DESIGNS ON CUBIC MULTIGRAPHS

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ABSTRACT

Necessary conditions are found for the existence of graph designs on cubic multigraphs. It is shown that, with a few given exceptions, these conditions are sufficient for all connected cubic multigraphs on six or fewer vertices, and for four of the five disconnected ones. Partial results are obtained for the remaining multigraph, which consists of a K_4 and a $3K_2$ component. Necessary and sufficient conditions for the existence of resolvable designs on all bipartite cubic multigraphs on six or fewer vertices are found. Graceful labellings are given for all cubic graphs on eight or ten vertices, and for all prisms on eighteen or fewer vertices. These are used to find some designs on these graphs, including some infinite classes. In addition, some small designs are found for the 5-prism and the Petersen graph, and some results are given for cubes.

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INTRODUCTION

A *graph design* (or G-design) is a generalization of a balanced incomplete block design (BIBD) in which a structure is imposed on each block. In graph theoretic terms, a G-design is a partitioning of the edges of a complete multigraph into copies of a given multigraph G. BIBDs then correspond to G-designs where G is a complete graph. A *graph decomposition* is more general than a graph design; it is a partition of the edges of *any* multigraph (not necessarily a complete multigraph) into copies of a given multigraph G. A (v,k,λ) G-design is defined to be a decomposition of the complete multigraph λK_v into copies of a given graph G, where G has k vertices. Graph designs were first introduced in 1972 by Hell and Rosa [19], who were interested in the case where G is the path P_3. Since then, classes of G-designs which have been studied include designs on paths, stars, cicuits, complete bipartite graphs and cubes. (See [6] for a survey.)

Although Wilson [36] has proved that asymptotically the necessary conditions for the existence of a G-design are sufficient, the exact conditions for small designs must still be determined. To do this we must look at values of v and λ satisfying the necessary conditions for the existence of a design on a given graph G, and either demonstrate that a design with these parameters exists, or prove that it does not. Generally, we must find smaller designs by direct construction, and for larger ones we can use recursive techniques.

In this paper we look at designs on cubic graphs and multigraphs. First we find necessary conditions for the existence of a G-design when G is a cubic multigraph. The smallest cubic multigraph is $3K_2$, and designs on this multigraph exist trivially when the necessary condition $3|\lambda$ is satisfied. Bialostocki and Roditty [8] have in effect shown that the necessary conditions are sufficient for the two disconnected multigraphs consisting of two and three triple edges respectively. Designs on K_4 are BIBDs, and Hanani [17] has shown that they exist whenever the necessary conditions are satisfied. Huang [21] has demonstrated that the necessary conditions are sufficient for the existence of $K_{3,3}$-designs, except when $v=10$ and $\lambda=1$, and when $v=6$ and $\lambda \equiv 3$ or 15 (mod 18). We go on to show that, with one exception, a design exists on each remaining connected cubic multigraph on six or fewer vertices whenever the necessary conditions on v and λ are satisfied. This exception is the multigraph on four vertices we have called

Cy (for cylinder). We prove that a (5,4,3) *Cy*-design does not exist. Then we examine the question for the remaining two cubic multigraphs on six or fewer vertices, which are both disconnected. For the first one, S_8 in Fig. 3.1, we show that the necessary conditions are sufficient except when $v = 6$ and $\lambda = 3$, where we prove a design does not exist. For the second, S_9 in Fig. 3.1, which consists of a K_4 and a $3K_2$ component, we present partial results, and state which designs are still needed to settle the question.

When constructing a design directly we most often find a set of *base blocks* which generate the whole design under a given automorphism. The recursive methods we use usually involve "gluing together" smaller designs to get larger ones, using as "glue" decompositions of complete bipartite multigraphs. In the case where *G* is not bipartite but tripartite we must use instead decompositions of complete tripartite multigraphs, and use results on triple systems to build the designs. If *G* is not tripartite, but quadripartite, we must use decompositions of complete quadripartite multigraphs, but we look at only one such multigraph and have been unable to completely determine the conditions for the existence of a *G*-design for this multigraph. From Theorem 1.8 the existence of a (v,k,λ_1)

and a (v,k,λ_2) *G*-design implies the existence of a $(v,k,r\lambda_1+s\lambda_2)$ *G*-design for all integers $r,s \geq 0$, so we need only look at designs with minimal values of λ.

A *G*-design is *resolvable* when the blocks can be partitioned into parallel classes. We look at resolvable designs in Chapter 8. We start by deriving necessary conditions for the existence of resolvable designs on cubic multigraphs. Huang [21] has shown that the necessary conditions for a resolvable $K_{3,3}$ design to exist are sufficient except when $v = 6$ and $\lambda \equiv 0 \pmod 3$. We show that the necessary conditions are sufficient for all other bipartite connected multigraphs on six or fewer vertices.

In Chapter 9 we present some results on larger cubic graphs, specifically cubic graphs on eight and ten vertices, prisms, and cubes. We find graceful labellings on all cubic graphs on eight and ten vertices, from which we can find some designs using the results in [31] and [26]. Similarly, we can find some designs, including an infinite class, from the graceful labellings on prisms given by Frucht and Gallian [14]. Next we examine more closely the prism on ten vertices, D_5, and give some small designs on this graph. We also find some small designs on the Petersen graph. Lastly we present some results due to Kotzig[24], who has used [31] and [26] to find two infinite classes of designs on cubes.

In the conclusion we summarize our results, and identify certain open problems and directions for further research.

CHAPTER 1

BASIC CONCEPTS

§1.1 Preliminaries

A *combinatorial (block) design* is a pair $(\mathcal{V},\mathcal{B})$ where $\mathcal{V}$ is a set of elements (or points, or varieties) and $\mathcal{B}$ is a collection of subsets of $\mathcal{V}$ which we call blocks (or lines). The design $\mathcal{D} = (\mathcal{V},\mathcal{B})$ can be represented by an incidence matrix $\mathcal{M} = (m_{i,j})$ of dimension $v \times b$ where

$$v = |\mathcal{V}|,$$

$$b = |\mathcal{B}|,$$

$$\mathcal{V} = (a_1,a_2,\ldots\ldots,a_v),$$

$$\mathcal{B} = (B_1,B_2,\ldots\ldots,B_v),$$

and $\mathcal{M}$ is defined by

$$m_{i,j} = \begin{cases} 1 \text{ if } a_i \in B_j, \\ 0 \text{ otherwise;} \end{cases}$$

e.g., $\mathcal{V} = \{0,1,2,3,4\}$,

$$\mathcal{B} = \{(0,1),(1,2,3),(0,3,4)\},$$

$$\mathcal{M} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Definition 1.1 Let $\mathcal{D}_1=(\mathcal{V}_1,\mathcal{B}_1)$ and $\mathcal{D}_2=(\mathcal{V}_2,\mathcal{B}_2)$ be two designs. Then a bijection

$$i:\mathcal{V}_1 \to \mathcal{V}_2$$

is an *isomorphism* if the induced mapping, $\hat{i}$, from $\mathcal{B}_1$ into $\mathcal{V}_2$ given by

$$\hat{i}(B) = \{i(\alpha):\alpha\in B\}$$

is also a bijection from $\mathcal{B}_1$ onto $\mathcal{B}_2$. In this case $\mathcal{D}_1$ and $\mathcal{D}_2$ are isomorphic, and if $\mathcal{D}_1=\mathcal{D}_2$ then i is an automorphism.

In terms of the incidence matrices of $\mathcal{D}_1$ and $\mathcal{D}_2$, say $\mathcal{M}_1$ and $\mathcal{M}_2$, $\mathcal{D}_1$ is isomorphic to $\mathcal{D}_2$ if there exist row and column permutations which transform $\mathcal{M}_1$ into $\mathcal{M}_2$, or equivalently there exist permutation matrices $\mathcal{P}$ and Q such that $\mathcal{P}\mathcal{M}_1 Q=\mathcal{M}_2$. The set of all automorphisms of a design forms a group, denoted Aut $\mathcal{D}$.

A *pairwise balanced design* (PBD) is a design in which each pair of elements is contained in the same number λ of blocks.

Definition 1.2 A design $\mathcal{D}=(\mathcal{V},\mathcal{B})$ is a *Balanced Incomplete Block Design*, or *BIBD*, with parameters v,k,λ if it satisfies

(i) $|\mathcal{V}| = v$,

(ii) $|B| = k$ for all $B\in\mathcal{B}$,

(iii) $|\{B:\{x,y\}\subset B, x\neq y, x,y\in V\}| = \lambda$

(iv) $k < v$

(i.e., all blocks have the same size k and any pair of distinct elements of $\mathcal{V}$ occurs in the same number λ of blocks. Condition (iv) is needed to exclude the possibility that $B = \mathcal{V}$ for every $B\in\mathcal{B}$.)

Definition 1.3 A *graph* G is a pair $(\mathcal{V},\mathcal{E})$ where $\mathcal{V}=\mathcal{V}(G)$ is a set of *points* or *vertices* , and $\mathcal{E}=\mathcal{E}(G)$ is a set of *edges* or *lines*, each of which corresponds to a unique unordered pair of distinct vertices and is said to join these two vertices, and no two of which correspond to the same pair of vertices. If there is an edge joining each pair of vertices then G is the *complete* graph on v vertices, K_v, where $v=|\mathcal{V}(G)|$.

Definition 1.4 A *multigraph* is a generalization of a graph in which repeated edges are allowed - i.e. two vertices may be joined by two or more edges. The *complete multigraph* on v vertices λK_v is the graph on v vertices in which every pair of distinct vertices is joined by exactly λ edges. A multigraph H is a *submultigraph* of a multigraph G if $\mathcal{V}(H)\subseteq\mathcal{V}(G)$ and $\mathcal{E}(H)\subset\mathcal{E}(G)$. An *n vertex clique* of a graph $(\mathcal{V},\mathcal{E})$ is a subgraph of G which is isomorphic to the complete graph on n vertices.

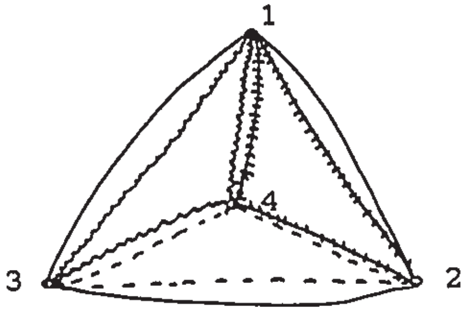
Definition 1.5 A *G-decomposition* of the graph $H = (\mathcal{V},\mathcal{E})$ is a partition of H into subgraphs $G_i = (\mathcal{V}_i,\mathcal{E}_i)$, $i=1,\ldots,q$, (where $\bigcup_i \mathcal{E}_i = \mathcal{E}$, $\mathcal{E}_i\cap\mathcal{E}_j = \emptyset$ for $i\neq j$) such that each G_i is isomorphic to a given graph G. We shall call these subgraphs *G-blocks*. When H is the complete multigraph λK_v these decompositions are known as *graph designs* or briefly *G-designs*.

Graph designs can be viewed either in terms of graphs or in terms of block designs. (Copies of the graph G in a graph design correspond to blocks in a block design.) Cases where G is a complete graph K_k correspond to balanced incomplete block designs (BIBD's) with blocks of size k. When G is not a complete graph, G-designs are generalizations of BIBD's in which a structure is imposed on each block - a pair of elements occurring in a block is "linked" in that block only if it is joined by an edge in the corresponding copy of G.

A *parallel class* in a design $\mathcal{D} = (\mathcal{V},\mathcal{B})$ is a set of blocks $P \subseteq \mathcal{B}$ which partitions the set of elements $\mathcal{V}$; i.e., if $P = \{B_1, B_2, \ldots, B_n\}$ is a parallel class then $B_i \cap B_j = \emptyset$ for $i \neq j$, and $\bigcup_{i=1}^{n} B_i = \mathcal{V}$.

A graph design is said to be *resolvable* if the blocks can be partitioned into parallel classes.

In Fig. 1.1 we see two examples of a graph design. The first is a decomposition of $2K_4$ into K_3's. This corresponds to a type of BIBD called a *Twofold Triple System*. (Each pair of distinct elements occurs in exactly two triples.) Here any two elements occurring together in a block are linked in that block since G in this case is a complete graph. Note that each element occurs in exactly three copies of K_3 and thus in three blocks of the block design. In the second example we have a decomposition of K_4



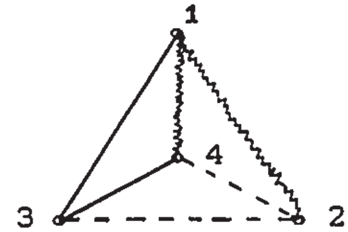


Fig. 1.1

into P_3's (paths with three vertices). P_3 is K_3 with one edge removed, and since it is not a complete graph, this is not a BIBD and not every pair of elements occurring in a block is linked in that block. In this particular case, where G is P_3, we have listed the blocks in such a way that only adjacent elements in a block are linked (i.e. not first and last). In general, when listing blocks in a G-design, we must specify the relationship between the block listing and the correspnding copy of G - i.e. which elements in a block are linked and by how many edges. This latter example with $G = P_3$ was the first generalization of BIBD's to graph designs, and in the resolvable case is known as the Handcuffed Prisoner's Problem (Hell and Rosa [19]). Note that in this case not every element occurs in the same number of blocks - three occur in two blocks and one occurs in three blocks.

We call a G-design on λK_v where G is a graph on k vertices a *(v,k,λ) G-design*. If each vertex of K_v occurs in the same number of copies of G (each element occurs in the same number of G-blocks) then the G-design is said to be *balanced*. The second example above, where G is P_3, is a case where the design is not balanced. In the case where G is a regular graph (all vertices have the same degree as in the first example above) it can easily be seen that the G-design is automatically balanced. Here is the main question regarding G-designs: Given a graph G on k vertices, for which values of v and λ does a G-design exist?

Some types of graph G which have been looked at with a view to finding G-designs are the following:

(1) the complete graph on k vertices K_k

(2) the star S_k

(3) the path P_k (with k vertices and k-1 edges)

(4) the cycle C_k

(5) the complete bipartite graph $K_{m,n}$

(6) small graphs with k vertices where $3 \leq k \leq 5$

(7) the cube Q_d.

When G is a complete graph K_k a G-design is the same thing as a BIBD. A lot of work has been done on these, see for instance Hanani [17]. Designs on cubes have been looked at by Kotzig [24]. For results on the other categories above see the survey by Bermond and Sotteau [6].

We shall be looking here only at G-designs on *regular* graphs, specifically at those on *cubic* graphs. Of the above, only complete graphs, cycles, and cubes are necessarily regular. (Categories (5) and (6) include some regular graphs.)

§1.2 Elementary Relations

First we derive necessary conditions for a (v,k,λ) G-design to exist when G is a d-regular graph on k vertices.

Theorem 1.6 Necessary conditions for a (v,k,λ) G-design to exist when G is a d-regular graph on k vertices are

$$\lambda v(v-1) \equiv 0 \pmod{kd} \qquad (1)$$

$$\lambda(v-1) \equiv 0 \pmod{d} \qquad (2)$$

$$\lambda \geq m \qquad (3)$$

$$v \geq k \qquad (4)$$

(where m is the greatest multiplicity of edges in G).

Proof The number of edges in G must divide the number of edges in λK so

$$[\lambda v(v-1)/2]/(kd/2) = \lambda v(v-1)/kd$$

must be an integer, which means we must have

$$\lambda v(v-1) \equiv 0 \pmod{kd}.$$

Also, since the design must be balanced, we have that each vertex of λK_v appears in the same number, say r, of G-blocks. Therefore $\lambda(v-1)/d$ must also be an integer, which means

$$\lambda(v-1) \equiv 0 \pmod{d}.$$

It is obvious that if G has multiple edges we must have $\lambda \geq m$ where m is the greatest multiplicity of edges in G $(1 \leq m \leq d)$, and of course we cannot have a block size greater than the number of points so we must have $v \geq k$.

Corollary If G is a cubic multigraph on k vertices with greatest multiplicity of edges m, then the necessary conditions for a (v,k,λ) G-design to exist are

$$\lambda v(v-1) \equiv 0 \pmod{3k} \qquad (1)$$

$$\lambda(v-1) \equiv 0 \pmod{3} \qquad (2)$$

$$\lambda \geq m \qquad (3)$$

$$v \geq k \qquad (4).$$

Lemma 1.7: The number of blocks in a (v,k,λ) G-design is given by

$$b = \lambda v(v-1)/kd.$$

Proof The number of blocks must equal the number of edges in λK_v divided by the number of edges in a G-block which is the number of edges in G. So we have

$$b = [\lambda v(v-1)/2]/[kd/2] = \lambda v(v-1)/kd.$$

Given a multigraph G, and values of v and λ satisfying the above necessary conditions, we must try to find a G-design on λK_v (or prove there is none). Asymptotically (for v sufficiently large) the necessary conditions are always sufficient (Wilson [36]). Therefore the real problem is to determine the exceptions in each case.

§1.3 Designs on 1-regular and 2-regular Multigraphs

The only connected 1-regular graph is the single edge () and all $(v,2,\lambda)$ designs exist on this graph trivially. Disconnected 1-regular graphs are of the form

so consist of m disjoint edges. If $m=v/2$ then such a G-design is equivalent to a partition of λK_v into 1-factors or matchings. For the case of arbitrary m, see for example Rees [29].

Connected 2-regular graphs are cycles C_k (C_k is the cycle on k vertices with k edges). The smallest of these is C_3 which is the same as K_3. Designs on this graph constitute the well known class of BIBDs called triple systems. These are known to exist whenever the necessary conditions of Theorem 1.6 are satisfied (Hanani [18]). For example, in the case $\lambda=1$, a $(v,3,\lambda)$ C_3-design, or Steiner Triple System, exists whenever $v \equiv 1,3 \pmod 6$ and $v \geq 3$. It has been shown that a (v,k,λ) C_k-design exists if

$k \equiv 0 \pmod 4$ and $v=2pk+1$ (Kotzig [25])

$k \equiv 2 \pmod 4$ and $v=2pk+1$ (Rosa [33])

k odd and $v=2pk+1$ or $v=2pk+k$ (Rosa [32]).

The necessary conditions of Theorem 1.6 have been shown to be sufficient for small values of k: by Huang and Rosa [34] for k = 4, 5, 6; and in [3], [4], and [5] for k = 4, 5, 6, 7, 8.

Disconnected 2-regular graphs are collections of disjoint cycles. The case where $k=v$ is equivalent to partitioning the edges of λK_v into isomorphic 2-factors. If $\lambda=1$, this is known as the *Oberwolfach problem* and is unsolved in general. (The Oberwolfach problem can be stated as follows: Given n participants and tables (T_i), each acommodating t_i people ($t_i \geq 3$) where the total number of people who can be accomodated is n, is it possible to seat these n people on m occasions so that each pair of participants are seated next to each other exactly once?) The smallest disconnected 2-regular graph is that consisting of two disjoint triangles, and among other configurations of pairwise disjoint triples has been looked at by Horák and Rosa [20]. Note that there are none of these on fewer than six vertices.

The next class of regular graphs is 3-regular or *cubic* graphs. The only examples of cubic graphs covered in the seven categories above are K_4, $K_{3,3}$ and the cube on eight vertices, Q_3. (The other cubes are all regular, but this is the only one which is cubic!) In this paper we shall look at G-designs on all cubic multigraphs on six or fewer vertices and we shall also consider some additional cases.

§1.4 Methods of Construction

The methods used to find G-designs fall into two general categories: direct and recursive. Typically we must find some designs directly for small values of v, and use recursion for larger values of v. We now give some definitions that will be needed in order to describe the techniques we use to find designs directly.

If $\mathcal{A}$ is an automorphism of a balanced G-design $\mathcal{D}$ with v elements, then two elements x and y of $\mathcal{D}$ are said to be in the same *orbit of elements* if $\mathcal{A}^t(x) = y$ for some $t \geq 1$. Two blocks $\mathcal{B}_m$ and $\mathcal{B}_n$ of $\mathcal{D}$ are in the same *orbit of blocks* if $\mathcal{A}^s(\mathcal{B}_m) = \mathcal{B}_n$ for some $s \geq 1$.

The property of being in the same orbit is an equivalence relation, so the elements and blocks of a design are partitioned into disjoint orbits by an automorphism $\mathcal{A}$ of $\mathcal{D}$. An orbit of blocks (or elements) can be generated by any one of its members. Therefore we need only give one block, which we shall call a *base block*, to represent each orbit of blocks in a design. A design with a given automorphism can then be completely determined by a set of base blocks, consisting of one block from each orbit.

To find a small design directly we first calculate (from Lemma 1.7) the number of blocks b that such a design would have. This tells us what type of design to look for. If $v|b$ we look for a cyclic design on Z_v, and if $(v-1)|b$ we look for a design on $\{Z_{v-1} \cup \infty\}$ (Z_{v-1} plus a "point at infinity", so called because it is fixed under all automorphisms). If $(v/2)|b$ but $v \nmid b$ then we try for a cyclic design on Z_v with one short orbit of blocks of length $v/2$ and the remaining orbits of length v.

The *order* of a base block B in a design under automorphism $\mathscr{A}$ is the smallest positive integer m such that

$$\mathscr{A}^m(B)=B.$$

A G-design $\mathfrak{D} = (\mathscr{V},\mathfrak{B})$ is said to be *cyclic* if its automorphism group Aut $\mathfrak{D}$ contains an element consisting of a single cycle of length $v = |\mathscr{V}|$. In this case we take the elements of the design to be the elements of Z_v, and as differences use the values

$$\delta_{i,j} = \min(|i-j|,\ v-|i-j|)$$

where $i,j \in Z_v$, and $\delta_{i,j}$ is called the *edgelength* between i and j. There are $\lfloor v/2 \rfloor$ edgelengths in Z_v, corresponding to the edgelengths in K_v.

It can easily be shown that the order m of a base block in a cyclic design must satisfy the conditions

$$1 \le m \le v, \quad mk \ge v \quad \text{and} \quad m|v.$$

In a cyclic design all differences are *pure differences*. A pure difference x_k-y_k is defined to be the difference between two distinct elements which are in the same orbit k of elements. In general the designs we construct will have more than one orbit of elements, and therefore besides pure differences we will have *mixed differences*, which are defined to be differences $x_k - y_l$ between elements which are in different orbits k and l of elements. The differences between the ∞ element and other elements belong to a special class of mixed differences. In constructions where we use an ∞ element, we have one "∞ difference", labelled ∞_k, between the orbit consisting of the ∞ element and each other orbit k of elements.

To find a design on Z_v we must construct base blocks on Z_v which between them use each edgelength from 1 to $\lfloor v/2 \rfloor$ λ times. If v is even the largest edgelength $v/2$ is needed only $\lambda/2$ times because it occurs half as often as the other edgelengths in K_v. Edgelenths occuring in a base block which generates a half orbit of length $v/2$ are counted once for every second time they occur in this block. If the element set is $Z_{v-1} \cup \infty$ the differences we must find are the edgelengths in K_{v-1} plus the difference with value ∞. Sometimes we have to use more complicated element sets for our base blocks - in general they will be of the form $\left\{ Z_\rho \times \{0,1,\ldots,q-1\} \right\}$ or $\left\{ (Z_\rho \times \{0,1,\ldots,q-1\} \cup \infty \right\}$ where $\rho \times q = v$ or $v-1$ respectively. In these cases we have mixed differences between each two copies of Z_ρ as well as pure ones within each copy.

For a design on $\{Z_\rho \times \{0,1,\ldots,q-1\}\}$ the differences we must find are the $\lfloor \rho/2 \rfloor q$ pure differences defined by

$$\delta^k_{i,j} = [\min(|i-j|,\ \rho-|i-j|)]_k \qquad i,j \in Z_\rho,\ i \neq j$$
$$k = 0,1,\ldots q-1$$

and the $\begin{bmatrix} q \\ 2 \end{bmatrix} \rho$ mixed differences defined by

$$\mu^{k,l}_{i,j} = [(i-j) \bmod \rho]_{k,l} \qquad i,j \in Z_\rho$$
$$k,l = 0,1,\ldots q-1,\ k \neq l.$$

If the element set is $\{(Z_\rho \times \{0,1,\ldots,q-1\} \cup \infty\}$ we need all the above differences plus the q "∞ differences" given by

$$\mu^{\infty,k}_{i} = \infty_k \qquad i \in Z_\rho$$
$$k=0,1,\ldots q-1.$$

Once we have found a set of base blocks which between them use each difference in the element set λ times, the required design is the set of all blocks contained in the orbits generated by these base blocks. Unless otherwise specified, the automorphism $\mathscr{A}$ used to generate a whole design with element set Z_ρ, $Z_\rho \cup \infty$, $Z_\rho \times \{0,1,\ldots,q-1\}$ or $(Z_\rho \times \{0,1,\ldots,q-1\}) \cup \infty$ from a set of base blocks will be that which adds 1 (mod ρ) to each element.

Thus, essentially, the method we have just described is a variation on Bose's method of "symmetrically repeated differences" (see for example [16]).

When we have found the first few cases directly, we can usually find the rest using recursion. The recursive method we shall be using most commonly in this thesis involves finding decompositions of complete bipartite or tripartite multigraphs into G-blocks, and using these to "glue together" smaller designs to make bigger ones. This will be explained in more detail when we come to an example of this in Chapter 2. We need only find designs with minimal values of λ since we can find ones with larger λ's using the following lemma.

Lemma 1.8 Given (resolvable) (v,k,λ) graph designs on a multigraph G for $\lambda=\lambda_1$ and $\lambda=\lambda_2$, we can find (resolvable) $(v,k,r\lambda_1+s\lambda_2)$ G-designs for all integers $r,s \geq 0$.

Proof The set of edges in the multigraph $(r\lambda_1+s\lambda_2)K_v$ is the union of the edge sets in the two multigraphs $r\lambda_1K_v$ and $s\lambda_2K_v$. The edges in $r\lambda_1K_v$ can be covered with r copies of the given decomposition of λ_1K_v, and the edges of $s\lambda_2K_v$ with s copies of the given decomposition of λ_2K_v. This covers all edges in $(r\lambda_1+s\lambda_2)K_v$ and thus gives us a $(v,k,r\lambda_1+s\lambda_2)$ G-design. If the (v,k,λ_1) and (v,k,λ_2) designs were resolvable then so is the $(v,k,r\lambda_1+s\lambda_2)$ design since the parallel classes are preserved.

Corollary If a (v,k,λ) G-design exists then a $(v,k,s\lambda)$ G-design exists for all positive integers s.

When finding decompositions of complete bipartite multigraphs we shall make use of the following result.

Lemma 1.9 If there exist decompositions of $\lambda K_{n,n}$ and $\lambda K_{m,n}$ into G-blocks for a given graph G, then there also exists a decomposition of $s\lambda K_{pn+qm,rn}$ into G-blocks for all positive integral values of p,q,r,s.

Proof The edge set of $s\lambda K_{pn+qm,rn}$ is the union of the edge sets of $s\lambda K_{pn,rn}$ and $s\lambda K_{qm,rn}$, so if we can decompose each of these into G-blocks we are through. $\lambda K_{pn,rn}$ can be decomposed into pr copies of $\lambda K_{n,n}$ and we are given that each of these can be decomposed into G-blocks. Therefore $\lambda K_{pn,rn}$ is also decomposable into G-blocks. Similarly $\lambda K_{qm,rn}$ can be decomposed into G-blocks. To get decompositions of $s\lambda K_{pn,rn}$ and $s\lambda K_{qm,rn}$ take s times the decompositions of $\lambda K_{pn,rn}$ and $\lambda K_{qm,rn}$.

Given a multigraph G for which we wish to examine the existence of G-designs we proceed as follows: First we find the necessary conditions for such designs to exist from Theorem 1.6. This gives us necessary conditions in terms of λ and v. Next we find the necessary conditions on v for different values of λ, and work out which values of λ we need to look at. If G is a simple graph, for instance, then we start with $\lambda=1$, but otherwise we must start with $\lambda=m$ (m is largest multiplicity of edges in G). The multigraphs we shall be considering all have $m = 1, 2,$ or 3 since they are cubic graphs. Once we have finished with the smallest λ, we proceed to the next for which all G-designs cannot be derived from this using Lemma 1.8. We continue looking at new λ's until all larger designs can be derived from the ones we have. Usually this is when we hit a λ for which there are no necessary conditions on v other than $v \geq k$.

If G has no multiple edges we start with $\lambda=1$. If we can solve the $\lambda=1$ case entirely, then we have solutions for all other cases with the same necessary conditions, so that for higher λ's we need only look at cases with weaker necessary conditions, and for these only at those v not covered in the $\lambda=1$ case. If G has $m>1$, then we must look first at the case $\lambda=m$, and then at cases where $m \nmid \lambda$ and/or the necessary conditions on v are weaker. We continue until designs with all subsequent λ's can be derived from the ones we have, which is usually at the smallest λ with no necessary conditions on v (except $v \geq k$).

Once we have decided which values of λ we need to look at, we take the first and try to find all (v,k,λ) G-designs for this λ, starting with the smallest v for which the necessary conditions are satisfied. Then we do the same for the other λ's. If there are any designs which cannot be found, we must also look for those designs with larger λ which we could otherwise have derived from these using Lemma 1.8. Usually the designs in the first few cases must be found directly.

When dealing with tripartite decompositions, which we must do if G is tripartite but not bipartite, we make use of some results on Steiner Triple Systems, and other related designs, to "glue" the designs together. In the case $k=6$, $\lambda=9$ we use some results on Latin Squares and Transversal Designs, and the more general result of Lemma 1.10 below.

Lemma 1.10 Let G_1, G_2, G_3 be three multigraphs with

$$|\mathcal{V}(G_1)|\leq|\mathcal{V}(G_2)|\leq|\mathcal{V}(G_3)|,$$

$$|\mathcal{E}(G_1)|\leq|\mathcal{E}(G_2)|\leq|\mathcal{E}(G_3)|.$$

If there exists a (resolvable) decomposition of G_3 into G_2-blocks, and a (resolvable) decomposition of G_2 into G_1-blocks, there also exists a (resolvable) decomposition of G_3 into G_1-blocks.

Proof To find such a decomposition, take the decomposition of G_3 into G_2-blocks, and cover each G_2-block with the decomposition of G_2 into G_1-blocks. It can be shown that resolvability is preserved here.

CHAPTER 2

DESIGNS ON CUBIC MULTIGRAPHS WITH FEWER THAN SIX VERTICES

§2.1 Introduction

There are only four cubic multigraphs on fewer than six vertices, as shown in Fig. 2.1:

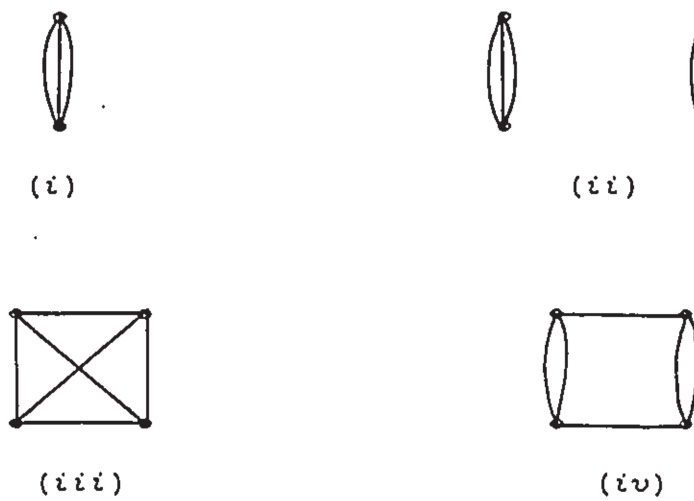


Fig. 2.1

The first case is trivial – a G-design exists on this graph whenever each edge multiplicity of the complete multigraph is a multiple of three – i.e. when $3|\lambda$. There is no restriction on v other than $v>1$. The second has, in effect, been solved by Bialostocki and Roditty [8], who have shown that the necessary conditions of Theorem 1.6 are sufficient. The third is K_4, and here it has been shown by

Hanani [17] that the necessary conditions are sufficient. Therefore the smallest unsolved case for cubic multigraphs is the fourth graph, which we shall call the *Cylinder*, abbreviated *Cy*.

