# CONTRIBUTIONS TO PARAMETRIC AND NONPARAMETRIC INFERENCE IN LIFE TESTING

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# Abstract

In this thesis, we consider both nonparametric and parametric inference in life-testing. Under nonparametric inference, we study the nonparametric precedence test and some alternatives. We first introduce a general maximal precedence test for testing the hypothesis that two distribution functions are equal, which is a variation of the precedence life-test discussed earlier in the literature. Next, we introduce three Wilcoxon-type rank-sum precedence tests, which in fact generalize the classical Wilcoxon rank-sum test for Type-II censored data. We propose the use of Edgeworth expansion to approximate the null distributions of these test statistics. Finally, we introduce another generalization of the precedence and maximal precedence tests, viz., weighted precedence and maximal precedence tests, by giving weights to the precedence failures. We also extend these tests to the case of progressive Type-II censoring. We examine the power properties of these test statistics and compare them with those of the precedence and maximal precedence tests under Lehmann alternative as well as the location-shift alternative.

Under parametric inference, we first propose the use of EM-algorithm to determine maximum likelihood estimators (MLE's) when the data are progressively Type-II censored. We explain how one could obtain the variances and covariances as well as the standard errors of the MLEs by using the missing information principle. Next, we discuss point and interval estimation for the normal distribution under progressive censoring. Then, we concentrate on the Weibull lifetime model under progressive censoring and discuss the determination of optimal censoring schemes. Finally, construction of progressively-censored reliability sampling plans is discussed.

Next, we discuss goodness-of-fit tests based on spacings under progressive censoring. We propose a test for exponentiality based on spacings from a progressively Type-II censored sample. We then generalize this test to a general location-scale family of distributions. We use a simulation study to investigate the power of this test under several different alternatives.

Finally, we study the estimation of the parameters of a two-parameter Birnbaum-Saunders distribution based on complete and Type-II censored samples. For the complete sample situation, we study the MLE's and propose and study modified moment estimators, simple bias-corrected estimators, and jackknifed estimators, and all their asymptotic distributions. For the Type-II censored sample situation, we discuss the MLE and derive the asymptotic variance-covariance matrix of the MLE's. A Monte Carlo EM-algorithm for the determination of the MLE's is discussed. We also propose a simple bias correction technique. Asymptotic confidence intervals based on these estimators and their probability coverages are examined.

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# Chapter 1

# Introduction

## 1.1 Reliability and Life Testing

Reliability theory has certainly played an important role in industrial and system engineering. Customers and users are always looking for high quality and long life products or systems. Therefore, continuous improvement of products or systems becomes essential and even critical. The term 'reliability' can be defined as the ability of a product or system to perform over a period of time according to design specifications or according to user expectations. Longer life is identified with greater reliability.

In measuring reliability, we are observing lifetimes of products or systems of interest. In order to gain a sound knowledge about product or system failure-time distributions, *life-testing and reliability experiments* are carried out before (and while) products are put on the market or systems are used. Life-testing experimentation is a practical way to determine the product reliability and product quality. A life-testing experiment is one in which a sample of the item of interest are subjected to stresses and environmental conditions that epitomize the intended operating conditions. During the life-test, successive times to failure are noted and lifetime data are collected. These lifetime data are used to estimate certain parameters, to make predictions, or to make decisions on accepting a batch of items. Life-testing is useful in many industrial environments including the automobile, materials, telecommunications, and electronics industries.

There is a huge body of literature on reliability and life testing. Present in it are a large number of books and volumes that describe different methods of analyzing lifetime data. These include Mann, Schafer and Singpurwalla (1974), Barlow and Proschan (1975), Lawless (1982), Nelson (1982), Cohen and Whitten (1988), Krishnaiah and Rao (1988), Bain and Engelhardt (1991), Crowder, Kimber, Smith and Sweeting (1991), Balakrishnan (1995), and Balakrishnan and Rao (2001). A brief review of life testing can also be found in Mann and Singpurwalla (1983).

In a life-testing experiment, if all the items under test are observed until failure, the life-test is said to be a *complete life test.* However, in order to save on test time or to save on the number of test items, it is common that not all the items under test will be observed until failure in an industrial setting. In other words, some of the test items will be withdrawn or removed from the life test. Such a life test is said to be a *censored-sample life test.* Some of the issues regarding the censoring mechanisms will be discussed in Section 1.4.

## 1.2 Order Statistics

In a life-testing experiment, since the observed failures occur in a naturally increasing order, the theory of order statistics plays an important role in the analysis of lifetime data, especially when censored data are observed.

Suppose that $X_1$, $X_2$, ..., $X_n$ are $n$ independent and identically distributed random variables from a distribution $F_X(x)$. The corresponding order statistics are obtained when $X_1, X_2, \ldots, X_n$ are arranged in ascending order of magnitude. We let $X_{1:n}$ denote the smallest observation, $X_{2:n}$ the second smallest, ..., and $X_{n:n}$ denote the largest observation. We call $X_{i:n}$ the $i$-th order statistic, $i = 1, 2, \ldots, n$. The $i$-th order statistic clearly will take on different values in different samples. In other words, it has a sampling distribution, which depends on the population distribution $F(x)$, $i$, and $n$. To obtain the density function of the $i$-th order statistic, we consider the probability of $x < X_{i:n} \leq x + h$ as $h$ tends to 0.

The event $x < X_{i:n} \leq x + h$ is essentially same as $X_r \leq x$ for $i - 1$ of the $X_r$'s,

$x < X_r \leq x + h$ for exactly one of the $X_r$'s, and $X_r > x + h$ for the remaining $n - i$ of the $X_r$'s. Since there are $\frac{n!}{(i-1)!(n-i)!}$ such arrangements possible, we can write

$$\begin{aligned} P(x &< X_{i:n} \leq x + h) \\ &= \frac{n!}{(i-1)!(n-i)!}\{F(x)\}^{i-1}\{1 - F(x+h)\}^{n-i}\{F(x+h) - F(x)\}. \end{aligned}$$

The density function of the $i$-th order statistic is then obtained as

$$\begin{aligned} f_{i:n}(x) &= \lim_{h \to 0}\left\{\frac{P(x < X_{i:n} \leq x + h)}{h}\right\} \\ &= \frac{n!}{(i-1)!(n-i)!}\{F(x)\}^{i-1}\{1 - F(x)\}^{n-i}f(x), \quad -\infty < x < \infty. \end{aligned} \tag{1.2.1}$$

By a similar argument, we can derive the joint density function of the $i$-th and $j$-th order statistics, $X_{i:n}$ and $X_{j:n}$, as

$$\begin{aligned} f_{i,j:n}(x, y) &= \frac{n!}{(i-1)!(j-i-1)!(n-j)!}\{F(x)\}^{i-1}\{F(y) - F(x)\}^{j-i-1}f(x) \\ &\quad \times\{1 - F(y)\}^{n-j}f(y), \ 1 \leq i < j \leq n, \ -\infty < x < y < \infty. \end{aligned} \tag{1.2.2}$$

The subject of order statistics generally deals with the properties and applications of these ordered random variables and of functions involving them. Order statistics are particularly useful in analysis of censored data, nonparametric and robust statistics, linear estimation, and characterizations of distributions. Thus, order statistics have a lot of important applications in reliability studies, life testing, and quality control. The books by David (1981), Arnold, Balakrishnan and Nagaraja (1992), and Balakrishnan and Rao (1998a, b) all illustrate many of these applications as well as provide theoretical background of order statistics.

## 1.3 Precedence-type Testing Procedure

### 1.3.1 Literature Review

The precedence test is a distribution-free two-sample life-test based on the order of early failures, and was first proposed by Nelson (1963). The precedence test allows a simple and robust comparison of two distribution functions. Suppose there are two failure time distributions $F_X$ and $F_Y$ and that we are interested in testing the hypotheses

$$H_0 : F_X = F_Y \text{ against } H_1 : F_X > F_Y. \tag{1.3.1}$$

Note that some specific alternatives such as the location-shift alternative and the Lehmann alternative are subclasses of the general alternative considered here. Precedence test, based on the number of $X$-failures that preceded the $r$-th $Y$-failure, will be useful (i) when life-tests involve expensive units since the units that had not failed could be used for some other testing purposes, and (ii) to make quick and reliable decisions early on in the life-testing experiment. For example, a manufacturer of electronic components wishes to compare two designs A and B with respect to life. Specifically, he wants to abandon design A if it gives an indication at the 0.05 level that is shorter life. In this situation, precedence life-test can be employed.

Nelson (1963) provided tables of critical values, which cover all combinations of sample sizes up to twenty for one-sided (two-sided) significance levels of 0.05 (0.10), 0.025 (0.05), and 0.005 (0.01). After Nelson (1963) first introduced the precedence life-test, many authors have studied the power properties of the precedence test and also proposed some alternatives. For example, Eilbott and Nadler (1965) investigated the properties of precedence tests under the assumption that both underlying distributions are exponential. They obtained the closed form expressions for the small sample and asymptotic power under the exponential assumption. Then, Shorack (1967) actually showed that these expressions are valid for a large class of distributions. Afterwards, Lin and Sukhatme (1992) considered the best precedence test and compared the power of the best precedence test with other nonparametric and parametric tests.

Nelson (1993) then examined the power of the precedence test when the underlying distributions were normal. After Lin and Sukhatme (1992) derived the best precedence test, van der Laan and Chakraborti (2001) used these results to derive the best precedence tests under Lehmann alternatives. For a comprehensive overview of precedence-type tests, one may refer to Nelson (1986) and Chakraborti and van der Laan (1996, 1997). Recently, upon noting that the precedence test suffers from a 'masking effect', a maximal precedence test was proposed by Balakrishnan and Frattina (2000). In this case, the test statistic is not the number of failures that preceded the $r$-th $Y$-failure, but it is the maximum of the numbers of failures that occurred before the first, between the first and second, ..., and between the $(r-1)$-th and $r$-th $Y$-failures. They derived the null distribution of the maximal precedence test statistic for $r = 2$ (observing only up to the second $Y$-failure). They also examined the power properties of the maximal precedence test and compared them with those of the precedence test and Wilcoxon's rank-sum test.

### 1.3.2 Two-sample Problem and Precedence Test

Assume that a random sample of size $n_1$ is from $F_X$, another independent sample of size $n_2$ is from $F_Y$, and that all these sample units are placed simultaneously on a life-testing experiment. We use $X_1, X_2, \ldots, X_{n_1}$ to denote the sample from distribution $F_X$, and $Y_1, Y_2, \ldots, Y_{n_2}$ to denote the sample from distribution $F_Y$. A natural null hypothesis is that the two distributions are equal and we are generally concerned with the alternative models where one distribution is stochastically larger than the other; for example, the alternative that $F_Y$ is stochastically larger than $F_X$ which is expressed in (1.3.1).

We denote the order statistics from the $X$-sample and the $Y$-sample by $X_{1:n_1} \leq X_{2:n_1} \leq \ldots \leq X_{n_1:n_1}$ and $Y_{1:n_2} \leq Y_{2:n_2} \leq \ldots \leq Y_{n_2:n_2}$, respectively. Without loss of generality, we assume that $n_1 \leq n_2$. Moreover, we let $M_1$ be the number of $X$-failures before $Y_{1:n_2}$ and $M_i$ be the number of $X$-failures between $Y_{i-1:n_2}$ and $Y_{i:n_2}$, $i = 2, 3, \ldots, r$. We denote the observed value of $M_i$ by $m_i$.

It is of interest to mention here that these $M_i$'s are related to the so called "exceedance statistics" whose distributional properties have been discussed, for example, by Fligner and Wolfe (1976) and Randles and Wolfe (1979).

The precedence test statistic $P_{(r)}$ is defined as the number of failures from the $X$-sample that precede the $r$-th failure from the $Y$-sample. That is,

$$P_{(r)} = \sum_{i=1}^{r} M_i. \tag{1.3.2}$$

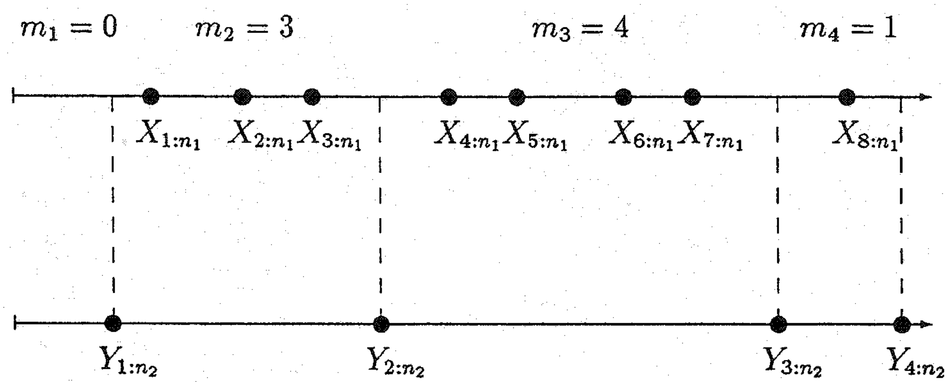



Figure 1.1: Schematic representation of a precedence life-test

For example, from Figure 1.1, with $r = 4$, the precedence test statistic takes on the value $P_{(4)} = 0 + 3 + 4 + 1 = 8$.

It is obvious that large values of $P_{(r)}$ lead to the rejection of $H_0$ and in favor of $H_1$ in (1.3.1).

For a fixed level of significance $\alpha$, the critical region will be $\{s, s+1, \ldots, n_1\}$, where

$$\alpha = \Pr(P_r \geq s | F_X = F_Y). \tag{1.3.3}$$

Note that the event $P_r = j$ maybe viewed as the number of ways of choosing $j$ from the first $x - r + 1$ failures for the $X$-sample and then $n_1 - j$ from the last $n_1 + n_2 - x - r + 1$ failures for the remaining $X$-sample. Therefore, for specified values of $n_1, n_2, s$ and $r$, an expression for $\alpha$ in (1.3.3) is given by

$$\alpha = \frac{\sum_{j=s}^{n_1} \binom{s+r-1}{j} \binom{n_1+n_2-s-r+1}{n_1-j}}{\binom{n_1+n_2}{n_2}}$$

Table 1.1: Near 5% upper critical values and exact levels of significance for the precedence test statistic $P_{(r)}$

| $n_1$ | $n_2$ | $r=1$ | $r=2$ | $r=3$ | $r=4$ | $r=5$ | $r=6$ |
|---|---|---|---|---|---|---|---|
| 10 | 10 | 4(0.04334) | 6(0.02864) | 7(0.03489) | 8(0.03489) | 9(0.02864) | 9(0.07043) |
| 15 | 15 | 4(0.04981) | 6(0.04004) | 7(0.05432) | 9(0.03022) | 10(0.03280) | 11(0.03280) |
| 20 | 20 | 4(0.05301) | 6(0.04574) | 8(0.03242) | 9(0.04118) | 10(0.04792) | 11(0.05267) |
| 30 | 30 | 4(0.05620) | 6(0.05139) | 8(0.03986) | 9(0.05208) | 10(0.06264) | 12(0.04202) |
| 30 | 50 | 3(0.04942) | 4(0.06293) | 5(0.06494) | 6(0.06226) | 7(0.05752) | 8(0.05190) |

with the summation terminating as soon as any of the factorials involve negative arguments.

The critical value $s$ and the exact level of significance $\alpha$ as close as possible to 5% for different choices of the sample sizes $n_1$ and $n_2$ and $r = 1(1)6$ are given in Table 1.1.

## 1.4 Censoring Mechanisms and Progressive Censoring

As we mentioned before, quite often some items on a life-test may not be observed until their failure. This situation is known as censoring. We can classify the censoring processes into different types by consider how the data are collected from the life-test. Here, we shall describe two common types of censoring, viz. Type-I and Type-II censoring, and a generalization called progressive censoring.

### 1.4.1 Type-I and Type-II Censoring

Consider a life-testing experiment in which $n$ items are placed on the test. Suppose the experimenter pre-fixes the total running time of the experiment. The life-test will be terminated at a prefixed time, say $T$; then the number of items that fail by time $T$, say $s$, is a random variable, where $s \leq n$. The lifetime of the first $s$ failures are observed and the remaining $(n-s)$ items are censored and are only known to be $> T$. In this case, the data are said to be *Type-I censored.*

Type-I censoring arises frequently in medical research wherein the study would be terminated at a pre-specified date on which not all individual's lifetimes would have been observed. The main advantage of Type-I censoring is that the experimenter can control the length of the experiment. However, there are two disadvantages of Type-I censoring: very few failures (even none) may be observed by time $T$, and the maximum likelihood estimation procedure becomes complicated in this case.

Instead of terminating the life-testing experiment at a prefixed time $T$, the experiment may be discontinued after the first $r$ failures are observed. In this case, the number of observations is fixed, but the length of the experiment is random. The lifetime of the first $r$ failures are observed and the remaining $(n-r)$ items are censored. In such a situation, the data are said to be *Type-II censored.*

Type-II censoring arises in life-testing problems in a natural way since an experimenter may plan to observe $r$ failures and then terminate the test as soon as the failure of the $r$-th item occurs. An advantage of Type-II censoring is that surviving items may be used for some other tests. Moreover, it is one of the economical sampling schemes in life testing since it would take a long time for all items to fail. The principal disadvantage of this censoring scheme is that the experimenter does not know in advance exactly how long it will take to complete the test.

Methods for analysis of Type-II censored data were originally proposed by Epstein and Sobel (1953). For additional details and references, one may refer to Mann, Schafer and Singpurwalla (1974), Lawless (1982), Nelson (1982), Cohen and Whitten (1988), Balakrishnan and Cohen (1991), and Cohen (1991).

### 1.4.2 Progressive Type-II Censoring

We consider a generalized censoring scheme called *Progressive Type-II censoring.* As we described above, a Type-II censored sample is one in which only the $r$ smallest observations in a random sample of $n$ items are observed $(1 \leq r \leq n)$. A generalization of Type-II censoring is progressive Type-II censoring. In this case, the first failure

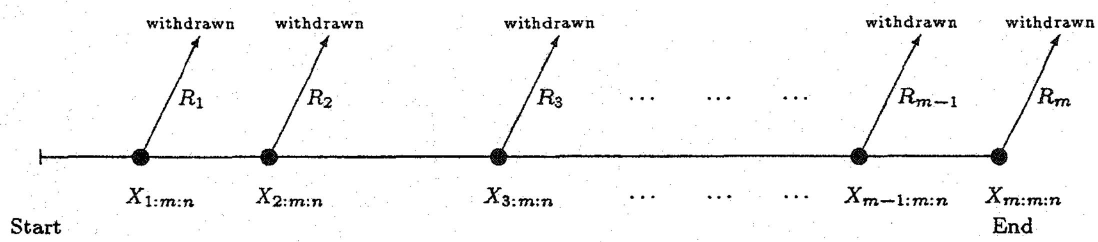



Figure 1.2: Schematic representation of a Type-II progressively censored life-test

in the sample is observed and a random sample of size $R_1$ is immediately withdrawn from the remaining $n-1$ unfailed items (or removed from the test), leaving only $n-1-R_1$ items in test. When the second item fails, $R_2$ of the still unfailed items are withdrawn from the test, and so on. The experiment terminates after some prefixed series of repetitions of this procedure. Suppose $n$ independent units are placed on a life-test with the corresponding lifetimes $X_1$, $X_2$, ..., $X_n$ being identically distributed. Prior to the experiment, a number $m < n$ is fixed and the censoring scheme $(R_1, R_2, \ldots, R_m)$ with $R_j > 0$ and $\sum_{j=1}^{m} R_j + m = n$ is specified. During the experiment, $j$-th failure is observed and immediately after that failure, $R_j$ functioning items are randomly removed from the test. We denote the $m$ completely observed (ordered) lifetimes by $X_{j:m:n}^{(R_1,\ldots,R_m)}$, $j = 1, 2, \ldots, m$. For convenience, sometimes we will suppress the censoring scheme in the notation of the $X_{j:m:n}$'s. A schematic representation of a Type-II progressively censored life-test is provided in Figure 1.2.

We may introduce a further generalization: Suppose the failure times of the first $r$ units are not observed. At the $(r+1)$-th failure, $R_{r+1}$ units are randomly removed. At successive failures, we remove units randomly as before. This is called *general progressive Type-II censoring.* If $r = 0$, this is the scheme outlined above.

Some early works on progressive censoring can be found in Cohen (1963), Mann (1969, 1971), and Thomas and Wilson (1972). A review of progressive censoring scheme and data analysis can be found in Sen (1986a, b). A practical application of progressive Type-II censoring on aging tests on solid insulating materials has been illustrated by Montanari and Cacciari (1988). Viveros and Balakrishnan (1994) proposed a conditional method of inference to derive exact confidence intervals. They also derived explicit expressions for the best linear unbiased estimates (BLUEs) of the parameters of the one- and two-parameter exponential distributions. Tse and Yuen (1996, 1998) and Tse, Yang and Yuen (2000) considered the statistical analysis for the Weibull distribution under Type-II progressive censoring with random and binomial removals, respectively. Recently, Balasooriya and Saw (1998), Balasooriya, Saw and Gadag (2000) and Balasooriya and Balakrishnan (2000) discussed progressively censored acceptance sampling plans for the exponential, Weibull and lognormal distributions, respectively. A book dedicated completely to progressive censoring has been prepared recently by Balakrishnan and Aggarwala (2000).

## 1.5 Scope of the Thesis

This thesis is divided into two parts. The first part consists of four chapters dealing with a nonparametric procedure and some alternatives. The second part consists of three chapters dealing with the parametric estimation and related issues based on progressively Type-II censored data, two chapters on goodness-of-fit tests based on spacings under progressive censoring, and two chapters on statistical inference for the two-parameter Birnbaum-Saunders distribution. Some possible future research problems are finally described in the last chapter of this thesis.

### 1.5.1 Nonparametric Inference

In Chapter 2, we introduce a general maximal precedence test for testing the hypothesis that two distribution functions are equal, which is a variation of the precedence life-test. We first point out the masking effect of the precedence test, which is the motivation for proposing the maximal precedence test, in Section 2.1. In Section 2.2, we propose the general maximal precedence test and derive the null distribution of the general maximal precedence test statistic $M_{(r)}$. Some critical values for $r = 2(1)6$ for some choices of sample sizes are presented. Next, we derive in Section 2.3 the exact power function of the general maximal precedence test under Lehmann alternative. Finally, we examine the power performance of the general maximal precedence test under a location-shift between the two populations through Monte Carlo simulations. Comparison and discussion of the power properties of the maximal precedence test with those of the precedence test and Wilcoxon's rank-sum test are presented in Section 2.4.

In Chapter 3, we introduce Wilcoxon-type rank-sum precedence tests, which are generalizations of the classical Wilcoxon rank-sum test to the case when the data are Type-II censored. Three Wilcoxon-type rank-sum precedence test statistics – the minimal, maximal and expected rank-sum statistics – are proposed. In Section 3.1, we review some results on the precedence test and Wilcoxon rank-sum test. In Section 3.2, we propose the Wilcoxon-type rank-sum precedence tests. We suggest three test statistics, which are the minimal, maximal and expected rank-sum statistics. The null distributions of these test statistics are derived and critical values for $r = 2(1)7$ for some choices of sample sizes are presented. In Section 3.3, we propose the use of Edgeworth expansion to approximate the null distributions of the Wilcoxon-type rank-sum statistics. We present a comparison of the exact probabilities with its Edgeworth approximation and Normal approximation for the three test statistics for some selected sample sizes. Then, we study the power function of these Wilcoxon-type rank-sum tests under Lehmann alternative in Section 3.4. Finally, we examine the power performance of the Wilcoxon-type rank-sum precedence tests under a location-shift between the two populations through Monte Carlo simulations in Section 3.5. Comparison and discussion of the power properties of the Wilcoxon-type rank-sum tests with those of the precedence test, maximal precedence test and Wilcoxon's rank-sum test are presented in Section 3.6 with two illustrative examples. Summary and conclusions are given in Section 3.7.

In Chapter 4, we next introduce another generalization of the precedence and maximal precedence tests. We first introduce weighted precedence and maximal precedence tests for Type-II censored data in Section 4.1. Then we present the test statistics and derive the null distributions of these test statistics in Section 4.2. In Section 4.3, exact power functions of these tests under Lehmann alternatives are derived. We then compare the power of the weighted precedence and maximal precedence tests with the original precedence and maximal precedence tests under Lehmann and location-shift alternatives in Sections 4.3 and 4.4, respectively. A numerical example is used to illustrate the weighted precedence and maximal precedence tests in Section 4.5.

In Chapter 5, we extend the weighted precedence and maximal precedence tests to progressive Type-II censoring situation. A brief introduction is given in Section 5.1. The test statistics, null distributions and critical values for $r = 2(1)5$ for some selected choices of sample sizes and progressive censoring schemes are presented in Section 5.2. We then study the exact power properties of the weighted precedence and maximal precedence tests for progressive Type-II censoring situation under Lehmann alternative in Section 5.3. We also examine the power properties of the weighted precedence and maximal precedence tests under a location-shift between the two populations through Monte Carlo simulations for different progressive censoring schemes in Section 5.4. Finally, an illustrative example is presented in Section 5.5.

### 1.5.2 Parametric Inference

In Chapter 6, EM-algorithm is used to determine the maximum likelihood estimates when the data are progressively Type-II censored. A brief introduction is given in Section 6.1. Section 6.2 explains how the EM-algorithm is used to determine the maximum likelihood estimates in a general setting. Section 6.3 describes how one could obtain the variances and covariance as well as the standard errors of the maximum likelihood estimates by means of the missing information principle. The methodology is illustrated with two popular models in lifetime analysis – the lognormal and Weibull lifetime distributions – in Sections 6.4 and 6.5, respectively. A discussion is provided in Section 6.6.

In Chapter 7, we discuss further the point and interval estimation for the normal distribution under progressive censoring. A brief introduction is given in Section 7.1. In Section 7.2, we discuss the maximum likelihood estimation of the parameters.

Section 7.3 provides explicit estimators obtained by appropriately approximating the likelihood function. The expressions for the observed Fisher information are provided in Section 7.4. In Section 7.5, we provide results of a simulation study to evaluate the performance of these approximate estimators compared to the MLEs. In Section 7.6, we determine the coverage probabilities for pivotal quantities based on asymptotic normality. In addition, we provide unconditional simulated percentage points of these pivotal quantities. Some conclusions and an illustrative example are finally provided in Section 7.7.

In Chapter 8, we consider the Weibull lifetime model under progressive censoring and consider the optimal censoring scheme. In Section 8.1, we give a brief introduction to this problem. In Section 8.2, we compute the expected Fisher information for a progressively Type-II censored sample and the asymptotic variance-covariance matrix of the MLEs by using direct calculation as well as by the missing information principle. Next, in Section 8.3, we discuss the determination of the optimal censoring plan subject to different optimality criteria. Some optimal censoring plans for selected sample sizes are also presented. By taking the data we considered earlier, we describe the design of the optimal censoring plan and also display the relative efficiency gained by the use of this optimal censoring scheme. In Section 8.4, the progressively censored reliability sampling plan is discussed. The procedure for determining the acceptance constant and the sample size is described. These are determined here based on exact MLEs and the exact expected Fisher information, while those computed by Balasooriya, Saw and Gadag (2000) are based on approximate estimators and the corresponding expressions. We also use the numerical data in Section 8.3 to illustrate the construction of a progressively censored reliability sampling plan. Finally, we conclude with a brief discussion in Section 8.5.

In Chapters 9 and 10, we discuss goodness-of-fit tests based on spacings under progressive censoring. In Chapter 9, we propose a test for exponentiality based on spacings under progressive Type-II censoring, which is a generalization of a test proposed by Tiku (1980). In Section 9.1, we briefly describe some ideas of goodness-of-fit tests. In Section 9.2, we propose a test statistic for exponentiality based on spacings. We derive the exact and asymptotic null distributions of the test statistic. In Section

9.3, we present results of a simulation study to display the power properties of this test under several different alternatives. We also discuss an approximation to the power and compare the approximate values with those obtained by simulations. In Section 9.4, we examine two standard tests (Cramer-von Mises $A^2$ and the Shapiro-Wilk $W_E$) for exponentiality discussed extensively in the literature, and compare the power performance of all three procedures via Monte Carlo simulations. Section 9.5 considers tests for the two-parameter exponential distribution. We illustrate the test procedures proposed here using some numerical examples in Section 9.6. Section 9.7 discusses the multi-sample extension of this procedure.

In Chapter 10, we propose a test based on spacings for a general location-scale family of distributions under progressive Type-II censoring. A brief introduction is given in Section 10.1. In Section 10.2, we propose a test statistic for the location-scale family based on spacings. We provide some heuristic arguments to justify the asymptotic null distribution of the test statistic. In Section 10.3, we present results of a simulation study to examine the power of this test for testing for the normal and extreme-value distributions under several different alternatives. We also discuss an approximation to the power and compare the approximate values with those obtained through simulations. We illustrate the test procedure proposed here using a numerical example in Section 10.4. Some conclusions are finally presented in Section 10.5.

In Chapters 11 and 12, we discuss the estimation of the parameters for a two-parameter Birnbaum-Saunders distribution based on complete and Type-II censored samples. In Chapter 11, we review some basic properties of Birnbaum-Saunders distribution in Sections 11.1 and 11.2. The maximum likelihood estimators are discussed in Section 11.3. Then, we propose modified moment estimators (MME's) in Section 11.4. The asymptotic distributions of the MME's are derived which are then used to construct confidence intervals for the unknown parameters. In Section 11.5, we propose a simple bias correction technique which performs quite well even for small sample sizes. Then, the jackknife procedure is also used to reduce the bias of the MLE's and MME's in Section 11.6. We have shown that the jackknife procedure works very well in this case; but, this procedure becomes computationally quite involved in case of large sample sizes. In Section 11.7, we evaluate the performance of all these estimators through simulations. The estimation procedures are illustrated with two examples in Section 11.8. Some concluding remarks are made in Section 11.9.

In Chapter 12, we first discuss the maximum likelihood estimation of the parameters based on Type-II right censored samples for a two-parameter Birnbaum-Saunders distribution in Section 12.2. We then derive the asymptotic variance-covariance matrix of the MLE's using which asymptotic confidence intervals are proposed with the use of asymptotic normality of the MLE's. We evaluate the performance of the MLE's through Monte Carlo simulations for various sample sizes and different degrees of censoring. Though the MLE's are asymptotically unbiased, these simulation results reveal that they are highly biased in case of small sample sizes particularly when the degree of censoring is high. Therefore, in Section 12.3 we propose a simple bias correction technique which performs quite well even for small sample sizes. Asymptotic confidence intervals based on these bias-corrected estimators are then proposed. In Section 12.4, a Monte Carlo EM-algorithm for the determination of the MLE's is discussed. A comparison of all the estimators and the probability coverages of confidence intervals based on inferential quantities associated with these estimators is made using Monte Carlo simulations in Section 12.5. We present in Section 12.6 two examples to illustrate all these methods of inference. Finally, some concluding remarks are made in Section 12.7.

In Chapter 13, we describe briefly some related problems which are worth considering for future research.

# Chapter 2

# A General Maximal Precedence Test

## 2.1 Introduction and Motivation

To begin with, we will point out the masking effect of the precedence test which was mentioned earlier in Section 1.3. From the critical values for the precedence test presented in Table 1.1, we can see that there is a masking effect when $r \geq 2$. For example, if we had $n_1 = n_2 = 20$ and we were using the precedence test with $r = 3$ and $s = 8$, then the null hypothesis will be rejected if there were at least 8 failures from the $X$-sample before the third failure from the $Y$-sample. If only 7 failures had occurred from the $X$-sample before the third failure from the $Y$-sample, then we will not reject the null hypothesis by $P_{(3)}$. Nevertheless, if all these 7 failures had occurred before the first failure from the Y-sample (the probability of this happening under $H_0$ is less than 1%), we would have suspected that there is a location-shift between the two populations. In fact, if we had used $P_{(1)}$ with $s = 4$ or $P_{(2)}$ with $s = 6$, we would have rejected the null hypothesis. Information given by $r = 3$ is thus getting masked in this case.

Maximal precedence test is proposed specifically to avoid this masking problem. It is a test procedure based on the maximum number of failures occurring from the $X$-sample before the first, between the first and the second, ..., between the

$(r-1)$-th and the $r$-th failures from the $Y$-sample. We present details of this test procedure in the following sections.

## 2.2 Test Statistic and Null Distribution

Maximal precedence test is a test procedure for testing the hypotheses (1.3.1) in a two-sample situation. We shall use the same notations as in Section 1.3. The general maximal precedence test statistic is defined as the maximum number of failures occurring from the $X$-sample before the first, between the first and the second, ..., between the $(r-1)$-th and the $r$-th failures from the $Y$-sample, that is, $M_{(r)} = \max(M_1, M_2, \ldots, M_r)$. For example, if we refer to Figure 1.1, with $r = 4$, the maximal precedence test statistic is $M_{(4)} = \max(0, 3, 4, 1) = 4$. Large values of $M_{(r)}$ lead to the rejection of $H_0$ and in favor of $H_1$ in (1.3.1). The null distribution of $M_{(r)}$ for the special case when $r = 2$ was derived by Balakrishnan and Frattina (2000). We will present here the null distribution function of the general maximal precedence test statistic $M_{(r)}$.

**Theorem 2.1** *The joint null probability mass function of $M_1$, $M_2$, ..., $M_r$ is given by*

$$\Pr\{M_1 = m_1, M_2 = m_2, \ldots, M_r = m_r \mid F_X = F_Y\}$$
$$= \frac{\dbinom{n_1 + n_2 - \sum_{i=1}^{r} m_i - r}{n_2 - r}}{\dbinom{n_1 + n_2}{n_2}}. \tag{2.2.1}$$

**Proof:** First, conditional on the $Y$-failures, we consider the probability that there are $m_1$ $X$-failures before $y_{1:n_2}$ and $m_i$ $X$-failures between $y_{i-1:n_2}$ and $y_{i:n_2}$, $i = 2, 3, \ldots, r$, given by

$$\Pr\{m_1 X's \leq y_{1:n_2},\ m_2 X's \in (y_{1:n_1}, y_{2:n_2}], \ldots,\ m_r X's \in (y_{r-1:n_2}, y_{r:n_2}],$$
$$\left(n_1 - \sum_{i=1}^{r} m_i\right) X's > y_{r:n_2} \,|Y_{1:n_2} = y_{1:n_2}, Y_{2:n_2} = y_{1:n_2}, \ldots, Y_{r:n_2} = y_{r:n_2}\}$$

$$= \frac{n_1!}{m_1!m_2!\cdots m_r!\left(n_1 - \sum\limits_{i=1}^{r} m_i\right)!}[F_X(y_{1:n_2})]^{m_1}$$
$$\times \left\{\prod_{i=2}^{r}[F_X(y_{i:n_2}) - F_X(y_{i-1:n_2})]^{m_i}\right\}[1 - F_X(y_{r:n_2})]^{\left(n_1 - \sum\limits_{i=1}^{r} m_i\right)}.$$

We also have the joint density of the first $r$ order statistics from the $Y$-sample as [see David (1981) and Arnold, Balakrishnan and Nagaraja (1992)]

$$f_{1,2,\ldots,r:n_2}(y_{1:n_2}, y_{2:n_2}, \ldots, y_{r:n_2})$$
$$= \frac{n_2!}{(n_2 - r)!} f_Y(y_{1:n_2}) f_Y(y_{2:n_2}) \cdots f_Y(y_{r:n_2})[1 - F_Y(y_{r:n_2})]^{n_2 - r}.$$

Then, we obtain the unconditional probability of $\{M_1 = m_1, M_2 = m_2, \ldots, M_r = m_r\}$ as

$$\Pr\{M_1 = m_1, M_2 = m_2, \ldots, M_r = m_r\}$$
$$= \frac{n_1!}{m_1!m_2!\cdots m_r!\left(n_1 - \sum\limits_{i=1}^{r} m_i\right)!}$$
$$\times \int_0^\infty \int_0^{y_{r:n_2}} \cdots \int_0^{y_{2:n_2}} [F_X(y_{1:n_2})]^{m_1} \left\{\prod_{i=2}^{r}[F_X(y_{i:n_2}) - F_X(y_{i-1:n_2})]^{m_i}\right\}$$
$$\times [1 - F_X(y_{r:n_2})]^{\left(n_1 - \sum\limits_{i=1}^{r} m_i\right)} f_{1,2,\ldots,r:n_2}(y_{1:n_2}, y_{2:n_2}, \ldots, y_{r:n_2})$$
$$\times dy_{1:n_2} dy_{2:n_2} \ldots dy_{r-1:n_2} dy_{r:n_2}$$
$$= C \int_0^\infty \int_0^{y_{r:n_2}} \cdots \int_0^{y_{2:n_2}} [F_X(y_{1:n_2})]^{m_1} \left\{\prod_{i=2}^{r}[F_X(y_{i:n_2}) - F_X(y_{i-1:n_2})]^{m_i}\right\}$$
$$\times [1 - F_X(y_{r:n_2})]^{\left(n_1 - \sum\limits_{i=1}^{r} m_i\right)} \left\{\prod_{i=1}^{r} f_Y(y_{i:n_2})\right\} [1 - F_Y(y_{r:n_2})]^{n_2 - r}$$
$$\times dy_{1:n_2} dy_{2:n_2} \ldots dy_{r-1:n_2} dy_{r:n_2}, \tag{2.2.2}$$

where

$$C = \frac{n_1! n_2!}{m_1!m_2!\cdots m_r!\left(n_1 - \sum\limits_{i=1}^{r} m_i\right)!(n_2 - r)!}.$$

Under null distribution, $H_0 : F_X = F_Y$, the expression in (2.2.2) becomes

$$
\begin{aligned}
&\Pr\{M_1 = m_1, M_2 = m_2, \ldots, M_r = m_r \mid F_X = F_Y\} \\
= &\; C \int_0^{\infty} \int_0^{y_{r:n_2}} \cdots \int_0^{y_{2:n_2}} [F_X(y_{1:n_2})]^{m_1} \left\{ \prod_{i=2}^{r} [F_X(y_{i:n_2}) - F_X(y_{i-1:n_2})]^{m_i} \right\} \\
& \times [1 - F_X(y_{r:n_2})]^{\left(n_1+n_2-\sum_{i=1}^{r} m_i - r\right)} \left\{ \prod_{i=1}^{r} f_X(y_{i:n_2}) \right\} \\
& \times dy_{1:n_2} dy_{2:n_2} \ldots dy_{r-1:n_2} dy_{r:n_2}.
\end{aligned}
$$

For notational convenience, let us now set

$$
u_i = F_X(y_{i:n_2}) \text{ and } du_i = f_X(y_{i:n_2}) dy_{i:n_2} \quad \text{for } i = 1, 2, \ldots, r.
$$

Then, the above expression for the unconditional probability becomes

$$
\begin{aligned}
&\Pr\{M_1 = m_1, M_2 = m_2, \ldots, M_r = m_r \mid F_X = F_Y\} \\
= &\; C \int_0^1 \int_0^{u_r} \cdots \int_0^{u_2} u_1^{m_1} \left\{ \prod_{i=2}^{r} (u_i - u_{i-1})^{m_i} \right\} \\
& \times (1 - u_r)^{\left(n_1+n_2-\sum_{i=1}^{r} m_i - r\right)} du_1 \ldots du_r.
\end{aligned}
$$

Using the transformation $w_1 = u_1/u_2$, we obtain

$$
\begin{aligned}
&\Pr\{M_1 = m_1, M_2 = m_2, \ldots, M_r = m_r \mid F_X = F_Y\} \\
= &\; C \int_0^1 w_1^{m_1} (1 - w_1)^{m_2} dw_1 \int_0^1 \int_0^{u_r} \cdots \int_0^{u_3} u_2^{m_1+m_2+1} \\
& \times \left\{ \prod_{i=3}^{r} (u_i - u_{i-1})^{m_i} \right\} (1 - u_r)^{\left(n_1+n_2-\sum_{i=1}^{r} m_i - r\right)} du_2 \ldots du_r \\
= &\; C B(m_1, m_2 + 1) \int_0^1 \int_0^{u_r} \cdots \int_0^{u_3} u_2^{m_1+m_2+1} \\
& \times \left\{ \prod_{i=3}^{r} (u_i - u_{i-1})^{m_i} \right\} (1 - u_r)^{\left(n_1+n_2-\sum_{i=1}^{r} m_i - r\right)} du_2 \ldots du_r,
\end{aligned}
$$

where $B(a, b) = \int_0^1 x^{a-1}(1 - x)^{b-1} dx$ denotes the complete beta function.

Similarly, upon performing the transformations $w_l = u_l/u_{l+1}$ for $l = 2, \ldots, r - 1$, we obtain

$$
\Pr\{M_1 = m_1, M_2 = m_2, \ldots, M_r = m_r \mid F_X = F_Y\}
$$

$$\begin{aligned}
&= C\left\{\prod_{j=1}^{r-1} B\left(\sum_{i=1}^{j} m_i + j, m_{j+1}+1\right)\right\} \\
&\quad \times \int_0^1 u_r^{\left(\sum_{i=1}^{r} m_i + r - 1\right)} (1-u_r)^{\left(n_1+n_2-\sum_{i=1}^{r} m_i - r\right)} \, du_r \\
&= C\left\{\prod_{j=1}^{r-1} B\left(\sum_{i=1}^{j} m_i + j, m_{j+1}+1\right)\right\} \\
&\quad \times B\left(\sum_{i=1}^{r} m_i + r, n_1+n_2-\sum_{i=1}^{r} m_i - r + 1\right) \\
&= \frac{n_1! n_2! \left(n_1+n_2-\sum_{i=1}^{r} m_i - r\right)!}{\left(n_1 - \sum_{i=1}^{r} m_i\right)!(n_2-r)!(n_1+n_2)!} \\
&= \frac{\binom{n_1+n_2-\sum_{i=1}^{r} m_i - r}{n_2-r}}{\binom{n_1+n_2}{n_2}}
\end{aligned}$$

which completes the proof of the theorem.

From the result of Theorem 2.1, the null cumulative distribution function of $M_{(r)} = \max(M_1, M_2, \ldots, M_r)$ is given by

$$\begin{aligned}
\Pr(M_{(r)} \le m \mid F_X = F_Y) &= \Pr(M_1 \le m, M_2 \le m, \ldots, M_r \le m \mid F_X = F_Y) \\
&= \sum_{\substack{m_i(i=1,2,\ldots,r)=0 \\ \sum_{i=1}^{r} m_i \le n_1}}^{m} \frac{\binom{n_1+n_2-\sum_{i=1}^{r} m_i - r}{n_2-r}}{\binom{n_1+n_2}{n_2}}. \qquad (2.2.3)
\end{aligned}$$

For specified values of $n_1, n_2, r$ and the level of significance $\alpha$, the critical value $s$ (corresponding to a level closest to $\alpha$) for the maximal precedence test can be found readily from (2.2.3) as

$$\begin{aligned}
\alpha &= \Pr(M_{(r)} \ge s \mid F_X = F_Y) \\
&= 1 - \Pr(M_{(r)} \le s-1 \mid F_X = F_Y)
\end{aligned}$$

Table 2.1: Near 5% upper critical values and exact levels of significance for the general maximal precedence test statistic $M_{(r)}$

| $n_1$ | $n_2$ | $r=2$ | $r=3$ | $r=4$ | $r=5$ | $r=6$ |
|---|---|---|---|---|---|---|
| 10 | 10 | 5(0.03250) | 5(0.04875) | 5(0.06498) | 6(0.02709) | 6(0.03251) |
| 15 | 15 | 5(0.04205) | 5(0.06292) | 6(0.03368) | 6(0.04209) | 6(0.05050) |
| 20 | 20 | 5(0.04691) | 6(0.03023) | 6(0.04026) | 6(0.05026) | 6(0.06025) |
| 30 | 30 | 5(0.05179) | 6(0.03540) | 6(0.04707) | 6(0.05868) | 7(0.03150) |
| 30 | 50 | 4(0.03445) | 4(0.05138) | 4(0.06810) | 5(0.02946) | 5(0.03529) |

$$= 1 - \sum_{\substack{m_i (i=1,2,\ldots,r)=0 \\ \sum_{i=1}^{r} m_i \le n_1}}^{s-1} \frac{\binom{n_1+n_2-\sum_{i=1}^{r} m_i - r}{n_2 - r}}{\binom{n_1+n_2}{n_2}}. \tag{2.2.4}$$

In Table 2.1, we have presented the critical value $s$ and the exact level of significance $\alpha$ as close as possible to 5% for some sample sizes $n_1$ and $n_2$ and $r = 2(1)6$.

These critical values are used later in the Monte Carlo simulations, presented in Section 2.4, for determining the rejection rates of the general maximal precedence test.

## 2.3 Exact Power Under Lehmann Alternative

In this section, we derive an explicit expression for the power function of the general maximal precedence test under the Lehmann alternative $H_1 : [F_X]^\gamma = F_Y$ for some $\gamma$, which was first proposed by Lehmann (1953). For more details on the Lehmann alternative, see Davies (1971), Lehmann (1975), Gibbons and Chakraborti (1992) and Hettmansperger and McKean (1998).

We note that $H_1 : [F_X]^\gamma = F_Y$ is a subclass of the alternative $H_1 : F_X > F_Y$ when $\gamma > 1$. The power of a test is the probability of rejecting the null hypothesis when the alternative hypothesis is indeed true. In the following theorem, we derive an explicit expression for the power function.

**Theorem 2.2** *Under the Lehmann alternative $H_1 : [F_X]^\gamma = F_Y$, the probability mass function of $M_1$, $M_2$, ..., $M_r$ is given by*

$$\begin{aligned}
&\Pr\{M_1 = m_1, M_2 = m_2, \ldots, M_r = m_r \mid [F_X]^\gamma = F_Y\} \\
&= \frac{n_1! n_2! \gamma^r}{m_1!(n_2-r)!} \left\{ \prod_{j=1}^{r-1} \frac{\Gamma\left(\sum_{i=1}^{j} m_i + j\gamma\right)}{\Gamma\left(\sum_{i=1}^{j+1} m_i + j\gamma + 1\right)} \right\} \\
&\times \sum_{k=0}^{n_2-r} \binom{n_2-r}{k} (-1)^k \frac{\Gamma\left(\sum_{i=1}^{r} m_i + (r+k)\gamma\right)}{\Gamma(n_1 + (r+k)\gamma + 1)}.
\end{aligned} \tag{2.3.1}$$

**Proof:** Under the Lehmann alternative $H_1 : [F_X]^\gamma = F_Y, \gamma > 1$, the expression in (2.2.2) can be simplified as follows:

$$\begin{aligned}
&\Pr\{M_1 = m_1, M_2 = m_2, \ldots, M_r = m_r \mid [F_X]^\gamma = F_Y\} \\
= \; & C \int_0^\infty \int_0^{y_{r:n_2}} \cdots \int_0^{y_{2:n_2}} [F_X(y_{1:n_2})]^{m_1} \left\{ \prod_{i=2}^{r} [F_X(y_{i:n_2}) - F_X(y_{i-1:n_2})]^{m_i} \right\} \\
& \times [1 - F_X(y_{r:n_2})]^{\left(n_1 - \sum_{i=1}^{r} m_i\right)} \left\{ \prod_{i=1}^{r} \gamma [F_X(y_{i:n_2})]^{\gamma-1} f_X(y_{i:n_2}) \right\} \\
& \times \{1 - [F_X(y_{r:n_2})]^\gamma\}^{n_2-r} \, dy_{1:n_2} dy_{2:n_2} \ldots dy_{r-1:n_2} dy_{r:n_2} \\
= \; & C\gamma^r \int_0^\infty \int_0^{y_{r:n_2}} \cdots \int_0^{y_{2:n_2}} [F_X(y_{1:n_2})]^{m_1+\gamma-1} \\
& \times \left\{ \prod_{i=2}^{r} [F_X(y_{i:n_2})]^{\gamma-1} [F_X(y_{i:n_2}) - F_X(y_{i-1:n_2})]^{m_i} \right\} \\
& \times [1 - F_X(y_{r:n_2})]^{\left(n_1 - \sum_{i=1}^{r} m_i\right)} \left\{ \prod_{i=1}^{r} f_X(y_{i:n_2}) \right\} \\
& \times \left\{ \sum_{k=0}^{n_2-r} \binom{n_2-r}{k} (-1)^k [F_X(y_{r:n_2})]^{\gamma k} \right\} dy_{1:n_2} \ldots dy_{r:n_2} \\
= \; & C\gamma^r \sum_{k=0}^{n_2-r} \binom{n_2-r}{k} (-1)^k \int_0^\infty \int_0^{y_{r:n_2}} \cdots \int_0^{y_{2:n_2}} [F_X(y_{1:n_2})]^{m_1+\gamma-1} \\
& \times \left\{ \prod_{i=1}^{r} f_X(y_{i:n_2}) \right\} \left\{ \prod_{i=2}^{r} [F_X(y_{i:n_2})]^{\gamma-1} [F_X(y_{i:n_2}) - F_X(y_{i-1:n_2})]^{m_i} \right\} \\
& \times [F_X(y_{r:n_2})]^{\gamma k} [1 - F_X(y_{r:n_2})]^{\left(n_1 - \sum_{i=1}^{r} m_i\right)} dy_{1:n_2} \ldots dy_{r:n_2}.
\end{aligned}$$

For notational convenience, let us now set

$$u_i = F_X(y_{i:n_2}) \text{ and } du_i = f_X(y_{i:n_2})dy_{i:n_2} \quad \text{for } i = 1, 2, \ldots, r.$$

Then, the above expression for the unconditional probability becomes

$$\begin{aligned}
&\Pr\{M_1 = m_1, M_2 = m_2, \ldots, M_r = m_r \mid [F_X]^\gamma = F_Y\} \\
= \;& C\gamma^r \sum_{k=0}^{n_2-r} \binom{n_2 - r}{k} (-1)^k \int_0^1 \int_0^{u_r} \cdots \int_0^{u_2} u_1^{m_1+\gamma-1} \\
& \times \left\{ \prod_{i=2}^{r} u_i^{\gamma-1}(u_i - u_{i-1})^{m_i} \right\} u_r^{\gamma k}(1-u_r)^{\left(n_1 - \sum_{i=1}^{r} m_i\right)} \, du_1 \ldots du_r.
\end{aligned}$$

Using the transformation $w_1 = u_1/u_2$, we obtain

$$\begin{aligned}
&\Pr\{M_1 = m_1, M_2 = m_2, \ldots, M_r = m_r \mid [F_X]^\gamma = F_Y\} \\
= \;& C\gamma^r \sum_{k=0}^{n_2-r} \binom{n_2 - r}{k} (-1)^k \int_0^1 w_1^{m_1+\gamma-1}(1-w_1)^{m_2} dw_1 \\
& \times \int_0^1 \int_0^{u_r} \cdots \int_0^{u_3} u_2^{m_1+m_2+2\gamma-1} \left\{ \prod_{i=3}^{r} u_i^{\gamma-1}(u_i - u_{i-1})^{m_i} \right\} \\
& \times u_r^{\gamma k}(1-u_r)^{\left(n_1 - \sum_{i=1}^{r} m_i\right)} \, du_2 \ldots du_r \\
= \;& C\gamma^r B(m_1 + \gamma, m_2 + 1) \sum_{k=0}^{n_2-r} \binom{n_2 - r}{k} (-1)^k \\
& \times \int_0^1 \int_0^{u_r} \cdots \int_0^{u_3} u_2^{m_1+m_2+2\gamma-1} \left\{ \prod_{i=3}^{r} u_i^{\gamma-1}(u_i - u_{i-1})^{m_i} \right\} \\
& u_r^{\gamma k}(1-u_r)^{\left(n_1 - \sum_{i=1}^{r} m_i\right)} \, du_2 \ldots du_r,
\end{aligned}$$

where $B(a, b)$ denotes the complete beta function as before.

Similarly, upon performing the transformations $w_l = u_l/u_{l+1}$ for $l = 2, \ldots, r-1$, we obtain

$$\begin{aligned}
&\Pr\{M_1 = m_1, M_2 = m_2, \ldots, M_r = m_r \mid [F_X]^\gamma = F_Y\} \\
= \;& C\gamma^r \left\{ \prod_{j=1}^{r-1} B\left( \sum_{i=1}^{j} m_i + j\gamma, m_{j+1} + 1 \right) \right\}
\end{aligned}$$

$$\times \sum_{k=0}^{n_2-r} \binom{n_2-r}{k} (-1)^k \int_0^1 u_r^{\sum_{i=1}^{r} m_i+(r+k)\gamma-1} (1-u_r)^{\left(n_1-\sum_{i=1}^{r} m_i\right)} du_r$$

$$= C\gamma^r \left\{ \prod_{j=1}^{r-1} B\left( \sum_{i=1}^{j} m_i + j\gamma, m_{j+1}+1 \right) \right\}$$

$$\times \sum_{k=0}^{n_2-r} \binom{n_2-r}{k} (-1)^k B\left( \sum_{i=1}^{r} m_i + (r+k)\gamma, n_1 - \sum_{i=1}^{r} m_i + 1 \right)$$

$$= \frac{n_1! n_2! \gamma^r}{m_1!(n_2-r)!} \left\{ \prod_{j=1}^{r-1} \frac{\Gamma\left(\sum_{i=1}^{j} m_i + j\gamma\right)}{\Gamma\left(\sum_{i=1}^{j+1} m_i + j\gamma + 1\right)} \right\}$$

$$\times \sum_{k=0}^{n_2-r} \binom{n_2-r}{k} (-1)^k \frac{\Gamma\left(\sum_{i=1}^{r} m_i + (r+k)\gamma\right)}{\Gamma(n_1+(r+k)\gamma+1)} \tag{2.3.2}$$

which completes the proof of the theorem.

From Theorem 2.2, the cumulative distribution function of $M_{(r)} = \max(M_1, M_2, \ldots, M_r)$ under the Lehmann alternative is given by

$$\Pr\left\{ M_{(r)} \le m \mid [F_X]^\gamma = F_Y \right\}$$

$$= \Pr\{ M_1 \le m, M_2 \le m, \ldots, M_r \le m \mid [F_X]^\gamma = F_Y \}$$

$$= \sum_{\substack{m_i (i=1,2,\ldots,r)=0 \\ \sum_{i=1}^r m_i \le n_1}}^{m} \Pr\{ M_1 = m_1, M_2 = m_2, \ldots, M_r = m_r \mid [F_X]^\gamma = F_Y \}$$

$$= \sum_{\substack{m_i (i=1,2,\ldots,r)=0 \\ \sum_{i=1}^r m_i \le n_1}}^{m} \frac{n_1! n_2! \gamma^r}{m_1!(n_2-r)!} \left\{ \prod_{j=1}^{r-1} \frac{\Gamma\left(\sum_{i=1}^{j} m_i + j\gamma\right)}{\Gamma\left(\sum_{i=1}^{j+1} m_i + j\gamma + 1\right)} \right\}$$

$$\times \sum_{k=0}^{n_2-r} \binom{n_2-r}{k} (-1)^k \frac{\Gamma\left(\sum_{i=1}^{r} m_i + (r+k)\gamma\right)}{\Gamma(n_1+(r+k)\gamma+1)}$$

Therefore, under the Lehmann alternative $H_1 : [F_X]^\gamma = F_Y$, the power function of the maximal precedence test is given by

$$1 - \Pr\left\{ M_{(r)} \le x - 1 \mid [F_X]^\gamma = F_Y \right\}$$

$$= 1 - \sum_{\substack{m_i(i=1,2,\ldots,r)=0 \\ \sum_{i=1}^{r} m_i \leq n_1}}^{x-1} \frac{n_1!n_2!\gamma^r}{m_1!(n_2-r)!} \left\{ \prod_{j=1}^{r-1} \frac{\Gamma\left(\sum_{i=1}^{j} m_i + j\gamma\right)}{\Gamma\left(\sum_{i=1}^{j+1} m_i + j\gamma + 1\right)} \right\}$$

$$\times \sum_{k=0}^{n_2-r} \binom{n_2-r}{k} (-1)^k \frac{\Gamma\left(\sum_{i=1}^{r} m_i + (r+k)\gamma\right)}{\Gamma\left(n_1 + (r+k)\gamma + 1\right)}. \qquad (2.3.3)$$

Here, we demonstrate the use of the expression in (2.3.3) as well as the Monte Carlo simulation method for the computation of the power of the maximal precedence test under the Lehmann alternative. In the simulation study of the power under Lehmann alternative, we consider the case when $X_1, X_2, \ldots, X_{n_1}$ is a random sample from a power function distribution with cumulative distribution function

$$F_X(x) = x^\gamma, \qquad 0 \leq x \leq 1, \gamma > 1$$

and $Y_1, Y_2, \ldots, Y_{n_2}$ is a random sample from a uniform distribution with cumulative distribution function

$$F_Y(x) = x, \qquad 0 \leq x \leq 1.$$

We generated 100,000 sets of data and found the test statistic $M_{(r)}$ for each set. Then, the power values are estimated by the rejection rates of the null hypothesis.

Under the Lehmann alternative and for $n_1 = n_2 = 10, r = 2, 3$ and $\gamma = 2(1)6$, the power values computed from the expression in (2.3.3) and those estimated through the Monte Carlo simulation method are presented in Table 2.2. We observe that the estimated values of the power determined from Monte Carlo simulations are quite close to the exact values. In addition to revealing the correctness of the expression in (2.3.3), these results also suggest that the Monte Carlo simulation method provides a simple and accurate way to estimate the power of the test.

Table 2.2: Power values under Lehmann alternative for $n_1 = n_2 = 10, r = 2, 3$ and $\gamma = 2(1)6$

| $r$ | $\gamma$ | Power computed from expression (2.3.3) | Simulated Power |
|---|---|---|---|
| r=2 | 2 | 0.220950 | 0.221520 |
| | 3 | 0.453424 | 0.454620 |
| | 4 | 0.636340 | 0.645573 |
| | 5 | 0.760590 | 0.760093 |
| | 6 | 0.841193 | 0.842160 |
| r=3 | 2 | 0.241783 | 0.241640 |
| | 3 | 0.468023 | 0.469117 |
| | 4 | 0.645608 | 0.645573 |
| | 5 | 0.766413 | 0.766080 |
| | 6 | 0.844902 | 0.845893 |

## 2.4 Monte Carlo Simulation and Power Comparison Under Location-Shift Alternative

In order to assess the power properties of the maximal precedence test, we consider the general maximal precedence test under the location-shift alternative $H_1 : F_X(x) = F_Y(x + \theta)$ for some $\theta > 0$, where $\theta$ is a shift in location. The power of the precedence test with $r = 1(1)6$, maximal precedence test with $r = 2(1)6$, and the Wilcoxon's rank-sum test were all estimated through Monte Carlo simulations when $\theta = 0.5$ and $\theta = 1.0$. The following lifetime distributions were used in the Monte Carlo simulations in order to demonstrate the power performance of the maximal precedence test under this location-shift alternative:

1. Standard normal distribution;
2. Standard exponential distribution;
3. Gamma distribution with shape parameter $a$ and standardized by mean $a$ and standard deviation $\sqrt{a}$;
4. Lognormal distribution with shape parameter $\sigma$ and standardized by mean $e^{\sigma^2/2}$ and standard deviation $\sqrt{e^{\sigma^2}(e^{\sigma^2} - 1)}$.

Table 2.3: Power of precedence tests, maximal precedence tests and Wilcoxon's rank-sum test for $n_1 = n_2 = 10$

| Test | $r$ | critical value | exact l.o.s. | location shift | Distribution | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | N(0,1) | Exp(1) | Gamma(2) | Gamma(10) | LN(0.1) | LN(0.5) |
| P | 1 | 4 | 0.0433 | 0.5 | 0.1736 | 0.7197 | 0.4141 | 0.2249 | 0.1936 | 0.3944 |
| | | | | 1.0 | 0.4208 | 0.9779 | 0.8806 | 0.5742 | 0.4817 | 0.8709 |
| | 2 | 6 | 0.0286 | 0.5 | 0.1496 | 0.3916 | 0.2525 | 0.1731 | 0.1603 | 0.2693 |
| | | | | 1.0 | 0.4243 | 0.8376 | 0.6879 | 0.5060 | 0.4571 | 0.7253 |
| | 3 | 7 | 0.0349 | 0.5 | 0.1779 | 0.3073 | 0.2343 | 0.1893 | 0.1828 | 0.2571 |
| | | | | 1.0 | 0.4813 | 0.7239 | 0.6148 | 0.5177 | 0.4964 | 0.6676 |
| | 4 | 8 | 0.0348 | 0.5 | 0.1771 | 0.2251 | 0.1936 | 0.1755 | 0.1777 | 0.2149 |
| | | | | 1.0 | 0.4799 | 0.5767 | 0.5114 | 0.4758 | 0.4783 | 0.5656 |
| | 5 | 9 | 0.0286 | 0.5 | 0.1473 | 0.1453 | 0.1380 | 0.1388 | 0.1444 | 0.1527 |
| | | | | 1.0 | 0.4245 | 0.3960 | 0.3725 | 0.3833 | 0.4038 | 0.4144 |
| | 6 | 9 | 0.0704 | 0.5 | 0.2659 | 0.2385 | 0.2344 | 0.2436 | 0.2567 | 0.2531 |
| | | | | 1.0 | 0.5832 | 0.4991 | 0.4954 | 0.5266 | 0.5545 | 0.5314 |
| M | 2 | 5 | 0.0325 | 0.5 | 0.1417 | 0.4923 | 0.2474 | 0.1611 | 0.1479 | 0.2543 |
| | | | | 1.0 | 0.3768 | 0.9177 | 0.7303 | 0.4569 | 0.4083 | 0.7327 |
| | 3 | 5 | 0.0487 | 0.5 | 0.1753 | 0.4949 | 0.2579 | 0.1870 | 0.1781 | 0.2678 |
| | | | | 1.0 | 0.4171 | 0.9178 | 0.7327 | 0.4765 | 0.4395 | 0.7360 |
| | 4 | 5 | 0.0650 | 0.5 | 0.1993 | 0.4974 | 0.2653 | 0.2040 | 0.1990 | 0.2768 |
| | | | | 1.0 | 0.4359 | 0.9181 | 0.7344 | 0.4853 | 0.4538 | 0.7376 |
| | 5 | 6 | 0.0270 | 0.5 | 0.0901 | 0.2684 | 0.1197 | 0.0895 | 0.0905 | 0.1297 |
| | | | | 1.0 | 0.2482 | 0.7791 | 0.5136 | 0.2833 | 0.2586 | 0.5284 |
| | 6 | 6 | 0.0325 | 0.5 | 0.0938 | 0.2690 | 0.1208 | 0.0921 | 0.0938 | 0.1309 |
| | | | | 1.0 | 0.2496 | 0.7791 | 0.5138 | 0.2839 | 0.2596 | 0.5286 |
| W | | 128 | 0.0446 | 0.5 | 0.2536 | 0.4332 | 0.3344 | 0.2663 | 0.2605 | 0.3508 |
| | | | | 1.0 | 0.6480 | 0.8163 | 0.7508 | 0.6713 | 0.6554 | 0.7819 |

Remark: P - Precedence Test; M - Maximal Precedence Test; W - Wilcoxon's Rank-sum Test

For a detailed discussion on various properties of these distributions, one may refer to Johnson, Kotz and Balakrishnan (1994). For different choices of sample sizes, we generated 100,000 sets of data, utilizing the IMSL subroutines RNNOR, RNEXP, RNGAM and RNLNL under Microsoft FORTRAN Visual Workbench Version 1.0, in order to obtain the estimated rejection rates.

In Tables 2.3–2.5, we have presented the estimated power values of the precedence tests with $r = 1(1)6$, maximal precedence tests with $r = 2(1)6$, and the Wilcoxon's rank-sum test for the underlying standard normal, standard exponential, standardized gamma and standardized lognormal distributions, with location-shift being equal to 0.5 and 1.0. For comparison proposes, the corresponding critical values and the exact levels of significance are also presented.

From Tables 2.3–2.5, we see that the power of all tests increase with increasing sample sizes as well as with increasing location-shift. When we compare the power

Table 2.4: Power of precedence tests, maximal precedence tests and Wilcoxon's rank-sum test for $n_1 = n_2 = 20$

| Test | $r$ | critical value | exact l.o.s. | location shift | Distribution N(0,1) | Exp(1) | Gamma(2) | Gamma(10) | LN(0.1) | LN(0.5) |
|---|---|---|---|---|---|---|---|---|---|---|
| P | 1 | 4 | 0.0530 | 0.5 | 0.2386 | 0.9887 | 0.7880 | 0.3552 | 0.2804 | 0.6855 |
| | | | | 1.0 | 0.5553 | 1.0000 | 0.9982 | 0.8026 | 0.6543 | 0.9943 |
| | 2 | 6 | 0.0457 | 0.5 | 0.2662 | 0.9355 | 0.6711 | 0.3638 | 0.3038 | 0.6453 |
| | | | | 1.0 | 0.6529 | 0.9998 | 0.9914 | 0.8371 | 0.7338 | 0.9901 |
| | 3 | 7 | 0.0324 | 0.5 | 0.2434 | 0.8066 | 0.5331 | 0.3161 | 0.2730 | 0.5480 |
| | | | | 1.0 | 0.6630 | 0.9972 | 0.9699 | 0.8121 | 0.7303 | 0.9755 |
| | 4 | 9 | 0.0412 | 0.5 | 0.2947 | 0.7499 | 0.5317 | 0.3583 | 0.3219 | 0.5614 |
| | | | | 1.0 | 0.7352 | 0.9932 | 0.9615 | 0.8418 | 0.7877 | 0.9716 |
| | 5 | 10 | 0.0479 | 0.5 | 0.3317 | 0.6904 | 0.5173 | 0.3808 | 0.3525 | 0.5555 |
| | | | | 1.0 | 0.7787 | 0.9864 | 0.9487 | 0.8532 | 0.8163 | 0.9648 |
| | 6 | 11 | 0.0527 | 0.5 | 0.3565 | 0.6291 | 0.4966 | 0.3920 | 0.3706 | 0.5369 |
| | | | | 1.0 | 0.8055 | 0.9733 | 0.9323 | 0.8558 | 0.8295 | 0.9540 |
| M | 2 | 5 | 0.0469 | 0.5 | 0.2434 | 0.9634 | 0.6580 | 0.3235 | 0.2741 | 0.5983 |
| | | | | 1.0 | 0.5994 | 1.0000 | 0.9934 | 0.7893 | 0.6788 | 0.9882 |
| | 3 | 6 | 0.0300 | 0.5 | 0.1847 | 0.9051 | 0.4878 | 0.2327 | 0.2016 | 0.4460 |
| | | | | 1.0 | 0.5235 | 0.9997 | 0.9787 | 0.6876 | 0.5896 | 0.9654 |
| | 4 | 6 | 0.0403 | 0.5 | 0.2138 | 0.9053 | 0.4949 | 0.2546 | 0.2273 | 0.4572 |
| | | | | 1.0 | 0.5605 | 0.9997 | 0.9790 | 0.7030 | 0.6170 | 0.9662 |
| | 5 | 6 | 0.0500 | 0.5 | 0.2360 | 0.9058 | 0.5004 | 0.2709 | 0.2472 | 0.4642 |
| | | | | 1.0 | 0.5833 | 0.9997 | 0.9793 | 0.7115 | 0.6327 | 0.9666 |
| | 6 | 6 | 0.0602 | 0.5 | 0.2545 | 0.9062 | 0.5044 | 0.2834 | 0.2626 | 0.4688 |
| | | | | 1.0 | 0.5977 | 0.9997 | 0.9796 | 0.7171 | 0.6423 | 0.9668 |
| W | | 472 | 0.0482 | 0.5 | 0.4403 | 0.7011 | 0.5699 | 0.4622 | 0.4462 | 0.5941 |
| | | | | 1.0 | 0.9136 | 0.9802 | 0.9622 | 0.9275 | 0.9179 | 0.9716 |

Remark: P - Precedence Test; M - Maximal Precedence Test; W - Wilcoxon's Rank-sum Test

Table 2.5: Power of precedence tests, maximal precedence tests and Wilcoxon's rank-sum test for $n_1 = n_2 = 30$

| Test | $r$ | critical value | exact l.o.s. | location shift | Distribution N(0,1) | Exp(1) | Gamma(2) | Gamma(10) | LN(0.1) | LN(0.5) |
|---|---|---|---|---|---|---|---|---|---|---|
| P | 1 | 4 | 0.0562 | 0.5 | 0.2690 | 0.9998 | 0.9359 | 0.4354 | 0.3247 | 0.8259 |
| | | | | 1.0 | 0.6135 | 1.0000 | 1.0000 | 0.8878 | 0.7300 | 0.9998 |
| | 2 | 6 | 0.0514 | 0.5 | 0.3181 | 0.9974 | 0.8832 | 0.4754 | 0.3754 | 0.8287 |
| | | | | 1.0 | 0.7355 | 1.0000 | 0.9999 | 0.9302 | 0.8277 | 0.9997 |
| | 3 | 8 | 0.0399 | 0.5 | 0.3166 | 0.9851 | 0.8073 | 0.4540 | 0.3702 | 0.7846 |
| | | | | 1.0 | 0.7755 | 1.0000 | 0.9995 | 0.9337 | 0.8525 | 0.9992 |
| | 4 | 9 | 0.0521 | 0.5 | 0.3898 | 0.9758 | 0.8088 | 0.5175 | 0.4393 | 0.8091 |
| | | | | 1.0 | 0.8455 | 1.0000 | 0.9992 | 0.9556 | 0.9014 | 0.9993 |
| | 5 | 11 | 0.0358 | 0.5 | 0.3468 | 0.9233 | 0.7081 | 0.4524 | 0.3867 | 0.7260 |
| | | | | 1.0 | 0.8375 | 0.9999 | 0.9965 | 0.9402 | 0.8882 | 0.9978 |
| | 6 | 12 | 0.0420 | 0.5 | 0.3888 | 0.8977 | 0.7007 | 0.4851 | 0.4263 | 0.7293 |
| | | | | 1.0 | 0.8718 | 0.9998 | 0.9953 | 0.9481 | 0.9104 | 0.9973 |
| M | 2 | 5 | 0.0518 | 0.5 | 0.2898 | 0.9990 | 0.8782 | 0.4284 | 0.3421 | 0.7880 |
| | | | | 1.0 | 0.6909 | 1.0000 | 1.0000 | 0.9019 | 0.7865 | 0.9996 |
| | 3 | 6 | 0.0350 | 0.5 | 0.2381 | 0.9956 | 0.7735 | 0.3435 | 0.2802 | 0.6766 |
| | | | | 1.0 | 0.6556 | 1.0000 | 0.9996 | 0.8577 | 0.7430 | 0.9983 |
| | 4 | 6 | 0.0471 | 0.5 | 0.2795 | 0.9957 | 0.7797 | 0.3761 | 0.3187 | 0.6910 |
| | | | | 1.0 | 0.7050 | 1.0000 | 0.9997 | 0.8748 | 0.7788 | 0.9984 |
| | 5 | 6 | 0.0587 | 0.5 | 0.3119 | 0.9957 | 0.7842 | 0.4013 | 0.3490 | 0.7007 |
| | | | | 1.0 | 0.7370 | 1.0000 | 0.9997 | 0.8849 | 0.8009 | 0.9985 |
| | 6 | 7 | 0.0315 | 0.5 | 0.2072 | 0.9866 | 0.6400 | 0.2690 | 0.2312 | 0.5460 |
| | | | | 1.0 | 0.6096 | 1.0000 | 0.9986 | 0.7844 | 0.6772 | 0.9943 |
| W | | 1027 | 0.0498 | 0.5 | 0.5868 | 0.8531 | 0.7339 | 0.6146 | 0.5909 | 0.7571 |
| | | | | 1.0 | 0.9809 | 0.9985 | 0.9951 | 0.9854 | 0.9822 | 0.9969 |

Remark: P - Precedence Test; M - Maximal Precedence Test; W - Wilcoxon's Rank-sum Test values of the maximal precedence tests with those of Wilcoxon's rank-sum test, we find that the Wilcoxon's rank-sum test performs better than the precedence tests and maximal precedence tests if the underlying distributions are close to symmetry, such as the normal distribution, gamma distribution with large values of shape parameter $a$, and lognormal distribution with small values of shape parameter $\sigma$. However, under some right-skewed distributions such as the exponential distribution, gamma distribution with shape parameter $a = 2.0$, and lognormal distribution with shape parameter $\sigma = 0.5$, the maximal precedence tests have higher power values than the Wilcoxon's rank-sum test. For example, in Table 2.4, when $n_1 = n_2 = 20$ and the location-shift equals 0.5, the power of the maximal precedence test with $r = 2$ is 0.6580 (exact level of significance is 0.0469) while the power of Wilcoxon's rank-sum test is 0.5699 (exact level of significance is 0.0482). From Tables 2.3–2.5, it is evident that the more right-skewed the underlying distribution is, the more powerful the maximal precedence test is.

Moreover, we can compare the power values of the precedence tests and the maximal precedence tests with the same value of $r$ since they are both test procedures based on failures from the $X$-sample occurring before the $r$-th failure from the $Y$-sample. The power values presented in Tables 2.3–2.5 show that the precedence test is more powerful than the maximal precedence test under the normal distribution. However, under the exponential distribution, the maximal precedence test performs better than the precedence test. From the power values of the precedence tests, we observe that the masking effect in the precedence test becomes more obvious when $r$ becomes larger. The maximal precedence test eliminates the masking effect present in precedence test. The power values reveal that larger the value of $r$, the more superior the maximal precedence test becomes as compared to the precedence test under an exponential distribution. For example, when $n_1 = n_2 = 20$ and the location-shift equals 0.5, we find the power of the precedence test to be 0.9355 (exact level of significance is 0.0457) while the power of the maximal precedence test is 0.9631 (exact level of significance is 0.0469) for $r = 2$; and the power of the precedence test is 0.6291 (exact level of significance is 0.0527) while the power of the maximal precedence test is 0.9062 (exact level of significance is 0.0602) for $r = 6$.

Through the simulation work, we demonstrated that the maximal precedence test does avoid the masking effect, but still suffers from a loss of power when compared to the classical Wilcoxon's rank-sum test (based on complete samples). Therefore, we will try to extend the precedence test in another way. We can see that even though we have suggested an alternative to the precedence test in the form of maximal precedence test, it too is based on frequencies of failures preceding the $r$-th failure. One possible extension may be to construct Wilcoxon-type rank based test that takes into account the magnitude of the failure times, which is the subject of discussion in the next chapter.

# Chapter 3

# Wilcoxon-type Rank-sum Precedence Tests

## 3.1 Introduction

First, we shall review some basic results of the Wilcoxon rank-sum test, which is a well-known nonparametric testing procedure for testing the hypothesis in (1.3.1) based on complete samples. We shall use the same notations as in Section 1.3. For testing the hypotheses in (1.3.1), if complete samples of size $n_1$ and $n_2$ were available, then the standard Wilcoxon's rank-sum statistic, which is proposed by Wilcoxon (1945), is [for more details, see Lehmann (1975), Gibbons and Chakraborti (1992) and Hettmansperger and McKean (1998)]

$$W_S = S_1 + S_2 + \ldots + S_{n_1},$$

where $(S_1, S_2, \ldots, S_{n_1})$ are the ranks of the $X$-failures, with small values of $W_S$ leading to the rejection of $H_0$. If there are no ties, the exact mean and variance of $W_S$ under the null hypothesis of equal distributions are given by

$$E(W_S) = \frac{n_1(n_1 + n_2 + 1)}{2}, \quad Var(W_S) = \frac{n_1 n_2(n_1 + n_2 + 1)}{12}, \tag{3.1.1}$$

and the statistic is symmetric about its mean.

In this chapter, we introduce Wilcoxon-type rank-sum precedence tests for testing the hypothesis in (1.3.1) when the $Y$-sample is Type-II censored. This test is a variation of the precedence life-test described in Section 1.3 and a generalization of the Wilcoxon rank-sum test. Three Wilcoxon-type rank-sum precedence test statistics - the minimal, maximal and expected rank-sum statistics - are proposed here. We derive the null distributions of these three Wilcoxon-type rank-sum precedence test statistics. Critical values for some combinations of sample sizes are presented. After noting that the large-sample normal approximation for the null distribution is not satisfactory in the case of small or moderate sample sizes, we develop an Edgeworth expansion to approximate the significance probabilities. Next, we derive the exact power function under the Lehmann alternative. We also examine the power properties of the Wilcoxon-type rank-sum precedence tests under a location-shift alternative through Monte Carlo simulations. Then, the power comparisons are made with the precedence test, the maximal precedence test and Wilcoxon's rank-sum test (based on complete samples). Finally, two examples are presented for illustration.

## 3.2 Test Statistics and Null Distributions

In this section, we propose three new test statistics of Wilcoxon-type for the precedence testing situation described in Section 1.3. Their null distributions are also derived.

### 3.2.1 Wilcoxon-type Rank-sum Precedence Test Statistics

In order to test the hypotheses in (1.3.1), instead of using the maximum of the frequencies of failures from the $X$-sample between the first $r$ failures of the $Y$-sample, one could use the sum of the ranks of those failures. More specifically, suppose $m_1, m_2, \ldots, m_r$ denote the number of $X$-failures that occurred before the first, between the first and the second, ..., between the $(r-1)$-th and the $r$-th $Y$-failures, respectively; see Figure 1.1. Let $W$ be the rank-sum of the $X$-failures that occurred before the $r$-th $Y$-failure. The Wilcoxon's test statistic will be smallest when all the remaining $\left(n_1 - \sum_{i=1}^{r} m_i\right)$ $X$-failures occur between the $r$-th and $(r+1)$-th $Y$-failures. The test statistic in this case would be

$$\begin{aligned} W_{\min} &= W + \left[\left(\sum_{i=1}^{r} m_i + r + 1\right) + \left(\sum_{i=1}^{r} m_i + r + 2\right) + \ldots + (n_1 + r)\right] \\ &= \frac{n_1(n_1 + 2r + 1)}{2} - (r+1)\sum_{i=1}^{r} m_i + \sum_{i=1}^{r} i m_i. \end{aligned}$$

This is called the *minimal rank-sum statistic.*

The Wilcoxon's test statistic will be the largest when all the remaining $\left(n_1 - \sum_{i=1}^{r} m_i\right)$ $X$-failures occur after the $n_2$-th $Y$-failure. Such a test statistic is called the *maximal rank-sum statistic* and is given by

$$\begin{aligned} W_{\max} &= W + \left[\left(\sum_{i=1}^{r} m_i + n_2 + 1\right) + \left(\sum_{i=1}^{r} m_i + n_2 + 2\right) + \ldots + (n_1 + n_2)\right] \\ &= \frac{n_1(n_1 + 2n_2 + 1)}{2} - (n_2+1)\sum_{i=1}^{r} m_i + \sum_{i=1}^{r} i m_i. \end{aligned}$$

We could similarly propose a rank-sum statistic using the expected rank sums of failures from the first sample between the $r$-th and the $(r+1)$-th, ..., after the $n_2$-th failures of the second sample, denoted by $W_E$. It can be shown that $W_E$ is simply the average of $W_{\min}$ and $W_{\max}$, and is given by

$$\begin{aligned} W_E &= W + \frac{1}{2}\left[\left(\sum_{i=1}^{r} m_i + r + 1\right) + \left(\sum_{i=1}^{r} m_i + r + 2\right) + \ldots + (n_1 + r)\right] \\ &\quad + \frac{1}{2}\left[\left(\sum_{i=1}^{r} m_i + n_2 + 1\right) + \left(\sum_{i=1}^{r} m_i + n_2 + 2\right) + \ldots + (n_1 + n_2)\right] \\ &= \frac{n_1(n_1 + n_2 + r + 1)}{2} - \left(\frac{n_2 + r}{2} + 1\right)\sum_{i=1}^{r} m_i + \sum_{i=1}^{r} i m_i. \end{aligned}$$

For example, from Figure 1.1, when $n_1 = n_2 = 10$ with $r = 4$, we have

$$\begin{aligned} W_{\min} &= 2+3+4+6+7+8+9+11+13+14 = 77, \\ W_{\max} &= 2+3+4+6+7+8+9+11+19+20 = 89, \\ W_E &= \frac{77+89}{2} = 83. \end{aligned}$$

It is evident that small values of $W_{\min}$, $W_{\max}$ and $W_E$ lead to the rejection of $H_0$ and in favor of $H_1$ in (1.3.1). Moreover, in the special case of $r = n_2$ (that is, when we observe all the failures from the $Y$-sample), we have $W_{\min} = W_{\max} = W_E$ and in this case they are all equivalent to the classical Wilcoxon's rank-sum statistic $W_S$ defined earlier in Section 3.1.

### 3.2.2 Null Distributions

The joint probability mass function of $M_1, M_2, \ldots, M_r$ under the null hypothesis $H_0 : F_X = F_Y$ is given in Theorem 2.1. For the minimal rank-sum test statistic, it is obvious that

$$\begin{aligned} & \Pr\{W_{\min} = w \mid M_1 = m_1, M_2 = m_2, \ldots, M_r = m_r\} \\ = & \begin{cases} 1, & \text{if } w = \frac{n_1(n_1+2r+1)}{2} - (r+1)\sum_{i=1}^{r} m_i + \sum_{i=1}^{r} i m_i \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Therefore, the null distribution of $W_{\min}$ is simply given by

$$\begin{aligned} \Pr\{W_{\min} &= w \mid F_X = F_Y\} \\ &= \Pr\left\{\left[\frac{n_1(n_1+2r+1)}{2} - (r+1)\sum_{i=1}^{r} M_i + \sum_{i=1}^{r} i M_i\right] = w \,\middle|\, F_X = F_Y\right\} \\ &= \sum_{\substack{m_i(i=1,2,\ldots,r)=0,\sum_{i=1}^{r} m_i \le n_1 \\ \left[\frac{n_1(n_1+2r+1)}{2} - (r+1)\sum_{i=1}^{r} m_i + \sum_{i=1}^{r} i m_i\right] = w}}^{n_1} \frac{\binom{n_1+n_2-\sum_{i=1}^{r} m_i - r}{n_2 - r}}{\binom{n_1+n_2}{n_2}}. \qquad (3.2.1) \end{aligned}$$

For specified values of $n_1, n_2, r$ and the level of significance $\alpha$, the critical value $s$ (corresponding to a level closest to $\alpha$) for the minimal rank-sum precedence test can then be found from Eq. (3.2.1) as

$$\begin{aligned} \alpha &= \Pr(W_{\min} \le s \mid F_X = F_Y) \\ &= \Pr\left\{\left[\frac{n_1(n_1+2r+1)}{2} - (r+1)\sum_{i=1}^{r} M_i + \sum_{i=1}^{r} i M_i\right] \le s \,\middle|\, F_X = F_Y\right\} \\ &= \sum_{m_i(i=1,2,\ldots,r)=0}^{n_1} \frac{\binom{n_1+n_2-\sum_{i=1}^{r} m_i - r}{n_2 - r}}{\binom{n_1+n_2}{n_2}} I_{\min,s}(m_1, m_2, \ldots, m_r), \qquad (3.2.2) \end{aligned}$$

where $I_{\min,s}$ is the indicator function defined by

$$I_{\min,s}(m_1, m_2, \ldots, m_r) = \begin{cases} 1, & \text{if } \left[\frac{n_1(n_1+2r+1)}{2} - (r+1)\sum_{i=1}^{r} m_i + \sum_{i=1}^{r} i m_i\right] \le s \\ & \quad\quad \text{and } \sum_{i=1}^{r} m_i \le n_1 \\ 0, & \text{otherwise.} \end{cases}$$

The null distributions of the maximal rank-sum statistic and the expected rank-sum statistic can be derived in a similar manner. The critical value $s$ for the maximal rank-sum statistic and expected rank-sum statistic can then be shown to be

$$\begin{aligned} \alpha &= \Pr(W_{\max} \le s \mid F_X = F_Y) \\ &= \Pr\left\{ \left[\frac{n_1(n_1+2n_2+1)}{2} - (n_2+1)\sum_{i=1}^{r} M_i + \sum_{i=1}^{r} i M_i\right] \le s \,\middle|\, F_X = F_Y \right\} \\ &= \sum_{m_i(i=1,2,\ldots,r)=0}^{n_1} \frac{\binom{n_1+n_2-\sum_{i=1}^{r} m_i - r}{n_2 - r}}{\binom{n_1+n_2}{n_2}} I_{\max,s}(m_1, m_2, \ldots, m_r), \end{aligned} \tag{3.2.3}$$

and

$$\begin{aligned} \alpha &= \Pr(W_E \le s \mid F_X = F_Y) \\ &= \Pr\left\{ \left[\frac{n_1(n_1+n_2+r+1)}{2} - \left(\frac{n_2+r}{2}+1\right)\sum_{i=1}^{r} M_i + \sum_{i=1}^{r} i M_i\right] \le s \,\middle|\, F_X = F_Y \right\} \\ &= \sum_{m_i(i=1,2,\ldots,r)=0}^{n_1} \frac{\binom{n_1+n_2-\sum_{i=1}^{r} m_i - r}{n_2 - r}}{\binom{n_1+n_2}{n_2}} I_{E,s}(m_1, m_2, \ldots, m_r), \end{aligned} \tag{3.2.4}$$

respectively, where $I_{\max,s}$ is the indicator function defined by

$$I_{\max,s}(m_1, m_2, \ldots, m_r) = \begin{cases} 1, & \text{if } \left[\frac{n_1(n_1+2n_2+1)}{2} - (n_2+1)\sum_{i=1}^{r} m_i + \sum_{i=1}^{r} i m_i\right] \le s \\ & \quad\quad \text{and } \sum_{i=1}^{r} m_i \le n_1 \\ 0, & \text{otherwise,} \end{cases}$$

and $I_{E,s}$ is the indicator function defined by

$$I_{E,s}(m_1, m_2, \ldots, m_r) = \begin{cases} 1, & \text{if } \left[\frac{n_1(n_1+n_2+r+1)}{2} - \left(\frac{n_2+r}{2}+1\right)\sum_{i=1}^{r} m_i + \sum_{i=1}^{r} i m_i\right] \le s \\ & \quad \text{and } \sum_{i=1}^{r} m_i \le n_1 \\ 0, & \text{otherwise,} \end{cases}$$

In Tables 3.1 and 3.2, we have presented the critical value $s$ and the exact level of significance $\alpha$ (as close as possible to 5% and 10%) for some sample sizes $n_1$ and $n_2$ and $r = 2(1)7$ for the minimal, maximal and expected rank-sum precedence test statistics, respectively.

Instead of testing the hypotheses in (1.3.1), we can also use the test statistics proposed here to test a two-sided alternative $H_1 : F_X \neq F_Y$. In this case, we will reject the null hypothesis for large and small values of the test statistics $W_{\min}$, $W_{\max}$ and $W_E$; that is, we reject the null hypothesis if the test statistic is at most $c_1$ or at least $c_2$, where $c_1 < c_2$. The critical values $(c_1, c_2)$ for specified values of $n_1, n_2, r$ and the level of significance $\alpha$ can be computed in a straightforward manner from the null distributions presented above.

However, for a specified value of $\alpha$, there may not be a critical region that the probability on both sides are close to $\alpha/2$. For example, when $n_1 = 30$ and $n_2 = 50$, the maximum possible value of $W_{\min}$ is 615 and the probability of $W_{\min} = 615$ is 0.0881 under the null distribution. Thus, it will be impossible to find a critical region corresponding to $\alpha = 10\%$ that has probability close to 5% on both sides. In this case, we may choose $c_1 = 594$ [$\Pr(W_{\min} \le 594 | H_0) = 0.0632$] and $c_2 = 615$ to attain a critical region with 15.31% level of significance. Similarly, we can construct the critical regions for $W_{\max}$ and $W_E$ in this example, which turn out to be $c_1 = 1671, c_2 = 1965$ (with 15.26% level of significance) and $c_1 = 1131, c_2 = 1290$ (with 15.26% level of significance), respectively.

Table 3.1: Near 5% upper critical values and exact levels of significance for the Wilcoxon's rank-sum precedence test statistics

| $n_1$ | $n_2$ | $r=2$ | $r=3$ | $r=4$ | $r=5$ | $r=6$ | $r=7$ |
|---|---|---|---|---|---|---|---|
| | | | | $W_{\text{min}}$ | | | |
| 10 | 10 | 67 (0.060) | 71 (0.046) | 75 (0.054) | 77 (0.045) | 80 (0.055) | 81 (0.050) |
| 10 | 15 | 68 (0.033) | 74 (0.045) | 79 (0.047) | 84 (0.056) | 87 (0.048) | 90 (0.048) |
| 15 | 15 | 141 (0.042) | 150 (0.047) | 158 (0.048) | 165 (0.050) | 171 (0.051) | 176 (0.050) |
| 15 | 20 | 143 (0.052) | 153 (0.053) | 162 (0.050) | 170 (0.047) | 178 (0.051) | 185 (0.053) |
| 20 | 20 | 241 (0.047) | 255 (0.053) | 267 (0.047) | 279 (0.052) | 289 (0.050) | 298 (0.048) |
| 30 | 30 | 516 (0.052) | 539 (0.046) | 561 (0.046) | 582 (0.047) | 602 (0.049) | 621 (0.052) |
| 30 | 50 | 518 (0.034) | 544 (0.047) | 569 (0.052) | 593 (0.052) | 616 (0.052) | 638 (0.049) |
| | | | | $W_{\text{max}}$ | | | |
| 10 | 10 | 107 (0.050) | 99 (0.051) | 93 (0.051) | 89 (0.051) | 86 (0.049) | 84 (0.047) |
| 10 | 15 | 147 (0.047) | 136 (0.050) | 129 (0.050) | 122 (0.051) | 116 (0.051) | 111 (0.049) |
| 15 | 15 | 270 (0.047) | 250 (0.049) | 238 (0.051) | 227 (0.050) | 218 (0.049) | 211 (0.049) |
| 15 | 20 | 340 (0.051) | 320 (0.049) | 302 (0.050) | 287 (0.050) | 274 (0.049) | 264 (0.050) |
| 20 | 20 | 510 (0.053) | 477 (0.050) | 458 (0.050) | 439 (0.050) | 423 (0.050) | 410 (0.051) |
| 30 | 30 | 1191 (0.051) | 1160 (0.051) | 1115 (0.050) | 1085 (0.050) | 1057 (0.051) | 1029 (0.050) |
| 30 | 50 | 1767 (0.048) | 1720 (0.049) | 1675 (0.050) | 1632 (0.050) | 1595 (0.050) | 1578 (0.050) |
| | | | | $W_E$ | | | |
| 10 | 10 | 87 (0.050) | 85 (0.047) | 84 (0.049) | 84 (0.052) | 83 (0.049) | 83 (0.049) |
| 10 | 15 | 108 (0.047) | 106 (0.050) | 105 (0.052) | 103 (0.051) | 102 (0.049) | 101 (0.049) |
| 15 | 15 | 205 (0.047) | 201 (0.049) | 199 (0.051) | 197 (0.052) | 195 (0.048) | 194 (0.049) |
| 15 | 20 | 241 (0.051) | 236 (0.051) | 232 (0.052) | 229 (0.050) | 227 (0.051) | 225 (0.049) |
| 20 | 20 | 375 (0.053) | 367 (0.050) | 363 (0.051) | 360 (0.051) | 357 (0.050) | 355 (0.049) |
| 30 | 30 | 855 (0.051) | 850 (0.051) | 841 (0.050) | 835 (0.050) | 830 (0.050) | 826 (0.050) |
| 30 | 50 | 1143 (0.048) | 1133 (0.049) | 1123 (0.050) | 1115 (0.050) | 1110 (0.050) | 1106 (0.050) |

Table 3.2: Near 10% upper critical values and exact levels of significance for the Wilcoxon's rank-sum precedence test statistics

| $n_1$ | $n_2$ | $r = 2$ | $r = 3$ | $r = 4$ | $r = 5$ | $r = 6$ | $r = 7$ |
|---|---|---|---|---|---|---|---|
| | | | | $W_{\min}$ | | | |
| 10 | 10 | 68 (0.086) | 73 (0.091) | 78 (0.111) | 81 (0.104) | 83 (0.095) | 85 (0.0980 |
| 10 | 15 | 70 (0.103) | 76 (0.097) | 82 (0.106) | 87 (0.109) | 91 (0.104) | 94 (0.094) |
| 15 | 15 | 143 (0.099) | 153 (0.109) | 161 (0.091) | 169 (0.099) | 176 (0.104) | 181 (0.095) |
| 15 | 20 | 144 (0.095) | 155 (0.096) | 165 (0.101) | 174 (0.100) | 182 (0.097) | 190 (0.104) |
| 20 | 20 | 243 (0.104) | 257 (0.088) | 271 (0.102) | 283 (0.097) | 295 (0.107) | 305 (0.104) |
| 30 | 30 | 518 (0.110) | 542 (0.096) | 565 (0.095) | 587 (0.096) | 608 (0.099) | 628 (0.102) |
| 30 | 50 | 520 (0.097) | 546 (0.094) | 572 (0.104) | 596 (0.092) | 620 (0.094) | 644 (0.105) |
| | | | | $W_{\max}$ | | | |
| 10 | 10 | 116 (0.103) | 107 (0.102) | 100 (0.100) | 95 (0.098) | 92 (0.101) | 90 (0.103) |
| 10 | 15 | 161 (0.111) | 148 (0.096) | 138 (0.099) | 132 (0.101) | 125 (0.098) | 120 (0.098) |
| 15 | 15 | 285 (0.100) | 264 (0.100) | 251 (0.098) | 240 (0.098) | 231 (0.099) | 223 (0.098) |
| 15 | 20 | 344 (0.093) | 330 (0.101) | 318 (0.100) | 305 (0.101) | 292 (0.102) | 281 (0.102) |
| 20 | 20 | 530 (0.107) | 497 (0.100) | 479 (0.102) | 460 (0.098) | 444 (0.100) | 430 (0.100) |
| 30 | 30 | 1220 (0.097) | 1190 (0.096) | 1160 (0.098) | 1121 (0.100) | 1093 (0.101) | 1066 (0.099) |
| 30 | 50 | 1816 (0.104) | 1768 (0.094) | 1723 (0.100) | 1680 (0.099) | 1668 (0.100) | 1629 (0.099) |
| | | | | $W_E$ | | | |
| 10 | 10 | 92 (0.103) | 90 (0.095) | 89 (0.100) | 88 (0.094) | 88 (0.103) | 88 (0.103) |
| 10 | 15 | 116 (0.111) | 112 (0.096) | 111 (0.099) | 109 (0.097) | 109 (0.105) | 108 (0.103) |
| 15 | 15 | 214 (0.100) | 209 (0.096) | 207 (0.099) | 205 (0.099) | 204 (0.099) | 203 (0.101) |
| 15 | 20 | 245 (0.093) | 245 (0.101) | 242 (0.102) | 239 (0.098) | 237 (0.099) | 236 (0.100) |
| 20 | 20 | 386 (0.107) | 378 (0.100) | 375 (0.101) | 372 (0.099) | 370 (0.101) | 368 (0.099) |
| 30 | 30 | 870 (0.097) | 866 (0.096) | 862 (0.101) | 856 (0.099) | 852 (0.101) | 848 (0.099) |
| 30 | 50 | 1168 (0.104) | 1157 (0.094) | 1148 (0.100) | 1140 (0.099) | 1140 (0.100) | 1135 (0.099) |

## 3.3 Large-sample Approximation for the Null Distributions

For small values of $r$, $n_1$ and $n_2$, the critical values and the exact significance probabilities of the Wilcoxon-type rank-sum precedence test statistics can be computed without any difficulty as displayed in Tables 3.1 and 3.2. However, for large values of $r$, $n_1$ or $n_2$, this would require a heavy computational effort and time. For this reason, we develop in this section some large-sample approximations for the null distributions of the Wilcoxon-type rank-sum statistics.

By the distributional properties of a linear rank statistic $T$, we know that $\frac{T-E(T)}{\sqrt{var(T)}}$ is asymptotically normally distributed with zero mean and unit variance [see Lehmann (1975)]. Hence, the distribution function of $\frac{T-E(T)}{\sqrt{var(T)}}$ can be written in a series form as

$$\begin{aligned}
&\Pr\left\{\frac{T-E(T)}{\sqrt{var(T)}} \le t\right\} \\
&= \Phi(t) + \phi(t)\left\{\frac{-\sqrt{\beta_1}}{6}(t^2-1)\right\} \\
&\quad +\phi(t)(-t)\left\{\frac{(\beta_2-3)}{24}(t^2-3) + \frac{\beta_1}{72}(t^4-10t^2+15)\right\} + \ldots, \qquad (3.3.1)
\end{aligned}$$

where $\sqrt{\beta_1}$ and $\beta_2$ are the coefficients of skewness and kurtosis of $T$, respectively, and $\Phi(t)$ is the c.d.f. of the standard normal distribution with corresponding p.d.f. $\phi(t)$. Formula (3.3.1) is known as the Edgeworth expansion [see, for example, Hall (1992) and Johnson, Kotz and Balakrishnan (1994)]. The approximation to the distribution of the classical Wilcoxon's rank-sum statistic by using Edgeworth expansion has been discussed earlier by many authors including Fix and Hodges (1955) and Bickel (1974).

Though $W_{\min}$, $W_{\max}$ and $W_E$ are not linear rank statistics, yet their distributions seem to be approximately normal even for values of $n_1$ and $n_2$ as small as 10 when $r$ is close to $n_2$; see Figure 3.1, where $q = \frac{r}{n_2}$. In other cases, however, the normal approximation does not seem to be reasonable; see Figure 3.2, for example. So, we consider the use of the second-order Edgeworth expansion to approximate the distribution of these test statistics, i.e.

$$F_W(w) \approx \Phi(w') - \phi(w')\left\{\frac{\sqrt{\beta_1}}{6}(w'^2-1)+\frac{(\beta_2-3)}{24}(w'^3-3w')+\frac{\beta_1}{72}(w'^5-10w'^3+15w')\right\}, \tag{3.3.2}$$

where $w' = \frac{w-E(W)}{\sqrt{var(W)}}$. In order to use the second-order Edgeworth approximation, we require the knowledge of the first four moments of $W_{\min}$, $W_{\max}$ and $W_E$. The exact explicit formulas for these moments are presented in Appendix A.

