MIXED WEIGHTED INEQUALITIES
FOR CLASSES OF OPERATORS

By

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ABSTRACT

This thesis is concerned with the study of weighted inequalities for operators defined on certain function spaces. If T is a linear - or sublinear operator, weakly bounded on some endpoint spaces, then it is shown that T is also bounded on weighted intermediate spaces. Since the weights govern the indices of the spaces, our results yield weighted extensions of known interpolation spaces and consequently weighted norm inequalities for many classical operators over an extended range of indices. Specifically we obtain new weighted estimates for certain generalizations of the Fourier- and Laplace-transforms, namely the Hankel-, $\mathcal{K}$- and $\mathcal{Y}$-transforms in Lebesgue and Lorentz spaces.

In Memory Of My Parents

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INTRODUCTION

The Marcinkiewicz interpolation theorem asserts that if T is a sublinear operator defined on suitable subspaces of L^p and if T is bounded from L^{p_i} to weak L^{q_i}, $1 \le p_i \le q_i \le \infty$, $i = 0, 1$; $p_0 < p_1$, $q_0 \neq q_1$, then T is bounded from L^p to L^q, where $1/p = t/p_0 + (1-t)/p_1$ and $1/q = t/q_0 + (1-t)/q_1$, $0 < t < 1$.

There are numerous generalizations and abstractions of this result with significant applications in harmonic analysis and other branches of mathematics and physics. See for example the monographs of Bergh and Löfström [5], Krein, Petunin and Semenov [25] and Triebel [39] as well as the literature cited there. We single out one extension of the Marcinkiewicz theorem which has recently been obtained by Heinig ([15], [18]). This generalization shows that under the hypotheses of the Marcinkiewicz theorem, there exist non-negative functions u and v, such that the operator T satisfies the inequality

$$(0.1)\quad \left\{\int_\Omega |u(x)Tf(x)|^q dx\right\}^{1/q} \le C\left\{\int_\Omega |v(x)f(x)|^p dx\right\}^{1/p}, \quad \Omega \subseteq \mathbb{R}^n,$$

where C is a constant independent of f and $1 \le p,q \le \infty$. For $u = v \equiv 1$, his result reduces to that of Marcinkiewicz, however, there are weights for which (0.1) holds not only for $p \in (p_0,p_1)$ and

$q \in (q_0, q_1)$, but also for p and q outside these intervals. In this sense, the result may also be viewed as an extrapolation theorem. The significance of this result lies in the fact that weighted conditions are deduced without strengthening the hypotheses. Moreover, it shows specifically that there exist translation invariant operators which map L_v^p to L_u^q, $q < p$, for some non-constant weights u and v. This contrasts with Hörmander's celebrated result [21] that for weights $u = v \equiv 1$, such operators can only be the trivial zero operators. In addition, if $1 \le p \le q < \infty$, then there are certain operators, such as the Fourier and Laplace transforms for which the weight conditions which imply (0.1) are near optimal ([1], [3], [4], [23], [28], [29]).

The object of this thesis is to generalize and abstract some of these results in several directions and specifically give examples which yield weighted inequalities of the form (0.1) for the Hankel-, $\mathcal{K}$- and $\mathcal{Y}$-transformations. The weight conditions are, as in the case of the Fourier transform ([3], [4]), easily verifiable and near optimal. For the Hankel transform the example gives another weighted estimate for the Fourier transform of a radial function on $\mathbb{R}^n$.

Our first generalization given in Chapter 2 is in the abstract setting of the intermediate spaces $(A_0, A_1)_{\theta,q}$, introduced by Lions and Peetre [26]. Here we obtain weight conditions for which weighted estimates for quasi-linear operators hold. Specifically, we generalize a result of Sagher [33] who proved a weighted estimate of the form

$$(0.2)\qquad \left\{\int_0^\infty [u(t)K(t,Tf;B)]^q dt\right\}^{1/q} \le C\left\{\int_0^\infty [v(t)K(t,f;A)]^q dt\right\}^{1/q},$$

where A and B are interpolation couples, C is a constant independent of f and $K(\cdot,\cdot;\cdot)$ is the Peetre K-functional. We extend this result to the case where the index q on the right side of (0.2) is replaced by p, $p \neq q$. For $q > p$, this is the content of Theorems 2.1 and 2.2, while Theorem 2.5 proves the case of $q < p$. Further, we study in this chapter interpolation spaces $(A_0,A_1)_{w,q}$, which are generalizations of the spaces $(A_0,A_1)_{\theta,q}$, $0 < \theta < 1$, in the sense that the power function with exponent θ is replaced by a more general function w. For these spaces a number of properties are deduced and it is shown that if T is bounded between weighted interpolation endpoint-spaces, then there exist weights u and v, such that, an inequality of the form (0.2) holds for $1 \le p \le q \le \infty$ (Theorem 2.12). The results are proved by utilizing an extension ([16]) of a result of Holmstedt [20].

In Chapter 3 we show (Theorems 3.5 and 3.6) that if T is a quasi-linear operator bounded between certain Lorentz spaces, then T satisfies an inequality of the form (0.1) with $1 \le p,q \le \infty$. These results are different from those in [15] and [18] in that the initial weak L^p spaces are replaced by Lorentz and variants of Lorentz spaces. The second section contains applications. We use the $F^*_{p,q}$-weight condition of [4] (see also [3]) to show that weighted $L^p - L^q$ estimates hold for the Hankel-, $\mathcal{K}$- and $\mathcal{Y}$-transformations. For the Hankel transform this complements and generalizes a result of Heywood and Rooney [19]. As in their case and the case of the Fourier and Laplace transforms ([1], [4]) the weight conditions are best possible in the range $1 < p \leq q < \infty$ whenever the weights are monotone. (In the domain space of T the weight is non-decreasing, while in the range space the weight is non-increasing.) In addition, the chapter contains a number of weighted inequalities for the Laplace transform. Utilizing and modifying results of Sawyer [34] it is shown that the weighted L^p-spaces in the range of the operator of Theorems 2.4 and 2.3 in [1] and [17], respectively may be replaced by weighted Lorentz spaces.

In Chapter 1 we introduce notations, definitions and state standard theorems and inequalities needed in the sequel. In the second section we give in Lemma 1.9 a result of Calderón and Scott [9]. The second part of the lemma is new but follows along the same line as in [9]. We give the proof only for sake of completeness.

Theorems, lemmas and corollaries are labeled by pairs of numbers, the first indicating the chapter and the second the order within the chapter. Similarly for equalities and inequalities.

CHAPTER 1

PRELIMINARIES

In this chapter we introduce notations, collect definitions and state results which are needed and applied in the sequel. This is done to make the thesis as self-contained as possible. The chapter is divided into four sections. The first section contains notations, definitions, and some unweighted and weighted inequalities. ~~In Section 1.2~~ we give in Lemma 1.9 a result of Calderón and Scott [9]. The second part of the lemma is new but follows along the same lines as in [9, Lemma 6.1]. We give the proof only for sake of completeness. In Section 1.3 the concept of weak and strong type boundedness of an operator is introduced and examples are given, while in the final section some interpolation theorems are stated.

1.1 Notations, Definitions And Basic Theory.

All inequalities appearing in this thesis are interpreted in the sense that, if the right sides are finite so are the left sides and the inequalities hold. C will denote a positive constant which is independent of the function involved, but may depend on the weight functions and the indices of the spaces. Moreover, the constants may be different at different appearances, and when it is evident from the context we use also other letters - possibly with subscripts - to denote constants.

If f and g are two positive functions then $f(x) \sim g(x)$ means that $f(x)/g(x)$, $x > 0$, is bounded above and below by two positive constants.

Sometimes we denote by $\mathbb{R}$ the interval $(-\infty,\infty)$ and by $\mathbb{R}^n$, n-copies of $\mathbb{R}$. We also write $\mathbb{R}^+ = (0,\infty)$ and $\mathbb{C}$ denotes the set of complex numbers. If E is a set in $\mathbb{R}^n$, then $|E|$ denotes the Lebesgue measure of E. For other measures μ we write $\mu(E)$.

Definition 1.1. If $0 < p \le \infty$, the Lebesgue spaces $L^p(\Omega)$, $\Omega \subseteq \mathbb{R}^n$ consists of all measurable functions f defined on Ω, such that

$$\|f\|_p = \begin{cases} \left(\int_\Omega |f(x)|^p dx\right)^{1/p}, & 0 < p < \infty \\ \operatorname*{ess\,sup}_{x\in\Omega} |f(x)|, & p = \infty \end{cases}$$

is finite. Here

$$\operatorname*{ess\,sup}_{x\in\Omega} |f(x)| = \inf\{y > 0 : |\{x : |f(x)| > y\}| = 0\}.$$

The letter p is called the index of the space. The conjugate index of p is as usual denoted by p' and is related to p by $1/p + 1/p' = 1$, if $p \neq 1$ and $p' = \infty$ if $p = 1$. Note that p' is negative if $0 < p < 1$. Similarly we write q' for the conjugate index of q and in the same way for other letters.

If $0 \le v(x) < \infty$, then we write $f \in L_v^p$ if $vf \in L^p$ and $\|f\|_{p,v} = \|vf\|_p$. L_v^p are called the weighted Lebesgue spaces with weights v.

Theorem 1.2 (Hölder's Inequality). If $f \in L^p$, $1 \le p \le \infty$ and $g \in L^{p'}$, then $fg \in L^1$ and

$$\int_\Omega |f(x)g(x)|dx \le \|f\|_p \|g\|_{p'}, \quad \Omega \subseteq R^n .$$

If $0 < p < 1$, the inequality is reversed.

Theorem 1.3 (Minkowski's Inequality). If f and g are in L^p, $1 \le p \le \infty$, then $f + g \in L^p$ and

$$\|f + g\|_p \le \|f\|_p + \|g\|_p .$$

If $0 < p < 1$, the inequality is reversed.

Theorem 1.4 (Minkowski's Integral Inequality). Let (X,μ) and (Y,ν) be measure spaces and f a $\mu \times \nu$ measurable function on $X \times Y$. If

$$\int_Y \left(\int_X |f(x,y)|^p \, d\nu(y) \right)^{1/p} d\mu(x)$$

is finite, $1 \le p \le \infty$, then $\int_Y |f(x,y)| \, d\mu(x)$ converges for a.e. y and

$$\left(\int_X \left(\int_Y |f(x,y)| \, d\mu(x) \right)^p d\nu(y) \right)^{1/p} \le \int_Y \left(\int_X |f(x,y)|^p \, d\nu(y) \right)^{1/p} d\mu(x) .$$

If $0 < p < 1$ the inequality is reversed.

<u>Theorem 1.5 (Fubini's Theorem).</u> Let (X,μ) and (Y,ν) be measure spaces and $\mu \times \nu$ the product measure on $X \times Y$. Assume that f is measurable on $X \times Y$. If at least one of the iterated integrals

$$\int_X \left[\int_Y |f(x,y)| \, d\nu(y) \right] d\mu(x) \qquad \text{or} \qquad \int_Y \left[\int_X |f(x,y)| \, d\mu(x) \right] d\nu(y)$$

exists, then $f \in L^1(\mu \times \nu)$ and

$$\int_{X \times Y} f(x,y) \, d(\mu \times \nu)(x,y) = \int_X \int_Y f(x,y) \, d\nu(y) \, d\mu(x)$$

$$= \int_Y \int_X f(x,y) \, d\mu(x) \, d\nu(y) .$$

The following result is the well-known weighted Hardy inequality and the dual inequality:

Lemma 1.6 ([6], [2], [27]). Suppose u, v and f are non-negative functions on $\mathbb{R}^+$ and $1 \le p \le q \le \infty$. Then

$$(1.1)\qquad \left\{\int_0^\infty \left[u(t)\int_0^t f(y)dy\right]^q dt\right\}^{1/q} \le C\left\{\int_0^\infty [v(t)f(t)]^p dt\right\}^{1/p},$$

if and only if

$$(1.2)\qquad \sup_{s>0}\left(\int_s^\infty u(t)^q dt\right)^{1/q}\left(\int_0^s v(t)^{-p'} dt\right)^{1/p'} \equiv A < \infty .$$

Moreover, $A \le C \le p^{1/q}(p')^{1/p'}A$ if $p \neq 1$, where C is the least constant for which (1.1) holds; if $p = 1$, then $A = C$. Note that in this case the second integral of (1.2) is interpreted to be the essential supremum of $1/v(t)$ for $t < s$. (A similar interpretation is made for the first integral of (1.2) if $q = \infty$.)

The dual inequality

$$\left\{\int_0^\infty \left[u(t)\int_t^\infty f(y)dy\right]^q dt\right\}^{1/q} \le C\left\{\int_0^\infty [v(t)f(t)]^p dt\right\}^{1/p}$$

holds if and only if

$$\sup_{s>0}\left(\int_0^s u(t)^q dt\right)^{1/q}\left(\int_s^\infty v(t)^{-p'}dt\right)^{1/p'} \equiv B < \infty .$$

Again $B \le C \le Bp^{1/q}(p')^{1/p'}$ if $p \neq 1$ and $B = C$ if $p = 1$.

Mazja [27] and independently Sawyer [34] extended the above result to the range $1 < q < p < \infty$, in fact Sawyer extended the result to the range $0 < q < p$, $p > 1$ with different but equivalent weight conditions. We state here their result in the form of the weight conditions of Mazja.

Lemma 1.7 ([27], [17], [34]).

a) Suppose u, v and f are non-negative functions on R^+ and $1 \le q < p < \infty$. Then

$$(1.3)\qquad \left\{\int_0^\infty \left[u(t)\int_0^t f(y)dy\right]^q dt\right\}^{1/q} \le C\left\{\int_0^\infty [v(t)f(t)]^p dt\right\}^{1/p},$$

if and only if

$$(1.4)\qquad \left\{\int_0^\infty \left[\left(\int_x^\infty u(y)^q dy\right)^{1/q}\left(\int_0^x v(y)^{-p'}dy\right)^{1/q'}\right]^r v(x)^{-p'}dx\right\}^{1/r} < \infty ,$$

where $1/r = 1/q - 1/p$.

In case $q = 1 < p$, condition (1.4) takes the form

$$\left[\int_0^\infty \left(\int_x^\infty u(y)dy\right)^{p'} v(x)^{-p'} dx\right]^{1/p'} < \infty .$$

b) If u, v, f and r are as above, then

$$\left\{\int_0^\infty [u(t)\int_t^\infty f(y)dy]^q dt\right\}^{1/q} \le C\left\{\int_0^\infty [v(t)f(t)]^p dt\right\}^{1/p} \tag{1.5}$$

if and only if

$$\left\{\int_0^\infty \left[\left(\int_0^x u(y)^q dy\right)^{1/q} \left(\int_x^\infty v(y)^{-p'} dy\right)^{1/q'}\right]^r v(x)^{-p'} dx\right\}^{1/r} < \infty . \tag{1.6}$$

If $q = 1 < p$, condition (1.5) is modified in a similar way as in a).

Note that in the limiting case $q = p$ the integral in (1.4) takes the form (1.2) and similarly the integral (1.6) takes the corresponding form of the weight conditions of the dual operator in Lemma 1.6.

1.2 Preliminary Lemmas.

First we state the following inequality which is needed frequently:

Lemma 1.8 ([22]). If f is a non-negative non-increasing function defined on R^+ then for α real and $0 < p \le q \le \infty$

$$\left\{\int_0^\infty [t^\alpha f(t)]^q t^{-1} dt\right\}^{1/q} \le C\left\{\int_0^\infty [t^\alpha f(t)]^p t^{-1} dt\right\}^{1/p}$$

holds.

We shall need the following lemma:

<u>Lemma 1.9 ([9])</u>. Let $0 < s < 1$ and f a non-negative and g a positive continuous function.

(i) If f is increasing on $[0,\infty)$ and g decreasing on $[0,\infty)$, such that $\lim_{t\to\infty} g(t) = 0$, then for $t \ge 0$

$$s\int_t^\infty f(x)d(-g(x)) \le \left[\int_t^\infty f(x)^s d(-g(x)^s)\right]^{1/s} \tag{1.7}$$

holds.

(ii) If f is decreasing on $[0,\infty)$ and g increasing on $[0,\infty)$ such that $\lim_{t\to 0} g(t) = 0$, then for $t \le \infty$

$$s\int_0^t f(x)d(g(x)) \le \left[\int_0^t f(x)^s d(g(x)^s)\right]^{1/s} \tag{1.8}$$

holds.

Part (i) follows from Calderón and Scott [9, Lemma 6.1] and Part (ii) follows essentially along the lines as in [9, Lemma 6.1]. We give the proof here only for completeness.

Proof. (i) Since the case $t = 0$ is known [9], we assume $t > 0$. Moreover, it suffices to prove the result when the integral on the right of (1.7) is finite. Let $I_{s,t}$ denote the right side of (1.7), then for $y \geq t > 0$

$$I_{s,t} \geq \left[\int_y^\infty f(x)^s d(-g(x)^s)\right]^{1/s} \geq f(y)g(y) ,$$

so that $f(y) \leq I_{s,t}/g(y)$. Using this estimate and integrating we get

$$\int_t^\infty f(x)d(-g(x)) = \int_t^\infty f(x)^s f(x)^{1-s} d(-g(x))$$

$$\leq \int_t^\infty f(x)^s \, I_{s,t}^{1-s} \, g(x)^{s-1} d(-g(x))$$

$$= s^{-1} \, I_{s,t}^{1-s} \int_t^\infty f(x)^s \, d(-g(x)^s) = s^{-1} \, I_{s,t} ,$$

which proves the first part of the lemma.

To prove the second part, denote the integral on the right of

(1.8) by $J_{s,t}$ and observe that for $0 < y \le t$

$$J_{s,t} \ge \left[\int_0^y f(x)^s d(g(x)^s)\right]^{1/s} \ge f(y)g(y) .$$

Therefore $f(y) \le J_{s,t}/g(y)$, $0 < y \le t$, and integrating we get

$$\int_0^t f(x)d(g(x)) = \int_0^t f(x)^s f(x)^{1-s} d(g(x))$$

$$\le \int_0^t f(x)^s \, J_{s,t}^{1-s} \, g(x)^{s-1} d(g(x))$$

$$= s^{-1} \, J_{s,t}^{1-s} \int_0^t f(x)^s d(g(x)^s) = s^{-1} J_{s,t} .$$

This proves the lemma. (Observe that the second part of the lemma follows at once from the first on substituting x by $1/x$.)

1.3 Lorentz Spaces And Operators.

Let f be a Lebesgue measurable function on Ω, $\Omega \subseteq R^n$. The distribution function f_* of f is defined by

$$f_*(y) = |\{x \in \Omega : |f(x)| > y\}| , \quad y > 0 .$$

The (non-negative) decreasing rearrangement of $|f|$ is essentially the inverse of f_*, or more precisely

$$f^*(t) = \inf\{y > 0,\ f_*(y) \le t\}\ ,\quad t > 0\ .$$

We also denote the integral average of f^* by f^{**}, that is,

$$f^{**}(x) = \frac{1}{x}\int_0^x f^*(t)dt\ .$$

It is clear that f_* and f^* are non-increasing functions, continuous from the right. Also

$$\int_\Omega |f(x)|dx = \int_0^\infty f^*(t)dt\ ,$$

and for any measurable f and g

$$\int_\Omega |f(x)g(x)|dx \le \int_0^\infty f^*(t)g^*(t)dt\ .$$

<u>Definition 1.10</u>. The Lorentz spaces $L(p,q)$, $0 < p,q \le \infty$ are defined to be the collection of all (Lebesgue) measurable functions f on $\Omega \subseteq \mathbb{R}^n$ for which the quasi-norm

$$\|f\|_{p,q} = \begin{cases} \left\{(q/p)\int_0^\infty [t^{1/p}f^*(t)]^q t^{-1}dt\right\}^{1/q}, & 0 < p \le \infty,\ 0 < q < \infty \\ \sup_{t>0} t^{1/p}f^*(t), & 0 < p \le \infty,\quad q = \infty \end{cases}$$

is finite.

If $0 < p \le q$, $0 < s \le q \le \infty$, then it follows from Lemma 1.8 that $L(p,s) \subseteq L(p,q)$ and $\|f\|_{p,q} \le C\|f\|_{p,s}$. Also if $p = q$ these spaces reduce to the L^p-spaces.

It must be noted that even for $1 < p < \infty$, $1 \le q \le \infty$ or $p = q = \infty$, $\|\cdot\|_{p,q}$ is not always a norm since the triangle inequality may fail. However, if we replace f^* by f^{**} that is $\|f\|_{p,q}$ is replaced by the norm

$$\|f\|^*_{p,q} = \begin{cases} \left\{(q/p)\int_0^\infty [t^{1/p}f^{**}(t)]^q t^{-1}dt\right\}^{1/q}, & 1 < p < \infty,\quad 1 \le q < \infty \\ \sup_{t>0} t^{1/p}f^{**}(t), & 1 < p \le \infty,\quad q = \infty, \end{cases}$$

then

$$\|f\|^*_{p,q} \le \|f\|_{p,q} \le (p/(p-1))\,\|f\|^*_{p,q}, \quad ([37,\ \text{Chapter V}]).$$

In either case $1 < p < \infty$ and $1 \le q < \infty$ or $p = q = \infty$, $L(p,q)$ is equivalent to a Banach space with quasi-norm $\|\cdot\|_{p,q}$ equivalent to the norm $\|\cdot\|^*_{p,q}$.

<u>Definition 1.11</u>. Let (X,μ) and (Y,ν) be two σ-finite measure spaces. An operator T, which maps functions on X into measurable functions on Y is called quasi-linear if $T(f + g)$ is uniquely defined whenever Tf and Tg are defined and

$$|T(f + g)| \le C(|Tf| + |Tg|) ,$$

holds for some constant $C > 0$. If $C = 1$ the operator is called sublinear.

T of course is linear, if for any $\alpha, \beta \in \mathbb{C}$,

$$T(\alpha f + \beta g) = \alpha Tf + \beta Tg .$$

Clearly linear operators are sublinear.

<u>Definition 1.12</u>. A (quasi-) linear operator T is said to be of (strong) type (p,q), $0 < p,q \le \infty$ if

$$\|Tf\|_q \le C\|f\|_p .$$

The (p,q) norm of T is equal to the infimum of all such C. Sometimes we also write

$$T: L^p \to L^q$$

if the above inequality holds.

An operator T is said to be of weak type (p,q), if for each $y > 0$

$$y[(Tf)_*(y)]^{1/q} \le C\|f\|_p, \quad q < \infty;$$

or equivalently if

$$\sup_{t>0} t^{1/q}(Tf)^*(t) \le C\|f\|_p, \quad q < \infty.$$

If $q = \infty$, weak and strong types are defined to coincide.

As a consequence of Fubini's theorem it is easy to see that strong type implies weak type, however, the converse is not true in general. Below are some examples.

The Hardy-Littlewood maximal operator M defined by

$$(Mf)(x) = \sup_{h>0} \frac{1}{2h} \int_{x-h}^{x+h} |f(y)|dy, \quad x \in R$$

and the Hilbert transform H defined by

$$(Hf)(x) = \lim_{\varepsilon\to 0^+} \frac{1}{\pi} \int_{|y|\geq\varepsilon} \frac{f(x-y)}{y}\,dy\,, \quad x \in \mathbb{R}$$

are both of weak type (1,1) and strong type (p,p), $p > 1$. Note that M is not linear, but sublinear.

Also, the operator T defined by

$$(Tf)(x) = x\,\hat{f}(x)\,,$$

where $\hat{f}$ denotes the Fourier transform of f:

$$\hat{f}(x) = \int_{-\infty}^{\infty} e^{-ixy} f(y)\,dy\,, \quad x \in \mathbb{R}\,,$$

is of weak type (1,1) with respect to the measure

$$\mu(E) = \int_E x^{-2}\,dx\,, \quad E \subset \mathbb{R}\backslash\{0\}\,,$$

but not of strong type (1,1). However

$$\int_{\infty}^{\infty} |(Tf)(x)|^p\,d\mu(x) \leq C\int_{-\infty}^{\infty} |f(x)|^p\,dx\,, \quad 1 < p \leq 2\,,$$

so that T is of strong type (p,p) (with respect to the μ-measure).

Finally we note that the fractional integral operator

$$(I_\alpha f)(x) = \Gamma(\alpha)^{-1}\int_{-\infty}^{\infty}|x-y|^{\alpha-1}f(y)dy\,, \quad 0<\alpha<1\,, \quad x\in\mathbb{R}$$

is of weak (but not strong) type $(1, (1-\alpha)^{-1})$ ([36]).

