ALMOST SELFCOMPLEMENTARY GRAPHS AND EXTENSIONS

By

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ABSTRACT

In this thesis the concept of selfcomplementary graphs is extended to almost selfcomplementary graphs. We define a p–vertex graph to be almost selfcomplementary if it is isomorphic to its complement with respect to K_p–e, the complete graph with one edge deleted. An almost selfcomplementary graph with p vertices exists if and only if $p \equiv 2$ or 3 (mod 4), i.e., precisely when selfcomplementary graphs do not exist. We investigate various properties of almost selfcomplementary graphs and examine the similarities and differences with those of selfcomplementary graphs.

The concepts of selfcomplementary and almost selfcomplementary graphs are combined to define so–called k–selfcomplementary graphs which include the former two classes as subclasses. Although a k–selfcomplementary graph may contain fewer edges than a selfcomplementary or an almost selfcomplementary graph it is found that the former preserves most of the properties of the latter graphs.

The notion of selfcomplementarity is further extended to combinatorial designs. In particular, we examine whether a Steiner triple system (twofold triple system, and a Steiner system S(2,4,v), respectively) can be partitioned into two isomorphic hypergraphs.

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CHAPTER 1
INTRODUCTION

A simple graph is called *selfcomplementary* (s.c.) if it is isomorphic with its complement. These graphs were first studied independently by G. Ringel [44] and H. Sachs [48] in 1962. In 1963, R. C. Read [43] enumerated the number of selfcomplementary graphs and digraphs. Since 1973, more than fifty papers have already appeared on these graphs dealing with the construction and the study of their various properties [1, 8–13, 30, 34, 36–41, 44, 48, 49]. The amount of work already done is a clear indication of the importance of this class of graphs.

There are apparently several reasons for studying this important class of graphs. First of all the number of selfcomplementary graphs is asymptotically much smaller than the number of all graphs with a given number of vertices. For instance, there are 720 s.c. graphs with 12 vertices whereas the number of all graphs with 12 vertices is over 165 billion [26] (A complete list of s.c. graphs with 12 vertices is given in [30]).

Furthermore, this class proved important from the view point of theoretical computational complexity. Colbourn and Colbourn [14] have proved that the graph isomorphism problem is polynomially equivalent to the isomorphism problem for s.c. graphs. The study of graphical invariants of a graph and its complement may be initiated from the study of similar invariants for s.c. graphs [27]. This list could be extended further.

The class of s.c. graphs, though interesting and useful, suffers from the drawback that a s.c. graph with p vertices can exist only if p is congruent to 0 or 1 (mod 4). It is clear from the definition that the corresponding complete graph must have an even number of edges, and thus a s.c. graph with p vertices does not exist when p is congruent to 2 or 3 (mod 4). This provided us with the motivation to remove this "trivial obstacle" by deleting some suitable odd number of edges (in our case one edge) from the complete graph, and then look for a partition of it into two isomorphic spanning subgraphs. We call this selfcomplementary–like graph an *almost selfcomplementary* (a.s.c.) *graph.*

There seem to be no known instances of this concept appearing in the literature except for a quite recent paper by Zelinka [55] where he briefly mentions a similar class of graphs and uses them in a different context. As for this thesis, we study the class of a.s.c. graphs along the lines of s.c. graphs and examine the similarities and differences between the two classes. Subsequently, the concept is further generalised to so called *k–selfcomplementary* (k–s.c.) *graphs* to include both of these classes, as well as further graphs of selfcomplementary nature. The edge set of the union of such a graph and its restricted complement possibly "misses" more than one edge from the corresponding complete graph. This is followed by a brief study of the idea of selfcomplementarity for combinatorial designs, in particular, for Steiner triple systems, twofold triple systems and Steiner systems S(2,4,v). For the terms not defined here, the reader is refered to see [3, 24, 25].

A more detailed outline of the present work is as follows:

In Chapter 2, we define an almost selfcomplementary (a.s.c.) graph to be one containing one edge less than its complement but which is isomorphic with the graph obtained after deletion of a suitable edge from its complement. Thus an a.s.c. graph is exactly a part of the partition of a graph, obtained after deleting an edge from the corresponding complete graph, into two isomorphic spanning subgraphs. We study various properties of the isomorphism involved, called *complementing permutation* (c.p.), and in the process, we provide a construction method for an a.s.c. graph with a given c.p.. Since there exist no regular a.s.c. graphs, we discuss two subclasses of a.s.c. graphs which are "nearly regular". This chapter also includes a decomposition of a.s.c. graphs.

Chapter 3 contains results on the existence of a hamiltonian path and of cycles of different lengths in an a.s.c. graph. We also prove that a hamiltonian a.s.c. graph is necessarily pancyclic.

In Chapter 4, we provide a construction of an a.s.c. graph with a given graphical degree sequence satisfying some prescribed necessary conditions. A lower bound on the number of triangles in an a.s.c. graph is obtained in terms of its number of vertices. It is also proved here that the diameter of a connected a.s.c. graph is 2 or 3.

The concept of selfcomplementary graphs is extended further in Chapter 5 to what we call k–selfcomplementary (k–s.c.) graphs. These include s.c. and a.s.c., but also other graphs. A k–s.c. graph is a graph isomorphic to its complement with respect to a complete graph from which a matching with k edges has been deleted. Thus k cannot exceed half the number of vertices. A construction of a k–s.c. graph for any k satisfying this obvious constraint is given, and several results on paths and cycles in such graphs are obtained.

Finally, in Chapter 6, we have attempted another extension of the notion of selfcomplementarity to combinatorial designs. Observing that a Steiner triple system with v elements (STS(v)) may be viewed as a partitioning of the edges of a complete graph with v vertices into triangles, partition of an STS into two isomorphic parts is essentially the same as finding a s.c. graph or a s.c.–like graph (according as the number of triples in STS(v) is even or odd respectively) with v vertices whose edge–set consists of a collection of edge–disjoint triangles. We call such a s.c. graph or s.c.–like graph a *Steiner selfcomplementary* (s.s.c.) *graph* or *almost Steiner selfcomplementary* (a.s.s.c.) *graph* respectively. We obtain a necessary and sufficient condition for the existence of a s.s.c. graph and an a.s.s.c. graph separately for which the complementing permutation is an automorphism of the corresponding Steiner triple system. Similar questions are also examined for twofold triple systems and Steiner designs S(2,4,v), with only partial answers.

CHAPTER 2

ALMOST SELFCOMPLEMENTARY GRAPHS

2.1. Introduction

A simple graph G is *selfcomplementary* (s.c.) if it is isomorphic with its complement $\overline{G}$ (cf., e.g., [44, 48]). For a s.c. graph with p vertices to exist, the number of edges in the complete graph K_p must be even, and thus any s.c. graph G with p vertices necessarily has p congruent to 0 or 1 (mod 4). Moreover, this simple necessary condition is also sufficient. A substantial amount of work has been done towards constructing and studying these graphs. [1, 8–13, 30, 34, 36–41, 44, 48].

The concept of s.c. graphs with p vertices can also be interpreted as partitioning the edge set of a complete graph K_p into two isomorphic halves, i.e., each half is isomorphic to the other, which is also its complement. So such a partitioning of the edge set of a K_p is not possible when K_p has an odd number of edges, i.e., when p is congruent to 2 or 3 (mod 4). However, after deleting some suitable odd number of edges from K_p the remaining graph may be partitioned into two isomorphic halves. In this chapter we discuss the simplest of these possibilities, i.e., we delete one edge from K_p where p is congruent to 2 or 3 (mod 4) and define a s.c. like graph. We always denote by e the edge deleted from K_p before the proposed partition and call it the *missing edge.*

Definition. A simple graph G with p vertices is *almost selfcomplementary* (a.s.c.) if it is isomorphic with its complement $\bar{G}$ with respect to the graph $\tilde{K}_p = K_p - e$, the complete graph from which one edge e has been deleted.

This definition fixes an edge e corresponding to a given a.s.c. graph G such that $G \cup \bar{G} = \tilde{K}_p = K_p - e$. Examples of a.s.c. graphs with 6 vertices are given in Fig. 2.1. We have the following simple result regarding existence of a.s.c. graphs.

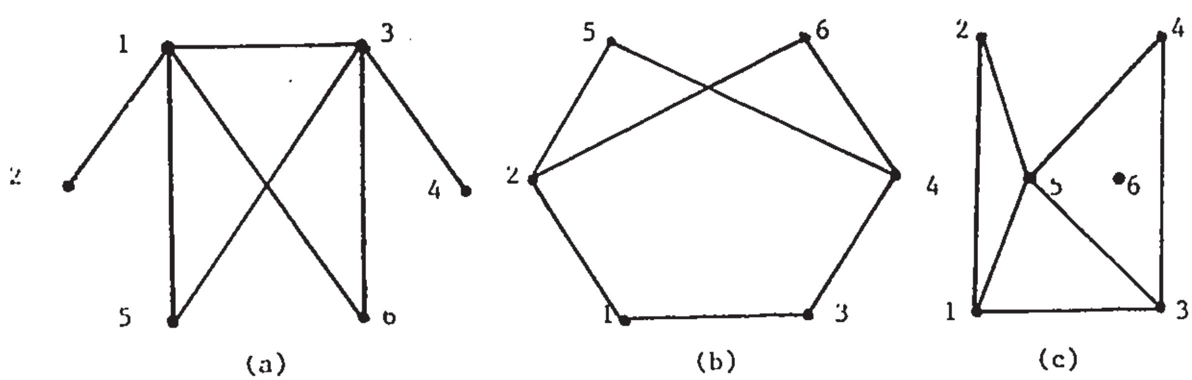
Theorem 2.1.1. An almost selfcomplementary graph with p vertices exists if and only if $p \equiv 2$ or $3 \pmod 4$.

Proof. Necessity is obvious. For sufficiency, first suppose $p \equiv 2 \pmod 4$. Take a s.c. graph G' with $p-1$ (odd) vertices. By the properties of s.c. graphs with odd numbers of vertices [48], such a G' always exists and there is a permutation of $V(G')$ taking G' to $\overline{G}'$ which fixes exactly one vertex, say v of G'. Now taking a vertex, say x, not in $V(G')$ and joining x to all those vertices of G' already joined to v, the resulting graph with vertex set $V(G') \cup \{x\}$ is an a.s.c. graph with p vertices where (x,v) is the missing edge. Next for $p \equiv 3 \pmod 4$, take a s.c. graph G_1 with $p-2$ vertices which again has a vertex, say u, fixed by a permutation of $V(G_1)$ taking G_1 to $\overline{G}_1$. Then take two vertices, say x and y, not in $V(G_1)$. Join both x and y to those vertices of G_1 which are already joined to u and also join u to one of x and y. The graph obtained on the vertex set $V(G_1) \cup \{x,y\}$ is an a.s.c. graph with p vertices, where (x,y) is the missing edge.

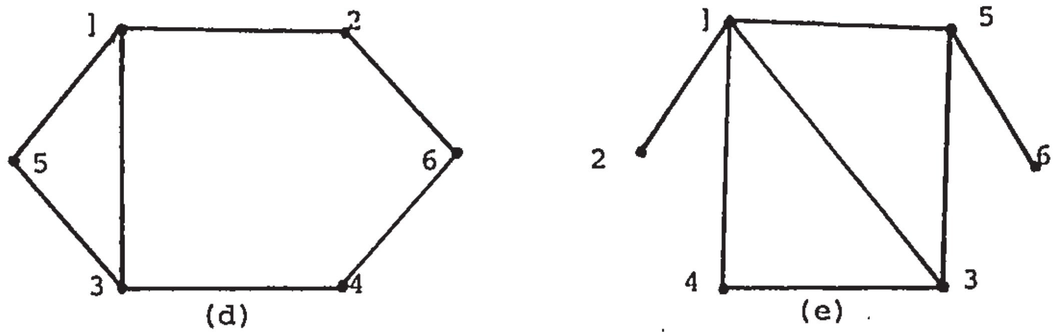
It may be remarked that not all a.s.c. graphs with given number of vertices can be constructed by the method discussed in the above proof.

2.2. Complementing permutation and its cycle structure.

Suppose G is an a.s.c. graph with p vertices and the missing edge e. Then, just like for s.c. graphs, an isomorphism between G and $\tilde{G}$ is given by a permutation $\tau : V(G) \longrightarrow V(G)$, called a *complementing permutation* (c.p.) of G. However (unlike in the case of s.c. graphs), there are two kinds of c.p.'s in the case of a.s.c. graphs, depending on whether or not the missing edge e is fixed by the c.p.. If $\tau(e) = e$ then τ is called a *strong* c.p., otherwise τ is a *weak* c.p.. Although these two kinds of isomorphisms are independent of the number of admissible vertices, the two have different cycle structures. In either case we assume that τ, being a permutation, can be written as a product of disjoint cycles. Note that a given a.s.c. graph G may admit more than one c.p., while, on the other hand, nonisomorphic a.s.c. graphs may have the same c.p. (this is precisely what happens to s.c. graphs as well). Fig. 2.1 below shows all a.s.c. graphs with 6 vertices together with their (strong or weak) c.p.'s.



(a), (b) and (c) are all with strong c.p. (1234)(56) or (1432)(56) and same missing edge (5,6)



(d) with strong c.p. (1234)(5)(6) or (1432)(5)(6) and missing edge (5,6).

(e) with weak c.p. (123456) or (163452) and missing edge (3,6) or (2,5) respectively.

Figure 2.1

The following elementary observations regarding the cycle structure of a complementing permutation τ of an a.s.c. graph G are parallel to those for s.c. graphs, and so are stated without proofs.

<u>Lemma 2.2.1</u>. τ has no cycle of odd length > 3. Also, if τ has a cycle of length 3 then τ is necessarily a weak c.p. and has no other cycle of odd length.

(A c.p. τ of a s.c. graph has no odd cycle of length > 1 [48].)

This is due to the fact that there exists an a.s.c. graph G with 3 vertices and a weak c.p. consisting of a single cycle, where E(G) contains exactly one edge. Moreover, this (disconnected) a.s.c. graph can always be taken as an induced a.s.c. subgraph of at least one a.s.c. graph with 4n+3 vertices such that deletion of these three vertices results in a s.c. graph.

<u>Lemma 2.2.2</u>. τ fixes at most two vertices of G. If τ fixes two vertices u,v then e = (u,v) is the missing edge, and τ is a strong c.p. of even degree.

(A c.p. of a s.c. graph fixes at most one vertex [48].)

<u>Lemma 2.2.3</u>. τ has at most one cycle of length $\ell > 1$ such that $\ell \equiv 2 \pmod 4$. If τ has a cycle of length $\ell \equiv 2 \pmod 4 > 2$ then τ fixes at most one vertex of the corresponding a.s.c. graph, and τ in this case is a weak c.p..

(For a s.c. graph, we have $\ell \equiv 0 \pmod 4$ for every cycle of length $\ell > 1$ of a c.p. [48].)

<u>Lemma 2.2.4</u>. The order of an a.s.c. graph with a strong c.p. containing two cycles of length 1 is always even.

<u>Remark</u>. An a.s.c. graph with more than 3 vertices is disconnected if and only if it has exactly two components of which one is pancyclic and the other is an isolated vertex. Also the associated c.p. is a strong c.p. containing a unique cycle of length 2. Fig. 1(c) is an example of such a graph. (A proof of this remark is given on page 15.)

The a.s.c. graphs with two or three vertices are always disconnected and we call these trivial a.s.c. graphs. Further, in view of the above remark every nontrivial a.s.c. graph with a weak c.p. is always connected. On the other hand every nontrivial connected a.s.c. graph with a strong c.p. satisfies the following condition.

"If G is an a.s.c. graph with a strong c.p. τ containing a unique cycle of length 2 then there is either at least one cycle τ_i of length > 2 in τ such that both vertices of the unique cycle of length 2 are adjacent to either of the fixed halves of the vertices of τ_i or one vertex of the two cycle is adjacent to all vertices of at least one cycle in τ and the other vertex is adjacent to all vertices of a different cycle in τ".

Henceforth an a.s.c. graph in our discussion will always mean a connected a.s.c. graph.

Lemma 2.2.5. If τ is a strong c.p. of an a.s.c. graph G then τ^2 is an automorphism of G. But if τ is a weak c.p. of an a.s.c. graph G then τ^2 is not an automorphism of G.

(For a c.p. τ of a s.c. graph, τ^2 is always an automorphism.)

This is due to the presence of a unique cycle of length either $\ell \equiv 2 \pmod 4 > 2$ or $\ell = 3$ in τ, where the image under τ^2 of the missing edge is not itself. But τ^2 is an automorphism if τ is a strong c.p..

2.3. Construction method

In this section we obtain a method of constructing an a.s.c. graph with a given complementing permutation and then note some immediate consequences. Here we do not distinguish between the symbols of the permutation and vertices of the graph.

A simple extension of the construction algorithm for s.c. graphs by Gibbs [21] yields a method of constructing all a.s.c. graphs with a given (strong/weak) c.p.. First, suppose τ is a weak c.p. with no odd cycle of length > 1, whose symbols are the numbers 1, 2, ..., p. Order the cycles (of lengths > 1) of τ in non-decreasing order of their lengths with the unique cycle of length 1 (if p = 4n+3) at the end. If $\tau_1 = (1\ 2\ 3\ \dots\ 4n_1)$ is the first cycle in this ordering then denote by S the set of all numbers 2, 3, ..., $2n_1+1$; the first $4n_1$ numbers of each subsequent cycle and p (if p = 4n+3). If $\tau_1 = (1\ 2\ \dots\ 4n_1+2)$, $n_1 \leq n$, is the first cycle then S consists of numbers 2, 3, ..., $2n_1+1$, $2n_1+2$; the first $4n_1+2$ numbers from each subsequent cycle and p (if p = 4n+3).

Now to construct an a.s.c. graph, say G, whose vertices are labelled 1, 2, ..., p and c.p. τ, decide arbitrarily whether the unordered pair (1,j), for every $j \in S$, is to be an edge or a nonedge in G. Then the same will be true for $(\tau^{2i}(1), \tau^{2i}(j))$ with

$$
\begin{aligned}
i &= 1,2,\dots,2n_{i_j}, && \text{if j is in a cycle of length } 4n_{i_j},\\
&= 1,2,\dots,2n_{i_j}+1, && \text{if j is in a cycle of length } 4n_{i_j}+2 \text{ and } n_{i_j} \neq n_1,\\
&= 1,2,\dots,2n_1+1, && \text{if } j \neq 2n_1+2 \text{ is in the cycle of length } 4n_1+2,\\
&= 1,2,\dots,n_1-1, && \text{if } j = 2n_1+2 \text{ is in the cycle of length } 4n_1+2,
\end{aligned}
$$

and i varies from 1 to $2n_1$ or $2n_1+1$ according as the first cycle is of length $4n_1$ or $4n_1+2$ and j = p(= 4n+3).

This gives all the edges of $G(\tau_1)$ and all the edges joining vertices in $G(\tau_1)$ with those in $G(\tau\backslash\tau_1)$. Then delete $G(\tau_1)$ and repeat the process for the c.p. $\tau\backslash\tau_1$ and continue till all cycles of τ are exhausted. Since p is finite the process will terminate after finitely many steps. Note that if the ℓth cycle is $(x_1x_2...x_{4n_\ell+2})$ then $(x_{2n_\ell+1}, x_{4n_\ell+2})$ is the missing edge provided $(x_1, x_{2n_\ell+2}) \in E(G)$ otherwise $(x_1, x_{2n_\ell+2})$ is the missing edge.

Suppose τ is a weak c.p. whose elements are the numbers 1,2,3,...p(=4n+3) and τ contains a cycle of length 3, say $\tau' = (p-2, p-1, p)$. By Lemma 2.2.1, τ in this case does not have any other cycle of odd length. First construct a s.c. graph G′ with p–3 vertices and a c.p. $\tau\backslash\tau'$ by the same procedure as above. Then join the vertices labelled p–2 and p–1 by an edge, and also join each vertex of τ' to every vertex in a complementary half of the vertices in G′. This results in an a.s.c. graph, say G, with p vertices, a weak c.p. τ and missing edge $e = (p-2, p)$.

Next suppose that τ is a strong c.p. whose symbols are the numbers 1, 2, ..., p. Then, besides the cycles of lengths divisible by 4, τ contains either

(i) two cycles of length 1 (if $p = 4n+2$)

or (ii) one cycle of length 2 (if $p = 4n+2$)

or (iii) one cycle of length 2 and one of length 1 (if $p = 4n+3$).

This is due to the fact that a strong c.p. of an a.s.c. graph fixes exactly one edge, say $e = (x,y)$, which is our missing edge, and each of the vertices x and y accounts for a 1–cycle or both together account for a 2–cycle in the strong c.p.

In either of the above cases the construction of an a.s.c. graph G with n vertices may be carried out by treating the cycle(s) of length 1 and length(s) > 2 in the same way as in case of a weak c.p.. For the cycle of length 2, one of the two vertices is treated like a cycle of length 1, and then the other vertex is joined to the same vertices of at least one cycle of length > 2 which are already joined to the first one while for the other cycles of lengths > 2 one only needs to maintain the self complimentarity (cf. remark on page 9). If there is a cycle of length 1 in addition to the cycle of length 2 then exactly one of the latter is joined to the vertex of the former. Here the missing edge is the edge joining the vertices in a unique 2–cycle or the two 1–cycles of the given strong c.p.

Now in both cases of weak and strong c.p. it can be easily checked that the resulting graph G is, in fact, an a.s.c. graph. For illustration of the above construction method take a weak c.p. $\tau = \tau_1\tau_2$, where $\tau_1 = (1\ 2\ 3\ 4)$ and $\tau_2 = (5\ 6\ 7\ 8\ 9\ 10)$. Then $S = \{2, 3, 5, 6, 7, 8\}$. Take the unordered (1,j) as an edge, for each $j \in S$. Then by the construction method, (1,2), (3,4), (1,3), (1,5), (1,6), (1,7), (1,8), (3,7), (1,9), (3,5), (3,9), (3,8), (1,10), (3,6), (3,10) are edges in $G(\tau_1)$ and edges joining the vertices in $G(\tau_1)$ with those in $G(\tau_2)$. In the second step of the construction we take the case of $G(\tau_2)$ by taking the "new" $S = \{6, 7, 8\}$ with $(5,j) \in E(G(\tau_2))$ for each $j \in S$. So the edges in $G(\tau_2)$ are (5,6), (5,7), (5,8), (7,8), (7,9), (9,10) and (9,5). The resulting graph is given in Fig. 2.2. Here the missing edge is (7,10).