In addition, we can modify the approximation by applying the continuity correction as all three test statistics are discrete. In other words, $w'$ in formula (3.3.2) should be modified as

$$w' = \frac{w - E(W) + 0.5}{\sqrt{var(W)}}. \tag{3.3.3}$$

In Tables 3.3-3.5, we provide a comparison of the normal approximation and the Edgeworth approximation (both with continuity correction) to the exact significance probabilities for the Wilcoxon-type rank-sum precedence test statistics $W_{\min}$, $W_{\max}$ and $W_E$. Specifically, the approximate values corresponding to the exact significance probabilities of 1%, 5%, 10%, 20% and 30% of the Wilcoxon-type rank-sum precedence test statistics (for some combinations of $n_1$ and $n_2$, and $r = 2(1)7$) are presented in these tables. From Tables 3.3-3.5,

From Tables 3.3-3.5, we find that even though the Edgeworth approximation may not always be better than the normal approximation, it improves the normal approximation considerably in most of the cases considered. We also note that the accuracy of the approximation does not in general increase with $n_1$ and $n_2$ for fixed values of $r$, and that the accuracy of the approximation usually depends on the value of $r$. Intuitively, this may be due to the fact that the null distributions of the Wilcoxon-type rank-sum statistics are highly skewed when $r$ is too small relative to $n_2$. From our extensive simulation results, we have observed that the normal approximation is quite reasonable in the practical range of sample sizes (like those considered here, say 10 to 50) as long as $r$ is at least 50% of $n_2$. For example, in Figure 3.1, we have presented the simulated histogram of the distribution of $W_E$ for $n_1 = n_2 = 10$ and 30 when $q = \frac{r}{n_2}$ equals 60%, 80% and 90%. These plots support this observation with the normal approximation being good even for sample sizes as small as 10.

Even though the normal approximation does not work for small or moderate values of $n_2$ for small values of $q$, we have observed that even when $q$ is small, the normal approximation works quite well for large values of $n_1$ and $n_2$. In Figure 3.2, this point is illustrated by the case when $q = 20\%$ for $n_1 = n_2 = 10, 30$ and 100. While the distributions are quite skewed (to the left) for sample sizes 10 and 30, the normal approximation seems to be quite good for sample sizes 100.

We conjecture that the asymptotic distributions of $W_{\min}$, $W_{\max}$ and $W_E$ are all normal for $q \in (0, 1)$ as $n_1, n_2 \to \infty$. We have not been able to prove this result and it seems to be an open problem!

Table 3.3a: Exact and approximate values of $\Pr(W_{\min} \leq s)$ by Normal approximation and Edgeworth approximation (near 1% critical values)

| Near 1% critical values | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $n_1$ | $n_2$ | | $r=2$ | $r=3$ | $r=4$ | $r=5$ | $r=6$ | $r=7$ |
| 10 | 10 | $s$ | 64 | 67 | 69 | 71 | 73 | 73 |
| | | (a) | 0.010836 | 0.010349 | 0.008530 | 0.009228 | 0.011258 | 0.009006 |
| | | (b) | 0.014773 | 0.011450 | 0.009140 | 0.009244 | 0.010870 | 0.008860 |
| | | (c) | 0.001291 | 0.001774 | 0.002220 | 0.003688 | 0.006387 | 0.006372 |
| 10 | 15 | $s$ | 66 | 70 | 74 | 77 | 80 | 82 |
| | | (a) | 0.009486 | 0.008457 | 0.008770 | 0.008721 | 0.009723 | 0.009371 |
| | | (b) | 0.014540 | 0.010208 | 0.010666 | 0.009646 | 0.010111 | 0.009501 |
| | | (c) | 0.000648 | 0.000669 | 0.001392 | 0.001881 | 0.003026 | 0.003676 |
| 15 | 15 | $s$ | 138 | 145 | 152 | 157 | 161 | 165 |
| | | (a) | 0.011589 | 0.009270 | 0.011336 | 0.010106 | 0.009076 | 0.009551 |
| | | (b) | 0.015129 | 0.011616 | 0.012825 | 0.010958 | 0.009533 | 0.009650 |
| | | (c) | 0.000805 | 0.000982 | 0.002224 | 0.002542 | 0.002831 | 0.003840 |
| 15 | 20 | $s$ | 140 | 148 | 156 | 163 | 169 | 174 |
| | | (a) | 0.012331 | 0.008702 | 0.009992 | 0.010201 | 0.010015 | 0.009375 |
| | | (b) | 0.016876 | 0.011348 | 0.012122 | 0.011802 | 0.011025 | 0.009998 |
| | | (c) | 0.000729 | 0.000634 | 0.001299 | 0.001932 | 0.002408 | 0.002683 |
| 20 | 20 | $s$ | 237 | 249 | 260 | 270 | 278 | 286 |
| | | (a) | 0.008314 | 0.009383 | 0.010053 | 0.010823 | 0.009721 | 0.010263 |
| | | (b) | 0.011628 | 0.012010 | 0.012262 | 0.012344 | 0.010684 | 0.010826 |
| | | (c) | 0.000356 | 0.000772 | 0.001419 | 0.002228 | 0.002357 | 0.003180 |
| 30 | 30 | $s$ | 512 | 533 | 553 | 572 | 590 | 606 |
| | | (a) | 0.010534 | 0.009225 | 0.009415 | 0.009837 | 0.010456 | 0.009884 |
| | | (b) | 0.016301 | 0.012620 | 0.012037 | 0.012045 | 0.012210 | 0.011151 |
| | | (c) | 0.000565 | 0.000631 | 0.000951 | 0.001433 | 0.002040 | 0.002212 |
| 30 | 50 | $s$ | 516 | 540 | 563 | 585 | 606 | 627 |
| | | (a) | 0.011837 | 0.011955 | 0.011020 | 0.010481 | 0.009825 | 0.010636 |
| | | (b) | 0.021007 | 0.017302 | 0.015263 | 0.013701 | 0.012350 | 0.012840 |
| | | (c) | 0.000678 | 0.000835 | 0.001031 | 0.001195 | 0.001306 | 0.001844 |

Remark: (a) - Exact probability; (b) - Edgeworth approximation; (c) - Normal approximation.

Table 3.3b: Exact and approximate values of $\Pr(W_{\min} \leq s)$ by Normal approximation and Edgeworth approximation (near 5% critical values)

| Near 5% critical values | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $n_1$ | $n_2$ | | $r=2$ | $r=3$ | $r=4$ | $r=5$ | $r=6$ | $r=7$ |
| 10 | 10 | $s$ | 67 | 71 | 75 | 77 | 80 | 81 |
| | | (a) | 0.059538 | 0.046386 | 0.053774 | 0.044616 | 0.054678 | 0.050077 |
| | | (b) | 0.050721 | 0.042527 | 0.049805 | 0.042107 | 0.052925 | 0.049029 |
| | | (c) | 0.032129 | 0.025736 | 0.035555 | 0.030737 | 0.043429 | 0.042195 |
| 10 | 15 | $s$ | 68 | 74 | 79 | 84 | 87 | 90 |
| | | (a) | 0.033160 | 0.044523 | 0.047266 | 0.055544 | 0.048403 | 0.047942 |
| | | (b) | 0.037555 | 0.041784 | 0.043322 | 0.052846 | 0.045866 | 0.045875 |
| | | (c) | 0.012363 | 0.021999 | 0.026107 | 0.037666 | 0.032446 | 0.034171 |
| 15 | 15 | $s$ | 141 | 150 | 158 | 165 | 171 | 176 |
| | | (a) | 0.042046 | 0.046543 | 0.048259 | 0.050005 | 0.050702 | 0.050462 |
| | | (b) | 0.042547 | 0.043279 | 0.045437 | 0.047417 | 0.048583 | 0.048702 |
| | | (c) | 0.019464 | 0.023841 | 0.028529 | 0.032495 | 0.035422 | 0.037221 |
| 15 | 20 | $s$ | 143 | 153 | 162 | 170 | 178 | 185 |
| | | (a) | 0.051866 | 0.053019 | 0.049504 | 0.046842 | 0.051492 | 0.053333 |
| | | (b) | 0.049987 | 0.048369 | 0.046683 | 0.044490 | 0.049218 | 0.051304 |
| | | (c) | 0.029718 | 0.028469 | 0.028264 | 0.027675 | 0.034065 | 0.037530 |
| 20 | 20 | $s$ | 241 | 255 | 267 | 279 | 289 | 298 |
| | | (a) | 0.046906 | 0.053347 | 0.046714 | 0.052337 | 0.049868 | 0.048164 |
| | | (b) | 0.046579 | 0.049516 | 0.044720 | 0.050050 | 0.047903 | 0.046464 |
| | | (c) | 0.023795 | 0.029956 | 0.026263 | 0.033710 | 0.032946 | 0.032910 |
| 30 | 30 | $s$ | 516 | 539 | 561 | 582 | 602 | 621 |
| | | (a) | 0.051787 | 0.045743 | 0.046115 | 0.047360 | 0.049444 | 0.051772 |
| | | (b) | 0.050083 | 0.045444 | 0.044751 | 0.045925 | 0.047917 | 0.050188 |
| | | (c) | 0.028797 | 0.023195 | 0.024603 | 0.027663 | 0.031195 | 0.034758 |
| 30 | 50 | $s$ | 518 | 544 | 569 | 593 | 616 | 638 |
| | | (a) | 0.034453 | 0.047051 | 0.052283 | 0.052450 | 0.051595 | 0.049401 |
| | | (b) | 0.044409 | 0.046485 | 0.049387 | 0.050450 | 0.049820 | 0.048017 |
| | | (c) | 0.012148 | 0.022901 | 0.028884 | 0.031308 | 0.031526 | 0.030443 |

Remark: (a) - Exact probability; (b) - Edgeworth approximation; (c) - Normal approximation.

Table 3.3c: Exact and approximate values of $\Pr(W_{\min} \leq s)$ by Normal approximation and Edgeworth approximation (near 10% critical values)

| Near 10% critical values | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $n_1$ | $n_2$ | | $r = 2$ | $r = 3$ | $r = 4$ | $r = 5$ | $r = 6$ | $r = 7$ |
| 10 | 10 | $s$ | 68 | 73 | 78 | 81 | 83 | 85 |
| | | (a) | 0.086330 | 0.091277 | 0.110806 | 0.104191 | 0.094931 | 0.098124 |
| | | (b) | 0.084350 | 0.085485 | 0.110303 | 0.102471 | 0.093664 | 0.097368 |
| | | (c) | 0.071777 | 0.071680 | 0.099463 | 0.091675 | 0.083969 | 0.089600 |
| 10 | 15 | $s$ | 70 | 76 | 82 | 87 | 91 | 94 |
| | | (a) | 0.103162 | 0.097446 | 0.105516 | 0.109229 | 0.104138 | 0.093967 |
| | | (b) | 0.104895 | 0.089679 | 0.105054 | 0.107394 | 0.102191 | 0.092556 |
| | | (c) | 0.101234 | 0.078193 | 0.094805 | 0.096997 | 0.091375 | 0.081679 |
| 15 | 15 | $s$ | 143 | 153 | 161 | 169 | 176 | 181 |
| | | (a) | 0.098517 | 0.108542 | 0.091454 | 0.098716 | 0.103816 | 0.094681 |
| | | (b) | 0.095878 | 0.103110 | 0.088323 | 0.096813 | 0.102358 | 0.093428 |
| | | (c) | 0.090315 | 0.094957 | 0.076064 | 0.085783 | 0.092362 | 0.083330 |
| 15 | 20 | $s$ | 144 | 155 | 165 | 174 | 182 | 190 |
| | | (a) | 0.094631 | 0.096413 | 0.101107 | 0.099555 | 0.096648 | 0.103928 |
| | | (b) | 0.080078 | 0.092818 | 0.097385 | 0.097363 | 0.094509 | 0.102528 |
| | | (c) | 0.073204 | 0.084411 | 0.087776 | 0.086892 | 0.083497 | 0.092808 |
| 20 | 20 | $s$ | 243 | 257 | 271 | 283 | 295 | 305 |
| | | (a) | 0.104391 | 0.088148 | 0.102114 | 0.097170 | 0.107160 | 0.103803 |
| | | (b) | 0.100900 | 0.083687 | 0.098877 | 0.094690 | 0.105569 | 0.102509 |
| | | (c) | 0.100469 | 0.073092 | 0.089936 | 0.084150 | 0.096652 | 0.093236 |
| 30 | 30 | $s$ | 518 | 542 | 565 | 587 | 608 | 628 |
| | | (a) | 0.110112 | 0.095906 | 0.095081 | 0.096080 | 0.098619 | 0.101776 |
| | | (b) | 0.105230 | 0.090470 | 0.090034 | 0.092755 | 0.096327 | 0.099980 |
| | | (c) | 0.111186 | 0.084505 | 0.081110 | 0.083253 | 0.087018 | 0.091128 |
| 30 | 50 | $s$ | 520 | 546 | 572 | 596 | 620 | 644 |
| | | (a) | 0.096856 | 0.093982 | 0.103598 | 0.091977 | 0.093880 | 0.104767 |
| | | (b) | 0.090392 | 0.082490 | 0.099533 | 0.087102 | 0.090588 | 0.102145 |
| | | (c) | 0.096615 | 0.077199 | 0.095847 | 0.077620 | 0.081004 | 0.094616 |

Remark: (a) - Exact probability; (b) - Edgeworth approximation; (c) - Normal approximation.

Table 3.3d: Exact and approximate values of $\Pr(W_{\min} \leq s)$ by Normal approximation and Edgeworth approximation (near 20% critical values)

| Near 20% critical values | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $n_1$ | $n_2$ | | $r=2$ | $r=3$ | $r=4$ | $r=5$ | $r=6$ | $r=7$ |
| 10 | 10 | $s$ | 70 | 76 | 81 | 85 | 88 | 90 |
| | | (a) | 0.205108 | 0.214867 | 0.212507 | 0.210770 | 0.205054 | 0.196475 |
| | | (b) | 0.222089 | 0.217024 | 0.215105 | 0.211841 | 0.205950 | 0.196684 |
| | | (c) | 0.245952 | 0.230501 | 0.222294 | 0.214487 | 0.205176 | 0.193505 |
| 10 | 15 | $s$ | 71 | 78 | 84 | 90 | 95 | 99 |
| | | (a) | 0.197431 | 0.181636 | 0.177512 | 0.197631 | 0.200406 | 0.193147 |
| | | (b) | 0.192605 | 0.191677 | 0.180167 | 0.199222 | 0.201774 | 0.194200 |
| | | (c) | 0.215053 | 0.205948 | 0.186202 | 0.206314 | 0.206489 | 0.195512 |
| 15 | 15 | $s$ | 145 | 155 | 165 | 174 | 181 | 188 |
| | | (a) | 0.215709 | 0.176883 | 0.196342 | 0.207945 | 0.193207 | 0.201213 |
| | | (b) | 0.231305 | 0.181280 | 0.198736 | 0.210372 | 0.194477 | 0.202230 |
| | | (c) | 0.270044 | 0.193632 | 0.210611 | 0.220795 | 0.198715 | 0.205598 |
| 15 | 20 | $s$ | 146 | 157 | 168 | 178 | 187 | 196 |
| | | (a) | 0.236739 | 0.179506 | 0.195956 | 0.196178 | 0.193628 | 0.208429 |
| | | (b) | 0.232633 | 0.181204 | 0.195546 | 0.198445 | 0.195180 | 0.209998 |
| | | (c) | 0.278563 | 0.198158 | 0.211211 | 0.210649 | 0.203508 | 0.218892 |
| 20 | 20 | $s$ | 245 | 260 | 275 | 288 | 301 | 312 |
| | | (a) | 0.220671 | 0.184738 | 0.206401 | 0.194308 | 0.209179 | 0.202079 |
| | | (b) | 0.234859 | 0.188728 | 0.208690 | 0.195922 | 0.210749 | 0.203197 |
| | | (c) | 0.282054 | 0.208768 | 0.228589 | 0.207902 | 0.222858 | 0.211123 |
| 30 | 30 | $s$ | 520 | 545 | 569 | 593 | 615 | 636 |
| | | (a) | 0.225428 | 0.192157 | 0.185385 | 0.206213 | 0.203271 | 0.202353 |
| | | (b) | 0.237605 | 0.195157 | 0.184854 | 0.207489 | 0.204001 | 0.203200 |
| | | (c) | 0.294004 | 0.223941 | 0.203696 | 0.228385 | 0.220307 | 0.216551 |
| 30 | 50 | $s$ | 521 | 548 | 575 | 601 | 625 | 650 |
| | | (a) | 0.181146 | 0.167004 | 0.205237 | 0.215847 | 0.188215 | 0.206000 |
| | | (b) | 0.169871 | 0.171555 | 0.206171 | 0.216526 | 0.188176 | 0.206664 |
| | | (c) | 0.204567 | 0.197348 | 0.237701 | 0.246081 | 0.205323 | 0.226211 |

Remark: (a) - Exact probability; (b) - Edgeworth approximation; (c) - Normal approximation.

Table 3.3e: Exact and approximate values of $\Pr(W_{\min} \leq s)$ by Normal approximation and Edgeworth approximation (near 30% critical values)

| Near 30% critical values | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $n_1$ | $n_2$ | | $r=2$ | $r=3$ | $r=4$ | $r=5$ | $r=6$ | $r=7$ |
| 10 | 10 | $s$ | 71 | 77 | 83 | 87 | 91 | 94 |
| | | (a) | 0.325851 | 0.274297 | 0.309836 | 0.284159 | 0.300661 | 0.308185 |
| | | (b) | 0.323663 | 0.279821 | 0.311200 | 0.286585 | 0.301682 | 0.308903 |
| | | (c) | 0.382247 | 0.310253 | 0.338084 | 0.301125 | 0.312094 | 0.314719 |
| 10 | 15 | $s$ | 72 | 79 | 86 | 92 | 98 | 103 |
| | | (a) | 0.283399 | 0.264812 | 0.275852 | 0.279124 | 0.304884 | 0.311808 |
| | | (b) | 0.311135 | 0.263167 | 0.284419 | 0.282752 | 0.307912 | 0.314157 |
| | | (c) | 0.380768 | 0.300774 | 0.318383 | 0.308693 | 0.332359 | 0.334005 |
| 15 | 15 | $s$ | 146 | 157 | 168 | 177 | 185 | 192 |
| | | (a) | 0.332375 | 0.286961 | 0.325041 | 0.306533 | 0.296568 | 0.289801 |
| | | (b) | 0.328416 | 0.291863 | 0.326512 | 0.309586 | 0.299214 | 0.291527 |
| | | (c) | 0.401445 | 0.337845 | 0.369382 | 0.340572 | 0.321997 | 0.308153 |
| 15 | 20 | $s$ | 147 | 159 | 170 | 181 | 191 | 200 |
| | | (a) | 0.326872 | 0.316787 | 0.285879 | 0.307426 | 0.312355 | 0.310850 |
| | | (b) | 0.350419 | 0.314069 | 0.289651 | 0.310203 | 0.315860 | 0.312967 |
| | | (c) | 0.438606 | 0.374318 | 0.332147 | 0.349306 | 0.349401 | 0.340613 |
| 20 | 20 | $s$ | 246 | 262 | 277 | 292 | 305 | 317 |
| | | (a) | 0.335412 | 0.292649 | 0.281439 | 0.315637 | 0.308384 | 0.304614 |
| | | (b) | 0.329978 | 0.296604 | 0.286975 | 0.318426 | 0.311030 | 0.306552 |
| | | (c) | 0.410723 | 0.351080 | 0.328276 | 0.358471 | 0.343210 | 0.332823 |
| 30 | 30 | $s$ | 521 | 547 | 572 | 596 | 619 | 641 |
| | | (a) | 0.338314 | 0.297963 | 0.288820 | 0.290723 | 0.292741 | 0.295377 |
| | | (b) | 0.330949 | 0.300462 | 0.293295 | 0.292793 | 0.294536 | 0.296992 |
| | | (c) | 0.419778 | 0.363924 | 0.343981 | 0.335598 | 0.331764 | 0.329904 |
| 30 | 50 | $s$ | 522 | 550 | 577 | 603 | 629 | 654 |
| | | (a) | 0.257989 | 0.309430 | 0.313154 | 0.293436 | 0.308437 | 0.307676 |
| | | (b) | 0.280217 | 0.313346 | 0.310391 | 0.294907 | 0.309308 | 0.308961 |
| | | (c) | 0.363262 | 0.390471 | 0.374809 | 0.347117 | 0.358570 | 0.353062 |

Remark: (a) - Exact probability; (b) - Edgeworth approximation; (c) - Normal approximation.

Table 3.4a: Exact and approximate values of $\Pr(W_{\max} \leq s)$ by Normal approximation and Edgeworth approximation (near 1% critical values)

| | | | Near 1% critical values | | | | | |
|---|---|---|---|---|---|---|---|---|
| $n_1$ | $n_2$ | | $r=2$ | $r=3$ | $r=4$ | $r=5$ | $r=6$ | $r=7$ |
| 10 | 10 | $s$ | 92 | 85 | 80 | 78 | 76 | 75 |
| | | (a) | 0.009883 | 0.009818 | 0.009218 | 0.010338 | 0.009656 | 0.009526 |
| | | (b) | 0.011075 | 0.009923 | 0.009118 | 0.010590 | 0.010239 | 0.010043 |
| | | (c) | 0.001611 | 0.003838 | 0.005961 | 0.009580 | 0.011313 | 0.012466 |
| 10 | 15 | $s$ | 130 | 118 | 109 | 103 | 98 | 95 |
| | | (a) | 0.009424 | 0.009843 | 0.010400 | 0.009894 | 0.009531 | 0.009759 |
| | | (b) | 0.014155 | 0.011534 | 0.009984 | 0.009747 | 0.009309 | 0.009777 |
| | | (c) | 0.001332 | 0.002450 | 0.003585 | 0.005214 | 0.006521 | 0.008422 |
| 15 | 15 | $s$ | 241 | 224 | 212 | 202 | 195 | 190 |
| | | (a) | 0.009720 | 0.009752 | 0.010186 | 0.009800 | 0.009963 | 0.010330 |
| | | (b) | 0.012842 | 0.010883 | 0.010272 | 0.009624 | 0.009773 | 0.010308 |
| | | (c) | 0.001180 | 0.002308 | 0.003840 | 0.005142 | 0.006842 | 0.008772 |
| 15 | 20 | $s$ | 302 | 284 | 267 | 253 | 242 | 234 |
| | | (a) | 0.010071 | 0.010219 | 0.009895 | 0.009859 | 0.009777 | 0.010177 |
| | | (b) | 0.012835 | 0.012589 | 0.011014 | 0.009982 | 0.009575 | 0.009952 |
| | | (c) | 0.000789 | 0.002024 | 0.002870 | 0.003678 | 0.004665 | 0.006116 |
| 20 | 20 | $s$ | 458 | 437 | 416 | 399 | 384 | 372 |
| | | (a) | 0.009828 | 0.010023 | 0.009896 | 0.010221 | 0.009983 | 0.009988 |
| | | (b) | 0.010342 | 0.011670 | 0.010733 | 0.010343 | 0.009855 | 0.009787 |
| | | (c) | 0.000536 | 0.001812 | 0.002823 | 0.003962 | 0.004897 | 0.006004 |
| 30 | 30 | $s$ | 1129 | 1080 | 1048 | 1013 | 982 | 955 |
| | | (a) | 0.009844 | 0.009830 | 0.010140 | 0.010041 | 0.010031 | 0.009944 |
| | | (b) | 0.012295 | 0.010701 | 0.011698 | 0.010863 | 0.010286 | 0.010027 |
| | | (c) | 0.000568 | 0.001008 | 0.002178 | 0.002807 | 0.003410 | 0.004086 |
| 30 | 50 | $s$ | 1671 | 1624 | 1578 | 1535 | 1495 | 1460 |
| | | (a) | 0.009987 | 0.010260 | 0.009877 | 0.010001 | 0.010062 | 0.010015 |
| | | (b) | 0.008197 | 0.010745 | 0.010793 | 0.010392 | 0.009950 | 0.009892 |
| | | (c) | 0.000171 | 0.000635 | 0.001126 | 0.001571 | 0.001969 | 0.002479 |

Remark: (a) - Exact probability; (b) - Edgeworth approximation; (c) - Normal approximation.

Table 3.4b: Exact and approximate values of $\Pr(W_{\max} \leq s)$ by Normal approximation and Edgeworth approximation (near 5% critical values)

| Near 5% critical values | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $n_1$ | $n_2$ | | $r=2$ | $r=3$ | $r=4$ | $r=5$ | $r=6$ | $r=7$ |
| 10 | 10 | $s$ | 107 | 99 | 93 | 89 | 86 | 84 |
| | | (a) | 0.049536 | 0.050965 | 0.050651 | 0.051078 | 0.049000 | 0.046618 |
| | | (b) | 0.041075 | 0.045758 | 0.047745 | 0.050099 | 0.049470 | 0.047712 |
| | | (c) | 0.024494 | 0.033200 | 0.038816 | 0.044524 | 0.047139 | 0.047949 |
| 10 | 15 | $s$ | 147 | 136 | 129 | 122 | 116 | 111 |
| | | (a) | 0.047431 | 0.050489 | 0.050350 | 0.051336 | 0.051481 | 0.049190 |
| | | (b) | 0.037632 | 0.039618 | 0.047414 | 0.049451 | 0.049475 | 0.048303 |
| | | (c) | 0.017946 | 0.024250 | 0.034535 | 0.039043 | 0.041535 | 0.042790 |
| 15 | 15 | $s$ | 270 | 250 | 238 | 227 | 218 | 211 |
| | | (a) | 0.047414 | 0.049450 | 0.050938 | 0.049910 | 0.049115 | 0.049094 |
| | | (b) | 0.048188 | 0.042414 | 0.046987 | 0.047543 | 0.047651 | 0.048319 |
| | | (c) | 0.029703 | 0.027123 | 0.034433 | 0.037573 | 0.040114 | 0.043021 |
| 15 | 20 | $s$ | 340 | 320 | 302 | 287 | 274 | 264 |
| | | (a) | 0.050737 | 0.048743 | 0.049698 | 0.050024 | 0.049116 | 0.049709 |
| | | (b) | 0.052739 | 0.053981 | 0.051190 | 0.049313 | 0.047284 | 0.047998 |
| | | (c) | 0.034351 | 0.037314 | 0.036562 | 0.036831 | 0.036881 | 0.039415 |
| 20 | 20 | $s$ | 510 | 477 | 458 | 439 | 423 | 410 |
| | | (a) | 0.053262 | 0.049792 | 0.050228 | 0.049673 | 0.049519 | 0.050532 |
| | | (b) | 0.055142 | 0.044032 | 0.048551 | 0.047806 | 0.047730 | 0.049100 |
| | | (c) | 0.037235 | 0.027125 | 0.034169 | 0.035603 | 0.037510 | 0.040652 |
| 30 | 30 | $s$ | 1191 | 1160 | 1115 | 1085 | 1057 | 1029 |
| | | (a) | 0.051395 | 0.050824 | 0.049641 | 0.049961 | 0.050515 | 0.050095 |
| | | (b) | 0.040255 | 0.049806 | 0.043367 | 0.046748 | 0.049107 | 0.048795 |
| | | (c) | 0.016852 | 0.031820 | 0.027140 | 0.032415 | 0.036336 | 0.037443 |
| 30 | 50 | $s$ | 1767 | 1720 | 1675 | 1632 | 1595 | 1578 |
| | | (a) | 0.048031 | 0.049454 | 0.050398 | 0.049652 | 0.050218 | 0.049752 |
| | | (b) | 0.042354 | 0.043165 | 0.042962 | 0.041659 | 0.042130 | 0.054496 |
| | | (c) | 0.016996 | 0.022494 | 0.024590 | 0.025007 | 0.026927 | 0.040620 |

Remark: (a) - Exact probability; (b) - Edgeworth approximation; (c) - Normal approximation.

Table 3.4c: Exact and approximate values of $\Pr(W_{\max} \leq s)$ by Normal approximation and Edgeworth approximation (near 10% critical values)

| Near 10% critical values | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $n_1$ | $n_2$ | | $r=2$ | $r=3$ | $r=4$ | $r=5$ | $r=6$ | $r=7$ |
| 10 | 10 | $s$ | 116 | 107 | 100 | 95 | 92 | 90 |
| | | (a) | 0.102941 | 0.102048 | 0.100359 | 0.098411 | 0.101128 | 0.102974 |
| | | (b) | 0.097726 | 0.100288 | 0.098048 | 0.096865 | 0.100974 | 0.103736 |
| | | (c) | 0.083405 | 0.086711 | 0.086788 | 0.088465 | 0.095203 | 0.100246 |
| 10 | 15 | $s$ | 161 | 148 | 138 | 132 | 125 | 120 |
| | | (a) | 0.111462 | 0.095923 | 0.098707 | 0.101296 | 0.098274 | 0.098435 |
| | | (b) | 0.099747 | 0.092948 | 0.090126 | 0.101186 | 0.096765 | 0.097613 |
| | | (c) | 0.088165 | 0.078888 | 0.076926 | 0.089936 | 0.087166 | 0.089830 |
| 15 | 15 | $s$ | 285 | 264 | 251 | 240 | 231 | 223 |
| | | (a) | 0.100383 | 0.099800 | 0.098176 | 0.098025 | 0.099481 | 0.098008 |
| | | (b) | 0.108353 | 0.089156 | 0.093616 | 0.095867 | 0.098248 | 0.097191 |
| | | (c) | 0.098960 | 0.075500 | 0.081355 | 0.085255 | 0.089374 | 0.090057 |
| 15 | 20 | $s$ | 344 | 330 | 318 | 305 | 292 | 281 |
| | | (a) | 0.093346 | 0.100692 | 0.100019 | 0.101377 | 0.101836 | 0.101594 |
| | | (b) | 0.062711 | 0.083472 | 0.100599 | 0.104788 | 0.101589 | 0.100156 |
| | | (c) | 0.046540 | 0.069506 | 0.088750 | 0.093967 | 0.091504 | 0.091148 |
| 20 | 20 | $s$ | 530 | 497 | 479 | 460 | 444 | 430 |
| | | (a) | 0.106871 | 0.100244 | 0.102209 | 0.098465 | 0.099591 | 0.099996 |
| | | (b) | 0.117278 | 0.088898 | 0.099892 | 0.098200 | 0.098593 | 0.098974 |
| | | (c) | 0.112829 | 0.075981 | 0.088453 | 0.087342 | 0.088809 | 0.090356 |
| 30 | 30 | $s$ | 1220 | 1190 | 1160 | 1121 | 1093 | 1066 |
| | | (a) | 0.097255 | 0.096046 | 0.098183 | 0.100450 | 0.101126 | 0.099490 |
| | | (b) | 0.068958 | 0.092703 | 0.105727 | 0.094046 | 0.097360 | 0.098011 |
| | | (c) | 0.055407 | 0.081931 | 0.096404 | 0.083045 | 0.087268 | 0.088622 |
| 30 | 50 | $s$ | 1816 | 1768 | 1723 | 1680 | 1668 | 1629 |
| | | (a) | 0.104303 | 0.093926 | 0.100452 | 0.099100 | 0.100467 | 0.099477 |
| | | (b) | 0.089094 | 0.090873 | 0.088134 | 0.083553 | 0.116357 | 0.108275 |
| | | (c) | 0.084692 | 0.081381 | 0.076549 | 0.071031 | 0.109208 | 0.099792 |

Remark: (a) - Exact probability; (b) - Edgeworth approximation; (c) - Normal approximation.

Table 3.4d: Exact and approximate values of $\Pr(W_{\max} \leq s)$ by Normal approximation and Edgeworth approximation (near 20% critical values)

| Near 20% critical values | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $n_1$ | $n_2$ | | $r=2$ | $r=3$ | $r=4$ | $r=5$ | $r=6$ | $r=7$ |
| 10 | 10 | $s$ | 125 | 116 | 109 | 103 | 99 | 96 |
| | | (a) | 0.186533 | 0.193498 | 0.204475 | 0.197617 | 0.196903 | 0.192535 |
| | | (b) | 0.206755 | 0.205500 | 0.205295 | 0.196120 | 0.195945 | 0.192331 |
| | | (c) | 0.212911 | 0.203961 | 0.199991 | 0.188516 | 0.188115 | 0.185332 |
| 10 | 15 | $s$ | 175 | 162 | 151 | 143 | 137 | 131 |
| | | (a) | 0.221146 | 0.208696 | 0.203828 | 0.197759 | 0.197978 | 0.196237 |
| | | (b) | 0.244856 | 0.216048 | 0.195855 | 0.193584 | 0.199898 | 0.195264 |
| | | (c) | 0.272213 | 0.223977 | 0.194897 | 0.189885 | 0.195234 | 0.189278 |
| 15 | 15 | $s$ | 300 | 285 | 268 | 256 | 247 | 238 |
| | | (a) | 0.198276 | 0.203640 | 0.201583 | 0.195744 | 0.201928 | 0.197447 |
| | | (b) | 0.223874 | 0.228968 | 0.199054 | 0.195356 | 0.202127 | 0.197055 |
| | | (c) | 0.245154 | 0.241113 | 0.199756 | 0.192924 | 0.198646 | 0.192126 |
| 15 | 20 | $s$ | 363 | 348 | 341 | 325 | 312 | 300 |
| | | (a) | 0.199866 | 0.199866 | 0.199630 | 0.197389 | 0.202580 | 0.197853 |
| | | (b) | 0.152535 | 0.172534 | 0.227708 | 0.210496 | 0.204581 | 0.197014 |
| | | (c) | 0.156334 | 0.175253 | 0.238579 | 0.214100 | 0.204886 | 0.194878 |
| 20 | 20 | $s$ | 550 | 519 | 501 | 484 | 469 | 454 |
| | | (a) | 0.203742 | 0.199638 | 0.201024 | 0.198499 | 0.202046 | 0.198283 |
| | | (b) | 0.231045 | 0.180918 | 0.192276 | 0.196779 | 0.201829 | 0.198338 |
| | | (c) | 0.261241 | 0.186405 | 0.196161 | 0.198607 | 0.202261 | 0.196819 |
| 30 | 30 | $s$ | 1275 | 1222 | 1194 | 1167 | 1139 | 1112 |
| | | (a) | 0.208965 | 0.195753 | 0.196767 | 0.200736 | 0.198829 | 0.200597 |
| | | (b) | 0.237195 | 0.175079 | 0.193214 | 0.204650 | 0.205799 | 0.204195 |
| | | (c) | 0.277210 | 0.184359 | 0.202660 | 0.213275 | 0.212078 | 0.208293 |
| 30 | 50 | $s$ | 1865 | 1817 | 1772 | 1731 | 1723 | 1682 |
| | | (a) | 0.201835 | 0.190367 | 0.199811 | 0.198692 | 0.202382 | 0.202046 |
| | | (b) | 0.223527 | 0.198225 | 0.179307 | 0.167527 | 0.224753 | 0.202165 |
| | | (c) | 0.264933 | 0.219399 | 0.189672 | 0.172143 | 0.240277 | 0.210736 |

Remark: (a) - Exact probability; (b) - Edgeworth approximation; (c) - Normal approximation.

Table 3.4e: Exact and approximate values of $\Pr(W_{\max} \leq s)$ by Normal approximation and Edgeworth approximation (near 30% critical values)

| | | | Near 30% critical values | | | | | |
|---|---|---|---|---|---|---|---|---|
| $n_1$ | $n_2$ | | $r=2$ | $r=3$ | $r=4$ | $r=5$ | $r=6$ | $r=7$ |
| 10 | 10 | $s$ | 128 | 121 | 115 | 109 | 114 | 101 |
| | | (a) | 0.291022 | 0.295666 | 0.302453 | 0.298740 | 0.505867 | 0.291861 |
| | | (b) | 0.254107 | 0.284365 | 0.303752 | 0.298193 | 0.505240 | 0.290442 |
| | | (c) | 0.273930 | 0.297696 | 0.310105 | 0.296944 | 0.509897 | 0.282827 |
| 10 | 15 | $s$ | 176 | 166 | 162 | 152 | 145 | 140 |
| | | (a) | 0.283399 | 0.301186 | 0.299563 | 0.297591 | 0.296672 | 0.306185 |
| | | (b) | 0.258047 | 0.263589 | 0.328091 | 0.298103 | 0.294991 | 0.306296 |
| | | (c) | 0.290184 | 0.283647 | 0.351599 | 0.308663 | 0.299559 | 0.307420 |
| 15 | 15 | $s$ | 303 | 291 | 279 | 268 | 258 | 249 |
| | | (a) | 0.298851 | 0.308238 | 0.304448 | 0.302282 | 0.300864 | 0.298520 |
| | | (b) | 0.253128 | 0.285225 | 0.296183 | 0.300931 | 0.301412 | 0.298524 |
| | | (c) | 0.284242 | 0.310929 | 0.314075 | 0.312198 | 0.307134 | 0.299726 |
| 15 | 20 | $s$ | 381 | 363 | 348 | 336 | 327 | 315 |
| | | (a) | 0.326872 | 0.295735 | 0.303999 | 0.299036 | 0.303263 | 0.303037 |
| | | (b) | 0.299314 | 0.283442 | 0.280383 | 0.289961 | 0.313560 | 0.304214 |
| | | (c) | 0.353774 | 0.316069 | 0.302640 | 0.307189 | 0.328576 | 0.313441 |
| 20 | 20 | $s$ | 553 | 535 | 518 | 502 | 486 | 472 |
| | | (a) | 0.302495 | 0.296852 | 0.299417 | 0.304199 | 0.300304 | 0.302784 |
| | | (b) | 0.252177 | 0.278306 | 0.293342 | 0.302329 | 0.300913 | 0.303229 |
| | | (c) | 0.289889 | 0.309430 | 0.318118 | 0.321474 | 0.314307 | 0.312458 |
| 30 | 30 | $s$ | 1278 | 1250 | 1222 | 1195 | 1169 | 1143 |
| | | (a) | 0.305977 | 0.302449 | 0.294528 | 0.295844 | 0.300539 | 0.301333 |
| | | (b) | 0.250778 | 0.280544 | 0.294853 | 0.303428 | 0.308462 | 0.307331 |
| | | (c) | 0.295835 | 0.320173 | 0.328174 | 0.331375 | 0.331794 | 0.326169 |
| 30 | 50 | $s$ | 1867 | 1865 | 1818 | 1774 | 1767 | 1727 |
| | | (a) | 0.314143 | 0.305277 | 0.292892 | 0.295196 | 0.300514 | 0.301537 |
| | | (b) | 0.230639 | 0.365295 | 0.312112 | 0.275739 | 0.348797 | 0.316151 |
| | | (c) | 0.274999 | 0.434357 | 0.358232 | 0.306691 | 0.387767 | 0.345579 |

Remark: (a) - Exact probability; (b) - Edgeworth approximation; (c) - Normal approximation.

Table 3.5a: Exact and approximate values of $\Pr(W_E \leq s)$ by Normal approximation and Edgeworth approximation (near 1% critical values)

| | | | Near 1% critical values | | | | | |
|---|---|---|---|---|---|---|---|---|
| $n_1$ | $n_2$ | | $r=2$ | $r=3$ | $r=4$ | $r=5$ | $r=6$ | $r=7$ |
| 10 | 10 | $s$ | 79 | 77 | 75 | 75 | 75 | 75 |
| | | (a) | 0.011669 | 0.010800 | 0.008914 | 0.009710 | 0.010944 | 0.010760 |
| | | (b) | 0.012888 | 0.011120 | 0.008951 | 0.010050 | 0.010886 | 0.011385 |
| | | (c) | 0.002124 | 0.004097 | 0.004900 | 0.007512 | 0.009672 | 0.011231 |
| 10 | 15 | $s$ | 98 | 94 | 93 | 91 | 90 | 89 |
| | | (a) | 0.009424 | 0.009565 | 0.010555 | 0.010164 | 0.009690 | 0.009452 |
| | | (b) | 0.013669 | 0.010622 | 0.011155 | 0.009934 | 0.009765 | 0.009280 |
| | | (c) | 0.001192 | 0.001934 | 0.003909 | 0.004708 | 0.005906 | 0.006651 |
| 15 | 15 | $s$ | 189 | 185 | 183 | 181 | 179 | 178 |
| | | (a) | 0.009720 | 0.009854 | 0.010448 | 0.010541 | 0.009515 | 0.009663 |
| | | (b) | 0.011887 | 0.010768 | 0.010788 | 0.010382 | 0.009588 | 0.009514 |
| | | (c) | 0.000966 | 0.002120 | 0.003792 | 0.005056 | 0.005767 | 0.006791 |
| 15 | 20 | $s$ | 221 | 216 | 212 | 209 | 207 | 205 |
| | | (a) | 0.010071 | 0.010112 | 0.010121 | 0.009787 | 0.010096 | 0.009537 |
| | | (b) | 0.012675 | 0.011823 | 0.010768 | 0.010140 | 0.010023 | 0.009629 |
| | | (c) | 0.000757 | 0.001702 | 0.002592 | 0.003481 | 0.004506 | 0.005212 |
| 20 | 20 | $s$ | 352 | 344 | 339 | 335 | 332 | 331 |
| | | (a) | 0.009828 | 0.009962 | 0.010100 | 0.009630 | 0.009633 | 0.010250 |
| | | (b) | 0.014636 | 0.011856 | 0.011225 | 0.010049 | 0.009622 | 0.010271 |
| | | (c) | 0.001137 | 0.001809 | 0.002707 | 0.003501 | 0.004295 | 0.005707 |
| 30 | 30 | $s$ | 821 | 811 | 801 | 794 | 788 | 783 |
| | | (a) | 0.009844 | 0.010077 | 0.010001 | 0.009981 | 0.010055 | 0.009927 |
| | | (b) | 0.012523 | 0.012675 | 0.011331 | 0.010863 | 0.010466 | 0.010208 |
| | | (c) | 0.000587 | 0.001432 | 0.001989 | 0.002707 | 0.003351 | 0.003968 |
| 30 | 50 | $s$ | 1095 | 1084 | 1072 | 1063 | 1055 | 1048 |
| | | (a) | 0.009987 | 0.010300 | 0.009877 | 0.010001 | 0.010100 | 0.009895 |
| | | (b) | 0.009093 | 0.009152 | 0.011143 | 0.011073 | 0.011041 | 0.010795 |
| | | (c) | 0.000207 | 0.000737 | 0.001193 | 0.001763 | 0.002308 | 0.002839 |

Remark: (a) - Exact probability; (b) - Edgeworth approximation; (c) - Normal approximation.

Table 3.5b: Exact and approximate values of $\Pr(W_E \leq s)$ by Normal approximation and Edgeworth approximation (near 5% critical values)

| | | | Near 5% critical values | | | | | |
|---|---|---|---|---|---|---|---|---|
| $n_1$ | $n_2$ | | $r=2$ | $r=3$ | $r=4$ | $r=5$ | $r=6$ | $r=7$ |
| 10 | 10 | $s$ | 87 | 85 | 84 | 84 | 83 | 83 |
| | | (a) | 0.049536 | 0.047100 | 0.048713 | 0.052031 | 0.049465 | 0.049205 |
| | | (b) | 0.041706 | 0.043331 | 0.046235 | 0.053103 | 0.049005 | 0.051162 |
| | | (c) | 0.024873 | 0.030409 | 0.036340 | 0.045524 | 0.044137 | 0.048107 |
| 10 | 15 | $s$ | 108 | 106 | 105 | 103 | 102 | 101 |
| | | (a) | 0.047431 | 0.050489 | 0.051937 | 0.050801 | 0.048957 | 0.048970 |
| | | (b) | 0.038802 | 0.043743 | 0.050586 | 0.048014 | 0.048865 | 0.047877 |
| | | (c) | 0.019073 | 0.027924 | 0.037100 | 0.036950 | 0.039815 | 0.040749 |
| 15 | 15 | $s$ | 205 | 201 | 199 | 197 | 195 | 194 |
| | | (a) | 0.047414 | 0.048909 | 0.051009 | 0.051789 | 0.048083 | 0.049019 |
| | | (b) | 0.044765 | 0.045261 | 0.049174 | 0.049611 | 0.047589 | 0.048093 |
| | | (c) | 0.025773 | 0.029633 | 0.036072 | 0.038776 | 0.038910 | 0.041207 |
| 15 | 20 | $s$ | 241 | 236 | 232 | 229 | 227 | 225 |
| | | (a) | 0.050737 | 0.051341 | 0.051610 | 0.050371 | 0.050944 | 0.048776 |
| | | (b) | 0.050175 | 0.050302 | 0.048977 | 0.048250 | 0.048958 | 0.048032 |
| | | (c) | 0.031158 | 0.033337 | 0.034147 | 0.035418 | 0.037876 | 0.038598 |
| 20 | 20 | $s$ | 375 | 367 | 363 | 360 | 357 | 355 |
| | | (a) | 0.053262 | 0.049792 | 0.051100 | 0.050705 | 0.050168 | 0.049410 |
| | | (b) | 0.052148 | 0.046491 | 0.054833 | 0.049223 | 0.048360 | 0.048778 |
| | | (c) | 0.033541 | 0.029514 | 0.033291 | 0.036584 | 0.037519 | 0.039514 |
| 30 | 30 | $s$ | 855 | 850 | 841 | 835 | 830 | 826 |
| | | (a) | 0.051395 | 0.050824 | 0.049718 | 0.049874 | 0.049882 | 0.049852 |
| | | (b) | 0.042261 | 0.049617 | 0.047340 | 0.047751 | 0.048058 | 0.048672 |
| | | (c) | 0.019457 | 0.031558 | 0.031096 | 0.033255 | 0.035061 | 0.036994 |
| 30 | 50 | $s$ | 1143 | 1133 | 1123 | 1115 | 1110 | 1106 |
| | | (a) | 0.048031 | 0.049500 | 0.050398 | 0.049652 | 0.049800 | 0.049955 |
| | | (b) | 0.042559 | 0.062072 | 0.043910 | 0.044327 | 0.054503 | 0.049841 |
| | | (c) | 0.017314 | 0.023519 | 0.025566 | 0.027668 | 0.031732 | 0.035745 |

Remark: (a) - Exact probability; (b) - Edgeworth approximation; (c) - Normal approximation.

Table 3.5c: Exact and approximate values of $\Pr(W_E \leq s)$ by Normal approximation and Edgeworth approximation (near 10% critical values)

| | | | Near 10% critical values | | | | | |
|---|---|---|---|---|---|---|---|---|
| $n_1$ | $n_2$ | | $r=2$ | $r=3$ | $r=4$ | $r=5$ | $r=6$ | $r=7$ |
| 10 | 10 | $s$ | 92 | 90 | 89 | 88 | 88 | 88 |
| | | (a) | 0.102941 | 0.094900 | 0.100435 | 0.093669 | 0.102719 | 0.102719 |
| | | (b) | 0.094876 | 0.095276 | 0.098786 | 0.095984 | 0.102400 | 0.106092 |
| | | (c) | 0.080528 | 0.081645 | 0.087380 | 0.086890 | 0.095409 | 0.100866 |
| 10 | 15 | $s$ | 116 | 112 | 111 | 109 | 109 | 108 |
| | | (a) | 0.111462 | 0.095923 | 0.098707 | 0.096702 | 0.104692 | 0.103481 |
| | | (b) | 0.106748 | 0.091273 | 0.101294 | 0.094900 | 0.105362 | 0.102754 |
| | | (c) | 0.096487 | 0.077225 | 0.088806 | 0.083398 | 0.095664 | 0.094392 |
| 15 | 15 | $s$ | 214 | 209 | 207 | 205 | 204 | 203 |
| | | (a) | 0.100383 | 0.096010 | 0.098905 | 0.098716 | 0.099350 | 0.100871 |
| | | (b) | 0.106643 | 0.093894 | 0.098414 | 0.097286 | 0.100111 | 0.100206 |
| | | (c) | 0.097154 | 0.080839 | 0.086556 | 0.086658 | 0.091016 | 0.092504 |
| 15 | 20 | $s$ | 245 | 245 | 242 | 239 | 237 | 236 |
| | | (a) | 0.093346 | 0.100692 | 0.102157 | 0.097702 | 0.099042 | 0.099956 |
| | | (b) | 0.069277 | 0.101315 | 0.102143 | 0.098106 | 0.097552 | 0.100815 |
| | | (c) | 0.054532 | 0.089931 | 0.090626 | 0.086840 | 0.087227 | 0.091700 |
| 20 | 20 | $s$ | 386 | 378 | 375 | 372 | 370 | 368 |
| | | (a) | 0.106871 | 0.100244 | 0.101000 | 0.099071 | 0.101144 | 0.099089 |
| | | (b) | 0.111721 | 0.093063 | 0.098375 | 0.098121 | 0.099939 | 0.099302 |
| | | (c) | 0.106146 | 0.080856 | 0.086747 | 0.087326 | 0.090219 | 0.090546 |
| 30 | 30 | $s$ | 870 | 866 | 862 | 856 | 852 | 848 |
| | | (a) | 0.097255 | 0.096046 | 0.101405 | 0.099315 | 0.101217 | 0.099133 |
| | | (b) | 0.073423 | 0.091985 | 0.101369 | 0.098786 | 0.099947 | 0.098998 |
| | | (c) | 0.061235 | 0.081149 | 0.091415 | 0.088466 | 0.090191 | 0.089711 |
| 30 | 50 | $s$ | 1168 | 1157 | 1148 | 1140 | 1140 | 1135 |
| | | (a) | 0.104303 | 0.093900 | 0.100452 | 0.099100 | 0.100000 | 0.099313 |
| | | (b) | 0.089095 | 0.093199 | 0.089461 | 0.087548 | 0.101409 | 0.102564 |
| | | (c) | 0.084723 | 0.080588 | 0.078170 | 0.075677 | 0.092896 | 0.093314 |

Remark: (a) - Exact probability; (b) - Edgeworth approximation; (c) - Normal approximation.

Table 3.5d: Exact and approximate values of $\Pr(W_E \leq s)$ by Normal approximation and Edgeworth approximation (near 20% critical values)

| Near 20% critical values | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $n_1$ | $n_2$ | | $r=2$ | $r=3$ | $r=4$ | $r=5$ | $r=6$ | $r=7$ |
| 10 | 10 | $s$ | 97 | 96 | 95 | 94 | 94 | 94 |
| | | (a) | 0.186533 | 0.204000 | 0.205839 | 0.194153 | 0.207517 | 0.207804 |
| | | (b) | 0.196023 | 0.206612 | 0.205778 | 0.198418 | 0.207554 | 0.213165 |
| | | (c) | 0.200197 | 0.206900 | 0.202774 | 0.193366 | 0.202573 | 0.208280 |
| 10 | 15 | $s$ | 123 | 120 | 118 | 117 | 116 | 115 |
| | | (a) | 0.221146 | 0.208696 | 0.203828 | 0.198677 | 0.192586 | 0.191806 |
| | | (b) | 0.239434 | 0.211905 | 0.199996 | 0.200792 | 0.197709 | 0.191890 |
| | | (c) | 0.265392 | 0.219754 | 0.201005 | 0.199605 | 0.194761 | 0.187725 |
| 15 | 15 | $s$ | 222 | 219 | 217 | 215 | 214 | 213 |
| | | (a) | 0.198276 | 0.199471 | 0.197684 | 0.195473 | 0.194378 | 0.195520 |
| | | (b) | 0.216184 | 0.206728 | 0.204052 | 0.196417 | 0.197492 | 0.195565 |
| | | (c) | 0.235431 | 0.214474 | 0.206860 | 0.195634 | 0.195471 | 0.192581 |
| 15 | 20 | $s$ | 255 | 255 | 253 | 251 | 249 | 248 |
| | | (a) | 0.199866 | 0.199866 | 0.196359 | 0.198194 | 0.195026 | 0.194907 |
| | | (b) | 0.164129 | 0.202660 | 0.204548 | 0.201502 | 0.195497 | 0.197267 |
| | | (c) | 0.171471 | 0.213668 | 0.211234 | 0.204628 | 0.195832 | 0.196604 |
| 20 | 20 | $s$ | 397 | 392 | 389 | 387 | 385 | 384 |
| | | (a) | 0.203742 | 0.199638 | 0.201000 | 0.199237 | 0.200565 | 0.202292 |
| | | (b) | 0.225113 | 0.205977 | 0.188184 | 0.203710 | 0.201422 | 0.204948 |
| | | (c) | 0.253521 | 0.218459 | 0.209330 | 0.207695 | 0.203051 | 0.205602 |
| 30 | 30 | $s$ | 897 | 885 | 882 | 880 | 876 | 874 |
| | | (a) | 0.208965 | 0.195753 | 0.196767 | 0.200485 | 0.197974 | 0.199957 |
| | | (b) | 0.233142 | 0.185265 | 0.194637 | 0.203941 | 0.198941 | 0.202228 |
| | | (c) | 0.271799 | 0.197675 | 0.204792 | 0.212926 | 0.204634 | 0.206783 |
| 30 | 50 | $s$ | 1193 | 1183 | 1174 | 1169 | 1173 | 1166 |
| | | (a) | 0.201835 | 0.190000 | 0.199811 | 0.198692 | 0.202000 | 0.202046 |
| | | (b) | 0.221851 | 0.159946 | 0.182579 | 0.182108 | 0.200004 | 0.201484 |
| | | (c) | 0.262622 | 0.221369 | 0.193991 | 0.190442 | 0.230548 | 0.210219 |

Remark: (a) - Exact probability; (b) - Edgeworth approximation; (c) - Normal approximation.

Table 3.5e: Exact and approximate values of $\Pr(W_E \leq s)$ by Normal approximation and Edgeworth approximation (near 30% critical values)

| Near 30% critical values | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $n_1$ | $n_2$ | | $r=2$ | $r=3$ | $r=4$ | $r=5$ | $r=6$ | $r=7$ |
| 10 | 10 | $s$ | 100 | 100 | 99 | 99 | 98 | 98 |
| | | (a) | 0.291022 | 0.296000 | 0.302680 | 0.311319 | 0.302810 | 0.303059 |
| | | (b) | 0.278562 | 0.311518 | 0.304729 | 0.319405 | 0.302805 | 0.309206 |
| | | (c) | 0.306937 | 0.332635 | 0.315066 | 0.325135 | 0.303159 | 0.307511 |
| 10 | 15 | $s$ | 124 | 124 | 124 | 122 | 122 | 121 |
| | | (a) | 0.283399 | 0.301186 | 0.299563 | 0.293916 | 0.299865 | 0.296660 |
| | | (b) | 0.263295 | 0.297340 | 0.320101 | 0.293798 | 0.306827 | 0.296924 |
| | | (c) | 0.297925 | 0.327243 | 0.344284 | 0.306492 | 0.316308 | 0.301684 |
| 15 | 15 | $s$ | 225 | 225 | 224 | 223 | 222 | 221 |
| | | (a) | 0.298851 | 0.308238 | 0.304448 | 0.307235 | 0.302488 | 0.301585 |
| | | (b) | 0.269626 | 0.303871 | 0.309352 | 0.309804 | 0.307028 | 0.301912 |
| | | (c) | 0.306866 | 0.335083 | 0.331444 | 0.324959 | 0.316818 | 0.307562 |
| 15 | 20 | $s$ | 264 | 261 | 260 | 260 | 258 | 258 |
| | | (a) | 0.326872 | 0.295735 | 0.303999 | 0.303519 | 0.298283 | 0.305808 |
| | | (b) | 0.303349 | 0.286659 | 0.295559 | 0.313221 | 0.300037 | 0.311237 |
| | | (c) | 0.359526 | 0.320856 | 0.322097 | 0.335873 | 0.315424 | 0.323947 |
| 20 | 20 | $s$ | 400 | 399 | 398 | 397 | 395 | 395 |
| | | (a) | 0.302495 | 0.296852 | 0.299000 | 0.302298 | 0.294995 | 0.304505 |
| | | (b) | 0.264617 | 0.286542 | 0.283450 | 0.304409 | 0.296576 | 0.307814 |
| | | (c) | 0.307052 | 0.320449 | 0.325027 | 0.325465 | 0.311472 | 0.320148 |
| 30 | 30 | $s$ | 900 | 899 | 897 | 895 | 893 | 892 |
| | | (a) | 0.305977 | 0.302449 | 0.294528 | 0.295582 | 0.297737 | 0.303000 |
| | | (b) | 0.259120 | 0.286145 | 0.294710 | 0.298521 | 0.299527 | 0.305077 |
| | | (c) | 0.307391 | 0.327587 | 0.328510 | 0.326323 | 0.322465 | 0.324827 |
| 30 | 50 | $s$ | 1195 | 1207 | 1197 | 1189 | 1195 | 1190 |
| | | (a) | 0.314143 | 0.305000 | 0.292892 | 0.295196 | 0.301000 | 0.301537 |
| | | (b) | 0.235710 | 0.340043 | 0.308960 | 0.279597 | 0.314861 | 0.312663 |
| | | (c) | 0.282244 | 0.427495 | 0.354566 | 0.311680 | 0.364550 | 0.342010 |

Remark: (a) - Exact probability; (b) - Edgeworth approximation; (c) - Normal approximation.

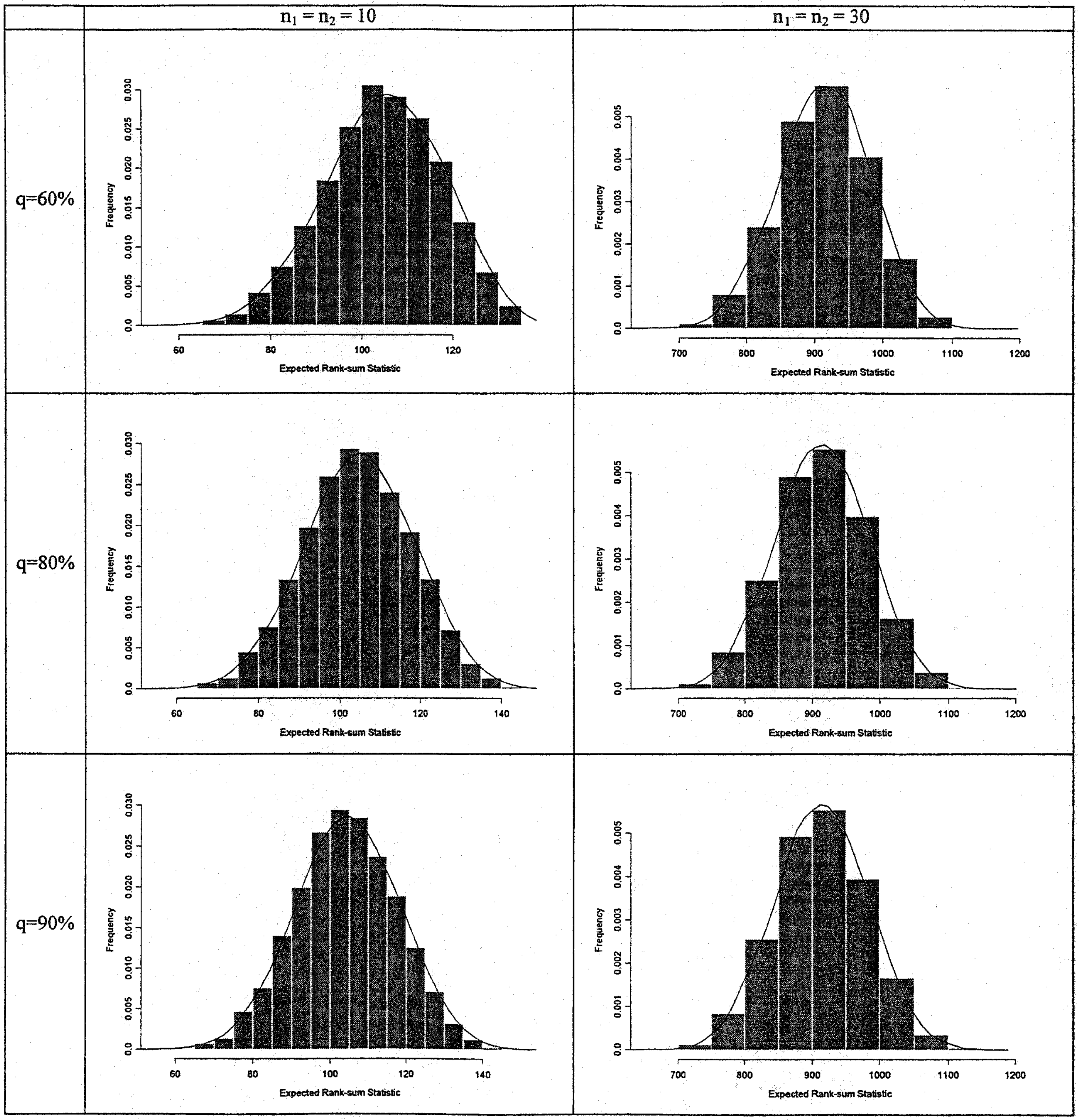



Figure 3.1: Null distribution of expected rank-sum statistic ($W_E$) for $n_1 = n_2 = 10$ and $n_1 = n_2 = 30$ when $q = \frac{r}{n_2}$ equals 60%, 80% and 90%

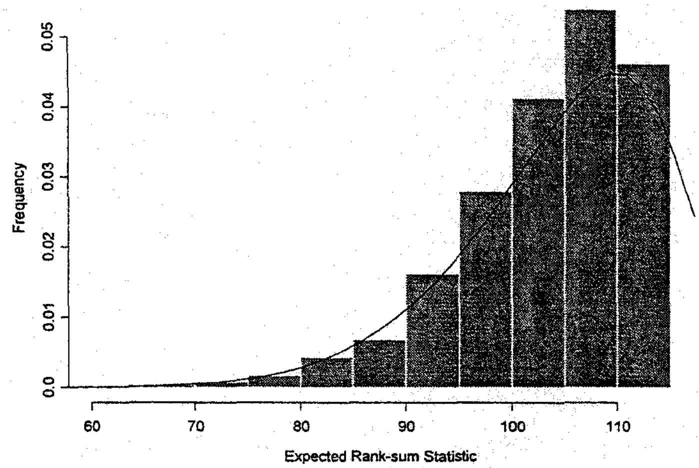
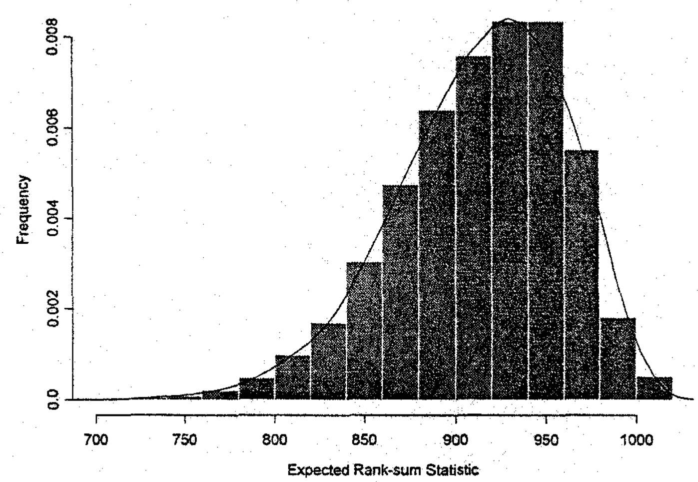
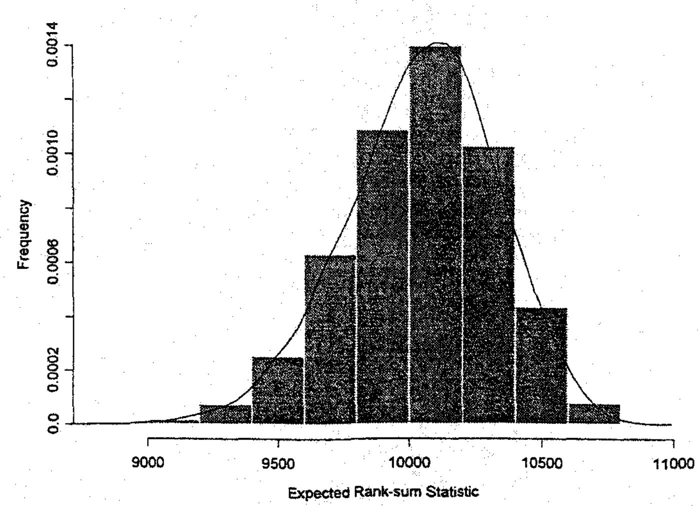
| | |
|---|---|
| $n_1 = n_2 = 10$ |  |
| $n_1 = n_2 = 30$ |  |
| $n_1 = n_2 = 100$ |  |







Figure 3.2: Null distribution of expected rank-sum statistic ($W_E$) for $n_1 = n_2 = 30$ and $n_1 = n_2 = 100$ when $q = \frac{r}{n_2}$ equals 20%

## 3.4 Exact Power Under Lehmann Alternative

In this section, we derive an explicit expression for the power functions of the minimal, maximal and expected rank-sum precedence tests under the Lehmann alternative $H_1 : [F_X]^\gamma = F_Y$ for some $\gamma$.

Earlier in Chapter 2, we presented the joint probability mass function of $M_1, M_2, \ldots, M_r$ under the Lehmann alternative as

$$\begin{aligned}
&\Pr\{M_1 = m_1, M_2 = m_2, \ldots, M_r = m_r \mid [F_X]^\gamma = F_Y\} \\
= &\frac{n_1! n_2! \gamma^r}{m_1!(n_2-r)!} \left\{ \prod_{j=1}^{r-1} \frac{\Gamma\left(\sum_{i=1}^{j} m_i + j\gamma\right)}{\Gamma\left(\sum_{i=1}^{j+1} m_i + j\gamma + 1\right)} \right\} \\
&\times \sum_{k=0}^{n_2-r} \binom{n_2-r}{k} (-1)^k \frac{\Gamma\left(\sum_{i=1}^{r} m_i + (r+k)\gamma\right)}{\Gamma\left(n_1 + (r+k)\gamma + 1\right)}.
\end{aligned}$$

The power of a test is the probability of rejecting the null hypothesis when the alternative hypothesis is indeed true. So under the Lehmann alternative, the power function for the minimal rank-sum precedence test is given by

$$\begin{aligned}
&\Pr\{W_{\min} \le s \mid [F_X]^\gamma = F_Y\} \\
= &\sum_{m_i(i=1,2,\ldots,r)=0}^{n_1} [\Pr\{M_1 = m_1, M_2 = m_2, \ldots, M_r = m_r \mid [F_X]^\gamma = F_Y\} \\
&\times I_{\min,s}(m_1, m_2, \ldots, m_r)]. \qquad (3.4.1)
\end{aligned}$$

Similarly, under the Lehmann alternative, the power function for the maximal and expected rank-sum precedence tests are given by

$$\begin{aligned}
&\Pr\{W_{\max} \le s \mid [F_X]^\gamma = F_Y\} \\
= &\sum_{m_i(i=1,2,\ldots,r)=0}^{n_1} [\Pr\{M_1 = m_1, M_2 = m_2, \ldots, M_r = m_r \mid [F_X]^\gamma = F_Y\} \\
&I_{\max,s}(m_1, m_2, \ldots, m_r)], \qquad (3.4.2)
\end{aligned}$$

and

$$\Pr\{W_E \le s \mid [F_X]^\gamma = F_Y\}$$

$$= \sum_{m_i(i=1,2,\ldots,r)=0}^{n_1} [\Pr\{M_1 = m_1, M_2 = m_2, \ldots, M_r = m_r \mid [F_X]^\gamma = F_Y\}$$

$$I_{E,s}(m_1, m_2, \ldots, m_r)], \tag{3.4.3}$$

respectively.

Now, we demonstrate the use of the formulas in (3.4.1) - (3.4.3) as well as the Monte Carlo simulation method for the computation of the power of the Wilcoxon-type rank-sum precedence tests under the Lehmann alternative. For this purpose, we generated 100,000 sets of data (from $F_X$ and $F_X^\gamma$) and computed the test statistics $W_{\min}$, $W_{\max}$ and $W_E$ for each set. The power values are estimated by the rejection rates of the null hypothesis for different values of $\gamma$. For $n_1 = n_2 = 10$, $r = 2, 3$ and $\gamma = 2(1)6$, the power values computed from (3.4.1) - (3.4.3) and those estimated through the above described Monte Carlo simulation method are presented in Table 3.6. We observe that the estimated values of the power determined from Monte Carlo simulations are quite close to the exact values. In addition to revealing the correctness of the formulas in (3.4.1) - (3.4.3), these results also suggest that the Monte Carlo simulation method provides a feasible and accurate way to estimate the power of the proposed test procedures.

Though the location shift in some cases (like extreme-value distribution) become a Lehmann alternative and hence the exact power can be determined, that is not the case for other distributions such as normal and exponential. Therefore, we examine the power using simulation in the next section.

## 3.5 Monte Carlo Simulation and Power Comparison Under Location-Shift Alternative

In order to assess the power properties of the proposed Wilcoxon-type rank-sum precedence tests, we consider now the location-shift alternative $H_1 : F_X(x) = F_Y(x + \theta)$ for some $\theta > 0$, where $\theta$ is a shift in location. The power of the precedence test with $r = 1(1)7$, maximal precedence test with $r = 2(1)7$, the three Wilcoxon-type rank-sum precedence tests with $r = 2(1)7$, and the classical Wilcoxon's rank-sum test (based on

Table 3.6: Power values under Lehmann alternative for $n_1 = n_2 = 10, r = 2, 3$ and $\gamma = 2(1)6$

| | | Power computed from formulas | | | Simulated Power | | |
|---|---|---|---|---|---|---|---|
| $r$ | $\gamma$ | $W_{\min}$ | $W_{\max}$ | $W_E$ | $W_{\min}$ | $W_{\max}$ | $W_E$ |
| r=2 | 1 | 0.05954 | 0.04954 | 0.04954 | 0.05852 | 0.04894 | 0.04894 |
| | 2 | 0.40396 | 0.35681 | 0.35681 | 0.40406 | 0.35538 | 0.35538 |
| | 3 | 0.69831 | 0.64389 | 0.64389 | 0.69941 | 0.64453 | 0.64453 |
| | 4 | 0.85319 | 0.80970 | 0.80970 | 0.85289 | 0.80999 | 0.80999 |
| | 5 | 0.92692 | 0.89619 | 0.89619 | 0.92651 | 0.89565 | 0.89565 |
| | 6 | 0.96212 | 0.94126 | 0.94126 | 0.96220 | 0.94057 | 0.94057 |
| r=3 | 1 | 0.04639 | 0.05096 | 0.04707 | 0.04515 | 0.04959 | 0.04713 |
| | 2 | 0.36428 | 0.35100 | 0.34624 | 0.36379 | 0.35127 | 0.34852 |
| | 3 | 0.66331 | 0.63139 | 0.62922 | 0.66492 | 0.63201 | 0.63094 |
| | 4 | 0.82952 | 0.79597 | 0.79513 | 0.83027 | 0.79593 | 0.79547 |
| | 5 | 0.91190 | 0.88402 | 0.88368 | 0.91160 | 0.88427 | 0.88415 |
| | 6 | 0.95263 | 0.93137 | 0.93123 | 0.95230 | 0.93096 | 0.93093 |

complete samples) were all estimated through Monte Carlo simulations when $\theta = 0.5$ and $\theta = 1.0$. The lifetime distributions used earlier in Section 2.4 were used here in the Monte Carlo simulations in order to demonstrate the power performance of the Wilcoxon-type rank-sum precedence tests under this location-shift alternative.

In Tables 3.7-3.9, we have presented the estimated power values of the precedence tests with $r = 1(1)7$, maximal precedence tests with $r = 2(1)7$, the three Wilcoxon-type rank-sum precedence tests with $r = 2(1)7$, and the classical Wilcoxon's rank-sum test for the underlying standard normal, standard exponential, standardized gamma and standardized lognormal distributions, with location-shift being equal to 0.5 and 1.0 (for the case $n_1 = n_2 = 10$, we present the estimated power values up to $r = 10$). For comparison purposes, the corresponding critical values and the exact levels of significance are also included.

From Tables 3.7-3.12, we see that the power of all tests increase with increasing sample sizes as well as with increasing location-shift. We can compare the power values of the precedence tests and the maximal precedence tests with the same value of $r$ since they are both test procedures based on failures from the $X$-sample occurring before the $r$-th failure from the $Y$-sample. When we compare the power values of the three Wilcoxon-type rank-sum precedence tests with those of precedence tests and maximal precedence tests, we find that the Wilcoxon-type rank-sum precedence tests perform better than the precedence tests and maximal precedence tests for same value of $r$. For example, in Table 3.12, when $n_1 = n_2 = 30$, the underlying distribution is standard normal and the location-shift equals 0.5, the powers of the minimal, maximal and expected rank-sum precedence tests with $r = 7$ are 0.446, 0.447 and 0.455 (exact levels of significance are 0.052, 0.050 and 0.050), respectively, while the power of precedence test is 0.423 (exact level of significance is 0.047) and the power of maximal precedence test is 0.221 (exact level of significance is 0.037).

From Tables 3.7-3.12, we can also see that when the underlying distributions are nearly symmetric, such as the normal distribution, gamma distribution with large values of shape parameter $a$, and lognormal distribution with small values of shape parameter $\sigma$, the power values of the Wilcoxon-type precedence tests increase with increasing values of $r$. However, under some right-skewed distributions such as the exponential distribution, gamma distribution with shape parameter $a = 2.0$, and lognormal distribution with shape parameter $\sigma = 0.5$, the power values of the Wilcoxon-type precedence tests may decrease with increasing values of $r$.

When we compare the power performance between the three Wilcoxon-type precedence tests, we find that the power values are almost the same for these three tests when the underlying distributions are nearly symmetric. Yet, the minimal rank-sum precedence test compares quite favorably with the maximal and expected rank-sum precedence tests under right-skewed distributions. For example, in Table 3.10, when $n_1 = 15, n_2 = 20$, the underlying distribution is gamma with shape parameter 2.0 and the location-shift equals 0.5, the power of the minimal rank-sum precedence test with $r = 4$ is 0.686 (exact level of significance is 0.050) while the power values of maximal and expected rank-sum precedence tests are 0.557 and 0.603 (exact level of significance is 0.050 and 0.052), respectively.

So, our overall recommendation will be to use the minimal rank-sum precedence test in general for testing $H_0 : F_X = F_Y$ against $H_1 : F_X > F_Y$. This test has better performance, in terms of power, than the precedence and maximal precedence tests, and also in comparison with the maximal and expected rank-sum precedence tests.

Table 3.7: Power of precedence tests, maximal precedence tests, minimal, maximal and expected rank-sum precedence tests and Wilcoxon's rank-sum test with $n_1 = n_2 = 10$

| Test | $r$ | $s$ | $\alpha$ | $\theta$ | Normal | Exp(1) | Gamma(2) | Gamma(5) | Gamma(10) | LN(0.25) | LN(0.5) |
|---|---|---|---|---|---|---|---|---|---|---|---|
| (i) | 1 | 4 | 0.04334 | 0.5 | 0.17355 | 0.71973 | 0.41408 | 0.26426 | 0.22491 | 0.23848 | 0.39436 |
| | | | | 1.0 | 0.42078 | 0.97785 | 0.88064 | 0.67354 | 0.57424 | 0.60775 | 0.87090 |
| | 2 | 6 | 0.02864 | 0.5 | 0.14958 | 0.39156 | 0.25252 | 0.19241 | 0.17311 | 0.18522 | 0.26933 |
| | | | | 1.0 | 0.42430 | 0.83764 | 0.68794 | 0.55814 | 0.50597 | 0.53318 | 0.72532 |
| | 3 | 7 | 0.03489 | 0.5 | 0.17793 | 0.30731 | 0.23431 | 0.20245 | 0.18929 | 0.19804 | 0.25708 |
| | | | | 1.0 | 0.48132 | 0.72393 | 0.61482 | 0.54595 | 0.51771 | 0.53939 | 0.66761 |
| | 4 | 8 | 0.03483 | 0.5 | 0.17708 | 0.22511 | 0.19357 | 0.18073 | 0.17548 | 0.18323 | 0.21493 |
| | | | | 1.0 | 0.47989 | 0.57667 | 0.51144 | 0.48487 | 0.47581 | 0.49217 | 0.56561 |
| | 5 | 9 | 0.02864 | 0.5 | 0.14725 | 0.14531 | 0.13799 | 0.13726 | 0.13884 | 0.14163 | 0.15265 |
| | | | | 1.0 | 0.42447 | 0.39603 | 0.37249 | 0.37815 | 0.38331 | 0.39065 | 0.41444 |
| | 6 | 9 | 0.07043 | 0.5 | 0.26586 | 0.23847 | 0.23441 | 0.24132 | 0.24361 | 0.24790 | 0.25305 |
| | | | | 1.0 | 0.58320 | 0.49909 | 0.49543 | 0.51311 | 0.52664 | 0.52909 | 0.53141 |
| | 7 | 9 | 0.13545 | 0.5 | 0.42308 | 0.36325 | 0.36545 | 0.38085 | 0.38607 | 0.38830 | 0.38359 |
| | | | | 1.0 | 0.73305 | 0.61390 | 0.62361 | 0.64824 | 0.66820 | 0.66626 | 0.64903 |
| | 8 | 9 | 0.24768 | 0.5 | 0.60199 | 0.51741 | 0.52933 | 0.54896 | 0.55762 | 0.55841 | 0.54376 |
| | | | | 1.0 | 0.85312 | 0.73126 | 0.74884 | 0.77783 | 0.79587 | 0.79120 | 0.76368 |
| | 9 | 9 | 0.39474 | 0.5 | 0.77887 | 0.69617 | 0.70765 | 0.72728 | 0.73847 | 0.73852 | 0.71894 |
| | | | | 1.0 | 0.93564 | 0.84599 | 0.86121 | 0.88478 | 0.89805 | 0.89279 | 0.86626 |
| | 10 | 9 | 0.52632 | 0.5 | 0.92282 | 0.87246 | 0.88038 | 0.89239 | 0.89989 | 0.89630 | 0.88168 |
| | | | | 1.0 | 0.98338 | 0.94081 | 0.94853 | 0.96066 | 0.96735 | 0.96452 | 0.94961 |
| (ii) | 2 | 5 | 0.03250 | 0.5 | 0.14174 | 0.49229 | 0.24736 | 0.17686 | 0.16108 | 0.17004 | 0.25426 |
| | | | | 1.0 | 0.37675 | 0.91770 | 0.73033 | 0.52376 | 0.45692 | 0.48601 | 0.73270 |
| | 3 | 5 | 0.04875 | 0.5 | 0.17526 | 0.49485 | 0.25785 | 0.19838 | 0.18700 | 0.19469 | 0.26781 |
| | | | | 1.0 | 0.41709 | 0.91784 | 0.73274 | 0.53569 | 0.47649 | 0.50380 | 0.73601 |
| | 4 | 5 | 0.06500 | 0.5 | 0.19933 | 0.49736 | 0.26527 | 0.21143 | 0.20397 | 0.21137 | 0.27682 |
| | | | | 1.0 | 0.43592 | 0.91810 | 0.73435 | 0.54141 | 0.48526 | 0.51146 | 0.73755 |
| | 5 | 6 | 0.02700 | 0.5 | 0.09008 | 0.26844 | 0.11968 | 0.09218 | 0.08952 | 0.09463 | 0.12970 |
| | | | | 1.0 | 0.24818 | 0.77912 | 0.51361 | 0.32886 | 0.28329 | 0.30676 | 0.52844 |
| | 6 | 6 | 0.03251 | 0.5 | 0.09378 | 0.26897 | 0.12081 | 0.09436 | 0.09210 | 0.09703 | 0.13085 |
| | | | | 1.0 | 0.24957 | 0.77914 | 0.51379 | 0.32931 | 0.28385 | 0.30725 | 0.52857 |
| | 7 | 6 | 0.03793 | 0.5 | 0.09648 | 0.26951 | 0.12190 | 0.09564 | 0.09399 | 0.09849 | 0.13153 |
| | | | | 1.0 | 0.25023 | 0.77920 | 0.51392 | 0.32961 | 0.28422 | 0.30743 | 0.52860 |
| | 8 | 6 | 0.04334 | 0.5 | 0.09819 | 0.27001 | 0.12278 | 0.09683 | 0.09520 | 0.09984 | 0.13222 |
| | | | | 1.0 | 0.25039 | 0.77923 | 0.51397 | 0.32971 | 0.28438 | 0.30751 | 0.52861 |
| | 9 | 6 | 0.04876 | 0.5 | 0.09921 | 0.27052 | 0.12341 | 0.09764 | 0.09614 | 0.10080 | 0.13276 |
| | | | | 1.0 | 0.25050 | 0.77928 | 0.51400 | 0.32983 | 0.28446 | 0.30761 | 0.52869 |
| | 10 | 6 | 0.05423 | 0.5 | 0.09986 | 0.27107 | 0.12408 | 0.09834 | 0.09676 | 0.10134 | 0.13313 |
| | | | | 1.0 | 0.25055 | 0.77930 | 0.51406 | 0.32991 | 0.28448 | 0.30763 | 0.52872 |
| (iii) | 2 | 67 | 0.05954 | 0.5 | 0.23942 | 0.75179 | 0.49375 | 0.34262 | 0.29921 | 0.31477 | 0.48316 |
| | | | | 1.0 | 0.55242 | 0.98116 | 0.91494 | 0.77416 | 0.69533 | 0.72338 | 0.91794 |
| | 3 | 71 | 0.04639 | 0.5 | 0.22288 | 0.62926 | 0.41788 | 0.30283 | 0.26819 | 0.28257 | 0.42193 |
| | | | | 1.0 | 0.55802 | 0.94955 | 0.86529 | 0.73787 | 0.67326 | 0.70033 | 0.88107 |
| | 4 | 75 | 0.05377 | 0.5 | 0.25717 | 0.62267 | 0.43420 | 0.33085 | 0.29946 | 0.31335 | 0.44410 |
| | | | | 1.0 | 0.61907 | 0.94681 | 0.86784 | 0.76259 | 0.71060 | 0.73585 | 0.88638 |
| | 5 | 77 | 0.04462 | 0.5 | 0.23764 | 0.52904 | 0.37798 | 0.29481 | 0.26907 | 0.28236 | 0.39265 |
| | | | | 1.0 | 0.60701 | 0.90048 | 0.81846 | 0.72485 | 0.67932 | 0.70318 | 0.84718 |
| | 6 | 80 | 0.05468 | 0.5 | 0.27557 | 0.54185 | 0.40483 | 0.32899 | 0.30405 | 0.31749 | 0.42206 |
| | | | | 1.0 | 0.66094 | 0.90138 | 0.83001 | 0.75389 | 0.71749 | 0.73689 | 0.85762 |
| | 7 | 81 | 0.05008 | 0.5 | 0.26530 | 0.49483 | 0.37369 | 0.30976 | 0.28812 | 0.30080 | 0.39274 |
| | | | | 1.0 | 0.65536 | 0.86798 | 0.79851 | 0.73116 | 0.69950 | 0.71745 | 0.82970 |
| | 8 | 82 | 0.05056 | 0.5 | 0.27085 | 0.47701 | 0.36709 | 0.30815 | 0.28815 | 0.30097 | 0.38469 |
| | | | | 1.0 | 0.66483 | 0.85289 | 0.78641 | 0.72710 | 0.69840 | 0.71506 | 0.81753 |
| | 9 | 82 | 0.04624 | 0.5 | 0.25841 | 0.44688 | 0.34403 | 0.29076 | 0.27256 | 0.28482 | 0.36066 |
| | | | | 1.0 | 0.65229 | 0.83064 | 0.76266 | 0.70628 | 0.67944 | 0.69529 | 0.79418 |
| | 10 | 82 | 0.04460† | 0.5 | 0.25363 | 0.43322 | 0.33441 | 0.28330 | 0.26633 | 0.27797 | 0.35084 |
| | | | | 1.0 | 0.64799 | 0.81627 | 0.75081 | 0.69701 | 0.67134 | 0.68684 | 0.78191 |

| Test | $r$ | $s$ | $\alpha$ | $\theta$ | Normal | Exp(1) | Gamma(2) | Gamma(5) | Gamma(10) | LN(0.25) | LN(0.5) |
|---|---|---|---|---|---|---|---|---|---|---|---|
| (iv) | 2 | 107 | 0.04954 | 0.5 | 0.21864 | 0.61386 | 0.40659 | 0.29714 | 0.26448 | 0.27814 | 0.41342 |
| | | | | 1.0 | 0.53289 | 0.94466 | 0.85125 | 0.71578 | 0.64977 | 0.67657 | 0.86760 |
| | 3 | 99 | 0.05096 | 0.5 | 0.23908 | 0.50740 | 0.36818 | 0.29194 | 0.26866 | 0.28068 | 0.38523 |
| | | | | 1.0 | 0.58076 | 0.88496 | 0.79624 | 0.69985 | 0.65486 | 0.67947 | 0.82888 |
| | 4 | 93 | 0.05065 | 0.5 | 0.24765 | 0.41622 | 0.33127 | 0.28210 | 0.26392 | 0.27619 | 0.35310 |
| | | | | 1.0 | 0.60476 | 0.79500 | 0.72522 | 0.66992 | 0.64209 | 0.66268 | 0.76894 |
| | 5 | 89 | 0.05108 | 0.5 | 0.25641 | 0.42763 | 0.32935 | 0.28533 | 0.26937 | 0.28094 | 0.35188 |
| | | | | 1.0 | 0.62957 | 0.81969 | 0.73843 | 0.67789 | 0.65597 | 0.67121 | 0.77665 |
| | 6 | 86 | 0.04900 | 0.5 | 0.25880 | 0.40750 | 0.32376 | 0.28292 | 0.26830 | 0.27868 | 0.34450 |
| | | | | 1.0 | 0.64022 | 0.79303 | 0.72400 | 0.67553 | 0.65529 | 0.67066 | 0.75990 |
| | 7 | 84 | 0.04662 | 0.5 | 0.25600 | 0.40644 | 0.32386 | 0.28095 | 0.26517 | 0.27624 | 0.34126 |
| | | | | 1.0 | 0.64532 | 0.77765 | 0.72302 | 0.67849 | 0.65873 | 0.67264 | 0.75783 |
| | 8 | 83 | 0.04650 | 0.5 | 0.25950 | 0.42334 | 0.33213 | 0.28493 | 0.26965 | 0.27980 | 0.34871 |
| | | | | 1.0 | 0.65223 | 0.80088 | 0.73981 | 0.69251 | 0.66899 | 0.68384 | 0.77149 |
| | 9 | 83 | 0.05064 | 0.5 | 0.27432 | 0.45219 | 0.35389 | 0.30321 | 0.28627 | 0.29770 | 0.37101 |
| | | | | 1.0 | 0.67193 | 0.82583 | 0.76506 | 0.71584 | 0.69200 | 0.70669 | 0.79468 |
| | 10 | 82 | 0.04460† | 0.5 | 0.25363 | 0.43322 | 0.33441 | 0.28330 | 0.26633 | 0.27797 | 0.35084 |
| | | | | 1.0 | 0.64799 | 0.81627 | 0.75081 | 0.69701 | 0.67134 | 0.68684 | 0.78191 |
| (v) | 2 | 87 | 0.04954 | 0.5 | 0.21864 | 0.61386 | 0.40659 | 0.29714 | 0.26448 | 0.27814 | 0.41342 |
| | | | | 1.0 | 0.53289 | 0.94466 | 0.85125 | 0.71578 | 0.64977 | 0.67657 | 0.86760 |
| | 3 | 85 | 0.04707 | 0.5 | 0.23359 | 0.50733 | 0.36738 | 0.28915 | 0.26483 | 0.27695 | 0.38366 |
| | | | | 1.0 | 0.57611 | 0.88496 | 0.79621 | 0.69928 | 0.65331 | 0.67820 | 0.82879 |
| | 4 | 84 | 0.04871 | 0.5 | 0.24586 | 0.48267 | 0.35581 | 0.29056 | 0.26953 | 0.28239 | 0.37451 |
| | | | | 1.0 | 0.60944 | 0.87470 | 0.78214 | 0.69957 | 0.66307 | 0.68524 | 0.81560 |
| | 5 | 84 | 0.05203 | 0.5 | 0.27860 | 0.48460 | 0.38023 | 0.32038 | 0.29960 | 0.31251 | 0.40119 |
| | | | | 1.0 | 0.65965 | 0.84538 | 0.78590 | 0.72757 | 0.70054 | 0.71729 | 0.82098 |
| | 6 | 83 | 0.04947 | 0.5 | 0.26307 | 0.46053 | 0.35292 | 0.29818 | 0.27934 | 0.29154 | 0.37322 |
| | | | | 1.0 | 0.64946 | 0.84060 | 0.77051 | 0.70977 | 0.68180 | 0.69872 | 0.80446 |
| | 7 | 83 | 0.04921 | 0.5 | 0.27844 | 0.46934 | 0.36525 | 0.31221 | 0.29334 | 0.30601 | 0.38478 |
| | | | | 1.0 | 0.67160 | 0.84411 | 0.77934 | 0.72432 | 0.69885 | 0.71501 | 0.81044 |
| | 8 | 83 | 0.05207 | 0.5 | 0.27729 | 0.46363 | 0.36281 | 0.30971 | 0.29138 | 0.30310 | 0.37985 |
| | | | | 1.0 | 0.67320 | 0.83989 | 0.77557 | 0.72302 | 0.69748 | 0.71369 | 0.80650 |
| | 9 | 83 | 0.05109 | 0.5 | 0.28397 | 0.47218 | 0.36976 | 0.31706 | 0.29789 | 0.31036 | 0.38830 |
| | | | | 1.0 | 0.68197 | 0.84352 | 0.78173 | 0.73106 | 0.70622 | 0.72115 | 0.81153 |
| | 10 | 82 | 0.04460† | 0.5 | 0.25363 | 0.43322 | 0.33441 | 0.28330 | 0.26633 | 0.27797 | 0.35084 |
| | | | | 1.0 | 0.64799 | 0.81627 | 0.75081 | 0.69701 | 0.67134 | 0.68684 | 0.78191 |
| (vi) | | 82 | 0.04460† | 0.5 | 0.25363 | 0.43322 | 0.33441 | 0.28330 | 0.26633 | 0.27797 | 0.35084 |
| | | | | 1.0 | 0.64799 | 0.81627 | 0.75081 | 0.69701 | 0.67134 | 0.68684 | 0.78191 |

Remarks: (i) - Precedence Test; (ii) - Maximal Precedence Test; (iii) - Minimal Rank-sum Precedence Test; (iv) - Maximal Rank-sum Precedence Test; (v) - Expected Rank-sum Precedence Test; (vi) - Wilcoxon's Rank-sum Test,
$s$ - critical value, $\alpha$ - exact level of significance, $\theta$ - location-shift

† - When $r = n_2 = 10$, $W_{\min} = W_{\max} = W_E$ and they are all equivalent to the Wilcoxon's rank sum test statistic. The level of significances and the power values are the same in this case.