Lemma 2.1 When G is a cubic multigraph on four vertices, the necessary conditions for the existence of a G-design are

$$\lambda v(v-1) \equiv 0 \pmod{12} \qquad (1)$$

$$\lambda(v-1) \equiv 0 \pmod{3} \qquad (2)$$

$$\lambda \geq 2 \qquad (3)$$

$$v \geq 4 \qquad (4)$$

and the necessary conditions on v for all the different possible values of λ are as follows:

$$v \equiv 1,4 \pmod{12} \text{ if } \lambda \equiv 1,\ 5 \pmod{6}$$

$$v \equiv 1 \pmod{3} \text{ if } \lambda \equiv 2,\ 4 \pmod{6}$$

$$v \equiv 0,\ 1 \pmod{4} \text{ if } \lambda \equiv 3 \pmod{6}$$

$$v \geq 4 \text{ if } \lambda \equiv 0 \pmod{6}.$$

Proof These results follow from Theorem 1.6.

§2.2 Cy-designs with $\lambda=2$

For the cylinder *Cy* we shall list blocks as shown below (Fig.2.2):

$(a,b;c,d)$

Fig. 2.2

The first two elements, separated by a comma in the block listing, are joined by a double edge in the *G*-block, as are the last two. The middle two elements are separated by a semi-colon in the block listing and are joined by a single edge in the *G*-block. The first and last elements in the block listing are also joined by a single edge in the *G*-block. The remaining pairs of edges in the block (i.e. non consecutive ones) are not joined by an edge at all.

If we can find *Cy*-designs in the cases $\lambda=2$, $\lambda=3$ and $\lambda=6$ then the designs for all other λ can be derived from these using the result of Lemma 1.8. We start with the smallest value $\lambda = 2$. The three smallest designs in this case, which have $v = 4$, 7 and 10 respectively, are found by direct construction.

Lemma 2.2 A (4,4,2) Cy-design exists.

Proof (by construction) The number of blocks in such a design is 2 from Lemma 1.7. A design of this type is shown in Fig. 2.3.

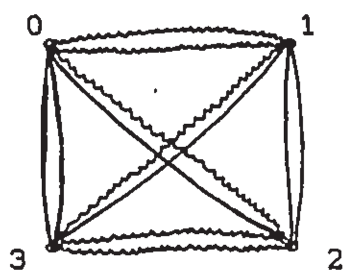


Fig. 2.3

We can represent this design by listing the two blocks

(0,1;3,2) and (1,2;0,3),

or since this is in fact a cyclic design on Z_4 with cycle of length two, we could just give one of the blocks, say the first one:

(0,1;3,2).

The second can then be obtained by adding 1 to each element in the first (mod 4). Adding 1 to each element in the second gives us the first again, so this base block generates a half orbit of length 2.

Lemma 2.3: A (7,4,2) *Cy*-design exists.

Proof The number of blocks b is 7 in this case. A cyclic design on Z_7 is generated by the block

$$(0,1;6,2).$$

This can be verified by checking that each of the differences 1,2 and 3 occurs twice in this block.

Lemma 2.4: A (10,4,2) *Cy*-design exists.

Proof: Here $b = 15$. A design of this type is generated on Z_{10} by the blocks

$$(0,8;7,1)$$

$$(0,3;8,5) \quad \text{(half orbit)}.$$

The designs in the remaining cases can be found recursively. To do this we must find decompositions of some complete bipartite multigraphs into *Cy*-blocks.

Lemma 2.5 The complete bipartite multigraph $2K_{3,3}$ can be decomposed into submultigraphs isomorphic to *Cy*.

Proof The required decomposition is shown in Fig. 2.4:

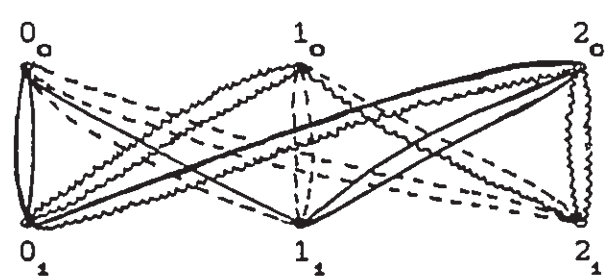


Fig. 2.4

The elements here are the vertices of the bipartition

$$(0_0,1_0,2_0;0_1,1_1,2_1),$$

and the blocks are

$$(0_0,0_1;2_0,1_1),\ (0_0,2_1;1_0,1_1),\ (1_0,0_1;2_0,2_1).$$

Lemma 2.6 The complete bipartite multigraphs $2K_{6,6}$ and $2K_{9,6}$ can be decomposed into *Cy*-blocks.

Proof Follows from previous lemma and Lemma 1.9.

Theorem 2.7 For a $(v,4,2)$ *Cy*-design to exist it is necessary and sufficient that $v \equiv 1 \pmod 3$.

Proof This condition is necessary from Lemma 2.1. To show it is also sufficient, we must find $(v,4,2)$ *Cy*-designs for all $v \equiv 1 \pmod 3$. Designs on $2K_4$, $2K_7$, and $2K_{10}$ have been found directly in Lemmas 2.2 to 2.4. All remaining $(v,4,2)$ *Cy*-designs can be constructed recursively using these three designs plus the decompositions of $2K_{6,6}$ and $2K_{9,6}$ into *Cy*-blocks which exist by Lemma 2.6. We separate them into those having $v \equiv 1 \pmod 6$ and those having $v \equiv 4 \pmod 6$, and show how to construct a design in each case.

Case I: $v \equiv 1 \pmod 6$.

Let $v \equiv 6m + 1$. Split the v points (or vertices) into m rows of 6 and one "point at infinity". Cover each row plus infinity point with the $2K_7$ *Cy*-design from Lemma

2.3 and each set of mixed differences (edges between rows) with the $2K_{6,6}$ decomposition from Lemma 2.6. This takes care of all the edges and so gives us a $(v,4,2)$ *Cy*-design.

Case II: $v \equiv 4 \pmod 6$

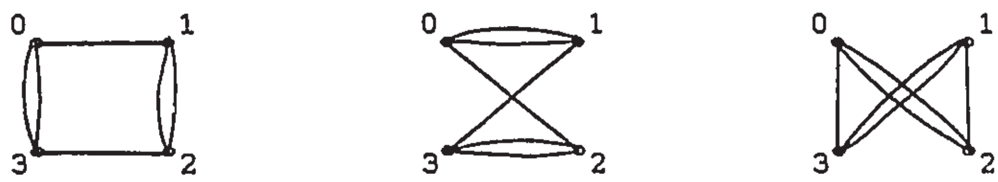
Let $v = 6m + 10$. Split the points into m rows of 6, one row of 9 and a point at "infinity". The row of 9 plus infinity point can be covered with the (10,4,2) *Cy*-design from Lemma 2.4, and each row of 6 plus infinity point can be covered with the (7,4,2) *Cy*-design (as in previous case). This leaves the edges between rows. Those between rows of 6 can be covered by the $2K_{6,6}$ decomposition, as above, and those between the row of 9 and each row of 6 can be covered using the decomposition of $2K_{9,6}$ into *Cy*-blocks from Lemma 2.6.

§2.3 Cy-designs with λ=3

We now proceed to the case $\lambda = 3$, where we show that the necessary conditions for the existence of a *Cy*-design are sufficient, except when $v = 5$. We first find designs in all other cases satisfying the necessary condition, and then give a proof of non-existence for a (5,4,3) *Cy*-design. The smallest designs to be found here (excluding the case $v = 5$) are those with $v = 4$, 8, 9 and 13. We find these four designs directly, and all larger ones using recursion.

Lemma 2.8 A (4,4,3) *Cy*-design exists.

Proof Such a design, which must have 3 blocks from Lemma 1.7, is shown in Fig. 2.5:



(0,3;2,1) (0,1;3,2) (0,2;1,3)

Fig. 2.5

Lemma 2.9 An (8,4,3) *Cy*-design exists.

Proof The number of blocks here is 14. A design on $\{Z_7 \cup \infty\}$ is generated by the two blocks

(0,1;2,5) and (0,2;∞,4).

Lemma 2.10 A (9,4,3) *Cy*-design exists.

Proof The number of blocks *b* is 18. A cyclic design on Z_9 is generated by the two blocks

(0,1;8,5) and (0,5;8,1).

Lemma 2.11 A (13,4,3) *Cy*-design exists.

Proof Here b=39. A cyclic design is generated by the three blocks

(0,3;8,4), (0,5;6,7), (0,6;8,10).

Now that we have found designs in these small cases, we need some decompositions of complete bipartite multigraphs for recursion, as in the $\lambda = 2$ case.

Lemma 2.12 There exists a decomposition of $3K_{4,4}$ into *Cy* blocks.

Proof The number of blocks here is 3x4x4/6=8. A decomposition on Z_4x{0,1} is generated by the two blocks

$$(0_0,0_1;2_0,1_1) \text{ and } (0_0,1_1,2_0,0_1).$$

Lemma 2.13 There exist decompositions of $3K_{8,8}$ and $3K_{12,8}$ into *Cy*-blocks.

Proof Follows from Lemma 1.9 and previous lemma.

Next we show, using recursive methods, that $(v,4,3)$ designs exist whenever $v \equiv 0$ or 1 (mod 4), and $v \neq 5$. We cannot use the obvious construction for values of v satisfying $v \equiv 1$ (mod 4) since we do not have a design for $v = 5$. Therefore we must split this case into the two sub cases $v \equiv 1$ (mod 8) and $v \equiv 5$ (mod 8).

Lemma 2.14 There exists a $(v,4,3)$ *Cy*-design whenever $v \equiv 0$ (mod 4).

Proof Let $v = 4m$. Arrange points in m rows of 4. Cover each row with the (4,4,3) *Cy*-design from Lemma 2.10 and edges between each pair of rows with the decomposition of $3K_{4,4}$ from Lemma 2.12.

Lemma 2.15 There exists a $(v,4,3)$ *Cy*-design whenever $v \equiv 1 \pmod 8$.

Proof Let $v = 8m + 1$. Arrange the points in m rows of 8 plus 1 point at "infinity". Cover each row plus ∞ point with the $(9,4,3)$ *Cy*-design from Lemma 2.10, and edges between each pair of rows with the decomposition of $3K_{8,8}$ from Lemma 2.13.

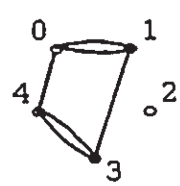
Lemma 2.16 There exists an $(v,4,3)$ *Cy*-design whenever $v \equiv 5 \pmod 8$ and $v \geq 13$.

Proof Let $v = 8m + 13$. Split the points into m rows of 8, one row of 12, and a point at "infinity". Cover each row of 8 plus ∞ point with the $(9,4,3)$ *Cy*-design from Lemma 2.10, the row of 12 plus ∞ point with the $(13,4,3)$ *Cy*-design from Lemma 2.11, all edges between rows of 8 with the decomposition of $3K_{8,8}$ from Lemma 2.13 and edges between the row of 12 and each other row with the decomposition of $3K_{12,8}$ from Lemma 2.13. This covers all edges and so gives us a $(v,4,3)$ *Cy*-design.

Now we have shown that a $(v,4,3)$ *Cy*-design exists for all values of v satisfying the necessary condition $v \equiv 0, 1 \pmod 4$, except for $v = 5$. To complete the case $\lambda = 3$ for the multigraph *Cy*, we must find a *Cy*-design with $\lambda = 3$ and $v = 5$ or prove that one does not exist.

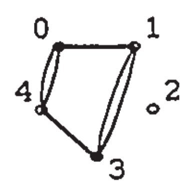
Lemma 2.17 A (5,4,3) *Cy*-design does not exist.

Proof We shall show that such a design cannot exist by finding certain conditions that it would have to satisfy, and then looking exhaustively at all ways of constructing it so as to satisfy these conditions. First we note that there are six different ways we can find a copy of *Cy* in $3K_5$. (We consider the *Cy*-blocks to be different only if one cannot be obtained from the other by rotation.) These are shown in Fig. 2.6, and indicated below for each is the number of edges with difference 1 and the number with difference 2.



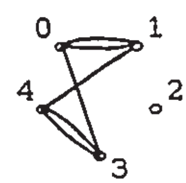
Difference 1: 5
2: 1

(a)



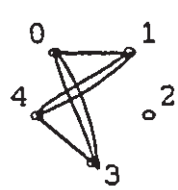
1: 4
2: 2

(b)



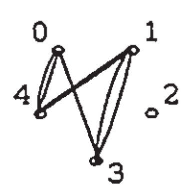
1: 4
2: 2

(c)



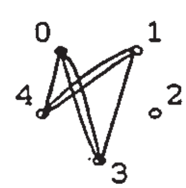
Difference 1: 2
2: 4

(d)



1: 2
2: 4

(e)



1: 1
2: 5

(f)

Fig. 2.6

If this design existed, then the number of blocks would be 5 from Lemma 1.7. Each of these blocks must be one of types (a) to (f). Let the number of blocks of type (a) be a, the number of type (b) be b and so on. Since $3K_5$ has a total of 30 edges of which half have difference 1 and half difference 2, we know that a,b,c,d,e,f must satisfy the following three equations.

$$(1) \quad a + b + c + d + e + f = 5$$

$$(2) \quad 5a + 4(b+c) + 2(d+e) + f = 15$$

$$(3) \quad a + 2(b+c) + 4(d+e) + 5f = 15.$$

Let $b+c = g$, $d+e = h$. Then we can rewrite these as follows:

$$(1) \quad a + g + h + f = 5$$

$$(2) \quad 5a + 4g + 2h + f = 15$$

$$(3) \quad a + 2g + 4h + 5f = 15.$$

We need only consider solutions to these equations in which $a \geq f$; any design with $a < f$ could be tranformed onto one where $a > f$ since there exists an isomorphism taking (a) blocks to (f) blocks. The only solutions to these equations having $a \geq$ f are:

$$\langle I \rangle \quad a = 1,\ g = 1,\ h = 3,\ f = 0,$$

$$\langle II \rangle \quad a = 2,\ g = 0,\ h = 2,\ f = 1.$$

Any (5,4,3) *Cy*-design would be balanced, since *Cy* is regular. Therefore each triple edge of $3K_5$ must have two edges in one block and one in another. (It is easily seen that we cannot have one in each of three different blocks.) Bearing this in mind, we try to construct a design first with the number of each kind of block specified in (I), and then with those specified in (II). We can put the first block anywhere we want W.L.O.G. so we choose to put as first (a) block (0,1;3,4) in each case.

(I) $a = 1,\ b + c = 1,\ d + e = 3,\ f = 0.$

Put (a) as specified above. Next we can add either a (b) or a (c). There are two nonisomorphic ways to add (b) and only one way up to isomorphism to add (c), shown in Figs. 2.7(i), (ii), and (iii) respectively. In (i) a (d) and an (e) configuration can each be added only once, and in (ii) neither (d) nor (e) will fit. In (iii) (d) can be added only once and (e) not at all. Therefore none of these work, and a design of type (I) cannot exist since we have exhausted all the possibilities for it.

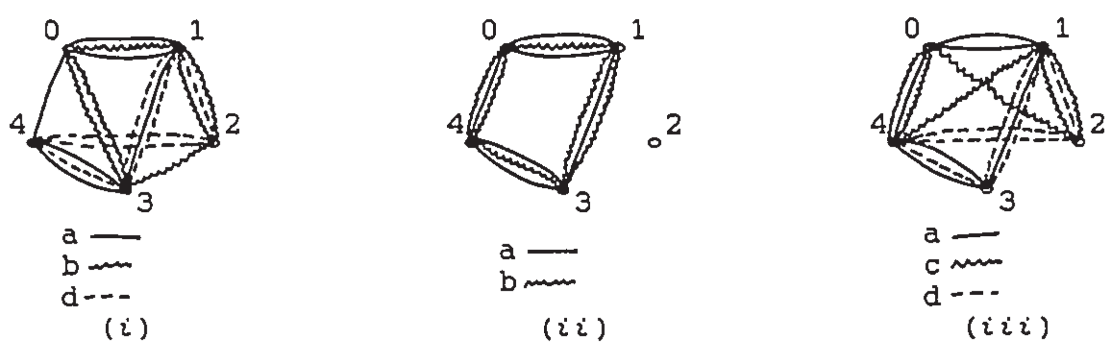


Fig. 2.7

(II) $a = 2$, $b + c = 0$, $d + e = 2$, $f = 1$.

Put first (a) as above. There is only one place up to isomorphism to put a second (a), and then only one way to add a (d). Having done this (Fig. 2.8) there is nowhere we can put an (e) or an (f). Therefore a design of type (II) cannot exist either, and since the existence of any (5,4,3) *Cy*-design implies the existence of either one of type (I) or one of type (II) we can conclude that no such design exists.

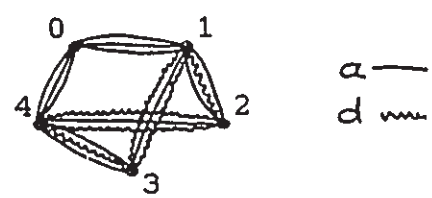


Fig. 2.8

Theorem 2.18 An $(v,4,3)$ *Cy*-design exists whenever the necessary condition $v \equiv 0, 1 \pmod 4$ from Lemma 2.1 is satisfied, *with the single exception* of the case $v = 5$.

Proof This follows from Lemmas 2.8 to 2.17.

§2.4 Cy-designs with λ=6 and 9

Next we look at (v,k,λ) *Cy*-designs with $\lambda = 6$. Here the only necessary condition is $v \geq 4$. All designs with $v \equiv 1 \pmod 3$ or $v \equiv 0, 1 \pmod 4$ (except $v = 5$) can be derived from the designs found with $\lambda=2$ and $\lambda=3$ respectively. Therefore we need look only at those with $v \equiv 2, 3, 6, 11 \pmod{12}$ and the one with $v = 5$. We give direct constructions for the three smallest of these designs, namely those with $v = 5$, 6, and 11.

Lemma 2.19 There exists a (5,4,6) *Cy*-design.

Proof The number of blocks is 6x5x4/12 = 10. A cyclic design on Z_5 is generated by the two blocks

$$(0,1;3,4) \quad \text{and} \quad (0,2;4,1).$$

Lemma 2.20 There exists a (6,4,6) *Cy*-design.

Proof The number of blocks here is 15. A design on $\{Z_5 \cup \infty\}$ is generated by the three base blocks

$$(0,1;3,4), \quad (0,2;\infty,4), \quad (0,2;4,\infty).$$

Lemma 2.21 There exists an (11,4,6) *Cy*-design.

Proof The number of blocks is 55. A cyclic design on Z_{11} is generated by the blocks

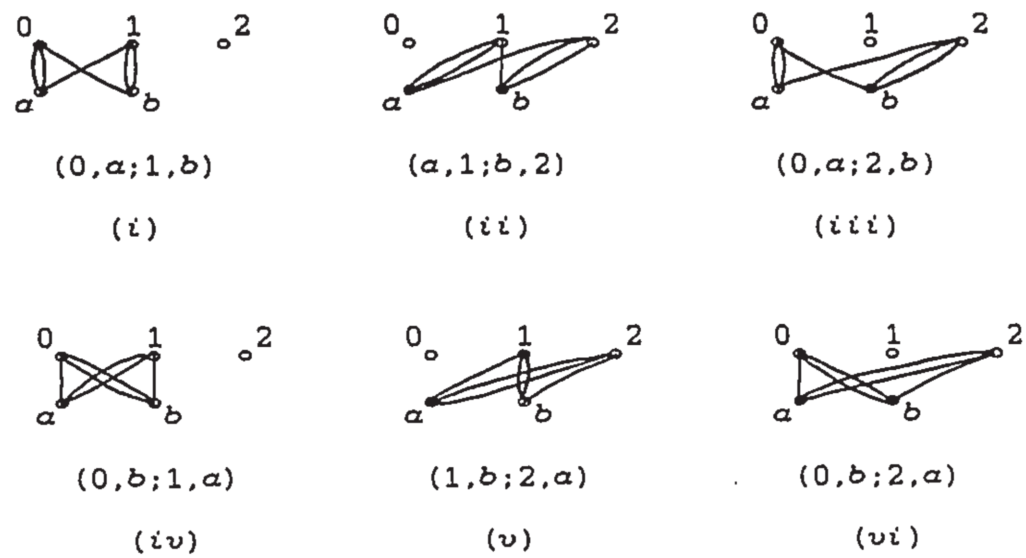
$$(0,1;3,8) \qquad \text{taken twice,}$$

$$\text{and} \quad (0,2;7,10), \quad (0,2;6,10), \quad (0,3;8,4).$$

Now that we have found these small designs directly, we need some decompositions of small bipartite multigraphs in order to construct the rest using recursion.

Lemma 2.22 There exists a decomposition of $6K_{3,2}$ into *Cy*-blocks.

Proof Such a decomposition on $(0,1,2;a,b)$ is shown in Fig. 2.9:



$(0,a;1,b)$ (*i*) $(a,1;b,2)$ (*ii*) $(0,a;2,b)$ (*iii*)

$(0,b;1,a)$ (*iv*) $(1,b;2,a)$ (*v*) $(0,b;2,a)$ (*vi*)

Fig. 2.9

Lemma 2.23 There exist decompositions of $6K_{3,3}$ and $6K_{6,6}$ into Cy-blocks.

Proof Follows from Lemmas 1.9 and 2.5.

Lemma 2.24 There exist decompositions of $6K_{6,5}$, $6K_{6,6}$, $6K_{6,8}$ and $6K_{9,6}$ into Cy-blocks.

Proof Follows from Lemmas 1.9, 2.22 and 2.23.

Lemma 2.25 There exists a $(v,4,6)$ Cy-design whenever $v \equiv 2 \pmod{12}$ and $v \geq 14$.

Proof Let $v = 12m + 14$. Arrange points in $2m+1$ rows of 6 plus one row of 8. Cover edges in the row of 8 with the $(8,4,6)$ Cy-design which can be found from Lemmas 2.9 and

1.8, and rows of 6 with the (6,4,6) design from Lemma 2.20. Cover edges joining each row of 6 to the row of 8 with the $6K_{6,8}$ decomposition from Lemma 2.23 and edges between rows of 6 with the $6K_{6,6}$ decomposition which exists by Lemma 2.24.

Lemma 2.26 There exists a $(v,4,6)$ *Cy*-design whenever $v \equiv 3 \pmod{12}$ and $v \geq 15$.

Proof Let $v = 12m + 15$. Arrange points in $2m+1$ rows of 6 and 1 row of 9. Cover edges in the row of 9 with the (9,4,6) *Cy*-design from Lemmas 2.10 and 1.8, and rows of 6 with (6,4,6) design from Lemma 2.20. Cover edges joining each row of 6 to the row of 9 with $6K_{6,9}$ decomposition from Lemma 2.24 and edges between rows of 6 with $6K_{6,6}$ decomposition from Lemma 2.23.

Lemma 2.27 There exists a $(v,4,6)$ *Cy*-design whenever $v \equiv 6 \pmod{12}$.

Proof Let $v = 6(2m+1)$. Arrange points into rows of 6. Cover each row of 6 with the (6,4,6) *Cy*-design from Lemma 2.20 and edges between rows with the $6K_{6,6}$ decomposition from Lemma 2.23.

Lemma 2.28 There exists a $(v,4,6)$ *Cy*-design whenever $n \equiv 11 \pmod{12}$.

Proof Let $v = 6(2m+1) + 5$. Arrange points in $2m+1$ rows of 6 and one row of 5. Cover the row of 5 with the (5,4,6) *Cy*-design from Lemma 2.19, the edges between each row of 6 and the row of 5 with the $6K_{6,5}$ decomposition from Lemma 2.24 and remaining edges as in the proof of the previous lemma.

Theorem 2.29 There exists a $(v,4,6)$ *Cy*-design for all $v \geq 4$.

Proof Follows from Lemmas 2.19 through 2.28 and Lemma 1.8.

We can now derive designs for all other values of v and λ satisfying the necessary conditions, except for those with $v = 5$, and $\lambda \equiv 3 \pmod 6$. These could have been derived from a (5,4,3) *Cy*-design, except that we have proved one does not exist. Therefore we use instead a (5,4,9)-design, for which a direct construction is given below.

Lemma 2.30 There exists a (5,4,9) *Cy*-design.

Proof Here $b = 15$. A cyclic design on Z_5 is generated by the blocks

(0,1;2,4), (0,2;1,3), (0,1;3,2).

Theorem 2.31 For the multigraph *Cy* (the "cylinder") a graph design exists whenever the necessary conditions are satisfied, with the exception of the case $\lambda=3$, $v=5$ where no (5,4,3) *Cy*-design exists.

Proof This follows from Theorems 2.7, 2.18, 2.29 and Lemmas 1.8 and 2.30.

CHAPTER 3

DESIGNS ON CUBIC MULTIGRAPHS WITH 6 VERTICES:

Bipartite Connected Multigraphs

§3.1 Introduction

Having looked at all cubic multigraphs on four or fewer vertices we now proceed to those on six vertices. First we find the necessary conditions on υ and λ for a G-design to exist when G is a cubic multigraph on six vertices:

Lemma 3.1 When G is a cubic multigraph on six vertices, the necessary conditions for the existence of a G-design are

$$\lambda\upsilon(\upsilon-1) \equiv 0 \pmod{18} \qquad (1)$$

$$\lambda(\upsilon-1) \equiv 0 \pmod{3} \qquad (2)$$

$$\lambda \geq m \qquad (3)$$

$$\upsilon \geq 6 \qquad (4)$$

Putting in values for λ we get necessary conditions on υ for each λ:

$$\upsilon \equiv 1 \pmod{9} \qquad \text{for } \lambda \equiv 1 \text{ or } 2 \pmod{3}$$

$$\upsilon \equiv 0 \text{ or } 1 \pmod{3} \qquad \text{for } \lambda \equiv 3 \text{ or } 6 \pmod{9}$$

$$\upsilon \geq 6 \qquad \text{for } \lambda \equiv 0 \pmod{9}.$$

Proof This result follows from Theorem 1.6.

There are nine cubic multigraphs on six vertices altogether, as shown in Fig. 3.1:

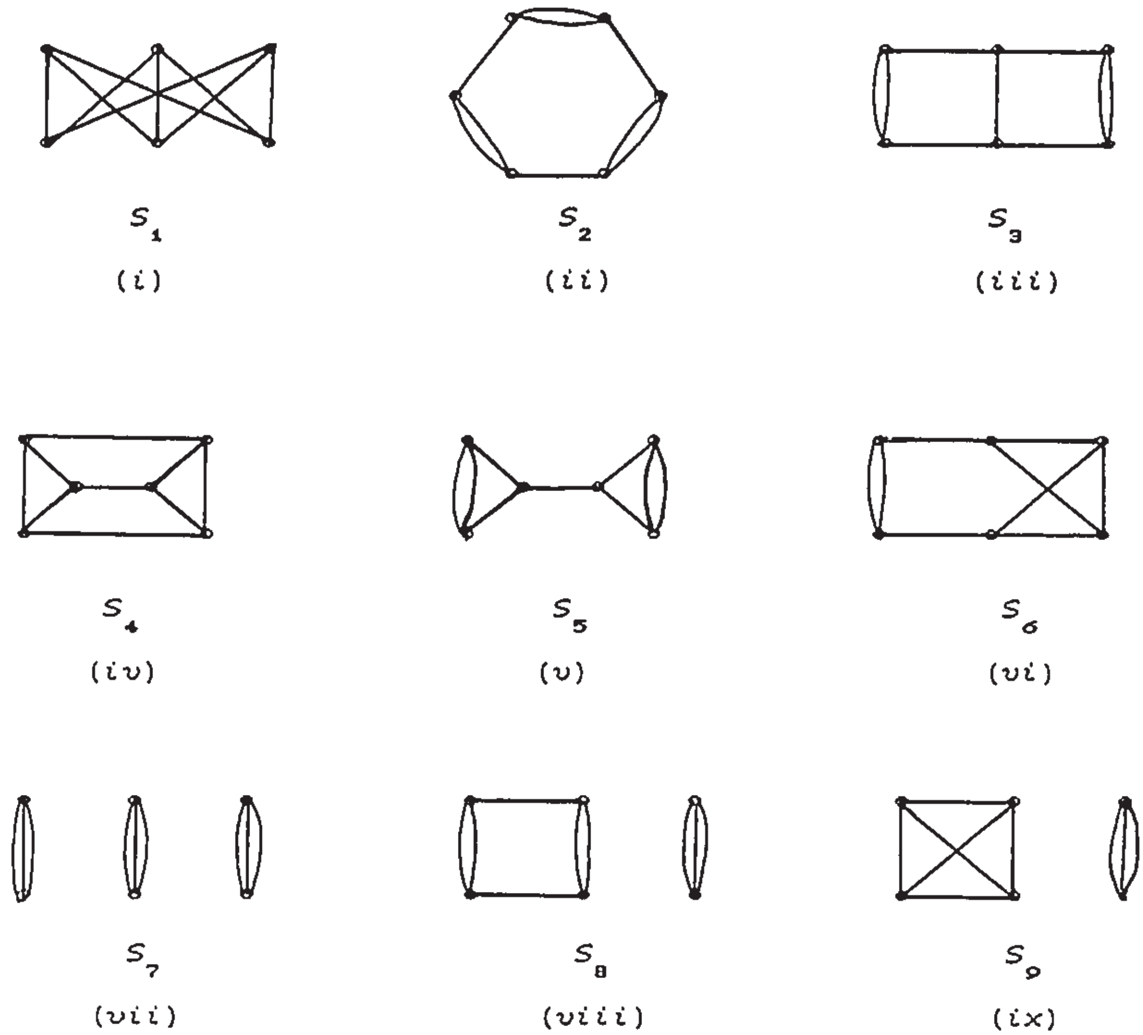


Fig. 3.1

We shall divide these multigraphs into three categories as follows: *Category I* contains all connected bipartite multigraphs, *Category II* contains all connected non bipartite multigraphs, and *Category III* contains all disconnected multigraphs. The first three multigraphs in Fig.3.1 fall into the first category, the second three into the second, and the last three into the third. In this chapter we look at those three in Category I. Categories II and III will be examined in Chapters 4, 5, 6 and 7.

The graph S_1 is the complete bipartite graph $K_{3,3}$, and the necessary conditions from Lemma 3.1 are

$$\begin{array}{ll} v \equiv 1 \pmod 9 & \text{for } \lambda \equiv 1 \text{ or } 2 \pmod 3 \\ v \equiv 0 \text{ or } 1 \pmod 3 & \text{for } \lambda \equiv 3 \text{ or } 6 \pmod 9 \\ v \geq 6 & \text{for } \lambda \equiv 0 \pmod 9). \end{array}$$

Huang [21] has established the nonexistence of (v,k,λ) S_1-designs for

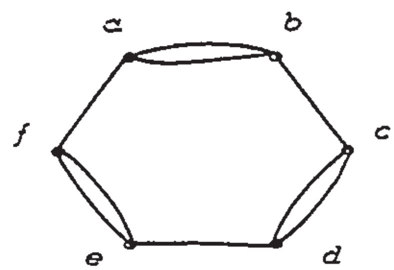
$$(v,k,\lambda) \in N = \{(10,6,1)\} \cup \{(6,6,6t+3) : t \geq 0\}$$

and has shown that in all other cases the necessary conditions are sufficient. Note that there is no (6,6,3) S_1-design.

§3.2 S_2-designs with $\lambda=2$

This leaves S_2 and S_3, both of which have maximum edge multiplicity $m = 2$, so the first possibility for a G-design to exist in either case is when $\lambda = 2$ and $v = 10$.

We start with S_2, for which we shall use the notation $(a,b;c,d;e,f)$ to describe a block, as illustrated in Fig.3.2:



$(a,b;c,d;e,f)$

Fig. 3.2

Lemma 3.2 There exists a (10,6,2) S_2-design.

