WELL-POSEDNESS AND WAVELET-BASED APPROXIMATIONS FOR HYPERSINGULAR INTEGRAL EQUATIONS

By

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Abstract

The main object of this thesis is to investigate the hypersingular integral equations which arise in Boundary Element (BE) models for crack problems. Although the associated hypersingular integrals are not defined in the usual sense, we interpret them in terms of Hadamard finite part integral operators or as pseudo-differential operators in the distribution sense.

By introducing weighted Sobolev spaces to regularize the equations, we have proved the well-posedness for such hypersingular integral equations. Global error estimates are obtained in the thesis.

A new approach to the numerical solution of the hypersingular equations based on the recently developed theory of wavelets is presented. Rather than applying wavelet bases directly to obtain new discretizations, we exploit the wavelet bases to obtain more efficient solution algorithms for the more classical discretizations. We discretize the hypersingular integral equation using the piecewise polynomial collocation method. The discrete wavelet transform is then used for the resulting dense algebraic system. This procedure involves $O(N^2)$ operations and leads to a sparse matrix problem. Solving this sparse matrix system requires only $O(N \log^2 N)$ operations whereas $O(N^3)$ operations are required for the traditional BE method. Exploiting an indexed storage structure, we reduce the memory requirements from $O(N^2)$ to $O(N \log N)$ words. Furthermore, the method is applicable to all operator

or matrix (with arbitrary geometries) problems as long as the operator or matrix possesses only a finite number of singularities in some rows or columns.

We demonstrate that the wavelet-based method can be extended to higher dimensional integral equation problems because these equations can also be discretized by the piecewise polynomial collocation approximation.

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Introduction

1.1 Motivation

Fracture-related damage is one of the major concerns for designing underground excavations, nuclear power plants, hydroelectric dams and offshore oil rigs, etc. [32] [79]. Effective inspection, preventive maintenance and possible control techniques cannot be established without powerful fracture analysis methods.

Given the complicated geometries and nonlinearities which characterize fracture mechanics, analytical methods are limited. The common trend is to use numerical procedures to solve these problems. Among the popular techniques are finite element (FE) and boundary element (BE) methods.

The finite element method can be used for stress analysis and crack propagation simulation [56] [32]. When the stress field at a crack front is singular, we need graded discretizations, i.e., small elements near the crack front and larger elements elsewhere. Although this domain discretization method ultimately reduces to a sparse influence matrix, the use of a refined grid near the crack will result in a model which involves an extremely large number of degrees of freedom. In the cases of multiple cracks [32] and cavity type problems in infinite domains [56], the computation costs of solving the resulting large system of equations tend to be prohibitive even with today's supercomputers.

By reformulating the crack interaction problem in terms of boundary integral equations, it is possible to only discretize the boundary of the domain instead of the whole domain. The so-called boundary element method effectively reduces the problem from that of discretizing on n dimensional volume to an $n-1$ dimensional surface. This technique is therefore very effective when it is applied to infinite domains – such as those typically encountered in models of mining excavations [56]. The boundary element technique is commonly used for modeling the interaction of large-scale tabular excavations with isolated fault-planes in 2D and 3D elastic media [57] [15] [60] [68]. There are, in fact, two distinct formulations of the BE method. The so-called direct BE method, which is commonly used in the stress analysis of cavities and machine parts, has difficulties in the representation of cracks [32] [56]. To overcome these difficulties, indirect BE methods [56] [15] [5] have been developed. One of the indirect BE methods, known as the displacement discontinuity (DD) method [15] [16], provides perhaps the most efficient representation of cracks, fractures and fault planes [59].

Consider a bounded region Ω in R^n and define $\bar{\Omega} = \mathrm{R}^n \backslash (\Omega \cup \partial\Omega)$, where $\partial\Omega$ is the boundary of Ω. Let σ_{ij}, u_i and $\bar{\sigma}_{ij}$, $\bar{u}_i$, be the stresses and displacements in Ω and $\bar{\Omega}$ respectively. For homogeneous, isotropic elasticity, the stresses in Ω satisfy the equilibrium equations of elastostatics:

$$\sigma_{ij,j} + f_i = 0 \ .$$

The stress-strain relation is given by Hooke's law:

$$\sigma_{ij} = \frac{E}{2(1+v)} \left[\frac{2v}{1-2v} \ \delta_{ij}\varepsilon_{kk} + 2\varepsilon_{ij} \right] \ ,$$

where the strain tensor ε_{ij} is defined by

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) ,$$

E is the Young's modulus and v is the Poisson's ratio for the elastic medium. The same system of partial differential equations holds for the stress and displacement field in $\bar{\Omega}$.

Applying Green's theorem, one can transform the above differential equations in Ω into the following integral equations defined on $\partial\Omega$ for u_k and $\sigma_{k\ell}$ [56] [6]:

$$u_k(p) = \int_{\partial\Omega} \{g(q_i, p_k)T_i(q) - G(q_{ij}, p_k)n_j(q)D_i(q)\} ds(q) , \tag{1.1.1}$$

$$\sigma_{k\ell}(p) = \int_{\partial\Omega} \{r(q_i, p_{k\ell})T_i(q) + \Gamma(q_{ij}, p_{k\ell})n_j(q)D_i(q)\} ds(q) . \tag{1.1.2}$$

Where n_j are the components of the normal directed toward the interior of Ω, $T_i(q) = (\bar{\sigma}_{ij}(q) - \sigma_{ij}(q))n_j(q)$ is the traction discontinuity, and $D_i(q) = \bar{u}_i(q) - u_i(q)$ is the displacement discontinuity between $\bar{\Omega}$ and Ω.

If we set the stresses and displacements $\bar{\sigma}_{ij} = \bar{u}_i = 0$, then (1.1.1) gives the classical direct BE formulation. By assuming that the tractions across $\partial\Omega$ are continuous, we obtain from (1.1.1) and (1.1.2) the indirect BE formulation known as the displacement discontinuity (DD) method [57].

The kernel $G(q_{ij}, p_k)$ [56] [58] in (1.1.1) is singular enough to require a Cauchy principal value to define the integral. Recall that the Cauchy integral exists in the principal value sense, which is defined as follows:

$$⨍_a^b \frac{u(t)}{t-x}\, dt = \lim_{\epsilon \to 0} \left\{ \int_a^{x-\epsilon} \frac{u(t)}{t-x}\, dt + \int_{x+\epsilon}^{b} \frac{u(t)}{t-x}\, dt \right\} , \tag{1.1.3}$$

where the notation $⨍_a^b$ denotes the Cauchy principal value integral. Compared with $G(q_{ij}, p_k)$, the influence function $\Gamma(q_{ij}, p_{k\ell})$ in (1.1.2) is more singular since $\Gamma(q_{ij}, p_{k\ell})$

is derived from $G(q_{ij}, p_k)$ by applying the appropriate derivative operations with respect to p [56] [57]. In particular, we consider a special case of (1.1.2) which represents a model of a crack located along the line $y = 0$ in a 2D elastic medium subjected to the normal stress distribution $p(x)$. The DD formulation leads to [15] [57] [39]:

$$\frac{1}{\pi}\int_{-\ell}^{\ell}\frac{u(t)}{(t-x)^2}\,dt + \frac{1}{\pi}\int_{-\ell}^{\ell}K(t,x)u(t)\,dt = p(x) + \sigma(x, u(x)) \,, \qquad (1.1.4)$$

where $u(x)$ is the convergence DD distribution and $K(t, x)$ is a smooth kernel. In general, the stress intensity distribution $\sigma(x, u(x))$ is nonlinear [58]. If we consider the case of the analytic stress intensity distribution [69]

$$\sigma(x) = \begin{cases} -1 & , \quad |x| < \ell \\ \frac{x}{\sqrt{x^2-\ell^2}} - 1 & , \quad |x| > \ell \,, \end{cases} \qquad (1.1.5)$$

then (1.1.4) becomes

$$\frac{1}{\pi}\int_{-\ell}^{\ell}\frac{u(t)}{(t-x)^2}\,dt + \frac{1}{\pi}\int_{-\ell}^{\ell}K(t,x)u(t)\,dt = f(x) \,. \qquad (1.1.6)$$

We call (1.1.6) a dominant equation if $K(t, x) \equiv 0$. If a collection of tabular crack-like excavations located on the plane $z = 0$ in a 3D elastic medium is subjected to an induced normal stress $p(x, y)$, then the DD formulation (1.1.2) gives [69] [39]:

$$\frac{E}{8(1-v^2)\pi}\iint\limits_{\partial\Omega}\frac{u(\xi,\eta)\,d\xi d\eta}{[(x-\xi)^2+(y-\eta)^2]^{3/2}} = p(x, y) \,, \qquad (1.1.7)$$

where $u(\xi, \eta)$ is the convergence DD distribution and $\partial\Omega$ is the boundary of a bounded region $\Omega \subset \mathbf{R}^3$.

Notice that the integrals in (1.1.4) and (1.1.7) are not defined in any usual sense and have been termed hypersingular. In addition, matrices that result from the discretizations of these integral equations are fully populated. When the effect of inelastic deformation is included and/or there are extensive discontinuity surfaces throughout the region $\partial\Omega$ [59], a large number N of discretization points is required. Therefore the computational costs and the computer memory requirements become prohibitive.

In brief, two difficulties we encountered when applying the BE technique to solve fracture mechanics problems:

(i) Treating the hypersingular integral equations;

(ii) Solving the system of algebraic equations with large fully populated matrices.

The work of this dissertation is therefore motivated by developing an analytical method to study the well-posedness of (i) and creating a novel wavelet approximation algorithm to handle (ii) which leads to more efficient solution procedures.

1.2 Review

The idea of finite part integrals was first introduced by Hadamard [34] in connection with the solution of hyperbolic partial differential equations for cylindrical waves. He encountered divergent integrals of the form

$$\int_a^x \frac{u(t)}{(x-t)^{p+\frac{1}{2}}}\,dt \tag{1.2.1}$$

where p is an integer. To explain Hadamard's basic idea we consider the following integral

$$I(x) = \int_x^b \frac{dt}{(t-x)^{1/2}} = 2(b-x)^{1/2}, \qquad x < b\ . \tag{1.2.2}$$

Differentiating both sides separately, we obtain

$$\frac{d}{dx}\,I(x) = \frac{1}{2}\int_x^b \frac{dt}{(t-x)^{3/2}} - \frac{1}{(t-x)^{1/2}}\Big|_{t=x} = -\frac{1}{(b-x)^{1/2}}\ . \tag{1.2.3}$$

Equation (1.2.3) tells us that the difference between a divergent integral and an unbounded integrated term remains finite. We may now consider the derivative of $I(x)$ as a finite part of the divergent integral and define

$$\oint_x^b \frac{dt}{(t-x)^{3/2}} = \lim_{\varepsilon\to 0}\left[\int_{x+\varepsilon}^b \frac{dt}{(t-x)^{3/2}} - \frac{2}{\varepsilon^{1/2}}\right] = -\frac{2}{(b-x)^{1/2}} . \quad (1.2.4)$$

Similarly, we can define $\oint_a^x \frac{dt}{(t-x)^{3/2}}$ and $\oint_a^b = \oint_a^x + \oint_x^b$. Thus (1.2.1) can be defined in the same way. Moreover, we can extend the definition of $\int_a^b \frac{1}{(x-t)^p}\, dt$ when p is an integer [34].

In general, we have (for $p = 2$) [34] [39]

$$\begin{aligned}\oint_a^b \frac{u(t)}{(t-x)^2}\, dt &= \int_a^b \frac{u(t)-u(x)-(t-x)u'(x)}{(t-x)^2}\, dt \\ &\quad + u(x)\oint_a^b \frac{1}{(t-x)^2}\, dt + u'(x)\fint_a^b \frac{1}{t-x}\, dt , \end{aligned} \quad (1.2.5)$$

where $\fint_a^b$ is the Cauchy integral defined by (1.1.3).

By definition, a finite part integral can be obtained by the direct differentiation of a Cauchy integral, i.e.,

$$\oint_a^b \frac{u(t)}{(t-x)^2}\, dt = \frac{d}{dx}\fint_a^b \frac{u(t)}{t-x}\, dt . \quad (1.2.6)$$

Alternately, we can explain the hypersingular integral $\oint_a^b \frac{u(t)}{(t-x)^2}\, dt$ by the theory of distributions [75] [46] and pseudo-differential operators [78] [51] [80].

The term pseudo-differential operator was first used by K. Friedrichs and P. Lax [28] and J.J. Kohn and L. Nirenberg [41] for the operators of the type

$$Au(x) = \int_{\mathbf{R}^n} a(x,\xi)\hat{u}(\xi)e^{ix\xi}\, d\xi , \quad (1.2.7)$$

where $\hat{u}(\xi)$ is the Fourier transform of $u(x)$ defined by

$$\hat{u}(\xi) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} u(x)\, e^{-ix\xi}\, dx . \quad (1.2.8)$$

The function $a(x,\xi)$ is called the symbol of the operator. If $a \in C^\infty$ and satisfies

$$|D_\xi^\alpha D_x^\beta a(x,\xi)| \le C_{\alpha\beta}(1+|\xi|)^{m-|\alpha|} \tag{1.2.9}$$

for all $\alpha, \beta \in \mathbf{Z}_+^n$, then the operator A is called a pseudo-differential operator of order m.

A representation for the operator A can also be obtained without using the Fourier transform. Substituting $\hat{u}(\xi) = \frac{1}{(2\pi)^n}\int_{\mathbf{R}^n} u(y)\, e^{-iy\xi}\, dy$ into (1.2.7), we have

$$Au(x) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} a(x,\xi)u(y)\, e^{i(x-y)\xi}\, dy\, d\xi\ .$$

If we formally write

$$k(x, x-y) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} a(x,\xi)\, e^{i(x-y)\xi}\, d\xi\ , \tag{1.2.10}$$

then

$$Au(x) = \int_{\mathbf{R}^n} k(x, x-y)u(y)\, dy\ . \tag{1.2.11}$$

And (1.2.10) shows that the symbol of the operator A can be viewed as the Fourier transform of the product $(2\pi)^n$ and the kernel $k(x,t)$ with respect to the second argument. Thus formally we have

$$a(x,\xi) = \int_{\mathbf{R}^n} k(x,t)\, e^{-i\xi t}\, dt\ . \tag{1.2.12}$$

Returning to the dominant part of the hypersingular integral equation (1.1.6):

$$\frac{1}{\pi}\int_{-\ell}^{\ell} \frac{u(t)}{(t-x)^2}\, dt = f(x)\ , \tag{1.2.13}$$

we define

$$Tu(x) = \int_{-\ell}^{\ell} k(x, x-t)\, u(t)\, dt\ , \tag{1.2.14}$$

where $k(x, x-t) = k(x-t) = \frac{1}{(x-t)^2}$.

Since the Fourier transform of $\frac{2\pi}{x^2}$ is $-\pi|\xi|$ [46] [57], the symbol of T is as follows

$$a(x,\xi) = -\pi|\xi| \ . \tag{1.2.15}$$

Notice that the symbol is homogeneous which makes it possible to drop the smoothness condition on the symbol at 0 [51]. Therefore by (1.2.9), we conclude that T is a pseudo-differential operator of order 1. Moreover T is elliptic since $|a(x,\xi)| = \pi|\xi|$. (Recall that a pseudo-differential operator of order m is called elliptic if the symbol $a(x,\xi)$ satisfies $|a(x,\xi)| \geq c|\xi|^m$, where c is a constant.)

We rewrite (1.2.13) as a pseudo-differential operator equation

$$Tu = f \ . \tag{1.2.16}$$

It is clear that the equation (1.2.16) can be solved provided

$$\text{the inverse } T^{-1} \text{ exists and is continuous.} \tag{1.2.17}$$

The above statement (1.2.17) can be interpreted as follows:

(i) A solution exists;

(ii) The solution is unique;

(iii) The solution depends continuously on the data $f(x)$.

Before outlining our study of these three problems, let us look at an example.

Since [30]

$$\frac{1}{\pi}⨍_{-1}^{1} \frac{1}{\sqrt{1-t^2}(t-x)}\, dt = 0 \ , \tag{1.2.18}$$

then by the Hadamard finite part,

$$\frac{1}{\pi}⨎_{-1}^{1} \frac{1}{\sqrt{1-t^2}(t-x)^2}\, dt = 0 \ . \tag{1.2.19}$$

Equation (1.2.19) implies that if the domain of operator T contains $u(t) = \frac{1}{\sqrt{1-t^2}}$, then the solution of (1.2.16) is not unique.

We shall introduce a Sobolev space to restrict the domain such that the inverse exists and is continuous.

It is difficult in general to find an analytic solution of (1.2.16) or higher dimensional cases such as (1.1.7). To solve (1.2.16) numerically, we will encounter an associated algebraic system with a fully populated $N \times N$ matrix. Directly solving such a system normally requires $O(N^3)$ operations. For a large-scale problem, the computation costs and the memory storage requirements are prohibitively expensive.

If the influence matrix is translation invariant, the spectral technique based on the fast Fourier transform can be applied to overcome these difficulties [57]. Recently, Peirce and Napier [59] have established a spectral multipole method to reduce memory requirements from $O(N^2)$ to $O(N)$ words and computational costs from $O(N^3)$ operations to $O(N^2 \log N)$ operations for problems with arbitrary geometries.

We shall introduce a novel wavelet approximation method for the conversion of dense matrices with arbitrary geometries to a sparse form. The conversion requires $O(N^2)$ operations and the algorithm for solving the resulting sparse system requires only $O(N \log^2 N)$ operations. This method reduces the memory requirement from $O(N^2)$ to $O(N \log N)$ words.

Up till now, only one paper [68] gives a local error estimate with order $O(h)$ for the piecewise constant collocation approximation to (1.2.16). We shall discuss the complete asymptotics of solutions on the whole interval and provide a global error analysis for piecewise polynomial approximations.

1.3 Organization of Dissertation

The objective of this dissertation is to study the hypersingular integral equations both theoretically and numerically.

By introducing weighted Sobolev space, we establish a global existence of the solution for the hypersingular integral equation (1.1.6) in Chapter 2. The uniqueness and continuous dependence of solutions on data $f(x)$ (see (1.1.6)) are also proved. Moreover, in the one-dimensional case, we obtain analytic solutions of (1.1.6) in terms of distributions.

In Chapter 3, we review some properties of wavelets which will be applied in the following chapter.

A fast wavelet approximation algorithm is presented in Chapter 4. Combining piecewise polynomial collocation approximations with direct wavelet approximations, we introduce the matrix problem which makes the extensions to high dimensional problems possible. The Schulz fast iterative method for matrix inversions is described.

We give the global error analysis in Chapter 5 for approximations based on piecewise polynomial collocation.

Chapter 6 illustrates the theoretical results by numerical examples.

The conclusions will be drawn in the last chapter.

Chapter 2

Well-posedness

Let us rewrite the dominant part of integral equations of the first kind (1.1.6) as

$$T\,u(x) = f(x), \qquad -\ell < x < \ell\ , \tag{2.1}$$

where

$$T\,u(x) = \frac{1}{\pi} \fint_{-\ell}^{\ell} \frac{u(t)}{(t-x)^2}\,dt, \qquad -\ell < x < \ell\ .$$

The symbol T can be viewed as an operator from a normed space X into a normed space Y. It is well known that integral equations of the first kind, in general, are ill-posed since compact operators cannot have a bounded inverse. Roughly speaking, we say that equation (2.1) is well-posed provided

i. A solution exists;

ii. The solution is unique;

iii. The solution depends continuously on the data $f(x)$.

Otherwise the equation (2.1) is ill-posed.

The ill-posedness of an equation will result in problems in its numerical solution. If the equation is ill-posed, then numerical approximations are sensitive to perturbed data. Therefore our work to establish well-posedness is motivated by the need to develop a framework for the analysis of numerical approximations. To this end we consider the following question:

> Can we find suitable spaces X and Y such that the integral equation (2.1) of the first kind is well-posed?

To answer this question, we start by introducing the appropriate weighted Sobolev spaces.

2.1 Weighted Sobolev Spaces

The Sobolev spaces $H^s(\Omega)$, $s \in \mathbf{R}$ – subsets of distribution spaces were first introduced by Sobolev [76], where Ω is an open set in $\mathbf{R}^n$. Let us review definitions of some basic properties of distributions. For more details, see Schwartz [75], Hörmander [35], and Egorov [23].

We denote by $C^\infty(\Omega)$ the set of infinitely differentiable functions on Ω. The class $C_0^\infty(\Omega)$ is a subset of $C^\infty(\Omega)$ consisting of function with compact support.

We shall say that $\varphi_j \to \varphi$ in $C_0^\infty(\Omega)$ provided

i. There is a compact subset $K \subset \Omega$ such that $\varphi_j = 0$ in $\Omega \backslash K$.

ii. For any α, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbf{Z}_+^n$, the sequence $D^\alpha \varphi_j$ converges to $D^\alpha \varphi$ uniformly in K.

The space $C_0^\infty(\Omega)$, endowed with such a topology, is called the space of test functions and is commonly denoted by $\mathcal{D}(\Omega)$. (This notation was introduced by L. Schwartz [75]).

Definition 2.1.1 *A linear functional u that is continuous on the space $\mathcal{D}(\Omega)$ is called a distribution.*

The set of all distributions is denoted by $\mathcal{D}'(\Omega)$.

The above distribution space is not the only possible one. We can consider the space $C^\infty(\Omega)$ as the space of test functions instead of $C_0^\infty(\Omega)$. We say that a sequence φ_j converges to φ in $C^\infty(\Omega)$, if for every α, $\alpha \in \mathbf{Z}_n^+$, and any compact subset $K \subset\subset \Omega$ the sequence $D^\alpha \varphi_j$ converges to $D^\alpha \varphi$ uniformly in K. $\mathcal{E}(\Omega)$ denotes the space $C^\infty(\Omega)$ equipped with such a topology [76].

Definition 2.1.2 *$\mathcal{E}'(\Omega)$ is the space of linear continuous functionals on the space $\mathcal{E}(\Omega)$.*

We know that Fourier transforms play an important role in developing analysis. One task is how to extend the definitions and results of Fourier transforms from functions on $\Omega = \mathbf{R}^n$ to distributions.

The Fourier transform of a function $f \in L^1$ is a function $\mathcal{F}(f) = \hat{f}$ on $\mathbf{R}^n$ defined by

$$\hat{f}(\xi) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} f(x) e^{-ix\xi}\, dx, \qquad \xi \in \mathbf{R}^n \tag{2.1.1}$$

where $x\xi = \sum_{j=1}^n x_i \xi_j$ and L^1 is a set of integrable functions on $\mathbf{R}^n$.

If $f,\ g \in L^1$, we have $f\,\hat{g} \in L^1$. By Fubini's theorem, we obtain

$$\int_{\mathbf{R}^n} f(x)\hat{g}(x)\, dx = \int_{\mathbf{R}^n} g(\xi)\hat{f}(\xi)\, d\xi\ . \tag{2.1.2}$$

We may try to extend the definition of the Fourier transformation according to the above relation (2.1.2) by taking one of the two functions as a test function and the other as a distribution. If we choose $g \in \mathcal{D}(\mathbf{R}^n)$ then (2.1.2) may be written as

$$\langle \hat{f}, g \rangle = \langle f, \hat{g} \rangle \tag{2.1.3}$$

which makes sense provided $\hat{g}$ is in $\mathcal{D}$. This is, however, not true in general. Since $\hat{g}$ is analytic when $g \in \mathcal{D}$ and cannot have compact support unless it is identically zero. This means that $\mathcal{D}$ is too small as a space of test functions, or, equivalently, that $\mathcal{D}'$ is too large for extension.

If we take g in $\mathcal{E}$, $\hat{g}$ may not exist. In other words, $\mathcal{E}$ is too big as a space of test functions.

Schwartz [75] solved the above problem by introducing tempered distribution.

A function $\varphi \in C^\infty(\Omega)$ is said to be rapidly decreasing if

$$\sup_{x\in\Omega} |x^\alpha \partial^\beta \varphi(x)| < \infty$$

for all pairs of multi-indices α and β, α, $\beta \in \mathbb{Z}_+^n$. This is equivalent to the condition that

$$\sup_{|\beta|\leq p} \sup_{x\in\Omega} (1+|x|^2)^p |\partial^\beta \varphi(x)| < \infty$$

for all $p \in \mathbb{Z}_+$. We shall denote by $\mathcal{S}$ the set of all rapidly decreasing functions. A sequence φ_j converges to 0 in $\mathcal{S}$ if and only if

$$x^\alpha \partial^\beta \varphi_j(x) \to 0$$

uniformly on Ω as $j \to \infty$.

Definition 2.1.3 *The space of continuous linear functionals on $\mathcal{S}$ is called the space of tempered distributions and denoted by $\mathcal{S}'$.*

It is clear that

$$\mathcal{D} \subset \mathcal{S} \subset \mathcal{E}$$

and

$$\mathcal{E}' \subset \mathcal{S}' \subset \mathcal{D}' .$$

We have:

Proposition 2.1.1 ([75],[77]) *If $\varphi \in \mathcal{S}$, then $\hat{\varphi} \in \mathcal{S}$.*

Definition 2.1.4 *For any $T \in \mathcal{S}'$ the Fourier transform $\mathcal{F}(T) = \hat{T}$ is defined by*

$$\hat{T}(\varphi) = T(\hat{\varphi}), \qquad \varphi \in \mathcal{S} \tag{2.1.4}$$

Proposition 2.1.2 ([75],[77]) *$\hat{T} \in \mathcal{S}'$ for every $T \in \mathcal{S}'$.*

Let $\Omega = \mathbf{R}^n$, $L^2(\mathbf{R}^n)$ be the set of square-integrable functions on $\mathbf{R}^n$.

Definition 2.1.5 *For any $s \in \mathbf{R}$, we define*

$$H^s(\mathbf{R}^n) = \{u \in \mathcal{S}'(\mathbf{R}^n) : (1+|\xi|^2)^{s/2}\hat{u}(\xi) \in L^2(\mathbf{R}^n)\}$$

and

$$\|u\|_s = \left(\int_{\mathbf{R}^n} (1+|\xi|^2)^s |\hat{u}(\xi)|^2 d\xi\right)^{1/2} .$$

If $s = m \in \mathbf{Z}_+$, then

$$H^m(\mathbf{R}^n) = \{u \in L^2(\mathbf{R}^n) : \partial^\alpha u \in L^2(\mathbf{R}),\ |\alpha| \le m\}$$

where $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbf{Z}_+^n$ and $D^\alpha u$ is weak derivative, i.e.:

$$\int_{\mathbf{R}^n} \varphi\, D^\alpha u\, dx = (-1)^{|\alpha|} \int_{\mathbf{R}^n} u\, D^\alpha \varphi\, dx, \qquad \varphi \in C_0^\infty(\mathbf{R}^n) .$$

The norm associated with H^m is defined by

$$\|u\|_m = \left[\sum_{|\alpha| \le m} \int_{\mathbf{R}^n} |D^\alpha u(x)|^2 dx\right]^{1/2} .$$

It is easy to see that

$$\mathcal{S} \subset H^s \subset H^t \subset \mathcal{S}', \qquad \text{for } s, t \in \mathbf{R},\ s > t .$$

When Ω is a smooth bounded open subset of $\mathbf{R}^n$, we define $H^s(\Omega)$ to be the space of restrictions to Ω of the elements of $H^s(\mathbf{R}^n)$, for any $s \in \mathbf{R}$.

If $s = m \in \mathbf{Z}_+$, then

$$H^m(\Omega) = \{u \in L^2(\Omega) : D^\alpha u \in L^2(\Omega),\ |\alpha| \le m\}$$

with norm

$$\|u\|_m = \left[\sum_{|\alpha|\le m} \int_\Omega |D^\alpha u(x)|^2 dx\right]^{1/2} .$$

We now define the weighted Sobolev spaces.

Let Ω be an open set of $\mathbf{R}^n$ and let m be a nonnegative integer, i.e., $m \in \mathbf{Z}_+^n$. Assume that

$$\sigma = \{\sigma_\alpha = \sigma_\alpha(x),\ x \in \Omega,\ |\alpha| \le m\}$$

is a vector of nonnegative (positive almost every where) measurable function on Ω, where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbf{Z}_+^n$ and $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$. We call σ a weight.

Definition 2.1.6 ([42]) *We define*

$$H^m(\Omega, \sigma) = \left\{u \in \mathcal{S}'(\Omega) :\ \int_\Omega |D^\alpha u(x)|^2 \sigma_\alpha(x)\, dx < \infty,\ |\alpha| \le m\right\} .$$

$H^m(\Omega, \sigma)$ is a normed linear space when equipped with the norm

$$\|u\|_{m,\sigma} = \left(\sum_{|\alpha|\le m} \int_\Omega |D^\alpha u(x)|^2 \sigma_\alpha(x)\right)^{1/2} .$$

If $m = 0$, then $H^0(\Omega, \sigma) = L^2(\Omega, \sigma)$. If $\sigma \equiv 1$, $H^m(\Omega, \sigma) = H^m(\Omega)$ is a "classical" Sobolev space.

Recall that

$$\frac{1}{\pi} \fint_{-\ell}^{\ell} \frac{u(t)}{(t-x)^2}\, dt = f(x), \qquad -\ell < x < \ell .$$

Without loss of generality, we assume $\ell = 1$ and $\Omega = (-1, 1) \subset \mathbf{R}$. For a crack in an infinite strip, the solution is assumed to be of the form $u(x) = F(x)w(x)$, where $w(x) = \sqrt{1 - x^2}$ for an internal crack [40].

Let $\rho = \sigma = \sqrt{1-x^2}$, we consider some special cases. By definition 2.1.6, we write

$$L^2(\rho) = \left\{ u : \|u\|_{0,\rho} = \left(\int_{-1}^{1} |u(x)|^2 \rho(x)\, dx \right)^{1/2} < \infty \right\},$$
$$L^2\left(\frac{1}{\rho}\right) = \left\{ u : \|u\|_{0,\frac{1}{\rho}} = \left(\int_{-1}^{1} |u(x)|^2 / \rho(x)\, dx \right)^{1/2} < \infty \right\},$$

and

$$H^1\left(\frac{1}{\rho}, \rho\right) = \left\{ u : u \in L^2\left(\frac{1}{\rho}\right) \text{ and } Du \in L^2(\rho) \right\}$$

equipped with the following norm:

$$\|u\|_{1,\frac{1}{\rho},\rho} = \left[\|u\|_{0,\frac{1}{\rho}}^2 + \|Du\|_{0,\rho}^2 \right]^{1/2}.$$

Since $\int_{-1}^{1} u^2(x)\rho(x)\, dx \le \int_{-1}^{1} u^2(x)\, dx \le \int_{-1}^{1} \frac{u^2(x)}{\rho(x)}\, dx$, We have

$$L^2\left(\frac{1}{\rho}\right) \subset L^2 \subset L^2(\rho).$$

It is clear that $H^1\left(\frac{1}{\rho}, \rho\right) \subset L^2$.

