UNIFORM MODULES OVER GOLDIE PRIME SERIAL RINGS

By

FRANCO GUERRIERO, B.Sc., M.Math.

A Thesis

Submitted to the School of Graduate Studies

in Partial Fulfilment of the Requirements

for the Degree

Doctor of Philosophy

McMaster University

UNIFORM MODULES OVER GOLDIE PRIME SERIAL RINGS

DOCTOR OF PHILOSOPHY (1996) McMaster University

(Mathematics) Hamilton, Ontario

TITLE: UNIFORM MODULES OVER GOLDIE PRIME SERIAL RINGS

AUTHOR: Franco Guerriero

B.Sc. (Brock University)

M.Math. (University of Waterloo)

SUPERVISOR: Professor B.J. Müller

NUMBER OF PAGES: vii, 108

Abstract

We investigate the uniseriality of uniform modules. Let R be any ring and fix a decomposition $1 = e_1 + e_2 + \cdots + e_n$ into orthogonal idempotents. Let V_R be uniform and injective; we prove that there exists $e = e_i$ such that $V_R \cong \hom_A(Re, Ve)$ where $A = eRe$. Moreover, Ve is a uniform injective A-module. If R is Goldie prime serial, we prove that V is uniserial if and only if Ve is uniserial as an A-module.

If R is Goldie prime serial, we know that such an A is a valuation on a division ring D. We prove that any uniform injective, E_A, is of the form $E = E(D/I)$ for some $I \leq A$. If D/I is injective, then E is uniserial. We give several necessary and sufficient conditions for D/I to be injective.

In this study of uniform injectives over Goldie prime serial rings we define a notion of generalized associated primes. This leads to a semiprime Goldie ideal, S, which can be associated to any uniform injective. We prove that for certain uniform injectives, $\mathcal{C}(S)$ (the set of elements regular modulo S) is the largest Ore set operating regularly on the module.

Acknowledgements

I would like to begin by thanking my supervisor Professor Bruno J. Müller. His guidance, encouragement, and patience are deeply appreciated and will never be forgotten. I am very proud to have been his student and will always consider him a friend. I would also like to wish him happiness and good health in his retirement.

The Mathematics department and NSERC deserve thanks for their financial support. I would like to thank Dr. M. Valeriote, Dr. T. Choe, and Dr. A. Rosa for serving on my supervisory committee. They have assisted me financially and were a great deal of help in preparing the final version of the thesis. I would also like to thank Dr. M. Clase, Dr. M. Lelwala, and Dr. S. Mohamed for the many hours they spent discussing mathematics with me.

My friends at McMaster deserve a great deal of thanks. Their friendship has made life much more enjoyable. I would to thank Rob Stamicar, Mike Phau, Paul Stephenson, Mary Ballyk, Sean Hill, Tarit Saha, Boris Brauckmann, Eric Derbez, and Michael Klemm for many "interesting" times. I would like give special thanks to my friend Spiro (Q-boy) Daoussis for building me a computer. Without his help I would not have been able to type this manuscript. His support will long be remembered.

My family deserves a great deal of thanks. I would like to thank my parents for their many years of encouragement and patience. My parents and my in-laws also deserve a great deal of thanks for looking after our two young boys while I spent many hours working.

Finally, it is with great pleasure that I thank my wife Elise. Her support and constant encouragement during the past several years has made this possible. I would also like to acknowledge our two sons, Franco Jr. and Nicholas. Even though they bring me to the point of exhaustion, they manage to keep me feeling young. They give me the feeling this whole thing was worthwhile.

Contents

Chapter 1

Introduction

1.1 Preliminaries

In this section we give some of the definitions and results which will be used in the subsequent chapters. We will normally recall definitions when needed.

All rings will be associative but not necessarily commutative. All rings will have an identity element which will usually be denoted by 1. Unless otherwise stated, modules are assumed to be unitary.

The notations $X \subseteq Y$ means that X is a subset (or submodule) of Y and $X \subset Y$ means that X is a proper subset (or submodule) of Y. We will also use $\leq$ and $<$ in the same way to stress that a submodule or right (or left) ideal is meant. The symbols $\triangleleft$ and $\trianglelefteq$ will be used to stress the fact that we mean a two sided ideal. We shall use $E(M)$ to denote the injective hull of a module M.

Definition 1 *A module is said to be* **uniserial** *if its submodules are linearly ordered by inclusion. We say that a module is* **serial** *if it is equal to the direct sum of uniserial submodules.*

Proposition 2 *A module, M_R, is uniserial if and only if for all $x, y \in M$ either $y = xr$ or $x = yr$ for some $r \in R$.*

Definition 3 *A ring, R, is* **right (left) serial** *if R_R (${}_RR$) is serial as a module. It is said to be* **serial** *if it is both left and right serial.*

A module, M, is said to be **uniform** if, for all $0 \neq X, Y \leq M$, $X \cap Y \neq 0$.

Let R be a right serial ring. Then the identity of R can be decomposed as $1 = e_1 + e_2 + \cdots + e_n$ where $e_1, e_2, \ldots, e_n$ is a collection of indecomposable orthogonal idempotents. For each i, the module e_iR is uniserial. Given such a decomposition, $R \cong (X_{ij})_{i,j=1}^n$ where $X_{ij} = e_iRe_j$ [cf. M2].

Idempotents e and f are said to be **isomorphic** if $eR \cong fR$ as right R-modules. Let R be serial and let $1 = e_1 + e_2 + \cdots + e_n$ where $e_1, e_2, \ldots, e_n$ is a collection of indecomposable orthogonal idempotents. Then R is said to be **basic** if the idempotents are pairwise nonisomorphic. A serial ring always has a basic subring and it is obtained from the basic subring by "blocking" [cf. M2]. We will always assume that our serial rings are basic.

Let R be a Goldie prime serial ring. It follows that $R \cong (X_{ij})$ is a tiled order in $M_n(D)$ for some division ring D. This means that the X_{ij} are subsets of D and that the operations are the natural ones obtained from the matrix operations. It follows that, for each i, $A_i = X_{ii}$ is a valuation on the division ring D [M1, Theorem 2].

Definition 4 *Let D be a division ring. A proper subring $A \subset D$ is said to be a* **valuation** *(or a valuation on D) if, for all $0 \neq d \in D$, either $d \in A$ or $d^{-1} \in A$.*

We now give some examples.

Example. Every semisimple module is serial.

Example. Let n be a non-negative integer and consider $M = Z/p^nZ$ as a Z-module. The only submodules of M are Z/p^kZ for $k = 0, 1, 2, \ldots, n$. Hence, M is a uniserial Z-module.

Example. Let $A \subset D$ be a valuation on a division ring. Let $0 \neq x, y \in D$. Then either $x^{-1}y \in A$ or $y^{-1}x \in A$. Thus, $y \in xA$ or $x \in yA$. By our proposition 2, D_A is uniserial. Similarly, ${}_AD$ is uniserial.

We give some examples of valuations.

(i) $Z_{(p)}$, the integers localized at a prime ideal is a valuation on the rationals.

(ii) Let k be a field and let

$$A = k\langle Q^+\rangle = \left\{ f = \sum a_\alpha x^\alpha \mid 0 \leq \alpha \in Q,\ a_\alpha \in k,\ supp(f) \text{ is well ordered} \right\}.$$

Then A is a valuation on its quotient ring, D, the Laurent series ring. We call A the ring of power series having well ordered support.

Example. Let A be a valuation on a division ring and let $\mathfrak{m}$ denote the maximal ideal of A. Then

$$R = \begin{pmatrix} A & A \\ \mathfrak{m} & A \end{pmatrix}$$

is a Goldie prime serial ring [M1].

Definition 5 *Let R be a ring. A proper ideal $P \lhd R$ is said to be a* **prime** *ideal if $a, b \in R$ and $aRb \subseteq P$ imply that $a \in P$ or $b \in P$. We say that P is* **completely prime** *if $a, b \in R$ and $ab \in P$ imply that $a \in P$ or $b \in P$.*

The collection of all prime ideals in a ring, R, is called the **spectrum of** R and will be denoted by $spec(R)$.

We shall use the following results. It is assumed that R is a serial ring and that $1 = e_1 + e_2 + \cdots + e_n$ is a fixed decomposition of the identity into indecomposable orthogonal idempotents:

(1) [MS1, lemma 3.1] If P and Q are incomparable prime ideals, then $P + Q = R$.

(2) [MS1, lemma 3.3] Let $P, Q \in spec(R)$. If $e_i \in Q$ implies that $e_i \in P$, then P and Q are comparable. If, in addition, there is some $e_i \in Q - P$, then $P \subset Q$.

For $P \in spec(R)$ we let $E(P) = \{e_i \mid e_i \notin P\}$. Since $P \neq R$, $E(P) \neq \emptyset$.

Proposition 6 *Let $P, Q \in spec(R)$. If $E(P) \cap E(Q) \neq \emptyset$, then P and Q are comparable.*

Proof. If $e \in E(P) \cap E(Q)$, then eP and eQ are in the uniserial module eR. Hence, eP and eQ are comparable. If $eP \subseteq eQ$, then $eP \subseteq Q$. Since Q is prime and $e \notin Q$, we conclude that $P \subseteq Q$.

For any subset $T \subseteq \{e_1, e_2, \ldots, e_n\}$, we define

$$P(T) = \{P \in spec(R) \mid P \cap \{e_1, e_2, \ldots, e_n\} = T\}.$$

In other words, $E(P) = T^c$. Using [MS1, lemma 3.3], we get that each $P(T)$ is a chain. It could be that some $P(T) = \emptyset$. If $P(T) \neq \emptyset$, then we call $P(T)$ a **tower** of $spec(R)$.

Let $P(T_1)$ and $P(T_2)$ be two towers. Then $T_1 \subset T_2$ if and only if $P_1 \subset P_2$ for some (and then for all) $P_1 \in P(T_1)$ and $P_2 \in P(T_2)$. We can now construct $spec(R)$ as follows. Construct a graph with vertices $V = \{T \mid P(T) \neq \emptyset\}$. Two distinct vertices, T_1 and T_2, are adjacent if they are comparable, say

$T_1 \subset T_2$, and there does not exist another $T \in V$ such that $T_1 \subset T \subset T_2$. The spectrum of R is obtained by replacing each vertex, T, with the chain $P(T)$. The interested reader is referred to [MS1] for a more detailed description. See also [M2] for diagrams of $spec(R)$.

1.1.1 The Morita Context

The following material is taken from [McR, 3.6].

Let R and S be rings and ${}_RV_S$ and ${}_SW_R$ be bimodules such that $VW \subseteq R$ and $WV \subseteq S$. We can then form a new ring

$$\begin{pmatrix} R & V \\ W & S \end{pmatrix}$$

by using the natural matrix operations. We call this a **Morita context.**

Theorem 7 *Let* $\begin{pmatrix} R & V \\ W & S \end{pmatrix}$ *be a Morita context. There is an order preserving* $1-1$ *correspondence between*

$$\{P \in spec(R) \mid VW \not\subseteq P\} \text{ and } \{Q \in spec(S) \mid WV \not\subseteq Q\}.$$

The correspondence is given by $P \mapsto \{s \in S \mid VsW \subseteq P\}$.

If P and Q are prime ideals which correspond under the above correspondence, we will write $P \rightleftharpoons Q$. We shall refer to this as the Morita context correspondence and we shall abbreviate this by MCC.

Let R be a ring and $e, f \in R$ be two idempotents in R. Let $A = eRe$ and $B = fRf$. Then we can form the following Morita contexts

$$\begin{pmatrix} R & Re \\ eR & A \end{pmatrix} \text{ and } \begin{pmatrix} A & eRf \\ fRe & B \end{pmatrix}.$$

Consider these contexts under the further assumption that R is serial and that e and f are indecomposable orthogonal idempotents. Then A and B are uniserial rings and eRf and fRe are uniserial as A and B modules [M2].

For the Morita context $\begin{pmatrix} R & Re \\ eR & A \end{pmatrix}$, the theorem implies that there is an order preserving $1-1$ correspondence between

$$\{P \in spec(R) \mid ReR \not\subseteq P\} \text{ and } \{\wp \in spec(A) \mid eRe \not\subseteq \wp\}.$$

But $ReR \not\subseteq P$ if and only if $e \notin P$ and $eRe \not\subseteq \wp$ is true for all $\wp \in spec(A)$. Therefore, in this case there is an order preserving $1-1$ correspondence between

$$\{P \in spec(R) \mid e \notin P\} \text{ and } spec(A).$$

In this case, if $P \in spec(R)$ and $e \notin P$, then $P \rightleftharpoons ePe$. If $\wp \in spec(A)$, then $\wp \rightleftharpoons \{r \in R \mid eRrRe \subseteq \wp\}$.

For the Morita context $\begin{pmatrix} A & eRf \\ fRe & B \end{pmatrix}$ the correspondence is between

$$\{\wp \in spec(A) \mid eRfRe \not\subseteq \wp\} \text{ and } \{\zeta \in spec(B) \mid fReRf \not\subseteq \zeta\}.$$

If A is uniserial, then $eRfRe \not\subseteq \wp$ if and only if $\wp \subset eRfRe$. Similarly for B. Thus, the Morita context correspondence (MCC) reduces to

$$\{\wp \in spec(A) \mid \wp \subset eRfRe\} \text{ and } \{\zeta \in spec(B) \mid \zeta \subset fReRf\}.$$

If $\wp \in spec(A)$ and $\wp \subset eRfRe$, then $\wp \rightleftharpoons \{b \in B \mid eRbRe \subseteq \wp\}$.

Let R be a serial ring and let e be an indecomposable idempotent and consider the Morita context $\begin{pmatrix} R & Re \\ eR & A \end{pmatrix}$ where $A = eRe$. The following is a consequence of the results in [McR, 3.6]. If $P \in spec(R)$ and $\wp \in spec(A)$ and $P \rightleftharpoons \wp$ under the Morita context correspondence, then P is Goldie if and only if $\wp$ is Goldie.

Definition 8 *An ideal $P \trianglelefteq R$ is said to be* **Goldie** *if R/P is Goldie.*

1.2 A Brief Survey

The study of serial rings, in the Artinian case, was initiated by Nakayama in the early 1940's under the name generalized uniserial rings [NK1, NK2, NK3]. He proved that an Artinian ring is serial if and only if every finitely generated module is serial. It was later shown that over an Artinian serial ring every module is serial [SK, EG1, EG2].

The structure of Artinian serial rings was studied by Kupish and he was able to show that these rings are completely determined (up to isomorphism) by a set of invariants [K1, K2, K3]. Murase continued the investigation into the structure of Artinian serial rings and described such rings in terms of matrix representations [MR1, MR2, MR3].

In 1975 Warfield began the study of general (that is, not necessarily Artinian) serial rings [W]. He proved that a ring is serial if and only if every finitely presented module is serial. In the same paper he also studied Noetherian serial rings and gave a fairly complete structure theory for such rings. He also showed that over a Noetherian serial ring every uniform module is uniserial. In the same year Ivanov characterized the rings having the property that all finitely generated modules are serial. He proved that every finitely generated R-module is serial if and only if R is left serial and every indecomposable injective R-module is uniserial [I]. In such a case, R is serial.

In 1984 Singh studied and gave the structure of a right Noetherian serial ring [S]. In a series of papers Wright (Upham) investigated the structure of serial rings with Krull dimension one and, in the prime case, rings having arbitrary Krull dimension [U, WR1, WR2, WR3]. Since Noetherian serial rings have Krull dimension one, her results, on one hand, can be viewed as generalizations of Warfield's and Singh's work. She first considered the case that the ring is nonsingular with Krull dimension one. In the next paper she removed the condition of nonsingularity. A result which proved to be important in this development is that certain uniform modules are uniserial. This is, in some sense, what motivates the question of when a uniform module over a particular serial ring is uniserial.

Chatters was able to extend some of the above results to rings having finite Krull dimension [C]. He described the structure of a prime serial ring in terms of blocked matrix rings. In the same paper he shows that if K is a Goldie semiprime ideal which doesn't contain any idempotents, then $C(K)$ is an Ore set.

In 1990 Müller and Singh also studied uniform modules over serial rings [MS1, MS2]. They showed that certain uniform modules are uniserial, and studied the spectrum of serial rings. Much information about the prime ideals of serial rings is compiled in their work.

In 1992 Müller determined the structure of Goldie prime serial rings [M1]. He also began the study of the structure of general serial rings [M2]. The structure theory for the latter is far from complete.

1.3 Thesis Overview

It is clear that a uniserial module is uniform but that the converse need not be true. Warfield has shown that over left Noetherian serial rings any uniform is uniserial [W]. Wright has also investigated conditions under which a uniform module is uniserial [U, WR2]. In [MS1] and [MS2] Müller and Singh have investigated the prime ideals of serial rings and introduced cliques. They have also shown certain uniform modules to be uniserial.

Our work begins with an examination of uniform modules. Since the injective hull of a uniform module is again uniform, we concentrate on uniform injective modules.

We begin with an arbitrary ring R and a fixed decomposition of the identity into orthogonal idempotents, say $1 = e_1 + e_2 + \cdots + e_n$. Let V be an R-module. We prove that V is uniform and injective if and only if there exists $e = e_i$ and, with $A = eRe$, a uniform injective A-module E such that $V_R \cong \hom_A(Re, E)_R$.

We use this description to study uniform injective modules over Goldie prime serial rings where the e_i are indecomposable idempotents. The main result of chapter 2 is that V_R is uniserial if and only if E_A is uniserial. It is still open whether this is true for an arbitrary serial ring.

In the case we have just described, the ring A is a valuation on a division ring [M1]. Since the uniseriality of V is equivalent to that of E, it makes sense to study uniform injective modules over valuation rings. In the third chapter it is shown that a uniform injective module over a valuation A on D is of the form $E(D/I)$ for some right ideal $I \leq A$. Furthermore, if D is a field, then E is uniserial if and only if $E = D/I$. Thus, when D is a field, the uniseriality of

E and the injectivity of D/I are equivalent. It remains unknown whether this true in general. In any case, if D/I is injective, then $E = D/I$ is uniserial and the module V is uniserial. This gives a sufficient condition for the uniseriality of V. The remainder of the chapter is devoted to determining when D/I is injective. The main result provides a list of conditions which are equivalent to the injectivity of D/I. Among others we prove that D/I is injective if and only if D is complete in certain linear topologies.

In the final chapter our goal is to attach a (semi) prime ideal to a uniform injective module over a Goldie prime serial ring. We consider a notion of generalized associated primes and show that for uniform injective modules over valuation rings these always exist. For a uniform injective module, V, over a Goldie prime serial ring R, we get a collection of idempotents $e_1, e_2, \ldots, e_k$ such that, with $A_i = e_iRe_i$, $V \cong \hom_{A_i}(Re, E_i)$. We have named such idempotents faithful. For each faithful idempotent, E_i is a uniform injective module over the valuation ring A_i. Hence, for each such idempotent we get a generalized associated prime ideal $\wp_i \in spec(A_i)$. Using the Morita context correspondence, we get (Goldie) prime ideals $P_1, P_2, \ldots, P_k \in spec(R)$ which correspond to the $\wp_i$. Among other things we prove that either all the P_i are equal, or the distinct P_i are incomparable Goldie prime ideals and any prime ideal which is properly contained in one of them is contained in all of them. In the case that all idempotents are faithful (to which we may reduce) this result implies that either all the P_i are equal and this Goldie prime is in the bottom tower of the spectrum of R, or the distinct P_i are the minimal primes above the "lowest fork".

Another characterization of the P_i is that they are the ideals which are maximal with respect to having empty intersection with the largest Ore set operating regularly on V. Furthermore, we prove that the largest Ore set operating regularly on V is $\mathcal{C}(\bigcap P_i)$.

Chapter 2

Uniseriality Of Uniform Modules

2.1 Uniform Injective Modules Over Arbitrary Rings

Let us recall a familiar fact: for rings R and A, and modules ${}_RM_A$ and N_A, $\hom_A(M, N)$ can be made into a right R-module. The action of R is given by

$$(\varphi r) : M \to N \quad \text{by} \quad (\varphi r)(m) = \varphi(rm)$$

for all $\varphi \in \hom_A(M, N)$, $r \in R$, $m \in M$ [H, theorem 4.8 p203].

In what follows we will be interested in rings having a decomposition of the identity $1 = e_1 + e_2 + \cdots + e_n$ into orthogonal idempotents. For any idempotent $e \in R$, $A = eRe$ is a subring of R and Re is an R-A bimodule. For any A-module, E_A, the above asserts that $V = \hom_A(Re, E)$ is a right R-module. We begin with some elementary results connecting certain properties of E_A and V_R. In the next several results we use A to denote eRe for a nonzero idempotent e of R and, unless otherwise stated, $\hom_A(Re, E)$ is taken to be a right R-module.

Proposition 9 *If E_A is injective, then $\hom_A(Re, E)_R$ is injective.*

Proof. Let $H = \hom_A(Re, E)$, I a right ideal of R, and $f : I \to H$ any R-module homomorphism. Note that I is a (perhaps not unitary) right A-module.

Define $\varphi : I \to E$ by

$$\varphi(x) = f(x)(e) \qquad \text{for all } x \in I.$$

An easy argument shows that $\varphi \in \hom_A(I, E)$; the injectivity of E allows us to extend φ to R. That is, there exists $\Phi \in \hom_A(R, E)$, such that, for all $x \in I$

$$\varphi(x) = \Phi(x) = f(x)(e).$$

Then $\hat{\Phi} \equiv \Phi\mid_{Re}$ is in H and the map $\Psi : R \to H$ defined by $\Psi(r) = \hat{\Phi}r$ for all $r \in R$ is in $\hom_R(R, H)$. For all $x \in I$, $z \in Re$

$$(\Psi(x))(z) = (\hat{\Phi}x)(z) = \hat{\Phi}(xz) = \Phi(xz) = f(xz)(e) = (f(x)z)(e) = (f(x))(z).$$

Baer's lemma now gives the result [H, theorem 3.8, p194].

Definition 10 *An R-module, M, is said to be* **uniform,** *if for all nonzero submodules $X, Y \leq M$, $X \cap Y \neq 0$.*

This is equivalent to the statement: for all $0 \neq x, y \in M$, there exist $r, s \in R$, such that $xr = ys \neq 0$.

Proposition 11 *If E_A is uniform, then $\hom_A(Re, E)_R$ is uniform.*

Proof. Let $H = \hom_A(Re, E)$, and $\varphi, \psi \in H$ be nonzero. Choose $u, v \in Re$ such that $\varphi(u), \psi(v) \neq 0$; the uniformity of E implies that $0 \neq \varphi(u)a = \psi(v)b = \varphi(ua) = \psi(vb)$, for some $a, b \in A$. Let $x = ua$ and $y = vb$. Then $x = xe$ and $y = ye$. Hence, for all $r \in R$

$$(\varphi x)(re) = \varphi(xre) = \varphi(ua)ere = \psi(vb)\, ere = \psi(yere) = (\psi y)(re).$$

Furthermore, $(\varphi x)(e) = \varphi(xe) = \varphi(x) = \varphi(ua) \neq 0$.

Definition 12 *Let R be a ring, V an R-module, and X a subset of R. The* **annihilator of X in V** *is $ann_V(X) = \{v \in V \mid vX = 0\}$.*

Lemma 13 *Let V be a uniform injective R-module and $e \in R$, an idempotent, such that $ann_V(Re) = 0$. Then Ve is a uniform injective A-module.*

Proof. Choose $ve, we \neq 0$ in Ve. Because V_R is uniform, there exist $r, s \in R$ such that $0 \neq ver = wes$. Since $verRe \neq 0$, we can choose $0 \neq x \in R$ such that

$$0 \neq ver(xe) = wes(xe) = ve(erxe) = we(esxe).$$

Thus, Ve is uniform.

Let I be any right ideal of A and $f \in \hom_A(I, Ve)$. Consider the right ideal $IR \leq R$. Define $F : IR \to V$ by

$$F\left(\sum_{i=1}^{k} x_i r_i\right) = \sum_{i=1}^{k} f(x_i)\, r_i = \sum_{i=1}^{k} f(x_i)\, er_i \text{ for } x_i \in I \text{ and } r_i \in R.$$

If $\sum_{i=1}^{k}(x_i r_i) = 0$, then, for all $s \in R$, $0 = \left(\sum_{i=1}^{k} x_i r_i\right) se = \sum_{i=1}^{k}(x_i er_i se)$. Because $er_i se \in A$ for all i,

$$0 = f\left(\sum_{i=1}^{k}(x_i er_i se)\right) = \sum_{i=1}^{k} f(x_i)\, er_i se = \left(\sum_{i=1}^{k} f(x_i)\, er_i\right) se.$$

Since $s \in R$ was arbitrary, $\left(\sum_{i=1}^{k} f(x_i) r_i\right) Re = 0$. Our assumption on e implies that $\sum_{i=1}^{k} f(x_i) r_i = F\left(\sum_{i=1}^{k} x_i r_i\right) = 0$. This proves that F is well-defined. We may now conclude that $F \in \hom_R(IR, V)$. Since V_R is injective,we may extend F to all of R. Consequently, there exists $v \in V$ such that $F(z) = vz$ for all $z \in IR$.

Define $\varphi : A \to Ve$ by $\varphi(a) = vea$. Then $\varphi \in \hom_A(A, Ve)$ and, for $x \in I \subseteq IR$, $\varphi(x) = vex = vx = F(x) = f(x)$. This proves that Ve is also injective.

Let R be any ring and fix a decomposition $1 = e_1 + e_2 + \cdots + e_n$ into orthogonal idempotents. This decomposition will remain fixed but is otherwise completely arbitrary.

Theorem 14 *Let V be an R-module, the following statements are equivalent.*

(1) *V is uniform and injective.*

(2) *There exists $e = e_j$, such that, with $A = eRe$, Ve is a uniform injective A-module and $V_R \cong \hom_A(Re, Ve)_R$.*

Proof. (1)⇒(2) Let V be a uniform injective R-module and, for each i, let $K(e_i) = ann_V(Re_i)$. Surely each $K(e_i)$ is an R-submodule of V. If $v \in \bigcap_{i=1}^{n} K(e_i)$, then $v = v(e_1 + e_2 + \cdots + e_n) = ve_1 + \cdots + ve_n = 0$. That is, $\bigcap_{i=1}^{n} K(e_i) = 0$. The uniformity of V, implies $K(e_j) = 0$ for some j. For this e_j we write e and set $A = eRe$. Lemma 13 implies that Ve is a uniform injective A-module.

For each $v \in V$, define

$$\varphi_v : Re \to Ve \text{ by } \varphi_v(re) = vre.$$

Clearly, $\varphi_v \in \hom_A(Re, Ve)$. It follows that the map $\Phi : V \to \hom_A(Re, Ve)$

defined by $\Phi(v) = \varphi_v$ is an R-module homomorphism. Furthermore,

$$\ker\Phi = \{v \in V \mid \varphi_v(Re) = vRe = 0\} = K(e) = 0.$$

That is, Φ is an injection. Since V is injective, $\hom_A(Re, Ve)_R = X \oplus Y$, for some submodule Y and $X \cong V$.

Because Ve is uniform, $\hom_A(Re, Ve)_R$ is uniform (proposition 11). We may now conclude that $Y = 0$ and $V_R \cong X_R = \hom_A(Re, Ve)_R$.

That (2)$\Rightarrow$(1) is a direct application of proposition 9 and proposition 11.