Table 3.8: Power of precedence tests, maximal precedence tests, minimal, maximal and expected rank-sum precedence tests and Wilcoxon's rank-sum test with $n_1 = 10, n_2 = 15$

| Test | $r$ | $s$ | $\alpha$ | $\theta$ | Normal | Exp(1) | Gamma(2) | Gamma(5) | Gamma(10) | LN(0.25) | LN(0.5) |
|---|---|---|---|---|---|---|---|---|---|---|---|
| (i) | 1 | 3 | 0.05217 | 0.5 | 0.19712 | 0.87328 | 0.57176 | 0.33812 | 0.27897 | 0.29466 | 0.51455 |
| | | | | 1.0 | 0.46480 | 0.99524 | 0.95367 | 0.78083 | 0.66569 | 0.69616 | 0.93954 |
| | 2 | 4 | 0.06403 | 0.5 | 0.26069 | 0.76379 | 0.52528 | 0.37730 | 0.33366 | 0.34764 | 0.52427 |
| | | | | 1.0 | 0.59416 | 0.98208 | 0.92419 | 0.80314 | 0.73197 | 0.75896 | 0.93088 |
| | 3 | 5 | 0.06185 | 0.5 | 0.27407 | 0.63397 | 0.45446 | 0.35675 | 0.32678 | 0.34122 | 0.47751 |
| | | | | 1.0 | 0.63349 | 0.94802 | 0.87349 | 0.77462 | 0.72702 | 0.75019 | 0.89333 |
| | 4 | 6 | 0.05327 | 0.5 | 0.25742 | 0.49333 | 0.37266 | 0.31205 | 0.29322 | 0.30474 | 0.40369 |
| | | | | 1.0 | 0.62596 | 0.87788 | 0.79245 | 0.71811 | 0.68466 | 0.70580 | 0.83081 |
| | 5 | 7 | 0.04158 | 0.5 | 0.22462 | 0.35667 | 0.28524 | 0.25207 | 0.24153 | 0.24962 | 0.31365 |
| | | | | 1.0 | 0.58636 | 0.75865 | 0.68107 | 0.63326 | 0.61399 | 0.62941 | 0.73112 |
| | 6 | 8 | 0.02861 | 0.5 | 0.17503 | 0.23232 | 0.19446 | 0.18170 | 0.17833 | 0.18244 | 0.21891 |
| | | | | 1.0 | 0.50964 | 0.58541 | 0.53156 | 0.50855 | 0.50514 | 0.51525 | 0.58444 |
| | 7 | 8 | 0.05727 | 0.5 | 0.27041 | 0.30909 | 0.27912 | 0.27008 | 0.26864 | 0.27224 | 0.30518 |
| | | | | 1.0 | 0.63565 | 0.65240 | 0.61900 | 0.61345 | 0.61475 | 0.62068 | 0.66583 |
| (ii) | 2 | 4 | 0.03316 | 0.5 | 0.15295 | 0.69192 | 0.36049 | 0.22587 | 0.19675 | 0.20721 | 0.34635 |
| | | | | 1.0 | 0.42252 | 0.97482 | 0.86261 | 0.64247 | 0.55219 | 0.58499 | 0.85073 |
| | 3 | 4 | 0.04968 | 0.5 | 0.19467 | 0.69459 | 0.37434 | 0.25466 | 0.23118 | 0.24214 | 0.36758 |
| | | | | 1.0 | 0.48194 | 0.97497 | 0.86588 | 0.66203 | 0.58416 | 0.61467 | 0.85587 |
| | 4 | 4 | 0.06615 | 0.5 | 0.22612 | 0.69714 | 0.38539 | 0.27538 | 0.25568 | 0.26619 | 0.38050 |
| | | | | 1.0 | 0.51596 | 0.97509 | 0.86789 | 0.67291 | 0.60049 | 0.62967 | 0.85837 |
| | 5 | 5 | 0.02371 | 0.5 | 0.09903 | 0.45191 | 0.18095 | 0.11995 | 0.10989 | 0.11562 | 0.18108 |
| | | | | 1.0 | 0.29035 | 0.90776 | 0.68311 | 0.43025 | 0.35977 | 0.38368 | 0.67065 |
| | 6 | 5 | 0.02845 | 0.5 | 0.10515 | 0.45248 | 0.18263 | 0.12333 | 0.11415 | 0.11965 | 0.18295 |
| | | | | 1.0 | 0.29478 | 0.90785 | 0.68336 | 0.43145 | 0.36168 | 0.38521 | 0.67090 |
| | 7 | 5 | 0.03320 | 0.5 | 0.11019 | 0.45314 | 0.18407 | 0.12615 | 0.11738 | 0.12261 | 0.18449 |
| | | | | 1.0 | 0.29732 | 0.90794 | 0.68364 | 0.43224 | 0.36301 | 0.38603 | 0.67113 |
| (iii) | 2 | 68 | 0.03316 | 0.5 | 0.17258 | 0.75476 | 0.47238 | 0.28885 | 0.24050 | 0.25464 | 0.44897 |
| | | | | 1.0 | 0.48197 | 0.98174 | 0.91591 | 0.75601 | 0.65882 | 0.68965 | 0.91761 |
| | 3 | 74 | 0.04452 | 0.5 | 0.22457 | 0.77202 | 0.52036 | 0.34847 | 0.29931 | 0.31468 | 0.50833 |
| | | | | 1.0 | 0.58015 | 0.98346 | 0.93114 | 0.80993 | 0.73106 | 0.75992 | 0.93635 |
| | 4 | 79 | 0.04727 | 0.5 | 0.24651 | 0.75878 | 0.51749 | 0.36307 | 0.31739 | 0.33307 | 0.51457 |
| | | | | 1.0 | 0.62674 | 0.98167 | 0.92722 | 0.82046 | 0.75432 | 0.78098 | 0.93482 |
| | 5 | 84 | 0.05554 | 0.5 | 0.28485 | 0.72092 | 0.52594 | 0.39376 | 0.35218 | 0.36865 | 0.53499 |
| | | | | 1.0 | 0.68328 | 0.96853 | 0.92159 | 0.83810 | 0.78649 | 0.80999 | 0.93467 |
| | 6 | 87 | 0.04840 | 0.5 | 0.27134 | 0.67670 | 0.48492 | 0.36620 | 0.33001 | 0.34367 | 0.49876 |
| | | | | 1.0 | 0.67934 | 0.95867 | 0.90213 | 0.81814 | 0.76999 | 0.79314 | 0.91918 |
| | 7 | 90 | 0.04794 | 0.5 | 0.27667 | 0.64783 | 0.46738 | 0.36189 | 0.32937 | 0.34222 | 0.48416 |
| | | | | 1.0 | 0.69276 | 0.95145 | 0.88969 | 0.81182 | 0.76966 | 0.79098 | 0.90894 |
| (iv) | 2 | 147 | 0.04743 | 0.5 | 0.22052 | 0.76278 | 0.51160 | 0.34271 | 0.29439 | 0.30920 | 0.49963 |
| | | | | 1.0 | 0.55191 | 0.98204 | 0.92325 | 0.79125 | 0.70872 | 0.73779 | 0.92864 |
| | 3 | 136 | 0.05049 | 0.5 | 0.24876 | 0.63347 | 0.44856 | 0.34054 | 0.30588 | 0.32052 | 0.46750 |
| | | | | 1.0 | 0.61105 | 0.94800 | 0.87319 | 0.76987 | 0.71819 | 0.74192 | 0.89271 |
| | 4 | 109 | 0.05035 | 0.5 | 0.25240 | 0.49328 | 0.37192 | 0.30930 | 0.28999 | 0.30132 | 0.40257 |
| | | | | 1.0 | 0.62314 | 0.87788 | 0.79244 | 0.71773 | 0.68385 | 0.70506 | 0.83079 |
| | 5 | 122 | 0.05134 | 0.5 | 0.27163 | 0.55230 | 0.41431 | 0.33315 | 0.31036 | 0.32282 | 0.43772 |
| | | | | 1.0 | 0.66561 | 0.90396 | 0.83769 | 0.76415 | 0.72805 | 0.74897 | 0.86830 |
| | 6 | 116 | 0.05148 | 0.5 | 0.27844 | 0.48610 | 0.37775 | 0.32441 | 0.30703 | 0.31678 | 0.40462 |
| | | | | 1.0 | 0.67769 | 0.87089 | 0.79086 | 0.73542 | 0.71425 | 0.72842 | 0.82587 |
| | 7 | 111 | 0.04919 | 0.5 | 0.27954 | 0.47535 | 0.37613 | 0.32015 | 0.30327 | 0.31222 | 0.39955 |
| | | | | 1.0 | 0.68871 | 0.84081 | 0.78889 | 0.74024 | 0.71832 | 0.73148 | 0.82441 |
| (v) | 2 | 108 | 0.04743 | 0.5 | 0.22052 | 0.76278 | 0.51160 | 0.34271 | 0.29439 | 0.30920 | 0.49963 |
| | | | | 1.0 | 0.55191 | 0.98204 | 0.92325 | 0.79125 | 0.70872 | 0.73779 | 0.92864 |
| | 3 | 106 | 0.05049 | 0.5 | 0.24876 | 0.63347 | 0.44856 | 0.34054 | 0.30588 | 0.32052 | 0.46750 |
| | | | | 1.0 | 0.61105 | 0.94800 | 0.87319 | 0.76987 | 0.71819 | 0.74192 | 0.89271 |
| | 4 | 105 | 0.05194 | 0.5 | 0.27863 | 0.67346 | 0.47908 | 0.36642 | 0.33454 | 0.34796 | 0.49356 |
| | | | | 1.0 | 0.66594 | 0.95871 | 0.89594 | 0.80260 | 0.75425 | 0.77758 | 0.91337 |
| | 5 | 103 | 0.05080 | 0.5 | 0.27487 | 0.56455 | 0.43060 | 0.34491 | 0.31919 | 0.33235 | 0.45415 |
| | | | | 1.0 | 0.67372 | 0.90624 | 0.84490 | 0.77711 | 0.74010 | 0.76193 | 0.87477 |
| | 6 | 102 | 0.04896 | 0.5 | 0.28794 | 0.57176 | 0.42875 | 0.35030 | 0.32738 | 0.33826 | 0.45150 |
| | | | | 1.0 | 0.69595 | 0.91616 | 0.85061 | 0.78256 | 0.75152 | 0.76827 | 0.87726 |
| | 7 | 101 | 0.04897 | 0.5 | 0.28496 | 0.53943 | 0.41136 | 0.34022 | 0.31777 | 0.32842 | 0.43439 |
| | | | | 1.0 | 0.70153 | 0.89715 | 0.83313 | 0.77338 | 0.74527 | 0.76006 | 0.86199 |
| (vi) | | 99 | 0.04550 | 0.5 | 0.29337 | 0.50201 | 0.39241 | 0.33335 | 0.31417 | 0.32272 | 0.41034 |
| | | | | 1.0 | 0.73058 | 0.86374 | 0.81379 | 0.76922 | 0.75008 | 0.76170 | 0.83985 |

Remarks: (i) - Precedence Test; (ii) - Maximal Precedence Test; (iii) - Minimal Rank-sum Precedence Test; (iv) - Maximal Rank-sum Precedence Test; (v) - Expected Rank-sum Precedence Test; (vi) - Wilcoxon's Rank-sum Test,

$s$ - critical value, $\alpha$ - exact level of significance, $\theta$ - location-shift

Table 3.9: Power of precedence tests, maximal precedence tests, minimal, maximal and expected rank-sum precedence tests and Wilcoxon's rank-sum test with $n_1 = n_2 = 15$

| Test | $r$ | $s$ | $\alpha$ | $\theta$ | Normal | Exp(1) | Gamma(2) | Gamma(5) | Gamma(10) | LN(0.25) | LN(0.5) |
|---|---|---|---|---|---|---|---|---|---|---|---|
| (i) | 1 | 4 | 0.04981 | 0.5 | 0.21166 | 0.93598 | 0.64016 | 0.37132 | 0.29851 | 0.31908 | 0.56758 |
| | | | | 1.0 | 0.50252 | 0.99955 | 0.98405 | 0.83848 | 0.72008 | 0.74977 | 0.97286 |
| | 2 | 6 | 0.04004 | 0.5 | 0.22061 | 0.76824 | 0.49059 | 0.32997 | 0.28478 | 0.30350 | 0.49083 |
| | | | | 1.0 | 0.57185 | 0.99273 | 0.94228 | 0.80772 | 0.72752 | 0.75944 | 0.94680 |
| | 3 | 7 | 0.05432 | 0.5 | 0.28472 | 0.69628 | 0.49085 | 0.37752 | 0.34087 | 0.35968 | 0.51584 |
| | | | | 1.0 | 0.67187 | 0.98192 | 0.92751 | 0.83281 | 0.77819 | 0.80630 | 0.94209 |
| | 4 | 9 | 0.03022 | 0.5 | 0.21126 | 0.44001 | 0.31724 | 0.25579 | 0.23896 | 0.25202 | 0.34941 |
| | | | | 1.0 | 0.60134 | 0.89959 | 0.79981 | 0.70956 | 0.66764 | 0.69747 | 0.84304 |
| | 5 | 10 | 0.03280 | 0.5 | 0.22635 | 0.36866 | 0.29113 | 0.25256 | 0.24100 | 0.25068 | 0.32262 |
| | | | | 1.0 | 0.62508 | 0.83183 | 0.74790 | 0.68407 | 0.66128 | 0.68471 | 0.79751 |
| | 6 | 11 | 0.03280 | 0.5 | 0.22719 | 0.30143 | 0.25289 | 0.23457 | 0.22839 | 0.23544 | 0.28335 |
| | | | | 1.0 | 0.62484 | 0.73930 | 0.67456 | 0.63918 | 0.63055 | 0.64836 | 0.73122 |
| | 7 | 11 | 0.06971 | 0.5 | 0.35144 | 0.40243 | 0.36276 | 0.34668 | 0.34421 | 0.35078 | 0.39436 |
| | | | | 1.0 | 0.75253 | 0.80155 | 0.76281 | 0.74719 | 0.74472 | 0.75525 | 0.80731 |
| (ii) | 2 | 5 | 0.04205 | 0.5 | 0.20162 | 0.84028 | 0.47509 | 0.29568 | 0.31930 | 0.27254 | 0.44924 |
| | | | | 1.0 | 0.51762 | 0.99729 | 0.95219 | 0.76861 | 0.76648 | 0.70499 | 0.93968 |
| | 3 | 5 | 0.06292 | 0.5 | 0.25626 | 0.84211 | 0.49078 | 0.33221 | 0.33340 | 0.31613 | 0.47444 |
| | | | | 1.0 | 0.58639 | 0.99732 | 0.95378 | 0.78817 | 0.77670 | 0.73674 | 0.94336 |
| | 4 | 6 | 0.03400 | 0.5 | 0.15784 | 0.68508 | 0.30398 | 0.19437 | 0.34526 | 0.19076 | 0.30140 |
| | | | | 1.0 | 0.43474 | 0.98852 | 0.87859 | 0.62283 | 0.78889 | 0.56833 | 0.85843 |
| | 5 | 6 | 0.04209 | 0.5 | 0.17235 | 0.68581 | 0.30797 | 0.20220 | 0.34436 | 0.20037 | 0.30553 |
| | | | | 1.0 | 0.44805 | 0.98857 | 0.87898 | 0.62551 | 0.79358 | 0.57248 | 0.85892 |
| | 6 | 6 | 0.05050 | 0.5 | 0.18348 | 0.68654 | 0.31089 | 0.20820 | 0.34596 | 0.20713 | 0.30876 |
| | | | | 1.0 | 0.45504 | 0.98857 | 0.87921 | 0.62745 | 0.79972 | 0.57515 | 0.85920 |
| | 7 | 6 | 0.05889 | 0.5 | 0.19223 | 0.68718 | 0.31355 | 0.21252 | 0.34827 | 0.21236 | 0.31145 |
| | | | | 1.0 | 0.45931 | 0.98859 | 0.87949 | 0.62837 | 0.80394 | 0.57640 | 0.85936 |
| (iii) | 2 | 141 | 0.04205 | 0.5 | 0.22459 | 0.88518 | 0.60499 | 0.37449 | 0.25543 | 0.32929 | 0.57076 |
| | | | | 1.0 | 0.57820 | 0.99832 | 0.97677 | 0.86393 | 0.67058 | 0.80054 | 0.97474 |
| | 3 | 150 | 0.04654 | 0.5 | 0.25997 | 0.87742 | 0.61237 | 0.40738 | 0.29928 | 0.36651 | 0.59499 |
| | | | | 1.0 | 0.65183 | 0.99821 | 0.97668 | 0.88705 | 0.70524 | 0.84036 | 0.97793 |
| | 4 | 158 | 0.04826 | 0.5 | 0.28632 | 0.83497 | 0.59952 | 0.42024 | 0.17648 | 0.38683 | 0.59747 |
| | | | | 1.0 | 0.69937 | 0.99570 | 0.97128 | 0.89459 | 0.53219 | 0.85993 | 0.97646 |
| | 5 | 165 | 0.05000 | 0.5 | 0.30607 | 0.81192 | 0.58975 | 0.42921 | 0.18621 | 0.39997 | 0.59452 |
| | | | | 1.0 | 0.73382 | 0.99463 | 0.96718 | 0.89760 | 0.53755 | 0.86956 | 0.97379 |
| | 6 | 171 | 0.05070 | 0.5 | 0.32068 | 0.77397 | 0.57326 | 0.43190 | 0.19376 | 0.40558 | 0.58443 |
| | | | | 1.0 | 0.75832 | 0.99013 | 0.96165 | 0.89771 | 0.54069 | 0.87448 | 0.96942 |
| | 7 | 176 | 0.05046 | 0.5 | 0.33025 | 0.74438 | 0.55487 | 0.42913 | 0.19943 | 0.40584 | 0.57106 |
| | | | | 1.0 | 0.77541 | 0.98709 | 0.95492 | 0.89453 | 0.54239 | 0.87493 | 0.96445 |
| (iv) | 2 | 270 | 0.04741 | 0.5 | 0.24479 | 0.86046 | 0.56513 | 0.37409 | 0.30890 | 0.33934 | 0.54855 |
| | | | | 1.0 | 0.60349 | 0.99782 | 0.96800 | 0.84783 | 0.77186 | 0.79679 | 0.96711 |
| | 3 | 250 | 0.04945 | 0.5 | 0.27457 | 0.69627 | 0.49010 | 0.37268 | 0.34484 | 0.35277 | 0.51347 |
| | | | | 1.0 | 0.66497 | 0.98192 | 0.92751 | 0.83231 | 0.81347 | 0.80493 | 0.94203 |
| | 4 | 238 | 0.05094 | 0.5 | 0.29630 | 0.62237 | 0.47011 | 0.37558 | 0.36465 | 0.36286 | 0.50003 |
| | | | | 1.0 | 0.70813 | 0.96207 | 0.90369 | 0.82992 | 0.83360 | 0.81423 | 0.92743 |
| | 5 | 227 | 0.04991 | 0.5 | 0.30688 | 0.59903 | 0.45016 | 0.37019 | 0.37839 | 0.35903 | 0.47857 |
| | | | | 1.0 | 0.73050 | 0.95887 | 0.89642 | 0.82583 | 0.84798 | 0.81470 | 0.91932 |
| | 6 | 218 | 0.04912 | 0.5 | 0.31516 | 0.57724 | 0.44076 | 0.36872 | 0.38593 | 0.35731 | 0.46871 |
| | | | | 1.0 | 0.74960 | 0.94373 | 0.89048 | 0.82710 | 0.85519 | 0.81906 | 0.91465 |
| | 7 | 211 | 0.04909 | 0.5 | 0.32803 | 0.54681 | 0.43359 | 0.36851 | 0.38762 | 0.36000 | 0.46137 |
| | | | | 1.0 | 0.76885 | 0.92760 | 0.87472 | 0.82602 | 0.85636 | 0.81903 | 0.90232 |
| (v) | 2 | 205 | 0.04741 | 0.5 | 0.24479 | 0.86046 | 0.56513 | 0.37409 | 0.31930 | 0.33934 | 0.54855 |
| | | | | 1.0 | 0.60349 | 0.99782 | 0.96800 | 0.84783 | 0.76648 | 0.79679 | 0.96711 |
| | 3 | 201 | 0.04891 | 0.5 | 0.27510 | 0.76629 | 0.51785 | 0.38108 | 0.33848 | 0.35865 | 0.53395 |
| | | | | 1.0 | 0.66937 | 0.99237 | 0.94713 | 0.84800 | 0.78914 | 0.81678 | 0.95475 |
| | 4 | 199 | 0.05101 | 0.5 | 0.30958 | 0.73329 | 0.52614 | 0.40228 | 0.36567 | 0.38500 | 0.54494 |
| | | | | 1.0 | 0.72412 | 0.98637 | 0.94489 | 0.86421 | 0.81757 | 0.84200 | 0.95521 |
| | 5 | 197 | 0.05179 | 0.5 | 0.32040 | 0.67778 | 0.50835 | 0.40157 | 0.36801 | 0.38439 | 0.53117 |
| | | | | 1.0 | 0.74916 | 0.97340 | 0.93081 | 0.86332 | 0.82499 | 0.84639 | 0.94640 |
| | 6 | 195 | 0.04808 | 0.5 | 0.32528 | 0.64703 | 0.48849 | 0.39599 | 0.36644 | 0.38154 | 0.51394 |
| | | | | 1.0 | 0.76584 | 0.96720 | 0.92083 | 0.85899 | 0.82760 | 0.84622 | 0.93778 |
| | 7 | 194 | 0.04902 | 0.5 | 0.33368 | 0.62344 | 0.47899 | 0.39464 | 0.36677 | 0.38073 | 0.50532 |
| | | | | 1.0 | 0.78084 | 0.95711 | 0.91423 | 0.85858 | 0.83034 | 0.84734 | 0.93366 |
| (vi) | | 192 | 0.04880 | 0.5 | 0.35996 | 0.59384 | 0.47190 | 0.40033 | 0.38005 | 0.38949 | 0.49266 |
| | | | | 1.0 | 0.82397 | 0.94063 | 0.90428 | 0.86491 | 0.84786 | 0.85733 | 0.92161 |

Remarks: (i) - Precedence Test; (ii) - Maximal Precedence Test; (iii) - Minimal Rank-sum Precedence Test; (iv) - Maximal Rank-sum Precedence Test; (v) - Expected Rank-sum Precedence Test; (vi) - Wilcoxon's Rank-sum Test,

$s$ - critical value, $\alpha$ - exact level of significance, $\theta$ - location-shift

Table 3.10: Power of precedence tests, maximal precedence tests, minimal, maximal and expected rank-sum precedence tests and Wilcoxon's rank-sum test with $n_1 = 15, n_2 = 20$

| Test | $r$ | $s$ | $\alpha$ | $\theta$ | Normal | Exp(1) | Gamma(2) | Gamma(5) | Gamma(10) | LN(0.25) | LN(0.5) |
|---|---|---|---|---|---|---|---|---|---|---|---|
| (i) | 1 | 3 | 0.06952 | 0.5 | 0.26103 | 0.98034 | 0.77808 | 0.47275 | 0.37926 | 0.39668 | 0.68983 |
| | | | | 1.0 | 0.56610 | 0.99994 | 0.99538 | 0.90770 | 0.80025 | 0.82145 | 0.99016 |
| | 2 | 5 | 0.04009 | 0.5 | 0.22799 | 0.87119 | 0.58158 | 0.37354 | 0.31301 | 0.33392 | 0.56444 |
| | | | | 1.0 | 0.58927 | 0.99777 | 0.97204 | 0.86035 | 0.77478 | 0.80473 | 0.97101 |
| | 3 | 6 | 0.04615 | 0.5 | 0.26977 | 0.79414 | 0.54880 | 0.39401 | 0.34462 | 0.36640 | 0.56539 |
| | | | | 1.0 | 0.66905 | 0.99367 | 0.95792 | 0.86573 | 0.80654 | 0.83116 | 0.96482 |
| | 4 | 7 | 0.04718 | 0.5 | 0.28676 | 0.70414 | 0.50231 | 0.38713 | 0.34864 | 0.37028 | 0.53471 |
| | | | | 1.0 | 0.70558 | 0.98285 | 0.93573 | 0.85376 | 0.80835 | 0.83120 | 0.94992 |
| | 5 | 8 | 0.04492 | 0.5 | 0.28890 | 0.60639 | 0.44743 | 0.36288 | 0.33594 | 0.35453 | 0.48667 |
| | | | | 1.0 | 0.71823 | 0.96025 | 0.90163 | 0.82691 | 0.79161 | 0.81410 | 0.92500 |
| | 6 | 9 | 0.04043 | 0.5 | 0.27957 | 0.50487 | 0.38805 | 0.33032 | 0.30925 | 0.32527 | 0.42887 |
| | | | | 1.0 | 0.71204 | 0.91834 | 0.85177 | 0.78776 | 0.76187 | 0.78186 | 0.88644 |
| | 7 | 10 | 0.03446 | 0.5 | 0.25718 | 0.40124 | 0.32210 | 0.28523 | 0.27515 | 0.28593 | 0.36215 |
| | | | | 1.0 | 0.69120 | 0.84953 | 0.78421 | 0.73317 | 0.71587 | 0.73249 | 0.82850 |
| (ii) | 2 | 4 | 0.05187 | 0.5 | 0.23931 | 0.93260 | 0.61612 | 0.37709 | 0.31680 | 0.33630 | 0.57112 |
| | | | | 1.0 | 0.57697 | 0.99933 | 0.98278 | 0.85651 | 0.76077 | 0.78992 | 0.97513 |
| | 3 | 5 | 0.02770 | 0.5 | 0.15938 | 0.82474 | 0.41274 | 0.23401 | 0.20145 | 0.21598 | 0.38511 |
| | | | | 1.0 | 0.46125 | 0.99674 | 0.94357 | 0.72230 | 0.61501 | 0.65157 | 0.92585 |
| | 4 | 5 | 0.03690 | 0.5 | 0.18512 | 0.82560 | 0.41999 | 0.25053 | 0.22094 | 0.23569 | 0.39570 |
| | | | | 1.0 | 0.49491 | 0.99675 | 0.94424 | 0.73044 | 0.63081 | 0.66553 | 0.92743 |
| | 5 | 5 | 0.04609 | 0.5 | 0.20491 | 0.82636 | 0.42531 | 0.26210 | 0.23570 | 0.24992 | 0.40293 |
| | | | | 1.0 | 0.51620 | 0.99675 | 0.94483 | 0.73513 | 0.63931 | 0.67290 | 0.92805 |
| | 6 | 5 | 0.05526 | 0.5 | 0.22093 | 0.82712 | 0.42947 | 0.27132 | 0.24728 | 0.26127 | 0.40826 |
| | | | | 1.0 | 0.52976 | 0.99677 | 0.94521 | 0.73832 | 0.64463 | 0.67774 | 0.92855 |
| | 7 | 5 | 0.06441 | 0.5 | 0.23365 | 0.82768 | 0.43317 | 0.27822 | 0.25643 | 0.26984 | 0.41227 |
| | | | | 1.0 | 0.53884 | 0.99678 | 0.94554 | 0.74029 | 0.64854 | 0.68099 | 0.92895 |
| (iii) | 2 | 143 | 0.05187 | 0.5 | 0.25985 | 0.95111 | 0.72250 | 0.45616 | 0.37273 | 0.39406 | 0.67415 |
| | | | | 1.0 | 0.62814 | 0.99960 | 0.99115 | 0.91802 | 0.83418 | 0.85783 | 0.98926 |
| | 3 | 153 | 0.05302 | 0.5 | 0.28617 | 0.94575 | 0.71835 | 0.47530 | 0.39655 | 0.42144 | 0.68614 |
| | | | | 1.0 | 0.68770 | 0.99948 | 0.99056 | 0.92903 | 0.86298 | 0.88455 | 0.99010 |
| | 4 | 162 | 0.04950 | 0.5 | 0.29737 | 0.90905 | 0.68584 | 0.47116 | 0.39971 | 0.42392 | 0.67168 |
| | | | | 1.0 | 0.72107 | 0.99860 | 0.98636 | 0.92796 | 0.87272 | 0.89252 | 0.98788 |
| | 5 | 170 | 0.04684 | 0.5 | 0.30454 | 0.88945 | 0.65930 | 0.46511 | 0.40060 | 0.42355 | 0.65678 |
| | | | | 1.0 | 0.74464 | 0.99818 | 0.98288 | 0.92547 | 0.87502 | 0.89601 | 0.98583 |
| | 6 | 178 | 0.05149 | 0.5 | 0.33433 | 0.86588 | 0.65908 | 0.48462 | 0.42481 | 0.44830 | 0.66554 |
| | | | | 1.0 | 0.78297 | 0.99670 | 0.98084 | 0.93135 | 0.89051 | 0.90908 | 0.98538 |
| | 7 | 185 | 0.05333 | 0.5 | 0.35208 | 0.85075 | 0.65027 | 0.49153 | 0.43604 | 0.45966 | 0.66309 |
| | | | | 1.0 | 0.80616 | 0.99592 | 0.97895 | 0.93276 | 0.89709 | 0.91387 | 0.98391 |
| (iv) | 2 | 340 | 0.05074 | 0.5 | 0.26086 | 0.93935 | 0.67390 | 0.43135 | 0.35907 | 0.38197 | 0.63716 |
| | | | | 1.0 | 0.62844 | 0.99940 | 0.98716 | 0.89894 | 0.81719 | 0.84402 | 0.98431 |
| | 3 | 320 | 0.04874 | 0.5 | 0.27939 | 0.86398 | 0.59185 | 0.41456 | 0.36043 | 0.38212 | 0.59491 |
| | | | | 1.0 | 0.68127 | 0.99755 | 0.97324 | 0.88498 | 0.82349 | 0.84752 | 0.97483 |
| | 4 | 302 | 0.04970 | 0.5 | 0.29838 | 0.80080 | 0.55660 | 0.41245 | 0.36698 | 0.38983 | 0.57586 |
| | | | | 1.0 | 0.72108 | 0.99403 | 0.96218 | 0.88079 | 0.83173 | 0.85374 | 0.96904 |
| | 5 | 287 | 0.05002 | 0.5 | 0.31665 | 0.75027 | 0.54798 | 0.41820 | 0.37895 | 0.39995 | 0.57100 |
| | | | | 1.0 | 0.75442 | 0.98727 | 0.95352 | 0.88593 | 0.84374 | 0.86472 | 0.96490 |
| | 6 | 274 | 0.04912 | 0.5 | 0.32750 | 0.67553 | 0.51837 | 0.41619 | 0.38002 | 0.40172 | 0.55136 |
| | | | | 1.0 | 0.77413 | 0.97046 | 0.93297 | 0.87499 | 0.84373 | 0.86265 | 0.94976 |
| | 7 | 264 | 0.04971 | 0.5 | 0.33957 | 0.63792 | 0.48901 | 0.40905 | 0.38100 | 0.39975 | 0.52510 |
| | | | | 1.0 | 0.78969 | 0.96524 | 0.91989 | 0.86293 | 0.83918 | 0.85464 | 0.93819 |
| (v) | 2 | 241 | 0.05074 | 0.5 | 0.26086 | 0.93935 | 0.67390 | 0.43135 | 0.35907 | 0.38197 | 0.63716 |
| | | | | 1.0 | 0.62844 | 0.99940 | 0.98716 | 0.89894 | 0.81719 | 0.84402 | 0.98431 |
| | 3 | 236 | 0.05134 | 0.5 | 0.28841 | 0.88129 | 0.62077 | 0.43135 | 0.37338 | 0.39637 | 0.61766 |
| | | | | 1.0 | 0.69158 | 0.99800 | 0.97798 | 0.89796 | 0.83638 | 0.85935 | 0.97942 |
| | 4 | 232 | 0.05161 | 0.5 | 0.31070 | 0.83120 | 0.60331 | 0.44114 | 0.38913 | 0.41291 | 0.61575 |
| | | | | 1.0 | 0.73815 | 0.99528 | 0.97111 | 0.90309 | 0.85398 | 0.87482 | 0.97678 |
| | 5 | 229 | 0.05037 | 0.5 | 0.32774 | 0.76175 | 0.57470 | 0.44037 | 0.39778 | 0.41912 | 0.59851 |
| | | | | 1.0 | 0.76806 | 0.98759 | 0.95769 | 0.89750 | 0.85834 | 0.87826 | 0.96846 |
| | 6 | 227 | 0.05094 | 0.5 | 0.33882 | 0.73913 | 0.55321 | 0.43890 | 0.39903 | 0.42158 | 0.58270 |
| | | | | 1.0 | 0.78779 | 0.98598 | 0.95181 | 0.89346 | 0.86080 | 0.87873 | 0.96346 |
| | 7 | 225 | 0.04878 | 0.5 | 0.34994 | 0.71183 | 0.54746 | 0.44049 | 0.40365 | 0.42436 | 0.57711 |
| | | | | 1.0 | 0.80544 | 0.97755 | 0.94720 | 0.89519 | 0.86713 | 0.88337 | 0.96087 |
| (vi) | | 220 | 0.0497 | 0.5 | 0.39901 | 0.64380 | 0.51830 | 0.44589 | 0.42322 | 0.43600 | 0.54447 |
| | | | | 1.0 | 0.87100 | 0.95604 | 0.92949 | 0.89928 | 0.88529 | 0.89288 | 0.94262 |

Remarks: (i) - Precedence Test; (ii) - Maximal Precedence Test; (iii) - Minimal Rank-sum Precedence Test; (iv) - Maximal Rank-sum Precedence Test; (v) - Expected Rank-sum Precedence Test; (vi) - Wilcoxon's Rank-sum Test,

$s$ - critical value, $\alpha$ - exact level of significance, $\theta$ - location-shift

Table 3.11: Power of precedence tests, maximal precedence tests, minimal, maximal and expected rank-sum precedence tests and Wilcoxon's rank-sum test with $n_1 = n_2 = 20$

| Test | $r$ | $s$ | $\alpha$ | $\theta$ | Normal | Exp(1) | Gamma(2) | Gamma(5) | Gamma(10) | LN(0.25) | LN(0.5) |
|---|---|---|---|---|---|---|---|---|---|---|---|
| (i) | 1 | 4 | 0.05301 | 0.5 | 0.23855 | 0.98866 | 0.78803 | 0.45093 | 0.35517 | 0.37686 | 0.68551 |
| | | | | 1.0 | 0.55525 | 1.00000 | 0.99817 | 0.91502 | 0.80255 | 0.82751 | 0.99428 |
| | 2 | 6 | 0.04574 | 0.5 | 0.26616 | 0.93554 | 0.67111 | 0.43567 | 0.36379 | 0.39035 | 0.64528 |
| | | | | 1.0 | 0.65292 | 0.99981 | 0.99140 | 0.91288 | 0.83709 | 0.86474 | 0.99011 |
| | 3 | 8 | 0.03242 | 0.5 | 0.24335 | 0.80663 | 0.53308 | 0.36703 | 0.31605 | 0.34027 | 0.54800 |
| | | | | 1.0 | 0.66299 | 0.99719 | 0.96991 | 0.87542 | 0.81209 | 0.83996 | 0.97553 |
| | 4 | 9 | 0.04118 | 0.5 | 0.29470 | 0.74992 | 0.53174 | 0.40110 | 0.35826 | 0.38078 | 0.56144 |
| | | | | 1.0 | 0.73517 | 0.99322 | 0.96147 | 0.88663 | 0.84176 | 0.86742 | 0.97162 |
| | 5 | 10 | 0.04792 | 0.5 | 0.33174 | 0.69035 | 0.51727 | 0.41772 | 0.38082 | 0.40343 | 0.55552 |
| | | | | 1.0 | 0.77868 | 0.98635 | 0.94874 | 0.88877 | 0.85320 | 0.87638 | 0.96482 |
| | 6 | 11 | 0.05267 | 0.5 | 0.35648 | 0.62910 | 0.49662 | 0.41973 | 0.39200 | 0.41012 | 0.53689 |
| | | | | 1.0 | 0.80552 | 0.97332 | 0.93225 | 0.88021 | 0.85581 | 0.87481 | 0.95402 |
| | 7 | 12 | 0.05548 | 0.5 | 0.36864 | 0.56701 | 0.46744 | 0.41259 | 0.39212 | 0.40861 | 0.51070 |
| | | | | 1.0 | 0.81861 | 0.95316 | 0.90856 | 0.86660 | 0.84911 | 0.86600 | 0.93674 |
| (ii) | 2 | 5 | 0.04691 | 0.5 | 0.24343 | 0.96341 | 0.65799 | 0.38935 | 0.32350 | 0.34864 | 0.59833 |
| | | | | 1.0 | 0.59939 | 0.99997 | 0.99337 | 0.88558 | 0.78925 | 0.82297 | 0.98815 |
| | 3 | 6 | 0.03000 | 0.5 | 0.18471 | 0.90506 | 0.48784 | 0.27136 | 0.23271 | 0.25067 | 0.44604 |
| | | | | 1.0 | 0.52351 | 0.99969 | 0.97866 | 0.79514 | 0.68755 | 0.72654 | 0.96538 |
| | 4 | 6 | 0.04026 | 0.5 | 0.21380 | 0.90533 | 0.49485 | 0.28888 | 0.25464 | 0.27237 | 0.45715 |
| | | | | 1.0 | 0.56051 | 0.99969 | 0.97904 | 0.80249 | 0.70298 | 0.74016 | 0.96618 |
| | 5 | 6 | 0.05000 | 0.5 | 0.23598 | 0.90583 | 0.50036 | 0.30132 | 0.27088 | 0.28786 | 0.46423 |
| | | | | 1.0 | 0.58325 | 0.99969 | 0.97931 | 0.80651 | 0.71154 | 0.74747 | 0.96660 |
| | 6 | 6 | 0.06025 | 0.5 | 0.25446 | 0.90621 | 0.50442 | 0.31157 | 0.28339 | 0.29944 | 0.46884 |
| | | | | 1.0 | 0.59768 | 0.99969 | 0.97955 | 0.80929 | 0.71709 | 0.75171 | 0.96683 |
| | 7 | 6 | 0.07021 | 0.5 | 0.26872 | 0.90653 | 0.50806 | 0.31961 | 0.29313 | 0.30885 | 0.47260 |
| | | | | 1.0 | 0.60667 | 0.99970 | 0.97970 | 0.81080 | 0.72051 | 0.75454 | 0.96700 |
| (iii) | 2 | 241 | 0.04691 | 0.5 | 0.26563 | 0.97529 | 0.76888 | 0.47784 | 0.38507 | 0.41115 | 0.70839 |
| | | | | 1.0 | 0.65211 | 0.99998 | 0.99717 | 0.94085 | 0.86307 | 0.88839 | 0.99603 |
| | 3 | 255 | 0.05335 | 0.5 | 0.31306 | 0.97323 | 0.78152 | 0.52251 | 0.43477 | 0.46196 | 0.74223 |
| | | | | 1.0 | 0.73361 | 0.99998 | 0.99743 | 0.95763 | 0.90239 | 0.92205 | 0.99728 |
| | 4 | 267 | 0.04671 | 0.5 | 0.31593 | 0.94911 | 0.73988 | 0.50644 | 0.42666 | 0.45624 | 0.71938 |
| | | | | 1.0 | 0.76021 | 0.99984 | 0.99567 | 0.95465 | 0.90690 | 0.92596 | 0.99599 |
| | 5 | 279 | 0.05234 | 0.5 | 0.35496 | 0.93654 | 0.74228 | 0.53457 | 0.45978 | 0.48976 | 0.73413 |
| | | | | 1.0 | 0.80732 | 0.99972 | 0.99501 | 0.96195 | 0.92479 | 0.94067 | 0.99618 |
| | 6 | 289 | 0.04987 | 0.5 | 0.36229 | 0.91749 | 0.71814 | 0.52715 | 0.45897 | 0.48704 | 0.71781 |
| | | | | 1.0 | 0.82412 | 0.99963 | 0.99344 | 0.96028 | 0.92623 | 0.94163 | 0.99508 |
| | 7 | 298 | 0.04816 | 0.5 | 0.36966 | 0.89261 | 0.69538 | 0.51957 | 0.45696 | 0.48555 | 0.70248 |
| | | | | 1.0 | 0.83961 | 0.99913 | 0.99143 | 0.95826 | 0.92733 | 0.94215 | 0.99393 |
| (iv) | 2 | 510 | 0.05326 | 0.5 | 0.28995 | 0.96892 | 0.73621 | 0.47815 | 0.39764 | 0.42488 | 0.69261 |
| | | | | 1.0 | 0.67845 | 0.99998 | 0.99595 | 0.93288 | 0.86214 | 0.88741 | 0.99463 |
| | 3 | 477 | 0.04979 | 0.5 | 0.31303 | 0.90068 | 0.67241 | 0.47535 | 0.40846 | 0.43704 | 0.67156 |
| | | | | 1.0 | 0.73784 | 0.99940 | 0.98864 | 0.93154 | 0.87985 | 0.90144 | 0.99091 |
| | 4 | 458 | 0.05023 | 0.5 | 0.33510 | 0.85888 | 0.64441 | 0.47383 | 0.41736 | 0.44318 | 0.65618 |
| | | | | 1.0 | 0.77776 | 0.99806 | 0.98423 | 0.93206 | 0.89030 | 0.91038 | 0.98864 |
| | 5 | 439 | 0.04967 | 0.5 | 0.34947 | 0.80467 | 0.59906 | 0.45986 | 0.41287 | 0.43694 | 0.62239 |
| | | | | 1.0 | 0.80024 | 0.99561 | 0.97622 | 0.92428 | 0.88658 | 0.90740 | 0.98374 |
| | 6 | 423 | 0.04952 | 0.5 | 0.36331 | 0.75011 | 0.57319 | 0.45612 | 0.41656 | 0.43729 | 0.60253 |
| | | | | 1.0 | 0.82318 | 0.99076 | 0.96701 | 0.92037 | 0.88866 | 0.90781 | 0.97779 |
| | 7 | 410 | 0.05053 | 0.5 | 0.37765 | 0.70899 | 0.55775 | 0.45879 | 0.42413 | 0.44447 | 0.59235 |
| | | | | 1.0 | 0.84235 | 0.98622 | 0.95727 | 0.91683 | 0.89187 | 0.90878 | 0.97186 |
| (v) | 2 | 375 | 0.05326 | 0.5 | 0.28995 | 0.96892 | 0.73621 | 0.47815 | 0.39764 | 0.42488 | 0.69261 |
| | | | | 1.0 | 0.67845 | 0.99998 | 0.99595 | 0.93288 | 0.86214 | 0.88741 | 0.99463 |
| | 3 | 367 | 0.04979 | 0.5 | 0.32336 | 0.90076 | 0.67643 | 0.48555 | 0.41876 | 0.44710 | 0.67790 |
| | | | | 1.0 | 0.74637 | 0.99940 | 0.98867 | 0.93312 | 0.88311 | 0.90416 | 0.99095 |
| | 4 | 363 | 0.05114 | 0.5 | 0.34072 | 0.86001 | 0.65310 | 0.48465 | 0.42571 | 0.45265 | 0.66558 |
| | | | | 1.0 | 0.78355 | 0.99806 | 0.98455 | 0.93512 | 0.89458 | 0.91428 | 0.98906 |
| | 5 | 360 | 0.05071 | 0.5 | 0.36530 | 0.83452 | 0.63627 | 0.49216 | 0.44006 | 0.46498 | 0.65793 |
| | | | | 1.0 | 0.81679 | 0.99754 | 0.98066 | 0.93694 | 0.90272 | 0.92136 | 0.98689 |
| | 6 | 357 | 0.05017 | 0.5 | 0.37277 | 0.80486 | 0.61393 | 0.48393 | 0.43713 | 0.46004 | 0.63891 |
| | | | | 1.0 | 0.83451 | 0.99571 | 0.97775 | 0.93532 | 0.90421 | 0.92214 | 0.98437 |
| | 7 | 355 | 0.04941 | 0.5 | 0.38888 | 0.78140 | 0.60740 | 0.48754 | 0.44576 | 0.46869 | 0.63603 |
| | | | | 1.0 | 0.85360 | 0.99323 | 0.97510 | 0.93648 | 0.91078 | 0.92578 | 0.98289 |
| (vi) | | 348 | 0.0482 | 0.5 | 0.44025 | 0.70110 | 0.56991 | 0.49170 | 0.46220 | 0.47674 | 0.59412 |
| | | | | 1.0 | 0.91358 | 0.98022 | 0.96218 | 0.93918 | 0.92747 | 0.93438 | 0.97157 |

Remarks: (i) - Precedence Test; (ii) - Maximal Precedence Test; (iii) - Minimal Rank-sum Precedence Test; (iv) - Maximal Rank-sum Precedence Test; (v) - Expected Rank-sum Precedence Test; (vi) - Wilcoxon's Rank-sum Test,

$s$ - critical value, $\alpha$ - exact level of significance, $\theta$ - location-shift

Table 3.12: Power of precedence tests, maximal precedence tests, minimal, maximal and expected rank-sum precedence tests and Wilcoxon's rank-sum test with $n_1 = n_2 = 30$

| Test | $r$ | $s$ | $\alpha$ | $\theta$ | Normal | Exp(1) | Gamma(2) | Gamma(5) | Gamma(10) | LN(0.25) | LN(0.5) |
|---|---|---|---|---|---|---|---|---|---|---|---|
| (i) | 1 | 4 | 0.05620 | 0.5 | 0.26896 | 0.99976 | 0.93586 | 0.57064 | 0.43542 | 0.45661 | 0.82589 |
| | | | | 1.0 | 0.61350 | 1.00000 | 0.99999 | 0.97496 | 0.88783 | 0.90367 | 0.99977 |
| | 2 | 6 | 0.05139 | 0.5 | 0.31807 | 0.99740 | 0.88320 | 0.58801 | 0.47543 | 0.50233 | 0.82866 |
| | | | | 1.0 | 0.73553 | 1.00000 | 0.99994 | 0.98092 | 0.93015 | 0.94616 | 0.99972 |
| | 3 | 8 | 0.03986 | 0.5 | 0.31657 | 0.98511 | 0.80734 | 0.54708 | 0.45403 | 0.48453 | 0.78458 |
| | | | | 1.0 | 0.77546 | 1.00000 | 0.99946 | 0.97586 | 0.93369 | 0.94924 | 0.99917 |
| | 4 | 9 | 0.05208 | 0.5 | 0.38975 | 0.97582 | 0.80882 | 0.59849 | 0.51749 | 0.54755 | 0.80911 |
| | | | | 1.0 | 0.84545 | 1.00000 | 0.99919 | 0.98279 | 0.95560 | 0.96631 | 0.99926 |
| | 5 | 10 | 0.06264 | 0.5 | 0.44400 | 0.96258 | 0.80528 | 0.62815 | 0.55909 | 0.58921 | 0.81648 |
| | | | | 1.0 | 0.88835 | 0.99998 | 0.99889 | 0.98487 | 0.96658 | 0.97464 | 0.99913 |
| | 6 | 12 | 0.04202 | 0.5 | 0.38877 | 0.89771 | 0.70069 | 0.54065 | 0.48511 | 0.51056 | 0.72929 |
| | | | | 1.0 | 0.87178 | 0.99975 | 0.99530 | 0.97181 | 0.94812 | 0.96000 | 0.99729 |
| | 7 | 13 | 0.04731 | 0.5 | 0.42346 | 0.86835 | 0.68778 | 0.55398 | 0.50542 | 0.52993 | 0.72543 |
| | | | | 1.0 | 0.89623 | 0.99941 | 0.99359 | 0.97179 | 0.95302 | 0.96425 | 0.99643 |
| (ii) | 2 | 5 | 0.05179 | 0.5 | 0.28975 | 0.99895 | 0.87822 | 0.53387 | 0.42839 | 0.45632 | 0.78800 |
| | | | | 1.0 | 0.69085 | 1.00000 | 0.99995 | 0.97159 | 0.90186 | 0.92235 | 0.99960 |
| | 3 | 6 | 0.03500 | 0.5 | 0.23805 | 0.99564 | 0.77353 | 0.42245 | 0.34348 | 0.36973 | 0.67661 |
| | | | | 1.0 | 0.65556 | 1.00000 | 0.99963 | 0.94524 | 0.85773 | 0.88517 | 0.99831 |
| | 4 | 6 | 0.04707 | 0.5 | 0.27951 | 0.99567 | 0.77973 | 0.44989 | 0.37610 | 0.40311 | 0.69102 |
| | | | | 1.0 | 0.70500 | 1.00000 | 0.99965 | 0.95053 | 0.87483 | 0.89984 | 0.99843 |
| | 5 | 6 | 0.05868 | 0.5 | 0.31191 | 0.99570 | 0.78415 | 0.46907 | 0.40128 | 0.42736 | 0.70067 |
| | | | | 1.0 | 0.73703 | 1.00000 | 0.99967 | 0.95344 | 0.88487 | 0.90823 | 0.99848 |
| | 6 | 7 | 0.03150 | 0.5 | 0.20718 | 0.98664 | 0.63997 | 0.32227 | 0.26895 | 0.28902 | 0.54597 |
| | | | | 1.0 | 0.60957 | 1.00000 | 0.99855 | 0.89387 | 0.78435 | 0.81916 | 0.99433 |
| | 7 | 7 | 0.03672 | 0.5 | 0.22067 | 0.98666 | 0.64208 | 0.32924 | 0.27778 | 0.29751 | 0.54915 |
| | | | | 1.0 | 0.62190 | 1.00000 | 0.99856 | 0.89513 | 0.78790 | 0.82183 | 0.99434 |
| (iii) | 2 | 516 | 0.05179 | 0.5 | 0.31188 | 0.99932 | 0.93259 | 0.62341 | 0.48957 | 0.51660 | 0.86694 |
| | | | | 1.0 | 0.73017 | 1.00000 | 0.99999 | 0.98797 | 0.94104 | 0.95550 | 0.99991 |
| | 3 | 539 | 0.04574 | 0.5 | 0.32940 | 0.99823 | 0.91639 | 0.62816 | 0.50446 | 0.53452 | 0.86850 |
| | | | | 1.0 | 0.78275 | 1.00000 | 0.99998 | 0.99034 | 0.95600 | 0.96833 | 0.99991 |
| | 4 | 561 | 0.04612 | 0.5 | 0.35996 | 0.99766 | 0.91039 | 0.64668 | 0.53028 | 0.56132 | 0.87551 |
| | | | | 1.0 | 0.82870 | 1.00000 | 0.99998 | 0.99229 | 0.96710 | 0.97700 | 0.99993 |
| | 5 | 582 | 0.04736 | 0.5 | 0.38952 | 0.99593 | 0.90346 | 0.66218 | 0.55459 | 0.58673 | 0.87887 |
| | | | | 1.0 | 0.86381 | 1.00000 | 0.99994 | 0.99331 | 0.97436 | 0.98249 | 0.99990 |
| | 6 | 602 | 0.04944 | 0.5 | 0.41984 | 0.99402 | 0.89795 | 0.67653 | 0.57692 | 0.60911 | 0.88247 |
| | | | | 1.0 | 0.89039 | 1.00000 | 0.99995 | 0.99437 | 0.97919 | 0.98602 | 0.99993 |
| | 7 | 621 | 0.05177 | 0.5 | 0.44591 | 0.99225 | 0.89331 | 0.68821 | 0.59549 | 0.62815 | 0.88459 |
| | | | | 1.0 | 0.91090 | 1.00000 | 0.99993 | 0.99516 | 0.98269 | 0.98856 | 0.99990 |
| (iv) | 2 | 1191 | 0.05139 | 0.5 | 0.31807 | 0.99740 | 0.88320 | 0.58801 | 0.47543 | 0.50233 | 0.82866 |
| | | | | 1.0 | 0.73553 | 1.00000 | 0.99994 | 0.98092 | 0.93015 | 0.94616 | 0.99972 |
| | 3 | 1160 | 0.05082 | 0.5 | 0.35905 | 0.99471 | 0.88022 | 0.62319 | 0.51761 | 0.54797 | 0.84977 |
| | | | | 1.0 | 0.81050 | 1.00000 | 0.99984 | 0.98753 | 0.95551 | 0.96747 | 0.99972 |
| | 4 | 1115 | 0.04964 | 0.5 | 0.38632 | 0.97582 | 0.80881 | 0.59764 | 0.51612 | 0.54591 | 0.80899 |
| | | | | 1.0 | 0.84425 | 1.00000 | 0.99919 | 0.98278 | 0.95551 | 0.96625 | 0.99926 |
| | 5 | 1085 | 0.04996 | 0.5 | 0.41155 | 0.96256 | 0.80349 | 0.61314 | 0.53704 | 0.56835 | 0.81224 |
| | | | | 1.0 | 0.87801 | 0.99998 | 0.99889 | 0.98452 | 0.96503 | 0.97385 | 0.99912 |
| | 6 | 1057 | 0.05052 | 0.5 | 0.43184 | 0.94598 | 0.78990 | 0.61683 | 0.54656 | 0.57582 | 0.80587 |
| | | | | 1.0 | 0.89821 | 0.99994 | 0.99838 | 0.98500 | 0.96772 | 0.97613 | 0.99909 |
| | 7 | 1029 | 0.05009 | 0.5 | 0.44701 | 0.92549 | 0.76833 | 0.60774 | 0.54629 | 0.57290 | 0.79040 |
| | | | | 1.0 | 0.91202 | 0.99983 | 0.99771 | 0.98464 | 0.96931 | 0.97745 | 0.99871 |
| (v) | 2 | 855 | 0.05139 | 0.5 | 0.31807 | 0.99740 | 0.88320 | 0.58801 | 0.47543 | 0.50233 | 0.82866 |
| | | | | 1.0 | 0.73553 | 1.00000 | 0.99994 | 0.98092 | 0.93015 | 0.94616 | 0.99972 |
| | 3 | 850 | 0.05082 | 0.5 | 0.37136 | 0.99472 | 0.88422 | 0.63686 | 0.53169 | 0.56212 | 0.85682 |
| | | | | 1.0 | 0.81850 | 1.00000 | 0.99984 | 0.98848 | 0.95845 | 0.96976 | 0.99975 |
| | 4 | 841 | 0.04972 | 0.5 | 0.38652 | 0.98690 | 0.83142 | 0.60707 | 0.52090 | 0.55166 | 0.82198 |
| | | | | 1.0 | 0.84642 | 1.00000 | 0.99968 | 0.98549 | 0.95888 | 0.96880 | 0.99954 |
| | 5 | 835 | 0.04987 | 0.5 | 0.41313 | 0.97687 | 0.81835 | 0.61906 | 0.54130 | 0.57239 | 0.82059 |
| | | | | 1.0 | 0.87942 | 1.00000 | 0.99932 | 0.98613 | 0.96658 | 0.97556 | 0.99941 |
| | 6 | 830 | 0.04988 | 0.5 | 0.43394 | 0.96516 | 0.80540 | 0.62462 | 0.55147 | 0.58215 | 0.81579 |
| | | | | 1.0 | 0.90042 | 1.00000 | 0.99911 | 0.98708 | 0.96999 | 0.97782 | 0.99931 |
| | 7 | 826 | 0.04985 | 0.5 | 0.45485 | 0.95444 | 0.79812 | 0.62965 | 0.56425 | 0.59209 | 0.81325 |
| | | | | 1.0 | 0.91759 | 0.99996 | 0.99889 | 0.98788 | 0.97401 | 0.98120 | 0.99922 |
| (vi) | | 803 | 0.0482 | 0.5 | 0.58676 | 0.85305 | 0.73394 | 0.64069 | 0.61460 | 0.62744 | 0.75711 |
| | | | | 1.0 | 0.98094 | 0.99847 | 0.99509 | 0.98947 | 0.98542 | 0.98806 | 0.99687 |

Remarks: (i) - Precedence Test; (ii) - Maximal Precedence Test; (iii) - Minimal Rank-sum Precedence Test; (iv) - Maximal Rank-sum Precedence Test; (v) - Expected Rank-sum Precedence Test; (vi) - Wilcoxon's Rank-sum Test,

$s$ - critical value, $\alpha$ - exact level of significance, $\theta$ - location-shift

## 3.6 Illustrative Examples

Nelson (1982, p. 462, Table 4.1) reported data on times to breakdown in minutes of an insulating fluid subjected to high voltage stress. The times are divided into six groups in the observed order. In analyzing these data, Nelson assumed an exponential model and compared the exponential means ($\theta_X$ and $\theta_Y$, say). He then constructed a two-sided confidence interval for the ratio $\theta_X/\theta_Y$. If 1 is not included in the $100(1-\alpha)\%$ confidence interval for the ratio $\theta_X/\theta_Y$, it is concluded that the two exponential means differ significantly at $\alpha\%$ level of significance. Similarly, a one-sided $100(1-\alpha)\%$ confidence interval for the ratio $\theta_X/\theta_Y$ can be used to test the hypothesis

$$\begin{aligned} & H_0 : \theta_X = \theta_Y \\ \text{against} \quad & H_1 : \theta_X < \theta_Y. \end{aligned} \tag{3.6.1}$$

If 1 is not included in the one-sided $100(1-\alpha)\%$ confidence interval for the ratio $\theta_X/\theta_Y$, then one could conclude that there is enough evidence in support of $H_1$ in (3.6.1). Note that testing the hypotheses in (3.6.1) is equivalent to testing the hypotheses in (1.3.1) under the exponential model.

### 3.6.1 Example 3.1

**Example 3.1:** For the first example, let us take our $X$-sample and $Y$-sample to be Sample 3 and Sample 2 in Nelson (1982, p.462), respectively. The observations are reported in Table 3.13.

For this example, if we had observed up to the 7-th breakdown from the $Y$-sample, then we would have observed only the first five observations from the $X$-sample. In this case, we have $n_1 = n_2 = 10$, $r = 7$, $m_1 = m_2 = 0$, $m_3 = m_4 = 1$, $m_5 = 0$, $m_6 = 3$, and $m_7 = 0$. Also, $\hat{\theta}_X = 1.872$, $\hat{\theta}_Y = 1.417$ and $\hat{\theta}_X/\hat{\theta}_Y = 1.321$. If the parametric testing procedure in Nelson (1982) (based on the assumption of exponentiality) is used, the $p$-value turns out to be 0.692, which will result in accepting $H_0$.

For the purpose of illustration of the nonparametric tests presented in the

Table 3.13: Times to insulating fluid breakdown data from Nelson (1982) for Samples 2 and 3

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $X$-sample(Sample 3) | **0.49** | **0.64** | **0.82** | **0.93** | **1.08** | 1.99 | 2.06 | 2.15 | 2.57 | 4.75 |
| $Y$-sample(Sample 2) | **0.00** | **0.18** | **0.55** | **0.66** | **0.71** | **1.30** | **1.63** | 2.17 | 2.75 | 10.60 |

earlier sections, we have given below the values of the test statistics and their corresponding $p$-values:

| Test | Test statistic | $p$-value |
|---|---|---|
| Precedence Test | $P_7 = 5$ | 0.672 |
| Maximal Precedence Test | $M_{(7)} = 3$ | 0.625 |
| Minimal Rank-sum Precedence Test | $W_{\text{min}} = 110$ | 0.848 |
| Maximal Rank-sum Precedence Test | $W_{\text{max}} = 125$ | 0.825 |
| Expected Rank-sum Precedence Test | $W_E = 117.5$ | 0.821 |

Based on these $p$-values, we conclude that there is a very strong evidence that the null hypothesis that the two samples are identically distributed is true.

### 3.6.2 Example 3.2

**Example 3.2:** For the second example, let us consider $X$-sample and $Y$-sample to be Sample 3 and Sample 6 in Nelson (1982), respectively. The observations are shown in Table 3.14.

For this data, if we had observed up to the 6-th breakdown from the $Y$-sample, then the first nine observations from the $X$-sample would have been observed. In this case, we have $n_1 = n_2 = 10$, $r = 6$, $m_1 = 5$, $m_2 = m_3 = 0$, $m_4 = 2$, $m_5 = 0$ and $m_6 = 2$. The following are the values of the test statistics and the corresponding $p$-values:

Table 3.14: Times to insulating fluid breakdown data from Nelson (1982) for Samples 3 and 6

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $X$-sample(Sample 3) | **0.49** | **0.64** | **0.82** | **0.93** | **1.08** | **1.99** | **2.06** | **2.15** | **2.57** | 4.75 |
| $Y$-sample(Sample 6) | **1.34** | **1.49** | **1.56** | **2.10** | **2.12** | **3.83** | 3.97 | 5.13 | 7.21 | 8.71 |

| Test | Test statistic | $p$-value |
|---|---|---|
| Precedence Test | $P_7 = 9$ | 0.065 |
| Maximal Precedence Test | $M_{(7)} = 5$ | 0.097 |
| Minimal Rank-sum Precedence Test | $W_{\min} = 77$ | 0.029 |
| Maximal Rank-sum Precedence Test | $W_{\max} = 81$ | 0.023 |
| Expected Rank-sum Precedence Test | $W_E = 79$ | 0.025 |

From these $p$-values, we observe that the data provide enough evidence to reject $H_0$.

The testing procedures we proposed here are applicable even though only a few early failures from the two samples are available as data. Furthermore, the decisions are reached without making any assumption on the life-time distributions underlying the two samples.

However, if one is willing to assume exponential distributions for the two samples, then we will find $\hat{\theta}_X = 1.7$, $\hat{\theta}_Y = 4.627$ and $\hat{\theta}_X/\hat{\theta}_Y = 0.367$, and the $p$-value of the test to be 0.027 [see Nelson (1982)] based on which we will decide to reject $H_0$ once again.

## 3.7 Summary and Conclusions

In this chapter, we introduced three Wilcoxon-type rank-sum precedence tests and derived their exact null distributions as well as their exact power functions under Lehmann alternatives. We simulated their power for various sample sizes and compared their performance with those of the precedence test, the maximal precedence test and the classical Wilcoxon's rank-sum test. Based on these results, we recommend the use of the Wilcoxon's minimal rank-sum precedence test. Finally, upon noting that the normal approximation for the null distribution is not generally satisfactory, we proposed an Edgeworth approximation and displayed that this method yields a very good approximation for the critical value.

# Chapter 4

# Weighted Precedence and Maximal Precedence Tests

## 4.1 Introduction

In this chapter, we introduce weighted precedence and maximal precedence tests for testing the hypothesis (1.3.1) that two distribution functions are equal, which is another logical extension of the precedence and maximal precedence tests. The motivation is explained by the following two cases in the example mentioned earlier in Section 1.3.

**Example 4.1:** A manufacturer of electronic components wishes to compare two designs A and B with respect to life. Specifically, he wants to abandon design A if there is evidence at the 0.05 level that it has shorter life. He places $n_1 = 10$ samples of design A and $n_2 = 10$ samples of design B simultaneously on a life-test. The test will be terminated when the 5-th failure from samples of design B occurs. Figure 4.1 and 4.2 show two possible outcomes of the life-testing experiment in this example.

The precedence and maximal precedence test statistics are equal in both cases. The test statistics are $P_{(5)} = 8$ and $M_{(5)} = 5$. In Tables 1.1 and 1.2, we have the critical values with $n_1 = n_2 = 10$ and $r = 5$ as 9 (with level of significance 0.02864) and 6 (with level of significance 0.02709) for the precedence and maximal precedence tests, respectively. Therefore, we will not reject the null hypothesis that two distributions

**Case 1:**

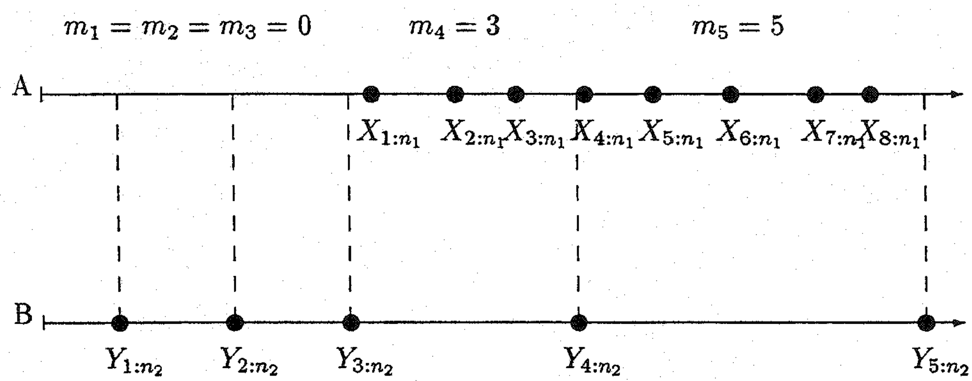



Figure 4.1: Case 1 of the precedence life-test for Example 4.1

**Case 2:**

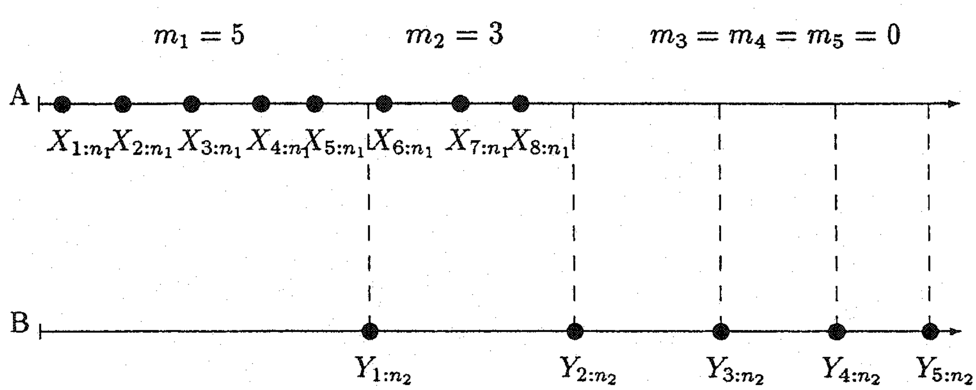



Figure 4.2: Case 2 of the precedence life-test for Example 4.1

are equal in both cases at the same level of significance. However, we feel that Case 2 provides much more evidence that the samples of design B is better than the samples of design A. This suggests that we should develop a testing procedure that distinguishes between Case 1 and Case 2. This is the basis for us to propose here the weighted precedence and maximal precedence tests.

## 4.2 Test Statistics and Null Distributions

In this section, we propose the weighted precedence and maximal precedence tests by giving a decreasing weight to $m_i$ as $i$ increases. The weighted precedence test statistic $P^*_{(r)}$ is defined as

$$P^*_{(r)} = \sum_{i=1}^{r} (n_2 - i + 1)\, m_i, \tag{4.2.1}$$

and the weighted precedence test statistic $M^*_{(r)}$ as

$$M^*_{(r)} = \max_{1 \le i \le r} \{(n_2 - i + 1)\, m_i\}. \tag{4.2.2}$$

It is obvious that large values of $P^*_{(r)}$ or $M^*_{(r)}$ would lead to the rejection of $H_0$ and in favor of $H_1$ in (1.3.1).

For a fixed level of significance $\alpha$, the critical region for the weighted precedence test will be $\{s, s+1, \ldots, n_1\}$, where

$$\alpha = \Pr(P^*_r \ge s | F_X = F_Y). \tag{4.2.3}$$

It follows from the result of Theorem 2.1 that, for specified values of $n_1, n_2, s$ and $r$, an expression for $\alpha$ in (4.2.3) is given by

$$\alpha = \sum_{\substack{m_i (i=1,2,\ldots,r)=0 \\ s \le \sum_{i=1}^{r}(n_2-i+1)m_i \le n_1}}^{n_1} \Pr(M_1 = m_1, M_2 = m_2, \ldots, M_r = m_r | F_X = F_Y). \tag{4.2.4}$$

Similarly, for a fixed level of significance $\alpha$, the critical region for the weighted maximal precedence test will be $\{t, t+1, \ldots, n_1\}$, where

$$\alpha = \Pr(M^*_{(r)} \ge t | F_X = F_Y)$$

$$= \sum_{\substack{m_i(i=1,2,\ldots,r)=0 \\ t \le \max\limits_{1\le i\le r}(n_2-i+1)m_i \le n_1}}^{n_1} \Pr(M_1 = m_1, M_2 = m_2, \ldots, M_r = m_r | F_X = F_Y), \tag{4.2.5}$$

where

$$\Pr(M_1 = m_1, M_2 = m_2, \ldots, M_r = m_r | F_X = F_Y) = \frac{\binom{n_1 + n_2 - \sum_{i=1}^{r} m_i - r}{n_2 - r}}{\binom{n_1+n_2}{n_2}}.$$

The critical values $s$ and $t$ and the exact level of significance $\alpha$ (as close as possible to 5%) for different choices of the sample sizes $n_1$ and $n_2$ and $r = 2(1)5$ are presented in Tables 4.1 and 4.2, respectively. It should be noted that for $r = 1$, the test statistics $P^*_{(1)}$ and $M^*_{(1)}$ become identical with $P_{(1)}$ and $M_{(1)}$, respectively.

For example, refer to Example 4.1, Case 1; in this case, the weighted precedence and maximal precedence test statistics are $P^*_{(5)} = (7 \times 3) + (6 \times 5) = 51$ and $M^*_{(5)} = \max\{(7 \times 3), (6 \times 5)\} = 30$. In Case 2, the weighted precedence and maximal precedence test statistics are $P^*_{(5)} = (10 \times 5) + (9 \times 3) = 77$ and $M^*_{(5)} = \max\{(10 \times 5), (9 \times 3)\} = 50$. Comparing these with the critical values in Tables 4.1 and 4.2, we will not reject the null hypothesis that two distributions are equal in Case 1, but we will conclude that design B is better than design A in Case 2 at close to 5% level of significance.

Table 4.1: Near 5% critical values and exact levels of significance for the weighted precedence test statistic $P^*_{(r)}$

| | | $r = 2$ | $r = 3$ | $r = 4$ | $r = 5$ |
|---|---|---|---|---|---|
| $n_1$ | $n_2$ | $s$ (exact l.o.s.) | $s$ (exact l.o.s.) | $s$ (exact l.o.s.) | $s$ (exact l.o.s.) |
| 10 | 10 | 48(0.04954) | 56(0.05096) | 62(0.05065) | 66(0.05108) |
| 15 | 15 | 75(0.04741) | 95(0.04945) | 107(0.05094) | 118(0.04991) |
| 20 | 20 | 100(0.05326) | 133(0.04979) | 152(0.05023) | 171(0.04967) |
| 30 | 30 | 174(0.05139) | 205(0.05082) | 250(0.04964) | 280(0.04996) |
| 30 | 50 | 198(0.04803) | 245(0.04945) | 290(0.05040) | 333(0.04965) |

Table 4.2: Near 5% critical values and exact levels of significance for the weighted maximal precedence test statistic $M^*_{(r)}$

| | | $r = 2$ | $r = 3$ | $r = 4$ | $r = 5$ |
|---|---|---|---|---|---|
| $n_1$ | $n_2$ | $t$ (exact l.o.s.) | $t$ (exact l.o.s.) | $t$ (exact l.o.s.) | $t$ (exact l.o.s.) |
| 10 | 10 | 40(0.05954) | 45(0.03792) | 42(0.04334) | 42(0.04489) |
| 15 | 15 | 70(0.04205) | 70(0.05042) | 72(0.04627) | 72(0.04942) |
| 20 | 20 | 95(0.04691) | 100(0.04359) | 100(0.05357) | 102(0.04439) |
| 30 | 30 | 145(0.05179) | 150(0.04943) | 162(0.04707) | 162(0.05225) |
| 30 | 50 | 150(0.06610) | 192(0.05138) | 196(0.04604) | 196(0.05181) |

## 4.3 Exact Power Under Lehmann Alternative

Here, we present an explicit expression for the power function of the weighted precedence and maximal precedence tests under the Lehmann alternative $H_1 : [F_X]^\gamma = F_Y$ for some $\gamma$ (see Section 2.3 for pertinent formulas).

The power functions under the Lehmann alternative for weighted precedence and maximal precedence tests are given by

$$\text{Power} = \sum_{\substack{m_i(i=1,2,\ldots,r)=0 \\ s\leq\sum_{i=1}^{r}(n_2-i+1)m_i\leq n_1}}^{n_1} \Pr(M_1 = m_1, M_2 = m_2, \ldots, M_r = m_r|[F_X]^\gamma = F_Y), \tag{4.3.6}$$

and

$$\text{Power} = \sum_{\substack{m_i(i=1,2,\ldots,r)=0 \\ t\leq \max\limits_{1\leq i\leq r}(n_2-i+1)m_i\leq n_1}}^{n_1} \Pr(M_1 = m_1, M_2 = m_2, \ldots, M_r = m_r|[F_X]^\gamma = F_Y), \tag{4.3.7}$$

respectively, where

$$\begin{aligned} &\Pr(M_1 = m_1, M_2 = m_2, \ldots, M_r = m_r|[F_X]^\gamma = F_Y) \\ &= \frac{n_1!n_2!\gamma^r}{m_1!(n_2-r)!}\left\{\prod_{j=1}^{r-1}\frac{\Gamma\left(\sum\limits_{i=1}^{j} m_i + j\gamma\right)}{\Gamma\left(\sum\limits_{i=1}^{j+1} m_i + j\gamma + 1\right)}\right\} \\ &\times\sum_{k=0}^{n_2-r}\binom{n_2-r}{k}(-1)^k\frac{\Gamma\left(\sum\limits_{i=1}^{r} m_i + (r+k)\gamma\right)}{\Gamma\left(n_1 + (r+k)\gamma + 1\right)}. \end{aligned} \tag{4.3.8}$$

Now, we compare the power under the Lehmann alternative for the precedence test (PT), maximal precedence test (MPT), the weighted precedence test (WPT), and the weighted maximal precedence test (WMPT). For $n_1 = n_2 = 10$, $r = 2, 3$ and $\gamma = 2(1)6$, the exact power values computed from (4.3.6)-(4.3.8) are presented in Table 4.3.

From Table 4.3, we see clearly that the weighted precedence and maximal precedence tests yield better power performance than the precedence and maximal

Table 4.3: Power comparison under Lehmann alternative for $n_1 = n_2 = 10, r = 2, 3$ and $\gamma = 2(1)6$

| $r$ | Test | $\gamma = 1$ | $\gamma = 2$ | $\gamma = 3$ | $\gamma = 4$ | $\gamma = 5$ | $\gamma = 6$ |
|---|---|---|---|---|---|---|---|
| 2 | PT | 0.02864 | 0.23970 | 0.49048 | 0.67299 | 0.78963 | 0.86216 |
| | MPT | 0.03250 | 0.22095 | 0.45342 | 0.63634 | 0.76059 | 0.84119 |
| | WPT | 0.04954 | 0.35681 | 0.64389 | 0.80970 | 0.89619 | 0.94126 |
| | WMPT | 0.05954 | 0.35732 | 0.62980 | 0.79435 | 0.88451 | 0.93342 |
| 3 | PT | 0.03489 | 0.24581 | 0.48087 | 0.65212 | 0.76460 | 0.83731 |
| | MPT | 0.04875 | 0.24178 | 0.46802 | 0.64561 | 0.76641 | 0.84490 |
| | WPT | 0.05096 | 0.35100 | 0.63139 | 0.79597 | 0.88402 | 0.93137 |
| | WMPT | 0.03792 | 0.22811 | 0.45815 | 0.63914 | 0.76225 | 0.84220 |

precedence tests under Lehmann alternative, with weighted precedence test being overall the best.

## 4.4 Simulated Power Under Location-Shift Alternative

In order to further examine the power properties of the precedence and maximal precedence tests with the corresponding weighted versions, we consider the location-shift alternative $H_1 : F_X(x) = F_Y(x + \theta)$ for some $\theta > 0$, where $\theta$ is a shift in location.

Power of all four tests were estimated through Monte Carlo simulations when $\theta = 0.5$ and 1.0. The following lifetime distributions were used in this study:

1. Standard normal distribution;

2. Standard exponential distribution;

3. Gamma distribution with shape parameter $a$ and standardized by mean $a$ and standard deviation $\sqrt{a}$;

4. Lognormal distribution with shape parameter $\sigma$ and standardized by mean $e^{\sigma^2/2}$ and standard deviation $\sqrt{e^{\sigma^2}(e^{\sigma^2} - 1)}$;

5. Standard extreme-value distribution standardized by mean $-0.5772156649$

(Euler's constant) and standard deviation $\pi/\sqrt{6}$.
For different choices of sample sizes, we generated 10,000 sets of data in order to obtain the estimated rejection rates. In Tables 4.4 and 4.5, we have presented the estimated power values of the precedence and maximal precedence tests and their weighted versions for $r = 2(1)6$ for the distributions listed above with location-shift being equal to 0.5 and 1.0. For comparison purposes, the corresponding exact levels of significance are also presented.

When we compare the power values of the weighted precedence and maximal precedence tests with those of precedence and maximal precedence tests, we find that the weighted precedence and maximal precedence tests perform better than the precedence and maximal precedence tests if the underlying distributions are right-skewed, such as the exponential distribution, gamma distribution with shape parameter $a = 2.0$, and lognormal distribution with shape parameter $\sigma = 0.5$. In addition, even if the underlying distributions are nearly symmetric, the power values of the weighted precedence and maximal precedence tests are close to those of usual precedence and maximal precedence tests. Overall, we find that the weighted precedence test outperforms the other tests in almost all cases considered.

Table 4.4: Power of precedence, maximal precedence, weighted precedence and maximal precedence tests for $n_1 = n_2 = 10$

| | | $\theta = 0.5$ | | | | $\theta = 1.0$ | | | |
|---|---|---|---|---|---|---|---|---|---|
| $r$ | Dist. | PT | MPT | WPT | WMPT | PT | MPT | WPT | WMPT |
| 2 | Exact l.o.s. | 0.0286 | 0.0325 | 0.0495 | 0.0595 | 0.0286 | 0.0325 | 0.0495 | 0.0595 |
| | N(0,1) | 0.1496 | 0.1417 | 0.2275 | 0.2309 | 0.4243 | 0.3768 | 0.5278 | 0.5067 |
| | Exp(1) | 0.3916 | 0.4923 | 0.6145 | 0.7197 | 0.8376 | 0.9177 | 0.9468 | 0.9784 |
| | Gamma(2) | 0.2525 | 0.2474 | 0.4080 | 0.4344 | 0.6879 | 0.7303 | 0.8525 | 0.8881 |
| | Gamma(10) | 0.1731 | 0.1611 | 0.2708 | 0.2735 | 0.5060 | 0.4569 | 0.6487 | 0.6313 |
| | LN(0.1) | 0.1603 | 0.1479 | 0.2379 | 0.2457 | 0.4571 | 0.4083 | 0.5803 | 0.5535 |
| | LN(0.5) | 0.2693 | 0.2543 | 0.4206 | 0.4301 | 0.7253 | 0.7327 | 0.8727 | 0.8801 |
| | EV | 0.1463 | 0.1388 | 0.2002 | 0.2034 | 0.3969 | 0.3573 | 0.4691 | 0.4504 |
| 3 | Exact l.o.s. | 0.0349 | 0.0487 | 0.0510 | 0.0379 | 0.0349 | 0.0487 | 0.0510 | 0.0379 |
| | N(0,1) | 0.1779 | 0.1753 | 0.2400 | 0.1542 | 0.4813 | 0.4171 | 0.5839 | 0.3942 |
| | Exp(1) | 0.3073 | 0.4949 | 0.5124 | 0.4886 | 0.7239 | 0.9178 | 0.8824 | 0.9164 |
| | Gamma(2) | 0.2343 | 0.2579 | 0.3707 | 0.2483 | 0.6148 | 0.7327 | 0.7987 | 0.7338 |
| | Gamma(10) | 0.1893 | 0.1870 | 0.2724 | 0.1739 | 0.5177 | 0.4765 | 0.6540 | 0.4610 |
| | LN(0.1) | 0.1828 | 0.1781 | 0.2518 | 0.1562 | 0.4964 | 0.4395 | 0.6087 | 0.4145 |
| | LN(0.5) | 0.2571 | 0.2678 | 0.3839 | 0.2559 | 0.6676 | 0.7360 | 0.8221 | 0.7289 |
| | EV | 0.1941 | 0.1869 | 0.2414 | 0.1584 | 0.5204 | 0.4398 | 0.5769 | 0.4007 |
| 4 | Exact l.o.s. | 0.0348 | 0.0650 | 0.0507 | 0.0449 | 0.0348 | 0.0650 | 0.0507 | 0.0449 |
| | N(0,1) | 0.1771 | 0.1993 | 0.2492 | 0.1715 | 0.4799 | 0.4359 | 0.6083 | 0.3984 |
| | Exp(1) | 0.2251 | 0.4974 | 0.4210 | 0.4854 | 0.5767 | 0.9181 | 0.7921 | 0.9155 |
| | Gamma(2) | 0.1936 | 0.2653 | 0.3360 | 0.2522 | 0.5114 | 0.7344 | 0.7172 | 0.7446 |
| | Gamma(10) | 0.1755 | 0.2040 | 0.2684 | 0.1791 | 0.4758 | 0.4853 | 0.6435 | 0.4711 |
| | LN(0.1) | 0.1777 | 0.1990 | 0.2560 | 0.1709 | 0.4783 | 0.4538 | 0.6165 | 0.4170 |
| | LN(0.5) | 0.2149 | 0.2768 | 0.3549 | 0.2671 | 0.5656 | 0.7376 | 0.7765 | 0.7381 |
| | EV | 0.2221 | 0.2228 | 0.2720 | 0.1799 | 0.5935 | 0.4862 | 0.6499 | 0.4330 |
| 5 | Exact l.o.s. | 0.0271 | 0.0271 | 0.0511 | 0.0449 | 0.0271 | 0.0271 | 0.0511 | 0.0449 |
| | N(0,1) | 0.1473 | 0.0901 | 0.2558 | 0.1638 | 0.4245 | 0.2482 | 0.6332 | 0.4047 |
| | Exp(1) | 0.1453 | 0.2684 | 0.4246 | 0.4857 | 0.3960 | 0.7791 | 0.8193 | 0.9111 |
| | Gamma(2) | 0.1380 | 0.1197 | 0.3372 | 0.2512 | 0.3725 | 0.5136 | 0.7373 | 0.7409 |
| | Gamma(10) | 0.1388 | 0.0895 | 0.2669 | 0.1739 | 0.3833 | 0.2833 | 0.6558 | 0.4684 |
| | LN(0.1) | 0.1444 | 0.0905 | 0.2609 | 0.1684 | 0.4038 | 0.2586 | 0.6334 | 0.4262 |
| | LN(0.5) | 0.1527 | 0.1297 | 0.3541 | 0.2702 | 0.4144 | 0.5284 | 0.7821 | 0.7464 |
| | EV | 0.2143 | 0.1131 | 0.2935 | 0.1699 | 0.6073 | 0.2945 | 0.7073 | 0.4223 |
| 6 | Exact l.o.s. | 0.0704 | 0.0325 | 0.0490 | 0.0449 | 0.0704 | 0.0325 | 0.0490 | 0.0449 |
| | N(0,1) | 0.2659 | 0.0938 | 0.2589 | 0.1715 | 0.5832 | 0.2496 | 0.6372 | 0.3984 |
| | Exp(1) | 0.2385 | 0.2690 | 0.4105 | 0.4854 | 0.4991 | 0.7791 | 0.7965 | 0.9155 |
| | Gamma(2) | 0.2344 | 0.1208 | 0.3255 | 0.2522 | 0.4954 | 0.5138 | 0.7200 | 0.7446 |
| | Gamma(10) | 0.2436 | 0.0921 | 0.2794 | 0.1791 | 0.5266 | 0.2839 | 0.6643 | 0.4711 |
| | LN(0.1) | 0.2567 | 0.0938 | 0.2597 | 0.1709 | 0.5545 | 0.2596 | 0.6380 | 0.4170 |
| | LN(0.5) | 0.2531 | 0.1309 | 0.3466 | 0.2671 | 0.5314 | 0.5286 | 0.7620 | 0.7381 |
| | EV | 0.3716 | 0.1203 | 0.3021 | 0.1799 | 0.7785 | 0.3000 | 0.7407 | 0.4330 |

Table 4.5: Power of precedence, maximal precedence, weighted precedence and maximal precedence tests for $n_1 = n_2 = 20$

| | | $\theta = 0.5$ | | | | $\theta = 1.0$ | | | |
|---|---|---|---|---|---|---|---|---|---|
| $r$ | Dist. | PT | MPT | WPT | WMPT | PT | MPT | WPT | WMPT |
| 2 | Exact l.o.s. | 0.0457 | 0.0469 | 0.0533 | 0.0469 | 0.0457 | 0.0469 | 0.0533 | 0.0469 |
| | N(0,1) | 0.2662 | 0.2434 | 0.2906 | 0.2401 | 0.6529 | 0.5994 | 0.6780 | 0.5985 |
| | Exp(1) | 0.9355 | 0.9634 | 0.9700 | 0.9642 | 0.9998 | 1.0000 | 0.9998 | 0.9998 |
| | Gamma(2) | 0.6711 | 0.6580 | 0.7352 | 0.6563 | 0.9914 | 0.9934 | 0.9958 | 0.9932 |
| | Gamma(10) | 0.3638 | 0.3235 | 0.3929 | 0.3240 | 0.8371 | 0.7893 | 0.8599 | 0.7905 |
| | LN(0.1) | 0.3038 | 0.2741 | 0.3201 | 0.2664 | 0.7338 | 0.6788 | 0.7575 | 0.6743 |
| | LN(0.5) | 0.6453 | 0.5983 | 0.6917 | 0.6048 | 0.9901 | 0.9882 | 0.9940 | 0.9885 |
| | EV | 0.1992 | 0.1865 | 0.2154 | 0.1845 | 0.4774 | 0.4412 | 0.5001 | 0.4481 |
| 3 | Exact l.o.s. | 0.0637 | 0.0302 | 0.0498 | 0.0436 | 0.0637 | 0.0302 | 0.0498 | 0.0436 |
| | N(0,1) | 0.3485 | 0.1811 | 0.3096 | 0.2328 | 0.7615 | 0.5249 | 0.7359 | 0.6088 |
| | Exp(1) | 0.9034 | 0.9061 | 0.9015 | 0.9626 | 0.9993 | 0.9996 | 0.9994 | 1.0000 |
| | Gamma(2) | 0.6767 | 0.4842 | 0.6741 | 0.6488 | 0.9886 | 0.9785 | 0.9886 | 0.9913 |
| | Gamma(10) | 0.4391 | 0.2333 | 0.4102 | 0.3169 | 0.8891 | 0.6880 | 0.8816 | 0.7941 |
| | LN(0.1) | 0.3833 | 0.2009 | 0.3486 | 0.2699 | 0.8210 | 0.5894 | 0.8047 | 0.6874 |
| | LN(0.5) | 0.6838 | 0.4468 | 0.6769 | 0.5879 | 0.9907 | 0.9644 | 0.9908 | 0.9857 |
| | EV | 0.2864 | 0.1549 | 0.2493 | 0.1933 | 0.6310 | 0.4285 | 0.5919 | 0.4823 |
| 4 | Exact l.o.s. | 0.0412 | 0.0403 | 0.0502 | 0.0536 | 0.0412 | 0.0403 | 0.0502 | 0.0536 |
| | N(0,1) | 0.2947 | 0.2138 | 0.3301 | 0.2576 | 0.7352 | 0.5605 | 0.7800 | 0.6381 |
| | Exp(1) | 0.7499 | 0.9053 | 0.8582 | 0.9617 | 0.9932 | 0.9997 | 0.9986 | 1.0000 |
| | Gamma(2) | 0.5317 | 0.4949 | 0.6453 | 0.6603 | 0.9615 | 0.9790 | 0.9829 | 0.9919 |
| | Gamma(10) | 0.3583 | 0.2546 | 0.4231 | 0.3442 | 0.8418 | 0.7030 | 0.8900 | 0.7923 |
| | LN(0.1) | 0.3219 | 0.2273 | 0.3695 | 0.2964 | 0.7877 | 0.6170 | 0.8298 | 0.7160 |
| | LN(0.5) | 0.5614 | 0.4572 | 0.6640 | 0.5993 | 0.9716 | 0.9662 | 0.9881 | 0.9872 |
| | EV | 0.2642 | 0.1963 | 0.2864 | 0.2304 | 0.6594 | 0.5061 | 0.6751 | 0.5459 |
| 5 | Exact l.o.s. | 0.0479 | 0.0503 | 0.0497 | 0.0444 | 0.0479 | 0.0503 | 0.0497 | 0.0444 |
| | N(0,1) | 0.3317 | 0.2360 | 0.3501 | 0.2251 | 0.7787 | 0.5833 | 0.7999 | 0.5641 |
| | Exp(1) | 0.6904 | 0.9058 | 0.8043 | 0.9038 | 0.9864 | 0.9997 | 0.9964 | 0.9993 |
| | Gamma(2) | 0.5173 | 0.5004 | 0.5978 | 0.4968 | 0.9487 | 0.9793 | 0.9749 | 0.9770 |
| | Gamma(10) | 0.3808 | 0.2709 | 0.4189 | 0.2631 | 0.8532 | 0.7115 | 0.8852 | 0.7071 |
| | LN(0.1) | 0.3525 | 0.2472 | 0.3816 | 0.2471 | 0.8163 | 0.6327 | 0.8401 | 0.6320 |
| | LN(0.5) | 0.5555 | 0.4642 | 0.6250 | 0.4574 | 0.9648 | 0.9666 | 0.9840 | 0.9641 |
| | EV | 0.3182 | 0.2276 | 0.3128 | 0.2010 | 0.7423 | 0.5564 | 0.7448 | 0.5358 |
| 6 | Exact l.o.s. | 0.0527 | 0.0602 | 0.0495 | 0.0485 | 0.0527 | 0.0602 | 0.0495 | 0.0485 |
| | N(0,1) | 0.3565 | 0.2545 | 0.3598 | 0.2252 | 0.8055 | 0.5977 | 0.8208 | 0.5729 |
| | Exp(1) | 0.6291 | 0.9062 | 0.7498 | 0.9023 | 0.9733 | 0.9997 | 0.9912 | 0.9997 |
| | Gamma(2) | 0.4966 | 0.5044 | 0.5678 | 0.4969 | 0.9323 | 0.9796 | 0.9642 | 0.9751 |
| | Gamma(10) | 0.3920 | 0.2834 | 0.4210 | 0.2667 | 0.8558 | 0.7171 | 0.8813 | 0.7015 |
| | LN(0.1) | 0.3706 | 0.2626 | 0.3777 | 0.2428 | 0.8295 | 0.6423 | 0.8604 | 0.6303 |
| | LN(0.5) | 0.5369 | 0.4688 | 0.5982 | 0.4581 | 0.9540 | 0.9668 | 0.9770 | 0.9668 |
| | EV | 0.3620 | 0.2560 | 0.3500 | 0.2254 | 0.8055 | 0.5974 | 0.8039 | 0.5609 |

Table 4.6: Times to insulating fluid breakdown data from Nelson (1982) for Samples 2 and 3

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $X$-sample(Sample 3) | **0.49** | **0.64** | **0.82** | **0.93** | **1.08** | 1.99 | 2.06 | 2.15 | 2.57 | 4.75 |
| $Y$-sample(Sample 2) | **0.00** | **0.18** | **0.55** | **0.66** | **0.71** | **1.30** | 1.63 | 2.17 | 2.75 | 10.60 |

## 4.5 Illustrative Example

We illustrate here the weighted precedence and maximal precedence tests proposed in Section 4.3.

**Example 4.2:** Refer to Example 3.1; let us take the $X$-sample and $Y$-sample to be Sample 3 and Sample 2 in Nelson (1982, p. 462), respectively. In this example, if we had observed only up to the 6-th breakdown from the $Y$-sample, then we would have observed only the first five $X$-failures. The resulting observations are presented in Table 4.6.

In this case, we would have $n_1 = n_2 = 10$, $r = 5$, $m_1 = m_2 = 0$, $m_3 = m_4 = 1$, $m_5 = 0$ and $m_6 = 3$. Also, $\hat{\theta}_X = 1.872$, $\hat{\theta}_Y = 1.417$ and $\hat{\theta}_X/\hat{\theta}_Y = 1.321$. If the parametric testing procedure in Nelson (1982) (based on the assumption of exponentiality) is used, the $p$-value will be 0.692, which will result in us accepting $H_0$.

For the purpose of illustration of the nonparametric tests discussed in the preceding sections, we present the following values of the test statistics and their corresponding $p$-values:

| Test | Test statistic | $p$-value |
|---|---|---|
| Precedence Test | $P_{(6)} = 5$ | 0.672 |
| Maximal Precedence Test | $M_{(6)} = 3$ | 0.552 |
| Weighted Precedence Test | $P^*_{(6)} = 30$ | 0.744 |
| Weighted Maximal Precedence Test | $M^*_{(6)} = 15$ | 0.756 |

Based on these $p$-values, we once again do not reject the null hypothesis that the two samples are identically distributed.

# Chapter 5

# An Extension to Progressive Censoring

## 5.1 Introduction

As mentioned earlier in Section 1.3, precedence and maximal precedence tests are particularly useful (i) when life-tests involve expensive units since the units that had not failed could be used for some other testing purposes, and (ii) to make quick and reliable decisions early on in the life-testing experiment. We note here that in a precedence test, since the life testing continues until the occurrence of the $r$-th $Y$-failure, it may be viewed as a test based on a Type-II right censored $Y$-sample. Therefore, if we want to save the testing units at an early stage of the life-test, a Type-II progressive censoring scheme may instead be employed on the $Y$-sample [see Balakrishnan and Aggarawala (2000)]. This will then be a logical extension of the precedence and maximal precedence tests.

## 5.2 Test Statistics and Null Distributions

### 5.2.1 Test Statistics

Suppose a Type-II progressive censoring scheme is to be adopted on the $Y$-sample which means that a number $r < n_2$ is prefixed for the number of complete failures to be observed and the censoring scheme $(R_1, R_2, \ldots, R_r)$ with $R_j > 0$ and $\sum_{j=1}^{r} R_j + r = n_2$ is to be employed at the failure times. During the life-testing experiment, at the time of the $j$-th failure, $R_j$ functioning items are randomly removed from the test. We denote such an observed ordered $Y$-sample by $Y_{1:r:n_2} \leq Y_{2:r:n_2} \leq \cdots \leq Y_{r:r:n_2}$. Moreover, we denote by $M_1$ the number of $X$-failures before $Y_{1:r:n_2}$, and by $M_i$ the number of $X$-failures between $Y_{i-1:r:n_2}$ and $Y_{i:r:n_2}$, $i = 2, 3, \ldots, r$.

Then, based on $M_1, M_2, \ldots, M_r$, we propose the weighted precedence test statistic $P^*_{(r)}$ [analogous to (4.2.1)] as

$$P^*_{(r)} = \sum_{i=1}^{r} \left[ n_2 - \left( \sum_{j=1}^{i-1} R_j \right) - i + 1 \right] m_i, \tag{5.2.1}$$

and the weighted maximal precedence test statistic $M^*_{(r)}$ [analogous to (4.2.2)] as

$$M^*_{(r)} = \max_{1 \leq i \leq r} \left\{ \left[ n_2 - \left( \sum_{j=1}^{i-1} R_j \right) - i + 1 \right] m_i \right\}. \tag{5.2.2}$$

For example, from Figure 5.1, with $n_2 = 10$, $r = 4$ and the censoring scheme $(2, 1, 1, 2)$, the weighted precedence test statistic becomes $P^*_{(4)} = (10 \times 0) + (7 \times 3) + (5 \times 4) + (3 \times 1) = 44$, while the weighted maximal precedence test statistic becomes $M^*_{(4)} = \max\{(10 \times 0), (7 \times 3), (5 \times 4), (3 \times 1)\} = 21$. It is clear that large values of $P^*_{(r)}$ or $M^*_{(r)}$ would lead to the rejection of $H_0$ and in favor of $H_1$ in (1.3.1).

It is important to mention here that the weighted precedence and maximal precedence test statistics defined earlier in (4.2.1) and (4.2.2) become special cases when we set the progressive censoring scheme as $R_1 = R_2 = \ldots = R_{r-1} = 0$ and $R_r = n_2 - r$.

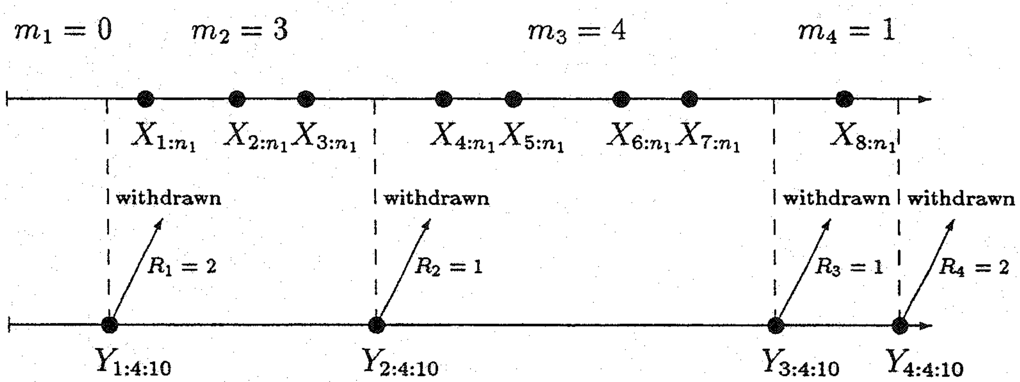



Figure 5.1: Schematic representation of a precedence life-test with progressive censoring

### 5.2.2 Null Distributions

**Theorem 5.1** *With a Type-II progressive censoring on the Y-sample, under the null hypothesis $F_X = F_Y$, the joint probability mass function of $M_1$, $M_2$, ..., $M_r$ is given by*

$$\begin{aligned} & \Pr(M_1 = m_1, M_2 = m_2, \ldots, M_r = m_r \mid F_X = F_Y) \\ = \; & C \sum_{\substack{j_k = 0 \\ (k=1,2,\ldots,r-1)}}^{\sum_{i=1}^{k} R_i - \sum_{i=1}^{k-1} j_i} \left\{ \prod_{l=1}^{r-1} \binom{\sum_{i=1}^{l} R_i - \sum_{i=1}^{l} j_i}{j_l} \Gamma(m_l + j_{l+1} + 1) \right\} \\ & \qquad \frac{\Gamma\left(n_1 + n_2 - r - \sum_{i=1}^{r} m_i - \sum_{i=1}^{r-1} j_i + 1\right)}{\Gamma(n_1 + n_2 + 1)}, \end{aligned} \tag{5.2.3}$$

*where*

$$C = \frac{n_1! n_2 (n_2 - R_1 - 1) \cdots (n_2 - R_1 - R_2 - \ldots - R_{r-1} - r + 1)}{m_1! m_2! \cdots m_r! \left(n_1 - \sum_{i=1}^{r} m_i\right)!}.$$

**Proof:** The derivation of the formula in (5.2.3) is presented in Appendix B.

For the weighted precedence test with specified values of $n_1, n_2, s, r$ and the

Table 5.1: Near 5% critical values and exact levels of significance for the modified precedence test statistic $P^*_{(r)}$ under progressive censoring

| | | $r=2$ | | $r=3$ | | $r=4$ | | $r=5$ | |
|---|---|---|---|---|---|---|---|---|---|
| $n_1$ | $n_2$ | censoring scheme | $s$ (exact l.o.s) | censoring scheme | $s$ (exact l.o.s) | censoring scheme | $s$ (exact l.o.s) | censoring scheme | $s$ (exact l.o.s) |
| 10 | 10 | (8,0) | 37(0.05108) | (7,0,0) | 42(0.04829) | (6,0,0,0) | 45(0.05166) | (5,0,0,0,0) | 49(0.05151) |
| | | (6,2) | 41(0.04782) | (5,1,1) | 46(0.04913) | (3,1,1,1) | 52(0.05349) | (3,0,0,0,2) | 56(0.05092) |
| | | (4,4) | 45(0.05037) | (2,2,3) | 51(0.04956) | (1,1,2,2) | 57(0.04929) | (1,1,1,1,1) | 60(0.04923) |
| 15 | 15 | (13,0) | 60(0.04981) | (12,0,0) | 66(0.04949) | (11,0,0,0) | 72(0.05051) | (10,0,0,0,0) | 79(0.04875) |
| | | (10,3) | 65(0.05164) | (10,1,1) | 72(0.04944) | (8,1,1,1) | 82(0.04943) | (6,1,1,1,1) | 93(0.04845) |
| | | (6,7) | 71(0.04903) | (4,4,4) | 85(0.05235) | (2,3,3,3) | 96(0.05039) | (2,2,2,2,2) | 105(0.04884) |
| 20 | 20 | (18,0) | 82(0.04955) | (17,0,0) | 90(0.05052) | (16,0,0,0) | 99(0.05036) | (15,0,0,0,0) | 108(0.05045) |
| | | (14,4) | 95(0.04691) | (14,2,1) | 102(0.04865) | (13,1,1,1) | 112(0.05105) | (11,1,1,1,1) | 126(0.05171) |
| | | (9,9) | 100(0.05680) | (5,6,6) | 119(0.05061) | (4,4,4,4) | 140(0.04559) | (3,3,3,3,3) | 152(0.05002) |
| 30 | 30 | (28,0) | 126(0.04951) | (27,0,0) | 139(0.05008) | (26,0,0,0) | 153(0.05047) | (25,0,0,0,0) | 168(0.04947) |
| | | (22,6) | 146(0.05004) | (22,3,2) | 162(0.05064) | (21,2,2,2) | 180(0.04980) | (20,2,1,1,1) | 196(0.05069) |
| | | (14,14) | 165(0.04498) | (9,9,9) | 190(0.05528) | (6,6,7,7) | 222(0.05051) | (5,5,5,5,5) | 252(0.04813) |
| 30 | 50 | (48,0) | 150(0.04942) | (47,0,0) | 153(0.05032) | (46,0,0,0) | 170(0.04982) | (45,0,0,0,0) | 185(0.04990) |
| | | (38,10) | 165(0.04862) | (38,5,4) | 188(0.04970) | (37,3,3,3) | 212(0.04965) | (36,3,2,2,2) | 230(0.04949) |
| | | (24,24) | 175(0.06151) | (15,16,16) | 219(0.05007) | (11,11,12,12) | 255(0.04929) | (9,9,9,9,9) | 290(0.05121) |

progressive censoring scheme $(R_1, R_2, \ldots, R_r)$, an expression for the level of significance $\alpha$ is obtained from (5.2.3) as

$$\alpha = \sum_{\substack{m_i(i=1,2,\ldots,r)=0 \\ s \le \sum_{i=1}^{r}\left[n_2-\left(\sum_{j=1}^{i-1} R_j\right)-i+1\right] m_i \le n_1}}^{n_1} \Pr(M_1 = m_1, M_2 = m_2, \ldots, M_r = m_r \mid F_X = F_Y). \tag{5.2.4}$$

Similarly, for the weighted maximal precedence test with specified values of $n_1, n_2, t, r$ and the censoring scheme $(R_1, R_2, \ldots, R_r)$, an expression for $\alpha$ is given by

$$\alpha = \sum_{\substack{m_i(i=1,2,\ldots,r)=0 \\ t \le \max\limits_{1\le i\le r}\left[n_2-\left(\sum_{j=1}^{i-1} R_j\right)-i+1\right] m_i \le n_1}}^{n_1} \Pr(M_1 = m_1, M_2 = m_2, \ldots, M_r = m_r \mid F_X = F_Y), \tag{5.2.5}$$

where $\Pr(M_1 = m_1, M_2 = m_2, \ldots, M_r = m_r \mid F_X = F_Y)$ is as given in (5.2.3).

The critical values $s, t$ and the exact level of significance $\alpha$ (as close as possible to 5%) for different choices of the sample sizes $n_1$ and $n_2$, $r = 2(1)5$, and different progressive censoring schemes, are presented in Tables 5.1 and 5.2.

