ABELIAN GROUPS

IN

A TOPOS OF SHEAVES ON A LOCALE

By

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ABSTRACT

This thesis is devoted to the study of Abelian Groups in the topos $Sh\mathcal{L}$ of sheaves on a locale $\mathcal{L}$. The main topics considered are: injectivity, essential extensions of torsion groups, divisibility, purity, internal hom-functor, tensor product and flatness.

We derive some general results about these notions. Also, we prove the Baer Criterion for injectivity in $AbSh\mathcal{L}$. For a well-ordered locale $\mathcal{L}$, we describe the injective hulls in $AbSh\mathcal{L}$ and for some special locales we characterize the injectives in $AbSh\mathcal{L}$.

We further discuss essential extensions of torsion groups and show amongst other things, that a first countable Hausdorff space X is discrete iff essential extensions in $AbShX$ preserve torsion.

Divisible groups are characterized here as absolutely pure groups. We discuss the internal adjointness between the tensor product and the internal hom-functor.

Finally, we consider the notion of flatness, and show that the flat groups in $AbSh\mathcal{L}$ are characterized the same way as in Ab, that is, flat = torsion free, and that A is flat in $AbSh\mathcal{L}$ iff $A^* = [A,P]$ is an injective group, where P is an injective cogenerator.

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To

My Husband, Ravender

and

My Son, Navin

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INTRODUCTION

This thesis deals with certain notions of Abelian group theory in the topos $Sh\mathcal{L}$ of sheaves on a locale $\mathcal{L}$. In particular, we are interested to see whether some well-known results concerning the category $Ab \cong AbSh2$ of abelian groups in the usual category Ens of sets hold in $AbSh\mathcal{L}$, for an arbitrary $\mathcal{L}$. The main topics considered here are: injectivity, essential extensions of torsion groups, Divisibility, Purity, Internal Hom-Functor, The tensor product and flatness.

Work in this direction was done earlier by B. Banaschewski, and indeed this thesis is motivated by his paper, "When are divisible abelian groups injective" [4] and the seminar talk on "Recovering a space from its abelian sheaves" [6]. The characterization of injectives for $\mathcal{L} = 3$, and the characterization of the locales for which divisible = injective in [4], is crucial to some of the results in the thesis as we shall see later.

We now sketch the process of development within the thesis.

Chapter 0 describes the special types of locales which we shall encounter, and deals with some background material required from sheaf theory and Abelian categories.

Chapter 1 deals with injectivity in $\mathrm{AbSh}\,\mathcal{L}$ where we first prove in proposition 1.1, the analogue of the Baer criterion for injectivity. We show injectivity is a local property. This was also shown by R. Harting [15] but by an entirely different method. She considers the preservation of maximal partial morphisms by the restriction functors R_U for $U \in \mathcal{L}$, where as, our approach uses the preservation of essentialness by the functors R_U.

For any injective $A \in \mathrm{AbSh}\,\mathcal{L}$ the characterization of injectives in AbSh3 [4], makes it easy to see that the restriction maps $AU \to AV$ are split epimorphisms between divisible groups in Ab. As a consequence of this, we describe in 1.6 the indecomposable injectives, and see that they are in one to one correspondence with the images of indecomposable injectives in Ab, under the local lattice homomorphisms from $\mathcal{L} \to 2$. We show that in general injectives in $\mathrm{AbSh}\,\mathcal{L}$, do not split into a direct sum of indecomposable injectives. For an inversely well-ordered locale, and for $\mathcal{L} = \mathcal{D}X$ where X is a discrete space, injectives do split into a direct sum of indecomposable injectives. Moreover, a sober T_1 space has this property iff X is discrete (1.9). The locales for which the torsion indecomposable injectives cogenerate $\mathrm{AbSh}\,\mathcal{L}$ will be characterized in 1.11, and they are exactly the spatial locales.

This is followed by a discussion on injective hulls, which is seen in 1.13, to be a local property, but not a

global one. However, the locales for which the global functor $A \rightsquigarrow AE$ preserves injective hulls will be characterized as exactly the finite, Boolean locales, that is the topologies of finite discrete spaces. For a well-ordered locale, we describe the injective hulls in $\mathrm{AbSh}\mathcal{L}$ in terms of injective hulls in Ab (1.15). Finally, we characterize in 1.21 and 1.23 the injectives in $\mathrm{AbSh}\mathcal{L}$ for the following locales:

1) $\mathcal{L}$ with descending chain conditions.

2) $\mathcal{L}$ inversely well-ordered.

thereby providing a partial solution to an open problem of Ebrahimi. ([14], page 68). As a consequence, we show that the direct sum of injectives in $\mathrm{AbSh}\mathcal{L}$ ($\mathcal{L}$ inversely well-ordered) is always injective. This does not hold for an arbitrary $\mathcal{L}$, and a counterexample is provided.

Chapter 2 studies the concept of torsion groups and their essential extensions. We show torsion is a local but not a global property (2.2). A number of properties of torsion groups are described which are the analogues of their counter parts in Ab . For example we see in (2.5), that every torsion group is a direct sum of its p-primary components, and A torsion implies that for all $B \in \mathrm{AbSh}\mathcal{L}$ [A,B] is reduced, that is, [A,B] has no non zero injective subgroups where [-,-] is the internal hom-functor of $\mathrm{AbSh}\mathcal{L}$. A counterexample showing that the torsion subgroup

of an injective group need not be injective is provided (2.14), and this holds in $\mathrm{AbSh}\mathcal{L}$ iff essential extensions preserve torsion (2.19). For any $\mathcal{L}$, if essential extensions preserve torsion in $\mathrm{AbSh}\mathcal{L}$, then the same holds for $\mathrm{AbSh}\uparrow U$ and $\mathrm{AbSh}\downarrow U$, for all $U \in \mathcal{L}$ (2.22). Finally, we see in 2.24 that a first countable Hausdorff space X has this property iff X is discrete.

Chapter 3 discusses divisibility and purity, where a number of results which hold for these notions in Ab are seen to be true in $\mathrm{AbSh}\mathcal{L}$. Divisible groups will be characterized as absolutely pure groups (3.18). A counter-example is provided to show that purity is not a local property (3.34). Using the known facts from (B. Banaschewski [6]) about pure subgroups of $Z_{\mathcal{L}}$, we characterize in 3.36 those locales for which the only pure and essential subgroup of $Z_{\mathcal{L}}$ is just $Z_{\mathcal{L}}$, as exactly the Boolean locales.

In Chapter 4, we show that the tensor product makes $\mathrm{AbSh}\mathcal{L}$ a closed monoidal category. We discuss a number of group theoretic properties of the Internal Hom-Functor and the Tensor Product which then lead to some open questions.

The last chapter concentrates on flatness in $\mathrm{AbSh}\mathcal{L}$, and flat groups will be characterized here the same way as they are in Ab, that is flat = torsion free (5.4). As a

consequence, we derive a number of corollaries and finally show that A is a flat group iff $A^* \doteq [A,P]$ (P an injective cogenerator) is an injective group (5.26), and A is a torsion group iff A^* is reduced (5.27).

In this thesis, we freely use the basic facts concerning our various topics which are available in the general literature. In particular, for Abelian group theory refer to [11], [12], for Abelian categories to [13], [19], for sheaf theory [9], [16] and [20], also [8] and [22] for general category theory.

We have used a single numbering throughout the thesis, where $m.n$ is the nth result of the mth chapter. The reference to Bibliography is enclosed in square brackets.

CHAPTER 0

PRELIMINARIES

Introduction: The purpose of this chapter is to provide the reader with a brief introduction to the categories of Presheaves and Sheaves on a locale, and to prove some new results which will be frequently used in this thesis.

In Section 1, we discuss the special types of locales where as in Section 2, we briefly review the notion of Abelian groups in any finitely complete category, and recall what is meant by torsion, torsion free and divisible abelian groups.

Section 3, concentrates on Presheaves and Sheaves of sets on any locale. Given a local lattice homomorphism, we discuss how a sheaf (Presheaf) on any one of the locales, produces in a natural way, a sheaf (Presheaf) on the other locale. A number of special local lattice homomorphisms will be considered. Finally, in Section 4, we study the category Ab$\mathbf{E}$, where $\mathbf{E} = \mathrm{Sh}\mathcal{L}$ or $\mathrm{PSh}\mathcal{L}$, and prove some results about some naturally arising functors for any element of the locale.

Section 1: Special Types of Locales

Definition 0.1: A locale, usually denoted by $\mathcal{L}$, is a complete lattice satisfying the following;

$$U \wedge \bigvee_{i \in I} U_i = \bigvee_{i \in I} U \wedge U_i$$

for all U, and any family $\{U_i\}_{i \in I}$ in $\mathcal{L}$. The zero (= bottom) of $\mathcal{L}$ will be denoted by 0, and the unit (= top) of $\mathcal{L}$ by E. A <u>morphism</u> of locales $h: \mathcal{L} \to \mathcal{M}$ (also called local lattice homomorphism) is a map which preserves arbitrary joins and finite meets (hence preserves the zero and the unit).

Any finite distributive lattice, any complete chain, or any complete Boolean algebra is an example of a locale. The other obvious example of a locale is the topology $\mathcal{O}X$ (that is, the lattice of open sets) of any topological space X with joins as unions and meets as intersections.

By the definition of continuity of maps between topological spaces, we get a contravariant functor $\mathcal{O}: \mathrm{TOP} \to \mathrm{LOC}$, where TOP is a category of topological spaces and continuous maps, and LOC is the category of locales and their morphisms. The functor $\mathcal{O}$ has an adjoint on the right, the contravariant functor $\Sigma: \mathrm{LOC} \to \mathrm{TOP}$, where $\Sigma\mathcal{L}$ is the space of completely prime filters F on $\mathcal{L}$, that is, F is a filter on $\mathcal{L}$ such that $\bigvee_{i \in I} U_i \in F$ for any family $\{U_i\}_{i \in I}$ in $\mathcal{L}$, implies $U_k \in F$ for some $k \in I$, and the sets $\Sigma_U = \{F \mid U \in F \in \Sigma\mathcal{L}\}$, $U \in \mathcal{L}$, form the open sets in this space. For any local lattice homomorphism $h: \mathcal{L} \to \mathcal{M}$, the corresponding continuous map

$\Sigma h: \Sigma\mathcal{M} \to \Sigma\mathcal{L}$ sends $F \to h^{-1}(F)$. The space $\Sigma\mathcal{L}$ is called the spectrum of $\mathcal{L}$.

<u>0.2</u>: A space X is called <u>sober</u> iff the adjunction $X \xrightarrow{V_X} \Sigma\mathcal{O}X$, $x \to \mathcal{O}(x)$ is a homeomorphism, where $\mathcal{O}(x)$ is the open neighbourhood filter of the point $x \in X$. This means, every completely prime filter F on $\mathcal{O}X$ is exactly the open neighbourhood filter of some unique point $x \in X$. Note that every Hausdorff space is sober but there are T_1 spaces which are not sober. For example, any infinite space with the cofinite topology.

<u>0.3</u>: A locale $\mathcal{L}$ is called <u>spatial</u> iff the adjunction $\mathcal{L} \xrightarrow{O_{\mathcal{L}}} \mathcal{O}(\Sigma\mathcal{L})$ is an isomorphism. Since $O_{\mathcal{L}}$ is always onto, a locale is spatial iff the completely prime filters separate points in $\mathcal{L}$. For more details refer to [5]. Note that any finite locale $\mathcal{L}$ is spatial by the fact that in any distributive lattice the prime filters separate the elements (Balbes-Dwinger, [17]), and for finite $\mathcal{L}$ the prime filters are completely prime. Also, any totally ordered locale is spatial since the $V > U$, $V \in \mathcal{L}$ form a completely prime filter on $\mathcal{L}$, for any $U \in \mathcal{L}$.

<u>0.4</u>: A locale is said to be <u>Noetherian</u> if every element of $\mathcal{L}$ is compact. This means that if $U = \bigvee_{i \in I} U_i$

is any cover of an element $U \in \mathcal{L}$, then there exists a finite subset $F \subseteq I$, such that $U = \bigvee_{i \in F} U_i$. This is equivalent to the Ascending Chain Condition on $\mathcal{L}$, that is every strictly ascending chain of elements in $\mathcal{L}$ terminates after a finite number of steps, or equivalently every non-empty subset of $\mathcal{L}$ has a maximal element. Note that any Noetherian locale $\mathcal{L}$ is spatial, because arbitrary joins are actually finite joins, and hence any prime filter is completely prime (see 0.3).

<u>0.5</u>: A locale $\mathcal{L}$ has <u>Descending Chain Condition</u> on its elements, iff every non-empty subset of $\mathcal{L}$ has a minimal element, or equivalently, every strictly descending chain of elements of $\mathcal{L}$, terminates after a finite number of steps.
Note that any $\mathcal{L}$ with DCC is spatial: If $U < V$ in $\mathcal{L}$ and $W \in \mathcal{L}$ is minimal such that $U < W \leq V$, then $F = \{S \mid S \in \mathcal{L}, U \vee S \geq W\}$ is a completely prime filter on $\mathcal{L}$ for which $U \notin F$, but $V \in F$.

<u>Remark</u>: A locale $\mathcal{L}$ satisfies both ACC and DCC iff $\mathcal{L}$ is finite. To prove the non-trivial implication ($\Rightarrow$), note that such an $\mathcal{L}$ is spatial, and if $\mathcal{L} \cong \mathcal{O}(X)$ and X is T_0, one has the following observations concerning X: Each $x \in X$ has a smallest open neighbourhood W_x, and for the partial order $\leq$ given, such that $x \leq y (x,y \in X)$

iff $\mathfrak{D}(x) \leq \mathfrak{D}(y)$ (and hence iff $W_y \leq W_x$), $W_x = \uparrow x = \{y \mid y \geq x\}$. Moreover, DCC for $\mathcal{L}$ then implies that $\uparrow x$ is finite, and since X is compact by ACC, X itself is finite. It follows that $\mathcal{L}$ is also finite.

<u>0.6</u>: A locale $\mathcal{L}$ is <u>Boolean</u> iff every element in $\mathcal{L}$ has a complement. This is equivalent to saying that $\mathcal{L}$ has no dense elements other than E. That is, there is no $W \neq E$ such that $U \wedge W = 0$ implies $U = 0$.
Note that a Boolean locale $\mathcal{L}$ is spatial iff $\mathcal{L}$ is atomic, that is, each element of $\mathcal{L}$ is the join of atoms (Balbes-Dwinger, [17]), the non-trivial implication since any completely prime filter in a Boolean $\mathcal{L}$ is a principal filter given by an atom.

<u>Section 2</u>: <u>Abelian Groups in a Category</u>

<u>Definition 0.7</u>: Let $\mathbf{E}$ be any finitely complete category, in particular, it has the terminal object 1 (= the product of the empty family of objects). Then an abelian group in $\mathbf{E}$ is an entity $A = (E,+,0,-)$, where E the underlying set of A (= $|A|$) is an object of $\mathbf{E}$, together with the maps

$$E \times E \xrightarrow{+} E, \qquad 1 \xrightarrow{0} E, \qquad E \xrightarrow{-} E$$

which make the following diagrams commute.

(i)
$$\begin{tikzcd} E\times E\times E \arrow[r, "(+)\times 1_E"] \arrow[d, "1_E\times(+)"'] & E\times E \arrow[d, "+"] \\ E\times E \arrow[r, "+"'] & E \end{tikzcd}$$

(ii)
$$\begin{tikzcd} E\times E \arrow[r, "1_{E\times E}"] \arrow[d, "\mathrm{flip}"'] & E\times E \arrow[d, "+"] \\ E\times E \arrow[r, "+"'] & E \end{tikzcd}$$

(iii)
$$\begin{tikzcd} E \arrow[r, "1_E\Pi(-)"] \arrow[dr, "\mathrm{unique}"'] & E\times E \arrow[r, "+"] & E \\ & 1 \arrow[ur, "0"'] & \end{tikzcd}$$

(iv)
$$\begin{tikzcd} E \arrow[r] \arrow[dr, "1_E"'] & 1\times E \arrow[r, "0\times 1_E"] & E\times E \arrow[dl, "+"] \\ & E & \end{tikzcd}$$

A homomorphism from an abelian group A to an abelian group B, is a map $h: |A| \to |B|$ which preserves these operations, that is, such that the following diagrams commute:

$$\begin{tikzcd} |A|\times|B| \arrow[r, "h\times h"] \arrow[d, "+"'] & |B|\times|B| \arrow[d, "+"] \\ |A| \arrow[r, "h"'] & |B| \end{tikzcd} \qquad \begin{tikzcd} & 1 \arrow[dl, "0"'] \arrow[dr, "0"] & \\ |A| \arrow[rr, "h"'] & & |B| \end{tikzcd} \qquad \begin{tikzcd} |A| \arrow[r, "h"] \arrow[d, "-"'] & |B| \arrow[d, "-"] \\ |A| \arrow[r, "h"'] & |B| \end{tikzcd}$$

The composition of homomorphisms and the identity morphism of any $|A| \in \mathbf{E}$ are homomorphisms, and thus one has a category $\mathrm{Ab}\mathbf{E}$, with objects as abelian groups in $\mathbf{E}$, and maps as homomorphisms between them.

In particular, if $\mathbf{E} = \mathbf{Ens}$, the category of sets and functions then $\mathrm{Ab}\mathbf{Ens}$ is simply the category Ab of abelian groups and homomorphisms between them.

Notation: For the sake of convenience we shall write A for the underlying set $|A| \in \mathbb{E}$.

<u>Definition 0.8</u>: For any $A \in \mathrm{Ab}\mathbb{E}$, and $0 \neq n \in N$,

(i) The diagonal map $\Delta_A : A \to A^n$ is the unique map such that $A \xrightarrow{\Delta_A} A^n \xrightarrow{p_i} A = 1$ for all $i = 1,2,\ldots n$, where $p_i : A^n \to A$ is the ith projection.

(ii) The sum $A^n \xrightarrow{+_A} A$ is the unique map such that $A \xrightarrow{q_i} A^n \xrightarrow{+_A} A = 1$ where $q_i : A \to A^n$ is the ith injection, and $i = 1,2,\ldots n$.

The composite $+_A\Delta_A : A \to A^n \to A$ is denoted by n_A and the kernel of n_A shall be denoted by $k_n : \mathrm{Ker}\, n_A \rightarrowtail A$.

<u>Definition 0.9</u>: For any $A \in \mathrm{Ab}\mathbb{E}$,

(i) A is called a <u>torsion free group</u> iff n_A is a monomorphism for all $0 \neq n \in N$.

(ii) A is called a <u>divisible group</u> iff n_A is an epimorphism for all $0 \neq n \in N$.

(iii) A is called a <u>torsion group</u> iff all k_n, $0 \neq n \in N$ are jointly epic, that is for any two homomorphisms f and g with domain A, if $fk_n = gk_n$ for all $0 \neq n \in N$, then $f = g$.

<u>Lemma 0.10</u>: If any subgroups $B_i \rightarrowtail A$ $(i \in I)$ have a join among all subgroups of A and if $\mathrm{Ab}\mathbb{E}$ has

co-equalizers, then $B_i \rightarrowtail A (i \in I)$ are jointly epic iff A is the join of all $B_i \rightarrowtail A$.

<u>Proof</u>: ($\Rightarrow$) Let $B \rightarrowtail A$ be the join of all $B_i \rightarrowtail A$. Consider the diagram $A \underset{0}{\overset{q}{\rightrightarrows}} A|B$ where q is the quotient map determined by $B \rightarrowtail A$. Then q and 0 co-equalize all $B_i \rightarrowtail A$, hence by hypothesis $q = 0$. Thus $B = A$.

($\Leftarrow$) Let $f,g: A \to C$ co-equalize $B_i \rightarrowtail A$ for all $i \in I$. Then there is a unique factorization $B_i \rightarrowtail Eq(f,g) \rightarrowtail A$ for all $i \in I$. Since A is the join of all $B_i \rightarrowtail A$, we get a factorization $A \rightarrowtail Eq(f,q) \rightarrowtail A$, hence $A = Eq(f,q)$. Thus $f = g$.

<u>Section 3</u>: <u>Presheaves and Sheaves on $\mathcal{L}$</u>

<u>0.11</u>: Since a locale $\mathcal{L}$ is a partially ordered set, it can be viewed as a category whose objects are the elements of $\mathcal{L}$, and there is a morphism $V \to U$ in $\mathcal{L}$ iff $V \leq U$. Then a presheaf of sets on $\mathcal{L}$ is a contravariant functor $B: \mathcal{L} \to Ens$, the category of sets and functions. Thus a presheaf assigns to each $U \in \mathcal{L}$ a set BU, and to each $V \leq U$ in $\mathcal{L}$, there is only one function $BU \xrightarrow{\rho_V^U} BV$ known as the restriction map, such that,

(i) $\rho_U^U = 1_{BU}$

(ii) if $V \le U \le W$, then
$$\begin{tikzcd} & BW \arrow[dl] \arrow[dr] & \\ BU \arrow[rr] & & BV \end{tikzcd}$$
commutes, that is

$$\rho_V^W = \rho_V^U \rho_U^W .$$

For any $b \in BU$, $\rho_V^U b$ is usually denoted by $b|V$.

A morphism of presheaves $h: B \to C$ is just a natural transformation between them, that is, a collection $\{h_U, U \in \mathcal{L}\}$ of functions $h_U: BU \to CU$, such that

$$\begin{tikzcd} BU \arrow[r, "h_U"] \arrow[d] & CU \arrow[d] \\ BV \arrow[r, "h_V"'] & CV \end{tikzcd}$$

commutes whenever $V \le U$ in $\mathcal{L}$. Then the presheaves on $\mathcal{L}$ as objects and natural transformations between them as morphisms, form a category denoted by $PSh\,\mathcal{L}$.

For any $B \in PSh\,\mathcal{L}$, and any cover $U = \bigvee_{i \in I} U_i$ of U in $\mathcal{L}$, we get a diagram

$$BU \xrightarrow{\alpha} \prod_{i \in I} BU_i \underset{\gamma}{\overset{\beta}{\rightrightarrows}} \prod_{i,j \in I \times I} B(U_i \wedge U_j)$$

where α, β and γ are induced by the restriction maps as follows: For any $b \in BU$, $\alpha(b) = (b|U_i)_{i \in I}$ and if $(b_i)_{i \in I} \in \prod_{i \in I} BU_i$, then the $(i,j)^{th}$ component of $\beta((b_i)_{i \in I})$ and $\gamma((b_i)_{i \in I})$ is respectively $b_j|U_i \wedge U_j$ and $b_i|U_i \wedge U_j$.

B is called a monopresheaf, if for any cover $U = \bigvee_{i\in I} U_i$, $U \in \mathcal{L}$, the corresponding α is a monomorphism. In addition, if α is the equalizer of β and γ, then B is called a sheaf. As a result we get a category with objects as sheaves on $\mathcal{L}$, and morphisms as natural transformations between them, denoted by $\mathrm{Sh}\mathcal{L}$, which forms a full subcategory of $\mathrm{PSh}\mathcal{L}$. If $\mathcal{L} = \mathcal{O}X$ for some topological space X, then we shall write $\mathrm{Sh}X$ for $\mathrm{Sh}\mathcal{O}X$.

<u>0.12</u>: The inclusion functor $i: \mathrm{Sh}\mathcal{L} \to \mathrm{PSh}\mathcal{L}$ has a left adjoint, the sheaf reflection functor $\simeq: \mathrm{PSh}\mathcal{L} \to \mathrm{Sh}\mathcal{L}$, which is obtained as follows: For any $B \in \mathrm{PSh}\mathcal{L}$, and any cover $C = (U_i)_{i\in I}$ of U in $\mathcal{L}$, denote by B_C the equalizer of the maps β and γ,

$$B_C \rightarrowtail \prod_{i\in I} BU_i \underset{\gamma}{\overset{\beta}{\rightrightarrows}} \prod_{i,j\in I\times I} B(U_i \wedge U_j)$$

where β and γ are determined by the restriction maps in the obvious manner. Since the covers of any $U \in \mathcal{L}$, form a directed set $\mathrm{Cov}U$ under refinement, we can form the direct limit of $\{B_C\}_{C\in \mathrm{Cov}U}$ in Ens. The sheaf reflection $\tilde{B}$ of B is defined by $\tilde{B}U = \widetilde{\varinjlim_{C\in \mathrm{Cov}U}} B_C$. Since $U = \{U\}$ is a cover of U, so the map $BU \to \tilde{B}U$ is clear, therefore for any $h: B \to C$ in $\mathrm{PSh}\mathcal{L}$, $\tilde{h}: \tilde{B} \to \tilde{C}$ follows from the properties of direct limits. Then $\tilde{B}$ is a sheaf, and $\simeq$ is left exact, left adjoint to i, follows from the properties of

limits and direct limits in Ens.

If A is defined as the sheaf reflection of the given presheaf B, then throughout this thesis we shall write $AU \doteq BU$.

<u>Remark</u>: It is easy to see that the limits and colimits in $PSh\mathcal{L}$ are computed pointwise. So, from the adjointness between the categories $Sh\mathcal{L}$ and $PSh\mathcal{L}$, we have the following constructions for the limits and colimits in $Sh\mathcal{L}$.

If $\{A_i\}_{i\in I}$ is any family in $Sh\mathcal{L}$, then for any $U \in \mathcal{L}$,

(1) $(\underset{i\in I}{\text{limit}}\, A_i)U = \underset{i\in I}{\text{limit}}(A_iU)$ with the obvious restriction maps.

(2) $(\underset{i\in I}{\text{colimit}}\, A_i)U \doteq \underset{i\in I}{\text{colimit}}(A_iU)$ with the restriction maps being the sheaf reflection of the corresponding restriction maps in $PSh\mathcal{L}$,

Note: $Sh2 \cong Ens$ for the two-element locale 2, and if X is a discrete topological space then $ShX \cong Ens^{|X|}$. Further, $Sh3$ for the three-element locale 3 is the same as $PSh2$, i.e. the arrow category of Ens; moreover, $Sh3$ is also $Sh\mathcal{S}$ for the Sierpinski space $\mathcal{S}$ with points 0 and 1 and non-trivial open set $\{1\}$.

<u>0.13</u>: If $\mathcal{L}$ and $\mathcal{M}$ are any two locales, and $\phi: \mathcal{L} \to \mathcal{M}$ a local lattice homomorphism, then it produces

a pair of adjoint functors $\mathrm{Sh}\mathcal{M} \underset{\phi^*}{\overset{\phi_*}{\rightleftarrows}} \mathrm{Sh}\mathcal{L}$ where $(\phi_* A)U = A(\phi(U))$ and for any $V \in \mathcal{M}$ $(\phi^* C)V \doteq \underset{\phi(W) \geq V}{\ell t}\, CW$ $(W \in \mathcal{L})$. Then ϕ^* is left exact, left adjoint to ϕ_*.

<u>0.14</u>: Special cases of local lattice homomorphisms:

(1) If $\phi: 2 \to \mathcal{L}$ is the unique local lattice homomorphism, then it gives $\mathrm{Sh}\mathcal{L} \underset{\phi^*}{\overset{\phi_*}{\rightleftarrows}} \mathrm{Sh}2 \cong \mathrm{Ens}$, where

$\phi_* A = AE$ and $(\phi^* B)U \doteq B$.

Notation: $\phi^* B = B_{\mathcal{L}}$, $\phi_* = \Gamma$

(2) Any local lattice homomorphism $\phi: \mathcal{L} \to 2$ produces

$\mathrm{Ens} \underset{\phi^*}{\overset{\phi_*}{\rightleftarrows}} \mathrm{Sh}\mathcal{L}$, where $(\phi_* A)U = \begin{cases} A & \text{if } \phi(U) = 1 \\ 0 & \text{if } \phi(U) = 0 \end{cases}$ and

$\phi^* A = \underset{\longrightarrow}{\ell t}\, AU$ (all U such that $\phi(U) = 1$).

(3) If $\mathcal{L} = \mathcal{O}X$ for some topological space X, and $x \in X$ is any point, then for the local lattice homomorphism $\mathfrak{x}: \mathcal{L} \to 2$ given by $\mathfrak{x}(U) = \mathrm{card}(U \cap \{x\})$, we get $(\mathfrak{x})^* A = \underset{\longrightarrow}{\ell t}\, AU (x \in U) = A_x$, the stalk of A at x.

(4) For any $U \in \mathcal{L}$, $\downarrow U = \{V \in \mathcal{L} \mid V \leq U\}$ is a locale and $\phi: \mathcal{L} \to \downarrow U$ given by $\phi(W) = W \wedge U$ is a locale lattice homomorphism. So, we get $\mathrm{Sh}\downarrow U \underset{\phi^*}{\overset{\phi_*}{\rightleftarrows}} \mathrm{Sh}\mathcal{L}$, where

$(\phi_* A)W = A(W \wedge U)$ and $(\phi^* B)W \doteq \underset{\phi(V) \geq W}{\underset{\longrightarrow}{\ell t}}\, BV = BW$, and so

$\phi^* B$ is just the restriction of B to the locale $\downarrow U$.

Notation: $\phi^*B = B|U = R_U B$.

Also, ϕ^* has a left adjoint denoted by E_U, where

$$(E_U A)V \doteq \begin{cases} AV & \text{if } V \leq U \\ 0 & \text{if } V \not\leq U \end{cases}$$

Then E_U is left adjoint left exact to R_U. Since R_U is both a right adjoint as well as a left adjoint, it preserves all limits and colimits.

(5) If $f: X \to Y$ is a continuous map of topological spaces, then it produces a local lattice homomorphism (also denoted by f) $f: \mathcal{O}Y \to \mathcal{O}X$, $V \to f^{-1}(V)$, and so correspondingly it gives $ShX \underset{f^*}{\overset{f_*}{\rightleftarrows}} ShY$. In particular for any topological space X, let $|X|$ be X with discrete topology, then the identity map $i: |X| \to X$ is continuous, hence it produces $ShX \underset{i_*}{\overset{i^*}{\rightleftarrows}} Sh|X| \cong Ens^{|X|}$.

Section 4: Abelian Groups in $PSh\mathcal{L}$ and $Sh\mathcal{L}$

0.15: In section 2, we discussed the notion of abelian groups in any finitely complete category $\mathbf{E}$. In particular if $\mathbf{E} = PSh\mathcal{L}$, or $\mathbf{E} = Sh\mathcal{L}$, then $AbPSh\mathcal{L}$ and $AbSh\mathcal{L}$ are, as observed long ago by Grothendieck, the same as the categories of Presheaves and Sheaves on $\mathcal{L}$ with values in Ab, the category of Abelian groups, that is, $B \in AbPSh\mathcal{L}$ is a contravariant functor $B: \mathcal{L} \to Ab$.

Further, the functor $\simeq: \mathrm{PSh}\mathcal{L} \to \mathrm{Sh}\mathcal{L}$ in 0.12 can actually be lifted to a functor also denoted by $\simeq: \mathrm{AbPSh}\mathcal{L} \to \mathrm{AbSh}\mathcal{L}$, such that $\simeq$ is left exact left adjoint to the functor $i: \mathrm{AbSh}\mathcal{L} \to \mathrm{AbPSh}\mathcal{L}$. Also note that $\mathrm{AbSh}2 \cong \mathrm{Ab}$, and for discrete topological space X, $\mathrm{AbSh}X \cong \mathrm{Ab}^{|X|}$.

<u>0.16</u>: For any local lattice homomorphism $\phi: \mathcal{L} \to \mathcal{M}$, the functors ϕ^*, ϕ_* in 0.13, can be lifted to functors denoted by the same letters, $\mathrm{AbSh}\mathcal{M} \underset{\phi^*}{\overset{\phi_*}{\rightleftarrows}} \mathrm{AbSh}\mathcal{L}$, where ϕ^* is left exact, left adjoint to ϕ_*.

<u>0.17</u>: The two categories $\mathbf{E} = \mathrm{Sh}\mathcal{L}$ and $\mathbf{E} = \mathrm{PSh}\mathcal{L}$ are Grothendieck topos (Johnstone [10], Section 0.4) and for such topos $\mathbf{E}$, $\mathrm{Ab}\mathbf{E}$ is a Grothendieck category with a generator ([10], Section 8.1) and hence has enough injective hulls. To understand this, we recall the following definitions:

(1) A monomorphism $h: A \to B$ in $\mathrm{Ab}\mathbf{E}$ is essential iff for any homomorphism $j: B \to C$, jh mono implies j is a monomorphism.