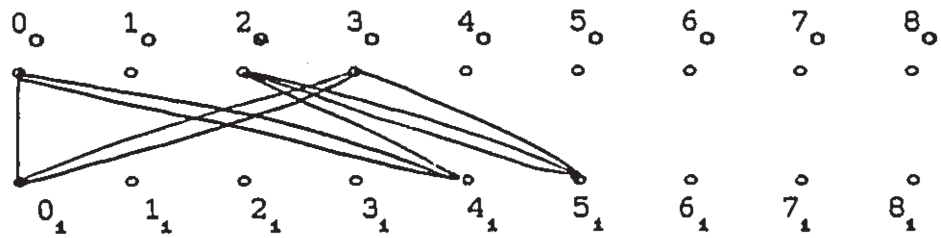
Proof For such a design, the number of blocks b = 10. A cyclic design on Z_{10} is generated by the block

(0,2;7,4;5,9).

To find S_2-designs with λ=2 for larger values of v we use a recursive construction, for which we need a decomposition of the complete bipartite multigraph $2K_{9,9}$:

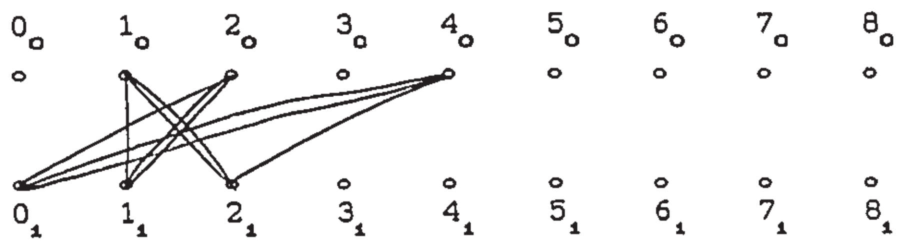
Lemma 3.3 There exists a decomposition of $2K_{9,9}$ into S_2-blocks.

Proof The number of blocks here is 2x9x9/9 = 18. A design on Z_9x{0,1}, generated by two base blocks, is shown in Fig. 3.3.



Base block: $(0_0, 4_1; 2_0, 5_1; 3_0, 0_1)$

(*i*)



Base block: $(1_0, 2_1; 4_0, 0_1; 2_0, 1_1)$

(*ii*)

Fig. 3.3

Theorem 3.4 There exists an $(v,6,2)$ S_2-design for all $v \equiv 1 \pmod 9$, $v \geq 10$.

Proof Arrange the $v = 9m + 1$ points into m rows of 9 and 1 point at ∞. Cover each row plus ∞ with the $(10,6,2)$ S_2-design from Lemma 3.2, and all edges between each pair of rows with the decomposition of $2K_{9,9}$ found in Lemma 3.3. This covers all edges of $2K_n$ and thus gives us an S_2-design on $2K_v$.

§3.3 S_2-designs with $\lambda=3$

Next we must look at S_2-designs with $\lambda = 3$. From Lemma 3.1 we know that the necessary condition here is $v \equiv 0, 1 \pmod 3$. Therefore the first designs to look for here have $v = 6$, $v = 7$, $v = 9$, and $v = 10$.

Lemma 3.5 There exists a (6,6,3) S_2-design.

Proof In this case $b = 5$, and a design is generated on $(Z_5 \cup \infty)$ by the block

$$(0,1;\infty,3;4,2).$$

Lemma 3.6 There exists a (7,6,3) S_2-design.

Proof Here $b = 7$. A cyclic design on Z_7 is generated by the block

$$(0,1;3,5;2,6).$$

Lemma 3.7 There exists a (9,6,3) S_2-design.

Proof Here $b = 12$. A design on $\{Z_8 \cup \infty\}$ is generated by the blocks

$$(0,1;5,4;7,3) \quad \text{and} \quad (0,2;\infty,4;3,6)$$

where the first block generates an orbit of length 4 (half orbit) and the second an orbit of length 8 (full orbit).

Lemma 3.8 There exists a (10,6,3) S_2-design.

Proof Here $b = 15$. A design on $Z_5 \times \{0,1\}$ is generated by the three blocks

$$(0_0,1_1;3_1,0_1;3_0,1_0)$$
$$(0_0,2_1;3_0,3_1;2_0,0_1)$$
$$(0_0,1_0;4_1,3_1;2_1,3_0).$$

Lemma 3.9 There exists a decompostion of $3K_{3,3}$ into copies of S_2.

Proof The number of blocks is 3. A decomposition on Z_3x{0,1} is generated by

$$(0_0,0_1;1_0,2_1;2_0,1_1).$$

Lemma 3.10 There exists a decomposition of $3K_{4,3}$ into copies of S_2.

Proof The number of blocks b = 4. This design on {0,1,2,3;a,b,c} is illustrated in Fig.3.4.

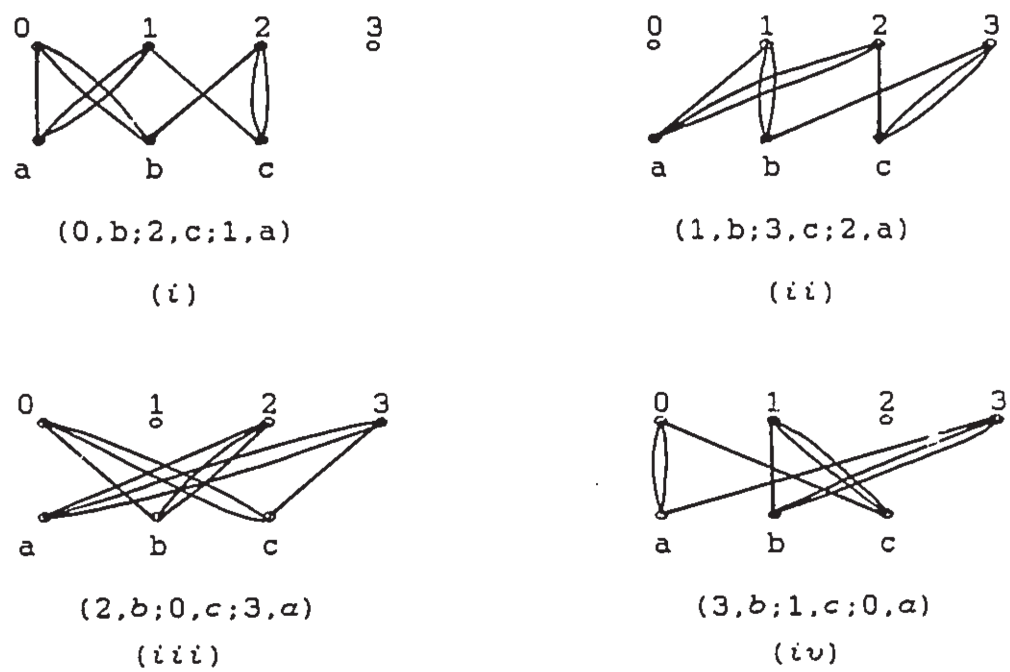


Fig. 3.4

Lemma 3.11 There exists a decomposition of $3K_{6,6}$, $3K_{9,6}$ and $3K_{10,6}$ into copies of S_2.

Proof Follows from previous two lemmas and Lemma 1.9.

Theorem 3.12 There exists a $(v,6,3)$ S_2-design whenever $v \equiv 0, 1 \pmod 3$ and $v \geq 6$.

Proof We split values of v satisfying the necessary conditions into the four cases $v \equiv 0, 1, 3,$ and $4 \pmod 6$ and construct a design for each case.

Case I: $v \equiv 0 \pmod 6$. Then $v = 6m$ for some $m \geq 1$. Split the v points into m rows of 6. Cover each row with the (6,6,3) design from Lemma 3.5, and edges between each pair of rows with the decomposition of $3K_{6,6}$ from Lemma 3.11.

Case II: $v \equiv 1 \pmod 6$. Here $v = 6m + 1$ for some $m \geq 1$. Split the v points into m rows of 6 and one point at ∞. Cover each row plus ∞ with the (7,6,3) S_2-design from Lemma 3.6 and edges between each pair of rows with the decomposition of $3K_{6,6}$ as above.

Case III: $v \equiv 3 \pmod 6$. Then $v = 6m + 9$ for some $m \geq 0$. Split the v points into m rows of 6 and one row of 9. Cover each row of 6 with the (6,6,3) S_2-design from Lemma 3.5 and the row of 9 with the (9,6,3) S_2-design from Lemma 3.7. Cover edges between each row of 6 and row of 9 with the decomposition of $3K_{9,6}$ into S_2-blocks from Lemma 3.11, and ones between rows of 6 with the decomposition of $3K_{6,6}$.

Case IV: $v \equiv 4 \pmod 6$. Then $v = 6m + 10$ for some $m \geq 0$. Split the v points into m rows of 6 and one row of 10. Cover row of 10 with the (10,6,3) S_2-design from Lemma 3.8, edges between each row of 6 and the row of 10 with the decomposition of $3K_{10,6}$ into S_2-blocks from Lemma 3.11, and all remaining edges as in the previous case.

§3.4 S_2-designs with λ=9

The only remaining S_2-designs we must find are those with $\lambda = 9$. Designs with all other values of λ can then be derived from those with $\lambda = 2$, 3, and 9 by Lemma 1.8. For $\lambda = 9$ there are no necessary conditions on v except that we must have $v \geq 6$. Designs with $v \equiv 0, 1 \pmod 3$ can be derived from the corresponding designs with $\lambda = 3$ by the corollary to Lemma 1.8, so we need only look at those with $v \equiv 2 \pmod 3$. The smallest example is $v = 8$.

Lemma 3.13 There exists an (8,6,9) S_2-design.

Proof: The number of blocks b is 28. A design on $\{Z_7 \cup \infty\}$ is generated by the blocks

$(0,5;4,1;\infty,6)$ 3 times,

and $(0,1;3,5;2,6)$.

Lemma 3.14 There exists an (11,6,9) S_2-design.

Proof The number of blocks is 55. A cyclic design on Z_{11} is generated by the blocks

(0,1;10,2;9,4) 3 times,

(0,1;3,8;5,9),

(0,9;5,8;3,1).

Lemma 3.15 There exist decompositions of $9K_{6,6}$, $9K_{8,6}$ and $9K_{11,6}$ into S_2-blocks.

Proof Follows from Lemmas 3.9, 3.10 and 1.9.

Theorem 3.16 There exists a $(v,6,9)$ S_2-design for all $v \geq 6$.

Proof If $v \equiv 0, 1 \pmod 3$ then a $(v,6,3)$ S_2-design exists from Theorem 3.12, and therefore a $(v,6,9)$ S_2-design exists by Lemma 1.8. For designs with $v \equiv 2 \pmod 3$, we look separately at those with $v \equiv 2 \pmod 6$ and those with $v \equiv 5 \pmod 6$.

Case I: $v \equiv 2 \pmod 6$. Therefore $v = 6m + 8$ for some $m \geq 0$. Split v into m rows of 6 and one row of 8, and use a $(6,6,9)$ S_2-design (see above), the $(8,6,9)$ S_2-design from Lemma 3.13 and decompositions of $9K_{6,6}$ and $9K_{11,6}$ from Lemma 3.15 to cover edges in each row and between rows as in the case where $\lambda = 3$.

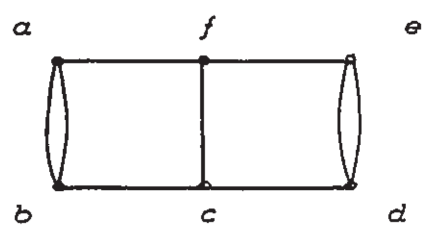
Case II: $v \equiv 5 \pmod 6$. Then $v = 6m + 11$ for some $m \geq 0$. Split the v points into m rows of 6 and one row of 11. Use the $(6,6,9)$ S_2-design from above, the $(11,6,9)$ S_2-design from Lemma 3.14, and decompositions of $9K_{6,6}$ and $9K_{11,6}$ as in Case I above to cover all edges.

Theorem 3.17 The necessary conditions for the existence of an S_2-design are also sufficient.

Proof Follows from Theorems 3.4, 3,12, 3.16, and Lemma 1.8.

§3.5 S_3-designs with λ=2

The only other multigraph in this category (connected, cubic, and bipartite on six vertices) is S_3 from Fig. 3.1. We shall use the block labelling system shown in Fig. 3.5 below for this multigraph.



$(a,b;c;d,e;f)$

Fig. 3.5

We must find exactly the same designs on this graph as in the previous case, since all the parameters are the same. First we look at those with $\lambda = 2$, where the necessary condition on v is $v \equiv 1 \pmod 9$.

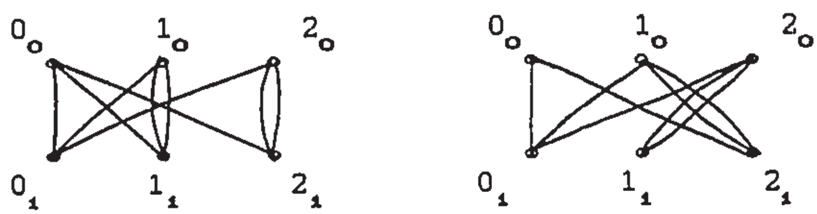
Lemma 3.18 A (10,6,2) S_3-design exists.

Proof Such a design (cyclic on Z_{10}) is generated by the block

$$(0,2;3;8,5;9).$$

Lemma 3.19 There exist decompositions of $2K_{3,3}$ and $2K_{9,9}$ into S_3-blocks.

Proof A decomposition of $2K_{3,3}$ is shown in Fig.3.6. It follows from Lemma 1.9 that a $2K_{9,9}$ decomposition also exists.



$(1_0,1_1;0_0;2_1,2_0;0_1)$ $(1_0,2_1;0_0;1_1,2_0;0_1)$

Fig. 3.6

Theorem 3.20 A $(v,6,2)$ S_3-design exists for all $v \geq 6$ satisfying the necessary condition $v \equiv 1 \pmod 9$.

Proof We have a (10,6,2) S_3-design and a decomposition of $2K_{9,9}$ into S_3-blocks, so we can construct a $(v,6,2)$ design for all $v \equiv 1 \pmod 9$ exactly as we constructed the corresponding S_2-designs in Theorem 3.4 – that is by gluing together copies of the (10,6,2) S_3-design using the decomposition of $2K_{9,9}$.

§3.6 S_3-designs with $\lambda=3$

The next value of λ we must look at here is $\lambda = 3$. The necessary condition for the existence of an $(v,6,3)$ S_3-design is $v \equiv 0,\ 1 \pmod 3$, so the first designs to find are those with $v = 6,\ 7,\ 9,\ 10$. We find these small designs directly, and then find decompositions of $3K_{3,3}$ and $3K_{4,3}$ into S_3-blocks for our recursive construction of all remaining S_3-designs with $\lambda=3$, exactly as we did for S_2.

Lemma 3.21 There exists a $(6,6,3)$ S_3-design.

Proof The number of blocks is $b = 5$. A $(6,6,3)$ S_3-design on $\{Z_5 \cup \infty\}$ is generated by the block

$$(0,1;4;2,\infty;3).$$

Lemma 3.22 There exists a $(7,6,3)$ S_3-design.

Proof Here $b = 7$. A cyclic design on Z_7 is generated by the block

$$(0,2;3,1,5;4).$$

Lemma 3.23 There exists a $(9,6,3)$ S_3-design.

Proof Here $b = 12$. A design on $\{Z_8 \cup \infty\}$ is generated by the two blocks

$$(0,2;3;4,6;7) \qquad \text{(half orbit)}$$

$$(0,3;5;\infty,7;4).$$

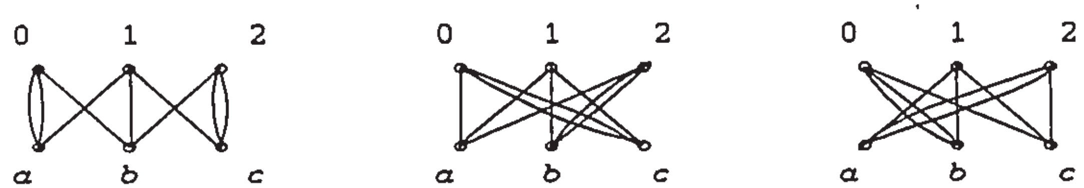
Lemma 3.24 There exists a $(10,6,3)$ S_3-design.

Proof Here $b = 15$. A design on $Z_5 \times \{0,1\}$ is generated by the three base blocks

$$(0_0,1_0;3_0;0_1,1_1;3_1)$$
$$(0_0,2_0;4_1;0_1,2_1;4_0)$$
$$(0_0,1_1;2_0;3_1,4_0;2_1).$$

Lemma 3.25 There exists a decomposition of $3K_{3,3}$ into S_3-blocks.

Proof The number of blocks is 3. Such a decomposition on $\{0,1,2;a,b,c\}$ is shown in Fig. 3.7.

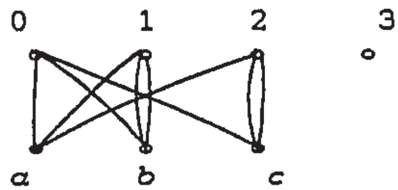


$(0,a;1;c,2;b)$ $(0,c;1;b,2;a)$ $(0,b;2;c,1;a)$

Fig. 3.7

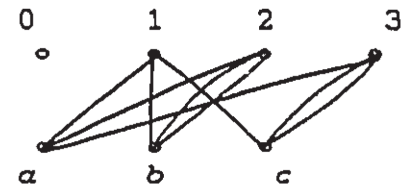
Lemma 3.26 There exists a decomposition of $3K_{4,3}$ into S_3-blocks.

Proof A decomposition on $\{0,1,2,3;a,b,c\}$ is shown in Fig. 3.8.



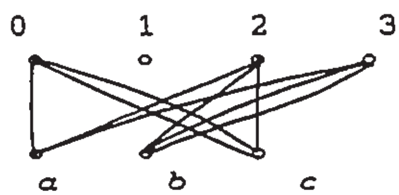
(1,*b*;0;*c*,2;*a*)

(*i*)



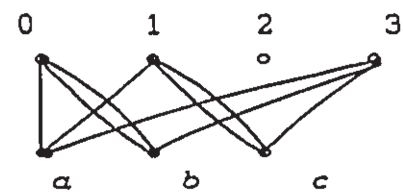
(2,*b*;1;*c*,3;*a*)

(*ii*)



(0,*c*;2;*b*,3;*a*)

(*iii*)



(0,*b*;3;*c*,1;*a*)

(*iv*)

Fig. 3.8

Theorem 3.27 There exists an $(v,6,3)$ S_3-design for all values of v satisfying the necessary condition $v \equiv 0, 1 \pmod 3$.

Proof From Lemma 1.9 and the previous two lemmas we have decompositions of $3K_{6,6}$, $3K_{9,6}$ and $3K_{10,6}$ into S_3-blocks. Using these and the designs we have found on $3K_6$, $3K_7$, $3K_9$, and $3K_{10}$ we can construct designs for all $v \equiv 0, 1 \pmod 3$ exactly as we did for S_2 in Theorem 3.12.

§3.7 S_3-designs with λ=9

Lastly we must look at S_3-designs with $\lambda = 9$. Here the only necessary condition is $v \geq 6$, and we can derive all designs with $v \equiv 0, 1 \pmod 3$ from those with $\lambda = 3$, so we start by looking at small values of v where $v \equiv 2 \pmod 3$.

Lemma 3.28 There exists an (8,6,9) S_3-design.

Proof The number of blocks is $b = 28$. A design on $\{Z_7 \cup \infty\}$ is generated by the four base blocks

(0,5;∞;3,6;1)

(0,4;3;6,∞;2)

(0,2;3;∞,5;4)

(0,4;3;6,1;2).

Lemma 3.29 There exists an (11,6,9) S_3-design.

Proof The number of blocks is 55. A cyclic design on Z_{11} is generated by the three blocks

(0,4;6;7,10;1) 3 times,

(0,8;2;6,1;7)

(0,1;8;6,4;3).

Theorem 3.30 A $(v,6,9)$ S_3-design exists for all $v \geq 6$.

Proof We can get decompositions of $9K_{6,6}$, $9K_{8,6}$, and $9K_{11,6}$ into S_3-blocks from the corresponding decompositions with $\lambda = 3$ and Lemma 1.9. Then proceed exactly as in the proof of Theorem 3.16 only substituting "S_3" and corresponding lemmas for "S_2".

Theorem 3.31 The necessary conditions for the existence of an S_3-design are also sufficient.

Proof This follows from Theorems 3.17, 3.27, 3.30 and Lemma 1.8.

CHAPTER 4

DESIGNS ON CUBIC MULTIGRAPHS WITH 6 VERTICES:

Non-Bipartite Connected Graphs

§4.1 Preliminary Results

In this chapter we look at the first multigraph in the second category from Fig. 3.1., non bipartite connected cubic multigraphs on six vertices. This graph, S_4, is the only simple graph in this category and is sometimes called the *envelope*.

Before we look at the graph S_4 we introduce some results on *partial triple systems*, which we will need to find graph designs in the general case for the three multigraphs in this category – S_4, S_5 and S_6. As mentioned before, a Steiner triple system is a decomposition of K_v into triangles, or a $(v,3,1)$ K_3-design. A Steiner triple system exists whenever $v \equiv 1, 3 \pmod 6$. A *partial triple system* is a set of triples such that each pair of points is contained in at most one triple (covers some of the edges of K_v, but not necessarily all). A partial triple system is a *maximal partial triple system* (*MPTS*) if no more triples can be added – i.e. there are no triangles left uncovered in K_v. The *leave* of a MPTS is defined to be the complement of the subgraph of K_v which is covered by the triples of the MPTS.

It is well known that for a MPTS with the maximum possible number of triples, called an MMPTS, there are only four possible types of leave, and that which type we get depends only on the value of υ (mod 6) (see for example [12]). These are shown in Table 4.1:

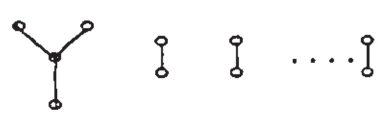
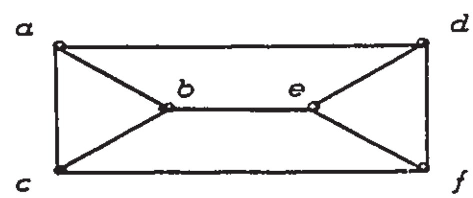
υ (mod 6)	*Leave of MPTS (υ)*	*Description*
0		$(\upsilon/2)$ K_2's
1		$\overline{K}_v$
2		$(\upsilon/2)$ K_2's
3		$\overline{K}_v$
4		$K_{1,3} \cup \frac{\upsilon-4}{2}$ K_2's
5		$C_4 \cup \overline{K}_{v-4}$

Table 4.1

§4.2 S_4-designs with $\lambda=1$

The labelling system we shall use for the envelope S_4 is shown in Fig. 4.2:



$(a,b,c;d,e,f)$

Fig. 4.2

Since S_4 is a simple graph we can start with the case $\lambda = 1$, in which we have the same necessary conditions for the existence of a G-design as in the case $\lambda = 2$ for the two multigraphs we have just looked at - i.e. $v \equiv 1 \pmod 9$. First we find all S_4-designs with $\lambda = 1$ up to the one with $v = 46$ by direct construction.

Lemma 4.1 A (10,6,1) S_4-design exists.

Proof The number of blocks b is 5. A cyclic design on Z_{10} is generated by the block

(0,1,8;6,5,3) (half orbit).

Lemma 4.2 A (19,6,1) S_4-design exists.

Proof The number of blocks is 19. A cyclic design on Z_{19} is generated by the block

(0,2,9;6,5,1).

Lemma 4.3 There exists a (28,6,1) S_4-design.

Proof Here $b = 42$. A cyclic design on Z_{28} is generated by the blocks

(0,1,6;15,17,26)

(0,3,7;14,21,17) (half orbit).

Lemma 4.4 There exists a (37,6,1) S_4-design.

Proof Here $b = 74$. A cyclic design on Z_{37} is generated by the two blocks

(0,1,3;5,32,14)

(0,7,20;33,19,4).

Lemma 4.5 There exists a (46,6,1) S_4-design.

Proof Here $b = 115$. A cyclic design on Z_{46} is generated by the three blocks

(0,1,3;36,43,8)

(0,13,30;21,33,6)

(0,6,14;37,29,23) (half orbit).

Now that we have found this type of design for the smallest orders, we must use a recursive method to show that such a design exists on any value of v satisfying the necessary condition $v \equiv 1 \pmod 9$. We cannot use the same method that we used for the previous two multigraphs, since they were bipartite and thus decompositions of bipartite multigraphs could be used. The graph S_4 is, however, *tripartite*, so we first find a decomposition of the complete tripartite graph $K_{9,9,9}$ into S_4-blocks. This decomposition was found with the aid of a computer.

Lemma 4.6 There exists a decomposition of $K_{9,9,9}$ into S_4-blocks.

Proof The number of blocks in such a decomposition is 3x9x9/9 = 27. A design on Z_9 x {0,1,2} is generated by the three base blocks

$$(0_0,0_1,0_2;3_2,1_0,2_1)$$
$$(0_0,2_1,4_2;6_1,5_2,8_0)$$
$$(0_0,4_1,1_2;5_1,0_2,2_0)$$

Using the above results we can find S_4-designs for all values of v satisfying the necessary condition $v \equiv 1$ (mod 9), although the procedure is more complicated than in the case of the bipartite multigraphs S_3 and S_4, which is why we had to find so many designs directly.

Theorem 4.7 There exists an $(v,6,1)$ S_4-design for all $v \equiv 1$ (mod 9).

Proof We divide the values of v satisfying the necessary condition into four cases, and demonstrate how to construct such a design for each one.

Case I: $v \equiv 10, 28 \pmod{54}$.

Split the v points into m rows of 9 and a point at ∞. The number of rows m will then satisfy the condition $m \equiv 1, 3$ (mod 6). Therefore there exists a *Steiner Triple*

System on m elements (see for example Hall [16]). Take such an STS(m) where here each element in the STS is a row of 9 points. Each triple in the STS then corresponds to three rows of points, and every pair of rows is in exactly one triple. Therefore if we cover the edges between each pair of rows in each triple with the decomposition of $K_{9,9,9}$ found in Lemma 4.6, we will have covered every edge between each pair of rows once. The edges within a row can be covered by taking each row plus the infinity point and covering it with the (10,6,1) S_4-design from Lemma 4.1. In this way we can find a $(v,6,1)$ S_4-design whenever $v \equiv 10$, 28 (mod 54).

Case II: $v \equiv 1,\ 19 \pmod{54}$.

Split into m rows of 9 and an ∞ point as above. In this case $m \equiv 0,\ 2 \pmod 6$. From Table 4.1 we see that the leave of a MMPTS(m) in this case consists of K_2's, or single edges. Take an MMPTS(m) where each of the m elements is a row of 9 in our configuration of the v points. Then we can find an S_4-design on the v points as follows. For each triple in the MPTS cover the edges between the three rows in the triple with the decomposition of $K_{9,9,9}$ from Lemma 4.6. Cover each pair of rows which corresponds to a K_2 in the leave of the MPTS plus the infinity point, including edges within as well as between rows, with the (19,6,1) S_4-design from Lemma 4.3.

Case III: $v \equiv 37 \pmod{54}$.

Arrange the points into m rows of 9 plus an ∞ point. In this case $m \equiv 4 \pmod 6$. The leave of a MPTS(m) in this case is of the form

We cannot use this directly since we cannot decompose the star graph $K_{1,3}$ into S_4's. However, if $g \equiv 5 \pmod 6$ we can partition the edge set of K_g into one K_5 and the rest K_3's (cf. e.g., [37]). If we then remove one of the points in the K_5, we have a partition of K_{g-1} into a K_4, K_3's and some K_2's. This tells us that we can take a MMPTS(m) where $m \equiv 4 \pmod 6$ with the above leave, and delete one triple from the MPTS in such a way that the leave of the new PTS(m) has a K_4 instead of the star with 3 edges in the leave of the MMPTS. We can then cover the K_4 plus the infinity point with the (37,6,1) S_4-design from Lemma 4.4, each $K_2 + \infty$ with the (19,6,1) S_4-design from Lemma 4.2. This leaves only edges between rows where the rows occur together in a triple of the PTS(m). These can be taken care of by covering each triple with the $K_{9,9,9}$ decomposition from Lemma 4.6.

Case IV: $v \equiv 46 \pmod{54}$

This time we get $m \equiv 5 \pmod 6$, and from the discussion in the previous case we know that we can partition the edge set of K_m into a unique K_5 and K_3's. Therefore there exists a PTS(m) with leave consisting of a K_5 and isolated points. Again, cover edges between rows occurring together in a triple in the PTS with the $K_{9,9,9}$ decomposition of Lemma 4.6 and each row plus infinity point with the (10,6,1) S_4-design from Lemma 4.1. Then the remaining edges can be taken care of by covering the rows in the K_5 plus the infinity point by the (46,6,1) S_4-design from Lemma 4.5.

§4.2 S_4-designs with $\lambda=3$

The next class of S_4-designs we must look at is the one containing those with $\lambda = 3$. Here the necessary condition is $v \equiv 0, 1 \pmod 3$. As in the case $\lambda = 1$, the recursive technique used is more complicated than that used for the bipartite multigraphs S_2 and S_3, and we must find many designs directly, up to and including the one with $v = 40$. We first construct all these designs which must be found directly, as well as a decomposition of $3K_{3,3,3}$ into S_4-blocks, and then explain how these can be used to find all other S_4-designs with $\lambda=3$.

Lemma 4.8 There exists a (6,6,3) S_4-design.

Proof A design on $\{Z_5 \cup \infty\}$ is generated by the block

$$(0,1,3;4,2,\infty).$$

Lemma 4.9 There exists a (7,6,3) S_4-design.

Proof Here $b = 7$. A cyclic design on Z_7 is generated by the block

$$(0,1,3;5,4,2).$$

Lemma 4.10 There exists a (9,6,3) S_4-design.

Proof Here $b = 12$. A design on $\{Z_8 \cup \infty\}$ is generated by the two blocks

$$(0,1,3;4,7,5) \quad \text{(half orbit)}$$

$$(0,1,3;\infty,5,2).$$

Lemma 4.11 There exists a (10,6,3) S_4-design.

Proof A design of this type can be obtained from the (10,6,1) S_4-design of Lemma 4.1 by Lemma 1.8.

Lemma 4.12 There exists a (12,6,3) S_4-design.

Proof Here $b = 22$. A design on $\{Z_{11} \cup \infty\}$ is generated by the two blocks

$$(0,1,3;6,2,8)$$

$$(0,4,8;10,2,\infty).$$

Lemma 4.13 There exists a (13,6,3) S_4-design.

Proof Here $b = 26$. A cyclic design on Z_{13} is generated by the two blocks

$$(0,1,3;8,5,7)$$

$$(0,1,5;6,8,11).$$

Lemma 4.14 There exists a (15,6,3) S_4-design.

Proof Here $b = 35$. A design on $(Z_7 \times \{0,1\}) \cup \infty$ is generated by the five blocks

$$(0_0,0_1,1_1;2_0,4_0,3_1)$$
$$(0_0,1_0,3_1;3_0,2_1,0_1)$$
$$(0_0,1_0,5_1;0_1,5_0,3_1)$$
$$(0_0,1_0,0_1;\infty,4_1,3_1)$$
$$(0_0,2_0,\infty;2_1,5_0,3_1).$$

Lemma 4.15 There exists a (16,6,3) S_4-design.

Proof Here $b = 40$. A cyclic design on Z_{16} is generated by the three blocks

(0,2,10;12,6,15)

(0,5,9;1,4,11)

(0,1,3;11,9,8)   (half orbit).

Lemma 4.16 There exists a (18,6,3) S_4-design.

Proof Here $b = 51$. A design on $\{Z_{17} \cup \infty\}$ is generated by the three blocks

(0,3,8;14,4,10)

(0,4,11;9,14,5)

$(0,1,3;\infty,6,5)$.

Lemma 4.17 There exists a (21,6,3) S_4-design.

Proof Here $b = 70$. A design on $Z_7 \times \{0,1,2\}$ is generated by the blocks

$(0_0,3_1,6_0;2_1,3_0,4_1)$ twice

$(0_1,3_2,6_1;2_2,3_1,4_2)$ twice

$(0_2,3_0,6_2;2_0,3_2,4_0)$ twice

$(0_0,3_0,1_1;1_2,5_0,5_1)$

$(0_1,3_1,1_2;1_0,5_1,5_2)$

$(0_0,3_0,6_0;5_2,2_2,1_2)$

$(0_0,2_1,3_1;0_2,5_1,2_2)$.