Table 5.2: Near 5% critical values and exact levels of significance for the maximal precedence test statistic $M^*_{(r)}$ under progressive censoring

| | | $r=2$ | | $r=3$ | | $r=4$ | | $r=5$ | |
|---|---|---|---|---|---|---|---|---|---|
| $n_1$ | $n_2$ | censoring scheme | $t$ (exact l.o.s) | censoring scheme | $t$ (exact l.o.s) | censoring scheme | $t$ (exact l.o.s) | censoring scheme | $t$ (exact l.o.s) |
| 10 | 10 | (8,0) | 40(0.04334) | (7,0,0) | 40(0.04334) | (6,0,0,0) | 40(0.04334) | (5,0,0,0,0) | 32(0.05294) |
| | | (6,2) | 40(0.04334) | (5,1,1) | 32(0.05294) | (3,1,1,1) | 40(0.05059) | (3,0,0,0,2) | 40(0.05263) |
| | | (4,4) | 40(0.04773) | (2,2,3) | 40(0.05567) | (1,1,2,2) | 40(0.06985) | (1,1,1,1,1) | 40(0.06994) |
| 15 | 15 | (13,0) | 60(0.04981) | (12,0,0) | 60(0.04981) | (11,0,0,0) | 60(0.04981) | (10,0,0,0,0) | 60(0.04994) |
| | | (10,3) | 60(0.04994) | (10,1,1) | 60(0.04994) | (8,1,1,1) | 60(0.05600) | (6,1,1,1,1) | 60(0.06338) |
| | | (6,7) | 60(0.05986) | (4,4,4) | 65(0.03202) | (2,3,3,3) | 64(0.04202) | (2,2,2,2,2) | 63(0.04669) |
| 20 | 20 | (18,0) | 80(0.05301) | (17,0,0) | 80(0.05301) | (16,0,0,0) | 80(0.05301) | (15,0,0,0,0) | 80(0.05306) |
| | | (14,4) | 80(0.05463) | (14,2,1) | 80(0.05463) | (13,1,1,1) | 80(0.05595) | (11,1,1,1,1) | 80(0.06578) |
| | | (9,9) | 90(0.03313) | (5,6,6) | 84(0.05222) | (4,4,4,4) | 90(0.05136) | (3,3,3,3,3) | 96(0.04975) |
| 30 | 30 | (28,0) | 120(0.05620) | (27,0,0) | 120(0.05620) | (26,0,0,0) | 120(0.05620) | (25,0,0,0,0) | 120(0.05621) |
| | | (22,6) | 120(0.06012) | (22,3,2) | 120(0.06012) | (21,2,2,2) | 120(0.06491) | (20,2,1,1,1) | 126(0.03521) |
| | | (14,14) | 135(0.04024) | (9,9,9) | 130(0.05092) | (6,6,7,7) | 144(0.04837) | (5,5,5,5,5) | 150(0.04488) |
| 30 | 50 | (48,0) | 150(0.04942) | (47,0,0) | 150(0.04942) | (46,0,0,0) | 150(0.04942) | (45,0,0,0,0) | 108(0.04999) |
| | | (38,10) | 150(0.05302) | (38,5,4) | 150(0.05302) | (37,3,3,3) | 150(0.05430) | (36,3,2,2,2) | 150(0.05571) |
| | | (24,24) | 150(0.06803) | (15,16,16) | 153(0.04330) | (11,11,12,12) | 156(0.04935) | (9,9,9,9,9) | 170(0.04295) |

## 5.3 Exact Power Under Lehmann Alternative

**Theorem 5.2** *With a Type-II progressive censoring on the $Y$-sample, under the Lehmann alternative $[F_X]^\gamma = F_Y, \gamma > 1$, the joint probability mass function of $M_1$, $M_2$, …, $M_r$ is given by*

$$\begin{aligned}
&\Pr(M_1 = m_1, M_2 = m_2, \ldots, M_r = m_r \mid [F_X]^\gamma = F_Y) \\
= \;& C\gamma^r \sum_{j_i(i=1,2,\ldots,r)=0}^{R_i} \left\{ \binom{R_1}{j_1}\binom{R_2}{j_2}\cdots\binom{R_r}{j_r}(-1)^{\left(\sum_{i=1}^{r} j_i\right)} \right. \\
& \left[\prod_{k=1}^{r-1} B\left(\sum_{i=1}^{k} m_i + \gamma\left(\sum_{i=1}^{k} j_i\right) + k\gamma, m_{k+1}+1\right)\right] \\
& \left. B\left(\sum_{i=1}^{r} m_i + \gamma\left(\sum_{i=1}^{r} j_i\right) + r\gamma, n_1 - \sum_{i=1}^{r} m_i + 1\right)\right\}.
\end{aligned} \tag{5.3.1}$$

**Proof:** The derivation of the formula in (5.3.1) is presented in Appendix B.

The power function of the precedence and maximal precedence tests under the Lehmann alternative are then given by

$$\sum_{\substack{m_i(i=1,2,\ldots,r)=0 \\ s \le \sum_{i=1}^{r}\left[n_2 - \left(\sum_{j=1}^{i-1} R_j\right) - i + 1\right] m_i \le n_1}}^{n_1} \Pr(M_1 = m_1, M_2 = m_2, \ldots, M_r = m_r \mid [F_X]^\gamma = F_Y) \tag{5.3.2}$$

Table 5.3: Power values under Lehmann alternative for $n_1 = n_2 = 10, r = 2$ and $\gamma = 2(1)6$

| Censoring Scheme | Test | $\gamma=1$ | $\gamma=2$ | $\gamma=3$ | $\gamma=4$ | $\gamma=5$ | $\gamma=6$ |
|---|---|---|---|---|---|---|---|
| (0,8) | WPT | 0.04954 | 0.35681 | 0.64389 | 0.80970 | 0.89619 | 0.94126 |
| | WMPT | 0.05954 | 0.35732 | 0.62980 | 0.79435 | 0.88451 | 0.93342 |
| (8,0) | WPT | 0.05108 | 0.34707 | 0.62563 | 0.79312 | 0.88424 | 0.93340 |
| | WMPT | 0.04334 | 0.31471 | 0.58850 | 0.76277 | 0.86219 | 0.91805 |
| (6,2) | WPT | 0.04782 | 0.35013 | 0.63832 | 0.80730 | 0.89596 | 0.94204 |
| | WMPT | 0.04334 | 0.31471 | 0.58850 | 0.76277 | 0.86219 | 0.91805 |
| (4,4) | WPT | 0.05037 | 0.36112 | 0.65009 | 0.81566 | 0.90104 | 0.94495 |
| | WMPT | 0.04773 | 0.32292 | 0.59495 | 0.76692 | 0.86471 | 0.91958 |

and

$$\sum_{\substack{m_i(i=1,2,\ldots,r)=0 \\ t\le \max_{1\le i\le r}\left[n_2-\left(\sum_{j=1}^{i-1} R_j\right)-i+1\right] m_i\le n_1}}^{n_1} \Pr(M_1 = m_1, M_2 = m_2, \ldots, M_r = m_r \mid [F_X]^\gamma = F_Y), \tag{5.3.3}$$

respectively, where $\Pr(M_1 = m_1, M_2 = m_2, \ldots, M_r = m_r \mid [F_X]^\gamma = F_Y)$ is as given in (5.3.1).

In Table 5.3, we have presented the exact power values computed from (5.3.2) and (5.3.3) for $n_1 = n_2 = 10$, $r = 2$, $\gamma = 2(1)6$, and four different progressive censoring schemes.

## 5.4 Monte Carlo Power Comparison Under Location-Shift Alternative

Power of the weighted precedence and maximal precedence tests with $r = 2(1)6$ were all estimated through Monte Carlo simulations for location-shift $\theta = 0.5$ and 1.0. The lifetime distributions listed earlier in Section 3.5 were used in the Monte Carlo simulations. Once again, we generated 10,000 sets of data in order to determine the estimated rejection rates. In Tables 5.4-5.7, we have presented these estimated power values as well as the corresponding exact levels of significance.

From Tables 5.4-5.7, upon comparing the power values of the weighted precedence and maximal precedence tests for different censoring schemes, we find that the Type-II progressive censoring with $R_1 = n_2 - r$ gives better power performance when the underlying distributions are right-skewed. For example, in Table 5.4, under the Gamma distribution with $a = 2.0$, the power of the weighted precedence test with $r = 3$, $\theta = 1.0$ and censoring scheme (7,0,0) is 0.8912 (with exact level of significance as 0.04829) while the corresponding power for the censoring scheme (0,0,7) is 0.7987 (with exact level of significance as 0.05096). Thus, in this case, we can withdraw 7 experimental units early on in the life-testing process and still have a significant gain in power. Moreover, though the progressive censoring schemes may not give a better power performance in all the cases when the underlying distribution is nearly symmetric or left-skewed, they can still retain good power to be useful, in addition to being cost-effective.

Table 5.4: Power of weighted precedence and maximal precedence tests under progressive censoring for $n_1 = n_2 = 10$ with location shift = 0.5

| $r$ | censoring scheme | Test | exact l.o.s. | N(0,1) | Exp(1) | Gamma(2) | Gamma(10) | LN(0.1) | LN(0.5) | EV |
|---|---|---|---|---|---|---|---|---|---|---|
| 2 | (0,8) | WPT | 0.04954 | 0.2275 | 0.6145 | 0.4080 | 0.2708 | 0.2379 | 0.4206 | 0.2002 |
| | | WMPT | 0.05954 | 0.2309 | 0.7197 | 0.4344 | 0.2735 | 0.2457 | 0.4301 | 0.2034 |
| | (8,0) | WPT | 0.05108 | 0.2059 | 0.7361 | 0.4524 | 0.2605 | 0.2218 | 0.4364 | 0.1742 |
| | | WMPT | 0.04334 | 0.1790 | 0.7173 | 0.4167 | 0.2293 | 0.1953 | 0.4031 | 0.1422 |
| | (6,2) | WPT | 0.04782 | 0.2056 | 0.6740 | 0.4379 | 0.2493 | 0.2219 | 0.4188 | 0.1836 |
| | | WMPT | 0.04334 | 0.1726 | 0.7189 | 0.4222 | 0.2198 | 0.1902 | 0.3858 | 0.1422 |
| | (4,4) | WPT | 0.05037 | 0.2191 | 0.6488 | 0.4322 | 0.2624 | 0.2370 | 0.4187 | 0.2055 |
| | | WMPT | 0.04773 | 0.1832 | 0.7191 | 0.4243 | 0.2306 | 0.2008 | 0.3897 | 0.1672 |
| 3 | (0,0,7) | WPT | 0.05096 | 0.2400 | 0.5124 | 0.3707 | 0.2724 | 0.2518 | 0.3839 | 0.2414 |
| | | WMPT | 0.03792 | 0.1542 | 0.4886 | 0.2483 | 0.1739 | 0.1562 | 0.2559 | 0.1584 |
| | (7,0,0) | WPT | 0.04829 | 0.1938 | 0.7194 | 0.4494 | 0.2585 | 0.2195 | 0.4215 | 0.1725 |
| | | WMPT | 0.04334 | 0.1672 | 0.7168 | 0.4204 | 0.2291 | 0.1915 | 0.3928 | 0.1427 |
| | (5,1,1) | WPT | 0.04913 | 0.2125 | 0.6738 | 0.4484 | 0.2725 | 0.2392 | 0.4283 | 0.2008 |
| | | WMPT | 0.05294 | 0.1926 | 0.7174 | 0.4252 | 0.2465 | 0.2133 | 0.3989 | 0.1854 |
| | (2,3,3) | WPT | 0.04956 | 0.2342 | 0.5605 | 0.4058 | 0.2756 | 0.2527 | 0.4061 | 0.2338 |
| | | WMPT | 0.05567 | 0.2049 | 0.7179 | 0.4301 | 0.2589 | 0.2261 | 0.4076 | 0.1949 |
| 4 | (0,0,0,6) | WPT | 0.05065 | 0.2492 | 0.4210 | 0.3360 | 0.2684 | 0.2560 | 0.3549 | 0.2720 |
| | | WMPT | 0.04334 | 0.1632 | 0.4847 | 0.2573 | 0.1693 | 0.1681 | 0.2618 | 0.1769 |
| | (6,0,0,0) | WPT | 0.05166 | 0.2159 | 0.7166 | 0.4691 | 0.2650 | 0.2373 | 0.4380 | 0.1893 |
| | | WMPT | 0.04334 | 0.1763 | 0.7167 | 0.4211 | 0.2187 | 0.1914 | 0.3907 | 0.1409 |
| | (3,1,1,1) | WPT | 0.05349 | 0.2546 | 0.6260 | 0.4452 | 0.2917 | 0.2704 | 0.4418 | 0.2509 |
| | | WMPT | 0.05059 | 0.2014 | 0.7173 | 0.4264 | 0.2390 | 0.2132 | 0.3996 | 0.1784 |
| | (1,1,2,2) | WPT | 0.04929 | 0.2490 | 0.5187 | 0.3849 | 0.2781 | 0.2589 | 0.3941 | 0.2611 |
| | | WMPT | 0.06985 | 0.2464 | 0.7198 | 0.4446 | 0.2811 | 0.2582 | 0.4301 | 0.2329 |
| 5 | (0,0,0,0,5) | WPT | 0.05108 | 0.2558 | 0.4246 | 0.3372 | 0.2669 | 0.2609 | 0.3541 | 0.2935 |
| | | WMPT | 0.04489 | 0.1638 | 0.4857 | 0.2512 | 0.1739 | 0.1684 | 0.2702 | 0.1699 |
| | (5,0,0,0,0) | WPT | 0.05151 | 0.2273 | 0.6963 | 0.4652 | 0.2800 | 0.2398 | 0.4547 | 0.2077 |
| | | WMPT | 0.05294 | 0.2004 | 0.7198 | 0.4185 | 0.2456 | 0.2138 | 0.4119 | 0.1836 |
| | (3,0,0,0,2) | WPT | 0.05092 | 0.2481 | 0.6031 | 0.4326 | 0.2873 | 0.2640 | 0.4384 | 0.2504 |
| | | WMPT | 0.05263 | 0.2021 | 0.7200 | 0.4206 | 0.2487 | 0.2162 | 0.4135 | 0.1842 |
| | (1,1,1,1,1) | WPT | 0.04923 | 0.2493 | 0.5207 | 0.3913 | 0.2765 | 0.2601 | 0.4068 | 0.2644 |
| | | WMPT | 0.06994 | 0.2428 | 0.7224 | 0.4384 | 0.2865 | 0.2593 | 0.4424 | 0.2309 |
| 6 | (0,0,0,0,0,4) | WPT | 0.04900 | 0.2589 | 0.4105 | 0.3255 | 0.2794 | 0.2597 | 0.3466 | 0.3021 |
| | | WMPT | 0.04495 | 0.1715 | 0.4854 | 0.2522 | 0.1791 | 0.1709 | 0.2671 | 0.1799 |
| | (4,0,0,0,0,0) | WPT | 0.05239 | 0.2417 | 0.6715 | 0.4606 | 0.2949 | 0.2533 | 0.4554 | 0.2249 |
| | | WMPT | 0.04886 | 0.1870 | 0.7157 | 0.4168 | 0.2421 | 0.2003 | 0.4052 | 0.1673 |
| | (0,0,0,1,2,1) | WPT | 0.05182 | 0.2668 | 0.4403 | 0.3473 | 0.2911 | 0.2722 | 0.3693 | 0.3002 |
| | | WMPT | 0.04348 | 0.1700 | 0.4853 | 0.2517 | 0.1782 | 0.1695 | 0.2671 | 0.1765 |
| | (0,1,1,1,1,0) | WPT | 0.05307 | 0.2673 | 0.4936 | 0.3783 | 0.3029 | 0.2777 | 0.4011 | 0.2840 |
| | | WMPT | 0.04111 | 0.1653 | 0.4852 | 0.2504 | 0.1768 | 0.1673 | 0.2660 | 0.1682 |

Table 5.5: Power of weighted precedence and maximal precedence tests under progressive censoring for $n_1 = n_2 = 10$ with location shift = 1.0

| $r$ | censoring scheme | Test | exact l.o.s. | N(0,1) | Exp(1) | Gamma(2) | Gamma(10) | LN(0.1) | LN(0.5) | EV |
|---|---|---|---|---|---|---|---|---|---|---|
| 2 | (0,8) | WPT | 0.04954 | 0.5278 | 0.9468 | 0.8525 | 0.6487 | 0.5803 | 0.8727 | 0.4691 |
| | | WMPT | 0.05954 | 0.5067 | 0.9784 | 0.8881 | 0.6313 | 0.5535 | 0.8801 | 0.4504 |
| | (8,0) | WPT | 0.05108 | 0.4708 | 0.9807 | 0.8981 | 0.6212 | 0.5286 | 0.8859 | 0.3828 |
| | | WMPT | 0.04334 | 0.4161 | 0.9784 | 0.8834 | 0.5768 | 0.4771 | 0.8695 | 0.3177 |
| | (6,2) | WPT | 0.04782 | 0.5128 | 0.9614 | 0.8752 | 0.6543 | 0.5556 | 0.8775 | 0.4173 |
| | | WMPT | 0.04334 | 0.4248 | 0.9774 | 0.8787 | 0.5857 | 0.4776 | 0.8643 | 0.3177 |
| | (4,4) | WPT | 0.05037 | 0.5443 | 0.9551 | 0.8656 | 0.6656 | 0.5836 | 0.8714 | 0.4761 |
| | | WMPT | 0.04773 | 0.4521 | 0.9774 | 0.8788 | 0.5939 | 0.4982 | 0.8652 | 0.3869 |
| 3 | (0,0,7) | WPT | 0.05096 | 0.5839 | 0.8824 | 0.7987 | 0.6540 | 0.6087 | 0.8221 | 0.5769 |
| | | WMPT | 0.03792 | 0.3942 | 0.9164 | 0.7338 | 0.4610 | 0.4145 | 0.7289 | 0.4007 |
| | (7,0,0) | WPT | 0.04829 | 0.4817 | 0.9741 | 0.8912 | 0.6197 | 0.5304 | 0.8802 | 0.3836 |
| | | WMPT | 0.04334 | 0.4233 | 0.9765 | 0.8777 | 0.5721 | 0.4734 | 0.8622 | 0.3193 |
| | (5,1,1) | WPT | 0.04913 | 0.5372 | 0.9591 | 0.8842 | 0.6595 | 0.5826 | 0.8860 | 0.4622 |
| | | WMPT | 0.05294 | 0.4618 | 0.9766 | 0.8779 | 0.5870 | 0.5034 | 0.8632 | 0.4167 |
| | (2,3,3) | WPT | 0.04956 | 0.5851 | 0.9060 | 0.8318 | 0.6695 | 0.6156 | 0.8490 | 0.5593 |
| | | WMPT | 0.05567 | 0.4930 | 0.9768 | 0.8792 | 0.6076 | 0.5299 | 0.8676 | 0.4402 |
| 4 | (0,0,0,6) | WPT | 0.05065 | 0.6083 | 0.7921 | 0.7172 | 0.6435 | 0.6165 | 0.7765 | 0.6499 |
| | | WMPT | 0.04334 | 0.4045 | 0.9173 | 0.7294 | 0.4611 | 0.4263 | 0.7391 | 0.4219 |
| | (6,0,0,0) | WPT | 0.05166 | 0.5047 | 0.9718 | 0.9001 | 0.6580 | 0.5704 | 0.9016 | 0.4083 |
| | | WMPT | 0.04334 | 0.4183 | 0.9765 | 0.8790 | 0.5770 | 0.4847 | 0.8726 | 0.3171 |
| | (3,1,1,1) | WPT | 0.05349 | 0.6060 | 0.9410 | 0.8681 | 0.7031 | 0.6407 | 0.8917 | 0.5737 |
| | | WMPT | 0.05059 | 0.4677 | 0.9765 | 0.8801 | 0.5970 | 0.5192 | 0.8747 | 0.4104 |
| | (1,1,2,2) | WPT | 0.04929 | 0.6133 | 0.8910 | 0.8021 | 0.6804 | 0.6319 | 0.8419 | 0.6225 |
| | | WMPT | 0.06985 | 0.5353 | 0.9768 | 0.8838 | 0.6404 | 0.5798 | 0.8842 | 0.5039 |
| 5 | (0,0,0,0,5) | WPT | 0.05108 | 0.6332 | 0.8193 | 0.7373 | 0.6558 | 0.6334 | 0.7821 | 0.7073 |
| | | WMPT | 0.04489 | 0.4047 | 0.9111 | 0.7409 | 0.4684 | 0.4262 | 0.7464 | 0.4223 |
| | (5,0,0,0,0) | WPT | 0.05151 | 0.5440 | 0.9658 | 0.8955 | 0.6830 | 0.5959 | 0.9016 | 0.4575 |
| | | WMPT | 0.05294 | 0.4626 | 0.9763 | 0.8829 | 0.5896 | 0.5102 | 0.8740 | 0.4172 |
| | (3,0,0,0,2) | WPT | 0.05092 | 0.6140 | 0.9320 | 0.8576 | 0.7035 | 0.6453 | 0.8747 | 0.5984 |
| | | WMPT | 0.05263 | 0.4744 | 0.9763 | 0.8837 | 0.5953 | 0.5170 | 0.8746 | 0.4247 |
| | (1,1,1,1,1) | WPT | 0.04923 | 0.6288 | 0.8993 | 0.8121 | 0.6862 | 0.6472 | 0.8434 | 0.6385 |
| | | WMPT | 0.06994 | 0.5356 | 0.9763 | 0.8881 | 0.6401 | 0.5735 | 0.8852 | 0.5066 |
| 6 | (0,0,0,0,0,4) | WPT | 0.04900 | 0.6372 | 0.7965 | 0.7200 | 0.6643 | 0.6380 | 0.7620 | 0.7407 |
| | | WMPT | 0.04495 | 0.3984 | 0.9155 | 0.7446 | 0.4711 | 0.4170 | 0.7381 | 0.4330 |
| | (4,0,0,0,0,0) | WPT | 0.05239 | 0.5749 | 0.9586 | 0.8939 | 0.7038 | 0.6194 | 0.8971 | 0.5133 |
| | | WMPT | 0.04886 | 0.4451 | 0.9781 | 0.8813 | 0.5935 | 0.4947 | 0.8721 | 0.3994 |
| | (0,0,0,1,2,1) | WPT | 0.05182 | 0.6536 | 0.8254 | 0.7539 | 0.6836 | 0.6530 | 0.7909 | 0.7316 |
| | | WMPT | 0.04348 | 0.3976 | 0.9155 | 0.7446 | 0.4709 | 0.4166 | 0.7381 | 0.4291 |
| | (0,1,1,1,1,0) | WPT | 0.05307 | 0.6492 | 0.8693 | 0.8002 | 0.7030 | 0.6597 | 0.8310 | 0.6831 |
| | | WMPT | 0.04111 | 0.3955 | 0.9153 | 0.7451 | 0.4698 | 0.4160 | 0.7381 | 0.4188 |

Table 5.6: Power of weighted precedence and maximal precedence tests under progressive censoring for $n_1 = n_2 = 20$ with location shift = 0.5

| $r$ | censoring scheme | Test | exact l.o.s. | N(0,1) | Exp(1) | Gamma(2) | Gamma(10) | LN(0.1) | LN(0.5) | EV |
|---|---|---|---|---|---|---|---|---|---|---|
| 2 | (0,18) | WPT | 0.05326 | 0.2906 | 0.9700 | 0.7352 | 0.3929 | 0.3201 | 0.6917 | 0.2154 |
| | | WMPT | 0.04691 | 0.2401 | 0.9642 | 0.6563 | 0.3240 | 0.2664 | 0.6048 | 0.1845 |
| | (18,0) | WPT | 0.04955 | 0.2401 | 0.9859 | 0.7805 | 0.3401 | 0.2675 | 0.6743 | 0.1555 |
| | | WMPT | 0.05301 | 0.2452 | 0.9891 | 0.7919 | 0.3490 | 0.2733 | 0.6834 | 0.1601 |
| | (14,4) | WPT | 0.04691 | 0.2631 | 0.9764 | 0.7557 | 0.3677 | 0.2928 | 0.6818 | 0.1977 |
| | | WMPT | 0.05463 | 0.2526 | 0.9891 | 0.7924 | 0.3547 | 0.2803 | 0.6844 | 0.1769 |
| | (9,9) | WPT | 0.05680 | 0.3037 | 0.9746 | 0.7657 | 0.4113 | 0.3336 | 0.7137 | 0.2299 |
| | | WMPT | 0.03313 | 0.1979 | 0.9637 | 0.6375 | 0.2715 | 0.2209 | 0.5607 | 0.1503 |
| 3 | (0,0,17) | WPT | 0.04979 | 0.3096 | 0.9015 | 0.6741 | 0.4102 | 0.3486 | 0.6769 | 0.2493 |
| | | WMPT | 0.04359 | 0.2328 | 0.9626 | 0.6488 | 0.3169 | 0.2699 | 0.5879 | 0.1933 |
| | (17,0,0) | WPT | 0.05052 | 0.2413 | 0.9841 | 0.7780 | 0.3661 | 0.2900 | 0.6939 | 0.1782 |
| | | WMPT | 0.05301 | 0.2309 | 0.9880 | 0.7854 | 0.3562 | 0.2800 | 0.6917 | 0.1629 |
| | (14,2,1) | WPT | 0.04865 | 0.2630 | 0.9741 | 0.7659 | 0.3842 | 0.3083 | 0.6943 | 0.2036 |
| | | WMPT | 0.05463 | 0.2397 | 0.9880 | 0.7860 | 0.3611 | 0.2885 | 0.6929 | 0.1819 |
| | (5,6,6) | WPT | 0.05061 | 0.3126 | 0.9461 | 0.7240 | 0.4175 | 0.3549 | 0.6989 | 0.2650 |
| | | WMPT | 0.05222 | 0.2504 | 0.9631 | 0.6525 | 0.3333 | 0.2870 | 0.5988 | 0.2216 |
| 4 | (0,0,0,16) | WPT | 0.05023 | 0.3301 | 0.8582 | 0.6453 | 0.4231 | 0.3695 | 0.6640 | 0.2864 |
| | | WMPT | 0.05357 | 0.2576 | 0.9617 | 0.6603 | 0.3442 | 0.2964 | 0.5993 | 0.2304 |
| | (16,0,0,0) | WPT | 0.05036 | 0.2582 | 0.9821 | 0.7812 | 0.3799 | 0.3028 | 0.6975 | 0.1901 |
| | | WMPT | 0.05301 | 0.2348 | 0.9878 | 0.7850 | 0.3625 | 0.2834 | 0.6882 | 0.1626 |
| | (13,1,1,1) | WPT | 0.05105 | 0.2921 | 0.9706 | 0.7773 | 0.4129 | 0.3354 | 0.7135 | 0.2388 |
| | | WMPT | 0.05595 | 0.2499 | 0.9878 | 0.7863 | 0.3722 | 0.2995 | 0.6919 | 0.1892 |
| | (4,4,4,4) | WPT | 0.04559 | 0.3201 | 0.9111 | 0.6855 | 0.4199 | 0.3582 | 0.6768 | 0.2763 |
| | | WMPT | 0.05136 | 0.2518 | 0.9616 | 0.6578 | 0.3424 | 0.2892 | 0.5992 | 0.2197 |
| 5 | (0,0,0,0,15) | WPT | 0.04967 | 0.3501 | 0.8043 | 0.5978 | 0.4189 | 0.3816 | 0.6250 | 0.3128 |
| | | WMPT | 0.04439 | 0.2251 | 0.9038 | 0.4968 | 0.2631 | 0.2471 | 0.4574 | 0.2010 |
| | (15,0,0,0,0) | WPT | 0.05045 | 0.2641 | 0.9809 | 0.7822 | 0.3922 | 0.3149 | 0.7017 | 0.2019 |
| | | WMPT | 0.05306 | 0.2317 | 0.9885 | 0.7839 | 0.3588 | 0.2810 | 0.6855 | 0.1625 |
| | (11,1,1,1,1) | WPT | 0.05171 | 0.3157 | 0.9617 | 0.7687 | 0.4387 | 0.3670 | 0.7193 | 0.2732 |
| | | WMPT | 0.06578 | 0.2810 | 0.9886 | 0.7905 | 0.4010 | 0.3288 | 0.7036 | 0.2338 |
| | (3,3,3,3,3) | WPT | 0.05002 | 0.3550 | 0.8921 | 0.6814 | 0.4441 | 0.3999 | 0.6860 | 0.3173 |
| | | WMPT | 0.04975 | 0.2535 | 0.9615 | 0.6544 | 0.3336 | 0.2859 | 0.5943 | 0.2167 |
| 6 | (0,0,0,0,0,14) | WPT | 0.04952 | 0.3598 | 0.7498 | 0.5678 | 0.4210 | 0.3777 | 0.5982 | 0.3500 |
| | | WMPT | 0.04852 | 0.2252 | 0.9023 | 0.4969 | 0.2667 | 0.2428 | 0.4581 | 0.2254 |
| | (14,0,0,0,0,0) | WPT | 0.05087 | 0.2750 | 0.9772 | 0.7853 | 0.3890 | 0.3219 | 0.7044 | 0.2183 |
| | | WMPT | 0.05464 | 0.2419 | 0.9887 | 0.7871 | 0.3537 | 0.2853 | 0.6847 | 0.1813 |
| | (7,2,2,1,1,1) | WPT | 0.05077 | 0.3440 | 0.9326 | 0.7315 | 0.4509 | 0.3881 | 0.7026 | 0.3180 |
| | | WMPT | 0.04917 | 0.2477 | 0.9636 | 0.6506 | 0.3221 | 0.2749 | 0.5743 | 0.2170 |
| | (2,2,3,3,2,2) | WPT | 0.04918 | 0.3634 | 0.8660 | 0.6659 | 0.4464 | 0.3931 | 0.6627 | 0.3373 |
| | | WMPT | 0.05024 | 0.2531 | 0.9636 | 0.6533 | 0.3298 | 0.2862 | 0.5817 | 0.2210 |

Table 5.7: Power of weighted precedence and maximal precedence tests under progressive censoring for $n_1 = n_2 = 20$ with location shift = 1.0

| $r$ | censoring scheme | Test | exact l.o.s. | N(0,1) | Exp(1) | Gamma(2) | Gamma(10) | LN(0.1) | LN(0.5) | EV |
|---|---|---|---|---|---|---|---|---|---|---|
| 2 | (0,18) | WPT | 0.05326 | 0.6780 | 0.9998 | 0.9958 | 0.8599 | 0.7575 | 0.9940 | 0.5001 |
| | | WMPT | 0.04691 | 0.5985 | 0.9998 | 0.9932 | 0.7905 | 0.6743 | 0.9885 | 0.4481 |
| | (18,0) | WPT | 0.04955 | 0.5512 | 0.9999 | 0.9977 | 0.7991 | 0.6508 | 0.9936 | 0.3517 |
| | | WMPT | 0.05301 | 0.5550 | 0.9999 | 0.9978 | 0.8026 | 0.6553 | 0.9942 | 0.3549 |
| | (14,4) | WPT | 0.04691 | 0.6338 | 0.9999 | 0.9969 | 0.8430 | 0.7237 | 0.9936 | 0.4687 |
| | | WMPT | 0.05463 | 0.5754 | 0.9999 | 0.9978 | 0.8078 | 0.6678 | 0.9942 | 0.4235 |
| | (9,9) | WPT | 0.05680 | 0.6938 | 0.9999 | 0.9968 | 0.8731 | 0.7682 | 0.9947 | 0.5275 |
| | | WMPT | 0.03313 | 0.5376 | 0.9998 | 0.9926 | 0.7468 | 0.6137 | 0.9843 | 0.4106 |
| 3 | (0,0,17) | WPT | 0.04979 | 0.7359 | 0.9994 | 0.9886 | 0.8816 | 0.8047 | 0.9908 | 0.5919 |
| | | WMPT | 0.04359 | 0.6088 | 1.0000 | 0.9913 | 0.7941 | 0.6874 | 0.9857 | 0.4823 |
| | (17,0,0) | WPT | 0.05052 | 0.5907 | 1.0000 | 0.9975 | 0.8161 | 0.6770 | 0.9941 | 0.3953 |
| | | WMPT | 0.05301 | 0.5598 | 1.0000 | 0.9975 | 0.7962 | 0.6504 | 0.9934 | 0.3403 |
| | (14,2,1) | WPT | 0.04865 | 0.6476 | 1.0000 | 0.9966 | 0.8523 | 0.7297 | 0.9949 | 0.4738 |
| | | WMPT | 0.05463 | 0.5813 | 1.0000 | 0.9975 | 0.8025 | 0.6652 | 0.9935 | 0.4103 |
| | (5,6,6) | WPT | 0.05061 | 0.7409 | 0.9999 | 0.9941 | 0.8882 | 0.8073 | 0.9938 | 0.6144 |
| | | WMPT | 0.05222 | 0.6216 | 1.0000 | 0.9915 | 0.7998 | 0.6942 | 0.9857 | 0.5157 |
| 4 | (0,0,0,16) | WPT | 0.05023 | 0.7800 | 0.9986 | 0.9829 | 0.8900 | 0.8298 | 0.9881 | 0.6751 |
| | | WMPT | 0.05357 | 0.6381 | 1.0000 | 0.9919 | 0.7923 | 0.7160 | 0.9872 | 0.5459 |
| | (16,0,0,0) | WPT | 0.05036 | 0.6049 | 1.0000 | 0.9972 | 0.8366 | 0.7021 | 0.9946 | 0.4098 |
| | | WMPT | 0.05301 | 0.5506 | 1.0000 | 0.9971 | 0.8022 | 0.6545 | 0.9937 | 0.3389 |
| | (13,1,1,1) | WPT | 0.05105 | 0.6900 | 0.9999 | 0.9969 | 0.8817 | 0.7760 | 0.9955 | 0.5481 |
| | | WMPT | 0.05595 | 0.5858 | 1.0000 | 0.9971 | 0.8121 | 0.6805 | 0.9938 | 0.4273 |
| | (4,4,4,4) | WPT | 0.04559 | 0.7700 | 0.9993 | 0.9907 | 0.8953 | 0.8271 | 0.9928 | 0.6696 |
| | | WMPT | 0.05136 | 0.6284 | 1.0000 | 0.9919 | 0.7857 | 0.7022 | 0.9873 | 0.5158 |
| 5 | (0,0,0,0,15) | WPT | 0.04967 | 0.7999 | 0.9964 | 0.9749 | 0.8852 | 0.8401 | 0.9840 | 0.7448 |
| | | WMPT | 0.04439 | 0.5641 | 0.9993 | 0.9770 | 0.7071 | 0.6320 | 0.9641 | 0.5358 |
| | (15,0,0,0,0) | WPT | 0.05045 | 0.6414 | 1.0000 | 0.9989 | 0.8499 | 0.7192 | 0.9955 | 0.4551 |
| | | WMPT | 0.05306 | 0.5584 | 1.0000 | 0.9983 | 0.8071 | 0.6510 | 0.9942 | 0.3617 |
| | (11,1,1,1,1) | WPT | 0.05171 | 0.7564 | 0.9998 | 0.9976 | 0.8987 | 0.8098 | 0.9959 | 0.6459 |
| | | WMPT | 0.06578 | 0.6504 | 1.0000 | 0.9984 | 0.8421 | 0.7218 | 0.9948 | 0.5233 |
| | (3,3,3,3,3) | WPT | 0.05002 | 0.8118 | 0.9991 | 0.9899 | 0.9101 | 0.8556 | 0.9919 | 0.7414 |
| | | WMPT | 0.04975 | 0.6293 | 1.0000 | 0.9933 | 0.7952 | 0.6926 | 0.9862 | 0.5356 |
| 6 | (0,0,0,0,0,14) | WPT | 0.04952 | 0.8208 | 0.9912 | 0.9642 | 0.8813 | 0.8604 | 0.9770 | 0.8039 |
| | | WMPT | 0.04852 | 0.5729 | 0.9997 | 0.9751 | 0.7015 | 0.6303 | 0.9668 | 0.5609 |
| | (14,0,0,0,0,0) | WPT | 0.05087 | 0.6552 | 1.0000 | 0.9974 | 0.8502 | 0.7476 | 0.9961 | 0.4873 |
| | | WMPT | 0.05464 | 0.5707 | 1.0000 | 0.9985 | 0.7962 | 0.6735 | 0.9939 | 0.4174 |
| | (7,2,2,1,1,1) | WPT | 0.05077 | 0.8007 | 0.9997 | 0.9939 | 0.9143 | 0.8602 | 0.9957 | 0.7336 |
| | | WMPT | 0.04917 | 0.6103 | 0.9999 | 0.9938 | 0.7729 | 0.6857 | 0.9873 | 0.5167 |
| | (2,2,3,3,2,2) | WPT | 0.04918 | 0.8281 | 0.9976 | 0.9868 | 0.9101 | 0.8773 | 0.9908 | 0.7850 |
| | | WMPT | 0.05024 | 0.6287 | 0.9999 | 0.9938 | 0.7869 | 0.7039 | 0.9882 | 0.5408 |

Table 5.8: Times to insulating fluid breakdown data from Nelson (1982) for Samples 3 and 6

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $X$-sample(Sample 3) | **0.49** | **0.64** | **0.82** | **0.93** | **1.08** | **1.99** | **2.06** | **2.15** | **2.57** | **4.75** |
| $Y$-sample(Sample 6) | **1.34** | **1.49** | **1.56** | 2.10 | **2.12** | 3.83 | 3.97 | **5.13** | 7.21 | 8.71 |

## 5.5 Illustrative Example

**Example 5.1:** For this example, we will illustrate the weighted precedence and maximal precedence tests under Type-II progressive censoring for the $Y$-sample. Referring to Example 4.2, let us consider $X$-sample and $Y$-sample to be Sample 3 and Sample 6 in Nelson (1982), respectively. If we had observed only up to the 5-th breakdown from the $Y$-sample with progressive censoring scheme (3,0,0,0,2), three items from $Y$-sample had to be randomly removed from the test at the time of the first $Y$-failure which resulted in the removal of breakdown times 2.10, 3.83 and 3.97, thus resulting in all ten observations from the $X$-sample being observed. The observations are shown in Table 5.8.

In this case, we have $n_1 = n_2 = 10$, $r = 5$, $m_1 = 5$, $m_2 = m_3 = 0$, $m_4 = 2$ and $m_5 = 3$. The following are the values of the test statistics and the corresponding $p$-values:

| Test | Test statistic | $p$-value |
|---|---|---|
| Precedence Test (with $r = 5$) | $P_{(5)} = 7$ | 0.185 |
| Maximal Precedence Test (with $r = 5$) | $M_{(5)} = 5$ | 0.081 |
| Weighted Precedence Test | $P^*_{(5)} = 67$ | 0.007 |
| Weighted Maximal Precedence Test | $M^*_{(5)} = 50$ | 0.006 |

From the small $p$-values of the weighted precedence and maximal precedence tests, we observe that the data provide strong evidence to reject $H_0$. In this example, since five $X$-failures occurred before the first $Y$-failure, it appears that this decision may indeed be a good one. However, if either the precedence or the maximal precedence test had been used instead, we would not have rejected $H_0$ at the usual 5%

level of significance. This demonstrates an advantage of the weighted precedence and maximal precedence tests.

It is of interest to note here that the proposed tests are applicable even though only a few early failures from the two samples are observed as data and the $Y$-sample is under progressive censoring. In addition, the decisions are reached without making any assumption on the underlying life-time distributions.

# Chapter 6

# Estimation of Parameters From Progressively Censored Data Using EM-Algorithm

## 6.1 Introduction

In this chapter, EM-algorithm is used to determine the maximum likelihood estimates when the data are progressively Type-II censored. The method is shown to be feasible and easy to implement. As mentioned earlier in Section 1.4.2, progressive Type-II censored sampling is economical in terms of both time and money, but it is not very popular in lifetime studies. It may be due to the complicated calculation of the likelihood function [Lawless (1982)]. Newton-Raphson algorithm is one of the standard methods to determine the maximum likelihood estimates of the parameters. To employ the algorithm, the second derivatives of the log-likelihood are required. Sometimes the calculations of the derivatives based on the progressively Type-II censored samples are complicated. For this reason, we propose to use the EM-algorithm instead.

EM-algorithm [Dempster, Laird and Rubin (1977); McLachlan and Krishnan (1997)] is a very powerful tool in handling the incomplete data problem. It is an iterative method which repeatedly fills in the missing data by estimated values and updates the parameter estimates. It is especially useful if the estimates for the complete data set are simple.

In this chapter, we shall use the same notation as in Section 1.4.2. We assume $n$ independent units are placed on a life-test with the corresponding lifetimes $X_1$, $X_2$, ..., $X_n$ being identically distributed. We assume that $X_i$, $i = 1, 2, \ldots, n$, are i.i.d. with p.d.f. $f_X(x; \boldsymbol{\theta})$ and c.d.f. $F_X(x; \boldsymbol{\theta})$, where $\boldsymbol{\theta}$ denotes the vector of parameters. Prior to the experiment, a number $m < n$ is determined and the censoring scheme $(R_1, R_2, \ldots, R_m)$ with $R_j \geq 0$ and $\sum\limits_{j=1}^{m} R_j + m = n$ is pre-specified. During the experiment, the $j$-th failure is observed and immediately after the failure, $R_j$ functioning items are randomly removed from the test.

Note that in the analysis of lifetime data, instead of working with the parametric model for $X_i$, it is often more convenient to work with the equivalent model for the log-lifetimes $W_i = \ln X_i$, for $i = 1, 2, \ldots, n$, which are i.i.d. with p.d.f. $f_W(w; \boldsymbol{\theta})$ and c.d.f. $F_W(w; \boldsymbol{\theta})$. We denote the $m$ completely observed (ordered) log-lifetimes by $Y_{j:m:n}$, $j = 1, 2, \ldots, m$. The likelihood function based on $Y_{j:m:n}$ is

$$L(\boldsymbol{\theta}) = c \prod_{j=1}^{m} f_W(y_{j:m:n}; \boldsymbol{\theta}) \left[1 - F_W(y_{j:m:n}; \boldsymbol{\theta})\right]^{R_j} \tag{6.1.1}$$

where

$$c = n(n - R_1 - 1) \cdots (n - R_1 - R_2 - \ldots - R_{m-1} - m + 1).$$

The maximum likelihood estimators are those values of $\boldsymbol{\theta}$ which maximize (6.1.1). In most cases, the estimators do not admit explicit expressions and some numerical procedures such as Newton-Raphson method have to be used to determine the estimates. Balakrishnan and Cohen (1991) and Balakrishnan and Aggarwala (2000) have discussed this problem extensively. To employ the Newton-Raphson method, the second derivatives of the log-likelihood function are required, which may sometimes be complicated. In addition, the Newton-Raphson method is not easy to extend to other forms of censored data.

## 6.2 EM-Algorithm

The progressive right censoring model problem can be viewed as an incomplete data problem and then the EM-algorithm is applicable to obtain the maximum likelihood estimates of the parameters. First of all, denote the observed and censored data by $\mathbf{Y} = (Y_{1:m:n}, Y_{2:m:n}, \ldots, Y_{m:m:n})$ and $\mathbf{Z} = (\mathbf{Z}_1, \mathbf{Z}_2, \ldots, \mathbf{Z}_m)$, respectively, where $\mathbf{Z}_j$ is a $1 \times R_j$ vector with $\mathbf{Z}_j = (Z_{j1}, Z_{j2}, \ldots, Z_{jR_j})$, for $j = 1, 2, \ldots, m$. The censored data vector $\mathbf{Z}$ can be thought of as the missing data. Combine $\mathbf{Y}$ and $\mathbf{Z}$ to form $\mathbf{W}$ which is the complete data set. The corresponding log-likelihood function is denoted by $\lambda(\mathbf{W}, \boldsymbol{\theta})$. We denote the estimate of $\boldsymbol{\theta}$ in the $h$-th iteration by $\boldsymbol{\theta}_{(h)}$. Then, the E-step of the algorithm requires the computation of the conditional expectation

$$E[\lambda(\mathbf{W}, \boldsymbol{\theta}_{(h+1)}) \mid \mathbf{Y} = \mathbf{y}, \boldsymbol{\theta}_{(h)}], \tag{6.2.1}$$

which mainly involves the computation of the conditional expectation of functions of $\mathbf{Z}$ conditional on the observed values $\mathbf{Y}$ and the current value of the parameters. Therefore, in order to facilitate the EM-algorithm, the conditional distribution of $\mathbf{Z}$, conditional on $\mathbf{Y}$ and the current value of the parameters, needs to be determined.

**Theorem 6.1** *Given $Y_{1:m:n} = y_{1:m:n}, Y_{2:m:n} = y_{2:m:n}, \ldots, Y_{j:m:n} = y_{j:m:n}$, the conditional distribution of $Z_{jk}, k = 1, 2, \ldots, R_j$, is*

$$\begin{aligned} & f_{Z|Y}(z_j \mid Y_{1:m:n} = y_{1:m:n}, Y_{2:m:n} = y_{2:m:n}, \ldots, Y_{j:m:n} = y_{j:m:n}) \\ = \; & f_{Z|Y}(z_j \mid Y_{j:m:n} = y_{j:m:n}) = \frac{f_W(z_j)}{1 - F_W(y_{j:m:n})}, \quad z_j > y_{j:m:n}, \end{aligned} \tag{6.2.2}$$

*and $Z_{jk}$ and $Z_{jl}$, $k \neq l$, are conditionally independent, given $Y_{j:m:n} = y_{j:m:n}$.*

**Proof:** The probability associated with the complete sample is

$$\prod_{j=1}^{m} \left[ f_W(y_{j:m:n}) \prod_{k=1}^{R_j} f_W(z_{jk}) \right].$$

The probability associated with the progressively censored sample $Y_{1:m:n} = y_{1:m:n}, Y_{2:m:n} = y_{2:m:n}, \ldots, Y_{j:m:n} = y_{j:m:n}$ is

$$\prod_{j=1}^{m} \left\{ f_W(y_{j:m:n}) \left[1 - F_W(y_{j:m:n})\right]^{R_j} \right\}.$$

Then the conditional probability $Z_{jk}, k = 1, 2 \ldots R_j$, given $Y_{1:m:n} = y_{1:m:n}$, $Y_{2:m:n} = y_{2:m:n}$, ..., $Y_{j:m:n} = y_{j:m:n}$, is given by

$$f_{\mathbf{Z}|\mathbf{Y}}(\mathbf{z}|\mathbf{y}) = \prod_{j=1}^{m} \prod_{k=1}^{R_j} \frac{f_W(z_{jk})}{1 - F_W(y_{j:m:n})}.$$

Therefore, by factorization theorem, $Z_{jk}$ are conditionally independent and follow the truncated distribution from the left at $y_{j:m:n}$, $k = 1, 2, \ldots, R_j$ and $j = 1, 2, \ldots, m$. This complete the proof of the theorem.

The theorem states that given $Y_{j:m:n} = y_{j:m:n}$, $\mathbf{Z}_j$ form a random sample from the truncated population and hence the expectations of functions of $\mathbf{Z}_j$ can be obtained.

In the M-step on the $(h+1)$-th iteration of the EM-algorithm, the value of $\boldsymbol{\theta}$ which maximizes $E[\lambda(\mathbf{W}, \boldsymbol{\theta}_{(h+1)}) \mid \mathbf{Y} = \mathbf{y}, \boldsymbol{\theta}_{(h)}]$ will be used as the next estimate of $\boldsymbol{\theta}_{(h+1)}$. The MLE of $\boldsymbol{\theta}$ can be obtained by repeating the E-step and M-step until convergence occurs. A reasonable starting value for $\boldsymbol{\theta}_{(0)}$ is the estimate of the parameter based on the "pseudo-complete" sample obtained by replacing the censored observations $\mathbf{Z}_j$ by $Y_{j:m:n}, j = 1, 2, \ldots, m$.

## 6.3 Asymptotic Variances and Covariances of the ML Estimates

Louis (1982) developed a procedure for extracting the observed information matrix when the EM-algorithm is used in order to find maximum likelihood estimates in incomplete data problem. The idea of the procedure can be expressed by the Missing Information Principle [Louis (1982), Tanner (1993)]:

Observed information = Complete information − Missing information.

We can use this procedure to compute the variances and covariances of the ML estimates under progressive Type-II right censoring. The observed information, complete information and missing information are denoted by $I_Y(\boldsymbol{\theta})$, $I_W(\boldsymbol{\theta})$ and $I_{Z|Y}(\boldsymbol{\theta})$, respectively.

The complete information $I_W(\boldsymbol{\theta})$ is given by

$$I_W(\boldsymbol{\theta}) = -E\left[\frac{\partial^2\lambda(\mathbf{W},\boldsymbol{\theta})}{\partial\boldsymbol{\theta}^2}\right]. \tag{6.3.1}$$

Based on the conditional distribution in (6.2.2), the Fisher information matrix in one observation which is censored at the time of the $j$-th failure can be computed as

$$I_{Z|Y}^{(j)}(\boldsymbol{\theta}) = E\left[\left(\frac{\partial \ln f_{Z_j}(z_j|y_{j:m:n},\boldsymbol{\theta})}{\partial\boldsymbol{\theta}}\right)^2\right] = -E\left[\frac{\partial^2 \ln f_{Z_j}(z_j|y_{j:m:n},\boldsymbol{\theta})}{\partial\boldsymbol{\theta}^2}\right] \tag{6.3.2}$$

Then the expected information for conditional distribution of $\mathbf{Z}$ given $\mathbf{Y}$ (missing information) is

$$I_{Z|Y}(\boldsymbol{\theta}) = \sum_{j=1}^{m} R_j I_{Z|Y}^{(j)}(\boldsymbol{\theta}). \tag{6.3.3}$$

Therefore, the observed information is

$$I_Y(\boldsymbol{\theta}) = I_W(\boldsymbol{\theta}) - I_{Z|Y}(\boldsymbol{\theta}). \tag{6.3.4}$$

The asymptotic covariance matrix of the ML estimate for $\boldsymbol{\theta}$ can be obtained by inverting the observed information matrix $I_Y(\hat{\boldsymbol{\theta}})$.

## 6.4 Lognormal Lifetime Data

Lognormal distribution is a commonly used lifetime model in reliability analysis since the logarithm of the lifetime variables are normally distributed. Because of the well-known properties of the normal distribution and because it is a location-scale model, the log-lifetimes will be used in the analysis.

The log-likelihood function based on the log-lifetimes $\mathbf{W}$ is

$$\begin{aligned}\lambda(\mathbf{W};\mu,\sigma) &= \text{constant} - n\ln\sigma - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(w_i-\mu)^2\\ &= \text{constant} - n\ln\sigma - \frac{1}{2\sigma^2}\sum_{j=1}^{m}(y_{j:m:n}-\mu)^2 - \frac{1}{2\sigma^2}\sum_{j=1}^{m}\sum_{k=1}^{R_j}(Z_{jk}-\mu)^2.\end{aligned} \tag{6.4.1}$$

### 6.4.1 The Algorithm

In the E-step, one requires to compute

$$\begin{aligned} -n\ln\sigma \quad &- \quad \frac{1}{2\sigma^2}\sum_{j=1}^{m}(y_{j:m:n}-\mu)^2 \\ &-\frac{1}{2\sigma^2}\sum_{j=1}^{m}\sum_{k=1}^{R_j}\left[E(Z_{jk}^2|Z_{jk}>y_{j:m:n})-2\mu E(Z_{jk}|Z_{jk}>y_{j:m:n})+\mu^2\right]. \end{aligned}$$

As a result of Theorem 6.1, the conditional distribution of $Z_{jk}$ given $Y_{j:m:n} = y_{j:m:n}$ follows a truncated normal distribution with left truncation at $y_{j:m:n}$.

The first and second moments of $Z_{jk}$ given $Z_{jk} > y_{j:m:n}$ can be found in Cohen (1991) as

$$E(Z_{jk} \mid Z_{jk} > y_{j:m:n}, \mu, \sigma) = \sigma Q_j + \mu, \tag{6.4.2}$$

$$E(Z_{jk}^2 \mid Z_{jk} > y_{j:m:n}, \mu, \sigma) = \sigma^2(1+\xi_j Q_j) + 2\sigma\mu Q_j + \mu^2, \tag{6.4.3}$$

where

$$\xi_j = \frac{y_{j:m:n}-\mu}{\sigma},$$

$$Q_j = \frac{\phi(\xi_j)}{1-\Phi(\xi_j)} \text{ is the hazard function of the standard normal distribution.}$$

From the standard results for complete data maximum likelihood estimation for normal distribution, the explicit formulas for the MLE of $\mu$ and $\sigma$ are

$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} w_i,$$

$$\hat{\sigma} = \left[\frac{1}{n}\sum_{i=1}^{n}(w_i-\hat{\mu})^2\right]^{1/2}.$$

Therefore, in the $(h+1)$-th iteration of the EM-algorithm, the value of $\mu_{(h+1)}$ and $\sigma_{(h+1)}$ are computed by the following formulas:

$$\hat{\mu}_{(h+1)} = \frac{1}{n}\left[\sum_{j=1}^{m} y_{j:m:n} + \sum_{j=1}^{m} R_j E(Z_j \mid Z_j > y_{j:m:n}, \mu_{(h)}, \sigma_{(h)})\right], \tag{6.4.4}$$

$$\hat{\sigma}_{(h+1)} = \left\{ \frac{1}{n} \left[ \sum_{j=1}^{m} y_{j:m:n}^2 + \sum_{j=1}^{m} R_j E(Z_j^2 \mid Z_j > y_{j:m:n}, \mu_{(h+1)}, \sigma_{(h)}) \right] - \hat{\mu}_{(h+1)}^2 \right\}^{1/2} . \tag{6.4.5}$$

### 6.4.2 Asymptotic Variances and Covariance

If the log-lifetime data follows normal distribution, it is well known that the information matrix based on the complete data is

$$I_W(\boldsymbol{\theta}) = \frac{n}{\sigma^2} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

with $\boldsymbol{\theta} = (\,\mu \quad \sigma\,)'$.

The logarithm of the truncated normal p.d.f. is

$$\begin{aligned} &\ln f_{Z_j}(z_j \mid Z_j > y_{j:m:n}, \mu, \sigma) \\ &= \text{constant} - \ln \sigma - \ln[1 - F_W(y_{j:m:n})] - \frac{1}{2\sigma^2}(z_j - \mu)^2. \end{aligned} \tag{6.4.6}$$

Differentiation of (6.4.6) with respect to $\mu$ and $\sigma$ yields

$$\frac{\partial \ln f_{Z_j}}{\partial \mu} = \frac{1}{\sigma}\left[\frac{z_j - \mu}{\sigma} - Q_j\right], \tag{6.4.7}$$

$$\frac{\partial \ln f_{Z_j}}{\partial \sigma} = \frac{1}{\sigma}\left[\left(\frac{z_j - \mu}{\sigma}\right)^2 - (1 + \xi_j Q_j)\right]. \tag{6.4.8}$$

It is easily shown that

$$E[(Z_j - \mu) \mid Z_j > y_{j:m:n}, \mu, \sigma] = \sigma Q_j, \tag{6.4.9}$$

$$E[(Z_j - \mu)^2 \mid Z_j > y_{j:m:n}, \mu, \sigma] = \sigma^2(1 + \xi_j Q_j), \tag{6.4.10}$$

$$E[(Z_j - \mu)^3 \mid Z_j > y_{j:m:n}, \mu, \sigma] = \sigma^3(2 + \xi_j^2), \tag{6.4.11}$$

$$E[(Z_j - \mu)^4 \mid Z_j > y_{j:m:n}, \mu, \sigma] = \sigma^4[3(1 + \xi_j Q_j) + \xi_j^3 Q_j]. \tag{6.4.12}$$

Using (6.4.9)-(6.4.12), the Fisher information matrix based on one observation which is censored at the time of the $j$-th failure can be computed by straightforward substitution. Then, the corresponding entries in (6.3.2) are

$$\begin{aligned}
E\left[\left(\frac{\partial \ln f_{Z_j}}{\partial \mu}\right)^2\right] &= \frac{1}{\sigma^2}\left[1+\xi_j Q_j - Q_j^2\right], \\
E\left[\left(\frac{\partial \ln f_{Z_j}}{\partial \sigma}\right)^2\right] &= \frac{1}{\sigma^2}\left[2+\xi_j Q_j(1-\xi_j Q_j+\xi_j^2)\right], \\
E\left[\left(\frac{\partial \ln f_{Z_j}}{\partial \mu}\right)\left(\frac{\partial \ln f_{Z_j}}{\partial \sigma}\right)\right] &= \frac{1}{\sigma^2}\left[Q_j+\xi_j Q_j(\xi_j - Q_j)\right].
\end{aligned}$$

Thus, the expected information for conditional distribution of $\mathbf{Z}$ given $\mathbf{Y}$ can be obtained using (6.3.3), and hence $I_Y(\boldsymbol{\theta})$. Inverting $I_Y(\boldsymbol{\theta})$ yields the variance-covariance matrix of $\hat{\boldsymbol{\theta}} = (\,\hat{\mu} \quad \hat{\sigma}\,)'$.

Instead, had we employed the Newton-Raphson method for finding the MLEs numerically, we would have solved the equations

$$\begin{aligned}
\frac{\partial \ln L(\mu,\sigma)}{\partial \mu} &= \frac{1}{\sigma}\left\{\sum_{j=1}^{m}\xi_j + \sum_{j=1}^{m} R_j Q_j\right\} = 0, \\
\frac{\partial \ln L(\mu,\sigma)}{\partial \sigma} &= -\frac{m}{\sigma} + \frac{1}{\sigma}\sum_{j=1}^{m}\xi_j^2 + \frac{1}{\sigma}\sum_{j=1}^{m} R_j \xi_j Q_j = 0
\end{aligned}$$

by using the second-order derivative forms

$$\begin{aligned}
\frac{\partial^2 \ln L(\mu,\sigma)}{\partial \mu^2} &= -\frac{m}{\sigma^2} + \frac{1}{\sigma^2}\sum_{j=1}^{m} R_j\left\{\frac{\xi_j\phi(\xi_j) - \xi_j\phi(\xi_j)\Phi(\xi_j) - \phi^2(\xi_j)}{[1-\Phi(\xi_j)]^2}\right\}, \\
\frac{\partial^2 \ln L(\mu,\sigma)}{\partial \sigma^2} &= \frac{m}{\sigma^2} - \frac{3}{\sigma^2}\sum_{j=1}^{m}\xi_j^2 - \frac{2}{\sigma^2}\sum_{j=1}^{m} R_j Q_j, \\
\frac{\partial^2 \ln L(\mu,\sigma)}{\partial \mu \partial \sigma} &= \frac{1}{\sigma^2}\sum_{j=1}^{m}\left\{2\xi_j + R_j Q_j + R_j \xi_j Q_j - \xi_j^2 Q_j\right\}.
\end{aligned}$$

Using the Newton-Raphson method, we will discuss the construction of confidence intervals for $\mu$ and $\sigma$ later in Chapter 7. It is important to mention here that Mi and Balakrishnan (2001) have shown that the MLEs of $\mu$ and $\sigma$ do exist and are unique in this case (by making use of the fact that the normal/lognormal density is log-concave). Due to this result, it is then evident that the EM-algorithm and the Newton-Raphson method would converge to the same values.

Table 6.1: Simulated progressively censored sample from standard normal distribution

| j | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| $y_{j:m:n}$ | -3.1538 | -0.84064 | -0.79798 | -0.65705 | -0.58301 | -0.12642 | -0.1145 |
| $R_j$ | 1 | 2 | 0 | 1 | 2 | 0 | 3 |

### 6.4.3 Illustrative Example

**Example 6.1:** To illustrate the method presented in this section, a progressively Type-II right censored sample of size $m = 7$ are generated from standard normal distribution. The data are presented in Table 6.1.

A computer program written in FORTRAN is used to execute the EM-algorithm. Since the mean and standard deviation of the 7 observed sample points equal to -0.89620 and 0.96127, respectively, we can set $\mu_{(0)} = -0.89620$ and $\sigma_{(0)} = 0.96127$ as the starting values of the EM-algorithm. After a few EM iterations, the estimates converge to $\mu_{(\infty)} = -0.10071$ and $\sigma_{(\infty)} = 1.14316$. To compare the convergence rate of EM-algorithm with that of Newton-Raphson method, we used the same initial values and fixed the level of accuracy at $5 \times 10^{-5}$. The EM-algorithm took 27 iterations while the Newton-Raphson method took 4 iterations to converge to the same values.

From these data, we have

$$
\begin{aligned}
I_W(\hat{\boldsymbol{\theta}}) &= \begin{pmatrix} 12.24340 & 0 \\ 0 & 24.48681 \end{pmatrix}, \\
I_{Z|Y}(\hat{\boldsymbol{\theta}}) &= \begin{pmatrix} 3.47011 & 4.81360 \\ 4.81360 & 12.04041 \end{pmatrix}, \\
I_Y(\hat{\boldsymbol{\theta}}) &= I_W(\hat{\boldsymbol{\theta}}) - I_{Z|Y}(\hat{\boldsymbol{\theta}}) = \begin{pmatrix} 8.77330 & -4.81360 \\ -4.81360 & 12.44640 \end{pmatrix}.
\end{aligned}
$$

The covariance matrix of $(\,\hat{\mu} \quad \hat{\sigma}\,)$ is thus

$$
I_Y^{-1}(\hat{\boldsymbol{\theta}}) = \begin{pmatrix} 0.14468 & 0.05596 \\ 0.05596 & 0.10199 \end{pmatrix}.
$$

## 6.5 Weibull Lifetime Data

Weibull distribution is another commonly used lifetime model in reliability studies. Similar to the lognormal case, it is more convenient to work with the log-lifetime which is distributed as extreme value.

The log-likelihood function based on the complete log-lifetime $\mathbf{W}$ is

$$\begin{aligned}
\lambda(\mathbf{W};\mu,\sigma) &= -n\ln\sigma + \sum_{i=1}^{n}\left(\frac{w_i-\mu}{\sigma}\right) - \sum_{i=1}^{n}\exp\left(\frac{w_i-\mu}{\sigma}\right) \\
&= -n\ln\sigma + \sum_{j=1}^{m}\left(\frac{y_{j:m:n}-\mu}{\sigma}\right) - \sum_{j=1}^{n}\exp\left(\frac{y_{j:m:n}-\mu}{\sigma}\right) \\
&\quad + \sum_{j=1}^{m}\sum_{k=1}^{R_j}\left(\frac{z_{jk}-\mu}{\sigma}\right) - \sum_{j=1}^{m}\sum_{k=1}^{R_j}\exp\left(\frac{z_{jk}-\mu}{\sigma}\right). \qquad (6.5.1)
\end{aligned}$$

### 6.5.1 The Algorithm

Using Theorem 6.1, the conditional distribution of $Z_j$ given $Y_{j:m:n} = y_{j:m:n}$ is a truncated extreme value distribution with left truncation at $y_{j:m:n}$ having p.d.f.

$$\begin{aligned}
f_Z(z_j \mid Z_j > y_{j:m:n},\mu,\sigma) &= \frac{\exp\left[\exp(\xi_j)\right]}{\sigma}\exp\left[\left(\frac{z_j-\mu}{\sigma}\right) - \exp\left(\frac{z_j-\mu}{\sigma}\right)\right], \\
&\quad y_{j:m:n} < z_j < \infty, \qquad (6.5.2)
\end{aligned}$$

where $\xi_j = \frac{y_{j:m:n}-\mu}{\sigma}$.

To evaluate the required conditional expectations in the E-step, the moment generating function for $\frac{Z_j-\mu}{\sigma}$, given $y_{j:m:n}$, is considered. It is given by

$$\begin{aligned}
M_{\frac{Z_j-\mu}{\sigma}}(t) = E\left[e^{t\left(\frac{Z_j-\mu}{\sigma}\right)}\right] &= e^{e^{\xi_j}}\Gamma(t+1,e^{\xi_j}) \\
&= \Gamma(t+1)\left[e^{e^{\xi_j}} - \sum_{p=0}^{\infty}\frac{e^{(t+p+1)\xi_j}}{\Gamma(t+p+2)}\right],
\end{aligned}$$

where $\Gamma(a,x) = \int_x^\infty u^{a-1}e^{-u}du$ is the incomplete Gamma function and $\Gamma(a) = \Gamma(a,0)$ is the complete Gamma function.

Then the conditional expectations of interest are

$$E\left[Z_j \mid \xi_j,\mu,\sigma\right] = E_{1,j}\sigma + \mu, \qquad (6.5.3)$$

$$E\left[e^{\frac{Z_j}{\sigma}}\middle|\,\xi_j,\mu,\sigma\right] = e^{\frac{\mu}{\sigma}}[e^{\xi_j}+1], \tag{6.5.4}$$

$$E\left[Z_j e^{\left(\frac{Z_j}{\sigma}\right)}\middle|\,\xi_j,\mu,\sigma\right] = e^{\frac{\mu}{\sigma}}[E_{2,j}\sigma+\mu(e^{\xi_j}+1)], \tag{6.5.5}$$

$$E\left[Z_j^2 e^{\left(\frac{Z_j}{\sigma}\right)}\middle|\,\xi_j,\mu,\sigma\right] = e^{\frac{\mu}{\sigma}}[E_{3,j}\sigma^2+2\mu\sigma E_2-\mu^2(e^{\xi_j}+1)], \tag{6.5.6}$$

where

$$\begin{aligned}
E_{1,j} &= E\left[\frac{Z_j-\mu}{\sigma}\middle|\,\xi_j,\mu,\sigma\right] \\
&= \psi(1)e^{e^{\xi_j}} + \sum_{p=0}^{\infty}\frac{e^{(p+1)\xi_j}\psi(p+2)}{\Gamma(p+2)} - [\xi_j+\psi(1)]\sum_{p=0}^{\infty}\frac{e^{(p+1)\xi_j}}{\Gamma(p+2)}, \\
E_{2,j} &= E\left[\left(\frac{Z_j-\mu}{\sigma}\right)e^{\left(\frac{Z_j-\mu}{\sigma}\right)}\middle|\,\xi_j,\mu,\sigma\right] \\
&= \psi(2)e^{e^{\xi_j}} + \sum_{p=0}^{\infty}\frac{e^{(p+2)\xi_j}\psi(p+3)}{\Gamma(p+3)} - [\xi_j+\psi(2)]\sum_{p=0}^{\infty}\frac{e^{(p+2)\xi_j}}{\Gamma(p+3)}, \\
E_{3,j} &= E\left[\left(\frac{Z_j-\mu}{\sigma}\right)^2 e^{\left(\frac{Z_j-\mu}{\sigma}\right)}\middle|\,\xi_j,\mu,\sigma\right] \\
&= [\psi'(2)+\psi^2(2)]e^{e^{\xi_j}} - [\xi_j^2+2\xi_j\psi(2)+\psi'(2)+\psi^2(2)]\sum_{p=0}^{\infty}\frac{e^{(p+2)\xi_j}}{\Gamma(p+3)} \\
&\quad +2[\xi_j+\psi(2)]\sum_{p=0}^{\infty}\frac{e^{(p+2)\xi_j}\psi(p+3)}{\Gamma(p+3)} + \sum_{p=0}^{\infty}\frac{e^{(p+2)\xi_j}[\psi'(p+3)-\psi^2(p+3)]}{\Gamma(p+3)},
\end{aligned}$$

$\psi(\cdot)$ is the digamma function and $\psi'(\cdot)$ is the trigamma function. The value of Gamma, digamma, and trigamma functions can be computed from the corresponding recursive formulae given in Abramowitz and Stegun (1964) in a straightforward manner.

Unlike the normal case, the maximum likelihood estimators of the parameters based on the complete data from the extreme value distribution cannot be solved explicitly. However, this problem has been well studied [see, for example, Lawless (1982)]. The estimators can be obtained by solving the equation

$$\hat{\sigma} = \frac{\sum_{i=1}^{n} w_i\exp(w_i/\hat{\sigma})}{\sum_{i=1}^{n}\exp(w_i/\hat{\sigma})} - \frac{1}{n}\sum_{i=1}^{n} w_i$$

and then $\hat{\mu}$ obtained explicitly as

$$\hat{\mu} = \hat{\sigma} \ln \left[ \frac{1}{n} \sum_{i=1}^{n} \exp\left(\frac{w_i}{\hat{\sigma}}\right) \right].$$

Thus, in the M-step of the $(h+1)$-th iteration of the EM-algorithm, the value of $\sigma_{(h+1)}$ is first obtained by solving the equation

$$\begin{aligned}
\sigma_{(h+1)} &= \frac{\sum_{j=1}^{m} y_{j:m:n} e^{\left(\frac{y_{j:m:n}}{\sigma_{(h+1)}}\right)} + \sum_{j=1}^{m} R_j E\left[ Z_j e^{\left(\frac{Z_j}{\sigma}\right)} \middle| \xi_j, \mu_{(h)}, \sigma_{(h)} \right]}{\sum_{j=1}^{m} e^{\left(\frac{y_{j:m:n}}{\sigma_{(h+1)}}\right)} + \sum_{j=1}^{m} R_j\ E\left[ e^{\frac{Z_j}{\sigma}} \middle| \xi_j, \mu_{(h)}, \sigma_{(h)} \right]} \\
&\quad - \frac{1}{n}\left[ \sum_{j=1}^{m} y_{j:m:n} + \sum_{j=1}^{m} R_j E\left[ Z_j \mid \xi_j, \mu_{(h)}, \sigma_{(h)} \right] \right]
\end{aligned} \tag{6.5.7}$$

and then $\mu_{(h+1)}$ is obtained as

$$\mu_{(h+1)} = \sigma_{(h+1)} \ln \left\{ \frac{1}{n} \left[ \sum_{j=1}^{m} e^{\frac{y_{j:m:n}}{\hat{\sigma}_{(h+1)}}} + \sum_{j=1}^{m} R_j E\left( e^{\left(\frac{Z_j}{\sigma}\right)} \middle| \xi_j, \mu_{(h)}, \sigma_{(h)} \right) \right] \right\}.$$

### 6.5.2 Asymptotic Variances and Covariance

From the classical results on the extreme value distribution, the complete data information matrix is [see, for example, Stephens (1977)]

$$I_Z \begin{pmatrix} \mu \\ \sigma \end{pmatrix} = \frac{n}{\sigma^2} \begin{pmatrix} 1 & 1-\gamma \\ 1-\gamma & c^2 \end{pmatrix}.$$

where $\gamma = 0.577215665$ is the Euler's constant and $c^2$ is $\frac{\pi^2}{6} + (1-\gamma)^2 = 1.823680661$.

The logarithm of the p.d.f. given in (6.5.2) is

$$\ln f_{Z_j}(z_j \mid Z_j > y_{j:m:n}, \mu, \sigma) = -\ln \sigma + e^{\xi_j} + \left(\frac{z_j - \mu}{\sigma}\right) - e^{\left(\frac{z_j-\mu}{\sigma}\right)}. \tag{6.5.8}$$

The three second partial derivatives of (6.5.8) with respect to $\mu$ and $\sigma$ are

$$\begin{aligned}
\frac{\partial^2 \ln f_{Z_j}}{\partial \mu^2} &= \frac{1}{\sigma^2}\left[ e^{\xi_j} - e^{\left(\frac{z_j-\mu}{\sigma}\right)} \right], \\
\frac{\partial^2 \ln f_{Z_j}}{\partial \sigma^2} &= \frac{1}{\sigma^2}\left[ e^{\xi_j} + \xi_j e^{\xi_j} - e^{\left(\frac{z_j-\mu}{\sigma}\right)} - \left(\frac{z_j-\mu}{\sigma}\right) e^{\left(\frac{z_j-\mu}{\sigma}\right)} \right],
\end{aligned}$$

$$\frac{\partial^2 \ln f_{Z_j}}{\partial \mu \partial \sigma} = \frac{1}{\sigma^2}\left[\xi_j^2 e^{\xi_j} + 2\xi_j e^{\xi_j} - \left(\frac{z_j-\mu}{\sigma}\right)^2 e^{\left(\frac{z_j-\mu}{\sigma}\right)} \right.$$
$$\left. -2\left(\frac{z_j-\mu}{\sigma}\right) e^{\left(\frac{z_j-\mu}{\sigma}\right)} + 2\left(\frac{z_j-\mu}{\sigma}\right) + 1\right].$$

We take the negative of the expected value of these three second partial derivatives by using the results given in (6.5.3)-(6.5.6). The Fisher information matrix in one observation that is censored at the time of the $j$-th failure can then be computed without any difficulty. Then, the observed information can be obtained by formulas (6.3.3) and (6.3.4) from which the variance-covariance matrix of the maximum likelihood estimate $\hat{\boldsymbol{\theta}} = (\,\hat{\mu} \quad \hat{\sigma}\,)$ can be determined.

In order to assess the accuracy of the approximation of the variances and covariance of the MLEs determined from the information matrix computed through the EM-algorithm described above, we carried out a simulation study (based on 1000 simulations) for different choices of $n, m$ and $(R_1, \ldots, R_m)$. Without loss of generality, we chose $\mu = 0$ and $\sigma = 1$. The simulated values of $Var(\hat{\mu})$, $Var(\hat{\sigma})$ and $Cov(\hat{\mu}, \hat{\sigma})$ as well as the approximate values determined by averaging the corresponding values obtained from the information matrix are presented in Table 6.2.

From this table, we observe that the approximate values determined from information matrix are quite close to the simulated values even for moderate values of $m$. Furthermore, we note that the approximation becomes quite accurate as $m$ increases.

Instead, had we employed Newton-Raphson method for finding the MLEs numerically, we would have solved the equations

$$\frac{\partial \ln L(\mu,\sigma)}{\partial \mu} = -\frac{m}{\sigma} + \frac{1}{\sigma}\sum_{j=1}^{m}(R_j+1)e^{\xi_j} = 0,$$
$$\frac{\partial \ln L(\mu,\sigma)}{\partial \sigma} = -\frac{m}{\sigma} - \frac{1}{\sigma}\sum_{j=1}^{m}\xi_j + \frac{1}{\sigma}\sum_{j=1}^{m}(R_j+1)\xi_j e^{\xi_j} = 0$$

by using the second-order derivative forms

$$\frac{\partial^2 \ln L(\mu,\sigma)}{\partial \mu^2} = -\frac{1}{\sigma^2}\sum_{j=1}^{m}(R_j+1)e^{\xi_j},$$

$$\begin{aligned}
\frac{\partial^2 \ln L(\mu,\sigma)}{\partial \sigma^2} &= \frac{1}{\sigma^2}\left\{m + 2\sum_{j=1}^{m}\xi_j - 2\sum_{j=1}^{m}(R_j+1)\xi_j e^{\xi_j} - 2\sum_{j=1}^{m}(R_j+1)\xi_j^2 e^{\xi_j}\right\}, \\
\frac{\partial^2 \log L(\mu,\sigma)}{\partial \mu \partial \sigma} &= \frac{1}{\sigma^2}\left\{m - \sum_{j=1}^{m}(R_j+1)e^{\xi_j} - \sum_{j=1}^{m}(R_j+1)\xi_j e^{\xi_j}\right\}.
\end{aligned}$$

It should be pointed out that the above second-order derivatives get used at every iteration in the Newton-Raphson method, but they only get used at the final stage of the EM-algorithm while computing the information measure. Clearly, this is one advantage of the EM-algorithm.

Though the Newton-Raphson method would yield exactly the same values for the MLEs and their asymptotic variances and covariance as the EM-algorithm, the convergence of the iterative process will be different than that of the EM-algorithm. Using the Newton-Raphson iterative procedure, Balakrishnan, Kannan, Lin and Wu (2001) have recently discussed the construction of confidence intervals for $\mu$ and $\sigma$.

Table 6.2: Variances and covariance of the MLEs

| n | m | Censoring Scheme | Simulated $Var(\hat{\mu})$ | Simulated $Var(\hat{\sigma})$ | Simulated $Cov(\hat{\mu}, \hat{\sigma})$ | From Information $Var(\hat{\mu})$ | From Information $Var(\hat{\sigma})$ | From Information $Cov(\hat{\mu}, \hat{\sigma})$ |
|---|---|---|---|---|---|---|---|---|
| 15 | 5 | (0, 0, 0, 0, 10) | 0.3676 | 0.1522 | 0.1549 | 0.3167 | 0.1507 | 0.1510 |
| 15 | 5 | (10, 0, 0, 0, 0) | 0.2194 | 0.0855 | -0.0004 | 0.1874 | 0.0816 | -0.0111 |
| 15 | 5 | (0, 0, 10, 0, 0) | 0.2558 | 0.0874 | 0.0550 | 0.1906 | 0.0750 | 0.0378 |
| 15 | 6 | (0, 0, 0, 0, 0, 9) | 0.2544 | 0.1182 | 0.0984 | 0.2147 | 0.1220 | 0.0974 |
| 15 | 6 | (9, 0, 0, 0, 0, 0) | 0.1751 | 0.0707 | -0.0046 | 0.1543 | 0.0713 | -0.0140 |
| 15 | 6 | (0, 9, 0, 0, 0, 0) | 0.1785 | 0.0660 | 0.0119 | 0.1482 | 0.0620 | 0.0020 |
| 20 | 6 | (0, 0, 0, 0, 0, 14) | 0.3320 | 0.1211 | 0.1390 | 0.2936 | 0.1256 | 0.1404 |
| 20 | 6 | (14, 0, 0, 0, 0, 0) | 0.1754 | 0.0652 | 0.0001 | 0.1559 | 0.0661 | -0.0092 |
| 20 | 6 | (0, 14, 0, 0, 0, 0) | 0.1835 | 0.0593 | 0.0190 | 0.1518 | 0.0554 | 0.0084 |
| 20 | 10 | (0, ..., 0, 10) | 0.1302 | 0.0817 | 0.0477 | 0.1183 | 0.0811 | 0.0440 |
| 20 | 10 | (10, 0, ..., 0) | 0.1051 | 0.0508 | -0.0107 | 0.1020 | 0.0502 | -0.0146 |
| 25 | 5 | (0, 0, 0, 0, 20) | 0.5928 | 0.1567 | 0.2450 | 0.5421 | 0.1563 | 0.2428 |
| 25 | 5 | (20, 0, 0, 0, 0) | 0.2232 | 0.0739 | 0.0084 | 0.1922 | 0.0707 | -0.0020 |
| 25 | 5 | (0, 20, 0, 0, 0) | 0.2535 | 0.0657 | 0.0418 | 0.1924 | 0.0561 | 0.0249 |
| 25 | 15 | (0, ..., 0, 10) | 0.0757 | 0.0530 | 0.0194 | 0.0679 | 0.0516 | 0.0167 |
| 25 | 15 | (10, 0, ..., 0) | 0.0713 | 0.0359 | -0.0095 | 0.0679 | 0.0349 | -0.0119 |
| 30 | 3 | (0, 0, 27) | 1.4423 | 0.1994 | 0.4612 | 1.2690 | 0.2047 | 0.4656 |
| 30 | 3 | (27, 0, 0) | 0.4693 | 0.0929 | 0.0730 | 0.3084 | 0.0772 | 0.0285 |
| 30 | 4 | (0, 0, 0, 26) | 0.9209 | 0.1819 | 0.3450 | 0.9019 | 0.1823 | 0.3606 |
| 30 | 4 | (26, 0, 0, 0) | 0.2876 | 0.0789 | 0.0177 | 0.2393 | 0.0720 | 0.0110 |
| 50 | 20 | (0, ..., 0, 30) | 0.0769 | 0.0444 | 0.0354 | 0.0763 | 0.0432 | 0.0346 |
| 50 | 20 | (30, 0, ..., 0) | 0.0511 | 0.0254 | -0.0079 | 0.0521 | 0.0250 | -0.0079 |
| 50 | 25 | (0, ..., 0, 25) | 0.0482 | 0.0330 | 0.0187 | 0.0485 | 0.0332 | 0.0181 |
| 50 | 25 | (25, 0, ..., 0) | 0.0397 | 0.0210 | -0.0064 | 0.0417 | 0.0211 | -0.0074 |

Table 6.3: Progressively censored sample presented by Viveros and Balakrishnan (1994)

| j | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| $y_{j:m:n}$ | -1.6608 | -0.2485 | -0.0409 | 0.2700 | 1.0224 | 1.5789 | 1.8718 | 1.9947 |
| $R_j$ | 0 | 0 | 3 | 0 | 3 | 0 | 0 | 5 |

### 6.5.3 Illustrative Example

**Example 6.2:** A progressively censored sample generated from the log-times to breakdown data on insulating fluid tested at 34 kilovolts by Viveros and Balakrishnan (1994) is used to demonstrate the above estimation procedure. The data are presented in Table 6.3.

For these data, we use EM-algorithm with starting values $\mu_{(0)} = 1.4127$ and $\sigma_{(0)} = 0.7912$, which are the estimates of the parameters based on the observed data points. The EM-algorithm converged to the values $\mu_{(\infty)} = 2.221960...$ and $\sigma_{(\infty)} = 1.0263807....$ The values of $\mu_{(h)}$ and $\sigma_{(h)}$ are plotted against $h$ and are presented in Figures 6.1 and 6.2. Note that the dotted line indicate the values of $\mu_{(\infty)}$ and $\sigma_{(\infty)}$. The results agreed with the MLEs of $\mu$ and $\sigma$ ($\hat{\mu} = 2.222$ and $\hat{\sigma} = 1.026$) computed by Viveros and Balakrishnan (1994) via numerical maximization, and also with the values obtained from Newton-Raphson method. In order to compare the convergence rate of the EM-algorithm with that of the Newton-Raphson method, same initial values were used and the level of accuracy was fixed at $5 \times 10^{-5}$. The EM-algorithm took 151 iterations while the Newton-Raphson method took 37 iterations to converge to the same values.

From these data, we have

$$\begin{aligned}
I_W(\hat{\boldsymbol{\theta}}) &= \begin{pmatrix} 18.03585 & 7.62528 \\ 7.62527 & 32.89163 \end{pmatrix}, \\
I_{Z|Y}(\hat{\boldsymbol{\theta}}) &= \begin{pmatrix} 10.44181 & 11.99591 \\ 11.99591 & 19.43904 \end{pmatrix}, \\
I_Y(\hat{\boldsymbol{\theta}}) &= I_W(\hat{\boldsymbol{\theta}}) - I_{Z|Y}(\hat{\boldsymbol{\theta}}) = \begin{pmatrix} 7.59404 & -4.37064 \\ -4.37064 & 13.45259 \end{pmatrix}.
\end{aligned}$$

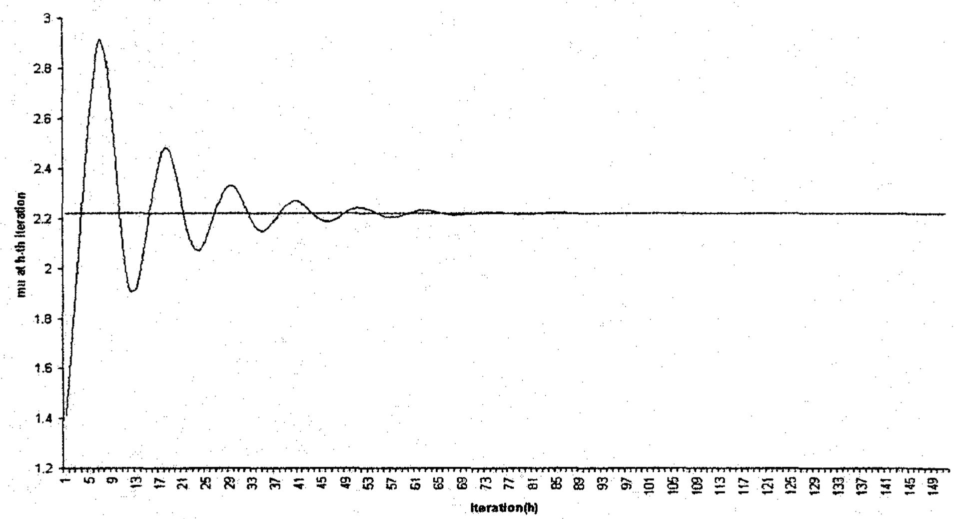



Figure 6.1: Trace plot for $\mu_{(h)}$ under EM-iterations (Weibull lifetime data)

By inverting the $I_Y(\hat{\boldsymbol{\theta}})$, we have

$$var(\hat{\mu}) = 0.16197, var(\hat{\sigma}) = 0.09143, cov(\hat{\mu}, \hat{\sigma}) = 0.05262.$$

## 6.6 Discussion

Although the maximum likelihood estimation method based on the progressively censored data has been studied extensively, traditionally Newton-Raphson method was used to obtain the estimates. In this chapter, the EM-algorithm is proposed to solve this problem. Two popular lifetime models, normal and extreme-value distributions, have been used to demonstrate how the algorithm works. For the normal case, the subsequent guesses of the parameters are in explicit form which is a perfect application of the EM-algorithm. Although this nice feature does not appear in the extreme-value case, the problem of obtaining the maximum likelihood estimates based on a complete sample has been studied extensively. The only problem in applying the EM-algorithm is in evaluating the moments of the truncated distribution. If the moments of the truncated distribution and the estimation based on complete sample case can be handled, the EM-algorithm can be applied.

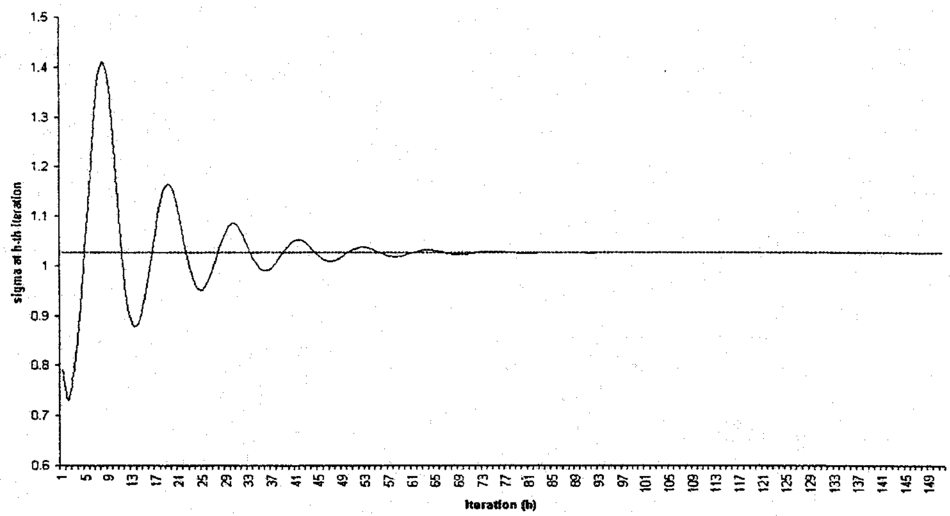



Figure 6.2: Trace plot for $\sigma_{(h)}$ under EM-iterations (Weibull lifetime data)

As pointed out by Little and Rubin (1983), the EM-algorithm will converge reliably but rather slowly (as compared to the Newton-Raphson method) when the amount of information in the missing data is relatively large. This has also been observed in the two examples considered in this chapter. However, the EM-algorithm possesses the advantages that (i) it gives a measure of information in the censored (missing) data in a natural way through the Missing Information Principle (which is not available in the Newton-Raphson method), and (ii) it can be generalized easily to other forms of censored data (such as Type-I censoring).

# Chapter 7

# Point and Interval Estimation for the Normal Distribution Under Progressive Censoring

## 7.1 Introduction

The likelihood equations based on a progressively Type-II censored sample from a normal distribution do not provide explicit solutions in any situation except the complete sample case. In the previous chapter, we showed that the EM-algorithm can be employed to obtain the MLEs in this case. In this chapter, we examine numerically the bias and mean squared error of the MLEs, and demonstrate that the probability coverages of the pivotal quantities (for location and scale parameters) based on asymptotic normality are unsatisfactory, and particularly so when the effective sample size is small. We, therefore, suggest the use of unconditional simulated percentage points of these pivotal quantities for the construction of confidence intervals. We also use an approximation of the normal hazard function to develop approximate estimators which are explicit and are almost as efficient as the MLEs; however, the probability coverages of the corresponding pivotal quantities based on asymptotic normality are also unsatisfactory. A wide range of sample sizes and progressive censoring schemes have been considered in this study. Finally, we present a numerical example to illustrate the methods of inference developed here.

## 7.2 MLEs for the Normal Distribution

Assume the log failure time distribution to be normal with probability density function

$$\phi(w;\mu,\sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(w-\mu)^2}{2\sigma^2}}, \qquad -\infty < w < \infty; -\infty < \mu < \infty; \sigma > 0$$

and corresponding cumulative distribution function

$$\Phi(w;\mu,\sigma) = \int_{-\infty}^{w} \phi(t;\mu,\sigma)dt.$$

From Eq. (6.1.1), the likelihood function based on a progressively Type-II censored sample is given by

$$L(\mu,\sigma) = C\prod_{j=1}^{m} \phi(y_{j:m:n};\mu,\sigma)[1-\Phi(y_{j:m:n};\mu,\sigma)]^{R_j}, \tag{7.2.9}$$

where

$$C = n(n-1-R_1)(n-2-R_1-R_2)\dots(n-m+1-R_1-\dots-R_{m-1}).$$

We shall use the same notations as in Chapter 6; let us denote $\xi_j = \frac{y_{j:m:n}-\mu}{\sigma}$ and $h(\xi_j) = Q_j = \frac{\phi(\xi_j)}{1-\Phi(\xi_j)}$. Then, $\frac{\partial \xi_j}{\partial \mu} = -\frac{1}{\sigma}$ and $\frac{\partial \xi_j}{\partial \sigma} = -\frac{\xi_j}{\sigma}$.

Now, let $\phi(\cdot)$ and $\Phi(\cdot)$ denote the standard normal p.d.f. and c.d.f., respectively. The log-likelihood function may then be written as

$$\ln L(\mu,\sigma) = \text{constant} - m\ln\sigma - \frac{1}{2}\sum_{j=1}^{m}\xi_j^2 + \sum_{j=1}^{m} R_j \ln\{1-\Phi(\xi_j)\}. \tag{7.2.10}$$

From Eq. (7.2.10), we derive the likelihood equations for $\mu$ and $\sigma$ as

$$\frac{\partial \ln L}{\partial \mu} = \frac{1}{\sigma}\sum_{j=1}^{m}\xi_j + \frac{1}{\sigma}\sum_{j=1}^{m} R_j Q_j = 0 \tag{7.2.11}$$

$$\Rightarrow \sum_{j=1}^{m}\xi_j + \sum_{j=1}^{m} R_j Q_j = 0 \tag{7.2.12}$$

$$\frac{\partial \ln L}{\partial \sigma} = -\frac{m}{\sigma} + \frac{1}{\sigma}\sum_{j=1}^{m}\xi_j^2 + \frac{1}{\sigma}\sum_{j=1}^{m} R_j \xi_j Q_j = 0 \tag{7.2.13}$$

$$\Rightarrow -m + \sum_{j=1}^{m}\xi_j^2 + \sum_{j=1}^{m} R_j \xi_j Q_j = 0 \tag{7.2.14}$$

Equations (7.2.12) and (7.2.14) do not yield explicit solutions for $\mu$ and $\sigma$ except in the case when $R_1 = \ldots = R_m = 0$, viz. the complete sample case. For the complete sample case, these equations lead to the standard formulae for the MLEs of the normal distribution. For all other censoring schemes, equations (7.2.12) and (7.2.14) will have to be solved numerically to obtain the MLEs of the two parameters. Cohen (1963) discussed maximum likelihood estimation under the assumption of progressive Type-I censoring. The likelihood equations will be identical, with the $y_{j:m:n}$'s being replaced by the corresponding censoring times $T_j$. Cohen discussed some numerical methods for obtaining the MLEs of the parameters; see also Cohen (1991).

## 7.3 Approximate Maximum Likelihood Estimators

In all cases, with the exception of the complete sample case, it is the presence of the term $h(\xi) = \frac{\phi(\xi)}{1-\Phi(\xi)}$ that makes the likelihood equations nonlinear. The function $h(\xi)$ is nothing but the hazard function of the normal distribution. Some approximate solutions have been discussed in the book by Tiku, Tan and Balakrishnan (1986).

We may approximate the hazard function by expanding in a Taylor series around $E(\Xi_{j:m:n}) = \nu_{j:m:n}$, where $\Xi_{j:m:n}$ is the $j$-th order statistic of a progressively Type-II censored sample from the standard normal distribution. From Balakrishnan and Sandhu (1995), we have

$$\Phi(\Xi_{j:m:n}) \stackrel{d}{=} U_{j:m:n},$$

where $U_{j:m:n}$ is the corresponding $j$-th order statistic of a progressively Type-II censored sample from the $U(0,1)$ distribution.

We then have

$$\Xi_{j:m:n} \stackrel{d}{=} \Phi^{-1}(U_{j:m:n}) \tag{7.3.15}$$

and hence

$$\nu_{j:m:n} = E(\Xi_{j:m:n}) \doteq \Phi^{-1}(\alpha_{j:m:n}) \tag{7.3.16}$$

where $\alpha_{j:m:n} = E(U_{j:m:n})$ and is given by [see Balakrishnan and Aggarwala (2000)]

$$\alpha_{j:m:n} = 1 - \prod_{k=m-j+1}^{m} \frac{k + R_{m-k+1} + \ldots + R_m}{k + 1 + R_{m-k+1} + \ldots + R_m}, \qquad j = 1, \ldots m. \tag{7.3.17}$$

Expanding $h(\xi_j)$ around $\nu_{j:m:n}$, we have (keeping only the first two terms),

$$h(\xi_j) \doteq h(\nu_{j:m:n}) + (\xi_j - \nu_{j:m:n}) h'(\xi)|_{\xi=\nu_{j:m:n}}, \tag{7.3.18}$$

where

$$h'(\xi) = \frac{\{1 - \Phi(\xi)\}\phi'(\xi) + \phi^2(\xi)}{\{1 - \Phi(\xi)\}^2}. \tag{7.3.19}$$

Replacing $\phi'(\xi)$ by $-\xi\phi(\xi)$, we can write

$$h'(\xi) = \frac{\phi(\xi)}{\{1 - \Phi(\xi)\}^2} \{\phi(\xi) - \xi\{1 - \Phi(\xi)\}\}. \tag{7.3.20}$$

Consider the second term in the above equation. We can show that

$$\begin{aligned}
\phi(\xi) - \xi\{1 - \Phi(\xi)\} &= \phi(\xi) - \xi \int_{\xi}^{\infty} \phi(t)dt \\
&\geq \phi(\xi) + \int_{\xi}^{\infty} -t\phi(t)dt \\
&= \phi(\xi) + \int_{\xi}^{\infty} \phi'(t)dt \geq 0 \\
\Longrightarrow h'(\xi) &\geq 0 \qquad \forall \xi.
\end{aligned}$$

Returning to the Taylor expansion, we have

$$h(\xi_j) \doteq h(\nu_{j:m:n}) + (\xi_j - \nu_{j:m:n}) h'(\nu_{j:m:n}) = \alpha_j + \beta_j \xi_j, \tag{7.3.21}$$

where

$$\alpha_j = h(\nu_{j:m:n}) - \nu_{j:m:n} h'(\nu_{j:m:n}), \qquad j = 1, \ldots, m \tag{7.3.22}$$

$$\beta_j = h'(\nu_{j:m:n}) \geq 0, \qquad j = 1, \ldots, m. \tag{7.3.23}$$

Using the above, we approximate the likelihood equations (7.2.12) and (7.2.14) by

$$\frac{\partial \ln L}{\partial \mu} \approx \sum_{j=1}^{m} \xi_j + \sum_{j=1}^{m} R_j\{\alpha_j + \beta_j \xi_j\} = 0, \tag{7.3.24}$$

$$\frac{\partial \ln L}{\partial \sigma} \approx -m + \sum_{j=1}^{m} \xi_j^2 + \sum_{j=1}^{m} R_j \xi_j \{\alpha_j + \beta_j \xi_j\} = 0. \tag{7.3.25}$$

Equation (7.3.24) may now be rewritten as

$$\sum_{j=1}^{m} \left(\frac{y_{j:m:n} - \mu}{\sigma}\right) + \sum_{j=1}^{m} R_j \left\{\alpha_j + \beta_j \left(\frac{y_{j:m:n} - \mu}{\sigma}\right)\right\} = 0 \tag{7.3.26}$$

which implies

$$\tilde{\mu} = \frac{1}{m + \sum_{j=1}^{m} R_j \beta_j} \left\{\sum_{j=1}^{m} y_{j:m:n} + \sum_{j=1}^{m} R_j \beta_j y_{j:m:n} + \sigma \sum_{j=1}^{m} R_j \alpha_j\right\} = K + L\sigma, \tag{7.3.27}$$

where

$$K = \frac{\sum_{j=1}^{m} (1 + R_j \beta_j) y_{j:m:n}}{m + \sum_{j=1}^{m} R_j \beta_j}, \tag{7.3.28}$$

$$L = \frac{\sum_{j=1}^{m} R_j \alpha_j}{m + \sum_{j=1}^{m} R_j \beta_j}. \tag{7.3.29}$$

Equation (7.3.25) may now be rewritten as

$$-m + \sum_{j=1}^{m} \left(\frac{y_{j:m:n} - \mu}{\sigma}\right)^2 + \sum_{j=1}^{m} R_j \alpha_j \left(\frac{y_{j:m:n} - \mu}{\sigma}\right) + \sum_{ji=1}^{m} R_j \beta_j \left(\frac{y_{j:m:n} - \mu}{\sigma}\right)^2 = 0 \tag{7.3.30}$$

$$\Rightarrow \sum_{j=1}^{m} (1 + R_j \beta_j)(y_{j:m:n} - \mu)^2 + \sigma \sum_{j=1}^{m} R_j \alpha_j (y_{j:m:n} - \mu) - m\sigma^2 = 0. \tag{7.3.31}$$

Replacing $\mu$ by $K + L\sigma$, we have

$$\begin{aligned} &\sum_{j=1}^{m}(1+R_j\beta_j)(y_{j:m:n}-K)^2 + \sigma\sum_{j=1}^{m}R_j\alpha_j(y_{j:m:n}-K) - m\sigma^2 \\ +\quad & L^2\sigma^2\sum_{j=1}^{m}(1+R_j\beta_j) - 2L\sigma\sum_{j=1}^{m}(1+R_j\beta_j)(y_{j:m:n}-K) - L\sigma^2\sum_{j=1}^{m}R_j\alpha_j = 0. \end{aligned} \tag{7.3.32}$$

The last three terms can be shown to vanish, leaving us with the equation

$$\sum_{j=1}^{m}(1+R_j\beta_j)(y_{j:m:n}-K)^2 + \sigma\sum_{j=1}^{m}R_j\alpha_j(y_{j:m:n}-K) - m\sigma^2 = 0 \tag{7.3.33}$$

or

$$m\sigma^2 - A_1\sigma - A_2 = 0,$$

where

$$\begin{aligned} A_1 &= \sum_{j=1}^{m}R_j\alpha_j(y_{j:m:n}-K) \\ A_2 &= \sum_{j=1}^{m}(1+R_j\beta_j)(y_{j:m:n}-K)^2 \geq 0. \end{aligned}$$

Equation (7.3.33) is a quadratic equation in $\sigma$, with the roots given by

$$\tilde{\sigma} = \frac{A_1 \pm \sqrt{A_1^2 + 4mA_2}}{2m}.$$

Since $A_2 \geq 0$, only one root is admissible, and hence the approximate MLE of $\sigma$ is given by

$$\tilde{\sigma} = \frac{A_1 + \sqrt{A_1^2 + 4mA_2}}{2m}. \tag{7.3.34}$$

The approximate MLEs may be easily obtained from (7.3.27) and (7.3.34).

