EINSTEIN METRICS ON BUNDLES

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A Thesis
Submitted to the School of Graduate Studies
in Partial Fulfilment of the Requirements
for the Degree
Doctor of Philosophy

McMaster University

Einstein Metrics on Bundles

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August 4, 1996

DOCTOR OF PHILOSOPHY(1996) McMaster University
(Mathematics) Hamilton, Ontario

TITLE: Einstein Metrics on Bundles
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SUPERVISOR: Professor McKenzie Y. Wang
NUMBER OF PAGES: vi, 101

ABSTRACT

Given a principal circle bundle with a nontrivial connection over a compact Kähler manifold, there is a Riemannian metric on the bundle by horizontal lift of the metric from the base. It is shown that, if the curvature form of the connection is of type (1,1), and the bundle is Einstein, then the base is a finite product of Kähler-Einstein manifolds with positive first Chern classes and the Euler class of the bundle must be a linear combination of the first Chern classes of the manifolds in the base. This is the uniqueness of a construction of Einstein metrics given by M.Y.Wang and W.Ziller. It is pointed out that the uniqueness theorem also holds for a class of principal torus bundles and some uniqueness for principal bundles with non-abelian structure groups are given as well.

The existence of Einstein metrics is given on some S^2-bundles. Given finitely many Kähler-Einstein manifolds with positive first Chern classes, we have principal circle bundles over their product whose Euler classes are linear combinations of their first Chern classes. To every such circle bundle an S^2-bundle can be associated. I show that there are at least two families of Einstein metrics on these S^2-bundles. For those with positive first Chern classes with respect to a natural complex structure on the total space, I present non-Kählerian Einstein metrics when the Futaki obstruction to the existence of Kähler-Einstein metrics does not vanish. An antipodal identification of every fibre of an S^2-bundle will yield an RP^2 bundle and the other family of Einstein metrics are lifts of those constructed on the associated RP^2 bundles.

Finally, the author suggests new constructions of Einstein metrics on certain principal S^1-bundles which allow the total spaces to be even dimensional.

AKNOWLEDGEMENT

I would like to express my whole-hearted gratitude to Professor McKenzie Y. Wang for his help and supervision.

Special thanks are given to Professor Min-Oo and Professor Nicas for their helpful talks and support.

I would also like to thank the Department of Mathematics & Statistics and the School of Graduate Studies for their generous financial support throughout my study at McMaster University.

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Introduction

The concept of Einstein manifolds has its origin in General Relativity. It provides models of vaccum state gravitational fields. For geometers, most researches are concentrated on homogeneous spaces, Kähler-Einstein manifolds and techniques of hyperKähler and quaternionic Kähler reductions.

Progress on homogeneous Einstein metrics was reviewed by Besse[Be](p6, p7 together with Chapter 6 and Chapter 7) and such metrics are abundant.

On the part of compact Kähler metrics, Yau showed that every Kähler class of a compact Kähler manifold with vanishing first Chern class admits a Kähler-Einstein metric. Aubin and Yau independently proved that the same conclusion holds for compact Kähler manifolds with negative first Chern classes. The remaining positive first Chern class case is more complicated because there are obstructions such as Futaki invariants and Matsushima's Theorem to the existence. In this case, in addition to several examples of Kähler-Einstein metrics, Tian showed that any complex surface with positive first Chern class has a Kähler-Einstein metric if and only if its Lie algebra of holomorphic fields is reductive. Progress in higher dimension is related to various notations of stability.

Einstein manifolds can also be constructed by hyperKähler and quaternionic Kähler reductions. We refer to [BGM1] for comments. These techniques can be applied to 3-Sasakian manifolds so as to give examples of inhomogeneous compact Einstein manifolds[BGM2].

In general, the existence problem for Einstein metrics is not solved although local existence of Einstein metrics is known[D].

The thesis is devoted to Einstein metrics on certain bundles. On this subject, Page gave on $CP^2\#\bar{CP}^2$ an inhomogeneous non-Kählerian Einstein metric with positive scalar curvature[P], which was generalized by Bérard Bergery so that the total space is an S^2-bundle and the base can be any compact Kähler-Einstein manifold with positive first Chern class[BB]. We will further generalize this result so that the base can be a product of Kähler-Einstein manifolds with positive first Chern classes.

Sakane and Koiso constructed Kähler-Einstein metrics with positive scalar curvature on the S^2-bundles over products of compact Kähler-Einstein manifolds with positive first Chern classes[S1][KS1][KS2]. The sufficient condition for their construction to yield Kähler-Einstein metrics is the vanishing of Futaki invariants. We find that there is a family of non-Kählerian Einstein metrics the sufficient condition for whose existence is in some sense the non-vanishing of Futaki invariants. In other words, although vanishing of Futaki invariants is 'good' for Kähler metrics, it is 'bad' for certain non-Kählerian metrics.

On the other hand, Wang and Ziller gave a general construction of Einstein metrics on principal torus bundles over a finite product of Kähler-Einstein manifolds with positive first Chern classes which yield compact inhomogeneous Einstein manifolds with positive scalar curvatures[WZ]. We will, on the one hand, establish uniqueness of their construction in a larger class of manifolds , and, on the other hand, propose new constructions of Einstein metrics based on their ideas.

Basic concepts and facts are given in Chapter 1. Most of the material is known although some of which has not been stated formally elsewhere. We give a detailed exposition of metric constructions on bundles in Section 1.3 which, together with Section 4.1, sets up the framework for geometric structures of this special type of Riemannian submersion.

Then following Wang and Ziller's construction, the auther shows in Chapter 2 that their construction of Einstein metrics on principal circle bundles is unique in a larger class of manifolds (Theorem 2.1). In particular, we prove a de Rham type decomposition theorem for foliations in a compact Kähler manifold with positive first Chern class (Proposition 2.2) which says such a manifold decomposes as a product if it admits two orthogonal holomorphic foliations which are mutually complementary. The proposition is of interest in itself and it is worth mentioning that it has no counterpart in the real case. Except the corollaries 2.1–2.3, the materials in Chapter 2 are essentially the content of [W] which is accepted by Differential Geometry and its Applications for publication.

Chapter 3 is devoted to the construction of new families of non-Kählerian Einstein metrics on the S^2-bundles over a finite product of Kähler-Einstein manifolds with positive first Chern classes(Theorem 3.2). We also include a proof of Koise and Sakane's existence of Kähler-Einstein metrics(Theorem 3.1) and a generalization of Bérard Bergery's construction of non-Kähler Einstein metrics(Theorem 3.3). Complete metrics are also discussed.

Chapter 4 discusses the uniqueness of Einstein metrics on compact principal bundles with non-abelian structure groups over 4-dimensional compact manifolds. For non-abelian structural groups there are few systematic works in this respect although this subject is important for the better understanding of geometry of fibre bundles[BGM2][Wa2]. Our investigation is preliminary for this topic and hence in some cases, we take different approaches to the same problem to exhibit geometric interactions. We omit proofs of Propositions 4.4 and 4.5 because the techniques are similar to the proofs of others. We establish suitable moving frames so as to do almost barehanded calculations. The main results are summarized in Section 4.1.

We will also generalize the uniqueness theorem of Chapter 2 to some principal torus bundles(Theorem 4.3). In the last section, based on Wang and Ziller's construction, we propose new constructions of Einstein metrics on certain principal circle bundles (Theorem 4.6 and Theorem 4.7, also Theorem 4.8 and Theorem 4.9), the importance of which, the author believes, is also their implication of more possible constructions of Einstein metrics along these lines and we can expect to greatly expand the collection of known compact manifolds that carry Einstein metrics. In the end, we include a variant of Wang and Ziller's construction (Theorem 4.10) which produces examples of compact odd dimensional non-homogeneous Einstein manifolds which admit circle actions with no fixed point and which are neither Sasakian nor total spaces of circle bundles.

Chapter 1

Fundamental Materials

1.1 Riemannian Geometry

A Riemannian manifold (M, g) is a manifold M equipped with an inner product g_p on the tangent space T_PM for every $p \in M$. To a given Riemannian metric, there corresponds a unique torsion-free metric connection ∇ known as the Levi-Civita connection.

The curvature tensor of (M, g) is defined to be

$$K(X, Y, Z, W) = g(\nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z - \nabla_{[Y,X]} Z, W)$$

In terms of a local orthonormal frame $\{e_A\}$, the Ricci tensor of M is

$$Ric(X, Y) = \sum K(X, e_A, Y, e_A)$$

The Ricci tensor is clearly independent of the choice of the basis.

The scalar curvature of the manifold is

$$S = \sum Ric(e_A, e_A)$$

A Killing vector field on M is a vector field so that its local 1-parameter group of local transformations consists of local isometries. X is a Killng vector field if and only if

$$g([X,Y],Z) + g(Y,[X,Z]) = Xg(Y,Z)$$

for all vector fields Y and Z, or, equivalently,

$$g(\nabla_Y X, Z) + g(Y, \nabla_Z X) = 0$$

for all Y and Z.

We now introduce the moving frame method of Cartan.

Given an orthonormal frame $\{e_i\}$, we define

$$h^i_{jk} = g(\nabla_{e_k} e_j, e_i)$$

For a vector field $X = \sum X_i e_i$(we will omit the summation expression hereafter), let

$$X_{ij} = g(\nabla_{e_j} X, e_i)$$

We can similarly define the expressions of any tensor and its covariant derivatives. For a vector X and a 2-tensor S,

$$X_{ijk} - X_{ikj} = K_{kjit} X_t$$

$$S_{ijkt} - S_{ijtk} = K_{tkjs} S_{is} + K_{tkis} S_{sj}$$

In particular, if X is Killing, then

$$X_{ij} + X_{ji} = 0$$

Proposition 1.1 *If X is a Killing vector field, then*

$$X_{ijk} = K_{ijkt} X_t$$

Proof For a vector field X, the following equalities hold:

$$X_{ijk} - X_{ikj} = K_{kjit}X_t$$

$$X_{jki} - X_{jik} = K_{ikjt}X_t$$

$$X_{kij} - X_{kji} = K_{jikt}X_t$$

Adding up the above equalities,

$$X_{ijk} + X_{jki} + X_{kij} = 0$$

So

$$X_{ijk} = -(X_{kij} - X_{kji}) = K_{ijkt}X_t$$

The Laplacian of a function f is defined as $\Delta f = f_{ii}$.

Definition 1.1 *(M, g) is called an Einstein manifold if there exists a constant c such that*

$$Ric(X, Y) = cg(X, Y)$$

for all vectors X and Y.

Einstein manifolds have the following regularity property due to [DK]

Proposition 1.2 *A C^2 compact Einstein manifold is C^∞.*

Given a function f, we know that $f_{ij} = f_{ji}$. Conversely by the Poincaré lemma, we know that if X is a vector field and $X_{ij} = X_{ji}$, then there exists locally a function f such that $\nabla f = X$.

Definition 1.2 *A closed 2-form ω is called harmonic if $\omega_{ijj} = 0$.*

A distribution F on M is a subbundle of TM over M. If for any vector fields X and Y in F, their Lie bracket is also in F, then F is called a foliation, and M is the union of disjoint submanifolds (called leaves) whose tangent spaces are the restriction of F on them. For a foliation F, we call a vector field X leaf-preserving if for the 1-parameter group $f(t,p)$ generated by X and for any fixed t, $f(t,p)$ maps a leaf into another wherever it is defined.

From the definition of Lie derivative we know that X is leaf-preserving if and only if for any vector field Y in F, $[X,Y]$ is also in F.

The following is adapted from the Local Stability Theorem of foliations due to Reeb(see [CL] for reference).

Proposition 1.3 *If a leaf L is compact and simply-connected, then there is a neighborhood of the leaf so that all the leaves in the neighborhood are diffeomorphic and they form a product space diffeomorphhic to $L \times D$ where D is an open set in $R^{dimM-dimL}$.*

Let (P,g) be a Riemannian manifold. If there are three mutually orthogonal unit Killing vector fields ξ_a, $a = 1,2,3$, such that

$$\nabla_X(\nabla\xi_a)(Y) = g(\xi_a, Y)X - g(X,Y)\xi_a$$

$$[\xi_a, \xi_b] = 2\epsilon_{abc}\xi_c$$

then we say (P,g) is a 3-Sasakian manifold.

Kashiwada observed the following important property[Ka]:

Proposition 1.4 *A 3-Sasakian manifold is Einstein.*

We can think of the curvature tensor as a mapping from Λ^2TM to itself (i.e. the curvature operator). If M is a 4-dimensional oriented Riemannian manifold, we have also a star operator $* : \Lambda^2TM \longrightarrow \Lambda^2TM$.

Definition 1.3 *If for an oriented 4-dimensional Riemannian manifold,*

$$*K = K* : \Lambda^2 TM \longrightarrow \Lambda^2 TM$$

we say M is self-dual.

1.2 Complex Geometry

An almost Hermitian structure J on (M, g) is an assignment of linear automorphisms J_p to the tangent spaces T_pM at all points such that

$$J^2 = -Id, \quad g(JX, JY) = g(X, Y)$$

Define the associated 2-form $\omega_J(X, Y)$ by $g(JX, Y)$.

Definition 1.4 *If ω_J is closed, we call (M, g) almost Kähler. If J is parallel, we call (M, g) Kähler. ω_J is called the Kähler form.*

For a Kähler manifold, since J and ∇ are commutative, the tensors defined by ∇ are invariant under J. In particular,

$$K(JX, JY, Z, W) = K(X, Y, Z, W)$$

$$Ric(X, Y) = Ric(JX, JY)$$

We usually choose a frame of the form

$$\{e_1, Je_1, e_2, Je_2, \ldots, e_n, Je_n\}.$$

We define

$$h^{\bar{i}}_{jk} = g(\nabla_{e_k} e_j, Je_i), \quad h^{\bar{i}}_{\bar{j}k} = g(\nabla_{e_k} Je_j, Je_i)$$

etc.

The Kählerian condition is characterized by

$$h^{\bar{i}}_{\bar{j}k} = h^{i}_{jk}, \quad h^{\bar{i}}_{jk} = -h^{i}_{\bar{j}k}$$

$$h^{\bar{i}}_{\bar{j}\bar{k}} = h^{i}_{j\bar{k}}, \quad h^{\bar{i}}_{j\bar{k}} = -h^{i}_{\bar{j}\bar{k}}$$

The Ricci form of M is $\rho_M(X,Y) = Ric(JX,Y)$. The following proposition is well-known:

Proposition 1.5 *The Ricci form of a Kähler manifold is closed. If the scalar curvature of M is constant, then the Ricci form is harmonic.*

Definition 1.5 *If a Kähler manifold is Einstein, we call it Kähler-Einstein.*

A Kähler manifold is Kähler-Einstein if and only if there exists a constant c such that $\rho_M = c\,\omega_J$.

For a compact Kähler manifold M, its integral cohomology group $H^2(M,\mathbf{Z})$ is identified with a subgroup of $H^2(M,\mathbf{R})$, the de Rham cohomology group. Let $c_1(M)$ be the first Chern class of M. If it can be represented by a positive definite real closed 2-form of type (1,1), we say that it is positive. The celebrated Calabi-Yau Theorem says that every form representing $2\pi c_1(M)$ is the Ricci form of one and only one Kähler metric in the Kähler class of ω_J. In particular, since the first Chern class of M is represented by

$$c_1(M) = \frac{1}{2\pi}[\rho_M] \in H^2(M;\mathbf{Z}),$$

the first Chern class is positive if and only if the Ricci tensor is positive definite.

If $c_1(M) > 0$, then M is simply-connected[Ko] and hence $H^2(M,\mathbf{Z})$ has no torsion. So there is an integer p_M and an indivisible $\alpha_M \in H^2(M;\mathbf{Z})$ such that

$$c_1(M) = p_M \alpha_M$$

The following result is due to [KO]:

Proposition 1.6 *If M is compact and the Ricci tensor is positive definite, then*

$$p_M \leq 1 + \frac{1}{2} dim M,$$

and the equality holds if and only if M is a complex projective space.

For complex projective space CP^n, there is a canonical Kähler metric, the Fubini-Study metric, which has constant holomorphic sectional curvature. $H^2(CP^2; \mathbf{Z})$ is isomorphic to $\mathbf{Z}$. Let α be its generator, then the first Chern class

$$c_1(CP^n) = (n+1)\alpha$$

Let E^* be the conjugate bundle of E, then the relation between the Chern classes of E and E^* is:

$$c_i(E) = (-1)^i c_i(E^*)$$

We know that Pontrjagin classes are differential topological invariants. Let E be a Hermitian vector bundle over a Kähler manifold M. then its first Pontrjagin class is

$$-c_2(E \oplus E^*)$$

In particular, the first Pontrjagin class of M is

$$-c_2(TM \oplus TM^*)$$

where TM denote the holomorphic tangent bundle of M. The first Pontrjagin class of CP^n is $(n+1)\alpha^2$.

Hitchin proved the following fact[H]:

Proposition 1.7 *If M is a 4-dimensional compact manifold which is Einstein and self-dual with positive scalar curvature, then M is isometric to either S^4 or CP^2.*

1.3 Riemannian Metrics on Bundles

Let P be a principal G-bundle over $(M, \bar{g})$, a Riemannian manifold, where G is a compact Lie group. A connection on P is a G-invariant distribution(called horizontal distribution) complementary to the fibres and can be described by an Ad_G-invariant g-valued 1-form θ where g is the Lie algebra of G and the kernel of θ is exactly the G-invariant distribution.

If we equip G with a bi-invariant metric, then we can define a Riemannian metric on P such that

- the metric along the fibres is induced by the G-action.
- the metric on the G-invariant distribution is the lift of that on the base.
- the vectors tangent to the fibres are orthogonal to the vectors along the horizontal distribution.

The projection from P to M is then a Riemannian submersion with all leaves totally geodesic.

We denote the new metric on P by g.

Given a point $p \in P$, the holonomy bundle with reference point p is the set of points in P which can be joined to p by a horizontal curve. We define a connection on P to be irreducible if P is identical with its holonomy bundle.

Proposition 1.8 *If the connection on P is irreducible, f is a function defined on P such that for any horizontal vector X, $X(f) = 0$, then f is constant on P.*

Proof Note that f is constant along any horizontal curve and therefore is constant on a holonomy bundle of P.

Proposition 1.9 *Let V be a globally defined vertical vector field induced by a vector in g, the Lie algebra of G, through right action of G on the fibres, and X be horizontal and basic, i.e. a lifting of a vector field from the base manifold. Then*

$$[V, X] = 0$$

$$\nabla_V X = \nabla_X V$$

In fact, since the horizontal distribution is G-invariant, $[V, X]$ is horizontal. On the other hand, it is obvious that for the foliation generated by the fibres, X is leaf-preserving. So $[V, X]$ is also vertical. Therefore $[V, X] = 0$. Recall that $\nabla_V X - \nabla_X V = [V, X]$. we find that

$$\nabla_V X = \nabla_X V$$

We can see from the definition of the metric on P that a vector field induced by an element in g, the Lie algebra of G, on P is a Killing vector field. In particular, every fibre is totally geodesic.

Since the Lie algebra given by all vertical Killing vector fields is isomorphic to g, we can interpret the connection by the Killing vector fields as follows:

The connection θ is defined by an orthonormal vertical Killing basis

$$\{V_1, V_2, \ldots, V_s\}$$

as

$$\theta(W) = g(W, V_m)V_m$$

for any vector W on P.

Then the curvature form Ω is, evaluated on horizontal vectors fields,

$$\Omega(X, Y) = -g([X, Y], V_m)V_m$$

Proposition 1.10 *For any vertical vector V and horizontal basic vector fields X, Y,*

$$g(\nabla_X Y, V) = -\frac{1}{2}g(\Omega(X,Y), V)$$

Proof Note the fact that we can think of V as a Killing vector field and apply $\nabla_X V = \nabla_V X$.

Let R^G denote the curvature tensor of Lie group G. The curvature tensor of P evaluated along the fibres is identified with R^G since the fibres are totally geodesic.

Let $V_\alpha, V_\beta, V_\gamma, \ldots$ denote the vertical Killing orthonormal frame, and $e_i, \ldots$ the basic horizontal orthonormal frame. Then

$$h^\alpha_{\beta\gamma} = h^\beta_{\gamma\alpha}$$

$$h^\alpha_{ij} = \frac{1}{2}g(\Omega(e_i, e_j), V_\alpha)$$

$$h^\alpha_{ij} + h^\alpha_{ji} = 0$$

$$h^j_{k\alpha} = h^\alpha_{kj}$$

Proposition 1.11 *Let P be a principal G-bundle over Riemannian manifold M where G is compact and is equipped with a bi-invariant metric. A connection will induce a Riemannian metric whose curvatures satisfy the following equalities:*

We think of h^α_{ij} as a 3-tensor so that it vanishes on other entries. Then

$$h^\alpha_{ijk} = e_k(h^\alpha_{ij}) - h^\alpha_{tj}h^t_{ik} - h^\alpha_{it}h^t_{jk}$$

$$K_{\alpha ijk} = h^\alpha_{jki}$$

$$K_{ijkt} = K^M_{ijkt} - 2h^\alpha_{ij}h^\alpha_{kt} - h^\alpha_{ik}h^\alpha_{jt} + h^\alpha_{it}h^\alpha_{jk}$$

$$K_{\alpha i\beta j} = h^{\alpha}_{jk}h^{\beta}_{ik} + h^{\alpha}_{\beta\gamma}h^{\gamma}_{ij} = h^{\beta}_{ji\alpha} + h^{\alpha}_{it}h^{\beta}_{jt}$$

$$K_{i\alpha\beta\gamma} = 0$$

These equations can be given by applying O'Neill's formulas. We will give an equation similar to the fourth equation in Chapter 4.

Proposition 1.12 *If for any given α, h^{α} is a lifting of some 2-form from M restricted on horizontal distribution, then either the connection is flat or the structure group is abelian.*

Proof The condition means that, under horizontal basic frame,

$$V_{\beta}(h^{\alpha}_{ij}) = 0$$

By the fourth equation in last proposition,

$$h^{\gamma}_{ij}h^{\gamma}_{\alpha\beta} = 0$$

Therefore the proposition is true.

Proposition 1.13 *If V_{ξ} is the Killing vector field generated by an element in the center of G, then h^{ξ} is the horizontal lifting of a 2-form from the base.*

Proof The proof is similar to the last one. By the fourth equation of Proposition 1.16,

$$V_{\alpha}(h^{\xi}_{ij}) = 0$$

and the proposition holds.

The above propositions indicate significant difference between principal bundles with abelian and non-abelian structure groups. In fact, by Theorem

1.2, we have Einstein metrics on principal circle bundles over $S^2 \times S^2$ as contrasted with Proposition 4.5 which asserts non-existence of Einstein metrics for non-abelian structure groups.

