UNIFORM MODULES OVER

SERIAL RINGS

BY

MENAKA LELWALA, B.Sc.Hons., M. Math.

A Thesis
Submitted to the School of Graduate Studies
in Partial Fulfilment of the Requirements
for the Degree
Doctor of Philosophy

McMaster University

UNIFORM MODULES OVER

SERIAL RINGS

DOCTOR OF PHILOSOPHY (1995) (MATHEMATICS)	McMASTER UNIVERSITY Hamilton, Ontario

TITLE:	Uniform Modules Over Serial Rings
AUTHOR:	Menaka Lelwala B.Sc.Hons. (University of Sri Jayewardenepura, Sri Lanka) M. Math. (University of Waterloo, Canada)
SUPERVISOR:	Professor Bruno J. Müller
NUMBER OF PAGES:	vi, 71

ABSTRACT

The main result of this thesis is a completely different proof via localization, of the following theorem due to Bruno J. Müller and Surjeet Singh: For any P-tame uniform module M over a serial ring, and the intersection T of the clique of P, the annihilator $ann_M(\cap T^n)$ is uniserial.

ACKNOWLEDGEMENTS

Many supported and helped me in many ways during the preparation of this thesis. Special mention should be made of my learned supervisor, Professor Bruno J. Müller. His kindness, patience, and guidance will always be appreciated. This work would have not been complete without the numerous hours which he devoted to assisting me. I am also grateful for his careful and critical review of this manuscript.

I thank Professor B. Banaschewski and Professor A. Rosa for serving on my supervisory committee, and also my fellow student Franco Guerriero for interesting conversations about ring theory. I again thank him and Sheree Cox, for helping me with Scientific Word.

I acknowledge financial support from the Commonwealth Scholarship and Fellowship Plan for the first three years of my study, and support from the Department of Mathematics and Statistics of McMaster University, which granted me a scholarship and teaching assistantship in the final year.

With immense pleasure I mention Dimuth, my husband. His encouragement, and the care for our little son, Nisaga, were invaluable during the final preparation of this thesis. Finally, my sincere thanks go to everyone, who guided, encouraged, or assisted me in completing this task.

TO AMMA

TABLE OF CONTENTS

CHAPTER 1

INTRODUCTION

The study of serial rings started with Nakayama in 1940 ([N1],[N2]). He studied Artinian serial rings, calling them generalized uniserial. He proved that a ring is Artinian and serial if and only if every finitely generated module is serial. Eisenbud and Griffith strengthened this, showing that an arbitrary module over an Artinian serial ring is serial [EG1]. In this situation, every indecomposable module is isomorphic to eR/eJ^k for some k (where J is the Jacobson radical of R), and therefore (up to isomorphism) there are only finitely many indecomposable modules.

Structure theorems for Artinian serial rings have been obtained by many authors, for instance Kupisch ([K1], [K2]), Murase ([Mu1], [Mu2], [Mu3]), Eisenbud and Griffith [EG2], and Ivanov [I].

The study of non-Artinian serial rings was initiated by Warfield [W]. He showed that serial rings are precisely the semiperfect rings for which every finitely presented module is a direct sum of uniserials. (This thesis contains an improvement of this result.) He obtained the structure of right and left Noetherian serial rings, and also proved that all uniform modules over such rings are uniserial. Singh determined the structure of right Noetherian serial rings [S].

Wright investigated under which conditions a uniform module over a serial ring is uniserial. She proved that every nonsingular uniform module over a serial ring is uniserial [U]. Also, assuming that the serial ring has Krull dimension, she showed that certain other uniform modules are uniserial [W3].

Chatters [C], and Wright [W4] developed a structure theory for prime serial rings with finite Krull dimension, and studied under which conditions an indecomposable serial ring with Krull dimension is prime.

Müller and Singh's joint publications ([MS1], [MS2]) investigated the relationship between prime ideals over serial rings, and introduced links. Their ([MS2] Theorem 2), which states "if M is a uniform module over a serial ring with an associated prime P which is Goldie, and if M is $P-$tame, then $ann_M \cap T^n$ is uniserial" (where T is the intersection of the clique of P), improves Wright's result. Our main contribution is to give a completely different proof of this theorem via localization. Relevant background material is taken mainly from the joint papers by Müller and Singh ([MS1], [MS2]), and from Chatters [C]. (Later in this introduction we shall include a sketch of our proof.)

Müller ([M2], [M3]) obtained a complete description of Goldie prime serial rings and some structural results for arbitrary serial rings.

The basic steps of the proof of our main theorem are as follows: As the injective hull of a $P-$tame module is $P-$tame, we may assumed that M_R is injective. If the clique of P is trivial, then $T = P$ is Goldie, by assumption. If the clique of P

is nontrivial, then T is Goldie by (Proposition 3.10), a result of Müller and Singh [MS1]. Since we are interested in $ann_M \cap T^n$, we may assume that $\cap T^n = 0$. The idea is to localize at $\zeta := \zeta(T)$. We obtain the Ore condition using one of Chatter's [C] results (Lemma 3.12), as T does not contain any non-zero idempotents.($e^2 = e \in T$ implies $e \in \cap T^n = 0$.) As far as reversibility concerned, we had some difficulties. One of the results (Proposition 3.15) of Müller and Singh [MS2] has helped us here. (We also have an improvement of this result, cf. Theorem 3.16.) It implies that, either $\cap T^n$ is Goldie prime , or T is nilpotent. In the first case we obtain that the ideal $K = \{r \in R / c_1 r c_2 = 0 \ for \ some \ c_1, c_2 \in \zeta\}$ is zero. In the second case we show $K = K' = {}'K$, where $K' = \{r \in R/rc = 0 \ for \ some \ c \in \zeta\}$ and ${}'K = \{r \in R/cr = 0 \ for \ some \ c \in \zeta\}$. This implies that there exists the localized ring Q, and that M is a $Q-$module (since M is $P-$tame hence $\zeta-$torsion free). If T is nilpotent, then the localized ring Q is Artinian: if $\cap T^n$ is prime, then Q is Noetherian. The $Q-$module M is indecomposable and injective. Thus Warfield's [W] result (Proposition 2.24) gives us that M is uniserial as $Q-$module. Now by Theorem 2.22 , which was originally proved for rings by Müller [M3], (and which we generalized to modules), we obtained that M is uniserial as $R-$module.

Let us proceed to an outline of this thesis. The second chapter provides some definitions and the preliminaries to be used in the subsequent chapters. Basic results about prime ideals in serial rings and localization techniques have been listed in the third and fourth chapters respectively. The results from research papers which were published only recently, or the ones that we used are in different form, were given with proofs. The fifth chapter is about our main contribution. Notations are given at the beginning of Chapter 2, and further notations can be found at the end of the thesis.

As usual, the thesis leaves a number of open questions. For instance, we conjecture, as a generalization of Proposition 3.15 ([MS2] Theorem 6), that every circular clique will incorporate all primes over the fork.

In regard to our main problem, namely when a uniform module is uniserial, much remains to be done. Our theorem gives a sufficient condition, and is valid over arbitrary serial rings. We believe that it is best possible, in this generality. But over Goldie prime serial rings, my fellow doctoral student Franco Guerriero has found more detailed criteria, criteria which are necessary and sufficient for an individual uniform injective modules.

CHAPTER 2

PRELIMINARIES

In this chapter we collect the known results we shall need. Results which are in books, and have been in the research papers for some time, are quoted without proofs.

For our purposes, rings will be associative, but not necessarily commutative, and will have an identity element,1. All modules are unitary right modules.

An *essential submodule* of a module B is any submodule A which has non-zero intersection with every non-zero submodule of B. We write $A \subseteq' B$ to denote this situation, and we also say that B is an *essential extension* of A. If A is a submodule of a module B over a ring R, then $A \subseteq' B$ if and only if for each element $0 \neq b \in B$, there exists $r \in R$ such that $0 \neq br \in A$. Also, if $A \subseteq' B$, then $b^{-1}A := \{r \in R / br \in A\}$ is an essential right ideal of R. A *uniform module* is a non-zero module M such that the intersection of any two nonzero submodules of M is non-zero, or, equivalently, such that every non-zero submodule of M is essential in M. The *singular submodule* of a module M is given by $Z(M) = \{m \in M/mI = 0 \text{ } for \text{ } some \text{ } I \subseteq' R_R\}$. If $Z(M) = M$, then M is called *singular*, if $Z(M) = 0$, then M is called *nonsingular.* A ring R is right nonsingular if the right singular ideal $Z(R_R)$ is zero. A ring R is nonsingular if it is both right and left nonsingular. We denote the right singular ideal of R by Z. An *injective hull* for a module M is any injective module which is an essential extension of M. We use the notation $E(M)$ for an injective hull of M.

J will always denote the Jacobson radical of the ring R. A ring R is called *semilocal* if R/J is semisimple. A ring R is *semiperfect* if R is semilocal and idempotents of R/J can be lifted to R. (We say that an idempotent $\varepsilon \in R/I$ can be lifted to R if there exists an idempotent $e \in R$ whose image under the canonical map $R \to R/I$ is ε.) R is right semihereditary if every finitely generated right ideal is projective. For further notations, see the list at the end of the thesis.

DEFINITION:

(i) A module is *uniserial* if its submodules are linearly ordered under inclusion.

(ii) A module is *serial* if it is a direct sum of uniserial submodules.

(iii) A ring is *right (uni)serial* if it is (uni)serial as a right module over itself.

(iv) A *serial ring* is a ring which is both left and right serial.

LEMMA 2.1. A module M is uniserial if and only if cyclic submodules are comparable.

Proof: Necessity is trivial. To prove the sufficiency, take any two submodules K and L of M. If K and L are not comparable, then there exist $x \in K - L$ and $y \in L - K$. By the hypothesis, xR and yR are comparable. We may assume $xR \subseteq yR$. Then $x \in xR \subseteq yR \subseteq L$, a contradiction. Therefore K and L are comparable, and hence M is uniserial. □

DEFINITION: A *non-zero module is local* if it has a unique maximal submodule which contains every proper submodule.

A non-zero *element* x of an arbitrary module is *local,* if xR is a local module.

Since every non-zero finitely generated module M has a maximal submodule and every proper submodule of M is contained in a maximal submodule, a non-zero finitely generated module is local, if it has a unique maximal submodule.

We first establish some useful facts about serial rings.

(i) Any right (or left) serial ring is semiperfect.

Proof: Suppose the ring R is right serial. Then as R is direct sum of uniserial right ideals, we get $1 = \sum_{i=1}^{n} e_i$ where the e_i are orthogonal indecomposable idempotents and each e_iR is uniserial. Hence e_iR has a unique maximal submodule, and is therefore local (ie. 1 is the sum of indecomposable orthogonal local idempotents). Thus the conclusion follows by the following old result: R is semiperfect if and only if the identity of R can be written as a sum of orthogonal, local idempotents in R ([M1] Theorem 1). □

(ii) Let I be an ideal of a right serial ring R. Then R/I is a right serial ring.

Proof: Suppose that $R = \sum_{i=1}^{n} \oplus e_i R$ where the e_i are orthogonal idempotents and each $e_i R$ is uniserial. The map $R/I \to \sum_{i=1}^{n} \oplus (e_i R/e_i I)$ given by $r+I \longmapsto (e_i r_i + e_i I)_i$, where $r = \sum_{i=1}^{n} e_i r_i$, can be routinely checked to be an $R/I-$ isomorphism. Each $e_i R/e_i I$ is a factor module of the uniserial $R-$ module $e_i R$, and so is itself uniserial as an $R-$ module. Hence, each $e_i R/e_i I$ is uniserial as an $R/I-$ module, since the $R-$submodules and $R/I-$ submodules of $e_i R/e_i I$ are the same. □

LEMMA 2.2.

(i) Over a right serial ring, any non-zero element of any module is the sum of local elements.

(ii) In a uniserial module, any non-zero element is local.

Proof: (i) Let M be a module over a right serial ring R and $0 \neq x \in M$. Right seriality of R implies $1 = \sum_{i=1}^{n} e_i$ with each $e_i R$ is uniserial. For a fixed i

with $xe_i \neq 0$, define the $R-$homomorphism, $e_iR \to xe_iR$ by $e_ir \longmapsto xe_ir$. Thus, $xe_iR \cong e_iR/ann_{e_iR}x$ is local, since it is a factor module of the uniserial module e_iR. Hence xe_i is local. Therefore the claim follows, as $x = \sum_{i=1}^{n} xe_i$.

(ii) Take any non-zero element y in a uniserial module. Since yR is uniserial and finitely generated, it has a unique maximal submodule and therefore is local. □

LEMMA 2.3. ([MS1] Lemma 1.5) If xR is a local module over a semiperfect ring R, then there is an indecomposable idempotent e such that $x = xe$.

Proof: Since a finitely generated module over a semiperfect ring has a projective cover, let P be projective such that $f : P \to xR$, is an $R-$epimorphism with $\ker f$ small in P. (A submodule X of M is small, if whenever $X + Y = M$, for any submodule Y in M, then $Y = M$.) Let $g : R \to xR$ given by $g(r) = xr$. As R is projective, there exists $h : R \to P$ such that $fh = g$. Then h is an epimorphism. For, take any $p \in P$. As the map g is onto, there exists $r \in R$ with $f(p) = g(r) = fh(r)$. Thus $p - h(r) \in \ker f$ and hence $p \in h(R) + \ker f$. Therefore we have $P = h(R) + \ker f$. As $\ker f$ is small in P, $P = h(R)$ and thus the map h is onto. The exact sequence $0 \to \ker h \to R \xrightarrow{h} P \to 0$ splits, and $R = \ker h \oplus X$ for some submodule X in R with $X \cong P$. Hence the decomposition of the identity into orthogonal idempotents $e_1 \in \ker h$, $e_2 \in X$ gives us $R = e_1R \oplus e_2R$. Note that $e_1R = \ker h$ and $e_2R = X$. (For any $x \in X$, $x = e_1r_1 + e_2r_2$; $r_i \in R$. Hence $x - e_2r_2 = e_1r_1 \in \ker h \cap X = 0$. Thus

$x = e_2 r_2$ and therefore $X = e_2 R$. Similarly we get, $\ker h = e_1 R$.) Let $e = e_2$.

Now we want to show that e is indecomposable and $x = xe$. Note that any maximal submodule M of P contains $\ker f$. (If not, $M + \ker f = P$, then, since $\ker f$ is a small submodule of P, we get $M = P$,a contradiction.) Therefore there is one-one correspondence between the maximal submodules of P and those of xR. As xR local, P has a unique maximal submodule L. Since $P(\cong eR)$ is finitely generated, every proper submodule of P is contained in L. If P is decomposable, say $P = A \oplus B$; A,$B \neq 0$, then $A, B \subseteq L$ implies $P \subseteq L$, a contradiction. Therefore P is indecomposable and so is eR. Now recalling that $e_1 \in \ker h$ we obtain, $x = g(1) = fh(1) = fh(e_1 + e_2) = fh(e_2) = (fh(1))e_2 = xe_2 = xe$. □

LEMMA 2.4. ([MS1] Lemma 1.8) Any proper factor module of a uniform(in particular uniserial) module is singular.

Proof: Let M be a uniform R−module and N be a proper submodule of M. Consider any $x + N \in M/N$. Recall that $x^{-1}N = \{r \in R/xr \in N\}$ is an essential right ideal of R and $(x + N)x^{-1}R = N$. Thus $x + N \in Z_R(M/N)$ and hence M/N is singular. □

LEMMA 2.5. ([MS1] Lemma 2.2) A non-zero projective module over a serial ring is not singular.