(2) An abelian group A is injective iff for any homomorphism $f: B \to A$ and any monomorphism $g: B \to C$, there exists a homomorphism $h: C \to A$ such that $hg = f$.

(3) By an injective hull of an abelian group A, we mean an essential monomorphism $A \to B$ with B injective.

(4) $\mathrm{Ab}\mathbf{E}$ has "enough injective hulls" means every abelian group $A \in \mathrm{Ab}\mathbf{E}$ has an injective hull.

Since $\mathrm{AbSh}\mathcal{L}$ has enough injectives it follows A is injective iff it has no proper essential extensions iff it is a direct summand in every extension.

<u>0.18</u>: Besides the obvious external Ab valued hom-functor $H = H_{\mathcal{L}} : \mathrm{AbSh}\mathcal{L}^{\mathrm{opp}} \times \mathrm{AbSh}\mathcal{L} \to \mathrm{Ab}$, $\mathrm{AbSh}\mathcal{L}$ also has an internal hom-functor $[-,-]: \mathrm{AbSh}\mathcal{L}^{\mathrm{opp}} \times \mathrm{AbSh}\mathcal{L} \to \mathrm{AbSh}\mathcal{L}$, for which $[A,B]U = H_{\downarrow U}(A|U, B|U)$, with the restriction maps $[A,B]U \to [A,B]V$ $(V \le U)$, given by $h = (h_W)_{W \le U} \to h|V = (h_W)_{W \le V}$ (see Ebrahimi [18]). A more detailed discussion of $[-,-]$ will be given in chapter 4.

For the following recall,

<u>Proposition 0.19</u>: If $F: A \to B$ has a left adjoint $G: B \to A$ which preserves monomorphisms, then F preserves injectives.

<u>Remark</u>: Thus it follows that, the inclusion $i: \mathrm{AbSh}\mathcal{L} \to \mathrm{AbPSh}\mathcal{L}$, and $\phi_*: \mathrm{AbSh}\mathcal{M} \to \mathrm{AbSh}\mathcal{L}$ (for any local lattice homomorphism $\phi: \mathcal{L} \to \mathcal{M}$) preserve injectives.

<u>0.20</u>: We produce some nontrivial injectives in $\mathrm{AbSh}\mathcal{L}$ as follows:

(1) For any topological space X and any injective P in Ab, the group $(P)_{x \in |X|}$ in $\mathrm{Ab}^{|X|}$ is injective, hence it follows from (0.16, 0.19) that $i_*((P)_{x \in |X|}) = P^{\#}$

is an injective group in AbShX . Clearly $P^{\#}U = \prod_{x \in U} P = P^U$, with restrictions as the projections. In particular, we shall be interested in $\mathbb{Q}^{\#}$ for the group $\mathbb{Q}$ of rational numbers.

(2) For any local lattice homomorphism $\phi: \mathcal{L} \to 2$, and an injective P in Ab , we get an injective $\phi_*(P)$ in $\mathrm{AbSh}\mathcal{L}$ (0.16, 0.19). In particular, if P is an indecomposable injective in Ab , then since $\phi^*\phi_*P \cong P$ it follows ϕ_*P is an indecomposable injective in $\mathrm{AbSh}\mathcal{L}$. Explicitly, ϕ_*P is given as follows:

$$(\phi_*P)U = \begin{cases} P & \text{if } \phi(U) = 1 \\ 0 & \text{if } \phi(U) = 0 \end{cases}$$

If $\mathcal{L} = \mathfrak{O}X$ for some space X and $x \in X$ then, for the local lattice homomorphism $\mathfrak{x}: \mathfrak{O}X \to 2$ given by x,

$$(\mathfrak{x}_*P)U = \begin{cases} P & \text{if } x \in U \\ 0 & \text{if } x \notin U \end{cases}$$

Recall that the possible choices of P are $\mathbb{Q}$ and the Prüfer groups $Z(p^\infty)$.

<u>Remark 0.21</u>:

(1) For any $A \in \mathrm{AbSh}\mathcal{L}$, the subgroup C of A generated by an element $a \in AU$, $U \in \mathcal{L}$, that is, the smallest subgroup $C \subseteq A$ such that $a \in CU$, is given by

$$CW \doteq \begin{cases} Z(a|W) & \text{if } W \leq U \\ 0 & \text{if } W \nleq U \end{cases}$$

(2) $B \geq A$ is an essential extension in $\mathrm{AbSh}\,\mathcal{L}$ iff for any $0 \neq b \in BU$, there exists $V \leq U$ in $\mathcal{L}$, $m \in Z$ such that $0 \neq mb|V \in AV$. To see this, one first notes that $B \geq A$ is essential iff $C \cap A \neq 0$ for any non-zero subgroup $C \leq B$ (since a homomorphism in $\mathrm{AbSh}\,\mathcal{L}$ is monic iff its kernel is 0), and then observes it is sufficient to consider subgroups generated by a single non-zero $b \in BU$ for any $U \in \mathcal{L}$.

<u>Proposition 0.22</u>: For any $U \in \mathcal{L}$, the functors $R_U: \mathrm{AbSh}\,\mathcal{L} \to \mathrm{AbSh}{\downarrow}U$, and $E_U: \mathrm{AbSh}{\downarrow}U \to \mathrm{AbSh}\,\mathcal{L}$ preserve essential extensions.

<u>Proof</u>: Consider any essential extension $B \geq A$ in $\mathrm{AbSh}\,\mathcal{L}$. Since R_U preserves all limits (0.14(4)), it follows $B|U \geq A|U$. We claim, this is an essential extension in $\mathrm{AbSh}{\downarrow}U$. So let $0 \neq b \in BW = (B|U)(W)$ for some $W \in {\downarrow}U$. Since B is an essential extension of A in $\mathrm{AbSh}\,\mathcal{L}$, there exists $V \leq W$, $m \in Z$ such that $0 \neq mb|V \in AV = (A|U)(V)$. Hence $B|U \geq A|U$ is an essential extension in $\mathrm{AbSh}{\downarrow}U$. To prove that E_U preserves essential extensions, consider an essential extension $P \geq Q$ in $\mathrm{AbSh}{\downarrow}U$. Since E_U preserves monomorphism (0.14(4)), it follows $E_UP \geq E_UQ$. Let $0 \neq a \in (E_UP)W$, then by the

definition of E_U, there exists a cover $W = \bigvee_{i \in I} W_i$, such that $0 \neq a|W_i \in PW_i$ for some $W_i \subseteq U$. But $P \supseteq Q$ is essential in $\mathrm{AbSh}\downarrow U$, so there exists $V \leq W_i$ and $m \in Z$ such that $0 \neq m(a|W_i)|V = ma|V \in QV$. Hence $E_U P \supseteq E_U Q$ is an essential extension in $\mathrm{AbSh}\mathcal{L}$.

<u>Corollary 0.23</u>: For any $U \in \mathcal{L}$, R_U preserves injectives and injective hulls.

<u>Proof</u>: Since R_U has a left adjoint which preserves monomorphisms (0.14(4)), it follows R_U preserves injectives (0.19). By the above proposition, it follows R_U preserves injective hulls.

<u>Corollary 0.24</u>: For an injective group $A \in \mathrm{AbSh}\mathcal{L}$, AU is an injective group in Ab, for all $U \in \mathcal{L}$.

<u>Proof</u>: By the above corollary, R_U preserves injectives, hence each $A|U$ is injective in $\mathrm{AbSh}\downarrow U$. From 0.14 (1) and 0.19, $\Gamma(A|U) = AU$ is an injective group in Ab.//

<u>Remark 0.25</u>: The composite $E_U R_U$ is denoted by T_U: $\mathrm{AbSh}\,\mathcal{L} \to \mathrm{AbSh}\,\mathcal{L}$, where

$$(T_U A)W \doteq \begin{cases} AW & \text{if } W \subseteq U \\ 0 & \text{if } W \nsubseteq U \end{cases}$$

Since both E_U and R_U preserve essential extensions, it follows T_U preserves essential extensions. Note, though, that T_U does not preserve injectives, as one can see by considering $\mathcal{L} = 3$.

<u>Proposition 0.26 (B. Banaschewski, [6])</u>: For any $U \in \mathcal{L}$, the functor T_U: AbSh $\mathcal{L} \to$ AbSh $\mathcal{L}$ has the following properties:

(i) T_U exact

(ii) $T_U \rightarrowtail 1_{\mathrm{AbSh}\mathcal{L}}$

(iii) If $A \subseteq B$, then $T_U A = A \cap T_U B$.

<u>Proof</u>:

(i) T_U is exact since both E_U and R_U are exact.

(ii) For $A \in$ AbSh $\mathcal{L}$, $(T_U A)W \doteq \begin{cases} AW & \text{if } W \subseteq U \\ 0 & \text{if } W \nsubseteq U \end{cases}$

So, the presheaf generating $T_U A$ is a subpresheaf of A. Since the sheaf reflection preserves monomorphism it follows $(T_U A)W \subseteq AW$ for all $W \in \mathcal{L}$. Hence $T_U A \rightarrowtail A$, thus $T_U \rightarrowtail 1_{\mathrm{AbSh}\mathcal{L}}$.

(iii) For any $W \in \mathcal{L}$, $(T_U B)W = \{a \in BW \mid W = (W \wedge U) \vee \xi_W(a)\}$ where $\xi_W(a) = \bigvee_{V \leq W} V(a|V = 0)$. Therefore

$$AW \cap (T_U B)W = \{a \in BW \mid a \in AW \text{ and } W = (W \wedge U) \vee \xi_W(a)\}$$
$$= (T_U A)W \text{ , hence the result.}$$

<u>Remark 0.27</u>: The family $\{T_U Z_{\mathcal{L}}\}_{U \in \mathcal{L}}$ where $Z_{\mathcal{L}}$ is the sheaf reflection of the constant presheaf Z is a generating family in the category $\mathrm{AbSh}\,\mathcal{L}$, as one can see from the following: For $0 \neq a \in AU$, $A \in \mathrm{AbSh}\,\mathcal{L}$, there exists a homomorphism $h_U: Z \to AU$ in Ab such that $h_U(1) = a$. For $V \leq U$ define $h_V: Z \to AV$ by $h_V(1) = a|V$. Then h is a morphism from the constant presheaf Z to A in the category $\mathrm{AbPSh}\downarrow U$. The sheaf reflection $\mathrm{AbPSh}\downarrow U \xrightarrow{\sim U} \mathrm{AbSh}\downarrow U$ produces a morphism of sheaves $Z_{\downarrow U} = Z_{\mathcal{L}}|U \xrightarrow{\tilde{h}^U} A|U$ in $\mathrm{AbSh}\downarrow U$. Applying the functor E_U, we get $E_U Z_{\mathcal{L}}|_U \xrightarrow{E_U(\tilde{h}^U)} E_U(A|U) = T_U Z_{\mathcal{L}} \to T_U A$. But by the above proposition $T_U A \overset{i}{\rightarrowtail} A$, hence there exists a morphism $f: T_U Z_{\mathcal{L}} \to T_U A \rightarrowtail A$, $f = i(E_U(\tilde{h}^U))$ such that $f_U(1_{Z_{\mathcal{L}}U}) = a$. Hence the family $\{T_U Z_{\mathcal{L}}\}_{U \in \mathcal{L}}$ forms a generating set in $\mathrm{AbSh}\,\mathcal{L}$. It follows that the coproduct $\bigoplus_{U \in \mathcal{L}} T_U Z_{\mathcal{L}}$ is a generator in the category $\mathrm{AbSh}\,\mathcal{L}$. Note that, in general, $Z_{\mathcal{L}}$ is not a generator.

<u>0.28</u>: In section 2, we described what is meant by torsion, torsion free and divisible groups in $\mathrm{Ab}\mathbf{E}$. For the case $\mathbf{E} = \mathrm{Sh}\,\mathcal{L}$, we have the following description of

those notions:

(1) $A \in \mathrm{AbSh}\mathcal{L}$ is said to be a torsion free group iff $n_A : A \to A$ is a monomorphism for all $0 \neq n \in N$. Here, this means that each AU is a torsion free group in the category Ab.

(2) $A \in \mathrm{AbSh}\mathcal{L}$ is said to be a divisible group iff n_A is an epimorphism for all $0 \neq n \in N$. This means for each $a \in AU$, there exists a cover $U = \bigvee_{i \in I} U_i$ such that for each $i \in I$, $a|U_i = nb_i$ with some $b_i \in AU_i$.

(3) $A \in \mathrm{AbSh}\mathcal{L}$ is said to be a torsion group iff $A = \varinjlim_{0 \neq n \in N} \mathrm{Ker}\, n_A$. This means for each $a \in AU$, there exists a cover $U = \bigvee_{i \in I} U_i$, and $0 \neq m_i \in \mathbb{Z}$, such that $m_i a|U_i = 0$ for all $i \in I$.

<u>Proposition 0.29</u>: For any $U \in \mathcal{L}$, the functors R_U, E_U and T_U preserve torsion groups.

<u>Proof</u>: Consider $A \in \mathrm{AbSh}\mathcal{L}$ which is a torsion group. Then $A = \varinjlim_{0 \neq n \in N} \mathrm{Ker}\, n_A$ since R_U preserves all colimits and limits (0.14(4)), it follows

$R_U A = \varinjlim_{0 \neq n \in N} R_U(\mathrm{Ker}\, n_A) = \varinjlim_{0 \neq n \in N} \mathrm{Ker}\, n_{A|U}$, hence $A|U$ is torsion in $\mathrm{AbSh}\downarrow U$.

Now consider $B \in \mathrm{AbSh}\downarrow U$ which is a torsion group. We claim $E_U B$ is torsion. By the same argument as above, (since E_U preserves all colimits and finite limits) it

follows $E_U B$ is a torsion group in $\mathrm{AbSh}\,\mathcal{L}$. Now $T_U = E_U R_U$, hence T_U preserves torsion groups.//

<u>Proposition 0.30</u>: For any $U \in \mathcal{L}$, the functors R_U, E_U and T_U preserve torsion free group.

<u>Proof</u>: If A is a torsion free group in $\mathrm{AbSh}\,\mathcal{L}$, then $\mathrm{Ker}\, n_A = 0$. Since R_U preserves kernels, it follows $R_U(\mathrm{Ker}\, n_A) = \mathrm{Ker}\, n_{A|U} = 0$, hence $A|U$ is torsion free. Also E_U preserves kernels, hence for a torsion free group $B \in \mathrm{AbSh}{\downarrow}U$, $E_U B$ is a torsion free group in $\mathrm{AbSh}\,\mathcal{L}$. From this it follows $T_U = E_U R_U$ preserves torsion free groups.//

CHAPTER 1

INJECTIVITY IN $\mathrm{AbSh}\mathcal{L}$

Introduction: This chapter is devoted to the study of injective groups in $\mathrm{AbSh}\mathcal{L}$. In the category Ab of abelian groups, the Baer criterion for injectivity shows that injectivity and divisibility are the same. Although we do prove the analogue of the Baer criterion in $\mathrm{AbSh}\mathcal{L}$, still the concept of injectivity and divisibility do not coincide for arbitrary $\mathcal{L}$. In fact, the two concepts do coincide iff $\mathcal{L}$ is Boolean (B. Banaschewski [4]).

We further show that injectivity is a local property. It is of course a global property, and hence if A is injective in $\mathrm{AbSh}\mathcal{L}$ then each AU is injective in Ab (0.24), for all $U \in \mathcal{L}$. The converse of this does not hold: for counterexample refer to (B. Banaschewski [3]). So A injective implies each AU is divisible in Ab, hence A is divisible in $\mathrm{AbSh}\mathcal{L}$. Thus for any $\mathcal{L}$, injectivity in $\mathrm{AbSh}\mathcal{L}$ always implies divisibility, but not conversely. We discuss what the indecomposable injectives look like in $\mathrm{AbSh}\mathcal{L}$. Also, we show that injectives in general do not split into direct sums of indecomposable injectives and the locales for which they do split will be discussed. For a sober T_1 space X, we show that

injectives in AbShX split into a direct sum of indecomposable injectives iff X is discrete. Moreover, for any $\mathcal{L}$, the indecomposable torsion injectives cogenerate AbSh$\mathcal{L}$ iff $\mathcal{L}$ is spatial.

Lastly, we discuss injective hulls, where we show that the injective hull is a local but not a global property. For a well-ordered locale $\mathcal{L}$, the injective hulls in AbSh$\mathcal{L}$ will be described, and for some special locales $\mathcal{L}$, we will characterize the injectives in AbSh$\mathcal{L}$.

Section 1: General Results

The Baer Criterion for Injectivity:

Proposition 1.1: $A \in \mathrm{AbSh}\mathcal{L}$ is injective iff it is injective relative to all $S \rightarrowtail Z_{\mathcal{L}}$.

Proof: ($\Rightarrow$) Being trivial one only has to show the converse.

($\Leftarrow$) Let A be injective relative to all $S \rightarrowtail Z_{\mathcal{L}}$. Consider the diagram

$$\begin{array}{ccc} B & \overset{f}{\rightarrowtail} & C \\ g\downarrow & & \\ A & & \end{array}$$

where we may assume that $B \subseteq C$, and $f: B \to C$ the natural embedding. Consider the family $\mathcal{A} = \{(B',g')\}$ where $B \subseteq B' \subseteq C$ and $g': B' \to A$ such that $g'|B = g$. Then this family is non-empty since $(B,g) \in \mathcal{A}$. As usual, we

introduce a partial ordering on this family by $(B',g') \leq (B'',g'')$ iff $B' \subseteq B''$ and $g''|B' = g'$. If $\{(B_i,g_i)\}_{i\in I}$ is a linearly ordered family in $\mathcal{A}$, then it has an upper bound in $\mathcal{A}$ given by (D,h), where D is the join of B_i in the subgroup lattice of C. That is, D is the sheaf reflection of the presheaf $U \rightsquigarrow \bigcup_I B_i U$ and h is the corresponding sheaf reflection of the morphism to which the g_{i_U} extend. By Zorn's Lemma, the family $\mathcal{A}$ has a maximal element (P,p). We claim $P = C$. If not, then there is a $U \in \mathcal{L}$ and $c \in CU$ such that $c \notin PU$. Let H be the subgroup of C generated by c. Then H is the sheaf reflection of the presheaf $W \rightsquigarrow \begin{cases} Z(c|W) & \text{if } W \subseteq U \\ 0 & \text{if } W \nsubseteq U. \end{cases}$

Since the presheaf defining $T_U Z_{\mathcal{L}}$ is given by $W \rightsquigarrow \begin{cases} Z & \text{if } W \subseteq U \\ 0 & \text{if } W \nsubseteq U \end{cases}$, therefore, there is an epimorphism of presheaves from that defining $T_U Z_{\mathcal{L}}$ to that defining H. The sheaf reflection preserves epimorphisms, and so $T_U Z_{\mathcal{L}} \xrightarrow{j} H$ is an epimorphism in $\mathrm{AbSh}\,\mathcal{L}$. The diagram

$$\begin{array}{ccc} & & T_U Z_{\mathcal{L}} \\ & & \downarrow j \\ P \cap H & \underset{i}{\rightarrowtail} & H \end{array}$$

can be completed to a pull back square,

$$\begin{tikzcd} \bar{H} \arrow[r, tail, "\bar{i}"] \arrow[d, "\bar{j}"'] & T_U Z_{\mathcal{L}} \arrow[d, "j"] \\ P \cap H \arrow[r, tail, "i"'] & H \end{tikzcd}$$

where $\bar{i}$ is a mono since i is. Moreover since j is an epimorphism $\bar{j}$ is an epimorphism, and the above diagram is actually a push out diagram [19, page 33]. But $T_U Z_{\mathcal{L}} \subseteq Z_{\mathcal{L}}$ and A is injective relative to all $\bar{H} \rightarrowtail Z_{\mathcal{L}}$, so there exists $\alpha: Z_{\mathcal{L}} \dashrightarrow A$ such that the outer triangle of the diagram

$$\begin{tikzcd}
\bar{H} \arrow[r, tail, "\bar{i}"] \arrow[d, tail, "\bar{j}"'] & T_U Z_{\mathcal{L}} \arrow[r, phantom, "\subseteq"] \arrow[d, "j"] & Z_{\mathcal{L}} \arrow[dd, dashed, bend left, "\alpha"] \\
P \cap H \arrow[r, tail, "i"'] \arrow[d, tail, "k"'] & H \arrow[dd, dashed, bend left, "\beta"] & \\
P \arrow[d, "p"'] & & \\
A & &
\end{tikzcd}$$

commutes, that is $(\alpha|T_U Z_{\mathcal{L}})\bar{i} = pk\bar{j}$. The inner square is also a push out square, and so there exists a unique $\beta: H \dashrightarrow A$ such that $\beta j = \alpha|T_U Z$, $\beta i = pk$. Define another presheaf M by $MU = PU + HU \subseteq CU$, with the obvious restriction maps, then

$$\begin{tikzcd}
P \cap H \arrow[r, tail, "i"] \arrow[d, tail, "k"'] & H \arrow[d, tail, "\tau"] \\
P \arrow[r, tail, "n"'] & M
\end{tikzcd}$$

is a push out in $\mathrm{AbPSh}\,\mathcal{L}$, because at each $U \in \mathcal{L}$ it is a push out in Ab. Since $P + H \doteq M$, and the sheaf reflection being a left adjoint preserves push outs, it follows

$$\begin{tikzcd} P \cap H \arrow[r, tail, "i"] \arrow[d, tail, "k"'] & H \arrow[d, tail, "\tau"] \\ P \arrow[r, tail, "n"] & P + H \end{tikzcd}$$

is a push out in $\mathrm{AbSh}\,\mathcal{L}$. Hence if we consider the diagram

$$\begin{tikzcd} P \cap H \arrow[r, tail, "i"] \arrow[d, tail, "k"'] & H \arrow[d, tail, "\tau"] \arrow[ddl, bend left, "\beta"] \\ P \arrow[r, tail, "n"] \arrow[d, "p"'] & P + H \\ A & \end{tikzcd}$$

then there is a unique $q: P + H \to A$ such that $qn = p$, and $q\tau = \beta$. Thus $(P + H, q) \in \mathcal{A}$, a contradiction, since $(P,p) \in \mathcal{A}$ is maximal and $P + H \supset P$. Thus $P = C$, and hence A is injective.//

<u>Lemma 1.2</u>: For any cover $E = \bigvee_{i \in I} U_i$ of the unit in $\mathcal{L}$, the functor $R: \mathrm{AbSh}\,\mathcal{L} \to \prod_{i \in I} \mathrm{AbSh}\downarrow U_i$ given by $RB = (B|U_i)_{i \in I}$, $Rh = (h|U_i)_{i \in I}$ for $h: A \to B$ in $\mathrm{AbSh}\,\mathcal{L}$, has the following two properties:

(a) R reflects and preserves monomorphisms.

(b) R is faithful.

<u>Proof</u>: (a) If $h: A \to B$ is a monomorphism in $\mathrm{AbSh}\,\mathcal{L}$, then each $h|U_i: A|U_i \to B|U_i$ is a mono in $\mathrm{AbSh}\downarrow U_i$ (0.16), hence $(h|U_i): RA \to RB$ is a monomorphism in

$\prod_{i\in I} \mathrm{AbSh}\downarrow U_i$. Now suppose Rh is given to be a monomorphism and we want to show that h is a monomorphism. Let $W \in \mathcal{L}$ be arbitrary, and suppose $h_W(a) = h_W(b)$ for some $a,b \in AW$. Since h is a morphism of sheaves, therefore

$$\begin{array}{ccc} AW & \longrightarrow & A(W \wedge U_i) \\ \downarrow & & \downarrow \\ BW & \longrightarrow & B(W \wedge U_i) \end{array}$$

commutes for all $i \in I$, and hence $h_{W\wedge U_i}(a|W \wedge U_i) = h_{W\wedge U_i}(b|W \wedge U_i)$. But $h_{W\wedge U_i}$ is a monomorphism in $\mathrm{AbSh}\downarrow U_i$ for all i, and therefore $a|W \wedge U_i = b|W \wedge U_i$ all i, which by the sheaf properties of A implies $a = b$. Hence h is a monomorphism. Thus R reflects and preserves monomorphisms.

(b) Suppose $Rf = Rg$ for some $f,g: A \to B$ in $\mathrm{AbSh}\,\mathcal{L}$. Then $f|U_i = g|U_i$ for all $i \in I$. We claim $f = g$, that is $f_W = g_W$ for all $W \in \mathcal{L}$. For any $a \in AW$, we have $g_W(a)|W\wedge U_i = g_{W\wedge U_i}(a|W\wedge U_i) = f_{W\wedge U_i}(a|W\wedge U_i) = f_W(a)|W\wedge U_i$, all $i \in I$. Thus for the cover $W = \bigvee_{i\in I} W \wedge U_i$, we have $f_W(a)|W \wedge U_i = g_W(a)|W \wedge U_i$, all $i \in I$, hence $f_W(a) = g_W(a)$. Thus $f_W = g_W$, for all $W \in \mathcal{L}$ implies $f = g$. //

<u>Proposition 1.3</u>: The functor R preserves and reflects injectives.

<u>Proof</u>: ($\Rightarrow$) If B is injective in $\mathrm{AbSh}\mathcal{L}$, then each $B|U_i$ is injective in $\mathrm{AbSh}{\downarrow}U_i$ (0.23), hence $RB = (B|U_i)_{i\in I}$ is injective in $\prod_{i\in I} \mathrm{AbSh}{\downarrow}U_i$.

($\Leftarrow$) Assume RB is injective and we want to show that B is an injective group in $\mathrm{AbSh}\mathcal{L}$. Consider an essential extension D of B. Since R_{U_i} preserves essential extensions (0.22) it follows each $D|U_i \supseteq B|U_i$ is an essential extension in $\mathrm{AbSh}{\downarrow}U_i$. So if $0 \neq S \subseteq RD$, then for some $i \in I$, $0 \neq S_i \subseteq D|U_i$, hence $S_i \cap B|U_i \neq 0$. This means $S \cap RB \neq 0$ which shows RD is an essential extension of RB. But RB is given to be injective, and so $RB = RD$. Since R is faithful it reflects epimorphisms, hence the natural embedding $B \rightarrowtail D$ is an epimorphism and therefore $B = D$. Thus B has no proper essential extensions in $\mathrm{AbSh}\mathcal{L}$, which means B is injective (0.17), hence the result.

<u>Corollary 1.4</u>: B is injective in $\mathrm{AbSh}\mathcal{L}$ iff there is a cover $E = \bigvee_{i\in I} U_i$ such that $B|U_i$ is injective in $\mathrm{AbSh}{\downarrow}U_i$, for all $i \in I$.

<u>Proof</u>: ($\Rightarrow$) Clear, since the restriction functors preserve injectives (0.23).

($\Leftarrow$) If each $B|U_i$ is injective in $\mathrm{AbSh}{\downarrow}U_i$, then $RB = (B|U_i)_{i\in I}$ is injective in $\prod_{i\in I} \mathrm{AbSh}{\downarrow}U_i$. By the

last proposition this implies that B is injective. The above corollary means injectivity is a local property.

<u>Lemma 1.5</u>: If A is injective in $\mathrm{AbSh}\mathcal{L}$, then for any $V \leq U$ in $\mathcal{L}$ the restriction $AU \to AV$ is a split epimorphism in Ab.

<u>Proof</u>: Consider the local lattice homomorphism $\phi: 3 \to \downarrow U$ with image $\begin{array}{c} U \\ \vee \\ 0 \end{array}$. Then since A is injective in $\mathrm{AbSh}\mathcal{L}$, it follows $\phi_* A = \begin{array}{c} AU \\ \downarrow \\ AV \end{array}$ is an injective group in $\mathrm{AbSh}3$ (0.19). But the injectives in $\mathrm{AbSh}3$ are exactly the projections $\begin{array}{c} P \times T \\ \downarrow \\ P \end{array}$ with divisible P and T, [4] hence $\begin{array}{c} AU \\ \downarrow \\ AV \end{array}$ is a split epimorphism in Ab. //

For the following, recall than an indecomposable injective in $\mathrm{AbSh}\mathcal{L}$ is an injective C such that $C = D \oplus E$ implies $D = 0$ or $E = 0$. Also remember that $\mathrm{Ab} \cong \mathrm{AbSh}2$.

<u>Section 2</u>: <u>Indecomposable Injectives</u>

<u>Proposition 1.6</u>: The indecomposable injectives in $\mathrm{AbSh}\mathcal{L}$, are upto isomorphism exactly the images of the indecomposable injectives in Ab with respect to the functors $\phi_*: \mathrm{Ab} \to \mathrm{AbSh}\mathcal{L}$ for the local lattice homomorphisms $\phi: \mathcal{L} \to 2$.

<u>Proof</u>: ($\Rightarrow$) Let C be an indecomposable injective in $\mathrm{AbSh}\,\mathcal{L}$. Define a presheaf $A: U \rightsquigarrow \prod_{W \le U} CW$ with restrictions $AU \to AV$ being the projections for $V \le U$. The morphism of presheaves $C \xrightarrow{i} A$ given by the maps $CU \to AU = \prod_{W \le U} CW$, $a \to (a|W)_{W \le U}$ is clearly a monomorphism. By 0.19, C is also injective in $\mathrm{AbPSh}\,\mathcal{L}$ and since $C \xrightarrow{i} A$ is a monomorphism, this implies that there exists $t: A \to C$ such that $ti = id_C$.

For any $W \in \mathcal{L}$, define presheaves C_W and A_W given by $C_W(U) = C(U \wedge W)$, $A_W(U) = A(U \wedge W)$ with the restrictions as given by A and C respectively. Denote by $i^W: C_W \to A_W$, and $t^W: A_W \to C_W$, the presheaf morphisms where $(i^W)U = i_{W \wedge U}: C(W \wedge U) \to A(W \wedge U)$, and $(t^W)_U = t_{W \wedge U}: A(W \wedge U) \to C(W \wedge U)$. Then since $ti = 1d_C$, it follows $t^W i^W = 1d_{C_W}$. Also let $r: C \to C_W$ and $s: A \to A_W$ be the morphisms in $\mathrm{AbPSh}\,\mathcal{L}$ given by the restriction maps of C and A respectively. Since C_W is a sheaf, from the way it is defined, r is actually a morphism of the sheaves. For any $U \in \mathcal{L}$,

$$\begin{tikzcd}
AU \arrow[r, "t_U"] \arrow[d, "s_U"'] & CU \arrow[d, "r_U"] \\
A(U \wedge W) \arrow[r, "t_{W \wedge U}"'] & C(U \wedge W)
\end{tikzcd}$$

commutes, since t is morphism of presheaves and hence

$rt = t^W s$. For $U \in \mathcal{L}$, define

$j_U: A_W U = A(W \wedge U) = \prod_{V \le W \wedge U} C_V \longrightarrow AU = \prod_{V' \le U} C_{V'}$ by

$$j_U(a) = \begin{cases} a(V') & \text{if } V' \le W \wedge U \\ 0 & \text{if } V' \nleq W \wedge U . \end{cases}$$

Then for $U' \le U$,

$$\begin{tikzcd} A_W U \arrow[r, "j_U"] \arrow[d] & AU \arrow[d] \\ A_W U' \arrow[r, "j_{U'}"'] & AU' \end{tikzcd}$$

commutes because of the nice way the j_U have been defined. Hence $j: A_W \to A$ is a morphism of presheaves and $s_U j_U = \mathrm{Id}_{A_W U}$ for all $U \in \mathcal{L}$ implies $sj = \mathrm{Id}_{A_W}$. This means s is a split epimorphism of presheaves. Now consider the composition of morphisms $tji^W: C_W \to C$, then $rtji^W = t_W sji^W = t_W \mathrm{Id}_{A_W} i^W = \mathrm{Id}_{C_W}$. This means r is a split epimorphism of sheaves. Hence $C \cong C_W \oplus K$ for some $K \in \mathrm{AbSh}\,\mathcal{L}$. But C is given to be an indecomposable group, so either $C_W = 0$ or $K = 0$. Therefore $C \to C_W$ is either an isomorphism or C_W is zero. Since $W \in \mathcal{L}$ was arbitrary, it follows $CE \to C_W E = CW$ is an isomorphism or $CW = 0$ for all $W \in \mathcal{L}$. Further, for any $V \le U$, $CU \to CV$ is an isomorphism or CV is zero according as $CE \to CV$ is isomorphism or $CV = 0$, this being because

$$\begin{tikzcd} & CE \arrow[dl] \arrow[dr] & \\ CU \arrow[rr] & & CV \end{tikzcd}$$

commutes.

