# HÖLDER'S INEQUALITY
# IN SPACES OF
# MEASURABLE FUNCTIONS

BY

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ABSTRACT

This thesis gives a historical account of the development of the theory of spaces of measurable functions. The study will be mainly centred around Holder's inequality and its many generalizations.

We present a formal axiomatization of the theory including a general form of Holder's inequality. Then we consider various families of spaces of measurable functions, namely the Orlicz spaces and the Lorentz spaces, both of which include the familiar $L^p$-spaces. In each of these special cases, Holder's inequality is interesting in its own right. As a particular example, the space $L\log^{+}L$ and its conjugate, the exponential space, are studied in detail as they are examples of both an Orlicz space and a Lorentz space.

THIS THESIS IS DEDICATED TO MY PARENTS

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# CHAPTER I

## INTRODUCTION

The purpose of this thesis is to discuss various generalizations of the classical Holder inequality and to consider the role they play in analysis. Just as the classical Holder inequality may be interpreted in terms of the Lebesgue spaces $L^p$, then so do its variants have formulations in terms of spaces which generalize $L^p$. Hence our account of the development of the Holder inequalities will go hand in hand with an account of the development of the theory of spaces of measurable functions.

There is an inequality due to Cauchy which asserts that [3]

$$(1.1) \qquad \sum_{i=1}^{n} a_i b_i \leq \left( \sum_{i=1}^{n} a_i^2 \right)^{\frac{1}{2}} \left( \sum_{i=1}^{n} b_i^2 \right)^{\frac{1}{2}}$$

where $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ are arbitrary n-tuples of non-negative real numbers. This inequality first appeared in 1821 and follows from the identity

$$\left( \sum_{i=1}^{n} a_i^2 \right) \left( \sum_{i=1}^{n} b_i^2 \right) - \left( \sum_{i=1}^{n} a_i b_i \right)^2 = \frac{1}{2} \sum_{i,j=1}^{n} ( a_i b_j - a_j b_i )^2 .$$

The following result, which contains (1.1) as the special case $p=2$, is known as Holder's inequality. It was published by L.J. Rogers in 1888 and a year later by O. Holder, [22], [10].

<u>Theorem 1.1.</u> For each real number $p$, $1<p<\infty$, let $p'$ be the conjugate of $p$ defined by $1/p + 1/p' = 1$. Let $(a_i)_{i=1}^{n}$ and $(b_i)_{i=1}^{n}$ be sequences of positive numbers. Then

$$(1.2) \qquad \sum_{i=1}^{n} a_i b_i \leq \left( \sum_{i=1}^{n} a_i^p \right)^{1/p} \left( \sum_{i=1}^{n} b_i^{p'} \right)^{1/p'} .$$

Proof. Let $\alpha=1/p$. If $0<a<b$, then by the mean value theorem, there exists $c$ between $a$ and $b$ such that

$$b^{1-\alpha} - a^{1-\alpha} = (1-\alpha)\ (b-a)\ c^{-\alpha}.$$

Since $0<\alpha<1$, it follows that

$$c^{-\alpha} < a^{-\alpha}$$

and so

(1.3) $$b^{1-\alpha} - a^{1-\alpha} \leq (1-\alpha)\ (b-a)\ a^{-\alpha}.$$

By rearranging (1.3), we obtain

(1.4) $$a^{\alpha} b^{1-\alpha} \leq \alpha a + (1-\alpha)\ b.$$

It is clear by symmetry that the condition $a<b$ may be dropped. Replacing $a$ by $a^p$, and $b$ by $b^p$, we obtain

(1.5) $$ab \leq \frac{a^p}{p} + \frac{b^{p'}}{p'}$$

Hence,

$$\frac{\sum_{i=1}^{n} a_i b_i}{\left(\sum_{i=1}^{n} a_i^p\right)^{1/p} \left(\sum_{i=1}^{n} b_i^{p'}\right)^{1/p'}} = \sum_{i=1}^{n} \left[\frac{a_i^p}{\sum_{j=1}^{n} a_j^p}\right]^{1/p} \left[\frac{b_i^{p'}}{\sum_{j=1}^{n} b_j^{p'}}\right]^{1/p'}$$

$$\leq \frac{1}{p}\left[\frac{\sum_{i=1}^{n} a_i^p}{\sum_{j=1}^{n} a_j^p}\right] + \frac{1}{p'}\left[\frac{\sum_{i=1}^{n} b_i^{p'}}{\sum_{j=1}^{n} b_j^{p'}}\right]$$

$$= \frac{1}{p} + \frac{1}{p'} = 1.$$

from which (1.2) follows.

As an immediate corollary of Holder's inequality, we obtain another classical inequality, namely Minkowski's inequality.

Theorem 1.2. Let $(a_i)_{i=1}^n$ and $(b_i)_{i=1}^n$ be two sequences of positive numbers and let $p>1$. Then

$$\left(\sum_{i=1}^n (a_i + b_i)^p\right)^{1/p} \leqslant \left(\sum_{i=1}^n a_i^p\right)^{1/p} + \left(\sum_{i=1}^n b_i^p\right)^{1/p}. \tag{1.6}$$

Proof. Let us write

$$S = \sum_{i=1}^n (a_i + b_i)^p$$

and note that

$$S = \sum_{i=1}^n a_i(a_i + b_i)^{p-1} + \sum_{i=1}^n b_i(a_i + b_i)^{p-1}. \tag{1.7}$$

By applying Theorem 1.1 to both terms on the right hand side of (1.7) and observing that $(p-1)p' = p$, we obtain

$$S \leqslant \left(\sum_{i=1}^n a_i^p\right)^{1/p}\left(\sum_{i=1}^n (a_i + b_i)^p\right)^{1/p'} + \left(\sum_{i=1}^n b_i^p\right)^{1/p}\left(\sum_{i=1}^n (a_i+b_i)^p\right)^{1/p'}$$
$$= \left[\left(\sum_{i=1}^n a_i^p\right)^{1/p} + \left(\sum_{i=1}^n b_i^p\right)^{1/p}\right] S^{1/p'}.$$

On dividing through by $S^{1/p'}$, we obtain (1.6).

H. Minkowski published this inequality in 1896, though his original proof[16] concerned convex bodies in n-dimensional Euclidean space. Both Theorem 1.1 and Theorem 1.2 can be generalized to infinite series and integrals, the proofs being similar to those of the finite

Theorem 1.3. Let $1 \leq p < \infty$ and $1/p + 1/p' = 1$. Then for every continuous linear functional $B$ on $L^p$, there exists a unique $g \in L^{p'}$ such that

$$B(f) = \int_A fg \, d\mu$$

for all $f$ belonging to $L^p$. The norm

$$\|B\| = \{\sup |B(f)| : \|f\|_p \leq 1\}$$

satisfies

$$\|B\| = \|g\|_{p'}.$$

The proof is a consequence of the Radon-Nikodym Theorem [23]. Thus, the Banach space dual $(L^p)^*$ of $L^p$ is isometrically isomorphic with $L^{p'}$ for all p, $1 \leq p < \infty$. F. Riesz's original proof was based on the characterization of the absolutely continuous functions whose derivatives belong to $L^p$, and did not include the case $p=1$. In 1918, H. Steinhaus published the result for $p=1$ [25]. Theorem 1.3 is not true when $p=\infty$. The dual space of $L^\infty$ cannot be identified with $L^1$ except in the case of $L^\infty$ being finite dimensional.

From Theorem 1.3 we obtain another characterization of $L^p$ spaces which, in a sense, is a converse of Holder's inequality.

Theorem 1.4. Let $1 < p < \infty$ and $1/p + 1/p' = 1$. Then $f$ belongs to $L^p$ if and only if

$$\int_A |fg| \, d\mu < \infty$$

for all $g$ in $L^{p'}$, and in that case

$$\|f\|_p = \sup\left\{\left|\int_A fg\,d\mu\right| : \|g\|_{p'} \le 1\right\}.$$

Also, there exists a function $g$ in $L^{p'}$ such that the supremum is attained. In this sense, Hölder's inequality is sharp.

Now let us consider a class of spaces more general than the $L^p$ spaces. Those spaces, known as Orlicz spaces, depend for their construction on the following inequality of W.H. Young [26].

<u>Theorem 1.5</u>. Let $\varphi$ be a continuous, strictly increasing function defined on the interval $[0,\infty)$ with the property that $\varphi(0) = 0$. Let $\psi$ be the inverse of $\varphi$. Then, for nonnegative $a$ and $b$,

(1.8) $$ab \le \Phi(a) + \Psi(b),$$

where $\Phi(x) = \int_0^x \varphi(u)\,du$ and $\Psi(y) = \int_0^y \psi(v)\,dv$. There is equality if and only if

$$\varphi(a) = b.$$

We will say that two such functions, $\Phi$ and $\Psi$, defined above, are complementary Young functions. Let us note that if

$$\varphi(u) = u^{1/p-1}$$

then (1.8) reduces to (1.5). Hence $x^p$ and $x^{p'}$ are complementary Young functions.

In 1932, W. Orlicz [18] showed that for each pair of complementary

Young functions there exist corresponding pairs of normed linear spaces of measurable functions which have many properties similar to those of the $L^p$ spaces. In particular, there is a Hölder inequality for Orlicz spaces which is based on the Young inequality (1.8). These matters will be discussed in Chapter III.

In the early fifties, G.G. Lorentz [11][12] introduced a new class of function spaces. These new spaces, the so-called Lorentz spaces, are the main topic of Chapter IV. They are generalizations of $L^p$ spaces, though they are related to $L^p$ spaces in a very different way from the way Orlicz spaces are related to $L^p$ spaces. The corresponding Hölder inequalities depend on a number of elementary results relating the averages of functions, due mainly to G.H. Hardy and J.E. Littlewood.

In the mid-1950's, a number of mathematicians including H.W. Ellis, I. Halperin, G.G. Lorentz and W.A.J. Luxemburg [4][8][11][12][13][14], developed an axiomatic theory of spaces of measurable functions which incorporates $L^p$ spaces, Orlicz spaces and Lorentz spaces. The resulting spaces, known as Banach function spaces, provide a broad framework within which the properties of the special classes can be studied in unison. Again we shall see that Hölder's inequality plays a decisive role. These general spaces will be considered in Chapter II.

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## CHAPTER II

## BANACH FUNCTION SPACES

### 1. Definitions and Elementary Properties

Much of the material in this chapter is taken from Luxemburg [14] and Bennett-Sharpley [2]. Throughout this chapter, $A$ will be a non-empty set, $\mathcal{A}$ will be a $\sigma$-algebra of subsets of $A$ and $\mu$ will be a nonnegative, countably additive measure. To avoid any solecisms, we shall assume that $\mu$ is neither identically zero nor identically infinite. We shall also assume that the measure space $(A, \mathcal{A}, \mu)$ is complete and totally $\sigma$-finite.

In that case, $A$ is the union of a countable collection of sets of finite measure. Let us choose, and fix once and for all, such a collection of sets of finite measure, say $(A_i)_{i=1}^{\infty}$, with $A_1 \subseteq A_2 \subseteq \ldots$ A set $E$ is defined to be bounded if it is contained in one of the sets $A_n$, for some $n$.

Let $\mathcal{M}^+$ be the set of $\mu$-measurable functions whose values lie in the interval $[0,\infty]$.

<u>Definition 2.1.1.</u> A mapping $\rho : \mathcal{M}^+ \to [0,\infty]$ is called a <u>Banach function norm</u>, or simply a function norm, if the following properties hold:

P1) $\rho(f) = 0$ if and only if $f = 0$ $\mu$-a.e.; $\rho(f + g) \leq \rho(f) + \rho(g)$;

where $C_E = \mu(E)^{1/p'}$.

Let $\mathfrak{M}$ denote the collection of all $\mu$-measurable, complex valued functions, modulo sets of measure zero.

Definition 2.1.2. Let $\rho$ be a Banach function norm. The set $X$ of all functions $f$ for which $\rho(|f|)<\infty$ is called a Banach function space. For each $f$ belonging to $X$ we define the norm of $f$ to be

$$\|f\|_X = \rho(|f|).$$

Theorem 2.1.3. Under the definitions above and under the usual vector space operations, $(X, \|\cdot\|_X)$ is a normed linear space.

The proof is evident. The next theorem shows that Fatou's Lemma has a direct analogue in Banach function spaces.

Theorem 2.1.4. Let $X$ be a Banach function space with norm $\|\ \|_X$ and suppose that $f_n$ belongs to $X$ for $n=1,2,\ldots$

i) If $0\leq f_n \uparrow f$ $\mu$-a.e., then either $f$ does not belong to $X$ and $\|f_n\|_X \uparrow \infty$ or $f$ belongs to $X$ and $\|f_n\|_X \uparrow \|f\|_X$.

ii) (Fatou's Lemma) If $f_n \to f$ $\mu$-a.e., and if

$$\liminf_{n\to\infty} \|f_n\|_X < \infty,$$

then $f$ is in $X$ and

$$\|f\|_X \leq \liminf_{n\to\infty} \|f_n\|_X.$$

Proof. The first part is just a restatement of P3). For the second part, let

$$h_n(x) = \inf_{m \geq n} |f_m(x)|$$

so that

$$0 \leq h_n \nearrow |f| \ \mu\text{-a.e.}$$

By the lattice property P2) and P3),

$$\rho(|f|) = \lim_{n\to\infty} \rho(h_n) \leq \lim_{n\to\infty} (\inf_{m\geq n} \rho(|f_m|)) = \liminf_{n\to\infty} \|f_n\|_X \quad .$$

Since $f$ is measurable, $f$ belongs to $X$ and

$$\|f\|_X \leq \liminf_{n\to\infty} \|f_n\|_X.$$

A consequence of Fatou's lemma is that all Banach function spaces are complete.

<u>Theorem 2.1.5.</u> Let $X$ be a Banach function space. Suppose $f_n$ belongs to $X$ for all natural numbers $n$ and

$$\sum_{n=1}^{\infty} \|f_n\|_X < \infty.$$

Then $\sum_{n=1}^{\infty} f_n$ converges in $X$ to a function $f$ in $X$ and

$$\|f\|_X \leq \sum_{n=1} \|f_n\|_X.$$

Proof. Let $t = \sum_{n=1}^{\infty} |f_n|$, $t_N = \sum_{n=1}^{N} |f_n|$ $(N = 1,2,\ldots)$

Since $\|t_N\|_X \leq \sum_{n=1}^{N} \|f_n\|_X \leq \sum_{n=1}^{\infty} \|f_n\|_X,$

it follows from Theorem 2.1.4 that $t$ belongs to $X$. $\sum_{n=1}^{\infty} |f_n(x)|$ converges pointwise $\mu$-a.e. and hence so does $\sum_{n=1}^{\infty} f_n(x)$. Thus, if

$$f = \sum_{n=1}^{\infty} f_n ,$$

and

$$S_N = \sum_{n=1}^{N} f_n$$

for $N = 1,2,\ldots$, then $S_N \to f$ $\mu$a.e. as $N\to\infty$. Furthermore,

$$\liminf_{N\to\infty} \|S_N - S_M\|_X \leq \liminf_{N\to\infty} \sum_{n=M+1}^{N} \|f_n\|_X = \sum_{n=M+1}^{\infty} \|f_n\|_X$$

which tends to $0$ as $M\to\infty$ because

$$\sum_{n=1}^{\infty} \|f_n\|_X$$

converges. It then follows from Fatou's lemma that $f-S_M$ belongs to $X$ (hence so does $f$ ) and $\|f-S_M\|_X \to 0$ as $M\to\infty$. So for $M = 1,2,\ldots,$ we have

$$\|f\|_X \leq \|f - S_M\|_X + \|S_M\|_X \leq \|f - S_M\|_X + \sum_{n=1}^{M} \|f_n\|_X .$$

By letting $M$ tend to infinity the desired result is obtained.

Corollary 2.1.6. All Banach function spaces are complete.