From the above construction method the following are immediate.

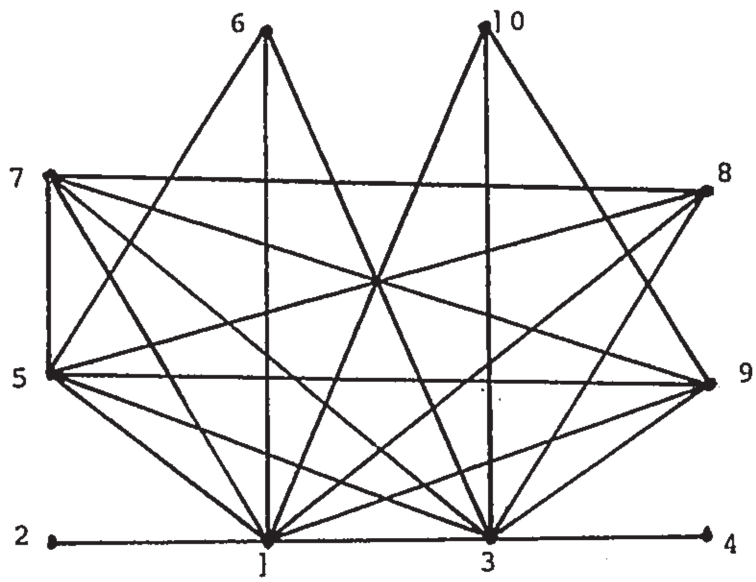


Figure 2.2

Lemma 2.3.1. The induced subgraph on the set of vertices of any subset of the cycles of a (strong/weak) c.p. is either a s.c. graph or an a.s.c. graph.

(For a s.c. graph, the induced subgraph on any subset of the cycles of a corresponding c.p. is always a s.c. graph.)

Lemma 2.3.2. The vertices of any cycle of length > 3 in a (weak/strong) c.p. of degree p alternate in vertex degree and the sum of the degree of any pair of complementary vertices is p–1 except exactly for one pair, in the cycle of length $\ell \equiv 2 \pmod 4$, which have degree sum p–2.

(In case of a c.p. of s.c. graph this holds without exception).

Lemma 2.3.3. The adjacencies among the vertices of two cycles of lengths $\ell \equiv 0 \pmod 4$ and $\ell' \equiv 2 \pmod 4$ with $\ell' \nmid \ell$ have exactly four possibilities, i.e., all the vertices of either cycle are adjacent to a fixed half of the other cycle.

Lemma 2.3.4. Every a.s.c. graph with more than three vertices and a weak c.p. consisting of a single cycle is connected.

Proof. Suppose G is an a.s.c. graph with $p > 3$ vertices and a weak c.p. $\tau = (1\ 2\ \ldots\ p)$, where $V(G) = \{1, 2, \ldots, p\}$. Clearly p is even and τ partitions V(G) into two disjoint parts, say $V_1 = \{x, \tau^2(x), \ldots\}$ and $V_2 = \{\tau(x), \tau^3(x), \ldots.\}$, for some $x \in V(G)$. Also one of $G[V_1]$ and $G[V_2]$ is complete while the other is totally disconnected, and $G[V_1,V_2]$ contains at least on edge. Without loss of generality, take $G[V_1] = K_{p/2}$, $G[V_2] = \overline{K}_{p/2}$ and $(u,v) \in E(G)$ for some $u \in V_1$ and $v \in V_2$. Then $(\tau^{2r}(u), \tau^{2r}(v)) \in E(G)$ for every positive integer r. This implies that every vertex of V_2 is joined in G to at least one vertex in V_1. Thus G is connected.

Now we give a proof of the remark on page 9.

Sufficiency of the remark is immediate. For necessity, suppose G is a disconnected a.s.c. graph. By the above lemma, and as every strong c.p. of an a.s.c. graph contains at least two cycles, any c.p. of G has at least two cycles. Let G′ be the maximal s.c. subgraph of G (using Lemma 2.3.1). Then the c.p. of G restricted to G\G′ has exactly one cycle and is of length $4n'+2$ (n' being a nonnegative integer). If $n' \neq 0$ then G\G′ is a connected a.s.c. graph. Also

$E(G[V(G'), V(G\backslash G')]) \neq \emptyset$. This means that G is connected – a contradiction. So $n' = 0$, i.e., $G\backslash G'$ has exactly two vertices. Again G being disconnected, only one of the two vertices of $G\backslash G'$ is joined to all the vertices of G' while the other vertex of $G\backslash G'$ is an isolated vertex of G.

Now the fact that every s.c. graph, and hence G', has a hamiltonian path, implies that the component of G containing G', by the above observation, is clearly pancyclic.

Theorem 2.3.1. Suppose $\tau(G) = \bar{G}$ and $\tau = (1\ 2\ 3\ \ldots\ p)$ is a weak c.p., where $p = 4n+2$ $(n \geq 1)$. Then

(a) there is a set of exactly n consecutive odd (even) labelled vertices each of which is adjacent to exactly n+1 even (odd) labelled vertices and the other set of consecutive n+1 odd (even) labelled vertices are each adjacent to n even (odd) labelled vertices.

(b) G has vertices of four degrees: for some r, $n \leq r \leq 2n$, there are n+1 vertices of degree r; n vertices of degree r+1; n+1 vertices of degree 4n–r, and n vertices of degree 4n+1–r.

Proof. (a) Suppose $(1, 2i) \in E(G)$ for $i \leq n$. Then $\tau^{4n+3-2i}(1,2i) = (1, 4n+4-2i) \notin E(G)$ for all $i \leq n$, as the image of an edge under every odd power of τ is a nonedge. This implies that the vertex labelled 1 is adjacent to at least n consecutive even labelled vertices. So every odd labelled vertex is adjacent to at least n consecutive even labelled vertices. Now we have two cases:

Case (i). $(1, 2n+2) \in E(G)$. Then $\tau^{2j}(1, 2n+2) \notin E(G)$ for all $j \geq n$ due to our construction of G, i.e., $2n+1$ is the first odd labelled vertex which is not adjacent to the same number of even labelled vertices as the vertex 1. So each of the first n odd labelled vertices 1, 3, 5, ..., $2n-1$ is adjacent to exactly $n+1$ even labelled vertices.

Case (ii). $(1, 2n+2) \notin E(G)$. Then $(2, 2n+3) \in E(G)$ and so, as above, $\tau^{2j}(2, 2n+3) \notin E(G)$ for all $j \geq n$. That is each of the first $n+1$ odd labelled vertices 1, 3, ..., $2n+1$ is adjacent to exactly n even labelled vertices and each of the remaining n odd labelled vertices is adjacent to exactly $n+1$ even labelled vertices.

(b) From (a) above each odd (even) labelled vertex in G is adjacent to n or $n+1$ even (odd) labelled vertices. Now for the adjacencies among the even labelled vertices only or the odd labelled vertices only consider an edge $(i, i+2)$ in G with i odd or even. The images of $(i, i+2)$ under different even powers of τ contribute degree 2 to each odd or even labelled vertex according as i is odd or even respectively such that the sum of the degrees of an odd labelled vertex and an even labelled vertex due to these adjacencies is 2n. Thus an odd (even) labelled vertex is adjacent to either none, two, four, ..., $2\lfloor n/2 \rfloor$ other odd (even) labelled vertices in G. Then the proof follows by taking $r = n+j$, where $j \in \{0,2,4,...,2\lfloor n/2 \rfloor\}$.

Remark. If $r = 2n$ in the proof of (b) then G has vertices of only two degrees: $2n+2$ vertices of degree 2n and 2n vertices are of degree $2n+1$ each. However, this is possible only when n is even.

For the a.s.c. graph considered in Theorem 2.3.1., an edge or a nonedge (1,j) contributes 2n+1 edges or nonedges for G through even powers of τ for $j \leq 2n+1$, whereas the edge or nonedge (1, 2n+2) contributes only n edges or nonedges. We call the first category the full orbits and the second the half orbits of τ. Now we have

Corollary 2.3.1.1. Suppose G and τ are as in Theorem 2.3.1. Further, if $d_1 \geq d_2 \geq \ldots \geq d_{4n+2}$ is the degree sequence of G then the ends of the missing edge have degrees

(i) d_{n+1} and d_{3n+2}, provided G has vertices of four degrees,

or, (ii) d_{2n+1} and d_{2n+2}, provided G has vertices of two degrees.

Proof. Consider the spanning subgraph G′ of G containing only the edges generated by the full orbits of τ. Then G′ is quasi regular or regular according as G has vertices of four or two different degrees. In the first case the vertex set V(G′) is partitioned into two classes as odd or even labelled vertices with degree difference of at least two between any two vertices of different parity. But any edge due to the half orbit always joins two vertices of opposite parity and so, exactly n vertices of each of the two classes of vertices of G′ will have an additional degree in G while the rest have the same degree as in G′. With this the proofs in both cases follow immediately.

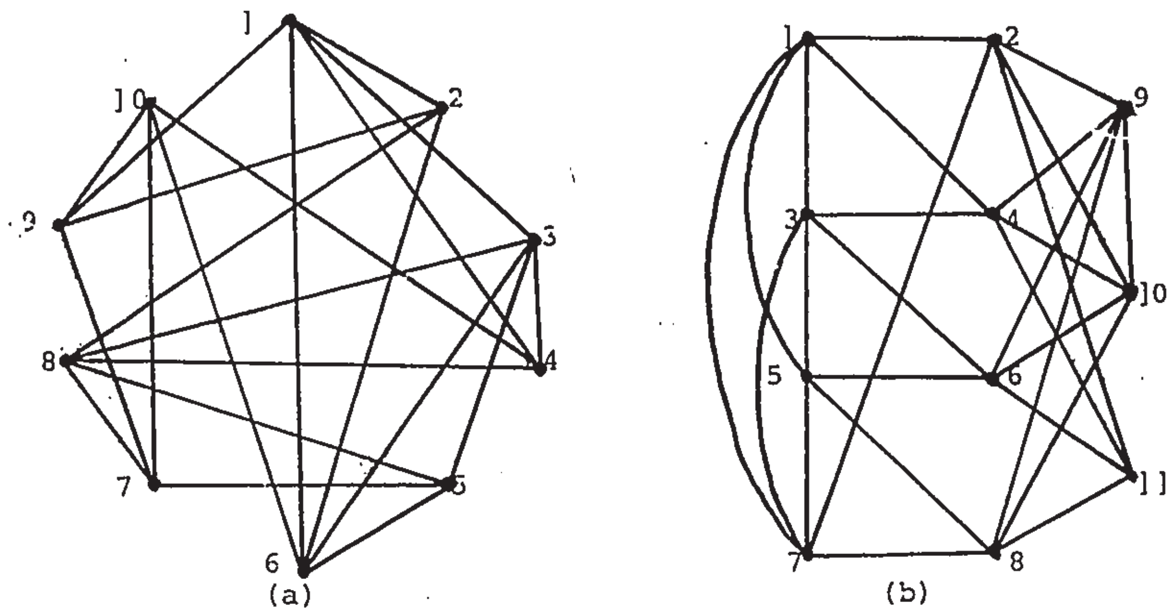
Corollary 2.3.1.2. Suppose G and τ are as in Theorem 2.3.1. Then the ends of the missing edge are adjacent to exactly n vertices in common.

We note that there are no parallel results in case of a strong c.p. since a strong c.p. of an a.s.c. graph always contains at least two cycles. However, a result regarding adjacency of the ends of the missing edge is following.

<u>Lemma 2.3.5</u>. If G is an a.s.c. graph with p (=4n+2 or 4n+3) vertices and a strong c.p. then the ends of the missing edge are adjacent to some j ($0 \leq j \leq 2n$) common vertices.

2.4. <u>Quasi regular and almost regular a.s.c. graphs</u>

The remark at the end of Theorem 2.3.1 guarantees the existence of an a.s.c. graph with 4n+2 vertices and a weak c.p. $\tau = (1\ 2\ \ldots\ 4n{+}2)$ which has vertices of only two different degrees (differing by 1) provided n is even. However, the restriction of n being even is not necessary in case of a strong c.p.. Such a.s.c. graphs with vertices of two different degrees, differing by 1 only, are called *quasi regular* a.s.c. graphs. By Lemma 2.3.2, every a.s.c. graph with p (= 4n+2) vertices has exactly one pair of complementing vertices with degree sum p–2 whereas this sum for all other complementing pairs is p–1. If such a G is quasi regular then take a new vertex, say x, and let it be fixed by a corresponding (weak/strong) c.p. of G. Now joining x to precisely all the vertices in a complementing half of the vertices of G of lower degree we obtain an a.s.c. graph with 4n+3 vertices of which 4n+2 vertices have degree 2n+1 each and one has degree 2n. Such a graph is called an *almost regular* a.s.c. graph. It may be noted that such a graph may be obtained through a different construction. For example, take a s.c. graph with 8 vertices and degree sequence (5,5,5,5,2,2,2,2), and an a.s.c. graph with 3 vertices and exactly one edge. Then joining every vertex of the latter to all the vertices of minimum degree of the former, an almost regular a.s.c. graph with 11 vertices is obtained in which 10 vertices have degree 5 each and one has degree 4. Examples of these two types of a.s.c. graphs are given in Fig. 2.3 below along with their c.p.'s.



(a) quasi regular a.s.c. graph with 10 vertices and weak c.p. = (12...10) (missing edge (5,10)).

(b) almost regular a.s.c. graph with 11 vertices and weak c.p. = (12...8)(9 10 11) (missing edge (9,11)).

Figure 2.3

The following are some of the results on the existence of such graphs.

<u>Lemma 2.4.1</u>. There exists no regular a.s.c. graph.

(For every positive integer n, there exists a regular s.c. graph with 4n+1 vertices).

Lemma 2.4.2. If n is even then there exists a quasi regular a.s.c. graph with 4n+2 vertices and a weak c.p. $\tau = (1\ 2\ \ldots\ 4n+2)$.

Lemma 2.4.3. There exists a quasi regular a.s.c. graph with 4n+2 vertices and an almost regular a.s.c. graph with 4n+3 vertices for every positive integer n and a strong c.p.

(There exists a quasi regular s.c. graph with 4n vertices for every positive integer n, but no almost regular s.c. graph).

So, in general, we have

Theorem 2.4.1. For every positive integer n, there exists at least one quasi regular a.s.c. graph with 4n+2 vertices and at least one almost regular a.s.c. graph with 4n+3 vertices.

Proof. Consider a quasi regular a.s.c. graph G' with 4n vertices (which always exists, Sachs [46]). Then G' will have 2n vertices of degree 2n and the rest 2n vertices of degree 2n–1 each. Also the vertices of the two kinds are complementary of one another. Take two new vertices x and y, and join both to all vertices of degree 2n–1 of G'. The resulting graph is a quasi regular a.s.c. graph.

Next, consider a regular s.c. graph G^* with 4n+1 vertices. Then G^* has a vertex, say x, fixed by some c.p. σ of G^*. Take two new vertices u and v not in

G^* and join u to some 2n vertices and v to the other 2n+1 vertices of G^* such that the self complementarity is preserved, i.e., $(u,y) \in E(G) \Leftrightarrow (v,\sigma(y)) \in E(G)$, $x \neq y \in V(G^*)$ and $(x,v) \in E(G)$, where G is the new graph. Notice that all the vertices except u of the new graph G with $V(G) = V(G^*) \cup \{u,v\}$ are of degree 2n+1 and u is of degree 2n. Hence the graph G is an almost regular a.s.c. graph.

Corollary 2.4.1.1. For every almost regular a.s.c. graph with 4n+3 vertices there corresponds a quasi regular a.s.c. graph with 4n+2 vertices but not conversely.

A trivial result may be stated that there is no quasi regular a.s.c. graph with an odd number of vertices and no almost regular a.s.c. graph with an even number of vertices.

2.5. Decomposition Theorem

Gibbs [21] has proved a decomposition theorem for s.c. graphs in terms of the smallest nontrivial induced s.c. subgraphs. To obtain a similar result for a.s.c. graphs, we need only to check the result for the induced a.s.c. subgraph on the vertex set of the cycle of length > 3 and not congruent to 0 (mod 4) in a weak c.p. of an a.s.c. graph.

Lemma 2.5.1. Suppose $\tau(G) = \bar{G}$ and $\tau = (1\ 2\ \ldots\ 4n+2)$. Then G contains a subgraph, induced by the vertex set $\{1, 2, 2i-1, 2i\}$ for some i, $2 \leq i \leq n+2$, which is isomorphic to the four-vertex s.c. graph.

Proof. Recall from Theorem 2.3.1(a) that the odd vertex 1 is adjacent to n+1 even vertices of G. So suppose i is the smallest positive integer such that exactly one of (1, 2i) and (1, 2) is in E(G). In particular, take $(1, 2) \in E(G)$ then $\tau^{2(i-1)}(1, 2) = (2i-1, 2i) \in E(G)$. Also exactly one of (1, 2i–1) and (2, 2i) is in E(G) by the definition of the c.p. τ. Finally $(2, 2i-1) \notin E(G)$, for otherwise $(1, 2i-2) \notin E(G)$, contradicting our choice of i. Thus the vertex set {1, 2, 2i–1, 2i} induces a four vertex s.c. subgraph. The other case when $(1, 2) \notin E(G)$ is similar.

∎

Lemma 2.5.2. If $\tau(G) = \bar{G}$ and $\tau = (1\ 2\ \ldots\ 4n+2)$ then G contains a collection of n disjoint induced four–vertex s.c. subgraphs.

Proof. By the previous lemma, G has an induced four–vertex s.c. subgraph on the vertex set {1, 2, 2i–1, 2i} for some i, $2 \le i \le n+2$. Then it follows from the construction method and the properties of the weak c.p. τ that the induced subgraph on the vertex set $\{\tau^{2r}(1), \tau^{2r}(2), \tau^{2r}(2i-1), \tau^{2r}(2i)\}$ is also a four–vertex s.c. graph, for each r = 1, 2, ..., n–1. This completes the proof of the lemma.

Recall that the withdrawal of the vertices of the unique cycle of length $\ell \not\equiv 0 \pmod 4$ or the two 1–cycles from a weak/strong c.p. of an a.s.c. graph results in a s.c. graph. So combining the above lemma with Gibbs' result, we have

Theorem 2.5.1. If G is an a.s.c. graph with 4n+2 vertices then G possesses a collection of n disjoint induced four–vertex s.c. subgraphs.

In passing it may be mentioned that the general result (Theorem 6) of Gibbs [21] regarding the (0, 1, −1)–adjacency matrices of s.c. graphs holds for a.s.c. graphs as well.

CHAPTER 3

PATHS AND CYCLES IN a.s.c. GRAPHS

3.1. Introduction

Suppose $\tau = (1\ 2\ 3\ ...\ p)$ is a (weak/strong) c.p. of an a.s.c. graph G, where $V(G) = \{1, 2, ..., p\}$. Also suppose that the numbering in every cycle of length $\geq$ 4 in τ begins with an odd integer. Without loss of generality, we assume that the edges $(i, i+2) \in E(G)$ for all i odd in any cycle of length ≥ 4 in τ. For if not, $(i+1, i+3) \in E(G)$ and we can relabel the vertices appropriately. Further, if $\tau = (1\ 2\ ...\ p)$ consists of a single cycle then we also assume that $(i, i+1) \in E(G)$ for all i odd. Otherwise (1, 2), in particular, is a nonedge and then τ may be replaced by $\tau^{-1} = (1p\ (p-1)\ ...2)$ and the vertices may be relabelled suitably. This assumption is frequently used throughout this chapter.

With the above assumption regarding existence of some edges within an individual cycle in τ, we next define a relation among different cycles of τ. Denote the cycles of lengths ≥ 4 of a c.p. (weak/strong c.p.) of a s.c. (an a.s.c.) graph G by $\tau_1, \tau_2, ..., \tau_m$. Then for any two cycles τ_i and τ_j of τ, define an order relation $\tau_i < \tau_j$ if some even vertex of τ_i is adjacent to some odd vertex of τ_j (cf. Clapham [9]). This, then, implies that every even vertex of τ_i is adjacent to some odd vertex of τ_j.

Now we state some results on s.c. graphs from Clapham [9] and Rao [37] which will be useful in what follows.

<u>Theorem 3.1.1</u>. Suppose G is a s.c. graph with 4n vertices and a c.p. τ whose cycles have an ordering $\tau_1 < \tau_2 < ... < \tau_m$. Then G has a hamiltonian path containing a pair of consecutive odd vertices of τ_1 appearing consecutively and whose end vertices are consecutive even vertices of τ_m. Further, such a hamiltonian path has some even vertex of τ_i and some odd vertex of τ_i+1 $(1 \leq i \leq m-1)$ appearing consecutively.

<u>Theorem 3.1.2</u>. If G is a s.c. graph with $4n \geq 8$ vertices with a c.p. τ, containing at least one cycle of length ≥ 8, then for every integer ℓ, $4n-4 \leq \ell \leq 4n-1$, G has a path of length ℓ containing a pair of consecutive odd vertices appearing consecutively, and whose end vertices are consecutive even vertices of τ.

It may also be noted that either the pair of consecutive odd vertices or the pair of consecutive even vertices mentioned in the above two theorems may be chosen arbitrarily (cf. Remark 2.1 [37]).

<u>Theorem 3.1.3</u>. If G is a s.c. graph with $p \geq 8$ vertices then G has an ℓ-cycle for each $3 \leq \ell \leq p-2$.

<u>Corollary 3.1.3.1</u>. If G is a s.c. graph with p (even) vertices and a c.p. $(\tau_1\tau_2...\tau_m)$, where $\tau_i < \tau_j$, for all $i < j$, then a $(p-2)$-cycle in G can be chosen containing a pair of consecutive odd vertices of τ_1 and a pair of consecutive odd vertices of τ_m appearing consecutively.