Definition 15 *An idempotent, e, such that* $\mathrm{ann}_V(Re) = 0$ *is said to be* **faithful** *(or faithful to V).*

Remark. In the next section we shall give an example to show that faithful idempotents need not be unique. Next we will show that any nonzero idempotent is faithful to some module.

Proposition 16 *Let e be a nonzero idempotent in R and let* $A = eRe$. *There exists a uniform injective R-module, V, such that e is faithful to V and* $V \cong \hom_A(Re, Ve)$.

Proof. Let E be any uniform injective A-module. Our previous results show that $V = \hom_A(Re, E)$ is a uniform injective R-module. Let $\varphi \in V$ and suppose that $\varphi Re = 0$. Then, for all $r \in R$, $0 = (\varphi re)(e) = \varphi(re)$. This implies that $\varphi = 0$, e is faithful, and $V \cong \hom_A(Re, Ve)$.

Note: Given A and $\mathfrak{m}$ a maximal right ideal of A, $A/\mathfrak{m}$ is simple. The injective hull of $A/\mathfrak{m}$ is a uniform injective A-module. Therefore, the module E that we needed in the above proof, always exists.

Definition 17 *A module is said to be* **uniserial** *if it submodules are linearly ordered by set inclusion.*

This is equivalent to the statement: an R-module, M, is uniserial if, for all $0 \neq x, y \in M$, either $x = yr$ or $y = xr$ for some $r \in R$.

Proposition 18 *Let R be a ring, e an idempotent, $A = eRe$, and E an A-module. If* $\hom_A(Re, E)$ *is a uniserial R-module, then E_A is uniserial.*

Proof. Let $0 \neq u, v \in E$. Define $\varphi_u : Re \to E$ by $\varphi_u(re) = u(ere)$ for all $r \in R$ and similarly for v define φ_v. Then $0 \neq \varphi_u, \varphi_v \in \hom_A(Re, E)$. By uniseriality, there exists $x \in R$, such that

$$\varphi_u x = \varphi_v \quad \text{or} \quad \varphi_v x = \varphi_u.$$

Suppose the latter, then $\varphi_u(e) = ue = u = (\varphi_v x)(e) = \varphi_v(xe) = v(exe)$. This shows that E is uniserial.

We have not yet proven the converse in general, even for serial rings. We have obtained positive results for Goldie prime serial rings. This is the topic of the next section.

2.2 The Goldie Prime Serial Case

We now turn our attention to Goldie prime serial rings. If R is Goldie prime serial, then $R = (X_{ij}) \subseteq M_n(D)$ for some division ring D. Furthermore, R is a tiled order and the $X_{ii} = A_i$ are valuation rings. We shall always assume that R is a proper tiled order and we will make use of results in [M1]. Recall,

Definition 19 *Let D be a division ring. A subring $A \subset D$ is a* valuation on D (a valuation ring) *if for all $0 \neq d \in D$, either $d \in A$ or $d^{-1} \in A$.*

It is immediate that if $A \subset D$ is a valuation , then D_A and ${}_AD$ are both uniserial.

Lemma 20 *Let $R = (X_{ij}) \subset M_n(D)$ be a Goldie prime serial ring. Then there exists a ring R_1 such that*

(1) $R \cong R_1 = (Y_{ij}) \subset M_n(D)$;

(2) $Y_{11} = X_{11} = A_1 \subset D$;

(3) *for all $j = 1, \cdots, n$, $Y_{j1} \cong X_{j1}$ as A_1-modules and $Y_{j1} \subseteq A_1$.*

Proof. Let j be arbitrary but fixed. Since $X_{j1} \subset D$, we may choose $0 \neq d_j \in D - X_{j1}$. By the uniseriality of D_{A_1}, $X_{j1} \subset d_jA_1$. Because A_1 is a valuation, either $d_j \in A_1$ or $d_j^{-1} \in A_1$. Therefore,

$$X_{j1} \subseteq A_1 \quad \text{or} \quad X_{j1} \cong d_j^{-1}X_{j1} \equiv Y_{j1} \subseteq A_1.$$

For each j, choose such a d_j, (if $X_{j1} \subseteq A_1$ choose $d_j = 1$).

Let $X = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & d_2^{-1} & 0 & \cdots & 0 \\ 0 & 0 & d_3^{-1} & & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ 0 & 0 & \cdots & 0 & d_n^{-1} \end{pmatrix}$. Then $X^{-1} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & 0 \\ 0 & 0 & d_3 & & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ 0 & 0 & \cdots & 0 & d_n \end{pmatrix}$
and

$$R_1 \equiv XRX^{-1} = \begin{pmatrix} A_1 & X_{12}d_2 & X_{13}d_3 & \cdots & X_{1n}d_n \\ d_2^{-1}X_{21} & d_2^{-1}A_2d_2 & d_2^{-1}X_{23}d_2 & \cdots & d_2^{-1}X_{2n}d_2 \\ d_3^{-1}X_{31} & d_3^{-1}X_{32}d_3 & d_3^{-1}A_3d_3 & \cdots & d_3^{-1}X_{3n}d_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ d_n^{-1}X_{n1} & d_n^{-1}X_{n2}d_2 & d_n^{-1}X_{n3}d_3 & \cdots & d_n^{-1}A_nd_n \end{pmatrix}$$

has the required properties.

Remark. If we consider the ring R_1 obtained in the lemma above, the Y_{ii} are of the form $d^{-1}Ad$, where $d \neq 0$ and A is a valuation on D. These diagonal entries are again orders since A is an order. This is because $D = d^{-1}Dd = \{d^{-1}rc^{-1}d \mid r, c \in A\} = \left\{d^{-1}rd(d^{-1}cd)^{-1} \mid r, c \in A\right\}$. Thus, R_1 is a proper tiled order in $M_n(D)$ which is isomorphic to R. Therefore, whenever we consider a Goldie prime serial ring $R = (X_{ij}) \subseteq M_n(D)$, we may suppose without loss that $X_{j1} \subseteq X_{11} = A_1$ for all $j = 1, 2, \cdots, n$.

Lemma 21 *Let A_R be an R-module and B an abelian group such that $A \cong B$ as groups. Then B can be made into an R-module in such a way that $A_R \cong B_R$.*

Proof. Let $\Phi : A \to B$ be a group isomorphism. For all $b \in B$, and $r \in R$, define $br = \Phi(\Phi^{-1}(b)r)$. A simple verification shows that B is then an R-module. That Φ is also an R isomorphism is just $\Phi(ar) = \Phi(\Phi^{-1}(\Phi(a))r) = \Phi(a)r$.

In such a situation, the action of R on B shall be called **the action on B induced by Φ**

Let $R = (X_{ij}) \subset M_n(D)$ be a Goldie prime serial ring. Let $e = e_1 = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$ and identify $eRe = \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$ with $A_1 = A$. Then $Re = \begin{pmatrix} A & 0 & \cdots & 0 \\ X_{21} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ X_{n1} & 0 & \cdots & 0 \end{pmatrix}$ is an A-module by $\begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ x_{21} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & 0 & \cdots & 0 \end{pmatrix} a = \begin{pmatrix} a_{11}a & 0 & \cdots & 0 \\ x_{21}a & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1}a & 0 & \cdots & 0 \end{pmatrix}$ for all $a \in A$ and $\begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ x_{21} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & 0 & \cdots & 0 \end{pmatrix} \in Re$.

As A-modules $Re \cong A \oplus X_{21} \oplus \cdots \oplus X_{n1}$. For an (injective) A-module, E, we have the following isomorphism of abelian groups

$$\hom_A(Re, E) \cong \hom_A(A, E) \oplus \hom_A(X_{21}, E) \oplus \cdots \oplus \hom_A(X_{n1}, E) \quad (1)$$

via the isomorphism $\Phi : \varphi \longmapsto (\varphi|_A, \varphi|_{X_{21}}, \cdots, \varphi|_{X_{n1}})$. (We have abused notation; instead of $\varphi|_{X_{j1}}$ it should be $\varphi \circ \iota_{X_{j1}}$, where $\iota_{X_{j1}} : X_{j1} \hookrightarrow Re$ is the inclusion).

The right side of (1) becomes an R-module via the action induced by Φ. We shall now describe this explicitly.

For each j, let $\hom_A(X_{j1}, E)$be denoted by $X_{j1}^{\#}$. Let $x \in X_{ij}$; define $\alpha_x : X_{j1} \to X_{i1}$ by $\alpha_x(y) = xy$ for all $y \in X_{j1}$. Surely $\alpha_x \in \hom_A(X_{j1}, X_{i1})$. If $\alpha_i \in X_{i1}^{\#}$, then $\alpha_i \circ \alpha_x \in X_{j1}^{\#}$. Denote the map $\alpha_i \circ \alpha_x$ by $\alpha_i \circ x$. Let $(\alpha_1, \alpha_2, \cdots, \alpha_n) \in \bigoplus_{j=1}^{n} X_{j1}^{\#}$ and $\overline{x} = (x_{ij}) \in R$. Define
$(\alpha_1, \alpha_2, \cdots, \alpha_n) \circ \overline{x} = (\beta_1, \beta_2, \cdots, \beta_n)$ where, for each $j = 1, 2, \cdots, n$,

$$\beta_j = \sum_{i=1}^{n} \alpha_i \circ x_{ij}.$$

By our earlier discussion, $\beta_j \in X_{j1}^{\#}$ for each j. Whence, we get a map

$$\circ : \left(\bigoplus_{j=1}^{n} X_{j1}^{\#} \right) \times R \longrightarrow \bigoplus_{j=1}^{n} X_{j1}^{\#}.$$

Lemma 22 *With all the previous notation,* $\circ$ *is the same map as the action induced by* Φ.

Proof. Let $(\alpha_1, \alpha_2, \cdots, \alpha_n) \in \bigoplus_{j=1}^{n} X_{j1}^{\#}$ and $\overline{x} = (x_{ij}) \in R$. Then there exists a unique $\alpha \in \hom_A(Re, E)$ such that $\alpha_i = \alpha|_{X_{i1}}$for each $i = 1, 2, \cdots, n$. (This is simply how Φ is defined).

The action induced by Φ is

$$(\alpha_1, \alpha_2, \cdots, \alpha_n)\overline{x} = \Phi(\alpha\overline{x}) = (\alpha\overline{x}|_A, \alpha\overline{x}|_{X_{21}}, \cdots, \alpha\overline{x}|_{X_{n1}})$$

We will show that $\beta_j = \alpha\overline{x}|_{X_{j1}}$ for each j. Let $y \in X_{j1}$, then

$$\left(\alpha\overline{x}|_{X_{j1}}\right)(y) = (\alpha\overline{x}) \left(\begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & 0 & \cdots & 0 \\ y & 0 & \cdots & 0 \\ \vdots & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{pmatrix} \right) = \alpha \left((x_{ij}) \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & 0 & \cdots & 0 \\ y & 0 & \cdots & 0 \\ \vdots & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{pmatrix} \right) =$$

$$\alpha\begin{pmatrix} x_{1j}y & 0 & \cdots & 0 \\ x_{2j}y & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ x_{nj}y & 0 & \cdots & 0 \end{pmatrix} = \alpha\begin{pmatrix} x_{1j}y & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} + \cdots + \alpha\begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ x_{nj}y & 0 & \cdots & 0 \end{pmatrix} =$$

$$\alpha\mid_A (x_{1j}y) + \cdots + \alpha\mid_{X_{nj}} (x_{nj}y) = \alpha_1 (x_{1j}y) + \cdots + \alpha_n (x_{nj}y) =$$

$$\alpha_1 \circ x_{1j}(y) + \cdots + \alpha_n \circ x_{nj}(y) = \left(\sum_{i=1}^{n} \alpha_i \circ x_{ij}\right)(y) = \beta_j(y).$$

Thus, the two multiplications are the same.

The action of R on $A^{\#} \oplus X_{21}^{\#} \oplus \cdots \oplus X_{n1}^{\#}$ is simply given by this formal matrix multiplication.

Lemma 23 *Let j be given and let $\varphi \in X_{j1}^{\#}$. Then, there exists $u_j \in E$, such that $\varphi(z) = u_j z$ for all $z \in X_{j1}$.*

Proof. Consider the diagram

$$\begin{array}{ccc} X_{j1} & \hookrightarrow & A \\ \varphi\downarrow & & \\ E & & \end{array}.$$

Because E is injective, there exists $\Phi \in \hom_A(A, E)$ such that, for all $z \in X_{j1}$, $\varphi(z) = \Phi(z)$. Let $\Phi(e) = u_j$. Then $\varphi(z) = \Phi(z) = \Phi(ez) = \Phi(e)z = u_j z$.

In the following theorem we assume that R is a Goldie prime serial ring with decomposition $1 = e_1 + e_2 + \cdots + e_n$ into indecomposable orthogonal idempotents. We let $e = e_1$, $A = eRe$, and E an arbitrary injective right A-module. For $\varphi_i \in X_{i1}^{\#}$, the above lemma shows that φ_i is multiplication by an element of E. We shall denote this by $\varphi_i = \alpha_{u_i}$ where $\varphi_i(z) = \alpha_{u_i}(z) = u_i z$.

Theorem 24 *Let R be a Goldie prime serial ring. Then* $\hom_A(Re, E)$ *is uniserial if and only if E_A is uniserial.*

Proof. ($\Rightarrow$) This is proposition 18.

($\Leftarrow$) This requires more work.

Let i and j be given, and suppose $\varphi = (0, \cdots, \varphi_i, 0, \cdots, 0)$ and $\psi = (0, \cdots, \psi_j, 0, \cdots, 0)$ are in $\bigoplus_{k=1}^{n} X_{k1}^{\#}$. Then $\varphi_i = \alpha_{u_i}$ and $\psi_j = \psi_{v_j}$ for some $u_i, v_j \in E$. Uniseriality implies that $u_i = v_j b$ or $v_j = u_i b$ for some $b \in A$. Suppose the latter:

Case(1): Suppose $b \in X_{ij}$. Let

$$B = (b_{lk}) \in R \quad \text{where } b_{lk} = \begin{cases} b \; if \; l = i \; and \; k = j \\ 0 \; otherwise \end{cases}.$$

Then $\varphi \circ B = (0, \cdots, \alpha_{u_i} \circ b, 0, \cdots 0)$. If $z \in X_{j1}$, then

$$(\alpha_{u_i} \circ b)(z) = \alpha_{u_i}(bz) = u_i(bz) = (u_i b) z = v_j z = \psi_j(z).$$

Therefore, $\varphi \circ B = \psi$.

Case(2): Suppose $b \notin X_{ij}$. Then $b^{-1} \in X_{ji}$ [M1,Theorem 2 and preliminaries]. Let

$$B = (b_{lk}) \in R \quad \text{where } b_{lk} = \begin{cases} b^{-1} \; if \; l = j \; and \; k = i \\ 0 \; otherwise \end{cases}.$$

Then $\psi \circ B = \left(0, \cdots, \alpha_{v_j} \circ b^{-1}, 0, \cdots, 0\right)$.

We claim that $\alpha_{v_j} \circ b^{-1} = \alpha_{u_i}$. To show this, let $z \in X_{i1} \subseteq A$. Then $z = (bb^{-1})z = b(b^{-1}z)$. Furthermore, $b^{-1}z \in X_{ji}X_{i1} \subseteq X_{j1} \subset A$. Hence, for all $z \in X_{i1}$,

$$\alpha_{u_i}(z) = u_i z = u_i\left(b\left(b^{-1}z\right)\right) = (u_i b)\left(b^{-1}z\right) = v_j\left(b^{-1}z\right) = \left(\alpha_{v_j} \circ b^{-1}\right)(z).$$

Consequently, $\psi \circ B = \varphi$.

These two cases allow us to conclude that φR and ψR are comparable.

Consider now an arbitrary $\varphi = (\varphi_1, \varphi_2, \cdots, \varphi_n) \in \bigoplus_{j=1}^{n} X_{j1}^{\#}$; let $\widehat{\varphi_i} \equiv (0, \cdots, \varphi_i, 0, \cdots, 0)$ for each $i = 1, 2, \cdots, n$. By the above, there exists i such that $\widehat{\varphi_j}R \subseteq \widehat{\varphi_i}R$ for all j. Thus,

$$\varphi R \subseteq \sum_{j=1}^{n} \widehat{\varphi_j}R \subseteq \widehat{\varphi_i}R = (\varphi e_i)R \subseteq \varphi R.$$

This implies $\varphi R = \widehat{\varphi_i}R$; similarly, for $\psi = (\psi_1, \psi_2, \cdots, \psi_n) \in \bigoplus_{j=1}^{n} X_{j1}^{\#}$, $\psi R = \widehat{\psi_j}R$ for some j. Therefore,

$$\psi R = \widehat{\psi_j}R \subsetneqq \widehat{\varphi_i}R = \varphi R$$

and the result obtains.

Example. This example shows that faithful idempotents need not be unique.

Let E_A be a uniform injective module over a valuation $A \subset D$. Denote the maximal ideal of A by m and suppose that $ann_E(\mathrm{m}) = 0$.

Let $R = \begin{pmatrix} A & A \\ \mathrm{m} & A \end{pmatrix}$ (which is a Goldie prime serial ring), $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, and $f = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. By previous results, $V_1 = \hom_A(Re, E)$ and

$V_2 = \hom_A(Rf, E)$ are both uniform injective R-modules. Furthermore, e is faithful for V_1 and f is faithful for V_2. Consider the map $\Psi : E \to \hom_A(\mathrm{m}, E)$ by $\Psi(u) = \alpha_u$, where α_u is multiplication by u. It is easy to show that Ψ is a group homomorphism; the injectivity of E implies that Ψ is surjective. Because $\ker\Psi = ann_E(\mathrm{m}) = 0$, $\hom_A(\mathrm{m}, E) \cong E$ as abelian groups. Note that $Re \cong A \oplus \mathrm{m}$ and $Rf \cong A \oplus A$ as A-modules. This implies

$$V_1 = \hom_A(Re, E) \cong \hom_A(A, E) \oplus \hom_A(\mathrm{m}, E) \cong E \oplus E$$

and

$$V_2 = \hom_A(Rf, E) \cong \hom_A(A, E) \oplus \hom_A(A, E) \cong E \oplus E$$

as abelian groups. Arguments in the previous section (see lemma 22) show that in both of the above cases the action of R on $E \oplus E$ is the formal matrix multiplication. This implies that $V_1 \cong V_2$ as R-modules; hence, e and f are both faithful for $V = V_1$.

To complete this example we need only show that there is such a valuation and uniform injective module. Let k be a field and let
$A = k\langle Q^+\rangle = \{f = \sum a_\alpha x^\alpha \mid Q \ni \alpha \geq 0, a_\alpha \in k, supp(f) \text{ is well ordered}\}$; that is , the ring of power series having well ordered support. Then A is a valuation in its quotient field, D, the Laurent series ring. The maximal ideal of A is $\mathrm{m} = \{f \in A \mid a_0 = 0\}$.

Let $I = xA$ and use $\overline{A}$ to denote A/I. it follows that $E = E\left(\overline{A}\right)$ is a uniform injective A-module. Let $\overline{p(x)}$ be nonzero in $\overline{A}$. This implies that the lowest power appearing in $p(x)$ is $\epsilon < 1$; choose $0 \neq \gamma$ such that $\epsilon + \gamma < 1$. Consequently, $x^\gamma \in \mathrm{m}$ but $p(x)x^\gamma \notin I$. That is, $\overline{p(x)} \notin ann_E(\mathrm{m})$.

Suppose there exists $0 \neq q \in ann_E(\mathrm{m})$. Because $\overline{A} \subseteq' E$, there is some $a \in A$ such that $0 \neq qa \in \overline{A}$. Since $ann_E(\mathrm{m})$ is a submodule of E, we infer that $0 \neq qa \in \overline{A} \cap ann_E(\mathrm{m})$. This contradiction implies $ann_E(\mathrm{m}) = 0$. This gives us the required example.

Chapter 3

Uniform Modules Over Valuation Rings

3.1 Uniform Injectives

Consider a uniform injective module, V, over a Goldie prime serial ring R. Our results from the previous chapter show that $V \cong \hom_A(Re, Ve)$ for some faithful idempotent e and $A = eRe$. Moreover, Ve is a uniform injective A-module and V is uniserial if and only if Ve is uniserial. When R is Goldie prime serial, A is a valuation on a division ring. Therefore, it makes sense to examine uniform injective modules over valuation rings.

Theorem 25 *Let $A \subset D$ be a valuation and E a uniform injective A-module. Then:*

(1) *$E = E(D/I)$ for some right ideal $I < A$;*

(2) *if D is a field, then E is uniserial if and only if $E = D/I$.*

Proof. (1) Let $0 \neq x \in E$ and define $\varphi_x : A \longrightarrow E$ by $\varphi_x(a) = xa$ for all $a \in A$. Then $\ker \varphi_x = ann_A(x) = x^0 \equiv I$ and $xA \cong A/I \equiv \overline{A}$. Define

$\varphi : \overline{A} \longrightarrow E$ by $\varphi(\overline{a}) = xa$. Surely $\varphi \in \hom_A\left(\overline{A}, E\right)$. The injectivity of E allows us to extend φ to D/I by Φ. Note that $\Phi\left(\overline{1}\right) = x$ and, for any $0 \neq d \in D$, $d \in A$ or $d^{-1} \in A$.

(i) *If* $d \in A$, *then* $\Phi\left(\overline{d}\right) = \Phi\left(\overline{1d}\right) = xd$.

(ii) *If* $d^{-1} \in A$, *then* $\Phi\left(\overline{1}\right) = x = \Phi\left(\overline{dd^{-1}}\right) = \Phi\left(\overline{d}d^{-1}\right) = \Phi\left(\overline{d}\right)d^{-1}$.

Suppose that $\Phi\left(\overline{d}\right) = 0$. Then (i) implies $0 = xd$; hence, $d \in x^0 = I$ and $\overline{d} = 0$. Option (ii) leads to the contradiction $x = 0$. Therefore, Φ is a monomorphism and $E(D/I) \subseteq E$. The uniformity of E gives the result.

(2) Since $\overline{D} \equiv D/I$ is uniserial (whether D is a field or not), one implication is clear.

Now assume that D is a field and that $E = E\left(\overline{D}\right)$ is uniserial. Let $x \in E - \overline{D}$; by essentiality, there exists $0 \neq a \in A$ such that $0 \neq xa \in \overline{D}$. Write $xa = \overline{d}$ for some $d \in D - I$. Let $r = \overline{da^{-1}} \in \overline{D}$. Then

$$ra = \overline{da^{-1}}a = \overline{d} = xa \neq 0.$$

Hence, $(r - x)a = 0$. Because E is uniserial, there is some $b \in A$ such that $(r - x)b = r$ or $rb = r - x$. The later implies that $x \in \overline{D}$ while the former implies that $0 \neq ra = (r - x)ba = (r - x)ab = 0$. Neither case being possible gives the result.

Remark. Given such a module, $E = E(D/I)$, if D/I is injective, then $E = D/I$ is uniserial. The above result showed that these are equivalent when D is a field. In the general case we have not yet been able to determine whether or not the uniseriality of E and the injectivity of D/I are equivalent. We therefore turn our attention to the question of when D/I is injective. This will then give sufficient conditions for the uniseriality of E. Before we do this, we need some background material about topological modules.

3.2 Topological Considerations

Definition 26 *A module, M_R, is said to be a* **topological module** *if M is also a topological space such that:*
(a) addition is a continuous map from $M \times M \longrightarrow M$;
(b) for all $r \in R$, the map $\hat{r} : M \longrightarrow M$ by $\hat{r}(m) = mr$ for all $m \in M$ is continuous.

For an element $m \in M$, a **base of neighborhoods about** m is a set $\mathcal{B}$, of neighborhoods of m, such that every neighborhood of m contains an element of $\mathcal{B}$. By a neighborhood of m we mean a set which contains an open set containing m.

Definition 27 *A topology on M is said to be* **linear** *if it has a base of neighborhoods about 0 consisting of submodules of M.*

When we consider linear topologies on modules, we will simply write "a base of neighborhoods about 0". It will be understood that this base consists of submodules. We shall usually use $\mathcal{B}$ to denote this base. When we write "let M have a linear topology", it should also be understood that M is a topological module.

In the definition of a linear topology, the elements of $\mathcal{B}$ are only assumed to be neighborhoods, not necessarily open. They are however open as the next proposition shows.

Proposition 28 *Let M have a linear topology and let $\mathcal{B}$ be a base of neighborhoods about 0. A submodule $W \subseteq M$ is open if and only if there is some $U \in \mathcal{B}$ such that $U \subseteq W$.*

Proof. That an open submodule contains such a $U \in \mathcal{B}$ is just the definition of a linear topology.

Conversely, suppose $W \supseteq U$ for some $U \in \mathcal{B}$. Then $W = \bigcup_{w \in W}(w + U)$; hence, if $U \in \mathcal{B}$ is open, then W is open. Since U is a neighborhood of 0, there is some open set O, such that $0 \in O \subseteq U$. Thus, $U = \bigcup_{u \in U}(u + O)$ is open.

Lemma 29 *Let M have a linear topology and let $\mathcal{B}$ be a base of neighborhoods about 0. The topology is Hausdorff if and only if $\bigcap_{U \in \mathcal{B}} U = \{0\}$.*

Proof. ($\Rightarrow$) Suppose there is some $0 \neq x \in \bigcap_{U \in \mathcal{B}} U$. Let V and W be open sets containing x and 0 respectively. By definition, there exists $U \in \mathcal{B}$, such that $U \subseteq W$. But then $x \in U \subseteq W$. This implies $x \in V \cap W \neq \emptyset$; this contradicts the hypothesis. Therefore, $\bigcap_{U \in \mathcal{B}} U = 0$.

($\Leftarrow$) Let $x \neq y$. If $x + U \cap y + U \neq \emptyset$, then $x - y \in U$. If this is the case for all $U \in \mathcal{B}$, then $0 \neq (x - y) \in \bigcap_{U \in \mathcal{B}} U = \{0\}$. Therefore, for some $U \in \mathcal{B}$, $(x + U) \cap (y + U) = \emptyset$.

We will be interested in topologies on uniserial modules; in particular, we will be interested in topologies on the division ring D and D/I from the previous section. We shall now restrict our attention to uniserial modules with linear topologies. Unless otherwise stated, $\mathcal{B}$ will denote a base of neighborhoods about 0 consisting of submodules. Note: without loss we may assume that $\mathcal{B}$ is equal to the set of all open submodules.

We will use Λ to denote an (upward) directed set (for all $\alpha, \beta \in \Lambda$, there exists $\sigma \in \Lambda$, such that $\sigma \geq \alpha, \beta$).

Definition 30 *Let M be a uniserial module with a linear topology. A net $(x_\alpha)_{\alpha\in\Lambda}$ is said to be a* **Cauchy net** *if for every $U \in \mathcal{B}$, there exists $\sigma = \sigma(U)$, such that $x_\alpha - x_\beta \in U$ whenever $\alpha, \beta \geq \sigma$. A net $(x_\alpha)_{\alpha\in\Lambda}$ is said to* **converge to** x *if for all $U \in \mathcal{B}$, there exists $\sigma = \sigma(U)$, such that $x_\alpha - x \in U$ whenever $\alpha \geq \sigma$.*

Let us consider nets which are indexed by elements of $\mathcal{B}$. The order on $\mathcal{B}$ is $U \leq V \Leftrightarrow V \subseteq U$. For uniserial modules, with this ordering, $\mathcal{B}$ is a totally ordered set; hence $\mathcal{B}$ is a directed set. To keep notation more simple we shall often omit the directed set Λ.