Proof: Suppose P is a projective R–module. As a projective over the semiperfect ring R, P is isomorphic to a direct sum of submodules of the form eR, where the e are indecomposable idempotents of R. Then $ann_R(e) = (1-e)R$ is not essential in R, and therefore $e \notin Z(eR)$. Consequently $Z(P) \subset P$. □

Lemma 2.6. ([W] Theorem 4.6) Over a serial nonsingular ring every finitely generated nonsingular module is projective.

Lemma 2.7. ([MS1] Lemma 2.4) Let R be serial, and let T be an ideal of R for which R/T is a nonsingular ring. If M is a uniserial R–module, and if $x \in M - MT$, then $xT = MT$.

Proof: Let $\overline{R} := R/T$. We only need to show that $xT \supseteq yT$ for all $y \in M$. This is true if $xR \supseteq yR$ or $yT = 0$. Thus we are left with the case $xR \subset yR$ and $yT \neq 0$. Since $y \in M$, y is local. By Lemma 2.3, choose an indecomposable idempotent e with $y = ye$. Then $eT \neq 0$. Define $f : eR \to yR$ by $er \longmapsto yer = yr$. As $f(eT) = yT \neq 0$, we conclude $\ker f \subset eT$, by uniseriality. Hence f induces an isomorphism, $eR/eT \cong yR/f(eT) = yR/yT$. Therefore yR/yT is a nonsingular $\overline{R}$–module. The inclusion $xR \subset yR$ induces a homomorphism $g : xR/xT \to yR/yT$. $Img = (x + yT)\overline{R}(\neq 0)$ is a cyclic submodule of the nonsingular $\overline{R}$–module, yR/yT. Hence it is projective by the previous lemma. Thus the homomorphism splits, and $xR/xT \cong \ker g \oplus Img$. By the uniseriality of xR/xT, it is indecomposable. Thus $0 = \ker g = yT/xT$. □

LEMMA 2.8. ([MS1] Corollary 2.5) Let R be serial and let T be an ideal of R for which R/T is a nonsingular ring. If M is a uniserial R–module with $MT \neq 0$, then M/MT is a nonsingular R/T–module.

Proof: Let $0 \neq \overline{x} = x + MT \in M/MT$. Then by (previous lemma), $xT = MT$. Since x is local, we choose an indecomposable idempotent e with $x = xe$. Define $f : eR \rightarrow xR$ by $er \longmapsto xer = xr$. Then $f(eT) = xT = MT \neq 0$. By uniseriality, $\ker f \subset eT$. Thus f induces the isomorphism $eR/eT \cong xR/f(eT) = xR/xT = xR/MT = \overline{x}R$. Since R/T nonsingular, eR/eT is nonsingular and thus $\overline{x}R$ is nonsingular as R/T–module, completing the proof.

Lemma 2.9. ([L] Proposition 21.20, 21.21) Let e, f be idempotents in a ring R. We say that e and f are isomorphic (write $e \cong f$), if they satisfy the following equivalent conditions:

(i) $eR \cong fR$ as right R–modules.

(ii) $\overline{eR} \cong \overline{fR}$ as right $\overline{R}$–modules, where $\overline{R} = R/J$.

(iii) There exist $a, b \in R$ such that $ab = e$ and $ba = f$.

Lemma 2.10. ([L] Proposition 21.22) If idempotents can be lifted modulo I, and $I \subsetneq J$, then an idempotent e is indecomposable in R if and only if $e + I$ is indecomposable in R/I.

LEMMA 2.11. ([L] Theorem 21.28) Idempotents can be lifted, modulo any nil ideal.

LEMMA 2.12. Let $1 = \sum_{i=1}^{n} e_i$ be a decomposition of the identity of the right serial ring R into indecomposable orthogonal idempotents. Then any indecomposable idempotent $e \in R$ is isomorphic to some e_i. In particular, eR is uniserial.

Proof: Let $\overline{R} = R/J$, and let $\overline{r}$ denote the image of $r \in R$ under the canonical map $R \to \overline{R}$. We have $\overline{1} = \sum_{i=1}^{n} \overline{e}_i$, with $\overline{e}_i$ non-zero orthogonal idempotents, as J contains no non-zero idempotents. Moreover, by Lemma 2.10 each $\overline{e}_i$ is indecomposable, since R is semiperfect. Therefore $\overline{e}\overline{R}$ is a simple $\overline{R}$–module, as $\overline{R}$ is semisimple. Thus by the Wedderburn-Artin Theorem, n is the same for any such decomposition $\overline{R} = \sum_{i=1}^{n} \oplus \overline{e}_i\overline{R}$.

Given e indecomposable, $\overline{e}$ is indecomposable (Lemma 2.10). Hence $\overline{e}\overline{R}$ is a simple $\overline{R}$-module. $\overline{1} = \overline{e} + \overline{(1-e)}$ gives us $\overline{R} = \overline{e}\overline{R} \oplus \overline{(1-e)}\,\overline{R}$. As a right ideal of $\overline{R}$, $\overline{(1-e)}\,\overline{R}$ is semisimple. Thus $\overline{(1-e)}\,\overline{R} = \sum \oplus I_i$ such that each I_i is simple, and there are $n-1$ simple components by the uniqueness of n. Any right ideal of a semisimple ring is generated by an idempotent, $I_i = \varepsilon_i\overline{R}$. Thus, $\overline{(1-e)}\,\overline{R} = \sum_{i=2}^{n} \oplus \varepsilon_i R$. Finally, we have $\overline{R} = \sum_{i=1}^{n} \oplus \overline{e}_i\overline{R} = \sum_{i=1}^{n} \oplus \varepsilon_i\overline{R}$; (where $\varepsilon_1 = \overline{e}$). Let $\pi_1 : \overline{R} \to \varepsilon_1\overline{R}$ be the projection on the first factor. Then $\varphi_{1i} : \overline{e}_i\overline{R} \to \varepsilon_1\overline{R}$, the restriction of π_1 to the summand $\overline{e}_i\overline{R}$, is either zero or an isomorphism for each i (Schur's Lemma). But it cannot be zero for every i, (because this would lead to the contradiction that $\pi_1 : \overline{R} \to \varepsilon_1\overline{R}$ is zero); thus there exists k such that, $\varphi_{1k} : \overline{e}_k\overline{R} \to \varepsilon_1\overline{R}$ is an isomorphism. Hence $\overline{e}\overline{R} = \varepsilon_1\overline{R} \cong \overline{e}_k\overline{R}$ and we conclude that $eR \cong e_kR$ (Lemma 2.9). Hence eR itself a uniserial module as claimed. □

DEFINITION:

(i) A module M is *finitely presented* (FP), if there is an R−epimorphism $f : F \to M$ with F finitely generated free and $\ker f$ finitely generated.

(ii) A module M is *locally presented* (LP), if there is an exact sequence $Q \to P \to M \to 0$ in which P and Q are both local projectives.

THEOREM 2.13 ([W] Theorem 2.6) The following properties of a semiperfect ring are equivalent:

(i) every FP left module is a direct sum of LP modules;

(ii) every FP left module is serial;

(iii) every FP right module is a direct sum of LP modules;

(iv)every FP left module and every FP right module is a direct sum of local modules.

Note that (iv) is left -right symmetric, so we can add the right-analogue of (ii): ie. every finitely presented right module is serial.

We have noted the following.

IMPROVEMENT: In Theorem 2.13, the assumption that the ring is semiperfect, is redundant.

Proof: By definition, R_R and $_RR$ are finitely presented. Our aim is to show that we can extract the semiperfectness of R from each of the conditions. Recall that R is semiperfect if and only if the identity can be written as a sum of orthogonal local idempotents.

By (i), $_RR$ is a direct sum of LP modules. Thus $R = \sum_{i=1}^{n} \oplus Re_i$ where $1 = \sum_{i=1}^{n} e_i$, the e_i are orthogonal idempotents, and the Re_i are locally presented modules. By definition, there exist P_i, Q_i local projectives such that $Q_i \to P_i \xrightarrow{f_i} Re_i \to 0$ exact. Hence , $Re_i \cong P_i / \ker f_i$. Since a factor module of a local module is local, we get Re_i is local. Therefore $1 = \sum_{i=1}^{n} e_i$, with each e_i is local. Hence R is semiperfect.

(ii) implies R is left serial. Hence it is semiperfect.

Semiperfectness of R from (iii), is obtained as in (i).

(iv) implies $R_R = \sum \oplus e_i R$ where $1 = \sum e_i$, each $e_i R$ is local, and the e_i are orthogonal. Again R is semiperfect. □

LEMMA 2.14. ([W] Corollary 3.4) A finitely presented module over a serial ring is a direct sum of local cyclic modules.

Combining this with Theorem 2.13, gives

LEMMA 2.15 The ring R is serial if and only if every finitely presented right (or left) module is serial.

LEMMA 2.16. ([W] Theorem 4.1) If R is right serial, then R is right nonsingular if only if R is right semihereditary.

DEFINITION: A module M *has finite Goldie dimension* n (write as $\dim M_R = n$), if M has an essential submodule which is a direct sum of n uniform submodules.

A ring R *is right Goldie,* if $\dim R_R$ is finite and R satisfies ascending chain condition on right annihilators.

(It is well known that if M has Goldie dimension n, then a direct sum of uniform submodules is essential in M if and only if it has n summands.)

If R is right serial, then since $R_R = \sum_{i=1}^{n} \oplus e_i R$, with each $e_i R$ is uniserial, we see that $\dim R_R = n$.

LEMMA 2.17. ([W] Lemma 4.3) If R is a ring that is either semiperfect or has left or right Goldie dimension is finite, then R is left semihereditary if and only if R is right semihereditary.

REMARK 2.18. As left or right serial rings are semiperfect, by Lemma 2.16 and Lemma 2.17 we see that a serial ring is right nonsingular if and only if it is left nonsingular. Therefore we can talk about the nonsingularity of a serial ring without mentioning the side.

LEMMA 2.19. ([W] Theorem 3.3). Let R be a serial ring, P be a finitely generated projective module, and M be a finitely generated submodule of P. Then there is a decomposition $P = \sum_{i=1}^{n} \oplus P_i$ into indecomposable projectives such that $M = \sum_{i=1}^{n} \oplus (M \cap P_i)$.

LEMMA 2.20.([U] Proposition 1.1) For any R-module M over a serial ring R, $M/Z(M)$ is a nonsingular $R/Z-$module.

In particular, R/Z is a nonsingular ring.

Proof: Let $\overline{M} = M/Z(M)$ and let $\overline{R} = R/Z$. It is straightforward to see that $MZ \subseteq Z(M)$. Hence $\overline{M}$ is an $\overline{R}-$module. Let $\overline{m}$ (respectively $\overline{r}$) denote the image of

$m \in M$ (respectively $r \in R$) under the canonical projection $M \to \overline{M}$ (respectively $R \to \overline{R}$). Let $e_1, \ldots, e_n$ be orthogonal idempotents such that $R = \sum_{i=1}^{n} \oplus e_i R$ and each $e_i R$ is uniserial. For a fixed i, let $e_i = e$.

Suppose $Z(\overline{M}_{\overline{R}}) \neq 0$. Then there exists $0 \neq \overline{m} \in Z(\overline{M}_{\overline{R}})$. As $\overline{me} \in Z(\overline{M}_{\overline{R}})$, take $\overline{I} = I/Z$, an essential right ideal of $\overline{R}$ such that $\overline{me}\overline{I} = 0$. Since $ann_R(e) = (1-e)R$ is not essential in R, $e \notin Z$. Hence $\overline{e}\overline{R}$ is non-zero. Therefore $\overline{I} \cap \overline{e}\overline{R}$ is non-zero. Simplifying, $(I \cap eR) + Z = I \cap (eR + Z)$ is not contained in Z. Hence $I \cap eR$ is not contained in Z. Thus $I \cap eR$ not contained in eZ, and therefore $eZ \subset I \cap eR$ by the uniseriality of eR. Now take any $y \in (I \cap eR) - eZ$. As $\overline{mey} \in \overline{me}\overline{I} = 0$, $mey \in Z(M)$. Let A in R be an essential right ideal such that $meyA = 0$. Since $y \notin Z$ (if $y \in Z$,then $y = ey \in eZ$, a contradiction), $yA \neq 0$. Thus $yA = eyA \subseteq' eR$ by the uniseriality. Thus $yA \oplus (1-e)R \subseteq' eR \oplus (1-e)R = R$. As the right ideal $yA \oplus (1-e)R$ annihilates me, we get $me \in Z(M)$. As this hold for any $e = e_i$, we obtain $m = \sum_{i=1}^{n} me_i \in Z(M)$ hence $\overline{m} = 0$, a contradiction. We conclude $Z(\overline{M}_{\overline{R}}) = 0$.

It follows in particular, that R/Z is a right nonsingular ring. As R/Z is serial, using Remark 2.18, R/Z is a nonsingular ring. □

LEMMA 2.21. ([U] Lemma 1.2(a)) Every nonsingular uniform module over a serial ring is uniserial.

Proof: Let M be a nonsingular uniform $R-$module. As $MZ \subseteq Z(M) = 0$, M is a uniform $R/Z-$module. Also as an $R/Z-$module, M is nonsingular. (For, if $m \in Z(M_{R/Z})$, then $m(I/Z) = 0$ for some essential right ideal I/Z of R/Z. Since I is an essential right ideal of R and $mI = 0$, we get $m \in Z(M) = 0$.) By Lemma 2.20, the serial ring R/Z is nonsingular. Since M is nonsingular uniform $R/Z-$module, for simplicity, we may assume that $Z = 0$. Consider any $x, y \in M$. As a submodule of M, $xR + yR$ is nonsingular and uniform. Hence by Lemma 2.6, $xR + yR$ is projective. The exact sequence $0 \to \ker \to R \oplus R \to xR + yR \to 0$ splits, hence $R \oplus R \cong (xR + yR) \oplus$kernel. Hence the kernel is finitely generated, and we get that $xR + yR$ is finitely presented. Thus by Lemma 2.15, $xR + yR$ is serial. Therefore $xR + yR$ is uniserial, since it is indecomposable as a uniform module. When this happens, xR and yR are comparable. □

The idea of the proof of the following lemma is derived from ([M2] Proposition 13).

THEOREM 2.22. Let $J(S) \subseteq R \subseteq S$ be rings. Assume that $S/J(S)$ is semisimple, $R/J(S)$ is serial, and that $R/J(S)$ is right-essential in $S/J(S)$. Then any uniserial right $S-$module is uniserial as an $R-$module.

Proof: We write $\overline{S} = S/J(S)$ and $\overline{R} = R/J(S)$. Let X be any simple $S-$module. As $XJ(S) = 0$, X is an $\overline{S}-$ and $\overline{R}-$module.

Claim (i):X is a nonsingular $\overline{R}-$module.

Proof: Suppose $x \in Z(X_{\overline{R}})$. Then $xE = 0$ for some $E \subseteq' \overline{R}$. Take any $0 \neq \overline{s} \in \overline{S}$. Then there exists $\overline{r} \in \overline{R}$ such that $0 \neq \overline{s}\overline{r} \in \overline{R}$, as $\overline{R} \subseteq' \overline{S}$. Since $E \subseteq' \overline{R}$, there exists $\overline{r_1} \in \overline{R}$ with $0 \neq \overline{s}\overline{r}\overline{r_1} \in E \subseteq E\overline{S}$. Hence $E\overline{S} \subseteq' \overline{S}$, and therefore $E\overline{S} = \overline{S}$, since $\overline{S}$ is semisimple and consequently has no proper essential submodules. Thus $0 = xE\overline{S} = x\overline{S}$, and we get $x = 0$.

Claim(ii): X is a uniform $\overline{R}-$module.

