COMBINATORIAL DESIGNS WITH PRESCRIBED AUTOMORPHISM TYPES

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ABSTRACT

In this thesis we deal with the following question: given a permutation α on a set V, does there exist a certain block design on V admitting α as an automorphism?

We are able to give a (complete or partial) answer to this question for the following:

1) 3- and 4-rotational Steiner triple systems,
2) 3-regular Steiner triple systems,
3) Steiner triple systems with an involution fixing precisely three elements,
4) 1-rotational triple systems,
5) cyclic extended triple systems,
6) 1-, 2- and 3-rotational extended triple systems,
7) 2-, 3- and 4-regular extended triple systems,
8) 1- and 3-rotational directed triple systems,
9) 1-rotational Mendelsohn triple systems,
10) cyclic extended Mendelsohn triple systems,
11) 1-rotational extended Mendelsohn triple systems.

We also present a recursive doubling construction for cyclic Steiner quadruple systems, and construct the latter for several orders.

Dedicated to

my father

and to the memory of my mother.

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TABLE OF CONTENTS

INTRODUCTION

Nowadays, combinatorics is the focus of much attention and it has become one of the fastest growing branches of mathematics, as witnessed by the number of published papers, textbooks and applications in applied sciences, computer science, economics, engineering, etc., as well as in other branches of mathematics, such as algebra, geometry, statistics, algorithms, coding theory, mathematical logic, etc.; yet, nowhere in the literature does there seem to be a satisfactory definition of this science that is both concise and complete.

Much combinatorics has arisen from games and puzzles. Among these are Euler's problem of the 36 officers [see 6, pp. 8-9], the Königsberg bridge problem [see 6, pp. 230] and Kirkman's schoolgirls problem [see 6, pp. 213-214]. Combinatorics has also its historical roots in mathematical recreations. For instance, many of the topics treated in the book Mathematical Recreations and Essays by Ball belong to combinatorics.

Combinatorial problems occur in every branch of mathematics. Roughly speaking, combinatorics is a study of arrangements of elements into sets. It deals with two general types of problems: existence of arrangements, and their enumeration or classification. To solve a combinatorial problem, often we need to use other richer structures of algebra and analysis. Conversely, often the crux of a problem of algebra or analysis reduces to a hard combinatorial question.

Combinatorial designs or block designs are collections of subsets of a finite set which meet certain requirements. They have arisen in the study of algebraic geometry, which was the source of Steiner's original problem [65]. They also occur in the theory of the design of experiments [see 46]. Finite geometrical systems are special kinds of combinatorial designs, as we see from fundamental papers by Bruck and Ryser [7] and by Chowla and Ryser [12].

This thesis is concerned with existence of certain combinatorial designs with prescribed automorphism types. The following problem has gained a lot of attention in the past few years: given a permutation α of a set V, does there exist a design on V admitting α as an automorphism? A large amount of work has been devoted to this question, and a great number of papers have resulted. These include papers dealing with cyclic designs [1, 13, 15, 16, 19, 21, 22, 30, 32, 39, 42, 43, 53, 54, 56, 57, 58, 61, 66], reverse designs [20, 23, 62, 68], automorphism-free designs [44, 48], and rotational designs [55, 59].

Broadly speaking, the methods of construction of designs are of two types: the <u>direct constructions</u>, in which a design is constructed directly, possibly and preferably from an algebraic structure, and the recursive constructions, in which a design is obtained from a collection of "smaller" designs.

One of the most important direct constructions comes from the application of the theory of difference families. An appealing feature of this style of proof is that correctness is easily verified. Let there be a collection of blocks formed from a given set V. In order to show that what we have is a t-design on V, we must prove two things:

(i) the number of blocks is correct, and

(ii) every t-subset of V is contained in at least λ blocks of the collection.

In most cases, (i) is easily verified by counting, while (ii) is straightforward (although sometimes tedious). For this reason, we often refrain from actually verifying (i) and (ii) in the course of the proof, as this follows a fairly standard pattern.

Aside from original results, this thesis attempts to provide a survey of the existence of some classes of 2-designs and of Steiner quadruple systems with a prescribed automorphism type. For this purpose, the well-known results on cyclic STS [19, 53, 54, 61, 66] and reverse STS [23, 62, 68] are included in Chapter 1. In Chapter 2, cyclic 2-designs with block size 3, which have been recently constructed by Colbourn and Colbourn [17], are given. Chapter 5 is a survey of known results that have appeared in [5, 8, 16, 19, 70]. Finally, Chapter 6 includes results on rotational SQS which appeared in [55].

A specific statement of the results which are obtained in the present work follows. In Chapter 1, we first survey what is known on cyclic, reverse, 1- and 2-rotational STSs, and present a self-contained proof of their existence. As our contribution, we obtain necessary and sufficient conditions for the existence of 3- and 4-rotational STSs, and give a new construction of 3-regular STS. In addition, we construct STS(v)'s with an involutory automorphism fixing precisely 3 elements for $v \equiv 3 \pmod 6$, which are different from Bose's [5].

In Chapter 2, after surveying what is known on cyclic triple systems with $\lambda > 1$, we proceed to deal with 1-rotational triple systems with $\lambda > 1$; we were able to completely determine the spectrum of rotational triple systems with $\lambda > 1$.

In Chapter 3, we construct cyclic extended triple systems (ETS) and obtain necessary and sufficient conditions for the existence of 1- and 2-rotational ETS. Further, we show that there exist 3-rotational ETS(v;p)'s for some values v and p. Also, we obtain necessary and sufficient conditions for the existence of 2- and 3-regular ETS, and show that there exist 4-regular ETS(v;p)'s for a certain

v and p. All results of this chapter are new.

In Chapter 4, we turn to directed triple systems (DTS); first of all, we completely determine the spectrum for k-rotational DTS. We also obtain a necessary and sufficient condition for the existence of 1-rotational Mendelsohn triple systems (MTS). Further, we completely determine cyclic extended Mendelsohn triple systems (EMTS) and 1-rotational EMTS(v;p)'s. Again, all results in this chapter, except for cyclic DTS and cyclic MTS, are new.

Chapter 5 surveys known results on cyclic $S(2,k,v)$ designs with $k > 3$.

Finally, in Chapter 6, we show that if a cyclic Steiner quadruple system SQS(v) exists, where $v \equiv 2, 10 \pmod{12}$, then there exists a cyclic SQS(2v). This appears to be the first recursive construction for cyclic SQS.

In the Appendices, we list a S-cyclic SQS(v) for v = 52, 68, 122, 130, 146, 170, 250, 290, 370, and a non-S-cyclic SQS(v) for v = 26, 28, 34, 50, 58, 76, 80, 88, 92, 98, 124. All these designs were constructed by hand.

CHAPTER 1. STEINER TRIPLE SYSTEMS

Section 1. Introduction.

A t-design, denoted $S_\lambda(t,k,v)$, is a pair (V,B) where V is a v-set and B is a collection of k-subsets (called blocks) of V such that every t-subset of V is contained in exactly λ blocks of B. The number v is called the order of $S_\lambda(t,k,v)$. A Steiner system of order v is a t-design $S_\lambda(t,k,v)$ with $\lambda = 1$. We write $S(t,k,v)$ instead of $S_1(t,k,v)$. Such systems were first defined by Woolhouse [71] in 1844 who asked: for which integers t,k,v does an $S(t,k,v)$ exist? In 1847, Kirkman [41] showed that $S(2,3,v)$ designs, known as Steiner triple systems of order v (STS(v)'s), exist if and only if $v \equiv 1$ or 3 (mod 6). Several years later, Steiner [65] asked for which values of v do $S(t,t+1,v)$ exist? Despite Woolhouse's and Kirkman's earlier papers, $S(t,k,v)$ systems are commonly referred to as Steiner systems.

Two designs $S_1 = (V_1,B_1)$ and $S_2 = (V_2,B_2)$ are isomorphic if there exists a bijection $\alpha: V_1 \to V_2$ such that $b \in B_1$ if and only if $\tilde{\alpha}(b) \in B_2$ (here $\tilde{\alpha}: B_1 \to B_2$ is the mapping induced by α; in what follows we will not distinguish between α and $\tilde{\alpha}$). The mapping α is called an isomorphism. If $S_1 = S_2$, then α is called an automorphism. Thus an automorphism of a design $S = (V,B)$ is a permutation acting on V and also on B , and the collection of all automorphisms of S constitutes a group.

Let (V,B) be a design with α as an automorphism and let Z denote the set of all integers. For a fixed block $b \in B$, the set

$$\{\alpha^n(b) \mid n \in Z\}$$

is called the <u>orbit</u> of b under α . Let us call an element of an orbit a <u>base block</u>. Then the whole set B is completely determined by a collection of base blocks containing one representative from each orbit. The number of elements of an orbit is called the <u>length</u> of the orbit. The <u>length</u> of a base block is the length of the orbit containing the base block.

The following problems have gained interest in the last decade.

First, given a finite abstract group G , does there exist a design whose automorphism group is isomorphic to G ? Lindner and Rosa [44] showed that for each $v \geq 15$ there is an STS(v) whose automorphism group is trivial (such systems are called <u>automorphism-free</u>), and Mendelsohn [48] gave an affirmative answer to the above question.

Second, given a permutation α acting on a v-set V , does there exist a design on V admitting α as an automorphism? We shall denote such a design by $S_\alpha(v)$.

If α has a single cycle of length v, then $S_\alpha(v)$ is called cyclic and α is a cyclic automorphism. It was shown first by Peltesohn [54] that a cyclic STS(v) exists if and only if $v \equiv 1$ or 3 (mod 6), except $v = 9$ [see also 19, 53, 61, 66].

If α has exactly one fixed element and k cycles of length $(v - 1)/k$, then $S_\alpha(v)$ is called k-rotational. Phelps and Rosa [59] obtained the necessary and sufficient conditions for the existence of a 1- and 2-rotational STSs.

If α is an involution with exactly one fixed element, then $S_\alpha(v)$, that is, $(v - 1)/2$-rotational, is called reverse; a necessary and sufficient condition for the existence of a reverse STS(v) is $v \equiv 1, 3, 9$ or 19 (mod 24) [23, 62, 68].

If α is an involution with exactly three fixed elements, then the existence of an STS $S_\alpha(v)$ has been conjectured for every $v \equiv 1$ or 3 (mod 6), except for $v = 1$ [see 24]. Such systems can be constructed by Bose's techniques [5] for every $v \equiv 3$ (mod 6).

This chapter considers STS with a given automorphism type. In Sections 2 and 3, we summarize results on cyclic STS and reverse STS, respectively. Section 4 provides rotational STS that are constructed by Phelps and Rosa [59]. Our principal results are also in Section 4. These results are reported in [11]. Section 5 contains regular STS that can be derived from cyclic STS easily. But we give a new construction of 3-regular STS . Also, Section 5 contains STS with an involutory automorphism fixing exactly 3 elements; we obtain a new construction of such systems.

Section 2. Cyclic Steiner Triple Systems.

It is elementary to establish that a necessary condition for the existence of an STS(v) is that $v \equiv 1$ or 3 (mod 6). Kirkman [41] and, later, Reiss [60] established that this condition is also sufficient. Even though the existence of STS is settled, one is still interested in the investigation of restricted classes of the systems. Typical restrictions which have been considered are those which constrain the automorphism group.

In this section, we consider an STS(v) whose automorphism group contains a v-cycle, that is, cyclic STS(v). In 1893, Netto [52] initiated the systematic investigation of cyclic STS. In this early paper, he demonstrated the existence of two infinite families of cyclic STS. The first is the case when $v = 6n + 1$ and prime. The second is for the case $v = 3p$ where p is a prime of the form $6n + 5$. Four years after the appearance of Netto's paper, Heffter [36] simplified Netto's second case. He constructed cyclic STS in the case where $v = 3p$ and p is a prime of the form $2n + 1$, except for $p = 3$. In the same paper, he also posed two difference problems:

Heffter's difference problem I. Can one partition the set $\{1, \ldots, 3n\}$ into 3-subsets such that in each

3-subset the sum of two numbers is equal to the third or the sum of the three is equal to $6n + 1$?

Heffter's difference problem II. Can one partition the set $\{1, \ldots, 2n, 2n + 2, \ldots, 3n + 1\}$ into 3-subsets such that in each 3-subset the sum of two numbers is equal to the third or the sum of the three is equal to $6n + 3$?

Heffter observed that a solution to his first difference problem would give a solution to the existence of cyclic STS(v) for $v \equiv 1 \pmod 6$. Further, he noted that a solution to his second difference problem (together with the triple $(2n + 1, 2n + 1, 2n + 1)$) would give a solution to the existence of cyclic STS(v) for $v \equiv 3 \pmod 6$.

Complete solutions to Heffter's difference problems were not known until Peltesohn's paper appeared in 1939 [54]. In that year, she constructed cyclic STS(v) for all $v \equiv 1$ or $3 \pmod 6$, except for $v = 9$. It is straightforward to demonstrate that the unique STS(9) is not cyclic. Continuing interest in these existence questions has involved restricted versions of the problems. In particular, Skolem [66, 67] examined an integer partitioning problem whose solutions correspond to cyclic STS. Various extensions of Skolem's original work have been investigated by O'Keefe [53] and Rosa [61]. Herein, we summarize the well-known results on cyclic STS by integer partitioning methods.

2.1 Definition [61]. An (A,k)-system is a set of ordered pairs $\{(a_r,b_r) \mid r = 1,\ldots, k\}$ such that $b_r - a_r = r$ for $r = 1,\ldots, k$ and $\bigcup_{r=1}^{k} \{a_r,b_r\} = \{1,\ldots, 2k\}$.

Let us remark that an (A,k)-system is essentially the same as what has been called in [63] a Skolem (2,k)-sequence. If such a system exists then the triples $(r,\ a_r + k,\ b_r + k)$, $r = 1,\ldots, k$, represent a solution to Heffter's difference problem I.

2.2 Lemma [19, 61, 66]. An (A,k)-system exists if and only if $k \equiv 0$ or $1 \pmod 4$.

Proof. A simple counting argument shows that $k \equiv 0$ or $1 \pmod 4$ is a necessary condition. For sufficiency, we distinguish two cases:

Case 1. $k = 4t$.

$(4t + r - 1,\ 8t - r + 1)$, $r = 1,\ldots, 2t$

$(r,\ 4t - r - 1)$, $r = 1,\ldots, t - 2$

$(t + r + 1,\ 3t - r)$, $r = 1,\ldots, t - 2$

$(t - 1,\ 3t)$, $(t,\ t + 1)$, $(2t,\ 4t - 1)$, $(2t + 1,\ 6t)$.

Case 2. $k = 4t + 1$.

$(4t + r + 1,\ 8t - r + 3)$, $r = 1,\dots, 2t$.

$(r,\ 4t - r + 1)$, $r = 1,\dots, t$

$(t + r + 2,\ 3t - r + 1)$, $r = 1,\dots, t - 2$

$(t + 1,\ t + 2)$, $(2t + 1,\ 6t + 2)$, $(2t + 2,\ 4t + 1)$.

In a t-design $S_\lambda(t,k,v)$, the blocks are also called triples, quadruples or quintuples, etc. if $k = 3, 4$ or 5, respectively.

Throughout this section, we will assume the set of elements of our cyclic STS(v) to be $V = Z_v$, the group of residue classes of Z modulo v, and the corresponding cyclic automorphism to be $\alpha = (0 \dots v - 1)$.

2.3 Theorem. If $v \equiv 1$ or $7 \pmod{24}$, then there exists a cyclic STS(v).

Proof. Let $v = 6k + 1$ and let $\{(a_r,b_r) \mid r = 1,\dots,k\}$ be an (A,k)-system for $k \equiv 0$ or $1 \pmod 4$. Then $\{0, r, b_r + k\}$, $r = 1,\dots, k$, are base triples of a cyclic STS(v).

2.4 Definition [61]. A (B,k)-system is a set of ordered pairs $\{(a_r,b_r) \mid r = 1,\dots, k\}$ such that $b_r - a_r = r$ for $r = 1,\dots, k$ and $\bigcup_{r=1}^{k} \{a_r,b_r\} = \{1,\dots, 2k - 1, 2k + 1\}$.

A (B,k)-system is essentially the same as what has been called in [63] a hooked Skolem $(2,k)$-sequence.

2.5 Lemma [19,53,61]. A (B,k)-system exists if and only if $k \equiv 2$ or $3 \pmod 4$.

Proof. A simple counting argument shows that $k \equiv 2$ or $3 \pmod 4$ is a necessary condition. For sufficiency, we distinguish two cases:

Case 1. $k = 4t + 2$.

$(r,\ 4t - r + 2)$, $\quad r = 1, \dots, 2t$

$(4t + r + 3,\ 8t - r + 4)$, $\quad r = 1, \dots, t - 1$

$(5t + r + 2,\ 7t - r + 3)$, $\quad r = 1, \dots, t - 1$

$(2t+1,\ 6t+2)$, $(4t+2,\ 6t+3)$, $(4t+3,\ 8t+5)$, $(7t+3,\ 7t+4)$.

Case 2. $k = 4t - 1$.

$(4t + r,\ 8t - r - 2)$, $\quad r = 1, \dots, 2t - 2$

$(r,\ 4t - r - 1)$, $\quad r = 1, \dots, t - 2$

$(t + r + 1,\ 3t - r)$, $\quad r = 1, \dots, t - 2$

$(t-1,\ 3t)$, $(t,\ t+1)$, $(2t,\ 4t-1)$, $(2t+1,\ 6t-1)$, $(4t,\ 8t-1)$.

2.6 Theorem. If $v \equiv 13$ or $19 \pmod{24}$, then there exists a cyclic $STS(v)$.

Proof. Let $v = 6k + 1$ and let $\{(a_r, b_r) \mid r = 1, \dots, k\}$ be a (B,k)-system for $k \equiv 2$ or $3 \pmod 4$. Then $\{0,\ r,\ b_r + k\}$, $r = 1, \dots, k$, are base triples of a cyclic $STS(v)$.

2.7 <u>Definition</u> [61]. A <u>(C,k)-system</u> is a set of ordered pairs $\{(a_r,b_r)\,|\,r = 1,\ldots, k\}$ such that $b_r - a_r = r$ for $r = 1,\ldots, k$ and

$$\bigcup_{r=1}^{k} \{a_r,b_r\} = \{1,\ldots, k, k+2,\ldots, 2k+1\} .$$

Let us remark that a (C,k)-system can be extended to cyclic STS(v) for the case $v \equiv 3 \pmod 6$. If such a system exists, then the triples $(r, a_r + k, b_r + k)$, $r = 1,\ldots, k$, are a solution to Heffter's difference problem II.

2.8 <u>Lemma</u> [19, 61]. A (C,k)-system exists if and only if $k \equiv 0$ or $3 \pmod 4$.

<u>Proof</u>. A simple counting argument shows that $k \equiv 0$ or $3 \pmod 4$ is a necessary condition. For sufficiency, we distinguish two cases:

<u>Case 1</u>. $k = 4t$.

$(r, 4t - r + 1)$,	$r = 1,\ldots, t-1$
$(t + r - 1, 3t - r)$,	$r = 1,\ldots, t-1$
$(4t + r + 1, 8t - r + 1)$,	$r = 1,\ldots, t-1$
$(5t + r + 1, 7t - r + 1)$,	$r = 1,\ldots, t-1$

$(2t-1, 2t)$, $(3t, 5t+1)$, $(3t+1, 7t+1)$, $(6t+1, 8t+1)$.

Case 2. $k = 4t - 1$.

$(r,\ 4t - r)$, $\quad r = 1, \ldots, 2t - 1$

$(4t + r + 1,\ 8t - r)$, $\quad r = 1, \ldots, t - 2$

$(5t + r,\ 7t - r - 1)$, $\quad r = 1, \ldots, t - 2$

$(2t,\ 6t-1)$, $(5t,\ 7t+1)$, $(4t+1,\ 6t)$, $(7t-1,\ 7t)$.

2.9 Theorem. If $v \equiv 3$ or 21 (mod 24) , then there exists a cyclic STS(v) .

Proof. Let $v = 6k + 3$ and let $\{(a_r,b_r) | r = 1,\ldots, k\}$ be a (C,k)-system for $k \equiv 0$ or 3 (mod 4) . Then $\{0,\ 2k + 1,\ 4k + 2\}$, $\{0,\ r,\ b_r + k\}$, $r = 1,\ldots, k$, are base triples of a cyclic STS(v) .

2.10 Definition [61]. A (D,k)-system is a set of ordered pairs $\{(a_r,b_r) | r = 1,\ldots, k\}$ such that $b_r - a_r = r$ for $r = 1,\ldots, k$ and

$$\bigcup_{r=1}^{k} \{a_r,b_r\} = \{1,\ldots,\ k,\ k + 2,\ldots,\ 2k,\ 2k + 2\} .$$

2.11 Lemma [19, 61]. A (D,k)-system exists if and only if $k \equiv 1$ or 2 (mod 4) , except for $k = 1$.

Proof. A simple counting argument shows that $k \equiv 1$ or 2 (mod 4) is a necessary condition. For sufficiency, we have:

$k = 2$: $(1,2)$, $(4,6)$.

$k = 5$: $(1,5)$, $(2,7)$, $(3,4)$, $(8,10)$, $(9,12)$.

$k = 4t + 1$:

$(r,\ 4t - r + 2)$, $\qquad r = 1, \ldots, 2t$

$(5t + r,\ 7t - r + 3)$, $\qquad r = 1, \ldots, t$

$(4t + r + 2,\ 8t - r + 3)$, $\qquad r = 1, \ldots, t - 2$

$(2t+1,\ 6t+2)$, $(6t+1,\ 8t+4)$, $(7t+3,\ 7t+4)$.

$k = 4t + 2$:

$(r,\ 4t - r + 3)$, $\qquad r = 1, \ldots, 2t$

$(4t + r + 4,\ 8t - r + 4)$, $\qquad r = 1, \ldots, t - 1$

$(5t + r + 3,\ 7t - r + 3)$, $\qquad r = 1, \ldots, t - 2$

$(2t+1,\ 6t+3)$, $(2t+2,\ 6t+2)$, $(4t+4,\ 6t+4)$, $(7t+3,\ 7t+4)$, $(8t+4,\ 8t+6)$.

2.12 <u>Theorem</u>. If $v \equiv 9$ or $15 \pmod{24}$, $v \neq 9$, then there exists a cyclic STS(v).

<u>Proof</u>. Let $v = 6k + 3$ and let $\{(a_r, b_r) \mid r = 1, \ldots, k\}$ be a (D,k)-system for $k \equiv 1$ or $2 \pmod 4$, except for $k = 1$. Then $\{0,\ 2k + 1,\ 4k + 2\}$, $\{0,\ r,\ b_r + k\}$, $r = 1, \ldots, k$, are base triples of a cyclic STS(v).

Summarizing, we have:

2.13 Theorem. A cyclic STS(v) exists if and only if $v \equiv 1$ or $3 \pmod 6$, except for $v = 9$.

Section 3. Reverse Steiner Triple Systems.

In 1972, Rosa [62] introduced the following problem: for what values of v does there exist an STS(v) with an involution fixing exactly one element as a automorphism, that is, a reverse STS(v) ? In the same paper, he showed that a necessary condition for the existence of a reverse STS(v) is $v \equiv 1, 3, 9$ or $19 \pmod{24}$. Also, he constructed a reverse STS(v) for every $v \equiv 1 \pmod{24}$, except for $v = 25$, for every $v \equiv 3$ or $9 \pmod{24}$ and for $v = 19$*. In the same year, Doyen [23] produced a reverse STS(v) for $v = 25$**, gave simpler constructions for $v \equiv 3$ or $9 \pmod{24}$ and proved that the necessary condition $v \equiv 19 \pmod{24}$ is asymptotically sufficient. One year later in 1973, Teirlinck [68] showed that the necessary conditions are also sufficient. In this section, we summarize these well-known results.

Throughout this thesis, an element (x,i) of $V = Z_v \times \{i\}$ will be written for brevity as x_i.

3.1 Lemma [62]. If there exists a reverse STS(v), then $v \equiv 1, 3, 9$ or $19 \pmod{24}$.

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* Recently, Denniston [20] proved that there are exactly 184 non-isomorphic reverse STS(19)'s.

** For a correction, see Zentralblatt für Mathematik und ihre Grenzgebiete, 272, 05013.

Proof. Let (V,B) be a reverse $STS(v)$, with α as an automorphism where $V = Z_2 \times Z_{(v-1)/2} \cup \{\infty\}$ and $\alpha = (\infty)(0_i 1_i)$, $i = 0,\ldots,(v-3)/2$. Then B contains all the triples of the form $\{\infty, 0_i, 1_i\}$, $i = 0,\ldots,(v-3)/2$, and does not contain any other triple involving ∞; this follows from the fact that, by the definition of an STS, the pair $0_i, 1_i$ occurs in exactly one triple of B, and therefore the third element of the triple containing $0_i, 1_i$ will necessarily be ∞. The $(v-1)/2$ triples containing ∞ are fixed under the action of α, while the remaining $(v-1)(v-3)/6$ triples in B are interchanged in pairs. The latter triples may be of one of the following forms:

(i) $\{0_i, 0_j, 0_k\}$, (iii) $\{0_i, 0_j, 1_k\}$,

(ii) $\{1_i, 1_j, 1_k\}$, (iv) $\{0_i, 1_j, 1_k\}$,

where clearly the number of triples of the forms (i) and (ii) is the same, and similarly for the triples (iii) and (iv). Denote the number of triples of the forms (i) and (iii) by m and n, respectively. Since $|B| = v(v-1)/6$, we have

$$(3.1.1) \qquad m + n = \tfrac{1}{2}\{\tfrac{1}{6}v(v-1) - \tfrac{1}{2}(v-1)\}.$$

Further, there are $\binom{(v-1)/2}{2}$ pairs of 0's; each triple of the form (i) contains three pairs of 0's, and each triple of the form (iii) contains one pair of 0's so that

$$(3.1.2) \qquad 3m + n = \binom{(v-1)/2}{2} .$$

Solving (3.1.1) and (3.1.2), we obtain

$$m = \frac{1}{48}(v-1)(v-3), \quad n = \frac{1}{16}(v-1)(v-3)$$

and since $v \equiv 1$ or $3 \pmod 6$ we observe that m and n are integers if and only if $v \equiv 1, 3, 9$ or $19 \pmod{24}$.