We shall claim that:

Theorem 2.1.1 *$H^1(\Omega) \subset H^1\left(\frac{1}{\rho}, \rho\right)$, where $\Omega = (-1, 1)$.*

Proof:

$$\begin{aligned} \|u\|_{0,\frac{1}{\rho}}^2 &= \int_{-1}^{1} \frac{|u(x)|^2}{\rho(x)}\, dx = \int_{-1}^{1} \frac{|u(x)|^2}{\sqrt{1-x^2}}\, dx \\ &= \int_{0}^{1} \frac{|u(x)|^2}{\sqrt{1-x^2}}\, dx + \int_{-1}^{0} \frac{|u(x)|^2}{\sqrt{1-x^2}}\, dx \end{aligned} \tag{2.1.5}$$

Without loss of generality, we consider the term $\int_0^1 \frac{|u(x)|^2}{\sqrt{1-x^2}}\, dx$.

$$\begin{aligned} &\left| \int_0^1 \frac{u^2(x)}{\sqrt{1-x^2}}\, dx \right| = \left| \int_0^1 \frac{u^2(x)}{\sqrt{1-x}\sqrt{1+x}}\, dx \right| \\ &\le \left| \int_0^1 \frac{u^2(x)}{\sqrt{1-x}}\, dx \right| \end{aligned}$$

$$
\begin{aligned}
&= \left| -2\sqrt{1-x}\,u^2(x)|_0^1 + 4\int_0^1 \sqrt{1-x}\,u(x)Du(x)\,dx \right| \\
&\le 2|u^2(0)| + 4\left| \int_0^1 \sqrt{1-x}\,u(x)Du(x)\,dx \right| \\
&= \mathrm{I} + \mathrm{II}\,. \qquad (2.1.6)
\end{aligned}
$$

Let us estimate part II first.

$$
\begin{aligned}
\mathrm{II} &\le c\left| \int_0^1 u(x)\,Du(x)\,dx \right| \\
&\le c\left(\int_0^1 |u(x)|^2\,dx \right)^{1/2} \left(\int_0^1 |Du(x)|^2 dx \right)^{1/2} \quad \text{(Cauchy inequality)} \\
&\le c\left(\int_{-1}^1 |u(x)|^2\,dx \right)^{1/2} \left(\int_{-1}^1 |Du(x)|^2\,dx \right)^{1/2} \\
&\le c\left[\int_{-1}^1 |u(x)|^2\,dx + \int_{-1}^1 |Du(x)|^2\,dx \right] \\
&= c\|u\|_{H^1}^2\,. \qquad (2.1.7)
\end{aligned}
$$

For the first part I, we take a function $\varphi(x) \in C_0^\infty(\Omega)$, such that

$$
\varphi(x) = \begin{cases} 1 & |x| < \frac{1}{4} \\ 0 & |x| > \frac{1}{2} \end{cases}
$$

We define $v(x) = u(x)\varphi(x)$, then

$$
v(x) = -\int_x^1 Dv(t)\,dt = \int_1^x Dv(t)\,dt\,,
$$

moreover, $u(0) = v(0)$.

Now,

$$
\begin{aligned}
|u(0)|^2 = |v(0)|^2 &= \left| \int_0^1 Dv(x)\,dx \right|^2 \\
&= \left| \int_0^1 (Du(x)\varphi(x) + u(x)D\varphi(x))\,dx \right|^2 \\
&\le \left| \left(\int_0^1 |Du(x)|^2\,dx \right)^{1/2} \left(\int_0^1 |\varphi(x)|^2\,dx \right)^{1/2} \right.
\end{aligned}
$$

$$
\begin{aligned}
&+ \left(\int_0^1 |u(x)|^2\,dx\right)^{1/2} \left(\int_0^1 |D\varphi(x)|^2\,dx\right)^{1/2}\Bigg|^2 \\
\leq\ & c'\left|\left(\int_{-1}^1 |Du(x)|^2\,dx\right)^{1/2} + \left(\int_{-1}^1 |u(x)|^2\,dx\right)^{1/2}\right|^2 \\
\leq\ & c\left(\int_{-1}^1 |u(x)|^2\,dx + \int_{-1}^1 |Du(x)|^2\,dx\right) \\
=\ & c\|u\|_{H^1}^2 \qquad (2.1.8)
\end{aligned}
$$

where c and c' are constants. By (2.1.6), (2.1.7) and (2.1.8), we obtain

$$\int_0^1 \frac{|u(x)|^2}{\sqrt{1-x^2}}\,dx \leq c\|u\|_{H^1}^2 .$$

Similarly, we have

$$\int_{-1}^0 \frac{|u(x)|^2}{\sqrt{1-x^2}}\,dx \leq c\|u\|_{H^1}^2 .$$

Therefore, $\|u\|_{0,\frac{1}{\rho}} \leq c\|u\|_{H^1}$. Obviously, $\|Du\|_{0,\rho} \leq \|Du\|_{L^2} \leq \|u\|_{H^1}$. Hence, we obtain

$$\|u\|_{1,\frac{1}{\rho},\rho} = \left[\|u\|_{0,\frac{1}{\rho}}^2 + \|Du\|_{0,\rho}^2\right]^{1/2} \leq c'\|u\|_{H^1} .$$

which completes the proof.

We shall now introduce the dual space of $H^m(\Omega,\sigma)$.

Definition 2.1.7 *For $0 \leq m < \infty$, by $(H^m(\Omega,\sigma))^*$ we denote the dual space of $H^m(\Omega,\sigma)$, i.e., the space of bounded linear functionals on $H^m(\Omega,\sigma)$.*

By Theorem 2.1.1, one immediately has that

$$\left(H^1\left(\frac{1}{\rho},\rho\right)\right)^* \subset (H^1)^* \subset \mathcal{S}' . \qquad (2.1.9)$$

To characterize $(H^1(\frac{1}{\rho},\rho))^*$, we note that the following facts [66], [30]:

1. The Chebyshev polynomials of the first kind

$$T_n(x) = \cos n\theta,\ \theta = \arccos x,\ -1 < x < 1,\ n = 0, 1, 2, \ldots$$

form an orthogonal basis for $L^2\left(\frac{1}{\rho}\right)$.

2. The Chebyshev polynomials of the second kind

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta},\ \theta = \arccos x,\ -1 < x < 1,\ n = 0, 1, 2, \ldots$$

constitute an orthogonal basis for $L^2(\rho)$.

3. $T_n'(x) = n\, U_{n-1}(x),\ n = 1, 2, \ldots$

Assume now that $u \in H^1\left(\frac{1}{\rho}, \rho\right)$, then

$$u(x) = \sum_{n=0}^{\infty} \langle u, T_n\rangle_{\frac{1}{\rho}} T_n(x) \tag{2.1.10}$$

and

$$Du(x) = \sum_{n=0}^{\infty} \langle Du, U_n\rangle_{\rho} U_n(x)\ . \tag{2.1.11}$$

Where

$$\langle u, T_n\rangle_{\frac{1}{\rho}} = \int_{-1}^{1} \frac{u(x)T_n(x)}{\rho(x)}\, dx$$

and

$$\langle Du, U_n\rangle_{\rho} = \int_{-1}^{1} Du(x)U_n(x)\rho(x)\, dx\ .$$

Integrating by parts, we obtain

$$\begin{aligned} \langle Du, U_n\rangle_{\rho} &= \int_{-1}^{1} Du(x) \cdot \frac{sin(n+1)\theta}{\sin\theta} \sqrt{1-x^2}\, dx\ ,\ \theta = \arccos x \\ &= \int_{-1}^{1} Du(x) \cdot \sin((n+1)\arccos x)\, dx \\ &= u(x)\sin((n+1)\arccos x)|_{-1}^{1} \\ &\quad +(n+1)\int_{-1}^{1} \frac{u(x)\cos(n+1)\theta}{\sqrt{1-x^2}}\, dx \\ &= (n+1)\langle u, T_{n+1}\rangle_{\frac{1}{\rho}}\ . \end{aligned}$$

The term $u(x)\sin(n+1)\theta|_{-1}^{1} = 0$, since $u(x) \in L^2\left(\frac{1}{\rho}\right)$. Hence, (2.1.11) can be rewritten as

$$Du(x) = \sum_{n=0}^{\infty} \langle u, T_{n+1}\rangle_{\frac{1}{\rho}} (n+1) U_n(x) \ . \tag{2.1.12}$$

By Plancherel's theorem

$$\begin{aligned} \|u\|_{1,\frac{1}{\rho},\rho} &= \left[\sum_{n=0}^{\infty} \langle u, T_n\rangle_{\frac{1}{\rho}}^2 + \sum_{n=0}^{\infty} \langle u, T_{n+1}\rangle_{\frac{1}{\rho}}^2 (n+1)^2\right]^{1/2} \\ &= \left(\sum_{n=0}^{\infty} (1+n^2) \langle u, T_n\rangle_{\frac{1}{\rho}}^2\right)^{1/2} \ . \end{aligned} \tag{2.1.13}$$

Theorem 2.1.2 *For $F \in \left(H^1\left(\frac{1}{\rho}, \rho\right)\right)^*$, we have*

$$\|F\|_{(H^1(\frac{1}{\rho},\rho))^*} = \left(\sum_{n=0}^{\infty} (1+n^2)^{-1} |c_n|^2\right)^{1/2}$$

where $c_n = F(T_n)$ and T_n, $n = 0, 1, \ldots$ are Chebyshev polynomials of the first kind. Conversely, to each sequence $\{c_n\}$ satisfying

$$\sum_{n=0}^{\infty} (1+n^2)^{-1} |c_n|^2 < \infty \tag{2.1.14}$$

there exists a bounded linear functional $F \in \left(H^1\left(\frac{1}{\rho}, \rho\right)\right)^*$ such that $F(T_n) = c_n$.

Proof: Assume that the sequence $\{c_n\}$ satisfies inequality (2.1.14) and define a functional $F : H^1\left(\frac{1}{\rho}, \rho\right) \to \mathbf{R}$ by

$$F(\varphi) := \sum_{n=0}^{\infty} a_n c_n \ ,$$

where $\varphi \in H^1\left(\frac{1}{\rho}, \rho\right)$ with $a_n = \langle \varphi, T_n\rangle_{\frac{1}{\rho}}$. By the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} |F(\varphi)|^2 &\le \sum_{n=0}^{\infty} (1+n^2)^{-1} |c_n|^2 \cdot \sum_{n=0}^{\infty} (1+n^2) |a_n|^2 \\ &= \|\varphi\|_{1,\frac{1}{\rho},\rho}^2 \sum_{n=0}^{\infty} (1+n^2)^{-1} |c_n|^2 \ . \text{ (by (2.1.13))} \end{aligned}$$

Therefore,

$$\|F\|_{(H^1(\frac{1}{\rho},\rho))^*} \le \left(\sum_{n=0}^{\infty}(1+n^2)^{-1}|c_n|^2\right)^{1/2} .$$

Conversely, if $F \in \left(H^1\left(\frac{1}{\rho},\rho\right)\right)^*$ with $c_n = F(T_n)$, we define for $m \in \mathbb{Z}_+$

$$\varphi_m := \sum_{n=0}^{m}(1+n^2)^{-1}c_nT_n .$$

Then

$$\begin{aligned}\|\varphi_m\|_{1,\frac{1}{\rho},\rho} &= \left(\|\varphi_m\|^2_{0,\frac{1}{\rho}} + \|D\varphi_m\|^2_{0,\rho}\right)^{1/2} \\ &= \left(\sum_{n=0}^{m}(1+n^2)^{-1}|c_n|^2\right)^{1/2} .\end{aligned}$$

Moreover,

$$F(\varphi_m) = \sum_{n=0}^{m}(1+n^2)^{-1}|c_n|^2 .$$

Therefore

$$\|F\|_{(H^1(\frac{1}{\rho},\rho))^*} \ge \frac{|F(\varphi_m)|}{\|\varphi_m\|_{1,\frac{1}{\rho},\rho}} = \left(\sum_{n=0}^{m}(1+n^2)^{-1}|c_n|^2\right)^{1/2} .$$

Hence, we have

$$\left(\sum_{n=0}^{\infty}(1+n^2)^{-1}|c_n|^2\right)^{1/2} \le \|F\|_{(H^1(\frac{1}{\rho},\rho))^*} .$$

The proof is complete.

Let

$$S_1 = \left\{\{c_n\} : \sum_{n=0}^{\infty}(1+n^2)^{-1}|c_n|^2 < \infty\right\} .$$

From theorem 2.1.2, we have

Lemma 2.1.1 *S_1 is isomorphic to $\left(H^1\left(\frac{1}{\rho},\rho\right)\right)^*$.*

And we define

$$S := \left\{u : u = \sum_{n=0}^{\infty} c_nU_n,\ \|u\|_S = \left(\sum_{n=0}^{\infty}(1+n)^{-2}|c_n|^2\right)^{1/2} < \infty\right\}$$

where U_n are the Chebyshev polynomials of the second kind. It is easy to see that

Lemma 2.1.2 *S is isomorphic to S_1.*

2.2 Existence, Uniqueness and Stability

At the beginning of this chapter, we pointed out that existence, uniqueness and stability are the three requirements of well-posedness. We shall make them more precise by introducing the following definitions.

Let us consider an operator equation

$$Au = f \ ,$$

and

$$A : X \to Y$$

where X, Y are normed spaces. The range $A(X)$ is defined by

$$A(X) := \{Au : u \in X\} \ .$$

Definition 2.2.1 *If for each $f \in A(X)$ there is only one element $u \in X$ with $Au = f$ then A is said to be injective.*

Definition 2.2.2 *If $A(X) = Y$ then A is said to be surjective.*

Definition 2.2.3 *The mapping A is called bijective if it is injective and surjective, that is, if the inverse mapping $A^{-1} : Y \to X$ exists.*

Definition 2.2.4 *The equation*

$$Au = f$$

is called well-posed if A is bijective and the inverse operator $A^{-1} : Y \to X$ is continuous. Otherwise the equation is called ill-posed.

It is clear now that existence and uniqueness of a solution to an operator equation can be equivalently expressed by the existence of the inverse operator. Stability is equivalent to continuity of the inverse operator.

Recall that

$$\frac{1}{\pi}\;=\!\!\!\!\!\!\int_{-1}^{1}\frac{v(t)}{(t-x)^2}\,dt = f(x), \qquad -1 < x < 1 \tag{2.2.1}$$

and the internal crack suggests solutions take the form [40]:

$$v(t) = \sqrt{1-t^2}\,u(t)\ .$$

Therefore, we define

$$Au(x) := \frac{1}{\pi}\;=\!\!\!\!\!\!\int_{-1}^{1}\frac{\sqrt{1-t^2}\,u(t)}{(t-x)^2}\,dt \tag{2.2.2}$$

where $=\!\!\!\!\!\int$ denotes the Hadamard finite part.

It is well known that [30], [39]

$$\frac{1}{\pi}\;-\!\!\!\!\!\!\int_{-1}^{1}\frac{\sqrt{1-t^2}\,U_n(t)}{(t-x)}\,dt = -T_{n+1}(x), \qquad n = 0,1,2,\ldots \tag{2.2.3}$$

where $T_n(x)$ and $U_n(x)$ are Chebyshev polynomials of the first and the second kind respectively. $-\!\!\!\!\!\int$ exists in the principal value sense which is defined as follows

$$-\!\!\!\!\!\!\int_{-1}^{1}\frac{f(t)}{t-x}\,dt = \lim_{\epsilon\to 0}\left(\int_{-1}^{x-\epsilon}\frac{f(t)}{t-x}\,dt + \int_{x+\epsilon}^{1}\frac{f(t)}{t-x}\,dt\right)\ .$$

By the Hadamard finite part and $T'_{n+1}(x) = (n+1)U_n(x)$, we obtain

$$\frac{1}{\pi}\;=\!\!\!\!\!\!\int_{-1}^{1}\frac{\sqrt{1-t^2}U_n(t)}{(t-x)^2}\,dt = -(n+1)U_n(x), \qquad n = 0,1,2,\ldots$$

i.e.,

$$AU_n(x) = -(n+1)U_n(x), \qquad n = 0,1,2,\ldots \tag{2.2.4}$$

Our main theorem, in this section, is as follows

Theorem 2.2.1 *Let*

$$Au = f \tag{2.2.5}$$

where A is defined by (2.2.2). Then A is bijective from $L^2(\rho)$ to S, where $\rho = \sqrt{1-x^2}$ and

$$S = \left\{ u : u = \sum_{n=0}^{\infty} c_n U_n,\ \|u\|_S = \left(\sum_{n=0}^{\infty} (1+n)^{-2} |c_n|^2 \right)^{1/2} < \infty \right\} ,$$

and A^{-1} continuous. That is, the equation (2.2.5) is well-posed.

Proof: Assume $u \in L^2(\rho)$, then

$$u(x) = \sum_{n=0}^{\infty} \langle u, U_n \rangle_\rho U_n(x) .$$

From Plancherel's theorem, we have

$$\|u\|_{L^2(\rho)} = \left(\sum_{n=0}^{\infty} \langle u, U_n \rangle_\rho^2 \right)^{1/2} .$$

Moreover, by formula (2.2.4), one has

$$\begin{aligned} Au(x) &= \sum_{n=0}^{\infty} \langle u, U_n \rangle_\rho AU_n \\ &= -\sum_{n=0}^{\infty} \langle u, U_n \rangle_\rho (n+1)\, U_n . \end{aligned}$$

Therefore,

$$\begin{aligned} \|Au\|_S &= \left(\sum_{n=0}^{\infty} \frac{\langle u, U_n \rangle_\rho^2 (1+n)^2}{(1+n)^2} \right)^{1/2} \\ &= \|u\|_{L^2(\rho)} . \end{aligned} \tag{2.2.6}$$

For $f \in S$, $f(x) = \sum_{n=0}^{\infty} c_n U_n(x)$, we define

$$Bf := -\sum_{n=0}^{\infty} \frac{c_n}{1+n}\, U_n .$$

Then

$$\|Bf\|_{L^2(\rho)} = \left(\sum_{n=0}^{\infty} \frac{|c_n|^2}{(1+n)^2} \right)^{1/2} = \|f\|_S \tag{2.2.7}$$

i.e., B is a bounded operator from S to $L^2(\rho)$. If we can show that B is the inverse operator of A, then the theorem is proved.

By (2.2.6), $Au(x) \in S$ when $u(x) \in L^2(\rho)$, hence

$$\begin{aligned} BAu(x) &= \sum_{n=0}^{\infty} \frac{\langle u, U_n\rangle_\rho (1+n)}{(1+n)} U_n(x) \\ &= \sum_{n=0}^{\infty} \langle u, U_n\rangle_\rho U_n(x) \\ &= u(x) \ . \end{aligned}$$

On the other hand, if $f \in S$, $f = \sum_{n=0}^{\infty} c_n U_n$, then

$$Bf(x) = -\sum_{n=0}^{\infty} \frac{c_n}{1+n} U_n(x) \ .$$

By (2.2.7), $Bf(x) \in L^2(\rho)$.

$$\begin{aligned} ABf(x) &= -\sum_{n=0}^{\infty} \frac{c_n}{1+n} AU_n(x) \\ &= \sum_{n=0}^{\infty} c_n U_n \\ &= f(x) \end{aligned}$$

i.e., $B = A^{-1}$, which completes the proof.

From Lemma 2.1.1 and Lemma 2.1.2, we have

Theorem 2.2.2 *Assume that A is the same operator as in theorem 2.2.1, then*

$$A : L^2(\rho) \to \left(H^1\left(\frac{1}{\rho}, \rho\right)\right)^*$$

is bijective and A^{-1} is continuous, i.e., the equation

$$Au(x) = f(x)$$

is well-posed.

Note that since $v(t) = \sqrt{1-t^2}\, u(t)$, it follows that

$$v(x) \in L^2\left(\frac{1}{\rho}\right) \Longleftrightarrow u(x) \in L^2(\rho) .$$

Therefore, we obtain

Theorem 2.2.3 *For $f(x) \in S$, there exists a unique solution $v(x) \in L^2\left(\frac{1}{\rho}\right)$ to equation (2.2.1).*

2.3 Analytic Solution of the One-Dimensional Problem

This section is devoted to the study of analytic solutions of the one-dimensional hypersingular integral equation:

$$\frac{1}{\pi} \oint_{-1}^{1} \frac{v(t)}{(t-x)^2}\, dt = f(x), \qquad -1 < x < 1 . \tag{2.3.1}$$

Let us review some results about Cauchy singular integral equations. Define

$$[H_\nu u](x) = \frac{1}{\pi} \int_{-1}^{1} \frac{w_\nu(t)\, u(t)}{(t-x)}\, dt, \qquad -1 < x < 1 , \tag{2.3.2}$$

where

$$w_\nu(x) = \begin{cases} \sqrt{1-x^2} , & \nu = -1 , \\ \frac{1}{\sqrt{1-x^2}} , & \nu = 1 . \end{cases} \tag{2.3.3}$$

Then we have (for proofs see [30]):

Lemma 2.3.1 *Let $T_n(x) = \cos(n \arccos x)$, $-1 \le x \le 1$, $n = 0, 1, 2, \ldots$ denote the Chebyshev polynomials of the first kind. Then*

$$\begin{cases} H_1 T_0 = 0, \\ H_1 T_n = U_{n-1}, \ n = 1, 2, \ldots, \ -1 \le x \le 1 , \end{cases} \tag{2.3.4}$$

where $U_n(x) = \sin((n+1)\arccos x)/\sin(\arccos x)$, $n = 0, 1, 2, \ldots$, *are the Chebyshev polynomials of the second kind.*

Lemma 2.3.2

$$H_{-1}U_n = -T_{n+1}, \qquad n \geq 0,\ -1 \leq x \leq 1\ . \tag{2.3.5}$$

Lemma 2.3.2 tells us that H_{-1} is a shift operator from the Sobolev space $L^2(\rho)$ to Sobolev space $L^2\left(\frac{1}{\rho}\right)$. H_{-1} is, therefore, invertible from the left [65], [29], [63].

In fact, suppose $u(x) \in L^2(\rho)$, then

$$u(x) = \sum_{n=0}^{\infty} \langle u, U_n \rangle_\rho U_n(x)\ .$$

Applying (2.3.5), we obtain

$$\begin{aligned} H_{-1}u(x) &= \sum_{n=0}^{\infty} \langle u, U_n \rangle_\rho H_{-1}U_n(x) \\ &= -\sum_{n=0}^{\infty} \langle u, U_n \rangle_\rho T_{n+1}(x)\ . \end{aligned} \tag{2.3.6}$$

From (2.3.4), we have

$$\begin{aligned} -H_1(H_{-1}u)(x) &= \sum_{n=0}^{\infty} \langle u, U_n \rangle_\rho H_1T_{n+1}(x) \\ &= \sum_{n=0}^{\infty} \langle u, U_n \rangle_\rho U_n \\ &= u(x)\ . \end{aligned} \tag{2.3.7}$$

Formula (2.3.7) shows that the operator $-H_1$ is a left inverse of H_{-1}, i.e., if

$$\frac{1}{\pi} \int_{-1}^{1} \frac{\sqrt{1-t^2}\,u(t)}{(t-x)}\,dt = g(x)\ , \qquad -1 < x < 1\ , \tag{2.3.8}$$

then

$$\frac{1}{\pi} \int_{-1}^{1} \frac{g(t)}{\sqrt{1-t^2}(x-t)}\,dt = u(x)\ , \qquad -1 < x < 1\ . \tag{2.3.9}$$

By the Hadamard finite part, formula (2.3.8) gives

$$\frac{1}{\pi}\oint_{-1}^{1}\frac{\sqrt{1-t^2}\,u(t)}{(t-x)^2}\,dt = g'(x)\,, \qquad -1 < x < 1\,, \tag{2.3.10}$$

where $g'(x)$ is a weak derivative defined by

$$\int_{\Omega}\varphi(x)g'(x)\,dx = -\int_{\Omega}g(x)\varphi'(x)\,dx, \qquad \varphi \in C_0^{\infty}(\Omega)\,.$$

where $\Omega = (-1, 1)$.

Comparing (2.3.10) with (2.3.1), we find that for any $f(x) \in S$, if we can choose $g(x)$ such that $g'(x) = f(x)$, then formula (2.3.9) gives a solution to (2.3.1). We previously mentioned $S \subset S' \subset \mathcal{D}'$. The above discussion therefore gives rise to the following question:

How can we define an antiderivative for a distribution?

We shall use some definitions and properties from [81].

Definition 2.3.1 *Let F, $G \in \mathcal{D}'$, G is said to be an antiderivative of F if $G' = F$, where G' is a weak derivative. A general form of all antiderivatives of F is called indefinite integral, denoted by $IntF$.*

We consider only the one dimension case.

Proposition 2.3.1 *For any $F \in \mathcal{D}'$, there exists an antiderivative G. Moreover $IntF = G + c$, where c is a constant.*

Proof: See [81].

Recall that

$$S = \left\{u(x) : u(x) = \sum_{n=0}^{\infty} c_n U_n(x),\ \|u\|_S = \left(\sum_{n=0}^{\infty}\frac{|c_n|^2}{(1+n)^2}\right)^{1/2} < \infty\right\}\,.$$

Again T_n and U_n, $n = 1, 2, \ldots$ are Chebyshev polynomials of the first and second kind respectively and they are related as follows:

$$T'_{n+1}(x) = (n+1)U_n(x), \qquad n = 0, 1, 2, \ldots .$$

We define an operator Intl such that

$$\begin{aligned} \operatorname{Intl} U_n(x) &= \int_{b_n}^{x} U_n(t)\, d(t) \\ &= \int_{b_n}^{x} \frac{T'_{n+1}(t)}{n+1}\, dt \\ &= T_{n+1}(x)/(n+1)\ , \end{aligned} \tag{2.3.11}$$

where the b_n are chosen such that

$$T_{n+1}(b_n) = 0\ .$$

Thus, if $u \in S$, and

$$u(x) = \sum_{n=0}^{\infty} c_n U_n(x)\ ,$$

we obtain

$$\begin{aligned} \operatorname{Intl} u(x) &= \sum_{n=0}^{\infty} c_n \operatorname{Intl} U_n(x) \\ &= \sum_{n=0}^{\infty} \frac{c_n}{n+1}\, T_{n+1}(x)\ . \end{aligned} \tag{2.3.12}$$

Therefore, by Plancherel's theorem, one has

$$\|\operatorname{Intl} u\|_{L^2(\frac{1}{\rho})} = \left(\sum_{n=0}^{\infty} \frac{|c_n|^2}{(n+1)^2} \right)^{1/2} = \|u\|_S\ , \tag{2.3.13}$$

i.e., the operator Intl is bounded from S to $L^2\left(\frac{1}{\rho}\right)$. Moreover, from (2.3.12), we have

$$\begin{aligned} (\operatorname{Intl} u(x))' &= \sum_{n=0}^{\infty} \frac{c_n}{(n+1)}\, T'_{n+1}(x) \\ &= \sum_{n=0}^{\infty} c_n U_n(x) \\ &= u(x)\ , \end{aligned} \tag{2.3.14}$$

which shows that $\operatorname{Intl} u(x)$ is an antiderivative of $u(x)$.

Assuming now the operator Int is defined by definition 2.3.1, we shall show that for any $u(x) \in S \subset \mathcal{D}'$

$$\operatorname{Int} u(x) - \operatorname{Intl} u(x) = c\,, \tag{2.3.15}$$

where c is a constant.

Lemma 2.3.3 *Let $F \in \mathcal{D}'$ and $F' = 0$ then $F =$ Constant.*

Proof: Choose $\varphi_0(x) \in C_0^\infty(\Omega)$, such that

$$\int_\Omega \varphi_0(x)\,dx = 1\ .$$

where $\Omega = (-1, 1)$.

For $\varphi \in C_0^\infty(\Omega)$, we define an operator J

$$(J\varphi)(x) := \int_{-1}^{x} \left(\varphi(t) - \left(\int_{-1}^{1} \varphi(\tau)\,d\tau\right) \varphi_0(t)\right) dt\ .$$

Then

$$\frac{d}{dx}\, J\varphi(x) = \varphi(x) - \int_{-1}^{1} \varphi(\tau) d\tau\, \varphi_0(x)\ .$$

Hence

$$\begin{aligned} \langle F, \varphi\rangle &= \langle F, \frac{d}{dx} J\varphi + \int_{-1}^{1} \varphi(\tau)\,d\tau\,\varphi_0\rangle \\ &= -\langle F', J\varphi\rangle + \int_{-1}^{1} \varphi(t)\,dt \langle F, \varphi_0\rangle\ . \end{aligned}$$

Since $F' = 0$, we have

$$\langle F, \varphi\rangle = \int_{-1}^{1} \langle F, \varphi_0\rangle\, \varphi(t)\,dt\ .$$

Hence,

$$F = \langle F, \varphi_0\rangle = \text{ Constant,}$$

which completes the Lemma.

By the definitions of the operators Int and Intl:

$$(\text{Int}\, u(x) - \text{Intl}\, u(x))' = u(x) - u(x) = 0 \ ,$$

and from Lemma 2.3.3 one immediately has formula (2.3.15). From above discussion, we obtain

Theorem 2.3.1 *For any $f(x) \in S$, there exists a unique solution $u(x) \in L^2(\rho)$*

$$u(x) = -\frac{1}{\pi} \fint_{-1}^{1} \frac{\text{Intl}\, f}{\sqrt{1-t^2}(t-x)}\, dt \tag{2.3.16}$$

for the equation

$$\frac{1}{\pi} \fint_{-1}^{1} \frac{\sqrt{1-t^2}\, u(t)}{(t-x)^2}\, dt = f(x) \ . \tag{2.3.17}$$

Here we demonstrate this theorem by means of some examples.

Example 2.3.1 To solve

$$\frac{1}{\pi} \fint_{-1}^{1} \frac{u(t)}{(t-x)^2}\, dt = 1, \qquad -1 < x < 1 \ . \tag{2.3.18}$$

By definition 2.3.1, one has Intl $= x + c$. According to formula (2.3.16), we obtain

$$u(x) = -\frac{1}{\pi} \fint_{-1}^{1} \frac{t+c}{\sqrt{1-t^2}(t-x)}\, dt \cdot \sqrt{1-x^2} \ . \tag{2.3.19}$$

This is a Cauchy singular integral, one can use standard complex variable techniques to solve it [30], [55], [64] and [1]. Applying the fact

$$\frac{1}{\pi} \fint_{-1}^{1} \frac{t+c}{\sqrt{1-t^2}(t-x)}\, dt = 1 \ ,$$

we have

$$u(x) = -\sqrt{1-x^2} \ , \tag{2.3.20}$$

which is a solution of (2.3.18)

Example 2.3.2 Consider

$$\frac{1}{\pi}\,⨍_{-1}^{1}\frac{u(t)}{(t-x)^2}\,dt = \frac{1}{1-x^2}\,, \qquad -1 < x < 1\,. \tag{2.3.21}$$

Assume $\varphi \in C_0^\infty(\Omega)$, $\Omega = (-1,1)$, then

$$\begin{aligned}
\langle\frac{1}{1-x^2},\varphi\rangle &= \int_{-1}^{1}\frac{1}{1-x^2}\,\varphi(x)\,dx \\
&= \frac{1}{2}\int_{-1}^{1}\left(\frac{1}{1+x}+\frac{1}{1-x}\right)\varphi(x)\,dx \\
&= -\frac{1}{2}\int_{-1}^{1}\ln\frac{1+x}{1-x}\,\varphi'(x)\,dx \\
&= -\langle\frac{1}{2}\ln\frac{1+x}{1-x},\varphi'\rangle\,.
\end{aligned}$$

Hence,

$$\mathrm{Intl}\,\frac{1}{1-x^2} = \frac{1}{2}\ln\frac{1+x}{1-x} + c$$

where c is a constant. From theorem 2.3.1, we have

$$u(x) = -\frac{1}{\pi}\,⨍_{-1}^{1}\frac{\frac{1}{2}\ln\frac{1+t}{1-t}+c}{\sqrt{1-t^2}(t-x)}\,dt\cdot\sqrt{1-x^2}\,.$$

Since [64]

$$\frac{1}{\pi^2}\,⨍_{-1}^{1}\frac{\ln\frac{1+t}{1-t}+c}{\sqrt{1-t^2}(t-x)}\,dt = \frac{1}{\sqrt{1-x^2}}, \qquad -1 < x < 1\,,$$

one immediately has

$$u(x) = -\frac{\pi}{2}\,.$$

Chapter 3

Wavelets

The term wavelets is used for a family of functions of the form

$$\psi_{a,b}(x) = |a|^{-1/2}\,\psi\left(\frac{x-b}{a}\right), \qquad a,b \in R,\ a \neq 0 \tag{3.1}$$

generated from a single function ψ by the operations of dilation and translation. The idea grew out of seismic analysis [31], [33]. Recent development has been led by A. Cohen, R. Coifman, I. Daubechies, S. Mallat, Y. Meyer and many others [9], [17], [52], [48], [13], [14].