Applying the structure equations for the submersion, we have

Proposition 1.14 *Let P be a principal circle bundle over a compact Riemannian manifold M with a connection so that with respect to the induced metric, P is S^{2n+1} with constant curvature. Then M is isometric to CP^n with a metric homothetic to Fubini-Study metric.*

Proof We can define complex structure on M by

$$g(X,Y) = \frac{1}{\sqrt{c}} h^{\alpha}(X, JY)$$

where c is the sectional curvature of S^{2n+1}. By equalities in Proposition 1.16, we can show that J is well-defined and parallel. M is of constant holomorphic sectional curvature.

We see that $Ric_{\alpha i} = 0$ if and only if $h^{\alpha}_{ij;j} = 0$. A connection with this property is called Yang-Mills. If $G = S^1$, the curvature form is the lifting of a closed 2-form from M. It is Yang-Mills if and only if the 2-form is harmonic.

The following well-known result is crucial for most of our constructions.

Proposition 1.15 *If (M, g) is a compact manifold, then the principal circle bundles over M are classified by $H^2(M, \mathbf{Z})$.*

Given a closed 2-form Ω such that

$$\frac{1}{2\pi}[\Omega] \in H^2(M, \mathbf{Z}),$$

then on the corresponding principal circle bundle (with Euler class $[\Omega]$), there exists a connection so that its curvature form is nothing but Ω.

For compact Kähler-Einstein manifolds (M_i, g_i) with positive first Chern classes, we know that

$$c_1(M_i) = \frac{1}{2\pi}[L_i\omega_i] = p_i\alpha_i$$

So

$$\frac{1}{2\pi}[\sum_{i=1}^{l}\frac{L_iq_i}{p_i}\omega_i] = \sum_{i=1}^{l}q_i\alpha_i \in H^2(\tilde{M}, \mathbb{Z})$$

where $\tilde{M} = M_1 \times M_2 \times \ldots \times M_l$. So we can construct over $\tilde{M}$ a principal circle bundle whose Euler class is $\sum q_i\alpha_i$ and the curvature form can be chosen as $\sum \frac{L_iq_i}{p_i}\omega_i$.

The following theorem is due to Wang and Ziller[WZ]:

Theorem 1.1 *There is an Einstein metric on every principal circle bundle so constructed over $\tilde{M}$.*

Let P be the principal circle bundle over $M_1 \times M_2 \times \ldots \times M_l$ constructed as above. Let $I = (a, b)$ be an interval and S^1 act on $S^1 \times I$ naturally. We construct a metric on $P \times_{S^1} (S^1 \times I)$ by

$$ds^2 = dr^2 + f^2\theta^2 + \sum_{i=1}^{l} g_i^2 ds_i^2$$

where r is the parameter of I, θ is the connection form and ds_i^2 is the lifting of the metric from M_i, f and g_i are functions of r. Let $\{e_{ij}\}$ be basic horizontal orthonormal frame and V_α be the unit Killing vector field along the fibres of P, then, the orthonormal frame of $P \times_{S^1} (S^1 \times I)$ is $\{\partial/\partial r, \frac{1}{f}V_\alpha, \frac{1}{g_i}e_{ij}\}$.

We now give some of the curvatures of the total space for future use in Chapter 3. The other unmentioned terms appearing in the computation of Ricci tensor all vanish. To simplify the notation, we assume $\{e_i, e_j, e_k, \ldots\}$ are from M_1 and $\{e_u, \ldots\}$ are from M_2, then by direct calculation we have

Proposition 1.16

$$K_{arar} = -\frac{f''}{f}$$

$$K_{rirj} = -\frac{g_1''}{g_1}\delta_{ij}$$

$$K_{aiaj} = -\frac{f'g_1'}{fg_1}\delta_{ij} + \frac{f^2}{g_1^4}h_{ki}^{\alpha}h_{kj}^{\alpha}$$

$$K_{ikjk} = \frac{1}{g_1^2}Ric_{ij}^{M_1} - (2n_1-1)\frac{(g_1)'}{g_1^2}\delta_{ij} - 3\frac{f^2}{g_1^4}h_{ik}^{\alpha}h_{jk}^{\alpha}$$

$$K_{iuju} = -2n_2\frac{g_1'g_2'}{g_1g_2}\delta_{ij}$$

We will collapse the boundaries of $P \times_{S^1} (S^1 \times I)$ to get S^2-bundles in Chapter 3. Then f and g_i should satisfy certain boundary conditions in order that the metric on the total space is smooth. We refer to [BB](p24 and p25) for discussion.

To illustrate the new existence results of the thesis, we give some simple examples as follows:

1. There are Einstein metrics on principal circle bundles over $(SU(3)/U(1)) \times \ldots \times (SU(3)/U(1))$ and $(SU(3)/U(1)) \times CP^2$.

2. There are Kähler-Einstein and non-Kählerian Einstein metrics on certain S^2-bundles over $CP^2 \times CP^2$. In this case, by the construction in Chapter 3, $p_1 = p_2 = 3$. There are two Kähler-Einstein metrics corresponding to $(q_1, q_2) = (1,1)$ and $(q_1, q_2) = (2,2)$ by Theorem 3.1, non-Kählerian Einstein metrics corresponding $1 \leq |q_1|, |q_2| \leq 2$ by Theorem 3.3, and non-Kählerian Einstein metrics correspoinding to $(|q_1|, |q_2|) = (1,2)$ by Theorem 3.2.

3. There are Einstein metrics on principal 2-torus bundles over $CP^2 \times CP^2$.

4. There is an Einstein metric on $P \times_{S^1} P_1$ where P is a principal circle bundle over CP^2 and P_1 is a 3-Sasakian manifold with a circle action specified by a Sasakian structure.

The above results follow from Theorems 3.1, 3.2, 3.3, 4.6, 4.7, 4.8 and 4.10.

Chapter 2

Einstein Metrics on Principal Circle Bundles

2.1 Introduction

If a connection is fixed on a principal circle bundle P over a Riemannian manifold M, a Riemannian metric will be uniquely determined on P by specifying a metric on fibres which is invariant under the circle action. Theorem 1.1 says that certain kinds of such metrics can be Einstein, when M is a product of Kähler-Einstein manifolds.

This chapter presents the uniqueness of Theorem 1.1 in a larger class of circle bundles over compact Kähler manifolds.

Theorem 2.1 *If M is a compact Kähler manifold, P, a principal circle bundle over M equipped with a metric induced by a connection, is Einstein, and the curvature form of the connection on P is of type $(1,1)$, then M is isometric to a product of Kähler-Einstein manifolds and P lies among those Einstein manifolds given by Theorem 1.1. In particular, the Euler class of P is a linear* *combination of the first Chern classes of the Kähler-Einstein manifolds in the base.*

The assumption that M is Kählerian is crucial. For otherwise, there are examples of Einstein metrics on principal circle bundles in which the bases are not products of Einstein manifolds[Wa1][S2]. Theorem 2.1 is the consequence of the following two propositions

Proposition 2.1 *Under the assumption of Theorem 2.1, TM decomposes globally as orthogonal sum of eigenspaces of the Ricci curvature. All eigenvalues of the Ricci curvature of M are constants. Any sum of the eigenspaces is integrable and invariant under J, the complex structure.*

Proposition 2.2 *Suppose M is a compact Kähler manifold with positive Ricci curvature and TM is orthogonal sum of two foliations both of which are invariant under J. Then M has a De Rham decomposition by the foliations.*

If the Ricci curvature of a compact Kähler manifold has all eigenvalues constant, the eigenspaces are not necessarily integrable even when the manifold is also homogeneous. But if the Ricci curvature of a compact Kähler manifold has exactly two constant eigenvalues, then the eigenspaces are both integrable. So by Proposition 2.2,

Corollary 2.1 *If (M, g) is a compact Kähler manifold whose Ricci tensor has exactly two positive constant eigenvalues, then M is a product of two Kähler-Einstein manifolds each of which corresponds to one eigenspace.*

Note that if F is a holomorphic foliation of complex codimension 1 in a Kähler manifold so that the induced metric is bundle-like, then the normal bundle to F is integrable and holomorphic. So by Proposition 2.2,

Corollary 2.2 *If (M, g) is a compact Kähler manifold with positive first Chern class, then it does not admit a complex codimension 1 foliation that bears an induced bundle-like metric and that is invariant under J, the complex structure, unless M is a product of a leaf of the foliation with a Riemann sphere.*

Examining the proof of Proposition 2.2 we can find the following result:

Corollary 2.3 *Let (M, g) be a compact Kähler manifold. If there is a holomorphic foliation F on M with integrable normal bundle such that the Ricci curvature of M, when restricted on F, is positively definite, then F is a Riemannian foliation with all leaves compact and simply-connected. In particular, M is a fibre bundle over M/F with the leaves of F as fibres.*

The second proposition says that each of the foliations has isometric leaves and that M is isometric to the Riemannian product of a leaf from each foliation.

Proposition 2.1 can be proved through careful calculation applying hamonicity of Ricci form of M and Einstein condition of P repeatedly, while Proposition 2.2, which suggests rigidity of Kählerian condition together with positivity of Ricci curvature, is derived by means of Bochner's technique on leaves which turn out to be all compact and all simply-connected. Unless additional conditions that can lead to triviality of holonomy of all leaves are imposed, there is no counterpart of the latter for compact Riemannian manifolds, even, for instance, in the case that all leaves of both foliations are compact and that one is lower dimensional, or in the case that both foliations are Riemannian with all leaves compact[BH].

The theorem follows once the above two propositions are established. In fact, (2) and (3) together with the propositions ensure that M is a product of compact Kähler-Einstein manifolds with positive first Chern classes and that the curvature form of the principal circle bundle is a linear combination of the Kähler forms of the Kähler-Einstein manifolds in the base. Therefore, the Euler class of the principal circle bundle, which can be represented by the curvature form, will be the homology class of a linear combination of the Ricci forms of the Kähler-Einstein manifolds in the base. We know that the Euler class lies in $2\pi H^2(M, \mathbf{Z})$ and that the Ricci forms of Kähler-Einstein manifolds with positive first Chern classes are related to their integral second homology groups as described before. Thus we see the principal circle bundle falls within those constructed in [WZ].

2.2 Facts and Notations

Fixing a connection θ on a circle bundle $\pi : P \to M$ and the standard metric on S^1, we can define a metric on P by

$$\langle U, V\rangle = \langle \pi_*(U), \pi_*(V)\rangle + \langle \theta(U), \theta(V)\rangle_{S^1}$$

where U, V are vectors on P, and ω by

$$\omega(\tilde{X}, \tilde{Y}) = \frac{1}{2}\langle [\tilde{Y}, \tilde{X}], N\rangle$$

where $\tilde{X}$, $\tilde{Y}$ are horizontal vector fields on P and N is the unit vector field along the fibres.

ω is $\frac{1}{2}$ of the curvature form Ω.

We will treat ω as a closed 2-form on M. Then the conditions for P to be Einstein become those on M as follows[Be]:

$$\omega \textit{ is a harmonic } 2-\textit{form} \tag{1}$$

$$|\omega|^2 \textit{ is constant on } M \tag{2}$$

$$Ric(X,Y) = |\omega|^2\langle X,Y\rangle + 2\langle\omega(X),\omega(Y)\rangle \tag{3}$$

The requirement that ω be of type $(1,1)$ ensures that the associated tensor S, defined by

$$S(X,Y) = \omega(X,JY)$$

is symmetric and of type $(1,1)$ as well. It follows that S will be a diagonal matrix under a local orthonormal basis which is invariant under J up to signs.

Such local basis will be denoted by $\{e_A\}$(sometimes by $\{e_A, Je_A\}$ if otherwise stated). Then, for e_A, e_B in the basis,

$$\omega(e_A, Je_B) = 0 \quad if \quad e_A \neq e_B \tag{4}$$

The following notations will also be used

$$\omega_{AB} = \omega(e_A, Je_B)$$

$$\rho_{AB} = -\sum_C \frac{1}{2} K_{ABC\bar{C}}$$

where $K_{ABC\bar{C}} = K(e_A, e_B, e_C, Je_C)$

$$h^A_{BC} = \langle\nabla_{e_C} e_B, e_A\rangle$$

$$h^{\bar{A}}_{BC} = \langle\nabla_{e_C} e_B, Je_A\rangle$$

etc.

The covariant derivative is written as

$$(\nabla_{e_C}\omega)(e_A, e_B) = \omega_{ABC}$$

i.e

$$\omega_{ABC} = e_C(\omega_{AB}) - \sum_T \omega_{AT} h^T_{BC} - \sum_T \omega_{TB} h^T_{AC}$$

We know from the relation between exterior differentiation and covariant derivative that ω is closed if and only if

$$\omega_{ABC} + \omega_{BCA} + \omega_{CAB} = 0$$

We rewrite (1) and (3) as

$$\omega_{ABC} + \omega_{BCA} + \omega_{CAB} = 0 \tag{5}$$

$$\sum_B \omega_{ABB} = 0 \tag{6}$$

$$\rho_{AB} = |\omega|^2 \delta_{AB} + 2\sum_C \omega_{AC}\omega_{BC} \tag{7}$$

Note that M is of constant scalar curvature as a consequence of (2) and (3)[Be], and $-\rho$ is the Ricci form. Therefore ρ is harmonic. So

$$\begin{aligned}
0 &= \rho_{ABC} + \rho_{BCA} + \rho_{CAB} \\
&= 2(\sum_D \omega_{AD}\omega_{\bar{B}D})_C + 2(\sum_D \omega_{BD}\omega_{\bar{C}D})_A + 2(\sum_D \omega_{CD}\omega_{\bar{A}D})_B \\
&= 2\sum_D \omega_{ADC}\omega_{\bar{B}D} + 2\sum_D \omega_{AD}\omega_{\bar{B}DC} + 2\sum_D \omega_{BDA}\omega_{\bar{C}D} \\
&\quad +2\sum_D \omega_{BD}\omega_{\bar{C}DA} + 2\sum_D \omega_{CDB}\omega_{\bar{A}D} + 2\sum_D \omega_{CD}\omega_{\bar{A}DB} \\
&= 2\sum_D \omega_{BD}(\omega_{\bar{C}DA} + \omega_{A\bar{D}C}) + 2\sum_D \omega_{AD}(\omega_{\bar{B}DC} + \omega_{C\bar{D}B}) \\
&\quad +2\sum_D \omega_{CD}(\omega_{\bar{A}DB} + \omega_{B\bar{D}A}) \\
&= 2\sum_D \omega_{BD}(-\omega_{C\bar{D}A} - \omega_{\bar{D}AC}) + 2\sum_D \omega_{AD}(-\omega_{B\bar{D}C} - \omega_{\bar{D}CB}) \\
&\quad +2\sum_D \omega_{CD}(-\omega_{A\bar{D}B} - \omega_{\bar{D}BA}) \\
&= 2\sum_D \omega_{BD}\omega_{AC\bar{D}} + 2\sum_D \omega_{AD}\omega_{CB\bar{D}} + 2\sum_D \omega_{CD}\omega_{BA\bar{D}}
\end{aligned}$$

i.e.

$$\sum_D (\omega_{BD}\omega_{AC\bar{D}} + \omega_{AD}\omega_{CB\bar{D}} + \omega_{CD}\omega_{BA\bar{D}}) = 0 \tag{8}$$

And

$$\begin{aligned} 0 &= \frac{1}{2}\sum_{B}\rho_{AB\bar{B}} = \sum_{B,C}(\omega_{AC}\omega_{\bar{B}\bar{C}})_{B} \\ &= \sum_{B,C}\omega_{\bar{B}\bar{C}}\omega_{ACB} + \sum_{B,C}\omega_{AC}\omega_{\bar{B}\bar{C}B} \\ &= \sum_{B,C}\omega_{\bar{B}\bar{C}}\omega_{ACB} + \sum_{B,C}\omega_{AC}\omega_{\bar{C}\bar{B}B} \\ &= \sum_{B,C}\omega_{\bar{B}\bar{C}}\omega_{ACB} \end{aligned}$$

i.e

$$\sum_{B,C}\omega_{\bar{B}\bar{C}}\omega_{ACB} = 0 \tag{9}$$

In deriving (8) and (9), we omitted the first term on the right hand side of (7) because it is the Kähler form up to a constant. And we repeatedly applied (5), (6) and the condition that ω is of type $(1,1)$.

We see from (3) or (7) that Ricci curvature is in the diagonal form under a local basis satisfying (4). To justify the global decomposition in Proposition 1, we have only to show that S has constant eigenvalues, or equivalently, ω has constant entries. This claim, together with the integrability conditions in Proposition 2.1, is proved simultaneously in the following three lemmas by elaborating the above established equalities (1)–(9).

2.3 Proof of Proposition 2.1

Lemma 2.1 *In local sense, any eigenspace of S, the symmetric tensor associated to ω, is integrable. And its eigenvalue is constant along the normal directions of the corresponding eigenspace.*

In other words, given $e_A, Je_A, e_B, Je_B, e_C, Je_C$ in a local basis satisfying (4) such that

$$\omega_{A\bar{A}} = \omega_{B\bar{B}}, \quad \omega_{A\bar{A}} \neq \omega_{C\bar{C}}$$

then

$$e_C(\omega_{A\bar{A}}) = (Je_C)(\omega_{A\bar{A}}) = 0$$

and

$$h^C_{AB} = h^C_{BA}$$

The last equality holds if we change A to $\bar{A}$, B to $\bar{B}$, or C to $\bar{C}$.

Proof Let $e_B = Je_A$ in (8) and use (4)

$$\begin{aligned} 0 &= \omega_{\bar{A}A}\omega_{AC\bar{A}} - \omega_{A\bar{A}}\omega_{C\bar{A}A} + \omega_{C\bar{C}}\omega_{A\bar{A}C} \\ &= \omega_{C\bar{C}}\omega_{A\bar{A}C} - \omega_{A\bar{A}}(\omega_{AC\bar{A}} + \omega_{C\bar{A}A}) \\ &= [\omega_{C\bar{C}} - \omega_{A\bar{A}}]\omega_{A\bar{A}C} \quad \textit{by } (5) \end{aligned}$$

So

$$\omega_{A\bar{A}C} = 0$$

i.e.

$$e_C(\omega_{A\bar{A}}) = 0$$

Similarly

$$(Je_C)(\omega_{A\bar{A}}) = 0$$

And

$$\omega_{AC\bar{A}} + \omega_{C\bar{A}A} = -\omega_{\bar{A}AC} = 0$$

Note that

$$\omega_{AC\bar{A}} = -\omega_{A\bar{A}}h^{\bar{A}}_{CA} - \omega_{\bar{C}C}h^{\bar{C}}_{A\bar{A}} = \omega_{A\bar{A}}h^{C}_{\bar{A}\bar{A}} - \omega_{C\bar{C}}h^{C}_{\bar{A}\bar{A}}$$

and that

$$\omega_{C\bar{A}A} = -\omega_{A\bar{A}}h^{A}_{CA} - \omega_{C\bar{C}}h^{\bar{C}}_{AA} = \omega_{A\bar{A}}h^{C}_{AA} - \omega_{C\bar{C}}h^{C}_{AA}$$

We find

$$[\omega_{A\bar{A}} - \omega_{C\bar{C}}](h^C_{\bar{A}\bar{A}} + h^C_{AA}) = 0$$

i.e.

$$h^{C}_{\bar{A}\bar{A}} + h^{C}_{AA} = 0 \quad or \quad h^{\bar{C}}_{A\bar{A}} = h^{\bar{C}}_{\bar{A}A}$$

Similarly

$$h^{\bar{C}}_{\bar{A}\bar{A}} + h^{\bar{C}}_{AA} = 0 \quad or \quad h^{C}_{A\bar{A}} = h^{C}_{\bar{A}A}$$

Suppose $e_A \neq e_B$. Then by (8)

$$\begin{aligned} 0 &= -\omega_{B\bar{B}}\omega_{ACB} - \omega_{A\bar{A}}\omega_{CBA} - \omega_{C\bar{C}}\omega_{BAC} \\ &= \omega_{A\bar{A}}[-\omega_{ACB} - \omega_{CBA}] - \omega_{C\bar{C}}\omega_{BAC} \quad since \quad \omega_{A\bar{A}} = \omega_{B\bar{B}} \\ &= (\omega_{A\bar{A}} - \omega_{C\bar{C}})\omega_{BAC} \quad by \quad (5) \end{aligned}$$

So

$$\omega_{BAC} = 0$$

$$\omega_{ACB} + \omega_{CBA} = 0$$

$$\begin{aligned} 0 &= -\omega_{A\bar{A}}h^{\bar{A}}_{CB} - \omega_{C\bar{C}}h^{\bar{C}}_{AB} - \omega_{C\bar{C}}h^{\bar{C}}_{BA} - \omega_{B\bar{B}}h^{\bar{B}}_{CA} \\ &= \omega_{A\bar{A}}h^{C}_{\bar{A}B} - \omega_{A\bar{A}}h^{C}_{\bar{B}A} - \omega_{C\bar{C}}h^{\bar{C}}_{BA} + \omega_{C\bar{C}}h^{C}_{AB} \\ &= \omega_{A\bar{A}}h^{\bar{C}}_{BA} - \omega_{A\bar{A}}h^{\bar{C}}_{AB} + \omega_{C\bar{C}}(h^{\bar{C}}_{AB} - h^{\bar{C}}_{BA}) \\ &= [\omega_{A\bar{A}} - \omega_{C\bar{C}}][h^{\bar{C}}_{BA} - h^{\bar{C}}_{AB}] \end{aligned}$$

i.e.

$$h^{\bar{C}}_{BA} = h^{\bar{C}}_{AB}$$

Similarly

$$h^{C}_{BA} = h^{C}_{AB}$$

etc.

