PRIME IDEALS AND LOCALIZATION IN NOETHERIAN ORE EXTENSIONS

By

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ABSTRACT

This thesis studies the prime ideals in a certain class of non-commutative polynomial rings known as Ore extensions. For a right Noetherian Ore extension $R[x;\sigma]$, the prime ideals are of three types: one type corresponds to the prime ideals of the coefficient ring R, another type corresponds to certain semiprime ideals of R, and the third type is in bijective correspondence with the irreducible polynomials in certain ordinary polynomial rings.

The second part of the thesis studies the question of the localizability of prime ideals in Ore extensions of commutative Noetherian rings. It is shown that these rings satisfy Jategaonkar's second layer condition and that the corresponding skew Laurent polynomial ring is Krull symmetric. Using these properties and the classification of prime ideals, a complete description of the obstructions to localizability - the links between prime ideals - is obtained.

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INTRODUCTION

In this thesis, we study the prime ideals of a certain class of non-commutative polynomial rings called Ore extensions or skew polynomial rings. Our goal is to analyze the structure of prime ideals in such rings and to apply this information to the problem of determining whether or not a given prime ideal is localizable.

For an arbitrary ring R with automorphism σ, the Ore extension $S = R[x;\sigma]$ consists of polynomials $r_0 + r_1x + \ldots + r_nx^n$ with $r_i \in R$ and $n \geq 0$. Addition is performed termwise and multiplication is determined by the rule $xr = r^\sigma x$.

Although these rings can be traced back to Hilbert, they were first studied for their own sake by Ore in 1933 in the case where R is a division ring [44]. Ten years later, Jacobson determined the two-sided ideals of such rings and analyzed the finitely-generated modules over them, again with R a division ring [21]. For the next twenty-five years, there was only sporadic interest in these Ore extensions and, with few exceptions (e.g. [11], [13]), they appeared primarily as a source of counterexamples.

In the past decade, however, there has been a flurry of activity as once again Ore extensions are being studied in their own right. Part of this interest stems from a connection between Ore extensions and group rings of polycyclic groups. Indeed, several papers have been devoted to proving known results for these group rings using Ore extension techniques ([15], [35], [36]). Therefore, one possible motivation for studying Ore extensions is that any result which holds for Ore extensions in general holds for polycyclic group rings in particular. On a simpler level, the Hilbert Basis Theorem shows that an Ore extension of a Noetherian ring is again Noetherian; thus we have a large class of non-commutative Noetherian rings in which to illustrate the rich theory of such rings.

The theory of non-commutative Noetherian rings can reasonably be said to have begun with Goldie's work on prime Noetherian rings and what is now usually known as Goldie's Theorem (cf. [14]). Many results and techniques from the theory of commutative Noetherian rings have been successfully adapted to non-commutative Noetherian rings. The notion of localization at a prime ideal, however, has not had an easy transition. Unlike the commutative case, not every prime ideal in a non-commutative Noetherian ring is localizable. This has led researchers to study the obstructions to localization - the links between prime ideals - and has shifted attention to families of prime ideals, namely those which are linked together ([39], [40]). The work of Jategaonkar figures prominently in these considerations and his recent survey [26] contains an extensive account of his module-theoretic treatment of the localization problem.

The goal of this thesis is to determine the mechanism by which prime ideals are linked in Noetherian Ore extensions. To this end, we first give a complete classification of the prime ideals in right Noetherian Ore extensions. Then, restricting our attention to Ore extensions of commutative Noetherian rings, we prove that they are "well-behaved" rings - Krull symmetric with the second layer condition.

Using the available results on localization in such rings, we are then able to classify the links and the localizable prime ideals in such Ore extensions.

Chapter I contains the definitions and basic properties of Ore extensions which we need in our investigations. In particular, we cite a version of the Hilbert Basis Theorem for Ore extensions and then, in the context of right Noetherian rings, we establish a connection between prime ideals of $R[x;\sigma]$ and so-called σ-prime ideals of R. Having given these preliminary results, in Chapter II we undertake the task of actually classifying the prime ideals of a right Noetherian Ore extension. Here we are motivated both by what one does in commutative polynomial rings [37] and by Jacobson's work on Ore extensions of division rings [21]. Somewhat more specifically, we establish a bijective correspondence between certain prime ideals - the upper primes - of the Ore extension $R[x;\sigma]$ and certain irreducible polynomials of a naturally-occurring polynomial ring over a field. Two other, more easily described, types of prime ideal - lower primes and primes containing x - complete our classification. Most of this work is accomplished by reducing to the situation where R is a direct sum of finitely-many copies of a simple Artinian ring cyclically permuted by σ. In this restricted setting, we also determine the center and consider some of the multiplicative ideal theory of $R[x;\sigma]$.

We next turn to the problem of the localizability of a prime ideal in a Noetherian Ore extension. Chapter III sets the stage for this by surveying the relevant results from the general theory of localization in non-commutative Noetherian rings. Our approach follows

Jategaonkar [26] but slightly reformulates his results in order to make more transparent the crucial role played by the links between prime ideals. In fact, there are several notions of "link" available and the precise relationship between them is not known in general; however, they all coincide in the setting which interests us - Krull symmetric Noetherian rings - and we show that the theory of localization is particularly satisfying for such rings. The chapter concludes with a few results which show that the presence of normal elements in a ring greatly reduces the possibilities for linked primes.

Chapter IV applies the foregoing theory to Ore extensions of commutative Noetherian rings. We completely determine the possibilites for links between prime ideals for each of the three classifications: primes containing x, lower primes, and upper primes. We first show that primes containing x are easily handled using the fact that x is a normal element. Lower primes, as it turns out, are all classical. Upper primes are harder to deal with but by passing to the skew Laurent polynomial ring $R[x, x^{-1}; \sigma]$ - a Krull symmetric Noetherian ring - we are able to develop a method which produces all the prime ideals linked to a given upper prime. More specifically, from the irreducible polynomial associated with a given upper prime P, we construct a new polynomial whose irreducible factors are the polynomials associated with the prime ideals which are linked to P. The chapter also contains several examples which illustrate the variety of possibilities for linked primes. It concludes with a proof that it is possible to localize at an infinite link component in an Ore extension of a commutative Noetherian algebra over an uncountable field.

In Appendix A of the thesis, we show that even for non-Noetherian Ore extensions, certain finiteness conditions are preserved by taking Ore extensions. Specifically, we show that Goldie dimension and non-singularity are each preserved in the above sense. Using these results, we then conclude that the Ore extension $R[x;\sigma]$ is a semiprime right Goldie ring if and only if R is a σ-semiprime right Goldie ring.

Appendix B contains an alternate proof of the fact that every lower prime ideal of the Ore extension $R[x;\sigma]$ is classical (in the special case where R is a commutative affine algebra over a field of characteristic zero). Although we have given a shorter and more general proof of this fact in Chapter IV, the techniques used in this section may be of interest.

CHAPTER I

PRELIMINARIES

Throughout this thesis, all rings will be associative, but not necessarily commutative, rings with identity elements. All modules will be unitary right modules unless indicated otherwise. An ideal is always assumed to be a two-sided ideal unless modified by "right" or "left". The same convention extends to concepts like Noetherian, Artinian, localizable, and classical - whenever there is a question of handedness, the unadorned adjective means the condition is assumed to hold on the right and on the left. A regular element of a ring is a non-zero-divisor.

For any undefined concepts from the theory of rings and modules, the reader is referred to the books by Lambek [30] and Renault [49].

§1. Definitions.

Let R be a ring with an automorphism σ. The Ore extension (skew polynomial ring) $S = R[x;\sigma]$ consists of elements of the form $r_0 + r_1x + \ldots + r_nx^n$, where $n \geq 0$ and $r_i \in R$ for $0 \leq i \leq n$. The underlying additive structure of S is that of $R[x]$, while multiplication in S is determined by the distributive law and the rule $xr = r^\sigma x$ for all $r \in R$.

We note that while the normal form of a polynomial $f(x) \in S$ has its coefficients written on the left of the indeterminate x, $f(x)$ may equivalently be written as a polynomial whose coefficients are on the right (since σ is an automorphism). Explicitly, $f(x) = r_0 + r_1x + \ldots + r_nx^n = r_0 + xr_1^{\sigma^{-1}} + \ldots + x^nr_n^{\sigma^{-n}}$. This observation will occasionally be useful and will be used without further mention.

The automorphism σ extends to an automorphism of S (which we also denote by σ) via $x^\sigma = x$.

Definition 1.1: A (right) ideal I of R is σ-invariant (or a σ-(right) ideal) if $I^\sigma \subseteq I$.

Observe that for a σ-ideal I of R, σ induces a ring endomorphism of R/I via $r + I \to r^\sigma + I$. We denote this endomorphism by σ as well, and note that σ is an automorphism of R/I if and only if $I^\sigma = I$. This latter condition holds in particular if R is right Noetherian: for if I is a σ-ideal, then $I \subseteq I^{\sigma^{-1}} \subseteq I^{\sigma^{-2}} \subseteq \ldots$ is an ascending chain of (right) ideals of R which must therefore become stationary at $I^{\sigma^{-k}} = I^{\sigma^{-k-1}}$, say. Whence $I^\sigma = I$.

Definition 1.2: (i) A σ-ideal I of R is σ-prime if, given an ideal A and a σ-ideal B of R such that $AB \subseteq I$, either $A \subseteq I$ or $B \subseteq I$.

(ii) A σ-ideal I of R is σ-semiprime if, for any ideal A of R and integer m such that $AA^{\sigma^i} \subseteq I$ for all $i \geq m$, then $A \subseteq I$.

(iii) The ring R is σ-semiprime (respectively σ-prime) if 0 is a σ-semiprime (respectively σ-prime) ideal of R.

We are interested primarily in right Noetherian rings, where these notions have several equivalent formulations. The following two propositions are modelled, in part, on Lemma 3.2 of [20] where R is assumed to be commutative.

<u>Proposition 1.3</u>: Let R be right Noetherian and I a σ-ideal of R. The following are equivalent:

(i) I is σ-prime.

(ii) Given ideals A and B of R and an integer m such that $AB^{\sigma^i} \subseteq I$ for all $i \geq m$, either $A \subseteq I$ or $B \subseteq I$.

(iii) Given elements $a, b \in R$ and an integer m such that $aRb^{\sigma^i} \subseteq I$ for all $i \geq m$, either $a \in I$ or $b \in I$.

(iv) Given σ-ideals A and B of R such that $AB \subseteq I$, either $A \subseteq I$ or $B \subseteq I$.

<u>Proof</u>: (i) $\Rightarrow$ (ii) Set $C = \sum_{i \geq m} B^{\sigma^i}$. Then C is a σ-ideal and $AC \subseteq I$. By (i), either $A \subseteq I$ or $C \subseteq I$. But since R is right Noetherian, $C^{\sigma^{-1}} = C$ and so $C = \sum_{i \in \mathbf{Z}} B^{\sigma^i} \supseteq B$. Thus either $A \subseteq I$ or $B \subseteq I$.

(ii) $\Rightarrow$ (iii) Set $A = RaR$ and $B = RbR$. Then $AB^{\sigma^i} \subseteq I$ for all $i \geq m$ and so either $A \subseteq I$ or $B \subseteq I$; that is, either $a \in I$ or $b \in I$.

(iii) $\Rightarrow$ (iv) Suppose $A \nsubseteq I$. Pick $a \in A$, $a \notin I$. For any $b \in B$, $b^{\sigma^i} \in B$ for all i and so $aRb^{\sigma^i} \subseteq AB \subseteq I$ for all i. By (iii), $b \in I$. Hence $B \subseteq I$.

(iv) $\Rightarrow$ (i) If A is an ideal and B a σ-ideal of R, then $B^{\sigma^i} = B$

for all i, since R is right Noetherian. So $A^{\sigma^i}B = (AB)^{\sigma^i} \subseteq I^{\sigma^i} = I$.
Set $C = \sum_{i\in\mathbb{Z}} A^{\sigma^i}$. Then C is a σ-ideal and $CB \subseteq I$. It follows that either $A \subseteq C \subseteq I$ or $B \subseteq I$. □

Proposition 1.4: Let R be right Noetherian and I a σ-ideal of R. The following are equivalent:

(i) I is σ-semiprime.

(ii) Given $a \in R$ and an integer m such that $aRa^{\sigma^i} \subseteq I$ for all $i \geq m$, then $a \in I$.

(iii) Given a σ-ideal A of R such that $A^2 \subseteq I$, then $A \subseteq I$.

Proof: An easy modification of the proof of Proposition 1.3. □

Proposition 1.3 (ii) shows that every σ-prime ideal is σ-semiprime in a right Noetherian ring. The next proposition shows that σ-semiprime ideals have a nice description in a right Noetherian ring – a description which makes them both easy to recognize and to construct.

Proposition 1.5: Let R be right Noetherian and I a σ-ideal of R. I is σ-semiprime if and only if it is semiprime and is the intersection of (finitely many) prime ideals which are permuted by σ. I is σ-prime if and only if it is the intersection of the σ-orbit of a prime ideal.

Proof: Sufficiency is easy in each case. The necessity of the conditions is an easy adaptation of [15] Properties 4* and 5* to σ-semiprime ideals. □

Thus, in a right Noetherian ring, σ-semiprime ideals are characterized as the intersection of finitely many σ-orbits while σ-prime ideals are the intersection of a single orbit.

<u>Example 1.6</u>: Let F be a field and $R = F \oplus F$. Define an automorphism σ of R via $(a,b)^{\sigma} = (b,a)$ for $a,b \in F$. R has two prime ideals $P_1 = F \oplus 0 = \{(a,0) | a \in F\}$ and $P_2 = 0 \oplus F = \{(0,a) | a \in F\}$, neither of which is σ-prime. On the other hand, $P_1 \cap P_2 = 0$ and $P_1^{\sigma} = P_2$ shows that 0 is a σ-prime ideal of R which is not prime.

In commutative algebra, a polynomial ring over a Noetherian ring is again a Noetherian ring. This is usually known as the Hilbert Basis Theorem. In our setting, we have the following version:

<u>Theorem 1.7</u>: If R is right Noetherian, then the Ore extension $S = R[x;\sigma]$ is right Noetherian.

<u>Proof</u>: A suitable adaptation of the usual proof of the Hilbert Basis Theorem works here as well. [cf. 20, Proposition 2.3]. □

For a discussion of the preservation of more general finiteness conditions, see Appendix A.

§2. Prime and Semiprime Ideals in Ore Extensions.

The polynomial ring $S = R[x]$ is a special case of an Ore extension, corresponding to $\sigma = id_R$, the identity automorphism of R. If P is a prime ideal of S, then its contraction $P \cap R$ is a prime ideal of R; if Q is a prime ideal of R, then its extension $QS = Q[x]$ is a prime ideal of S. Neither of these implications is true for Ore extensions in general as the following example shows.

Example 1.8: Let R and σ be as in Example 1.6 and set $S = R[x;\sigma]$. Then S is a prime ring (i.e. 0 is a prime ideal) but R is not a prime ring. (cf. Proposition 1.11). On the other hand, $P_1 = F \oplus 0$ is a prime ideal of R but P_1S is not a prime ideal of S.

In this section, we describe the extension and contraction relations which hold in right Noetherian Ore extensions. Most of what follows is due to Goldie and Michler [15]. For commutative but not necessarily Noetherian coefficient rings, these results may be found in Irving [20].

Lemma 1.9: ([15] Lemma 1.1) Let R be right Noetherian and $S = R[x;\sigma]$. If A is a σ-ideal of S, then $S/(A \cap R)S \cong (R/(A \cap R))[x;\sigma]$. □

Definition 1.10: Let A be an ideal of S. Define $\tau_n(A) = \{r \in R \mid (rx^n + \text{lower terms}) \in A\} \cup \{0\}$.

Proposition 1.11: Let $S = R[x;\sigma]$. Then:

(i) S is semiprime if and only if R is σ-semiprime.

(ii) S is prime if and only if R is σ-prime.

Proof: (i) Suppose S is semiprime. Let A be an ideal of R and m an integer such that $AA^{\sigma^i} = 0$ for all $i \geq m$. If we define $I = \{a_m x^m + \ldots + a_n x^n | a_i \in A,\ n \geq m\}$, then I is a right ideal of S and $(SI)^2 = 0$. Hence $SI = 0$ and so $I = 0$. Therefore $A = 0$.

Conversely, assume R is σ-semiprime and let B be an ideal of S such that $B^2 = 0$. Suppose $B \neq 0$ and let b be any non-zero element of $\tau(B)$, an ideal of R. Then $bx^m + \ldots + b_0 \in B$ and since $B^2 = 0$, $(bx^m + \ldots + b_0)rx^i(bx^m + \ldots + c_0) = 0$ for all $i \geq 0$ and $r \in R$. Thus $bRb^{\sigma^{m+i}} = 0$ for all $i \geq 0$ and so $(RbR)(RbR)^{\sigma^i} = 0$ for all $i \geq m$. Since R is σ-semiprime, $RbR = 0$ and so $b = 0$, a contradiction. We conclude that $B = 0$ and S is semiprime.

(ii) The proof of (ii) is similar to that of (i) and is left to the reader. □

The next proposition says that prime ideals of S which contain x are in one-to-one correspondence with prime ideals of R and so, in a sense, are known.

Proposition 1.12: (i) Let P be a prime ideal of S, $x \in P$. Then $P = (P \cap R) + xS$ and $P \cap R$ is a prime ideal of R.

(ii) If Q is a prime ideal of R, then $Q + xS$ is a prime ideal of S.

Proof: Use $S/((P \cap R) + xS) \cong R/P \cap R$ and $S/(Q + xS) \cong R/Q$, respectively. □

For the most part, we will be concerned with those prime ideals of S which do not contain x. For these, the following result gives the correct extension and contraction relations. For an ideal I of R, $\mathcal{C}_R(I) = \{r \in R \mid r + I$ is a regular element of $R/I\}$.

Proposition 1.13: Let R be right Noetherian and $S = R[x;\sigma]$.

(i) If A is a semiprime ideal of S, none of whose minimal prime ideals contains x, then $x \in \mathcal{C}_S(A)$, $A^\sigma = A$, $A \cap R$ is a σ-semiprime ideal of R, and $(A \cap R)S$ is a semiprime ideal of S. If A is prime, then $A \cap R$ is σ-prime and $(A \cap R)S$ is again prime.

(ii) If I is a σ-semiprime ideal of R, then IS is a semiprime of S. If I is σ-prime, IS is prime.

Proof: (i) Let $A = \bigcap_{i=1}^{n} P_i$ where $P_1, \ldots, P_n$ are the minimal prime ideals of S over A. Suppose $f(x) \in S$ and $xf(x) \in A$. Then $xSf(x) = Sxf(x) \subseteq P_i$ for $1 \leq i \leq n$. Since $x \notin P_i$, $f(x) \in P_i$, for all i and so $f(x) \in A$. Similarly $f(x)x \in A$ implies $f(x) \in A$. Thus $x \in \mathcal{C}_S(A)$. Let $f(x) \in A$. Then $f^\sigma(x)x = xf(x) \in A$ and so $f^\sigma(x) \in A$. Thus $A^\sigma \subseteq A$ and the right Noetherian property gives $A^\sigma = A$. The argument of Proposition 1.11(i) may be used to show that $A \cap R$ is σ-semiprime. By Lemma 1.9, $S/(A \cap R)S \cong (R/(A \cap R))[x;\sigma]$. Proposition 1.11(i) then shows that $(A \cap R)S$ is semiprime.

The last statement of (i) is proved in the same way. (cf. [15], Lemmata 1.2, 1.3, 1.7.)

(ii) Use the isomorphism $S/IS \cong (R/I)[x;\sigma]$ and Proposition 1.11. □

Remark 1.14: In Example 1.8, R is σ-prime. This is the prototypical example for this situation.

CHAPTER II

CLASSIFICATION OF PRIME IDEALS IN ORE EXTENSIONS

Our goal in this chapter is to analyze the prime ideals of the right Noetherian Ore extension $S = R[x;\sigma]$. Since the prime ideals of S which contain x are in one-to-one correspondence with the prime ideals of R, which we may assume are known, we concern ourselves with the prime ideals of S which do not contain x. By Proposition 1.13, such a prime ideal contracts to a σ-prime ideal of R. Thus, to determine the prime ideals of S, it suffices to determine the prime ideals of S lying over a given σ-prime ideal I of R. Using Lemma 1.9, there is no loss of generality in assuming $I = 0$ (i.e. that R is σ-prime). We must therefore determine the prime ideals of S, not containing x, which contract to 0 in R.

For R a division ring, this was done by Jacobson [21] and for R a commutative ring, by Irving [20]. Our results will include elements of both of these characterizations. But it is perhaps most instructive to keep in mind the characterization of prime ideals in commutative polynomial rings. A nice account of this can be found in McAdam [37] or is easily deduced from material in Kaplansky [29].

Let us summarize this result for future reference. R is commutative and $S = R[x]$. In this setting, a prime ideal of S contracts to a prime ideal of R, which we may assume is 0 (i.e. R is an integral domain). Let $Q(R)$ be the quotient field of R. It is not difficult to prove the following:

Theorem 2.1: There is a one-to-one correspondence between the non-zero prime ideals of S which contract to 0 in R, and the monic irreducible polynomials of $Q(R)[x]$. □

Our objective, then, is to obtain an analogue of this result for right Noetherian Ore extensions.

§1. Initial Reductions.

Recall that a multiplicative set C of non-zero elements of a ring R is said to be a right Ore set (or, R satisfies the right Ore condition with respect to C) if for all $r \in R$ and $c \in C$, there exist $r' \in R$ and $c' \in C$ such that $rc' = cr'$. For $C = C_R(0)$, the set of regular elements of R, R has a classical right quotient ring $Q(R)$ if and only if C is a right Ore set. Moreover, Goldie has given necessary and sufficient conditions for R to have a classical right quotient ring which is semisimple Artinian (cf. [14]). This fundamental result, usually known as Goldie's Theorem, implies in particular that a right Noetherian semiprime ring has a semisimple Artinian classical right quotient ring $Q(R)$.

Throughout this section, R denotes a right Noetherian ring which, for the purposes of analyzing the prime ideals of its Ore extension $S = R[x;\sigma]$, we assume to be σ-prime. By Proposition 1.5, R is then semiprime and so Goldie's Theorem applies to give $Q(R)$ is semisimple Artinian.

We can say a bit more. Proposition 1.5 actually shows that the finitely many minimal primes of R – $P = P_1, P_2, \ldots, P_n$ say – form a full orbit under σ. Renumbering if necessary, we have $P_i^\sigma = P_{i+1}$ and $P_n^\sigma = P_1$. Since $\mathcal{C}_R(0) = \bigcap_{i=1}^{n} \mathcal{C}_R(P_i)$, the right ideals $P_i\mathcal{Q}(R)$ are all two-sided and are precisely the prime ideals of $\mathcal{Q}(R)$ (cf. [14]). Now σ extends uniquely to an automorphism of $\mathcal{Q}(R)$ via $(rc^{-1})^\sigma = r^\sigma(c^\sigma)^{-1}$, $r \in R$, $c \in \mathcal{C}(0)$. Then $\mathcal{Q}(R) = \mathcal{Q}(R/\bigcap_{i=1}^{n} P_i) \cong \bigoplus_{i=1}^{n} \mathcal{Q}(R/P_i) \cong \bigoplus_{i=1}^{n} (\mathcal{Q}(R)/P_i\mathcal{Q}(R)) \cong (\mathcal{Q}(R)/P\mathcal{Q}(R))^n$, where $\mathcal{Q}(R)/P\mathcal{Q}(R)$ is isomorphic to an $m \times m$ matrix ring over a division ring and these n blocks of $\mathcal{Q}(R)$ are cyclically permuted by σ.