The subject of "wavelet analysis" has drawn much attention from both mathematicians and engineers alike. There are two important mathematical entities in wavelet analysis: the "integral wavelet transform" and the "wavelet series". Unlike Fourier analysis, the wavelet transform and the wavelet series are intimately related, see [8].

Wavelets need not be orthogonal. Both orthogonal and non-orthogonal bases are important. Here we pay particular attention to orthonormal wavelet bases since we would like to represent our hypersingular integral operator in terms of an orthonormal basis of wavelets in a compact way. The standard method to construct orthonormal bases of wavelets is by multiresolution analysis which was introduced by Y. Meyer and S. Mallat [52] [49]. Though not every orthonormal wavelet basis is derived from a multiresolution analysis, see [49], [19], however, if the basis has some regularity and decay, it can necessarily be obtained by a multiresolution analysis [44], [4], [38].

With the goal of applying wavelets to the numerical approximation of hypersingular integral equations, we survey the properties of wavelets that we will require later in the thesis. We start by discussing wavelet transforms which are a tool for decomposing functions and operators in various applications. In the second section, we introduce multiresolution analysis which provides a natural framework for the understanding the wavelet bases. Daubechies' compactly supported wavelets, which have had a great impact on the development in this field, will be presented in the third section. Bases on finite intervals and wavelet algorithms for decomposition and reconstruction will also be covered in this chapter.

The development of this chapter will mainly follow Daubechies [17] [49], Meyer [54], Mallat [48] and Chui [8].

3.1 Wavelet Transforms

3.1.1 The Continuous Wavelet Transform

Given a "mother wavelet" ψ with $\psi \in L^2(\mathrm{R})$, we define the wavelet transform Wf of a function $f(x) \in L^2(\mathrm{R})$ by

$$\begin{aligned}(Wf)(a,b) &= |a|^{-\frac{1}{2}} \int_{-\infty}^{\infty} f(x)\, \overline{\psi\left(\frac{x-b}{a}\right)}\, dx \\ &= 2\pi\, |a|^{1/2} \int_{-\infty}^{\infty} \hat{f}(\xi)\, \overline{\hat{\psi}(a\xi)}\, e^{ib\xi}\, d\xi \\ &\qquad \text{(by Parseval's formula),} \qquad (3.1.1)\end{aligned}$$

where $a, b \in \mathbf{R}$, $a \neq 0$ and "$\widehat{\ }$" denotes Fourier transform of a function defined by

$$\widehat{f}(\xi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x)\, e^{-i\xi x}\, dx\ . \tag{3.1.2}$$

If we define

$$\psi_{a,b}(x) = |a|^{-1/2} \psi\left(\frac{x-b}{a}\right)$$

then

$$(Wf)(a,b) = \langle f, \psi_{a,b} \rangle\ . \tag{3.1.3}$$

The inverse of a wavelet transform for an L^2-function can be obtained through the following proposition.

Proposition 3.1.1 *For all* $f, g \in L^2(\mathbf{R})$,

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (Wf)(a,b)\, \overline{(Wg(a,b))}\, \frac{da\, db}{a^2} = c_\psi \langle f, g \rangle\ , \tag{3.1.4}$$

where

$$c_\psi = \int_{-\infty}^{\infty} |\xi|^{-1} |\widehat{\psi}(\xi)|^2 d\xi < \infty\ . \tag{3.1.5}$$

Proof:

$$\begin{aligned} &\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (Wf)(a,b)\, \overline{(Wg)(a,b)}\, \frac{da\, db}{a^2} \\ &\quad = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[2\pi \int_{-\infty}^{\infty} |a|^{\frac{1}{2}} \widehat{f}(\xi)\, e^{ib\xi}\, \overline{\widehat{\psi}(a\xi)}\, d\xi\right] \\ &\qquad \times \left[2\pi \int_{-\infty}^{\infty} |a|^{\frac{1}{2}}\, \overline{\widehat{g}(\xi')}\, e^{-ib\xi'}\, \widehat{\psi}(a\xi')\, d\xi'\right] \frac{da\, db}{a^2}\ . \end{aligned} \tag{3.1.6}$$

If we define

$$F_a(\xi) = |a|^{\frac{1}{2}} \widehat{f}(\xi)\, \overline{\widehat{\psi}(a\xi)}$$

and

$$G_a(\xi) = |a|^{\frac{1}{2}} \widehat{g}(\xi)\, \overline{\widehat{\psi}(a\xi)}\ .$$

By Parseval's formula it follows that

$$\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(Wf)(a,b)\,\overline{(Wg)(a,b)}\,\frac{da\,db}{a^2}\\
&= 2\pi\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F_a(\xi)\,\overline{G_a(\xi)}\,d\xi\,\frac{da}{a^2}\\
&= 2\pi\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\widehat{f}(\xi)\,\overline{\widehat{g}(\xi)}\,|\widehat{\psi}(a\xi)|^2\,d\xi\,\frac{da}{|a|}\\
&= 2\pi\int_{-\infty}^{\infty}|\xi|^{-1}|\widehat{\psi}(\xi)|^2\,d\xi\int_{-\infty}^{\infty}\widehat{f}(\xi)\,\overline{\widehat{g}(\xi)}\,d\xi\\
&= c_\psi\langle f,g\rangle\,.
\end{aligned}$$

With weak convergence in the L^2-sense, we can rewrite (3.1.4) as

$$f(x)=\frac{1}{c_\psi}\int_{-\infty}^{\infty}(Wf)(a,b)\;\psi_{a,b}\,\frac{da\,db}{a^2}\,. \tag{3.1.7}$$

We shall only consider "admissible" wavelets, i.e., functions ψ which satisfy the admissibility condition (3.1.5). To ensure (3.1.5), we require that ψ has mean zero,

$$\widehat{\psi}(0)=\int_{-\infty}^{\infty}\psi(x)\,dx=0\,. \tag{3.1.8}$$

Remark: We discussed only one-dimensional case, higher dimensional versions are straightforward. For details, see [19], [20] and [18].

3.1.2 The Discrete Wavelet Transform

The formula (3.1.7) shows that a function $f\in L^2(\mathrm{R})$ can be easily reconstructed from its wavelet transform $(Wf)(a,b)$, provided these transforms are known for all values of the parameters a,b. In practice, however, the parameters are restricted to discrete data. Similar to the continuous version, we consider the family

$$\psi_{a,b}(x)=|a|^{-\frac{1}{2}}\,\psi\left(\frac{x-b}{a}\right) \tag{3.1.9}$$

where $a \in R_+$, $b \in R$, $a \neq 0$, a, b are discrete values. Since a is restricted to positive values only (just for convenience), the admissibility condition (3.1.5) becomes

$$c_\psi = \int_0^\infty \xi^{-1}|\widehat{\psi}(\xi)|^2\, d\xi = \int_{-\infty}^0 |\xi|^{-1}|\widehat{\psi}(\xi)|^2\, d\xi < \infty \,. \tag{3.1.10}$$

We choose the dilation parameter $a = a_0^m$, where $a_0 \neq 1$, $m \in$ Z. Without loss of generality, we assume $a_0 > 1$ fixed. For the translation parameter b we choose $b = nb_0a_0^m$, where $n \in$ Z and b_0 is fixed. Formulae (3.1.9) and (3.1.3) can be rewritten as

$$\begin{aligned} \psi_{m,n}(x) &= a_0^{-m/2}\psi\left(\frac{x - nb_0a_0^m}{a_0^m}\right) \\ &= a_0^{-m/2}\psi(a_0^{-m}x - nb_0) \,, \end{aligned} \tag{3.1.11}$$

and

$$W_{m,n}(f) = \langle f, \psi_{m,n}(x)\rangle \,. \tag{3.1.12}$$

In the discrete case, there does not exist, in general, a reconstruction formula analogous to the role played by (3.1.7) for the continuous case. Reconstruction of f from the $W_{m,n}(f)$ must therefore be done using alternative methods.

One would expect that the reconstruction for the discrete case is possible if the discrete lattice has a very fine mesh. It is natural to ask what the threshold for the lattice parameters is? Thus we shall introduce the concept of "frames" to study the possibility of finding an inversion procedure for the discrete wavelet transform

$$W_{m,n}(f) = a_0^{-m/2}\int_{-\infty}^{\infty} f(t)\psi(a_0^{-m}t - nb_0)\, dt \,. \tag{3.1.13}$$

Duffin and Schaeffer [22] introduced Frames for nonharmonic Fourier analysis, see also [82].

Definition 3.1.1 *A family of functions* $(\varphi_j)_{j\in J}$ *in a Hilbert space* H *is called a frame if there exist* $0 < A,\ B < \infty$ *such that, for all* $f \in H$,

$$A\|f\|^2 \le \sum_{j\in\mathbf{Z}} |\langle f, \varphi_j\rangle|^2 \le B\|f\|^2 . \tag{3.1.14}$$

We call A *and* B *the frame bounds. If* $A = B$, *i.e.,*

$$\sum_{j\in\mathbf{Z}} |\langle f, \varphi_j\rangle|^2 = A\|f\|^2 , \tag{3.1.15}$$

then we say the $(\varphi_j)_{j\in J}$ *constitute a tight frame.*

Definition 3.1.2 *If* $(\varphi_j)_{j\in\mathbf{Z}}$ *is a frame in* H, *then the frame operator* T *is the linear operator from* H *to*

$$\ell^2(\mathbf{Z}) = \left\{ c = (c_j)_{j\in\mathbf{Z}} \mid \|c\|^2 = \sum_{j\in\mathbf{Z}} |c_j|^2 < \infty \right\}$$

defined by

$$(Tf)_j = \langle f, \varphi_j\rangle . \tag{3.1.16}$$

The formula (3.1.14) ensures that T is bounded, i.e., $\|Tf\|^2 \le B\|f\|^2$. The adjoint operator T^* of T is given by

$$T^*c = \sum_{j\in\mathbf{Z}} c_j\varphi_j \tag{3.1.17}$$

in the weak sense since

$$\begin{aligned} \langle T^*c, f\rangle &= \langle c, Tf\rangle \\ &= \sum_{j\in\mathbf{Z}} c_j\overline{\langle f, \varphi_j\rangle} \\ &= \sum_{j\in\mathbf{Z}} c_j\langle \varphi_j, f\rangle . \end{aligned}$$

Hence

$$T^*c = \sum_{j\in\mathbf{Z}} c_j\varphi_j \ .$$

Using the standard notation $T_1 \geq T_2$ for two operators $T_1,\ T_2$ if

$$\langle f, T_1 f\rangle \geq \langle f, T_2 f\rangle$$

for all $f \in H$, one can easily check that (3.1.14) can be rewritten as

$$AI \leq T^*T \leq BI \ , \tag{3.1.18}$$

where I denotes the unit operator, $If = f$. Since the constant $A > 0$, (3.1.18) implies that the operator T^*T is invertible [38], we define

$$\tilde{\varphi}_j = (T^*T)^{-1}\varphi_j \ ,$$

then $(\tilde{\varphi}_j)_{j\in\mathbf{Z}}$ also constitutes a frame, with frame bounds B^{-1} and A^{-1}, for details see [38]. We shall call $(\tilde{\varphi}_j)_{j\in\mathbf{Z}}$ the dual frame of $(\varphi_j)_{j\in\mathbf{Z}}$. Therefore, one has

$$\begin{aligned} f &= (T^*T)^{-1}(T^*T)f \\ &= (T^*T)^{-1}\sum_{j\in\mathbf{Z}}(Tf)_j\varphi_j \\ &= (T^*T)^{-1}\sum_{j\in\mathbf{Z}}\langle f, \varphi_j\rangle\varphi_j \\ &= \sum_{j\in\mathbf{Z}}\langle f, \varphi_j\rangle\tilde{\varphi}_j \ . \end{aligned} \tag{3.1.19}$$

Similarly, we derive

$$f = \sum_{j\in\mathbf{Z}}\langle f, \tilde{\varphi}_j\rangle\varphi_j \ . \tag{3.1.20}$$

Formula (3.1.19) shows how to reconstruct f from its inner product $\langle f, \varphi_j\rangle$, and (3.1.20) provides a formula to expand f in terms of a frame φ_j.

In order to have a numerically stable reconstruction algorithm for f from the inner product $\langle f, \psi_{mn} \rangle$, we require that ψ_{mn} constitute a frame.

Note that an orthonormal basis is a frame with frame constants $A = B = 1$. Conversely, a frame φ_j with frame bounds $A = B = 1$ and $\|\varphi_j\| = 1$ for every $j \in \mathbf{Z}$ is necessarily an orthonormal basis. This statement can be verified from the following two facts:

(i) φ_j span all of H. Since $\langle f, \varphi_j \rangle = 0$ for all $j \in \mathbf{Z}$ implies $f = 0$.

(ii) $\langle \varphi_i, \varphi_j \rangle = \delta_{ij}$. Since $A = B = 1$ and for any $j \in \mathbf{Z}$

$$\begin{aligned} \|\varphi_i\|^2 &= \sum_{j \in \mathbf{Z}} |\langle \varphi_i, \varphi_j \rangle|^2 \qquad \text{(by (3.1.15))} \\ &= \|\varphi_i\|^2 + \sum_{\substack{j \neq i \\ j \in \mathbf{Z}}} |\langle \varphi_i, \varphi_j \rangle|^2 \end{aligned}$$

From $\|\varphi_j\| = 1$ and the above formula, we obtain the fact (ii).

We shall specifically discuss orthonormal bases in the following sections.

3.2 Orthonormal Wavelet Bases and Multiresolution Analysis

A multiresolution analysis compresses a sequence $(V_j)_{j \in \mathbf{Z}}$ of closed linear subspaces of $L^2(\mathbf{R})$ which satisfies the following four conditions:

(i) $V_j \subset V_{j+1}$, $\bigcap_{j \in \mathbf{Z}} V_j = \{0\}$ and $\overline{\bigcup_{j \in \mathbf{Z}} V_j} = L^2(\mathbf{R})$; (3.2.1)

(ii) $f \in V_j \iff f(2^j \cdot) \in V_0$ (dilation invariance); (3.2.2)

(iii) $f \in V_0 \iff f(\cdot - k) \in V_0$ for all $k \in \mathbf{Z}$ (translation invariance); (3.2.3)

(iv) V_0 has a shift-invariant Riesz basis $(g(x-k))_{k \in \mathbf{Z}}$. (3.2.4)

The function g is called a scaling function for $(V_j)_{j\in \mathbf{Z}}$.

Let us recall that $\{e_k\}$ form a Riesz basis of a Hilbert space H if both of the following properties are satisfied [49] [54]:

(i) The linear space $\{e_k,\ k \in \mathbf{Z}\}$ is dense in H. (3.2.5)

(ii) There exist positive constants A and B, with $0 < A \leq B < \infty$ such that

$$A\|\{c_k\}\|^2_{\ell^2} \leq \Big\|\sum_{k\in \mathbf{Z}} c_k e_k\Big\|^2_H \leq B\|\{c_k\}\|^2_{\ell^2}$$

for all $\{c_k\} \in \ell^2(\mathbf{Z})$. (3.2.6)

In other words, if the mapping defined by

$$T : \{c_k\} \rightarrow \sum_{k\in \mathbf{Z}} c_k e_k$$

is an isomorphism between $\ell^2(\mathbf{Z})$ and H, then $\{e_k\}_{k\in \mathbf{Z}}$ is called a Riesz basis. If this mapping is a unitary isomorphism, $\{e_k\}_{k\in \mathbf{Z}}$ is an orthonormal basis. The Riesz basis $g(x-k)$, $k \in \mathbf{Z}$ is easily transformed into an orthonormal basis $\varphi(x-k)$ for V_0 [54]. Thus the last requirement of a multiresolution analysis can be replaced by the following:

> There exists a function $\varphi \in V_0$ such that $\{\varphi(x-k)\}_{k\in \mathbf{Z}}$ is an orthonormal basis of V_0. (3.2.7)

In order to show how orthonormal wavelet bases can be constructed from a multiresolution analysis, let us consider the example of the Harr basis.

Let

$$V_m = \{f \in L^2(\mathbf{R}); f \text{ constant on } [2^{-m}n, 2^{-m}(n+1)) \text{ for all } n \in \mathbf{Z}\}, \qquad m \in \mathbf{Z}\ .$$

The conditions (3.2.1), (3.2.2) and (3.2.3) are clearly satisfied. We choose $\varphi(x)$ to be the characteristic function of the interval $[0,1)$, i.e.:

$$\varphi(x) = \begin{cases} 1 & 0 \leq x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

It can be shown that $\varphi \in V_0$, and

$$\{\varphi_{0n};\ n \in \mathbb{Z}\} \text{ is an orthonormal basis of } V_0 \ , \tag{3.2.8}$$

where $\varphi_{0n}(x) = \varphi(x-n)$, $n \in \mathbb{Z}$. Thus $\{V_m\}_{m\in\mathbb{Z}}$ constitutes a multiresolution analysis.

Note that (3.2.2) and (3.2.8) imply that $\{\varphi_{mn},\ n \in \mathbb{Z}\}$ is an orthonormal basis of V_m for all $m \in \mathbb{Z}$, where $\varphi_{mn}(x) = 2^{m/2}\varphi(2^m x - n)$. We define the projections P_m

$$P_m f\Big|_{[2^{-m}n,2^{-m}(n+1))} = 2^m \int_{2^{-m}n}^{2^{-m}(n+1)} f(x)\,dx$$

and

$$c_{mn}(f) = \langle f, \varphi_{mn}\rangle = 2^{m/2} \int_{2^{-m}n}^{2^{-m}(n+1)} f(x)\,dx \ .$$

Then

$$P_m f = \sum_{n\in\mathbb{Z}} c_{mn}(f)\varphi_{mn} \ .$$

We define Q_m to be the difference between P_{m+1} and P_m, i.e.,

$$Q_m f = P_{m+1} f - P_m f \ .$$

One can easily check that

$$\varphi_{mn} = \frac{1}{\sqrt{2}}\ (\varphi_{m+1\,2n} + \varphi_{m+1\,2n+1})$$

and

$$c_{mn}(f) = \frac{1}{\sqrt{2}}\ (c_{m+1\,2n}(f) + c_{m+1\,2n+1}(f)) \ .$$

Hence

$$Q_m f = \frac{1}{2} \sum_{n \in \mathbf{Z}} [c_{m+1\,2n}(f) - c_{m+1\,2n+1}(f)] \, [\varphi_{m+1\,2n} - \varphi_{m+1\,2n+1}] \ . \qquad (3.2.9)$$

We choose

$$\psi(x) = \varphi(2x) - \varphi(2x-1) = \begin{cases} 1 & 0 \le x < \frac{1}{2}, \\ -1 & \frac{1}{2} \le x < 1, \\ 0 & \text{otherwise} \ . \end{cases} \qquad (3.2.10)$$

Then

$$\begin{aligned} \psi_{mn}(x) &= 2^{m/2}\psi(2^m x - n) \\ &= \frac{1}{\sqrt{2}} (\varphi_{m+1\,2n} - \varphi_{m+1\,2n+1}) \ . \end{aligned}$$

If we define

$$d_{mn}(f) = \langle f, \psi_{nm} \rangle = \frac{1}{\sqrt{2}} (c_{m+1\,2n}(f) - c_{m+1\,2n+1}(f)) \ ,$$

then (3.2.9) can be rewritten as

$$Q_m f = \sum_{n \in \mathbf{Z}} d_{mn}(f) \, \psi_{mn} \ . \qquad (3.2.11)$$

From (3.2.10), we know that for fixed m the ψ_{mn} are orthonormal. The projection Q_m is from $L^2(\mathbf{R})$ onto the orthogonal complement of V_m in V_{m+1}. We denote by W_m the complement. Above discussions together with (3.2.1) imply that the W_m are mutually orthogonal. One can therefore show that $\{\psi_{mn}, \ n, m \in \mathbf{Z}\}$ is an orthonormal wavelet basis for $L^2(\mathbf{R})$, for details see [19]. In fact $\{\psi_{mn}\}$ is the Harr basis.

We now outline the steps that can be used to generalize this procedure:

Assume that $\{V_m\}_{m\in\mathbf{Z}}$ is a multiresolution analysis and W_m is the orthogonal complement of V_m in V_{m+1}.

$$V_{m+1} = V_m \oplus W_m, \qquad W_m \perp V_m,\ m \in \mathbf{Z}\ . \tag{3.2.12}$$

By (3.2.1), one has

$$L^2(\mathbf{R}) = \bigoplus_{m\in\mathbf{Z}} W_m\ . \tag{3.2.13}$$

We can show that there exists a function ψ such that [48] [50]:

$$W_0 = \overline{\text{span}\{\psi_{0n}\}}\ . \tag{3.2.14}$$

Note that since

$$f \in W_m \Longleftrightarrow f(2^m\cdot) \in W_0\ . \tag{3.2.15}$$

We have

$$W_m = \overline{\text{span}\{\psi_{mn}\}}, \qquad m \in \mathbf{Z}\ . \tag{3.2.16}$$

In fact, the embedding $V_0 \subset V_1$ immediately implies

$$\varphi(x) = \sum_{n\in\mathbf{Z}} c_n\, \varphi(2x-n) \tag{3.2.17}$$

where the c_n decay rapidly for k tending to infinity. The corresponding wavelet $\psi(x)$ is defined by

$$\psi(x) = \sum_{n\in\mathbf{Z}} (-1)^n \bar{c}_{1-n}\, \varphi(2x-n)\ . \tag{3.2.18}$$

Then $\psi(x-n)$, $n \in \mathbf{Z}$, together with $\varphi(x-n)$, $n \in \mathbf{Z}$, constitute an orthonormal basis for V_1. Thus $\psi(x-n)$, $n \in \mathbf{Z}$ is an orthonormal basis for the orthogonal complement W_0 of V_0 in V_1. Because of (3.2.13) and (3.2.15), $\{\psi_{mn},\ m,n \in \mathbf{Z}\}$ constitute an orthonormal basis of wavelets for $L^2(\mathbf{R})$.

Up to now, we have discussed only the one dimensional case. It is easy, however, to extend the orthonormal bases to higher dimensions. There are two natural ways to construct wavelet bases [9] [52] [36] [37]. The first is using the one dimensional construction. The second way is simply by the tensor product. We illustrate this procedure in two dimensions.

Let $\mathcal{V}_m$ be the subspace of $L^2(\mathrm{R})$ defined by $\mathcal{V}_m = V_m \otimes V_m$. Then

$$\Phi_{m(n_1,n_2)}(x,y) = \varphi_{mn_1}(x)\varphi_{mn_2}(y)$$

are an orthonormal basis for $\mathcal{V}_m$. Define $\mathcal{W}_m$ as the orthogonal complement of $\mathcal{V}_m$ in $\mathcal{V}_{m+1}$. We have

$$\begin{aligned}\mathcal{V}_{m+1} &= V_{m+1} \otimes V_{m+1} \\ &= (V_m \oplus W_m) \otimes (V_m \oplus W_m) \\ &= (V_m \otimes V_m) \oplus (W_m \otimes V_m) \oplus (V_m \otimes W_m) \\ &\qquad \oplus (W_m \otimes W_m) \ .\end{aligned}$$

so that

$$\mathcal{W}_m = (W_m \otimes V_m) \oplus (V_m \otimes W_m) \oplus (W_m \otimes W_m) \ .$$

A basis of $\mathcal{W}_m$ is given by the function

$$\psi^{(j)}_{m,n} = \psi^{(j)}_{m,(n_1,n_2)} = 2^m \, \psi^{(j)} \, (2^m x - n_1, 2^m x - n_2)$$

where

$$\begin{cases} \psi^{(1)}(x,y) = \psi(x)\varphi(y), \\ \psi^{(2)}(x,y) = \varphi(x)\psi(y), \\ \psi^{(3)}(x,y) = \psi(x)\psi(y) \ . \end{cases}$$

It is clear that $\{\psi^{(j)}_{mn}\}$, $m \in \mathrm{Z}$, $n = (n_1, n_2)$, $n_1, n_2 \in \mathrm{Z}$, $j = 1,2,3$ constitute an orthonormal bases of wavelets for $L^2(\mathrm{R}^2)$.

The second possible construction of a wavelet basis for $L^2(\mathbf{R}^2)$ is given by $\psi_{m_1n_1}(x)\psi_{m_2n_2}(y)$. This basis, unlike the above one which we have just constructed, has unrelated localizations in x, y directions. (Here $m_1 \neq m_2$.)

3.3 Compactly Supported Orthonormal Wavelets

Recall from the previous section that

$$\varphi(x) = \sum_{n\in\mathbf{Z}} c_n \, \varphi(2x-n) \tag{3.3.1}$$

and the corresponding wavelet

$$\psi(x) = \sum_{n\in\mathbf{Z}} (-1)^n \bar{c}_{1-n} \, \varphi(2x-n) \ , \tag{3.3.2}$$

where

$$c_n = \int_{-\infty}^{\infty} \varphi(x) \, \varphi(2x-n) \, dx \ .$$

It is clear that if $\varphi(x)$ has compact support, then only finitely many c_n are nonzero. It follows that ψ is a finite linear combination of compactly supported functions, and therefore it has compact support as well.

Let us start by considering the scaling function $\varphi(x)$. Taking the Fourier transform of both sides of (3.3.1), we have

$$\widehat{\varphi}(\xi) = \frac{1}{2} \sum_{n\in\mathbf{Z}} c_n \ e^{-in\frac{\xi}{2}} \ \widehat{\varphi}\left(\frac{\xi}{2}\right) \ . \tag{3.3.3}$$

The above formula can be rewritten as

$$\widehat{\varphi}(\xi) = m_0\left(\frac{\xi}{2}\right) \ \widehat{\varphi}\left(\frac{\xi}{2}\right) \ , \tag{3.3.4}$$

where

$$m_0(\xi) = \frac{1}{2} \sum_{n\in\mathbf{Z}} c_n \ e^{-in\xi} \ . \tag{3.3.5}$$

Moreover

$$\widehat{\varphi}(\xi) = \widehat{\varphi}(0) \prod_{j=1}^{\infty} m_0(2^{-j}\xi), \text{ with } \widehat{\varphi}(0) \neq 0 . \tag{3.3.6}$$

Thus $\int_{-\infty}^{\infty} \varphi(x)\, dx \neq 0$, and for orthonormality we require $\int_{-\infty}^{\infty} \varphi(x)\, dx = 1$. Since $\varphi(x-k)$ are orthonormal, one has

$$\begin{aligned} \delta_{k0} &= \int_{-\infty}^{\infty} \varphi(x)\, \overline{\varphi(x-k)}\, dx \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} |\widehat{\varphi}(\xi)|^2\, e^{ik\xi}\, d\xi \\ &= \frac{1}{2\pi} \int_0^{2\pi} \sum_{\ell \in \mathbf{Z}} |\widehat{\varphi}(\xi + 2\pi\ell)|^2\, e^{ik\xi}\, d\xi . \end{aligned} \tag{3.3.7}$$

Note that the series $\sum_{\ell \in \mathbf{Z}} |\widehat{\varphi}(\xi + 2\pi\ell)|^2$ converges a.e. to an $L^1(0, 2\pi)$ function because of $\varphi \in L^2(\mathrm{R})$ and therefore $\widehat{\varphi} \in L^2(\mathrm{R})$. Define

$$G(x) = \sum_{\ell=-\infty}^{\infty} |\widehat{\varphi}(x + 2\pi\ell)|^2 .$$

Then (3.3.7) can be interpreted as the kth Fourier coefficient of G. Thus

$$\sum_{\ell \in \mathbf{Z}} |\widehat{\varphi}(\xi + 2\pi\ell)|^2 = 1, \qquad \text{a.e. .} \tag{3.3.8}$$

Substituting (3.3.4) into (3.3.8)

$$\sum_{\ell \in \mathbf{Z}} |m_0(\xi + \pi\ell)|^2 |\widehat{\varphi}(\xi + \pi\ell)|^2 = 1, \qquad \text{a.e. .}$$

Since m_0 is a 2π periodic function, we have

$$\begin{aligned} &|m_0(\xi)|^2 \sum_{k \in \mathbf{Z}} |\widehat{\varphi}(\xi + 2\pi k)|^2 \\ &\quad + |m_0(\xi + \pi)|^2 \sum_{k \in \mathbf{Z}} |\widehat{\varphi}(\xi + \pi + 2k\pi)|^2 = 1, \text{ a.e. .} \end{aligned}$$

From (3.3.8), one immediately has

$$|m_0(\xi)|^2 + |m_0(\xi + \pi)|^2 = 1, \qquad \text{a.e. .} \tag{3.3.9}$$

If φ has compact support, then there are only a finite number of terms in (3.3.5), which implies $m_0(\xi)$ is a trigonometric polynomial. It is clear now that to construct the compactly supported wavelet ψ, we can start from m_0 rather than from φ.

There are some necessary and sufficient conditions on m_0 which were discovered mainly by Cohen [10] and Lawton [43]. A detailed discussion can be found in Daubechies [19].