We may now ask the logical question: How good are the approximate estimators? It would naturally be of interest to evaluate the efficiency of these estimators compared to the MLEs obtained by solving the full likelihood equations numerically.

Furthermore, the approximate solution may provide us with an excellent starting value for the iterative solution of the likelihood equations.

It is important to mention here that a similar approximate MLE has been presented by Tiku, Tan and Balakrishnan (1986) for progressively censored samples, wherein linear approximations used are different (not based on properties of uniform order statistics).

## 7.4 Observed Fisher Information

In this section, we compute the observed Fisher information from the full and approximate equations. These will enable us to develop pivotal quantities based on the limiting normal distribution and examine the probability coverages through simulation. We have already shown that the observed Fisher information can be obtained by the missing information principle via EM-algorithm in Chapter 6.

We now derive the observed Fisher Information from the full likelihood using equations (7.2.11) and (7.2.13). We have

$$\frac{\partial^2 \ln L(\mu,\sigma)}{\partial \mu^2} = -\frac{m}{\sigma^2} + \frac{1}{\sigma^2}\sum_{j=1}^{m} R_j \left\{ \frac{\xi_j \phi(\xi_j) - \xi_j \phi(\xi_j)\Phi(\xi_j) - \phi^2(\xi_j)}{[1-\Phi(\xi_j)]^2} \right\}, \tag{7.4.35}$$

$$\frac{\partial^2 \ln L(\mu,\sigma)}{\partial \sigma^2} = \frac{m}{\sigma^2} - \frac{3}{\sigma^2}\sum_{j=1}^{m} \xi_j^2 - \frac{2}{\sigma^2}\sum_{j=1}^{m} R_j Q_j, \tag{7.4.36}$$

$$\frac{\partial^2 \ln L(\mu,\sigma)}{\partial \mu \partial \sigma} = \frac{1}{\sigma^2}\sum_{j=1}^{m} \left\{ 2\xi_j + R_j Q_j + R_j \xi_j Q_j - \xi_j^2 Q_j \right\}. \tag{7.4.37}$$

Similarly, from the approximate likelihood equations, we obtain

$$\frac{\partial^2 \ln L(\mu,\sigma)}{\partial \mu^2} \approx -\frac{m}{\sigma^2} - \frac{1}{\sigma^2}\sum_{j=1}^{m} R_j \beta_j, \tag{7.4.38}$$

$$\frac{\partial^2 \ln L(\mu,\sigma)}{\partial \mu \partial \sigma} \approx -\frac{2}{\sigma^2}\sum_{j=1}^{m} \xi_j - \frac{1}{\sigma^2}\sum_{j=1}^{m} R_j \beta_j \xi_j - \frac{1}{\sigma^2}\sum_{j=1}^{m} R_j(\alpha_j + \beta_j \xi_j), \tag{7.4.39}$$

$$\frac{\partial^2 \ln L(\mu,\sigma)}{\partial \sigma^2} \approx \frac{m}{\sigma^2} - \frac{3}{\sigma^2}\sum_{j=1}^{m} \xi_j^2 - \frac{2}{\sigma^2}\sum_{j=1}^{m} R_j \xi_j(\alpha_j + \beta_j \xi_j) - \frac{1}{\sigma^2}\sum_{j=1}^{m} R_j \beta_j \xi_j^2. \tag{7.4.40}$$

Let $-\frac{\partial^2 \ln L}{\partial \mu^2} = \frac{V_1}{\sigma^2}$, $-\frac{\partial^2 \ln L}{\partial \mu \partial \sigma} = \frac{V_2}{\sigma^2}$, $-\frac{\partial^2 \ln L}{\partial \sigma^2} = \frac{V_3}{\sigma^2}$. The observed information matrix can be inverted to obtain the asymptotic variance-covariance matrix of the estimators as

$$\frac{1}{\sigma^2}\begin{bmatrix} V_1 & V_2 \\ V_2 & V_3 \end{bmatrix}^{-1} = \sigma^2 \begin{bmatrix} V^{11} & V^{12} \\ V^{12} & V^{22} \end{bmatrix}, \tag{7.4.41}$$

where $V^{11} = \frac{V_3}{V_1 V_3 - V_2^2}$, $V^{12} = -\frac{V_2}{V_1 V_3 - V_2^2}$, $V^{22} = \frac{V_1}{V_1 V_3 - V_2^2}$.

Similarly, we may obtain the terms $V_*^{11}, V_*^{12}, V_*^{22}$ from the observed Fisher information from the approximate equations. These are not presented here for conciseness.

## 7.5 Monte Carlo Simulation Results

We conducted a simulation study to compare the performance of the approximate estimators with the MLEs. We generated progressively Type-II censored samples from the standard normal distribution using the algorithm presented by Balakrishnan and Sandhu (1995). We computed the approximate MLEs from equations (7.3.34) and (7.3.27). The MLEs of the parameters were obtained by solving the nonlinear equations (7.2.12) and (7.2.14) using the IMSL nonlinear equation solver, in which the approximate MLEs were used as starting values for the iterations. Tables 7.1 and 7.2 provide the average values of the estimates, their variances, and covariance. We also averaged the values of variances and covariances determined from the observed Fisher information matrix and those values are presented in the table for comparison. All the averages were computed over 1000 simulations. It may be noted that the MSEs of these estimators are comparable to the variances of the BLUEs presented by Balakrishnan and Aggarwala (2000).

From Tables 7.1 and 7.2, we observe that the approximate MLEs and MLEs are almost identical in terms of both bias and variance. The approximate MLEs are almost as efficient as the MLEs for all sample sizes and censoring schemes. As the effective sample proportion $m/n$ increases, the bias and variance of the estimators reduce significantly. For a fixed $n$ and $m$, we can determine the censoring scheme that is most efficient. For almost all choices, the censoring scheme $R_1 = n - m$, $R_2 =$

$\ldots = R_m = 0$ seems to provide the smallest bias and variance for the estimates. This point has also been noticed in terms of BLUEs by Balakrishnan and Aggarwala (2000).

## 7.6 Evaluation of Coverage Probabilities

Using the asymptotic normality of $\begin{pmatrix} \hat{\mu} \\ \hat{\sigma} \end{pmatrix}$, we have the asymptotic distribution of

$$P_1 = \frac{\hat{\mu} - \mu}{\hat{\sigma}\sqrt{V^{11}}} \qquad P_2 = \frac{\hat{\mu} - \mu}{\sigma\sqrt{V^{11}}} \qquad P_3 = \frac{\hat{\sigma} - \sigma}{\hat{\sigma}\sqrt{V^{22}}} \tag{7.6.42}$$

to be standard normal. Through Monte Carlo simulations, we simulate the probability coverages of

$$P(-1.65 \leq P_i \leq 1.65) \text{ and } P(-1.96 \leq P_i \leq 1.96), \qquad i = 1, 2, 3,$$

which we expect to be approximately 90 % and 95 %, respectively. We repeat this process for the approximate estimators as well.

Tables 7.3 and 7.4 provide results of the simulation for the MLEs and the approximate estimators. It is clear that when $\sigma$ is unknown, the probability coverages are extremely unsatisfactory especially when the effective sample percentage, viz., $m/n$ is small. If $\sigma$ is known, the coverage probabilities for $P_2$ are close to the required levels. In most practical situations, however, $\sigma$ is unknown and hence using the normal approximation for the corresponding pivotal quantity is not advisable. The distribution of the pivotal quantities are extremely skewed.

In Tables 7.5 and 7.6, we provide the unconditional percentage points for these pivotal quantities determined through simulation. It is clear for small sample sizes, the percentage points are very different from what might be expected if the distribution were normal. These percentage points will be helpful to compute confidence intervals for $\mu$ and $\sigma$.

Table 7.1: Variances and covariance of the MLEs

| $n$ | $m$ | Scheme | $\hat{\mu}$ | $\hat{\sigma}$ | $Var(\hat{\mu})$ | $Var(\hat{\sigma})$ | $Cov(\hat{\mu},\hat{\sigma})$ | $\hat{I}^{11}$ | $\hat{I}^{22}$ | $\hat{I}^{12}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 15 | 2 | (0,13) | -0.7232 | 0.4821 | 0.4385 | 0.1639 | 0.2112 | 0.2920 | 0.1426 | 0.1810 |
| 15 | 2 | (13,0) | -0.4513 | 0.7031 | 0.4262 | 0.1353 | 0.1660 | 0.2799 | 0.0899 | 0.1044 |
| 15 | 3 | (2*0,12) | -0.4016 | 0.6626 | 0.3055 | 0.1562 | 0.1681 | 0.2274 | 0.1381 | 0.1453 |
| 15 | 3 | (12,2*0) | -0.2452 | 0.8221 | 0.2680 | 0.1004 | 0.0929 | 0.2226 | 0.0787 | 0.0690 |
| 15 | 4 | (3*0,11) | -0.2504 | 0.7571 | 0.1961 | 0.1305 | 0.1069 | 0.1696 | 0.1190 | 0.1055 |
| 15 | 4 | (11,3*0) | -0.1627 | 0.8704 | 0.1980 | 0.0762 | 0.0470 | 0.1795 | 0.0681 | 0.0464 |
| 15 | 5 | (4*0,10) | -0.1690 | 0.8182 | 0.1594 | 0.1128 | 0.0858 | 0.1340 | 0.1028 | 0.0772 |
| 15 | 5 | (10,4*0) | -0.1184 | 0.9002 | 0.1641 | 0.0695 | 0.0392 | 0.1536 | 0.0615 | 0.0324 |
| 15 | 5 | (0,10,3*0) | -0.1330 | 0.8758 | 0.1591 | 0.0746 | 0.0542 | 0.1398 | 0.0646 | 0.0460 |
| 15 | 5 | (2*0,10,2*0) | -0.1442 | 0.8591 | 0.1578 | 0.0847 | 0.0650 | 0.1355 | 0.0734 | 0.0554 |
| 15 | 5 | (2,2,2,2,2) | -0.1468 | 0.8464 | 0.1523 | 0.0946 | 0.0675 | 0.1288 | 0.0837 | 0.0577 |
| 15 | 5 | (4,4,2,2*0) | -0.1301 | 0.8759 | 0.1562 | 0.0775 | 0.0535 | 0.1376 | 0.0680 | 0.0451 |
| 20 | 2 | (0,18) | -0.8037 | 0.4797 | 0.4867 | 0.1649 | 0.2360 | 0.3462 | 0.1456 | 0.2058 |
| 20 | 2 | (18,0) | -0.4732 | 0.7212 | 0.4477 | 0.1302 | 0.1775 | 0.2979 | 0.0870 | 0.1133 |
| 20 | 3 | (2*0,17) | -0.4592 | 0.6598 | 0.3502 | 0.1585 | 0.1950 | 0.2722 | 0.1419 | 0.1713 |
| 20 | 3 | (17,2*0) | -0.2604 | 0.8367 | 0.2759 | 0.0945 | 0.1002 | 0.2311 | 0.0745 | 0.0758 |
| 20 | 4 | (3*0,16) | -0.2947 | 0.7545 | 0.2257 | 0.1339 | 0.1325 | 0.2020 | 0.1230 | 0.1293 |
| 20 | 4 | (16,3*0) | -0.1763 | 0.8829 | 0.1988 | 0.0710 | 0.0533 | 0.1838 | 0.0639 | 0.0522 |
| 20 | 5 | (4*0,15) | -0.2037 | 0.8155 | 0.1824 | 0.1161 | 0.1082 | 0.1565 | 0.1067 | 0.0986 |
| 20 | 5 | (15,4*0) | -0.1305 | 0.9113 | 0.1649 | 0.0645 | 0.0445 | 0.1556 | 0.0573 | 0.0376 |
| 20 | 5 | (10,5,3*0) | -0.1370 | 0.8959 | 0.1610 | 0.0684 | 0.0540 | 0.1455 | 0.0606 | 0.0459 |
| 20 | 5 | (5,5,5,2*0) | -0.1504 | 0.8783 | 0.1611 | 0.0763 | 0.0660 | 0.1409 | 0.0669 | 0.0561 |
| 20 | 5 | (3,3,3,3,3) | -0.1730 | 0.8468 | 0.1637 | 0.0953 | 0.0834 | 0.1384 | 0.0845 | 0.0718 |
| 20 | 5 | (0,15,3*0) | -0.1490 | 0.8841 | 0.1639 | 0.0700 | 0.0621 | 0.1445 | 0.0608 | 0.0530 |
| 20 | 5 | (5,10,3*0) | -0.1437 | 0.8884 | 0.1622 | 0.0696 | 0.0590 | 0.1442 | 0.0610 | 0.0504 |
| 20 | 10 | (9*0,10) | -0.0667 | 0.9204 | 0.0733 | 0.0600 | 0.0296 | 0.0689 | 0.0560 | 0.0273 |
| 20 | 10 | (10,9*0) | -0.0591 | 0.9506 | 0.0855 | 0.0403 | 0.0110 | 0.0865 | 0.0381 | 0.0103 |
| 25 | 5 | (4*0,20) | -0.2294 | 0.8139 | 0.2061 | 0.1183 | 0.1252 | 0.1796 | 0.1094 | 0.1151 |
| 25 | 5 | (20,4*0) | -0.1377 | 0.9191 | 0.1662 | 0.0610 | 0.0478 | 0.1576 | 0.0543 | 0.0410 |
| 25 | 10 | (9*0,15) | -0.0788 | 0.9191 | 0.0774 | 0.0622 | 0.0400 | 0.0722 | 0.0583 | 0.0371 |
| 25 | 10 | (15,9*0) | -0.0650 | 0.9561 | 0.0844 | 0.0379 | 0.0134 | 0.0861 | 0.0359 | 0.0126 |
| 25 | 15 | (14*0,10) | -0.0422 | 0.9369 | 0.0498 | 0.0392 | 0.0148 | 0.0467 | 0.0363 | 0.0132 |
| 25 | 15 | (10,14*0) | -0.0349 | 0.9551 | 0.0595 | 0.0289 | 0.0057 | 0.0591 | 0.0270 | 0.0046 |
| 50 | 20 | (19*0,30) | -0.0450 | 0.9569 | 0.0372 | 0.0320 | 0.0193 | 0.0377 | 0.0305 | 0.0194 |
| 50 | 20 | (30,19*0) | -0.0367 | 0.9766 | 0.0433 | 0.0194 | 0.0040 | 0.0452 | 0.0191 | 0.0050 |
| 50 | 25 | (24*0,25) | -0.0278 | 0.9656 | 0.0278 | 0.0234 | 0.0112 | 0.0290 | 0.0236 | 0.0115 |
| 50 | 25 | (25,24*0) | -0.0207 | 0.9791 | 0.0344 | 0.0159 | 0.0025 | 0.0369 | 0.0164 | 0.0029 |

Table 7.2: Variances and covariance of the approximate MLEs

| $n$ | $m$ | Scheme | $\tilde{\mu}$ | $\tilde{\sigma}$ | $Var(\tilde{\mu})$ | $Var(\tilde{\sigma})$ | $Cov(\tilde{\mu},\tilde{\sigma})$ | $\tilde{I}^{11}$ | $\tilde{I}^{22}$ | $\tilde{I}^{12}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 15 | 2 | (0,13) | -0.7234 | 0.4820 | 0.4383 | 0.1639 | 0.2111 | 0.2943 | 0.1427 | 0.1815 |
| 15 | 2 | (13,0) | -0.4613 | 0.7096 | 0.4215 | 0.1378 | 0.1655 | 0.2720 | 0.0889 | 0.1114 |
| 15 | 3 | (2*0,12) | -0.4019 | 0.6625 | 0.3053 | 0.1562 | 0.1680 | 0.2297 | 0.1383 | 0.1458 |
| 15 | 3 | (12,2*0) | -0.2566 | 0.8312 | 0.2653 | 0.1028 | 0.0925 | 0.2170 | 0.0765 | 0.0762 |
| 15 | 4 | (3*0,11) | -0.2509 | 0.7570 | 0.1960 | 0.1305 | 0.1069 | 0.1715 | 0.1191 | 0.1060 |
| 15 | 4 | (11,3*0) | -0.1741 | 0.8803 | 0.1970 | 0.0782 | 0.0463 | 0.1754 | 0.0657 | 0.0526 |
| 15 | 5 | (4*0,10) | -0.1694 | 0.8180 | 0.1593 | 0.1128 | 0.0857 | 0.1356 | 0.1030 | 0.0777 |
| 15 | 5 | (10,4*0) | -0.1306 | 0.9115 | 0.1634 | 0.0715 | 0.0384 | 0.1504 | 0.0588 | 0.0382 |
| 15 | 5 | (0,10,3*0) | -0.1381 | 0.8782 | 0.1586 | 0.0750 | 0.0539 | 0.1376 | 0.0643 | 0.0475 |
| 15 | 5 | (2*0,10,2*0) | -0.1480 | 0.8600 | 0.1574 | 0.0849 | 0.0647 | 0.1340 | 0.0734 | 0.0560 |
| 15 | 5 | (2,2,2,2,2) | -0.1528 | 0.8522 | 0.1514 | 0.0960 | 0.0672 | 0.1317 | 0.0829 | 0.0601 |
| 15 | 5 | (4,4,2,2*0) | -0.1393 | 0.8855 | 0.1553 | 0.0792 | 0.0531 | 0.1385 | 0.0668 | 0.0479 |
| 20 | 2 | (0,18) | -0.8038 | 0.4796 | 0.4866 | 0.1649 | 0.2359 | 0.3481 | 0.1457 | 0.2062 |
| 20 | 2 | (18,0) | -0.4823 | 0.7272 | 0.4433 | 0.1324 | 0.1774 | 0.2913 | 0.0860 | 0.1199 |
| 20 | 3 | (2*0,17) | -0.4595 | 0.6598 | 0.3500 | 0.1584 | 0.1950 | 0.2740 | 0.1420 | 0.1717 |
| 20 | 3 | (17,2*0) | -0.2709 | 0.8453 | 0.2733 | 0.0966 | 0.1000 | 0.2264 | 0.0724 | 0.0826 |
| 20 | 4 | (3*0,16) | -0.2950 | 0.7544 | 0.2256 | 0.1339 | 0.1325 | 0.2036 | 0.1231 | 0.1297 |
| 20 | 4 | (16,3*0) | -0.1870 | 0.8927 | 0.1978 | 0.0728 | 0.0528 | 0.1803 | 0.0614 | 0.0583 |
| 20 | 5 | (4*0,15) | -0.2040 | 0.8154 | 0.1823 | 0.1161 | 0.1081 | 0.1578 | 0.1068 | 0.0990 |
| 20 | 5 | (15,4*0) | -0.1424 | 0.9227 | 0.1640 | 0.0665 | 0.0439 | 0.1527 | 0.0545 | 0.0434 |
| 20 | 5 | (10,5,3*0) | -0.1487 | 0.9092 | 0.1599 | 0.0706 | 0.0537 | 0.1468 | 0.0587 | 0.0499 |
| 20 | 5 | (5,5,5,2*0) | -0.1579 | 0.8886 | 0.1602 | 0.0781 | 0.0660 | 0.1427 | 0.0658 | 0.0586 |
| 20 | 5 | (3,3,3,3,3) | -0.1780 | 0.8543 | 0.1629 | 0.0970 | 0.0835 | 0.1426 | 0.0835 | 0.0742 |
| 20 | 5 | (0,15,3*0) | -0.1539 | 0.8867 | 0.1633 | 0.0704 | 0.0619 | 0.1423 | 0.0604 | 0.0545 |
| 20 | 5 | (5,10,3*0) | -0.1509 | 0.8982 | 0.1613 | 0.0711 | 0.0589 | 0.1446 | 0.0601 | 0.0530 |
| 20 | 10 | (9*0,10) | -0.0670 | 0.9203 | 0.0733 | 0.0600 | 0.0296 | 0.0694 | 0.0561 | 0.0276 |
| 20 | 10 | (10,9*0) | -0.0686 | 0.9609 | 0.0858 | 0.0413 | 0.0104 | 0.0858 | 0.0366 | 0.0127 |
| 25 | 5 | (4*0,20) | -0.2296 | 0.8138 | 0.2060 | 0.1183 | 0.1252 | 0.1807 | 0.1095 | 0.1154 |
| 25 | 5 | (20,4*0) | -0.1490 | 0.9302 | 0.1653 | 0.0628 | 0.0473 | 0.1550 | 0.0515 | 0.0466 |
| 25 | 10 | (9*0,15) | -0.0791 | 0.9190 | 0.0773 | 0.0622 | 0.0400 | 0.0727 | 0.0584 | 0.0373 |
| 25 | 10 | (15,9*0) | -0.0753 | 0.9675 | 0.0847 | 0.0388 | 0.0128 | 0.0854 | 0.0342 | 0.0154 |
| 25 | 15 | (14*0,10) | -0.0424 | 0.9368 | 0.0498 | 0.0392 | 0.0148 | 0.0470 | 0.0364 | 0.0133 |
| 25 | 15 | (10,14*0) | -0.0423 | 0.9638 | 0.0597 | 0.0295 | 0.0053 | 0.0590 | 0.0262 | 0.0059 |
| 50 | 20 | (19*0,30) | -0.0452 | 0.9568 | 0.0372 | 0.0320 | 0.0193 | 0.0379 | 0.0306 | 0.0194 |
| 50 | 20 | (30,19*0) | -0.0452 | 0.9872 | 0.0437 | 0.0203 | 0.0034 | 0.0452 | 0.0182 | 0.0061 |
| 50 | 25 | (24*0,25) | -0.0279 | 0.9656 | 0.0278 | 0.0234 | 0.0112 | 0.0291 | 0.0237 | 0.0116 |
| 50 | 25 | (25,24*0) | -0.0288 | 0.9900 | 0.0344 | 0.0164 | 0.0023 | 0.0370 | 0.0158 | 0.0037 |

Table 7.3: 95 % and 90 % coverage probabilities for the MLEs

| $n$ | $m$ | Scheme | $\tilde{\mu}, \sigma$ unknown | | $\tilde{\mu}, \sigma$ known | | $\tilde{\sigma}$ | |
|---|---|---|---|---|---|---|---|---|
| 15 | 2 | (0,13) | 43.3 | 39.8 | 93.7 | 84.8 | 41.1 | 37.7 |
| 15 | 2 | (13,0) | 66.0 | 61.7 | 89.6 | 82.0 | 60.8 | 56.0 |
| 15 | 3 | (2*0,12) | 60.1 | 56.1 | 92.7 | 86.2 | 59.1 | 54.4 |
| 15 | 3 | (12,2*0) | 79.4 | 75.1 | 93.1 | 87.6 | 72.8 | 67.4 |
| 15 | 4 | (3*0,11) | 72.7 | 67.7 | 94.9 | 89.1 | 67.2 | 62.0 |
| 15 | 4 | (11,3*0) | 85.0 | 78.5 | 95.3 | 88.7 | 78.6 | 74.2 |
| 15 | 5 | (4*0,10) | 77.4 | 72.1 | 94.3 | 88.2 | 72.8 | 68.1 |
| 15 | 5 | (10,4*0) | 87.7 | 81.6 | 94.3 | 89.4 | 81.0 | 75.9 |
| 15 | 5 | (0,10,3*0) | 85.3 | 79.3 | 93.4 | 89.1 | 78.0 | 72.7 |
| 15 | 5 | (2*0,10,2*0) | 82.9 | 76.9 | 94.1 | 88.5 | 76.6 | 71.1 |
| 15 | 5 | (2,2,2,2,2) | 81.3 | 75.3 | 94.0 | 88.4 | 75.2 | 70.7 |
| 15 | 5 | (4,4,2,2*0) | 85.1 | 79.7 | 93.1 | 89.1 | 77.4 | 73.4 |
| 20 | 2 | (0,18) | 42.3 | 38.6 | 95.0 | 85.8 | 41.2 | 37.7 |
| 20 | 2 | (18,0) | 66.1 | 62.1 | 88.4 | 81.1 | 62.9 | 58.3 |
| 20 | 3 | (2*0,17) | 59.1 | 54.5 | 93.5 | 86.1 | 59.2 | 54.3 |
| 20 | 3 | (17,2*0) | 79.4 | 74.7 | 92.2 | 87.1 | 74.5 | 70.0 |
| 20 | 4 | (3*0,16) | 70.7 | 65.3 | 94.9 | 88.1 | 66.8 | 61.6 |
| 20 | 4 | (16,3*0) | 85.0 | 78.6 | 94.6 | 88.5 | 80.1 | 75.4 |
| 20 | 5 | (4*0,15) | 74.4 | 70.9 | 94.3 | 88.3 | 72.7 | 68.1 |
| 20 | 5 | (15,4*0) | 87.4 | 81.3 | 93.6 | 89.3 | 82.4 | 77.1 |
| 20 | 5 | (10,5,3*0) | 86.0 | 80.3 | 93.4 | 88.6 | 80.6 | 75.4 |
| 20 | 5 | (5,5,5,2*0) | 83.6 | 77.4 | 93.1 | 88.2 | 78.3 | 73.5 |
| 20 | 5 | (3,3,3,3,3) | 79.5 | 73.6 | 93.5 | 87.2 | 75.1 | 70.9 |
| 20 | 5 | (0,15,3*0) | 84.8 | 78.8 | 93.5 | 88.4 | 79.0 | 73.8 |
| 20 | 5 | (5,10,3*0) | 85.3 | 79.3 | 93.3 | 88.3 | 78.8 | 74.4 |
| 20 | 10 | (9*0,10) | 88.5 | 82.8 | 95.1 | 89.9 | 84.8 | 78.8 |
| 20 | 10 | (10,9*0) | 92.4 | 87.4 | 95.3 | 90.5 | 88.7 | 83.6 |
| 25 | 5 | (4*0,20) | 73.6 | 69.0 | 94.9 | 88.1 | 72.3 | 67.8 |
| 25 | 5 | (20,4*0) | 87.1 | 81.4 | 93.4 | 88.9 | 83.6 | 78.1 |
| 25 | 10 | (9*0,15) | 87.8 | 82.4 | 94.9 | 89.4 | 85.0 | 78.8 |
| 25 | 10 | (15,9*0) | 92.6 | 87.3 | 95.1 | 90.2 | 89.5 | 83.9 |
| 25 | 15 | (14*0,10) | 91.6 | 85.1 | 94.5 | 90.3 | 86.6 | 81.1 |
| 25 | 15 | (10,14*0) | 92.3 | 87.4 | 95.7 | 90.3 | 88.4 | 83.1 |
| 50 | 20 | (19*0,30) | 92.0 | 86.7 | 95.5 | 90.2 | 88.4 | 83.3 |
| 50 | 20 | (30,19*0) | 93.3 | 88.8 | 95.1 | 89.9 | 91.1 | 87.0 |
| 50 | 25 | (24*0,25) | 92.6 | 87.9 | 95.8 | 90.2 | 90.4 | 85.6 |
| 50 | 25 | (25,24*0) | 94.8 | 91.0 | 96.7 | 92.2 | 92.8 | 88.0 |

Table 7.4: 95 % and 90 % coverage probabilities for the approximate MLEs

| $n$ | $m$ | Scheme | $\tilde{\mu}, \sigma$ unknown | | $\tilde{\mu}, \sigma$ known | | $\tilde{\sigma}$ | |
|---|---|---|---|---|---|---|---|---|
| 15 | 2 | (0,13) | 43.5 | 40.0 | 93.8 | 85.1 | 41.1 | 37.7 |
| 15 | 2 | (13,0) | 65.2 | 61.0 | 88.2 | 80.5 | 60.9 | 56.3 |
| 15 | 3 | (2*0,12) | 60.1 | 56.2 | 92.8 | 86.4 | 59.1 | 54.5 |
| 15 | 3 | (12,2*0) | 78.8 | 74.2 | 92.0 | 87.1 | 72.9 | 67.9 |
| 15 | 4 | (3*0,11) | 72.8 | 68.0 | 95.1 | 89.6 | 67.2 | 62.0 |
| 15 | 4 | (11,3*0) | 84.3 | 77.8 | 94.6 | 87.2 | 78.6 | 74.1 |
| 15 | 5 | (4*0,10) | 77.8 | 72.4 | 94.5 | 88.4 | 72.8 | 68.1 |
| 15 | 5 | (10,4*0) | 87.1 | 80.8 | 93.8 | 88.6 | 80.9 | 75.9 |
| 15 | 5 | (0,10,3*0) | 85.1 | 79.2 | 93.3 | 88.2 | 78.2 | 72.8 |
| 15 | 5 | (2*0,10,2*0) | 82.8 | 76.4 | 93.8 | 87.9 | 76.7 | 71.1 |
| 15 | 5 | (2,2,2,2,2) | 81.5 | 75.1 | 94.0 | 88.4 | 75.3 | 70.8 |
| 15 | 5 | (4,4,2,2*0) | 84.9 | 79.4 | 92.8 | 88.4 | 77.8 | 73.6 |
| 20 | 2 | (0,18) | 42.4 | 38.6 | 95.3 | 85.8 | 41.2 | 37.7 |
| 20 | 2 | (18,0) | 65.4 | 61.5 | 87.8 | 80.2 | 63.1 | 58.3 |
| 20 | 3 | (2*0,17) | 59.3 | 54.7 | 93.6 | 86.3 | 59.2 | 54.3 |
| 20 | 3 | (17,2*0) | 78.9 | 73.9 | 91.1 | 86.6 | 74.7 | 70.4 |
| 20 | 4 | (3*0,16) | 70.7 | 65.5 | 95.1 | 88.4 | 66.8 | 61.6 |
| 20 | 4 | (16,3*0) | 84.6 | 77.7 | 93.8 | 87.0 | 80.1 | 75.5 |
| 20 | 5 | (4*0,15) | 74.7 | 70.9 | 94.3 | 88.3 | 72.7 | 68.1 |
| 20 | 5 | (15,4*0) | 87.0 | 81.1 | 93.1 | 88.0 | 82.6 | 77.3 |
| 20 | 5 | (10,5,3*0) | 86.0 | 80.1 | 93.0 | 87.8 | 80.8 | 75.4 |
| 20 | 5 | (5,5,5,2*0) | 83.7 | 77.1 | 93.1 | 87.8 | 78.5 | 74.1 |
| 20 | 5 | (3,3,3,3,3) | 79.8 | 73.8 | 93.6 | 87.2 | 75.2 | 70.7 |
| 20 | 5 | (0,15,3*0) | 84.4 | 78.3 | 93.2 | 87.7 | 79.1 | 73.9 |
| 20 | 5 | (5,10,3*0) | 85.1 | 79.0 | 93.1 | 87.8 | 79.1 | 74.8 |
| 20 | 10 | (9*0,10) | 88.5 | 82.7 | 95.1 | 89.9 | 84.8 | 78.8 |
| 20 | 10 | (10,9*0) | 92.0 | 86.8 | 94.7 | 89.5 | 88.8 | 83.9 |
| 25 | 5 | (4*0,20) | 73.6 | 69.0 | 94.9 | 88.3 | 72.3 | 67.8 |
| 25 | 5 | (20,4*0) | 86.8 | 81.2 | 92.9 | 87.8 | 83.9 | 77.9 |
| 25 | 10 | (9*0,15) | 88.1 | 82.5 | 95.0 | 89.4 | 85.0 | 78.8 |
| 25 | 10 | (15,9*0) | 91.9 | 86.8 | 94.7 | 89.1 | 89.8 | 84.2 |
| 25 | 15 | (14*0,10) | 91.6 | 85.1 | 94.7 | 90.5 | 86.6 | 81.1 |
| 25 | 15 | (10,14*0) | 92.1 | 87.0 | 95.1 | 89.6 | 88.7 | 83.4 |
| 50 | 20 | (19*0,30) | 92.0 | 86.9 | 95.6 | 90.2 | 88.4 | 83.3 |
| 50 | 20 | (30,19*0) | 92.9 | 88.5 | 94.6 | 89.4 | 91.0 | 86.2 |
| 50 | 25 | (24*0,25) | 92.6 | 88.0 | 96.0 | 90.3 | 90.4 | 85.6 |
| 50 | 25 | (25,24*0) | 94.7 | 91.1 | 96.0 | 91.7 | 92.6 | 87.7 |

Table 7.5: (2.5, 97.5) and (5.0, 95.0) percentage points of the pivotal quantities based on MLEs

| $n$ | $m$ | Scheme | $\tilde{\mu}, \sigma$ unknown | | $\tilde{\mu}, \sigma$ known | | $\tilde{\sigma}$ | |
|---|---|---|---|---|---|---|---|---|
| 15 | 2 | (0,13) | (-86.90,0.66) | (-44.41, 0.47) | (-2.23,0.85) | (-2.00,0.58) | (-97.22,0.58) | (-46.55,0.38) |
| 15 | 2 | (13,0) | (-23.13,1.13) | (-12.97,0.85) | (-2.52,1.27) | (-2.27,0.92) | (-32.76,0.84) | (-15.97,0.71) |
| 15 | 3 | (2*0,12) | (-17.74,0.95) | (-11.44,0.72) | (-2.25,1.16) | (-2.08,0.87) | (-19.68,0.77) | (-12.27,0.58) |
| 15 | 3 | (12,2*0) | (-7.92,1.31) | (-5.04,1.10) | (-2.39,1.44) | (-2.01,1.17) | (-9.45,0.96) | (-6.52,0.82) |
| 15 | 4 | (3*0,11) | (-8.38,1.14) | (-5.83,0.91) | (-2.16,1.39) | (-1.95,0.99) | (-10.10,0.89) | (-7.19,0.77) |
| 15 | 4 | (11,3*0) | (-3.94,1.58) | (-3.18,1.28) | (-2.18,1.60) | (-1.92,1.24) | (-5.94,1.05) | (-4.53,0.91) |
| 15 | 5 | (4*0,10) | (-6.03,1.38) | (-4.29,1.13) | (-2.18,1.67) | (-1.94,1.21) | (-8.15,0.98) | (-5.49,0.82) |
| 15 | 5 | (10,4*0) | (-3.59,1.59) | (-2.69,1.35) | (-2.20,1.68) | (-1.90,1.32) | (-5.27,1.19) | (-3.88,0.98) |
| 15 | 5 | (0,10,3*0) | (-4.06,1.55) | (-3.17,1.27) | (-2.21,1.65) | (-1.97,1.35) | (-5.54,1.11) | (-4.34,0.92) |
| 15 | 5 | (2*0,10,2*0) | (-4.96,1.46) | (-3.66,1.23) | (-2.22,1.66) | (-1.93,1.35) | (-6.22,1.08) | (-4.84,0.89) |
| 15 | 5 | (2,2,2,2,2) | (-5.03,1.45) | (-3.78,1.20) | (-2.22,1.61) | (-1.92,1.29) | (-6.72,1.02) | (-5.12,0.87) |
| 15 | 5 | (4,4,2,2*0) | (-4.04,1.54) | (-3.21,1.27) | (-2.22,1.65) | (-1.99,1.37) | (-5.56,1.13) | (-4.37,0.92) |
| 20 | 2 | (0,18) | (-90.56,0.63) | (-45.90,0.45) | (-2.14,0.79) | (-1.96,0.58) | (-96.88,0.57) | (-47.37,0.37) |
| 20 | 2 | (18,0) | (-20.87,1.09) | (-11.98,0.80) | (-2.55,1.26) | (-2.29,0.87) | (-28.46,0.86) | (-14.17,0.73) |
| 20 | 3 | (2*0,17) | (-18.38,0.85) | (-11.91,0.67) | (-2.18,1.13) | (-2.02,0.79) | (-19.65,0.77) | (-12.29,0.58) |
| 20 | 3 | (17,2*0) | (-7.63,1.23) | (-4.89,1.04) | (-2.44,1.45) | (-2.06,1.19) | (-8.62,0.97) | (-6.03,0.85) |
| 20 | 4 | (3*0,16) | (-9.21,1.02) | (-6.37,0.77) | (-2.13,1.31) | (-1.91,0.99) | (-10.19,0.89) | (-7.15,0.77) |
| 20 | 4 | (16,3*0) | (-4.04,1.48) | (-3.33,1.19) | (-2.21,1.58) | (-1.94,1.20) | (-5.47,1.09) | (-4.26,0.93) |
| 20 | 5 | (4*0,15) | (-6.84,1.17) | (-4.66,0.97) | (-2.17,1.60) | (-1.96,1.16) | (-8.15,0.97) | (-5.47,0.82) |
| 20 | 5 | (15,4*0) | (-3.56,1.52) | (-2.76,1.25) | (-2.23,1.66) | (-1.99,1.32) | (-5.07,1.22) | (-3.61,1.02) |
| 20 | 5 | (10,5,3*0) | (-3.92,1.48) | (-3.03,1.22) | (-2.24,1.64) | (-1.99,1.37) | (-5.00,1.19) | (-3.88,0.98) |
| 20 | 5 | (5,5,5,2*0) | (-4.52,1.43) | (-3.51,1.16) | (-2.24,1.63) | (-1.97,1.35) | (-5.48,1.12) | (-4.34,0.93) |
| 20 | 5 | (3,3,3,3,3) | (-5.56,1.29) | (-3.93,1.12) | (-2.22,1.66) | (-1.99,1.19) | (-6.69,1.03) | (-5.10,0.87) |
| 20 | 5 | (0,15,3*0) | (-4.23,1.46) | (-3.26,1.19) | (-2.25,1.60) | (-1.98,1.36) | (-5.41,1.13) | (-4.11,0.94) |
| 20 | 5 | (5,10,3*0) | (-3.99,1.46) | (-3.17,1.20) | (-2.25,1.60) | (-1.98,1.38) | (-5.08,1.15) | (-4.09,0.96) |
| 20 | 10 | (9*0,10) | (-3.11,1.72) | (-2.58,1.29) | (-2.08,1.73) | (-1.83,1.43) | (-4.14,1.24) | (-3.19,1.05) |
| 20 | 10 | (10,9*0) | (-2.42,1.77) | (-2.09,1.43) | (-2.07,1.79) | (-1.77,1.40) | (-3.59,1.38) | (-2.66,1.13) |
| 25 | 5 | (4*0,20) | (-7.20,1.13) | (-4.90,0.88) | (-2.14,1.56) | (-1.93,1.15) | (-8.09,0.95) | (-5.57,0.81) |
| 25 | 5 | (20,4*0) | (-3.52,1.49) | (-2.77,1.20) | (-2.24,1.65) | (-2.00,1.34) | (-4.86,1.24) | (-3.44,1.04) |
| 25 | 10 | (9*0,15) | (-3.56,1.48) | (-2.82,1.25) | (-2.11,1.74) | (-1.84,1.37) | (-4.18,1.23) | (-3.24,1.04) |
| 25 | 10 | (15,9*0) | (-2.53,1.72) | (-2.13,1.37) | (-2.08,1.74) | (-1.80,1.36) | (-3.47,1.42) | (-2.58,1.15) |
| 25 | 15 | (14*0,10) | (-2.92,1.74) | (-2.31,1.46) | (-2.23,1.74) | (-1.90,1.45) | (-3.57,1.34) | (-2.92,1.08) |
| 25 | 15 | (10,14*0) | (-2.55,1.81) | (-2.06,1.57) | (-2.11,1.72) | (-1.76,1.48) | (-3.20,1.43) | (-2.63,1.15) |
| 50 | 20 | (19*0,30) | (-2.81,1.52) | (-2.34,1.26) | (-2.19,1.68) | (-1.83,1.34) | (-3.01,1.38) | (-2.72,1.17) |
| 50 | 20 | (30,19*0) | (-2.50,1.76) | (-1.99,1.45) | (-2.13,1.69) | (-1.81,1.43) | (-2.65,1.45) | (-2.31,1.25) |
| 50 | 25 | (24*0,25) | (-2.58,1.64) | (-2.13,1.29) | (-2.02,1.76) | (-1.77,1.39) | (-3.02,1.43) | (-2.49,1.14) |
| 50 | 25 | (25,24*0) | (-2.10,1.65) | (-1.78,1.48) | (-1.92,1.72) | (-1.64,1.46) | (-2.81,1.50) | (-2.26,1.22) |

Table 7.6: (2.5, 97.5) and (5.0, 95.0) percentage points of the pivotal quantities based on approximate MLEs

| $n$ | $m$ | Scheme | $\tilde{\mu}, \sigma$ unknown | | $\tilde{\mu}, \sigma$ known | | $\tilde{\sigma}$ | |
|---|---|---|---|---|---|---|---|---|
| 15 | 2 | (0,13) | (-86.55,0.66) | (-44.24,0.47) | (-2.22,0.85) | (-1.99,0.58) | (-97.18,0.58) | (-46.54,0.38) |
| 15 | 2 | (13,0) | (-23.48,1.13) | (-13.18,0.84) | (-2.58,1.29) | (-2.33,0.92) | (-32.92,0.87) | (-16.03,0.74) |
| 15 | 3 | (2*0,12) | (-17.65,0.95) | (-11.37,0.71) | (-2.24,1.15) | (-2.06,0.87) | (-19.66,0.77) | (-12.26,0.58) |
| 15 | 3 | (12,2*0) | (-7.96,1.29) | (-5.15,1.09) | (-2.45,1.45) | (-2.09,1.17) | (-9.58,1.02) | (-6.58,0.87) |
| 15 | 4 | (3*0,11) | (-8.29,1.14) | (-5.79,0.89) | (-2.14,1.38) | (-1.94,0.99) | (-10.09,0.89) | (-7.18,0.77) |
| 15 | 4 | (11,3*0) | (-4.06,1.57) | (-3.26,1.26) | (-2.28,1.59) | (-1.95,1.24) | (-6.03,1.15) | (-4.55,0.97) |
| 15 | 5 | (4*0,10) | (-5.99,1.37) | (-4.26,1.11) | (-2.17,1.65) | (-1.93,1.19) | (-8.14,0.98) | (-5.49,0.82) |
| 15 | 5 | (10,4*0) | (-3.69,1.59) | (-2.75,1.32) | (-2.29,1.69) | (-2.02,1.32) | (-5.35,1.25) | (-3.93,1.06) |
| 15 | 5 | (0,10,3*0) | (-4.09,1.55) | (-3.19,1.27) | (-2.23,1.65) | (-1.99,1.36) | (-5.53,1.13) | (-4.34,0.93) |
| 15 | 5 | (2*0,10,2*0) | (-4.89,1.47) | (-3.69,1.21) | (-2.26,1.67) | (-1.95,1.35) | (-6.21,1.08) | (-4.84,0.89) |
| 15 | 5 | (2,2,2,2,2) | (-5.02,1.42) | (-3.75,1.18) | (-2.23,1.58) | (-1.92,1.26) | (-6.70,1.05) | (-5.11,0.89) |
| 15 | 5 | (4,4,2,2*0) | (-4.03,1.52) | (-3.18,1.25) | (-2.25,1.60) | (-2.02,1.36) | (-5.51,1.17) | (-4.36,0.98) |
| 20 | 2 | (0,18) | (-90.32,0.63) | (-45.78,0.45) | (-2.13,0.79) | (-1.95,0.58) | (-96.85,0.57) | (-47.35,0.37) |
| 20 | 2 | (18,0) | (-21.12,1.09) | (-12.14,0.79) | (-2.61,1.26) | (-2.35,0.87) | (-28.60,0.89) | (-14.23,0.75) |
| 20 | 3 | (2*0,17) | (-18.28,0.84) | (-11.85,0.67) | (-2.17,1.13) | (-2.01,0.79) | (-19.64,0.76) | (-12.29,0.58) |
| 20 | 3 | (17,2*0) | (-7.67,1.21) | (-4.99,1.03) | (-2.52,1.45) | (-2.11,1.18) | (-8.74,1.04) | (-6.08,0.89) |
| 20 | 4 | (3*0,16) | (-9.14,1.02) | (-6.35,0.77) | (-2.12,1.31) | (-1.89,0.99) | (-10.18,0.89) | (-7.15,0.77) |
| 20 | 4 | (16,3*0) | (-4.05,1.46) | (-3.33,1.19) | (-2.26,1.57) | (-1.98,1.21) | (-5.57,1.18) | (-4.29,0.99) |
| 20 | 5 | (4*0,15) | (-6.79,1.17) | (-4.65,0.96) | (-2.16,1.60) | (-1.96,1.16) | (-8.14,0.97) | (-5.47,0.82) |
| 20 | 5 | (15,4*0) | (-3.66,1.52) | (-2.81,1.23) | (-2.30,1.65) | (-2.03,1.31) | (-5.11,1.29) | (-3.66,1.09) |
| 20 | 5 | (10,5,3*0) | (-3.92,1.46) | (-3.05,1.19) | (-2.29,1.63) | (-2.04,1.33) | (-5.02,1.23) | (-3.91,1.04) |
| 20 | 5 | (5,5,5,2*0) | (-4.51,1.43) | (-3.48,1.13) | (-2.24,1.64) | (-1.99,1.34) | (-5.50,1.18) | (-4.37,0.99) |
| 20 | 5 | (3,3,3,3,3) | (-5.48,1.26) | (-3.92,1.09) | (-2.21,1.65) | (-1.99,1.17) | (-6.66,1.05) | (-5.09,0.89) |
| 20 | 5 | (0,15,3*0) | (-4.26,1.46) | (-3.26,1.19) | (-2.27,1.61) | (-2.00,1.36) | (-5.39,1.15) | (-4.11,0.96) |
| 20 | 5 | (5,10,3*0) | (-4.01,1.46) | (-3.19,1.18) | (-2.24,1.59) | (-2.00,1.37) | (-5.09,1.19) | (-4.12,1.03) |
| 20 | 10 | (9*0,10) | (-3.10,1.72) | (-2.59,1.29) | (-2.08,1.72) | (-1.82,1.42) | (-4.13,1.24) | (-3.19,1.05) |
| 20 | 10 | (10,9*0) | (-2.45,1.76) | (-2.16,1.40) | (-2.09,1.78) | (-1.85,1.40) | (-3.65,1.47) | (-2.68,1.21) |
| 25 | 5 | (4*0,20) | (-7.17,1.13) | (-4.89,0.88) | (-2.13,1.55) | (-1.92,1.14) | (-8.09,0.95) | (-5.57,0.81) |
| 25 | 5 | (20,4*0) | (-3.62,1.46) | (-2.84,1.19) | (-2.33,1.64) | (-2.06,1.32) | (-4.94,1.31) | (-3.49,1.12) |
| 25 | 10 | (9*0,15) | (-3.55,1.47) | (-2.82,1.25) | (-2.11,1.73) | (-1.84,1.36) | (-4.18,1.23) | (-3.24,1.04) |
| 25 | 10 | (15,9*0) | (-2.57,1.71) | (-2.18,1.31) | (-2.14,1.74) | (-1.87,1.36) | (-3.47,1.49) | (-2.54,1.25) |
| 25 | 15 | (14*0,10) | (-2.91,1.73) | (-2.31,1.46) | (-2.21,1.73) | (-1.90,1.44) | (-3.57,1.34) | (-2.91,1.08) |
| 25 | 15 | (10,14*0) | (-2.55,1.81) | (-2.11,1.56) | (-2.20,1.73) | (-1.80,1.48) | (-3.16,1.51) | (-2.63,1.23) |
| 50 | 20 | (19*0,30) | (-2.81,1.51) | (-2.34,1.25) | (-2.20,1.68) | (-1.83,1.34) | (-3.01,1.38) | (-2.72,1.17) |
| 50 | 20 | (30,19*0) | (-2.53,1.72) | (-2.02,1.42) | (-2.21,1.69) | (-1.90,1.40) | (-2.58,1.65) | (-2.33,1.39) |
| 50 | 25 | (24*0,25) | (-2.58,1.63) | (-2.12,1.30) | (-2.01,1.76) | (-1.77,1.39) | (-3.02,1.43) | (-2.49,1.14) |
| 50 | 25 | (25,24*0) | (-2.13,1.63) | (-1.86,1.45) | (-1.97,1.68) | (-1.71,1.44) | (-2.86,1.60) | (-2.22,1.33) |

## 7.7 Illustrative Example and Conclusions

**Example 7.1:** Refer to the illustrative example in Section 6.5.3. For this example, $n = 19, m = 8$. Assuming the data has come from a normal distribution, we obtain the MLEs of $\mu$ and $\sigma$ as

$$\hat{\mu} = 1.882 \qquad \text{and} \qquad \hat{\sigma} = 1.615.$$

We simulated the percentage points for a 90 % interval to obtain $P(-3.04 \leq P_1 \leq 1.24) = 0.90$ and $P(-3.63 \leq P_3 \leq 1.04) = 0.9$, from which we obtain the 90 % confidence interval for $\mu$ to be $(1.26, 3.41)$ and the 90 % interval for $\sigma$ to be $(1.17, 3.16)$, respectively.

In this chapter, we considered point and interval estimation for parameters of the normal distribution based on progressively Type-II censored data. We developed explicit estimators based on an approximation to the hazard function of the normal distribution. Results of a simulation study show that the approximate MLEs and MLEs are almost identical in terms of bias and variance.

We also conducted a simulation study to examine the coverage probabilities of the pivotal quantities based on asymptotic normality. The results show the coverages are extremely unsatisfactory especially when the effective sample size is small. Based on these observations, we recommend the use of simulated percentage points to construct confidence intervals for the parameters.

# Chapter 8

# Optimal Progressive Censoring Scheme for the Weibull Distribution

## 8.1 Introduction

In this chapter, we discuss the determination of optimal censoring schemes for the Weibull distribution based on progressively Type-II censored samples. Under progressive censoring, Balasooriya, Saw and Gadag (2000) constructed sampling plans for the Weibull distribution, while Balasooriya and Balakrishnan (2000) constructed similar plans for the lognormal distribution. Under the assumption that $m$ and the progressive censoring scheme $(R_1, R_2, \ldots, R_m)$ are pre-specified by the experimenter, Viveros and Balakrishnan (1994) and Balasooriya, Saw and Gadag (2000) have developed inferential methods. However, realizing that this progressive censoring is a versatile censoring scheme in that there is flexibility in the choice of $R_1, R_2, \ldots, R_m$, we study here the problem of optimal progressive censoring for the Weibull distribution using the EM-algorithm for the MLEs. Specifically, for specified values of the sample size $n$ and the effective sample size (the number of complete failures) $m$, we discuss the determination of the optimal $(R_1, R_2, \ldots, R_m)$ that minimizes (i) the trace of the variance-covariance matrix of the MLEs, (ii) the determinant of the variance-covariance matrix of the MLEs, and (iii) the trace of the missing information matrix. We show that significant gain in efficiency can be achieved by using these optimal censoring plans. Finally, we also describe the construction of progressively censored reliability sampling plans based on these optimal censoring schemes.

It is important to mention here that progressive censoring is typical of failure-time data obtained from field units. In the early works of Cohen (1963), for example, progressive censoring was thought of as a model to accommodate accidental loss of units or for the purpose of saving some live units for other tests. Thus, in a planned life-testing experiment, it is more common to observe a singly censored data as it is easier to implement. However, we feel that such a restrictive censoring scheme may result in a substantial loss in efficiency in the ensuing inferential methods. But, a properly planned progressively censored life-test can result in highly efficient methods. With this specific goal in mind, we tackle in this chapter two related but distinct problems: (i) optimal progressively censored test planning for the purpose of estimating the parameters, and (ii) optimal progressively censored test planning for the purpose of demonstration testing. In the first case, once the experimenter specifies $n$ (the number of units available for test) and $m$ (the number of complete failures allowed), the methodology developed in this chapter will assist in the designing of an appropriate life-test that would result in the optimal estimation of life parameters. In the second case, once the experimenter specifies $m$ and the progressive censoring scheme in terms of proportions, the results developed here will enable the determination of the minimum sample size $n$ for a demonstration testing at pre-fixed points $(p_\alpha, 1-\alpha)$ and $(p_\beta, \beta)$ on the operating characteristic (OC) curve, where $\alpha$ and $\beta$ are producer's and consumer's risks, respectively.

## 8.2 Expected Fisher Information

Let us now consider the lifetime of the units in the life-test $X_i$, $i = 1, 2, \ldots, n$, to be independently distributed as Weibull with p.d.f.

$$f(x; \alpha, \beta) = \frac{\beta}{\alpha}\left(\frac{x}{\alpha}\right)^{\beta-1} \exp\left[-\left(\frac{x}{\alpha}\right)^{\beta}\right], \qquad x > 0,$$

where $\alpha > 0$ is the scale parameter and $\beta > 0$ is the shape parameter. Then the logarithm $W_i = \ln X_i$ follows the extreme value distribution with parameters $\mu = \ln \alpha$ and $\sigma = 1/\beta$. The p.d.f. of $W_i$, $i = 1, 2, \ldots, n$, is

$$f_W(w_i; \mu, \sigma) = \frac{1}{\sigma} \exp\left[\left(\frac{w_i - \mu}{\sigma}\right) - \exp\left(\frac{w_i - \mu}{\sigma}\right)\right], \qquad -\infty < w_i < \infty, \quad (8.2.1)$$

and the c.d.f. is

$$F_W(w_i) = 1 - \exp\left[-\exp\left(\frac{w_i - \mu}{\sigma}\right)\right], \qquad -\infty < w_i < \infty. \qquad (8.2.2)$$

A major advantage in working with this extreme value distribution is that $\mu$ and $\sigma$ are location and scale parameters; see Lawless (1982). Therefore, we will work with the log-lifetimes $W_i$ instead of the actual lifetimes $X_i$ throughout this chapter.

The maximum likelihood estimators of the parameters $\mu$ and $\sigma$ based on progressively Type-II censored data are studied by many authors; see Balakrishnan and Aggarwala (2000) for details. Since the likelihood equations cannot be solved analytically, estimation procedures often use Taylor series approximations. In Chapter 6, we proposed the usage of the EM-algorithm in the determination of the maximum likelihood estimates when the data are progressively Type-II censored. The major advantage of the EM-algorithm is in its easy implementation [see Dempster et al. (1977)] and also in the fact that the variances and covariance of the ML estimates can be readily obtained by the Missing Information Principle as shown already in Chapter 6. In this section, we will provide a computational procedure for the expected Fisher information matrix for a specified censoring scheme via the Missing Information Principle and direct computation from the log-likelihood functions.

### 8.2.1 Missing Information Principle

As mentioned in Chapter 6, the Missing Information Principle states that

Observed information $=$ Complete information $-$ Missing information.

By using the missing information principle, Louis (1982) developed a procedure for extracting the observed information matrix when the EM-algorithm is used in order to find the maximum likelihood estimates for the incomplete data problem. For each progressively censored sampling plan, we can compute the expected missing information as well as the asymptotic variances and covariance of the ML estimates.

Let us denote the observed information, complete information and missing information for the parameter $\boldsymbol{\theta}$ by $I_Y(\boldsymbol{\theta})$, $I_W(\boldsymbol{\theta})$ and $I_{Z|Y}(\boldsymbol{\theta})$, respectively. From well-known results on the extreme value distribution, the complete information matrix for $\boldsymbol{\theta} = (\,\mu \quad \sigma\,)'$ is

$$I_W(\boldsymbol{\theta}) = \frac{n}{\sigma^2}\begin{bmatrix} 1 & 1-\gamma \\ 1-\gamma & c^2 \end{bmatrix}, \tag{8.2.3}$$

where $\gamma = 0.577215665$ is the Euler's constant and $c^2$ is $\frac{\pi^2}{6} + (1-\gamma)^2 = 1.823680661$; see, for example, Lawless (1982) and Johnson, Kotz and Balakrishnan (1995).

To compute the expected missing information matrix under progressive Type-II right censoring, we need the distribution of the observed progressively censored order statistics $Y_{r:m:n}$ for $r = 1, 2, \ldots, m$ [see Balakrishnan and Aggarwala (2000)]. The cumulative distribution function of $Y_{r:m:n}$ is

$$F_{Y_{r:m:n}}(y) = 1 - c_{r-1}\sum_{i=1}^{r}\frac{a_{i,r}}{\gamma_i}\left[1 - F_W(y)\right]^{\gamma_i}, \qquad -\infty < y < \infty, r = 1, 2, \ldots, m, \tag{8.2.4}$$

where

$$\begin{aligned}
\gamma_r &= m - r + 1 + \sum_{i=r}^{m} R_i, \qquad r = 1, 2, \ldots, m, \\
c_{r-1} &= \prod_{i=1}^{r}\gamma_i, \qquad r = 1, 2, \ldots, m, \\
a_{i,r} &= \prod_{\substack{j=1 \\ j\neq i}}^{r}\frac{1}{\gamma_j - \gamma_i}, \qquad 1 \le i \le r \le m.
\end{aligned}$$

We readily obtain the probability density function of $Y_{r:m:n}$, from (8.2.1), (8.2.2) and (8.2.4), as

$$\begin{aligned} f_{Y_{r:m:n}}(y) &= c_{r-1} \sum_{i=1}^{r} a_{i,r} \left[1 - F_W(y)\right]^{\gamma_i - 1} f_W(y) \\ &= c_{r-1} \sum_{i=1}^{r} a_{i,r} \left[\frac{1}{\sigma} e^{\left(\frac{y-\mu}{\sigma}\right) - \gamma_i e^{\left(\frac{y-\mu}{\sigma}\right)}}\right], \qquad -\infty < y < \infty, r = 1, 2, \ldots, m. \end{aligned}$$

The expected Fisher information matrix for $\boldsymbol{\theta}$ in one observation which is censored at the time of the $r$-th failure ($Y_{r:m:n}$) is

$$E_{Y_{r:m:n}}\left[I_{Z|Y}^{(r)}(\boldsymbol{\theta})\right] = \frac{1}{\sigma^2}\begin{bmatrix} M_{11}^r(n) & M_{12}^r(n) \\ M_{21}^r(n) & M_{22}^r(n) \end{bmatrix}.$$

The computational formulas for $M_{11}^r(n), M_{12}^r(n), M_{21}^r(n)$ and $M_{22}^r(n)$ are given in the Appendix C.

Then the expected missing information is

$$E_{Y_{1:m:n},\ldots,Y_{m:m:n}}\left[I_{Z|Y}(\boldsymbol{\theta})\right] = \sum_{r=1}^{m} R_r E_{Y_{r:m:n}}\left[I_{Z|Y}^{(r)}(\boldsymbol{\theta})\right]. \tag{8.2.5}$$

Based on this expected missing information, we can compute the asymptotic variance-covariance matrix of the ML estimate of $\boldsymbol{\theta}$ by inverting the expected observed information matrix given by

$$E_{Y_{1:m:n},\ldots,Y_{m:m:n}}\left[I_Y(\boldsymbol{\theta})\right] = I_Z(\boldsymbol{\theta}) - E_{Y_{1:m:n},\ldots,Y_{m:m:n}}\left[I_{Z|Y}(\boldsymbol{\theta})\right]. \tag{8.2.6}$$

### 8.2.2 Direct Computation

Form (6.1.1), the log-likelihood equations for the parameters $\mu$ and $\sigma$ are given by

$$\frac{\partial \ln L}{\partial \mu} = -\frac{1}{\sigma}\left[m - \sum_{j=1}^{m}(R_j + 1)e^{\xi_j}\right] = 0, \tag{8.2.7}$$

$$\frac{\partial \ln L}{\partial \sigma} = -\frac{1}{\sigma}\left[m + \sum_{j=1}^{m}\xi_j - \sum_{j=1}^{m}(R_j + 1)\xi_j e^{\xi_j}\right] = 0, \tag{8.2.8}$$

where $\xi_j = \frac{Y_{j:m:n} - \mu}{\sigma}$.

Then the expected Fisher information matrix can be computed as

$$E_{Y_{1:m:n},\ldots,Y_{m:m:n}}\left[I_Y(\boldsymbol{\theta})\right] = \frac{1}{\sigma^2}\begin{bmatrix} O_{11}(n) & O_{12}(n) \\ O_{21}(n) & O_{22}(n) \end{bmatrix}$$

with entries

$$\begin{aligned}
O_{11}(n) &= \sum_{j=1}^{m}(R_j+1)E_{Y_{j:m:n}}(e^{\xi_j}) = m, \\
O_{12}(n) &= O_{21}(n) = -m + \sum_{j=1}^{m}(R_j+1)E_{Y_{j:m:n}}(e^{\xi_j}) + \sum_{j=1}^{m}(R_j+1)E_{Y_{j:m:n}}(\xi_j e^{\xi_j}) \\
&= m + \sum_{j=1}^{m} E_{Y_{j:m:n}}(\xi_j), \\
O_{22}(n) &= -m - 2\sum_{j=1}^{m} E_{Y_{r:m:n}}(\xi_j) + 2\sum_{j=1}^{m}(R_j+1)E_{Y_{j:m:n}}(\xi_j e^{\xi_j}) + \sum_{j=1}^{m}(R_j+1)E_{Y_{j:m:n}}(\xi_j^2 e^{\xi_j}) \\
&= m + \sum_{j=1}^{m}(R_j+1)E_{Y_{j:m:n}}(\xi_j^2 e^{\xi_j}),
\end{aligned}$$

where the expressions for those expected values are given in Appendix C.

## 8.3 Optimal Progressive Censoring Scheme

For a specific censoring scheme $[n, m, (R_1, R_2, \ldots, R_m)]$, we can compute the expected missing information matrix and the asymptotic variance-covariance matrix of the MLEs. When the values of $n$ and $m$ are chosen in advance depending on the availability of units, experimental facilities and cost considerations, we can find the optimal censoring scheme $(R_1, R_2, \ldots, R_m)$.

### 8.3.1 Joint Estimation

When $\boldsymbol{\theta}$ is two-dimensional, optimality can be defined in terms of
(1) minimizing the trace of the variance-covariance matrix $(V(\boldsymbol{\theta}))$ of the ML estimators, or equivalently

$$tr[V(\boldsymbol{\theta})] = V_{11}(n) + V_{22}(n); \tag{8.3.1}$$

(2) minimizing the determinant of the variance-covariance matrix ($V(\boldsymbol{\theta})$) of the ML estimators, or equivalently

$$det[V(\boldsymbol{\theta})] \quad = \quad V_{11}(n)V_{22}(n) - V_{12}^2(n); \tag{8.3.2}$$

(3) maximizing the trace of the Fisher information matrix of the ML estimators.

In the case of the extreme value distribution, the element of the missing information matrix corresponding to the parameter $\mu$ is independent of the censoring scheme (see Appendix C). Thus, the third optimality criterion can be expressed equivalently as

(3′) maximizing the element of the Fisher information matrix corresponding to the parameter $\sigma$ (or equivalently, $O_{22}(n)$).

### 8.3.2 Estimation of Quantiles

In reliability context, we may often be interested in estimating the $p$-th upper quantile of the population. The ML estimate of the $p$-th upper quantile is given by

$$\hat{Q}_p = \hat{\mu} + \hat{\sigma}F^{-1}(1-p), \tag{8.3.3}$$

where $F^{-1}(\cdot)$ is the inverse c.d.f. of the standard extreme value distribution, that is,

$$F^{-1}(1-p) = \ln[-\ln(p)].$$

In this case, optimality can be simply defined in terms of minimizing the variance of the estimate for the $p$-th upper quantile, or equivalently

$$Var\left[\hat{\mu} + \hat{\sigma}F^{-1}(1-p)\right] \quad = \quad V_{11}(n) + [F^{-1}(1-p)]^2 V_{22}(n) + 2F^{-1}(1-p)V_{12}(n).$$

In the finite sample situation, we can list all possible choices of censoring schemes and compute the corresponding objective function, and then determine the optimal censoring scheme through an extensive search.

Some selected combinations of $m$ and $n$ and the corresponding optimal censoring schemes thus determined are presented in Tables 8.1-8.4. When optimal censoring schemes are used in place of other censoring schemes, we would expect a gain in efficiency. We will illustrate this point and also given an idea about the amount of gain in efficiency with the following example.

Table 8.1: Optimal progressive censoring schemes for $n = 10, m = 5(1)9$

| $n$ | $m$ | degree of censoring | optimality criterion | $R_1$ | $R_2$ | $R_3$ | $R_4$ | ... | $R_m$ | $tr[V(\boldsymbol{\theta})]$ | $det[V(\boldsymbol{\theta})]$ | $O_{22}(n)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 10 | 5 | 50.00% | (1),(2) | 0 | 5 | 0 | 0 | | 0 | 0.2918 | 0.0183 | 10.9057 |
| | | | (3) | 5 | 0 | 0 | 0 | | 0 | 0.2948 | 0.0184 | 10.9822 |
| | 6 | 40.00% | (1),(2) | 0 | 4 | 0 | 0 | ... | 0 | 0.2494 | 0.0136 | 12.3719 |
| | | | (3) | 4 | 0 | 0 | 0 | ... | 0 | 0.2549 | 0.0138 | 12.4332 |
| | 7 | 30.00% | (1) | 0 | 0 | 3 | 0 | ... | 0 | 0.2205 | 0.0108 | 13.4105 |
| | | | (2) | 0 | 3 | 0 | 0 | ... | 0 | 0.2212 | 0.0106 | 13.8381 |
| | | | (3) | 3 | 0 | 0 | 0 | ... | 0 | 0.2261 | 0.0108 | 13.8841 |
| | 8 | 20.00% | (1) | 0 | 0 | 2 | 0 | ... | 0 | 0.1994 | 0.0087 | 15.0193 |
| | | | (2) | 0 | 2 | 0 | 0 | ... | 0 | 0.2005 | 0.0086 | 15.3044 |
| | | | (3) | 2 | 0 | 0 | 0 | ... | 0 | 0.2039 | 0.0087 | 15.3350 |
| | 9 | 10.00% | (1) | 0 | 0 | 0 | 1 | ... | 0 | 0.1836 | 0.0072 | 16.4107 |
| | | | (2) | 0 | 1 | 0 | 0 | ... | 0 | 0.1846 | 0.0072 | 16.7706 |
| | | | (3) | 1 | 0 | 0 | 0 | ... | 0 | 0.1862 | 0.0072 | 16.7859 |

Table 8.2: Optimal progressive censoring schemes for $n = 15, m = 5(1)14$

| $n$ | $m$ | degree of censoring | optimality criterion | $R_1$ | $R_2$ | $R_3$ | $R_4$ | $R_5$ | $R_6$ | ... | $R_m$ | $tr[V(\boldsymbol{\theta})]$ | $det[V(\boldsymbol{\theta})]$ | $O_{22}(n)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 15 | 5 | 66.67% | (1) | 10 | 0 | 0 | 0 | 0 | – | | – | 0.2788 | 0.0157 | 12.7210 |
| | | | (2),(3) | 0 | 10 | 0 | 0 | 0 | – | | – | 0.2819 | 0.0154 | 13.2944 |
| | 6 | 60.00% | (1),(2),(3) | 0 | 9 | 0 | 0 | 0 | 0 | | – | 0.2353 | 0.0114 | 14.7005 |
| | 7 | 53.33% | (1),(2),(3) | 0 | 8 | 0 | 0 | 0 | 0 | | 0 | 0.2051 | 0.0089 | 16.1065 |
| | 8 | 46.67% | (1) | 0 | 0 | 7 | 0 | 0 | 0 | ... | 0 | 0.1831 | 0.0073 | 17.2201 |
| | | | (2),(3) | 0 | 7 | 0 | 0 | 0 | 0 | ... | 0 | 0.1835 | 0.0072 | 17.5126 |
| | 9 | 40.00% | (1) | 0 | 0 | 6 | 0 | 0 | 0 | ... | 0 | 0.1658 | 0.0060 | 18.6680 |
| | | | (2),(3) | 0 | 6 | 0 | 0 | 0 | 0 | ... | 0 | 0.1671 | 0.0060 | 18.9187 |
| | 10 | 33.33% | (1) | 0 | 0 | 0 | 5 | 0 | 0 | ... | 0 | 0.1524 | 0.0051 | 19.6400 |
| | | | (2) | 0 | 0 | 5 | 0 | 0 | 0 | ... | 0 | 0.1525 | 0.0051 | 20.1158 |
| | | | (3) | 0 | 5 | 0 | 0 | 0 | 0 | ... | 0 | 0.1541 | 0.0051 | 20.3248 |
| | 11 | 26.67% | (1) | 0 | 0 | 0 | 4 | 0 | 0 | ... | 0 | 0.1415 | 0.0044 | 21.1830 |
| | | | (2) | 0 | 0 | 4 | 0 | 0 | 0 | ... | 0 | 0.1420 | 0.0044 | 21.5637 |
| | | | (3) | 0 | 4 | 0 | 0 | 0 | 0 | ... | 0 | 0.1435 | 0.0044 | 21.7309 |
| | 12 | 20.00% | (1) | 0 | 0 | 0 | 0 | 3 | 0 | ... | 0 | 0.1327 | 0.0039 | 22.3309 |
| | | | (2) | 0 | 0 | 3 | 0 | 0 | 0 | ... | 0 | 0.1334 | 0.0038 | 23.0116 |
| | | | (3) | 0 | 3 | 0 | 0 | 0 | 0 | ... | 0 | 0.1346 | 0.0038 | 23.1370 |
| | 13 | 13.33% | (1) | 0 | 0 | 0 | 0 | 2 | 0 | ... | 0 | 0.1255 | 0.0034 | 24.0056 |
| | | | (2) | 0 | 0 | 2 | 0 | 0 | 0 | ... | 0 | 0.1261 | 0.0034 | 24.4595 |
| | | | (3) | 0 | 2 | 0 | 0 | 0 | 0 | ... | 0 | 0.1269 | 0.0034 | 24.5430 |
| | 14 | 6.67% | (1) | 0 | 0 | 0 | 0 | 0 | 1 | ... | 0 | 0.1195 | 0.0030 | 25.5192 |
| | | | (2) | 0 | 0 | 1 | 0 | 0 | 0 | ... | 0 | 0.1199 | 0.0030 | 25.9073 |
| | | | (3) | 0 | 1 | 0 | 0 | 0 | 0 | ... | 0 | 0.1203 | 0.0030 | 25.9491 |

Table 8.3: Optimal progressive censoring schemes for $n = 20, m = 5(1)13$

| $n$ | $m$ | degree of censoring | optimality criterion | Optimal censoring scheme $R_1$ | $R_2$ | $R_3$ | $R_4$ | $R_5$ | $\ldots$ | $R_m$ | $tr[V(\theta)]$ | $det[V(\theta)]$ | $O_{22}(n)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 20 | 5 | 75.00% | (1) | 15 | 0 | 0 | 0 | 0 | | – | 0.2706 | 0.0141 | 14.2037 |
| | | | (2) | 0 | 15 | 0 | 0 | 0 | | – | 0.2793 | 0.0136 | 15.4986 |
| | | | (3) | 0 | 0 | 15 | 0 | 0 | | – | 0.3153 | 0.0154 | 15.5317 |
| | 6 | 70.00% | (1),(2) | 0 | 14 | 0 | 0 | 0 | | 0 | 0.2298 | 0.0100 | 16.8969 |
| | | | (3) | 0 | 0 | 14 | 0 | 0 | | 0 | 0.2438 | 0.0106 | 16.9279 |
| | 7 | 65.00% | (1),(2) | 0 | 13 | 0 | 0 | 0 | $\ldots$ | 0 | 0.1981 | 0.0078 | 18.2953 |
| | | | (3) | 0 | 0 | 13 | 0 | 0 | $\ldots$ | 0 | 0.2034 | 0.0080 | 18.3240 |
| | 8 | 60.00% | (1),(2) | 0 | 12 | 0 | 0 | 0 | $\ldots$ | 0 | 0.1758 | 0.0063 | 19.6936 |
| | | | (3) | 0 | 0 | 12 | 0 | 0 | $\ldots$ | 0 | 0.1772 | 0.0064 | 19.7201 |
| | 9 | 55.00% | (1),(2),(3) | 0 | 0 | 11 | 0 | 0 | $\ldots$ | 0 | 0.1586 | 0.0053 | 21.1162 |
| | 10 | 50.00% | (1),(2),(3) | 0 | 0 | 10 | 0 | 0 | $\ldots$ | 0 | 0.1446 | 0.0044 | 22.5124 |
| | 11 | 45.00% | (1) | 0 | 0 | 0 | 9 | 0 | $\ldots$ | 0 | 0.1335 | 0.0039 | 23.5270 |
| | | | (2),(3) | 0 | 0 | 9 | 0 | 0 | $\ldots$ | 0 | 0.1335 | 0.0038 | 23.9085 |
| | 12 | 40.00% | (1) | 0 | 0 | 0 | 8 | 0 | $\ldots$ | 0 | 0.1241 | 0.0034 | 24.9655 |
| | | | (2),(3) | 0 | 0 | 8 | 0 | 0 | $\ldots$ | 0 | 0.1246 | 0.0033 | 25.3046 |
| | 13 | 35.00% | (1) | 0 | 0 | 0 | 0 | 7 | $\ldots$ | 0 | 0.1164 | 0.0030 | 25.9034 |
| | | | (2).(3) | 0 | 0 | 7 | 0 | 0 | $\ldots$ | 0 | 0.1171 | 0.0030 | 26.7007 |

Table 8.4: Optimal progressive censoring schemes for $n = 10(5)40, m = 5$

| $n$ | $m$ | degree of censoring | optimality criterion | Optimal censoring scheme $R_1$ | $R_2$ | $R_3$ | $R_4$ | $R_5$ | $tr[V(\theta)]$ | $det[V(\theta)]$ | $O_{22}(n)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 10 | 5 | 50.00% | (1),(2) | 0 | 5 | 0 | 0 | 0 | 0.2918 | 0.0183 | 10.9057 |
| | | | (3) | 5 | 0 | 0 | 0 | 0 | 0.2948 | 0.0184 | 10.9822 |
| 15 | 5 | 66.67% | (1) | 10 | 0 | 0 | 0 | 0 | 0.2788 | 0.0157 | 12.7210 |
| | | | (2),(3) | 0 | 10 | 0 | 0 | 0 | 0.2819 | 0.0154 | 13.2944 |
| 20 | 5 | 75.00% | (1) | 15 | 0 | 0 | 0 | 0 | 0.2706 | 0.0141 | 14.2037 |
| | | | (2) | 0 | 15 | 0 | 0 | 0 | 0.2793 | 0.0136 | 15.4986 |
| | | | (3) | 0 | 0 | 15 | 0 | 0 | 0.3153 | 0.0154 | 15.5317 |
| 25 | 5 | 80.00% | (1) | 20 | 0 | 0 | 0 | 0 | 0.2657 | 0.0130 | 15.4846 |
| | | | (2) | 0 | 20 | 0 | 0 | 0 | 0.2787 | 0.0124 | 17.4778 |
| | | | (3) | 0 | 0 | 20 | 0 | 0 | 0.3231 | 0.0141 | 17.9893 |
| 30 | 5 | 83.33% | (1) | 25 | 0 | 0 | 0 | 0 | 0.2623 | 0.0121 | 16.6117 |
| | | | (2) | 0 | 25 | 0 | 0 | 0 | 0.2789 | 0.0115 | 19.2617 |
| | | | (3) | 0 | 0 | 0 | 25 | 0 | 0.4461 | 0.0176 | 20.3399 |
| 35 | 5 | 85.71% | (1) | 30 | 0 | 0 | 0 | 0 | 0.2598 | 0.0115 | 17.6192 |
| | | | (2) | 0 | 30 | 0 | 0 | 0 | 0.2793 | 0.0108 | 20.8834 |
| | | | (3) | 0 | 0 | 0 | 30 | 0 | 0.4662 | 0.0168 | 22.7797 |
| 40 | 5 | 87.50% | (1) | 35 | 0 | 0 | 0 | 0 | 0.2579 | 0.0110 | 18.5313 |
| | | | (2) | 0 | 35 | 0 | 0 | 0 | 0.2799 | 0.0102 | 22.3702 |
| | | | (3) | 0 | 0 | 0 | 0 | 35 | 0.9995 | 0.0331 | 25.1846 |

Table 8.5: Comparison of optimal censoring scheme and the censoring scheme used by Viveros and Balakrishnan (1994)

| optimality criterion | | Censoring Scheme | $tr[V(\theta)]$ | $det[V(\theta)]$ | $O_{22}(n)$ | $Var(\hat{Q}_{0.05})$ |
|---|---|---|---|---|---|---|
| (1), (2) and (3) | Optimal | $(0,11,0,0,0,0,0,0)$ | 0.176956 | 0.006489 | 19.270694 | 0.184169 |
| | Example | $(0,0,3,0,3,0,0,5)$ | 0.239682 | 0.010806 | 14.191945 | 0.366111 |
| | | Efficiency | 1.355249 | 1.665405 | 1.357861 | 1.987905 |
| $Var(\hat{Q}_{0.05})$ | Optimal | $(11,0,0,0,0,0,0,0)$ | 0.182561 | 0.006924 | 18.366109 | 0.169801 |
| | | Efficiency | 1.313641 | 1.560729 | 1.294122 | 2.156117 |

### 8.3.3 Illustrative Examples

**Example 8.1:** Refer to the illustrative example in Section 6.5.3 with data presented in Table 6.3. For $n = 19$ and $m = 8$, we can find the corresponding optimal censoring schemes subject to the three optimality criteria described above. In Table 8.5, we have presented comparisons of the three optimal censoring schemes and the optimal censoring scheme for estimation of the upper 5-th quantile with the censoring scheme $(0,0,3,0,3,0,0,5)$ considered by Viveros and Balakrishnan (1994). We have presented the relative efficiency which is simply the ratio of the objective function values corresponding to the censoring scheme used by Viveros and Balakrishnan (1994) and the optimal censoring scheme. It is clear from these values that the optimal censoring schemes result in substantial gain in efficiency no matter which criteria is used.

It is also heartening to observe, for example, if one uses the 'trace optimal' censoring scheme, in addition to obtaining the maximal gain in efficiency with respect to the trace, the gain in efficiency resulting with respect to the determinant or $O_{22}(n)$ is also optimal.

**Example 8.2:** In order to illustrate the implementation of the optimal censoring schemes, we use the data set generated by Viveros and Balakrishnan (1994) as well as another Type-II progressively censored sample of size $m = 8$ generated from the $n = 19$ observations given by Nelson (1982, p. 228, Table 6.1) using the optimal censoring scheme. These data are presented in Table 8.6, and a comparison of the estimates determined from these two progressively censored samples are presented in

Table 8.6: Progressively censored sample generated from Nelson (1982, Table 6.1) using the optimal censoring scheme

| r | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| $y_{r:m:n}$ | -1.6608 | -0.2485 | 0.2700 | 1.1505 | 1.5411 | 2.0806 | 3.4578 | 3.6030 |
| $R_r$ | 0 | 11 | 0 | 0 | 0 | 0 | 0 | 0 |

Table 8.7: Comparison of the estimates based on the data generated using the optimal censoring scheme and the progressively censored data used by Viveros and Balakrishnan (1994)

| | ML Estimates | | Observed Information | | | Variance-covariance of MLEs | | |
|---|---|---|---|---|---|---|---|---|
| Censoring Scheme | $\hat{\mu}$ | $\hat{\sigma}$ | $I_Y^{11}$ | $I_Y^{22}$ | $I_Y^{12}$ | $var(\hat{\mu})$ | $var(\hat{\sigma})$ | $cov(\hat{\mu},\hat{\sigma})$ |
| (0, 11, 0, 0, 0, 0, 0, 0) | 2.4398 | 1.2056 | 8.0000 | 20.1614 | 0.2655 | 0.1251 | 0.0496 | -0.0016 |
| (0, 0, 3, 0, 3, 0, 0, 5) | 2.2220 | 1.0264 | 7.9998 | 13.7581 | -4.6543 | 0.1556 | 0.0905 | 0.0527 |

Table 8.7.

We observe that the data generated using the optimal censoring scheme yields smaller standard errors for the ML estimates. Although this may not be true in every case, we would expect on an average the use of optimal censoring scheme to result in estimates with smaller standard errors.

# 8.4 Reliability Sampling Plans

## 8.4.1 Determination of Acceptance Constant and Sample Size

Suppose $L$ is the actual one-sided lower specification limit of a product with Weibull lifetime distribution. Then the fraction of defectives, $p$, is given by the probability $\Pr(W \leq L')$, where $L' = \ln L$.

Design of acceptance sampling plans require agreement between the consumer and the producer. Lots for which the fraction of defectives $p$ does not exceed $p_\alpha$ are presumed to be good and the consumer accepts these lots with probability at least

$1-\alpha$, where $\alpha$ is the producer's risk; moreover, lots for which the fraction of defectives $p$ exceeds $p_\beta$ are presumed to be unacceptable and the consumer rejects these lots with probability at least $1-\beta$, where $\beta$ is the consumer's risk. For more details, one may refer to the book by Schilling (1982).

Here, we employ the well-known Lieberman and Resnikoff (1955) procedure; thus, for a specified lower limit $L'$, a lot under testing is accepted if

$$\hat{\mu} - l\hat{\sigma} > L',$$

where $l$ is the acceptance constant. By asymptotic theory of MLEs, we can approximate the distribution of $\hat{\mu}-l\hat{\sigma}$ by a normal distribution with mean $\mu-l\sigma$ and variance $\sigma^2[V_{11}(n) + l^2V_{22}(n) - 2lV_{12}(n)]$. Schneider (1989) showed that the acceptance constant $l$ is then given by

$$l = \frac{s_{p_\alpha} z_{1-\beta} - s_{p_\beta} z_\alpha}{z_\alpha - z_{1-\beta}} \tag{8.4.1}$$

and the sample size $n$ is the solution to

$$S(n) = \left[\frac{z_\alpha - z_{1-\beta}}{s_{p_\alpha} - s_{p_\beta}}\right]^2 [V_{11}(n) + l^2V_{22}(n) - 2lV_{12}(n)] = 1, \tag{8.4.2}$$

where $s_\delta = \ln[-\ln(1-\delta)]$ and $z_\delta = \Phi^{-1}(\delta)$ with $\Phi(\cdot)$ denoting the standard normal cumulative distribution function.

For any two points $(p_\alpha, 1-\alpha)$ and $(p_\beta, \beta)$ on the OC curve together with the degree of censoring at the time of the $r$-th failure $q_r = \frac{R_r}{n}$, $r = 1, 2, \ldots, m$, one could compute the required progressively censored acceptance sampling plan from (8.4.1) and (8.4.2).

We summarize the procedure of obtaining a progressively censored acceptance sampling plan step-by-step as follows:

Step 1. Choose two points $(p_\alpha, 1-\alpha)$ and $(p_\beta, \beta)$ on the OC curve and the degrees of censoring $q_1, q_2, \ldots, q_m$.

Step 2. Compute the acceptance constant $l$ using (8.4.1).

Step 3. Give an initial guess of $n$ (one may even start with a moderately large $n$, say 25 or 30, and then decide to move up or down according to Step 5).

Step 4. Compute the value of $S(n)$ from (8.4.2) and formulas (C.6)-(C.8) in Appendix C.

Step 5. If $S(n) > 1$, then increase $n$ by 1; if $S(n) < 1$, then decrease $n$ by 1, and repeat Step 4 until $n$ satisfies the two conditions $S(n+1) > 1$ and $S(n) \leq 1$.

For the purpose of comparison, we determined sampling plans with $p_\alpha =$ 0.00041 and $p_\beta = 0.01840$, which are chosen to match with the specifications in MIL-STD-105D [see Schneider (1989)]. Table 8.8 provides a comparison between sample sizes computed by applying the procedure proposed here with the corresponding values obtained by Balasooriya, Saw and Gadag (2000) using approximate MLE's. Note that the sample sizes obtained by the procedure proposed here are slightly larger than those of Balasooriya, Saw and Gadag (2000); this may be due to the fact that the approximate MLE's tend to be less precise with increased degree of censoring. To demonstrate this point, we generated 10,000 progressively censored samples from the extreme value distribution in (8.2.1) with $\mu = 0$ and $\sigma = 1$. For each of the sampling plans, we simulated the values of the two levels of probability of acceptance ($\alpha$ and $\beta$), and these results are presented in Table 8.8. We can see from these values that the sample sizes obtained from the approximate MLEs of Balasooriya, Saw and Gadag (2000) result in larger values of $\alpha$ and $\beta$ than the pre-fixed ones, while the sample sizes obtained by the exact MLE's can attain closer levels of probabilities of acceptance.

We also determined sampling plans with $p_\alpha$ and $p_\beta$ that are chosen to match with the specifications in MIL-STD-105D by using better censoring schemes. These results are presented in Table 8.9. It should be mentioned that under the same degree of censoring and exactly the same specifications, if a better censoring scheme is employed, the required sample size gets reduced significantly.

### 8.4.2 Expected Total Test Time

In practical applications, it is useful to have an idea on the total test time for a particular reliability sampling plan. When a progressively Type-II censored sampling

Table 8.8: Progressively censored reliability sampling plans with $p_\alpha = 0.00041$ and $p_\beta = 0.01840$

| | | | $q_1, q_2, q_m$ (%) | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | 42,00,28 | | 36,00,24 | | 30,00,20 | | 24,00,16 | | 18,00,12 | | 12,00,08 | |
| $\alpha$ | $\beta$ | | [A] | [B] | [A] | [B] | [A] | [B] | [A] | [B] | [A] | [B] | [A] | [B] |
| 0.05 | 0.10 | $n$ | 35 | 27 | 29 | 23 | 26 | 21 | 21 | 19 | 19 | 17 | 17 | 16 |
| ($l$ = 5.6560) | | Sim. $\alpha$ | 0.0531 | 0.0587 | 0.0454 | 0.0622 | 0.0432 | 0.0512 | 0.0454 | 0.0520 | 0.0413 | 0.0491 | 0.0409 | 0.0476 |
| | | Sim. $\beta$ | 0.1306 | 0.1612 | 0.1281 | 0.1748 | 0.1207 | 0.1551 | 0.1325 | 0.1602 | 0.1288 | 0.1575 | 0.1309 | 0.1426 |
| 0.05 | 0.05 | $n$ | 50 | 43 | 42 | 36 | 36 | 31 | 31 | 27 | 27 | 23 | 24 | 22 |
| ($l$ = 5.8926) | | Sim. $\alpha$ | 0.0482 | 0.0586 | 0.0440 | 0.0540 | 0.0449 | 0.0510 | 0.0423 | 0.0474 | 0.0428 | 0.0545 | 0.0430 | 0.0480 |
| | | Sim. $\beta$ | 0.0492 | 0.0689 | 0.0487 | 0.0745 | 0.0498 | 0.0650 | 0.0501 | 0.0655 | 0.0533 | 0.0777 | 0.0565 | 0.0712 |
| | | | $q_1, q_2, q_m$ (%) | | | | | | | | | | | |
| | | | 24,23,23 | | 20,20,20 | | 17,17,16 | | 14,13,13 | | 10,10,10 | | 07,07,06 | |
| 0.05 | 0.10 | $n$ | 29 | 27 | 27 | 23 | 24 | 21 | 22 | 19 | 19 | 18 | 17 | 15 |
| ($l$ = 5.6560) | | Sim. $\alpha$ | 0.0430 | 0.0542 | 0.0423 | 0.0579 | 0.0432 | 0.0560 | 0.0433 | 0.0460 | 0.0446 | 0.0465 | 0.0399 | 0.0483 |
| | | Sim. $\beta$ | 0.1241 | 0.1526 | 0.1230 | 0.1742 | 0.1303 | 0.1521 | 0.1241 | 0.1434 | 0.1316 | 0.1477 | 0.1292 | 0.1594 |
| 0.05 | 0.05 | $n$ | 45 | 40 | 37 | 33 | 34 | 30 | 30 | 26 | 27 | 23 | 23 | 22 |
| ($l$ = 5.8926) | | Sim. $\alpha$ | 0.0469 | 0.0556 | 0.0462 | 0.0588 | 0.0442 | 0.0514 | 0.0439 | 0.0500 | 0.0452 | 0.0468 | 0.0447 | 0.0461 |
| | | Sim. $\beta$ | 0.0565 | 0.0704 | 0.0530 | 0.0807 | 0.0487 | 0.0636 | 0.0513 | 0.0663 | 0.0556 | 0.0677 | 0.0587 | 0.0650 |

Remarks: [A] – Procedure proposed in this chapter; [B] – Procedure proposed by Balasooriya, Saw and Gadag (2000)

Table 8.9: Progressively censored reliability sampling plans with $p_\alpha$ and $p_\beta$ matching with MIL-STD-105D

| Degree of censoring | | | 70% | | 60% | | 50% | | 40% | | 30% | | 20% | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | $\alpha = 0.05, \beta = 0.10$ | | | | | | | |
| $p_\alpha$ | $p_\beta$ | $l$ | | | | | $n$ (censoring scheme) | | | | | | | |
| 0.00041 | 0.0184 | 5.6560 | 22 | ($R_3 = 15$) | 20 | ($R_3 = 12$) | 19 | ($R_3 = 10$) | 18 | ($R_3 = 7$) | 17 | ($R_3 = 5$) | 16 | ($R_3 = 3$) |
| 0.00284 | 0.0311 | 4.5092 | 36 | ($R_4 = 25$) | 33 | ($R_4 = 20$) | 31 | ($R_4 = 16$) | 30 | ($R_4 = 12$) | 27 | ($R_4 = 8$) | 26 | ($R_4 = 5$) |
| 0.00654 | 0.0426 | 3.9629 | 46 | ($R_5 = 32$) | 42 | ($R_5 = 25$) | 40 | ($R_5 = 20$) | 37 | ($R_5 = 15$) | 35 | ($R_5 = 11$) | 34 | ($R_5 = 7$) |
| 0.01090 | 0.0535 | 3.6070 | 54 | ($R_6 = 38$) | 49 | ($R_6 = 29$) | 46 | ($R_6 = 23$) | 43 | ($R_6 = 17$) | 41 | ($R_6 = 12$) | 39 | ($R_6 = 8$) |
| | | | | | | | $\alpha = 0.05, \beta = 0.05$ | | | | | | | |
| $p_\alpha$ | $p_\beta$ | $l$ | | | | | $n$ (censoring scheme) | | | | | | | |
| 0.00041 | 0.0184 | 5.8926 | 32 | ($R_3 = 22$) | 28 | ($R_3 = 17$) | 26 | ($R_3 = 13$) | 24 | ($R_3 = 10$) | 23 | ($R_3 = 7$) | 21 | ($R_3 = 4$) |
| 0.00284 | 0.0311 | 4.6587 | 52 | ($R_4 = 36$) | 47 | ($R_4 = 28$) | 42 | ($R_4 = 21$) | 40 | ($R_4 = 16$) | 37 | ($R_4 = 11$) | 35 | ($R_4 = 7$) |
| 0.00654 | 0.0426 | 4.0804 | 66 | ($R_5 = 46$) | 59 | ($R_5 = 35$) | 54 | ($R_5 = 27$) | 51 | ($R_5 = 20$) | 47 | ($R_5 = 14$) | 45 | ($R_5 = 9$) |
| 0.01090 | 0.0535 | 3.7071 | 76 | ($R_6 = 53$) | 69 | ($R_6 = 41$) | 63 | ($R_6 = 32$) | 59 | ($R_6 = 24$) | 56 | ($R_6 = 17$) | 54 | ($R_6 = 11$) |

plan is used, one can estimate the total test time by

$$\begin{aligned} E(X_{m:m:n}) &\approx \exp\left[E(Y_{m:m:n})\right] \\ &= \exp\left\{\mu + \sigma c_{m-1} \sum_{i=1}^{m} \frac{a_{i,m}}{\gamma_i}\left[\psi(1) - \ln \gamma_i\right]\right\}. \end{aligned} \tag{8.4.3}$$

Usually, we will have some information on $\mu$ and $\sigma$ from past data or prior experience. Therefore, we can approximate the expected test length provided that the values of $n$ and $m$ and the reliability sampling plan are specified.

### 8.4.3 Illustrative Example

**Example 8.3:** Refer to Example 8.1. Suppose we wish to construct a progressively censored sampling plan for given specifications of $p_\alpha = 0.00284$, $1 - \alpha = 0.95$, $p_\beta = 0.03110$, $\beta = 0.10$ with total degree of censoring equal to 58%. Then, we consider the following two censoring schemes with $q_i$'s denoting the proportion of censoring at the $i$-th failure:

(i) $q_3 = q_5 = 0.16, q_m = 0.26$ and $q_i = 0, i \neq 3$ or 5 or $m$;

(ii) $q_4 = 0.58$ and $q_i = 0, i \neq 4$;

(iii) $q_1 = 0.2$, $q_2 = q_m = 0.19$ and $q_i = 0, i \neq 1$ or 2 or $m$.

Note that, when $n = 19$, censoring scheme (i) is precisely the censoring scheme used by Viveros and Balakrishnan (1994). By using (8.4.1) and (8.4.2), we obtain $l = 4.50920$, $n = 43$ for (i), $n = 33$ for (ii), and $n = 46$ for (iii) (see Table 8.9); that is,

(i) $n = 43$, censoring scheme: $(0, 0, 7, 0, 7, 0, \ldots, 0, 11)$,

(ii) $n = 33$, censoring scheme: $(0, 0, 0, 19, 0, \ldots, 0)$.

(iii) $n = 46$, censoring scheme: $(9, 9, 0, \ldots, 0, 9)$.

It is important to note that when $n = 33$, the censoring scheme $(0, 0, 0, 19, 0, \ldots, 0)$ is optimal under all three optimality criteria discussed earlier in Section 8.3.1. We thus observe that when a better censoring scheme is determined and adopted, the sample size required for the reliability sampling plan gets reduced substantially.

Furthermore, we may use the estimates of the parameters $\mu$ and $\sigma$ determined from the progressively censored data provided by Viveros and Balakrishnan (1994) as preliminary estimates of $\mu$ and $\sigma$ in order to estimate the expected total test time. The MLEs to be used as preliminary estimates in this case are $\hat{\mu} = 2.22196$ and $\hat{\sigma} = 1.02638$; then from (8.4.3), we estimate the expected total test time to be 7.916 for plan (i), 26.814 for plan (ii), and 9.637 for plan (iii). Clearly, the censoring scheme (ii) results in a much longer test which is the price paid for improved efficiency in inference as well as reduced sample size for censored sampling plan.

## 8.5 Conclusion

In this chapter, we have applied the EM-algorithm and direct computation for the MLEs to determine optimal censoring schemes for the Weibull distribution. For this purpose, we have used minimization of the trace of the variance-covariance matrix of the MLEs, the determinant of the variance-covariance matrix of the MLEs, and the trace of the Fisher information matrix of the MLEs. We have also discussed the estimation of quantiles. We have shown that the use of these optimal censoring schemes result in a substantial gain in efficiency. We have further shown that the optimal censoring scheme with respect to one of these three criteria also result in a gain in efficiency that is almost optimal with respect to the other two criteria. With an illustrative example, we have shown that if one uses the optimal censoring scheme as an experimental design to produce data, then one would get estimates with smaller standard errors. Finally, we have described the construction of progressively censored reliability sampling plans for given specifications of $\alpha$ (producer's risk), $p_\alpha$, $\beta$ (consumer's risk), $p_\beta$, and the censoring proportions $q_1, \ldots, q_m$. We have further shown that the sample size required for the reliability sampling plan gets reduced when a better censoring scheme is adopted.

# Chapter 9

# A Test of Exponentiality Based on Spacings Under Progressive Type-II Censoring

## 9.1 Introduction

The exponential distribution is one of the most widely used life-time models in the areas of life testing and reliability. The volume by Balakrishnan and Basu (1995) [see also Johnson, Kotz, and Balakrishnan (1994, Chapter 19)] provides an extensive review of the genesis of the distribution and its properties, including several characterization results. Because of its wide applicability and its relations to other distributions like the gamma and Weibull, there have been numerous tests proposed in the literature to determine whether or not an exponential model is indeed appropriate for a given sample.

The history of goodness-of-fit tests originated with the seminal paper by Karl Pearson in 1900 on the chi-squared test. Tests based on the empirical distribution function (EDF) like the Kolmogorov-Smirnov, Cramer-von Mises and their variants are applicable for testing the hypothesis that the random sample comes from some arbitrary distribution. The properties of these "omnibus" tests under various scenarios have been investigated by several authors; see D'Agostino and Stephens (1986) for a detailed bibliography. These tests are intuitive, and easily modified in the event of censored data.

However, if the experimenter is interested in testing whether a particular model like the normal or exponential is appropriate, it may be appropriate to use the properties of the underlying distribution to derive a more specific (hopefully, more powerful) test. For the exponential distribution, one can exploit the fact that the hazard function is constant or that the logarithm of the survival function is linear. Shapiro (1995) and Stephens (1986) provide a fairly extensive review of the literature on tests for the exponential distribution. Spinelli and Stephens (1987) discuss tests for the two-parameter exponential distribution when the parameters are unknown.