Proof. Let $f_n$ be a Cauchy sequence in a Banach space $X$. For each integer $k$ there is an integer $n_k$ such that

$$\|f_n - f_m\|_X < 2^{-k}$$

for all $n,m \geq n_k$, and the $n_k$'s may be chosen so that $n_{k+1} > n_k$. Then $(f_{n_k})_{k=1}^{\infty}$ is a subsequence of $(f_n)$ and we set

$$g_1 = f_{n_1}, \quad g_k = f_{n_k} - f_{n_{k-1}} \qquad (k>1).$$

Then $(g_k)$ is a sequence whose partial sums are $f_{n_k}$. But

$$\|g_k\|_X \leq 2^{-k+1}$$

if $k>1$ and so

$$\sum_{k=1}^{\infty} \|g_k\|_X \leq \|g_1\|_X + \sum_{k=2}^{\infty} 2^{-k+1} = \|g_1\|_X + 1.$$

Therefore,

$$\sum_{k=1}^{\infty} \|g_k\|_X$$

converges. By Theorem 2.1.5 there is an element $f$ in $X$ to which the partial sums of $g_k$ converge. But the $k^{th}$ partial sum is $f_{n_k}$ and so the subsequence $(f_{n_k})$ converges to $f$. The sequence $(f_n)$ is Cauchy and hence, given $\varepsilon > 0$, there is an $N$ such that

$$\|f_n - f_m\|_X < \tfrac{1}{2}\varepsilon$$

for all $n,m \geq N$. Since $f_{n_k} \to f$, there is a $K$ such that for all $k \geq K$ we have

$$\|f_{n_k} - f\|_X < \tfrac{1}{2}\varepsilon.$$

Let $K$ be sufficiently large so that $k \geq K$ and $n_k \geq N$. Then

$$\|f_n - f\|_X \leq \|f_n - f_{n_k}\|_X + \|f_{n_k} - f\|_X \leq \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon.$$

Thus we have shown that

$$\|f_n - f\|_X < \varepsilon$$

for all $n \geq N$ and so $f_n \to f$. Therefore, $X$ is complete.

Theorem 2.1.5 states that a Banach function space has the Riesz-Fischer property. In addition to Theorem 2.1.5, all that was used in the corollary was that $X$ is normed and linear. In fact, in any normed linear space, the Riesz-Fischer property is equivalent to completeness.

We round out this section with an observation about Banach function space topologies.

Theorem 2.1.7. Let $X$ and $Y$ be Banach function spaces over the same measure space. If $X \subseteq Y$, then in fact the inclusion map from $X$ to $Y$ is continuous, or, equivalently

$$\|f\|_Y \leq C\|f\|_X, \tag{2.1}$$

for all $f$ in $X$ and some constant $C$ independent of $f$.

Proof. Suppose $X \subseteq Y$ but (2.1) fails to hold. Then there exist functions $f_n$ in $X$ for which

$$\|f_n\|_X \leq 1, \quad \|f_n\|_Y > n^3 \qquad (n = 1,2,\ldots).$$

Replacing each $f_n$ with its absolute value, we may assume that $f_n \geq 0$ for all $n$. It follows that

$$\sum_{n=1}^{\infty} n^{-2} f_n$$

converges in $X$ to some function $f$, by the Riesz-Fischer property. By hypothesis, $f$ also belongs to $Y$. But this is impossible because

$$0 \leq n^{-2} f_n \leq f$$

and so

$$\|f\|_Y \geq n^{-2} \|f_n\|_Y \geq n,$$

for all n. Hence, (2.1) must hold for some $C$ independent of $f$. This asserts that the inclusion map from $X$ to $Y$ is continuous.

Corollary 2.1.8. If two Banach function spaces consist of the same set of functions, then their norms are equivalent.

Proof. If $X=Y$ then $X \subseteq Y$ and by Theorem 2.1.7 there exists $C_1$ independent of $f$ such that

$$\|f\|_Y \leq C_1 \|f\|_X$$

for all $f$ in X. Also $Y \subseteq X$ and hence there exists $C_2$ independent of $f$ such that

$$\|f\|_X \leq C_2 \|f\|_Y$$

for all $f$ in X. This says that the norms are equivalent.

## 2. The Associate Space

Definition 2.2.1. If $\rho$ is a Banach function norm then its associate function norm $\rho'$ is defined on $\mathfrak{M}^+$ by

$$\rho'(g) = \sup\left\{\int_A fg\, d\mu : f \in \mathfrak{M}^+, \rho(f) \le 1\right\}$$

for $g$ in $\mathfrak{M}^+$.

The next theorem shows that the associate norm defined above is in fact a Banach function norm.

Theorem 2.2.2. Let $\rho$ be a Banach function norm. Then the associate norm $\rho'$ is itself a Banach function norm.

Proof.

P1) If $\rho(f) \le 1$ then $f$ is finite $\mu$-a.e. Hence $g = 0$ $\mu$-a.e. implies

$$\int_A fg\, d\mu = 0$$

and thus $\rho'(g) = 0$. Conversely, if $\rho'(g) = 0$ then

$$\int_A fg\, d\mu = 0$$

for all $f$ in $\mathfrak{M}^+$ with $\rho(f) \le 1$. If $E$ is any bounded subset of $A$ then $\mu(E) < \infty$ and if $0 < \mu(E)$, properties P1) and P4) for $\rho$ give $0 < \rho(\chi_E) < \infty$. Setting

$$f = \frac{\chi_E}{\rho(\chi_E)}$$

gives

$$\int_A fg\,d\mu > \xi ,$$

by definition of $\rho'(g)$. Since $0 \le fg_n \uparrow fg$ $\mu$-a.e., the monotone convergence theorem shows that

$$\int_A fg_n\,d\mu \uparrow \int_A fg\,d\mu .$$

Hence, there is an integer $N$ such that

$$\int_A fg_n\,d\mu > \xi$$

for all $n \geq N$ and so for such values of $n$, $\rho'(g_n) > \xi$. This shows that $\rho'(g_n) \uparrow \rho'(g)$.

P4) If $E$ is a bounded set, then P5) for $\rho$ gives us a constant $C_E < \infty$ for which

$$\int_A \chi_E f\,d\mu \le C_E \rho(f).$$

This gives

$$\rho'(\chi_E) \le C_E \rho(f) \le C_E$$

if $\rho(f) \le 1$. This establishes P4).

P5) Let us fix a bounded set $E$ and so $\mu(E) < \infty$. Without loss of generality, we may assume that $\mu(E) > 0$. Hence P1) and P4) for show that $C_E{}'$ ( $= \rho(\chi_E)$ ) satisfies $0 < C_E{}' < \infty$. The function

$$f = \chi_E / \rho(\chi_E)$$

clearly has the property that $\rho(f) = 1$ and so for any $g$ in $\mathfrak{M}^+$,

$$\int_E g\,d\mu = C_E{}' \int_A fg\,d\mu \le C_E{}' \rho'(g)$$

and so P5) holds.

Definition 2.2.3. Let $\rho$ be a Banach function norm and let $X$ be the Banach function space induced by $\rho$. Let $\rho'$ be the associate norm of $\rho$. The Banach function space $X'$ induced by $\rho'$ is called the associate space of $X$.

From Definition 2.2.3, the norm of a $\mu$-measurable function $g$ in $X'$ is given by

$$\|g\|_{X'} = \sup \left\{ \int_A |fg| \, d\mu : f \in X, \ \|f\|_X \leqslant 1 \right\}.$$

Theorem 2.2.4. ( Hölder's Inequality ) Let $X$ be a Banach function space with associate space $X'$. If $f$ belongs to $X$ and $g$ belongs to $X'$ then $fg$ is integrable and

$$\int_A |fg| \, d\mu \leqslant \|f\|_X \cdot \|g\|_{X'}. \tag{2.2}$$

Proof. If $\|f\|_X = 0$ then the result is trivial. If $\|f\|_X \neq 0$ then the norm of the function $f/\|f\|_X$ is 1, and so

$$\int_A \left|(f/\|f\|_X)g\right| \, d\mu \leqslant \|g\|_{X'}.$$

Multiplying through by $\|f\|_X$ gives (2.2), the desired result.

By the definition of $\|g\|_{X'}$, this inequality is sharp in the sense that if $g$ belongs to $X'$ and $\varepsilon > 0$, there exists an element $f$ of $X$ such that $\|f\|_X = 1$ and

$$0 \leqslant \|g\|_{X'} - \left| \int_A fg \, d\mu \right| = \|g\|_{X'} \cdot \|f\|_X - \left| \int_A fg \, d\mu \right| < \varepsilon.$$

The next theorem shows that every Banach function space coincides with its second associate space. First, we need a lemma which is a partial converse of Hölder's inequality, analogous to Theorem 1.4.

Lemma 2.2.5. If $fg$ is integrable for every $f$ in $X$ then $g$ belongs to $X'$.

Proof. Assume the lemma is false. Then we would have a function $g$ not in $X'$ ( i.e. $\rho'(g) = \infty$ ), but $fg$ is integrable for every $f$ in $X$. We can find nonnegative functions $f_n$ such that $\|f_n\|_X \leq 1$ and

$$\int_A |f_n g| \, d\mu > n^3$$

for every natural number n. Therefore, by Theorem 2.1.5, the function

$$f = \sum_{n=1}^{\infty} f_n/n^2$$

is in $X$. But

$$\int_A |fg| \, d\mu \geq n^{-2} \int_A |f_n g| \, d\mu > n.$$

If this is true for every natural number $n$ then $fg$ cannot be integrable. This contradiction proves the lemma.

Theorem 2.2.6. ( G.G. Lorentz - W.A.J. Luxemburg ) A function $f$ belongs to $X$ if and only if it belongs to $X''$ ( $= (X')'$ ) and in this case $\|f\|_X = \|f\|_{X''}$ .

Proof. If $f$ is in $X$ then $fg$ is integrable for every $g$ in $X'$. Therefore, by the preceding lemma, $f$ belongs to $X''$ also, and

$$\|f\|_{X''} = \sup\left\{\int_A |fg|\,d\mu : \|g\|_{X'} \le 1\right\} \le \|f\|_X ,$$

the inequality resulting from Hölder's inequality. Hence $X \subseteq X''$ and $\|f\|_{X''} \le \|f\|_X$ . For the converse, let us recall the collection $(A_i)_{i=1}^{\infty}$ of increasing sets of finite measure whose union is A. For each natural number n and each f in X", let

$$f_n(x) = \min(|f(x)|, n)\cdot\chi_{A_n}(x).$$

Since $0 \le f_n \le n\chi_{A_n}$ , and the latter function belongs to X, $f_n$ belongs to X and thus to X". By the way the functions $f_n$ are defined, $0 \le f_n \uparrow f$ and so it will suffice to show that

$$\|f_n\|_X \le \|f_n\|_{X''} \qquad (n = 1,2,\ldots).$$

Let us fix n and f. We may assume that $\|f_n\|_X > 0$. Let $L_n^1$ and $L_n^{\infty}$ denote the spaces of all $\mu$-integrable functions and bounded functions, respectively, having supports in $A_n$. $L_n^1$ is a Banach space with the norm given by

$$\|g_n\| = \int_{A_n} |g|\,d\mu , \qquad (g \in L_n^1).$$

If S denotes the closed unit ball in X then the set $U = S \cap L_n^1$ is a closed, convex subset of $L_n^1$. If $\lambda > 1$ is any real number then the function

$$g = \frac{\lambda f_n}{\|f_n\|_X} .$$

belongs to $L_n^1$ but not U. By the Hahn-Banach theorem, a closed hyperspace separates $U$ and $g$. Therefore, there exists a non-zero $\varphi \varepsilon L_n^\infty$ such that

$$\mathrm{Re}\left(\int_{A_n} \varphi h \, d\mu\right) < \gamma < \mathrm{Re}\left(\int_{A_n} \varphi g \, d\mu\right)$$

for some real number $\gamma$ and all $h$ in $U$. This can be written as

$$\sup_{h \varepsilon U} \int_{A_n} |\varphi h| \, d\mu \leq \gamma < \mathrm{Re}\left(\int_{A_n} \varphi g \, d\mu\right) \leq \int_{A_n} |\varphi g| \, d\mu.$$

Now, any $h$ in $S$ is the pointwise limit of the increasing sequence of truncations

$$h_n(x) = \min\,(h(x), n) \cdot \chi_{A_n}(x),$$

each of which is in $L_n^1$ and hence in $U$. By the monotone convergence theorem, we can replace the supremum over $U$ by the supremum over $S$, obtaining

$$\|\varphi\|_{X'} = \sup_{h \varepsilon S} \int_{A_n} |\varphi h| \, d\mu \leq \gamma < \frac{\lambda}{\|f_n\|_X} \int_{A_n} |\varphi f_n| \, d\mu .$$

Rearranging terms gives

$$\|f_n\|_X < \lambda \int_{A_n} \left| f_n \frac{\varphi}{\|\varphi\|_{X'}} \right| d\mu \leq \lambda \|f_n\|_{X''}$$

by Holder's inequality. By letting $\lambda$ tend to 1, the desired result is obtained.

For $p > 1$, the associate space of the Banach function space $L^p$ is $L^{p'}$. This raises the question of when the associate space of an arbitrary Banach function space is equal to the Banach space dual, since the Banach space dual $L^{p*}$ of $L^p$ is also $L^{p'}$. The next theorem gives a partial answer but first we need a lemma.

Lemma 2.2.7. The norm of a function $g$ in the associate space $X'$ is given by

$$\|g\|_{X'} = \sup\left\{\left|\int_A fg\, d\mu\right| : f\in X,\ \|f\|_X \leq 1\right\}.$$

Proof. Since

$$\left|\int_A fg\, d\mu\right| \leq \int_A |fg|\, d\mu,$$

it is always the case that

$$\|g\|_{X'} \geq \sup\left\{\left|\int_A fg\, d\mu\right| : f\in X,\ \|f\|_X \leq 1\right\},$$

Hence it remains only to prove the reverse inequality which is equivalent to showing

$$\sup_{f\in S} \int_A |fg|\, d\mu \leq \sup_{f\in S}\left|\int_A fg\, d\mu\right|,$$

where both suprema extend over the unit ball $S$ of $X$. On the set

$$E = \{x\in A : g(x) \neq 0\}$$

we can write $g$ in the polar form

$$g(x) = |g(x)|\, \phi(x),$$

where $|\phi(x)| = 1$. Hence

$$|g(x)| = g(x)\, \overline{\phi(x)}$$

on $E$. Thus, for any $f$ in $S$,

$$\int_A |fg|\, d\mu = \int_E |fg|\, d\mu = \int_E |f|\overline{\phi} g\, d\mu .$$

If $h = |f|\overline{\phi}$ on $E$ and $h = 0$ off $E$, then $|h| \leq |f|$ on $A$ and also, $h$ is in $S$. Hence,

$$\int_A |fg|\, d\mu = \int_A hg\, d\mu \leq \left|\int_A hg\, d\mu\right| \leq \sup_{f\in S}\left|\int_A fg\, d\mu\right|.$$

We establish the result by taking the supremum on the left over all functions $f$ in $S$.

Recall that a closed linear subspace $W$ of the dual space $X^*$ of a Banach space $X$ is said to be norm-fundamental if

$$\|f\|_X = \sup\left\{|L(f)| : L \in W, \|L\|_{X^*} \leq 1\right\}$$

for every $f$ in $X$. In other words, $W$ must contain sufficiently many functionals to reproduce the norm of every function in $X$.

Theorem 2.2.8. The associate space $X'$ of a Banach function space $X$ is canonically isometrically isomorphic to a closed norm-fundamental subspace of the Banach space dual $X^*$ of $X$.

Proof. For each $g$ in $X'$, the linear functional $L_g$ on $X$, defined by

$$L_g(f) = \int_A fg \, d\mu, \qquad (f \in X), \tag{2.2}$$

is bounded. By Holder's inequality,

$$L_g = \sup_{\|f\|_X \leq 1} \left| \int_A fg \, d\mu \right| = \sup_{\|f\|_X \leq 1} \int_A |fg| \, d\mu \leq \|g\|_{X'} .$$

Furthermore, if $L_g(f) = 0$ for all $f$ in $X$, then $\|g\|_{X'} = 0$ and hence $g = 0$ $\mu$-a.e. Therefore, the map $g \to L_g$ is an isomorphism of $X'$ onto a linear subspace of $X^*$. It is isometric because

$$\|L_g\|_{X^*} = \sup\left\{|L_g(f)| : \|f\|_X \leq 1\right\} = \|g\|_{X'} . \tag{2.3}$$

Since $X'$ is complete, the isomorphism has closed range in $X^*$. Finally, if $f$ belongs to $X$, then

$$\|f\|_X = \|f\|_{X''} = \sup\left\{\left|\int_A fg\, d\mu\right| : g \in X',\ \|g\|_{X'} \leq 1\right\}$$
$$= \sup\left\{\left|L_g(f)\right| : g \in X',\ \|L_g\|_{X^*} \leq 1\right\},$$

by Theorem 2.2.6 and Lemma 2.2.7. Using (2.2) and (2.3), we obtain

$$\|f\|_X = \sup\left\{\left|L_g(f)\right| : g \in X',\ \|L_g\|_{X^*} \leq 1\right\}.$$

This shows that the image of $X'$ is norm-fundamental in $X^*$.

