90	9	37	4012	47	3942	1.785
10	100	10	37	4184	47	4109	1.821
10	110	11	36	4349	47	4269	1.878
10	120	12	35	4511	47	4423	1.975
10	130	13	34	4670	47	4572	2.150
10	140	14	33	4833	47	4717	2.448
10	150	15	31	5002	47	4861	2.900
10	160	16	29	5181	47	5007	3.481
10	170	17	27	5366	47	5155	4.083
10	180	18	25	5547	47	5306	4.557
10	190	19	24	5716	47	5454	4.787
10	200	20	23	5862	47	5596	4.748

$T = 20$							
μ	s_1	s_1/μ	S_{best}	$NPV_T(S_{best})$	S	$NPV_T(S)$	% Increase
20	0	0	78	4537	92	4432	2.363
20	20	1	78	4937	92	4832	2.168
20	40	2	78	5337	92	5232	2.002
20	60	3	78	5737	92	5632	1.860
20	80	4	77	6138	92	6032	1.762
20	100	5	77	6545	92	6431	1.776
20	120	6	76	6939	92	6819	1.759
20	140	7	75	7319	92	7193	1.756
20	160	8	74	7686	92	7553	1.765
20	180	9	73	8041	92	7900	1.786
20	200	10	72	8383	92	8233	1.822
20	220	11	71	8714	92	8553	1.880
20	240	12	70	9036	92	8861	1.980
20	260	13	67	9356	92	9158	2.158
20	280	14	64	9681	92	9448	2.462
20	300	15	61	10021	92	9737	2.920
20	320	16	57	10379	92	10028	3.506
20	340	17	53	10749	92	10325	4.108
20	360	18	49	11112	92	10626	4.575
20	380	19	47	11448	92	10924	4.798
20	400	20	45	11739	92	11207	4.750

$T = 20$							
μ	s_1	s_1/μ	S_{best}	$NPV_T(S_{best})$	S	$NPV_T(S)$	% Increase
30	0	0	117	6814	138	6657	2.360
30	30	1	117	7414	138	7257	2.165
30	60	2	117	8014	138	7857	2.000
30	90	3	117	8614	138	8457	1.858
30	120	4	116	9217	138	9057	1.763
30	150	5	115	9827	138	9655	1.776
30	180	6	114	10417	138	10237	1.759
30	210	7	113	10987	138	10797	1.756
30	240	8	111	11538	138	11338	1.765
30	270	9	110	12069	138	11857	1.787
30	300	10	108	12582	138	12357	1.823
30	330	11	106	13079	138	12837	1.881
30	360	12	104	13562	138	13299	1.982
30	390	13	101	14042	138	13744	2.161
30	420	14	96	14529	138	14180	2.467
30	450	15	90	15040	138	14612	2.927
30	480	16	84	15578	138	15049	3.514
30	510	17	78	16133	138	15495	4.116
30	540	18	73	16677	138	15947	4.582
30	570	19	70	17180	138	16393	4.801
30	600	20	67	17617	138	16818	4.751

$T = 20$							
μ	s_1	s_1/μ	S_{best}	$NPV_T(S_{best})$	S	$NPV_T(S)$	% Increase
40	0	0	156	9092	184	8882	2.359
40	40	1	156	9892	184	9682	2.164
40	80	2	156	10692	184	10482	1.999
40	120	3	156	11492	184	11282	1.857
40	160	4	154	12295	184	12082	1.764
40	200	5	153	13109	184	12880	1.776
40	240	6	151	13895	184	13655	1.759
40	280	7	150	14655	184	14402	1.756
40	320	8	148	15389	184	15122	1.765
40	360	9	146	16098	184	15815	1.787
40	400	10	144	16782	184	16481	1.823
40	440	11	141	17444	184	17121	1.882
40	480	12	138	18088	184	17737	1.983
40	520	13	134	18727	184	18331	2.163
40	560	14	128	19378	184	18911	2.469
40	600	15	120	20059	184	19488	2.931
40	640	16	112	20777	184	20071	3.519
40	680	17	104	21517	184	20665	4.120
40	720	18	98	22243	184	21267	4.585
40	760	19	92	22912	184	21862	4.803
40	800	20	88	23494	184	22429	4.751

$T = 20$							
μ	s_1	s_1/μ	S_{best}	$NPV_T(S_{best})$	S	$NPV_T(S)$	% Increase
50	0	0	194	11369	230	11107	2.358
50	50	1	194	12369	230	12107	2.163
50	100	2	194	13369	230	13107	1.998
50	150	3	194	14369	230	14107	1.857
50	200	4	193	15374	230	15107	1.764
50	250	5	191	16391	230	16105	1.776
50	300	6	189	17374	230	17073	1.759
50	350	7	187	18323	230	18007	1.756
50	400	8	185	19241	230	18907	1.765
50	450	9	183	20127	230	19773	1.787
50	500	10	180	20982	230	20606	1.823
50	550	11	177	21809	230	21406	1.882
50	600	12	173	22614	230	22175	1.983
50	650	13	167	23413	230	22917	2.164
50	700	14	159	24227	230	23642	2.470
50	750	15	150	25078	230	24363	2.933
50	800	16	140	25976	230	25092	3.521
50	850	17	130	26901	230	25835	4.123
50	900	18	122	27808	230	26588	4.587
50	950	19	115	28645	230	27332	4.804
50	1000	20	110	29372	230	28039	4.751