Lemma 4.18 There exists (22,6,3) S_4-design.

Proof Here $b = 77$. A cyclic design on Z_{22} is generated by the four blocks

(0,1,7;4,2,5)

(0,5,15;9,14,17)

(0,6,14;13,18,2)

(0,3,7;18,14,11) (half orbit).

Lemma 4.19 There exists a (33,6,3) S_4-design.

Proof Here $b = 176$. A design on $\{Z_{32} \cup \infty\}$ is generated by the six blocks

$(0,4,5;8,\infty,7)$

(0,6,11;3,10,13)

(0,13,20;5,6,8)

(0,8,19;14,23,1)

(0,10,16;8,25,2)

(0,9,21;16,5,25) (half orbit).

Lemma 4.20 There exists a (34,6,3) S_4-design.

Proof Here b = 187. A cyclic design on Z_{34} is generated by the six blocks

(0,1,8;2,4,3)

(0,4,10;9,13,16)

(0,2,5;8,14,28)

(0,7,23;14,25,4)

(0,4,19;13,21,31)

(0,9,22;5,26,17) (half orbit).

Lemma 4.21 There exists a (39,6,3) S_4-design.

Proof Here b = 247. A design on $(Z_{38} \cup \infty)$ is generated by the seven blocks

(0,4,∞;1,2,7)

(0,1,7;3,5,10)

(0,2,8;29,5,18)

(0,12,21;5,28,17)

(0,10,20;14,27,5)

(0,8,21;18,26,2)

(0,12,23;31,19,4) (half orbit).

Lemma 4.22 There exists a (40,6,3) S_4-design.

Proof Here b = 260. A cyclic design on Z_{40} is generated by the seven blocks

(0,5,7;1,4,3)

(0,4,11;6,14,8)

(0,3,15;8,12,20)

(0,5,23;15,31,2)

(0,13,22;10,27,3)

(0,10,23;24,30,2)

(0,15,29;9,35,20) (half orbit).

To summarize the above, we have found $(v,6,3)$ S_4-designs directly for $v = 6, 7, 9, 10, 12, 13, 15, 16, 18, 21, 22, 33, 34, 39$ and 40. To find the remaining S_4-designs for $\lambda = 3$ we need the following bipartite decomposition:

Lemma 4.23 There exists a decomposition of $3K_{3,3,3}$ into S_4-blocks.

Proof Here $b = 3x3x3x3/9 = 9$. A design on $Z_3 \times \{0,1,2\}$ is generated by the three blocks

$$(0_0,0_1,0_2;1_1,1_2,1_0)$$

$$(0_0,1_1,2_2;2_1,0_2,1_0)$$

$$(0_0,1_2,2_1;0_1,1_0,2_2).$$

Before we give a recursive construction for all other S_4-designs with $\lambda = 3$, we need some results due to Rees [29].

Definition 4.24 A *Uniformly Resolvable Pairwise Balanced Design* (*URPBD*) is a PBD (see Chapter 1) in which the blocks can be resolved into parallel classes in such a way that all blocks in a given parallel class have the same size.

Definition 4.25 A *URD*(ρ, r) is a uniformly resolvable PBD on ρ points with repetition number r, in which each block has size 2 or 3 – i.e. a resolution of K_p into t 1-factors and $r-t$ *triangle factors*, where a triangle factor is a 2-factor consisting of triangles.

Theorem 4.26 If $\rho \equiv 0 \pmod 6$ and

$$\rho/2+1 \leq r \leq \rho-2$$

then there exists a URD(ρ, r). Furthermore, the replications consist of t 1-factors and $r-t$ triangle factors where r and t satisfy the relation

$$2r - t = \rho - 1.$$

Proof Rees [29].

Theorem 4.27 There exists an $(v,6,3)$ S_4-design whenever the necessary condition $v \equiv 0, 1 \pmod 3$ is satisfied.

Proof We divide the orders v satisfying the necessary condition into five cases and deal with each in turn.

Case I: $v \equiv 0, 1, 6, 7 \pmod{18}$

Arrange the v points in rows of three, with a point at infinity if $v \equiv 1$ or $7 \pmod{18}$. Then the number of rows m satisfies $m \equiv 0$ or $2 \pmod 6$, and therefore from Table 4.1 we know that a MMPTS(m) has a leave consisting of K_2's. Take an MMPTS(m) where each element is a row of three. Cover all mixed edges (edges between rows) in each triple in the MPTS with the decomposition of $3K_{3,3,3}$ into S_4-blocks from Lemma 4.23. This covers all mixed edges except those between pairs of rows not occurring in a triple - i.e. those in the leave of the MMPTS. Cover each pair of rows in the leave with the (6,6,3) S_4-design from Lemma 4.8 (or each pair plus infinity with the (7,6,3) S_4-design from Lemma 4.9 if $v \equiv 1$ or 7 (mod 18)). This takes care of all pure edges (edges within rows) and remaining mixed edges, and thus gives us the required $(v,6,3)$ S_4-design.

Case II: $v \equiv 9, 10 \pmod{18}$

Arranging into m rows of three we get $m \equiv 3$ (mod 6). Therefore there exists a KTS(m), or resolvable STS(m) (see Hall[16]). Take one parallel class of the KTS(m) (set of triples covering each point once, or in this case each row once) and cover each triple in it with a (9,6,3) or (10,6,3) S_4-design from Lemma 4.10 or 4.11 respectively (using infinity point if $v \equiv 10$ (mod 18)). This takes care of all pure edges. Remaining mixed edges can be covered using the $3K_{3,3,3}$ decomposition of Lemma 4.23 on all remaining triples in the KTS.

Case III: $v \equiv 12, 13 \pmod{18}$.

Arrange points in m rows of three, with a point at infinity if $v \equiv 13$ (mod 18). This time $m \equiv 4$ (mod 6), so the leave of a MMPTS(m) is a single $K_{1,3}$ plus K_2's. As in the case $\lambda = 1$ we can use the result by Wilson[37] to show that this MMPTS(m) can be reduced to a PTS(m) having K_4 and K_2's as leave, by deleting one triple from the MMPTS(m) (and thus adding a triple to the leave). Then we cover the K_4 (four rows of three points if $v \equiv 12 \pmod{18}$ and four rows plus infinity point if $v \equiv 13 \pmod{18}$) with the (12,6,3) or (13,6,3) S_4-designs from Lemmas 4.12 and 4.13 respectively. Each K_2 in the leave can be covered by the (6,6,3) or (7,6,3) S_4-designs from Lemmas 4.8 and 4.9 respectively. All remaining edges in K_v can be taken care of by covering each triple in the PTS(m) with the decomposition of $3K_{3,3,3}$ into S_4-blocks from Lemma 4.23.

Case IV: $v \equiv 15, 16 \pmod{18}$.

Again, arrange points in rows of three, with a point at infinity if $v \equiv 16 \pmod{18}$. Here we get $m \equiv 5 \pmod{6}$, and from Wilson[37] we know there exists a PBD(v,[3,5*],1) (partition of the edge set of K_m into a unique K_5 and K_3's). In the $\lambda = 1$ case this was enough, but in this case we have rows of three instead of rows of nine, so that we cannot cover a single row with an S_4-design as it has fewer than 6 points. Therefore, in order to cover all the pure edges, the above PBD must have a parallel class containing the unique K_5. Let $m = 6q + 5$, and $\rho = 6q$, so we have $m = \rho + 5$. Then by Theorem 4.26 we can find a URD(p,k) on ρ vertices where $\rho = m - 5 \equiv 0 \pmod{6}$, and as long as $\rho \geq 12$ we can take $t = 5$. Then we construct an S_4-design as follows. Match each of the 5 1-factors with one of the 5 points excluded from the URD. Cover the resulting triangles (triples of rows) with the $3K_{3,3,3}$ decomposition from Lemma 4.23. Cover the 5 points with the (15,6,3) (or (16,6,3), as appropriate) S_4-design from Lemma 4.14. We are left with triangle factors. Cover one of these with copies of the (9,6,3) S_4-design (or the (10,6,3) S_4-design in case $v \equiv 16 \pmod{18}$) and the rest with the $3K_{3,3,3}$ decomposition of Lemma 4.23. In this way all edges within and between rows are covered, so we have the required $(v,6,3)$ S_4-design. This works for all $v \equiv 15,16 \pmod{18}$ except for $v = 15$, 16, 33, 34. Designs on these values have been found directly, so we have S_4-designs for all $v \equiv 15, 16 \pmod{18}$.

Case V: $v \equiv 3, 4 \pmod{18}$.

Arrange in rows of three as above to get $m \equiv 1 \pmod 6$. Proceed exactly as in previous case except this time take $p = m - 7$, and the number of 1-factors t to be 7. Use the (21,6,3) S_4-design (or the (22,6,3) S_4-design in the case $v \equiv 4 \pmod{18}$) to cover the K_7. The remaining edges can be covered as above. This covers all $v \equiv 3, 4 \pmod{18}$ except $v = 21, 22, 39, 40$ where designs have been found directly.

§4.4 S_4-designs with $\lambda=9$

Now that we have looked at S_4-designs with $\lambda = 1$ and $\lambda = 3$, to settle the existence question for designs on this graph we need only find when they exist with $\lambda = 9$. This is because S_4-designs on all other values of λ can be derived from these three cases by Lemma 1.8, excepting only any cases where a design with $\lambda = 1$, 3, or 9 does not exist when the necessary conditions on v are satisfied. We have already determined (Theorems 4.7 and 4.27) that there are no such exceptions for $\lambda = 1$ or 3, so now we look at S_4-designs with $\lambda = 9$.

With $\lambda = 9$ the necessary conditions (of Theorem 1.5) for the existence of an S_4-design are reduced to the single condition: $v \geq 6$. If $v \equiv 0,1 \pmod 3$ then a $(v,6,9)$ S_4-design can be derived from smaller designs.

Lemma 4.28 A $(v,6,9)$ S_4-design exists whenever $v \geq 6$ and $v \equiv 0, 1 \pmod 3$.

Proof From Theorem 4.24 we have a $(v,6,3)$ S_4-design, and from Lemma 1.8 we can therefore find a $(v,6,9)$ S_4-design.

We are left with those values of v satisfying $v \equiv 2 \pmod 3$. The smallest designs we must find are therefore those with $v = 8$ and 11.

Lemma 4.29 An (8,6,9) S_4-design exists.

Proof A design on $\{Z_7 \cup \infty\}$ is generated by the two blocks

$$(0,1,5;4,3,6)$$
$$\text{and} \quad (0,5,\infty;4,6,3) \quad 3 \text{ times}.$$

Lemma 4.30 An (11,6,9) S_4-design exists.

Proof A cyclic design on Z_{11} is generated by the three blocks

$$(0,5,6;2,8,7) \quad \text{twice},$$
$$(0,3,7;9,1,5) \quad \text{twice},$$
$$(0,1,7;3,4,6).$$

In fact we did not need to find the above design directly, since an (11,6,9) S_4-design exists by the following theorem:

Lemma 4.31 A $(v,6,9)$ S_4-design exists whenever $v \geq 6$ and v is *prime*.

Proof The number of blocks in such a design is $v(v-1)/2$. If v is prime then $\{Z_v^+, \times\}$ is a cyclic group. The number of base blocks in a cyclic design on Z_v would be $(v-1)/2$. We can find a set of base blocks by taking *any* S_4-block in $9K_v$ and multiplying it successively by any primitive element α of Z_v until we have all $(v-1)/2$ base blocks. (A *primitive element* α in a group is one which generates the whole group under the group multiplication, so that each group element can be written as α^t where $1 \leq t \leq v-1$.) Each difference occurs the same number of times (nine) in these blocks since multiplying any difference in the first block by α $(v-1)/2$ times must take us through each possible difference once (the differences come in pairs, ±1, $\pm2, \ldots, \pm(v-1)/2$, and we consider differences in a pair to be the same since they represent the same edgelength in K_v, so we get all required differences with only $(v-1)/2$ base blocks).

For example, an alternate $(11,6,9)$ S_4-design on Z_{11} can be found by taking the first base block to be $(0,1,2;3,4,5)$. Then the remaining four base blocks are found by successively multiplying the elements of this first one by 2, which is a primitive root of $\{Z_{11}^+, \times\}$. This gives a cyclic $(11,6,9)$ S_4-design on Z_{11} with base blocks

$$(0,1,2;,3,4,5)$$
$$(0,2,4;6,8,10)$$
$$(0,4,8;1,5,9)$$
$$(0,8,5;2,10,7)$$
$$(0,5,10;4,9,3).$$

We will use a result involving transversal designs and Latin Squares to find all remaining $(v,6,9)$ S_4-designs with $v \geq 50$, but this method does not work for smaller designs. Therefore we must use direct methods to find designs on all $v \equiv 2 \pmod 3$ where $v < 50$ and v is not prime, i.e. for $v = 14, 20, 26, 32, 35, 38$, and 44.

Lemma 4.32 There exists a $(14,6,9)$ S_4-design.

Proof Here $b = 91$. Such a design is generated on $\{Z_{13} \cup \infty\}$

by the blocks

$(0,1,8;3,6,10)$ 4 times

$(0,1,\infty;4,3,2)$

$(0,3,4;9,5,\infty)$

$(0,2,7;4,\infty,5)$.

Lemma 4.33 There exists a $(20,6,9)$ S_4-design.

Proof Here $b = 190$. A design on $\{Z_{19} \cup \infty\}$ is generated by the blocks

$(0,1,16;2,13,7)$ 7 times

$(0,8,13;7,\infty,3)$ twice

$(0,1,3;\infty,2,5)$

Lemma 4.34 There exists a $(26,6,9)$ S_4-design.

Proof Here $b = 13 \times 25 = 325$. A cyclic design on Z_{25} is generated by the blocks

$(0,6,13;4,20,3)$ 8 times

$(0,3,5;11,6,8)$

$(0,2,5;9,6,4)$

$(0,2,\infty;17,7,5)$

$(0,2,7;\infty,5,10)$

$(0,3,5;2,8,\infty)$.

Lemma 4.35 There exists a $(32,6,9)$ S_4-design.

Proof Here $b = 16 \times 31 = 496$. A design on $\{Z_{31} \cup \infty\}$ is generated by the blocks

$(0,3,20;16,26,7)$ 9 times

$(0,1,7;4,6,11)$ 4 times

$(0,6,7;\infty,11,13)$

$(0,1,2;6,\infty,8)$

$(0,2,4;6,\infty,5)$.

Lemma 4.36 There exists a (35,6,9) S_4-design.

Proof Here b = 17x35. A design on Z_{35} is generated by the blocks

(0,10,18;11,26,4)	9 times
(0,4,9;12,16,7)	4 times
(0,3,6;1,4,7)	twice
(0,1,2;6,4,8)	
(0,6,12;9,8,14).	

Lemma 4.37 There exists a (38,6,9) S_4-design.

Proof Here b = 19 x 37. A design on $(Z_{37} \cup \infty)$ is generated by the blocks

(0,10,19;22,27,35)	9 times
(0,4,11;14,16,17)	7 times
(0,4,11;12,18,∞)	twice
(0,1,3;∞,2,5).	

Lemma 4.38 There exists a (44,6,9) S_4-design.

Proof Here b = 22 x 43. A design on $(Z_{43} \cup \infty)$ is generated by the blocks

(0,15,27;17,36,7)	9 times
(0,13,18;11,5,14)	8 times
(0,11,18;7,6,5)	
(0,2,9;7,3,1)	
(0,1,2;7,∞,9)	twice
(0,1,2;7,∞,4).	

Lemma 4.39 There exists a $(v,6,9)$ S_4-design for all $v \le 49$.

Proof This result follows from Lemmas 4.28 to 4.38.

Next we look at $(v,6,9)$ S_4-designs with $v \ge 50$, but first we need some more theory.

A *Latin square of order* n is an $n \times n$ matrix on n symbols such that each symbol occurs exactly once in each row and once in each column. Two Latin squares $A = \{a_{i,j}\}$, $B = \{b_{i,j}\}$ are said to be *orthogonal* if the n^2 ordered pairs $\{(a_{i,j}, b_{i,j})\}_{i,j=1}^{n}$ are all distinct. A set of t *MOLS*(n) is a set of t Latin squares of order n where each pair is orthogonal.

Theorem 4.40 If n is a prime power there exist $n-1$ MOLS(n).

Proof See Dénes, Keedwell [13].

Definition 4.41 A *Transversal Design* $TD(r,n)$ is a triple $(\mathcal{V},\mathcal{G},\mathcal{B})$ where $\mathcal{V}$ is a set of v elements, $\mathcal{B}$ is a collection of subsets of $\mathcal{V}$ called *blocks*, and $\mathcal{G}$ is a collection of subsets of $\mathcal{V}$ called groups such that

(*i*) $|b| = r$ for all $b \in \mathcal{B}$, $|\mathcal{G}| = r$

$|g| = n$ for all $g \in \mathcal{G}$, $|\mathcal{V}| = nr$.

(*ii*) $b \in \mathcal{B}$, $g \in \mathcal{G} \rightarrow |b \cap g| = 1$.

(*iii*) If x, $y \in g_i$ for some i, they are not contained in any block.

(*iv*) If $x \in g_i$, $y \in g_j$ $i \neq j$ then the pair $\{x,y\}$ is contained in exactly one block.

Theorem 4.42 There exists a set of $r-2$ MOLS(n) if and only if there exists a TD(r,n).

Proof See Hanani [16].

Theorem 4.43 For $n \geq 63$ there exists a TD$(7,n)$.

Proof There exist 5 MOLS(n) whenever $n \geq 63$ (see [6]), and therefore by Theorem 4.42 there also exists a TD$(7,n)$.

We can use the above results to find all remaining required S_4-designs. If $v = nr$ where $n, r \geq 6$ and there exists a TD(r,n) then we can decompose $9K_v$ into copies of $9K_n$ and $9K_r$ using the TD(r,n), and then provided we have S_4-designs on $9K_n$ and $9K_r$ we have a $(v,6,9)$ S_4-design by Lemma 1.10. If v does not satisfy these conditions we must find m such that $v+m = nr$ satisfies the conditions and such that we can delete m points from the TD(r,n) in such a way that each block and each group still have a number of elements satisfying the conditions on n and r above. For $v \geq 384$ we can use the result in Theorem 4.43 to find a $(v,6,9)$ S_4-design, but for smaller values we must show the existence of suitable parameters n, r, and m.

Lemma 4.44 For all $v \geq 6$, either there exists a $(v,6,9)$ S_4-design or there exist n, r, m satisfying

(*i*) $n, r \geq 6$, $-1 \leq m \leq (n-6)(r-6)$

(*ii*) $v = nr - m$

(*iii*) There exist $r-2$ MOLS(n).

(Note (*iii*) $\Rightarrow n \geq r-1$ since $\mathcal{N}(n) \leq n-1$).

In order to prove this result, we first need the results of Lemmas 4.45 to Lemma 4.47.

Lemma 4.45 For all $v \geq 384$ there exist n, r, m satisfying the conditions of Lemma 4.44.

Proof Take $r = 7$. We know from Theorem 4.43 that for $n \geq 63$ there exists 5 MOLS(n). When $n = 63$ we have $v = 441 - m$ where $m \leq (n-6)(r-6)$, so the smallest value of v we can have here corresponds to the largest value of m. Substituting 63 for n and 7 for r this gives us $m \leq 57$. Therefore we must have $v \geq 7 \times 63 - 57 = 384$.

Lemma 4.46 For all $50 \leq v \leq 83$ not satisfying the conditions of Lemmas 4.28 or 4.31 there exist n, r, m satisfying the conditions of Lemma 4.44.

Proof These values are shown in Table 4.4 below.

v	n	r	m
50	7	7	−1
56	8	7	0
62	9	7	1
65	8	8	−1
68	9	8	4
74	11	7	3
77	11	7	0
80	9	9	1

Table 4.4

Lemma 4.47 For all $84 \le v \le 384$ there exist n, r, m satisfying the conditions of Lemma 4.44.

Proof For $84 \le v \le 182$ put

$$n = 13,$$
$$r = \lceil v/n \rceil = \lceil v/13 \rceil,$$
$$m = nr - v = 13r - v.$$

Since 13 is prime there exists 12 MOLS(n) by Theorem 4.40, and m as defined above is always non negative. Therefore the conditions of Lemma 4.43 reduce to conditions on r and m:

$$6 \le r \le 14, \qquad (1)$$

$$m \le 7(r-6). \qquad (2)$$

The smallest v we are considering here is $v = 84$. In this case we get

$$r = \lceil 84/13 \rceil = 7,$$

$$m = 13x7 - 84 = 7.$$

Therefore the condition $r \ge 6$ is satisfied for all $v \ge 84$. Similarly the condition $r \le 14$ is satisfied for all $v \le 182$. Condition (2) is satisfied whenever $r \ge 8$ since in this case it suffices that $m < 13$, which condition always holds from the way we have defined m. And when $r = 7$, the largest m is obtained when v is 84 where we get $m = 7$, and equality in condition (2). Therefore the conditions of Lemma 4.44 are satisfied for all $84 \le v \le 182$.

For $180 \le v \le 384$ proceed as above with $n = 29$ instead of 13. Since 29 is prime there exists 28 MOLS(29) by

Theorem 4.40. Here conditions (1) and (2) become:

$$6 \le r \le 30, \qquad (1)$$

$$m \le 23(r-6). \qquad (2)$$

These are satisfied whenever $180 \le v \le 870$, and therefore also for $182 \le v \le 384$.

Proof of Lemma 4.44 There exists a $(v,6,9)$ S_9-design for all $6 \le v \le 49$ by Lemma 4.39, and for $v \equiv 0,1 \pmod 3$ or v prime by Lemmas 4.28 and 4.31 respectively. For all other v there exist n, r, m satisfying conditions (i), (ii) and (iii) by Lemmas 4.45, 4.46, and 4.47.

Theorem 4.48 There exists $(v,6,9)$ S_4-design for all $v \ge 6$.

Proof Use recursion on v. From Lemma 4.39 we know that such a design exists for all $v \le 49$. This gives us the first step of the recursion and also enables us to assume that $v \ge 49$ in the remaining step. For all $v \ge 49$ either a $(v,6,9)$ S_9-design exists or there exist n, r and m which satisfy conditions (i) to (iii) of Lemma 4.44. Assume the theorem is true for $v \le q-1$, then we can find a $(q,6,9)$ S_4-design as follows:

Take n, r, m satisfying the conditions of Lemma 4.44. Since there exists $r-2$ MOLS(n), by Theorem 4.42 there exists a TD(n,r,1). Take each row and column of this TD$(n,r,1)$ nine times. If $m = 0$ then $q = nr$ so we have a TD on q points and can proceed as follows:

Cover each block of the repeated TD(n,r,1) with an (r,6,9) S_4-design, and each row with an (n,6,9) S_4-design. These S_4-designs exist by assumption of recursive proof since n, $r < q$. We have then covered all edges in $9K_q$ and therefore have a (q,6,9) S_4-design.

If $m = -1$ then $q = nr + 1$. Take the above TD(n,r,1) and add a "point at infinity" to bring the total number of points to q. Put this ∞ point in each row. Then multiply all rows and columns by nine and proceed as above, the only difference being that now each row has size n+1 instead of n. An (n+1,6,9) S_4-design must exist by assumption since n+1 is certainly less than q, so we can use it to cover each row of the repeated TD(n,r,1).

The remaining possibility is that $1 \le m \le (n-6)(r-6)$. In this case we must delete m points from the TD(n,r,1) to bring the total number of points to q. Proceed as follows:
(i) If $m \le n-6$, delete m points from the first row, otherwise delete $n-6$ points.
(ii) If we have deleted m points we are finished. Otherwise continue deleteting points from each row in succession as in (i) until we have deleted m points.

Since $m \le (n-6)(r-6)$ we must be able to delete points in this fashion and be left with at least six points in each row and six points in each block. Then take each row and each block nine times and cover all resulting rows and blocks with the appropriate S_4-design. Since each row and block must have size s where $6 \le s < q$ we have such a design in each case (*i.e.* an $(s,6,9)$ S_4-design) by our induction hypothesis. We have thus covered all edges of $9K_q$ and therefore we have found a $(q,6,9)$ S_4-design as required.

Theorem 4.49 There exists a $(v,6,\lambda)$ S_4-design for all v and λ satisfying the necessary conditions of Theorem 1.5.

Proof This follows from Theorems 4.7, 4.27, 4.48, and Lemma 1.8.

CHAPTER 5

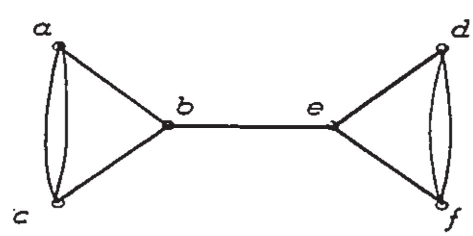
DESIGNS ON CUBIC MULTIGRAPHS ON 6 VERTICES:

Non-Bipartite Connected Multigraphs

§5.1 S_5-designs with λ=2

Now we look at the remaining two multigraphs in category II of Fig. 3.1, S_5 and S_6. Both these multigraphs have maximum edge multiplicity $m = 2$, so we must first look for designs with $\lambda = 2$, then for ones with $\lambda=3$ and $\lambda=9$, and if we can find these whenever the necessary conditions are satisfied then we are finished by Lemma 1.8. We can use exactly the same constructions that we used for the envelope S_4, so that all we need to find are some small designs and some decompositions of complete tripartite multigraphs.

We start with the multigraph S_5. For this multigraph we use the labelling system shown in Fig. 5.1:



$(a,b,c;d,e,f)$

Fig. 5.1

For $\lambda=2$ the necessary conditions for the existence of an S_5-design are $v \equiv 1 \pmod 9$ and $v \geq 6$. We can use exactly the same constructions that were used for S_4 with $\lambda=1$, the only difference being that since $\lambda=2$ now all edges in the complete multigraphs we must decompose are double edges. From Lemma 1.8, since the necessary conditions are the same in both cases, the techniques used with $\lambda=1$ are also valid with $\lambda=2$. Therefore we must find S_5 designs directly for the same values of v for which we had to find S_4-designs directly with $\lambda=1$ - i.e. $v = 10, 19, 28, 37$, and 46.

Lemma 5.1 There exists a $(10,6,2)$ S_5-design.

Proof Here $b = 10$. A cyclic design on Z_{10} is generated by the block

$$(0,7,8;1,2,5).$$

Lemma 5.2 There exists a $(19,6,2)$ S_5-design.

Proof Here $b = 38$. A cyclic design on Z_{19} is generated by the two blocks

$$(0,16,4;13,7,14)$$
$$(0,16,5;13,7,15).$$

Lemma 5.3 There exists a $(28,6,2)$ S_5-design.

Proof Here $b = 84$. A design on Z_{28} is generated by the three blocks

$$(0,24,9;12,25,19)$$
$$(0,22,23;15,5,17)$$
$$(0,17,3;11,7,19).$$

Lemma 5.4 A (37,6,2) S_5-design exists.

Proof Here b = 148. A cyclic design on Z_{37} is generated by the four blocks

(0,20,2;26,13,35)
(0,21,3;4,28,14)
(0,12,1;3,24,7)
(0,15,29;25,20,31).

Lemma 5.5 There exists a (46,6,2) S_5-design.

Proof Here b = 230. A design on Z_{46} is generated by the blocks

(0,25,3;34,17,45)
(0,26,4;18,34,19)
(0,21,41;37,7,39)
(0,17,7;29,11,38)
(0,36,13;3,30,15)

In order to find all S_5-designs with $\lambda = 2$ we now need only find a decomposition of $2K_{9,9,9}$ into S_5-blocks:

Lemma 5.6 There exists a decomposition of $2K_{9,9,9}$ into S_5-blocks.

Proof Here b = 54. A decomposition on Z_9x {0,1,2} is generated by the blocks

$(2_1,0_0,5_2;3_0,1_1,7_2)$
$(2_2,0_1,5_0;3_1,1_2,7_0)$
$(2_0,0_2,5_1;3_2,1_0,7_1)$
$(0_0,2_2,0_1;7_1,3_0,6_2)$
$(0_1,2_0,0_2;7_2,3_1,6_0)$
$(0_2,2_1,0_0;7_0,3_2,6_1)$.

Theorem 5.7 There exists a $(v,6,2)$ S_5-design for all values of v satisfying the necessary condition $v \equiv 1 \pmod 9$.

Proof Take Lemma 1.8, and the proof of Theorem 4.7 with "S_4" replaced by "S_5" and Lemmas 4.1 to 4.6 replaced by Lemmas 5.1 to 5.6 respectively.

§5.2 S_5-designs with $\lambda=3$

Next we must look at S_5-designs with $\lambda=3$. The necessary conditions here are $v \equiv 0, 1 \pmod 3$ and $v \geq 6$. Here we can use exactly the same constructions as we did for the envelope, S_4. Therefore we must find designs directly for the same values of v as we did for the envelope. We must also find an S_5-design directly for $v = 10$. (For S_4, designs on v points where $v \equiv 1 \pmod 9$ could be derived from the corresponding $(v,6,1)$ S_4-designs, but for S_5 they cannot, since S_5 has double edges and therefore S_5 designs exist only with $\lambda \geq 2$. The only such value of v which is too small to be found using recursion is $v = 10$.) Altogether, we need direct constructions of $(v,6,3)$ S_5-designs for v = 6, 7, 9, 10, 12, 13, 15, 16, 21, 22, 33, 34, 39, and 40.

Lemma 5.8 There exists a $(6,6,3)$ S_5-design.

Proof Here $b = 5$. A design is generated on $\{Z_5 \cup \infty\}$ by the block

$$(0,3,1;2,4,\infty).$$

Lemma 5.9 There exists a $(7,6,3)$ S_5-design.

Proof Here $b = 7$. A cyclic design on Z_7 is generated by the block

$$(0,4,1;3,2,5).$$

Lemma 5.10 There exists a $(9,6,3)$ S_5-design.

Proof Here $b = 12$. A design on $\{Z_8 \cup \infty\}$ is generated by the two blocks

$$(0,5,6;2,1,4) \quad \text{(half cycle)}$$
$$(0,1,\infty;3,5,6).$$

Lemma 5.11 There exists a $(10,6,3)$ S_5-design.

Proof A design on $Z_5 \times \{0,1\}$ is generated by the three blocks

$$(0_0,1_0,2_0;0_1,1_1,2_1)$$
$$(1_1,0_0,2_0;1_0,0_1,2_1)$$
$$(0_0,4_0,2_1;0_1,4_1,2_0).$$

Lemma 5.12 There exists a $(12,6,3)$ S_5-design.

Proof Here $b = 22$. A design on $\{Z_{11} \cup \infty\}$ is generated by the two blocks

$$(0,2.10;1,7,5)$$
$$(10,0,7;4,6,\infty).$$

Lemma 5.13 There exists a $(13,6,3)$ S_5-design.

Proof Here $b = 26$. A cyclic design on Z_{13} is generated by the two blocks

$$(0,4,12;5,11,8)$$
$$(0,2.11;4,8,9).$$

Lemma 5.14 There exists a $(15,6,3)$ S_5-design.

Proof Here $b = 35$. A design on $(Z_7 \times \{0,1\}) \cup \infty$ is generated by the five blocks

$$(0_0,2_1,0_1;5_0,3_0,6_1)$$

$(0_0, 4_0, 2_1; 5_0, 4_1, 6_0)$

$(0_0, 3_1, 5_0; 0_1, 4_1, 5_1)$

$(0_0, 3_0, \infty; 5_0, 6_0, 2_1)$

$(6_1, 5_0, 2_1; 5_1, 4_1, \infty)$.

Lemma 5.15 There exists a (16,6,3) S_5-design.

Proof Here $b = 40$. A design on Z_{16} is generated by the three blocks

(0,3,7;15,11,8) (half orbit)

(0,10,3;7,2,13)

(0,5,4;6,7,8).

Lemma 5.16 There exists a (21,6,3) S_5-design.