In general, the canonical mapping from $X'$ to $X^*$ is not surjective. If $X = L^\infty$ then $X' = L^1$ is a proper subspace of $X^*$.

## 3. Absolute Continuity of the Norm

Throughout this section $(E_n)_{n=1}^{\infty}$ will denote an arbitrary sequence of $\mu$-measurable subsets of A. The notation $E_n \to \emptyset$ $\mu$-a.e. will mean that the characteristic function $\chi_{E_n}$ converges to 0 pointwise $\mu$-a.e. Recall that $(A_n)_{n=1}^{\infty}$ is the sequence of increasing subsets of A whose union is A.

Definition 2.3.1. A $\mu$-measurable function f in a Banach function space X is said to have absolutely continuous norm in X if the following hold:

a) If E is bounded, and $E_n$ is a sequence of $\mu$-measurable subsets of E such that $E_n \to \emptyset$ as $n \to \infty$, then $\|f\chi_{E_n}\|_X \to 0$ as $n \to \infty$.

b) $\|f_{A-A_n}\|_X \to 0$ as $n \to \infty$.

As it turns out, the definition of absolute continuity is independent of the sets $A_n$. This claim is verified by the following lemma.

Lemma 2.3.2. An element f belonging to X has absolutely continuous norm if and only if f satisfies the following conditions :

a) Given $\varepsilon > 0$, there exists a number $\delta > 0$ such that $\mu(E) < \delta$ implies

$$\|f\chi_E\|_X < \varepsilon .$$

b) If the sequence of sets $(E_n)_{n=1}^{\infty}$ converges to a set of measure 0, then

$$\|f\chi_{E_n}\|_X \to 0$$

as $n\to\infty$.

Proof. If $f\varepsilon X$ satisfies the conditions of Lemma 2.3.2 then obviously $f$ has absolutely continuous norm. Conversely, let $f\varepsilon X$ have absolutely continuous norm and let $\mu(E_n)\to 0$. Then, if $\varepsilon > 0$ is given, there exists an integer $N$, depending on $\varepsilon$, such that

$$\|f\chi_{A-A_N}\|_X < \tfrac{1}{2}\varepsilon, \qquad (n\varepsilon N). \tag{2.4}$$

Write $E_n = E_n' \cup E_n''$, where $E_n' = E_n \cap A_n$ and hence, $E_n'' = E_n - E_n'$. Then

$$\|f\chi_{E_n}\|_X \leq \|f\chi_{E_n'}\|_X + \|f\chi_{E_n''}\|_X \leq \|f\chi_{E_n'}\|_X + \tfrac{1}{2}\varepsilon$$

for $n$ sufficiently large, since $E_n' \subseteq A_n$ and $\mu(E_n')\to 0$.

Next, let $E_n$ converge to a set of measure zero. Given $\varepsilon > 0$, and $N$ such that (2.4) holds, we have

$$\|f\chi_{E_n}\|_X + \|f\chi_{E_n\cap A_n}\|_X + \|f\chi_{E_n\cap(A-A_n)}\|_X \leq \|f\chi_{E_n\cap A_n}\|_X + \tfrac{1}{2}\varepsilon .$$

Since the sequence $E_n \cap A_n$ of subsets of $A_n$ converge to a set of measure zero, we obtain

$$\|f\chi_{E_n}\|_X < \varepsilon$$

for $n$ sufficiently large.

Definition 2.2.3. Let $X^{\alpha}$ consist of all f in X which possess absolutely continuous norm.

The space X is said to have absolutely continuous norm whenever every $f \in X$ has absolutely continuous norm, i.e. $X = X^{\alpha}$. Clearly it is true that $f \in X^{\alpha}$, $g \in X$ and $|g| \leq |f|$ $\mu$-a.e. implies $g \in X^{\alpha}$. Therefore, if $f \in X$, then $f \chi_E \in X^{\alpha}$ for every $\mu$-measurable set E.

Theorem 2.3.4. The subspace X of functions of absolutely continuous norm is a closed linear subspace of the Banach function space X.

Proof. All that is needed to show is that $X^{\alpha}$ is closed in X. Suppose $f_n \in X^{\alpha}$ for natural numbers n and that $f_n \to f$ in X. Then, given $\varepsilon > 0$, it must be the case that

$$\|f - f_N\|_X < \tfrac{1}{2}\varepsilon$$

for some sufficiently large N. Suppose $(E_m)_{m=1}^{\infty}$ is a decreasing sequence satisfying $E_m \to \emptyset$ $\mu$-a.e. Since $f_N$ has absolutely continuous norm, for all m sufficiently large,

$$\|f_N \chi_{E_m}\|_X < \tfrac{1}{2}\varepsilon.$$

Hence,

$$\|f \chi_{E_m}\|_X \leq \|(f - f_N)\chi_{E_m}\|_X + \|f_N \chi_{E_m}\|_X$$
$$\leq \|(f - f_N)\|_X + \|f_N \chi_{E_m}\|_X \leq \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon,$$

for all sufficiently large $m$. This shows that

$$\|f\chi_{E_m}\|_X \searrow 0 \quad .$$

and hence that $f \in X^{\alpha}$. This shows that $X^{\alpha}$ is closed.

<u>Theorem 2.3.5</u>. If $f_n \in X$ for all natural numbers $n$, then $f_n$ converges strongly (i.e. in the norm topology) to an element $f$ in $X$ if and only if $f_n$ converges in measure to $f$ on each set $E$ of finite measure, and the norms of the $f_n$ are uniformly absolutely continuous.

Proof. In order to prove that the conditions are necessary, we have only to show that the norms of the $f_n$ are uniformly absolutely continuous. This is because the proofs of Theorem 2.1.5 and Corollary 2.1.6 show that strong convergence in $X$ implies convergence in measure on sets of finite measure. Given $\varepsilon > 0$, there exists an integer $N$ such that

$$\|f - f_n\|_X < \tfrac{1}{2}\varepsilon$$

for all $n \geq N$. Now let $E$ be bounded and $E_m \subseteq E$ such that $\mu(E_m) \to 0$ as $m \to \infty$. The norms of the $f_n$ are absolutely continuous and so, by Theorem 2.3.3, the norm of $f$ is also absolutely continuous. Thus, there exists an integer $M$ such that, for $m > M$,

$$\|f\chi_{E_m}\|_X < \tfrac{1}{2}\varepsilon \, .$$

This, together with

$$\|(f_n - f)\chi_{E_m}\|_X < \tfrac{1}{2}\varepsilon$$

for $n = 1,2,\ldots,N$ imply that, if $m > M$ and $n$ is arbitrary, then

$$\|f_n\chi_{E_m}\|_X < \|(f_n - f)\chi_{E_m}\|_X + \|f\chi_{E_m}\|_X < \varepsilon .$$

Showing that

$$\|f - f\chi_{A-A_n}\|_X \to 0$$

as $n \to \infty$ is similar.

We now turn to the proof that the conditions are sufficient. The hypothesis that the norms of the $f_n$ are uniformly absolutely continuous implies that, given $\varepsilon > 0$, there exists an integer $N$ such that

$$\|f_n\chi_{A-A_N}\|_X < \tfrac{1}{2}\varepsilon$$

for $n = 1,2,\ldots$ Hence, for all $m,n > N$,

$$\|(f_m - f_n)\chi_{A-A_N}\|_X \leq \|f_m\chi_{A-A_N}\|_X + \|f_n\chi_{A-A_N}\|_X < \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon .$$

For the same $\varepsilon$, let us write

$$E_{m,n} = \left\{x : |f_m(x) - f_n(x)| > \varepsilon\right\} \cap A_n .$$

Then,

$$\|(f_m - f_n)\chi_{A_N}\|_X \leq \|(f_m - f_n)\chi_{A_N - E_{m,n}}\|_X + \|(f_m - f_n)\chi_{E_{m,n}}\|_X$$

$$\leq \varepsilon\|\chi_{A_N}\|_X + \|(f_m - f_n)\chi_{E_{m,n}}\|_X .$$

By the convergence in measure on $A_N$ and the fact that the norms of of the $f_n$ are uniformly absolutely continuous,

$$\|(f_m - f_n)\chi_{A_n}\|_X$$

can be made arbitrarily small for sufficiently large m and n, so that

$$\limsup_{m,n} \|(f_m - f_n)\chi_{A_N}\|_X \leq \varepsilon \|\chi_{A_N}\|_X .$$

Since $\varepsilon > 0$ was arbitrary,

$$\|(f_m - f_n)\chi_{A_N}\|_X \to 0$$

as $m,n \to \infty$. Finally,

$$\|f_m - f_n\|_X \leq \|(f_m - f_n)\chi_{A-A_N}\|_X + \|(f_m - f_n)\chi_{A_N}\|_X$$

implies

$$\limsup_{m,n} \|f_m - f_n\|_X < \varepsilon .$$

Since $\varepsilon$ was chosen to be arbitrary,

$$\|f_m - f_n\|_X \to 0$$

as $m,n \to \infty$. But then $f_n$ converges strongly to some $g \varepsilon X$, and, from the convergence in measure of $f_n$ to $g$ on each set of finite measure, as well as to $f$, we can conclude that $f = g$ $\mu$-a.e. Hence,

$$\|f - f_n\|_X \to 0$$

as $n \to \infty$.

We come now to the main result of this section, that is, classification ~~of~~ the Banach function spaces whose associate spaces are equal to their Banach space duals. These are precisely the spaces which have absolutely continuous norm.

<u>Theorem 2.3.6</u>. In order that the dual space $X^*$ of a Banach function space $X$ be isometrically isomorphic with the associate space $X'$ of $X$ it is necessary and sufficient that $X = X^{\wedge}$.

Proof. By Theorem 2.2.8 and Holder's inequality, for $g$ in $X'$, every form $B_g$ defined by

$$B_g(f) = \int_A fg \, d\mu$$

is a bounded linear functional on $X$. In order to prove the sufficiency we need to show that every bounded linear functional on the Banach function space $X = X^{\wedge}$ may be uniquely written in this manner. Assume that $B$ is a bounded linear functional on $X$ and define the set function $F$ by

$$F(E) = B(\chi_E)$$

for all $\mu$-measurable subsets $E \subseteq A_1$. Since

$$|F(E)| \leq \|A\| \cdot \|\chi_E\|_X \to 0$$

as $\mu(E) \to 0$, there exists a $\mu$-integrable function $g$ on $A_1$ such that

$$B(\chi_E) = F(E) = \int_{A_1} \chi_E g \, d\mu ,$$

by the Radon-Nikodym Theorem. We extend $F(E)$ and the corresponding $g$ in the obvious way to every set $A_n$. Hence, for every bounded set $E$,

$$B(\chi_E) = F(E) = \int_A \chi_E g \, d \quad ,$$

from which it follows that

$$B(f) = \int_A fg \, d\mu$$

for all step functions $f$ vanishing outside a bounded set. For any $\mu$-measurable, nonnegative, bounded function $f$, vanishing outside a bounded set, there exists a sequence of step functions $f_n \geqslant 0$ such that $f_n \to f$ uniformly on $A$. By Theorem 2.3.4,

$$\|f - f_n\|_X \to 0$$

as $n \to \infty$. This implies that

$$B(f) = \int_A fg \, d\mu$$

for every such $f$ and the same is true if we remove the condition that $f$ be nonnegative. We claim that the function $g$ is in $X'$. Let $f$ be in $X$, and write

$$f_n(x) = |f(x)| / \operatorname{sgn} g(x)$$

for all $x \epsilon A_n$ such that $|f(x)| \leqslant n$, and $f_n(x) = 0$ otherwise. Then

$$\|f_n\|_X \leqslant \|f\|_X,$$

so

$$|B(f_n)| \leqslant \|B\| \cdot \|f_n\|_X \leqslant \|B\| \cdot \|f\|_X .$$

But

$$B(f_n) = \int_A f_n g \, d\mu = \int_A |f_n g| d\mu ,$$

so that, since

$$|f_n g| \nearrow |fg| \ \mu\text{-a.e.},$$

we have

$$\int_A |fg|\,d\mu = \lim_{n\to\infty} \int_A |f_n g|\,d\mu = \lim_{n\to\infty} B(f_n) \leq \|B\| \cdot \|f\|_X < \infty.$$

It follows that $fg$ is $\mu$-integrable for every $f$ in $X$. Hence, by Lemma 2.2.5, $g$ is in $X'$. All that remains to be shown is that for any $f$ belonging to $X^{\alpha} = X$,

$$B(f) = \int_A fg\,d\mu .$$

Let us take $f \in X^{\alpha} = X$ such that $f$ vanishes outside some $A_n$. For $n = 1,2,\ldots$, write $f_n = f$ for $|f| \leq n$, and $f_n = 0$ otherwise. Then, letting

$$E_n = \left\{ x : f_n(x) \neq f(x) \right\} ,$$

we have $\mu(E_n) \to 0$ as $n \to \infty$. Hence,

$$\|f - f_n\|_X = \|f \chi_{E_n}\|_X \to 0 ,$$

and

$$B(f) = \lim_{n\to\infty} B(f_n) = \lim_{n\to\infty} \int_A f_n g\,d\mu = \int_A fg\,d\mu$$

by dominated convergence. Next, let $f$ in $X^{\alpha} = X$ be arbitrary, and write $f_n = f$ on $A_n$ and $f_n = 0$ otherwise. Then

$$\|f - f_n\|_X = \|f \chi_{A-A_n}\|_X \to 0$$

as $n \to \infty$, and hence, once again,

$$B(f) = \int_A fg\,d\mu .$$

The uniqueness is clear from construction. The conclusion is that there exists an isometric isomorphism between $X^*$ and $X'$.

Now we will prove that the condition $X^{\alpha} = X$ is sufficient. It is always the case that $X^{\alpha} \subseteq X$. Assuming the inclusion to be proper, there exists a $\mu$-measurable function $f$ in $X$ which does not have absolutely continuous norm. Without loss of generality, we may assume that $f \geq 0$, for any $f$ is a linear combination of nonnegative functions. Since $f$ does not have absolutely continuous norm, it must be the case that either

a) $\|f \chi_{E_n}\|_X \to 0$ for every sequence $(E_n)_{n=1}^{\infty}$ with $E_n \to \emptyset$ $\mu$-a.e., or

b) $\|f \chi_{A-A_n}\|_X \to 0$ as $n \to \infty$,

fails to hold. Let us assume that a) fails to hold. The proof is similar in the case that b) is false. Thus, for some bounded set $E$ and some $\varepsilon > 0$, there exist subsets $E_n$ of $E$ such that

$$\mu(E_n) < n^{-2}$$

and

$$\|f \chi_{E_n}\|_X > \varepsilon$$

for all $n = 1,2,3,\ldots$ Writing

$$F_n = \bigcup_{i=n}^{\infty} E_i$$

gives a decreasing sequence $(F_n)_{n=1}^{\infty}$ with the properties that $\mu(F_n) \to 0$ and

$$\|f \chi_{F_n}\|_X > \varepsilon$$

for all $n$. Consider now the sequence $(G_n)_{n=1}^{\infty}$ defined by

$$G_n = \left\{ g \in X' : \left| \int_A f \chi_{F_n} g \, d\mu \right| < \varepsilon \right\} .$$

These sets $B_n$ constitute a weak* open covering of $X'$. But, by Alaoglu's theorem, the unit sphere $S'$ of $X'$ is compact in the weak* topology and so there exist a finite number of indices $n_1, n_2, \ldots, n_k$, with the property that for any $g \in S'$ there exists an index $n_i$, $1 \leq i \leq k$, depending on $g$ such that

$$\left| \int_A f \chi_{F_{n_i}} g \, d\mu \right| < \varepsilon .$$

Then, since $f \geq 0$ and $g \in S'$ implies $|g| \in S'$, it is also the case that

$$\int_A \left| f \chi_{F_{n_j}} g \right| d\mu < \varepsilon$$

for some $n_j$, $1 \leq j \leq k$. If $N = \max\,(n_1, n_2, \ldots, n_k)$, then for $n \geq N$,

$$\int_A \left| f \chi_{F_N} g \right| d\mu < \varepsilon$$

for every $g \in S'$ because $(F_n)_{n=1}^{\infty}$ is decreasing. Therefore,

$$\left\| f \chi_{F_n} \right\|_X \leq \varepsilon$$

for every $n \geq N$. This contradicts our assumption that

$$\left\| f \chi_{F_n} \right\|_X > \varepsilon$$

for all natural numbers $n$. Hence, Theorem 2.3.5 is proved.