3.2 Lemma [23, 62, 69]. If $v \equiv 1 \pmod{24}$, then there exists a reverse STS(v).

Proof. Let $v = 24t + 1$

Elements: $V = Z_{v-3} \cup \{\infty, a, b\}$

Automorphism: $\alpha = (\infty)(0 \ldots v-4)(ab)$

Base triples: $B = B_1 \cup B_2$

where

B_1: $\{\{\infty, a, b\}, \{\infty, 0, 12t-1\}, \{a, 0, 12t-3\}\}$

B_2: $\{\{0, r, b_r + 4t - 1\} \mid r = 1, \ldots, 4t-1\}$

where $\{(a_r,b_r)|r = 1,\ldots, 4t-1\}$ is a $(B, 4t-1)$-system. Then (V,B) is a reverse $STS(v)$ with $\alpha^{(v-3)/2}$ as an involutory automorphism fixing exactly one element.

3.3 Lemma [23, 59, 62]. If $v \equiv 3$ or $9 \pmod{24}$, then there exists a reverse $STS(v)$.

Proof. Let $v \equiv 3$ or $9 \pmod{24}$.

Elements: $V = Z_{v-1} \cup \{\infty\}$

Automorphism: $\alpha = (\infty)(0\ldots v-2)$

Base triples: $B = B_1 \cup B_2$

where

$$B_1: \{\{\infty, 0, (v-1)/2\}\}$$

$$B_2: \{\{0, r, b_r + k\}|r = 1,\ldots, k\}$$

where $\{(a_r,b_r)|r = 1,\ldots, k\}$ is an (A,k)-system with $k = (v-1)/6$; since $v \equiv 3$ or $9 \pmod{24}$, $k \equiv 0$ or $1 \pmod 4$ and so an (A,k)-system exists. Then (V,B) is a reverse $STS(v)$ with $\alpha^{(v-1)/2}$ as an involutory automorphism fixing exactly one element.

3.4 Lemma [62]. There is a reverse $STS(19)$.

<u>Proof</u>. Elements: $V = Z_2 \times Z_9 \cup \{\infty\}$

Automorphism: $\alpha = (\infty)(0_i 1_i)$, $i = 0,\ldots, 8$

Base triples B:

$\{\infty, 0_i, 1_i\}$, $i = 0,\ldots, 8$

$\{0_0,0_3,0_6\}, \{0_1,0_4,0_7\}, \{0_2,0_5,0_8\}, \{0_0,0_4,0_8\}, \{0_1,0_5,0_6\}, \{0_2,0_3,0_7\}$

$\{0_0,0_1,1_5\}, \{0_0,0_2,1_4\}, \{0_1,0_2,1_3\}, \{0_3,0_4,1_8\}, \{0_3,0_5,1_7\}, \{0_4,0_5,1_6\}$

$\{0_6,0_7,1_2\}, \{0_6,0_8,1_1\}, \{0_7,0_8,1_0\}, \{0_0,0_5,1_2\}, \{0_0,0_7,1_6\}, \{0_5,0_7,1_4\}$

$\{0_1,0_3,1_0\}, \{0_1,0_8,1_7\}, \{0_3,0_8,1_5\}, \{0_2,0_4,1_1\}, \{0_2,0_6,1_8\}, \{0_4,0_6,1_3\}$.

Then (V,B) is a reverse STS(19) .

3.5 <u>Definition</u>. A <u>(W,k)-system</u> is a set of ordered pairs $\{(a_r,b_r) \mid r = 1,\ldots, k\}$ such that $b_r - a_r = r$ for $r = 1,\ldots, k$ and

$$\bigcup_{r=1}^{k} \{a_r,b_r\} = \{2,\ldots k/2,\ k/2 + 2,\ k/2 + 4,\ldots,\ 2k + 2,\ 2k + 4\} .$$

3.6 <u>Lemma</u>. A (W,k)-system exists if and only if $k \equiv 0 \pmod 4$.

<u>Proof</u>. ($\Rightarrow$) Let $\{(a_r,b_r) \mid r = 1,\ldots, k\}$ be a (W,k)-system. Then, since $k/2$ is an integer, k is even. On the other hand,

$$(3.6.1)\qquad \sum_{r=1}^{k} b_r - \sum_{r=1}^{k} a_r = \frac{1}{2}k(k+1)\,,$$

$$(3.6.2)\qquad \sum_{r=1}^{k} b_r + \sum_{r=1}^{k} a_r = \frac{1}{2}(2k+4)(2k+5)$$

$$- 1 + (k/2 + 1) + (k/2 + 3) + (2k + 3)\,.$$

Adding both sides of (3.6.1) and (3.6.2) yields $5k^2 + 13k + 4 \equiv 0 \pmod 4$ and hence $k \equiv 0$ or $3 \pmod 4$; but since k is even, we have $k \equiv 0 \pmod 4$.

(⇐) (see [68]). $k = 4$: (2,6), (4,7), (10,12), (8,9).

$k = 4t$

$(2 + r,\ 4t + 2 - r)$, $\quad r = 0, \ldots, 2t - 2$

$(4t + 4 + r,\ 8t + 1 - r)$, $\quad r = 0, \ldots, t - 2$

$(5t + 2 + r,\ 7t + 1 - r)$, $\quad r = 1, \ldots, t - 2$

$(2t+2,\ 6t+1)$, $(4t+3,\ 6t+2)$, $(7t+1,\ 7t+2)$, $(8t+2,\ 8t+4)$.

3.7 Lemma [68]. If $v \equiv 19 \pmod{24}$, $v \neq 19$, then there exists a reverse STS(v).

Proof. Let $v = 24t + 19$, $t \geq 1$.

Elements: $V = Z_{24t+10} \cup \{\infty, a_i, b_i, c_i, d_i \mid i = 1, 2\}$

Automorphism: $\alpha = (\infty)(a_1a_2)(b_1b_2)(c_1c_2)(d_1d_2)(0 \ldots 24t+9)$

Base triples: $B = B_1 \cup B_2 \cup B_3 \cup B_4$

where

B_1: $\{\{\infty, 0, 12t+5\}$

B_2: $\{\{0, 4t+1, a_1\}, \{0, 6t+1, b_1\}, \{0, 6t+3, c_1\}, \{0, 12t+3, d_1\}\}$

B_3: the collection of all base triples of a reverse STS(9) based on $\{\infty, a_i, b_i, c_i, d_i \mid i = 1, 2\}$ with $(\infty)(a_1a_2)(b_1b_2)(c_1c_2)(d_1d_2)$ as an involutory automorphism,

B_4: $\{\{0, r, b_r+4t\} \mid r = 1, \ldots, 4t\}$

where $\{(a_r, b_r) \mid r = 1, \ldots, 4t\}$ is a (W,4t)-system. Then (V,B) is a reverse STS(v) with α^{12t+5} as the required automorphism.

Summarizing, we have

3.8 Theorem. A reverse STS(v) exists if and only if $v \equiv 1, 3, 9$ or $19 \pmod{24}$.

Section 4. Rotational Steiner Triple Systems.

Recall that an STS(v) is k-rotational if it admits an automorphism consisting of exactly one fixed element and k disjoint cycles of the same length. Phelps and Rosa [59] showed that there is a 1-rotational STS(v) if and only if $v \equiv 3$ or 9 (mod 24) and there is a 2-rotational STS(v) if and only if $v \equiv 1, 3, 7, 9, 15$ or 19 (mod 24). Also, they showed that there are exactly 10 non-isomorphic 2-rotational STS(19)'s and there are exactly 35 non-isomorphic 1-rotational STS(27)'s.

In this section, we summarize Phelps' and Rosa's results [59] and obtain the necessary and sufficient conditions for the existence of 3- and 4-rotational STS.

4.1 Lemma [59]. If there exists a 1-rotational STS(v), then $v \equiv 3$ or 9 (mod 24).

Proof. Let $V = Z_{v-1} \cup \{\infty\}$, and let $\alpha = (\infty)(0 \ldots v - 2)$ be an automorphism of a 1-rotational STS(v) (V,B). Since $\{\infty, i, j\} \in B$ implies $\{\infty, i + 1, j + 1\} \in B$, it follows that $\{\infty, i, j\} \in B$ if and only if $i - j \equiv (v - 1)/2 \pmod{v - 1}$; in other words, any 1-rotational STS(v) contains $(v - 1)/2$ triples of the form $\{\infty, i, i + (v - 1)/2\}$ mod $v - 1$. All 3-subsets

of V not containing the element ∞ are partitioned into orbits under α all of which are of length $v - 1$ except possibly a single orbit Q_0 of length $(v - 1)/3$ of triples $\{0, (v - 1)/3, 2(v - 1)/3\}$. It is easily seen that no 1-rotational STS(v) contains triples of Q_0: this would require $v \equiv 1 \pmod 6$, and at the same time, there would be need for further
$v(v - 1)/6 - (v - 1)/2 - (v - 1)/3 = (v - 1)(v - 5)/6$ triples in B which would then necessarily have to be partitioned into $(v - 5)/6$ orbits of length $v - 1$; this is obviously impossible as $(v - 5)/6$ is not an integer. Thus the remaining $v(v - 1)/6 - (v - 1)/2 = (v - 1)(v - 3)/6$ triples of B fall into $(v - 3)/6$ orbits of length $v - 1$. If $\{a, b, c\}$ is a triple in one such orbit then clearly the six differences $\pm(a - b)$, $\pm(a - c)$, $\pm(b - c)$ are all distinct, and if $\{a_1, b_1, c_1\}$, $\{a_2, b_2, c_2\}$ are two triples from two distinct orbits in B then the corresponding 12 differences are all distinct. Since there are still $v - 3$ non-zero differences "available" it follows that $(v - 3)/6$ must be an integer, and so we must have

$$v \equiv 3 \pmod 6 . \tag{4.1.1}$$

On the other hand, since v is odd, the automorphism $\alpha^{(v-1)/2}$ is an involution fixing exactly one element, and so (V,B) is a reverse STS(v). It follows from Section 3 that

(4.1.2) $v \equiv 1, 3, 9$ or $19 \pmod{24}$.

The congruences (4.1.1) and (4.1.2) together yield $v \equiv 3$ or $9 \pmod{24}$.

4.2 Lemma [59]. If $v \equiv 3$ or $9 \pmod{24}$, then there exists a 1-rotational STS(v).

Proof. cf. Lemma 3.3 in Section 3.

Lemmas 4.1 and 4.2 together yield

4.3 Theorem. A 1-rotational STS(v) exists if and only if $v \equiv 3$ or $9 \pmod{24}$.

4.4 Lemma [59]. If a 2-rotational STS(v) exists, then $v \equiv 1, 3, 7, 9, 15$ or $19 \pmod{24}$.

Proof. Let $V = (Z_{(v-1)/2} \times Z_2) \cup \{\infty\}$ and let $\alpha = (\infty)(0_i \ldots ((v-1)/2)_i)$, $i \in Z_2$, be an automorphism of a 2-rotational STS(v). If $(v-1)/2 \equiv 0 \pmod{2}$ then $\alpha^{(v-1)/4}$ is an involution fixing exactly one element so that the STS(v) is a reverse STS(v). But a reverse STS(v) cannot exist for $v \equiv 13$ or $21 \pmod{24}$ thus we have $v \equiv 1, 3, 7, 9, 15$ or $19 \pmod{24}$.

4.5 Lemma. If $v \equiv 3$ or 9 (mod 24), then there exists a 2-rotational STS(v).

Proof. For $v \equiv 3$ or 9 (mod 24), there exists a 1-rotational STS(v) and $v - 1 \equiv 0$ (mod 2).

4.6 Lemma [59]. There is a 2-rotational STS(19).

Proof. (see No. 1 in [59]).

Elements: $V = (Z_9 \times Z_2) \cup \{\infty\}$

Automorphism: $\alpha = (\infty)(0_i \ldots 8_i)$, $i \in Z_2$

Base triples B: $\{\infty, 0_0, 0_1\}$, $\{0_0, 3_0, 6_0\}$, $\{0_1, 1_1, 3_1\}$, $\{5_0, 0_1, 4_1\}$, $\{3_0, 4_0, 0_1\}$, $\{6_0, 8_0, 0_1\}$, $\{2_0, 7_0, 0_1\}$.

Then (V,B) is a 2-rotational STS(19).

4.7 Lemma [59]. If $v \equiv 7, 15$ or 19 (mod 24), then there exists a 2-rotational STS(v).

Proof. A 2-rotational STS(19) exists by Lemma 4.6. Let $u \equiv 1$ or 3 (mod 6), $u \neq 9$, and let $U = Z_u$ and (U,W) be a cyclic STS(u) with $\beta = (0 \ldots u-1)$ its cyclic automorphism. Put $V = (Z_u \times Z_2) \cup \{\infty\}$ and define a set of triples B on V as follows:

$$B = B_1 \cup B_2 \cup B_3$$

where

B_1: $\{\{\infty, a_0, a_1\} | a \in Z_u\}$,

B_2: $\{\{a_0, (a-b)_1, (a+b)_1\} | a \in Z_u, b = 1, \ldots, (u-1)/2\}$,

B_3: $\{\{a_0, b_0, c_0\} | \{a, b, c\} \in W\}$.

Then (V,B) is an $STS(2u + 1)$ with $\alpha = (\infty)(0_i \ldots (u - 1)_i)$, $i \in Z_2$, as an automorphism. Set $v = 2u + 1$. Then $v \equiv 3, 7, 15$ or $19 \pmod{24}$, $v \neq 19$.

Let k be a natural number, and let

$$S(k) = \{1, \ldots, 2k - 1, 2k + 1, \ldots, 4k - 1\} ,$$

$$T(k) = \begin{cases} \{2, \ldots, 2k\} & \text{if } k \text{ is odd,} \\ \{1, 3, \ldots, 2k\} & \text{if } k \text{ is even.} \end{cases}$$

4.8 <u>Definition</u> [59]. A <u>(F,k)-system</u> is a set of ordered pairs $\{(a_r,b_r) | r \in T(k)\}$ such that $b_r - a_r = r$ for all $r \in T(k)$ and $\cup_{r \in T(k)} \{a_r,b_r\} = S(k)$.

4.9 <u>Lemma</u> [59]. A (F,k)-system exists if and only if $k \neq 2$.

<u>Proof</u>. We have $T(2) = \{1, 3, 4\}$, but it is easily

seen that $S(2) = \{1, 2, 3, 5, 6, 7\}$ cannot be partitioned into three pairs having differences 1, 3, 4 .

$k = 1$: (1,3)

$k = 3$: (1,3), (8,11), (5,9), (2,7), (4,10)

$k = 4$: (13,14), (3,6), (11,15), (2,7), (4,10), (5,12), (1,9)

$k = 5$: (6,8), (16,19), (14,18), (12,17), (1,7), (2,9), (3,11), (4,13), (5,15) .

Let now $k \geq 6$; distinguish three cases:

Case 1. $k = 2t$, $t \geq 3$.

$(r+1,\ 2k-r)$, $r = 1, \ldots, k-2$

$(2k+1+r,\ 4k-1-r)$, $r = 1, \ldots, t-2$

$(5t-1+r,\ 7t-1-r)$, $r = 1, \ldots, t-2$

$(1, 2k+1)$, $(k, 3k-2)$, $(k+1, 3k)$, $(3k-1, 4k-1)$, $(7t-1, 7t)$.

Case 2. $k = 4t + 3$, $t \geq 1$.

$(r+1,\ 2k-r)$, $r = 1, \ldots, k-2$

$(2k+2r,\ 4k-2-2r)$, $r = 1, \ldots, 2t$

$(2k+1+2r,\ 4k+1-2r)$, $r = 1, \ldots, t$

$(2k+2t+1+2r,\ 3k+2t-2r)$, $r = 1, \ldots, t-1$ $(t \geq 2)$

$(1, 2k+1)$, $(k, 3k-2)$, $(k+1, 3k)$, $(3k-1, 4k-2)$, $(3k+2t, 3k+2t+2)$.

Case 3. $k = 4t + 1$, $t \geq 2$.

$(r,\ 2k-1-r)$, $r = 1,\ldots,k-2$

$(k-1,\ 3k-2)$, $(k,\ 3k)$, $(2k-1,\ 4k-3)$

and (i) for $t = 2$

(19,29), (20,32), (21,35), (22,24), (23,31),
(26,30), (28,34) ;

(ii) for $t \geq 3$

$(2k-1+2r,\ 4k-3-2r)$, $r = 1,\ldots,t$

$(2k+2r,\ 4k-2r)$, $r = 1,\ldots,t$

$(2k+2t+3+2r,\ 3k+2t-2r)$, $r = 1,\ldots,t-3$ $(t>3)$

$(2k+2t+2+4r,\ 3k+2t+3-4r)$, $r = 1,\ldots,[(t-1)/2]$ $(t>2)$

$(2k+2t+4r,\ 3k+2t-3-4r)$, $r = 1,\ldots,[(t-2)/2]$ $(t>3)$

$(3k+2,\ 4k-1)$, $(2k+2t+1,\ 2k+2t+3)$

and

$$\begin{cases} (3k-3,\ 3k+1) & \text{if } t \text{ is odd} \\ (3k-1,\ 3k+3) & \text{if } t \text{ is even.} \end{cases}$$

4.10 Lemma [59]. If $v \equiv 1 \pmod{24}$, then there exists a 2-rotational STS(v) .

Proof. Let $v = 24t + 1$.

Elements: $V = (Z_{12t} \times Z_2) \cup \{\infty\}$

Automorphism: $\alpha = (\infty)(0_i \ldots (12t-1)_i)$, $i \in Z_2$

Base triples: $B = B_1 \cup B_2 \cup B_3 \cup B_4 \cup B_5 \cup B_6$

where

B_1: $\{\{\infty, 0_i, (6t)_i\} | i \in Z_2\}$,

B_2: $\{\{0_0, (4t)_0, (8t)_0\}\}$,

B_3: $\{\{0_0, r_0, (b_r-1)_1\} | r = 1,\ldots, 6t-1;\ r \neq 4t\}$,

where $\{(a_r,b_r) | r = 1,\ldots, 6t-1\}$ is an (A, 6t-1)-system or a (B, 6t-1)-system depending on whether t is odd or even.

B_4: $\{\{0_0, (a_{4t}-1)_1, (b_{4t}-1)_1\}\}$,

$$B_5: \begin{cases} \{\{0_1, 1_1, 2_0\}\} & \text{if } t \text{ is odd,} \\ \{\{0_1, 2_1, 3_0\}\} & \text{if } t \text{ is even,} \end{cases}$$

$$B_6: \begin{cases} \{\{0_1, 1_1, 10_1\}, \{0_1, 5_1, 11_1\}, \{0_1, 3_1, 7_1\}\} & \text{if } t = 2, \\ \{\{0_1, (c_r+2t)_1, (d_r+2t)_1\} | r \in T(t)\} & \text{if } t \neq 2 \end{cases}$$

where $\{(c_r, d_r) | r \in T(t)\}$ is a (F,t)-system. Then (V,B) is a 2-rotational STS(v).

Lemmas 4.4, 4.5, 4.6, 4.7 and 4.10 together yield

4.11 Theorem. A 2-rotational STS(v) exists if and only if $v \equiv 1, 3, 7, 9, 15$ or $19 \pmod{24}$.

4.12 Lemma. If a 3-rotational STS(v) exists, then $v \equiv 1$ or $19 \pmod{24}$.

Proof. Let $V = (Z_{(v-1/3} \times Z_3) \cup \{\infty\}$ and let $\alpha = (\infty)(0_i \ldots ((v-1)/3 - 1)_i)$, $i \in Z_3$, be an automorphism of a 3-rotational STS(v). If $(v-1)/3 \equiv 0 \pmod 2$ then $\alpha^{(v-1)/6}$ is an involution fixing exactly one element so that the STS(v) is a reverse STS(v). But a reverse STS(v) cannot exist for $v \equiv 7$ or $13 \pmod{24}$ thus we have $v \equiv 1$ or $19 \pmod{24}$ because of $v \equiv 1$ or $3 \pmod 6$ and $v \equiv 1 \pmod 3$.

4.13 Definition. An (E,k)-system is a set of ordered pairs $\{(a_r, b_r) | r = 1, \ldots, k\}$ such that $b_r - a_r = r$ for $r = 1, \ldots, k$ and

$$\bigcup_{r=1}^{k} \{a_r, b_r\} = \{1, \ldots, (k+1)/2 - 1, (k+1)/2 + 1, \ldots, 2k+1\}.$$

4.14 Lemma. An (E,k)-system exists if and only if k is odd.

<u>Proof</u>. (→) Since $(k + 1)/2$ is an integer, k must be odd.

(←) $k = 4t + 1$

$(4t + 1 + r, 8t + 4 - r)$, $r = 1, \ldots, 2t + 1$

$(r, 4t + 2 - r)$, $r = 1, \ldots, 2t$

$k = 4t - 1$

$(4t - 1 + r, 8t - r)$, $r = 1, \ldots, 2t$

$(r, 4t - r)$, $r = 1, \ldots, 2t - 1$.

4.15 <u>Lemma</u>. If $v \equiv 1 \pmod{24}$, then there exists a 3-rotational STS(v).

<u>Proof</u>. Let $v = 24t + 1$, $t \geq 1$.

Elements: $V = (Z_{8t} \times Z_3) \cup \{\infty\}$

Automorphism: $\alpha = (\infty)(0_i \ldots (8t - 1)_i)$, $i \in Z_3$

Base triples: $B = B_1 \cup B_2 \cup B_3 \cup B_4$

where

B_1: $\{\{\infty, 0_i, (4t)_i\} | i \in Z_3\}$,

B_2: $\{\{0_0, 0_1, 0_2\}, \{0_0, (2t)_1, (6t)_2\}\}$,

B_3: $\{\{0_0, r_0, (b_r)_1\}, \{0_2, r_2, (b_r)_0\} | r = 1, \ldots, 4t - 1\}$

where $\{(a_r, b_r) | r = 1, \ldots, 4t - 1\}$ is an $(E, 4t-1)$-system,

B_4: $\{\{0_1, r_1, (b_r)_2\} | r = 1, \ldots, 4t - 1\}$

where $\{(a_r, b_r) | r = 1, \ldots, 4t - 1\}$ is a $(C, 4t-1)$-system. Then (V,B) is a 3-rotational $STS(v)$.

4.16 <u>Definition</u>. A <u>$(-B,k)$-system</u> is a set of ordered pairs $\{(a_r, b_r) | r = 1, \ldots, k\}$ such that $b_r - a_r = r$ for $r = 1, \ldots, k$ and $\bigcup_{r=1}^{k} \{a_r, b_r\} = \{1, 3, \ldots, 2k + 1\}$.

4.17 <u>Lemma</u>. A $(-B,k)$-system exists if and only if $k \equiv 2$ or $3 \pmod 4$.

<u>Proof</u>. If we replace each $i = 1, \ldots, 2k - 1, 2k + 1$ in a (B,k)-system by $2k + 2 - i$, then we will get a $(-B,k)$-system.

4.18 <u>Lemma</u>. If $v \equiv 19 \pmod{24}$, then there exists a 3-rotational $STS(v)$.

<u>Proof</u>. Let $v = 24t + 19$, $t \geq 0$.

Elements: $V = (Z_{8t+6} \times Z_3) \cup \{\infty\}$

Automorphism: $\alpha = (\infty)(0_i \ldots (8t + 5)_i)$, $i \in Z_3$

Base triples: $B = B_1 \cup B_2 \cup B_3 \cup B_4$

where

B_1: $\{\{\infty, 0_i, (4t+3)_i\} | i \in Z_3\}$,

B_2: $\{\{0_0, (4t+3)_1, (4t+3)_2\}, \{0_0, (8t+5)_1, 1_2\}\}$,

B_3: $\{\{0_0, r_0, (b_r)_1\}, \{0_2, r_2 (b_r)_0\} | r = 1,\ldots, 4t+2\}$

where $\{(a_r,b_r) | r = 1,\ldots, 4t+2\}$ is a (D, 4t+2)-system,

B_4: $\{\{0_1, r_1, (b_r)_2\} | r = 1,\ldots, 4t+2\}$

where $\{(a_r,b_r) | r = 1,\ldots, 4t+2\}$ is a (-B, 4t+2)-system. Then (V,B) is a 3-rotational STS(v) .

Lemmas 4.12, 4.15 and 4.18 together yield

4.19 <u>Theorem</u>. A 3-rotational STS(v) exists if and only if $v \equiv 1$ or 19 (mod 24) .

4.20 <u>Lemma</u>. If a 4-rotational STS(v) exists, then $v \equiv 1, 9, 13$ or 21 (mod 24) .

<u>Proof</u>. We have $v \equiv 1$ or 3 (mod 6) and $v \equiv 1$ (mod 4) .

4.21 <u>Lemma</u>. If $v \equiv 1$ or 9 (mod 24) , then there exists a 4-rotational STS(v) .

Proof. For $v \equiv 1$ or $9 \pmod{24}$, there exists a 2-rotational STS(v) and so does a 4-rotational STS(v).

4.22 Lemma. There is a 4-rotational STS(37).

Proof. Elements: $V = (Z_9 \times Z_4) \cup \{\infty\}$
Automorphism: $\alpha = (\infty)(0_i \dots 8_i)$, $i \in Z_4$
Base triples B:

$\{\infty, 0_0, 7_2\}$, $\{\infty, 0_1, 0_3\}$

$\{0_i, 3_i, 6_i\}$, $i = 2, 3$

$\{0_2, 1_2, 4_0\}$, $\{0_2, 2_2, 0_3\}$, $\{0_2, 4_2, 5_1\}$, $\{0_1, 0_2, 8_3\}$,

$\{0_3, 1_3, 5_0\}$, $\{0_3, 2_3, 7_1\}$, $\{0_3, 4_3, 8_1\}$, $\{0_0, 0_1, 6_3\}$,

$\{0_0, 1_0, 2_1\}$, $\{0_0, 2_0, 7_1\}$, $\{0_0, 3_0, 6_1\}$, $\{0_0, 4_0, 8_1\}$,

$\{0_1, 1_1, 2_2\}$, $\{0_1, 2_1, 7_2\}$, $\{0_1, 3_1, 6_2\}$, $\{0_1, 4_1, 7_3\}$,

$\{0_0, 0_2, 1_3\}$, $\{0_0, 1_2, 3_3\}$, $\{0_0, 8_2, 2_3\}$, $\{0_0, 3_2, 7_3\}$,

$\{0_0, 4_2, 0_3\}$, $\{0_0, 2_2, 8_3\}$.