In general, orthonormal wavelet bases obtained in this way cannot be written in a closed analytic form. Daubechies [17] constructed the first family of wavelets with compact support (different from the Harr basis) by starting from the coefficients c_n, where c_n satisfy

$$m_0(\xi) = \frac{1}{2} \sum_{n=0}^{2M-1} c_n \, e^{-in\xi} \, .$$

Table 3.1 lists the $h_n = c_n/\sqrt{2}$ for $M = 2$ through $M = 10$. Figures 3.1 and 3.2 show Daubechies wavelets for $M = 2$. The corresponding scaling function φ

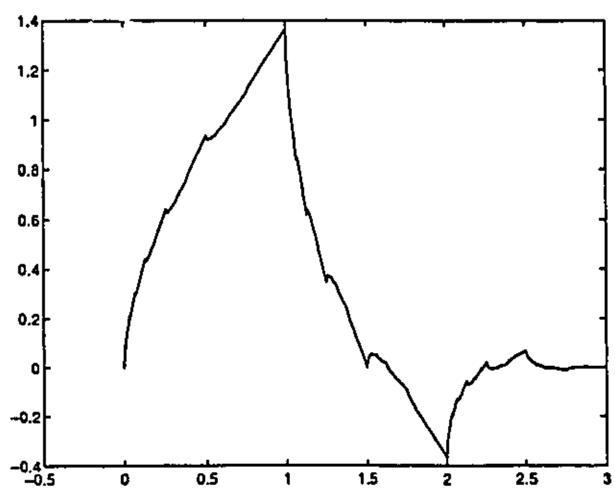


Figure 3.1: The Scaling Function of Daubechies Wavelets ($M = 2$)

satisfies

$$\int_{-\infty}^{\infty} \varphi(x) \, dx = 1$$

M	n	h_n	M	n	h_n
2	0	.482962913145	8	0	.054415842243
	1	.836516303738		1	.312871590914
	2	.224143868042		2	.675630736297
	3	-.129409522551		3	.585354683654
3	0	.332670552950		4	-.015829105256
	1	.806891509311		5	-.284015542962
	2	.459877502118		6	.000472484574
	3	-.135011020010		7	.128747426620
	4	-.085441273882		8	-.017369301002
	5	.035226291882		9	-.044088253931
4	0	.230377813309		10	.013981027917
	1	.714846570553		11	.008746094047
	2	.630880767930		12	-.004870352993
	3	-.027983769417		13	-.000391740373
	4	-.187034811719		14	.000675449406
	5	.030841381836		15	-.000117476784
	6	.032883011667	9	0	.038077947364
	7	-.010597401785		1	.243834674613
5	0	.160102397974		2	.604823123690
	1	.603829259797		3	.657288078051
	2	.724308528438		4	.133197385825
	3	.138428145901		5	-.293273783279
	4	-.242294887066		6	-.096840783223
	5	-.032244869585		7	.148540749338
	6	.077571493840		8	.030725681479
	7	-.006241490213		9	-.067632829061
	8	-.012580751999		10	.000250947115
	9	.003335725285		11	.022361662124
6	0	.111540743350		12	-.004723204758
	1	.494623890398		13	-.004281503682
	2	.751133908021		14	.001847646883
	3	.315250351709		15	.000230385764
	4	-.226264693965		16	-.000251963189
	5	-.129766867567		17	.000039347320
	6	.097501605587	10	0	.026670057901
	7	.027522865530		1	.188176800078
	8	-.031582039318		2	.527201188932
	9	.000553842201		3	.688459039454
	10	.004777257511		4	.281172343661
	11	-.001077301085		5	-.249846424327
7	0	.077852054085		6	-.195946274377
	1	.396539319482		7	.127369340336
	2	.729132090846		8	.093057364604
	3	.469782287405		9	-.071394147166
	4	-.143906003929		10	-.029457536822
	5	-.224036184994		11	.033212674059
	6	.071039219267		12	.003606553567
	7	.080612609151		13	-.010733175483
	8	-.038029936935		14	.001395351747
	9	-.016574541631		15	.001992405295
	10	.012550998556		16	-.000685856695
	11	.000429577973		17	-.000116466855
	12	-.001801640704		18	.000093588670
	13	.000353713800		19	-.000013264203

Table 3.1: The Coefficients h_n $(n = 0, \cdots, 2M-1)$, $M = 2, \cdots, 10$

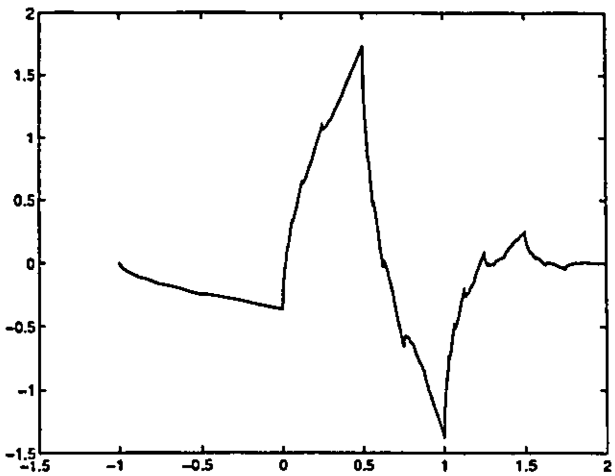


Figure 3.2: The Mother Function of Daubechies Wavelets ($M = 2$)

and the "mother wavelet" ψ has M vanishing moments, i.e.,

$$\int_{-\infty}^{\infty} x^k \, \psi(x) \, dx = 0, \qquad k = 0, 1, \ldots, M-1 \; .$$

and

$$\text{supp}\varphi = [0, 2M-1], \qquad \text{supp}\psi = [-(M-1), M] \; .$$

3.4 Wavelet Decompositions and Reconstructions

The main question now is how to decompose a function into its wavelet coefficients, and how to reconstruct the function from these coefficients. We shall introduce the Mallat's pyramid algorithm [48] which makes these steps simple and fast.

Suppose we are given a fine-scale approximation to f, $f^j = P_j f$ (recall that P_j is the orthogonal projection onto V_j). Since $V_j = V_{j-1} \oplus W_{j-1}$, we decompose as follows: $f^j = f^{j-1} + \tilde{f}^{j-1}$, where $f^{j-1} = P_{j-1} f^j = P_{j-1} f$ is the next coarser approximation of f, $\tilde{f}^{j-1} = Q_{j-1} f^j = Q_{j-1} f$ (Q_j is the orthogonal projection onto W_j). In each of these V_j, W_j, we have orthonormal bases $(\varphi_{jk})_{k \in \mathbf{Z}} = \left(2^{j/2} \, \varphi(2^j x - k)\right)_{k \in \mathbf{Z}}$

and $(\psi_{jk})_{k\in\mathbf{Z}} = \left(2^{j/2}\ \psi(2^j x - k)\right)_{k\in\mathbf{Z}}$ respectively, so that

$$f^j = \sum_k s_k^j\ \varphi_{jk} \tag{3.4.1}$$

and

$$\tilde{f}^j = \sum_k d_k^j\ \psi_{jk}\ . \tag{3.4.2}$$

Where

$$s_k^j = \langle f^j, \varphi_{jk}\rangle, \qquad d_k^j = \langle \tilde{f}^j, \psi_{jk}\rangle\ .$$

Substituting $h_n = 2^{-\frac{1}{2}} c_n$ and $g_n = 2^{-\frac{1}{2}}(-1)^n \bar{c}_{1-n}$ into (3.3.1), (3.3.2) separately, we have

$$\varphi(x) = \sqrt{2}\ \sum_n h_n\ \varphi(2x-n)$$

and

$$\psi(x) = \sqrt{2}\ \sum_n g_n\ \varphi(2x-n)\ .$$

Consequently,

$$\begin{aligned}
\varphi_{jk}(x) &= 2^{j/2}\ \varphi(2^j x - k) \\
&= 2^{j/2}\ 2^{\frac{1}{2}}\ \sum_n h_n\ \varphi(2(2^j x - k) - n) \\
&= 2^{\frac{j+1}{2}}\ \sum_n h_n\ \varphi(2^{j+1}x - (2k+n)) \\
&= \sum_n h_n\ \varphi_{j+1\,2k+n}(x) \\
&= \sum_n h_{n-2k}\ \varphi_{j+1\,n}(x)\ .
\end{aligned} \tag{3.4.3}$$

Similarly,

$$\psi_{jk}(x) = \sum_n g_{n-2k}\ \varphi_{j+1\,n}(x)\ . \tag{3.4.4}$$

Hence,

$$\begin{aligned} s_k^j &= \langle f^j, \varphi_{jk}\rangle = \langle f^{j+1}, \varphi_{jk}\rangle \\ &= \sum_n \overline{h_{n-2k}}\, \langle f^{j+1}, \varphi_{j+1\,n}\rangle \qquad \text{(by (3.4.2))} \\ &= \sum_n \overline{h_{n-2k}}\, s_n^{j+1} \end{aligned} \tag{3.4.5}$$

and

$$\begin{aligned} d_k^j &= \langle \tilde{f}^j, \psi_{jk}\rangle \\ &= \langle f^{j+1}, \psi_{jk}\rangle \\ &= \sum_n \overline{g_{n-2k}}\, \langle f^{j+1}, \varphi_{j+1\,n}\rangle \qquad \text{(by (3.4.4))} \\ &= \sum_n \overline{g_{n-2k}}\, s_n^{j+1} \ . \end{aligned} \tag{3.4.6}$$

The formulae (3.4.5) and (3.4.6) can be interpreted as a decomposition algorithm.

Returning to $f^{j+1} = f^j + \tilde{f}^j$, one has

$$f^{j+1} = \sum_k s_k^j\ \varphi_{jk} + \sum_k d_k^j\ \psi_{jk}\ .$$

Therefore,

$$\begin{aligned} s_n^{j+1} &= \sum_k s_k^j \langle \varphi_{jk}, \varphi_{j+1\,n}\rangle + \sum_k d_k^j \langle \psi_{jk}, \varphi_{j+1\,n}\rangle \\ &= \sum_n [h_{n-2k} s_k^j + g_{n-2k} d_k^j]\ . \\ &\qquad \text{(by (3.4.2), (3.4.4))}\ . \end{aligned} \tag{3.4.7}$$

This is a reconstruction formula. If we define

$$(La)_k = \sum_n a_{2k-n} b_n$$

where $a = (a_n)_{n\in\mathbf{Z}}$, $\bar{a} = (\overline{a_{-n}})_{n\in\mathbf{Z}}$.

It follows that

Decomposition: Initialize $s^J = f$. For $j = J, \ldots, 1$,

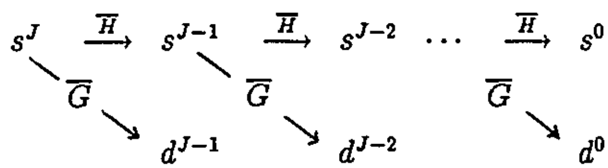
$$s^{j-1} = \overline{H} s^j \qquad \text{and} \qquad d^{j-1} = \overline{G} s^j \,. \tag{3.4.8}$$

Reconstruction: Start with $s^0, d^0, \ldots, d^{J-1}$.

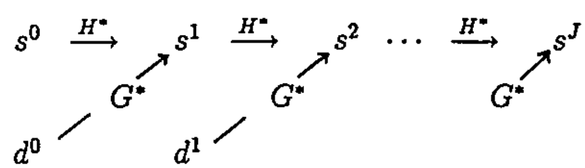
$$s^j = H^* s^{j-1} + G^* d^{j-1}$$

where H^*, G^* are adjoint operators of H and G, respectively.

Summarizing the above we obtain the decomposition algorithm



and the reconstruction algorithm



Both of these are fast algorithms which take a total of $O(N)$ multiplications. Let us consider the decomposition case. If we start with N points s^J, $2^J = N$, then we compute $K\,\frac{N}{2}$ multiplications for s^{J-1}, $K\,\frac{N}{2}$ for d^{J-1}, $K\,\frac{N}{4}$ for s^{J-2}, $K\,\frac{N}{4}$ for d^{J-2}, and so on, where K is the number of coefficients of the filter, for example $K = 2$ for the Harr basis. The total number of multiplications is therefore $2K(\frac{N}{2} + \frac{N}{4} + \cdots + 1) = 2K(N-1)$. If we choose compactly supported orthonormal bases, then only a finite number of the h_n are nonzero. Let K be the number of nonzero coefficients, then the full decomposition takes $O(N)$ multiplications.

3.5 Orthonormal Wavelet Bases on Finite Intervals

The wavelets we have discussed so far are for L^2 functions on infinite intervals. In many practical situations, however, people are interested in only a compact subinterval. The signal to be analyzed is restricted to an interval, or a rectangle in the case of image. Numerical computation is usually related to finite intervals. Without loss of generality, we consider only the one-dimensional interval $[0,1]$.

Let us start by illustrating the basic ideas in Meyer's [53] construction which is based on orthonormal wavelet bases with compact support.

Since $L^2[0,1]$ is not invariant under translations and dilations, one needs to redefine a multiresolution analysis. In particular, we only consider $j \geq j_0 > 0$ and define V_j^I to be the space of restrictions to I of the functions in V_j, where $I = [0,1]$. Thus $V_j^I \subset V_{j+1}^I$ and $\bigcup_{j \geq j_0} V_j^I = L^2[0,1]$. Suppose that $\varphi(x-k)$, $k \in \mathbf{Z}$ are an orthonormal basis of V_0, and $\varphi(x)$ with support $[0, 2M-1]$. (Recall that M is the number of vanishing moments. See §3.3). The j_0 is chosen so that for $j \geq j_0 > 0$, $\text{supp}(\varphi_{j_0}) = [0, 2^{-j_0}(2M-1)]$ is included in $[0, 1/2]$. By this assumption the wavelets or scaling functions near the left edge are supported away from the right edge. Instead of considering the two edges of $[0,1]$, therefore, we just look at a half line, i.e., one edge only. Let

$$\varphi_{jk}^{\text{half}}(x) = \begin{cases} \varphi_{jk}(x) & \text{if } \quad x \geq 0, \\ 0 & \text{if } \quad x < 0 \,. \end{cases}$$

$$V_j^{\text{half}} = \overline{\text{span}\{\varphi_{jk}^{\text{half}};\ k \in \mathbf{Z}\}} \,.$$

For convenience, we shift $\psi(x)$ so that $\text{supp}\psi = [0, 2M-1]$ ($\text{supp}\psi = [-M+1, M]$ in the construction of §3.3). Then $\varphi_{jk}^{\text{half}}(x) = 0$ if $k \leq -2M+1$. It is clear that all but $2M-2$ of these $\varphi_{jk}^{\text{half}}(x)$ are untouched by the restriction. (We view V_j^{half} as the space of restrictions to $[0, \infty)$ of functions in V_j). These functions $\{\varphi_{jk}^{\text{half}}(x),\ k \geq 0\}$ are therefore still orthonormal. The $2M-2$ functions $\varphi_{jk}^{\text{half}}$, $k = -2M+2, \ldots, -1$ are all independent, and orthogonal to the interior $\varphi_{jk}^{\text{half}}$, $k \geq 0$. Similarly, we can define ψ_{jk}^{half} and introduce $W_j^{\text{half}} = V_{j+1}^{\text{half}} \cap (V_j^{\text{half}})^{\perp}$. Meyer [53], Lemarié and Malgouyres [45] showed that:

(i) The interior ψ_{jk}^{half}, $k \geq 0$ are all in W_j^{half} since they are orthogonal to all the $\varphi_{jk}^{\text{half}}$ and lie in V_{j+1}^{half}.

(ii) The ψ_{jk}^{half} with $k = -2M+2, \ldots, -M$ are in V_j^{half}, i.e., they are orthogonal to W_j^{half}.

(iii) The ψ_{jk}^{half} with $k = -M+1, \ldots, -1$ are in W_j^{half}. They are independent of each other and orthogonal to the interior ψ_{jk}^{half}, $k \geq 0$.

Hence we have that

$$\{\varphi_{jk}^{\text{half}};\ k \geq -2M+2\} \cup \{\psi_{jk}^{\text{half}};\ k \geq -M+1\}$$

constitute a non-orthogonal basis for V_{j+1}^{half}.

By a Gram-Schmidt procedure, we orthonormalize this basis.

1. Orthonormalize the $\varphi_{0k}^{\text{half}}$, $k = -2M+2, \ldots, -1$, then the resulting functions $\tilde{\varphi}_k$, $k = -2M+2, \ldots, -1$, together with φ_{0k}, $k \geq 0$, constitute an orthonormal basis for V_0^{half}. Define

$$\tilde{\varphi}_{jk}(x) = 2^{j/2}\tilde{\varphi}_k(2^j x),\ j \in \mathrm{Z},\ k = -2M+2, \ldots, -1,$$

then $\{\varphi_{jk};\ k \geq 0\} \cap \{\tilde{\varphi}_{jk};\ k = -2M+2, \ldots, -1\}$ is an orthonormal basis for V_j^{half}, $j \in \mathrm{Z}$.

2. Orthonormalize $\mathrm{Proj}_{W_0^{\mathrm{half}}}\psi_{0k}^{\mathrm{half}}$, $k=-M+1,\ldots,-1$, where

$$\mathrm{Proj}_{W_0^{\mathrm{half}}}\psi_{0k}^{\mathrm{half}} = \psi_{0k}^{\mathrm{half}} - \sum_{\ell=0}^{2M-2}\langle\psi_{0k}^{\mathrm{half}},\tilde{\varphi}_\ell\rangle\tilde{\varphi}_\ell$$

then the resulting functions $\tilde{\varphi}_k$, $k=-M+1,\ldots,-1$, together with the $\psi_{0k}^{\mathrm{half}}$, $k\geq 0$, constitute an orthonormal basis for W_0^{half}. Again, we define

$$\tilde{\psi}_{jk}(x) = 2^{j/2}\tilde{\psi}_k(2^j x), \qquad j\in\mathbf{Z},\ k=-M+1,\ldots,-1\ .$$

Then $\{\psi_{jk};\ k\geq 0\}\cup\{\tilde{\psi}_{jk};\ k=-M+1,\ldots,-1\}$ is an orthonormal basis for W_j^{half}.

The union of all these bases

$$\begin{aligned}\{\varphi_{0k}^{\mathrm{half}},\ \psi_{jk}^{\mathrm{half}};\ j\geq 0,\ k\geq 0\} \ &\cup\ \{\tilde{\varphi}_{0k};\ k=-2M+2,\ldots,-1\}\\ &\cup\ \{\tilde{\psi}_{jk};\ j\geq 0,\ k=-M+1,\ldots,-1\}\end{aligned}$$

is an orthonormal basis for $L^2[0,\infty)$.

Similarly, we can consider the right edge (corresponding to $(-\infty,1]$). Note that

$$L^2[0,1] = V_{j_0}^I \oplus W_{j_0}^I \oplus W_{j_0+1}^I \oplus\cdots W_j^I \oplus\cdots$$

(where j_0 is chosen, such that $\mathrm{supp}\varphi_{j_0}\subset[0,1/2]$).

We have that

$$\begin{aligned}&\{\varphi_{j_0k}^I;\ -2m+2\leq k\leq 2^{j_0}-1\}\\ &\quad\cup\{\psi_{jk};\ j\geq j_0,\ 0\leq k\leq 2^j-2M+1\}\\ &\quad\cup\{\psi_{jk}^0;\ j\geq j_0,\ -M+1\leq k\leq -1\}\\ &\quad\cup\{\psi_{jk}^1;\ j\geq j_0;\ 2^j-2M+2\leq k\leq 2^j-1\}\end{aligned}$$

where we define

$$\begin{aligned}\psi_{jk}^0(x) &= 2^{j/2}\psi_{0k}^{\mathrm{left}}(2^j x), &&\text{(left edge)}\\ \psi_{jk}^1(x) &= 2^{j/2}\psi_{0k}^{\mathrm{right}}(2^j(x-1)), &&\text{(right edge)}\ .\end{aligned}$$

and $\{\varphi^I_{j_0k};\ -2M+2 \le k \le 2^{j_0}-1\}$ is an orthonormal basis for V^I_0.

We know that the fast algorithm for the wavelet decomposition and reconstruction must have the same format for the approximation. In Meyer's construction, however, there are $2M-2$ scaling functions, but only $M-1$ wavelets at each edge. Hence we have that $\dim(V^I_j) = 2^j - 2M + 2$ and $\dim(W^I_j) = 2^j$. Some difficulties in the computation will occur. It is hard to find adaptive filters at the edges in the algorithm.

Cohen, Daubechies, Jawerth and Vial introduced another construction in [11] [12] to avoid this problem. We state the results without proofs.

For more details, see [11] [12]. Let

$$\begin{cases} \sum_k \varphi(x-k) = 1 = p_0(x) \\ \sum_k k\,\varphi(x-k) = p_1(x) \\ \sum_k k^{M-1}\varphi(x-k) = p_{M-1}(x) \end{cases}$$

where $p_i(x)$ is a polynomial of degree i and $\text{supp}\varphi = [0, 2M-1]$.

We define, for $0 \le i \le M-1$

$$\begin{aligned} \tilde{\varphi}^{\ell,i}(x) &= \left(p_i(x) - \sum_{k>0} k^i\varphi(x-k)\right)\chi_{[0,+\infty)} \\ &= \sum_{k\le 0} k^i\varphi(x-k)\chi_{[0,+\infty)} \end{aligned}$$

and

$$\begin{aligned} \tilde{\varphi}^{r,i}(x) &= \left(p_i(x) - \sum_{k<1-2M} k^i\varphi(x-k)\right)\chi_{(-\infty,1]} \\ &= \sum_{k\ge 1-2M} k^i\varphi(x-k)\chi_{(-\infty,1]} \end{aligned}$$

Theorem 3.5.1 *For all j_0 (j_0 is chosen as before), $V^I_j \subset V^I_{j+1}$ and*

$$\{\tilde{\varphi}^{\ell,i}(2^jx), \tilde{\varphi}^{r,i}(2^j(x-1))\}_{0\le i\le M-1} \cup \{\varphi_{jk}\}_{1\le k\le 2^j-2M}$$

constitute a basis (non-orthogonal) for V^I_j.

Again, we define, for $0 \leq i \leq M-1$,

$$\tilde{\psi}^{\ell,i}(2^j x) = \varphi^{\ell,i}(2^{j+1}x) - \mathrm{Proj}_{V_j}(\varphi^{\ell,i}(2^{j+1}x))$$

and

$$\tilde{\psi}^{r,i}(2^j(x-1)) = \varphi^{r,i}(2^{j+1}(x-1)) - \mathrm{Proj}_{V_j}(\varphi^{r,i}(2^{j+1}(x-1)) \,.$$

Theorem 3.5.2

$$\{\tilde{\psi}^{\ell,i}(2^j x), \tilde{\psi}^{r,i}(2^j(x-1)),\ 0 \leq i \leq M-1\} \cup \{\psi_{jk};\ 1 \leq k \leq 2^j - 2M\}$$

constitute a basis for W_j^I.

Similar to Meyer's construction, they orthonormalize these bases by a Gram-Schmidt process and obtain an orthonormal basis for $L^2[0,1]$.

Theorem 3.5.1 and Theorem 3.5.2 imply that $\dim(V_j^I) = \dim(W_j^I) = 2^j$. It follows that the fast decomposition and reconstruction algorithms corresponding to these wavelets have the same pyramidal structure as the standard algorithms. However, for every step for the M first and last coefficients, one needs to use special filters which correspond to the scaling relations and wavelets at both edges.

Chapter 4

Wavelet Approximations

Because of the associated dense matrices, numerical algorithms for solving integral equations usually require order $O(N^3)$ operations, where N is the number of degrees of freedom in the model. For large-scale problems, the computational costs and the memory storage tends to be prohibitive. In applications, however, the requirement for models with a large number of degrees of freedom N is inevitable [59] [2].

In [7], Beylkin, Coifman and Rokhlin showed that a large class of operators such as the Calderon-Zygmund and pseudo-differential operators have sparse representations in wavelet bases. We have seen in Chapter 3 that there are two significant properties of compactly supported wavelet bases which lead to sparsity. One is the vanishing moment condition. And the other is the narrow interval of support of most of the basis functions. Therefore, by using wavelet bases in numerical approximations to achieve sparse representations of the operators, it becomes possible to consider models with large numbers of degrees of freedom which would otherwise be intractable due to the fully populated matrices.

A gap remains, however, in the representation of an integral operator in $\mathbf{R}$ (let us consider one dimensional case first) for solving the integral equation on a finite interval.

Since for wavelets with M vanishing moments ($M \geq 2$), the basis functions overlap between intervals, and the basis functions need to be modified at the interval end points. As we point out in Chapter 3, Meyer's modification [53] is not suitable for implementing a fast algorithm because of $2M-2$ scaling functions and the fact that there are only $M-1$ wavelets at each interval edge. For Cohen, Daubechies, Jawerth and Vial's construction, although one can find special filters at each edge of the interval [11] [12], one must calculate inner products involving wavelets and our hypersingular kernel in two dimensional space by costly integrations.

Another issue for solving integral equations is that the projection of the hypersingular integral operator onto the basis functions requires appropriate integration quadratures. Since analytical expressions for the Daubechies compactly supported wavelets are generally not available, the recursive form of wavelets will make those integrations more costly.

To eliminate the above difficulties, Alpert, Beylkin, Coifman and Rokhlin [2] introduced wavelet-like bases for the fast solution of second-kind integral equations. They mentioned that, although conceptually the generalization of the wavelet-like bases to high dimensions is quite straightforward, actual procedures to perform the required orthogonalizations have not been developed. As a result, this issue remains unresolved for two and three dimensional problems.

We shall consider matrix problems in this chapter to investigate the above difficulties. First, we extend the solutions of the hypersingular integral equations periodically outside the finite interval. Secondly, rather than applying wavelet bases directly, we use the discrete wavelet transform to the dense influence matrix which is obtained from the discretization by a piecewise polynomial collocation method. The sparsity of the resulting matrix enables us to calculate the inverse matrix by

Schulz's method [74] which involves only $O(N\log^2 N)$ operations. Therefore, we can reduce the cost for solution of the discretized problem to $O(N\log^2 N)$ operations.

In addition, for certain practical problems it is necessary to solve a system of nonlinear equations by performing a sequence of iterations each involving a matrix vector product. The sparse matrix representation makes it possible to perform one of these iterations in $O(N\log N)$ operations rather than the $O(N^2)$ operations that would be required to perform one iteration directly.

Since all numerical approximations to higher dimensional problems reduce to a matrix problem, we can treat these problems similarly to the one dimensional case provided that the matrix possess only a finite number of singularities in some rows or columns. The convolutional structure of the matrix is not necessary for our method.

4.1 Piecewise Polynomial Collocation Approximations

Let P_h^d, $d \in \mathbb{Z}_+$, be the space of piecewise polynomials of degree d on a uniform mesh Δ of the interval $[a,b]$:

$$\Delta : \; a = x_0 < x_1 < \cdots < x_N = b \; ,$$

where $x_j = a + jh$, $j = 0, 1, \ldots, N$ and $h = 1/N$:

$$\begin{aligned} P_h^d \;&=\; \{\varphi(x) : \varphi(x)|_{(jh,(j+1)h)} \text{ is a polynomial} \\ &\qquad \text{of degree } \le d, \; j = 0, 1, \ldots, N-1\} \, . \end{aligned} \qquad (4.1.1)$$

We seek an approximate solution $u_N \in P_h^d$ of the following hypersingular integral equation:

$$Au(x) = \fint_a^b \frac{u(t)}{(t-x)^2}\, dt = f(x), \qquad a < x < b\ . \tag{4.1.2}$$

Without loss of generality, we choose $a = 0$ and $b = 1$. Assume that $\{H_{ij}(x), i = 1, 2, \ldots, N,\ j = 1, \ldots, d+1\}$ is a basis of P_h^d, then

$$u_N(x) = \sum_{i=1}^{N} \sum_{j=1}^{d+1} a_{ij} H_{ij}(x)\ . \tag{4.1.3}$$

Substituting (4.1.3) to (4.1.2), we obtain the coefficients a_{ij} to the following equation.

$$(Au_N)(x_{ij}) = f(x_{ij}), \qquad i = 1, \ldots, N,\ j = 1, \ldots, d+1\ , \tag{4.1.4}$$

where $x_{ij} \in (x_{i-1}, x_i),\ i = 1, \ldots, N,\ j = 1, \ldots, d+1$ are collocation points. Since interpolation at zeros of Chebyshev polynomials of the first kind is nearly optimal [21] [62], we choose the zeros of the Chebyshev polynomial of degree $d+1$ as collocation points, i.e.,

$$\begin{aligned} x_{ij} &= \left[(x_i + x_{i-1}) - (x_i - x_{i-1})\cos((2j-1)\pi/(2(d+1))\right]/2\ , \\ &\qquad i = 1, \ldots, N,\ j = 1, \ldots, d+1\ . \end{aligned} \tag{4.1.5}$$

The algebraic equation (4.1.4) can be rewritten as a projection equation:

$$Q_h A u_N = Q_h f\ . \tag{4.1.6}$$

where Q_h is an interpolation operator. Let $Q_h A = A_N^P$ and $Q_h f = f_N$, so that (4.1.6) becomes

$$A_N^P u_N = f_N\ . \tag{4.1.7}$$

The matrix associated with piecewise constant polynomials is as follows:

$$A_N^P(i,j) = \frac{1}{h[(i-j)^2 - \frac{1}{4}]}, \qquad i,j = 1,\ldots,N\,. \tag{4.1.8}$$

For piecewise linear polynomial collocation approximations, we choose

$$H_{ij}(x) = \begin{cases} \frac{x-x_{ik}}{x_{ij}-x_{ik}} & , \; x \in (x_{i-1}, x_i)\,, \\ 0 & , \; \text{otherwise}\,. \end{cases}$$
$$i = 1,2,\ldots,N,\; \{j=1,\; k=2\} \text{ or } \{j=2,\; k=1\}$$

as a basis of P_h^1, where x_{ij}, $i = 1,2,\ldots,N$, $j = 1,2$ are collocation points. The associated matrix A_N^P is not translation invariant, see Appendix A.

Since the matrices resulting from piecewise polynomial collocation approximations are fully populated, direct solution of (4.1.7) for a large N is prohibitive. We therefore consider wavelet approximations with the view to obtaining a sparse representation of the matrices, which can then be used in iterative solution algorithms.

4.2 Wavelet Approximations

In this section we outline the procedures that are necessary for obtaining wavelet approximations to the hypersingular integral equation (4.1.2). In addition, we establish the required framework for a sparse representation of the resulting system of algebraic equations by means of the fast wavelet transform. Although such direct wavelet approximations turn out to be inconvenient due to the hypersingular kernel, the framework established in this section forms the basis for the fast solution of the matrix problem outlined in the next section.

Wavelet approximations are similar to piecewise polynomial collocation approximations, in which, instead of representing a function by a piecewise polynomial basis, we expand the function in a wavelet basis.

Recall from the previous chapter that the scaling function $\varphi(x)$ and the corresponding wavelet $\psi(x)$, of the Daubechies compactly supported wavelets with M vanishing moments, satisfy the following two recursive formulae respectively:

$$\varphi(x) = \sqrt{2}\sum_{k=1}^{2M} h_k\,\varphi(2x-k+1) \tag{4.2.1}$$

and

$$\psi(x) = \sqrt{2}\sum_{k=1}^{2M} g_k\,\varphi(2x-k+1) \tag{4.2.2}$$

where

$$g_k = (-1)^{k-1}h_{2M-k+1}, \qquad k = 1,\ldots,2M \tag{4.2.3}$$

and h_k, $k = 1,\ldots,2M$ are given in Table 3.1 of Chapter 3. Moreover

$$\int \varphi(x)\,dx = 1 \tag{4.2.4}$$

and

$$\int x^k\psi(x)\,dx = 0, \qquad k = 0,\ldots,M-1\,. \tag{4.2.5}$$

Define

$$V_j = \overline{\text{span}\{\varphi_{jk}(x) = 2^{j/2}\varphi(2^jx-k+1),\ k\in \mathbb{Z}\}}$$

and

$$W_j = \overline{\text{span}\{\psi_{jk}(x) = 2^{j/2}\psi(2^jx-k+1),\ k\in \mathbb{Z}\}}\,,$$

then (see Chapter 3)

$$V_j \subset V_{j+1} \qquad \text{and} \qquad \bigcup_{j\in\mathbb{Z}} V_j = L^2(\mathbb{R})$$

and

W_j is the orthogonal complement of V_j in V_{j+1}.

Since supp$\varphi = [0, 2M-1]$ and supp$\psi = [-(M-1), M]$, $\varphi_{jk}(x)$ and $\psi_{jk}(x)$ overlap between intervals. In order to represent a function $u(x) \in L^2(0,1)$, we extend the function periodically outside of the interval $[0,1]$. This is equivalent to imposing periodic conditions on the basis functions. We denote by $\varphi_k^j(x)$ and $\psi_k^j(x)$ the periodic versions of $\varphi_{jk}(x)$ and $\psi_{jk}(x)$ respectively and define:

$$V_j^{I'} = \overline{\text{span}\{\varphi_k^j(x),\ k = 1, 2, \ldots, 2^j\}}$$

and

$$W_j^{I'} = \overline{\text{span}\{\psi_k^j(x),\ k = 1, 2, \ldots, 2^j\}}\ ,$$

where $I' = (0,1)$.