There have been numerous tests proposed in the literature to determine whether or not an exponential model is appropriate for a given data set. These procedures range from graphical techniques, to tests that exploit characterization results for the exponential distribution. In this chapter, we propose a goodness-of-fit test for the exponential distribution based on general progressively Type-II censored data. This test based on spacings generalizes a test proposed by Tiku (1980). In Section 9.2, we will propose a test statistic for exponentiality based on spacings. We will derive the exact and asymptotic null distributions of the test statistic. In Section 9.3, we present results of a simulation study to investigate the power of this test under several different alternatives. We also discuss an approximation to the power and compare the approximate values with those obtained by simulations. In Section 9.4, we examine two standard tests (Cramer-von Mises $A^2$ and the Shapiro-Wilk $W_E$) for exponentiality discussed extensively in the literature, and compare the power performance of all three procedures. Section 9.5 considers tests for the two-parameter exponential distribution. We illustrate the test procedures proposed here using some numerical examples in Section 9.6. Section 9.7 discusses the multi-sample extension of this procedure. Finally, we conclude with some comments and suggestions for further research in Section 9.8.

# 9.2 Test for Exponentiality

## 9.2.1 Test Statistic

Let us assume that the failure times have an exponential distribution with probability density function (p.d.f.)

$$f(x;\sigma) = \frac{1}{\sigma}\exp\left(-\frac{x}{\sigma}\right), \qquad x > 0, \tag{9.2.1}$$

and cumulative distribution function (c.d.f.)

$$F(x;\sigma) = 1 - \exp\left(-\frac{x}{\sigma}\right), \qquad x > 0, \tag{9.2.2}$$

where $\sigma > 0$ is an unknown scale parameter.

Let $X_{1:m:n}^{(R_1,\ldots,R_m)}, X_{2:m:n}^{(R_1,\ldots,R_m)}, \ldots, X_{m:m:n}^{(R_1,\ldots,R_m)}$ denote a progressively Type-II right censored sample. We would like to test whether such a sample comes from an exponential distribution with p.d.f. (9.2.1) with $\sigma$ being unknown. In other words, we want to test the hypotheses

$$\begin{aligned} & H_0 : X \stackrel{d}{=} Exp(\sigma) \\ \text{against} \quad & H_1 : X \stackrel{d}{\neq} Exp(\sigma). \end{aligned} \tag{9.2.3}$$

For convenience, we will suppress the censoring scheme in the notation of the $X_{i:m:n}$'s.

Define the normalized spacings $S_1$, $S_2$, ..., $S_m$ as

$$\begin{aligned} S_1 &= nX_{1:m:n}^{(R_1,\ldots,R_m)}, \\ S_2 &= (n - R_1 - 1)(X_{2:m:n}^{(R_1,\ldots,R_m)} - X_{1:m:n}^{(R_1,\ldots,R_m)}), \\ S_3 &= (n - R_1 - R_2 - 2)(X_{3:m:n}^{(R_1,\ldots,R_m)} - X_{2:m:n}^{(R_1,\ldots,R_m)}), \\ \ldots & \quad \ldots \\ S_m &= (n - R_1 - \ldots - R_{m-1} - m + 1)(X_{m:m:n}^{(R_1,\ldots,R_m)} - X_{m-1:m:n}^{(R_1,\ldots,R_m)}). \end{aligned} \tag{9.2.4}$$

If the underlying distribution is exponential, $S_1$, $S_2$, ..., $S_m$ defined in (9.2.4) are all independent and identically distributed as exponential with scale parameter $\sigma$; see Balakrishnan and Aggarwala (2000) for details.

Consider the test statistic given by

$$T = \frac{\sum_{i=1}^{m-1} (m-i) S_i}{(m-1) \sum_{i=1}^{m} S_i}. \tag{9.2.5}$$

The numerator of the test statistic is a linear combination of the spacings with decreasing weights, and the denominator is the sum of the spacings. The test statistic is clearly scale invariant, with small and large values of $T$ leading to the rejection of $H_0$. The statistic $T$ was suggested by Tiku (1980) for complete and doubly Type-II censored samples. Balakrishnan (1983) studied the power of the test against a variety of alternatives, and showed that the test (for complete samples) performs well compared to standard tests in the literature.

### 9.2.2 Null Distribution

To derive the null distribution of the test statistic $T$, we first write $T$ in the following form:

$$\begin{aligned} T &= \frac{1}{m-1}\left[\frac{S_1}{\sum_{i=1}^{m} S_i} + \frac{S_1+S_2}{\sum_{i=1}^{m} S_i} + \frac{S_1+S_2+S_3}{\sum_{i=1}^{m} S_i} + \ldots + \frac{S_1+\ldots+S_{m-1}}{\sum_{i=1}^{m} S_i}\right] \\ &= \frac{1}{m-1} \sum_{j=1}^{m-1} Z_j, \end{aligned} \tag{9.2.6}$$

where

$$Z_j = \frac{\sum_{i=1}^{j} S_i}{\sum_{i=1}^{m} S_i}, \qquad j = 1, 2, \ldots, m-1.$$

Since $S_1$, $S_2$, $\ldots$, $S_m$ are all independent and identically distributed as exponential with scale parameter $\sigma$, the joint p.d.f. of $S_1$, $S_2$, $\ldots$, $S_m$ is given by

$$f_{S_1,\ldots,S_m}(s_1,\ldots,s_m) = \frac{1}{\sigma^m} \exp\left(-\frac{1}{\sigma}\sum_{i=1}^{m} s_i\right), \qquad s_i > 0, i = 1, 2, \ldots, m.$$

Consider the transformation

$$\begin{aligned} Z_j &= \frac{\sum_{i=1}^{j} S_i}{\sum_{i=1}^{m} S_i}, \qquad j = 1, 2, \ldots, m-1, \\ Z_m &= \sum_{i=1}^{m} S_i. \end{aligned}$$

We then have

$$\begin{aligned} S_1 &= Z_1 Z_m, \\ S_2 &= Z_2 Z_m - Z_1 Z_m, \\ S_3 &= Z_3 Z_m - Z_2 Z_m, \\ \cdots & \quad \cdots \\ S_{m-1} &= Z_{m-1} Z_m - Z_{m-2} Z_m, \\ S_m &= Z_m - Z_{m-1} Z_m. \end{aligned}$$

The Jacobian of this transformation is

$$|J| = \det \begin{bmatrix} z_m & 0 & 0 & 0 & \cdots & \cdots & 0 & z_1 \\ -z_m & z_m & 0 & 0 & \cdots & \cdots & 0 & z_2 - z_1 \\ 0 & -z_m & z_m & 0 & \cdots & \cdots & 0 & z_3 - z_2 \\ \vdots & & \ddots & \ddots & \ddots & & \vdots & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & & & & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & -z_m & z_m & z_{m-1} - z_{m-2} \\ 0 & \cdots & 0 & \cdots & \cdots & 0 & -z_m & 1 - z_{m-1} \end{bmatrix}$$

which can be shown to equal $z_m^{m-1}$.

Therefore, the joint density of $Z_1$, $Z_2$, $\ldots$, $Z_m$ is given by

$$\begin{aligned} & f_{Z_1,\ldots,Z_m}(z_1, \ldots, z_m) \\ &\quad = \frac{1}{\sigma^m} e^{-z_m/\sigma} z_m^{m-1}, \qquad 0 < z_1 < z_2 < \ldots < z_{m-1} < 1, z_m > 0, \end{aligned}$$

which yields the joint density of $Z_1$, $Z_2$, $\ldots$, $Z_{m-1}$ to be

$$\begin{aligned} f_{Z_1,\ldots,Z_{m-1}}(z_1, \ldots, z_{m-1}) &= \int_0^\infty \frac{1}{\sigma^m} e^{-z_m/\sigma} z_m^{m-1} \, dz_m \\ &= (m-1)!, \qquad 0 < z_1 < z_2 < \ldots < z_{m-1} < 1. \end{aligned}$$

The joint distribution of $Z_1, Z_2, \ldots, Z_{m-1}$ is thus the same as the joint distribution of the $(m-1)$ order statistics (say, $U_{(1)}, \ldots, U_{(m-1)}$) obtained from a random sample of size $(m-1)$ from the Uniform (0,1) distribution (say, $U_1, \ldots, U_{m-1}$). Hence, we immediately have

$$(m-1)T \stackrel{d}{=} \sum_{i=1}^{m-1} Z_i \stackrel{d}{=} \sum_{i=1}^{m-1} U_{(i)} \stackrel{d}{=} \sum_{i=1}^{m-1} U_i.$$

This implies that the null distribution of the test statistic $T$ is exactly the same as the average of $m-1$ i.i.d. Uniform(0,1) random variables. Therefore, the null distribution of $T$ tends to normality very rapidly as $m$ increases. It is readily verified that the mean of the limiting distribution is $E(T) = \frac{1}{2}$ and variance $Var(T) = \frac{1}{12(m-1)}$.

**Remark:** *The above expressions of $E(T) = \frac{1}{2}$ and $Var(T) = \frac{1}{12(m-1)}$ can also be derived by taking expectations on both sides of*

$$T(m-1)\sum_{i=1}^{m} S_i = \sum_{i=1}^{m-1}(m-i)S_i$$

*and using Basu's theorem with the facts that $2\sum_{i=1}^{m} S_i/\sigma$ is distributed as $\chi^2_{2m}$, $2S_i/\sigma$ is distributed as $\chi^2_2$, and that the ancillary statistic $T$ is independent of the complete sufficient statistic $\sum_{i=1}^{m} S_i$.*

# 9.3 Power Function Approximation and Simulation Results

## 9.3.1 Approximation of Power Function

The power function of the test is given by

$$\Pr\left(T \geq \frac{1}{2} + z_{\alpha/2}\sqrt{\frac{1}{12(m-1)}}\,\middle|\, H_1\right) + \Pr\left(T \leq \frac{1}{2} - z_{\alpha/2}\sqrt{\frac{1}{12(m-1)}}\,\middle|\, H_1\right).$$

To compute the power under different alternatives, we need to compute probabilities of the form $\Pr(T \geq c)$, for $c$ being some constant. Since $(m-1)\sum_{i=1}^{m} S_i$ is a positive quantity, we may write

$$\begin{aligned}\Pr(T \geq c) &= \Pr\left[\frac{\sum\limits_{i=1}^{m-1}(m-i)S_i}{(m-1)\sum\limits_{i=1}^{m} S_i} \geq c\right] \\ &= \Pr(L \geq 0),\end{aligned}$$

where

$$L = \sum_{i=1}^{m-1}(m-i)S_i - c(m-1)\sum_{i=1}^{m} S_i.$$

From (9.2.4), $L$ may be written as a linear combination of the progressively Type-II right censored order statistics as

$$\begin{aligned}L &= \sum_{i=1}^{m-1}(m-i)S_i - c(m-1)\sum_{i=1}^{m} S_i \\ &= \sum_{i=1}^{m} a_i X_{i:m:n}^{(R_1,\ldots,R_m)},\end{aligned}$$

where

$$\begin{aligned}a_i &= [(m-i) - c(m-1)](R_i+1) + (n-i-R_1-\ldots-R_i), \\ &\qquad\qquad\qquad\qquad i = 1,\ldots,m-1, \\ a_m &= -c(m-1)(R_m+1).\end{aligned}$$

For large values of $m$, we may approximate the probability by

$$\Pr(L \geq 0) \approx \Pr\left[Z \geq -\frac{\theta}{\tau}\right], \tag{9.3.1}$$

where $Z$ is a standard normal random variable, and

$$\begin{aligned}\theta &= E(L) = \sum_{i=1}^{m} a_i E\left[X_{i:m:n}^{(R_1,\ldots,R_m)}\right], \\ \tau^2 &= Var(L) = \sum_{i=1}^{m} a_i^2 Var\left[X_{i:m:n}^{(R_1,\ldots,R_m)}\right] \\ &\quad +2\sum_{i=1}^{m-1}\sum_{j=i+1}^{m} a_i a_j Cov\left[X_{i:m:n}^{(R_1,\ldots,R_m)}, X_{j:m:n}^{(R_1,\ldots,R_m)}\right].\end{aligned}$$

The single and product moments of progressively Type-II right censored order statistics occurring in the above expression may be obtained by first-order approximations; see Balakrishnan and Rao (1997). The idea is to use the probability integral transformation

$$X_{i:m:n}^{(R_1,\ldots,R_m)} \stackrel{d}{=} F^{-1}(U_{i:m:n}^{(R_1,\ldots,R_m)}), \tag{9.3.2}$$

where $U_{i:m:n}^{(R_1,\ldots,R_m)}$ is the $i^{th}$ progressively Type-II right censored order statistic from the uniform $U(0,1)$ distribution, and $F^{-1}$ is the inverse c.d.f. of the underlying distribution.

The mean, variance, and covariance for progressively Type-II order statistics from the Uniform $U(0,1)$ distribution are given by [see Balakrishnan and Aggarwala (2000)]

$$\begin{aligned}
E(U_{i:m:n}) &= \alpha_{i:m:n} = 1 - b_i, \qquad i = 1, \ldots, m, && (9.3.3)\\
Var(U_{i:m:n}) &= a_i b_i, \qquad i = 1, \ldots, m, && (9.3.4)\\
Cov(U_{i:m:n}, U_{j:m:n}) &= a_i b_j, \qquad 1 \le i \le j \le m, && (9.3.5)
\end{aligned}$$

where

$$\begin{aligned}
b_i &= \prod_{k=1}^{i} \frac{R_k + R_{k+1} + \ldots + R_m + m - k + 1}{R_k + R_{k+1} + \ldots + R_m + m - k + 2},\\
a_i &= \prod_{k=1}^{i} \frac{R_k + R_{k+1} + \ldots + R_m + m - k + 2}{R_k + R_{k+1} + \ldots + R_m + m - k + 3}\\
&\quad - \prod_{k=1}^{i} \frac{R_k + R_{k+1} + \ldots + R_m + m - k + 1}{R_k + R_{k+1} + \ldots + R_m + m - k + 2}.
\end{aligned}$$

Expanding $F^{-1}(U_{i:m:n})$ in a Taylor series (keeping only the first term), we have

$$\begin{aligned}
E(X_{i:m:n}) &\approx F^{-1}(\alpha_{i:m:n}), && (9.3.6)\\
Var(X_{i:m:n}) &\approx \left\{F^{-1(1)}(\alpha_{i:m:n})\right\}^2 Var(U_{i:m:n}), && (9.3.7)\\
Cov(X_{i:m:n}, X_{j:m:n}) &\approx \left\{F^{-1(1)}(\alpha_{i:m:n})\right\}\left\{F^{-1(1)}(\alpha_{j:m:n})\right\}\\
&\quad \times Cov(U_{i:m:n}, U_{j:m:n}), && (9.3.8)
\end{aligned}$$

where $F^{-1(1)}(u) = \frac{dF^{-1}(u)}{du} = \frac{1}{f(F^{-1}(u))}$. Balakrishnan and Rao (1997) used these results to derive expressions for the approximate best linear unbiased estimators for an arbitrary location-scale family of distributions.

We would like to point out that even though limiting results for linear combinations of regular order statistics are available [see, for example, David (1981)], such results under progressive censoring have not been studied yet. It is unclear whether the results in the regular case can be easily extended to progressive censoring.

Instead of rewriting the test statistic as a linear combination of the progressively censored order statistics, we may directly approximate the power by considering the test statistic $T$. We may write

$$T = \frac{\sum_{i=1}^{m-1}(m-i)S_i}{(m-1)\sum_{i=1}^{m} S_i} = \frac{W_1}{W_2}. \tag{9.3.9}$$

We then have

$$E(T) = E\left[\frac{W_1}{W_2}\right] \approx \frac{E(W_1)}{E(W_2)}, \tag{9.3.10}$$

$$Var(T) \approx \left[\frac{E(W_1)}{E(W_2)}\right]^2 \left[\frac{Var(W_1)}{E^2(W_1)} + \frac{Var(W_2)}{E^2(W_2)} - \frac{2Cov(W_1, W_2)}{E(W_1)E(W_2)}\right]; \tag{9.3.11}$$

see Kendall and Stuart (1969) for details. We may then approximate the distribution of $T$ by a normal distribution with mean and variance given by the above expressions.

### 9.3.2 Monte Carlo Power Comparison

In order to assess the power properties of the test statistic $T$, a Monte Carlo simulation study was conducted to determine the power under different alternatives. The following lifetime distributions were used as alternatives to the exponential distribution:

1. Weibull distribution with shape parameter 0.5, 2.0;
2. Lomax distribution with shape parameter 0.5, 2.0;

3. Lognormal distribution with shape parameter 0.5, 1.0;
4. Gamma distribution with shape parameter 0.75, 2.0.

For a detailed discussion on various properties of these distributions, one may refer to Johnson, Kotz, and Balakrishnan (1994). For different choices of sample sizes and progressive censoring schemes, we generated 100,000 sets of data in order to obtain the estimated power values. These values are tabulated in Tables 9.2–9.5 for $n = 20$ $(m = 8, 12, 16), 40$ $(m = 10, 20, 30)$ and 60 $(m = 20, 40, 50)$ with three different progressive censoring schemes in each case. For convenience, Table 9.1 lists the different censoring schemes (c.s.) used in the simulation study.

The power values presented in Tables 9.2–9.5 clearly show that the test proposed performs very well for all the alternatives considered. The power increases with $m$ for a fixed $n$, and also increases as $n$ increases. We also calculated the power values of $T$ from the normal approximation using the two methods and found them to be close to the simulated power values for large values of $m$. The approximations are also presented in Tables 9.2–9.5 for comparison. It is important to note from these tables that the approximation in (9.3.1) does not work well for small values of $m$, even when the value of $n$ is large.

To demonstrate the accuracy of the Monte Carlo simulations, we also tabulate in Table 9.6 the null probabilities for the exponential distribution at levels 2.5(2.5)50%. Since the critical values are independent of $n$ and the progressive censoring schemes, we only present the values for different values of $m$. We can see that the simulated probabilities under the null distribution are very close to the pre-fixed levels, which suggests that the Monte Carlo method provides a very good approximation. The results in Table 9.6 also provide ample evidence to the accuracy of the normal approximation to the null distribution of the test statistic. If we have to report the p-value of the test, we are then justified in computing tail probabilities using the normal approximation.

Table 9.1: Progressive censoring schemes used in Monte Carlo simulation study

| $n$ | $m$ | $(R_1, R_2, \ldots, R_m)$ | Scheme No. |
|---|---|---|---|
| 20 | 8 | $R_1 = 12$, $R_i = 0$ for $i \neq 1$ | [1] |
| | | $R_8 = 12$, $R_i = 0$ for $i \neq 8$ | [2] |
| | | $R_1 = R_8 = 6$, $R_i = 0$ for $i \neq 1, 8$ | [3] |
| | 12 | $R_1 = 8$, $R_i = 0$ for $i \neq 1$ | [4] |
| | | $R_{12} = 8$, $R_i = 0$ for $i \neq 12$ | [5] |
| | | $R_3 = R_5 = R_7 = R_9 = 2$, $R_i = 0$ for $i \neq 3, 5, 7, 9$ | [6] |
| | 16 | $R_1 = 4$, $R_i = 0$ for $i \neq 1$ | [7] |
| | | $R_{16} = 4$,$R_i = 0$ for $i \neq 16$ | [8] |
| | | $R_5 = 4$, $R_i = 0$ for $i \neq 5$ | [9] |
| 40 | 10 | $R_1 = 30$, $R_i = 0$ for $i \neq 1$ | [10] |
| | | $R_{10} = 30$, $R_i = 0$ for $i \neq 10$ | [11] |
| | | $R_1 = R_5 = R_{10} = 10$, $R_i = 0$ for $i = 1, 5, 10$ | [12] |
| | 20 | $R_1 = 20$, $R_i = 0$ for $i \neq 1$ | [13] |
| | | $R_{20} = 20$, $R_i = 0$ for $i \neq 20$ | [14] |
| | | $R_i = 1$, for $i = 1, 2, \ldots, 20$ | [15] |
| | 30 | $R_1 = 10$, $R_i = 0$ for $i \neq 1$ | [16] |
| | | $R_{30} = 10$, $R_i = 0$ for $i \neq 30$ | [17] |
| | | $R_1 = R_{30} = 5$, $R_i = 0$ for $i \neq 1, 30$ | [18] |
| 60 | 20 | $R_1 = 40$, $R_i = 0$ for $i \neq 1$ | [19] |
| | | $R_{20} = 40$, $R_i = 0$ for $i \neq 20$ | [20] |
| | | $R_1 = R_{20} = 10$, $R_{10} = 20$ , $R_i = 0$ for $i \neq 1, 10, 20$ | [21] |
| | 40 | $R_1 = 20$, $R_i = 0$ for $i \neq 1$ | [22] |
| | | $R_{40} = 20$, $R_i = 0$ for $i \neq 40$ | [23] |
| | | $R_{2i-1} = 1$, $R_{2i} = 0$ for $i = 1, 2, \ldots, 20$ | [24] |
| | 50 | $R_1 = 10$, $R_i = 0$ for $i \neq 1$ | [25] |
| | | $R_{50} = 10$, $R_i = 0$ for $i \neq 50$ | [26] |
| | | $R_1 = R_{50} = 5$, $R_i = 0$ for $i \neq 1, 50$ | [27] |

Table 9.2: Monte Carlo power estimates for Weibull distribution at 10% and 5% levels of significance

| | Weibull(0.5) | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | 10% | | | | | 5% | | | | |
| c.s. | $T$ | App(L) | App(W) | $A^2$ | $W_E$ | $T$ | App(L) | App(W) | $A^2$ | $W_E$ |
| [1] | 0.71672 | 0.79033 | 0.70467 | 0.60746 | 0.55796 | 0.61452 | 0.69872 | 0.59668 | 0.52883 | 0.45663 |
| [2] | 0.51847 | 0.66998 | 0.57311 | 0.37713 | 0.33930 | 0.39990 | 0.55197 | 0.45694 | 0.29668 | 0.24614 |
| [3] | 0.57377 | 0.72616 | 0.63247 | 0.43776 | 0.39414 | 0.45581 | 0.61273 | 0.51555 | 0.35465 | 0.29455 |
| [4] | 0.83379 | 0.83901 | 0.81327 | 0.66873 | 0.63183 | 0.76001 | 0.77399 | 0.72796 | 0.59327 | 0.53617 |
| [5] | 0.69796 | 0.78209 | 0.73798 | 0.47934 | 0.44554 | 0.59111 | 0.68995 | 0.63459 | 0.39233 | 0.34245 |
| [6] | 0.85449 | 0.85515 | 0.83683 | 0.68503 | 0.65829 | 0.79102 | 0.80064 | 0.76174 | 0.61718 | 0.56993 |
| [7] | 0.90389 | 0.87568 | 0.88596 | 0.71755 | 0.68749 | 0.85230 | 0.82815 | 0.82318 | 0.64605 | 0.59762 |
| [8] | 0.84024 | 0.85959 | 0.86140 | 0.59537 | 0.56385 | 0.76609 | 0.80119 | 0.78752 | 0.51014 | 0.46244 |
| [9] | 0.92164 | 0.89050 | 0.90996 | 0.74722 | 0.72349 | 0.87656 | 0.84947 | 0.85511 | 0.68152 | 0.63910 |
| [10] | 0.80630 | 0.83480 | 0.79047 | 0.69662 | 0.64007 | 0.72293 | 0.76423 | 0.69720 | 0.61928 | 0.54240 |
| [11] | 0.58011 | 0.69950 | 0.62607 | 0.39267 | 0.36022 | 0.46239 | 0.58494 | 0.51114 | 0.31052 | 0.26233 |
| [12] | 0.70747 | 0.81095 | 0.75718 | 0.51728 | 0.48377 | 0.60430 | 0.72898 | 0.65652 | 0.43550 | 0.38432 |
| [13] | 0.95650 | 0.91312 | 0.94612 | 0.80383 | 0.77610 | 0.92708 | 0.88138 | 0.90818 | 0.74337 | 0.69770 |
| [14] | 0.86101 | 0.87312 | 0.88381 | 0.56856 | 0.54223 | 0.78663 | 0.81655 | 0.81548 | 0.48023 | 0.43650 |
| [15] | 0.94605 | 0.91715 | 0.95127 | 0.75547 | 0.73113 | 0.91204 | 0.88713 | 0.91625 | 0.68957 | 0.64844 |
| [16] | 0.99036 | 0.95235 | 0.98797 | 0.87649 | 0.86136 | 0.98199 | 0.93601 | 0.97634 | 0.83067 | 0.80121 |
| [17] | 0.97063 | 0.94666 | 0.97797 | 0.74441 | 0.72546 | 0.94722 | 0.92471 | 0.95751 | 0.67077 | 0.63439 |
| [18] | 0.97969 | 0.95122 | 0.98410 | 0.79352 | 0.77495 | 0.96253 | 0.93245 | 0.96856 | 0.72763 | 0.69231 |
| [19] | 0.95994 | 0.91637 | 0.95033 | 0.82456 | 0.79270 | 0.93151 | 0.88588 | 0.91435 | 0.76548 | 0.71624 |
| [20] | 0.83761 | 0.85969 | 0.86431 | 0.53713 | 0.50973 | 0.75680 | 0.79580 | 0.78887 | 0.44811 | 0.40481 |
| [21] | 0.94038 | 0.92336 | 0.95982 | 0.72449 | 0.70207 | 0.90314 | 0.89470 | 0.92723 | 0.65344 | 0.61401 |
| [22] | 0.99840 | 0.97427 | 0.99792 | 0.93655 | 0.92493 | 0.99638 | 0.96575 | 0.99528 | 0.90704 | 0.88510 |
| [23] | 0.98985 | 0.97129 | 0.99328 | 0.80242 | 0.78826 | 0.97958 | 0.95890 | 0.98529 | 0.73386 | 0.70346 |
| [24] | 0.99819 | 0.97611 | 0.99834 | 0.92981 | 0.91833 | 0.99601 | 0.96820 | 0.99615 | 0.89855 | 0.87707 |
| [25] | 0.99970 | 0.98529 | 0.99961 | 0.96329 | 0.95653 | 0.99916 | 0.98049 | 0.99901 | 0.94347 | 0.92864 |
| [26] | 0.99872 | 0.98636 | 0.99921 | 0.90278 | 0.89330 | 0.99691 | 0.98104 | 0.99797 | 0.85950 | 0.83900 |
| [27] | 0.99924 | 0.98660 | 0.99949 | 0.92795 | 0.92069 | 0.99818 | 0.98174 | 0.99866 | 0.89362 | 0.87540 |

| | Weibull(2.0) | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | 10% | | | | | 5% | | | | |
| c.s. | $T$ | App(L) | App(W) | $A^2$ | $W_E$ | $T$ | App(L) | App(W) | $A^2$ | $W_E$ |
| [1] | 0.81945 | 0.89734 | 0.93811 | 0.25905 | 0.23759 | 0.68849 | 0.81533 | 0.84627 | 0.17926 | 0.14879 |
| [2] | 0.49956 | 0.55783 | 0.55755 | 0.14362 | 0.13726 | 0.34470 | 0.39873 | 0.39878 | 0.08634 | 0.07574 |
| [3] | 0.60582 | 0.68462 | 0.68882 | 0.16607 | 0.15746 | 0.44619 | 0.52574 | 0.52552 | 0.10594 | 0.09065 |
| [4] | 0.91172 | 0.94854 | 0.97258 | 0.30676 | 0.25619 | 0.82316 | 0.89889 | 0.92478 | 0.21770 | 0.16360 |
| [5] | 0.72826 | 0.78185 | 0.78672 | 0.19354 | 0.16742 | 0.58402 | 0.64632 | 0.64752 | 0.12369 | 0.09740 |
| [6] | 0.87935 | 0.92728 | 0.94502 | 0.24654 | 0.21070 | 0.76253 | 0.85062 | 0.86532 | 0.16674 | 0.12852 |
| [7] | 0.95772 | 0.97482 | 0.98732 | 0.34094 | 0.27326 | 0.90236 | 0.94434 | 0.96109 | 0.24467 | 0.17720 |
| [8] | 0.89272 | 0.92350 | 0.93404 | 0.26365 | 0.21288 | 0.79847 | 0.84933 | 0.85772 | 0.17796 | 0.13134 |
| [9] | 0.96362 | 0.97831 | 0.98915 | 0.33296 | 0.27218 | 0.91216 | 0.94911 | 0.96429 | 0.23641 | 0.17514 |
| [10] | 0.93802 | 0.95734 | 0.99257 | 0.42401 | 0.35878 | 0.87462 | 0.92656 | 0.97241 | 0.32530 | 0.24949 |
| [11] | 0.58036 | 0.63285 | 0.63318 | 0.15917 | 0.14561 | 0.42602 | 0.47993 | 0.47986 | 0.09810 | 0.08196 |
| [12] | 0.73598 | 0.79830 | 0.80971 | 0.19894 | 0.17756 | 0.58615 | 0.66508 | 0.66851 | 0.12964 | 0.10439 |
| [13] | 0.99378 | 0.99339 | 0.99923 | 0.53410 | 0.40719 | 0.98161 | 0.98593 | 0.99659 | 0.42401 | 0.29094 |
| [14] | 0.91227 | 0.93098 | 0.93719 | 0.28490 | 0.22011 | 0.83312 | 0.86453 | 0.86957 | 0.19431 | 0.13708 |
| [15] | 0.98015 | 0.98502 | 0.99220 | 0.37401 | 0.29222 | 0.94816 | 0.96359 | 0.97446 | 0.27154 | 0.19144 |
| [16] | 0.99956 | 0.99932 | 0.99994 | 0.61063 | 0.45501 | 0.99800 | 0.99806 | 0.99964 | 0.49531 | 0.33160 |
| [17] | 0.99342 | 0.99455 | 0.99691 | 0.45004 | 0.32440 | 0.98097 | 0.98533 | 0.98956 | 0.33607 | 0.21861 |
| [18] | 0.99723 | 0.99742 | 0.99910 | 0.51169 | 0.37044 | 0.99111 | 0.99282 | 0.99636 | 0.39531 | 0.25746 |
| [19] | 0.99673 | 0.99372 | 0.99978 | 0.63063 | 0.48642 | 0.98981 | 0.98830 | 0.99890 | 0.52833 | 0.36543 |
| [20] | 0.89434 | 0.91226 | 0.91695 | 0.26949 | 0.20967 | 0.80681 | 0.83526 | 0.83849 | 0.18194 | 0.12919 |
| [21] | 0.97705 | 0.98067 | 0.98895 | 0.37580 | 0.29183 | 0.94247 | 0.95524 | 0.96683 | 0.27360 | 0.19119 |
| [22] | 1.00000 | 0.99994 | 1.00000 | 0.76301 | 0.57640 | 0.99993 | 0.99982 | 0.99999 | 0.66509 | 0.45036 |
| [23] | 0.99878 | 0.99896 | 0.99953 | 0.54811 | 0.38065 | 0.99612 | 0.99674 | 0.99805 | 0.42790 | 0.26786 |
| [24] | 1.00000 | 0.99994 | 1.00000 | 0.67704 | 0.50448 | 0.99984 | 0.99977 | 0.99997 | 0.56469 | 0.37931 |
| [25] | 1.00000 | 1.00000 | 1.00000 | 0.81038 | 0.61641 | 1.00000 | 0.99999 | 1.00000 | 0.71880 | 0.49141 |
| [26] | 0.99999 | 0.99996 | 0.99999 | 0.69585 | 0.49341 | 0.99988 | 0.99985 | 0.99996 | 0.58462 | 0.36861 |
| [27] | 0.99999 | 0.99998 | 1.00000 | 0.74535 | 0.54158 | 0.99997 | 0.99994 | 0.99999 | 0.64051 | 0.41547 |

Table 9.3: Monte Carlo power estimates for Lomax distribution at 10% and 5% levels of significance

| | Lomax(0.5) | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | 10% | | | | | 5% | | | | |
| c.s. | $T$ | App(L) | App(W) | $A^2$ | $W_E$ | $T$ | App(L) | App(W) | $A^2$ | $W_E$ |
| [1] | 0.82303 | 0.93658 | 0.79763 | 0.68983 | 0.68715 | 0.77238 | 0.91681 | 0.73874 | 0.64701 | 0.63292 |
| [2] | 0.21607 | 0.31514 | 0.23585 | 0.12015 | 0.11848 | 0.13322 | 0.23460 | 0.15311 | 0.06864 | 0.06345 |
| [3] | 0.35568 | 0.48739 | 0.37394 | 0.17834 | 0.17364 | 0.25550 | 0.37055 | 0.27518 | 0.11914 | 0.10764 |
| [4] | 0.93605 | 0.96461 | 0.92302 | 0.82777 | 0.81768 | 0.91141 | 0.95555 | 0.89321 | 0.79650 | 0.77966 |
| [5] | 0.50119 | 0.61830 | 0.51626 | 0.22339 | 0.20745 | 0.39396 | 0.50200 | 0.41055 | 0.15846 | 0.13678 |
| [6] | 0.87040 | 0.93725 | 0.83274 | 0.73201 | 0.71268 | 0.83437 | 0.92106 | 0.78946 | 0.69530 | 0.66573 |
| [7] | 0.97759 | 0.97571 | 0.97297 | 0.90291 | 0.89338 | 0.96663 | 0.97001 | 0.95990 | 0.88231 | 0.86585 |
| [8] | 0.84985 | 0.88405 | 0.85203 | 0.54634 | 0.51649 | 0.79228 | 0.84954 | 0.79410 | 0.47630 | 0.43396 |
| [9] | 0.97492 | 0.97467 | 0.97028 | 0.89796 | 0.88618 | 0.96332 | 0.96873 | 0.95639 | 0.87614 | 0.85794 |
| [10] | 0.89205 | 0.95408 | 0.87313 | 0.76905 | 0.76184 | 0.85680 | 0.94141 | 0.83023 | 0.73191 | 0.71581 |
| [11] | 0.15319 | 0.22198 | 0.16594 | 0.10395 | 0.10412 | 0.08547 | 0.15746 | 0.09761 | 0.05345 | 0.05197 |
| [12] | 0.30073 | 0.38308 | 0.31154 | 0.14420 | 0.13878 | 0.20851 | 0.28029 | 0.22031 | 0.08717 | 0.07877 |
| [13] | 0.99203 | 0.98155 | 0.99126 | 0.94559 | 0.93672 | 0.98753 | 0.97742 | 0.98616 | 0.93202 | 0.91870 |
| [14] | 0.51911 | 0.57796 | 0.53041 | 0.17396 | 0.15807 | 0.40612 | 0.45920 | 0.41917 | 0.11173 | 0.09385 |
| [15] | 0.87879 | 0.88714 | 0.84641 | 0.63073 | 0.57919 | 0.83686 | 0.85845 | 0.80122 | 0.57361 | 0.51157 |
| [16] | 0.99947 | 0.98876 | 0.99965 | 0.98655 | 0.98275 | 0.99906 | 0.98639 | 0.99934 | 0.98181 | 0.97631 |
| [17] | 0.95434 | 0.91384 | 0.95475 | 0.61017 | 0.55853 | 0.92672 | 0.88995 | 0.92805 | 0.53408 | 0.46811 |
| [18] | 0.98875 | 0.94983 | 0.98856 | 0.83422 | 0.80025 | 0.98088 | 0.93788 | 0.98073 | 0.79008 | 0.74271 |
| [19] | 0.99196 | 0.98152 | 0.99125 | 0.94550 | 0.93663 | 0.98750 | 0.97739 | 0.98614 | 0.93192 | 0.91861 |
| [20] | 0.27506 | 0.31672 | 0.29002 | 0.11468 | 0.10877 | 0.18182 | 0.21508 | 0.19538 | 0.06215 | 0.05715 |
| [21] | 0.67153 | 0.72728 | 0.67173 | 0.29673 | 0.26005 | 0.58254 | 0.64608 | 0.58271 | 0.22258 | 0.18253 |
| [22] | 0.99998 | 0.99205 | 0.99999 | 0.99707 | 0.99574 | 0.99997 | 0.99041 | 0.99998 | 0.99578 | 0.99347 |
| [23] | 0.95153 | 0.90840 | 0.95330 | 0.49868 | 0.43662 | 0.92107 | 0.87992 | 0.92351 | 0.41043 | 0.33957 |
| [24] | 0.99844 | 0.97332 | 0.99211 | 0.96215 | 0.94623 | 0.99740 | 0.96775 | 0.98879 | 0.95045 | 0.92816 |
| [25] | 1.00000 | 0.99392 | 1.00000 | 0.99937 | 0.99884 | 1.00000 | 0.99269 | 1.00000 | 0.99906 | 0.99835 |
| [26] | 0.99923 | 0.96646 | 0.99920 | 0.91593 | 0.88434 | 0.99846 | 0.95875 | 0.99839 | 0.88664 | 0.84007 |
| [27] | 0.99997 | 0.97647 | 0.99992 | 0.98098 | 0.97197 | 0.99987 | 0.97147 | 0.99982 | 0.97274 | 0.95854 |

| | Lomax(2.0) | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | 10% | | | | | 5% | | | | |
| c.s. | $T$ | App(L) | App(W) | $A^2$ | $W_E$ | $T$ | App(L) | App(W) | $A^2$ | $W_E$ |
| [1] | 0.30907 | 0.28451 | 0.21716 | 0.17958 | 0.16996 | 0.22452 | 0.21132 | 0.14050 | 0.12285 | 0.10901 |
| [2] | 0.11154 | 0.18329 | 0.12257 | 0.09971 | 0.09993 | 0.05553 | 0.13127 | 0.06576 | 0.05001 | 0.05019 |
| [3] | 0.12760 | 0.19612 | 0.13394 | 0.10209 | 0.10199 | 0.06827 | 0.14152 | 0.07401 | 0.05164 | 0.05101 |
| [4] | 0.40318 | 0.34265 | 0.28542 | 0.21513 | 0.19097 | 0.31774 | 0.24740 | 0.19945 | 0.15517 | 0.12782 |
| [5] | 0.14348 | 0.18665 | 0.14423 | 0.10234 | 0.10098 | 0.07840 | 0.12598 | 0.08144 | 0.05257 | 0.05097 |
| [6] | 0.34351 | 0.26750 | 0.22272 | 0.19374 | 0.17011 | 0.26190 | 0.18905 | 0.14520 | 0.13441 | 0.11023 |
| [7] | 0.48634 | 0.41182 | 0.35544 | 0.24258 | 0.20713 | 0.39831 | 0.30515 | 0.26333 | 0.17987 | 0.14128 |
| [8] | 0.24019 | 0.25206 | 0.21992 | 0.11499 | 0.11005 | 0.15713 | 0.16879 | 0.14033 | 0.06245 | 0.05625 |
| [9] | 0.47963 | 0.40199 | 0.34740 | 0.24079 | 0.20562 | 0.39190 | 0.29672 | 0.25631 | 0.17800 | 0.13977 |
| [10] | 0.36079 | 0.30957 | 0.25018 | 0.19619 | 0.18124 | 0.27491 | 0.22425 | 0.16867 | 0.13741 | 0.11851 |
| [11] | 0.10710 | 0.16402 | 0.11660 | 0.09873 | 0.10080 | 0.05239 | 0.11186 | 0.06150 | 0.04911 | 0.04951 |
| [12] | 0.12625 | 0.17748 | 0.12956 | 0.10021 | 0.10212 | 0.06505 | 0.12231 | 0.07084 | 0.05093 | 0.05065 |
| [13] | 0.55863 | 0.47640 | 0.42100 | 0.27115 | 0.22498 | 0.46974 | 0.36730 | 0.32598 | 0.20418 | 0.15610 |
| [14] | 0.13945 | 0.16562 | 0.14154 | 0.10118 | 0.09983 | 0.07666 | 0.10342 | 0.07927 | 0.05174 | 0.05031 |
| [15] | 0.28328 | 0.23982 | 0.21548 | 0.13414 | 0.12020 | 0.20092 | 0.15828 | 0.13758 | 0.07975 | 0.06556 |
| [16] | 0.69864 | 0.60886 | 0.56620 | 0.32850 | 0.26260 | 0.61953 | 0.51160 | 0.47313 | 0.25704 | 0.18883 |
| [17] | 0.28986 | 0.29036 | 0.27554 | 0.11381 | 0.10897 | 0.19623 | 0.19331 | 0.18472 | 0.06078 | 0.05647 |
| [18] | 0.40740 | 0.39444 | 0.37317 | 0.13461 | 0.12268 | 0.30562 | 0.28362 | 0.27218 | 0.07802 | 0.06664 |
| [19] | 0.55835 | 0.47602 | 0.42058 | 0.27110 | 0.22499 | 0.46945 | 0.36690 | 0.32560 | 0.20421 | 0.15603 |
| [20] | 0.11311 | 0.14384 | 0.11934 | 0.09941 | 0.09869 | 0.05773 | 0.08832 | 0.06338 | 0.05016 | 0.04953 |
| [21] | 0.17575 | 0.19212 | 0.16926 | 0.10483 | 0.10222 | 0.10522 | 0.12260 | 0.10054 | 0.05452 | 0.05210 |
| [22] | 0.79492 | 0.70105 | 0.67852 | 0.37994 | 0.29782 | 0.72959 | 0.62078 | 0.59503 | 0.30652 | 0.21959 |
| [23] | 0.26916 | 0.26786 | 0.25884 | 0.10682 | 0.10408 | 0.17633 | 0.17449 | 0.16974 | 0.05611 | 0.05310 |
| [24] | 0.59787 | 0.49216 | 0.46968 | 0.23909 | 0.18454 | 0.50840 | 0.38396 | 0.36923 | 0.17237 | 0.11847 |
| [25] | 0.86092 | 0.76694 | 0.76334 | 0.42547 | 0.33020 | 0.80912 | 0.70166 | 0.69209 | 0.34902 | 0.24792 |
| [26] | 0.51287 | 0.49757 | 0.48655 | 0.13395 | 0.12176 | 0.40291 | 0.38241 | 0.37653 | 0.07531 | 0.06508 |
| [27] | 0.65010 | 0.61974 | 0.60888 | 0.17696 | 0.15089 | 0.55128 | 0.51589 | 0.50570 | 0.11144 | 0.08636 |

Table 9.4: Monte Carlo power estimates for lognormal distribution at 10% and 5% levels of significance

| | Lognormal(0.5) | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | 10% | | | | | 5% | | | | |
| c.s. | $T$ | App(L) | App(W) | $A^2$ | $W_E$ | $T$ | App(L) | App(W) | $A^2$ | $W_E$ |
| [1] | 0.97317 | 0.99822 | 0.99953 | 0.57405 | 0.45279 | 0.93250 | 0.99138 | 0.99521 | 0.45287 | 0.32599 |
| [2] | 0.92186 | 0.95576 | 0.95264 | 0.36626 | 0.30817 | 0.82028 | 0.87119 | 0.86690 | 0.26666 | 0.20415 |
| [3] | 0.95509 | 0.98294 | 0.98435 | 0.44054 | 0.35978 | 0.88607 | 0.93814 | 0.93878 | 0.32760 | 0.24556 |
| [4] | 0.98369 | 0.99916 | 0.99951 | 0.61419 | 0.40693 | 0.95757 | 0.99580 | 0.99662 | 0.48638 | 0.28805 |
| [5] | 0.97790 | 0.99171 | 0.99046 | 0.49165 | 0.34131 | 0.93753 | 0.96818 | 0.96493 | 0.37131 | 0.23011 |
| [6] | 0.98599 | 0.99917 | 0.99931 | 0.53921 | 0.37542 | 0.95939 | 0.99506 | 0.99516 | 0.41481 | 0.25898 |
| [7] | 0.98804 | 0.99948 | 0.99947 | 0.62071 | 0.37729 | 0.96865 | 0.99725 | 0.99697 | 0.49201 | 0.26345 |
| [8] | 0.99050 | 0.99799 | 0.99747 | 0.57392 | 0.35703 | 0.97141 | 0.99078 | 0.98897 | 0.44477 | 0.24460 |
| [9] | 0.99008 | 0.99964 | 0.99960 | 0.60367 | 0.37816 | 0.97302 | 0.99787 | 0.99752 | 0.47486 | 0.26274 |
| [10] | 0.99883 | 0.99999 | 1.00000 | 0.91263 | 0.73497 | 0.99648 | 0.99993 | 1.00000 | 0.84945 | 0.61977 |
| [11] | 0.98880 | 0.99641 | 0.99603 | 0.58133 | 0.43831 | 0.96171 | 0.98239 | 0.98083 | 0.45922 | 0.31393 |
| [12] | 0.99844 | 0.99971 | 0.99988 | 0.70643 | 0.54012 | 0.99226 | 0.99791 | 0.99862 | 0.58682 | 0.40452 |
| [13] | 0.99968 | 1.00000 | 1.00000 | 0.93916 | 0.63184 | 0.99916 | 0.99999 | 1.00000 | 0.88579 | 0.50824 |
| [14] | 0.99983 | 0.99998 | 0.99998 | 0.83711 | 0.52330 | 0.99918 | 0.99985 | 0.99980 | 0.73616 | 0.39745 |
| [15] | 0.99995 | 1.00000 | 1.00000 | 0.88454 | 0.58418 | 0.99977 | 0.99999 | 1.00000 | 0.80170 | 0.45680 |
| [16] | 0.99987 | 1.00000 | 1.00000 | 0.94584 | 0.59018 | 0.99970 | 1.00000 | 1.00000 | 0.89330 | 0.46627 |
| [17] | 0.99999 | 1.00000 | 1.00000 | 0.92174 | 0.56214 | 0.99993 | 1.00000 | 0.99999 | 0.85183 | 0.43561 |
| [18] | 0.99999 | 1.00000 | 1.00000 | 0.93584 | 0.57814 | 0.99993 | 1.00000 | 1.00000 | 0.87549 | 0.45259 |
| [19] | 0.99998 | 1.00000 | 1.00000 | 0.99074 | 0.80103 | 0.99993 | 1.00000 | 1.00000 | 0.97817 | 0.70248 |
| [20] | 0.99997 | 1.00000 | 1.00000 | 0.90574 | 0.60691 | 0.99986 | 0.99998 | 0.99998 | 0.83300 | 0.48040 |
| [21] | 1.00000 | 1.00000 | 1.00000 | 0.95619 | 0.70913 | 1.00000 | 1.00000 | 1.00000 | 0.91205 | 0.58757 |
| [22] | 0.99999 | 1.00000 | 1.00000 | 0.99518 | 0.73613 | 0.99999 | 1.00000 | 1.00000 | 0.98564 | 0.62309 |
| [23] | 1.00000 | 1.00000 | 1.00000 | 0.98858 | 0.69547 | 1.00000 | 1.00000 | 1.00000 | 0.96990 | 0.57676 |
| [24] | 1.00000 | 1.00000 | 1.00000 | 0.99267 | 0.73544 | 0.99999 | 1.00000 | 1.00000 | 0.97930 | 0.62046 |
| [25] | 1.00000 | 1.00000 | 1.00000 | 0.99540 | 0.71991 | 1.00000 | 1.00000 | 1.00000 | 0.98692 | 0.60573 |
| [26] | 1.00000 | 1.00000 | 1.00000 | 0.99435 | 0.71123 | 1.00000 | 1.00000 | 1.00000 | 0.98352 | 0.59598 |
| [27] | 1.00000 | 1.00000 | 1.00000 | 0.99510 | 0.71781 | 1.00000 | 1.00000 | 1.00000 | 0.98578 | 0.60334 |

| | Lognormal(1.0) | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | 10% | | | | | 5% | | | | |
| c.s. | $T$ | App(L) | App(W) | $A^2$ | $W_E$ | $T$ | App(L) | App(W) | $A^2$ | $W_E$ |
| [1] | 0.20500 | 0.29174 | 0.27413 | 0.11028 | 0.10874 | 0.12145 | 0.21294 | 0.18012 | 0.05800 | 0.05538 |
| [2] | 0.24330 | 0.27591 | 0.26382 | 0.10210 | 0.10222 | 0.13892 | 0.18235 | 0.16012 | 0.05253 | 0.05076 |
| [3] | 0.23727 | 0.28077 | 0.26728 | 0.10238 | 0.10244 | 0.13645 | 0.18965 | 0.16506 | 0.05240 | 0.05095 |
| [4] | 0.18549 | 0.24289 | 0.22190 | 0.11502 | 0.10925 | 0.11137 | 0.17729 | 0.14109 | 0.06320 | 0.05743 |
| [5] | 0.21975 | 0.25508 | 0.24096 | 0.09901 | 0.09883 | 0.12839 | 0.17135 | 0.14707 | 0.05157 | 0.04913 |
| [6] | 0.20836 | 0.27502 | 0.26113 | 0.11233 | 0.10829 | 0.12532 | 0.19362 | 0.16736 | 0.06123 | 0.05592 |
| [7] | 0.18059 | 0.20926 | 0.18788 | 0.12156 | 0.11289 | 0.11081 | 0.15041 | 0.11520 | 0.06750 | 0.05970 |
| [8] | 0.16825 | 0.21182 | 0.19219 | 0.10035 | 0.10005 | 0.09605 | 0.14480 | 0.11486 | 0.05062 | 0.04966 |
| [9] | 0.18455 | 0.21834 | 0.19881 | 0.12050 | 0.11267 | 0.11222 | 0.15713 | 0.12356 | 0.06690 | 0.05911 |
| [10] | 0.29002 | 0.41242 | 0.40955 | 0.13184 | 0.12372 | 0.19056 | 0.30891 | 0.29732 | 0.07706 | 0.06719 |
| [11] | 0.40977 | 0.43709 | 0.43629 | 0.11688 | 0.11248 | 0.27017 | 0.30055 | 0.29513 | 0.06533 | 0.05891 |
| [12] | 0.42399 | 0.46854 | 0.46800 | 0.11977 | 0.11430 | 0.28509 | 0.32841 | 0.32411 | 0.06761 | 0.05954 |
| [13] | 0.21872 | 0.28042 | 0.27187 | 0.14004 | 0.12703 | 0.13898 | 0.20647 | 0.18414 | 0.08305 | 0.06956 |
| [14] | 0.40377 | 0.42535 | 0.42453 | 0.11755 | 0.10848 | 0.27285 | 0.29961 | 0.29391 | 0.06296 | 0.05608 |
| [15] | 0.33082 | 0.40361 | 0.40213 | 0.11992 | 0.11126 | 0.21963 | 0.28892 | 0.28084 | 0.06599 | 0.05802 |
| [16] | 0.20384 | 0.21442 | 0.20436 | 0.15210 | 0.13327 | 0.13046 | 0.15171 | 0.12945 | 0.09250 | 0.07404 |
| [17] | 0.26370 | 0.29442 | 0.28827 | 0.11259 | 0.10725 | 0.16711 | 0.20260 | 0.18821 | 0.06001 | 0.05530 |
| [18] | 0.20548 | 0.24718 | 0.23741 | 0.11441 | 0.10902 | 0.12479 | 0.17094 | 0.15149 | 0.06116 | 0.05656 |
| [19] | 0.27206 | 0.37697 | 0.37584 | 0.16762 | 0.14182 | 0.18277 | 0.28612 | 0.27495 | 0.10273 | 0.08003 |
| [20] | 0.59741 | 0.61445 | 0.61335 | 0.14869 | 0.12681 | 0.45061 | 0.46773 | 0.46762 | 0.08561 | 0.06788 |
| [21] | 0.54944 | 0.59360 | 0.59258 | 0.14769 | 0.12547 | 0.41023 | 0.45363 | 0.45334 | 0.08552 | 0.06683 |
| [22] | 0.21695 | 0.22832 | 0.22276 | 0.18074 | 0.14920 | 0.14186 | 0.16210 | 0.14505 | 0.11487 | 0.08614 |
| [23] | 0.41991 | 0.44414 | 0.44395 | 0.13418 | 0.11751 | 0.29660 | 0.32185 | 0.31788 | 0.07462 | 0.06233 |
| [24] | 0.27574 | 0.35465 | 0.35287 | 0.15682 | 0.13262 | 0.18403 | 0.25781 | 0.24838 | 0.09450 | 0.07436 |
| [25] | 0.22026 | 0.19908 | 0.19353 | 0.18767 | 0.15237 | 0.14658 | 0.13547 | 0.12114 | 0.12045 | 0.08813 |
| [26] | 0.25109 | 0.28162 | 0.27666 | 0.12797 | 0.11442 | 0.16094 | 0.19489 | 0.18231 | 0.06991 | 0.05983 |
| [27] | 0.19203 | 0.22635 | 0.21912 | 0.13294 | 0.11846 | 0.11469 | 0.15524 | 0.13924 | 0.07404 | 0.06270 |

Table 9.5: Monte Carlo power estimates for gamma distribution at 10% and 5% levels of significance

| | Gamma(0.75) | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | 10% | | | | | 5% | | | | |
| c.s. | $T$ | App(L) | App(W) | $A^2$ | $W_E$ | $T$ | App(L) | App(W) | $A^2$ | $W_E$ |
| [1] | 0.18415 | 0.20186 | 0.14326 | 0.13881 | 0.13226 | 0.10768 | 0.14541 | 0.08019 | 0.08196 | 0.07277 |
| [2] | 0.17285 | 0.24856 | 0.18925 | 0.13105 | 0.12493 | 0.09870 | 0.18408 | 0.11582 | 0.07548 | 0.06725 |
| [3] | 0.17698 | 0.24464 | 0.18837 | 0.13428 | 0.12747 | 0.10133 | 0.17894 | 0.11471 | 0.07845 | 0.06911 |
| [4] | 0.20126 | 0.18905 | 0.15029 | 0.13406 | 0.12826 | 0.12070 | 0.12841 | 0.08523 | 0.07908 | 0.07091 |
| [5] | 0.19227 | 0.23571 | 0.20012 | 0.12946 | 0.12390 | 0.11468 | 0.16427 | 0.12364 | 0.07521 | 0.06779 |
| [6] | 0.22455 | 0.19843 | 0.15971 | 0.13788 | 0.13169 | 0.14011 | 0.13668 | 0.09233 | 0.08220 | 0.07366 |
| [7] | 0.21915 | 0.18877 | 0.16068 | 0.13048 | 0.12508 | 0.13728 | 0.12449 | 0.09287 | 0.07550 | 0.06775 |
| [8] | 0.21378 | 0.22985 | 0.20544 | 0.12798 | 0.12310 | 0.13250 | 0.15524 | 0.12734 | 0.07413 | 0.06615 |
| [9] | 0.23260 | 0.19744 | 0.17014 | 0.13328 | 0.12758 | 0.14705 | 0.13109 | 0.10003 | 0.07760 | 0.06965 |
| [10] | 0.20925 | 0.20162 | 0.15722 | 0.14563 | 0.13917 | 0.12582 | 0.13937 | 0.08970 | 0.08834 | 0.07907 |
| [11] | 0.19293 | 0.25732 | 0.21050 | 0.13239 | 0.12763 | 0.11360 | 0.18612 | 0.13223 | 0.07732 | 0.07132 |
| [12] | 0.21813 | 0.27227 | 0.23039 | 0.14202 | 0.13576 | 0.13313 | 0.19523 | 0.14796 | 0.08415 | 0.07732 |
| [13] | 0.25790 | 0.20852 | 0.18919 | 0.13505 | 0.12917 | 0.16655 | 0.13580 | 0.11353 | 0.07895 | 0.07154 |
| [14] | 0.24734 | 0.27371 | 0.25698 | 0.12966 | 0.12466 | 0.15974 | 0.18714 | 0.16810 | 0.07488 | 0.06859 |
| [15] | 0.28922 | 0.26257 | 0.24672 | 0.13907 | 0.13300 | 0.19353 | 0.17766 | 0.15884 | 0.08176 | 0.07444 |
| [16] | 0.30315 | 0.23667 | 0.22608 | 0.13627 | 0.13220 | 0.20500 | 0.15465 | 0.14195 | 0.07859 | 0.07327 |
| [17] | 0.29555 | 0.30082 | 0.29251 | 0.13444 | 0.13005 | 0.19685 | 0.20602 | 0.19660 | 0.07629 | 0.07161 |
| [18] | 0.29813 | 0.29032 | 0.28190 | 0.13498 | 0.13071 | 0.19872 | 0.19721 | 0.18755 | 0.07702 | 0.07198 |
| [19] | 0.26838 | 0.21659 | 0.19826 | 0.14554 | 0.13880 | 0.17535 | 0.14130 | 0.12004 | 0.08561 | 0.07798 |
| [20] | 0.25713 | 0.28799 | 0.27132 | 0.13654 | 0.13307 | 0.16713 | 0.19870 | 0.17981 | 0.07880 | 0.07305 |
| [21] | 0.30867 | 0.32241 | 0.30814 | 0.14807 | 0.14217 | 0.20850 | 0.22575 | 0.21093 | 0.08878 | 0.08112 |
| [22] | 0.35995 | 0.28396 | 0.27819 | 0.13843 | 0.13502 | 0.25210 | 0.19031 | 0.18315 | 0.08067 | 0.07448 |
| [23] | 0.35395 | 0.36277 | 0.35837 | 0.13581 | 0.13221 | 0.24655 | 0.25689 | 0.25220 | 0.07848 | 0.07290 |
| [24] | 0.39902 | 0.33282 | 0.32834 | 0.14345 | 0.13933 | 0.29018 | 0.23045 | 0.22488 | 0.08515 | 0.07792 |
| [25] | 0.40398 | 0.32213 | 0.31855 | 0.13907 | 0.13471 | 0.29295 | 0.22116 | 0.21665 | 0.07973 | 0.07439 |
| [26] | 0.39963 | 0.39085 | 0.38809 | 0.13739 | 0.13324 | 0.29200 | 0.28087 | 0.27803 | 0.07866 | 0.07312 |
| [27] | 0.40108 | 0.37789 | 0.37505 | 0.13829 | 0.13378 | 0.29301 | 0.26930 | 0.26625 | 0.07910 | 0.07378 |

| | Gamma(2.0) | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | 10% | | | | | 5% | | | | |
| c.s. | $T$ | App(L) | App(W) | $A^2$ | $W_E$ | $T$ | App(L) | App(W) | $A^2$ | $W_E$ |
| [1] | 0.46065 | 0.62475 | 0.62450 | 0.12548 | 0.12036 | 0.31030 | 0.46339 | 0.46149 | 0.07321 | 0.06460 |
| [2] | 0.31230 | 0.35326 | 0.35024 | 0.10908 | 0.10780 | 0.18855 | 0.23024 | 0.22497 | 0.05885 | 0.05508 |
| [3] | 0.35993 | 0.42015 | 0.41776 | 0.11336 | 0.11036 | 0.22623 | 0.27977 | 0.27830 | 0.06245 | 0.05763 |
| [4] | 0.51783 | 0.67413 | 0.67629 | 0.13134 | 0.11943 | 0.37470 | 0.52369 | 0.52329 | 0.07698 | 0.06509 |
| [5] | 0.42380 | 0.46831 | 0.46754 | 0.11795 | 0.11029 | 0.28395 | 0.32517 | 0.32399 | 0.06544 | 0.05888 |
| [6] | 0.50328 | 0.64337 | 0.64437 | 0.12347 | 0.11446 | 0.35268 | 0.47912 | 0.47894 | 0.07051 | 0.06077 |
| [7] | 0.57175 | 0.71129 | 0.71348 | 0.13545 | 0.12040 | 0.42260 | 0.56653 | 0.56661 | 0.07713 | 0.06545 |
| [8] | 0.52100 | 0.58340 | 0.58317 | 0.12623 | 0.11473 | 0.37236 | 0.43239 | 0.43223 | 0.07143 | 0.06223 |
| [9] | 0.58871 | 0.72468 | 0.72717 | 0.13433 | 0.11948 | 0.43735 | 0.57774 | 0.57791 | 0.07667 | 0.06468 |
| [10] | 0.64455 | 0.79434 | 0.81290 | 0.18295 | 0.15373 | 0.50515 | 0.67882 | 0.68469 | 0.11693 | 0.09002 |
| [11] | 0.40708 | 0.44438 | 0.44338 | 0.12129 | 0.11380 | 0.27068 | 0.30636 | 0.30449 | 0.06812 | 0.06056 |
| [12] | 0.49898 | 0.56138 | 0.56073 | 0.13200 | 0.12107 | 0.34671 | 0.40423 | 0.40411 | 0.07695 | 0.06556 |
| [13] | 0.76435 | 0.86731 | 0.87863 | 0.19476 | 0.15400 | 0.64243 | 0.77634 | 0.78295 | 0.12190 | 0.08749 |
| [14] | 0.64708 | 0.67861 | 0.67864 | 0.15059 | 0.13037 | 0.50255 | 0.53621 | 0.53617 | 0.08910 | 0.07071 |
| [15] | 0.74438 | 0.81444 | 0.81902 | 0.16688 | 0.14117 | 0.60845 | 0.69161 | 0.69312 | 0.10085 | 0.07851 |
| [16] | 0.84495 | 0.91453 | 0.92076 | 0.19910 | 0.15700 | 0.74340 | 0.84214 | 0.84661 | 0.12385 | 0.08902 |
| [17] | 0.79250 | 0.82753 | 0.82892 | 0.17654 | 0.14476 | 0.67360 | 0.71617 | 0.71654 | 0.10737 | 0.08076 |
| [18] | 0.81585 | 0.85752 | 0.86045 | 0.18534 | 0.14961 | 0.70285 | 0.75773 | 0.75903 | 0.11441 | 0.08376 |
| [19] | 0.83133 | 0.90917 | 0.92771 | 0.24927 | 0.18278 | 0.73048 | 0.84521 | 0.86022 | 0.16829 | 0.11168 |
| [20] | 0.67630 | 0.70367 | 0.70371 | 0.16159 | 0.13807 | 0.53490 | 0.56499 | 0.56494 | 0.09688 | 0.07688 |
| [21] | 0.78997 | 0.83131 | 0.83647 | 0.18907 | 0.15461 | 0.66470 | 0.71712 | 0.71917 | 0.11754 | 0.08968 |
| [22] | 0.93598 | 0.96845 | 0.97375 | 0.25943 | 0.18517 | 0.87880 | 0.93434 | 0.94018 | 0.17081 | 0.11083 |
| [23] | 0.90123 | 0.91524 | 0.91700 | 0.21660 | 0.16524 | 0.82293 | 0.84274 | 0.84371 | 0.13600 | 0.09562 |
| [24] | 0.94513 | 0.96927 | 0.97321 | 0.24655 | 0.18208 | 0.89032 | 0.93273 | 0.93696 | 0.15958 | 0.10905 |
| [25] | 0.96211 | 0.98182 | 0.98469 | 0.27168 | 0.19690 | 0.92165 | 0.95854 | 0.96208 | 0.18012 | 0.11732 |
| [26] | 0.94769 | 0.95990 | 0.96196 | 0.24890 | 0.18512 | 0.89636 | 0.91673 | 0.91854 | 0.16102 | 0.10871 |
| [27] | 0.95310 | 0.96790 | 0.97045 | 0.25895 | 0.19005 | 0.90658 | 0.93157 | 0.93412 | 0.16873 | 0.11246 |

Table 9.6: Monte Carlo null probabilities of $T$ for exponential distribution at levels 2.5 (2.5) 50%

| $m$ | 8 | 10 | 12 | 16 | 20 | 30 | 40 | 50 | 60 |
|---|---|---|---|---|---|---|---|---|---|
| 2.5% | 0.02253 | 0.02328 | 0.02371 | 0.02301 | 0.02429 | 0.02413 | 0.02476 | 0.02507 | 0.02462 |
| 5% | 0.04890 | 0.04808 | 0.04835 | 0.04923 | 0.04911 | 0.04898 | 0.05009 | 0.04965 | 0.04871 |
| 7.5% | 0.07512 | 0.07483 | 0.07351 | 0.07483 | 0.07470 | 0.07397 | 0.07582 | 0.07520 | 0.07412 |
| 10% | 0.10062 | 0.10068 | 0.09847 | 0.09988 | 0.09974 | 0.10001 | 0.10157 | 0.10074 | 0.09971 |
| 12.5% | 0.12675 | 0.12631 | 0.12446 | 0.12603 | 0.12449 | 0.12483 | 0.12693 | 0.12671 | 0.12484 |
| 15% | 0.15306 | 0.15266 | 0.14985 | 0.15147 | 0.15003 | 0.15098 | 0.15134 | 0.15170 | 0.14946 |
| 17.5% | 0.17857 | 0.17859 | 0.17529 | 0.17705 | 0.17558 | 0.17612 | 0.17663 | 0.17680 | 0.17457 |
| 20% | 0.20468 | 0.20510 | 0.20045 | 0.20283 | 0.20115 | 0.20173 | 0.20129 | 0.20140 | 0.19986 |
| 22.5% | 0.23018 | 0.23117 | 0.22611 | 0.22773 | 0.22700 | 0.22683 | 0.22697 | 0.22716 | 0.22475 |
| 25% | 0.25656 | 0.25657 | 0.25116 | 0.25349 | 0.25133 | 0.25243 | 0.25266 | 0.25172 | 0.25019 |
| 27.5% | 0.28182 | 0.28228 | 0.27619 | 0.27841 | 0.27594 | 0.27754 | 0.27726 | 0.27639 | 0.27496 |
| 30% | 0.30769 | 0.30775 | 0.30175 | 0.30400 | 0.30082 | 0.30235 | 0.30175 | 0.30168 | 0.30020 |
| 32.5% | 0.33301 | 0.33277 | 0.32711 | 0.32783 | 0.32573 | 0.32753 | 0.32655 | 0.32668 | 0.32499 |
| 35% | 0.35814 | 0.35827 | 0.35161 | 0.35277 | 0.35141 | 0.35271 | 0.35235 | 0.35113 | 0.35056 |
| 37.5% | 0.38273 | 0.38346 | 0.37726 | 0.37720 | 0.37649 | 0.37799 | 0.37732 | 0.37617 | 0.37520 |
| 40% | 0.40781 | 0.40829 | 0.40274 | 0.40147 | 0.40152 | 0.40335 | 0.40204 | 0.40129 | 0.40072 |
| 42.5% | 0.43366 | 0.43375 | 0.42809 | 0.42668 | 0.42666 | 0.42830 | 0.42760 | 0.42633 | 0.42546 |
| 45% | 0.45883 | 0.45921 | 0.45420 | 0.45230 | 0.45167 | 0.45283 | 0.45339 | 0.45039 | 0.44989 |
| 47.5% | 0.48439 | 0.48401 | 0.47919 | 0.47614 | 0.47604 | 0.47765 | 0.47838 | 0.47578 | 0.47557 |
| 50% | 0.50943 | 0.50819 | 0.50365 | 0.50109 | 0.50159 | 0.50268 | 0.50383 | 0.50129 | 0.50035 |

## 9.4 Modified EDF and Shapiro-Wilk Statistics

As we mentioned in Section 9.1, there have been several goodness-of-fit tests for exponentiality proposed in the literature. Spinelli and Stephens (1987) compared the performance of several test procedures based on the EDF as well as those based on regression methods. They concluded that, in particular, two statistics, viz. Cramer-von Mises $A^2$ and the Shapiro-Wilk $W_E$, had overall better power performance. In this section, we will modify the two statistics in the case of progressively Type-II censored data, and compare their performance with the test based on spacings proposed in Section 9.2.

Testing the null hypothesis that the sample comes from an exponential distribution is equivalent to testing the hypothesis that the spacings $S_1, S_2, \ldots, S_m$ are distributed as scaled exponential. We can then apply the procedures in Spinelli and Stephens (1987) to the $S_i$'s as follows: Let $S_{(1)}, S_{(2)}, \ldots, S_{(m)}$ be the ordered spacings. Let $\hat{\sigma} = \bar{S} = \frac{1}{m}\sum_{i=1}^{m} S_i$ denote the estimator of $\sigma$ based on generalized least squares. Define

$$w_i = S_{(i)}/\hat{\sigma} \qquad \text{and} \qquad z_i = 1 - \exp(-w_i), \qquad i = 1, \ldots, m.$$

The test statistic $A^2$ is then defined as

$$A^2 = -\frac{1}{m}\sum_{i=1}^{m}(2i-1)\{\ln z_i + \ln(1-z_i)\} - m. \tag{9.4.1}$$

Large values of $A^2$ lead to rejection of the null hypothesis that the sample comes from an exponential distribution.

An alternative test was introduced by Shapiro and Wilk (1972) that compares the generalized least squares estimator of $\sigma$ with the estimator obtained from the sample variance. The resulting test statistics $W_E$ is defined as

$$W_E = \bar{S}^2 / \sum_{i=1}^{m} S_i^2. \tag{9.4.2}$$

This test is a two-tailed test.

A Monte Carlo simulation study was conducted to compare the three procedures. We simulated the 5 and 10 percentage points for the test statistics $A^2$ and $W_E$, and used them to compute power for different alternatives. The results presented in Tables 9.2–9.5 show that for all the alternatives considered, the test based on spacings performs significantly better than either of the other two procedures.

## 9.5 Two-Parameter Exponential Case

We may also consider a test for the two-parameter exponential distribution (location-scale model) with p.d.f.

$$f(x;\mu,\sigma) = \frac{1}{\sigma}\exp\left[-\left(\frac{x-\mu}{\sigma}\right)\right], \qquad x > \mu, \tag{9.5.1}$$

where the scale $\sigma > 0$ and the location $\mu$ are unknown parameters. In this case, the progressively Type-II right censored spacings $S_1^*$, $S_2^*$, ..., $S_m^*$ are defined as

$$\begin{aligned} S_1^* &= n(X_{1:m:n}^{(R_1,\ldots,R_m)} - \mu), \\ S_i^* &= S_i, \qquad i = 2,3,\ldots,m, \end{aligned} \tag{9.5.2}$$

where $S_i$'s are as defined earlier in (9.2.4). Once again, $S_1^*$, $S_2^*$, ..., $S_m^*$ are all independent and identically distributed as exponential with scale parameter $\sigma$. Since the first spacing $S_1^*$ involves the unknown parameter $\mu$, the test statistic $T$ proposed earlier in (9.2.5) may be modified as

$$T^* = \frac{\sum\limits_{i=2}^{m-1}(m-i)S_i^*}{(m-2)\sum\limits_{i=2}^{m} S_i^*}. \tag{9.5.3}$$

Following the same procedure outlined in Section 9.2, the null distribution of the test statistic $T^*$ can be derived. The distribution of $T^*$ is the same as the distribution of the average of $(m-2)$ i.i.d. Uniform(0,1) random variables. Hence, the asymptotic null distribution of $T^*$ is normal with mean $E(T^*) = \frac{1}{2}$ and variance $Var(T^*) = \frac{1}{12(m-2)}$. Furthermore, the power approximation procedure discussed in Section 9.3 can also be adapted to this two-parameter exponential case.

## 9.6 Illustrative Examples

In this section, we present two examples to illustrate the use of the statistics $T$ and $T^*$ in testing for the validity of the one- and two-parameter exponential distributions for an observed progressively Type-II right censored sample.

### 9.6.1 One-parameter Exponential Case

**Example 9.1:** Refer to Example 6.2 wherein we considered the progressively censored sample presented by Viveros and Balakrishnan (1994). The data presented in Table 6.3 with the spacings computed from Eq. (9.2.4) are as follows:

Progressively censored sample presented by Viveros and Balakrishnan (1994)

| $i$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| $x_{i:m:n}$ | 0.18999 | 0.77997 | 0.95993 | 1.30996 | 2.77986 | 4.84962 | 6.49999 | 7.35000 |
| $R_i$ | 0 | 0 | 3 | 0 | 3 | 0 | 0 | 5 |
| $S_i$ | 3.60975 | 10.61969 | 3.05924 | 4.55051 | 17.63873 | 16.55808 | 11.55257 | 5.10007 |

Nelson (1982) and Viveros and Balakrishnan (1994) considered a Weibull model for these data and constructed confidence intervals for the Weibull shape and scale parameters based on the complete sample and the progressively censored sample, respectively. In both cases, the confidence interval for the shape parameter contained the value of 1 (the shape parameter value for the exponential case) leading to the conclusion that the data are consistent with an exponential distribution.

In this example, we have $n = 19$, $m = 8$. The test statistic is computed as

$$T = \frac{\sum_{i=1}^{m-1}(m-i)S_i}{(m-1)\sum_{i=1}^{m} S_i} = \frac{220.06957}{508.82052} = 0.43251,$$

and the $p$-value is

$$2\Phi\left(\frac{0.43251 - 0.5}{\sqrt{1/84}}\right) = 2 \times 0.26810 = 0.53620.$$

Based on this $p$-value, we fail to reject the null hypothesis that the random sample is from an exponential distribution. This is consistent with the findings of Nelson (1982) and Viveros and Balakrishnan (1994).

### 9.6.2 Two-parameter Exponential Case

**Example 9.2:** Spinelli and Stephens (1987) reported data with 32 observations on measurements of modulus of repute (a measure of the breaking strength of lumber) of wood beams. For the purpose of illustrating the test procedure outlined in Section 9.4, a progressively Type-II right censored sample of size $m = 20$ has been randomly generated from the $n = 32$ observations in Table 3 of Spinelli and Stephens (1987). The observations, the removal pattern applied and the corresponding spacings computed from Eq. (9.4.2) are as follows:

Progressively censored sample generated from the measurements of modulus of repute of wood beams data by Spinelli and Stephens (1987)

| $i$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $x_{i:m:n}$ | 43.19 | 49.44 | 51.55 | 56.63 | 67.27 | 78.47 | 86.59 | 90.63 | 94.38 | 98.21 |
| $R_i$ | 0 | 2 | 0 | 0 | 2 | 0 | 0 | 0 | 0 | 0 |
| $S_i^*$ | — | 193.75 | 59.08 | 137.16 | 276.64 | 257.60 | 178.64 | 84.84 | 75.00 | 72.77 |

| $i$ | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
|---|---|---|---|---|---|---|---|---|---|---|
| $x_{i:m:n}$ | 98.39 | 99.74 | 100.22 | 103.48 | 105.54 | 107.13 | 108.14 | 108.94 | 110.81 | 116.39 |
| $R_i$ | 2 | 2 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 2 |
| $S_i^*$ | 3.24 | 20.25 | 5.76 | 35.86 | 20.60 | 14.31 | 8.08 | 4.80 | 7.48 | 16.74 |

Spinelli and Stephens (1987) studied tests based on regression and the empirical distribution function for testing the null hypothesis of exponentiality using the complete sample. They found that all the test statistics were highly significant (with $p$-value $< 0.01$) and rejected the null hypothesis that the data are exponentially distributed with p.d.f. (9.4.1).

The test statistic in (9.5.3) for testing the validity of a two-parameter exponential distribution is computed as

$$T^* = \frac{\sum\limits_{i=2}^{m-1}(m-i)S_i}{(m-2)\sum\limits_{i=2}^{m} S_i} = \frac{19983.72}{26506.8} = 0.75391,$$

and the $p$-value is

$$2\left[1 - \Phi\left(\frac{0.75391 - 0.5}{\sqrt{1/216}}\right)\right] = 2 \times 0.0000951 = 0.00019026.$$

From this $p$-value, we observe that the data provide enough evidence to reject the null hypothesis that the progressively censored sample comes from a two-parameter exponential distribution, which agrees with the conclusion of Spinelli and Stephens (1987) drawn from the complete sample.

## 9.7 Multi-Sample Extension

To test that $k$ independent progressively censored samples $X_{1:m_i:n_i}^{(R_{1i},\ldots,R_{m_i i})}, \ldots, X_{m_i:m_i:n_i}^{(R_{1i},\ldots,R_{m_i i})}, i = 1, 2, \ldots, k$, come from exponential populations $E(\mu_i, \sigma_i)$, we can generalize the test statistic $T^*$ in (9.5.3) as follows:

$$T^* = \frac{\sum_{i=1}^{k}(m_i - 2)T_i^*}{\sum_{i=1}^{k}(m_i - 2)}, \tag{9.7.1}$$

where $T_i^*$ is the test statistic computed from the $i^{th}$ sample. Small and large values of $T^*$ indicate the non-exponentiality of at least one of the $k$ samples.

If we wish to test that the samples come from one-parameter exponential populations $E(\sigma_i)$, we can generalize the test statistic $T$ in (9.2.5) as follows:

$$T_* = \frac{\sum_{i=1}^{k}(m_i - 1)T_i}{\sum_{i=1}^{k}(m_i - 1)}, \tag{9.7.2}$$

where $T_i$ is the test statistic computed from the $i^{th}$ sample. Small and large values of $T_*$ indicate the non-exponentiality of at least one of the $k$ samples. Note that, in both cases we may have the censoring schemes ($R_{ij}$'s), sample sizes ($n_i$), and effective sample sizes ($m_i$) for the $k$ samples to be all different.

The null distribution of $T^*$ ($T_*$) may once again be shown to be equivalent to the distribution of the average of $\sum_{i=1}^{k}(m_i - 2)$ ($\sum_{i=1}^{k}(m_i - 1)$) Uniform $U(0,1)$ random variables. To compute the power under different alternatives, we may use an approximation similar to the one discussed in Section 9.3. In this case, however, we will not be able to write $P(T^* > c)$ in terms of a probability involving linear combinations of progressively Type-II right censored order statistics from each sample. We rely on the expressions in (9.3.10) and (9.3.11) to compute the moments of $T^*$ and $T_*$, and the corresponding normal approximations to the probabilities.

Table 9.7: Simulated and approximate values of the power of $T_*$ at 10% and 5% levels of significance

| | | | | Lognormal(1.0) | | | | Lomax(2.0) | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | 10% | | 5% | | 10% | | 5% | |
| $k$ | $n_j$ | $m_j$ | c.s. | Sim. | App. | Sim. | App. | Sim. | App. | Sim. | App. |
| 2 | 40 | 20 | [13][14] | 0.44296 | 0.51678 | 0.35086 | 0.39493 | 0.49618 | 0.37621 | 0.39880 | 0.27749 |
| | | | [13][15] | 0.42930 | 0.49896 | 0.33882 | 0.38034 | 0.59646 | 0.43641 | 0.50350 | 0.33563 |
| | | | [14][14] | 0.65778 | 0.68772 | 0.52172 | 0.55493 | 0.16830 | 0.17085 | 0.09696 | 0.10069 |
| | 60 | 40 | [22][24] | 0.25098 | 0.38508 | 0.16614 | 0.28199 | 0.81572 | 0.78496 | 0.76808 | 0.71048 |
| | | | [23][24] | 0.52032 | 0.61590 | 0.43032 | 0.49254 | 0.64546 | 0.53579 | 0.54524 | 0.42705 |
| | | | [23][23] | 0.66720 | 0.69734 | 0.54014 | 0.57330 | 0.39242 | 0.38314 | 0.28358 | 0.27471 |
| 3 | 40 | 20 | [13][14][15] | 0.53794 | 0.70703 | 0.44136 | 0.59081 | 0.57268 | 0.42501 | 0.47108 | 0.32097 |
| | | | [14][14][15] | 0.75916 | 0.82885 | 0.64342 | 0.72754 | 0.29140 | 0.24573 | 0.20196 | 0.15959 |
| | | | [14][14][14] | 0.81718 | 0.84408 | 0.71218 | 0.74542 | 0.19724 | 0.19967 | 0.12016 | 0.12231 |
| | 60 | 40 | [22][23][24] | 0.44540 | 0.60647 | 0.35110 | 0.49042 | 0.78820 | 0.81008 | 0.74146 | 0.73500 |
| | | | [23][23][24] | 0.65518 | 0.79441 | 0.56334 | 0.69244 | 0.70282 | 0.61208 | 0.60718 | 0.50176 |
| | | | [23][23][23] | 0.82042 | 0.84665 | 0.72174 | 0.75498 | 0.50552 | 0.49186 | 0.38716 | 0.37461 |

Table 9.7 presents some simulation results for $k = 2, 3$ in the case of the one-parameter exponential model. The approximate values of power are reasonably close to the simulated values for most cases considered. It is of interest to note that combinations of censoring schemes for the $k$ samples provide distinctly different power values.

## 9.8 Conclusions

In this chapter, we have proposed goodness-of-fit tests for the one- and two-parameter exponential models under general progressive Type-II censoring. These tests are based on normalized spacings, generalizing tests proposed by Tiku (1980). The exact and asymptotic null distribution of the test statistics have been derived. Further, two approximations to compute the power under different alternatives have been suggested.

Results of the simulation study for a wide range of sample sizes and censoring schemes show that the test performs well in detecting departures from exponentiality. If the alternative model is distinctly different from exponential, the power values are close to 1. The approximations for the power are very close to the values obtained through simulations. The proposed test procedures are illustrated on some real data for the one- and two-parameter exponential models. The conclusions drawn from these tests are consistent with those drawn by other authors using different procedures. Finally, some extensions to the multi-sample case have been suggested.

There are several theoretical aspects that still need to be looked at carefully. In particular, it would be useful to develop limit theorems for linear combinations of progressively Type-II censored order statistics. This would provide theoretical justification for the normal approximations suggested in this chapter. Finally, it would also be interesting to develop analogous goodness-of-fit tests for the general location-scale family of distributions which is the subject matter of the following chapter.

# Chapter 10

# Goodness-of-Fit Tests Based on Spacings for General Location-Scale Distributions Under Progressive Type-II Censoring

## 10.1 Introduction

The problem of testing for the adequacy of model specifications is extremely important. The validity of inferential procedures depends in great part on the assumption of a particular underlying distribution for the data. There have been numerous procedures developed in the literature for determining whether a random sample comes from some specific distribution. All these procedures have been broadly classified as 'goodness-of-fit' tests. There has been extensive research on goodness-of-fit procedures for testing whether or not a sample comes from a specified distribution.

In the previous chapter, we proposed goodness-of-fit tests for exponential distribution. For the normal distribution, departures from normality have been detected using the third and fourth moments. Shapiro and Wilk (1965) developed a test statistic based on the ratio of the best linear unbiased estimate (BLUE) of $\sigma^2$ to the sample variance. The Shapiro-Wilk $W$ statistic is one of the most widely used tests for normality in the literature. D'Agostino and Stephens (1986) provide an extensive review of tests for other distributions including the uniform, Weibull, and extreme-value.

In this chapter, we propose a test based on spacings for a general location-scale family of distributions under progressive Type-II censoring. The test statistic is based on sample spacings and generalizes a test procedure proposed by Tiku (1980). We will first propose a test statistic for the location-scale family based on sample spacings in Section 10.2. We will provide some heuristic arguments to justify the asymptotic null distribution of the test statistic. In Section 10.3, we present results of a simulation study to examine the power of this test for testing for the normal and extreme-value distributions under several different alternatives. We also discuss an approximation to the power and compare the approximate values with those obtained through simulations. We illustrate the test procedure proposed here using a numerical example in Section 10.4. Some conclusions are finally made in Section 10.5.

## 10.2 Omnibus Test for a General Location-Scale Family

### 10.2.1 Test Statistic

Assume that the failure times have a location-scale distribution with probability density function (p.d.f.)

$$\frac{1}{\sigma} f\left(\frac{x-\mu}{\sigma}\right). \tag{10.2.1}$$

The functional form $f$ is completely specified but the location and scale parameters, viz. $\mu$ and $\sigma > 0$, are unknown. This is a rather wide family of distributions and includes the normal and extreme-value models as special cases.

In the previous chapter, we used $X_{1:m:n}^{(R_1,\ldots,R_m)}, X_{2:m:n}^{(R_1,\ldots,R_m)}, \ldots, X_{m:m:n}^{(R_1,\ldots,R_m)}$ to denote the progressively Type-II right censored sample. In this chapter, for convenience, we will suppress the censoring scheme in the notation of the $X_{i:m:n}$'s. We are interested in testing the null hypothesis $H_0$: the sample comes from a specified location-scale distribution with p.d.f. (10.2.1) with $\mu$ and $\sigma$ being unknown.

Define the spacings

$$\begin{aligned}
S_1 &= n(X_{1:m:n} - \mu), \\
S_2 &= (n - R_1 - 1)(X_{2:m:n} - X_{1:m:n}), \\
S_3 &= (n - R_1 - R_2 - 2)(X_{3:m:n} - X_{2:m:n}), \\
\dots & \quad \dots \\
S_m &= (n - R_1 - \ldots - R_{m-1} - m + 1)(X_{m:m:n} - X_{m-1:m:n}). \qquad (10.2.2)
\end{aligned}$$

If the underlying distribution is exponential, $S_1$, $S_2$, $\ldots$, $S_m$ are independent and identically distributed as exponential with scale parameter $\sigma$; see Balakrishnan and Aggarwala (2000) for details.

Let us denote the expected value of the $i$-th order statistic from progressive Type-II censored sample for the standard case (with $\mu = 0$ and $\sigma = 1$) under the null distribution by $\mu_{i:m:n}$. The $i$-th standardized sample spacing $G_i$ is then defined as

$$G_i = \frac{S_i}{E(S_i)} = \frac{X_{i:m:n} - X_{i-1:m:n}}{\mu_{i:m:n} - \mu_{i-1:m:n}}, \qquad \text{for } i = 2, 3, \ldots, m.$$

Let us define the test statistic

$$T = \frac{\sum_{i=2}^{m-1} (m - i) G_i}{(m - 2) \sum_{i=2}^{m} G_i}. \qquad (10.2.3)$$

The numerator of the test statistic is a linear combination of the sample spacings with decreasing weights, and the denominator is the sum of the sample spacings. The test statistic is clearly location and scale invariant, with small and large values of $T$ leading to the rejection of $H_0$. The statistic $T$ was suggested by Tiku (1980) for complete and doubly Type-II censored samples. In Chapter 9, we constructed a similar test for the exponential distribution under progressive Type-II censoring.

In this chapter, we will specifically consider the following two distributions, both of which are extensively used as lifetime models:

(1) Normal Distribution $N(\mu, \sigma)$

$$f(x; \mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right], \quad -\infty < x < \infty; \tag{10.2.4}$$

(2) Extreme-value Distribution $EV(\mu, \sigma)$

$$f(x; \mu, \sigma) = \frac{1}{\sigma} \exp\left[\left(\frac{x-\mu}{\sigma}\right) - \exp\left(\frac{x-\mu}{\sigma}\right)\right], \quad -\infty < x < \infty. \tag{10.2.5}$$

### 10.2.2 Null Distribution of $T$

If the underlying distribution was exponential, the null distribution of the test statistic may be easily derived using the distribution of spacings mentioned earlier. In Chapter 9, we have shown that for testing exponentiality, the null distribution of the test statistic $T$ is exactly the same as the average of $m-2$ i.i.d. Uniform(0,1) random variables. Therefore, the null distribution of $T$ tends to normality very rapidly as $m$ increases.

However, in most other cases, the distribution of spacings is intractable making the derivation of the exact null distribution extremely difficult. We conducted a Monte Carlo simulation study to examine the null distribution of $T$. We simulated the coefficient of skewness ($\sqrt{\beta_1}$) and kurtosis ($\beta_2$) of the statistic $T$ for normal and extreme value distributions under different choices of sample sizes and censoring schemes. For all the cases considered, the values of $\sqrt{\beta_1}$ and $\beta_2$ of $T$ were found to be close to 0 and 3, respectively (see Table 10.1). This suggests that the null distribution of $T$ can be approximated closely by the normal distribution with mean $E(T)$ and variance $Var(T)$.

The mean and variance of $T$ may be obtained by first-order approximations as follows. Let

$$W_1 = \sum_{i=2}^{m-1}(m-i)G_i \text{ and } W_2 = (m-2)\sum_{i=1}^{m} G_i,$$

using which we can write $T = \frac{W_1}{W_2}$.

Under the null distribution, we have

$$E(W_1) \;=\; \sum_{i=2}^{m-1}(m-i)E(G_i)$$

Table 10.1: Simulated coefficients of skewness and kurtosis of $T$ under null distribution (normal and extreme-value distributions)

| | | | Normal | | Extreme-value | |
|---|---|---|---|---|---|---|
| $n$ | $m$ | Censoring Scheme | $\sqrt{\beta_1}$ | $\beta_2$ | $\sqrt{\beta_1}$ | $\beta_2$ |
| 40 | 10 | $R_1 = 30$, $R_i = 0$ for $i \neq 1$ | 0.04415 | 2.90772 | 0.05183 | 2.91273 |
| | | $R_{10} = 30$, $R_i = 0$ for $i \neq 10$ | -0.03182 | 2.85491 | -0.01040 | 2.84169 |
| | | $R_1 = R_5 = R_{10} = 10$, $R_i = 0$ for $i = 1, 5, 10$ | -0.05356 | 2.90184 | -0.04586 | 2.88812 |
| 40 | 20 | $R_1 = 20$, $R_i = 0$ for $i \neq 1$ | 0.00443 | 2.99052 | 0.04509 | 2.96579 |
| | | $R_{20} = 20$, $R_i = 0$ for $i \neq 20$ | -0.04189 | 2.97800 | 0.00236 | 2.95176 |
| | | $R_i = 1$, for $i = 1, 2, \ldots, 20$ | -0.00588 | 2.97094 | 0.04992 | 2.95614 |
| 40 | 30 | $R_1 = 10$, $R_i = 0$ for $i \neq 1$ | 0.02134 | 2.98785 | 0.06562 | 2.99508 |
| | | $R_{30} = 10$, $R_i = 0$ for $i \neq 30$ | -0.00839 | 2.98491 | 0.03241 | 2.98470 |
| | | $R_1 = R_{30} = 5$, $R_i = 0$ for $i \neq 1, 30$ | -0.00222 | 2.98893 | 0.03558 | 2.98881 |
| 60 | 20 | $R_1 = 40$, $R_i = 0$ for $i \neq 1$ | 0.01268 | 2.99593 | 0.04637 | 2.96573 |
| | | $R_{20} = 40$, $R_i = 0$ for $i \neq 20$ | -0.03957 | 2.95463 | 0.00077 | 2.94980 |
| | | $R_1 = R_{20} = 10$, $R_{10} = 20$ , $R_i = 0$ for $i \neq 1, 10, 20$ | -0.06477 | 3.00562 | -0.03273 | 2.97755 |
| 60 | 40 | $R_1 = 20$, $R_i = 0$ for $i \neq 1$ | 0.00222 | 2.97723 | 0.04691 | 2.97487 |
| | | $R_{40} = 20$, $R_i = 0$ for $i \neq 40$ | -0.03331 | 2.97898 | 0.01149 | 2.97863 |
| | | $R_{2i-1} = 1$, $R_{2i} = 0$ for $i = 1, 2, \ldots, 20$ | -0.00613 | 2.97977 | 0.05202 | 2.98409 |
| 60 | 50 | $R_1 = 10$, $R_i = 0$ for $i \neq 1$ | -0.00360 | 2.97529 | 0.04373 | 2.98314 |
| | | $R_{50} = 10$, $R_i = 0$ for $i \neq 50$ | -0.02474 | 2.97823 | 0.01532 | 2.97937 |
| | | $R_1 = R_{50} = 5$, $R_i = 0$ for $i \neq 1, 50$ | -0.01845 | 2.97845 | 0.02195 | 2.98016 |
| 90 | 70 | $R_1 = 20$, $R_i = 0$ for $i \neq 1$ | -0.00043 | 2.96448 | 0.03964 | 2.99355 |
| | | $R_{70} = 20$, $R_i = 0$ for $i \neq 70$ | -0.02444 | 2.97135 | 0.01037 | 2.99736 |
| | | $R_1 = R_{70} = 10$, $R_i = 0$ for $i \neq 1, 70$ | -0.01778 | 2.97075 | 0.01542 | 2.99711 |
| 90 | 80 | $R_1 = 10$, $R_i = 0$ for $i \neq 1$ | -0.00031 | 2.98708 | 0.03220 | 3.01821 |
| | | $R_{80} = 10$, $R_i = 0$ for $i \neq 80$ | -0.01509 | 2.98709 | 0.01108 | 3.01350 |
| | | $R_1 = R_{80} = 5$, $R_i = 0$ for $i \neq 1, 80$ | -0.00965 | 2.98694 | 0.01716 | 3.01474 |

$$
\begin{aligned}
&= \sum_{i=2}^{m-1}(m-i) = \frac{1}{2}(m-2)(m-1), \\
E(W_2) &= (m-2)\sum_{i=2}^{m} E(G_i) \\
&= (m-2)(m-1),
\end{aligned}
$$

and consequently

$$
E(T) = E\left(\frac{W_1}{W_2}\right) \approx \frac{E(W_1)}{E(W_2)} = \frac{1}{2}. \tag{10.2.6}
$$

Now we write $W_1$ and $W_2$ in the form of linear combinations of progressive Type-II censored order statistics as

$$
\begin{aligned}
W_1 &= \sum_{i=2}^{m-1}(m-i)\left[\frac{X_{i:m:n} - X_{i-1:m:n}}{\mu_{i:m:n} - \mu_{i-1:m:n}}\right] = \sum_{i=1}^{m} a_i X_{i:m:n}, \\
W_2 &= (m-2)\sum_{i=2}^{m}\left[\frac{X_{i:m:n} - X_{i-1:m:n}}{\mu_{i:m:n} - \mu_{i-1:m:n}}\right] = \sum_{i=1}^{m} b_i X_{i:m:n},
\end{aligned}
$$

where

$$
\begin{aligned}
a_1 &= \frac{-(m-2)}{\mu_{2:m:n} - \mu_{1:m:n}}, \\
a_i &= \frac{(m-i)}{\mu_{i:m:n} - \mu_{i-1:m:n}} - \frac{(m-i-1)}{\mu_{i+1:m:n} - \mu_{i:m:n}}, \qquad i = 2, 3, \ldots, m-1, \\
a_m &= 0, \\
b_1 &= \frac{-(m-2)}{\mu_{2:m:n} - \mu_{1:m:n}}, \\
b_i &= (m-2)\left[\frac{1}{\mu_{i:m:n} - \mu_{i-1:m:n}} - \frac{1}{\mu_{i+1:m:n} - \mu_{i:m:n}}\right], \qquad i = 2, 3, \ldots, m-1, \\
b_m &= \frac{(m-2)}{\mu_{m:m:n} - \mu_{m-1:m:n}}.
\end{aligned}
$$

The variances and covariance of $W_1$ and $W_2$ are then given by

$$
\begin{aligned}
Var(W_1) &= \sum_{i=1}^{m} a_i^2 Var(X_{i:m:n}) + 2\sum_{i=1}^{m-1}\sum_{j=i+1}^{m} a_i a_j Cov(X_{i:m:n}, X_{j:m:n}), \\
Var(W_2) &= \sum_{i=1}^{m} b_i^2 Var(X_{i:m:n}) + 2\sum_{i=1}^{m-1}\sum_{j=i+1}^{m} b_i b_j Cov(X_{i:m:n}, X_{j:m:n}), \\
\text{and } Cov(W_1, W_2) &= \sum_{i=1}^{m}\sum_{j=1}^{m} a_i b_j Cov(X_{i:m:n}, X_{j:m:n}).
\end{aligned}
$$

The single and product moments of progressively Type-II right censored order statistics occurring in the above expressions may be obtained by the first-order approximations presented earlier in (9.3.2) - (9.3.8). Then, the variance of $T$ can be approximated by [Kendall and Stuart (1969) p. 232]

$$Var(T) \approx \left[\frac{E(W_1)}{E(W_2)}\right]^2 \left\{\frac{Var(W_1)}{[E(W_1)]^2} + \frac{Var(W_2)}{[E(W_2)]^2} - \frac{Cov(W_1, W_2)}{E(W_1)E(W_2)}\right\}. \qquad (10.2.7)$$

# 10.3 Power Function Approximation and Simulation Results

## 10.3.1 Approximation of Power Function

The asymptotic power function of the test is given by

$$\Pr(T \geq c_1 | H_1) + \Pr(T \leq c_2 | H_1), \qquad (10.3.1)$$

where $c_1 = \frac{1}{2} + z_{(\alpha/2)}\sqrt{Var(T)}$ and $c_2 = \frac{1}{2} - z_{(\alpha/2)}\sqrt{Var(T)}$ with $Var(T)$ as given by (10.2.7), and $z_{(\alpha/2)}$ is the upper $\alpha/2$ percentage point of the standard normal distribution.

To compute the power under different alternatives, we therefore need to compute probabilities of the form $\Pr(T \geq c)$, with $c$ being some constant. Since $(m-2)\sum_{i=2}^{m} G_i$ is a positive quantity, we may write

$$\Pr(T \geq c) = \Pr\left[\frac{\sum_{i=2}^{m-1}(m-i)G_i}{(m-2)\sum_{i=2}^{m} G_i} \geq c\right] = \Pr(L \geq 0), \qquad (10.3.2)$$

where

$$L = \sum_{i=2}^{m-1}(m-i)G_i - c(m-2)\sum_{i=2}^{m} G_i = \sum_{i=1}^{m} e_i X_{i:m:n},$$

where

$$e_1 = \frac{(c-1)(m-2)}{\mu_{2:m:n} - \mu_{1:m:n}},$$

$$\begin{aligned} e_i &= \frac{(m-i)-c(m-2)}{\mu_{i:m:n}-\mu_{i-1:m:n}} - \frac{(m-i-1)-c(m-2)}{\mu_{i+1:m:n}-\mu_{i:m:n}}, \qquad i=2,3,\ldots,m-1, \\ e_m &= -\frac{c(m-2)}{\mu_{m:m:n}-\mu_{m-1:m:n}}. \end{aligned}$$

Once again, we may write the test statistic as a ratio of two random variables $W_1$ and $W_2$, and determine as before

$$E(T) \;=\; E\left[\frac{W_1}{W_2}\right] \approx \frac{E(W_1)}{E(W_2)}. \tag{10.3.3}$$

But the expected value of the spacing $G_i$ under the alternative hypothesis is no longer 1, since the denominator $E(S_i)$ is computed under the null hypothesis. The variance of $T$ may also be obtained in a similar fashion from (10.2.7). Then, under the alternative model considered, after computing $E(T)$ and $Var(T)$ [or that of $E(L)$ and $Var(L)$], we may use normal approximation to compute the corresponding power from (10.3.1) and (10.3.2). These are denoted in the tables by App($W$) and App($L$), respectively. However, theoretical justification of the normal approximation to the null and non-null distributions of the test statistic still needs to be given.