Now define $\mathcal{F} = \{U \in \mathcal{L} \mid CU \neq 0\}$. We claim $\mathcal{F}$ is a completely prime filter on $\mathcal{L}$.

(i) If $V \in \mathcal{F}$ and $V \leq U$, then certainly $U \in \mathcal{F}$ because $CU \to CV$ is an isomorphism.

(ii) Let $U,V \in \mathcal{F}$ and $U \wedge V \notin \mathcal{F}$. Then $C(U \wedge V) = 0$. Now, by the sheaf properties the diagram

$$\begin{tikzcd} & C(U \vee V) \arrow[dl, "f"'] \arrow[dr, "g"] & \\ CU \arrow[dr] & & CV \arrow[dl] \\ & C(U \wedge V) = 0 & \end{tikzcd}$$

is a pull back in the category Ab. So if $C(U \wedge V) = 0$, then $C(U \vee V) = CU \times CV$. But f and g are isomorphisms, and so either $CU = 0$ or $CV = 0$, a contradiction, since $U,V \in \mathcal{F}$. Hence $C(U \wedge V) \neq 0$, and therefore $U \wedge V \in \mathcal{F}$.

(iii) Let $\bigvee_{i \in I} U_i \in \mathcal{F}$ for some family $\{U_i\}_{i \in I}$ in $\mathcal{L}$. Then $C(\bigvee_I U_i) \neq 0$. But $C(\bigvee_I U_i) \rightarrowtail \prod_{i \in I} CU_i$ and so for at least some $i \in I$, $CU_i \neq 0$. This means $U_i \in \mathcal{F}$ and hence $\mathcal{F}$ is a completely prime filter on $\mathcal{L}$. Let $\phi: \mathcal{L} \to 2$ be the associated local lattice homomorphism, then $\phi_*(CE): U \rightsquigarrow \begin{cases} CE & \text{if } U \in \mathcal{F} \\ 0 & \text{if } U \notin \mathcal{F} \end{cases}$ (0.20(2)) is

isomorphic to C. Finally, of course CE is injective since C is (0.24); moreover CE is indecomposable since C is, for otherwise we get a non-trivial decomposition of C.

($\leftarrow$) Conversely for any local lattice homomorphism $\phi: \mathcal{L} \to 2$ and any indecomposable injective $P \in Ab$, $\phi_*(P)$ is an indecomposable injective in $AbSh\mathcal{L}$ (0.20).//

Recall, that the space X is called <u>sober</u> iff for any completely prime filter $\mathcal{F} \subseteq \mathcal{O}(X)$, $\mathcal{F} = \mathcal{O}(x)$ for some unique $x \in X$.

<u>Corollary 1.7</u>: For a sober space X, the indecomposable injectives in $AbShX$ are upto isomorphism, exactly given as follows: For an indecomposable injective P in Ab, that is $P = Z(p^\infty)$ for some prime p, or $P = \mathbb{Q}$, and any $x \in X$,

$$C(P,x)U = \begin{cases} P & \text{if } x \in U \\ 0 & \text{if } x \notin U \end{cases}$$

with $C(P,x)U \to C(P,x)V$ the identity map if $x \in V$ and the zero map if $x \notin V$.

<u>Proof</u>: ($\rightarrow$) Suppose A is an indecomposable injective in $AbShX$. Then $\mathcal{F} = \{U \in \mathcal{O}(X) \mid AU \neq 0\}$ is a completely prime filter on $\mathcal{O}(X)$ (1.6). Hence $\mathcal{F} = \mathcal{O}(x)$ for some unique $x \in X$. Then, clearly $A \cong \phi_*(AE)$ where

$\phi = \mathfrak{X}$, that is $A \cong C(AE,x)$, AE an indecomposable injective in Ab (1.6).

($\Leftarrow$) True since $C(P,x)$ for an indecomposable injective P in Ab and $x \in X$, form an indecomposable injective in AbShX (0.20).

<u>Definition 1.8</u>: The category AbSh$\mathcal{L}$ is said to have enough indecomposable (torsion free) injectives iff every (torsion free) injective is a direct sum of indecomposable injectives.

<u>Proposition 1.9</u>: For a sober T_1 space X , the following are equivalent:

(1) AbShX has enough indecomposable injectives.

(2) AbShX has enough indecomposable torsion free injectives.

(3) $U \rightsquigarrow \mathbb{Q}^U$ is a direct sum of indecomposable injectives.

(4) X is discrete.

<u>Proof</u>: (1) $\Rightarrow$ (2) Clear, since every torsion free injective is a direct sum of indecomposable injectives which are also necessarily torsion free.

(2) $\Rightarrow$ (3) Since the group $A: U \rightsquigarrow \mathbb{Q}^U$ is torsion free injective (0.20), it is direct sum of indecomposable torsion free injectives.

(3) $\Rightarrow$ (4) By hypothesis the given group $A: U \rightsquigarrow \mathbb{Q}^U$ is direct sum of indecomposable injectives, that is $A = \oplus A_i$.

By the corollary above $A_i = C(A_iE,x_i)$ for some $x_i \in X$, for all $i \in I$. Consider $a \in AX = \mathbb{Q}^X$ the constant function with value 1. So $a \in AX = (\oplus C(A_iE,x_i))X$ and therefore there is a cover $X = \bigcup_{k\in K} U_k$ such that $a|U_k = \sum_{s=1}^{n} a_{i_s}$ where $a_{i_s} \neq 0$, $a_{i_s} \in A_{i_s}U_k$. Since $a_{i_s} \neq 0$, for $s = 1,2,\ldots n$ this means $x_{i_1},\ldots x_{i_n} \in U_k$. We claim $U_k = \{x_{i_1},\ldots x_{i_n}\}$. If not, then there is a $y \in U_k$ such that $y \neq x_{i_1},\ldots x_{i_n}$. Since X is T_1, we can find an open set $V \ni y$ such that $x_{i_s} \notin V$, $s = 1,2,\ldots n$. This means $A_{i_s}V = 0$ for $s = 1,2,\ldots n$, and so $a_{i_s}|(U_k \cap V) = 0$, hence $a|(U_k \cap V) = 0$. But a is a constant function with value 1, so $a|(U_k \cap V) = 0$. implies $V \cap U_k = \phi$, a contradiction, since $y \in V \cap U_k$. Hence our supposition that $y \in U_k$ is false, therefore $U_k = \{x_{i_1},\ldots x_{i_n}\}$. Thus X has a cover of open sets which are finite. if $x \in X$ is an arbitrary point, then $x \in U_k$ for some finite open set U_k. But X is T_1, and so there exists an open set $W \ni x$, such that $W \cap U_k = \{x\}$. Hence X is discrete.

(4) $\Rightarrow$ (1) If X is discrete, then $Ab^{|X|} \cong AbShX$ (0.15) and hence $AbShX$ has enough indecomposable injectives.

<u>Corollary 1.10</u>: If X is a sober T_1 space, then the functor $G: Ab^{|X|} \to AbShX$ preserves co-products iff X is discrete.

<u>Proof</u>: ($\leftarrow$) Clear, since $Ab^{|X|} \cong AbShX$.

($\rightarrow$) Consider $(\mathbb{Q})_{x\in|X|}$ in $Ab^{|X|}$, then $(\mathbb{Q})_{x\in|X|} = \bigoplus_x (\mathbb{Q}^x)$ where $(\mathbb{Q}^x) \in Ab^{|X|}$ is given by $\mathbb{Q}^x_z = \begin{cases} \mathbb{Q} & \text{if } z = x \\ 0 & \text{if } z \neq x \end{cases}$. Since G preserves co-products, it follows $G(\mathbb{Q})_{x\in|X|}) = \bigoplus_{x\in|X|} G(\mathbb{Q}^x)$. But for $A = G((\mathbb{Q})_{x\in|X|})$, $AU = \mathbb{Q}^U$, hence $A = \bigoplus_{x\in|X|} C(\mathbb{Q},x)$, and $G(\mathbb{Q}^x) = C(\mathbb{Q},x)$. By the last proposition it follows that X is discrete.

The obvious examples of non-discrete sober T_1-spaces are the non-discrete Hausdorff spaces. Hence, for any such space X, the above group $A \in AbShX$ where $AU = \mathbb{Q}^U$ is injective, but not the direct sum of indecomposable injectives.//

Recall, A family $\{A_i\}_{i\in I}$ of objects in any category $\mathcal{K}$, forms a cogenerating set, if for any $g \neq f: B \to C$ in $\mathcal{K}$, there exists an $i \in I$, and $h: C \to A_i$ such that $hg \neq hf$.

<u>Proposition 1.11</u>: The torsion indecomposable injectives in the category $AbSh\mathcal{L}$ form a cogenerating set iff $\mathcal{L}$ is spatial.

<u>Proof</u>: ($\rightarrow$) In order to show that $\mathcal{L}$ is spatial it is enough to show that the completely prime filters separate the elements of $\mathcal{L}$ (0.3). Let $U,V \neq U$ be arbitrary distinct elements of $\mathcal{L}$. Without loss of generality we may assume that $U \nleq V$. Then $S = U \wedge V < U$. Consider $\sigma: \mathcal{L} \to \uparrow S = \{W \in \mathcal{L} \mid W \geq S\}$ given by $\sigma(W) = W \vee S$. Then

σ is a local lattice homomorphism and so it produces $\sigma_*: \mathrm{AbSh}\uparrow S \to \mathrm{AbSh}\,\mathcal{L}$ (0.16), where $(\sigma_* B)W = B(W \vee S)$.

For any $0 \neq A$ in Ab, the group $A_{\mathcal{L}}$ in $\mathrm{AbSh}\,\mathcal{L}$ (0.16) has the property that $A_{\mathcal{L}} W \neq 0$ for all $W > 0$, $W \in \mathcal{L}$. In particular, for the locale $\mathcal{M} = \uparrow S$, we have $A_{\mathcal{M}} S = 0$, and $A_{\mathcal{M}} W \neq 0$ for all $W > S$. Therefore $(\sigma_* A_{\mathcal{M}})U = A_{\mathcal{M}}(U \vee S) = A_{\mathcal{M}}(U) \neq 0$ and $(\sigma_* A_{\mathcal{M}})S = A_{\mathcal{M}}(S \vee S) = A_{\mathcal{M}}(S) = 0$. If $0 \neq C \in A_{\mathcal{M}} U = (\sigma_* A_{\mathcal{M}})U$, then there exists $f: T_U Z_{\mathcal{L}} \to \sigma_* A_{\mathcal{M}}$ with $f_U(1_U) = C$, 1_U being the identity of $Z_{\mathcal{L}} U$. Since $f \neq 0$, so by hypothesis there exists an indecomposable torsion injective B and $h: \sigma_* A_{\mathcal{M}} \to B$ such that $hf \neq 0$. Hence $h_U f_U(1_U) = h_U(C) \neq 0$, implies $BU \neq 0$. We now show that $BS = 0$. Assume $BS \neq 0$, since h is a morphism of sheaves, we have a commutative diagram

$$\begin{tikzcd}
 & (\sigma_* A_{\mathcal{M}})U \arrow[r, "h_U"] \arrow[d] & BU \arrow[d] \\
0 = & (\sigma_* A_{\mathcal{M}})S \arrow[r, "h_S"'] & BS
\end{tikzcd}$$

and so $h_U(C)|S = h_S(C|S) = h_S(0) = 0$. But $BS \neq 0$, implies $BU \to BS$ is an isomorphism (1.6), and therefore $h_U(C)|S = 0$ means $h_U(C) = 0$, a contradiction, hence $BS = 0$. Since B is an indecomposable injective, the set of all $W \in \mathcal{L}$, such that $BW \neq 0$ forms a completely prime filter F on $\mathcal{L}$ (1.6). Therefore $U \in F$, and $S \notin F$, the latter further implies $V \notin F$ for otherwise $U \wedge V = S \in F$.

Hence for $V \neq U$ in $\mathcal{L}$, we can find a completely prime filter on $\mathcal{L}$ which separates U and V, therefore $\mathcal{L}$ is spatial.

($\Leftarrow$) Let $\mathcal{L} = \mathcal{O}(X)$, for some space X, then $\mathrm{AbSh}\,\mathcal{L} = \mathrm{AbSh}X$. If $A \in \mathrm{AbSh}X$, and $0 \neq b \in AU$, $U \in \mathcal{L}$, then for some $x \in U$, $0 \neq b_x \in A_x$. Let $\phi = \mathfrak{x}$, and $h: A \to \phi_*\phi^*A (= \phi_*(A_x))$ be the adjunction map (0.14). Since the family $\{Z(p^\infty)\}_{p,\text{prime}}$ cogenerates the category Ab, therefore for some prime p there exists a homomorphism $g: A_x \to Z(p^\infty)$, such that $g(b_x) \neq 0$. Therefore $(\phi_* g)h: A \xrightarrow[h]{} \phi_*\phi^*A (= \phi_*(A_x)) \xrightarrow[\phi_* g]{} \phi_*(Z(p^\infty))$ satisfies the condition that $(\phi_* g)_U h_U(b) = (\phi_* g)_U(b_x) = g(b_x) \neq 0$. Hence the family $\{\phi_*(Z(p^\infty))\}$ of torsion indecomposable injectives cogenerates the category $\mathrm{AbSh}\,\mathcal{L}$.

<u>Remark 1.12</u>: Suppose $\mathcal{L}$ is a locale such that every injective in $\mathrm{AbSh}\,\mathcal{L}$ is a direct sum of indecomposable injectives. Since $A \in \mathrm{AbSh}\,\mathcal{L}$ has an embedding into an injective B, by hypothesis $B = \oplus B_i$, where each B_i is an indecomposable injective in $\mathrm{AbSh}\,\mathcal{L}$. From proposition 1.6, each $B_i = \phi_*(BE)$ for some $\phi: \mathcal{L} \to 2$ where either $B_iE = Z(p^\infty)$ or $\mathbb{Q}$. If $B_iE = \mathbb{Q}$, then since $\mathbb{Q} \rightarrowtail \Pi Z(p^\infty)$, it follows by (0.16), that $\phi_*\mathbb{Q} \rightarrowtail \Pi\phi_*(Z(p^\infty))$. Hence every A in $\mathrm{AbSh}\,\mathcal{L}$ has an embedding into the product of groups of the type $\phi_*(Z(p^\infty))$, p prime where $\phi: \mathcal{L} \to 2$ is a local lattice homomorphism. That is the torsion

indecomposable injectives cogenerate the category $\mathrm{AbSh}\,\mathcal{L}$. Hence by the last proposition $\mathcal{L}$ is spatial.

Recall that a Boolean locale is spatial iff it is atomic. Thus if B is a non-atomic, Boolean locale then in the category $\mathrm{AbSh}B$ the torsion indecomposable injectives do not cogenerate $\mathrm{AbSh}B$. The standard example of a non-atomic Boolean locale is the complete Boolean algebra $\mathcal{R}X$ of regular open sets of a regular Hausdorff space X without isolated point. In fact $\mathcal{R}X$ is a Boolean algebra without any atoms, and hence $\mathrm{AbSh}\,\mathcal{R}X$ contains no indecomposable injectives.

Section 3: Injective Hulls

Given $A, B \in \mathrm{AbSh}\,\mathcal{L}$, recall that B is an injective hull of A iff it is an essential injective extension of A.

Proposition 1.13: B is an injective hull of A in $\mathrm{AbSh}\,\mathcal{L}$, iff there is a cover $E = \bigvee_{i \in I} U_i$, such that $B|U_i$ is an injective hull of $A|U_i$ in $\mathrm{AbSh}\downarrow U_i$.

Proof: ($\Rightarrow$) Clear, by taking the trivial cover. ($\Leftarrow$) Given that $B|U_i$ is the injective hull of $A|U_i$, all $i \in I$, it follows by 1.4 that B is injective in $\mathrm{AbSh}\,\mathcal{L}$. So, it only remains to show that B is an essential extension of A. Let $D \subseteq B$ be a non zero subgroup of B. Then $DU \neq 0$ for some $U \in \mathcal{L}$. Since $U = \bigvee_I U \wedge U_i$, it

follows $0 \neq DU \rightarrowtail \Pi D(U \wedge U_i)$ and so for some $i \in I$, $0 \neq D(U \wedge U_i) = (D|U_i)(U \wedge U_i)$. But $B|U_i$ is an essential extension of $A|U_i$ in $\mathrm{AbSh}\downarrow U_i$, so $0 \neq D|U_i \subseteq B|U_i$ implies $D|U_i \cap A|U_i \neq 0$, and therefore $D \cap A \neq 0$. Hence B is an essential extension of A, and also being injective, it follows B is the injective hull of A. //

<u>Remark 1.14</u>: In our next result, we describe the injective hull of any A in $\mathrm{AbSh}\,\mathcal{L}$ where $\mathcal{L}$ is well-ordered, and so it might be appropriate to describe the topology of the spectrum of a well-ordered locale. If $\mathcal{L}$ is well-ordered, then without loss of generality we may assume $\mathcal{L} = \lambda + 1$, for some ordinal λ. We now show that the sets $W_\alpha = \{\gamma: \gamma$ not a limit ordinal, $0 < \gamma \leq \alpha\}$ for each $\alpha \in \lambda + 1$, form a topology $\mathcal{O}$ on the set X consisting of all the non zero, non limit ordinals $\gamma \leq \lambda$. Now $W_0 = \phi$, $W_\lambda = |X|$, $W_\alpha \cap W_\beta = W_{\alpha\wedge\beta}$ since for $\alpha \leq \beta$, $W_\alpha \subseteq W_\beta$. To check $W_{\bigvee_I \alpha_i} = \bigcup_I W_{\alpha_i}$ for any family $\{\alpha_i\}_I$ in $\lambda + 1$, we consider $\gamma \in W_{\bigvee_I \alpha_i}$. Then $\gamma \leq \bigvee_I \alpha_i$, so if $\gamma \notin \bigcup_I W_{\alpha_i}$ then we must have $\alpha_i < \gamma$, for all $i \in I$. But $\gamma \in X$ and so $\gamma = \beta + 1$ for some $\beta < \lambda$. Therefore $\alpha_i < \gamma$ implies $\alpha_i \leq \beta$ for all $i \in I$, hence $\bigvee_I \alpha_i \leq \beta < \gamma$ a contradiction, since $\gamma \leq \bigvee_I \alpha_i$. Thus, there is some $i \in I$ such that $\gamma \leq \alpha_i$ and so $\gamma \in \bigcup_I W_{\alpha_i}$. Therefore $W_{\bigvee_I \alpha_i} \subseteq \bigcup_I W_{\alpha_i}$.

Moreover, for all $i \in I$, $\alpha_i \leq \bigvee_I \alpha_i$ implies $\bigcup W_{\alpha_i} \subseteq W_{\bigvee_I \alpha_i}$ and hence $W_{\bigvee_I \alpha_i} = \bigcup_I W_{\alpha_i}$. Therefore, $\mathfrak{D}$ is indeed a topology on X. Now let $W_\alpha = W_\beta$ for some $\alpha, \beta \in \lambda + 1$, and suppose $\alpha < \beta$. Then $\alpha + 1 \leq \beta$, and so $\alpha + 1 \in W_\beta = W_\alpha$ which means $\alpha + 1 \leq \alpha$, a contradiction. Hence $W_\alpha = W_\beta$ implies $\alpha = \beta$. Therefore $\mathcal{L} = \lambda + 1$ is isomorphic to $\mathfrak{D}$ by $\alpha \to W_\alpha$. Note that the space $(X, \mathfrak{D})$ is actually the spectrum of $\mathcal{L}$, since the sets $\uparrow r \subseteq \lambda + 1$, $r \in X$ are exactly the completely prime filters of $\lambda + 1$.

<u>Proposition 1.15</u>: For a well-ordered locale $\mathcal{L}$, the injective hull of any $A = A_\lambda \to \ldots \to A_2 \xrightarrow{h_2} A_1 \xrightarrow{h_1} A_0 (= 0)$ in $\mathrm{AbSh}\,\mathcal{L}$ is given by the group $C = C_\lambda \to \ldots \to C_2 \to C_1 \to C_0 (= 0)$ where $C_\beta = CW_\beta = \prod_{\alpha \in W_\beta} E(\mathrm{Ker}\, h_\alpha)$ for all $\beta \in \lambda + 1$.

<u>Proof</u>: Define a family $(B_\alpha)_{\alpha \in |X|}$ in $\mathrm{Ab}^{|X|}$ by $B_\alpha = E(\mathrm{Ker}\, h_\alpha)$ for all $\alpha \in X$. Since the functor $F: \mathrm{Ab}^{|X|} \to \mathrm{AbSh}X = \mathrm{AbSh}\,\mathcal{L}$ preserves injectives (0.20), it produces an injective C in $\mathrm{AbSh}\,\mathcal{L}$, where $C = F((B_\alpha)_{\alpha \in |X|})$, and so $C_\beta = CW_\beta = \prod_{\alpha \in W_\beta} E(\mathrm{Ker}\, h_\alpha)$, $\beta \in \lambda + 1$ with restrictions $CU_\beta \to CU_\gamma$ as projections for all $\gamma \leq \beta$. The morphism from A to C is obtained by induction as follows:

For $n = 1$, $A_1 \to C_1 = E(\operatorname{Ker} h_1) = E(A_1)$ is the natural embedding. Assume $A_\alpha \xrightarrow{\tau_\alpha} B_\alpha$ already defined for all $\alpha < \beta$. Then there are two possibilities

Case (i) β is not a limit ordinal, that is $\beta = \gamma + 1$ for some $\gamma \in \lambda + 1$.

Case (ii) β is a limit ordinal.

For the case (i) we are given $A_\gamma \to C_\gamma$ and since $C_{\gamma+1} = C_\gamma \times E(\operatorname{Ker} h_{\gamma+1})$ with $C_{\gamma+1} \to C_\gamma$ the projection, we can define $\tau_{\gamma+1}: A_{\gamma+1} \to C_\gamma \times E(\operatorname{Ker} h_{\gamma+1})$ as $(\tau_\gamma h_{\gamma+1}) \sqcap \overline{h_{\gamma+1}}$ where $\overline{h_{\gamma+1}}: A_{\gamma+1} \to E(\operatorname{Ker} h_{\gamma+1})$ is an extension of the natural embedding $\operatorname{Ker} h_{\gamma+1} \to E(\operatorname{Ker} h_{\gamma+1})$ to $A_{\gamma+1}$: then as required $p_\gamma \tau_{\gamma+1} = p_\gamma(\tau_\gamma h_{\gamma+1} \sqcap \overline{h_{\gamma+1}}) = \tau_\gamma h_{\gamma+1}$, that is,

$$\begin{tikzcd}
A_{\gamma+1} \arrow[r, "\tau_{\gamma+1}"] \arrow[d, "h_{\gamma+1}"'] & C_{\gamma+1} = C_\gamma \times E(\operatorname{Ker} h_{\gamma+1}) \arrow[d, "p_\gamma"] \\
A_\gamma \arrow[r, "\tau_\gamma"'] & C_\gamma
\end{tikzcd}$$

commutes.

Case (ii) $\beta = \bigvee_{\alpha<\beta} \alpha$, and so $C_\beta = \varprojlim_{\alpha<\beta} C_\alpha$. Since $A_\beta = \varprojlim_{\alpha<\beta} A_\alpha$, and by assumption all $A_\alpha \xrightarrow{\tau_\alpha} C_\alpha$ $(\alpha < \beta)$ are defined, therefore we get a family of maps

$A_\beta \to A_\alpha \xrightarrow{\tau_\alpha} C_\alpha$ $(\alpha < \beta)$ and so by the definition of limit there is a unique $\tau_\beta: A_\beta \to C_\beta$ such that

$$\begin{tikzcd} A_\beta \arrow[r, "\tau_\beta"] \arrow[d] & C_\beta \arrow[d] \\ A_\alpha \arrow[r, "\tau_\alpha"'] & C_\alpha \end{tikzcd}$$

commutes for all $\alpha < \beta$.

Hence we can define a morphism $\tau: A \to C$ with components $\tau_\alpha: A_\alpha \to C_\alpha$ as defined above.

Now to check that τ is a monomorphism. Clearly τ_1 is a monomorphism, so assume τ_α is a mono for all $\alpha < \beta$.

Case (i) If β is not a limit ordinal, then $\beta = \gamma + 1$. So if $\tau_{\gamma+1}(a) = 0$, then $\tau_\gamma h_{\gamma+1}(a) = 0 = \overline{h_{\gamma+1}}(a)$. This means $h_{\gamma+1}(a) = 0$, that is $a \in \operatorname{Ker} h_{\gamma+1}$. Hence $a = 0$, since $\overline{h_{\gamma+1}}(a) = a$ for $a \in \operatorname{Ker} h_{\gamma+1}$. Thus τ_β is a monomorphism.

Case (ii) If β is a limit ordinal, and $\tau_\beta(a) = 0$, then $\tau_\alpha(a|\alpha) = 0$ for all $\alpha < \beta$. But τ_α is a monomorphism for each $\alpha < \beta$, so $a|\alpha = 0$ which by the sheaf properties implies that $a = 0$. Thus the morphism $\tau: A \to C$ is indeed a monomorphism.

Finally, we want to show that C is an essential extension of A, and so consider $0 \neq D \subseteq C$. Since $\mathcal{L}$ is well ordered we can find a smallest $\alpha \in \mathcal{L}$ such that $D_\alpha \neq 0$. Then α is not a limit ordinal, since otherwise we get a contradiction $0 \neq D_\alpha \rightarrowtail \prod_{\gamma<\alpha} D_\gamma = 0$. For δ such that $\alpha = \delta + 1$, we have a commutative diagram

$$\begin{array}{ccccccc} D_\alpha & \subseteq & C_\alpha & = & \prod_{\gamma\in W_\alpha} E(\operatorname{Ker} h_\gamma) & = & C_\delta \times E(\operatorname{Ker} h_\alpha) \\ \downarrow & & \downarrow & & & & \\ 0=D_\delta & \subseteq & C_\delta & = & \prod_{\gamma\in W_\delta} E(\operatorname{Ker} h_\gamma) & & \end{array}$$

and so we conclude that $D_\alpha = 0 \times \overline{D_\alpha}$ where $\overline{D_\alpha} \subseteq E(\operatorname{Ker} h_\alpha)$. Hence there exists $0 \neq x \in \operatorname{Ker} h_\alpha$ such that $(0,x) \in D_\alpha$. Now $\operatorname{Im}\tau_\alpha = \operatorname{Im}(\tau_{\alpha-1} h_\alpha \sqcap \bar{h}_\alpha)$ and $(0,x) \in D_\alpha$ where $x \in \operatorname{Ker} h_\alpha$ implies $(0,x) \in \operatorname{Im}(\tau_{\alpha-1} h_\alpha \sqcap \bar{h}_\alpha) = \operatorname{Im}\tau_\alpha$. Hence $\tau_\alpha(A_\alpha) \cap D_\alpha \neq 0$ which means $\tau(A) \cap D \neq 0$. Thus $\tau: A \to C$ is an essential monomorphism. Also C is an injective group and therefore C describes the injective hull of A. //

Applied to the special case $\mathcal{L} = 3$, proposition 1.15 leads to the following:

<u>Corollary 1.16</u>: The injective hull of $\begin{array}{c} A \\ \downarrow h \\ B \end{array}$ is

$$\begin{array}{ccc} A & \xrightarrow{v} & E(B)\times E(\operatorname{Ker} h) \\ h \downarrow & & \downarrow \\ B & \xrightarrow[u]{} & E(B) \end{array}$$

where u embeds B into its injective hull and $v = (uh) \sqcap k$ where $k: A \to E(\operatorname{Ker} h)$ extends the natural embedding $\operatorname{Ker}(h) \to A$.

<u>Counterexample</u>: Injective hull is not a global property; a counterexample is given by $\mathcal{L} = 3$. The above

corollary shows $\begin{smallmatrix}\mathbb{Q}\\ \downarrow 1\\ \mathbb{Q}\end{smallmatrix}$ is the injective hull of $\begin{smallmatrix}0\\ \downarrow\\ \mathbb{Z}\end{smallmatrix}$ in AbSh3, although $\mathbb{Q}$ is not the injective hull of 0 in Ab.

However, we do have the following:

<u>Lemma 1.17</u>: The Global functor $\Gamma: \mathrm{AbSh}\,\mathcal{L} \to \mathrm{Ab}$ preserves injective hulls iff it preserves essential extensions.

<u>Proof</u>: ($\Leftarrow$) Clear, by the hypothesis and the fact that the functor Γ preserves injectives (since the functor $-_{\mathcal{L}}$ is an exact left adjoint of Γ, (0.16)).

($\Rightarrow$) Let $A \overset{i}{\rightarrowtail} B$ be an essential monomorphism in $\mathrm{AbSh}\,\mathcal{L}$, and $A \overset{j}{\rightarrowtail} A'$ be the natural embedding of A into its injective hull A'. Then there exists $f: B \to A'$ such that $fi = j$, which is actually a monomorphism (since i is essential). By hypothesis $AE \overset{i_E}{\rightarrowtail} A'E = AE \overset{i_E}{\rightarrowtail} BE \overset{f_E}{\rightarrowtail} A'E$ is an essential monomorphism in Ab, hence $i_E: AE \to BE$ is an essential monomorphism.//

<u>Lemma 1.18</u>: For a Boolean locale $\mathcal{L}$, if E is not compact, then there exists $A,B \in \mathrm{AbSh}\,\mathcal{L}$ such that $A \subseteq B$ is essential but $AE \subseteq BE$ is not essential in Ab.

<u>Proof</u>: Let $E = \bigvee_{i\in I} U_i$ where I is an infinite set. Then there exists a countable subset, say J of I,

so that we can write $E = (\bigvee_{i\in J} U_i) \vee S$ where $S = (\bigvee_{i\in J} U_i)'$. Since $\mathcal{L}$ is a Boolean, we can find a sequence $\{U_n\}_{n\in\omega}$ such that $U_n \wedge U_m = 0$ and $E = \bigvee_{n\in\omega} U_n$.

Define $A_n, B_n \in \mathrm{AbSh}{\downarrow}U_n$ by $A_nU \doteq Z/P_nZ$, $B_nU \doteq Z/P_n^2Z$ for some prime P_n where $P_n \neq P_m$ if $n \neq m$. Then $B_n \geq A_n$ is essential in $\mathrm{AbSh}{\downarrow}U_n$, for if $0 \neq \phi \in B_nU$ but $\notin A_nU$ then $\phi(a) \neq 0$ for some $a \in Z|(P_n^2)$ where order $a = P_n^2$ and so $0 \neq P_n\phi|\phi(a) \in A(\phi(a))$ which shows $A_n \leq B_n$ is essential. If $A,B \in \mathrm{AbSh}\mathcal{L}$ are defined by $A = \prod_{n\in\omega} (\alpha_n)_*A_n$, $B = \prod_{n\in\omega} (\alpha_n)_*B_n$, where $(\alpha_n)_*: \mathrm{AbSh}{\downarrow}U_n \to \mathrm{AbSh}\mathcal{L}$ corresponds to the morphism $\alpha_n: \mathcal{L} \to {\downarrow}U_n$, $U \rightsquigarrow U \wedge U_n$ (0.14 (4)), then $AU = \prod_{n\in\omega} A_n(U \wedge U_n)$ and $BU = \prod_{n\in\omega} B_n(U \wedge U_n)$, $U \in \mathcal{L}$. We claim $A \leq B$ is essential in $\mathrm{AbSh}\mathcal{L}$. If $0 \neq \phi = (\phi_n) \in BU$, then for some $m \in \omega$, $0 \neq \phi_m \in B_m(U \wedge U_m)$, and so by the above argument for some $a \in Z|(P_m^2)$, $P_m\phi_m|\phi_m(a) \in A_m(\phi_m(a))$. Since $U_m \wedge U_n = 0$ for all $n \neq m$, we get $P_m\phi|\phi_m(a) = (P_m\phi_n|\phi_m(a))_{n\in\omega} \in AU$, since all components are zero except when $n = m$. Hence $A \leq B$ is essential. To show now that $AE \leq BE$ is not essential, consider $\phi = (\phi_n) \in BE = \prod_{n\in\omega} B_nU_n$ where ϕ_n is of order P_n^2. If $AE \leq BE$ was essential, then there exists $k \in Z$ such that $k\phi_n \in A_nU_n$ for all $n \in \omega$. This means $P_n|k$ for all $n \in \omega$, hence $k = 0$. Hence the result.//

<u>Proposition 1.19</u>: The functor $\Gamma: \mathrm{AbSh}\,\mathcal{L} \to \mathrm{Ab}$ preserves injective hulls iff $\mathcal{L}$ is a finite Boolean locale.