3.2. <u>Paths and cycles in a special class of a.s.c. graphs</u>

In this section we consider a.s.c. graphs each of which possesses a weak c.p. consisting of a single cycle.

Lemma 3.2.1. Every a.s.c. graph with 4n+2 ($n > 1$) vertices and a weak c.p. consisting of a single cycle is hamiltonian.

Proof. Suppose G is an a.s.c. graph with 4n+2 vertices and a weak c.p. $\tau = (1\ 2\ 3\ \ldots\ 4n+2)$. Then, by the assumption at the beginning of this chapter, the edges (i, i+1) and (i, i+2) $\in$ E(G) for all i odd. So, in particular, the edges (1,2), (1,3), $\in$ E(G). Again, for $n > 1$, the edge (1,4) $\in$ E(G) iff (2,5) $\notin$ E(G). Since $4n+2 > 4$, we have in G,

either (a) the edges (1,4), (3,6), ..., (4n+1, 2)

or (b) the edges (2,5), (4,7), ..., (4n, 1), (4n+2, 3).

In case (a), a hamiltonian cycle in G is

$$2, 1, 4, 3, 6, 5, \ldots, 4n, 4n-1, 4n+2, 4n+1 \tag{1}$$

(starting with 2 then subtracting 1 and adding 3 alternately).

In case (b), it is

$$1, 2, 5, 6, \ldots, 4n-2, 4n+1, 4n+2; 3, 4, 7, 8, \ldots, 4n-4, 4n-1, 4n \tag{2}$$

(starting with 1 and then adding 1 and 3 alternately),

where the numbers are taken residues modulo 4n+2.

This completes the proof.

This result can not include the case $n = 1$ for there is an a.s.c. graph G with V(G) = {1, 2, 3, 4, 5, 6} and edge set E(G) = {(1,2), (3,4), (5,6), (1,3), (3,5), (5,1), (1,4)} which is not hamiltonian.

In the above Lemma we already have (i, i+1) $\in$ E(G) for all i odd. So, switching the last two vertices in (1), a hamiltonian path in G in the case (a) above is given by

2, 1, 4, 3, 6, 5, ..., 4n, 4n–1, 4n+1, 4n+2.

Also (1,3) $\in$ E(G) and so reversing the order of the vertices from 1 to 4n+2 in (2), a hamiltonian path in the case (b) is given by

4n+2, 4n+1, 4n–2, 4n–3, ..., 6, 5, 2, 1; 3, 4, 7, 8, ..., 4n–4, 4n–1, 4n.

Thus a direct checking for the case n = 1 completes the proof of the following lemma.

Lemma 3.2.2. Every a.s.c. graph with p = 4n+2 (n $\geq$ 1) vertices and a weak c.p. τ consisting of a single cycle has a hamiltonian path containing a pair of consecutive odd vertices of τ appearing consecutively and consecutive even vertices of τ at the ends.

Theorem 3.2.1. If G is an a.s.c. graph with p = 4n+2 (n > 1) and a weak c.p. consisting of a single cycle then G is pancyclic.

Proof. Suppose V(G) = {1, 2, ..., p} and τ = (1 2 ... p) is a weak c.p. of G. Then (i, i+1) and (i, i+2) $\in$ E(G) for all i odd. Again, by the construction method of a.s.c. graphs discussed in Chapter 2, (1,j) $\in$ E(G) with j < 2n+2 implies that the image of the edge (1,j) under any even power of τ is also in E(G).

<u>Case I</u>. $(1,5) \in E(G) \Rightarrow (i, i+4) \in E(G)$ for all i odd (since $5 < 2n+2$). Then G has induced subgraph G′ with the former's $2n+1$ odd vertices in which the edges $(i, i+2)$, $(i, i+4)$, $(i, i-2)$, $(i, i-4) \in E(G')$ for all i in $V(G')$, where the vertices are to be taken residues modulo $4n+2$. So G′ is a 4 regular graph and it can be easily checked that G′ on $2n+1$ vertices is pancyclic.

<u>Case I(i)</u>. $(1,4) \in E(G) \Rightarrow (i, i+3) \in E(G)$ for all i odd. Then every pair of alternate odd vertices in the hamiltonian cycle in case (a) of Lemma 3.2.1. are adjacent in G. So by withdrawing one, two, ..., $(2n+1)$ even vertices successively from that hamiltonian cycle in G one gets ℓ-cycles, $2n+1 \leq \ell \leq 4n+2$.

<u>Case I(ii)</u>. $(1,4) \notin E(G) \Rightarrow (2,5) \in E(G) \Rightarrow (i, i+3) \in E(G)$ for all i even. Also we have $(i, i+4) \in E(G)$ for all i odd. Then by the same procedure the desired ℓ-cycles, $2n+1 \leq \ell \leq 4n+2$, in G can be obtained.

<u>Case II</u>. $(1,5) \notin E(G) \Rightarrow (2,6) \in E(G) \Rightarrow (i, i+4) \in E(G)$ for all i even (since $5 < 2n+2$). Here again we consider two possibilities:

<u>Case II(i)</u>. $(1,4) \in E(G) \Rightarrow (i, i+3) \in E(G)$ for all i odd. For even j, and $4 \leq j \leq 2n$, consider the cycle

$$(1, 4, 3, 5, 8, 7, 9 \ldots 2j, 2j-1, 2j+1; 2j+2, 2j-2, 2j-6, \ldots, 6, 2, 1)$$

of length $2j+2$. Since $(i, i+2) \in E(G)$ for all i odd, one can withdraw successively one, two, three, ..., $j/2$ even vertices, whose labellings are multiples of 4, from this cycle to get cycles of lengths $2j+1$, $2j$, ..., $3j/2+2$ for each even j. Now varying even j within its chosen bounds one gets ℓ-cycles, $8 \leq \ell \leq 4n+2$. The remaining cycles of lengths 7, 6, 5, 4, and 3 are (1, 3, 5, 7, 10, 6, 2), (1, 4, 3, 5, 6, 2), (1, 3, 5, 6, 2), (1, 3, 6, 2) and (1, 3, 4) respectively.

<u>Case II(ii)</u>. $(1,4) \notin (E(G) \Rightarrow (2,5) \in E(G) \Rightarrow (i, i+3) \in E(G)$ for all i even. For even j, and $4 \leq j \leq 2n$, consider the cycle

$$(4, 7, 8, 11, 12, \ldots, 2j-5, 2j-4, 2j, 2j-1, 2j+1, 2j+2; 2j-2, 2j-3, 2j-6, 2j-7, \ldots 2, 1, 3, 4)$$

of length 2j+2. Since $(i, i+3) \in E(G)$ for all i even, one can withdraw successively one, two, ..., (j–1) vertices from this cycle with vertex labels in $\{5, 7, 9, \ldots, 2j-3\} \cup \{2j, 2j+2\}$ to get cycles of lengths 2j+1, 2j, ..., j+3 for each even j. Then varying j within its chosen bounds one gets ℓ-cycles in G for $7 \leq \ell \leq 4n+2$. The remaining cycles of lengths 6, 5, 4, and 3 are (7, 9, 10, 6, 2, 5), (6, 2, 1, 3, 5), (5, 2, 1, 3) and (2, 5, 6) respectively. ∎

From the above proof along with a direct checking for n = 1 follows:

<u>Theorem 3.2.2</u>. If G is an a.s.c. graph with 4n+2 ($n \geq 1$) vertices and a weak c.p. τ consisting of a single cycle then for every integer ℓ, $3 \leq \ell \leq 4n$, G has an ℓ-cycle. Moreover, each such ℓ-cycle in G with $4 \leq \ell \leq 4n$, has a pair of consecutive odd vertices of τ appearing consecutively.

Now employing the withdrawal technique of Theorem 3.2.1 in Lemma 3.2.2 one can easily prove:

Corollary 3.2.2.1. If G is an a.s.c. graph with p (≥ 6) vertices and a weak c.p. τ consisting of a single cycle then for every integer ℓ, $p-4 \leq \ell \leq p-1$, G has a path of length ℓ containing consecutive even vertices of τ at the ends and in which a pair of consecutive odd vertices of τ appear consecutively.

3.3. Paths in a.s.c. graphs

In this section we show the existence of a hamiltonian path (and of other paths) in any given a.s.c. graph by an explicit construction.

Theorem 3.3.1. Every a.s.c. graph G with a weak c.p. τ containing no cycle of length 3 has a hamiltonian path.

Proof. First suppose that $|V(G)| = p$ (=4n+2). Also suppose that τ is a weak c.p. of G whose cycles are σ, τ_1, τ_2, ..., τ_m such that σ is the unique cycle of length not divisible by 4. We consider the following three possibilities with respect to the ordering of the cycles of τ defined earlier.

(i) $\sigma < \tau_1 < \tau_2 < \ldots < \tau_m$,

(ii) $\tau_1 < \tau_2 < \ldots < \tau_m < \sigma$

and (iii) $\tau_1 < \tau_2 < \ldots < \tau_j < \sigma < \tau_{j+1} < \ldots < \tau_m$.

Let us take a hamiltonian path P in G(σ) by Lemma 3.2.2. But the ends of P are consecutive even vertices of σ each of which is adjacent to some odd vertex of τ_1 in (i) above, P can be inserted into a hamiltonian path (Theorem 3.1.1 and the note following theorem 3.1.2) of $G(\tau\backslash\sigma)$ containing a pair of consecutive odd vertices of τ_1 appearing consecutively which are, in turn, adjacent to the ends of P to get a hamiltonian path in G. The case with possibility (ii) is proved similarly.

For (iii), the two segments of P obtained after deleting the edge between the consecutive odd vertices of σ may be inserted into suitably constructed hamiltonian paths of $G(\tau_1\tau_2...\tau_j)$ and $G(\tau_{j+1}\tau_{j+2}...\tau_m)$ to get a required hamiltonian path in G as shown in the Fig. 3.1 below.

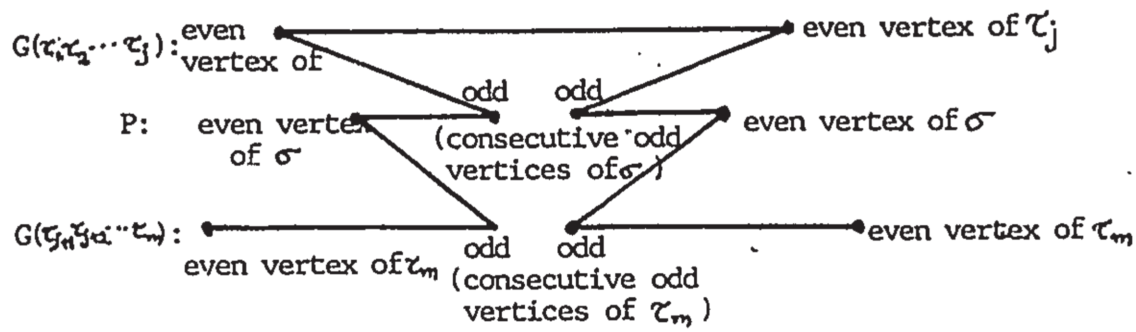


Figure 3.1

Next, suppose that $|V(G)| = p$ (= 4n+3). In this case a weak c.p. of G has a unique cycle of length 1, say σ' with the vertex labelled x. Suppose the other cycles of τ have an ordering $\tau_1 < \tau_2 < ... < \tau_{m'}$. Take a hamiltonian path P′ of $G(\tau\backslash\sigma')$ as above. Note that for every cycle τ_i of τ, vertex x is adjacent to every odd or to every even vertex of τ_i. Then the following are the possibilities.

(i) If x is adjacent to the odd vertices of τ_1 then a hamiltonian path in G is obtained by inserting x between the adjacent consecutive odd vertices of τ_1 in P′.

(ii) If x is adjacent to the even vertices of $\tau_{m'}$, then a hamiltonian path in G is obtained by joining x to either end of P′.

(iii) If neither (i) nor (ii) holds then x is adjacent to the even vertices of τ_i and the odd vertices of τ_{i+1}, for some i, $1 \leq i \leq m'-1$. Then using theorem 3.1.1, x can be inserted suitably into P′ to get a hamiltonian path in G. ∎

Remark. If G is an a.s.c. graph with a strong c.p. then it may not have a hamiltonian path because such a graph may be disconnected. For example, take a strong c.p. $\tau = \tau_1\tau_2$ where $\tau_1 = (1234)$ and $\tau_2 = (56)$. Construct a s.c. graph G′ with 4 vertices and a c.p. τ_1. Then join the vertex labelled 5 to all vertices of G′. The resulting graph, say G, is an a.s.c. graph with six vertices and strong c.p. τ. The vertex labelled 6 is an isolated vertex of G and so G is disconnected.

Further, there always exists an a.s.c. graph with an odd number of vertices and a weak c.p. such that this graph does not have a hamiltonian path. For example, take a s.c. graph G_1 with 4k vertices of which exactly two vertices are of degree 1 each. Then adjoin an a.s.c. graph G_2 with vertex set $\{x_1, x_2, x_3\}$ and edge set $\{(x_1, x_2)\}$ to G_1 by joining each vertex of the former to every vertex in a complementary half of vertices of maximum degree of the latter. Clearly this resulting graph does not have a hamiltonian path.

Corollary 3.3.1.1. Suppose G is an a.s.c. graph with $p = 4n+2$ $(n \geq 1)$ vertices and a weak c.p. τ. Then for every integer ℓ, $p-4 \leq \ell \leq p-1$, G has a path of length ℓ containing a pair of consecutive odd vertices of τ appearing consecutively, and whose end vertices are consecutive even vertices of τ.

Proof. We have a path P of length p–1 (hamiltonian path) constructed in the proof of the above theorem. Also the weak c.p. τ contains a unique cycle of length $4n'+2$, $1 \leq n' \leq n$. If $n = n'$ then the result follows from Corollary 3.2.2.1. If $2 \leq n' < n$ then one may follow the withdrawal technique (described in Theorem 3.2.1) and withdraw suitable intermediary even or odd vertices from the segment(s) in P of the unique cycle of length $4n'+2$ in τ to get paths of lengths n–2, n–3 and n–4 in G. If $n' = 1$, i.e., τ has unique cycle, say τ', of length 6 then one can withdraw only one intermediate even vertex of τ' from P to get a path of length p–2. But if at least one cycle of τ is of length ≥ 8 then also the result would follow from Theorem 3.1.2. So we need to consider the case have that τ has exactly one cycle of length 6 and each of the other is of length 4. To construct paths of lengths p–3 and p–4 in this case we consider two possibilities:

Suppose τ' is not the first cycle in our ordering of the cycles in τ. Then by Lemma 2.3.3 every even vertex of the cycle, say τ'', preceding τ' is adjacent to every odd vertex of τ' (Lemma 2.3.3). Notice that the above path of length p–2 in G contains two consecutive odd vertices, say x and y, of τ' appearing consecutively and one of them, say x, being adjacent to an even vertex of τ''. Deleting x and joining y to this even vertex one gets a path of length p–3 with the required property. Again by the same Lemma 2.3.3, either every even vertex or every odd vertex of τ'' is adjacent to every odd vertex of τ'. Necessary adjustments for these two alternatives are shown below in Fig. 3.2 and 3.3 respectively.

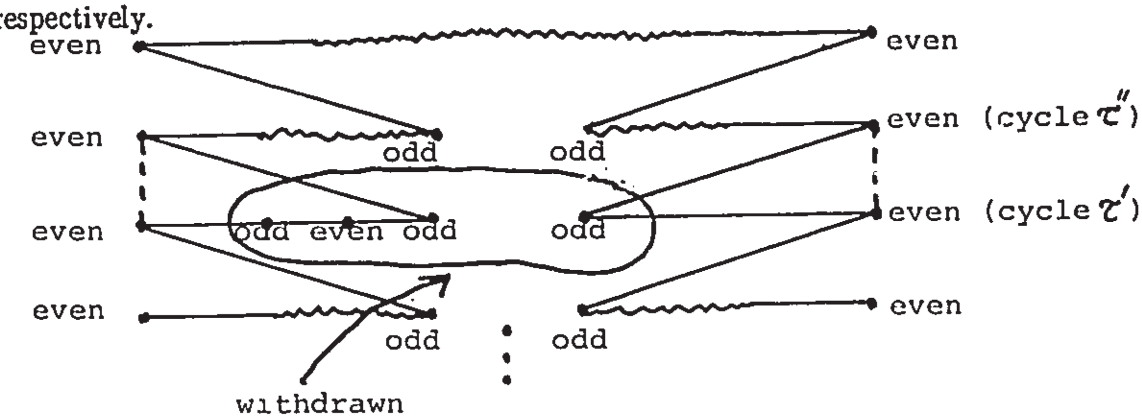


Figure 3.2

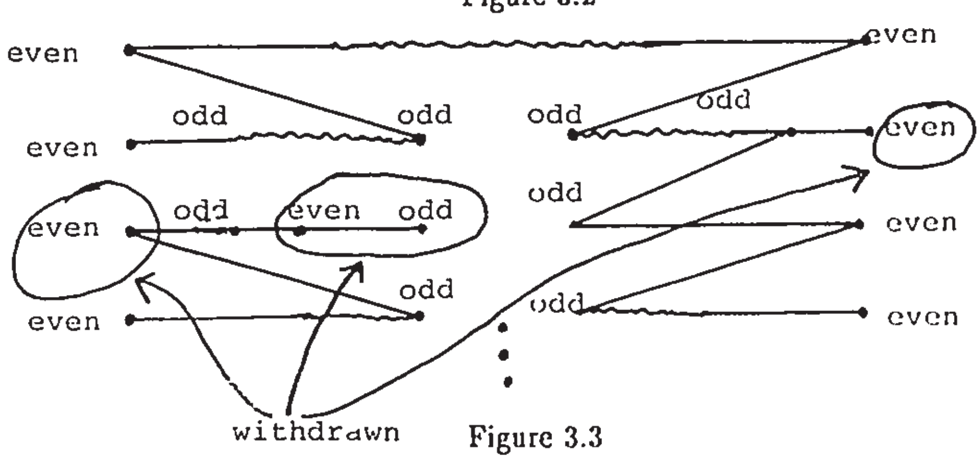


Figure 3.3

Finally, if τ' is the first cycle in the ordering, similar adjustments can be made with the vertices of τ' and the cycle τ'' following it (Fig. 3.4).

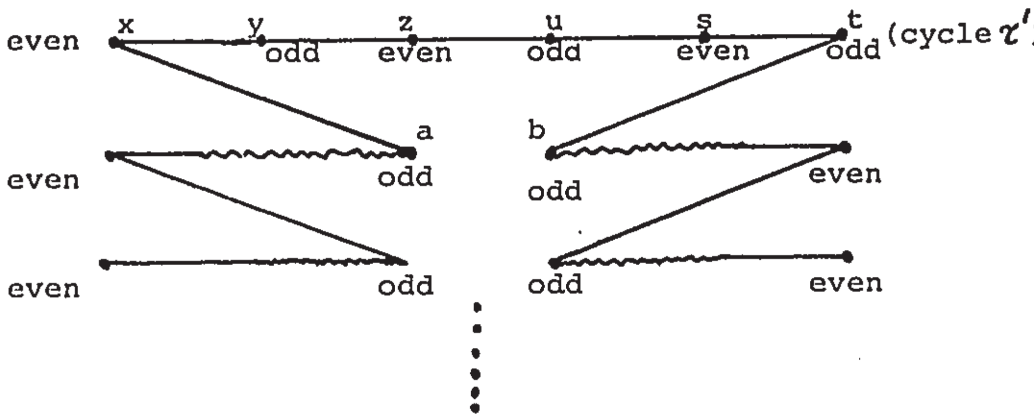


Figure 3.4

In the above Fig.,	(x)	deletion of even z and joining y with u gives a (p–2)–path.
	(xx)	deletion of z and u, and joining y with s gives a (p–3)–path.
	(xxx)	deletion of u, s, t, and joining z with b gives a (p–4)–path.

3.4. Cycles in a.s.c. graphs

We now use the results obtained in the previous sections to construct cycles of different lengths in a given a.s.c. graph.

Theorem 3.4.1. If G is an a.s.c. graph with $p = 4n+2$ ($n \geq 1$) vertices and a weak c.p. τ then for every integer ℓ, $3 \leq \ell \leq 4n$, G has an ℓ–cycle.

Proof. (By induction on the number of cycles in τ.) Suppose $\tau = \tau_1\tau_2 \ldots \tau_m$ where $\tau_i < \tau_{i+1}$, $1 \leq i \leq m-1$. For $m = 1$ the result follows from Theorem 3.2.2. So let $m \geq 2$.

<u>Case I</u>. Every cycle except the unique cycle of length $4n'+2$ ($n' < n$) which is not τ_1, is of length 4.

Then the induced subgraph $G' = G(\tau\backslash\tau_1)$ is an a.s.c. graph with $4n-2$ vertices whose weak c.p. has $m-1$ cycles. So by induction, G' and hence G has cycles of length ℓ, $3 \leq \ell \leq 4n-4$.

<u>Case I(i)</u>. Suppose τ_2 is of length 4. Note that the above $(4n-4)$–cycle in G' contains all the vertices of the cycle τ_2 in a single segment. So cycles of lengths $4n-3$, $4n-2$, $4n-1$ and $4n$ in G can be constructed by suitably inserting one, two, three and all four vertices of τ_1 respectively into the segment containing all vertices of τ_2 in the above $(4n-4)$–cycle in G.

<u>Case I(ii)</u>. Suppose τ_2 is the unique cycle of length $4n'+2$. Note that the above $(4n-4)$–cycle in G' contains as a segment a hamiltonian path of $G(\tau_2)$ whose end vertices are a pair of consecutive even vertices, say $\tau_{2,2r}$ and $\tau_{2,2r+2}$. Now take paths of lengths $4n'+2$, $4n'+3$, $4n'+4$, $4n'+5$ with vertices from $G(\tau_1 \cup \tau_2)$ whose end vertices are $\tau_{2,2r}$ and $\tau_{2,2r+2}$ (such paths exist by Corollary 3.3.1.1). Then inserting these paths in place of the hamiltonian path of $G(\tau_2)$ in the above $(4n-4)$–cycle of G' one gets cycles of lengths $4n-3$, $4n-2$, $4n-1$ and $4n$ in G.