Proof Here $b = 70$. A design on $\{Z_{20} \cup \infty\}$ is generated by the four blocks

(0,4,15;5,14,10) (half orbit)

(0,9,6;2,19,10)

(0,15,7;4,2,8)

$(0,3,2;5,4,\infty)$.

Lemma 5.17 There exists a (22,6,3) S_5-design.

Proof Here $b = 77$. A design on Z_{22} is generated by the four blocks

(0,10,6;11,21,17) (half orbit)

(0,15,10;12,4,21)

(0,17,8;7,11,14)

(0,3,2;9,7,10).

Lemma 5.18 There exists a (33,6,3) S_5-design.

Proof Here $b = 176$. A design on $\{Z_{32} \cup \infty\}$ is generated by the six blocks

(0,24,15;16,8,31) (half orbit)

(0,22,14;17,6,30)
(0,12,25;3,29,15)
(0,4,11;22,26,31)
(0,10,6;9,11,14)
(0,3,2;7,4,∞).

Lemma 5.19 There exists a (34,6,3) S_5-design.

Proof Here b = 187. A design on Z_{34} is generated by the six blocks

(0,9,16;17,26,33) (half orbit)
(0,26,15;3,9,24)
(0,26,14;12,6,24)
(0,10,11;21,18,31)
(0,4,9;3,32,30)
(0,5,4;9,8,11).

Lemma 5.20 There exists a (39,6,3) S_5-design.

Proof Here b = 247. A design on $\{Z_{38} \cup \infty\}$ is generated by the seven blocks

(0,8,18;19,27,37) (half orbit)
(0,28,17;18,9,34)
(0,18,15;13,35,20)
(0,28,14;8,16,21)
(0,8,12;13,15,24)
(0,9,6;20,15,19)
(0,3,2;12,7,∞).

Lemma 5.21 There exists a (40,6,3) S_5-design.

Proof Here b = 260. A design on Z_{40} is generated by the seven blocks

(0,10,19;20,30,39) (half orbit)
(0,36,18;10,16,33)

$(0,32,16;3,13,28)$
$(0,28,14;7,1,34)$
$(0,7,12;12,18,23)$
$(0,7,37;14,15,23)$
$(0,5,4;8,7,10)$.

Lemma 5.22 There exists a decomposition of $3K_{3,3,3}$ into S_5-blocks.

Proof The number of blocks here is 9. A decomposition on $Z_3 \times \{0,1,2\}$ is generated by the blocks

$$(0_0,0_2,0_1;2_1,2_0,2_2)$$
$$(0_0,2_2,1_1;0_1,2_0,1_2)$$
$$(2_1,0_0,1_2;1_0,2_2,0_1).$$

Theorem 5.23 A $(v,6,3)$ S_5-design exists for all values of v satisfying the necessary condition $v \equiv 0, 1 \pmod 3$.

Proof Take the proof of Theorem 4.27 (page 73), replace "S_4" by "S_5", and Lemmas 4.8 to 4.23 by Lemmas 5.8 to 5.21 respectively.

§5.3 S_5-designs with $\lambda=9$

For $\lambda=9$, we can use exactly the same methods to find S_5-designs as we did to find S_4-designs. Therefore all we need do is find S_5-designs directly for the same values of v as was done for the envelope S_4 – *i.e.* for values of v satisfying $v \equiv 2 \pmod 3$, $v < 50$, and v is not prime. These values are $v = 8$, 14, 20, 26, 32, 35, 38 and 44.

Lemma 5.24 There exists a (8,6,9) S_5-design.

Proof Here $b = 28$. A design on $\{Z_7 \cup \infty\}$ is generated by the blocks

(0,4,1;3,2,5)
(0,4,1;3,∞,5) 3 times.

Lemma 5.25 There exists a (14,6,9) S_5-design.

Proof Here $b = 91$. A design on $\{Z_{13} \cup \infty\}$ is generated by the blocks

(0,4,10;7,9,∞)
(0,2,12;3,5,∞)
(0,2,12;3,4,∞)
(0,1,5;3,12,9) 4 times.

Lemma 5.26 There exists a (20,6,9) S_5-design.

Proof Here $b = 190$. A design on $\{Z_{19} \cup \infty\}$ is generated by the blocks

(0,5,8;2,17,11) 4 times
(0,6,1;8,18,10)
(0,13,7;5,18,12)
(0,1,4;13,15,18)
(0,∞,2;12,16,13) 3 times.

Lemma 5.27 There exists a (26,6,9) S_5-design.

Proof Here $b = 325$. A design on $\{Z_{25} \cup \infty\}$ is generated by the blocks

(0,18,12;8,3,19) 4 times
(0,9,4;13,3,21) 4 times
(0,10,3;14,2,16)
(0,5,3;15,9,18)
(0,3,1;6,4,∞) 3 times.

Lemma 5.28 There exists a (32,6,9) S_5-design.

Proof Here $b = 496$. A design on $\{Z_{31} \cup \infty\}$ is generated by the blocks

(0,24,15;5,11,19)	4 times
(0,21,11;5,30,17)	4 times
(0,3,5;4,11,8)	4 times
(0,20,1;16,2,18)	
(0,23,2;21,14,∞)	
(0,6,1;9,5,∞)	
(0,3,1;5,4,∞).	

Lemma 5.29 There exists a (35,6,9) S_5-design.

Proof Here $b = 595$. A design on Z_{35} is generated by the blocks

(0,11,20;5,30,22)	4 times
(0,10,21;19,26,32)	4 times
(0,9,4;10,17,22)	4 times
(0,19,32;21,1,22)	
(0,9,1;30,20,32)	
(0,3,6;11,10,13)	
(0,4,6;12,7,13)	
(0,3,2;7,6,9).	

Lemma 5.30 There exists a (38,6,9) S_5-design.

Proof Here $b = 703$. A design on $\{Z_{37} \cup \infty\}$ is generated by the blocks

(0,11,21;1,28,19)	4 times
(0,25,13;16,8,31)	4 times
(0,3,7;22,17,28)	4 times
(0,5,2;17,9,18)	3 times
(0,26,9;15,5,23)	
(0,25,3;20,18,∞)	
(0,2,4;20,25,∞)	
(0,32,1;18,19,∞).	

Lemma 5.31 There exists a (44,6,9) S_5-design.

Proof Here $b = 946$. A design on $\{Z_{43} \cup \infty\}$ is generated by the blocks

(0,10,22;20,29,43)	4 times
(0,24,13;15,41,33)	4 times
(0,31,16;11,3,17)	4 times
(0,11,7;25,21,30)	4 times
(21,0,24;37,20,38)	
(9,0,11;25,33,26)	
(5,0,6;36,39,∞)	
(14,0,16;25,13,28)	
(2,0,3;38,41,∞)	
(0,1,41;39,42,∞).	

Lemma 5.32 There exists a $(v,6,9)$ S_5-design whenever $v \equiv 0,1 \bmod 3$, v is prime, or $v \le 49$.

Proof This result follows from Lemma 1.8 and Theorem 5.23 ($v \equiv 0,1 \bmod 3$), Lemma 4.31 with S_4 replaced by S_5 (v prime) and Lemmas 5.24 to 5.31 (all remaining $v \le 49$).

Theorem 5.33 There exists a $(v,6,9)$ S_5-design for all v satisfying the necessary condition $v \ge 6$.

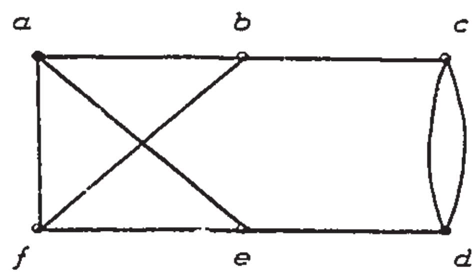
Proof Take Theorem 4.48, replace "S_4" by "S_5", and Lemmas 4.28, 4.31 and 4.39 by Lemma 5.32.

Theorem 5.34 There exist a $(v,6,\lambda)$ S_5-designs for all v and λ satisfying the necessary conditions of Theorem 1.6.

Proof This result follows from Lemma 1.8, Theorems 5.7, 5.23 and 5.33.

§5.4 S_6-designs with λ=2

Now we look at the final multigraph S_6 in category II of Fig. 3.1: connected non-bipartite cubic multigraphs on six vertices. Here we need only find the same small designs directly that we did for the other two multigraphs in this category: S_4 and S_5. Then we can use the same techniques exactly to show the existence of all larger cases satisfying the necessary conditions. For S_6 we use the block labelling system shown in Fig. 5.2 below.



$(a,b,c;d,e,f)$

Fig. 5.2

For λ=2 the necessary conditions for the existence of an S_6-design are $v \equiv 1 (\mathrm{mod}\ 9)$, $v \geq 6$. We must find direct constructions for v = 10, 19, 28, 37, 46, and we must also find a decomposition of $2K_{9,9,9}$ into S_6-blocks.

Lemma 5.35 There exists a (10,6,2) S_6-design.

Proof Here b = 10. A design on Z_{10} is generated by the block

$$(0,1,9;6,4,5).$$

Lemma 5.36 There exists a (19,6,2) S_6-design.

Proof Here $b = 38$. A design on Z_{19} is generated by the two blocks

(0,18,14;11,7,6)

(0,14,4;2,11,6).

Lemma 5.37 There exists a (28,6,2) S_6-design.

Proof Here $b = 84$. A design on Z_{28} is generated by the three blocks

(0,14,2;15,5,8)

(0,10,1;12,24,2)

(0,9,10;3,1,4).

Lemma 5.38 There exists a (37,6,2) S_6-design.

Proof Here $b = 148$. A design on Z_{37} is generated by the four blocks

(0,13,31;11,29,22)

(0,15,29;8,14,27)

(0,12,22;33,4,5)

(0,9,4;2,36,3).

Lemma 5.39 There exists a (46,6,2) S_6-design.

Proof Here $b = 230$. A design on Z_{46} is generated by the five blocks

(0,23,2;24,32,11)

(0,19,38;12,17,32)

(0,17,33;5,40,10)

(0,12,27;36,3,10)

(0,6,44;2,3,1).

Lemma 5.40 There exists a decomposition of $2K_{3,3,3}$ into S_6-blocks.

Proof The number of blocks here is 6. A design on $Z_3 \times \{0,1,2\}$ is generated by the two blocks

$$(0_0,1_2,1_0;0_1,2_2,1_1)$$
$$(0_0,0_2,2_1;2_0,1_2,1_1).$$

Lemma 5.41.. There exists a decomposition of $2K_{p,p,p}$ into S_6-blocks.

Proof This follows from Lemma 5.40 and Lemma 1.9.

Theorem 5.42 There exists a $(v,6,2)$ S_6-design for all values of v satisfying the necessary conditions $v \equiv 1 \pmod 9$ and $v \geq 6$.

Proof Take the proof of Theorem 4.7. replace "S_4" by "S_6" and Lemmas.4.1 to 4.6 by Lemmas 5.35 to 5.41 respectively.

§5.5 S_6-designs with λ=3

For $\lambda = 3$ we must find direct constructions for v = 6, 7, 9, 10, 12, 13, 15, 16, 21, 22, 33, 34, 39, 40, and we must also find a decomposition of $3K_{3,3,3}$ into S_6-blocks, as we did for S_4 and S_5.

Lemma 5.43 There exists a $(6,6,3)$ S_6-design.

Proof A design on $\{Z_5 \cup \infty\}$ is generated by the block

$$(0,3,2;\infty,4,1).$$

Lemma 5.44 There exists a (7,6,3) S_6-design.

Proof A design on Z_7 is generated by the block

(0,1,6;5,2,4).

Lemma 5.45 There exists a (9,6,3) S_6-design.

Proof Here b = 12. A design on $\{Z_8 \cup \infty\}$ is generated by the two blocks

(0,1,6;2,5,4) (half orbit)

(0,2,∞;4,6,7).

Lemma 5.46 There exists a (10,6,3) S_6-design.

Proof Here b = 15. A design on Z_{10} is generated by the two blocks

(0,2,1;6,7,5) (half orbit)

(0,1,4;2,6,7).

Lemma 5.47 There exists a (12,6,3) S_6-design.

Proof Here b = 22. A design on $\{Z_{11} \cup \infty\}$ is generated by the two blocks

(3,0,4;9,5,8)

(3,1,0;10,∞,5).

Lemma 5.48 There exists a (13,6,3) S_6-design.

Proof Here b = 26. A design on Z_{13} is generated by the two blocks

(0,6,11;2,7,10)

(0,5,6;4,1,12).

Lemma 5.49 There exists a (15,6,3) S_σ-design.

Proof Here b = 35. A design on $(Z_7 \times \{0,1\}) \cup \infty$ is generated by the five blocks

$$(2_0,3_1,4_1;6_1,3_0,0_1)$$
$$(2_0,3_0,4_0;6_0,3_1,0_1)$$
$$(1_0,6_0,3_0;5_1,3_1,0_1)$$
$$(3_1,0_0,0_1;\infty,4_1,4_0)$$
$$(3_0,0_0,\infty;4_0,2_1,3_1).$$

Lemma 5.50 There exists a (16,6,3) S_σ-design.

Proof Here b = 40. A design on Z_{16} is generated by the three blocks

(0,6,1;9,14,8) (half orbit)

(0,5,1;13,6,15)

(0,1,15;12,5,2)

Lemma 5.51 There exists a (21,6,3) S_σ-design.

Proof Here b = 70. A design on $(Z_{20} \cup \infty)$ is generated by the four blocks

(0,7,15;25,17,10) (half orbit)

(0,7,6;17,8,2)

(0,5,1;15,7,3)

(0,1,2;∞,3,5).

Lemma 4.52 There exists a (22,6,3) S_σ-design.

Proof Here b = 154. A design on Z_{22} is generated by the four blocks

(0,7,17;6,18,11) (half orbit)
(0,5,3;15,6,9)
(0,5,14;21,16,8)
(0,3,5;4,8,2).

Lemma 5.53 There exists a (33,6,3) S_σ-design.

Proof Here b = 176. A design on $(Z_{32} \cup \infty)$ is generated by the six blocks

(0,11,25;9,27,16) (half orbit)
(0,12,25;8,18,27)
(0,11,23;4,22,2)
(0,4,26;18,7,30)
(0,1,7;14,6,5)
(0,3,∞;7,4,1).

Lemma 5.54 There exists a (34,6,3) S_σ-design.

Proof Here b = 187. A design on Z_{34} is generated by the six blocks

(0,12,25;8,29,17) (half orbit)
(0,14,29;11,26,32)
(0,12,25;5,15,3)
(0,8,29;6,10,1)
(0,8,19;26,2,33)
(0,3,8;2,4,33).

Lemma 5.55 There exists a (39,6,3) S_σ-design.

Proof Here b = 217. A design on $(Z_{38} \cup \infty)$ is generated by the seven blocks

(0,12,28;9,31,19) (half orbit)
(0,17,33;13,29,11)
(0,15,29;8,23,12)
(0,13,27;2,14,4)

$$(0,9,20;30,37,7)$$
$$(0,5,11;3,35,2)$$
$$(0,5,\infty;6,2,1).$$

Lemma 5.56 There exists a (40,6,3) S_6-design.

Proof A design on Z_{40} is generated by the seven blocks

$$(0,13,30;10,33,20) \quad \text{(half orbit)}$$
$$(0,12,28;7,17,26)$$
$$(0,11,30;8,26,15)$$
$$(0,13,30;6,15,25)$$
$$(0,12,25;33,3,11)$$
$$(0,6,15;8,3,2)$$
$$(0,5,11;9,6,1).$$

Lemma 5.57 There exists a decomposition of $3K_{3,3,3}$ into S_6-blocks.

Proof A decomposition on $Z_3 \times \{0,1,2\}$ is generated by the three blocks

$$(0_0,0_1,1_0;2_2,1_1,0_2)$$
$$(1_0,2_1,2_0;1_2,0_1,2_2)$$
$$(1_0,1_1,2_0;2_2,2_1,0_2).$$

Theorem 5.58 A $(v,6,3)$ S_6-design exists for all values of v satisfying the necessary conditions $v \equiv 0,1 \pmod 3$ and $v \geq 6$.

Proof Take the proof of Theorem 4.27. Replace "S_4" by "S_6" and Lemmas 4.8 to 4.23 by Lemmas 5.43 to 5.57 respectively.

§5.6 S_6-designs with $\lambda=9$

For $\lambda=9$ we need to find the same small designs directly as we did for the previous two multigraphs S_4 and S_5, that is for v = 8, 14, 20, 26, 32, 35, 38 and 44.

Lemma 5.59 There exists an (8,6,9) S_6-design.

Proof Here b = 28. A design on $\{Z_7 \cup \infty\}$ is generated by the blocks

$(0,1,\infty;2,5,6)$ twice

$(0,1,6;4,3,\infty)$

$(0,2,5;1,4,6)$.

Lemma 5.60 There exists a (14,6,9) S_6-design.

Proof A design on $\{Z_{13} \cup \infty\}$ is generated by the blocks

$(0,4,9;2,5,3)$ 4 times

$(0,4,\infty;7,11,2)$

$(0,6,\infty;8,4,1)$

$(0,2,\infty;11,12,1)$.

Lemma 5.61 There exists a (20,6,9) S_6-design.

Proof Here b = 190. A design on $\{Z_{19} \cup \infty\}$ is generated by the blocks

$(0,6,15;4,14,2)$ 4 times

$(0,6,9;10,7,2)$ 3 times

$(0,5,\infty;3,9,2)$

$(0,7,\infty;13,5,1)$

$(0,3,\infty;5,4,1)$.

Lemma 5.62 There exists a (26,6,9) S_6-design.

Proof Here $b = 325$. A design on $\{Z_{25} \cup \infty\}$ is generated by the blocks

(0,9,19;6,11,4)	4 times
(0,6,17;9,19,22)	4 times
(0,9,8;1,12,2)	
(0,8,12;10,7,1)	
(∞,4,0;1,3,8)	
(∞,4,0;2,3,5)	
(∞,4,0;1,3,5).	

Lemma 5.63 There exists a (32,6,9) S_6-design.

Proof Here $b = 49$. A design on $\{Z_{31} \cup \infty\}$ is generated by the blocks

(0,13,21;5,24,6)	4 times
(0,11,23;6,26,5)	4 times
(0,10,14;5,3,2)	4 times
(0,15,4;7,13,1)	
(0,10,∞;4,8,1)	
(0,4,∞;5,2,1)	
(0,4,∞;9,5,1).	

Lemma 5.64 There exists a (35,6,9) S_6-design.

Proof Here $b = 595$. A design on Z_{35} is generated by the blocks

(0,12,25;7,26,20)	4 times
(0,13,28;7,16,23)	4 times
(0,6,16;5,8,1)	4 times
(0,4,9;7,3,34)	twice
(0,17,2;4,14,8)	
(0,16,3;6,11,4)	
(0,3,4;2,33,34).	

Lemma 5.65 There exists a (38,6,9) S_6-design.

Proof A design on $\{Z_{37} \cup \infty\}$ is generated by the blocks

(0,14,30;11,20,26)	4 times
(0,13,30;3,8,6)	4 times
(0,16,28;13,26,30)	4 times
(0,9,12;11,3,5)	4 times
(0,18,∞;2,17,8)	
(0,16,∞;26,12,5)	
(0,6,∞;16,3,2).	

Lemma 5.66 There exists a (44,6,9) S_6-design.

Proof Here b = 940. A design on $\{Z_{43} \cup \infty\}$ is generated by the blocks

(0,17,31;10,19,32)	4 times
(0,16,26;3,9,33)	4 times
(0,16,31;6,37,30)	4 times
(0,12,1;9,4,5)	4 times
(0,16,37;40,17,31)	
(0,19,37;39,7,9)	
(0,4,17;14,6,1)	
(0,4,∞;5,3,2)	
(0,4,∞;5,3,1)	
(0,4,∞;6,2,1).	

Lemma 5.67 There exists a $(v,6,9)$ S_6-design whenever $v \equiv 0,1 \bmod 3$, v is prime, or $v \le 49$.

Proof This result follows from Lemma 1.8 and Theorem 5.57 ($v \equiv 0,1 \bmod 3$), Lemma 4.31 with S_4 replaced by S_6 (v prime) and Lemmas 5.58 to 5.65 (all remaining $v \le 49$).

Theorem 5.68 There exists a $(v,6,9)$ S_6-design for all v satisfying the necessary condition $v \geq 6$.

Proof Take Theorem 4.48, replace "S_4" by "S_6", and Lemmas 4.28, 4.31 and 4.39 by Lemma 5.67.

Theorem 5.69 There exist $(v,6,\lambda)$ S_6-designs for all v and λ satisfying the necessary conditions of Theorem 1.6.

Proof This result follows from Lemma 1.8, and Theorems 5.42, 5.58, and 5.68.

CHAPTER 6

DESIGNS ON CUBIC MULTIGRAPHS ON 6 VERTICES:

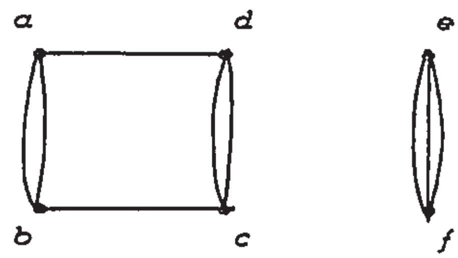
Bipartite Disconnected Multigraphs

§6.1 S_7-designs

Now we look at the final category of cubic multigraphs on six vertices, disconnected ones. The first multigraph in this category, S_7 in Fig. 3.1, consists of three triple edges. This case has been solved in effect by Bialostocki and Roditty [8], who have shown that a $(v,6,\lambda)$ S_7-design exists whenever the necessary conditions of Theorem 1.6 are satisfied.

§6.2 S_8-designs with $\lambda=3$

The next multigraph, S_8, consists of the cylinder *Cy* from Chapter 2 along with a triple edge. This multigraph is bipartite, and we shall use the same methods that were used for the other bipartite multigraphs S_2 and S_3. In this case, however, we have $m = 3$, so that the smallest λ we need consider is $\lambda=3$, and we must therefore look separately at designs with $\lambda = 4$ and $\lambda = 5$ since we cannot obtain these from designs with smaller λ as we could when we were looking at S_2 and S_3. The block labelling system we shall use for this multigraph is shown in Fig. 6.1.



(a,b;c,d)(e,f)

Fig. 6.1

For $\lambda = 3$ the necessary conditions of Theorem 3.1 are $v \geq 6$ and $v \equiv 0, 1 \pmod 3$. The smallest v satisfying these conditions is $v = 6$. We were unable to find a (6,6,3) S_8-design and in fact such a design cannot exist by the following lemma.

Lemma 6.1 There is no (6,6,3) S_8-design.

Proof Here the number of blocks $b = 5$. Therefore in order to find the required design we must put five copies of S_8 on $3K_6$ in such a way that each edge is covered exactly once. We shall try to do this one copy at a time and show that it is impossible.

For our first copy, we can obviously choose any position we want. Therefore W.L.O.G. choose the first block to be

(0,1;2,5)(3,4).

This leaves ten triple edges, eight of which are adjacent to the first copy of $3K_2$ - that is, contain vertex 3 or vertex 4. Since we need four more copies of $3K_2$, we must use at at least two of these as $3K_2$'s. Therefore we can choose the triple edge in the next S_8-block to be (2,3) W.L.O.G. Then there are two ways to choose the remaining component (up to symmetry):

(i) (0,5;4,1), which has two edges in common with the first block

or (ii) (0,5;1,4), which has one edge in common with the first block.

(Note that we cannot choose the remaining component to be (0,4;1,5), since a triple edge in $3K_6$ obviously cannot be contained in three different S_8-blocks.)

Fig. 6.2 shows the edges remaining in each case.

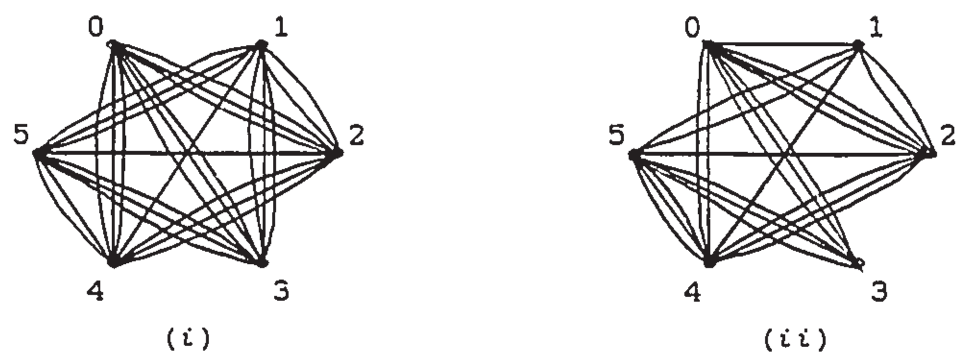


Fig. 6.2

Now we shall try to fit in three more copies of S_8, first for (i) and then for (ii). For (i) we look at all the different ways in which we can use the double edge (4,5). There are only three S_8-blocks which fit on (i) and contain this edge, namely

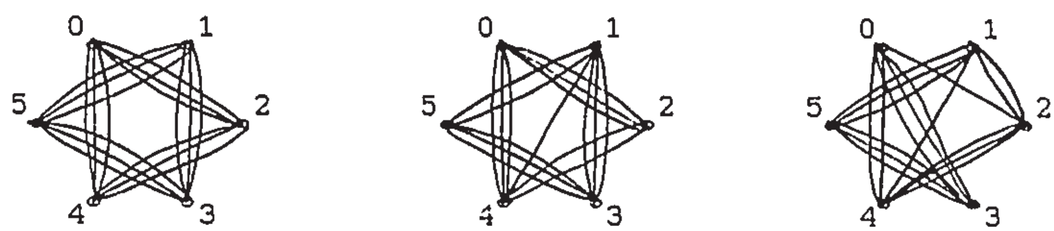
(a) (1,2;5,4)(0,3),

(b) (1,2;4,5)(0,3),

(c) (0,2;5,4)(1,3).

(The S_8-block (1,3;5,4)(0,2) also fits but gives the same thing as (c) up to automorphism.)

We now look at each of these in turn and show that in no case can we find two more S_8-blocks. The edges remaining are shown in Fig. 6.3:



(a) (b) (c)

Fig. 6.3

The configuration in (a) is $\{3K_3 \cup 3K_3\}$ on which there obviously cannot exist an S_8-design. In (b) the only way to use the single and double edges would be to make the first component of the design (1,5;2,4), but then we need the triple edge (0,3) which has already been used. For (c) we must use the S_8-block (0,4;1,2)(3,5) since this is the only way we can cover the single and double edges. But this leaves three triple edges which clearly cannot be covered by an S_8-block.

Having shown that we cannot find three more S_8-blocks in (i), we now examine (ii). Here we look at the different ways we can cover the (single) edge (0,1). There are four ways we can choose the first component so that it contains this edge, three of which also contain the double edge (1,5):

(a) (0,2;5,1),

(b) (0,4;5,1),

(c) (0,3;5,1),

(d) (0,4;2,1).

We cannot have (a) or (b) because the triple edges (3,4) and (2,3) have already been used. If we choose (c) then one of the last two S_8-blocks must have (1,2;5,4) as first component in order to use the double edge (1,2), but then we need the triple edge (0,3) which is not available. With (d) we would be forced to have (1,5;2,4)(0,3) as our fourth block which leaves three triple edges. This exhausts all the possibilities, so we conclude that a (6,6,3) S_8-design does not exist.

The next value for v satisfying the conditions necessary for an S_8-design to exist is 7. We shall give direct constructions for v = 7, 9, 10, 12, 13, 15, 16, 18, and 19.

Lemma 6.2 There exists a (7,6,3) S_8-design.

Proof Such a design is given by the blocks

(0,1;6,5)(2,3)

(0,6;5,1)(2,4)

(0,2;6,4)(1,3)

(0,4;6,2)(3,5)

(0,5;1,6)(3,4)

(1,2;5,4)(0,3)

(1,4;5,2)(3,6).

Lemma 6.3 There exists a (9,6,3) S_8-design.

Proof A design on $\{Z_8 \cup \infty\}$ is generated by the blocks

(0,6;4,2)(1,5) (half cycle)

$(0,1;4,5)(2,\infty)$ (half cycle)

$(0,5;4,1)(6,\infty)$ (half cycle).

Lemma 6.4 A (10,6,3) S_8-design exists.

Proof Here $b = 15$. A design on $Z_5 \times \{0,1\}$ is generated by the blocks

$(1_0,2_1;4_0,3_1)(0_0,0_1)$

$(0_0,3_0;2_1,1_1)(0_1,3_1)$

$(0_0,2_1;1_1,3_0)(1_0,2_0)$.

Lemma 6.5 There exists a (12,6,3) S_8-design.

Proof Here $b = 22$. A design on $\{Z_{11} \cup \infty\}$ is generated by the blocks

$(0,7;10,9)(1,\infty)$

(0,3;2,4)(1,6).

Lemma 6.6 There exists a (13,6,3) S_8-design.

Proof Here $b = 26$. A cyclic design on Z_{13} is generated by the blocks

$$(0,3;12,1)(4,9)$$

$$(0,1;12,3)(4,10).$$

Lemma 6.7 There exists a (15,6,3) S_8-design.

Proof Here $b = 35$. A design on $(Z_7 \times \{0,1\}) \cup \{\infty\}$ is generated by the blocks

$$(1_0,2_0;3_1,0_1)(4_0,6_0)$$

$$(1_0,3_1;0_1,2_0)(4_1,5_1)$$

$$(1_0,5_1;0_1,4_0)(6_0,6_1)$$

$$(1_0,4_0;0_1,5_1)(5_0,\infty)$$

$$(1_0,0_1;2_0,3_1)(4_1,\infty).$$

Lemma 6.8 There exists a (16,6,3) S_8-design.

Proof Here $b = 40$. A design on Z_{16} is generated by the blocks

$$(2,6;10,14)(4,12) \qquad \text{(half orbit)}$$

$$(1,14;11,6)(3,4)$$

$$(0,2;8,14)(4,11).$$

Lemma 6.9 There exists an (18,6,3) S_8-design.

Proof Here $b = 51$. A design on $\{Z_{17} \cup \infty\}$ is generated by the blocks

$$(0,1;16,3)(4,\infty)$$

$$(2,16;4,15)(6,14)$$

$$(0,2;13,1)(4,11).$$

Lemma 6.10 There exists a (19,6 3) S_8-design.

Proof Here b = 57. A design on Z_{19} is generated by the blocks

(0,1;18,3)(7,14)

(0,3;18,5)(8,16)

(0,5;18,1)(7,16).

We now need some decompositions of complete bipartite multigraphs in order to settle the question of the existence of S_8-designs when λ = 3, just as we did for the other bipartite multigraphs S_2 and S_3.

Lemma 6.11 There exists a decomposition of $3K_{3,3}$ into S_8-blocks.

Proof Such a decomposition (on Z_3x{0,1}) is given by the three blocks

$(0_0,0_1;1_0,1_1)(2_0,2_1)$

$(0_0,1_1;2_0,0_1)(1_0,2_1)$

$(1_0,0_1;2_0,1_1)(0_0,2_1)$.

Lemma 6.12 There exists a decomposition of $3K_{4,3}$ into S_8-blocks.

Proof Such a decomposition is shown in Fig. 6.4.

(*i*) (*ii*)

(*iii*) (*iv*)

Fig. 6.4

Lemma 6.13 There exist decompositions of $3K_{12,12}$, $3K_{15,12}$, $3K_{10,12}$, and $3K_{16,12}$ into S_8-blocks.

Proof Lemmas 6.11, 6.12 and 1.9.

Since we could not find an S_8-design on $3K_6$ we cannot use exactly the same constructions as we did for the multigraphs S_2 and S_3, but must use larger components, or building blocks. The general idea, however, is the same.

Theorem 6.14 There exist $(v,6,3)$ S_8-designs for all v satisfying the necesary conditions $v \geq 6$, $v \equiv 0,1 \pmod 3$, *with the single exception of the case* $v = 6$.

Proof A $(6,6,3)$ S_8-design does not exist from Lemma 6.1. For $v > 6$ we divide the v satifying $v \equiv 0,1 \pmod 3$ into four cases, and give a construction for each.