This, then, is our setting. In order to facilitate our study of the prime ideals of S, we first make some observations and reductions.

<u>Lemma 2.2</u>: ([15], Lemma 1.4) If R is a right Noetherian σ-prime ring, then S satisfies the right Ore condition with respect to $\mathcal{C}_R(0)$. □

This says, in other words, that the partial right quotient ring $S_{\mathcal{C}} = \mathcal{Q}(R)[x;\sigma]$ exists.

<u>Lemma 2.3</u>: ([15], Lemma 1.5) Let R be a right Noetherian σ-prime ring. If I is a non-zero σ-ideal of R, then $I \cap \mathcal{C}_R(0) \neq \phi$ (i.e. I contains a regular element of R). □

<u>Definition 2.4</u>: A ring R with automorphism σ is called <u>σ-simple</u> if R has no proper σ-ideals.

Corollary 2.5: If R is a right Noetherian σ-prime ring, then $Q(R)$ is σ-simple Artinian.

Proof: If A is a non-zero σ-ideal of $Q(R)$, then $A \cap R$ is a non-zero σ-ideal of R, hence contains a regular element of R. Therefore A contains an invertible element of $Q(R)$ and so $A = Q(R)$. □

Corollary 2.6: Let R be a right Noetherian σ-prime ring and let P be a prime ideal of S such that $P^{\sigma} = P$. Then $P \cap R = 0$ if and only if $P \cap C_R(0) = \phi$. □

This is true in particular for those prime ideals P of S which do not contain x. Thus the prime ideals of S which we want to classify "survive" in the partial right quotient ring $S_C = Q(R)[x;\sigma]$: if P is a prime ideal of S, $x \notin P$, $P \cap R = 0$, then PS_C is a prime ideal of S_C, $x \notin PS_C$, and $PS_C \cap Q(R) = 0$. The next lemma says that we can formally recover the original prime P from PS_C.

Lemma 2.7: Let R be right Noetherian, σ-prime. If P is a prime ideal of S such that $x \notin P$ and $P \cap R = 0$, then $PS_C \cap S = P$.

Proof: Clearly $P \subseteq PS_C \cap S$. Conversely if $g(x) = f(x)c^{-1} \in PS_C \cap S$ where $f(x) \in P$ and $c \in C = C_R(0)$, then $g(x)c = f(x) \in P$. Now $C \subseteq C_S(P)$ ([15], Lemma 1.6). Hence $g(x) \in P$. □

Lemma 2.8: ([15], Lemma 1.10) Let R be right Noetherian, σ-prime. Let P be a non-zero prime ideal of S such that $P \cap R = 0$. If A is an ideal of S properly containing P, then $A \cap \mathcal{C}_R(0) \neq \phi$. □

Corollary 2.9: Let R be right Noetherian. Then there does not exist a chain of three distinct prime ideals of S with the same contraction in R.

Proof: Suppose $P_1 \subsetneqq P_2 \subsetneqq P_3$ are prime ideals of S with $P_1 \cap R = P_2 \cap R = P_3 \cap R = I$, say. If $x \in P_1$, then $P_1 = P_2 = P_3 = I + xS$ by Proposition 1.12(i). If $x \notin P_1$, then I is σ-prime and we may assume without loss of generality that $I = 0$ and $P_1 = 0$. Now if $x \notin P_3$ then $P_3^\sigma = P_3$; otherwise $P_3 = xS$ and again $P_3^\sigma = P_3$. So, by Corollary 2.6, $P_3 \cap \mathcal{C}_R(0) = \phi$. This contradicts Lemma 2.8. □

Definition 2.10: A prime ideal P of S is called an upper prime if $x \notin P$ and $P \neq (P \cap R)S$. In this situation, if $P \cap R = I$, we say P is upper to I. P is called a lower prime if $P = (P \cap R)S$.

In Theorem 2.39, we shall give necessary and sufficient conditions for the existence of prime ideals of S upper to a given σ-prime ideal of R. Before proceeding with this analysis, let us summarize the reductions we have made.

Proposition 2.11: Let R be a right Noetherian, σ-prime ring and set $\mathcal{C} = \mathcal{C}_R(0)$. Then $Q(R)$ is σ-simple Artinian and there is a one-to-one correspondence between the prime ideals of S, not containing x, contracting to 0 in R and the prime ideals of $S_C = Q(R)[x;\sigma]$, not containing x, contracting to 0 in $Q(R)$.

Proof: Combine Corollaries 2.5, 2.6, and Lemma 2.7. The correspondence is $P \to PS_C$ for P prime in S, $x \notin P$, $P \cap R = 0$; $Q \to Q \cap S$ for Q prime in S_C, $x \notin Q$, $Q \cap Q(R) = 0$. □

Thus, to determine the prime ideals of S in question, we may assume without loss of generality that R is σ-simple Artinian.

§2. σ-Simple Artinian Coefficient Rings.

In practice, the σ-simple Artinian rings in this section arise as the semisimple Artinian classical right quotient rings $Q(R)$ of right Noetherian σ-prime rings R. Even without this information we still have the following:

Lemma 2.12: Let R be a σ-simple Artinian ring. Then R is semisimple and σ permutes the simple Artinian factors.

Proof: R is certainly σ-prime, hence semiprime by Proposition 1.5. Let $P_1, \ldots, P_n$ be the minimal primes of R with $P_i^\sigma = P_{i+1}$. Then P_i is maximal for $1 \le i \le n$ since R is Artinian, and so $R \cong \bigoplus_{i=1}^{n} (R/P_i)$ with R/P_i simple Artinian for $1 \le i \le n$. The last statement is obvious. □

We now set out to prove our analogue of Theorem 2.1.

Lemma 2.13: ([15], Lemma 1.9) Let R be a σ-simple Artinian ring. Let $f(x) = f_0 + f_1x + \ldots + f_nx^n \in S$ with f_n invertible in R, and let $g(x) \in S$. There exist $q(x), r(x) \in S$ such that $g(x) = f(x)g(x) + r(x)$ and $\deg(r(x)) < \deg(f(x))$ or $r(x) = 0$. $\square$

We will refer to this lemma as the Division Algorithm. Using it, we see immediately that the prime ideals of S which we are considering are principal.

Proposition 2.14: Let R be σ-simple Artinian and let P be a non-zero prime ideal of S such that $P^\sigma = P$. Then $P = p(x)S = Sp(x)$ for a unique monic polynomial $p(x) \in P$.

Proof: Let n be the minimal degree of polynomials in P. Recall that $\tau_n(P)$ denotes the ideal of R consisting of the leading coefficients of degree n polynomials in P, together with 0. Since $P^\sigma = P$, $\tau_n(P)$ is a non-zero σ-ideal of R, hence $\tau_n(P) = R$, by the σ-simplicity of R. We conclude that there exists a monic polynomial $p(x)$ of minimal degree n in P. By minimality, this polynomial is unique and therefore $p^\sigma(x) = p(x)$. Let $f(x) \in P$. The Division Algorithm gives $q(x), r(x) \in S$ such that $f(x) = p(x)q(x) + r(x)$. Thus $r(x) \in P$ with $\deg(r(x)) < \deg(p(x))$, and so $r(x) = 0$. Therefore $P = p(x)S$. This also shows that $Sp(x) \subseteq p(x)S$. To get the reverse inclusion, it suffices to show that $p(x)x = xp(x)$ and that for all $r \in R$, there exists $r' \in R$ such that $p(x)r = r'p(x)$. The former equality is trivially true since $p^{\sigma}(x) = p(x)$. For the latter, let $r \in R$ and set $a = r^{\sigma^n}$. Then $ap(x) = p(x)b(x)$ for some $b(x) \in S$. But a comparison of degrees shows that $b(x) = b \in R$ and the fact that $p(x)$ is monic shows that $a = b^{\sigma^n}$. Therefore $p(x)r = r^{\sigma^n}p(x)$ for all $r \in R$. Consequently $Sp(x) = p(x)S = P$. □

Remark 2.15: The preceding proposition applies in particular to those prime ideals of S which do not contain x. The principal generator $p(x)$ of P is a normal element of S; one might ask whether or not it is possible to find a central element of S which also generates P. As the following corollary shows, the answer is yes. We denote the center of a ring R by $C(R)$.

Corollary 2.16: Let R be σ-simple Artinian. If P is a non-zero prime ideal of S such that $x \notin P$, then $P = p'(x)S$ for some central, centrally-irreducible polynomial $p'(x) \in P$ of minimal degree.

Proof: Let $p(x) = p_0 + p_1x + \ldots + p_{n-1}x^{n-1} + x^n \in P$ as in Proposition 2.14. By Proposition 1.13, $x \in C_s(P)$ and so the minimality of n guarantees that $p_0 \neq 0$. Since $p^{\sigma}(x) = p(x)$, $p_0^{\sigma} = p_0$. From the condition $p(x)r = r^{\sigma^n}p(x)$ for all $r \in R$, we get $p_0r = r^{\sigma^n}p_0$ for all $r \in R$ and so $p_0R = Rp_0 = Rp_0R$, a two-sided non-zero σ-ideal of R. Hence $p_0R = R$ and we conclude that p_0 is invertible in R. Set $p'(x) = p_0^{-1}p(x)$. We claim that $p'(x)$ is central. Clearly $p'(x)x = xp'(x)$. For all $r \in R$, $r^{\sigma^n} = p_0rp_0^{-1}$. Thus

$p'(x)r = p_0^{-1}p(x)r = p_0^{-1}r^{\sigma^n}p(x) = p_0^{-1}p_0rp_0^{-1}p(x) = rp'(x)$.

Clearly $p'(x)$ also has minimal degree n . Suppose $p'(x) = f(x)g(x)$ for some central polynomials $f(x)$ and $g(x)$. Then $f(x)Sg(x) = f(x)g(x)S = p'(x)S = P$ and hence either $f(x) \in P$ or $g(x) \in P$. But since $\deg(f(x)) + \deg(g(x)) = \deg(p'(x))$ either $f(x) \in R \cap C(S)$ or $g(x) \in R \cap C(S)$. Note that $R \cap C(S) = (C(R))^{\sigma}$, the fixed subring of the center of R - a field. Thus either $f(x)$ or $g(x)$ is invertible, and so $p'(x)$ is irreducible. □

We would like to be able to describe more explicitly these central polynomials which arise. With this in mind, we turn our attention to the center $C(S)$ of the Ore extension $S = R[x;\sigma]$ of the σ-simple Artinian ring R . We will give an internal description of $C(S)$ in terms of R and σ .

R being σ-simple Artinian, hence semisimple, its center $C(R)$ is a direct sum of finitely many copies of a field F cyclically permuted by σ . Let $K = (C(R))^{\sigma}$ be the fixed subring of $C(R)$. Then K is a field: indeed it is isomorphic to the fixed subfield of F under the isomorphism induced by σ , and K is diagonally embedded in $C(R)$. As we have already noted, it is clear that $C(S) \cap R = K$.

Definition 2.17: $C(S)$ is said to be <u>trivial</u> if $C(S) = K$. (In other words, the center of the Ore extension S is trivial if it contains no non-constant polynomials.)

<u>Theorem 2.18</u>: Let R be a σ-simple Artinian ring. The following are equivalent for the Ore extension $S = R[x;\sigma]$:

(i) $C(S)$ is non-trivial.

(ii) σ^m is an inner automorphism of R for some $m \geq 1$.

(iii) σ has finite order on $C(R)$.

(iv) $C(S) = K[t]$ for any non-constant polynomial $t = t(x) \in C(S)$ of minimal degree.

(An automorphism α of R is called <u>inner</u> if there exists an invertible element ε of R such that $r^\alpha = \varepsilon r \varepsilon^{-1}$ for all $r \in R$. In this case we denote α by ι_ε . The group of all inner automorphisms of R is denoted by $\mathrm{Inn}(R)$.)

<u>Proof</u>: (iii) $\Rightarrow$ (ii) is a special case of the Skolem-Noether Theorem ([49], page 110). The converse implication (ii) $\Rightarrow$ (iii) is trivial.

We shall prove (ii) $\Rightarrow$ (i) $\Leftrightarrow$ (iv) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (i) Suppose $\sigma^m \in \mathrm{Inn}(R)$ for some $m \geq 1$. Say $\sigma^m = \iota_\varepsilon$, where ε is an invertible element of R . For all $i \in \mathbf{Z}$, $r^{\sigma^i\sigma^m\sigma^{-i}} = (\varepsilon r^{\sigma^i}\varepsilon^{-1})^{\sigma^{-i}} = \varepsilon^{\sigma^{-i}} r (\varepsilon^{\sigma^{-i}})^{-1}$. Thus $\iota_\varepsilon = \sigma^m = \sigma^i\sigma^m\sigma^{-i} = \iota_{\varepsilon^{\sigma^{-i}}}$ for all $i \in \mathbf{Z}$. It follows that $\sigma^{m^2} = (\iota_{\varepsilon^{\sigma^{m-1}}})(\iota_{\varepsilon^{\sigma^{m-2}}})\ldots(\iota_{\varepsilon^\sigma})(\iota_\varepsilon) = \iota_{\varepsilon_0}$ where $\varepsilon_0 = \varepsilon^{\sigma^{m-1}}\varepsilon^{\sigma^{m-2}}\ldots\varepsilon^\sigma\varepsilon$. We observe that $\varepsilon_0^\sigma = \varepsilon^{\sigma^m}\varepsilon^{\sigma^{m-1}}\ldots\varepsilon^\sigma$. Since $\varepsilon^{\sigma^m} = \varepsilon\varepsilon\varepsilon^{-1} = \varepsilon$, we see that $\varepsilon_0^{\sigma^m} = \varepsilon\varepsilon_0\varepsilon^{-1} = \varepsilon\varepsilon^{\sigma^{m-1}}\ldots\varepsilon^\sigma = \varepsilon_0^\sigma$ but, using $\varepsilon^{\sigma^m} = \varepsilon$ again, $\varepsilon_0^{\sigma^m} = \varepsilon_0$. Hence $\varepsilon_0^\sigma = \varepsilon_0$. Now set

$t(x) = \varepsilon_0^{-1}x^{m^2}$ - we claim that $t(x) \in C(S)$. We have $t(x)x = \varepsilon_0^{-1}x^{m^2+1} = x(\varepsilon_0^{\sigma})^{-1}x^{m^2} = x\varepsilon_0^{-1}x^{m^2} = xt(x)$ and for any $r \in R$, $t(x)r = \varepsilon_0^{-1}x^{m^2}r = \varepsilon_0^{-1}r^{\sigma^{m^2}}x^{m^2} = \varepsilon_0^{-1}\varepsilon_0 r\varepsilon_0^{-1}x^{m^2} = r\varepsilon_0^{-1}x^{m^2} = rt(x)$. This establishes our claim to show that $C(S)$ is non-trivial.

(i) $\Rightarrow$ (iv) Assume $C(S)$ to be non-trivial and let $t = t(x)$ be any non-constant central polynomial of minimal degree. Clearly $K[t] \subseteq C(S)$. Let $f(x) \in C(S)$. We intend to show that $f(x) \in K[t]$ by induction on $\deg(f(x))$.

If $\deg(f(x)) = 0$, then $f(x) \in C(S) \cap R = K \subseteq K[t]$. Suppose the result is true for central polynomials of degree less than n and assume $\deg(f(x)) = n$. Since $t(x)$ is central, its leading coefficient is invertible in R and so, by the Division Algorithm, there exist $q(x), r(x) \in S$ such that $f(x) = t(x)q(x) + r(x)$ with $r(x) = 0$ or $\deg(r(x)) < \deg(t(x))$.

We claim that $q(x)$ and $r(x)$ are central polynomials. Since $f(x)$ and $t(x)$ are central, $f(x)a = af(x)$ and $t(x)a = at(x)$ for all $a \in R$. Thus $t(x)q(x)a + r(x)a = f(x)a = af(x) = at(x)q(x) + ar(x) = t(x)aq(x) + ar(x)$, and so $t(x)[q(x)a - aq(x)] = ar(x) - r(x)a$. If $r(x) = 0$, then $t(x)[q(x)a - aq(x)] = 0$ implies $q(x)a = aq(x)$ (since $t(x)$ is central and S is a prime ring). Otherwise, $\deg(r(x)) < \deg(t(x))$. Suppose $q(x)a - aq(x) \neq 0$. Then $\deg(r(x)) \geq \deg(ar(x) - r(x)a) = \deg\{t(x)[q(x)a - aq(x)]\} \geq \deg(t(x)) > \deg(r(x)$, a contradiction. Therefore $q(x)a = aq(x)$ and $r(x)a = ar(x)$ for all $a \in R$.

Similarly, $f(x)x = xf(x)$ and $t(x)x = xt(x)$ imply $t(x)[q(x)x - xq(x)] = xr(x) - r(x)x$. If $r(x) = 0$, then $q(x)x = xq(x)$. Otherwise, suppose $q(x)x \neq xq(x)$. Then $\deg[q(x)x - xq(x)] = \deg[(q(x) - q^{\sigma}(x))x] \geq 1$ and so $\deg(xr(x) - r(x)x) = \deg[(r(x) - r^{\sigma}(x))x] \leq \deg(r(x)) + 1 < \deg(t(x)) + 1 \leq \deg\{t(x)[q(x)x - xq(x)]\} = \deg(xr(x) - r(x)x)$. We again get a contradiction and conclude that $q(x)x = xq(x)$ and $r(x)x = xr(x)$. Consequently, $q(x)$ and $r(x)$ are central, which proves the claim.

Now, either $r(x) = 0 \in K$ or, by the minimality of $t(x)$, $r(x) \in C(S) \cap R = K$. In any event, either $q(x) = 0$ in which case $f(x) = r(x) \in K$, or $\deg(q(x)) < \deg(f(x))$ in which case $q(x) \in K[t]$ by the induction hypothesis. It follows that $f(x) \in K[t]$.

(iv) $\Rightarrow$ (i) This is evident.

(i) $\Rightarrow$ (ii) Let $t(x) = t_0 + t_1x + \ldots + t_mx^m \in C(S)$, where $m \geq 1$ and $t_m \neq 0$. Since $t(x)x = xt(x)$, $t^{\sigma}(x) = t(x)$ and so $t_m^{\sigma} = t_m$. For all $r \in R$, $t(x)r = rt(x)$ and so $t_mr^{\sigma^m} = rt_m$ for all $r \in R$. It follows that $t_mR = Rt_m$ is a non-zero, two-sided σ-ideal of R, hence equal to R since R is σ-simple and so t_m is invertible in R. Thus $r^{\sigma^m} = t_m^{-1}rt_m$ for all $r \in R$. In other words $\sigma^m \in \mathrm{Inn}(R)$. $\square$

Our problem now is to find a non-constant central polynomial of minimal degree. The theorem tells us that if $\sigma^m \in \mathrm{Inn}(R)$ then there exists a central polynomial of degree m^2 - but this may not be the minimal degree of polynomials in $C(S)$. We must tighten our grasp on the inner automorphism σ^m and by so doing we shall establish that the least m for which $\sigma^m \in \mathrm{Inn}(R)$ is precisely the minimal degree of non-constant central polynomials.

Proposition 2.19: Let R be σ-simple Artinian and suppose that $\sigma^m \in \mathrm{Inn}(R)$ where $m \geq 1$ and no smaller power of σ is inner. There exists an invertible element ε_0 of R such that $\varepsilon_0^\sigma = \varepsilon_0$ and $\sigma^m = \iota_{\varepsilon_0}$.

Proof: For all $r \in R$, $r^{\sigma^m} = \varepsilon r \varepsilon^{-1}$. Hence $\varepsilon^\sigma r (\varepsilon^\sigma)^{-1} = (\varepsilon r^{\sigma^{-1}} \varepsilon^{-1})^\sigma = (r^{\sigma^{-1}})^{\sigma^m \sigma} = r^{\sigma^m} = \varepsilon r \varepsilon^{-1}$ and as a result $\varepsilon^{-1}\varepsilon^\sigma r = r \varepsilon^{-1} \varepsilon^\sigma$ for all $r \in R$. Set $a = \varepsilon^{-1}\varepsilon^\sigma$. Then a is an invertible element of $C(R)$ and the norm of a, $N(a) = aa^\sigma \ldots a^{\sigma^{m-1}} = 1$. (Note that the order of σ, as an automorphism of $C(R)$, is m, by Theorem 2.18.) The following lemma will show that there exists an invertible element $b \in C(R)$ such that $a = b(b^\sigma)^{-1}$. Replacing ε by $b\varepsilon = \varepsilon b$, we obtain $(b\varepsilon) r (b\varepsilon)^{-1} = b \varepsilon r \varepsilon^{-1} b^{-1} = \varepsilon r \varepsilon^{-1} = r^{\sigma^m}$ for all $r \in R$. Also $(\varepsilon b)^\sigma = \varepsilon^\sigma b^\sigma = \varepsilon a \cdot a^{-1} b = \varepsilon b$. Therefore $\varepsilon_0 = \varepsilon b$ is the required element.

The lemma we require is a generalization of Hilbert's Theorem 90 (cf. [22], Theorem 4.28).

Lemma 2.20: Let R be a commutative ring with an automorphism σ of order m. If R is σ-simple and a is an invertible element of R of norm 1, then there exists an invertible element $b \in R$ such that $a = b(b^\sigma)^{-1}$.

Proof: Define a sequence of invertible elements of R as follows: $a_0 = 1$, $a_1 = a$, $a_{i+1} = aa^{\sigma}\ldots a^{\sigma^i}$ for $0 \le i \le m-2$ (since $a_m = N(a) = 1 = a_0$). For all i, $a_{i+1} = aa_i^{\sigma}$. For $r \in R$, define $\phi(r) = \sum_{i=0}^{m-1} a_i r^{\sigma^i}$. We claim that there exists $r \in R$ such that $\phi(r) \neq 0$.