<u>Case II</u>. τ has cycles not satisfying the condition of case I.

Consider the induced subgraph $G' = G(\tau\backslash\tau_m)$ of G and suppose $|V(G')| = p'$. If G' is an a.s.c. graph, then, by induction or otherwise by Theorem 3.1.3., G' and hence G has ℓ–cycles, $3 \leq \ell \leq p'-2$. Also by Corollary 3.3.1.1. if G' is a.s.c. and by Theorem 3.1.2 if G' is s.c., G' has paths of lengths

$p'-4$, $p'-3$, $p'-2$, and $p'-1$ whose end vertices are $\tau_{m-1,2r}$ and $\tau_{m-1,2r+2}$ (a pair of consecutive even vertices of τ_{m-1}). Choose a pair of consecutive odd vertices $\tau_{m,2s+1}$ and $\tau_{m,2s+3}$ of τ_m such that $(\tau_{m-1,2r}, \tau_{m,2s+1})$ and $(\tau_{m-1,2r+2}, \tau_{m,2s+3}) \in E(G)$. Now adjoining the path $(\tau_{m-1,2r}, \tau_{m,2s+1}, \tau_{m,2s+3}, \tau_{m-1,2r+2})$ to the above paths of lengths $p'-4$, $p'-3$, $p'-2$ and $p'-1$, one gets cycles of lengths $p'-1$, p', $p'+1$ and $p'+2$ respectively in G. If τ_m has four vertices then we are done. If τ_m is of length ≥ 6 then we take a 4–cycle in $G(\tau_m)$ containing a pair of consecutive odd vertices appearing consecutively. Notice that this pair of consecutive odd vertices can be decided arbitrarily so that they are adjacent to the fixed pair of consecutive even vertices $\tau_{m-1,2r}$ and $\tau_{m-1,2r+2}$, in the way needed, provided $G(\tau_m)$ is a s.c. graph. Otherwise, the pair of consecutive even vertices of τ_{m-1} at the ends of the paths of lengths p–i, $1 \leq i \leq 4$ in G′ can be suitably decided, for in that case $G(\tau_{m-1})$ is a s.c. graph. Now inserting the paths of lengths $p'-2$ and $p'-1$ into the 4–cycle of $G(\tau_m)$ suitably one gets $(p'+3)$ and $(p'+4)$–cycles respectively in G. This process can be repeated till all the required cycles in G are obtained. ∎

<u>Theorem 3.4.2</u>. Suppose G is an a.s.c. graph with $p \geq 7$ vertices and a weak c.p.. Then for every integer ℓ, $3 \leq \ell \leq p-2$, G has an ℓ–cycle.

<u>Proof</u>. Suppose $V(G) = \{1,2,3,\ldots,p\}$ and τ is a weak c.p. of G. Then the edges $(i, i+2) \in E(G)$ for all i odd in every cycle of lengths > 2 in τ. If $p = 4n+2$, i.e., p is even, then the result follows from Theorem 3.4.1. So take $p = 4n+3$. In this case τ has either a unique cycle of length 1 or a unique cycle of length of 3. In the first case, the induced subgraph G′ of G with 4n+2 vertices obtained by excluding the vertex, say p, representing the unique cycle of length 1 in τ, is an a.s.c. graph. Then by above theorem, G′ and hence G has an ℓ–cycle for every ℓ, $3 \leq \ell \leq 4n$.

Now we need only show the existence of a (4n+1)–cycle in G. Suppose $\tau' = \tau_1\tau_2...\tau_m$ is a weak c.p. of G′ such that $\tau_i < \tau_{i+1}$ for all i, $1 \leq i \leq m-1$ and $\tau = \tau'(p)$ where (p) denotes the unique cycle of length 1 with vertex labelled p. Then G′ has a 4n–cycle, say C, in which a pair of consecutive odd vertices of τ_1 appear consecutively. Also some even vertex of τ_i and some odd vertex of τ_{i+1} appear consecutively in C for $1 \leq i \leq m-1$. Further note that the vertex p of the unique 1–cycle in τ is adjacent to either all even vertices or all odd vertices of any other cycle of τ. If the vertex p is adjacent to either all odd vertices of τ_1 or, all even vertices of τ_i and all odd vertices of τ_{i+1} for some i, $1 \leq i \leq m-1$, then the vertex p can be inserted suitably into C to get a (4n+1)–cycle in G. Suppose the vertex p is adjacent to all even vertices of every other cycle of τ. Take a path of length 4n–1 of G′ (using Corollary 3.3.1.1) whose end vertices are consecutive even vertices of τ_m and join these ends to the vertex p to get a (4n+1)–cycle in G.

Next, suppose τ has a unique cycle, say $\tau' = (x_1\ x_2\ x_3)$, of length 3. Then τ has no other cycle of odd length (Lemma 2.2.1). Take the induced s.c. subgraph $G_1 = G(\tau\backslash\tau')$ with 4n vertices and a c.p. $\tau\backslash\tau' = \tau_1\tau_2...\tau_m$ such that $\tau_i < \tau_{i+1}$ for all i, $1 \leq i \leq m-1$. Then by Theorem 3.1.2, G_1, and hence G, has ℓ–cycles, $3 \leq \ell \leq 4n-2$. Now to construct cycles of lengths 4n–1, 4n, 4n+1 in G, take a hamiltonian path P (say) following Theorem 3.1.1. in G_1 and the induced a.s.c. subgraph $G_2 = G(\tau')$ where $V(G_2) = \{x_1, x_2, x_3\}$, $E(G_2) = \{(x_1, x_2)\}$ and (x_1, x_3) is the missing edge. Note that all vertices of G_2 in this case are adjacent to precisely all the vertices in a complementary half of the vertices in every cycle of $\tau\backslash\tau'$. Then we have the following different possibilities:

(i) If every vertex in G_2 is adjacent to all odd vertices of τ_1 and τ_m, then delete both even vertices of τ_m at the ends of P. Join the ends of the resulting path of length 4n–3 to x_2 to get a cycle C (say) of length 4n–1 in G. Then inserting the vertex x_1 or x_2, and the edge (x_1, x_2) between the two consecutive odd vertices of τ_1 appearing consecutively in C one gets cycles of lengths 4n and 4n+1 in G respectively.

(ii) If every vertex of G_2 is adjacent to all even vertices of τ_i and all odd vertices of τ_{i+1} for some i, $1 \leq i \leq m-1$ then take a cycle C of length 4n–2 obtained by deleting the ends of P and joining the ends of the resulting path. Note that an even vertex of τ_i and an odd vertex of τ_{i+1} appear consecutively at two different places in C. So inserting one, two and three vertices of G_2 at these two places in C suitably one gets cycles of lengths 4n–1, 4n and 4n+1 respectively in G.

(iii) If every vertex of G_2 is adjacent to all odd vertices of τ_i, $1 \leq i \leq m-1$, and all even vertices of τ_m then take a cycle C of length 4n–2 in G as in (ii) above. From C one obtains cycles of lengths 4n–1 and 4n by inserting any one vertex and the edge (x_1, x_2) in G_2 respectively between the pair of consecutive odd vertices of τ_1 in C. Also a cycle of length 4n+1 is obtained by joining the ends of P to any one vertex of G_2.

(iv) If every vertex of G_2 is adjacent to all even vertices of every cycle in $\tau \backslash \tau'$ then a cycle of length 4n+1 is obtained as in (iii). Next, delete two vertices from one of the ends of P. Then joining the ends of the resulting path with any one vertex and with the ends of the only edge in G_2 one gets in G cycles of lengths 4n–1 and 4n respectively. ∎

The result in the above theorem can also be extended to the a.s.c. graphs with strong c.p.. Combining the two we have the following theorem.

Theorem 3.4.3. Every a.s.c. graph with $p \geq 10$ vertices has ℓ-cycles, $3 \leq \ell \leq p-2$.

Proof. In view of Theorem 3.4.2 and Remark on page 9, we need only to prove the result for connected a.s.c. graphs with strong c.p..

Suppose G is a connected a.s.c. graph with $p \geq 10$ vertices and a strong c.p. τ. Then G has an induced s.c. subgraph G′ with p–2 vertices and so, by Theorem 3.1.3, G′, and hence G, has ℓ-cycles for $3 \leq \ell \leq p-4$. Let us order the cycles of lengths ≥ 4 in τ as $\tau_1 < \tau_2 < \ldots < \tau_m$. Then, by Corollary 3.3.1.1, $G' = G(\tau_1, \tau_2, \ldots, \tau_m)$ has a cycle, say C, of length p–4 in which a pair of consecutive odd vertices of τ_1 and also a pair of consecutive odd vertices of τ_m appear consecutively. Also for any i, $1 \leq i \leq m-1$, this cycle C has some even vertex of τ_i and some odd vertex of τ_{i+1} appearing consecutively exactly twice.

Case I. Suppose $p = 4n+2$ and τ has a unique cycle of length 2 with vertices, say x and y. In order to keep G connected, assume that either both the vertices x and y of the unique 2–cycle of τ are adjacent to all the vertices of a complementary half of the vertices of at least one cycle τ_i for some i, $1 \leq i \leq m$ or the vertex x is adjacent to all vertices of one cycle of τ and the vertex y is adjacent to all vertices of a different cycle of τ. We, then, consider the following:

(i) If x and y are both adjacent to some even vertex of τ_i and some odd vertex of τ_{i+1} for some i, $1 \leq i \leq m-1$, then these two vertices can be inserted into C suitably to get a (4n–1)–cycle and a 4n–cycle in G.

(ii) If x and y are adjacent to only odd vertices of every cycle τ_i, $1 \leq i \leq m$, then one can also get a (4n–1)– and a 4n–cycle in G by inserting x and y into C.

(iii) If x and y are adjacent to only even vertices of every cycle τ_i then, by Theorem 3.1.1, take a hamiltonian path P in G′ containing a pair of consecutive odd vertices of τ_1 appearing consecutively and whose end vertices are consecutive even vertices of τ_m. Deleting the adjacent pair of consecutive odd vertices of τ_1 from P and joining the resulting segments by x and y suitably one gets a 4n–cycle in G. Again, deleting a pair of consecutive vertices from one end of P and then joining the ends of the resulting path to either x or y one gets a (4n–1)–cycle in G.

(iv) If x and y are both adjacent to all odd vertices of every cycle τ_i, $i < m'$ for some $m' < m$ and to all even vertices of the cycles τ_j, $m' \leq j \leq m$, then a (4n–1)–cycle in G is same as that in (i) above. Next, take hamiltonian paths P_1 and P_2 in $G(\tau_1\tau_2 \ldots \tau_{m'-1})$ and $G(\tau_{m'}, \tau_{m'+1} \cdots \tau_m)$, respectively and delete the two even vertices at the ends of P_1. Then the two ends of P_2 and those of the truncated P_1 can be joined to x and y separately to get a 4n–cycle in G.

For the second possibility the verification of the result is straightforward.

Case II. Suppose $p = 4n+3$. Then τ has a unique cycle of length 2 and another unique cycle of length 1. So as in the previous case one can have cycles of lengths p–3 and p–2 in G.

Case III. Suppose $p = 4n+2$ and τ has two cycles of length 1 each. Denote by G′ the maximal induced s.c. subgraph of G obtained by deleting the vertex representing one of the two cycles of length 1 in τ. Then G′, and hence G, has ℓ–cycles, $3 \leq \ell \leq 4n-1$ (since $n > 1$). Finally, a 4n–cycle in G can be constructed by considering different possibilities of adjacencies of the deleted vertex.

□

From this theorem and with a direct checking for a.s.c. graphs with 6 and 7 vertices, we have

Corollary 3.4.3.1. There is no nontrivial a.s.c. bipartite graph.

Theorem 3.4.4. Every hamiltonian a.s.c. graph with a c.p. is pancyclic.

Proof (by contradiction). Suppose G is an a.s.c. graph with $V(G) = \{1,2,3,...,p\}$ and a hamiltonian cycle $C = (1,2,3,4,...,p)$. Then by Theorem 3.4.3, G has ℓ-cycles, $3 \le \ell \le p-2$. So it remains to show the existence of a (p–1)–cycle in G. Assume that G, and hence $\bar{G}$, has no (p–1)–cycle. This implies that the edges $(i, i+2) \in E(\bar{G})$ for all i except possibly one edge which may be the missing edge e.

Case I: p odd.

Clearly $\bar{G}$ has a hamiltonian path irrespective of the situation whether an edge (i, i+2) is the missing edge e or not, for some i. Let us take such a hamiltonian path

$$i_0+2,\ i_0+4,\ \dots,\ p+i_0-1,\ i_0+1,\ i_0+3,\ \dots,\ p+i_0-2,\ i_0$$

(where the vertices are to be taken residues modulo p) in $\bar{G}$ in which (i_0, i_0+2) may be the missing edge e for some fixed i_0, $1 \le i_0 \le p$. Then $(i_0, i_0+4) \notin E(\bar{G})$ for otherwise

$$\left[i_0+4,\ i_0+6,\ \dots,\ p+i_0-1,\ i_0+1,\ i_0+3,\ \dots,\ p+i_0-2,\ i_0,\ i_0+4\right]$$

is a (p–1)–cycle in $\bar{G}$ — a contradiction. We can choose one i_0 such that (i_0, i_0+4) is not the missing edge. Then $(i_0, i_0+4) \in E(G)$. This implies that none of the edges $(i_0+2, p+i_0-1)$, (i_0+2, i_0+5), $(i_0+3, p+i_0-2)$ can be in G, for the presence of any of these in G would give a (p–1)–cycle in G. Note that the vertices i_0+5, i_0+6, $p+i_0-1$, $p+i_0-2$ are all distinct if $p > 7$. Again at most one of the above four edges can be the missing edge. So assume $p > 7$ and (i_0+1, i_0+6) and $(i_0+2, p+i_0-1) \in E(\bar{G})$. Then

$$\left[p+i_0-1,\ p+i_0-3,\ \ldots,\ i_0+6,\ i_0+1,\ i_0+3,\ \ldots,\ p+i_0-2,\ i_0,\ i_0+2,\ p+i_0-1\right]$$

is a (p–1)–cycle in $\bar{G}$ (missing the vertex i_0+4), which is a contradiction. In case $p = 7$ and (1,2,3,4,5,6,7) is a hamiltonian cycle in G, neither (1,5) nor (3,6) is in $E(\bar{G})$. Also at least one of these two edges is different from e. Suppose $(1,5) \neq e$. Then neither (3,6) nor (3,7) is in E(G). So at least one of (3,6) and (3,7) is in $E(\bar{G})$, which immediately gives a 6–cycle in $\bar{G}$, a contradiction. Hence G must have a (p–1)–cycle.

<u>Case II</u>: p even

<u>Claim</u>. $(i, i+j) \notin E(G)$ for all i,j; $1 \leq i,j \leq p$ and j even.

Suppose (i, i+2) is the missing edge e for some fixed odd vertex $i = i_0$. Then

$$i_0,\ p+i_0-2,\ p+i_0-4,\ \ldots,\ i_0+4,\ i_0+2$$

is at least a path in $\bar{G}$ containing all odd vertices, and the vertices

$$i_0+1,\ i_0+3,\ \ldots,\ p+i_0-1,\ i_0+1$$

from a cycle in $\bar{G}$ containing all even vertices of $\bar{G}$. Suppose $(i_0, i_0+j) \in E(G)$. Then none of the edges $(i_0+2, p+i_0-1)$, (i_0+2, i_0+j+1), (i_0+1, i_0+j+2) and $(i_0+j-1, p+i_0-2)$ is in E(G), for the presence of any such edge in G gives a (p–1)–cycle in G, which is not possible. So at least three of these edges are in $\bar{G}$. However, one can always choose either the edges (i_0+2, i_0+j+1), $(i_0+j-1, p+i_0-2)$ or the edges $(i_0+2, p+i_0-1)$, (i_0+1, i_0+j-2) to be in $\bar{G}$. For both the cases $\bar{G}$ contains a (p–1)–cycle, a contradiction. It may be noted that if no (i, i+2) is the missing edge e, that the above checking may be started with any odd i; otherwise with that odd i for which e = (i, i+2). So, by symmetry of odd and even vertices, the above claim is established. But then G is bipartite, which is a contradiction.

∎

CHAPTER 4

SOME OTHER RESULTS ON a.s.c. GRAPHS

4.1. Degree sequence of an a.s.c. graph

Suppose the p nonnegative integers $d_1, d_2, ..., d_p$ denote the degrees of the vertices of an a.s.c. graph G with p vertices. Then, as usual, we call the sequence, $\pi = (d_1, d_2, ..., d_p)$, the *degree sequence* of G. Recall that the results following the construction method in Chapter 2 imply some structural properties of an a.s.c. graph in terms of its degree sequence. Here again, we discuss some more results involving the degree sequence of the said graph. To start with, the following are a few simple observations.

Lemma 4.1.1. If G is a quasi regular a.s.c. graph with 4n+2 vertices and a weak c.p. then the degree sequence of G realizes an a.s.c. graph with a strong c.p. and conversely, provided n is even.

Proof. The degree sequence of G, written in a nondecreasing order, is of the form $d_1 = d_2 = ... = d_{2n} = 2n+1$; $d_{2n+1} = ... = d_{4n+2} = 2n$. Then the degree sequence

$$d_1-2,\ d_2-2,\ ...,\ d_{2n}-2,\ d_{2n+3},\ d_{2n+4},\ ...,\ d_{4n+2} \tag{1}$$

satisfies the suitability condition (Clapham and Kleitman [13]) for a s.c. graph with 4n vertices, and also is clearly graphical since G is already a graph. So the degree sequence (1) realizes a quasi regular s.c. graph, say G'. Then by taking two new vertices, not in G', of degree $d_{2n+1} = d_{2n+2} = 2n$ each and joining both to the vertices of degree $d_1-2, d_2-2, \ldots, d_{2n}-2$ in G' one gets an a.s.c. graph with a strong c.p. containing a unique cycle of length 2 or two 1–cycles.

For the converse, let us label the vertices as $v_1, v_2, \ldots, v_{4n+2}$ such that the vertices with the first n odd subscripts have degree 2n+1 each, the vertices with the remaining odd subscripts have degree 2n each, the vertices with the first n even subscripts together with the last vertex v_{4n+2} have degree 2n each and each of the remaining vertices with the even subscripts has degree 2n+1. Now it can be easily checked that the given degree sequence of a quasi–regular a.s.c. graph with a strong c.p. also realizes an a.s.c. graph with a weak c.p. $(v_1v_2 \ldots v_{4n+2})$.

Further, since all the edges of this graph joining the vertices with subscripts of the same parity are divided into two equal halves, one joining only odd subscript vertices and the other joining only even subscript vertices, the requirement of n being even is necessary. ∎

In Chapter 2, we already had the simple observation that a given a.s.c. graph may have more than one complementing permutation and a given complementing permutation may produce more than one a.s.c. graph. But the above result implies that a given degree sequence may produce different a.s.c. graphs depending upon whether the degree sequence is taken as one with a strong c.p. or with a weak c.p.. However, this result fails when the number of vertices is odd. This is due to the fact that there is no quasi regular a.s.c. graph with an odd number of vertices and a weak c.p..

Lemma 4.1.2. If G is a quasi regular a.s.c. graph with 4n+3 vertices and a strong c.p. then the degree sequence of G does not realize an a.s.c. graph with a weak c.p.

Lemma 4.1.3. If G is a quasi regular graph with 4n+2 vertices and a weak c.p. τ then the induced subgraph on the vertex set of the unique cycle of length 4n′+2 (n′ $\leq$ n) is also quasi regular, provided n′ is even and the length of no other cycle in τ is divisible by 4n′+2.

Proof. Consider the induced subgraph G′ of G on the vertex set of the unique cycle of length 4n′+2 in τ. If n = n′ then G = G′, and there is nothing to prove. So take n′ < n. If G′ is not quasi regular then it has vertices of four different degrees (Theorem 2.3.1(b)). This induces a partition of V(G′) into four different sets as A, B, C and D of cardinality n′, n′+1, n′ and n′+1 respectively, in which the degree of any vertex in the sets is 4n′+1–r′, 4n′–r′, r′+1 and r′(n′ $\leq$ r′ $\leq$ 2n′) respectively, when restricted to G′ (Lemma 2.2.3). Now looking back to G, every vertex of G\G′ is adjacent to exactly a complementary half of the vertices (all of A and B, or all of C and D) of G′. Notice that in the process of getting G from G′ the degrees of the vertices in C and D can be equal to those in A and B respectively, provided 4n′–2r′ $\equiv$ 0 (mod 4), i.e., r′ is even. But r′ is even only when n′ is even, for r′ = n′+an even integer ($\geq$0) (by Theorem 2.3.1(b)). Thus the vertices of G′ must remain as vertices of four different degrees in G, when n′ is odd. This implies that G is not quasi regular, which contradicts the hypothesis. ∎

Now we are all set for the main result of this section dealing with the construction of a nontrivial a.s.c. graph from a given degree sequence satisfying prescribed necessary conditions for a.s.c. realization.

The degree sequence $\pi = (d_1, d_2, \ldots, d_p)$, $d_i \geq d_{i+1}$ for $1 \leq i \leq p-1$, of every a.s.c. graph with p vertices satisfies the following necessary conditions (cf. Lemma 2.3.2).

(A) If the number of vertices is $p = 4n+2$ then

(1) (i) $d_i + d_{4n+3-i} = 4n+1$, for $i = 1, 2, \ldots, m-1, m+1, \ldots, 2n+1$

(ii) $d_m + d_{4n+3-m} = 4n$,

and

(2) if m is odd

(i) $d_{2j} = d_{2j-1}$, for $j = 1, 2, \ldots, \lfloor m/2 \rfloor$

(ii) $d_{2j} = d_{2j+1}$, for $j = \lfloor m/2 \rfloor + 1, \ldots, n$

(iii) $d_m = d_{m+1}$

and if m is even

(i) $d_{2j} = d_{2j-1}$ for $j = 1, 2, \ldots, (m/2)-1$

(ii) $d_{2j} = d_{2j+1}$ for $j = m/2, (m/2)+1, \ldots, n$

(iii) $d_{m-1} = d_m + 1$.