1.4 Interpolation Theory.

We first recall the Marcinkiewicz interpolation theorem mentioned in the introduction.

Theorem 1.13 ([32]). Let T be a sublinear operator of weak types (p_i,q_i), $1\le p_i\le q_i\le\infty$, $i=0,1$; $p_0<p_1$, $q_0\ne q_1$. Then T is of strong type (p,q), where $1/p=(1-\theta)/p_0+\theta/p_1$ and $1/q=(1-\theta)/q_0+\theta/q_1$, $0<\theta<1$.

There are many generalizations and abstractions of this result. We sketch briefly the abstraction obtained first by J.L. Lions and J. Peetre [26].

Let $\mathcal{A}$ be a Hausdorff topological vector space and let A_i, $i=0,1$, be Banach spaces continuously imbedded in $\mathcal{A}$. We call such a pair (A_0, A_1) an interpolation couple and define the sum by

$$A_0 + A_1 = \{f \in \mathcal{A} : f = f_0 + f_1 ; \quad f_i \in A_i , \quad i = 0,1\} .$$

The space $A_0 + A_1$ is a Banach space under the norm

$$\|f\|_{A_0+A_1} = \inf_{f=f_0+f_1} \{\|f_0\|_{A_0} + \|f_1\|_{A_1} ; \quad f_i \in A_i , \quad i = 0,1\} .$$

Definition 1.14. If (A_0, A_1) is an interpolation couple and $0 < t < \infty$, then the Peetre K-functional is defined by

$$K(t,f; A_0,A_1) = \inf(\|f_0\|_{A_0} + t\|f_1\|_{A_1}) ,$$

where the infimum is taken over all decompositions $f = f_0 + f_1$, $f \in A_0 + A_1$ and $f_i \in A_i$, $i = 0,1$.

If it is clear from the context that the interpolation couple is (A_0, A_1), then we also denote $K(t,f; A_0,A_1)$ by $K(t,f)$.

Note that $K(t,f) = tK(t^{-1},f)$, $K(t,f)$ is a continuous non-negative concave and increasing function ([7, p. 167]), while $t^{-1}K(t,f)$ is decreasing.

If we take $A_0 = L^p$, $0 < p < \infty$, and $A_1 = L^\infty$, then

$$K(t,f; L^p,L^\infty) \sim \left(\int_0^{t^p} f^*(y)dy \right)^{1/p}$$

with equality when $p = 1$ ([5, p. 109]).

<u>Definition 1.15</u>. Let (A_0, A_1) be an interpolation couple and $0 < \theta < 1$. The space $\mathcal{A}_{\theta,p} \equiv (A_0, A_1)_{\theta,p}$, $0 < p < \infty$, is the space of all $f \in A_0 + A_1$, such that

$$\|f\|_{\mathcal{A}_{\theta,p}} = \begin{cases} \left\{ \int_0^\infty [t^{-\theta} K(t,f)]^p t^{-1} dt \right\}^{1/p}, & 0 < p < \infty \\ \sup_{t>0} t^{-\theta} K(t,f), & p = \infty, \end{cases}$$

is finite. The space $(\mathcal{A}_{\theta,p}, \|\cdot\|_{\mathcal{A}_{\theta,p}})$ is called an interpolation or intermediate space.

If $1 \le p \le \infty$, then $\|\cdot\|_{\mathcal{A}_{\theta,p}}$ defines a norm and $\mathcal{A}_{\theta,p}$ is a Banach space ([7, Prop. 3.2.5]).

The following example shows that the interpolation spaces between Lebesgue spaces are Lorentz spaces. Specifically if $A_0 = L^{p_0}$, $A_1 = L^{p_1}$, $0 < p_0 < p_1 < \infty$, and $q > p_0$, then $(L^{p_0}, L^{p_1})_{\theta,q} = L(p,q)$, where $1/p = (1-\theta)/p_0 + \theta/p_1$, $0 < \theta < 1$, ([5, Theorem 5.2.1]).

The next result, due to Holmstedt, gives useful estimates of the K-functional of interpolation spaces.

<u>Theorem 1.16 ([20]).</u> Let (A_0, A_1) be an interpolation couple and $E_i = (A_0, A_1)_{\theta_i, p_i}$, $i = 0,1$, $0 < \theta_0 < \theta_1 < 1$ and $0 < p_0 < p_1 \le \infty$. If $\lambda = \theta_1 - \theta_0$, then

$$K(t,f;\, E_0,E_1) \sim \left\{\int_0^{t^{1/\lambda}} [y^{-\theta_0} K(y,f;\, A_0,A_1)]^{p_0}\, y^{-1}dy\right\}^{1/p_0} + t\left\{\int_{t^{1/\lambda}}^{\infty} [y^{-\theta_1} K(y,f;\, A_0,A_1)]^{p_1}\, y^{-1}dy\right\}^{1/p_1}.$$

Specifically, if $E_0 = L(p_0, q_0)$, $E_1 = L(p_1, q_1)$, $0 < p_0 < p_1 \le \infty$, $0 < q_0, q_1 \le \infty$ and $1/\lambda = 1/p_0 - 1/p_1$, then it follows that

$$(1.9)\quad K(t,f;\, L(p_0,q_0),\, L(p_1,q_1)) \sim \left\{\int_0^{t^{\lambda}} [y^{1/p_0} f^*(y)]^{q_0}\, y^{-1}dy\right\}^{1/q_0} + t\left\{\int_{t^{\lambda}}^{\infty} [y^{1/p_1} f^*(y)]^{q_1}\, y^{-1}dy\right\}^{1/q_1}.$$

<u>Theorem 1.17 ([5], [20]) (Reiteration Theorem).</u> Let (A_0, A_1) be an interpolation couple and (E_0, E_1) is the pair of interpolation spaces with $E_i = (A_0, A_1)_{\theta_i, q_i}$, $0 \le \theta_i \le 1$, $0 < q_i \le \infty$, $i = 0,1$. Then

$$(E_0, E_1)_{\lambda,p} = (A_0, A_1)_{\theta,p}$$

where $0 < p \le \infty$ and $\theta = (1-\lambda)\theta_0 + \lambda\theta_1$, $0 < \lambda < 1$. Moreover,

$$\|f\|_{(E_0,E_1)_{\lambda,p}} \sim \|f\|_{(A_0,A_1)_{\theta,p}} .$$

CHAPTER 2

REAL INTERPOLATION WITH WEIGHTS

Suppose (A_0, A_1) and (B_0, B_1) are two interpolation couples, such that $T: A_i \to B_i$, $i = 0,1$, where T is a linear operator. More precisely

$$\|Tf\|_{B_i} \le M_i \|f\|_{A_i}, \quad f \in A_i, \quad i = 0,1.$$

If $0 < \theta < 1$ and $0 < p \le \infty$ then $T: (A_0, A_1)_{\theta,p} \to (B_0, B_1)_{\theta,p}$ and

$$\|Tf\|_{(B_0,B_1)_{\theta,p}} \le M_0^{1-\theta} M_1^{\theta} \|f\|_{(A_0,A_1)_{\theta,p}}$$

([26], [30]).

In this chapter we give various generalizations of the following weighted interpolation theorem of Sagher:

<u>Theorem A ([33])</u>. Suppose $\mathcal{A} = (A_0, A_1)$, $\mathcal{B} = (B_0, B_1)$ are two interpolation couples. Let $0 < \theta_i, \bar{\theta}_i < 1$, $0 < q_i, \bar{q}_i \le \infty$, $i = 0,1$; $\lambda = \theta_1 - \theta_0 \ne 0$, $\bar{\lambda} = \bar{\theta}_1 - \bar{\theta}_0 > 0$ and T a quasi-linear operator such that $T: \mathcal{A}_{\theta_i,q_i} \to \mathcal{B}_{\bar{\theta}_i,\bar{q}_i}$. If $q > \max(q_0,q_1)$ and u, v non-negative locally integrable weight functions satisfying

$$\text{(2.1)}\quad \sup_{s>0}\left\{\int_{s^{1/\bar{\lambda}}}^{\infty}\left[u(t)t^{\bar{\theta}_0}\right]^q dt\right\}^{1/q} \times$$

$$\left\{\int_0^s\left[v(t^{1/\lambda})t^{(1/q+\theta_0)/\lambda}\right]^{-qq_0/(q-q_0)} t^{-1}dt\right\}^{(q-q_0)/(qq_0)} < \infty$$

and

$$\text{(2.2)}\quad \sup_{s>0}\left\{\int_0^{s^{1/\bar{\lambda}}}\left[u(t)t^{\bar{\theta}_1}\right]^q dt\right\}^{1/q} \times$$

$$\left\{\int_s^{\infty}\left[v(t^{1/\lambda})t^{(1/q+\theta_1)/\lambda}\right]^{-qq_1/(q-q_1)} t^{-1}dt\right\}^{(q-q_1)/(qq_1)} < \infty ,$$

then

$$\text{(2.3)}\quad \left\{\int_0^{\infty}[u(t)K(t,Tf;\mathcal{B})]^q dt\right\}^{1/q} \le C\left\{\int_0^{\infty}[v(t)K(t,f;\mathcal{A})]^q dt\right\}^{1/q} .$$

In case $q_1 < q \le q_0$ or $q_0 < q \le q_1$, (2.3) still holds, only now, in the first case (2.1) is replaced by

$$\sup_{s>0}\left\{\int_{s^{1/\bar{\lambda}}}^{\infty}\left[u(t)t^{\bar{\theta}_0}\right]^q dt\right\}^{1/q} \operatorname*{ess\,sup}_{0<t<s}\left[v(t^{1/\lambda})t^{(1/q+\theta_0)/\lambda}\right]^{-1} < \infty$$

and in the second case, (2.2) is replaced by

$$\sup_{s>0}\left\{\int_{0}^{s^{1/\bar{\lambda}}}\left[u(t)t^{\bar{\theta}_1}\right]^q dt\right\}^{1/q} \operatorname*{ess\,sup}_{s<t<\infty}\left[v(t^{1/\lambda})t^{(1/q+\theta_1)/\lambda}\right]^{-1} < \infty .$$

In this chapter we prove theorems of this type, where the index q on the right side of (2.3) is replaced by p, $p \neq q$. If $p < q$, this is done in Theorems 2.1 and 2.2 of Section 2.1. Theorem 2.3 of Section 2.2 contains the case $q < p$.

Later, in Section 2.3 we prove also similar theorems where the parameters θ_i and $\bar{\theta}_i$, $i = 0, 1$ are replaced by certain weight functions w_i and $\bar{w}_i$, $i = 0, 1$ belonging to some function class B_ψ.

2.1 Extension of Sagher's Theorem.

Theorem 2.1. Suppose $\mathcal{A} = (A_0, A_1)$, $\mathcal{B} = (B_0, B_1)$ are interpolation couples, $0 < \theta_i, \bar{\theta}_i < 1$, $0 < q_i, \bar{q}_i \leq \infty$, $i = 0, 1$. Let $\lambda = \theta_1 - \theta_0 \neq 0$, $\bar{\lambda} = \bar{\theta}_1 - \bar{\theta}_0 > 0$ and $T: \mathcal{A}_{\theta_i, q_i} \to \mathcal{B}_{\bar{\theta}_i, \bar{q}_i}$ be a quasi-linear operator. If $\lambda > 0$ and u, v non-negative locally integrable weight functions satisfying any of the conditions of (a), (b) or (c):

(a) If $\max(q_0,q_1) \le p \le q \le \infty$

$$\text{(2.4)} \qquad \sup_{s>0} \left\{ \int_{s^{\lambda/\bar{\lambda}}}^{\infty} \left[u(t) t^{\bar{\theta}_0} \right]^q dt \right\}^{1/q} \times$$

$$\left\{ \int_0^s \left[v(t) t^{1/p+\theta_0} \right]^{-pq_0/(p-q_0)} t^{-1} dt \right\}^{(p-q_0)/(pq_0)} < \infty$$

and

$$\text{(2.5)} \qquad \sup_{s>0} \left\{ \int_0^{s^{\lambda/\bar{\lambda}}} \left[u(t) t^{\bar{\theta}_1} \right]^q dt \right\}^{1/q} \times$$

$$\left\{ \int_s^{\infty} \left[v(t) t^{1/p+\theta_1} \right]^{-pq_1/(p-q_1)} t^{-1} dt \right\}^{(p-q_1)/(pq_1)} < \infty ;$$

(b) If $q_1 \le p \le q \le q_0 \le \infty$

$$\text{(2.6)} \qquad \sup_{s>0} \left\{ \int_{s^{\lambda/\bar{\lambda}}}^{\infty} \left[u(t) t^{\bar{\theta}_0} \right]^q dt \right\}^{1/q} \operatorname*{ess\,sup}_{0<t<s} \left[v(t) t^{1/p+\theta_0} \right]^{-1} < \infty$$

and (2.5);

(c) If $q_0 \le p \le q \le q_1 \le \infty$

$$\text{(2.7)} \qquad \sup_{s>0} \left\{ \int_0^{s^{\lambda/\bar\lambda}} \left[u(t) t^{\bar\theta_1} \right]^q dt \right\}^{1/q} \operatorname*{ess\,sup}_{s<t<\infty} \left[v(t) t^{1/p+\theta_1} \right]^{-1} < \infty$$

and (2.4), then there is a $C > 0$, such that

$$\text{(2.8)} \qquad \left\{ \int_0^\infty [u(t) K(t, Tf; \mathcal{B})]^q dt \right\}^{1/q} \le C \left\{ \int_0^\infty [v(t) K(t, f; \mathcal{A})]^p dt \right\}^{1/p},$$

holds.

If $\lambda < 0$, (2.8) also holds, only now the ranges of the first integrals in (2.4) - (2.7) are changed from $(s^{\lambda/\bar\lambda}, \infty)$ to $(0, s^{\lambda/\bar\lambda})$ and from $(0, s^{\lambda/\bar\lambda})$ to $(s^{\lambda/\bar\lambda}, \infty)$.

Proof. We write $X_i = \mathcal{A}_{\theta_i, q_i}$, $i = 0, 1$, $(X_0, X_1) = \mathcal{C}$ and $Y_i = \mathcal{B}_{\bar\theta_i, \bar q_i}$, $(Y_0, Y_1) = \mathcal{D}$. Since $T: X_i \to Y_i$ is bounded, there exist numbers $M_i > 0$, such that for $f = f_0 + f_1 \in X_0 + X_1$, $f_i \in X_i$, $\|Tf_i\|_{Y_i} \le M_i \|f_i\|_{X_i}$. Therefore, $K(t, Tf; \mathcal{D}) \le M_0 K(tM_1/M_0, f; \mathcal{C})$, and since $K(t,f)$ is increasing, while $K(t,f)/t$ is decreasing, we may take without loss of generality, $M_1/M_0 = 1$ so that

$$\text{(2.9)} \qquad K(t, Tf; \mathcal{D}) \le C K(t, f; \mathcal{C}).$$

Since $K(s,Tf)/s$ is decreasing we obtain

$$\left\{\int_0^{t^{1/\bar\lambda}} \left[s^{-\bar\theta_0} K(s,Tf;\mathcal{B})\right]^{\bar q_0} s^{-1} ds\right\}^{1/\bar q_0}$$

$$\geq K(t^{1/\bar\lambda},Tf;\mathcal{B})t^{-1/\bar\lambda}\left[\int_0^{t^{1/\bar\lambda}} s^{(1-\bar\theta_0)\bar q_0-1} ds\right]^{1/\bar q_0}$$

$$= CK(t^{1/\bar\lambda},Tf;\mathcal{B})t^{-\bar\theta_0/\bar\lambda} .$$

Also since $K(s,Tf)$ is increasing and $\bar\lambda = \bar\theta_1 - \bar\theta_0$ we get

$$t\left\{\int_{t^{1/\bar\lambda}}^{\infty} \left[s^{-\bar\theta_1} K(s,Tf;\mathcal{B})\right]^{\bar q_1} s^{-1} ds\right\}^{1/\bar q_1}$$

$$\geq K(t^{1/\bar\lambda},Tf;\mathcal{B})\, t\left[\int_{t^{1/\bar\lambda}}^{\infty} s^{-\bar\theta_1\bar q_1-1} ds\right]^{1/\bar q_1}$$

$$= CK(t^{1/\bar\lambda},Tf;\mathcal{B})t^{1-\bar\theta_1/\bar\lambda} = CK(t^{1/\bar\lambda},Tf;\mathcal{B})t^{-\bar\theta_0/\bar\lambda} .$$

Combining these inequalities we obtain

$$t^{-\bar{\theta}_0/\bar{\lambda}} K(t^{1/\bar{\lambda}},Tf;\mathcal{B}) \le C\left\{\left[\int_0^{t^{1/\bar{\lambda}}} (s^{-\bar{\theta}_0} K(s,Tf;\mathcal{B}))^{\bar{q}_0} s^{-1} ds\right]^{1/\bar{q}_0}\right.$$

$$\left. + t\left[\int_{t^{1/\bar{\lambda}}}^{\infty} (s^{-\bar{\theta}_1} K(s,Tf;\mathcal{B}))^{\bar{q}_1} s^{-1} ds\right]^{1/\bar{q}_1}\right\}$$

$$\le CK(t,Tf;\mathcal{D}) \le CK(t,f;\mathcal{C}) .$$

$$\le C\left\{\left[\int_0^{t^{1/\lambda}} (s^{-\theta_0} K(s,f;\mathcal{A}))^{q_0} s^{-1} ds\right]^{1/q_0}\right.$$

$$\left. + t\left[\int_{t^{1/\lambda}}^{\infty} (s^{-\theta_1} K(s,f;\mathcal{A}))^{q_1} s^{-1} ds\right]^{1/q_1}\right\} ,$$

where the second inequality and the last inequality follow from Theorem 1.16, while the third inequality follows from (2.9). Replacing $t^{1/\bar{\lambda}}$ by t in the inequality and then multiply the resulting inequality by $t^{\bar{\theta}_0} u(t)$ and integrating, we obtain

$$\left\{\int_0^\infty [u(t)K(t,Tf;\mathcal{B})]^q dt\right\}^{1/q}$$

$$\le C\left\{\int_0^\infty u(t)^q t^{\bar\theta_0 q}\left[\left(\int_0^{t^{\bar\lambda/\lambda}} (s^{-\theta_0}K(s,f;\mathcal{A}))^{q_0} s^{-1}ds\right)^{1/q_0}\right.\right.$$

$$\left.\left.+ t^{\bar\lambda}\left(\int_{t^{\bar\lambda/\lambda}}^\infty (s^{-\theta_1}K(s,f;\mathcal{A}))^{q_1} s^{-1}ds\right)^{1/q_1}\right]^q dt\right\}^{1/q}$$

$$\le C\left\{\left[\int_0^\infty u(t)^q t^{\bar\theta_0 q+1}\left(\int_0^{t^{\bar\lambda/\lambda}} (s^{-\theta_0}K(s,f;\mathcal{A}))^{q_0} s^{-1}ds\right)^{q/q_0} t^{-1}dt\right]^{1/q}\right.$$

$$\left.+\left[\int_0^\infty u(t)^q t^{\bar\theta_0 q+\bar\lambda q+1}\left(\int_{t^{\bar\lambda/\lambda}}^\infty (s^{-\theta_1}K(s,f;\mathcal{A}))^{q_1} s^{-1}ds\right)^{q/q_1} t^{-1}dt\right]^{1/q}\right\}$$

$\equiv C\{J_0 + J_1\}$, respectively. Here the last inequality follows from Minkowski's inequality if $1 \le q \le \infty$, and is trivial if $0 < q < 1$.

We now estimate J_0 and J_1 in the case $\lambda > 0$, using Lemma 1.6.

(a) If $\max(q_0,q_1) \le p \le q \lneq \infty$, then $1 \le p/q_0 \le q/q_0$ and $1 \le p/q_1 \le q/q_1$. On making the substitution $t^{\bar\lambda/\lambda} \to t$ in the integral of J_0, then an application of Lemma 1.6 shows for $1 < p/q_0$

$$J_0^{q_0} = C\left\{\int_0^\infty \left[u\left(t^{\lambda/\bar{\lambda}}\right)^{q_0} t^{\bar{\theta}_0 q_0(\lambda/\bar{\lambda})+q_0(\lambda/\bar{\lambda})/q}\right.\right.$$

$$\left.\left.\times \int_0^t \left(s^{-\theta_0} K(s,f;\mathcal{A})\right)^{q_0} s^{-1} ds\right]^{q/q_0} t^{-1} dt\right\}^{q_0/q}$$

$$\le C\left\{\int_0^\infty [v(t)K(t,f;\mathcal{A})]^p \, dt\right\}^{q_0/p} ,$$

if and only if

$$\sup_{s>0}\left\{\int_s^\infty u\left(t^{\lambda/\bar{\lambda}}\right)^q t^{\bar{\theta}_0 q(\lambda/\bar{\lambda})+\lambda/\bar{\lambda}} t^{-1} dt\right\}^{q_0/q} \times$$

$$\left\{\int_0^s \left[v(t)t^{1/p+\theta_0}\right]^{-pq_0/(p-q_0)} t^{-1} dt\right\}^{(p-q_0)/p} < \infty .$$

For $1 = p/q_0$ the inequality holds if and only if

$$\sup_{s>0}\left\{\int_s^\infty u\left(t^{\lambda/\bar{\lambda}}\right)^q t^{\bar{\theta}_0 q(\lambda/\bar{\lambda})+\lambda/\bar{\lambda}} t^{-1} dt\right\}^{1/q} \operatorname*{ess\,sup}_{0<t<s}\left[v(t)t^{1/p+\theta_0}\right]^{-1} < \infty .$$

Replacing $t^{\lambda/\bar{\lambda}}$ by t, we see that these conditions are equivalent to (2.4).

Similarly, the substitution $t^{\bar{\lambda}/\lambda}$ by t in the integral of J_1 and an application of Lemma 1.6 yields

$$J_1^{q_1} = C\left\{\int_0^\infty\left[u\left(t^{\lambda/\bar{\lambda}}\right)^{q_1} t^{\lambda q_1+\bar{\theta}_0 q_1(\lambda/\bar{\lambda})+q_1(\lambda/\bar{\lambda})/q}\right.\right.,$$

$$\left.\left.\times \int_t^\infty\left(s^{-\theta_1}K(s,f;A)\right)^{q_1} s^{-1}ds\right]^{q/q_1} t^{-1}dt\right\}^{q_1/q}$$

$$\le C\left\{\int_0^\infty[v(t)K(t,f;A)]^p dt\right\}^{q_1/p},$$

if and only if

$$\sup_{s>0}\left\{\int_0^s u\left(t^{\lambda/\bar{\lambda}}\right)^q t^{\lambda q+\bar{\theta}_0 q(\lambda/\bar{\lambda})+\lambda/\bar{\lambda}} t^{-1}dt\right\}^{1/q} \times$$

$$\left\{\int_s^\infty\left[v(t)t^{1/p+\theta_1}\right]^{-pq_1/(p-q_1)} t^{-1}dt\right\}^{(p-q_1)/(pq_1)} < \infty,$$

for $1 < p/q_1$. In case $1 = p/q_1$, the inequality above holds if and only if

$$\sup_{s>0}\left\{\int_0^s u\left(t^{\lambda/\bar\lambda}\right)^{\bar q} t^{\lambda q+\bar\theta_0 q(\lambda/\bar\lambda)+\lambda/\bar\lambda}\, t^{-1}dt\right\}^{1/q} \operatorname*{ess\,sup}_{s<t<\infty}\left[v(t)t^{1/p+\theta_1}\right]^{-1} < \infty$$

Replacing $t^{\lambda/\bar\lambda}$ by t in the first integrals we see that these conditions are equivalent to (2.5).

If $q = \infty$ the argument is the same and hence omitted. This proves the theorem for (a).