Case I: $v \equiv 0,1 \pmod{12}$. Then $v = 12m$, or $12m+1$ for some $m \geq 1$. Split the v points into m rows of 12, with a point at ∞ if $v \equiv 1 \pmod{12}$. Cover each row of 12 with the $(12,6,3)$ S_8-design from Lemma 6.5 if $v \equiv 0 \pmod{12}$, or each row plus infinity point with the $(13,6,3)$ S_8-design from Lemma 6.6 if

$v \equiv 1 \pmod{12}$. Then cover edges between each pair of rows with the decomposition of $3K_{12,12}$ into S_8-blocks which we have from Lemma 6.13.

Case II: $v \equiv 6, 7 \pmod{12}$. A (7,6,3) S_8-design exists from Lemma 6.2, so we can assume $v \geq 18$. Then $v = 12m+6$, or $12m+7$ for some $m \geq 1$. Split the v points into one row of 18 and $m-1$ rows of 12, with a point at infinity if $v \equiv 7 \pmod{12}$. Cover rows with the (18,6,3) S_8-design from Lemma 6.9 and the (12,6,3) S_8-design from Lemma 6.5 if $v \equiv 6 \pmod{12}$, or rows plus ∞ point with the (19,6,3) and (13,6,3) S_8-designs from Lemmas 6.10 and 6.6 respectively if $v \equiv 7 \pmod{12}$. Cover edges between each pair of rows with the decompositions of $3K_{18,12}$ and $3K_{12,12}$ into S_8-blocks which we have from Lemma 6.13.

Case III: $v \equiv 3,4 \pmod{12}$. Here $v = 12m+3$, or $12m+4$ with $m \geq 1$. Split the v points into one row of 15 and $m-1$ rows of 12, with an ∞ point if $m \equiv 4 \pmod{12}$. Cover rows with (15,6,3) or (12,6,3) S_8-designs from Lemmas 6.7 and 6.5, or rows plus ∞ with (16,6,3) or (13,6,3) S_8-designs from Lemmas 6.8 and 6.6 respectively. Then edges between rows can be covered with the decompositions of $3K_{15,12}$ and $3K_{12,12}$ into S_8-blocks from Lemma 6.13.

Case IV: $v \equiv 9,10 \pmod{12}$. Here $v = 12m+9$, or $12m+10$ for $m \geq 0$. Split into one row of 9 and m rows of 12, with an ∞ point if $v \equiv 10 \pmod{12}$. Cover row of 9 with the S_8-design from Lemma 6.3, or row of 9 plus ∞ with the (10,6,3) S_8-design from Lemma 6.4. Cover remaining edges exactly as in the previous case, except that here we need a decomposition of $3K_{9,12}$ into S_8-blocks, which also comes from Lemma 6.13.

§6.3 S_8-designs with $\lambda=4$ and $\lambda=5$

Next we must look at S_8-designs with $\lambda=4$ and $\lambda=5$. The necessary conditions here are $v\equiv1 \pmod 9$, $v\geq6$. For both cases we find the smallest design, with $v=10$, and a decomposition of $4K_{9,9}$ directly, and then all the others can be found using recursion.

Lemma 6.15 There exists a (10,6,4) S_8-design.

Proof Here $b = 20$. A design on Z_{10} is generated by the two blocks

(0,3;9,2)(4,8),
(3,4;9,8)(5,7).

Lemma 6.16 There exists a decomposition of $4K_{9,9}$ into S_8-blocks.

Proof A decomposition on $Z_9 \times \{0,1\}$ is generated by the blocks

$(0_0,2_1;3_0,0_1)(4_0,3_1)$,
$(0_0,3_1;5_0,0_1)(7_0,5_1)$,
$(0_0,2_1;6_0,0_1)(8_0,4_1)$,
$(0_0,4_1;3_0,0_1)(4_0,5_1)$.

Theorem 6.17 There exists a $(v,6,4)$ S_8-design for all v satisfying the necessary conditions $v \equiv 1 \pmod 9$ and $v \geq 6$.

Proof If $v \equiv 1 \pmod 9$ and $v \geq 6$ then $v = 9m+1$ for some $m \geq 1$. Split the v points into m rows of 9 and a point at ∞. Cover each row+∞ with the (10,6,4) design from Lemma 6.15, and all edges between rows with the decomposition of $4K_{9,9}$ into S_8-blocks from Lemma 6.16.

Lemma 6.18 There exists a (10,6,5) S_8-design.

Proof Here $b = 25$. A design on $Z_5 \times \{0,1\}$ is generated by the five blocks

$$(1_0,2_0;3_1,0_1)(0_0,2_1)$$
$$(1_0,2_0;3_1,0_1)(4_0,4_1)$$
$$(1_0,0_1;3_1,2_0)(1_1,2_1)$$
$$(0_0,3_0;2_1,1_1)(1_0,4_1)$$
$$(1_0,3_1;3_0,1_1)(2_0,4_0).$$

Lemma 6.19 There exists a decomposition of $5K_{9,9}$ into S_8-blocks.

Proof Here $b = 45$. A design on $Z_9 \times \{0,1\}$ is generated by the five blocks

$$(0_0,1_1;3_0,2_1)(2_0,6_1)$$
$$(0_0,4_1;7_0,3_1)(8_0,8_1)$$
$$(0_0,2_1;2_0,0_1)(7_0,3_1)$$
$$(0_0,3_1;5_0,2_1)(6_0,7_1)$$
$$(0_0,3_1;5_0,2_1)(7_0,6_1).$$

Theorem 6.20 There exists a $(v,6,5)$ S_8-design for all v satisfying the necessary conditions $v \equiv 1 \pmod 9$ and $v \geq 6$.

Proof Let $v=9m+1$. Split the points into m rows of 9 and a point at ∞. Cover each row+∞ with the $(10,6,5)$ S_8-design from Lemma 6.18, and edges between rows with the decomposition of $5K_{9,9}$ from Lemma 6.19.

§6.4 S_8-designs with $\lambda=6$ and $\lambda=9$

We can derive all but one S_8-design with $\lambda=6$ from the corresponding designs with $\lambda=3$, but since a $(6,6,3)$ S_8-design does not exist, we must find a $(6,6,6)$ S_8-design by direct construction.

Lemma 6.21 There exists a $(6,6,6)$ S_8-design.

Proof Here $b = 10$. A design on $\{Z_5 \cup \infty\}$ is generated by the blocks

$$(0,1;2,4)(3,\infty) \quad \text{and} \quad (0,2;1,4)(3,\infty).$$

Finally we look at S_8-designs with $\lambda=9$, where the only necessary condition is $v \geq 6$. If $v \equiv 0, 1 \pmod 3$ then a $(v,6,9)$ S_8-design can be derived from the $(v,6,3)$ S_8-design of Theorem 6.14 by the Corollary to Lemma 1.8, except for when $v=6$, as there is no $(6,6,3)$ S_8-design. We give direct constructions for v = 6, 8 and 11, then a recursive method for finding the larger designs.

Lemma 6.22 There exists a (6,6,9) S_8-design.

Proof Here $b = 15$. A design on $(Z_5 \cup \infty)$ is generated by the three blocks

$$(0,4;3,1)(2,\infty)$$
$$(0,1;4,3)(2,\infty)$$
$$(0,3;1,4)(2,\infty).$$

Lemma 6.23 There exists an (8,6,9) S_8-design.

Proof Here $b = 28$. A design on $(Z_7 \cup \infty)$ is generated by the four blocks

$$(0,6;3,1)(2,\infty)$$
$$(0,6;4,1)(2,\infty)$$
$$(0,5;2,6)(3,\infty)$$
$$(0,6;4,2)(1,5).$$

Lemma 6.24 There exists an (11,6,9) S_8-design.

Proof Here $b = 55$. A cyclic design on Z_{11} is generated by the five blocks

$$(0,1;10,3)(4,8)$$
$$(0,2;10,4)(5,8)$$
$$(0,3;10,5)(7,8)$$
$$(0,4;10,2)(7,8)$$
$$(0,5;10,1)(7,9)$$

Lemma 6.25 There exist decompositions of $9K_{6,6}$, $9K_{8,6}$, and $9K_{11,6}$ into S_8-blocks.

Proof This result follows from Lemmas 6.11, 6.12, and 1.9.

Theorem 6.26 There exists a $(v,6,9)$ S_8-design for all v satisfying the necessary condition $v \geq 6$.

Proof For $v = 6$ we have a design from Lemma 6.21. All other designs with $v \equiv 0,1 \mod(3)$ can be obtained from Theorem 6.14 and Lemma 1.8. We have direct constructions for $v=8$ and $v=11$ from Lemmas 6.23 and 6.24 respectively. For the remaining designs with $v \equiv 2 \pmod 3$ we proceed as follows:

Case I: $v \equiv 2 \pmod 6$. Then $v = 6m + 2$ for some $m \geq 1$. Split the v points into $m-1$ rows of six and one row of eight. Cover row of 8 with the (8,6,9) S_8-design from Lemma 6.23, each row of six with the (6,6,9) S_8-design from Lemma 6.22, and edges between rows with the decompositions of $9K_{6,6}$ and $9K_{8,6}$ into S_8-blocks from Lemma 6.25.

Case II: $v \equiv 5 \pmod 6$. Then $v = 6m + 5$ for some $m \geq 1$. Split the v points into $m-1$ rows of 6 and one row of eleven. Cover row of eleven with the (11,6,9) S_8-design from Lemma 6.24, each row of six with the (6,6,9) S_8-design from Lemma 6.22, and edges between rows with the decompositions of $9K_{6,6}$ and $9K_{11,6}$ into S_8-blocks from Lemma 6.25.

Theorem 6.27 There exists a $(v,6,\lambda)$ S_8-design for all v and λ satisfying the necessary conditions of Theorem 1.6 *except for* $v=6$, $\lambda=3$.

Proof This result follows from Theorems 6.14, 6.17, 6.20, 6.26 and Lemma 1.8.

CHAPTER 7

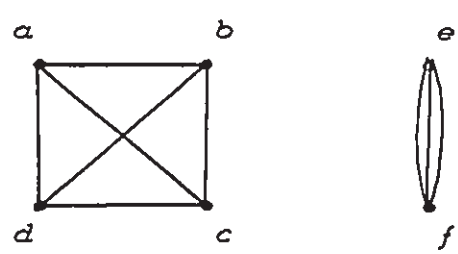
DESIGNS ON CUBIC MULTIGRAPHS ON 6 VERTICES:

Non-Bipartite Disconnected Multigraphs

§7.1 Introduction

There remains one cubic multigraph on six vertices – S_9 in Fig. 3.1. This multigraph is $K_4 \cup 3K_2$, and is the most difficult of these multigraphs to find graph designs on, especially in the general case, since it is not bipartite nor even tripartite. In fact we have not been able to solve the general question of the existence of G-designs for this multigraph, so we present here the partial results we have been able to establish and indicate which designs are still needed to settle this case completely.

For S_9 we shall use the block labelling system shown in Fig. 7.1.



$(a,b,c,d)(e,f)$

Fig. 7.1

For this multigraph some of the designs we shall find will have the first component (K_4) repeated with different copies of the second component ($3K_2$). (Since the first component has no multiple edges and the second is a triple edge, we can take each first component three times, with three different labellings for the second component.) For the sake of brevity, when the same first component occurs with different second components we shall list the first component only once, and list the accompanying second components in the same row. The first example of this occurs in Lemma 7.4.

§7.2 S_9-designs with $\lambda=3$

As with the previous multigraph S_8 we must start with $\lambda=3$, and the necessary conditions for the existence of a $(v,6,3)$ S_9-design are $v \equiv 0,\ 1 \pmod 3$ and $v \geq 6$.

Lemma 7.1 There exists a $(6,6,3)$ S_9-design.

Proof Here $b = 5$. A design on $\{Z_5 \cup \infty\}$ is generated by the block

$$(0,1,2,3)(4,\infty).$$

The next value of v satisfying the necessary conditions for the existence of a $(v,6,3)$ S_9-design is $v = 7$. A $(7,6,3)$ S_9-design does not exist, as we prove in the following lemma.

Lemma 7.2 There does not exist a (7,6,3) S_p-design.

Proof If such a design existed it would have 7 blocks. Try to find such a design on Z_7. Arrange the seven points as a column of four points $(0_0,1_0,2_0,3_0)$ on the left and a column of three points $(0_1,1_1,2_1)$ on the right. Assuming that there is a (7,6,3) S_p-design, let one of the S_p-blocks be $(0_0,1_0,2_0,3_0)(0_1,1_1)$. Consider the K_4 components of the other S_p-blocks. Each of these must contain at least two points on the left since one of the triple edges on the right is in the given S_p-block, so we cannot have all 3 points on the right in one of the remaining K_4 components. Let a, b, c, be the number of K_4's that contain precisely 4, 3, 2 respectively, of the vertices on the left. Then since there are six remaining blocks, and twelve remaining edges on the left which must all be contained in a K_4 component, we have

$$a + b + c = 6$$

$$6a + 3b + c = 12$$

These equations have only one solution in nonnegative integers, namely, $a = 0$, $b = 3$, and $c = 3$. Therefore, 3 of the 6 remaining K_4 components must contain 3 points on the left, and the other 3 contain 2 points each. We can assume WLOG that the first 3 contain points $(0_0,2_0,3_0)$, $(0_0,1_0,3_0)$, and $(1_0,2_0,3_0)$ respectively. The 3 containing 2 points must all contain the same two points on the right in order for the joining edge to be used exactly 3 times. Therefore we can assume WLOG that these are

$$(0_0, 1_0, 0_1, 1_1)$$
$$(0_0, 2_0, 0_1, 1_1)$$
$$(1_0, 2_0, 0_1, 1_1)$$

Now we try to complete the K_4 components containing 3 points on the left by adding a point on the right to each. They must contain distinct points on the right since otherwise one of the mixed edges would be used 4 times, or if all three contained the point 2_1 there would not be enough triple edges left over for the remaining $3K_2$ components. But there is no way to do this and be left with only triple edges as required. Therefore it is impossible to find a (7,6,3) $S_\wp$-design.

We have been unable to find, or to prove the nonexistence of, a $(v,6,3)$ $S_\wp$-design with the next admissible value of v, $v=9$. The difficulties we encountered when trying to find such a design do however lead us to make the following conjecture.

Conjecture 7.3 There does not exist a (9,6,3) $S_\wp$-design.

We were able to find $(v,6,3)$ $S_\wp$-designs for all other $v \leq 19$ satisfying the necessary conditions, and these are given in the following seven lemmas.

Lemma 7.4 There exists a (10,6,3) S_p-design.

Proof Here $b = 15$. A design on $Z_5 \times \{0,1\}$ is generated by the blocks

$(0_0, 1_0, 0_1, 2_1)$ taken 3 times,

with $(3_0, 1_1)$, $(2_0, 4_0)$, $(3_1, 4_1)$.

Lemma 7.5 There exists a (12,6,3) S_p-design.

Proof Here $b = 22$. The obvious thing here would be to look for a construction on $\{Z_{11} \cup \infty\}$, but we can show that one does not exist. Instead we found a design on $Z_4 \times \{0,1,2\}$ which is generated by the six blocks

$(0_0, 2_0, 0_1, 2_1)(0_2, 2_2)$ (half orbit)

$(0_0, 1_0, 2_0, 2_1)(3_0, 3_2)$

$(0_0, 1_0, 1_1, 2_1)(3_0, 2_2)$

$(0_1, 2_1, 0_2, 1_2)(1_0, 3_2)$

$(0_1, 1_1, 0_2, 1_2)(2_0, 3_2)$

$(0_1, 1_1, 2_2, 3_2)(3_0, 2_1)$.

Lemma 7.6 There exists a (13,6,3) S_p-design.

Proof Here $b = 26$. A cyclic design on Z_{13} is generated by the two blocks

(0,2,4,11)(5,10),

(0,3,6,10)(7,8).

Lemma 7.7 There exists a (15,6,3) S_p-design.

Proof Here $b = 35$. A design on $Z_7 \times \{0,1\} \cup \infty$ is generated by the five blocks

$$(0_0,3_0,1_1,2_1)(4_1,6_1)$$
$$(0_0,3_0,1_1,2_1)(3_1,6_1)$$
$$(0_0,2_0,4_0,2_1)(6_0,3_1)$$
$$(0_0,1_0,0_1,1_1)(3_1,\infty)$$
$$(0_0,1_0,2_0,\infty)(3_0,6_1)$$

Lemma 7.8 There exists a (16,6,3) S_9-design.

Proof Here $b = 40$. A design on $Z_8 \times \{0,1\}$ is generated by the blocks

$$(2_0,4_0,0_1,1_1)(1_0,5_0) \qquad \text{(half orbit)}$$
$$(0_0,6_0,4_1,5_1)(2_1,6_1) \qquad \text{(half orbit)}$$
$$(0_0,1_0,2_0,7_1)(3_1,6_1)$$
$$(0_0,2_0,3_0,4_1)(0_1,2_1)$$
$$(0_0,3_0,1_1,2_1)(4_0,7_1)$$
$$(0_0,3_0,4_1,5_1)(6_0,6_1).$$

Lemma 7.9 There exists an (18,6,3) S_9-design.

Proof Here $b = 51$. A design on $\{Z_{17} \cup \infty\}$ is generated by the three blocks

$$(0,3,7,15)(5,6)$$
$$(0,3,7,15)(8,14)$$
$$(0,3,7,15)(4,\infty).$$

Lemma 7.10 There exists a (19,6,3) S_9-design.

Proof Here $b = 57$. A cyclic design on Z_{19} is generated by the blocks

$$(0,3,7,17)(4,5)$$
$$(0,3,7,17)(8,16)$$
$$(0,3,7,17)(9,15).$$

Lemma 7.11 There exists a decomposition of $3K_{6,6,6,6}$ into S_9-blocks.

Proof The number of blocks here is 72. A decomposition on Z_6x{0,1,2,3} is generated by the 12 blocks

$(0_0,2_1,5_2,2_3)$ 3 times, with $(5_0,5_1)$, $(0_1,0_2)$, $(2_2,0_3)$
$(0_0,3_1,1_2,0_3)$ 3 times, with $(4_0,4_2)$, $(5_0,4_3)$, $(3_2,5_3)$
$(0_0,1_1,2_2,3_3)$ 3 times, with $(2_0,5_2)$, $(4_0,5_3)$, $(3_1,4_3)$
$(5_0,4_1,3_2,3_3)$ 3 times, with $(3_0,1_1)$, $(0_1,2_2)$, $(2_1,0_3)$.

Lemma 7.12 There exists a decomposition of $3K_{9,9,9,9}$ into S_9-blocks.

Proof A decomposition on Z_9x{0,1,2,3} is generated by the eighteen blocks

$(0_0,0_1,0_2,0_3)$	3 times	with	$(5_0,1_1)$,	$(8_0,5_1)$,	$(5_2,1_3)$
$(0_0,1_1,2_2,3_3)$	"	"	"	$(7_0,3_2)$,	$(8_1,4_2)$,	$(8_2,7_3)$
$(0_0,2_1,4_2,6_3)$	"	"	"	$(3_0,6_2)$,	$(6_0,8_3)$,	$(8_0,7_3)$
$(0_0,3_1,7_2,1_3)$	"	"	"	$(6_0,5_2)$,	$(8_0,7_1)$,	$(8_2,5_3)$
$(0_0,4_1,1_2,5_3)$	"	"	"	$(5_1,8_3)$,	$(6_1,2_3)$,	$(8_1,6_2)$
$(0_0,7_1,6_2,4_3)$	"	"	"	$(2_0,0_3)$,	$(0_1,3_2)$,	$(8_1,7_3)$.

We do not have enough to solve this case completely, but using the above decompositions we can find some infinite classes of S_9-designs with $\lambda=3$.

Lemma 7.13 There exist $(\upsilon,6,3)$ S_9-designs for all $\upsilon \equiv 6,\ 24 \pmod{72}$.

Proof Split the v points into g rows of six points, then we have $g \equiv 1,\ 4 \pmod{12}$. From Hanani [17] there exists a Steiner system $S(2,4,g)$, that is, a $(g,4,1)$ K_4-design. Take each block from this, which contains four rows of six points, and cover all edges between rows using the decomposition of $3K_{6,6,6,6}$ from Lemma 7.11. Then each row can be covered using the $(6,6,3)$ S_9-design from Lemma 7.1.

Note that the previous lemma could have been used to give S_9-designs when $v \equiv 7,\ 25 \pmod{72}$, by adding a point at infinity, but for the fact that we do not have a $(7,6,3)$ S_9-design, as was proved in Lemma 7.2.

Lemma 7.14 There exists a $(v,6,3)$ S_9-design for all $v \equiv 12,\ 13,\ 48,\ 49 \pmod{72}$.

Proof Split the v points into g rows of six, with a point at infinity if $v \equiv 13$ or $49 \pmod{72}$. Then from Brouwer [11] there exists a dense packing of K_4's into K_g which covers all edges in K_g (that is, pairs of rows of six) except for $\frac{1}{2}g$ disjoint pairs. Therefore we can take each block from this packing, which is a set of four rows of six, and cover all edges between rows with the decomposition of $3K_{6,6,6,6}$. We are left with the edges within rows, and the edges between each of the disjoint pairs not covered by the packing. These pairs must include each of the g rows exactly once, so we can cover all remaining edges by covering each of these pairs with the (12,6,3) S_9-design from Lemma 7.5, or if $v \equiv 13$ or 49 (mod 72) then cover each pair plus the infinity point with the (13,6,3) S_9-design from Lemma 7.6.

Lemma 7.15 There exists a $(v,6,3)$ S_9-design whenever $v \equiv 10$, 37 (mod 108).

Proof Split the v points into g rows of 9 plus an infinity point. Then $g \equiv 1$, 4 (mod 12) and there exists a decomposition of K_g into K_4's as in Lemma 7.14. Take such a decomposition, where each point is a row of 9 of the v points, and cover all edges between the 4 rows of 9 in the block with the decomposition of $3K_{9,9,9,9}$ into S_9-blocks from Lemma 7.12. Then cover each row plus infinity with the (10,6,3) S_9-design from Lemma 7.2.

Lemma 7.16 There exists a $(v,6,3)$ S_9-design whenever $v \equiv$ 18, 19, 72, 73 (mod 108).

Proof Arrange points into g rows of nine as above, then $g \equiv$ 2, 8 (mod 12) and there is a packing of K_4's into K_g with a 1-factor left over as in Lemma 7.14. Cover the K_4's, which are each four rows of nine, with the $3K_{9,9,9,9}$ decomposition from Lemma 7.12, and each pair of rows from the 1-factor with the (18,6,3) S_9-design from Lemma 7.9, or if $v \equiv 19$ or 73 (mod 108) with the (19,6,3) S_9-design from Lemma 7.10.

Combining the above results we get the following:

Lemma 7.17 There exists a $(v,6,3)$ S_ρ-design whenever $v \equiv 6$, 10, 12, 13, 18, 19, 24, 37, 48, 49, 72, 73, 78, 84, 85, 96, 118, 120, 121, 126, 127, 145, 150, 156, 157, 168, 180, 181, 192, 193 (mod 216).

Proof This result follows from Lemmas 7.13 to 7.16.

The above gives us designs for 30 of the 144 values of v satisfying the necessary conditions (mod 216). We have been unable to construct designs for the remaining ones. If we could find a decomposition of $3K_{3,3,3,3}$ into S_ρ-blocks we would be able to find contructions for larger infinite classes, but it does not look as if such a decomposition exists. Similarly we would have more if we could find a $(7,6,3)$ S_ρ-design or a $(9,6,3)$ S_ρ-design, but we have proved the nonexistence of the former and conjectured the nonexistence of the latter.

§7.3 S_ρ-designs with λ=4 and λ=5

Next we look at S_ρ-designs with $\lambda = 4$, where the necessary conditions are $v \equiv 1 \pmod 9$ and $v \geq 6$.

Lemma 7.18 There exists a $(10,6,4)$ S_ρ-design.

Proof Here $b = 20$. A design on Z_{10} is generated by the two blocks

$$(0,1,3,8)(4,5),$$
$$(0,2,4,7)(1,5).$$

Lemma 7.19 There exists a (19,6,4) S_9-design.

Proof *Here* $b = 76$. A cyclic design on Z_{19} is generated by the four blocks

$$(0,3,8,13)(1,2),$$
$$(0,1,10,13)(2,9),$$
$$(0,4,6,14)(1,3),$$
$$(0,3,6,11)(12,16).$$

In order to find an infinite class of S_9-designs with $\lambda=4$ we need a decomposition of a complete quadripartite multigraph – $4K_{9,9,9,9}$ would be best – but so far we have been unable to find any.

For $\lambda = 5$, where the necessary conditions are the same as for $\lambda = 4$, we have the following results.

Lemma 7.20 There exists a (10,6,5) S_9-design.

Proof A design on $Z_5 \times \{0,1\}$ is generated by the blocks

$$(0_0,1_0,0_1,3_1)(4_0,4_1)$$
$$(0_0,1_0,2_0,2_1)(3_0,4_1)$$
$$(1_0,3_0,0_1,2_1)(3_1,4_1)$$
$$(2_0,3_0,0_1,1_1)(2_1,4_1)$$
$$(3_0,4_0,1_1,2_1)(0_0,2_0).$$

Lemma 7.21 There exists a (19,6,5) S_9-design.

Proof Here $b = 95$. A cyclic design on Z_{19} is generated by the blocks

$$(0,2,5,10)(13,17)$$
$$(0,2,5,10)(11,17)$$
$$(0,1,5,7)(10,17)$$

$(0,3,6,7)(9,17)$
$(0,1,2,3)(8,17)$.

As with $\lambda=4$, we were unable to find any decompositions of complete quadripartite multigraphs with $\lambda=5$, and therefore do not have any infinite classes here but only the above small designs.

§7.4 S_9-designs with $\lambda=6$ and $\lambda=9$

For $\lambda=6$ we need find only designs that did not exist with $\lambda=3$ when the necessary conditions were satisfied, since the rest can be derived from the $\lambda=3$ designs by Lemma 1.8. The smallest values for which we could not find a design with $\lambda=3$ were $v = 7$ and $v = 9$. We were unable to find, or prove the nonexistence of a $(7,6,6)$ S_9-design, but are led to make the following conjecture:

Conjecture 7.22 There does not exist a $(7,6,6)$ S_9-design.

We were, however, able to find a $(v,6,6)$ S_9-design with $v=9$, as shown in the following lemma.

Lemma 7.23 There exists a $(9,6,6)$ S_9-design.

Proof Here $b = 24$. A design on $\{Z_8 \cup \infty\}$ is generated by the blocks

$(0,2,4,7)(1,\infty)$ twice,
$(0,2,4,7)(5,6)$.

Finally we look at designs on S_9 with $\lambda=9$, where the only necessary condition is $v\geq 6$. If we had completely solved the question of existence with $\lambda=3$ then we would need only look at values of v where $v\equiv 2 \pmod 3$, but since we have not we shall try to find designs for all values $v\geq 6$. Of course, we can still use the results that we do have with $\lambda=3$, and we can find designs with $v \equiv 2 \pmod 3$ exactly as we did for the other non bipartite cubic multigraphs on six vertices, such as S_4, the envelope.

Lemma 7.24 There exists a $(9,6,9)$ S_9-design.

Proof Here $b = 36$. A design on Z_9 is generated by the blocks

$$(0,1,7,8)(2,6)$$
$$(0,2,5,7)(3,4)$$
$$(0,1,4,5)(6,8)$$
$$(0,1,4,7)(3,6).$$

Lemma 7.25 There exist $(v,6,9)$ S_9-designs for all $v \equiv 9, 36 \pmod{108}$.

Proof Split the v points into s rows of nine. From Brouwer [11] there exists a packing of K_4's into K_s when $s \equiv 1, 4 \pmod{12}$. Take such a packing on the rows of nine. Cover each block with the decomposition of $3K_{9,9,9,9}$ from Lemma 7.12 and each row of nine with the above $(9,6,9)$ S_9-design.

Lemma 7.26 There exist $(v,6,9)$ S_9-designs for all $v \equiv 6$, 9, 10, 12, 13, 18, 19, 24, 36, 37, 48, 49, 72, 73, 78, 84, 85, 96, 117, 118, 120, 121, 126, 127, 144, 145, 150, 156, 157, 168, 180, 181, 192, 193 (mod 216).

Proof This result follows from Lemmas 7.26, 7.17 and Lemma 1.8.

Lemma 7.27 There exists a $(v,6,9)$ S_9-design whenever $v \geq 6$ and v is *prime*.

Proof Take the proof of Theorem 4.31 and replace "S_4" by "S_9".

Next we give direct constructions for some designs on S_9 with $\lambda = 9$ and $v \equiv 2$ (mod 3). These are the same designs as we found on S_4 with $\lambda = 9$, that is, designs with $v = 8$, 14, 20, 26, 32, 35, 38, 44.

Lemma 7.28 There exists an $(8,6,9)$ S_9-design.

Proof Here $b = 28$. A design on $\{Z_7 \cup \infty\}$ is generated by the four blocks

$$(0,1,3,6)(4,5)$$
$$(0,1,3,6)(2,\infty)$$
$$(0,1,3,\infty)(2,5)$$
$$(0,1,3,\infty)(4,6).$$

Lemma 7.29 There exists a $(14,6,9)$ S_9-design.

Proof Here $b = 91$. A design on $\{Z_{13} \cup \infty\}$ is generated by the blocks

$$(0,3,7,11)(8,10)$$
$$(0,3,7,11)(9,10)$$

(0,3,7,11)(2,∞)
(0,1,3,∞)(6,12)
(0,1,4,∞)(6,12)
(0,1,3,4)(6,11) twice.

Lemma 7.30 There exists a (20,6,9) S_9-design.

Proof Here b = 190. A design on $\{Z_{19} \cup \infty\}$ is generated by the blocks

(0,4,10,12)(11,16) 3 times
(0,4,10,12)(5,14)
(0,4,10,12)(6,14)
(0,4,10,12)(7,14)
(0,1,3,4)(9,15)
(0,1,3,4)(15,∞)
(0,1,4,∞)(12,15)
(0,1,3,∞)(6,7).

Lemma 7.31 There exists a (26,6,9) S_9-design.

Proof Here b = 325. A design on $\{Z_{25} \cup \infty\}$ is generated by the blocks

(0,6,13,17)(12,21) 3 times,
(0,6,13,17) 3 times, with (19,21) twice, and (20,21)
(0,5,10,13) 3 times, with (14,19), (14,20), and (14,∞)
(0,1,3,4)(10,20) twice
(0,1,4,∞)(6,13)
(0,1,3,∞)(10,21).

Lemma 7.32 There exists a (32,6,9) S_9-design.

Proof Here b = 496. A design on $\{Z_{31} \cup \infty\}$ is generated by the blocks

(0,8,15,21)(14,28) 3 times

(0,8,15,21)(16,28) twice

(0,8,15,21)(17,28) twice

(0,8,15,21)(19,28) twice

(0,1,3,12)(15,20) 3 times

(0,1,3,4)(5,9)

(0,1,3,4)(5,∞)

(0,1,3,∞)(5,7)

(0,1,4,∞)(5,9).

Lemma 7.33 There exists a (35,6,9) S_p-design.

Proof Here $b \equiv 595$. A design on Z_{35} is generated by the blocks

(0,2,6,15)(17,27) 3 times

(0,2,6,15)(19,27) 3 times

(0,2,6,15)(24,27) 3 times

(0,1,12,17) 3 times, with (10,27), (10,26), (10,22)

(0,1,12,17) 3 times, with (10,21), (10,15), (10,11)

(0,7,14,21) twice, with (10,24), (10,17).

Lemma 7.34 There exists a (38,6,9) S_p-design.

Proof Here $b = 721$. A design on $\{Z_{37} \cup \infty\}$ is generated by the blocks

(0,9,19,26)(6,22) 3 times

(0,9,19,26)(7,22) 3 times

(0,9,19,26)(8,22) 3 times

(0,2,3,8)(9,22) 3 times

(0,4,6,12) 3 times, with (10,22) twice, and (13,∞)

(0,1,3,4)(10,15) twice

(0,1,4,∞)(10,18)

(0,1,3,∞)(10,14).