<u>Proof</u>: ($\Leftarrow$) By lemma 1.17, it is enough to show that Γ preserves essential extensions. If $\mathcal{L}$ is finite Boolean then $\mathcal{L} = \mathcal{D}(X)$ for a finite discrete space X, therefore $\mathrm{AbSh}\,\mathcal{L} = \mathrm{Ab}^{|X|}$. So $A \subseteq B$ essential in $\mathrm{AbSh}\,\mathcal{L}$ implies $A\{x\} \subseteq B\{x\}$ is essential in Ab for all $x \in |X|$. Therefore $\prod_{x \in |X|} A\{x\} = \Gamma A \subseteq \Gamma B = \prod_{x \in |X|} B\{x\}$ is essential in Ab, since finite product in Ab preserve essential extension.

($\Rightarrow$) By lemma 1.17 it follows that the functor $\Gamma: \mathrm{AbSh}\,\mathcal{L} \to \mathrm{Ab}$ preserves essential extensions. We first show that $\mathcal{L}$ is Boolean. If not, then there exists $W \in \mathcal{L}$ such that W is dense. Let $AU \subseteq \mathcal{Q}_{\mathcal{L}} U$ be the subgroup consisting of all $\phi \in \mathcal{Q}_{\mathcal{L}} U$ such that $\bigvee_{0 \neq a \in \mathcal{Q}} \phi(a) \leq U \wedge W$. Then A is a subgroup of $\mathcal{Q}_{\mathcal{L}}$ ([4]). Define $B \in \mathrm{AbSh}\,\mathcal{L}$ by $BU = A(U \wedge W)$, with the restrictions as given by A. Then $h: A \to B$ given by the restriction map of A is a monomorphism, since W is dense in $\mathcal{L}$. Moreover, this monomorphism is essential, for if $0 \neq \phi \in BU = A(U \wedge W)$, then clearly $\phi|U \wedge W = h_{U \wedge W}(\phi) = \phi \neq 0$. By hypothesis $AE \overset{h_E}{\rightarrowtail} BE = AW = \mathcal{Q}_{\mathcal{L}} W$ is an essential monomorphism. Consider $\phi \in \mathcal{Q}_{\mathcal{L}} W$ with $\phi(1) = W$. By essentialness, there

exists $\psi \in AE$ such that $0 \neq h_E(\psi) = \psi|W = m\phi$ for some $m \in Z$. Then $(m\phi)(m) = W$, so $\psi(m) \wedge W = W$, that is $W \leq \psi(m)$, which means $\psi(m)$ is dense in $\mathcal{L}$. So if $k \neq m$, then $\psi(m) \wedge \psi(k) = 0$ implies $\psi(k) = 0$, therefore $\psi(m) = E$. But $\psi \in AE$, so $\bigvee_{k \neq 0} \psi(k) \leq W$, and $m \neq 0$ implies $\psi(m) \leq W$ that is $E = W$, hence $\mathcal{L}$ is Boolean. By lemma 1.18 it then follows that E is compact. But $\mathcal{L}$ Boolean implies each $U \in \mathcal{L}$ is compact, hence $\mathcal{L}$ is spatial. Therefore $\mathcal{L} = \mathfrak{D}(X)$ for some discrete space X, which by compactness of E means X is a finite discrete space. Hence the result.//

<u>Corollary 1.20</u>: For any $U \in \mathcal{L}$, the functor Γ_U: $\mathrm{AbSh}\,\mathcal{L} \to \mathrm{Ab}$, $A \to AU$ preserves injective hulls iff $\downarrow U$ is a finite Boolean locale.

<u>Proof</u>: ($\Rightarrow$) An argument similar to one in lemma 1.17 shows that the functor Γ_U preserves injective hulls iff it preserves essential extensions. Consider an essential extension $h: A \rightarrowtail B$ in $\mathrm{AbSh}\downarrow U$, then $E_U A \xrightarrow{E_U h} E_U B$ is an essential extension in $\mathrm{AbSh}\,\mathcal{L}$ (0.22). By hypothesis $(E_U A)U \xrightarrow{(E_U h)_U} (E_U B)U = AU \xrightarrow{h_U} BU$ is an essential extension in Ab, and hence by proposition 1.19, $\downarrow U$ is a finite Boolean locale.

($\Leftarrow$) Let D be the injective hull of C in $\mathrm{AbSh}\,\mathcal{L}$. Then

$R_U D$ is the injective hull of $R_U C$ in $AbSh{\downarrow}U$. By proposition 1.19, we get that DU is the injective hull of CU in Ab. //

Section 4: Characterizing Injectives for some Special Locales

We have seen in our previous discussion that an injective $A \in AbSh\mathcal{L}$ has the following two properties:

(a) For all $U \in \mathcal{L}$, each AU is an injective abelian group in Ab.

(b) Whenever $V \leq U$ in $\mathcal{L}$, then the restriction $AU \to AV$ is a split epimorphism in Ab.

Hence it is reasonable to ask whether the properties (a) and (b) characterize injectives in $AbSh\mathcal{L}$. The answer is yes, for some special locales, which we shall soon discuss, although the question still remains open for an arbitrary $\mathcal{L}$.

Recall that for $\mathcal{L} = 3$, B. Banaschewski has shown that injectives in AbSh3 are exactly those groups which satisfy the conditions (a) and (b), [4]. This fact is crucial in the following proofs.

Proposition 1.21: If $\mathcal{L}$ satisfies the Descending chain condition then $A \in AbSh\mathcal{L}$ is injective iff it satisfies the conditions (a) and (b).

Proof: To prove the remaining implication, consider

any essential extension $B \geq A$. If $A \subsetneq B$, then since $\mathcal{L}$ has DCC, we can find a minimal $S \in \mathcal{L}$ such that $AS \subset BS$. Clearly, for all $U < S$, $AU = BU$. If $W = \vee U(U < S)$ then $AW = BW$, since for any $b \in BW$, $b|U \in BU = AU$ for $U < S$ implies $b \in AW$, hence $W < S$. Consider the commutative diagram,

$$\begin{array}{ccc} AS & \subset & BS \\ \downarrow & & \downarrow \\ AW & = & BW \end{array}$$

in AbSh3. If $0 \neq b \in BS$, then by essentialness there exists $V \leq S$ and $m \in Z$ such that $0 \neq mb|V \in AV$. Now either $V = S$ which means $0 \neq mb \in AS$, or $V < S$ and then $V \leq W$ so $0 \neq mb|V = (mb|W)|V$ implies $0 \neq mb|W \in BW = AW$. Thus $\begin{smallmatrix} BS \\ \downarrow \\ BW \end{smallmatrix}$ is an essential extension of $\begin{smallmatrix} AS \\ \downarrow \\ AW \end{smallmatrix}$ in AbSh3. But, by the given hypothesis $\begin{smallmatrix} AS \\ \downarrow \\ AW \end{smallmatrix}$ is injective in AbSh3 [4], and hence $AS = BS$. Thus $A = B$, which means A is injective in AbSh$\mathcal{L}$. Therefore (a) and (b) characterize injectives in AbSh$\mathcal{L}$ where $\mathcal{L}$ has DCC on its elements.

<u>Corollary 1.22</u>: If $\mathcal{L}$ is finite or well-ordered then the conditions (a) and (b) characterize injectives in AbSh$\mathcal{L}$.

<u>Proof</u>: Clear, since these locales have descending chain conditions.

<u>Proposition 1.23</u>: For any inversely well-ordered $\mathcal{L}$, $A \in \mathrm{AbSh}\,\mathcal{L}$ is injective iff it satisfies the conditions (a) and (b).

<u>Proof</u>: If $\mathcal{L}$ is inversely well-ordered, then the elements of $\mathcal{L}$ may be arranged in the form $E = U_0 > U_1 > U_2 \ldots > U_\lambda = 0$, so that $\mathcal{L}^{opp} \cong \lambda + 1$ for some ordinal λ. Since each non-empty subset of $\mathcal{L}$ has a largest element, it follows that every element in $\mathcal{L}$ has only trivial covers, hence every presheaf on $\mathcal{L}$ is also a sheaf on $\mathcal{L}$. In particular, $Z_{\mathcal{L}} U_\alpha = Z$ for all α. If $A \in \mathrm{AbSh}\,\mathcal{L}$ satisfies the conditions (a) and (b), then we claim that A is injective. The proof will use the Baer Criterion, so consider a diagram

$$\begin{array}{c} C \subseteq Z_{\mathcal{L}} \\ h \downarrow \\ A \end{array}$$

Our aim is to extend h to all of $Z_{\mathcal{L}}$.

If $C = 0$, then we are done. If $C \neq 0$, then we can pick the first α_0 such that $CU_{\alpha_0} \neq 0$. If $U_\alpha > U_\beta$, then the commutativity of the diagram

$$\begin{array}{ccc} CU_\alpha \subseteq Z_{\mathcal{L}} U_\alpha & = & Z \\ \downarrow & & \downarrow \mathrm{Id} \\ CU_\beta \subseteq Z_{\mathcal{L}} U_\beta & = & Z \end{array}$$

implies $CU_\alpha \subseteq CU_\beta$. Let U_{α_1} be the first element in $\mathcal{L}$ such that $CU_{\alpha_0} \subsetneq CU_{\alpha_1}$. Proceeding in the same fashion we obtain a strictly ascending chain of subgroups of Z given by $0 \neq CU_{\alpha_0} \subset CU_{\alpha_1} \subset CU_{\alpha_2} \subset \ldots$. Since Z is noetherian, this chain must terminate after a finite number of steps and so for some n $CU_{\alpha_n} = CU_\alpha$ for all $\alpha \geq \alpha_n$.

If we consider the finite chain $F = U_{\alpha_0} > U_{\alpha_1} > \ldots > U_{\alpha_n}$ (which has only trivial covers), then the presheaf $AU_{\alpha_0} \to AU_{\alpha_1} \to \ldots \to AU_{\alpha_n}$ satisfies the conditions (a) and (b) and so by our last result it is an injective group in AbShF. Hence there exists morphisms $g_{U_{\alpha_i}} : Z \to AU_{\alpha_i}$, such that $g_{U_{\alpha_i}}|CU_{\alpha_i} = h_{U_{\alpha_i}}$, and $g_{U_{\alpha_{i+1}}} = g_{U_{\alpha_i}}|U_{\alpha_{i+1}}$ for all $i = 0,1,\ldots n$.

For any $U_\alpha \in \mathcal{L}$, where $\alpha \neq \alpha_0, \alpha_1, \ldots \alpha_n$ define g_{U_α} as follows:

$g_{U_\alpha} = i_\alpha g_{U_{\alpha_0}}$ if $0 \le \alpha < \alpha_0$ where $i_\alpha : AU_{\alpha_0} \to AU_\alpha$ is the inclusion into the product $AU_{\alpha_0} \to AU_0$ followed by the restriction map $AU_0 \to AU_\alpha$

$g_{U_\alpha} = g_{U_{\alpha_0}}|U_\alpha$ if $\alpha_0 \le \alpha < \alpha_1$

$g_{U_\alpha} = g_{U_{\alpha_1}}|U_\alpha$ if $\alpha_1 \le \alpha < \alpha_2$

$\vdots$

$g_{U_\alpha} = g_{U_{\alpha_n}}|U_\alpha$ if $\alpha \ge \alpha_n$

Since $g_{U_{\alpha_{i+1}}} = g_{U_{\alpha_i}}|U_{\alpha_{i+1}}$, it follows for all $U_\alpha \le U_{\alpha_0}$, we have $g_{U_\alpha} = g_{U_{\alpha_0}}|U_\alpha$. It remains to show, that g is a morphism of presheaves and that g extends h. So let U_α, U_β be arbitrary elements of $\mathcal{L}$ such that $U_\alpha \le U_\beta$. Then there are three cases:

(i) $U_{\alpha_0} \ge U_\beta \ge U_\alpha$

(ii) $U_\beta \ge U_{\alpha_0} \ge U_\alpha$

(iii) $U_\beta \ge U_\alpha \ge U_{\alpha_0}$.

Case (i). In this case $g_{U_\beta} = g_{U_{\alpha_0}}|U_\beta$, so

$g_{U_\beta}|U_\alpha = \left(g_{U_{\alpha_0}}|U_\beta\right)|U_\alpha = g_{U_{\alpha_0}}|U_\alpha = g_{U_\alpha}$ hence

$$\begin{tikzcd} & Z \arrow[dl, "g_{U_\beta}"'] \arrow[dr, "g_{U_\alpha}"] & \\ AU_\beta \arrow[rr] & & AU_\alpha \end{tikzcd}$$

commutes.

Case (ii). $g_{U_\alpha} = i_\beta g_{U_{\alpha_0}}$ hence

$g_{U_\beta}|U_\alpha = \left(g_{U_\beta}|U_{\alpha_0}\right)|U_\alpha = \left(\left(i_\beta g_{U_{\alpha_0}}\right)|U_{\alpha_0}\right)|U_\alpha = g_{U_{\alpha_0}}|U_\alpha = g_{U_\alpha}$.

Case (iii). $g_{U_\beta} = i_\beta g_{U_{\alpha_0}}$, therefore

$g_{U_\beta}|U_\alpha = \left(i_\beta g_{U_{\alpha_0}}\right)|U_\alpha = i_\alpha g_{U_{\alpha_0}} = g_{U_\alpha}$.

Hence, we conclude that g is indeed a morphism of sheaves. Now, to check that g extends h, we consider any $U_\alpha \in \mathcal{L}$. If $U_\alpha > U_{\alpha_0}$, then $CU_\alpha = 0$, so $g_{U_\alpha}|C_{U_\alpha} = 0 = h_{U_\alpha}$. So let us suppose now, that $U_{\alpha_0} \geq U_\alpha$. Then $g_{U_\alpha}|CU_\alpha = g_{U_\alpha}|CU_{\alpha_0}$ (if $CU_\alpha = CU_{\alpha_0}$) $= \left(g_{U_{\alpha_0}}|CU_{\alpha_0}\right)|U_\alpha = h_{U_{\alpha_0}}|U_\alpha = h_{U_\alpha}$. If $CU_\alpha \neq CU_{\alpha_0}$, then $\alpha_1 \leq \alpha$. If $CU_\alpha = CU_{\alpha_1}$, then again we are done by the same argument with α_1 in place of α_0 since $g_{U_{\alpha_1}}|CU_{\alpha_1} = h_{U_{\alpha_1}}$. Otherwise, $CU_\alpha \supset CU_{\alpha_1}$ and in that case $\alpha_2 \leq \alpha$ and one can proceed as before. Continuing in the

same way one sees that $g_{U_\alpha}|CU_\alpha = h_{U_\alpha}$ for all α, that is g extends h. That shows A is injective.

<u>Corollary 1.24</u>: In $\mathrm{AbSh}\,\mathcal{L}$ where $\mathcal{L}$ is inversely well-ordered, the direct sum of injectives is injective.

<u>Proof</u>: Let $A = \bigoplus_I A_i$, where each A_i is an injective group in $\mathrm{AbSh}\,\mathcal{L}$. Then each A_iU is divisible in Ab, all $U \in \mathcal{L}$. Therefore $AU = \oplus A_iU$ is divisible in Ab. For any $V \leq U$ in $\mathcal{L}$, each $A_iU \to A_iV$ is a split epi, therefore $\bigoplus_I A_iU \to \bigoplus_I A_iV$, that is, $AU \to AV$ is a split epi in Ab. By proposition 1.23, A is injective, hence the result.

<u>Counterexample 1.25</u>: Here is a counterexample showing that the direct sum of injectives in $\mathrm{AbSh}\,\mathcal{L}$ is not always injective for an arbitrary $\mathcal{L}$: Consider an infinite space X, with the topology given by $U \in \mathfrak{O}X$ iff $U = X$, ϕ or $U \not\ni x$ where x is a fixed point of X. Then $\{y\} \in \mathfrak{O}X$ iff $y \neq x$. For all $z \in |X|$, define $A_z = \phi_*(\mathbb{Q})$ where $\phi: \mathcal{L} \to 2$ is the local lattice homomorphism corresponding to the point $z \in |X|$ (0.20). Then $A_zU \subseteq B: U \rightsquigarrow \mathbb{Q}^U$ consists of all $a \in BU$, with support contained in $\{z\}$. Let $A = \bigoplus_{z \in |X|} A_z$. We claim A is not injective, although each A_z is an injective group (0.20).

Note that A can be taken as a subgroup of B, and $f \in BX$ belongs to AX iff there exists a cover $X = \bigcup_{i \in I} U_i$ such that $f|U_i$ is of finite support for all $i \in I$. Since X has only trivial covers it follows $X = U_i$ for some $i \in I$. Hence AX consists of all f in $\mathbb{Q}^X$ of finite support, and so $AX \subset BX$. Note that for $X \neq U$, $AU = BU$. Now let $0 \neq a \in BX$. If $a(y) \neq 0$ for any $y \neq x$ then $0 \neq a|\{y\} \in A\{y\}$, and otherwise $a(y) = 0$ for all $y \neq x$ so that $a \in AX$. Hence B is an essential extension of A and therefore, A is not injective. Since B is also injective (0.20), it follows B is the injective hull of A.

<u>Lemma 1.26</u>: If $\mathcal{L}$ is an inversely well-ordered locale, then for any injective $0 \neq C$ in $\mathrm{AbSh}\,\mathcal{L}$, there exists an indecomposable injective $0 \neq D \subseteq C$.

<u>Proof</u>: Let C be expressed as

$$CE = CU_0 \xrightarrow{h_0} CU_1 \to \ldots \to CU_\alpha \xrightarrow{h_\alpha} CU_{\alpha+1} \to \ldots \to 0$$

where the restrictions are split epi as C is given to be injective. Since $\mathcal{L}$ is inversely well-ordered we can pick the first α such that h_α is not an isomorphism. Let $K_\alpha = \mathrm{Ker}\, h_\alpha$ and K_β be the inverse image of K_α under the isomorphism $CU_\beta \to CU_\alpha$ for all $\beta < \alpha$. Consider now the non-zero injective group K_0 in Ab. If Q_0 is

an indecomposable injective such that $Q_0 \subseteq K_0$, then we get $D \in \mathrm{AbSh}\,\mathcal{L}$ given by

$$Q_0 \overset{\sim}{\to} Q_1 \overset{\sim}{\to} \ldots \overset{\sim}{\to} Q_\alpha \to 0 \to \ldots \to 0$$

where $Q_\beta = h_{\beta-1}(Q_{\beta-1})$ for $0 < \beta \leq \alpha$. Since $\{\beta \mid \beta \leq \alpha\}$ is a completely prime filter in $\mathcal{L}$, D is an indecomposable injective by proposition 1.6.

<u>Proposition 1.27</u>: If $\mathcal{L}$ is inversely well-ordered, then every injective in $\mathrm{AbSh}\,\mathcal{L}$ is a direct sum of indecomposable injectives.

<u>Proof</u>: Let $A \in \mathrm{AbSh}\,\mathcal{L}$ be an injective group. Let $\mathcal{A}$ be the collection of all sets $\mathfrak{A}$ of indecomposable injective subgroups of A whose sum is direct, partially ordered by inclusion. Consider any chain $\mathcal{C}$ in $\mathcal{A}$. We claim that $\mathcal{B} = \cup\mathfrak{A}(\mathfrak{A} \in \mathcal{C})$ is an upper bound of $\mathcal{C}$ in $\mathcal{A}$, that is, the sum $\sum B(B \in \mathcal{B})$ is direct. For this we have to show that

$$B_0 \cap \sum_{B \in \mathcal{B} - \{B_0\}} B = 0$$

for any $B_0 \in \mathcal{B}$. Let $\mathfrak{A}_0 \in \mathcal{C}$ be such that $B_0 \in \mathfrak{A}_0$. Then

$$\sum_{B \in \mathcal{B} - \{B_0\}} B = \Big(\bigcup_{B_1, \ldots, B_n \in \mathcal{B} - \{B_0\}} (B_1 + \ldots + B_n) \Big)^{\sim} = \Big(\bigcup_{\mathfrak{A}_0 \subseteq \mathfrak{A} \in \mathcal{C}} \; \bigcup_{B_1, \ldots, B_n \in \mathfrak{A} - \{B_0\}} (B_1 + \ldots + B_n) \Big)^{\sim}$$

and in $\mathrm{AbPSh}\mathcal{L}$ we have

$$B_0 \cap \bigcup_{B_1,\ldots,B_n \in \mathcal{B}-\{B_0\}} (B_1 + \ldots + B_n) = B_0 \cap \bigcup_{\mathfrak{A}_0 \subseteq \mathfrak{A} \in \mathcal{C}} \bigcup_{B_1,\ldots,B_n \in \mathfrak{A}-\{B_0\}} (B_1 + \ldots + B_n)$$

$$= \bigcup_{\mathfrak{A}_0 \subseteq \mathfrak{A} \in \mathcal{C}} \bigcup_{B_1,\ldots,B_n \in \mathfrak{A}-\{B_0\}} B_0 \cap (B_1 + \ldots + B_n) = 0$$

since all $B_0 \cap (B_1 + \ldots + B_n) \subseteq B_0 \cap \sum B(B \in \mathfrak{A} - \{B_0\}) = 0$

Therefore

$$0 = (B_0 \cap \bigcup_{B_1,\ldots,B_n \in \mathcal{B}-\{B_0\}} (B_1 + \ldots + B_n))^{\sim} = B_0 \cap \sum_{B \in \mathcal{B}-\{B_0\}} B .$$

By Zorn's lemma, there exists a maximal such family $\mathfrak{A}$ in $\mathcal{A}$. By corollary 1.24, $C = \sum_{B \in \mathfrak{A}} B$ is an injective group, and $C \subset A$ implies $A = H \oplus C$ for some $0 \neq H \subseteq A$.

By lemma 1.26, there exists an indecomposable injective $0 \neq D \subseteq H$, and therefore the family $\mathfrak{A} \cup \{D\}$ belong to $\mathcal{A}$. This contradicts the maximality of $\mathfrak{A}$, hence $A = C = \oplus B(B \in \mathfrak{A})$, which proves the proposition.//

CHAPTER 2

ESSENTIAL EXTENSIONS OF TORSION GROUPS

Introduction: This chapter concentrates on torsion groups in $\mathrm{AbSh}\mathcal{L}$ and their essential extensions which are not always torsion groups for an arbitrary $\mathcal{L}$. In section 1, we show that torsion is a local property but not a global one, that is, A torsion in $\mathrm{AbSh}\mathcal{L}$ does not necessarily imply that AE is a torsion group in Ab. However if $\mathcal{L}$ has ACC, then torsion implies global torsion. As a consequence of the sheaf reflection being a left adjoint, we will show that every torsion group is a direct sum of its p-primary components. Recall that in the category Ab, the torsion subgroup of an injective group is always injective. We show by giving an example that this does not hold in $\mathrm{AbSh}\mathcal{L}$ for an arbitrary $\mathcal{L}$.

In section 2, we show that in $\mathrm{AbSh}\mathcal{L}$, essential extensions of torsion groups are torsion iff every injective group splits into a direct sum of a torsion group and a torsion free group. For Boolean locales and any finite locale, the above result holds. After proving some more results about essential extensions of torsion groups, we conclude our chapter by showing that, for a first countable Hausdorff space X, essential extensions of torsion groups in AbShX are torsion groups iff X is discrete.

Section 1: Torsion Groups

Recall from Chapter 0 that $A \in \mathrm{AbSh}\mathcal{L}$ is said to be a torsion group if $A = \bigcup_{0 \neq n \in N} \mathrm{Ker}\, n_A$. This means, for each $a \in AU$, there exists a cover $U = \bigvee_{i \in I} U_i$ and $0 \neq n_i \in N$, such that $n_i a|U_i = 0$ for all $i \in I$.

Definition 2.1: (i) By the p-primary component of the group $A \in \mathrm{AbSh}\mathcal{L}$ we mean the subgroup of A given by $\bigcup_{0 \neq n \in N} \mathrm{Ker}\, p_A^n$. We denote the p-primary component of A by Ap.

(ii) $A \in \mathrm{AbSh}\mathcal{L}$ is called a p-primary group if $A = Ap$.

Proposition 2.2: $A \in \mathrm{AbSh}\mathcal{L}$ is torsion iff there is a cover $E = \bigvee_{i \in I} U_i$ such that $A|U_i$ is torsion in $\mathrm{AbSh}{\downarrow}U_i$ for all $i \in I$.

Proof: ($\Rightarrow$) Clear by taking the trivial cover of E. On the other hand, if all $A|U_i$ are torsion groups in $\mathrm{AbSh}{\downarrow}U_i$, we claim A is torsion. So, consider an arbitrary $b \in AU$, $U \in \mathcal{L}$. Then $U = \bigvee_{i \in I} U \wedge U_i$, and $b|U \wedge U_i \in A(U \wedge U_i) = A|U_i(U \wedge U_i)$ for all $i \in I$. But $A|U_i$ is torsion in $\mathrm{AbSh}{\downarrow}U_i$, and so for each $i \in I$, there is a cover $U \wedge U_i = \bigvee_{j \in J_i} W_{ji}$ and $0 \neq n_{ji} \in N$ such that $n_{ji} b|W_{ji} = 0$, $j \in J_i$. Hence for $b \in AU$, we can find

a cover $U = \bigvee_{\substack{j\in J_i \\ i\in I}} W_{ji}$ such that $n_{ji}b|W_{ji} = 0$ for all i, j where $0 \neq n_{ji} \in N$, which shows that A is a torsion group.

<u>Counterexample 2.3</u>: Proposition 1 shows that torsion is a local property. However it is not a global property as we shall see from the following counterexample: Consider $\mathcal{L} = \omega$ and $A \in \mathrm{AbSh}\mathcal{L}$ given by

$$\begin{array}{ccccccccccc} \prod_{n<\omega} Z/nZ & \to & \dots & \to & Z/2Z \times Z/3Z & \to & Z/2Z & \to & 0 \left(= \frac{Z}{Z}\right) & \longrightarrow & 0 \\ \omega & > & \dots\dots & > & 3 & > & 2 & > & 1 & > & 0 \end{array}$$

By proposition 2.2, A is torsion, since for the cover $\omega = \bigvee_{n<\omega} n$, $A|n = \prod_{k\leq n} Z/kZ$ is torsion in $\mathrm{AbSh}\downarrow n$ for all $n < \omega$. But certainly $A\omega = \prod_{n<\omega} Z/nZ$ is not torsion in Ab, as the element $(1 + nZ)_{n<\omega}$ does not have a finite order.

<u>Definition 2.4</u>: By the torsion subgroup B of an arbitrary $A \in \mathrm{AbSh}\mathcal{L}$, we mean the subgroup of A given by $B = \bigcup_{0\neq n\in N} \mathrm{Ker}\, n_A$.

<u>Proposition 2.5</u>: Every torsion group is a direct sum of its p-primary components.

<u>Proof</u>: Let A be a torsion group and denote by B

the presheaf $BU = t(AU)$, the torsion subgroup of AU. Then A is the sheaf reflection of $B (A = \tilde{B})$. Now $BU = t(AU) = \bigoplus_p (t(AU))_p$ where $(t(AU))_p$ denotes the p-primary component of $t(AU)$. If $B_p \subseteq B$ is the subpresheaf $B_pU = (t(AU))_p$, then clearly $B = \bigoplus_p B_p$ in $\mathrm{AbPSh}\,\mathcal{L}$. The sheaf reflection being a left adjoint preserves co-limits, in particular direct sums, and so $A = \tilde{B} = (\bigoplus_p B_p)^{\sim} = \bigoplus_p \tilde{B}_p$. But $\tilde{B}_p = A_p$, and hence we get $A = \bigoplus_p A_p$.

<u>Definition 2.6</u>: By the <u>torsion type</u> of a group A we mean the set of all prime numbers p such that $A_p \neq 0$.

<u>Proposition 2.7</u>: If A is a torsion group and $B \supseteq A$ is an essential extension, then B and A have the same torsion type.

<u>Proof</u>: Since $A \subseteq B$, it follows $A_p \subseteq B_p$ and therefore $A_p \subseteq A \cap B_p$ for all p. Consider any $U \in \mathcal{L}$, then

$$\begin{aligned}
(A \cap B_p)U = AU \cap B_pU &= AU \cap (\cup \operatorname{Ker} p_B^n)U \\
&\doteq AU \cap \Big(\bigcup_{0 \neq n \in N} \operatorname{Ker} p_{BU}^n\Big) \\
&\doteq \bigcup_{0 \neq n \in N} AU \cap \operatorname{Ker} p_{BU}^n \\
&\doteq \bigcup_{0 \neq n \in N} (A \cap \operatorname{Ker} p_B^n)U \\
&\doteq \bigcup_{0 \neq n \in N} (\operatorname{Ker} p_A^n)U = A_pU
\end{aligned}$$

Hence $A \cap B_p = A_p$ for all primes p.

We now want to show that $A_p \subseteq B_p$ is an essential extension. If $0 \neq C \subseteq B_p$, then since $A \subseteq B$ is essential it follows $A \cap C \neq 0$. This means $0 \neq A \cap C \cap B_p = A_p \cap C$, thereby showing that $A_p \subseteq B_p$ is essential. Hence $B_p \neq 0$ iff $A_p \neq 0$ which means that A and B have the same torsion type.

<u>Definition 2.8</u>: Let us call a group P in $\mathrm{AbPSh}\mathcal{L}$ to be a pre-torsion group if for each $a \in PU$, there exists a cover $U = \bigvee_{i \in I} U_i$, $0 \neq n_i \in N$ such that $n_i a|U_i = 0$ for all $i \in I$.

<u>Lemma 2.9</u>: If $P \in \mathrm{AbPSh}\mathcal{L}$ is pretorsion, then its sheaf reflection $\tilde{P}$ is a torsion group in $\mathrm{AbSh}\mathcal{L}$.

<u>Proof</u>: Consider any $a \in \tilde{P}U$, then there is a cover $U = \bigvee_{k \in K} U_k$ such that $a|U_k \in PU_k$ for all $k \in K$. Since P is a pretorsion presheaf, it follows for each $k \in k$ there exists a cover $U_k = \bigvee_{j \in J_k} W_{jk}$, and $0 \neq n_{jk} \in N$ such that $0 = n_{jk}(a|U_k)|W_{jk} = n_{jk}a|W_{jk}$ for all $j \in J_k$, $k \in K$. Hence for $a \in \tilde{P}U$, we can find a cover $U = \bigvee_{\substack{j \in J_k \\ k \in K}} W_{jk}$ and $0 \neq n_{jk} \in N$ such that $n_{jk}a|W_{jk} = 0$ for all j, k. This shows P is torsion in $\mathrm{AbSh}\mathcal{L}$.

<u>Proposition 2.10</u>: The direct limit of a family of torsion groups is again a torsion group.

<u>Proof</u>: Let $\{A_i\}_{i\in I}$ be a directed family of torsion groups in $\mathrm{AbSh}\mathcal{L}$. We shall show that the presheaf $B: U \rightsquigarrow \underrightarrow{\ell t}\, A_i U$ is a pre-torsion presheaf. So consider an $a \in \underrightarrow{\ell t}\, A_i U$, then there is some $k \in I$ such that $a \in \mathrm{Image}(\tau_k)_U$, where $(\tau_k)_U: A_k U \to \underrightarrow{\ell t}\, A_i U$ is the limit map. Hence $a = (\tau_k)_U(a_k)$ where $a_k \in A_k U$. Since A_k is a torsion group, there exists a cover $U = \bigvee_{j\in J_k} W_j$ and $0 \neq n_j \in N$ such that $n_j a_k|W_j = 0$ for all $j \in J_k$. This means $n_j a|W_k = (\tau_k)_U(n_j a_k|W_j) = 0$. Hence for $a \in \underrightarrow{\ell t}\, A_i U$, we can find a cover $U = \bigvee_{j\in J_k} W_j$, and $0 \neq n_j \in N$, such that $n_j a|W_j = 0$ for all $j \in J_k$. Hence B is a pretorsion presheaf. Since $\underrightarrow{\ell t}\, A_i = \tilde{B}$, it follows by the lemma above that $\underrightarrow{\ell t}\, A_i$ is a torsion group in $\mathrm{AbSh}\mathcal{L}$. //

<u>Proposition 2.11</u>: If $\{A_i\}_{i\in I}$ is any family of torsion groups in $\mathrm{AbSh}\mathcal{L}$, then $\bigoplus_{i\in I} A_i$ is also a torsion group.