This theorem also shows that the $L^p$ spaces for $1 \leq p < \infty$ have absolutely continuous norm. For the space $L^\infty$, however, the absolute continuity depends on the underlying measure space. For example, if the measure space is nonatomic then $(L^\infty)^\chi$ consists only of the zero function. If the measure space is completely atomic, as in the case of the natural numbers with counting measure, then $(\ell^\infty)^\chi$ consists of those sequences which converge to zero at infinity.

## CHAPTER III

## ORLICZ SPACES

### 1. Introduction to Orlicz Spaces

As particular examples of Banach function spaces we shall consider the Orlicz spaces. The Orlicz spaces are generalizations of the $L^p$ spaces. For simplicity, we shall take the measure space to be the interval $[0,1]$ of the real line with Lebesgue measure m.

Given two complementary Young functions $\Phi$, $\Psi$, defined in Theorem 1.5, let the Young class $P_\Phi$ be defined by

$$P_\Phi = \left\{ f : \int_0^1 \Phi|f|\, dm < \infty \right\} .$$

The class $P_\Psi$ is defined similarly. In 1926, W.H. Young [26], using his famous inequality (Theorem 1.5), found many properties of these classes. In general, the classes $P_\Phi$ and $P_\Psi$ are not linear. However, if the function $\Phi$ ( or $\Psi$ ) satisfies a certain growth condition, the $\Delta_2$-property, then the class is linear. In this case the class $P_\Phi$ consists of the same functions as the space $L_\Phi$, a normed linear space which was first defined by W. Orlicz [18] in 1932. In this setting, Holder's inequality becomes the following.

Theorem 3.1.1. Let $\Phi$ and $\Psi$ be complementary Young functions. Then for f belonging to $L_\Phi$ and g in $L_\Psi$, fg is integrable and

$$\int_0^1 |fg|\, dm \leq \|f\|_{L_\Phi} \cdot \|g\|_{L_\Psi} .$$

While the inequality is true, it is not sharp, in the sense that

$$\|f\|_{L^{\Phi}} \neq \sup\left\{\int_0^1 |fg|\,dm : \|g\|_{L^{\Psi}} \leq 1\right\} .$$

Another defect of the spaces introduced by Orlicz was that such spaces did not include spaces of the $L^1$ or $L^{\infty}$ type. This was remedied in 1949 by A.C. Zaanen [27] who extended Young's inequality to the case where $\Psi$ may jump to infinity. Finally, in 1955, W.A.J. Luxemburg renormed the spaces with the so-called Luxemburg norms $\|\cdot\|_{M\Phi}$ and $\|\cdot\|_{M\Psi}$. He showed that the following refinement of Holder's inequality is sharp.

<u>Theorem 3.1.2.</u> For all m-measurable functions $f$ belonging to $L_{\Phi}$ and $g$ belonging to $L_{\Psi}$,

$$\int_0^1 |fg|\,dm \leq \|f\|_{M\Phi}\|g\|_{\Psi}.$$

Also,

$$\|f\|_{M\Phi} = \sup\left\{\int_0^1 |fg|\,dm : \|g\|_{\Psi} \leq 1\right\} .$$

As a corollary to Theorem 3.1.2 we obtain

$$(L_{M\Phi})' = L_{\Psi} .$$

## 2. Preliminary Definitions

For $u \geq 0$, let $\varphi(u)$ be a nondecreasing function of $u$ such that $\varphi(0) = 0$. Then, for every $u \geq 0$, the number

$$\varphi(u+) = \lim_{h \downarrow 0} \varphi(u+h)$$

exists. Also, for every $u > 0$,

$$\varphi(u-) = \lim_{h \uparrow 0} \varphi(u+h)$$

exists. The function $\varphi$ is continuous precisely when $\varphi(u+) = \varphi(u-)$. If $\varphi(u+) \neq \varphi(u-)$ then $\varphi$ is said to have a jump at $u$. We shall assume from here on that

$$\varphi(u) = \varphi(u-)$$

for every $u \geq 0$, where, by convention, $\varphi(0-) = 0$; that is to say, $\varphi$ is left continuous. Then $\varphi$ is Lebesgue measurable on $[0,\infty)$, since the set on which $\varphi \leq a$ is either empty when $a<0$, or when $a \geq 0$, is a closed interval $[0,b]$ or the whole set $[0,\infty)$. The inverse $\psi$ of $\varphi$ is a well defined Lebesgue measurable function if we use the convention that $\psi \circ \varphi(u) = u$ when $\varphi$ is continuous at $u$ and if $\varphi$ makes a jump at $u = a$, then $\psi(v) = a$ for

$$\varphi(a-) < v \leq \varphi(a+).$$

Under these conditions, $\psi(0) = 0$ and if

$$\lim_{u \to \infty} \varphi(u) = \ell < \infty ,$$

then $\psi(v) = \infty$ for $v > \ell$. Hence, $\psi$ is nondecreasing for $v \geq 0$ and left continuous for the values of $v$ at which $\psi(v)$ is finite.

Definition 3.2.1. If the nondecreasing functions $\varphi$ and $\psi$, inverse to each other, satisfy the above conditions, then the functions $\Phi$ and $\Psi$, defined for $u \geq 0$, $v \geq 0$ by the integrals

$$\Phi(u) = \int_0^u \varphi(x)\, dx,$$

$$\Psi(v) = \int_0^v \psi(y)\, dy$$

are called complementary Young functions.

For completeness, let us restate Theorem 1.5 of the Introduction.

Theorem 3.2.2. ( Young's Inequality )
If $\Phi$ and $\Psi$ are complementary Young functions, then

$$ab \leq \Phi(a) + \Psi(b) \tag{3.1}$$

for all $a,b \geq 0$. There is equality if and only if $\varphi(a) = b$.

Interpreting the integrals as areas under curves, the theorem is readily apparent.

Since a necessary and sufficient condition that a function defined on an interval be convex is that it be the integral of a nondecreasing function, every function $\Phi$, defined on the nonnegative real numbers, which is convex and nonnegative and satisfies $\Phi(0) = 0$

may be considered as a Young function. That is to say, for every such function, there corresponds another function $\Psi$ with similar properties such that (3.1) holds.

For complementary Young functions, $\Phi$ and $\Psi$ , it is possible that $\Psi(v) = \infty$ for all $v > \ell$, where $\lambda$ is some finite number. However, this will occur only if $\lim_{u\to\infty} \varphi(u) = \ell$ . If $\Phi$ is such that

$$\frac{\Phi(u)}{u} \to \infty$$

as $u \to \infty$, then $\varphi(u) \to \infty$ also and this anomaly could not occur. Also, if $f$ is a m-measurable function, the function $\varphi|f|$ is also m-measurable, since

$$\{ x : \varphi|f(x)| \leqslant a \}$$

is empty for $a<0$, and equal to a set

$$\{ x : |f(x)| \leqslant b \}$$

or to

$$\{ x : |f(x)| < \infty \}$$

for any $a \geqslant 0$. In the same way, $\psi|f|$, $\Phi|f|$, and $\Psi|f|$ are also measurable.

<u>Definition 3.2.3</u>. By the Young class $P_\Phi$ we shall mean the class of all m-measurable functions $f$ such that

$$M_\Phi(f) = \int_0^1 \Phi|f|\,dm$$

is finite. The Young class $P_\Psi$ is defined similarly.

If $\Phi(u) = u^p/p$, $1\leq p<\infty$, the class $P_\Phi$ consists of the same functions as the Lebesgue space $L^p$. For $p>1$, $\Psi(v) = v^{p'}/p'$, so that the complementary class $L_\Psi$ consists of the same functions as $L^{p'}$. For $p=1$, $\Psi(v) = 0$ for $0\leq v\leq 1$ and $\Psi(v) = \infty$ for $v>1$. In this case, $P_\Psi$ consists of all m-measurable functions $f$ for which $|f| \leq 1$ $\mu$-a.e. Thus the classes $P_\Phi$ and $P_\Psi$ are, as a rule, not linear. We shall try to find necessary and sufficient conditions for $P_\Phi$ and $P_\Psi$ to be linear.

<u>Lemma 3.2.4</u>. Let $\Psi$ be a complementary Young function and $\Psi_1$ any other complementary Young function. Then, if $\Psi(v) = 0$ for $0\leq v\leq \ell<\infty$ and $\Psi(v) = \infty$ for $v>\ell$, we have $P_\Psi \subseteq P_{\Psi_1}$ if and only if $\Psi_1(\ell)<\infty$. In all other cases $P_\Psi \subseteq P_{\Psi_1}$ if and only if there exist two positive constants $a$ and $b$ such that $\Psi(a)<\infty$ and $\Psi_1(v)\leq b\Psi(v)$ for all $v>a$.

Proof. If $\Psi(v) = 0$ for $0\leq v\leq \ell<\infty$ and $\Psi(v) = \infty$ for $v>\ell$ then $P_\Psi$ consists of all measurable functions $f$ such that $|f|\leq \ell$ a.e. Hence, if $\Psi_1(\ell)<\infty$, then

$$\int_0^1 \Psi_1|f|\ dm \leq \int_0^1 \Psi_1(\ell)\ dm = \Psi_1(\ell)<\infty,$$

Also, if $\Psi_1(\ell) = \infty$, then $M_\Psi(f) = \infty$. In all other cases, the sufficiency of the condition is evident. To prove the necessity, assume that the condition is not satisfied. That is, there exists an increasing sequence $(v_n)_{n=1}^{\infty}$ such that

$$0 < \Psi(v_n) < \infty$$

and

$$\Psi_1(v_n) > 2^n \Psi(v_n).$$

Let $E_n$ be a sequence of disjoint m-measurable subsets of a set $E_o$ contained in $[0,1]$ such that

$$m(E_n) = \frac{m(E_o)\,\Psi(v_1)}{2^n \Psi(v_n)} \quad . \tag{3.2}$$

If $E_o$ is any set of positive measure then clearly the sets $E_n$ can be found to satisfy (3.2) since

$$\sum_{n=1}^{\infty} m(E_n) \leq m(E_o) \quad .$$

If $f(x) = v_n$ for $x \in E_n$ and vanishes elsewhere, then

$$M_{\Psi}(f) = \int_0^1 \Psi|f|\,dm = \sum_{n=1}^{\infty} \Psi(v_n)\; m(E_n)$$

$$= \sum_{n=1}^{\infty} 2^{-n}\, m(E_o)\Psi(v_1) = m(E_o)\Psi(v_1) < \infty \;,$$

and

$$M_{\Psi_1}(f) = \int_0^1 \Psi_1|f|\;dm = \sum_{n=1}^{\infty} \Psi_1(v_n)\quad m(E_n)$$

$$\geq \sum_{n=1}^{\infty} 2^n \Psi(v_n)\; m(E_n) = \sum_{n=1}^{\infty} m(E_0)\,\Psi(v_1) = \infty \quad .$$

Therefore, $f$ belongs to $P_{\Psi}$ but not to $P_{\Psi_1}$, which is a contradiction.

If $1 \leq q < p < \infty$ then Lemma 3.2.4 shows that $L^p[0,1] \subseteq L^q[0,1]$ since $v^p \leq v^q$ for $v$ belonging to this interval.

<u>Theorem 3.2.5</u>. The Young class $P_\Psi$ is linear if and only if

$$P_{\Psi(v)} \subseteq P_{\Psi(2v)} ,$$

and in this case,

$$P_{\Psi(v)} = P_{\Psi(2v)} .$$

Proof. If $P_{\Psi(v)}$ is linear, then

$$\int_0^1 \Psi|f| \, dm < \infty$$

implies

$$\int_0^1 \Psi|2f| \, dm < \infty$$

and hence

$$P_{\Psi(v)} \subseteq P_{\Psi(2v)} .$$

Since $\Psi$ is convex, it is always the case that

$$P_{\Psi(2v)} \subseteq P_{\Psi(v)} .$$

Conversely, if the other inclusion holds and $f \in P_\Psi$ , then $2^k f \in P_\Psi$ for any integer $k \geq 1$. Hence $af \in P_\Psi$ for any complex constant $a$. Furthermore, if $f_1, f_2$ belong to $P_{\Psi(v)} \subseteq P_{\Psi(2v)}$, then

$$\int_0^1 \Psi|f_1 + f_2|\,dm = \int_0^1 \Psi\left(\tfrac{1}{2}(2|f_1| + 2|f_2|)\right)\,dm$$

$$\leq \tfrac{1}{2}\int_0^1 \Psi|2f_1|\,dm + \tfrac{1}{2}\int_0^1 \Psi|2f_2|\,dm < \infty,$$

by the convexity of $\Psi$ . Hence $f_1 + f_2$ belongs to $P_\Psi$ and it follows that $P_\Psi$ is linear.

If we choose $\Psi_1(v) = \Psi(2v)$ in Lemma 3.1.4 then we have necessary and sufficient conditions for the linearity of $P_\Psi$ . For reasons of convenience, we introduce some notation.

<u>Definition 3.2.6</u>. A Young function $\Psi$ is said to have the property $\Delta_2$ if there exist two positive constants a and b such that $\Psi(a)<\infty$ and

$$\Psi(2v) \leq b\Psi(v)$$

for all $v \geq a$. Hence, in this case, $\Psi(v)$ is finite for all v.

Combining Lemma 3.2.4, Theorem 3.2.5 and Definition 3.2.6, we obtain the following theorem.

<u>Theorem 3.2.10</u>. The Young class $P_\Psi$ is linear if and only if $\Psi$ satisfies $\Delta_2$.

3. Properties of Orlicz Spaces

In keeping with the theme of this thesis, we will present the Orlicz spaces as they were defined historically. In 1932, W. Orlicz [18] defined the following spaces.

Definition 3.3.1. Let $\Phi$ and $\Psi$ be complementary Young functions. If $f$ is any measurable function, define the numbers $\|f\|_\Phi$ and $\|f\|_\Psi$ by

$$\|f\|_\Phi = \sup\left\{\int_0^1 \Phi|fg|\,dm : M_\Psi(g) \leq 1\right\},$$

$$\|f\|_\Psi = \sup\left\{\int_0^1 \Psi|fg|\,dm : M_\Phi(g) \leq 1\right\}.$$

The Orlicz spaces $L_\Phi$ and $L_\Psi$ are defined to be the collection of all functions $f$ for which $\|f\|_\Phi$ and $\|f\|_\Psi$, respectively, are finite.

It is clear that both $L_\Phi$ and $L_\Psi$ are both linear spaces. If $f$ and $g$ are measurable, then by Young's inequality,

$$\int_0^1 |fg|\,dm \leq \int_0^1 \Phi|f|\,dm + \int_0^1 \Psi|g|\,dm \leq \int_0^1 \Phi|f|\,dm + 1,$$

provided that

$$\int_0^1 \Psi|g|\,dm \leq 1.$$

This shows that $P_\Phi \subseteq L_\Phi$. We wish to obtain conditions for the Young class $P_\Phi$ to be equal to the Orlicz space $L_\Phi$. A necessary and sufficient condition is that $\Phi$ have the $\Delta_2$-property.

Theorem 3.3.2. If $f$ belongs to $L_\Phi$ then there exists a constant $p>0$ such that $pf$ belongs to $P_\Phi$. More precisely, if $f$ is in $L_\Phi$ and $f = 0$ on a set of positive measure, then

$$\int_0^1 \Phi\left[|f|/\|f\|_\Phi\right] dm \leq 1.$$

A similar theorem holds for any $g$ in $L_\Psi$.