Proof: Let U be an arbitrary non-zero $\overline{R}-$submodule of X. Since $U\overline{S}$ is a non-zero submodule of the simple module X, we have $U\overline{S} = X$. Consider any $0 \neq \sum_{i=1}^{n} u_i s_i \in U\overline{S} = X$. Without loss of generality we may assume that $s_i \neq 0$ for all $i = 1, \ldots, n$. As $\overline{R} \subseteq' \overline{S}$, $\{\overline{r} \in \overline{R}/s_i\overline{r} \in \overline{R}\} = s_i^{-1}\overline{R} \subseteq' \overline{R}$. Hence $E = \cap_{i=1}^{n} s_i^{-1}\overline{R} \subseteq' \overline{R}$, with $s_i E \subseteq \overline{R}$. Therefore $(\sum_{i=1}^{n} u_i s_i)E \subseteq \sum_{i=1}^{n} u_i\overline{R} \subseteq U$. Moreover $(\sum_{i=1}^{n} u_i s_i)E \neq 0$, since X is nonsingular. This proves $U \subseteq' X$, for any non-zero $\overline{R}-$submodule U of X, and therefore X is a uniform $\overline{R}-$module.

By Lemma 2.21, we know that any nonsingular uniform module over a serial ring is uniserial. Thus, any simple $S-$module X is uniserial as an $\overline{R}-$module, hence uniserial as an $R-$module.

Now let M be any uniserial $S-$module. Take any $x, y \in M$. Then xS and yS

are comparable. If $yS \subset xS$, then, since $xJ(S)$ is the unique maximal submodule of xS which contains every proper submodule of xS, we have $xS \supseteq xR \supseteq xJ(S) \supseteq yS \supseteq yR$. Similarly we get $xR \subseteq yR$, if $xS \subset yS$. If $xS = yS$, then $xJ(S) = J(xS) = J(yS) = yJ(S)$. What we have shown above implies the simple $S-$module $xS/xJ(S) = yS/yJ(S)$ is uniserial as an $R-$module and therefore $xR/xJ(S)$ and $yR/yJ(S)$ are comparable. Hence again we get the comparability of xR and yR, completing the proof of the uniseriality of M as an $R-$module. □

PROPOSITION 2.23.([S] Lemma 2.1) Let R be a serial ring with Jacobson radical J. If $\cap_{n\in\mathbf{N}} J^n = 0$, then R is Noetherian.

Proof: Let $e_1, \ldots, e_n$ be orthogonal idempotents such that $R = \sum_{i=1}^{n} \oplus e_i R$ and each $e_i R$ is uniserial. For convenience, let $e_i = e$ for a fixed i.

Claim: For any $x \neq 0$ in eR, there exists $n \in \mathbf{N}$ such that $xR = eJ^n$.

Proof of claim: Since $\cap eJ^n = e(\cap J^n) = 0$, there exists n such that $x \in eJ^n - eJ^{n+1}$ (because if $x \in eJ^n$ for all n, then $0 \neq x \in \cap eJ^n = 0$, a contradiction). Thus $eJ^n \supseteq xR \supset eJ^{n+1}$. As R/J semisimple, eJ^n/eJ^{n+1} is semisimple as a right $R/J-$module and hence also as an $R-$module. Since eJ^n/eJ^{n+1} is a factor module of a uniserial, it is a uniserial $R-$module. Hence we get that eJ^n/eJ^{n+1} is a simple $R-$module. Therefore $eJ^n = xR$.

Now take any non-zero submodule Y of eR. Then $Y = \sum_{i \in I} y_i R$ for $y_i \in Y$. By the claim, for all non-zero y_i, $y_i R = eJ^{n_i}$; $n_i \in \mathbf{N}$. Let $S = \{n_i \in \mathbf{N} / 0 \neq y_i R = eJ^{n_i}\}$. Note that S is non-empty, since $Y \neq 0$. Take n to be the least element of S. Because eR is uniserial, $eJ^{n_i} \subseteq eJ^n = yR$ for some $0 \neq y \in Y$. Thus $Y = eJ^n = yR$. Thus every submodule of eR is finitely generated, and we get eR is Noetherian. Therefore as a finite direct sum of Noetherian right ideals, R is right Noetherian. Similarly R is left Noetherian. □

PROPOSITION 2.24. ([W] Lemma 5.1) Let R be a right Noetherian serial ring. Then every uniform $R-$module is uniserial.

In particular, an indecomposable injective $R-$module is uniserial.

Proof: Let M be a uniform $R-$module. Given any $x, y \in M$, $0 \to \ker \to R \oplus R \to xR \oplus yR \to 0$ an exact sequence with ker is finitely generated, as a submodule of the Noetherian $R-$module $R \oplus R$. Thus $xR + yR$ is a finitely presented module and by Lemma 2.15, $xR + yR$ is serial. This implies $xR + yR$ is uniserial, because as a uniform module it is indecomposable. Hence $xR \subseteq yR$ or $yR \subseteq xR$.

The last statement follows from the fact that, an injective module is uniform if and only if it is indecomposable. □

DEFINITION:

(i) A *regular element* in R is any element $x \in R$ such that $r(x) = 0$ and $l(x) = 0$.

(ii) A *multiplicative set* in a ring R is any subset $\zeta \subseteq R$ such that $0 \notin \zeta$, $1 \in \zeta$ and ζ is closed under multiplication.

(iii) A multiplicative set ζ satisfies *right Ore condition* if for given $r \in R$ and $c \in \zeta$, there exist $r' \in R$ and $c' \in \zeta$ such that $rc' = cr'$.

(iv) A *right Ore set* is a multiplicative set satisfying the right Ore condition.

GOLDIE'S THEOREM (2.25). In a semiprime Goldie ring R, the set of regular elements satisfies the Ore condition.

LEMMA 2.26. ([GW] Corollary 5.4) If R is a semiprime ring with ascending chain condition on right annihilators, then R is a right nonsingular ring.

LEMMA 2.27. ([CH] Lemma 1.14) If R is a right nonsingular ring with finite right Goldie dimension, then R satisfies the ascending and descending chain condition for right annihilators. (ie.R is a right Goldie ring.)

LEMMA 2.28. ([CH] Theorem 1.35) If R be a right Goldie ring, then each nil subring of R is nilpotent.

LEMMA 2.29. ([CH] Corollary 6.6) Let R be a ring which is a direct sum of uniform right ideals and let I be an ideal of R which has no proper essential extension in R as right R–module. Then $I = eR$ for some idempotent element e of R.

LEMMA 2.30. ([CH] Lemma 6.13) Let R be a right nonsingular right serial ring and let N be the prime radical of R. Then R/N is a right Goldie ring.

(We include the proof, taken from ([CH] Lemma 6.13), since in this reference the definition of the ideal N is somewhat ambiguous.)

Proof: Since the serial ring R has finite Goldie dimension, by Lemma 2.27, R satisfies the descending chain condition for right annihilators. This implies that R has ascending chain conditions for left annihilators. Hence the non-empty set $X = \{l(I)/\, 0 \neq {}_R I \subseteq_R R\}$ has a maximal element $P = l(I)$. We conclude $P = lr(P)$. It is well known that this maximal element P is prime.

Let us now show that $r(P)$ has no proper essential extension in R as right R–module. (ie. $r(P)$ is a complement right ideal of R.) Suppose $r(P) \subseteq' K \subseteq R$. Then $K/r(P)$ is a singular R–module. If $K/r(P)$ is non-zero, then there exists $0 \neq x + r(P) \in K/r(P)$ with $(x + r(P))A = 0$ for some $A_R \subseteq' R_R$. This implies $xA \subseteq r(P)$, and therefore $PxA = 0$. This leads to the contradiction $0 \neq Px \subseteq Z(R_R) = 0$. Thus $K/r(P)$ must be zero.

Now apply Lemma 2.29 to get $r(P) = eR$ for some idempotent $e \in R$. Set $f = 1 - e$. Since $eR = r(P)$ and $Rf = l(eR)$, we get that eR is an ideal, and hence Rf is also an ideal. Thus $eR = ReR$, hence $eRe = ReRe = Re$. Similarly we get $fR = fRf$.

Using $eRe = Re$, it is straightforward to verify that the map $R \rightarrow eRe$ given by $r \longmapsto ere$ is a surjective ring homomorphism and hence $R/Rf \cong eRe$ as rings. Since $Rf = l(eR) = lr(P) = P$, we get $eRe \cong R/P$. Thus eRe is a prime right serial ring. Thus the Goldie dimension of eRe is finite. As a subring of R, eRe inherits the ascending chain condition for right annihilators. It follows that the ring eRe is Goldie.

Using $fRf = fR$, similarly, we obtain the ring isomorphism $R/eR \cong fRf$. Hence fRf is right serial. We want to show that fRf is nonsingular. Take any $fxf \in Z(fRf_{fRf})$. Then $fxfK = 0$ for some essential right ideal K of $fRf(= fR)$. Note that K is essential in fR as right R−modules. Therefore $K \oplus eR \subseteq'_R fR \oplus eR = R$, and $fxf(K \oplus eR) = 0$. Since R is nonsingular, $fxf = 0$, and we conclude that fRf is nonsingular.

Since $1 = e+f$ where e and f are orthogonal idempotents, the ring R is isomorphic to the matrix ring $\begin{pmatrix} eRe & eRf \\ fRe & fRf \end{pmatrix} = \begin{pmatrix} eRe & eRf \\ 0 & fRf \end{pmatrix}$. (As $Re = eRe$, $fRe = 0$.) Now let us describe the prime ideals of this matrix ring. As $\begin{pmatrix} 0 & eRf \\ 0 & 0 \end{pmatrix}$ is nilpotent, it is contained in every prime ideal. Hence any prime ideal has the form $\begin{pmatrix} P_1 & eRf \\ 0 & fRf \end{pmatrix}$ or $\begin{pmatrix} eRe & eRf \\ 0 & P_2 \end{pmatrix}$ where P_1 and P_2 are prime ideals of eRe and fRf respectively.

Hence the prime radical of the matrix ring is $\begin{pmatrix} N_1 & eRf \\ 0 & N_2 \end{pmatrix}$ where $N_1$and N_2 are prime radicals of eRe and fRf respectively. $N_1 = 0$ as eRe is prime. Thus $R/N \cong eRe \oplus fRf/N_2$.

If dim$R$=1, then R is uniserial. If $e = 0$, then $f = 1$, hence $P = Rf = R$, a contradiction and we have $f = 0$. Thus $e \neq 0$. Thus $R/N \cong eRe$, is Goldie. Therefore inductively, let us assume that the theorem is true for all right nonsingular right serial rings with Goldie dimension less than that of R.

If $f = 0$, then we are again in the previous situation, so say $R/N \cong eRe$, which is Goldie. If $f \neq 0$, then as $fRf = fR$ is not essential in R, $\dim R_R > \dim fRf_R \geq \dim fRf_{fRf}$. Thus induction applies to the ring fRf, and we get that fRf/N_2 is Goldie. Therefore R/N is Goldie, as ring direct sum of two Goldie rings. □

LEMMA 2.31. A finite intersection $\cap_{i=1}^{t} P_i$ of prime ideals is irredundant if and only if the P_i are pairwise incomparable.

In fact these P_i 's are precisely the prime ideals minimal over the intersection.

Proof: Necessity is clear, since if $P_i \subset P_j$, then P_j is redundant. To prove the sufficiency, suppose that all P_i *'s* incomparable, and that P_j is redundant. Then $P_j \supseteq \cap_{i=1}^{t} P_i = \cap_{i=1, i\neq j}^{t} P_i \supseteq \prod_{i=1, i\neq j}^{t} P_i$. Hence $P_j \supseteq P_k$ for some $k \neq j$, as P_j is prime. This is a contradiction.

To show that P_i *'s* are minimal over the intersection, let P be a prime such that

$\cap_{i=1}^t P_i \subseteq P \subseteq P_i$. Then $\sqcap_{i=1}^t P_i \subseteq P$ implies, there exists k with $P_k \subseteq P \subseteq P_i$. Therefore $P_k = P_i = P$, by incomparability.

Now consider an arbitrary minimal prime P over $\cap_{i=1}^t P_i$. As before $P \supseteq \sqcap_{i=1}^t P_i$ implies $P \supseteq P_k \supseteq \cap_{i=1}^t P_i$ for some k. Hence $P = P_k$ by the minimality of P over the intersection. □

DEFINITION: Let I be an ideal in a ring R. An element $x \in R$ is said to be *regular modulo* I, provided the coset $x + I$ is a regular element of the ring R/I. The set of all such x is denoted by $\zeta(I)$.

LEMMA 2.32. ([GW] Lemma 6.4 and [J] Lemma) Let N be a Goldie semiprime ideal, and suppose $N = \cap_{i=1}^t P_i$ where the primes P_i are incomparable. Then,

(i) $\zeta(N) = \cap_{i=1}^t \zeta(P_i)$,

(ii) for all $c_i \in \zeta(P_i)$, $i = 1, \ldots, t$, there exist $r_i \in R$ with $\sum c_i r_i \in \zeta(N)$.

Over a semiprime Goldie ring, the torsion submodule of a module M defined to be $t(M) = \{m \in M / mr = 0 \; for\ some\ regular\ element\ r \in R\}$. Then M is a torsion module if and only if $t(M) = M$, and M is torsionfree if and only if $t(M) = 0$.

LEMMA 2.33. Let R be a semiprime Goldie ring, and let M be a uniform module which contains a torsionfree non-zero submodule N. Then M itself is torsionfree.

Proof: Suppose $t(M) \neq 0$. Then $t(M) \cap N \neq 0$, since M is uniform. $t(M) \cap N$ is torsion as a submodule of $t(M)$, and it is torsion free as a submodule of M, a contradiction. Thus $t(M) = 0$. □

To conclude this section, let us note the following easy result.

PROPOSITION 2.34. Let E be an injective module over a ring R and let I be an ideal of R.

(i) $ann_E I$ is an injective $R/I-$module.

(ii) If $EI = 0$, then E is an injective $R/I-$module.

Proof: (i) Suppose we are given $R/I-$homomorphisms $A \xrightarrow{g} B$, $A \xrightarrow{f} ann_E I \subseteq E$, with g is 1-1.Then there exists $B \xrightarrow{h} E$ such that $hg = f$, since E is injective. As $BI = 0$, for any $b \in B$, $h(b)x = h(bx) = 0$ for all $x \in I$. Hence $h(B) \subseteq ann_E I$, proving that $B \xrightarrow{h} ann_E I$ is an $R/I-$homomorphism with $hg = f$, and the conclusion follows.

(ii) trivially follows from (i). □

CHAPTER 3

FRIME IDEALS IN SERIAL RINGS

The results of this section are taken from the papers by B.J. Müller and S. Singh ([MS1] and [MS2]) and from the lecture notes given by B.J. Müller in 1992/1993.

Let us first discuss some known results about maximal ideals of a semilocal ring R. By definition, R/J is semisimple and therefore $\overline{R} = R/J = \sum_{i=1}^{m} \oplus I_i$ where the I_i are simple Artinian rings. Decompose the identity into indecomposable idempotents, $\overline{1} = \sum_{i=1}^{n} \varepsilon_i$ such that for $i = 1, \ldots, m$, $\varepsilon_i \in \mathrm{I}_i$ for $m+1 \leq j \leq n$, ε_j is isomorphic to one of $\varepsilon_1, \ldots, \varepsilon_m$. (We can do this, because each I_i is isomorphic to a full matrix ring $M_{n_i}(D_i)$ over a division ring D_i, where $n_1 + \ldots + n_m = n$.) It is clear that the maximal ideals of $\overline{R}$ are precisely the $\overline{M}_j = \sum_{i=1, i\neq j}^{m} \oplus I_i; j = 1, \ldots, m$. Let $M_j$be the full inverse image under the canonical map $R \to \overline{R}$ given by $r \longmapsto r + J$. Thus $M_1, \ldots, M_m$ are maximal ideals of R which contains J. Since every maximal ideal is primitive, it contains the Jacobson radical J of R. Therefore $M_1, \ldots, M_m$ are precisely the maximal ideals of the semilocal ring R. Moreover, as $\cap_{i=1}^{m} \overline{M}_i = \overline{0}$, we have $\cap_{i=1}^{m} M_i = J$. (ie. The Jacobson radical of a semilocal ring is the intersection of the maximal ideals.)