Definition 31 *Let M be a uniserial module with a linear topology. A net $(y_U)_{U\in\mathcal{B}}$ is said to be a* **special Cauchy net** (SC) *if for all $U \in \mathcal{B}$, $y_V - y_W \in U$ whenever $V, W \in \mathcal{B}$ and $V, W \subseteq U$. A net $(y_U)_{U\in\mathcal{B}}$ is* **special convergent to** y *if for all $U \in \mathcal{B}$, $y_V - y \in U$ whenever $V \in \mathcal{B}$ and $V \subseteq U$.*

Proposition 32 *Let M_A be a uniserial module over a valuation A.*

(1) *If a net is SC, then it is also Cauchy.*

(2) *If a net is special convergent, then it is convergent with the same limit.*

(3) *A convergent net is Cauchy.*

(4) *A special convergent net is SC.*

(5) *If a net is SC and convergent, then it is also special convergent.*

Proof. We only prove (5). Let $(y_U)_{U\in\mathcal{B}}$ be SC and convergent to y. Let $U \in \mathcal{B}$. Then $y_V - y_W \in U$ whenever $V, W \subseteq U$ and, there exists $\sigma(U) \in \mathcal{B}$, such that $y_W - y \in U$ whenever $W \subseteq \sigma(U)$. Let $W = \sigma(U) \cap U \in \mathcal{B}$. Then, for all $V \subseteq U$,

$$y_V - y = (y_V - y_W) + (y_W - y) \in U + U = U.$$

Definition 33 *A module is* **complete** *if every Cauchy net converges and is* **special complete** *if every special Cauchy net is special convergent.*

It is now immediate that a complete module is special complete. To see this, let $(y_U)_{U\in\mathcal{B}}$ be SC. Then it is Cauchy by (1) of proposition 32. Completeness implies that it is also convergent; by proposition 32 part (5), it is special convergent. In fact, as we shall now show, these are actually equivalent.

Lemma 34 *Let M be a uniserial module with a linear topology and $\mathcal{B}$ a base of neighborhoods about 0. Then M is complete if and only if M is special complete.*

Proof. The comment above establishes one direction.

Suppose that M is special complete and that $(x_\alpha)_{\alpha\in\Lambda}$ is Cauchy. For all $U \in \mathcal{B}$, there exists $\sigma = \sigma(U)$, such that $x_\alpha - x_\beta \in U$ whenever $\alpha, \beta \geq \sigma$. Define a new net $(y_U)_{U\in\mathcal{B}}$, where $y_U = x_{\sigma(U)}$ for all $U \in \mathcal{B}$. We claim that this net is SC.

Let U be given and let $V, W \in \mathcal{B}$ with $V, W \subseteq U$. Choose $\sigma' \geq \sigma(W), \sigma(V)$. Then

$$y_V - y_W = x_{\sigma(V)} - x_{\sigma(W)} = \left(x_{\sigma(V)} - x_{\sigma'}\right) + \left(x_{\sigma'} - x_{\sigma(W)}\right) \in V + W \subseteq U.$$

This proves our claim.

By hypothesis, (y_U) is special convergent to some y. That is, for all $U \in \mathcal{B}$, $y_V - y \in U$ whenever $V \in \mathcal{B}$ and $V \subseteq U$; in particular $y_U - y \in U$. We'll show that $(x_\alpha)_{\alpha\in\Lambda}$ is convergent to y. If $U \in \mathcal{B}$, then, for all $\beta \geq \sigma(U)$,

$$x_\beta - y = \left(x_\beta - x_{\sigma(U)}\right) + \left(x_{\sigma(U)} - y\right) = \left(x_\beta - x_{\sigma(U)}\right) + (y_U - y) \in U + U = U.$$

Definition 35 *A collection of nonempty sets is said to have the* **finite intersection property** *(fip) if any finite subset of them has nonempty intersection.*

The collection is said to have the **intersection property** *(ip) if any subset of them has nonempty intersection.*

We now establish a relationship between intersection properties and completeness of modules.

Theorem 36 *Let M be a uniserial module having a linear topology and let $\mathcal{B}$ be a base of neighborhoods about 0. The following are equivalent*

(1) *M is complete;*

(2) *any collection $\mathcal{C} = \{x_U + U \mid U \in \mathcal{B}, x_U \in M\}$ having the fip also has the ip.*

Proof. (2) $\Rightarrow$ (1). By lemma 34, it is enough to show that M is special complete. Let $(y_U)_{U \in \mathcal{B}}$ be SC and let $\mathcal{C} = \{y_U + U \mid U \in \mathcal{B}\}$. We claim that $\mathcal{C}$ has the fip. Let $\mathcal{F} \subseteq \mathcal{B}$ be finite and set $U = \bigcap_{V \in \mathcal{F}} V$. By uniseriality $U \in \mathcal{B}$ (in fact $U = V$ for one of the $V \in \mathcal{F}$). For each $V \in \mathcal{F}$, $y_U - y_V \in V$ (because $(y_U)$is special Cauchy and $V, U \subseteq V$). This shows that $y_U \in y_V + V$ for each $V \in \mathcal{F}$. This establishes our claim that $\mathcal{C}$ has the fip.

By hypothesis, there exists $y \in \bigcap_{U \in \mathcal{B}} (y_U + U)$. Let $U \in \mathcal{B}$ and let $V \in \mathcal{B}$ with $V \subseteq U$. Now $y \in y_V + V$ implies that $y_V - y \in V \subseteq U$. This means that M is special complete.

(1) $\Rightarrow$ (2). Let $\mathcal{C} = \{x_U + U \mid U \in \mathcal{B}, x_U \in M\}$ have the fip. We will show that $(x_U)_{U \in \mathcal{B}}$ is SC and then that $\mathcal{C}$ has the ip. Let $U \in \mathcal{B}$ be given and let $V, W \in \mathcal{B}$ with $V, W \subseteq U$. By hypothesis, there is some $x \in (x_V + V) \cap (x_W + W)$. Therefore,

$$x_V - x_W = (x_V - x) + (x - x_W) \in V + W \subseteq U.$$

That is, (x_U) is SC. Completeness implies special completeness; thus, (x_U) is special convergent to z say. Hence, for all $U \in \mathcal{B}$, $x_U - z \in U$. In other words, $z \in x_U + U$ and $\mathcal{C}$ has the ip.

3.2.1 The K-topology

We now consider the construction of linear topologies on modules. Let M be a uniserial module and $K < M$ a submodule. Let $\mathcal{B} = \{U \leq M \mid K \subset U\}$, and let $\Omega = \{m + U \mid m \in M, U \in \mathcal{B}\}$. Using the uniseriality of M, it is not hard to show that Ω is a basis for a topology on M [cf. MK, p78]. The open sets of the topology generated by Ω are precisely the sets which are unions of elements of Ω [MK, Lemma 2.1, p80]. We will call this topology, which is induced by the submodule K, the **K-topology** (on M).

At times we may consider modules over more than one ring. In cases where the ring may be in question, we will write K_R–topology to stress which ring we are considering.

Notation. Let M be an R-module and let $S \subsetneq M$ be a subset. For any $r \in R$, we use the following notation:

$$Sr^{-1} = \{m \in M \mid mr \in S\}.$$

Proposition 37 *Let M be a uniserial R-module and $K < M$. With the K-topology, M is a topological module and the topology is linear. A base of neighborhoods of 0 consisting of submodules is $\mathcal{B} = \{U \leq M \mid U \supset K\}$.*

Proof. To show that M is a topological module, we must show:

(1) $p : M \times M \longrightarrow M$ by $p(x, y) = x + y$ for all $x, y \in M$ is continuous;

(2) $\widehat{r} : M \longrightarrow M$ by $\widehat{r}(m) = mr$ for all $m \in M$ is continuous for each $r \in R$.

For each $x, y \in M$ and $U \in \mathcal{B} = \{U \leq M \mid U \supset K\}$, let $U_{x,y} = (y + U) \times ((x - y) + U)$. Thus, each $U_{x,y}$ is open in $M \times M$. Let $x + U$

be an arbitrary element in $\Omega = \{m + U \mid m \in M,\ U \in \mathcal{B}\}$; an easy argument shows that

$$p^{-1}(x+U) = \bigcup_{y \in M} U_{x,y}.$$

This shows that p is continuous.

Let $r \in R$ and let $x + U$ be an arbitrary element in Ω. We claim that

$$\hat{r}^{-1}(x+U) = \bigcup_{z \in (x+U)r^{-1}} (z+U).$$

This would then show that $\hat{r}$ is continuous.

If $m \in \hat{r}^{-1}(x+U)$, then $\hat{r}(m) = mr \in (x+U)$. Thus, $m \in (x+U)r^{-1}$. It now follows that

$$m \in m + U \subseteq \bigcup_{z \in (x+U)r^{-1}} (z+U).$$

Conversely, let $z \in (x+U)r^{-1}$; then $zr = x + u_0$ for some $u_0 \in U$. Hence,

$$\hat{r}(z+U) \subseteq zr + Ur = x + u_0 + Ur \subseteq x + U.$$

Therefore, $z+U \subseteq \hat{r}^{-1}(x+U)$ and we have proven the claim. This concludes the proof that M is a topological module.

To show that the topology is linear, let W be any open set which contains 0. By definition, there is some $(m+U) \in \Omega$, such that $0 \in (m+U) \subseteq W$. Since U is a submodule, we conclude that $U = m + U \subseteq W$. This shows that the topology is linear.

Let M be a uniserial module with a linear topology and $\mathcal{B}$ a base of neighborhoods of 0. Let $K = \bigcap_{U \in \mathcal{B}} U$. Then $K \subseteq U$ for all $U \in \mathcal{B}$ and K is a submodule of M. We examine some cases.

(1) If $K = M$, then $\mathcal{B} = \{M\}$ and the topology is trivial. In this case the topology is complete.

(2) If $K < M$, then there are two cases:

$$\text{(i)} \quad K \in \mathcal{B}$$
$$\text{(ii)} \quad K \notin \mathcal{B}$$

Consider first $K \in \mathcal{B}$. Let (x_α) be a Cauchy net. Since $K \in \mathcal{B}$, there exists $\sigma = \sigma(K)$ such that $x_\alpha - x_\beta \in K$ whenever $\alpha, \beta \geq \sigma$. Choose $x = x_\sigma$ and, for any $U \in \mathcal{B}$, choose $\sigma(U) = \sigma$. Then whenever $\beta \geq \sigma(U)$, $x_\beta - x = x_\beta - x_\sigma \in K \subseteq U$. This shows that the topology is complete.

Consider now the case $K \notin \mathcal{B}$. Then $K < U$ for each $U \in \mathcal{B}$ and $K = \bigcap_{U \in \mathcal{B}} U$. We claim that the given topology is simply the K-topology on M.

Denote the original topology on M by $\mathcal{T}$, the K-topology by $\mathcal{T}_K$ and the base of neighborhoods of 0 by $\mathcal{B}_K$. Since $K \subset U$ for all $U \in \mathcal{B}$, $\mathcal{B} \subseteq \mathcal{B}_K$. It is then easy to show that $\mathcal{T} \subseteq \mathcal{T}_K$.

Note next that if $V \leq M$ and $K \subset V$, then there exists $U \in \mathcal{B}$ such that $U \subseteq V$. Otherwise, uniseriality implies that $V \subseteq U$ for all $U \in \mathcal{B}$. This now implies that $V \subseteq \bigcap_{U \in \mathcal{B}} U = K \subset V$. A contradiction.

Let $V \in \mathcal{B}_K$; by the above, there is some $U \in \mathcal{B}$ such that $U \subseteq V$. A previous proposition implies that $V \in \mathcal{T}$. Let $O \in \mathcal{T}_K$; then, by definition, O is a union of sets of the form $m + V$ where $m \in M$ and $V \in \mathcal{B}_K$. This implies that $O \in \mathcal{T}$. We may now conclude that $\mathcal{T} = \mathcal{T}_K$.

Reduction. The above discussion gives us the following:

When considering the completeness of a linear topology with, $\mathcal{B}$, a base of neighborhoods about 0 on a uniserial module we may assume that it is the K-topology. Further we may assume that $K \notin \mathcal{B}$ and that $K = \bigcap_{U \in \mathcal{B}} U$; hence $K < U$ for all $U \in \mathcal{B}$. In the construction of the K-topology a base of neighborhoods about 0 was given by $\mathcal{B}_K = \{V \leq M \mid K < V\}$. Consequently,

$\mathcal{B} \subseteq \mathcal{B}_K$. Therefore,

$$K \subseteq \bigcap_{V \in \mathcal{B}_K} V \subseteq \bigcap_{U \in \mathcal{B}} U = K.$$

Definition 38 *Let M be a module and $K < M$ a submodule. We say that K is* **friendly** *(or* **a friendly submodule***) if*

$$K = \bigcap_{K < U \leq M} U.$$

To rephrase the above, we may assume that the topology is the K-topology for a friendly submodule K.

To make a further reduction we observe the following. The K-topology has a base of neighborhoods about 0, $\mathcal{B} = \{U \leq M \mid K < U\}$. Suppose that $\mathcal{B}' \subseteq \mathcal{B}$ such that $K = \bigcap_{V \in \mathcal{B}'} V$. Then $\mathcal{B}'$ is also a base of neighborhoods about 0 which consists of submodules. To see this, let $U \in \mathcal{B}$ remain fixed. For all $V \in \mathcal{B}'$, $V$and U are comparable (by uniseriality). If $U \subset V$ for all $V \in \mathcal{B}'$, then

$$U \subseteq \bigcap_{V \in \mathcal{B}'} V = K < U.$$

This contradiction implies that for all $U \in \mathcal{B}$ there exists $V \in \mathcal{B}'$ such that $V \subseteq U$.

Therefore, to show that a linear topology on a uniserial module, M, is complete we may proceed as follows:

(1) Assume that it is the K-topology for a friendly submodule $K < M$. A base of neighborhoods about 0 consisting of submodules is $\mathcal{B} = \{U \leq M \mid K < U\}$ and $K = \bigcap_{U \in \mathcal{B}} U$.

(2) If $\mathcal{B}' \subseteq \mathcal{B}$ and $K = \bigcap_{V \in \mathcal{B}'} V$, then we can (if we want) choose $\mathcal{B}'$ to be a base of neighborhoods about 0.

(3) Let $\mathcal{B}'$ be a base of neighborhoods of 0 that we have chosen in step (2). We then consider an arbitrary collection $\mathcal{C} = \{x_U + U \mid x_U \in M,\ U \in \mathcal{B}\}$ which has the fip. If $\mathcal{C}$ has the ip, then M is complete.

3.3 Main Theorem

Let $A \subset D$ be a valuation and I a right ideal of A. We return to the question of when D/I is injective. Since D is injective, we assume that $I \neq 0$. First we set up some notation. Let $I \leq A$ and $\mathfrak{m}$ be the maximal ideal of A. Then $I^* = \{d \in D \mid d^{-1} \notin I\} \cong \hom_A(I, \mathfrak{m})$ as left A-modules. Furthermore, $\wp = I^* I$ is a completely prime ideal of A [cf. M1]. This implies that $A - \wp$ is an Ore set, and we localize at $\wp$ to get $B = A_\wp = \{c^{-1}a \mid c \in A - \wp,\ a \in A\} = \{ac^{-1} \mid c \in A - \wp,\ a \in A\} = {}_\wp A$. This is because $A - \wp$ is a two sided Ore set and consists of regular elements.

Notation. Throughout this section we will keep the above notation.

Proposition 39 *Let $K \leq D$ be an A-submodule. Then*

(1) *K is a B-submodule $\Leftrightarrow$ $K = Kc$ for all $c \in A - \wp$;*

(2) *I is a B-submodule of D.*

Proof. (1) Let K be a B-submodule of D; we need to show that $K \subseteq Kc$ for all $c \in A - \wp$. If $c \in A - \wp$, then c^{-1} is in B. Thus $Kc^{-1} \subseteq K$; multiplying by c gives $K \subseteq Kc$.

Conversely, let $c^{-1}a \in B$ be arbitrary. By hypothesis $Kc^{-1}a = (Kc)\,c^{-1}a = Ka \subseteq K$.

(2) Let $c \in A - \wp$, and $x \in I$. Then $x = (xc^{-1})c$ (in D). If $xc^{-1} \notin I$, then $cx^{-1} \in I^*$ and $c = cx^{-1}x \in I^*I = \wp$. Since this is not the case, $xc^{-1} \in I$ and $I \subseteq Ic \subseteq I$. Applying (1) proves (2).

If $B = D$, then $D = I \subseteq A$. Since this is not the case, we get that $A \subseteq B \subset D$. Thus, B is also a valuation on D. Consider the set $C = \{d \in D \mid dI \subseteq I\}$. Using the uniseriality of D_A, it is easy to see that C

is a valuation on D. Furthermore, I and D are left modules over C. We now have the following bimodule structures:

$$_{C}D_{B},\ _{C}I_{B},\ \textit{and}\ _{C}(D/I)_{B}.$$

Since B and C are valuations on D, the above (bi)modules are all uniserial whenever the module structure makes sense. We will use $\overline{D}$ to denote D/I.

In this section we will prove that the injectivity of D/I (as an A-module) is equivalent to $_{C}(D/I)$ being complete in all linear topologies. As one would expect, this is rather complicated. The proof will be broken up into a sequence of results which are easier to follow.

Lemma 40 *$\overline{D}_A$ is injective if and only if $\overline{D}_B$ is injective.*

Proof. Suppose that $\overline{D}_A$ is injective. Let $L \leq B$ be a right ideal and consider the following diagram

$$\begin{array}{ccc} L & \hookrightarrow & B \\ \varphi\downarrow & & \\ \overline{D}_B & & \end{array}$$

where $\varphi \in \hom_B\left(L_B, \overline{D}_B\right)$. Since L, B, and $\overline{D}_B$ are also A-modules we get that $\varphi \in \hom_A\left(L_A, \overline{D}_A\right)$. Using the injectivity of $\overline{D}_A$, there exists $\Phi \in \hom_A\left(B_A, \overline{D}_A\right)$ which extends φ. We show that Φ is also a B-module homomorphism.

Recall that $B = A_{\wp} = \{c^{-1}a \mid a \in A,\ c \in A - \wp\}$ and that $B = Bc$ for all $c \in A - \wp$. Let $x, b \in B$ and write $b = c^{-1}a$ for some $a \in A$, $c \in A - \wp$. Now write $x = zc$ for some $z \in B$. Then

$$\Phi(xb) = \Phi(za) = \Phi(z)a = \Phi(z)cc^{-1}a = \Phi(zc)c^{-1}a = \Phi(x)b.$$

Therefore, $\Phi \in \hom_B\left(B_B, \overline{D}_B\right)$ and extends φ. That is, $\overline{D}_B$ is injective.

Assume that $\overline{D}_B$ is injective. Let $K \leq A$ be a right ideal and consider the following diagram

$$\begin{array}{ccc} K & \hookrightarrow & A \\ \varphi \downarrow & & \\ \overline{D}_A & & \end{array}$$

where $\varphi \in \hom_A\left(K_A, \overline{D}_A\right)$.

Consider the B-modules KB and AB. Using the (left) uniseriality of B as an A-module, it is easily proven that $KB = \{kb \mid k \in K,\ b \in B\}$. Define $\widehat{\varphi} : KB \longrightarrow \overline{D}_B$ by $\widehat{\varphi}(kb) = \varphi(k)\, b$. We show that $\widehat{\varphi}$ is well defined. Suppose that $k_1 b_1 = k_2 b_2$. By uniseriality we can assume that $b_2 = ab_1$ for some $a \in A$. Then $k_1 b_1 = k_2 a b_1$ and so $k_1 = k_2 a$. Therefore,

$$\widehat{\varphi}(k_1 b_1) \equiv \varphi(k_1)\, b_1 = \varphi(k_2 a)\, b_1 = \varphi(k_2)\, ab_1 = \varphi(k_2)\, b_2 = \widehat{\varphi}(k_2 b_2).$$

This proves that $\widehat{\varphi}$ is well defined; that $\widehat{\varphi} \in \hom_B\left(KB_B, \overline{D}_B\right)$ is now easy to prove. We now have the following diagram

$$\begin{array}{ccc} KB & \hookrightarrow & AB \\ \widehat{\varphi} \downarrow & & \\ \overline{D}_B & & \end{array} .$$

Since $\overline{D}_B$ is injective, there exists $\widehat{\Phi} \in \hom_B\left(AB_B, \overline{D}_B\right)$ which extends $\widehat{\varphi}$. Because $A \subseteq B$, we can set $\Phi = \widehat{\Phi}\mid_A$ and get that $\Phi \in \hom_A\left(A, \overline{D}_A\right)$. For any $k \in K \subseteq KB$

$$\Phi(k) = \widehat{\Phi}(k) = \widehat{\varphi}(k) = \varphi(k).$$

We may now conclude that $\overline{D}_A$ is injective.

Definition 41 *A module is said to be* **linearly compact** (*lc*) *if it is complete in every linear topology. It is said to be* **almost linearly compact** (*alc*) *if it is complete in every non-Hausdorff linear topology.*

Lemma 42 *${}_C D$ is almost linearly compact if and only if ${}_C\overline{D}$ is linearly compact.*

Proof. ($\Rightarrow$)Suppose that ${}_C D$ is alc. Let $\overline{D}$ have a linear topology with a base of neighborhoods about 0 denoted by $\overline{\mathcal{B}}$. As we argued in a previous section we may assume that this topology is the ${}_C\overline{K}$-topology where $\overline{K} = \bigcap_{\overline{U}\in\overline{\mathcal{B}}} \overline{U}$. We can therefore assume that $\overline{\mathcal{B}} = \left\{\overline{U} \leq \overline{D} \mid \overline{K} \subset \overline{U}\right\}$. Here we assume that, for all $\overline{U} \in \overline{\mathcal{B}}$, $I \subseteq U$; hence, $I \subseteq K$ where $K = \bigcap_{\overline{U}\in\overline{\mathcal{B}}} U$.

Consider ${}_C D$ with the ${}_C K$-topology. Then this topology has a base (of neighborhoods about 0) $\mathcal{B} = \{{}_C W \leq D \mid K \subset W\}$ and $I \subseteq K \subseteq \bigcap_{W\in\mathcal{B}} W$. This shows that the topology is non-Hausdorff. The hypothesis implies that this topology is complete.

Let $(\overline{x_\alpha})$ be a Cauchy net in $\overline{D}$ and consider the net (x_α) in D. If $W \in \mathcal{B}$, then $\overline{W} \in \overline{\mathcal{B}}$. Hence there exists σ such that $\overline{x_\alpha} - \overline{x_\beta} \in \overline{W}$ whenever $\alpha, \beta \geq \sigma$. This implies that $x_\alpha - x_\beta = w + a$ for some $w \in W$ and $a \in I$. Since $I \subseteq W$, we get that $x_\alpha - x_\beta \in W$. Therefore, (x_α) is Cauchy in D; completeness implies that it converges to some $x \in D$.

If $\overline{U} \in \overline{\mathcal{B}}$, then $U \in \mathcal{B}$ and there exists some σ such that $\alpha \geq \sigma$ implies that $x_\alpha - x \in U$. Therefore,

$$\overline{x_\alpha} - \overline{x} = \overline{x_\alpha - x} \in \overline{U} \text{ whenever } \alpha \geq \sigma.$$

That is, $(\overline{x_\alpha})$ converges and ${}_C\overline{D}$ is complete.

($\Leftarrow$) Suppose that ${}_C\overline{D}$ is linearly compact and let ${}_C D$ have a non-Hausdorff linear topology. As before, we may assume that this is the ${}_C K$-topology for some K, and that the base of neighborhoods about 0 is $\mathcal{B} = \{{}_C W \leq D \mid K \subset W\}$. Furthermore, $K = \bigcap_{W\in\mathcal{B}} W \neq \{0\}$ (since the topology is non-Hausdorff).

Case (1): Assume that $I \subseteq K$. Then $\overline{K} = K/I$ is a C-submodule of $\overline{D}$ and, by hypothesis, $\overline{D}$ is complete in the ${}_C\overline{K}$-topology. Also note that the base of neighborhoods is $\overline{\mathcal{B}} = \left\{\overline{W} \mid W \in \mathcal{B}\right\}$.

Let (x_α) be a Cauchy net in D. It is a simple matter to show that $(\overline{x_\alpha})$ is Cauchy in ${}_C\overline{D}$. Completeness implies that $(\overline{x_\alpha})$ converges to some $\overline{x}$. Because $I \subseteq W$ for each $W \in \mathcal{B}$, it is equally easy to show that (x_α) is convergent to x.

Case (2): Assume that $K \subset I$. Then $I \in \mathcal{B}$ and $K \subset I \subseteq A$. Pick $0 \neq a \in D$ such that $I \subseteq C \subseteq Ka$ ($a = k^{-1}$ for any $0 \neq k \in K$ will do). By case (1) D is complete in the Ka-topology. Note that
$Ka = \left(\bigcap_{W \in \mathcal{B}} W\right) a = \bigcap_{W \in \mathcal{B}} Wa$. Hence, a base of neighborhoods about 0 for the Ka-topology is $\mathcal{B}_{Ka} = \{Wa \mid W \in \mathcal{B}\}$.

Let (x_α) be Cauchy in D with the K-topology. We claim that $(x_\alpha a)$ is Cauchy in D with the Ka-topology. Let $Wa \in \mathcal{B}_{Ka}$ be given. Choose σ such that $x_\alpha - x_\beta \in W$ whenever $\alpha, \beta \geq \sigma$. Then $x_\alpha a - x_\beta a \in Wa$; that is, $(x_\alpha a)$ is Cauchy in D with the Ka-topology.

Since D is complete in the Ka-topology, $(x_\alpha a)$ converges to some x. We will show that (x_α) converges to xa^{-1} in the K-topology. Let $W \in \mathcal{B}$ and choose $\sigma = \sigma(Wa)$ such that $\alpha \geq \sigma$ implies $x_\alpha a - x \in Wa$. Then $\alpha \geq \sigma$ implies that $x_\alpha - xa^{-1} \in W$. Therefore, the claim is true and the proof is complete.

Lemma 43 *${}_CD$ is almost linearly compact if and only if ${}_CC$ is almost linearly compact.*

Proof. ($\Rightarrow$) Suppose that ${}_CD$ is alc and let ${}_CC$ have a non-Hausdorff linear topology with a base (of neighborhoods about 0) $\mathcal{B}_C$. We may therefore assume that the given topology on C is the ${}_CK$-topology for $K = \bigcap_{W \in \mathcal{B}_C} W \neq \{0\}$. We may also assume that $\mathcal{B}_C = \{{}_CW \leq C \mid K \subset W\}$.

Consider ${}_CK$ as a C-submodule of D and endow D with the ${}_CK$-topology.

With this topology on D a base of neighborhoods about 0 will be $\mathcal{B}_D = \{_C U \leq D \mid K \subset U\}$. Moreover, the $_C K$-topology is a linear non-Hausdorff topology. By hypothesis, it is complete. Note also that $\mathcal{B}_C \subseteq \mathcal{B}_D$.

If $U \in \mathcal{B}_D$, then there is some $W \in \mathcal{B}_C$ such that $W \subseteq U$. Otherwise, by uniseriality, $U \subset W$ for all $W \in \mathcal{B}_C$. This implies that $K \subset U \subseteq \bigcap_{W \in \mathcal{B}_C} W = K$; a contradiction.