Lemma 2.2 *Any eigenvalue of S is constant along its eigenspace.*

Equivalently, if e_A, Je_A, e_B, Je_B are in a local basis satisfying (4), and if $\omega_{A\bar{A}} = \omega_{B\bar{B}}$, then

$$e_B(\omega_{A\bar{A}}) = (Je_B)(\omega_{A\bar{A}}) = 0$$

Proof We choose a basis $\{e_B, Je_B\}$ satisfying (4) with B ranging from 1 to the complex dimension of M. Then (9) becomes

$$\sum_B \omega_{B\bar{B}}\omega_{A\bar{B}B} - \sum_B \omega_{B\bar{B}}\omega_{AB\bar{B}} = 0$$

or

$$\sum_B \omega_{B\bar{B}}(\omega_{A\bar{B}B} + \omega_{AB\bar{B}}) = 0 \qquad (10)$$

If $\omega_{A\bar{A}} = \omega_{B\bar{B}}$ and $B \neq A$, then

$$\begin{aligned} \omega_{A\bar{B}B} &= -\omega_{A\bar{A}}h^{\bar{A}}_{\bar{B}B} - \omega_{B\bar{B}}h^{\bar{B}}_{AB} \\ &= -\omega_{A\bar{A}}h^{\bar{A}}_{\bar{B}B} + \omega_{B\bar{B}}h^{\bar{A}}_{\bar{B}B} = 0 \end{aligned}$$

and

$$\begin{aligned} \omega_{AB\bar{B}} &= -\omega_{A\bar{A}}h^{\bar{A}}_{B\bar{B}} - \omega_{B\bar{B}}h^{B}_{A\bar{B}} \\ &= -\omega_{A\bar{A}}h^{A}_{\bar{B}B} + \omega_{B\bar{B}}h^{A}_{\bar{B}B} = 0 \end{aligned}$$

If $\omega_{A\bar{A}} \neq \omega_{B\bar{B}}$,

$$\begin{aligned} &\omega_{A\bar{B}B} + \omega_{AB\bar{B}} = \\ &-\omega_{A\bar{A}}h^{\bar{A}}_{\bar{B}B} - \omega_{B\bar{B}}h^{\bar{B}}_{AB} \\ &-\omega_{A\bar{A}}h^{\bar{A}}_{B\bar{B}} - \omega_{B\bar{B}}h^{B}_{A\bar{B}} \\ = \;& -\omega_{A\bar{A}}(h^{\bar{A}}_{\bar{B}B} + h^{\bar{A}}_{B\bar{B}}) + \omega_{B\bar{B}}(h^{A}_{\bar{B}B} - h^{A}_{B\bar{B}}) \\ = \;& -\omega_{A\bar{A}}(h^{A}_{\bar{B}B} - h^{A}_{B\bar{B}}) + \omega_{B\bar{B}}(h^{A}_{\bar{B}B} - h^{A}_{B\bar{B}}) = 0 \quad \textit{by Lemma 1.} \end{aligned}$$

Substituting the above discussion into (10), we have

$$\omega_{A\bar{A}}\omega_{A\bar{A}A} = 0$$

i.e.

$$\frac{1}{2}(Je_A)[(\omega_{A\bar{A}})^2] = 0$$

Similarly

$$\frac{1}{2}e_A[(\omega_{A\bar{A}})^2] = 0$$

If $e_B \neq e_A$ and $\omega_{BB} = \omega_{A\bar{A}}$, then

$$\begin{aligned} 0 &= \omega_{A\bar{A}B} + \omega_{\bar{A}BA} + \omega_{BA\bar{A}} \\ &= \omega_{A\bar{A}B} - \omega_{\bar{A}A}h^{A}_{BA} - \omega_{\bar{B}B}h^{\bar{B}}_{\bar{A}A} - \omega_{B\bar{B}}h^{\bar{B}}_{A\bar{A}} - \omega_{\bar{A}A}h^{\bar{A}}_{B\bar{A}} \\ &= \omega_{A\bar{A}B} - \omega_{A\bar{A}}h^{B}_{AA} + \omega_{BB}h^{B}_{AA} + \omega_{BB}h^{B}_{\bar{A}\bar{A}} - \omega_{A\bar{A}}h^{B}_{\bar{A}\bar{A}} \\ &= \omega_{A\bar{A}B} \end{aligned}$$

i.e.

$$e_B(\omega_{A\bar{A}}) = 0$$

Similarly

$$(Je_B)(\omega_{A\bar{A}}) = 0$$

Lemma 2.3 *The sum of any two eigenspaces of S is integrable.*

Proof Choosing the same basis as in Lemma 2, we need only to prove that , if

$$\omega_{A\bar{A}} \neq \omega_{B\bar{B}}, \quad \omega_{BB} \neq \omega_{CC}, \quad \omega_{CC} \neq \omega_{A\bar{A}},$$

then,

$$h^{A}_{BC} = h^{A}_{CB}$$

and the last equality still holds if we change A to $\bar{A}$, B to $\bar{B}$ or C to $\bar{C}$.

By (5),

$$\begin{aligned} 0 &= \omega_{AB\bar{C}} + \omega_{B\bar{C}A} + \omega_{\bar{C}AB} \\ &= -\omega_{A\bar{A}} h^{\bar{A}}_{B\bar{C}} - \omega_{B\bar{B}} h^{\bar{B}}_{A\bar{C}} \\ &\quad -\omega_{B\bar{B}} h^{B}_{\bar{C}A} - \omega_{C\bar{C}} h^{C}_{BA} \\ &\quad -\omega_{\bar{C}C} h^{C}_{AB} - \omega_{\bar{A}A} h^{\bar{A}}_{\bar{C}B} \\ &= \omega_{A\bar{A}}[h^{A}_{\bar{C}B} - h^{\bar{A}}_{B\bar{C}}] + \omega_{B\bar{B}}[h^{\bar{B}}_{A\bar{C}} - h^{B}_{\bar{C}A}] + \omega_{C\bar{C}}[h^{C}_{AB} - h^{C}_{BA}] \end{aligned}$$

i.e.

$$\omega_{A\bar{A}}[h^{A}_{\bar{C}B} - h^{\bar{A}}_{B\bar{C}}] + \omega_{B\bar{B}}[h^{\bar{B}}_{A\bar{C}} - h^{B}_{\bar{C}A}] + \omega_{C\bar{C}}[h^{C}_{AB} - h^{C}_{BA}] = 0 \tag{11}$$

By (8),

$$\begin{aligned} 0 &= -\omega_{B\bar{B}}\omega_{A\bar{C}B} - \omega_{A\bar{A}}\omega_{\bar{C}BA} + \omega_{\bar{C}C}\omega_{BA\bar{C}} \\ &= \omega_{B\bar{B}}\omega_{A\bar{A}} h^{\bar{A}}_{\bar{C}B} + \omega_{B\bar{B}}\omega_{C\bar{C}} h^{C}_{AB} + \omega_{A\bar{A}}\omega_{\bar{C}C} h^{C}_{BA} \\ &\quad +\omega_{A\bar{A}}\omega_{B\bar{B}} h^{\bar{B}}_{\bar{C}A} - \omega_{\bar{C}C}\omega_{B\bar{B}} h^{B}_{A\bar{C}} - \omega_{\bar{C}C}\omega_{\bar{A}A} h^{\bar{A}}_{B\bar{C}} \\ &= \omega_{B\bar{B}}\omega_{C\bar{C}}[h^{\bar{B}}_{A\bar{C}} - h^{A}_{\bar{C}B}] \\ &\quad +\omega_{C\bar{C}}\omega_{A\bar{A}}[h^{B}_{\bar{C}A} - h^{\bar{A}}_{B\bar{C}}] + \omega_{A\bar{A}}\omega_{B\bar{B}}[h^{A}_{\bar{C}B} - h^{B}_{\bar{C}A}] \end{aligned}$$

i.e.

$$\begin{aligned} &\omega_{B\bar{B}}\omega_{C\bar{C}}[h^{A}_{\bar{C}B} - h^{\bar{A}}_{B\bar{C}}] + \omega_{C\bar{C}}\omega_{A\bar{A}}[h^{\bar{B}}_{A\bar{C}} - h^{B}_{\bar{C}A}] \\ &\quad +\omega_{A\bar{A}}\omega_{B\bar{B}}[h^{C}_{AB} - h^{C}_{BA}] = 0 \end{aligned} \tag{12}$$

Note that

$$(h^{A}_{\bar{C}B} - h^{\bar{A}}_{B\bar{C}}) + (h^{\bar{B}}_{A\bar{C}} - h^{B}_{\bar{C}A}) + (h^{C}_{AB} - h^{C}_{BA}) = 0 \tag{13}$$

Since

$$\det \begin{pmatrix} \omega_{A\bar{A}} & \omega_{B\bar{B}} & \omega_{C\bar{C}} \\ \omega_{B\bar{B}}\omega_{C\bar{C}} & \omega_{C\bar{C}}\omega_{A\bar{A}} & \omega_{A\bar{A}}\omega_{B\bar{B}} \\ 1 & 1 & 1 \end{pmatrix}$$

$$= (\omega_{A\bar{A}} - \omega_{B\bar{B}})(\omega_{B\bar{B}} - \omega_{C\bar{C}})(\omega_{C\bar{C}} - \omega_{A\bar{A}}) \neq 0$$

then $h^{A}_{BC} - h^{\bar{A}}_{\bar{B}\bar{C}}$, $h^{\bar{B}}_{A\bar{C}} - h^{B}_{CA}$, $h^{C}_{AB} - h^{C}_{BA}$ which are variables in linear system (11), (12), (13), must all be zero.

Combining the above three lemmas together with (3) or (7) confirms that the eigenvalues of Ricci curvature are all constant locally. And a topological argument about closed set and open set yields Proposition 1.

2.4 Proof of Proposition 2.2

We divided the proof of Proposition 2. into four steps

Step 1 *Under condition of Proposition 2, the integral submanifolds of the foliations are all compact and simply-connected.*

Proof We want to show that any leaf of a foliation is a complete Kähler manifold with positive Ricci curvature bounded below by a positive number.

Since M is compact, any open covering of M compatible with the foliation has a finite subcovering. Similar to the proof that a compact Riemannian manifold is complete, we see all leaves are complete. On the other hand, each leaf is Kählerian with the induced Kähler structure. If we know that the Ricci curvature of any leaf is bounded below by a positive number, we can see by Myers' theorem that the leaves are all compact and by Kobayashi's Theorem[Ko] that the leaves are also all simply-connected. So there remains only to show that the Ricci curvature of any leaf is bounded below by a positive number to complete the proof. We will deduce the required equality (18) as follows:

Suppose P and Q are foliations whose sum is the tangent bundle TM, and suppose a local basis of P is represented by $\{e_i, Je_i\}$ and that of Q by

$\{e_a, Je_a\}$. We will use i, j, k, t, *etc.* to denote vector fields on P and a, b, c, *etc.* to denote those on Q.

Regard h^a_{ij} as a Q-valued symmetric tensor on P. Its covariant derivative along Q is defined by

$$h^a_{ijb} = e_b(h^a_{ij}) - \sum_t (h^a_{tj}h^t_{ib} + h^a_{\bar{t}j}h^{\bar{t}}_{ib}) - \sum_t (h^a_{it}h^t_{jb} + h^a_{i\bar{t}}h^{\bar{t}}_{jb}) - \sum_c (h^c_{ij}h^c_{ab} + h^{\bar{c}}_{ij}h^{\bar{c}}_{ab})$$

We mention several equalities such as

$$h^a_{ij} + h^a_{\bar{i}\bar{j}} = 0, \quad h^a_{ija} + h^a_{\bar{i}\bar{j}a} = 0, \ etc. \tag{14}$$

$$\begin{aligned}\sum_a (K_{aiaj} + K_{\bar{a}i\bar{a}j}) &= \sum_a (h^a_{ija} + h^{\bar{a}}_{ij\bar{a}}) \\ &\quad -2\sum_{t,a}(h^a_{ti}h^a_{tj} + h^{\bar{a}}_{ti}h^{\bar{a}}_{tj}) - 2\sum_{a,b}(h^i_{ab}h^j_{ab} + h^{\bar{i}}_{ab}h^{\bar{j}}_{ab})\end{aligned}$$

On the other hand

$$\begin{aligned}\sum_a (K_{a\bar{i}a\bar{j}} + K_{\bar{a}\bar{i}\bar{a}\bar{j}}) &= \sum_a (h^a_{\bar{i}\bar{j}a} + h^{\bar{a}}_{\bar{i}\bar{j}\bar{a}}) \\ &\quad -2\sum_{t,a}(h^a_{t\bar{i}}h^a_{t\bar{j}} + h^a_{ti}h^a_{tj}) - 2\sum_{a,b}(h^{\bar{i}}_{ab}h^{\bar{j}}_{ab} + h^i_{ab}h^j_{ab})\end{aligned}$$

And the left hand side terms are equal in the above two equalities. So

$$\sum_a (h^a_{ija} + h^{\bar{a}}_{ij\bar{a}}) = \sum_a (h^a_{\bar{i}\bar{j}a} + h^{\bar{a}}_{\bar{i}\bar{j}\bar{a}})$$

By (14)

$$\sum_a (h^a_{ija} + h^{\bar{a}}_{ij\bar{a}}) = 0 \tag{15}$$

$$\sum_a (K_{aiaj} + K_{\bar{a}i\bar{a}j}) = -2\sum_{t,a}(h^a_{ti}h^a_{tj} + h^a_{t\bar{i}}h^a_{t\bar{j}}) - 2\sum_{a,b}(h^{\bar{i}}_{ab}h^{\bar{j}}_{ab} + h^i_{ab}h^j_{ab}) \tag{16}$$

$$\sum_t (K_{titj} + K_{\bar{t}i\bar{t}j}) = Ric_{ij} - \sum_a (K_{aiaj} + K_{\bar{a}i\bar{a}j}) \tag{17}$$

Let Ric^P be Ricci curvature of the leaves of P and apply Gauss Equation to submanifolds. We find

$$Ric^P_{ij} = \sum_t (K_{titj} + K_{\bar{t}i\bar{t}j}) - 2\sum_t (h^a_{ti}h^a_{tj} + h^a_{\bar{t}i}h^a_{\bar{t}j}) \tag{18}$$

Substituting (16) and (17) into (18)

$$Ric^P_{ij} = Ric_{ij} + 2\sum_{a,b}(h^{\bar{i}}_{ab}h^{\bar{j}}_{ab} + h^i_{ab}h^j_{ab}) \tag{19}$$

We conclude that each leaf of P is of positive Ricci curvature with a positive lower bound by (19) and thus finish the proof.

We will call a vector field in Q leaf-preserving for P if the local diffeomorphism generated by its flow maps any piece of a leaf of P into another wherever it is defined.

Step 2 *If $\xi = \sum_a(\xi_a e_a + \xi_{\bar{a}} J e_a)$ is defined around a leaf L of P and is leaf-preserving for P, then ξ lies in the kernel of the second fundamental form of Q along L.*

Proof We will apply Bochner's technique to $|\xi|^2$ on L.

That ξ is leaf -preserving means that $[\xi, e_i]$ and $[\xi, Je_i]$ are both vector fields in P. The covariant derivative of ξ along P is

$$\begin{aligned} \xi_{ai} &= \langle \nabla_{e_i}\xi, e_a\rangle \\ &= e_i(\xi_a) - \sum_b(\xi_b h^b_{ai} - \xi_{\bar{b}} h^{\bar{b}}_{ai}) \end{aligned}$$

Then we find

$$\xi_{ai} = -\sum_b (h^i_{ab}\xi_b + h^i_{a\bar{b}}\xi_{\bar{b}}) \tag{20}$$

We have only to show that $\xi_{ai} = 0$ to finish the argument.

Think of ξ_{ai} as a Q-valued 1-form on P and its covariant derivative along P is

$$\xi_{aij} = e_j(\xi_{ai}) - \sum_t(\xi_{at}h^t_{ij} + \xi_{a\bar{t}}h^{\bar{t}}_{ij}) - \sum_b(\xi_{bi}h^b_{aj} + \xi_{\bar{b}i}h^{\bar{b}}_{aj})$$

The laplacian of $\frac{1}{2}|\xi|^2$ (with positive sign) on the leaf L is

$$\triangle_L[\frac{1}{2}\sum_a(\xi_a)^2 + \frac{1}{2}\sum_a(\xi_{\bar{a}})^2] = \sum_{a,i}[(\xi_{ai})^2 + (\xi_{\bar{a}i})^2 + (\xi_{a\bar{i}})^2 + (\xi_{\bar{a}\bar{i}})^2]$$

$$+\sum_{a,i}[\xi_a\xi_{aii} + \xi_{\bar{a}}\xi_{\bar{a}ii} + \xi_a\xi_{a\bar{i}\bar{i}} + \xi_{\bar{a}}\xi_{\bar{a}\bar{i}\bar{i}}] \tag{21}$$

By (20) and the dual of (15), i.e.

$$\sum_i(h^i_{abi} + h^{\bar{i}}_{ab\bar{i}}) = 0$$

we find

$$\begin{aligned}
&\sum_{a,i}[\xi_a\xi_{aii} + \xi_a\xi_{a\bar{i}\bar{i}}] \\
&= \sum_{a,i,b}\xi_a[-h^i_{abi}\xi_b - h^i_{ab}\xi_{bi} \\
&-h^i_{a\bar{b}i}\xi_{\bar{b}} - h^i_{a\bar{b}}\xi_{\bar{b}i} - h^{\bar{i}}_{ab\bar{i}}\xi_b \\
&-h^{\bar{i}}_{ab}\xi_{b\bar{i}} - h^{\bar{i}}_{a\bar{b}\bar{i}}\xi_{\bar{b}} - h^{\bar{i}}_{a\bar{b}}\xi_{\bar{b}\bar{i}}] \\
&= \sum_{a,i,b}[(-h^i_{ba}\xi_a)\xi_{bi} + (-h^i_{\bar{b}a}\xi_a)\xi_{\bar{b}i} + (-h^{\bar{i}}_{ba}\xi_a)\xi_{b\bar{i}} + (-h^{\bar{i}}_{\bar{b}a}\xi_a)\xi_{\bar{b}\bar{i}}]
\end{aligned}$$

Similarly

$$\sum_{a,i}[\xi_{\bar{a}}\xi_{\bar{a}ii} + \xi_{\bar{a}}\xi_{\bar{a}\bar{i}\bar{i}}] = \sum_{a,i,b}[(-h^i_{b\bar{a}}\xi_{\bar{a}})\xi_{bi} + (-h^i_{\bar{b}\bar{a}}\xi_{\bar{a}})\xi_{\bar{b}i} + (-h^{\bar{i}}_{b\bar{a}}\xi_{\bar{a}})\xi_{b\bar{i}} + (-h^{\bar{i}}_{\bar{b}\bar{a}}\xi_{\bar{a}})\xi_{\bar{b}\bar{i}}]$$

By (20) again, (21) becomes

$$\triangle_L(\frac{1}{2}|\xi|^2) = 2\sum_{a,i}[(\xi_{ai})^2 + (\xi_{\bar{a}i})^2 + (\xi_{a\bar{i}})^2 + (\xi_{\bar{a}\bar{i}})^2]$$

All terms on the right hand side of the above equality must be zero since L is compact.

Step 3 *Leaves of P and Q are all totally geodesic.*

Proof We know that all leaves of P are compact and simply-connected. By Local Stability Theorem of compact leaves of foliations(see [CL] for reference), for any leaf L of P, there exists an open set containing L which is the union of some leaves of P and which is diffeomorphic to $L \times D$ with D an open set in R^{dimQ} such that any leaf in U corresponds to a level set in $L \times D$. Then we can construct a set of linearly independent leaf-preserving vector fields for P on U. And the number of the vector fields is equal to the dimension of Q. They all lie in the kernel of the second fundamental form of Q by Step 2. Then Q is totally geodesic. So is P.

Step 4 *M has a De Rham decomposition by P and Q.*

Proof Let ξ and η be leaf-preserving vector fields for P. Then by (20) and Step 3.

$$\xi_{ai} = \eta_{ai} = 0$$

So for any vector field X in P

$$X(\langle \xi, \eta \rangle) = 0 \tag{22}$$

On the other hand, it is easy to see that a leaf-preserving vector field multiplied by a function constant along any leaf is also leaf-preserving. We conclude from this fact and (22) that we can find a leaf-preserving orthonormal basis for Q around any leaf L of P from the set of linearly independent leaf-preserving vector fields given in Step 3. The same argument holds if we alternate P and Q. In summary, we have local orthonormal bases $\{v_i\}$ for P and $\{v_a\}$ for Q such that v_i is leaf-preserving for Q and v_a is for P. Then

$$[v_a, v_i] = 0$$

Since P and Q are both totally geodesic,

$$\nabla_{v_a} v_i = 0, \quad \nabla_{v_i} v_a = 0$$

This implies that M has a De Rham decomposition by P and Q in Kähler version.

The above proof can be applied to prove Corollary 2.3. In fact, Step 1 implies that the Ricci curvature of F is positive difinite with respect to the induced metric and hence each leaf of F is compact and simply-connected and M/F is a manifold. Step 2 and Step 3 guarantee that the normal bundle to F is totally geodesic. Therefore F is a Riemannian foliation.

Chapter 3

Einstein Metrics on S^2-Bundles

3.1 Introduction

Among the attempts to find inhomogeneous compact Einstein manifolds with positive scalar curvatures, Page[P] constructed a first example on the total space of the non-trivial S^2-bundle over S^2, which was later generalized by Bérard Bergery[BB] to the case where the base is a Kähler-Einstein manifold with positive first Chern class.

On the other hand, Koiso and Sakane found Einstein metrics for the bases being product of two irreducible hermitian symmetric spaces of compact type[S1] or two copies of the same Kähler-Einstein manifolds with positive first Chern class if the Futaki invariant vanishes[KS1]. While the examples by Page and Berárd Bergery are non-Kählerian, those by Koiso and Sakane are all Kählerian.

The main purpose of this chapter is to further generalize Bérard Bergery's construction to the case where the base is a finite product of compact Kähler-Einstein manifolds with positive first Chern classes. The S^2-bundles discussed here are similarly constructed. There is a natural complex structure on the total space of every S^2-bundle.

Koiso and Sakane[KS1] showed that there exists a Kähler-Einstein metric on a total space with positive first Chern class if the Futaki invariant vanishes. We will show that there exists a new family of non-Kählerian Einstein metrics even when the Futaki invariant does not vanishes.

An antipodal identification of every fibre will yield an RP^2-bundle. It is shown that Berárd Bergery's construction can be generalized to these bundles and therefore the Einstein metrics will be lifted to the corresponding S^2-bundles. This kind of Einstein metrics can be found on every S^2-bundle with non-negative first Chern class.

3.2 Construction and Main Results

Suppose $M_i^{2n_i}$, $i = 1, 2, \ldots, l$ are compact Kähler-Einstein manifolds with positive first Chern class. Let ρ_i be Ricci forms of M_i and ω_i Kähler forms. There are positive numbers L_i, positive integers p_i, and indivisible elements $\alpha_i \in H^2(M_i; \mathbf{Z})$, $i = 1, 2, \ldots, l$, such that

$$\rho_i = L_i \omega_i$$

and that

$$c_1(M_i) = \frac{1}{2\pi}[\rho_i] = p_i \alpha_i$$

We may regard α_i, ρ_i and ω_i as defined over $\bar{M} = M_1 \times M_2 \times \ldots \times M_l$. Then

$$\frac{1}{2\pi}[\sum_{i=1}^{l} \frac{L_i q_i}{p_i} \omega_i] = \sum_{i=1}^{l} q_i \alpha_i \in H^2(\bar{M}; \mathbf{Z})$$

We assume that the q_i, $i = 1, 2, \ldots, l$, are nonzero integers. Then $\sum_{i=1}^{l} q_i \alpha_i$

determines over $\bar{M}$ a principal S^1-bundle P on which there exists some connection whose curvature form is $\sum_{i=1}^{l} \frac{L_i q_i}{p_i}\omega_i$. P is Riemannian with the metric

$$ds_P^2 = \theta^2 + \sum_{i=1}^{l} ds_i^2$$

where θ is the connection form of P and ds_i^2 are horizontal lifts of the metric on M_i, $i = 1, 2, \ldots, l$.