Now let us make more assumptions on R. Suppose that idempotents of R/J can be lifted to R (ie.R is semiperfect). Since any countable set of orthogonal idempotents in R/J can be lifted orthogonally ([L] Proposition 21.25), we let $e_i \in R$ be orthogonal idempotents such that $\overline{e}_i = e_i + J = \varepsilon_i$. Then $\sum_{i=1}^{n} e_i = e$ is an idempotent with $\overline{e} = \sum_{i=1}^{n} \varepsilon_i = \overline{1}$. Since the idempotent $1 - e \in J$, we conclude $1 - e = 0$. Therefore $1 = \sum_{i=1}^{n} e_i$. By (Lemma 2.9) $e_1, \ldots, e_m$ are non-isomorphic and each e_i $(i > m)$ is isomorphic to one of $e_1, \ldots, e_m$. Moreover the e_i are indecomposable (an idempotent $e \in R$ is indecomposable if eR cannot be written as a direct sum of two nontrivial submodules), by Lemma 2.10. For $j = 1, \ldots, m$, $\varepsilon_j \notin \overline{M}_j = M_j/J$ and for $i \neq j, i = 1, \ldots, m$, $\varepsilon_i \in \overline{M}_j$ We deduce $e_j \notin M_j$ and $e_i \in M_j$ $(i \neq j)$.

Let us turn to a right serial ring R. As R is semiperfect and hence semilocal, the ideals $M_1, \ldots, M_m$ are precisely the maximal ideals of R.

We fix the decomposition $1 = \sum_{i=1}^{n} e_i$ of the identity into indecomposable orthogonal idempotents, that we obtained before. Suppose $1 = \sum_{i=1}^{k} f_i$ is another decomposition into indecomposable orthogonal idempotents. For any indecomposable idempotent e, eR is uniserial, and therefore e is local. By ([L], Exercise 21.17) we get k=n and $f_{\sigma(i)} = ue_iu^{-1}$ for all i, where σ is a permutation of $\{1, \ldots, n\}$. Therefore $f_{\sigma(i)}R = ue_iR \simeq e_iR$ for all i. After reindexing we may assume that e_i is isomorphic to f_i. Hence $f_1, \ldots, f_m$ are non isomorphic idempotents, every f_j with

$j > m$ is isomorphic to one of $f_1, \ldots, f_m$: For any ideal $I, f_i \in I$ *if and only if* $e_i \in I$. Therefore the results stated in this section do not depend on the choice of the decomposition, $1 = \sum_{i=1}^{n} e_i$.

Let us introduce some notation.

DEFINITION Let $\tilde{N} = \{1, \ldots, m\}$ where $e_1, \ldots, e_m$ are the non-isomorphic idempotents in the fixed decomposition of the identity into indecomposable idempotents.

For a prime ideal P, define $E(P) = \{i \in \tilde{N} / e_i \notin P\}$.

NOTE: (1) $E(M_j) = \{j\}$.

(2) Given any prime ideal P, $E(P)$ is non-empty. (Since $P \subseteq M_k$ for some $k \in \tilde{N}$, $e_k \notin P$. Thus $k \in E(P)$.)

LEMMA 3.1. Let P, Q be prime ideals of the right serial ring R. Then the following are true:

(1) If $E(P) = E(Q)$, then P and Q are comparable.

(2) If $E(P) \subset E(Q)$, then $Q \subset P$.

Proof: We prove : (i) If $E(P) \subseteq E(Q)$, then P and Q are comparable.

(ii) If $E(P) \subset E(Q)$, then $Q \subset P$.

To see (i), suppose P is not contained in Q. If $e_iP \subseteq e_iQ$ for all $i = 1, \ldots, n$, then we get the contradiction $P = \sum_{i=1}^{n} e_iP \subseteq \sum_{i=1}^{n} e_iQ = Q$. Thus there exists e_i such that $e_iQ \subset e_iP$. Since $e_iQ \subset e_iP \subseteq P$ we get either $e_i \in P$ or $Q \subseteq P$ by the primeness of P. Note that $e_i \notin Q$.(If $e_i \in Q$, then $e_iR = e_iQ \subset e_iP \subseteq e_iR$, a contradiction.) Thus $i \in E(Q) \subseteq E(P)$ and therefore $e_i \notin P$. Hence $Q \subseteq P$ as required.

If $E(P) \subset E(Q)$, then there exists $i \in E(Q) - E(P)$. Thus $e_i \in P - Q$ by definition. Hence $Q \subset P$ by (i). □

COROLLARY 3.2. Let P be a prime ideal of the right serial ring R and let $k \in \tilde{N}$. Then $P \subseteq M_k$ *if and only if* $e_k \notin P$.

Proof: Since $e_k \notin M_k$, $P \subseteq M_k$ implies $e_k \notin P$. Conversely, $e_k \notin P$ implies $E(M_k) = \{k\} \subseteq E(P)$. Thus Lemma 3.1 implies the comparability of P and M_k. The result follows from the maximality of M_k. □

PROPOSITION 3.3. ([MS1] Lemma 3.1) Any two incomparable prime ideals of a right serial ring are comaximal.

Proof: Let P and Q are incomparable prime ideals of R. As e_iR is uniserial, $e_iP \subseteq e_iQ$ or $e_iQ \subseteq e_iP$. Hence $e_iP \subseteq Q$ or $e_iQ \subseteq P$. Incomparability of P and Q gives $e_i \in Q$ or $e_i \in P$, hence $e_i \in P + Q$ for all $i \in \{1, \ldots, n\}$. Therefore $1 = \sum_{i=1}^{n} e_i \in P + Q$; and we get $P + Q = R$. □

LEMMA 3.4. ([MS1] Corollary 3.2) Let P,Q be prime ideals of a right serial ring R. Assume there is a uniserial module M such that $MP \subset M$ and $MQ \subset M$. Then P and Q are comparable.

Proof : Suppose P and Q are not comparable. Then $P + Q = R$ by Proposition 3.3. Hence $M = M(P+Q) \subseteq MP+MQ$. By the uniseriality of M, the sum $MP+MQ$ is MP or MQ, and therefore properly contained in M. Thus $M \subseteq MP + MQ \subset M$, a contradiction. □

LEMMA 3.5. Let P,Q be prime ideals of the right serial ring R. Then $E(P)$ and $E(Q)$ are either comparable or disjoint.

Proof : Suppose $E(P)$ and $E(Q)$ are not disjoint. Let $k \in E(P) \cap E(Q)$. By Corollary 3.2, $P \subseteq M_k$ and $Q \subseteq M_k$. Thus $P + Q \subseteq M_k$ and hence P and Q are not comaximal. By Lemma 3.3, P and Q are comparable, and so are the sets $E(P)$ and $E(Q)$. □

PROPOSITION 3.6. Any set of incomparable primes is finite.

Proof : Let $\Im$ be a set of incomparable primes P_i. Also let $\xi = \{ E(P_i) \,/\, P_i \in \Im \}$. By Lemma 3.1 the sets $E(P_i)$ are incomparable, and therefore $E(P_i)$ is disjoint from $E(P_j)$ whenever $i \neq j$ (Lemma 3.5). As each $E(P_i)$ is a subset of the finite set $\tilde{N} = \{1, \ldots, m\}$, ξ is finite, and so is $\Im$. □

DEFINITION : If P and Q are primes in a serial ring R , then we say that there is a *(right) link from P to Q*, written $P \rightsquigarrow Q$, if

(i) P and Q are equal or incomparable,

(ii) $PQ \subset P \cap Q$.

Link connectivity components are called *cliques.*

REMARK (3.7) : The conditions in the above definition are symmetric in P and Q in the following sence. If there is a (right) link $P \rightsquigarrow Q$, then we may think that there is a left link from Q to P.

The following frequently used fact has been obtained from the proof of ([MS1], Lemma 3.5).

NOTE (3.8) :If P and Q are incomparable primes of the serial ring R such that $ePQ \subset eP \subset eR$, then $xP = eP$ holds for all $x \in eR - eP$.

Proof :By the choice of x , $eP \subset xR \subseteq eR$. It is clear that $xP \subseteq eP$. Suppose $xP \subset eP$; we show that this assumption leads to a contradiction. Observe that $eP^2 \subseteq xRP = xP \subset eP$. Since eP^2,ePQ are properly contained in the uniserial module eP, Lemma 3.4 applied to $M = eP$, shows that P and Q are comparable, contrary to the incomparability assumption. Hence we conclude $xP = eP$. □

LEMMA 3.9. Let P and Q be incomparable prime ideals in the serial ring R such that $P \rightsquigarrow Q$. Then $ePQ \subset eP \subset eR$ holds for all indecomposable idempotents $e \notin P$.

Proof: $PQ \subset P \cap Q$ implies there exists an indecomposable idempotent $e \in R$ such that $ePQ \subset e(P{\cap}Q)$. (This is because, if $ePQ = e(P{\cap}Q)$ for all indecomposable idempotents, then we get the contradiction $PQ = \sum_{i=1}^{n} e_iPQ = \sum_{i=1}^{n} e_i(P \cap Q) = P \cap Q$.) Therefore $ePQ \subset e(P \cap Q) \subseteq eP \cap eQ \subseteq eP \subsetneq eR$. If $eP = eR$, then $eQ = ePQ \subset eP \cap eQ = eQ$, a contradiction. Hence $ePQ \subset eP \subset eR$, and $e \notin P$ (because if $e \in P$, then $eP = eR$).

Now consider any other $e' \notin P$. Observe that eRe' is not contained in P. (If $eRe' \subseteq P$, then by the primeness of P, e or $e' \in P$, contrary to the assumption.)

Hence there exists $x_0 \in eRe' - P$. As $x_0 \in eR - eP$, $x_0P = eP$, by Note (3.8). Thus $eP = x_0P \subseteq eRe'P \subseteq eP$, and therefore $eRe'P = eP$. Hence we conclude $e'PQ \subset e'P$, because $e'PQ \subseteq e'P$, but $e'PQ \neq e'P$. (If $e'PQ = e'P$, then $eP = eRe'P = eRe'PQ = ePQ \subset eP$, a contradiction.) Therefore we have $e'PQ \subset e'P \subset e'R$ for any $e' \notin P$. □

PROPOSITION 3.10.([MS1] Corollary 3.8) Let P and Q be incomparable primes in a serial ring R such that $P \rightsquigarrow Q$. Then

(i) P and Q contain the same prime ideals properly.

(ii) P and Q are Goldie, and determine each other uniquely.

Proof: (i) Consider any prime $Q' \subset P$. By Lemma 3.9, there exists an indecomposable idempotent $e \notin P$ such that $ePQ \subset eP \subset eR$. If $ePQ' = eP$, then $eP = ePQ' \subseteq Q'$. Then by the primeness of Q', we get $e \in Q'$. This implies the contradiction $e \in P$. Therefore we conclude $ePQ' \subset eP$. By Lemma 3.4, applied to $M = eP$,we get that Q and Q' are comparable. Since $Q \subseteq Q'$ implies the contradiction $Q \subseteq Q' \subset P$ we conclude, $Q' \subset Q$.

Similarly, if $Q' \subset Q$, then by the symmetry between P and Q explained in Remark(3.7), we get $Q' \subset P$.

(ii) To show that P is Goldie, let us show that R/P is a nonsingular ring. As R/P is serial , it has finite Goldie dimension, and therefore it then follows that R/P

is a Goldie ring. Observe that $R/P \cong \sum_{i=1}^{n} \oplus e_iR/e_iP = \sum_{e_i \notin P} \oplus e_iR/e_iP$, since $e_iP = e_iR$ for $e_i \in P$.

Consider any $e \notin P$. By Lemma 3.9, $ePQ \subset eP \subset eR$, hence $eP \neq 0$. Note (3.8) implies $xP = eP$ for all $x \in eR - eP$. By Lemma (2.3), choose an indecomposable idempotent e' such that $x = xe'$. We define the maps $\Psi : e'R \to xR$ by $\Psi(e'r) = xe'r = xr$ and $\Phi : xR \to xR/xP$ by $\Phi(xr) = xr + xP$. Obviously the map $\Phi\Psi$ is a surjective R–homomorphism. We claim that $\ker(\Phi\Psi) = e'P$. $e'P \subseteq \ker(\Phi\Psi)$ is clear, since for any $e'r \in e'P$ with $r \in P$, $\Phi\Psi(e'r) = xr + xP = 0$. Conversely, take any $e'r \in \ker(\Phi\Psi)$. Then $xr + xP = 0$ implies $xr = xy$ for some $y \in P$. Thus $x(r-y) = 0$ and therefore $e'(r-y) \in \ker\Psi$. As $e'R$ uniserial, $\ker\Psi$ and $e'P$ are comparable. If $e'P \subset \ker\Psi$, then $0 = \Psi(e'P) = xP = eP$, a contradiction as $eP \neq 0$. Therefore $\ker\Psi \subseteq e'P$; we get $e'(r-y) = e'p$ for some $p \in P$. Hence $e'r = e'(y+p) \in e'P$ for any $e'r \in \ker(\Phi\Psi)$. Thus $\ker(\Phi\Psi) \subseteq e'P \subseteq \ker(\Phi\Psi)$; hence $e'P = \ker(\Phi\Psi)$. Then we get $e'R/e'P \cong xR/xP = xR/eP$. As a direct summand of R/P, $e'R/e'P$ is a projective R/P–module. Therefore we have proved that an arbitrary cyclic submodule xR/eP of eR/eP is projective as an R/P–module.

Now take any $\overline{y} \in Z_{R/P}(eR/eP)$. Then $\overline{y}\overline{R}$ is a singular $\overline{R}$–module (where $\overline{R} = R/P$). As a cyclic submodule of eR/eP, $\overline{y}\overline{R}$ is projective by what we have proved before. Therefore by Lemma 2.5, it is not singular. This can only happen if $\overline{y} = 0$. Therefore we get $Z_{R/P}(eR/eP) = 0$, to say that, eR/eP is a nonsingular R/P–module for every $e \notin P$. Therefore $R/P \cong \sum_{e_i \notin P} \oplus e_iR/e_iP$ is a nonsingular ring , as required. Thus P is Goldie.

Similarly we obtain that Q is Goldie, by the Remark(3.7).

To prove the uniqueness, first let $P \rightsquigarrow P$ and $P \rightsquigarrow Q$; we prove $Q = P$. Suppose $Q \neq P$. Then by definition, P and Q are incomparable. By Lemma 3.9, $eP^2 \subset eP \subset eR$ and $ePQ \subset eP \subset eR$ for all $e \notin P$. By Lemma 3.4, applied to $M = eP$, gives the contradiction that P and Q are comparable. Thus Q must equal to P. Similarly if $Q \rightsquigarrow Q$ and $P \rightsquigarrow Q$, then by the symmetry given in Remark(3.7), we get $P = Q$.