Suppose, by way of contradiction, that $\phi(r) = 0$ for all $r \in R$. Then $\{1, \sigma, \ldots, \sigma^{m-1}\}$ is a set of "linearly dependent" automorphisms over R. Let

$$(*) \qquad \sum_{j=0}^{\ell} b_{i_j} r^{\sigma^{i_j}} = 0 \qquad \begin{array}{l}(0 \le \ell \le m-1,\ 0 \le i_j \le m-1 \text{ for all } j, \\ i_j \neq i_k \text{ for } j \neq k,\ b_{i_j} \in R)\end{array}$$

be a non-trivial dependence relation of minimal length ℓ. Then $b_{i_j} \neq 0$ for $0 \le j \le \ell$. Since σ has order m, $\sigma^{i_0} \neq \sigma^{i_1}$ and so there exists $s \in R$ such that $s^{\sigma^{i_0}} \neq s^{\sigma^{i_1}}$. Replacing r by rs in (*), we obtain:

$$b_{i_0} r^{\sigma^{i_0}} s^{\sigma^{i_0}} + b_{i_1} r^{\sigma^{i_1}} s^{\sigma^{i_1}} + \ldots + b_{i_\ell} r^{\sigma^{i_\ell}} s^{\sigma^{i_\ell}} = 0 \quad \text{for all } r \in R.$$

Multiplying (*) by $s^{\sigma^{i_0}}$, we obtain:

$$b_{i_0} r^{\sigma^{i_0}} s^{\sigma^{i_0}} + b_{i_1} r^{\sigma^{i_1}} s^{\sigma^{i_0}} + \ldots + b_{i_\ell} r^{\sigma^{i_\ell}} s^{\sigma^{i_0}} = 0 \quad \text{for all } r \in R.$$

Subtracting these last two relations gives:

$$b_{i_1}(s^{\sigma^{i_1}} - s^{\sigma^{i_0}}) r^{\sigma^{i_1}} + \ldots + b_{i_\ell}(s^{\sigma^{i_\ell}} - s^{\sigma^{i_0}}) r^{\sigma^{i_\ell}} = 0 \quad \text{for all } r \in R.$$

On the other hand, if we apply σ^k to (*) we obtain $\sum_{j=0}^{\ell} b_{i_j}^{\sigma^k} r^{\sigma^{i_j+k}} = 0$ for all $r \in R$, $k \in \mathbf{Z}$. This is equivalent to:

$$(**) \qquad \sum_{j=0}^{\ell} b_{i_j}^{\sigma^k} r^{\sigma^{i_j}} = 0 \text{ for all } r \in R, \ k \in \mathbf{Z}.$$

If we repeat the above elimination procedure on (**), we arrive at:

$$b_{i_1}^{\sigma^k}(s^{\sigma^{i_1}} - s^{\sigma^{i_0}})r^{\sigma^{i_1}} + \ldots + b_{i_\ell}^{\sigma^k}(s^{\sigma^{i_\ell}} - s^{\sigma^{i_0}})r^{\sigma^{i_\ell}} = 0 \text{ for all } r \in R, k \in \mathbf{Z}.$$

Now $b_{i_1}^{\sigma^k}(s^{\sigma^{i_1}} - s^{\sigma^{i_0}}) \neq 0$ for some $k \in \mathbf{Z}$. (Otherwise, since R is in particular a commutative σ-prime ring, either $b_{i_1} = 0$ or $s^{\sigma^{i_0}} = s^{\sigma^{i_1}}$ – a contradiction.) We have therefore produced a dependence relation of length less than ℓ. This is a contradiction and so we have proved our claim.

Set $b = \phi(r) \neq 0$ for the particular $r \in R$ just found. Then

$$b^\sigma = (\sum_{i=0}^{m-1} a_i r^{\sigma^i})^\sigma = \sum_{i=0}^{m-1} a_i^\sigma r^{\sigma^{i+1}} = \sum_{i=0}^{m-1} a^{-1} a_{i+1} r^{\sigma^{i+1}} = a^{-1} \sum_{i=0}^{m-1} a_i r^{\sigma^i} = a^{-1} b.$$

Therefore $(bR)^\sigma = bR \neq 0$ and so $bR = R$ since R is σ-simple. It follows that b is invertible in R. Moreover $(b^\sigma)^{-1} = (b^{-1})^\sigma \in R$ and thus $a = b(b^\sigma)^{-1}$. □

To complete the proof of Proposition 2.19, we need only observe that the center of a σ-simple Artinian ring is again σ-simple by Lemma

2.12. As noted, the hypotheses imply that the order of σ on $C(R)$ is m. The lemma now applies and the proof of the proposition is complete. □

We can now extend Theorem 2.18 to give a more precise description of the center of the Ore extension S.

<u>Theorem 2.21</u>: Let R be a σ-simple Artinian ring, $S = R[x;\sigma]$, and set $K = (C(R))^{\sigma}$. If $C(S)$ is non-trivial, then some power of σ is inner and there exists an integer $m \geq 1$ and an invertible element $\varepsilon \in R$ such that $\varepsilon^{\sigma} = \varepsilon$, $\sigma^m = \iota_{\varepsilon}$, and no smaller power of σ is inner. In this case, $C(S) = K[\varepsilon^{-1}x^m]$.

<u>Proof</u>: By Theorem 2.18, $C(S)$ is non-trivial if and only if some power of σ is inner. Let $m \geq 1$ be the least integer such that $\sigma^m \in \mathrm{Inn}(R)$. Proposition 2.19 them furnishes the required invertible element ε. To prove the last statement, we must show that $\varepsilon^{-1}x^m$ is a central polynomial of minimal degree. It is easily checked that $\varepsilon^{-1}x^m$ is central. Let $f(x) = f_0 + f_1x + \ldots + f_nx^n \in C(S)$, $f_n \neq 0$. As in the proof of Theorem 2.18 (i) $\Rightarrow$ (ii), we see that f_n is an invertible element of R and $r^{\sigma^n} = f_n^{-1}rf_n$ for all $r \in R$. By the minimality of m, this implies that $m \leq n$, and hence m is also the minimal degree of non-constant central polynomials, witness $\varepsilon^{-1}x^m$. The result follows from Theorem 2.18(iv). □

§3. <u>Further Ideal Theory</u>.

In this section, we continue our investigation of prime ideals in Ore extensions of σ-simple Artinian rings. We give some additional information about the arithmetic of prime ideals and extend some of the results of the previous section. Our main result is a structure theorem for the ideals of these Ore extensions. To conclude, we return to our original setting (R σ-prime right Noetherian) and give the analogue of Theorem 2.1 that was announced in the introduction to this chapter.

Lemma 2.22: Let R be a σ-simple Artinian ring and let $f(x) = f_0 + f_1x + \ldots + x^n \in S = R[x;\sigma]$. Then $f(x)S$ is a two-sided σ-ideal of S if and only if $f(x)x = xf(x)$ and, for all $r \in R$, $f(x)r = r^{\sigma^n} f(x)$.

Proof: The sufficiency of the conditions is clear. To see that they are also necessary, suppose that $f(x)S$ is a σ-ideal. Then $(f(x)S)^{\sigma} = f(x)S$ and so $f^{\sigma}(x) = f(x)g(x)$ for some $g(x) \in S$. Since both $f(x)$ and $f^{\sigma}(x)$ are monic of degree n , it follows that $g(x) = 1$ and $f^{\sigma}(x) = f(x)$ or, equivalently, $f(x)x = xf(x)$. The remaining condition is shown to hold exactly as in the last part of the proof of Proposition 2.14. □

We may therefore generalize Proposition 2.14 as:

Proposition 2.23: Let R be σ-simple Artinian. Then every non-zero σ-ideal I of S is principal, generated by the unique monic polynomial $f(x) \in I$ of minimal degree. □

Remark 2.24: (i) If R is actually simple, then the above proposition holds for all ideals of S . This is essentially what

Jacobson showed in the case where R is a division ring.

(ii) There is a (usually simpler) version of most of the results of this chapter for the <u>skew Laurent polynomial ring</u> $R[x, x^{-1}; \sigma]$ consisting of polynomials of the form $f(x) = f_m x^m + \ldots + f_n x^n$, $m,n \in Z$, $m \le n$. Every ideal I of $R[x, x^{-1}; \sigma]$ is a σ-ideal since $I^\sigma = I^\sigma_x x^{-1} = xIx^{-1} \subseteq I$. Define the <u>length</u> of $f(x) \neq 0$ to be $\ell(f(x)) = n - m$ where f_n and f_m are non-zero above; then Proposition 2.23 holds for all <u>ideals</u> of $R[x, x^{-1}; \sigma]$, with "minimal degree" replaced by "minimal length".

Recall that an ideal I of a ring R is said to be <u>invertible</u> if there is an overring T of R such that if $A = \{t \in T | tI \subseteq R\}$ and $B = \{t \in T | It \subseteq R\}$ then $AI = IB = R$. In this situation, we have $A = B$ and we write $I^{-1} = A = B$.

<u>Corollary 2.25</u>: In the situation of Proposition 2.23, every non-zero σ-ideal of S is invertible.

<u>Proof</u>: Let $I = f(x)S = Sf(x)$ be a non-zero σ-ideal. We know that S is a prime ring (Proposition 1.11(i)) and it is obvious that $f(x)$, a normal element in a prime ring, is regular. It is well-known that the principal ideal generated by such an element is invertible. (cf. [9] page 51.) □

<u>Corollary 2.26</u>: In the situation of Proposition 2.23, if I is a σ-ideal of S then $\bigcap_{n \ge 0} I^n = 0$.

Proof: If $I \neq 0$ then $I = f(x)S = Sf(x)$, $f(x)$ monic, as in Proposition 2.23. Then for all $n \geq 0$, $I^n = f^n(x)S$ and $\deg(f^n(x)) = n\cdot\deg(f(x))$. If $0 \neq g(x) \in \bigcap_{n\geq 0} I^n$, then for all $n \geq 0$ there exists $h_n(x) \in S$ such that $g(x) = f^n(x)h_n(x)$. Hence $\deg(g(x)) \geq n\cdot\deg(f(x))$ for all $n \geq 0$. This is impossible and so we conclude $g(x) = 0$. □

Corollary 2.27: Let R be σ-simple Artinian and let I and B be σ-ideals of S with $I \subseteq B$. There exists a σ-ideal A of S with $I = AB$.

Proof: Set $A = IB^{-1}$. (Or, directly, write $I = f(x)S$, $B = b(x)S$, and from $f(x) = a(x)b(x)$ for some $a(x) \in S$, deduce that $a(x)$ is monic and satisfies the conditions of Lemma 2.22. Conclude that $A = a(x)S$ is the desired σ-ideal.) □

Lemma 2.28: Let R be σ-simple Artinian, I a maximal σ-ideal of S (i.e. maximal among σ-ideals of S) with $x \notin I$. Then I is a maximal ideal.

Proof: It is easy to see that I is σ-prime. For if A and B are σ-ideals of S such that $AB \subseteq I$, then $(A + I)(B + I) \subseteq I$. Suppose $A \nsubseteq I$; then $I \subsetneq A + I$, a σ-ideal. Hence $A + I = S$ and so $B \subseteq B + I = S(B + I) \subseteq I$ as required. Now, by Proposition 1.5, there exists a prime ideal P of S, minimal over I, such that $I = \bigcap_{i=1}^{n} P^{\sigma^i}$

for some integer $n \geq 1$. Since $x \notin I$, it follows that $x \notin P$. Hence $P^{\sigma} = P$ and $I = P$ is prime.

Suppose M is a maximal ideal of S with $I \subseteq M$. Then $I \subseteq \bigcap_{i \in \mathbf{Z}} M^{\sigma^i}$, a σ-ideal. By the maximality of I, $I = \bigcap_{i \in \mathbf{Z}} M^{\sigma^i}$. If $x \in M$, then $x \in M^{\sigma^i}$ for all $i \in \mathbf{Z}$ and so $x \in \bigcap_{i \in \mathbf{Z}} M^{\sigma^i} = I$, a contradiction. Therefore $x \notin M$ and $M^{\sigma} = M$ by Proposition 1.13. Hence $I = \bigcap_{i \in \mathbf{Z}} M^{\sigma^i} = M$ and I is maximal. □

<u>Proposition 2.29</u>: Let R be σ-simple Artinian, I a non-zero σ-ideal of S. Then I is a prime ideal if and only if I is a maximal ideal.

<u>Proof</u>: If I is maximal, it is prime. Conversely let I be a prime σ-ideal of S and assume $I \subsetneqq B$ for some ideal B of S. Since $I^{\sigma} = I$, $I \cap R$ is a σ-ideal of R and hence $I \cap R = 0$. Lemma 2.8 may now be applied to show that $B \cap \mathcal{C}_R(0) \neq \phi$. Since R is Artinian, this means B contains an invertible element of R and so $B = S$. Therefore I is maximal. □

Let us now generalize Corollary 2.16 to arbitrary σ-ideals.

<u>Proposition 2.30</u>: Let R be σ-simple Artinian. If I is a non-zero σ-ideal of S then $I = x^m g(x)S$ for some integer $m \geq 0$ and $g(x) \in C(S)$.

Proof: Let $I = f(x)S = Sf(x)$ where $f(x)$ is the unique monic polynomial of minimal degree in I. Write $f(x) = x^m f'(x)$ where $f'(x) \notin xS$. Observe that $f'(x)$ is also monic, $f'(x)S = Sf'(x)$, and $[f'(x)]^\sigma = f'(x)$ since $f^\sigma(x) = f(x)$. If $f'(x) = f_0 + f_1 x + \ldots + f_n x^n$, then $f_0 \neq 0$ because $f'(x) \notin xS$. The proof of Corollary 2.16 may now be used mutatis mutandis to show that f_0 is invertible in R and $g(x) = f_0^{-1} f'(x) \in C(S)$. Hence $I = x^m g(x)S$ as desired. □

Corollary 2.31: Let R be σ-simple Artinian, P a non-zero σ-invariant prime ideal of S. Then either $P = xS$ or $P = p(x)S$ for some irreducible polynomial $p(x) \in C(S)$. □

Corollary 2.32: Let R be as above and let I be a non-zero σ-ideal of S. If $\sigma^i \notin \mathrm{Inn}(R)$ for all $i \geq 1$, then $I = x^m S$ for some $m \geq 1$.

Proof: The center of S is trivial in this case (Theorem 2.18). □

We next give some structure theorems for ideals in the Ore extension S. First we consider the σ-ideals of S.

Lemma 2.33: Let R be σ-simple Artinian, P and Q distinct σ-invariant prime ideals of S. Then $PQ = QP = P \cap Q$.

Proof: $PQ = QP$ follows from the fact that P and Q are either centrally generated or equal to xS. But since P and Q are maximal,

$P \cap Q = (P \cap Q)S = (P \cap Q)(P + Q) \subseteq QP + PQ \subseteq P \cap Q$ and hence $P \cap Q = PQ + QP = PQ$. $\square$

Theorem 2.34: Let R be σ-simple Artinian. Every non-zero σ-ideal of S can be written uniquely (up to order) as a finite product of pairwise commuting σ-invariant maximal ideals.

Proof: Let I be a non-zero σ-ideal of S. If I is a maximal ideal, we are done. Otherwise there is a prime ideal $M_1 \supseteq I$ which is maximal among those ideals M such that $M \supsetneq I$ and $M \cap \mathcal{C}_R(0) = \phi$. Clearly $M_1^\sigma = M_1$ and therefore $M_1 \cap R = 0$. By Proposition 2.29, M_1 is actually a maximal ideal of S. Using Corollary 2.27, there exists a σ-ideal I_1 of S such that $I = I_1 M_1$. An inductive use of the above procedure yields an ascending chain $I = I_0 \subseteq I_1 \subseteq \cdots$ of σ-ideals, and maximal ideals M_j such that $I_{j-1} = I_j M_j$ for all $j \geq 1$ such that I_j is not maximal. The ascending chain must become stationary at $I_{n-1} = I_n$, say, and consequently $I_n = I_{n-1} = I_n M_n$. The invertibility of I_n implies $M_n = S$, a contradiction unless I_{n-1} is already a maximal ideal. Setting $M_n = I_{n-1}$, we have $I = M_n M_{n-1} \cdots M_1$ where the M_i are maximal by construction and pairwise commuting by Lemma 2.33.

The proof of the uniqueness of this decomposition is routine - we will only briefly outline the argument. Suppose $I = \prod_{i=1}^{n} M_i = \prod_{j=1}^{s} N_j$ are two such decompositions. For all j' $(1 \leq j' \leq s)$ there exists i' $(1 \leq i' \leq n)$ such that $M_{i'} \subseteq N_{j'}$ and hence $M_{i'} = N_{j'}$ by maximality. Using the commutativity of the products and the invertibility of σ-ideals, we can "cancel" this common factor to obtain $\prod_{i\neq i'} M_i = \prod_{j\neq j'} N_j$. Repeating this procedure forces us to conclude that $n = s$ and $\{M_i | 1 \le i \le n\} = \{N_j | 1 \le j \le n\}$. □

We now obtain our decomposition theorem for ideals. There are two versions.

<u>Theorem 2.35</u>: Let R be σ-simple Artinian. Every non-zero ideal I of S can be written uniquely (up to order) in the form $I = A \cdot \prod_{i=1}^{n} M_i$, where the M_i are pairwise commuting, σ-invariant, maximal ideals of S and A is an ideal of S which is contained in no σ-ideal of S.

<u>Proof</u>: Let I be a non-zero ideal of S. If $I^\sigma = I$, the preceding theorem gives the desired result. If $I^\sigma \neq I$, set $B = \sum_{i\in Z} I^{\sigma^i}$ the smallest σ-ideal containing I. If $B \neq S$, then B is invertible, $IB^{-1} \subseteq BB^{-1} = S$, and IB^{-1} is an ideal of S. Now IB^{-1} is contained in no proper σ-ideal of S: for if M is a σ-ideal of S with $IB^{-1} \subsetneqq M \subsetneqq S$, then $I \subsetneqq MB \subsetneqq B$ with MB a σ-ideal - this is a contradiction. Setting $A = IB^{-1}$ gives $I = AB$ and the preceding theorem gives $B = \prod_{i=1}^{n} M_i$, with the M_i pairwise commuting, σ-invariant maximal ideals.

If $B = S$, then I itself is contained in no proper σ-ideal and the result follows trivially. □

Our alternate version of this result is patterned after [7] Theorem 5.7.

Theorem 2.36: Let R be σ-simple Artinian with n simple Artinian factors, cyclically permuted by σ. Every non-zero σ-ideal I of S can be written uniquely (up to order) in the form $I = B\prod_{i=1}^{m} M_i$, where the M_i are pairwise commuting maximal ideals of S, $x \notin M_i$ for $1 \le i \le m$, and B is an ideal of S such that $x^{\ell+n-1}S \subseteq B \subseteq x^{\ell}S$ for some $\ell \ge 0$.

Proof: Let $N_1, \ldots, N_n$ be the n maximal ideals of R, arranged so that $N_i^{\sigma} = N_{i+1}$, $N_n^{\sigma} = N_1$. These are all the prime ideals of R. Let $M_1, \ldots, M_m$ be those maximal ideals in Theorem 2.35 which do not contain x; the rest of the M_i's must therefore all equal xS – let h be their multiplicity. Now, with A as in the preceding theorem, $x \in A$; otherwise, A would be maximal by Lemma 2.28, and therefore $A^{\sigma} = A$, a contradiction. Now any maximal ideal containing A must contain x and hence corresponds to one of the n maximal ideals $N_1, \ldots, N_n$ of R. Let $C = \bigcap_{i=1}^{t} P_i$ be the intersection of all the maximal ideals containing A. If $t = n$, then $A \subseteq C = \bigcap_{i=1}^{n} (N_i + xS) = (\bigcap_{i=1}^{n} N_i) + xS = xS$, a σ-ideal. This contradicts the construction of A. Therefore $t \le n - 1$. For some $k \ge 1$, $C^k \subseteq A \subseteq C$, and thus $x^{t+k} \in (\prod_{i=1}^{t} P_i)^k \subseteq C^k \subseteq A$. Therefore $x^{n-1+k}S \subseteq A$ and if we set $\ell = k + h$, $B = Ax^{\ell}S$, we obtain $x^{\ell+n-1}S \subseteq B \subseteq x^{\ell}S$ as required. □

We conclude this chapter by giving the analogue of Theorem 2.1: a complete characterization of the prime ideals of the Ore extension $S = R[x;\sigma]$ of the right Noetherian ring R in terms of certain irreducible polynomials. Recall that there is no loss of generality in assuming R to be σ-prime and that the prime ideals of S we are considering contract to 0 in R. We have already proved most of the theorem – the only missing ingredient is provided by the following lemma.

Lemma 2.37: Let R be a σ-simple Artinian ring, $S = R[x;\sigma]$. Assume that $\sigma^m = \iota_\varepsilon \in \mathrm{Inn}(R)$ where $\varepsilon^\sigma = \varepsilon$ and no smaller power of σ is inner. Let $p(x)$ be a centrally-irreducible central polynomial which is not an associate of $\varepsilon^{-1}x^m$. Then $p(x)S$ is a prime ideal of S which does not contain x. (Two elements p, q of a commutative ring are called associates if there is an invertible element u of the ring such that $p = qu$.)

Proof: The center of S is $C(S) = K[\varepsilon^{-1}x^m]$ where $K = [C(R)]^\sigma$. Since $p^\sigma(x) = p(x)$, $P = p(x)S = Sp(x)$ is a σ-ideal. Suppose $I = x^i f(x)S$ is a σ-ideal containing P ($i \geq 0$, $f(x) \in C(S)$). Then $p(x) = x^i f(x)g(x)$ for some $g(x) \in S$. Since $p(x)$ and $f(x)$ are central, so is $x^i g(x)$ and thus the irreducibility of $p(x)$ implies that $p(x)$ is an associate of either $f(x)$ or $x^i g(x)$. In the first case, $uf(x) = p(x) = x^i g(x)f(x)$ implies $x^i g(x) = u \in K$ and so $i = 0$ and $g(x) = u \in K$. Then $I = f(x)S = p(x)u^{-1}S = p(x)S = P$. In the second case, $vx^i g(x) = p(x) = f(x)x^i g(x)$ implies $f(x) = v \in K$ and so $p(x) = vx^i g(x) \in C(S)$. It follows that $i = km$ for some $k \geq 0$ and hence $P \subseteq I = x^{km}S = (\varepsilon^{-1}x^m)^k S$. Writing $p(x) = p_0 + p_1(\varepsilon^{-1}x^m) + \ldots + p_n(\varepsilon^{-1}x^m)^n$, $p_i \in K$, we see that $p_0 \neq 0$ (for otherwise, either $p(x)$ is not irreducible or it is an associate of $\varepsilon^{-1}x^m$, contradictions in both cases). Moreover, if $i \neq 0$, then $p_0 = x^{km}vg(x) - p_1(\varepsilon^{-1}x^m) - \ldots - p_n(\varepsilon^{-1}x^m)^n \in x^m S$ which is impossible. Therefore $i = 0$ and $I = S$. The foregoing shows that P is a maximal σ-ideal.

Suppose $x \in P$. Then $\varepsilon^{-1}x^m \in P$ and thus $\varepsilon^{-1}x^m = p(x)r(x)$ for some $r(x) \in S$. Since $\varepsilon^{-1}x^m$ and $p(x)$ are central, we must have $r(x) \in C(S)$ also. But now $\deg(p(x)) \leq m$, forcing equality since $p(x)$ is a non-constant polynomial in $\varepsilon^{-1}x^m$. Hence $r(x) = r_0 \in K$ and $p(x)$ is an associate of $\varepsilon^{-1}x^m$. This contradiction proves that $x \notin P$. We now invoke Lemma 2.28 to get that P is a maximal (or, equivalently, a prime) ideal of S. □

<u>Theorem 2.38</u>: Let R be a right Noetherian σ-prime ring, $\mathcal{Q}(R)$ its σ-simple Artinian classical right quotient ring, and $S = R[x;\sigma]$. The following are equivalent:

(i) There exist non-zero prime ideals of S, not containing x, contracting to 0 in R;

(ii) σ^m is an inner automorphism on $\mathcal{Q}(R)$ for some $m \geq 1$.