(b) Now $q_1 \le p \le q \le q_0 < \infty$ and recall that

(2.10)
$$J_0 = C\left\{\int_0^\infty\left[u\left(t^{\lambda/\bar\lambda}\right)^{q_0} t^{\bar\theta_0 q(\lambda/\bar\lambda)+q_0(\lambda/\bar\lambda)/q}\right. \times\right.$$

$$\left.\left.\int_0^t\left(s^{-\theta_0} K(s,f;\mathcal{A})\right)^{q_0} s^{-1}ds\right]^{q/q_0} t^{-1}dt\right\}^{1/q} .$$

But by Lemma 1.8 the inner integral of (2.10) is

$$\left\{\int_0^t\left[s^{1-\theta_0} K(s,f;\mathcal{A})/s\right]^{q_0} s^{-1}ds\right\}^{1/q_0} \le C\left\{\int_0^t\left[s^{-\theta_0} K(s,f;\mathcal{A})\right]^{p} s^{-1}ds\right\}^{1/p} ,$$

so that

$$J_0^p \le C\left\{\int_0^\infty \left[u\left(t^{\lambda/\bar\lambda}\right)^p t^{\bar\theta_0 p(\lambda/\bar\lambda)+p(\lambda/\bar\lambda)/q}\right.\right.$$

$$\left.\left.\times \int_0^t \left(s^{-\theta_0}K(s,f;\mathcal{A})\right)^{\bar p} s^{-1}ds\right]^{q/p} t^{-1}dt\right\}^{p/q}$$

$$\le C\left\{\int_0^\infty [v(t)K(t,f;\mathcal{A})]^p dt\right\} .$$

Here the last inequality follows from Lemma 1.6 if and only if

$$\sup_{s>0}\left\{\int_s^\infty u\left(t^{\lambda/\bar\lambda}\right)^q t^{\bar\theta_0 q(\lambda/\bar\lambda)+\lambda/\bar\lambda} t^{-1}dt\right\}^{1/q} \operatorname*{ess\,sup}_{0<t<s}\left[v(t)t^{1/p+\theta_0}\right]^{-1} < \infty .$$

Replacing $t^{\lambda/\bar\lambda}$ by t we see that this is again equivalent to (2.6).

If $q_0 = +\infty$ the argument is the same, since Lemma 1.6 still applies.

For J_1 we have by Lemma 1.6

$$J_1^{q_1} = C\left\{\int_0^\infty \left[u\left(t^{\lambda/\bar\lambda}\right)^{q_1} t^{\lambda q_1+\bar\theta_0 q_1(\lambda/\bar\lambda)+q_1(\lambda/\bar\lambda)/q}\right.\right.$$

$$\times \int_t^{\infty} \left(s^{-\theta_1} K(s,f;\mathcal{A}) \right)^{q_1} s^{-1} ds \Bigg]^{q/q_1} t^{-1} dt \Bigg\}^{q_1/q}$$

$$\leq C \left\{ \int_0^{\infty} [v(t) K(t,f;\mathcal{A})]^p dt \right\}^{q_1/p} ,$$

if and only if (2.5) holds.

This proves the theorem for (b).

(c) Now if $q_0 \leq p \leq q \leq q_1 < \infty$, then the estimate of J_0 is the same as in (a), provided (2.4) holds.

Since

$$J_1 = C \left\{ \int_0^{\infty} \left[u\left(t^{\lambda/\bar{\lambda}}\right)^{q_1} t^{\lambda q_1 (\lambda/\bar{\lambda}) + q_1 (\lambda/\bar{\lambda})/q} \right. \right.$$

$$\times \int_t^{\infty} \left(s^{-\theta_1} K(s,f;\mathcal{A}) \right)^{q_1} s^{-1} ds \Bigg]^{q/q_1} t^{-1} dt \Bigg\}^{1/q} ,$$

we apply Lemma 1.9(i) with $s = p/q_1$ to the inner integral, so that

$$\int_t^\infty \left[s^{-\theta_1} K(s,f;\mathcal{A})\right]^{q_1} s^{-1}ds = \int_t^\infty K(s,f;\mathcal{A})^{q_1}\, d\left(-\int_s^\infty x^{-\theta_1 q_1 -1}dx\right)$$

$$\le (q_1/p)\left\{\int_t^\infty K(s,f;\mathcal{A})^p\, d\left[-\left(\int_s^\infty x^{-\theta_1 q_1 -1}dx\right)^{p/q_1}\right]\right\}^{q_1/p}$$

$$= C\left\{\int_t^\infty s^{-\theta_1 p} K(s,f;\mathcal{A})^p\, s^{-1}ds\right\}^{q_1/p} .$$

Therefore

$$J_1^p \le C\left\{\int_0^\infty \left[u\left(t^{\lambda/\bar\lambda}\right)^p t^{\lambda p+\bar\theta_0 p(\lambda/\bar\lambda)+p(\lambda/\bar\lambda)/q}\right.\right.$$

$$\left.\left.\times \int_t^\infty \left(s^{-\theta_1} K(s,f;\mathcal{A})\right)^p s^{-1}ds\right]^{q/p} t^{-1}dt\right\}^{p/q}$$

$$\le C\left\{\int_0^\infty [v(t)K(t,f;\mathcal{A})]^p dt\right\} ,$$

where the last inequality follows from Lemma 1.6, if and only if

$$\sup_{s>0}\left\{\int_0^s u\left(t^{\lambda/\bar\lambda}\right)^q t^{\lambda q+\bar\theta_0 q(\lambda/\bar\lambda)+\lambda/\bar\lambda}\, t^{-1}dt\right\}^{1/q} \operatorname*{ess-sup}_{s<t<\infty}\left[v(t)t^{1/p+\theta_1}\right]^{-1} < \infty .$$

But this is clearly equivalent to (2.7).

The case $q_1 = \infty$ is similar and hence omitted.

If $\lambda < 0$ the arguments are essentially the same only now the ranges of the first integrals in (2.4) - (2.7) are reversed.

This completes the proof of the theorem.

Theorem 2.1 of course shows that in case $q_i = \bar{q}_i = 1$, $i = 0, 1$, the conclusion holds if (2.4) and (2.5) are satisfied. That is, (2.8) is valid if both

$$(2.11) \quad \sup_{s>0} \left\{ \int_{s^{\lambda/\bar{\lambda}}}^{\infty} \left[u(t) t^{\bar{\theta}_0} \right]^q dt \right\}^{1/q} \left\{ \int_0^s \left[v(t) t^{1+\theta_0} \right]^{-p'} dt \right\}^{1/p'} < \infty$$

and

$$(2.12) \quad \sup_{s>0} \left\{ \int_0^{s^{\lambda/\bar{\lambda}}} \left[u(t) t^{\bar{\theta}_1} \right]^q dt \right\}^{1/q} \left\{ \int_s^{\infty} \left[v(t) t^{1+\theta_1} \right]^{-p'} dt \right\}^{1/p'} < \infty ,$$

hold for $\lambda > 0$. (If $\lambda < 0$ one simply interchanges the limits of integration of the first two integrals in (2.11) and (2.12).)

The next theorem shows that (2.11) and (2.12) are sufficient to insure (2.8) in the case $q_i, \bar{q}_i > 1$, $i = 0, 1$.

<u>Theorem 2.2</u>. If T satisfies the hypotheses of Theorem 2.1 with $1 \le q_i, \bar{q}_i \le \infty$, then for $\lambda > 0$, (2.11) and (2.12) imply (2.8).

In case $\lambda < 0$, then again (2.8) holds provided in the weight conditions (2.11) and (2.12) the ranges of integrations of the first two integrals are interchanged.

<u>Proof</u>. Let $\lambda > 0$.

As in the proof of Theorem 2.1 one obtains the following inequality

$$\left\{\int_0^\infty [u(t)K(t,Tf;\mathcal{B})]^q dt\right\}^{1/q}$$

$$\le C\left\{\left[\int_0^\infty u\left(t^{\lambda/\bar{\lambda}}\right)^q t^{\bar{\theta}_0 q\lambda/\bar{\lambda}+\lambda/\bar{\lambda}}\left(\int_0^t \left(s^{-\theta_0}K(s,f;\mathcal{A})\right)^{q_0} s^{-1}ds\right)^{q/q_0} t^{-1}dt\right]^{1/q}\right.$$

$$\left.+\left[\int_0^\infty u\left(t^{\lambda/\bar{\lambda}}\right)^q t^{\lambda q+\bar{\theta}_0 q\lambda/\bar{\lambda}+\lambda/\bar{\lambda}}\left(\int_t^\infty \left(s^{-\theta_1}K(s,f;\mathcal{A})\right)^{q_1} s^{-1}ds\right)^{q/q_1} t^{-1}dt\right]^{1/q}\right\}.$$

But by Lemma 1.8 the inner integral of the first term shows that

$$(2.13)\quad \left\{\int_0^t \left[s^{-\theta_0}K(s,f;\mathcal{A})\right]^{q_0} s^{-1}ds\right\}^{1/q_0} \le C\int_0^t s^{-\theta_0-1}K(s,f;\mathcal{A})ds \;,$$

$1 \le q_0 \le \infty$.

Similarly, Lemma 1.9(i) applied to the inner integral of the second term yields

$$\int_t^\infty \left[s^{-\theta_1}K(s,f;\mathcal{A})\right]^{q_1} s^{-1}ds \le C\left\{\int_t^\infty s^{-\theta_1-1}K(s,f;\mathcal{A})ds\right\}^{1/q_1} ,$$

$1 \le q_1 \le \infty$. Therefore

$$(2.14)\quad \left\{\int_t^\infty \left[s^{-\theta_1}K(s,f;\mathcal{A})\right]^{q_1} s^{-1}ds\right\}^{1/q_1} \le C\int_t^\infty s^{-\theta_1-1}K(s,f;\mathcal{A})ds \;.$$

From (2.13) and (2.14) we have

$$\left\{\int_0^\infty [u(t)K(t,Tf;\mathcal{B})]^q dt\right\}^{1/q}$$

$$\le C\left\{\left[\int_0^\infty \left[u(t^{\lambda/\bar{\lambda}})t^{\bar{\theta}_0\lambda/\bar{\lambda}+\lambda/(\bar{\lambda}q)}\int_0^t s^{-\theta_0}K(s,f;\mathcal{A})s^{-1}ds\right]^q t^{-1}dt\right]^{1/q}\right.$$

$$+\left[\int_0^\infty\left[u(t^{\lambda/\bar\lambda})t^{\lambda+\bar\theta_0\lambda/\bar\lambda+\lambda/(\bar\lambda q)}\int_t^\infty s^{-\theta_1}K(s,f;A)s^{-1}ds\right]^q t^{-1}dt\right]^{1/q}\Bigg\}$$

$\equiv C\{I_0 + I_1\}$, respectively.

Now by Lemma 1.6

$$I_0 \le C\left\{\int_0^\infty [v(t)K(t,f;A)]^p dt\right\}^{1/p},$$

if and only if

$$\sup_{s>0}\left\{\int_s^\infty u\left(t^{\lambda/\bar\lambda}\right)^q t^{\bar\theta_0\lambda/\bar\lambda+\lambda/\bar\lambda} t^{-1}dt\right\}^{1/q}\left\{\int_0^s\left[v(t)t^{1/p+\theta_0}\right]^{-p'} t^{-1}dt\right\}^{1/p'} < \infty .$$

Replacing $t^{\lambda/\bar\lambda}$ by t one gets

$$\sup_{s>0}\left\{\int_{s^{\lambda/\bar\lambda}}^\infty\left[u(t)t^{\bar\theta_0}\right]^q dt\right\}^{1/q}\left\{\int_0^s\left[v(t)t^{1/p+\theta_0}\right]^{-p'} t^{-1}dt\right\}^{1/p'} < \infty ,$$

which is equivalent to (2.11).

Similarly

$$I_1 \le C\left\{\int_0^\infty [v(t)K(t,f;A)]^p dt\right\}^{1/p},$$

if and only if

$$\sup_{s>0}\left\{\int_0^s u\left(t^{\lambda/\bar{\lambda}}\right)^q t^{\lambda q+\bar{\theta}_0 q\lambda/\bar{\lambda}+\lambda/\bar{\lambda}} t^{-1} dt\right\}^{1/q} \times$$

$$\left\{\int_s^\infty \left[v(t)t^{1/p+\theta_1}\right]^{-p'} t^{-1} dt\right\}^{1/p'} < \infty$$

and replacing $t^{\lambda/\bar{\lambda}}$ by t in the first integral this implies

$$\sup_{s>0}\left\{\int_0^{s^{\lambda/\bar{\lambda}}} \left[u(t)t^{\bar{\theta}_1}\right]^q dt\right\}^{1/q} \left\{\int_s^\infty \left[v(t)t^{1/p+\theta_1}\right]^{-p'} t^{-1} dt\right\}^{1/p'} < \infty,$$

which is equivalent to (2.12).

If $\lambda < 0$ the result also holds under the obvious modifications.

This completes the proof of the theorem.

We now give two examples of pairs of weights for which Theorem 2.2 holds.

Example 2.3. The weight functions $u(t) = t^{-(\bar{\theta}_0+\alpha+1/q)}$, $v(t) = t^{-(\theta_0+\lambda\alpha/\bar{\lambda}+1/p)}$, $\lambda = \theta_1 - \theta_0 > 0$, $\bar{\lambda} = \bar{\theta}_1 - \bar{\theta}_0 > 0$, $0 < \alpha < \bar{\lambda}$ satisfy the weight conditions (2.11) and (2.12) for $1 < p \le q < \infty$.

Example 2.4. $u(t) = e^{-t}t^{-\bar{\theta}_0}$, $v(t) = t^{-(1+\theta_0)}e^{t}$, $\lambda = \theta_1 - \theta_0 < 0$, $\bar{\lambda} = \bar{\theta}_1 - \bar{\theta}_0 > 0$ satisfy (2.11) and (2.12) and hence satisfy Theorem 2.2.

2.2 The case $1 \le q < p < \infty$.

The purpose of this section is to prove the result of Theorem 2.2 for $1 \le q < p < \infty$.

Theorem 2.5. Suppose $\mathcal{A} = (A_0,A_1)$, $\mathcal{B} = (B_0,B_1)$ are two interpolation couples, $0 < \theta_i$, $\bar{\theta}_i < 1$, $1 \le q_i$, $\bar{q}_i \le \infty$, $i = 0, 1$. Let $\lambda = \theta_1 - \theta_0 \ne 0$, $\bar{\lambda} = \bar{\theta}_1 - \bar{\theta}_0 > 0$ and $T: \mathcal{A}_{\theta_i,q_i} \to \mathcal{B}_{\bar{\theta}_i,\bar{q}_i}$, be a quasi-linear operator. If $\lambda > 0$, $1 \le q < p < \infty$ and $1/r = 1/q - 1/p$; u, v non-negative weights satisfying

$$(2.15)\qquad \left\{\int_0^\infty \left[\left(\int_{s^{\lambda/\bar{\lambda}}}^\infty \left[u(t)t^{\bar{\theta}_0}\right]^q dt\right)^{1/q} \left(\int_0^s \left[v(t)t^{1+\theta_0}\right]^{-p'} dt\right)^{1/q'}\right]^r \times\right.$$

$$\left(v(s)s^{1+\theta_0}\right)^{-p'} ds\Bigg\}^{1/r} < \infty$$

and

$$\text{(2.16)} \quad \left\{\int_0^\infty \left[\left(\int_0^{s^{\lambda/\bar\lambda}} \left[u(t)t^{\bar\theta_1}\right]^q dt\right)^{1/q} \left(\int_s^\infty \left[v(t)t^{1+\theta_1}\right]^{-p'} dt\right)^{1/q'}\right]^r \times\right.$$

$$\left(v(s)s^{1+\theta_1}\right)^{-p'} ds\Bigg\}^{1/r} < \infty ,$$

then (2.8) holds.

If $\lambda < 0$, (2.8) still holds provided the ranges of the first inner integrals in (2.15) and (2.16) are changed from $(s^{\lambda/\bar\lambda},\infty)$ to $(0,s^{\lambda/\bar\lambda})$ and from $(0,s^{\lambda/\bar\lambda})$ to $(s^{\lambda/\bar\lambda},\infty)$. (In case $q = 1 < p$, the conditions (2.15) and (2.16) take the forms

$$\left\{\int_0^\infty \left(\int_{s^{\lambda/\bar\lambda}}^\infty u(t)t^{\bar\theta_0}dt\right)^{p'} \left(v(s)s^{1+\theta_0}\right)^{-p'} ds\right\}^{1/p'} < \infty ,$$

respectively

$$\left\{\int_0^\infty \left(\int_0^{s^{\lambda/\bar\lambda}} u(t)t^{\bar\theta_1}dt\right)^{p'} \left(v(s)s^{1+\theta_1}\right)^{-p'} ds\right\}^{1/p'} < \infty ,$$

whenever $\lambda > 0$.

If $\lambda < 0$, then the inner integrals in the two last weight conditions are to be taken from 0 to $s^{\lambda/\bar\lambda}$, respectively from $s^{\lambda/\bar\lambda}$ to ∞.) .

__Proof.__ Proceeding as in the proof of Theorem 2.2, it follows that

$$\left\{\int_0^\infty [u(t)K(t,Tf;\mathcal{B})]^q dt\right\}^{1/q}$$

$$\le C\left\{\left[\int_0^\infty \left[u(t^{\lambda/\bar\lambda})t^{\bar\theta_0\lambda/\bar\lambda+\lambda/(\bar\lambda q)} \int_0^t s^{-\theta_0}K(s,f;\mathcal{A})s^{-1}ds\right]^q t^{-1}dt\right]^{1/q}\right.$$

$$\left.+\left[\int_0^\infty \left[u(t^{\lambda/\bar\lambda})t^{\lambda+\bar\theta_0\lambda/\bar\lambda+\lambda/(\bar\lambda q)} \int_t^\infty s^{-\theta_1}K(s,f;\mathcal{A})s^{-1}ds\right]^q t^{-1}dt\right]^{1/q}\right\}$$

$\equiv C\{I_0 + I_1\}$, respectively. Now using Lemma 1.7(a) and (b) one estimates I_0 and I_1 to obtain the result.

We omit the details.

Remark 2.6. The weight functions $u(t) = t^{-(\bar{\theta}_0+\alpha+1/q)}$, $v(t) = t^{-(1+\theta_0)}e^t$, $0 < \alpha < \min(\bar{\lambda}/(r\lambda), \bar{\lambda})$, $1 < q < p < \infty$, $1/r = 1/q - 1/p$, λ and $\bar{\lambda}$ as before satisfy (2.15) and (2.16) and therefore the conclusion of Theorem 2.5.

The calculations are straightforward and we omit the details.

2.3 Weighted Intermediate Spaces.

Suppose (A_0,A_1) is an interpolation couple, then instead of constructing the intermediate spaces $(A_0,A_1)_{\theta_i,q_i}$, $i = 0, 1$ as in the previous section, we follow Kalugina [24] and Gustavsson [13] and define interpolation spaces $(A_0,A_1)_{w_i,q_i}$, where w_i are weight functions of a certain function class B_Ψ:

These interpolation spaces are larger than those considered previously, since $w_i(t) = t^{\theta_i}$ is a function in B_Ψ. Before we define these interpolation spaces and give the main result (Theorem 2.12) we discuss the weight classes B_K and B_Ψ.

Definition 2.7 ([24]). A continuous non-decreasing function $w: \mathbb{R}^+ \to \mathbb{R}^+$ belongs to the class B_K, if

$$\int_0^\infty \min(1,1/t)\tilde{w}(t)t^{-1}dt < \infty,$$

where $\tilde{w}(s) = \sup_{y>0} \frac{w(sy)}{w(y)}$ and $\tilde{w}(s) < \infty$ for $s > 0$.

<u>Proposition 2.8 ([13]).</u> If $w \in B_K$ then

(i) $\underset{\sim}{w}(s)\tilde{w}(1/s) = 1$, where $\underset{\sim}{w}(s) = \inf_{t>0} \frac{w(st)}{w(t)}$.

(ii) $0 < \underset{\sim}{w}(s)w(t) \le w(st) \le \tilde{w}(s)w(t)$.

(iii) $\tilde{w}$ and $\underset{\sim}{w}$ are non-decreasing and $\tilde{w}(1) = \underset{\sim}{w}(1) = 1$.

(iv) For any $p > 0$.

$$\left\{\int_0^\infty [\min(1,1/t)\tilde{w}(t)]^p \, t^{-1}dt\right\}^{1/p} < \infty ,$$

with the usual modification if $p = \infty$.

(v) There exist constants $A, B > 0$ such that for all $s > 0$,

$$A \le s^{-1}w(s)\left\{\int_0^s [t/w(t)]^p \, t^{-1}dt\right\}^{1/p} \le B, \qquad p > 0 .$$

In fact $A = p^{-1/p}$ and $B = \left\{\int_0^\infty [\tilde{w}(t)/t]^p \, t^{-1}dt\right\}^{1/p}$, if $p < \infty$.

(vi) There are positive constants C, D such that

$$C \le w(s)\left\{\int_s^\infty [1/w(t)]^p\, t^{-1}dt\right\}^{1/p} \le D\,, \qquad p > 0\,.$$

Here $C = \left\{\int_1^\infty [1/\tilde{w}(t)]^p\, t^{-1}dt\right\}^{1/p}$ and $D = \left\{\int_0^1 \tilde{w}(t)^p t^{-1}dt\right\}^{1/p}$, if $p < \infty$.

Clearly, if $0 < \theta < 1$ then $w(t) = t^\theta \in B_K$. Also, as Gustavsson has shown, the function $t^\beta/\log(1+t^\alpha) \in B_K$ if $0 < \alpha < \beta < 1$.

The weight class B_Ψ is now defined as follows:

<u>Definition 2.9</u>. B_Ψ consists of all non-negative continuously differentiable functions w on R^+ such that

$$\sup_{t>0} tw'(t)/w(t) = \beta < 1 \quad \text{and} \quad \inf_{t>0} tw'(t)/w(t) = \alpha > 0\,.$$

It is not difficult to see that $B_\Psi \subset B_K$.

Again $w(t) = t^\theta$, $0 < \theta < 1$ is in B_Ψ and also $w(t) = t^\alpha(\log(1+t^\gamma))^\theta$, $0 < \alpha < 1$, θ real and γ in a sufficiently small neighbourhood of zero (Gustavsson [13]).

<u>Definition 2.10</u>. Let $w \in B_\Psi$ and (A_0,A_1) an interpolation couple. The weighted intermediate spaces $\mathcal{A}_{w,p} = (A_0,A_1)_{w,p}$, $0 < p \le \infty$ with

weight w consist of all $f \in A_0 + A_1$ such that

$$\|f\|_{\mathcal{A}_{w,p}} = \begin{cases} \left\{ \int_0^\infty [K(t,f;A_0,A_1)/w(t)]^p \, t^{-1} dt \right\}^{1/p}, & 0 < p < \infty \\ \sup_{t>0} K(t,f;A_0,A_1)/w(t), & p = \infty \end{cases}$$

is finite.

Remark 2.11. If $w_i \in B_\Psi$, $i = 0, 1$ and $\tau(t) = w_1(t)/w_0(t)$ satisfying $t\tau'(t)/\tau(t) \geq \alpha > 0$ for all $t > 0$, then τ has an inverse and $\lim_{t\to 0} \tau(t) = 0$, $\lim_{t\to\infty} \tau(t) = \infty$.

Clearly the above inequality implies $\tau'(t) > 0$ so τ has an inverse. Also if $0 < t < 1$ then the inequality implies

$$\log \tau(1) - \log \tau(t) = \int_t^1 \tau'(s)/\tau(s)ds \geq \alpha \int_t^1 s^{-1} ds = -\alpha \log t$$

which implies $\tau(t) \leq \tau(1)t^\alpha$ so that $\tau(t) \to 0$ as $t \to 0$. Similarly, if $1 < t < \infty$ then

$$\log(\tau(t)/\tau(1)) = \int_1^t \tau'(s)/\tau(s)ds \geq \alpha \log t$$

which implies $\tau(t) \to \infty$ as $t \to \infty$.

The main result of this section is the following theorem:

Theorem 2.12. Suppose $\mathcal{A} = (A_0,A_1)$, $\mathcal{B} = (B_0,B_1)$ are interpolation couples, $w_i,\bar{w}_i \in B_\psi$, $i = 0, 1$ and $\tau = w_1/w_0$, $\bar{\tau} = \bar{w}_1/\bar{w}_0$ satisfy $|t\tau'(t)/\tau(t)| \geq \alpha > 0$ and $t\bar{\tau}'(t)/\bar{\tau}(t) \geq \bar{\alpha} > 0$. Further, let $\eta = \tau^{-1}$, $\bar{\eta} = \bar{\tau}^{-1}$, $\sigma = \bar{\eta}\circ\tau$. Assume T is a quasi-linear operator satisfying

$$T: \mathcal{A}_{w_i,q_i} \to \mathcal{B}_{\bar{w}_i,\bar{q}_i},$$

$i = 0, 1$, $1 \leq q_i, \bar{q}_i \leq \infty$.