Let S^1 act on S^2 by rotation and define an S^2-bundle over $\bar{M}$ by

$$W_{q_1, q_2, \ldots, q_l} = P \times_{S^1} S^2 \tag{23}$$

If we regard S^2 as the Riemann sphere, there exists a natural complex structure on $W_{q_1, q_2, \ldots, q_l}$ and its first Chern class is positive if and only if $|q_i| < |p_i|$, $i = 1, 2, \ldots, l$. But, to involve the non-Kählerian case in our discussion, we will take another approach.

We think of S^2 as the quotient space of $S^1 \times [0, a]$ by collapsing each boundary component to a point. We consider the metric on $W_{q_1, q_2, \ldots, q_l}$ of the form

$$ds^2 = dr^2 + f^2\theta^2 + \sum_{i=1}^{l} g_i^2 ds_i^2 \tag{24}$$

where r is the parameter of interval $[0, a]$ and f, g_i, $i = 1, 2, \ldots, l$, are smooth functions of r which are positive in $(0, a)$. Such a metric is smooth if the following boundary conditions hold:

1. $f'(0) = 1$, $f'(a) = -1$, and f is odd, i.e. $f^{(k)}(0) = f^{(k)}(a) = 0$ for k even.
2. $g_i > 0$ and g_i is even, i.e. $g_i^{(k)}(0) = g_i^{(k)}(a) = 0$ for k odd.

We sometimes omit the lower index of $W_{q_1, q_2, \ldots, q_l}$ for brevity. The metric on W is Einstein with Einstein constant c if the following equations hold:

$$-\frac{f''}{f} - \sum_{i=1}^{l} 2n_i \frac{g_i''}{g_i} = c \tag{25}$$

$$-\frac{f''}{f} - \sum_{i=1}^{l} 2n_i \frac{f'g_i'}{fg_i} + \sum_{i=1}^{l} 2n_i {\lambda_i}^2 \frac{f^2}{g_i^4} = c \tag{26}$$

$$-\frac{g_i''}{g_i} - \frac{f'g_i'}{fg_i} - \sum_{j\neq i} 2n_j \frac{g_i'g_j'}{g_ig_j} - (2n_i - 1)(\frac{g_i'}{g_i})^2 + \frac{L_i}{g_i^2} - 2{\lambda_i}^2 \frac{f^2}{g_i^4} = c \tag{27}$$

where $n_i = dim_C M_i$, $\lambda_i = \frac{L_i g_i}{2p_i}$, $i = 1, 2, \ldots, l$.

We impose the condition that the sectional curvatures of a horizontal direction with any two vertical directions are equal. This is equivalent to saying that

$$f^2 = \frac{g_i^2(g_i')^2}{A_i g_i^2 + \lambda_i^2} \quad i = 1, 2, \ldots, l \tag{28}$$

where A_i are constants.

Seen from the equations, the above condition is also desirable. In fact, subtracting the left hand side of (25) from that of (26), we find

$$\sum_{i=1}^{l} 2n_i (\frac{g_i''}{g_i} - \frac{f'g_i'}{fg_i} + \lambda_i^2 \frac{f^2}{g_i^4}) = 0$$

And letting each factor in the summation be 0 yields (28).

It is seen from (24) that $\{\partial/\partial r, \frac{1}{f}e\}$ is an orthonormal frame of the fibres where e is the vector field dual to θ. We introduce a complex structure on W which is the horizontal lift of that from the base on horizontal distribution and which is defined by

$$J(\partial/\partial r) = \frac{1}{f}e$$

on fibres.

Let X be the lift of a unit vector field on (M_i, ω_i), then $\bar{X} = \frac{1}{g_i}X$ is a unit vector field on W. If J is Kählerian,

$$< \nabla_{\bar{X}} \bar{X}, \partial/\partial r > = < \nabla_{\bar{X}} J\bar{X}, \frac{1}{f}e >$$

From (24), we see

$$f = \frac{g_i g_i'}{\lambda_i}$$

which is a special case of (28) with $A_i = 0$. It is easy to see that the condition is also sufficient for the metric to be Kählerian.

The following theorem is due to Koiso and Sakane[KS1] (Theorem 4.2). We shall give a proof of it in Section 3 from the point of view of solving (25)–(28) in the special case where $A_i = 0$. We can then compare this with the situations of Theorems 3.2 and 3.3.

Theorem 3.1 *There exists a Kähler-Einstein metric on* $W_{q_1,q_2,\ldots,q_l}$ *of form (28) such that all* $A_i = 0$ *if*

$$0 < |q_i| < |p_i|,$$

and if

$$\int_{-1}^{1} (\frac{p_1}{q_1} + x)^{n_1} (\frac{p_2}{q_2} + x)^{n_2} \ldots (\frac{p_l}{q_l} + x)^{n_l} x dx = 0$$

Note that the left hand side term of the integral equality in Theorem 3.1 is Futaki's Functional computed on the (real) holomorphic vector field $f\partial/\partial r$ along the fibres as explained by Besse[Be](p 475).

We remark that if we assume that all the q_i are equal and that the Ricci tensor of the Kählerian base manifold has positive constant eigenvalues instead that the base manifold is the product of Kähler-Einstein manifolds, then everything else in Theorem 3.1 will follows. And this is equivalent to Koiso and Sakane's construction. Therefore, we see their construction also starts from first Chern classes. In particular, if the Ricci tensor of the base manifold has only two different eigenvalues, then the base is a product of Kähler-Einstein manifolds according to Corollary 2.1.

Among the large number of S^2-bundles that can be constructed as above from Kähler-Einstein Fano manifolds, very few of them satisfy the integral equality of Theorem 3.1. However, in the Riemannian category, we can prove the less restrictive

Theorem 3.2 *There exists a non-Kählerian Einstein metric on* $W_{q_1,q_2,\ldots,q_l}$ *of form (6) such that each* A_i *is nonzero and takes the same sign as* ε_i *if*

$$0 < |q_i| < |p_i|, \quad i = 1, 2, \ldots, l$$

and if

$$\int_{-1}^{1} (|\frac{p_1}{q_1}| + \varepsilon_1 x)^{n_1} (|\frac{p_2}{q_2}| + \varepsilon_2 x)^{n_2} \ldots (|\frac{p_l}{q_l}| + \varepsilon_l x)^{n_l} x dx < 0$$

where $|\varepsilon_i| = 1$, $i = 1, 2, \ldots, l$ *and at least one of* ε_i *is positive.*

It is easy to see that $l > 1$. For otherwise, the integral will be positive.

We remark that the second inequality is not as restrictive as it may appear at first glance. Indeed, if the integral is positive for a group of ε_i, $i = 1, 2, \ldots, l$, changing the signs of all ε_i will make the integral negative. Hence

Corollary 3.1 *If for all* $0 < |q_i| < |p_i|$, *the integral in Theorem 3.1 does not vanish, there exists a non-Kählerian Einstein metric. In particular, there always exists on* W *an Einstein metric of type (28) for* $l > 1$ *such that* A_i *are not all negative.*

For example, if $M_1 = S^2$ and $M_2 = CP^n$ with $n > 1$, there is no Kähler-Einstein metric on the bundles[S1](Corollary 2). But we can find many Einstein metrics by Theorem 3.2.

We can also construct an RP^2-bundle from W by proper antipodal identification of the fibres of W. In fact, on a fibre there is a one -to-one correspondance between the orbits of the S^1 action and $[0, a]$, the domain of r. If we choose f and g_i, $i = 1, 2, \ldots, l$ to be symmetric with respect to $a/2$, then we can identify the part of the fibre corresponding to $[a/2, a]$ with that corresponding to $[0, a/2]$ and produce an RP^2-bundle with smooth induced metric. It is therefore obvious that once f and g_i are given on $[0, a/2]$ so that, among other things, they are extendable beyond $a/2$ smoothly and symmetrically, there correspond unique metrics both on S^2-bundle and RP^2-bundle so that the natural covering mapping from the former to the latter is locally isometric.

We remark that the Einstein metrics constructed in the above two theorems do not factor through to the RP^2-bundle. We will show in Section 3.6 that Einstein metrics exist on the RP^2-bundle.

Theorem 3.3 *There exists a non-Kählerian Einstein metric on $W_{q_1,q_2,\ldots,q_l}$ of form (6) such that all the A_i are negative and the metric can be reduced to that on an RP^2-bundle if* $0 < |q_i| < |p_i|$.

This generalizes Theorem 1.10 in [BB] in that the bases are allowed to be products of compact Kähler-Einstein manifolds with positive first Chern classes instead of a single one.

Some changes of the boundary conditions are necessary for the proof of Theorem 3.3. These will be stated in Section 3.6.

Unlike the Kählerian case, the formal solutions f and g_i for Theorems 3.2 and 3.3 are independent of the signs of q_i. If there is an Einstein metric on $W_{q_1,q_2,\ldots,q_l}$ given by either Theorem 3.2 or Theorem 3.3, we can immediately get as many the Einstein manifolds as the number of the diffeomorphic types of $\{W_{\bar{q}_1,\bar{q}_2,\ldots,\bar{q}_l};\ |\bar{q}_1| = |q_1|, |\bar{q}_2| = |q_2|, \ldots, |\bar{q}_l| = |q_l|\}$.

For application of Theorems 3.2 and 3.3, we show in Section 3.7 that there are infinitely many diffeomorphic types of S^2-bundles over $CP^n \times CP^m$. So we indeed get nontrivial new examples of non-Kählerian Einstein metrics.

Throughout the following 3 sections, for a given i, we define a function of g_i by

$$Y(g) = (g_i')^2$$

and for any function X, define

$$\dot{X} = \frac{dX}{dg_i} \quad \ddot{X} = \frac{d^2X}{dg_i^2}$$

Then we see

$$g_i'' = \frac{1}{2}\dot{Y} \quad g_i''' = \frac{1}{2}\ddot{Y}g_i'$$

etc.

If we assume that (M_i, g_i), $i = 1, 2, \ldots, l$ are only compact Einstein manifolds with Ricci constants $L_i > 0$ and that there are harmonic 2-forms ω_i defined on M_i so that

$$\frac{1}{2\pi}[\omega_i] \in H^2(M_i; Z)$$

then we can construct a principal circle bundle P over $\bar{M} = M_1 \times M_2 \times \ldots \times M_l$ such that the curvature form of a connection on P is

$$\frac{1}{2\pi}[\sum_{i=1}^{l} q_i\omega_i] \in H^2(\bar{M}; Z)$$

where q_i are nonzero integers.

Then we can similarly construct an S^2-bundle as

$$W = P \times_{S^1} S^2$$

We define a symmetric tensor $< \omega_i, \omega_i >$ as

$$< \omega_i, \omega_i > (X, Y) = trace(\omega_i(X), \omega_i(Y))$$

and can generalize Theorems 3.2 and 3.3 as follows

Theorem 3.4 *Suppose*

$$< \omega_i, \omega_i >= (2\lambda_i)^2 g_i$$

where λ_i are positive constants. Then the results of Theorems 2 and 3 hold where $|\frac{p_i}{q_i}|$ are replaced by $\frac{L_i}{2\lambda_i q_i}$.

The proof of Theorem 3.4 is similar to those of Theorems 3.2 and 3.3. We can see that each M_i is almost Kähler.

Theorem 3.1 can also be similarly generalized.

3.3 Proof of Theorem 3.1

In this section, we choose $A_i = 0$, $i = 1, 2, \ldots, l$ and replace (28) by

$$f = \frac{g_i g_i'}{\lambda_i}, \quad i = 1, 2, \ldots, l \tag{29}$$

Then the metric on W is Kählerian.

The proof of the theorem is based on the following two lemmas concerned with the solutions of the system (25)-(27) and (29).

Lemma 3.1 *The solutions of the system (25)-(27) and (29) satisfy the following conditions:*

$$g_i^2 (g_i')^2 = \frac{1}{\prod_{j=1}^{l} g_j^{2n_j}} \int (\prod_{j=1}^{l} g_j^{2n_j})(L_i - c g_i^2) g_i dg_i \tag{30}$$

$$\frac{1}{\lambda_i}(L_i - c g_i^2) = \frac{1}{\lambda_j}(L_j - c g_j^2), \quad i, j = 1, 2, \ldots, l \tag{31}$$

Proof Think of g_j as a function of g_i, substitute Y into (27), and replace f according to (29)

$$\dot{Y} = -[\frac{2}{g_i} + \sum_{j=1}^{l} \frac{2n_j \dot{g}_j}{g_j}]Y + (\frac{L_i}{g_i} - c g_i) \tag{32}$$

So

$$(g_i')^2 = \frac{1}{g_i^2 \prod_{j=1}^{l} g_j^{2n_j}} \int (\prod_{j=1}^{l} g_j^{2n_j})(L_i - cg_i^2) g_i dg_i$$

We see from (29) and above that

$$\frac{1}{\lambda_i^2} \int (\prod_{j=1}^{l} g_j^{2n_j})(L_i - cg_i^2) g_i dg_i = \frac{1}{\lambda_k^2} \int (\prod_{j=1}^{l} g_j^{2n_j})(L_k - cg_k^2) g_k dg_k$$

i.e.

$$\frac{1}{\lambda_i}(L_i - cg_i^2) = \frac{1}{\lambda_k}(L_k - cg_k^2)$$

Lemma 3.2 *The functions determined by (29), (30) and (31) are solutions of the system (25)–(27) and (29).*

Proof We need only to show that the functions determined by (29), (30) and (31) satisfy (25), noting that the left hand side terms of (25) and (26) are equal.

(25) can be written as

$$\ddot{Y} = -[\frac{3}{g_i} + \sum_{j=1}^{l} 2n_j \frac{\dot{g}_j}{g_j}]\dot{Y} - \sum_{j=1}^{l} 4n_j \frac{\ddot{g}_j}{g_j} Y - 2c$$

On the other hand, by (32)

$$\dot{Y} = -[\frac{2}{g_i} + \sum_{j=1}^{l} 2n_j \frac{\dot{g}_j}{g_j}]Y + (\frac{L_i}{g_i} - cg_i)$$

or

$$\ddot{Y} = -[\frac{2}{g_i} + \sum_{j=1}^{l} 2n_j \frac{\dot{g}_j}{g_j}]\dot{Y} - [-\frac{2}{g_i^2} + \sum_{j=1}^{l} 2n_j \frac{\ddot{g}_j}{g_j} - \sum_{j=1}^{l} 2n_j (\frac{\dot{g}_j}{g_j})^2]Y - (\frac{L_i}{g_i^2} + c)$$

Therefore (25) is equivalent to

$$\frac{1}{g_i}\dot{Y} = -[\frac{2n_i + 2}{g_i^2} + \sum_{j \neq i} 2n_j [\frac{\ddot{g}_j}{g_j} + (\frac{\dot{g}_j}{g_j})^2]]Y + \frac{L_i}{g_i^2} - c$$

Also it is clear that

$$\frac{\ddot{g}_j}{g_j} + (\frac{\dot{g}_j}{g_j})^2 = \frac{1}{g_i}\frac{\dot{g}_j}{g_j}$$

We find that (25) can be reduced to (32).

We conclude that the system (25)–(27) and (29) is equivalent to its subsystem (27) and (29). And the solutions are smooth. There remains the discussion of the boundary conditions to complete the proof of the theorem.

We assume that at boundaries,

$$f_0' = 1 \tag{33}$$

$$f_1' = -1 \tag{34}$$

$$f_0 = f_1 = 0 \tag{35}$$

$$g_{i0}' = g_{i1}' = 0 \tag{36}$$

$$g_{i0} > 0 \tag{37}$$

$$g_{i1} > 0 \tag{38}$$

$$f_0'' = f_1'' = 0 \tag{39}$$

where f_0, f_1, g_{i0}, g_{i1}, $i = 1, 2, \ldots, l$ are values of f and g at the boundaries respectively. If these boundary conditions are satisfied, the metric on W is C^2.

Substituting (33)–(38) into (27) and (29)

$$g_{i0}^2 = \frac{L_i - 2\lambda_i}{c}$$

$$g_{i1}^2 = \frac{L_i + 2\lambda_i}{c}$$

It follows that

$$|\lambda_i| < \frac{L_i}{2}$$

i.e.

$$|q_i| < |p_i|$$

By (30) and (31),

$$\int_{\sqrt{\frac{L_i-2\lambda_i}{c}}}^{\sqrt{\frac{L_i+2\lambda_i}{c}}} [\prod_{j=1}^{l}(L_j - \frac{\lambda_j}{\lambda_i}L_i + c\frac{\lambda_j}{\lambda_i}x^2)^{n_j}](L_i - cx^2)x dx = 0$$

Changing variables in the above we deduce the equality

$$\int_{-1}^{1} [\prod_{j=1}^{l}(\frac{p_j}{q_j} + y)^{n_j}] y dy = 0$$

Conversely, if the above equality holds, imposing the boundary conditions (33)–(38) on the functions determined by (29), (30) and (31) will yield the Kähler-Einstein metric on W as is done in [S1].

To justify this claim, we observe that the defining term for $(g_i')^2$ is positive in the inteval. In fact, without loss of generality, we can assume $g_{i0} < g_{i1}$ and if, for some $y \in (g_{i0}, g_{i1})$,

$$\int_{g_{i0}}^{y} (\prod_{j=1}^{l} g_j^{2n_j})(L_i - cg_i^2) g_i dg_i = 0$$

then for any $g_i \in (y, g_{i1})$, $L_i - cg_i^2 < 0$. On the other hand,

$$\int_{g_{i0}}^{g_{i1}} (\prod_{j=1}^{l} g_j^{2n_j})(L_i - cg_i^2) g_i dg_i = 0$$

So

$$\int_{y}^{g_{i1}} (\prod_{j=1}^{l} g_j^{2n_j})(L_i - cg_i^2) g_i dg_i = 0$$

which is impossible since the integrated term is negative.

Therefore g_i' is nowhere vanishing within the interval and well-defined.

The metric is C^1 at the points corresponding to the boundaries of the interval.

Note that

$$f = \frac{g_i g_i'}{\lambda_i}$$

and

$$g_i''' = \frac{1}{2}\ddot{Y} g_i'$$

It is evident that

$$f'' = \frac{6g_i'' + g_i \ddot{Y}}{2\lambda_i} g_i'$$

In particular, f'' is vanishing at the boundaries which implies that the metric is C^2. A theorem of DeTurck and Kazdan[DK] ensures that the C^2 Einstein metric is smooth.

We remark that a similar statement holds if c and L_i, $i = 1, 2, \ldots, l$ are negative. And the Einstein metrics cannot be reconstructed on an RP^2-bundle since g_i are monotone.

3.4 Non-Kählerian Einstein Equations

We will solve the system (25)–(28) with $A_i \neq 0, i = 1, 2, \ldots, l$

Lemma 3.3 *The solution to the system (25)–(28) with $A_i \neq 0$ satisfies, for some constant E, the following conditions*

$$E = \frac{L_i}{A_i} + c(\frac{\lambda_i}{A_i})^2 = \frac{L_j}{A_j} + c(\frac{\lambda_j}{A_j})^2, \quad i, j = 1, 2, \ldots, j \tag{40}$$

$$\frac{L_i - cg_i^2}{A_i} = \frac{L_j - cg_j^2}{A_j} \tag{41}$$

$$(g_i')^2 = \frac{(A_i g_i^2 + \lambda_i^2)^{3/2}}{g_i^2 \prod_{j=1}^{l} g_j^{2n_j}} \int [\prod_{j=1}^{l} g_j^{2n_j}](A_i g_i^2 + \lambda_i^2)^{-3/2}(L_i - cg_i^2) g_i dg_i \tag{42}$$

Proof By (27)

$$\dot{Y} = -[\frac{2n_i+2}{g_i} + \sum_{j\neq i}\frac{2n_j\dot{g}_j}{g_j} - \frac{3A_ig_i}{A_ig_i^2+\lambda_i^2}]Y + (\frac{L_i}{g_i} - cg_i) \tag{43}$$

Solving the equation we can derive (42).

We introduce the functions

$$\varphi_i = \frac{1}{A_i}(A_ig_i^2+\lambda_i^2)^{1/2}$$

Then

$$g_i^2 = A_i\varphi_i^2 - \frac{\lambda_i^2}{A_i}$$

We can rewrite (42) as

$$(\varphi_i')^2 = \frac{\varphi_i}{A_i\prod_{j=1}^l g_j^{2n_j}}\int[\prod_{j=1}^l g_j^{2n_j}]\varphi_i^{-2}(L_i - cg_i^2)d\varphi_i \tag{44}$$

So by (28)

$$\frac{\varphi_i}{A_i}\int(\prod_{j=1}^l g_j^{2n_j})\varphi_i^{-2}(L_i-cg_i^2)d\varphi_i = \frac{\varphi_k}{A_k}\int(\prod_{j=1}^l g_j^{2n_j})\varphi_k^{-2}(L_k-cg_k^2)d\varphi_k$$

Note that

$$\varphi_k' = \varepsilon_k\varphi_i', \quad \varepsilon_k = \pm 1$$

and the differentiation of the above equality yields

$$[\frac{1}{\varphi_i} - \frac{\varepsilon_k}{\varphi_k}](\varphi_i')^2 = \frac{\varepsilon_k[L_k - cg_k^2]}{A_k\varphi_k} - \frac{L_i - cg_i^2}{A_i\varphi_i}$$

Write

$$\varphi_k - \varepsilon_k\varphi_i = D_{k,i}, \quad E_k = \frac{L_k}{A_k} + c\frac{\lambda_k^2}{A_k^2}$$

Then

$$\frac{D_{k,i}}{\varphi_i\varphi_k}(\varphi_i')^2 = \frac{\varepsilon_k[L_k - cg_k^2]}{A_k\varphi_k} - \frac{L_i - cg_i^2}{A_i\varphi_i}$$

So

$$\frac{D_{k,i}}{A_i}\int(\prod_{j=1}^l g_j^{2n_j})\varphi_i^{-2}(L_i - cg_i^2)d\varphi_i = (\prod_{j=1}^l g_j^{2n_j})[\varepsilon_k(E_k - c\varphi_k^2) - \frac{E_k\varphi_k}{\varphi_i} + c\varphi_i\varphi_k]$$

Differentiating φ_i again, we find that either

$$D_{k,i} = 0 \tag{45}$$

or

$$\frac{E_i - E_k}{\varphi_i^2}\varphi_i' + (\frac{E_k}{\varphi_i} + c\varepsilon_k\varphi_k)(\sum_j 2n_j\frac{g_j'}{g_j}) = 0 \tag{46}$$

The latter is impossible however since φ_i are nonconstant functions. Therefore

$$D_{k,i} = 0$$

and

$$E_k = E_i$$

Lemma 3.4 *The functions given by (28), (40), (41) and (42) are solutions to the system (25)–(28) with* $A_i \neq 0$.