In these equivalent cases, there is an invertible element $\varepsilon \in \mathcal{Q}(R)$ and a bijective correspondence between prime ideals as in (i) and non-constant irreducible polynomials of $K[\varepsilon^{-1}x^m]$ which are not associates of $\varepsilon^{-1}x^m$. (Here $K = [C(\mathcal{Q}(R))]^{\sigma}$.)

Proof: The equivalence of (i) and (ii) is given by Proposition 2.11, Corollary 2.16, and Theorem 2.18. For $S_C = Q(R)[x;\sigma]$, Theorem 2.21 shows that $C(S_C) = K[\varepsilon^{-1}x^m]$. Let P denote the set of non-zero prime ideals of S, not containing x, and contracting to 0 in R; let I denote the set of non-constant centrally-irreducible central polynomials in S_C which are not associates of $\varepsilon^{-1}x^m$. If $P \in P$, then consider the mapping $P \mapsto p(x)$, where $PS_C = p(x)S_C$ for $p(x)$ irreducible in $C(S_C)$ (Proposition 2.11, Corollary 2.16.) Clearly $p(x)$ is not an associate of $\varepsilon^{-1}x^m$ and so we have a mapping $\phi: P \to I$. It is also evident that ϕ is one-to-one. Define $\psi: I \to P$ by $p(x) \to p(x)S_C \cap S$, $p(x) \in I$, using Lemma 2.37 and Proposition 2.11. It is easy to check that $\psi\phi = id_P$ and $\phi\psi = id_I$ and so the correspondence given by ϕ is bijective. □

In general, we have the following characterization:

Theorem 2.39: Let R be a right Noetherian ring, I a σ-prime ideal of R, $S = R[x;\sigma]$. There is a bijective correspondence between prime ideals of S which are upper to I and non-constant irreducible polynomials of $K[\varepsilon^{-1}x^m]$ which are not associates of $\varepsilon^{-1}x^m$. (Here $K = [C(Q(R/I))]^{\bar{\sigma}}$, m is the order of $\bar{\sigma}$ on $C(Q(R/I))$, and ε is a suitably-chosen invertible element of $Q(R/I)$.)

Proof: Apply Theorem 2.38 to $\bar{R} = R/I$ and $\bar{S} = S/IS \cong \bar{R}[x;\bar{\sigma}]$. □

Henceforth, we will use the notation $[I,p]$ for the upper prime ideal of S determined by the σ-prime ideal I of R and the polynomial $p = p(\varepsilon^{-1}x^m)$ of $K[\varepsilon^{-1}x^m]$.

Let us summarize our classification of prime ideals in right Noetherian Ore extensions:

Theorem 2.40: Let R be a right Noetherian ring, $S = R[x;\sigma]$. The prime ideals of S are of three types:

(i) $P + xS$, where P is a prime ideal of R,

(ii) IS, where I is a σ-prime ideal of R, and

(iii) $[I,p]$ where I is a σ-prime ideal of R. $\square$

CHAPTER III

LOCALIZATION IN NOETHERIAN RINGS

The technique of localization at a prime (or semiprime) ideal, well-known and always possible in the case of commutative rings, has been studied in some detail for non-commutative Noetherian rings (cf. [25], [26], [31], [32], [33]). Based on work of Jategaonkar, Müller developed the fundamental concept of a link between prime ideals, showing how the presence of links is an obstruction to the localizability of a prime ideal. The related notions of a classical set of prime ideals and a minimal such set - a clan - can then be explained in terms of links ([39], [40]).

Recently, Jategaonkar [26] has given a fairly extensive survey of the theory of localization. In this chapter we summarize and reformulate parts of this material. After giving the necessary definitions, we present Jategaonkar's criteria for localizability and classicality in a way which makes the role of links more transparent. In the third section, we set the stage for our study of localization in Ore extensions by considering the special cases of the theory of localization in that class of Noetherian rings to which our Ore extensions will belong - Krull symmetric Noetherian rings. To this end, we discuss briefly the use of (Gabriel-Rentschler) Krull dimension.

We also discuss some of the various definitions of "link" which appear in the literature and show the connections between them in our setting. In the final section, we include a few useful results on normal elements and links which we will need in the next chapter.

§1. Definitions.

If S is a semiprime ideal of the right Noetherian ring R, we say that S is right localizable if $\mathcal{C}_R(S)$ satisfies the right Ore condition. The S-torsion submodule $\tau_S(R) = \{r \in R \mid rc = 0$ for some $c \in \mathcal{C}(S)\}$ is actually a two-sided ideal of R and we may form the right localization of R at S, denoted by R_S, whose elements are of the form $\bar{r}(\bar{c})^{-1}$ where $r \in R$, $c \in \mathcal{C}(S)$, and $(\bar{\ })$ denotes coset modulo $\tau_S(R)$. If I is a right ideal of R, $\bar{I}R_S$ can be identified with $\{\bar{a}(\bar{c})^{-1} \mid a \in I,\ c \in \mathcal{C}(S)\}$ and is a right ideal of R_S. If I is a two-sided ideal of R, $\bar{I}R_S$ is a two-sided ideal of R_S. The Jacobson radical of R_S is $\bar{S}R_S$ and R_S is semilocal (i.e. $R_S/J(R_S)$ is semisimple Artinian).

For a right ideal I of R, the S-closure of I is $\gamma_S(I) = \{r \in R \mid rc \in I$ for some $c \in \mathcal{C}(S)\}$; I is S-closed if $\gamma_S(I) = I$. There is a bijective correspondence between the right ideals of R_S and the S-closed right ideals of R. If $S = \bigcap_{i=1}^{n} P_i$, where $P_1, \ldots, P_n$ are the minimal prime ideals over S, then there is a bijective correspondence between the prime ideals of R_S and the prime ideals of R contained in $\bigcup_{i=1}^{n} P_i$. (In fact, if a prime ideal is contained in $\bigcup_{i=1}^{n} P_i$, it must be contained in some P_j, $1 \le j \le n$.)

An ideal I of R is said to have the right AR-property if for any right ideal E of R there is a natural number n such that $E \cap I^n \subseteq EI$. If S is a right localizable semiprime ideal of R, then S is said to be right classical if $J(R_S)$ has the right AR-property in R_S. A finite set of mutually incomparable prime ideals $\{P_1, \ldots, P_n\}$

is said to be <u>right classical</u> if $\bigcap_{i=1}^{n} P_i$ is a right classical semiprime ideal. If R is Noetherian, $\{P_1, \ldots, P_n\}$ is called a <u>clan</u> if it is a minimal classical set (i.e. it is left and right classical and no proper subset is such.) (cf. [39], [40].)

For a uniform right R-module U over a right Noetherian ring R, recall that $\mathrm{ass}(U)$ is defined to be the (unique) maximal member of the set of annihilators in R of non-zero submodules of U – in other words $P = \mathrm{ass}(U)$ if $P = r_R(V)$ for all non-zero $V_R \subseteq U$. P is necessarily a prime ideal. Recall also that over a right Noetherian ring, every injective right R-module is a direct sum of indecomposable injectives and for any prime ideal P of R, the R-module injective hull of R/P satisfies $E(R/P) \cong E_P^n$ where E_P is an indecomposable injective right R-module with $\mathrm{ass}(E_P) = P$ and $n = \mathrm{Gdim}(R/P)$. (See Appendix A for the definition of Goldie dimension.)

A uniform module U_R is said to be <u>tame</u> (or <u>P-tame</u>) if $E(U) \cong E_P$ for some prime ideal P of R; otherwise U is said to be <u>wild</u>. U is P-tame if and only if $\mathrm{ann}_U(P)$ is R/P-torsion free. A module M_R is tame if all of its non-zero uniform submodules are tame. If U is a uniform module with $P = \mathrm{ass}(U)$, the <u>first layer</u> of U is defined to be $\mathrm{ann}_U(P)$; the <u>second layer</u> of U consists of the isomorphism types of uniform submodules of $U/\mathrm{ann}_U(P)$. U is said to satisfy the <u>second layer condition</u> if its second layer is tame. A prime ideal P is said to satisfy the <u>right second layer condition</u> if E_P satisfies the second layer condition as a right R-module. A set $\mathcal{P}$ of prime ideals satisfies the second layer condition if every $P \in \mathcal{P}$ satisfies the right second layer condition. A ring R satisfies the right second layer condition if all of its prime ideals do. (cf. [26].)

If P and Q are prime ideals of R, we say there is a <u>bimodule link</u> $Q \sim P$ if there are ideals A and B of R, $A \subsetneqq B$, such that $QB + BP \subseteq A$, B/A is right R/P-torsion free and $\ell_R(D) = Q$ for all non-zero subbimodules D of B/A. We sometimes specify $Q \sim P$ via B/A. As a special case of this, we say there is a <u>second layer link</u> $Q \underset{2}{\sim} P$ if $Q \sim P$ via $(Q \cap P)/A$ for some ideal $A \subsetneqq Q \cap P$. A <u>right link</u> $P \rightsquigarrow Q$ exists if there exists a non-zero uniform submodule U of $E_P/\mathrm{ann}_{E_P}(P)$ with $E(U_R) \cong E_Q$.

§2. <u>Localizability Criteria.</u>

Most of the material in this section can be found in [26]. We will only give proofs of those results whose proofs are either short or strategically different from those in [26]. The ring R is always at least right Noetherian.

<u>Lemma 3.1</u>: If R is Noetherian and P and Q are prime ideals of R, the following are equivalent:

(i) $Q \sim P$ via B/A

(ii) B/A is left R/Q-torsion free and right R/P-torsion free.

<u>Proof</u>: (i) $\Rightarrow$ (ii) Let $T = \tau(B/A)$, the torsion submodule of of the left R/Q-module B/A. Since R is right Noetherian. $T = \sum_{i=1}^{n} t_i R$ for some $t_1, \ldots, t_n \in B/A$. Since $\bar{R} = R/Q$ is left Noetherian, $\mathcal{C}_{\bar{R}}(0)$ is a left Ore set and so, by the left common multiple property, there exists $\bar{c} \in \mathcal{C}_{\bar{R}}(0)$ such that $\bar{c}t_i = 0$ for $1 \le i \le n$; therefore $\bar{c}T = 0$.

But by assumption $\ell_R(T) = Q$, and so $\bar{c} \in \ell_{\bar{R}}(T) = 0$, a contradiction unless $T = 0$.

(ii) $\Rightarrow$ (i) Let D be a non-zero subbimodule of B/A. By hypothesis, $QB \subseteq A$; suppose $Q \subsetneq \ell_R(D)$. Then $\ell_R(D) \cap \mathcal{C}_R(Q) \neq \emptyset$ and so $cD = 0$ for some $c \in \mathcal{C}(Q)$. Therefore D is left R/Q-torsion, a contradiction. Thus $\ell_R(D) = Q$. $\square$

<u>Definition 3.2</u>: Let $\mathcal{P}$ be a set of prime ideals of the right Noetherian ring R. We say $\mathcal{P}$ is <u>bimodule link-closed</u> (respectively <u>second layer link-closed</u>, respectively <u>right link-closed</u>) if $P \in \mathcal{P}$ and $Q \sim P$ (respectively $Q \underset{2}{\sim} P$, respectively $P \rightsquigarrow Q$) implies $Q \in \mathcal{P}$.

<u>Proposition 3.3</u>: Let Σ be a right Ore set in the right Noetherian ring R and let $\mathcal{P} = \{P \mid P$ is a prime ideal with $P \cap \Sigma = \emptyset\}$. Then $\mathcal{P}$ is bimodule link-closed and $Q \sim P \in \mathcal{P}$ via B/A if and only if $Q_\Sigma \sim P_\Sigma$ via B_Σ/A_Σ.

<u>Proof</u>: [26], Proposition 5.3. $\square$

<u>Lemma 3.4</u>: ([26], Lemma 6.1) Let R be right Noetherian, P and Q prime ideals of R, I an ideal of R with $QI + IP = 0$ and I right R/P-torsion free. Let $E = E(R/P)_R$ and set $F = \operatorname{ann}_E(I)$. Then $(E/F)Q = 0$, $E/F \cong \operatorname{Hom}({}_R I_R, E_R)$ as right R-modules, and $d \in R$ acts regularly on $(E/F)_R$ if and only if it acts regularly on ${}_R I$. $\square$

Jategaonkar uses the foregoing lemma to prove his so-called Main Lemma:

<u>Lemma 3.5</u>: ([26], Lemma 6.3) Let P be a prime ideal in the right Noetherian ring R. Let M be a P-tame uniform right R-module and set $N = \mathrm{ann}_M(P)$. Assume that (i) M/N is a non-zero uniform module with $\mathrm{ass}(M/N) = r_R(M/N) = Q$(say), and (ii) for every submodule $X \leq M$, either $X \leq N$ or $r_R(X) = r_R(M)$. Then either (a) $Q \underset{2}{\sim} P$ via $(Q \cap P)/r(M)$, or (b) $r(M) = Q \subsetneq P$. Moreover, if R is Noetherian then (a) and (b) are characterized by M/N being Q-tame or wild, respectively. □

This lemma, the cornerstone of Jategaonkar's approach to localization, is more transparent than it seems at first glance. Viewed properly, it becomes a statement about links. To see this, let us first extend the definition of a right link.

<u>Definition 3.6</u>: Let R be right Noetherian.

(i) The <u>right spectrum</u> of R, rt-Spec(R), is the collection of isomorphism types of indecomposable injective right R-modules.

(ii) For $E, F \in$ rt-Spec(R), there is a <u>right link</u> $E \rightsquigarrow F$ if there exists a non-zero uniform submodule U of $E/\mathrm{ann}_E(\mathrm{ass}(E))$ with $E(U) \cong F$.

To see that this definition generalizes our previous definition of a right link, observe that every prime ideal of R is an element of rt-Spec(R) via the Gabriel correspondence $P \to E_P$ which is always a one-to-one map from Spec(R), the set of prime ideals of R, to rt-Spec(R). (Recall that this correspondence is bijective if and only if R is right fully bounded [6].) When $E = E_P$ and $F = E_Q$, we write $P \rightsquigarrow Q$ instead of $E_P \rightsquigarrow E_Q$.

Lemma 3.7: Let R be right Noetherian, $E, F \in$ rt-Spec(R) with ass(E) = P, ass(F) = Q. Then $E \rightsquigarrow F$ if and only if there exists a cyclic uniform right R-module M such that $E(M) \cong E$ and for $N = \mathrm{ann}_M(P)$ we have:

(i) M/N is uniform with $E(M/N) \cong F$,

(ii) $(M/N)Q = 0$, and

(iii) for every submodule $X \subseteq M$, either $X \subseteq N$ or $r_R(X) = r_R(M)$.

Proof: Suppose $E \rightsquigarrow F$. We claim first that there exists a uniform right R-module M such that $E(M) \cong E$ and for $N = \mathrm{ann}_M(P)$, M/N is uniform with $E(M/N) \cong F$. By the definition of a right link, there exists a uniform submodule $U \subseteq E/\mathrm{ann}_E(P)$ with $E(U) \cong F$. Let M be any uniform module with $E(M) \cong E$ and set $N = \mathrm{ann}_M(P)$. Then M/N embeds in $E/\mathrm{ann}_E(P)$ and we may identify M/N with its image under this embedding to obtain a submodule $M' \subseteq M$, $N \subsetneq M'$, $M'/N = (M/N) \cap U$. Then $N' = \mathrm{ann}_{M'}(P) = N$ and M'/N' is uniform with $E(M'/N') = E(U) \cong F$. This establishes the claim.

We now proceed to modify M to obtain the remaining assertions. Without loss of generality, the module M we have produced is cyclic: for any $m \in M$, $m \notin N$, we have $mR \cap N = \mathrm{ann}_{mR}(P)$, $mR/(mR \cap N) \cong (mR + N)/N \subseteq M/N$ and hence $mR/\mathrm{ann}_{mR}(P)$ is uniform with injective hull of type F. Now $Q = \mathrm{ass}(M/N)$ so there exists $N \subsetneq M' \subseteq M$ with $r_R(M'/N) = Q$. Then $N' = \mathrm{ann}_{M'}(P) = N$ and so without loss of generality $r_R(M/N) = Q$.

Next, pick $M' \leq M$ such that $r(M')$ is maximal with respect to not containing P. Then $M' \neq 0$ for otherwise $r(M') = R \supseteq P$, and hence $E(M') = E(M) \cong E$ and $\mathrm{ass}(M') = P$. Set $N' = \mathrm{ann}_{M'}(P) = N \cap M'$: $N' \subsetneqq M'$ since $N' = M'$ implies $M'P = 0$ and therefore $r(M') \supseteq P$. We have $0 \neq M'/N'$ embedded in M/N; consequently $E(M'/N') \cong F$ and $(M'/N')Q = 0$. Let X be a submodule of M' and suppose $X \nsubseteq N'$. Then $XP \neq 0$ and hence $P \nsubseteq r(X)$. But $r(X) \supseteq r(M')$ and so maximality forces $r(X) = r(M')$.

This establishes (ii) and (iii) for M' which need only be finitely generated at this point. However $M' = \sum_{i=1}^{n} M_i$ where $M_1, \ldots, M_n$ are cyclic submodules of M'. Since $P \nsubseteq r(M') = \bigcap_{i=1}^{n} r(M_i)$, there exists an M_i $(1 \leq i \leq n)$ such that $P \nsubseteq r(M_i)$. But $r(M_i) \supseteq r(M)$ and hence $r(M_i) = r(M)$ by the maximality of $r(M)$. With $N_i = \mathrm{ann}_{M_i}(P)$ we are done.

The converse is immediate from (i) and the definition of a right link. □

Using this characterization, Jategaonkar's Main Lemma takes the following form:

<u>Proposition 3.8</u>: Let R be right Noetherian, P, Q prime ideals of R, $F \in$ rt-Spec(R) with $\mathrm{ass}(F) = Q$. If $P \rightsquigarrow F$ (i.e. $E_P \rightsquigarrow F$) then either (a) $Q \underset{2}{\sim} P$, or (b) $Q \subsetneqq P$. If R is Noetherian, then (a) and (b) are characterized by $F = E_Q$ and $F \neq E_Q$ respectively. □

The usefulness of the second layer condition is now apparent:

Corollary 3.9: Let R be a Noetherian ring satisfying the right second layer condition. Then $\mathrm{Spec}(R)$, the set of all prime ideals of R, is right link-closed.

Proof: Let P be a prime ideal of R, $F \in \text{rt-Spec}(R)$, with $P \rightsquigarrow F$. By definition, there is a uniform module $U \subseteq E_P/\mathrm{ann}_{E_P}(P)$ with $E(U) \cong F$. But the right second layer condition implies that U is tame. Therefore $\mathrm{ass}(U) = \mathrm{ass}(F) = Q$, say, is a prime ideal and $E(U) \cong E_Q$. That is, $P \rightsquigarrow Q$. □

We have the following equivalent characterizations of a right link between prime ideals P and Q:

Theorem 3.10: Let P and Q be prime ideals of the Noetherian ring R. The following are equivalent:

(i) $Q \underset{2}{\sim} P$ and $Q \nsubseteq P$,

(ii) $P \rightsquigarrow Q$,

(iii) there exists a short exact sequence of finitely generated uniform right R-modules $0 \to N \to M \to L \to 0$ with N a right ideal of R/P, L a right ideal of R/Q, and $N = \mathrm{ann}_M(P)$.

Proof: (i) ⇒ (ii) We are given $Q \sim P$ via $(Q \cap P)/A$ where $QP \subseteq A$. Without loss of generality $A = 0$. Set $I = Q \cap P$, $E = E(R/P)_R$, $F = \mathrm{ann}_E(I)$. Since I is left R/Q-torsion free by

Lemma 3.1, $\mathcal{C}(Q)$ acts regularly on ${}_R I$. Hence $\mathcal{C}(Q)$ acts regularly on $(E/F)_R$, by Lemma 3.4.

Let $e \in \mathrm{ann}_E(I)$ and suppose $eP \neq 0$. Then $ePQ \subseteq e(Q \cap P) = 0$ and so $Q \subseteq r(eP) \subseteq \mathrm{ass}(E) = P$. $Q \neq P$ for otherwise $0 \neq eP = e(Q \cap P) = 0$, an absurdity. Hence $Q \subsetneqq P$, a contradiction. Therefore $e \in \mathrm{ann}_E(P)$ and we see that $\mathrm{ann}_E(I) = \mathrm{ann}_E(Q \cap P) = \mathrm{ann}_E(P)$. It follows that $E/F = E/\mathrm{ann}_E(P)$ is right R/Q-torsion free and thus contains a Q-tame uniform submodule. Consequently $P \rightsquigarrow Q$.

(ii) $\Rightarrow$ (iii) From $P \rightsquigarrow Q$ and Lemma 3.7 we obtain the short exact sequence $0 \to N \to M \to M/N \to 0$ of finitely generated uniform right R-modules, with $N = \mathrm{ann}_M(P)$, $(M/N)Q = 0$, $E(N) = E(M) \cong E_P$ and $E(M/N) \cong E_Q$. Thus N is a uniform, torsion free R/P-module and M/N is a uniform, torsion free R/Q-module. It is well-known that over a prime Noetherian ring, finitely generated, uniform, torsion free modules are isomorphic to right ideals.

(iii) $\Rightarrow$ (i) N is uniform, R/P-torsion free and so $E(M) = E(N) \cong E_P$. Similarly M/N is uniform with $(M/N)Q = 0$ and $E(M/N) \cong E_Q$. Therefore $M/N = M/\mathrm{ann}_M(P)$ is a uniform Q-tame module which embeds into $E_P/\mathrm{ann}_{E_P}(P)$. By definition, this means $P \rightsquigarrow Q$ but then Proposition 3.8 shows $Q \nsubseteq P$ and hence $Q \underset{2}{\sim} P$. $\square$

We now state Jategaonkar's criteria for a semiprime ideal to be localizable ([26], Theorems 4.10, 4.11).

<u>Theorem 3.11</u>: Let S be a semiprime ideal of the right Noetherian ring R and set $E = E(R/S)_R$. The following are equivalent:

(i) S is right localizable,

(ii) $\mathrm{ann}_E(S)$ is S-closed in E,

(iii) If M is a finitely generated uniform right R-module containing a non-zero submodule N such that $NS = 0$, N is R/S-torsion free and M/N is S-torsion, then $MS = 0$. □

For a semiprime ideal S we denote by min(S) the set of minimal prime ideals over S. For a collection P of prime ideals, $\mathrm{her}(P) = \{P \mid P \text{ prime}, P \subseteq P' \text{ for some } P' \in P\}$.

<u>Lemma 3.12</u>: Let S be a right localizable semiprime ideal of the right Noetherian ring R. Then her(min(S)) is bimodule link-closed.

<u>Proof</u>: It follows from remarks in §1 that $\mathrm{her}(\min(S)) = \{P \mid P \text{ prime}, P \cap C(S) = \emptyset\}$. Since $C(S)$ is a right Ore set in R, the result is clear from Proposition 3.3. □

If R is Noetherian and S is localizable, there is a sharper version of this.