In case $t\tau'(t)/\tau(t) \geq \alpha > 0$ and the weights u, v satisfy

$$(2.17) \quad \sup_{s>0} \left\{ \int_{\sigma(s)}^{\infty} [u(t)\bar{w}_0(t)]^q dt \right\}^{1/q} \left\{ \int_0^s [v(t)tw_0(t)]^{-p'} dt \right\}^{1/p'} < \infty$$

and

$$(2.18)\quad \sup_{s>0}\left\{\int_0^{\sigma(s)}[u(t)w_1(t)]^q dt\right\}^{1/q}\left\{\int_s^{\infty}[v(t)tw_1(t)]^{-p'}dt\right\}^{1/p'} < \infty$$

for $1 \le p \le q \le \infty$, then there is a $C > 0$ such that

$$(2.19)\quad \left\{\int_0^{\infty}[u(t)K(t,Tf;\mathcal{B})]^q dt\right\}^{1/q} \le C\left\{\int_0^{\infty}[v(t)K(t,f;\mathcal{A})]^p dt\right\}^{1/p}.$$

If $-t\tau'(t)/\tau(t) \ge \alpha > 0$, (2.19) holds provided the ranges of integrations of the first two integrals in (2.17) and (2.18) are interchanged.

<u>Proof.</u> The proof is quite similar to those of the previous two theorems, only now we utilize instead of Holmstedt's theorem (Theorem 1.16) the following result of Heinig [16]:

Let $X_i = \mathcal{A}_{w_i,q_i}$, $i = 0, 1$; $(X_0,X_1) = \mathcal{D}$, then

$$(2.20)\quad K(t,f;\mathcal{D}) \sim \left\{\int_0^{\eta(t)}[K(s,f;\mathcal{A})/w_0(s)]^{q_0}\, s^{-1}ds\right\}^{1/q_0}$$

$$+\, t\left\{\int_{\eta(t)}^{\infty}[K(s,f;\mathcal{A})/w_1(s)]^{q_1}\, s^{-1}ds\right\}^{1/q_1}.$$

Writing $Y_i = \mathcal{B}_{\bar{w}_i,\bar{q}_i}$, $i = 0, 1$ and $(Y_0,Y_1) = \mathcal{E}$, this shows that

$$(2.21) \qquad K(t,f;\mathcal{E}) \sim \left\{ \int_0^{\bar{\eta}(t)} [K(s,f;\mathcal{B})/\bar{w}_0(s)]^{\bar{q}_0} s^{-1} ds \right\}^{1/\bar{q}_0} + t \left\{ \int_{\bar{\eta}(t)}^{\infty} [K(s,f;\mathcal{B})/\bar{w}_1(s)]^{\bar{q}_1} s^{-1} ds \right\}^{1/\bar{q}_1} .$$

Because some of the details are different from the proof of the previous two theorems we continue with the proof.

Since $T: X_i \to Y_i$ is bounded, there exist numbers $M_i > 0$ such that, $\|Tf_i\|_{Y_i} \le M_i \|f_i\|_{X_i}$, $f_i \in X_i$, $i = 0, 1$ and therefore we have $K(t,Tf;\mathcal{E}) \le M_0 K(tM_1/M_0, f;\mathcal{D})$. Again we may take without loss of generality $M_1/M_0 = 1$, so that

$$(2.22) \qquad K(t,Tf;\mathcal{E}) \le C\, K(t,f;\mathcal{D}) .$$

We now prove the estimates

$$K(\bar{\eta}(t),Tf;\mathcal{B})/\bar{w}_0(\bar{\eta}(t)) \le C \left\{ \int_0^{\bar{\eta}(t)} [K(s,Tf;\mathcal{B})/\bar{w}_0(s)]^{\bar{q}_0} s^{-1} ds \right\}^{1/\bar{q}_0}$$

and

$$K(\bar{\eta}(t),Tf;\mathcal{B})/\bar{w}_0(\bar{\eta}(t)) \le Ct\left\{\int_{\bar{\eta}(t)}^{\infty} [K(s,Tf;\mathcal{B})/\bar{w}_1(s)]^{\bar{q}_1} s^{-1}ds\right\}^{1/\bar{q}_1} .$$

From Proposition 2.8(v) with p, s and w(s) are replaced by $\bar{q}_0$, $\bar{\eta}(s)$ and $\bar{w}_0(s)$, respectively, we get

$$A \le \bar{w}_0(\bar{\eta}(s))/\bar{\eta}(s)\left\{\int_0^{\bar{\eta}(s)} [t/\bar{w}_0(t)]^{\bar{q}_0} t^{-1}dt\right\}^{1/\bar{q}_0} \le B ,$$

that is

$$\int_0^{\bar{\eta}(s)} [t/\bar{w}_0(t)]^{\bar{q}_0} t^{-1}dt \ge C[\bar{\eta}(s)/\bar{w}_0(\bar{\eta}(s))]^{\bar{q}_0} .$$

Since K(s,Tf)/s is decreasing then the previous inequality shows that

$$\int_0^{\bar{\eta}(t)} [K(s,Tf;\mathcal{B})/\bar{w}_0(s)]^{\bar{q}_0} s^{-1}ds$$

$$\ge C[K(\bar{\eta}(t),Tf;\mathcal{B})/\bar{\eta}(t)]^{\bar{q}_0} \int_0^{\bar{\eta}(t)} [s/\bar{w}_0(s)]^{\bar{q}_0} s^{-1}ds$$

$$\geq C[K(\bar{\eta}(t),Tf;\mathcal{B})/\bar{w}_0(\bar{\eta}(t))]^{\bar{q}_0}$$

and therefore

$$(2.23)\quad K(\bar{\eta}(t),Tf;\mathcal{B})/\bar{w}_0(\bar{\eta}(t)) \leq C\left\{\int_0^{\bar{\eta}(t)} [K(s,Tf;\mathcal{B})/\bar{w}_0(s)]^{\bar{q}_0}\, s^{-1}ds\right\}^{1/\bar{q}_0}.$$

Similarly, from (vi) of Proposition 2.8 with p, s and w(s) are replaced by $\bar{q}_1$, $\bar{\eta}(s)$ and $\bar{w}_1(s)$, respectively, we obtain

$$C \leq \bar{w}_1(\bar{\eta}(s))\left\{\int_{\bar{\eta}(s)}^{\infty} [1/\bar{w}_1(t)]^{\bar{q}_1}\, t^{-1}dt\right\}^{1/\bar{q}_1} \leq D,$$

that is

$$\int_{\bar{\eta}(s)}^{\infty} [1/\bar{w}_1(t)]^{\bar{q}_1}\, t^{-1}dt \geq C[1/\bar{w}_1(\bar{\eta}(s))]^{\bar{q}_1}.$$

Also since K(t,Tf) is decreasing we have

$$\int_{\bar{\eta}(t)}^{\infty} [K(s,Tf;\mathcal{B})/\bar{w}_1(s)]^{\bar{q}_1}\, s^{-1}ds$$

$$\geq K(\bar{\eta}(t),Tf;\mathcal{B})^{\bar{q}_1} \int_{\bar{\eta}(t)}^{\infty} [1/\bar{w}_1(s)]^{\bar{q}_1}\, s^{-1}ds$$

$$\geq C[K(\bar{\eta}(t),Tf;\mathcal{B})/\bar{w}_1(\bar{\eta}(t))]^{\bar{q}_1}$$

and hence we obtain

$$K(\bar{\eta}(t),Tf;\mathcal{B})/\bar{w}_1(\bar{\eta}(t)) \leq C\left\{\int_{\bar{\eta}(t)}^{\infty} [K(s,Tf;\mathcal{B})/\bar{w}_1(s)]^{\bar{q}_1}\, s^{-1}ds\right\}^{1/\bar{q}_1}.$$

But $\bar{\tau}(t) = \bar{w}_1(t)/\bar{w}_0(t)$, $(t = \bar{\tau}(\bar{\eta}(t)) = \bar{w}_1(\bar{\eta}(t))/\bar{w}_0(\bar{\eta}(t)))$, thus one gets

(2.24) $$K(\bar{\eta}(t),Tf;\mathcal{B})/\bar{w}_0(\bar{\eta}(t))$$

$$\leq C\,t\left\{\int_{\bar{\eta}(t)}^{\infty} [K(s,Tf;\mathcal{B})/\bar{w}_1(s)]^{\bar{q}_1}\, s^{-1}ds\right\}^{1/\bar{q}_1}.$$

By combining (2.23) and (2.24) one obtains

$$K(\bar{\eta}(t),Tf;\mathcal{B})/\bar{w}_0(\bar{\eta}(t)) \leq C\left\{\left[\int_0^{\bar{\eta}(t)} [K(s,Tf;\mathcal{B})/\bar{w}_0(s)]^{\bar{q}_0}\, s^{-1}ds\right]^{1/\bar{q}_0}\right.$$

$$+ t\left[\int_{\bar{\eta}(t)}^{\infty}[K(s,Tf;\mathcal{B})/\bar{w}_1(s)]^{\bar{q}_1} s^{-1}ds\right]^{1/\bar{q}_1}\Bigg\}$$

$$\le C\,K(t,Tf;\mathcal{E}) \le C\,K(t,f;\mathcal{D})$$

$$\le C\Bigg\{\left[\int_0^{\eta(t)}[K(s,f;\mathcal{A})/w_0(s)]^{q_0} s^{-1}ds\right]^{1/q_0}$$

$$+ t\left[\int_{\eta(t)}^{\infty}[K(s,f;\mathcal{A})/w_1(s)]^{q_1} s^{-1}ds\right]^{1/q_1}\Bigg\},$$

where the second, the third and the last inequalities follow from (2.21), (2.22) and (2.20), respectively. Hence we get

(2.25) $$K(\bar{\eta}(t),Tf;\mathcal{B})/\bar{w}_0(\bar{\eta}(t))$$

$$\le C\Bigg\{\left[\int_0^{\eta(t)}[K(s,f;\mathcal{A})/w_0(s)]^{q_0} s^{-1}ds\right]^{1/q_0}$$

$$+ t\left[\int_{\eta(t)}^{\infty}[K(s,f;\mathcal{A})/w_1(s)]^{q_1} s^{-1}ds\right]^{1/q_1}\Bigg\}.$$

Lemma 1.9(ii) applied to the first integral of (2.25) and one gets

$$(2.26)\qquad \int_0^t [K(s,f;A)/w_0(s)]^{q_0}\, s^{-1}ds.$$

$$= \int_0^t [K(s,f;A)/s]^{q_0}\, d\left\{\int_0^s [x/w_0(x)]^{q_0}\, x^{-1}dx\right\}$$

$$\le C\left\{\int_0^t s^{-1}K(s,f;A)\, d\left[\int_0^s [x/w_0(x)]^{q_0}\, x^{-1}dx\right]^{1/q_0}\right\}^{q_0}$$

$$= C\left\{\int_0^t s^{-1}K(s,f;A)[s/w_0(s)]^{q_0}\left[\int_0^s [x/w_0(x)]^{q_0}x^{-1}dx\right]^{1/q_0-1} s^{-1}ds\right\}^{q_0}.$$

Next, from Proposition 2.8(v), it follows that

$$[s/w_0(s)]^{q_0} \le C\int_0^s [t/w_0(t)]^{q_0}\, t^{-1}dt$$

and since $1/q_0 - 1 < 0$ we get

$$\left\{\int_0^s [t/w_0(t)]^{q_0}\, t^{-1}dt\right\}^{1/q_0-1} \le C[s/w_0(s)]^{q_0(1/q_0-1)}\,.$$

Substituting this estimate in the inner integral in the right side of (2.26) one sees that (2.26) yields

$$(2.27)\qquad \left\{\int_0^t [K(s,f;\mathcal{A})/w_0(s)]^{q_0}\, s^{-1}ds\right\}^{1/q_0}$$

$$\le C\int_0^t [K(s,f;\mathcal{A})/w_0(s)]s^{-1}ds\,, \qquad 1 < q_0 < \infty\,.$$

Similarly, Lemma 1.9(i) applied to the second integral of (2.25) and we obtain

$$\int_t^\infty [K(s,f;\mathcal{A})/w_1(s)]^{q_1}\, s^{-1}ds = \int_t^\infty K(s,f;\mathcal{A})^{q_1}\, d\left(-\int_s^\infty [1/w_1(x)]^{q_1}\, x^{-1}dx\right)$$

$$\le C\left\{\int_t^\infty K(s,f;\mathcal{A})\; d\left[-\left(\int_s^\infty [1/w_1(x)]^{q_1}\, x^{-1}dx\right)^{1/q_1}\right]\right\}^{q_1}$$

$$= C\left\{\int_t^\infty K(s,f;\mathcal{A})[1/w_1(s)]^{q_1}\left(\int_s^\infty [1/w_1(x)]^{q_1}\, x^{-1}dx\right)^{1/q_1-1} s^{-1}ds\right\}^{q_1}$$

$$\le C\left\{\int_t^\infty [K(s,f;\mathcal{A})/w_1(s)]s^{-1}ds\right\}^{q_1},$$

where the last inequality follows from Proposition 2.8(vi). Thus

$$(2.28)\qquad \left\{\int_t^\infty [K(s,f;\mathcal{A})/w_1(s)]^{q_1} s^{-1}ds\right\}^{1/q_1}$$

$$\le C\int_t^\infty [K(s,f;\mathcal{A})/w_1(s)]s^{-1}ds\,, \qquad 1 < q_1 < \infty .$$

Therefore from (2.25), (2.27) and (2.28) we get

$$(2.29)\qquad K(\bar{\eta}(t),Tf;\mathcal{B})/\bar{w}_0(\bar{\eta}(t)) \le C\left\{\int_0^{\eta(t)} [K(s,f;\mathcal{A})/w_0(s)]s^{-1}ds\right.$$

$$\left. + t\int_{\eta(t)}^\infty [K(s,f;\mathcal{A})/w_1(s)]s^{-1}ds\right\},$$

$1 \le q_0$, $q_1 \le \infty$, where the case $q_0 = q_1 = 1$ follows directly from (2.25) and the case $q_0 = q_1 = \infty$, follows from these two estimations

$$\sup_{0\le s<t} K(s,f;\mathcal{A})/w_0(s) = \sup_{0<s<t} [K(s,f;\mathcal{A})/s]s/w_0(s)$$

$$\le \sup_{0<s<t} [K(s,f;\mathcal{A})/s] \int_0^s [x/w_0(x)]x^{-1}dx$$

$$\le \sup_{0<s<t} \int_0^s [K(x,f;\mathcal{A})/w_0(x)]x^{-1}dx$$

$$\le \int_0^t [K(x,f;\mathcal{A})/w_0(x)]x^{-1}dx$$

and

$$\sup_{t<s<\infty} K(s,f;\mathcal{A})/w_1(s) = \sup_{t<s<\infty} K(s,f;\mathcal{A})\int_s^\infty [1/w_1(x)]x^{-1}dx$$

$$\le \sup_{t<s<\infty} \int_s^\infty [K(x,f;\mathcal{A})/w_1(x)]x^{-1}dx$$

$$\le \int_t^\infty [K(x,f;\mathcal{A})/w_1(x)]x^{-1}dx .$$

Here we applied (v) and (vi) of Proposition 2.8 and the monotonicity properties of the K-functional.

On replacing t by $\bar{\tau}(t)$ in (2.29) and then on multiplying the resulting inequality by $u(t)\bar{w}_0(t)$ and by integrating, we get

$$\left\{\int_0^\infty [u(t)K(t,Tf;\mathcal{B})]^q dt\right\}^{1/q}$$

$$\le C\left\{\int_0^\infty u(t)^q \bar{w}_0(t)^q \left[\int_0^{\bar{\sigma}(t)} [K(s,f;\mathcal{A})/w_0(s)]s^{-1}ds\right.\right.$$

$$+ \bar{\tau}(t) \int_{\bar{\sigma}(t)}^{\infty} [K(s,f;\mathcal{A})/w_1(s)]s^{-1}ds \Bigg]^q dt \Bigg\}^{1/q} ,$$

where $\bar{\sigma}(t) = \eta(\bar{\tau}(t))$. From $t\tau'(t)/\tau(t) \geq \alpha > 0$, it follows that $\tau(t) \to 0$ as $t \to 0$; $\tau(t) \to \infty$ as $t \to \infty$ (Remark 2.11) and similarly, since $t\bar{\tau}'(t)/\bar{\tau}(t) \geq \bar{\alpha} > 0$, $\bar{\tau}(t) \to 0$ as $t \to 0$ and $\bar{\tau}(t) \to \infty$ as $t \to \infty$. The change of variable $\bar{\sigma}(t) = y$ $(t = \sigma(y)$, $dt = \sigma'(y)dy)$ on the right side of the above integral inequality yields

$$\left\{ \int_0^{\infty} [u(t)K(t,Tf;\mathcal{B})]^q dt \right\}^{1/q}$$

$$\leq C \Bigg\{ \int_0^{\infty} u(\sigma(y))^q \bar{w}_0(\sigma(y))^q \Bigg[\int_0^{y} [K(s,f;\mathcal{A})/w_0(s)]s^{-1}ds$$

$$+ \bar{\tau}(\sigma(y)) \int_y^{\infty} [K(s,f;\mathcal{A})/w_1(s)]s^{-1}ds \Bigg]^q \sigma'(y)dy \Bigg\}^{1/q} .$$

By Minkowski's inequality for $q \geq 1$ and directly if $0 < q < 1$ we obtain

$$\left\{ \int_0^{\infty} [u(t)K(t,Tf;\mathcal{B})]^q dt \right\}^{1/q}$$

$$\leq C \Bigg\{ \Bigg[\int_0^{\infty} \Bigg[u(\sigma(y))\bar{w}_0(\sigma(y)) \int_0^{y} [K(s,f;\mathcal{A})/w_0(s)]s^{-1}ds \Bigg]^q \sigma'(y)dy \Bigg]^{1/q}$$

$$+ \left[\int_0^\infty \left[u(\sigma(y))\bar{w}_1(\sigma(y)) \int_y^\infty [K(s,f;A)/w_1(s)]s^{-1}ds\right]^q \sigma'(y)dy\right]^{1/q}\Bigg\}$$

$\equiv C\{L_0 + L_1\}$, respectively. Here we used the hypothesis $\bar{\tau}(\sigma(t))\bar{w}_0(\sigma(t)) = \bar{w}_1(\sigma(t))$ (recall $\bar{\tau}(t)\bar{w}_0(t) = \bar{w}_1(t)$) . Since the estimates for L_0 and L_1 proceed precisely as in the proof of Theorem 2.2 we omit the details.

If $-t\tau'(t)/\tau(t) \geq \alpha > 0$, the result also holds under the obvious modifications.

This completes the proof of the theorem.

CHAPTER 3

FURTHER RESULTS AND APPLICATIONS

On applying and modifying the arguments of Chapter 2, specific weighted L^p estimates for functions satisfying certain Lorentz spaces data can be deduced. Specifically, we state in Section 3.1 a result which corresponds to Theorem 2.1 and generalizes the corresponding result of Sagher [33]. Then the Lorentz spaces $L^{p,w}$ – with $w \in B_\psi$ are defined and some properties are deduced. Moreover, norm inequalities of the form $\|Tf\|_{q,u} \le C\|f\|_{p,v}$, $1 \le p, q \le \infty$ are obtained whenever the weights u, v satisfy certain weight conditions and $T: L^{q_i,w_i} \to L^{\bar{q}_i,\bar{w}_i}$, $1 \le q_i, \bar{q}_i \le \infty$, $i = 0, 1$ with w_i and $\bar{w}_i$ in B_ψ, the initial data. Again for specific values of w_i, $i = 0, 1$ one obtains the results in [15] and [18].

Section 3.2 contains weighted norm inequalities for the Hankel-, $\mathcal{K}$-, and $\mathcal{Y}$-transformations. For these results, we require the same $F^*_{p,q}$-weight conditions as for corresponding results for the Fourier transform. These results seem to be new, although for the Hankel transform, Heywood and Rooney [19] established similar results in case $1 < p \le q < \infty$ with different, but equivalent weight conditions. Also, as in the case for the Fourier transform our weighted conditions are essentially sharp. For specific values of the parameter of the Hankel transform we obtain as a special case another estimate for the Fourier operator for a radial function on $\mathbb{R}^n$.

The final section contains some weighted results for the Laplace transform in weighted Lebesgue-Lorentz spaces.

3.1 $L^{p,w}$-Spaces And Interpolation.

The first result below, corresponds to Theorem 2.1 when T satisfy certain Lorentz spaces data.

Theorem 3.1. Let $0 < p_i, \bar{p}_i, q_i, \bar{q}_i \le \infty$, $i = 0, 1$; and $T: L(p_i,q_i) \to L(\bar{p}_i,\bar{q}_i)$ is a quasi-linear operator and $\lambda = 1/p_0 - 1/p_1 > 0$, $\bar{\lambda} = 1/\bar{p}_0 - 1/\bar{p}_1 \neq 0$. Suppose $\bar{\lambda} > 0$, u and v non-negative locally integrable functions satisfying

(a) If $\max(q_0,q_1) \le p \le q \le \infty$,

$$(3.1)\qquad \sup_{s>0}\left\{\int_{s^{\lambda/\bar{\lambda}}}^{\infty}\left[u(t)t^{-1/\bar{p}_0}\right]^q dt\right\}^{1/q} \times$$

$$\left\{\int_0^s\left[v(t)t^{1/p-1/p_0}\right]^{-pq_0/(p-q_0)} t^{-1}dt\right\}^{(p-q_0)/(pq_0)} < \infty$$

and

$$(3.2)\qquad \sup_{s>0}\left\{\int_0^{s^{\lambda/\bar{\lambda}}}\left[u(t)t^{-1/\bar{p}_1}\right]^q dt\right\}^{1/q} \times$$

$$\left\{\int_s^\infty \left[v(t)t^{1/p-1/p_1}\right]^{-pq_1/(p-q_1)} t^{-1}dt\right\}^{(p-q_1)/(pq_1)} < \infty ;$$

(b) If $q_1 \le p \le q \le q_0 \le \infty$, (3.2) and

$$\sup_{s>0}\left\{\int_{s^{\lambda/\bar{\lambda}}}^\infty \left[u(t)t^{-1/\bar{p}_0}\right]^q dt\right\}^{1/q} \operatorname*{ess\,sup}_{0<t\le s}\left[v(t)t^{1/p-1/p_0}\right]^{-1} < \infty ;$$

(c) If $q_0 \le p \le q \le q_1 \le \infty$, (3.1) and

$$\sup_{s>0}\left\{\int_0^{s^{\lambda/\bar{\lambda}}} \left[u(t)t^{-1/\bar{p}_1}\right]^q dt\right\}^{1/q} \operatorname*{ess\,sup}_{0<s<t}\left[v(t)t^{1/p-1/p_1}\right]^{-1} < \infty ,$$

then

$$(3.3) \qquad \left\{\int_0^\infty [u(t)(Tf)^*(t)]^q dt\right\}^{1/q} \le C\left\{\int_0^\infty [v(t)f^*(t)]^p dt\right\}^{1/p} ,$$

where $C > 0$.

If $\bar{\lambda} < 0$, (3.3) holds provided the ranges of the first integrals in (a), (b) and (c) are reversed.

The idea of the proof is essentially as in Theorem 2.1. First one shows that

$$(Tf)^*(t^{1/\bar\lambda})t^{1/(\bar p_0\bar\lambda)} \le C\,K(t,Tf;L(\bar p_0,\bar q_0),L(\bar p_1,\bar q_1))$$

then one applies Holmstedt's estimate (1.9). Now by Minkowski's inequality and the appropriate weighted Hardy inequalities one obtains the desired estimate (3.3). We omit the details.

If one writes $u(t) = W(t^{\bar\lambda})t^{1/\bar p_0 - 1/q}$, $v(t) = V(t^{\lambda})t^{1/p_0 - 1/p}$ and $p = q$ then Theorem 3.1 reduces to a result of Sagher [33].

Definition 3.2. Let $w \in B_K$, then $L^{p,w}$, $0 < p \le \infty$ consists of all measurable functions f on R^+ such that

$$\|f\|_{w,p} = \begin{cases} \left\{\int_0^\infty [tf^*(t)/w(t)]^p t^{-1}dt\right\}^{1/p}, & 0 < p < \infty, \\ \operatorname*{ess\,sup}_{t>0} tf^*(t)/w(t), & p = \infty, \end{cases}$$

is finite.

Here, of course, f^* is the equimeasurable decreasing rearrangement of f (with respect to Lebesgue measure).

Note that $L^{p,w} = L^{p,q}$ if $w(t) = t^{1-1/q}$, $0 < q \le \infty$.