Let (x_α) be a Cauchy net in C. By the fact above, it is easily seen that (x_α) is Cauchy in D. By completeness, (x_α) converges, in D, to some x say. We will prove that $x \in C$.

Let $W \in \mathcal{B}_C \subseteq \mathcal{B}_D$, then there exists σ such that $x_\alpha - x \in W$ whenever $\alpha \geq \sigma$. But $W \subseteq C$ and $x_\alpha \in C$; therefore, $x \in C$.

($\Leftarrow$) Suppose that $_C C$ is alc. Let $_C D$ have a non-Hausdorff linear topology which we assume to be the $_C K$-topology. We can assume that a base of neighborhoods of 0 is $\mathcal{B}_D = \{_C U \leq D \mid K \subset U\}$ and that $0 \neq K = \bigcap_{U \in \mathcal{B}_D} U \neq D$.

Choose $U_0 \in \mathcal{B}_D$ such that $U_0 \neq D$. Pick $d \in D - U_0$. For all $V \in \mathcal{B}_D$ with $V \subseteq U_0$, $d \notin V$; thus, by uniseriality, $Vd^{-1} \subset C$. Let

$$K' = \bigcap_{V \subseteq U_0} Vd^{-1}.$$

Then $0 \neq {}_C K' < C$ $(0 \neq Kd^{-1} \subseteq K')$ and so the K'-topology on C is a non-Hausdorff linear topology. By assumption, it is complete. Since $K' = \bigcap_{V \subseteq U_0} Vd^{-1}$, the set $\mathcal{B}_C = \{Vd^{-1} \mid V \subseteq U_0\}$ is a base (of neighborhoods about 0) for the K'-topology.

Let $(x_\alpha)_{\alpha \in \Lambda}$ be a Cauchy net in D. There exists $\sigma = \sigma(U_0) \in \Lambda$ such that $x_\alpha - x_\beta \in U_0$ whenever $\alpha, \beta \geq \sigma$. For all $\alpha \geq \sigma$ define $y_\alpha = x_\alpha - x_\sigma \in U_0$. Thus, $y_\alpha d^{-1} \in U_0 d^{-1} \subseteq C$ whenever $\alpha \geq \sigma$. Therefore, $(y_\alpha d^{-1})_{\alpha \geq \sigma}$ is a net in C. We will show that it is a Cauchy net.

Let $Vd^{-1} \in \mathcal{B}_C$ be given; then there is some $\sigma_1 = \sigma(V) \in \Lambda$ such that $x_\alpha - x_\beta \in V$ whenever $\alpha, \beta \geq \sigma_1$. Thus $(x_\alpha - x_\beta) d^{-1} \in Vd^{-1}$ for all $\alpha, \beta \geq \sigma_1$. Pick $\sigma' \in \Lambda$ such that $\sigma' \geq \sigma, \sigma_1$ (recall Λ is upward directed). Therefore, $\alpha, \beta \geq \sigma'$ implies that

$$y_\alpha d^{-1} - y_\beta d^{-1} = (x_\alpha - x_\sigma - x_\beta + x_\sigma) d^{-1} = (x_\alpha - x_\beta) d^{-1} \in Vd^{-1}.$$

This shows that $(y_\alpha d^{-1})_{\alpha \geq \sigma}$ is a Cauchy net in C. Since C is complete, it must converge to some $y \in C$.

We claim that $(x_\alpha)_{\alpha \in \Lambda}$ is convergent (in D) to $x_\sigma + yd$. Let $W \in \mathcal{B}_D$. By uniseriality, $V \equiv W \cap U_0$ is equal either W or U_0. Thus, $V \in \mathcal{B}_D$ and $V \subseteq U_0$. Therefore, there exists $\sigma(Vd^{-1})$ such that $y_\alpha d^{-1} - y \in Vd^{-1}$ whenever $\alpha \geq \sigma(Vd^{-1})$. This now implies that

$$\left(y_\alpha d^{-1} - y\right) d = x_\alpha - x_\sigma - yd = x_\alpha - (x_\sigma + yd) \in V \subseteq W$$

whenever $\alpha \geq \sigma(Vd^{-1})$. This proves the claim and the lemma.

Recall that for any $x \in D$ we use $Ix^{-1} = \{d \in D \mid dx \in I\}$. For $x = 0$, $Ix^{-1} = D$. Since D is a division ring, for any $0 \neq x \in D$, $Ix^{-1} = \{ax^{-1} \mid a \in I\}$. Let ${}_CK < D$, we let

$$X(K) \equiv \left\{x \in D \mid K \subset Ix^{-1}\right\}.$$

Notation. To keep notation simple we will use X to denote $X(K)$ throughout the remainder of this section. If there is any confusion about the submodule K, we will write $X(K)$.

Proposition 44 *Let $K < D$ be a C-submodule of D. Then X is a right B-submodule of D.*

Proof. Since $K < D$, $0 \in X$. Let $r, s \in X$. Then Ir^{-1} and Is^{-1} are left C-submodules of D; by uniseriality, they are comparable. Suppose that $Ir^{-1} \subseteq Is^{-1}$. If $z \in Ir^{-1}$, then $zr, zs \in I$; hence, $z(r+s) \in I$. Consequently,

$$K < Ir^{-1} \subseteq I(r+s)^{-1}.$$

and $r+s$ is in X.

Because $Ia \subseteq I$ for all $a \in A$, $I \subseteq Ia^{-1}$ for all $a \in A$ Let $0 \neq x \in X$ and $0 \neq b \in B$. Write $b = ac^{-1} \in B = A_{\wp} = \{ac^{-1} \mid a \in A,\ c \in A - \wp\}$. Then, by a previous result, $I = Ic$ and

$$I(xb)^{-1} = Ib^{-1}x^{-1} = Ica^{-1}x^{-1} = Ia^{-1}x^{-1} \supseteq Ix^{-1} \supset K.$$

Therefore, $xb \in X$ aud X is a right B-submodule of D.

Lemma 45 *Let $K < D$ be a left C-submodule. If $a \in (\bigcap_{x \in X} Ix^{-1}) - K$, then $I = aX$.*

Proof. Let $x \in X$. By hypothesis $a \in Ix^{-1}$; thus, $ax \in I$. This implies that $aX \subseteq I$.

Conversely, suppose that there exists $z \in I - aX$. Then $a^{-1}z \notin X$. By the uniseriality of ${}_CD$, we must have $I(a^{-1}z)^{-1} \subseteq K$. This implies that $z \in I \subseteq K(a^{-1}z)$. We conclude that $z = ka^{-1}z$ for some $k \in K$. But then $a = k \in K$; since this is not the case, no such z can exist.

Lemma 46 *Let ${}_CK < D$. Then K is friendly if and only if $K = \bigcap_{x \in X} Ix^{-1}$.*

Proof. ($\Leftarrow$) This is clear.

($\Rightarrow$) Suppose that there is some $a, b \in (\bigcap_{x \in X} Ix^{-1}) - K$. Then, by the previous lemma, $aX = I = bX$. From this it follows that $ab^{-1}I = I = ba^{-1}I$;

hence $ab^{-1}, ba^{-1} \in C$. Let $\mathfrak{n}$ denote the maximal ideal of C. Since ab^{-1} is a unit in C, $\mathfrak{n} = \mathfrak{n}ab^{-1}$. Consequently, $\mathfrak{n}a = \mathfrak{n}b$.

Because $a \notin K$ and $\mathfrak{n}a$ is maximal in Ca, we can conclude that $K \subseteq \mathfrak{n}a \subset Ca$. If $K = \mathfrak{n}a$, then, because K is friendly,

$$\mathfrak{n}a = K = \bigcap_{K <_C U \leq D} U = Ca.$$

This contradiction implies that $K \subset \mathfrak{n}a \subset Ca$.

Choose $z \in \mathfrak{n}$ such that $za \notin K$. Since $(\bigcap_{x \in X} Ix^{-1})$ is a left C-submodule, $za \in (\bigcap_{x \in X} Ix^{-1}) - K$. By the above, $\mathfrak{n}a = \mathfrak{n}za$. In particular, $za = cza$ for some $c \in \mathfrak{n}$. But this implies $c = 1 \in \mathfrak{n}$. Because this is not the case, our original assumption is wrong and the proof is complete.

We are now reaǔy to state the main theorem of this chapter.

Theorem 47 *Let $A \subset D$ be a valuation on a division ring and $I \leq A$ a right ideal of A. Let $\mathfrak{p} = I^{\bullet}I$ (which is a completely prime ideal of A), and let $B = A_{\mathfrak{p}}$. Let $C = \{d \in D \mid dI \subseteq I\}$ and $\overline{D} = D/I$. Then the following are equivalent.*

(1) *$\overline{D}_A$ is injective.*

(2) *$\overline{D}_B$ is injective.*

(3) *${}_C\overline{D}$ is linearly compact.*

(4) *${}_CD$ is almost linearly compact.*

(5) *${}_CC$ is almost linearly compact.*

Proof. It remains only to prove that (2) and (3) are equivalent.

(2) $\Rightarrow$(3) Let ${}_C\overline{D}$ have a linear topology with a base of neighborhoods about 0, $\mathcal{B}$ say. By our reduction we may assume that this is the $\overline{K}$-topology for some friendly $\overline{K} < \overline{D}$ and $\overline{K} = \bigcap_{\overline{U} \in \mathcal{B}} \overline{U}$. We may also suppose that $\overline{K} < \overline{U}$ for each $\overline{U} \in \mathcal{B}$. We are also assuming that $I \subseteq U$ for each $\overline{U} \in \mathcal{B}$.

Let $K = \bigcap_{\overline{U} \in \mathcal{B}} U \subset D$. Then $I \subseteq K$ and ${}_C K$ is a friendly C-submodule of D. By lemma 46,

$$I \subseteq K = \bigcap_{x \in X} Ix^{-1} \quad where \quad X = X(K) \equiv \left\{x \in D \mid K \subset Ix^{-1}\right\}.$$

It now follows that $\overline{K} = \bigcap_{x \in X} \overline{Ix^{-1}}$. By our reduction we need to show that any collection $\mathcal{C} = \left\{\overline{d_x} + \overline{Ix^{-1}} \mid x \in X\right\}$, where $d_x \in D$, which has the fip also has the ip.

Suppose that $\mathcal{C} = \left\{\overline{d_x} + \overline{Ix^{-1}} \mid x \in X\right\}$, where $d_x \in D$, has the fip. Recall that X is a B-submodule of D_B. Define $\varphi : X \longrightarrow \overline{D}_B$ by $\varphi(x) = \overline{d_x x}$. We will show that φ is a B-module homomorphism.

Let $x, y \in X$; by the fip, there exists

$$\overline{a} = \overline{d_x} + \overline{z_x} = \overline{d_y} + \overline{z_y} = \overline{d_{x+y}} + \overline{z_{x+y}}$$

where $z_x \in Ix^{-1}$, $z_y \in Iy^{-1}$, $z_{x+y} \in I(x+y)^{-1}$. Thus, $\overline{ax} = \overline{d_x x} + \overline{z_x x} = \overline{d_x x}$. We have used the facts that $z_x \in Ix^{-1}$ implies that $z_x x \in I$, and that $I \subseteq K \subset Ix^{-1}$. Similar statements hold for y and $x + y$. Therefore,

$$\varphi(x+y) \equiv \overline{d_{x+y}(x+y)} = \overline{a(x+y)} = \overline{ax} + \overline{ay} = \overline{d_x x} + \overline{d_y y} = \varphi(x) + \varphi(y).$$

Let $x \in X$ and $b \in B$; using the same argument as above with x and xb there is some $a \in D$ such that $\overline{ax} = \overline{d_x x}$ and $\overline{d_{xb}(xb)} = \overline{axb}$. Therefore,

$$\varphi(xb) \equiv \overline{d_{xb}(xb)} = \overline{axb} = \overline{ax}b = \overline{d_x x}b = \varphi(x)\, b.$$

Therefore, φ is a B-module homomorphism.

Because $\overline{D}_B$ is injective, there exists $\Phi \in \hom_B\left(D_B, \overline{D}_B\right)$ which extends φ. Let $\Phi(1) = \overline{u}$. Then, for $0 \neq x \in X$, we have the following (recall B is a valuation):

(i) $x \in B$ implies that $\varphi(x) = \overline{d_x x} = \Phi(x) = \overline{ux}$.

(ii) $x^{-1} \in B$ implies that $\overline{u} = \Phi(xx^{-1}) = \Phi(x)x^{-1} = \varphi(x)x^{-1} = \overline{d_x x}x^{-1} = \overline{d_x}$.

Case (i) implies that $ux = d_x x + a$ for some $a \in I$. Thus,
$u = d_x + ax^{-1} \in d_x + Ix^{-1}$. It then follows that $\overline{u} \in \overline{d_x} + \overline{Ix^{-1}}$. Case (ii) immediately implies that $\overline{u} \in \overline{d_x} + \overline{Ix^{-1}}$.

If $x = 0$, then $\overline{Ix^{-1}} = \overline{D}$. Hence, $\overline{u} = \overline{d_x} + (\overline{u - d_x}) \in \overline{d_x} + \overline{Ix^{-1}}$. Therefore, $\overline{u} \in \bigcap_{x \in X}\left(\overline{d_x} + \overline{Ix^{-1}}\right)$. That is, C has the ip.

(3)⇒(2) Let $0 \neq L_B$ be a right ideal of B and consider the diagram

$$\begin{array}{ccc} L_B & \hookrightarrow & B \\ \varphi \downarrow & & \\ \overline{D}_B & & \end{array}$$

where $\varphi \in \hom_B\left(L_B, \overline{D}_B\right)$. We will show that we can extend φ to B by $\Phi \in \hom_B\left(B, \overline{D}\right)$.

Let

$$K = \bigcap_{r \in L} Ir^{-1}.$$

Then K is a left C-submodule of D. Because $L \subseteq B$, we get that $IL \subseteq I$. This implies that $I \subseteq K$.

If $0 \neq r \in L$, then $Ir^{-1} \neq D$ (otherwise, $I = Dr = D$). Thus $K < D$. Suppose that $K = Ir^{-1}$ for some $0 \neq r \in L$; we will show how to extend φ. Let $\varphi(r) = \overline{x_r}$ for some $x_r \in D$ and let $u = x_r r^{-1}$. Define $\Phi : B \longrightarrow \overline{D}$ by $\Phi(b) = \overline{u}b$. Surely $\Phi \in \hom_B\left(B, \overline{D}\right)$.

Let $0 \neq s \in L$, then, by uniseriality, $s = ra$ or $r = sa$ for some $a \in B$.

(i) If $s = ra$, then $a = r^{-1}s \in B$. Thus,

$$\varphi(s) = \varphi(ra) = \varphi(r)a = \overline{x_r}\left(r^{-1}s\right) = \overline{x_r(r^{-1}s)} = \overline{u}s = \Phi(s).$$

(ii) If $r = sa$, then $Is^{-1} \subseteq Ir^{-1} = K \subseteq Is^{-1}$. Thus, $Is^{-1} = Ir^{-1}$. Write $\varphi(s) = \overline{x_s}$; then $\varphi(r) = \varphi(sa) = \overline{x_r} = \overline{x_s a}$. This implies that $x_r + b = x_s a$ for some $b \in I$. Using the identity $a^{-1} = r^{-1}s$, we obtain $x_s = x_r r^{-1}s + br^{-1}s$. Since $br^{-1} \in Ir^{-1} = Is^{-1}$, $br^{-1}s \in I$. Therefore,

$$\varphi(s) = \overline{x_s} = \overline{x_r r^{-1}s} + \overline{br^{-1}s} = \overline{x_r r^{-1}s} = \overline{u}s = \Phi(s).$$

This shows that we can extend φ. Thus we may now assume that $K \subset Ir^{-1}$ for all $r \in L$.

We now have ${}_C\overline{K} < {}_C\overline{D}$ and we consider ${}_C\overline{D}$ with the ${}_C\overline{K}$-topology. This topology is complete by hypothesis. By our above arguments and our reduction, we may take $B = \left\{\overline{Ir^{-1}} \mid r \in L\right\}$ as a base of neighborhoods about 0.

For each $0 \neq r \in L$, let $\varphi(r) = \overline{x_r}$ and let

$$C = \left\{\overline{x_r r^{-1}} + \overline{Ir^{-1}} \mid r \text{ and } x_r \text{ as above}\right\}.$$

(For $r = 0$, $\overline{Ir^{-1}} = \overline{D}$ and so we simply add $\overline{x_0} + \overline{D} = \overline{D}$ to C). Suppose that we can show that C has the fip. Then C has the ip (because ${}_C\overline{K}$-topology is complete). Choose

$$\overline{u} \in \bigcap_{r \in L} \left(\overline{x_r r^{-1}} + \overline{Ir^{-1}}\right)$$

and define $\Phi : B \longrightarrow \overline{D}$ by $\Phi(b) = \overline{u}b$.

Now for each $0 \neq r \in L$ write $\overline{u} = \overline{x_r r^{-1}} + \overline{z_r}$ for some $z_r \in Ir^{-1}$. Then

$$\Phi(r) = \overline{u}r = \left(\overline{x_r r^{-1}} + \overline{z_r}\right)r = \overline{x_r} + \overline{z_r r} = \overline{x_r} = \varphi(r).$$

This shows that Φ extends φ. Thus we need only show that C has the fip.

Consider $\overline{x_r r^{-1}} + \overline{Ir^{-1}}$ and $\overline{x_s s^{-1}} + \overline{Is^{-1}}$ for some $r, s \in L$. Without loss we may assume that $r = sb$ for some $b \in B$. Then $\overline{Is^{-1}} \subseteq \overline{Ir^{-1}}$ and $\varphi(r) = \overline{x_r} = \varphi(s)b = \overline{x_s b}$. This implies that $x_r + a = x_s b$ for some $a \in I$. Since $r^{-1} = b^{-1}s^{-1}$, we get $x_r r^{-1} + ar^{-1} = x_s s^{-1}$. It follows that

$$\overline{x_s s^{-1}} + \overline{Is^{-1}} \subseteq \overline{x_s s^{-1}} + \overline{Ir^{-1}} = \left(\overline{x_r r^{-1}} + \overline{ar^{-1}}\right) + \overline{Ir^{-1}} = \overline{x_r r^{-1}} + \overline{Ir^{-1}}.$$

This shows that C has the fip.

Chapter 4

Associated Primes

4.1 Generalized Associated Primes

Definition 48 *Let V be an R-module. An ideal $P \trianglelefteq R$ is said to be an* **associated prime** *if there exists some $0 \neq W \leq V$ such that $P = ann(X)$ for all $0 \neq X \leq W$.*

Note that such a P is always a prime ideal. If the module is uniform and there exists an associated prime ideal, then it must be unique. Such an ideal is sometimes called a **classical associated prime** and will be denoted by $P = class(E)$. The submodule W is called a P-**prime submodule.** Obviously this submodule need not be unique.

Let $A \subset D$ be a valuation on a division ring and E_A a uniform injective A-module. By a previous result $E = E(D/I)$ for some $I \leq A \subset D$. As left A-modules, $I^* = \{d \in D \mid d^{-1} \notin I\} \cong \hom_A(I, \mathrm{m})$, where m is the maximal ideal of A. Furthermore, $(I^*)I$ is a completely prime (equivalently Goldie prime) ideal of A [cf. M1].

For any module V_R and $v \in V$, $ann_R(v) = \{r \in R \mid vr = 0\}$ will be denoted by v^0. This is always a right ideal of R.

Theorem 49 *Let $A \subset D$ be a valuation and E_A a uniform injective A-module. Then*

(1) $E = E\left({}^{D}/_{I}\right)$ *for some* $I \leq A$;

(2) $(I^{\bullet})\, I = \bigcup_{0 \neq u \in {}^{D}/_{I}} u^0 = \bigcup_{0 \neq v \in E} v^0$.

Proof. The first statement has already appeared.

Denote D/I by $\overline{D}$. Note first that $\bigcup_{0\neq u \in \overline{D}} u^0$ is a right ideal since it is a union of a chain of right ideals. Let $b \in I^{\bullet}$ and $a \in I$. Then $b^{-1} \notin I$ and $\overline{b^{-1}}$ is nonzero in $\overline{D}$. Consequently,

$$\overline{b^{-1}}(ba) = \overline{b^{-1}(ba)} = \overline{a} = \overline{0}.$$

Therefore, $ba \in \bigcup_{0\neq u \in \overline{D}} u^0$. It follows that $I^{\bullet}I \subseteq \bigcup_{0\neq u \in \overline{D}} u^0$.

Conversely, suppose $x \in u^0$ for some $0 \neq u \in \overline{D}$. Write $u = \overline{a}$ where $a \notin I$. This implies that $a^{-1} \in I^{\bullet}$ and $0 = ux = \overline{a}x = \overline{ax}$. Therefore, $ax \in I$ and $x = a^{-1}(ax) \in I^{\bullet}I$. The first equality in (2) is proved.

Surely $\bigcup_{0\neq u \in \overline{D}} u^0 \subseteq \bigcup_{0\neq v \in E} v^0 \equiv \zeta$. Let $a \in \zeta$ and $0 \neq v \in E$ with $va = 0$. Since $\overline{D} \subseteq' E$, there is an element $b \in A$ such that $0 \neq vb \in \overline{D}$. Uniseriality implies $b = as$ or $a = bs$ for some $s \in A$. The former leads to the contradiction $0 \neq vb = vas = 0$. Hence, $0 = va = (vb)s$ and $0 \neq vb \in \overline{D}$. Thus $s \in I^{\bullet}I$ (by the above); since $I^{\bullet}I$ is a two sided ideal in A, $a = bs \in I^{\bullet}I$.

In the theorem, the choice of I is not unique. However we always get a unique completely prime ideal $\wp = I^{\bullet}I$.

Definition 50 *The completely prime ideal in the above theorem will be called the* **generalized associated prime ideal** *of E. We shall denote this by $gass(E)$.*

Suppose A is a valuation and E is a uniform injective A-module. We then get the completely prime ideal $\wp = I^{\bullet}I = gass(E)$. It is clear that if E also has an associated prime, P, then $P \subseteq \wp$. There are some obvious questions that arise at this point.

(1) Let V be a uniform injective module over a Goldie prime serial ring R. Suppose that there are two distinct faithful idempotents e and f. Then Ve and Vf are uniform injectives over $A = eRe$ and $B = fRf$ respectively. Hence, each will have a generalized associated prime. Let $\wp = gass(Ve)$ and $\zeta = gass(Vf)$. Then $\wp \rightleftharpoons P \in spec(R)$ via the Morita context correspondence (MCC) between A and R, and $\zeta \rightleftharpoons Q \in spec(R)$ via the Morita context correspondence between B and R.

(i) How are $\wp$ and ζ related? Is there a MCC between $\wp$ and ζ?

(ii) Could P and Q be comparable?

(iii) Is it the case that $P = Q$?

(2) For an arbitrary valuation $A \subset D$. Let E_A be a uniform injective A-module in which $class(E)$ exists. Does $class(E) = gass(E)$?

A partial answer to (2) is given in the next result.

Proposition 51 *Let A be a valuation on a field D and E a uniform injective A-module. If class(E) exists, then $class(E) = gass(E)$.*

Proof. Let $P = class(E)$ and $0 \neq W \leq E$ be a P-prime submodule. Let $a \in \wp = gass(E)$ and $0 \neq u \in E$ such that $a \in u^0$. Because A is commutative, $(uA)\,a = u\,(aA) = 0$. Therefore, $a \in ann_A(uA) \subseteq ann_A(uA \cap W) = P$.

Since $P \subseteq \wp$ is always true, we are done.

To handle the general case we will need some definitions and a preliminary result.

Definition 52 *Let R be a ring. A right ideal $I < R$ ($I \neq R$) is said to be a* prime right ideal *if $aRb \subseteq I$ implies that $a \in I$ or $b \in I$. For a right ideal $I \leq R$, the* bound of I, *denoted by bd(I), is the largest ideal of R which is contained in I.*

Note that $bd(I)$ always exists; it is the sum of all two-sided ideals in I. Thus, it also contains all the two-sided ideals which are contained in I.

Lemma 53 *Let $A \subset D$ be a valuation and E_A a uniform injective A-module. Then $class(E) = P$ exists if and only if there is some prime right ideal I such that $E = E(D/I)$. In this case $P = bd(I)$.*

Proof. ($\Leftarrow$) Suppose that $E = E(\overline{D})$ where $\overline{D} = D/I$ and I is a prime right ideal of A. Let $P = bd(I) \subseteq I$ so that $P \triangleleft A$. We will first show that $P = ann_A(\overline{A})$.

Let $\overline{A} = A/I$. Surely $ann_A(\overline{A}) \subseteq ann_A(\overline{1}) \subseteq I$ and $ann_A(\overline{A})$ is a two-sided ideal in A. Thus, $ann_A(\overline{A}) \subseteq P$. Conversely, for any $p \in P$, and $\overline{a} \in \overline{A}$, $ap \in P \subseteq I$. Therefore, $P = ann_A(\overline{A})$.

Let $0 \neq \overline{N} \leq \overline{A}$, where $I \subset N$ and $\overline{N} = N/I$. Then $P = ann_A\left(\overline{A}\right) \subseteq ann_A(\overline{N}) \trianglelefteq A$. Fix $x \in N - I$. Let $p \in ann_A(\overline{N})$ and let $r \in A$ be arbitrary. Then $\overline{0} = \overline{xr}p = \overline{xrp}$. Thus, $xAp \subseteq I$. Because I is prime and $x \notin I$, we conclude that $p \in I$. Therefore, $ann_A(\overline{N}) \subseteq I$. Because $P = bd(I)$, we may conclude that $P = ann_A(\overline{N})$; therefore, $P = class(E)$.

($\Rightarrow$) Let $P = class(E)$ and let W be a P-prime submodule. Suppose that we can find some $I < A$ such that $E = E(D/I)$ and $P \subseteq I$. We claim that I is a prime right ideal. If $aAb \subseteq I$ and $a \notin I$, then $0 \neq \overline{a}A \subseteq E$. Since $aAb \subseteq I$, it follows that $b \in ann_A(\overline{a}A \cap W) = P \subseteq I$. This shows that I is a prime right ideal. To complete the proof we need only find such a right ideal.

Pick any $0 \neq x \in W$, then, by the proof of Theorem 25, $E = E(D/I)$ where $I = x^0 < A$. The map

$$\varphi : A \to E \quad by \quad \varphi(r) = xr \quad for\ all \quad r \in A$$

induces an isomorphism from $\overline{A} = A/I$ to $xA \subseteq W$. Therefore,
$P = ann_A(xA) = ann_A\left(\overline{A}\right) \subseteq I$.

Remark. In the commutative case the above lemma implies that
$P = class(E) = bd(I) = I$ for some prime ideal $I \lhd A$ and
$E = E(D/I)$. In this case, $\wp = gass(E) = I^{*}I = P = I$.

Using the lemma, we are in a position to answer question (2). Suppose there exists a valuation, A, possessing a prime right ideal, I, that is not a two-sided ideal. Then, with $E = E(D/I)$, $P = bd(I) = class(E) \neq I$. Furthermore, $1 \notin I$ implies that $1 \in I^{*}$; hence,
$class(E) = P \subset I \subseteq I^{*}I = gass(E)$. This would give a negative answer to question (2).

In a private communication, Dr. H.H. Brungs has furnished us with such an example. We would like to thank him for this contribution.

We will now begin the task of answering the questions in (1). We begin with part(iii).

Example. This example shows that P and Q need not be equal.