<u>Theorem 3.13</u>: ([26], Theorem 5.10) Let S be a localizable semiprime ideal in the Noetherian ring R. Then min(S) is bimodule link-closed. □

We remark that the one-sided version of Theorem 3.13 is not true.

We now turn to the question of classical localization. If $P_1, P_2, \ldots, P_n$ are the minimal prime ideals over the semiprime ideal S, it is not hard to see that the second layer of $E(R/S)_R$ is the union of the second layers of the $E(R/P_i)_R$ $(1 \le i \le n)$. We say that a uniform module is S-tame if it is P-tame for some $P \in \min(S)$.

There is the following useful characterization of a classical semiprime ideal:

Theorem 3.14: Let S be a semiprime ideal of the right Noetherian ring R and set $E = E(R/S)_R$. The following are equivalent:

(i) S is right classical,

(ii) $E = \bigcup_{n=1}^{\infty} \operatorname{ann}_E(S^n)$,

(iii) the second layer of E is S-tame.

Proof: [26] Propositions 8.3, 8.4, and Theorem 8.5. □

In terms of links, this result may be reformulated as:

Corollary 3.15: Let S be a semiprime ideal of the right Noetherian ring R. S is right classical if and only if $\min(S)$ is right link-closed (in rt-Spec(R)).

Proof: Suppose S is right classical and let $P \rightsquigarrow F$, $P \in \min(S)$, $F \in$ rt-Spec(R). By definition, there is a uniform right R-module U in the second layer of E_P (hence in the second layers of

$E(R/P)$ and $E(R/S))$ with $E(U) \cong F$. By Theorem 3.14, U is S-tame – $E(U) \cong E_Q$ for some $Q \in \min(S)$. Therefore $F = E_Q$ and $P \rightsquigarrow Q \in \min(S)$ as desired.

Conversely, assume that $\min(S)$ is right link-closed. Let U be a uniform right R-module in the second layer of $E(R/S)$ (hence in the second layer of $E(R/P)$ for some $P \in \min(S)$). This gives a uniform module U' in the second layer of E_P and so, by definition, $P \rightsquigarrow E(U')$. By assumption $E(U') = E_Q$ for some $Q \in \min(S)$. Hence $E(U) = E_Q$ and U is Q-tame. By Theorem 3.14 again, S is right classical. □

The criteria for a semiprime ideal to be right classical may now be proved (cf. [26], Theorem 8.6).

<u>Theorem 3.16</u>: Let S be a semiprime ideal of the right Noetherian ring R. The following are equivalent:

(i) S is right classical,

(ii) $\min(S)$ satisfies the right second layer condition and is bimodule link-closed,

(iii) $\min(S)$ satisfies the right second layer condition and is second layer link-closed,

(iv) $\min(S)$ satisfies the right second layer condition and is right link-closed in $\mathrm{Spec}(R)$.

<u>Proof</u>: (i) $\Rightarrow$ (ii) Let $P \in \min(S)$ and let U be a uniform right R-module contained in the second layer of E_P. Then $P \rightsquigarrow E(U)$ and so $E(U) \cong E_Q$ for some $Q \in \min(S)$ by Corollary 3.15. In other

words, U is tame and hence $\min(S)$ satisfies the right second layer condition.

Now let $Q \sim P \in \min(S)$ via B/A. Choose a right ideal K of R maximal with respect to $B \cap K = A$. Then $B/A = B/(B \cap K) \cong (B + K)/K$ is essential in R/K and hence there is an embedding $R/K \rightarrowtail E(B/A)_R \cong E_P^n \rightarrowtail E(R/S)^n$ (since B/A is a torsion-free right R/P-module). Since S is right classical, Proposition 3.14 yields a natural number m such that $(R/K)S^m = 0$. Therefore $S^m \subseteq K$ and hence $S^m B \subseteq S^m \cap B \subseteq K \cap B = A$ so that $S^m \subseteq \ell(B/A) = Q$. Then $S \subseteq Q$ and hence $P' \subseteq Q$ for some $P' \in \min(S)$. Now $Q \in \mathrm{her}(\min(S))$ by Lemma 3.12 so there exists $P'' \in \min(S)$ with $Q \subseteq P''$. It follows that $Q = P' = P'' \in \min(S)$ as desired.

(ii) $\Rightarrow$ (iii) This is trivial.

(iii) $\Rightarrow$ (iv) Assume $P \in \min(S)$ and $P \rightsquigarrow Q$. Then by Lemma 3.7, there exists a uniform cyclic P-tame module M and for $N = \mathrm{ann}_M(P)$, M/N is uniform, Q-tame with $r(M/N) = Q$. If $Q \not\supseteq P = r(N)$ then N is a right R/Q-module and $P \cap \mathcal{C}(Q) \neq \emptyset$ implies that N is R/Q-torsion (since a uniform module over a prime ring is either torsion or torsion free). Therefore M (and hence M/N) is Q-torsion. But M/N is Q-tame (i.e. Q-torsion free) so this is a contradiction. We now apply Proposition 3.8 to see that $Q \underset{2}{\sim} P$. By (iii), $Q \in \min(S)$.

(iv) $\Rightarrow$ (i) Let $P \in \min(S)$, $F \in \text{rt-Spec}(R)$ with $P \rightsquigarrow F$. The right second layer condition implies that $F = E_Q$ for some prime ideal Q. Hence $P \rightsquigarrow Q$ and (iv) implies that $Q \in \min(S)$. By Corollary 3.15, S is right classical. □

Any of the notions of "link" we have defined produces a directed graph on Spec(R) with the prime ideals as vertices and links between them as directed edges. The connected component of a prime ideal P in the underlying undirected graph is called the <u>link component</u> of P and is denoted by $L(P)$. (If it is not clear from context which type of link is intended, we shall refer to the <u>bimodule link component</u>, etc.) In general, it is not known whether the different notions of link produce the same component.

<u>Corollary 3.17</u>: Let R be a Noetherian ring. The clans in R are precisely the finite link components which satisfy the second layer condition. □

<u>Corollary 3.18</u>: A prime ideal in a Noetherian ring belongs to at most one clan. □

§3. <u>Krull Dimension Considerations</u>.

In this section, we consider only Noetherian rings and we examine certain subclasses of Noetherian rings which will arise in Chapter IV. In particular, we give special cases of some of the material from the preceding section. Most of the results in this section make use of Krull dimension (in the sense of Gabriel and Rentschler); for definitions and basic properties, the reader is referred to [16]. The Krull dimension of a right R-module M is denoted by kdim(M) , if it exists, and is referred to as the <u>right Krull dimension</u> of M ; left Krull dimension of a left R-module M is denoted by ℓdim(M) .

The ring R is said to be Krull symmetric (or K-symmetric) if $\ell\dim(B/A) = \mathrm{kdim}(B/A)$ for all pairs of ideals $A \subsetneq B$. R is called weakly Krull symmetric (or weakly K-symmetric) if $\ell\dim(R/A) = \mathrm{kdim}(R/A)$ for every ideal A. It is easy to show that R is weakly K-symmetric if and only if $\ell\dim(R/P) = \mathrm{kdim}(R/P)$ for every prime ideal P. An ideal T of R is said to be right weakly prime ideal invariant (rt-w.p.i.i.) if $\mathrm{kdim}(T/PT) < \mathrm{kdim}(R/T)$ whenever P is a prime ideal of R with $\mathrm{kdim}(R/P) < \mathrm{kdim}(R/T)$. R is called right weakly prime ideal invariant if every ideal of R is rt-w.p.i.i.

Lemma 3.19: Let R be right Noetherian. R is rt-w.p.i.i. if and only if every prime ideal of R is rt-w.p.i.i.

Proof: The proof of [5] Theorem 2.3 can be adapted mutatis mutandis. □

Left weak prime ideal invariance is defined analogously. R is said to be right smooth if, for all finitely generated right R-modules M containing an essential submodule N, $\mathrm{kdim}(R/r(M)) = \mathrm{kdim}(R/r(N))$. Left smoothness is defined analogously. R is said to be a right poly-AR ring if for every pair of prime ideals $Q \subsetneq P$, there exists an ideal I of R with $Q \subsetneq I \subseteq P$ such that I/Q has the right AR-property in R/Q.

These properties are connected in the following:

Proposition 3.20: Let R be a Noetherian poly-AR ring. The following are equivalent:

(i) R is K-symmetric,

(ii) R is weakly K-symmetric and weakly prime ideal invariant.

Proof: (i) $\Rightarrow$ (ii) If R is K-symmetric, it is certainly weakly K-symmetric. Indeed, let A be an ideal of R. Then for any ideal $B \supsetneq A$, $\mathrm{kdim}(R/A) = \sup\{\mathrm{kdim}(B/A), \mathrm{kdim}(R/B)\} = \sup\{\ell\mathrm{dim}(B/A), \ell\mathrm{dim}(R/B)\} = \ell\mathrm{dim}(R/A)$, using K-symmetry and a Noetherian induction. For the other assertion, it suffices to check rt-w.p.i.i. Let T be an ideal of R, P a prime ideal, with $\mathrm{kdim}(R/P) < \mathrm{kdim}(R/T)$. Then $\mathrm{kdim}(T/PT) = \ell\mathrm{dim}(T/PT) \le \ell\mathrm{dim}(R/P) = \mathrm{kdim}(R/P) < \mathrm{kdim}(R/T)$.

(ii) $\Rightarrow$ (i) [2] Theorem 3.2 implies that R is smooth. [3] Lemma 2.1 then shows that R is K-symmetric. □

A set P of prime ideals is said to satisfy the incomparability condition (INC) if for any two prime ideals P and Q of P, $P \subseteq Q$ only if $P = Q$.

Proposition 3.21: Let R be a Noetherian poly-AR ring. Then R satisfies the second layer condition and (bimodule) link components satisfy INC.

Proof: R has the second layer condition by [26] Proposition 7.13 and Corollary 7.24. The other assertion is [26] Theorem 7.26. □

A module is said to be <u>K-homogeneous</u> (or <u>α-homogeneous</u>) if $\mathrm{kdim}(N) = \mathrm{kdim}(M) = \alpha$ for every non-zero submodule N of M. A set $\mathcal{P}$ of prime ideals of a (right) Noetherian ring R is called α-homogeneous if $\mathrm{kdim}(R/P) = \alpha$ for every $P \in \mathcal{P}$.

<u>Proposition 3.22</u>: Let R be a Noetherian ring satisfying the second layer condition. If every second layer link component of R is K-homogeneous then R is w.p.i.i.

<u>Proof</u>: This is a corollary of [26] Theorem 9.11. □

<u>Theorem 3.23</u>: Let R be a K-symmetric Noetherian ring with prime ideals P and Q. The following are equivalent:

(i) $Q \underset{2}{\sim} P$

(ii) $P \rightsquigarrow Q$

(iii) $\mathrm{kdim}(R/Q) = \mathrm{kdim}((Q \cap P)/QP) = \mathrm{kdim}(R/P)$

(iv) $\ell_R((Q \cap P)/QP) = Q$ and $r_R((Q \cap P)/QP) = P$

(v) $Q \underset{\ell}{\rightsquigarrow} P$

<u>Proof</u>: First observe that $Q \sim P$ via B/A implies $\mathrm{kdim}(R/Q) = \ell\mathrm{dim}(R/Q) = \ell\mathrm{dim}(R/\ell(B/A)) = \ell\mathrm{dim}(B/A) = \mathrm{kdim}(B/A) = \mathrm{kdim}(R/r(B/A)) = \mathrm{kdim}(R/P)$. Hence $Q \nsubseteq P$ and so the equivalence of (i) and (ii) follows from Theorem 3.10. The left-hand version of this shows (i) $\Leftrightarrow$ (v).

(i) $\Rightarrow$ (iii) Let $Q \underset{2}{\sim} P$ via $(Q \cap P)/A$. Then $P = r_R((Q \cap P)/A)$ and so $\mathrm{kdim}(R/P) = \mathrm{kdim}((Q \cap P)/A) \le \mathrm{kdim}((Q \cap P)/QP) \le \mathrm{kdim}(R/P)$, the first inequality because $QP \subseteq A$ implies $(Q \cap P)/A$ is a factor module of $(Q \cap P)/QP$, and the second because $(Q \cap P)/QP$ is a right R/P-module. Hence equality holds throughout. Similarly $\mathrm{kdim}(R/Q) = \ell\mathrm{dim}(R/Q) = \ell\mathrm{dim}((Q \cap P)/QP) = \mathrm{kdim}((Q \cap P/QP)$.

(iii) $\Rightarrow$ (iv) Clearly $P \subseteq r((Q \cap P)/QP) = I$; suppose $P \subsetneqq I$. Then $I \cap C(P) \neq \emptyset$ and so $\mathrm{kdim}(R/I) < \mathrm{kdim}(R/P)$ by [16] Proposition 6.1. Therefore $\mathrm{kdim}((Q \cap P)/QP) = \mathrm{kdim}(R/I) < \mathrm{kdim}(R/P)$ contradicting (iii). A symmetric argument proves $\ell((Q \cap P)/QP) = Q$.

(iv) $\Rightarrow$ (i) Statement (iv) says that Q is bimodule-linked to P via $(Q \cap P)/QP$ in the sense of Müller [41]. By Lemma 12 of [43], this implies that $Q \sim P$ via $(Q \cap P)/A$ for some ideal A with $QP \subseteq A \subsetneqq Q \cap P$. In other words $Q \underset{2}{\sim} P$. □

A module M_R with Krull dimension is said to be <u>critical</u> if every proper factor module has Krull dimension strictly less than $\mathrm{kdim}(M)$. Following Brown, we call a critical module M_R of <u>type I</u> if $\mathrm{kdim}(M) = \mathrm{kdim}(R/r_R(M))$. Following Müller, we call a critical module M <u>non-singular</u> if $\mathrm{ass}(M) = r_R(M)$ and M is of type I. The following crucial result is Lemma 5 of [43].

<u>Lemma 3.24</u>: Over a K-symmetric Noetherian ring R, every type I critical right R-module is non-singular. □

There is an analogue of the Jordan-Hölder Theorem which applies to finitely generated modules over a Noetherian ring and in which the critical modules play the role of the simple modules. A <u>Krull composition series</u> of M (k.c.s.) is an ascending chain $0 = M_0 \subsetneqq M_1 \subsetneqq \ldots \subsetneqq M_n = M$

of submodules of M such that the <u>Krull factors</u> M_i/M_{i-1} are all critical and the sequence $\{\mathrm{kdim}(M_i/M_{i-1}) \mid 1 \le i \le n\}$ is non-decreasing. The "Jordan-Hölder" Theorem then states that any two k.c.s. for M have the same length n and the injective hulls of the Krull factors are unique up to order and isomorphism. Therefore the collection $\{\mathrm{ass}(M_i/M_{i-1}) \mid 1 \le i \le n\}$ of prime ideals is independent of our choice of k.c.s. We call these prime ideals the <u>right Krull primes</u> of M and denote the collection by $\text{rt-KSpec}(M)$. If M is an R-bimodule we write $\mathrm{KSpec}(M) = \text{rt-KSpec}(M) \cup \ell\text{t-KSpec}(M)$. (See also [2], [43].)

The next two results are due to Müller and are taken directly from [43] which contains a more extensive discussion of the relationship between Krull primes and localizability.

<u>Lemma 3.25</u>: Let R be a Noetherian K-symmetric ring and M an R-bimodule which is α-homogeneous as a right module. Then all right Krull factors of M are non-singular, $\text{rt-KSpec}(M)$ is α-homogeneous, and $\text{rt-KSpec}(M) = \min(r_R(M))$. □

<u>Lemma 3.26</u>: Let R be a Noetherian K-symmetric ring.

(i) If P and Q are prime ideals of R with $Q \sim P$ via B/A and $\mathrm{kdim}(B/A) = \alpha$, then $\{Q\} = \alpha\text{-}\ell\text{t-KSpec}(B/A)$ and $\{P\} = \alpha\text{-rt-KSpec}(B/A)$.

(ii) Conversely, if $A \subsetneqq B$ are ideals of R and $P \in \text{rt-KSpec}(B/A)$, then there exist ideals A' and B' of R with $A \subseteq A' \subsetneqq B' \subseteq B$ such that $Q \sim P$ via B'/A' for some $Q \in \ell\text{t-KSpec}(B/A)$. □

§4. Normal Elements, the AR-property, and Links.

For future reference, we collect in this section some further results on links which exploit the presence of normal elements. Although, in some cases, the results as stated are not the best possible, they will suffice for our purposes.

An element x of a ring R is said to be a normal element if $xR = Rx$. If R is a prime ring, the ideal xR generated by a normal element is well-known to be invertible (cf. [9] page 51) and hence has the (right) AR-property ([9] Lemma 3.3). However, the latter property holds even if R is not prime as the following result shows.

Lemma 3.27: ([38]) In a (right) Noetherian ring R, an ideal generated by normal elements has the (right) AR-property. □

The next two results will be crucial.

Lemma 3.28: ([26] Proposition 7.9) Let R be a right Noetherian ring and I an ideal with the right AR-property. If $Q \sim P$ and $I \subseteq P$, then $I \subseteq Q$. □

Corollary 3.29: Let x be a normal element of the right Noetherian ring R. If $Q \sim P$ and $x \in P$, then $x \in Q$. □

Definition 3.30: For a normal element x of a ring R and a prime ideal P, $P^x = \{r \in R \mid rx \in xP\}$.

An equivalent description is $P^x = \ell_R(xR/xP)$. Observe that $R/r(x) \cong R/\ell(x)$ via $r + r(x) \to r' + \ell(x)$ where $xr = r'x$. If $r(x) \subseteq P$, then under this isomorphism $P/r(x)$ corresponds to $P^x/\ell(x)$ and it is easy to see that P^x is a prime ideal in this case.

<u>Proposition 3.31</u>: Let R be a Noetherian ring with normal element x and prime ideal P. The following are equivalent:

(i) $r(x) \subseteq P$,

(ii) P^x is a prime ideal such that $P^x \sim P$ via xR/xP.

<u>Proof</u>: (i) $\Rightarrow$ (ii) We have already remarked that P^x is a prime ideal. Suppose $xrc \in xP$ for $r \in R$, $c \in \mathcal{C}(P)$. Then $rc - p \in r(x) \subseteq P$ for some $p \in P$. Hence $rc \in P$ and so $r \in P$. Therefore xR/xP is right P-torsion free. Now let $d \in \mathcal{C}(P^x)$, $r \in R$ with $dxr \in xP$. Let $dx = xd'$ and $d'a \in P$ for some $a \in R$. Then $da'x = dxa = xd'a \in xP$ and so $da' \in P^x$. Hence $a' \in P^x$ and so $xa = a'x = xp$ for some $p \in P$. Thus $a - p \in r(x) \subseteq P$ and so $a \in P$. It follows that $d' \in \mathcal{C}(P)$. Using this, we see that $xd'r = dxr \in xP$ and so $d'r - p' \in r(x) \subseteq P$ for some $p' \in P$. Hence $d'r \in P$ and so $r \in P$. We conclude that xR/xP is left P^x-torsion free. By Lemma 3.1, this implies $P^x \sim P$ via xR/xP.

(ii) $\Rightarrow$ (i) Suppose $r \in R$ such that $xr = 0$. Then $xRr = Rxr = 0 \subseteq xP$ and hence $r \in r_R(xR/xP) = P$, using Lemma 3.1 and the definition of a bimodule link. □

If $x \in P$, we can characterize those prime ideals which are second layer linked to P. We will use the following lemma:

<u>Lemma 3.32</u>: Let R be a right Noetherian ring, P and Q prime ideals of R, I an ideal of R with the right AR-property, and $I \subseteq P$. Suppose $Q \underset{2}{\sim} P$ via $(Q \cap P)/A$. Then either $Q/I \underset{2}{\sim} P/I$ or $Q \sim P$ via $I/(A \cap I)$.

<u>Proof</u>: By Corollary 3.29, $I \subseteq Q$. Set $\bar{R} = R/I$. If $I \subseteq A$, then it is obvious that $\bar{Q} \underset{2}{\sim} \bar{P}$ via $(\bar{Q} \cap \bar{P})/\bar{A}$. If $I \nsubseteq A$, then $A \cap I \subsetneqq Q \cap P \cap I = I$. Moreover $I/(A \cap I) \cong (A + I)/A \subseteq (Q \cap P)/A$ as $(R/Q, R/P)$-bimodules. By Lemma 3.1, $Q = \ell(D)$ and $P = r(D)$ for every subbimodule D of $I/(A \cap I)$. Hence $Q \sim P$ via $I/(A \cap I)$. □

<u>Proposition 3.33</u>: Let R be a Noetherian ring with prime ideals P and Q and a normal element $x \in P$. If $Q \underset{2}{\sim} P$ via $(Q \cap P)/A$ then either $Q/xR \underset{2}{\sim} P/xR$ or $A \cap xR = xP$ and $Q = P^x \sim P$ via xR/xP.

<u>Proof</u>: With $I = xR$, Lemma 3.32 gives either $Q/xR \underset{2}{\sim} P/xR$ or $Q \sim P$ via $xR/(A \cap xR)$. Let $r \in r(x)$. Then $xRr = Rxr = 0 \subseteq A \cap xR$ and hence $r \in r_R(xR/(A \cap xR)) = P$. P^x is therefore a prime ideal by Proposition 3.31. If $r \in P^x$ then $rx \in xP \subseteq A \cap xR$ and so $r \in \ell_R(xR/(A \cap xR)) = Q$; hence $P^x \subseteq Q$. Suppose $P^x \neq Q$: since P^x is prime, $Q \cap \mathcal{C}(P^x) \neq \emptyset$. That is, there exists $c \in \mathcal{C}(P^x)$ such that $cx \in A \cap xR$. Let $c' \in R$ with $xc' = cx$. Then $xRc' = Rxc' = Rcx \subseteq A \cap xR$ and so $c' \in r(xR/(A \cap xR)) = P$. Thus $c \in P^x \cap \mathcal{C}(P^x)$ which is impossible. We conclude that $Q = P^x$.

Clearly $xP \subseteq A \cap xR$; suppose the inclusion is strict. Then, using the isomorphism $xR \cong R/r(x) = \bar{R}$, we have $\bar{P} \subsetneqq \overline{A + r(x)}$ and so

$A + r(x)$ contains some $c \in \mathcal{C}(P)$. Write $c = a + p$ where $a \in A$ and $p \in r(x) \subseteq P$. Then $xc = x(a + p) = xa \in A \cap xR$ and thus $xRc = Rxc \subseteq A \cap xR$. Therefore $c \in P \cap \mathcal{C}(P)$ and we conclude from this contradiction that $A \cap xR = xP$. □

For more information related to normal elements, the AR-property, and localization, the reader is referred to the work of McConnell [38], Smith [53], Cozzens and Sandomierski [10], Heinicke [17], and Jategaonkar [26].