Lemma 7.35 There exists a (44,6,9) S_p-design.

Proof Here $b = 946$. A design on $\{Z_{43} \cup \infty\}$ is generated by the blocks

(0,8,18,27)(10,31) 3 times
(0,8,18,27)(10,25) 3 times
(0,8,18,27)(10,24) 3 times
(0,5,11,31)(10,23) 3 times
(0,5,11,31)(10,17) 3 times
(0,5,11,31)(10,14) twice
(0,5,11,31)(10,∞)
(0,1,3,4)(6,8) twice
(0,1,4,∞)(7,10)
(0,1,3,∞)(4,5).

To show that $(v,6,9)$ S_p-designs exist whenever the necessary conditions are satisfied, we would need to find designs with v = 22, 25, 27, 28, 29, 30, 31, 33, 34 and 40. We could find these either by constructing a design with $\lambda=9$ directly, or by constructing one with $\lambda=3$ and deriving one with $\lambda=9$ from Lemma 1.8.

CHAPTER 8

RESOLVABLE DESIGNS

§8.1 Preliminary Results

In this chapter we shall look at *resolvable* designs on small cubic multigraphs. As defined in Chapter 1, a resolvable design is one where the blocks can be partitioned into parallel classes, that is, sets of blocks containing each of the v vertices exactly once. First we derive the necessary conditions for the existence of a resolvable design.

Lemma 8.1 In order for a resolvable G-design to exist on v points the necessary conditions of Theorem 1.6 must be satisfied, and in addition v, k, must satisfy the condition

$$v \equiv 0 \pmod{k}.$$

Proof Obviously the necessary conditions from Theorem 1.6 must be satisfied, and we must have $k|v$ since otherwise it would be impossible to partition the blocks into parallel classes.

We now look at each multigraph G for which we have already examined the question of the existence of G-designs in general, that is, all cubic multigraphs on 6 or fewer vertices, and try to determine when a resolvable design exists.

For the smallest cubic multigraph, $3K_2$, a resolvable design exists on λK_v whenever $3|\lambda$ and $v \equiv 0 \pmod 2$. This is because when v is even there exists a decomposition of K_v into 1-factors (see for example [9]) which immediately gives us a resolvable decomposition into K_2's, and from this we can obtain a decomposition of λK_v into $3K_2$'s if $3|\lambda$ (Lemmas 1.8 and 1.10). We now give the necessary conditions for the existence of a resolvable G-design if G is a cubic multigraph on 4 vertices.

Lemma 8.2 The necessary conditions for the existence of a resolvable G-design when G is a cubic multigraph on 4 vertices are

(a) $\lambda \geq m$,

(b) $v \geq 4$,

(c) $v \equiv 0 \pmod 4$ if $\lambda \equiv 0 \pmod 3$

$v \equiv 4 \pmod{12}$ if $\lambda \equiv 1, 2 \pmod 3$.

Proof The necessary conditions (from Chapter 2) for a G-design to exist on a cubic multigraph on 4 vertices are

$\lambda v(v-1) \equiv 0 \pmod{12}$ (1)

$\lambda(v-1) \equiv 0 \pmod 3$ (2)

$v \geq 4$ (3)

$\lambda \geq m$ (4)

For a resolvable design we have the added condition that

$v \equiv 0 \pmod 4$ (5).

Note that (5) $\Rightarrow$ (1), and if $\lambda \equiv 1, 2 \pmod 3$ then combining

(2) and (5) we get $v \equiv 4 \pmod{12}$. If $\lambda \equiv 0 \pmod 3$ then condition (2) always holds, and we are left with (3), (4), and (5).

Recall from Chapter 2 that there are 3 cubic multigraphs on 4 vertices; $3K_2 \cup 3K_2$, K_4 and the cylinder *Cy*. Resolvable K_4-designs exist whenever the necessary conditions are satisfied [17], as do resolvable *G*-designs where *G* is $3K_2 \cup 3K_2$ [8].

§8.2 Resolvable Cy-designs

We shall now show that the necessary conditions are also sufficient for the existence of resolvable designs on the remaining cubic multigraph on 4 vertices, the cylinder *Cy*.

Lemma 8.3 There exists a resolvable (4,4,2) *Cy*-design.

Proof Here $b = 2$. A design on Z_4 is generated by the block

(0,1;3,2) (half orbit).

Theorem 8.4 There exists a resolvable $(v,4,2)$ *Cy*-design for all v satisfying the necessary condition $v \equiv 4 \pmod{12}$.

Proof From the previous lemma, we have a (resolvable) decomposition of $2K_4$ into *Cy*-blocks, and therefore from Lemma 1.10, there exists a resolvable $(v,4,2)$ *Cy*-design whenever there exists a resolvable $(v,4,2)$ K_4-design. But a resolvable $(v,4,1)$ K_4-design exists whenever the necessary condition $v \equiv 4 \pmod{12}$ is satisfied (Hanani [17]), and the necessary condition remains the same for $\lambda=2$ as for $\lambda=1$. Therefore from Lemma 1.8 a resolvable $(v,4,2)$ *Cy*-design exists whenever the necessary condition is satisfied.

Lemma 8.5 There exists a resolvable $(4,4,3)$ *Cy*-design.

Proof A design on Z is given by the blocks

$$(0,1;2,3),$$
$$(0,2;3,1),$$
$$(0,3;1,2).$$

Note that a *G*-design on λK_k, where k is the number of vertices in *G*, must always be resolvable since in this case each block constitutes a parallel class. Therefore we could have deduced the existence of the resolvable designs in Lemmas 8.2 and 8.4 from the fact that we found *Cy*-designs with these parameters in Chapter 2.

Lemma 8.6 There exists a resolvable $(v,4,3)$ *Cy*-design whenever $v \equiv 4 \pmod{12}$.

Proof If $v \equiv 4 \pmod{12}$ then a resolvable $(v,4,1)$ $3K_4$-design exists from Hanani [17]. Therefore using the above result plus Lemmas 1.8 and 1.10 we can find a resolvable $(v,4,3)$ *Cy*-design.

We cannot use the results on resolvable K_4-designs for the remaining values of v satisfying the necessary conditions (that is, for $v \equiv 0, 8 \pmod{12}$), so we proceed to find some of these directly.

Lemma 8.7 There exists a resolvable (8,4,3) *Cy*-design.

Proof Here $b = 14$. A resolvable design on $\{Z_7 \cup \infty\}$ is generated by the two blocks

$$(0,6;4,1),$$

$$(2,\infty;3,5).$$

Lemma 8.8 There exists a resolvable (12,4,3) *Cy*-design.

Proof Here $b = 33$. A design on $\{Z_{11} \cup \infty\}$ is given by the blocks

$$(0,2;4,8),$$

$$(3,9;10,7),$$

$$(1,\infty;5,6).$$

Lemma 8.9 There exists a resolvable decomposition of $3K_{2,2}$ into *Cy*-blocks.

Proof A design on $Z_2 \times \{0,1\}$ is given by the blocks

$$(0_0,0_1;1_0,1_1), \quad (0_0,1_1;1_0,0_1)$$

where each block is a parallel class, since $v = k$.

Lemma 8.10 There exists a decomposition of K_v into 1-factors whenever v is even.

Proof See for example [9].

Theorem 8.11 There exists a resolvable $(v,4,3)$ *Cy*-design whenever the necessary condition $v \equiv 0 \pmod 4$ is satisfied.

Proof Since $v \equiv 0 \pmod 4$, $v = 2q$ for some $q \equiv 0 \pmod 2$. Split the v points into q rows of 2. Then from the previous lemma there exists a decomposition of K_q into 1-factors.

Take such a decomposition, and cover each pair of rows in the first 1-factor with the resolvable (4,4,3) *Cy*-design from Lemma 8.5. This gives us the first parallel class. Then all pairs in all remaining 1-factors can be covered with the (automatically resolvable) decomposition of $3K_{2,2}$ into *Cy*-blocks from Lemma 8.9, thus covering all edges in $3K_{\upsilon}$ and giving us a parallel class for each 1-factor.

Theorem 8.12 There exists a resolvable $(\upsilon,4,\lambda)$ *Cy*-design whenever the necessary conditions of Lemma 8.1 are satisfied.

Proof From Theorems 8.3 and 8.10, the necessary conditions are sufficient when $\lambda=2$, and $\lambda=3$. Designs with all other values of λ can be derived from these two by Lemma 1.8.

§8.3 Resolvable Designs on Cubic Multigraphs on 6 Vertices

Next we try to find resolvable designs on the nine cubic multigraphs on six vertices from Chapter 3. We start with the first three multigraphs, which are all bipartite, but first we derive the necessary conditions.

Lemma 8.13 The necessary conditions for the existence of a resolvable *G*-design when *G* is a cubic multigraph on six vertices are

(1) $\lambda \equiv 0 \pmod 3$

(2) $\upsilon \equiv 0 \pmod 6$.

Proof Combining the conditions for the existence of a G-design when G is a cubic multigraph on six vertices from Chapter 3 with the added requirement for resolvability that $v \equiv 0 \pmod{k}$, we find that a resolvable design can exist only if $\lambda \equiv 0 \pmod{3}$. In this case the conditions from Chapter 3 are satisfied whenever the resolvability condition $v \equiv 0 \pmod{6}$ is satisfied.

For the graph S_1, which is $K_{3,3}$, Huang [21] has shown that a necessary and sufficient condition for the existence of a resolvable $(v,6,\lambda)$ S_1-design is

$v \equiv 0 \pmod{6}$, $v \neq 6$, for $\lambda \equiv 0 \pmod{3}$

$v \equiv 0 \pmod{6}$ for $\lambda \equiv 0 \pmod{6}$.

§8.4 Resolvable S_2-designs

The next multigraph on our list is S_2 from Fig. 3.1:

Lemma 8.14 There exists a resolvable $(6,6,3)$ S_2-design.

Proof We have a $(6,6,3)$ S_2-design from Lemma 3.5, and since $v = k$ this design is automatically resolvable.

Lemma 8.15 There exists a resolvable decomposition of $3K_{3,3}$ into S_2-blocks.

Proof From Lemma 3.8 there exists a decomposition of $3K_{3,3}$ into S_2-blocks, and since $v=k$ here, this decomposition must be resolvable.

Theorem 8.16 There exists a resolvable $(v,6,3)$ S_2-design whenever the necessary condition $v \equiv 0 \pmod 6$ is satisfied.

Proof Split the v points into q rows of 3. Then we have $q \equiv 0 \pmod 2$, and therefore there is a decomposition of K_q into 1-factors. Cover pairs in one 1-factor with the resolvable $(6,6,3)$ S_2-design from Lemma 8.14, and pairs in all remaining 1-factors with the resolvable decomposition of $3K_{3,3}$ into S_2-blocks from Lemma 8.15. Each 1-factor gives a parallel class and we obtain a resolvable $(v,6,3)$ S_2-design.

Theorem 8.17 There exists a resolvable S_2-design whenever the necessary conditions $\lambda \equiv 0 \pmod 3$ and $v \equiv 0 \pmod 6$ are satisfied.

Proof This result follows from Theorem 8.16 and Lemma 1.8.

§8.5 Resolvable S_3-designs

We can solve the question of resolvable designs on the remaining bipartite multigraph S_3 in exactly the same way we did for S_2.

Lemma 8.18 There exists a resolvable $(6,6,3)$ S_3-design.

Proof We have a $(6,6,3)$ S_3-design from Lemma 3.21, and since $v = k$ this design is automatically resolvable.

Lemma 8.19 There exists a resolvable decomposition of $3K_{3,3}$ into S_3-blocks.

Proof From Lemma 3.25 there exists a decomposition of $3K_{3,3}$ into S_3-blocks, and since $v=k$ here, this decomposition must be resolvable.

Theorem 8.20 There exists a resolvable S_3-design whenever the necessary conditions $\lambda \equiv 0 \pmod 3$ and $v \equiv 0 \pmod 6$ are satisfied.

Proof From Lemmas 8.18 and 8.19 we have resolvable decompositions of $3K_6$ and $3K_{3,3}$ into S_3-blocks. Therefore we can find a resolvable S_3-design whenever $\lambda=3$ and $v=0 \pmod 6$ from Theorem 8.16 with S_3 substituted for S_2, and hence for all cases satisfying the necessary conditions by Lemma 1.8.

§8.6 Resolvable Designs on S_4, S_5, and S_6

We now look at resolvable designs on the three multigraphs in the second category of cubic multigraphs on six vertices, connected non-bipartite ones. Here we cannot obtain all the required resolvable decompositions from results in the non-resolvable case since we do not have decompositions of $3K_{3,3}$. Instead we must look for decompositions of complete tripartite multigraphs, such as $3K_{6,6,6}$, in order to solve the question in general. We can, however, use the results previously obtained to find the smallest resolvable design for each multigraph (that is, the one on $3K_6$), since this design again has $v = k$.

Lemma 8.21 There exists a resolvable (6,6,3) S_4-design.

Proof We have a (6,6,3) S_4-design from Lemma 4.8, and since $v = k$ this design is automatically resolvable.

Lemma 8.22 There exists a resolvable (6,6,3) S_5-design.

Proof We have a (6,6,3) S_5-design from Lemma 5.8, and since $v = k$ this design is automatically resolvable.

Lemma 8.23 There exists a resolvable (6,6,3) S_6-design.

Proof We have a (6,6,3) S_6-design from Lemma 5.41, and since $v = k$ this design is automatically resolvable.

CHAPTER 9

DESIGNS ON LARGER CUBIC GRAPHS, PRISMS, AND CUBES

§9.1 Preliminary Results

In this chapter we shall give some results on larger cubic graphs and also look at designs on cubes and prisms. From now on we confine our attention to *simple graphs*, that is, finite undirected graphs with no multiple edges. (We are no longer looking at the more general category of multigraphs as we were in previous chapters.) First we explain what graceful labellings are and how they can be used to find graph designs.

A *valuation*, or *labelling*, $\mathcal{L}_G$ of a graph G with k vertices and e edges is a numbering of the vertices v_i of G by distinct natural numbers $\{a_i\}_{i=1}^{k}$, with induced edge numbering for the edge $h_j = \{v_p, v_q\}$ given by $b_j = |a_p - a_q|$. Denote the set of numbers a_i in the valuation $\mathcal{L}_G$ of G by $V_{\mathcal{L}a}$ and the set of edge values b_i by $H_{\mathcal{L}a}$.

Definition 9.1 A *graceful labelling*, or *β-valuation*, of a graph G with v vertices and e edges is a labelling $\mathcal{L}_G$ of G satisfying the conditions

(1) $V_{\mathcal{L}a} \subset \{0, 1, \ldots, e\}$;

(2) $H_{\mathcal{L}a} = \{1, 2, \ldots, e\}$.

A graceful labelling is called an *α-labelling* if there exists a natural number x so that for each edge (v_p, v_q) either $a_p \le x < a_q$ or $a_q \le x < a_p$. It can easily be shown that this number x must be the smaller of the two vertex labels a_p and a_q which yield the edge label 1.

Lemma 9.2 If G is a graph with k vertices and e edges which has a graceful labelling, then there exists a cyclic $(2e+1,k,1)$ G-design. If in addition G has an α-labelling, then there exist cyclic $(2ce+1,k,1)$ G-designs for all positive integers c.

Proof These results are proved in [32].

Lemma 9.3 If G is a graph with k vertices and e edges which has a graceful labelling, then there exists a cyclic$(e+1,k,2)$ G-design.

Proof This is proved in [26].

Next we give graceful labellings for all cubic graphs on eight or ten vertices, and thus prove the existence of a cyclic $(2e+1,k,1)$ and $(e+1,k,2)$ G-design on each one.

§9.2 Designs on Cubic Multigraphs on 8 Vertices

First we derive necessary conditions for the existence of a G-design when G is a cubic multigraph on 8 vertices:

Lemma 9.4 The necessary conditions for the existence of a (v,k,λ) G-design when G is a cubic graph on 8 vertices are

$v \equiv 1,\ 16 \pmod{24}$ if $\lambda \equiv 1,5,7,11 \pmod{12}$

$v \equiv 1,\ 4 \pmod{12}$ if $\lambda \equiv 2,10 \pmod{12}$

$v \equiv 0,\ 1 \pmod{8}$ if $\lambda \equiv 3,9 \pmod{12}$

$v \equiv 1 \pmod{3}$ if $\lambda \equiv 4,8 \pmod{12}$

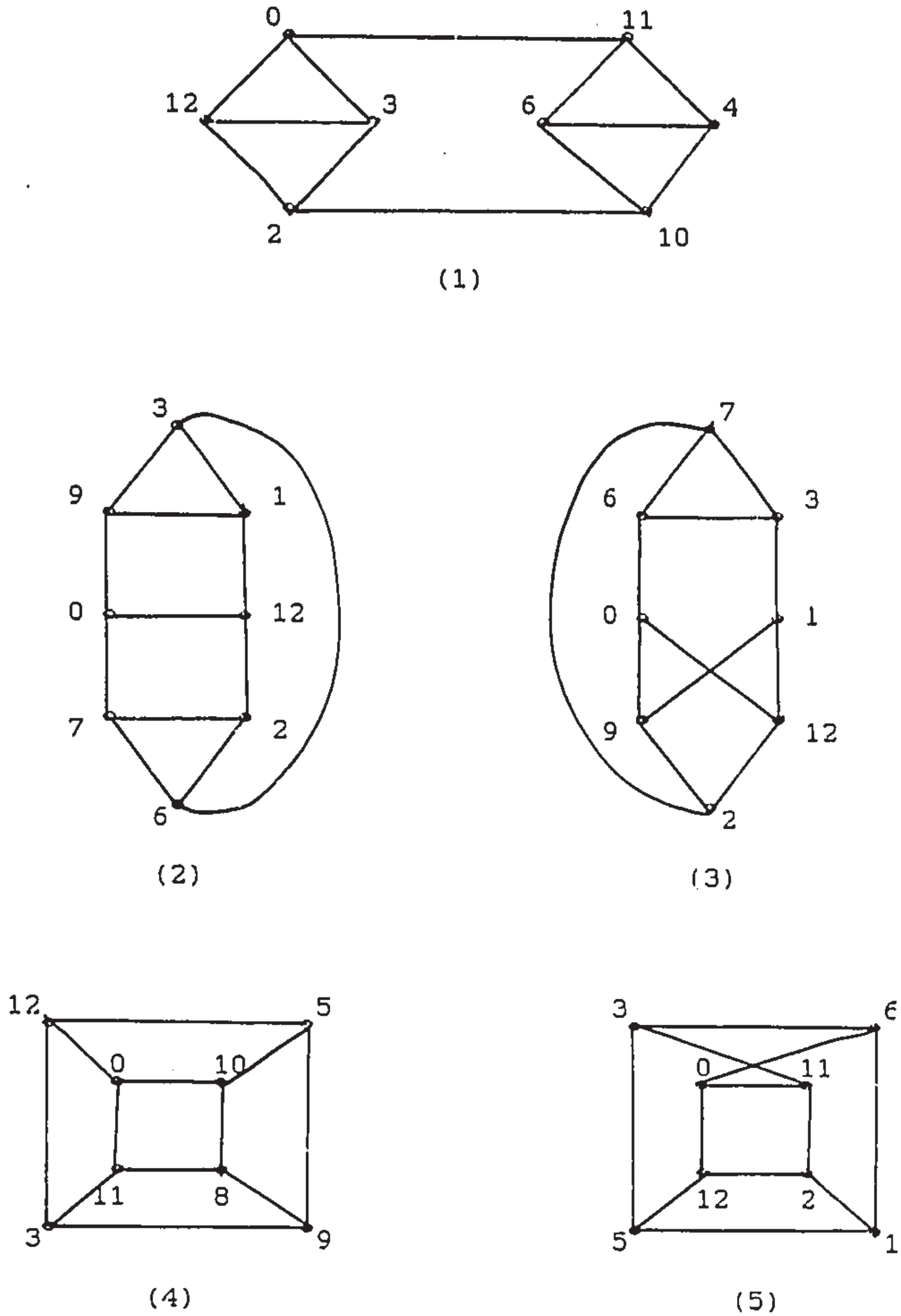
$v \equiv 0,\ 1 \pmod{4}$ if $\lambda \equiv 6 \pmod{12}$

$v \geq 8$ if $\lambda \equiv 0 \pmod{12}$.

Proof This result follows from Theorem 1.6.

Lemma 9.5 There exists a graceful labelling for every cubic graph on eight vertices.

Proof From [2] there are exactly five cubic graphs on eight vertices. A graceful labelling for each is shown in Fig 9.1. These graphs are presented in the same sequence as they are given in [2], and with the same numbering.


0
11
12
3
6
4
2
10
(1)
3
9
1
0
12
7
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6
(2)
7
6
3
0
1
9
12
2
(3)
12
5
0
10
11
8
3
9
(4)
3
6
0
11
12
2
5
1
(5)

Fig. 9.1

Theorem 9.6 For every cubic graph G on eight vertices there exist (25,8,1) and (13,8,2) G-designs.

Proof This result follows from Lemmas 9.2, 9.3 and 9.5.

§9.3 Designs on Cubic Graphs on 10 Vertices

Next we find necessary conditions for a G-design to exist when G is cubic on 10 vertices. Then we give graceful labellings for all cubic graphs on 10 vertices, and indicate which designs can be derived from these.

Lemma 9.7 The necessary conditions for the existence of a G-design when G is a cubic graph on ten vertices are

$v \equiv 1, 10 \pmod{15}$ if $\lambda \equiv 1,2,4,7,8,11,13,14 \pmod{15}$

$v \equiv 0,1 \pmod 5$ if $\lambda \equiv 3,6,9,12 \pmod{15}$

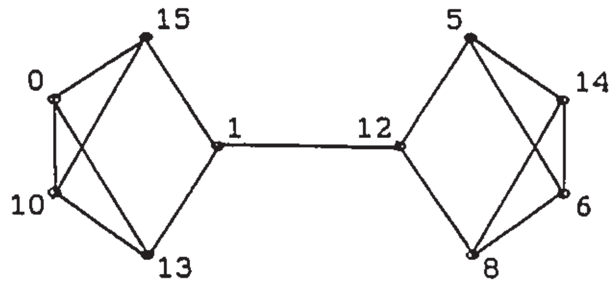
$v \equiv 1 \pmod 3$ if $\lambda \equiv 5,10 \pmod{15}$

$v \geq 10$ if $\lambda \equiv 0 \pmod{15}$.

Proof This result follows from Theorem 1.6.

Lemma 9.8 There exists a graceful labelling for every cubic graph on ten vertices.

Proof From [2] there are exactly nineteen cubic graphs on ten vertices. These are shown below in the same order as in [2] and with a graceful labelling for each one.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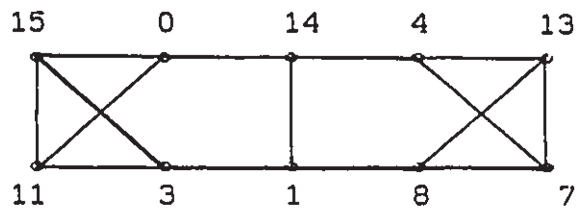
(1)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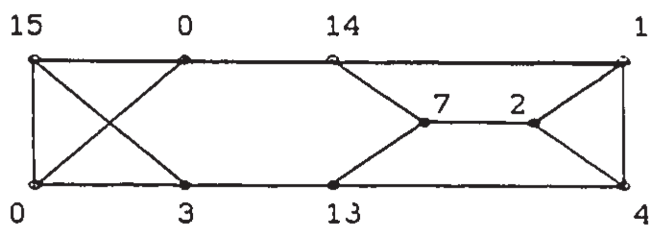
(2)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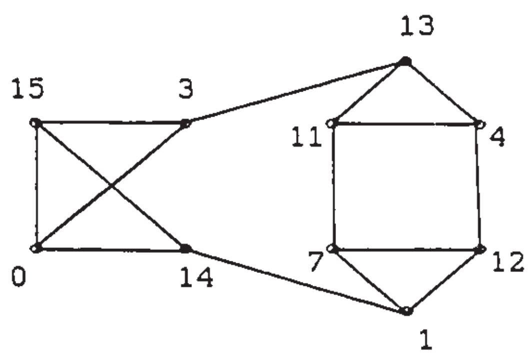
4

(3)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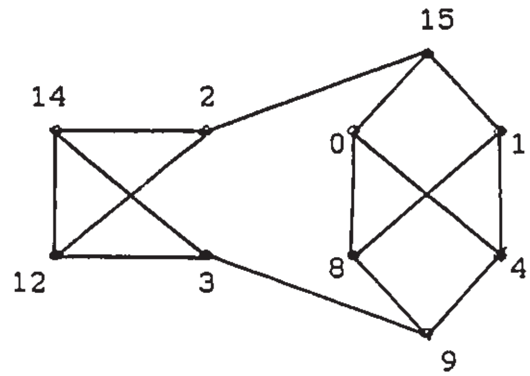
14
1

(4)



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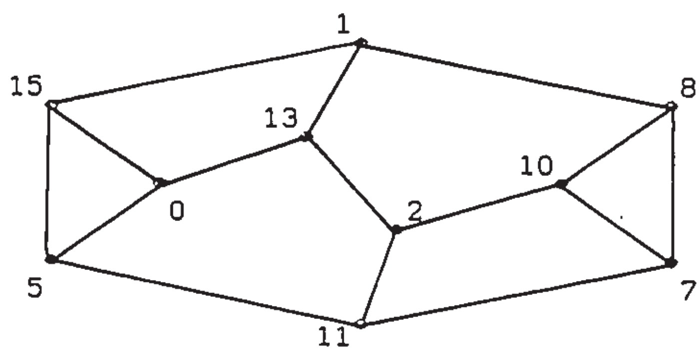

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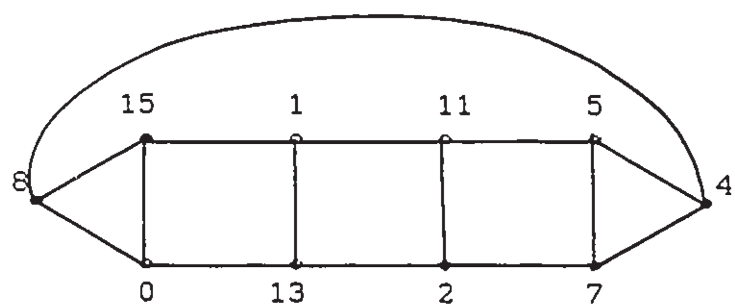

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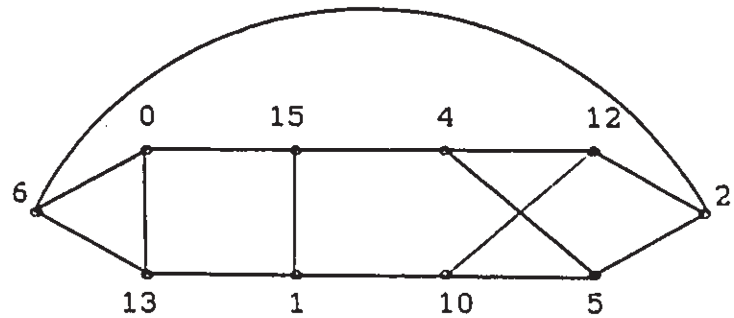

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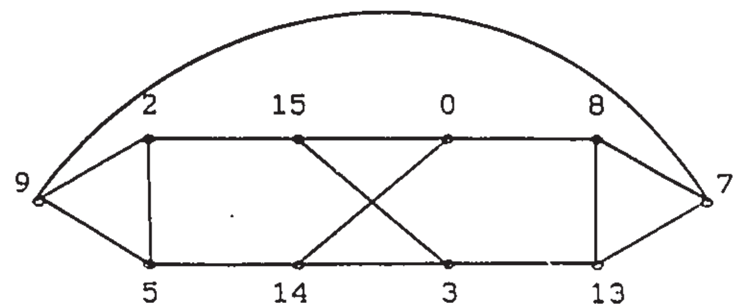

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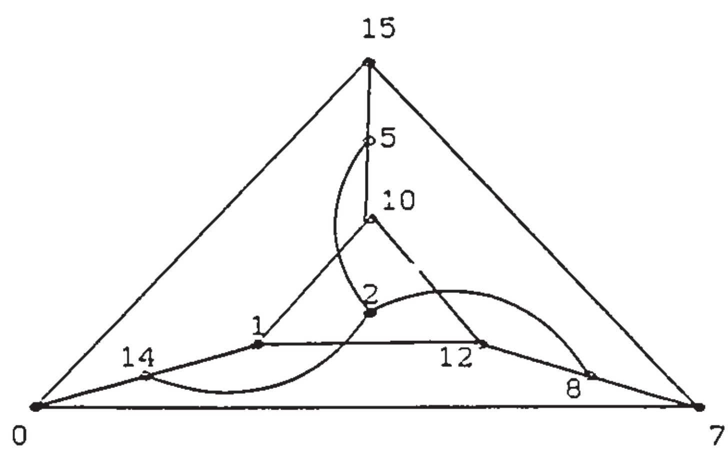

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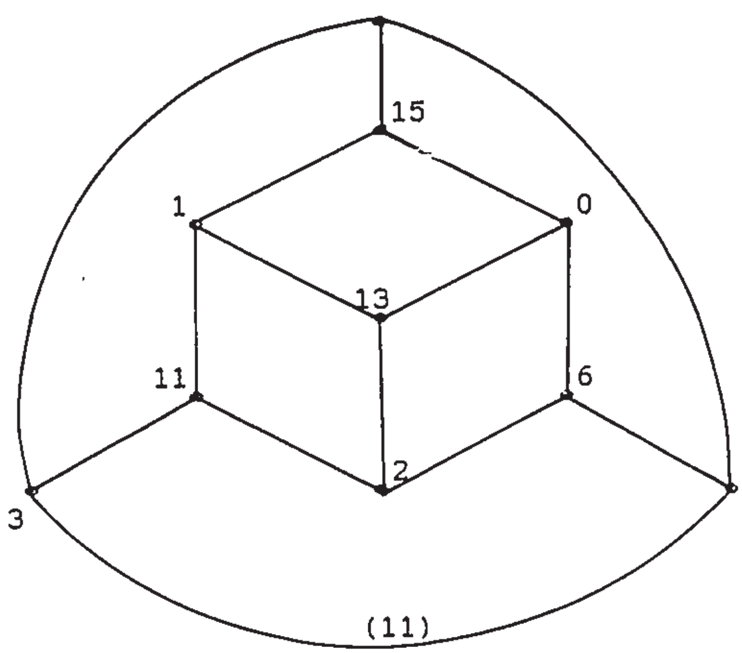

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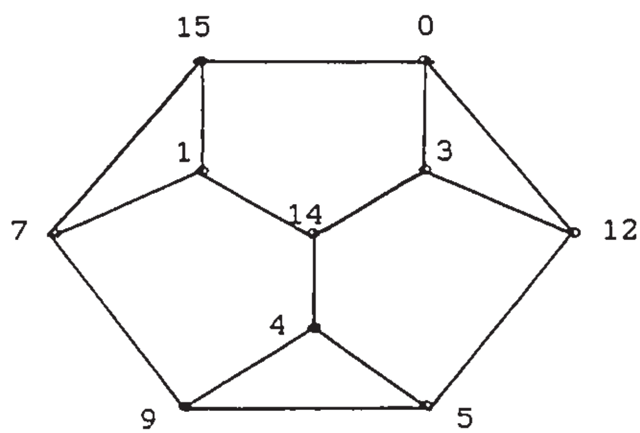

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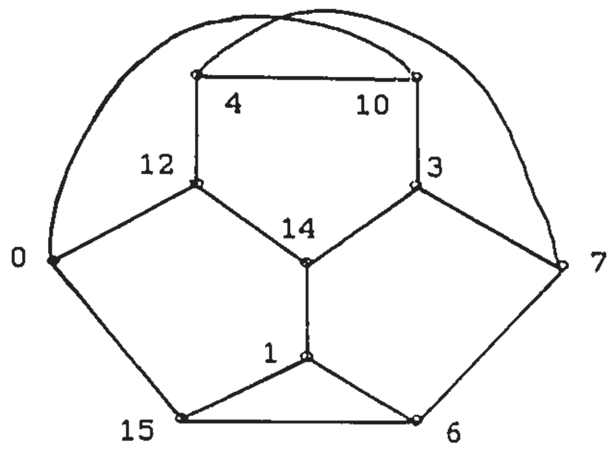

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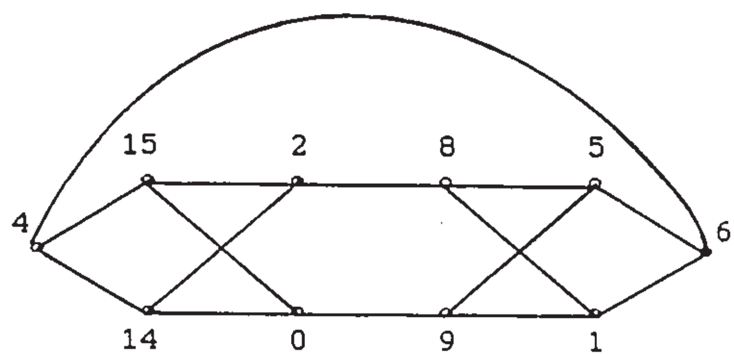

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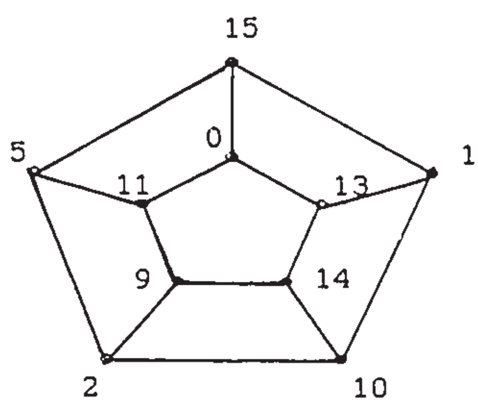

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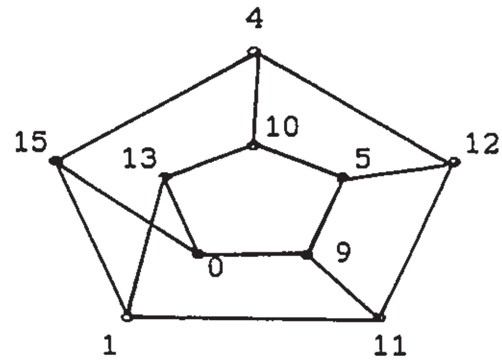

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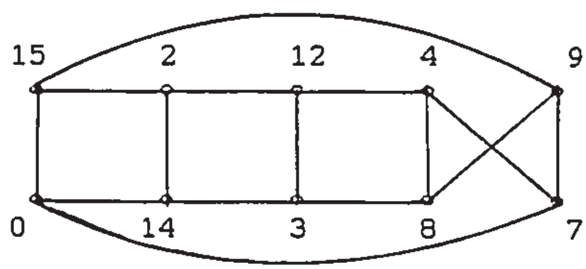
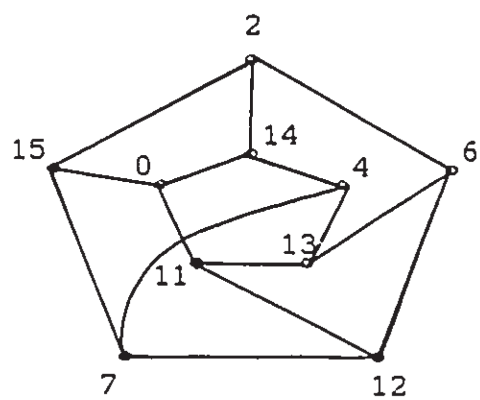
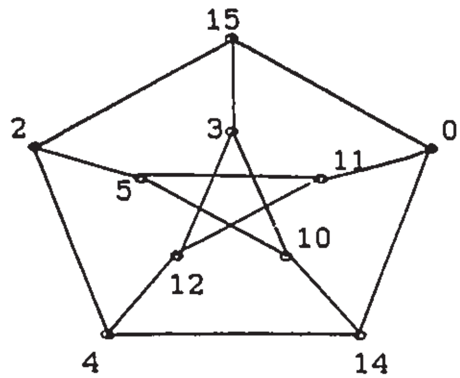


(17)



(18)



(19)

Fig. 9.2

Lemma 9.9 For every cubic graph G on ten vertices there exist (31,10,1) and (16,10,2) G-designs.