Then (V,B) is a 4-rotational STS(37).

4.23 Lemma. If $v \equiv 13 \pmod{24}$, then there exists a 4-rotational STS(v).

Proof. A 4-rotational STS(37) exists by Lemma 4.22. Let $v = 24t + 13$, $t \neq 1$.

Elements: $V = (Z_{6t+3} \times Z_4) \cup \{\infty\}$

Automorphism: $\alpha = (\infty)(0_i \ldots (6t+2)_i)$, $i \in Z_4$

Base triples: $B = B_1 \cup B_2 \cup B_3 \cup B_4 \cup B_5 \cup B_6 \cup B_7$

where we distinguish two cases.

Case 1. $t \equiv 0$ or $1 \pmod 4$.

Let $\{(a_r, b_r) \mid r = 1, \ldots, 3t+1\}$ be an (A, 3t+1)-system.

B_1: $\{\{\infty, 0_0, (2t+1-b_{2t+1})_2\}, \{\infty, 0_1, (-b_{2t+1})_3\}\}$,

B_2: the collection of all base triples of a cyclic STS(6t + 3) based on $Z_{6t+3} \times \{2\}$,

B_3: $\{\{0_3, (2t+1)_3, (4t+2)_3\}\}$,

B_4: $\{\{0_3, r_3, (b_r)_1\} \mid r = 1, \ldots, 2t, 2t+2, \ldots, 3t+1\}$,

B_5: $\{\{0_0, r_0, (b_r)_1\}, \{0_1, r_1, (b_r)_2\} \mid r = 1, \ldots, 3t+1\}$,

B_6: $\{\{0_0, 0_1, (2t+1-b_{2t+1})_3\}, \{0_1, 0_2, 0_3\}\}$,

B_7: $\{\{(b_{2t+1}-2t-1-r)_0, 0_2, r_3\} \mid r = 1, \ldots, 6t+2\}$.

Case 2. $t \equiv 2$ or $3 \pmod 4$.

Let $\{(a_r, b_r) \mid r = 1, \ldots, 3t+1\}$ be a $(B, 3t+1)$-system.

B_1: $\{\{\infty, 0_0, (2t-2-b_{2t+1})_2\}, \{\infty, 0_1, (-b_{2t+1})_3\}\}$,

B_2, B_3, B_4 and B_5 are the same as Case 1,

B_6: $\{\{0_0, (6t+2)_1, (2t-b_{2t+1})_3\}, \{0_1, (6t+2)_2, 1_3\}\}$,

B_7: $\{\{(b_{2t+1}-2t+2-r)_0, 0_2, (2+r)_3\} \mid r = 1, \ldots, 6t+2\}$.

Then (V,B) is a 4-rotational STS(v).

4.24 Lemma. If $v \equiv 21 \pmod{24}$, then there exists a 4-rotational STS(v).

Proof. $v = 24t + 21$, $t \geq 0$.

Elements: $V = (Z_{6t+5} \times Z_4) \cup \{\infty\}$

Automorphism: $\alpha = (\infty)(0_i \ldots (6t+4)_i)$, $i \in Z_4$

Base triples: $B = B_1 \cup B_2 \cup B_3 \cup B_4 \cup B_5 \cup B_6$

where we distinguish three cases.

Case 1. $t \equiv 0 \pmod 4$.

B_1: $\{\{\infty, 0_0, (6t+2)_3\}, \{\infty, 0_1, (6t+4)_2\}\}$,

B_2: $\{\{0_0, r_0, (b_r)_1\}, \{0_1, r_1, (b_r)_2\}, \{0_3, r_3 (b_r)_1\} |$
$r = 1, \ldots, 3t + 2\}$

where $\{(a_r, b_r) | r = 1, \ldots, 3t+2\}$ is a $(B, 3t+2)$-system,

B_3: $\{\{0_0, (6t+4)_1, 0_3\}\}$,

B_4: $\{\{0_2, r_2, (b_r+t)_2\} | r = 1, \ldots, t\}$

where $\{(a_r, b_r) | r = 1, \ldots, t\}$ is an (A, t)-system,

B_5: $\{\{0_2, (3t+1)_2, 0_0\}, \{0_2, (3t+2)_2, (6t+4)_3\}\}$,

B_6: $\{\{0_0, (3t+2)_2, (6t+3)_3\}, \{0_0, (6t+4)_2, (6t+4)_3\},$
$\{0_0, r_2, (2r)_3\} | r = 1, \ldots, 3t, 3t+3, \ldots, 6t+3\}$.

<u>Case 2</u>. $t \equiv 1 \pmod 4$.

B_1: $\{\{\infty, 0_0, (6t+2)_3\}, \{\infty, 0_1, 0_2\}\}$,

B_2: $\{\{0_0, r_p, (b_r)_1\}, \{0_1, r_1, (b_r)_2\}, \{0_3, r_3, (b_r)_1\} |$
$r = 1, \ldots, 3t + 2\}$

where $\{(a_r, b_r) | r = 1, \ldots, 3t + 2\}$ is an $(A, 3t+2)$-system,

B_3: $\{\{0_0, 0_1, 0_3\}\}$,

B_4, B_5 and B_6 are the same as Case 1.

<u>Case 3</u>. $t \equiv 2$ or $3 \pmod 4$.

B_1: $\{\{\infty, 0_0, (6t)_3\}, \{\infty, 0_1, (3t+3)_2\}\}$

B_2: $\{\{0_0, r_0, (b_r)_1\}, \{0_1, r_1, (b_r)_2\}, \{0_3, r_3, (b_r)_1\} \mid$
$r = 1, \ldots, 3t+2\}$

where $\{(a_r, b_r) \mid r = 1, \ldots, 3t+2\}$ is a $(C, 3t+2)$-system,

B_3: $\{\{0_0, (3t+3)_1, 0_3\}\}$,

B_4: $\{\{0_2, r_2, (b_r+t)_2\} \mid r = 1, \ldots, t\}$

where $\{(a_r, b_r) \mid r = 1, \ldots, t\}$ is a (B, t)-system,

B_5: $\{\{0_2, (3t)_2, (3t+1)_0\}, \{0_2, (3t+2)_2, (6t+2)_3\}\}$,

B_6: $\{\{0_0, 0_2 (6t+3)_3\}, \{0_0, 1_2, (6t+2)_3\},$
$\{0_0, (3t+3)_2, (6t+4)_3\}, \{0_0, (r+1)_2, (2r-1)_3\} \mid$
$r = 1, \ldots, 3t+1, 3t+4, \ldots, 6t+2\}$.

Then (V, B) is a 4-rotational $STS(v)$.

Lemmas 4.20, 4.21, 4.22, 4.23 and 4.24 together yield

4.25 <u>Theorem</u>. A 4-rotational STS(v) exists if and only if $v \equiv 1, 9, 13$ or $21 \pmod{24}$.

4.26 <u>Corollary</u>. For each order $v \equiv 1$ or $3 \pmod{6}$, there exists a k-rotational STS(v) for some $k \le 4$.

4.27 <u>Lemma</u>. If a 5-rotational STS(v) exists, then $v \equiv 1, 51, 81$ or $91 \pmod{120}$.

<u>Proof</u>. Let $V = (Z_{(v-1)/5} \times Z_5) \cup \{\infty\}$ and let $\alpha = (\infty)(0_i \ldots ((v - 1)/5 - 1))$, $i \in Z_5$, be an automorphism of a 5-rotational STS(v). If $(v - 1)/5 \equiv 0 \pmod{2}$ then $\alpha^{(v-1)/10}$ is an involution fixing exactly one element so that the STS(v) is a reverse STS(v). But a reverse STS(v) cannot exist for $v \equiv 7, 13, 15$ or $21 \pmod{21}$ thus we have $v \equiv 1, 51, 81$ or $91 \pmod{120}$ since $v \equiv 1$ or $3 \pmod{6}$ and $v \equiv 1 \pmod{5}$.

4.28 <u>Lemma</u>. If $v \equiv 51$ or $81 \pmod{120}$, then there exists a 5-rotational STS(v).

<u>Proof</u>. For $v \equiv 51$ or $81 \pmod{120}$, there exists a 1-rotational STS(v) and so does a 5-rotational STS(v).

For $v \equiv 1$ or $91 \pmod{120}$, the existence problem for 5-rotational STS(v) remains open.

4.29 <u>Theorem</u>. A 6-rotational STS(v) exists if and only if $v \equiv 1, 7$ or $19 \pmod{24}$.

<u>Proof</u>. (⇒) Let $V = (Z_{(v-1)/6} \times Z_6) \cup \{\infty\}$ and let $\alpha = (\infty)(0_i \ldots ((v-1)/6 - 1)_i)$, $i \in Z_6$, be an automorphism of a 6-rotational STS(v). If $(v-1)/6 \equiv 0 \pmod{2}$ then $\alpha^{(v-1)/12}$ is an involution fixing exactly one element so that the STS(v) is a reverse STS(v). But a reverse STS(v) cannot exist for $v \equiv 13 \pmod{24}$ thus we have $v \equiv 1, 7$ or $19 \pmod{24}$.

(⇐) For $v \equiv 1, 7$ or $19 \pmod{24}$, there exists a 2-rotational STS(v) and so does a 6-rotational STS(v).

Section 5. Regular Steiner Triple Systems and Steiner Triple Systems with an Involution Fixing Exactly 3 Elements

5.1 Definition. A design is k-regular if it admits an automorphism α consisting of k disjoint cycles of the same length.

Note that k must be a divisor of the degree of α. We may discard the trivial case when k equals the degree or when $k = 1$ since this is the case of cyclic designs.

In this section, we consider k-regular STS(v)'s. Since $v \equiv 1$ or $3 \pmod 6$, $k = 3$ or $v/3$. It is easy to see that the unique STS(9) is 3-regular. From Section 1, we have immediately the following:

5.2 Theorem. Let $k = 3$ or $v/3$. Then a k-regular STS(v) exists if and only if $v \equiv 3 \pmod 6$, $v \neq 3$.

Using integer partitions, we obtain a new construction of 3-regular STS.

5.3 Construction. Let $v = 6t + 3$, $t \geq 1$.

Elements: $V = Z_{2t+1} \times Z_3$

Automorphism: $\alpha = (0_i \ldots (2t)_i)$, $i \in Z_3$.

We distinguish two cases.

<u>Case 1</u>. $t \equiv 0$ or $1 \pmod 4$.

Base triples: $B = B_1 \cup B_2$

where

$$B_1: \ \{\{0_0,\ 0_1,\ 0_2\}\},$$

$$B_2: \ \{\{0_i,\ r_i,\ (b_r)_{i+1}\} \mid i \in Z_3,\ r = 1, \ldots, t\}$$

where $\{(a_r,b_r) \mid r = 1, \ldots, t\}$ is an (A,t)-system.

<u>Case 2</u>. $t \equiv 2$ or $3 \pmod 4$.

Base triples: $B = B_1 \cup B_2 \cup B_3$

where

$$B_1: \ \{\{0_0,\ (2t)_1,\ 1_2\}\},$$

$$B_2: \ \{\{0_0,\ r_0,\ (b_r)_1\},\ \{0_2,\ r_2,\ (b_r)_0\} \mid r = 1, \ldots, t\}$$

where $\{(a_r,b_r) \mid r = 1, \ldots, t\}$ is a (B,t)-system,

$$B_3: \ \{\{0_1,\ r_1,\ (b_r)_2\} \mid r = 1, \ldots, t\}$$

where $\{(a_r,b_r) \mid r = 1, \ldots, t\}$ is a $(-B,t)$-system. Then (V,B) is a 3-regular $STS(v)$.

Now, we consider STS with an involution fixing exactly 3 elements as an automorphism. The existence of such systems have been conjectured for every $v \equiv 1$ or $3 \pmod 6$, $v \geq 3$. In the case $v \equiv 3 \pmod 6$, such STS(v)'s can be constructed by Bose's techniques [5]. In the case $v \equiv 1 \pmod 6$, the problem is still open and we have not succeeded in this case yet. However, we give a new construction for the case $v \equiv 3 \pmod 6$.

5.4 Construction [5]. Let G be a finite multiplicative abelian group of order $2t + 1$. Set $V = G \times Z_3$ and define a collection of 3-subsets B on V as follows:

(i) $\{a_0, a_1, a_2\}$ for every $a \in G$,

(ii) $\{a_i, b_i, c_{i+1}\}$ for every $i \in Z_3$ and $a,b,c \in G$ such that $a \neq b$ and $ab = c^2$.

Then (V,B) is a $STS(6t + 3)$.

5.5 Theorem [see 24]. If $v \equiv 3 \pmod 6$, then there exists an STS(v) with an involutory automorphism fixing exactly 3 elements.

Proof. In Construction 5.4 above, consider the automorphism α of (V,B) defined by $\alpha(a_i) = (a^{-1})_i$ for every $a \in G$ and $i \in Z_3$.

5.6 Construction. Let $v = 6t + 3$.

Elements: $V = (Z_{2t} \times Z_3) \cup \{a, b, c\}$

Automorphism: $\alpha = (a)(b)(c)(0_i \ldots (2t-1)_i)$, $i \in Z_3$

Base triples: $B = B_1 \cup B_2 \cup B_3 \cup B_4$

where

B_1: $\{\{a, b, c\}, \{0_0, 0_1, 0_2\}\}$,

B_2: $\{\{0_0, t_0, a\}, \{0_1, t_1, b\}, \{0_2, t_2, c\}\}$,

B_3: $\{\{0_i, r_i, (b_r)_{i+1}\} | r = 1, \ldots, t-1, i \in Z_3\}$

where $\{(a_r, b_r) | r = 1, \ldots, t-1\}$ is an (A, t-1)-system or a (B, t-1)-system depending on whether $t \equiv 1, 2 \pmod 4$ or $t \equiv 0, 3 \pmod 4$; distinguish two cases:

Case 1. $t \equiv 1$ or $2 \pmod 4$.

B_4: $\{\{0_0, (2t-1)_1, c\}, \{0_1, (2t-1)_2, a\}, \{0_2, (2t-1)_0, b\}\}$.

Case 2. $t \equiv 0$ or $3 \pmod 4$.

B_4: $\{\{0_0, (2t-2)_1, c\}, \{0_1, (2t-2)_2, a\}, \{0_2, (2t-2)_0, b\}\}$.

Then (V, B) is an STS(v) with α^t as an involutory automorphism fixing exactly 3 elements.

CHAPTER 2. TRIPLE SYSTEMS WITH $\lambda > 1$

Section 1. Introduction.

A triple system with v elements and balance factor λ ($TS_\lambda(v)$) is a 2-design $S_\lambda(2, 3, v)$. A system $TS_\lambda(v)$ with $\lambda = 1$ is an $STS(v)$. As mentioned in Chapter 1, Kirkman [41] determined $TS_1(v)$. Bhattacharya [4] used techniques suggested by Bose [5] to completely determine $TS_2(v)$; $TS_\lambda(v)$ for every λ was determined first by Hanani [34].

1.1 Theorem. A $TS_\lambda(v)$ exists if and only if

(i) $\lambda \equiv 1, 5 \pmod 6$ and $v \equiv 1, 3 \pmod 6$ or

(ii) $\lambda \equiv 2, 4 \pmod 6$ and $v \equiv 0, 1 \pmod 3$ or

(iii) $\lambda \equiv 3 \pmod 6$ and $v \equiv 1 \pmod 2$ or

(iv) $\lambda \equiv 0 \pmod 6$ and $v \geq 3$.

Hanani's proof employs recursive construction techniques; direct proofs have been given by Nash-Williams [51] and Hwang and Lin [38].

In this chapter, we provide cyclic $TS_\lambda(v)$'s with $\lambda > 1$ determined by Colbourn and Colbourn [17] (Section 2).

In Section 3, we obtain the necessary and sufficient conditions for the existence of 1-rotational $TS_\lambda(v)$ with every $\lambda > 1$ obtained in [10].

Section 2. Cyclic Triple Systems.

Throughout this section, we will assume the set of elements of our cyclic $TS_\lambda(v)$ to be $V = Z_v$ and the corresponding cyclic automorphism to be $\alpha = (0 \ldots v-1)$.

Finding a cyclic $TS_\lambda(v)$ is equivalent to finding a suitable collection of base triples. Again, this problem can be recast as follows. Consider a collection of base triples; each base trible $\{a, b, c\}$ is represented as the collection of six differences $\{a-b, b-a, b-c, c-b, c-a, a-c\}$. To represent this set, it suffices to retain only the difference triple for the base triple, which is $\{\min(a-b, b-a), \min(b-c, c-b), \min(c-a, a-c)\}$. Let $\{x, y, z\}$ be a difference triple obtained in this manner. It is evident that either x, y and z sum to v, or one is the sum of the other two. It is further the case that none of x, y or z exceeds $v/2$. A difference triple is taken to be a triple satisfying these properties.

Following Colbourn and Colbourn [17] we denote by $D(v,\lambda)$ the multiset containing each i for $0 \le i < v/2$ λ times when v is odd. When v is even, $D(v,\lambda)$ contains in addition the difference $v/2$ $\lambda/2$ times. Thus $D(v,\lambda)$ is not defined for v even and λ odd. When $v \equiv 0 \pmod 3$, define $D_0(v,\lambda) = D(v,\lambda)$ and $D_m(v,\lambda) = D_{m-1}(v,\lambda) - \{v/3\}$. Heffter's difference problems (see Chapter 1) give a solution to the existence of cyclic $TS_\lambda(v)$'s with $\lambda \leq 1$. Colbourn and Colbourn [17] generalize Heffter's difference problems for arbitrary λ:

I. If $v \equiv 1$ or $2 \pmod 3$, can $D(v,\lambda)$ be partitioned into difference triples?

II. If $v \equiv 0 \pmod 3$, is there an m for which $D_m(v,\lambda)$ can be partitioned into difference triples?

They showed that the resolution of these two difference problems would be equivalent to a complete determination of cyclic $TS_\lambda(v)$'s.

2.1 <u>Lemma</u> [17]. If $v \equiv 2 \pmod 4$ and $\lambda \equiv 2 \pmod 4$, then there is no cyclic $TS_\lambda(v)$.

<u>Proof</u>. Since v is even, every difference triple uses either zero or two odd differences. Now $D(v,\lambda)$ contains an odd number of odd differences; in fact, for $v = 4m + 2$, it contains $2\lambda m + \lambda/2$ odd differences - this is odd since $\lambda/2$ is odd. This completes the proof when $v \equiv 1$ or $2 \pmod 3$. In the case $v \equiv 0 \pmod 3$, $v = 12m + 6$. But then the difference used by the base triple(s) of length $v/3$ is $4m + 2$ which is even. Hence the difference triples must use an odd number of odd differences and this cannot be.

Theorem 1.1 and Lemma 2.1 together yield

2.2 Lemma. If a cyclic $TS_\lambda(v)$ exists, then

(i) $\lambda \equiv 1, 5, 7, 11 \pmod{12}$ and $v \equiv 1, 3 \pmod 6$ or

(ii) $\lambda \equiv 2, 10 \pmod{12}$ and $v \equiv 0, 1, 3, 4, 7, 9 \pmod{12}$ or

(iii) $\lambda \equiv 3, 9 \pmod{12}$ and $v \equiv 1 \pmod 2$ or

(iv) $\lambda \equiv 4, 8 \pmod{12}$ and $v \equiv 0, 1 \pmod 3$ or

(v) $\lambda \equiv 6 \pmod{12}$ and $v \equiv 0, 1, 3 \pmod 4$ or

(vi) $\lambda \equiv 0 \pmod{12}$ and $v \geq 3$.

Except for $TS_1(9)$ and $TS_2(9)$, this necessary condition is sufficient. From Chapter 1, a cyclic $TS_1(v)$ exists if and only if $v \equiv 1$ or $3 \pmod 6$, $v \neq 9$. Thus we may assume that $\lambda > 1$. A simple argument demonstrates that there is no cyclic $TS_2(9)$. Further, since the existence of a cyclic $TS_\lambda(v)$ implies the existence of a cyclic $TS_{t\lambda}(v)$ for all $t \geq 1$, we consider only cyclic $TS_\lambda(v)$'s for

$\lambda = 2$, $v \equiv 0, 1, 3, 4, 7$ or $9 \pmod{12}$, $v \neq 9$,

$\lambda = 3$, $v \equiv 1 \pmod 2$,

$\lambda = 4$, $v \equiv 0$ or $1 \pmod 3$,

$\lambda = 6$, $v \equiv 0, 1$ or $3 \pmod 4$,

$\lambda = 12$, $v \geq 3$,

$\lambda \equiv 1$ or $5 \pmod 6$, $\lambda > 1$, and $v \equiv 1$ or $3 \pmod 6$.

To determine cyclic $TS_2(v)$, Colbourn and Colbourn [17] used difference triples. We present a new construction here employing integer partitions.

2.3 Lemma. If $v \equiv 0, 1, 3, 4, 7$ or $9 \pmod{12}$, $v \neq 9$, then there exists a cyclic $TS_2(v)$.

Proof. We distinguish three cases.

Case 1. $v \equiv 1, 3, 7$ or $9 \pmod{12}$, $v \neq 9$.
In this case, we have a cyclic $TS_1(v)$.

Case 2. $v \equiv 0 \pmod{12}$.
Let $v = 12t$.
Base triples: $\{\{0,4t,8t\}, \{0,4t,8t\}, \{0,r,b_r\} \mid r = 1,\ldots, 4t-1\}$

where $\{(a_r,b_r) \mid r = 1,\ldots, 4t-1\}$ is a (C, 4t-1)-system.

Case 3. $v \equiv 4 \pmod{12}$.
Let $v = 12t + 4$.
Base triples: $\{\{0,r,b_r\} \mid r = 1,\ldots, 4t+1\}$

where $\{(a_r,b_r) \mid r = 1,\ldots, 4t+1\}$ is an (A, 4t+1)-system.

2.4 <u>Theorem</u>. A cyclic $TS_2(v)$ exists if and only if $v \equiv 0, 1, 3, 4, 7$ or $9 \pmod{12}$, $v \neq 9$.

2.5 <u>Lemma</u> [17]. If $v \equiv 1 \pmod 2$, then there exists a cyclic $TS_3(v)$.

<u>Proof</u>.

<u>Case 1</u>. $2t + 1 \equiv 1$ or $2 \pmod 3$.

$\{(r,r,\min(2r,2t+1-2r)) \mid r = 1,\ldots, t\}$

partitions $D(2t+1, 3)$.

<u>Case 2</u>. $2t + 1 \equiv 0 \pmod 3$.

$\{(r,r,\min(2r,2t+1-2r)) \mid r = 1,\ldots, t, r \neq (2t+1)/3\}$

partitions $D_3(2t+1, 3)$.

In the case $\lambda = 3$, Hwang and Lin's determination of $TS_3(v)$'s [38] also gives cyclic $TS_3(v)$'s.

2.6 <u>Theorem</u>. A cyclic $TS_3(v)$ exists if and only if $v \equiv 1 \pmod 2$.

2.7 <u>Lemma</u>. If $v \equiv 0, 1, 3, 4, 7$ or $9 \pmod{12}$, then there exists a cyclic $TS_4(v)$.

<u>Proof</u>. For $v \equiv 0, 1, 3, 4, 7$ or $9 \pmod{12}$, $v \neq 9$, we have a cyclic $TS_2(v)$.

A cyclic $TS_4(9)$ has base triples $\{0,1,3\}$, $\{0,1,2\}$, $\{0,2,5\}$, $\{0,2,5\}$, $\{0,1,5\}$ and $\{0,3,6\}$.

2.8 <u>Lemma</u> [17]. If $v \equiv 6 \pmod{12}$, then there exists a cyclic $TS_4(v)$.

<u>Proof</u>. Let $v = 12t + 6$, $t \geq 1$. Consider the following difference triples:

$(2r+1,\ 3t-r+1,\ 3t+r+2)$, $\quad r = 0,\ldots, t-1$

$(2r+2,\ 5t-r+2,\ 5t+r+4)$, $\quad r = 0,\ldots, t-2$

$(2r+2,\ 3t-r-1,\ 3t+r+1)$, $\quad r = 0,\ldots, t-2$

$(2r+1,\ 5t-r+2,\ 5t+r+3)$, $\quad r = 0,\ldots, t-1$

$(2r+2,\ 3t-r+1,\ 3t+r+3)$, $\quad r = 0,\ldots, t-2$

$(2r+1,\ 5t-r+3,\ 5t+r+4)$, $\quad r = 0,\ldots, t-1$

$(2r+2,\ 3t-r,\ 3t+r+2)$, $\quad r = 0,\ldots, t-2$

$(2r+1,\ 5t-r+2,\ 5t+r+3)$, $\quad r = 0,\ldots, t-1$

$(2t,\ 2t+1,\ 4t+1)$, $\quad$ taken twice

$(3t+1,\ 3t+2,\ 6t+3)$,

$(2t+1,\ 2t+2,\ 4t+3)$,

$(2t,\ 2t,\ 4t)$,

$(3t,\ 4t+3,\ 5t+3)$.

These triples partition $D_4(12t+6, 4)$.

A cyclic $TS_4(6)$ has base triples $\{0,1,2\}$, $\{0,1,3\}$, $\{0,1,3\}$ and $\{0,2,4\}$.

2.9 Lemma [17]. If $v \equiv 10 \pmod{12}$, then there exists a cyclic $TS_4(v)$.

Proof. Let $v = 12t + 10$, $t \geq 0$. Consider the following difference triples:

$(2r+1,\ 3t-r+3,\ 3t+r+4)$, $r = 0,\ldots, t-1$

$(2r+2,\ 5t-r+4,\ 5t+r+6)$, $r = 0,\ldots, t-1$

$(2r+1,\ 3t-r+2,\ 3t+r+3)$, $r = 0,\ldots, t-1$

$(2r+2,\ 5t-r+3,\ 5t+r+5)$, $r = 0,\ldots, t-1$

$(2r+2,\ 3t-r,\ 3t+r+2)$, $r = 0,\ldots, t-1$

$(2r+1,\ 5t-r+5,\ 5t+r+6)$, $r = 0,\ldots, t-1$

$(2r+2,\ 3t-r+1,\ 3t+r+3)$, $r = 0,\ldots, t-1$

$(2r+1,\ 5t-r+3,\ 5t+r+4)$, $r = 0,\ldots, t-1$

$(2t+1,\ 2t+3,\ 4t+4)$,

$(2t+1,\ 2t+2,\ 4t+3)$,

$(4t+2,\ 4t+3,\ 4t+5)$,

$(2t+1,\ 4t+3,\ 6t+4)$,

$(3t+1,\ 4t+4,\ 5t+5)$,

$(2t+2,\ 3t+2,\ 5t+4)$.