CHAPTER IV

LOCALIZATION IN ORE EXTENSIONS OF COMMUTATIVE NOETHERIAN RINGS

We now return to our study of Ore extensions and, equipped with the concepts of the preceding chapter, we examine the localizability of prime ideals. In particular, we seek to understand the mechanism by which prime ideals are linked to one another and to give an internal characterization of the link component of a prime ideal.

To facilitate matters, we restrict our attention in this chapter to Ore extensions $S = R[x;\sigma]$ whose coefficient ring R is commutative Noetherian. In particular, this guarantees that every σ-prime ideal I of R is a classical semiprime ideal and that $\mathcal{C}_R(I)$ is an Ore set in both R and S. If σ has finite order n then, since it is commutative, R is integral over its fixed subring R^σ and it is easy to deduce from this that S is then finitely-generated as a module over its center $R^\sigma[x^n]$; in this setting we know that every prime ideal of S belongs to a clan (S has <u>enough clans</u>) and the clans are precisely the sets of prime ideals lying over given central primes (cf. [39]). Although σ need not have finite order in general, the existence of upper prime ideals (Definition 2.10) is equivalent to σ having finite order on some σ-prime factor ring R/I or R; $S/IS \cong (R/I)[x;\sigma]$ is then module-finite over its center and hence fully bounded, a fact which we will frequently exploit.

Let us set the stage for our investigation of localization in Ore extensions by considering the situation studied by Jacobson [21]: $R = K^\sigma$

is a field with automorphism σ and $S = K[x;\sigma]$. The only prime ideals of S are 0 (i.e. S is a prime ring), $p(x)S$ where $p(x)$ is a central, centrally-irreducible polynomial, and xS. 0 is classical by Goldie's Theorem; $p(x)S$ and xS are each generated by a regular normal element and, as we have already remarked, such ideals are invertible. In a Noetherian ring, invertible prime ideals are classical [8]. Therefore all prime ideals of $K[x;\sigma]$ are classical.

The general situation is by no means so trivial, even for R a commutative σ-simple Artinian ring (i.e. a direct sum of fields forming a σ-orbit). As we shall see, an Ore extension of a commutative Noetherian ring may possess classical prime ideals, primes which belong to clans, and primes whose link component is infinite.

As in our classification of prime ideals, we split our attention to prime ideals which contain x and those which do not. In each case, we determine the second layer link components (or, equivalently, the right link components) of a prime ideal. In the process, we establish some of the Krull dimension theoretic properties of §3.3 ~~for~~ the Ore extension S and the corresponding skew Laurent polynomial ring $R[x, x^{-1}; \sigma]$ which will be used in our investigation.

§1. Prime Ideals Containing x.

We begin by determining the second layer link component of a prime ideal which contains the indeterminate x. Later, we will show that this coincides with the (right or left) link component.

Theorem 4.1: Let R be a commutative Noetherian ring,

$S = R[x;\sigma]$. If P is a prime ideal of S containing x then the second layer link component $L_2(P)$ of P is $\{P^{\sigma^i} \mid i \in \mathbb{Z}\}$. Also $x \in Q$ for every $Q \in L_2(P)$.

Proof: Since x is a normal element of S, the last statement is a consequence of Corollary 3.29 (and its left-hand version). By Proposition 3.33, if $Q \underset{2}{\sim} P$ then either $Q/xS \sim P/xS$ or $Q = P^x$. But since $x \in Q \cap P$, $Q/xS \cong Q \cap R$ and $P/xS \cong P \cap R$; hence the former case implies $Q \cap R = P \cap R$ since R is commutative and moreover $Q = (Q \cap R) + xS = (P \cap R) + xS = P$. On the other hand, since $P^\sigma x = xP$ and x is regular, $P^\sigma = P^x$. It now follows that $L_2(P) = \{P^{\sigma^i} \mid i \in \mathbb{Z}\}$ using the symmetricized version of Proposition 3.33. □

Example 4.2: We are now in a position to see that if we generalize the "Jacobson situation" mentioned in the introduction, we can get non-localizable prime ideals. Specifically, let R be a commutative σ-simple Artinian ring (i.e. $R = \sum_{i=1}^{n} \oplus F_i$, $F_i \cong F$ a field, $F_i^\sigma = F_{i+1}$, $F_n^\sigma = F_1$). Then the only prime ideals of $S = R[x;\sigma]$ are 0 (which is classical by Goldie's Theorem), $p(x)S$ for central, centrally-irreducible $p(x)$ (classical, as before), and $Q_i = P_i + xS$ for P_i one of the n prime ideals of R. Since $P_1, \ldots, P_n$ form a full σ-orbit in R, Theorem 4.1 implies that $\{Q_1, \ldots, Q_n\}$ is the only non-trivial link component - a clan.

For a specific example of this, take $R = F \oplus F$ and

$\sigma: (a,b) \to (b,a)$ as in Example 1.6. Then for the prime ideals $P_1 = F \oplus 0$ and $P_2 = 0 \oplus F$ of R, the only non-trivial clan of S is $\{P_1 + xS, P_2 + xS\}$ and $xS = (P_1 + xS) \cap (P_2 + xS)$ is therefore a classical semiprime ideal of S.

Theorem 4.1 also shows us how to construct an infinite link component - find an automorphism σ and a prime ideal P of S such that σ-orbit of P is infinite. We give an explicit example.

Example 4.3: Let $R = K[y]$, the commutative polynomial ring in one indeterminate over a field K of characteristic 0. Let σ be the automorphism of R given by $y^{\sigma} = y - 1$ and $S = R[x;\sigma]$. Take $P = \langle x,y\rangle = yS + xS = yR + xS$, a prime ideal of S. Clearly $L_2(P) = \{\langle x, y + n\rangle \mid n \in \mathbf{Z}\}$ is (countably) infinite.

This example arises as follows: S is the enveloping algebra of the two-dimensional non-Abelian Lie algebra $F = \langle x,y \mid [y,x] = x\rangle$ and P is its augmentation ideal. (cf. [4], [39])

Remark 4.4: We know of no example in the literature, explicit or implicit, of the link component of a prime ideal P of an Ore extension where $x \notin P$. In §3 we will develop the theory of links for such primes and then give several examples in §4.

§2. Lower Prime Ideals.

We now turn our attention to the prime ideals of S which do not contain x. Recall that every such prime ideal contracts to a σ-prime ideal of R (Proposition 1.13) and lying over every σ-prime ideal of R there is at most a two-chain of prime ideals of S (Corollary 2.9). For a prime ideal P with $x \notin P$, recall that P is said to be a lower prime if $P = (P \cap R)S$ and an upper prime otherwise. In this section, we concentrate on the lower primes, proving that they are all classical.

Using Roseblade's technique for checking the AR-property in non-commutative rings - an extension of the idea of the proof of the usual Artin-Rees Lemma - we show that lower prime ideals have the AR-property.

<u>Lemma 4.5</u>: [51] Let A be a ring and T an ideal of A. Let $A^*(T)$ be the subring of the polynomial ring $A[t]$ given by $A^*(T) = A + Tt + T^2t^2 + \dots$.

(i) If A is (right) Noetherian and T is centrally generated then $A^*(T)$ is (right) Noetherian.

(ii) If $A^*(T)$ is (right) Noetherian then T has the (right) AR-property. $\square$

<u>Proposition 4.6</u>: Let R be a commutative Noetherian ring and P a lower prime ideal of $S = R[x;\sigma]$. Then P has the AR-property.

<u>Proof</u>: Set $I = P \cap R$. Then, by Lemma 4.5(i), $R^*(I)$ is Noetherian. In $S[t]$, we have $S^*(P) = \langle S, Pt\rangle = \langle S, It\rangle = \langle R, x, It\rangle = \langle R, It, x\rangle = \langle R^*(I), x\rangle$ (where $\langle\dots\rangle$ denotes "the ring generated by..."). Extend σ to an automorphism of $R^*(I)$ by setting $t^\sigma = t$. Then for $r^* \in R^*(I)$ we have $xr^* = (r^*)^\sigma x$ so that $S^*(P) = R^*(I)[x;\sigma]$. Hence ,

$S^*(P)$ is Noetherian by the Hilbert Basis Theorem (Theorem 1.7) and therefore P has the AR-property by Lemma 4.5(ii). □

The notion of a poly-AR ring was used in §3.3; equipped with Proposition 4.6, it is now a simple task to show that the Ore extension S has this property.

Theorem 4.7: The Ore extension $S = R[x;\sigma]$ of a commutative Noetherian ring R is a poly-AR ring.

Proof: We must show that for every pair of prime ideals $P \supsetneq Q$ of S, there is an ideal I of S with $P \supseteq I \supsetneq Q$ such that I/Q is AR in S/Q. We consider three cases.

Case (i): $x \in Q$. Then $x \in P$ also and P/Q is an ideal of $S/Q = S/((Q \cap R) + xS) \cong R/(Q \cap R)$, a commutative Noetherian ring. Hence P/Q has the AR-property.

Case (ii): $x \notin Q$, $x \in P$. Here $P/Q = [(P \cap R) + xS]/Q \supseteq (Q + xS)/Q \cong xS/(Q \cap xS) = xS/xQ$ which has the AR-property since it is an ideal generated by a normal element (Lemma 3.27).

Case (iii): $x \notin P$. If P is a lower prime then it has the AR-property (Proposition 4.6) and so P/Q is therefore AR as well. Otherwise let $I = P \cap R$ and $J = Q \cap R$. Then $J \subseteq I$ and $IS \subseteq JS \subsetneq P$. The only remaining case is therefore $I = J$ (i.e. $P \supsetneq Q = (P \cap R)S$). Without loss of generality, we may assume $I = 0$ (and therefore R is σ-prime). Set $\mathcal{C} = \mathcal{C}_R(0)$. Then in $S_{\mathcal{C}} = Q(R)[x;\sigma]$ we have the prime ideal $PS_{\mathcal{C}} = p(x)S_{\mathcal{C}}$ where $p(x)$ is a central, centrally-irreducible polynomial of S_C (Corollary 2.16). Since R is commutative, σ has finite order by Theorem 2.18. Moreover, $Q(R) = R_{C*}$, the quotient ring obtained by inverting elements of $C* = R^{\sigma} \sim \{0\}$ ([20], Lemma 5.8). Hence $p(x) = p'(x)c^{-1}$ for some $p'(x) \in S$ and $c \in C*$. Now $c \in C(S) = R^{\sigma}[x^n]$, where n is the order of σ, and so for any $f(x) \in S$ we have $p'(x)f(x) = p'(x)c^{-1} \cdot cf(x) = p(x)cf(x) = cf(x)p(x) = cf(x) \cdot c^{-1}p'(x) = f(x)c \cdot c^{-1}p'(x) = f(x)p'(x)$. Thus $p'(x)$ is central and consequently $p'(x)S$ is a non-zero two-sided ideal of S which has the AR-property (Lemma 3.27). Since $p'(x)S = c^{-1}p(x)S \subseteq PS_C \cap S = P$, this concludes the proof. □

<u>Corollary 4.8</u>: If R is a commutative Noetherian ring, S has the second layer condition.

<u>Proof</u>: A Noetherian poly-AR ring has the second layer condition by [26] Proposition 7.13 and Corollary 7.24. □

<u>Corollary 4.9</u>: If R is a commutative Noetherian ring, S satisfies INC for (bimodule) link components.

<u>Proof</u>: The second layer condition guarantees this ([26] Theorem 7.26). □

<u>Proposition 4.10</u>: Let R be a commutative Noetherian ring, $S = R[x;\sigma]$. For any prime ideal P of S, the (bimodule) link component $\mathcal{L}(P)$ is K-homogeneous.

Proof: If $x \in P$ and $Q \sim P$ then $P = (P \cap R) + xS$ and $Q = P^{\sigma} = (P \cap R)^{\sigma} + xS$. Therefore $\mathrm{kdim}(S/Q) = \mathrm{kdim}(R/(Q \cap R)) = \mathrm{kdim}(R/(P \cap R)^{\sigma}) = \mathrm{kdim}(R/(P \cap R)) = \mathrm{kdim}(S/P)$.

If P is a lower prime then P has the AR-property (Proposition 4.6) and so $Q \sim P$ implies $P \subseteq Q$ by Lemma 3.28. Corollary 4.9 then gives $P = Q$ and so $\mathrm{kdim}(S/Q) = \mathrm{kdim}(S/P)$ trivially.

If P is an upper prime, say $P = [I, p(x)]$; and $Q \sim P$ then $Q \supseteq IS$ since IS has the AR-property. Hence $Q \cap R \supseteq I$ and $Q \cap R$ is a σ-ideal (Theorem 4.1, Proposition 1.13). Now if $Q \cap R \neq I$, then $Q \cap \mathcal{C}_R(I) \neq \emptyset$ (Lemma 2.3). However $\mathcal{C}_R(I)$ is an Ore set in S and $P \cap \mathcal{C}_R(I) = \emptyset$ by Corollary 2.6; hence $Q \cap \mathcal{C}_R(I) = \emptyset$ by Proposition 3.3. This is a contradiction and so $Q \cap R = I$. Now, as mentioned in the introduction to this chapter, since P is upper to I and R is commutative, S/IS is fully bounded Noetherian. Hence Krull dimension coincides with classical Krull dimension and we conclude that $\mathrm{kdim}(S/Q) = \mathrm{kdim}(S/IS) - 1 = \mathrm{kdim}(S/P)$, since P and Q are both upper to I. It follows that $\mathcal{L}(P)$ is K-homogeneous. □

The second paragraph of the above proof, combined with Corollary 3.17, proves the following result which we single out as:

Theorem 4.11: Let R be a commutative Noetherian ring. Then every lower prime ideal of S is classical. □

Another consequence of Proposition 4.10 is that second layer links and right links are the same in the Ore extension S:

Corollary 4.12: Let R be a commutative Noetherian ring, P and Q prime ideals of S. Then $Q \underset{2}{\sim} P$ if and only if $P \rightsquigarrow Q$ if and only if $Q \underset{\ell}{\rightsquigarrow} P$.

Proof: If $Q \underset{2}{\sim} P$ then $Q \subsetneq P$ is impossible, by Proposition 4.10. Apply Theorem 3.10 and its left-hand version. □

From now on, we shall use these notions interchangably and simply refer to a "link" for any of the three equivalent forms in Corollary 4.12.

Corollary 4.13: If R is a commutative Noetherian ring then S is weakly prime ideal invariant.

Proof: Combine Corollary 4.8, Proposition 4.10, and Proposition 3.22. □

§3. Upper Prime Ideals.

In this section, we concern ourselves with the upper prime ideals of the Ore extension $S = R[x;\sigma]$. Recall that a prime ideal P is an upper prime if $x \notin P$ and $P \neq (P \cap R)S$. In this case $I = P \cap R$ is σ-prime, σ has finite order on R/I, and P is completely determined by I and a polynomial $p(x)$ which is central and centrally-irreducible in $Q(R/I)[x;\sigma]$. We write $P = [I, p(x)]$.

Now $X = \{x^i \mid i \geq 0\}$ is an Ore set of regular elements in S (since $f(x)x^i = x^i f^{\sigma^{-1}}(x)$ for all $f(x) \in S$, $i \geq 0$.) and so we may form the partial quotient ring S_X of S. Observe that this localization

S_X coincides with the skew Laurent polynomial ring $R[x, x^{-1}; \sigma]$. Moreover, since $x \notin P$, $P \cap X = \emptyset$ and so P "survives" under the localization. By Proposition 3.3, the link component of P is the same in S as in S_X and so, without loss of generality, we may take P to be an upper prime ideal of $R[x, x^{-1}; \sigma]$ which we now denote by S for ease of notation.

It follows from the preceding section that the skew Laurent polynomial ring S is a poly-AR ring, has the second layer condition, has K-homogeneous link components, and is weakly prime ideal invariant. In addition, we found that all lower prime ideals of S are classical. We can now say more:

<u>Proposition 4.14</u>: If R is a commutative Noetherian ring then $S = R[x, x^{-1}; \sigma]$ is weakly K-symmetric.

<u>Proof</u>: Since R is commutative, the map $*: S \to S$ given by $rx^i \to r^{\sigma^{-1}} x^{-i}$ $(r \in R, \; i \in \mathbb{Z})$ is an involution. Hence the lattice of right ideals of S is isomorphic to the lattice of left ideals of S and it follows from this that $\mathrm{kdim}(S) = \ell\mathrm{dim}(S)$.

To show that S is weakly K-symmetric, it suffices to show that $\mathrm{kdim}(S/P) = \ell\mathrm{dim}(S/P)$ for every prime ideal P. If P is a lower prime ideal, then $P = (P \cap R)S$ and $S/P \cong (R/(P \cap R))[x, x^{-1}; \sigma]$ whence $\mathrm{kdim}(S/P) = \ell\mathrm{dim}(S/P)$ by the remarks in the first paragraph.

If P is an upper prime ideal then σ has finite order on $R/(P \cap R)$ and so $S/(P \cap R)S$ is fully bounded; since $(P \cap R)S \subsetneq P$ all the moreso S/P is fully bounded. Hence $\mathrm{kdim}(S/P) = \ell\mathrm{dim}(S/P)$, a well-known property of FBN rings. □

<u>Corollary 4.15</u>: If R is a commutative Noetherian ring, then $S = R[x, x^{-1}; \sigma]$ is K-symmetric.

<u>Proof</u>: S is a poly-AR ring by Theorem 4.7. It is p.i.i. by Corollary 4.13 and weakly K-symmetric by Proposition 4.14. Proposition 3.20 then shows that S is K-symmetric. $\square$

We now have at our disposal the results of [43]; the ones which we shall use were stated in §3.3. With these in hand, we return to the problem of determining the link component of an upper prime ideal.

Let $P = [I, p(x)]$ be an upper prime ideal of S ($I = P \cap R$ is σ-prime, $\bar{\sigma}$ has finite order n on R/I, and for $K = [Q(R/I)]^{\bar{\sigma}}$, $p(x)$ is irreducible in $K[x^n, x^{-n}]$.) We have already shown that $\mathcal{L}(P)$ is K-homogeneous and that if $P \rightsquigarrow Q$ then Q is an upper prime to I also (Proposition 4.10). Our goal is to determine the possibilities for Q in terms of P and σ: we will describe the irreducible polynomial $g(x) \in K[x^n, x^{-n}]$ which uniquely determines Q. By iteration and symmetry this will give an internal characterization of $\mathcal{L}(P)$.

We begin by making a few reductions. Since $\mathcal{C}_R(I)$ is trivially a σ-invariant Ore set in R (and hence in S), $P \rightsquigarrow Q$ in S if and only if $PS_I \rightsquigarrow QS_I$ in S_I (by Proposition 3.3). So without loss of generality R is semilocal with semimaximal σ-cyclic $J(R) = I$ (i.e. $I = \bigcap_{i=1}^{m} M_i$, M_i maximal, $M_i^{\sigma} = M_{i+1}$, $M_m^{\sigma} = M_1$) and P and Q are maximal ideals of S. Moreover, it is clear from the definition of a link that $P \rightsquigarrow Q$ in S if and only if $P/I^2S \rightsquigarrow Q/I^2S$ in

$S/I^2S \cong (R/I^2)[x, x^{-1}; \sigma]$ (use Theorem 3.23). Thus there is no loss of generality in assuming $I^2 = 0$. Note that R is now Artinian, by Hopkins' Theorem, and so $\mathrm{kdim}(S) = 1$. Note also that the field K remains unchanged by these reductions.

Having made these modifications, we now set out to determine precisely which prime ideals are linked to a given prime ideal P.

<u>Lemma 4.16</u>: Let R be a commutative Noetherian ring, $S = R[x, x^{-1}; \sigma]$, P and Q prime ideals of S. Suppose P is upper to the σ-prime ideal I of R and $P \rightsquigarrow Q$. Then either $Q = P$ or Q is a left Krull prime of the bimodule subfactor IS/IP of S.

<u>Proof</u>: Make the above reductions. By Theorem 3.23, $P \rightsquigarrow Q$ if and only if $Q \underset{2}{\sim} P$ via $(Q \cap P)/QP$. Since $IS \subseteq Q \cap P$ and IS has the AR-property (Proposition 4.6), Lemma 3.32 implies that either $Q/IS \underset{2}{\sim} P/IS$ or $Q \sim P$ via $IS/(QP \cap IS)$.

In the former case, we have a link between non-zero prime ideals of $\bar{S} = S/IS \cong (R/I)[x, x^{-1}; \sigma] = \bar{R}[x, x^{-1}; \sigma]$ which is therefore preserved under localization at the Ore set $\mathcal{C} = \mathcal{C}_{\bar{R}}(0)$ of $\bar{S}$ (Proposition 3.3). The partial quotient ring $\bar{S}_{\mathcal{C}}$ is then in the setting of Example 4.2 where there are no non-trivial link components. The same is therefore true of $\bar{S}$ and we conclude that $Q = P$ in this case.

In the latter case, since we may assume that P and Q are maximal ideals of S, the bimodules $IS/(QP \cap IS)$ and $(QP \cap IS)/IP$ are both Artinian and so we may splice together their composition series to obtain one for IS/IP. But then, since $\{Q\} = \ell t\text{-KSpec}(IS/(QP \cap IS))$

by Lemma 3.26(i), $Q \in \ell t\text{-KSpec}(IS/IP)$. □

By Lemma 3.26(ii), every left Krull prime Q' of IS/IP is bimodule-linked to P (since P is the only right Krull prime of IS/IP). We would like to know whether this is a second layer link (i.e. $P \rightsquigarrow Q'$). The next lemma says that this is the case.

<u>Lemma 4.17</u>: Let R , S and P be as above. If $Q \in \ell t\text{-KSpec}(IS/IP)$ then $P \rightsquigarrow Q$.

<u>Proof</u>: It suffices to find an element $e \in E_P$ such that $eQP = 0$ but $eP \neq 0$. By adapting the proof of [41] Theorem 15, we see that there is a chain of (right or left) links from P to Q (i.e. $Q \in \mathcal{L}(P)$) . This means that there is a non-zero $e \in E_P$ such that $P = P_1, P_2, \ldots, P_n = Q$ are the Krull primes of eS and we may assume that they are all maximal and distinct. By construction, $eQP_{n-1} \cdots P_2P = 0$. We may also take $e \in \mathrm{ann}_{E_P}(IS)$. Now it follows from Lemma 2.33 that $eP_{n-1}P_{n-2} \cdots P_2QP = e(QP_{n-1} \cdots P_2P + IS) = e(Q \ldots P) = 0$. Hence there exists $e' \in eP_{n-1} \cdots P_2$ such that $e'P \neq 0$, otherwise $eP_{n-1} \cdots P_2P = 0 \subseteq eQ$ would imply $P_{n-1} \cdots P_2P \subseteq Q$ and $P_i = Q$ for some $i = 1, \ldots, n - 1$, a contradiction. Since $e'QP = 0$, this completes the proof. □

The last two lemmas effectively characterize those prime ideals Q such that $P \rightsquigarrow Q$ as the left Krull primes of IS/IP . For reference, let us summarize this as:

Proposition 4.18: Let R be a commutative Noetherian ring, $S = R[x, x^{-1}; \sigma]$, P and Q prime ideals of S with P upper to the σ-prime ideal I of R. Then $P \rightsquigarrow Q$ if and only if either $Q = P$ or $Q \in \ell t\text{-KSpec}(IS/IP)$. □

The foregoing provides us with an abstract characterization of the link component of an upper prime ideal of S. We will now make this concrete by describing these prime ideals explicitly.