(B) If the number of vertices is $p = 4n+3$ then

(1) (i) $d_i + d_{4n+4-i} = 4n+2$, for $i = 1, 2, \ldots, m-1, m+1, \ldots, 2n+2$

(ii) $d_m + d_{4n+4-m} = 4n+1$,

and

(2) if m is odd

(i) $d_{2j} = d_{2j-1}$, for $j = 1, 2, \ldots, \lfloor m/2 \rfloor$

(ii) $d_{2j} = d_{2j+1}$, for $j = \lfloor m/2 \rfloor + 1, \ldots, n$

(iii) $d_m = d_{m+1}$

and if m is even

(i) $d_{2j} = d_{2j-1}$ for $j = 1, 2, \ldots, (m/2)-1$

(ii) $d_{2j} = d_{2j+1}$ for $j = m/2, (m/2)+1, \ldots, n$

(iii) $d_{m-1} = d_m+1$.

Remark. If the unique cycle of length $\ell \equiv 2 \pmod 4$ with $\ell > 2$ also satisfies $\ell \equiv 2 \pmod 8$ or the unique cycle is of length 3 in a weak c.p. then m is odd, otherwise m is even. If there is a unique cycle of length 2 or two cycles of length 1 in the c.p. then m is always odd.

The following Lemma has been proved by Wang and Kleitman for general graphs.

Lemma 4.1.4. [53]. If $\pi = (d_1, d_2, \ldots, d_p)$, $d_1 \geq d_2 \geq \ldots \geq d_p$ is a realizable graphical sequence then the residual sequence, after connecting the vertex of degree d_k to the first d_k vertices in π other than itself, is also graphical.

From the above Lemma, one can easily prove

Corollary 4.1.4.1. A nonincreasing sequence $\pi = (d_1, d_2, \ldots, d_p)$ of positive integers is realizable as the degree sequence of a simple graph if and only if the residual sequence

$$\pi' = (d_1-2, d_2-2, \ldots, d_{d_{j+1}}-2, d_{d_{j+1}+1}-1, \ldots, d_{d_j}-1, d_{d_j+1}, \ldots, d_{j-1}, d_{j+2}, \ldots, d_p)$$

is so, provided the sequence

$$\pi' = (d_1-1, d_2-1, \ldots, d_{d_j}-1, d_{d_j+1}, \ldots, d_{j-1}, d_{j+1}, \ldots, d_p)$$

is nonincreasing.

Theorem 4.1.1. If $\pi = (d_1, d_2, \ldots, d_p)$, $p = 4n+2$, is a nonincreasing sequence of positive integers which is graphical and satisfies the necessary condition for an a.s.c. realization then there is an a.s.c. graph with degree sequence π.

Proof. The proof is divided into three cases:

Case 1. $m = 2n+1$. Then, by (A), $d_{2n+1} = d_{2n+2} = 2n$. Taking $j = m$ in (A), the sequence π satisfies the requirement in the above Corollary, and so, π being graphical, the residual degree sequence

$$d_1-2, d_2-2, \ldots, d_{2n}-2, d_{2n+3}, d_{2n+4}, \ldots, d_{4n+2}$$

is also graphical. But the latter sequence satisfies the suitability condition for s.c. realization, and so realizes a s.c. graph, say G′ (Clapham and Kleitman [13]). Then, by taking two new vertices of degree $d_{2n+1} = d_{2n+2} = 2n$ and joining both to all the vertices of degree d_i-2, $1 \leq i \leq 2n$, one obtains an a.s.c. graph.

Case 2. $m = n+1$ and π contains only four distinct integers $\geq n$. In this case the degrees are $d_1 = d_2 = \ldots = d_n = 4n+1-r$, $d_{n+1} = \ldots = d_{2n+1} = 4n-r$, $d_{2n+2} = \ldots = d_{3n+1} = r+1$ and $d_{3n+2} = \ldots = d_{4n+2} = r$, where $n \leq r \leq 2n$. Rearrange the second half of the sequence π so that the sum of each pair of numbers equidistant from the beginning and the end, excepting the middle pair, is 4n+1 and that of the middle pair is 4n. Then label the first 2n+1 vertices by 1, 2, ..., 2n+1 starting from the beginning and the rest 2n+1 vertices by 1′, 2′, 3′, ..., (2n+1)′ starting from the end. With this labelling the following procedure gives the required graph.

(i) Join vertex pairs with labels i and (i+j)′ by an edge for each i = 1, 2, ..., 2n+1, and j = 0, 1, ..., n–1.

(ii) Join the vertex pairs labelled i and (i+n)′ by an edge for i = 1, 2, ..., n.

After these connections, the remaining degree sequence is

$$\underbrace{3n-r,\ 3n-r,\ \ldots,\ 3n-r}_{2n+1},\ \underbrace{r-n,\ r-n,\ \ldots,\ r-n}_{2n+1}$$

(iii) Join the vertex pairs with labels i and i+j by an edge for all i = 1, 2, ..., 2n+1, and j = 1, 2, ..., 3n–r.

This completes the construction if r = n.

(iv) If r > n then join each of the vertex pairs labelled i′ and (3n–r+i+j)′ by an edge for all i = 1, 2, ..., 2n+1 and j = 1, 2, ..., r–n,

where the numbers are to be taken residues modulo 2n+1.

Case 2. (Alternative method) (cf. [13]).

Arrange the degree sequence in nonincreasing order. Then

1. connect the last vertex with degree d_p to the first d_p vertices of maximum degrees and then reduce the degrees accordingly.
2. connect the first vertex with degree d_1-1 of the reduced degree sequence to the first d_1-1 vertices of maximum degrees, and then reduce the degrees accordingly.

Continue this process from the end and the beginning alternately till all the degrees are exhausted. This process ultimately produces an almost s.c. graph with degree sequence π.

Case 3. Cases different from Case 1 and Case 2 above, i.e., when there is a unique cycle τ' of length $4n'+2$, $(n' < n)$ in τ. Note that the part of the degree sequence π corresponding to the degrees in τ' may appear in one segment or in two different segments of equal lengths in π. In either of these cases (since both differ from case 1), the subsequence of π corresponding to τ' must have four different degrees as $4n+1-r$, $4n-r$, $r+1$ and r such that the pairs of vertices with degree sum $p-2$ are equidistant from the beginning and the end, and are of degrees $4n-r$ and r, where $r \geq n'$. Moreover, these degrees appear at the middle of the two equal segments due to τ'. This enables one to recognize the part(s) of the degree sequence π corresponding to τ'. Now, by Theorem 2.3.1(b), to construct an a.s.c. graph in the present case we consider the following subcases:

Subcase (i). Suppose the degrees corresponding to the vertices in τ' are at the beginning and the end in π, i.e.,

$$d_1 = \dots = d_{n'} = 4n+1-r,$$
$$d_{n'=1} = \dots = d_{2n'+1} = 4n-r,$$
$$d_{p-2n'} = d_{p-2n'+1} = \dots = d_{p-n'+1} = r+1,$$
$$d_{p-n'} = d_{p-n'+1} = \dots = d_p = r.$$

Denote $\lfloor (r-n')/2 \rfloor$ by s, which is ≥ 0. Connect each of the last $2n'+1$ vertices to the first 2s vertices by an edge and also connect each of the first $2n'+1$ vertices to the first $4(n-n')(2n'+1)$ vertices by an edge, in both cases starting with the vertex of degree $d_{2n'+1}$. Then the sequence with the reduced degrees is divided into two parts, one consisting of the first $2n'+1$ and the last $2n'+1$ degrees in π giving an a.s.c. subgraph (by Case 2) with a weak c.p. τ' and the other giving a s.c. subgraph with $4(n-n')$ vertices by the method in Clapham and Kleitman [13].

Subcase (ii). Suppose π does not begin with degrees corresponding to τ'. Then a s.c. subgraph with vertices of degrees d_1, d_2, d_{p-1}, and d_p together with the connections of these vertices with the rest of the vertices can be constructed by the method in Clapham and Kleitman [13]. Then delete these four vertices and repeat the process till we reach subcase (i), i.e., the reduced degree sequence begins with the degrees corresponding to τ'. ∎

It may be remarked that the above construction procedure could have been simplified further provided the induced a.s.c. subgraph, with $4n'+2$ vertices and a weak c.p. consisting of a single cycle, would have another weak c.p. consisting of more than one cycle. In this sense it is worth mentioning that an a.s.c. graph G, with 10 vertices and a weak c.p. consisting of a single cycle, does not have a weak c.p. consisting of more than one cycle. We believe that this situation for an a.s.c. graph may be checked with the help of a computer.

Next, the case of odd p $(= 4n+3, n \geq 1)$ is simple since deleting the degree d_{2n+2} and reducing each degree preceding d_{2n+1} in π by 1, the new degree sequence satisfies conditions (A). So one can first construct an a.s.c. graph G′ with the new degree sequence and 4n+2 vertices. Then taking an additional vertex not in G′ and joining it to each of the first d_{2n+1} vertices of maximum degree in G′ one gets an a.s.c. graph with p vertices.

4.2. Triangles in a.s.c. graphs

In 1958, Bostwick [5] posed the following problem:

Prove that at a gathering of any six people, some three of them are either mutual acquaintances or complete strangers to each other.

Goodman [23] considered this problem in a more general setting and proved

Theorem 4.2.1. (Goodman) In any 2–edge colouring of a complete graph K_p, the number of monochromatic triangles is at least

$$\begin{cases} \frac{1}{3}n(n-1)(n-2) & \text{, if } p = 2n, \\ \frac{2}{3}n(n-1)(4n+1) & \text{, if } p = 4n+1, \\ \frac{2}{3}n(n+1)(4n-1) & \text{, if } p = 4n+3, \end{cases}$$

and this number is best possible.

Clapham [12], using the above result, determined the minimum number of triangles in a s.c. graph. Here again through Goodman's result we determine the lower bound on the number of triangles in an a.s.c. graph. For this, suppose t(G) denote the number of triangles in a graph G.

<u>Theorem 4.2.2</u>. (a) If G is an a.s.c. graph with $p > 3$ vertices and a weak c.p. then

$$t(G) \geq \begin{cases} \dfrac{8n^3 - 4n+3}{6} & , \text{if } p = 4n+2, \\ \dfrac{n}{3}(4n^2+3n-4) & , \text{if } p = 4n+3; \end{cases}$$

(b) If G is an a.s.c. graph with p vertices and a strong c.p. then

$$t(G) \geq \begin{cases} \dfrac{4n}{3}(n-1)(n+1) & , \text{if } p = 4n+2, \\ \dfrac{n}{3}(4n^2+3n-4) & , \text{if } p = 4n+3; \end{cases}$$

and these results are best possible.

<u>Proof (a)</u>. Note that $t(G) = t(\bar{G})$ for an a.s.c. graph G, where $\bar{G}$ is the restricted complement of G (cf. the definition of a.s.c. graph). So colouring the edges of G and $\bar{G}$ blue and red respectively, the number of blue triangles is equal to the number of red triangles in $K_p - e = G \cup \bar{G}$, where e is the missing edge. First take p even. Then, by Corollary 2.3.1.2, joining the ends of the missing edge in G (or in $\bar{G}$) with a blue (or a red) edge increases the number of triangles in G (or in $\bar{G}$), and hence in K_p, by n provided the corresponding c.p. (say τ) consists of

exactly one cycle. Since τ may consist of more than one cycle, let us suppose that the length of the unique cycle τ', not divisible by 4, in τ is $4n'+2$ ($n' \geq 1$). Clearly the ends of the missing edge are in τ' and are adjacent to exactly n' common vertices of τ'. So this number of common adjacencies of the ends of the missing edge can be at most $2n-n'$, for both ends together may be adjacent to at most half of the vertices in every other cycle in τ, i.e., $2(n-n')$ vertices in rest of the cycles of τ. Thus t(G) is at least half the number obtained after subtracting $2n-n'$ from the number in Goodman's theorem. But, in order to make our result independent of n', i.e., independent of the length of τ', we may take τ' to be of minimum possible length which is 6. This gives $n' = 1$ and correspondingly we obtain the bound in our theorem.

Next, for odd p, there are two ways of obtaining an a.s.c. graph with an odd number of vertices and a weak c.p.. One is by adding a new vertex suitably to an a.s.c. graph with p–1 vertices and a weak c.p.. The other is by taking a s.c. graph with p–3 vertices and an a.s.c. graph with three vertices and a weak c.p., and adjoining them suitably. It can be easily checked that the ends of the missing edge are together adjacent to at most 2n–1 (or 2n) common vertices in the first (or second) construction of the graph. So the required bound for t(G) is obtained by taking the second construction.

Then (b) follows from Lemma 2.3.5 and the argument in (a) above.

4.3. Diameter of an a.s.c. graph

For any graph G, the diameter of G, denoted by diam (G), is the maximum of the distances between pairs of vertices of G. The class of s.c. graphs and that of a.s.c. graphs share the same result regarding their diameters. Here we prove the result for a.s.c. graphs.

<u>Theorem 4.3.1</u>. Every nontrivial connected a.s.c. graph has diameter 2 or 3.

<u>Proof</u>. (By contradiction). Suppose G is a nontrivial connected a.s.c. graph. Clearly diam (G) $\neq$ 1, since G is not complete. Take diam (G) $\geq$ 4. Then by definition of a.s.c. graph, and diameter being a graphical invariant, diam $(\tilde{G}) \geq 4$. Let u and v be any pair of vertices in G (and hence in $\tilde{G}$) such that $dist_{\tilde{G}}(u,v) = 4$, i.e., there is a shortest path, say u, x, y, z, v of length 4 in $\tilde{G}$ joining u and v. Then none of the edges (u,y), (u,z), (x,z), (x,v), (y,v) is in $\tilde{G}$. Note that at most one of these edges may be the missing edge e, where $\overline{G} = \tilde{G} + e$. In any case, we get $dist_{\overline{G}}(u,v) > 2$, which is a contradiction because diam(G) $\geq$ 4 implies diam$(\overline{G}) \leq 2$ (see [4]). Hence diam(G) = 2 or 3. ∎

<u>Theorem 4.3.2</u>. For all admissible p $\geq$ 6 and for all D, $2 \leq D \leq 3$, there exists an a.s.c. graph with diameter D.

<u>Proof</u>. Take any a.s.c. graph G′ with p–4 vertices and a (weak/strong) c.p. τ'. Suppose the vertices of G′ are 1,2,..,p–4. Then construct a graph G with

$$V(G) = V(G') \cup \{x,y,z,t\}$$

$$\text{and } E(G) = E(G') \cup \{(x,y), (y,z), (z,t), (x,i), (t,i) : 1 \leq i \leq p-4\}.$$

Then the a.s.c. graph G has a (weak/strong) c.p. $\tau'(yxzt)$ which map G onto $\bar{G}$. Now it is easy to check that diam (G) = 2.

The construction of an a.s.c. graph G of diameter 3 follows by taking edges (y,i) and (z,i) in place of the edges (x,i) and (t,i) above, $1 \leq i \leq p-4$.

∎

CHAPTER 5

k-SELFCOMPLEMENTARY GRAPHS

5.1. Introduction

In the preceding chapters we considered the class of a.s.c. graph which is defined by extending the notion of s.c. graphs. It requires deleting an edge from the complete graph K_p, for suitable p, and then partitioning it into two isomorphic parts. The basic question here is whether it is possible to partition a (not necessarily complete) graph into two isomorphic parts. Thus, examining the concept of s.c. and a.s.c. graphs, we observe that a graph, obtained after deleting some suitable small number of edges from K_p, can always be partitioned into two isomorphic parts. This leads us to the following definition.

Definition. A simple graph G with p vertices is called *k–selfcomplementary* (k–s.c.) if and only if it is isomorphic to its complement $\tilde{G}$ in K_p', where K_p' is obtained by deleting k mutually nonadjacent edges from K_p. The deleted k edges are called the *missing edges* of G.

The following are some simple observations from the above definition.

(1) If $k = 0$ then we have 0–s.c. or simply the s.c. graphs and if $k = 1$ then we have 1–s.c. or the a.s.c. graphs.

(2) Since the k edges being deleted from K_p are mutually nonadjacent, we have $0 \leq k \leq p/2$. Also for any $\lfloor p/2 \rfloor$–s.c. graph with p vertices, the set of missing edges forms a 1–factor or a near 1–factor.

(3) If k is even then in order for a k–s.c. graph with p vertices to exist, the number of edges in K_p must be even. This implies that $p \equiv 0, 1 \pmod 4$.

(4) If k is odd then in order for a k–s.c. graph with p vertices to exist, the number of edges in K_p is odd. This implies that $p \equiv 2,3 \pmod 4$.

<u>Theorem 5.1.1</u>. A k–s.c. graph with p vertices ($0 \leq k \leq p/2$) exists if and only if $p \equiv 0,1 \pmod 4$, provided k is even and $p \equiv 2,3 \pmod 4$, provided k is odd.

<u>Proof</u>. Necessity follows from the above observations (3) and (4).

For sufficiency, we consider the two cases separately.

<u>Case I</u>: k even.

Suppose $k = 2r$. Then take a s.c. graph G' with $p-4r$ vertices, which always exists since $p \equiv 0,1 \pmod 4$ and hence $p-4r \equiv 0,1 \pmod 4$. Also take r 4–vertex 2–s.c. graphs G_i on the vertex set $V_i = \{x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}\}$ each of which contains only a pair of edges (x_{i_1}, x_{i_2}) and (x_{i_3}, x_{i_4}), for $1 \leq i \leq r$. Then join all the vertices of each G_i, $1 \leq i \leq r$, to a complementary half of the vertices of G'. Also join all the vertices of G_i to a complementary half of the vertices of G_j for all $1 \leq i \leq r$ and $j > i$. The graph obtained is a required k–s.c. graph.

<u>Case II</u>: k odd.

Take a (k–1)–s.c. graph G' with p–2 vertices as in Case I. Then take a K_2 and connect both vertices of $\overline{K}_2$, to a complementary half of the vertices of G'. The resulting graph with p vertices is a required k–s.c. graph.

5.2. Complementing permutation

If G is a k–s.c. graph with p vertices then an isomorphism from G onto $\bar{G}$, where $G \cup \bar{G} = K_p - \{e_1, e_2, ..., e_k\}$, can be expressed as a permutation τ of V(G). Such a permutation τ is, as usual, called a *complementing permutation* (c.p.) of G. Further, following the terminology of a.s.c. graphs, a c.p. τ of a k–s.c. graph G is called a *strong* c.p. if $\tau(U) = U$, where U is the set of k missing edges for G. And, if a c.p. τ is not strong then it is called a *weak* c.p.. Also we assume that any (strong/weak) c.p. of a k–s.c. graph can be expressed as a product of disjoint cycles.

Note that in case of a 0–s.c. graph (i.e., a s.c. graph) there is no distinction between a strong and a weak c.p. and thus the notion of a complementing permutation here clearly agrees with the well–known notion of c.p. for s.c. graphs.

Suppose G is a k–s.c. graph with p vertices and a strong c.p. τ. Then, by definition, τ fixes exactly k mutually nonadjacent edges, say $e_1, e_2, ..., e_k$ of G. If T denotes the set of end vertices of the edges $e_1, e_2, ..., e_k$ then $\tau(T) = T$. So $\tau' : G[V\backslash T] \to \overline{G}[V\backslash T]$ is a c.p. of $G[V\backslash T]$ and the induced subgraph $G[V\backslash T]$ of G is a s.c. subgraph with p–2k vertices irrespective of whether k is even or odd, where τ' is the restriction of τ to $G[V\backslash T]$. By the properties of s.c. graphs, the cycle lengths of τ' are multiples of 4 together with at most one cycle of length 1. But the lengths of the cycles of the part $\tau\backslash\tau'$ ($= \tau^*$, say) of the c.p τ may be any combination of the following possibilities:

(i) τ^* may have at most 2 cycles of length 1 each.

(ii) τ^* may have at most k cycles of length 2 each. If it has k cycles of length 2 each then it has no other cycle.

(iii) τ' may have either cycles of lengths divisible by 4 only.

(iv) τ' may have cycles of lengths which are odd multiple of 2 only.

Hence the class of k–s.c. graphs with p vertices with strong c.p. may be divided into three categories.

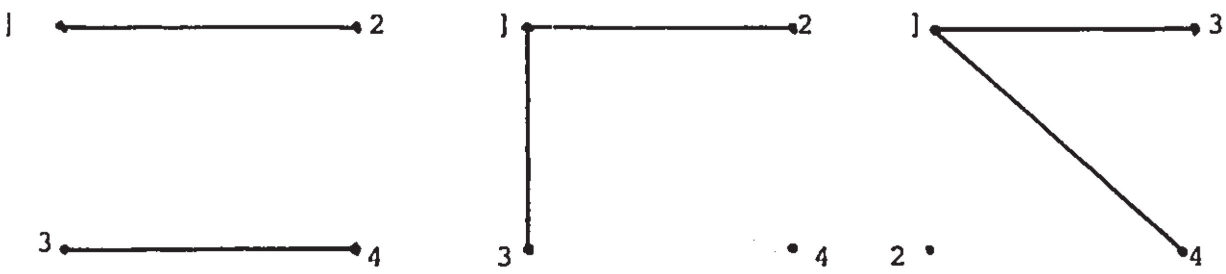
(1) $p = 2k$. Then K_p' is obtained after deleting a 1–factor from K_p and there is no subcomplementing permutation like τ' mentioned above.

(2) $p = 2k+1$. Here K_p' is obtained from K_p after deleting a near 1–factor and so τ' in this case consists of exactly one cycle which is of length 1.

(3) $p > 2k+1$. In this case $p-(2k+1) \geq 4$, for otherwise we can not have a k–s.c. graph. Suppose T is the set of 2k vertices fixed by a strong c.p. τ of a k–s.c. graph G. Then G consists of two parts; one a s.c. graph on the vertex set $V(G)\backslash T$ and the other a k–s.c. graph with the vertex set T. Note that the second part is same as the situation (1) above.

Next, suppose G is a k–s.c. graph with p vertices and a weak c.p. τ. By definition, τ fixes at most (k–1) edges and the cycle structure of τ is dependent upon the number of edges fixed by it. For instance, if τ does not fix any edge then it can neither have a 2–cycle nor more than one 1–cycle. This implies that there is no weak c.p. for a 0–s.c. (or simply s.c.) graph. Similarly, if τ fixes exactly one edge then it can have either exactly one 2–cycle or exactly two 1–cycles. Thus $p > 2k+1$, whenever τ is a weak c.p. of a k–s.c. graph with p vertices.