Then

$$V_j^{I'} \subset V_{j+1}^{I'} \qquad \text{and} \qquad \bigcup_{j \in \mathbb{Z}_+} V_j^{I'} = L^2(0,1) \tag{4.2.6}$$

and

$$W_j^{I'} \oplus V_j^{I'} = V_{j+1}^{I'}\ . \tag{4.2.7}$$

Instead of applying the Meyer [53] or the Cohen et al. [11] [12] constructions, we use the above construction which makes algorithms for solving our hypersingular integral equations easier (see discussions at the beginning of this chapter). Again let Δ be a uniform mesh on the interval $[0,1]$.

$$\Delta : 0 = x_0 < x_1 < \cdots < x_N = 1\ .$$

For simplicity of the algorithms, we always assume $N = 2^j$ for some $j \in \mathbb{Z}_+$. We now seek an approximate solution $u_N \in V_n^{I'}$, i.e.,

$$u_N = \sum_{k=1}^{N} s_k^n(u)\varphi_k^n\ , \tag{4.2.8}$$

where $N = 2^n$. Since there are no analytic forms of φ_k^n, $k = 1, \ldots, 2^n$, we have to divide every interval (x_i, x_{i+1}) into small subintervals, then to calculate $\neq\!\!\!\!\int_0^1 \frac{\varphi_k^n(x)}{(t-x)^2}\,dt$. Therefore due to the hypersingular kernel $\frac{1}{(x-t)^2}$, it is hard to evaluate the integral at a collocation point when it is located in the interval (x_i, x_{i+1}).

Hence we apply the Galerkin method to determine coefficients s_k^n, $k = 1, \ldots, N$, i.e.,

$$A_N^\varphi u_N^\varphi = f_N^\varphi \ , \tag{4.2.9}$$

where

$$\begin{aligned} A_N^\varphi(i,k) &= \int\!\!\int \frac{\varphi_i^n(t)\varphi_k^n(x)}{(t-x)^2}\,dt\,dx \\ &\triangleq s_{i,k}^n \ . \end{aligned} \tag{4.2.10}$$

$$u_N^\varphi(k) = \int u(x)\varphi_k^n(x)\,dx \triangleq s_k^n(u) \ , \tag{4.2.11}$$

and

$$f_N^\varphi(k) = \int f(x)\varphi_k^n(x)\,dx \triangleq s_k^n(f),\ i,k = 1,2,\ldots,N \ . \tag{4.2.12}$$

Remark: $\int\!\int$ are integrations on $\text{supp}\varphi_i^n \times \text{supp}\varphi_k^n$ and $\int$ are integrations on $\text{supp}\varphi_k^n$. For convenience, we write $\int\!\int$ and $\int$ instead of $\int_{\text{supp}\varphi_i^n}\int_{\text{supp}\varphi_k^n}$ and $\int_{\text{supp}\varphi_k^n}$ respectively in this section.

Note that

$$V_n^{I'} = W_{n-1}^{I'} \oplus W_{n-1}^{I'} \oplus \cdots \oplus W_0^{I'} \oplus V_o^{I'} \ .$$

Hence $\{\varphi_1^0(x),\ \psi_k^j(x),\ k = 1, \ldots, 2^j,\ j = 0, \ldots, n-1\}$ is a basis of $V_n^{I'}$. We can express $u_N \in V_n^{I'}$ in the form

$$u_N = \sum_{j=0}^{n-1} \sum_{k=1}^{2^j} d_k^j(u)\psi_k^j + s_1^0(u)\varphi_1^0 \ . \tag{4.2.13}$$

And (4.2.9) becomes

$$A_N^\psi \, u_N^\psi = f_N^\psi \, , \tag{4.2.14}$$

where

$$A_N^\psi(m_1, m_2) = \int\int \frac{\psi_i^j(t)\psi_k^\ell(x)}{(t-x)^2} \, dt \tag{4.2.15}$$

($m_1, m_2 = 1, \ldots, N$, and m_1, m_2 are associated with (i,j) and (k,ℓ) respectively, for details see (4.2.18) - (4.2.24))

$$u_N^\psi = \left[d_1^{n-1}(u)\, d_2^{n-1}(u) \cdots d_{N/2}^{n-1}(u)\, d_1^{n-2}(u) \cdots d_{N/4}^{n-2}(u) \cdots d_1^0(u) s_1^0(u)\right]^T \tag{4.2.16}$$

and

$$f_N^\psi = \left[d_1^{n-1}(f)\, d_2^{n-1}(f) \cdots d_{N/2}^{n-1}(f)\, d_1^{n-2}(f) \cdots d_{N/4}^{n-2}(f) \cdots d_1^0(f) s_1^0(f)\right]^T , \tag{4.2.17}$$

where 'T' denotes transpose of a vector.

More explicitly, if we write

$$d^j = [d_1^j \, d_2^j \cdots d_{2^j}^j]^T \, ,$$

then (4.2.14) can be rewritten as:

$$\begin{bmatrix} A_{n-1} & B_{n-1}^n & B_{n-1}^{n-1} & \cdots & B_{n-1}^2 & B_{n-1}^1 \\ \Gamma_{n-1}^n & A_{n-2} & B_{n-2}^{n-1} & \cdots & B_{n-2}^2 & B_{n-2}^1 \\ \Gamma_{n-1}^{n-1} & \Gamma_{n-2}^{n-1} & A_{n-3} & \cdots & B_{n-3}^2 & B_{n-3}^1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \Gamma_{n-1}^2 & \Gamma_{n-2}^2 & \Gamma_{n-3}^2 & \cdots & A_0 & B_0^1 \\ \Gamma_{n-1}^1 & \Gamma_{n-2}^1 & \Gamma_{n-3}^1 & \cdots & \Gamma_0^1 & T_0 \end{bmatrix} \begin{bmatrix} d^{n-1}(u) \\ d^{n-2}(u) \\ d^{n-3}(u) \\ \vdots \\ d^0(u) \\ s^0(u) \end{bmatrix} = \begin{bmatrix} d^{n-1}(f) \\ d^{n-2}(f) \\ d^{n-3}(f) \\ \vdots \\ d^0(f) \\ s^0(f) \end{bmatrix} \tag{4.2.18}$$

where A_j, $j = 0, \ldots, n-1$, are $2^j \times 2^j$ matrices and

$$A_j(i,\ell) = 2^j \int\int \frac{\psi(2^j t - i + 1)\psi(2^j x - \ell + 1)}{(t-x)^2} \, dt \, dx \, ; \tag{4.2.19}$$

$B_j^{j'}$, $j = 1, \ldots, n-1$, $j' = 2, \ldots, n$ are $2^j \times 2^{j'-2}$ matrices and

$$B_j^{j'}(i,\ell) = 2^{\frac{i+j'-2}{2}} \int\int \frac{\psi(2^j t - i + 1)\psi(2^{j'-2}x - \ell + 1)}{(t-x)^2}\, dt\, dx\ ; \qquad (4.2.20)$$

$\Gamma_j^{j'}$, $j = 1, \ldots, n-1$, $j' = 2, \ldots, n$ are $2^{j'-2} \times 2^j$ matrices and

$$\Gamma_j^{j'}(i,\ell) = 2^{\frac{i+j'-2}{2}} \int\int \frac{\psi(2^{j'-2} t - i + 1)\psi(2^j x - \ell + 1)}{(t-x)^2}\, dt\, dx\ ; \qquad (4.2.21)$$

B_j^1, $j = 0, \ldots, n-1$, are $2^j \times 1$ matrices and

$$B_j^1(i,\ell) = 2^{\frac{j}{2}} \int\int \frac{\psi(2^j t - i + 1)\varphi(x)}{(t-x)^2}\, dt\, dx\ ; \qquad (4.2.22)$$

Γ_j^1, $j = 0, \ldots, n-1$, are 1×2^j matrices and

$$\Gamma_j^1(i,\ell) = 2^{\frac{j}{2}} \int\int \frac{\varphi(t)\psi(2^j x - \ell + 1)}{(t-x)^2}\, dt\, dx\ , \qquad (4.2.23)$$

and

$$T_0 = \int\int \frac{\varphi(t)\varphi(x)}{(t-x)^2}\, dt\, dx\ . \qquad (4.2.24)$$

It is easy to see now that A_N^ψ (4.2.18) is the representation of the operator A in the two dimensional tensor product basis, for details see Section 3.2. We call this representation the standard form [7].

We will show in section 4.4 that A_n^ψ is sparse. However, if we discretize the equation (4.1.2) directly using the wavelet basis $\{\varphi_1^0(x), \psi_k^j(x),\ k = 1, \ldots, 2^j,\ j = 0, \ldots, n-1\}$, calculations of (4.2.18) - (4.2.24) make the problem more difficult. Since $\psi_{ji}(x) \times \psi_{j'\ell}(x)$ supported on the rectangle $I \times J$ ($I = \text{supp}\psi_{ji}(x) = \left[\frac{i-M-1}{2^j}, \frac{i+M}{2^j}\right]$ and $J = \text{supp}\psi_{j'\ell}(x) = \left[\frac{\ell-M-1}{2^{j'}}, \frac{\ell+M}{2^{j'}}\right]$), the length of the two intervals is not equal if $j \neq j'$, we have to calculate $A_N^\psi(m_1, m_2)$ and $A_N^\psi(n_1, n_2)$ using different integration quadratures if $m_1 \neq n_1$, $m_2 \neq n_2$, whereas $A_N^\varphi(i,k)$ (4.2.10)

$i, k = 1, \ldots, N$ can be evaluated in a uniform integration quadrature because of the lengths $|\mathrm{supp}\varphi_i^n| = |\mathrm{supp}\varphi_k^n|$ for $i, k = 1, 2, \ldots, N$. However, A_n^φ is not sparse since there are no vanishing moments on $\varphi_j^k(x)$.

We would like to discretize (4.1.2) by (4.2.9) and to solve the problem using the sparse system (4.2.14). Let us consider the relation between (4.2.9) and (4.2.14).

We define

$$\beta_{i,\ell}^j = 2^j \int\int \frac{\psi(2^j t - i + 1)\varphi(2^j x - \ell + 1)}{(t-x)^2}\, dt\, dx\ ,$$

$$\gamma_{i,\ell}^j = 2^j \int\int \frac{\varphi(2^j t - i + 1)\psi(2^j x - \ell + 1)}{(t-x)^2}\, dt\, dx$$

and

$$s_{i,\ell}^j = 2^j \int\int \frac{\varphi(2^j t - i + 1)\varphi(2^j x - \ell + 1)}{(t-x)^2}\, dt\, dx\ ,$$
$$i, \ell = 1, \ldots, 2^j,\ j = 0, \ldots, n\ .$$

Then from the recursive formulae (4.2.1) and (4.2.2), we have

$$\beta_{i,\ell}^j = \sum_{k,m=1}^{2M} g_k h_m s_{k+2i-2,m+2\ell-2}^{j+1}\ , \tag{4.2.25}$$

$$\gamma_{i,\ell}^j = \sum_{k,m=1}^{2M} h_k g_m s_{k+2i-2,m+2\ell-2}^{j+1}\ , \tag{4.2.26}$$

and

$$s_{i,\ell}^j = \sum_{k,m=1}^{2M} h_k h_m s_{k+2i-2,m+2\ell-2}^{j+1}\ . \tag{4.2.27}$$

Note that the above integrations are calculated on the rectangle $I \times J$ with $|I| = |J|$. For the case $|I| \neq |J|$, we can calculate the first step at $j = n$, then move to the next step $n - 1$ by recursion. For example, from (4.2.20), one has

$$B_j^{j+1}(i, \ell) = 2^{\frac{i+j-1}{2}} \int\int \frac{\psi(2^j t - i + 1)\psi(2^{j-1}x - \ell + 1)}{(t-x)^2}\, dt\, dx$$

$$
\begin{aligned}
&= 2^j \sum_{m=1}^{2M} g_m \int\int \frac{\psi(2^j t - i + 1)\varphi(2^j x - (m + 2\ell - 2) + 1)}{(t-x)^2}\, dt\, dx \\
&= \sum_{m=1}^{2M} g_m\, \beta^j_{i,m+2\ell-2}\,. \qquad (4.2.28)
\end{aligned}
$$

We define

$$
\begin{aligned}
SB_j^{j+1}(i,\ell) &= 2^{\frac{i+j-1}{2}} \int\int \frac{\psi(2^j t - i + 1)\varphi(2^{j-1} x - \ell + 1)}{(t-x)^2}\, dt\, dx \\
&= \sum_{m=1}^{2M} h_m \beta^j_{i,m+2\ell-2}\,. \qquad (4.2.29)
\end{aligned}
$$

Comparing with (3.4.5) and (3.4.6), (4.2.28) and (4.2.29) can be viewed as a one dimensional fast scheme for a fixed index i.

Similarly, we have

$$\Gamma_j^{j+1}(i,\ell) = \sum_{m=1}^{2M} g_m \beta^j_{m+2i-2,\ell} \qquad (4.2.30)$$

and define

$$
\begin{aligned}
S\Gamma_j^{j+1}(i,\ell) &= 2^{\frac{i+j-1}{2}} \int\int \frac{\varphi(2^{j-1} t - i + 1)\psi(2^j x - \ell + 1)}{(t-x)^2}\, dt\, dx \\
&= \sum_{m=1}^{2m} h_m \beta^j_{m+2i-2,\ell}\,. \qquad (4.2.31)
\end{aligned}
$$

Moreover, applying (4.2.1) and (4.2.2) to (4.2.19), we obtain

$$A_j(i,\ell) = \sum_{k=1}^{2M} \sum_{m=1}^{2M} g_k g_m s^{j+1}_{k+2i-2,m+2\ell-2}\,. \qquad (4.2.32)$$

Note that if, $A_N^{\varphi}(i,\ell) = s^n_{i,\ell}$, $i,\ell = 1,\ldots,2^n$ (from (4.2.19)), are given, then $A_j(i,\ell)$, $B_j^{j'}(i,\ell)$ and $\Gamma_j^{j'}(i,\ell)$ and therefore A_N^{ψ} can be obtained via (4.2.25) - (4.2.32) recursively. More precisely, A_N^{ψ} can be obtained by transforming A_N^{φ} on its first index (row or column) then on its second index (column or row).

We denote by "$\sim$" the wavelet transform, then (4.2.14) can be rewritten as

$$\tilde{A}_N^{\varphi}\, \tilde{u}_N = \tilde{f}_N^{\varphi}\,. \qquad (4.2.33)$$

Clearly, the transformation from A_N^{φ} to A_N^{ψ} requires $O(N^2)$ operations. However, since the structure of singularities of the kernel $\frac{1}{(t-x)^2}$ is known, the procedure requires only $O(N\log(N))$ operations (we will show this in section 4.4).

We now focus on calculations of $A_N^{\varphi}(i,\ell)$. From (4.2.10), we have

$$A_N^{\varphi}(i,\ell) = 2^n \int\int \frac{\varphi(2^n t - i + 1)\varphi(2^n x - \ell + 1)}{(t-x)^2}\, dt\, dx \, .$$

Because there is no analytic form of $\varphi(x)$, since $\varphi(x)$ is given recursively, the numerical computations of $A_N^{\varphi}(i,\ell)$ involving hypersingular kernel $\frac{1}{(t-x)^2}$ tend to be tedious. Therefore, we seek another approximation which we call the matrix problem approximation.

Remark: In formulae (4.2.25) - (4.2.32), we assume that all indices are periodic with period N. Since the periodic extensions are performed on $\varphi_{jk}(x)$ and $\psi_{jk}(x)$, it is equivalent to performing periodic extensions on those indices with period N.

4.3 The Matrix Problem

For convenience, we write $\{\varphi_k^n(x),\ k = 1,\ldots,2^n\}$ and $\{\varphi_1^0(x),\ \psi_k^j(x),\ j = 0,\ldots,n-1\}$ as $\{\varphi_j(x),\ j = 1,\ldots,N\}$ and $\{\psi_j(x),\ j = 1,\ldots,N\}$ respectively, where $N = 2^n$. Since both of these are orthonormal bases of V_n^I, there exists an invertible $N \times N$ matrix $B = (b_{ij})$ with $B^{-1} = B^T$ such that

$$\begin{bmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_N \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1N} \\ b_{21} & b_{22} & \cdots & b_{2N} \\ \vdots & \vdots & & \vdots \\ b_{N1} & b_{N2} & \cdots & b_{NN} \end{bmatrix} \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_N \end{bmatrix} . \tag{4.3.1}$$

From (4.2.15), we have

$$A_N^{\psi}(i,j) \;=\; \int\int \frac{\psi_i(t)\psi_j(x)}{(t-x)^2}\, dt\, dx$$

$$
\begin{aligned}
&= \int\int \frac{\sum_{k=1}^{N} b_{ik}\varphi_k \cdot \sum_{\ell=1}^{N} b_{j\ell}\varphi_\ell}{(t-x)^2} \\
&= \sum_{\ell=1}^{N} b_{j\ell} \sum_{k=1}^{N} b_{ik} A_N^{\varphi}(k,\ell) \qquad (4.3.2)
\end{aligned}
$$

i.e.,

$$A_N^{\psi} = B\,A_N^{\varphi}\,B^T . \qquad (4.3.3)$$

Similarly, we have

$$u_N^{\psi} = B\,u_N^{\varphi} \qquad (4.3.4)$$

and

$$f_N^{\psi} = B\,f_N^{\varphi} . \qquad (4.3.5)$$

Formulae (4.3.3), (4.3.4) and (4.3.5), together with (4.2.14) and (4.2.33), imply that

$$
\begin{aligned}
\tilde{A}_N^{\varphi} &= B\,A_N^{\varphi}\,B^T \qquad (4.3.6) \\
\tilde{u}_N^{\varphi} &= B\,u_N^{\varphi} \qquad (4.3.7)
\end{aligned}
$$

and

$$\tilde{f}_N^{\varphi} = B\,f_N^{\varphi} . \qquad (4.3.8)$$

Now we consider piecewise polynomial collocation approximations

$$A_N^P\,u_N^P = f_N^P . \qquad (4.3.9)$$

And assume $dN_1 = 2^n$, where dN_1 is the dimension of the space P_h^d with $h = \frac{1}{N_1}$.

Multiplying both sides of (4.3.9) by B, one has

$$B\,A_N^P\,B^T B\,u_N^P = B f_N^P . \qquad (4.3.10)$$

Clearly, this is equivalent to applying discrete wavelet transforms to both sides of (4.3.9). Hence (4.3.10) can be rewritten as

$$\tilde{A}_N^P \, \tilde{u}_N^P = \tilde{f}_N^P \, , \tag{4.3.11}$$

where

$$\tilde{A}_N^P = B \, A_N^P \, B^T \, , \tag{4.3.12}$$

$$\tilde{u}_N^P = B \, u_N^P \tag{4.3.13}$$

and

$$\tilde{f}_N^P = B \, f_N^P \, . \tag{4.3.14}$$

We shall show in the next section that the wavelet representation $\tilde{A}_N^P$ resulting from piecewise constant and piecewise linear approximations is sparse. Since the discretization by piecewise polynomials is much easier than by wavelet bases, we would like to discretize our hypersingular integral equation by piecewise polynomial collocation approximations rather than directly by wavelet approximations.

The generalization of this method to high dimensional problems is straightforward since any high dimensional problem can also be expressed in the following matrix form

$$A_N \, u_N = f_N \, .$$

The wavelet transform $\tilde{A}_N$ of the matrix A_N is sparse provided that A_N has a finite number of singularities in some rows or columns.

4.4 Sparsity and Error Analysis

Beylkin, Coifman and Rokhlin [7] have shown that the integral operators with the kernel $k(t,x)$ satisfying

$$\begin{aligned} |k(t,x)| &\leq \frac{1}{|t-x|}, \\ |\partial_t^M k(t,x)| + |\partial_x^M k(t,x)| &\leq \frac{c_0}{|t-x|^{1+M}} \end{aligned}$$

where c_0 is a constant, for some $M \geq 1$ have sparse representations in the Daubechies compactly supported orthonormal wavelet bases. We can establish similar results for the kernel $k(t,x) = \frac{1}{(t-x)^2}$.

Note that our wavelet approximations for solving integral equations are based on the discretization by the piecewise polynomial collocation method. This is equivalent to using wavelet bases for the resulting approximate discrete kernels, such as the kernels in (4.1.8) and those in Appendix A instead of applying wavelet bases directly to the kernel $k(t,x) = \frac{1}{(t-x)^2}$. One can see that the kernels in (4.1.8) and Appendix A are more complicated than $k(t,x) = \frac{1}{(t-x)^2}$. Moreover, the kernel resulting from the piecewise linear approximation is not translation invariant.

We must consider the question of whether the kernels based on piecewise polynomial approximation have sparse representations in terms of wavelet bases. To answer this question, we start by demonstrating the sparsity of the operator with kernel $k(t,x) = \frac{1}{(t-x)^2}$.

Let us consider the standard form of A_N^ψ in (4.2.18). Since $\text{supp}\psi(x) = [-(M-1), M]$, $\text{supp}\psi(2^j x - i + 1) = \left[\frac{i-M-1}{2^j}, \frac{i+M}{2^j}\right]$ for some i, j. We define the distance between two intervals $I = \left[\frac{i-M-1}{2^j}, \frac{i+M}{2^j}\right]$ and $J = \left[\frac{\ell-M-1}{2^k}, \frac{\ell+m}{2^k}\right]$ by

$$d(I,J) = \min_{t\in I, x\in J} |t-x| ,$$

for some i, j, k, ℓ. And A_{IJ} denotes the element of A_N^ψ which is associated with intervals I and J. Hence

$$A_{IJ} = 2^{\frac{j+k}{2}} \int\int \frac{\psi(2^j t - i + 1)\psi(2^k x - \ell + 1)}{(t-x)^2}\, dt\, dx\ , \tag{4.4.1}$$

except for the Nth row and the Nth column of A_N^{ψ}.

Remark: We wrap $\psi(2^j t - i + 1)$ and $\psi(2^k x - \ell + 1)$ around the right edge $x = 1$ and the left edge $x = 0$ when I, J spill out of the interval $[0, 1]$.

Without loss of generality, we assume $|I| \leq |J|$. For fixed x and $d(I, J) > 0$, we expand $\frac{1}{(t-x)^2}$ into a Taylor series around the center t_I of I. Applying the moment conditions

$$\int x^m \psi(x)\, dx = 0, \qquad m = 0, 1, \ldots, M-1\ ,$$

we obtain

$$A_{IJ} = c'\, 2^{\frac{j+k}{2}} \int_J \int_I \frac{(t-t_I)^M}{(\xi - x)^{M+2}}\ \psi(2^j t - i + 1)\psi(2^k x - \ell + 1)\ dt\, dx\ , \tag{4.4.2}$$

where $\xi \in I$ and $c' = (-1)^M(M+1)$.

Therefore

$$\begin{aligned} |A_{IJ}| &\leq c'\, 2^{\frac{j+k}{2}} \left(\frac{|I|}{2}\right)^M \frac{|I|}{(d(I,J))^{M+1}} (\sup_x |\psi|)^2 \\ &\leq c\, (|I|/|J|)^{\frac{1}{2}}\ \frac{|I|^M}{(d(I,J))^{M+1}}\ , \end{aligned} \tag{4.4.3}$$

where c is a constant, and $|I|$ and $|J|$ are the lengths of intervals I and J respectively.

We now compress the standard form A_N^{ψ} according to the estimate (4.4.3). Let $A_N^{\psi,b}$ denote the approximation to A_n^{ψ} which is obtained from A_n^{ψ} by setting to zero all coefficients of matrices A_j, $B_j^{j'}$, $\Gamma_j^{j'}$, $j = 1, \ldots, n-1$, $j' = 2, \ldots, n$ outside of bands of width b around their diagonals. Similar to Beylkin, Coifman and Rokhlin's estimate for the non-standard form, we can show that (refer to Appendix B)

$$\|h(A_N^{\psi,b} - A_N^{\psi})\| \leq \frac{c'}{b^M}\ \log_2 N\ , \tag{4.4.4}$$

where c' is a constant, $h = \frac{1}{N}$ and the norm $\|\cdot\|$ is defined by the row-sum norm

$$\|A\| = \max_i \sum_{j=1}^{N} |A(i,j)| \ . \tag{4.4.5}$$

From (4.4.2) it can be seen that near the diagonals, $|h A_{IJ}| > c_1$, where $c_1 > 0$ is a constant and $h = \frac{1}{N}$. Hence

$$\|h A_N^{\psi}\| > c_1 \ . \tag{4.4.6}$$

The above inequality, together with (4.4.4) implies that

$$\frac{\|A_N^{\psi,b} - A_N^{\psi}\|}{\|A_N^{\psi}\|} \le \frac{c}{b^M} \ \log_2 N \ , \tag{4.4.7}$$

where c is a constant.

Given a threshold $\varepsilon > 0$ and a fixed M, b has to be chosen such that

$$\frac{\|A_N^{\psi,b} - A_N^{\psi}\|}{\|A_N^{\psi}\|} \le \frac{c}{b^M} \ \log_2 N < \varepsilon \ , \tag{4.4.8}$$

i.e.,

$$b \ge \left(\frac{c}{\varepsilon} \ \log_2 N\right)^{1/M} \ . \tag{4.4.9}$$

The resulting compressed matrix $A_N^{\psi,b}$ contains at most $O(N \log N)$ non-zero elements, which is similar to the non-standard case (for details see [2]).

Remark: Since $|A_N^{\psi}(i,j)| = O(h^{-1})$, we can always multiply by h both sides of (4.1.7) before applying discrete wavelet transforms to A_N^P (see kernels (4.1.8) and Appendix A). Otherwise we need to compare $A_N^{\psi}(i,j)$ to $a = \max_{i,j} |A_N^{\psi}(i,j)|$. If $|A_N^{\psi}(i,j)|/a < \varepsilon$, then we set the element $A_N^{\psi}(i,j) = 0$. It is clear that this procedure makes the algorithm complicated.

Clearly, the key fact which results in sparsity is not only the narrow support of most basis functions and vanishing moments, but also the decay rates of the Mth derivatives of the matrix elements with distance from the diagonal.

We can show that the matrices resulting from piecewise constant approximations (see (4.1.8)) and piecewise linear approximations (see Appendix A) possess the above-mentioned property. Therefore they can be compressed (to any fixed accuracy) with only $O(N\log(N))$ non-zero elements. See Figures 4.1 and 4.2.

Suppose $\tilde{A}_N^{P,b}$ is the compressed form of $\tilde{A}_N^P$. We consider the following algebraic equation:

$$\tilde{A}_N^{P,b}\,\tilde{u}_N^{P,b} = \tilde{f}_N^P\,, \tag{4.4.10}$$

where "$\sim$" denotes the wavelet transform. Let u be the exact solution of (4.1.2). We shall ask:

$$\text{Does the approximate solution } \tilde{u}_N^{P,b} \text{ converge to } \tilde{u} \text{ as } N \to \infty? \tag{4.4.11}$$

Recall from (4.3.9) and (4.3.11) that

$$A_N^P\,u_N^P = f_N^P\,,$$

and

$$\tilde{A}_N^P\,\tilde{u}_N^P = \tilde{f}_N^P\,.$$

We shall show in the next chapter that

$$\|u_N^P - u\|_{L^2} \to 0 \qquad \text{as } N \to \infty$$

for the piecewise constant and the piecewise linear approximations. Therefore by (4.3.13)

$$\begin{aligned} \|\tilde{u}_N^P - \tilde{u}\|_{L^2} &= \|B(u_N^P - u)\|_{L^2} \\ &\leq \|B\|\;\|u_N^P - u\|_{L^2} \to 0 \qquad \text{as } N \to \infty\,. \end{aligned} \tag{4.4.12}$$

Thus to establish the convergence described in (4.4.11), we only need to show that for a given $\varepsilon > 0$,

$$\|\tilde{u}_N^{P,b} - \tilde{u}_N^P\|_{L^2} < \varepsilon\,. \tag{4.4.13}$$

Figure 4.1: The Standard Form of PWC Influence Matrix (Entries above Threshold 10^{-6}, $N = 256$)

Figure 4.2: The Standard Form of PWL Influence Matrix (Entries above Threshold 10^{-6}, $N = 256$)

provided $N > N_\varepsilon$.

We will make use of the following theorem:

Theorem 4.4.1 (Rudin [67]) *Let X be a Banach space and $A : X \to X$ be a bounded linear operator with $\|A\| < 1$. Then $I - A$ is invertible and*

$$\|(I-A)^{-1}\| \le \frac{1}{1-\|A\|}\,, \tag{4.4.14}$$

where I denotes the identity operator.

We refer to Rudin [67], Chapter 10 for the proof.

We shall also show in the next chapter that $(A_N^P)^{-1}$ exists for the piecewise constant and the piecewise linear approximations and that there exists a constant c such that

$$\|(A_N^P)^{-1}\| \le c\,. \tag{4.4.15}$$

Therefore, from (4.3.12), $(\tilde{A}_N^P)^{-1}$ exists and

$$\|(\tilde{A}_N^P)^{-1}\| \le c'\,, \tag{4.4.16}$$

where c' is a constant. We define the condition number of $\tilde{A}_N^P$ by

$$\mathrm{cond}(\tilde{A}_N^P) = \|\tilde{A}_N^P\|\;\|(\tilde{A}_N^P)^{-1}\|\,. \tag{4.4.17}$$

For $\varepsilon_1 > 0$ with $\mathrm{cond}(\tilde{A}_N^P)\varepsilon_1 < 1$, we choose $b > 0$, such that

$$\frac{\|\tilde{A}_N^{P,b} - \tilde{A}_N^P\|}{\|\tilde{A}_N^P\|} < \varepsilon_1\,. \tag{4.4.18}$$

Hence

$$\begin{aligned}
&\|(\tilde{A}_N^P)^{-1}(\tilde{A}_N^P - \tilde{A}_N^{P,b})\| \\
\le\;& \|(\tilde{A}_N^P)^{-1}\|\;\|\tilde{A}_N^P\|\left(\frac{\|\tilde{A}_N^{P,b} - \tilde{A}_N^P\|}{\|\tilde{A}_N^P\|}\right) \\
\le\;& \mathrm{cond}(\tilde{A}_N^P)\varepsilon_1 < 1\,.
\end{aligned} \tag{4.4.19}$$

Applying theorem 4.4.1, we have that $[I-(\tilde{A}_N^P)^{-1}(\tilde{A}_N^P-\tilde{A}_N^{P,b})]^{-1}$ exists and

$$\left\|[I-(\tilde{A}_N^P)^{-1}(\tilde{A}_N^P-\tilde{A}_N^{P,b})]^{-1}\right\| \leq \frac{1}{1-\|(\tilde{A}_N^P)^{-1}(\tilde{A}_N^P-\tilde{A}_N^{P,b})\|}\,. \tag{4.4.20}$$

Since $[I-(\tilde{A}_N^P)^{-1}(\tilde{A}_N^P-\tilde{A}_N^{P,b})]^{-1}(\tilde{A}_N^P)^{-1}=(\tilde{A}_N^{P,b})^{-1}$, we obtain

$$\|(\tilde{A}_N^{P,b})^{-1}\| \leq \frac{\|(\tilde{A}_N^P)^{-1}\|}{1-\|(\tilde{A}_N^P)^{-1}(\tilde{A}_N^P-\tilde{A}_N^{P,b})\|}\,. \tag{4.4.21}$$

Therefore, by (4.3.11) and (4.4.10),

$$\begin{aligned}
\tilde{A}_N^{P,b}(\tilde{u}_N^{P,b}-\tilde{u}_N^P) &= \tilde{A}_N^{P,b}\tilde{u}_N^{P,b}-\tilde{A}_N^P\tilde{u}_N^P+\tilde{A}_N^P\tilde{u}_N^P-\tilde{A}_N^{P,b}\tilde{u}_N^P \\
&= (\tilde{A}_N^P-\tilde{A}_N^{P,b})\tilde{u}_N^P\,.
\end{aligned}$$

Consequently, from (4.4.18), (4.4.19), and (4.4.21), one has

$$\begin{aligned}
\|\tilde{u}_N^{P,b}-\tilde{u}_N^P\|_{L^2} &\leq \frac{\|(\tilde{A}_N^P)^{-1}\|\;\|\tilde{A}_N^P-\tilde{A}_N^{P,b}\|\;\|\tilde{u}_N^P\|}{1-\|(\tilde{A}_N^P)^{-1}(\tilde{A}_N^P-\tilde{A}_N^{P,b})\|} \\
&= \frac{\operatorname{cond}(\tilde{A}_N^P)\,\frac{\|\tilde{A}_N^P-\tilde{A}_N^{P,b}\|}{\|\tilde{A}_N^P\|}\,\|\tilde{u}_N^P\|}{1-\|(\tilde{A}_N^P)^{-1}(\tilde{A}_N^P-\tilde{A}_N^{P,b}\|} \\
&\leq \frac{\operatorname{cond}(\tilde{A}_N^P)\|\tilde{u}_N^P\|}{1-\operatorname{cond}(\tilde{A}_N^P)\cdot\varepsilon_1}\,\varepsilon_1\,.
\end{aligned} \tag{4.4.22}$$

And (4.4.22) yields (4.4.13). Moreover, (4.4.22) shows that the compression rates depend on the condition number of the operator.