Now suppose $P \rightsquigarrow Q_1$ and $P \rightsquigarrow Q_2$ and both Q_1 and Q_2 are incomparable with P. Then for all $e \notin P$, $ePQ_i \subset eP \subset eR$ $(i = 1, 2)$. Again we apply Lemma 3.4 to $M = eP$ and we get the comparability of Q_1 and Q_2. If $Q_2 \subset Q_1$, then by (i), $Q_2 \subset P$ contradicting the incomparability of Q_2 and P. If $Q_1 \subset Q_2$, then we get a similar contradiction. Thus we conclude $Q_1 = Q_2$. Similarly the uniqueness of P, whenever $P \rightsquigarrow Q$, follows by Remark(3.7). □

Therefore we have the following:

The non-trivial cliques contain incomparable primes, and thus cliques are finite (Proposition 3.6 and 3.10). Each prime has at most one incoming and outgoing link (Proposition 3.10). Consequently, cliques are linear or circular.

LEMMA 3.11. ([MS1] Proposition 3.4) Every right serial semiprime ring R is a direct sum of prime rings.

Proof: Since R is right serial, it has only finite number of minimal primes, $P_1, \ldots, P_s$. (This is because minimal primes are incomparable by definition and, any set of incomparable primes is finite (Proposition 3.6). For $i \neq j$, $P_i + P_j = R$ (Proposition 3.3). We claim that $P_i + \cap_{j\neq i} P_j = R$. If this is not true, $P_i + \cap_{j\neq i} P_j \subseteq M$ for some maximal ideal M. Hence $P_i \subseteq M$ and $\prod_{j\neq i} P_j \subseteq \cap_{j\neq i} P_j \subseteq M$, which implies for some $k \neq i$, $P_k \subseteq M$. Thus $R = P_i + P_k \subseteq M$, a contradiction.

Hence by Chinese Remainder Theorem, $R = R/(\cap_{i=1}^{s} P_i) \cong \oplus_{i=1}^{s} R/P_i$ as required. □

DEFINITION:

(i) Let ζ be a multiplicative set in a ring R. We say that a right $R-$module M is $\zeta-divisible$ if and only if $Mc = M$ for all $c \in \zeta$.

(ii) Let ζ be a right Ore set (cf. the definition after Proposition 2.24) in R. For any right $R-$module M, the submodule $t_\zeta(M) = \{m \in M/mc = 0 \ for\ some\ c \in \zeta\}$ is called the $\zeta - torsion\ submodule\ of\ M$. The module M is $\zeta - torsion$ if and only if $t_\zeta(M) = M$, and $\zeta - torsionfree$ if and only if $t_\zeta(M) = 0$.

The following lemma due to Chatters, is crucial for our discussion.

LEMMA 3.12. ([C] Lemma 3.2) Let R be a serial ring and let N be a semiprime Goldie ideal of R such that N contains no non-zero idempotents. Then $N = Nc = cN$ for all $c \in \zeta(N)$, and $\zeta(N)$ is a left and right Ore set in R.

Proof: Let $c \in \zeta(N)$. Since cR is a submodule of a projective module R_R, by Lemma 2.19, there exists a decomposition $R = \sum_{i=1}^{n} \oplus I_i$ such that $cR = \sum_{i=1}^{n} \oplus (I_i \cap cR)$. Then $R = \sum_{i=1}^{n} \oplus e_i R$, $I_i = e_i R$, where $1 = \sum_{i=1}^{n} e_i$ with e_i's are orthogonal idempotents. Hence $cR = \sum_{i=1}^{n} \oplus(e_iR \cap cR)$. Then right ideal $\overline{cR} = (cR + N)/N$ is essential in $\overline{R} = R/N$. (To see this, note that, since $\overline{c}$ is regular in $\overline{R}$, $\overline{c}\overline{R} \cong \overline{R}$. Hence $\dim\overline{c}\overline{R} = \dim\overline{R}$. This implies, $\overline{c}\overline{R} \subseteq' \overline{R}$, since $\overline{R}$ has finite Goldie dimension.) We fix an integer j with $1 \leq j \leq n$. By the hypothesis, $e_j \notin N$. Thus $\overline{e_jR} = (e_jR+N)/N \neq 0$ and hence $\overline{e_jR} \cap \overline{cR} \neq 0$, by the essentiality. Hence $e_jR \cap (cR+N)$ is not contained in N. Therefore $e_jN \subset e_jR \cap (cR+N)$, by the uniseriality of e_jR. But $e_jR \cap (cR+N) = e_jR \cap (\sum \oplus(cR \cap e_iR) + \sum \oplus e_iN) = e_jR \cap (\sum \oplus((cR \cap e_iR) + e_iN)) = (cR \cap e_jR) + e_jN$. Hence $e_jN \subset (cR \cap e_jR) + e_jN$. Therefore $e_jN \subset cR \cap e_jR$, again by the uniseriality of e_jR. Hence $e_jN \subseteq cR$ for all j. Since $N = \sum_{i=1}^{n} e_iN$, $N \subseteq cR$. Now take any $x \in N$. Then $x = cr$ for some $r \in R$. Since $(c+N)(r+N) = 0$ in R/N and $c \in \zeta(N)$; we get $r + N = 0$. Hence $r \in N$ and $x = cr \in cN$. Therefore $N = cN$. Imitating the same proof for left modules, we obtain $N = Nc$.

Because R/N is a semiprime Goldie ring, the set of regular elements of R/N, that is $\overline{\zeta(N)} = (\zeta(N) + N)/N$ satisfies Ore condition (Goldie Theorem). Thus for given $r \in R$, $c \in \zeta(N)$, there exist $r' \in R$ and $c' \in \zeta(N)$ such that $\overline{r}\overline{c'} = \overline{c}\overline{r'}$. It follows that $rc' = cr' + y$ for some $y \in N$. As $N = cN$, $y = cy'$ for some $y' \in N$. Thus $rc' = c(r' + y')$ and therefore $\zeta(N)$ satisfies the right Ore condition. Hence $\zeta(N)$ is an Ore set as desired. □

The following two results were proved in ([C] Lemma 3.1 and Theorem 3.3), under the additional assumption that the ring R has Krull dimension. However it turns out, on the basis of the previous results, that this extra assumption can be avoided.

LEMMA 3.13.([C] Lemma 3.1) Let R be a serial ring and let N be the prime radical of R. Let K be a non-zero uniserial right $R-$module with $KN = 0$. Then there is a unique minimal prime ideal P of R such that $KP = 0$. Also $KQ = K$ for every minimal prime ideal Q of R with $Q \neq P$.

Proof: Let $P_1, \ldots, P_s$ be the minimal primes of R. Since $N = \cap_{i=1}^{s} P_i \supseteq \prod_{i=1}^{s} P_i$, we have $K(\prod_{i=1}^{s} P_i) = 0$. If $KP_i = K$ for all $i = 1, \ldots, s$, then we get the contradiction $0 = KP_1 \ldots P_s = K$. Hence there exists P_i such that $KP_i \subset K$. For $j \neq i$, $P_i + P_j = R$ (Proposition 3.3). Hence $K = K(P_i + P_j) \subseteq KP_i + KP_j \subseteq K$. Thus $K = KP_i + KP_j$. By the uniseriality of K, $KP_i + KP_j$ is KP_i or KP_j. Since $KP_i \subset K$, we deduce $K = KP_j$ for all $j \neq i$, $i = 1, \ldots, s$. Thus $0 = KP_1 \ldots P_s = KP_i$ as required. □

LEMMA 3.14.([MS2] Proposition 5) Let R be an indecomposable serial ring whose prime radical is Goldie and nilpotent. Suppose R has a non-nilpotent ideal X such that $\cap_{n \in \mathbf{N}} X^n = 0$. Then R is prime.

Proof: Let $P_1, \ldots, P_s$ be minimal primes of R. Since N is Goldie, these P_i's are Goldie. If $P_i \supseteq X$ for $i = 1, \ldots, s$, then $N \supseteq X$. This is a contradiction as N is nilpotent and X is not. Therefore there exists P_i such that P_i does not contain X. Let $P = P_i$. Let e be an indecomposable idempotent of R. We want to show that

$eN = eNP$, and this is trivial if $eN = 0$.

Suppose that $eN \neq 0$ and that $eN \neq eNP$. Hence $eN \supset eNP \supseteq eN^2$, and we have that $K = eN/eN^2$ is non-zero. As a factor module of the uniserial module eR, K is uniserial. Lemma 3.13 applied to $K = eN/eN^2$, shows that there exists a unique minimal prime Q such that $KQ = 0$ and $KQ' = K$ for every minimal prime Q' of R with $Q' \neq Q$. Then $Q = P$.(To see this, note that if $P \neq Q$, then $KP = K$. But this is impossible since, $eNP \subset eN$ implies $KP \subseteq eNP/eN^2 \subset eN/eN^2 = K$.) Thus $KP = 0$ and therefore $eNP \subseteq eN^2$. Thus $eNP = eN^2$.

Since $X + P \supset P$, $(X+P)/P$ is a non-zero ideal of the prime Goldie ring R/P. Hence $(X+P)/P$ is essential in R/P, and therefore by Goldie theorem there is a regular element $c_i + P \in (X+P)/P$ with $c_i \in \zeta(P) \cap (X+P)$. Recall $P = P_i$. Also for $j \neq i$, $(X+P)+P_j \supset P_j$ and similarly there exists a regular element $c_j + P_j \in ((X+P)+P_j)/P_j$ with $c_j \in (X+P) \cap \zeta(P_j)$. Now by Lemma 2.32 there exist $r_k \in R$ such that $c = \sum_{i=1}^{s} c_k r_k \in \zeta(N)$. Note that $c \in X + P$. But $Nc = N$ by Lemma 3.12. Hence $eN = eNc \subseteq eN(X+P) \subseteq eNX + eNP \subseteq eNX + eN^2 \subseteq eN$. Thus $eN = eNX + eN^2$. By the uniseriality of eR, eNX and eN^2 are comparable and therefore $eN = eNX$ or $eN = eN^2$. But $eN = eN^2$ is impossible as N is nilpotent. Thus $eN = eNX$. This implies $eN = eNX^n$ for all $n \in \mathbf{N}$. This is a contradiction, because $eN \neq 0$ and $\cap X^n = 0$. Therefore we conclude that $eN = eNP$ for any indecomposable idempotent e. Seriality of R implies $1 = \sum_{i=1}^{n} e_i$ where e_i are indecomposable orthogonal idempotents. Thus, $N = \sum eN = \sum eNP = NP$ as required.

By Lemma 3.11, $R/N \cong \oplus R/P_i$ as rings. Thus the ideal $\overline{P} := P/N$ is generated by a central idempotent, say ε, of $\overline{R} := R/N$. As idempotents can be lifted, there exists an idempotent f of R such that $\overline{f} = \varepsilon$. Since $\overline{P} = \varepsilon\overline{R} = \overline{f}\,\overline{R}$, we get $P = fR + N = Rf + N$. Thus $N = NP$ gives $N \subseteq N(Rf + N) \subseteq Nf + N^2 \subseteq N$ and therefore $N = Nf + N^2$. Repeating, $N = Nf + N^2 = Nf + N^2P = Nf + N^2(Rf + N) \subseteq N$; we get $N = Nf + N^3$. Hence $N = Nf + N^m$ for any $m \in \mathbf{N}$ and therefore $N = Nf$, since N is nilpotent. This implies $P = Rf + N = Rf + Nf = Rf$. By symmetry we get $P = Rf = fR$. Thus f is a central idempotent of R. As R is indecomposable, $f = 0$. Hence $P = 0$, and therefore R is prime. □

The next proposition is crucial for the proof of our main theorem.

PROPOSITION 3.15. ([MS2] Theorem 6) Let R be an indecomposable serial ring, P a Goldie prime ideal of R, and T the intersection of the clique of P. If the clique is linear, $P_1 \rightsquigarrow \ldots \rightsquigarrow P_t$, then $\cap_{n\in\mathbf{N}} T^n = T^t = P_1 \ldots P_t$ is idempotent. If the clique is circular, then either T is nilpotent or $\cap_{n\in\mathbf{N}} T^n$ is a Goldie prime ideal.

Proof: We fix a decomposition $1 = \sum_{i=1}^n e_i$ into indecomposable orthogonal idempotents.

Case (i): The clique is linear.

Since there are no self links, $P_i^2 = P_i$ holds, and since there are no links from P_{i+1} to P_i, we have $P_{i+1}P_i = P_i \cap P_{i+1}$. As $P_i \rightsquigarrow P_{i+1}$,we have $P_iP_{i+1} \subset P_i \cap P_{i+1}$. For $k \neq i-1, i+1$, $P_kP_i = P_i \cap P_k = P_iP_k$. Therefore $T = \cap_{i=1}^t P_i \supseteq \prod_{i=1}^t P_i$, and the above information implies $T^n \supseteq (P_1 \dots P_t)^n \supseteq P_1^n \dots P_t^n = P_1 \dots P_t \supseteq T^t$ for all n. Hence $\cap T^n = T^t = P_1 \dots P_t$, as required.

Case (ii): The clique is circular, $P_1 \rightsquigarrow \dots P_t \rightsquigarrow P_1$.

We prove:

(a) If for each $e_i \notin T$, there are m_i, n_i such that $e_i T^{m_i} = 0 = T^{n_i} e_i$, then T is nilpotent.

(b) If there is $e = e_i \notin T$ such that $eT^n \neq 0$ for all n, then (our aim is to prove that $\cap T^n$ is prime Goldie), we show by induction, $eT^n \supset eT^{n+1}$ for all n, and there exists a unique P_i such that $P_i = ann_R(eT^n/eT^{n+1})$ and that $P_{i+1} = ann_R(eT^{n+1}/eT^{n+2})$. (ie. The P_i reoccur periodically as the annihilators of the eT^m/eT^{m+1}.) Moreover $eR/e(\cap T^n)$ is a nonsingular $R/\cap T^n$–module.

Proof of (a): Let $\varepsilon = \sum_{e_i \in T} e_i$. Then $\varepsilon \in T$ and $\varepsilon \neq 1$ as $T \subset R$. For $n = \max\{m_i, n_i\}$, since $\sum_{e_i \notin T} e_i T^n = 0$, we have $T^n = \sum_{e_i \in T} e_i T^n = \varepsilon T^n$. Similarly $T^n = T^n \varepsilon$. Therefore $\varepsilon R = \varepsilon T^n = T^n = T^n \varepsilon = R\varepsilon$, and hence ε is a central idempotent. As R is indecomposable, $\varepsilon = 0$, hence $T^n = 0$ as required.

Proof of (b): $eT^n \supset eT^{n+1}$ is true for $n = 0$. There exists a unique P_i which annihilates eT^n/eT^{n+1}, and $(eT^n/eT^{n+1})P_j = eT^n/eT^{n+1}$ for all $j \neq i$ (cf. Lemma 3.13). Since R/T is Goldie, by Lemma 2.26 R/T is a nonsingular ring, and therefore by (Lemma 2.8) applied to $M = eT^n$, we get that eT^n/eT^{n+1} is a nonsingular R/T–module. Consequently, eT^n/eT^{n+1} is a nonsingular R/P_i–module. (If not, take any $0 \neq x \in Z_{R/P_i}(eT^n/eT^{n+1})$. Then $x\overline{I} = 0$ for $\overline{I} \subseteq' R/P_i$. Recall that

$R/T \cong R/P_1 \times \ldots \times R/P_t$ (Lemma 3.11). Let $Y = R/P_1 \times \ldots \times \bar{I} \times \ldots \times R/P_t$. Then $Y \subseteq' R/T$ and $xY = 0$. Hence $0 \neq x \in Z_{R/T}(eT^n/eT^{n+1}) = 0$, a contradiction.) If $P_i \subset ann_R(eT^n/eT^{n+1}) =: L$, then $L/P_i \subseteq' R/P_i$. Thus $(eT^n/eT^{n+1})(L/P_i) = 0$ implies that eT^n/eT^{n+1} is a singular R/P_i−module, which is impossible. Therefore $P_i = ann_R(eT^n/eT^{n+1})$.