Proof. We shall first prove that

$$\Psi(qv) \geqslant q\Psi(v)$$

for $q \geqslant 1$, $v \geqslant 0$. We may assume that $q \geqslant 1$, $v \geqslant 0$, and then

$$\Psi(qv) - \Psi(v) = \int_v^{qv} \psi(x)dx \geqslant (q - 1)v\,\psi(v)$$

$$= (q - 1)\int_0^v \psi(x)dx$$

$$= (q - 1)\Psi(v).$$

Hence $\Psi(qv) \geqslant q\Psi(v)$. Let us assume now that $f$ belongs to $L_\Phi$ and $g$ belongs to $P_\Psi$. Then

$$\int_0^1 |fg|dm \leqslant \|f\|_\Phi ,$$

provided that $M_\Psi(g) \leq 1$. If $1 < M_\Psi(g) < \infty$, then by what we have just shown,

$$\Psi\ |g/M_\Psi(g)| \leq \Psi|g|/M_\Psi(g),$$

so that

$$\int_0^1 \Psi|g/M_\Psi(g)|\ dm \leqslant 1.$$

Hence,

$$\int_0^1 |fg|\,dm \leqslant \|f\|_\Phi\, M_\Psi(g)$$

because

$$\int_0^1 |fg/M_\Psi(g)|\ dm \leq \|f\|_\Phi.$$

Therefore, in any case,

$$\int_0^1 |fg|\,dm \leqslant \|f\|_\Phi\, M'_\Psi(g),$$

where $M'_\Psi(g) = \max(M_\Psi(g), 1)$. If $f$ is a bounded function then the bounded functions

$$\Phi\left[|f|/\|f\|_\Phi\right] \qquad \text{and } \Psi\left[\varphi\{|f|/\|f\|_\Phi\}\right]$$

are integrable over $[0,1]$. Hence Young's inequality becomes an equality for

$$g = \varphi\{|f|/\|f\|_\Phi\},$$

and

$$M'_\Psi(g) \geqslant \int_0^1 \left|\frac{f}{\|f\|_\Phi}\, g\right|\ dm = \int_0^1 \Phi\left[\frac{|f|}{\|f\|_\Phi}\right]\ dm + M_\Psi(g).$$

If $M'_\Psi(g) = M_\Psi(g)$, then we have

$$\int_0^1 \Phi\left[|f|/\|f\|_\Phi\right] dm = 0.$$

If $M'_{\Psi}(g) < M_{\Psi}(g)$, then $M'_{\Psi}(g) = 1$, so that

$$\int_0^1 \Phi\left[|f| / \|f\|_{\Phi}\right] dm \leqslant 1.$$

This is the desired result.

<u>Corollary 3.3.3</u>. If $\Phi$ has the $\Delta_2$-property, then the class $P_{\Phi}$ contains the same functions as $L_{\Phi}$.

Proof. We know already that $P_{\Phi} \subseteq L_{\Phi}$. Let $f$ belong to $L_{\Phi}$, where we may suppose that $\|f\|_{\Phi} = 0$. Since $\Phi$ has the $\Delta_2$-property, there exist constants $a$ and $b$ such that

$$\Phi(2u) \leqslant b\Phi(u)$$

for all $u \geqslant a$. We write $f = f_1 + f_2$, where $f_1 = f$ for all values of $x$ such that $|f| / \|f\|_{\Phi} < a$ and $f_1 = 0$ elsewhere. The function $\Phi|f_1|$ is bounded and therefore integrable. If $\|f_2\|_{\Phi} = 0$ then we are done. If $\|f_2\|_{\Phi} \neq 0$ then $f_2/\|f_2\|_{\Phi}$ belongs to $P_{\Phi}$ ; hence, since $\|f\|_{\Phi} \leqslant 2^p$ for a suitable nonnegative integer $p$ , we have

$$\Phi|f_2| \leqslant b^p \Phi\left[|f_2| / \|f_2\|_{\Phi}\right],$$

which shows that $f_2$ belongs to $P_{\Phi}$. Since

$$\Phi|f| = \Phi|f_1| + \Phi|f_2| ,$$

it must be the case that $f$ also belongs to $P_{\Phi}$.

Theorem 3.3.4. ( Hölder's Inequality )

If $L_\Phi$ and $L_\Psi$ are complementary spaces, f belongs to $L_\Phi$ and g belongs to $L_\Psi$, then fg is integrable over $[0,1]$, and

$$\left|\int_0^1 fg\, dm\right| \leq \int_0^1 |fg|\, dm \leq \|f\|_\Phi \cdot \|g\|_\Psi .$$

Proof. If $\|g\|_\Psi = 0$, the proof is trivial. If $\|g\|_\Psi \neq 0$, then

$$\int_0^1 |fg|\, dm = \int_0^1 \left| f \frac{g}{\|g\|_\Psi} \right| dm \cdot \|g\|_\Psi \leq \|f\|_\Phi \cdot \|g\|_\Psi,$$

by the definition of $\|f\|_\Phi$, since

$$\int_0^1 \Psi\left[|g| / \|g\|_\Psi\right] dm \leq 1.$$

However, as we have pointed out in the introduction to this chapter, this version of the Hölder inequality, using the so-called Orlicz norm, is not sharp. To get a sharp version, we need to introduce another equivalent norm on the Orlicz space, the Luxemburg norm. This norm makes each of the Orlicz spaces a Banach function space.

For every f belonging to $\mathfrak{M}^+$, we have defined

$$M_\Psi(f) = \int_0^1 \Psi|f|\, dm.$$

Some of the properties of M are:

a) $M_\Psi(0) = 0$;

b) $M_\Psi(kf) = 0$ for all $k \geq 0$ iff $f = 0$ a.e.;

c) $M_\Psi(kf) \leq 1$ for all $k \geq 0$ iff $f = 0$ a.e.;

d) If $\Psi(v)>0$ for all $v>0$, then $M_\Psi(f) = 0$ iff $f = 0$ a.e. ;

e) $M_\Psi( e^{ic} f ) = M_\Psi(f)$ for all real $c$;

f) If $0\leq a\leq 1$, then

$$M_\Psi( af + (1-a)g ) \leq aM_\Psi(f) + (1-a)M_\Psi(g);$$

g) $M_\Psi(kf)$ is a convex, left-continuous function of $k$ for $k>0$, and if $M_\Psi(af)<\infty$ for some $a>0$, then $M_\Psi(kf)$ is a finite, convex, continuous function of $k$ for $0\leq k\leq a$.

Definition 3.3.5. For every measurable function $f$, let us assign the nonnegative number $\rho(f)$ to $f$ by

$$\rho(f) = \inf k^{-1} ,$$

where the infimum is taken over all $k>0$ such that $M_\Psi(kf)\leq 1$. This number is called the Minkowski functional.

Theorem 3.3.6. The Minkowski functional $\rho$ is a Banach function norm.

Proof.

P1) $\rho(f) = 0$ iff $M_\Psi(kf)\leq 1$ for all $k>0$ iff $f = 0$ a.e. ; If $a=0$, then clearly $(af) = a\,(f)$. If $a\neq 0$, then

$$\rho(af) = \inf k^{-1} = \inf a(ak)^{-1} = a \inf (ak)^{-1} \tag{3.3}$$

where the infimum is taken over all positive numbers $k$ such that $M_\Psi(kf)\leq 1$. (3.3) shows that $\rho(af) = a\rho(f)$.

To prove the triangle inequality, let us assume that

$$\varrho(f_1) + \varrho(f_2) = c \quad .$$

If $c = 0$ the result is obvious. Otherwise, let $\varrho(f_1) = ac$, with $0 \leq a \leq 1$, and hence $\varrho(f_2) = (1-a)c$. Then

$$\begin{aligned} M_\Psi\left[\frac{f_1 + f_2}{c}\right] &= M_\Psi\left( a\left[\frac{f_1}{ac}\right] + (1-a)\left[\frac{f_2}{(1-a)c}\right] \right) \\ &\leq a\, M_\Psi\left[\frac{f_1}{ac}\right] + (1-a)\, M_\Psi\left[\frac{f_2}{(1-a)c}\right] \\ &\leq a\, M_\Psi\left[\frac{f_1}{\varrho(f_1)}\right] + (1-a)\, M_\Psi\left[\frac{f_2}{\varrho(f_2)}\right] . \\ &\leq a + (1-a) \\ &= 1. \end{aligned}$$

Therefore,

$$\varrho(f_1 + f_2) \leq C = \varrho(f_1) + \varrho(f_2).$$

P2). Assume $0 \leq g \leq f$ a.e. Since $\Psi$ is convex,

$$\Psi(kg) \leq \Psi(kf) \qquad (k \geq 0)$$

and hence, if $M_\Psi(kf) \leq 1$ then $M_\Psi(kg) \leq 1$. Then

$$\begin{aligned} \varrho(g) &= \inf_{k>0} k^{-1} \qquad (M_\Psi(kg) \leq 1) \\ &\leq \inf_{\ell>0} \ell^{-1} \qquad (M_\Psi(\ell f) \leq 1) \\ &= \varrho(f) . \end{aligned}$$

P3) Assume $0 \leq f_n \nearrow f$ a.e. and $\varrho(f_n) = a_n \nearrow a$. Without loss of generality, we may assume that $0 < a < \infty$. Since $a \geq a_n$ and

$$M_\Psi( f_n/a_n ) \leq 1,$$

it is also the case that

$$M_\Psi( f_n/a ) \leq 1$$

and hence

$$M_\Psi( f/a ) \leq 1$$

by Fatou's Lemma. Therefore,

$$\rho(f) \leq a = \lim_{n\to\infty} \rho(f_n) \leq \rho(f),$$

and hence

$$\rho(f_n) \nearrow \rho(f).$$

P4) If E is a bounded set and $\chi_E$ its characteristic function, then

$$\rho(\chi_E) = \inf k^{-1} ,$$

where the infimum is taken over all $k>0$ such that

$$\int_E \Psi(k)\ dm \leq 1.$$

If the range of $\Psi$ covers the whole interval $[0,\infty)$ then

$$\rho(\chi_E) = ( \Psi^{-1}( m(E)^{-1}))^{-1}.$$

However, if there exists a number $\ell>0$ such that $0 \leq \Psi(\ell)<\infty$ and $\Psi(v) = \infty$ for all $v>\ell$, then $\rho(\chi_E) = \ell^{-1}$ if

$$m(E)\cdot\Psi(\ell) \leq 1$$

and

$$\ell^{-1} < \rho(\chi_E) = (\Psi^{-1}( m(E)^{-1}))^{-1}$$

if

$$m(E)\cdot\Psi(\ell) > 1.$$

In any case, $\rho(\chi_E)=\infty$.

P5) Clearly, we may assume $0<\rho(f)<\infty$. If $k = 1/\rho(f)$ then

$$\int_0^1 \Psi(kf)\ dm \leq 1$$

and by Jensen's inequality,

$$\Psi\left[\frac{1}{m(E)}\int_E kf\ dm\right] \leq \frac{1}{m(E)}\int_E \Psi(kf)\ dm \leq \frac{1}{m(E)},$$

where $E$ is any bounded set. If the range of $\Psi$ is the whole interval $[0,\infty)$ then

$$\frac{1}{m(E)}\int_E kf\ dm \leq \Psi^{-1}(\,m(E)^{-1}) \ .$$

Hence,

$$\int_E f\cdot dm \leq \Psi^{-1}(\,m(E)^{-1})\cdot\, m(E)\cdot\,(1/k) = \Psi^{-1}(\,m(E)^{-1})\cdot\, m(E)\cdot\rho(f).$$

Choosing

$$C_E = \Psi^{-1}(\,m(E)^{-1})\ m(E),$$

we have that P5) holds. If, however, there does not exist a number $\ell>0$ such that $0\leq\Psi(\ell)<\infty$, and $\Psi(v)=\infty$ for all $v>\ell$, we again distinguish between the cases when $m(E)\Psi(\ell)>1$ and when $m(E)\Psi(\ell)\leq 1$. In the former case,

$$m(E)^{-1}<\Psi(\ell)$$

so $C_E$ can be chosen to be

$$C_E = \Psi^{-1}( m(E)^{-1} ) m(E)$$

and hence

$$\int_E f \, dm \leqslant \Psi^{-1}( m(E)^{-1} ) m(E) \, \varrho(f) = C_E \varrho(f).$$

In the latter case,

$$m(E)^{-1} \int_E kf \, dm \leqslant \ell .$$

Choosing $C_E$ to be $\ell \cdot m(E)$ gives the desired result.

Definition 3.3.7. The Banach function space $L_{M\Psi}$ is defined to be the set of all complex valued measurable functions such that

$$\|f\|_{M\Psi} = \varrho(|f|) < \infty ,$$

where, as usual, functions equal a.e. are identified.

If $\Psi(v) = v^p/p$ for $1<p<\infty$, then $L_{M\Psi} = L^p$, the Lebesgue space and we have our old example of a Banach function space. In this case,

$$M_\Psi(f) = \frac{1}{p}\int_0^1 |f|^p \, dm,$$

and hence,

$$M_\Psi(kf) = \frac{1}{p}\int_0^1 |kf|^p \, dm = \frac{k^p}{p} \int_0^1 |f|^p \, dm \leqslant 1.$$

This implies that

$$\left[\frac{1}{p}\right]^{1/p} \cdot \left[\int_0^1 |f|^p \, dm\right]^{1/p} \leqslant \frac{1}{k} .$$

From this we obtain

$$\|f\|_{M\Psi} = \left(\frac{1}{p}\right)^{1/p} \left[\int_0^1 |f|^p \, dm\right]^{1/p} = \left(\frac{1}{p}\right)^{1/p} \|f\|_p .$$

Here we notice that the complementary Young function is $\Phi(u) = u^{p'}/p'$. Therefore, $L_M = L^{p'}$. This is not by chance. We shall see that any pair of complementary Young functions generate Banach function spaces which are associate.

<u>Lemma 3.3.8</u>. $\|f\|_{M\Phi} \leqslant 1$ if and only if $M_\Psi(f) \leqslant 1$. More precisely, if $\|f\|_{M\Psi} \leqslant 1$, then

$$M_\Psi(f) \leqslant \|f\|_{M\Psi} ,$$

and if $\|f\|_{M\Psi} > 1$, then

$$M_\Psi(f) \geqslant \|f\|_{M\Psi}.$$

Proof. Let $k^{-1} = \|f\|_{M\Psi} \leqslant 1$. Then

$$k\, M_\Psi(f) \leqslant M_\Psi(kf) = M_\Psi\left[\frac{f}{\|f\|_{M\Psi}}\right] \leqslant 1 .$$

Hence,

$$M_\Psi(f) \leqslant k^{-1} = \|f\|_{M\Psi} .$$

Conversely, let $\|f\|_{M\Psi} > 1$. Then for any constant $k$ such that $1 < k < \|f\|_{M\Psi}$, we have

$$M_\Psi( k^{-1} f ) > 1$$

by the definition of $\|f\|_{M\Psi}$. Hence,

$$k^{-1} M_\Psi(f) \geq M_\Psi( k^{-1}f ) > 1,$$

and so,

$$M_\Psi(f) > k.$$

Since this is true for every $k < \|f\|_M$, we obtain

$$M_\Psi(f) \geq \|f\|_{M\Psi} .$$

If $\Phi$ has the $\Delta_2$-property then $P_\Phi = L_\Phi$. That is, $L_\Phi$ consists of those functions $f$ for which

$$\int_0^1 \Phi|f|\, dm < \infty .$$

It is routine to verify that $L_\Phi$ is also a Banach function space. As we will see from the following theorem, $L_\Phi$ is precisely the Banach function space $L_{M\Phi}$.

<u>Theorem 3.3.9</u>. $L_\Phi = L_{M\Phi}$, with equivalent norms.