<u>Proof</u>: Obviously, if A and B are torsion groups in $\mathrm{AbSh}\mathcal{L}$ then $A \times B$ is a torsion group; hence $\bigoplus_{i\in I} A_i$ is a torsion group, as direct limit of the torsion group $A_{i_1} \times \ldots \times A_{i_n}$, by Proposition 2.10.

<u>Proposition 2.12</u>: Any epimorphic image of a torsion group is torsion.

<u>Proof</u>: Consider an epimorphism $h: A \to B$ in $\mathrm{AbSh}\mathcal{L}$ where A is given to be a torsion group. By definition of epimorphism $BU \doteq \mathrm{Image}\, h_U(AU)$. If $b \in \mathrm{Im}h_U(AU)$, then $b = h_U(a)$ for some $a \in AU$. Since A is torsion, therefore there exists a cover $U = \bigvee_{i \in I} U_i$, and $0 \neq n_i \in N$ such that $n_i a|U_i = 0$ all $i \in I$. Hence, we get $n_i b|U_i = n_i h_U(a)|U_i = h_{U_i}(n_i a|U_i) = 0$ for all $i \in I$. Thus the presheaf defining B is pretorsion, and so by lemma 2.9 it follows that B is a torsion group in $\mathrm{AbSh}\mathcal{L}$.

<u>Proposition 2.13</u>: If $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is an exact sequence in $\mathrm{AbSh}\mathcal{L}$ where A is a torsion group, then B is torsion iff C is a torsion group.

<u>Proof</u>: ($\Rightarrow$) Clear by proposition 2.12.
For the converse, consider an arbitrary $b \in BU$, $U \in \mathcal{L}$, then $g_U(b) \in CU$. Since C is given to be a torsion group, there exists a cover $U = \bigvee_{i \in I} U_i$, $0 \neq n_i \in N$ such that $n_i g_U(b)|U_i = 0$ for all $i \in I$. That is $g_{U_i}(n_i b|U_i) = 0$, hence $n_i b|U_i \in \mathrm{Ker}\, g_{U_i}$, $i \in I$. But $\mathrm{Im} f_U = \mathrm{Ker}\, g_U$, $U \in \mathcal{L}$, and so for each $i \in I$, there exists $a_i \in AU_i$ such that $n_i b|U_i = f_{U_i}(a_i)$. Since A is torsion, there

exists a cover $U_i = \bigvee_{j \in J_i} W_{ji}$, and $0 \neq n_{ji} \in N$ such that $n_{ji} a_i | W_{ji} = 0$ for all $j \in J_i$, $i \in I$. Hence $n_i n_{ji} b | W_{ji} = n_{ji} f_{W_{ji}}(a_i | W_{ji}) = 0$ for all $i \in I$, $j \in J_i$, which shows that B is a torsion group in $\mathrm{AbSh}\mathcal{L}$.

As in 2.8, we call a group $P \in \mathrm{AbPSh}\mathcal{L}$ to be a pre-p-primary group, if for each $a \in PU$, there exists a cover $U = \bigvee_{i \in I} U_i$, $0 \neq n_i \in N$ such that $P^{n_i} a | U_i = 0$, for all $i \in I$.

Then the analogue of 2.9, 2.10, 2.11, 2.12 and 2.13 hold for p-primary groups, that is, the following are true:

(i) If $P \in \mathrm{AbPSh}\mathcal{L}$ is a pre-p-primary group, then its sheaf reflection $\tilde{P}$ is a p-primary group in $\mathrm{AbSh}\mathcal{L}$.

(ii) The direct limit of a family of p-primary groups is again a p-primary group.

(iii) If $\{A_i\}_{i \in I}$ is any family of p-primary groups in $\mathrm{AbSh}\mathcal{L}$, then $\bigoplus_{i \in I} A_i$ is also a p-primary group.

(iv) Any epimorphic image of a p-primary group is p-primary.

(v) If $0 \to A \to B \to C$ is an exact sequence in $\mathrm{AbSh}\mathcal{L}$, where A is a p-primary group, then B is p-primary iff C is a p-primary group.

Recall that in the category Ab, the torsion subgroup of an injective group is always injective. We show in the following example that, for an arbitrary $\mathcal{L}$, the

torsion subgroup of an injective group need not be injective, except for some special locales which we shall discuss in the next section.

<u>Example 2.14</u>: Consider the locale $\mathcal{L} = \omega + 1$ and $A \in \mathrm{AbSh}\mathcal{L}$ given by

$$P_1 \rightarrowtail \prod_{n<\omega} P_n \to \ldots \to P_2 \times P_1 \to P_1 \to 0$$

$$\omega+1 > \omega > \ldots > 2 > 1 > 0$$

where the P_i are finite groups with increasing exponent. By (1.15) the injective hull of A is given by the group

$$B = \prod_{n<\omega} E(P_n) \xrightarrow{\mathrm{ld}} \prod_{n<\omega} E(P_n) \to \ldots \to E(P_2) \times E(P_1) \to E(P_1) \to 0$$

where $E(P_i)$ denotes the injective hull of the group P_i in Ab. If TB denotes the torsion subgroup of B, then clearly $(TB)n = Bn$ all $n < \omega$, and so $(TB)\omega = B\omega = \prod_{n<\omega} E(P_n)$, but $(TB(\omega + 1)) = T(B(\omega + 1)) = \bigoplus_{n<\omega} E(P_n)$. Hence $TB \subset B$ and since $A \subseteq TB$, it follows TB is not injective since B, being the injective hull of A, is the minimal injective extension of A. Hence the result.

<u>Definition 2.15</u>: We call an $A \in \mathrm{AbSh}\mathcal{L}$ to be a <u>reduced</u> group if it has no non-zero injective subgroups. Recall that in the category Ab, for any torsion group B, the group $\mathrm{Hom}(B,K)$ is reduced for all $K \in \mathrm{Ab}$. We shall

prove the analogue of this for the Ab-valued hom-functor H and the internal hom-functor $[-,-]$ of $\mathrm{AbSh}\mathcal{L}$ (0.18).

<u>Lemma 2.16</u>: If $A \in \mathrm{AbSh}\mathcal{L}$ is a torsion group, then $H(A,P)$ is reduced in Ab for all $P \in \mathrm{AbSh}\mathcal{L}$.

<u>Proof</u>: Let $0 \neq C \subseteq H(A,P)$ be an injective subgroup. Consider any $0 \neq \alpha \in C$, then for some $U \in \mathcal{L}$, and $a \in AU$, $\alpha_U(a) \neq 0$. Since A is torsion and $a \in AU$, there exists a cover $U = \bigvee_{i \in I} U_i$, and $0 \neq n_i \in N$ such that $n_i a|U_i = 0$ for all $i \in I$. But $\alpha_U(a) \neq 0$, implies that $\alpha_{U_k}(a|U_k) \neq 0$ for some $k \in I$. Consider now $0 \neq n_k \in N$, then C an injective hence divisible group in Ab implies that there exists some $\beta \in C$ such that $n_k\beta = \alpha$. Therefore $n_k\beta_{U_k}(a|U_k) = \beta_{U_k}(n_k a|U_k) = \beta_{U_k}(0) = 0$, which means $\alpha_{U_k}(a|U_k) = 0$, a contradiction. Hence $C = 0$, which shows $\mathrm{AbSh}\mathcal{L}(A,P)$ is reduced in the category Ab.

<u>Proposition 2.17</u>: If A is a torsion group in $\mathrm{AbSh}\mathcal{L}$, then $[A,P]$ is reduced for all $P \in \mathrm{AbSh}\mathcal{L}$.

<u>Proof</u>: Let $0 \neq B \subseteq [A,P]$ be an injective subgroup. Then for some $U \in \mathcal{L}$, $BU \neq 0$ is an injective subgroup of $[A,P]U = H_{\downarrow U}(A|U,P|U)$. Since A is torsion it follows $A|U$ is torsion (0.29) in $\mathrm{AbSh}{\downarrow}U$ and so by last lemma

$H_{\downarrow U}(A|U,P|U)$ is reduced in Ab. Thus $BU = 0$ for all $U \in \mathcal{L}$, hence $B = 0$ which means $[A,P]$ is reduced in $\mathrm{AbSh}\,\mathcal{L}$.

Section 2: Essential Extensions of Torsion Groups

If for any torsion group $A \in \mathrm{AbSh}\,\mathcal{L}$, all essential extensions of A are torsion, then we say that essential extensions in $\mathrm{AbSh}\,\mathcal{L}$ preserve torsion. The following proposition shows that "essential extensions preserve torsion" is a local property.

Proposition 2.18: Essential extensions preserve torsion in $\mathrm{AbSh}\,\mathcal{L}$ iff there exists a cover $E = \bigvee_{i \in I} U_i$ such that essential extensions preserve torsion in $\mathrm{AbSh}{\downarrow}U_i$ for all $i \in I$.

Proof: ($\Rightarrow$) Clear by taking the trivial cover of E. ($\Leftarrow$) For the converse, consider any essential extension B of the torsion group A in $\mathrm{AbSh}\,\mathcal{L}$. Since for each $i \in I$, the functor $R_{U_i}: \mathrm{AbSh}\,\mathcal{L} \to \mathrm{AbSh}{\downarrow}U$, preserves essential extensions and torsion (0.22, 0.29), it follows $B|U_i$ is an essential extension of the torsion group $A|U_i$ in $\mathrm{AbSh}{\downarrow}U_i$. By hypothesis, $B|U_i$ is torsion in $\mathrm{AbSh}{\downarrow}U_i$ all $i \in I$, hence by proposition 2.2 B is torsion in $\mathrm{AbSh}\,\mathcal{L}$.

Proposition 2.19: For any $\mathcal{L}$, essential extensions

in $\mathrm{AbSh}\mathcal{L}$ preserve torsion iff every injective splits into a torsion group and a torsion free group.

<u>Proof</u>: ($\Rightarrow$) Let B denote the torsion subgroup of an injective group $A \in \mathrm{AbSh}\mathcal{L}$. If $C \supseteq B$ is any essential extension, then by hypothesis C is a torsion group. Since A is injective, we may assume that $C \subseteq A$, so C torsion implies $C \subseteq B$ and hence $C = B$. Thus B has no proper essential extensions which means that B is injective. Therefore $A = B \oplus E$ for some subgroup E of A. If TE denotes the torsion subgroup of E, then $TE \subseteq B$ and so $TE \subseteq B \cap E = 0$, hence $TE = 0$. Thus E is torsion free. ($\Leftarrow$) Let P be a torsion group and H the injective hull of P. By hypothesis $H = T \oplus F$ where T is a torsion group and F is a torsion free group. If $F \neq 0$, then since H is an essential extension of P, it follows that $P \cap F \neq 0$, a contradiction, since P is torsion. Hence $F = 0$ which shows that H is a torsion group. Since every essential extension of P has an embedding into H, it follows all essential extensions of P are torsion, hence the result.

<u>Proposition 2.20</u>: For a Boolean locale, essential extensions in $\mathrm{AbSh}\mathcal{L}$ preserve torsion.

<u>Proof</u>: Consider an essential extension B of the

torsion group $A \in \mathrm{AbSh}\,\mathcal{L}$. Let C denote the torsion subgroup of B (2.4). For any $U \in \mathcal{L}$ consider an arbitrary element $b \in BU$. Let $W \leq U$ be the largest element in $\downarrow U$ such that $b|W \in CW$. We claim W is dense in $\downarrow U$. If not, then there exists $S \in \downarrow U$, $S \neq 0$ such that $S \wedge W = 0$. Now for any $V \leq S$, $b|V \in CV$ gives $V \leq W$ and so $V = V \wedge W \leq S \wedge W = 0$ implies $V = 0$. In particular $b|S \neq 0$. Since $B \supseteq A$ is an essential extension therefore there exists a $V \leq S$ and $m \in Z$ such that $0 \neq mb|V \in AV \subseteq CV$. Now C is the torsion subgroup of B and $0 \neq mb|V \in CV$, therefore $b|V \in CV$. But then $V = 0$, a contradiction, since $0 \neq mb|V \in AV$. Hence W is dense in $\downarrow U$. Since $\mathcal{L}$ is Boolean we have $W = U$. Thus $BU \subseteq CU$ for all $U \in \mathcal{L}$ and so $B = C$. Hence B is a torsion group.

<u>Remark</u>: On the other hand, one can see that if essential extensions preserve torsion in $\mathrm{AbSh}\,\mathcal{L}$, then it does not necessarily follow that $\mathcal{L}$ is Boolean. Here is a counterexample: Consider $\mathcal{L} = 3$. If $B = \begin{array}{c} B_1 \\ \downarrow h \\ B_2 \end{array}$ is torsion in AbSh3, then both B_1 and B_2 are torsion in Ab. By 1.15 the injective hull of B is given by

$$A = \begin{array}{c} E(B_1) \times E(\mathrm{Ker}\, h) \\ \downarrow \\ E(B_1) \end{array}$$

which is torsion in AbSh3. Hence all essential extensions of B are torsion, although

$\mathcal{L} = 3$ is not Boolean. Of course the remark is a special case of the following more general result which shows that there are non-Boolean $\mathcal{L}$ such that essential extensions in $\mathrm{AbSh}\mathcal{L}$ preserve torsion.

<u>Proposition 2.21</u>: For any finite $\mathcal{L}$ essential extensions in $\mathrm{AbSh}\mathcal{L}$ preserve torsion.

<u>Proof</u>: Let B be any essential extension of the torsion group A. Then for an arbitrary $a \in AU$, $U \in \mathcal{L}$, A being torsion implies that there is a cover $U = U_1 \vee U_2 \vee \ldots \vee U_k$ and $0 \neq n_i \in N$ such that $n_i a|U_i = 0$ for all $i = 1,2,\ldots,k$. If $m = n_1 n_2 \ldots n_k$, then $ma|U_i = 0$ for all $i = 1\ldots k$ and therefore $ma = 0$ and $m \neq 0$. This shows for each $U \in \mathcal{L}$, AU is a torsion group in the category Ab. Now, if there are $V \in \mathcal{L}$ such that BV is not a torsion group then let S be minimal such that BS is not torsion. Then $S \neq 0$ and for all $U < S$, BU is a torsion group in Ab. If $W = \bigvee_{U<S} U$, then since each BU is torsion it follows by proposition 2.2 that $B|W$ is torsion in $\mathrm{AbSh}\downarrow W$. By the same argument as above it follows BW is torsion and hence $W < S$. Consider an arbitrary $b \in BS$ of infinite order. Since $B \supseteq A$ is an essential extension, there exists a $V \leq S$ and $0 \neq m \in Z$ such that $0 \neq mb|V \in AV$. Then $V \neq S$, for otherwise $0 \neq mb \in AV$ has finite order and so b will

have finite order, a contradiction, since b has infinite order. Hence $V \leq W$. This implies $b|W \neq 0$. But BW is torsion and so for some $0 \neq n \in N$, $nb|W = 0$. But $0 \neq nb \in BS$ is again of infinite order and so by the same argument $0 \neq nb|W$, a contradiction. Hence BS is a torsion group which contradicts the definition of S. This shows B is a torsion group in $\mathrm{AbSh}\mathcal{L}$.

<u>Remark</u>: Recall from Chapter 0 that the finite locales $\mathcal{L}$ are exactly those $\mathcal{L}$ in which both ACC and DCC hold. It is therefore of interest to note that there exists an $\mathcal{L}$ which satisfies DCC but for which essential extensions in $\mathrm{AbSh}\mathcal{L}$ do not preserve torsion. Here is an <u>example</u> which is actually the same as that considered in 2.14 for a different purpose: If A and its injective hull $B \supseteq A$ are as in 2.14, then B is not torsion because its torsion subgroup is proper.

<u>Proposition 2.22</u>: If essential extensions preserve torsion in $\mathrm{AbSh}\mathcal{L}$, then for all $U \in \mathcal{L}$, the following is true:

(i) Essential extensions preserve torsion in $\mathrm{AbSh}{\downarrow}U$.

(ii) Essential extensions preserve torsion in $\mathrm{AbSh}{\uparrow}U$.

<u>Proof</u>: (i) Let B be any essential extension of the torsion group A in $\mathrm{AbSh}{\downarrow}U$. Since the functor

E_U: AbSh$\downarrow U \to$ AbSh$\mathcal{L}$ preserves essential extensions (0.22) and also torsion (0.29), it follows $E_U B$ is an essential extension of the torsion group $E_U A$. By hypothesis $E_U B$ is torsion in AbSh$\mathcal{L}$. Therefore $R_U(E_U B) = B$ is again torsion since the functor R_U preserves torsion (0.29), hence the result.

(ii) Consider the local lattice homomorphism $\phi: \mathcal{L} \to \uparrow U$ given by $\phi(W) = W \vee U$. Then ϕ produces ϕ_*: AbSh$\uparrow U \to$ AbSh$\mathcal{L}$ (0.16) where $(\phi_* A)W = A(U \vee W)$, $W \in \mathcal{L}$. Let B be any essential extension of the torsion group A in AbSh$\uparrow U$. We claim that B is torsion. We first show that ϕ_* preserves torsion. Let $0 \neq a \in (\phi_* A)W = A(U \vee W)$. Since A is torsion, there is a cover $U \vee W = \bigvee_{i \in I} U_i$ in $\uparrow U$, and $0 \neq n_i \in N$ such that $n_i a | U_i = 0$ for all $i \in I$. So we can form a cover $W = (U \vee W) \wedge W = \bigvee_{i \in I} (U_i \wedge W)$ in $\mathcal{L}$ such that $n_i a | (U_i \wedge W) = 0$ all $i \in I$. Hence for $0 \neq a \in (\phi_* A)W$, we can always find a cover $W = \bigvee_{i \in I} (U_i \wedge W)$ in $\mathcal{L}$, such that $0 = n_i a | (U_i \wedge W)$, which proves that $\phi_* A$ is torsion in AbSh$\mathcal{L}$.

Our next step is to show that ϕ_* preserves essential extensions, and so let $0 \neq b \in (\phi_* B)W = B(W \vee U)$, $W \in \mathcal{L}$. Since $B \geq A$ is essential in AbSh$\uparrow U$, there exists $V \leq W \vee U$, and $m \in Z$ such that $0 \neq mb | V \in AV$. But $U \leq V$ and $V \leq W \vee U$ implies $V = (V \wedge W) \vee U$ and therefore $0 \neq mb | (V \wedge W) \vee U \in A((V \wedge W) \vee U)$. Thus for

$0 \neq b \in (\phi_* B)W$, there exists $V \wedge W \leq W$ such that $0 \neq mb|V \wedge W \in (\phi_* A)(V \wedge W)$ for some $m \in Z$. This shows $\phi_* B$ is an essential extension of $\phi_* A$.

Finally we show that ϕ_* reflects torsion. So, let $\phi_* P$ be a torsion group in $\mathrm{AbSh}\mathcal{L}$, for some $P \in \mathrm{AbSh}\uparrow U$. If $0 \neq a \in PW$, $W \in \uparrow U$, then $0 \neq a \in (\phi_* P)W = P(W \vee U) = P(W)$, and so $\phi_* P$ being a torsion group implies, that there is a cover $W = \bigvee_{i \in I} W_i$ in $\mathcal{L}$, and $0 \neq n_i \in N$ such that $n_i a|W_i = 0$ for all $i \in I$, where $a|W_i \in (\phi_* P)W_i = P(W_i \vee U)$. If we consider the cover $W = \bigvee_{i \in I} W_i \vee U$ in $\uparrow U$, then we get $0 = n_i a|(W_i \vee U)$ for all $i \in I$, which proves that P is torsion in $\mathrm{AbSh}\uparrow U$. Now in order to prove (ii), we consider an essential extension D of the torsion group C in $\mathrm{AbSh}\uparrow U$. Then by the above argument $\phi_* D$ is an essential extension of $\phi_* C$ in $\mathrm{AbSh}\mathcal{L}$. But $\phi_* C$ is torsion since C is torsion, hence by hypothesis $\phi_* D$ is torsion. Now ϕ_* reflects torsion and that proves D is torsion in $\mathrm{AbSh}\uparrow U$. Hence the result.

<u>Remark</u>: As a special case, if $\mathcal{L} = \mathcal{O}X$ for some topological space X and $Y \subseteq X$ is a closed subspace then $\uparrow \mathbb{C}Y \cong \mathcal{O}Y$, the isomorphism being given by $U \rightsquigarrow U \cap Y$, $U \in \uparrow \mathbb{C}Y$. Hence, by the last proposition, essential extensions preserve torsion in $\mathrm{AbSh}Y$, if they do in $\mathrm{AbSh}X$.

<u>Lemma 2.23</u>: On the space $X = \{0\} \cup \{\frac{1}{n} | n = 1,2,\ldots\} \subseteq \mathbb{R}$ there is a torsion group C with a non-torsion essential extension.

<u>Proof</u>: Consider $A \in Ab^{|X|}$ by $A\{0\} = 0$, $A(\frac{1}{n}) = Z(p^{\infty})$ for all $n \neq 0$. Then the functor $F: Ab^{|X|} \to AbShX$ (0.20), produces $B = FA$, $(FA)U = \prod_{x \in U} A\{x\} = \{\phi: U \to Z(p^{\infty}), \phi(0) = 0 \text{ if } 0 \in U\}$ in $AbShX$. Let C be the torsion subgroup of B. Assume $C = B$, then $CX = BX$ and so the function $\phi \in BX$ given by $\phi(0) = 0$, $\phi(\frac{1}{n}) = a_n$ where a_n has order p^n, $n = 1,2\ldots$ is in CX. This means there exists a cover $X = \bigcup_{i \in I} U_i$ and $0 \neq k_i \in N$ such that $k_i\phi|U_i = 0$ for all $i \in I$. Since $0 \in U_j$ for some $j \in I$ and hence U_j contains infinitely many $\{\frac{1}{n}, n \in N\}$, thus $k_j\phi|U_j = 0$ a contradiction. Hence $\phi \notin CX$, which shows B is not a torsion group in $AbShX$. We now show that B is an essential extension of C. Let $0 \neq \alpha \in BU$, then $\alpha(\frac{1}{n}) \neq 0$ for some $\frac{1}{n} \in U$. If $W = \{\frac{1}{n}\}$, then $\alpha|W \neq 0$ is of finite order since $\alpha(\frac{1}{n}) \in Z(p^{\infty})$, hence $0 \neq \alpha|W \in CW$. Thus B is an essential extension of C which is torsion, although B itself is not torsion.

<u>Proposition 2.24</u>: If X is a first countable Hausdorff space, then essential extensions preserve torsion in $AbShX$ iff X is discrete.

<u>Proof</u>: ($\Rightarrow$) Suppose on the contrary, that X is not discrete. Then there is a point $x_0 \in X$ for which $\{x_0\}$ is not open. Let the countable basic neighbourhoods of x_0 be arranged in the form $U_1 \supset U_2 \supset \ldots$ and for each $n \in N$ pick an element $x_n \in U_n - U_{n+1}$. Denote by X_0 the subspace of X consisting of the points $\{x_0, x_1, x_2, \ldots\}$. Since the sequence $\{x_k\}_{k \in N}$ converges to the point x_0, it follows that the space X_0 is compact in X. But X is Hausdorff and therefore X_0 is closed in X. For any x_n, $n \neq 0$ the subset $X_0 - \{x_n\}$ also being compact, is also closed in X. Hence $\{x_n\} = \{X - (X_0 - \{x_n\})\} \cap X_0$ is open in the space X_0. It is then easy to see that the subspace $X_0 = \{x_0, x_1, \ldots\}$ is homeomorphic to the space $\{0\} \cup \{\frac{1}{n} | n \in N\} \subseteq \mathbb{R}$. By the above lemma, essential extensions of torsion groups need not be torsion groups in $AbShX_0$, a contradiction to proposition 2.22, hence X is discrete.

($\Leftarrow$) If X is discrete then $AbShX \cong Ab^{|X|}$ (0.15). If $A \in AbShX$ is torsion then clearly each $A\{x\}$, $x \in X$ is a torsion group in Ab. So, if $B \supseteq A$ is an essential extension in $AbShX$, then by 0.22 $B|\{x\} = B\{x\} \supseteq A|\{x\} = A\{x\}$ is essential in Ab, hence each $B\{x\}$ is a torsion group in Ab. Thus B is torsion in $AbShX$.

<u>Corollary 2.25</u>: If $X = \prod_{\alpha \in I} X_\alpha$, where each X_α is a first countable, Hausdorff space, and essential extensions

in AbShX preserve torsion, then X is discrete.

<u>Proof</u>: If $X = \prod_{\alpha \in I} X_\alpha$, then each $X_\alpha \cong$ closed subspace of X, hence by proposition 2.22, essential extensions preserve torsion in $\mathrm{AbSh}X_\alpha$. But X_α is given to be first countable and Hausdorff, therefore by proposition 2.24, each X_α is discrete. Suppose X_α is non-trivial for infinitely many α, then $2^\omega \cong$ subspace of X. But 2^ω is compact, hence closed in X. Also 2^ω is first countable, Hausdorff. But it is not discrete, hence only finitely many X_α are non-trivial which implies that X is discrete.//

<u>Remark</u>: All finite $\mathcal{L}$ are spatial, and for all finite $\mathcal{L}$, essential extensions in $\mathrm{AbSh}\mathcal{L}$ preserve torsion. Hence there are many non-discrete spaces X such that essential extensions preserve torsion in AbShX.//

CHAPTER 3

DIVISIBILITY AND PURITY

Introduction: This chapter is a study of divisibility and purity in $\mathrm{AbSh}\mathcal{L}$. We start by showing that divisibility is a local property, and is preserved under epimorphic images. Divisible groups are characterized to be exactly those groups which are pure in every extension. We also discuss what we mean by p-divisibility and prove some results in $\mathrm{AbSh}\mathcal{L}$ in analogue with their counter parts in Ab. The notion of purity and p-purity is discussed, and we provide a counterexample to show purity is not a local property. The locales for which the only pure and essential subgroup of $Z_{\mathcal{L}}$ is just $Z_{\mathcal{L}}$ are characterized, and they are exactly the Boolean locales.

Section 1: Divisibility

Recall from Chapter 0 (0.28) that $A \in \mathrm{AbSh}\mathcal{L}$ is said to be divisible if $nA = A$ for all $0 \neq n \in N$.

Proposition 3.1: $A \in \mathrm{AbSh}\mathcal{L}$ is divisible iff there exists a cover $E = \bigvee_{i \in I} U_i$, such that $A|U_i$ is divisible in $\mathrm{AbSh}\downarrow U_i$, for all $i \in I$.

<u>Proof</u>: To prove, the non-trivial implication, we consider the functor $R: \mathrm{AbSh}\mathcal{L} \to \prod_{i\in I} \mathrm{AbSh}{\downarrow}U_i$ described in 1.2. Since each $n_{A|U_i}$ is given to be an epimorphism in $\mathrm{AbSh}{\downarrow}U_i$, it follows $R(n_A) = (n_{A|U_i})_{i\in I}: (A|U_i)_{i\in I} \to (A|U_i)_{i\in I}$ is an epimorphism in $\prod_{i\in I} \mathrm{AbSh}{\downarrow}U_i$. Since R is faithful, it reflects epimorphisms, hence n_A is an epimorphism, which means A is divisible in $\mathrm{AbSh}\mathcal{L}$.

<u>Proposition 3.2</u>: Any epimorphic image of a divisible group is divisible.

<u>Proof</u>: Consider an epimorphism $h: A \to B$ where A is a divisible group. Then for all $0 \neq n$, the commutativity of the square

$$\begin{tikzcd} A \arrow[r, "n_A"] \arrow[d, "h"'] & A \arrow[d, "h"] \\ B \arrow[r, "n_B"'] & B \end{tikzcd}$$

implies $n_B h = h n_A$. Now A divisible means n_A is an epimorphism, hence $n_B h = h n_A$ is an epimorphism. Thus n_B is an epimorphism and therefore B is divisible.

<u>Remark</u>: It is not true in general that any epimorphic image of an injective group is injective in $\mathrm{AbSh}\mathcal{L}$. Here

is a counterexample for $\mathcal{L} = 3$. Consider $A, B \in \mathrm{AbSh}3$ given by

$$A = \begin{array}{c} \mathbb{Q} \\ \downarrow 1_{\mathbb{Q}} \\ \mathbb{Q} \end{array} \qquad B = \begin{array}{c} \mathbb{Q} \\ \downarrow v \\ \mathbb{Q}/Z \end{array}$$

where v is the quotient map. Then $k: A \to B$ given by

$$\begin{tikzcd} \mathbb{Q} \arrow[r, "1_{\mathbb{Q}}"] \arrow[d, "1_{\mathbb{Q}}"'] \arrow[dr, phantom, "\neg"] & \mathbb{Q} \arrow[d, "v"] \\ \mathbb{Q} \arrow[r, "v"'] & \mathbb{Q}/Z \end{tikzcd}$$

is an epimorphism, where B is not an injective group although A is (B. Banaschewski [4]).
Since an injective group is always divisible, it follows by proposition 3.2 that any epimorphic image of an injective is always divisible. For a Boolean locale, the two concepts coincide [4]; hence if $\mathcal{L}$ is Boolean, any epimorphic image of an injective group is injective.//

It still remains to be settled whether the property 'Any Epimorphic image of an injective in $\mathrm{AbSh}\,\mathcal{L}$ is injective' makes the locale Boolean, but we can show that this is a local property.

<u>Proposition 3.3</u>: Any epimorphic image of an injective group is injective in $\mathrm{AbSh}\,\mathcal{L}$ iff there exists a cover

$E = \bigvee_{i \in I} U_i$ such that, any epimorphic image of an injective group is injective in $\mathrm{AbSh}{\downarrow}U_i$ for all $i \in I$.

<u>Proof</u>: To prove the non-trivial implication, consider an epimorphism $h: A \to B$ where A is an injective group in $\mathrm{AbSh}\mathcal{L}$. Since the restriction functors $R_{U_i}: \mathrm{AbSh}\mathcal{L} \to \mathrm{AbSh}{\downarrow}U_i$ preserve epimorphic images and injectives (0.14, 0.23), it follows that $B|U_i$ is an epimorphic image of an injective group $A|U_i$ in $\mathrm{AbSh}{\downarrow}U_i$, hence $B|U_i$ is injective for all $i \in I$.
By (1.4) it follows B is an injective group in $\mathrm{AbSh}\mathcal{L}$.

<u>Definition 3.4</u>: $A \in \mathrm{AbSh}\mathcal{L}$ is said to be p-divisible for some prime p, if $p^n A = A$ for all $0 \neq n \in N$. This means that the homomorphism $p^n_A : A \to A$ is an epimorphism, that is, for each $a \in AU$ there exists a cover $U = \bigvee_{i \in I} U_i$ such that $p^n a_i = a|U_i$ for some $a_i \in AU_i$ and all $i \in I$.
Now $p^n A = p.p...pA$ and so p-divisibility is implied by $pA = A$.
Then the analogue of 3.1 and 3.2 hold for p-divisibility, that is, the following are true:

(i) p-divisibility is a local property.

(ii) Any epimorphic image of a p-divisible group is a p-divisible group.

Recall, $A \in \mathrm{AbSh}\mathcal{L}$ is said to be p-primary if $A = \bigcup_{0 \neq n \in N} \mathrm{Ker}\, p_A^n$, that is, for each $a \in AU$ there exists a cover $U = \bigvee_{i \in I} U_i$ such that $p^{n_i} a|U_i = 0$ for some $0 \neq n_i \in N$, and all $i \in I$

<u>Proposition 3.5</u>: A p-primary group is q-divisible for all primes $q \neq p$.