Take any $x \in eT^n - eT^{n+1}$. By Lemma 2.7 applied to $M = eT^n$, we get $xT = eT^{n+1}$. Since xR is local, by Lemma 2.3 choose an indecomposable idempotent e' with $x = xe'$. Define $f : e'R \to xR$ by $e'r \longmapsto xe'r = xr$. Then f is an epimorphism with $f(e'T^m) = xT^m = eT^{m+n} \neq 0$ for all $m \geq 1$. Thus $\ker f$ does not contain $e'T^m$, and therefore, by the uniseriality, $\ker f \subset e'T^m$ for all $m \geq 1$. Hence $\ker f \subseteq \cap e'T^m = e'(\cap T^m)$. Hence f induces an isomorphism $e'R/e'(\cap T^m) \cong xR/f(e'(\cap T^m)) = xR/x(\cap T^m)$. Under this isomorphism, $e'(\cap T^m)$ maps to $x(\cap T^m)$. Since $e'(\cap T^m) = \cap e'T^m$, we conclude $x(\cap T^m) = \cap xT^m$. Note that $\cap xT^m = \cap eT^{m+n} = e(\cap T^m)$. In consequence of this we have $e'R/e'(\cap T^m) \cong xR/x(\cap T^m) = xR/\cap xT^m = xR/e(\cap T^m)$. Therefore as a direct summand of $R/\cap T^m$, $xR/e(\cap T^m)$ is projective and therefore by Lemma 2.5 it is not singular as an $R/\cap T^m$−module. Thus any cyclic submodule $xR/e(\cap T^m)$ of $eR/e(\cap T^m)$ is not singular as $R/\cap T^n$−module. Hence $eR/e(\cap T^m)$ is a nonsingular $R/\cap T^n$−module, as required.

Since $e' \notin P_i$ (if $e' \in P_i$, then we get the contradiction $x = xe' \in eT^nP_i \subsetneq eT^{n+1}$), $e'P_iP_{i+1} \subset e'P_i$ by Lemma 3.9. Since $f(e'P_iP_{i+1}) = xP_iP_{i+1} \supseteq xT^2 = eT^{n+2} \neq 0$, by uniseriality, $\ker f \subset e'P_iP_{i+1}$. Thus $f(e'P_iP_{i+1}) \subset f(e'P_i)$. Therefore we deduce that, $eT^{n+2} = xT^2 \subseteq xP_iP_{i+1} = f(e'P_iP_{i+1}) \subset f(e'P_i) = xP_i = xT = eT^{n+1}$.

Arguing as before, there exists P_j with $P_j = ann_R(eT^{n+1}/eT^{n+2})$. Therefore

$eT^{n+1}P_j \subseteq eT^{n+2} \subset eT^{n+1}$ and $eT^{n+1}P_{i+1} = xP_iP_{i+1} \subset xP_i = eT^{n+1}$, which implies the comparability of P_j and P_{i+1}, by Lemma 3.4. Thus $P_j = P_{i+1}$, completing the proof of (b).

Now take any $e' \notin T$. We show that $e'T^n \neq 0$ for all n. $e' \notin T$ implies $e' \notin P_k$ for some k. By (b), there is n such that $P_k = ann_R(eT^n/eT^{n+1})$. There is $y \in eT^n - eT^{n+1}$ such that $ye' \in eT^n - eT^{n+1}$. (If $ye' \in eT^{n+1}$ for all $y \in eT^n - eT^{n+1}$, then $e' \in ann_R(eT^n/eT^{n+1}) = P_k$, a contradiction. Let $x = ye'$. We define the epimorphism $f : e'R \longrightarrow xR$ by $e'r \longmapsto xe'r = xr$. Then $f(e'T^m) = xT^m = eT^{m+n} \neq 0$ for all $m \geq 1$. Therefore $e'T^m \neq 0$ for all m.

Therefore all $e' \notin T$ satisfy the properties given in (b), as the original e. Therefore $e'R/e'(\cap T^n)$ is a nonsingular $R/\cap T^n-$module for all $e' \notin T$. If $e_j \in T$, then $e_j = e_j^n \in T^n$ and we have $e_j \in \cap T^n$. Hence $e_jR/e_j(\cap T^n) = 0$. Thus $R/\cap T^n \cong \sum_{e \notin T} \oplus eR/e(\cap T^n)$ is a right nonsingular $R/\cap T^n-$module, and therefore $R/\cap T^n$ is a nonsingular ring. By Lemma 2.27 the ring $R/\cap T^n$ is Goldie, and thus the ideal $\cap T^n$ is Goldie as required.

To show that $\cap T^n$ is prime, let us prove that $R/\cap T^n$ satisfies all the assumptions of Lemma 3.14. By (Lemma 2.28 and 2.30), the prime radical of $R/\cap T^n$ is Goldie and nilpotent. $T/\cap T^n$ is a non-nilpotent ideal of $R/\cap T^n$. (If it is nilpotent, $T^m \subseteq \cap T^n$ for some m, and therefore $T^m = T^{m+1}$, which gives the contradiction $eT^m = eT^{m+1}$.)

To prove that $R/\cap T^n$ is indecomposable, let us first note that the clique of P is preserved in the factor ring. (If $P \rightsquigarrow Q$, then since $PQ \supseteq T^2$, we have $PQ + \cap T^n =$

$PQ \subset P \cap Q$. Thus $\overline{PQ} = (P/\cap T^n)(Q/\cap T^n) \subseteq (PQ + \cap T^n)/\cap T^n = PQ/\cap T^n \subset (P \cap Q)/\cap T^n = \overline{P} \cap \overline{Q}$. We conclude $\overline{P} \rightsquigarrow \overline{Q}$.) Suppose that we have a ring decomposition $\overline{R} = R/\cap T^n = S_1 \times \ldots \times S_m$ into indecomposable rings, and $\overline{P} \rightsquigarrow \overline{Q}$. Then since $\overline{P} = S_1 \times \ldots \times \overline{P}_i \times \ldots \times S_m$ for some prime ideal $\overline{P}_i$ of S_i, using the condition $\overline{P}\,\overline{Q} \subset \overline{P} \cap \overline{Q}$ we see that $\overline{Q}$ is also of the form $\overline{Q} = S_1 \times \ldots \times \overline{Q}_i \times \ldots \times S_m$ for some prime ideal $\overline{Q}_i$ of S_i. Hence $\cap T^n = 0$ in $R/\cap T^n$ gives us $0 = S_1 \times \ldots \times (\cap_{n \in \mathbf{N}} Y^n) \times \ldots \times S_m$, where $Y = \cap_{i=1}^{t} \overline{P}_i$. Hence $S_j = 0$ for all $j \neq i$, and we get $\overline{R} = S_i$ and $\overline{R}$ is indecomposable. Hence by Lemma 3.14, $R/\cap T^n$ is a prime ring, and therefore $\cap T^n$ is a prime ideal. □

Under the assumptions of the above proposition, if the clique of P is circular, we have that either T is nilpotent or $\cap T^n$ is (Goldie) prime. For the case that $\cap T^n$ is prime, we have the following improvement:

THEOREM 3.16. $\cap T^n$ is the largest prime properly contained in T.

Proof: Consider any prime $Q \subset T$. Then, since Q and $\cap T^n$ are not comaximal (if they are comaximal, then $R = Q + \cap T^n \subseteq T \subset R$, a contradiction), Q and $\cap T^n$ are comparable (Proposition 3.3). If $Q \subseteq \cap T^n$, then there is nothing to prove. Suppose this is not the case. Then for some m, Q is not contained in T^m. Hence there exists an indecomposable idempotent e such that $eQ \supset eT^m$. As $Q \supseteq eQ \supset eT^m$, by the primeness of Q, $e \in Q$ or $T^m \subseteq Q$. If $e \in Q$, then $e = e^m \in T^m$, and therefore

$eT^m = eR = eQ$, a contradiction. On the other hand if $T^m \subseteq Q$, then we get the contradiction $T \subseteq Q$. Therefore we conclude, $Q \subseteq \cap T^n$. □

CHAPTER 4

LOCALIZATION

DEFINITION:

A *right ring of fractions (or right Ore quotient ring or right Ore localization)* for R with respect to ζ is a ring homomorphism $\alpha : R \to R_\zeta$ such that

(a) $\alpha(c)$ is invertible in R_ζ for all $c \in \zeta$.

(b) Each element of R_ζ has the form $\alpha(r)\alpha(c)^{-1}$ for some $r \in R$, $c \in \zeta$.

(c) $\ker \alpha = \{r \in R / rc = 0 \textit{ for some } c \in \zeta\}$.

DEFINITION: Let ζ be a multiplicative set in R. Then ζ is *right reversible* if, given $r \in R$ and $c \in \zeta$ such that $cr = 0$, then there exists $c' \in \zeta$ such that $rc' = 0$.

A *right denominator set* is a right reversible right Ore set.

We list now several results quoted from [GW] which are used in this chapter.

LEMMA 4.1. ([GW] Theorem 9.7, Corollary 9.5) Let ζ be a multiplicative set in a ring R. Then there exists a right ring of fractions for R with respect to ζ if and only if ζ is a right denominator set.

When the right ring of fractions exists, it is unique up to a unique isomorphism over R.

PROPOSITION 4.2. ([GW] Proposition 9.8) If ζ is a right and left denominator set in a ring R, then any right (left) ring of fractions for R with respect to ζ is also a left (right) ring of fractions.

For the moment, we take the ring R serial and $T = \cap_{i=1}^{t} P_i$ where the P_i are incomparable Goldie primes such that T does not contain any non-zero idempotents. Then T is Goldie semiprime. By Chatters result (Lemma 3.12), we get that $\zeta = \zeta(T)$ is a left and right Ore set in R and $T = Tc = cT$ for all $c \in \zeta$. (This set ζ is not reversible in general.)

Define $\mathbf{K} = \{x \in R / c_1 x c_2 = 0 \ for\ some\ c_1, c_2 \in \zeta\}$. This set K is an ideal of R.(To prove this, take any $k_1, k_2 \in K$ and $r \in R$. There exist $c_i, c_i' \in \zeta (i = 1, 2)$ with $c_1 k_1 c_1' = 0 = c_2 k_2 c_2'$. By the Ore condition, there exist $x, x' \in R$ and $c, c' \in \zeta$ such that $xc_1 = cc_2$ and $c_1' x' = c_2' c'$. Hence $cc_2(k_1 + k_2)c_2' c' = 0$; we get $k_1 + k_2 \in K$. Again by the left Ore condition, $ac_1 = dr$ for some $a \in R$ and $d \in \zeta$. Hence, $d(rk_1)c_2 = ac_1 k c_2 = 0$, and therefore $rk_1 \in K$. Similarly using right Ore condition, we get $k_1 r \in K$.)

Let $\overline{R} := R/K$. Since the Ore condition is inherited by any factor ring, $\overline{\zeta} = (\zeta + K)/K$ is a left and right Ore set. Moreover $\overline{\zeta}$ consists of regular elements of

$\overline{R}$. (To see this suppose $\overline{rc} = 0$; $\overline{r} \in \overline{R}$, $\overline{c} \in \overline{\zeta}$. Then $rc \in K$. Hence by definition of K, $c_1rcc_2 = 0$ for some $c_i \in \zeta$. As $cc_2 \in \zeta$, we get $r \in K$ and thus $\overline{r} = 0$. Similarly we prove that if $\overline{cs} = 0; \overline{s} \in \overline{R}$, then $\overline{s} = 0$.) Since the right ring of fractions exists with respect to any right denominator set, $\overline{R}_{\overline{\zeta}}$ exists (Lemma 4.1). Observe that, as $\overline{\zeta}$ consists of regular elements, $\ker\alpha = 0$. Hence we identify $\overline{R}$ with $\alpha(\overline{R})$, and consider $\overline{R}$ as a subring of $\overline{R}_{\overline{\zeta}}$. Since $\overline{\zeta}$ is also a left denominator set, the left ring of fractions ${}_{\overline{\zeta}}\overline{R}$ also exists and ${}_{\overline{\zeta}}\overline{R} = \overline{R}_{\overline{\zeta}}$ (Lemma 4.2).

Let $Q := \overline{R}_{\overline{\zeta}}$. Then Q is a serial ring. (We show that Q is right serial. Left seriality follows similarly. As $\overline{R}$ is serial, let $\overline{R} = \sum_{i=1}^{n} \oplus \varepsilon_i\overline{R}$ where ε_i are orthogonal idempotents, $\overline{1} = \sum_{i=1}^{n} \varepsilon_i$ and each $\varepsilon_i\overline{R}$ is uniserial. We claim that $Q = \sum_{i=1}^{n} \oplus\varepsilon_i Q$ and each $\varepsilon_i Q$ is uniserial. To see this, first note that, as any element of Q has the form $\overline{r}\,\overline{c}^{-1}$ for some $\overline{r} \in \overline{R}$ and $\overline{c} \in \overline{\zeta}$, using $\overline{r} = \varepsilon_1\overline{r} + \ldots + \varepsilon_n\overline{r}$ it is immediate that $Q = \sum_{i=1}^{n} \varepsilon_i Q$. To show the sum is direct, suppose $\sum_{i=1}^{n} \overline{q}_i = 0$, where $\overline{q}_i = \varepsilon_i\overline{r}_i\overline{c}_i^{-1} \in \varepsilon_i Q$, $\overline{r}_i \in \overline{R}$ and $\overline{c}_i \in \overline{\zeta}$. Thus $-\overline{q}_i = -\varepsilon_i\overline{r}_i\overline{c}_i^{-1} = \varepsilon_1\overline{r}_1\overline{c}_1^{-1} + \ldots + \varepsilon_{i-1}\overline{r}_{i-1}\overline{c}_{i-1}^{-1} + \varepsilon_{i+1}\overline{r}_{i+1}\overline{c}_{i+1}^{-1} + \ldots + \varepsilon_n\overline{r}_n\overline{c}_n^{-1}$. Multiplying both sides by ε_i, we get $\overline{q}_i = 0$. It remains to show that, each summand is uniserial. Consider $\alpha, \beta \in \varepsilon_i Q$. Thus $\alpha = \varepsilon_i\overline{r}_1\overline{c}_1^{-1}$, $\beta = \varepsilon_i\overline{r}_2\overline{c}_2^{-1}$ for some $\overline{r}_i \in \overline{R}$ and $\overline{c}_i \in \overline{\zeta}$. As $\alpha\overline{c}_1 = \varepsilon_i\overline{r}_1$ and $\beta\overline{c}_2 = \varepsilon_i\overline{r}_2$ are in $\varepsilon_i\overline{R}$ which is uniserial, $\alpha\overline{c}_1\overline{R}$ and $\beta\overline{c}_2\overline{R}$ are comparable. We may assume that $\alpha\overline{c}_1\overline{R} \subseteq \beta\overline{c}_2\overline{R}$. Then $\alpha\overline{c}_1 = \beta\overline{c}_2\overline{x}$ for some $\overline{x} \in \overline{R}$. Hence $\alpha = \beta\overline{c}_2\overline{x}\,\overline{c}_1^{-1}$, and therefore $\alpha Q \subseteq \beta Q$. □)

Our main objective is to describe the maximal ideals of the localized ring Q and obtain the Jacobson radical $J(Q)$.

To fulfill our objective, we discuss the (right) ideals of the localized ring R_ζ for an *arbitrary ring* R with respect to an *arbitrary regular Ore set* ζ. As before we consider R as a subring of $R_\zeta(=_\zeta R)$.

NOTE: Let I be a right ideal of R. Then $IR_\zeta = \{xc^{-1}/x \in I \text{ and } c \in \zeta\}$.