Proof. It is a routine exercise to show that the norm $\|\cdot\|_\Phi$ makes $L_\Phi$ into a Banach function space. Therefore, in view of Corollary 2.1.8, to prove the theorem, it suffices to show that $L_\Phi$ and $L_{M\Phi}$ consist of the same set of functions. That is, $f$ belongs to $L_\Phi$ if and only if $M_\Phi(kf)<\infty$ for some constant $k>0$. Let $f$ be a measurable function such that $M_\Phi(kf)<\infty$ for some constant $k>0$ and $M_\Phi(g) \leq 1$. Then

$$\int_0^1 k|fg|\,dm \leqslant M_\Phi(kf) + 1,$$

by Young's inequality, and so

$$\|f\|_\Phi \leqslant k^{-1}\ ( M_\Phi(kf) + 1 ) < \infty .$$

Conversely, we shall show that

$$M_\Phi( f/\|f\|_\Phi ) \leqslant 1$$

for every $f$ belonging to $L_\Phi$ which does not vanish identically. Suppose that $f$ satisfies the above conditions. Then

$$\int_0^1 |fg|\,dm \leqslant \|f\|_\Phi \cdot M_\Psi(g)$$

if $M_\Psi(g) > 1$ by Holder's inequality and Lemma 3.3.8. Hence,

$$\int_0^1 |fg|\,dm \leqslant \|f\|_\Phi \cdot M'_\Psi(g),$$

where

$$M'_\Psi(g) = \max ( M_\Psi(g),\ 1 ).$$

First, let $f$ be bounded and vanishing outside the set $A_n$. Then

$$M_\Phi( f/\|f\|_\Phi )$$

and

$$M_\Psi \left(\varphi( |f| /\|f\|_\Phi)\right) < \infty .$$

Hence, since Young's inequality becomes equality for

$$g = \varphi( |f| / \|f\|_\Phi ),$$

it will be the case that

$$M'_\Psi(g) \geqslant \int_0^1 \left[f/\|f\|_\Phi\right] g\ dm = M_\Phi( f/\|f\|_\Phi ) + M_\Psi(g).$$

If

$$M'_{\Psi}(g) = M_{\Psi}(g),$$

then

$$M_{\Phi}(\ f/\|f\|_{\Phi}\ ) = 0 \leqslant 1\ ;$$

if

$$M'_{\Psi}(g) = 1,$$

then

$$M\ (\ f/\|f\|_{\Phi}\ ) + M\ (g) \leqslant 1,$$

and hence

$$M\ (\ f/\|f\|_{\Phi}\ ) \leqslant 1\ .$$

Now let $f$ in $L_{\Phi}$ be arbitrary. For natural numbers $n$, define $f_n$ by $f_n(x) = 0$ outside $A_n$ and $f_n(x) = f(x)$ for $x$ in $A_n$ if $|f(x)| \leqslant n$. If $|f(x)| > n$ for $x$ in $A_n$, then define $f_n(x) = n$. In this case,

$$|f_n| \nearrow |f|\ ,$$

$$M_{\Phi}(\ f_n/\|f_n\|_{\Phi}\ ) \leqslant 1,$$

and

$$\|f_n\|_{\Phi} \leqslant \|f\|_{\Phi}\ .$$

Hence,

$$M_{\Phi}(\ f_n/\|f\|_{\Phi}\ ) \leqslant 1,$$

and so

$$M_{\Phi}(\ f/\|f\|_{\Phi}\ ) \leqslant 1$$

by Fatou's lemma.

In Definition 3.3.1, we could have let

$$\|f\|_{\Phi} = \sup\left\{\int_0^1 |fg|\,dm : \|g\|_{M\Psi} \leq 1\right\},$$

in view of Lemma 3.3.8. This gives us the following theorem.

<u>Theorem 3.3.10</u>. Let $\Phi$ and $\Psi$ be complementary Young functions with $\Phi$ having the $\Delta_2$-property. Then the Orlicz space $L_\Phi$ is equal to $(L_{M\Psi})'$, the associate space of $L_{M\Psi}$, as defined in Chapter II.

Therefore, the following version of Hölder's inequality is sharp.

<u>Theorem 3.3.11</u>. For all measurable functions $f$ and $g$ belonging to $L_\Phi$ and $L_\Psi$, respectively, $fg$ is integrable and

$$\int_0^1 |fg|\,dm \leq \|f\|_{\Phi} \cdot \|g\|_{M\Psi} .$$

The reason for this version being the sharp version of Hölder's inequality and not the previous one is that $L_\Phi$ and $L_\Psi$ are not associate in the sense of Definition 2.2.3. That is, even though the spaces $L_\Phi = L_{M\Phi}$ are equivalent as Banach function spaces, the respective norms of an arbitrary function are not necessarily equal.

## 4. The Space $L \log^+ L$

Let us now consider the particular case where $\Psi(v) = v \log^+ v$; that is, $\Psi(v) = v \log v$ for $v$ in $(1,\infty)$ and $\Psi(v) = 0$ for $v$ in $[0,1]$. Then $\Psi$ is nonnegative, convex, it satisfies the condition $\Psi(0) = 0$ and we also have that $\Psi(v)/v$ tends to infinity as $v$ does. Therefore, $\Psi$ may be considered as a Young function and it is clear that $\Psi$ has the $\Delta_2$-property.

An easy calculation shows that its complementary Young function is

$$\Phi(u) = e^{u-1} - e^{-1} .$$

For any measurable function $f$ ,

$$M_\Phi(f) = e^{-1} \int_0^1 ( e^{|f|} - 1 ) \, dm$$

and so,

$$M_\Phi(kf) = e^{-1} \int_0^1 ( e^{k|f|} - 1 ) \, dm.$$

Thus, the Minkowski functional of $f$ is

$$\rho(f) = \inf_{k>0} k^{-1} ,$$

where

$$\int_0^1 ( e^{k|f|} - 1 ) \, dm \leqslant e. \tag{3.4}$$

So the associate space of $L \log^+ L$ is the class of all measurable functions $f$ for which (3.4) holds for some positive number $k$.

Accordingly, we make the following definition.

Definition 3.4.1. Let $L_{exp}$ be the class of all measurable functions whose exponent is integrable. More precisely, a measurable function $f$ will belong to $L_{exp}$ if

$$\int_0^1 e^{k|f|} dm < \infty$$

for some constant $k>0$.

From the previous section, it is clear that $L_{exp}$, with the Orlicz norm $\|\cdot\|_{M\Psi}$ is a normed linear space under the obvious operations.

Theorem 3.4.2. The Banach function space $L_{exp}$ is equal to the associate space of $L \log^+ L$.

Proof. It suffices to show that for any measurable function $f$, $f$ belongs to $L_{exp}$ if and only if it is in the associate space of $L \log^+ L$. Suppose that $f$ is in $(L\log^+ L)'$ ; that is, there exists a number $k>0$ such that

$$\int_0^1 ( e^{k|f|} - 1 ) \, dm \leq e.$$

This is equivalent to

$$\int_0^1 e^{k|f|}\,dm \leqslant e + 1 < \infty ,$$

and so $f$ belongs to $( L \log^+ L )'$.

Conversely, suppose that $f$ is in $L_{exp}$. Then by Holder's inequality, if $p>1$ and $1/p + 1/p' = 1$, then

$$\int_0^1 e^{k|f|}\,dm \leqslant \left(\int_0^1 \left[e^{k|f|}\right]^p dm\right)^{1/p}\left(\int_0^1 dm\right)^{1/p'} = \left(\int_0^1 e^{kp|f|}\,dm\right)^{1/p}.$$

Since $f$ is in $L_{exp}$, there exists $\lambda>0$ such that

$$\int_0^1 e^{\lambda|f|}\,dm = M < \infty.$$

Letting $\lambda=kp$ gives

$$\int_0^1 e^{\lambda|f|}\,dm \leqslant M^{1/p}.$$

We can always choose $p$ such that $M^{1/p} \leqslant e$. Therefore, $f$ belongs to $( L \log^+ L )'$.

More on the spaces $L \log^+ L$ and $L_{exp}$ will be discussed in Chapter IV.

## 5. Absolute Continuity of the Norm

The main purpose of this section is to investigate what conditions are necessary and sufficient in order that $L_\Phi$ have absolutely continuous norm.

Definition 3.5.1. A measurable function $f$ is said to be a finite element of $L$ if

$$M_\Phi(kf) < \infty$$

for every constant $k>0$. The class of all finite elements is denoted by $L_\Phi^f$.

A similar definition may be given for the complementary Orlicz space $L_\Psi$. It may happen that $\Psi(v) = \infty$ for all $v>\ell$, where $\ell<\infty$. That is to say, $\Psi$ jumps. This will happen if and only if $L_\Psi^f$ contains only the zero function. In order to prove this statement, let $\Psi(v) = \infty$ for $v>\ell$ and assume that there exists an element $f$ in $L_\Phi^f$ such that $f \neq 0$ on a set of positive measure. Then there exists a number $\varepsilon>0$ such that $|f|>\varepsilon$ on a set $E$ of positive measure and so

$$g = \varepsilon \chi_E$$

belongs to $L_\Psi^f$. But

$$M_\Psi(kg) = \Psi(k\varepsilon)$$

and so

$$M_\Psi(kg) = \infty$$

for all $k>\ell/\varepsilon$, which is a contradiction.

Conversely, if $L_{\Phi}^{f}$ consists only of the zero function and E is a set of positive measure, then

$$M_{\Psi}(\ k\chi_E\ ) = \infty$$

for some $k>0$; hence $\Psi(k) = \infty$. This shows that $\Psi(v) = \infty$ for sufficiently large v. If $\Psi$ does not jump then the theory for the function $\Psi$ is no different than for the function $\Phi$ . In view of the above, we shall assume from this point on that $\Psi$ does not have any jump discontinuities.

<u>Theorem 3.5.2</u>. $L_{\Phi}^{f} = L_{\Phi}^{\chi}$ .

Proof. Assuming that f belongs to $L_{\Phi}^{f}$ , we shall prove that the norm of f is absolutely continuous. Let $E_n$ be any decreasing sequence of subsets of $[0,1]$ such that $m(E_n)\to 0$. Then

$$g_n = \left| f\chi_{E_n} \right| \searrow 0 \text{ a.e.}$$

Hence, since

$$\Phi(\ kg_n\ ) \leq \Phi(\ kf\ ),$$

we have

$$M_{\Phi}(kg_n) \to 0$$

for any $k>0$ by dominated convergence. Therefore,

$$\|f\chi_{E_n}\|_{\Phi} = \|g_n\|_{\Phi} \to 0 \ .$$

It remains to prove that the same is true if $E_n$ is not necessarily decreasing. Assuming it to be false, there exists a number $\varepsilon > 0$ such that

$$\| f \chi_{E_n} \|_{\Phi} > \varepsilon$$

for some sequence $E_n$ contained in $[0,1]$ satisfying $m(E_n) \to 0$. We may assume that

$$m(E_n) < n^{-2}.$$

Then, if

$$F_n = \bigcup_{i=n}^{\infty} E_i ,$$

the sequence $F_n$ is decreasing, $m(F_n) \to 0$ and

$$\| f \chi_{F_n} \|_{\Phi} > \varepsilon ,$$

in contradiction to what has already been proved. The proof that

$$\| f \chi_{[0,1]-E_n} \|_{\Phi} \to 0$$

is similar.

The converse is evident since

$$M_{\Phi}(kf) \leq \Phi( kN )$$

for any $k>0$, where $N = \sup_{x \in [0,1]} |f(x)|$.

For any Young function $\Phi$, we have shown that $L^{f}_{\Phi} = L^{\chi}_{\Phi}$ ; hence the Orlicz space $L_{\Phi}$ has absolutely continuous norm if and only if all elements of $L_{\Phi}$ are finite elements. This, however, is equivalent to the linearity of the Young class $P_{\Phi}$ . On account of what we have proved in the first section of this chapter, we obtain the following theorem.

<u>Theorem 3.5.3</u>. Let $\Phi$ be a Young function. Then the Orlicz space $L_{\Phi}$ has absolutely continuous norm if and only if $\Phi$ has the $\Delta_2$-property.

Therefore, to investigate whether or not the Orlicz space $L_{\Phi}$ has its Banach space dual equal to its associate space, we need only know the Young function $\bar{\Phi}$ .

# CHAPTER IV

## LORENTZ SPACES

### 1. Preliminaries

In this chapter, we will discuss another class of Banach function spaces, namely the Lorentz spaces [11][12]. For simplicity, let us assume that the measure space is the interval $[0,1]$ of the real line with Lebesgue measure. First we need some preliminary definitions [24].

Definition 4.1.1. For $f$ belonging to $\mathcal{M}$, the distribution function of $f$ is

$$D_f(k) = m(\{x : |f(x)| > k\}),$$

for $k \geq 0$.

Note that it may be the case that $D_f(k)$ is infinite for some values of $k$.

Definition 4.1.2. Two functions $f$ and $g$ are said to be equimeasurable if

$$D_f(k) = D_g(k)$$

for all $k \geq 0$.

Definition 4.1.3. For a measurable function $f$, with $D_f(k)$ its distribution function, the decreasing rearrangement of $f$ is defined for $t \geq 0$ by

$$f^*(t) = \inf \left\{ k : D_f(k) \leq t \right\} .$$

Here we can see that measurable functions $f$ which have the property that $f^*(t) < \infty$ for all $t > 0$ are precisely those functions $f$ for which $D_f(k)$ tends to zero as $k$ tends to infinity.

Proposition 4.1.4. Let $f$ be a measurable function, $D_f$ its distribution and $f^*$ its decreasing rearrangement. Then the following hold:

a) $D_f$ and $f^*$ are both nonincreasing and right continuous;

b) $f$ and $f^*$ have the same distribution function, i.e. for all $k \geq 0$,

$$D_f(k) = D_{f^*}(k) .$$

Proof. It is clear that the function $D_f$ is nonincreasing since

$$\left\{ x \in A : |f(x)| > s_1 \right\} \subseteq \left\{ x \in A : |f(x)| > s_2 \right\}$$

when $s_1 > s_2$. Then it must be the case that $f^*$ is also nonincreasing, by its very definition. Continuity on the right for $D_f$ follows from the fact that

$$\lim_{s \uparrow s_0} \left\{ x \in A : |f(x)| > s \right\} = \left\{ x \in A : |f(x)| \geq s_0 \right\} .$$

In order to show that $f^*$ is continuous on the right at $t_0$ we first observe that this is immediate if $f^*(t_0) = 0$ ( since, in this case, $f^*(t) = 0$ for $t > t_0$ because $f^*$ is nonnegative and nonincreasing ). If $f^*(t_0)$ is positive, let $\alpha$ be such that

$$f^*(t_0) > \alpha > 0$$

and $(\varepsilon_n)_{n=1}^{\infty}$ a sequence of positive real numbers decreasing to 0. From the definition of $f^*$, it is the case that

$$D_f( f^*(t_0) - \alpha ) > t_0 .$$

Hence, there exists an integer $N$ such that

$$D_f( f^*(t_0) - \alpha ) > t_0 + \varepsilon_n$$

for all $n \geq N$. But this implies

$$f^*(t_0) - \alpha < f^*( t_0 + \varepsilon_n )$$

for all natural numbers $n$. Otherwise, for some $n$, we would have

$$f^*(t_0) - \alpha \geq f^*( t_0 + \varepsilon_n) .$$

Since $D_f$ and $f^*$ are both nonincreasing and $D_f$ is right continuous, it follows that

$$D_{f^*}(t) \leq t$$

for all $t > 0$ and this implies

$$D_f( f^*(t_0) - \alpha ) \leq D_{f^*}( t_0 + \varepsilon_n ) \leq t_0 + \varepsilon_n ,$$

which is a contradiction. Thus,

$$f^*(t_o) - \alpha < f^*(t_o + \varepsilon_n) \leqslant f^*(t_o)$$

for all natural numbers n. This shows that $f^*$ is continuous from the right, which proves a).

To prove b), let us first observe that it follows from the definition of $f^*$ that $f^*(t) > s$ if and only if $t < D_f(s)$. Thus,

$$E_s^* = \left\{ t > 0 : f^*(t) > s \right\}$$

is precisely the interval $(0, D_f(s))$. This implies that the value of the distribution function of $f^*$ at s is the Lebesgue measure of the set $E_s^*$, and the result follows.

Now we are ready to define the Lorentz $L^{pq}$-spaces.

<u>Definition 4.1.5</u>. If f is a measurable function then we define

$$\|f\|_{p,q} = \begin{cases} \left[ \int_0^\infty \left( f^*(t)\ t^{1/p} \right)^q dt/t \right]^{1/q}, & (1 \leq p < \infty, 1 \leq q < \infty) \\ \sup_{t > 0} f^*(t)\ t^{1/p} & (1 \leq p \leq \infty, q = \infty). \end{cases}$$

The <u>Lorentz space $L^{pq}$</u> consists of those functions for which $\|f\|_{p,q}$ is finite.