Recall our example from a previous section 2.2. We have $R = \begin{pmatrix} A & A \\ \mathrm{m} & A \end{pmatrix}$,

a uniform injective V_R, such that $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $f = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ are both faithful. We showed that $V_R \cong E \oplus E$, where E is a uniform injective A-module and that $ann_E(\mathrm{m}) = 0$. In this case $Ve \cong E \cong Vf$. Suppose that we can show $\wp = gass(Ve) = gass(E) = \mathrm{m} = gass(Vf) = \zeta$. By the MCC $\wp \rightleftharpoons P = \{r \in R \mid eRrRe \subseteq \wp\}$ and $\zeta \rightleftharpoons Q = \{r \in R \mid fRrRf \subseteq \zeta\}$. An elementary calculation shows that

$$P = \begin{pmatrix} \mathrm{m} & A \\ \mathrm{m} & A \end{pmatrix} \neq \begin{pmatrix} A & A \\ \mathrm{m} & \mathrm{m} \end{pmatrix} = Q.$$

Therefore, $P \neq Q$ nor are P and Q comparable. We show $gass(E) = \mathrm{m}$.

Recall that

$$A = k\langle Q^+\rangle = \left\{f = \sum a_\alpha x^\alpha \mid Q \ni \alpha \geq 0, a_\alpha \in k, supp\,(f) \text{ is well ordered}\right\}$$

and that $E = E\,(D/I)$, where $I = xA$. By uniseriality and the maximality of m, $gass(E) \subseteq \mathrm{m}$.

Conversely, let $0 \neq f \in \mathrm{m}$. Then $f = a_\alpha x^\alpha + g$ for some $g \in A$ and $\alpha = \min\{supp(f)\}$. Note that $\alpha \neq 0$ ($f \in \mathrm{m}$ implies the constant term of f is zero). If $\alpha \geq 1$, then $f \in I$. Hence, $f \in gass(E)$. If $0 < \alpha < 1$, then $0 < 1 - \alpha < 1$. Choose $p(x) = x^{1-\alpha}$. Then $0 \neq \overline{p} \in D/I$ and $\overline{p}f = 0$ in D/I. Thus, $\mathrm{m} = gass(E)$.

The rest of this section is devoted to obtaining a relationship between the generalized associated primes (of different faithful idempotents) and how the primes in R (which arise under the MCC) are related.

Notation. Again we fix a decomposition $e_1 + e_2 + \cdots + e_n = 1 \in R$ into orthogonal idempotents and V will denote a uniform injective R-module. We are interested in the case that there are at least two distinct faithful idempotents. When we are dealing with only two such idempotents, we use e and f to denote them. Also, we use the following notation: $A = eRe$, $B = fRf$, $X_{12} = eRf$, and $X_{21} = fRe$. We will, unless otherwise stated, use $\wp = gass(Ve)$ and $\zeta = gass(Vf)$.

Lemma 54 *Let R be a ring and V a uniform injective R-module. Let e and f be faithful to V and let $E = Ve$. Then $Vf \cong \hom_A(X_{21}, E)$ as right B-modules.*

Proof. Let $H = \hom_A(X_{21}, E)$. Then H is a B-module by

$$(\varphi \circ b)(x) = \varphi(bx) \text{ for all } \varphi \in H,\ b \in B,\ x \in X_{21}.$$

Because $Vf \subseteq V \cong \hom_A(Re, E)$, we will simply consider Vf as a subset of $\hom_A(Re, E)$. Define

$$\Phi : Vf \to H \text{ by } \Phi(\varphi) = \varphi|_{fRe} = \varphi|_{X_{21}}.$$

Using the decomposition $Re = fRe \oplus (1-f)Re$, one can show that Φ is a B-module isomorphism.

We need the following technical lemma.

Lemma 55 *Let R be a ring and V a uniform injective R-module. Let e and f be faithful to V and $b \in B$. The following are equivalent:*

(1) *There exists $0 \neq \Phi \in \hom_A(X_{21}, Ve)$ such that $\Phi \circ b = 0$.*

(2) *There exists $0 \neq \Phi \in \hom_A(X_{21}, Ve)$ such that $\Phi(bRe) = 0$.*

(3) *There exists $0 \neq \Psi \in \hom_A\left(\overline{X_{21}}, Ve\right)$ where $\overline{X_{21}} = X_{21}/bRe$.*

(4) *There exists $x \in X_{21} - bRe$ and $0 \neq y \in Ve$, such that, with $\overline{x} = x + bRe$, $\overline{x}^0 \subseteq y^0$ in A.*

Proof. Since $b \in B = fRf$, $b = fb = bf$ and $bX_{21} = bRe \subseteq X_{21}$.

(1) $\Leftrightarrow$ (2) Clear by definition.

(2) $\Rightarrow$ (3) Define $\Psi : \overline{X_{21}} \to Ve$ by $\Psi(\overline{x}) = \Phi(x)$. This map is well-defined since $\Phi(bRe) = 0$. Because Φ is nonzero, Ψ is not zero.

(3) $\Rightarrow$ (2) Define $\Phi : X_{21} \to Ve$ by $\Phi(fre) = \Psi\left(\overline{fre}\right)$.

(3) $\Rightarrow$ (4) Let $x \in X_{21} - bRe$ such that $0 \neq \Psi(\overline{x}) = y$. Then it is clear that $\overline{x}^0 \subseteq y^0$.

(4) $\Rightarrow$ (3) Since $\overline{x}^0 \subseteq y^0$, the map $\varphi : \overline{x}A \to yA$ by $\varphi(\overline{x}a) = ya$ is well-defined. Consider the diagram

$$\begin{array}{lcl} \overline{x}A & \hookrightarrow & \overline{X_{21}} \\ \varphi\downarrow & & \\ yA & & \\ \iota\downarrow & & \\ Ve & & \end{array}$$

where $\overline{X_{21}} = X_{21}/bRe$ and ι is the inclusion map. Since Ve is injective, we can extend to $\overline{X_{21}}$ with a map Ψ. Surely Ψ is nonzero.

4.1.1 The Goldie Prime Serial Case

We now consider R, a Goldie prime serial (GPS) ring. All of our previous notation remains in place. In the statements of results we may, at times omit stating the obvious assumptions. In this case A and B are valuations on the same division ring and are left and right uniserial. This implies

$$X_{12}X_{21} \not\subseteq \wp \Leftrightarrow \wp \subset X_{12}X_{21} \text{ and } X_{21}X_{12} \not\subseteq \zeta \Leftrightarrow \zeta \subset X_{21}X_{12}.$$

Recall the Morita context correspondence between A and B. For the Morita context

$$\begin{pmatrix} A & X_{12} \\ X_{21} & B \end{pmatrix}$$

there is a $1-1$ order preserving correspondence between

$$\{\rho \in spec(A) \mid X_{12}X_{21} \not\subseteq \rho\} \text{ and } \{\varsigma \in spec(B) \mid X_{21}X_{12} \not\subseteq \varsigma\}.$$

By our comment above, this simply becomes a $1-1$ order preserving correspondence between

$$\{\rho \in spec(A) \mid \rho \subset X_{12}X_{21}\} \text{ and } \{\varsigma \in spec(B) \mid \varsigma \subset X_{21}X_{12}\}.$$

Recall that $\rho \in spec(A)$ corresponds to $\{b \in B \mid X_{12}bX_{21} \subseteq \rho\}$. We will simply say that $\rho \in spec(A)$ corresponds to $\varsigma \in spec(B)$ under the MCC (Morita context correspondence). We will usually write $\rho \rightleftharpoons \varsigma$ or $\rho \rightleftharpoons \rho'$ to denote this.

Lemma 56 *Let R be a Goldie prime serial ring, V a uniform injective R-module and $e \neq f$ faithful idempotents. Then $\mathfrak{p} = gass(Ve) \subseteq X_{12}X_{21}$ and $\zeta = gass(Vf) \subseteq X_{21}X_{12}$.*

Proof. Suppose that $X_{12}X_{21} \subset \wp$ and choose $a \in \wp - X_{12}X_{21}$. By uniseriality, $X_{12}X_{21} \subset aA \subseteq \wp$. Pick $0 \neq u \in Ve$ such that $ua = 0$. Then

$$uX_{12}X_{21} = u\,(eRfRe) = uRfRe \subseteq u\,(aA) = 0.$$

This contradicts the assumption that e and f are faithful. Obviously, the statement about ζ is proven the same way.

Lemma 57 *Let R be a GPS ring and $e \neq f$ faithful idempotents for a uniform injective module V. If $\wp_o = class\,(Ve)$ exists, then $\wp_o \subset X_{12}X_{21}$.*

Proof. Let $0 \neq Y \subseteq Ve$ be a $\wp_o$-prime submodule. Lemma 56 implies that $\wp_o \subseteq \wp \subseteq X_{12}X_{21}$. If $\wp_o = X_{12}X_{21}$, then, for some $0 \neq u \in Y$,

$$0 = u\wp_o = uX_{12}X_{21} = uRfRe.$$

This contradicts the faithfulness of e and f.

The next result is crucial.

Theorem 58 *Let R be a GPS ring, V a uniform injective R-module, and $e \neq f$ faithful idempotents. Then $\wp \subseteq \{a \in A \mid X_{21}aX_{12} \subseteq \zeta\}$ and $\zeta \subseteq \{b \in B \mid X_{12}bX_{21} \subseteq \wp\}$.*

Proof. By lemma 54, $Vf \cong \hom_A(X_{21}, Ve)$ as B-modules. If $b \in \zeta = gass\,(Vf)$, then there is some $0 \neq \Phi \in \hom_A(X_{21}, Ve)$ such that $\Phi \circ b = 0$. Applying lemma 55, we choose $x \in X_{12} - bRe$ and $0 \neq y \in Ve$, such that $\overline{x}^0 \subseteq y^0$ (where $\overline{x} = x + bRe$ in $X_{12}/_{bRe}$). Since $0 \neq x \in X_{21} \subseteq D$, x^{-1} exists in D. It follows that $\overline{x}^0 = x^{-1}bRe \cap A$. Thus,

$$\overline{x}^0 = x^{-1}bRe \cap A = x^{-1}bX_{21} \cap A \subseteq y^0 \subset A.$$

Because D_A is uniserial, $x^{-1}bX_{21} \subset A$. Thus, $\overline{x}^0 = x^{-1}bRe \subseteq y^0$. Therefore, $x^{-1}bX_{21} \subseteq \wp$ (recall $0 \neq y \in Ve$). Hence, $bX_{21} \subseteq x\wp \subseteq X_{21}\wp$ (recall $x \in X_{21}$). Therefore, $X_{12}bX_{21} \subseteq X_{12}X_{21}\wp \subseteq \wp$. The argument for $\wp$ is symmetrical.

Under the MCC, when $\rho \in spec(A)$ and $\rho \subset X_{12}X_{21}$, we will use ρ' to denote the corresponding ideal in B. Thus $\rho' \subset X_{21}X_{12}$. In this way

$$\underset{\text{in } A}{\rho} \rightleftharpoons \underset{\text{in } B}{\rho'} \rightleftharpoons \underset{\text{in } A}{(\rho')'} = \rho .$$

Definition 59 *By the* **extended Morita context correspondence** (*EMCC*) *we mean the usual Morita context correspondence between A and B together with the additional correspondence* $X_{12}X_{21} \rightleftharpoons X_{21}X_{12}$.

Corrollary 60 *Let R be a GPS ring, V a uniform injective module and* $e \neq f$ *faithful idempotents. Then* $\wp \rightleftharpoons \zeta$ *under the EMCC.*

Proof. By lemma 56, $\wp \subseteq X_{12}X_{21}$.

Case(i): If $\wp \subset X_{12}X_{21}$, then the theorem implies $\zeta \subseteq \wp' \subset X_{21}X_{12}$. Consequently, $\zeta' \subseteq (\wp')' = \wp \subseteq \zeta'$ (again by the theorem). Therefore, $\wp = \zeta'$ and $\zeta = \wp'$. That is, $\wp$ and ζ correspond via the usual MCC.

Case(ii): Suppose that $\wp = X_{12}X_{21}$. If $\zeta \subset X_{21}X_{12}$, then, by the theorem, $\wp \subseteq \zeta' \subset X_{12}X_{21}$. This contradiction implies that $\wp = X_{12}X_{21}$ if and only if $\zeta = X_{21}X_{12}$.

At this point we have essentially answered our questions. We now continue the exploration of the EMCC we have just obtained and see what this means in a Goldie prime serial ring R.

Lemma 61 *Let R be a GPS ring, V a uniform injective module, and e a faithful idempotent. Let $P = class(V)$ and Y be P-prime. Suppose that $P \rightleftharpoons \rho$, where $\rho \in spec(A)$. Then Ye is ρ-prime and $\rho = class(Ve)$.*

Proof. We first show that $\rho = ann_A(Ye)$. Because $\rho = ePe$, we get that $Ye(\rho) = Y(ePe) \subseteq YPe = 0$. Conversely, if $ere \in ann_A(Ye)$, then $Ye(ere) = Y(ere) = 0$. Thus $ere \in P$. This implies that $e(ere)e = ere \in ePe = \rho$. Therefore, $\rho = ann_A(Ye)$.

Let $0 \neq v \in Ye \leq Y$. Then $0 \neq vR \leq Y$ and $ann_R(vR) = P$. Furthermore, $vA = v(eRe) = vRe \leq Ye$. If $a \in ann_A(vA) = ann_A(vRe)$, then $0 = (vRe)a = (vR)a$. This implies that $a \in P$; hence, $a = eae \in ePe = \rho$. Therefore, $ann_A(vA) \subseteq ann_A(Ye) \subseteq ann_A(vA)$ (because $vA \leq Ye$). This proves that Ye is ρ-prime and that $\rho = class(Ve)$.

Lemma 62 *Let R be a GPS ring, V a uniform injective module, and e faithful to V. Let $\wp = gass(Ve)$ and let $\wp \rightleftharpoons P$ via the MCC between A and R. Then:*

(*i*) *$P = class(V)$ if and only if $\wp = class(Ve)$*

Furthermore, in this case

(*ii*) *if $X \leq Ve$ is $\wp$-prime, then XR is P-prime;*

(*iii*) *if $Y \leq V$ is P-prime, then Ye is $\wp$-prime.*

Proof. We first prove (i). ($\Rightarrow$) This is the previous lemma.

($\Leftarrow$) Suppose that $\wp = class(Ve)$ and that $0 \neq X \leq Ve$ is $\wp$-prime. We first show that $ann_R(XR) = P$.

If $r \in P$, then $eRrRe \subseteq \wp$. Thus, $(XRr)Re = X(eRrRe) \subseteq X\wp = 0$. Since e is faithful, $XRr = 0$; whence, $P \subseteq ann_R(XR)$. Conversely, if $r \in ann_R(XR)$, then $X(eRrRe) \subseteq (XRr)Re = 0$. Therefore, $eRrRe \subseteq ann_A(X) = \wp$. We conclude that $P = ann_R(XR)$.

Let $0 \neq u \in XR$ and let $r \in ann_R(uR)$. Since e is faithful, we can choose $s \in R$ such that $use \neq 0$. Then

$$use(eRrRe) \subseteq uR(eRrRe) \subseteq uRrRe = 0.$$

This implies $eRrRe \subseteq gass(Ve) = \wp$. Thus, $r \in P = ann_R(XR)$. It follows that $ann_R(uR) \subseteq ann_R(XR) \subseteq ann_R(uR)$ (because $uR \leq XR$). Therefore, $P = ann_R(uR)$ and XR is P-prime.

The proofs of (ii) and (iii) are contained in the proof (i).

Suppose that e and f are two faithful idempotents for a uniform injective module V. Suppose further that $gass(Ve) = class(Ve)$. An obvious question is whether or not $class(Vf)$ exists. If $class(Vf)$ exists, then is $gass(Vf) = class(Vf)$? The next theorem answers this question.

Theorem 63 *Let R be a GPS ring, V a uniform injective module and e faithful. Suppose that $\wp = gass(Ve) = class(Ve)$. Then, for any other faithful idempotent f, $\zeta = gass(Vf) = class(Vf)$.*

Proof. Let $0 \neq X \leq Ve$ be $\wp$-prime and let $\wp \rightleftharpoons P$ via the MCC between $A = eRe$ and R. Application of lemma 62 implies that $P = class(V)$ and XR is P-prime. Because f faithful, $XRf \neq 0$; thus, $f \notin P$. Consequently, there is a MCC $P \rightleftharpoons \zeta_o$ between R and $B = fRf$. By our lemma 61, $\zeta_o = class(Vf)$ and XRf is ζ_o-prime. Hence, $gass(Vf) \equiv \zeta \supseteq \zeta_o$.

By lemma 57, $\wp = gass(Ve) \subset X_{12}X_{21}$. Thus, $\wp \rightleftharpoons \zeta$ via the MCC between A and B. Hence, $\zeta = \{b \in B \mid X_{12}bX_{21} \subseteq \wp\}$. Let $b \in \zeta$, then

$$(XRfb)Re = (XeRfb)Re = X(eRbRe) = X(X_{12}bX_{21}) \subseteq X\wp = 0.$$

Because e is faithful, $XRfb = 0$. This implies that $\zeta \subseteq ann_B(XRf) = \zeta_o$.

Lemma 64 *Let R be GPS, V a uniform injective module, and e faithful. Let f be an idempotent that is not faithful. Then there exists $0 \neq u \in Ve$ such that $uRf = 0$. Furthermore, if $\wp = gass\,(Ve) \subseteq eRfRe$, then $\wp = class\,(Ve)$.*

Proof. Choose $0 \neq v \in V$ such that $vRf = 0$ and choose $r \in R$ such that $vre \neq 0$. This is possible because e is faithful and f is not faithful. Let $u = vre$. Then $uRf = vreRf \subseteq vRf = 0$.

Assume that $\wp \subseteq eRfRe$. Consider the submodule $uRe = ueRe \leq Ve$, and note that $uRe \neq 0$. Then

$$(uRe)\,\wp \subseteq uRe\,(eRfRe) \subseteq uRfRe = 0.$$

Thus, $\wp \subseteq ann_A\,(uRe) \subseteq gass\,(Ve) = \wp$. That is, $\wp = ann_A\,(uRe)$.

Let $0 \neq use \in uRe$ and $x \in ann_A\,(useRe)$. Then $x \in gass\,(Ve) = \wp$. Therefore,

$$\wp = ann_A\,(uRe) \subseteq ann_A\,(useRe) \subseteq \wp.$$

This proves that $\wp = class\,(Ve)$ and that uRe is $\wp$-prime.

The following result allows us to determine exactly which of the idempotents are faithful provided we know one of the faithful idempotents.

Theorem 65 *Let R be GPS, V a uniform injective module, and let e be a faithful idempotent. Let f be another idempotent.*

(1) *If $\wp = gass\,(Ve) = class\,(Ve)$, then the following are equivalent:*

(*i*) *f is faithful;*

(*ii*) *$\wp \subset eRfRe$;*

(*iii*) *$f \notin P$, where $\wp \rightleftharpoons P$ and $P \in spec(R)$.*

(2) *If $\wp = gass\,(Ve) \neq class\,(Ve)$, then:*

(*iv*) *f is faithful if and only if $\wp \subseteq eRfRe$;*

(*v*) *$\wp = eRfRe$ if and only if f is faithful and $\zeta = gass\,(Vf) = fReRf$.*

Proof. (1) (i)$\Rightarrow$(ii) This is lemma 57.

(ii)$\Rightarrow$(iii) If $f \in P$, then $eRfRe \subseteq \wp \subset eRfRe$.

(iii)$\Rightarrow$(i) With the given assumptions, it follows that $P = class(V)$ (this is lemma 62). Let $Y \leq V$ be a P-prime submodule. If f is not faithful, then there exists $0 \neq v \in V$ such that $vRf = 0$. Then $0 \neq N = vR \cap Y \leq Y$ and $Nf = 0$. Hence, $f \in ann_R(N) = P$. This contradiction completes the proof of (1).

(2) (iv) ($\Rightarrow$) This is lemma 56.

($\Leftarrow$) If f is not faithful, then lemma 64 forces us to conclude that $\wp = class(Ve)$. This contradiction allows to conclude that f is faithful.

(v) This just uses (iv), and really just restates the fact that $\wp$ and ζ correspond via the EMCC.

Theorem 66 *Let R be a GPS ring, V a uniform injective module, and e and f faithful. Suppose that $\wp \rightleftharpoons P$ and $\zeta \rightleftharpoons Q$ where $P, Q \in spec(R)$. If $\wp \subset X_{12}X_{21}$, then $P = Q$.*

Proof. If $\wp \subset X_{12}X_{21}$, then $\zeta \subset X_{21}X_{12}$, and $\wp \rightleftharpoons \zeta$ under the MCC. Recall that $\zeta = \{b \in B \mid X_{12}bX_{21} \subseteq \wp\}$, $P = \{r \in R \mid eRrRe \subseteq \wp\}$, and that $\zeta = fQf \subseteq Q$. This immediately shows that $\zeta \subseteq P$. Let $p \in P$. Then $fpf \in B$ and $eR(fpf)Re \subseteq eRpRe \subseteq \wp$. Therefore, $fPf \subseteq \zeta \subseteq Q$. Because $f \notin Q$ and Q is prime, it follows that $P \subseteq Q$. By symmetry, $Q \subseteq P$.

Recall again our notation:

Notation. R is a Goldie prime serial ring and V is a uniform injective R-module. Idempotents, e and f are faithful to V, $A = eRe$, and $B = fRf$. We use $\wp = gass(Ve)$, $\zeta = gass(Vf)$, $X_{12} = eRf$, and $X_{21} = fRe$. Finally, we let $\wp \rightleftharpoons P$ and $\zeta \rightleftharpoons Q$, where $P, Q \in spec(R)$. We will keep this fixed in the next several results.

Lemma 67 *Let R, V, e, f, $\wp$, ζ, P, and Q be as described above.*

(*i*) *P and Q contain all nonfaithful idempotents.*

(*ii*) *If $\wp = X_{12}X_{21}$, then $f \in P$ and $e \in Q$; hence, P and Q are not comparable.*

Proof. (i) Recall that $P = \{r \in R \mid eRrRe \subseteq \wp\}$. Let g be any nonfaithful idempotent. Choose $0 \neq v \in V$ such that $vRg = 0$ and pick $r \in R$ such that $vre \neq 0$. This is possible since e is faithful and g is not faithful. Thus, $vre\,(eRgRe) \subseteq vRgRe = 0$. This implies that $eRgRe \subseteq \wp$. Therefore, $g \in P$. The proof for Q is the same.

(ii) Since $\wp = eRfRe$, we see that $f \in P$. Similarly, $e \in Q$.

Lemma 68 *Let R, V, e, f, $\wp$, ζ, P, and Q be as described above. Then*

$$P = Q \; if \text{ and only } if \; \wp \subset X_{12}X_{21} \;\; (if \text{ and only } if \; \zeta \subset X_{21}X_{12}).$$

Proof. ($\Leftarrow$) This is theorem 66.

($\Rightarrow$) If $P = Q$, then $f \notin P$. Thus, $X_{12}X_{21} = eRfRe \not\subseteq \wp$. Uniseriality implies that $\wp \subset X_{12}X_{21}$.

Consider the set of faithful idempotents for a given uniform injective. For each faithful idempotent, e, we get a corresponding ideal $gass(Ve) = \wp \in spec(eRe)$. Using the MCC we get $P \in spec(R)$, where $\wp \rightleftharpoons P$. Using this setup, we make the following definition.

Definition 69 *For faithful idempotents, define the relation $e \sim f$ if and only if $P = Q$ (if and only if $\wp \subset X_{12}X_{21}$)*

It is clear that $\sim$ is an equivalence relation on the set of faithful idempotents. We shall denote the equivalence class of e by $[e]$.

Lemma 70 *Let R, V, e, $\wp$, and P be as we have described. Then $P \cap [e] = \emptyset$ and P contains all other idempotents.*

Proof. That P contains the nonfaithful idempotents is lemma 67. Let f be any faithful idempotent, $\zeta = gass(Vf)$, and $\zeta \rightleftharpoons Q$, where $Q \in spec(R)$.

If $e \sim f$, then $f \notin Q = P$. This shows that $P \cap [e] = \emptyset$.

If $f \notin [e]$, then $P \neq Q$. Therefore, $\wp = eRfRe$. By definition of the MCC, $f \in P$.

If there is only one class of faithful idempotents, then there is a unique prime ideal P associated with V. We now assume that there are at least two classes. If e and f are faithful and $f \notin [e]$, then $\wp = eRfRe = X_{12}X_{21} = gass(Ve)$ and $\zeta = fReRf = X_{21}X_{12} = gass(Vf)$. We also have the following: If $P \rightleftharpoons \wp$ and $Q \rightleftharpoons \zeta$, where $P, Q \in spec(R)$, then

$$[f] \subseteq P \quad \text{and} \quad [e] \subseteq Q.$$

Lemma 71 *Let R be a GPS ring, V a uniform injective module, and e a faithful idempotent. Let $\wp_o \in spec(A)$ with $\wp_o \subset \wp$. Let $\wp_o \rightleftharpoons S$ under the MCC between A and R. Then, for any other faithful idempotent f,*

(*i*) $S \cap [f] = \emptyset$;

(*ii*) *if $gass(Vf) = \zeta$ and $\zeta \rightleftharpoons Q$, then $S \subset Q$.*

Proof. Since $\wp_o \subset \wp$, we get $S \subset P$.

If $f \sim$ e, then $P = Q$. Hence (ii) is clear. Statement (i) is true by lemma 70.

Suppose that f is faithful and $f \notin [e]$. By definition, $\wp = eRfRe$. If $f \in S$, then $eRfRe \subseteq \wp_o \subset \wp = eRfRe$. Therefore, (i) obtains.

The above lemma asserts that Q contains all idempotents except those in $[f]$. Therefore, every idempotent in S is in Q. This implies that S and Q are comparable [MS1]. Because $e \in Q - S$, we are left to conclude that $S \subset Q$.

We now reach the main result of this section. We have named this the "Fork theorem". The expression the "P_i sit on top of a fork" means that in the spectrum of R, any prime ideal which is properly contained in one P_j must be contained in all of the P_i.