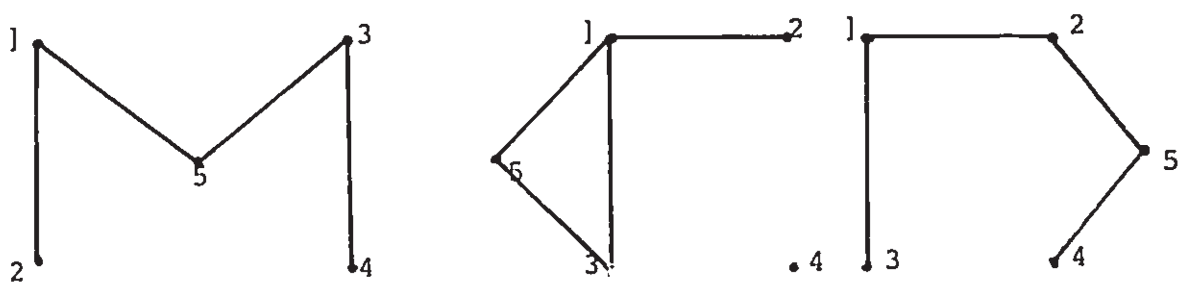
Some examples of k–s.c. graphs along with their (weak\strong) c.p.'s are given below:



Strong c.p. (1234) or (1432) or (13) (2) (4) and missing edges (1,3) (2,4).

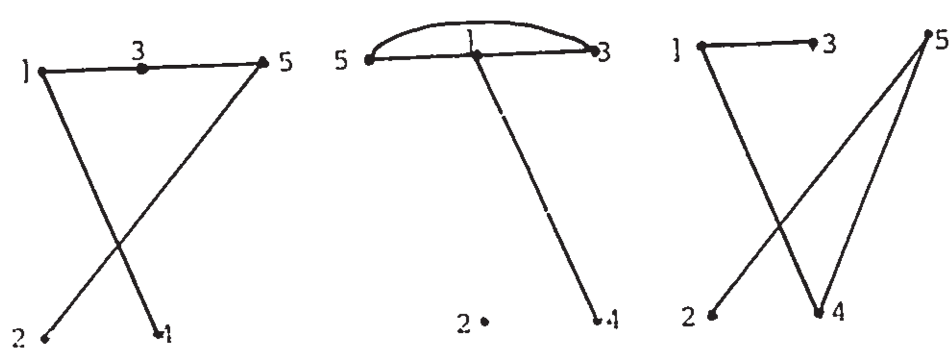
Weak c.p. (1234) and missing edges (3,4), (4,1).

Strong c.p. (12) (34) or (12) (3) (4) and missing edges (1,2), (3,4)



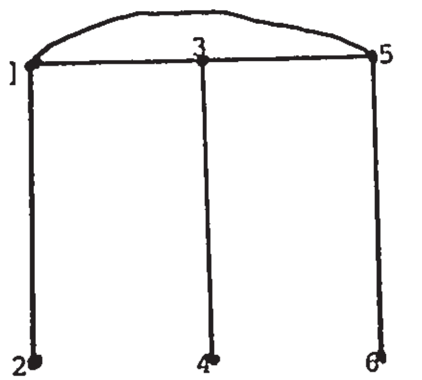
Strong c.p. (1234)(5) or (1432)(5) and missing edges (1,3),(2,4).

Both with weak c.p. (1234)(5) and missing edges (3,4),(4,1).

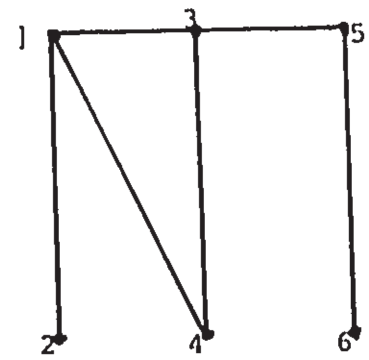


All with strong c.p. (12)(34)(5) and missing edges (1,2), (3,4)

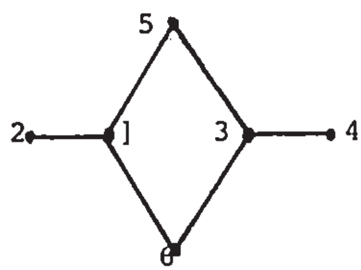
Fig. 5.1. (2–s.c. graphs with 4 and 5 vertices)



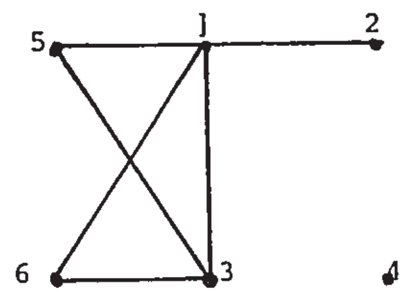
Strong c.p. (123456) and
missing edges (1,4),(2,5),(3,6).



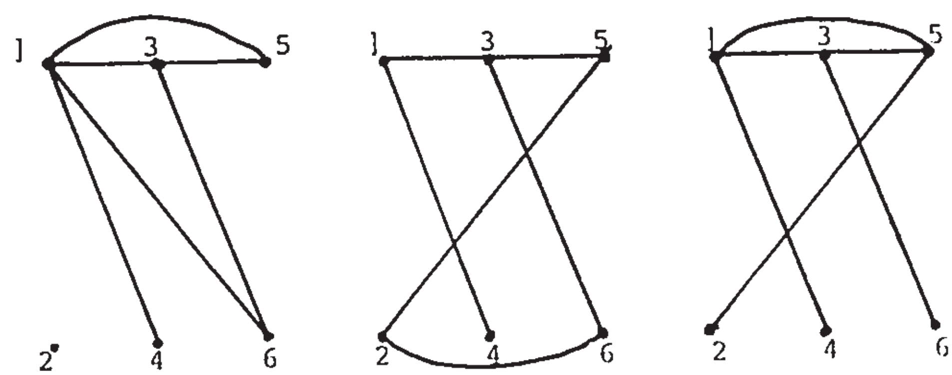
Weak c.p. (123456) and
missing edges (1,5),(2,6),(3,6).



Strong c.p. (1234)(56) and
missing edges (1,3),(2,4),(5,6).



Weak c.p. (1234)(56) and
missing edges (3,4),(4,1),(5,6).



All with strong c.p. (12)(34)(56) and
missing edges (1,2),(3,4),(5,6).

Fig. 5.2 (3–s.c. graphs with 6 vertices)

The following are some of the elementary properties of a (strong/weak) c.p. τ for a k–s.c. graph G with p vertices.

(a) τ has no cycle of odd length $> 2k+1$ provided τ is a weak c.p., otherwise it has no odd cycle of length > 1.

(b) The number of 1–cycles in a strong c.p. τ is even or odd according as p is even or odd.

(c) The set of vertices in any subset of cycles of τ induces a k′–s.c. subgraph of G where $0 \leq k' \leq k$.

(d) If τ contains some 1–cycles then G can not be extended to another k–s.c. graph with p+1 vertices. On the other hand, a k–s.c. subgraph with p–1 vertices can be obtained from G if and only if τ has exactly one 1–cycle.

(e) If τ is a strong c.p. then every cycle of length 2i in τ fixes either exactly i edges or no edge at all. (This is due to the definition of strong c.p..)

(f) If τ is a weak c.p. then it has at least one cycle of length 2i ($i > 1$) which fixes j edges, where $0 < j < i$.

It is clear from the definition and examples of k–s.c. graphs that such graphs may be disconnected. Also the disconnectedness is due to the presence of 2–cycles in the c.p. Thus in case we wish to restrict the discussion to connected k–s.c. graphs then an assumption like the one for almost s.c. graphs (Chapter 2) is necessary.

5.3. <u>Construction of a k–s.c. graph</u>

Here we describe a method of constructing a k–s.c. graph with a given (strong/weak) c.p.. First take a strong c.p. τ with p elements to construct a k–s.c. graph with the c.p. τ. By definition of strong c.p., $p \geq 2k$. Suppose $\tau = \tau'\tau''\tau'''$, where τ' is a product of all 1–cycles (if any), τ'' is the product of all 2–cycles (if any) and τ''' is the product of all cycles of lengths > 2 in τ. We assume that each of τ', τ'' and τ''' contains at least one cycle for otherwise the method of construction discussed below can be modified suitably.

Suppose that τ', τ'' and τ''' contain respectively $p' = t+1$, p'' and $p - p'' - t - 1$ elements, where $t = 0$ or 1 according as τ contains one or two 1–cycles. Note that $k \geq t+(p''/2)$. Take $k' = k-t-(p''/2)$. Now τ being a strong c.p., τ''' contains a product τ_1''' of cycles with exactly $2k'$ elements together. Take $\tau''' = (\tau_1''' \; \tau_2''')$, where $|\tau_2'''| \geq 0$ and τ_2''' is a product of those cycles of length > 2 in τ which have no edge in the set of missing edges. Also suppose $\tau_1''' = \left[\tau_{1n_1} \tau_{1n_2} \cdots \tau_{1n_m}\right]$ where $|\tau_{1_{n_i}}| \leq |\tau_{1_{n_j}}|$ for all $i < j$ and $\tau_{1n_i} = (1 \; 2 \; 3 \; \ldots \; 2n_i)$. Denote by S the set of integers $2,3,\ldots,n_1$ and the first $2n_1$ integers from each subsequent cycle in τ_1'''.

Now to construct a k'–s.c. subgraph $G(\tau_1''')$ with $2k'$ vertices and a strong c.p. τ_1''', denote the $2k'$ elements of τ_1''' as the vertices of $G(\tau_1''')$. Then for $1 \in \tau_1'''$, designate arbitrarily the unordered pair $(1,j)$ to be an edge or a nonedge in $G(\tau_1''')$ for every $j \in S$. Then the same adjacency will be true for

$$\left[{\tau_1'''}^{2r}(1), {\tau_1'''}^{2r}(j)\right] = \left[\tau^{2r}(1), \tau^{2r}(j)\right]$$

with $r = 1,2,\ldots,n_i$, if j is from a cycle of length $2n_i$. This gives all the edges of $G(\tau_{1n_1})$ and the edges joining the vertices in $G(\tau_{1n_1})$ and those in $G(\tau_1''' \backslash \tau_{1n_1})$. Then delete $G(\tau_{1n_1})$ and repeat the process for $G(\tau_1''' \backslash \tau_{1n_1})$. This ultimately gives a k'–s.c. graph $G(\tau_1''')$.

On the other hand, construct a s.c. graph $G(\tau_2''')$ and then join each of the vertices in $V(G(\tau_1'''))$ (or, $V(G(\tau_2''')))$ to each of the vertices in a complementary half of $V(G(\tau_2'''))$ (or, $V(G(\tau_1''')))$. This results in a k′–s.c. graph $G(\tau''')$.

Finally, take $G(\tau')$, which consists of either a vertex or two vertices but no edge, and $G(\tau'')$, which consists of $p''/2$ copies of $\overline{K}_2$. Join the vertex in each 1–cycle of τ' to a complementary half of $V(G(\tau'''))$ and to exactly one vertex of each $\overline{K}_2$ (or one vertex of each 2–cycle in τ''). Then name the 2–cycles in τ'' in any fixed order, say $\tau'' = \sigma_1, \sigma_2 ... \sigma_r$ and join both vertices of each σ_i to exactly one vertex of each σ_j, for all $j > i$ and to each vertex of a complementary half of $V(G(\tau'''))$.

This results in a required k–s.c. graph with a strong c.p. τ.

Next take a weak c.p. τ with p elements to construct a k–s.c. graph G with c.p. τ having no odd cycle of length > 1. Clearly $p > 2k$. As in earlier case, take $\tau = (\tau' \tau'' \tau''')$, where τ', τ'' and τ''' have their usual meaning and cardinality. Unlike the case with a strong c.p., now $k > t+p''/2$. Take $k' = k-t-p''/2$. Choose a subset of cycles in τ''' containing $2k''$ elements together such that $k' > k''$ and $k' - k''$ is as small as possible. Then choose a cycle from the rest of the cycles in τ''' containing $2(k'-k'')+2k'''$ (k''' being even, and > 0) elements. Thus $\tau''' = \sigma_1\sigma_2\sigma_3$ (say), where σ_1 is the product of the cycles which together contain $2k''$ elements, σ_2 is a single cycle with $2(k'-k''+k''')$ elements and σ_3 is the product of the rest of the cycles in τ'''. If there is no such cycle σ_2 with required number of elements after the choice of σ_1 then σ_1 may be chosen differently so that a suitable σ_2 can be chosen.

To construct a required k–s.c. graph G, denote the vertices of G by the elements of τ. Then the construction of $G(\sigma_1)$, which is a $k^{\bullet}$–s.c. graph with a strong c.p. σ_1 is the same as the construction of $G(\tau_1''')$ and that of $G(\sigma_3)$ is the same as that of $G(\tau_2''')$ as in the case of a strong c.p. above. Next, we have $|\sigma_2|$ is even or odd multiple of 2 according as $k'-k^{\bullet}$ is an even or odd. So it readily satisfies the necessary condition for the existence of a $(k'-k^{\bullet})$–s.c. graph $G(\sigma_2)$ with $2(k'-k^{\bullet}+k''')$ vertices. Denote the elements of σ_2 by the integers $1,2,3,\ldots,2(k'-k^{\bullet}+k''')$ and by the set S the integers $2,3,\ldots,k'-k^{\bullet}+k'''$, $k'-k^{\bullet}+k'''+1$, $k'-k^{\bullet}+k'''+2$. For $1 \in \sigma_2$, arbitrarily designate the unordered pair (1,j) to be an edge or a non–edge in $G(\sigma_2)$ for $j \in S$. Then the same will be true for $\left[\sigma_2^{2r}(1),\ \sigma_2^{2r}(j)\right]$ with $r = 1,2,\ldots,k'-k^{\bullet}+k'''$ provided $2 \le j \le k'-k^{\bullet}+k'''+1$, and with $r = 1,2,\ldots,k'''$ in case $j = k'-k^{\bullet}+k'''+2$. This results in a $(k'-k^{\bullet})$–s.c. graph $G(\sigma_2)$ with $2(k'-k^{\bullet}+k''')$ vertices. Then join all the vertices of $G(\sigma_\ell)$ to each vertex of a complementary half of the vertices of $G(\sigma_{\ell'})$, for $1 \le \ell < \ell' \le 3$. This completes the construction of $G(\tau''')$.

Finally the edges among the vertices of $G(\tau')$, $G(\tau^{\bullet})$ and $G(\tau''')$ can be joined as in case of a strong c.p. above. This ultimately produces a k–s.c. graph G with a weak c.p. τ.

Now the case left is that of a weak c.p. τ which contains some odd cycles of length > 1. But such an odd cycle in τ may contribute one or more edges to the set of missing edges. Thus in this case we obviously cannot present a construction method along the lines of that outlined above.

It may be remarked that the construction of a k–s.c. graph with a strong and a weak c.p. described above does not produce all possible k–s.c. graphs with a given number of vertices and a fixed suitable k. In particular, a k–s.c. graph with a weak c.p. may not have the symmetry used in the latter construction. This is due to the fact that the missing edges for a k–s.c. graph ($k > 1$) with a weak c.p. may not be restricted to a single orbit of edges even when the c.p. consists of only one cycle. So studying the properties of a k–s.c. graph with a weak c.p. through the cycle structure of the latter for $k > 1$ seems very complicated. On the other hand, the missing edges of a k–s.c. graph with a strong c.p. always constitute all edges in one or more orbits of edges (observation (e), page 65). Hence the rest of our discussion is restricted to the k–s.c. graphs with strong c.p.

5.4. Hamiltonian path in a k–s.c. graph

In this section we discuss the existence of a hamiltonian path in a given k–s.c. graph and give an explicit construction whenever one exists. In view of the observation at the end of Section 5.2, we begin with the construction of a disconnected k–s.c. graph and hence conclude that such graphs cannot have a hamiltonian path.

Lemma 5.4.1. For every positive integer $p \geq 4$ there exists a k–s.c. graph (for suitable $k \geq 1$) with p vertices without a hamiltonian path.

Proof. Suppose there is a k–s.c. graph with p vertices. Then an even k implies $p \equiv 0,1 \pmod 4$, i.e., $p-2k \equiv 0,1 \pmod 4$, and an odd k implies $p \equiv 2,3 \pmod 4$, i.e., $p-2k \equiv 0,1 \pmod 4$. Take a s.c. graph G_0 with $p-2k$ vertices and a set of k K_2's. Join all the vertices of G_0 to one of the vertices in any one K_2. The resulting graph, say G_1, containing one isolated vertex is a 1–s.c. graph. Then take another K_2 and join one of its vertices to all the vertices of G_1 to get a graph G_2. This graph G_2 is a 2–s.c. graph containing an isolated vertex. Continue this process until all the k K_2's have been used. This finally gives a k–s.c. graph G_k which is disconnected. Hence G_k has no hamiltonian path. ∎

Lemma 5.4.2. Every k–s.c. graph with p (≥ 10) vertices and a strong c.p. consisting of a single cycle is hamiltonian, provided p is an odd multiple of 2.

Proof. Suppose G is a k–s.c. graph with p (≥ 10) vertices and a strong c.p. $\tau = (1\ 2\ 3\ \dots\ p)$ where p is an odd multiple of 2. By the construction of such a G discussed Section 5.3 and for $p \geq 10$, G has at least three full orbits of edges beginning with the edge or nonedge (1,2),(1,3) and (1,4). As in Chapter 2, we assume (1,2) and (1,3) $\in E(G)$. This implies (i, i+1) and (i, i+2) $\in E(G)$ for all i odd. Also $(1,4) \in E(G) \Leftrightarrow (2,5) \notin E(G)$, i.e,$(i, i+3) \in E(G) \Leftrightarrow (i+1, i+4) \notin E(G)$ for all i odd. Hence we can construct a hamiltonian cycle in G as in Lemma 3.2.1.

∎

We note that the above proof works for even multiples of 4 provided (1,4) $\in E(G)$. But the restriction in the Lemma implies that if p is an even multiple of 4 then there always exists a nontrivial k–s.c. nonhamiltonian graph (for some suitable k) with p vertices and a strong c.p. consisting of a single cycle. For example, take p = 8, k = 4. Then the graph with the edges

(1,2), (3,4), (5,6), (7,8), (1,3), (3,5), (5,7), (7,1), (2,5), (4,7), (6,1) and (8,3)

is a 4–s.c. nonhamiltonian graph with c.p. = (12345678).

Corollary 5.4.2.1. Every k–s.c. graph with p ($\geq$ 8 and even) vertices and a strong c.p. consisting of a single cycle has a hamiltonian path containing a pair of consecutive odd vertices of the c.p. appearing consecutively and whose end vertices are consecutive even vertices of the c.p.

Further, either the consecutive even vertices at the ends or the pair of consecutive odd vertices appearing consecutively in the hamiltonian path may be chosen arbitrarily.

Proof. Suppose that G is a k–s.c. graph with $p(\geq 8)$ vertices and a strong c.p. τ consisting of a single cycle. As before we assume that $(i, i+1)$ and $(i, i+2) \in E(G)$ for all i odd.

Case (i). $(1,4) \in E(G) \Rightarrow (i, i+3) \in E(G)$ for all i odd. Then G has a hamiltonian cycle

$$(2, 1, 4, 3, 6, 5, \ldots, p, p-1, 2).$$

Suppose we wish to have a pair of consecutive odd vertices, say i and i+2, appearing consecutively in a hamiltonian path of G. We start by breaking the above cycle into a hamiltonian path, say P, so that the vertices i and i+1 appear at the ends. Then, by switching the pair of vertices at the end containing the vertex i of P, one gets a required hamiltonian path containing the pair of consecutive odd vertices i and i+2 appearing consecutively. Notice that this hamiltonian path has a pair of consecutive even vertices i+1 and i+3 at its ends.

The case of a chosen pair of consecutive even vertices to appear at the ends of a hamiltonian path of G is similar.

<u>Case (ii)</u>. $(2,5) \in E(G) \Rightarrow (i, i+3) \in E(G)$ for all i even.

First suppose that p is not a multiple of 4. Then G has a hamiltonian path

$$1, 2, 5, 6, \ldots, p-1, p; 3, 4, 7, 8, \ldots, p-3, p-2. \quad (1)$$

In order to construct a hamiltonian path containing a chosen pair of consecutive odd vertices, say i and i+2 appearing consecutively, we consider two possibilities:

(I) If i occurs in the first half of (1), take the segment to the left of the vertex i (if any) into the right end of (1) so that the ends of the resulting hamiltonian path are i and i–3. This is always possible since $(1, p-2) \in E(G)$. Then reversing the segment from the vertex i to the vertex preceding the vertex i+2, one gets a desired path.

(II) If i occurs in the last half of (1) then interchanging the vertices i and i+2 in (I) above one gets a desired path.

Next, suppose that p is a multiple of 4. Construct two disjoint paths of G by writing the vertices in increasing order such that one contains all vertices of the form 4r+1 and 4r+2, and the other contains all vertices of the form 4r+3 and 4r+4, where r is a nonnegative integer. It may be observed that these two paths in fact form two disjoint cycles, each containing exactly half the vertices of G. Also, no pair of consecutive odd or even vertices are in the same cycle. Since every pair of consecutive odd vertices are adjacent (by the assumption at the beginning), these two cycles can be broken up and then the two resulting paths can be joined together suitably to get a desired hamiltonian path.

Again, the case of a chosen pair of consecutive even vertices appearing at the ends of a hamiltonian path of G is similar. ∎

The result of this corollary may also be extended to a k–s.c. graph with an odd number of vertices in a restricted sense.

Corollary 5.4.2.2. Every k–s.c. graph with p (odd) vertices and a strong c.p. consisting of one cycle of length p–1 and another of length 1 has a hamiltonian path.