These triples partition $D(12t+10, 4)$.

2.10 Theorem. A cyclic $TS_4(v)$ exists if and only if $v \equiv 0$ or $1 \pmod 3$.

2.11 Lemma. If $v \equiv 1$ or $3 \pmod 4$, then there exists a cyclic $TS_6(v)$.

Proof. For $v \equiv 1$ or $3 \pmod 4$, we have a cyclic $TS_3(v)$.

2.12 Lemma [17]. If $v \equiv 0 \pmod 4$, then there exists a cyclic $TS_6(v)$.

Proof. Let $v = 4t$, $t \geq 0$. Consider the following difference triples:

$$(2r-1, 2r, \min(4r-1, 4t-4r+1)), \quad r = 1, \ldots, t, \text{ taken twice}$$
$$(r, r, \min(2r, 4t-2r)), \quad r = 1, \ldots, 2t-1.$$

These triples partition $D(4t,6)$.

2.13 Theorem. A cyclic $TS_6(v)$ exists if and only if $v \equiv 0, 1$ or $3 \pmod 4$.

2.14 Lemma. If $v \not\equiv 2 \pmod{12}$, then there exists a cyclic $TS_{12}(v)$.

Proof. For $v \equiv 0, 1$ or $3 \pmod 4$, there exists a cyclic $TS_6(v)$. For $v \equiv 6$ or $10 \pmod{12}$, we have a cyclic $TS_4(v)$.

2.15 Lemma [17]. If $v \equiv 2 \pmod{12}$, then there exists a cyclic $TS_{12}(v)$.

Proof. Let $v = 12t + 2$, $t \geq 1$. Consider the following difference triples:

$(2r+2,\ 3t-r,\ 3t+r+2)$, $r = 0,\dots,t-2$, taken six times
$(2r+1,\ 5t-r+1,\ 5t+r+2)$, $r = 0,\dots,t-2$, taken six times
$(2r+1,\ 3t-r,\ 3t+r+1)$, $r = 0,\dots,t-1$, taken twice
$(2r+2,\ 5t-r,\ 5t+r+2)$, $r = 0,\dots,t-2$, taken twice
$(2r+1,\ 3t-r+1,\ 3t+r+2)$, $r = 0,\dots,t-1$, taken twice
$(2r+2,\ 5t-r,\ 5t+r+2)$, $r = 0,\dots,t-2$, taken twice
$(2r+1,\ 3t-r,\ 3t+r+1)$, $r = 0,\dots,t-1$
$(2r+2,\ 5t-r,\ 5t+r+2)$, $r = 0,\dots,t-2$
$(2r+1,\ 3t-r-1,\ 3t+r)$, $r = 0,\dots,t-1$
$(2r+2,\ 5t-r-1,\ 5t+r+1)$, $r = 0,\dots,t-2$
$(2t,\ 2t+1,\ 4t+1)$, taken seven times
$(2t,\ 3t+1,\ 5t+1)$, taken four times
$(2t+1,\ 4t+1,\ 6t)$,
$(3t+1,\ 4t,\ 5t+1)$,
$(3t+1,\ 4t+1,\ 5t)$.

These triples partition $D(12t+2, 12)$.

2.16 <u>Theorem</u>. A cyclic $TS_{12}(v)$ exists if and only if $v \geq 3$.

2.17 <u>Lemma</u>. If $\lambda \equiv 2$ or $10 \pmod{12}$, $\lambda \neq 2$, then there exists a cyclic $TS_\lambda(9)$.

<u>Proof</u>. <u>Case 1</u>. $\lambda = 12t + 2$, $t \geq 1$.
We have a cyclic $TS_{12t-8}(9)$ and a cyclic $TS_8(9)$.

<u>Case 2</u>. $\lambda = 12t + 10$.
We have a cyclic $TS_{12t+4}(9)$ and a cyclic $TS_6(9)$.

2.18 <u>Lemma</u>. If $v \equiv 1$ or $3 \pmod 6$ and $\lambda \equiv 1$ or $5 \pmod 6$, $\lambda > 1$, then there exists a cyclic $TS_\lambda(v)$.

<u>Proof</u>. <u>Case 1</u>. $\lambda = 12t + 5$, $t \geq 1$.
We have a cyclic $TS_{12t+2}(v)$ and a cyclic $TS_3(v)$.
A cyclic $TS_5(9)$ has base triples $\{0,1,2\}$, $\{0,1,3\}$, $\{0,1,4\}$, $\{0,1,5\}$, $\{0,2,4\}$, $\{0,2,5\}$, $\{0,3,6\}$ and $\{0,3,6\}$.

<u>Case 2</u>. $\lambda = 12t + 7$, $t \geq 0$.
We have a cyclic $TS_{12+4}(v)$ and a cyclic $TS_3(v)$.

<u>Case 3</u>. $\lambda = 12t + 11$, $t \geq 0$.
We have a cyclic $TS_{12t+2}(v)$ and a cyclic $TS_9(v)$.

Case 4. $\lambda = 12t + 1$, $t \geq 1$.

We have a cyclic $TS_{12t-2}(v)$ and a cyclic $TS_3(v)$.

2.19 Theorem. Let $\lambda \equiv 1$ or $5 \pmod 6$, $\lambda \neq 1$. Then a cyclic $TS_\lambda(v)$ exists if and only if $v \equiv 1$ or $3 \pmod 6$.

Summarizing we have

2.20 Theorem. The necessary condition for the existence of a cyclic $TS_\lambda(v)$

(i) $\lambda \equiv 1, 5, 7, 11 \pmod{12}$ and $v \equiv 1, 3 \pmod 6$ or

(ii) $\lambda \equiv 2, 10 \pmod{12}$ and $v \equiv 0, 1, 3, 4, 7, 9 \pmod{12}$ or

(iii) $\lambda \equiv 3, 9 \pmod{12}$ and $v \equiv 1 \pmod 2$ or

(iv) $\lambda \equiv 4, 8 \pmod{12}$ and $v \equiv 0, 1 \pmod 3$ or

(v) $\lambda \equiv 6 \pmod{12}$ and $v \equiv 0, 1, 3 \pmod 4$ or

(vi) $\lambda \equiv 0 \pmod{12}$ and $v \geq 3$,

is also sufficient with only two exceptions: There are no cyclic $TS_1(9)$ and $TS_2(9)$.

Section 3. Rotational Triple Systems.

In this section, we demonstrate that the necessary condition for the existence of $TS_\lambda(v)$ is also sufficient for the existence of 1-rotational $TS_\lambda(v)$, except for $\lambda = 1$; in the latter case, we have $v \equiv 3$ or $9 \pmod{24}$ from Chapter 1. Since the existence of a 1-rotational $TS_\lambda(v)$ also implies the existence of a 1-rotational $TS_{t\lambda}(v)$ for all $t \geq 1$, we need only to construct 1-rotational triple systems $TS_\lambda(v)$'s for

$\lambda = 2$, $v \equiv 0$ or $1 \pmod 3$,

$\lambda = 3$, $v \equiv 1 \pmod 2$,

$\lambda = 6$, $v \equiv 2 \pmod 6$,

$\lambda \equiv 1$ or $5 \pmod 6$, $\lambda > 1$, and $v \equiv 1$ or $3 \pmod 6$.

In what follows, we will always assume the set of elements of our 1-rotational $TS_\lambda(v)$ to be $V = Z_{v-1} \cup \{\infty\}$, and the corresponding automorphism to be $\alpha = (\infty)(0 \ldots v-2)$.

3.1 Lemma. If $v \equiv 0 \pmod 6$, then there exists a 1-rotational $TS_2(v)$.

Proof. Let $v = 6t$, $t \geq 1$.

Case 1. $t \equiv 1 \pmod 2$.

Base triples: $\{\{\infty,0,2t\}, \{0,r,b_r\} \mid r = 1,\ldots, 2t-1\}$

where $\{(a_r,b_r) \mid r = 1,\ldots, 2t-1\}$ is an (A, 2t-1)-system.

Case 2. $t \equiv 0 \pmod 2$.

Base triples: $\{\{\infty,0,4t-2\}, \{0,r,b_r\} \mid r = 1,\ldots, 2t-1\}$

where $\{(a_r,b_r) \mid r = 1,\ldots, 2t-1\}$ is a (B, 2t-1)-system.

3.2 Lemma. If $v \equiv 4 \pmod 6$, then there exists a 1-rotational $TS_2(v)$.

Proof. Let $v = 6t + 4$, $t \geq 0$.

Case 1. $t \equiv 0 \pmod 2$.

Base triples: $\{\{\infty,0,2t+2\}, \{0,2t+1,4t+2\}, \{0,r,b_r\} \mid r = 1,\ldots, 2t\}$

where $\{(a_r,b_r) \mid r = 1,\ldots, 2t\}$ is an (A, 2t)-system.

Case 2. $t \equiv 1 \pmod 2$.

Base triples: $\{\{\infty,0,4t\}, \{0,2t+1,4t+2\}, \{0,r,b_r\} \mid r = 1,\ldots, 2t\}$

where $\{(a_r,b_r) \mid r = 1,\ldots, 2t\}$ is a (B, 2t)-system.

3.3 Lemma. If $v \equiv 1 \pmod 6$, then there exists a 1-rotational $TS_2(v)$.

<u>Proof</u>. Let $v = 6t + 1$, $t \geq 1$.

<u>Case 1</u>. $t \equiv 1 \pmod 2$.

(i) $t = 1$:

Base triples: $\{\infty,0,3\}$, $\{\infty,0,3\}$, $\{0,2,4\}$, $\{0,1,2\}$.

(ii) $t \geq 1$:

Base triples: $\{\{\infty,0,2t+1\}, \{0,2t,4t\}, \{0,r,b_r\} \mid r = 1,\ldots, 2t-1\}$

where $\{(a_r,b_r) \mid r = 1,\ldots, 2t-1\}$ is an (A, 2t-1)-system.

<u>Case 2</u>. $t \equiv 0 \pmod 2$.

(i) $t = 2$:

Base triples: $\{\infty,0,6\}$, $\{\infty,0,6\}$, $\{0,4,8\}$, $\{0,1,3\}$, $\{0,2,7\}$, $\{0,3,4\}$.

(ii) $t > 2$:

Base triples: $\{\{\infty,0,4t-2\}, \{0,2t,4t\}, \{0,r,b_r\} \mid r = 1,\ldots, 2t-1\}$

where $\{(a_r,b_r) \mid r = 1,\ldots, 2t-1\}$ is a (B, 2t-1)-system.

3.4 <u>Lemma</u>. If $v \equiv 3 \pmod 6$, then there exists a 1-rotational $TS_2(v)$.

<u>Proof</u>. Let $v = 6t + 3$, $t \geq 0$.

<u>Case 1</u>. $t \equiv 0 \pmod 2$.

(i) $t = 0$:

Base triples: $\{\infty,0,1\}$, $\{\infty,0,1\}$.

(ii) $t > 0$:

Base triples: $\{\{\infty,0,2t+1\}, \{0,r,b_r\} \mid r = 1,\dots, 2t\}$

where $\{(a_r,b_r) \mid r = 1,\dots, 2t\}$ is an $(A, 2t)$-system.

Case 2. $t \equiv 1 \pmod 2$.

(i) $t = 1$:

Base triples: $\{\infty,0,4\}$, $\{\infty,0,4\}$, $\{0,1,2\}$, $\{0,2,5\}$.

(ii) $t > 1$:

Base triples: $\{\{\infty,0,4t\}, \{0,r,b_r\} \mid r = 1,\dots, 2t\}$

where $\{(a_r,b_r) \mid r = 1,\dots, 2t\}$ is a $(B, 2t)$-system.

The 1-rotational $TS_2(v)$'s constructed above have no repeated triples, except for $v = 3, 7, 9$ and 13.

3.5 Theorem. A 1-rotational $TS_2(v)$ exists if and only if $v \equiv 0$ or $1 \pmod 3$.

3.6 Corollary. A 1-rotational $TS_4(v)$ exists if and only if $v \equiv 0$ or $1 \pmod 3$.

3.7 Corollary. If $v \equiv 0$ or $1 \pmod 3$, then there exists a 1-rotational $TS_6(v)$.

3.8 Lemma. If $v \equiv 1 \pmod 2$, then there exists a 1-rotational $TS_3(v)$.

Proof. Let $v = 2t + 1$, $t \geq 1$.

Case 1. $t \equiv 1$ or $2 \pmod 4$.

(i) $t = 1$:

Base triples: $\{\infty,0,1\}$, $\{\infty,0,1\}$, $\{\infty,0,1\}$.

(ii) $t > 1$:

Base triples: $\{\{\infty,0,1\}, \{\infty,0,t\}, \{0,r,b_r\} \mid r = 1,\ldots, t-1\}$

where $\{(a_r,b_r) \mid r = 1,\ldots, t-1\}$ is an $(A, t-1)$-system.

Case 2. $t \equiv 0$ or $3 \pmod 4$.

Base triples: $\{\{\infty,0,2\}, \{\infty,0,t\}, \{0,r,b_r\} \mid r = 1,\ldots, t-1\}$

where $\{(a_r,b_r) \mid r = 1,\ldots, t-1\}$ is an $(B, t-1)$-system.

The (trivial) 1-rotational $TS_3(3)$ has, of course, repeated triples; the 1-rotational $TS_3(v)$'s of other orders constructed above have no repeated triples.

3.9 Theorem. A 1-rotational $TS_3(v)$ exists if and only if $v \equiv 1 \pmod{2}$.

3.10 Corollary. If $v \equiv 1 \pmod{2}$, then there exists a 1-rotational $TS_6(v)$.

3.11 Lemma. If $v \equiv 2 \pmod{6}$, then there exists a 1-rotational $TS_6(v)$.

Proof. Let $v = 6t + 2$, $t \geq 1$.

Case 1. $t \equiv 2$ or $3 \pmod{4}$.

Base triples: $\{\infty,0,2\}$, $\{\infty,0,2\}$, $\{\infty,0,1\}$, $\{0,3t,3t+1\}$,
$\{\{0,r,b_r\}, \{0,r,b_r\} | r = 1,\ldots, 3t-1\}$
where $\{(a_r,b_r) | r = 1,\ldots, 3t-1\}$ is an (A, 3t-1)-system.

Case 2. $t \equiv 0$ or $1 \pmod{4}$.

Base triples: $\{\infty,0,3\}$, $\{\infty,0,3\}$, $\{\infty,0,1\}$, $\{0,3t,3t+1\}$,
$\{\{0,r,b_r\}, \{0,r,b_r\} | r = 1,\ldots, 3t-1\}$
where $\{(a_r,b_r) | r = 1,\ldots, 3t-1\}$ is a (B, 3t-1)-system.

Corollaries 3.7, 3.10 and Lemma 3.11 together yield

3.12 Theorem. A 1-rotational $TS_6(v)$ exists if and only if $v \geq 3$.

3.13 Lemma. If $v \equiv 1$ or $3 \pmod 6$ and $\lambda \equiv 1$ or $5 \pmod 6$, $\lambda > 1$, then there exists a 1-rotational $TS_\lambda(v)$.

Proof. Case 1. $\lambda = 6t + 1$, $t \geq 1$. Then we have a 1-rotational $TS_{6t-2}(v)$ and a 1-rotational $TS_3(v)$.

Case 2. $\lambda = 6t + 5$, $t \geq 0$. Then we have a 1-rotational $TS_{6t+2}(v)$ and a 1-rotational $TS_3(v)$.

Summarizing, we have

3.14 Theorem. A 1-rotational $TS_\lambda(v)$ exists if and only if

$\lambda = 1$ and $v \equiv 3$ or $9 \pmod{24}$ or

$\lambda \equiv 1$ or $5 \pmod 6$, $\lambda > 1$, and $v \equiv 1$ or $3 \pmod 6$ or

$\lambda \equiv 2$ or $4 \pmod 6$ and $v \equiv 0$ or $1 \pmod 3$ or

$\lambda \equiv 3 \pmod 6$ and $v \equiv 1 \pmod 2$ or

$\lambda \equiv 0 \pmod 6$ and $v \geq 3$.

CHAPTER 3. EXTENDED TRIPLE SYSTEMS

Section 1. Introduction.

The concept of an extended triple system was introduced by Johnson and Mendelsohn [40]. An extended triple system is a pair (V,B) where B is a finite set and B is a collection of 3-subsets of V (called blocks or triples), where each triple may have repeated elements, such that every pair of elements of V, not necessarily distinct, is contained in exactly one triple of B. The triples of B are of three types:

$$\{a,a,a\},\ \{b,b,c\},\ \{x,y,z\}$$

where the element a is called an idempotent and b a non-idempotent of the system (V,B).

We will denote by $ETS(v;p)$ an extended triple system on v elements which has p idempotents. Obviously, we have $0 \le p \le v$. Johnson and Mendelsohn [40] obtained necessary conditions for the existence of $ETS(v;p)$'s and Bennett and Mendelsohn [3] showed that these necessary conditions were also sufficient.

1.1 Theorem [3]. Let $0 \le p \le v$. Then an $ETS(v;p)$ exists if and only if

(i) if $v \equiv 0 \pmod 3$ then $p \equiv 0 \pmod 3$,

(ii) if $v \equiv 1$ or $2 \pmod 3$ then $p \equiv 1 \pmod 3$,

(iii) if v is even then $p \leq v/2$,

(iv) if $p = v - 1$ then $v = 2$.

In this chapter, we provide ETS(v;p) with prescribed automorphism types. In Section 2, we determine completely cyclic ETS(v;p)'s. In Section 3, we obtain necessary and sufficient conditions for the existence of 1- and 2-rotational ETS(v;p)'s. Further, we obtain necessary conditions for the existence of 3-rotational ETS(v;p) and show that these conditions are also sufficient, except possibly for $v \equiv 0 \pmod{18}$, $p = (v + 2)/3$ and $v \equiv 37$ or $55 \pmod{72}$, $p = (v + 2)/3$ or $(2v + 1)/3$. In Section 4, we determine completely 2- and 3-regular ETS(v;p)'s and 4-regular ETS(v;p)'s, except possibly for $v \equiv 12$ or $20 \pmod{24}$ and $p = v/2$.

Section 2. Cyclic Extended Triple Systems.

Let us assume in this section the set of elements of our cyclic ETS(v;p) to be Z_v and the corresponding cyclic automorphism to be $\alpha = (0 \ldots v-1)$. If $\{a, b, c\}$ is a block of a cyclic ETS(v;p) , then $\{a+1, b+1, c+1\}$ is also a block of the cyclic ETS(v;p) and hence we have $p = 0$ or v .

2.1 Lemma. Necessary conditions for the existence of a cyclic ETS(v;p) are

(i) if $p = v$ then $v \equiv 1$ or $3 \pmod 6$, $v \neq 9$,

(ii) if $p = 0$ then $v \equiv 3 \pmod 6$.

Proof. (i) It follows from the fact that a system obtained by deleting all blocks containing idempotents of a cyclic ETS(v;p) is a cyclic STS(v) .

(ii) If $p = 0$ then $v \equiv 0 \pmod 3$ by the existence of an ETS(v;0) . In the case $v \equiv 0 \pmod 6$, the possible lengths of orbits of a cyclic ETS(v;0) are v or $v/3$. Let m, n be the number of a cyclic ETS(v;0) whose lengths are v , $v/3$, respectively. Then we have

$$(2.1.1) \qquad mv + n(v/3 = \binom{v}{2} \Big/ \binom{3}{2} + (2v)/3 \, .$$

and hence

$$(2.1.2) \qquad 6m + 2n = v + 3 \, .$$

But there are no such integers m, n satisfying (2.1.2) because the left-hand side of (2.1.2) is even and its right-hand side is odd since v is even.

For $v \equiv 1$ or $3 \pmod 6$, $v \neq 9$, there exists a cyclic STS(v). The next theorem follows immediately:

2.2 <u>Theorem</u>. A cyclic ETS(v;v) exists if and only if $v \equiv 1$ or $3 \pmod 6$, $v \neq 9$.

2.3 <u>Lemma</u>. There exists a cyclic ETS(9;0).

<u>Proof</u>. {0, 0, 2} and {0, 1, 4} form base triples of a cyclic ETS(9;0).

Note that if $v \equiv 3 \pmod 6$, then a cyclic STS(v) always contains the base triple $\{0, v/3, 2v/3\}$.

2.4 <u>Lemma</u>. If $v \equiv 3 \pmod 6$, $v \neq 9$, then there exists a cyclic ETS(v;0).

Proof. Let $v = 6t + 3$, $t \neq 1$.

Base triples: $B = B_1 \cup B_2$

where

B_1: $\{\{0, 0, 2t+1\}\}$

B_2: the collection of all base triples except $\{0, 2t+1, 4t+2\}$ of a cyclic $STS(6t+3)$.

Then (V,B) is a cyclic $ETS(v;0)$.

2.5 Theorem. A cyclic $ETS(v;0)$ exists if and only if $v \equiv 3 \pmod 6$.

Section 3. Rotational Extended Triple Systems.

An extended triple system ETS(v;p) is *k-rotational* if it admits an automorphism consisting of a single fixed element and k disjoint cycles of the same length. By an elementary argument, we obtain easily the following lemma:

3.1 *Lemma*. If a k-rotational ETS(v;p) exists, then $p = t\{(v - 1)/k\} + 1$, $t = 0,\ldots, k$.

3.2 *Lemma*. Necessary conditions for the existence of a 1-rotational ETS(v;p) are

(i) if $p = v$ then $v \equiv 3$ or 9 (mod 24),

(ii) if $p = 1$ then $v \equiv 1$ or 2 (mod 3),

Proof. (i) Follow from the existence of 1-rotational STS's.

(ii) By the existence of ETS(v;1)'s.

Immediately, we have the following theorem:

3.3 *Theorem*. A 1-rotational ETS(v;v) exists if and only if $v \equiv 3$ or 9 (mod 24).

Throughout this section, we will assume the set of elements of our k-rotational ETS(v;p) to be $V = (Z_{(v-1)/k} \times Z_k) \cup \{\infty\}$, and the corresponding automorphism to be $\alpha = (\infty)(0_i \ldots ((v-1)/k - 1)_i)$, $i \in Z_k$. In the case $k = 1$, we write for brevity $V = Z_{(v-1)} \cup \{\infty\}$ instead of $V = (Z_{(v-1)} \times Z_1) \cup \{\infty\}$.

3.4 Lemma. There is no 1-rotational ETS(10;1).

Proof. If there were a 1-rotational ETS(10;1), then it must contain base triples of the forms $\{\infty, \infty, \infty\}$ and $\{\infty, 0, 0\}$. Deleting these base triples would yield a cyclic STS(9) which does not exist.

3.5 Lemma. If $v \equiv 4 \pmod 6$, $v \neq 10$, then there exists a 1-rotational ETS(v;1).

Proof. Let $v = 6t + 4$; $t \neq 1$.

Base triples: $B = B_1 \cup B_2$

where

B_1: $\{\{\infty, \infty, \infty\}, \{\infty, 0, 0\}\}$,

B_2: the collection of all base triples of a cyclic STS(6t + 3) based on Z_{6t+3}.

Then (V,B) is a 1-rotational ETS(v;1).

3.6 Definition. A (H,k)-system is a set of ordered pairs $\{(a_r,b_r) \mid r = 1,\ldots, k\}$ such that $b_r - a_r = r$ for $r = 1,\ldots, k$ and $\bigcup_{r=1}^{k} \{a_r,b_r\} = \{1,\ldots, k+1, k+3,\ldots, 2k+1\}$.

3.7 Lemma. A (H,k)-system exists if and only if $k \equiv 1$ or $2 \pmod 4$.

Proof. ($\Rightarrow$) Let $\{(a_r,b_r) \mid r = 1,\ldots, k\}$ be a (H,k)-system. Then we have

(3.7.1) $$\sum_{r=1}^{k} (b_r - a_r) = \frac{k(k + 1)}{2}$$

and

(3.7.2) $$\sum_{r=1}^{k} (b_r + a_r) = \frac{(2k + 1)(2k + 2)}{2} - (k + 2) .$$

Adding both sides of (3.7.1) and (3.7.2), respectively, we get

(3.7.3) $$2\sum_{r=1}^{k} b_r = \frac{5k^2 + 5k - 2}{2} .$$

Since $\sum_{r=1}^{k} b_r$ is an integer, $5k^2 + 5k - 2 \equiv 0 \pmod 4$ and hence we have $k \equiv 1$ or $2 \pmod 4$.

(⇐) Before giving the general constructions we present the solutions for $k = 1, 2, 5$ and 6.

$k = 1$: (1,2) .

$k = 2$: (1,2), (3,5) .

$k = 5$: (10,11), (2,4), (6,9), (1,5), (3,8) .

$k = 6$: (11,12), (3,5), (10,13), (2,6), (4,9), (1,7) .

$k = 4t + 1$, $t \geq 2$.

$(r,\ 4t+2-r)$, $r = 1,\dots,\ 2t$

$(4t+3+r,\ 8t+4-r)$, $r = 1,\dots,\ t-1$

$(5t+2+r,\ 7t+3-r)$, $r = 1,\dots,\ t-1$

$(2t+1,\ 6t+2)$, $(4t+2,\ 6t+3)$, $(7t+3,\ 7t+4)$.

$k = 4t + 2$, $t \geq 2$.

$(r,\ 4t+4-r)$, $r = 1,\dots,\ 2t+1$

$(4t+4+r,\ 8t+5-r)$, $r = 1,\dots,\ t-1$

$(5t+3+r,\ 7t+4-r)$, $r = 1,\dots,\ t-1$

$(2t+2,\ 6t+3)$, $(6t+4,\ 8t+5)$, $(7t+4,\ 7t+5)$.

3.8 Lemma. If $v \equiv 13$ or $19 \pmod{24}$, then there exists a 1-rotational ETS$(v;1)$.

Proof. Let $v = 6t + 1$ and $t \equiv 2$ or $3 \pmod 4$.

Base triples: $B = B_1 \cup B_2$

where

B_1: $\{\{\infty,\infty,\infty\}, \{\infty,0,3t\}, \{0,2t,4t\}, \{0,0,3t-1\}$,

B_2: $\{\{0,r,b_r+t-1\} \mid r = 1,\ldots, t-1\}$

where $\{(a_r,b_r) \mid r = 1,\ldots, t-1\}$ is a $(H, t-1)$-system. Then (V,B) is a 1-rotational ETS$(v;1)$.