<u>Proof</u>: Let B be a p-primary group, then for each $b \in BU$, there exists a cover $U = \bigvee_{i \in I} U_i$ such that $p^{n_i} b|U_i = 0$ for $0 \neq n_i \in N$ and all $i \in I$. Since $(q, p^{n_i}) = 1$, we have $qr_i + p^{n_i} s_i = 1$ for some $r_i, s_i \in Z$, all $i \in I$. Therefore, $qr_i b|U_i + p^{n_i} s_i b|U_i = b|U_i$, that is $b|U_i = qr_i b|U_i$ for all $i \in I$. That shows $qB = B$, hence B is q-divisible.

<u>Proposition 3.6</u>: $B \in \mathrm{AbSh}\mathcal{L}$ is divisible iff it is p-divisible for all primes p.

<u>Proof</u>: Since divisibility always implies p-divisibility for all p, we need to check only the converse. For any $n \in N$, $n = p_1 p_2 \ldots p_k$, and since B is p-divisible for all primes, it follows $nB = p_1 p_2 \ldots p_k B = B$, therefore B is divisible in $\mathrm{AbSh}\mathcal{L}$.

<u>Proposition 3.7</u>: A p-primary group is divisible iff it is p-divisible.

<u>Proof</u>: ($\Rightarrow$) Clear. For the converse, consider a p-primary group B which is given to be p-divisible. By 3.5, B is q-divisible for all primes $q \neq p$, hence by 3.6 it follows that B is divisible.//

For the following recall,
$A \in \mathrm{AbSh}\mathcal{L}$ is a torsion free divisible group iff for all $U \in \mathcal{L}$ each AU is a torsion free divisible group in Ab [4].

<u>Proposition 3.8</u>: In a torsion free group, the intersection of any family of divisible subgroups is divisible.

<u>Proof</u>: Let $\{A_i\}_{i \in I}$ be a family of divisible subgroups of a torsion free group A. Then A_i is also a torsion free group (being a subgroup of a torsion free group) and so A_iU is torsion free divisible in Ab for all $i \in I$, all $U \in \mathcal{L}$. Hence $(\bigcap_{i \in I} A_iU) = (\bigcap_{i \in I} A_i)U$ is divisible, that is $\bigcap_{i \in I} A_i$ is a divisible subgroup of A.

<u>Lemma 3.9</u>: If $F: \mathrm{AbSh}\mathcal{L} \to \mathrm{AbSh}\mathcal{M}$ preserves finite products, finite coproducts and epimorphisms then F preserves divisible groups.

Proof: Consider a divisible group $A \in \mathrm{AbSh}\mathcal{L}$. Then $n_A : A \to A$ is an epimorphism for all $0 \neq n \in N$. Hence $F(n_A)$ is an epimorphism. Since F preserves finite products and coproducts we have $F(n_A) = n_{FA}$ and thus FA is divisible.

Proposition 3.10: The sheaf reflection $\sim : \mathrm{AbPSh}\mathcal{L} \to \mathrm{AbSh}\mathcal{L}$ preserves divisible groups.

Proof: Being an exact left adjoint, the functor $\sim$ preserves finite products, coproducts and epimorphisms. Hence for $A \in \mathrm{AbPSh}\mathcal{L}$, if n_A is an epimorphism, then $(n_A)^{\sim} = n_{\tilde{A}}$ is also an epimorphism, that is, $\tilde{A}$ is divisible in $\mathrm{AbSh}\mathcal{L}$.//

Proposition 3.11: For any $U \in \mathcal{L}$, the functors R_U, E_U, T_U preserve divisible groups.

Proof: Clear since each of these functors satisfies the conditions of the lemma 3.9.//

If we now replace divisibility by p-divisibility for an arbitrary prime p, we can see that 3.9, 3.10 and 3.11 hold.

Proposition 3.12: If $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is an exact sequence in $\mathrm{AbSh}\mathcal{L}$ where A is divisible, then B is divisible iff C is divisible.

<u>Proof</u>: Consider the diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "n_A"] & B \arrow[r, "g"] \arrow[d, "n_B"] & C \arrow[r] \arrow[d, "n_C"] & 0 \\ 0 \arrow[r] & A \arrow[r, "f"'] & B \arrow[r, "g"'] & C \arrow[r] & 0 \end{tikzcd}$$

in $\mathrm{AbSh}\mathcal{L}$ where n_A is given to be an epimorphism, $0 \neq n \in N$.

($\Rightarrow$) If B is divisible then by proposition 3.2, C is divisible.

($\Leftarrow$) Suppose now C is given to be divisible, that is, n_C is an epimorphism. We claim n_B is an epimorphism, so let $\alpha n_B = 0$ for some $\alpha: B \to D$. Then $0 = \alpha n_B f = \alpha f n_A$, but n_A is an epimorphism and so $\alpha f = 0$. Now $g = \operatorname{coker} f$, therefore there exists unique $h: C \dashrightarrow D$ such that $hg = \alpha$. Since $hn_C g = hgn_B = \alpha n_B = 0$ and $n_C g$ is an epimorphism, it follows $h = 0$, hence $\alpha = 0$. Thus n_B is an epimorphism which proves B is divisible.

<u>Proposition 3.13</u>: The sum of any family $\{B_i\}_{i \in I}$ of divisible subgroups of A in $\mathrm{AbSh}\mathcal{L}$ is again a divisible subgroup.

<u>Proof</u>: Let $C = \sum_{i \in I} B_i$, and $0 \neq n \in N$. Since $B_i \subseteq C$ for all i, it follows $nB_i \subseteq nC$, and therefore $B_i \subseteq nC$ (since B_i is divisible). Thus $C = \sum B_i \subseteq nC$,

hence $C = nC$ which proves the result.

<u>Proposition 3.14</u>: The direct sum of any family of groups is divisible iff each component is divisible.

<u>Proof</u>: ($\Rightarrow$) Clear, since any epimorphic image of a divisible group is divisible.

($\Leftarrow$) Suppose $\{B_i\}$ is a family of divisible groups. Then each $B_i \cong B_i'$ where B_i' is the image of B_i under the natural embedding $B_i \rightarrowtail \prod_{i \in I} B_i$. Since $\oplus B_i = \sum B_i'$, and B_i' is divisible for all $i \in I$, it follows by proposition 3.13, that $\oplus_I B_i$ is a divisible group.

<u>Section 2</u>: <u>Purity</u>

<u>Definition 3.15</u>: A monomorphism $h: A \rightarrowtail B$ in $\mathrm{AbSh}\mathcal{L}$ is said to be pure, if for all $n \in N$, the diagram

$$\begin{tikzcd} nA \arrow[r, tail, "\bar{h}"] \arrow[d, tail, "i_A"'] & nB \arrow[d, tail, "i_B"] \\ A \arrow[r, tail, "h"'] & B \end{tikzcd}$$

is a pull back diagram.

If A is a subgroup of B and h is a natural embedding then $A \subseteq B$ is pure if $nA = A \cap nB$, that is, $(nA)U = AU \cap (nB)U$ for all $U \in \mathcal{L}$.

<u>Proposition 3.16</u>: If $A \in \mathrm{AbSh}\mathcal{L}$ is divisible, then for all extensions B of A, the monomorphism $h: A \rightarrowtail B$ is pure.

<u>Proof</u>: A divisible means $nA = A$. Hence, it is clear that the diagram

$$\begin{tikzcd} nA \arrow[r, tail, "\bar{h}"] \arrow[d, tail, "i_A"'] & nB \arrow[d, tail, "i_B"] \\ A \arrow[r, tail, "h"'] & B \end{tikzcd}$$

is a pull back, that is, h is a pure map.

<u>Proposition 3.17</u>: If $h: A \rightarrowtail B$ is pure with B a divisible group, then A is also a divisible group.

<u>Proof</u>: By the given hypothesis, we have a pull back square

$$\begin{tikzcd} nA \arrow[r, tail, "\bar{h}"] \arrow[d, tail, "i_A"'] & nB \arrow[d, tail, "i_B"] \\ A \arrow[r, tail, "h"'] & B \end{tikzcd}$$

for all $n \in N$. So, there exists a unique $\alpha: A \dashrightarrow nA$ such that in the diagram,

$$\begin{tikzcd}
A \arrow[dr, dashed, "\alpha"] \arrow[drr, bend left, "h"] \arrow[ddr, bend right, "1_A"'] & & \\
& nA \arrow[r, tail, "\bar{h}"] \arrow[d, tail, "i_A"'] & nB = B \arrow[d, tail, "i_B"] \\
& A \arrow[r, tail, "h"'] & B
\end{tikzcd}$$

$\bar{h}\alpha = h$ and $i_A\alpha = 1_A$. So i_A is an epimorphism, hence and isomorphism and therefore $A = nA$, that is A is divisible.

Thus we obtain the following:

<u>Corollary 3.18</u>: $A \in \mathrm{AbSh}\mathcal{L}$ is divisible iff it is pure in every extension.

<u>Proof</u>: ($\Rightarrow$) Clear from proposition 3.16. For the converse, since $\mathrm{AbSh}\mathcal{L}$ has enough injectives, there exists an injective $B \in \mathrm{AbSh}\mathcal{L}$ and a monomorphism $h: A \rightarrowtail B$, and since injective implies divisible, the result now follows from proposition 3.17.

<u>Remark 3.19</u>: It is clear that $A \in \mathrm{AbSh}\mathcal{L}$ is injective iff A is an absolute retract. Since the category $\mathrm{AbSh}\mathcal{L}$ has this special property that injective always implies divisibility, hence by the above corollary we have that injective implies absolute purity. Of course in $\mathrm{AbSh}\mathcal{L}$,

A an absolutely pure group does not necessarily imply that A is injective. This is so because for a non Boolean locale $\mathcal{L}$, there are divisible (= absolutely pure) groups which are not injective [4].

Here is an example of an Abelian category, where we show that injective does not imply divisible, which also shows that absolutely pure does not imply divisible.

Consider the category $\mathbb{P}$ of elementary p-groups, that is $A \in \mathbb{P}$ is an abelian group in which $pa = 0$ for all $a \in A$. Then $\mathbb{P}$ is an abelian category and is the same as the category of vector spaces over the field Z/pZ. Therefore each $A \in \mathbb{P}$ is an injective group, hence absolutely pure but no non-zero A is divisible, since $0 = pA \neq A$.

<u>Proposition 3.20</u>: For $A \overset{h}{\to} B$ in $\mathrm{AbPSh}\,\mathcal{L}$, if each $AU \overset{h_U}{\rightarrowtail} BU$ ~~is~~ pure in Ab, then $\tilde{A} \overset{\tilde{h}}{\rightarrowtail} \tilde{B}$ is pure in $\mathrm{AbSh}\,\mathcal{L}$.

<u>Proof</u>: Since the sheaf reflection preserves finite limits, and colimits it preserves pull backs and satisfies the condition $(nA)^{\sim} = n\tilde{A}$. Hence if

$$\begin{tikzcd} n(AU) \arrow[r, tail] \arrow[d, tail] & n(BU) \arrow[d, tail] \\ AU \arrow[r, tail, "h_U"'] & BU \end{tikzcd}$$

is a pull back in Ab at each $U \in \mathcal{L}$, then

$$\begin{tikzcd} n\tilde{A} \arrow[r, tail] \arrow[d, tail] & n\tilde{B} \arrow[d, tail] \\ \tilde{A} \arrow[r, tail, "\tilde{h}"'] & \tilde{B} \end{tikzcd}$$

is a pull back in $\mathrm{AbSh}\,\mathcal{L}$, that is $\tilde{h}$ is pure.

<u>Proposition 3.21</u>: The torsion subgroup of a group is a pure subgroup.

<u>Proof</u>: Let T denote the torsion subgroup of a given group $A \in \mathrm{AbSh}\,\mathcal{L}$. Then $T = \bigcup_{0 \neq n \in N} \mathrm{Ker}\, n_A$, which is the same as saying that $TU \doteq T(AU)$, where $T(AU)$ is the torsion subgroup of AU. Now at each $U \in \mathcal{L}$, the diagram

$$\begin{tikzcd} nT(AU) \arrow[r, tail] \arrow[d, tail] & nAU \arrow[d, tail] \\ T(AU) \arrow[r, tail] & AU \end{tikzcd}$$

is a pull back in Ab, therefore

$$\begin{tikzcd} nT_0 \arrow[r, tail] \arrow[d, tail] & nA \arrow[d, tail] \\ T_0 \arrow[r, tail] & A \end{tikzcd}$$

is a pull back in $\mathrm{AbPSh}\,\mathcal{L}$, where T_0 is the presheaf $U \rightsquigarrow T(AU)$. By Proposition 3.20, it follows that the square

$$\begin{tikzcd} nT \arrow[r, tail] \arrow[d, tail] & nA \arrow[d, tail] \\ T \arrow[r, tail] & A \end{tikzcd}$$

is a pull back square in $\mathrm{AbSh}\,\mathcal{L}$, hence T is a pure subgroup of A.//

Alternate proof of proposition 2.20 using the notion of purity. Consider the injective hull B of the torsion group $A \in \mathrm{AbSh}\,\mathcal{L}$, and let T be the torsion subgroup of B. Then by 3.21 T is pure in B, hence divisible (3.17), and therefore injective because $\mathcal{L}$ is Boolean. Hence $T = B$, since B is the minimal injective extension of A and $T \supseteq A$. Therefore B is torsion, which shows that all essential extensions of A are torsion.

<u>Proposition 3.22</u>: If $h: A \to B$ is retractable, then h is pure.

<u>Proof</u>: Let $g: B \to A$ be such that $gh = 1_A$. Consider the diagram

$$\begin{tikzcd} C \arrow[rr, bend left, "t"] \arrow[dr, bend right, "s"'] & & \\ & nA \arrow[r, tail, "\bar{h}"] \arrow[d, tail, "i_A"'] & nB \arrow[d, tail, "i_B"] \\ & A \arrow[r, tail, "h"'] & B \end{tikzcd}$$

where $hs = i_B t$. Now $gh = 1_A$ implies $\bar{g}\bar{h} = 1_{nA}$ can be seen as follows: (where $\bar{g}: nB \to nA$ is the unique map such that $i_A\bar{g} = gi_B$).

Since $i_B\bar{h} = hi_A$, $gi_B = i_A\bar{g}$ it follows $gi_B\bar{h} = ghi_A = i_A$.

Now $i_A\bar{g}\bar{h} = gi_B\bar{h} = i_A = i_A 1_{nA}$. Since i_A is a monomorphism it follows $\bar{g}\bar{h} = 1_{nA}$.

Consider the map $\alpha := \bar{g}t: C \longrightarrow nA$. We claim this is the desired map, that is,

α is unique such that $i_A\alpha = s$ and $\bar{h}\alpha = t$.

That α is unique is clear, since i_A is a monomorphism.

Also $i_A\alpha = i_A\bar{g}t = gi_B t = ghs = s$, and

$i_B\bar{h}\bar{g}t = hi_A\bar{g}t = hs = i_B t$. Since i_B is a monomorphism, it follows $\bar{h}\bar{g}t = \bar{h}\alpha = t$, hence $h: A \rightarrowtail B$ is pure.

<u>Corollary 3.23</u>: Any direct summand is pure.

<u>Proof</u>: Since direct summands are retracts, the result follows from the last proposition.//

<u>Lemma 3.24</u>: The composition of pure maps is pure.

<u>Proof</u>: Let $g: A \rightarrowtail B$ and $h: B \rightarrowtail C$ be pure maps in $\mathrm{AbSh}\,\mathcal{L}$. Then we have the pull back squares,

$$\begin{tikzcd} nA \arrow[r, tail, "\bar{g}"] \arrow[d, tail, "i_A"'] & nB \arrow[d, tail, "i_B"] \\ A \arrow[r, tail, "g"'] & B \end{tikzcd} \qquad \begin{tikzcd} nB \arrow[r, tail, "\bar{h}"] \arrow[d, tail, "i_B"'] & nC \arrow[d, tail, "i_C"] \\ B \arrow[r, tail, "h"'] & C \end{tikzcd}$$

and hence the outer square of the diagram

$$\begin{tikzcd} nA \arrow[r, tail, "\bar{g}"] \arrow[d, tail] & nB \arrow[r, tail, "\bar{h}"] \arrow[d, tail] & nC \arrow[d, tail] \\ A \arrow[r, tail, "g"'] & B \arrow[r, tail, "h"'] & C \end{tikzcd}$$

is a pull back square, that is, hg is a pure map.

<u>Proposition 3.25</u>: The p-primary component of any $A \in \mathrm{AbSh}\mathcal{L}$ is a pure subgroup.

<u>Proof</u>: If A_p denotes the p-primary component of A, then A_p is a direct summand of the torsion subgroup T of A. By proposition 3.21 T is pure in A, and by corollary 3.23 A_p is pure in T, hence by 3.24 A_p is pure in A.

<u>Lemma 3.26</u>: If m and n are relatively prime, then $mnG = mG \cap nG$.

<u>Proof</u>: Let $x \in (mG)U \cap (nG)U$. Since $x \in (mG)U$, there exists a cover $U = \bigvee_{j \in J} U_j$ such that $x|U_j = ms_j$, where $s_j \in GU_j$, $j \in J$. Also, $x|U_j \in (nG)U_j$, and so there exists a cover $U_j = \bigvee_{k \in T_j} W_{kj}$ such that $x|W_{kj} = nt_{kj}$, $k \in T_j$ and $t_{kj} \in GW_{kj}$. Therefore $x|W_{kj} = x|U_j|W_{kj} = ms_i|W_{kj} = nt_{kj}$ for all $k \in T_j$, $j \in J$.

But $(m,n) = 1$, so $mr + ns = 1$ for some r,s Z. Hence $s_j|W_{kj} = mrs_j|W_{kj} + nss_j|W_{kj} = nrt_{kj} + nss_j|W_{kj}$ $= n(rt_{kj} + ss_j|W_{kj})$. Thus $x|W_{kj} = mn(rt_{kj} + ss_j|W_{kj})$, hence $x \in mnGU$ which show $(mG)U \cap (nG)U \subseteq mnGU$, that is $mG \cap nG \subseteq mnG$. Also $mnG \subseteq mG \cap nG$ is clear, hence $mnG = mG \cap nG$.

<u>Definition 3.27</u>: A monomorphism $h: A \rightarrowtail B$ in $\mathrm{AbSh}\,\mathcal{L}$ is p-pure if for all $n \in N$, the diagram

$$\begin{tikzcd} p^nA \arrow[r, tail] \arrow[d, tail] & p^nB \arrow[d, tail] \\ A \arrow[r, tail, "h"'] & B \end{tikzcd}$$

is a pull back.

If A is a subgroup of B, and h is the natural embedding then $A \subseteq B$ is p-pure if for all $n \in N$, $p^nA = A \cap p^nB$.

<u>Proposition 3.28</u>: If G is p-pure in A for all primes p, then G is pure in A.

<u>Proof</u>: Let G be p-pure in A for all p. Consider $n \in Z$, then $n = p_1^{r_1}p_2^{r_2}\ldots p_k^{r_k}$, and so $nG = p_1^{r_1}\ldots p_k^{r_k}G$. By lemma 3.26,

$$nG = p_1^{r_1}G \cap \ldots \cap p_k^{r_k}G$$

$= (G \cap p_1^{r_1}A) \cap \ldots \cap (G \cap p_k^{r_k}A)$, (since G is p-pure in A).

$= G \cap (p_1^{r_1}A \cap \ldots \cap p_k^{r_k}A)$

$= G \cap (p_1^{r_1} \ldots p_k^{r_k})A$

$= G \cap nA$.

Hence G is pure in A.

<u>Lemma 3.29</u>: If G is a p-divisible subgroup of A, then G is p-pure in A.

<u>Proof</u>: Since G is p-divisible therefore $p^kG = G$ for all $k \in N$. Hence $G = p^kG \subseteq p^kA$ which means $p^kG = G \cap p^kA$, that is G is p-pure in A.

<u>Proposition 3.30</u>: A p-pure, p-primary subgroup of a group is pure.

<u>Proof</u>: Let G be a p-pure, p-primary subgroup of a group A. By proposition 3.5, G is q-divisible for all primes $q \neq p$. Again by lemma 3.29, G is q-pure in A for all $q \neq p$, and since G is also p-pure, by 3.28, it follows G is pure in A.

<u>Proposition 3.31</u>: If G is pure in A, then nG

is pure in nA for all $n \in Z$.

<u>Proof</u>: G pure in A implies $kG = G \cap kA$ for all k, and so $knG = G \cap knA$. Since $nG \subseteq G$, we get $nG \cap knA \subseteq G \cap knA = knG$. Hence $knG = nG \cap knA$, that shows nG is pure in nA.

<u>Proposition 3.32</u>: For any $U \in \mathcal{L}$, the functors R_U, E_U and T_U preserve pure maps.

<u>Proof</u>: Clear, since each of these functors preserves finite limits, finite coproducts and epimorphisms. In particular, they preserve pull backs and images, hence they preserve pure maps.

<u>Proposition 3.33</u>: If $A \in \mathrm{AbSh}\,\mathcal{L}$ is a torsion free group, then a subgroup B of A is a pure subgroup iff $\frac{A}{B}$ is a torsion free group.

<u>Proof</u>: ($\Leftarrow$) We are given that $(\frac{A}{B})U$ is a torsion free group in Ab for all $U \in \mathcal{L}$. Since the presheaf $U \rightsquigarrow \frac{AU}{BU}$ generating $\frac{A}{B}$ is a monopresheaf, $\frac{AU}{BU} \subseteq (\frac{A}{B})U$, hence $\frac{AU}{BU}$ is a torsion free group in Ab. Thus BU is a pure subgroup of AU for all $U \in \mathcal{L}$, and by proposition 3.20, it follows B is pure in A.

($\Rightarrow$) Since A is torsion free, we have $(nA)U = n(AU)$, for

all $U \in \mathcal{L}$, and hence if B is pure in A, then

$$\begin{tikzcd} nB \arrow[r, tail] \arrow[d, tail] & nA \arrow[d, tail] \\ B \arrow[r, tail] & A \end{tikzcd}$$

a pull back, implies

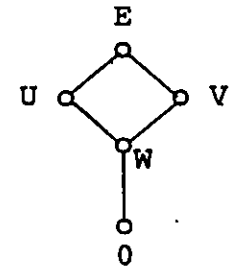
$$\begin{tikzcd} n(BU) = (nB)U \arrow[r, tail] \arrow[d, tail] & (nA)U = n(AU) \arrow[d, tail] \\ BU \arrow[r, tail] & AU \end{tikzcd}$$

is a pull back,

that is, BU is pure in AU, for all $U \in \mathcal{L}$. Therefore $\frac{AU}{BU}$ is a torsion free group for all $U \in \mathcal{L}$, and since direct limit of torsion free groups in Ab is torsion free, it follows $(\frac{A}{B})U$ is torsion free, hence $\frac{A}{B}$ is torsion free.

<u>Counterexample 3.34</u>: The following counterexample shows that purity is not a local property:
That is, there is a locale $\mathcal{L}$ and $A,B \in AbSh\mathcal{L}$ such that for some cover $E = \bigvee_{i \in I} U_i$, $A|U_i \subseteq B|U_i$ is pure in $AbSh{\downarrow}U_i$ for all $i \in I$, but $A \subseteq B$ is not pure in $AbSh\mathcal{L}$.
Consider the locale

$\mathcal{L}$



and $A,B \in AbSh\mathcal{L}$ given by,

$$A = \begin{tikzcd} & Z_2 \arrow[dl, tail, "i"'] \arrow[dr] & \\ Z_4 \arrow[dr, "\alpha"'] & & 0 \arrow[dl] \\ & Z_2 & \end{tikzcd} \qquad B = \begin{tikzcd} & Z_4 \arrow[dl, tail, "j"'] \arrow[dr] & \\ Z_4 \oplus Z_2 \arrow[dr, "q"'] & & 0 \arrow[dl] \\ & Z_2 & \end{tikzcd}$$

where α is multiplication by 2 and the other maps are the obvious maps. Then $A|U \subseteq' B|U$ and $A|V \subseteq B|V$ are both pure maps in $\mathrm{AbSh}\downarrow U$ and $\mathrm{AbSh}\downarrow V$ respectively, but $A \subseteq B$ is not pure in $\mathrm{AbSh}\,\mathcal{L}$. For, if this was pure, then

$$\begin{tikzcd} 2A \arrow[r] \arrow[d] & 2B \arrow[d] \\ A \arrow[r] & B \end{tikzcd}$$

has to be a pull back which implies that

$$\begin{tikzcd} 0 = (2A)E \arrow[r] \arrow[d] & (2B)E = Z_2 \arrow[d] \\ Z_2 = AE \arrow[r] & BE = Z_4 \end{tikzcd}$$

is a pull back, a contradiction, hence the conclusion.

The concepts of purity in $\mathrm{AbSh}\,\mathcal{L}$ was earlier discussed by B. Banaschewski in [6] where he proved that the pure subgroups of $Z_{\mathcal{L}} \in \mathrm{AbSh}\,\mathcal{L}$ are exactly given by $T_U Z_{\mathcal{L}}$ for the different $U \in \mathcal{L}$. The aim is now to characterize those $U \in \mathcal{L}$ for which the subgroup $T_U Z_{\mathcal{L}}$ is an essential subgroup of $Z_{\mathcal{L}}$.

<u>Lemma 3.35</u>: For any $U \in \mathcal{L}$, $T_U Z_{\mathcal{L}} \subseteq Z_{\mathcal{L}}$ is an essential subgroup iff U is dense in $\mathcal{L}$.

<u>Proof</u>: ($\Rightarrow$) Consider $0 \neq V \in \mathcal{L}$ and $0 \neq a \in Z_{\mathcal{L}} V$. By hypothesis there exists $W \leq V$ and $m \in Z$ such that $0 \neq ma|W \in (T_U Z_{\mathcal{L}})W$. By definition of $T_U Z_{\mathcal{L}}$, this means, there is a cover $W = \bigvee_{i \in I} W_i$ such that for some $W_i \subseteq U$, $0 \neq ma|W|W_i = ma|W_i \in Z_{\mathcal{L}} W_i$, that shows $0 \neq W_i \leq U \wedge V$, that is $U \wedge V \neq 0$, and therefore U is dense in $\mathcal{L}$.
($\Leftarrow$) Consider any $0 \neq \phi \in Z_{\mathcal{L}} V$ for some $V \in \mathcal{L}$. Then $V = \bigvee_{n \in Z} \phi(n)$, and since U is dense in $\mathcal{L}$, it follows $0 \neq U \wedge V = \bigvee_{n \in Z} U \wedge \phi(n)$, therefore $U \wedge \phi(n) \neq 0$ for some n. Thus $0 \neq \phi|U \wedge \phi(n) \in Z_{\mathcal{L}}(U \wedge \phi(n)) = (T_U Z_{\mathcal{L}})(U \wedge \phi(n))$ which shows $T_U Z_{\mathcal{L}} \subseteq Z_{\mathcal{L}}$ is essential.

<u>Proposition 3.36</u>: A locale $\mathcal{L}$ is Boolean iff the only pure and essential subgroup of $Z_{\mathcal{L}}$ is $Z_{\mathcal{L}}$.

<u>Proof</u>: ($\Rightarrow$) If $\mathcal{L}$ is Boolean, then $\mathcal{L}$ has no dense elements, so the result follows by the lemma 3.35.
($\Leftarrow$) The given conditions imply that $\mathcal{L}$ has no dense element, hence $\mathcal{L}$ is Boolean.//

CHAPTER 4

TENSOR PRODUCT AND INTERNAL HOM-FUNCTOR

Introduction: We begin this chapter by discussing the notion of tensor product, internal hom-functor and their adjointness in the category $\mathrm{AbSh}\mathcal{L}$. We shall prove some properties of these functors which are analogous to their counterparts in the category Ab, and which will be used in the next chapter on the concept of flatness in $\mathrm{AbSh}\mathcal{L}$.

Section 1: The Tensor Product

Definition 4.1: For any A, B in $\mathrm{AbSh}\mathcal{L}$, their tensor product $A \otimes B$ in $\mathrm{AbSh}\mathcal{L}$ is the sheaf reflection of the presheaf $U \rightsquigarrow AU \otimes BU$ with the obvious restriction maps. The bi-additive map $f_U: AU \times BU \to AU \otimes BU$, given by $f_U(a,b) = a \otimes b$ is a bi-additive morphism in $\mathrm{PSh}\mathcal{L}$ from $A \times B$ to the presheaf defining $A \otimes B$. Hence we get a bi-additive morphism $g: A \times B \to A \otimes B$ (that is each component g_U is a bi-additive map in Ab) in $\mathrm{Sh}\mathcal{L}$ where $g_U = \tilde{f_U}$, which has the following universal property: For any $C \in \mathrm{AbSh}\mathcal{L}$ and any bi-additive map $A \times B \xrightarrow{k} C$ in $\mathrm{Sh}\mathcal{L}$, there exists a unique $\bar{k}: A \otimes B \dashrightarrow C$ in $\mathrm{AbSh}\mathcal{L}$ such that the following diagram

$$\begin{tikzcd} A \times B \arrow[r, "g"] \arrow[d, "k"'] & A \otimes B \arrow[dl, dashed, "\bar{k}"] \\ C & \end{tikzcd}$$

commutes. (Note: Here $\bar{k}$ is the sheaf reflection of the unique morphism from the presheaf defining $A \otimes B$ to C).

<u>Remark</u>: The fact that the tensor product is unique up to isomorphism follows directly from this universal property. Also, for $A,B \in \mathrm{AbSh}\mathcal{L}$, and any $U \in \mathcal{L}$, $(A \otimes B)|U = A|U \otimes B|U$, since the presheaves defining them are the same.

<u>Notation</u>: For any P,Q in $\mathrm{AbPSh}\mathcal{L}$, $P \boxtimes Q$ is their tensor product in the category $\mathrm{AbPSh}\mathcal{L}$, where $(P \boxtimes Q)U = PU \otimes QU$.

<u>Lemma 4.2</u>: If $P,Q \in \mathrm{AbPSh}\mathcal{L}$, then $(P \boxtimes Q)^{\sim} \cong \tilde{P} \otimes \tilde{Q}$, natural in P and Q.

<u>Proof</u>: The morphism $r: P \times Q \to P \boxtimes Q$ given by $r_U(a,b) = a \otimes b$ produces a commutative square

$$\begin{tikzcd} P \times Q \arrow[r, "r"] \arrow[d, "\alpha"'] & P \boxtimes Q \arrow[d, "\beta"] \\ \tilde{P} \times \tilde{Q} \arrow[r, "\tilde{r}"] & (P \boxtimes Q)^{\sim} \end{tikzcd}$$

in $PSh\mathcal{L}$, where $(P \times Q)^{\sim} = \tilde{P} \times \tilde{Q}$ by exactness of the sheaf reflection functor. Now for any $C \in AbSh\mathcal{L}$ and a bi-additive morphism $f: \tilde{P} \times \tilde{Q} \to C$, the morphism $f\alpha: P \times Q \to C$ is bi-additive in $PSh\mathcal{L}$, so the universal property of r implies that there is a unique $g: P \boxtimes Q \to C$ in $AbPSh\mathcal{L}$ such that $gr = f\alpha$. Since $(P \boxtimes Q)^{\sim}$ is the sheaf reflection of $P \boxtimes Q$, there exists a unique $k: (P \otimes Q)^{\sim} \to C$ in $AbSh\mathcal{L}$ such that $k\beta = g$. Now $k\tilde{r}\alpha = k\beta r = gr = f\alpha$, and since $\tilde{P} \times \tilde{Q}$ is the sheaf reflection of $P \times Q$ with reflection map α, $k\tilde{r} = f$. Hence $\tilde{r}$ has the universal property of the tensor product for the pair $\tilde{P}, \tilde{Q}$, which implies that $(P \boxtimes Q)^{\sim} \cong \tilde{P} \otimes \tilde{Q}$. We now consider the specific isomorphism i_{PQ} from $(P \boxtimes Q)^{\sim}$ to $\tilde{P} \otimes \tilde{Q}$ such that the diagram

$$\begin{tikzcd} \tilde{P} \times \tilde{Q} \arrow[r, "\tilde{r}"] \arrow[d, "\tau_{\tilde{P}\tilde{Q}}"'] & (P \boxtimes Q)^{\sim} \arrow[dl, "i_{PQ}"] \\ \tilde{P} \otimes \tilde{Q} & \end{tikzcd}$$

commutes, where τ is the natural transformation giving the

universal bi-additive maps in $Sh\mathcal{L}$, and $i_{PQ} = k$ as above for $f = \tau_{\tilde{P}\tilde{Q}}$. The claim is that the isomorphism i_{PQ} is natural in P and Q. Consider $f: P \to P'$ and $g: Q \to Q'$ in $AbPSh\mathcal{L}$, then we get the following diagram

$$\begin{tikzcd}
P\boxtimes Q \arrow[rrr, "\beta"] \arrow[ddd, "f\boxtimes g"'] & & & (P\boxtimes Q)^{\sim} \arrow[r, "i_{PQ}"] \arrow[ddd, "(f\boxtimes g)^{\sim}"] \arrow[dd, phantom, "[2]"] & \tilde{P}\otimes\tilde{Q} \arrow[ddd, "\tilde{f}\otimes\tilde{g}"] \arrow[dd, phantom, "[1]"] \\
& P\times Q \arrow[ul, "\sigma_{PQ}"'] \arrow[r, "\alpha"] \arrow[d, "f\times g"] \arrow[dr, phantom, "[5]"] & \tilde{P}\times\tilde{Q} \arrow[r, "\tilde{r}"] \arrow[d, "\tilde{f}\times\tilde{g}"] \arrow[dr, phantom, "[3]"] & & \\
& P'\times Q' \arrow[r, "\alpha'"'] \arrow[dl, "\sigma_{P'Q'}"] & \tilde{P}'\times\tilde{Q}' \arrow[r, "\tilde{r}'"'] & (P'\boxtimes Q')^{\sim} \arrow[r, "i_{P'Q'}"'] & \tilde{P}'\otimes\tilde{Q}' \\
P'\boxtimes Q' \arrow[urrr, "\beta'"'] & & & &
\end{tikzcd}$$

where we want to prove that the square [1] commutes. So we need to show that:

(a) the outer square of [2][1] commutes

(b) the square [2] commutes.