4.5 Numerical Solution

In this section, we seek numerical solutions of sparse algebraic equations. Due to Schulz's method [74], the numerical algorithms require only $O(N\log^2 N)$ operations.

4.5.1 Schulz's Fast Iterative Method

Following Alpert, Beylkin, Coifman and Rokhlin [2], we apply Schulz's method to solve the sparse linear systems that result from the discretization of the integral equation (4.1.2) and the subsequent wavelet representation.

For convenience, we write the resulting sparse matrix problems as follows:

$$Au = f \; . \tag{4.5.1}$$

Assume the matrix A_0 is an approximate inverse of the matrix A and let us start the iterative procedure with this matrix A_0. Define the residual matrix R of A_0 as

$$R = I - A_0 A \; . \tag{4.5.2}$$

Therefore

$$A_0 A = I - R \; . \tag{4.5.3}$$

We consider

$$\begin{aligned} A^{-1} &= A^{-1}(A_0^{-1}A_0) \\ &= (A_0 A)^{-1} A_0 \\ &= (I - R)^{-1} A_0 \\ &= (I + R + R^2 + \cdots + R^n + \cdots) A_0 \; . \end{aligned} \tag{4.5.4}$$

And we define

$$A_n = (I + R + R^2 + \cdots + R^n) A_0 \tag{4.5.5}$$

and

$$u_n = A_n f \; . \tag{4.5.6}$$

Then

$$\begin{aligned} u_{n+1} &= (I + R + \cdots + R^{n+1})A_0 f \\ &= (I - (I - R)(I + R + \cdots + R^n) + (I + R + \cdots + R^n))A_0 f \\ &= u_n + (A_0 f - A_0 A u_n) \\ &= u_n + A_0(f - A u_n) \ . \end{aligned} \tag{4.5.7}$$

It is interesting that

$$\begin{aligned} &2A_n - A_n A A_n \\ &= 2(I + R + \cdots + R^n)A_0 - (I + R + \cdots + R^n)(I - R)(I + R + \cdots + R^n)A_0 \\ &= (2I - (I + R + \cdots + R^n) + R(I + R + \cdots + R^n))(I + R + \cdots + R^n)A_0 \\ &= (I + R^{n+1})(I + R + \cdots + R^n)A_0 \\ &= (I + R + \cdots + R^n + R^{n+1} + \cdots + R^{2n+1})A_0 \\ &= A_{2n+1} \qquad n = 0, 1, 3, 7, \ldots \ . \end{aligned} \tag{4.5.8}$$

We can apply either the iterative formula (4.5.7) or the iterative formula (4.5.8) to solve equation (4.5.1). However, the algorithm (4.5.8) converges quadratically [74]. To show this, we define the following norm of a matrix A:

$$\|A\|_2 = (\text{largest eigenvalue of } A^T A)^{1/2} \ . \tag{4.5.9}$$

And denote by $k(A) = \|A\|_2 \|A^{-1}\|_2$ the condition number of A.

It is convenient that we rewrite (4.5.8) as follows:

$$A_{m+1} = 2A_m - A_m A A_m \ . \tag{4.5.10}$$

Then we have

Lemma 4.5.1 ([2] [74]) *Suppose that A is an invertible matrix, A_0 is a matrix given by $A_0 = A^T/\|A^TA\|_2$, and for $m = 0, 1, 2, \ldots$ the matrix A_{m+1} is defined by the recursion (4.5.9). Then A_{m+1} satisfies the formula*

$$I - A_{m+1}A = (I - A_mA)^2 \tag{4.5.11}$$

Furthermore, $A_m \to A^{-1}$ and $m \to \infty$ and for any $\varepsilon > 0$, we have

$$\|I - A_mA\|_2 < \varepsilon$$

provided

$$m \geq 2\log_2 k(A) + \log_2 \log(1/\varepsilon) \ . \tag{4.5.12}$$

Proof: see [2].

Clearly, if the algorithm (4.5.8) converges at the mth step, then (4.5.7) will converge at the 2^mth step. However, we should notice that (4.5.8) involves two matrix multiplications at each iteration. Each matrix multiplication requires N^3 additions and multiplies when the matrices are fully populated. Therefore the recursion (4.5.8) is practical only when the matrices have some special properties which will allow these matrix multiplications to be performed efficiently. Our sparse matrices of the standard form (4.2.18) possess these properties.

4.5.2 The Algorithm Complexity Analysis

Returning to the standard form of our operator in (4.2.18), we see that the standard form is a representation in an orthonormal basis (a two dimensional tensor product basis). Using Beylkin, Coifman and Rokhlin's notation, we write the standard form as follows

$$T = \left\{A_j, \{B_j^{j'}\}_{j'=2}^{j'=n}, \{\Gamma_j^{j'}\}_{j'=2}^{j'=n}, B_j^1, \Gamma_j^1, B_0^1, \Gamma_0^1, A_0, T_0\right\}_{j=1,\ldots,n-1} \tag{4.5.13}$$

Suppose that $\tilde{T}$ and $\hat{T}$ are also two standard forms, i.e.,

$$\tilde{T} = \left\{\tilde{A}_j, \{\tilde{B}_j^{j'}\}_{j'=2}^{j'=n}, \{\tilde{\Gamma}_j^{j'}\}_{j'=2}^{j'=n}, \tilde{B}_j^1, \tilde{\Gamma}_j^1, \tilde{B}_0^1, \tilde{\Gamma}_0^1, \tilde{A}_0, \tilde{T}_0\right\}_{j=1,\dots,n-1} \tag{4.5.14}$$

and

$$\hat{T} = \left\{\hat{A}_j, \{\hat{B}_j^{j'}\}_{j'=2}^{j'=n}, \{\hat{\Gamma}_j^{j'}\}_{j'=2}^{j'=n}, \hat{B}_j^1, \hat{\Gamma}_j^1, \hat{B}_0^1, \hat{\Gamma}_0^1, \hat{A}_0, \hat{T}_0\right\}_{j=1,\dots,n-1} \tag{4.5.15}$$

and suppose that we wish to compute $T = \tilde{T}\hat{T}$.

Since the result of the multiplication of two standard forms is also a representation in the same basis, $T = \tilde{T}\hat{T}$ must have the same structure as that given in (4.5.13).

Given a threshold of accuracy $\varepsilon > 0$, T_ε denotes the matrix obtained by setting all the elements of T below the threshold to zero. Thus, the corresponding blocks A_j^ϵ, $B_j^{\epsilon j'}$, $\Gamma_j^{\epsilon j'}$ (except for the Nth row Γ_j^1, Nth column B_j^Γ, $j = 0, \dots, n-1$ and T_0) are banded. T_ε therefore contains no more than $O(N \log N)$ non-zero elements. It is easy to check now that the multiplication of two such sparse matrices $\tilde{T}_\varepsilon$ and $\hat{T}_\varepsilon$ requires at most $O(N \log^2 N)$ operations.

Since for a given ε and the sparse representation of the matrix A, the iterative step m can be determined from (4.5.12). Therefore, applying Schulz's fast iterative method (4.5.10), we take only $O(N \log^2 N)$ operations to solve the system (4.5.1).

By using indexed storage of sparse matrices, the algorithm needs only $O(N \log N)$ words of computer memory. Moreover, this data structure of indexed storage allows us to avoid having to multiply by zeros at each step of the algorithm, thereby losing the advantage of the sparse structure of the standard form.

Chapter 5

Error Analysis

It is clear from the previous chapter that the framework of our wavelet collocation approximations is based on the piecewise constant and piecewise linear collocation methods. Therefore, we shall focus on error estimates for the piecewise polynomial collocation approximations.

Let s_h^d, $d \in \mathbb{Z}_+$, be the space of $d-1$ times continuously differentiable splines of degree d on the uniform mesh of the interval $[a, b]$, $a = x_0 < x_1 < x_2 < \cdots < x_n = b$, with $x_j = a + jh$ and $h = \frac{1}{n}$:

$$s_h^d = \{\varphi \in C^{d-1} : \varphi|_{(jh,(j+1)h)} \text{ is a polynomial of degree } \leq d,\ j = 0, \ldots, n-1\}\ . \tag{5.1}$$

The spline s_h^d has a basis $\{B_{j,d}\}$, $j = 1, \ldots, n-1$ of B-splines which are defined as follows (deBoor [21], S. Prösdorff and B. Silbermann [65])

$$B_{j,0}(x) = \begin{cases} 1, & x_j \leq x \leq x_{j+1}\ , \\ 0, & \text{otherwise} \end{cases}$$

For $d \geq 1$, we use the recurrence

$$B_{j,d}(x) = \frac{x - x_j}{x_{j+d} - x_j}\ B_{j,d-1}(x) + \frac{x_{j+d+1} - x}{x_{j+d+1} - x_{j+1}}\ B_{j+1,d-1}(x)\ .$$

We can easily see that piecewise constant polynomials belong to s_h^0. And s_h^1 is a special case of the piecewise linear polynomial space P_h^1. In this chapter, we consider error estimates in spline space s_h^d.

From Chapter 2, we know that our hyper-singular integral equations have an intimate relationship with Cauchy singular integral equations. There are some papers [70] [24] [71] [72] [25] [73] [26] which deal with error estimates of spline collocation methods and Galerkin methods with splines for Cauchy singular integral equations. We will review a few related results in section 5.1 and establish our error estimates in section 5.2.

We mentioned in the introduction that only one paper [68] gives a local error estimate of order $O(h)$ for the piecewise-constant collocation method. We will show, however, that if we take into account the complete asymptotics of solutions on the whole interval, the rates of convergence are different. With our estimates, we can explain the phenomena which were observed by Ryder and Napier in their paper [68].

5.1 Review of Some Results in Spline Approximations

By H^s, $s \in \mathbf{R}$, we denote the usual Sobolev space of order s on an open set Ω. (See the definition in Chapter 2.) Let H_0^s be the closed subspace of $H^2(\mathbf{R}^n)$ consisting of functions with support in $\overline{\Omega}$. Without loss of generality, we consider $\Omega = (-1, 1)$ for the one-dimensional case.

We introduce the following properties which we shall apply in the next section. For proofs see [24] [27] [47].

Property 5.1.1 $H_0^s = H^s$ when $|s| < \frac{1}{2}$, and H_0^s is a subset of H^s when $s \geq 1/2$.

Property 5.1.2 For any $k \in \mathbb{C}$ and $x \in \mathbb{R}$, $(1+x)^k I$, $(1-x)^k I$ and $(1+x)^k(1-x)^k I$ are continuous maps of H_0^s into $H^{s+\min(0,\mathrm{Re}k)}$, where I is the identity operator.

Property 5.1.3 ([47]) Let $\Omega = (-1,1)$ and $s > \frac{1}{2}$. Then the following two conditions are equivalent:

$$u \in H_0^s \tag{5.1.1}$$

and

$$\begin{cases} u \in H^s \\ \left.\frac{d^j u}{dx^j}\right|_{x=-1} = 0 \ , \left.\frac{d^j u}{dx^j}\right|_{x=1} = 0 \ , \ 0 \le j < s - \frac{1}{2} \ . \end{cases} \tag{5.1.2}$$

where $\frac{d^j}{dx^j}$ is the weak derivative.

Let Δ be any partition on the interval $[-1,1]$:

$$\Delta = \{x_0 = -1 < x_1 < \cdots < x_n = 1\}$$

and

$$\bar{h} = \max(x_j - x_{j-1}), \underline{h} = \min(x_j - x_{j-1}), \ j = 1, \ldots, n \ .$$

A mesh Δ is said to be γ-quasiuniform ($\gamma \ge 1$) if $\overline{h} \le \gamma \underline{h}$ and the set of all γ-quasiuniform meshes is denoted by $\mathcal{D}_\gamma$. We denote by s_Δ^d the space of $d-1$ times continuous differentiable splines of degree d on the mesh Δ. We have $s_\Delta^d \subset H^s$ if and only if $s < d + \frac{1}{2}$ [65] [71]. Here we introduce two well-known approximations and inverse properties of splines [65] [24] [3].

Approximation property (AP): Let $s \le r \le d+1$, $s < d + \frac{1}{2}$ and $\sigma < s$. Then for any $u \in H^r$ and any partition Δ there exists $u_\Delta \in s_\Delta^d$ such that

$$\|u - u_\Delta\|_t \le c(t)\overline{h}^{r-t}\|u\|_r \tag{5.1.3}$$

for all $t \in [\sigma, s]$, where $c(t)$ denotes a constant independent of u and Δ, and $\|\cdot\|_t$ is the norm in H^t.

Inverse property (IP): Let $t \leq s < d + \frac{1}{2}$ and $\gamma \geq 1$. Then there exists a constant c such that

$$\|\varphi\|_s \leq c\overline{h}^{t-s}\|\varphi\|_t \tag{5.1.4}$$

for all $\varphi \in s^d_\Delta$ and $\Delta \in \mathcal{D}_\gamma$.

In order to approximate an operator equation in spline spaces, we use the following notation and properties.

Let X and Y be Banach spaces and let $\{X_n\}_{n=0}^\infty$ and $\{Y_n\}_{n=0}^\infty$ be sequences of closed subspaces of X and Y, respectively. Assume $\{P_n\}_{n=0}^\infty$ is a sequence of projections: $P_n : Y \to Y_n$. We consider the equation

$$Ax = y\ , \tag{5.1.5}$$

where A is an operator from X to Y. The projection method is defined by seeking a sequence of elements $x_n \in X_n$, such that

$$A_n x_n = P_n y \qquad x_n \to x \text{ in } X\ .$$

where $A_n = P_n A$.

We say the operator sequence $\{A_n\}$ is stable if the operators A_n are invertible for all n large enough, and if there is a positive number c independent of n such that

$$\|A_n x_n\|_Y \geq c\|x_n\|_X\ .$$

for all $x_n \in X_n$. Prössdorf and Silbermann [65] proved that:

Property 5.1.4 If $\{A_n\}$ is stable, then the sequence $x_n = A_n^{-1} P_n y$ converges quasioptimally to the solution $x = A^{-1} y$, i.e.

$$\|x - x_n\|_X \le c \inf_{v_n \in X_n} \|x - v_n\|_X \; . \tag{5.1.6}$$

We now define spline collocation methods for the approximate solution of equation (5.1.5):

$$Au = f \; .$$

Let Δ be a uniform mesh, i.e., $h = \underline{h} = \bar{h}$. Similar to piecewise polynomial collocation methods, we find $u_h \in s_h^d$ (defined in (5.1)) such that

$$(Au_h)(x_{ij}) = f(x_{ij}) \; , \qquad i = 1, \ldots, n; \; j = 1, \ldots, d+1 \; . \tag{5.1.7}$$

The above collocation method can be interpreted as a projection method by means of the interpolation operator $Q_h : Y \to s_h^d$,

$$Q_h A u_h = Q_h f \; . \tag{5.1.8}$$

Returning to our operator equation:

$$Au = \frac{1}{\pi} \neq\!\!\!\!\!\int_{-1}^{1} \frac{u(t)}{(t-x)^2} \, dt = f(x) \; . \tag{5.1.9}$$

We mentioned in the introduction that the symbol of the operator A is $|\xi|$, i.e., A is an elliptic pseudo-differential operator of order 1. Schmidt in his paper [71] proved that

Lemma 5.1.1 *Let A be a classical elliptic pseudo-differential operator of order $m \in \mathbf{Z}$ with convolutional principal part (i.e., the principal symbol $a_m(x, \xi) \neq 0$ depends only on the Fourier transformed variable ξ and is C^∞ for $\xi \neq 0$). Furthermore, assume $A : X \to Y$, where $X \cdot Y$ are Banach spaces. Define*

$$kerA = \{u \in X : \; Au = 0\}$$

and

$$a = (a_m(+1) + a_m(-1))/2 \ .$$

If $a \neq 0$ *and* $\dim \ker A = 0$, *then* $\left(Q_h A\big|_{s_h^d}\right)^{-1}$ *exists, moreover*

$$\left\| \left(Q_h A\big|_{s_h^d}\right)^{-1} \right\|_{Y \to X} \leq c \ , \tag{5.1.10}$$

where c is a constant.

From theorem 2.2.3, we have that equation (5.1.9) has a unique solution in $L^2(\frac{1}{\rho})$ for any $f \in S$, where $L^2(\frac{1}{\rho})$ and S are defined in Chapter 2. This means $\dim \ker A = 0$, where A is defined in (5.1.9). Furthermore the symbol associated with A : $a_m = -\pi|\xi|$. Hence $a = -\pi(|+1| + |-1|)/2 = -\pi \neq 0$. Therefore, by Lemma 5.1.1, $Q_h A|_{s_h^d}^{-1}$ exists and

$$\left\| \left(Q_h A\big|_{s_h^d}\right)^{-1} \right\|_{S \to L^2(\frac{1}{\rho})} \leq c \ .$$

Note that if $u \in s_h^d$, then $u \in L^2(\frac{1}{\rho})$. Hence, we obtain

$$\begin{aligned} \|u_h\|_{L^2(\frac{1}{\rho})} &= \left\| Q_h A\big|_{s_h^d}^{-1} Q_h f \right\|_{L^2(\frac{1}{\rho})} \\ &\leq c\|Q_h f\|_S \\ &= c\|Q_h A u_h\|_S \ . \end{aligned} \tag{5.1.11}$$

The above discussion implies that the sequence $\{Q_h A\}$ is stable. (5.1.12)

5.2 Error Estimates in Weighted Sobolev Spaces

Motivated by the crack problem introduced in Chapter 1 we consider [39]

$$\frac{1}{\pi} \int_{-1}^{1} \frac{u(t)}{(t-x)^2}\, dt = f(x), \ -1 < x < 1 \ , \tag{5.2.1}$$

$$u(1) = u(-1) = 0. \tag{5.2.2}$$

Theorem 2.2.3 tells us that for any $f(x) \in S$, equation (5.2.1) has a unique solution $L^2(\frac{1}{\rho})$. Furthermore, we have

Lemma 5.2.1 *If $f(x) \in L^2(\rho)$, then the solution $u(x)$ of (5.2.1) and (5.2.2) belongs to $H^1(\frac{1}{\rho}, \rho)$.*

Proof: It is clear that $u(x) \in L^2(\frac{1}{\rho})$ since $L^2(\rho) \in S$. We only need to show that $u'(x) \in L^2(\rho)$.

Integrating by parts, one has

$$\frac{1}{\pi} \int_{-1}^{1} \frac{u(t)}{(t-x)^2}\, dt = \frac{1}{\pi} \int_{-1}^{1} \frac{u'(t)}{t-x}\, dt \qquad (\text{since } u(1) = u(-1) = 0).$$

Therefore

$$\frac{1}{\pi} \int_{-1}^{1} \frac{u'(t)}{t-x}\, dt = f(x), \qquad -1 < x < 1\ ,$$

or

$$\frac{1}{\pi} \int_{-1}^{1} \frac{\sqrt{1-t^2}\, u'(t)}{\sqrt{1-t^2}\,(t-x)}\, dt = f(x), \qquad -1 < x < 1\ .$$

Then [30]

$$\sqrt{1-x^2}\, u'(x) = -\frac{1}{\pi} \int_{-1}^{1} \frac{\sqrt{1-t^2}\, f(t)}{t-x}\, dt + \frac{c}{\pi} \in L^2\left(\frac{1}{\rho}\right)\ .$$

This is equivalent to

$$u'(x) \in L^2(\rho)$$

which completes the proof.

Lemma 5.2.2 *$u(x) \in H^1(\frac{1}{\rho}, \rho)$ and $u(1) = u(-1) = 0$ imply $\rho^\alpha u(x) \in H^1$ for fixed $\alpha > 1$. Moreover $\rho^\alpha u(x) \in H_0^1$.*

Proof: We only need to show that $(\rho^\alpha u(x))' \in L^2$. Note that

$$\begin{aligned} (\rho^\alpha u(x))' &= \left((1-x^2)^{\frac{\alpha}{2}} u(x)\right)' \\ &= -\alpha x(1-x^2)^{\frac{\alpha}{2}-1} u(x) + (1-x^2)^{\frac{\alpha}{2}} u'(x)\ . \end{aligned} \qquad (5.2.3)$$

Since $\alpha > 1$, we have

$$\begin{aligned}
\int_{-1}^{1}\left((1-x^2)^{\frac{\alpha}{2}}u'(x)\right)^2 dx &\leq \int_{-1}^{1}\sqrt{1-x^2}\{u'(x)\}^2 dx \\
&= \|u'\|^2_{L^2(\rho)} . \qquad (5.2.4)
\end{aligned}$$

We still need to prove that the first term (5.2.3) belongs to L^2. In fact,

$$\begin{aligned}
&\int_{-1}^{1}\left(-\alpha x(1-x^2)^{\frac{\alpha}{2}-1}u(x)\right)^2 dx \\
&\leq \alpha^2\int_{-1}^{1}(1-x^2)^{\alpha-2}u^2(x)\,dx \\
&= \alpha^2\left(\int_{0}^{1}(1-x^2)^{\alpha-2}u^2(x)\,dx + \int_{-1}^{0}(1-x^2)^{\alpha-2}u^2(x)\,dx\right) \\
&= \alpha^2(\mathrm{I}+\mathrm{II}) . \qquad (5.2.5)
\end{aligned}$$

Integrating by parts and using $u(1)=0$, one has

$$\begin{aligned}
\mathrm{I} &= \int_{0}^{1}((1-x)(1+x))^{\alpha-2}u^2(x)\,dx \\
&\leq 2^{\alpha-2}\int_{0}^{1}(1-x)^{\alpha-2}u^2(x)\,dx \\
&\leq \frac{2^{\alpha-1}}{\alpha-1}\left(\frac{|u^2(0)|}{2} + \int_{0}^{1}|u(x)u'(x)|\,dx\right) \\
&= \frac{2^{\alpha-1}}{\alpha-1}(\mathrm{I}_1+\mathrm{I}_2) .
\end{aligned}$$

We estimate part I_2 first.

$$\begin{aligned}
\mathrm{I}_2 &\leq c\left(\int_{0}^{1}\frac{(u(x))^2}{\sqrt{1-x^2}}\,dx\right)^{\frac{1}{2}}\left(\int_{0}^{1}\sqrt{1-x^2}\,(u'(x))^2\,dx\right)^{\frac{1}{2}} \\
&\qquad\qquad \text{(Cauchy inequality)} \\
&\leq c\left(\int_{-1}^{1}\frac{(u(x))^2}{\sqrt{1-x^2}}\,dx\right)^{\frac{1}{2}}\left(\int_{-1}^{1}\sqrt{1-x^2}\,(u'(x))^2\,dx\right)^{\frac{1}{2}} \\
&\leq c'\left(\int_{-1}^{1}\frac{(u(x))^2}{\sqrt{1-x^2}}\,dx + \int_{-1}^{1}\sqrt{1-x^2}\,(u'(x))^2\,dx\right) \\
&= c'\left(\|u\|^2_{L(\frac{1}{\rho})} + \|u'\|^2_{L^2(\rho)}\right) \\
&= c'\,\|u\|^2_{H^1(\frac{1}{\rho},\rho)} .
\end{aligned}$$

For the first part I_1, we take a function $\varphi(x) \in C_0^\infty(\Omega)$, such that

$$\varphi(x) = \begin{cases} 1 & |x| < \frac{1}{4} \\ 0 & |x| > \frac{1}{2} \end{cases}$$

We define $v(x) = u(x)\varphi(x)$, then

$$v(x) = -\int_x^1 v'(t)\,dt = \int_1^x v'(t)\,dt\ ,$$

moreover, $u(0) = v(0)$.

Now,

$$\begin{aligned}
2\mathrm{I}_1 = |\mathrm{u}(0)|^2 = |\mathrm{v}(0)|^2 &= \left|\int_0^1 \mathrm{v}'(\mathrm{x})\,\mathrm{dx}\right|^2 \\
&= \left|\int_0^1 (u'(x)\varphi(x) + u(x)\varphi'(x))\,dx\right|^2 \\
&\le c'\left|\left(\int_0^1 \sqrt{1-x^2}|u'(x)|^2\,dx\right)^{1/2}\left(\int_0^1 \frac{|\varphi(x)|^2}{\sqrt{1-x^2}}\,dx\right)^{1/2}\right. \\
&\quad \left. + \left(\int_0^1 \frac{|u(x)|^2}{\sqrt{1-x^2}}\,dx\right)^{1/2}\left(\int_0^1 \sqrt{1-x^2}|\varphi'(x)|^2\,dx\right)^{1/2}\right|^2 \\
&\le c'\left|\left(\int_{-1}^1 \sqrt{1-x^2}|u'(x)|^2\,dx\right)^{1/2} + \left(\int_{-1}^1 \frac{|u(x)|^2}{\sqrt{1-x^2}}\,dx\right)^{1/2}\right|^2 \\
&\le c\left(\int_{-1}^1 \frac{|u(x)|^2}{\sqrt{1-x^2}}\,dx + \int_{-1}^1 \sqrt{1-x^2}|u'(x)|^2\,dx\right) \\
&= c\|u\|^2_{H^1(\frac{1}{\rho},\rho)},
\end{aligned}$$

Hence,we have

$$\mathrm{I} \le c(\alpha)\|u\|^2_{H^1(\frac{1}{\rho},\rho)}\ . \tag{5.2.6}$$

Similarly, we obtain

$$\mathrm{II} \le c(\alpha)\|u\|^2_{H^1(\frac{1}{\rho},\rho)}\ . \tag{5.2.7}$$

By (5.2.3), (5.2.4), (5.2.5), (5.2.6) and (5.2.7), we obtain

$$\int_{-1}^{1}[(\rho^\alpha u(x))']^2\,dx \le c(\alpha)\|u\|^2_{H^1(\frac{1}{\rho},\rho)}$$

where $c(\alpha)$ is a constant depending on α. Therefore $\rho^\alpha u(x) \in H^1$. This fact, together with property 5.1.3, implies $\rho^\alpha u(x) \in H_0^1$. Hence lemma 5.2.2 is proved.

We now apply similar techniques to those used in [65] [71] [25]. Let $X = L^2(\frac{1}{\rho})$, $Y = S$ and $A_n = Q_h A$. It is clear that $u_h \in L^2(\frac{1}{\rho})$ when $u_h \in s_h^d$. The facts (5.1.11) and (5.1.12), together with property 5.1.4, imply the following

Lemma 5.2.3 *For any $f \in S$, the approximate solution u_h converges in $L^2(\frac{1}{\rho})$ to the exact solution $u(x)$ of $Au = f$ with*

$$\|\rho^{-\frac{1}{2}}(u-u_h)\|_0 \le c\min_{v\in s_h^d}\|\rho^{-\frac{1}{2}}(u-v)\|_0\ , \tag{5.2.8}$$

where

$$\|\rho^{-\frac{1}{2}}u\|_0 = \|\rho^{-\frac{1}{2}}u\|_{L^2} = \|u\|_{L^2(\frac{1}{\rho})}\ .$$

Theorem 5.2.1 *Assume $f(x) \in L^2(\rho)$, then the approximate solutions $u_h \in s_h^d$ defined by*

$$Q_h A u_h = Q_h f$$

converge to the exact solution $u(x)$ of (5.2.1) and (5.2.2) at the rate

$$\|u-u_h\|_{L^2} \le ch^{\frac{1}{4}}\|f\|_{L^2(\rho)}\ . \tag{5.2.9}$$

Proof: For any $\alpha > 1$, lemma 5.2.1 and 5.2.2 imply $\rho^\alpha \in H_0^1$. By property 5.1.2, and $\rho = \sqrt{1-x^2}$, we have

$$\|u\|_{1-\frac{\alpha}{2}} \le c\|\rho^\alpha u\|_{H^1}\ . \tag{5.2.10}$$

For any $v \in s_h^d$, one has

$$\begin{aligned} \|v-u_h\|_0 &\le \|\rho^{-\frac{1}{2}}(v-u_h)\|_0 \\ &\le \|\rho^{-\frac{1}{2}}(u-v)\|_0 + \|\rho^{-\frac{1}{2}}(u-u_h)\|_0 \\ &\le \|\rho^{-\frac{1}{2}}(u-v)\|_0 + c_1 \min_{\varphi\in s_h^d} \|\rho^{-\frac{1}{2}}(u-\varphi)\|_0 \quad \text{(by lemma 5.2.3)} \\ &\le c\|\rho^{-\frac{1}{2}}(u-v)\|_0 \end{aligned} \tag{5.2.11}$$

From property 5.1.2, we further have

$$\|\rho^{-\frac{1}{2}}(u-v)\|_0 \le c\|u-v\|_{\frac{1}{4}} \,. \tag{5.2.12}$$

By the approximation property, we can choose $v \in s_h^d$ such that

$$\|u-v\|_0 \le h^{1-\frac{\alpha}{2}}\|u\|_{1-\frac{\alpha}{2}} \tag{5.2.13}$$

and

$$\|u-v\|_{\frac{1}{4}} \le h^{\frac{3}{4}-\frac{\alpha}{2}}\|u\|_{1-\frac{\alpha}{2}} \,. \tag{5.2.14}$$

Therefore, we substitute (5.2.11), (5.2.12), (5.2.13) and (5.2.14) into the following inequality and obtain

$$\begin{aligned} \|u-u_h\|_0 &\le \|u-v\|_0 + \|v-u_h\|_0 \\ &\le ch^r\ \|u\|_{1-\frac{\alpha}{2}} \\ &\le ch^r\ \|\rho^\alpha u\|_{H^1} \qquad \text{(by (5.2.10))} \\ &\le ch^r\ \|u\|_{H^1(\frac{1}{\rho},\rho)} \qquad \text{(by lemma 5.2.2)} \\ &\le ch^r\ \|f\|_{L^2(\rho)} \qquad \text{(by lemma 5.2.1)} \end{aligned} \tag{5.2.15}$$

where $r = \frac{3}{4} - \frac{\alpha}{2}$.