### 10.3.2 Monte Carlo Power Comparison

In order to assess the power properties of the test statistic $T$, a Monte Carlo simulation study was conducted to estimate the power values under different alternative distributions. For the case of testing the goodness-of-fit of the normal distribution, the following lifetime distributions were used as alternatives in order to demonstrate the power performance of the test statistic:

1. Student-$t$ distribution with degrees of freedom 2.0, 3.0 and 4.0; the corresponding p.d.f. with $\nu$ degrees of freedom is

$$f(y) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\pi\nu}\,\Gamma\left(\frac{\nu}{2}\right)}\left(1+\frac{y^2}{\nu}\right)^{-\frac{\nu+1}{2}} \qquad \text{for } -\infty < y < \infty;$$

2. Chi-square distribution with degrees of freedom 1.0, 2.0 and 4.0; the corresponding p.d.f. with $\nu$ degrees of freedom is

$$f(y) = \frac{1}{2^{\nu/2}\Gamma(\nu/2)} y^{\frac{\nu-2}{2}} \exp\left(-\frac{y}{2}\right) \qquad \text{for } y > 0;$$

3. Lognormal distribution with shape parameter 0.5, 1.0 and 2.0; the corresponding p.d.f. with $\alpha$ as the shape parameter is

$$f(y) = \frac{1}{\sqrt{2\pi}y\alpha} \exp\left[-\frac{(\ln y)^2}{2\alpha^2}\right] \qquad \text{for } y > 0.$$

In the case of testing the goodness-of-fit of the extreme-value distribution, the following lifetime distributions were used in the Monte Carlo simulations:

1. Gamma distribution with shape parameter 0.75, 2.0 and 4.0; the corresponding p.d.f. with $\alpha$ as the shape parameter is

$$f(y) = \frac{1}{\Gamma(\alpha)} y^{\alpha-1} \exp(-y) \qquad \text{for } y > 0;$$

2. Log-gamma distribution with shape parameter 4.0, 8.0 and $\infty$ (normal distribution); the corresponding p.d.f. with $\kappa$ as the shape parameter is

$$f(y) = \frac{\kappa^{\kappa-\frac{1}{2}}}{\Gamma(\kappa)} \exp\left[\sqrt{\kappa}y - \kappa \exp\left(\frac{y}{\sqrt{\kappa}}\right)\right] \qquad \text{for } -\infty < y < \infty;$$

3. Lognormal distribution with shape parameter 0.5, 1.0 and 2.0; the corresponding p.d.f. is as given above.

For a detailed discussion on various properties of these distributions, one may refer to Johnson, Kotz and Balakrishnan (1994, 1995). For different choices of sample sizes and progressive censoring schemes, we generated 100,000 sets of data, in order to obtain the estimated power values. These values are presented in Tables 10.2-10.7 for $n = 20$ ($m = 8,12,16$), 40 ($m = 10,20,30$) and 60 ($m = 20,40,50$) with three different progressive censoring schemes. The different censoring schemes (c.s.) used in the simulation study are listed in Table 9.1.

The power values presented in Tables 10.2-10.7 clearly show that the test proposed performs very well for all the alternatives considered. The power increases with $m$ for a fixed $n$, and also increases as $n$ increases. We also calculated the power values of $T$ from the normal approximation using the two methods and found them to be close to the simulated power values, especially for large values of $m$. These approximate power values are also presented in Tables 10.2-10.7 for the purpose of comparison.

Table 10.2: Monte Carlo power estimates for Student-$t$ distribution at 10% level of significance for test of fit of normal distribution

| | t(2) | | | t(3) | | | t(4) | | |
|---|---|---|---|---|---|---|---|---|---|
| c.s. | Sim. | App(L) | App(W) | Sim. | App(L) | App(W) | Sim. | App(L) | App(W) |
| [1] | 0.36182 | 0.33007 | 0.29689 | 0.25674 | 0.23354 | 0.20258 | 0.20558 | 0.19498 | 0.16481 |
| [2] | 0.35112 | 0.35034 | 0.27855 | 0.24605 | 0.24269 | 0.19019 | 0.19650 | 0.20466 | 0.15529 |
| [3] | 0.37052 | 0.35339 | 0.29416 | 0.25432 | 0.23476 | 0.19443 | 0.20184 | 0.19445 | 0.15629 |
| [4] | 0.42236 | 0.37091 | 0.34914 | 0.30746 | 0.26086 | 0.23832 | 0.24898 | 0.21269 | 0.19034 |
| [5] | 0.45394 | 0.42118 | 0.35759 | 0.31899 | 0.26483 | 0.23192 | 0.24840 | 0.20837 | 0.18036 |
| [6] | 0.45720 | 0.43918 | 0.38606 | 0.33862 | 0.29811 | 0.26669 | 0.27270 | 0.23666 | 0.21071 |
| [7] | 0.46502 | 0.40231 | 0.38617 | 0.34084 | 0.28310 | 0.26503 | 0.27548 | 0.22798 | 0.20979 |
| [8] | 0.46322 | 0.42261 | 0.37343 | 0.33087 | 0.26976 | 0.24516 | 0.26068 | 0.21072 | 0.19063 |
| [9] | 0.47112 | 0.41706 | 0.39577 | 0.34787 | 0.29463 | 0.27463 | 0.28102 | 0.23697 | 0.21768 |
| [10] | 0.44460 | 0.39384 | 0.37156 | 0.30979 | 0.26103 | 0.24252 | 0.24320 | 0.20802 | 0.18823 |
| [11] | 0.45140 | 0.44661 | 0.36162 | 0.31701 | 0.28782 | 0.23858 | 0.24642 | 0.22779 | 0.18558 |
| [12] | 0.53590 | 0.52789 | 0.44410 | 0.37034 | 0.32153 | 0.28130 | 0.28188 | 0.23900 | 0.20933 |
| [13] | 0.52108 | 0.44881 | 0.43910 | 0.37975 | 0.30782 | 0.29716 | 0.30178 | 0.24144 | 0.22983 |
| [14] | 0.67392 | 0.62770 | 0.56273 | 0.48203 | 0.39542 | 0.36169 | 0.37288 | 0.28131 | 0.26156 |
| [15] | 0.65796 | 0.65011 | 0.58425 | 0.48218 | 0.42932 | 0.39251 | 0.37966 | 0.31033 | 0.28924 |
| [16] | 0.56186 | 0.48670 | 0.48016 | 0.41520 | 0.33952 | 0.33129 | 0.32808 | 0.26509 | 0.25623 |
| [17] | 0.68862 | 0.63458 | 0.58409 | 0.49981 | 0.40780 | 0.38234 | 0.38936 | 0.29082 | 0.27703 |
| [18] | 0.61546 | 0.56633 | 0.52077 | 0.44388 | 0.36015 | 0.34066 | 0.34614 | 0.26324 | 0.25162 |
| [19] | 0.54296 | 0.46739 | 0.45895 | 0.39040 | 0.31426 | 0.30646 | 0.30730 | 0.24338 | 0.23386 |
| [20] | 0.71536 | 0.66974 | 0.60626 | 0.52079 | 0.43601 | 0.39722 | 0.40210 | 0.30926 | 0.28646 |
| [21] | 0.78246 | 0.73296 | 0.68981 | 0.58261 | 0.50132 | 0.46381 | 0.44958 | 0.35369 | 0.33186 |
| [22] | 0.59794 | 0.52244 | 0.51845 | 0.44462 | 0.36465 | 0.35934 | 0.35274 | 0.28195 | 0.27590 |
| [23] | 0.83142 | 0.75795 | 0.73505 | 0.63514 | 0.53308 | 0.50879 | 0.49288 | 0.37859 | 0.36448 |
| [24] | 0.68980 | 0.67822 | 0.63526 | 0.52629 | 0.47463 | 0.45011 | 0.41674 | 0.35115 | 0.33714 |
| [25] | 0.61642 | 0.54158 | 0.53816 | 0.46426 | 0.38160 | 0.37689 | 0.36388 | 0.29520 | 0.28994 |
| [26] | 0.76196 | 0.70119 | 0.66710 | 0.57538 | 0.47744 | 0.45632 | 0.44468 | 0.34183 | 0.33068 |
| [27] | 0.67292 | 0.62005 | 0.58385 | 0.49977 | 0.41082 | 0.39441 | 0.38712 | 0.30003 | 0.29118 |

Table 10.3: Monte Carlo power estimates for chi-square distribution at 10% level of significance for test of fit of normal distribution

| | Chi-square(1) | | | Chi-square(2) | | | Chi-square(4) | | |
|---|---|---|---|---|---|---|---|---|---|
| c.s. | Sim. | App(L) | App(W) | Sim. | App(L) | App(W) | Sim. | App(L) | App(W) |
| [1] | 0.80512 | 0.82981 | 0.81096 | 0.59896 | 0.60996 | 0.56038 | 0.39942 | 0.38819 | 0.35126 |
| [2] | 0.54516 | 0.67314 | 0.59744 | 0.25820 | 0.32736 | 0.29005 | 0.12794 | 0.18670 | 0.15957 |
| [3] | 0.61986 | 0.73288 | 0.67177 | 0.33504 | 0.40570 | 0.36398 | 0.17204 | 0.22609 | 0.20164 |
| [4] | 0.92440 | 0.89303 | 0.92540 | 0.75780 | 0.72483 | 0.71476 | 0.53284 | 0.49149 | 0.46603 |
| [5] | 0.79268 | 0.89303 | 0.82786 | 0.46620 | 0.72483 | 0.49282 | 0.23102 | 0.49149 | 0.26175 |
| [6] | 0.95644 | 0.92260 | 0.96698 | 0.79806 | 0.76037 | 0.76132 | 0.54216 | 0.49331 | 0.47209 |
| [7] | 0.97210 | 0.92905 | 0.97110 | 0.85402 | 0.80000 | 0.81603 | 0.63080 | 0.57393 | 0.55955 |
| [8] | 0.92848 | 0.90653 | 0.94286 | 0.69120 | 0.70099 | 0.69904 | 0.39650 | 0.42494 | 0.41230 |
| [9] | 0.98096 | 0.93908 | 0.98228 | 0.87546 | 0.81996 | 0.84320 | 0.64486 | 0.58824 | 0.57603 |
| [10] | 0.91072 | 0.89045 | 0.91433 | 0.74758 | 0.73093 | 0.71242 | 0.53374 | 0.51051 | 0.47646 |
| [11] | 0.68458 | 0.77119 | 0.72686 | 0.32340 | 0.38537 | 0.35661 | 0.14102 | 0.19313 | 0.17630 |
| [12] | 0.84782 | 0.88017 | 0.89122 | 0.51596 | 0.58100 | 0.54749 | 0.24684 | 0.29816 | 0.27998 |
| [13] | 0.99468 | 0.96219 | 0.99460 | 0.94344 | 0.88935 | 0.92480 | 0.77666 | 0.71572 | 0.71858 |
| [14] | 0.96364 | 0.93677 | 0.97301 | 0.70968 | 0.72749 | 0.73331 | 0.35994 | 0.40670 | 0.39917 |
| [15] | 0.99568 | 0.96850 | 0.99792 | 0.91868 | 0.88624 | 0.92157 | 0.65674 | 0.64673 | 0.64481 |
| [16] | 0.99968 | 0.98587 | 0.99970 | 0.98702 | 0.95180 | 0.98165 | 0.89150 | 0.83138 | 0.85210 |
| [17] | 0.99738 | 0.98126 | 0.99859 | 0.92942 | 0.90632 | 0.93546 | 0.65458 | 0.67083 | 0.67338 |
| [18] | 0.99848 | 0.98359 | 0.99917 | 0.95444 | 0.92726 | 0.95760 | 0.74058 | 0.73725 | 0.74572 |
| [19] | 0.99584 | 0.96516 | 0.99584 | 0.95248 | 0.90039 | 0.93703 | 0.80066 | 0.74084 | 0.74711 |
| [20] | 0.96374 | 0.93697 | 0.97314 | 0.68100 | 0.70397 | 0.70687 | 0.31574 | 0.36339 | 0.35727 |
| [21] | 0.99610 | 0.97060 | 0.99875 | 0.89920 | 0.87713 | 0.91242 | 0.56954 | 0.60490 | 0.59969 |
| [22] | 0.99998 | 0.99521 | 0.99999 | 0.99858 | 0.98216 | 0.99717 | 0.95994 | 0.91593 | 0.94099 |
| [23] | 0.99986 | 0.99335 | 0.99991 | 0.97324 | 0.95461 | 0.97718 | 0.75170 | 0.76054 | 0.77041 |
| [24] | 0.99998 | 0.99610 | 1.00000 | 0.99852 | 0.98514 | 0.99805 | 0.94450 | 0.91012 | 0.93347 |
| [25] | 1.00000 | 0.99818 | 1.00000 | 0.99952 | 0.99230 | 0.99943 | 0.98256 | 0.95169 | 0.97224 |
| [26] | 1.00000 | 0.99798 | 1.00000 | 0.99634 | 0.98604 | 0.99698 | 0.90856 | 0.89421 | 0.91365 |
| [27] | 1.00000 | 0.99817 | 1.00000 | 0.99812 | 0.98925 | 0.99838 | 0.94134 | 0.92141 | 0.94217 |

Table 10.4: Monte Carlo power estimates for lognormal distribution at 10% level of significance for test of fit of normal distribution

| | Lognormal(0.5) | | | Lognormal(1.0) | | | Lognormal(2.0) | | |
|---|---|---|---|---|---|---|---|---|---|
| c.s. | Sim. | App(L) | App(W) | Sim. | App(L) | App(W) | Sim. | App(L) | App(W) |
| [1] | 0.38426 | 0.37777 | 0.33535 | 0.69936 | 0.74203 | 0.66926 | 0.92406 | 0.94498 | 0.94636 |
| [2] | 0.09892 | 0.16052 | 0.12930 | 0.19324 | 0.27071 | 0.23636 | 0.47260 | 0.62272 | 0.53115 |
| [3] | 0.13506 | 0.18905 | 0.16384 | 0.29748 | 0.38488 | 0.33798 | 0.63436 | 0.77699 | 0.69378 |
| [4] | 0.51460 | 0.48756 | 0.45349 | 0.85640 | 0.83059 | 0.82761 | 0.98624 | 0.96130 | 0.99063 |
| [5] | 0.17076 | 0.22057 | 0.20754 | 0.43378 | 0.50956 | 0.47654 | 0.83108 | 0.86382 | 0.86844 |
| [6] | 0.48356 | 0.44838 | 0.42111 | 0.83834 | 0.81847 | 0.81433 | 0.98288 | 0.96123 | 0.99316 |
| [7] | 0.61386 | 0.57126 | 0.54886 | 0.93052 | 0.87593 | 0.90915 | 0.99766 | 0.96890 | 0.99828 |
| [8] | 0.32558 | 0.36878 | 0.35547 | 0.74366 | 0.75935 | 0.76140 | 0.97862 | 0.94244 | 0.98669 |
| [9] | 0.61430 | 0.57214 | 0.55134 | 0.93618 | 0.87994 | 0.91775 | 0.99816 | 0.97010 | 0.99891 |
| [10] | 0.50690 | 0.49250 | 0.44942 | 0.83022 | 0.82413 | 0.80459 | 0.97572 | 0.96031 | 0.98477 |
| [11] | 0.09728 | 0.15070 | 0.12811 | 0.19200 | 0.25749 | 0.23704 | 0.49088 | 0.60190 | 0.54578 |
| [12] | 0.16164 | 0.21293 | 0.19667 | 0.39096 | 0.47492 | 0.43607 | 0.78034 | 0.84244 | 0.83118 |
| [13] | 0.75602 | 0.69944 | 0.69515 | 0.97906 | 0.91993 | 0.96786 | 0.99980 | 0.97671 | 0.99984 |
| [14] | 0.23476 | 0.28525 | 0.28017 | 0.61384 | 0.66056 | 0.65674 | 0.95834 | 0.92371 | 0.97119 |
| [15] | 0.53162 | 0.53332 | 0.52290 | 0.91300 | 0.87569 | 0.91632 | 0.99808 | 0.96377 | 0.99890 |
| [16] | 0.87624 | 0.81217 | 0.83188 | 0.99746 | 0.95509 | 0.99452 | 1.00000 | 0.98281 | 1.00000 |
| [17] | 0.52614 | 0.56693 | 0.56285 | 0.94062 | 0.91142 | 0.95051 | 0.99968 | 0.97607 | 0.99989 |
| [18] | 0.65344 | 0.66975 | 0.67097 | 0.97618 | 0.93806 | 0.97884 | 0.99992 | 0.97897 | 0.99998 |
| [19] | 0.77786 | 0.72062 | 0.71936 | 0.98184 | 0.92490 | 0.97251 | 0.99982 | 0.97776 | 0.99988 |
| [20] | 0.17810 | 0.22744 | 0.22362 | 0.48548 | 0.54438 | 0.53468 | 0.90704 | 0.88714 | 0.92752 |
| [21] | 0.38742 | 0.43954 | 0.42970 | 0.83180 | 0.82918 | 0.85784 | 0.99452 | 0.95879 | 0.99824 |
| [22] | 0.94912 | 0.89545 | 0.92459 | 0.99988 | 0.97525 | 0.99931 | 1.00000 | 0.98711 | 1.00000 |
| [23] | 0.58346 | 0.62501 | 0.62517 | 0.96962 | 0.94558 | 0.97576 | 0.99998 | 0.98679 | 0.99999 |
| [24] | 0.89376 | 0.85419 | 0.87572 | 0.99924 | 0.97376 | 0.99844 | 1.00000 | 0.98477 | 1.00000 |
| [25] | 0.97574 | 0.93430 | 0.96112 | 1.00000 | 0.98470 | 0.99989 | 1.00000 | 0.98938 | 1.00000 |
| [26] | 0.82762 | 0.83220 | 0.84825 | 0.99870 | 0.98460 | 0.99889 | 1.00000 | 0.99218 | 1.00000 |
| [27] | 0.90162 | 0.88427 | 0.90610 | 0.99962 | 0.98709 | 0.99966 | 1.00000 | 0.99139 | 1.00000 |

Table 10.5: Monte Carlo power estimates for gamma distribution at 10% level of significance for test of fit of extreme-value distribution

| | Gamma(0.75) | | | Gamma(2.0) | | | Gamma(4.0) | | |
|---|---|---|---|---|---|---|---|---|---|
| c.s. | Sim. | App(L) | App(W) | Sim. | App(L) | App(W) | Sim. | App(L) | App(W) |
| [1] | 0.81940 | 0.82641 | 0.82551 | 0.62335 | 0.63813 | 0.59828 | 0.50363 | 0.51578 | 0.47544 |
| [2] | 0.43400 | 0.53409 | 0.47961 | 0.18545 | 0.24976 | 0.22830 | 0.12490 | 0.18894 | 0.16785 |
| [3] | 0.54130 | 0.63941 | 0.58899 | 0.27202 | 0.34153 | 0.31391 | 0.18370 | 0.24745 | 0.22895 |
| [4] | 0.93650 | 0.89395 | 0.93931 | 0.79150 | 0.75969 | 0.76551 | 0.67078 | 0.64845 | 0.63439 |
| [5] | 0.72495 | 0.76146 | 0.76187 | 0.39070 | 0.45414 | 0.43566 | 0.26248 | 0.32198 | 0.31100 |
| [6] | 0.95458 | 0.91229 | 0.96531 | 0.79583 | 0.76574 | 0.77634 | 0.65583 | 0.63423 | 0.62254 |
| [7] | 0.97870 | 0.93037 | 0.97934 | 0.88380 | 0.83253 | 0.86407 | 0.77958 | 0.73697 | 0.74517 |
| [8] | 0.91653 | 0.88847 | 0.93164 | 0.66858 | 0.69064 | 0.69220 | 0.49890 | 0.54297 | 0.53297 |
| [9] | 0.98423 | 0.93692 | 0.98604 | 0.89400 | 0.84183 | 0.87790 | 0.78603 | 0.74374 | 0.75475 |
| [10] | 0.97870 | 0.89083 | 0.92991 | 0.88380 | 0.76138 | 0.75950 | 0.77958 | 0.65675 | 0.63505 |
| [11] | 0.53852 | 0.61606 | 0.57904 | 0.20798 | 0.26673 | 0.25295 | 0.13124 | 0.18928 | 0.17589 |
| [12] | 0.76750 | 0.80574 | 0.80991 | 0.39780 | 0.47202 | 0.44612 | 0.25790 | 0.32451 | 0.30899 |
| [13] | 0.99670 | 0.96356 | 0.99709 | 0.96300 | 0.91223 | 0.95549 | 0.90322 | 0.85263 | 0.88482 |
| [14] | 0.93138 | 0.90349 | 0.94363 | 0.60438 | 0.64612 | 0.64595 | 0.40476 | 0.46931 | 0.46272 |
| [15] | 0.99396 | 0.96309 | 0.99697 | 0.90404 | 0.87755 | 0.91505 | 0.76726 | 0.76729 | 0.78234 |
| [16] | 0.99990 | 0.98634 | 0.99990 | 0.99386 | 0.96466 | 0.99244 | 0.97216 | 0.93272 | 0.96552 |
| [17] | 0.99638 | 0.97771 | 0.99795 | 0.91762 | 0.89861 | 0.93020 | 0.77818 | 0.79107 | 0.80775 |
| [18] | 0.99850 | 0.98247 | 0.99925 | 0.96010 | 0.93265 | 0.96549 | 0.86972 | 0.85938 | 0.88548 |
| [19] | 0.99762 | 0.96652 | 0.99787 | 0.96984 | 0.92109 | 0.96401 | 0.91790 | 0.86776 | 0.90237 |
| [20] | 0.91270 | 0.88975 | 0.92712 | 0.51954 | 0.57059 | 0.56551 | 0.32074 | 0.38645 | 0.38139 |
| [21] | 0.99158 | 0.96034 | 0.99600 | 0.84132 | 0.83596 | 0.86641 | 0.64804 | 0.68909 | 0.69334 |
| [22] | 1.00000 | 0.99535 | 1.00000 | 0.99948 | 0.98787 | 0.99928 | 0.99478 | 0.97461 | 0.99351 |
| [23] | 0.99956 | 0.99188 | 0.99979 | 0.96136 | 0.94385 | 0.96920 | 0.85384 | 0.85613 | 0.87657 |
| [24] | 1.00000 | 0.99604 | 1.00000 | 0.99900 | 0.98766 | 0.99898 | 0.98780 | 0.96908 | 0.98879 |
| [25] | 1.00000 | 0.99820 | 1.00000 | 0.99993 | 0.99508 | 0.99991 | 0.99860 | 0.98859 | 0.99840 |
| [26] | 1.00000 | 0.99798 | 1.00000 | 0.99713 | 0.98747 | 0.99789 | 0.97508 | 0.96101 | 0.98002 |
| [27] | 1.00000 | 0.99827 | 1.00000 | 0.99920 | 0.99189 | 0.99932 | 0.98990 | 0.97632 | 0.99169 |

Table 10.6: Monte Carlo power estimates for log-gamma distribution at 10% level of significance for test of fit of extreme-value distribution

| | Log-gamma(4.0) | | | Log-gamma(8.0) | | | Log-gamma($\infty$) | | |
|---|---|---|---|---|---|---|---|---|---|
| c.s. | Sim. | App(L) | App(W) | Sim. | App(L) | App(W) | Sim. | App(L) | App(W) |
| [1] | 0.12992 | 0.14753 | 0.12930 | 0.15016 | 0.16495 | 0.14926 | 0.21988 | 0.22813 | 0.21348 |
| [2] | 0.08488 | 0.14347 | 0.09717 | 0.08242 | 0.14233 | 0.09871 | 0.08102 | 0.14369 | 0.10627 |
| [3] | 0.08915 | 0.13624 | 0.10202 | 0.09008 | 0.13798 | 0.10659 | 0.09598 | 0.14859 | 0.12340 |
| [4] | 0.14715 | 0.15664 | 0.14506 | 0.17895 | 0.18517 | 0.17619 | 0.29036 | 0.28519 | 0.27602 |
| [5] | 0.09653 | 0.13060 | 0.10579 | 0.09388 | 0.13550 | 0.11382 | 0.11078 | 0.15703 | 0.14234 |
| [6] | 0.14280 | 0.15230 | 0.13505 | 0.16658 | 0.17529 | 0.16181 | 0.26462 | 0.25879 | 0.25101 |
| [7] | 0.16015 | 0.16666 | 0.15813 | 0.20355 | 0.20480 | 0.19894 | 0.34518 | 0.33550 | 0.32927 |
| [8] | 0.10783 | 0.13866 | 0.12482 | 0.12023 | 0.15592 | 0.14543 | 0.17540 | 0.21922 | 0.21409 |
| [9] | 0.15578 | 0.16418 | 0.15458 | 0.19730 | 0.20078 | 0.19447 | 0.33818 | 0.32891 | 0.32397 |
| [10] | 0.15905 | 0.16639 | 0.15510 | 0.19630 | 0.19906 | 0.18889 | 0.30804 | 0.30595 | 0.29114 |
| [11] | 0.08765 | 0.13607 | 0.09712 | 0.08103 | 0.13509 | 0.09861 | 0.08168 | 0.13685 | 0.10621 |
| [12] | 0.09485 | 0.13405 | 0.10777 | 0.09348 | 0.13906 | 0.11552 | 0.11036 | 0.15941 | 0.14198 |
| [13] | 0.21098 | 0.20820 | 0.20424 | 0.27848 | 0.27348 | 0.26993 | 0.49028 | 0.46678 | 0.45987 |
| [14] | 0.09648 | 0.12919 | 0.11580 | 0.10192 | 0.14128 | 0.13084 | 0.13622 | 0.18574 | 0.18098 |
| [15] | 0.14034 | 0.16357 | 0.15474 | 0.17280 | 0.20041 | 0.19467 | 0.29354 | 0.32404 | 0.32116 |
| [16] | 0.25086 | 0.24722 | 0.24515 | 0.34546 | 0.33841 | 0.33663 | 0.61568 | 0.58666 | 0.58536 |
| [17] | 0.12878 | 0.16566 | 0.16082 | 0.15724 | 0.20690 | 0.20436 | 0.28088 | 0.34145 | 0.33981 |
| [18] | 0.15302 | 0.18950 | 0.18620 | 0.19876 | 0.24612 | 0.24442 | 0.36844 | 0.42184 | 0.41920 |
| [19] | 0.22598 | 0.22284 | 0.21903 | 0.30422 | 0.29523 | 0.29104 | 0.52468 | 0.49891 | 0.49151 |
| [20] | 0.09284 | 0.12393 | 0.10827 | 0.09400 | 0.13135 | 0.11837 | 0.11328 | 0.16034 | 0.15319 |
| [21] | 0.11602 | 0.15207 | 0.14251 | 0.13392 | 0.17986 | 0.17311 | 0.21504 | 0.27251 | 0.26909 |
| [22] | 0.32184 | 0.31457 | 0.31342 | 0.45106 | 0.43958 | 0.43808 | 0.75640 | 0.72406 | 0.72923 |
| [23] | 0.13518 | 0.17818 | 0.17516 | 0.17530 | 0.22904 | 0.22775 | 0.32114 | 0.38786 | 0.38655 |
| [24] | 0.25452 | 0.26622 | 0.26442 | 0.35654 | 0.37015 | 0.36926 | 0.64580 | 0.64162 | 0.64265 |
| [25] | 0.35428 | 0.35332 | 0.35254 | 0.51000 | 0.49699 | 0.49616 | 0.81836 | 0.79365 | 0.80173 |
| [26] | 0.18964 | 0.24506 | 0.24389 | 0.27720 | 0.33668 | 0.33599 | 0.52680 | 0.58584 | 0.58574 |
| [27] | 0.23060 | 0.28165 | 0.28075 | 0.33913 | 0.39247 | 0.39160 | 0.62754 | 0.66873 | 0.67089 |

Table 10.7: Monte Carlo power estimates for lognormal distribution at 10% level of significance for test of fit of extreme-value distribution

| | Lognormal(0.5) | | | Lognormal(1.0) | | | Lognormal(2.0) | | |
|---|---|---|---|---|---|---|---|---|---|
| c.s. | Sim. | App(L) | App(W) | Sim. | App(L) | App(W) | Sim. | App(L) | App(W) |
| [1] | 0.60695 | 0.63444 | 0.58463 | 0.82714 | 0.84726 | 0.83187 | 0.95726 | 0.96009 | 0.98213 |
| [2] | 0.13961 | 0.20719 | 0.18576 | 0.26284 | 0.35792 | 0.32070 | 0.54652 | 0.69485 | 0.61077 |
| [3] | 0.21822 | 0.28899 | 0.26560 | 0.41056 | 0.52027 | 0.46507 | 0.71448 | 0.83018 | 0.77731 |
| [4] | 0.77985 | 0.75740 | 0.75650 | 0.94628 | 0.90222 | 0.94470 | 0.99495 | 0.97130 | 0.99846 |
| [5] | 0.31721 | 0.38804 | 0.37154 | 0.59819 | 0.66700 | 0.64642 | 0.89499 | 0.89684 | 0.92694 |
| [6] | 0.75408 | 0.73826 | 0.73573 | 0.93725 | 0.89840 | 0.94403 | 0.99409 | 0.97233 | 0.99939 |
| [7] | 0.87791 | 0.82755 | 0.85708 | 0.98431 | 0.92868 | 0.98143 | 0.99938 | 0.97656 | 0.99986 |
| [8] | 0.61011 | 0.65206 | 0.64675 | 0.88945 | 0.86593 | 0.90886 | 0.99206 | 0.95549 | 0.99667 |
| [9] | 0.88219 | 0.83119 | 0.86402 | 0.98629 | 0.93067 | 0.98481 | 0.99953 | 0.97726 | 0.99993 |
| [10] | 0.76315 | 0.75453 | 0.74311 | 0.92868 | 0.89866 | 0.93094 | 0.99041 | 0.97107 | 0.99704 |
| [11] | 0.13905 | 0.20038 | 0.18676 | 0.27118 | 0.34827 | 0.32569 | 0.57207 | 0.67669 | 0.62913 |
| [12] | 0.29195 | 0.36702 | 0.34673 | 0.54439 | 0.63548 | 0.60244 | 0.85584 | 0.88286 | 0.90357 |
| [13] | 0.95784 | 0.90234 | 0.94862 | 0.99749 | 0.95423 | 0.99684 | 0.99998 | 0.98251 | 1.00000 |
| [14] | 0.47568 | 0.53957 | 0.53285 | 0.80271 | 0.81064 | 0.83688 | 0.98343 | 0.94352 | 0.99040 |
| [15] | 0.84782 | 0.83318 | 0.86294 | 0.98173 | 0.93620 | 0.98660 | 0.99948 | 0.97246 | 0.99993 |
| [16] | 0.99258 | 0.95503 | 0.99003 | 0.99993 | 0.97391 | 0.99984 | 1.00000 | 0.98678 | 1.00000 |
| [17] | 0.87183 | 0.86361 | 0.89267 | 0.99153 | 0.96191 | 0.99454 | 0.99993 | 0.98128 | 0.99999 |
| [18] | 0.94263 | 0.91462 | 0.95024 | 0.99810 | 0.97117 | 0.99869 | 1.00000 | 0.98328 | 1.00000 |
| [19] | 0.96520 | 0.91074 | 0.95717 | 0.99792 | 0.95700 | 0.99754 | 0.99998 | 0.98336 | 1.00000 |
| [20] | 0.35882 | 0.42662 | 0.42021 | 0.67657 | 0.71548 | 0.72268 | 0.95455 | 0.91892 | 0.96756 |
| [21] | 0.72663 | 0.75387 | 0.76749 | 0.95240 | 0.91771 | 0.96845 | 0.99877 | 0.96908 | 0.99984 |
| [22] | 0.99937 | 0.98124 | 0.99878 | 1.00000 | 0.98528 | 1.00000 | 1.00000 | 0.98996 | 1.00000 |
| [23] | 0.91920 | 0.90949 | 0.93622 | 0.99767 | 0.98138 | 0.99854 | 1.00000 | 0.98990 | 1.00000 |
| [24] | 0.99717 | 0.97840 | 0.99705 | 0.99998 | 0.98582 | 0.99998 | 1.00000 | 0.98793 | 1.00000 |
| [25] | 0.99988 | 0.99087 | 0.99980 | 1.00000 | 0.99063 | 1.00000 | 1.00000 | 0.99156 | 1.00000 |
| [26] | 0.99825 | 0.98029 | 0.99534 | 1.00000 | 0.99448 | 0.99999 | 1.00000 | 0.99364 | 1.00000 |
| [27] | 0.99825 | 0.98751 | 0.99865 | 1.00000 | 0.99444 | 1.00000 | 1.00000 | 0.99289 | 1.00000 |

## 10.4 Illustrative Example

In this section, we present an example to illustrate the use of the statistic $T$ for testing the validity of the extreme-value distribution for an observed progressively Type-II right censored sample.

**Example 10.1:** Refer to Example 6.2 and Example 9.1 wherein we considered the progressively censored sample presented by Viveros and Balakrishnan (1994). The data presented in Table 6.3 with the sample spacings $G_i$ are as follows:

Progressively censored sample presented by Viveros and Balakrishnan (1994)

| i | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| $x_{i:m:n}$ | -1.6608 | -0.2485 | -0.0409 | 0.2700 | 1.0224 | 1.5789 | 1.8718 | 1.9947 |
| $R_i$ | 0 | 0 | 3 | 0 | 3 | 0 | 0 | 5 |
| $G_i$ | — | 1.9620 | 0.4790 | 0.8275 | 2.5822 | 1.7598 | 1.0929 | 0.5118 |

Nelson (1982) and Viveros and Balakrishnan (1994) considered a Weibull model for the original time scale data so that the log-times to breakdown can be treated as extreme-value observations.

In this example, we have $n = 19$, $m = 8$. The test statistic computed from (10.2.3) is

$$T = \frac{\sum\limits_{i=2}^{m-1}(m-i)G_i}{(m-2)\sum\limits_{i=2}^{m} G_i} = \frac{29.83637}{55.29182} = 0.53962.$$

From Eqs. (10.2.6) and (10.2.7), the null distribution of $T$ in this example can be approximated by a normal distribution with mean 0.5 and variance 0.013597. So, the $p$-value of the test can be computed as

$$2\left[1-\Phi\left(\frac{0.53962-0.5}{\sqrt{0.013597}}\right)\right] = 2 \times 0.367025 = 0.73405.$$

Based on this large $p$-value, we fail to reject the null hypothesis that the observed progressively Type-II censored sample is from an extreme-value distribution. Incidentally, this is consistent with the informal findings of Nelson (1982) and Viveros and Balakrishnan (1994).

## 10.5 Conclusions

In this chapter, we have proposed goodness-of-fit tests for the general location-scale family of distributions when the data are progressively Type-II censored. The null distribution of the test statistic has been studied, mainly through an examination of the approximate values of coefficients of skewness and kurtosis based on Monte Carlo method, and has been shown that it can be approximated closely by a normal distribution. The performance of the testing procedure for testing normal and extreme-value distributions have been evaluated. In addition, two methods for approximating the power have also been suggested.

Results of the simulation study for a wide range of sample sizes, censoring schemes and different alternatives reveal that the proposed test has good power properties in detecting departures from the normal and extreme-value distributions. Also, the suggested approximations to the power values are close to the simulated power values. It is important to mention here that the testing procedure proposed in this chapter can be extended to multi-sample situations in a straightforward manner. One may refer to Chapter 7 wherein such a generalization of the test for exponentiality has been discussed.

# Chapter 11

# Estimation for Birnbaum-Saunders Distribution

## 11.1 Introduction

The two-parameter Birnbaum-Saunders distribution was originally proposed [Birnbaum and Saunders (1969a)] as a failure time distribution for fatigue failure caused under cyclic loading. It was also assumed that the failure is due to the development and growth of a dominant crack. A more general derivation was provided by Desmond (1985) based on a biological model. Desmond (1985) also strengthened the physical justification for the use of this distribution by relaxing the assumptions made by Birnbaum and Saunders (1969a). Desmond (1986) investigated the relationship between the Birnbaum-Saunders distribution and the inverse Gaussian distribution. Some recent works on Birnbaum-Saunders distribution can be found in Chang and Tang (1993, 1994), Dupuis and Mills (1998) and Rieck (1995, 1999), and a review can be found in Johnson, Kotz and Balakrishnan (1995).

The cumulative distribution function (c.d.f.) of a two-parameter Birnbaum-Saunders random variable $T$ can be written as

$$F_T(t;\alpha,\beta) = \Phi\left[\frac{1}{\alpha}\left\{\left(\frac{t}{\beta}\right)^{\frac{1}{2}} - \left(\frac{\beta}{t}\right)^{\frac{1}{2}}\right\}\right], \qquad 0 < t < \infty, \qquad \alpha,\beta > 0, \quad (11.1.1)$$

where $\Phi(\cdot)$ is the standard normal c.d.f.. The parameters $\alpha$ and $\beta$ are the shape and the scale parameters, respectively. It is known that the density function of the Birnbaum-Saunders distribution is unimodal and although the hazard rate is not an increasing function of $T$, the average hazard rate is nearly a non-decreasing function of $t$ [Mann, Schafer and Singpurwalla (1974), p.155]. The maximum likelihood estimators (MLE's) were discussed originally by Birnbaum and Saunders (1969b) and their asymptotic distributions were obtained by Engelhardt, Bain and Wright (1981).

Although the MLE's have several optimal properties, one still needs to solve a non-linear equation in $\beta$ to obtain the solution; for this purpose, Birnbaum and Saunders (1969b) suggested some iterative schemes to solve the required non-linear equation. Also, the exact distributions of the MLE's are not available. Therefore, for constructing confidence intervals for the unknown parameters $\alpha$ and $\beta$, the asymptotic distributions of the MLE's need to be used. However, it is not known how these asymptotic confidence intervals behave in case of small sample sizes. Moreover, the conventional moment estimators also have a difficulty in that they may not always exist and even if they do, they may not be unique.

In this chapter, we first review some results for the two parameter Birnbaum-Saunders distribution. Then, we propose modified moment estimators (MME's) for $\alpha$ and $\beta$. The MME's are very easy to compute as they are explicit in terms of the sample observations. Unlike the moment estimators, MME's always exist uniquely. The asymptotic distributions of the MME's are derived which are then used to construct confidence intervals for the unknown parameters. We evaluate the performance of all these estimators through simulations. Even though the MLE's and MME's are asymptotically unbiased, these simulation results reveal that they are highly biased in case of small sample sizes. We propose a simple bias correction technique which performs quite well even for small sample sizes. Jackknife procedure is also used to reduce the bias of the MLE's and MME's and is shown to work very well in this case; but, this procedure becomes computationally quite involved in case of large sample sizes.

## 11.2 The Birnbaum-Saunders Distribution

The c.d.f. of a two-parameter Birnbaum-Saunders random variable $T$ is given by (11.1.1) and the corresponding probability density function (p.d.f.) is

$$f_T(t;\alpha,\beta) = \frac{1}{2\sqrt{2\pi}\alpha\beta}\left[\left(\frac{\beta}{t}\right)^{\frac{1}{2}} + \left(\frac{\beta}{t}\right)^{\frac{3}{2}}\right]\exp\left[-\frac{1}{2\alpha^2}\left(\frac{t}{\beta}+\frac{\beta}{t}-2\right)\right],$$
$$t > 0, \alpha, \beta > 0. \quad (11.2.1)$$

Consider the following monotone transformation

$$X = \frac{1}{2}\left[\left(\frac{T}{\beta}\right)^{\frac{1}{2}} - \left(\frac{T}{\beta}\right)^{-\frac{1}{2}}\right]$$

or

$$T = \beta\left[1 + 2X^2 + 2X(1+X^2)^{\frac{1}{2}}\right];$$

then, from (11.1.1), we know $X$ is normally distributed with mean zero and variance $\frac{1}{4}\alpha^2$. Using the above transformation, the expected value, variance, and coefficients of skewness and kurtosis can be easily obtained as

$$E(T) = \beta\left(1 + \frac{1}{2}\alpha^2\right), \quad (11.2.2)$$

$$Var(T) = (\alpha\beta)^2\left(1 + \frac{5}{4}\alpha^2\right), \quad (11.2.3)$$

$$\beta_1(T) = \frac{16\alpha^2(11\alpha^2+6)}{(5\alpha^2+4)^3}, \quad (11.2.4)$$

$$\beta_2(T) = 3 + \frac{6\alpha^2(93\alpha^2+41)}{(5\alpha^2+4)^2}. \quad (11.2.5)$$

Moreover, if $T$ has a Birnbaum-Saunders distribution with parameters $\alpha$ and $\beta$, then $T^{-1}$ also has a Birnbaum-Saunders distribution with the corresponding parameters $\alpha$ and $\beta^{-1}$, respectively [Birnbaum and Saunders (1969a)]. Therefore, we also readily have

$$E(T^{-1}) = \beta^{-1}\left(1 + \frac{1}{2}\alpha^2\right) \quad (11.2.6)$$

and

$$Var(T^{-1}) = \alpha^2\beta^{-2}\left(1 + \frac{5}{4}\alpha^2\right). \quad (11.2.7)$$

## 11.3 Maximum Likelihood Estimators

Let $\{t_1, t_2, \ldots, t_n\}$ be a random sample of size $n$ from the Birnbaum-Saunders distribution with p.d.f. as in (11.2.1). The sample arithmetic and harmonic means are defined by

$$s = \frac{1}{n}\sum_{i=1}^{n} t_i, \qquad r = \left[\frac{1}{n}\sum_{i=1}^{n} t_i^{-1}\right]^{-1}.$$

Let us further define the harmonic mean function $K$ by

$$K(x) = \left[\frac{1}{n}\sum_{i=1}^{n}(x+t_i)^{-1}\right]^{-1} \qquad \text{for } x \geq 0$$

so that $r \equiv K(0)$.

The MLE of $\beta$ (denoted by $\hat{\beta}$) can be obtained as the unique positive root of the equation

$$\beta^2 - \beta[2r + K(\beta)] + r[s + K(\beta)] = 0. \tag{11.3.1}$$

Once $\hat{\beta}$ is obtained as a solution of (11.3.1), the MLE of $\alpha$ (denoted by $\hat{\alpha}$) can be obtained explicitly as

$$\hat{\alpha} = \left[\frac{s}{\hat{\beta}} + \frac{\hat{\beta}}{r} - 2\right]^{\frac{1}{2}}.$$

Since (11.3.1) is a non-linear equation in $\beta$, one needs to use an iterative procedure to solve for $\hat{\beta}$. Birnbaum and Saunders (1969b) proposed two iterative procedures (one simple and one complicated) to compute $\hat{\beta}$, but noted that the simpler one works very well for small $\alpha$ ($< \frac{1}{2}$) but may not work at all for large $\alpha$ ($> 2$). The complicated one also does not work in certain range of the sample space.

Engelhardt, Bain and Wright (1981) showed that the asymptotic joint distribution of $\hat{\alpha}$ and $\hat{\beta}$ is bivariate normal and is given by

$$\begin{pmatrix} \hat{\alpha} \\ \hat{\beta} \end{pmatrix} \sim N\left[\begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \begin{pmatrix} \frac{\alpha^2}{2n} & 0 \\ 0 & \frac{\beta^2}{n[0.25+\alpha^{-2}+I(\alpha)]} \end{pmatrix}\right], \tag{11.3.2}$$

where

$$I(\alpha) = 2\int_0^{\infty} \left\{[1 + g(\alpha x)]^{-1} - 0.5\right\}^2 d\Phi(x),$$

$$g(y) = 1 + \frac{y^2}{2} + y\left(1 + \frac{y^2}{4}\right)^{\frac{1}{2}}.$$

It is interesting to observe that $\hat{\alpha}$ and $\hat{\beta}$ are asymptotically independent of each other. The asymptotic confidence interval of $\alpha$ can be easily obtained from (11.3.2). Moreover, the asymptotic confidence interval of $\beta$, for a given $\alpha$, can also be obtained from (11.3.2).

## 11.4 Modified Moment Estimators

For the usual moment estimators in a two-parameter case, the first and second population moments are equated with the corresponding sample moments. In this case, the sample mean and the sample variance can be equated to the right hand sides of (11.2.2) and (11.2.3), respectively, and the corresponding moment estimators of $\alpha$ and $\beta$ can then be obtained as solutions of $\alpha$ and $\beta$ to these equations. It can be easily seen from these equations that if the sample coefficient of variation is greater than $\sqrt{5}$, then the moment estimators do not exist. If the sample coefficient of variation is less than $\sqrt{5}$, the moment estimators exist; however, the moment estimator of $\beta$ may not be unique.

Instead of using (11.2.2) and (11.2.3), we propose to use (11.2.2) and (11.2.6) and equate them with the corresponding sample estimates to obtain the MME's. In this case, we have the following two moment equations:

$$s = \beta\left(1 + \frac{1}{2}\alpha^2\right), \tag{11.4.1}$$

$$r^{-1} = \beta^{-1}\left(1 + \frac{1}{2}\alpha^2\right). \tag{11.4.2}$$

Solving Equations (11.4.1) and (11.4.2) for $\alpha$ and $\beta$, we obtain the MME's for $\alpha$ and $\beta$, denoted by $\tilde{\alpha}$ and $\tilde{\beta}$, as

$$\tilde{\alpha} = \left\{2\left[\left(\frac{s}{r}\right)^{\frac{1}{2}} - 1\right]\right\}^{\frac{1}{2}},$$

$$\tilde{\beta} = (sr)^{\frac{1}{2}}.$$

The asymptotic joint distribution of $\tilde{\alpha}$ and $\tilde{\beta}$ is bivariate normal and is given by

$$\begin{pmatrix} \tilde{\alpha} \\ \tilde{\beta} \end{pmatrix} \sim N\left[\begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \begin{pmatrix} \frac{\alpha^2}{2n} & 0 \\ 0 & \frac{(\alpha\beta)^2}{n}\left(\frac{1+\frac{3}{4}\alpha^2}{(1+\frac{1}{2}\alpha^2)^2}\right) \end{pmatrix}\right]. \tag{11.4.3}$$

The proof of this result is presented in Appendix D. Note that the MME's $\tilde{\alpha}$ and $\tilde{\beta}$ are also asymptotically independent of each other, just as in the case of the MLE's.

## 11.5 Bias-reduced Estimators

Based on the results of an extensive Monte Carlo simulation study, we observed that the MLE's and the MME's performed very similarly in terms of both bias and mean square error, especially for small values of $\alpha$. Upon inspecting the pattern of the bias of the MLE's and MME's, we observed that

$$\begin{aligned} Bias(\hat{\alpha}) &\approx Bias(\tilde{\alpha}) \approx -\frac{\alpha}{n}, \\ Bias(\hat{\beta}) &\approx Bias(\tilde{\beta}) \approx \frac{\alpha^2}{4n}. \end{aligned}$$

Then, by employing a standard bias reduction method, we can simply construct almost unbiased maximum likelihood estimators (UMLE's, denoted by $\hat{\alpha}^*$ and $\hat{\beta}^*$) and almost unbiased modified moment estimators (UMME's, denoted by $\tilde{\alpha}^*$ and $\tilde{\beta}^*$) of $\alpha$ and $\beta$. These bias-reduced estimators are given by

$$\hat{\alpha}^* = \left(\frac{n}{n-1}\right)\hat{\alpha}, \quad \hat{\beta}^* = \left(1+\frac{\alpha^2}{4n}\right)^{-1}\hat{\beta} \approx \left(1+\frac{\hat{\alpha}^{*2}}{4n}\right)^{-1}\hat{\beta}, \tag{11.5.1}$$

$$\tilde{\alpha}^* = \left(\frac{n}{n-1}\right)\tilde{\alpha}, \quad \tilde{\beta}^* = \left(1+\frac{\alpha^2}{4n}\right)^{-1}\tilde{\beta} \approx \left(1+\frac{\tilde{\alpha}^{*2}}{4n}\right)^{-1}\tilde{\beta}. \tag{11.5.2}$$

From the distributional results presented in (11.3.2), we readily have the asymptotic joint distribution of $\hat{\alpha}^*$ and $\hat{\beta}^*$ to be bivariate normal and is given by

$$\begin{pmatrix} \hat{\alpha}^* \\ \hat{\beta}^* \end{pmatrix} \sim N\left[\begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \begin{pmatrix} \frac{n\alpha^2}{2(n-1)^2} & 0 \\ 0 & \frac{16n\beta^2}{(4n+\alpha^2)^2[0.25+\alpha^{-2}+I(\alpha)]} \end{pmatrix}\right]; \tag{11.5.3}$$

similarly, from (11.4.3), the asymptotic joint distribution of $\tilde{\alpha}^*$ and $\tilde{\beta}^*$ is bivariate normal and is given by

$$\begin{pmatrix} \tilde{\alpha}^* \\ \tilde{\beta}^* \end{pmatrix} \sim N\left[\begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \begin{pmatrix} \frac{n\alpha^2}{2(n-1)^2} & 0 \\ 0 & \frac{16n(\alpha\beta)^2}{(4n+\alpha^2)^2}\left(\frac{1+\frac{3}{4}\alpha^2}{(1+\frac{1}{2}\alpha^2)^2}\right) \end{pmatrix}\right]. \tag{11.5.4}$$

## 11.6 Jackknifed Estimators

Jackknifing is based on sequentially deleting one sample point $t_i$ and recomputing the MLE's and MME's from the reduced sample of size $n-1$. We remove the point $t_j$ from the data set, and then recompute $r$ and $s$ and also the function $K$ as

$$\begin{aligned} s_{(j)} &= \frac{1}{n-1}\sum_{\substack{i=1 \\ i\neq j}}^{n} t_i = \frac{ns-t_j}{n-1}, \\ r_{(j)} &= \left[\frac{1}{n-1}\sum_{\substack{i=1 \\ i\neq j}}^{n} t_i^{-1}\right]^{-1} = \frac{nr-t_j^{-1}}{n-1}, \\ K_{(j)}(x) &= \left[\frac{1}{n-1}\sum_{\substack{i=1 \\ i\neq j}}^{n} (x+t_i)^{-1}\right]^{-1} = \frac{nK(x)-(x+t_j)^{-1}}{n-1}. \end{aligned}$$

Then, we obtain $\hat{\beta}_{(j)}$ as the unique positive root of the equation

$$\beta^2 - \beta[2r_{(j)} + K_{(j)}(\beta)] + r_{(j)}[s_{(j)} + K_{(j)}(\beta)] = 0$$

and

$$\hat{\alpha}_{(j)} = \left[\frac{s_{(j)}}{\hat{\beta}_{(j)}} + \frac{\hat{\beta}_{(j)}}{r_{(j)}} - 2\right]^{\frac{1}{2}}.$$

Similarly, we find

$$\begin{aligned} \tilde{\alpha}_{(j)} &= \left\{2\left[\left(\frac{s_{(j)}}{r_{(j)}}\right)^{\frac{1}{2}} - 1\right]\right\}^{\frac{1}{2}}, \\ \tilde{\beta}_{(j)} &= \left(s_{(j)}r_{(j)}\right)^{\frac{1}{2}}. \end{aligned}$$

Let us now define

$$\hat{\alpha}_{(\cdot)} = \frac{1}{n}\sum_{j=1}^{n}\hat{\alpha}_{(j)}, \qquad \hat{\beta}_{(\cdot)} = \frac{1}{n}\sum_{j=1}^{n}\hat{\beta}_{(j)},$$

$$\tilde{\alpha}_{(\cdot)} = \frac{1}{n}\sum_{j=1}^{n}\tilde{\alpha}_{(j)}, \qquad \tilde{\beta}_{(\cdot)} = \frac{1}{n}\sum_{j=1}^{n}\tilde{\beta}_{(j)}.$$

Then, the bias-corrected jackknifed maximum likelihood estimates (JMLE's) of $\alpha$ and $\beta$ [see, for example, Efron (1982)] are given by

$$\begin{aligned}\hat{\alpha}_J &= n\hat{\alpha} - (n-1)\hat{\alpha}_{(\cdot)},\\ \hat{\beta}_J &= n\hat{\beta} - (n-1)\hat{\beta}_{(\cdot)};\end{aligned}$$

similarly, the bias-corrected jackknifed modified moment estimates (JMME's) of $\alpha$ and $\beta$ are given by

$$\begin{aligned}\tilde{\alpha}_J &= n\tilde{\alpha} - (n-1)\tilde{\alpha}_{(\cdot)},\\ \tilde{\beta}_J &= n\tilde{\beta} - (n-1)\tilde{\beta}_{(\cdot)}.\end{aligned}$$

## 11.7 Monte Carlo Simulation Results

In order to compare the performance of all the above estimators, we performed a simulation study for different sample sizes and for different parameter values. We took the sample size as $n = 5$, 10, 20, 50, 100, and the shape parameter as $\alpha = .10$, .20, .25, .35, .50, 1.00, 2.00. The scale parameter $\beta$ was kept fixed at 1.0, without loss of any generality. All the results were based on 10,000 Monte Carlo runs. We computed the MLE, MME, UMLE, UMME, JMLE and JMME for each run, and then computed the average estimates and the standard deviations over the 10,000 runs for all these estimators. The results so obtained are reported in Tables 11.1 and 11.2. We also computed the 90% and 95% probability coverages of confidence intervals based on inferential quantities associated with all these estimators using the asymptotic distributions given earlier. Specifically, the $100(1-\gamma)\%$ confidence intervals for $\alpha$ and $\beta$ based on the MLE's and UMLE's are given by

$$\left[\hat{\alpha}\left(\frac{z_{\gamma/2}}{\sqrt{2n}}+1\right)^{-1}, \hat{\alpha}\left(\frac{z_{1-\gamma/2}}{\sqrt{2n}}+1\right)^{-1}\right], \left[\hat{\beta}\left(\frac{z_{\gamma/2}}{\sqrt{nh_1(\hat{\alpha})}}+1\right)^{-1}, \hat{\beta}\left(\frac{z_{1-\gamma/2}}{\sqrt{nh_1(\hat{\alpha})}}+1\right)^{-1}\right],$$

and

$$\left[\hat{\alpha}^*\left(\sqrt{\frac{n}{2}}\frac{z_{\gamma/2}}{(n-1)}+1\right)^{-1}, \hat{\alpha}^*\left(\sqrt{\frac{n}{2}}\frac{z_{1-\gamma/2}}{(n-1)}+1\right)^{-1}\right],$$

$$\left[\hat{\beta}^*\left(\sqrt{\frac{n}{h_1(\hat{\alpha})}}\frac{4z_{\gamma/2}}{(4n+\hat{\alpha}^2)}+1\right)^{-1}, \hat{\beta}^*\left(\sqrt{\frac{n}{h_1(\hat{\alpha})}}\frac{4z_{1-\gamma/2}}{(4n+\hat{\alpha}^2)}+1\right)^{-1}\right],$$

where $h_1(x) = 0.25 + x^{-2} + I(x)$ and $z_p$ is the $100p$-th percentile of the standard normal distribution. Similarly, the $100(1-\gamma)\%$ confidence intervals for $\alpha$ and $\beta$ based on the MME's and UMME's are given by

$$\left[\tilde{\alpha}\left(\frac{z_{\gamma/2}}{\sqrt{2n}}+1\right)^{-1}, \tilde{\alpha}\left(\frac{z_{1-\gamma/2}}{\sqrt{2n}}+1\right)^{-1}\right], \left[\tilde{\beta}\left(\frac{z_{\gamma/2}}{\sqrt{nh_2(\tilde{\alpha})}}+1\right)^{-1}, \tilde{\beta}\left(\frac{z_{1-\gamma/2}}{\sqrt{nh_2(\tilde{\alpha})}}+1\right)^{-1}\right],$$

and

$$\left[\tilde{\alpha}^*\left(\sqrt{\frac{n}{2}}\frac{z_{\gamma/2}}{(n-1)}+1\right)^{-1}, \tilde{\alpha}^*\left(\sqrt{\frac{n}{2}}\frac{z_{1-\gamma/2}}{(n-1)}+1\right)^{-1}\right],$$

$$\left[\tilde{\beta}^*\left(\sqrt{\frac{n}{h_2(\tilde{\alpha})}}\frac{4z_{\gamma/2}}{(4n+\tilde{\alpha}^2)}+1\right)^{-1}, \tilde{\beta}^*\left(\sqrt{\frac{n}{h_2(\tilde{\alpha})}}\frac{4z_{1-\gamma/2}}{(4n+\tilde{\alpha}^2)}+1\right)^{-1}\right],$$

where

$$h_2(x) = \frac{1+\frac{3}{4}x^2}{\left(1+\frac{1}{2}x^2\right)^2}.$$

These results are reported in Tables 11.3 and 11.4, respectively.

From the simulation results, it is clear that the performance of the MLE's and MME's are almost identical for different sample sizes and if the shape parameter $\alpha$ is not too large. The average estimates of the MLE's and MME's and their standard deviations coincide up to four decimal places if $\alpha$ is less than 0.5. It is also evident from these results that the MLE's and the MME's are both highly biased if $n$ is small and $\alpha$ is large. The bias reduction method works very well in this case for both the parameters even for small samples; but as expected, it increases the corresponding standard deviation of the estimates. The performance of the JMLE and JMME are almost identical at least for small values of $\alpha$, and they perform better than UMLE and UMME in terms of bias.

Table 11.1: Means of estimates based on Monte Carlo simulation ($\beta = 1.0$)

| | | Estimate of $\alpha$ | | | | | | Estimate of $\beta$ | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| n | $\alpha$ | MLE | MME | UMLE | UMME | JMLE | JMME | MLE | MME | UMLE | UMME | JMLE | JMME |
| 5 | 0.10 | 0.0842 | 0.0842 | 0.1053 | 0.1053 | 0.1015 | 0.1015 | 1.0007 | 1.0007 | 1.0004 | 1.0004 | 0.9997 | 0.9997 |
| | 0.20 | 0.1683 | 0.1683 | 0.2104 | 0.2104 | 0.2029 | 0.2029 | 1.0033 | 1.0033 | 1.0014 | 1.0014 | 0.9993 | 0.9993 |
| | 0.25 | 0.2102 | 0.2102 | 0.2629 | 0.2629 | 0.2536 | 0.2536 | 1.0054 | 1.0054 | 1.0022 | 1.0022 | 0.9992 | 0.9992 |
| | 0.35 | 0.2939 | 0.2939 | 0.3675 | 0.3675 | 0.3550 | 0.3550 | 1.0109 | 1.0109 | 1.0042 | 1.0042 | 0.9989 | 0.9989 |
| | 0.50 | 0.4186 | 0.4186 | 0.5236 | 0.5236 | 0.5068 | 0.5068 | 1.0225 | 1.0225 | 1.0081 | 1.0081 | 0.9984 | 0.9984 |
| | 1.00 | 0.8273 | 0.8270 | 1.0351 | 1.0348 | 1.0115 | 1.0112 | 1.0835 | 1.0832 | 1.0241 | 1.0242 | 0.9927 | 0.9922 |
| | 2.00 | 1.6199 | 1.6142 | 2.0279 | 2.0210 | 2.0251 | 2.0182 | 1.2524 | 1.2524 | 1.0422 | 1.0444 | 0.9190 | 0.9193 |
| 10 | 0.10 | 0.0924 | 0.0924 | 0.1023 | 0.1023 | 0.1005 | 0.1005 | 1.0002 | 1.0002 | 1.0005 | 1.0005 | 0.9997 | 0.9997 |
| | 0.20 | 0.1848 | 0.1848 | 0.2046 | 0.2046 | 0.2010 | 0.2010 | 1.0014 | 1.0014 | 1.0014 | 1.0014 | 0.9994 | 0.9994 |
| | 0.25 | 0.2310 | 0.2310 | 0.2557 | 0.2557 | 0.2513 | 0.2513 | 1.0024 | 1.0024 | 1.0020 | 1.0020 | 0.9993 | 0.9993 |
| | 0.35 | 0.3231 | 0.3231 | 0.3577 | 0.3577 | 0.3517 | 0.3517 | 1.0050 | 1.0050 | 1.0035 | 1.0035 | 0.9990 | 0.9990 |
| | 0.50 | 0.4610 | 0.4610 | 0.5103 | 0.5103 | 0.5024 | 0.5024 | 1.0104 | 1.0104 | 1.0061 | 1.0061 | 0.9985 | 0.9985 |
| | 1.00 | 0.9168 | 0.9166 | 1.0150 | 1.0148 | 1.0046 | 1.0046 | 1.0380 | 1.0379 | 1.0148 | 1.0150 | 0.9955 | 0.9953 |
| | 2.00 | 1.8181 | 1.8140 | 2.0134 | 2.0089 | 2.0124 | 2.0110 | 1.1017 | 1.1029 | 1.0025 | 1.0056 | 0.9730 | 0.9738 |
| 20 | 0.10 | 0.0964 | 0.0964 | 0.1011 | 0.1011 | 0.1002 | 0.1002 | 1.0001 | 1.0001 | 1.0000 | 1.0000 | 0.9999 | 0.9999 |
| | 0.20 | 0.1927 | 0.1927 | 0.2021 | 0.2021 | 0.2005 | 0.2005 | 1.0007 | 1.0007 | 1.0003 | 1.0003 | 0.9997 | 0.9997 |
| | 0.25 | 0.2408 | 0.2408 | 0.2525 | 0.2525 | 0.2506 | 0.2506 | 1.0012 | 1.0012 | 1.0005 | 1.0005 | 0.9997 | 0.9997 |
| | 0.35 | 0.3370 | 0.3370 | 0.3534 | 0.3534 | 0.3508 | 0.3508 | 1.0025 | 1.0025 | 1.0010 | 1.0010 | 0.9995 | 0.9995 |
| | 0.50 | 0.4811 | 0.4811 | 0.5046 | 0.5046 | 0.5012 | 0.5012 | 1.0053 | 1.0053 | 1.0021 | 1.0021 | 0.9994 | 0.9994 |
| | 1.00 | 0.9596 | 0.9595 | 1.0064 | 1.0063 | 1.0023 | 1.0023 | 1.0190 | 1.0190 | 1.0059 | 1.0059 | 0.9986 | 0.9986 |
| | 2.00 | 1.9122 | 1.9099 | 2.0057 | 2.0032 | 2.0056 | 2.0053 | 1.0466 | 1.0479 | 0.9952 | 0.9966 | 0.9937 | 0.9941 |
| 50 | 0.10 | 0.0984 | 0.0984 | 0.1005 | 0.1005 | 0.0999 | 0.0999 | 0.9999 | 0.9999 | 1.0001 | 1.0001 | 0.9998 | 0.9998 |
| | 0.20 | 0.1967 | 0.1967 | 0.2010 | 0.2010 | 0.1998 | 0.1998 | 0.9999 | 0.9999 | 1.0002 | 1.0002 | 0.9995 | 0.9995 |
| | 0.25 | 0.2459 | 0.2459 | 0.2513 | 0.2513 | 0.2497 | 0.2497 | 1.0000 | 1.0000 | 1.0004 | 1.0004 | 0.9994 | 0.9994 |
| | 0.35 | 0.3442 | 0.3442 | 0.3517 | 0.3517 | 0.3496 | 0.3496 | 1.0004 | 1.0004 | 1.0007 | 1.0007 | 0.9992 | 0.9992 |
| | 0.50 | 0.4915 | 0.4915 | 0.5023 | 0.5023 | 0.4994 | 0.4994 | 1.0013 | 1.0013 | 1.0012 | 1.0012 | 0.9989 | 0.9989 |
| | 1.00 | 0.9820 | 0.9820 | 1.0035 | 1.0035 | 0.9987 | 0.9987 | 1.0059 | 1.0059 | 1.0030 | 1.0030 | 0.9980 | 0.9980 |
| | 2.00 | 1.9615 | 1.9605 | 2.0044 | 2.0034 | 1.9976 | 1.9976 | 1.0151 | 1.0157 | 0.9981 | 0.9987 | 0.9966 | 0.9966 |
| 100 | 0.10 | 0.0992 | 0.0992 | 0.1002 | 0.1002 | 0.0999 | 0.0999 | 0.9999 | 0.9999 | 1.0001 | 1.0001 | 0.9998 | 0.9998 |
| | 0.20 | 0.1983 | 0.1983 | 0.2003 | 0.2003 | 0.1998 | 0.1998 | 0.9999 | 0.9999 | 1.0003 | 1.0003 | 0.9997 | 0.9997 |
| | 0.25 | 0.2479 | 0.2479 | 0.2504 | 0.2504 | 0.2498 | 0.2498 | 0.9999 | 0.9999 | 1.0004 | 1.0004 | 0.9996 | 0.9996 |
| | 0.35 | 0.3470 | 0.3470 | 0.3505 | 0.3505 | 0.3497 | 0.3497 | 1.0001 | 1.0001 | 1.0006 | 1.0006 | 0.9995 | 0.9995 |
| | 0.50 | 0.4957 | 0.4957 | 0.5007 | 0.5007 | 0.4996 | 0.4996 | 1.0004 | 1.0004 | 1.0010 | 1.0010 | 0.9992 | 0.9992 |
| | 1.00 | 0.9909 | 0.9909 | 1.0008 | 1.0008 | 0.9992 | 0.9992 | 1.0025 | 1.0025 | 1.0021 | 1.0021 | 0.9986 | 0.9986 |
| | 2.00 | 1.9805 | 1.9800 | 2.0003 | 1.9998 | 1.9984 | 1.9984 | 1.0067 | 1.0070 | 0.9996 | 1.0001 | 0.9978 | 0.9978 |

The asymptotic confidence intervals do not work very well when the sample size is very small as the coverage probabilities are much lower than the corresponding nominal levels. But for sample sizes 20 or more, the performances are quite satisfactory for confidence intervals for both $\alpha$ and $\beta$. The bias reduction technique definitely helps to improve the coverage probabilities in both cases to a certain extent.

Table 11.2: Standard deviations of estimates based on Monte Carlo simulation ($\beta = 1.0$)

| | | Estimate of $\alpha$ | | | | | | Estimate of $\beta$ | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| n | $\alpha$ | MLE | MME | UMLE | UMME | JMLE | JMME | MLE | MME | UMLE | UMME | JMLE | JMME |
| 5 | 0.10 | 0.0306 | 0.0306 | 0.0383 | 0.0383 | 0.0373 | 0.0373 | 0.0449 | 0.0449 | 0.0445 | 0.0445 | 0.0449 | 0.0449 |
| | 0.20 | 0.0612 | 0.0612 | 0.0766 | 0.0766 | 0.0748 | 0.0748 | 0.0900 | 0.0900 | 0.0890 | 0.0890 | 0.0897 | 0.0897 |
| | 0.25 | 0.0764 | 0.0764 | 0.0957 | 0.0957 | 0.0935 | 0.0935 | 0.1126 | 0.1126 | 0.1112 | 0.1112 | 0.1120 | 0.1120 |
| | 0.35 | 0.1069 | 0.1069 | 0.1338 | 0.1338 | 0.1312 | 0.1312 | 0.1579 | 0.1579 | 0.1555 | 0.1556 | 0.1564 | 0.1564 |
| | 0.50 | 0.1524 | 0.1524 | 0.1908 | 0.1908 | 0.1883 | 0.1883 | 0.2263 | 0.2263 | 0.2216 | 0.2216 | 0.2225 | 0.2223 |
| | 1.00 | 0.3038 | 0.3034 | 0.3805 | 0.3800 | 0.3858 | 0.3851 | 0.4583 | 0.4583 | 0.4367 | 0.4368 | 0.4464 | 0.4382 |
| | 2.00 | 0.6202 | 0.6131 | 0.7764 | 0.7678 | 0.8319 | 0.8164 | 0.9511 | 0.9553 | 0.8657 | 0.8684 | 1.0513 | 0.9397 |
| 10 | 0.10 | 0.0219 | 0.0219 | 0.0246 | 0.0246 | 0.0239 | 0.0239 | 0.0314 | 0.0314 | 0.0315 | 0.0315 | 0.0313 | 0.0313 |
| | 0.20 | 0.0438 | 0.0438 | 0.0493 | 0.0493 | 0.0478 | 0.0478 | 0.0626 | 0.0626 | 0.0628 | 0.0628 | 0.0625 | 0.0625 |
| | 0.25 | 0.0547 | 0.0547 | 0.0616 | 0.0616 | 0.0597 | 0.0597 | 0.0782 | 0.0782 | 0.0784 | 0.0784 | 0.0780 | 0.0780 |
| | 0.35 | 0.0765 | 0.0765 | 0.0861 | 0.0861 | 0.0837 | 0.0837 | 0.1092 | 0.1092 | 0.1092 | 0.1092 | 0.1086 | 0.1086 |
| | 0.50 | 0.1092 | 0.1092 | 0.1230 | 0.1230 | 0.1200 | 0.1200 | 0.1550 | 0.1550 | 0.1544 | 0.1544 | 0.1534 | 0.1533 |
| | 1.00 | 0.2185 | 0.2183 | 0.2459 | 0.2457 | 0.2435 | 0.2434 | 0.2979 | 0.2979 | 0.2912 | 0.2915 | 0.2904 | 0.2882 |
| | 2.00 | 0.4445 | 0.4410 | 0.5001 | 0.4961 | 0.5060 | 0.5036 | 0.5213 | 0.5261 | 0.4887 | 0.4939 | 0.5177 | 0.4874 |
| 20 | 0.10 | 0.0155 | 0.0155 | 0.0165 | 0.0165 | 0.0162 | 0.0162 | 0.0225 | 0.0225 | 0.0223 | 0.0223 | 0.0225 | 0.0225 |
| | 0.20 | 0.0311 | 0.0311 | 0.0331 | 0.0331 | 0.0324 | 0.0324 | 0.0449 | 0.0449 | 0.0444 | 0.0444 | 0.0449 | 0.0449 |
| | 0.25 | 0.0388 | 0.0388 | 0.0414 | 0.0414 | 0.0405 | 0.0405 | 0.0560 | 0.0560 | 0.0554 | 0.0554 | 0.0559 | 0.0559 |
| | 0.35 | 0.0544 | 0.0544 | 0.0579 | 0.0579 | 0.0567 | 0.0567 | 0.0780 | 0.0780 | 0.0771 | 0.0771 | 0.0778 | 0.0778 |
| | 0.50 | 0.0776 | 0.0776 | 0.0827 | 0.0827 | 0.0812 | 0.0812 | 0.1101 | 0.1101 | 0.1088 | 0.1088 | 0.1095 | 0.1095 |
| | 1.00 | 0.1554 | 0.1553 | 0.1655 | 0.1654 | 0.1635 | 0.1635 | 0.2053 | 0.2056 | 0.2020 | 0.2022 | 0.2019 | 0.2016 |
| | 2.00 | 0.3139 | 0.3125 | 0.3344 | 0.3329 | 0.3323 | 0.3320 | 0.3280 | 0.3340 | 0.3153 | 0.3206 | 0.3182 | 0.3156 |
| 50 | 0.10 | 0.0099 | 0.0099 | 0.0102 | 0.0102 | 0.0101 | 0.0101 | 0.0142 | 0.0142 | 0.0140 | 0.0140 | 0.0142 | 0.0142 |
| | 0.20 | 0.0199 | 0.0199 | 0.0204 | 0.0204 | 0.0202 | 0.0202 | 0.0284 | 0.0284 | 0.0280 | 0.0280 | 0.0284 | 0.0284 |
| | 0.25 | 0.0249 | 0.0249 | 0.0255 | 0.0255 | 0.0253 | 0.0253 | 0.0354 | 0.0354 | 0.0349 | 0.0349 | 0.0353 | 0.0353 |
| | 0.35 | 0.0348 | 0.0348 | 0.0357 | 0.0357 | 0.0354 | 0.0354 | 0.0492 | 0.0492 | 0.0486 | 0.0486 | 0.0491 | 0.0491 |
| | 0.50 | 0.0497 | 0.0497 | 0.0509 | 0.0509 | 0.0505 | 0.0505 | 0.0693 | 0.0693 | 0.0684 | 0.0684 | 0.0691 | 0.0691 |
| | 1.00 | 0.0994 | 0.0994 | 0.1018 | 0.1018 | 0.1014 | 0.1014 | 0.1273 | 0.1275 | 0.1258 | 0.1258 | 0.1263 | 0.1264 |
| | 2.00 | 0.1997 | 0.1994 | 0.2044 | 0.2040 | 0.2037 | 0.2037 | 0.1935 | 0.1979 | 0.1904 | 0.1934 | 0.1901 | 0.1930 |
| 100 | 0.10 | 0.0070 | 0.0070 | 0.0072 | 0.0072 | 0.0071 | 0.0071 | 0.0100 | 0.0100 | 0.0099 | 0.0099 | 0.0100 | 0.0100 |
| | 0.20 | 0.0140 | 0.0140 | 0.0144 | 0.0144 | 0.0141 | 0.0141 | 0.0200 | 0.0200 | 0.0197 | 0.0197 | 0.0200 | 0.0200 |
| | 0.25 | 0.0176 | 0.0176 | 0.0180 | 0.0180 | 0.0177 | 0.0177 | 0.0249 | 0.0249 | 0.0246 | 0.0246 | 0.0249 | 0.0249 |
| | 0.35 | 0.0246 | 0.0246 | 0.0252 | 0.0252 | 0.0248 | 0.0248 | 0.0346 | 0.0346 | 0.0341 | 0.0341 | 0.0346 | 0.0346 |
| | 0.50 | 0.0351 | 0.0351 | 0.0360 | 0.0360 | 0.0354 | 0.0354 | 0.0487 | 0.0487 | 0.0480 | 0.0480 | 0.0486 | 0.0486 |
| | 1.00 | 0.0702 | 0.0702 | 0.0720 | 0.0720 | 0.0709 | 0.0709 | 0.0890 | 0.0891 | 0.0880 | 0.0880 | 0.0887 | 0.0887 |
| | 2.00 | 0.1407 | 0.1406 | 0.1443 | 0.1441 | 0.1421 | 0.1421 | 0.1341 | 0.1366 | 0.1323 | 0.1344 | 0.1330 | 0.1349 |

Table 11.3: Probability coverages of 90% confidence intervals based on Monte Carlo simulation ($\beta = 1.0$)

| | | Probability coverages for $\alpha$ | | | | Probability coverages for $\beta$ | | | |
|---|---|---|---|---|---|---|---|---|---|
| n | $\alpha$ | MLE | MME | UMLE | UMME | MLE | MME | UMLE | UMME |
| 5 | 0.10 | 86.25 | 86.25 | 91.12 | 91.12 | 78.57 | 78.57 | 86.11 | 86.11 |
| | 0.20 | 86.25 | 86.25 | 91.12 | 91.12 | 78.48 | 78.47 | 86.08 | 86.08 |
| | 0.25 | 86.25 | 86.25 | 91.14 | 91.14 | 78.45 | 78.45 | 86.03 | 86.03 |
| | 0.35 | 86.24 | 86.24 | 91.20 | 91.20 | 78.36 | 78.37 | 85.98 | 85.98 |
| | 0.50 | 86.15 | 86.15 | 91.32 | 91.32 | 78.33 | 78.36 | 85.64 | 85.64 |
| | 1.00 | 85.57 | 85.58 | 91.45 | 91.49 | 78.20 | 78.16 | 84.25 | 84.32 |
| | 2.00 | 83.53 | 83.72 | 91.02 | 91.41 | 77.40 | 77.57 | 80.52 | 81.74 |
| 10 | 0.10 | 88.69 | 88.69 | 90.43 | 90.43 | 84.91 | 84.91 | 88.14 | 88.14 |
| | 0.20 | 88.73 | 88.73 | 90.47 | 90.47 | 84.94 | 84.94 | 88.09 | 88.09 |
| | 0.25 | 88.73 | 88.73 | 90.45 | 90.45 | 85.00 | 85.00 | 88.13 | 88.13 |
| | 0.35 | 88.71 | 88.71 | 90.52 | 90.52 | 84.84 | 84.83 | 88.06 | 88.07 |
| | 0.50 | 88.72 | 88.72 | 90.56 | 90.56 | 84.78 | 84.76 | 88.05 | 88.03 |
| | 1.00 | 88.35 | 88.36 | 90.73 | 90.75 | 84.72 | 84.87 | 87.30 | 87.38 |
| | 2.00 | 87.53 | 87.70 | 90.13 | 90.34 | 83.96 | 84.35 | 85.39 | 85.99 |
| 20 | 0.10 | 89.87 | 89.87 | 90.31 | 90.31 | 87.09 | 87.09 | 89.29 | 89.29 |
| | 0.20 | 89.90 | 89.90 | 90.31 | 90.31 | 86.96 | 86.96 | 89.25 | 89.24 |
| | 0.25 | 89.89 | 89.89 | 90.32 | 90.32 | 87.09 | 87.09 | 89.27 | 89.27 |
| | 0.35 | 89.85 | 89.85 | 90.36 | 90.36 | 86.99 | 86.99 | 89.22 | 89.25 |
| | 0.50 | 89.83 | 89.83 | 90.39 | 90.39 | 87.14 | 87.18 | 89.07 | 89.08 |
| | 1.00 | 89.62 | 89.64 | 90.42 | 90.45 | 86.97 | 87.03 | 88.75 | 88.73 |
| | 2.00 | 89.16 | 89.21 | 89.97 | 90.18 | 86.66 | 86.74 | 87.63 | 88.22 |
| 50 | 0.10 | 89.33 | 89.33 | 90.14 | 90.14 | 88.58 | 88.58 | 89.96 | 89.96 |
| | 0.20 | 89.33 | 89.33 | 90.15 | 90.15 | 88.61 | 88.61 | 89.94 | 89.94 |
| | 0.25 | 89.33 | 89.33 | 90.14 | 90.14 | 88.66 | 88.66 | 89.84 | 89.84 |
| | 0.35 | 89.31 | 89.31 | 90.14 | 90.14 | 88.66 | 88.67 | 89.71 | 89.73 |
| | 0.50 | 89.28 | 89.28 | 90.19 | 90.19 | 88.78 | 88.79 | 89.72 | 89.68 |
| | 1.00 | 89.25 | 89.24 | 90.11 | 90.11 | 88.67 | 88.80 | 89.41 | 89.56 |
| | 2.00 | 89.04 | 89.10 | 89.93 | 89.98 | 88.18 | 88.53 | 88.81 | 89.24 |
| 100 | 0.10 | 90.05 | 90.05 | 89.65 | 89.65 | 89.43 | 89.43 | 90.54 | 90.54 |
| | 0.20 | 90.06 | 90.06 | 89.66 | 89.66 | 89.45 | 89.45 | 90.51 | 90.52 |
| | 0.25 | 90.06 | 90.06 | 89.67 | 89.67 | 89.45 | 89.44 | 90.51 | 90.52 |
| | 0.35 | 90.03 | 90.03 | 89.68 | 89.68 | 89.39 | 89.37 | 90.52 | 90.50 |
| | 0.50 | 90.04 | 90.04 | 89.64 | 89.64 | 89.36 | 89.38 | 90.43 | 90.47 |
| | 1.00 | 90.03 | 90.03 | 89.64 | 89.64 | 89.21 | 89.40 | 90.11 | 90.26 |
| | 2.00 | 89.96 | 89.98 | 89.69 | 89.70 | 88.95 | 89.33 | 89.46 | 90.01 |

Table 11.4: Probability coverages of 95% confidence intervals based on Monte Carlo simulation ($\beta = 1.0$)

| | | Probability coverages for $\alpha$ | | | | Probability coverages for $\beta$ | | | |
|---|---|---|---|---|---|---|---|---|---|
| n | $\alpha$ | MLE | MME | UMLE | UMME | MLE | MME | UMLE | UMME |
| 5 | 0.10 | 93.86 | 93.86 | 95.60 | 95.60 | 84.89 | 84.89 | 90.56 | 90.56 |
| | 0.20 | 93.85 | 93.85 | 95.63 | 95.63 | 84.88 | 84.88 | 90.44 | 90.44 |
| | 0.25 | 93.87 | 93.87 | 95.68 | 95.68 | 84.75 | 84.75 | 90.43 | 90.43 |
| | 0.35 | 93.87 | 93.87 | 95.69 | 95.69 | 84.70 | 84.70 | 90.18 | 90.18 |
| | 0.50 | 93.79 | 93.79 | 95.70 | 95.70 | 84.38 | 84.38 | 89.81 | 89.81 |
| | 1.00 | 93.65 | 93.67 | 95.94 | 96.05 | 83.37 | 83.34 | 88.48 | 88.51 |
| | 2.00 | 92.24 | 92.38 | 95.91 | 96.10 | 81.54 | 81.89 | 86.15 | 86.63 |
| 10 | 0.10 | 94.46 | 94.46 | 95.43 | 95.43 | 90.33 | 90.33 | 93.15 | 93.15 |
| | 0.20 | 94.44 | 94.44 | 95.44 | 95.44 | 90.39 | 90.39 | 93.14 | 93.14 |
| | 0.25 | 94.46 | 94.46 | 95.46 | 95.46 | 90.36 | 90.36 | 93.17 | 93.17 |
| | 0.35 | 94.46 | 94.46 | 95.49 | 95.49 | 90.34 | 90.34 | 93.16 | 93.15 |
| | 0.50 | 94.46 | 94.46 | 95.50 | 95.50 | 90.25 | 90.27 | 92.79 | 92.80 |
| | 1.00 | 94.25 | 94.25 | 95.45 | 95.50 | 89.60 | 89.72 | 92.09 | 92.12 |
| | 2.00 | 93.45 | 93.56 | 95.17 | 95.34 | 88.49 | 88.68 | 90.54 | 90.99 |
| 20 | 0.10 | 95.19 | 95.19 | 95.32 | 95.32 | 92.90 | 92.90 | 94.12 | 94.12 |
| | 0.20 | 95.17 | 95.17 | 95.32 | 95.32 | 92.85 | 92.85 | 94.19 | 94.19 |
| | 0.25 | 95.17 | 95.17 | 95.34 | 95.34 | 92.78 | 92.78 | 94.16 | 94.15 |
| | 0.35 | 95.17 | 95.17 | 95.37 | 95.37 | 92.74 | 92.74 | 94.11 | 94.11 |
| | 0.50 | 95.16 | 95.16 | 95.41 | 95.41 | 92.59 | 92.55 | 94.23 | 94.21 |
| | 1.00 | 95.06 | 95.07 | 95.41 | 95.42 | 92.32 | 92.36 | 93.68 | 93.71 |
| | 2.00 | 94.73 | 94.80 | 95.17 | 95.26 | 91.44 | 91.73 | 92.69 | 93.02 |
| 50 | 0.10 | 94.72 | 94.72 | 95.08 | 95.08 | 93.81 | 93.81 | 94.65 | 94.65 |
| | 0.20 | 94.70 | 94.70 | 95.08 | 95.08 | 93.80 | 93.80 | 94.60 | 94.60 |
| | 0.25 | 94.70 | 94.70 | 95.07 | 95.07 | 93.82 | 93.82 | 94.61 | 94.61 |
| | 0.35 | 94.69 | 94.69 | 95.06 | 95.06 | 93.82 | 93.79 | 94.56 | 94.56 |
| | 0.50 | 94.72 | 94.72 | 95.04 | 95.04 | 93.77 | 93.80 | 94.57 | 94.54 |
| | 1.00 | 94.59 | 94.59 | 95.04 | 95.05 | 93.56 | 93.63 | 94.20 | 94.26 |
| | 2.00 | 94.45 | 94.47 | 94.98 | 95.03 | 93.35 | 93.33 | 93.88 | 93.88 |
| 100 | 0.10 | 95.00 | 95.00 | 94.70 | 94.70 | 94.44 | 94.44 | 95.01 | 95.01 |
| | 0.20 | 94.99 | 94.99 | 94.70 | 94.70 | 94.48 | 94.48 | 94.94 | 94.94 |
| | 0.25 | 94.99 | 94.99 | 94.71 | 94.71 | 94.46 | 94.46 | 94.93 | 94.94 |
| | 0.35 | 95.00 | 95.00 | 94.71 | 94.71 | 94.46 | 94.46 | 94.96 | 94.97 |
| | 0.50 | 94.98 | 94.98 | 94.71 | 94.71 | 94.44 | 94.42 | 95.02 | 95.00 |
| | 1.00 | 94.96 | 94.96 | 94.73 | 94.73 | 94.46 | 94.40 | 94.70 | 94.85 |
| | 2.00 | 94.94 | 94.95 | 94.71 | 94.73 | 94.21 | 94.39 | 94.27 | 94.67 |

Table 11.5: Fatigue lifetime data presented by Birnbaum and Saunders (1969b)

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 70 | 90 | 96 | 97 | 99 | 100 | 103 | 104 | 104 | 105 | 107 | 108 | 108 | 108 | 109 |
| 109 | 112 | 112 | 113 | 114 | 114 | 114 | 116 | 119 | 120 | 120 | 120 | 121 | 121 | 123 |
| 124 | 124 | 124 | 124 | 124 | 128 | 128 | 129 | 129 | 130 | 130 | 130 | 131 | 131 | 131 |
| 131 | 131 | 132 | 132 | 132 | 133 | 134 | 134 | 134 | 134 | 134 | 136 | 136 | 137 | 138 |
| 138 | 138 | 139 | 139 | 141 | 141 | 142 | 142 | 142 | 142 | 142 | 142 | 144 | 144 | 145 |
| 146 | 148 | 148 | 149 | 151 | 151 | 152 | 155 | 156 | 157 | 157 | 157 | 157 | 158 | 159 |
| 162 | 163 | 163 | 164 | 166 | 166 | 168 | 170 | 174 | 196 | 212 | | | | |

Table 11.6: Point estimates of $\alpha$ and $\beta$ for Example 11.1

| Estimator | $\alpha$ | $\beta$ |
|---|---|---|
| MLE | 0.170385 | 131.818792 |
| MME | 0.170385 | 131.819255 |
| UMLE | 0.172089 | 131.809130 |
| UMME | 0.172089 | 131.809593 |
| JMLE | 0.172006 | 131.798227 |
| JMME | 0.172006 | 131.798661 |

## 11.8 Illustrative Examples

Practical application of the above estimators is illustrated here with two examples with one involving a large sample and the other with a small sample.
**Example 11.1:** The data set is given by Birnbaum and Saunders (1969b) on the fatigue life of 6061-T6 aluminum coupons cut parallel to the direction of rolling and oscillated at 18 cycles per second. The data set consists of 101 observations with maximum stress per cycle 31000 psi. The data are presented in Table 11.5.

In summary, we have in this case $n = 101$, $s = 133.73267$, and $r = 129.93321$. For this example, the point estimates of $\alpha$ and $\beta$ obtained by all the methods are summarized in Table 11.6.

From Equations (11.3.2), (11.4.3), (11.5.3) and (11.5.4), the asymptotic variance of the estimators can be readily obtained, and also the confidence intervals for $\alpha$

Table 11.7: Standard deviations of estimates and interval estimates of $\alpha$ and $\beta$ for Example 11.1

| | $\alpha$ | | | $\beta$ | | |
|---|---|---|---|---|---|---|
| Estimator | S.D. | 90% C.I. | 95% C.I. | S.D. | 90% C.I. | 95% C.I. |
| MLE | 0.0120 | (0.1527,0.1927) | (0.1497,0.1976) | 2.2267 | (128.2552,135.5861) | (127.5944,136.3325) |
| MME | 0.0120 | (0.1527,0.1927) | (0.1497,0.1976) | 2.2267 | (128.2556,135.5866) | (127.5948,136.3330) |
| UMLE | 0.0122 | (0.1541,0.1949) | (0.1511,0.1999) | 2.2487 | (128.2116,135.6143) | (127.5448,136.3685) |
| UMME | 0.0122 | (0.1541,0.1949) | (0.1511,0.1999) | 2.2487 | (128.2121,135.6148) | (127.5452,136.3690) |

Table 11.8: Fatigue lifetime data presented by McCool (1974)

| | | | | |
|---|---|---|---|---|
| 152.7 | 172.0 | 172.5 | 173.3 | 193.0 |
| 204.7 | 216.5 | 234.9 | 262.6 | 422.6 |

and $\beta$ based on the MLE's, MME's, UMLE's and UMME's can be readily constructed using the asymptotic normality. The results so obtained are presented in Table 11.7.

**Example 11.2:** This example is from McCool (1974) on the fatigue life in hours of ten bearings of a certain type. These data were used as an illustrative example for the three-parameter Weibull distribution by Cohen, Whitten and Ding (1984). The data are presented in Table 11.8.

In this case, we find $n = 10$, $s = 220.48$ and $r = 203.8853$. The point estimates and interval estimates obtained from this data are summarized in Tables 11.9 and 11.10, respectively.

It is heartening to observe in both these examples that the MME's are very nearly the same as the MLE's and also the corresponding confidence intervals. Also, while the unbiased estimators correct for the bias, they do result in a larger standard deviation.

Table 11.9: Point estimates of $\alpha$ and $\beta$ for Example 11.2

| Estimator | $\alpha$ | $\beta$ |
|---|---|---|
| MLE | 0.282489 | 212.049084 |
| MME | 0.282489 | 212.020378 |
| UMLE | 0.313877 | 211.528097 |
| UMME | 0.313877 | 211.499462 |
| JMLE | 0.323068 | 211.470377 |
| JMME | 0.323068 | 211.446452 |

Table 11.10: Standard deviations of estimates and interval estimates of $\alpha$ and $\beta$ for Example 11.2

| | $\alpha$ | | | $\beta$ | | |
|---|---|---|---|---|---|---|
| Estimator | S.D. | 90% C.I. | 95% C.I. | S.D. | 90% C.I. | 95% C.I. |
| MLE | 0.0632 | (0.2065,0.4468) | (0.1964,0.5029) | 18.7527 | (185.1207,248.1452) | (180.7241,256.5101) |
| MME | 0.0632 | (0.2065,0.4468) | (0.1964,0.5029) | 18.7504 | (185.0954,248.1121) | (180.6993,256.4760) |
| UMLE | 0.0780 | (0.2228,0.5308) | (0.2111,0.6118) | 20.7356 | (182.2191,252.0726) | (177.5074,261.6814) |
| UMME | 0.0780 | (0.2228,0.5308) | (0.2111,0.6118) | 20.7332 | (182.1940,252.0394) | (177.4828,261.6472) |

## 11.9 Concluding Remarks

Since the maximum likelihood estimators and the modified moment estimators behave very similarly in almost all cases considered, we recommend the use of the modified moment estimators because they are explicit estimators and are very easy to compute. Although the jackknifed estimators also work very well, they can not be recommended for large sample sizes as they are computationally quite involved. If the sample size is very small (say, less than 10), then bias corrected modified moment estimators or bias corrected jackknifed estimators should be used as in this case the original estimators can be highly biased. The asymptotic confidence intervals behave very well for large sample sizes (at least 20), but not for small sample sizes. In the latter case, one may rely on simulated percentage points rather than on the asymptotic normality.

# Chapter 12

# Estimation for Birnbaum-Saunders Distribution Under Type-II Censoring

## 12.1 Introduction

In the last chapter, we discussed the point and interval estimation for the two-parameter Birnbaum-Saunders Distribution based on complete samples. In this chapter, we first discuss the maximum likelihood estimation of the parameters $\alpha$ and $\beta$ based on Type-II right censored samples. We then derive the asymptotic variance-covariance matrix of the MLE's using which asymptotic confidence intervals for $\alpha$ and $\beta$ are proposed with the use of asymptotic normality of the MLE's. We evaluate the performance of the MLE's through Monte Carlo simulations for various sample sizes and degrees of censoring. Though the MLE's are asymptotically unbiased, these simulation results reveal that they are highly biased in case of small sample sizes particularly when the degree of censoring is high. We, therefore, propose a simple bias correction technique which performs quite well even for small sample sizes. Asymptotic confidence intervals for $\alpha$ and $\beta$ based on these bias-corrected estimators are then proposed. Next, a Monte Carlo EM-algorithm for the determination of the MLE's is discussed. A comparison of all the estimators and the probability coverages of confidence intervals based on inferential quantities associated with these estimators is made using Monte Carlo simulations. We present two examples to illustrate all the methods of inference discussed here. Finally, we make some concluding remarks.

## 12.2 Maximum Likelihood Estimators

Let $\{t_{(1)}, t_{(2)}, \ldots, t_{(r)}\}$ be an ordered Type-II right censored random sample obtained from $n$ units placed on a life-testing experiment wherein each unit has its lifetime following the Birnbaum-Saunders distribution with p.d.f. as in (11.2.1), with the largest $(n-r)$ lifetimes having been censored. Then, the likelihood function is given by [Balakrishnan and Cohen (1991)]

$$\begin{aligned} L &= \frac{n!}{(n-r)!}\left\{1-\Phi\left[\frac{1}{\alpha}\xi\left(\frac{t_{(r)}}{\beta}\right)\right]\right\}^{n-r} \\ &\times\left\{\frac{1}{(\sqrt{2\pi}\alpha\beta)^r}\left[\prod_{i=1}^{r}\xi'\left(\frac{t_{(i)}}{\beta}\right)\right]\exp\left[-\frac{1}{2\alpha^2}\sum_{i=1}^{r}\xi^2\left(\frac{t_{(i)}}{\beta}\right)\right]\right\}, \end{aligned} \tag{12.2.1}$$

and the log-likelihood function is

$$\begin{aligned} \ln L &= \text{constant} + (n-r)\ln\left\{1-\Phi\left[\frac{1}{\alpha}\xi\left(\frac{t_{(r)}}{\beta}\right)\right]\right\} - r\ln\alpha - r\ln\beta \\ &+\sum_{i=1}^{r}\xi'\left(\frac{t_{(i)}}{\beta}\right) - \frac{1}{2\alpha^2}\sum_{i=1}^{r}\xi^2\left(\frac{t_{(i)}}{\beta}\right); \end{aligned} \tag{12.2.2}$$

here,

$$\begin{aligned} \xi(t) &= t^{\frac{1}{2}} - t^{-\frac{1}{2}}, \\ \xi^2(t) &= t + t^{-1} - 2, \\ \xi'(t) &= \frac{1}{2t}\left(t^{\frac{1}{2}} + t^{-\frac{1}{2}}\right) = \frac{1}{2\xi(t)}\left(1 - \frac{1}{t^2}\right), \\ \frac{t\xi''(t)}{\xi'(t)} &= -\frac{1}{2} - \frac{1}{t+1}. \end{aligned}$$

For notational convenience, let us denote

$$t^*_{(i)} = \frac{t_{(i)}}{\beta}$$

and the hazard function of the standard normal distribution as

$$H(x) = \frac{\phi(x)}{1 - \Phi(x)}.$$

Then, we find from (12.2.2)

$$\begin{aligned} -\frac{\alpha^3}{r}\frac{\partial \ln L}{\partial \alpha} &= -\frac{\alpha(n-r)}{r} H\left[\frac{1}{\alpha}\xi\left(t^*_{(r)}\right)\right]\xi\left(t^*_{(r)}\right) + \alpha^2 \\ &\quad -\frac{1}{r}\sum_{i=1}^{r}\xi^2\left(t^*_{(i)}\right), \end{aligned} \tag{12.2.3}$$

$$\begin{aligned} -\frac{\alpha^2\beta}{r}\frac{\partial \ln L}{\partial \beta} &= -\frac{\alpha(n-r)}{r} H\left[\frac{1}{\alpha}\xi\left(t^*_{(r)}\right)\right] t^*_{(r)}\xi'\left(t^*_{(r)}\right) + \alpha^2 \\ &\quad +\frac{\alpha^2}{r}\sum_{i=1}^{r}\frac{t^*_{(i)}\xi''\left(t^*_{(i)}\right)}{\xi'\left(t^*_{(i)}\right)} - \frac{1}{r}\sum_{i=1}^{r} t^*_{(i)}\xi\left(t^*_{(i)}\right)\xi'\left(t^*_{(i)}\right). \end{aligned} \tag{12.2.4}$$

By equating (12.2.3) and (12.2.4) to zero, we obtain the likelihood equations as

$$\alpha^2 - g(\alpha,\beta)h_1(\beta) + h_2(\beta) = 0, \tag{12.2.5}$$

$$\alpha^2 - g(\alpha,\beta)h_3(\beta) + h_4(\beta) = 0, \tag{12.2.6}$$

where

$$\begin{aligned} g(\alpha,\beta) &= \frac{\alpha(n-r)}{r} H\left[\frac{1}{\alpha}\xi\left(t^*_{(r)}\right)\right], \\ h_1(\beta) &= \xi\left(t^*_{(r)}\right), \\ h_2(\beta) &= -\frac{1}{r}\sum_{i=1}^{r}\xi^2\left(t^*_{(i)}\right), \\ h_3(\beta) &= \left[1 + \sum_{i=1}^{r}\frac{t^*_{(i)}\xi''\left(t^*_{(i)}\right)}{\xi'\left(t^*_{(i)}\right)}\right]^{-1} t^*_{(r)}\xi'\left(t^*_{(r)}\right), \\ h_4(\beta) &= \left[1 + \sum_{i=1}^{r}\frac{t^*_{(i)}\xi''\left(t^*_{(i)}\right)}{\xi'\left(t^*_{(i)}\right)}\right]^{-1}\left[-\frac{1}{r}\sum_{i=1}^{r} t^*_{(i)}\xi\left(t^*_{(i)}\right)\xi'\left(t^*_{(i)}\right)\right]. \end{aligned}$$

From (12.2.5) and (12.2.6), $\alpha$ can be written as a pure function of $\beta$ as

$$\alpha^2 = \frac{h_2(\beta)h_3(\beta) - h_1(\beta)h_4(\beta)}{h_1(\beta) - h_3(\beta)} \quad (= \varphi^2(\beta),\ \text{say}). \tag{12.2.7}$$

Let

$$
\begin{aligned}
u^* &= \frac{1}{r}\sum_{i=1}^{r} t^*_{(i)}, \\
v^* &= \left[\frac{1}{r}\sum_{i=1}^{r} {t^*_{(i)}}^{-1}\right]^{-1}, \\
K^*(\beta) &= \left[\frac{1}{r}\sum_{i=1}^{r}(1+t^*_{(i)})^{-1}\right]^{-1}, \\
K'^*(\beta) &= [K^*(\beta)]^2\left[\frac{1}{r}\sum_{i=1}^{r}(1+t^*_{(i)})^{-2}\right];
\end{aligned}
$$

then (12.2.4) can be rewritten as

$$
\begin{aligned}
Q(\beta) &= \varphi^2(\beta)\left[\frac{1}{2}-\frac{1}{K^*(\beta)}\right]-\frac{u^*}{2}+\frac{1}{2v^*} \\
&\quad -\frac{\varphi(\beta)(n-r)}{r}H\left[\frac{1}{\varphi(\beta)}\xi\left(t^*_{(r)}\right)\right]t^*_{(r)}\xi'\left(t^*_{(r)}\right). \qquad (12.2.8)
\end{aligned}
$$

The maximum likelihood estimate of $\beta$ is the solution of $Q(\beta)=0$. Since $Q(\beta)=0$ is a non-linear equation, one needs to use a numerical procedure to solve for $\beta$. Once we have the maximum likelihood estimate $\hat{\beta}$ of $\beta$, the maximum likelihood estimate $\hat{\alpha}$ of $\alpha$ can be obtained as the positive root of the right-hand side of (12.2.7).