3.9 Definition. An (I,k)-system is a set of ordered pairs $\{(a_r,b_r) \mid r = 1,\ldots, k\}$ such that $b_r - a_r = r$ for $r = 1,\ldots, k$ and $\bigcup_{r=1}^{k} \{a_r,b_r\} = \{1,\ldots, k+1, k+3,\ldots, (3k+1)/2 + 1, (3k+1)/2 + 3,\ldots, 2k+2\}$.

3.10 Lemma. An (I,k)-system exists if and only if k is odd.

Proof. ($\Rightarrow$) It follows that $(3k+1)/2$ is an integer.

($\Leftarrow$) $k = 4t + 3$

$(r, 4t+5-r)$, $\quad r = 1,\ldots, 2t+2$

$(4t+5+r, 8t+9-r)$, $\quad r = 1,\ldots, 2t+1$.

$k = 4t + 1$

$(r, 4t+3-r)$, $\quad r = 1,\ldots, 2t+1$,

$(4t+3+r, 8t+5-r)$, $\quad r = 1,\ldots, 2t$

3.11 Lemma. If $v \equiv 1 \pmod{24}$, then there exists a 1-rotational $ETS(v;1)$.

Proof. Let $v = 6t + 1$ and $t \equiv 0 \pmod 4$, $t \geq 0$. Base triples: $B = B_1 \cup B_2$

where

$$B_1: \{\{\infty,\infty,\infty\}, \{\infty,0,3t\}, \{0,2t,4t\}, \{0,0,(5t)/2\}\},$$

$$B_2: \{\{0,r,b_r+t-1\} \mid r = 1,\ldots, t-1\}$$

where $\{(a_r,b_r) \mid r = 1,\ldots, t-1\}$ is an (I, t-1)-system. When $t = 0$, take $B = \{\{\infty, \infty, \infty\}\}$. Then (V,B) is a 1-rotational $ETS(v;1)$.

3.12 Definition. A (J,k)-system is a set of ordered pairs $\{(a_r,b_r) \mid r = 1,\ldots, k\}$ such that $b_r - a_r = r$ for $r = 1,\ldots, k$ and $\bigcup_{r=1}^{k} \{a_r,b_r\} = \{1,\ldots, k/2, k/2+2,\ldots, k+1, k+3,\ldots, 2k+2\}$.

3.13 Lemma. A (J,k)-system exists if and only if k is even.

Proof. ($\Rightarrow$) It follows that $k/2$ is an integer.

(-) $k = 4t + 2$.

$(r,\ 4t+4-r)$, $r = 1, \ldots, 2t + 1$,

$(4t+4+r,\ 8t+7-r)$, $r = 1, \ldots, 2t + 1$.

$k = 4t$.

$(r,\ 4t+2-r)$, $r = 1, \ldots, 2t$,

$(4t+2+r,\ 8t+3-r)$, $r = 1, \ldots, 2t$.

3.14 Lemma. If $v \equiv 7 \pmod{24}$, then there exists a 1-rotational $ETS(v;1)$.

Proof. Let $v = 6t + 1$ and $t \equiv 1 \pmod 4$, $t > 1$. Base triples: $B = B_1 \cup B_2$

where

$$B_1: \{\{\infty,\infty,\infty\}, \{\infty,0,3t\}, \{0,2t,4t\}, \{0,0,3(t-1)/2+1\}\},$$

$$B_2: \{\{0,r,b_r+t-1\} \mid r = 1, \ldots, t-1\}$$

where $\{(a_r,b_r) \mid r = 1, \ldots, t-1\}$ is a $(J, t-1)$-system. When $t = 1$, take $B = B_1$. Then (V,B) is a 1-rotational $ETS(v;1)$.

3.15 Lemma. If $v \equiv 2 \pmod 6$, then there exists a 1-rotational $ETS(v;1)$.

<u>Proof</u>. Let $v = 6t + 2$.

Base triples: $B = B_1 \cup B_2$

where

B_1: $\{\{\infty, \infty, \infty\}, \{\infty, 0, 0\}\}$,

B_2: the collection of all base triples of a cyclic $STS(6t + 1)$ based on Z_{6t+1}.

Then (V,B) is a 1-rotational $ETS(v;1)$.

3.16 <u>Lemma</u>. If $v \equiv 5 \pmod 6$, then there exists a 1-rotational $ETS(v;1)$.

<u>Proof</u>. Let $v = 6t + 5$.

Base triples: $B = B_1 \cup B_2$.

where

$$B_1: \begin{cases} \{\{\infty,\infty,\infty\}, \{\infty,0,3t+2, \quad 0,0,3t+1\}\} & \text{if } t \equiv 0 \text{ or } 1 \pmod 4; \\ \{\{\infty,\infty,\infty\}, \{\infty,0,3t+2\}, \{0,0,3t\} & \text{if } t \equiv 2 \text{ or } 3 \pmod 4, \end{cases}$$

B_2: $\{\{0,r,b_r+t\} \mid r = 1,\ldots, t\}$

where $\{(a_r,b_r) \mid r = 1,\ldots, t\}$ is an (A,t)-system or a (B,t)-system depending whether $t \equiv 0, 1 \pmod 4$ or $t \equiv 2, 3$.

(mod 4) . Then (V,B) is a 1-rotational $ETS(v;1)$.

3.17 <u>Theorem</u>. A 1-rotational $ETS(v;1)$ exists if and only if $v \equiv 1$ or $2 \pmod 3$, $v \neq 10$.

Let us now construct 2-rotational $ETS(v;p)$'s . From Lemma 3.1, if a 2-rotational $ETS(v;p)$ exists then $p = 1$, $(v + 1)/2$ or v .

3.18 <u>Lemma</u>. Necessary conditions for the existence of a 2-rotational $ETS(v;p)$ are

(i) if $p = v$ then $v \equiv 1, 3, 7, 9, 15$ or $19 \pmod{24}$,

(ii) if $p = (v + 1)/2$ then $v \equiv 1 \pmod 6$,

(iii) if $p = 1$ then $v \equiv 1$ or $5 \pmod 6$.

<u>Proof</u>. (i) Follow from the existence of 2-rotational STS's .

(ii) Let $p = (v + 1)/2$. Then $(v + 1)/2 \equiv 0$ or $1 \pmod 3$ since the existence of $ETS(v;p)$'s implies $p \equiv 0$ or $1 \pmod 3$. If $(v + 1)/2 \equiv 0 \pmod 3$ then $v \equiv 5 \pmod 6$. Since $p \equiv 0 \pmod 3$ implies $v \equiv 0 \pmod 3$, $v \equiv 5 \pmod 6$ is impossible. So we only have $(v + 1)/2 \equiv 1 \pmod 3$ and hence $v \equiv 1 \pmod 6$.

(iii) If $p = 1$ then $v \equiv 1$ or $2 \pmod 3$ and hence $v \equiv 1$ or $5 \pmod 6$ since $(v - 1)/2$ is an integer, i.e., v is odd.

From the existence of 2-rotational $STS(v)$'s and 1-rotational $ETS(v;1)$'s, respectively, we have easily the following two theorems.

3.19 Theorem. A 2-rotational $ETS(v;v)$ exists if and only if $v \equiv 1, 3, 7, 9, 15$ or $19 \pmod{24}$.

3.20 Theorem. A 2-rotational $ETS(v;1)$ exists if and only if $v \equiv 1$ or $5 \pmod 6$.

3.21 Lemma. There is a 2-rotational $ETS(19;10)$.

Proof. Base triples: $B = B_1 \cup B_2$

where

B_1: $\{\{\infty,\infty,\infty\}, \{0_0,0_0,0_0\}, \{\infty,0_0,0_1\}, \{0_1,0_1,4_1\}, \{0_1,2_1,8_1\}$

B_2: $\{\{0_0,r_0,(b_r)_1\} \mid r = 1, 2, 3, 4\}$

where $\{(a_r,b_r) \mid r = 1, 2, 3, 4\}$ is an $(A,4)$-system. Then (V,B) is a 2-rotational $ETS(19;10)$.

3.22 <u>Lemma</u>. If $v \equiv 7 \pmod{12}$, $v \neq 19$, then there exists a 2-rotational $ETS(v; (v+1)/2)$.

<u>Proof</u>. Let $v = 12t + 7$, $t \neq 1$.

Base triples: $B = B_1 \cup B_2 \cup B_3$

where

$$B_1: \begin{cases} \{\{\infty,\infty,\infty\}, \{0_0,0_0,0_0\}, \{\infty,0_0,0_1\}, \{0_1,0_1,(2t+1)_1\}\} & \text{if } t \equiv 0 \text{ or } 1 \pmod 4; \\ \{\{\infty,\infty,\infty\}, \{0_0,0_0,0_0\}, \{\infty,0_0,(6t+2)_1\}, \{0_1,0_1,(2t+1)_1\}\} & \text{if } t \equiv 2 \text{ or } 3 \pmod 4, \end{cases}$$

B_2: $\{\{0_0, r_0, (b_r)_1\} \mid r = 1, \ldots, 3t + 1\}$

where $\{(a_r, b_r) \mid r = 1, \ldots, 3t + 1\}$ is an $(A, 3t+1)$-system or a $(B, 3t+1)$-system depending on whether $t \equiv 0, 1 \pmod 4$ or $t \equiv 2, 3 \pmod 4$,

B_3: the collection of all base triples of a cyclic $STS(6t+3)$ except the base triple $\{0_1, (2t+1)_1, (4t+2)_1\}$ based on $Z_{6t+3} \times \{1\}$.

Then (V,B) is a 2-rotational $ETS(v; (v+1)/2)$.

3.23 <u>Definition</u>. A <u>(K,k)-system</u> is a set of ordered pairs $\{(a_r, b_r) \mid r = 1, \ldots, k\}$ such that $b_r - a_r = r$ for

$r = 1, \ldots, k$ and $\bigcup_{r=1}^{k} \{a_r, b_r\} = \{1, \ldots, k-1, k+1, \ldots, 2k+1\}$.

3.24 Lemma. A (K,k)-system exists if and only if $k \equiv 1$ or $2 \pmod 4$.

Proof. ($\Rightarrow$) Let $\{(a_r, b_r) \mid r = 1, \ldots, k\}$ be a (K,k)-system. Then we have

$$\sum_{r=1}^{k} (b_r - a_r) = \frac{k(k+1)}{2} \tag{3.24.1}$$

and

$$\sum_{r=1}^{k} (b_r + a_r) = \frac{(2k+1)(2k+2)}{2} - k. \tag{3.24.2}$$

Adding both sides of (3.24.1) and (3.24.2), respectively, gives

$$2\sum_{r=1}^{k} b_r = \frac{5k^2 + 5k + 2}{2}. \tag{3.24.3}$$

Since $\sum_{r=1}^{k} b_r$ is an integer, $5k^2 + 5k + 2 \equiv 0 \pmod 4$ and hence $k \equiv 1$ or $2 \pmod 4$.

(⟸) $k = 1$: $(2,3)$.

$k = 2$: $(4,5)$, $(1,3)$.

$k = 4t + 1$, $t \geq 1$.

$(r, 4t+1-r)$, $r = 1,\ldots, t-1$,

$(t+1+r, 3t+2-r)$, $r = 1,\ldots, t-1$,

$(4t+2+r, 8t+4-r)$, $r = 1,\ldots, 2t$;

$(2t+1, 4t+2)$, $(2t+2, 6t+3)$, $(t, t+1)$.

$k = 4t + 2$, $t \geq 1$.

$(r, 4t+2-r)$, $r = 1,\ldots, 2t$,

$(4t+3+r, 8t+6-r)$, $r = 1,\ldots, t$,

$(5t+3+r, 7t+4-r)$, $r = 1,\ldots, t-1$,

$(2t+1, 6t+3)$, $(4t+3, 6t+4)$, $(7t+4, 7t+5)$.

3.25 Lemma. If $v \equiv 13$ or $25 \pmod{48}$, then there exists a 2-rotational ETS$(v; (v+1)/2)$.

Proof. Let $v = 12t + 1$ and $t \equiv 1$ or $2 \pmod 4$.

Base triples: $B = B_1 \cup B_2 \cup B_3$

where

B_1: $\{\{\infty,\infty,\infty\}, \{\infty,0_0,(3t)_0\}, \{\infty,0_1,(3t)_1\}, \{0_0,0_0,0_0\},$
$\{0_0,0_1,(3t-1)_1\}\}$,

B_2: $\{\{0_0, r_0, (b_r)_1\} \mid r = 1, \ldots, 3t - 1\}$

where $\{(a_r, b_r) \mid r = 1, \ldots, 3t - 1\}$ is a (K, 3t-1)-system,

B_3: $\{\{0_1, r_1, (b_r + t - 1)_1\} \mid r = 1, \ldots, t - 1\}$

where $\{(a_r, b_r) \mid r = 1, \ldots, t - 1\}$ is an (A, t-1)-system.
Then (V,B) is a 2-rotational ETS(v; (v+1)/2).

3.26 <u>Definition</u>. A <u>(G,k)-system</u> is a set of ordered pairs (a_r, b_r) $r = 1, \ldots, k$ such that $b_r - a_r = r$ for $r = 1, \ldots, k$ and $\bigcup_{r=1} \{a_r, b_r\} = \{1, \ldots, k/2, k/2 + 2, \ldots, 2k + 1\}$.

3.27 <u>Lemma</u>. A (G,k)-system exists if and only if k is even.

<u>Proof</u>. (⇒) Since $k/2$ is an integer, k is even.

(⇐) $k = 4t$.

$(4t+1+r,\ 8t+2-r)$, $\quad r = 1, \ldots, 2t$,

$(r,\ 4t+2-r)$, $\quad r = 1, \ldots, 2t$.

$k = 4t + 2$.

$(4t+3+r,\ 8t+6-r)$, $\qquad r = 1,\dots,\ 2t + 1$,

$(r,\ 4t+4-r)$, $\qquad r = 1,\dots,\ 2t + 1$.

3.28 <u>Definition</u>. An <u>(L,k)-system</u> is a set of ordered pairs $\{(a_r,b_r) \mid r = 1,\dots,\ k\}$ such that $b_r - a_r = r$ for $r = 1,\dots,\ k$ and $\bigcup_{r=1}^{k} \{a_r,b_r\} = \{1,\dots,\ k/2 + 1,\ k/2 + 3,\dots,\ k + 2,\ k + 4,\dots,\ 2k + 2\}$.

3.29 <u>Lemma</u>. An (L,k)-system exists if and only if k is even.

<u>Proof</u>. ($\Rightarrow$) Since $k/2$ is an integer, k is even.

($\Leftarrow$) k is even.

$(2+r,\ k+3-r)$, $\qquad r = 1,\dots,\ k/2 - 1$,

$(k+3+r,\ 2k+3-r)$, $\qquad r = 1,\dots,\ k/2 - 1$,

$(1,2)$, $(k/2 + 3,\ 3k/2 + 3)$.

3.30 <u>Lemma</u>. If $v \equiv 37 \pmod{48}$, then there exists a 2-rotational $ETS(v;\ (v+1)/2)$.

<u>Proof</u>. Let $v = 12t + 1$ and $t \equiv 3 \pmod 4$.

Base triples: $B = B_1 \cup B_2 \cup B_3$

where

B_1: $\{\{\infty,\infty,\infty\},\ \{0_0,0_0,0_0\},\ \{\infty,0_0,(3t)_0\},\ \{\infty,0_1,(3t)_1\},$
$\{0_0,0_1,((3t+1)/2)_1\},\ \{0_1,0_1,(2t+1)_1\}\}$,

B_2: $\{\{0_0,r_0,(b_r)_1\} \mid r = 1,\ldots, 3t-1\}$

where $\{(a_r,b_r) \mid r = 1,\ldots, 3t-1\}$ is a $(G, 3t-1)$-system,

B_3: $\{\{0_1,r_1,(b_r+t-1)_1\} \mid r = 1,\ldots, t-1\}$,

where $\{(a_r,b_r) \mid r = 1,\ldots, t-1\}$ is a $(L, t-1)$-system. Then (V,B) is a 2-rotational ETS$(v; (v+1)/2)$.

3.31 <u>Definition</u>. A <u>(M,k)-system</u> is a set of ordered pairs $\{(a_r,b_r) \mid r = 1,\ldots, k\}$ such that $b_r - a_r = r$ for $r = 1,\ldots, k$ and $\bigcup_{r=1}^{k} \{a_r,b_r\} = \{1,\ldots, (k+1)/2,$ $(k+1)/2 + 2,\ldots, k+1, k+3,\ldots, 2k+2\}$.

3.32 <u>Lemma</u>. A (M,k)-system exists if and only if $k > 1$ is odd.

<u>Proof</u>. Since $(k+1)/2$ is an integer, k is odd. Obviously, there is no $(M,1)$-system.

Conversely, let $k > 1$ be odd and take the following ordered pairs:

$(2+r,\ k+2-r)$, $\quad r = 1, \ldots, (k-1)/2 - 1$

$(k+2+r,\ 2k+3-r)$, $\quad r = 1, \ldots, (k-1)/2$

$(1,2)$, $((k+1)/2 + 2,\ (3k+1)/2 + 2)$.

3.33 Lemma. If $v \equiv 1 \pmod{48}$, then there exists a 2-rotational ETS$(v; (v+1)/2)$.

Proof. Let $v = 12t + 1$ and $t \equiv 0 \pmod 4$, $t \neq 0$. Base triples: $B = B_1 \cup B_2 \cup B_3$

where

$$B_1: \{\{\infty,\infty,\infty\}, \{0_0,0_0,0_0\}, \{\infty,0_0,(3t)_0\}, \{\infty,0_1,(3t)_1\}, \{0_0,0_1,((3t)/2)_1\}, \{0_1,0_1,(2t)_1\}\},$$

$$B_2: \{\{0_0,r_0,(b_r)_1\} \mid r = 1,\ldots, 3t-1\}$$

where $\{(a_r,b_r) \mid r = 1,\ldots, 3t-1\}$ is an $(E, 3t-1)$-system (see, Lemma 4.14 in Chapter 1),

$$B_3: \{\{0_1,r_1,(b_r+t-1)_1\} \mid r = 1,\ldots, t-1\}$$

where $\{(a_r,b_r) \mid r = 1,\ldots, t-1\}$ is a $(M, t-1)$-system. Then (V,B) is a 2-rotational ETS$(v; (v+1)/2)$.

We now obtain the following theorem:

3.34 Theorem. A 2-rotational ETS(v; (v+1)/2) exists if and only if $v \equiv 1 \pmod 6$.

In the remainder of this section, we will construct 3-rotational extended triple systems. From Lemma 3.1, if a 3-rotational ETS(v;p) exists then $p = 1$, $(v + 2)/3$, $(2v + 1)/3$ or v.

3.35 Lemma. Necessary conditions for the existence of a 3-rotational ETS(v;p) are

(i) if $p = v$ then $v \equiv 1$ or $19 \pmod{24}$,

(ii) if $p = 1$ then $v \equiv 1 \pmod 3$,

(iii) if $p = (v + 2)/3$ then $v \equiv 1 \pmod 9$,

(iv) if $p = (2v + 1)/3$ then $v \equiv 1 \pmod{18}$.

Proof. (i) Follow from the existence of 3-rotational Steiner triple systems.

(ii) If $p = 1$ then $v \equiv 1 \pmod 3$ since $(v - 1)/3$ is an integer.

(iii) If $p = (v + 2)/3$ then $(v + 2)/3 \equiv 1 \pmod 3$ since $v \equiv 1 \pmod 3$. So we have $v \equiv 1 \pmod 9$.

(iv) If $p = (2v + 1)/3$ then $(2v + 1)/3 \equiv 1 \pmod 3$ and hence $v \equiv 1 \pmod 9$. But if $v \equiv 10 \pmod{18}$ then

v is even; so we must have $p \le v/2$. However, $(2v+1)/3 > v/2$. Thus $v \equiv 10 \pmod{18}$ is impossible.

Obviously, the existence of 3-rotational STS's implies the following theorem:

3.36 Theorem. A 3-rotational ETS(v;v) exists if and only if $v \equiv 1$ or $19 \pmod{24}$.

3.37 Lemma. There is a 3-rotational ETS(10;1).

Proof. Base triples: $B = B_1 \cup B_2$

where

$$B_1: \{\{0_i, 1_i, 2_{i+1}\} \mid i \in Z_3\}$$

$$B_2: \{\{\infty,\infty,\infty\}, \{\infty, 0_2, 0_2\}, \{\infty, 0_0, 0_1\}, \{0_1, 0_1, 0_2\}, \{0_0, 0_0, 0_2\}\}.$$

Then (V,B) is a 3-rotational ETS(10;1).

Lemma 3.37 and the existence of 1-rotational ETS(v;1)'s together yield

3.38 Theorem. A 3-rotational ETS(v;1) exists if and only if $v \equiv 1 \pmod 3$.

3.39 Definition. A (N, 3t-1)-system is a set of ordered pairs $\{(a_r,b_r) \mid r = 1,\ldots, 2t-1, 2t+1,\ldots, 3t-1\}$ such that $b_r - a_r = r$ for $r = 1,\ldots, 2t-1, 2t+1,\ldots, 3t-1$

$$\bigcup_{\substack{r=1 \\ r\neq 2t}}^{3t-1} \{a_r,b_r\} = \{-(t/2),\ldots, -1, 1,\ldots, 3t-1, 3t+1,\ldots, 4t-1,$$

$$4t+1,\ldots, 5t-1, 5t+1,\ldots, 6t - t/2 - 1\} .$$

3.40 Lemma. A (N, 3t-1)-system exists if and only if $t \equiv 0 \pmod 4$, $t > 0$.

Proof. ($\Rightarrow$) Let $\{(a_r,b_r) \mid r = 1,\ldots, 2t-1, 2t+1,\ldots, 3t-1\}$ be a (N, 3t-1)-system. Then we have

$$(3.40.1) \qquad \sum_{\substack{r=1 \\ r\neq 2t}}^{3t-1} (b_r-a_r) = \frac{(3t-1)(3t)}{2} - 2t = \frac{9t^2-7t}{2}$$

and

$$(3.40.2) \qquad \sum_{\substack{r=1 \\ r\neq 2t}}^{3t-1} (b_r+a_r) = \frac{(6t-t/2-1)(6t-t/2)}{2}$$

$$- \left(3t+4t+5t + \frac{(t/2)(t/2+1)}{2}\right) = \frac{30t^2-30t}{2} .$$

Adding both sides of (3.40.1) and (3.40.2), respectively, and taking into account that $\sum b_r$ is an integer, we get

$t(39t - 37) \equiv 0 \pmod 4$; so $t \equiv 0$ or $3 \pmod 4$. But since $t/2$ is an integer, t is even and hence $t \equiv 0 \pmod 4$.

($\Leftarrow$) Let $t \equiv 0 \pmod 4$, $t > 0$.

$t = 4$:

$(16-r,\ 16+r)$, $r = 1, 2, 3, 5$.

$(-2,9)$, $(-1,8)$, $(1,4)$, $(2,7)$, $(3,10)$, $(5,6)$.

$t > 4$:

$(4t-r,\ 4t+r)$, $r = 1,\ldots,t-1,t+1,\ldots,(3t-2)/2$,

$(-(t+2)/2+r,\ 5t/2-r)$, $r = 1,\ldots,\ t/2$,

$(1+r,\ 2t-r)$, $r = 1,\ldots,\ (t-2)/2$,

$((t+4)/2+r,\ (3t+2)/2-r)$, $r = 1,\ldots,\ (t-4)/4$,

$(t-r,\ t+1+r)$, $r = 1,\ldots,\ (t-8)/4$ where $t>8$,

$(1,t)$, $((t+2)/2, 5t/2)$, $((t+4)/2, t+1)$, $(5t/4, (5t+4)/4)$.

3.41 Lemma. If $v \equiv 1 \pmod{72}$, then there exists a 3-rotational $ETS(v;\ (v+2)/3)$.

Proof. Let $v = 18t + 1$ and $t \equiv 0 \pmod 4$.

Base triples: $B = B_1 \cup B_2 \cup B_3 \cup B_4 \cup B_5$

where

B_1: $\{\{0_0, 0_0, (2t)_0\}\}$,

B_2: $\{\{\infty,\infty,\infty\},\ \{0_1,0_1,0_1\},\ \{0_1,(2t)_1,(4t)_1\},$
$\{0_2,0_2,(2t)_3\}\}$,

B_3: $\{\{\infty,0_i,(3t)_i\} \mid i \in Z_3\}$,

B_4: $\{\{0_i, r_i, (b_r)_{i+1}\} \mid i \in Z_3,\ r = 1,\ldots,\ 2t-1,\ 2t+1,\ldots,$
$3t-1\}$

where $\{(a_r,b_r) \mid r = 1,\ldots,\ 2t-1,\ 2t+1,\ldots,\ 3t-1\}$ is a $(N,\ 3t-1)$-system,

B_5: $\{\{0_0,0_1,0_2\},\ \{(3t)_1,t_2,0_0\},\ \{(4t)_1,(3t)_2,0_0\},$
$\{(5t)_1,(2t)_2,0_0\}\}$.

Then (V,B) is a 3-rotational $ETS(v;\ (v+2)/3)$.

3.42 <u>Corollary</u>. If $v \equiv 1 \pmod{72}$, then there exists a 3-rotational $ETS(v;\ (2v+1)/3)$.

<u>Proof</u>. Replace B_1 in Lemma 3.41 by

B_1': $\{\{0_0,0_0,0_0\},\ \{0_0,(2t)_0,(4t)_0\}\}$.