Since (5.2.15) holds for any $\alpha > 1$, let $\alpha \to 1$, and we obtain

$$\|u-u_h\|_0 \le ch^{\frac{1}{4}}\|f\|_{L^2(\rho)} \,.$$

which completes the proof.

Since piecewise constant polynomials are in s_h^0, it follows that

Corollary 5.2.1 *Assume $f(x) \in L^2(\rho)$. Then the piecewise constant approximate solutions u_h converge to the exact solution $u(x)$ of (5.2.1) and (5.2.2) with the rate*

$$\|u - u_h\|_{L^2} = O(h^{\frac{1}{4}}) \ . \tag{5.2.16}$$

Furthermore, the same rate holds for the linear spline approximate solutions.

It is not surprising that the estimate (5.2.16), in general, cannot be improved even if higher order elements are used. Since the approximation property tells us that the error estimates of spline collocation methods depend on both the regularity of the solutions and the degree of splines. Usually for an operator equation $Au = f$, the smoothness of the solution $u(x)$ depends on the regularity of the function $f(x)$. For our hyper-singular integral equations, however, this is not true. In the example 2.3.1 $f(x) = 1 \in C^\infty$, while solution $u(x) = -\sqrt{1-x^2} \notin H^1$. In another example 2.3.1, $f(x) = \frac{1}{1-x^2}$, which does not even belong to $L^2(\rho)$, but $u(x) = 1 \in C^\infty$.

For special cases, however, the convergence rates can be improved.

Let us consider an example:

Example 5.2.1 Consider the solution of

$$\frac{1}{\pi} \int_{-1}^{1} \frac{u(t)}{(t-x)^2}\, dt = x, \qquad -1 < x < 1 \ . \tag{5.2.17}$$

By theorem 2.3.1, we obtain

$$\begin{aligned} u(x) &= -\frac{1}{\pi} \int_{-1}^{1} \frac{\frac{t^2}{2} + c}{\sqrt{1-t^2}(t-x)}\, dt \\ &= -\frac{x}{2}\sqrt{1-x^2} \\ &= v(x)\sqrt{1-x^2} \end{aligned}$$

where $v(x) = -\frac{x}{2} \in H^m$, any $m \in \mathbf{R}$.

Now we deal with the following general case:

Assume that the solution $u(x)$ of the problems 5.2.1 and 5.2.2 has the form

$$u(x) = \sqrt{1-x^2}\, v(x) \tag{5.2.18}$$

where $v(x) \in H^m$ with $m \geq 1$.

It is easy to see that $\rho^\alpha u(x) \in H^1$ for any $\alpha > 0$. Assume $u_h \in s_h^d$ are the approximate solutions. We apply the same techniques with the proof of theorem 5.2.1. Then

$$\|u - u_h\|_0 = O(h^r)$$

where $r = \frac{3}{4} - \frac{\alpha}{2}$, for any $\alpha > 0$. Let $\alpha \to 0$, and we obtain

$$\|u - u_h\|_0 = O\left(h^{\frac{3}{4}}\right) . \tag{5.2.19}$$

Therefore, we have proved that

Corollary 5.2.2 *If the solutions $u(x)$ of (5.2.1) and (5.2.2) have the form $u(x) = \sqrt{1-x^2}\, v(x)$, with $v(x) \in H^m$, $m \geq 1$, then the piecewise constant approximate solutions $u_h \in s_h^0$ converge to the exact solution $u(x)$ at the rate*

$$\|u - u_h\|_0 = O(h^{\frac{3}{4}}) . \tag{5.2.20}$$

Furthermore, the same estimate $O(h^{\frac{3}{4}})$ holds for the linear spline approximations.

We are now ready to explain the phenomena which were observed by Ryder and Napier in their paper [68]. First, we note that the special $\sqrt{x}$ elements at the edge grids (near $x = 0$) are equivalent to the weights $\sqrt{1-x}$ (near $x = 1$) and $\sqrt{1+x}$ (near $x = -1$). Therefore, we introduce weighted splines. (See [68]) Let $\tilde{s}_h^d = \rho\, s_h^d$, where $\rho = \sqrt{1-x^2}$.

Suppose that the exact solution of (5.2.1) $u(x) = \sqrt{1-x^2}\,v(x)$. For any approximate solutions $u_h \in \tilde{s}_h^d$, we have

$$\|u - u_h\|_0 = \|\sqrt{1-x^2}(v - v_h)\|_0 \ , \tag{5.2.21}$$

where $v_h \in s_h^d$. In fact, $v(x)$ is a solution of the following equation

$$\frac{1}{\pi} \int_{-1}^{1} \frac{\sqrt{1-t^2}\,v(t)}{(t-x)^2}\,dt = f(x), \qquad -1 < x < 1 \ . \tag{5.2.22}$$

By theorem 2.2.1, for any $f(x) \in S$, there exists a unique solution $v(x) \in L^2(\rho)$. If we choose $X = L^2(\rho)$ and $Y = X$, (5.1.10) implies (similar to (5.1.11))

$$\|v_h\|_{L^2(\rho)} \le c\|Q_h A v_h\|_S \tag{5.2.23}$$

where A is defined as

$$(Av)(x) = \frac{1}{\pi} \int_{-1}^{1} \frac{\sqrt{1-t^2}\,v(t)}{(t-x)^2}\,dt \ .$$

Then the inequality (5.2.23) and property 5.1.4 imply that

Lemma 5.2.4 *For any $f(x) \in S$, the approximate solution $v_h \in s_h^d$ converges in $L^2(\rho)$ to the exact solution $v(x)$ of (5.2.22) with*

$$\|\rho^{\frac{1}{2}}(v - v_h)\|_0 \le c \min_{\varphi \in s_h^d} \|\rho^{\frac{1}{2}}(v - \varphi)\|_0 \ . \tag{5.2.24}$$

Theorem 5.2.2 *If the solution $u(x)$ of (5.2.1) and (5.2.2) has the form $u(x) = \sqrt{1-x^2}\,v(x)$ with $v(x) \in H^m$, $m \ge 1$, then the approximate solutions $u_h \in \tilde{s}_h^d = \rho\, s_h^d$ converge to the exact solution at the rate*

$$\|u - u_h\|_0 = O(h^r)$$

where $r = \min\left\{d + \frac{3}{2}, m\right\}$.

Proof: By (5.2.21), we have

$$\|u - u_h\|_0 \le \|v - v_h\|_0 \,, \tag{5.2.25}$$

where $v \in s_h^d$.

For any $\varphi \in s_h^d$, from lemma 5.2.4, one has

$$\begin{aligned} \|\rho^{\frac{1}{2}}(\varphi - v_h)\|_0 &\le \|\rho^{\frac{1}{2}}(v-\varphi)\|_0 + \|\rho^{\frac{1}{2}}(v - v_h)\| \\ &\le \|\rho^{\frac{1}{2}}(v-\varphi)\|_0 + c_1 \min_{\tilde{\varphi}\in s_h^d} \|\rho^{\frac{1}{2}}(v-\tilde{\varphi})\|_0 \\ &\le c\|\rho^{\frac{1}{2}}(v-\varphi)\|_0 \\ &\le c\|v-\varphi\|_0 \,. \end{aligned} \tag{5.2.26}$$

From the inverse property and the property 5.1.2, we obtain for any $\varphi \in s_h^d$

$$\begin{aligned} \|\varphi - v_h\|_0 &\le ch^{-\frac{1}{4}}\|\varphi - v_h\|_{-\frac{1}{4}} \\ &\le ch^{\frac{1}{4}}\|\rho^{\frac{1}{2}}(\varphi - v_h)\|_0 \,. \end{aligned} \tag{5.2.27}$$

By the approximation property, we can choose $\varphi \in s_h^d$ such that

$$\|v - \varphi\|_0 \le h^s\|v\|_s \tag{5.2.28}$$

where $s \le d+1$ and $s \le m$.

Thus using (5.2.25), (5.2.26), (5.2.27) and (5.2.28), we have

$$\begin{aligned} \|u-u_h\|_0 &\le \|v - v_h\|_0 \\ &\le \|v-\varphi\|_0 + \|\varphi - v_h\|_0 \\ &\le h^s\|v\|_s + h^{s-\frac{1}{4}}\|v\|_s \\ &= O(h^r) \qquad (\text{choose } s = \min\{d+1, m\}) \,, \end{aligned}$$

where $r = \min\{d + \frac{3}{4}, m\}$ and the theorem is proved.

From the above theorem, one easily obtains:

Corollary 5.2.3 *If the solution $u(x)$ of (5.2.1) and (5.2.2) has the form $u(x) = \sqrt{1-x^2}\,v(x)$ with $v(x) \in H^m$, $m \geq 2$, then piecewise constant approximate solutions $u_h \in \bar{s}_h^0$ converge to the exact solution at the rate*

$$\|u - u_h\|_0 = O\left(h^{\frac{3}{4}}\right) . \tag{5.2.29}$$

While for the linear spline approximations, the rate is

$$\|u - u_h\|_0 = O\left(h^{1\frac{3}{4}}\right) . \tag{5.2.30}$$

Therefore by using $\sqrt{x}$ elements at the edges of the interval in conjunction with the standard piecewise constant approximation within the interval, the gain in accuracy is only from $O(h^r)$, $r < \frac{3}{4}$ and $r \to \frac{3}{4}$ (5.2.20) to $O(h^{\frac{3}{4}})$ (5.2.29). This is why Ryder and Napier found only modest improvement in this case. They observed that in the case of slits, using linear displacement discontinuity variations with average-stress collocation (no special $\sqrt{x}$ elements are used at the edge grids) is ineffective. Our explanation of this phenomenon is that there is almost no gain in accuracy because of corollary 5.2.2. However, by using the special $\sqrt{x}$ elements at the edge grids for piecewise linear approximations, we can gain from $O(h^{\frac{3}{4}})$ (5.2.20) to $O(h^{1\frac{3}{4}})$ (5.2.30). This coincides with their observation that special $\sqrt{x}$ elements with linear splines give meaningful gains in accuracy, while $\sqrt{x}$ edge elements with piecewise constant approximation yield only a marginal improvement in accuracy.

5.3 Numerical Results

We consider the numerical solution of Example 2.3.1:

$$\frac{1}{\pi}\int_{-1}^{1}\frac{u(t)}{(t-x)^2}\,dt = 1, \qquad -1 < x < 1 . \tag{5.3.1}$$

Since (5.3.1) can be solved analytically, it is easy to check the accuracy of the solutions obtained by the PWC or PWL collocation method.

The numerical convergence rates for PWC and PWL approximate solutions are plotted in Figure 5.1 and Figure 5.2 respectively. The numerical solutions converge to the exact solution $u(x) = -\sqrt{1-x^2}$ at the rate $O(h^{0.91})$ for the PWC approximate and at the rate $O(h^{0.82})$ for the PWL approximate. These numerical results are consistent with the error estimate $\|u - u_h\|_{L^2} \leq ch^{0.75}$, where c is a constant. In addition, because of Corollary 5.2.1 and 5.2.2, it is not surprising that the accuracy is not improved with higher order elements. Since the effect of round-off error propagation depends on the condition number of influence matrices, the convergence rate of the PWL approximate is slightly lower than that of the PWC approximate.

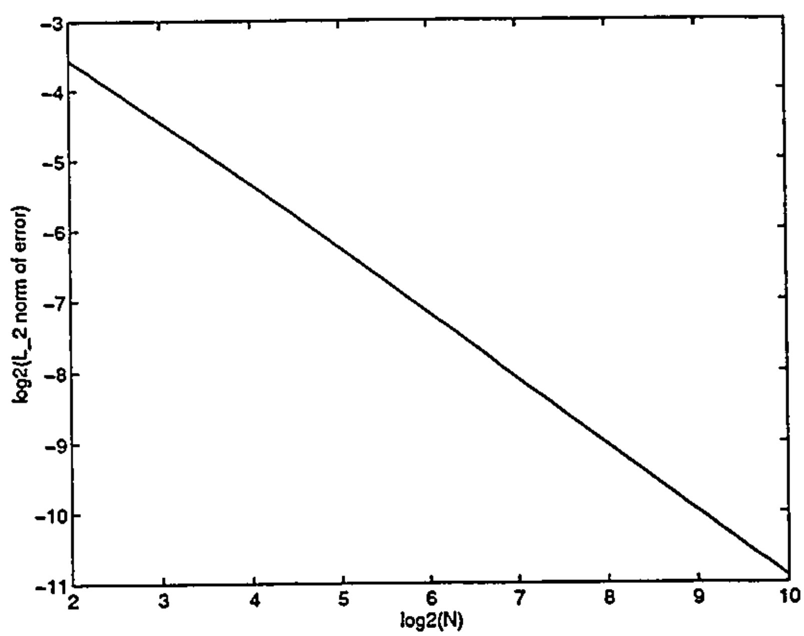


Figure 5.1: The Convergence Rate for PWC Approximate Solution

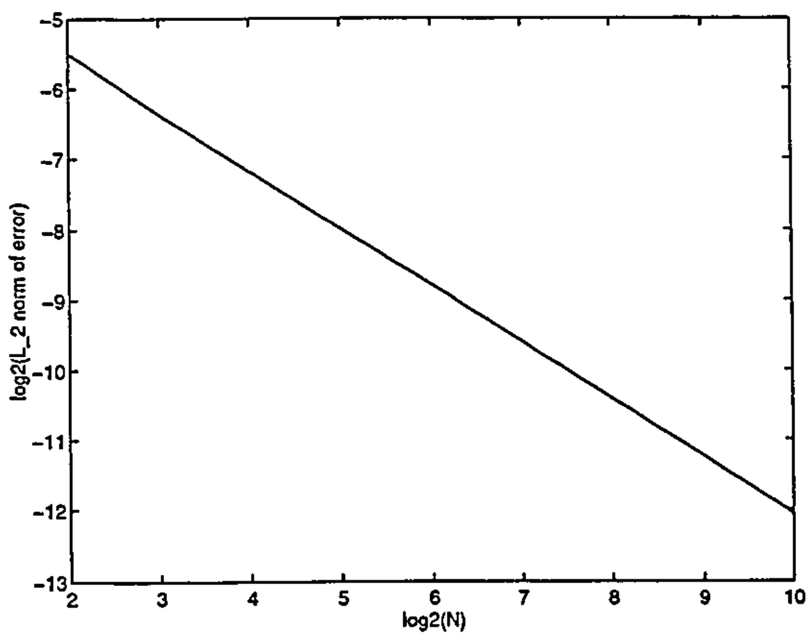


Figure 5.2: The Convergence Rate for PWL Approximate Solution

Chapter 6

Applications

In this chapter we apply the wavelet approximation for matrix problems to illustrate the performance of the algorithm. The first set of examples use wavelet approximations to piecewise constant and piecewise linear discretizations to the one-dimensional crack problem (4.1.2). The matrix of the piecewise constant discretization is translationally invariant – so a wavelet procedure to speed up a matrix multiply by such an operator does not provide us with new tools as this can be done by means of the fast Fourier transform (FFT). The matrix for the piecewise linear discretization is not translationally invariant and therefore a matrix multiply by such an operator cannot be performed efficiently using the FFT. The wavelet approximation for this operator makes it possible to perform matrix-vector multiplications efficiently in $O(N \log N)$ operations. This shows that the wavelet approximation method represents a new device that can be used to solve large systems of equations associated with BE models with arbitrary geometries. In order to establish the applicability of the wavelet approximation methods, we solve a system of equations that arise in a two-dimensional piecewise linear displacement discontinuity model for elasto-statics. We start by the description of the algorithm.

6.1 Description of the Algorithm

Step 1. Set up the influence matrix A_N^P by using the piecewise constant or the piecewise linear collocation methods for the integral equation $Au = f$ (one or higher dimensions). This step requires $O(N^2)$ operations.

Step 2. Derive the sparse matrix representation.

METHOD (I). Directly apply discrete wavelet transforms to the matrix A_N^P, i.e., apply one-dimensional wavelet transforms to every row (column) then to every column (row). And discard elements below a threshold ε_1 determined by the accuracy criterion (4.4.22). This step also requires $O(N^2)$ operations.

METHOD (II). Use the estimate (4.4.3) and the estimates for the non-standard form (see Beylkin, Coifman, Rokhlin [7]) and only calculate elements of $A_j^{(k,\ell)}$, $B_j^{j'(k,\ell)}$, $\Gamma_j^{j'(k,\ell)}$ for k, ℓ satisfying $|k - \ell| < b$, where b is determined by (4.4.7). Note that there are interactions between scales for different bands. Therefore we need to calculate those boundary elements involving different bands. Since the resulting matrix contains only $O(N \log N)$ elements, this step requires no more than $O(N \log N)$ operations. However, the program (II) is more complicated than (I).

We denote the resulting sparse matrix from Step 2 by A_s, and we use an indexed data structure to store the matrix A_s.

Step 3. Solve the sparse matrix problem $A_s\tilde{u} = \tilde{f}$.

METHOD (I). Obtain an inverse A_{m_s} by Schulz's method and calculate $A_{m_s}\tilde{f} = \tilde{u}_{m_s}$.

Use the row sum norm instead of $\|\cdot\|_2$ norm in lemma 4.5.1. Start by computing

$A_0 = \alpha A_s^T$, $\alpha = \frac{1}{\|A_s^T A_s\|}$ and discard elements whose absolute value is smaller than ε_1. Then use iterations:

do $m = 0, 1, \ldots$
while $\|I - A_m A_s\| \geq \varepsilon_1$
$A_{m+1} = 2A_m - A_m A_s A_m$,
using the same data structure for $m = 0, 1, 2, \ldots$ to store the inverse matrix A_m.
enddo

Since the multiplication of two sparse matrices of this type requires $O(N \log^2 N)$ operations and iterative steps m_s, which are independent of N and can be determined by (4.5.11), this procedure requires only $O(N \log^2 N)$ operations in total.

Although calculating the inverse matrix is not common practice in numerical computations due to conditioning and storage problems, in this case conditioning does not present a problem and the representation of the inverse operator is also in the sparse standard form.

Note: Perhaps the most important application of this algorithm will be in the iterative solution of nonlinear problems. Typically the most expensive component of such iterative algorithms is the matrix-vector multiply that needs to be performed to determine the influence of one element on another. In this case the wavelet approximation can be used to reduce the operation count for a matrix-vector multiply from $O(N^2)$ to $O(N \log N)$.

METHOD (II). Apply the preconditioned biconjugate gradient method (PBCG).

The PBCG algorithm can be stated as follows:

(i) Initialization

Given $\tilde{u}_0$ the initial guess, Solve $Cz_0 = (\tilde{f} - A_s\tilde{u}_0) = r_0$,

$\bar{r}_0 = z_0 = \bar{z}_0 = p_0 = \bar{p}_0 = r_0$.

(ii) $k = 0$,

If $\|A_s\tilde{u}_k - \tilde{f}\|/\|\tilde{f}\| > \varepsilon$ do

(a) $\tilde{u}_{k+1} = \tilde{u}_k + \alpha_k p_k$, where $\alpha_k = \frac{\bar{r}_k^T z_k}{\bar{p}_k^T A_s p_k}$,

(b) $r_{k+1} = r_k - \alpha_k A_s p_k$, $\bar{r}_{k+1} = \bar{r}_k - \alpha_k A_s^T \bar{p}_k$,

(c) Solve $Cz_{k+1} = r_{k+1}$, $C^T \bar{z}_{k+1} = \bar{r}_{k+1}$,

(d) $p_{k+1} = z_{k+1} + \beta_k p_k$, $\bar{p}_{k+1} = \bar{z}_{k+1} + \beta_k \bar{p}_k$, where $\beta_k = \frac{\bar{r}_{k+1}^T z_{k+1}}{\bar{r}_k^T z_k}$,

$k = k + 1$,

enddo

Here the matrix C is the preconditioner and can be regarded as an approximate inverse of A_s. We simply choose the trivial diagonal part of A_s as a preconditioner C. Note that the above iterative scheme only involves the multiplication of the form $A_s p$ and $A_s^T p$. Since the sparse matrix A_s contains no more than $O(N \log N)$ non-zero elements, only $O(N \log N)$ operations are required in each iteration. This method requires $O(N_i N \log N)$ operations, where N_i is the number of iteration steps. In our numerical examples for solving PWC and PWL models, we note that the number of iterations for PBCG remains constant ($\sim O(\log N)$) so that the operation count increases with $O(N \log^2 N)$ in these examples.

Step 4. Take inverse wavelet transform to $\tilde{u}_k$ which gives a numerical solution u_k. Clearly, this step requires $O(N)$ operations.

6.2 One-dimensional Fracture Mechanics Problem

The interval $[a, b]$ is divided into N elements over each of which the solution is assumed to be constant or linear (as described in section 4.1). For the model problem (4.1.2), we assume that $a = 0$, $b = 1$ and $f(x) = -\pi$ and vary the number N of boundary elements in the discretization.

(I) Accuracy of the approximation

(1) Solution of PWC crack models

In Figure 6.1 the solution to the piecewise constant algebraic equation $Au = f$ using a direct elimination method is compared with the sparse matrix approximation for $n = 256$, $M = 6$ and $\varepsilon = 10^{-6}$. This matrix is the one whose sparse wavelet approximation is shown in figure 4.1. The analytic solution $\sqrt{x(1-x)}$ is also plotted for comparison. The direct solution and the sparse matrix solution are indistinguishable on this plot. In Figure 6.2 we plot the pointwise difference between the direct solution and the sparse wavelet solution. The L_2-norm of these errors $\|u_d - u_w\|_{L^2}$ is 5.819×10^{-5} which is in agreement with the estimate (4.2.22). (Here $\text{cond}(A) = 221$, $N = 256$.) Thus despite the terms that were discarded in the process of obtaining a sparse wavelet representation, the approximate operator contains sufficient information about the original operator to reproduce the solution to the desired accuracy.

The L_2-norm and L_∞-norm of the errors for other values of N are given in Table 6.1.

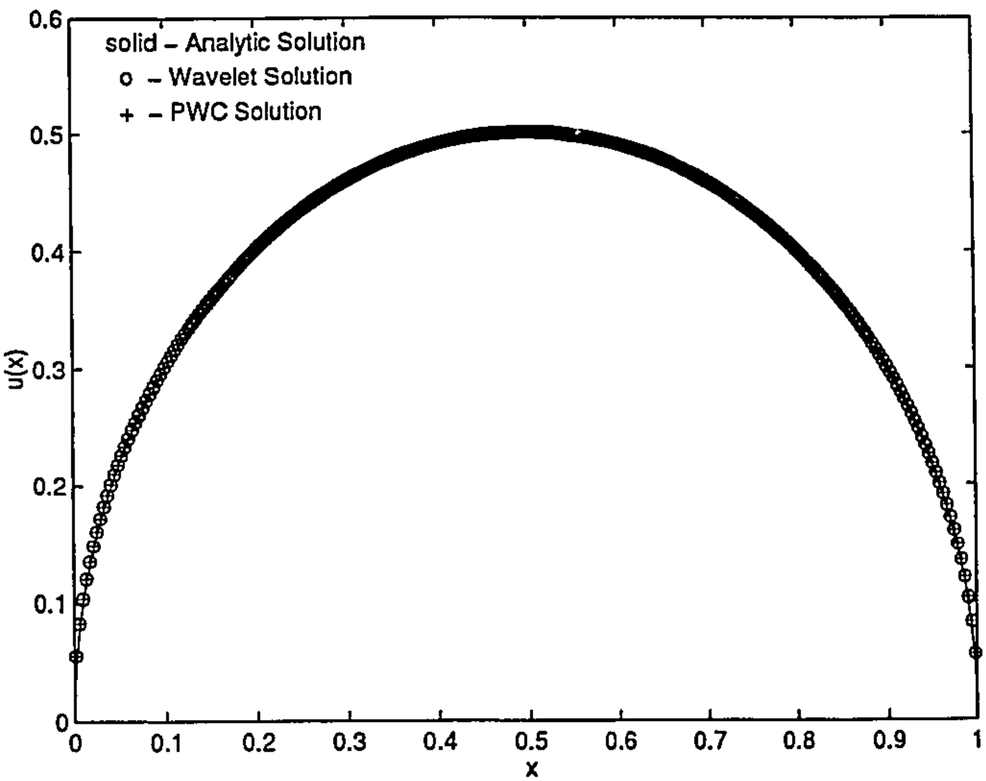


Figure 6.1: Solutions of the PWC Crack Model ($N = 256$)

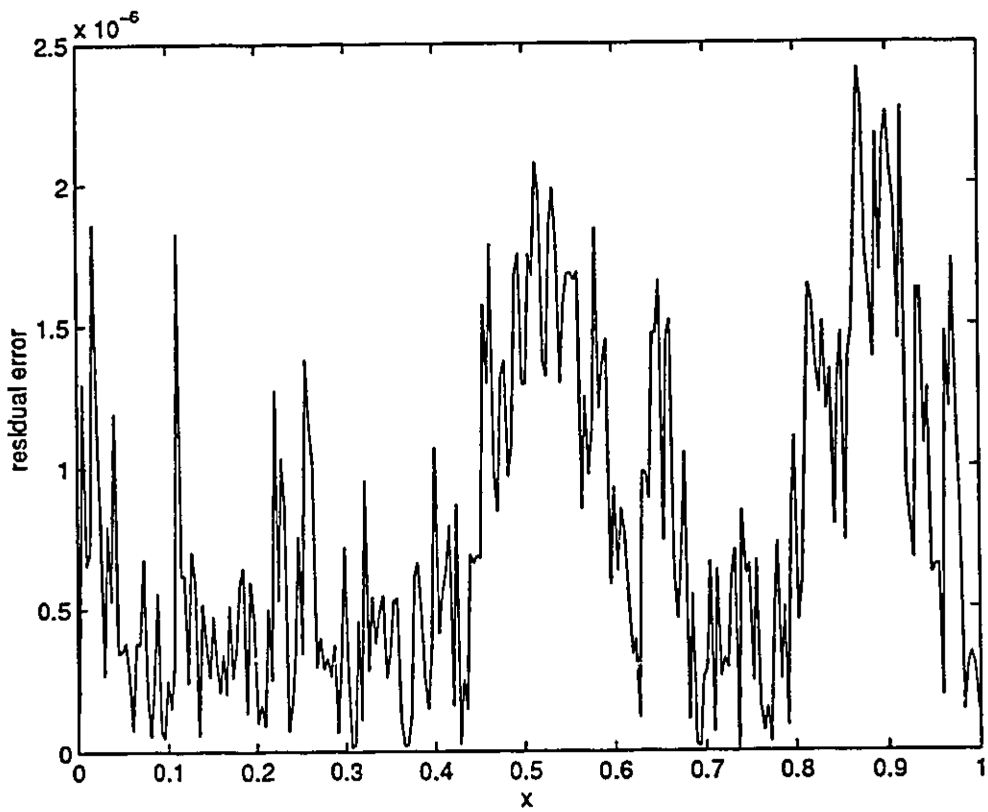


Figure 6.2: Residual Error | $u_d - u_w$ | for the PWC Crack Model

(2) Solution of PWL crack models

Similar to (1), in Figure 6.3 we plot the direct solution and the sparse matrix approximation for $N = 256$, $M = 6$ and $\varepsilon = 10^{-6}$ to the piecewise linear algebraic equation $Au = f$.

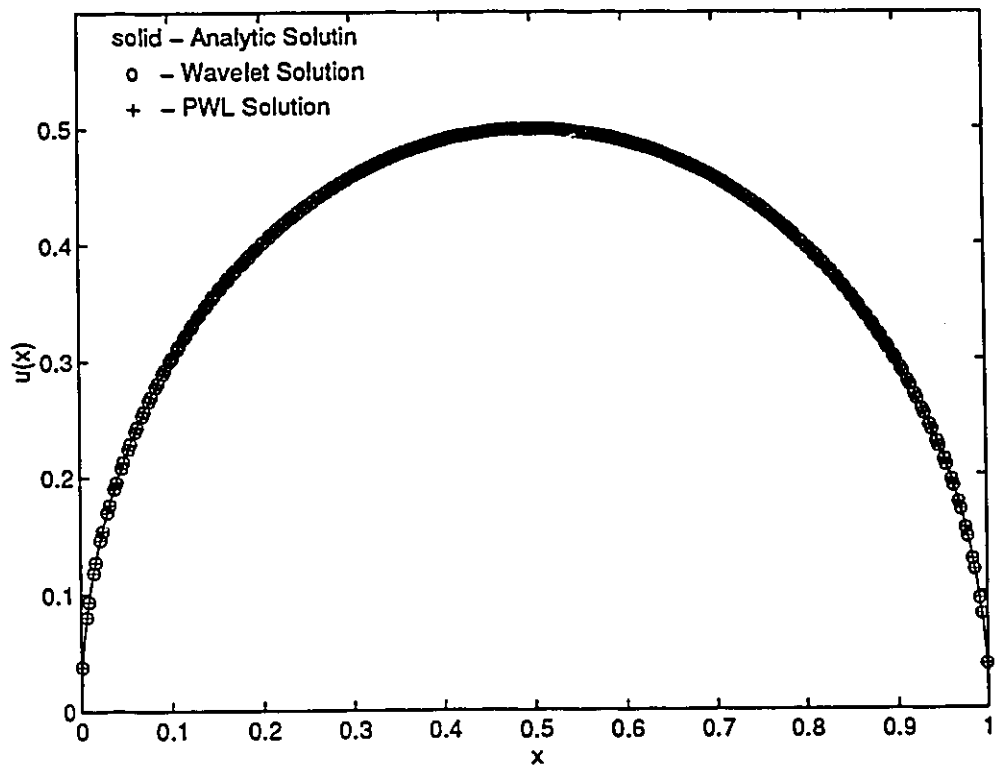


Figure 6.3: Solutions of the PWL Crack Model ($N = 256$)

The standard sparse form of the matrix A is shown in figure 4.2. The analytic solution $\sqrt{x(1-x)}$ is also plotted in this figure for comparison. The pointwise difference between the direct solution and the sparse wavelet solution is shown in Figure 6.4.

The L_2-norm of $u_d - u_w$ is 2.868×10^{-5}.