Theorem 72 [Fork Theorem] *Let R be a Goldie prime serial ring and let V be a uniform injective R-module such there exist $k \geq 2$ classes of faithful idempotents. Let $e_1, \cdots, e_k$ be faithful idempotents such that each comes from a distinct class. For each $i = 1, \ldots, k$ let $\wp_i = gass(Ve_i)$ and $P_i \rightleftharpoons \wp_i$ where $P_i \in spec(R)$. Then*

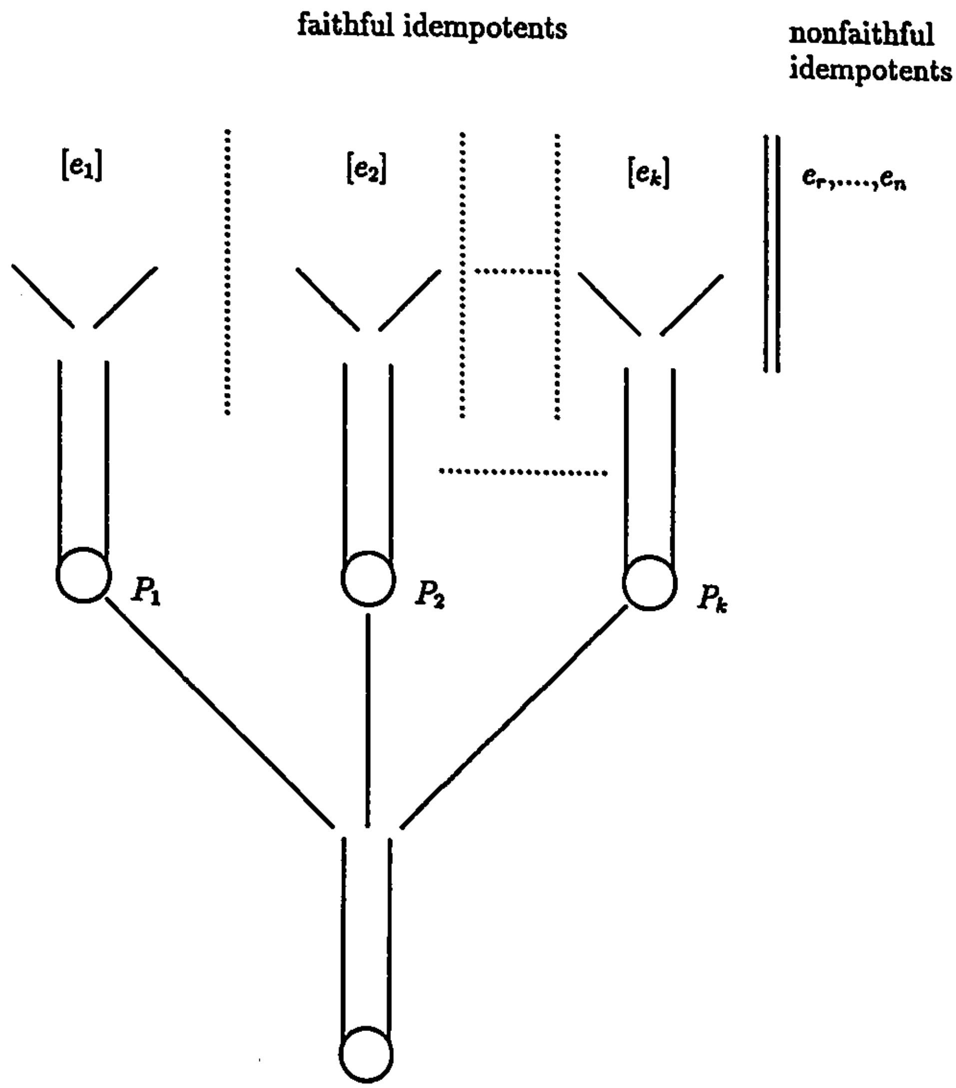
(*i*) *each P_i is minimal in its tower;*

(*ii*) *the P_i "sit on top of a fork" in spec(R).*

Proof. (i) If P_1 is not minimal in its tower, then we can choose $S \subset P_1$ in this tower. Since S is in the same tower, S contains the same idempotents as P_1. In particular, all the other faithful idempotents are in S. Let $S \rightleftharpoons \wp_o$, where $\wp_o \in spec(e_1Re_1)$. Then $\wp_o \subset \wp_1$. By lemma 71, $S \subset P_i$ for each $i = 1, 2, \ldots, k$. This contradicts the fact that $P_i \cap [e_i] = \emptyset$. Therefore, each P_i is minimal.

(ii) The argument in (i) shows that if $S \in spec(R)$ and $S \subset P_1$, then $S \subset P_i$ for each $i = 1, 2, \ldots, k$. This is precisely what (ii) asserts.

The following is an illustration of the Fork Theorem.



4.2 The Structure Of Goldie Prime Ideals

Let R be a serial ring and fix a decomposition $e_1+e_2+\cdots+e_n = 1$ into indecomposable orthogonal idempotents. Let P be a prime ideal in R and let $E(P) = \{e_i \mid e_i \notin P\}$. Suppose that $\{e_1, e_2, \ldots, e_r\} = E(P)$ and $e_{r+1}, e_{r+2}, \ldots, e_n \in P$. For each $k = 1, 2, \ldots, r$ let $A_k = e_k A e_k$. Then, for each $k = 1, 2, \ldots r$, there exists $\wp_k \in spec(A_k)$ such that $P \rightleftharpoons \wp_k$ under the MCC. Because R is serial, each of the A_k are uniserial rings [cf. M2].

Remark. The idempotents in P should not be confused with the faithful idempotents of the previous sections, nor should the $\wp_k$ be confused with the generalized associated primes that we studied earlier.

In many of the following arguments it will only be necessary to consider two idempotents in $E(P)$ at a time. To simplify notation, we again use e and f to denote these idempotents. We use $X = eRf$, $Y = fRe$, $A = eRe$, $B = fRf$, to denote these rings and bimodules. Finally, we will use the following notation to denote the MCC $P \rightleftharpoons \wp \in spec(A)$, and $P \rightleftharpoons \zeta \in spec(B)$. When we need to consider the more general case, we will use X_{ij} to denote the bimodule $e_i R e_j$ (for any $1 \leq i, j \leq n$) and $P \rightleftharpoons \wp_k \in spec(A_k)$ when such a correspondence exists.

Proposition 73 *Let R be a serial ring, P a prime ideal in R, and let $e, f \in E(P)$. Then $\wp \rightleftharpoons \zeta$ via the MCC.*

Proof. Because $f \notin P$, and $P \rightleftharpoons \wp$, we conclude that $eRfRe \not\subseteq \wp$. The uniseriality of A implies that $\wp \subset XY$. The same is obviously true of ζ.

Under the MCC between A and B, $\wp \rightleftharpoons \wp' = \{b \in B \mid eRbRe \subseteq \wp\}$. Note that $\zeta = fPf \subseteq P = \{r \in R \mid eRrRe \subseteq \wp\}$ (because $P \rightleftharpoons \zeta$). Thus, $\zeta \subseteq \wp'$. By symmetry $\wp \subseteq \zeta'$. Therefore, $\zeta \subseteq \wp' \subseteq (\zeta')' = \zeta$.

Lemma 74 *Let R be a serial ring, P a prime ideal in R and let $E(P) = \{e_1, e_2, \ldots, e_r\}$. Then $P = (P_{ij})$ where, for each $1 \leq i, j \leq n$, $P_{ij} = \{x \in X_{ij} \mid X_{ki} x X_{jk} \subseteq \wp_k \text{ for all } k = 1, 2, \ldots, r\}$.*
Furthermore,

(1) *if $i > r$, then $P_{ij} = X_{ij}$ for all j;*
(2) *if $j > r$, then $P_{ij} = X_{ij}$ for all i;*
(3) *if $i \leq r$, then $P_{ii} = \wp_i$;*
(4) *if $i, j \leq r$, then $\wp_i X_{ij}$, $X_{ij} \wp_j \subseteq P_{ij}$.*

Proof. Let $(x_{st}) \in P$ and let i,j be fixed but otherwise arbitrary. We will show that $x_{ij} \in P_{ij}$. Now, $e_i (x_{st}) e_j = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & x_{ij} & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \equiv \widehat{x}_{ij} \in P$ and $e_i \widehat{x}_{ij} e_j = \widehat{x}_{ij}$. For all $k = 1, 2, \ldots, r$, $X_{ki} x_{ij} X_{jk} = e_k R e_i \widehat{x}_{ij} e_j R e_k = e_k R \widehat{x}_{ij} R e_k \subseteq \wp_k$. This shows that $P \subseteq (P_{ij})$.

Conversely, let $X = (x_{ij}) \in (P_{ij})$. Then $e_1 R X R e_1 = \begin{pmatrix} C_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$, where

$$C_{11} = \left(\sum_{i=1}^{n} X_{1i} x_{i1}\right) X_{11} + \left(\sum_{i=1}^{n} X_{1i} x_{i2}\right) X_{21} + \cdots + \left(\sum_{i=1}^{n} X_{1i} x_{in}\right) X_{n1}.$$

By the definition of each P_{ij}, we get that $C_{11} \subseteq \wp_1$. This shows that $X \in P$.

(1) If $i > r$, then $e_i \in P$. Thus, for all j, $e_i R e_j = X_{ij} \subseteq P = (P_{ij})$. Thus, $X_{ij} \subseteq P_{ij} \subseteq X_{ij}$.

(2) This is similar to part (1).

(3) If $i \leq r$, then $\wp_i = e_i P e_i = e_i (P_{ij}) e_i = P_{ii}$.

(4) Let $i, j \leq r$ be given and let $1 \leq k \leq r$ be arbitrary. By our proposition 73, $\wp_i \rightleftharpoons \wp_k$ under the MCC between A_i and A_j. Hence, $X_{ki}(\wp_i X_{ij}) X_{jk} \subseteq X_{ki}\wp_i (X_{ij}X_{jk}) \subseteq X_{ki}\wp_i X_{ik} \subseteq \wp_k$. The first part of this lemma implies that $\wp_i X_{ij} \subseteq P_{ij}$. The other part is the same.

To illustrate the lemma, we write P in the following way

$$P = \begin{pmatrix} \wp_1 & & P_{ij} & \vdots & \\ & \ddots & & \vdots & X_{ij} \\ P_{ij} & & \wp_r & \vdots & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ & X_{ij} & & \vdots & X_{ij} \end{pmatrix}.$$

Suppose P is a Goldie prime ideal and $E(P) = \{e_1, e_2, \ldots, e_r\}$. The Morita context is $\begin{pmatrix} R & Re_k \\ e_k R & A_k \end{pmatrix}$ and $P \rightleftharpoons \wp_k \in spec(A_k)$ for each $1 \leq k \leq r$. Known results, [McR, section 3.6.7], imply that $\wp_k$ is Goldie prime for each k. Because A_k is uniserial, each $\wp_k$ is a completely prime ideal in A_k.

Lemma 75 *Let A be a uniserial ring and $\wp$ a completely prime ideal of A. Then $a\wp = \wp a = \wp$ for all $a \in A - \wp$.*

Proof. Let $a \in A - \wp$ and $x \in \wp$. By uniseriality, $x = as$ or $a = xs$ for some $s \in A$. The latter is not possible because $a \notin \wp$. We conclude that $x = as \in \wp$. It follows that $s \in \wp$ ($\wp$ is completely prime and $a \notin \wp$). Therefore, $x \in a\wp$.

Theorem 76 *Let R be a Goldie prime serial ring, $P = (P_{ij})$ a Goldie prime ideal of R, and let $E(P) = \{e_1, e_2, \ldots, e_r\}$. Then $P_{ij} = \wp_i X_{ij} = X_{ij}\wp_j$ for all $i, j \leq r$.*

Proof. Fix $i, j \leq r$. Write $e = e_i$ and $f = e_j$ and use the notation we set up earlier. Our lemma 74 implies that $\wp X \subseteq P_{ij}$. Thus, it remains to show that $P_{ij} \subseteq \wp X$. We first show that $\wp X = X\zeta$.

Because $\wp$ is completely prime and $\wp \subset XY$, the above lemma implies that $XY\wp = \wp$. Since $\wp \rightleftharpoons \zeta$, we know that $Y\wp X \subseteq \zeta$. Therefore, $XY\wp X = \wp X \subseteq X\zeta$. By symmetry, $\wp X = X\zeta$.

Recall that $P_{ij} = \{x \in X_{ij} = X \mid X_{ki}xX_{jk} \subseteq \wp_k \ \mathit{for\ all}\ k = 1, 2, \ldots, r\}$. In particular, for $k = i$ $(\leq r)$ we get that $P_{ij} = \{x \in X \mid xY \subseteq \wp\}$. Let $x \in P_{ij}$ and choose $b \in YX - \zeta$. Then $\zeta b = \zeta$ and $xYX = (xY)X \subseteq \wp X = X\zeta = X\zeta b$. In particular, $xb = zb$ for some $z \in X\zeta$. Since R is Goldie prime serial, x, z, and b are in a division ring. Therefore, $x = z \in X\zeta = \wp X$.

Again we illustrate the result: if P is a Goldie prime ideal in a Goldie prime serial ring and $E(P) = \{e_1, e_2, \ldots, e_r\}$, then

$$P = \left(\begin{array}{cccccc}
\wp_1 & & & \wp_i X_{ij} & \vdots & \\
& \wp_2 & & & \vdots & X_{ij} \\
& & \ddots & & \vdots & \\
X_{ij}\wp_j & & & \wp_r & \vdots & \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
& X_{ij} & & & \vdots & X_{ij}
\end{array}\right).$$

In general, if $E(P) = \{e_r, \ldots e_s\}$, then

$$P = \left(\begin{array}{ccccccc}
X_{ij} & \vdots & & X_{ij} & & \vdots & X_{ij} \\
\cdots & \vdots & \cdots & \cdots & \cdots & \vdots & \cdots \\
& \vdots & \wp_r & & \wp_i X_{ij} & \vdots & \\
X_{ij} & \vdots & & \ddots & & \vdots & X_{ij} \\
& \vdots & X_{ij}\wp_j & & \wp_s & \vdots & \\
\cdots & \vdots & \cdots & \cdots & \cdots & \vdots & \cdots \\
X_{ij} & \vdots & & X_{ij} & & \vdots & X_{ij}
\end{array}\right).$$

4.2.1 Other Ideals Containing Idempotents

We now examine arbitrary ideals containing idempotents. Let R be a Goldie prime serial ring and, as usual, fix $e_1+e_2+\cdots+e_n=1$. Recall that $R=(X_{ij})$ where $X_{ij}\cong e_iRe_j$. Let $\varepsilon=e_1+e_2+\cdots+e_m$ for some $1\leq m<n$. Then $R\varepsilon R=(C_{ij})$ where

$$C_{ij}=\sum_{k=1}^{m}X_{ik}X_{kj}\quad \textit{for all } 1\leq i,j\leq n. \tag{2}$$

The following facts are easy to prove.

Proposition 77 *With the notation above*

(1) *if* $i\leq m$, *then* $C_{ii}=A_i$ *where* $A_i=X_{ii}$;

(2) *if* $i>m$, *then* C_{ii} *is a completely prime ideal of* A_i;

(3) *if* $i\leq m$, *then* $C_{ij}=X_{ij}$ *for all* j;

(4) *if* $j\leq m$, *then* $C_{ij}=X_{ij}$ *for all* i.

Proof. (1) If $i\leq m$, then the index $k=i$ appears in the right hand side of equation(2). Thus, $A_i\subseteq C_{ii}\subseteq A_i$.

(2) Let $i>m$. Then, for each $k\leq m$, $X_{ik}X_{ki}$ is a completely prime ideal of A_i [M1]. By uniseriality, there exists $k(i)\leq m$ such that $X_{ik}X_{ki}\subseteq X_{ik(i)}X_{k(i)i}$ for all $k\leq m$. Using equation (2), $C_{ii}=\sum_{k=1}^{m}X_{ik}X_{kj}=X_{ik(i)}X_{k(i)i}$. As we noted, $X_{ik(i)}X_{k(i)i}$ is completely prime.

(3) If $i\leq m$, then the index $k=i$ occurs in the right hand side of equation (2). Hence, $X_{ii}X_{ij}=A_iX_{ij}=X_{ij}\subseteq C_{ij}\subseteq X_{ij}$.

(4) Same as (3).

Notation. In part (2) of the proposition, we will use $\mathfrak{p}_i$ to denote the completely prime ideal $C_{ii} = X_{ik(i)}X_{k(i)i}$. If $i > m$, then $C_{ii} = \sum_{k=1}^{m} X_{ik}X_{ki} = \mathfrak{p}_i$. Therefore, a partial description of $R\varepsilon R$ is

$$R\varepsilon R = \begin{pmatrix} X_{ij} & \vdots & & X_{ij} & \\ \cdots & \vdots & \cdots & \cdots & \cdots \\ & \vdots & \mathfrak{p}_{m+1} & & C_{ij} \\ X_{ij} & \vdots & & \ddots & \\ & \vdots & C_{ij} & & \mathfrak{p}_n \end{pmatrix}$$

Lemma 78 *If $i, j > m$, then $\mathfrak{p}_i X_{ij},\ X_{ij}\mathfrak{p}_j \subseteq C_{ij}$.*

Proof. Fix $i,j > m$. Using equation (2),

$$C_{ij} = \sum_{k=1}^{m} X_{ik}X_{kj} \supseteq \sum_{k=1}^{m} X_{ik}(X_{ki}X_{ij}) = \sum_{k=1}^{m} (X_{ik}X_{ki})\,X_{ij} = \left(\sum_{k=1}^{m} X_{ik}X_{ki}\right) X_{ij} = \mathfrak{p}_i X_{ij}.$$

Similarly for the other containment.

4.2.2 Ideals Arising From Nonfaithful Idempotents

Let us consider a Goldie prime serial ring R and a uniform injective R-module V. Then some of the idempotents are faithful. Suppose that $e_1, \ldots, e_{m-1}$ are not faithful and that $e_m, \ldots, e_n$ are faithful. If $\varepsilon = e_1 + \cdots + e_{m-1}$, then the results of the previous section (section 4.2.1) imply that

$$R\varepsilon R = \begin{pmatrix} X_{ij} & \vdots & & X_{ij} & \\ \cdots & \vdots & \cdots & \cdots & \cdots \\ & \vdots & \wp_m & & C_{ij} \\ X_{ij} & \vdots & & \ddots & \\ & \vdots & C_{ij} & & \wp_n \end{pmatrix}.$$

Remark. The $\wp_i$ that appear here need not be the $gass(Ve_i)$ that we examined in the previous chapter. Hopefully, the notation does not cause too much confusion. Throughout the remainder of this section, we will assume that the idempotents $e_1, \ldots e_{m-1}$ are not faithful and that $e_m, \ldots, e_n$ are faithful.

Lemma 79 *For all $i, j \geq m$, $\wp_i \subset X_{ij}X_{ji}$.*

Proof. Fix $i, j \geq m$. There exists $k(i) \leq m-1$ such that $\wp_i = X_{ik(i)}X_{k(i)i}$. Since $k(i) \leq m-1$, the idempotent $e_{k(i)}$ is not faithful. Denote $e_{k(i)}$ by f and choose $0 \neq v \in V$ such that $vRf = 0$. Because e_i is faithful, there is some $r \in R$ such that $vre_i \neq 0$. Thus, $vre_i \neq 0$ while $vrX_{ik(i)} \subseteq vRf = 0$.

If $X_{ij}X_{ji} \subseteq \wp_i$, then

$$vre_iRe_jRe_i = vrX_{ij}X_{ji} \subseteq vr\wp_i = vrX_{ik(i)}X_{k(i)i} = 0.$$

But this is a contradiction since $vre_i \neq 0$ and e_i and e_j are both faithful. We are left to conclude that $\wp_i \subset X_{ij}X_{ji}$.

Lemma 80 *For each $i, j \geq m$, there is a MCC $\wp_i \rightleftharpoons \wp_j$.*

Proof. Let $i, j \geq m$ be fixed but arbitrary. Recall that $\wp_i = X_{ik(i)}X_{k(i)i}$ and $\wp_j = X_{jk(j)}X_{k(j)j}$ for some $k(j)$, $k(i) \leq m-1$ and that $X_{ik(i)}X_{k(i)i} \supseteq X_{it}X_{ti}$ for all $t \leq m-1$. Using the above lemma, there is a prime ideal $\wp_i' \in spec(A_j)$, such that $\wp_i \rightleftharpoons \wp_i'$. Furthermore, $\wp_i' = \{a \in A_j \mid X_{ij}aX_{ji} \subseteq \wp_i\}$. Hence,

$$X_{ij}\wp_j X_{ji} = X_{ij}\left(X_{jk(j)}X_{k(j)j}\right)X_{ji} \subseteq X_{ik(j)}X_{k(j)i} \subseteq X_{ik(i)}X_{k(i)i} \subseteq \wp_i.$$

Therefore, $\wp_j \subseteq \wp_i'$. Symmetry implies that $\wp_i \subseteq \wp_j'$. Consequently, $\wp_j \subseteq \wp_i' \subseteq \left(\wp_j'\right)' = \wp_j$.

Proposition 81 *Let R be a Goldie prime serial ring and V a uniform injective R-module. Suppose that $e_1, \ldots, e_{m-1}$ are not faithful and that $e_m, \ldots, e_n$ are faithful to V. Then, with the notation from above:*

(1) $\wp_i X_{ij} = X_{ij}\wp_j$ *for all $i, j \geq m$.*

(2) $\wp_i X_{ij}X_{ji} = \wp_i$ *for all $i, j \geq m$.*

(3) $X_{ik}X_{ki} \subseteq \wp_i$ *for all $1 \leq k \leq m-1$ and $i \geq m$.*

(4) $X_{ik}X_{kj} \subseteq \wp_i X_{ij}$ *for all $1 \leq k \leq m-1$ and $i, j \geq m$.*

Proof. Parts (1) and (2) were (essentially) proven earlier in section 4.2 which dealt with Goldie prime ideals (see proof of theorem 76). Part (3) is simply the definition of $\wp_i$.

To prove (4) suppose that $\wp_i X_{ij} \subset X_{ik}X_{kj}$ for some $1 \leq k \leq m-1$ and $i, j \geq m$. Choose $b \in X_{ik}X_{kj} - \wp_i X_{ij}$ and choose $a_j \in X_{ji}X_{ij} - \wp_j$. If $bX_{ji} \subseteq \wp_i$, then

$$bX_{ji}X_{ij} \subseteq \wp_i X_{ij} = X_{ij}\wp_j = X_{ij}\wp_j a_j.$$

Thus, $ba_j = za_j$ for some $z \in X_{ji}\wp_j$. This implies that $b = z \in X_{ji}\wp_j = \wp_i X_{ij}$. This contradiction implies that $\wp_i \subset bX_{ji}$. Therefore,

$$\wp_i X_{ij}X_{ji} \subseteq \wp_i \subset bX_{ji} \subseteq X_{ik}X_{kj}X_{ji} \subseteq X_{ik}X_{ki}$$

where $1 \leq k \leq m-1$. This is contrary to definition of $\wp_i$. Therefore, $X_{ik}X_{kj} \subseteq \wp_i X_{ij}$ whenever $1 \leq k \leq m-1$ and $i, j \geq m$.

Theorem 82 *Let R be a Goldie prime serial ring and V a uniform injective R-module such that $e_1, \ldots, e_{m-1}$ are not faithful and $e_m, \ldots, e_n$ are faithful to V. Let $\varepsilon = e_1 + \cdots + e_{m-1}$. Then*

$$R\varepsilon R = \begin{pmatrix} X_{ij} & \vdots & & X_{ij} & \\ \cdots & \vdots & \cdots & \cdots & \cdots \\ & \vdots & \wp_m & & \wp_i X_{ij} \\ X_{ij} & \vdots & & \ddots & \\ & \vdots & X_{ij}\wp_j & & \wp_n \end{pmatrix}.$$

Proof. Use $P = (P_{ij})$ to denote the right-hand side of the above. We will show that P is an ideal of R. To show this we need only to show that $(RP)_{ij} \subseteq \wp_i X_{ij}$ and $(PR)_{ij} \subseteq \wp_i X_{ij}$ for all $i, j \geq m$. We will show the first inclusion and note that the other is proven in the same way. For $i, j \geq m$

$$(RP)_{ij} = \sum_{k=1}^{n} X_{ik}P_{kj} = \sum_{k=1}^{m-1} X_{ik}P_{kj} + \sum_{k=m}^{n} X_{ik}P_{kj} = \sum_{k=1}^{m-1} X_{ik}X_{kj} + \sum_{k=m}^{n} X_{ik}\wp_k X_{kj}.$$

Application of proposition 81 now proves that P is an ideal of R.

By lemma 78, $P \subseteq R\varepsilon R$. Since P is an ideal and $\varepsilon \in P$, we also get that $R\varepsilon R \subseteq P$.

Proposition 83 *Let R be a Goldie prime serial ring and e, f two idempotents in R (taken from a fixed decomposition of the identity). Let $A = eRe$, $B = fRf$, and let $\wp \in spec(A)$, $\zeta \in spec(B)$ such that $\wp \rightleftharpoons \zeta$ via the MCC. Let $\wp \rightleftharpoons P$ and $\zeta \rightleftharpoons Q$ where $P, Q \in spec(R)$. Then $P = Q$.*

Proof. Because $\wp \rightleftharpoons \zeta$, we get that $\zeta \subset fReRf$.
Thus,$e \notin Q = \{r \in R \mid fRrRf \subseteq \zeta\}$. Similarly, $f \notin P$. We now have that

$fR(ePe)Rf = fR\wp Rf \subseteq \zeta$ (recall $ePe = \wp$). This implies that $ePe \subseteq Q$. Since Q is prime and $e \notin Q$, we conclude that $P \subseteq Q$. By symmetry $Q \subseteq P$.

Theorem 84 *Let R be a Goldie prime serial ring and V a uniform injective R-module such that $e_1, \ldots, e_{m-1}$ are not faithful and $e_m, \ldots, e_n$ are faithful to V. Let $\varepsilon = e_1 + \cdots + e_{m-1}$. Then $R\varepsilon R$ is a Goldie prime ideal.*

Proof. Let $P = R\varepsilon R$. By theorem 82,

$$P = R\varepsilon R = \begin{pmatrix} X_{ij} & \vdots & & X_{ij} & \\ \cdots & \vdots & \cdots & \cdots & \cdots \\ & \vdots & \wp_m & & \wp_i X_{ij} \\ X_{ij} & \vdots & & \ddots & \\ & \vdots & X_{ij}\wp_j & & \wp_n \end{pmatrix}.$$

For each $i, j \geq m$, the ideal $\wp_i$ is a Goldie prime ideal in A_i. Let $P_i \in spec(R)$ such that $\wp_i \rightleftharpoons P_i$. By previous results (iemma 80), for each $i, j \geq m$, there is also a correspondence $\wp_i \rightleftharpoons \wp_j$. Proposition 83 implies $P_i = P_j$ for all i and j. Let this ideal be denoted by Q. It follows from the results in [McR, 3.6] that Q is a Goldie prime ideal.

Now $e_m P e_m = \wp_m = e_m Q e_m \subseteq Q$. Because Q is prime and $e_m \notin Q$, we get that $P \subseteq Q$. Consequently, $\{e_1, e_2, \ldots, e_{m-1}\} \subseteq P \subseteq Q$. It follows that $E(Q) = \{e_m, \ldots, e_n\}$. Our previous results on Goldie prime ideals (theorem 76) can now be invoked to give that $P = Q$.

4.3 Ore Sets In Serial Rings

We begin with a general discussion about Ore sets.

Definition 85 *Let R be a ring. A subset, $\Sigma \subseteq R$, is said to be* **multiplicatively closed** *if $a,b \in \Sigma$ implies that $ab \in \Sigma$. A multiplicatively closed set, Σ, is said to be a* **right Ore set** *if $0 \notin \Sigma$ and if for any $r \in R$ and $\sigma \in \Sigma$, there exists $r' \in R$ and $\sigma' \in \Sigma$ such that $r\sigma' = \sigma r'$. A* **left Ore set** *is defined analogously. A set is said to be an* **Ore set** *if it is both right and left Ore.*

When we speak of Ore sets we will always assume that $1 \in \Sigma$. For an Ore set Σ, set $K(\Sigma) = \{r \in R \mid \textit{there exists } \sigma_1, \sigma_2 \in \Sigma, \textit{ such that } \sigma_1 r \sigma_2 = 0\}$. In the next proposition we list some elementary, but important, facts. For completeness, we include (some of) the proof.