After the initial reductions, we have a maximal ideal P of S which is upper to the semimaximal σ-cyclic ideal I of R and R is semilocal Artinian with $J(R) = I$. Also $R/I = Q(R/I)$ and σ has finite order m on R/I. We set $K = (R/I)^{\bar{\sigma}}$, the fixed subfield of R/I. $I = I/I^2$ is therefore a finite-dimensional K-vector space with basis $a_1, \ldots, a_n$ say. Setting $t = x^m$, we have $P = [I, p(t)]$ for some irreducible polynomial $p(t) \in K[t, t^{-1}]$. By multiplying by an appropriate power of t, we may take $p(t) \in K[t]$. (Note that every irreducible $p(t) \in K[t]$ gives rise to such a prime ideal P by Theorem 2.40). Since $I^\sigma = I$, for each $i = 1, \ldots, n$ we must have

$a_i^\sigma = \sum_{j=1}^{n} u_{ij} a_j$ with $u_{ij} \in K$. Set $\sigma_I = (u_{ij}) \in M_n(K)$ and let $\tau = \sigma_I^m$. Denote by $\mu_\tau(t)$ the minimum polynomial of τ in $K[t]$.

The prime ideals we want to characterize are the left Krull primes of IS/IP - these are precisely the minimal prime ideals over $\ell_S(IS/IP)$ and they are all maximal. Indeed, these ideals are all upper to I and are thus of the form $[I, q(t)]$ for $q(t)$ irreducible in $K[t]$. Let $f(t) = \sum_{i=0}^{d} f_i t^i$ be the product of the $q(t)$'s for the left

Krull primes in question. It follows that $f(t)IS \subseteq IP = IS(p(t)S + IS) = ISp(t)$.

Let $p(t) = (t - \alpha_1)\dots(t - \alpha_s)$ where $\alpha_1,\dots,\alpha_s$ are the zeroes of $p(t)$ in $\bar{K}$, the algebraic closure of K. Let $\mu_\tau(t) = (t - \varepsilon_1)\dots(t - \varepsilon_r)$ where $\varepsilon_1,\dots,\varepsilon_r$ are eigenvalues of τ in $\bar{K}$. For every $j = 1,\dots,n$ we obtain $f(t)a_j = \sum_{i=0}^{d} f_i t^i a_j = \sum_{i=0}^{d} f_i a_j^{\sigma^{mi}} t^i = \sum_{i=0}^{d} f_i \left(\sum_{k=1}^{n} u_{jk}^{mi} a_k\right) t^i = \sum_{k=1}^{n} a_k \sum_{i=0}^{d} f_i u_{jk}^{mi} t^i \in ISp(t)$ by assumption. (Here $u_{jk}^{mi} = (\tau^i)_{jk}$.) Consequently, for $\alpha \in \{\alpha_1,\dots,\alpha_s\}$ we have $f(\alpha)a_j = \sum_{k=1}^{n} a_k \sum_{i=0}^{d} f_i u_{jk}^{mi} \alpha^i = 0$ for $j = 1,\dots,n$.

Since $a_1,\dots,a_n$ are a basis for I over K (and $\bar{K}$) we have $\sum_{i=0}^{d} f_i u_{jk}^{mi} \alpha^i = 0$ for $1 \le j \le n$, $1 \le k \le n$. Hence the matrix $\sum_{i=0}^{d} f_i u^{mi} \alpha^i = \sum_{i=0}^{d} f_i \tau^i \alpha^i = \sum_{i=0}^{d} f_i (\tau\alpha)^i = 0$. Therefore, by the Cayley-Hamilton Theorem, $\mu_{\tau\alpha}(t)$ must divide $f(t)$ for all $\alpha \in \{\alpha_1,\dots,\alpha_s\}$. Thus the least common multiple of the $\mu_{\tau\alpha_i}(t)$, $1 \le i \le s$, must divide $f(t)$ which in turn must divide the product of the characteristic polynomials $\chi_{\tau\alpha_i}(t)$ (since this product is in $\ell_s(IS/IP)$). Hence the distinct irreducible factors are the same in each case and so the prime ideals which are linked to P are therefore completely determined by the irreducible factors of $f(t)$. These are also the same as the irreducible factors of $\prod_{i=1}^{s} \mu_{\tau\alpha_i}(t)$. Note that

$\mu_{\tau\alpha_j}(t) = (t - \varepsilon_1\alpha_j)(t - \varepsilon_2\alpha_j)\ldots(t - \varepsilon_r\alpha_j)$. If we set $f_{ij}(t) = \mathrm{irr}_K(\varepsilon_i\alpha_j)$, the irreducible polynomial for $\varepsilon_i\alpha_j$ in $K[t]$, then the distinct $f_{ij}(t)$ are the polynomials we want.

We summarize the above discussion in the following theorem:

<u>Theorem 4.19</u>: Let R be a commutative Noetherian ring, $S = R[x;\sigma]$, P and Q distinct prime ideals of S with $P = [I, p(t)]$ upper to the σ-prime ideal I of R . Then $P \rightsquigarrow Q$ if and only if there exists a zero α of $p(t)$ in $\bar{K}$ and an eigenvalue ε of σ_I^m in $\bar{K}$ such that $Q = [I, \mathrm{irr}_K(\varepsilon\alpha)]$. □

<u>Corollary 4.20</u>: Let R be a commutative affine algebra over an algebraically closed field k , $S = R[x;\sigma]$, I a maximal σ-prime ideal of R , P and Q distinct maximal ideals of S with $P = [I, t - \alpha]$. Then $P \rightsquigarrow Q$ if and only if there exists an eigenvalue ε of σ_I^m in k such that $Q = [I, t - \varepsilon\alpha]$.

<u>Proof</u>: If I is a maximal σ-prime ideal then it is semimaximal and σ-cyclic. Hence the field K is an algebraic extension of the base field k and so $K = k$. P necessarily has the form $[I, t - \alpha]$ if it is upper to I . Theorem 4.19 then gives the result. □

<u>Corollary 4.21</u>: In the setting of Theorem 4.19, P has a finite link component if and only if all eigenvalues of σ_I are roots of unity.

Proof: Iterating the procedure of Theorem 4.19 serves to multiply eigenvalues of σ_I: $Q \in L(P)$ if and only if $Q = [I, \mathrm{irr}_K(\prod_{i \in F} \varepsilon_i^{n_i} \alpha_j)]$ where $\{\varepsilon_i | i \in F\}$ is some finite subset of the set of eigenvalues of σ_I^m and $n_i \in \mathbb{Z}$. Clearly there are only finitely many possibilities for Q if and only if all eigenvalues of σ_I^m, and equivalently of σ_I, are roots of unity. □

In general, it may be difficult to determine the zeros of the polynomial $p(t)$ and the eigenvalues of σ_I^m. The statement of Theorem 4.19 is therefore unsatisfactory inasmuch as it leads one to believe that our technique is apparently dependent on a knowledge of those quantities. We can at least remove this dependence from the statement of Theorem 4.19.

Theorem 4.22: Let R be a commutative Noetherian ring, $S = R[x;\sigma]$, and $P = [I, p(t)]$ upper to the σ-prime ideal I of R (where $t = x^m$ for m the order of σ on R/I). Set $\tau = \sigma_I^m \in M_n(K)$. Then the prime ideals of S other than P which are linked to P are determined by the irreducible factors over K of $\det(\tau^d(p(\tau^{-1}t)))$ where d is the degree of $p(t)$.

Proof: Let $p(t) = (t - \alpha_1)(t - \alpha_2)\ldots(t - \alpha_\alpha)$ in $\bar{K}[t]$. By Theorem 4.19, the irreducible factors of $\prod_{i=1}^{d} \mu_{\tau\alpha_i}(t)$ over K determine the prime ideals of S which are linked to P. However, these are precisely the irreducible factors of the product of characteristic polynomials $\prod_{i=1}^{d} \chi_{\tau\alpha_i}(t) = \prod_{i=1}^{d} \det(tE - \tau\alpha_i) = \det(\prod_{i=1}^{d} (tE - \tau\alpha_i)) = f(t)$, say. (Here E denotes the identity matrix of $M_n(\bar{K})$.) Now $\prod_{i=1}^{d} (tE - \tau\alpha_i) = \tau^d \prod_{i=1}^{d} (\tau^{-1}t - \alpha_i D) = \tau^d p(\tau^{-1}t)$. Therefore $f(t) = \det(\tau^d p(\tau^{-1}t))$ and the result follows. □

§4. Examples.

In this section we give several examples to illustrate the techniques of the preceding section.

Example 4.23: Let $R = k[y]$, k a field, $\alpha, \beta \in k$, and let σ be the automorphism of R given by $y^\sigma = \alpha y$. Let $S = R[x;\sigma]$. $I = yR$ is a prime σ-ideal of R, and $P = [I, x - \beta] = IS + (x - \beta)S$ is a prime ideal of S, upper to I. σ has order 1 on $R/I \cong k$ and the only eigenvalue of σ_I is α. The link component of P is $\mathcal{L}(P) = \{[I, x - \alpha^i\beta] \mid i \in \mathbf{Z}\}$ which is finite if and only if α is a root of unity. Indeed, if this is so then σ has finite order and S actually has enough clans.

Example 4.24: Let $R = k[y,z]$, $y^\sigma = y$, $z^\sigma = y + z$, and take $I = yR$. Then σ has order 1 on R/I and σ_I is the identity. Hence all upper primes $P = [I, p(x)]$ of S are classical. In fact, it can be shown that for any σ-prime ideal I of R, 1 is the only eigenvalue of σ_I. Hence all prime ideals of S which do not contain x are classical.

If we modify this example by taking $R = k[y, y^{-1}, z, z^{-1}]$, $y^\sigma = y$, $z^\sigma = yz$, and $S = R[x, x^{-1}; \sigma]$ then it can be shown that every prime ideal of S is classical. Note that in this case $S \cong kG$, the group algebra over k of the semidirect product $G = \mathbb{Z}^2 \times_\sigma \mathbb{Z}$. G is a nilpotent group of class 2 and group ring techniques will also show that every prime ideal of S is classical.

Example 4.25: Let $R = \mathbb{Q}[y,z]$, σ the automorphism which interchanges y and z. Since σ has order 2, $S = R[x;\sigma]$ has enough clans. Take $I_1 = (y + z)R$, $I_2 = (y - z)R$ and $I_3 = (y - \alpha)R + (z - \alpha)R$. These are all σ-invariant prime ideals of R. Then σ has order 2 on R/I_1, $K = \mathbb{Q}(y^2)$ and $\sigma_{I_1}^2$ is the identity matrix in $M_2(K)$; all upper primes to I_1 are thus classical. The order of σ on R/I_3 is 1, $K = \mathbb{Q}$ and $\sigma_{I_3} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in M_2(\mathbb{Q})$ with eigenvalues 1 and -1; hence the prime ideal $[I_3, x - \alpha]$ is linked to $[I_3, x + \alpha]$ and these form a two-element clan. Thus not all link components of upper primes are of the same length. In fact, there can be clans of different cardinalities lying over the same σ-prime ideal. For example, σ has order 1 on R/I_2, $K = \mathbb{Q}(y)$ and $\sigma_{I_2} = (-1)$. Let $p_1(x) = x - y$ and $p_2(x) = x^2 - y$, two irreducible polynomials in $K[x]$. Then $[I_2, p_1] \longleftrightarrow [I_2, x + y]$ gives a two-element clan while $[I_2, p_2]$ is classical since $p_2(x) = \operatorname{irr}_K(\sqrt{y}) = \operatorname{irr}_K(-\sqrt{y})$.

Example 4.26: The preceding example could have been analyzed using the fact that S was module-finite over its center. There σ had finite order and hence the eigenvalues of σ_I were trivially roots of unity. In this example, σ has infinite order but σ_I has eigenvalues which are roots of unity.

Take $R = Q[y,z]$, $y^\sigma = -y$, $z^\sigma = y - z$, and $I = yR + zR$. Then σ has order 1 on R/I, $K = Q$, and $\sigma_I = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \in M_2(K)$ for the K-basis $\{\bar{y}, \bar{z}\}$ of I/I^2. Then, for example, $[I, x + 2] \longleftrightarrow [I, x - 2]$ gives a two-element clan while $[I, x^2 - 2]$ is classical.

Example 4.27: This example shows that there may be both classical upper primes and upper primes with an infinite link component. Indeed, we shall produce a classical maximal ideal which contains a prime ideal with an infinite link component.

Set $R = Q[y, y^{-1}, z]$, $y^\sigma = y$, $z^\sigma = yz$, $I = zR \subseteq M = (y - 1)R + zR$ and $S = R[x;\sigma]$. I is a prime σ-ideal and M a maximal σ-ideal of R. $R/I \cong Q[y]$ and $R/M \cong Q$ so that σ is trivial in each factor ring. Now $\{\bar{z}\}$ is a basis for the $Q(y)$-space I/I^2; then $\sigma_I(\bar{z}) = \overline{yz} = y\bar{z}$ so that $\sigma_I = (y)$. Also $\{\overline{y - 1}, \bar{z}\}$ is a basis for the Q-space M/M^2; then $\sigma_M(\overline{y - 1}) = \overline{y - 1}$ and $\sigma_M(\bar{z}) = \overline{yz} = \overline{(y - 1)z} + \bar{z} = \bar{z}$ so that σ_M is the identity matrix of $M_2(Q)$. Now let $P = [I, x - \alpha]$ and $Q = [M, x - \alpha]$ be uppers to I and M respectively for $\alpha \in Q$. Then $P \subseteq Q$, Q is classical since 1 is the only eigenvalue of σ_M, and $\mathcal{L}(P) = \{[I, x - \alpha y^i] \mid i \in \mathbb{Z}\}$.

<u>Example 4.28</u>: This sort of behaviour - a classical prime ideal which contains a prime ideal with an infinite component - can also happen in a group ring. Using the techniques of this chapter, it is not too difficult to construct such an example.

Let $R = Q[y_i, y_i^{-1} \mid 1 \le i \le 4]$ and take σ to be the automorphism of R given by $y_1^\sigma = y_1^2 y_2$, $y_2^\sigma = y_1 y_2$, $y_3^\sigma = y_3$, $y_4^\sigma = y_4$. Let $I = (y_1 - 1)R + (y_2 - 1)R$ and $M = \sum_{i=1}^{4} (y_i - 1)R$, σ-prime ideals of R. Then σ is trivial on R/I and we compute $(y_1 - 1)^\sigma = y_1^2 y_2 - 1 = (y_1^2 - 1)(y_2 - 1) + (y_1 + 1)(y_1 - 1) + (y_2 - 1)$ and $(y_2 - 1)^\sigma = y_1 y_2 - 1 = (y_1 - 1)(y_2 - 1) + (y_1 - 1) + (y_2 - 1)$. Hence, with respect to the $Q(y_3,y_4)$-basis $\{\overline{y_1 - 1}, \overline{y_2 - 1}\}$ of I/I^2, $\sigma_I(\overline{y_1 - 1}) = 2(\overline{y_1 - 1}) + (\overline{y_2 - 1})$ and $\sigma_I(\overline{y_2 - 1}) = (\overline{y_1 - 1}) + (\overline{y_2 - 1})$ and so $\sigma_I = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ with eigenvalues $(3 \pm \sqrt{5})/2$. Let $S = R[x, x^{-1}; \sigma]$ and take $p(x) = (y_3 - 1)x + (y_4 - 1) \in Q(y_3, y_4)[x]$. Then $P = [I, p(x)]$ has an infinite link component but $P \subseteq MS$ which is classical by Theorem 4.11. Here $S \cong QG$ where $G = \mathbf{Z}^4 \times_\sigma \mathbf{Z}$.

§5. <u>Localizing at an Infinite Link Component</u>.

If R is a Noetherian ring satisfying the second layer condition and P is a prime ideal of R belonging to a clan $\mathcal{C} = \{P = P_1, P_2, \ldots, P_n\}$, then $\mathcal{C}$ is the link component $\mathcal{L}(P)$ of P and $\bigcap_{i=1}^{n} P_i$ is a classical semiprime ideal (Theorem 3.16, Corollary 3.17).

In particular, $\Sigma = \bigcap_{i=1}^{n} \mathcal{C}(P_i)$ is an Ore set in R. If a prime ideal P has an infinite link component $\mathcal{L}(P)$, it is not known in general whether $\Sigma = \bigcap_{Q \in \mathcal{L}(P)} \mathcal{C}(Q)$ is an Ore set; affirmative results have only been obtained in certain special cases (cf. [4], [43]). In this section, we shall show that for Ore extensions of commutative Noetherian algebras, it is frequently possible to localize at an infinite link component.

Let $\mathcal{P}$ be a non-empty set of prime ideals in a Noetherian ring R and set $\Sigma = \bigcap_{P \in \mathcal{P}} \mathcal{C}(P)$. $\mathcal{P}$ is said to satisfy the <u>right intersection condition</u> if for any right ideal A, $A \cap \mathcal{C}(P) \neq \emptyset$ for every $P \in \mathcal{P}$ if and only if $A \cap \Sigma \neq \emptyset$. The left intersection condition is analogous.

$\mathcal{P}$ is said to be <u>right classical</u> if Σ is a right Ore set, R_Σ / PR_Σ is simple Artinian for every $P \in \mathcal{P}$, every right primitive ideal of R_Σ is of the form PR_Σ for some $P \in \mathcal{P}$, and for any simple right R_Σ-module M, $E_{R_\Sigma}(M)$ is the union of its socle sequence. Again, left classical is defined analogously. (cf. [26] Chapter 8.)

Our interest is in knowing when a link component is classical. We will use the following result:

<u>Theorem 4.29</u>: ([26] Theorem 8.36) Let P be a prime ideal in a Noetherian ring R. $\mathcal{L}(P)$ is classical if and only if $\mathcal{L}(P)$ satisfies the intersection condition and the second layer condition. □

For Ore extensions of commutative Noetherian rings, we have already established the second layer condition (Corollary 4.8). To obtain the intersection condition, we need the following:

Lemma 4.30: ([26] Theorem 8.44) Let R be a Noetherian P.I. algebra over an infinite field F. If $\mathcal{P}$ is a non-empty set of prime ideals of R with $\mathrm{card}(\mathcal{P}) < \mathrm{card}(F)$, then $\mathcal{P}$ satisfies the intersection condition. □

For an Ore extension of a commutative Noetherian algebra over an uncountable field, it is now easy to prove that link components are classical.

Theorem 4.31: Let R be a commutative Noetherian algebra over an uncountable field F and let $S = R[x;\sigma]$. Then every link component in S is classical.

Proof: Lower prime ideals are classical by Theorem 4.11. More generally, any finite link component is a clan and hence classical (Corollary 3.17). In any case, a link component $\mathcal{L}(P)$ of S is at most countable (Theorems 4.1, 4.19) and so $\mathrm{card}(\mathcal{L}(P)) < \mathrm{card}(F)$. We need only show that if A is a right ideal with $A \cap \mathcal{C}(Q) \neq \emptyset$ for all $Q \in \mathcal{L}(P)$, then $A \cap \Sigma \neq \emptyset$ where $\Sigma = \bigcap_{Q \in \mathcal{L}(P)} \mathcal{C}(Q)$. There are two cases: $x \in P$ and P an upper prime ideal.

In the first case $\mathcal{L}(P) = \{P^{\sigma^i} \mid i \in \mathbb{Z}\}$ and $x \in P^{\sigma^i}$ for all i. Therefore, for every $Q \in \mathcal{L}(P)$, $\bar{A} \cap \mathcal{C}(\bar{Q}) \neq \emptyset$ in $\bar{S} = S/xS \cong R$. Since R is commutative, Lemma 4.30 applies trivially to give $\bar{A} \cap \bar{\Sigma} \neq \emptyset$ and hence $A \cap \Sigma \neq \emptyset$.

In the second case, if P is upper to I, say, then so is every $Q \in \mathcal{L}(P)$ by Proposition 4.10. We also know that σ necessarily has finite order on R/I and consequently $\bar{S} = S/IS$ is finitely-generated as a module over its center; $\bar{S}$ is therefore a Noetherian P.I. algebra over F. For every $Q \in L(P)$, $\bar{A} \cap \mathcal{C}(\bar{Q}) \neq \emptyset$ and so we may use Lemma 4.30 to obtain $\bar{A} \cap \bar{\Sigma} \neq \emptyset$ and thus $A \cap \Sigma \neq \emptyset$.

This establishes the intersection condition and so we may appeal to Theorem 4.29 to complete the proof. □

APPENDIX A

GOLDIE CONDITIONS IN ORE EXTENSIONS

Goldie's Theorem gives necessary and sufficient conditions for a ring R to have a semisimple Artinian classical right quotient ring. These "Goldie conditions" are, in a sense, finiteness conditions on R and they are satisfied, for example, if R is semiprime and right Noetherian. Now, the Hilbert Basis Theorem (1.7) shows that an Ore extension of a right Noetherian ring R is again right Noetherian. We shall show in this section that, even if R is not right Noetherian, the Goldie conditions are preserved by taking Ore extensions.

A ring R has <u>finite right Goldie dimension</u> if it contains no infinite direct sum of non-zero right ideals; if it is finite, we denote the right Goldie dimension of R by $\mathrm{Gdim}(R)$. $\mathrm{Gdim}(R) = n$ if and only if R contains an essential direct sum of n uniform right ideals. A right ideal of R is a <u>right annihilator</u> if it is of the form $r_R(A) = \{r \in R | ar = 0$ for all $a \in A\}$, for some non-empty subset A of R. The right annihilator in R of an element $a \in R$ is denoted by $r_R(a)$, or $r(a)$ if no confusion will arise. A ring R is called <u>right Goldie</u> if it has finite right Goldie dimension and the ascending chain condition on right annihilators. The <u>right singular ideal</u> $Z(R)$ of a ring R is defined as $Z(R) = \{a \in R | r_R(a)$ is an essential right ideal$\}$ and is a two-sided ideal. If $Z(R) = 0$, R is said to be <u>right nonsingular</u>.

Clearly, every right Noetherian ring is right Goldie; the converse is not true since any commutative integral domain is trivially right Goldie. Goldie's Theorem states that R has a semisimple Artinian classical right quotient ring if and only if R is semiprime right Goldie. Moreover, a semiprime ring R is right Goldie if and only if R has finite right Goldie dimension and is right nonsingular. What we shall in fact prove is that $R[x;\sigma]$ is semiprime right Goldie if and only if R is σ-semiprime right Goldie. We accomplish this by proving that right Goldie dimension and nonsingularity are each preserved by an Ore extension.

These results, or variants of them, appear to be known [54]. However, they are not easily accessible in the literature so we have included them here for reference. Although we cannot claim the results, our proofs may be ~~new~~. Versions of these results for commutative coefficient rings and monomorphisms σ may be found in [20] - this paper was, in fact, our motivation for trying to find proofs of these theorems which worked for non-commutative coefficient rings.