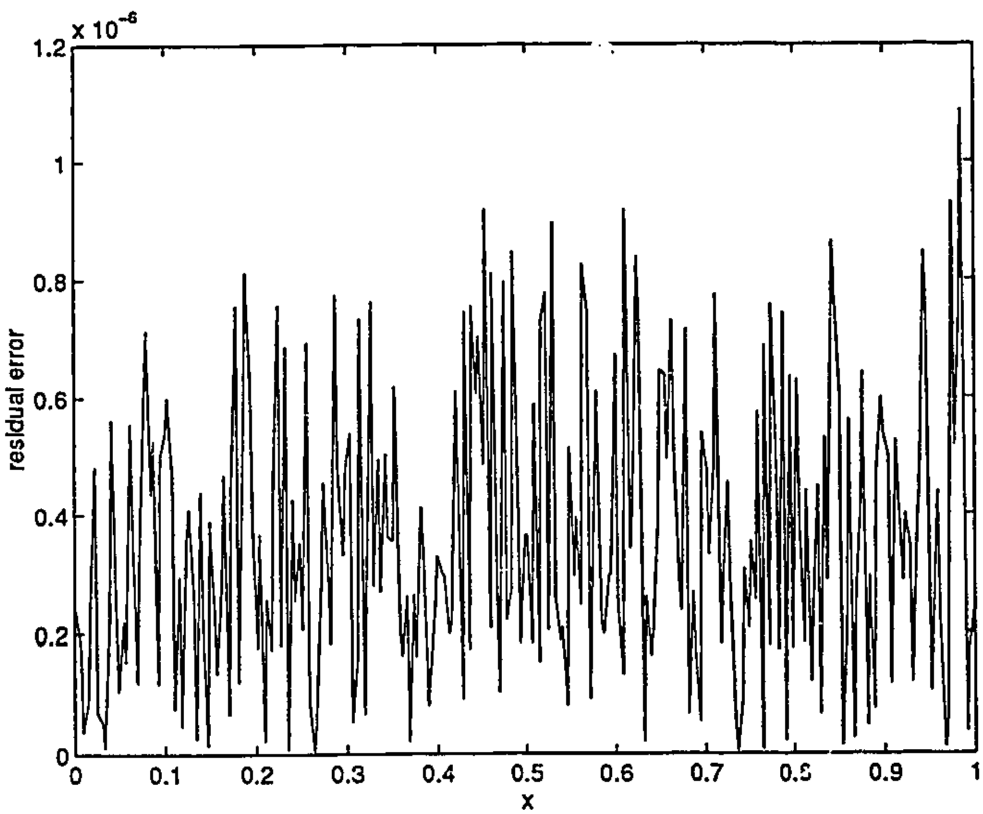


Figure 6.4: Residual Error $| u_d - u_w |$ for the PWL Crack Model

(II) Efficiency of the algorithm

We define the compression rates C_{comp} by the ratio of N^2 to the number of non-zero elements in the standard form of A. In Table 6.1 we provide the solution times and memory requirements using the PBCG method and the direct elimination method

N	Memory (words)		Ccomp	Run Time (sec.)			Errors of $u_d - u_w$	
	direct	PBCG		direct	PBCG	WT	L_2-norm	L_∞-norm
32	1024	901	1.14	0.0253	0.0089	0.0238	3.16e-4	1.63e-4
64	4096	2309	1.77	0.2753	0.0263	0.1064	1.07e-4	5.42e-4
128	16384	8255	1.98	4.3120	0.2328	1.0145	1.73e-4	4.23e-5
256	65536	19317	3.39	25.286	0.7995	2.9119	5.82e-5	1.41e-5
512	262144	50549	5.19	213.71	1.4576	13.540	1.80e-5	2.99e-6
1024	1048576	106111	9.88	1809.0	3.3644	67.224	3.27e-5	3.15e-6

Table 6.1: Run Time and Memory Statistics for PWC Crack Models

respectively for the PWC algebraic equation. Column 7 in this table records run times of wavelet transform which is required to transform the dense matrix to a sparse representation. The run time and memory statistics for the PWL algebraic equation are summarized in Table 6.2. The L_2-norm and L_∞-norm of $u_d - u_w$

N	Memory (words)		Ccomp	Run Time (sec.)			Errors of $u_d - u_w$	
	direct	PBCG		direct	PBCG	WT	L_2-norm	L_∞-norm
32	1024	870	1.18	0.0251	0.0163	0.0238	4.14e-4	2.05e-4
64	4096	2631	1.56	0.2814	0.0383	0.1030	5.86e-4	3.01e-4
128	16384	8654	1.89	3.1129	0.2759	1.0900	2.05e-4	6.00e-6
256	65536	21725	3.01	25.238	0.8163	2.8400	2.87e-5	6.62e-6
512	262144	56556	4.64	214.82	2.3371	13.170	1.92e-5	3.37e-6
1024	1048576	128511	8.16	1800.6	4.1171	66.841	2.75e-5	2.91e-6

Table 6.2: Run Time and Memory Statistics for PWL Crack Models

are given in these two tables for different numbers of elements N.

In Figure 6.5 we plot run times (for PWL) of the PBCG method and the direct method for comparison.

We also use Schulz's method to solve sparse matrix problems for $N = 32$, 64, 128, 256, 512 and 1024. However, the Schulz's method takes relatively more time than the PBCG method. The reasons are that (1) the compression rates C_{comp} for

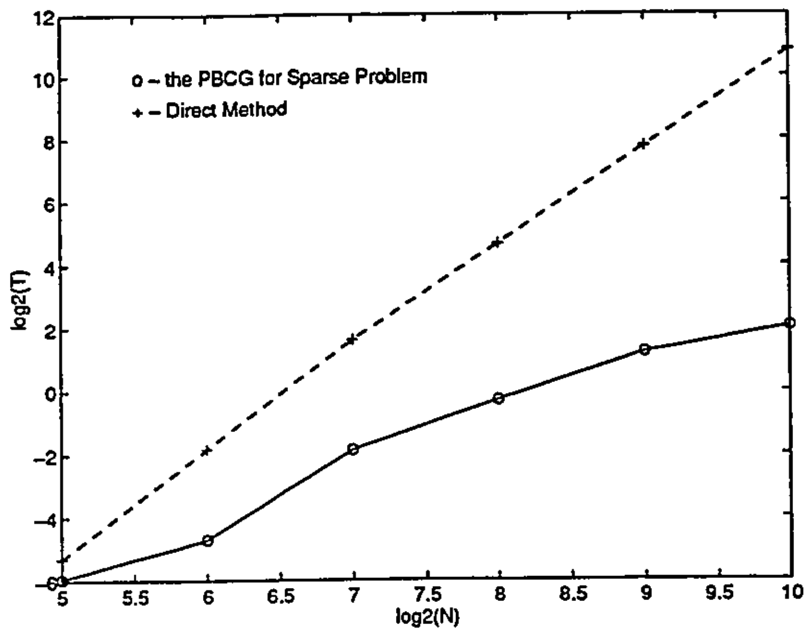


Figure 6.5: Comparison Run Times of the PBCG and Direct Methods

the numbers of elements N considered are not large enough and (2) accessing the link lists for the indexed data structure in each iteration takes time. Since for a given ε and sparse representation of the matrix A, the number of iterative steps m in Schulz's method can be determined from (4.5.12), we can expect better performance from Schulz's method for large N (in which case the compression rates are large). In addition, we can apply Schulz's method to obtain a preconditioner using a small number of iterative steps m. In table 6.3 we summarize the iterative steps m for

N	Theoretical Results (m >)	Numerical Results (m =)
32	10.64	11
64	12.64	13
128	14.65	15
256	16.64	17
512	18.64	19
1024	20.58	21

Table 6.3: The Iterative Steps of Constructing a Preconditioner by Schulz's Method

different values of N (for the PWL model). The theoretical results are determined by (4.5.12). The sparsity and representation of A and Schulz's method provide the possibility of obtaining a preconditioner which will be efficient in reducing run times even for non-linear problems. .

6.3 Two-dimensional Fracture Mechanics Problem

Figures 6.6 and 6.7 illustrate the standard forms of the two-dimensional PWL influence matrix for $N = 256$ and $N = 512$ respectively. The direct solutions and the sparse matrix approximations are plotted for $N = 256$ and $N = 512$ respectively in Figure 6.8 and 6.9.

We summarize the run time and memory statistics for $2D$ problems in Table 6.4. Comparing with the one-dimensional case, we find that the compression

N	Memory (words)		Ccomp	Run Time (sec.)			Errors of $u_d - u_w$	
	direct	PBCG		direct	PBCG	WT	L_2-norm	L_∞-norm
256	65536	42158	1.55	22.20	5.04	2.58	6.70e-3	1.17e-3
512	262144	149931	1.75	188.46	28.91	16.02	2.48e-2	3.38e-3

Table 6.4: Run Time and Memory Statistics for 2D Crack Models

rate of the $2D$ problem for $N = 256$ is similar to the rate of the $1D$, $N = 64$ case and the rate of $2D$, $N = 512$ case is similar to the rate of $1D$, $N = 128$ case. So we can expect that in order to be substantially more efficient than direct methods for two dimensional problems, the algorithm requires $N = 1024$ or more elements.

Remark. The data files of the two-dimensional crack problems are derived from a piecewise linear two dimensional displacement discontinuity formulation for solving the equations of elastostatics. The model comprises two perpendicular interacting

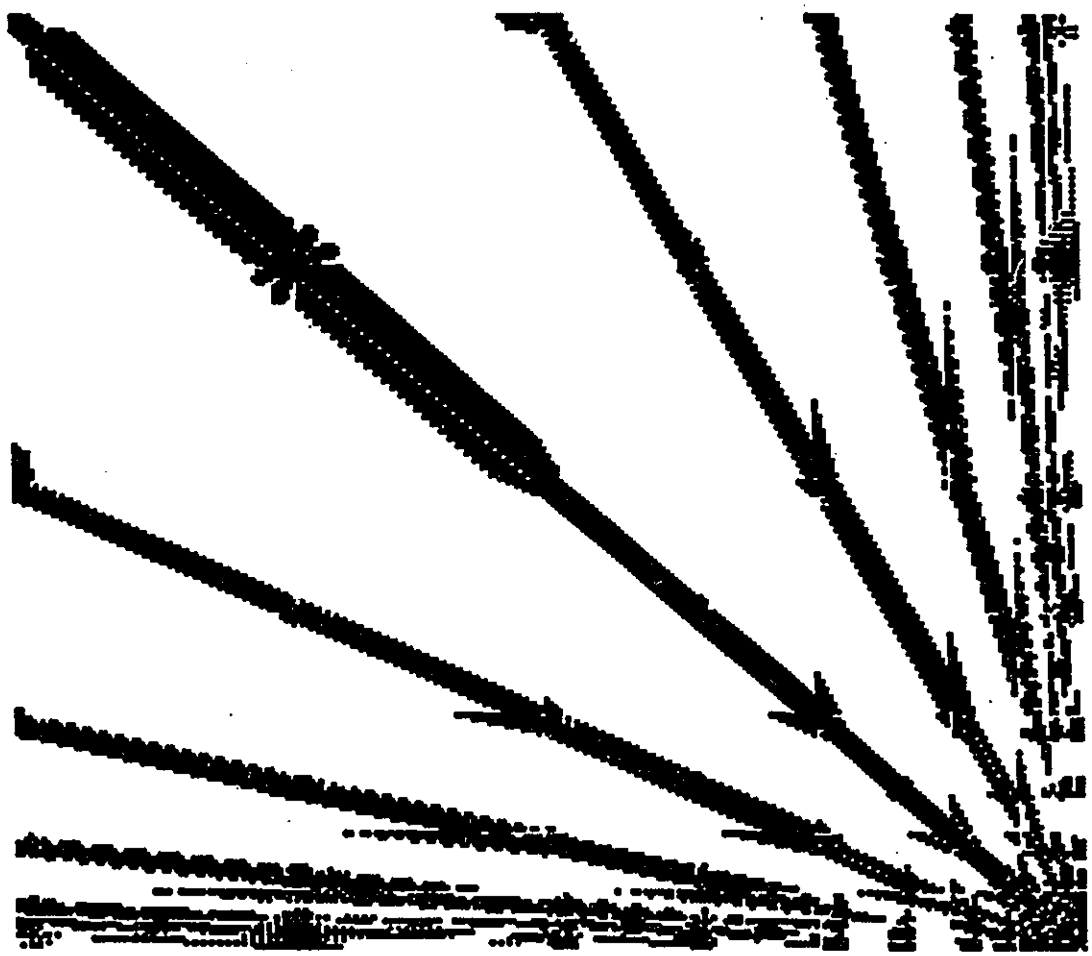


Figure 6.6: The Standard Form of the 2D Influence Matrix (Entries above Threshold 0.002, $N = 256$)

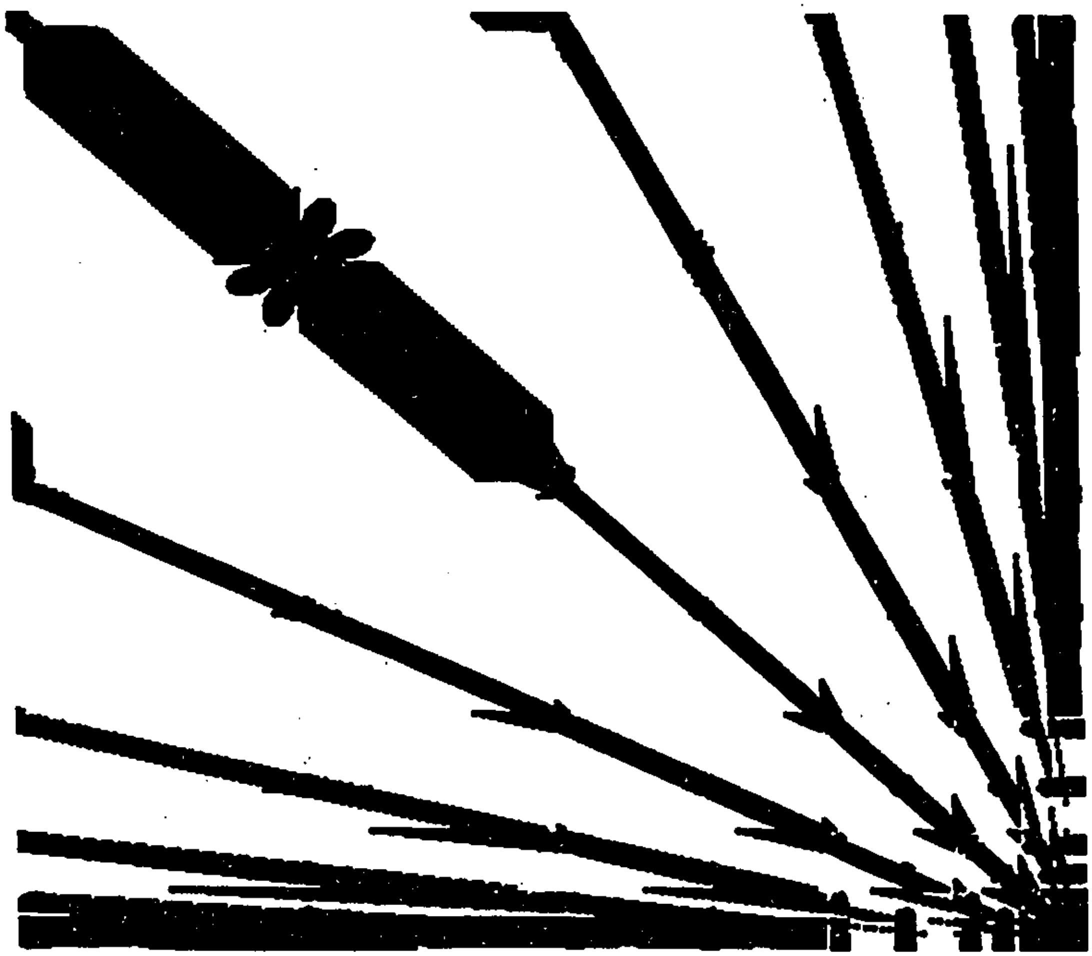


Figure 6.7: The Standard Form of the 2D Influence Matrix (Entries above Threshold 5×10^{-5}, $N = 512$)

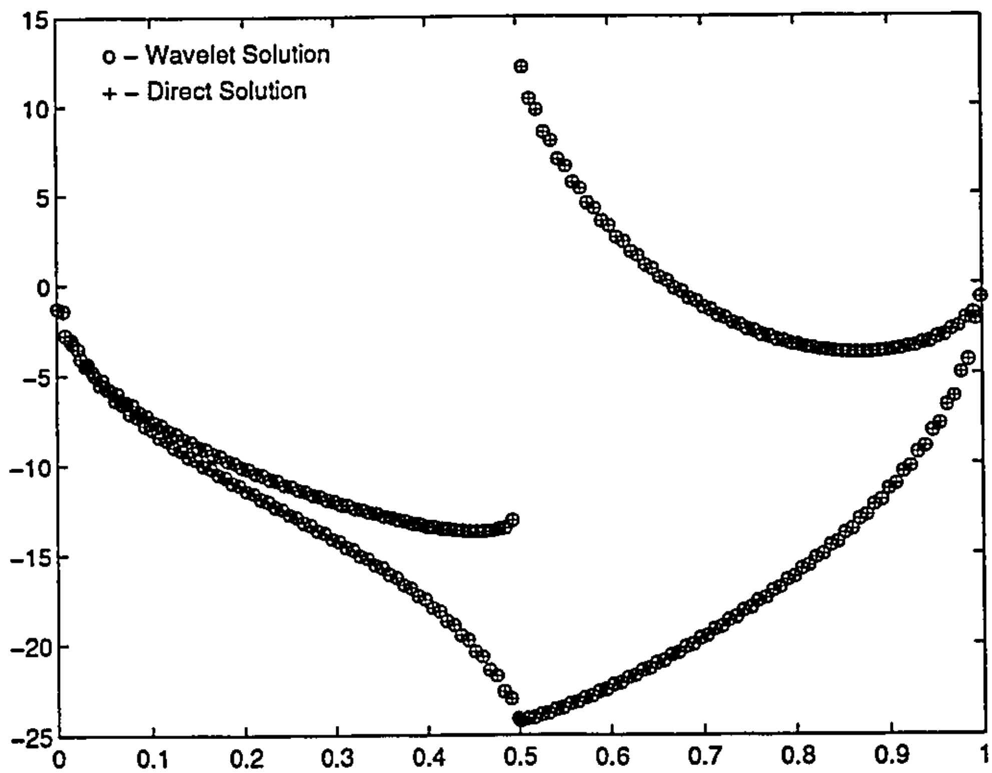


Figure 6.8: Solutions of the 2D Crack Model ($N = 256$)

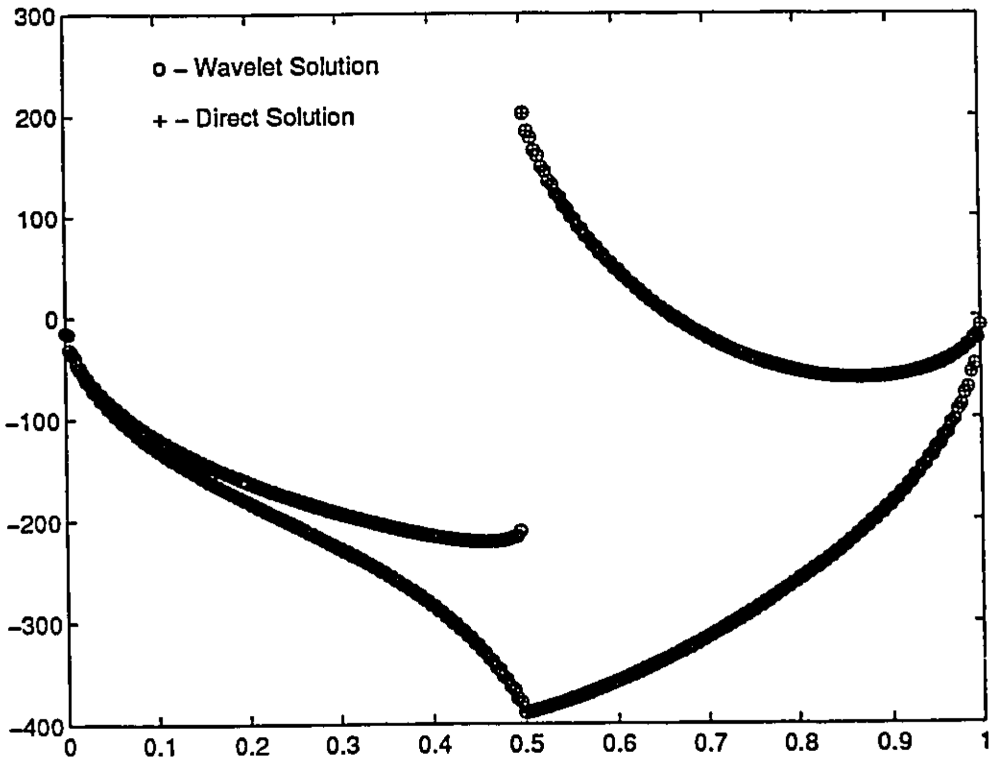


Figure 6.9: Solutions of the 2D Crack Model ($N = 512$)

cracks as shown in Figure 6.10. The model assumes a Young's Modulus of $70GPa$ and a Poisson's Ratio of 0.2 and that the boundaries of the cracks are subjected to uniform normal and shear stresses.

Geometry of 2D crack problem with $N = 16$

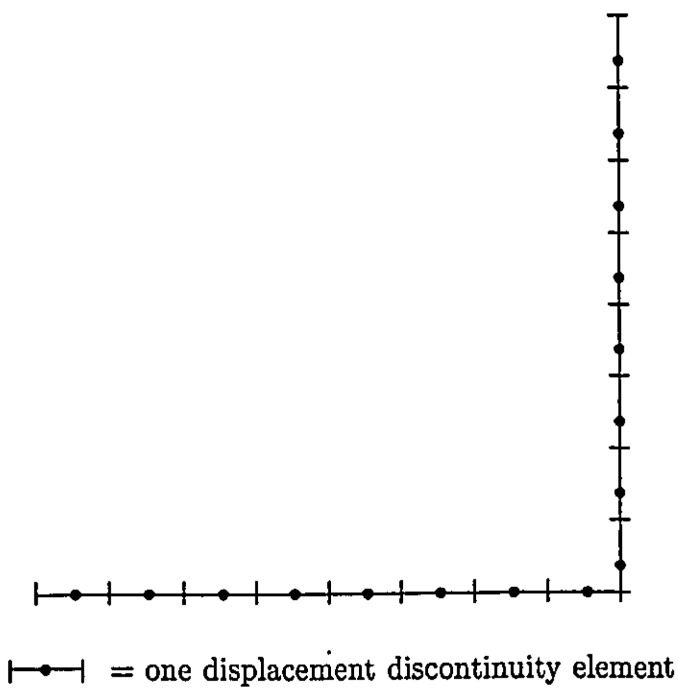


Figure 6.10: Geometry of the 2D Crack Model for the case $N = 16$

Chapter 7

Conclusions

A hypersingular integral equation associated with the BE technique that is used for designing underground excavations, nuclear power plants, hydroelectric dams and offshore oil rigs, etc. has been studied theoretically and numerically.

One of the major contributions of this thesis is that the global existence of solutions for such hypersingular integral equations has been established. As we mentioned in the introduction these hypersingular integral equations can be viewed as elliptic pseudo-differential operator equations. The local solvability has been investigated by many authors [23] for elliptic equations. However, no global solvability has been obtained so far for the hypersingular integral equations considered in this thesis.

By introducing weighted Sobolev spaces to regularize the equations, we also proved that the solution is unique and stable (i.e. the solution depends continuously on the given data). The existence, uniqueness and stability imply that it is possible to obtain stable numerical procedures for solving the integral equation (Chapter 5).

Furthermore, the global solvability provides a theoretical basis for error estimates. Up till now only one paper [68] has investigated the error characteristics of collocation approximations. This paper was restricted to deriving local error estimates for the simplest piecewise constant collocation method. We have shown in this thesis that if we take into account the global asymptotics of solutions, the convergent rates are different. We also derive estimates, by which we can explain the phenomena which were observed by Ryder and Napier in their paper [68].

Another major accomplishment of this thesis is the establishment of a new highly efficient wavelet approximate method for solving integral equations. Similar to other numerical methods, wavelet approximations are based on the discretization of equations by expanding operators and vectors into a basis. Although the Carlderon-Zygmund and pseudo-differential operators have sparse representations in wavelet bases, which will result in memory savings and reduced computational costs, a number of difficulties are encountered. The first difficulty is the overlap of the wavelet basis (with M vanishing moments, $M \geq 2$) functions between intervals or domains. The second difficulty arises from the special filters required at the endpoints of the domain, which makes it necessary to compute expensive inner products involving the wavelets and the integral kernel. The third difficulty is that the projection of the integral operator onto the basis functions requires the evaluation of the appropriate quadratures. Since analytical expressions for the Daubechies compactly supported wavelets are not available, the recursive procedures necessary to evaluate the wavelet basis functions make these integrations costly.

To eliminate the above difficulties, Alpert, Beylkin, Coifman and Rokhlin [2] introduced wavelet-like bases for the fast solution of one-dimensional integral equations of the second kind. However, since the extension of these procedures to perform the required orthogonalizations for higher dimensional bases have not been developed, the two and three dimensional problems remain unsolved.

The above difficulties are even worse for the hypersingular integral equations.

However, instead of using wavelet bases directly, we discretize the hypersingular integral equations by the piecewise polynomial collocation method and then we apply discrete wavelet transforms to the resulting discretized algebraic equations. This procedure involves $O(N^2)$ operations and lead to a sparse matrix problem. The sparsity of the matrix enables us to solve the problem in only $O(N \log^2 N)$ operations, whereas traditional iterative and elimination methods usually require $O(N^3)$ operations. The memory requirement has been reduced from $O(N^2)$ to $O(N \log N)$ words by this method. We have demonstrated several numerical examples in Chapter 6.

Since higher dimensional boundary integral equations can be discretized by the piecewise polynomial collocation method, the fast wavelet approximation technique developed in this thesis can be extended to higher dimensional problems. This extension was demonstrated in the two dimensional case in Chapter 6.

Directions for Future Work

The compact representation of the discretized Boundary Integral operators by means of the fast wavelet approximation has a number of interesting applications to industrial problems.

1. Nonlinear problems in geomechanics.

Peirce, Ryder and Napier, in their recent paper [58], have studied the performance of a number of iterative schemes that can be used to solve the nonlinear algebraic equations that arise in the modeling of tabular mining excavations with backfill, the mobilization of faults, and the growth of fractures. The most costly component of each iteration in these schemes involves a multiplication of the influence matrix with a vector. We can consider the application of wavelet-based methods to such iterative schemes. This procedure will reduce the operation count from $O(N^2)$ to $O(N \log N)$ operations for each iteration. In addition, the compact representation of an approximate inverse of the influence matrix opens the possibility of reducing the number of iterations that need to be performed by using a preconditioned conjugate gradient algorithm.

2. Electromagnetic problems.

Many problems of analysis of time-harmonic electromagnetic scattering can be formulated in terms of an integral equation of the following form [61]

$$\int_\Omega k(x,t)u(t)\,dt = f(x), \qquad x \in \Omega, \tag{$*$}$$

where u is the unknown field, f is a known source, and $k(x,t)$ has a singularity at $x = t$. We can therefore consider the application of our wavelet-based technique directly to the discrete equations that result from an application of the method of moments to $(*)$. Current 'fast' methods for such problems make use of the Fast Fourier Transform which requires a regular discretization grid and therefore suffers from staircase errors in the discretization of oblique boundaries. The fast wavelet approximation can then be used to speed up accurate discretizations that are not restricted to a regular grid. This technique can be used in the efficient and accurate modeling of microstrip array antennae for example.

Appendix A

The following matrix A_N^P is associated with approximation of (4.1.2) by piecewise linear polynomials.

$$
\begin{aligned}
A_N^P(2i-1,2j-1) &= -\frac{1}{2\varepsilon h}\left[\log\left|\frac{2(j-i)+2-\alpha}{2(j-i)-\alpha}\right| - \frac{\alpha}{2(j-i)+2-\alpha}\right.\\
&\qquad \left. -\frac{2-\alpha}{2(j-i)-\alpha}\right] ;\\
A_N^P(2i-1,2j) &= \frac{1}{2\varepsilon h}\left[\log\left|\frac{2(j-i)+2-\alpha}{2(j-i)-\alpha}\right| - \frac{2-\alpha}{2(j-i)+2-\alpha}\right.\\
&\qquad \left. -\frac{\alpha}{2(j-i)-\alpha}\right] ;\\
A_N^P(2i,2j-1) &= -\frac{1}{2\varepsilon h}\left[\log\left|\frac{2(j-i)+\alpha}{2(j-i)-2+\alpha}\right| - \frac{\alpha}{2(j-i)+\alpha}\right.\\
&\qquad \left. -\frac{2-\alpha}{2(j-i)-2+\alpha}\right] ;\\
A_N^P(2i,2j) &= \frac{1}{2\varepsilon h}\left[\log\left|\frac{2(j-i)+\alpha}{2(j-i)-2+\alpha}\right| - \frac{2-\alpha}{2(j-i)+\alpha}\right.\\
&\qquad \left. -\frac{\alpha}{2(j-i)-2+\alpha}\right] ;
\end{aligned}
$$

$$i,j = 1,2,\ldots,N\ .$$

where $\varepsilon = \cos\pi/4$, $\alpha = 1-\varepsilon$ if we choose zeros of Chebyshev polynomials as collocation points, see (4.1.5).

Appendix B

We want to derive the estimate:

$$\|h(A_N^{\psi,b} - A_N^{\psi})\| \le \frac{c}{b^M} \log_2 N \ , \qquad (4.4.4)$$

where $\|\cdot\|$ is defined by the row sum norm.

Recall that $A_n^{\psi,b}$ is obtained from A_N^{ψ} by setting to zero all the elements of matrices, $A_j, B_j^{j'}, \Gamma j'_j$, $j = 1, \ldots, n-1$, $j' = 2, \ldots, n$ outside of bands of width b around their diagonals. And the elements of A_N^{ψ} satisfy the following estimate:

$$|A_{IJ}| \le c(|I|/|J|)^{1/2} \frac{|I|^M}{d(I,J)^{M+1}} \ , \qquad (B.1)$$

where c is a constant and $|I| < |J|$.

Without loss of generality, we consider the sum of the first row.

Note that the remaining terms in the sum satisfy $d(I,J) \ge \frac{b}{2^k}$, where $I = \left[\frac{i-M-1}{2^j}, \frac{i+M}{2^j}\right]$ and $J = \left[\frac{\ell-M-1}{2^k}, \frac{\ell+M}{2^k}\right]$ with $|I| \le |J|$. Therefore

$$\begin{aligned}
& h\sum_{j=1}^{N} \left|A_N^{\psi,b}(1,j) - A_N^{\psi}(1,j)\right| \\
\le\ & \frac{c'h}{b^M} \left[\left(\frac{1}{\frac{b}{2^{n-1}}} + \frac{1}{\frac{b+1}{2^{n-1}}} + \cdots + \frac{1}{\frac{2^{n-1}-b}{2^{n-1}}}\right)\right. \\
& + \left(\frac{1}{\frac{b}{2^{n-2}}} + \frac{1}{\frac{b+1}{2^{n-2}}} + \cdots + \frac{1}{\frac{2^{n-2}-b}{2^{n-2}}}\right) \\
& \vdots \\
& + \left(\frac{1}{\frac{b}{2^{n-k}}} + \frac{1}{\frac{b+1}{2^{n-k}}} + \cdots + \frac{1}{\frac{2^{n-k}-b}{2^{n-k}}}\right) \\
& \left. \vdots \quad (\quad \cdots \quad) \quad \right] \\
\le\ & \frac{c'}{b^M}\left[\frac{1}{2}\left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{n-1}}\right)\right. \\
& + \frac{1}{2^2}\left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{n-2}}\right)
\end{aligned}$$

$$
\begin{aligned}
&\vdots \\
&+ \frac{1}{2^k}\left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{n-k}}\right) \\
&\vdots \\
&+ \frac{1}{2^n} \qquad \Big] \\
\leq\ & \frac{c}{b^M} \log_2 N \ , \qquad \text{(B.2)}
\end{aligned}
$$

where c is a constant.

Similarly, we have the estimate for $i = 2, \ldots, n$. Consequently (4.4.4) holds.

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