(a) holds since $i_{PQ}\tilde{r} = \tau_{\tilde{P}\tilde{Q}}$ and τ is a natural transformation.

To see (b), first note that [4] commutes from universal bi-additivity of σ, and $\tilde{r}\alpha = \beta\sigma_{PQ}$, $\tilde{r}'\alpha' = \beta'\sigma_{P'Q'}$, $(f\boxtimes g)^{\sim}\beta = \beta'(f\boxtimes g)$ hold because of the naturality of the sheaf reflection. Therefore

$$(f\boxtimes g)^{\sim}\tilde{r}\alpha = (f\boxtimes g)^{\sim}\beta\sigma_{PQ} = \beta'(f\boxtimes g)\sigma_{PQ} = \beta'\sigma_{P'Q'}(f\times g) = \tilde{r}'\alpha'(f\times g),$$

which shows the outer square of [3][2] commutes. Since α

is sheaf reflection and [3] commutes, it follows that [2] commutes. Now by universal property of $\tilde{\gamma}$, it follows that [1] commutes. Hence i_{PQ} is natural in P and Q. //

<u>Corollary 4.3</u>: For any $A \in \mathrm{AbSh}\mathcal{L}$, $Z_{\mathcal{L}} \otimes A \cong A$,

<u>Proof</u>: Let Z stand for the constant presheaf Z in $\mathrm{AbPSh}\mathcal{L}$. Then $(Z \boxtimes A)U = Z \otimes AU \cong AU$, and for $V \leq U$ in $\mathcal{L}$, the diagram

$$\begin{array}{ccc} Z \otimes AU & \cong & AU \\ \downarrow & & \downarrow \\ Z \otimes AV & \cong & AV \end{array}$$

commutes. Therefore $Z \boxtimes A \cong A$, and this isomorphism is natural in A, for if $f: A \to A'$ is a morphism in $\mathrm{AbPSh}\mathcal{L}$, then the diagram

$$\begin{array}{ccc} Z \otimes AU & \cong & AU \\ {\scriptstyle \mathrm{Id}_Z \otimes f_U}\downarrow & & \downarrow{\scriptstyle f_U} \\ Z \otimes A'U & \cong & A'U \end{array}$$

commutes at each $U \in \mathcal{L}$. By above lemma $A \cong (Z \boxtimes A)^{\sim} \cong \tilde{Z} \otimes A = Z_{\mathcal{L}} \otimes A$. Hence $Z_{\mathcal{L}} \otimes A \cong A$ for all $A \in \mathrm{AbSh}\mathcal{L}$. //

We shall see in the following that the formation of tensor product is associative.

<u>Proposition 4.4</u>: If $A,B,C \in \mathrm{AbSh}\mathcal{L}$, then there is an isomorphism $\tilde{\alpha}: A \otimes (B \otimes C) \to (A \otimes B) \otimes C$ natural in each variable, where $\tilde{\alpha}_U: (A \otimes (B \otimes C))U \to ((A \otimes B) \otimes C)U$ is the sheaf reflection of the isomorphism $AU \otimes (BU \otimes CU) \xrightarrow{\alpha_U} (AU \otimes BU) \otimes CU$ defined via $\alpha_U(a \otimes (b \otimes c)) = (a \otimes b) \otimes c$.

<u>Proof</u>: We know from the category Ab, that there is an isomorphism $\alpha_U: AU \otimes (BU \otimes CU) \to (AU \otimes BU) \otimes CU$ mapping $a \otimes (b \otimes c) \to (a \otimes b) \otimes c$ which is natural, hence gives a natural isomorphism $A \boxtimes (B \boxtimes C) \to (A \boxtimes B) \boxtimes C$ in $\mathrm{AbPSh}\mathcal{L}$. Now applying the sheaf reflection functor, and using lemma 4.2 we get the result.

<u>Proposition 4.5</u>: For $A,B \in \mathrm{AbSh}\mathcal{L}$, there is an isomorphism $\tilde{\alpha}: A \otimes B \to B \otimes A$ natural in each variable, where $\tilde{\alpha}_U: (A \otimes B)U \to (B \otimes A)U$ is the sheaf reflection of the isomorphism $AU \otimes BU \xrightarrow{\alpha_U} BU \otimes AU$ defined via $\alpha_U(a \otimes b) = b \otimes a$.

<u>Proof</u>: Exactly the same as Proposition 4.4.//

Hence from 4.3 and 4.5, it follows that $A \cong Z_{\mathcal{L}} \otimes A \cong A \otimes Z_{\mathcal{L}}$, thus $\otimes$ is a monoidal structure on the category $\mathrm{AbSh}\mathcal{L}$ ([22], page 158).

<u>Section 2</u>: <u>The Internal Adjointness</u>

Recall from chapter 0, that the internal hom functor $[-,-]: \mathrm{AbSh}\mathcal{L}^{opp} \times \mathrm{AbSh}\mathcal{L} \to \mathrm{AbSh}\mathcal{L}$ is

given by $[A,B]U = H_{\downarrow U}(A|U, B|U)$ and for any $V \le U$ in $\mathcal{L}$, the restriction $[A,B]U \to [A,B]V$ sends $h = (h_W)_{W \le U} \to (h_W)_{W \le V}$. For any $V \in \mathcal{L}$, $[A,B]|V = [A|V, B|V]$.

<u>Proposition 4.6</u>: For any $B \in \mathrm{AbSh}\mathcal{L}$, the functor $- \otimes B: \mathrm{AbSh}\mathcal{L} \to \mathrm{AbSh}\mathcal{L}$ is internally left adjoint to the functor $[B,-]: \mathrm{AbSh}\mathcal{L} \to \mathrm{AbSh}\mathcal{L}$, that is for any $A, C \in \mathrm{AbSh}\mathcal{L}$ there is an isomorphism $[A \otimes B, C] \overset{\alpha}{\simeq} [A,[B,C]]$ which is natural in each of the variables A, B and C.

<u>Proof</u>: In the following, we use the notion $(-,-)$ for external hom-functors. For any $U \in \mathcal{L}$, and $f: (A \otimes B)|_U = A|_U \otimes B|_U \to C|_U$ in $\mathrm{AbSh}{\downarrow}U$, $\alpha_U f \in (A|_U, [B,C]|_U)$ is defined by $(\alpha_U f)_V(a): B|_V \to C|_V$, where $((\alpha_U f)_V(a))_W(b) = f_W((A|W) \otimes b)$ for each $W \le V$. Then for $W' \le W$ in $\downarrow U$, the diagram

$$\begin{tikzcd}
BW \arrow[r, "((\alpha_U f)_V(a))_W"] \arrow[d] & CW \arrow[d] \\
BW' \arrow[r, "((\alpha_U f)_V(a))_{W'}"'] & CW'
\end{tikzcd}$$

commutes since $f: A|_U \otimes B|_U \to C|_U$ is a morphism of sheaves and $((a|W) \otimes b)\, W' = (a|W') \otimes (b|W')$. Also for $V' \le V$, the diagram

$$\begin{tikzcd} AV \arrow[r, "(\alpha_U f)_V"] \arrow[d] & (B|V, C|V) \arrow[d] \\ AV' \arrow[r, "(\alpha_U f)_{V'}"'] & (B|V', C|V') \end{tikzcd}$$

where $(\alpha_U f)_V(a) = (f_W(a|W \otimes -))_{W \le V}$, $a \in AV$, $V \le U$, commutes since $(a|V')|W = a|W$. In addition, direct calculation shows that each α_U is a group homomorphism. Now let $V \le U$ in $\mathcal{L}$, then from the way α_U is defined, we get that the diagram

$$\begin{tikzcd} (A|U \otimes B|U, C|U) \arrow[r, "\alpha_U"] \arrow[d] & (A|U, [B|U, C|U]) \arrow[d] \\ (A|V \otimes B|V, C|V) \arrow[r, "\alpha_V"'] & (A|V, [B|V, C|V]) \end{tikzcd}$$

commutes, that is, α is a morphism of sheaves. Hence α is a map in $\mathrm{AbSh}\mathcal{L}$. To obtain the inverse of α, define $\phi\colon [A,[B,C]] \to [A \otimes B, C]$, with components $\phi_U\colon (A|U,[B|U,C|U]) \to (A|U \otimes B|U, C|U)$ given by $(\phi_U h) = \bar{h}$, where $\bar{h}$ is the sheaf reflection of the unique morphism $A|U \otimes B|U \to C|U$ associated with the bi-additive map $A|U \times B|U \to C|U$ with components $AV \times BV \to CV$,

$(a,b) \to (h_V(a))_V(b)$, $V \le U$. It is then easy to see that ϕ is a morphism of sheaves. We now want to show that, ϕ is the inverse of α , that is, for any $U \in \mathcal{L}$, $\phi_U\alpha_U = 1$, and $\alpha_U\phi_U = 1$. If $f \in (A|U \otimes B|U, C|U)$, then $\phi_U(\alpha_U f): A|U \otimes B|U \to C|U$, has components given by $(\phi_U(\alpha_U f))_V: AV \otimes BV \to CV$ which send $a \otimes b \to ((\alpha_U f)_V(a))_V(b) = f_V(a \otimes b)$, hence $(\phi_U(\alpha_U f))_V = f_V$ and therefore $\phi_U\alpha_U f = f$.

Now consider $h: A|U \to [B|U, C|U]$, then $\alpha_U(\phi_U(h)): A|U \to [B|U, C|U]$ has components given by $(\alpha_U(\phi_U h))_V: AV \to (B|V, C|V)$ where $(\alpha_U(\phi_U h))_V(a))_W(b) = (\phi_U(h))_W((a|W) \otimes b) = (h_W(a|W))_W(b) = (h_V(a)|W)_W(b) = (h_V(a))_W(b)$, for all $b \in BW$ and $W \le V$.

Hence $((\alpha_U(\phi_U(h))_V(a))_W = (h_V(a))_W$ for all $a \in AV$ and all $W \le V$, that is $\alpha_U\phi_U(h) = h$, hence α_U is an isomorphism for each $U \in \mathcal{L}$. Thus, we have $[A \otimes B, C] \xrightarrow[\alpha]{\sim} [A,[B,C]]$. We now claim that this isomorphism is natural in each of the three variables. If $g: A \to A'$ is in $\mathrm{AbSh}\mathcal{L}$, then to see that the diagram

$$\begin{tikzcd}
{[A \otimes B, C]} \arrow[r, "\alpha"] & {[A,[B,C]]} \\
{[A' \otimes B, C]} \arrow[u, "{[g \otimes 1_B, 1_C]}"] \arrow[r, "\alpha'"] & {[A',[B,C]]} \arrow[u, "{[g,[1_B,1_C]]}"']
\end{tikzcd}$$

commutes. For any $U \in \mathcal{L}$, and $\beta \in [A' \otimes B, C]U = (A'|U \otimes B|U, C|U)$ let $[g \otimes 1_B, 1_C]_U(\beta) = \tilde{\beta}$, where $\tilde{\beta}_W : (A|U \otimes B|U)W \to CW$ is the sheaf reflection of the unique morphism $AW \otimes BW \to CW$ corresponding to the bi-additive map $AW \times BW \to CW$, $(a,b) \to \beta_W((g_W(a) \otimes b)^{\sim})$. Then $\alpha_U(\tilde{\beta}) : (A|U, (B|U, C|U))$ is given by $((\alpha_U\tilde{\beta})_V(a))_W(b) = \tilde{\beta}_W(a|W \otimes b)$, where $W \leq V$, $a \in AV$, $b \in BW$. Also $\alpha'_U(\beta)$ with components $(\alpha'_U\beta)_V$ is given by $((\alpha'_U\beta)_V(a'))_W(b) = \beta_V(a'|W \otimes b)$, and $[g,[B,C]]_U(\alpha'_U\beta) = \beta^*$ where $((\beta^*_V)a)_W(b) = ((\alpha'_U\beta)_V(g_V(a)))_W(b) = \beta_V(g_V(a)|W \otimes b) = \beta_W((g_W(a) \otimes b)^{\sim})$. Hence $\alpha[g \otimes 1_B, 1_C] = [g,[1_B,1_C]]\alpha'$, that is α is natural in A. Similarly we can show that α is also natural in the variables B and C respectively. All this shows that the functor $- \otimes B$ is internally left adjoint to the functor $[B,-]$ for any $B \in \mathrm{AbSh}\mathcal{L}$. Also α being an isomorphism implies α_E is an isomorphism, that is $(A \otimes B, C) \simeq (A,[B,C])$, therefore $- \otimes B$ is adjoint to $[B,-]$.

Hence we conclude that the tensor product makes $\mathrm{AbSh}\mathcal{L}$ an internally closed monoidal category.

<u>Lemma 4.7</u>: For any $P,T \in \mathrm{AbSh}\mathcal{L}$, $[P,-] \cong [T,-]$, or $[-,P] \cong [-,T]$, implies $P \cong T$.

<u>Proof</u>: $[P,-] \cong [T,-]$ obviously implies $(P,-) \cong (T,-)$ which is known to imply $P \cong T$, and the second part follows similarly.

<u>Corollary 4.8</u>: For any $A \in \mathrm{AbSh}\,\mathcal{L}$, $A \cong [Z_{\mathcal{L}}, A]$.

<u>Proof</u>: For any $B \in \mathrm{AbSh}\,\mathcal{L}$, $B \otimes Z_{\mathcal{L}} \cong B$, hence $[B,A] \cong [B \otimes Z_{\mathcal{L}}, A] \cong [B,[Z_{\mathcal{L}},A]]$ and therefore by lemma 4.7 $A \cong [Z_{\mathcal{L}},A]$.

<u>Section 3</u>: <u>Group Theoretic Properties of the Internal Hom-Functor</u>

<u>4.9</u>: For $A,C \in \mathrm{AbSh}\,\mathcal{L}$, the group $[A,C] = 0$ in the following cases:

(i) A is a torsion group and C is a torsion free group.

(ii) A is p-primary and C is q-primary for distinct primes p and q.

<u>Proof</u>: (i) let $f \in [A,C]U = (A|U, C|U)$, $U \in \mathcal{L}$. Since the restriction functor R_U preserves torsion and torsion freeness (0.28, 0.29), it follows $A|U$ is torsion and $C|U$ is a torsion free group. Also, since any epimorphic image of a torsion group is torsion (2.12) $f(A|U)$ is torsion group contained in $C|U$, hence $f(A|U) = 0$. Thus $f = 0$ which means $[A,C]U = 0$, that is $[A,C] = 0$.

(ii) A p-primary means $A = \bigcup_{0 \neq n \in N} \mathrm{Ker}\, p^n A$, and similarly $C = \bigcup_{0 \neq n \in N} \mathrm{Ker}\, q^n A$, let $f \in [A,C]U$, $U \in \mathcal{L}$. Since $f(\mathrm{Ker}\, p^n(A|U)) \subseteq \mathrm{Ker}\, p^n(C|U)$ for all $0 \neq n \in N$, and $\mathrm{Ker}\, p^n(E|U) \cap \mathrm{Ker}\, q^m(C|U) = 0$ for all $m,n \neq 0$, it follows

$f(\operatorname{Ker} p^n(A|U)) = 0$. Hence $f(A|U) = 0$, that is $f = 0$, which implies $[A,C] = 0$.

<u>4.10</u>: If $[A,C] = 0$, for all torsion free groups C, then A is a torsion group.

<u>Proof</u>: If tA denotes the torsion subgroup of A, then $\frac{A}{tA}$ is torsion free, so by hypothesis $[A,\frac{A}{tA}] = 0$. Hence the quotient map $h: A \to \frac{A}{tA}$ is zero, that is $\frac{A}{tA} = 0$, therefore $A = tA$.

<u>4.11</u>: If $[A,C] = 0$ for all torsion groups A, then C is torsion free.

<u>Proof</u>: Let tC denote the torsion subgroup of A. If $tC \neq 0$, then $[tC,C] \neq 0$, a contradiction to the hypothesis. Hence $tC = 0$, that is C is a torsion free group.

For any $A \in \mathrm{AbSh}\mathcal{L}$, since the functor $[A,-]$ is a right adjoint, it preserves all limits, hence it takes the morphism $n_C: C \to C^n \to C$ to $n_{[A,C]}: [A,C] \to [A,C^n] = [A,C]^n \to [A,C]$ and so we have the following:

<u>4.12</u>: If C is a torsion free group, then so is $[A,C]$ for all $A \in \mathrm{AbSh}\mathcal{L}$.

Proof: C torsion free means, n_C is a monomorphism for all $0 \neq n \in N$. The functor $[A,-]$ preserves monos, hence $[A,n_C] = n_{[A,C]}$ is a mono, that is $[A,C]$ is torsion free.

4.13: If B is a torsion free divisible group, then so is $[A,B]$ for all $A \in \mathrm{AbSh}\mathcal{L}$.

Proof: B torsion free divisible means each n_B is a monomorphism as well as an epimorphism, hence an isomorphism all $0 \neq n \in N$. Since $[A,-]$ takes n_B to $n_{[A,B]}$ and functors preserve isomorphisms, we get $n_{[A,B]}$ is an isomorphism, hence $[A,B]$ is a torsion free divisible group.

Remark 4.14: For any $C \in \mathrm{AbSh}\mathcal{L}$, we have the contravariant functor $[-,C]: \mathrm{AbSh}\mathcal{L} \to \mathrm{AbSh}\mathcal{L}$, and if $f: A' \to A$ is in $\mathrm{AbSh}\mathcal{L}$, then $[f,C]: [A,C] \to [A',C]$ is just composition by $f|U$. Then if $A,B \in \mathrm{AbSh}\mathcal{L}$, we have $(B,[A,C]) \cong (B \otimes A,C) \cong (A \otimes B,C) \cong (A,[B,C])$, which means $[-,C]$ is adjoint to itself on the right, hence the contravariant functor $[-,C]$ converts colimits to limits, in particular transforms epimorphisms to monomorphisms. Thus we have the following.

4.15: If A is a divisible group, then $[A,C]$ is torsion free for all $C \in \mathrm{AbSh}\mathcal{L}$.

<u>Proof</u>: A divisible means $n_A : A \to A$ is a epimorphism. The functor $[-,C]$ takes n_A to $n_{[A,C]}$ and transforms epis to monos. Hence $[A,C] \xrightarrow{n_{[A,C]}} [A,C]$ is a monomorphism, that is, $[A,C]$ is torsion free.

<u>4.16</u>: If A is a torsion free divisible group, then so is the group $[A,C]$, for all $C \in \mathrm{AbSh}\mathcal{L}$.

<u>Proof</u>: Clear, since A torsion free divisible implies n_A is an isomorphism. The functor $[-,C]$ preserves isomorphisms and takes n_A to $n_{[A,C]}$, it follows $[A,C]$ is torsion free divisible.

<u>Counterexample 4.17</u>: Recall that in the category Ab, $\mathrm{Hom}(A,C) = 0$ whenever A is an injective group and C a reduced group. This is essentially due to the fact that in Ab, any epimorphic image of an injective group is injective. Since this no longer remains true in an arbitrary $\mathcal{L}$, we have a counterexample showing that $[A,C] \neq 0$, even if A is injective and C a reduced group. Consider the locale $\mathcal{L} = 3$, and $A = \begin{smallmatrix}\mathbb{Q}\\ \downarrow \\ \mathbb{Q}\end{smallmatrix}\ \mathrm{Id}$, $C = \begin{smallmatrix}\mathbb{Q}\\ \downarrow \\ \mathbb{Q}/Z\end{smallmatrix}\ v$. Then A is injective and C is reduced, but $[A,C] \neq 0$ since

$h: A \to C$ given by
$$\begin{tikzcd} \mathbb{Q} \arrow[r, "\mathrm{Id}"] \arrow[d, "\mathrm{Id}"'] & \mathbb{Q} \arrow[d, "v"] \\ \mathbb{Q} \arrow[r, "v"'] & \mathbb{Q}/Z \end{tikzcd}$$
(where v is the quotient map) is a non zero morphism in AbSh3 . //

<u>4.18</u>: However, if $\mathcal{L}$ is a Boolean locale then $[A,C] = 0$ whenever A is injective and C is reduced, because for such $\mathcal{L}$ divisible = injective (B. Banaschewski [4]), hence any epimorphic image of an injective is injective. Let C be a reduced group in $\mathrm{AbSh}\mathcal{L}$; then $C|U$ is also reduced in $\mathrm{AbSh}{\downarrow}U$. For if $0 \neq B$ is an injective subgroup of $C|U$, then $0 \neq E_U(B)$ is divisible in $\mathrm{AbSh}\mathcal{L}$, hence injective. Thus $0 \neq E_U(B) \leq E_U(C|U) = T_U C \leq C$, but C reduced implies $E_U B = 0$, that is $B = 0$. So all this shows that $C|U$ is reduced in $\mathrm{AbSh}{\downarrow}U$, whenever C is reduced in $\mathrm{AbSh}\mathcal{L}$. Now, suppose there exists a non zero $f \in [A,C]U$. Then $f: A|U \to C|U$ and since ${\downarrow}U$ is also Boolean, $\mathrm{Im}(f)$ is again injective, therefore zero because $C|U$ is reduced. Therefore we get that, for a Boolean locale $[A,C] = 0$ whenever A is injective and C is reduced.

<u>4.19</u>: For any $A \in \mathrm{AbSh}\mathcal{L}$, $[A,-]$ always preserves zero and reflects zeroes if A is generator.

<u>Proof</u>: Since the functor $[A,-]$ is right adjoint, for any $A \in \mathrm{AbSh}\mathcal{L}$, it preserves zero. Moreover, if A is a generator then $(A,-)$ is faithful, hence $[A,-]$ is faithful, and therefore $[A,-]$ reflects zeroes.

4.20: For any $C \in \mathrm{AbSh}\mathcal{L}$, the functor $[-,C]$ preserves zeroes, and reflects zeroes if C is a cogenerator.

Proof: Dual of the last result.

Remark: From 4.13, it is natural to ask if the functor $[A,-]$ preserves torsion free injectives. Since for a Boolean locale divisible = injective (B. Banaschewski [4]), it follows $[A,B]$ is a torsion free injective group whenever B is .
Also for an arbitrary $\mathcal{L}$, the functor $[A,-]$ will preserve (torsion free) injectives if its left adjoint $A \otimes -$ preserves monomorphisms (0.19). In the next chapter we shall discuss those groups $A \in \mathrm{AbSh}\mathcal{L}$ for which this is the case.

Proposition 4.21: If $\mathcal{L}$ is a well ordered locale, then the functor $[A,-]$ for any $A \in \mathrm{AbSh}\mathcal{L}$, preserves torsion free injectives.

Proof: Consider a torsion free injective $B \in \mathrm{AbSh}\mathcal{L}$. Then we know from 4.13, that $[A,B]$ is a torsion free divisible group, hence each $[A,B]U$ is a torsion free divisible group in Ab. We now claim that for any $V \leq U$ in $\mathcal{L}$, $[A,B]U \to [A,B]V$ is a onto homomorphism in Ab.

It is enough to consider $U = E$. Then for any $\phi: A|V \to B|V$, let $\bar{\phi}: A \to B$ be any extension of $T_V A = E_V(A|V) \to E_V(B|V) \rightarrowtail B$ which exists since B is injective. Then $\bar{\phi}|V = \phi$, hence $[A,B]U \to [A,B]V$ is onto for all $V \leq U$. Since each $[A,B]U$ is torsion free divisible it is a module over $\mathbb{Q}$ and therefore $[A,B]U \to [A,B]V$ is a split epi in Ab. By (1.22) it follows that $[A,B]$ is injective in $AbSh\mathcal{L}$. //

However for an arbitrary $\mathcal{L}$, it still remains open whether the functor $[A,-]$ preserves torsion free injectives.

Section 4: Group Theoretic Properties of the Tensor Product

In this section we list some properties of tensor product in $AbSh\mathcal{L}$.

Lemma 4.22: For any $B,C \in AbSh\mathcal{L}$, and $0 \neq n \in Z$, $nB \otimes C = B \otimes nC = n(B \otimes C)$.

Proof: By definition

$$\begin{aligned}(B \otimes nC)U &\doteq BU \otimes (nC)U \\ &\doteq BU \otimes n(CU) \\ &\doteq n(BU) \otimes CU\end{aligned}$$

Also $(nB \otimes C)U \doteq (nB)U \otimes CU$

$\doteq n(BU) \otimes CU$

and $n(B \otimes C)U \doteq n((B \otimes C)U)$

$\doteq n(BU \otimes CU)$

$\doteq nBU \otimes CU$.

Thus $B \otimes nC = nB \otimes C = n(B \otimes C)$.

<u>4.23</u>: If $A,B \in \mathrm{AbSh}\mathcal{L}$, then $A \otimes B = 0$ in the following cases.

(i) A is p-divisible and B is p-primary for some prime p .

(ii) A is p-primary and B is q-primary for distinct primes p and q .

(iii) A is divisible and B is torsion.

<u>Proof</u>: (i) Since A is p-divisible, it follows $A = p^n A$ for all $n \in N$. Also B is p-primary, so $B = \underrightarrow{\ell t}\, \mathrm{Ker}\, p^n B$ . . By the above lemma

$A \otimes \mathrm{Ker}\, p^n{}_B = p^n A \otimes \mathrm{Ker}\, p^n B = A \otimes p^n \mathrm{Ker}\, p^n B = A \otimes 0 = 0$

hence $A \otimes B = A \otimes \underrightarrow{\ell t}\, \mathrm{Ker}\, p^n B$. But tensor product being a left adjoint commutes with direct limits, therefore

$A \otimes B = \underrightarrow{\ell t}(A \otimes \mathrm{Ker}\, p^n B) = 0$.

(ii) Since A is p-primary, it follows that A is q-divisible (3.5) for all primes $q \neq p$. Now apply part (i), we get $A \otimes B = 0$.

(iii) A divisible means $A = nA$ for all $n \in N$, and B

a torsion group implies $B = \underset{0 \neq n \in N}{\underrightarrow{\ell t}} \operatorname{Ker} n_B$. Hence by the above lemma $A \otimes \operatorname{Ker} n_B = nA \otimes \operatorname{Ker} n_B = A \otimes n\operatorname{Ker} n_B = 0$. Thus $A \otimes B = \underrightarrow{\ell t}(A \otimes \operatorname{Ker} n_B) = 0$.

<u>4.24</u>: If A is a torsion group, then so is the group $A \otimes C$ for all $C \in \mathrm{AbSh}\mathcal{L}$.

<u>Proof</u>: A torsion implies $A = \underset{0 \neq n \in N}{\underrightarrow{\ell t}} \operatorname{Ker} n_A$, and so $A \otimes C = (\underset{0 \neq n \in N}{\underrightarrow{\ell t}} \operatorname{Ker} n_A) \otimes C = \underset{0 \neq n \in N}{\underrightarrow{\ell t}} (\operatorname{Ker} n_A \otimes C)$. It is easy to see that the presheaf $U \rightsquigarrow \operatorname{Ker} n_{AU} \otimes CU$ defining $\operatorname{Ker} n_A \otimes C$ is a pretorsion presheaf, so by 2.9, $\operatorname{Ker} n_A \otimes C$ is a torsion group in $\mathrm{AbSh}\mathcal{L}$. Now applying 2.10, we get $\underset{0 \neq n \in N}{\underrightarrow{\ell t}} (\operatorname{Ker} n_A \otimes C) = A \otimes C$ is a torsion group in $\mathrm{AbSh}\mathcal{L}$, hence the result.

<u>Proposition 4.25</u>: If A is p-primary, so is $A \otimes C$ for all $C \in \mathrm{AbSh}\mathcal{L}$.

<u>Proof</u>: Clear, since each $\operatorname{Ker} p_A^n \otimes C$ is a p-primary group, and therefore $\underrightarrow{\ell t}(\operatorname{Ker} p_A^n \otimes C) = A \otimes C$ is a p-primary group in $\mathrm{AbSh}\mathcal{L}$.

<u>4.26</u>: If A is a p-divisible group, then so is $A \otimes C$, for all $C \in \mathrm{AbSh}\mathcal{L}$.

<u>Proof</u>: A p-divisible implies that $A = p^n A$, $0 \neq n \in N$, that is $A \xrightarrow{p_A^n} A$ is an epimorphism. Since tensor product preserves co-limits, it preserves epimorphisms and finite products, therefore $p_A^n \otimes C = p^n_{(A\otimes C)}$ and $p^n_{(A\otimes C)}$ is an epimorphism. This shows $A \otimes C$ is p-divisible for all $C \in \mathrm{AbSh}\mathcal{L}$.

<u>4.27</u>: If A is a divisible group, then so is $A \otimes C$, for all $C \in \mathrm{AbSh}\mathcal{L}$.

<u>Proof</u>: A divisible means $n_A : A \to A$ is an epimorphism for all $0 \neq n \in N$. By the same argument as above $n_A \otimes C = n_{A\otimes C}$ is an epimorphism, hence $A \otimes C$ is divisible.//

As an immediate consequence one has: If $\mathcal{L}$ is Boolean then, for any $C \in \mathrm{AbSh}\mathcal{L}$, the functor $- \otimes C$ preserves injectives.

<u>Counterexample 4.28</u>: Since injective and divisible are not the same in $\mathrm{AbSh}\mathcal{L}$ for an arbitrary $\mathcal{L}$, we have the question whether $A \otimes C$ is injective if A is an injective group. Here is a counterexample: Consider the locale $\mathcal{L} = 3$, and $A, C \in \mathrm{AbSh}3$ as follows:

$$A = \mathrm{Id}\begin{array}{c}\mathbb{Q}\\ \downarrow\\ \mathbb{Q}\end{array}, \quad C = \begin{array}{c}\mathbb{Q}/\mathbb{Z}\\ \downarrow 0\\ \mathbb{Z}\end{array}$$

then A is an injective group (Banaschewski [4]) but

$$A \otimes C = \begin{array}{c}\mathbb{Q} \times \mathbb{Q}/\mathbb{Z} \cong 0\\ \downarrow 0\\ \mathbb{Q} \otimes \mathbb{Z} \cong \mathbb{Q}\end{array}$$

is certainly not injective in AbSh3 . However, if we take <u>both</u> A and C to be injective groups in AbSh3 ,

$$A = \begin{array}{c}P \times H\\ \downarrow p\\ P\end{array}, \quad C = \begin{array}{c}R \times T\\ \downarrow q\\ T\end{array}$$

where P, H, R, T are divisible groups in Ab , then

$$A \otimes C = \begin{array}{c}(P\times H)\otimes(R\times T)\\ \downarrow p\otimes q\\ P\otimes T\end{array} = \begin{array}{c}P\otimes R\times H\otimes R\times P\otimes T\times H\otimes T\\ \downarrow p\otimes q\\ P\otimes T\end{array}$$

which is a split epimorphism between divisible groups in Ab , and hence injective in AbSh3 . That shows if

$A, C \in \mathrm{AbSh}3$ are injective, then so is $A \otimes C$.