3.43 Definition. An (0, 3t-1)-system is a set of ordered pairs $\{(a_r,b_r) \mid r = 1,\ldots, 2t-1, 2t+1,\ldots, 3t-1\}$ such that $b_r - a_r = r$ for $r = 1,\ldots, 2t-1, 2t+1,\ldots, 3t-1$ and

$$\bigcup_{\substack{r=1 \\ r\neq 2t}}^{3t-1} \{a_r,b_r\} = \{-(t-1)/2,\ldots, -1, 1,\ldots, 3t-1, 3t+1,\ldots, 4t-1, 4t+1,\ldots, 5t-1, 5t+1,\ldots, 6t - (t+1)/2\} .$$

3.44 Lemma. An (0, 3t-1)-system exists if and only if $t \equiv 1 \pmod 4$.

Proof. ($\Rightarrow$) Let $\{(a_r,b_r) \mid r = 1,\ldots, 2t-1, 2t+1,\ldots, 3t-1\}$ be an (0, 3t-1)-system. Then we have

$$(3.44.1) \qquad \sum_{\substack{r=1 \\ r\neq 2t}}^{3t-1} (b_r-a_r) = \frac{(3t-1)(3t)}{2} - 2t = \frac{9t^2-7t}{2}$$

and

$$(3.44.2) \qquad \sum_{\substack{r=1 \\ r\neq 2t}}^{3t-1} (b_r+a_r) = \frac{(6t-(t+1)/2)(6t-(t-1)/2)}{2}$$

$$- \left(3t+4t+5t + \frac{(t-1)/2\cdot(t+1)/2}{2}\right)$$

$$= \frac{30t^2-24t}{2} .$$

Adding both sides of (3.44.1) and (3.44.2), respectively, gives $t(39t - 31) \equiv 0 \pmod 4$; so $t \equiv 0$ or $1 \pmod 4$. But since $(t-1)/2$ is an integer, t is odd and hence $t \equiv 1 \pmod 4$.

($\Leftarrow$) Let $t \equiv 1 \pmod 4$.

$t = 1$: $(1,2)$.

$t = 5$:

$(20-r,\ 20+r)$, $r = 1, 2, 3, 4, 6, 7$

$(-2,11)$, $(-1,10)$, $(1,8)$, $(2,7)$, $(3,12)$, $(4,5)$, $(6,9)$.

$t > 5$:

$(4t-r, 4t+r)$, $r = 1,\dots,t-1,t+1,\dots,(3t-1)/2$,

$(-(t+1)/2+r, (5t-1)/2-r)$, $r = 1,\dots,(t-1)/2$,

$(1+r, 2t-r)$, $r = 1,\dots,(t-3)/2$,

$((t+3)/2+r, (3t+3)/2-r)$, $r = 1,\dots,(t-1)/4$,

$(t-r, t+1+r)$, $r = 1,\dots,(t-9)/4$ where $t > 9$,

$(1,t+1)$, $((t+1)/2, (5t-1)/2)$, $((t+3)/2, t)$,

$((5t-1)/4, (5t+3)/4)$.

3.45 <u>Corollary</u>. If $v \equiv 19 \pmod{72}$, then there exists a 3-rotational $ETS(v;\ (v+2)/3)$.

<u>Proof</u>. Replace $t \equiv 0 \pmod 4$ and $(N,\ 3t-1)$-system in Lemma 3.41 by $t \equiv 1 \pmod 4$ and $(0,\ 3t-1)$-system, respectively.

3.46 <u>Corollary</u>. If $v \equiv 19 \pmod{72}$, then there exists a 3-rotational $ETS(v; (2v+1)/3)$.

<u>Proof</u>. Replace $t \equiv 0 \pmod 4$ and $(N, 3t-1)$-system in Corollary 3.42 by $t \equiv 1 \pmod 4$ and $(0, 3t-1)$-system, respectively.

In the existence problem for 3-rotational $ETS(v;p)$'s, the following cases are still open:

(i) If $v \equiv 10 \pmod{18}$, does there exist a 3-rotational $ETS(v; (v+2)/3)$?

(ii) Let $p = (v + 2)/3$ or $(2v + 1)/3$. Then if $v \equiv 37$ or $55 \pmod{72}$, does there exist a 3-rotational $ETS(v;p)$?

Section 4. Regular Extended Triple Systems.

An extended triple system $ETS(v;p)$ is k-regular if it admits an automorphism consisting of k disjoint cycles of the same length v/k. In Section 2 of this chapter, we determined completely 1-regular $ETS(v;p)$'s, that is, cyclic $ETS(v;p)$'s. In this section, we obtain necessary and sufficient conditions for the existence of 2- and 3-regular $ETS(v;p)$'s, and 4-regular $ETS(v;p)$ except possibly for $v \equiv 12, 20 \pmod{24}$ and $p = v/2$.

Throughout this section, we will assume the set of elements of our k-regular $ETS(v;p)$ to be $V = Z_{v/k} \times Z_k$ and corresponding regular automorphism to be $\alpha = (0_i \ldots (v/k - 1)_i)$, $i \in Z_k$. By simple arguments, we have easily seen the following lemma.

4.1 Lemma. If a k-regular $ETS(v;p)$ exists, then we have

(i) $p = t(v/k)$, $t = 0, \ldots, k$,

(ii) v/k is odd.

Let us construct 2-regular $ETS(v;p)$'s. By above Lemma 4.1, if a 2-regular $ETS(v;p)$ exists then $p = 0$, $v/2$ or v. But since $v/2$ is an integer, v is even; so $0 \le p \le v/2$ and hence $p = 0$ or $v/2$.

4.2 Lemma. Necessary conditions for the existence of a 2-regular ETS(v;p) are

(i) if $p = 0$ then $v \equiv 6 \pmod{12}$,

(ii) if $p \neq v/2$ then $v \equiv 2$ or $6 \pmod{12}$.

Proof. (i) If $p = 0$ we have $v \equiv 0 \pmod 3$. Since v is also even and $v/2$ is odd, $v \equiv 6 \pmod{12}$.

(ii) If $p \neq v/2$ then $v/2 \equiv 0$ or $1 \pmod 3$ and hence $v \equiv 0$ or $2 \pmod 6$; so $v \equiv 2$ or $6 \pmod{12}$ since $v/2$ is odd.

4.3 Lemma. There is a 2-regular ETS(18;0).

Proof. Base triples: $B = B_1 \cup B_2$ where

$$B_1: \{\{0_0, 0_0, 0_1\}, \{0_1, 0_1, 2_1\}, \{0_1, 1_1, 4_1\}\},$$

$$B_2: \{\{0_0, r_0, (b_r)_1\} \mid r = 1, 2, 3, 4\}$$

where $\{(a_r, b_r) \mid r = 1, 2, 3, 4\}$ is an (A,4)-system. Then (V,B) is a 2-regular ETS(18;0).

4.4 Lemma. If $v \equiv 6 \pmod{12}$, then there exists a 2-regular ETS(v;0).

Proof. The case $v = 18$ has been treated in Lemma 4.3. Let $v = 12t + 6$, $t \neq 1$.

Base triples: $B = B_1 \cup B_2 \cup B_3$

where

$$B_1: \begin{cases} \{\{0_0, 0_0, 0_1\},\ \{0_1, 0_1, (2t+1)_1\}\} & \text{if } t \equiv 0 \text{ or } 1 \pmod 4 ; \\ \{\{0_0, 0_0, (6t+2)_1\}, \{0_1, 0_1, (2t+1)_1\}\} & \text{if } t \equiv 2 \text{ or } 3 \pmod 4 , \end{cases}$$

B_2: $\{\{0_0, r_0, (b_r)_1\} \mid r = 1, \ldots, 3t + 1\}$

where $\{(a_r, b_r) \mid r = 1, \ldots, 3t + 1\}$ is an $(A, 3t+1)$-system or a $(B, 3t+1)$-system depending on whether $t \equiv 0, 1 \pmod 4$ or $t \equiv 2, 3 \pmod 4$,

B_3: the collection of all base triples of a cyclic $STS(6t + 3)$ except the base triple $\{0_1, (2t+1)_1, (4t+2)_1\}$ based on $Z_{6t+3} \times \{1\}$.

Then (V, B) is a 2-regular $ETS(v; 0)$.

Thus, we have the following theorem

4.5 Theorem. A 2-regular $ETS(v; 0)$ exists if and only if $v \equiv 6 \pmod{12}$.

4.6 Lemma. If $v \equiv 2 \pmod{12}$, then there exists a 2-regular $ETS(v; v/2)$.

Proof. Let $v = 12t + 2$.

Base triples: $B = B_1 \cup B_2 \cup B_3$

where

B_1: $\{\{0_0,0_0,0_0\},\ \{0_1,0_1,0_0\}\}$ if $t \equiv 0$ or $3 \pmod 4$;

$\{\{0_0,0_0,0_0\},\ \{0_1,0_1,(6t)_0\}\}$ if $t \equiv 1$ or $2 \pmod 4$,

B_2: the collection of all base triples of a cyclic STS(6t + 1) based on $Z_{6t+1} \times \{0\}$,

B_3: $\{\{0_1, r_1, (b_r)_0\} \mid r = 1,\ldots, 3t\}$

where $\{(a_r,b_r) \mid r = 1,\ldots, 3t\}$ is an (A,3t)-system or a (B,3t)-system depending on whether $t \equiv 0,\ 3 \pmod 4$ or $t \equiv 1$ or $2 \pmod 4$. Then (V,B) is a 2-regular ETS(v; v/2) .

4.7 Lemma. There is a 2-regular ETS(18;9) .

Proof. Base triples B: $\{0_0,0_0,0_0\}$, $\{0_0,4_1,4_1\}$, $\{0_0,2_0,8_0\}$, $\{0_0,4_0,0_1\}$, $\{0_1,3_1,6_1\}$, $\{0_1,1_1,2_0\}$, $\{0_1,2_1,8_0\}$, $\{0_1,4_1,7_0\}$.

Then (V,B) is a 2-regular ETS(18;9) .

4.8 Lemma. If $v \equiv 6 \pmod{12}$, then there exists a 2-regular ETS(v; v/2) .

Proof. Again, $v = 18$ is handled as in the previous lemma. Let $v = 12t + 6$, $t \neq 1$.

Base triples: $B = B_1 \cup B_2 \cup B_3$

where

$$B_1: \begin{cases} \{\{0_0,0_0,0_0\},\ \{0_1,0_1,0_0\}\} & \text{if } t \equiv 0 \text{ or } 1 \pmod 4; \\ \{\{0_0,0_0,0_0\},\ \{0_1,0_1,(6t+2)_0\} & \text{if } t \equiv 2 \text{ or } 3 \pmod 4), \end{cases}$$

B_2: the collection of all base triples of a cyclic STS(6t + 3) based on $Z_{6t+3} \times \{0\}$,

B_3: $\{\{0_1,r_1,(b_r)_0\} \mid r = 1,\ldots, 3t + 1\}$

where $\{(a_r,b_r) \mid r = 1,\ldots, 3t + 1\}$ is an (A, 3t+1)-system or a (B, 3t+1)-system depending on whether $t \equiv 0, 1 \pmod 4$ or $t \equiv 2, 3 \pmod 4$. Then (V,B) is a 2-regular ETS(v; v/2).

We have the following theorem.

4.9 Theorem. A 2-regular ETS(v; v/2) exists if and only if $v \equiv 2$ or $6 \pmod{12}$.

Before constructing 3-regular ETS(v;p)'s, note that if a 3-regular ETS(v;p) exists then $p = 0$, $v/3$, $2v/3$ or v by Lemma 4.1.

4.10 Lemma. Necessary conditions for the existence of a 3-regular ETS(v;p) are

(i) if $p = 0$ then $v \equiv 3 \pmod 6$,

(ii) if $p = v/3$ then $v \equiv 9 \pmod{18}$,

(iii) if $p = 2v/3$, then $v \equiv 9 \pmod{18}$,

(iv) if $p = v$ then $v \equiv 3 \pmod 6$.

Proof. (i) If $p = 0$ we have $v \equiv 0 \pmod 3$ and hence $v \equiv 3 \pmod 6$ since $v/3$ is odd.

(ii) If $p = v/3$ we have $v/3 \equiv 0 \pmod 3$; so $v \equiv 9 \pmod{18}$ since $v/3$ is odd.

(iii) Similar to the case (ii).

(iv) It follows from the existence of 3-regular STS's.

The next theorem is obtained directly from results for 3-regular STS's.

4.11 Theorem. A 3-regular ETS(v;v) exists if and only if $v \equiv 3 \pmod 6$.

4.12 Lemma. If $v \equiv 3 \pmod 6$, $v \neq 9$, then there exists a 3-regular ETS(v;0).

<u>Proof</u>. Elements: $V' = Z_v$, $v \neq 9$.

Automorphism: $\alpha' = (0 \ldots v-1)$

Base triples: $B' = B_1 \cup B_2$

where

B_1: $\{\{0, 0, v/3\}\}$,

B_2: the collection of all base triples of a cyclic $STS(v)$ except the base triple $\{0, v/3, 2v/3\}$

Then (V',B') is a 3-regular $ETS(v;0)$ with $(\alpha')^3$ as a required automorphism.

A 3-regular $ETS(9;0)$ has base triples

$$\{\{0_i, 0_i, 1_i\} | i \in Z_3\}$$

and

$$\{\{0_0, i_1, (2_i)_2\} | i \in Z_3\}.$$

Thus, we have the following theorem.

4.13 <u>Theorem</u>. A 3-regular $ETS(v;0)$ exists if and only if $v \equiv 3 \pmod 6$.

4.14 <u>Lemma</u>. There is a 3-regular $ETS(27;9)$.

<u>Proof</u>. Base triples: $B = B_1 \cup B_2 \cup B_3$

where

B_1: $\{\{0_0, 0_0, 0_0\}, \{0_0, 3_0, 6_0\}, \{0_0, 3_1, 6_2\}, \{0_0, 6_1, 3_2\}\}$,

B_2: $\{\{0_i, 0_i, 3_i\} | i \in Z_3 \setminus \{0\}\}$,

B_3: $\{\{0_i, 1_i, 2_{i+1}\}, \{0_i, 2_i, 7_{i+1}\}, \{0_i, 4_i, 8_{i+1}\} | i \in Z_3\}$.

Then (V,B) is a 3-regular ETS(27;9).

4.15 <u>Lemma</u>. If $v \equiv 9 \pmod{18}$, then there exists a 3-regular ETS(v; v/3).

<u>Proof</u>. The case $v = 27$ is treated in Lemma 4.14.
Let $v = 18t + 9$, $t \neq 1$.
Base triples: $B = B_1 \cup B_2 \cup B_3 \cup B_4$

where

B_1: $\{\{0_0, 0_0, 0_0\}, \{0_1, 0_1, (2t+1)_1\}, \{0_2, 0_2 (2t+1)_2\}\}$,

B_2: the collection of all base triples of a cyclic STS(6t + 3) based on $Z_{6t+3} \times \{0\}$,

B_3: the collection of all base triples of a cyclic STS(6t + 3) except the base triples $\{0_i, (2t+1)_i, (4t+2)_i\}$ based on $Z_{6t+3} \times \{i\}$, $i \in Z_3 \setminus \{0\}$,

B_4: $\{\{0_0, r_1, (2r)_2\} | r \in Z_{6t+3}\}$.

Then (V,B) is a 3-regular ETS(v; v/3).

Thus, we have

4.16 Theorem. A 3-regular ETS(v; v/3) exists if and only if $v \equiv 9 \pmod{18}$.

4.17 Lemma. There is a 3-regular ETS(27;18).

Proof. Base triples: $B = B_1 \cup B_2 \cup B_3$

where

B_1: $\{\{0_i, 0_i, 0_i\}, \{0_i, 3_i, 6_i\} | i \in Z_3 \setminus \{2\}\}$,

B_2: $\{\{0_2, 0_2, 3_2\}, \{0_0, 3_1, 6_2\}, \{0_0, 6_1, 3_2\}\}$,

B_3: $\{\{0_i, 1_i, 2_{i+1}\}, \{0_i, 2_i, 7_{i+1}\}, \{0_i, 4_i, 8_{i+1}\} | i \in Z_3\}$.

Then (V,B) is a 3-regular ETS(27;18).

4.18 Lemma. If $v \equiv 9 \pmod{18}$, then there exists a 3-regular ETS(v; 2v/3).

Proof. An ETS(27;18) exists by Lemma 4.17. Let $v = 18t + 9$, $t \neq 1$.

Base triples: $B = B_1 \cup B_2 \cup B_3 \cup B_4$

where

B_1: $\{\{0_i, 0_i, 0_i\}, \{0_2, 0_2, (2t+1)_2\} \mid i \in Z_3 \setminus \{2\}\}$,

B_2: $\{\{0_0, r_1, (2r)_2\} \mid r \in Z_{6t+3}\}$,

B_3: the collection of all base triples of a cyclic STS(6t + 3) based on $Z_{6t+3} \times \{i\}$, $i \in Z_3 \setminus \{2\}$,

B_4: the collection of all base triples of a cyclic STS(6t + 3) except the base triple $\{0_2, (2t+1)_2, (4t+2)_2\}$ based on $Z_{6t+3} \times \{2\}$.

Then (V,B) is a 3-regular ETS(v; 2v/3).

Thus, we have

4.19 Theorem. A 3-regular ETS(v; 2v/3) exists if and only if $v \equiv 9 \pmod{18}$.

In the remainder of this section we construct 4-regular ETS(v;p)'s. By Lemma 4.1, if such a system exists then $p = 0$, $v/4$ or $v/2$.

4.20 Lemma. Necessary conditions for the existence of a 4-regular ETS(v;p) are

(i) if $p = 0$ then $v \equiv 12 \pmod{12}$,

(ii) if $p = v/4$ then $v \equiv 4$ or $12 \pmod{24}$,

(iii) if $p = v/2$ then $v \equiv 12$ or $20 \pmod{24}$.

Proof. (i) If $p = 0$ we have $v \equiv 0 \pmod 3$; so $v \equiv 12 \pmod{24}$ since $v/4$ is an odd integer.

(ii) If $p = v/4$ we have $v/4 \equiv 0$ or $1 \pmod 3$, that is, $v \equiv 0$ or $4 \pmod{12}$ and hence $v \equiv 4$ or $12 \pmod{24}$ since $v/4$ is odd.

(iii) If $p = v/2$ we have $v/2 \equiv 0$ or $1 \pmod 3$, that is, $v \equiv 0$ or $2 \pmod 6$ and hence $v \equiv 12$ or $20 \pmod{24}$ since $v/4$ is an odd integer.

4.21 Lemma. There is a 4-regular ETS(36;0).

Proof. Base triples: $B = B_1 \cup B_2 \cup B_3$

where

B_1: $\{\{0_3,0_3,4_3\}, \{0_3,2_3,8_3\}, \{0_i,0_i,0_3\} | i \in Z_4\backslash\{3\}\}$,

B_2: $\{\{0_0, r_1, (2r)_2\} | r \in Z_9\}$,

B_3: $\{\{0_i,r_i,(b_r)_3\} | i \in Z_4\backslash\{3\}, r = 1, 2, 3, 4\}$

where $\{(a_r,b_r) | r = 1, 2, 3, 4\}$ is an (A,4)-system. Then (V,B) is a 4-regular ETS(36;0).

4.22 <u>Lemma</u>. If $v \equiv 12 \pmod{24}$, then there exists a 4-regular ETS(v;0).

<u>Proof</u>. The case $v = 36$ is treated in Lemma 4.21. Let $v = 24t + 12$, $t \neq 1$.
Base triples: $B = B_1 \cup B_2 \cup B_3 \cup B_4$

where

$$B_1: \begin{cases} \{\{0_3,0_3,(2t+1)_3\}, \{0_i,0_i,0_3\} | i \in Z_4\backslash\{3\}\} & \text{if } t \equiv 0,1 \pmod 4; \\ \{\{0_3,0_3,(2t+1)_3\}, \{0_i,0_i,(6t+2)_3\} | i \in Z_4\backslash\{3\}\} & \text{if } t \equiv 2,3 \pmod 4), \end{cases}$$

B_2: $\{\{0_i,r_i,(b_r)_3\} | i \in Z_4\backslash\{3\}, r = 1,\ldots, 3t + 1\}$

where $\{(a_r,b_r) | r = 1,\ldots, 3t + 1\}$ is an (A, 3t+1)-system or a (B, 3t+1)-system depending on whether $t \equiv 0, 1 \pmod 4)$

or $t \equiv 2, 3 \pmod 4$,

B_3: $\{\{0_0, r_1, (2r)_2\} | r \in Z_{6t+3}\}$,

B_4: the collection of all base triples of a cyclic STS(6t + 3) except the base triple $\{0_3, (2t+1)_3, (4t+2)_3\}$ based on $Z_{6t+3} \times \{3\}$.

Then (V,B) is a 4-regular ETS(v;0).

Thus, we have

4.23 Theorem. A 4-regular ETS(v;0) exists if and only if $v \equiv 12 \pmod{24}$.

4.24 Lemma. There is a 4-regular ETS(36;9).

Proof. Base triples: $B = B_1 \cup B_2 \cup B_3$

where

B_1: $\{\{0_0, 0_0, 0_0\}, \{0_0, 2_0, 8_0\}, \{0_0, 4_0, 4_3\}, \{0_3, 1_3, 2_0\}, \{0_3, 2_3, 8_0\}, \{0_3, 4_3, 7_0\}, \{0_3, 3_3, 6_3\}, \{0_3, 0_3, 4_0\}, \{0_1, 0_1, 0_3\}, \{0_2, 0_2, 0_3\}$,

B_2: $\{\{0_0, r_1, (2r)_2\} | r \in Z_9\}$,

B_3: $\{\{0_i, r_i, (b_r)_3\} | i = 1, 2, \quad r = 1, 2, 3, 4\}$

where $\{(a_r, b_r) | r = 1, 2, 3, 4\}$ is an $(A,4)$-system. Then (V,B) is a 4-regular ETS(36;9).

4.25 Lemma. If $v \equiv 12 \pmod{24}$, then there exists a 4-regular ETS(v; $v/4$).

Proof. A 4-regular ETS(36;9) exists by Lemma 4.24. Let $v = 24t + 12$, $t \neq 1$.
Base triples: $B = B_1 \cup B_2 \cup B_3 \cup B_4$

where

$$B_1: \begin{cases} \{\{0_0, 0_0, 0_0\}, \{0_1, 0_1, 0_3\}, \{0_2, 0_2, 0_3\}, \{0_3, 0_3, 0_0\}\} \\ \text{if } t \equiv 0, 1 \pmod 4; \\ \\ \{\{0_0, 0_0, 0_0\}, \{0_1, 0_1, (6t+2)_3\}, \{0_2, 0_2, (6t+2)_3\}, \\ \{0_3, 0_3, (6t+2)_0\}\} \text{ if } t \equiv 2, 3 \pmod 4, \end{cases}$$

B_2: the collection of all base triples of a cyclic STS($6t + 3$) based on $Z_{6t+3} \times \{0\}$,

B_3: $\{\{0_0, r_1, (2r)_2\} | r \in Z_{6t+3}\}$,

B_4: $\{\{0_1, r_1, (b_r)_3\}, \{0_2, r_2, (b_r)_3\}, \{0_3, r_3, (b_r)_0\} \mid$
$r = 1, \ldots, 3t + 1\}$

where $\{(a_r, b_r) \mid r = 1, \ldots, 3t + 1\}$ is an (A, 3t+1)-system or a (B, 3t+1)-system depending on whether $t \equiv 0, 1 \pmod 4$ or $t \equiv 2, 3 \pmod 4$. Then (V, B) is a 4-regular $ETS(v; v/4)$.

4.26 <u>Lemma</u>. If $v \equiv 4 \pmod{24}$, then there exists a 4-regular $ETS(v; v/4)$.

<u>Proof</u>. Let $v = 24t + 4$.

Base triples: $B = B_1 \cup B_2 \cup B_3 \cup B_4$

where

$$B_1: \begin{cases} \{\{0_0, 0_0, 0_0\}, \{0_3, 0_3, 0_0\}, \{0_i, 0_i, 0_3\} \mid i = 1, 2\} \\ \text{if } t \equiv 0, 3 \pmod 4; \\ \\ \{\{0_0, 0_0, 0_0\}, \{0_3, 0_3, (6t)_0\}, \{0_i, 0_i, (6t)_3\} \mid i = 1, 2\} \\ \text{if } t \equiv 1, 2 \pmod 4, \end{cases}$$

B_2: the collection of all base triples of a cyclic $STS(6t + 1)$ based on $Z_{6t+1} \times \{0\}$,

B_3: $\{\{0_0, r_1, (2r)_2\} | r \in Z_{6t+1}\}$,

B_4: $\{\{0_3, r_3, (b_r)_0\}, \{0_i, r_i, (b_r)_3\} | i = 1, 2,$
$r = 1, \ldots, 3t\}$

where $\{(a_r, b_r) | r = 1, \ldots, 3t\}$ is an (A, 3t)-system or a (B, 3t)-system depending on whether $t \equiv 0, 3 \pmod 4$ or $t \equiv 1, 2 \pmod 4$. Then (V, B) is a 4-regular ETS(v; v/4).

Thus, we have

4.27 Theorem. A 4-regular ETS(v, v/4) exists if and only if $v \equiv 4$ or $12 \pmod{24}$.

We end this section with the statement of an open problem: if $v \equiv 12$ or $20 \pmod{24}$ does there exist a 4-regular ETS(v; v/2)?

CHAPTER 4. DIRECTED TRIPLE SYSTEMS AND MENDELSOHN TRIPLE SYSTEMS

Section 1. Directed Triple Systems.

Directed triple systems were introduced by Hung and Mendelsohn [37] as a generalization of Steiner triple systems. Throughout Sections 1 and 2, in what follows an ordered pair will always be an ordered pair (a,b) where $a \neq b$. In this section, when we write a triple with square brackets as $[a,b,c]$ we mean that it contains the ordered pairs (a,b), (a,c), (b,c) but not (b,a), (c,a), (c,b). A directed triple system of order v (DTS(v)) is a pair (V,B) where V is a v-set and B is a collection of 3-subsets of the form $[a,b,c]$ of V (called triples or blocks) such that each ordered pair of distinct elements of V appears in precisely one triple in B. The existence of DTS's has been settled in [37].

1.1 Theorem. A DTS(v) exists if and only if $v \equiv 0$ or $1 \pmod 3$.

In this section, we provide cyclic directed triple systems that have been determined by Colbourn and Colbourn [18] and construct completely k-rotational DTS(v)'s.

1.2 Remark. Every cyclic DTS(v) has only base triples of length v.

Proof. This follows from the fact that in a DTS any two cyclic shifts of a block are distinct (there exists at least one ordered pair that is contained in one but not in the other).

1.3 Lemma [18]. If a cyclic DTS(v) exists, then $v \equiv 1, 4$ or $7 \pmod{12}$.

Proof. If each block [a,b,c] in a cyclic DTS(v) is regarded as containing unordered pairs {a,b}, {a,c} and {b,c} then we obtain a cyclic triple system $TS_2(v)$; so $v \equiv 0, 1, 3, 4, 7$ or $9 \pmod{12}$, $v \neq 9$, since this is the spectrum of a cyclic $TS_2(v)$ (see, Lemma 2.2 in Chapter 2). Since $v(v-1)/3$ is the total number of blocks, $v \equiv 1 \pmod 3$ by Remark 1.2. Thus we have only $v \equiv 1, 4$ or $7 \pmod{12}$.