Holmstedt's K-functional estimate in this context takes the following form:

<u>Theorem 3.3 [16].</u> Suppose $w_i \in B_\psi$, $i = 0, 1$ with $\tau(t) = w_1(t)/w_0(t)$ and $\eta(t) = \tau^{-1}(t)$, such that

$$t\tau'(t)/\tau(t) \ge \alpha > 0, \tag{3.4}$$

holds. Then for $0 < q_0, q_1 \le \infty$

$$K\left(t,f;L^{q_0,w_0},L^{q_1,w_1}\right) \sim \left\{\int_0^{\eta(t)} [sf^*(s)/w_0(s)]^{q_0} s^{-1} ds\right\}^{1/q_0} + t\left\{\int_{\eta(t)}^{\infty} [sf^*(s)/w_1(s)]^{q_1} s^{-1} ds\right\}^{1/q_1}. \tag{3.5}$$

We require the following result:

<u>Lemma 3.4.</u> Suppose $w_i \in B_\psi$, $i = 0, 1$ with $\tau(t) = w_1(t)/w_0(t)$ and $\eta(t) = \tau^{-1}(t)$, such that (3.4) holds. Then for $1 \le q_0, q_1 \le \infty$

$$(3.6)\qquad K\left(t,f;L^{q_0,w_0},L^{q_1,w_1}\right) \le C\left\{\int_0^{n(t)} [f^*(s)/w_0(s)]ds + t\int_{n(t)}^{\infty} [f^*(s)/w_1(s)]ds\right\}.$$

Proof. Since $f^*(s)/w_0(s)$ is non-increasing it follows directly from Lemma 1.8 that

$$(3.7)\qquad \left\{\int_0^t [sf^*(s)/w_0(s)]^{q_0} s^{-1} ds\right\}^{1/q_0} \le C\int_0^t [f^*(s)/w_0(s)]ds ,$$

$1 \le q_0 \le \infty$. Now, define g by

$$g(s) = \begin{cases} f^*(t)\left\{\int_t^{\infty} [1/w_1(y)]^{q_1} y^{-1} dy\right\}^{1/q_1}, & \text{if } 0 < s \le t , \\ f^*(s)\left\{\int_s^{\infty} [1/w_1(y)]^{q_1} y^{-1} dy\right\}^{1/q_1}, & \text{if } s > t , \end{cases}$$

where $t > 0$ is fixed. Then g is non-increasing and from (vi) of Proposition 2.8 and Lemma 1.8 we obtain for fixed $t > 0$

$$\left\{\int_t^\infty [sf^*(s)/w_1(s)]^{q_1} s^{-1} ds\right\}^{1/q_1}$$

$$\le C\left\{\int_t^\infty s^{q_1} f^*(s)^{q_1}\left(\int_s^\infty [1/w_1(y)]^{q_1} y^{-1} dy\right) s^{-1} ds\right\}^{1/q_1}$$

$$\le C\left\{\int_0^t s^{q_1} f^*(t)^{q_1}\left(\int_t^\infty [1/w_1(y)]^{q_1} y^{-1} dy\right) s^{-1} ds\right.$$

$$\left. + \int_t^\infty s^{q_1} f^*(s)^{q_1}\left(\int_s^\infty [1/w_1(y)]^{q_1} y^{-1} dy\right) s^{-1} ds\right\}^{1/q_1}$$

$$= C\left\{\int_0^\infty [sg(s)]^{q_1} s^{-1} ds\right\}^{1/q_1} \le C\int_0^\infty g(s) ds$$

$$\le C\left\{\int_0^t f^*(t)\left(\int_t^\infty [1/w_1(y)]^{q_1} y^{-1} dy\right)^{1/q_1} ds\right.$$

$$\left. + \int_t^\infty f^*(s)\left(\int_s^\infty [1/w_1(y)]^{q_1} y^{-1} dy\right)^{1/q_1} ds\right\}$$

$$\le C\left\{\int_0^t f^*(s)\left(\int_t^\infty [1/w_1(y)]^{q_1} y^{-1} dy\right)^{1/q_1} ds\right.$$

$$\left. + \int_t^\infty f^*(s)\left(\int_s^\infty [1/w_1(y)]^{q_1} y^{-1} dy\right)^{1/q_1} ds\right\} .$$

Here the second inequality is obtained by adding the first integral term. On applying (vi) of Proposition 2.8 with p and w(s) replaced by q_1 and $w_1(s)$, respectively, then the right side of the previous inequality is dominated by

$$C\left\{w_1(t)^{-1}\int_0^t f^*(s)ds + \int_t^\infty [f^*(s)/w_1(s)]ds\right\}$$

$$\le C\left\{[w_0(t)/w_1(t)]\int_0^t [f^*(s)/w_0(s)]ds + \int_t^\infty [f^*(s)/w_1(s)]ds\right\}$$

$$= C\left\{[1/\tau(t)]\int_0^t [f^*(s)/w_0(s)]ds + \int_t^\infty [f^*(s)/w_1(s)]ds\right\} .$$

Here

$$\text{(3.8)} \qquad \left\{\int_t^\infty [sf^*(s)/w_1(s)]^{q_1} s^{-1} ds\right\}^{1/q_1}$$

$$\le C\left\{[1/\tau(t)]\int_0^t [f^*(s)/w_0(s)]ds\right.$$

$$\left. + \int_t^\infty [f^*(s)/w_1(s)]ds\right\} ,$$

$1 \le q_1 \le \infty$. From (3.5), (3.7) and (3.8) one gets the desired result, which proves the lemma.

We now state and prove our main interpolation theorems of this section.

Theorem 3.5. Suppose $w_i, \bar{w}_i \in B_\psi$, $i = 0, 1$ with $\tau = w_1/w_0$, $\bar{\tau} = \bar{w}_1/\bar{w}_0$ and $\eta = \tau^{-1}$, $\bar{\eta} = \bar{\tau}^{-1}$ satisfy $t\tau'(t)/\tau(t) \ge \alpha > 0$, and $|t\bar{\tau}'(t)/\bar{\tau}(t)| \ge \bar{\alpha} > 0$. Let $\sigma = \bar{\eta}\circ\tau$ and $T: L^{q_i,w_i} \to L^{\bar{q}_i,\bar{w}_i}$, $1 \le q_i, \bar{q}_i \le \infty$, $i = 0, 1$ be a quasi-linear operator.

If $t\bar{\tau}'(t)/\bar{\tau}(t) \ge \bar{\alpha} > 0$, $1 \le p \le q \le \infty$ and u, v satisfying the two conditions

$$(3.9)\qquad \sup_{s>0}\left\{\int_{\sigma(s)}^{\infty}[u^*(t)\bar{w}_0(t)/t]^q dt\right\}^{1/q} \times$$

$$\left\{\int_0^s [1/(1/v)^*(t)w_0(t)]^{-p'}dt\right\}^{1/p'} < \infty$$

and

$$(3.10)\qquad \sup_{s>0}\left\{\int_0^{\sigma(s)} [u^*(t)\bar{w}_1(t)/t]^q dt\right\}^{1/q} \times$$

$$\left\{\int_s^\infty [1/(1/v)^*(t)w_1(t)]^{-p'}dt\right\}^{1/p'} < \infty ,$$

then for all simple functions f

$$(3.11) \qquad \left\{\int_0^\infty |u(x)(Tf)(x)|^q dx\right\}^{1/q} \le C\left\{\int_0^\infty |v(x)f(x)|^p dx\right\}^{1/p} ,$$

where C is independent of f.

If $-t\bar{\tau}'(t)/\bar{\tau}(t) \ge \bar{\alpha} > 0$, (3.11) still holds, provided the ranges of the first integrals in (3.9) and (3.10) are changed from $(\sigma(s),\infty)$ to $(0,\sigma(s))$ and from $(0,\sigma(s))$ to $(\sigma(s),\infty)$, respectively.

Proof. For the case $t\bar{\tau}'(t)/\bar{\tau}(t) \ge \bar{\alpha} > 0$ we apply Proposition 2.8(v) with s, w and p replaced by $\bar{\eta}$, $\bar{w}_0$ and $\bar{q}_0$, respectively and use (3.5) to obtain

$$(Tf)^*(\bar{\eta}(t))\bar{\eta}(t)/\bar{w}_0(\bar{\eta}(t))$$

$$\le C(Tf)^*(\bar{\eta}(t))\left\{\int_0^{\bar{\eta}(t)} [s/\bar{w}_0(s)]^{\bar{q}_0} s^{-1} ds\right\}^{1/\bar{q}_0}$$

$$\le C\left\{\int_0^{\bar{\eta}(t)} [s(Tf)^*(s)/\bar{w}_0(s)]^{\bar{q}_0} s^{-1} ds\right\}^{1/\bar{q}_0}$$

$$\le C\,K\left(t, Tf; L^{q_0,\bar{w}_0}, L^{q_1,\bar{w}_1}\right).$$

Now by hypothesis

$$(Tf)^*(\bar{\eta}(t))\bar{\eta}(t)/\bar{w}_0(\bar{\eta}(t)) \le C\,K\left(tM_1/M_0, f; L^{q_0,w_0}, L^{q_1,w_1}\right)$$

and again since $K(t,f)$ is increasing whereas $t^{-1}K(t,f)$ is decreasing we may take without loss of generality $M_1/M_0 = 1$. Therefore

$$(Tf)^*(\bar{\eta}(t)) \le C[\bar{w}_0(\bar{\eta}(t))/\bar{\eta}(t)]\; K\left(t,f;L^{q_0,w_0}, L^{q_1,w_1}\right).$$

Hence, it follows from (3.6) of Lemma 3.4 that

$$(Tf)^*(\bar{\eta}(t)) \le C[\bar{w}_0(\bar{\eta}(t))/\bar{\eta}(t)]\left\{\int_0^{\eta(t)} [f^*(s)/w_0(s)]ds\right.$$

$$\left. + \, t\int_{\eta(t)}^{\infty} [f^*(s)/w_1(s)]ds\right\}.$$

Now let $\bar{\eta}(t) = y$, then $t = \bar{\tau}(y)$ and

$$(Tf)^*(y) \le C[\bar{w}_0(y)/y]\left\{\int_0^{\bar{\sigma}(y)} [f^*(s)/w_0(s)]ds + \bar{\tau}(y)\int_{\bar{\sigma}(y)}^{\infty} [f^*(s)/w_1(s)]ds\right\},$$

where $\bar{\sigma}(y) = \eta(\bar{\tau}(y))$. Utilizing properties of rearrangement of functions and Minkowski's inequality one obtains from this estimate

$$\left\{\int_0^{\infty} |u(x)(Tf)(x)|^q dx\right\}^{1/q} \le \left\{\int_0^{\infty} [u^*(y)(Tf)^*(y)]^q dy\right\}^{1/q}$$

$$\le C\left\{\int_0^{\infty} u^*(y)^q \bar{w}_0(y)^q y^{-q}\left[\int_0^{\bar{\sigma}(y)} [f^*(s)/w_0(s)]ds + \bar{\tau}(y)\int_{\bar{\sigma}(y)}^{\infty} [f^*(s)/w_1(s)]ds\right]^q dy\right\}^{1/q}$$

$$\le C\left\{\left[\int_0^{\infty} u^*(y)^q \bar{w}_0(y)^q y^{-q}\left(\int_0^{\bar{\sigma}(y)} [f^*(s)/w_0(s)]ds\right)^q dy\right]^{1/q}\right.$$

$$+\left[\int_0^\infty u^*(y)^q \bar{w}_1(y)^q y^{-q}\left(\int_{\bar{\sigma}(y)}^\infty [f^*(s)/w_1(s)]ds\right)^q dy\right]^{1/q}\Bigg\}$$

$\equiv C\{Z_1 + Z_2\}$, respectively. Here we used in the second inequality the assumption that $\bar{\tau}(y)\bar{w}_0(y) = \bar{w}_1(y)$.

Now let $\bar{\sigma}(y) = t$, then by definition of $\bar{\sigma}$, $\eta(\bar{\tau}(y)) = t$ or $y = \bar{\tau}^{-1}(\eta^{-1}(t))$. But since $\bar{\eta}(t) = \bar{\tau}^{-1}(t)$ and $\tau(t) = \eta^{-1}(t)$ one obtains $y = \bar{\eta}(\eta^{-1}(t)) = \bar{\eta}(\tau(t)) = \sigma(t)$, and similarly $\bar{\tau}(y) = \tau(t)$. Also by Remark 2.11, $\tau(t)$ and $\bar{\tau}(t)$ tend to zero and infinity as $t \to 0$, respectively $t \to \infty$. Also $t\bar{\tau}'(t)/\bar{\tau}(t) \geq \bar{\alpha} > 0$ implies $\bar{\tau}'(t) > 0$ and since $\tau(t) = \bar{\tau}(\sigma(t))$ we obtain $\tau'(t) = \bar{\tau}'(\sigma(t))\sigma'(t)$, which implies $\sigma'(t) > 0$. Therefore, Lemma 1.6 shows that

$$Z_1 = C\left\{\int_0^\infty \left(u^*(\sigma(t))\bar{w}_0(\sigma(t))\sigma(t)^{-1}\int_0^t [f^*(s)/w_0(s)]ds\right)^q \sigma'(t)dt\right\}^{1/q}$$

$$\leq C\left\{\int_0^\infty [1/(1/v)^*(s)f^*(s)]^p ds\right\}^{1/p},$$

if and only if

$$\sup_{s>0}\left\{\int_s^\infty [u^*(\sigma(t))\bar{w}_0(\sigma(t))/\sigma(t)]^q d\sigma(t)\right\}^{1/q} \times$$

$$\left\{\int_0^s [1/(1/v)^*(t)w_0(t)]^{-p'}dt\right\}^{1/p'} < \infty .$$

Now let $\sigma(t) = y$, then the last supremum becomes (3.9).

The same substitution $\bar{\sigma}(y) = t$ and the dual part of Lemma 1.6 give

$$Z_2 = C\left\{\int_0^\infty \left(u^*(\sigma(t))\bar{w}_1(\sigma(t))\sigma(t)^{-1}\int_t^\infty [f^*(s)/w_1(s)]ds\right)^q \sigma'(t)dt\right\}^{1/q}$$

$$\le C\left\{\int_0^\infty [1/(1/v)^*(s)f^*(s)]^p ds\right\}^{1/p},$$

if and only if

$$\sup_{s>0}\left\{\int_0^s [u^*(\sigma(t))\bar{w}_1(\sigma(t))/\sigma(t)]^q d\sigma(t)\right\}^{1/q} \times$$

$$\left\{\int_s^\infty [1/(1/v)^*(t)w_1(t)]^{-p'}dt\right\}^{1/p'} < \infty .$$

Again if $\sigma(t) = y$ in the first integral we see that this condition is (3.10).

Therefore

$$\left\{\int_0^\infty |u(x)(Tf)(x)|^q dx\right\}^{1/q} \le C\left\{\int_0^\infty [1/(1/v)^*(x)f^*(x)]^p dx\right\}^{1/p}$$

$$\le C\left\{\int_0^\infty |v(x)f(x)|^p dx\right\}^{1/p} ,$$

where the second inequality follows from the integral analogue of [14, Theorem 368] obtained by approximating v by appropriate simple function and using Lebesgue's theorem of monotone convergence.

If $-t\bar{\tau}'(t)/\bar{\tau}(t) \ge \bar{\alpha} > 0$ let $\mu(t) = 1/\bar{\tau}(t)$, then the inequality takes the form

(3.12) $$t\mu'(t)/\mu(t) \ge \bar{\alpha} > 0 .$$

From (3.12) it follows that μ has an inverse ρ (see Remark 2.11), $\mu(t) \to 0$ as $t \to 0$, and $\mu(t) \to \infty$ as $t \to \infty$. Now let $\bar{\tau}(x) = 1/t$, then $\bar{\eta}(1/t) = x = \mu^{-1}(\mu(x)) = \mu^{-1}(1/\bar{\tau}(x)) = \rho(t)$. Hence (3.5) with $\eta(t)$ and q_i, $i = 0, 1$ replaced by $\rho(t)$ and $\bar{q}_i$, respectively, yields

$$K\left(t,f;L^{\bar{q}_1,\bar{w}_1},L^{\bar{q}_0,\bar{w}_0}\right) \geq C\left\{\int_0^{\rho(t)} [sf^*(s)/\bar{w}_1(s)]^{\bar{q}_1} s^{-1} ds\right\}^{1/\bar{q}_1}$$

$$= C\left\{\int_0^{\bar{\eta}(1/t)} [sf^*(s)/\bar{w}_1(s)]^{\bar{q}_1} s^{-1} ds\right\}^{1/\bar{q}_1}$$

and

$$K\left(t,f;L^{\bar{q}_0,\bar{w}_0},L^{\bar{q}_1,\bar{w}_1}\right) = tK\left(t^{-1},f;L^{\bar{q}_1,\bar{w}_1},L^{\bar{q}_0,\bar{w}_0}\right)$$

$$\geq Ct\left\{\int_0^{\bar{\eta}(t)} [sf^*(s)/\bar{w}_1(s)]^{\bar{q}_1} s^{-1} ds\right\}^{1/\bar{q}_1} .$$

Replacing f by Tf then Proposition 2.8(v) with s , w and p replaced by $\bar{\eta}$, $\bar{w}_1$ and $\bar{q}_1$, respectively and (3.5) give

$$(Tf)^*(\bar{\eta}(t))\bar{\eta}(t)/\bar{w}_0(\bar{\eta}(t)) = (Tf)^*(\bar{\eta}(t))t\bar{\eta}(t)/\bar{w}_1(\bar{\eta}(t))$$

$$\leq C(Tf)^*(\bar{\eta}(t))\left\{\int_0^{\bar{\eta}(t)} [s/\bar{w}_1(s)]^{\bar{q}_1} s^{-1} ds\right\}^{1/\bar{q}_1}$$

$$\leq C\left\{\int_0^{\bar{\eta}(t)} [s(Tf)^*(s)/\bar{w}_1(s)]^{\bar{q}_1} s^{-1} ds\right\}^{1/\bar{q}_1}$$

$$\le CK\left(t, Tf; L^{\bar{q}_0,\bar{w}_0}, L^{\bar{q}_1,\bar{w}_1}\right).$$

The rest of the proof is similar to the case $t\tau'(t)/\tau(t) \ge \alpha > 0$ and therefore omitted. This completes the proof of the theorem.

We now extend Theorem 3.5 to the case $1 < q < p < \infty$.

Theorem 3.6. Suppose $w_i, \bar{w}_i \in B_\psi$, $i = 0, 1$ with $\tau = w_1/w_0$, $\bar{\tau} = \bar{w}_1/\bar{w}_0$ and $\eta = \tau^{-1}$, $\bar{\eta} = \bar{\tau}^{-1}$ satisfy $t\tau'(t)/\tau(t) \ge \alpha > 0$, and $|t\bar{\tau}'(t)/\bar{\tau}(t)| \ge \bar{\alpha} > 0$. Let $\sigma = \bar{\eta}\circ\tau$ and $T: L^{q_i,w_i} \to L^{\bar{q}_i,\bar{w}_i}$, $1 \le q_i, \bar{q}_i \le \infty$, $i = 0, 1$ be a quasi-linear operator.

If $t\bar{\tau}'(t)/\bar{\tau}(t) \ge \bar{\alpha} > 0$, $1 < q < p < \infty$, $1/r = 1/q - 1/p$; u and v are non-negative weight functions satisfying

$$(3.13) \qquad \left\{\int_0^\infty \left[\left(\int_{\sigma(s)}^\infty [u^*(t)\bar{w}_0(t)/t]^q dt\right)^{1/q} \times \left(\int_0^s [1/(1/v)^*(t)w_0(t)]^{-p'} dt\right)^{1/q'}\right]^r [1/(1/v)^*(s)w_0(s)]^{-p'} ds\right\}^{1/r} < \infty$$

and

$$(3.14)\qquad \left\{\int_0^\infty \left[\left(\int_0^{\sigma(s)} [u^*(t)\bar{w}_1(t)/t]^q dt\right)^{1/q} \times \right.\right.$$

$$\left.\left.\left(\int_s^\infty [1/(1/v)^*(t)w_1(t)]^{-p'} dt\right)^{1/q'}\right]^r [1/(1/v)^*(s)w_1(s)]^{-p'} ds\right\}^{1/r} < \infty ,$$

then (3.11) yields.

If $-t\bar{\tau}'(t)/\bar{\tau}(t) \geq \bar{\alpha} > 0$, (3.11) still holds provided the ranges of the first inner integrals in (3.13) and (3.14) are interchanged.

<u>Proof</u>. Proceeding as in the proof of Theorem 3.5, we get

$$\left\{\int_0^\infty |u(x)(Tf)(x)|^q dx\right\}^{1/q}$$

$$\leq C\left\{\left[\int_0^\infty \left(u^*(\sigma(t))\bar{w}_0(\sigma(t))\sigma(t)^{-1}\int_0^t [f^*(s)/w_0(s)]ds\right)^q \sigma'(t)dt\right]^{1/q}\right.$$

$$\left.+\left[\int_0^\infty \left(u^*(\sigma(t))\bar{w}_1(\sigma(t))\sigma(t)^{-1}\int_t^\infty [f^*(s)/w_1(s)]ds\right)^q \sigma'(t)dt\right]^{1/q}\right\}$$

$\equiv C\{Z_1 + Z_2\}$, respectively. Now using Lemma 1.7(a) and (b) and the integral analogue of [14, Theorem 368] one estimates Z_1 and Z_2 to obtain the result. We omit the details.

The following corollaries are consequences of Theorem 3.5, respectively Theorem 3.6 with $w_i(t) = t^{1-1/p_i}$ and $\bar{w}_i(t) = t^{1-1/\bar{p}_i}$, $i = 0, 1$:

<u>Corollary 3.7</u>. Let $0 < p_i, \bar{p}_i \le \infty$, $1 \le q_i, \bar{q}_i \le \infty$, $i = 0, 1$ and $T: L(p_i,q_i) \to L(\bar{p}_i,\bar{q}_i)$ be a quasi-linear operator and $\lambda = 1/p_0 - 1/p_1 > 0$, $\bar{\lambda} = 1/\bar{p}_0 - 1/\bar{p}_1 \neq 0$.

If $\bar{\lambda} > 0$, $1 \le p \le q \le \infty$; u, v satisfying

$$(3.15)\qquad \sup_{s>0}\left\{\int_{s^{\lambda/\bar{\lambda}}}^{\infty}\left[u^*(t)t^{-1/\bar{p}_0}\right]^q dt\right\}^{1/q} \times$$

$$\left\{\int_0^s\left[1/(1/v)^*(t)t^{1/p-1/p_0}\right]^{-p'} t^{-1}dt\right\}^{1/p'} < \infty$$

and

$$(3.16)\qquad \sup_{s>0}\left\{\int_0^{s^{\lambda/\bar{\lambda}}}\left[u^*(t)t^{-1/\bar{p}_1}\right]^q dt\right\}^{1/q} \times$$

$$\left\{\int_s^{\infty}\left[1/(1/v)^*(t)t^{1/p-1/p_1}\right]^{-p'} t^{-1}dt\right\}^{1/p'} < \infty ,$$

then (3.11) holds.

If $\bar{\lambda} < 0$, (3.11) still holds provided the ranges of the first integrals in (3.15) and (3.16) will be taken from 0 to $s^{\lambda/\bar{\lambda}}$ and from $s^{\lambda/\bar{\lambda}}$ to ∞, respectively.

Observe that if $q_i = p_i$, $\bar{q}_i = \infty$, $i = 0, 1$ then this result is essentially that of [18, Theorem 2.4].

Corollary 3.8. Let $0 < p_i, \bar{p}_i \le \infty$, $1 \le q_i, \bar{q}_i \le \infty$, $i = 0, 1$ and $T: L(p_i,q_i) \to L(\bar{p}_i,\bar{q}_i)$ be a quasi-linear operator and $\lambda = 1/p_0 - 1/p_1 > 0$, $\bar{\lambda} = 1/\bar{p}_0 - 1/\bar{p}_1 \ne 0$.

If $\bar{\lambda} > 0$, $1 < q < p < \infty$; u, v satisfying

$$(3.17)\qquad \left\{ \int_0^\infty \left[\left(\int_{s^{\lambda/\bar{\lambda}}}^\infty \left[u^*(t) t^{-1/\bar{p}_0} \right]^q dt \right)^{1/q} \times \right.\right.$$

$$\left.\left. \left(\int_0^s \left[1/(1/v)^*(t) t^{1/p-1/p_0} \right]^{-p'} t^{-1} dt \right)^{1/q'} \right]^r \left[1/(1/v)^*(s) s^{1/p-1/p_0} \right]^{-p'} s^{-1} ds \right\}^{1/r} < \infty$$

and

$$(3.18)\qquad \left\{\int_0^\infty \left[\left(\int_0^{s^{\lambda/\bar\lambda}} \left[u^*(t)t^{-1/\bar p_1}\right]^q dt\right)^{1/q} \times \right.\right.$$

$$\left.\left.\left(\int_s^\infty \left[1/(1/v)^*(t)t^{1/p-1/p_1}\right]^{-p'} t^{-1}dt\right)^{1/q'}\right]^r \left[1/(1/v)^*(s)s^{1/p-1/p_1}\right]^{-p'} s^{-1}ds\right\}^{1/r} < \infty,$$

then (3.11) holds.