Proof Given (28), the left hand side terms of (25) and (26) are equal. So we have only to check (25).

It follows from (25) that

$$\frac{1}{2}\ddot{Y} = -[\frac{3}{2g_i} - \frac{3A_ig_i}{2(A_ig_i^2+\lambda_i^2)} + \sum_{j=1}^{l} n_j\frac{\dot{g}_j}{g_j}]\dot{Y} + [\frac{3A_i\lambda_i^2}{(A_ig_i^2+\lambda_i^2)^2} - \sum_{j\neq i} 2n_j\frac{\ddot{g}_j}{g_j}]Y - c$$

On the other hand, by (43)

$$\frac{1}{2}\ddot{Y} = -[\frac{1}{g_i} - \frac{3A_ig_i}{2(A_ig_i^2+\lambda_i^2)} + \sum_{j=1}^{l} n_j\frac{\dot{g}_j}{g_j}]\dot{Y}$$

$$-[-\frac{n_i+1}{g_i^2} + \sum_{j\neq i}[n_j\frac{\ddot{g}_j}{g_j} - n_j\frac{(\dot{g}_j)^2}{h_j^2}] - \frac{3A_i}{2(A_ig_i^2+\lambda_i^2)} + \frac{3A_i^2g_i^2}{(A_ig_i^2+\lambda_i^2)^2}]Y$$

$$+\frac{1}{2}(-\frac{L_i}{g_i^2} - c)$$

So (25) is equivalent to the following

$$\frac{\dot{Y}}{2g_i} = [-\frac{1}{g_i^2} - \sum_{j=1}^{l} n_j[\frac{\ddot{g}_j}{g_j} + (\frac{\dot{g}_j}{g_j})^2] + \frac{3A_i}{2(A_i g_i^2 + \lambda_i^2)}]Y + \frac{1}{2}(\frac{L_i}{g_i^2} - c)$$

The above equality is nothing but (43) since we can easily show that

$$\frac{\ddot{g}_j}{g_j} + (\frac{\dot{g}_j}{g_j})^2 = \frac{A_j}{A_i g_j^2}$$

The functions given by (28), (40)–(42) are all smooth.

3.5 Proof of Theorem 3.2

We assume the boundary conditions (33)–(39) for the system determined by (28), (40)–(42). Given $E > 0$ and $c > 0$, we can determine boundary values of g_i and hence φ_i by (27) and (28). We can also determine A_i by (40). Substituting these values into (44), the equivalent of (42), we will see that a necessary condition for the existence of a smooth Einstein metric is the vanishing of an integral involving E and c. Since c is the Einstein constant and can be varied by homothety, we think of it as fixed and will show that there is a proper E such that the integral vanishes as long as the conditions in Theorem 2 are satisfied. Finally, we will show that the vanishing of the integral with the given A_i and boundary values of φ_i is sufficient for the existence of an Einstein metric.

Note that $f^2 = (\varphi_i')^2$ by (28), we will specify their relation by

$$f = \epsilon_i \varphi_i'$$

so as to be compatible with (41). Substituting (35)–(38) into (27) and (28), at the boundaries,

$$\frac{L_i - cg_i^2}{(A_i g_i^2 + \lambda_i^2)^{1/2}} = 2f'\epsilon_i$$

Then by this and the boundary conditions (33) and (34),

$$\frac{E}{\varphi_{i0}} - c\varphi_{i0} = 2\epsilon_i, \quad \frac{E}{\varphi_{i1}} - c\varphi_{i1} = -2\epsilon_i \tag{47}$$

where φ_{i0} and φ_{i1} are values of φ_i at the boundaries.

That the metric on W is smooth implies, by(42) or (44),

$$\int_{\varphi_{i0}}^{\varphi_{i1}} \left(\prod_{j=1}^{l}[\varphi_i^2 - (\frac{\lambda_j}{A_j})^2]^{n_j}\right)[E - c\varphi_i^2]\varphi_i^{-2} d\varphi_i = 0 \tag{48}$$

subject to (40) and the following conditions

$$g_{i0}^2 = A_i\varphi_{i0}^2 - \frac{\lambda_i^2}{A_i} > 0$$

$$g_{i1}^2 = A_i\varphi_{i1}^2 - \frac{\lambda_i^2}{A_i} > 0$$

which are equivalent to that $|q_i| < |p_i|$.

We will require $E > 0$ and $c > 0$ in this section. Then by (40),

$$\frac{1}{A_i} = \frac{\varepsilon_i(L_i^2 + 4Ec\lambda_i^2)^{1/2} - L_i}{2c\lambda_i^2}$$

where $\varepsilon_i = \frac{A_i}{|A_i|}$, $\quad i = 1, 2, \ldots, l$. We have shown that all φ_i have the same absolute values and will denote $|\varphi_i|$ by φ. So by (47), (48) can be written as

$$\int_{\frac{(1+cE)^{1/2}-1}{c}}^{\frac{(1+cE)^{1/2}+1}{c}} [\prod_{j=1}^{l}(\varphi^2 - (\frac{\lambda_j}{A_j})^2)^{n_j}]\frac{E - c\varphi^2}{\varphi^2} d\varphi = 0$$

Changing variable by $y = c\varphi - (1 + cE)^{1/2}$,

$$\int_{-1}^{1} [\prod_{j=1}^{l}(y^2 + 2(1 + cE)^{1/2}y + 1 - \frac{L_j^2}{2\lambda_j^2}$$

$$+ \frac{\varepsilon_j L_j(L_j^2 + 4cE\lambda_j^2)^{1/2}}{2\lambda_j^2})^{n_j}]\frac{y^2 + 2(1 + cE)^{1/2}y + 1}{(y + (1 + cE)^{1/2})^2} dy = 0$$

Let the left hand side term of the above equality be $F(E)$. We want to find $\lim_{E\to 0} F(E)$. Note that

$$\int_{-1}^{1} \frac{y^2 + 2(1 + cE)^{1/2}y + 1}{(y + (1 + cE)^{1/2})^2} dy = 0$$

We can write

$$F(E) = F(E) - \int_{-1}^{1} \{\prod(\frac{\epsilon_j L_j (L_j^2 + 4cE\lambda_j^2)^{1/2} - L_j^2}{2\lambda_j^2} - cE)^{n_j}\}$$

$$\frac{y^2 + 2(1+cE)^{1/2} y + 1}{(y + (1+cE)^{1/2})^2} dy$$

Then

$$\lim_{E\to 0} F(E) = (-1)^{\sum_{\epsilon_j=-1} n_j} \int_{-1}^{1} [\prod_{\epsilon_j=1} (1+y)^{2n_j}][\prod_{\epsilon_j=-1} (\frac{4p_j^2}{q_j^2} - (y+1)^2)^{n_j}] dy$$

It is clear that

$$\lim_{E\to 0} F(E) > 0, \quad if \quad \sum_{\epsilon_j=-1} n_j \quad is \quad even$$

$$\lim_{E\to 0} F(E) < 0, \quad if \quad \sum_{\epsilon_j=-1} n_j \quad is \quad odd$$

On the other hand,

$$\lim_{E\to\infty} [F(E) E^{\frac{1}{2} - \sum_j \frac{n_j}{2}}] = \int_{-1}^{1} [\prod_{j=1}^{l} (2c^{1/2} y + c^{1/2} \varepsilon_j \frac{L_j}{|\lambda_j|})^{n_j}] \frac{2}{c^{1/2}} y dy$$

$$= (-1)^{\sum_{\epsilon_j=-1} n_j} T \int_{-1}^{1} \prod_{j=1}^{l} (|\frac{p_j}{q_j}| + \frac{A_j}{|A_j|} y)^{n_j} y dy$$

where T is a positive constant.

Therefore if

$$\int_{-1}^{1} [\prod_{j=1}^{l} (|\frac{p_j}{q_j}| + \frac{A_j}{|A_j|} y)^{n_j}] y dy < 0$$

there exists some $E > 0$ such that $F(E) = 0$.

Conversely, choosing such E, all A_i and the boundary values of φ_i will be determined as above. Since g_i are functions of φ_i, f will be given by (44). Note that in the integral of (44), $L_i - cg_i^2$ is monotone and so the integral is positive within the interval. Therefore f is well-defined. (36)–(38) are satisfied. On the other hand, since $f = \epsilon_i \varphi_i'$ and φ_i are given by (44) with the above boundary conditions, we can check by direct calculation that (33)–(35) hold. By (628) we can show that

$$f'' = F(g_i, Y, \dot{Y}, \ddot{Y})g_i'$$

where F is a smooth function.

In particular, f'' is vanishing at the boundaries, i.e. (39) holds. Therefore the Einstein metric is C^2 and hence smooth[DK]. So the theorem is true.

3.6 Proof of Theorem 3.3

We see that the Einstein metric constructed in Theorem 1 and Theorem 2 can not be reduced to an RP^2-bundle since g_i are monotone. To prove Theorem 3, we have only to consider the following change of the boundary conditions while retaining the other boundary conditions not mentioned among (33)–(39):

$$g_{i1}^2 = -\frac{\lambda_i^2}{A_i} \tag{49}$$

$$f_1 > 0, f_1' = 0 \tag{50}$$

$$f_1'' \textit{ is not necessarily } 0 \tag{51}$$

Since $\varphi_i = \varphi_j$, we denote this function by φ and choose

$$f = \varphi'$$

where f_1 and g_{i1} correspond to the boundary values of f and g_i as expained in Section 2.

Note that φ differs from that defined in Section 3.5 by a sign.

Substituting the above equality and the boundary conditions up to the first order differentiations into (27),

$$\frac{E}{\varphi_0} - c\varphi_0 = 2 \tag{52}$$

$$\varphi_0 < 0 \tag{53}$$

$$f_1^2 = -E \tag{54}$$

According to the calculation in Section 3.3,

$$f^2 = \frac{\varphi}{\prod_{j=1}^{l} g_j^{2n_j}} \int_{\varphi_0}^{\varphi} [\prod_j (A_j x^2 - \frac{\lambda_j^2}{A_j})^{n_j}] \frac{E - cx^2}{x^2} dx$$

where $g_j^2 = A_j \varphi^2 - \frac{\lambda_j^2}{A_j}$.

(49) is trivial since it is equivalent to $\varphi_1 = 0$. It is easy to check that (54) is true. And $f_1' = 0$ if

$$\lim_{\varphi \to 0} [\int_{\varphi_0}^{\varphi} [\prod_j (A_j x^2 - \frac{\lambda_j^2}{A_j})^{n_j}] \frac{E - cx^2}{x^2} dx + \frac{E}{\varphi} \prod_j g_j^{2n_j}] = 0$$

The left hand side term is

$$\int_0^1 [\prod_{j=1}^{l} (A_j \varphi_0^2 (1-y)^2 - \frac{\lambda_j^2}{A_j})^{n_j} - \prod_{j=1}^{l} (-\frac{\lambda_j^2}{A_j})^{n_j}] \frac{-E + c\varphi_0^2 (1-y)^2}{\varphi_0 (1-y)^2} dy$$

$$+2(1 + c\varphi_0) \prod_{j=1}^{l} (-\frac{\lambda_j^2}{A_j})^{n_j}$$

Let the above term be $G(E)$ and choose

$$\varphi_0 = \frac{-1 - (1 + cE)^{1/2}}{c}$$

and

$$\frac{1}{A_i} = \frac{-L_i - (L_i^2 + 4\lambda_i^2 cE)^{1/2}}{2c\lambda_i^2}$$

(52) and (53) are satisfied and

$$\varphi_0^2 < (\frac{\lambda_i^2}{A_i})^2$$

for $-\frac{1}{c} < E < 0$.

Then

$$\lim_{E\to 0} G(E) = -2\int_0^1 [\prod(\frac{L_j}{c} - \frac{4\lambda_j^2}{cL_j}(1-y)^2)^{n_j}]dy < 0$$

and

$$\lim_{E\to -\frac{1}{c}} G(E) = \int_0^1 [\prod(\frac{L_j + (L_j^2 - 4\lambda_j^2)^{1/2}}{2c})^{n_j}$$

$$-\prod(\frac{L_j + (L_j^2 - 4\lambda_j^2)^{1/2}}{2c} - \frac{2\lambda_j^2}{c(L_j + (L_j^2 - 4\lambda_j^2)^{1/2})}(1-y)^2)^{n_j}]\frac{1+(1-y)^2}{(1-y)^2}dy > 0$$

Therefore there is some E lying between $-1/c$ and 0 such that $G(E) = 0$.

Thus we see that the boundary conditions are satisfied. Then the metric is C^2 at the points corresponding to φ_1 automatically and at the points corresponding to φ_0 as is proved in the previous section. So the metric is smooth[DK].

3.7 Cohomology of W

We refer to [SS] for topological manipulations.

Let P be the principal S^1-bundle with Euler class

$$\alpha = \sum_{i=1}^{l} q_i\alpha_i \in H^2(\bar{M}; \mathbf{Z})$$

We identify S^1 with $U(1)$ and define a connection on $P \times_{U(1)} C$ by

$$\nabla v = -i\theta v$$

Then $P \times_{U(1)} C$ has a holomorphic structure and the first Chern class of the line bundle is α.

There is a natural imbedding

$$P \hookrightarrow P \times_{U(1)} C$$

Hence there exists a diffeomorphism between W and the projective bundle

$$Proj(P \times_{U(1)} (C \oplus \{Trivial\ line\ bundle\}))$$

For the projection

$$\pi : Proj(E) \longrightarrow \bar{M}$$

where $E = P \times_{U(1)} (C \oplus \{Trivial\ line\ bundle\})$, $\pi^{-1}E$ is the vector bundle over $Proj(E)$ whose fibre at l_p is E_p. When restricted to fibre $\pi^{-1}(p)$ it becomes the trivial bundle

$$\pi^{-1}E|_{Proj(E)_p} = Proj(E)_p \times E_p$$

The universal subbundle S over $Proj(E)$ is defined by

$$S = \{(l_p, v) \in \pi^{-1}E | v \in l_p\}$$

Its fibre at l_p consists of all the points in l_p.

Let s be the first Chern class of S^*.

On the other hand, according to [SS], let $\pi_1 : P^*(E) \longrightarrow \bar{M}$ be the bundle over $\bar{M}$ so that $P^*(E_p)$ is the set of 1-dimensional subspaces of E_p. Let F be the tautological rank 1 subbundle of $\pi_1^{-1}E$ given by

$$F_y = (\pi_1^{-1}E)_y \quad for \quad y \in P^*(E)$$

We define $\xi = \pi_1^{-1}E/F$. Then

$$c_1(\xi) = \alpha - s$$

Applying Leray-Hirsch Theorem,

$$H^*(W; \mathbf{Z}) = \{H^*(\bar{M}; \mathbf{Z}) \oplus H^*(\bar{M}; \mathbf{Z})s\}/\{s^2 + \alpha s\}$$

Denote by L the tangent bundle of the fibres on W. L is also a holomorphic line bundle.

We can also show that

$$c_1(L)^2 = \alpha^2 - 8\alpha s$$

In fact, from (5.58) of [SS],

$$detL^* = \pi^* det(P \times_{u(1)} (C \oplus \{Trivial\ line\ bundle\})) \otimes \xi^{-2}$$

So

$$c_1(L) = 2(\alpha - s) - \alpha$$

We can see that W is a Kähler manifold when identified with

$$Proj(P \times_{U(1)} C \oplus \{Trivial\ line\ bundle\})$$

Since $f\partial/\partial r$ is (real) holomorphic and $|f\partial/\partial r|^2 = f^2$, $-id'd''logf^2$ is the Ricci form of L. By (24),

$$-id'd''logf^2 = -\frac{f''}{f}dr \wedge f\theta - 2\lambda_i f'\omega_i$$

So

$$c_1(L) = \frac{1}{2\pi}[-\frac{f''}{f}(fdr \wedge \theta) - 2\lambda_i f'\omega_i]$$

Note that $c_1(M_i) = p_i\alpha_i$ and $q_i\alpha_i = \frac{1}{2\pi}[2\lambda_i\omega_i]$ We find

$$c_1(L) = \frac{1}{2\pi}[-\frac{f''}{f}(fdr \wedge \theta)] - f'q_i\frac{1}{p_i}c_1(M_i)$$

Therefore,the first Chern class of W is represented by

$$-\frac{f''}{2\pi f}(fdr \wedge d\theta) + \sum_{i=1}^{l}[1 - f'\frac{q_i}{p_i}]c_1(M_i)$$

Note that the minimum range of f' is [-1,1] considering the boundary conditions. We see that its first Chern class is positive if $|q_i| < |p_i|$, $i = 1, 2, \ldots, l$.

To see that our construction gives nontrivial Einstein metrics, we show the following proposition:

Proposition 3.1 *Given $\bar{M} = CP^n \times CP^m$,*

1. *For $n \neq m$, W_{q_1,q_2} is not diffeomorphic to $W_{\bar{q}_1,\bar{q}_2}$ if either $|q_1| \neq |\bar{q}_1|$, or $|q_2| \neq |\bar{q}_2|$.*

2. *For $n = m$, W_{q_1,q_2} is not diffeomorphic to $W_{\bar{q}_1,\bar{q}_2}$ if either*

$$(q_1^2 - \bar{q}_1^2)^2 + (q_2^2 - \bar{q}_2^2)^2 \neq 0$$

or

$$(q_1^2 - \bar{q}_2^2)^2 + (q_2^2 - \bar{q}_1^2)^2 \neq 0$$

Proof In fact, in this case,

$$H^*(W; \mathbf{Z}) = \mathbf{Z}[\alpha_1, \alpha_2, s]/\{\alpha_1^{n+1}, \alpha_2^{m+1}, s^2 + (q_1\alpha_1 + q_2\alpha_2)s\}$$

where α_1 is a generator of $H^2(CP^n; \mathbf{Z})$ and α_2 is that of $H^2(CP^m; \mathbf{Z})$. Then the first Pontrjagin class of $\bar{M}$ is

$$p_1(\bar{M}) = (n+1)\alpha_1^2 + (m+1)\alpha_2^2$$

On the other hand,

$$p_1(L) = c_1(L)^2 = \alpha^2 - 8\alpha s$$

where $\alpha = q_1\alpha_1 + q_2\alpha_2$. So

$$p_1(L) = q_1^2\alpha_1^2 + q_2^2\alpha_2^2 + 2q_1q_2\alpha_1\alpha_2 - 8q_1\alpha_1 s - 8q_2\alpha_2 s$$

Therefore, the first Pontrjagin class of W is

$$p_1(W) = (n + q_1^2 + 1)\alpha_1^2 + (m + q_2^2 + 1)\alpha_2^2 + 2q_1q_2\alpha_1\alpha_2 - 8q_1\alpha_1 s - 8q_2\alpha_2 s$$

Suppose there is some isomorphic mapping F between

$$Z[\alpha_1, \alpha_2, s]/\{\alpha_1^{n+1}, \alpha_2^{m+1}, s^2 + (q_1\alpha_1 + q_2\alpha_2)s\}$$

and

$$Z[\beta_1, \beta_2, t]/\{\beta_1^{n+1}, \beta_2^{m+1}, t^2 + (r_1\beta_1 + r_2\beta_2)t\}$$

sending

$$(n + q_1^2 + 1)\alpha_1^2 + (m + q_2^2 + 1)\alpha_2^2 + 2q_1q_2\alpha_1\alpha_2 - 8q_1\alpha_1 s - 8q_2\alpha_2 s$$

to

$$(n + r_1^2 + 1)\beta_1^2 + (m + r_2^2 + 1)\beta_2^2 + 2r_1r_2\beta_1\beta_2 - 8r_1\beta_1 t - 8r_2\beta_2 t$$

We assume

$$F(\alpha_1) = u_1\beta_1 + u_2\beta_2 + u_3 t$$

$$F(\alpha_2) = v_1\beta_1 + v_2\beta_2 + v_3 t$$

$$F(s) = w_1\beta_1 + w_2\beta_2 + w_3 t$$

Substituting into $F(s^2) = (F(s))^2$ and the invariance of the first Pontrjagin class under F, we can derive the following equations if $n > 1$ and $m > 1$.

$$w_1[w_1 + q_1u_1 + q_2v_1] = 0$$

$$w_2[w_2 + q_1u_2 + q_2v_2] = 0$$

$$2w_1w_2 + w_1[q_1u_2 + q_2v_2] + w_2[q_1u_1 + q_2v_1] = 0$$

$$2w_1w_3 + w_1[q_1u_3 + q_2v_3] + w_3[q_1u_1 + q_2v_1]$$
$$= r_1w_3[w_3 + q_1u_3 + q_2v_3]$$

$$2w_2w_3 + w_2[q_1u_3 + q_2v_3] + w_3[q_1u_2 + q_2v_2]$$

$$= r_2 w_3[w_3 + q_1 u_3 + q_2 v_3]$$

$$n + r_1^2 + 1 = (n + q_1^2 + 1)u_1^2 + (m + q_2^2 + 1)v_1^2$$
$$+2q_1 q_2 u_1 v_1 - 8(q_1 u_1 + q_2 v_1)w_1$$

$$m + r_2^2 + 1 = (n + q_1^2 + 1)u_2^2 + (m + q_2^2 + 1)v_2^2$$
$$+2q_1 q_2 u_2 v_2 - 8(q_1 u_2 + q_2 v_2)w_2$$

$$r_1 r_2 = (n + q_1^2 + 1)u_1 u_2 + (m + q_2^2 + 1)v_1 v_2$$
$$+q_1 q_2(u_1 v_2 + u_2 v_1) - 4q_1(u_1 w_2 + u_2 w_1) - 4q_2(v_1 w_2 + v_2 w_1)$$

$$-8r_1 = -r_1(n + q_1^2 + 1)u_3^2 - r_1(m + q_2^2 + 1)v_3^2 - 2r_1 q_1 q_2 u_3 v_3$$
$$+2(n + q_1^2 + 1)u_1 u_3 + 2(m + q_2^2 + 1)v_1 v_3 + 2q_1 q_2(u_1 v_3 + u_3 v_1)$$
$$-8q_1(u_1 w_3 + u_3 w_1) - 8q_2(v_1 w_3 + v_3 w_1) + 8r_1 q_1 u_3 w_3 + 8r_1 q_2 v_3 w_3$$

$$-8r_2 = -r_2(n + q_1^2 + 1)u_3^2 - r_2(m + q_2^2 + 1)v_3^2 - 2r_2 q_1 q_2 u_3 v_3$$
$$+2(n+q_1^2+1)u_2 u_3 + 2(m+q_2^2+1)v_2 v_3 + 2q_1 q_2(u_2 v_3 + u_3 v_2) - 8q_1(u_2 w_3 + u_3 w_2)$$
$$-8q_2(v_2 w_3 + v_3 w_2) + 8r_2 q_1 u_3 w_3 + 8r_2 q_2 v_3 w_3$$

Noting that both the matrices associated with F and its inverse have integral entries, we can derive from the above equations that

$$w_1 = w_2 = 0, \quad w_3^2 = 1$$

and furthermore

$$|q_i| = |\bar{q}_i|, \quad i = 1, 2$$

if $n \neq m$. If $n = m$, then either

$$|q_i| = |\bar{q}_i|, \quad i = 1, 2$$

or

$$|q_1| = |\bar{q}_2|, \quad |q_2| = |\bar{q}_1|$$

which imply the claim.