$T = 20$							
μ	s_1	s_1/μ	S_{best}	$NPV_T(S_{best})$	S	$NPV_T(S)$	% Increase
60	0	0	233	13647	276	13333	2.357
60	60	1	233	14847	276	14533	2.163
60	120	2	233	16047	276	15733	1.998
60	180	3	233	17247	276	16933	1.856
60	240	4	231	18453	276	18133	1.764
60	300	5	229	19672	276	19329	1.775
60	360	6	227	20852	276	20491	1.759
60	420	7	224	21992	276	21612	1.756
60	480	8	222	23093	276	22692	1.765
60	540	9	219	24155	276	23731	1.787
60	600	10	216	25181	276	24730	1.824
60	660	11	212	26174	276	25690	1.882
60	720	12	207	27140	276	26613	1.984
60	780	13	200	28099	276	27504	2.164
60	840	14	191	29075	276	28374	2.471
60	900	15	180	30097	276	29239	2.935
60	960	16	168	31174	276	30113	3.523
60	1020	17	156	32284	276	31006	4.124
60	1080	18	146	33373	276	31909	4.588
60	1140	19	138	34377	276	32801	4.804
60	1200	20	132	35249	276	33650	4.751

$T = 20$							
μ	s_1	s_1/μ	S_{best}	$NPV_T(S_{best})$	S	$NPV_T(S)$	% Increase
70	0	0	272	15925	322	15558	2.357
70	70	1	272	17325	322	16958	2.162
70	140	2	272	18725	322	18358	1.998
70	210	3	272	20125	322	19758	1.856
70	280	4	269	21531	322	21158	1.765
70	350	5	267	22954	322	22554	1.775
70	420	6	265	24330	322	23910	1.759
70	490	7	262	25660	322	25217	1.756
70	560	8	259	26944	322	26477	1.765
70	630	9	255	28184	322	27689	1.787
70	700	10	252	29381	322	28855	1.824
70	770	11	247	30538	322	29974	1.883
70	840	12	241	31667	322	31051	1.984
70	910	13	233	32785	322	32090	2.165
70	980	14	223	33924	322	33105	2.472
70	1050	15	210	35116	322	34114	2.936
70	1120	16	196	36373	322	35135	3.524
70	1190	17	182	37668	322	36176	4.125
70	1260	18	170	38938	322	37230	4.589
70	1330	19	161	40110	322	38271	4.805
70	1400	20	154	41126	322	39261	4.751

$T = 20$							
μ	s_1	s_1/μ	S_{best}	$NPV_T(S_{best})$	S	$NPV_T(S)$	% Increase
80	0	0	310	18202	367	17783	2.357
80	80	1	311	19802	367	19383	2.162
80	160	2	310	21402	367	20983	1.997
80	240	3	310	23002	367	22583	1.856
80	320	4	308	24610	367	24183	1.765
80	400	5	305	26236	367	25778	1.775
80	480	6	302	27809	367	27328	1.759
80	560	7	299	29328	367	28822	1.756
80	640	8	296	30796	367	30262	1.765
80	720	9	292	32213	367	31647	1.787
80	800	10	287	33580	367	32979	1.824
80	880	11	282	34903	367	34258	1.883
80	960	12	276	36193	367	35488	1.984
80	1040	13	267	37471	367	36676	2.165
80	1120	14	255	38772	367	37837	2.473
80	1200	15	240	40135	367	38990	2.936
80	1280	16	223	41572	367	40156	3.525
80	1360	17	208	43052	367	41346	4.126
80	1440	18	194	44503	367	42550	4.590
80	1520	19	184	45842	367	43740	4.805
80	1600	20	176	47004	367	44872	4.751

$T = 20$							
μ	s_1	s_1/μ	S_{best}	$NPV_T(S_{best})$	S	$NPV_T(S)$	% Increase
90	0	0	349	20480	413	20008	2.357
90	90	1	349	22280	413	21808	2.162
90	180	2	349	24080	413	23608	1.997
90	270	3	349	25880	413	25408	1.856
90	360	4	346	27688	413	27208	1.765
90	450	5	343	29518	413	29003	1.775
90	540	6	340	31287	413	30746	1.759
90	630	7	336	32996	413	32427	1.756
90	720	8	332	34647	413	34046	1.765
90	810	9	328	36241	413	35605	1.787
90	900	10	323	37780	413	37103	1.824
90	990	11	317	39268	413	38543	1.883
90	1080	12	310	40719	413	39926	1.984
90	1170	13	300	42156	413	41263	2.165
90	1260	14	286	43621	413	42568	2.473
90	1350	15	270	45154	413	43865	2.937
90	1440	16	251	46770	413	45178	3.526
90	1530	17	234	48436	413	46516	4.127
90	1620	18	219	50068	413	47871	4.590
90	1710	19	207	51574	413	49210	4.806
90	1800	20	198	52881	413	50483	4.751

$T = 20$							
μ	s_1	s_1/μ	S_{best}	$NPV_T(S_{best})$	S	$NPV_T(S)$	% Increase
100	0	0	388	22757	459	22233	2.356
100	100	1	388	24757	459	24233	2.162
100	200	2	388	26757	459	26233	1.997
100	300	3	388	28757	459	28233	1.856
100	400	4	384	30767	459	30233	1.765
100	500	5	381	32800	459	32228	1.775
100	600	6	378	34765	459	34164	1.759
100	700	7	374	36665	459	36032	1.756
100	800	8	369	38499	459	37831	1.765
100	900	9	365	40270	459	39563	1.787
100	1000	10	359	41980	459	41228	1.824
100	1100	11	353	43633	459	42827	1.883
100	1200	12	344	45245	459	44364	1.984
100	1300	13	333	46842	459	45849	2.166
100	1400	14	318	48470	459	47300	2.474
100	1500	15	300	50173	459	48741	2.937
100	1600	16	279	51969	459	50199	3.526
100	1700	17	260	53820	459	51686	4.128
100	1800	18	243	55634	459	53192	4.591
100	1900	19	230	57307	459	54679	4.806
100	2000	20	220	58759	459	56094	4.751

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