If $\bar\lambda < 0$, (3.11) still holds provided the ranges of the first inner integrals in (3.17) and (3.18) are interchanged.

Our next lemma yields a result of Benedetto and Heinig [3] when $p_0 = 1$, $\bar p_0 = \infty$, $\bar p_1 = p_1 = 2$; $\lambda = 1/2$, $\bar\lambda = -1/2$.

Lemma 3.9. Let u and v be non-negative functions defined on $\mathbb{R}^+$ such that u is non-increasing and v non-decreasing. If $\lambda = 1/p_0 - 1/p_1 > 0$, $\bar\lambda = 1/\bar p_0 - 1/\bar p_1 < 0$, $1 \le p_i, \bar p_i \le \infty$, $i = 0, 1$, $1 \le p \le q \le \infty$ and p, q are not both equal to p_1 or $\bar p_1$, then

$$(3.19)\qquad \sup_{s>0}\left\{\int_0^{s^{\lambda/\bar\lambda}} \left[u(t)t^{-1/\bar p_0}\right]^q dt\right\}^{1/q} \times$$

$$\left\{\int_0^s \left[v(t)t^{1/p-1/p_0}\right]^{-p'} t^{-1}dt\right\}^{1/p'} < \infty,$$

implies

$$\text{(3.20)} \qquad \sup_{s>0}\left\{\int_{s^{\lambda/\bar\lambda}}^{\infty}\left[u(t)t^{-1/\bar p_1}\right]^q dt\right\}^{1/q} \times$$

$$\left\{\int_s^{\infty}\left[v(t)t^{1/p-1/p_1}\right]^{-p'} t^{-1}dt\right\}^{1/p'} < \infty .$$

(Note that (3.19) and (3.20) are the weight conditions of Corollary 3.7 if $\bar\lambda < 0$, $1 \le p_i, \bar p_i \le \infty$, since $u^* = u$ and $(1/v)^* = 1/v$.)

<u>Proof</u>. If $p_0 < p < p_1$, $p \le q < \bar p_0$, then (3.19) implies

$$u(s^{\lambda/\bar\lambda})s^{\lambda/\bar\lambda(-1/\bar p_0+1/q)} \le C\left\{\int_0^s\left[v(y)y^{1/p-1/p_0}\right]^{-p'} y^{-1}dy\right\}^{1/p'} .$$

Substitute in this estimate $s^{\lambda/\bar\lambda} = t$ and since v is non-decreasing the integrals in (3.20) are dominated by

$$\left\{\int_{s^{\lambda/\bar\lambda}}^{\infty} t^{\bar\lambda q-1}\left[\int_0^{t^{\bar\lambda/\lambda}} v(y)^{-p'} y^{-(1/p-1/p_0)p'} y^{-1}dy\right]^{-q/p'} dt\right\}^{1/q}$$

$$\times\left\{\int_s^\infty \left[v(t)^{-p'} t^{-(1/p-1/p_1)p'} t^{-1} dt\right\}^{1/p'}\right.$$

$$\le Cv(s)^{-1} s^{-(1/p-1/p_1)} \left\{\int_{s^{\lambda/\bar\lambda}}^\infty t^{\bar\lambda q-1} v(t^{\bar\lambda/\lambda})^q t^{\bar\lambda/\lambda(1/p-1/p_0)q} dt\right\}^{1/q}$$

$$\le Cv(s)^{-1} s^{-1/p+1/p_0-\lambda} v(s) s^{\lambda+1/p-1/p_0} = C .$$

If $q = \infty$, $p_0 < p < p_1$ or $1 = p \le q < \bar p_0$ the argument is the same and hence omitted.

If $p_0 < p \le q$, $\max(p_1, \bar p_1) < q < \bar p_0$, then (3.19) implies

$$v(s)^{-1} s^{-1/p+1/p_0} \le C\left\{\int_0^{s^{\lambda/\bar\lambda}} \left[u(t) t^{-1/\bar p_0}\right]^q dt\right\}^{-1/q} .$$

On substituting this estimate in the left side of (3.20) one obtains

$$\left\{\int_{s^{\lambda/\bar\lambda}}^\infty \left[u(t) t^{\bar\lambda-1/\bar p_0}\right]^q dt\right\}^{1/q} \left\{\int_s^\infty t^{-\lambda p'-1} \left(\int_0^{t^{\lambda/\bar\lambda}} \left[u(y) y^{-1/\bar p_0}\right]^q dy\right)^{-p'/q} dt\right\}^{1/p'}$$

$$\le Cu(s^{\lambda/\bar\lambda})s^{\lambda-\lambda/(\bar\lambda\bar p_0)+\lambda/(\bar\lambda q)} \left\{\int_s^\infty t^{-\lambda p'-1}u(t^{\lambda/\bar\lambda})^{-p'}t^{-p'\lambda(-1/\bar p_0+1/q)/\bar\lambda}dt\right\}^{1/p'}$$

$$\le Cu(s^{\lambda/\bar\lambda})s^{\lambda-\lambda/(\bar\lambda\bar p_0)+\lambda/(\bar\lambda q)}u(s^{\lambda/\bar\lambda})^{-1}s^{-\lambda+\lambda/(\bar\lambda\bar p_0)-\lambda/(\bar\lambda q)} = C .$$

This proves the lemma.

3.2 Weighted Estimates For The Hankel-, $\mathcal{K}$- And $\mathcal{Y}$-Transformations.

In this section, we apply our previous results - or alternately the results of Benedetto, Heinig and Johnson ([3], [4], [18]) to prove weighted norm inequalities for the Hankel-, $\mathcal{K}$- and $\mathcal{Y}$-transformations. These norm inequalities in weighted Lebesgue spaces with different indices are also sharp for $1 < p \le q < \infty$, provided the weights satisfy certain monotonicity conditions ([10]). In the case of the Hankel transform the results are similar to those recently established by Heywood and Rooney [19] but with different weight conditions. As in their case, we obtain a weighted estimate for the Fourier transform of a radial function on R^n. The results for the $\mathcal{K}$-transform constitute generalizations of weighted norm inequalities for the Laplace transform. The section concludes with examples of some specific weights which satisfy our weight conditions.

First we recall the weight conditions and some results:

Definition 3.10 ([4]). Let u and v be non-negative functions defined on $\mathbb{R}^+$ and u^*, $(1/v)^*$ be the equimeasurable decreasing rearrangement of u and $1/v$. We write $(u,v) \in F^*_{p,q}$, $1 < p, q < \infty$, if

$$\text{(3.21)} \qquad \sup_{s>0} \left\{ \int_0^{1/s} u^*(t)^q dt \right\}^{1/q} \left\{ \int_0^s (1/v)^*(t)^{p'} dt \right\}^{1/p'} < \infty$$

holds for $1 < p \le q < \infty$ and in the case $1 < q < p < \infty$ the conditions

$$\text{(3.22)} \qquad \left\{ \int_0^\infty \left[\left(\int_0^{1/x} u^*(t)^q dt \right)^{1/q} \times \left(\int_0^x [(1/v)^*(t)]^{p'} dt \right)^{1/q'} \right]^r (1/v)^*(x)^{p'} dx \right\}^{1/r} < \infty$$

and

$$\text{(3.23)} \qquad \left\{ \int_0^\infty \left[\left(\int_{1/x}^\infty [u^*(t) t^{-1/2}]^q dt \right)^{1/q} \times \left(\int_x^\infty [(1/v)^*(t) t^{-1/2}]^{p'} dt \right)^{1/q'} \right]^r [(1/v)^*(x) x^{-1/2}]^{p'} dx \right\}^{1/r} < \infty$$

hold, where $1/r = 1/q - 1/p$.

If (3.21), (3.22) and (3.23) hold without rearrangements on u and $1/v$, then we write $(u,v) \in F_{p,q}$.

If u and $1/v$ are non-increasing, then $u^* = u$ and $(1/v)^* = 1/v$.

It is easily seen that for $1 < p \leq q < \infty$, u and $1/v$ non-increasing, one has $(u,v) \in F_{p,q}$ if and only if $(1/v, 1/u) \in F_{q',p'}$.

<u>Theorem 3.11 ([4, Theorem 1.1 and Proposition 3.1])</u>. Suppose T is a sublinear operator defined on simple functions such that T is of type $(1,\infty)$ and $(2,2)$. If $(u,v) \in F^*_{p,q}$, $1 < p, q < \infty$, then

$$\left\{\int_0^\infty |u(x)(Tf)(x)|^q dx\right\}^{1/q} \leq C\left\{\int_0^\infty |v(x)f(x)|^p dx\right\}^{1/p} .$$

A corollary of this result is the following:

<u>Corollary 3.12</u>. Let T be as in Theorem 3.11 and B defined by $(Bf)(x) = w(x)(Tg)(x)$, where $g(x) = w(x)f(x)$. If $(uw, v/w) \in F^*_{p,q}$, $1 < p, q < \infty$, then there is a constant C independent of f such that for all simple functions f

$$\left\{\int_0^\infty |u(x)(Bf)(x)|^q dx\right\}^{1/q} \le C\left\{\int_0^\infty |v(x)f(x)|^p dx\right\}^{1/p} .$$

For $w \equiv 1$ this is of course Theorem 3.11.

In the first application we consider the Hankel transform defined by

(3.24) $$(\mathcal{H}_\alpha f)(x) = \int_0^\infty J_\alpha(xt)(xt)^{1/2} f(t)dt , \quad x > 0 , \quad \alpha \ge -1/2 ,$$

where f is a function - say simple - for which the integral converges and $J_\alpha(x)$, $\alpha > -1/2$ is the Bessel function of the first kind of order α, with representation

(3.25) $$J_\alpha(x) = 2/(\pi^{1/2}\Gamma(\alpha+1/2))(x/2)^\alpha \int_0^{\pi/2} \cos(x\cos\theta)\sin^{2\alpha}\theta d\theta ,$$

([12, P.952(4)]), where Γ is the gamma function and

$$J_{-1/2}(x) = (2/(\pi x))^{1/2}\cos x .$$

The reciprocal formula of (3.24) is given by

$$f(x) = \int_0^\infty J_\alpha(xt)(xt)^{1/2}(\mathcal{H}_\alpha f)(t)dt, \; f \in L^2(\mathbb{R}^+) ,$$

([38, Theorem 129]).

Now, we state and prove the weighted norm inequality for the Hankel transform.

<u>Theorem 3.13</u>. Let $\alpha \geq -1/2$, u and v non-negative functions on $\mathbb{R}^+$ such that $u_\alpha(x) = x^{\alpha+1/2}u(x)$, $v_\alpha(x) = x^{-\alpha-1/2}v(x)$ and $f \in L_v^p(\mathbb{R}^+)$.

(a) If $(u_\alpha, v_\alpha) \in F_{p,q}^*$, $1 < p, q < \infty$ and $\lim_{k\to\infty} \|f_k - f\|_{p,v} = 0$ for a sequence $\{f_k\}$ of simple functions, then $\{\mathcal{H}_\alpha f_k\}$ converges in $L_u^q(\mathbb{R}^+)$ to a function $\mathcal{H}_\alpha f \in L_u^q(\mathbb{R}^+)$. $\mathcal{H}_\alpha f$ is independent of the sequence $\{f_k\}$ and is called the Hankel transform of f.

(b) If $(u_\alpha, v_\alpha) \in F_{p,q}^*$, $1 < p, q < \infty$, then there is a $C > 0$ such that for all $f \in L_v^p(\mathbb{R}^+)$

$$\|\mathcal{H}_\alpha f\|_{q,u} \leq C\|f\|_{p,v} . \tag{3.26}$$

(c) If (3.26) holds then $(u_\alpha, v_\alpha) \in F_{p,q}$, $1 < p \leq q < \infty$.

Proof. First, we verify (3.26) for simple functions f. If $\alpha = -1/2$ the result reduces to a weighted estimate for the Fourier cosine transform which follows from [4] or [18] and is therefore omitted. We only look at the case $\alpha > -1/2$.

From (3.24) and (3.25) we get

$$(\mathcal{H}_\alpha f)(x) = 2^{1-\alpha} x^{\alpha+1/2} / (\pi^{1/2}\Gamma(\alpha+1/2)) \times$$

$$\int_0^\infty t^{\alpha+1/2} f(t) \left(\int_0^{\pi/2} \cos(xt\cos\theta)\sin^{2\alpha}\theta\, d\theta \right) dt$$

$$= 2^{1-\alpha} x^{\alpha+1/2} / (\pi^{1/2}\Gamma(\alpha+1/2)) \times$$

$$\int_0^{\pi/2} \sin^{2\alpha}\theta \left(\int_0^\infty \cos(xt\cos\theta) t^{\alpha+1/2} f(t)\, dt \right) d\theta$$

$$= x^{\alpha+1/2}(T_\alpha g)(x), \quad \text{where} \quad g(t) = t^{\alpha+1/2} f(t) \quad \text{and}$$

$$(T_\alpha g)(x) = 2^{1-\alpha} / (\pi^{1/2}\Gamma(\alpha+1/2)) \times$$

$$\int_0^{\pi/2} \sin^{2\alpha}\theta \left(\int_0^\infty \cos(xt\cos\theta) g(t)\, dt \right) d\theta .$$

The interchange of order of integration above is justified by Fubini's theorem. For by Hölder's inequality and the fact that f vanishes outside an interval say $(0,a)$, for some $a > 0$ it follows that

$$\int_0^\infty |t^{\alpha+1/2} v_\alpha(t) f(t) v_\alpha(t)^{-1}| \int_0^{\pi/2} |\cos(xt\cos\theta)\sin^{2\alpha}\theta|\,d\theta\,dt$$

$$\leq \|f\|_{p,v}\left\{\int_0^a (1/v_\alpha)^*(t)^{p'}\left(\int_0^{\pi/2}|\cos(xt\cos\theta)\sin^{2\alpha}\theta|\,d\theta\right)^{p'} dt\right\}^{1/p'} < \infty,$$

since the inner integral is dominated by $B(\alpha+1/2,1/2)$, where B is the beta function.

We now claim that $T_\alpha: L^1(\mathbb{R}^+) \to L^\infty(\mathbb{R}^+)$ and $T_\alpha: L^2(\mathbb{R}^+) \to L^2(\mathbb{R}^+)$. If $\alpha > -1/2$ then clearly

$$|T_\alpha g(x)| = 2^{1-\alpha}/(\pi^{1/2}\Gamma(\alpha+1/2))\left|\int_0^{\pi/2} \sin^{2\alpha}\theta \int_0^\infty \cos(xt\cos\theta) g(t)\,dt\,d\theta\right|$$

$$\leq CB(\alpha+1/2,1/2)\int_0^\infty |g(t)|\,dt,$$

since

$$2\int_0^{\pi/2} \sin^{2\alpha}\theta d\theta = B(\alpha+1/2,1/2)\ , \qquad ([12,\ P.948(2)]).$$

Also, by Minkowski's integral inequality we get

$$\left\{\int_0^\infty |T_\alpha g(x)|^2 dx\right\}^{1/2} \leq 2^{1-\alpha}/(\pi^{1/2}\Gamma(\alpha+1/2)) \times$$

$$\int_0^{\pi/2} \sin^{2\alpha}\theta \left\{\int_0^\infty \left| \int_0^\infty \cos(xt\cos\theta)g(t)dt\right|^2 dx\right\}^{1/2} d\theta$$

and if $x\cos\theta = y$, $\theta \in (0,\pi/2)$, then the above integral is dominated by

$$C\left(\int_0^{\pi/2} \sin^{2\alpha}\theta\cos^{-1/2}\theta d\theta\right)\left\{\int_0^\infty \left| \int_0^\infty \cos(yt)g(t)dt\right|^2 dy\right\}^{1/2}$$

$$= CB(\alpha+1/2,1/4)\left\{\int_0^\infty |g(t)|^2 dt\right\}^{1/2} ,$$

where the last equation follows from Plancherel's theorem for the Fourier cosine transform and

$$2\int_0^{\pi/2} \sin^{2\alpha}\theta\cos^{-1/2}\theta d\theta = B(\alpha+1/2, 1/4)\,, \quad ([12,\ P.948(2)])\,.$$

Now if $(u_\alpha, v_\alpha) \in F^*_{p,q}$, then by Corollary 3.12 with $(Bf)(x) \equiv (\mathcal{H}_\alpha f)(x)$, $w(x) = x^{\alpha+1/2}$ and $(Tg)(x) \equiv (T_\alpha g)(x)$ we obtain

$$\left\{\int_0^\infty |u(x)(\mathcal{H}_\alpha f)(x)|^q dx\right\}^{1/q} \le C\left\{\int_0^\infty |v(x)f(x)|^p dx\right\}^{1/p}.$$

Since $L^p_v(R^+)$ is a Banach space and simple functions are dense in $L^p_v(R^+)$, then by (3.26), the mapping $f \to \mathcal{H}_\alpha f$ is a continuous transformation from a dense subspace $L^p_v(R^+)$ to $L^q_u(R^+)$. Consequently, the transformation has a unique extension to all $L^p_v(R^+)$. The extension is also denoted by $\mathcal{H}_\alpha f$. Thus, the proof of part (a) as well as part (b) is complete.

To prove (c), fix $s > 0$ and set

$$f(x) = x^{(\alpha+1/2)/(p-1)} v(x)^{-p'} \chi_{(0,s)}(x)\,.$$

Because (3.26) can be written as

$$\left\{\int_0^\infty |u(x)(\mathcal{H}_\alpha f)(x)|^q dx\right\}^{1/q} \le C\left\{\int_0^s v(x)^{-p'} x^{(\alpha+1/2)p'} dx\right\}^{1/p},$$

we obtain the inequality

$$(3.27)\quad \left\{\int_0^{1/s} u(x)^q \left|\int_0^s (xt)^{1/2} J_\alpha(xt)\, t^{(\alpha+1/2)/(p-1)} v(t)^{-p'} dt\right|^q dx\right\}^{1/q}$$

$$\le C\left\{\int_0^s v(x)^{-p'} x^{(\alpha+1/2)p'} dx\right\}^{1/p}.$$

But since $0 < xt < 1$ it follows that $\cos(xt\cos\theta) \ge 1/2$, $0 \le \theta \le \pi/2$. Hence (3.25) yields $J_\alpha(xt) \ge C(xt)^\alpha$. Substituting this into (3.27) shows that

$$\left\{\int_0^{1/s} u_\alpha(t)^q dt\right\}^{1/q} \left\{\int_0^s [v_\alpha(t)]^{-p'} dt\right\}^{1/p'} \le C .$$

This proves the theorem.

The following pairs of functions satisfy the above weight conditions:

(i) $u(x) = x^{-\alpha-1/2+\beta-1/q}$ and $v(x) = x^{\alpha+1/2-\beta+1/p'}$, $\alpha \geq -1/2$, where $1 < p \leq q < \infty$ and $0 < \beta < \min(1/q, 1/p')$.

For the proof we write $u_\alpha(x) = x^{\beta-1/q}$ and $v_\alpha(x) = x^{-\beta+1/p'}$. The functions u_α and $1/v_\alpha$ are decreasing, and therefore

$$\left\{\int_0^{1/s} u_\alpha^*(t)dt\right\}^{1/q} \left\{\int_0^s [(1/v_\alpha)^*(t)]^{p'} dt\right\}^{1/p'}$$

$$= \left\{\int_0^{1/s} t^{\beta q-1} dt\right\}^{1/q} \left\{\int_0^s t^{\beta p'-1} dt\right\}^{1/p'} = C < \infty .$$

(ii) $u(x) = x^{-\alpha-1/2+\beta-1/q}$ and $v(x) = x^{\alpha+1/2} e^x$, $\alpha \geq -1/2$ where $1 < q < p < \infty$, $1/r = 1/q - 1/p$ and $0 < \beta < \min(1/2, 1/r)$.

To prove (ii), we must show that the integrals

$$\text{(3.28)} \quad \left\{\int_0^\infty \left[\left(\int_0^{1/x} u_\alpha^*(t)^q dt\right)^{1/q} \times \left(\int_0^x [(1/v_\alpha)^*(t)]^{p'} dt\right)^{1/q'}\right]^r (1/v_\alpha)^*(x)^{p'} dx\right\}^{1/r}$$

and

$$(3.29)\quad \left\{\int_0^\infty \left[\left(\int_{1/x}^\infty [u_\alpha^*(t)t^{-1/2}]^q dt\right)^{1/q} \times\right.\right.$$

$$\left.\left.\left(\int_x^\infty [(1/v_\alpha)^*(t)t^{-1/2}]^{p'} dt\right)^{1/q'}\right]^r [(1/v_\alpha)^*(x)x^{-1/2}]^{p'} dx\right\}^{1/r}$$

are finite. Since $\int_0^{1/x} u_\alpha^*(t)^q dt = \int_0^{1/x} t^{\beta q-1} dt = (1/(\beta q))x^{-\beta q}$ and

$$\int_0^x [(1/v_\alpha)^*(t)]^{p'} dt = \int_0^x e^{-tp'} dt = (1/p')(1-e^{-xp'}) \le 1/p'$$

we obtain on substituting this in (3.28)

$$(\beta q)^{-1/q}(p')^{-1/q'}\left\{\int_0^\infty x^{-r\beta}e^{-xp'}dx\right\}^{1/r} = (\beta q)^{-1/q}(p')^{\beta-1/q'-1/r}\{\Gamma(1-r\beta)\}^{1/r},$$

where $u_\alpha^* = u_\alpha$, $(1/v_\alpha)^* = 1/v_\alpha$.

Similarly, since

$$\int_{1/x}^{\infty} [u_{\alpha}^{*}(t)t^{-1/2}]^{q} dt = \int_{1/x}^{\infty} t^{\beta q-q/2-1} dt = (\beta q-q/2)^{-1} x^{q/2-\beta q}$$

and

$$\int_{x}^{\infty} [(1/v_{\alpha})^{*}(t)t^{-1/2}]^{p'} dt = \int_{x}^{\infty} t^{-p'/2} e^{-tp'} dt$$

$$\le x^{-p'/2}\int_{x}^{\infty} e^{-tp'} dt = (1/p')x^{-p'/2} e^{-xp'} ,$$

we get by substituting that in (3.29)

$$(\beta q-q/2)^{-1/q}(p')^{-1/q'}\left\{\int_{0}^{\infty} e^{-xp'(r/q'+1)} x^{r/2-r\beta-p'(r/q'+1)/2} dx\right\}$$

$$= (\beta q-q/2)^{-1/q}(p')^{-1/q'}\left\{\int_{0}^{\infty} e^{-xr} x^{-r\beta} dx\right\}^{1/r}$$

$$= (\beta q-q/2)^{-1/q}(p')^{-1/q'} r^{-1/r}\{\Gamma(1-r\beta)\}^{1/r} ,$$

Since $1/r = 1/q - 1/p = 1/p' - 1/q'$, so that $r = p'(r/q'+1)$. This completes the proof of (ii).

The Muckenhoupt weight conditions [29] for the weighted Fourier estimates are the following: Let u and v be non-negative functions and A, B positive constants, such that for all $r > 0$

$$\left\{\int_{x^{p'/q}u^*(x)^{p'}\ge Brx} u^*(x)^q\,dx\right\}^{1/q}\left\{\int_{v(x)<r^{1/p'}} v(x)^{-p'}\,dx\right\}^{1/p'} \le A, \quad q \le p'$$

(3.30)

$$\left\{\int_{u(x)>r^{-1/q}} u(x)^q\,dx\right\}^{1/q}\left\{\int_{x^{q/p'}(1/v)^*(x)^{q/(p-1)}>Brx} (1/v)^*(x)^{p'}\,dx\right\}^{1/p'} \le A, \quad p' < q .$$

Moreover, Muckenhoupt has shown [28] that (3.30) is equivalent to (3.21). It follows therefore from Theorem 3.13 that in case $1 < p \le q < \infty$, (3.30) with u and v replaced by u_α and v_α, implies $\|\mathcal{H}_\alpha f\|_{q,u} \le C\|f\|_{p,v}$. Conditions similar to (3.30) were used by Heywood and Rooney [20] to prove weighted norm inequalities for the Hankel transformation.

In the next theorem we find integral representations for $\mathcal{H}_\alpha$ which corresponds to the Fourier integral representation in [3].