Proof This result follows from Lemmas 9.2, 9.3 and 9.7.

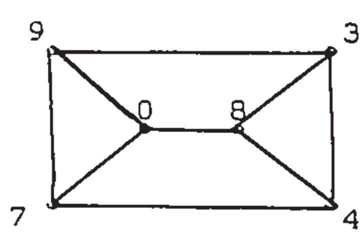
§9.4 Designs on Prisms

The *prism* D_n is the cubic graph obtained by taking the Cartesian product $P_2 \times C_n$ of the path of length 2 and the cycle of length n. The smallest prism D_3 is the same graph as the envelope S_4, and we have of course completely solved the question of the existence of designs on this graph in Chapter 4. We showed that these designs exist for all v and λ satisfying the necessary conditions. The second prism D_4 is the same graph as the cube Q_3, which is graph number (4) in our list of cubic graphs on 8 vertices in Fig.9.1, and has been looked by Kotzig [24] along with the other cubes.

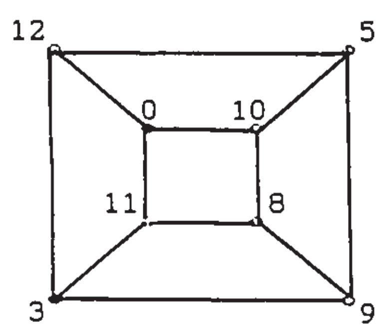
Frucht and Gallian [14] have shown that if n is *even*, the prism D_n has an α-labelling. This means that for all prisms D_n with n even there exist cyclic $(6cn+1, 2n, 1)$ (where c is any positive integer) and $(3n+1, 2n, 2)$ D_n-designs from Lemma 9.2. Gallian and Prout[15] have further shown that all vertex deleted and edge deleted prisms are graceful. (By vertex deleted and edge deleted prisms we mean the graphs obtained by deleting one vertex or one edge respectively from a prism.) It should be noted that only a bipartite graph can have an α-labelling. Therefore the prism D_n with n odd, which is not bipartite since it contains cycles of odd length, cannot have an α-labelling. It can of course still be graceful (have an β-labelling) and we shall in fact give graceful labellings for all prisms with $n \leq 9$.

Lemma 9.10 There exist graceful labellings for all prisms D_n with $n \leq 9$.

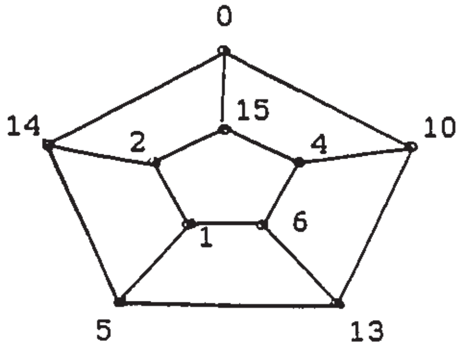
Proof These labellings are shown in Fig. 9.3 below:



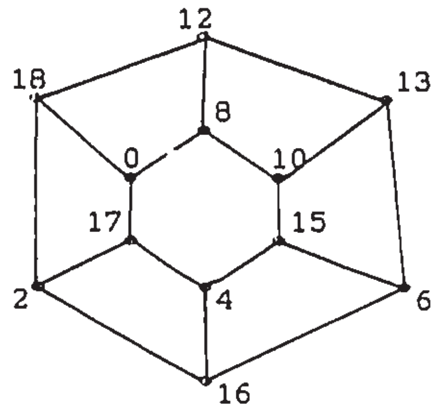
D_3



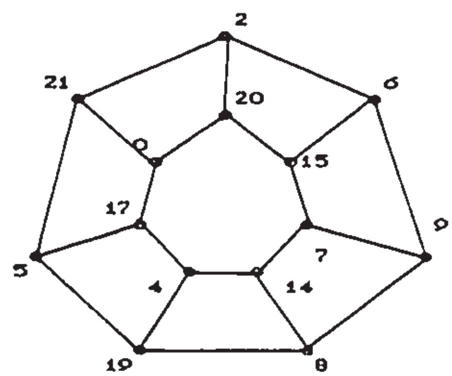
D_4


0
14
15
10
2
4
1
6
5
13

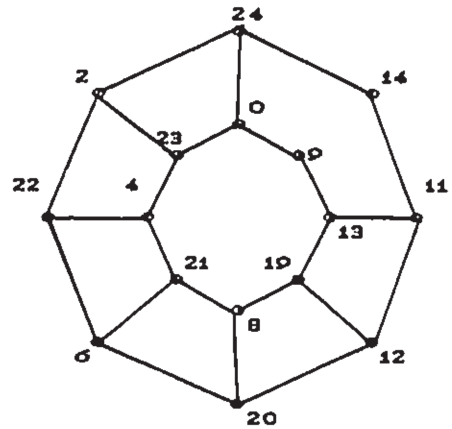
D_5


12
18
13
8
0
10
17
15
2
4
6
16

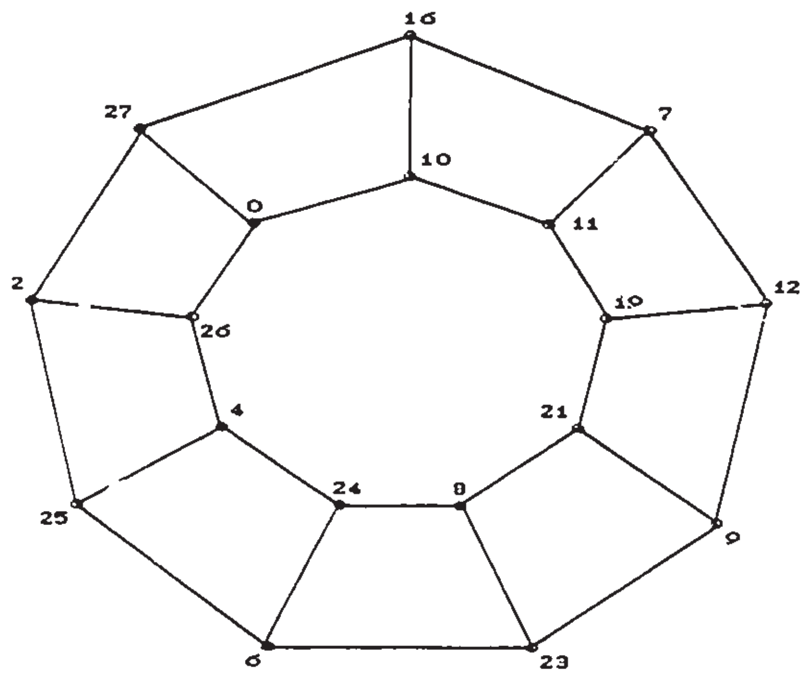
D_6


2
21
20
6
0
15
17
9
7
5
4
14
19
8

D_7


24
14
2
0
23
9
22
4
11
13
21
19
8
6
12
20

D_8


16
27
7
10
0
11
2
12
10
26
21
4
24
8
25
9
6
23

D_9

Fig. 9.3

Lemma 9.11 There exists an α-valuation for all prisms D_n where n is even.

Proof This result is proved by Frucht and Gallian in [14].

Lemma 9.12 For all prisms D_n with $3 \le n \le 9$ there exist $(6n+1,2n,1)$ and $(3n+1,2n,2)$ D_n-designs.

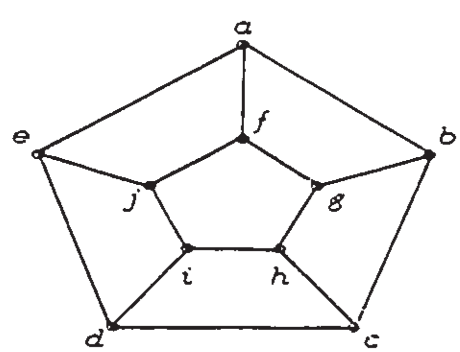
Proof This result follows from Lemmas 9.2, 9.3 and 9.10.

Lemma 9.13 For all prisms D_n with n even there exists a $(6cn+1,2n,1)$ D_n-design for each positive integer c.

Proof This result follows from Lemmas 9.2 and 9.11.

§9.5 Designs on the 5-prism

The existence of prism designs in general is an open question. We do however have a few results on the prism D_5, which is also graph number (15) in our listing of cubic graphs on ten vertices in Fig.9.2. We will list blocks for D_5 as shown below:



$(a,b,c,d,e;f,g,h,i,j)$

Fig. 9.4

The smallest design to look for on D_5 is a (10,10,1) design. We prove that such a design cannot exist.

Lemma 9.14 A (10,10,1) D_5-design does not exist.

Proof (By contradiction). If such a design existed the number of blocks in it would be 3. Assume the design exists, and let the element set be $Z_5 \times \{0,1\}$. WLOG let the first block B_1 be

$$(0_0,2_0,4_0,1_0,3_0;0_1,2_1,4_1,1_1,3_1).$$

Then the remaining edges in K_{10}, or the complement of B_1 in K_{10} (see Fig. 9.5 below), must be decomposable into two more copies of D_5. Call these two remaining blocks B_2 and B_3.

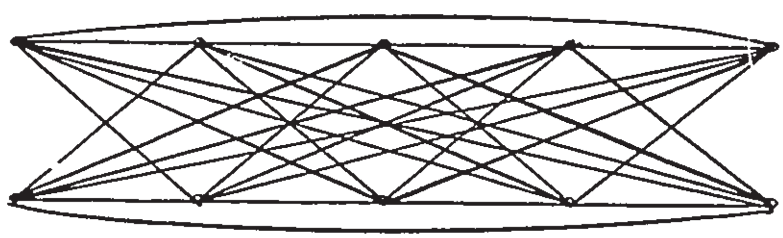


Fig. 9.5

Clearly, both B_2 and B_3 must use for each pentagon either 5 or 3 or 1 edges of the pentagons $(0_0,1_0,2_0,3_0,4_0)$ and $(0_1,1_1,2_1,3_1,4_1)$. We now look at each of these possibilities for B_2 separately.

Case I Graph B_2 uses all 5 edges of $(0_0,1_0,2_0,3_0,4_0)$ for its first pentagon, and therefore necessarily all edges of $(0_1,1_1,2_1,3_1,4_1)$ for its second pentagon. But then B_3 is bipartite, a contradiction.

Case II One of the pentagons in B_2 uses 3 edges from the two pentagons $(0_0,1_0,2_0,3_0,4_0)$ and $(0_1,1_1,2_1,3_1,4_1)$. Then WLOG there are six possibilities for the first pentagon:

(i) $(0_0,1_0,2_1,3_0,4_0)$

(ii) $(0_0,1_0,0_1,3_0,4_0)$

(iii) $(0_0,1_0,2_0,3_1,2_1)$

(iv) $(0_0,1_0,2_0,3_1,4_1)$

(v) $(0_0,1_0,2_0,4_1,3_1)$

(vi) $(0_0,1_0,2_0,0_1,1_1)$.

We look at each of these separately to see if we can complete the design.

(i) In this case the two pentagons of B_2 must be

$$(0_0,1_0,2_1,3_0,4_0) \text{ and } (0_1,1_1,2_0,3_1,4_1).$$

Clearly, the remaining edges in B_2 must include either the edge $(1_0,2_0)$ or the edge $(2_0,3_0)$ (but not both). Assume WLOG that they include the edge $(1_0,2_0)$. Then the other edges in B_2 must be $(0_0,1_1)$, $(3_0,4_1)$, $(4_0,0_1)$ and $(2_1,3_1)$. The edges remaining constitute the last G-block B_3. But B_3 then contains a triangle, which is a contradiction.

(ii) Here the two pentagons of B_2 must be

$$(0_0,1_0,0_1,3_0,4_0) \text{ and } (2_0,1_1,2_1,3_1,4_1).$$

Clearly one of the edges $(1_0,2_0)$, $(2_0,3_0)$, $(2_0,0_1)$ must belong to B_2. There is one way to complete B_2 using each of these so we have the following three possibilitites for B_2:

(a) $(0_0, 1_0, 0_1, 3_0, 4_0; 4_1, 2_0, 1_1, 2_1, 3_1)$

(b) $(0_0, 1_0, 0_1, 3_0, 4_0; 3_1, 4_1, 2_0, 1_1, 2_1)$

(c) $(0_0, 1_0, 0_1, 3_0, 4_0; 2_1, 3_1, 4_1, 2_0, 1_1)$.

For (a) and (c) the edges remaining for B_3 form the graph shown in Fig. 9.6 below, which is T_{16} of Fig. 9.2, and for (b) they contain a triangle, so in no case do they form T_{15} (or D_5) as required.

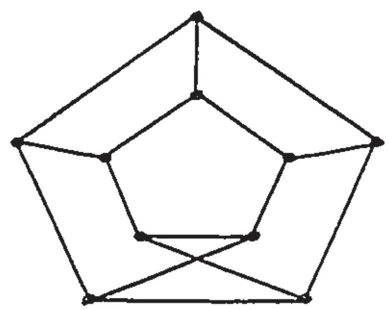


Fig. 9.6

(iii) The pentagons in B_2 must be either

(a) $(0_0, 1_0, 2_0, 2_1, 3_1)$ and $(3_0, 4_0, 1_1, 0_1, 4_1)$

or (b) $(0_0, 1_0, 2_0, 2_1, 3_1)$ and $(3_0, 4_1, 0_1, 4_0, 1_1)$.

If (a) then clearly B_2 must contain either the edge $(3_0, 2_0)$ or the edge $(3_0, 2_1)$. But if the former then B_2 must contain either $(1_0, 1_1)$ or $(2_0, 5_0)$, a contradiction and if the latter then it must contain $(0_0, 0_1)$ or $(1_0, 1_1)$, also a contradiction.

If the pentagons are those in (b) then B_2 must again contain either $(3_0, 2_0)$ or $(3_0, 2_1)$. If the former then it must also contain $(1_0,1_1$ or $(1_1,3_1)$, and if the latter $(1_1,3_1)$ or $(1_0,4_0)$. In either case we have a contradiction.

(iv) The pentagons in B_2 are $(0_0,1_0,2_0,3_1,4_1)$ and one of

(a) $(3_0,4_0,2_1,1_1,0_1)$

(b) $(3_0,4_0,0_1,1_1,2_1)$

(c) $(3_0,0_1,1_1,4_0,2_1)$

(d) $(3_0,1_1,0_1,4_0,2_1)$

(e) $(3_0,0_1,4_0,1_1,2_1)$

(f) $(3_0,0_1,4_0,2_1,1_1)$.

Clearly B_2 must contain either the edge $(1_0,0_1)$ or the edge $(1_0,2_1)$. A close examination reveals that it is impossible to complete B_2 without using an edge from B_1 when the second pentagon is (a), (c), (d), (e) or (f). With pentagon (b) there are two possibilities for B_2;

$$(0_0,1_0,2_0,3_1,4_1;4_0,0_1,1_1,2_1,3_0)$$

and $$(0_0,1_0,2_0,3_1,4_1;1_1,2_1,3_0,4_0,0_1).$$

In either case B contains a triangle, a contradiction.

(v) The pentagons in B_2 are $(0_0,1_0,2_0,4_0,3_0)$ and one of

(a) $(3_0,4_0,2_1,1_1,0_1)$

(b) $(3_0,4_0,0_1,1_1,2_1)$

(c) $(3_0,0_1,1_1,4_0,2_1)$

(d) $(3_0,1_1,0_1,4_0,2_1)$

(e) $(3_0,0_1,4_0,1_1,2_1)$

(f) $(3_0,0_1,4_0,2_1,1_1)$.

As in (iv) B_2 must contain either the edge $(1_0,0_1)$ or the edge $(1_0,2_1)$. Further examination shows that we can complete B_2 only when the second pentagon is (b), and in this case there are two possibilities for B_2:

$$(0_0,1_0,2_0,3_1,4_1;4_0,0_1,1_1,3_0,2_1)$$

$$\text{and}\quad (0_0,1_0,2_0,3_1,4_1;1_1,2_1,3_0,0_1,4_0).$$

In either case B_3 contains a triangle, a contradiction.

(vi) Here the pentagons in B_2 are (0,1,2,0,1) and either

$$(3_0,4_0,2_1,3_1,4_1) \quad\text{or}\quad (3_0,2_1,4_0,3_1,4_1).$$

There are four ways to complete B_2, three using the first pentagon and one the second, to obtain the following possibilities for B_2:

(a) $(0_0,1_0,2_0,0_1,1_1;4_0,2_1,3_1,4_1,3_0)$

(b) $(0_0,1_0,2_0,0_1,1_1;2_1,3_1,4_1,3_0,4_0)$

(c) $(0_0,1_0,2_0,0_1,1_1;3_1,4_1,3_0,4_0,2_1)$

(d) $(0_0,1_0,2_0,0_1,1_1;4_0,3_1,4_1,3_0,2_1)$.

For (b) and (d) B_3 contains a triangle, for (a) B_3 is T_{16} from Fig. 9.2, and for (c) B_3 is T_{18}, so in no case is B_3 isomorphic to T_{15} (or D_5) as required.

Case III Graph B_2 uses one edge from the two pentagons in B_1, say from $(0_0,1_0,2_0,3_0,4_0)$, for its first pentagon. Its second pentagon must then use one edge from the other pentagon $(0_1,1_1,2_1,3_1,4_1)$, since we have already dimissed Case II, and it is impossible for both to use just one edge from the same pentagon since they would then both contain 3 vertices from this pentagon. The same argument applies to B_3. Therefore the remaining three edges in the pentagon $(0_0,1_0,2_0,3_0,4_0)$ must be used as joining edges in B_2 and B_3, two in one and one in the other. Assume two of them are used as connecting edges on B_2. Let the edge used in the first pentagon in B_2 be $(0_0,4_0)$. Then we can assume WLOG that the two connecting edges are $(1_0,2_0)$ and $(3_0,4_0)$, since if they were for instance $(0_0,1_0)$ and $(3_0,4_0)$, then $(1_0,3_0)$ would have to be included in B_2 as well, which is a contradiction.

There are then six possibilities for B_2 (found by exhaustive examination):

(i) $(0_0,4_0,0_1,1_0,4_1;2_1,3_0,1_1,2_0,3_1)$

(ii) $(0_0,4_0,0_1,1_0,2_1;4_1,3_0,1_1,2_0,3_1)$

(iii) $(0_0,4_0,3_1,2_0,1_1;4_1,3_0,2_1,1_0,0_1)$

(iv) $(0_0,4_0,3_1,2_0,4_1;1_1,3_0,2_1,1_0,0_1)$

(v) $(0_0,4_0,0_1,2_0,3_1;1_1,3_0,4_1,1_0,2_1)$

(vi) $(0_0,4_0,1_1,2_0,4_1;2_1,3_0,0_1,1_0,3_1)$.

In (i), (ii), (iii) and (iv) the graph B_3 (what is left) contains a triangle, a contradiction. In (v) the graph B_3 is the graph T_{18} from Fig. 9.2 and in (vi) it is the graph T_{16}. Therefore there is no decomposition of K_{10} into D blocks in either case, and none in general.

Lemma 9.14 There exists a (16,10,1) D_5-design.

Proof Here $b = 8$. A design on Z_{16} is generated by the block

$$(0,14,9,3,15;7,11,1,6,8).$$

Lemma 9.15 There exists a (31,10,1) D_5-design.

Proof This result follows from both Lemma 9.9 and Lemma 9.12.

Lemma 9.16 There exists a (10,10,2) D_5-design.

Proof Here $b = 6$. The required design is generated on $Z_5 \times \{0,1\}$ by the two blocks

$(0_0,1_0,2_0,3_0,4_0;0_1,1_1,2_1,3_1,4_1)$ (1/5 orbit = 1 block)

$(0_0,1_0,3_1,1_1,3_0;0_1,2_1,4_0,2_0,4_1)$.

Lemma 9.17 There exists a (16,10,2) D_5-design.

Proof This result follows from Lemma 9.9 or Lemma 9.12.

Lemma 9.18 There exists a (10,10,3) D_5-design.

Proof Here $b = 9$. A design on $\{Z_9 \cup \infty\}$ is generated by the block

$$(0,2,5,8,4;\infty,3,1,7,6).$$

Lemma 9.19 There exists an (11,10,3) D_5-design.

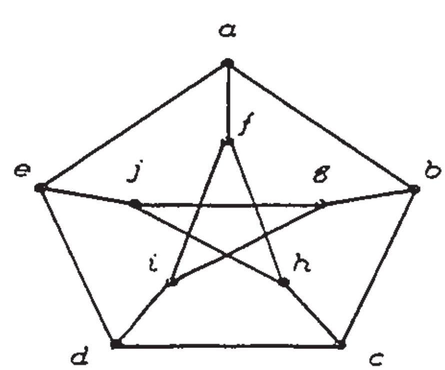
Proof Here $b = 11$. A design on Z_{11} is generated by the block

$$(0,1,6,2,8;9,5,4,7,10).$$

These are all the D_5-designs we have so far. We cannot prove that a (10,10,1) D_5-design does not exist as we do for the Petersen graph below, but have not been able to find one either. This graph is not bipartite, but it is *tripartite*, so we could try to find decompositions of complete tripartite multigraphs, such as $K_{5,5,5}$, into D_5-blocks, in order to find some infinite classes of D_5-designs by recursion. Also open is the question of the existence of D_5-designs when $\lambda \equiv 0$, 5, or 10 (mod 15), and $v \not\equiv 0,1 \pmod 5$.

§9.6 Designs on the Petersen Graph

We have also found a few designs on the Petersen graph, which is number (19) in our list of cubic graphs on ten vertices. We shall therefore call it T_{10}. The block labelling system we shall use for the Petersen graph is shown in Fig. 9.7.



$(a,b,c,d,e;f,g,h,i,j)$

Fig. 9.7

The *diameter* of a graph G is defined to be the maximum distance between two vertices of G. The following result relates the diameter of a graph G to the existence of designs on G.

Lemma 9.20 If K_n is decomposable into three factors of diameter 2, then $n \geq 12$.

Proof This result is proved in [10].

Lemma 9.21 There does not exist a (10,10,1) T_{10}-design.

Proof The Petersen graph has diameter 2, therefore such a design cannot exist by the previous lemma.

Lemma 9.22 There exists a (16,10,1) T_{10}-design.

Proof Here $b = 8$. A design on $(Z_5 \times \{0,1,2\}) \cup \{\infty\}$ is generated by the four blocks

$(0_0, 1_1, 2_0, 4_1, 3_2; 3_1, \infty, 4_2, 1_2, 4_0)$

$(0_0, 1_0, 2_0, 3_0, 4_0; 0_1, 1_1, 2_1, 3_1, 4_1)$ 1 block

$(0_1, 1_1, 2_1, 3_1, 4_1; 0_2, 1_2, 2_2, 3_2, 4_2)$ 1 block

$(0_2, 1_2, 2_2, 3_2, 4_2; 4_0, 0_0, 1_0, 2_0, 3_0)$ 1 block.

Lemma 9.23 There exists a (31,10,1) T_{10}-design.

Proof This result follows from Lemma 9.9.

Lemma 9.24 There exists a (10,10,2) T_{10}-design.

Proof Here $b = 6$. A design on $Z_5 \times \{0,1\}$ is generated by the two blocks

$(0_0, 1_0, 2_0, 3_0, 4_0; 1_1, 2_1, 3_1, 4_1, 0_1)$ 1 block only

$(0_0, 1_1, 2_0, 4_1, 3_1; 2_1, 1_0, 4_0, 0_1, 3_0)$.

Lemma 9.25 There exists a (16,10,2) T_{10}-design.

Proof This result follows from Lemma 9.9.

Lemma 9.26 There exists a (10,10,3) T_{10}-design.

Proof Here $b = 9$. A design on $\{Z_9 \cup \infty\}$ is generated by the block

$$(0,2,5,6,8;1,7,\infty,4,3).$$

Lemma 9.27 A (10,10,λ) T_{10}-design exists if and only if $\lambda \geq 2$.

Proof This result follows from Lemmas 9.24, 9.26 and 1.8.

Lemma 9.28 There exists an (11,10,3) T_{10}-design.

Proof Here $b = 11$. A design on Z_{11} is generated by the block

$$(0,2,1,7,3;6,10,8,9,4).$$

Lemma 9.29 There exists a (15,10,3) T_{10} design.

Proof Here $b = 21$. A design on $(Z_7 \times \{0,1\}) \cup \{\infty\}$ is generated by the three blocks

$$(0_0,1_1,2_0,4_1,6_0;2_1,3_0,5_1,5_0,3_1)$$
$$(0_0,1_1,\infty,4_1,6_0;3_1,2_1,3_0,5_1,5_0)$$
$$(0_0,2_1,5_1,6_0,3_0;2_0,0_1,5_0,\infty,6_1).$$

These are all the designs we have been able to find on the Petersen graph to date. This graph is not bipartite but is tripartite, so, as in the case of D_5, we could look next for some decompositions of complete tripartite multigraphs such as $K_{5,5,5}$. We could also look for designs with different values of λ.

§9.7 Designs on Cubes

Finally we should like to mention some results which have been obtained regarding designs on the *cube* Q_d, which is defined by

$$Q_d = K_2 \times K_2 \times \ldots \times K_2 \qquad (d \text{ times}).$$

All cubes are bipartite, and they therefore form a class of graphs on which graph designs can be found relatively easily. Because of this, they offer a promising direction for future research. As mentioned before, the only cube which is cubic is the 3-dimensional cube Q_3 which is also the prism D_4, and is graph number (4) in our list of cubic graphs on eight vertices in Fig. 9.1. We already know that this graph has an α-labelling and therefore there exists a (13,8,2) Q_3-design, and a $(24c+1,8,1)$ Q_3-design for each positive integer c (Lemma 9.11). Kotzig [24] has shown further that every cube Q_d has an α-labelling so we have the following.

Lemma 9.30 An $(n,2^d,1)$ Q_d-design exists if and only if $n \equiv 1 \pmod{d2^d}$.

Proof This result is due to Kotzig [24].

Lemma 9.30 For any positive integer d, there exists a $(d2^{d-1}+1,2^d,2)$ Q_d-design.

Proof From Kotzig [24] every cube Q_d has an α-labelling. The number of edges e in a cube is $d2^{d-1}$, and the number of vertices k is 2^d. Therefore the required designs exist by Lemma 9.3.

CONCLUSION

In this thesis we have taken multigraphs from several different categories, and for each multigraph G, where G has k vertices, have tried to find the values of v and λ for which a (v,k,λ) G-design exists. Complete results have been obtained for all cubic multigraphs on six or fewer vertices with one exception, this being the disconnected multigraph $K_4 \cup 3K_2$, which we have also called S_9. We have further obtained partial results for all simple graphs on eight and ten vertices, and for all prisms on eighteen or fewer vertices. For each of these we have found a graceful labelling, from which some designs can be found by [32] and [10]. Some additional designs have been found for the prism D_5 and the Petersen graph. Partial results on all prisms D_n with n even, and on all cubes, have been similarly obtained by Frucht and Gallian [14] and Kotzig [24] respectively.

The existence of a (9,6,3) S_9-design is an open question, so is the existence of most S_9-designs with $\lambda=3$, and $\lambda=9$. We could, however, find all S_9-designs with $\lambda=9$ if we could find those with v = 22, 25, 27, 28, 30, 33, 39, 36, 40 and 42. Finding these designs with $\lambda=3$ would also suffice since their existence would imply the existence of the corresponding designs with $\lambda=9$. Further research into the existence of graph designs on cubic graphs could include the existence of graph designs on cubic graphs could include looking for more complete results on the cubic graphs on eight vertices, and those on ten vertices. The bipartite graphs are probably the most hopeful candidates in this respect. It would be of great interest to find all designs on the 5-prism and the Petersen graph. For prisms the next step is to show that *all* prisms, not just those with n even, have a graceful labelling. Then one could try to find those prism designs which cannot be derived from the graceful labellings, and this could also be done for designs on cubes.

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