Proof: Take any $\sum_{i=1}^{n} x_i s_i \in IR_\zeta$, where $x_i \in I$ and $s_i \in R_\zeta$. Then $s_i = r_i c_i^{-1}$ for some $r_i \in R$ and $c_i \in \zeta$. Thus $\sum x_i s_i = \sum x_i r_i c_i^{-1} = \sum y_i c_i^{-1}$ for some $y_i \in I$. It is enough to prove that $y_1 c_1^{-1} + y_2 c_2^{-1} = yc^{-1}$, for some $y \in I$ and $c \in \zeta$. By the right Ore condition, there exist $r \in R$ and $d \in \zeta$ such that $c_1 d = c_2 r$. Since $c_2 r = c_1 d \in \zeta$, $c_2 r$ is invertible in R_ζ. Therefore $y_1 c_1^{-1} + y_2 c_2^{-1} = y_1 d(c_1 d)^{-1} + y_2 r(c_2 r)^{-1} = (y_1 d + y_2 r)(c_1 d)^{-1}$. The reverse inclusion is a triviality. □

DEFINITION: Let I be a right ideal of R. Then we define *right closure* of I to be

$$rt - cl(I) = \{r \in R/rc \in I \; for \; some \; c \in \zeta\}.$$

I is said to be *right closed* if $I = rt - cl(I)$.

Similar definition for *left closure* and *left closed.*

An ideal I is said to be *2-closed* if is left closed and right closed.

Before we relate the (right) ideals of R and (right) ideals of R_ζ, let us first state the following result, which will be used later.

LEMMA 4.3. If I is 2-closed ideal, then $IR_\zeta = R_\zeta I$.

Proof: If $sx \in R_\zeta I$; $s \in R_\zeta$ and $x \in I$, then since $s = c^{-1}r$, $sx = c^{-1}(rx) = y_1 c_1^{-1}$where $(rx)c_1 = cy_1$ by the right Ore condition. Since $cy_1 = rxc_1 \in I$, we get $y_1 \in lt - cl(I) = I$. Thus, $sx = y_1 c_1^{-1} \in IR_\zeta$ and therefore $R_\zeta I \subseteq IR_\zeta$ and the symmetry gives the equality. □

LEMMA 4.4.

(i) There is one to one correspondence between the right closed right ideals of R, and the right ideals of R_ζ, given as follows: for any right closed right ideal I, IR_ζ is a right ideal of R_ζ, and conversely, for any right ideal A of R, $A \cap R$ is right closed right ideal of R.

(ii) There is one to one correspondence between 2-closed ideals of R and the ideals of R_ζ given as follows:

if an ideal I is 2-closed, then $IR_\zeta = R_\zeta I$ is an ideal of R_ζ, and conversely, if A is an ideal of R_ζ, then $A \cap R$ is 2-closed ideal of R.

Proof: (i) To prove that $A \cap R$ is right closed, first observe that $A \cap R \subseteq$

$rt - cl(A \cap R)$. Choose any $x \in rt - cl(A \cap R)$. Then $xc \in A \cap R$ for some $c \in \zeta$. Hence $x = (xc)c^{-1} \in A$ and therefore $rt - cl(A \cap R) \subseteq A \cap R$.

One to one correspondence is given by the following:

(a) $IR_\zeta \cap R = rt - cl(I)$

(b) $(A \cap R)R_\zeta = A$

To obtain (a), choose $r \in IR_\zeta \cap R$. Then $r = xc^{-1}$ for some $x \in I$ and $c \in \zeta$. Thus $rc = x \in I$ and therefore $r \in rt - cl(I)$. Conversely, if $r \in rt - cl(I)$, then there exists some $c \in \zeta$ with $rc \in I$. Thus $r = (rc)c^{-1} \in IR_\zeta \cap R$; we get $rt - cl(I) \subseteq IR_\zeta \cap R$.

Take any $rc^{-1} \in A$. Then $r = (rc^{-1})c \in A \cap R$. Thus $rc^{-1} \in (A \cap R)R_\zeta$, we have $A \subseteq (A \cap R)R_\zeta$. The converse inclusion follows, as A is a right ideal of R_ζ.

To prove (ii), first note that by Lemma 4.3, IR_ζ is an ideal of R_ζ. Part (i) and symmetry gives that $A \cap R$ is 2-closed ideal of R, and the one to one correspondence as required. □

LEMMA 4.5. The maximal 2-closed ideals are precisely the ideals I which are maximal with respect to $I \cap \zeta = \emptyset$.

Proof: Let I be an ideal maximal with respect to $I \cap \zeta = \emptyset$. If I is not right closed, then since rt-cl(I) is an ideal, $rt - cl(I) \cap \zeta \neq \emptyset$. Take $c \in rt - cl(I) \cap \zeta$. Then for some $c' \in \zeta$, $cc' \in I \cap \zeta$. This is a contradiction. We get a similar contradiction if I is not left closed.

Suppose L is 2-closed ideal such that $I \subset L \subseteq R$. Then by the maximality of I, $L \cap \zeta \neq \emptyset$. Take $c_1 \in L \cap \zeta$. Then $1 \in rt-cl(L) = L$, and therefore $L = R$. Thus I is maximal 2-closed.

Conversely let I be a maximal 2-closed, proper ideal of R. Then $I \cap \zeta = \emptyset$. (If $c \in I \cap \zeta$, then $1 \in rt-cl(I) = I$, and hence the contradiction, $I = R$.) Suppose I is not maximal with the property $I \cap \zeta = \emptyset$. Consider the set $\Im = \{L/\ I \subseteq L \trianglelefteq R\ and\ L \cap \zeta = \emptyset\}$. As $I \in \Im$, $\Im$ is non-empty and every chain L_α has an upper bound $\cup L_\alpha$. As I is not maximal in $\Im$, by Zorn's Lemma we pick a maximal element $L \in \Im$ such that $I \subset L$. This L is not 2-closed, as I is maximal 2-closed. Say, L is not right closed (ie.$L \subset rt-cl(L)$). Then $rt-cl(L) \cap \zeta \neq \emptyset$, by the maximality of L. Take $c \in rt-cl(L) \cap \zeta$. Then for some $c' \in \zeta$, $cc' \in L \cap \zeta = \emptyset$, a contradiction. Therefore I is maximal with respect to $I \cap \zeta = \emptyset$. □

LEMMA 4.6. If I_α is a family of 2-closed ideals of R, then $\cap I_\alpha R_\zeta = (\cap I_\alpha)R_\zeta$.

Proof : $(\cap I_\alpha R_\zeta) \cap R = \cap(I_\alpha R_\zeta \cap R) = \cap rt-cl(I_\alpha) = \cap I_\alpha$. (The second equality follows from (a), given in the proof of Lemma 4.4, and the last equality is because I_α is 2-closed.) Therefore $(\cap I_\alpha)R_\zeta = [(\cap I_\alpha R_\zeta) \cap R]R_\zeta = \cap I_\alpha R_\zeta$. (The last equality is from (b), again given in the proof of Lemma 4.4.) □

Now let us go back to our previous case that R is serial, $\overline{R} = R/K$, $\overline{\zeta} = (\zeta+K)/K$ where $\zeta = \zeta(T)$ and $Q = \overline{R}_{\overline{\zeta}}$. Recall that $T = \cap_{i=1}^{t} P_i$. Also let $\overline{r}$ denotes the image of $r \in R$ under the canonical projection $R \to \overline{R}$.

Since for any ideal $K \subseteq I$ we have the isomorphism $\overline{R}/\overline{I} \to R/I$ given by $\overline{r}+\overline{I} \longmapsto r+I$, we see that $r \in \zeta(I)$ if and only if $\overline{r} \in \zeta(\overline{I})$, and therefore $\overline{\zeta(I)} = (\zeta(I)+K)/K = \zeta(\overline{I})$.

LEMMA 4.7. $\overline{P}_1, \ldots, \overline{P}_t$ are precisely the ideals which are maximal with respect to the property; $\overline{P}_i \cap \overline{\zeta} = \emptyset$.

Proof : Suppose $\overline{x} \in \overline{P}_i \cap \overline{\zeta}$. Then this leads to the contradiction, $x \in P_i \cap \zeta \subseteq P_i \cap \zeta(P_i) = \emptyset$.(Note that $\zeta = \zeta(T) = \cap_{i=1}^{t} \zeta(P_i)$.) Hence $\overline{P}_i \cap \overline{\zeta} = \emptyset$.

Now let us prove that $\overline{P}_i$ is maximal with respect to $\overline{P_i} \cap \overline{\zeta} = \emptyset$. Let i be fixed. Suppose there exists $\overline{M}$ such that $\overline{P}_i \subset \overline{M} \subseteq \overline{R}$ and $\overline{M} \cap \overline{\zeta} = \emptyset$. Then $\overline{M}/\overline{P}_i$ is a non-zero ideal of the prime Goldie ring $\overline{R}/\overline{P}_i$. Hence $\overline{M}/\overline{P}_i$ is an essential right ideal and hence by Goldie Theorem it contains a regular element $\overline{c}_i + \overline{P}_i$ of $\overline{R}/\overline{P}_i$. Hence $c_i + P_i$ is regular in R/P_i and therefore $c_i \in \zeta(P_i) \cap M$. For $j \neq i$, $(\overline{M} + \overline{P}_j)/\overline{P}_j$ is non-zero (because, if it is zero, then $P_i \subset M \subseteq P_j$ contradicts the incomparability of P_i and P_j). Therefore as before $(\overline{M} + \overline{P}_j)/\overline{P}_j$ contains a regular element $\overline{c}_j + \overline{P}_j$ where $c_j \in M \cap \zeta(P_j)$ for $1 \leq j \leq t$ and $j \neq i$. Now by Lemma 2.32, there exist r_k such that $\sum_{k=1}^{t} c_k r_k \in \zeta$. As $\sum_{k=1}^{t} c_k r_k \in M$, we get $M \cap \zeta \neq \emptyset$, and thus the contradiction

$\overline{M} \cap \overline{\zeta} \neq \emptyset$. Therefore for each i, $\overline{P}_i$ must be maximal with respect to $\overline{P}_i \cap \overline{\zeta} = \emptyset$.

Take any ideal $\overline{M}$ of $\overline{R}$ such that $\overline{M}$ is maximal with respect to the property $\overline{M} \cap \overline{\zeta} = \emptyset$. First let us show that $\overline{M}$ is prime. If this is not true, then there exist ideals $\overline{A}, \overline{B}$ such that $\overline{AB} \subseteq \overline{M} \subset \overline{A}, \overline{B}$. By the maximality of $\overline{M}, \overline{A} \cap \overline{\zeta}$ and $\overline{B} \cap \overline{\zeta}$ are non-empty. Choose $\overline{c}_1 \in \overline{A} \cap \overline{\zeta}$ and $\overline{c}_2 \in \overline{B} \cap \overline{\zeta}$. Then $\overline{c}_1\overline{c}_2 \in \overline{AB} \cap \overline{\zeta} \subseteq \overline{M} \cap \overline{\zeta} = \emptyset$, contradiction. Therefore $\overline{M}$ is prime.

Suppose $\overline{M}$ is different from all $\overline{P}_j$. Then $\overline{P}_j \subset \overline{M} + \overline{P}_j$ and therefore as before $(\overline{M}+\overline{P}_j)/\overline{P}_j$ contains a regular element $\overline{c}_j+\overline{P}_j$ where $c_j \in M \cap \zeta(P_j)$. Thus there exist r_k such that $c = \sum_{i=1}^{t} c_k r_k \in \zeta$. Since $c \in M$, we get the contradiction $\overline{c} \in \overline{M} \cap \overline{\zeta} = \emptyset$. Therefore there exists $\overline{P}_i$ such that $\overline{M} = \overline{P}_i$. □

We immediately conclude that $\overline{P}_1, \ldots, \overline{P}_t$ are precisely the maximal 2-closed ideals of $\overline{R}$ (Lemma 4.5). Therefore by the one to one correspondence given in Lemma 4.4 we get,

LEMMA 4.8. $\overline{P}_1 Q, \ldots, \overline{P}_t Q$ are precisely the maximal ideals of Q.

PROPOSITION 4.9. $J(Q) = T/K$.

Proof : Since Q is serial, it is semilocal. Hence the Jacobson radical is the intersection of the maximal ideals. Thus $J(Q) = \cap_{i=1}^{t}(\overline{P}_i Q) = (\cap_{i=1}^{t}\overline{P}_i)Q = \overline{T}Q$ (by Lemma 4.6 and 4.8). We claim that $\overline{T}Q = \overline{T}$. To see this, take any element of the form

$\bar{t}\,\bar{c}^{-1} \in \overline{T}Q$. As $T = Tc$, there exists $t_1 \in T$ such that $t = t_1c$. Thus $\bar{t}\bar{c}^{-1} = \bar{t}_1\overline{cc}^{-1} \in \overline{T}$. As any element of $\overline{T}Q$ is a sum of elements of the form $\bar{t}\bar{c}^{-1}$, we conclude $\overline{T}Q \subseteq \overline{T}$. Hence $\overline{T}Q = \overline{T}$ and $J(Q) = \overline{T}$ as required. □

CHAPTER 5

MAIN THEOREM

In this chapter we give a new proof of a Theorem due to Müller and Singh.([MS2] Theorem 2).

DEFINITION: Let M be a uniform module. M has an *assassinator* P, if there is a non-zero submodule N of M such that $P = ann_R N'$ for all $0 \neq N' \subseteq N$.

REMARK: It is known that the assassinator P of a uniform module M is prime, and is unique maximal among the annihilators of non-zero submodules.

DEFINITION: A uniform module M is *P-tame* if M has assassinator, P, which is right Goldie, and $ann_M P$ is a nonsingular (=torsionfree) $R/P-$module.

PROPOSITION 5.1. Every essential extension of a $P-$tame module is $P-$tame.

Proof: Suppose M is $P-$tame and $M \subseteq' N$. Then N is uniform and has the assassinator P. Since $ann_N P$ is a uniform module containing the torsion free $R/P-$submodule $ann_M P$,we obtain that $ann_N P$ itself a torsion free (=nonsingular) $R/P-$module, by Lemma 2.33. □

Let us recall that if R serial and $T = \cap_{i=1}^{t} P_i$, where the P_i are incomparable Goldie primes such that T does not contain any non-zero idempotents, then T is semiprime Goldie and $\zeta := \zeta(T)$ is an Ore set. We defined $K = \{r \in R/c_1 r c_2 = 0 \; for \; some \; c_1, c_2 \in \zeta\}$ and showed that K is an ideal of R. Analogously, using the Ore condition we can show that $K' := \{r \in R/rc = 0 \; for \; some \; c \in \zeta\}$ and $'K = \{r \in R/cr = 0 \; for \; some \; c \in \zeta\}$ are ideals of R.

MAIN THEOREM. ([MS2] Theorem 2) Let M be a uniform $P-tame$ module over a serial ring, and let T be the intersection of the clique of P. Then $ann_M(\cap_{n \in \mathbb{N}} T^n)$ is uniserial.

Proof: Let E be the injective hull of M. Then E is $P-tame$ by Proposition 5.1. By Proposition 2.34, $ann_E(\cap T^n)$ is an injective $R/\cap T^n-$module. Our aim is to show that $ann_E(\cap T^n)$ is uniserial as an $R/\cap T^n-$module, which implies that $ann_M(\cap T^n)$ is uniserial as an $R/\cap T^n-$module, and hence as an $R-$module.

P is Goldie, and every prime linked to a Goldie prime, is Goldie, by Proposition

3.10. Since in a serial ring, any set of incomparable primes is finite (Proposition 3.6), the clique of P is finite. Therefore T is Goldie semiprime as a finite intersection of Goldie primes.