Proof. Suppose G is a k–s.c. graph satisfying the hypothesis, i.e., G has a strong c.p. $\tau = \tau_1\tau_2$, where $\tau_1 = (12...p-1)$ and $\tau_2 = (p)$. Take $G_1 = G(\tau_1)$. Then by the above result, G_1 has a hamiltonian path, say P, containing a pair of consecutive odd vertices of τ_1 appearing consecutively and whose end vertices are consecutive even vertices of τ_1. Now the vertex labelled p of G is adjacent to either all even or all odd vertices of G_1. So in either case the vertex p may be adjoined suitably to P to get a hamiltonian path of G. ∎

Theorem 5.4.3. Suppose G is a k–s.c. graph with p vertices and a strong c.p. $\tau = (\tau_1...\tau_m)$ such that τ has at most one 1–cycle but no 2–cycle, and each $G(\tau_i)$ has a hamiltonian path. Then G has a hamiltonian path.

Proof. Suppose G is a k–s.c. graph satisfying the hypothesis.

Case I. p even.

In this case $|\tau_i| \geq 4$ for all i = 1,2,...,m. Also, by Corollary 5.4.2.1, the induced subgraph $G(\tau_i)$ of G has a hamiltonian path containing a pair of consecutive odd vertices of τ_i and whose end vertices are consecutive even vertices of τ_i, for each i provided that $|\tau_i| \geq 8$. But this property also holds for the induced subgraph $G(\tau_i)$ with $|\tau_i| = 4$ since $G(\tau_i)$ has a hamiltonian path which implies that G (τ_i) is a s.c. graph. Now noting that τ has no cycle of length 6, order the cycles of τ as $\tau_i < \tau_j$ if some even vertex of τ_i is adjacent to some odd vertex of τ_j, for all i and j (see Chapter 3). Then the technique of Theorem 3.3.1 gives a required hamiltonian path of G.

<u>Case II</u>. p odd.

This case is similar to that of Theorem 3.3.1.

<u>Corollary 5.4.3.1</u>. (cf. Clapham [13]) Every s.c. graph has a hamiltonian path.

5.5. <u>Cycles in a k–s.c. graph</u>

This section deals with the existence of cycles of different lengths in a k–s.c. graph. To this end we first prove a few results relating to the existence of some paths and cycles in special cases.

<u>Lemma 5.5.1</u>. Suppose G is a k–s.c. graph with p ($\geq$ 10 and even) vertices and a strong c.p. τ consisting of a single cycle. Then for every integer ℓ, $p-4 \leq \ell \leq p-1$, G has a path of length ℓ in which a pair of consecutive odd vertices of τ appear consecutively and whose end vertices are consecutive even vertices of τ.

Further, either the pair of consecutive odd vertices within or the consecutive even vertices of τ at the ends of such paths can be chosen arbitrarily.

This lemma may be proved from Corollary 5.4.2.1 and using the withdrawal technique in Chapter 3. It may be noted that the above lemma does not hold for the case $p = 8$. For, it can be checked directly that paths of lengths 5,6,7 exist in a k–s.c. graph with 8 vertices and a strong c.p.. But the arbitrary choice of consecutive even vertices at the ends or a pair of consecutive odd vertices within such paths is not always possible.

Lemma 5.5.2. Suppose G is a k–s.c. graph with p ($\geq$ 10 and even) vertices and a strong c.p. τ such that the induced subgraph on each cycle of τ has a hamiltonian path. Then for every integer ℓ, $p-4 \leq \ell \leq p-1$, G has a path of length ℓ containing a pair of consecutive odd vertices of τ appearing consecutively and whose end vertices are consecutive even vertices of τ.

Proof. Suppose G is a k–s.c. graph with p vertices and a strong c.p. τ satisfying the hypothesis. We note that τ has no 6–cycle, and consider the following cases:

Case I. Suppose that at least one cycle τ_i of τ has length ≥ 10. Then by Lemma 5.5.1, $G(\tau_i)$ has paths of lengths ℓ_i, $p_i-4 \leq \ell_i \leq p_i-1$, satisfying the hypothesis, where $|V(G(\tau_i))| = p_i$. Now ordering the cycles $\tau_1, \tau_2, \ldots, \tau_m$ of τ as before one can join hamiltonian paths of $G(\tau_j)$, $j \neq i$ and paths of length ℓ_i of $G(\tau_i)$ suitably to get paths of lengths ℓ, $p-4 \leq \ell \leq p-1$, in G satisfying the required conditions.

Case II. Suppose that every cycle of τ is of length < 10. Also assume that $k > 0$, for otherwise G is a s.c. graph and there is nothing to prove. Then τ has at least one 8–cycle. Take $\tau = (\tau_1\tau_2\ldots\tau_m)$, where $\tau_i < \tau_{i+1}$, $1 \leq i \leq m-1$.

<u>Case II(i)</u>. Suppose again that τ has exactly one cycle, say τ_i, of length 8 for some fixed i, and each τ_j is of length 4, $1 \le j \le m$ and $j \ne i$. Take $\tau_i = (u_1\ u_2\ u_3\ u_4\ u_5\ u_6\ u_7\ u_8)$. By assumption, we have (u_1,u_2) and $(u_1,u_3) \in E(G(\tau_i))$. If $(u_1,u_4) \in E(G(\tau_i))$ then $G(\tau_i)$ has paths of lengths 4,5,6 and 7. These paths and a hamiltonian path of each $G(\tau_j)$ may be joined suitably to get desired paths in G. So take $(u_1,u_4) \notin E(G)$. This implies $(u_2,u_5) \in E(G)$. In this situation it is enough to check that the induced subgraph $G(\tau_i \cup \tau_{i+1})$ or $G(\tau_i \cup \tau_{i-1})$, according as $i \ne m$ or $i = m$, has paths of lengths 11,10,9 and 8 satisfying the hypothesis. First, take $G(\tau_i \cup \tau_{i+1})$, $i \ne m$ and $\tau_{i+1} = (v_1v_2v_3v_4)$. Since $\tau_i < \tau_{i+1}$, each even vertex of τ_i is adjacent to some odd vertex of τ_{i+1}. Suppose $(u_2,v_1) \in E(G)$. This implies (u_4,v_3), (u_6,v_1), $(u_8,v_3) \in E(G)$. Then a path of length 11 in $G(\tau_i \cup \tau_{i+1})$ satisfying the required conditions is

$$v_2\ v_1\ u_6\ u_5\ u_2\ u_1\ u_3\ u_4\ u_7\ u_8\ v_3\ v_4.$$

Further, if $(u_7,v_3) \in E(G)$, which implies (u_1,v_1), (u_3,v_3), $(u_5,v_1) \in E(G)$, then paths of lengths 10,9 and 8 in $G(\tau_i \cup \tau_{i+1})$ are respectively

$$v_2\ v_1\ u_6\ u_5\ u_2\ u_1\ u_3\ u_4\ u_7\ v_3\ v_4,$$
$$v_2\ v_1\ u_5\ u_2\ u_1\ u_3\ u_4\ u_7\ v_3\ v_4,$$
$$\text{and}\quad v_2\ v_1\ u_1\ u_3\ u_4\ u_7\ u_8\ v_3\ v_4.$$

If $(u_7,v_3) \notin E(G)$ then (u_2,v_2), (u_4,v_4), (u_6,v_2), $(u_8,v_4) \in E(G)$ and the paths of lengths 10,9 and 8 in $G(\tau_i \cup \tau_{i+1})$ are

$$v_2\ v_1\ u_6\ u_5\ u_2\ u_1\ u_3\ u_4\ u_7\ u_8\ v_4,$$
$$v_2\ v_1\ u_2\ u_1\ u_3\ u_4\ u_7\ u_8\ v_3\ v_4,$$
$$\text{and}\quad v_2\ u_2\ u_1\ u_3\ u_4\ u_7\ u_8\ v_3\ v_4 \qquad \text{respectively.}$$

Suppose $(u_2,v_1) \notin E(G)$ then (u_2,v_3), (u_4,v_1), (u_6,v_3), $(u_8,v_1) \in E(G)$ and again the result may be checked directly as above.

Also the required checking is similar to that for the case $G(\tau_i \cup \tau_{i-1})$, $i = m$.

Case II(ii). Suppose each τ_i is of length 8. Here we only need to show that the induced subgraph $G(\tau_1 \cup \tau_2)$ has paths of lengths 15, 14, 13 and 12 satisfying the required conditions. But the required constructions are the same as above.

∎

Lemma 5.5.3. Suppose G is a k–s.c. graph with p (> 8 and even) vertices and a strong c.p. τ consisting of a single cycle. Then for every integer ℓ, $3 \leq \ell \leq p-2$, G has an ℓ–cycle. Further, each such ℓ–cycle of G has a pair of consecutive odd vertices of τ appearing consecutively provided $4 \leq \ell \leq p-2$.

Proof. Suppose $|V(G)| = p(= 2r > 8)$ and $\tau = (1\ 2\ ....\ p)$ is a strong c.p. for G. Then, by the assumption at the beginning, we have (i, i+1) and (i, i+2) $\in E(G)$ for all i odd in τ. Also p > 8 implies that both the orbits starting with the edge or the nonedge (1,4), and the edge or the nonedge (1,5) are full orbits each giving r edges in G. Now consider the following different possibilities:

Case I. $(1,5) \in E(G) \Rightarrow (i, i+4) \in E(G)$ for all i odd. Then the induced subgraph on the r odd vertices of G is a 4–regular graph with the edges (i, i+2), (i, i+4), (i, i–2) and (i, i–4) being incident to i where the vertices are to be taken residues modulo p. One can easily check that this subgraph is pancyclic.

<u>Case I(i)</u>. $(1,4) \in E(G) \Rightarrow (i, i+3) \in E(G)$ for all i odd. Take the hamiltonian cycle

$$(2, 1, 4, 3, 6, 5, ..., p, p-1, 2)$$

in G. Since consecutive odd vertices are adjacent in G and they occur alternately in the above cycle, one can withdraw one, two, ..., r even vertices successively from this cycle to get an ℓ-cycle, $p \geq \ell \geq r$, in G.

<u>Case I(ii)</u>. $(1,4) \notin E(G) \Rightarrow (2,5) \in E(G) \Rightarrow (i, i+3) \in E(G)$ for all i even. If p is not a multiple of 4 then

$$(1, 2, 5, 6, 9, 10, ..., p-1, p; 3, 4, 7, 8, ..., p-3, p-2, 1) \qquad (1)$$

is a hamiltonian cycle of G. But, if p is a multiple of 4 then G may not have a hamiltonian cycle. In this case G does have two disjoint cycles together containing all the vertices of G. One contains all the vertices of G of the form 4n+1 and 4n+2 in order and the other contains all the vertices of the form 4n+3 and 4n+4 $(0 \leq n \leq (p/4)-1)$ in order. Then deleting the vertex labelled p–2 from the first cycle and the vertex p from the second cycle, and joining each of the pairs of vertices 1,3 and 4n–3, 4n–1 by an edge we get a cycle

$$(1, 3, 4, ..., p-1, p-3, p-6, ..., 6, 5, 2, 1) \qquad (2)$$

of length p–2 in G. Since $(i, i+4) \in E(G)$ for all i odd, withdrawing none, one, two, ..., r even vertices from (1) successively we get an ℓ-cycle, $p \geq \ell \geq r$, in G. Similarly withdrawing none, one, two, ..., r–2 even vertices from (2) successively one gets an ℓ-cycle, $p-2 \geq \ell \geq r$, in G.

<u>Case II</u>. $(1,5) \notin E(G) \Rightarrow (2,6) \in E(G) \Rightarrow (i, i+4) \in E(G)$ for all i even. Here again we have the following two cases:

<u>Case II (i)</u>. $(1,4) \in E(G) \Rightarrow (i, i+3) \in E(G)$ for all i odd. For j even, and $4 \leq j \leq r-1$, consider the cycle

$$(1, 4, 3, 5, 8, 7, \ldots, 2j, 2j-1, 2j+1; 2j+2, 2j-2, 2j-6, \ldots, 6, 2, 1) \tag{3}$$

of length 2j+2. Since $(i, i+2) \in E(G)$ for all i odd, withdrawing none, one, two, ..., (j/2) even vertices, whose labellings are multiples of 4, successively from the cycle (3) one gets cycles of lengths 2j+1, 2j, ..., 3(j/2)+2 for each even j. Now varying j within its chosen bounds one gets ℓ-cycles, $8 \leq \ell \leq p$, in G. The remaining cycles of lengths 7, 6, 5, 4, and 3 are, respectively, (1,3,5,7,10,6,2), (1,4,3,5,6,2), (1,3,5,6,2), (1,3,6,2) and (1,3,4).

<u>Case II (ii)</u>. $(1,4) \notin E(G) \Rightarrow (2,5) \in E(G) \Rightarrow (i, i+3) \in E(G)$ for all i even. For even j, and $4 \leq j \leq r-1$, take the cycle

$$(4, 7, 8, 11, 12, \ldots, 2j-5, 2j-4; 2j, 2j-1, 2j+1, 2j+2;$$

$$2j-2, 2j-3, 2j-6, 2j-7, \ldots, 2, 1, 3, 4) \tag{4}$$

of length 2j+2. Since (i, i+3) and $(i, i+4) \in E(G)$ for all i even, withdrawing none, one, two, ..., (j–1) vertices from this cycle (4) with vertex labels in $\{5,7,9,\ldots,2j-3\} \cup \{2j, 2j+2\}$ one gets cycles of lengths 2j+1, 2j, ..., (j+3) for each j. Now varying even j within its chosen bounds one gets ℓ-cycles in G for $7 \leq \ell \leq p$.

The remaining cycles of lengths 6,5,4 and 3 are (7,9,10,6,2,5), (6,2,1,3,5), (5,2,1,3) and (2,5,6) respectively.

The second part of the lemma is quite clear from the construction in each case above. ∎

<u>Theorem 5.5.1</u>. Suppose G is a k–s.c. graph with p (even and ≥ 8) vertices and a strong c.p. τ such that the induced subgraph of G on each cycle of τ has a hamiltonian path. Then for every integer ℓ, $3 \leq \ell \leq p-2$, G has an ℓ–cycle.

<u>Proof</u>. (By induction on the number of cycles in τ).

Suppose $\tau = \tau_1\tau_2...\tau_m$ such that $\tau_i < \tau_{i+1}$ for $1 \leq i \leq m-1$. Then every even vertex of τ_i is adjacent to some odd vertex of τ_{i+1} (by definition of the ordering ' < ' in Chapter 3). If m = 1, the result follows from Lemma 5.5.3. So take m > 1 and, by the induction hypothesis, suppose that the result holds for the induced subgraph $G' = G(\tau_1\tau_2...\tau_{m-1})$ of G, i.e., G′, and hence G, has ℓ–cycles, $3 \leq \ell \leq p-r-2$ provided $|V(G')| \geq 8$, where $|\tau_m| = r$. Since τ is a strong c.p. and each $G(\tau_i)$ has a hamiltonian path, $|\tau_i| \neq 6$ for all i. So we divide the proof into two cases.

<u>Case I</u>. r = 4. If $|V(G')| \geq 10$ then, by Lemma 5.5.2, take paths P_1, P_2, P_3 and P_4 of lengths p–8, p–7, p–6 and p–5 respectively containing a pair of consecutive even vertices of τ_{m-1} at the ends. Then adjoining the edge with ends the odd vertices of τ_m to the above paths one gets cycles of lengths p–5, p–4, p–3 and p–2 respectively. If $|V(G')| \leq 8$ the result may be checked directly.

<u>Case II</u>. $r \neq 4$. Then $r \geq 8$ and the result may be proved using the Lemmas 5.5.2 and 5.5.3.

Thus the result holds by induction and this completes the proof.

The above result also holds when G contains an additional vertex which accounts for a 1–cycle for the c.p. τ. Thus we state the following without proof which includes both even and odd p.

<u>Theorem 5.5.2</u>. Suppose G is a k–s.c. graph with $p(\geq 8)$ vertices and a strong c.p. τ containing at most one 1–cycle such that the induced subgraph of G on each cycle (of length > 1) of τ has a hamiltonian path. Then for every integer ℓ, $3 \leq \ell \leq p-2$, G has an ℓ–cycle.

CHAPTER 6
HALVING COMBINATORIAL DESIGNS

6.1. Introduction

In this Chapter the notion of selfcomplementarity is extended to combinatorial designs. In particular, we examine whether a Steiner triple system (i.e., a BIBD (v,b,r,k,λ) with $k = 3$ and $\lambda = 1$) or a corresponding truncated system (obtained by deleting a triple if the total number of triples is odd) can be partitioned into two isomorphic sets of triples, with v elements each. This is done by obtaining a necessary condition for such existence, and then providing a method of construction in each case. Similar questions for twofold triple systems and Steiner systems S(2,4,v) are also discussed.

6.2. Steiner selfcomplementary graphs

The problem of partitioning the set of all triples in a Steiner triple system with v elements (STS(v)) into two isomorphic sets of triples, with v elements each, is the same as finding the class of selfcomplementary graphs with v vertices such that the edge set of each of these s.c. graphs is a collection of edge–disjoint triangles. This leads us to the following definition.

Definition. A simple graph G with v vertices is called *Steiner selfcomplementary* (s.s.c.) if it is isomorphic with its complement $\overline{G}$, and E(G) is a set of edge–disjoint triangles.

For example, take v = 9, i.e., V(G) = {1,2,...,9} and E(G) = {(1,2), (1,4), (2,4); (3,4), (3,6), (4,6); (5,6), (5,8), (6,8); (7,8), (8,,2), (7,2); (1,5), (5,9), (1,9); (3,7), (3,9), (7,9)}.

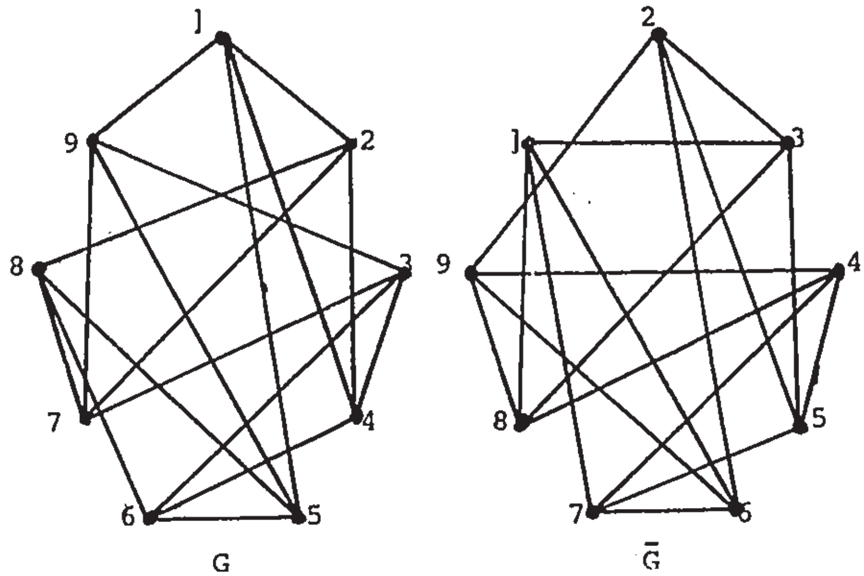


Figure 6.1

Note that the graph G in this example is a regular graph. Since the number of vertices in a s.s.c. graph is odd, every complementing permutation (c.p.) of such a graph always contains exactly one fixed vertex. The fixed vertex of a c.p. is usually denoted by ∞.

A simple numerical argument gives the following necessary condition for the existence of a s.s.c. graph.

Lemma 6.2.1. If there is a s.s.c. graph with v vertices then v $\equiv$ 1, 9, 13, 21 (mod 24).

The class of s.s.c. graphs being a subclass of s.c. graphs, every s.s.c. graph is connected. It can be illustrated by simple examples that if τ is a c.p. of a s.s.c. graph G then a subgraph of G induced on the vertex set of a subset of cycles of τ is not necessarily a s.s.c. graph.

Lemma 6.2.2. Suppose τ is a c.p. of a s.s.c. graph. Then for any two cycles (of length > 1) in τ, the length of one is an integral multiple of the length of that of the other.

Proof. Suppose τ_1 and τ_2 are any two cycles (of length > 1) in τ such that the length of neither is an integral multiple of the other. Now we use the structural property of the corresponding STS to complete the proof.

By definition of STS, every element from τ_1 must occur exactly once with every element of τ_2 in the triples of the STS. This can happen in three different ways. These are: two elements from τ_1 with one element of τ_2 and/or one element from τ_1 with two elements of τ_2 and/or one element from τ_1 and one element from τ_2 with an element from some third cycle (say, τ_3) of τ in the triples of the STS. Looking at the orbit containing such a triple, it can easily be checked that the first two cases violate the condition $\lambda = 1$ for an STS. For the last case the length of the corresponding orbit must be the l.c.m. of the lengths of the three cycles. But this orbit again contains an element of τ_3 with either an element of τ_1 or τ_2 appearing at least twice, which is not allowed in an STS. ∎

Construction of a s.s.c. graph with v vertices for v ≡ 1,9 (mod 24) follows from the construction of 1–and 2–rotational STS(v) given in Phelps and Rosa [35]. This is done by taking alternate triples in every orbit of the STS as triangles in the s.s.c. graph since each orbit contains an even number of triples.

Before proceeding to consider the remaining two classes of values for v, we note that a c.p. of a s.s.c. graph is not necessarily an automorphism of the corresponding STS. For example, the s.s.c. graph G with 13 vertices whose edge set consists of the edge–disjoint triangles

0,1,3; 1,2,10; 2,5,6; 3,5,12; 3,4,10; 6,9,10; 6,11,12; 4,8,12; 4,9,7; 0,9,11; 1,8,11; 2,7,8; 0,5,7

has a c.p. (0) (1 2 3 4 5 6 7 8 9 10 11 12) which is not an automorphism of the STS = $G \cup \overline{G}$.

Since we do not have a method, in general, to deal with the construction of s.s.c. graphs with v ≡ 13, 21 (mod 24) vertices at this point, we restrict our discussion of these s.s.c. graphs to those whose c.p.'s are always automorphisms of the corresponding STS(v) and call this class *strictly s.s.c. graphs.* This immediately implies that there are no strictly s.s.c. graphs with 13 vertices since neither of the two nonisomorphic STS(13) has an automorphism which satisfies the cycle length restriction in a complementing permutation of a s.c. graph (cf. [33]).