If $n = 1$, some of the above equalities will not hold, but we can get other equalities from the fact that

$$(F(\alpha_1))^2 = 0$$

and can similarly prove the result.

If we apply the classification theorem of 6-dimensional manifolds given by Jupp[J], we can show:

Proposition 3.2 *If $M = CP^1 \times CP^1$, then W_{q_1,q_2} is uniquely determined by $|q_1|$ and $|q_2|$ up to orientation-preserving homeomorphism.*

3.8 Complete Metrics

We now consider the line bundle constructed in last section with the first Chern class α. We can think of the fibre as R^2 and then give a metric

$$ds^2 = dr^2 + f^2\theta^2 + \sum_{i=1}^{l} g_i^2 ds_i^2$$

The metric is complete and non-compact if f and g_i are positive on $(0, \infty)$ and the following boundary conditions hold:

$$f'(0) = 1, \;\; g_i(0) > 0, \;\; f \text{ is odd at } 0, \;\; g \text{ is even at } 0$$

The Einstein equations are given by (25)–(27). We assume (28) holds.

Lemma 3.5 *If there is a complete and non-compact Einstein metric, then* $c \leq 0$. g_i *is monotonely increasing and* $\lim_{r\to\infty} g_i = \infty$.

Proof Since f is positive on $(0,\infty)$, g_i is strictly monotone. If $c > 0$, then g_i is bounded. For otherwise, $(g_i')^2$ will tend to $-\infty$ as $r \to \infty$ by (30) and (42).

Therefore $\lim_{r\to\infty} g_i$ is a finite number. So $\lim_{r\to\infty}(g_i')^2 = 0$. In this case, we will get a metric on either a compact manifold or a compact orbifold, and, in particular, the domain of g_i is bounded. This contradiction shows that c cannot be positive.

If $c = 0$ and the metric is Kählerian, then we see from Section 3 that

$$\frac{L_i}{\lambda_i} = \frac{L_j}{\lambda_j}$$

$$\frac{g_i^2}{\lambda_i} - \frac{g_j^2}{\lambda_j} \textit{ are constants}$$

Checking the boundary conditions with (30), we can find that $q_i = p_i$ is the sufficient and necessary condition for existence of Kähler Ricci-flat metric on the line bundle.

If $c < 0$ and the metric is Kählerian, then, we can similarly find that $q_i > p_i$ is both sufficient and necessary for the existence of a Kähler-Einstein metric.

If $c = 0$ and the metric is not Kähler, we check the results in Section 4 and find that there is a Ricci-flat metric if and only if $q_i < p_i$.

If $c < 0$ and the metric is not Kähler, on any line bundle there is an Einstein metric.

If we consider the following boundary conditions

$$f \textit{ and } g_i \textit{ are odd at } 0, \quad f'(0) = g_i'(0) = 1$$

then the total space is diffeomorphic to $R^{2+2\sum_{i=1}^{l} n_i}$.

If $c = 0$ and the metric is Kählerian, then without loss of generality we can assume $\lambda_i = \lambda_j$ and the metric is reduced to the form

$$ds^2 = dr^2 + f^2\theta^2 + g^2 ds^2_{\bar{M}}$$

which has been explored in [BB] and $\bar{M}$ can be only a complex projective space.

If $c < 0$ and the metric is Kählerian, we find that

$$p_i = (1 + \sum_{j=1}^{l} n_j) q_i$$

But we know that $p_i \leq n_i + 1$ and the equality holds if and only if M_i is CP^{n_i}. Hence $\bar{M}$ is a complex projective space.

If $c \leq 0$ and the metric is not Kähler, we can similarly show that $\bar{M}$ is a complex projective space.

Chapter 4

Einstein Metrics on Principal Bundles

4.1 Introduction

Let P be a principal G-bundle over a compact Riemannian manifold M^n with an irreducible connection. We define a Riemannian metric on P as before. We will discuss the cases where the bases are 4-dimensional and the structure groups are non-abelian. We will also discuss principal torus bundles over Kähler manifolds. Finally, we will suggest new constructions of Einstein metrics based on the known examples.

There are very few examples of Einstein metrics on principal bundles with non-abelian groups except for some homogeneous examples(see [Je], [Wa2] and [BGM2]). The difficulty is likely to lie in the fact that the influence of Lie group structure on the total space is hard to be fully explored, noting that even for homogeneous bundles over homogeneous spaces, the basic horizontal vector fields are not natural in the sense that they are not in general invariant under the Lie group action.

We will try to choose proper horizontal frames in our work so as to draw as much information as possible from the Lie group structure. We hope that our initial investigation could serve as motivation for further research on this subject.

Let V_α denote vertical Killing vector field and $\{e_i\}$ be a horizontal orthonormal frame. We now drop the requirement that e_i be basic. Then, since the connection is G-invariant, $[V_\alpha, e_i]$ is horizontal. So

$$h^\alpha_{ij} + h^\alpha_{ji} = 0$$

But, in general, $h^\alpha_{ji} \neq h^i_{j\alpha}$. We can similarly derive the following equations

$$h^\alpha_{ijk} + h^\alpha_{jki} + h^\alpha_{kij} = 0 \tag{55}$$

$$K_{\alpha i\beta j} = h^\alpha_{ij\beta} + h^\alpha_{it}h^\beta_{jt} = h^\alpha_{jt}h^\beta_{it} + h^\alpha_{\beta\gamma}h^\gamma_{ij} \tag{56}$$

In fact, (55) follows from Proposition 1.1 and the fact that for Killing vector field $V = V_\alpha$,

$$V_{ijk} = -h^\alpha_{ijk}$$

We now derive (56). We can compute that

$$K_{\alpha i\beta j} = h^\beta_{ji\alpha} + h^\alpha_{ik}h^\beta_{jk} \quad K_{\alpha j\beta i} = h^\beta_{ij\alpha} + h^\alpha_{jk}h^\beta_{ik} \quad K_{ij\alpha\beta} = -h^\alpha_{ik}h^\beta_{jk} + h^\alpha_{jk}h^\beta_{ik} + 2h^\alpha_{\beta\gamma}$$

By Bianchi's identity

$$K_{ij\alpha\beta} + K_{j\alpha i\beta} - K_{\alpha i\beta j} = 0$$

and the fact that

$$h^\beta_{ij\alpha} + h^\beta_{ji\alpha} = 0$$

we find

$$K_{\alpha j\beta i} = h^\beta_{ij\alpha} + h^\beta_{ik}h^\alpha_{jk} = h^\alpha_{ik}h^\beta_{jk} + h^\beta_{\alpha\gamma}h^\gamma_{ij}$$

Interchanging α and β yields (56).

Letting K^M be lift of the curvature tensor from M,

$$K_{ijkt} = K^M_{ijkt} - 2h^\alpha_{ij}h^\alpha_{kt} - h^\alpha_{ik}h^\alpha_{jt} + h^\alpha_{it}h^\alpha_{jk} \tag{57}$$

(57) is true by Proposition 1.11 if the horizontal basis is basic. However since all the terms in it are well-defined tensors, it is also true for any horizontal basis. This observation is critical for our treatment in the subsequent sections. The same argument applies to the remaining equations in Proposition 1.11. In particular, the Einstein equations are

$$h^\alpha_{ijj} = 0 \tag{58}$$

$$h^\alpha_{ij}h^\beta_{ij} + Ric^G_{\alpha\beta} = c\delta_{\alpha\beta} \tag{59}$$

$$Ric^M_{ij} - 2h^\alpha_{it}h^\alpha_{jt} = c\delta_{ij} \tag{60}$$

where Ric^G is the Ricci curvature of thr Lie group G and c is a constant.

If a basis V_α is chosen properly so that Ric^G is diagonal, then by (59),

$$h^\alpha_{ij}h^\beta_{ij} = 0, \quad \alpha \neq \beta$$

Therefore

Proposition 4.1 *If the metric on P is Einstein, then*

$$\dim G \leq \frac{n(n-1)}{2}$$

Sections 4.2–4.4 discuss the cases when the bases are Kählerian and Sections 4.5, 4.6 assume the bases are Einstein. In particular, we find the following equivalent conditions from Sections 4.2, 4.3, 4.5, 4.6:

Let P be a principal G-bundle over a 4-dimensional compact manifold M where G is non-abelian and $dimG = 3$. If there is an irreducible connection on P, then the following are equivalent:

- The induced metric on P is Einstein and the base is Einstein.
- P is homogeneous and is one of S^7 with structure group $Sp(1)$ over S^4, RP^7 with structure group $SO(3)$ over S^4 or $SU(3)/U(1)$ with structure group $SO(3)$ over CP^2.
- P is 3-Sasakian.

More specifically, we also have the following equivalent conditions:

- P is Einstein, M is Kählerian and the curvature form of the connection is of type (1,1).
- P is homogeneous and is $SU(3)/U(1)$.

Let P be a principal G-bundle over a 4-dimensional compact manifold M where G is non-abelian and $dimG = 4$. If there is an irreducible connection on P , then the following are equivalent:

- The induced metric on P is Einstein and the base is Einstein.
- P is $SU(3)$.
- The induced metric on P is Einstein, the base is Kählerian and the curvature form of the connection is of type (1,1).

In Section 4.4, we point out that the theorem given in Chapter 2 can be generalized to principal torus bundles. In Section 4.7,the author suggests new constructions of Einstein metrics on certain principal S^1-bundles, which, the author believes, can be further generalized.

4.2 7-dimensional Einstein manifolds

Proposition 4.2 *Suppose P is a principal G-bundle over a Kähler manifold of complex dimension 2. If P is Einstein with respect to a metric induced by a connection and the curvature form of the connection is of type (1,1), then $\dim G \leq 4$.*

Proof The proof is similar to that of the last proposition.

The proposition tells that we need only to discuss the cases when $\dim G = 3$ and $\dim G = 4$. We will assume $\dim G = 3$ in this section and assume $\dim G = 4$ in next section.

Let $\dim G = 3$ and M be a compact Kähler manifold. $\mathfrak{g}$ is the same as $\mathfrak{so}(3)$ or $\mathfrak{su}(2)$. We require that the curvature form be of type (1,1), i.e. for J, the horizontal lifting of complex structure from the base,

$$h^\alpha(JX, JY) = h^\alpha(X, Y)$$

We represent the three vertical Killing vector fields by $\{V_\alpha, V_\beta, V_\gamma\}$ and assume $h^\alpha_{\beta\gamma} = 1$. Then $Ric^G = 2g$ when restricted on fibres.

We also have local horizontal basis $\{e_1, Je_1, e_2, Je_2\}$. We apply the following notations:

$$h^\alpha_{ij} = h^\alpha(e_i, e_j), \quad h^\alpha_{i\bar j} = h^\alpha(e_i, Je_j)$$

Then $h^\alpha_{ij} = h^\alpha_{\bar i\bar j}$, *etc.*

We choose the horizontal basis so that

$$h^\alpha_{12} = h^\alpha_{1\bar 2} = 0$$

and let

$$h^\alpha_{1\bar 1} = \lambda, \quad h^\alpha_{2\bar 2} = \mu$$

We can also assume that $\lambda \neq \mu$

Lemma 4.1 *λ and μ are constants.*

Proof By (58),

$$h^{\alpha}_{1\bar{1}\bar{1}} + h^{\alpha}_{1\bar{2}\bar{2}} + h^{\alpha}_{122} = 0$$

i.e.

$$\lambda_{\bar{1}} = (\lambda - \mu)(h^{1}_{2\bar{2}} + h^{\bar{1}}_{22})$$

By (55),

$$h^{\alpha}_{2\bar{2}\bar{1}} + h^{\alpha}_{\bar{2}\bar{1}2} + h^{\alpha}_{\bar{1}2\bar{2}} = 0$$

i.e.

$$\mu_{\bar{1}} = (\mu - \lambda)(h^{\bar{1}}_{22} + h^{\bar{1}}_{\bar{2}\bar{2}})$$

On the other hand, it follows from (59) that

$$2\lambda^2 + 2\mu^2 + 2 = c$$

i.e.

$$\lambda\lambda_{\bar{1}} + \mu\mu_{\bar{1}} = 0$$

So

$$(\lambda - \mu)^2(h^{\bar{1}}_{22} + h^{\bar{1}}_{\bar{2}\bar{2}}) = 0$$

Therefore,

$$h^{\bar{1}}_{22} + h^{\bar{1}}_{\bar{2}\bar{2}} = 0$$

and hence $\lambda_{\bar{1}} = \mu_{\bar{1}} = 0$.

Similarly, we can show that $\lambda_1 = \lambda_2 = \lambda_{\bar{2}} = 0$. Since the connection on P is assumed to be irreducible, we see that λ is a constant. And so is μ.

Lemma 4.2 $h^{\gamma}_{1\bar{1}} = h^{\gamma}_{2\bar{2}} = h^{\beta}_{1\bar{1}} = h^{\beta}_{2\bar{2}} = 0$

Proof We apply (56) to prove. In fact,

$$h^{\alpha}_{1\bar{1}\beta} + h^{\alpha}_{1\bar{1}} h^{\beta}_{\bar{1}\bar{1}} = h^{\alpha}_{\bar{1}1} h^{\beta}_{11} + h^{\alpha}_{\beta\gamma} h^{\gamma}_{1\bar{1}}$$

So we find $h^{\gamma}_{1\bar{1}} = 0$.

Similarly, $h^{\gamma}_{2\bar{2}} = h^{\beta}_{1\bar{1}} = h^{\beta}_{2\bar{2}} = 0$

We can choose proper $\{e_1, Je_1\}$ such that $h^{\beta}_{1\bar{2}} = 0$.

Lemma 4.3

$$\lambda + \mu = 0, \quad h^{\gamma}_{12} = 0$$

h^{β}_{12} and $h^{\gamma}_{1\bar{2}}$ are constants.

Proof By (59) h^{β}_{12} is constant.

By (59) again, $h^{\beta}_{ij} h^{\gamma}_{ij} = 0$. It follows that $h^{\gamma}_{12} = 0$. and therefore $h^{\gamma}_{1\bar{2}}$ is constant.

By (56)

$$h^{\beta}_{1\bar{1}\gamma} + h^{\beta}_{12} h^{\gamma}_{\bar{1}2} = h^{\beta}_{\bar{1}\bar{2}} h^{\gamma}_{1\bar{2}} + h^{\beta}_{\gamma\alpha} h^{\alpha}_{1\bar{1}}$$

i.e.

$$h^{\beta}_{12}(h^{\gamma}_{1\bar{2}} + h^{1}_{\bar{2}\gamma}) + \lambda h^{\alpha}_{\beta\gamma} = 0$$

By (56) again

$$h^{\beta}_{2\bar{2}\gamma} + h^{\beta}_{\bar{2}1} h^{\gamma}_{\bar{2}1} = h^{\beta}_{\bar{2}\bar{1}} h^{\gamma}_{2\bar{1}} + h^{\beta}_{\gamma\alpha} h^{\alpha}_{2\bar{2}}$$

i.e.

$$h^{\beta}_{12}(h^{\gamma}_{1\bar{2}} + h^{1}_{\bar{2}\gamma}) - \mu h^{\alpha}_{\beta\gamma} = 0$$

Comparing the above equalities we see $\lambda + \mu = 0$.

Note that, by (59), $|h^{\beta}_{12}| = |h^{\gamma}_{1\bar{2}}|$. We will assume that $h^{\beta}_{12} = h^{\gamma}_{1\bar{2}}$ and the other case is similar. We compute (56) for all its possible entries and can conclude:

Lemma 4.4 *Assume* $\lambda \neq \mu$, $h^{\beta}_{1\bar{2}} = 0$ *and* $h^{\beta}_{12} = h^{\gamma}_{1\bar{2}}$. *Then*

$$\lambda + \mu = 0, \quad h^{\beta}_{12} = h^{\gamma}_{1\bar{2}}, \quad h^{\bar{2}}_{2\beta} = h^{\bar{1}}_{1\beta}, \quad h^{\bar{2}}_{2\gamma} = h^{\bar{1}}_{1\gamma}, \quad h^{1}_{\bar{2}\gamma} = h^{1}_{2\beta}$$

$$h^{\bar{1}}_{2\beta} = h^{1}_{2\gamma} = h^{\bar{1}}_{2\alpha} = h^{1}_{2\alpha} = 0, \quad \lambda^2 = (h^{\beta}_{12})^2 = (h^{\gamma}_{1\bar{2}})^2$$

$$(\lambda - \mu)(h^{\gamma}_{1\bar{2}} + h^{1}_{2\gamma}) + 2h^{\alpha}_{\beta\gamma}h^{\beta}_{12} = 0$$

$$h^{\beta}_{12}(h^{\bar{2}}_{2\alpha} - h^{\bar{1}}_{1\alpha}) - (\lambda - \mu)h^{1}_{2\beta} = 0$$

$$h^{\gamma}_{1\bar{2}}(h^{\beta}_{12} + h^{1}_{2\beta}) + \lambda h^{\alpha}_{\beta\gamma} = 0$$

Lemma 4.5 *Under the assumption of Lemma 4,*

$$h^{1}_{22} = h^{1}_{\bar{2}\bar{2}} = h^{\bar{1}}_{22} = h^{\bar{1}}_{\bar{2}\bar{2}} = h^{2}_{11} = h^{2}_{\bar{1}\bar{1}} = h^{\bar{2}}_{11} = h^{\bar{2}}_{\bar{1}\bar{1}} = 0$$

$$h^{\bar{1}}_{11} = h^{\bar{2}}_{21}, \quad h^{1}_{\bar{1}\bar{1}} = h^{2}_{\bar{2}\bar{1}}, \quad h^{\bar{2}}_{22} = h^{\bar{1}}_{12}, \quad h^{2}_{\bar{2}\bar{2}} = h^{1}_{1\bar{2}}$$

Proof We compute (55) for all its possible entries and can derive the equalities.

Lemma 4.6 *M is Kähler-Einstein and self-dual. Therefore M is* CP^2 *with Einstein constant* $c + 6\lambda^2$.

Proof By Lemma 4.4,

$$h^{\alpha}_{ik}h^{\alpha}_{jk} = h^{\beta}_{ik}h^{\beta}_{jk} = h^{\gamma}_{ik}h^{\gamma}_{jk} = \lambda^2\delta_{ij}$$

We see from (60) that M is Einstein with Einstein constant $c + 6\lambda^2$. It follows that

$$K^{M}_{1\bar{1}1\bar{1}} = K^{M}_{2\bar{2}2\bar{2}}$$

On the other hand, the other curvatures of M can be derived as follows:

We first compute K_{ijkl} by means of the conditions given in the last two lemmas. Then apply (57) to find the corresponding curvatures of M. We can get

$$K^{M}_{1\bar{1}12} = K^{M}_{1\bar{1}1\bar{2}} = K^{M}_{1\bar{1}\bar{1}2} = K^{M}_{1\bar{1}\bar{1}\bar{2}} = K^{M}_{121\bar{2}} = 0$$

$$K^M_{2\bar{2}\bar{1}2} = K^M_{2\bar{2}2\bar{1}} = K^M_{2\bar{2}\bar{2}1} = K^M_{2\bar{2}\bar{2}\bar{1}} = K^M_{212\bar{1}} = 0$$

By the above relations we see M is self-dual and hence is CP^2.

We want to calculate the value of λ.

Since $M = CP^2$,

$$K^M_{1\bar{1}1\bar{1}} = K^M_{2\bar{2}2\bar{2}} = 4K^M_{1212} = 4K^M_{1\bar{2}1\bar{2}}$$

$$K^M_{1212} = K_{1212} + 2(h^\xi_{12})^2 + h^\xi_{11}h^\xi_{22} - h^\xi_{12}h^\xi_{21} = K_{1212} + 3\lambda^2$$

On the other hand, by direct calculation and applying the relations given by Lemmas 4.4 and 4.5, we see

$$K_{1212} = -3\lambda^2 - 2\lambda$$

So by (57),

$$K^M_{1212} = -2\lambda$$

$$Ric^M_{11} = 6K^M_{1212} = -12\lambda$$

On the other hand we know by last lemma that the Einstein constant is $c+6\lambda^2$ So

$$6\lambda^2 + 12\lambda + c = 0$$

Note that by (59),

$$4\lambda^2 + 2 = c$$

Therefore $\lambda = -1$ or $\lambda = -1/5$.

Lemma 4.7 *If there is an Einstein metric on P such that $\lambda = -\frac{1}{5}$, then by a homothety of the metric on the base, we can get another Einstein metric on P such that the new parameter $\lambda = -1$.*

Proof Choose new horizontal orthonormal basis as

$$\tilde{e}_i = \sqrt{5}e_i, \quad J\tilde{e}_i = \sqrt{5}Je_i$$

Then we can check that (58)–(60) hold. And the correspoinding $\lambda = 1$.

So without loss of generality, we assume $\lambda = -1$. Then $c = 6$. We can prove

Lemma 4.8 *P is a 3-Sasakian manifold.*

To prove the lemma, we need only to check that for Killing vector field $X = V_\eta$, $\eta = \alpha, \beta, \gamma$,

$$X_{ABC} = \delta_{AC}\delta_{B\eta} - \delta_{BC}\delta_{A\eta}$$

For example, let $X = V_\alpha$, then

$$X_{ij\alpha} = K_{ij\alpha\alpha} = 0$$

$$X_{ij\beta} = K_{ij\beta\alpha} = -K_{j\beta i\alpha} - K_{\beta ij\alpha} = h^\alpha_{ik}h^\beta_{jk} - h^\alpha_{jk}h^\beta_{ik} - 2h^\alpha_{\beta\gamma}h^\gamma_{ij} = 0$$

$$X_{ijk} = K_{ijk\alpha} = -h^\alpha_{ijk} = 0, \quad X_{\alpha ij} = -K_{\alpha i\alpha j} = -h^\alpha_{ik}h^\alpha_{jk} = -\delta_{ij}$$

etc.