Despite the notation, the functionals $\|\cdot\|_{p,q}$ are not norms because they do not satisfy the triangle inequality unless $p \geqslant q$.

However, if $p > 1$, we can introduce an equivalent functional which does have this property.

Definition 4.1.6. For a measurable function $f$ with decreasing rearrangement $f^*$, define the averaged function $f^{**}$ by

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\,ds .$$

Definition 4.1.7. For a measurable function $f$, define

$$\|f\|_{(p,q)} = \begin{cases} \left[\int_0^\infty \left(f^{**}(t)\, t^{1/p}\right)^q dt/t\right]^{1/q} & (1<p<\infty, 1\leq q<\infty) \\ \sup_{t>0} f^{**}(t)\, t^{1/p} . & (1\leq p\leq\infty, q=\infty). \end{cases}$$

Theorem 4.1.8. Let $1<p<\infty$, $1\leq q\leq\infty$. Then a necessary and sufficient condition for a measurable function $f$ to belong to $L^{p,q}$ is that $\|f\|_{(p,q)}<\infty$. More precisely,

$$\|f\|_{p,q} \leq \|f\|_{(p,q)} \leq \left[p'\|f\|_{p,q}\right]^{1/q}.$$

Proof. Since $f^*$ is decreasing, $f^* \leq f^{**}$ and so

$$\|f\|_{p,q} \leq \|f\|_{(p,q)} .$$

On the other hand,

$$\left[\|f\|_{(p,q)}\right]^q \leq p/(p-1)\ \|f\|_{p,q}$$

is just an application of Hardy's inequality [28,p.20]. The inequality is also valid for $q=\infty$. Indeed, if $\|f\|_{p,q} = k$, then

$$f^*(t) \leqslant k\, t^{-1/p},$$

and thus

$$f^{**}(t) = \frac{1}{t}\int_0^1 f^*(s)\,ds \leqslant \frac{k}{t}\int_0^1 s^{-1/p}\,ds = k\,p'\,t^{-1/p}.$$

Therefore,

$$f^{**}(t)\,t^{1/p} \leqslant p'\,k\,,$$

from which we obtain

$$\|f\|_{(p,q)} \leqslant p'\,\|f\|_{p,\infty}\,.$$

Theorem 4.1.9. $\|\cdot\|_{(p,q)}$ is a Banach function norm for $1<p<\infty$, $1\leqslant q\leqslant\infty$.

Proof. First, let us consider the case $1<q<\infty$.

P1) $\|f\|_{(p,q)} = 0$ if and only if $f^{**} = 0$ and this is true if and only if $f = 0$ a.e. Clearly,

$$\|af\|_{(p,q)} = a\,\|f\|_{(p,q)}$$

and the triangle inequality follows from the subadditivity of $f^{**}$ and Minkowski's inequality on the real line. The operation

$$f \rightarrow f^{**}$$

is subadditive because, for $s,t>0$,

$$f^{**}(s+t) = \frac{1}{s+t}\int_0^{s+t} f^*(u)\,du$$

$$= \frac{1}{s+t}\int_0^{t} f^*(u)\,du + \frac{1}{s+t}\int_0^{s} f^*(u+s)\,du$$

$$\leqslant \frac{1}{t}\int_0^{t} f^*(u)\,du + \frac{1}{s}\int_0^{s} f^*(u+s)\,du\ .$$

Since $f^*$ is nonincreasing,

$$f^*(u+s) \leq f^*(u)$$

and hence

$$f^{**}(s+t) \leqslant f^{**}(t) + f^{**}(s).$$

P2) This property is obvious since $0 \leq g \leq f$ implies

$$g^{**}(t) \leq f^{**}(t)$$

for all $t \geqslant 0$.

P3) Let $0 \leq f_n \nearrow f$. Then clearly the sequence $f_n^*$ is increasing. Let

$$g = \lim_{n\to\infty} f_n^*\ .$$

Then $g$ is decreasing because the $f_n^*$ are decreasing and furthermore,

$$0 \leq f_n^* \leq f^*\ .$$

Hence, $g \leq f^*$. Now if $k > 0$ and $m$ denotes Lebesgue measure, then

$$m\left(\left\{t : g(t) > k\right\}\right) = m\left(\bigcup_{n=1}^{\infty}\left\{t : f_n^*(t) > k\right\}\right)$$

$$= \lim_{n\to\infty} m\left(\left\{t : f_n^*(t) > k\right\}\right) = \lim_{n\to\infty} m\left(\left\{x \in [0,1] : |f_n(x)| > k\right\}\right)$$

$$= m\left(\bigcup_{n=1}^{\infty}\left\{x \in [0,1] : |f_n(x)| > k\right\}\right) = m\left(\left\{x \in [0,1] : |f(x)| > k\right\}\right).$$

Hence, $g$ and $f$ are equimeasurable, so $g$ and $f^*$ are equimeasurable with $0 \le g \le f^*$. This implies $g = f^*$ a.e. By the monotone convergence theorem, $f_n^{**} \nearrow f^{**}$ a.e., from which we obtain that

$$\|f_n\|_{(p,q)} \nearrow \|f\|_{(p,q)} .$$

P4) Let $E$ be a bounded set. Then

$$\|\chi_E\|_{(p,q)} = \left(\int_0^{\infty}\left[\chi_E^{**}(t)\ t^{1/p}\right]^q dt/t\right)^{1/q} .$$

But

$$\chi_E^*(t) = \begin{cases} 1 & \text{if } 0 < t \le m(E) \\ 0 & \text{if } t > m(E) \end{cases} ,$$

and so

$$\chi_E^{**}(t) = \begin{cases} 1 & \text{if } 0 < t \le m(E) \\ \dfrac{m(E)}{t} & \text{if } t > m(E) \end{cases} .$$

Therefore,

$$\text{(4.1)} \qquad \|\chi_E\|_{(p,q)} = \left(\int_0^{m(E)} t^{q/p}\frac{dt}{t} + \int_{m(E)}^{\infty}\frac{m(E)}{t}^q t^{q/p}\frac{dt}{t}\right)^{1/q} .$$

The first integral in (4.1) converges because

$$q/p - 1 > -1 .$$

The second integral reduces to

$$\left[m(E)\right]^q \int_{m(E)}^{\infty} t^{-q+q/p-1}\, dt$$

and this converges because

$$-q + \frac{q}{p} - 1 < -1$$

precisely when $p > 1$. Therefore $\|\chi_E\|_{(p,q)}$ is finite.

P5) Let $E$ be a bounded set and assume that $q > 1$, for if $q=1$ the result is clear. Then

$$\int_E f\, dm = \int_0^1 f\chi_E\, dm = \int_0^{\infty} (f\chi_E)^*\, dt$$

$$= \int_0^{m(E)} f^*(t)\, dt =$$

$$= \int_0^{\infty} f^* \chi_E^*\, dt$$

$$= \int_0^{\infty} \left[f^*(t)t^{1/p-1/q}\right]\left[\chi_E^*(t)t^{1/q-1/p}\right] dt$$

If we apply Holder's inequality, we obtain

$$\int_E f\, dm \leqslant \left[\int_0^{\infty}\left(f^*(t)t^{1/p}\right)^q dt/t\right]^{1/q} \cdot \left[\int_0 \chi_E(t)t^{q'/q - q'/p} dt\right]^{1/q'}$$

where $q' = q/(q-1)$.

Therefore,

$$\int_E f\, dm \leqslant \|f\|_{p,q} \left[\int_0^{m(E)} t^{q'/q - q'/p}\, dt\right]^{1/q'}$$

$$= C_E \|f\|_{p,q} \leq C_E \|f\|_{(p,q)} .$$

Note that the integral converges because, when $p>1$, we have

$$\frac{q'}{q} - \frac{q'}{p} > -1 .$$

This shows that $\|\cdot\|_{(p,q)}$ is a Banach function norm when $1<p<\infty$, $1\leq q<\infty$.

Now, let us show that $\|\cdot\|_{(p,q)}$ is a Banach function norm when $q=\infty$. The proofs of the first three properties are similar to the corresponding proofs when $q<\infty$.

P4) Let $E$ be a bounded set. Then

$$\|\chi_E\|_{(p,\infty)} = \sup_{t>0} \chi_E^{**}(t)\, t^{1/p} = \sup_{t>0} \begin{cases} t^{1/p} & \text{if } 0\leq t\leq m(E) \\ m(E)\, t^{1/p-1}, & t>m(E) \end{cases}$$

$$= [m(E)]^{1/p} < \infty .$$

P5) Let $E$ be a bounded set. Then

$$\int_E f\, dm = \int_0^1 f\chi_E\, dm \leq \int_0 f^*(t)\, \chi_E^*(t)\, dt$$

$$\leq \int_0^\infty f^{**}(t)\, \chi_E^{**}(t)\, dt$$

$$= \int_0 \left[f^{**}(t)\, t^{1/p}\right]\left[\chi_E^{**}(t)\, t^{-1/p}\right] dt$$

$$\leq \sup_{x>0} f^{**}(x)\, x^{1/p} \cdot \int_0^\infty \chi_E^{**}(t)\, t^{-1/p}\, dt$$

$$= \|f\|_{(p,\infty)} \left[ \int_0^{m(E)} t^{-1/p} dt + m(E) \int_{m(E)}^{\infty} t^{-1-1/p}\, dt \right]$$

$$= C_E \|f\|_{(p,\infty)} ,$$

since both of the integrals converge.

We can thus define a family $L^{pq}$ of Banach function spaces for $1<p<\infty$, $1\leq q\leq\infty$, or in the cases where $p=1$ and $q=1$ or $p=\infty$ and $q=\infty$.

<u>Definition 4.1.10</u>. For $1\leq p\leq\infty$, $1\leq q\leq\infty$, define norms by

a) $$\|f\|_{L^{pq}} = \left(\frac{q}{pp'}\right)^{1/q} \|f\|_{(p,q)} \qquad \text{if } 1<p<\infty \text{ and } 1\leq q<\infty,$$

where, as usual, $p' = \dfrac{p}{p-1}$ ;

b) $$\|f\|_{L^{p\infty}} = \|f\|_{(p,\infty)} \qquad \text{if } 1<p<\infty;$$

c) $$\|f\|_{L^{11}} = \|f\|_1 , \qquad \|f\|_{L^{\infty\infty}} = \|f\|_{\infty} ,$$

where the norms on the right hand sides of the equations in c) denote the Lebesgue space norm.

Clearly, all of the above are Banach space norms. The factor

$$\left(\frac{q}{pp'}\right)^{1/q}$$

has been introduced because if $f$ is the characteristic function of a set or if $f$ is a simple function, then

$$\|f\|_{L^{pp}} = \|f\|_p .$$

This can be seen by a simple calculation. Under the norms defined above, the Lorentz spaces are Banach function spaces.

2. The Associate Space

Through a sequence of lemmata and Holder's inequality, it will be shown that the associate space of $L^{pq}$ is $L^{p'q'}$, where, as before,

$$p' = \frac{p}{p-1}, \qquad q' = \frac{q}{q-1}.$$

Certainly this is true if $p=q$ by the remark at the end of the last section. Most of the results in this section are due to G. G Lorentz [12].

Lemma 4.2.1. If $1<p<\infty$, $1\leq q<\infty$, then

$$\|f\|_{(p,q)} = \sup_g \left| \int_0^1 fg\, dm \right| ,$$

where $g$ varies over the set of measurable functions such that, for some measurable function $h$,

$$g^*(t) = \int_t^\infty \frac{h(s)}{s}\, ds ,$$

with

$$\left( \int_0 \left[h(t)\, t^{1/p'}\right]^{q'} dt/t \right)^{1/q'} \leq 1 .$$

Proof. For such a function $g$,

$$\left| \int_0^1 fg\, dm \right| \leq \int_0^\infty f^*(t)\, g^*(t)\, dt = \int_0^\infty f^*(t) \int_t^\infty \frac{h(s)}{s}\, ds\, dt$$

$$= \int_0^\infty \frac{h(s)}{s} \int_0^s f^*(t)\, dt\, ds = \int_0^\infty h(s)\, f^{**}(s)\, ds$$

$$= \int_0^\infty \left[f^{**}(s)\, s^{1/p}\right] \left[h(s)\, s^{-1/p}\right] ds$$

$$\leqslant \left[\int_0^\infty \left(f^{**}(s)\, s^{1/p}\right)^q ds/s\right]^{1/q} \left[\int_0^\infty \left(h(s) s^{1/p'}\right)^{q'} ds/s\right]^{1/q'}$$

$$\leqslant \|f\|_{(p,q)} .$$

Now choose $0<a<b<\infty$ and $h(s)$ such that

$$h(s)\, s^{1/p'} = \left[f^{**}(s)\, s^{1/p}\right]^{q-1} c^{1-q} ,$$

where

$$c^q = \int_a^b \left[f^{**}(s)\, s^{1/p}\right]^q ds/s .$$

It can be shown [12] that there exists a $g$ such that

$$\left|\int_0^1 fg\, dm\right| = \int_0^\infty f^*(t)\, g^*(t)\, dt.$$

In this case,

$$\left|\int_0^1 fg\, dm\right| = \int_0^\infty f^*(t)\, g^*(t)\, dt = \int_0^\infty h(s)\, f^{**}(s)\, ds$$

$$= \left[\int_a^b f^{**}(s)\, s^{1/p\, q}\, ds/s\right] c^{1-q}$$

$$= \left[\int_a^b f^{**}(s)\, s^{1/p\, q}\, ds/s\right]^{1/q} .$$

Moreover, for this function $h$, we have

$$\left(\int_a^\infty \left[h(s)\ s^{1/p'}\right]^{q'} ds/s\right)^{1/q'} = c^{1-q}\left(\int_a^b \left[f^{**}(s)\ s^{1/p}\right]^q ds/s\right)^{1/q}$$

$$= 1\ .$$

Since a and b were arbitrary, by letting a tend to zero and b tend to infinity, the result is obtained.

<u>Lemma 4.2.2</u>. If $1<p<\infty$, $1<q<\infty$, then

$$q^{-1/q} q'^{-1/q'} \|f\|_{L^{pq}} \leqslant \sup\left\{\left|\int_0^1 fg\ dm\right| : \|g\|_{L^{p'q'}} = 1\right\} \tag{4.1}$$

$$\leqslant pp' q^{-1/q} q'^{-1/q'} \|f\|_{L^{pq}}\ .$$

Proof. If $g^*(t) = \int_t^\infty \frac{h(s)}{s}\, ds$ , then

$$g^{**}(t) = \frac{1}{t}\int_0^t g^*(t)\ dt = \frac{1}{t}\int_0^t \int_r^\infty \frac{h(s)}{s}\, ds\ dr$$

$$= \frac{1}{t}\int_0^t \frac{h(s)}{s}\int_0^s dr\ ds + \frac{1}{t}\int_t^\infty \frac{h(s)}{s}\int_0^t dr\ ds$$

$$= \frac{1}{t}\int_0^t h(s)\ ds + \int_t^\infty \frac{h(s)}{s}\, ds\ .$$

If

$$\left(\int_0^\infty \left[h(t)\right]^{q'} t^{q'/(p-1)}\ dt\right)^{1/q'} \leqslant 1,$$

then by Lemma 4.2.1 and Hardy's inequality [28],

$$\|g\|_{L^{p'q'}} = \left(\frac{q'}{pp'}\int_0^\infty\left[\frac{1}{t}\int_0^t h(s)ds + \int_t^\infty \frac{h(s)}{s}\,ds\right]^{q'} t^{q'/(p'-1)}dt\right)^{1/q'}$$

$$\leqslant \left(\frac{q'}{pp'}\int_0^\infty \frac{1}{t}\left[\int_0^t h(s)\,ds\right]^{q'} t^{q'/(p-1)}dt\right)^{1/q'}$$

$$+\left(\frac{q'}{pp'}\int_0^\infty \left[\int_t^\infty \frac{h(s)}{s}\,ds\right]^{q'} t^{q'/(p-1)}dt\right)^{1/q'}$$

$$\leqslant p\left(\frac{q'}{pp'}\int_0^\infty \left[h(t)\right]^{q'} t^{q'/(p'-1)}\,dt\right)^{1/q'}$$

$$+\ p'\left(\frac{q'}{pp'}\int_0^\infty \left[h(t)\right]^{q'} t^{q'/(p-1)}\,dt\right)^{1/q'}$$

$$\leqslant (p + p')\left(\frac{q'}{pp'}\right)^{1/q'} = pp'\left(\frac{q'}{pp'}\right)^{1/q'}$$

$$= (pp')^{1/q}\, q'^{1/q'} .$$

Consequently, the set of those functions $g$ whose rearrangement arises from an $h$ is contained in the set of those $g$ such that

$$\|g\|_{L^{p'q'}} \leqslant (pp')^{1/q}\, q'^{1/q'} .$$

Therefore, by Theorem 4.1.8, we obtain

$$\|f\|_{(p,q)} \leqslant \sup\left\{\left|\int_0^1 fg\ dm\right| : \|g\|_{L^{p'q'}} \leqslant (pp')^{1/q}\, q'^{1/q'}\right\}$$

$$= (pp')^{1/q}\, q'^{1/q'} \sup\left\{\left|\int_0^1 fg\ dm\right| : \|g\|_{L^{p'q'}} \leqslant 1\right\},$$

from which results the first inequality of (4.1). On the other hand,

$$\left|\int_0^1 fg\ dm\right| \leq \int_0^\infty f^*(t)\ g^*(t)\ dt \leq \int_0^\infty f^{**}(t)\ g^*(t)\ dt$$

$$= \int_0^\infty \left[f^{**}(t)\ t^{1/p}\right]\left[g^{**}(t)\ t^{1/p'}\right] dt/t$$

$$\leq \|f\|_{(p,q)} \cdot \|g\|_{(p',q')}$$

$$= pp'q^{-1/q}\ q'^{-1/q'}\ \|f\|_{L^{pq}}\ \|g\|_{L^{p'q'}}\ ,$$

which proves the second inequality.