Proposition 86 *Let R be a serial ring and let Σ be an Ore set. Then:*

(1) *$K(\Sigma)$ is a two sided ideal of R.*

(2) *If $P \trianglelefteq R$ and $P \cap \Sigma = \emptyset$, then $(K(\Sigma) + P) \cap \Sigma = \emptyset$.*

(3) $K(\Sigma) \cap \Sigma = \emptyset$.

(4) *There exist ideals $P_i \trianglelefteq R$ $(i \in \Lambda)$ such that*

(*a*) *each P_i is maximal with respect to having empty intersection with Σ;*

(*b*) *each P_i is prime;*

(*c*) *if $i \neq j$, then P_i and P_j are incomparable (hence, comaximal);*

(*d*) $|\Lambda| < \infty$;

(*e*) *for all i, $K(\Sigma) \subseteq P_i$.*

Proof. (1) This is known to be the case for any ring.

(2) Denote $K(\Sigma)$ by K. Suppose there is $p \in P$ and some $k \in K$ such that $p + k \in (P + K) \cap \Sigma$. Pick $\sigma_1, \sigma_2 \in \Sigma$, such that $\sigma_1 k \sigma_2 = 0$. Then $\sigma_1 (p + k) \sigma_2 = \sigma_1 p \sigma_2 \in P \cap \Sigma = \emptyset$. Therefore, no such elements exist.

(3) Let $P = \{0\}$ in part (2).

(4) (a) An easy Zorn's lemma argument proves that such ideals exist.

(b) Let P be one of these ideals and suppose that P is not prime. Then we can find ideals A and B of R such that $AB \subseteq P \subset A, B$. By the maximality of P, there exists $a \in A \cap \Sigma$ and $b \in B \cap \Sigma$. Then $ab \in AB \subseteq P \cap \Sigma$. This contradiction implies that P is a prime ideal.

(c) If two (distinct) P_i were comparable, then, by maximality, they would be equal. Hence, the (distinct) P_i are pairwise incomparable. In serial rings any two prime ideals are either comparable or comaximal [M1].

(d) In any serial ring, any collection of incomparable prime ideals must be finite [cf. M1]. We will use $P_1, P_2, \ldots, P_k$ to denote these ideals.

(e) By part (2), $(P_i + K(\Sigma)) \cap \Sigma = \emptyset$ for each $i = 1, 2, \ldots, k$. Maximality of P_i implies that $P_i = P_i + K(\Sigma)$.

For an Ore set $\Sigma \subseteq R$, we consider the ring R_Σ (the localization of R at Σ). Recall the construction of R_Σ. First we construct the ring $\overline{R} = R/K$, where $K = K(\Sigma)$. Then $\overline{\Sigma} = \{\sigma + K \mid \sigma \in K\}$ is Ore and consists of regular elements in $\overline{R}$. Hence, we get the quotient ring $\overline{R}_{\overline{\Sigma}}$. In this way $\overline{R} \subseteq \overline{R}_{\overline{\Sigma}} = R_\Sigma$ and elements of Σ become invertible in R_Σ (that is, $\overline{\sigma} \in \overline{\Sigma}$ is invertible in $\overline{R}_{\overline{\Sigma}}$). Since $\overline{\Sigma}$ is two sided Ore, $\overline{R}_{\overline{\Sigma}} \cong {}_{\overline{\Sigma}}\overline{R} = \{\overline{r}\,\overline{\sigma}^{-1} \mid r \in R,\ \sigma \in \Sigma\} = \{\overline{\sigma}^{-1}\overline{r} \mid r \in R,\ \sigma \in \Sigma\}$. For a discussion on Ore localizations see [GW, chapter 9].

For any ring R, we will use $J(R)$ to denote the **Jacobson radical** of R.

Proposition 87 *Let R be a serial ring and Σ an Ore set. Then*

(1) *R_Σ is a serial ring and $J(R_\Sigma) = \bigcap$ maximal ideals of R_Σ;*

(2) *the maximal ideals of R_Σ are in 1-1 correspondence with the ideals of R which are maximal with respect to having empty intersection with Σ.*

Proof. (1) Let $Q = R_\Sigma = \{\overline{r}\,\overline{\sigma}^{-1} \mid r \in R,\ \sigma \in \Sigma\}$ and note that $\overline{R} = R/K$, where $K = K(\Sigma)$, is serial. Thus, $\overline{R} = \oplus_{i=1}^k \varepsilon_i \overline{R}$ where $\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_k = \overline{1}$ is a decomposition into orthogonal indecomposable idempotents. The right ideals, $\varepsilon_i \overline{R}$, are uniserial. We claim that $Q = \oplus_{i=1}^k \varepsilon_i Q$ and that each $\varepsilon_i Q$ is uniserial as a right Q-module.

Surely $Q = \sum_{i=1}^k \varepsilon_i Q$. If $q_i \in \varepsilon_i Q \bigcap \sum_{j \neq i}^k \varepsilon_j Q$, then $q_i = \varepsilon_i q_i = 0$. Therefore, $Q = \oplus_{i=1}^k \varepsilon_i Q$.

Let i be given and denote ε_i by ε. Let $\alpha, \beta \in \varepsilon Q$ and write $\alpha = \varepsilon \overline{r}\,\overline{c}^{-1}$ and $\beta = \varepsilon \overline{s}\,\overline{\sigma}^{-1}$. This implies that $\alpha \overline{c} = \varepsilon \overline{r}$ and $\beta \overline{\sigma} = \varepsilon \overline{s}$. Using the uniseriality of $\varepsilon \overline{R}$, $\varepsilon \overline{r x} = \varepsilon \overline{s}$ or $\varepsilon \overline{r} = \varepsilon \overline{s x}$ for some $\overline{x} \in \overline{R}$. If we assume the latter, then

$$\alpha = \varepsilon \overline{r}\,\overline{c}^{-1} = \varepsilon \overline{s x c}^{-1} = \beta \overline{\sigma x c}^{-1}.$$

This shows that εQ_Q is uniserial.

The second part of part (1) is true of any serial ring.

(2) This is a known result and the proof will not be presented. We recall the correspondence. If P is maximal with respect to having empty intersection with Σ, then P corresponds to the ideal $\overline{P} R_\Sigma = R_\Sigma \overline{P} = \{\overline{\sigma}^{-1} \overline{r} \mid r \in P,\ \sigma \in \Sigma\}$. Moreover, $\overline{P} = \overline{P} R_\Sigma \bigcap \overline{R}$.

Let $P_1, P_2, \ldots, P_k$ denote the ideals which are maximal with respect to having empty intersection with Σ. Then

$$J(R_\Sigma) = \bigcap_{i=1}^k \overline{P_i} R_\Sigma = \left(\bigcap_{i=1}^k \overline{P_i} \right) R_\Sigma = \overline{S} R_\Sigma$$

where $S = \bigcap_{i=1}^{k} P_i$. Here the bar refers to modulo $K = K(\Sigma)$.

Lemma 88 *Let R be a serial ring, $\Sigma \subseteq R$ an Ore set, and let $P_1, P_2, \ldots, P_k$ denote the ideals which are maximal with respect to having empty intersection with Σ. Then P_i is Goldie for each i.*

Proof. Since R_Σ is serial, it is semiperfect. Thus,

$$^{R_\Sigma}/_{J(R_\Sigma)} = {}^{R_\Sigma}/_{\overline{S}R_\Sigma} \cong \left({}^{R}/_{S}\right)_{\overline{\Sigma}}$$

is semisimple Artinian. Goldie's theorem implies that S is Goldie semiprime. Because $S = \bigcap_{i=1}^{k} P_i$ and the P_i are comaximal, we get that each P_i is Goldie prime.

We really should mention something about the isomorphism which occurs in the above proof. On the left hand side $\overline{S}$ means modulo $K = K(\Sigma)$ while on the right hand side $\overline{\Sigma}$ means modulo S. We will state a result which gives us the isomorphism above.

Notation. Let R be a ring and S an ideal of R. We will use $C(S)$ to denote the set of elements which are regular modulo S. That is,

$$C(S) = \{r \in R \mid rs \in S \text{ implies } s \in S \text{ and } sr \in S \text{ implies } s \in S\}.$$

Lemma 89 *Let R be a serial ring, $\Sigma \subseteq R$ an Ore set, and $K = K(\Sigma)$. Let $J(R_\Sigma) = \overline{S}R_\Sigma$. Then $\Sigma \subseteq C(S)$ and*

$$R_\Sigma / J(R_\Sigma) \cong (R/S)_{\overline{\Sigma}} \quad \text{where } \overline{\Sigma} = \{\sigma + S \mid \sigma \in \Sigma\}.$$

Proof. Note first that $S \supseteq K$ (because $S = \bigcap_{i=1}^{k} P_i$ and $P_i \supseteq K$ for each i). Let $a \in \Sigma$ and suppose that $as \in S$ for some $s \in R$. We will prove that $s \in S$.

We know that $\overline{S}R_\Sigma = R_\Sigma\overline{S} = \{\overline{\sigma}^{-1}\overline{x} \mid \sigma \in \Sigma,\ x \in S\}$ and $\overline{S} = \overline{S}R_\Sigma \cap \overline{R}$ (here we mean modulo K). Consequently, $\overline{s} = \overline{a}^{-1}(\overline{as}) \in \overline{S}R_\Sigma \cap \overline{R} = \overline{S}$. This implies that $s = s_1 + k$ for some $s_1 \in S$ and $k \in K$. Since $K \subseteq S$, we get that $s \in S$. Similarly, $sa \in S$ implies that $s \in S$. Therefore, $\Sigma \subseteq \mathcal{C}(S)$.

The above allows us to conclude that $\overline{\Sigma} = \{\sigma + S \mid \sigma \in \Sigma\}$ is Ore and consists of regular elements in $\overline{R} = R/S$. Localization gives

$$\overline{R} \hookrightarrow \overline{R}_{\overline{\Sigma}} = \left\{\overline{r}\,\overline{c}^{-1} \mid r \in R,\ c \in \Sigma\right\} \equiv Q_1.$$

On the other hand, the ring $R_\Sigma = \{\overline{r}\,\overline{\sigma}^{-1} \mid r \in R,\ \sigma \in \Sigma\}$ (here we mean modulo K). Thus,

$$R_\Sigma / J(R_\Sigma) = \left\{\overline{\overline{r}\,\overline{c}^{-1}} \mid r \in R,\ \sigma \in \Sigma\right\} \equiv Q$$

where the lower bar is modulo K and the upper bar is modulo $J(R_\Sigma)$.

Define

$$\alpha : Q_1 \to Q \quad by \quad \alpha\left(\overline{r}\,\overline{c}^{-1}\right) = \overline{\overline{r}\,\overline{c}^{-1}}. \tag{3}$$

It is a straightforward, but rather tedious, task to show that α is an isomorphism. It really only uses the Ore conditions. The rest of the proof is omitted.

4.3.1 Ore Sets In Which K Does Not Contain Any Idempotents

Again we consider a serial ring R with a fixed decomposition of the identity, $1 = e_1 + e_2 + \cdots + e_n$ into indecomposable orthogonal idempotents. Let Σ be an Ore set such that $K(\Sigma)$ does not contain any idempotents. Throughout this section we will keep this notation.

Definition 90 *Let R be a serial ring and $P_1, P_2, \ldots, P_k$ be a collection of incomparable prime ideals of R. We say that $P_1, P_2, \ldots, P_k$ form a* **complete cross-section** *of spec(R) if $\{e_1, e_2, \ldots, e_n\} = \bigcup_{i=1}^{k} E(P_i)$.*

Since the ideals are incomparable, the union in the above must be disjoint.

Proposition 91 *Let R and Σ be as described above and let $P_1, P_2, \ldots, P_k$ be all the ideals which are maximal with respect to having empty intersection with Σ. Then $P_1, P_2, \ldots, P_k$ form a complete cross-section of spec(R).*

Proof. If $P_1, P_2, \ldots, P_k$ don't form a complete cross-section of *spec*(R), then $S = \bigcap_{i=1}^{k} P_i$ contains an idempotent, e say. Since $e \notin K = K(\Sigma)$, we conclude that $0 \neq \overline{e} \in \overline{S} = S/K \subseteq R/K$. Thus, $0 \neq \overline{e} \in \overline{S}R_\Sigma = J(R_\Sigma)$. This simply cannot occur.

Remark. At this point we note that $C(S)$, the set of elements regular modulo S, is Ore and $Sc = cS = S$ for all $c \in C(S)$. This is simply a result of Chatters[C].

For any Ore set Σ, the set $\{r \in R \mid \overline{r} \in R/K$ is invertible in $R_\Sigma\}$ is called the saturation of Σ. This set will be denoted by $sat(\Sigma)$. It is always the case that $\Sigma \subseteq sat(\Sigma)$ and that $sat(\Sigma)$ is also an Ore set. It follows that $R_\Sigma \cong R_{sat(\Sigma)}$.

In the setting above, we start with an Ore set Σ such that $K(\Sigma)$ doesn't contain any idempotents. We then get Ore sets, $\mathcal{C}(S)$, via the Chatters result, and $sat(\Sigma)$. Our aim is to prove that $sat(\Sigma)= \mathcal{C}(S)$. This then shows that for this type of Ore set Σ, the localized ring R_Σ can be thought of as having come from a "Chatters type" localization. That is $R_\Sigma \cong R_{sat(\Sigma)} = R_{\mathcal{C}(S)}$.

Lemma 92 *Let R and Σ be as described above and let $P_1, P_2, \ldots, P_k$ be all the ideals which are maximal with respect to having empty intersection with Σ. Let $S = \bigcap_{i=1}^k P_i$ and let $\overline{\Sigma} = \{\sigma + S \mid \sigma \in \Sigma\}$. Then the set $U_1 = \{\overline{r}\,\overline{\sigma}^{-1} \mid r \in \mathcal{C}(S),\ \sigma \in \Sigma\}$ is the set of units in $(R/S)_{\overline{\Sigma}}$.*

Proof. Let $\overline{R} = R/S$ and let $Q_1 = \left(\overline{R}\right)_{\overline{\Sigma}}$. Since $Q_1 \cong R_\Sigma/J(R_\Sigma)$ is semisimple Artinian, the set of units is precisely the set of regular elements.

If $q = \overline{r}\,\overline{\sigma}^{-1}$ is regular in Q_1, then $\overline{r} = q\overline{\sigma}$ is regular in R/S. Thus, $r \in \mathcal{C}(S)$.

Conversely, if $r \in \mathcal{C}(S)$ and $\sigma \in \Sigma$, then $\overline{r}$ and $\overline{\sigma}^{-1}$ are both regular in Q_1. This implies that $\overline{r\sigma^{-1}}$ is also regular.

Lemma 93 *Let R and Σ be as described above and let $P_1, P_2, \ldots, P_k$ be all the ideals which are maximal with respect to having empty intersection with Σ. Let $S = \bigcap_{i=1}^k P_i$. The set $U = \left\{\overline{\overline{r}\,\overline{\sigma}^{-1}} \mid r \in \mathcal{C}(S),\ \sigma \in \Sigma\right\}$, where the lower bars are modulo K and the upper bar is modulo $J(R_\Sigma)$, is the set of units in $R_\Sigma/J(R_\Sigma)$.*

Proof. Let $Q = R_\Sigma / J(R_\Sigma)$. Since $Q \cong Q_1$, (Q_1 from above) the set of units in Q is the set $U = \alpha(U_1)$, where α is the isomorphism in (3). The definition of α gives the result.

Lemma 94 *An element $q \in R_\Sigma$ is invertible if and only if $\overline{q} \in R_\Sigma / J(R_\Sigma)$ is invertible.*

Proof. It is well known that invertibility in a ring is equivalent to invertibility modulo the Jacobson radical.

We will make use of the following fact: If $a \in C(S)$ and $s \in S$, then $(a+s) \in C(S)$.

Lemma 95 *The set of units in R_Σ is $U(R_\Sigma) = \{\overline{r}\,\overline{\sigma}^{-1} \mid r \in C(S),\ \sigma \in \Sigma\}$ where the bar is mod K.*

Proof. Let $r \in C(S)$, $\sigma \in \Sigma$ and let $q = \overline{r}\,\overline{\sigma}^{-1} \in R_\Sigma$. By lemma 93, $\overline{q}$ is invertible in $R_\Sigma / J(R_\Sigma)$. Lemma 94 implies that q is unit in R_Σ.

Conversely, let $q \in R_\Sigma$ be a unit. Then, modulo $J = J(R_\Sigma)$, $\overline{q}$ is a unit. By lemma 93, $\overline{q} = \overline{\overline{r}\,\overline{\sigma}^{-1}}$ for some $r \in C(S)$, $\sigma \in \Sigma$. This means $q = \overline{r}\,\overline{\sigma}^{-1} + k$ where $k \in J = \overline{S}R_\Sigma = \{\overline{s}\,\overline{c}^{-1} \mid s \in S,\ c \in \Sigma\}$. Consequently,

$$q = \overline{r}\,\overline{\sigma}^{-1} + \overline{s}\,\overline{\sigma_1}^{-1} \quad \text{where } r \in C(S),\ s \in S, \text{ and } \sigma, \sigma_1 \in \Sigma.$$

Using the Ore condition, we can write $\overline{r}\,\overline{\sigma}^{-1} = \overline{r_1}\,\overline{\sigma_2}^{-1}$ and $\overline{s}\,\overline{\sigma_1}^{-1} = \overline{s_1}\,\overline{\sigma_2}^{-1}$ where $\sigma_2 \in \Sigma$. Then $r_1 \in C(S)$ and $s_1 \in S$. It follows that $(r_1 + s_1) \in C(S)$. Therefore,

$$q = \overline{r}\,\overline{\sigma}^{-1} + \overline{s}\,\overline{\sigma_1}^{-1} = \overline{r_1}\,\overline{\sigma_2}^{-1} + \overline{s_1}\,\overline{\sigma_2}^{-1} = (\overline{r_1 + s_1})\,\overline{\sigma_2}^{-1} \in U(R_\Sigma).$$

Lemma 96 *Let R, Σ, S, and K be as above. Let $a \in C(S)$, $\sigma \in \Sigma$. If $\overline{a}\,\overline{\sigma}^{-1} \in R/K$, then $\overline{a}\,\overline{\sigma}^{-1} \in \overline{C(S)}$ where the bar is modulo K.*

Proof. Let $\overline{R} = R/K$ and suppose that $\overline{r} = \overline{a}\,\overline{\sigma}^{-1} \in \overline{R}$. We will show that $r \in \mathcal{C}(S)$.

If $zr \in S$, then $\overline{z}\,\overline{r} = \overline{z}\,\overline{a}\,\overline{\sigma}^{-1} \in \overline{S}$. Rewriting, this becomes $\overline{z}\,\overline{a} = \overline{s}\,\overline{\sigma}$ for some $s \in S$. This implies that $za = s\sigma + k$ for some $k \in K \subseteq S$. Thus, $za \in S$. Because $a \in \mathcal{C}(S)$, we get that $z \in S$.

If $rz \in S$, then $\overline{r}\,\overline{z} = \overline{a}\,\overline{\sigma}^{-1}\overline{z} \in S$. Write $\overline{\sigma}^{-1}\overline{z} = \overline{z_1}\,\overline{\sigma_1}^{-1}$. Then $\overline{a}\,\overline{z_1}\,\overline{\sigma_1}^{-1} \in \overline{S}$. As above, we get that $z_1 \in S$. Using the above equality, $\overline{z\sigma_1} = \overline{\sigma z_1} \in \overline{S}$. This implies $z\sigma_1 = s + k$ for some $s \in S$ and $k \in K$. Therefore, $z \in S$ (since $K \subseteq S$ and $\Sigma \subseteq \mathcal{C}(S)$).

Lemma 97 *Let R, Σ, S, and K be as above and let $r \in R$. Then $\overline{r}$ is invertible in R_Σ if and only if $r \in \mathcal{C}(S)$. That is, $sat(\Sigma) = \mathcal{C}(S)$.*

Proof. Let $r \in \mathcal{C}(S)$ and $\sigma \in \Sigma$. Then $r\sigma \in \mathcal{C}(S)$ and $\overline{r} = (\overline{r\sigma})\overline{\sigma}^{-1} \in U(R_\Sigma)$.

Conversely, suppose that $r \in R$ and $\overline{r}$ is a unit. Then $\overline{r} = \overline{a}\,\overline{\sigma}^{-1} \in R/K$ for some $a \in \mathcal{C}(S)$, $\sigma \in \Sigma$. Lemma 96 implies $\overline{r} \in \overline{\mathcal{C}(S)}$. Therefore, $r = x + k$ where $x \in \mathcal{C}(S)$ and $k \in K \subseteq S$. We conclude that $r \in \mathcal{C}(S)$.

We collect all of the above information in the following theorem.

Theorem 98 *Let R be a serial ring and $\Sigma \subseteq R$ an Ore set such that $K(\Sigma)$ does not contain any nonzero idempotents. Then there exist ideals $P_1, P_2, \ldots P_k$ of R, for some $1 \leq k \leq n$, such that:*

(1) *P_i is Goldie prime for each i.*

(2) *Each P_i is maximal with respect to having empty intersection with Σ.*

(3) *The P_i are pairwise incomparable.*

(4) *$P_1, P_2, \ldots, P_k$ form a complete cross-section of spec(R).*

(5) *With $S = \bigcap_{i=1}^{k} P_i$, $\mathcal{C}(S)$ is Ore and $cS = Sc = S$ for all $c \in \mathcal{C}(S)$.*

(6) $\mathcal{C}(S) = sat(\Sigma)$.

4.3.2 The Facchini And Puninski Ore Set

We now consider a particular type of Ore set. The following is based on results of Facchini and Puninski [FP]. They have considered localizable systems which is a more general case than what we require. The theorem and proof that we give is based on their work.

Let R be a serial ring and let $1 = e_1 + e_2 + \cdots + e_n$ be a decomposition of the identity into indecomposable orthogonal idempotents. Then $R \cong (X_{ij})$ where $X_{ij} \cong e_i R e_j$ for all $1 \leq i, j \leq n$. In each of the (uniserial) subrings, $A_i = e_i R e_i$, let P_i be a completely prime ideal. Suppose further that this collection of ideals has the following property:

$$\textit{if } x \in X_{ij} \textit{ and } y \in X_{ji}, \textit{ and } xy \notin P_i, \textit{ then } yx \notin P_j.$$

This property may seem somewhat artificial, but, in the next section when we return to faithful idempotents and consider the EMCC which exists, this property will be satisfied. We will refer to this property as the **FPMC** (Facchini Puninski Morita Context) property.

Consider the set

$$\Sigma = \begin{pmatrix} A_1 - P_1 & & & 0 \\ & A_2 - P_2 & & \\ & & \ddots & \\ 0 & & & A_n - P_n \end{pmatrix}.$$

Since each P_i is a completely prime ideal, Σ is multiplicatively closed. It is also clear that $0 \notin \Sigma$ and $1 \in \Sigma$.

We will show that Σ is actually an Ore set. Only the right Ore condition is discussed. The proof of the left Ore condition is the same. We need to show the following:

for all $\sigma \in \Sigma$ and $r \in R$, there exist $\sigma' \in \Sigma$ and $r' \in R$ such that $\sigma r' = r\sigma'$.

We call an element of the form $\begin{pmatrix} 1 & & & & 0 \\ & \ddots & & & \\ & & \sigma_i & & \\ & & & \ddots & \\ 0 & & & & 1 \end{pmatrix} \in \Sigma$ where $\sigma_i \in A_i - P_i$ a basic element of Σ. Given such a basic element, we will use $\widehat{\sigma}_i$ to denote it.

We will prove that Σ is Ore in two steps. The next result will show that we need only verify the right Ore condition for the basic elements. Then we will show that the basic elements do satisfy the right Ore condition.

Proposition 99 *Let Σ be as shown above. If the basic elements of Σ satisfy the right Ore condition, then Σ is right Ore.*

Proof. Suppose that all basic elements do satisfy the right Ore condition. Let $\sigma \in \Sigma$ and $r \in R$ be arbitrary. Let

$$\sigma = \begin{pmatrix} \sigma_1 & & & 0 \\ & \sigma_2 & & \\ & & \ddots & \\ 0 & & & \sigma_n \end{pmatrix} \text{ where each } \sigma_i \in A_i - P_i.$$

Then $\sigma = \prod_{i=1}^n [(e_i\sigma e_i) + (1 - e_i)] = \prod_{i=1}^n \widehat{\sigma}_i$. By assumption, there exist elements $c_1 \in \Sigma$ and $r_1 \in R$ such that $\widehat{\sigma}_1 r_1 = rc_1$. If c_{i-1} and r_{i-1} have been chosen in this way, then choose $c_i \in \Sigma$ and $r_i \in R$ such that $\widehat{\sigma}_i r_i = r_{i-1}c_i$ (this can be done by the hypothesis). Let $\sigma' = c_1c_2\cdots c_n$ and $r' = r_n$. Then

$$r\sigma' = (r_1c_1)\, c_2\cdots c_n = \widehat{\sigma}_1(r_1c_2)c_3\cdots c_n = \cdots = \widehat{\sigma}_1\widehat{\sigma}_2\cdots\widehat{\sigma}_n r_n = \sigma r'.$$

This shows that Σ is right Ore.

Theorem 100 *The basic elements of Σ satisfy the right Ore condition. Hence, Σ is right Ore.*

Proof Let i be arbitrary but fixed and, in keeping with our notation let $\widehat{\sigma}_i$ be a basic element of Σ. Let $r \in R$ be arbitrary and let $1 \leq j \leq n$. Note that $e_i\widehat{\sigma}_i e_i = \sigma_i$ and $e_i r e_j \equiv r_{ij}$ are in the uniserial R-module $e_i R$. Hence, either

$$r_{ij} \in \sigma_i R \ \ or \ \ \sigma_i \in r_{ij} R.$$

(1) Suppose that $r_{ij} \in \sigma_i R$. Then $r_{ij} = \sigma_i t_j = \sigma_i (e_i t_j e_j)$ for some $t_j \in R$. For such a j, let $y_j = e_j$. Then

$$r_{ij} = r_{ij} y_j = \sigma_i t_j$$

(2) Suppose that $\sigma_i \in r_{ij} R$. Then $\sigma_i = r_{ij} h_j = (e_i r e_j)(e_j h_j e_i)$ for some $h_j \in R$. Since $\sigma_i \notin P_i$, it follows that $y_j = (e_j h_j e_i) r_{ij} \notin P_j$ (this is the FPMC property). In this case set $t_j = r_{ij}$.

Using the two lines above, we get that, for all j, there exist elements $t_j \in e_i R e_j \subseteq R$ and $y_j \in A_j - P_j$ such that

$$r_{ij} y_j = \sigma_i t_j.$$

Set $y = \sum_{j=1}^n y_j$ and $t = \left(\sum_{j=1}^n t_j\right) + (1 - e_i) r y$. Then $y \in \Sigma$ and

$$\widehat{\sigma}_i t = (\sigma_i + (1 - e_i))\, t = \sum_{j=1}^n \sigma_i t_j + (1 - e_i) r y = \sum_{j=1}^n r_{ij} y_j + (1 - e_i) r y.$$

But $\sum_{j=1}^n r_{ij} y_j = \sum_{j=1}^n e_i r e_j y_j = e_i r \left(\sum_{j=1}^n e_j y_j e_j\right) = e_i r y$. Therefore,

$$\widehat{\sigma}_i t = \sum_{j=1}^n r_{ij} y_j + (1 - e_i) r y = e_i r y + r y - e_i r y = r y.$$

4.4 Ore Sets Which Operate Regularly

Let R be a ring and V an R-module. An Ore set, Σ, is said to **operate regularly** on V if $v \in V$, $\sigma \in \Sigma$ and $v\sigma = 0$ implies that $v = 0$.