<u>Definition A.1</u>: A polynomial $f(x) = f_0 + f_1x + \ldots + f_nx^n \in S$ $(f_n \neq 0)$ satisfies condition (*) if $r_R(f_i^{\sigma^{-1}}) = r_R(f_n^{\sigma^{-n}})$ for all i such that $f_i \neq 0$.

<u>Lemma A.2</u>: Let $f(x)$ be a non-zero polynomial in S. There exists $r \in R$ such that $f(x)r$ is non-zero and satisfies (*).

Proof: Let $f(x) = f_0 + f_1x + \ldots + f_nx^n$, $f_n \neq 0$. If $f(x)$ already satisfies (*) we may take $r = 1$. Otherwise, for some $i,j \in \{0, 1, \ldots, n\}$, $r(f_i^{\sigma^{-i}}) \neq r(f_j^{\sigma^{-j}})$ and so there exists $r_1 \in R$ such that $f_i^{\sigma^{-i}} r_1 = 0$ but $f_j^{\sigma^{-j}} r_1 \neq 0$. Consequently $f(x)r_1 \neq 0$ and has fewer non-zero coefficients. If $f(x)r_1$ does not satisfy (*), we repeat the procedure. Eventually we find an integer $k \geq 1$ and $r_1, \ldots, r_k \in R$ such that $f(x)r_1r_2\ldots r_k$ is a non-zero polynomial satisfying (*). □

Lemma A.3: Let $f(x) = f_0 + \ldots + f_nx^n$ be a non-zero polynomial satisfying (*) and let $g(x) = g_0 + \ldots + g_mx^m$. Then $f(x)g(x) = 0$ if and only if

$$(**) \qquad f_i g_j^{\sigma^i} = 0 \quad \text{for } 0 \leq i \leq n, \ 0 \leq j \leq m.$$

Proof: $f(x)g(x) = \sum_{k=0}^{n+m} \left(\sum_{i+j=k} f_i g_j^{\sigma^i} \right) x^k$ so the sufficiency of the condition (**) is obvious. Conversely, suppose $f(x)g(x) = 0$. We will prove the necessity of condition (**) by induction on j. For $j = 0$, we note that $f_0g_0 = 0$ implies $g_0 \in r(f_0) = r(f_i^{\sigma^{-i}})$ for all i such that $f_i \neq 0$. Hence $f_i^{\sigma^{-i}} g_0 = 0$ and so $f_i g_0^{\sigma^i} = 0$ for all i. Assume the condition is true for $j < k$. The coefficient of x^k in $f(x)g(x)$ is $\sum_{i+j=k} f_i g_j^{\sigma^i} = 0$. By the induction hypothesis, $f_i g_j^{\sigma^i} = 0$ for $j < k$, for all i, and so $f_0g_k = 0$. Thus $g_k \in r(f_0) = r(f_i^{\sigma^{-i}})$

for all i and, as above, we have $f_i g_k^{\sigma^i} = 0$ for all i. The result follows. □

<u>Theorem A.4</u>: Let I be a uniform right ideal of R. Then IS is a uniform right ideal of S.

<u>Proof</u>: If not, then there exist non-zero polynomials $f(x), g(x) \in S$ such that $f(x)S + g(x)S$ is a direct sum and $f(x)S \oplus g(x)S \subseteq IS$. By Lemma A.2, we may assume that $f(x)$ and $g(x)$ satisfy (*). We may also assume that $\deg(f(x)) \leq \deg(g(x))$ and that, for a fixed $f(x)$, $g(x)$ has been chosen of minimal degree. Let $f(x) = \sum_{i=0}^{n} f_i x^i$ and $g(x) = \sum_{j=0}^{m} g_j x^j$ with all f_i, $g_j \in I$. Since I is uniform, there exist $a, b \in R$ such that $f_n a^{\sigma^n} = g_m b^{\sigma^m} \neq 0$. Set $h(x) = f(x)\ a\ x^{m-n} - g(x)b$. Then $h(x) \neq 0$ (since $f(x)S \cap g(x)S = 0$) and $\deg(h(x)) < \deg(g(x))$.

Now if $f(x)S + h(x)S$ is a direct sum then, by Lemma A.2, we can find some $r \in R$ such that $h_1(x) = h(x)r \neq 0$ and satisfies (*). We then have $f(x)S + h_1(x)S$ is a direct sum, $f(x)$ and $h_1(x)$ satisfy (*), and $\deg(h_1(x)) \leq \deg(h(x)) < \deg(g(x))$. This contradicts the minimality of $g(x)$. Hence $f(x)S \cap h(x)S \neq 0$ and there exist $s(x), t(x) \in S$ such that $f(x)s(x) = h(x)t(x) \neq 0$.

If $g(x)bt(x) = 0$, then $bt(x)$ satisfies condition (**) with respect to $g(x)$: if $t(x) = \sum_{j=0}^{\ell} t_j x^j$, then for $0 \leq i \leq m$, $0 \leq j \leq \ell$, $g_i(bt_j)^{\sigma^i} = 0$. Now $g(x)b = \sum_{i=0}^{m} g_i b^{\sigma^i} x^i$, so $t(x)$ satisfies (**) with respect to $g(x)b$. Then $0 \neq h(x)t(x) = (f(x)ax^{m-n} - g(x)b)t(x) = f(x)ax^{m-n}t(x)$. However, $r(f_n a^{\sigma^n}) = r(g_m b^{\sigma^m})$ and from (**), $t_j \in r(g_i^{\sigma^{-i}} b)$ for all i, j. Thus $t_j \in r(g_m^{\sigma^{-m}} b) = r(f_n^{\sigma^{-m}} a^{\sigma^{n-m}})$ and so $a^{\sigma^{n-m}} t_j \in r(f_n^{\sigma^{-m}}) = r(f_n^{\sigma^{-n}})^{\sigma^{n-m}} = r(f_n^{\sigma^{-i}})^{\sigma^{n-m}}$ for all $0 \le i \le n$, $0 \le j \le \ell$. It follows that $f_i^{\sigma^{-i}} a t_j^{\sigma^{m-n}} = 0$ for all i, j. Consequently, $f(x)ax^{m-n}t(x) = \sum_{k=0}^{m+\ell} (\sum_{i+j=k} f_i a^{\sigma^i} t_j^{\sigma^{i+m-n}})x^k = 0$, a contradiction. Hence $g(x)bt(x) \neq 0$.

We now see that $0 \neq g(x)bt(x) = f(x)ax^{m-n}t(x) - h(x)t(x) = f(x)ax^{m-n}t(x) - f(x)s(x) = f(x)[ax^{m-n}t(x) - s(x)] \in f(x)S \cap g(x)S$, contradicting our initial assumption that $f(x)S + g(x)S$ was a direct sum. It follows that IS is a uniform right ideal of S. $\square$

It is well-known that if R is a domain, then R has finite right Goldie dimension if and only if $\mathrm{Gdim}(R) = 1$ if and only if R is a right Ore domain. It is also well-known that an Ore extension of a right Ore domain is again a right Ore domain (cf. [11], [18]). We will now generalize this result to show that Goldie dimension is preserved by taking Ore extensions. First a lemma.

<u>Lemma A.5</u>: Let I be a right ideal of R. Then I is essential in R if and only if IS is essential in $S = R[x;\sigma]$.

<u>Proof</u>: Assume that I is essential in R and let $s = s_0 + s_1x + \ldots + s_nx^n$ be a non-zero element of S, $s_n \neq 0$. There

exists $a_n \in R$ such that $0 \neq s_n a_n^{\sigma^n} \in I$. If $s_{n-1}a_n^{\sigma^{n-1}} \neq 0$, then there exists $a_{n-1} \in R$ such that $0 \neq s_{n-1}a_n^{\sigma^{n-1}} a_{n-1}^{\sigma^{n-1}} \in I$. Continuing in this manner, we eventually obtain $a_0, a_1, \ldots, a_n \in R$ such that for $a = a_n a_{n-1} \cdots a_0$, $0 \neq s(x)a = \sum_{i=0}^{n} s_i a^{\sigma^i} x^i \in IS$. Hence IS is essential in S. The converse is trivial. □

Theorem A.6: R has finite Goldie dimension if and only if $S = R[x;\sigma]$ has finite Goldie dimension. In this case, $\mathrm{Gdim}(S) = \mathrm{Gdim}(R)$.

Proof: If $E = \sum_{i \in I} \oplus E_i$ is a direct sum of right ideals of R, then the sum $ES = \sum_{i \in I} E_i S$ is also direct. Hence if S has finite right Goldie dimension, so must R.

Conversely, suppose $\mathrm{Gdim}(R) = n$ and choose $E = \sum_{i=1}^{n} \oplus E_i$ to be a direct sum of uniform right ideals E_i of R, with E essential in R. As above, the sum $ES = \sum_{i=1}^{n} E_i S$ is direct and the preceding lemma shows that ES is essential in S. Theorem A.4 gives us that $E_i S$ is a uniform right ideal of S for $1 \leq i \leq n$. Therefore $\mathrm{Gdim}(S) = n$. □

This partially sharpens a result of Irving who proved a version of the above theorem in the case where R is a commutative σ-cyclic ring, σ a monomorphism. (R is <u>σ-cyclic</u> if $0 = P_1 \cap \ldots \cap P_t$ with the P_i distinct prime ideals such that $P_i^{\sigma} = P_{i+1}$, $P_t^{\sigma} = P_1$. If R is right Noetherian, σ-cyclic is equivalent to σ-prime by Proposition 1.5.)

Corollary A.7: ([20], Theorem 6.2) Let R be a commutative σ-cyclic ring with $\mathrm{Gdim}(R) = n$. Then $\mathrm{Gdim}(S) = n$ for $S = R[x;\sigma]$. □

Proposition A.8: $Z(S) = Z(R)S$.

Proof: Let $f(x) = \sum_{i=0}^{n} f_i x^i \in Z(R)S$. Then for all $i = 0, 1, \ldots, n$, $r_R(f_i)$ is an essential right ideal of R. Hence $r_R(f_i^{\sigma^{-i}}) = (r_R(f_i))^{\sigma^{-i}}$ is essential and therefore so is $I = \bigcap_{i=0}^{n} r_R(f_i^{\sigma^{-i}})$. Lemma A.5 implies that IS is essential in S but, since $IS \subseteq r_S(f(x))$, this shows that $r_S(f(x))$ is essential in S and hence $f(x) \in Z(S)$.

To get the reverse inclusion, let $f(x) \in Z(S)$ and write $f(x) = g(x) + h(x)$ where $g(x) \in Z(R)S \subseteq Z(S)$ and $h(x) \in (R \sim Z(R))S$. Then $h(x) = f(x) - g(x) \in Z(S)$. Let $h(x) = f_0 + \ldots + f_m x^m$, $f_m \neq 0$, so that $r_R(f_m)$ is not essential in R. Moreover, $r_R(f_m^{\sigma^{-m}})$ is not essential and there exists a non-zero right ideal A of R such that $r_R(f_m^{\sigma^{-m}}) \cap A = 0$. In particular, for any non-zero $a \in A$, $r_R(f_m^{\sigma^{-m}}) \cap aR = 0$. Suppose $r_S(h(x)) \cap aS \neq 0$ and let $s(x) = \sum_{i=0}^{t} s_i x^i$ with $as(x) \neq 0$ and $h(x)as(x) = 0$. Without loss of generality $as_t \neq 0$. Then $f_m a^{\sigma^m} s_t^{\sigma^m} = 0$, $as_t \in r_R(f_m^{\sigma^{-m}})$, and hence $r_R(f_m^{\sigma^{-m}}) \cap aR \neq 0$, a contradiction. Therefore $r_S(h(x)) \cap aS = 0$, contradicting the fact that $h(x) \in Z(S)$. So we must have $h(x) = 0$ and $f(x) = g(x) \in Z(R)S$. □

<u>Corollary A.9</u>: R is right nonsingular if and only if S is right nonsingular. □

<u>Corollary A.10</u>: ([20], Theorem 6.3) Let R be a commutative σ-cyclic ring. Then $S = R[x;\sigma]$ is right nonsingular.

<u>Proof</u>: A commutative σ-cyclic ring is clearly semiprime, hence nonsingular ([30], page 108). □

With Goldie's Theorem in mind, we can now prove:

<u>Theorem A.11</u>: $S = R[x;\sigma]$ is a semiprime right Goldie ring if and only if R is a σ-semiprime right Goldie ring.

<u>Proof</u>: If S is semiprime right Goldie, then R is σ-semiprime (Proposition 1.11) and S is right nonsingular with finite right Goldie dimension [14]. Hence R is right nonsingular with finite right Goldie dimension (Corollary A.9, Theorem A.6) and it is well-known that this implies that R is right Goldie (see, for example, [9] Lemma 1.14).

Conversely if R is σ-semiprime right Goldie, then S is semiprime by Proposition 1.11. Since R is actually semiprime [34], it is right nonsingular with finite right Goldie dimension. It follows that S inherits these properties and hence is right Goldie. □

APPENDIX B

LOCALIZATION AND EXTENSION OF THE BASE FIELD

We proved in Theorem 4.11 that in an Ore extension $S = R[x;\sigma]$ of a commutative Noetherian ring R every lower prime ideal IS is classical. In this section, we will give an alternate proof of this result in the special case where R is a commutative affine algebra over a field k of characteristic zero. The technique of the proof relies heavily on results of Yammine [55], [56] and may be of independent interest.

Recall that an ideal I of a ring R is said to be <u>polynormal</u> (or has a <u>normalizing set of generators</u>) if $I = a_1R + a_2R + \ldots + a_nR$ where a_1 is a normal element of R and for $Q \le i \le n$, a_i is normal modulo $a_1R + \ldots + a_{i-1}R$. The definition implies that $I = Ra_1 + Ra_2 + \ldots + Ra_n$ also. A semiprime ideal I is <u>classically polynormal</u> if I is polynormal on generators $a_1, \ldots, a_n$ and for every minimal prime P of I and every $i = 1, \ldots, n$, $r_R(\bar{a}_i) \subseteq P$ implies $P^{\bar{a}_i}$ is a minimal prime of I. (Here $\bar{a}_i = a_i + a_1R + \ldots + a_{i-1}R \in R/(a_1R + \ldots + a_{i-1}R)$ and $P^{\bar{a}_i} = \{r \in R | r\bar{a}_i \in \bar{a}_iP\}$.) Consult [10], [17], [26], [38], and [53] for information on polynormal ideals.

Let $P = IS$ be a lower prime ideal of S. Then I is σ-prime in R and, since R is Noetherian, $I = \bigcap_{i=1}^{n} P_i$ where $P_1, \ldots, P_n$ are the minimal prime ideals over I and $P_i^\sigma = P_{i+1}$, $P_n^\sigma = P_1$.

$\mathcal{C}_R(I) = R - \bigcup_{i=1}^{n} P_i$ is trivially a σ-invariant Ore set in R, hence is an Ore set in S. P is localizable (respectively classical) in S if and only if PS_I is localizable (respectively classical) in S_I. Therefore there is no loss of generality in assuming that I is semi-maximal, σ-cyclic, and R is semilocal with $J(R) = I$. Let K_i denote the field R/P_i so that $R/I \cong K_1 \oplus \dots \oplus K_n$ and $K_i \cong K$ for all i. Notice that K is a field extension of k and $k \subseteq K$ is separable since the characteristic is zero.

Now I/I^2 is a finitely-generated R/I-module and hence a finite-dimensional K-vector space. Let $a_1, \dots, a_m$ be a K-basis for I/I^2. Since $I^\sigma = I$, for $1 \le i \le m$ we have $a_i^\sigma = \sum_{j=1}^{m} u_{ij} a_j$ for $u_{ij} \in K$. As in §4.3, we set $\sigma_I = (u_{ij}) \in M_m(K)$. If K is algebraically closed, then all eigenvalues of σ_I are in K and hence σ_I is triangularizable. In general this is not the case so we pass to $\bar{K}$, the algebraic closure of K. Let $(\ )' = (\ ) \otimes_k \bar{K}$.

<u>Lemma B.1</u>: ([49], Proposition 2.11) Let $k \subseteq K$ be a separable field extension and S a k-algebra. If I is a semiprime ideal of S then $I \otimes_k K$ is a semiprime ideal of $S \otimes_k K$. □

<u>Corollary B.2</u>: Let R be a commutative affine k-algebra where $\mathrm{char}(k) = 0$. If P is a lower prime ideal of $S = R[x;\sigma]$, then $(P/P^2)'$ is a polynormal semiprime ideal of $(S/P^2)'$.

Proof: Let $P = IS$ for a σ-prime ideal I of R. $P' = P \otimes_k \bar{K}$ is a semiprime ideal of S' by Lemma B.1. Hence $(P/P^2)' \cong P'/(P')^2$ is a semiprime ideal of $S'/(P')^2 \cong (S/P^2)'$. By the remarks preceding Lemma B.1, $\sigma_{I'}$ is triangularizable over $\bar{K}$ for some $\bar{K}$-basis $a_1,\ldots, a_m$ of $(I/I^2)'$. But $a_1,\ldots, a_m$ also generate $(P/P^2)'$ as an ideal, whence the triangular form of $\sigma_{I'}$ for $a_1,\ldots, a_m$ says that they form a normalizing set of generators for $(P/P^2)'$; in other words, $(P/P^2)'$ is polynormal. □

If $(P/P^2)'$ is a prime ideal of $(S/P^2)'$ (for example, if the base field k is algebraically closed ([57] Proposition 2), then $(P/P^2)'$ is classical by [10] Theorem 4.5. For the general case, we need the semiprime version of this result ([26] Theorem 10.10).

Proposition B.3: For every lower prime ideal $P = IS$ of S, $(P/P^2)'$ is a classical semiprime ideal of $(S/P^2)'$.

Proof: We may assume without loss of generality that $I^2 = P^2 = 0$ and $P' = \sum_{i=1}^{m} a_i S'$ is a polynormal semiprime ideal. By abuse of notation we have identified a_i with $a_i \otimes 1$. We shall show that P' is classically polynormal.

$S' = S \otimes \bar{K} = R[x;\sigma] \otimes \bar{K} = (R \otimes \bar{K})[x; \sigma \otimes 1] = R'[x; \sigma \otimes 1]$ so that S' is an Ore extension. Since R' is an affine K-algebra, R' and S' are Noetherian by the Hilbert Basis Theorem. Let $P' = \bigcap_{i=1}^{t} Q_i'$ where $Q_1',\ldots, Q_t'$ are the minimal primes of S' over the semiprime ideal

P'. Set $P_i' = Q_i' \cap R'$ for $1 \leq i \leq t$. Then P_i' is a σ-prime ideal of R' for each i by Proposition 1.13. (Here we use the fact that $x \notin \bigcup_{i=1}^{t} Q_i'$ since $x \notin P'$.) Set $I' = P' \cap R'$; then $I' = (\bigcap_{i=1}^{t} Q_i') \cap R' = \bigcap_{i=1}^{t} (Q_i' \cap R') = \bigcap_{i=1}^{t} P_i'$ so that I' is σ-semiprime. From $Q_i' \cap R' = P_i'$, it follows that $P_i'S' \subseteq Q_i'$. But $I' \subseteq P_i'$ only if $I'S' \subseteq P_i'S' \subseteq Q_i'$ and so $P_i'S' = Q_i'$ since $P_i'S'$ is a prime ideal (Proposition 1.13) and Q_i' is minimal over $P' = (IS)' = I'S'$.

To check that P' is classically polynormal, it suffices to check the case where P' is generated by a single normal element a – the proof in the general case is similar. Suppose $r_{S'}(a) \subseteq Q_i'$. We claim that $(Q_i')^a \cap R' = P_i'$. If $p \in P_i'$ then $pa = ap \in aP_i'S = aQ_i'$ and so $p \in (Q_i')^a \cap R'$. For the reverse inclusion, let $r \in (Q_i')^a \cap R'$. Then $ra \in aQ_i' \cap R'$ – say $ra = a\sum_{j=0}^{n} q_j x^j = aq_0$. Then $q_j \in r_{R'}(a) \subseteq r_{S'}(a) \cap R' \subseteq Q_i' \cap R' = P_i'$ for $1 \leq j \leq n$. Therefore $q_0 \in (Q_i' + P_i'S) \cap R' = Q_i' \cap R' = P_i'$ and so $ar = ra = aq_0$ gives $r - q_0 \in r_{R'}(a) \subseteq P_i'$ and consequently $r \in P_i'$. We conclude that $(Q_i')^a \cap R' = P_i'$ as claimed.

Now $r_{S'}(a) \subseteq Q_i'$ implies $(Q_i')^a \sim Q_i'$ via aS'/aQ_i by Proposition 3.31. As in the proof of Proposition 4.10, $(Q_i')^a \cap R' = Q_i' \cap R' = P_i'$ and so either $(Q_i')^a = Q_i' = P_i'S' \in \min(P')$ or $Q_i' \subsetneqq (Q_i')^a$. But the latter is impossible by Corollary 4.9. Therefore P' is classically polynormal and hence is classical by [26] Theorem 10.10. □

This gives the desired result in $S' = S \otimes \bar{K}$. To pull it back into S, we need the following result which is due to Yammine [56]:

<u>Lemma B.4</u>: Let $k \subseteq K$ be a separable field extension, S a k-algebra and I a semiprime ideal of S. If $S \otimes_k K$ is (right) Noetherian and $I \otimes_k K$ is localizable in $S \otimes_k K$, then I is localizable in S.

<u>Proof</u>: We must show that $\{A \mid A$ is an ideal of S with $s^{-1}A \cap \mathcal{C}(I) \neq \emptyset$ for all $s \in S\} = \{A \mid A$ is an ideal of S with $A \cap \mathcal{C}(I) \neq \emptyset\}$. (Here $s^{-1}A = \{t \in S \mid st \in A\}$.)

The inclusion $\subseteq$ is trivial. For the inclusion $\supseteq$, take an ideal A of S with $c \in A \cap \mathcal{C}(I)$. Since $c \otimes 1 \in \mathcal{C}(I \otimes K)$ [56], $A \otimes K \cap \mathcal{C}(I \otimes K) \neq \emptyset$. Let $s \in S$. Since $I \otimes K$ is localizable, $(s \otimes 1)^{-1}(A \otimes K) \cap \mathcal{C}(I \otimes K) \neq \emptyset$ and hence $(s^{-1}A \otimes K) \cap \mathcal{C}(I \otimes K) \neq \emptyset$. It follows that $[(s^{-1}A + I)/I] \otimes K \cong [(s^{-1}A \otimes K + I \otimes K)/I \otimes K]$ is an essential (right) ideal of $S \otimes K/I \otimes K \cong (S/I) \otimes K$. This clearly implies that $(s^{-1}A + I)/I$ is essential in S/I and hence $s^{-1}A \cap \mathcal{C}(I) \neq \emptyset$ by Goldie's Theorem. □

<u>Corollary B.5</u>: With R and S as above and P a lower prime ideal of S, P is classical.

<u>Proof</u>: $(P/P^2) \otimes_k \bar{K}$ is localizable by Proposition B.3. Hence P/P^2 is localizable by Lemma B.4. Using Theorem 2.4 of [10], this gives that P/P^n is localizable for all $n \geq 1$. But P has the AR-property (Proposition 4.6) and so P is localizable by [53]. Moreover, the AR-property actually guarantees that P is classical. □

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