<u>Theorem 3.14</u>. Let $\alpha \geq -1/2$, u and v be non-negative functions on $\mathbb{R}^+$ such that $u_\alpha(x) = x^{\alpha+1/2}u(x)$ is non-increasing and $v_\alpha(x) = x^{-\alpha-1/2}v(x)$ is non-decreasing. Suppose that $(u_\alpha, v_\alpha) \in F_{p,q}$, $1 < p \leq q < \infty$ and $f \in L^p_v(\mathbb{R}^+)$.

(a) If $g \in L^{q'}_{1/u}(\mathbb{R}^+)$ then

(3.31) $$\int_0^\infty (\mathcal{H}_\alpha f)(x)g(x)dx = \int_0^\infty f(x)(\mathcal{H}_\alpha g)(x)dx$$

(b) $(\mathcal{H}_\alpha f)(x)$ has the pointwise representations

(i) $$(\mathcal{H}_\alpha f)(x) = x^{-(\alpha+1/2)} \frac{d}{dx} x^{\alpha+1/2} \times$$

$$\int_0^\infty (xt)^{1/2} J_{\alpha+1}(xt)f(t)t^{-1}dt \quad \text{a.e.}$$

and

(ii) $$(\mathcal{H}_\alpha f)(x) = \frac{d}{dx}\int_0^\infty f(t)\int_0^x (ty)^{1/2}J_\alpha(ty)dydt \quad \text{a.e.}$$

<u>Proof</u>. If $\alpha = -1/2$, the result was proved in [3] for the Fourier transform.

We begin with (a).

If $f \in L^p_v(R^+)$ and $g \in L^{q'}_{1/u}(R^+)$, then using Hölder's inequality and Theorem 3.13(b) give

(3.32)
$$\left|\int_0^\infty (\mathcal{H}_\alpha f)(x)g(x)dx\right| \le \|\mathcal{H}_\alpha f\|_{q,u}\,\|g\|_{q',1/u}$$
$$\le C\|f\|_{p,v}\,\|g\|_{q',1/u}$$

and

(3.33)
$$\left|\int_0^\infty f(x)(\mathcal{H}_\alpha g)(x)dx\right| \le \|f\|_{p,v}\,\|\mathcal{H}_\alpha g\|_{p',1/v}$$
$$\le C\|f\|_{p,v}\,\|g\|_{q',1/u} ,$$

where in the second inequality of (3.33) we used the fact that $(u_\alpha, v_\alpha) \in F_{p,q}$, implies $(1/v_\alpha, 1/u_\alpha) \in F_{q',p'}$.

Let $\{f_k\}$ and $\{g_k\}$ be sequences of simple functions satisfying

$$\lim_{k\to\infty} \|f_k - f\|_{p,v} = \lim_{k\to\infty} \|g_k - g\|_{q',1/u} = 0 .$$

Then (3.31) follows from Parseval's formula for the Hankel transform ([31]) with f and g replaced by f_k and g_k respectively. Since

$$(\mathcal{H}_\alpha f)g - f(\mathcal{H}_\alpha g) = (\mathcal{H}_\alpha f)(g-g_k) + g_k((\mathcal{H}_\alpha f) - (\mathcal{H}_\alpha f_k)) + ((\mathcal{H}_\alpha f_k)g_k$$
$$- f_k(\mathcal{H}_\alpha g_k)) + f_k((\mathcal{H}_\alpha g_k) - (\mathcal{H}_\alpha g)) + (\mathcal{H}_\alpha g)(f_k - f) ,$$

then

$$\left|\int_0^\infty [(\mathcal{H}_\alpha f)(x)g(x) - f(x)(\mathcal{H}_\alpha g)(x)]dx\right|$$
$$\le \left|\int_0^\infty (\mathcal{H}_\alpha f)(x)(g(x)-g_k(x))dx\right| + \left|\int_0^\infty g_k(x)\mathcal{H}_\alpha(f(x)-f_k(x))dx\right|$$
$$+ \left|\int_0^\infty [(\mathcal{H}_\alpha f_k)(x)g_k(x) - f_k(x)(\mathcal{H}_\alpha g_k)(x)]dx\right|$$
$$+ \left|\int_0^\infty f_k(x)\mathcal{H}_\alpha(g_k(x)-g(x))dx\right| + \left|\int_0^\infty (\mathcal{H}_\alpha g)(x)(f_k(x)-f(x))dx\right|$$
$$\le \|\mathcal{H}_\alpha f\|_{q,u}\, \|g_k-g\|_{q',1/u} + \|\mathcal{H}_\alpha(f-f_k)\|_{q,u}\, \|g_k\|_{q',1/u} + 0$$
$$+ \|f_k\|_{p,v}\, \|\mathcal{H}_\alpha(g_k-g)\|_{p',1/v} + \|f_k-f\|_{p,v}\, \|\mathcal{H}_\alpha g\|_{p',1/v}$$

$$\le C\Big\{\|f\|_{p,v}\,\|g_k-g\|_{q',1/u} + \|f-f_k\|_{p,v}\,\|g_k\|_{q',1/u}$$

$$+ \|f_k\|_{p,v}\,\|g_k-g\|_{q',1/u} + \|f_k-f\|_{p,v}\,\|g\|_{q',1/u}\Big\} .$$

Here we used (3.32) and (3.33) repeatedly. Now let $k \to \infty$ then (a) follows.

To prove (b) we assume $\alpha > -1/2$. Consider first (i). Fix $x > 0$, define

$$q_{\alpha,x}(y) = \begin{cases} y^{\alpha+1/2} , & 0 < y \le x , \\ 0 , & y > x , \end{cases}$$

and $r_{\alpha,x}(y) = x^{\alpha+1}y^{-1/2}J_{\alpha+1}(xy)$. Since

$$(\mathcal{H}_\alpha q_{\alpha,x})(t) = \int_0^x (ty)^{1/2}J_\alpha(ty)y^{\alpha+1/2}dy$$

$$= x^{\alpha+1}y^{-1/2}J_{\alpha+1}(xy) = r_{\alpha,x}(y) ,$$

where the second equality follows from [11, 7.7.1(1)], (see also [31]). Since $q_{\alpha,x} \in L^2(\mathbb{R}^+)$ it follows from [38, Theorem 129] that

$$\mathcal{H}_\alpha r_{\alpha,x} = \mathcal{H}_\alpha^2 q_{\alpha,x} = q_{\alpha,x} .$$

Also since $u_\alpha(x)^{-q'} = \left(x^{\alpha+1/2}u(x)\right)^{q'}$ is non-decreasing

$$\|q_{\alpha,x}\|_{q',1/u}^{q'} = \int_0^x y^{(\alpha+1/2)q'} u(y)^{-q'} dy$$

$$= \int_0^x y^{2(\alpha+1/2)q'} y^{-(\alpha+1/2)q'} u(y)^{-q'} dy$$

$$\le x\left(x^{\alpha+1/2}u(x)\right)^{-q'} \int_0^x y^{2(\alpha+1/2)q'-1} dy$$

$$= Cx^{(\alpha+1/2)q'-1} u(x)^{-q'} < \infty ,$$

so that $q_{\alpha,x} \in L^{q'}_{1/u}(\mathbb{R}^+)$. But from part (a)

$$\int_0^x y^{\alpha+1/2} (\mathcal{H}_\alpha f)(y) dy = \int_0^\infty q_{\alpha,x}(y) (\mathcal{H}_\alpha f)(y) dy$$

$$= \int_0^\infty f(y)(\mathcal{H}_\alpha q_{\alpha,x})(y)dy = \int_0^\infty r_{\alpha,x}(y)f(y)dy$$

$$= x^{\alpha+1/2}\int_0^\infty (xy)^{1/2}J_{\alpha+1}(xy)f(y)y^{-1}dy$$

and the result follows on differentiation.

To prove (b) (ii), define $g_x(t) = \chi_{(0,x)}(t)$, then

$$(\mathcal{H}_\alpha g_x)(t) = \int_0^x (ty)^{1/2}J_\alpha(ty)dy$$

and since u_α is non-increasing $g_x \in L^{q'}_{1/u}(R^+)$. Now apply part (a) to get

$$\int_0^x (\mathcal{H}_\alpha f)(t)dt = \int_0^\infty (\mathcal{H}_\alpha f)(t)g_x(t)dt$$

$$= \int_0^\infty f(t)(\mathcal{H}_\alpha g_x)(t)dt$$

$$= \int_0^\infty f(t)\int_0^x (ty)^{1/2}J_\alpha(ty)dydt ,$$

and (b) (ii) follows on differentiating. This completes the proof of the theorem.

Let $f(x)$ be a radial function on $\mathbb{R}^n$, $x \in \mathbb{R}^n$, $t = |x|$ and let θ_n be the volume of the unit n-sphere in $\mathbb{R}^n$ then $n\theta_n$ is the surface area of the unit n-sphere and

$$\int_{\mathbb{R}^n} |f(x)|\,dx = n\theta_n \int_0^\infty |f(t)|\,t^{n-1}\,dt \ .$$

Since the Fourier transform

$$\hat{f}(x) = \int_{\mathbb{R}^n} e^{-ix\cdot y} f(y)\,dy\ , \qquad x \in \mathbb{R}^n$$

with f radial can be written in terms of the Hankel transformation, we obtain the following corollary from Theorem 3.13.

<u>Corollary 3.15</u>. Let u and v be non-negative radial functions on $\mathbb{R}^n$ and suppose f is radial. If $\left(r^{(n-1)/q}u(r), r^{-(n-1)/p'}v(r)\right) \in F^*_{p,q}$, $1 < p,\ q < \infty$ then

$$\left\{\int_{\mathbb{R}^n} |u(x)\hat{f}(x)|^q dx\right\}^{1/q} \le C\left\{\int_{\mathbb{R}^n} |v(x)f(x)|^p dx\right\}^{1/p} .$$

Proof. Let $f(x) = g(t)$, $|x| = t$, then $\hat{f}$ is radial and writing $\hat{f}(y) = \phi(r)$, $r = |y|$ one obtains ([35, V.3])

$$\phi(r) = (2\pi)^{1-n/2} r^{1-n/2} \int_0^\infty t^{n/2} J_{(n-2)/2}(rt) g(t)\,dt$$

$$= (2\pi)^{1-n/2} r^{(1-n)/2} (\mathcal{H}_{(n-2)/2} h)(r) ,$$

where $h(t) = t^{(n-1)/2} g(t)$ and $\mathcal{H}_{(n-2)/2}$ is the Hankel transform. Now

$$\left\{\int_{R^n} |u(y)\hat{f}(y)|^q dy\right\}^{1/q} = C\left\{\int_0^\infty |u(r)\phi(r)|^q r^{n-1} dr\right\}^{1/q}$$

$$= C\left\{\int_0^\infty |r^{(n-1)/q} r^{(1-n)/2} u(r) (\mathcal{H}_{(n-2)/2} h)(r)|^q dr\right\}^{1/q} .$$

But by Theorem 3.13 (b) with $\alpha = (n-2)/2$ this is dominated by

$$C\left\{\int_0^\infty |r^{(n-2)/2+1/2} r^{-(n-1)/p'} v(r) h(r)|^p dr\right\}^{1/p}$$

$$= C\left\{\int_0^\infty |r^{(n-1)(1-1/p')} v(r) g(r)|^p dr\right\}^{1/p}$$

$$= C\left\{\int_{\mathbb{R}^n} |v(x)f(x)|^p dx\right\}^{1/p} .$$

This completes the proof of the corollary.

In the next application we consider the $\mathcal{K}$-transform defined by

(3.34) $$(\mathcal{K}_\alpha f)(x) = \int_0^\infty K_\alpha(xt)(xt)^{1/2} f(t)dt , \quad x > 0 , \quad \alpha \geq -1/2 ,$$

where K_α is the modified Bessel function of third kind of order α. This function has the integral representations ([12, P.958(3), P.959(5)])

$$K_\alpha(x) = (2^{-\alpha}\pi^{1/2}x^\alpha/\Gamma(\alpha+1/2))\int_1^\infty e^{-xt}(t^2-1)^{\alpha-1/2}dt , \quad \alpha \geq -1/2 ,$$

$$K_\alpha(x) = (2^\alpha\Gamma(\alpha+1/2)x^{-\alpha}/\pi^{1/2})\int_0^\infty [\cos xt/(1+t^2)^{\alpha+1/2}]dt , \quad \alpha > -1/2 .$$

If $\alpha = \pm 1/2$ the $\mathcal{K}$-transform reduces to the Laplace transform:

$$(\mathcal{K}_{\pm 1/2} f)(x) = (\pi/2)^{1/2}\int_0^\infty e^{-xy} f(y)dy .$$

Substituting these kernels into (3.34) one obtains

(3.35) $$(\mathcal{K}_\alpha f)(x) = (2^{-\alpha}\pi^{1/2}/\Gamma(\alpha+1/2)) \times$$

$$\int_0^\infty y^{\alpha+1/2} f(y)\left[\int_1^\infty e^{-xyt}(t^2-1)^{\alpha-1/2}dt\right]dy$$

(3.36) $$(\mathcal{K}_\alpha f)(x) = 2^\alpha\pi^{-1/2}\Gamma(\alpha+1/2)x^{-\alpha+1/2} \times$$

$$\int_0^\infty y^{-\alpha+1/2} f(y)\left\{\int_0^\infty [\cos xyt/(1+t^2)^{\alpha+1/2}]dt\right\}dy$$

which are needed to prove the weighted norm inequality for $\mathcal{K}_\alpha$.

<u>Theorem 3.16</u>. Let $\alpha \geq -1/2$, $\alpha \neq 0$, u, v non-negative functions on $\mathbb{R}^+$ such that $u_\alpha(x) = x^{1/2-|\alpha|}u(x)$ and $v_\alpha(x) = x^{-1/2+|\alpha|}v(x)$. Suppose that $(u_\alpha, v_\alpha) \in F^*_{p,q}$, $1 < p, q < \infty$ and $f \in L^p_v(\mathbb{R}^+)$.

(a) If $\lim_{k\to\infty} \|f_k - f\|_{p,v} = 0$ for a sequence $\{f_k\}$ of simple functions, then $\{\mathcal{K}_\alpha f_k\}$ converges in $L^q_u(\mathbb{R}^+)$ to a function $\mathcal{K}_\alpha f \in L^q_u(\mathbb{R}^+)$. $\mathcal{K}_\alpha f$ is independent of the sequence $\{f_k\}$ and is called the $\mathcal{K}$-transform of f.

(b) There is a constant $C > 0$, independent of f such that

(3.37) $$\|\mathcal{K}_\alpha f\|_{q,u} \le C\|f\|_{p,v} .$$

Proof. Since $(\mathcal{K}_{1/2}f)(x) = (\mathcal{K}_{-1/2}f)(x)$ we may assume that $\alpha > -1/2$.

The proof of (a) is the same as in Theorem 3.13 and hence omitted. To prove (b) we take f simple, since a standard limiting argument proves the result for $f \in L_v^p(\mathbb{R}^+)$.

Consider first the case $-1/2 < \alpha < 0$, then an interchange of order of integration shows that (3.35) can be written in the form

$$(\mathcal{K}_\alpha f)(x) = (2^{-\alpha}\pi^{1/2}x^{\alpha+1/2}/\Gamma(\alpha+1/2))\int_1^\infty (t^2-1)^{\alpha-1/2}\left[\int_0^\infty e^{-xyt}y^{\alpha+1/2}f(y)dy\right]dt$$

$$= x^{\alpha+1/2}(T'_\alpha g)(x) , \quad \text{where} \quad g(y) = y^{\alpha+1/2}f(y) , \quad \text{and}$$

$$(T'_\alpha g)(x) = (2^{-\alpha}\pi^{1/2}/\Gamma(\alpha+1/2))\int_1^\infty (t^2-1)^{\alpha-1/2}\left[\int_0^\infty e^{-xyt}g(y)dy\right]dt .$$

Since f vanishes outside $(0,a)$ for some $a > 0$, Hölder's inequality shows that

$$\int_0^\infty y^{\alpha+1/2} v_\alpha(y)\,|f(y)|\,v_\alpha(y)^{-1}\int_1^\infty e^{-xyt}(t^2-1)^{\alpha-1/2}\,dt\,dy$$

$$\le \|f\|_{p,v}\left\{\int_0^a (1/v_\alpha)^*(y)^{p'}\left|\int_1^\infty (t^2-1)^{\alpha-1/2}dt\right|^{p'}dy\right\}^{1/p'} < \infty ,$$

so that the above interchange of order of integration is justified by Fubini's theorem. Since $\int_1^\infty (t^2-1)^{\alpha-1/2}dt < \infty$, it is clear that $\|T_\alpha' g\|_\infty \le C\|g\|_1$. Also since $e^{-xyt} \le e^{-xy}$ if $t \ge 1$, and since the Laplace transform maps $L^2(\mathbb{R}^+)$ to $L^2(\mathbb{R}^+)$ it follows that

$$\left\{\int_0^\infty |(T_\alpha' g)(x)|^2 dx\right\}^{1/2}$$

$$\le (2^{-\alpha}\pi^{1/2}/\Gamma(\alpha+1/2))\left\{\int_0^\infty \left|\int_1^\infty (t^2-1)^{\alpha-1/2}dt\int_0^\infty e^{-xy}g(y)dy\right|^2 dx\right\}^{1/2}$$

$$\le (2^{-\alpha+1}\pi^{1/2}/\Gamma(\alpha+1/2))\, B(\alpha+1/2, -\alpha)\left\{\int_0^\infty |g(y)|^2 dy\right\}^{1/2} .$$

Hence T_α': $L^1(\mathbb{R}^+) \to L^\infty(\mathbb{R}^+)$ and T_α': $L^2(\mathbb{R}^+) \to L^2(\mathbb{R}^+)$ and if $(u_\alpha, v_\alpha) \in F^*_{p,q}$ then by Corollary 3.12 with $(Bf)(x) \equiv (\mathcal{K}_\alpha f)(x)$, $w(x) = x^{\alpha+1/2}$ and $(Tg)(x) \equiv (T_\alpha' g)(x)$ we obtain

$$\left\{\int_0^\infty |u(x)(\mathcal{K}_\alpha f)(x)|^q dx\right\}^{1/q} \le C\left\{\int_0^\infty |v(x)f(x)|^p dx\right\}^{1/p},$$

which is the result for the case $-1/2 < \alpha < 0$.

If $\alpha > 0$ we use the integral representation (3.36) so that

$$(\mathcal{K}_\alpha f)(x) = 2^\alpha \pi^{-1/2}\Gamma(\alpha+1/2)x^{-\alpha+1/2} \times$$

$$\int_0^\infty (1+t^2)^{-\alpha-1/2}\left[\int_0^\infty \cos xyt\, y^{-\alpha+1/2} f(y)dy\right]dt$$

$$= x^{-\alpha+1/2}(T''_\alpha g)(x), \quad \text{where} \quad g(y) = y^{-\alpha+1/2}f(y) \quad \text{and}$$

$$(T''_\alpha g)(x) = 2^\alpha \pi^{-1/2}\Gamma(\alpha+1/2)\int_0^\infty (1+t^2)^{-\alpha-1/2}\left[\int_0^\infty \cos xyt\, g(y)dy\right]dt\,.$$

Again it is easily seen that the interchange of order of integration is justified by Fubini's theorem.

Next we show that $T''_\alpha: L^1(\mathbb{R}^+) \to L^\infty(\mathbb{R}^+)$ and $T''_\alpha: L^2(\mathbb{R}^+) \to L^2(\mathbb{R}^+)$. Clearly $T''_\alpha: L^1(\mathbb{R}^+) \to L^\infty(\mathbb{R}^+)$ and by Minkowski's integral inequality and Plancherel's theorem for the Fourier cosine transform we obtain

$$\left\{\int_0^\infty |(T''_\alpha g)(x)|^2 dx\right\}^{1/2}$$

$$\le 2^\alpha \pi^{-1/2}\Gamma(\alpha+1/2)\int_0^\infty \left\{\left(\int_0^\infty (1+t^2)^{-\alpha-1/2}\left|\int_0^\infty \cos xty\, g(y)dy\right|\right)^{1/2} dx\right\}^{1/2} dt$$

$$= 2^\alpha \pi^{-1/2}\Gamma(\alpha+1/2)\left\{\int_0^\infty (1+t^2)^{-\alpha-1/2} t^{-1/2} dt\right\}\left\{\int_0^\infty \left|\int_0^\infty \cos sy\, g(y)dy\right|^2 ds\right\}^{1/2}$$

$$= 2^{\alpha-1}\pi^{-1/2}\Gamma(\alpha+1/2)B(1/4,\alpha+1/4)\left\{\int_0^\infty |g(y)|^2 dy\right\}^{1/2} .$$

Finally applying Corollary 3.12, one obtains with $(u_\alpha, v_\alpha) \in F^*_{p,q}$, $(Bf)(x) \equiv (\mathcal{K}_\alpha f)(x)$, $w(x) = x^{-\alpha+1/2}$ and $(Tg)(x) \equiv (T''_\alpha g)(x)$

$$\left\{\int_0^\infty |u(x)(\mathcal{K}_\alpha f)(x)|^q dx\right\}^{1/q} \le C\left\{\int_0^\infty |v(x)f(x)|^p dx\right\}^{1/p} .$$

This completes the proof of the theorem.

Note that as in the case of $\mathcal{H}_\alpha$ we obtain also here a partial converse: Define

$$f(x) = \begin{cases} x^{-\xi} v(x)^{-p'} , & \text{if } 0 < x < s , \\ 0 & \text{otherwise} , \end{cases}$$

where $\xi = (1/2-\alpha)/(1-p)$, then by reducing the left side of (3.37) we get

$$\left\{ \int_0^{1/s} u(x)^q \left| \int_0^s (xt)^{1/2} \mathcal{K}_\alpha(xt) t^{-\xi} v(t)^{-p'} dt \right|^q dx \right\}^{1/q}$$

$$\le C \left\{ \int_0^\infty v(x)^{-p'} x^{-\xi p} dx \right\}^{1/p} .$$

Now

$$K_\alpha(xt) = C_\alpha (xt)^{-\alpha} \int_0^\infty \cos xty \, (1+y^2)^{-\alpha-1/2} dy , \qquad \alpha > -1/2 ,$$

where $C_\alpha = 2^\alpha \Gamma(\alpha+1/2)/\pi^{1/2}$, and since the integral converges and is positive for all xt , $0 < xt \le 1$, there exists some η , $0 < \eta < 1$ such that both

$$\int_0^{\eta} \cos xty\,(1+y^2)^{-\alpha-1/2}dy \quad \text{and} \quad \int_{\eta}^{\infty} \cos xty\,(1+y^2)^{-\alpha-1/2}dy$$

are positive. But then for $0 < xt \le 1$

$$K_\alpha(xt) \ge C_\alpha(xt)^{-\alpha}\int_0^{\eta} \cos y\,(1+y^2)^{-\alpha-1/2}dy = C(xt)^{-\alpha} .$$

Substituting this estimate in the left side of the previous inequality we obtain with $\xi = (1/2-\alpha)/(1-p)$

$$\left\{\int_0^{1/s} u(x)^q \left[\int_0^s (xt)^{1/2-\alpha} t^{(1/2-\alpha)/(p-1)} v(t)^{-p'}dt\right]^q dx\right\}^{1/q}$$

$$\le C\left\{\int_0^s v(x)^{-p'} x^{(1/2-\alpha)p'}dx\right\}^{1/p}$$

or

$$\left\{\int_0^{1/s}\left[x^{1/2-\alpha}u(x)\right]^q dx\right\}^{1/q}\left\{\int_0^s t^{(1/2-\alpha)(1+1/(p-1))}v(t)^{-p'}dt\right\} .$$

$$\le C\left\{\int_0^s \left[v(x)x^{\alpha-1/2}\right]^{-p'}dx\right\}^{1/p}$$

and this implies

$$\left\{\int_0^{1/s}\left[x^{1/2-\alpha}u(x)\right]^q dx\right\}^{1/q}\left\{\int_0^{s}\left[x^{-1/2+\alpha}v(x)\right]^{-p'} dx\right\}^{1/p'} \le C ,$$

for all $s > 0$ and $\alpha > -1/2$.