Then according to [BGM1](Corollary 3.12),

Theorem 4.1 *Let P be a principal G-bundle over a Kähler manifold of complex dimension 2 with G non-abelian and dimG = 3. If there is an irreducible connection on P such that its curvature form is of type (1,1), and the induced metric on P is Einstein, then P is homogeneous and is $SU(3)/U(1)$. Hence the base M is CP^2.*

4.3 Homogeneous Einstein Metrics

We assume $dimG = 4$ and G is not abelian. We also assume that the curvature form is of type (1,1).The Lie algebra of G is the same as that of $U(2)$. There is a vertical Killing vector field V_ξ which corresponds to the center of G and h^ξ corresponds to a closed 2-form on M. We can as is done in the last section, construct a vertical Killing frame $\{V_\xi, V_\alpha, V_\beta, V_\gamma\}$ such that Ric^G is diagonalized and $h^\alpha_{\beta\gamma} = 1$.

Lemma 4.9 *Lemmas 4.4 and 4.5 hold.*

Lemma 4.10 h^ξ *is proportional to lifting of Kähler form from* M.

Proof It follows from (59).

Lemma 4.11 $M = CP^2$.

Without loss of generality, we assume $h^\xi_{1\bar{1}} = -\frac{1}{2}\sqrt{c}$ since by (59), $4(h^\xi_{1\bar{1}})^2 = c$. Then

$$Ric^M_{1\bar{1}} = c + 6\lambda^2 + \frac{c}{2} = 6\lambda^2 + \frac{3c}{2}$$

On the other hand, we find

$$Ric^M_{1\bar{1}} = -12\lambda$$

as is done in the last section. So

$$6\lambda^2 + 12\lambda + \frac{3c}{2} = 0$$

Note that by (59),

$$c = 4\lambda^2 + 2$$

Then we find $\lambda = -1/2$. Applying (56),

Lemma 4.12

$$h^{\alpha}_{1\bar{1}} = -1/2, \quad h^{\alpha}_{2\bar{2}} = 1/2, \quad h^{\xi}_{1\bar{1}} = h^{\xi}_{2\bar{2}} = -\frac{\sqrt{3}}{2}, \quad c = 3$$

$$h^{\beta}_{12} = h^{\gamma}_{1\bar{2}} = h^{1}_{2\beta} = h^{1}_{\bar{2}\gamma}, \quad h^{\bar{2}}_{2\alpha} - h^{\bar{1}}_{1\alpha} = 2\lambda$$

$$h^{\bar{2}}_{2\beta} = h^{\bar{1}}_{1\beta}, \quad h^{\bar{2}}_{2\gamma} = h^{\bar{1}}_{1\gamma}, \quad h^{\bar{2}}_{2\xi} = h^{\bar{1}}_{1\xi}$$

$$h^{\bar{1}}_{2\beta} = h^{1}_{2\gamma} = h^{1}_{2\alpha} = h^{\bar{1}}_{2\alpha} = h^{1}_{2\xi} = h^{\bar{1}}_{2\xi} = 0$$

Lemma 4.13 *P is locally symmetric. i.e.*

$$\nabla K = 0$$

Proof It follows from calculation applying the above lemmas.

We can also show that P has a locally symmetric Lie group structure with bi-invariant metric.

In fact, we can define a vector by

$$\theta_1 = -h^{\bar{1}}_{11}, \quad \theta_{\bar{1}} = h^{1}_{1\bar{1}}, \quad \theta_2 = -h^{\bar{2}}_{22}, \quad \theta_{\bar{2}} = h^{2}_{2\bar{2}}$$

$$\theta_\alpha = -\lambda - h^{\bar{1}}_{1\alpha}, \quad \theta_\beta = -h^{\bar{1}}_{1\beta}, \quad \theta_\gamma = -h^{\bar{1}}_{1\gamma}, \quad \theta_\xi = -\sqrt{3} - h^{\bar{1}}_{1\xi}$$

Checking by the integrability conditions, we see they are in fact the curvature relations. For example,

$$\theta_{\alpha\beta} = \theta_{\beta\alpha}$$

if and only if

$$K_{\alpha\beta 1\bar{1}} = 0$$

But we can show that $K_{\alpha 1\beta\bar{1}} = K_{\alpha\bar{1}\beta 1} = 0$. So the equality holds.

Therefore there is a function θ so that $\nabla\theta$ is defined as above locally.

We construct a new frame

$$\tilde{e}_1 = cos\theta e_1 + sin\theta J e_1$$

$$\tilde{e}_2 = cos\theta e_2 + sin\theta Je_2$$

Then $\{V_\alpha, V_\beta, V_\gamma, V_\xi, \tilde{e}_1, \tilde{e}_2, J\tilde{e}_1, J\tilde{e}_2\}$ satisfies the Lie Algebra structure conditions of Lie Group $SU(3)$. In particular, if P is simply-connected, $P = SU(3)$.

Theorem 4.2 *Let P be a principal G bundle over a 4-dimensional Kähler manifold so that G is non-abelian and $dimG = 4$. If there is an irreducible connection so that the curvature form is of type (1,1) and the corresponding metric on P is Einstein, then M can be only CP^2 and P is locally isometric to $SU(3)$. In particular, if P is simply-connected, then P is $SU(3)$.*

We will see that P is $SU(3)$ in Section 4.6.

4.4 Principal Torus Bundles

Suppose P is a principal torus bundle over a compact Kähler manifold M with a connection θ. We can define a metric on P. The Einstein condition is

$$h^\alpha \text{ is a harmonic } 2-form$$

$$|h^\alpha|^2 = c$$

$$Ric^M(X,Y) = cg(X,Y) + 2\sum_\beta g(h^\beta, h^\beta)(X,Y)$$

In this case we need only to consider the horizontal basic frames, or can think of all the component of the curvature form as 2-tensors over M. The proof in this section is very brief because a similar one has been given in Chapter 2.

We say the curvature form is diagonal if there is a local horizontal frame $\{e_A\}$ such that

$$h^\alpha_{AB} = 0 \quad if \quad e_B \neq \pm Je_A$$

Theorem 4.13 *If the curvature form of the connection on P is of type (1,1) and diagonal, then M is a product of Kähler-Einstein manifolds.*

Proof The proof is similar to that in Chapter 2. For convenience. we call a type (1,1) tensor diagonalized if its associated symmetric tensor is diagonal.

Lemma 4.14 *For fixed α, the eigenvalues of h^α are all constants.*

Proof Let $\{e_A, Je_A\}$ be a local frame diagnolizing the curvature form. Since for fixed α, h^α is a cclosed 2-form on M, we have

$$h^\alpha_{i\bar{i}j} + h^\alpha_{\bar{i}ji} + h^\alpha_{ji\bar{i}} = 0$$

i.e.

$$e_j(h^\alpha_{i\bar{i}}) = (h^\alpha_{i\bar{i}} - h^\alpha_{j\bar{j}}) + (h^j_{ii} - h^j_{\bar{i}\bar{i}})$$

Therefore $e_j(h^\alpha_{i\bar{i}}) = 0$ if $h^\alpha_{i\bar{i}} = h^\alpha_{j\bar{j}}$.

Note that the Ricci form of M is closed. So

$$\sum_\beta e_j[(h^\beta_{i\bar{i}})^2] = \sum_\beta [(h^\beta_{i\bar{i}})^2 - (h^\beta_{j\bar{j}})^2](h^j_{ii} - h^j_{\bar{i}\bar{i}}) = 0$$

Therefore $\{\sum(h^\beta_{i\bar{i}} - h^\beta_{j\bar{j}})^2\}(h^j_{ii} + h^j_{\bar{i}\bar{i}}) = 0$.

Therefore if there is some β such that $h^\beta_{i\bar{i}} \neq h^\beta_{j\bar{j}}$, then $h^j_{ii} + h^j_{\bar{i}\bar{i}} = 0$. And for this β, $e_j(h^\beta_{i\bar{i}}) = 0$.

It follows that all $h^\beta_{i\bar{i}}$ are constants.

Then TM is decomposed as the direct sum of spaces such that for any eigenspace of h^α, each component is either a subset of it or their intersection is $\{0\}$.

Lemma 4.15 *Each component in the decomosition of TM is integrable.*

Proof The proof is similar.

Lemma 4.16 *any sum of the components of the decomposition is integrable.*

Proof For h^η, since it is closed,

$$h^\eta_{i\bar{i}}(h^i_{kj} - h^{\bar{i}}_{j\bar{k}}) + h^\eta_{j\bar{j}}(h^{\bar{j}}_{i\bar{k}} - h^j_{ki}) + h^\eta_{k\bar{k}}(h^k_{ij} - h^k_{ji}) = 0$$

Note that the tensor

$$\tilde{\omega}_{ij} = \sum_\xi h^\xi_{ik} h^\xi_{jk}$$

is also closed. We find

$$[\sum_\xi (h^\xi_{i\bar{i}})^2](h^i_{kj} - h^{\bar{i}}_{j\bar{k}}) + [\sum_\xi (h^\xi_{j\bar{j}})^2](h^{\bar{j}}_{i\bar{k}} - h^j_{ki}) + [\sum_\xi (h^\xi_{k\bar{k}})^2](h^k_{ij} - h^k_{ji}) = 0$$

It is easy to check that

$$(h^i_{kj} - h^{\bar{i}}_{j\bar{k}}) + (h^{\bar{j}}_{i\bar{k}} - h^j_{ki}) + (h^k_{ij} - h^k_{ji}) = 0$$

We think of $(h^i_{kj} - h^{\bar{i}}_{j\bar{k}}), (h^{\bar{j}}_{i\bar{k}} - h^j_{ki}), (h^k_{ij} - h^k_{ji})$ as variables in the above equations and by discussing the rank of the coefficient matrics we can show that if there are α, β and γ such that

$$h^\alpha_{i\bar{i}} \neq h^\alpha_{\bar{i}\bar{i}}, \quad h^\beta_{j\bar{j}} \neq h^\beta_{k\bar{k}}, \quad h^\gamma_{\bar{i}} \neq h^\gamma_{\bar{i}\bar{i}}$$

then either each h^α has at most 2 eigenspaces or the variables are all 0. And hence the theorem is true by Proposition 2.2.

For example, let G be a semi-simple compact Lie group, T its maximal torus and H a subgroup of T. If there is a homogeneous metric on G/H so that it is Einstein, and the induced metric on G/T by submersion from G/H to G/T is Kähler, then G/T is either Einstein or a product of Einstein manifolds.

4.5 3-Sasakian Manifolds

For the principal $SO(3)$-bundle $SO(4)$ over S^3, we know there are 2 homogeneous Einstein metrics on the total space. In general, applying the same techniques in Sections 4.2 and 4.3, we can easily prove:

Proposition 4.3 *Let P be aprincipal G-bundle over a 3-dimensional manifold M. If G is non-abelian and there is an irreducible connection on P such that the induced metric is Einstein, then M is of positive constant sectional curvature and P is locally isometric to $SO(4)$ with one of the above mentioned metrics.*

We now generalize this result to the case where the base is 4-dimensional and Einstein.

Let P be a principal G bundle over Einstein manifold M where G is either $SU(2)$ or $SO(3)$. Let $V_\alpha, V_\beta, V_\gamma$ be the vertical Killing basis. We choose a proper horizontal basis such that

$$h^\alpha_{13} = h^\alpha_{14} = h^\alpha_{23} = h^\alpha_{24} = 0$$

Let

$$h^\alpha_{12} = \lambda, \quad h^\alpha_{34} = \mu$$

Let $Ric^M = c_0 g$ when restricted on horizontal distribution.

Then we have

$$\lambda h^\beta_{12} + \mu h^\beta_{34} = 0, \quad \lambda h^\gamma_{12} + \mu h^\gamma_{34} = 0$$

We assume

$$h^\beta_{12} = \mu F, \quad h^\beta_{34} = -\lambda F, \quad h^\gamma_{12} = \mu G, \quad h^\gamma_{34} = -\lambda G$$

Substituting into the Einstein equation (60),

$$c_0 = c + 2\lambda^2 + 2(h^\beta_{12})^2 + 2(h^\beta_{13})^2 + 2(h^\beta_{14})^2 + 2\lambda^2 + 2(h^\gamma_{12})^2 + 2(h^\gamma_{13})^2 + 2(h^\gamma_{14})^2$$

$$c_0 = c + 2\lambda^2 + 2(h^\beta_{12})^2 + 2(h^\beta_{23})^2 + 2(h^\beta_{24})^3 + 2\lambda^2 + 2(h^\gamma_{12})^2 + 2(h^\gamma_{23})^2 + 2(h^\gamma_{24})^2$$

$$c_0 = c + 2\mu^2 + 2(h^\beta_{13})^2 + 2(h^\beta_{23})^2 + 2(h^\beta_{34})^2 + 2\lambda^2 + 2(h^\gamma_{13})^2 + 2(h^\gamma_{23})^2 + 2(h^\gamma_{34})^2$$

$$c_0 = c + 2\mu^2 + 2(h^\beta_{14})^2 + 2(h^\beta_{24})^2 + 2(h^\beta_{34})^2 + 2\lambda^2 + 2(h^\gamma_{14})^2 + 2(h^\gamma_{24})^2 + 2(h^\gamma_{34})^2$$

Therefore

$$(\lambda^2 - \mu^2)(1 - F^2 - G^2) = 0$$

So either $\lambda^2 = \mu^2$ or $F^2 + G^2 = 1$.

If $F^2 + G^2 = 1$, noting that

$$\lambda_\beta = 2h^\alpha_{\beta\gamma} G\mu, \quad \lambda_\gamma = -2h^\alpha_{\beta\gamma} G\lambda$$

so

$$|\nabla\lambda|^2 = 2(h^\alpha_{\beta\gamma})^2 G^2 |h^\alpha|^2$$

Then $|\nabla\lambda|^2 + |\nabla\mu|^2 = 2(h^\alpha_{\beta\gamma})^2|h^\alpha|^2$ which is impossible. Therefore $\lambda^2 = \mu^2$. WLOG we assume $\lambda = \mu$, i.e. $g(h^\alpha, h^\alpha) = \lambda^2 g$.

Since we can interchange α, β, γ,

$$g(h^\beta, h^\beta) = g(h^\gamma, h^\gamma) = \lambda^2 g$$

Note that $\lambda_\beta = 0$ and hence $G = 0$. Similarly $F = 0$. i.e.

$$h^\beta_{12} = h^\beta_{34} = h^\gamma_{12} = h^\gamma_{34} = 0$$

Then we see $g(h^\alpha, h^\beta) = 0$. And similarly $g(h^\gamma, h^\beta) = 0$.

We can then choose proper horizontal basis such that

$$h^\beta_{14} = h^\beta_{23} = h^\gamma_{13} = h^\gamma_{24} = 0$$

$$(h^{\beta}_{13})^2 = (h^{\beta}_{24})^2 = (h^{\gamma}_{14})^2 = (h^{\gamma}_{23})^2 = \lambda^2$$

We assume $h^{\beta}_{13} = h^{\gamma}_{14} = h^{\gamma}_{23} = \lambda, \ h^{\beta}_{24} = -\lambda$

Applying equations (55) and (58) we can derive the following relation:

$$h^1_{24} + h^3_{44} = 0, \quad h^3_{22} + h^4_{12} = 0, \quad h^3_{11} = h^4_{21}, \quad h^4_{33} + h^2_{13} = 0$$

$$h^4_{11} + h^3_{21} = 0, \quad h^4_{22} = h^3_{12}, \quad h^1_{22} + h^3_{42} = 0, \quad h^1_{44} + h^2_{34} = 0$$

$$h^1_{33} = h^2_{43}, \quad h^2_{11} + h^4_{31} = 0, \quad h^2_{44} = h^1_{34}, \quad h^2_{33} + h^1_{43} = 0$$

We can then show that P is a 3-Sasakian manifold by direct calculation. According to [BGM1](Corollary 3.12), P is one of $S^7, RP^7, SU(3)/U(1)$ and the base manifold is S^4 or CP^2.

Theorem 4.4 *Let P be a principal G-bundle over 4-dimensional compact Einstein manifold M with G non-abelian and $dimG = 3$. If there is an irreducible connection on P such that the induced metric on P is Einstein, then P is homogeneous and is one of S^7, RP^7 or $SU(3)/U(1)$.*

Applying the similar technique, we can also show

Proposition 4.4 *Let P be a principal G-bundle over $S^2 \times S^2$ where the spheres are allowed to have different constant curvatures. Then any metric on P induced by an irreducible connection is not Einstein.*

4.6 Principal $SO(3)$-Bundles

Suppose P and M are Einstein manifolds admitting an irreducible connection. If $dimG = 4$ and V_ξ is the Killing vector field corresponding to the center of G. We can locally diagonalize h^{ξ}_{ij} such that

$$h^{\xi}_{13} = h^{\xi}_{14} = h^{\xi}_{23} = h^{\xi}_{24} = 0, \quad h^{\xi}_{12} = \tilde{\lambda}, \quad h^{\xi}_{34} = \tilde{\mu}$$

Lemma 4.17 *$g(h^\xi, h^\xi)$ is proportional to g when restricted on the horizontal distribution.*

Proof By Einstein equations,

$$c_0 = c + 2\tilde{\lambda}^2 + 2\sum[(h^\alpha_{12})^2 + (h^\alpha_{13})^2 + (h^\alpha_{14})^2]$$

$$c_0 = c + 2\tilde{\lambda}^2 + 2\sum[(h^\alpha_{12})^2 + (h^\alpha_{23})^2 + (h^\alpha_{24})^2]$$

$$c_0 = c + 2\tilde{\mu}^2 + 2\sum[(h^\alpha_{13})^2 + (h^\alpha_{23})^2 + (h^\alpha_{34})^2]$$

$$c_0 = c + 2\tilde{\mu}^2 + 2\sum[(h^\alpha_{14})^2 + (h^\alpha_{24})^2 + (h^\alpha_{34})^2]$$

So

$$\tilde{\lambda}^2 + \sum_{\alpha\neq\xi}(h^\alpha_{12})^2 = \tilde{\mu}^2 + \sum_{\alpha\neq\xi}(h^\alpha_{34})^2$$

Note that $\tilde{\lambda}h^\alpha_{12} + \tilde{\mu}h^\alpha_{34} = 0$ for $\alpha \neq \xi$.Let

$$h^\alpha_{12} = \tilde{\mu}F^\alpha, \quad h^\alpha_{34} = -\tilde{\mu}F^\alpha$$

Substituting into the above equation we see that if $\tilde{\lambda}^2 \neq \tilde{\mu}^2$, then

$$\sum_{\alpha\neq\xi}(F^\alpha)^2 = 1, \quad \tilde{\lambda}^2 = \sum_{\alpha\neq\xi}(h^\alpha_{34})^2, \quad \tilde{\mu}^2 = \sum_{\alpha\neq\xi}(h^\alpha_{12})^2$$

We can choose α such that $\tilde{\lambda}^2 = (h^\alpha_{34})^2$ and $h^\beta_{34} = h^\gamma_{34} = 0$. Then $F^\beta = F^\gamma = 0$. So $h^\beta_{12} = h^\gamma_{12} = 0$ and $\tilde{\mu}^2 = (h^\alpha_{12})^2$.

Note that

$$2(h^\alpha_{12})^2 + 2(h^\alpha_{34})^2 + 2(h^\alpha_{13})^2 + 2(h^\alpha_{14})^2 + 2(h^\alpha_{23})^2 + 2(h^\alpha_{24})^2 + 2 = c$$

and

$$2\tilde{\lambda}^2 + 2\tilde{\mu}^2 = c$$

which yields an obvious contradiction.

Therefore $\tilde{\lambda}^2 = \tilde{\mu}^2$ and $g(h^\xi, h^\xi)$ is proportional to the metric on the horizontal distribution by a constant.

By the above lemma, $Ric^M - 2g(h^\xi, h^\xi)$ is diagonal. But

$$Ric^M - 2g(h^\xi, h^\xi) = cg + 2g(h^\alpha, h^\alpha) + 2g(h^\beta, h^\beta) + 2g(h^\gamma, h^\gamma)$$

Similar to the proof in Section 5, there is a proper Killing basis $V_\xi, V_\alpha, V_\beta, V_\gamma$ such that

$$h^\alpha_{13} = h^\alpha_{14} = h^\alpha_{23} = h^\alpha_{24} = h^\beta_{12} = h^\beta_{14} = h^\beta_{23} = h^\gamma_{12} = h^\gamma_{13} = h^\gamma_{24} = 0$$

$$h^\alpha_{12} = \lambda, \quad h^\alpha_{34} = \mu, \quad (h^\beta_{13})^2 = (h^\beta_{24})^2 = (h^\gamma_{14})^2 = (h^\gamma_{23})^2$$

We assume

$$\lambda + \mu = 0, \quad h^\beta_{13} = h^\beta_{24} = h^\gamma_{14} = \lambda, \quad h^\gamma_{23} = -\lambda$$

From the Einstein equations we can find

$$c_0 = 3c - 3, \quad \lambda^2 = \frac{c-2}{4}, \quad \tilde{\lambda}^2 = \frac{c}{4}$$

$$h^\beta_{13} = h^\beta_{24}, \quad h^\gamma_{14} + h^\gamma_{23} = 0, \quad h^2_{3\beta} = h^1_{4\beta}, \quad h^2_{4\gamma} + h^1_{3\gamma} = 0$$

$$h^1_{2\beta} = h^3_{4\beta}, \quad h^1_{2\gamma} + h^3_{4\gamma} = 0, \quad h^2_{3\alpha} = h^1_{4\alpha}, \quad h^1_{3\alpha} + h^2_{4\alpha} = 0$$

$$h^\xi_{12} = h^\xi_{34}, \quad h^\xi_{13} + h^\xi_{24} = 0, \quad h^\xi_{14} = h^\xi_{23}$$

$$\lambda(h^1_{3\beta} + h^2_{4\beta} + 2h^\beta_{13}) + 2h^\gamma_{14} = 0, \quad \lambda(h^1_{4\gamma} - h^2_{3\gamma} + 2h^\gamma_{14}) + 2h^\beta_{24} = 0$$

$$h^\alpha_{12} = -h^\alpha_{34} = \lambda, \quad h^2_{3\xi} = h^1_{4\xi}, \quad h^2_{4\xi} + h^1_{3\xi} = 0, \quad h^1_{2\xi} = h^3_{4\xi}$$

$$h^\beta_{13}(h^1_{2\alpha} - h^3_{4\alpha} + 2\lambda) + 2h^\gamma_{14} = 0$$

$$h^3_{44} = h^1_{24}, \quad h^3_{11} = -h^4_{21}, \quad h^3_{22} = h^4_{12}, \quad h^4_{33} = h^2_{13}$$

$$h^4_{11} = h^3_{21}, \quad h^4_{22} = -h^3_{12}, \quad h^1_{22} = h^3_{42}, \quad h^1_{44} = h^2_{34}$$

$$h^1_{33} = -h^2_{43}, \quad h^2_{11} = h^4_{31}, \quad h^2_{44} = -h^1_{34}, \quad h^2_{33} = h^1_{43}$$

We can find the symmetric properties from the above equalities. Calculating the curvature tensors we can also get

$$K^M_{1212} - K^M_{1234} = 4\lambda^2 + 2\lambda(h^1_{2\alpha} - h^3_{4\alpha})$$

$$K^M_{1313} - K^M_{1342} = 4\lambda^2 + 2h^\beta_{13}(h^2_{4\beta} - h^3_{1\beta})$$

$$K^M_{1414} - K^M_{1423} = 4\lambda^2 + 2h^\gamma_{14}(h^3_{2\gamma} - h^4_{1\gamma})$$

We can then find that

$$c_0 = -12\lambda, \quad \lambda = -1/2, \quad c = 3$$

$$K^M_{1314} = K^M_{1323}, \quad K^M_{2412} = K^M_{2434}$$

$$K^M_{2323} = K^M_{1414}, \quad K^M_{4242} = K^M_{1313}$$

$$K^M_{1212} = K^M_{3434}, \quad K^M_{1313} - K^M_{1342} = 2, \quad K^M_{1414} - K^M_{1423} = 2$$

$$K^M_{1213} + K^M_{1224} = 0, \quad K^M_{1212} - K^M_{1234} = 2, \quad K^M_{1214} = K^M_{1223}$$

etc. In particular,

$$Ric^M = 6g_M$$

Therefore M is self-dual, and hence is either CP^2 or S^4.