1.4 Lemma. If $v \equiv 1 \pmod 6$, then there exists a cyclic DTS(v).

Proof. We obtain a cyclic DTS(v) from a cyclic STS(v) by replacing each block {a,b,c} of the STS with the blocks [a,b,c] and [c,b,a].

1.5 Lemma [18]. If $v \equiv 4 \pmod{12}$, then there exists a cyclic DTS(v) .

Proof. Let $v = 12t + 4$, $t \geq 0$.

Elements: $V = Z_{12t+4}$,

Automorphism: $\alpha = (0 \ldots 12t + 3)$,

Base triples: $B = \{[0, r, b_r+4t+1] \mid r = 1,\ldots, 4t + 1\}$.

where $\{(a_r,b_r) \mid r = 1,\ldots, 4t + 1\}$ is an (A, 4t+1)-system. Then (V,B) is a cyclic DTS(v) .

Lemmas 1.3, 1.4 and 1.5 together yield

1.6 Theorem. A cyclic DTS(v) exists if and only if $v \equiv 1, 4$ or $7 \pmod{12}$.

We now consider rotational directed triple systems. From now on we assume that the set of elements of our k-rotational DTS(v) is $V = (Z_{(v-1)/k} \times Z_k) \cup \{\infty\}$ and the corresponding automorphism is $\alpha = (\infty)(0_i \ldots ((v-1)/k - 1)_i)$, $i \in Z_k$. In the case $k = 1$, we write for brevity $V = Z_{(v-1)} \cup \{\infty\}$ instead of $V = (Z_{(v-1)} \times Z_1) \cup \{\infty\}$

Analogously to Remark 1.2, we have

1.7 Remark. If a k-rotational DTS(v) exists, then it consists of all base triples of the same length $(v - 1)/k$.

1.8 Remark. If a k-rotational DTS(v) exists then $kv \equiv 0 \pmod 3$.

Proof. Since the total number of blocks is $v(v-1)/3$ and each base triple has length $(v-1)/k$,

$$\frac{1}{3}v(v-1)\Big/\frac{1}{k}(v-1) = \frac{kv}{3}$$

must be an integer.

Remark 1.8 yields the following necessary conditions for the existence of k-rotational DTS's.

1.9 Lemma. The necessary conditions for the existence of a k-rotational DTS(v) are

(i) if $k \equiv 0 \pmod 3$ then $v \equiv 1 \pmod k$,

(ii) if $k \equiv 1$ or $2 \pmod 3$ then $v \equiv 0 \pmod 3$ and $v \equiv 1 \pmod k$.

Lemma 1.9 tells us that we need only to construct k-rotational DTS(v)'s for

$$k = 1, \quad v \equiv 0 \pmod 3,$$

$$k = 3, \quad v \equiv 1 \pmod 3.$$

1.10 Lemma. If $v \equiv 3$ or $6 \pmod{12}$, then there exists a 1-rotational DTS(v) .

Proof. Let $v = 3t$ and $t \equiv 1$ or $2 \pmod 4$.

Base triples: $B = B_1 \cup B_2$

where

B_1: $\{[0,\infty,1]\}$,

B_2: $\{[0, r, b_r+t-1] | r = 1,\ldots, t-1\}$

where $\{(a_r,b_r)|r = 1,\ldots, t-1\}$ is an (A, t-1)-system. Then (V,B) is a 1-rotational DTS(v) .

1.11 Lemma. If $v \equiv 0$ or $9 \pmod{12}$, then there exists a 1-rotational DTS(v) .

Proof. Let $v = 3t$ and $t \equiv 0$ or $3 \pmod 4$.

Base triples: $B = B_1 \cup B_2$

where

B_1: $\{[0,\infty,2]\}$,

B_2: $\{[0, r, b_r+t-1] | r = 1,\ldots, t-1\}$

where $\{(a_r,b_r)|r = 1,\ldots, t - 1\}$ is a $(B, t-1)$-system. Then (V,B) is a 1-rotational DTS(v).

Lemmas 1.9, 1.10 and 1.11 together yield

1.12 <u>Theorem</u>. A 1-rotational DTS(v) exists if and only if $v \equiv 0 \pmod 3$.

1.13 <u>Corollary</u>. Let $k \equiv 1$ or $2 \pmod 3$. Then a k-rotational DTS(v) exists if and only if $v \equiv 0 \pmod 3$ and $v \equiv 1 \pmod k$.

1.14 <u>Lemma</u>. If $v \equiv 16 \pmod{18}$, then there exists a 3-rotational DTS(v).

<u>Proof</u>. Let $v = 18t + 16$, $t \geq 0$.

Base triples: $B = B_1 \cup B_2 \cup B_3$

where

$$B_1: \{[0_i,\infty,(3t+2)_i]|i \in Z_3\},$$

$$B_2: \{[0_i,r_i,(b_r+2t+1)_i]|i \in Z_3,\ r = 1,\ldots, 2t + 1\}$$

where $\{(a_r,b_r)|r = 1,\ldots, 2t + 1\}$ is an $(E, 2t+1)$-system.

$$B_3: \{[0_0,r_1,(2r)_2],\ [(2r)_2,r_1,0_0]|r \in Z_{6t+5}\}.$$

Then (V,B) is a 3-rotational $DTS(v)$.

1.15 <u>Lemma</u>. If $v \equiv 4 \pmod{18}$, then there exists a 3-rotational $DTS(v)$.

<u>Proof</u>. Let $v = 18t + 4$, $t \geq 0$.

Base triples: $B = B_1 \cup B_2 \cup B_3$

where

B_1: $\{[\infty,0_0,0_1], [0_0,\infty,0_2], [0_2,0_1,\infty], [0_1,0_2,0_0]\}$

B_2: the collection of the base triples obtained by replacing each base triple $\{a_i,b_i,c_i\}$ of a cyclic $STS(6t+1)$ based on $Z_{6t+1} \times \{i\}$ with the base triples $[a_i,b_i,c_i]$ and $[c_i,b_i,a_i]$, $i \in Z_3$,

B_3: $\{[0_0,r_1,(2r)_2], [(2r)_2,r_1,0_0] \mid r \in Z_{6t+1} \setminus \{0\}\}$.

Then (V,B) is a 3-rotational $DTS(v)$.

1.16 <u>Lemma</u>. There exists a 3-rotational $DTS(28)$.

<u>Proof</u>. Base triples: $B = B_1 \cup B_2 \cup B_3 \cup B_4$

where

B_1: $\{[1_i,\infty,0_i] | i \in Z_3\}$,

B_2: $\{[3_0,0_0,0_1], [3_1,0_1,0_2], [0_0,3_2,0_2], [6_2,0_1,0_0],$
$[0_2,6_1,0_0], [3_2,0_0,3_1], [3_1,0_0,6_2]\}$,

B_3: $\{[0_i,1_i,4_i], [0_i,2_i,7_i]\ i \in Z_3\}$,

B_4: $\{[0_0,r_1,(9-r)_2], [(9-r)_2,r_1,0_0] | r = 1,2,4,5,7,8\}$.

Then (V,B) is a 3-rotational DTS(28) .

1.17 Lemma. If $v \equiv 10 \pmod{18}$, then there exists a 3-rotational DTS(v) .

Proof. For $v = 28$, see the previous lemma.
Let $v = 18t + 10$, $t \neq 1$.
Base triples: $B = B_1 \cup B_2 \cup B_3 \cup B_4$

where

B_1: $\{[0_i,\infty,(2t+1)_i] | i \in Z_3\}$,

B_2: $\{[(2t+1)_0,0_0,0_1], [(2t+1)_1,0_1,0_2], [0_0,(2t+1)_2,0_2],$
$[(4t+2)_2,0_1,0_0], [0_2,(4t+2)_1,0_0],$
$[(2t+1)_2,0_0,(2t+1)_1], [(2t+1)_1,0_0,(4t+2)_2]\}$,

B_3: the collection of the base triples obtained by replacing each base triple $\{a_i, b_i, c_i\}$ except the base triple $\{0_i, (2t+1)_i, (4t+2)_i\}$ of a cyclic STS(6t + 3) based on $Z_{6t+3} \times \{i\}$ with the base triples $[a_i, b_i, c_i]$ and $[c_i, b_i, a_i]$, $i \in Z_3$,

B_4: $\{[0_0, r_1, (6t+3-r)_2], [(6t+3-r)_2, r_1, 0_0] \mid r = 1, \ldots, 2t, 2t+2, \ldots, 4t+1, 4t+3, \ldots, 6t+2\}$.

Then (V,B) is a 3-rotational DTS(v).

1.18 <u>Lemma</u>. If $v \equiv 1$ or 19 (mod 24), then there exists a 3-rotational DTS(v).

<u>Proof</u>. We obtain a 3-rotational DTS(v) from a 3-rotational STS(v) constructed in Section 4 of Chapter 1 by replacing each triple $\{a,b,c\}$ not containing ∞ of the STS(v) with the triples $[a,b,c]$ and $[c,b,a]$, $\{\infty,a,b\}$ of the STS(v) with $[a,\infty,b]$.

1.19 <u>Lemma</u>. If $v \equiv 7$ (mod 24), then there exists a 3-rotational DTS(v).

<u>Proof</u>. Let $v = 24t + 7$, $t \geq 0$.

Base-triples: $B = B_1 \cup B_2 \cup B_3$

where

B_1: $\{[0_1,\infty,0_0], [0_2,\infty,(4t+1)_1], [0_0,\infty,0_2]\}$,

B_2: $\{[0_i,r_i,(b_r)_{i+1}], [(b_r)_{i+1},r_i,0_i] | i \in Z_3,$
$r = 1,\ldots, 4t\}$

where $\{(a_r,b_r) | r = 1,\ldots, 4t\}$ is a (C, 4t)-system,

B_3: $\{[0_2,0_1,(4t+1)_0], [0_2,0_0,(4t+1)_2],$
$[0_i,(4t+1)_i,0_{i+1}] | i = 0, 1\}$.

Then (V,B) is a 3-rotational DTS(v) .

1.20 Lemma. There exists a 3-rotational DTS(37) .

Proof. Base triples: $B = B_1 \cup B_2 \cup B_3 \cup B_4$

where

B_1: $\{[0_i,\infty,6_i], [0_i,1_i,8_i], [8_i,1_i,0_i] | i = 0, 1\}$

B_2: $\{[0_2,\infty,4_2], [0_2,1_2,11_2], [0_2,2_2,8_2]\}$

B_3: $\{[0_0,2_0,9_1], [9_1,2_0,0_0], [0_0,3_0,11_1],$
$[11_1,3_0,0_0], [0_1,2_1,2_2], [2_2,2_1,0_1], [0_1,3_1,11_2],$
$[11_2,3_1,0_1], [0_2,5_2,5_0], [5_0,5_2,0_2], [0_2,3_2,10_0],$
$[10_0,3_2,0_2]\}$

B_4: $\{[0_0,0_1,4_2], [0_0,1_i,11_2], [0_0,2_1,8_2],$
$[0_0,3_1,10_2], [4_2,0_1,0_0], [11_2,1_1,0_0], [8_2,2_1,0_0],$
$[10_2,3_1,0_0], [0_0,4_1,1_2], [0_0,5_1,6_2], [0_0,6_1,9_2],$
$[0_0,10_1,3_2], [1_2,4_1,0_0], [6_2,5_1,0_0], [9_2,6_1,0_0],$
$[3_2,10_1,0_0]\}$.

Then (V,B) is a 3-rotational $DTS(37)$.

1.21 <u>Definition</u>. A <u>(P,k)-system</u> is a set of ordered pairs $\{(a_r,b_r) \mid r = 1,\ldots, k\}$ such that $b_r - a_r = r$ for $r = 1,\ldots, k$, $\bigcup_{r=1}^{k} \{a_r,b_r\} = \{1,\ldots, (3k+3)/2 - 1, (3k+3)/2 + 1,\ldots, 2k + 1\}$, and when $r = (k + 1)/2$, $a_r = (k + 1)/2$ and $b_r = k + 1$.

1.22 <u>Lemma</u>. A (P,k)-system exists if and only if $k \equiv 1 \pmod 4$ and $k \neq 5$.

Proof. (→) Let $\{(a_r, b_r) \mid r = 1, \ldots, k\}$ be a (P,k)-system. Then we have

(1.22.1) $$\sum_{r=1}^{k} (b_r - a_r) = \frac{k(k+1)}{2}$$

and

(1.22.2) $$\sum_{r=1}^{k} (b_r + a_r) = \frac{(2k+1)(2k+2)}{2} - \frac{3k+3}{2} .$$

Adding both sides of (1.22.1) and (1.22.2), respectively, we get

(1.22.3) $$2 \sum_{r=1}^{k} b_r = \frac{5k^2 + 4k - 1}{2} .$$

Since $\sum_{r=1}^{k} b_r$ is an integer, $5k^2 + 4k - 1 \equiv 0 \pmod 4$ and hence $k \equiv 1 \pmod 4$.

(←) Let $k = 4t + 1$.

$t \equiv 0, 2 \pmod 4$.

$(2t+2-r,\ 4t+1+r)$, $\quad r = 1, \ldots, t+1$,

$(t+1-r,\ 2t+1+r)$ $\quad r = 1, \ldots, \frac{t}{2}$,

$(r,\ 4t+2-r)$, $\quad r = 1, \ldots, \frac{t}{2}$,

$(\frac{5t}{2}+1+r,\ \frac{7t}{2}+2-r)$, $r = 1,\ldots,\frac{t}{2}$,

$(5t+2+r,\ 8t+4-r)$, $r = 1,\ldots,\frac{t}{2}$,

$(\frac{11t}{2}+2+r,\ \frac{15t}{2}+3-r)$, $r = 1,\ldots,\frac{t}{2}$,

$(6t+3+r,\ 7t+3-r)$, $r = 1,\ldots,\frac{t}{2}-1$,

$(\frac{13t}{2}+3,\ \frac{15t}{2}+3)$.

$t \equiv 1, 3 \pmod 4$.

It is easy to check that there is no (P,5)-system.

t = 3: (24,25), (9,11), (19,22), (23,27), (3,8),
(20,26), (7,14), (10,18), (6,15), (2,12),
(5,16), (1,13), (4,17) .

For $t > 3$, we distinguish 4 cases and each case contains the following ordered pairs:

$(r,\ 4t+2-r)$, $r = 1,\ldots,\frac{t+1}{2}$,

$(\frac{t+1}{2}+r,\ \frac{5t-1}{2}+2-r)$, $r = 1,\ldots,\frac{t-1}{2}$,

$(2t+2-r,\ 4t+1+r)$, $r = 1,\ldots,t+1$,

$(\frac{5t-1}{2}+1+r,\ \frac{7t-1}{2}+2-r)$, $r = 1,\ldots,\frac{t-1}{2}$

$(3t+1,\ 5t+3)$,

$(5t+3+r,\ 8t+4-r)$, $\quad r = 1,\ldots,\frac{t-1}{2}-1$,

$(\frac{11t-1}{2}+2+r,\ \frac{15t+1}{2}+3-r)$, $\quad r = 1,\ldots,\frac{t+1}{2}$.

<u>Case 1</u>. $t \equiv 3 \pmod 4$.

$t = 7$: (47,48), (46,49), (51,56), (50,57).

$t = 11$: (71,72), (73,76), (74,79), (70,77), (78,87), (75,86).

$t \geq 15$:

$(\frac{13t+1}{2}+3+r,\ \frac{15t+1}{2}+5-r)$, $\quad r = 1, 2$,

$(6t+3+r,\ 7t+1-r)$, $\quad r = 1,\ldots,\frac{t-7}{4}$,

$(\frac{25t-7}{4}+3+r,\ \frac{25t-7}{4}+8-r)$, $\quad r = 1, 2$,

$(\frac{25t-7}{4}+7+r,\ \frac{27t+7}{4}+1-r)$, $\quad r = 1,\ldots,\frac{t-7}{4}-2$,

$(\frac{13t-7}{2}+5+r,\ 7t+3-r)$, $\quad r = 1, 2$.

<u>Case 2</u>. $t \equiv 5 \pmod{12}$.

$(\frac{13t+1}{2}+3+r,\ \frac{15t+1}{2}+5-r)$, $\quad r = 1, 2$,

$(6t+5+r,\ 7t+3-r)$, $\quad r = 1,\ldots,\frac{t-8}{3}$,

$(\frac{19t+7}{3}+1,\ \frac{19t+7}{3}+4)$,

$(6t+3+r,\ \frac{19t+7}{3}+4-r)$, $\quad r = 1,\ 2,$

$(\frac{13t+1}{2}+6,\ \frac{13t+1}{2}+7)$,

$(\frac{19t+7}{3}+4+r,\ \frac{20t+17}{3}-r)$, $\quad r = 1,\dots,\ \frac{t-17}{6}$.

Case 3. $t \equiv 1 \pmod{12}$.

$(\frac{13t+1}{2}+4,\ \frac{15t+1}{2}+4)$,

$(6t+3+r,\ 7t+3-r)$, $\quad r = 1,\dots,\ \frac{t-1}{6}-1$,

$(\frac{41t+1}{6}+3,\ \frac{15t+1}{2}+3)$,

$(\frac{37t-1}{6}+3,\ \frac{13t+1}{2}+3)$,

$(\frac{37t-1}{6}+3+r,\ \frac{41t+1}{6}+3-r)$, $\quad r = 1,\dots,\ \frac{t-1}{6}-1$,

$(\frac{19t-1}{3}+3,\ \frac{19t-1}{3}+4)$,

$(\frac{13t+1}{2}+3-r,\ \frac{13t+1}{2}+4+r)$, $\quad r = 1,\dots,\ \frac{t-1}{6}-1$.

Case 4. $t \equiv 9 \pmod{12}$.

$(\frac{13t+1}{2}+3,\ \frac{15t+1}{2}+3)$,

$(\frac{41t+3}{6}+3,\ \frac{15t+1}{2}+4)$,

$(\frac{13t+1}{2}+2,\ \frac{41t+3}{6}+2)$,

$(6t+3+r, 7t+3-r)$, $r = 1,\ldots, \frac{t-9}{6}$,

$(\frac{37t-3}{6}+2+r, \frac{41t+3}{6}+2-r)$, $r = 1,\ldots, \frac{t-3}{6}$,

$(\frac{19t}{3}+2, \frac{19t}{3}+3)$,

$(\frac{19t}{3}+3+r, \frac{20t}{3}+3-r)$, $r = 1,\ldots, \frac{t-9}{6}$.

1.23 Lemma. If $v \equiv 13 \pmod{24}$, then there exists a 3-rotational DTS(v) .

Proof. The case $v = 37$ has been treated in Lemma 1.20. Let $v = 24t + 13$ and $t \neq 1$.

Base triples: $B = B_1 \cup B_2 \cup B_3 \cup B_4$

where

B_1: $\{[0_0,\infty,0_1], [0_2,\infty,0_0], [0_1,\infty,0_2]\}$,

B_2: $\{[0_i,(2t+1)_i,(6t+3)_i] \mid i \in Z_3\}$,

B_3: $\{[0_i,r_i,(b_r)_{i+1}], [(b_r)_{i+1},r_i,0_i] \mid i \in Z_3;$
$r = 1,\ldots, 2t, 2t+2,\ldots, 4t+1\}$

where $\{(a_r,b_r) \mid r = 1,\ldots, 4t-1\}$ is a (P, 4t+1)-system,

B_4: $\{[0_1, 0_0, (2t+1)_2], [0_0, (4t+2)_2, (4t+2)_1],$
$[(6t+3)_2 (4t+2)_1, 0_0], [0_0, (2t+1)_1, 0_2],$
$[(4t+2)_2, 0_0, (6t+3)_1], [(2t+1)_1, 0_0, (6t+3)_2],$
$[(2t+1)_2, (6t+3)_1, 0_0]\}$.

Then (V,B) is a 3-rotational DTS(v) .

Summarizing, we have

1.24 Theorem. A 3-rotational DTS(v) exists if and only if $v \equiv 1 \pmod 3$.

1.25 Corollary. Let $k \equiv 0 \pmod 3$. Then a k-rotational DTS(v) exists if and only if $v \equiv 1 \pmod k$.

Section 2. Mendelsohn Triple Systems.

A cyclic triple is a collection b of three ordered pairs such that an element occurs as a first coordinate of an ordered pair in b if and only if it occurs as a second coordinate of an ordered pair in b. We will denote the cyclic triple {(a,b), (b,c), (c,a)} by (a,b,c), (b,c,a) or (c,a,b). A Mendelsohn triple system of order v (MTS(v)) is a pair (V,B) where V is a v-set and B is a collection of cyclic triples of elements of V such that every ordered pair of distinct elements of V belongs to exactly one cyclic triple in B. In 1971, Mendelsohn [49] proved that the spectrum for MTS's is the set of all $v \equiv 0$ or 1 (mod 3) except $v = 6$. Mendelsohn himself called such systems cyclic triple systems. This vernacular, however, can be a bit confusing since cyclic Steiner triple systems (see [54]) are also called cyclic triple systems. The terminology "Mendelsohn triple system" is due to Mathon and Rosa [47]. It is well taken since it not only eliminates some ambiguity but recognizes, as well, the fact that Mendelsohn was the first to determine the spectrum for such systems. We remark, as is well known, that an MTS is equivalent to a quasigroup satisfying the identities $a^2 = a$ and $a(ba) = b$. However, in what follows, we will use design vernacular exclusively.

In this section, we give cyclic Mendelsohn triple systems which have been settled by Colbourn and Colbourn [14]. We show that a necessary and sufficient condition for the existence of a 1-rotational MTS(v) is $v \equiv 1, 3$ or $4 \pmod 6$.

By some simple observations concerning the structure of a cyclic MTS(v), the existence of a cyclic MTS(v) for $v \equiv 1 \pmod 3$ is equivalent to a partitioning of the set $Z_v \setminus \{0\}$ into difference triples $\{a, b, c\}$ for which $a + b + c \equiv 0 \pmod v$. When $v \equiv 0 \pmod 3$, a cyclic MTS(v) is equivalent to a partitioning of $Z_v \setminus \{0, v/3, 2v/3\}$ into difference triples. These simple observations enable us to prove the following theorem:

2.1 <u>Theorem</u> [14]. A cyclic MTS(v) exists if and only if $v \equiv 1$ or $3 \pmod 6$, $v \neq 9$.

<u>Proof</u>. ($\rightarrow$) A basic necessary condition for the existence of a cyclic MTS(v) is that $v \equiv 0, 1, 3, 4, 7$ or $9 \pmod{12}$, and $v \neq 9$, since this is the spectrum of cyclic $TS_2(v)$'s, and removing the directions from a cyclic MTS(v) gives a cyclic $TS_2(v)$. A stronger necessary condition is obtained as follows. Consider a set of difference triples for a cyclic MTS(v). Since v divides the sum of the differences in each difference triple, it therefore divides the sum of all of the differences being partitioned by the

triple. In case $v \equiv 0 \pmod 3$, we omit two differences $\{v/3, 2v/3\}$ from the set $\{1, \ldots, v - 1\}$; in case $v \equiv 1 \pmod 3$, we omit none. In either event, v divides the sum of the integers 1 through $v - 1$, that is, $v | v(v - 1)/2$. Thus v is odd and hence $v \equiv 1$ or $3 \pmod 6$, $v \neq 9$.

($\Leftarrow$) We obtain a cyclic MTS(v) from a cyclic STS(v) by replacing each block $\{a, b, c\}$ of the STS(v) with the cyclic triples (a, b, c) and (a, c, b).

Let us assume the set of elements of our 1-rotational MTS(v) to be $V = Z_{v-1} \cup \{\infty\}$ and the corresponding automorphism to be $\alpha = (\infty)(0 \ldots v - 2)$.

2.2 Lemma. If a 1-rotational MTS(v) exists, then $v \equiv 1, 3$ or $4 \pmod 6$.

Proof. First of all we have $v \equiv 0$ or $1 \pmod 3$, and $v \neq 6$, since this is the spectrum of MTS(v). In case $v \equiv 0 \pmod 6$, $v \neq 6$, the existence of a 1-rotational MTS(v) is equivalent to a partitioning of the set $\{1, \ldots, v - 2\} \setminus \{k\}$ for some $1 \le k \le v - 2$ into difference triples $\{a, b, c\}$ for which $a + b + c \equiv 0 \pmod{v - 1}$. Since $v - 1$ divides the sum of the differences in each difference triple, it divides the sum of all of the differences being partitioned by the triple. Thus $v - 1$ divides the sum of the integers 1 through $v - 2$ except exactly one number, that is, $(v - 1)(v - 2)/2 - k \equiv 0 \pmod{v - 1}$ for some $1 \le k \le v - 2$ but there is no such a k in $\{1, \ldots, v - 2\}$.

2.3 Lemma [26]. There is no 1-rotational MTS(10).

2.4 Lemma. If $v \equiv 4 \pmod 6$, $v \neq 10$, then there exists a 1-rotational MTS(v).

Proof. Let $v = 6t + 4$, $t \neq 1$.
Base cyclic triples: $B = B_1 \cup B_2$

where

B_1: $\{(0, \infty, 2t+1), (0, 2t+1, 4t+2)\}$

B_2: the collection of the base cyclic triples obtained by replacing each base block $\{a, b, c\}$ except the base block of the form $\{0, 2t+1, 4t+2\}$ of a cyclic STS(6t + 3) based on Z_{6t+3} with the cyclic triples (a, b, c) and (a, c, b).

Then (V,B) is a 1-rotational MTS(v).

2.5 Lemma. If $v \equiv 7$ or $13 \pmod{18}$, then there exists a 1-rotational MTS(v).

Proof. Let $v = 6t + 1$ and $t \equiv 1$ or $2 \pmod 3$.
Base cyclic triples: $B = B_1 \cup B_2 \cup B_3$

where

B_1: $\{(0, \infty, t), (0, 4t, 2t)\}$,

B_2: $\{(0, 3r, 2t-3+6r) \mid r = 1, \ldots, t\}$,

B_3: $\{(0, 3r, 6r-4t) \mid r = t+1, \ldots, 2t-1\}$ where $t > 1$.

Then (V,B) is a 1-rotational $MTS(v)$.