Now with the nonexistence of a strictly s.s.c graph with 13 vertices in hand, we look at the question of the existence of a strictly s.s.c. graph with

$v \equiv 13, 21 \pmod{24}$ vertices. Recall that every c.p. in either case has a fixed point, ∞. If $v \equiv 13, 21 \pmod{24}$ then any c.p. of a strictly s.s.c. graph with v vertices contains either an odd number of cycles of length 4 or a cycle with length an odd multiple of 4. But the second possibility may be reduced to the first by taking some suitable odd power of the c.p.. Thus, without loss of generality, assume that every c.p. of a strictly s.s.c. graph of order $v \equiv 13, 21 \pmod{24}$ has an odd number (say r) of cycles of length 4 each. If $r = 1$, each vertex of the 4–cycle in τ must occur in pair with each of the other three vertices of this 4–cycle exactly once in the triples of STS(v). Two of the six such possible pairs may be combined with ∞ to give two triples and the remaining four pairs must combine with vertices of some other cycle of length > 4 in τ. But then τ would produce a pair of vertices of the 4–cycle appearing in more than one triple, violating the definition of STS.

Next, take odd $r \geq 3$. Suppose $\tau_1, \tau_2, \ldots, \tau_r$ are the 4–cycles in τ. There are two ways of forming triples by taking a pair of vertices from some one 4–cycle and one vertex from another 4–cycle. One way is taking an orbit by combining two vertices from τ_i (τ_j) and one vertex from τ_j (τ_i) which together produce 8 triples. These together with the four triples each with a pair of vertices either in τ_i or τ_j and ∞ as the third vertex account for all triples with vertices from τ_i, τ_j and (∞). This reduces the number of 4–cycles to be taken care of in terms of its vertex pairs to (r–2). It ultimately reduces to the case $r = 1$, and so, is not possible.

The other way of forming orbits with vertices from the two 4–cycles τ_i and τ_j is to combine a pair of vertices from τ_i and one vertex from τ_j, and then combine a pair of vertices from τ_j with one vertex from some 4–cycle other than τ_i. In this case each vertex of τ_i has been paired with two vertices of τ_j by the above combination. So each vertex of τ_i must be paired with the remaining two vertices of τ_j in order to satisfy the definition of STS(v). But this requires vertices from some other cycle of τ. However, this third cycle is neither (∞) nor of length > 4, for the latter would violate the condition $\lambda = 1$ for an STS. So the third cycle in τ required above must also be of length 4. Thus we are reduced to the case that the subset of all 4–cycles in a c.p. τ of a s.s.c. graph together with ∞ form a s.s.c. subgraph. Let us denote by τ' the subset of all cycles of length 4 in τ and the cycle (∞). Then $(\tau')^2$ is an involution consisting of 4r+1 (r odd) elements. But there exists no reverse STS(4r+1), r odd, by Rosa [45]. Hence we have the following result.

Theorem 6.2.1. There exists no strictly s.s.c. graph of order v ≡ 13, 21 (mod 24).

Combining Lemma 6.2.1 and the above theorem, we have the following theorem:

Theorem 6.2.2. A strictly s.s.c graph with v vertices exists if and only if v ≡ 1, 9 (mod 24).

Corollary 6.2.2.1. If Γ is an automorphism of an STS(v), v ≡ 13, 21 (mod 24), with block orbits $O_1, O_2, \ldots, O_s$ then $|O_i|$ is odd for at least one i, $1 \leq i \leq s$.

Although by Theorem 6.2.1 there exists no strictly s.s.c. graph with v ≡ 13, 21 (mod 24) vertices, this still leaves a possibility that a s.s.c. graph with such a number of vertices may exist. In fact we have the following theorem.

Theorem 6.2.3. Let $v = p^{\alpha} \equiv 13 \pmod{24}$ be a prime power. Then there exists a s.s.c. graph with v vertices.

Proof. If $v = p^{\alpha} \equiv 13 \pmod{24}$, take Bose's construction of a cyclic STS(v) (cf. Theorem 15.3.4 [24]). The base blocks are $(x^i, x^{2t+i}, x^{4t+i})$, $i = 0, 1, \ldots, t-1$, where x is a primitive root of $GF(p^{\alpha})$. In our case the number of orbits (i.e., of base blocks) is even. Include in G all orbits with even i. Then τ, defined by $\tau(u) = xu$ for all $u \in V(G)$, is an isomorphism between G and $\overline{G}$. ∎

Remark. We do not know whether there exist non–strict s.s.c. graphs for non–prime–power orders $v \equiv 13 \pmod{24}$ or any orders $v \equiv 21 \pmod{24}$.

6.3. Almost Steiner selfcomplementary graphs

It is well known that an STS(v) exists if and only if $v \equiv 1, 3 \pmod 6$. Every STS(v) with $v \equiv 1, 9, 13, 21, \pmod{24}$ has an even number of triples whereas those with $v \equiv 3, 7, 15, 19 \pmod{24}$ have an odd number of triples each. So the blocks of an STS in the second class cannot be partitioned into two isomorphic parts. However, by analogy with almost selfcomplementary graphs, we may proceed for this class of STS(v) as follows:

Consider an STS(v) with $v \equiv 3, 7, 15, 19 \pmod{24}$ and delete one of the triples. Then we may ask whether the remaining triples can be partitioned into two isomorphic sets of triples, each with v elements. It is a trivial observation that any STS with the above mentioned order may be seen as a complete graph whose edge set is a collection of edge–disjoint triangles. Thus we are led to the following definition.

Definition. A simple graph G with v vertices is called *almost Steiner selfcomplementary* (a.s.s.c.) if it is isomorphic with its restricted complement $\bar{G} = \bar{K}_v \backslash G$, where $\bar{K}_v = K_v \backslash T$, T is a triangle in K_v, and E(G) is a set of edge-disjoint triangles.

The triangle T in K_v is called the *missing triangle*. An example of such a graph is given below

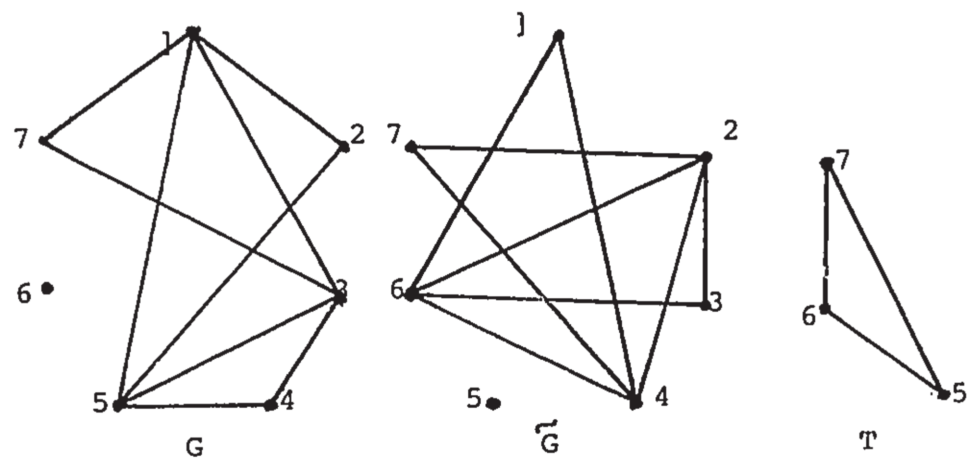


Figure 6.2

Note that unlike a s.s.c. graph, an a.s.s.c. graph is not always connected. It is clear from the definition that an a.s.s.c. graph is in a way similar to a 3–s.c. graph with an odd number of vertices. This gives us the following obvious result.

Lemma 6.3.1. If there exists an a.s.s.c. graph with v vertices then $v \equiv 3, 7, 15, 19 \pmod{24}$.

As before, an isomorphism $\tau : G \to \bar{G}$ can be considered as a mapping of the vertex set which maps edges of G onto edges of $\bar{G}$ and nonedges of G onto nonedges of $\bar{G}$. Thus τ is a c.p. of the a.s.s.c. graph and for the above figure, τ may be taken as (1 2 3 4)(5 6)(7). In keeping with the terminology of a.s.c. graphs, here again we distinguish two types of c.p.'s of an a.s.s.c. graph. If a c.p. fixes the missing triangle then it is called a *strong* c.p., otherwise it is called a *weak* c.p.. So the c.p. τ = (1 2 3 4)(5 6)(7) mentioned above is a strong c.p.

The following are some of the properties of a c.p. of an a.s.s.c. graph.

Lemma 6.3.2. A (weak/strong) c.p. τ of an a.s.s.c. graph G has at most 3 fixed elements (i.e., τ can fix at most three vertices of G).

Proof. By definition of a.s.s.c. graph, τ can fix at most the three sides of the missing triangle. This implies that τ cannot fix more than three vertices of G.

∎

The following is an example of an a.s.s.c. graph with 15 vertices and a c.p. τ fixing three vertices.

$$V(G) = \{1,2,\ldots,12\} \cup \{x,y,z\}, \quad \tau = (1\ 2\ 3\ 4)(5\ 6\ 7\ 8)(9\ 10\ 11\ 12)\ (x)(y)(z)$$

and the set of edge-disjoint triangles constituting E(G) is

{1,3,x; 5,7,y; 9,11,z; 1,9,y; 3,11,y; 1,5,z; 1,7,z; 5,9,x; 7,11,x; 1,2,7; 3,4,5; 5,6,11;

7,8,9; 9,10,3; 11,12,1; 1,8,10; 3,6,12},

where the missing triangle is obviously x,y,z.

Lemma 6.3.3. τ has at most one cycle of length 2.

Proof. Suppose τ has more than one cycle of length 2. Then each of the edges with ends in the same 2–cycle of τ is fixed by τ. Moreover, these edges being pairwise nonadjacent, at most one of them can be an edge of the missing triangle. Hence τ can have at most one cycle of length 2. ∎

Lemma 6.3.4. For any two cycles (of lengths > 1) in τ, the length of one is an integral multiple of that of the other.

Proof. Similar to the proof of Lemma 6.2.2.

The rest of the section deals with the construction of an a.s.s.c. graph with a given number of vertices. This is done by considering the different cases separately.

Lemma 6.3.5. If v ≡ 3 (mod 24) then there exists an a.s.s.c. graph of order v.

Proof. For $v \equiv 3 \pmod{24}$, there exists a 1–rotational STS(v) and so the construction of an a.s.s.c. graph with v vertices follows from Theorem 2.2 of Phelps and Rosa [35], for an STS obtained there contains exactly one orbit of odd length. Note that the c.p. here is a weak c.p. consisting of only two cycles, one of which is a 1–cycle.

Lemma 6.3.6. If $v \equiv 7, 15 \pmod{24}$ then there exists an a.s.s.c. graph with v vertices.

Proof. In theorem 4.19 of [20], Gardner gives a construction of an STS(v), $v \equiv 7, 15 \pmod{24}$, with an automorphism $\tau = (\infty)(a\ b)(0\ 1\ 2\ \ldots\ (v-4))$. Since each of the orbits (except the one containing the single triple $\{\infty, a, b\}$) of triples in the said construction is of even length, an a.s.s.c. graph with v vertices can be constructed by taking alternate triples from each of the orbits of even lengths of the corresponding STS(v). We note that the (strong) c.p. of the resulting a.s.s.c. graph is the automorphism τ of the STS(v) and the triple $\{\infty, a, b\}$ is the missing triangle.

Now to complete the construction of a.s.s.c. graphs with v vertices, for admissible v, we only need to demonstrate the construction of an a.s.s.c. graph with $v \equiv 19 \pmod{24}$ vertices. For this we first construct a corresponding STS(v) with an automorphism such that each orbit of (except one containing exactly one triple) of triples is of even length.

Theorem 6.3.1. There exists an STS(v) on the set $\{Z_{12t+8} \times (1,2)\} \cup \{\infty,a,b\}$ of v (=24t+19) elements admitting $\tau = (\infty)(a\ b)(0_1 1_1 \ldots (12t+7)_1)(0_2 1_2 \ldots (12t+7)_2)$ as an automorphism.

Proof. We use a direct construction applying Bose's method of "symmetrically repeated differences" (cf., e.g., [24]).

The pure differences from each of the cycles $\tau_1 = (0_1 1_1 \ldots (12t+7)_1)$ and $\tau_2 = (0_2 1_2 \ldots (12t+7)_2)$ of τ are

$$1, 2, 3, \ldots, 6t+3, 6t+4$$

while the mixed differences are

$$1, 2, 3, \ldots, 12t+7.$$

It is clear from the cycle structure of τ that $\{\infty,a,b\}$ is a fixed block of the system. Then taking the pure difference 6t+4 from each of the cycles τ_1 and τ_2 we get a set of blocks

$$B_1 = \left\{\{i_1, (6t+4)_1, \infty\}, \{i_2, (6t+4)_2, \infty\} \mid i = 0, 1, 2, \ldots, 6t+3\right\}.$$

For the construction of the remaining blocks we consider two cases:

Case I: t even.

Partition the pure differences 1,2,...,6t+3 from the cycle τ_1 into 2t+1 triples (a_i,b_i,c_i) satisfying $a_i+b_i = c_i$, i = 1,2,...,2t+1, by using, say, Skolem sequences (cf., e.g., [24]). This is always possible for 2t+1 ≡ 1 (mod 4). These difference triples are used to form 2t+1 base triples. Denote by B_2 the set of all triples developed from these base triples. Next, partition the mixed differences 0,1,2,...,12t+4,12t+6 into 6t+3 pairs using hooked Skolem sequence. These pairs with the pure differences 1,2,3,...,6t+3 from the cycle τ_2 form 6t+3 difference triples. Suppose B_3 is the set of all triples obtained by developing the corresponding 6t+3 base triples. Then the remaining mixed differences 12t+5 and 12t+7 are used to get the set of blocks

$$B_4 = \Big\{\{i_1, (12t+5+i)_2, a\}, \{i_1, (12t+7+i)_2, b\} \mid i = 0, 1, ..., 12t+7\Big\}.$$

Now $B_1 \cup B_2 \cup B_3 \cup B_4 \cup \{\infty,a,b\}$ is the block set of the STS(v), with τ as an automorphism.

Case II: t odd.

Since t is odd, there are odd number of odd differences among the pure differences 1,2,...,6t+3 of the cycle τ_1. So these cannot be partitioned into triples as above. However, it is easily seen that one can partition all the above pure differences except the differences 1, 4 and 6 into difference triples (e.g., by taking an appropriate hooked Skolem sequence) [50]. Thus the pure differences 2,3,5,7,8,...,6t+3 are partitioned into 2t difference triples which are used to form 2t base triples for our STS. Denote by B_2' the set of all triples developed from these base triples. Also let B_3' be the set of all triples developed by the base triple formed from the difference triple (2,4,6) from the cycle τ_2.

Next, partition the mixed differences 1,2,....,12t+1,12t+3 into 6t+1 pairs with differences 3,4,...,6t+3. This can be done by using hooked Langford sequence with d = 3, (see [50]). These pairs together with the pure differences {3,5,7,8,...,6t+3} from the cycle τ_2 and {4,6} from the cycle τ_1 form 6t+1 difference triples, which, in turn, produce 6t+1 base triples for the system. Suppose B_4' is the set of all triples generated by these base triples.

Finally, the pure difference 1 from each of the cycles τ_1 and τ_2, and the mixed differences 0, 12t+2, 12t+4, 12t+5, 12t+6 and 12t+7 give the set of triples

$$B_5' = \Big\{\{i_1,(i+1)_1,(12t+6)_2\},\{i_2,(i+1)_2,(12t+8)_1\},\{i_1,i_2,a\},\{i_1,(12t+2+i)_2,b\} \;\Big|\; i = 0, 1, ..., 12t+7\Big\}.$$

So $B_1 \cup B_2' \cup B_3' \cup B_4' \cup B_5' \cup \{\infty,a,b\}$ is the block set of the required STS(v).

∎

Now taking the automorphism τ in the above theorem as a strong c.p. for an a.s.s.c. graph, we have the next Corollary.

Corollary 6.3.1.1. If $v \equiv 19 \pmod{24}$ then there exists an a.s.s.c. graph with v vertices.

6.4. Twofold triple systems

A balanced incomplete block design BIBD with $k = 3$ and $\lambda = 2$ is called a *twofold triple system* (TTS(v)). It is well known that a TTS(v) exists if and only if $v \equiv 0, 1 \pmod 3$. Here again we attempt to partition the set of triples in a TTS(v) into two isomorphic sets of triples. For this, we note that, unlike an STS, the total number of triples in a TTS is always even. Thus, every TTS satisfies the necessary condition for our desired partition.

Since for $v \equiv 1, 3 \pmod 6$ there exists an STS(v), we may obtain for such v the required partition in a trivial manner by simply taking two isomorphic or even identical copies of an STS. Thus, unless we want to impose additional conditions, which we will not do, we may dismiss this case. This leaves us to consider only the values of $v \equiv 0, 4 \pmod 6$.

In order to check the existence of at least one partition of a TTS(v), $v \equiv 0, 4 \pmod 6$, in the sense mentioned above, we first consider the class of cyclic TTSs. To this end, we have the following useful result due to Colbourn and Colbourn.

Lemma 6.4.1 [16]. A cyclic TTS(v) exists if and only if $v \equiv 0,1,3,4,7,9 \pmod{12}$.

Thus, restricted to $v \equiv 0,4 \pmod 6$, this lemma guarantees that there is at least one TTS of order v, with $v \equiv 0$ or $4 \pmod{12}$ which can be partitioned into two isomorphic sets of triples. This further reduces our problem to orders $v \equiv 6$ or $10 \pmod{12}$.

As usual, let us call an isomorphism defining a desired partition of a TTS(v) a complementing permutation (c.p.) for the partition. But the fact that every pair of distinct elements in a TTS occurs exactly twice allows a c.p., in this case, to contain cycles of various odd and even lengths. This presents difficulties which we are not able to overcome at present. Thus, we remain far from a complete answer to our problem in this case. However, it can be checked directly that the unique TTS(6) can not be partitioned into two isomorphic sets of triples.

6.5. Steiner systems S(2,4,v)

In this section we ask questions similar to those asked in the preceding sections concerning partitioning of a Steiner system S(2,4,v) into two isomorphic sets of quadruples. It is well known that a Steiner system S(2,4,v) exists if and only if $v \equiv 1$ or 4 (mod 12). Further, the number of quadruples in a Steiner system S(2,4,v) is even or odd according as $v \equiv 1, 16 \pmod{24}$ or $v \equiv 4, 13 \pmod{24}$ respectively.

Considerably less is known at present about Steiner systems S(2,4,v) than about STSs. For instance, although many infinite classes of cyclic S(2,4,v)'s are known (cf. [17, 24, 31]), a necessary and sufficient condition for the existence of cyclic S(2,4,v)'s remains unknown. Cyclic S(2,4,v) for $v = 16,25,28$ are known not to exist and it is conjectured [31] that they exist for all other orders $v \equiv 1$ or 4 (mod 12). It is obvious that if this conjecture were true it would instantly imply that for each order $v \equiv 16 \pmod{24}$, except for $v = 16$, there exists a S(2,4,v) whose blocks can be partitioned into two isomorphic sets.

The unique S(2,4,16) (i.e., the affine plane of order 4) is easily seen to be partitionable in the above sense. For this it suffices to take as set of elements $V = Z_8 \times \{1,2\}$, and as base blocks

$$\{0_1, 4_1, 0_2, 4_2\}, \{0_1, 1_1, 2_2, 3_1\} \{0_2, 2_2, 3_2, 5_1\} \pmod 8.$$

Thus we make the conjecture of our own:

Conjecture. For every $v \equiv 16 \pmod{24}$ there exists an S(2,4,v) whose blocks can be partitioned into two isomorphic sets.

Let us remark that the conjecture is true for v = 16,40,64.

We can say even less about S(2,4,v)'s if $v \equiv 1 \pmod{24}$. Of the 16 known nonisomorphic S(2,4,25)'s, only one (no. 1 in the listing of [29]) admits partitioning into two isomorphic "halves". We can say virtually nothing about large orders, except perhaps that the (conjectured) existence of cyclic S(2,4,v)'s is of no help for our problem in this case.

CONCLUSION

The study of selfcomplementary (s.c.) graphs has generated considerable interest though such a graph exists with v vertices only for $v \equiv 0,1 \pmod 4$, i.e., when K_v has an even number of edges. Besides the interesting properties of this class of graphs, the basic idea involved in the definition of s.c. graphs is to partition a complete graph K_v, for admissible v, into two isomorphic spanning subgraphs. In this thesis, this notion is extended to a partitioning of an almost complete graph $\tilde{K}_v$ (obtained after deleting one edge from the complete graph K_v with an odd number of edges) and the properties of the new class of graphs, called almost selfcomplementary (a.s.c.) graphs are studied.

It is found that the a.s.c. graphs and s.c. graphs have many properties in common. In addition, it is also proved that many of the properties of the s.c. graphs are preserved by the graph (the so called k–s.c. graph) obtained by partitioning a certain subgraph of a complete graph (from which some or all edges of a matching contained in the complete graph have been deleted) into two isomorphic subgraphs.

The enumeration problem for a.s.c. graphs is one of the open problems that remain. Another possible extension of the idea of selfcomplementarity, along the lines of a.s.c. graphs, is to study partitions of an r–partite complete graph with an odd number of edges, and their properties. The class of r–partite s.c. graphs has been studied in [19, 42].

The class of strongly regular s.c. graphs has been studied by many authors [32, 39, 46, 47], due to the connection of these graphs with combinatorial designs. But there exist no regular a.s.c. graphs and so it seems that nothing can be done in this direction. However, in Chapter 6, the notion of selfcomplementarity is extended to combinatorial designs by examining whether a Steiner triple system (twofold triple system, and a Steiner system S(2,4,v), respectively) can be partitioned into isomorphic hypergraphs. This chapter also contains many questions about the existence of such a partition still to be answered.

Above all, the feasibility of selfcomplementary or "nearly" selfcomplementary partitioning of a complete graph or of a complete multigraph, possibly with some additional conditions imposed, implies that the notion of such partitioning may be extended to many other complete configurations.

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