<u>Lemma 4.2.3</u>. If $1 \leq p < \infty$, then

a) $$\|f\|_{L^{p\infty}} = \sup\left\{\left|\int_0^1 fg\ dm\right| : \|g\|_{L^{p'1}} \leq 1\right\}\ ;$$

b) $$\|f\|_{L^{p1}} = \sup\left\{\left|\int_0^1 fg\ dm\right| : \|g\|_{L^{p'\infty}} \leq 1\right\}\ .$$

Proof. For $p=1$, the result has already been proven, that is

$$\|f\|_{L} = \|f\|_{L^{11}} = \|f\|_{L^1}\ ,$$

and

$$\|f\|_{L^{\infty\infty}} = \|f\|_{L^{\infty}} = \|f\|_{L^{\infty,1}}\ , \text{ by definition.}$$

If $f$ belongs to $L^{p\infty}$ for $1 < p < \infty$, and $g = \chi_E$, where $m(E) = k$,

$$\left|\int_0^1 fg\ dm\right| \leq \int_0^\infty f^*(t)\ g^*(t)\ dt = \int_0^k f^*(t)\ dt$$

$$= k^{1/p'} \left( k^{1/p} f^{**}(k) \right)$$

$$\leqslant \|f\|_{L^{p\infty}} \cdot \|g\|_{L^{p'1}} .$$

If $g$ is nonegative and simple, $g$ can be expressed as

$$g = \sum_{i=1}^{n} c_i g_i ,$$

where $g_i = \chi_{B_i}$ , $B_1 \subseteq B_2 \ldots \subseteq B_n$, so that

$$g^*(t) = \sum_{i=1}^{n} c_i g_i^*(t)$$

and

$$\|g\|_{L^{p'1}} = \sum_{i=1}^{n} c_i \|g\|_{L^{p'1}} .$$

Thus,

$$\left| \int_0^1 fg \, dm \right| \leqslant \int_0^{\infty} f^*(t) \, g^*(t) \, dt$$

$$= \sum_{i=1}^{n} c_i \int_0^{\infty} f^*(t) \, g_i^*(t) \, dt$$

$$\leqslant \|f\|_{L^p} \sum_{i=1}^{n} c_i \|g_i\|_{L^{p'1}}$$

$$= \|f\|_{L^{p\infty}} \cdot \|g\|_{L^{p'1}} .$$

If $g$ is arbitrary, then it is possible to select simple functions $h_n$ such that

$$h_n \nearrow g.$$

Then,

$$h_n^* \nearrow g^*$$

and so, in general,

$$\left|\int_0^1 fg\ dm\right| \leqslant \|f\|_{L^{p\infty}} \cdot \|g\|_{L^{p'1}} .$$

Thus,

$$\sup\left\{\left|\int_0^1 fg\ dm\right| : \|g\|_{L^{p'1}} \leqslant 1\right\} \leqslant \|f\|_{L^{p\infty}} .$$

If the roles of $f$ and $p$ are reversed with those of $g$ and $p'$ then we also have

$$\sup\left\{\left|\int_0^1 fg\ dm\right| : \|g\|_{L^{p\infty}} \leqslant 1\right\} \leqslant \|f\|_{L^{p1}} .$$

If $f$ belongs to $L^p$, let $g$ be such that $g^*(t) = 0$ if $t<k$ and $g^*(t) = k^{1/p'}$ if $0<t<k$, and so

$$\left|\int_0^1 fg\ dm\right| = \int_0^\infty f^*(t)\ g^*(t)\ dt = k^{-1/p'}\int_0^k f^*(t)\ dt$$

$$= k^{1/p}\ f^{**}(k).$$

Also, for this $g$,

$$\|g\|_{L^{p'1}} = \frac{1}{p'}\ k^{-1/p} \int_0^k t^{1/p' - 1}\ dt = 1 ;$$

If we select $k_n$ such that

$$k_n^{1/p}\ f^{**}(k_n) \to \|f\|_{L^{p\infty}} ,$$

then statement a) is proved. If $f$ belongs to $L^{p1}$, let $g$ be such that

$$g^*(t) = \frac{1}{p} t^{1/p - 1}$$

and

$$\left|\int_0^1 fg\, dm\right| = \int_0^\infty f^*(t)\, g^*(t)\, dt = \frac{1}{p}\int_0^\infty f^*(t)\, t^{1/p - 1}\, dt$$
$$= \|f\|_{L^{p1}} .$$

For this $g$ we have

$$g^{**}(t) = \frac{1}{pt}\int_0^t s^{1/p - 1}\, ds = t^{-1/p'}$$

from which

$$\|g\|_{L^{p'\infty}} = \sup g^{**}(t)\, t^{1/p'} = 1,$$

and statement b) of the lemma is proved.

Putting together the last three lemmata, we can characterize the associate space of a Lorentz space.

Theorem 4.2.4. If $1<p<\infty$ and $1\leq q\leq\infty$, then the associate space $(L^{pq})'$ of $L^{pq}$ is $L^{p'q'}$.

Now it will be shown that the associate space $L^{p'q'}$ of $L^{pq}$ is also the Banach space dual, provided $q<\infty$.

Theorem 4.2.5. If $1<p<\infty$ and $1\leq q<\infty$, then the Lorentz space $L^{pq}$ has absolutely continuous norm.

Proof. Let E be bounded, and $(E_n)_{n=1}^{\infty}$ a sequence of measurable subsets of E such that $E_n \to \emptyset$. Let f belong to $L^{pq}$ and let g be such that $g^*(t) = f^*(t)$ for $t\leq m(E_n)$ and $g^*(t)=0$ if $t>m(E_n)$. Then, for $t\geq 0$,

$$(f\chi_{E_n})^*(t) \leq g^*(t)$$

It follows that

$$(f\chi_{E_n})^{**} \leq g^{**} .$$

Then

$$(4.2) \qquad \|f\chi_{E_n}\|_{(p,q)} \leq \left(\int_0^{\infty} g^{**}(t)\ t^{1/p\ q}\ dt/t\right)^{1/q}$$

$$= \left(\int_0^{m(E_n)} \left[f^{**}(t)t^{1/p}\right]^q \frac{dt}{t} + \int_{m(E_n)}^{\infty} \left[\frac{m(E_n)}{t}\ f^{**}(m(E_n))t^{1/p}\right]^q \frac{dt}{t}\right)^{1/q}$$

Consider the first integral in the last term of (4.2).

$$\int_0^{m(E_n)} \left[f^{**}(t)\ t^{1/p}\right]^q dt/t = \int_0^{m(E_n)} \left[f^{**}(t)\right]^q\ t^{q/p\ -\ 1}\ dt$$

$$\leq M^q \int_0^{m(E_n)} t^{q/p\ -\ 1}\ dt\ ,$$

where

$$M = \max f^{**}(t) \qquad (\ 0\ t\ m(E_n)\ )$$

The integral converges and equals

$$\frac{p}{q}\, m(E_n)^{q/p}$$

For the second integral,

$$(4.3)\quad \int_{m(E_n)}^{\infty} \left[\frac{m(E_n)}{t}\, f^{**}(\, m(E_n)\, )\, t^{1/p}\right]^q dt/t$$

$$= \left[m(E_n)\, f^{**}(\, m(E_n)\, )\right]^q \int_{m(E_n)}^{\infty} t^{q/p - q - 1} dt\ .$$

The integral converges precisely when $p > 1$, and hence (4.3) becomes

$$\left[m(E_n)\ f^{**}(m(E_n))\right]^q\ \frac{p'}{q}\ m(E_n)^{-q/p'} = \frac{p'}{q}\ m(E_n)^{q/p}\left[f^{**}(m(E_n))\right]^q\ .$$

Therefore,

$$(4.4)\qquad \|f\chi_{E_n}\|_{(p,q)} \leq \left(M^q\ \frac{p}{q}\ m(E_n)^{q/p} + \frac{p'}{q} m(E_n)^{q/p}\left[f^{**}(m(E_n))\right]^q\right)^{\frac{1}{q}}$$

$$= 2^{1/q} M \left(\frac{p}{q}\right)^{1/q} m(E_n)^{1/p}$$

$$+\ 2^{1/p}\left(\frac{p'}{q}\right)^{1/q} m(E_n)\ f^{**}(m(E_n))\,.$$

Since $E_n \to \emptyset$, we may assume that $E_{n+1} \subseteq E_n$. In this case, let

$$M' = \max f^{**}(t)\ ,$$

where $t$ is between $0$ and $m(E)$. Then (4.4) becomes

$$(4.5) \qquad \|f\chi_{E_n}\|_{(p,q)} \leqslant m(E_n)^{1/p}\left[M\left(\frac{2p}{q}\right)^{1/q} + f^{**}(m(E))\left(\frac{2p}{q}\right)^{1/q}\right].$$

From (4.5) we see that

$$\|f\chi_{E_n}\|_{(p,q)} \to 0$$

as $E_n \to \phi$. Also, it is clear from the above that

$$\lim_{n\to\infty} \|f\chi_{A-A_n}\|_{(p,q)} \quad 0 .$$

Theorem 4.2.5 is not true when $q=\infty$. We may now conclude that the following holds.

Theorem 4.2.6. For $1<p<\infty$ and $1\leq q<\infty$ or when $p=1$ and $q=1$, the Banach space dual of $L^{pq}$ is $L^{p'q'}$.

$$= \int_0^1 f^{**}(u)\ du \quad .$$

<u>Lemma 4.3.1</u>. The associate space of $\Lambda(\log(1/t), 1)$ is the space of all measurable functions $g$ for which

$$\|g\|_{M} = \sup \frac{g^{**}(t)}{1+\log(1/t)}$$

is finite, the supremum being taken over the interval $[0,1]$.

Proof. We will use Hardy's lemma [15] which states that if

$$\int_0^t f(x)\ dx \leqslant \int_0^t g(x)\ dx$$

and $h$ is a decreasing function on $(0,t)$, then

$$\int_0^t f(x)h(x)\ dx \leqslant \int_0^t g(x)h(x)\ dx \ .$$

It is easily proved by an integration by parts. We have

$$\int_0^u g^{*}(t)\ dt = u\, g^{**}(u) = u\ (1 + \log(1/u))\ \frac{g^{**}(u)}{1+\log(1/u)}$$

$$= \frac{g^{**}(u)}{1+\log(1/u)} \int_0^u \log(1/t)\ dt$$

$$\leqslant \sup_{0 \leqslant t \leqslant 1} \frac{g^{**}(t)}{1+\log(1/t)} \int_0^u \log(1/t)\ dt \ .$$

Hence, by Hardy's lemma,

$$(4.6) \qquad \int_0^1 fg\,dm \leqslant \int_0^1 f^*(t)g^*(t)\,dt$$

$$\leqslant \sup_{0\leq t\leq 1} \frac{g^{**}(t)}{1+\log(1/t)} \int_0^1 f^*(t)\log(1/t)\,dt$$

$$= \|g\|_M \cdot \|f\|_\Lambda .$$

We still need to show that there exists a measurable function $f$ such that equality holds in (4.6). Let

$$f_s(t) = \frac{\chi_{(0,s)}(t)}{s(1+\log(1/s)} .$$

Then for all $s$, $0<s<1$,

$$\|f_s\|_\Lambda = \int_0^1 f_s^*(t)\log(1/t)\,dt$$

$$= \frac{1}{s(1+\log(1/s))}\int_0^1 \chi_{(0,s)}^*(t)\log(1/t)\,dt$$

$$= \frac{1}{s(1+\log(1/s))}\int_0^s \log(1/t)\,dt = 1 .$$

Let $v$ be a value such that the supremum is attained, and let $f$ be such that

$$f^*(t) = f_v(t) .$$

Then,

$$\int_0^1 fg\,dm = \int_0^1 f_v(t)\,g^*(t)\,dt$$

$$= \frac{1}{v(1+\log(1/v))}\int_0^v g^*(t)\,dt = \|g\|_M .$$

## 3. Lorentz $\Lambda$-Spaces

In his paper [11], Lorentz discusses properties of more general spaces $\Lambda(\varphi,q)$, where the Lorentz spaces $L^{pq}$ are the particular case when

$$\varphi(t) = \frac{1}{p}\, t^{1/p'} \qquad (1<p<\infty).$$

A measurable function $f$ on a finite measure space ( we will again assume that we are dealing with Lebesgue measure on the unit interval ) belongs to the class $\Lambda(\varphi,q)$ provided the norm $\|f\|_\Lambda$, defined by

$$\|f\|_\Lambda = \left(\int_0^1 \varphi(t)\left[f^*(t)\right]^q dt\right)^{1/q} \qquad (q \geq 1)$$

is finite. Here, $\varphi$ is a given nonnegative integrable function on $[0,1]$, not identically zero. Only in the case where $\varphi$ is a decreasing function will the norm satisfy the triangle inequality, making the classes $\Lambda(\varphi,q)$ into Banach function spaces.

Let us consider the space $\Lambda(\log(1/t),1)$ . Thus $f$ belongs to $\Lambda(\log(1/t),1)$ if and only if

$$\int_0^1 f^*(t)\log(1/t)\,dt$$

is finite. Note that by an integration by parts

$$\int_0^1 f^*(t)\,dt = \left(\int_0^t f^*(u)\,du\right)\log(1/t)\,\Big]_0^1 + \int_0^1 \frac{1}{t}\int_0^t f^*(u)\,du\,dt$$

Now we will show that the space $\Lambda(\log(1/t),1)$ coincides with the Orlicz space $L \log^{+} L$, discussed in the previous chapter.

Lemma 4.3.2. Suppose $F$ is a real valued, positive, decreasing function on the interval $[0,1]$. Then

$$\int_0^1 F(t) \log F(t)\, dt$$

is finite if and only if

$$\int_0^1 F(t) \log(1/t)\, dt$$

is finite.

Proof. Let us assume that

$$\int_0^1 F(t) \log F(t)\, dt$$

is finite. Let

$$E = \{t : F(t) > 1/\sqrt{t}\}$$

and

$$G = \{t : F(t) \leq 1/\sqrt{t}\}.$$

Then

$$\int_0^1 F(t) \log(1/t)\, dt = \int_E F(t)\log(1/t)dt + \int_G F(t)\log(1/t)dt$$

$$\leq \int_E F(t) \log\left[F(t)\right]^2 dt + \int_G \frac{1}{\sqrt{t}} \log \frac{1}{t}\, dt$$

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