In order to construct confidence intervals for $\alpha$ and $\beta$ using the MLE's, we need the variance-covariance matrix of the MLE's. We define

$$
\begin{aligned}
Q'(\beta) &= \frac{1}{\beta}\Bigg\{-\varphi^2(\beta)\left[\frac{1}{K^*(\beta)}-\frac{1}{r}\sum_{i=1}^{r}(1+t^*_{(i)})^{-2}\right]+\frac{u^*}{2}+\frac{1}{2v^*} \\
&\quad +\frac{\varphi(\beta)(n-r)}{r}t^*_{(r)}\Bigg[H\left[\frac{1}{\varphi(\beta)}\xi\left(t^*_{(r)}\right)\right]\left[t^*_{(r)}\xi''\left(t^*_{(r)}\right)+\xi'\left(t^*_{(r)}\right)\right] \\
&\quad +\frac{t^*_{(r)}}{\varphi(\beta)}\left[\xi'\left(t^*_{(r)}\right)\right]^2 H'\left[\frac{1}{\varphi(\beta)}\xi\left(t^*_{(r)}\right)\right]\Bigg]\Bigg\} \\
&= \frac{1}{\beta}A_{31}\text{ (say)},
\end{aligned}
$$

where $H'(x)=-xH(x)+H^2(x)$. For the observed information matrix for $\alpha$ and $\beta$, we find

$$
-\frac{\partial^2\ln L}{\partial\alpha^2} = \frac{(n-r)}{\alpha^4}\xi\left(t^*_{(r)}\right)\left\{2\alpha H\left[\frac{1}{\alpha}\xi\left(t^*_{(r)}\right)\right]+\xi\left(t^*_{(r)}\right)H'\left[\frac{1}{\alpha}\xi\left(t^*_{(r)}\right)\right]\right\}
$$

$$
\begin{aligned}
& -\frac{r}{\alpha^2}+\frac{3}{\alpha^4}\sum_{i=1}^{r}\xi^2\left(t^*_{(r)}\right)=A_1,\\
-\frac{\partial^2 \ln L}{\partial\alpha\partial\beta} &= \frac{1}{\alpha^3\beta}\left\{(n-r)\left[\alpha t^*_{(r)}\xi'\left(t^*_{(r)}\right)H\left[\frac{1}{\alpha}\xi\left(t^*_{(r)}\right)\right]\right.\right.\\
&\qquad \left.\left.+t^*_{(r)}\xi\left(t^*_{(r)}\right)\xi'\left(t^*_{(r)}\right)H'\left[\frac{1}{\alpha}\xi\left(t^*_{(r)}\right)\right]\right]+2\sum_{i=1}^{r}t^*_{(i)}\xi\left(t^*_{(i)}\right)\xi'\left(t^*_{(i)}\right)\right\}\\
&= \frac{1}{\beta}A_2,\\
-\frac{\partial^2 \ln L}{\partial\beta^2} &= \frac{r}{(\alpha\beta)^2}[Q(\beta)+A_{31}]=\frac{1}{\beta^2}A_3.
\end{aligned}
$$

Then the observed information matrix is given by

$$
\begin{pmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{pmatrix}=\begin{pmatrix} A_1 & \frac{1}{\beta}A_2 \\ \frac{1}{\beta}A_2 & \frac{1}{\beta^2}A_3 \end{pmatrix},
$$

so that the variance-covariance matrix may be approximated as

$$
\begin{aligned}
\mathbf{V}=\begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix} &= \begin{pmatrix} A_1 & \frac{1}{\beta}A_2 \\ \frac{1}{\beta}A_2 & \frac{1}{\beta^2}A_3 \end{pmatrix}^{-1}\\
&= \frac{\beta^2}{A_1A_3-A_2^2}\begin{pmatrix} \frac{1}{\beta^2}A_3 & -\frac{1}{\beta}A_2 \\ -\frac{1}{\beta}A_2 & A_1 \end{pmatrix}.
\end{aligned}
$$

The asymptotic joint distribution of $\hat{\alpha}$ and $\hat{\beta}$ is then approximately bivariate normal, and is given by

$$
\begin{pmatrix} \hat{\alpha} \\ \hat{\beta} \end{pmatrix} \overset{appr.}{\sim} N\left[\begin{pmatrix} \alpha \\ \beta \end{pmatrix},\begin{pmatrix} V_{11} & V_{12} \\ V_{12} & V_{22} \end{pmatrix}\right]. \tag{12.2.9}
$$

Since $\mathbf{V}$ involves the parameters $\alpha$ and $\beta$, we replace the parameters by the corresponding MLE's in order to obtain an estimate of $\mathbf{V}$, which is denoted by

$$
\hat{\mathbf{V}}=\begin{pmatrix} \hat{V}_{11} & \hat{V}_{12} \\ \hat{V}_{12} & \hat{V}_{22} \end{pmatrix} = \frac{\hat{\beta}^2}{\hat{A}_1\hat{A}_3-\hat{A}_2^2}\begin{pmatrix} \frac{1}{\hat{\beta}^2}\hat{A}_3 & -\frac{1}{\hat{\beta}}\hat{A}_2 \\ -\frac{1}{\hat{\beta}}\hat{A}_2 & \hat{A}_1 \end{pmatrix}.
$$

By using (12.2.9), approximate $100(1-\gamma)\%$ confidence intervals for $\alpha$ and $\beta$ are determined as

$$
\left[\hat{\alpha}-z_{\gamma/2}\sqrt{\hat{V}_{11}},\hat{\alpha}+z_{\gamma/2}\sqrt{\hat{V}_{11}}\right],
$$

and

$$\left[\hat{\beta}\left(z_{\gamma/2}\sqrt{\frac{\hat{A}_1}{\hat{A}_1\hat{A}_3-\hat{A}_2^2}}+1\right)^{-1}, \hat{\beta}\left(z_{1-\gamma/2}\sqrt{\frac{\hat{A}_1}{\hat{A}_1\hat{A}_3-\hat{A}_2^2}}+1\right)^{-1}\right], \tag{12.2.10}$$

where $z_p$ is the upper $p$-th percentile of the standard normal distribution.

## 12.3 Bias-corrected Estimators

Based on the results of an extensive Monte Carlo simulation study and inspecting the pattern of the bias of the MLE of $\alpha$, we observed that

$$Bias(\hat{\alpha}) \approx -\frac{\alpha}{n}\left[1+2.5\left(1-\frac{r}{n}\right)\right].$$

Then, by employing a standard bias-reduction method, we can simply construct an almost unbiased maximum likelihood estimator (MULE, denoted by $\hat{\alpha}^*$) of $\alpha$. The bias-corrected estimator thus obtained is given by

$$\hat{\alpha}^* = \hat{\alpha}\left\{1-\frac{1}{n}\left[1+2.5\left(1-\frac{r}{n}\right)\right]\right\}^{-1}. \tag{12.3.1}$$

From the distributional results presented in (12.2.9), we then readily have the asymptotic joint distribution of $\hat{\alpha}^*$ and $\hat{\beta}$ to be bivariate normal and is given by

$$\begin{pmatrix}\hat{\alpha}^*\\ \hat{\beta}\end{pmatrix} \overset{appr.}{\sim} N\left[\begin{pmatrix}\alpha\\ \beta\end{pmatrix}, \begin{pmatrix}W^2V_{11} & WV_{12}\\ WV_{12} & V_{22}\end{pmatrix}\right]. \tag{12.3.2}$$

where

$$W = \left\{1-\frac{1}{n}\left[1+2.5\left(1-\frac{r}{n}\right)\right]\right\}^{-1}$$

**Remark:** *Though the results presented here are for right-censored samples, they are also useful when the observed sample is Type-II left-censored. This is due to the fact that if* $t_{(n-r+1)}, t_{(n-r+2)}, \ldots, t_{(n)}$ *is a left-censored sample from Birnbaum-Saunders distribution with parameters* $\alpha$ *and* $\beta$, *then* $t_1^* = \frac{1}{t_{(n)}}$, $t_2^* = \frac{1}{t_{(n-1)}}$, $\ldots$, $t_r^* = \frac{1}{t_{(n-r+1)}}$ *form a right-censored sample from Birnbaum-Saunders distribution with parameters* $\alpha$ *and* $\frac{1}{\beta}$.

## 12.4 Monte-Carlo EM-Algorithm

The Type-II right censored data can be viewed as an incomplete data problem, and consequently the EM-algorithm is applicable to determine the maximum likelihood estimates of the parameters. First of all, denote the observed and censored data by $\mathbf{T} = (T_{(1)}, T_{(2)}, \ldots, T_{(r)})$ and $\mathbf{Y} = (Y_1, Y_2, \ldots, Y_{n-r})$, respectively. The censored data vector $\mathbf{Y}$ can be thought of as the missing data. Combine $\mathbf{T}$ and $\mathbf{Y}$ to form $\mathbf{X}$ which is the complete data. In order to facilitate the EM-algorithm, the conditional distribution of $\mathbf{Y}$, conditional on $\mathbf{T}$ and the current guess of the parameters, needs to be determined. Due to the Markovian property of order statistics, the conditional density of $Y_j, j = 1, 2 \ldots, n-r$, conditioned on $T_{(1)} = t_{(1)}, \ldots, T_{(r)} = t_{(r)}$, is [see David (1981) and Arnold, Balakrishnan and Nagaraja (1992)]

$$
\begin{aligned}
& f_{Y|T}(y_j \mid T_{(1)} = t_{(1)}, T_{(2)} = t_{(2)}, T_{(r)} = t_{(r)}) \\
= \; & f_{Y|T}(y_j \mid T_{(r)} = t_{(r)}) = \frac{f_T(y_j)}{1 - F_T(t_{(r)})}, \qquad y_j > t_{(r)}; \qquad (12.4.1)
\end{aligned}
$$

in other words, given $T_{(r)} = t_{(r)}$, $\mathbf{Y}$ forms a random sample from the left-truncated Birnbaum-Saunders distribution, and hence the expectations of functions of $\mathbf{Y}$ can be derived.

Let

$$
s = \frac{1}{n}\sum_{i=1}^{n} x_i, \qquad r = \left[\frac{1}{n}\sum_{i=1}^{n} x_i^{-1}\right]^{-1},
$$

$$
K(\beta) = \left[\frac{1}{n}\sum_{i=1}^{n}(\beta + x_i)^{-1}\right]^{-1} \qquad \text{for } \beta \geq 0,
$$

and

$$
K'(\beta) = [K(\beta)]^2 \left[\frac{1}{n}\sum_{i=1}^{n}(\beta + x_i)^{-1}\right]^{-2} \qquad \text{for } \beta \geq 0.
$$

For the complete sample case, the maximum likelihood estimate of $\beta$ can be obtained as the unique positive root of the equation [see Birnbaum and Saunders (1969b) and Engelhardt, Bain and Wright (1981)]

$$
g(\beta) = \beta^2 - \beta[2v + K(\beta)] + v[u + K(\beta)] = 0. \qquad (12.4.2)
$$

Once $\hat{\beta}$ is obtained as a solution of (12.4.2), the MLE of $\alpha$ can be obtained explicitly as

$$\hat{\alpha} = \left[\frac{u}{\hat{\beta}} + \frac{\hat{\beta}}{v} - 2\right]^{\frac{1}{2}}.$$

If Newton-Raphson method is instead used to solve (12.4.2), we would require

$$g'(\beta) = 2\beta - 2v + (v - \beta)K'(\beta) - K(\beta).$$

E-step

In the (h+1)-th iteration, we need

$$\begin{aligned}
E_1 &= E\left[Y \,|\, t_{(r)}, \alpha_{(h)}, \beta_{(h)}\right], \\
E_2 &= E\left[\frac{1}{Y}\middle| t_{(r)}, \alpha_{(h)}, \beta_{(h)}\right], \\
E_3 &= E\left[\frac{1}{(\beta_{(h)} + Y)}\middle| t_{(r)}, \alpha_{(h)}, \beta_{(h)}\right], \\
E_4 &= E\left[\frac{1}{(\beta_{(h)} + Y)^2}\middle| t_{(r)}, \alpha_{(h)}, \beta_{(h)}\right],
\end{aligned}$$

which can be estimated as follows using the Monte Carlo method:

1. Simulate $l$ random variates from the left-truncated Birnbaum-Saunders distribution with parameters $\alpha = \alpha_{(h)}$ and $\beta = \beta_{(h)}$ and truncation point at $t_{(r)}$, and denote them by $Y_1, Y_2, \ldots, Y_l$.

2. Compute the average values of the required functions as estimates of these functions

$$\hat{E}_1 = \frac{1}{l}\sum_{i=1}^{l} Y_i, \quad \hat{E}_2 = \frac{1}{l}\sum_{i=1}^{l}\frac{1}{Y_i}, \quad \hat{E}_3 = \frac{1}{l}\sum_{i=1}^{l}\frac{1}{(\beta_{(h)} + Y_i)}, \quad \hat{E}_4 = \frac{1}{l}\sum_{i=1}^{l}\frac{1}{(\beta_{(h)} + Y_i)^2}.$$

3. Compute

$$\begin{aligned}
\hat{u}_{(h)} &= \frac{1}{n}\left[\sum_{i=1}^{r} t_i + (n-r)\hat{E}_1\right], \\
\hat{v}_{(h)} &= \left\{\frac{1}{n}\left[\sum_{i=1}^{r} t_i^{-1} + (n-r)\hat{E}_2\right]\right\}^{-1}, \\
\hat{K}(\beta_{(h)}) &= \left\{\frac{1}{n}\left[\sum_{i=1}^{r}(\beta_{(h)} + t_i)^{-1} + (n-r)\hat{E}_3\right]\right\}^{-1}, \\
\hat{K}'(\beta_{(h)}) &= \frac{\left[\hat{K}(\beta_{(h)})\right]^2}{n}\left[\sum_{i=1}^{r}(\beta_{(h)} + t_i)^{-2} + (n-r)\hat{E}_4\right].
\end{aligned}$$

M-step

In the (h+1)-th iteration, $\beta_{(h+1)}$ can be obtained as the unique positive root of the equation [see Eq. (12.4.2)]

$$g(\beta) = \beta^2 - \beta[2\hat{v}_{(h)} + K(\beta)] + \hat{v}_{(h)}[\hat{u}_{(h)} + K(\beta)] = 0. \tag{12.4.3}$$

Once $\beta_{(h+1)}$ is obtained, $\alpha_{(h+1)}$ can be obtained explicitly as

$$\alpha_{(h+1)} = \left[\frac{\hat{u}_{(h)}}{\beta_{(h+1)}} + \frac{\beta_{(h+1)}}{\hat{v}_{(h)}} - 2\right]^{\frac{1}{2}}.$$

If Newton-Raphson method is instead used to solve Eq. (12.4.3), we would require

$$g'(\beta) = 2\beta_{(h)} - 2\hat{v}_{(h)} + (\hat{v}_{(h)} - \beta_{(h)})\hat{K}'(\beta_{(h)}) - \hat{K}(\beta_{(h)}).$$

Birnbaum and Saunders (1969b) showed that the maximum likelihood estimates are equivalent to the modified moment estimates (MME's) under certain conditions, in the case of complete samples. Therefore, instead of solving Eq. (12.4.3) by a numerical procedure, we can obtain the estimates of $\alpha$ and $\beta$ (using the modified moment estimates) in the M-step as

$$\begin{aligned}
\alpha_{(h+1)} &= \left\{2\left[\left(\frac{\hat{u}_{(h)}}{\hat{v}_{(h)}}\right)^{\frac{1}{2}} - 1\right]\right\}^{\frac{1}{2}}, \\
\beta_{(h+1)} &= \left(\hat{u}_{(h)}\hat{v}_{(h)}\right)^{\frac{1}{2}}.
\end{aligned}$$

The EM iterations are repeated for $I = 3000$ times and the burn-in period is taken as $J = 800$, with the final estimate of $\alpha$ and $\beta$ taken as

$$\begin{aligned}
\hat{\alpha} &= \frac{1}{I-J}\sum_{i=J+1}^{I} \alpha_{(i)}, \\
\hat{\beta} &= \frac{1}{I-J}\sum_{i=J+1}^{I} \beta_{(i)}.
\end{aligned}$$

## 12.5 Monte Carlo Simulation Results

In order to compare the performance of all the above estimators, we performed a simulation study for different sample sizes and different parameter values with varying degrees of censoring. We took the sample size as $n = 20, 30, 50$, the shape parameter as $\alpha = .10, .30, .50, 1.00, 2.00$, and the degree of censoring as 0(10)60 %. The scale parameter $\beta$ was kept fixed at 1.0, without loss of any generality. All the results were based on 10,000 Monte Carlo runs.

In Tables 12.1-12.3, we have presented the average values, standard errors and covariances of the estimates $\hat{\alpha}$, $\hat{\alpha}^*$ and $\hat{\beta}$. From these tables, we readily observe that while the bias of $\hat{\alpha}$ increases as the degree of censoring increases for all sample sizes considered, the simple bias-corrected estimator $\hat{\alpha}^*$ remains almost unbiased in all cases. The unbiasedness of $\hat{\alpha}^*$ comes with a slightly larger standard deviation than that of $\hat{\alpha}$. Also, the estimator $\hat{\beta}$ is almost unbiased in all cases considered.

In Tables 12.4 and 12.5, we have presented the probability coverages of 90% and 95% confidence intervals for $\alpha$ and $\beta$ based on MLE's as well as UMLE's. From these tables, we note that the probability coverages of confidence intervals based on MLE's become considerably smaller than the nominal levels particularly when the degree of censoring increases; however, the probability coverages of confidence intervals based on UMLE's remain close to the nominal levels.

Table 12.1: Means, standard deviations and covariance of estimates based on Monte Carlo simulations for $n = 20$ ($\beta = 1.0$)

| $\alpha$ | d.o.c. | $\hat{\alpha}$ | SD($\hat{\alpha}$) | $\hat{\alpha}^*$ | SD($\hat{\alpha}^*$) | $\hat{\beta}$ | SD($\hat{\beta}$) | Cov($\hat{\alpha}$, $\hat{\beta}$) | Cov($\hat{\alpha}^*$, $\hat{\beta}$) |
|---|---|---|---|---|---|---|---|---|---|
| 0.1 | 0% | 0.0964 | 0.0155 | 0.1014 | 0.0164 | 1.0001 | 0.0225 | 0.0000 | 0.0000 |
| | 10% | 0.0956 | 0.0169 | 0.1020 | 0.0180 | 0.9997 | 0.0227 | 0.0000 | 0.0000 |
| | 20% | 0.0949 | 0.0182 | 0.1026 | 0.0197 | 0.9991 | 0.0233 | 0.0001 | 0.0001 |
| | 30% | 0.0941 | 0.0198 | 0.1031 | 0.0217 | 0.9984 | 0.0241 | 0.0001 | 0.0001 |
| | 40% | 0.0930 | 0.0218 | 0.1034 | 0.0243 | 0.9974 | 0.0255 | 0.0002 | 0.0002 |
| | 50% | 0.0914 | 0.0243 | 0.1030 | 0.0273 | 0.9956 | 0.0275 | 0.0003 | 0.0003 |
| | 60% | 0.0887 | 0.0272 | 0.1014 | 0.0311 | 0.9923 | 0.0310 | 0.0005 | 0.0005 |
| 0.3 | 0% | 0.2889 | 0.0466 | 0.3041 | 0.0491 | 1.0018 | 0.0671 | 0.0000 | 0.0000 |
| | 10% | 0.2866 | 0.0506 | 0.3057 | 0.0540 | 1.0005 | 0.0678 | 0.0002 | 0.0002 |
| | 20% | 0.2845 | 0.0546 | 0.3076 | 0.0591 | 0.9990 | 0.0694 | 0.0005 | 0.0006 |
| | 30% | 0.2822 | 0.0595 | 0.3092 | 0.0652 | 0.9970 | 0.0720 | 0.0010 | 0.0011 |
| | 40% | 0.2790 | 0.0658 | 0.3100 | 0.0731 | 0.9940 | 0.0762 | 0.0017 | 0.0019 |
| | 50% | 0.2742 | 0.0733 | 0.3089 | 0.0826 | 0.9891 | 0.0823 | 0.0027 | 0.0031 |
| | 60% | 0.2661 | 0.0826 | 0.3041 | 0.0944 | 0.9800 | 0.0923 | 0.0043 | 0.0050 |
| 0.5 | 0% | 0.4811 | 0.0776 | 0.5065 | 0.0817 | 1.0053 | 0.1101 | 0.0000 | 0.0000 |
| | 10% | 0.4773 | 0.0843 | 0.5091 | 0.0899 | 1.0030 | 0.1116 | 0.0006 | 0.0007 |
| | 20% | 0.4738 | 0.0912 | 0.5122 | 0.0986 | 1.0007 | 0.1146 | 0.0016 | 0.0017 |
| | 30% | 0.4698 | 0.0996 | 0.5148 | 0.1091 | 0.9977 | 0.1193 | 0.0029 | 0.0032 |
| | 40% | 0.4645 | 0.1107 | 0.5161 | 0.1229 | 0.9932 | 0.1267 | 0.0050 | 0.0056 |
| | 50% | 0.4565 | 0.1240 | 0.5144 | 0.1397 | 0.9857 | 0.1372 | 0.0081 | 0.0091 |
| | 60% | 0.4430 | 0.1405 | 0.5063 | 0.1606 | 0.9719 | 0.1542 | 0.0129 | 0.0147 |
| 1.0 | 0% | 0.9596 | 0.1554 | 1.0101 | 0.1635 | 1.0190 | 0.2053 | -0.0004 | -0.0005 |
| | 10% | 0.9514 | 0.1690 | 1.0148 | 0.1803 | 1.0140 | 0.2108 | 0.0030 | 0.0032 |
| | 20% | 0.9439 | 0.1839 | 1.0204 | 0.1988 | 1.0098 | 0.2197 | 0.0076 | 0.0082 |
| | 30% | 0.9356 | 0.2030 | 1.0253 | 0.2225 | 1.0051 | 0.2323 | 0.0142 | 0.0156 |
| | 40% | 0.9249 | 0.2295 | 1.0276 | 0.2550 | 0.9988 | 0.2514 | 0.0247 | 0.0275 |
| | 50% | 0.9092 | 0.2632 | 1.0244 | 0.2965 | 0.9883 | 0.2783 | 0.0405 | 0.0456 |
| | 60% | 0.8826 | 0.3080 | 1.0087 | 0.3520 | 0.9695 | 0.3221 | 0.0670 | 0.0766 |
| 2.0 | 0% | 1.9054 | 0.3177 | 2.0057 | 0.3344 | 1.0462 | 0.3288 | -0.0035 | -0.0037 |
| | 10% | 1.8925 | 0.3436 | 2.0150 | 0.3666 | 1.0383 | 0.3496 | 0.0116 | 0.0132 |
| | 20% | 1.8719 | 0.3791 | 2.0254 | 0.4124 | 1.0329 | 0.3748 | 0.0340 | 0.0378 |
| | 30% | 1.8481 | 0.4297 | 2.0213 | 0.4722 | 1.0208 | 0.4094 | 0.0678 | 0.0746 |
| | 40% | 1.8207 | 0.5054 | 2.0106 | 0.5584 | 1.0079 | 0.4576 | 0.1268 | 0.1343 |
| | 50% | 1.7886 | 0.6158 | 1.9976 | 0.6842 | 1.0058 | 0.5578 | 0.2342 | 0.2529 |
| | 60% | 1.7493 | 0.8341 | 1.9992 | 0.9532 | 1.0481 | 0.9194 | 0.6046 | 0.6910 |

Table 12.2: Means, standard deviations and covariance of estimates based on Monte Carlo simulations for $n = 30$ ($\beta = 1.0$)

| $\alpha$ | d.o.c. | $\hat{\alpha}$ | SD($\hat{\alpha}$) | $\hat{\alpha}^*$ | SD($\hat{\alpha}^*$) | $\hat{\beta}$ | SD($\hat{\beta}$) | Cov($\hat{\alpha}$, $\hat{\beta}$) | Cov($\hat{\alpha}^*$, $\hat{\beta}$) |
|---|---|---|---|---|---|---|---|---|---|
| 0.1 | 0% | 0.0975 | 0.0127 | 0.1008 | 0.0132 | 1.0000 | 0.0185 | 0.0000 | 0.0000 |
| | 10% | 0.0970 | 0.0138 | 0.1012 | 0.0144 | 0.9997 | 0.0187 | 0.0000 | 0.0000 |
| | 20% | 0.0965 | 0.0150 | 0.1016 | 0.0158 | 0.9993 | 0.0191 | 0.0000 | 0.0000 |
| | 30% | 0.0961 | 0.0163 | 0.1020 | 0.0173 | 0.9989 | 0.0198 | 0.0001 | 0.0001 |
| | 40% | 0.0953 | 0.0180 | 0.1021 | 0.0193 | 0.9982 | 0.0209 | 0.0001 | 0.0001 |
| | 50% | 0.0941 | 0.0200 | 0.1017 | 0.0216 | 0.9969 | 0.0226 | 0.0002 | 0.0002 |
| | 60% | 0.0924 | 0.0227 | 0.1008 | 0.0247 | 0.9948 | 0.0258 | 0.0003 | 0.0004 |
| 0.3 | 0% | 0.2923 | 0.0382 | 0.3024 | 0.0395 | 1.0010 | 0.0551 | 0.0000 | 0.0000 |
| | 10% | 0.2910 | 0.0415 | 0.3036 | 0.0433 | 1.0002 | 0.0557 | 0.0001 | 0.0002 |
| | 20% | 0.2894 | 0.0451 | 0.3047 | 0.0475 | 0.9991 | 0.0569 | 0.0004 | 0.0004 |
| | 30% | 0.2881 | 0.0490 | 0.3060 | 0.0520 | 0.9980 | 0.0590 | 0.0007 | 0.0007 |
| | 40% | 0.2859 | 0.0542 | 0.3063 | 0.0581 | 0.9959 | 0.0624 | 0.0011 | 0.0012 |
| | 50% | 0.2822 | 0.0605 | 0.3051 | 0.0654 | 0.9922 | 0.0676 | 0.0018 | 0.0020 |
| | 60% | 0.2771 | 0.0688 | 0.3022 | 0.0751 | 0.9864 | 0.0773 | 0.0031 | 0.0034 |
| 0.5 | 0% | 0.4869 | 0.0636 | 0.5034 | 0.0663 | 1.0032 | 0.0904 | 0.0000 | 0.0000 |
| | 10% | 0.4846 | 0.0691 | 0.5055 | 0.0720 | 1.0018 | 0.0916 | 0.0004 | 0.0005 |
| | 20% | 0.4821 | 0.0753 | 0.5080 | 0.0787 | 1.0001 | 0.0940 | 0.0011 | 0.0011 |
| | 30% | 0.4799 | 0.0821 | 0.5089 | 0.0867 | 0.9984 | 0.0978 | 0.0020 | 0.0021 |
| | 40% | 0.4761 | 0.0911 | 0.5090 | 0.0968 | 0.9952 | 0.1038 | 0.0034 | 0.0035 |
| | 50% | 0.4700 | 0.1023 | 0.5066 | 0.1094 | 0.9895 | 0.1128 | 0.0055 | 0.0058 |
| | 60% | 0.4614 | 0.1173 | 0.5023 | 0.1261 | 0.9810 | 0.1296 | 0.0092 | 0.0097 |
| 1.0 | 0% | 0.9721 | 0.1273 | 1.0056 | 0.1317 | 1.0119 | 0.1674 | -0.0001 | -0.0001 |
| | 10% | 0.9671 | 0.1387 | 1.0091 | 0.1447 | 1.0090 | 0.1722 | 0.0022 | 0.0023 |
| | 20% | 0.9616 | 0.1522 | 1.0122 | 0.1602 | 1.0058 | 0.1792 | 0.0054 | 0.0057 |
| | 30% | 0.9571 | 0.1676 | 1.0164 | 0.1780 | 1.0036 | 0.1897 | 0.0098 | 0.0104 |
| | 40% | 0.9494 | 0.1895 | 1.0172 | 0.2031 | 0.9989 | 0.2051 | 0.0169 | 0.0181 |
| | 50% | 0.9372 | 0.2176 | 1.0132 | 0.2353 | 0.9904 | 0.2278 | 0.0276 | 0.0298 |
| | 60% | 0.9207 | 0.2591 | 1.0044 | 0.2826 | 0.9802 | 0.2712 | 0.0484 | 0.0528 |
| 2.0 | 0% | 1.9388 | 0.2585 | 2.0056 | 0.2674 | 1.0300 | 0.2579 | -0.0011 | -0.0012 |
| | 10% | 1.9274 | 0.2811 | 2.0109 | 0.2929 | 1.0222 | 0.2767 | 0.0088 | 0.0101 |
| | 20% | 1.9123 | 0.3142 | 2.0168 | 0.3279 | 1.0152 | 0.2992 | 0.0249 | 0.0263 |
| | 30% | 1.8985 | 0.3567 | 2.0141 | 0.3770 | 1.0124 | 0.3324 | 0.0490 | 0.0517 |
| | 40% | 1.8788 | 0.4231 | 2.0090 | 0.4474 | 1.0092 | 0.3813 | 0.0910 | 0.0932 |
| | 50% | 1.8536 | 0.5198 | 1.9966 | 0.5573 | 1.0084 | 0.4634 | 0.1673 | 0.1784 |
| | 60% | 1.8369 | 0.7114 | 2.0039 | 0.7761 | 1.0418 | 0.7461 | 0.4301 | 0.4692 |

Table 12.3: Means, standard deviations and covariance of estimates based on Monte Carlo simulations for $n = 50$ ($\beta = 1.0$)

| $\alpha$ | d.o.c. | $\hat{\alpha}$ | SD($\hat{\alpha}$) | $\hat{\alpha}^*$ | SD($\hat{\alpha}^*$) | $\hat{\beta}$ | SD($\hat{\beta}$) | Cov($\hat{\alpha}$, $\hat{\beta}$) | Cov($\hat{\alpha}^*$, $\hat{\beta}$) |
|---|---|---|---|---|---|---|---|---|---|
| 0.1 | 0% | 0.0984 | 0.0099 | 0.1004 | 0.0101 | 0.9999 | 0.0142 | 0.0000 | 0.0000 |
| | 10% | 0.0981 | 0.0108 | 0.1006 | 0.0111 | 0.9997 | 0.0144 | 0.0000 | 0.0000 |
| | 20% | 0.0975 | 0.0127 | 0.1008 | 0.0121 | 0.9992 | 0.0152 | 0.0000 | 0.0000 |
| | 30% | 0.0970 | 0.0139 | 0.1010 | 0.0132 | 0.9988 | 0.0160 | 0.0001 | 0.0000 |
| | 40% | 0.4848 | 0.0703 | 0.1011 | 0.0145 | 0.9963 | 0.0796 | 0.0020 | 0.0001 |
| | 50% | 0.0964 | 0.0157 | 0.1009 | 0.0164 | 0.9981 | 0.0174 | 0.0001 | 0.0001 |
| | 60% | 0.0953 | 0.0177 | 0.1003 | 0.0187 | 0.9967 | 0.0199 | 0.0002 | 0.0002 |
| 0.3 | 0% | 0.2950 | 0.0298 | 0.3010 | 0.0304 | 1.0002 | 0.0423 | 0.0000 | 0.0000 |
| | 10% | 0.2942 | 0.0324 | 0.3017 | 0.0332 | 0.9997 | 0.0428 | 0.0001 | 0.0001 |
| | 20% | 0.2933 | 0.0351 | 0.3024 | 0.0362 | 0.9991 | 0.0437 | 0.0002 | 0.0002 |
| | 30% | 0.2923 | 0.0383 | 0.3029 | 0.0397 | 0.9982 | 0.0454 | 0.0004 | 0.0004 |
| | 40% | 0.2910 | 0.0418 | 0.3031 | 0.0436 | 0.9970 | 0.0479 | 0.0007 | 0.0007 |
| | 50% | 0.2891 | 0.0473 | 0.3027 | 0.0496 | 0.9951 | 0.0523 | 0.0011 | 0.0012 |
| | 60% | 0.2858 | 0.0538 | 0.3008 | 0.0566 | 0.9914 | 0.0599 | 0.0019 | 0.0020 |
| 0.5 | 0% | 0.4915 | 0.0497 | 0.5023 | 0.0509 | 1.0013 | 0.0693 | 0.0001 | 0.0000 |
| | 10% | 0.4901 | 0.0540 | 0.5037 | 0.0549 | 1.0004 | 0.0704 | 0.0003 | 0.0002 |
| | 20% | 0.4886 | 0.0587 | 0.5050 | 0.0602 | 0.9994 | 0.0721 | 0.0007 | 0.0006 |
| | 30% | 0.4870 | 0.0642 | 0.5059 | 0.0668 | 0.9981 | 0.0752 | 0.0012 | 0.0012 |
| | 40% | 0.4848 | 0.0703 | 0.5055 | 0.0738 | 0.9963 | 0.0796 | 0.0020 | 0.0020 |
| | 50% | 0.4817 | 0.0801 | 0.5041 | 0.0830 | 0.9934 | 0.0873 | 0.0034 | 0.0034 |
| | 60% | 0.4761 | 0.0918 | 0.5019 | 0.0961 | 0.9879 | 0.1006 | 0.0057 | 0.0058 |
| 1.0 | 0% | 0.9820 | 0.0994 | 1.0020 | 0.1015 | 1.0059 | 0.1273 | 0.0001 | 0.0001 |
| | 10% | 0.9789 | 0.1083 | 1.0040 | 0.1111 | 1.0041 | 0.1314 | 0.0015 | 0.0016 |
| | 20% | 0.9757 | 0.1186 | 1.0059 | 0.1223 | 1.0023 | 0.1370 | 0.0034 | 0.0035 |
| | 30% | 0.9722 | 0.1313 | 1.0074 | 0.1361 | 1.0003 | 0.1455 | 0.0062 | 0.0064 |
| | 40% | 0.9677 | 0.1464 | 1.0080 | 0.1525 | 0.9975 | 0.1570 | 0.0101 | 0.0105 |
| | 50% | 0.9616 | 0.1710 | 1.0070 | 0.1791 | 0.9938 | 0.1764 | 0.0172 | 0.0180 |
| | 60% | 0.9508 | 0.2033 | 1.0008 | 0.2140 | 0.9868 | 0.2099 | 0.0297 | 0.0313 |
| 2.0 | 0% | 1.9643 | 0.2003 | 2.0044 | 0.2044 | 1.0184 | 0.1940 | -0.0006 | -0.0006 |
| | 10% | 1.9537 | 0.2191 | 2.0087 | 0.2220 | 1.0108 | 0.2069 | 0.0065 | 0.0053 |
| | 20% | 1.9447 | 0.2448 | 2.0112 | 0.2497 | 1.0068 | 0.2259 | 0.0163 | 0.0151 |
| | 30% | 1.9342 | 0.2807 | 2.0117 | 0.2906 | 1.0035 | 0.2525 | 0.0316 | 0.0315 |
| | 40% | 1.9222 | 0.3293 | 2.0053 | 0.3438 | 1.0016 | 0.2904 | 0.0558 | 0.0573 |
| | 50% | 1.9113 | 0.4162 | 1.9984 | 0.4265 | 1.0054 | 0.3568 | 0.1073 | 0.1066 |
| | 60% | 1.9029 | 0.5656 | 2.0068 | 0.5880 | 1.0262 | 0.5133 | 0.2437 | 0.2501 |

Table 12.4: Probability coverages of 90% confidence intervals based on Monte Carlo simulations ($\beta = 1.0$)

| | | Probability coverages for $\alpha$ | | | | | | Probability coverages for $\beta$ | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | $n = 20$ | | $n = 30$ | | $n = 50$ | | $n = 20$ | | $n = 30$ | | $n = 50$ | |
| $\alpha$ | d.o.c. | MLE | UMLE | MLE | UMLE | MLE | UMLE | MLE | UMLE | MLE | UMLE | MLE | UMLE |
| 0.1 | 0% | 85.65 | 93.06 | 87.19 | 92.01 | 87.56 | 90.58 | 87.09 | 88.85 | 87.67 | 88.97 | 88.58 | 89.40 |
| | 10% | 84.56 | 93.08 | 85.90 | 91.98 | 87.45 | 91.17 | 86.64 | 89.04 | 87.41 | 89.06 | 88.21 | 89.18 |
| | 20% | 83.80 | 93.77 | 85.61 | 92.46 | 86.64 | 91.50 | 86.14 | 88.77 | 87.02 | 88.90 | 87.73 | 89.00 |
| | 30% | 82.92 | 94.02 | 85.36 | 92.96 | 86.41 | 92.17 | 85.51 | 88.49 | 86.50 | 88.48 | 87.81 | 88.95 |
| | 40% | 81.02 | 93.81 | 83.54 | 92.88 | 86.34 | 92.28 | 84.53 | 88.10 | 86.35 | 88.43 | 87.80 | 88.93 |
| | 50% | 79.18 | 93.48 | 82.40 | 92.68 | 85.19 | 92.08 | 83.25 | 86.37 | 84.88 | 87.24 | 87.03 | 88.28 |
| | 60% | 75.90 | 91.23 | 80.60 | 91.86 | 83.81 | 91.54 | 80.91 | 84.04 | 83.37 | 85.43 | 85.79 | 86.93 |
| 0.3 | 0% | 85.59 | 93.00 | 87.17 | 92.05 | 87.52 | 90.56 | 87.05 | 89.02 | 87.62 | 88.86 | 88.67 | 89.39 |
| | 10% | 84.51 | 93.11 | 85.89 | 91.95 | 87.43 | 91.15 | 86.62 | 88.89 | 87.56 | 88.88 | 88.26 | 89.19 |
| | 20% | 83.74 | 93.78 | 85.57 | 92.44 | 87.17 | 91.47 | 85.99 | 88.81 | 87.02 | 88.70 | 88.06 | 89.04 |
| | 30% | 82.90 | 93.95 | 85.34 | 92.95 | 86.63 | 92.08 | 85.38 | 88.55 | 86.46 | 88.59 | 87.75 | 88.88 |
| | 40% | 80.95 | 93.76 | 83.44 | 92.83 | 86.38 | 92.28 | 84.67 | 88.12 | 86.51 | 88.58 | 87.74 | 89.00 |
| | 50% | 79.07 | 93.41 | 82.31 | 92.61 | 85.24 | 92.04 | 83.30 | 86.56 | 85.20 | 87.25 | 87.02 | 88.28 |
| | 60% | 75.67 | 91.04 | 80.50 | 91.68 | 83.77 | 91.43 | 81.27 | 84.25 | 83.67 | 85.64 | 85.99 | 86.87 |
| 0.5 | 0% | 85.49 | 92.97 | 87.06 | 91.80 | 87.49 | 91.03 | 87.15 | 89.00 | 87.56 | 89.63 | 88.79 | 89.78 |
| | 10% | 84.35 | 93.04 | 85.81 | 92.38 | 87.36 | 91.73 | 86.66 | 89.00 | 87.33 | 89.49 | 88.36 | 90.04 |
| | 20% | 83.57 | 93.76 | 85.43 | 92.88 | 87.11 | 92.01 | 86.05 | 88.79 | 86.83 | 89.60 | 88.06 | 90.13 |
| | 30% | 82.71 | 93.94 | 85.13 | 93.00 | 86.69 | 91.95 | 85.31 | 88.75 | 86.42 | 89.14 | 87.81 | 89.74 |
| | 40% | 80.86 | 93.67 | 83.38 | 92.93 | 86.34 | 92.03 | 84.84 | 88.01 | 86.52 | 88.93 | 87.80 | 89.54 |
| | 50% | 78.98 | 93.24 | 82.12 | 92.24 | 85.08 | 91.86 | 83.44 | 86.79 | 85.31 | 87.73 | 86.96 | 88.99 |
| | 60% | 75.33 | 90.71 | 80.35 | 90.91 | 83.61 | 91.05 | 81.52 | 84.38 | 83.84 | 85.91 | 85.99 | 87.60 |
| 1.0 | 0% | 85.25 | 92.78 | 86.66 | 91.81 | 87.31 | 90.53 | 86.99 | 89.00 | 87.29 | 88.74 | 88.75 | 89.49 |
| | 10% | 83.80 | 92.84 | 85.50 | 91.70 | 87.10 | 90.94 | 86.83 | 89.13 | 87.41 | 88.91 | 88.45 | 89.47 |
| | 20% | 82.96 | 93.50 | 84.97 | 92.23 | 86.90 | 91.26 | 86.03 | 88.89 | 86.84 | 88.67 | 88.10 | 89.05 |
| | 30% | 82.33 | 93.73 | 84.72 | 92.81 | 86.41 | 91.95 | 85.23 | 88.47 | 86.54 | 88.62 | 87.83 | 88.78 |
| | 40% | 80.19 | 93.02 | 82.97 | 92.56 | 86.12 | 91.99 | 84.99 | 87.99 | 86.45 | 88.69 | 87.62 | 88.90 |
| | 50% | 77.94 | 92.29 | 81.23 | 91.73 | 84.64 | 91.49 | 83.69 | 86.59 | 85.37 | 87.10 | 87.17 | 88.32 |
| | 60% | 74.02 | 89.36 | 78.94 | 90.11 | 83.07 | 90.46 | 81.94 | 84.01 | 84.25 | 85.51 | 86.18 | 86.71 |
| 2.0 | 0% | 83.28 | 91.32 | 85.51 | 91.38 | 87.47 | 90.82 | 86.90 | 89.04 | 87.40 | 89.37 | 88.41 | 89.61 |
| | 10% | 82.81 | 91.99 | 85.04 | 91.70 | 86.62 | 91.48 | 86.61 | 89.26 | 87.39 | 89.42 | 88.36 | 89.90 |
| | 20% | 81.59 | 91.96 | 84.05 | 91.89 | 86.37 | 91.64 | 86.46 | 89.08 | 87.14 | 89.69 | 88.26 | 89.77 |
| | 30% | 80.63 | 91.77 | 83.63 | 91.78 | 85.93 | 91.18 | 85.58 | 88.37 | 86.51 | 89.03 | 87.92 | 89.76 |
| | 40% | 78.39 | 90.40 | 82.19 | 90.83 | 85.58 | 91.07 | 84.64 | 87.68 | 86.06 | 88.54 | 87.69 | 89.29 |
| | 50% | 76.21 | 88.45 | 79.81 | 88.18 | 84.43 | 89.66 | 83.77 | 85.81 | 85.38 | 86.88 | 87.36 | 88.84 |
| | 60% | 73.19 | 84.66 | 78.67 | 85.38 | 83.52 | 87.08 | 81.64 | 81.26 | 84.84 | 83.77 | 86.33 | 85.93 |

Table 12.5: Probability coverages of 95% confidence intervals based on Monte Carlo simulations ($\beta = 1.0$)

| | | Probability coverages for $\alpha$ | | | | | | Probability coverages for $\beta$ | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | $n = 20$ | | $n = 30$ | | $n = 50$ | | $n = 20$ | | $n = 30$ | | $n = 50$ | |
| $\alpha$ | d.o.c. | MLE | UMLE | MLE | UMLE | MLE | UMLE | MLE | UMLE | MLE | UMLE | MLE | UMLE |
| 0.1 | 0% | 90.49 | 95.87 | 91.89 | 95.68 | 92.42 | 93.95 | 92.90 | 94.08 | 93.25 | 95.18 | 93.81 | 94.26 |
| | 10% | 89.26 | 96.00 | 90.75 | 95.70 | 92.37 | 94.07 | 92.45 | 93.98 | 93.05 | 95.36 | 93.68 | 94.31 |
| | 20% | 88.69 | 96.25 | 90.26 | 95.92 | 91.63 | 94.01 | 91.90 | 93.78 | 92.67 | 95.43 | 93.05 | 94.28 |
| | 30% | 87.84 | 96.20 | 89.96 | 95.89 | 91.26 | 93.70 | 91.13 | 93.30 | 92.06 | 95.89 | 92.76 | 93.84 |
| | 40% | 86.07 | 95.92 | 88.44 | 95.71 | 91.18 | 93.09 | 90.25 | 92.73 | 91.59 | 95.80 | 92.86 | 93.67 |
| | 50% | 83.78 | 95.41 | 87.14 | 95.16 | 90.38 | 92.23 | 88.69 | 91.27 | 90.44 | 95.43 | 92.30 | 93.13 |
| | 60% | 80.90 | 93.80 | 85.60 | 94.26 | 89.03 | 90.27 | 86.13 | 88.37 | 88.79 | 94.57 | 90.78 | 91.73 |
| 0.3 | 0% | 90.42 | 95.87 | 91.83 | 95.65 | 92.42 | 95.16 | 92.78 | 94.03 | 93.14 | 94.03 | 93.85 | 94.21 |
| | 10% | 89.25 | 96.00 | 90.73 | 95.68 | 92.40 | 95.31 | 92.33 | 94.06 | 93.02 | 94.13 | 93.77 | 94.37 |
| | 20% | 88.63 | 96.21 | 90.25 | 95.87 | 91.82 | 95.42 | 91.90 | 93.82 | 92.65 | 94.06 | 93.49 | 94.42 |
| | 30% | 87.70 | 96.17 | 89.92 | 95.85 | 91.62 | 95.88 | 91.13 | 93.47 | 92.12 | 93.70 | 93.18 | 93.93 |
| | 40% | 85.99 | 95.86 | 88.46 | 95.73 | 91.24 | 95.76 | 90.43 | 92.86 | 91.70 | 93.17 | 92.71 | 93.75 |
| | 50% | 83.77 | 95.32 | 87.04 | 95.07 | 90.27 | 95.36 | 89.07 | 91.62 | 90.64 | 92.40 | 92.38 | 93.21 |
| | 60% | 80.68 | 93.70 | 85.40 | 94.20 | 88.87 | 94.47 | 86.49 | 88.78 | 89.08 | 90.47 | 91.04 | 91.90 |
| 0.5 | 0% | 90.28 | 95.82 | 91.78 | 95.56 | 92.35 | 95.38 | 92.59 | 93.90 | 93.07 | 94.20 | 93.77 | 94.59 |
| | 10% | 89.17 | 95.92 | 90.67 | 95.87 | 92.28 | 95.81 | 92.41 | 94.06 | 93.00 | 94.37 | 93.80 | 94.58 |
| | 20% | 88.43 | 96.19 | 90.10 | 95.96 | 91.78 | 95.65 | 91.71 | 93.83 | 92.68 | 94.38 | 93.61 | 94.82 |
| | 30% | 87.65 | 96.12 | 89.84 | 95.87 | 91.58 | 95.65 | 91.21 | 93.50 | 92.16 | 94.20 | 93.14 | 94.60 |
| | 40% | 85.87 | 95.75 | 88.30 | 95.39 | 91.18 | 95.47 | 90.64 | 92.97 | 91.73 | 93.71 | 92.86 | 94.36 |
| | 50% | 83.50 | 95.20 | 86.77 | 95.04 | 90.08 | 95.32 | 89.23 | 91.71 | 90.82 | 92.48 | 92.32 | 94.01 |
| | 60% | 80.32 | 93.38 | 85.16 | 93.94 | 88.54 | 94.38 | 86.88 | 89.04 | 89.41 | 91.06 | 91.27 | 92.36 |
| 1.0 | 0% | 89.94 | 95.72 | 91.59 | 95.54 | 92.25 | 95.03 | 92.34 | 93.71 | 92.72 | 93.83 | 93.62 | 94.20 |
| | 10% | 88.73 | 95.68 | 90.33 | 95.61 | 92.17 | 95.18 | 92.02 | 93.75 | 92.56 | 93.76 | 93.70 | 94.33 |
| | 20% | 88.14 | 96.04 | 89.69 | 95.58 | 91.68 | 95.34 | 91.45 | 93.66 | 92.56 | 93.91 | 93.54 | 94.26 |
| | 30% | 86.97 | 95.77 | 89.47 | 95.55 | 91.24 | 95.64 | 91.22 | 93.41 | 92.26 | 93.60 | 93.04 | 94.11 |
| | 40% | 85.19 | 95.22 | 87.70 | 95.20 | 90.73 | 95.47 | 90.74 | 92.99 | 91.94 | 93.47 | 92.96 | 93.80 |
| | 50% | 82.37 | 94.47 | 85.70 | 94.35 | 89.62 | 94.71 | 89.66 | 91.81 | 90.97 | 92.30 | 92.58 | 93.37 |
| | 60% | 78.74 | 92.10 | 83.71 | 92.93 | 87.49 | 93.59 | 87.41 | 89.17 | 89.78 | 90.75 | 91.55 | 92.06 |
| 2.0 | 0% | 88.58 | 94.59 | 90.70 | 95.11 | 92.40 | 95.14 | 91.61 | 93.12 | 92.35 | 93.44 | 93.51 | 94.13 |
| | 10% | 87.92 | 94.79 | 89.67 | 95.38 | 91.80 | 95.58 | 91.86 | 93.47 | 92.31 | 93.70 | 93.70 | 94.30 |
| | 20% | 86.93 | 95.03 | 88.83 | 95.31 | 91.16 | 95.36 | 91.74 | 93.55 | 92.28 | 94.00 | 93.46 | 94.42 |
| | 30% | 85.10 | 94.30 | 88.23 | 94.76 | 90.49 | 95.03 | 91.00 | 93.23 | 92.26 | 94.00 | 93.12 | 94.41 |
| | 40% | 83.09 | 93.07 | 86.17 | 93.41 | 89.78 | 94.05 | 90.58 | 92.71 | 91.96 | 93.71 | 93.07 | 94.30 |
| | 50% | 80.47 | 91.61 | 84.10 | 91.50 | 88.60 | 92.61 | 89.71 | 90.89 | 91.16 | 92.35 | 92.68 | 93.73 |
| | 60% | 77.44 | 88.31 | 82.97 | 88.86 | 87.72 | 90.25 | 87.43 | 87.23 | 90.67 | 89.81 | 91.86 | 91.57 |

Table 12.6: Point estimates of $\alpha$ and $\beta$ for Example 12.1

| | Newton-Raphson | | | MCEM (MLE) | | MCEM (MME) | |
|---|---|---|---|---|---|---|---|
| $r$ | $\hat{\alpha}$ | $\hat{\alpha}^*$ | $\hat{\beta}$ | $\hat{\alpha}$ | $\hat{\beta}$ | $\hat{\alpha}$ | $\hat{\beta}$ |
| 101 | 0.1704 | 0.1721 | 131.8188 | 0.1704 | 131.8188 | 0.1704 | 131.8188 |
| 100 | 0.1698 | 0.1716 | 131.7871 | 0.1698 | 131.7876 | 0.1698 | 131.7871 |
| 95 | 0.1691 | 0.1710 | 131.7686 | 0.1691 | 131.7691 | 0.1691 | 131.7687 |
| 90 | 0.1706 | 0.1728 | 131.8776 | 0.1706 | 131.8781 | 0.1706 | 131.8776 |
| 80 | 0.1751 | 0.1777 | 132.2527 | 0.1751 | 132.2531 | 0.1751 | 132.2525 |
| 70 | 0.1735 | 0.1766 | 132.1069 | 0.1735 | 132.1074 | 0.1735 | 132.1062 |
| 60 | 0.1829 | 0.1866 | 133.2411 | 0.1829 | 133.2416 | 0.1829 | 133.2403 |
| 50 | 0.1848 | 0.1891 | 133.5043 | 0.1848 | 133.5051 | 0.1848 | 133.5038 |
| 40 | 0.2112 | 0.2165 | 137.5925 | 0.2111 | 137.5921 | 0.2111 | 137.5879 |

## 12.6 Illustrative Examples

Practical application of the inferential methods discussed in the preceding sections is illustrated here with two examples, with one involving a large sample and the other with a small sample.

**Example 12.1:** Refer to Example 11.1 and the data presented in Table 11.5. Under different censoring proportions, the point estimates of $\alpha$ and $\beta$ obtained by Newton-Raphson method and the Monte Carlo EM-algorithm (using MLE's and MME's) are summarized in Table 12.6. From (12.2.9), the asymptotic variances of the estimators can be obtained, and also the asymptotic confidence intervals for $\alpha$ and $\beta$ based on the MLE's can be readily constructed using (12.2.10). The results so obtained are presented in Table 12.7.

**Example 12.2:** Refer to Example 11.2 and the data presented in Table 11.8. In this example, if we assume a Type-II right censored sample of size $r = 8$, we obtain the maximum likelihood estimates of $\alpha$ and $\beta$ to be $\hat{\alpha} = 0.1792$ and $\hat{\beta} = 200.7262$, and the bias-corrected estimate of $\alpha$ to be $\hat{\alpha}^* = 0.2108$. The corresponding standard deviations and the confidence intervals are presented in Table 12.8.

Table 12.7: Standard deviations of estimates and confidence intervals for $\alpha$ and $\beta$ for Example 12.1

| | | $\alpha$ | | | $\beta$ | | |
|---|---|---|---|---|---|---|---|
| $r$ | Estimator | S.D. | 90% C.I. | 95% C.I. | S.D. | 90% C.I. | 95% C.I. |
| 101 | MLE | 0.0120 | (0.1507,0.1901) | (0.1469,0.1939) | 2.2267 | (128.2552,135.5861) | (127.5944,136.3325) |
| | UMLE | 0.0124 | (0.1517,0.1925) | (0.1478,0.1964) | 2.2492 | (128.2203,135.6251) | (127.5532,136.3796) |
| 100 | MLE | 0.0120 | (0.1500,0.1897) | (0.1462,0.1934) | 2.2203 | (128.2336,135.5432) | (127.5746,136.2873) |
| | UMLE | 0.0124 | (0.1512,0.1919) | (0.1473,0.1958) | 2.2432 | (128.1979,135.5831) | (127.5325,136.3354) |
| 95 | MLE | 0.0125 | (0.1486,0.1895) | (0.1446,0.1935) | 2.2213 | (128.2136,135.5265) | (127.5543,136.2710) |
| | UMLE | 0.0128 | (0.1499,0.1921) | (0.1459,0.1961) | 2.2470 | (128.1735,135.5713) | (127.5071,136.3250) |
| 90 | MLE | 0.0131 | (0.1491,0.1921) | (0.1450,0.1962) | 2.2596 | (128.2627,135.7021) | (127.5927,136.4602) |
| | UMLE | 0.0136 | (0.1503,0.1952) | (0.1460,0.1995) | 2.2886 | (128.2176,135.7526) | (127.5396,136.5210) |
| 80 | MLE | 0.0145 | (0.1512,0.1989) | (0.1466,0.2035) | 2.3763 | (128.4561,136.2801) | (127.7536,137.0799) |
| | UMLE | 0.0151 | (0.1529,0.2025) | (0.1482,0.2073) | 2.4124 | (128.4001,136.3432) | (127.6876,137.1560) |
| 70 | MLE | 0.0156 | (0.1478,0.1992) | (0.1429,0.2041) | 2.4435 | (128.2056,136.2516) | (127.4845,137.0756) |
| | UMLE | 0.0163 | (0.1498,0.2034) | (0.1446,0.2086) | 2.4856 | (128.1405,136.3253) | (127.4077,137.1645) |
| 60 | MLE | 0.0182 | (0.1530,0.2128) | (0.1473,0.2185) | 2.7506 | (128.8646,137.9236) | (128.0589,138.8586) |
| | UMLE | 0.0191 | (0.1552,0.2180) | (0.1492,0.2240) | 2.8018 | (128.7858,138.0139) | (127.9663,138.9677) |
| 50 | MLE | 0.0204 | (0.1512,0.2185) | (0.1448,0.2249) | 3.0475 | (128.6725,138.7120) | (127.7866,139.7565) |
| | UMLE | 0.0216 | (0.1536,0.2246) | (0.1468,0.2314) | 3.1049 | (128.5848,138.8140) | (127.6836,139.8799) |
| 40 | MLE | 0.0270 | (0.1667,0.2555) | (0.1582,0.2640) | 4.1182 | (131.1319,144.7126) | (129.9636,146.1625) |
| | UMLE | 0.0286 | (0.1694,0.2636) | (0.1604,0.2726) | 4.1876 | (131.0283,144.8390) | (129.8424,146.3162) |

Table 12.8: Standard deviations of estimates and confidence intervals for $\alpha$ and $\beta$ for Example 12.2 with $r = 8$

| | $\alpha$ | | | $\beta$ | | |
|---|---|---|---|---|---|---|
| Estimator | S.D. | 90% C.I. | 95% C.I. | S.D. | 90% C.I. | 95% C.I. |
| MLE | 0.0471 | (0.1017,0.2566) | (0.0868,0.2715) | 11.6871 | (183.1828,221.9857) | (180.1662,226.5831) |
| UMLE | 0.0719 | (0.0925,0.3290) | (0.06984,0.3517) | 13.7416 | (180.4109,226.1973) | (176.9795,231.8331) |

## 12.7 Concluding Remarks

In this chapter, we have discussed the MLE's of parameters $\alpha$ and $\beta$ of the Birnbaum-Saunders distribution based on Type-II right-censored samples. As the estimator of $\alpha$ becomes quite biased especially when the degree of censoring is high, we have proposed a simple bias-corrected estimator based on MLE which has been shown to be almost unbiased in all situations considered. We have also described the Monte Carlo EM-algorithm for the determination of the MLE's. The probability coverages of confidence intervals based on these estimators have been evaluated. While the probability coverages in the case of MLE's turn out to be considerably below the nominal levels, the confidence intervals based on bias-corrected MLE's seem to have probability coverages much closer to the nominal levels.

# Chapter 13

# Possible Future Research

In this chapter, we will outline some possible future research that could be considered following the research work in this thesis. We first discuss some future research directions for nonparametric precedence tests. Then, we will discuss some future research directions for parametric inference based on progressively censored samples and for the Birnbaum-Saunders distribution.

## 13.1 Nonparametric Inference

1. Even though we have suggested some alternatives to the precedence test in the form of maximal precedence test, Wilcoxon-type rank-sum test, and weighted precedence and maximal precedence tests, they all deal with the two-sample problem. One possible extension may be to construct these tests for a $k$-sample problem since comparative studies usually involve the simultaneous comparison not just of two, but three or more treatments or conditions. All the test procedures we studied in this thesis can be generalized to test the hypothesis of no difference between the $k$ samples.

2. In the $k$-sample problem just mentioned, if the hypothesis of no difference between the $k$ samples is rejected, we may want to know which treatment or population is the best. Therefore, the precedence statistics can be utilized to

select the best sample.

3. If we take a careful look at the precedence and maximal precedence tests, they belong to a class of nonparametric precedence tests. This class of nonparametric precedence tests can be defined as follows:

   Let us use the same notation as in Chapters 2 - 5. For a certain value of $r$, define the test statistics

   $$P_{(r,k)} = \sum_{i=1}^{k} M_{(r-i+1)}, \qquad k = 1, 2, \ldots, r,$$

   where $M_{(j)}$ is the $j$-th order statistic from $\{M_1, M_2, \ldots, M_r\}$.

   For $k = 1$, $P_{(r,1)} = M_{(r)} = \max\{M_1, M_2, \ldots, M_r\}$, which is the maximal precedence test statistic.

   For $k = r$, $P_{(r,r)} = \sum\limits_{i=1}^{r} M_{(r-i+1)} = \sum\limits_{i=1}^{r} M_i$, which is the precedence test statistic. We can then study and compare the power properties of the test statistics belonging to this general class of precedence tests. A similar class of weighted precedence tests can be also defined and studied.

4. The idea of weighted precedence and maximal precedence test can be extended to the Wilcoxon-type rank based test that takes into account the magnitude of the failure times.

## 13.2 Parametric Inference

1. In Chapter 8, we discussed the optimal progressive censoring scheme for the Weibull distribution. Instead of the Weibull lifetime model, we can consider the optimal progressive censoring scheme under other lifetime distributions, for example, the lognormal and gamma distributions. Progressively censored reliability sampling plans can also be studied for these distributions.

2. In order to determine the optimal censoring scheme, exhaustive search of all possible progressive censoring schemes for fixed $n$ and $m$ have been used. Instead

of using the exhaustive search method, we can develop some efficient algorithms to search for the optimal censoring scheme. By using these algorithms, one can obtain the optimal censoring scheme without going through all possible censoring schemes.

3. Table 8.8 showed simulation values of the two levels of probability of acceptance ($\alpha$ and $\beta$) for the sampling plans determined by the procedure described in Section 8.4. From this table, we see that our procedure sometimes result in larger values of $\alpha$ and $\beta$ than the pre-fixed ones. The reason may be due to the non-normality of the asymptotic distribution of the linear combination of the MLE's under heavy censoring. Therefore, adjustments to the sample sizes can be examined to try to obtain smaller values of $\alpha$ and $\beta$. Another possible direction to the work presented here is to consider the construction of the reliability sampling plans by means of best linear unbiased estimators (BLUE's) instead of MLE's.

4. In Chapter 10, we conducted a Monte Carlo simulation study to examine the null distribution of $T$ and the result suggests that the null distribution of $T$ can be approximated closely by the normal distribution with certain mean and variance. However, the derivations of the exact null distribution as well as the exact distribution of the sample spacings are still open problems.

5. In Chapters 11 and 12, we discussed the point and interval estimation procedures for the two-parameter Birnbaum-Saunders distribution for complete and Type-II censored samples. One may also develop statistical inferential procedures for the Birnbaum-Saunders distribution when the data are progressively censored or grouped.

6. Finally, it will be of interest to develop some goodness-of-fit tests for the Birnbaum-Saunders distribution and examine their power properties.

# Appendix A

# Computational Formulas for the Moments of Test Statistics $W_{\min}$, $W_{\max}$, $W_E$

The moments of $M_i$'s are denoted by

$$\begin{aligned}
\mu_a &= E(M_i^a), \\
\mu_{a,b} &= E(M_i^a M_j^b), \\
\mu_{a,b,c} &= E(M_i^a M_j^b M_k^c), \\
\mu_{a,b,c,d} &= E(M_i^a M_j^b M_k^c M_j^d).
\end{aligned}$$

To derive the first, second, third and fourth moments of the Wilcoxon-type rank-sum statistics, we require the following:

$$\begin{aligned}
\mu_1 &= \frac{n_1}{n_2+1}, \\
\mu_2 &= \frac{n_1(n_2+2n_1)}{(n_2+1)(n_2+2)}, \\
\mu_3 &= \frac{3n_1(n_1-1)(n_1-2)}{n_2+3} - (3n_1^2-6n_1+2)\mu_1 + 3(n_1-1)\mu_2, \\
\mu_4 &= -\frac{4n_1(n_1-1)(n_1-2)(n_1-3)}{n_2+4} + 2(2n_1^3-9n_1^2+11n_1-3)\mu_1 \\
&\quad -(6n_1^2-18n_1+11)\mu_2 + 2(2n_1-3)\mu_3,
\end{aligned}$$

$$
\begin{aligned}
\mu_{1,1} &= \frac{n_1(n_1-1)}{(n_2+1)(n_2+2)}, \\
\mu_{2,1} &= \frac{1}{n_2}\left(n_1\mu_2 - \mu_3\right), \\
\mu_{3,1} &= \frac{1}{n_2}\left(n_1\mu_3 - \mu_4\right), \\
\mu_{2,2} &= -\frac{2n_1(n_1-1)(n_1-2)(n_1-3)(2n_2+3)}{3(n_2+3)(n_2+4)} + \frac{2}{3}(2n_1^3 - 9n_1^2 + 11n_1 - 3)\mu_1 \\
&\quad -\frac{1}{3}(6n_1^2 + 18n_1 + 11)(\mu_2 + \mu_{1,1}) + \frac{2}{3}(2n_1 - 3)(\mu_3 + 3\mu_{2,1}) - \frac{4}{3}\mu_{3,1} - \frac{1}{3}\mu_4, \\
\mu_{1,1,1} &= \frac{1}{n_2(n_2^2-1)}[n_1^3 - (n_2+1)\mu_3 - 3n_2(n_2+1)\mu_{2,1}], \\
\mu_{2,1,1} &= -\frac{n_1(n_1-1)(n_1-2)(n_1-3)(n_2^2+2n_2+2)}{3(n_2+2)(n_2+3)(n_2+4)} \\
&\quad +\frac{1}{6}(2n_1^3 - 9n_1^2 + 11n_1 - 3)\mu_1 - \frac{1}{12}(6n_1^2 - 18n_1 + 11)(\mu_2 + 2\mu_{1,1}) \\
&\quad +\frac{1}{6}(2n_1 - 3)(\mu_3 + 6\mu_{2,1} + 2\mu_{1,1,1}) - \frac{1}{6}(4\mu_{3,1} + 3\mu_{2,2}) - \frac{1}{12}\mu_4, \\
\mu_{1,1,1,1} &= \frac{1}{n_2(n_2^2-1)(n_2-2)} \\
&\quad \times[n_1^4 - (n_2+1)\mu_4 - n_2(n_2+1)(4\mu_{3,1} + 3\mu_{2,2}) - 6n_2(n_2^2-1)\mu_{2,1,1}].
\end{aligned}
$$

Since $M_1, M_2, \ldots, M_r$ are exchangeable random variables, we readily have

$$
\begin{aligned}
\mu_{2,1} &= \mu_{1,2}, \\
\mu_{3,1} &= \mu_{1,3}, \\
\mu_{2,1,1} &= \mu_{1,2,1} = \mu_{1,1,2}.
\end{aligned}
$$

We further define the following quantities:

$$
\begin{aligned}
S_r(1) &= \sum_{i=1}^{r} i = \frac{r(r+1)}{2}, \\
S_r(2) &= \sum_{i=1}^{r} i^2 = \frac{r(r+1)(2r+1)}{6}, \\
S_r(3) &= \sum_{i=1}^{r} i^3 = \frac{r^2(r+1)^2}{4}, \\
S_r(4) &= \sum_{i=1}^{r} i^4 = \frac{r(r+1)(2r+1)(3r^2+3r-1)}{30},
\end{aligned}
$$

$$\begin{aligned}
S_r(1,1) &= \sum_{i=1}^{r}\sum_{\substack{j=1\\ i\neq j}}^{r} ij = [S_r(1)]^2 - S_r(2) = \frac{r(r^2-1)(3r+2)}{12},\\
S_r(2,1) &= \sum_{i=1}^{r}\sum_{\substack{j=1\\ i\neq j}}^{r} i^2 j = S_r(1)S_r(2) - S_r(3) = \frac{r^2(r+1)^2(r-1)}{6},\\
S_r(2,2) &= \sum_{i=1}^{r}\sum_{\substack{j=1\\ i\neq j}}^{r} i^2 j^2 = [S_r(2)]^2 - S_r(4) = \frac{r(r^2-1)[3r^2(5r+7)-4]}{120},\\
S_r(3,1) &= \sum_{i=1}^{r}\sum_{\substack{j=1\\ i\neq j}}^{r} i^3 j = S_r(1)S_r(3) - S_r(4) = \frac{r(r^2-1)(4r^2-1)(5r+6)}{180},\\
S_r(1,1,1) &= \sum_{i=1}^{r}\sum_{\substack{j=1\\ i\neq j\neq k}}^{r}\sum_{k=1}^{r} ijk\\
&= [S_r(1)]^3 - 3S_r(2,1) - S_r(3) = \frac{r^2(r+1)^2(r-1)(r-2)}{8},\\
S_r(2,1,1) &= \sum_{i=1}^{r}\sum_{\substack{j=1\\ i\neq j\neq k}}^{r}\sum_{k=1}^{r} i^2 jk = [S_r(1)]^2 S_r(2) - 2S_r(3,1) - S_r(2,2) - S_r(4),\\
S_r(1,1,1,1) &= \sum_{i=1}^{r}\sum_{\substack{j=1\\ i\neq j\neq k\neq l}}^{r}\sum_{k=1}^{r}\sum_{l=1}^{r} ijkl\\
&= [S_r(1)]^4 - 6S_r(2,1,1) - 3S_r(2,2) - 4S_r(3,1) - S_r(4),
\end{aligned}$$

and

$$\nu_{a,b} = E\left[\left(\sum_{i=1}^{r} M_i\right)^a \left(\sum_{i=1}^{r} iM_i\right)^b\right].$$

Now we can express all the required $\nu_{a,b}$ in terms of $S_r$'s and $\mu$'s as follows:

$$\begin{aligned}
\nu_{1,0} &= r\mu_1,\\
\nu_{0,1} &= S_r(1)\mu_1,\\
\nu_{2,0} &= r\mu_2 + r(r-1)\mu_{1,1},\\
\nu_{0,2} &= S_r(2)\mu_2 + S_r(1,1)\mu_{1,1},\\
\nu_{1,1} &= S_r(1)\mu_2 + (r-1)S_r(1)\mu_{1,1},\\
\nu_{3,0} &= r\mu_3 + 3r(r-1)\mu_{2,1} + r(r-1)(r-2)\mu_{1,1,1},
\end{aligned}$$

$$
\begin{aligned}
\nu_{0,3} &= S_r(3)\mu_3 + 3S_r(2,1)\mu_{2,1} + S_r(1,1,1)\mu_{1,1,1}, \\
\nu_{2,1} &= S_r(1)\mu_3 + 3(r-1)S_r(1)\mu_{2,1} + (r-1)(r-2)S_r(1)\mu_{1,1,1}, \\
\nu_{1,2} &= S_r(2)\mu_3 + 2S_r(1,1)\mu_{2,1} + (r-1)S_r(2)\mu_{2,1} + (r-2)S_r(1,1)\mu_{1,1,1}, \\
\nu_{4,0} &= r\mu_4 + 4r(r-1)\mu_{3,1} + 3r(r-1)\mu_{2,2} + 6r(r-1)(r-2)\mu_{2,1,1} \\
&\quad + r(r-1)(r-2)(r-3)\mu_{1,1,1,1}, \\
\nu_{0,4} &= S_r(4)\mu_4 + 4S_r(3,1)\mu_{3,1} + 3S_r(2,2)\mu_{2,2} + 6S_r(2,1,1)\mu_{2,1,1} \\
&\quad + S_r(1,1,1,1)\mu_{1,1,1,1}, \\
\nu_{3,1} &= S_r(1)\mu_4 + 4(r-1)S_r(1)\mu_{3,1} + 3(r-1)S_r(1)\mu_{2,2} + 6(r-1)(r-2)S_r(1)\mu_{2,1,1} \\
&\quad + (r-1)(r-2)(r-3)S_r(1)\mu_{1,1,1,1}, \\
\nu_{1,3} &= S_r(3)\mu_4 + (r-1)S_r(3)\mu_{3,1} + 3S_r(2,1)\mu_{3,1} + 3S_r(2,1)\mu_{2,2} \\
&\quad + 3(r-2)S_r(2,1)\mu_{2,1,1} + 3S_r(1,1,1)\mu_{2,1,1} + (r-3)S_r(1,1,1)\mu_{1,1,1,1}, \\
\nu_{2,2} &= S_r(2)\mu_4 + 2(r-1)S_r(2)\mu_{3,1} + 2S_r(1,1)\mu_{3,1} + (r-1)S_r(2)\mu_{2,2} \\
&\quad + 2S_r(1,1)\mu_{2,2} + (r-1)(r-2)S_r(2)\mu_{2,1,1} + (r-2)S_r(1,1)\mu_{2,1,1} \\
&\quad + 4(r-2)S_r(1,1)\mu_{2,1,1} + (r-2)(r-3)S_r(1,1)\mu_{1,1,1,1}.
\end{aligned}
$$

Now by taking expectation of the conditional expectation, we obtain the first, second, third and fourth moments of the minimal rank-sum statistic as follows:

$$
\begin{aligned}
E\left[W_{\min}\right] &= \frac{n_1(n_1+2r+1)}{2} - (r+1)\nu_{1,0}, \\
E\left[W_{\min}^2\right] &= \left[\frac{n_1(n_1+2r+1)}{2}\right]^2 + (r+1)^2\nu_{2,0} + \nu_{0,2} - 2(r+1)\nu_{1,1} \\
&\quad + n_1(n_1+2r+1)\left\{\nu_{0,1} - (r+1)\nu_{1,0}\right\}, \\
E\left[W_{\min}^3\right] &= \left[\frac{n_1(n_1+2r+1)}{2}\right]^3 + (r+1)^3\nu_{3,0} + \nu_{0,3} \\
&\quad + 3\left\{\left[\frac{n_1(n_1+2r+1)}{2}\right]^2 \left[\nu_{0,1} - (r+1)\nu_{1,0}\right]\right. \\
&\quad \left. + \frac{n_1(n_1+2r+1)}{2}\left[\nu_{0,2} + (r+1)^2\nu_{2,0}\right] + (r+1)^2\nu_{2,1} - (r+1)\nu_{1,2}\right\} \\
&\quad - 3n_1(n_1+2r+1)(r+1)\nu_{1,1}, \\
E\left[W_{\min}^4\right] &= \left[\frac{n_1(n_1+2r+1)}{2}\right]^4 + (r+1)^4\nu_{4,0} + \nu_{0,4}
\end{aligned}
$$

$$+4\left\{\left[\frac{n_1(n_1+2r+1)}{2}\right]^3[\nu_{0,1}-(r+1)\nu_{1,0}]\right.$$

$$\left.+\frac{n_1(n_1+2r+1)}{2}\left[\nu_{0,3}-(r+1)^3\nu_{3,0}\right]-(r+1)^3\nu_{3,1}-(r+1)\nu_{0,3}\right\}$$

$$+6\left\{\left[\frac{n_1(n_1+2r+1)}{2}\right]^2\left[\nu_{0,2}+(r+1)^2\nu_{2,0}\right]+(r+1)^2\nu_{2,2}\right\}$$

$$+6n_1(n_1+2r+1)(r+1)\left[(r+1)\nu_{2,1}-\nu_{1,2}-\frac{n_1(n_1+2r+1)}{2}\nu_{1,1}\right].$$

Similarly, the first, second, third and fourth moments of the maximal rank-sum statistic are found to be the above expressions with $r$ replaced by $n_2$, and those of the expected rank-sum statistic are with $r$ replaced by $(n_2+r)/2$.

# Appendix B

# Proof of Theorems 5.1 and 5.2

First, conditional on the $Y$-failures, we consider the probability that there are $m_1$ $X$-failures before $y_{1:r:n_2}$ and $m_i$ $X$-failures between $y_{i-1:r:n_2}$ and $y_{i:r:n_2}$, $i = 2, 3, \ldots, r$, given by

$$\begin{aligned}
&\Pr\{m_1 X's \leq y_{1:r:n_2},\ m_2 X's \in (y_{1:r:n_1}, y_{2:r:n_2}], \ldots,\ m_r X's \in (y_{r-1:r:n_2}, y_{r:r:n_2}], \\
&\left(n_1 - \sum_{i=1}^{r} m_i\right) X's > y_{r:r:n_2} \,|Y_{1:r:n_2} = y_{1:r:n_2}, Y_{2:r:n_2} = y_{1:r:n_2}, \ldots, Y_{r:r:n_2} = y_{r:r:n_2}\} \\
= \ & \frac{n_1!}{m_1! m_2! \cdots m_r! \left(n_1 - \sum\limits_{i=1}^{r} m_i\right)!} [F_X(y_{1:r:n_2})]^{m_1} \left\{\prod_{i=2}^{r} [F_X(y_{i:r:n_2}) - F_X(y_{i-1:r:n_2})]^{m_i}\right\} \\
&\times [1 - F_X(y_{r:r:n_2})]^{\left(n_1 - \sum\limits_{i=1}^{r} m_i\right)}.
\end{aligned}$$

Next, we have the joint density of the Type-II progressively censored order statistics from the $Y$-sample as [Kamps (1995a, 1995b) and Balakrishnan and Aggarwala (2000)]

$$\begin{aligned}
& f_{1,2,\ldots,r:r:n_2}(y_{1:r:n_2}, y_{2:r:n_2}, \ldots, y_{r:r:n_2}) \\
= \ & n_2(n_2 - R_1 - 1) \cdots (n_2 - R_1 - R_2 - \ldots - R_{r-1} - r + 1) \\
& \times \prod_{j=1}^{r} f_Y(y_{j:r:n_2}) [1 - F_Y(y_{j:r:n_2})]^{R_j}, \qquad y_{1:r:n_2} \leq y_{2:r:n_2} \leq \ldots \leq y_{r:r:n_2}.
\end{aligned}$$

Then, the unconditional probability of $\{M_1 = m_1, M_2 = m_2, \ldots, M_r = m_r\}$ is simply given by

$$\begin{aligned}
& \Pr\{M_1 = m_1, M_2 = m_2, \ldots, M_r = m_r\} \\
= \; & C \int_0^{\infty} \int_0^{y_{r:r:n_2}} \cdots \int_0^{y_{2:r:n_2}} [F_X(y_{1:r:n_2})]^{m_1} \left\{ \prod_{i=2}^{r} [F_X(y_{i:r:n_2}) - F_X(y_{i-1:r:n_2})]^{m_i} \right\} \\
& \times [1 - F_X(y_{r:r:n_2})]^{\left(n_1 - \sum\limits_{i=1}^{r} m_i\right)} \left\{ \prod_{i=1}^{r} f_Y(y_{i:r:n_2}) [1 - F_Y(y_{i:r:n_2})]^{R_i} \right\} \\
& \times dy_{1:r:n_2} dy_{2:r:n_2} \ldots dy_{r-1:r:n_2} dy_{r:r:n_2},
\end{aligned} \tag{B.1}$$

where

$$C = \frac{n_1! n_2 (n_2 - R_1 - 1) \cdots (n_2 - R_1 - R_2 - \ldots - R_{r-1} - r + 1)}{m_1! m_2! \cdots m_r! \left(n_1 - \sum\limits_{i=1}^{r} m_i\right)!}$$

# B.1 Under Lehmann Alternative

Under Lehmann alternative $H_1 : [F_X]^{\gamma} = F_Y, \gamma > 1$, the expression in (B.1) can be simplified as

$$\begin{aligned}
& \Pr\{M_1 = m_1, M_2 = m_2, \ldots, M_r = m_r \mid [F_X]^{\gamma} = F_Y\} \\
= \; & C \int_0^{\infty} \int_0^{y_{r:r:n_2}} \cdots \int_0^{y_{2:r:n_2}} [F_X(y_{1:r:n_2})]^{m_1} \left\{ \prod_{i=2}^{r} [F_X(y_{i:r:n_2}) - F_X(y_{i-1:r:n_2})]^{m_i} \right\} \\
& \times [1 - F_X(y_{r:r:n_2})]^{\left(n_1 - \sum\limits_{i=1}^{r} m_i\right)} \left\{ \prod_{i=1}^{r} \gamma [F_X(y_{i:r:n_2})]^{\gamma - 1} f_X(y_{i:r:n_2}) \, [1 - (F_X(y_{i:r:n_2}))^{\gamma}]^{R_i} \right\} \\
& \times dy_{1:r:n_2} dy_{2:r:n_2} \ldots dy_{r-1:r:n_2} dy_{r:r:n_2} \\
= \; & C \gamma^r \sum_{j_1=0}^{R_1} \binom{R_1}{j_1} (-1)^{j_1} \int_0^{\infty} \int_0^{y_{r:r:n_2}} \cdots \int_0^{y_{2:r:n_2}} [F_X(y_{1:r:n_2})]^{m_1 + \gamma j_1 + \gamma - 1} \left\{ \prod_{i=1}^{r} f_X(y_{i:r:n_2}) \right\} \\
& \times \left\{ \prod_{i=2}^{r} [F_X(y_{i:r:n_2}) - F_X(y_{i-1:r:n_2})]^{m_i} \left[ \sum_{j_i=0}^{R_i} \binom{R_i}{j_i} (-1)^{j_i} [F_X(y_{i:r:n_2})]^{\gamma j_i + \gamma - 1} \right] \right\} \\
& \times [1 - F_X(y_{r:r:n_2})]^{\left(n_1 - \sum\limits_{i=1}^{r} m_i\right)} \; dy_{1:r:n_2} \ldots dy_{r:r:n_2}.
\end{aligned}$$

Setting now

$$u_i = F_X(y_{i:r:n_2}) \text{ and } du_i = f_X(y_{i:r:n_2}) dy_{i:r:n_2} \quad \text{for } i = 1, 2, \ldots, r,$$

we get

$$
\begin{aligned}
&\Pr\{M_1 = m_1, M_2 = m_2, \ldots, M_r = m_r \mid [F_X]^\gamma = F_Y\} \\
= \;& C\gamma^r \sum_{j_1=0}^{R_1} \binom{R_1}{j_1} (-1)^{j_1} \int_0^1 \int_0^{u_r} \cdots \int_0^{u_2} u_1^{m_1+\gamma j_1+\gamma-1} \\
&\times \left\{ \prod_{i=2}^{r} (u_i - u_{i-1})^{m_i} \left[ \sum_{j_i=0}^{R_i} \binom{R_i}{j_i} (-1)^{j_i} u_i^{\gamma j_i+\gamma-1} \right] \right\} (1-u_r)^{\left(n_1 - \sum_{i=1}^{r} m_i\right)} du_1 \ldots du_r.
\end{aligned}
$$

Using the transformation $w_1 = u_1/u_2$, we readily obtain

$$
\begin{aligned}
&\Pr\{M_1 = m_1, M_2 = m_2, \ldots, M_r = m_r \mid [F_X]^\gamma = F_Y\} \\
= \;& C\gamma^r \sum_{j_1=0}^{R_1} \sum_{j_2=0}^{R_2} \binom{R_1}{j_1} \binom{R_2}{j_2} (-1)^{j_1+j_2} B(m_1 + \gamma j_1 + \gamma, m_2 + 1) \\
&\times \int_0^1 \int_0^{u_r} \cdots \int_0^{u_3} u_2^{m_1+m_2+\gamma(j_1+j_2)+2\gamma-1} \\
&\times \left\{ \prod_{i=3}^{r} (u_i - u_{i-1})^{m_i} \left[ \sum_{j_i=0}^{R_i} \binom{R_i}{j_i} (-1)^{j_i} u_i^{\gamma j_i+\gamma-1} \right] \right\} (1-u_r)^{\left(n_1 - \sum_{i=1}^{r} m_i\right)} du_1 \ldots du_r,
\end{aligned}
$$

where $B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx$ denotes the complete beta function.

Proceeding similarly upon performing the transformations $w_l = u_l/u_{l+1}$ for $l = 2, 3, \ldots, r-1$, we arrive at the expression given in Theorem 5.2.

## B.2 Under Null Hypothesis

Under the null hypothesis $H_0 : F_X = F_Y$, the expression in (B.1) can be simplified by employing the same technique as presented above by setting $\gamma = 1$. Upon simplifying the resulting expression, we complete the proof of Theorem 5.1.

# Appendix C

# Computation of Asymptotic Variance-Covariance Matrix of MLEs for Weibull Distribution

For the Weibull lifetime data, the moment generating function of $\xi_r = \frac{Y_{r:m:n}-\mu}{\sigma}$ is obtained from (8.2.4) to be

$$E_{Y_{r:m:n}}\left(e^{t\xi_r}\right) = c_{r-1}\Gamma(t+1)\sum_{i=1}^{r} a_{i,r}\gamma_i^{-(t+1)}. \tag{C.1}$$

The first and second derivatives of (C.1) with respect to $t$ are

$$\begin{aligned}
\frac{d}{dt}E_{Y_{r:m:n}}\left(e^{t\xi_r}\right) &= c_{r-1}\Gamma(t+1)\sum_{i=1}^{r} a_{i,r}\gamma_i^{-(t+1)}\left[\psi(t+1)-\log\gamma_i\right],\\
\frac{d^2}{dt^2}E_{Y_{r:m:n}}\left(e^{t\xi_r}\right) &= c_{r-1}\Gamma(t+1)\sum_{i=1}^{r} a_{i,r}\gamma_i^{-(t+1)}\left\{\psi'(t+1)+\left[\psi(t+1)-\log\gamma_i\right]^2\right\},
\end{aligned}$$

where $\psi(z) = \frac{d}{dz}\ln\Gamma(z)$ and $\psi'(z) = \frac{d}{dz}\psi(z)$ are the digamma and trigamma functions, respectively; see, Abramowitz and Stegun (1964). Then we have,

$$\begin{aligned}
E_{Y_{r:m:n}}\left(e^{k\xi_r}\right) &= c_{r-1}\Gamma(k+1)\sum_{i=1}^{r} a_{i,r}\gamma_i^{-(k+1)},\\
E_{Y_{r:m:n}}\left(\xi_r e^{k\xi_r}\right) &= c_{r-1}\Gamma(k+1)\sum_{i=1}^{r} a_{i,r}\gamma_i^{-(k+1)}\left[\psi(k+1)-\log\gamma_i\right],\\
E_{Y_{r:m:n}}\left(\xi_r^2 e^{k\xi_r}\right) &= c_{r-1}\Gamma(k+1)\sum_{i=1}^{r} a_{i,r}\gamma_i^{-(k+1)}\left\{\psi'(k+1)+\left[\psi(k+1)-\log\gamma_i\right]^2\right\},
\end{aligned}$$

and

$$E_{Y_{r:m:n}}\left(e^{e^{\xi_r}}\right) = c_{r-1}\sum_{i=1}^{r} a_{i,r}(\gamma_i - 1)^{-1}. \tag{C.2}$$

Note that it is possible that some $\gamma_i$ will equal 1 when $R_i = 0$. In this case, (C.2) cannot be computed. However, we will see in the subsequent derivations that when $R_i = 0$, the above computation can be avoided since the resulting matrix needs to be multiplied by $R_i$.

From the results of Chapter 6, we have

$$\begin{aligned}
E_{1,r} &= E_{Y_{r:m:n}}\left\{E_{Z_r}\left[\frac{Z_r-\mu}{\sigma}\middle|\xi_r,\mu,\sigma\right]\right\} \\
&= \psi(1)E_{Y_{r:m:n}}\left[e^{e^{\xi_r}}\right] + \sum_{p=0}^{\infty}\frac{\psi(p+2)}{\Gamma(p+2)}E_{Y_{r:m:n}}\left[e^{(p+1)\xi_r}\right] \\
&\quad -\sum_{p=0}^{\infty}\frac{1}{\Gamma(p+2)}\left\{E_{Y_{r:m:n}}\left[\xi_r e^{(p+1)\xi_r}\right] + \psi(1)E_{Y_{r:m:n}}\left[e^{(p+1)\xi_r}\right]\right\}, \\
E_{2,r} &= E_{Y_{r:m:n}}\left\{E_{Z_r}\left[\left(\frac{Z_r-\mu}{\sigma}\right)e^{\left(\frac{Z_r-\mu}{\sigma}\right)}\middle|\xi_r,\mu,\sigma\right]\right\} \\
&= \psi(2)E_{Y_{r:m:n}}\left[e^{e^{\xi_r}}\right] + \sum_{p=0}^{\infty}\frac{\psi(p+3)}{\Gamma(p+3)}E_{Y_{r:m:n}}\left[e^{(p+2)\xi_r}\right] \\
&\quad -\sum_{p=0}^{\infty}\frac{1}{\Gamma(p+3)}\left\{E_{Y_{r:m:n}}\left[\xi_r e^{(p+2)\xi_r}\right] + \psi(1)E_{Y_{r:m:n}}\left[e^{(p+2)\xi_r}\right]\right\}, \\
E_{3,r} &= E_{Y_{r:m:n}}\left\{E_{Z_r}\left[\left(\frac{Z_r-\mu}{\sigma}\right)^2 e^{\left(\frac{Z_r-\mu}{\sigma}\right)}\middle|\xi_r,\mu,\sigma\right]\right\} \\
&= [\psi'(2)+\psi^2(2)]E_{Y_{r:m:n}}\left[e^{e^{\xi_r}}\right] - \sum_{p=0}^{\infty}\frac{1}{\Gamma(p+3)}\left\{E_{Y_{r:m:n}}\left[\xi_r^2 e^{(p+2)\xi_r}\right]\right. \\
&\quad \left. +2\psi(2)E_{Y_{r:m:n}}\left[\xi_r e^{(p+2)\xi_r}\right] + [\psi^2(2)+\psi'(2)]E_{Y_{r:m:n}}\left[e^{(p+2)\xi_r}\right]\right\} \\
&\quad +\sum_{p=0}^{\infty}\frac{2\psi(p+3)}{\Gamma(n+3)}\left\{E_{Y_{r:m:n}}\left[\xi_r e^{(p+2)\xi_r}\right] + \psi(2)E_{Y_{r:m:n}}\left[e^{(p+2)\xi_r}\right]\right\} \\
&\quad +\sum_{p=0}^{\infty}\frac{\psi'(p+3)-\psi^2(p+3)}{\Gamma(p+3)}E_{Y_{r:m:n}}\left[e^{(p+2)\xi_r}\right].
\end{aligned}$$

The expected missing Fisher information matrix in one observation which is censored at the time of the $r$-th failure can then be computed as

$$E_{Y_{r:m:n}}\left[I_{Z|Y}^{(j)}(\boldsymbol{\theta})\right] = E_{Y_{r:m:n}}\left\{-E_{Z_r}\left[\frac{\partial^2 \log f_{Z_r}(z_r|y_{r:m:n},\boldsymbol{\theta})}{\partial \boldsymbol{\theta}^2}\right]\right\}$$

$$= \frac{1}{\sigma^2}\begin{bmatrix} M_{11}^r(n) & M_{12}^r(n) \\ M_{21}^r(n) & M_{22}^r(n) \end{bmatrix},$$

where

$$\begin{aligned}
M_{11}^r(n) &= E_{Y_{r:m:n}}\left\{E_{Z_r}\left[-\frac{\partial^2 \log f_{Z_r}}{\partial \mu^2}\right]\right\} = E_{Y_{r:m:n}}\left\{E_{Z_r}\left[e^{\left(\frac{z_r-\mu}{\sigma}\right)}\right] - e^{\xi_r}\right\} \\
&= E_{Y_{r:m:n}}\left[e^{\xi_r} + 1 - e^{\xi_r}\right] = 1, && \text{(C.3)} \\
M_{12}^r(n) &= M_{21}^r(n) = E_{Y_{r:m:n}}\left\{E_{Z_r}\left[-\frac{\partial^2 \log f_{Z_r}}{\partial \mu \partial \sigma}\right]\right\} \\
&= E_{2,r} + E_{3,r} - E_{Y_{r:m:n}}\left[e^{\xi_r}\right] - E_{Y_{r:m:n}}\left[\xi_r e^{\xi_r}\right] - 1, && \text{(C.4)} \\
M_{22}^r(n) &= E_{Y_{r:m:n}}\left\{E_{Z_r}\left[-\frac{\partial^2 \log f_{Z_r}}{\partial \sigma^2}\right]\right\} \\
&= E_{3,r} + 2E_{2,r} - 2E_{1,r} - 2E_{Y_{r:m:n}}\left[\xi_r e^{\xi_r}\right] - E_{Y_{r:m:n}}\left[\xi_r^2 e^{\xi_r}\right] - 1. && \text{(C.5)}
\end{aligned}$$

From (8.2.3) and the missing information principle, the expected value of the observed Fisher information matrix is then

$$E_{Y_{1:m:n},\ldots,Y_{m:m:n}}\left[I_Y(\boldsymbol{\theta})\right] = \frac{1}{\sigma^2}\begin{bmatrix} O_{11}(n) & O_{12}(n) \\ O_{21}(n) & O_{22}(n) \end{bmatrix} = \begin{bmatrix} I_Y^{11} & I_Y^{12} \\ I_Y^{21} & I_Y^{22} \end{bmatrix}, \qquad \text{(C.6)}$$

where

$$\begin{aligned}
O_{11}(n) &= m, \\
O_{12}(n) &= O_{21}(n) = n(1-\gamma) - \sum_{r=1}^{m} R_r M_{12}^r(n), \\
O_{22}(n) &= nc^2 - \sum_{r=1}^{m} R_r M_{22}^r(n). && \text{(C.7)}
\end{aligned}$$

Then the expected asymptotic variance-covariance matrix of the MLEs is

$$\begin{aligned}
\sigma^2 V(\boldsymbol{\theta}) &= \sigma^2 \begin{bmatrix} V_{11}(n) & V_{12}(n) \\ V_{21}(n) & V_{22}(n) \end{bmatrix} \\
&= \sigma^2 \begin{bmatrix} O_{11}(n) & O_{12}(n) \\ O_{21}(n) & O_{22}(n) \end{bmatrix}^{-1}. && \text{(C.8)}
\end{aligned}$$

# Appendix D

# Derivation of Asymptotic Distribution of MMEs for Birnbaum-Saunders Distribution

Let $T, T_1, \ldots, T_n$ be independent and identically distributed Birnbaum-Saunders random variables with p.d.f. as in (10.2.1). Let us define the random variables

$$S = \frac{1}{n}\sum_{i=1}^{n} T_i \quad \text{and} \quad R = \left[\frac{1}{n}\sum_{i=1}^{n} T_i^{-1}\right]^{-1}.$$

From the strong law of large numbers, it is known that $S$ and $R^{-1}$ converge almost surely to $E(T)$ and $E(T^{-1})$, respectively. Also, from the central limit theorem, we have for large $n$

$$\frac{\sqrt{n}[S - E(T)]}{\sqrt{Var(T)}} \sim N(0,1)$$

and

$$\frac{\sqrt{n}[R^{-1} - E(T^{-1})]}{\sqrt{Var(T^{-1})}} \sim N(0,1).$$

Further, we have $(S\ R^{-1})'$ to be asymptotically distributed as bivariate normal. Consequently, any linear combination of the form

$$aS + bR^{-1} = \frac{1}{n}\sum_{i=1}^{n}\left[aT_i + bT_i^{-1}\right]$$

is also asymptotically normally distributed for all $a$ and $b$. Therefore, we have

$$\sqrt{n}\begin{pmatrix} S - E(T) \\ R^{-1} - E(T^{-1}) \end{pmatrix} \sim N\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \Sigma\right],$$

where

$$\begin{aligned}
\Sigma &= \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}, \\
\sigma_{11} &= Var(T) = (\alpha\beta)^2\left(1 + \frac{5}{4}\alpha^2\right), \\
\sigma_{12} &= \sigma_{21} = Cov(T, T^{-1}) = E(1) - E(T)E(T^{-1}) = 1 - \left(1 + \frac{1}{2}\alpha^2\right)^2, \\
\sigma_{22} &= Var(T^{-1}) = \alpha^2\beta^{-2}\left(1 + \frac{5}{4}\alpha^2\right).
\end{aligned}$$

We now need to find the asymptotic joint distribution of

$$\begin{pmatrix} \tilde{\alpha} \\ \tilde{\beta} \end{pmatrix} = \begin{bmatrix} f_1(S, R) \\ f_2(S, R) \end{bmatrix},$$

where

$$\begin{aligned}
f_1(x, y) &= \left\{2\left[\left(\frac{x}{y}\right)^{\frac{1}{2}} - 1\right]\right\}^{\frac{1}{2}}, \\
f_2(x, y) &= (xy)^{\frac{1}{2}}.
\end{aligned}$$

By using the Taylor series expansion, we obtain

$$\sqrt{n}\begin{pmatrix} \tilde{\alpha} - \alpha \\ \tilde{\beta} - \beta \end{pmatrix} \sim N\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tilde{\Sigma}\right],$$

where

$$\begin{aligned}
\tilde{\Sigma} &= \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} \Sigma \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix}^T \Bigg|_{x=E(T), y=E(T^{-1})} \\
&= \begin{bmatrix} \frac{\alpha^2}{2} & 0 \\ 0 & (\alpha\beta)^2\left(\frac{1+\frac{3}{4}\alpha^2}{(1+\frac{1}{2}\alpha^2)^2}\right) \end{bmatrix}.
\end{aligned}$$

Of course, we are using here the following expressions:

$$\begin{aligned}
\left.\frac{\partial f_1}{\partial x}\right|_{x=E(T),y=E(T^{-1})} &= \frac{1}{2\alpha\beta}, \\
\left.\frac{\partial f_1}{\partial y}\right|_{x=E(T),y=E(T^{-1})} &= \frac{\beta}{2\alpha}, \\
\left.\frac{\partial f_2}{\partial x}\right|_{x=E(T),y=E(T^{-1})} &= \frac{1}{2+\alpha^2}, \\
\left.\frac{\partial f_2}{\partial y}\right|_{x=E(T),y=E(T^{-1})} &= -\frac{\beta^2}{2+\alpha^2}.
\end{aligned}$$

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