The final application involves the $\mathcal{Y}$-transform, defined for $0 < |\alpha| \le 1/2$ by

$$(3.38)\qquad (\mathcal{Y}_\alpha f)(x) = \int_0^\infty Y_\alpha(xt)(xt)^{1/2}f(t)dt , \quad x > 0 .$$

Here Y_α is the Bessel function of second kind or Neuman's function. If $0 < |\alpha| < 1/2$, the kernel Y_α can be written as

$$Y_\alpha(x) = (2(x/2)^\alpha/(\Gamma(1/2+\alpha)\Gamma(1/2)))\left\{\int_0^{\pi/2}\sin(x\sin\theta)\cos^{2\alpha}\theta d\theta - \int_0^\infty [e^{-xy}/(1+y^2)^{-\alpha+1/2}]dy\right\} , \quad ([12, \text{P}.955(5)])$$

or

$$Y_\alpha(x) = -(2(x/2)^{-\alpha}/\Gamma(1/2-\alpha)\Gamma(1/2))\int_1^\infty [\cos xt/(t^2-1)^{\alpha+1/2}]dt \ ,$$

([12, P.955(2)]). If $\alpha = 1/2$, $Y_{1/2}(x) = -(2/(\pi x))^{1/2}\cos x$, ([12, P.967(1)]) and if $\alpha = -1/2$, $Y_{-1/2}(x) = (2/(\pi x))^{1/2}\sin x$, ([12, P.967 (2)]).

It follows that (3.38) has the two integral representations

$$(3.39)\quad (\mathscr{Y}_\alpha f)(x) = (2^{1-\alpha}x^{\alpha+1/2}/(\Gamma(1/2+\alpha)\Gamma(1/2)))\int_0^\infty t^{\alpha+1/2}f(t)\ \times$$

$$\left\{\int_0^{\pi/2} \sin(xt\sin\theta)\cos^{2\alpha}\theta d\theta - \int_0^\infty [e^{-xty}/(1+y^2)^{-\alpha+1/2}]dy\right\}dt$$

and

$$(3.40)\quad (\mathscr{Y}_\alpha f)(x) = -(2^{1+\alpha}x^{1/2-\alpha}/(\Gamma(1/2-\alpha)\Gamma(1/2)))\int_0^\infty t^{1/2-\alpha}f(t)\ \times$$

$$\left\{\int_1^\infty [\cos xty/(y^2-1)^{\alpha+1/2}]dy\right\}dt$$

which are needed to prove the weighted norm inequality for $\mathcal{Y}_\alpha$.

<u>Theorem 3.17</u>. Let $0 < |\alpha| \le 1/2$, u, v non-negative functions on $\mathbb{R}^+$ such that $u_\alpha(x) = x^{1/2-|\alpha|}u(x)$ and $v_\alpha(x) = x^{-1/2+|\alpha|}v(x)$. Suppose that $(u_\alpha, v_\alpha) \in F^*_{p,q}$, $1 < p, q < \infty$ and $f \in L^p_v(\mathbb{R}^+)$.

(a) If $\lim_{k\to\infty} \|f_k - f\|_{p,v} = 0$ for a sequence $\{f_k\}$ of simple functions, then $\{\mathcal{Y}_\alpha f_k\}$ converges in $L^q_u(\mathbb{R}^+)$ to a function $\mathcal{Y}_\alpha f \in L^q_u(\mathbb{R}^+)$. $\mathcal{Y}_\alpha f$ is independent of the sequence $\{f_k\}$ and is called the $\mathcal{Y}$-transform of f .

(b) There is a constant $C > 0$, independent of f such that

$$\|\mathcal{Y}_\alpha f\|_{q,u} \le C\|f\|_{p,v} .$$

<u>Proof</u>. If $\alpha = \pm 1/2$ the result reduces to weighted estimates for the Fourier cosine-, and sine-transformations, which follows from [4] or [18]. Also the proof of (a) is the same as in Theorem 3.13 and hence omitted.

We prove (b) by arguing as in the proof of Theorems 3.13 and 3.16.

First letting $-1/2 < \alpha \le 0$, then by (3.39)

$$(\mathcal{Y}_\alpha f)(x) = (2^{1-\alpha}x^{\alpha+1/2}/(\Gamma(1/2+\alpha)\Gamma(1/2)))\left\{\int_0^{\pi/2}\cos^{2\alpha}\theta\left[\int_0^\infty \sin(xt\sin\theta)\right.\right.$$

$$\left.\left.\times\, t^{\alpha+1/2}f(t)dt\right]d\theta - \int_0^\infty (1+y^2)^{\alpha-1/2}\left[\int_0^\infty e^{-xty}t^{\alpha+1/2}f(t)dt\right]dy\right\}$$

$$\equiv x^{\alpha+1/2}(F'_\alpha g)(x)\ , \quad \text{where} \quad g(t) = t^{\alpha+1/2}f(t) \quad \text{and}$$

$$(F'_\alpha g)(x) = (2^{1-\alpha}/(\Gamma(\alpha+1/2)\Gamma(1/2)))\left\{\int_0^{\pi/2}\cos^{2\alpha}\theta\left[\int_0^\infty \sin(xt\sin\theta)\right.\right.$$

$$\left.\left.\times\, g(t)dt\right]d\theta - \int_0^\infty (1+y^2)^{\alpha-1/2}\left[\int_0^\infty e^{-xty}g(t)dt\right]dy\right\}.$$

The interchange of the order of integrations being justified by Fubini's theorem and the fact that f vanishes outside $(0,a)$, for some $a > 0$. Thus Hölder's inequality shows that

$$\int_0^\infty |t^{\alpha+1/2}v_\alpha(t)f(t)v_\alpha(t)^{-1}|\left\{\int_0^{\pi/2}|\sin(xt\sin\theta)\cos^{2\alpha}\theta|d\theta\right.$$

$$\left.+\int_0^\infty [e^{-xty}/(1+y^2)^{-\alpha+1/2}]dy\right\}dt$$

$$\le \|f\|_{p,v} \left\{ \int_0^a (1/v_\alpha)^*(t)^{p'} \left[\int_0^{\pi/2} |\sin(xt \sin\theta)\cos^{2\alpha}\theta|\, d\theta \right.\right.$$

$$\left.\left. + \int_0^\infty [e^{-xty}/(1+y^2)^{-\alpha+1/2}]\, dy \right]^{p'} dt \right\}^{1/p'} < \infty .$$

Since the sum of the two inner integrals is dominated by

$$B(\alpha+1/2,1/2) + \int_0^\infty (1-y^2)^{\alpha-1/2} dy = B(\alpha+1/2,1/2) + B(-\alpha,1/2)/2 < \infty ,$$

where again B denotes the beta function, it follows that

$$|(F_\alpha' g)(x)| \le (2^{1-\alpha}/(\Gamma(1/2+\alpha)\Gamma(1/2))) \int_0^\infty |g(t)| \left\{ \left| \int_0^{\pi/2} \sin(xt \sin\theta) \right.\right.$$

$$\left.\left. \times \cos^{2\alpha}\theta\, d\theta \right| + \int_0^\infty [e^{-xty}/(1+y^2)^{-\alpha+1/2}]\, dy \right\} dt$$

$$\le C \int_0^\infty |g(t)|\, dt .$$

Also, by using Minkowski's integral inequality

$$\left\{\int_0^\infty |(F'_\alpha g)(x)|^2 dx\right\}^{1/2} \le (2^{1-\alpha}/(\Gamma(1/2+\alpha)\Gamma(1/2))) \left\{\int_0^{\pi/2} \cos^{2\alpha}\theta \right.$$

$$\times \left[\int_0^\infty \left|\int_0^\infty \sin(xt\sin\theta)g(t)dt\right|^2 dx\right]^{1/2} d\theta$$

$$\left. + \int_0^\infty (1+t^2)^{\alpha-1/2} \left[\int_0^\infty \left|\int_0^\infty e^{-xty} g(y)dy\right|^2 dt\right]^{1/2} dt\right\} .$$

If we let $x\sin\theta = z$, $\theta \in (0,\pi/2)$ in the first inner integral and $xt = z$ in the second inner integral, then the above integrals are dominated by

$$C\left\{\int_0^{\pi/2} \cos^{2\alpha}\theta \sin^{-1/2}\theta d\theta \left[\int_0^\infty \left|\int_0^\infty \sin(tz)g(t)dt\right|^2 dz\right]^{1/2}\right.$$

$$\left. + \int_0^\infty (1+t^2)^{\alpha-1/2} t^{-1/2} \left[\int_0^\infty \left|\int_0^\infty e^{-yz} g(y)dy\right|^2 dz\right]^{1/2} dt\right\}$$

$$\le C\{B(\alpha+1/2,1/2) + B(-\alpha+1/4,1/4)/2\} \left\{\int_0^\infty |g(y)|^2 dy\right\}^{1/2} ,$$

where the last inequality follows from Plancherel's theorem for the

Fourier sine transform and the fact that the Laplace transform maps $L^2(\mathbb{R}^+)$ to $L^2(\mathbb{R}^+)$.

Now if $(u_\alpha, v_\alpha) \in F^*_{p,q}$, $1 < p, q < \infty$, then by Corollary 3.12 with $(\mathscr{Y}_\alpha f)(x) \equiv (Bf)(x)$, $w(x) = x^{\alpha+1/2}$ and $(Tg)(x) \equiv (F'_\alpha g)(x)$ we obtain

$$\left\{\int_0^\infty |u(x)(\mathscr{Y}_\alpha f)(x)|^q dx\right\}^{1/q} \le C\left\{\int_0^\infty |v(x)f(x)|^p dx\right\}^{1/p},$$

which proves the theorem for $-1/2 < \alpha < 0$.

If $0 < \alpha < 1/2$ we use the integral representation (3.40) so that

$$(\mathscr{Y}_\alpha f)(x) = -(2^{1+\alpha}x^{-\alpha+1/2}/(\Gamma(1/2-\alpha)\Gamma(1/2)))\int_1^\infty (t^2-1)^{-\alpha-1/2}$$

$$\times \left\{\int_0^\infty \cos(xyt)y^{-\alpha+1/2}f(y)dy\right\}dt$$

$$= x^{-\alpha+1/2}(F''_\alpha g)(x), \text{ where } g(y) = y^{-\alpha+1/2}f(y) \text{ and}$$

$$(F''_\alpha g)(x) = -(2^{1+\alpha}/(\Gamma(1/2-\alpha)\Gamma(1/2)))\int_1^\infty (t^2-1)^{-\alpha-1/2}\left\{\int_0^\infty \cos(xyt)g(y)dy\right\}dt .$$

Again it is seen that the interchange of order of integration is justified by Fubini's theorem. Since F''_{α} is of type $(1,\infty)$ and $(2,2)$ the result follows in this case also.

3.3 A Weighted Lebesgue-Lorentz Inequality For The Laplace Transformation.

Recall that the Laplace transform of f is defined by

$$(\mathcal{L}f)(x) = \int_0^\infty e^{-xt} f(t)\,dt\ , \qquad x > 0\ .$$

Our chief aim in this section is to find conditions on non-negative functions (u,v) which are sufficient for the weighted inequality

$$\|\mathcal{L}f\|_{p,q,u} \le C\|f\|_{r,v}\ , \tag{3.41}$$

$1 \le r,\ q < \infty$, $0 < p < \infty$, where $\|\cdot\|_{p,q,u}$ denotes the weighted Lorentz quasi-norm.

Definition 3.18. Let u and v be non-negative functions defined on $\mathbb{R}^+$. We write $(u,v) \in L^{\beta}_{r,q}$, for some $\beta \in [0,1]$, $1 \le r,\ q < \infty$, if

(a)

$$\text{(3.42)} \qquad \sup_{y>0}\left\{\int_0^{1/y} u(x)^q dx\right\}^{1/q}\left\{\int_0^{y} v(x)^{-r'} dx\right\}^{1/r'} < \infty ,$$

$$\text{(3.43)} \quad \sup_{y>0}\left\{\int_{1/y}^{\infty} e^{-\beta sqx} u(x)^q dx\right\}^{1/q}\left\{\int_y^{\infty} e^{-(1-\beta)r'x/y} v(x)^{-r'} dx\right\}^{1/r'} < \infty ,$$

hold for $1 \le r \le q < \infty$.

(b) If $1 \le q < r < \infty$ we require that the conditions

$$\int_0^{\infty}\left[\left(\int_0^{1/y} u(x)^q dx\right)^{1/q}\left(\int_0^{y} v(x)^{-r'} dx\right)^{1/q'}\right]^s v(y)^{-r'} dy < \infty ,$$

$$\int_0^{\infty}\left[\left(\int_{1/y}^{\infty} e^{-xyq} u(x)^q dx\right)^{1/q}\left(\int_y^{\infty} v(x)^{-r'} dx\right)^{1/q'}\right]^s v(y)^{-r'} dy < \infty ,$$

hold, where $1/s = 1/q - 1/r$.

Observe that if u is non-increasing and v non-decreasing then (3.42) implies (3.43) ([1, Theorem 2.4]).

In [1] and [17] Andersen and Heinig showed that if $(u,v) \in L^{\beta}_{r,q}$, then

(3.44) $$\|\mathcal{L}f\|_{q,u} \le C\|f\|_{r,v} .$$

Moreover if u is non-increasing and v non-decreasing then (3.42) is both necessary and sufficient for (3.44).

We extend these results to the case where the left side of (3.44) is replaced by weighted Lorentz quasi-norm.

Definition 3.19 [35]. Let f and $w(x) \ge 0$ be defined on $\mathbb{R}^+$ then the distribution function of f relative to the measure $w(x)dx$ is defined by

$$f_w(s) = \int_{\{x:|f(x)|>s\}} w(x)dx = w(\{x: |f(x)| > s\}) ,$$

where $s > 0$ and the decreasing rearrangement of $|f|$ relative to $w(x)dx$ is defined by $f^w(t) = \inf\{s: f_w(s) \le t\}$. Further, if $0 < p, q \le \infty$, then the weighted Lorentz spaces $L(p,q,w)$ are defined by

$$L(p,q,w) = \{f: \|f\|_{p,q,w} < \infty\} ,$$

where

$$(3.45)\quad \|f\|_{p,q,w} = \begin{cases} \left\{\frac{q}{p}\int_0^\infty \left[t^{1/p} f^w(t)\right]^q t^{-1}dt\right\}^{1/q}, & 0 < p,\ q < \infty \\ \sup_{t>0} t^{1/p} f^w(t)\ , \quad 0 < p \le \infty, & q = \infty\ . \end{cases}$$

In case either $1 < p < \infty$ and $1 \le q < \infty$ or $p = q = \infty$, $L(p,q,w)$ is a Banach space with norm equivalent to the quasi-norm $\|f\|_{p,q,w}$.

Clearly if $w \equiv 1$ then $L(p,q,w) = L(p,q)$, where $L(p,q)$ are the usual Lorentz spaces. Note that $L(q,q,w) = L^q_w(\mathbb{R}^+)$.

<u>Lemma 3.20</u>. Suppose $0 < p,\ q < \infty$. Then

(a)
$$\|f\|^q_{p,q,w} = q\int_0^\infty f_w(s)^{q/p} s^{q-1} ds\ .$$

(b) If f is non-negative and non-increasing on $\mathbb{R}^+$ and if $g(x) = \int_0^x w(s)ds < \infty$, whenever $w(x) > 0$, then

$$\|f\|^q_{p,q,w} = \frac{q}{p}\int_0^\infty f(x)^q g(x)^{q/p-1} w(x)dx .$$

Part (a) is due to Sawyer [34, Lemma 1] and part (b) follows essentially along the same lines as in [34, Lemma 1].

Proof. Part (a) follows, on evaluating the two integrals of $qs^{q-1}(q/p)t^{q/p-1}$ over the set $\{(t,s): 0 < s < f^w(t), 0 < t\}$. By performing the s integration first we obtain the right side of (3.45) to the q-th power, i.e.,

$$\int_{\{(t,s):0<s<f^w(t),0<t\}} qs^{q-1}(q/p)t^{q/p-1}dsdt$$

$$= q\int_0^\infty s^{q-1}\left(\int_0^{f_w(s)} (q/p)t^{q/p-1}dt\right)ds = q\int_0^\infty s^{q-1}f_w(s)^{q/p}ds .$$

Hence part (a) follows.

(b) is established by evaluating the two iterated integrals of

(3.46) $$qx^{q-1}(q/p)g(y)^{q/p-1}w(y) ,$$

over the set $M = \{(x,y): 0 < x < f(y); 0 < y\}$. Performing the x integration over M on (3.46) first yields the right side of (b)

$$\int_0^\infty \left[q \int_0^{f(y)} x^{q-1} dx \right] (q/p) g(y)^{q/p-1} w(y) dy$$

$$= (q/p) \int_0^\infty f(y)^q g(y)^{q/p-1} w(y) dy$$

and performing the y integration first yields the right side of (a)

$$\int_0^\infty \left[q \int_0^{f(y)} x^{q-1} dx \right] (q/p) g(y)^{q/p-1} w(y) dy \tag{3.47}$$

$$= (q^2/p) \int_0^\infty x^{q-1} dx \int_0^\infty \chi_{(0,f(y))}(x) g(x)^{q/p-1} w(y) dy .$$

But since $(0,f(y)) = \{0 < x < f(y); y > 0\} = (0,S(x))$ where $S(x) = \inf\{y: f(y) \le x\}$. Therefore, the right side of (3.47) is equal to

$$\int_0^\infty x^{q-1} dx \int_{(0,S(x))} g(y)^{q/p-1} w(y) dy \tag{3.48}$$

$$= (q^2/p)\int_0^\infty x^{q-1}dx \int_0^{S(x)} \left(\int_0^y w(t)dt\right)^{q/p-1} d\left(\int_0^y w(t)dt\right)$$

$$= q\int_0^\infty x^{q-1} g(S(x))^{q/p} dx .$$

But

(3.49) $$f_w(x) = w(\{y: f(y) > x\}) = \int_{\{y:f(y)>x\}} w(y)dy$$

$$= \int_0^{S(x)} w(y)dy = g(S(x)) .$$

So that, from (3.48) and (3.49) we obtain

$$\|f\|^q_{p,q,w} = q\int_0^\infty x^{q-1} f_w(x)^{q/p} dx$$

$$= (q/p)\int_0^\infty f(x)^q g(x)^{q/p-1} w(x)dx .$$

This completes the proof of the lemma.

We can now state and prove the main result of this section.

<u>Theorem 3.21</u>. Let u and v be defined on $\mathbb{R}^+$. If $1 \le r, q < \infty$, $0 < p < \infty$ and $(\tilde{u},v) \in L^{\beta}_{r,q}$, for some β, $\beta \in [0,1]$, where

$$\tilde{u}(x) = \left\{\int_0^x u(s)^p ds\right\}^{1/p-1/q} u(x)^{p/q} ,$$

then

$$\|\mathcal{L}f\|_{p,q,u^p} \le C\|f\|_{r,v} .$$

<u>Proof</u>. From Lemma 3.20 (b), for $1 \le q < \infty$, $0 < p < \infty$ we have for all $f(x) \ge 0$

$$\|\mathcal{L}f\|^q_{p,q,u^p} = (q/p)\int_0^\infty (\mathcal{L}f)^q(x)\tilde{u}(x)^q dx .$$

Thus from this equality and (3.44) one obtains

$$\|\mathcal{L}f\|_{p,q,u^p} = (q/p)^{1/q}\left\{\int_0^\infty (\mathcal{L}f)^q(x)\tilde{u}(x)^q dx\right\}^{1/q}$$

$$\le C\left\{\int_0^\infty [v(x)f(x)]^r dx\right\}^{1/r} = C\|f\|_{r,v} ,$$

provided $(\tilde{u},v) \in L^{\beta}_{r,q}$, $1 \le r, q < \infty$, $0 < p < \infty$, $\beta \in [0,1]$.

This completes the proof of the theorem.

BIBLIOGRAPHY

1. K.F. Andersen and H.P. Heinig; Weighted norm inequalities for certain integral operators; Siam J. Math. Anal. 14 (4) (1983), 834-844.

2. K.F. Andersen and B. Muckenhoupt; Weighted weak type Hardy inequalities with applications to Hilbert transform and maximal functions; Studia Math. 72 (1981), 9-26.

3. J.J. Benedetto and H.P. Heinig; Weighted Hardy spaces and the Laplace transform; Lecture Notes In Math. 992 (1983), 240-277.

4. J.J. Benedetto, H.P. Heinig and R. Johnson; Weighted Hardy spaces and the Laplace transform II; Math. Nachrichten (to appear).

5. J. Bergh and J. Löfström; Interpolation spaces; Springer-Verlag (1976).

6. J.S. Bradley; Hardy inequalities with mixed norms; Canad. Math. Bull. 21 (1978), 405-408.

7. P.L. Butzer and H. Berens; Semi-groups of operators and approximations; Springer-Verlag (1967).

8. A.P. Calderón; Intermediate spaces and interpolation, the complex method; Studia Math. 24 (1964), 113-190.

9. A.P. Calderón and R. Scott; Sobolev type inequalities for $p > 0$; Studia Math. 62 (1978), 73-92.

10. S.A. Emara and H.P. Heinig; Weighted norm inequalities for the Hankel- and K-transformations; Proc. Royal Soc. Edinburgh (to appear).

11. A. Erdélyi et al; Higher transcendental functions, V.II; New York (1953).

12. I.G. Gradshteyn and I.M. Ryzhik; Tables of integrals, series and products; Academic Press (1980).

13. J. Gustavsson; A function parameter in connection with interpolation of Banach spaces; Math. Scand. 42 (1978), 289-305.

14. G.H. Hardy, J.E. Littlewood and G. Polya; Inequalities; Cambridge Univ. Press (1967).

15. H.P. Heinig; Estimate for operators in mixed L^p-spaces; Trans. A.M.S. 287 (2) (1985), 483-493.

16. __________; Interpolation of quasi-normed spaces involving weights; Canad. Math. Soc. Conf. Proc. V.1 (1981), 245-267.

17. __________; Weighted norm inequalities for certain integral operators II; Proc. Amer. Math. Soc. (3) 95 (1985), 387-396.

18. __________; Weighted norm inequalities for classes of operators; Indiana Univ. Math. J. 33 (4) (1984), 573-582.

19. P. Heywood and P.G. Rooney; A weighted norm inequality for the Hankel transformation; Proc. Royal Soc. Edinburgh 99 A (1984), 50-55.

20. T. Holmstedt; Interpolation of quasi-normed spaces; Math. Scand. 26 (1970), 177-199.

21. L. Hörmander; Estimates for translation invariant operators in L^p-spaces; Acta Math. 103 (1960), 93-140.

22. R. Johnson; Temperatures, Riesz potentials, and the Lipshitz spaces of Herz; Proc. London Math. Soc. (3), 27 (1973), 290-316.

23. W.B. Jurkat and G. Sampson; On rearrangement and weight inequalities for the Fourier transform; Indiana Univ. Math. J. 33 (1984), 257-270.

24. T.F. Kalugina; Interpolation of Banach spaces with functional parameter. The Reiteration theorem; Vestnik Moskovskoyo University, Ser. 1, Math. Mech. 30 (6) (1975), 68-77 (Engl. Transl. Moscow Univ. Math. Bull. 30 (6) (1975), 108-116.)

25. S.G. Krein, Ju. I. Petunin and E.M. Semenov; Interpolation of linear operator; Transl. Math. Monographs, A.M.S. 54 (1982).

26. J.L. Lions and J. Peetre; Sur une class d'éspaces d'interpolation; Inst. Hautes Études Sci. Publ. Math. 19 (1964), 5-68.

27. W. Mazja; Einbettungassätze für Sobolevsche Räume; Teil 1, Teubner Texte zur Math., Teubner Verl. Leipzig (1979).

28. B. Muckenhoupt; A note on two weight function conditions for a Fourier transform inequality; Proc. A.M.S. 88 (1983), 97-100.

29. ____________; Weighted norm inequalities for the Fourier transform; Trans. A.M.S. 276 (1983), 729-742.

30. J. Peetre; Espaces d'interpolation, généralisations, applications; Rend. Sem. Math. Fis. Milano 34 (1964), 133-164.

31. P.G. Rooney; On the $\mathcal{Y}_v$ and $\mathcal{H}_v$, transformations; Canad. J. Math. 32 (1980), 1021-1044.

32. C. Sadosky; Interpolation of operators and singular integrals, An introduction to harmonic analysis; New York and Basel (1979).

33. Y. Sagher; Real interpolation with weights; Indiana Univ. Math. J. 30 (1981), 113-121.

34. E. Sawyer; Weighted Lebesgue and Lorentz norm inequalities for the Hardy operator; Trans. A.M.S. 281 (1) (1984), 329-337.

35. L. Schwartz; Mathematics for physical sciences; Hermann, Paris (1966).

36. E.M. Stein; Singular integrals and differentiability properties of functions; Princeton Univ. Press (1970).

37. E.M. Stein and G. Weiss; Fourier analysis on euclidean spaces; Princeton Univ. Press (1971).

38. E.C. Titchmarch; Introduction to the theory of Fourier integrals; Oxford Univ. Press (1948).

39. H. Triebel; Interpolation theory, Function spaces, Differential operators; VED Deutscher Verlag der Wissenschaften, Berlin (1978).