Proposition 4.5 *Let P be a principal G-bundle over 4-dimensional compact Einstein maniold M. G is non-abelian and is 4-dimensional. If there is an irreducible connection on P so that the induced metric is Einstein, then M is CP^2.*

Proof We need only to prove that M cannot be S^4. In fact, if $M = S^4$, note that h^ξ is a harmonic 2-form on M. Calculating $\Delta(\frac{1}{2}|\widehat{h^\xi}|^2)$ on M, we find it is positive unless h^ξ vanishes everywhere. But h^ξ is nowhere vanishing by (59). We conclude that M cannot be S^4.

Note that we can think of h^ξ as a harmonic 2-form on M and $H(M;\mathbf{R}) = \mathbf{R}$. We find:

Proposition 4.6 *h^ξ is proportional to horizontal lifting of Kähler form from M by a constant.*

We now consider M', the quotient space of P by the 1-dimensional foliation generated by V_ξ. M' is obviously a principal bundle over M. We change the metric on the horizontal distribution by defining a horizontal orthonormal basis by

$$e_i' = \sqrt{2}e_i, \; i = 1, 2, 3, 4.$$

Then under the new metric, we find that M' is Einstein by means of submersion equations. According to the conclusion in Section 4.5 and the fact that the base is CP^2, we see M' is $SU(3)/U(1)$ and is a principal $SO(3)$-bundle over CP^2. Therefore P is $SU(3)$.

Theorem 4.5 *Let P be a principal G-bundle over 4-dimensional compact Einstein maniold M. G is non-abelian and is 4-dimensional. If there is an irreducible connection on P so that the induced metric is Einstein, then P is $SU(3)$.*

4.7 New Constructions

We know that both Wang and Ziller's examples of Einstein metrics on principal circle bundles[WZ] and those of inhomogeneous 3-Sasakian Einstein manifolds[BGM2] admit circle action with no fixed point. In this section, inspired by the proof of Theorem 4.5, we will suggest new constructions bearing this property. To start with, we define a class of principal bundles.

Definition 4.1 *We call P almost strongly Einstein if P satisfies the following conditions:*

- *P is a principal G-bundle over M.*
- *M is a compact Kähler-Einstein manifold with positive first Chern class and its Kähler form is ω_M.*
- *G is a compact Lie group equipped with a bi-invariant metric and is Einstein.*
- *There is a Yang-Mills connection on P so that we induce a metric on P.*
- *The following conditions hold with respect to this metric:*

$$g(h^\alpha, h^\alpha) = a^2 g$$

$$trace\, g(h^\alpha, h^\beta) = 0, \quad if\ \alpha \neq \beta$$

$$trace\, g(\omega_M, h^\alpha) = 0$$

when restricted on horizontal distribution where a is some positive constant. Note that $g(h^\alpha, h^\beta)$ is a 2-tensor defined as

$$g(h^\alpha, h^\beta)(X, Y) = trace\{h^\alpha(X)h^\beta(Y)\}$$

We write

$$Ric^G_{\alpha\beta} = c^G \delta_{\alpha\beta}, \quad Ric^M_{ij} = c^M \delta_{ij}$$

$$\dim G = m, \quad \dim_C M = n$$

Then similar to Proposition 4.1,

$$m \leq \frac{n(n-1)}{2}$$

It is not clear how large this class of principal bundles is at this stage. It might be helpful to explore this class because of its strong conditions and because the Einstein metrics on principal bundles are largely unknown.

By Kobayashi's construction[Ko] we can construct principal S^1-bundles over M with connections so that its curvature form is proportional to the first Chern class of M by a constant. We can lift it to be a principal S^1-bundle over P by projection and similarly define a metric on the new bundle. We have

Theorem 4.6 *If P is almost strongly Einstein and*

$$(c^M)^2 \geq \frac{8(n+1)(n+m+1)}{n} a^2 c^G$$

then by homotheties of G and M if necessary, there is an Einstein metric on every principal S^1-bundle constructed over P as above.

Proof We assume the curvature form of the principal S^1-bundle over M is $2b\omega_M$ where ω_M is the Kähler form of M.

We define the new metric on the total space of the S^1-bundle over P by changing the metrics on M and G by constant factors x^2 and y^2 respectively. Then by (56),(57), the metric is Einstein if

$$\frac{c^M}{x^2} - \frac{2ma^2y^2}{x^4} = 2(n+1)\frac{b^2}{x^4}$$

$$\frac{c^G}{y^2} + \frac{2na^2y^2}{x^4} = 2n\frac{b^2}{x^4}$$

The question is reduced to finding positive solution of

$$[a^2 + \frac{ma^2}{(n+1)}](\frac{x}{y})^4 - \frac{c^M}{2(n+1)}(\frac{x}{y})^2 + \frac{c^G}{2n} = 0$$

and it is easy to see the condition given in the theorem is sufficient.

We can check that the principal bundles with non-abelian structure groups in Theorem 4.5 satisfy the exact equality in the theorem.

If $G = S^1$ and $M = N_1 \times N_2$ where N_1 and N_2 are Kähler-Einstein manifolds, we can choose an almost strongly Einstein P to be a principal circle bundle whose Euler class is proportional to $\omega_{N_1} - \omega_{N_2}$ by a constant. Following the construction as above, we can find an Einstein metric on a principal circle bundle over P, and the total space is a principal 2-torus bundle over M. This is a special case of Wang and Ziller's construction of Einstein metrics on principal torus bundles[WZ].

Corollary 4.1 *Given compact Fano Kähler-Einstein manifolds N_1 and N_2, there are Einstein metrics on certain non-trivial principal 2-torus bundles over $N_1 \times N_2$.*

We now further generalize the construction. Suppose N is a compact Kähler-Einstein manifold with positive first Chern class and P is a principal G-bundle as defined before Theorem 4.6. We can construct a principal S^1-bundle over $M \times N$ by means of their first Chern classes and lift it to be a principal S^1-bundle over $P \times N$. We can also construct a Riemannian metric on the total space.

Theorem 4.7 *Let P be almost strongly Einstein. Then there is an Einstein metric on any principal S^1-bundle over $P \times N$ constructed as above.*

Proof Assume the curvature form of the connection on the bundle is $2\lambda_1\omega_M + 2\lambda_2\omega_N$ where ω_M and ω_N are Kähler forms. We also use the following notations:

$$dimM = n_1, \quad dimN = n_2, \quad dimG = m$$

$$g(h^a, h^a) = a^2 g$$

when restricted on the horizontal distribution of P.

c^M, c^N, c^G are Einstein constants.

We change the metric on the total space as follows:

$$\tilde{g}^M = x^2 g^M, \quad \tilde{g}^N = y^2 g^N, \quad \tilde{g}^G = z^2 g^G$$

Then the Einstein condition is

$$\frac{2n_1\lambda_1^2}{x^4} + \frac{2n_2\lambda_2^2}{y^4} = \frac{c^G}{z^2} + \frac{2n_1a^2}{x^4}z^2 = \frac{c^M}{x^2} - \frac{2ma^2}{x^4}z^2 - \frac{2\lambda_1^2}{x^4} = \frac{c^N}{y^2} - \frac{2\lambda_2^2}{y^4}$$

From the above equations, we find

$$\frac{1}{x^4} = \frac{c^N}{2n_1\lambda_1^2y^2} - \frac{(n_2+1)\lambda_2^2}{n_1\lambda_1^2y^4}$$

$$z^2 = \frac{c^M x^2}{2ma^2} - \frac{\lambda_1^2}{ma^2} - \frac{c^N x^4}{2ma^2y^2} + \frac{\lambda_2^2x^4}{ma^2y^4}$$

and we need only to find a solution of the equation

$$\frac{c^G}{z^2} + \frac{2n_1a^2z^2}{x^4} - \frac{c^N}{y^2} + \frac{2\lambda_2^2}{y^4} = 0$$

Let $w = \frac{1}{y^2}$ such that the solution w lies between 0 and $\frac{c^N}{2(n_2+1)\lambda_2^2}$. Let the left hand side term of the equation be $F(w)$, then

$$\lim_{w\to 0} \frac{F(w)}{\sqrt{w}} > 0$$

$$\lim_{w\to \frac{c^N}{2(n_2+1)\lambda_2^2}} F(w) < 0$$

Therefore there is a solution satisfying the condition.

The new constructions can be generalized in many ways. For example, we can replace the condition in Definition 4.1 that M is Kählerian by requiring M admit a principal circle bundle carrying a Yang-Mills connection over it satisfying following conditions and there are examples of homogeneous 3-Sasakian manifolds that fit well[BGM2]:

- Let ω be the curvature form on the circle bundle, then $g(\omega,\omega)$ is proportional to g when restricted on the horizontal distribution with respect to the induced metric on the bundle.

- When the circle bundle is lifted over P,

$$trace\, g(h^a,\omega) = 0$$

It is not our intention to give all the possible constructions implied by the above theorems in the thesis. Instead, we illustrate our claim by the following two theorems. we will try to decrease the lower bound of c^M in Theorem 4.6 by adding more copies as follows:

Theorem 4.8 *Let P be almost strongly Einstein. We construct a principal circle bundle over Einstein manifold $M \times M \times \ldots \times M$ (p copies) so that its Euler class is proportional to the Kähler form of the base by a constant, and lift it to be a circle bundle over $P \times P \times \ldots \times P$ (p copies) through the corresponding projection. If*

$$(c^M)^2 \geq \frac{8(np+1)(np+mp+1)}{np^2} a^2 c^G$$

then there is an Einstein metric on the total space of the principal circle bundle over $P \times P \times \ldots \times P$ (p copies).

Proof We define a new metric on the total space by changing the metric on the base and G by constant factors x^2 and y^2 respectively. We assume the curvature form of the principal circle bundle over $M \times M \times \ldots \times M$ is 2λ times the Kähler form of the base. Then the Einstein condition is:

$$\frac{2np\lambda^2}{x^4} = 2na^2\frac{y^2}{x^4} + \frac{c^G}{y^2} = \frac{c^M}{x^2} - 2ma^2\frac{y^2}{x^4} - \frac{2\lambda^2}{x^4}$$

So

$$\frac{2\lambda^2}{x^4} = 2a^2\frac{y^2}{px^4} + \frac{c^G}{npy^2}$$

$$\frac{2\lambda^2}{x^4} = \frac{c^M}{(np+1)x^2} - 2ma^2\frac{y^2}{(np+1)x^4}$$

The question is then reduced to finding positive solution of

$$2a^2(\frac{1}{p} + \frac{m}{np+1})(\frac{y}{x})^4 - \frac{c^M}{np+1}(\frac{y}{x})^2 + \frac{c^G}{np} = 0$$

and we can see that the condition in the theorem is sufficient.

More generally, it is easy to find by calculating the limit of the lower bound as $p \longrightarrow \infty$ in the above theorem:

Corollary 4.2 *If P is almost strongly Einstein and*

$$(c^M)^2 > 8(n+m)a^2c^G$$

then there is some p_0 such that for any $p \geq p_0$, there are Einstein metrics on certain non-trivial circle bundles over $P \times P \times \ldots \times P$ (p copies).

We remark that P admits an Einstein metric if and only if $(c^M)^2 \geq 8(n+m)a^2c^G$. So if P admits no Einstein metrics, there is no Einstein metrics on the principal circle bundles over $P \times P \times \ldots \times P$ constructed as above.

Let P_1 and P_2 be almost strongly Einstein principal bundles over M_1 and M_2 respectively. The structure group of P_1 and P_2 are G and S^1 respectively with $dimG = m$. We can lift a principal circle bundle over $M_1 \times M_2$ to be a principal circle bundle over $P_1 \times P_2$.

Theorem 4.9 *Let P_1 and P_2 be almost strongly Einstein. Then there is an Einstein metric on the principal bundle over $P_1 \times P_2$.*

Proof Let $2\lambda_1\omega_{M_1}+2\lambda_2\omega_{M_2}$ be the curvature form of the principal circle bundle over $M_1\times M_2$. Let $a_1, n_i, i=1,2$ be the data correspoinding to a, n given in the definition of P for P_1, P_2. We change the metrics on M_1, M_2, G, S^1(structure group of P_2) by constant factors $x_1^2, x_2^2, y_1^2, y_2^2$. Then the Einstein condition is

$$\frac{2n_1\lambda_1^2}{x_1^4}+\frac{2n_2\lambda_2^2}{x_2^4}=2n_1a_1^2\frac{y_1^2}{x_1^4}+\frac{c^G}{y_1^2}$$

$$=2n_2a_2^2\frac{y_2^2}{x_2^4}=\frac{c^{M_1}}{x_1^2}-2ma_1^2\frac{y_1^2}{x_1^4}-\frac{2\lambda_1^2}{x_1^4}=\frac{c^{M_2}}{x_2^2}-2a_2^2\frac{y_2^2}{x_2^4}-\frac{2\lambda_2^2}{x_2^4}$$

We can find

$$y_2^2=\frac{c^{M_2}}{2(n_2+1)a_2^2}x_2^2-\frac{\lambda_2^2}{(n_2+1)a_2^2}$$

So

$$x_2^2>\frac{2\lambda_2^2}{c^{M_2}}$$

and the Einstein condition is reduced to

$$\frac{2n_1\lambda_1^2}{x_1^4}+\frac{2n_2\lambda_2^2}{x_2^4}=2n_1a_1^2\frac{y_1^2}{x_1^4}+\frac{c^G}{y_1^2}=\frac{c^{M_1}}{x_1^2}-2ma_1^2\frac{y_1^2}{x_1^4}-\frac{2\lambda_1^2}{x_1^4}$$

$$=\frac{n_2c^{M_2}}{(n_2+1)x_2^2}-\frac{2(n_2+2)\lambda_2^2}{(n_2+1)x_2^4}$$

Similar to the proof of Theorem 4.7, we can show the theorem is true.

So far, we have given constructions of Einstein metrics on principal circle bundles so that the bases can be non-Kählerian. We end the thesis by giving a variant of Wang and Ziller's construction so that the total spaces may not be total spaces of some principal circle bundles:

Let $N_1, N_2, \ldots, N_l$ be compact Fano Kähler-Einstein manifolds. Let P_1 be a compact 3-Sasakian manifold. Then there is a circle action on P_1 induced by a Sasakian structure. Suppose P is a principal circle bundle over $N_1\times N_2\times \ldots N_l$ such that its Euler class is determined by the first Chern classes of $N_1, N_2, \ldots N_l$. Then there is an Einstein metric on $P\times_{S^1} P_1$. The proof is similar to that given by Wang and Ziller[WZ] since the quotient of P_1 by the circle action is a Kähler-Einstein orbifold. Note that there are many inhomogeneous 3-Sasakian manifolds[BGM2]. This construction produces examples of Einstein manifolds that are neither Sasakian nor total spaces of circle bundles

Theorem 4.10 *Let P be a principal circle bundle over $N_1 \times N_2 \times \ldots \times N_l$ so that its Euler class is a linear combination of the first Chern classes of $N_i, i = 1, 2, \ldots, l$. Let P_1 be a compact 3-Sasakian manifold. Then there is an Einstein metric on $P \times_{S^1} P_1$.*

Bibliography

[A] T.Aubin, Equations du type de Monge-Ampère sur les variétés kähleriennes compactes, C.R.Acad.Sci.Paris 283, 119-121(1976).

[BB] L.Bérard Bergery, Sur de nouvelles variétés riemanniennes d'Einstein. Publications de l'Institut E.Cartan n°4(Nancy),1-60(1982).

[Be] A.Besse, Einstein Manifolds, Erg. der Math. und Ihrer Grenzebiete,3. Folge, Band 10, Springer-Verlag, (1987).

[BH] R.A.Blumenthal & J.J.Hebda, De Rham decomposition theorems for foliated manifolds, Ann.Inst.Fourier, Grenoble 33,2(1983), 183-198.

[BGM1] C.P.Boyer, K.Galicki & B.M.Mann, Quaternionic Reduction and Einstein Manifolds. Comm. in Analysis & Geom. 1(2),(1993) 229-280.

[BGM2] C.P.Boyer, K.Galicki & B.M.Mann, The geometry and topology of 3-Sasakian manifolds,preprint,.(1993).

[C] E.Calabi, On Kähler manifolds with vanishing canonical class, in Algebraic geometry and topology, a symposium in honor of S.Lefschetz, Princeton Univ. Press, 79-89(1955).

[CL] C.Camcho and A.Lins Neto, Geometric Theory of Foliations, Progress in Math. Birkhäuser(1985), Boston,Basel,Stuttgart.

[D] D.DeTurck, Existence of Metrics with prescribed Ricci curvature: Local theory, Invent. Math. 65, 179-207(1981).

[DK] D.DeTurck &J.L.Kazdan, Some regularity theorems in Riemannian geometry, Ann.Sci.École Norm.Sup.(4) 14(1981), no 3, 249-260.

[EW] J.H.Eschenburg & M.Y.Wang, The initial value problem for cohomogeneity one Einstein metrics, preprint, October, (1995).

[F1] A.Futaki, An obstruction to the existence of Einstein Kähler metrics, Invent. Math. 73, 437-443(1983).

[F2] A.Futaki, On compact Kähler manifolds of constant scalar curvature, Proc.Japan Acad.Sci. 59, 401-40291983).

[H] N.Hitchin, Kählerian twistor spaces, Proc. London Math. Soc.(3) 43(1981), no.1, 133-150.

[GPP] G.Gibbons, D.Page, C.Pope, Einstein metrics on $S3$, R^3 and R^4 bundles, Comm.Math.Phys. 127, 529-553(1990).

[Je] G.Jensen, Einstein metrics on principal fibre bundles, J.Diff.Geom. 8(1973), 599-614.

[J] P.E.Jupp, Classification of certain 6-manifolds. Proc. Camb. Phil. Soc.(1973).73, 293-300.

[Ka] T.Kashiwada, A note on a Riemannian space with Sasakian 3-structure, Nat.Sci.Reps. Ochanomizu Univ. 22(1971),1-2.

[KW] J.L.Kazdan & F.W.Warner, Curvature functions for open 2-manifolds, Ann.of Math. 99(1974) 203-219.

[Ko] S.Kobayashi, Topology of Positively Pinched Kähler Manifolds, Tôhoku Math.J. 15(1963) 121-139

[KO] S.Kobayashi & T.Ochiai, Characterizations of complex projective spaces and hyperquadrics, J.Math.Kyoto Univ. 13(1973),31-47.

[KS1] N.Koiso &Y.Sakane, Nonhomogeneous Kähler-Einstein metrics on compact complex manifolds. Curvature and topology of Riemannian manifolds(Katata),(1985), 165-179 Lecture Notes in Math.1201, Springer, Berlin-New York, (1986).

[KS2] N.Koiso &Y.Sakane, Non-homogeneous Kähler-Einstein metrics on compact complex manifolds(II), Osaka J. Math.25(1988), 933-959.

[N] A.Nadel, Multiplier ideal sheaves and existence of Kähler-Einstein metrics of positive scalar curvature. Proc.Natl.Acad. Sci. USA, V.86, No.19(1989).

[P] D.Page, A compact rotating gravitational instanton,Phys.Lett.79B, 235-238(1979).

[PP] D.Page & C.Pope, Einstein metrics on quaternionic line bundles, Classical Quantum Gravity 3, 249-259(1986).

[PP1] D.Page & C.Pope, Inhomogeneous Einstein metrics on complex line bundles, Classical Quantum Gravity 4(1987) 213-225.

[PZ] P.Petersen & S. Zhu, U(2) invariant four dimensoonal Einstein metrics, preprint

[S] Y.Sakane, Examples of compact Kähler-Einstein manifolds with positive Ricci curvatures, Osaka J. Math.23(1986) 585-617.

[S2] Y. Sakane, Homogeneous Einstein metrics on a principal circle bundles. Complex geometry (Osaka, 1990) 161-178, Lecture Notes in Pure and Appl. Math., 143, Dekker.

[SS] B.Shiffman & A.J.Sommese, Vanishing theorems on complex manifolds, Progress in Math. 56, Birkhäuser, (1985).

[T] G.Tian, On Kähler-Einstein metrics on certain Kähler manifolds with $c_1(M) > 0$, Invent.Math. 89(1987), 225-246.

[T1] G.Tian, On Calabi's conjecture for complex surfaces with positive first Chern class, Inv.Math. Vol.101, No.1(1990),101-172.

[TY] G.Tian & S.T.Yau, Kähler-Einstein metrics on complex surfaces with $c_1(M)$ positive, Comm. Math. Phy. 112(1987).

[W] J.Wang, Einstein metrics on principal circle bundles, submitted to Diff. Geom. and its Appli. in August,(1995).

[Wa1] M.Y. Wang, Some examples of homogeneous Einstein manifolds in dimension seven, Duke Math. J. 49(1982), 23-28.

[Wa2] M.Y. Wang, Einstein metrics and quaternionic Kähler manifolds, Math.Z. 210, 305-326(1992).

[WZ1] M.Y.Wang and W.Ziller, Existence and non-existence of homogeneous Einstein metrics, Invent.Math. 84, 177-194(1986).

[WZ] M.Y.Wang and W.Ziller, Einstein Metrics on Principal Torus Bundles, J.Differ. Geom. 31, 215-248(1990).

[Wo] J.Wolf, The geometry and structure of isotropy irreducible homogeneous spaces, Acta Math.120,59-148(1968) and Erratum. Acta Math.152, 141-142(1984).

[Y1] S.T.Yau, Calabi's conjecture and some new results in algebraic geometry. Proc. Natl. Acad. Sci. USA, 74(1977), No.5, 1798-1799.

[Y2] S.T.Yau, On the Ricci cuevature of a compact Kähler manifold and the complex Monge-Ampère equation I, Comm. Pure.Appl. Math. 31(1978) 339-411.