<u>Remark 4.29</u>: If we now consider an arbitrary locale $\mathcal{L}$, and A, C injective groups in $\mathrm{AbSh}\mathcal{L}$, then it is clear that, $A \otimes C$ has the following two properties.

(i) $AU \otimes CU$ is divisible in Ab for all $U \in \mathcal{L}$, since AU and CU are both divisible.

(ii) If $V \leq U$ in $\mathcal{L}$, then $AU \otimes CU \to AV \otimes CV$ is a split epi in Ab, because $AU \to AV$, $CU \to CV$ are split epis in Ab (1.5). Therefore, if $\mathcal{L}$ is inversely well ordered, then it has only trivial covers and hence every presheaf is a sheaf. So $(A \otimes C)U = AU \otimes CU$ is injective in Ab for each $U \in \mathcal{L}$, and $(A \otimes C)U \to (A \otimes C)V = AU \otimes CU \to AU \otimes CU$ is a split epimorphism whenever $V \leq U$ in $\mathcal{L}$. By (1.23), it follows $A \otimes C$ is an injective group in $\mathrm{AbSh}\mathcal{L}$. Hence, we get the following:

<u>Proposition 4.30</u>: If $A, B \in \mathrm{AbSh}\mathcal{L}$ are injective and $\mathcal{L}$ is inversely well ordered then $A \otimes B$ is injective.//

CHAPTER 5

FLATNESS IN THE CATEGORY $\mathrm{AbSh}\mathcal{L}$

Introduction: In this chapter we undertake the study of flat groups in $\mathrm{AbSh}\mathcal{L}$, and prove the counterparts of some of the results for flat groups in Ab. For example, we show that $A \in \mathrm{AbSh}\mathcal{L}$ is flat iff A is a torsion free group iff each AU, $U \in \mathcal{L}$ is a torsion free (= flat) group in Ab. From this we get a number of corollaries, in particular, we show that for any local lattice homomorphism $\phi: \mathcal{L} \to \mathcal{M}$, the functors ϕ_* as defined in (0.16) preserve flatness. Finally, we show that $A \in \mathrm{AbSh}\mathcal{L}$ is flat (= torsion free) iff the group $A^* = [A,P]$ is injective where P is an injective cogenerator in $\mathrm{AbSh}\mathcal{L}$, and A is a torsion group, iff A^* is a reduced group, that is A^* has no non-zero injective subgroups.

Section 1: Characterizing Flat Groups

5.1: Recall from the last chapter, that any $A \in \mathrm{AbSh}\mathcal{L}$ produces a pair of functors $[A,-]: \mathrm{AbSh}\mathcal{L} \to \mathrm{AbSh}\mathcal{L}$ and $A \otimes -: \mathrm{AbSh}\mathcal{L} \to \mathrm{AbSh}\mathcal{L}$ where $A \otimes -$ is internally left adjoint to $[A,-]$, and where $[A,B]U = (A|U, B|U)$, $(A \otimes B)U \doteq AU \otimes BU$ for $U \in \mathcal{L}$. Since left adjoints preserves epimorphisms, it follows that for any exact sequence

$0 \to B \xrightarrow{f} C \xrightarrow{g} D \to 0$ in $\mathrm{AbSh}\,\mathcal{L}$, the sequence

$A \otimes B \xrightarrow{1_A \otimes f} A \otimes C \xrightarrow{1_A \otimes g} A \otimes D \to 0$ is always exact, where

$(1_A \otimes f)_U = (1d_{AU} \otimes f_U)^{\sim}$.

<u>Definition 5.2</u>: $A \in \mathrm{AbSh}\,\mathcal{L}$ is called a flat group iff the functor $A \otimes -: \mathrm{AbSh}\,\mathcal{L} \to \mathrm{AbSh}\,\mathcal{L}$ preserves monomorphisms. That is, if $B \xrightarrow{f} C$ is a monomorphism then $A \otimes B \xrightarrow{1_A \otimes f} A \otimes C$ is also a monomorphism, on equivalently A is flat iff the tensor product with A preserves exact sequences.

<u>Lemma 5.3</u>: The group $Z_{\mathcal{L}} \in \mathrm{AbSh}\,\mathcal{L}$ is a flat group.

<u>Proof</u>: From the last chapter, we have that for any $B \in \mathrm{AbSh}\,\mathcal{L}$, $Z_{\mathcal{L}} \otimes B \overset{\psi_B}{\cong} B$ (natural). So, if $B \xrightarrow{f} C$ is a monomorphism, then the naturality of ψ_B implies that the diagram,

$$\begin{tikzcd}
B \arrow[r, "f"] & C \\
Z_{\mathcal{L}} \otimes B \arrow[u, "\psi_B"] \arrow[r, "1_{Z_{\mathcal{L}}} \otimes f"'] & Z_{\mathcal{L}} \otimes C \arrow[u, "\psi_C"']
\end{tikzcd}$$

commutes. Hence $Z_{\mathcal{L}} \otimes B \to Z_{\mathcal{L}} \otimes C$ is a monomorphism, which shows that $Z_{\mathcal{L}}$ is a flat group.

For the following, we say $B \in \mathrm{AbSh}\,\mathcal{L}$ is flat relative to $Z_{\mathcal{L}}$ iff for any monomorphism $D \overset{f}{\to} Z_{\mathcal{L}}$ in $\mathrm{AbSh}\,\mathcal{L}$, $B \otimes D \xrightarrow{1_B \otimes f} B \otimes Z_{\mathcal{L}}$ is also a monomorphism.

<u>Proposition 5.4</u>: For $B \in \mathrm{AbSh}\,\mathcal{L}$, the following are equivalent,

(i) B is flat relative to $Z_{\mathcal{L}}$.

(ii) B is a torsion free group.

(iii) B is a flat group.

(iv) There is a cover $E = \bigvee_{i \in I} U_i$ such that $B|U_i$ is a flat group in $\mathrm{AbSh}\downarrow U_i$ for all $i \in I$.

<u>Proof</u>: <u>(i)</u> implies (ii) given B flat relative to $Z_{\mathcal{L}}$, we want to show $\mathrm{Ker}\, n_B = 0$ where $n_B : B \to B$ is the composite $B \overset{\Delta_B}{\to} B^n \overset{+_B}{\to} B$.

If $i : nZ_{\mathcal{L}} \to Z_{\mathcal{L}}$ and $j : \mathrm{Ker}(n_B) \to B$ are the natural embeddings, consider in $\mathrm{AbPSh}\,\mathcal{L}$ the composite map

$$(nZ_{\mathcal{L}}) \boxtimes \mathrm{Ker}(n_B) \xrightarrow[g = 1 \boxtimes j]{} (nZ_{\mathcal{L}}) \boxtimes B \xrightarrow[f = i \boxtimes 1]{} Z_{\mathcal{L}} \boxtimes B.$$

One easily sees, by looking at elements for each $U \in \mathcal{L}$, that $fg = 0$. Since the sheaf reflection preserves zero, it follows that $\tilde{f}\tilde{g} = 0$ in $\mathrm{AbSh}\,\mathcal{L}$. By hypothesis, B flat relative to $Z_{\mathcal{L}}$ implies $\tilde{f}$ is a monomorphism, therefore $\tilde{f}\tilde{g} = 0$ gives $\tilde{g} = 0$. $nZ_{\mathcal{L}} \cong Z_{\mathcal{L}}$ is flat, hence

$\mathfrak{g} = 0$ implies $\operatorname{Ker} n_B = 0$, that is B is a torsion free group.

<u>(ii)</u> implies (iii) B a torsion free group in $\mathrm{AbSh}\mathcal{L}$ means each BU is torsion free hence a flat group in Ab. Consider a monomorphism $C \xrightarrow{f} D$ in $\mathrm{AbSh}\mathcal{L}$, then each $CU \xrightarrow{fu} DU$ is a monomorphism in Ab, therefore the flatness of BU implies $BU \otimes CU \to BU \otimes DU$ is a monomorphism. Thus the morphism $B \boxtimes C \to B \boxtimes C$ is a monomorphism, and since the sheaf reflection preserves monomorphisms, it follows $B \otimes C \to B \otimes D$ is a monomorphism. Hence B is flat.

(iii) $\Rightarrow$ (iv) clear, by taking the trivial cover of E.

(iv) $\Rightarrow$ (i) consider a monomorphism $S \xrightarrow{h} Z_{\mathcal{L}}$ in $\mathrm{AbSh}\mathcal{L}$. Since the restriction functors preserves monomorphisms (0.14 (4)), it follows $S|U_i \xrightarrow{h|U_i} Z_{\mathcal{L}}|U_i$ is a monomorphism in $\mathrm{AbSh}{\downarrow}U_i$ for all $i \in I$. But $B|U_i$ flat in $\mathrm{AbSh}{\downarrow}U_i$ implies $B|U_i \otimes S|U_i \xrightarrow{1_{B|U_i} \otimes h|U_i} B|U_i \otimes Z_{\mathcal{L}}|U_i$ is a monomorphism for all $i \in I$. Let $U \in \mathcal{L}$ be arbitrary, then $U = \bigvee_{i\ I} U \wedge U_i$ and the diagram

$$\begin{tikzcd}
(B \otimes S)U \arrow[r] \arrow[d, "(1_B \otimes h)_U"'] & \prod_{i \in I} (B \otimes S)U \wedge U_i \arrow[d, "(1_B \otimes h)_{U \wedge U_i}"] \\
(B \otimes Z_{\mathcal{L}})U \arrow[r] & \prod_{i \in I} (B \otimes Z_{\mathcal{L}})U \wedge U_i
\end{tikzcd}$$

commutes, since it commutes at each $i \in I$. It then follows that $(1_B \otimes h)_U$ is a monomorphism for all $U \in \mathcal{L}$, which implies $1_B \otimes h$ is a monomorphism. Hence B is flat relative to $Z_{\mathcal{L}}$.

Corollaries:

5.5: If $A \in \mathrm{AbSh}\mathcal{L}$ is a torsion free group, then the functor $[A,-]$ preserves injectives.

Proof: By the above proposition, A is a flat group, so the functor $A \otimes -$ preserves monomorphism. Since $[A,-]$ is right adjoint to $A \otimes -$, it follows by (0.19) that $[A,-]$ preserves injectives.

5.6: If $A \in \mathrm{AbSh}\mathcal{L}$ is flat, and $B \geq A$ is an essential extension, then B is flat.

Proof: Let $tB \subseteq B$ be the torsion subgroup of B. Then $A \cap tB = 0$ by hypothesis hence $tB = 0$ by essentialness, therefore B is torsion free and hence flat by 5.4.

5.7: If A is flat then any subgroup C is flat.

Proof: Any subgroup of a torsion free group is torsion free.

5.8: For any $A \in \mathrm{AbSh}\mathcal{L}$, the functor $[A,-]: \mathrm{AbSh}\mathcal{L} \to \mathrm{AbSh}\mathcal{L}$ preserves flatness.

Proof: Since flat = torsion free, this follows by 4.12.

5.9: For any $U \in \mathcal{L}$, the functors R_U, E_U and T_U preserve flatness.

Proof: Since flat = Torsion free, the result follows by 0.30.

5.10: If $P \in \mathrm{AbPSh}\mathcal{L}$ has the property that each PU is torsion free group in Ab, then its sheaf reflection $\tilde{P} \in \mathrm{AbSh}\mathcal{L}$ is a flat group.

Proof: For any cover $C = (U_i)_{i \in I}$ of the element $U \in \mathcal{L}$, the equalizer P_C of the maps

$$\prod_{i \in I} PU_i \underset{\gamma}{\overset{\beta}{\rightrightarrows}} \prod_{(i,j) \in I \times I} P(U_i \wedge U_j)$$

is also a torsion free group in Ab. Hence $\underset{C \in \mathrm{Cov}U}{\overrightarrow{\ell t}} P_C = \tilde{P}U$ is a torsion free group since a direct limit of torsion free group is torsion free. Hence $\tilde{P}$ is torsion free, and by proposition 5.4, $\tilde{P}$ is then a flat group.

5.11: If $\phi: \mathcal{L} \to \mathcal{M}$ is a local lattice homomorphism, then the functors $\phi^*: \mathrm{AbSh}\mathcal{L} \to \mathrm{AbSh}\mathcal{M}$, $\phi_*: \mathrm{AbSh}\mathcal{M} \to \mathrm{AbSh}\mathcal{L}$ preserve flatness.

Proof: If $A \in \mathrm{AbSh}\mathcal{L}$ is flat = torsion free, then $(\phi^*A)V \doteq \underset{\phi(\overrightarrow{W}) \geq V}{\ell t}\, AW$, $V \in \mathcal{M}$, is clearly the sheaf reflection of a presheaf which is torsion free. By 5.10, it follows ϕ^*A is torsion free, hence flat in $\mathrm{AbSh}\mathcal{M}$. If $B \in \mathrm{AbSh}\mathcal{M}$ is flat, then $(\phi_*B)U = B(\phi(U))$ is torsion free in Ab for

all $U \in \mathcal{L}$, hence $\phi_* B$ is torsion free = flat in $\mathrm{AbSh}\,\mathcal{L}$.

Note: In the above corollaries, 5.8, 5.9, and 5.11 we are dealing with functors F such that $F(n_A) = n_{FA}$, $F(0) = 0$ and $F(\mathrm{Ker}\, n_A) = \mathrm{Ker} F(n_A)$, and for any such F, $\mathrm{Ker}\, n_A = 0$ implies $\mathrm{Ker}\, n_{FA} = 0$, that is F preserves flatness.

5.12: For any $A \in \mathrm{AbSh}\,\mathcal{L}$, the functor $(A,-) = \mathrm{AbSh}\,\mathcal{L}(A,-) : \mathrm{AbSh}\,\mathcal{L} \to \mathrm{Ab}$ preserves flatness.

Proof: The functor $(A,-)$ is the composite of $[A,-]$ and $B \rightsquigarrow BE$, both of which preserve flatness.

5.13: The direct limit of any family of flat groups is flat.

Proof: Let $\{A_i\}_{i \in I}$ be any directed family of flat groups in $\mathrm{AbSh}\,\mathcal{L}$. Then $(\varinjlim A_i)U \doteq \varinjlim A_i U$, so the presheaf generating $\varinjlim A_i$ is a torsion free presheaf. By 5.10, it follows $\varinjlim A_i$ is a torsion free and hence a flat group.

5.14: The product of any family of groups is flat iff each component is flat.

Proof: ($\Rightarrow$) let $\prod_{i \in I} A_i$ be flat in $\mathrm{AbSh}\,\mathcal{L}$. Then each $A_i U \subseteq (\prod A_i)U = \prod A_i U$ is torsion free, hence a flat group in $\mathrm{AbSh}\,\mathcal{L}$.

($\Leftarrow$) suppose $\{A_i\}_{i \in I}$ is a family of flat groups. Then each

A_iU, $i \in I$, $U \in \mathcal{L}$ is a torsion free group in Ab. Hence $\prod_{i\in I} A_iU$ is torsion free in Ab, that is ΠA_i is torsion free, hence by 5.4, ΠA_i is a flat group in $\mathrm{AbSh}\,\mathcal{L}$.

<u>5.15</u>: The direct sum of any family of groups is flat iff each component is flat.

<u>Proof</u>: ($\Leftarrow$) If $\{A_i\}_{i\in I}$ is a family of flat groups, then by the above corollary, ΠA_i is flat in $\mathrm{AbSh}\,\mathcal{L}$. Since $\bigoplus_{i\in I} A_i \subseteq \prod_I A_i$, it follows by 5.7, that $\bigoplus_{i\in I} A_i$ is a flat group.

($\Rightarrow$) Suppose now $\bigoplus_{i\in I} A_i$ is a flat group. Then $A_iU \subseteq (\bigoplus_{i\in I} A_i)U$, so A_iU is torsion free in Ab, all $i \in I$, $U \in \mathcal{L}$. That is each A_i is a torsion free, hence flat in $\mathrm{AbSh}\,\mathcal{L}$.

<u>5.16</u>: If $A, B \in \mathrm{AbSh}\,\mathcal{L}$ are flat, then so is $A \otimes B$.

<u>Proof</u>: By definition $(A \otimes B)U \doteq AU \otimes BU$, $U \in \mathcal{L}$. Since A and B are torsion free, each AU and BU is a torsion free group in Ab, therefore $AU \otimes BU$ is torsion free. Hence the presheaf defining $A \otimes B$ is a torsion free presheaf, by 5.10 it follows $A \otimes B$ is torsion free and hence flat in $\mathrm{AbSh}\,\mathcal{L}$.

<u>5.17</u>: The functor $-_{\mathcal{L}}: \mathrm{Ab} \to \mathrm{AbSh}\,\mathcal{L}$ preserves flatness.

<u>Proof</u>: If P is flat in Ab, then P is torsion

free. But $(P_{\mathcal{L}})U \doteq P$, so by 5.10 it follows $P_{\mathcal{L}}$ is a flat group in $\mathrm{AbSh}\,\mathcal{L}$.

5.18: If $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is an exact sequence in $\mathrm{AbSh}\,\mathcal{L}$, where C is a flat group, then A is flat iff B is flat.

Proof: consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "n_A"'] & B \arrow[r, "g"] \arrow[d, "n_B"'] & C \arrow[r] \arrow[d, "n_C"'] & 0 \\
0 \arrow[r] & A \arrow[r, "f"'] & B \arrow[r, "g"'] & C \arrow[r] & 0
\end{tikzcd}$$

in $\mathrm{AbSh}\,\mathcal{L}$, where C a flat group implies n_C is a monomorphism.

($\Rightarrow$) If B is a flat group then it is torsion free, and so n_B is a monomorphism. Therefore $n_B f = f n_A$ a monomorphism, implies n_A is a monomorphism. Hence A is torsion free, that is, flat in $\mathrm{AbSh}\,\mathcal{L}$.

($\Leftarrow$) A flat means n_A is a monomorphism. We claim that n_B is a monomorphism, so let $n_B\alpha = 0$ for some $\alpha: D \to B$. Then $0 = g n_B \alpha = n_C g \alpha$, which by the flatness of C implies that $g\alpha = 0$. Since $f = \mathrm{Ker}\, g$, so $g\alpha = 0$ implies that there exists a unique $\gamma: D \to A$, such that $f\gamma = \alpha$. Then $f n_A \gamma = n_B f \gamma = n_B \alpha = 0$, and since $f n_A$ is a monomorphism, it follows $\gamma = 0$, that shows $\alpha = f\gamma = 0$. Hence n_B is a monomorphism, by 5.4 it follows B is flat.

5.19: If tA denotes the torsion subgroup of a group $A \in \mathrm{AbSh}\,\mathcal{L}$, then the quotient group $\frac{A}{tA}$ is flat.

Proof: By definition, for any $U \in \mathcal{L}$, $\left(\frac{A}{tA}\right)U = \frac{AU}{(tA)U}$. Consider an $a \in AU$ such that $na \in (tA)U$ for some $0 \neq n \in \mathbb{Z}$. Then there exists a cover $U = \bigvee_{i \in I} U_i$, and $0 \neq m \in \mathbb{Z}$ such that $m_i na|U_i = 0$ for all $i \in I$. This shows $a \in (tA)U$, hence the presheaf $U \quad \frac{AU}{(tA)U}$ is torsion free. By 5.10, it follows that $\frac{A}{tA} \in \mathrm{AbSh}\,\mathcal{L}$ is a flat group.

Proposition 5.20: A functor $F: \mathrm{AbSh}\,\mathcal{L} \to \mathrm{AbSh}\,\mathcal{L}$ which preserves monomorphisms, and has an internal right adjoint preserves flatness.

Proof: Let $G: \mathrm{AbSh}\,\mathcal{L} \to \mathrm{AbSh}\,\mathcal{L}$ be the internal right adjoint of F. Then for $A,B,C \in \mathrm{AbSh}\,\mathcal{L}$, we have isomorphisms natural in A, B, C

$$(FB \otimes C, A) \cong (C, [FB, A]) \cong (C, [B, GA]) \cong (B \otimes C, GA) \cong (F(B \otimes C), A).$$

Hence $FB \otimes C \cong F(B \otimes C)$ natural in B and C. If $h: C \rightarrowtail D$ is a monomorphism, then the diagram

$$\begin{tikzcd} FB\otimes C \arrow[r, "\sim"] \arrow[d, "1_{FB}\otimes h"'] & F(B\otimes C) \arrow[d, "F(1_B\otimes h)"] \\ FB\otimes D \arrow[r, "\sim"'] & F(B\otimes D) \end{tikzcd}$$

commutes. So if B is flat, then $1_B \otimes h$ is a monomorphism. By hypothesis $F(1_B \otimes h)$ is a mono and therefore $1_{FB} \otimes h$ is a mono. That shows FB is a flat group in $\mathrm{AbSh}\mathcal{L}$.

<u>Remark</u>: 5.16 can also be proved as a consequence of the above proposition. If A is flat, then $A \otimes -$ preserves monos and has an internal right adjoint $[A,-]$, so by above proposition $A \otimes -$ preserves flatness, hence B flat implies $A \otimes B$ is flat.

<u>Section 2</u>: <u>Character Group</u>

For the following, note that $\mathrm{AbSh}\mathcal{L}$ has an injective cogenerator obtained as follows:
let H be the product of all the quotients of the different groups $T_U Z_{\mathcal{L}}$, $U \in \mathcal{L}$, and P be the injective hull of H. Then P is an injective cogenerator in $\mathrm{AbSh}\mathcal{L}$ as can be seen from the following: For any $A \in \mathrm{AbSh}\mathcal{L}$, let $0 \neq a \in AU$. Then there is an epimorphism from $T_U Z_{\mathcal{L}} \to [a]$, where $[a]$ is the subgroup of A generated by the element $a \in AU$. So $[a] \cong H_1$, where H_1 is a quotient of $T_U Z_{\mathcal{L}}$. Hence $[a] \rightarrowtail H \rightarrowtail P$. Since P is an injective group, there exists $f: A \dashrightarrow P$ such that the diagram

$$\begin{tikzcd}
{[a]} \arrow[r, tail] \arrow[d, tail] & A \arrow[dl, dashed, "f"] \\
H \arrow[d, tail] & \\
P &
\end{tikzcd}$$

commutes. Therefore $f_U(a) \neq 0$, hence P is an injective cogenerator in $\mathrm{AbSh}\mathcal{L}$.

<u>Definition 5.21</u>: For any $A \in \mathrm{AbSh}\mathcal{L}$, denote by A^*, the character group of A, given by the group $[A,P]$.

<u>Lemma 5.22</u>: The functor $(-)^* = [-,P]: \mathrm{AbSh}\mathcal{L} \to \mathrm{AbSh}\mathcal{L}$ has the following properties.

(i) $(-)^*$ is faithful

(ii) $(-)^*$ reflects zeros.

<u>Proof</u>: (i) let $f,g: A \to B$ be such that $f^* = g^*$. We claim $f = g$. If not, then since P is a cogenerator, there exists $h: B \to P$ such that $hf \neq hg$, that is for some $U \in \mathcal{L}$, $h_U f_U \neq h_U g_U$. Also $f^* = g^*$ implies $f^*_U = g^*_U$ and so $h_U f_U = h_U g_U$, a contradiction, hence $f = g$.

(ii) See 4.20.

<u>Note</u>: In the above lemma, we only need that P is a cogenerator of course for any P, as shown in chapter 4, the functor $[-,P]$ preserves zeros.

Recall from chapter 4, that for any $B \in \mathrm{AbSh}\mathcal{L}$, $[-,B]$ is adjoint to itself on the right, and the adjunction $A \xrightarrow{\alpha} [[A,B],B]$ for any $A \in \mathrm{AbSh}\mathcal{L}$ is given by, $(\alpha_U(a))_W: f \to f_W(a|W)$ where $W \leq U$, $a \in AU$, $f \in (A|W,B|W)$.

<u>Lemma 5.23</u>: If B is a cogenerator then

(i) The adjunction $A \xrightarrow{\alpha} [[A,B],B]$ is a monomorphism, for any A in $\mathrm{AbSh}\,\mathcal{L}$.

(ii) For any $f: A \to C$ in $\mathrm{AbSh}\,\mathcal{L}$, $[C,B] \xrightarrow{[f,B]} [A,B]$ an epimorphism, implies f is a monomorphism.

<u>Proof</u>: (i) For any $U \in \mathcal{L}$, suppose $\alpha_U(a) = 0$, $a \in AU$. Then the components $(\alpha_U(a))_W = 0$ for all $W \leq U$. If $a \neq 0$, there exists $h: A \to B$ such that $h_U(a) \neq 0$ (since B is a cogenerator). Therefore $(\alpha_U(a))_U h|U = h_U(a) \neq 0$, a contradiction since $\alpha_U(a) = 0$. Hence $a = 0$, that is, α_U is a monomorphism for all $U \in \mathcal{L}$, so α is a monomorphism.

(ii) If $f^* = [f,B]$ is epi, then f^{**} is a mono, since $[-,B]$ is adjoint to itself on the right. Now $\alpha_A: A \to A^{**}$ is also a monomorphism by (i), hence the commutativity of the diagram

$$\begin{array}{ccc} A & \xrightarrow{\alpha_A} & A^{**} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f^{**}} \\ C & \xrightarrow[\alpha_C]{} & C^{**} \end{array}$$

implies f is a monomorphism.

<u>Lemma 5.24</u>: If B is injective, then the functor

$[-,B]$ transforms monomorphisms to epimorphisms.

Proof: Let $A \overset{f}{\rightarrowtail} C$ be a monomorphism. If $g \in [A,B]U$, then $g: A|U \to B|U$ is in $AbSh{\downarrow}U$. Since the restriction functor $-|U: AbSh\mathcal{L} \to AbSh{\downarrow}U$, preserves monomorphisms and injectives (0.14 (4), 0.23), it follows $f|U: A|U \to C|U$ is a mono, and $B|U$ is injective in $AbSh{\downarrow}U$. Hence there exists $h: C|U \to B|U$ such that $h(f|U) = g$. All this shows that $[C,B]U \xrightarrow{[f,B]_U} [A,B]U$ is an epimorphism, and therefore $[f,B]$ is an epimorphism.

Hence we have the following:

Proposition 5.25: $0 \to A \overset{f}{\rightarrowtail} C$ is exact in $AbSh\mathcal{L}$ iff the sequence $C^* \to A^* \to 0$ is exact where $C^* = [C,P]$, for an injective cogenerator P.

Proof: Follows directly from lemma 5.23, and 5.24.

Proposition 5.26: $A \in AbSh\mathcal{L}$ is a flat group iff the character group $A^* = [A,P]$ is an injective group.

Proof: ($\Rightarrow$) since flat = torsion free, it follows by 5.5 that $A^* = [A,P]$ is an injective group.
($\Leftarrow$) consider a monomorphism $f: C \to D$ in $AbSh\mathcal{L}$. Since $[A,P]$ is given to be injective, it follows by lemma 5.24,

that $[D,[A,P]] \xrightarrow{[f,[A,P]]} [C,[A,P]]$ is an epimorphism. The naturality of the isomorphism between the functors $[A\otimes -,P] \simeq [-,[A,P]]$ implies the commutativity of the diagram

$$\begin{array}{ccc} [A\otimes D,P] & \xrightarrow{[1_A\otimes f,P]} & [A\otimes C,P] \\ \wr\downarrow & & \downarrow\wr \\ [D,[A,P]] & \xrightarrow{[f,[A,P]]} & [C,[A,P]] \to 0 \end{array}$$

and so, $[1_A\otimes f,P]$ is an epimorphism. By lemma 5.23 (ii) it follows that $A\otimes C \xrightarrow{1_A\otimes f} A\otimes D$ is a monomorphism, hence A is a flat group.

<u>Remark</u>. This proposition is the exact counterpart for $\mathrm{AbSh}\mathcal{L}$ to a well-known result, see B. Banaschewski [7].

<u>Corollary 5.27</u>: $A \in \mathrm{AbSh}\mathcal{L}$ is a torsion group iff the character group A^* is a reduced group.

<u>Proof</u>: ($\Rightarrow$) clear from chapter 2, proposition 2.17. ($\Leftarrow$) for the converse, consider the torsion free group $\frac{A}{tA}$, where tA is the torsion subgroup of A. Since $A \to \frac{A}{tA}$ is an epimorphism in $\mathrm{AbSh}\mathcal{L}$ it follows by (4.13) that $[\frac{A}{tA},P] \rightarrowtail [A,P]$ is a monomorphism. But $\frac{A}{tA}$ is a flat group (5.19), hence by proposition 5.26 it follows $[\frac{A}{tA},P]$ is an

injective group. By hypothesis $A^* = [A,P]$ is a reduced group, therefore $[\frac{A}{tA},P]$ injective implies $[\frac{A}{tA},P] = 0$. From 5.22, it follows $\frac{A}{tA} = 0$, therefore $A = tA$, that is A is a torsion group.//

OPEN QUESTIONS

Chapter 1 is the most important chapter of the thesis, where we generate enough tools useful for the subsequent chapters. In fact, some of the results in the thesis, especially in chapter 1 lead to a number of open questions for further research in this direction. We believe that the tools developed in this chapter would provide a possible approach to attack these problems. Here are a few of them:

1) Are the injectives in $\mathrm{AbSh}\mathcal{L}$ characterized by the following;
   a) For all $U \in \mathcal{L}$, AU is injective in Ab.
   b) For all $V \leq U$ in $\mathcal{L}$, $AU \to AV$ is a split epi in Ab.
2) Exactly which spaces X satisfy the condition that every injective in $\mathrm{AbSh}X$ is a direct sum of indecomposable injectives?
3) Characterize those locales $\mathcal{L}$ for which any epimorphic image of an injective in $\mathrm{AbSh}\mathcal{L}$ is injective.
4) Characterize those locales $\mathcal{L}$ for which the functors $E_U: \mathrm{AbSh}{\downarrow}U \to \mathrm{AbSh}\mathcal{L}$, preserves injectives for all $U \in \mathcal{L}$.
5) Describe the injective hulls in $\mathrm{AbSh}\mathcal{L}$ for any $\mathcal{L}$ in terms of injective hulls in Ab.

6) Characterize those spaces X, for which $\bigoplus_{x \in |X|} A_x$ is an injective group where $A_x U = \begin{cases} \mathbb{Q} & \text{if } x \in U \\ 0 & \text{if } x \notin U \end{cases}$.

7) If $A, B \in \mathrm{AbSh}\mathcal{L}$ are injective, then is $A \otimes B$ injective in $\mathrm{AbSh}\mathcal{L}$?

8) For which locales $\mathcal{L}$ does the functor $[A,-]$ preserve torsion free injectives?

9) When is an arbitrary product of divisible groups divisible?

It may be worth mentioning that the characterization divisible = injective in a Boolean locale (B. Banaschewski [4] shows that the conditions stated in 3, 4, 8, and 9 indeed hold for a Boolean locale. Thus we are motivated to ask whether the stated conditions make the locale Boolean.//

NOTATIONS

$\mathcal{L}$	a locale.
Ens	category of sets.
$PSh\mathcal{L}$	category of presheaves on $\mathcal{L}$.
$Sh\mathcal{L}$	category of sheaves on $\mathcal{L}$.
Ab	category of abelian groups.
$AbPSh\mathcal{L}$	abelian groups in a category of presheaves on $\mathcal{L}$.
$AbSh\mathcal{L}$	abelian groups in a category of sheaves on $\mathcal{L}$.
$\mathbb{Q}$	rational numbers.
$\mathbb{Z}$	integers
$\mathbb{Z}_{\mathcal{L}}$	sheaf reflection of the constant presheaf $U \to \mathbb{Z}$.
$\boxtimes$	tensor product in $AbPSh\mathcal{L}$.
$\simeq$	sheaf reflection functor.
Γ	global functor of $\mathcal{L}$, $A \to AE$.
Γ_U	global functor of $\downarrow U$, $A \to AU$.
$\underset{\rightarrow}{\ell t}$	direct limit.
$\rightarrowtail$	a monomorphism.
$\twoheadrightarrow$	an epimorphism.

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