Lemma 101 *Let R be a ring and V an R-module. There exists an Ore set which operates regularly on V and contains every other Ore set which operates regularly on V.*

Proof. Since $\{1\}$ is an Ore set which operates regularly on V, such sets do exist. Surely the multiplicative closure of two such Ore sets is again an Ore set which operates regularly on V. The product of all such Ore sets (defined to be the collection of all finite products) is then the unique maximal one.

Let us return again to the case of a Goldie prime serial ring, R, and a uniform injective R-module, V. Fix a decomposition of the identity $1 = e_1 + e_2 + \cdots + e_n$ into indecomposable orthogonal idempotents. Then there are some faithful idempotents among these, and possibly some which are not faithful. Let ε be the sum of all the nonfaithful idempotents. In a previous section we showed that $R\varepsilon R = P$ is a Goldie prime ideal. Therefore, either all idempotents are faithful and $R\varepsilon R = 0$, or there is at least one nonfaithful idempotent and $R\varepsilon R = P \neq 0$. In the latter case, we consider $W = ann_V(P)$. Then W is an (R/P)-module and R/P is a Goldie prime serial ring.

Lemma 102 *With the notation above, W is a nonzero uniform injective (R/P)-module and all idempotents of R/P are faithful to W.*

Proof. Suppose $\{e_1, e_2, \ldots, e_k\}$ is the set of nonfaithful idempotents and let $\varepsilon = e_1 + e_2 + \cdots + e_k$. For each i, choose $0 \neq v_i \in V$ such that $v_i R e_i = 0$. By the uniformity of V, there is some $0 \neq v \in \bigcap_{i=1}^{k} v_i R$. It is easy to show that $v(R\varepsilon R) = vP = 0$. Thus $W \neq 0$.

Let $R/P = \overline{R}$ and recall that W is an $\overline{R}$-module via $w\overline{r} = wr$ for all $r \in R$ and $w \in W$. An easy argument shows that the set of $\overline{R}$-submodules (of W) is the same as the set of R-submodules of W. Therefore, W is uniform as an $\overline{R}$-module.

To show that W is injective, consider the diagram

$$\begin{array}{ccc} \overline{I} & \hookrightarrow & \overline{R} \\ \varphi \downarrow & & \\ W & & \end{array}$$

where φ is an $\overline{R}$-module homomorphism and I is a right ideal of R containing P. Consider the diagram

$$\begin{array}{ccc} I & \hookrightarrow & R \\ f \downarrow & & \\ W & & \\ \iota \downarrow & & \\ V & & \end{array}$$

where $f : I \to W$ by $f(x) = \varphi(\overline{x})$ and ι is the inclusion. We can easily show that f is an R-module homomorphism. Since V is injective, we can extend f to R by F (actually we extend $\iota \circ f$). Letting $F(1) = v$ implies that $F(r) = vr$ for all $r \in R$. Thus,

$$\varphi(\overline{x}) = f(x) = F(x) = vx \text{ for all } x \in I.$$

We claim that $v \in W$. Let $p \in P \subseteq I$. Then $vp = f(p) = \varphi(\overline{p}) = 0$. This shows that $vP = 0$; therefore, $v \in W$. Define $\Phi : \overline{R} \to W$ by $\Phi(\overline{r}) = v\overline{r} = vr$.

Then Φ is an $\overline{R}$-module homomorphism. Furthermore, for all $x \in I$, $\Phi(\overline{x}) = vx = f(x) = \varphi(\overline{x})$. This shows that W is injective.

In $\overline{R}$, we have that $\overline{1} = \overline{e_1 + e_2 + \cdots + e_n} = \overline{e_{k+1}} + \cdots + \overline{e_n}$. If $\overline{e_i}$, for some $k+1 \le i \le n$, is not faithful, then $w\overline{Re_i} = wRe_i = 0$ for some $0 \neq w \in W$. This cannot occur since e_i is faithful.

Because of the above result, when we consider a uniform injective module, V, over a Goldie prime serial ring, R, we may assume (by passing to W if necessary) that all idempotents are faithful.

Lemma 103 *Let R be a Goldie prime serial ring and V a uniform injective module such that all idempotents are faithful. For each i, let $\wp_i = gass(Ve_i)$. Then*

$$\Sigma = \begin{pmatrix} A_1 - \wp_1 & & & 0 \\ & A_2 - \wp_2 & & \\ & & \ddots & \\ 0 & & & A_n - \wp_n \end{pmatrix}$$

is an Ore set which operates regularly on V.

Proof. Write $R = (X_{ij})$. Recall that there is an EMCC between the $\wp_i = gass(Ve_i)$. We will show that $\wp_1, \wp_2, \ldots, \wp_n$ satisfy the FPMC property we described in the Facchini and Puninski Ore set section. This then shows that Σ is Ore.

If $\wp_i = X_{ij}X_{ji}$, then $\wp_j = X_{ji}X_{ij}$. Hence, that $x \in X_{ij}$, $y \in X_{ji}$ and $xy \notin \wp_i$ implies that $yx \notin \wp_j$ is true vacuously.

If $\wp_i \subset X_{ij}X_{ji}$, then $\wp_j \subset X_{ji}X_{ij}$ and $\wp_j \rightleftharpoons \wp_j$ under the usual MCC. Let $x \in X_{ij}$ and $y \in X_{ji}$. If $yx \in \wp_j = \{a \in A_j \mid X_{ij}aX_{ji} \subseteq \wp_i\}$, then $(xy)^2 \in X_{ij}yxX_{ji} \subseteq \wp_i$. Since $\wp_i$ is completely prime and $xy \in A_i$, we conclude that $xy \in \wp_i$. This allows us to conclude that $xy \notin \wp_i$ implies $yx \notin \wp_j$. Therefore, Σ is an Ore set.

We have observed in section 2.2 that

$$V \cong \hom_{A_1}(A_1, Ve_1) \oplus \hom_{A_1}(X_{21}, Ve_1) \oplus \cdots \oplus \hom_{A_1}(X_{n1}, Ve_1).$$

Since all idempotents are faithful, $\hom_{A_1}(X_{j1}, Ve_1) \cong Ve_j$ as A_j-modules for each j (see lemma 54). Hence, $V \cong Ve_1 \oplus Ve_2 \oplus \cdots \oplus Ve_n$ and the action of R is given by the formal matrix multiplication. Because $\wp_i = gass(Ve_i)$, it is clear that $A_i - \wp_i$ acts regularly on Ve_i. It is now obvious that Σ acts regularly on V.

Since all the idempotents are faithful, we get a partition of the idempotents. This comes from the equivalence relation we defined in definition 69. Let $[e_1] = \{e_1, e_2, \ldots, e_r\}$, then there exists $P \in spec(R)$ such that $\wp_i \rightleftharpoons P$ for all $i = 1, 2, \ldots, r$. Moreover, P contains all other idempotents. That is, $E(P) = [e_1] = \{e_1, e_2, \ldots, e_r\}$. Because each $\wp_i$ is Goldie, so is P (this follows from results in [McR, 3.6]. Our results on Goldie prime ideals imply that

$$P = \begin{pmatrix} \wp_1 & & \wp_i X_{ij} & \vdots & \\ & \ddots & & \vdots & X_{ij} \\ X_{ij}\wp_j & & \wp_r & \vdots & \\ \cdots & \cdots & \cdots & \vdots & \cdots \\ & X_{ij} & & \vdots & X_{ij} \end{pmatrix}$$

Clearly, $P \cap \Sigma = \emptyset$.

Lemma 104 *Let R be a Goldie prime serial ring and V a uniform injective R-module such that all idempotents are faithful. Let P be one of the ideals which is obtained as described above. Then P is maximal with respect to having empty intersection with Σ.*

Proof. Consider the P above as a typical case. Suppose that $P \subseteq Q$ and $Q \cap \Sigma = \emptyset$. Let $q = (q_{ij}) \in Q$. Suppose that for some $i \leq r$, $q_{ii} \in A_i - \wp_i$.

Then

$$e_i q e_i = \begin{pmatrix} 0 & & & & 0 \\ & \ddots & & & \\ & & q_{ii} & & \\ & & & \ddots & \\ 0 & & & & 0 \end{pmatrix} \in Q \quad \textit{where } q_{ii} \in A_i - \wp_i.$$

Let $1 \leq j \leq r$ be arbitrary. Then, $X_{ji} q_{ii} X_{ij} \not\subseteq \wp_j$ ($\wp_i \rightleftharpoons \wp_j$). Choose $x_j \in X_{ji}$ and $y_j \in X_{ij}$ such that $b_j = x_j q_{ii} y_j \notin \wp_j$. As a typical case consider $j = 1$. Let

$$\widehat{x}_1 = \begin{pmatrix} 0 & \cdots & x_1 & \cdots & 0 \\ \vdots & & 0 & & \vdots \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix} \text{ (column } (i)\text{)} \quad \text{and} \quad \widehat{y}_1 = \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ \vdots & & & \vdots \\ y_1 & 0 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix} \cdots (i).$$

Hence,

$$\widehat{x}_1 e_i q e_i \widehat{y}_1 = \begin{pmatrix} b_1 & & & 0 \\ & 0 & & \\ & & \ddots & \\ 0 & & & 0 \end{pmatrix} \in Q.$$

A similar statement is true for all $j \leq r$.

Because $P \subseteq Q$, $e_k \in Q$ for all $k > r$. Therefore,

$$\begin{pmatrix} b_1 & & & & & 0 \\ & \ddots & & & & \\ & & b_r & & & \\ & & & 1 & & \\ & & & & \ddots & \\ 0 & & & & & 1 \end{pmatrix} \in Q \cap \Sigma.$$

This contradiction shows that $q_{ii} \in \wp_i$ for all $i \leq r$.

Now let $i \neq j$ and $1 \leq i, j \leq r$. Recall that $\wp_i X_{ij} = \{x \in X_{ij} \mid xX_{ji} \subseteq \wp_i\}$. If $q_{ij} \notin \wp_i X_{ij}$, then we can choose $y \in X_{ji}$ such that $b_i = q_{ij}y \notin \wp_i$. Define $\widehat{y} = (y_{st}) \in R$ where

$$y_{st} = \begin{cases} y & s = j,\ t = i \\ 0 & otherwise \end{cases}.$$

Then,

$$(e_i q e_j)\,\widehat{y} = \begin{pmatrix} 0 & & & 0 \\ & \ddots & & \\ & & b_i & \\ & & & \ddots \\ 0 & & & 0 \end{pmatrix} \in Q \quad where\ b_i = q_{ij}y \notin \wp_i.$$

By the first part of the proof, this is not possible. Therefore, $q_{ij} \in \wp_i X_{ij}$ for all $1 \leq i, j \leq r$ and $i \neq j$. This proves that $Q \subseteq P$. Therefore P is maximal with respect to having empty intersection with Σ.

Lemma 105 *Let R be a Goldie prime serial ring, V a uniform injective R-module such that all idempotents are faithful and that there are k classes of idempotents. Let $P_1, P_2, \ldots, P_k$ be the Goldie prime ideals in spec(R) which correspond to the generalized associated primes via the MCC. Then $P_1, P_2, \ldots, P_k$ is a complete list of the ideals which are maximal with respect to having empty intersection with Σ.*

Proof. By the fork theorem we get that each P_i is minimal in its tower and the P_i sit on top of a fork. Since all idempotents are faithful, $P_1, P_2, \ldots, P_k$ form a complete cross-section of $spec(R)$. Thus, any prime ideal of R must be comparable to some P_i.

Let Q be an ideal which is maximal with respect to having empty intersection with Σ. Then Q is prime, and so would have to be comparable to some P_i. Since both are maximal, equality holds.

Lemma 106 *Let R be a GPS ring and $X \subseteq R$ any Ore set. Then $K(X) = 0$. In particular, $K(X)$ does not contain any idempotents.*

Proof. Let $K' = \{r \in R \mid \text{there exists } \sigma \in X \text{ such that } \sigma r = 0\}$. Because X is Ore, K' is an ideal in R. Since R is a Goldie prime ring, if $K' \neq 0$, then it must contain a regular element. This contradicts the definition of K'. Thus, $K' = 0$. Similarly, ${}'K = \{r \in R \mid \text{there exists } \sigma \in X \text{ such that } r\sigma = 0\} = \{0\}$. From these two facts it follows easily that $K(X) = 0$.

Definition 107 *A multiplicatively closed set X is said to be* **right reversible** *if $r \in R$ and $\sigma \in X$, such that $\sigma r = 0$, then there is $\sigma' \in X$ such that $r\sigma' = 0$.* **Left reversible** *is defined similarly. A right (left) Ore set which is also right (left) reversible is said to be a* **right (left) denominator set.** *A set which is both a left and right denominator set is simply called a* **denominator set.**

Using the above lemma, our results on Ore sets, and a little more work we get the following result.

Theorem 108 *Let R be a Goldie prime serial ring, V a uniform injective R-module such that all idempotents are faithful and suppose that there are k classes of idempotents. Let $P_1, P_2, \ldots, P_k$ be the Goldie prime ideals in spec(R) which correspond under the MCC with $\mathfrak{p}_i = gass(Ve_i)$ for each i. Let*

$$\Sigma = \begin{pmatrix} A_1 - \mathfrak{p}_1 & & & 0 \\ & A_2 - \mathfrak{p}_2 & & \\ & & \ddots & \\ 0 & & & A_n - \mathfrak{p}_n \end{pmatrix}.$$

Then:

(1) Σ *is a denominator set.*

(2) *With* $S = \bigcap_{i=1}^{k} P_i$, $C(S) = sat(\Sigma)$.

(3) $C(S)$ *operates regularly on* V.

Proof. In the lemma above we actually showed that, in a Goldie prime serial ring, Ore sets consist of regular elements. Thus, (1) is clear.

Since $K(\Sigma) = 0$, our results about Ore sets for which $K(\Sigma)$ doesn't contain idempotents apply (see section 4.3.1). Using the fact that $P_1, P_2, \ldots, P_k$ are all the ideals which are maximal with respect to having empty intersection with Σ, we apply theorem 98 to get (2).

To prove (3) we first note that $R \hookrightarrow R_\Sigma$ (since $K(\Sigma) = 0$) and so elements of Σ are invertible in R_Σ. There exists a right module of fractions for V with respect to Σ, V_Σ say. Furthermore, $V \hookrightarrow V_\Sigma$ [cf. GW, chapter 9]. If $\sigma \in C(S) = sat(\Sigma)$, then $\sigma^{-1} \in R_\Sigma$. If $v\sigma = 0$ for some $v \in V$, then $0 = (v\sigma)\sigma^{-1} = v$. This completes the proof.

An earlier result asserted the existence of a largest Ore set which operates regularly on V. We will denote this set by Σ_1. Recall that Σ_1 contains every Ore set which operates regularly on V. The above implies that

$$\Sigma \subsetneq sat(\Sigma) = C(S) \subseteq \Sigma_1.$$

Proposition 109 *Let R be a Goldie prime serial ring and V a uniform injective R-module. Let Σ_1 denote the largest Ore set which operates regularly on V. There exist Goldie prime ideals $Q_1, Q_2, \ldots, Q_m$ such that, with $S_1 = \bigcap_{i=1}^{m} Q_i$, $\Sigma_1 = C(S_1)$.*

Proof. Since R is Goldie prime, $K(\Sigma_1) = 0$; in particular, $K(\Sigma_1)$ does not contain any idempotents. Our results show that there exist Goldie prime ideals $Q_1, Q_2, \ldots, Q_m$ such that $\Sigma_1 \subseteq sat(S_1) = \mathcal{C}(S_1)$ where $S_1 = \bigcap_{i=1}^{m} Q_i$. The proof of (3) in theorem 108, can be used to show that $\mathcal{C}(S_1)$ also operates regularly on V. The maximality of Σ_1 gives equality.

Lemma 110 *Let Q be a Goldie prime ideal in a Goldie prime serial ring R and let $E(Q) = \{e_1, e_2, \ldots, e_s\}$. For each $i = 1, 2, \ldots, s$ let $Q \rightleftharpoons \zeta_i$ via the MCC between R and $A_i = e_i R e_i$. Then*

$$\begin{pmatrix} A_1 - \zeta_1 & & & & & 0 \\ & \ddots & & & & \\ & & A_s - \zeta_s & & & \\ & & & A_{s+1} & & \\ & & & & \ddots & \\ 0 & & & & & A_n \end{pmatrix} \subseteq \mathcal{C}(Q).$$

Proof. Recall that

$$Q = \begin{pmatrix} \zeta_1 & & \zeta_i X_{ij} & \vdots & \\ & \zeta_2 & & \vdots & X_{ij} \\ & & \ddots & \vdots & \\ X_{ij}\zeta_j & & \zeta_s & \vdots & \\ \cdots & \cdots \quad \cdots & \cdots & \cdots & \cdots \\ & X_{ij} & & \vdots & X_{ij} \end{pmatrix}.$$

Let

$$a = \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix} \in \begin{pmatrix} A_1 - \zeta_1 & & & & & 0 \\ & \ddots & & & & \\ & & A_s - \zeta_s & & & \\ & & & A_{s+1} & & \\ & & & & \ddots & \\ 0 & & & & & A_n \end{pmatrix}.$$

Assume that $ax \in Q$ for some $x = (x_{ij}) \in R$. We will prove that $x \in Q$.

Let $1 \leq i \leq s$. Then $a_i x_{ii} \in \zeta_i$. Since $a_i \notin \zeta_i$ and ζ_i is completely prime, $x_{ii} \in \zeta_i$.

Let $i \neq j$ and $1 \leq i, j \leq s$. Then $ax \in Q$ implies that $a_i x_{ij} \in \zeta_i X_{ij}$. Because $a_i \notin \zeta_i$, it follows from lemma 75 that $a_i \zeta_i = \zeta_i$. This implies $a_i x_{ij} \in a_i \zeta_i X_{ij}$. This shows that $x_{ij} \in \zeta_i X_{ij}$. Therefore, $x \in Q$.

In general, if $E(Q) = \{e_r, \ldots, e_s\}$, then

$$\begin{pmatrix} A_1 & & & & & & 0 \\ & \ddots & & & & & \\ & & A_r - \zeta_r & & & & \\ & & & \ddots & & & \\ & & & & A_s - \zeta_s & & \\ & & & & & \ddots & \\ 0 & & & & & & A_n \end{pmatrix} \subseteq \mathcal{C}(Q).$$

Going back to the previous proposition, we have that $S_1 = \bigcap_{i=1}^{m} Q_i$, and that the Q_i are Goldie prime ideals which form a complete cross-section of *spec*(R). Using the above lemma and the fact that $\mathcal{C}(S_1) = \mathcal{C}(\bigcap_{i=1}^{r} Q_i) = \bigcap_{i=1}^{r} \mathcal{C}(Q_i)$, we conclude that

$$\begin{pmatrix} A_1 - \zeta_1 & & & 0 \\ & A_2 - \zeta_2 & & \\ & & \ddots & \\ 0 & & & A_n - \zeta_n \end{pmatrix} \subseteq \mathcal{C}(S_1) = \Sigma_1$$

where each ζ_i comes from a MCC with some Q_j.

We now have the following situation:

R is a Goldie prime serial ring and V is a uniform injective R-module such that all idempotents are faithful. For each i, $A_i = e_i R e_i$ is a valuation ring and there is a completely prime ideal $\wp_i = gass(Ve_i) \in spec(A_i)$. Then

$$\Sigma = \begin{pmatrix} A_1 - \wp_1 & & & 0 \\ & A_2 - \wp_2 & & \\ & & \ddots & \\ 0 & & & A_n - \wp_n \end{pmatrix}$$

is an Ore set and operates regularly on V. We also get a list of Goldie prime ideals $P_1, P_2, \ldots, P_k$ in $spec(R)$ via the MCC. We have shown that $P_1, P_2, \ldots, P_k$ is a complete list of the ideals which are maximal with respect to having empty intersection with Σ. Furthermore, $P_1, P_2, \ldots, P_k$ form a complete cross-section of $spec(R)$ and $sat(\Sigma) = \mathcal{C}(S)$ where $S = \bigcap_{i=1}^{k} P_i$.

On the other hand, we also have a set Σ_1, which is the largest Ore set operating regularly on V. Then Σ_1 contains Σ and $\mathcal{C}(S)$, and $\Sigma_1 = \mathcal{C}(S_1)$ where $S_1 = \bigcap_{i=1}^{m} Q_i$. The Q_i are Goldie prime, form a complete cross-section of $spec(R)$ and are maximal with respect to having empty intersection with Σ_1.

Lemma 111 *With respect to the discussion above let $Q \in \{Q_1, Q_2, \ldots, Q_m\}$ and $P \in \{P_1, P_2, \ldots, P_k\}$. If P and Q are comparable, then $P = Q$.*

Proof. Let $E(P) = \{e_1, e_2, \ldots, e_r\}$. Suppose that P and Q are comparable and that $Q \subset P$. For each $i = 1, 2, \ldots, r$, $e_i \notin Q$. Thus, there exists $\zeta_i \in spec(A_i)$ such that $Q \rightleftharpoons \zeta_i$ under the MCC. Because the MCC is $1-1$ and order preserving, $\zeta_i \subset \wp_i$ for each i. Pick $a_1 \in \wp_1 - \zeta_1$. Then

$$\sigma = \begin{pmatrix} a_1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix} \in \mathcal{C}(S_1) = \Sigma_1.$$

Since $\wp_i = gass\,(Ve_i)$, there exists $0 \neq ve_1 \in Ve_1$ such that $ve_1a_1 = va_1 = 0$. Now $0 \neq (ve_1, 0, \ldots, 0) = w \in V$ and $w\sigma = 0$. This contradicts the fact that Σ_1 operates regularly on V. Therefore, if P and Q are comparable, then $P \subseteq Q$.

Since $\Sigma \subseteq \Sigma_1$ and $Q \cap \Sigma_1 = \emptyset$, it is also the case that $Q \cap \Sigma = \emptyset$. The maximality of P allows us to conclude that $P = Q$.

Lemma 112 *With the notation above, $S = S_1$.*

Proof. Let Q_i be given. Since $P_1, P_2, \ldots, P_k$ form a cross-section of *spec*(R), there is some $j = j(i)$ such that Q_i and P_j are comparable. The above lemma implies that $P_j = Q_i$. Since $Q_1, Q_2, \ldots, Q_m$ also form a cross-section of *spec*(R), we are left to conclude that $\{P_1, P_2, \ldots, P_k\} = \{Q_1, Q_2, \ldots, Q_m\}$. Therefore $S = \bigcap_{i=1}^{k} P_i = \bigcap_{i=1}^{m} Q_i = S$.

We have proven the following theorem.

Theorem 113 *Let R be a Goldie prime serial ring and V a uniform injective R-module such that all idempotents are faithful to V. For each i, let $\wp_i = gass\,(Ve_i)$, and let $P_1, P_2, \ldots, P_k$ be the Goldie prime ideals in R which arise from the MCC with the $\wp_i$. The ideal $S = \bigcap_{i=1}^{k} P_i$ is a Goldie semiprime ideal in R, and $C\,(S)$ is the largest Ore set which operates regularly on V.*

We call the ideal S from the above theorem the **generalized associated semiprime** of V.

References

[C] Chatters, A.W., *Serial Rings with Krull Dimension*, Glasgow Math. J. 32 (1990), 71-78.

[EG1] Eisenbud, D. and Griffith, P., *Serial Rings*, J. Algebra 17 (1971), 389-400.

[EG2] Eisenbud, D. and Griffith, P., *The Sructure of Serial Rings*, Pacific J. Math. 36 (1971), 109-121.

[FP] Facchini, A. and Puninski, G., *Σ-Pure-Injective Modules over Serial Rings*, in Abelian Groups and Modules, A. Facchini and C. Menini eds., Kluwer Acad. Publishers, Dodecht, 1995, 145-162.

[GW] Goodearl, K.R. and Warfield, R.B., *An Introduction to Noncommutative Rings*, London Mathematical Society, Student Texts 16, 1989.

[H] Hungerford, T.W., *Algebra*, Graduate Texts in Mathematics, Springer Verlag, 1989.

[I] Ivanov, G., *Decomposition of Modules over Serial Rings*, Comm. Algebra 3(11) (1975), 1031-1036.

[K1] Kupish, H., *Beiträge zur Theorie nichthalbeinfacher Ringe mit Minimalbedingung*, J. Reine Angew. Math. 201 (1959), 100-112.

[K2] Kupish, H., *Über eine Klasse von Ringen mit Minimalbedingung I*, Arch. Math. 17 (1966), 20-35.

[K3] Kupish, H., *Über eine Klasse von Artin-Ringen II,* Arch. Math. 26 (1975), 22-35.

[McR] McConnell, J.C. and Robson, J.C., *Noncommutative Noetherian Rings*, Wiley 1987.

[M1] Müller, B.J., *Goldie-Prime Serial Rings*, Contemporary Mathematics 130 (1992), 301-310.

[M2] Müller, B.J., *The Structure of Serial Rings*, Methods in Module Theory, Marcel Dekker, Inc. (1990), 249-270.

[MS1] Müller, B.J. and Singh, S., *Uniform Modules over Serial Rings*, J. Algebra 144 (1991), 94-109.

[MS2] Müller, B.J. and Singh, S., *Uniform Modules over Serial Rings II*, Proc. Conf. on Rings and Modules, (Athens, Ohio, 1989), Lecture Notes in Math. 1448, Springer Verlag 1990, 25-32.

[MK] Munkres, J.R., *Topology A First Course*, Prentice-Hall, Inc., 1975.

[MR1] Murase, I., *On the Structure of Generalized Uniserial Rings I*, Sci. Papers College Gen. Ed. Univ. Tokyo 13 (1963), 1-22.

[MR2] Murase, I., *On the Structure of Generalized Uniserial Rings II*, Sci. Papers College Gen. Ed. Univ. Tokyo 13 (1963), 131-158.

[MR3] Murase, I., *On the Structure of Generalized Uniserial Rings III*, Sci. Papers College Gen. Ed. Univ. Tokyo 14 (1964), 11-25.

[N1] Nakayama, T., *On Frobeniusean Algebras I*, Ann. Math. 40 (1939), 611-633.

[N2] Nakayama, T., *On Frobeniusean Algebras II*, Ann. Math. 42 (1941), 1-21.

[N3] Nakayama, T., *Note on Uni-serial and Generalized Uni-serial Rings*, Proc. Imp. Acad. Toyko 16 (1940), 285-289.

[S] Singh, S., *Serial right Noetherian Rings*, Can. J. Math. 36 (1984), 22-37.

[SK] Skornjakov, L.A., *When are all Modules Semi-chained?*, Mat. Zametki 5 (1969), 173-182.

[U] Upham, M.H., *Serial Rings with right Krull dimension one*, J. Algebra 109 (1987), 319-333.

[W] Warfield, R.B., *Serial Rings and Finitely Presented Modules*, J. Algebra 37 (1975), 187-222.

[WR1] Wright, M.H, *Serial Rings with Right Krull Dimension One II*, J. Algebra 117 (1988), 99-116.

[WR2] Wright, M.H., *Certain Uniform Modules over Serial Rings are Uniserial*, Comm. Algebra 17(2) (1989), 441-469.

[WR3] Wright, M.H., *Prime Serial Rings with Krull Dimension*, Comm. Algebra 18(8) (1990), 2559-2572.