Recall that, links are preserved in the factor ring $R/\cap T^n$ (cf. proof of Proposition 3.15), and there we also proved that $R/\cap T^n$ is indecomposable.

Without loss of generality, we may assume that $\cap T^n = 0$.

Claim (i): $\zeta = \zeta(T)$ is a left and a right Ore set in R, and T is $\zeta-$divisible.

Proof: Note that the semiprime Goldie ideal T does not contain any non-zero idempotent. (if $0 \neq e^2 = e \in T$, then $e \in T^n$ for all n, hence $0 \neq e \in \cap T^n = 0$, a contradiction.) Thus we may apply (Lemma 3.12), with $N = T$.

Claim (ii): E is $\zeta-$torsionfree.

Proof: Suppose $t_\zeta(E) \neq 0$. Take any $0 \neq x \in t_\zeta(E)$. Then there exists $c \in \zeta$ such that $xc = 0$. Since $ann_E P \subseteq' E$, there exists $r \in R$ such that $0 \neq xr \in ann_E P$. By the Ore condition, there exist $r' \in R$ and $c' \in \zeta$ with $rc' = cr'$. Thus $0 = xcr' = xrc' = xr(c' + P)$ and $c' \in \zeta \subseteq \zeta(P)$ (Lemma 2.32). Since E is $P-$tame, $ann_E P$ is torsionfree as an $R/P-$module. Thus we get the contradiction $xr = 0$. We conclude $t_\zeta(E) = 0$, and E is $\zeta-$ torsionfree.

By (Proposition 3.15), either $0 = \cap T^n$ is Goldie prime, or T is nilpotent.

Claim (iii): If $0 = \cap T^n$ is prime, then $K = 0$.

Proof: Take any $k \in K$. Then $c_1 k c_2 = 0$ for some $c_i \in \zeta$, $(i = 1, 2)$. Hence $0 = Tc_1kc_2 = Tkc_2$ by the divisibility of T. Thus, $T = 0$ or $kc_2 = 0$, since 0 is prime.

If $T = 0$, then ζ is the set of regular elements in R. This forces $K = 0$ as required, by the definition of K. Therefore suppose $T \neq 0$. Then $kc_2 = 0$ gives $0 = kc_2T = kT$ again by the divisibility. As $T \neq 0$, $k = 0$. Hence $K = 0$.

Claim (iv): If T is nilpotent, then $K = K' = {}'K$

Proof: First we prove that $K = K'$. Suppose $K \neq K'$. Then there exists m such that KT^m is not contained in K', but $KT^{m+1} \subseteq K'$. Then the R/K'−module $X = (KT^m + K')/K'$ is non-zero. Notice that $TX = 0 = XT$. ($XT = 0$, since $KT^{m+1} \subseteq K'$. To see $TX = 0$, take any $t \in T$, $x \in X$. Then $x = \sum k_it_i + K'$, where $k_i \in K$ and $t_i \in T^m$. By definition there exist $c_i, c_i' \in \zeta$ such that $c_ik_ic_i' = 0$. By the ζ divisibility of T, $t = t_i'c_i$ and $t_i = c_i't_i''$ for some $t_i', t_i'' \in T$. Thus $t(\sum k_it_i) = \sum t_i'c_ik_ic_i't_i'' = 0$ implies $tx = 0$.) Therefore X is R/T−bimodule.

Let S be the Goldie quotient ring of the semiprime Goldie ring R/T. By Goldie's Theorem, the ring S is semisimple. We claim that X has following properties;

(a) As a left R/T−module, X is ζ− divisible and torsion.

(b) As a right R/T−module, X is ζ− divisible and torsionfree.

(c) X is a finitely generated semisimple right S−module, to say $X = \sum_{i=1}^{l} \oplus x_iS$ for some $x_i \in X$.

Proof: (a) To show the divisibility, it is enough to show that $K = cK$ for any $c \in \zeta$. To see this, take any $k \in K$. Then there exist $c_1, c_2 \in \zeta$ such that $c_1kc_2 = 0$. As $K \subseteq T = cT$, $k = ct$ for some $t \in T$. Thus $0 = c_1(ct)c_2 = (c_1c)tc_2$ and $c_1c \in \zeta$ implies $t \in K$. Hence $k = ct \in cK$ for any $k \in K$. Therefore $K \subseteq cK$.

Take any $x \in X$. Then $x = \sum k_i t_i + K'$ for some $k_i \in K,\ t_i \in T^m$. Let $c_i, c_i' \in \zeta$ with $c_i k_i c_i' = 0$. By the divisibility, there exist $t_i' \in T$ such that $t_i = c_i' t_i'$. Thus $c_i k_i t_i = c_i k_i c_i' t_i' = 0$. By the left Ore condition, there is a common multiple $c = d_i c_i$ for some $d_i \in R$ and $c \in \zeta$. Therefore $0 = \sum d_i c_i k_i t_i = \sum c k_i t_i = c(\sum k_i t_i)$ implies $cx = 0$, ie.X is left torsion.

(b) Take any $x \in t_{R/T}(X) = \{x \in X / xc = 0 \ for \ some \ c \in \zeta\}$, say, $x = \sum k_i t_i + K'; k_i \in K, t_i \in T^m$. Then $xc = 0$ for some $c \in \zeta$ implies $(\sum k_i t_i)c \in K'$. By the definition of K', $(\sum k_i t_i)cc' = 0$ for some $c' \in \zeta$ and therefore $\sum k_i t_i \in K'$, concluding $x = 0$.

To show the divisibility, take any $c \in \zeta$ and $x = \sum k_i t_i + K' \in X$. Each $t_i \in T^m$ is a sum of the elements of the form $t_{i1} \dots t_{im}$, $t_{ij} \in T$. Since each $t_{im} = y_{im} c$ for some $y_{im} \in T$ we get $t_i = t_i' c$ for some $t_i' \in T^m$, and therefore $x = \sum k_i t_i' c + K' = (\sum k_i t_i' + K')c \in Xc$. Hence $X = Xc$.

(c) Since X is a torsionfree divisible right $R/T-$module, X is a right $S-$module (cf.[GW] Proposition 6.13(b)), and therefore semisimple.

As a right ideal of the serial ring R/K', X has finite right Goldie dimension. Therefore, as an $R-$module, hence as an $R/T-$module, and then as an $S-$module, X has finite right Goldie dimension. Therefore X is a finite direct sum of simple $S-$modules. Say $X = \sum_{i=1}^{l} \oplus X_i$, and $X_i = x_i Q$ for $0 \neq x_i \in X_i$.

As X is left torsion, there exist $c_i \in \zeta$ such that $c_i x_i = 0$. As before there exist $d_i \in R,\ c \in \zeta$ with $d_i c_i = c$. Thus $0 = d_i c_i x_i = c x_i$ for all i. Hence $cX = 0$. On the other hand, $X = cX$ by (a). This is a contradiction as $X \neq 0$. Thus the assumption $K \neq K'$ leads to a contradiction, and we conclude $K = K'$. We prove $K = {}'K$ analogously.

We have now shown that, in both cases (iii) and (iv), ζ is a denominator set.

Recall that, in Chapter 4, we showed that $\overline{\zeta} = (\zeta + K)/K$ consists of regular elements of $\overline{R} := R/K$, and $\overline{\zeta}$ is an Ore set. Moreover $J(Q) = T/K$, where $Q := \overline{R}_{\overline{\zeta}}$.

If T is nilpotent, say, $T^m = 0$, then $J(Q)^m = 0$. Hence the ring Q is Artinian. If $0 = \cap T^n$, then since $K = 0$, we have $\cap J(Q)^n = \cap T^n = 0$. Hence Q is Noetherian for this case, by seriality of Q and Proposition 2.23. Consequently, in any case, Q is a Noetherian ring.

Claim (v): $E = E_{\overline{\zeta}}$ is an injective $Q-$module.

Proof: We first observe that E is $\zeta-$ divisible. For if $x \in E$ and $c \in \zeta$, then $r_R(c) \subseteq r_R(x)$, because $cr = 0$ implies, $rc' = 0$, hence $xrc' = 0$, and thus $xr = 0$, since ζ is right reversible and E is $\zeta-$ torsionfree. Hence there is a homomorphism from cR to E sending c to x. By injectivity, this map extends $R \rightarrow E$. Take $y \in E$ with $1 \longmapsto y$. Then we have $yc = x$.

Note that $EK = 0$. (To prove this take any $x \in E$ and $k \in K$. Then as $K = {}'K$, $ck = 0$ for some $c \in \zeta$. By the divisibility, $x = yc$ for some $y \in E$. Hence $xk = yck = 0$.) Therefore E is an injective $R/K-$module (Proposition 2.34). Since the kernel $t_\zeta(E)$ of the map $E \to E_{\overline{\zeta}}$ is zero. (by claim (ii)), we may consider E as a submodule of $E_{\overline{\zeta}}$. To show $E = E_{\overline{\zeta}}$, take any $x\overline{c}^{-1} \in E_{\overline{\zeta}}$; $(x \in E, \overline{c} \in \overline{\zeta})$. Since $x = yc = y\overline{c}$ for some $y \in E$, we have $x\overline{c}^{-1} = (y\overline{c})\overline{c}^{-1} \in E$. Hence $E_{\overline{\zeta}} \subseteq E$. Since E is $\overline{\zeta}$−torsionfree injective R/K−module, we get, as Q−module, $E_{\overline{\zeta}}$ is injective (cf. [GW] Corollary 9.16(b)).

Since E_R is uniform, E is indecomposable as an R−module, and hence as an $\overline{R} = R/K$−module. Since every Q−submodule of E is also an $\overline{R}$−submodule of E, E_Q is also indecomposable. Thus we obtain that E is an indecomposable injective module over the Noetherian serial ring Q. Thus, E is uniserial as an Q−module by Proposition 2.24.

We use the notations $\overline{R} = R/K$, $\overline{T} = T/K$ and $\overline{r}$ to the image of $r \in R$ under the canonical projection $R \to \overline{R}$. Now our aim is to show that $\overline{R}/\overline{T}$ is essential in $Q/\overline{T}$ as right $\overline{R}/\overline{T}$−module. To show this, take any $0 \neq q + \overline{T} \in Q/\overline{T}$: then $q = \overline{r}\,\overline{c}^{-1}$ for some $\overline{r} \in \overline{R}$ and $\overline{c} \in \overline{\zeta}$. Then $q\overline{c} = \overline{r}$ is non-zero in $\overline{R}$. If $(q + \overline{T})(\overline{c} + \overline{T}) = \overline{r} + \overline{T} \neq 0$, then we are done. Suppose $\overline{r} + \overline{T} = 0$. Then $0 = \overline{r} + \overline{T} = (q + \overline{T})(\overline{c} + \overline{T})$ implies $q + \overline{T} = 0$, since $\overline{c} \in \zeta(\overline{T})$. Thus $q \in \overline{T}$,a contradiction.

Consequently, Theorem 2.22 applies with $S = Q$, and we obtain that E is uniserial as an $\overline{R}$−module, and therefore as an R−module. This completes the proof. □

NOTATIONS

$A \subseteq B$	A is a submodule of B
$A \subset B$	A is a proper submodule of B
$A \subseteq' B$	A is an essential submodule of B
$I \trianglelefteq R$	I is an ideal of R
$\zeta(I)$	elements in R which are regular modulo I
J	Jacobson radical of the ring R
$J(R)$	Jacobson radical of the ring S, when the ring has to be specified.
Z	right singular ideal of R.
$Z(M)$	singular submodule of M
$r_R(X), l_R(X)$	right and left annihilator in R respectively, of the set X
$E(M)$	injective hull of M

REFERENCES

[AF] Anderson, Frank W., and Fuller, Kent R. Rings and Categories of Modules, Second edition, Graduate Texts in Mathematics, Vol 13, Springer-Verlag, 1992.

[C] Chatters, A.W. Serial Rings with Krull Dimension, Glassglow Math.J. 32 (1990),71-78.

[CH] Chatters, A.W., and Hajarnavis,C.R. Rings with chain conditions, Re search Notes in Mathematics 44, Pitman(1980).

[EG1] Eisenbud, D. and Griffith, P. Serial rings, J. Algebra 17(1971)389-400.

[EG2] Eisenbud, D. and Griffith, P. The structure of serial rings, Pacific J. Math. 36(1971)109-121.

[GW] Goodearl, K.R. and Warfield, R.B. An Introduction to Noncommutative Noetherian Rings, London Mathematical Society, Student Texts16.

[H] Hungerford, Thomas W. Algebra, Graduate texts in Mathematics, Vol.73, Springer-Verlag , New York-Heidelberg-Berlin,1989.

[I] Ivanov, G. Left generalized uniserial rings, J. Algebra, 31(1974)166-181.

[J] Jategaonkar, A.V. The Torsion Theory at a Semi-Prime Ideal, An. Acad. Brasil Ci.,45(1973)197-200.

[K1] Kupisch, H. Beiträge zur Theorie nichthalbeinfacher Ringe mit Mini malbedingung, J. Reine Angew. Math. 201(1959),100-112

[K2] Kupisch, H. *Ü*ber eine Klasse von Ringen mit Minimalbedingung I, Arch. Math. 17(1966),20-35.

[L] Lam, T.Y. A first Course in Noncommutative Rings,Graduate Texts in Mathematics, Vol.131,Springer-Verlag,1991.

[M1] Müller, B.J. On Semiperfect Rings, Illinois J. Math.14(1970),464-467.

[M2] Müller, B.J. The Structure of Serial Rings, Methods in module theory, Marcel Dekker, Inc.(1992)pp 249-270.

[M3] Müller, B.J. Goldie-Prime Serial Rings, Contemporary Mathematics 130(1992),301-310.

[MS1] Müller, B.J. and Singh, Surjeet Uniform Modules over Serial Rings, J.Algebra 144(1991),94-109.

[MS2] Müller, B.J. and Singh, Surjeet Uniform Modules over Serial Rings II, Proc. Conf.on Rings and Modules, Athens, Ohio,1989, Lecture Notes in Math. 1448, Springer Verlag.1990,25-32

[Mu1] Murase, I. On the structure of generalized uniserial rings I, Sci. Pap. Coll.Gen. Educ., Univ. Tokyo, 13(1963),1-22.

[Mu2] Murase, I. On the structure of generalized uniserial rings II, Sci. Pap. Coll. Gen. Educ., Univ. Tokyo, 13(1963),131-158.

[Mu3] Murase, I. On the structure of generalized uniserial rings III, Sci. Pap. Coll. Gen. Educ., Univ. Tokyo, 14(1964),12-25.

[N1] Nakayama, T. Note on uni-serial and generalized uni-serial rings, Proc. Imp. Acad. Tokyo, 16(1940),285-289.

[N2] Nakayama, T. On Frobeniusean algebra II, Ann. of Math., 42(1941),1-21.

[S] Singh, Surjeet Serial Right Noetherian rings, Can. J. Math 36(1984),22-37.

[U] Upham, Mary H. Serial rings with right Krull dimension one, J. Algebra 109(1987),319-333.

[W] Warfield, R.B. Serial rings and finitely presented modules, J. Algebra 37(1975),187-222.

[W1] Wright, Mary H. Serial rings with right Krull dimension one II, J. Algebra 117(1988)99-116.

[W2] Wright, Mary H. Krull dimension in serial rings, J. Algebra 124(1989),317-328.

[W3] Wright, Mary H. Certain uniform modules over serial rings are uniserial. Commun. in Alg., 17(2),(1989)441-469.

[W4] Wright, Mary H. Prime serial rings with Krull dimension, Commun. in Alg., 18(8),(1990)2559-2572.