2.6 Lemma. If $v \equiv 1 \pmod{18}$, then there exists 1-rotational $MTS(v)$.

Proof. Let $v = 6t + 1$ and $t \equiv 0 \pmod 3$.
Base cyclic triples: $B = B_1 \cup B_2 \cup B_3$

where

B_1: $\{(\infty, 0, t), (0, 2t, 4t)\}$,

B_2: $\{(0, 3t+1-r, r) \mid r = 1, \ldots, t\}$,

B_3: $\{(0, r, 7t-r) \mid r = t+1, \ldots, 2t-1\}$.

Then (V,B) is a 1-rotational $MTS(v)$.

2.7 <u>Lemma</u>. If $v \equiv 3$ or $9 \pmod{24}$, then there exists 1-rotational $MTS(v)$.

<u>Proof</u>. Let $v = 6t + 3$ and $t \equiv 0$ or $1 \pmod 4$:
Base cyclic triples: $B = B_1 \cup B_2$

where

B_1: $\{(\infty, 0, 3t+1)\}$,

B_2: $\{(0, r, b_r+t), (0, b_r+t, r) \mid r = 1, \ldots, t\}$

where $\{(a_r, b_r) \mid r = 1, \ldots, t\}$ is an (A,t)-system. Then (V,B) is a 1-rotational $MTS(v)$.

2.8 <u>Lemma</u>. If $v \equiv 15$ or $21 \pmod{24}$, then there exists 1-rotational $MTS(v)$.

<u>Proof</u>. Let $v = 6t + 3$ and $t \equiv 2$ or $3 \pmod 4$.
Base cyclic triples: $B = B_1 \cup B_2$

where

B_1: $\{(\infty, 0, 3t+1)\}$,

B_2: $\{(0, r, 3t+1-r), (0, 5t+2-r, r) | r = 1,\ldots, t\}$.

Then (V,B) is a 1-rotational MTS(v) .

Lemmas 2.2, 2.3, 2.4, 2.5, 2.6, 2.7 and 2.8 together yield:

2.9 Theorem. A 1-rotational MTS(v) exists if and only if $v \equiv 1, 3$ or $4 \pmod 4$ and $v \neq 10$.

Note that a 1-rotational MTS(v) exists for all admissible orders v which are the spectrum for the existence of a MTS(v) , except for $v \equiv 0 \pmod 6$ and $v = 10$. If $v \equiv 0 \pmod 6$ and $v \not\equiv 6 \pmod{30}$, then $v - 1$ is prime. Thus, for the orders $v \equiv 0 \pmod 6$ and $v \not\equiv 6 \pmod{30}$, only $(v - 1)$-rotational MTS(v)'s are considered; clearly such systems exist as their existence follows trivially from the existence of MTS (the $(v - 1)$-rotational automorphism is exactly the identity automorphism). In addition, a 3-rotational MTS(10) has base cyclic triples $(\infty,1,0)$, $(\infty,4,3)$, $(\infty,7,6)$, $(0,1,3)$, $(3,4,6)$, $(0,6,7)$, $(0,4,8)$, $(0,8,4)$, $(0,3,6)$ and $(0,7,5)$ with $\alpha = (\infty)(0\ 1\ 2)(3\ 4\ 5)(6\ 7\ 8)$ as an automorphism. Therefore, the only unsettled problem for the existence of rotational MTS is: if $v = 30t + 6$ and $t \neq 0$, does there exist a 5-rotational or $(6t + 1)$-rotational MTS(v) ?

Section 3. Extended Mendelsohn Triple Systems.

An *extended Mendelsohn triple system* (EMTS) is a pair (V,B) where V is a finite set and B is a collection of cyclic triples of elements (not necessarily distinct) of V such that every ordered pair of elements (not necessarily distinct) of V is contained in exactly one cyclic triple in B. Like the triples of extended triple systems, the cyclic triples of B are of three types:

$$(a,a,a)\ ,\quad (b,b,c)\ ,\quad (x,y,z)\ .$$

The element a is called an *idempotent* and b a non-idempotent of (V,B). We will denote by $EMTS(v;p)$ an extended Mendelsohn triple system on v elements which has p idempotents. The existence of extended Mendelsohn triple systems has been settled by Bennett [2]. Although Bennett himself called such systems extended cyclic triple systems, it is natural that we should call those systems extended Mendelsohn triple systems.

3.1 *Theorem* [2]. The necessary and sufficient conditions for the existence of an $EMTS(v;p)$ with $0 \le p \le v$ are

(i) if $v \equiv 0 \pmod 3$ then $p \equiv 0 \pmod 3$,

(ii) if $v \equiv 1$ or $2 \pmod 3$ then $p \equiv 1 \pmod 3$,

(iii) if $v = 6$ then $p \leq 3$.

In this section, we obtain necessary and sufficient conditions for the existence of cyclic EMTS(v;p)'s, and those of 1-rotational EMTS(v;p)'s. Also, we will assume that $V = Z_v$ is the set of elements of our cyclic EMTS(v;p) and $\alpha = (0 \ldots v-1)$ is the corresponding cyclic automorphism. In the case of 1-rotational EMTS(v;p), $V = Z_{v-1} \cup \{\infty\}$ and $\alpha = (\infty)(0 \ldots v-2)$, respectively.

3.2 Remark. If a cyclic EMTS(v;p) exists, then $p = 0$ or v.

Proof. Obvious.

3.3 Lemma. Necessary conditions for the existence of a cyclic EMTS(v;p) are

(i) if $p = v$ then $v \equiv 1$ or $3 \pmod 6$, $v \neq 9$,

(ii) if $p = 0$ then $v \equiv 3 \pmod 6$.

Proof. (i) Follow from the existence of a cyclic MTS.

(ii) If $p = 0$ we have $v \equiv 0 \pmod 3$ from the existence of an EMTS(v;0). In case $v \equiv 0 \pmod 6$, the existence of a cyclic EMTS$(v;0)$ is equivalent to a partitioning of the set $\{1,\ldots, v-1\}\setminus\{x,y\}$ for some $1 \le x < y \le v-1$ such that $x + y = v$, into difference triples $\{a,b,c\}$ for which $a + b + c \equiv 0 \pmod{v}$. Since v divides the sum of the differences in each difference triple, it divides the sum of all of the differences being partitioned by the triple. Thus v divides the sum of the integers 1 through $v - 1$ except exactly two numbers whose sum is v, that is, $v(v-1)/2 - v \equiv 0 \pmod{v}$. Equivalently, $(v-1)/2$ is an integer which is impossible for $v \equiv 0 \pmod{6}$.

As a consequence of Theorem 2.1, we have

3.4 Theorem. A cyclic EMTS$(v;v)$ exists if and only if $v \equiv 1$ or $3 \pmod{6}$, $v \neq 9$.

3.5 Lemma. If $v \equiv 3 \pmod{6}$, then there exists a cyclic EMTS$(v;0)$.

Proof. Let $v = 6t + 3$, $t \neq 1$.

Base cyclic triples: $B = B_1 \cup B_2$

where

B_1: $\{(0,\ 0,\ 2t+1)\}$,

B_2: the collection of the base cyclic triples obtained by replacing each base block $\{a,b,c\}$ except the base block of the form $\{0, 2t+1, 4t+2\}$ of a cyclic $STS(6t+3)$ based on Z_{6t+3} with the cyclic triples (a,b,c) and (a,c,b).

When $t = 1$, $B = \{(0,0,2), (0,1,6), (3,0,8)\}$.

Then (V,B) is a cyclic $EMTS(v;0)$.

Lemmas 3.3 and 3.5 together yield

3.6 Theorem. A cyclic $EMTS(v;0)$ exists if and only if $v \equiv 3 \pmod 6$.

In the rest of this section, we will consider 1-rotational EMTS's.

3.7 Remark. If a 1-rotational $EMTS(v;p)$ exists, then $p = 1$ or v.

Proof. Obvious.

3.8 Lemma. Necessary conditions for the existence of a 1-rotational $EMTS(v;p)$ are

(i) if $p = v$ then $v \equiv 1, 3$ or $4 \pmod 6$,

(ii) if $p = 1$ then $v \equiv 1$ or $2 \pmod 3$.

Proof. (i) Follow from the fact that the system obtained by deleting all cyclic triples of the form (a,a,a) of a 1-rotational EMTS(v;v) is a 1-rotational MTS(v).

(ii) By Theorem 3.1.

3.9 Lemma. There is no 1-rotational EMTS(10;1).

Proof. This is easily seen from the fact that there is no cyclic MTS(9) (see Theorem 2.1).

3.10 Lemma. If $v \equiv 2$ or $4 \pmod 6$, $v \neq 10$, then there exists a 1-rotational EMTS(v;1).

Proof. Let $v \equiv 2$ or $4 \pmod 6$, $v \neq 10$.
Base cyclic triples: $B = B_1 \cup B_2$

where

B_1: $\{(\infty,\infty,\infty), (\infty,0,0)\}$,

B_2: the collection of all base cyclic triples obtained by replacing each base block $\{a,b,c\}$ of a cyclic STS(v - 1) based on Z_{v-1} with the cyclic triples (a,b,c) and (a,c,b).

Then (V,B) is a 1-rotational EMTS$(v;1)$.

3.11 <u>Lemma</u>. If $v \equiv 1$ or $5 \pmod 6$, then there exists a 1-rotational EMTS$(v;1)$.

<u>Proof</u>. For $v \equiv 1$ or $5 \pmod 6$, let (V,B) be a 1-rotational ETS$(v;1)$ constructed in Section 3 of Chapter 3. Then B should contain blocks of the forms $\{\infty,\infty,\infty\}$, $\{\infty,0,a\}$ and $\{0,0,b\}$ for some $a, b \in Z_v \setminus \{0\}$, $a \neq b$. Set $B' = B_1 \cup B_2$ where

$$B_1: \{(\infty,\infty,\infty), (\infty,i,a+i), (i,i,b+i) \mid i = 0,\ldots, v-1\},$$

$$B_2: \{(x,y,z), (x,z,y) \mid \{x,y,z\} \in B \setminus B_1'\}$$

where B_1' is the collection of all members of B_1 with the cyclic order disregarded. Then (V,B') is a 1-rotational EMTS$(v;1)$.

Summarizing, we have:

3.12 <u>Theorem</u>. A 1-rotational EMTS$(v;1)$ exists if and only if $v \equiv 1$ or $2 \pmod 3$ and $v \neq 10$.

In the case $p = v$, the existence of k-rotational EMTS$(v;p)$'s is, in effect, equivalent to the existence of k-rotational MTS(v)'s . Thus, the following theorem immediately follows from Theorem 2.9 in this chapter.

3.13 Theorem. A 1-rotational EMTS(v;v) exists if and only if $v \equiv 1, 3$ or $4 \pmod 6$ and $v \neq 10$.

CHAPTER 5. STEINER 2-DESIGNS S(2,k,v) WITH k > 3

Section 1. Introduction.

There is nothing new in this chapter. However, our aim is to summarize known results about 2-designs with prescribed automorphism types not included in previous chapters that may give some information to the reader for further research.

As mentioned in Chapter 1, a system $S(2,k,v)$ is a 2-design which is a Steiner system, a so-called Steiner 2-design. Very little is known about cyclic $S(2,k,v)$ systems when $k > 3$; the existence problem for them remains open. A necessary condition for the existence of a $S(2,4,v)$ system is that $v \equiv 1$ or $4 \pmod{12}$. For $S(2,5,v)$ systems, the necessary condition is that $v \equiv 1$ or $5 \pmod{20}$. Hanani [34] demonstrated that these conditions are also sufficient. When cyclic Steiner 2-designs are considered, these conditions are not sufficient as for some small orders cyclic Steiner 2-designs are known not to exist [see 16]. For example, there is no cyclic $S(2,4,v)$ for $v = 16, 25$ or 28. For higher values of v, it remains unknown whether $v \equiv 1, 4 \pmod{12}$ is a sufficient condition for the existence of cyclic $S(2,4,v)$ designs. A similar situation exists in the case $k = 5$. In this chapter, we present a survey of known results for $k > 3$.

It appears that virtually nothing is known about ℓ-rotational $S(2,k,v)$ designs if $k > 3$. The only exception, as far as we can tell, is the well-known fact that affine planes of order v are 1-rotational $S(2,v,v^2)$'s [cf. 6, pp. 196-204].

Section 2. Cyclic Steiner 2-designs $S(2,k,v)$.

Bose [5] has constructed two infinite families of cyclic $S(2,k,v)$ systems. The first is for $k = 4$. In what follows $GF(p)$ denotes the Galois field of order p.

2.1 Theorem [5]. Let v be prime of the form $12t + 1$. Let x be a primitive element of $GF(v)$, which satisfies $x^{4t} - 1 = x^q$ for some odd q. Then there exists a cyclic $S(2,4,v)$.

Proof. The t base blocks are

$$\{\{0, x^{2i}, x^{4t+2i}, x^{8t+2i}\} | i = 0,\ldots, t - 1\}.$$

The second construction is for $k = 5$. The construction is very similar to the first.

2.2 Theorem [5]. Let v be a prime of the form $20t + 1$. Let x be a primitive element of $GF(v)$ satisfying $x^{4t} + 1 = x^q$ where q is odd. Then there exists a cyclic $S(2,5,v)$.

Proof. The t base blocks are

$\{\{x^{2i}, x^{4t+2i}, x^{8t+2i}, x^{12t+2i}, x^{16t+2i}\} | i = 0, \ldots, t-1\}$.

The next two constructions are due to Colbourn and Mathon [19].

2.3 <u>Theorem</u> [19]. Let $v = 4p$, where p is a prime of the form $12t + 1$. Let x be a primitive element of $GF(p)$ satisfying $x \equiv 3 \pmod 4$. Then the $4t + 1$ blocks

$$\{0, x^{4i}, x^{4i+3}, x^{4i+6}\}, \qquad i = 0, \ldots, 3t-1$$

$$\{0, x^{4i+1}, x^{4t+4i+1}, x^{8t+4i+1}\} \qquad i = 0, \ldots, t-1$$

$$\{0, p, 2p, 3p\},$$

are the base blocks of a cyclic $S(2,4,v)$ system.

2.4 <u>Theorem</u> [19]. Let $v = 5p$, where p is a prime of the form $4t + 1$. Let x be a primitive element of $GF(p)$ satisfying $x \equiv 4 \pmod 5$ and such that $(x^a + 1) = x^b(x^a - 1)$ for some odd integers a, b. Then the $t + 1$ blocks

$$\{0, x^{2i}, x^{2i+a}, x^{2t+2i}, x^{2t+2i+a}\}, \quad i = 0, \ldots, t-1$$

$$\{0, p, 2p, 3p, 4p\}$$

are the base blocks of a cyclic $S(2,5,v)$ system.

The following two general constructions are applicable to various values of block size k.

2.5 <u>Theorem</u> [5, 8, pp. 56]. If $v = (q^{d+1} - 1)/(q - 1)$, $d \geq 2$ and q is a prime power, then there exists a cyclic $S(2, q+1, v)$.

The proof of this theorem constructs projective geometries which are cyclic designs with these parameters.

2.6 <u>Theorem</u> [70]. If $v \equiv 1 \pmod{k(k-1)}$ is a prime power and $v > (k(k-1))^{k(k-1)}$, then there exists a cyclic $S(2,k,v)$.

The above constructions rely on primality. We would prefer a general construction technique which does not depend on a primality condition.

So far we have only direct constructions. Hereafter, we present recursive constructions due to Colbourn [16]. We will assume the set of elements of our cyclic $S_\lambda(2,k,v)$ to be Z_v and corresponding cyclic automorphism to be $(0 \ldots v-1)$.

2.7 <u>Construction</u> [16]. Let $v \not\equiv 0 \pmod{k}$ and $\{B_1, \ldots, B_m\}$ be the set of base blocks for a cyclic

$S_\lambda(2,k,v)$. Let $\{B_1',\ldots, B_r'\}$ be the set of base blocks for a cyclic $S_\lambda(2,k,m)$ with m relatively prime to $(k-1)!$. Then the set of base blocks for a cyclic $S_\lambda(2,k,mv)$ is constructed as follows:

(i) For each $B_j = \{0, b_1,\ldots, b_{k-1}\}$, take the m base blocks

$\{0, b_1+iv, b_2+2iv,\ldots, b_{k-1} + (k-1)iv\}$,

$i = 0,\ldots, m-1$.

(ii) For each $B_j' = \{0, b_1',\ldots, b_{k-1}'\}$, take the single base block $\{0, vb_1',\ldots, vb_{k-1}'\}$.

2.8 Construction [16]. Let $\{B_1,\ldots, B_n\}$ be the set of base blocks of length kv for a cyclic $S(2,k,kv)$. Let $\{B_1',\ldots, B_r'\}$ be the set of base blocks for a cyclic $S(2,k,km)$ with m relatively prime to $(k-1)!$. Then the set of base blocks for a cyclic $S(2,k,kmv)$ is constructed as follows:

(i) For each $B_j = \{0, b_1,\ldots, b_{k-1}\}$, take the m base blocks

$\{0, b_1+ikv, b_2+2ikv,\ldots, b_{k-1} + (k-1)ikv\}$,

$i = 0,\ldots, m-1$.

(ii) For each $B_j' = \{0, b_1', \ldots, b_{k-1}'\}$, take the single base block $\{0, vb_1', \ldots, vb_{k-1}'\}$.

From Constructions 2.7 and 2.8, we have

2.9 Theorem. Let m be relatively prime to $(k-1)!$, and $v \not\equiv 0 \pmod{k}$. Then if a cyclic $S_\lambda(2,k,v)$ and a cyclic $S_\lambda(2,k,m)$ exist then there exists a cyclic $S_\lambda(2,k,mv)$.

2.10 Theorem. Let m be relatively prime to $(k-1)!$. Then if a cyclic $S(2,k,kv)$ and a cyclic $S(2,k,km)$ exist then there exists a cyclic $S(2,k,kmv)$.

Combining Theorems 2.1 and 2.9, we have

2.11 Theorem. Let $p_1, \ldots, p_s$ be primes which are all 1 (mod 12). In addition, suppose that for each p_i there exists a primitive element y_i of $GF(p_i)$ satisfying $y_i^{4t} - 1 = y_i^q$ for some odd q. Then a cyclic $S(2,4,v)$ exists for all $v = p_1^{x_1} \cdots p_s^{x_s}$ for all $x_i \geq 0$, $i = 1, \ldots, s$.

2.12 Theorem. If $v \equiv 1 \pmod{12}$ and there exists a cyclic $S(2,4,v)$, then there exists a cyclic $S(2,4,49v)$ and a cyclic $S(2,4,85v)$.

Proof. There exists a cyclic $S(2,4,49)$ and a cyclic $S(2,4,85)$ [16] and hence Theorem 2.9 can be applied.

Since there exists a cyclic $S(2,4,76)$ [16], we have

2.13 Theorem. If $v = 4(3t + 1)$ and there exists a cyclic $S(2,4,v)$, then there exists a cyclic $S(2, 4, 76(3t + 1))$ for all $t \geq 0$.

A similar situation exists for $k = 5$. Using Theorems 2.2 and 2.10, we obtain

2.14 Theorem. Let $p_1, \ldots, p_s$ be primes which are all $1 \pmod{20}$. In addition, for each p_i there exists a primitive element y_i of $GF(p_i)$ satisfying $y_i^{4t} + 1 = y_i^q$ for some odd q. Then there exists a cyclic $S(2,5,v)$ for all $v = p_1^{x_1} \cdots p_s^{x_s}$ for all $x_i \geq 0$, $i = 1, \ldots, s$.

2.15 Theorem. If $v \equiv 1 \pmod{20}$ and there exists a cyclic $S(2,5,v)$, then there exists a cyclic $S(2,5,5v)$.

Proof. 5 and 4! are relatively prime; apply Theorem 2.9.

Since there exists a cyclic $S(2,6,31)$ and a cyclic $S(2,6,91)$ [16], we have

2.16 Theorem. There exists a cyclic $S(2,6,v)$ for all $v = 31^x \cdot 91^y$, $x, y \geq 0$.

For individual constructions of cyclic $S_\lambda(2,k,v)$ for small values of v, see [16, 41]. Although many infinite families have been obtained it is not known whether, for a fixed $k > 3$, a cyclic $S(2,k,v)$ system exists for each admissible order of v.

CHAPTER 6. STEINER QUADRUPLE SYSTEMS

Section 1. Introduction.

A Steiner quadruple system of order v (SQS(v)) is a $S(3,4,v)$ design. One obtains immediately that $v \equiv 2$ or $4 \pmod 6$ is a necessary condition for the existence of an SQS(v); and the total number of quadruples is $\frac{1}{24}v(v-1)(v-2)$, the number of quadruples containing a given element is $\frac{1}{6}(v-1)(v-2)$, and the number of quadruples containing a given pair of elements is $\frac{1}{2}(v-2)$. In 1847, Kirkman [41] first investigated the existence of SQS, showing that an SQS(v) exists whenever $v = 2^n$ for every n. The existence of SQS was settled by Hanani [33] in 1960, when he proved, with the aid of recursive constructions, that the necessary condition is also sufficient.

As long as we consider cyclic SQS, we always assume the set of elements of our cyclic SQS(v) to be $V = Z_v$, and its corresponding cyclic automorphism to be $\alpha = (0 \ldots v-1)$.

The investigation of cyclic SQS initially focused on small values of v. Barrau [1] found that the unique SQS(10) is cyclic and its quadruples are determined by the three base quadruples

$$\{0,1,3,4\}, \{0,1,2,6\}, \{0,2,4,7\}.$$

Cyclic SQS were investigated further by Fitting [25] who constructed cyclic SQS(26) and cyclic SQS(34). Much later, with assistance of a computer, Guregová and Rosa [27] showed that cyclic SQS(v) do not exist for $v = 8, 14$ or 16.

Before going further, following Lindner and Rosa [45] we partition the admissible orders for SQS(v) into four classes:

A. $v \equiv 2$ or $10 \pmod{24}$,

B. $v \equiv 4$ or $20 \pmod{24}$,

C. $v \equiv 14$ or $22 \pmod{24}$,

D. $v \equiv 8$ or $16 \pmod{24}$.

Cyclic systems in classes B and D necessarily contain the unique orbit of length $\frac{v}{4}$, while those in C and D contain an odd number of orbits of length $\frac{v}{2}$.

A cyclic SQS(20) has been constructed first by Jain [39]. He has shown that his is the unique S-cyclic system (i.e., each orbit is invariant under the mapping $\beta: x \to -x \pmod{20}$) which has the following 15 base quadruples:

{0,1,3,4} , {0,1,2,11} , {0,1,5,16} ,
{0,2,6,8} , {0,2,4,12} , {0,3,6,12} ,
{0,3,9,14} , {0,1,6,7} , {0,1,9,12} ,
{0,1,8,13} , {0,2,7,9} , {0,2,5,17} ,
{0,3,7,16} , {0,4,8,14} , {0,5,10,15} .

Later, Phelps [56], Griggs and Grannell [32] and myself [9] constructed other cyclic SQS(20)'s. More recently, Phelps [58] has made a complete enumeration of cyclic SQS(20)'s. There are exactly 29 non-isomorphic such systems including one which is S-cyclic [39] and there is a total of 152 distinct cyclic SQS(20)'s.

In the class C, the only orders v for which a cyclic SQS(v) is known to exist are 22 and 38. For $v = 22$ Phelps [57] has constructed 7 non-isomorphic cyclic systems and Diener [21] has enumerated all cyclic SQS(22)'s. There are exactly 21 non-isomorphic such systems and there is a total of 210 distinct cyclic SQS(22)'s. For the case $v = 38$, a cyclic SQS(38) has been constructed by Colbourn and Phelps [15]. An example of a cyclic SQS(22) which is #1 as given in [21] is:

{0,1,1[illegible] , {0,1,2,4} , {0,1,5,6} ,
{0,1,7,[illegible] , {0,1,8,9} , {0,1,10,13} ,
{0,1,16,20} , {0,2,5,10} , {0,2,6,14} ,
{0,2,7,17} , {0,2,8,12} , {0,2,9,16} ,

{0,2,11,15} ,	{0,2,13,19} ,	{0,3,6,17} ,
{0,3,7,11} ,	{0,3,9,15} ,	{0,4,9,17} .

In the class D , Colbourn and Phelps [15] constructed a cyclic SQS(40) and very recently Grannell and Griggs [30] determined a cyclic SQS(32) which guarantees the existence of a cyclic $SQS(2^n)$ for every $n \geq 5$. An example of a cyclic SQS(32) as given in [30] is:

{0,8,16,24} ,	{0,2,16,18} ,	{0,5,16,21} ,
{0,6,16,22} ,	{0,1,2,17} ,	{0,1,3,21} ,
{0,1,4,6} ,	{0,1,5,27} ,	{0,1,7,26} ,
{0,1,8,25} ,	{0,1,9,10} ,	{0,1,11,29} ,
{0,1,12,22} ,	{0,1,13,14} ,	{0,1,15,18} ,
{0,1,28,30} ,	{0,2,6,12} ,	{0,2,7,9} ,
{0,2,8,10} ,	{0,2,11,23} ,	{0,2,13,15} ,
{0,2,14,22} ,	{0,3,6,19} ,	{0,3,7,10} ,
{0,3,9,12} ,	{0,3,13,22} ,	{0,3,15,20} ,
{0,4,8,22} ,	{0,4,9,13} ,	{0,4,11,25} ,
{0,4,12,16} ,	{0,4,15,19} ,	{0,5,11,26} ,
{0,5,12,25} ,	{0,5,14,19} ,	{0,5,15,22} ,
{0,6,14,20} ,	{0,6,15,23} ,	{0,7,14,23} ,
{0,3,8,27} ,	{0,3,11,24} ,	

Further cyclic SQS were constructed by Köhler [42, 43] and Colbourn and Colbourn [13]. Grannell and Griggs [29] showed that there were exactly 18 nonisomorphic S-cyclic SQS(26)'s. The orders v less than 100 for which the existence of a cyclic SQS(v) is in doubt, are v = 46, 56, 62, 70, 86 and 94 which are all in class C except v = 56 in class D.

The first infinite families of cyclic SQS were found by Phelps [57] who exploited the structure of inversive planes, which are $S(3, q+1, q^2+1)$ designs.

6.1 Theorem [57]. If there exists a SQS(q + 1), where q is a prime power, then there exists a cyclic $SQS(q^2 + 1)$ containing SQS(q + 1) as a subsystem.

6.2 Theorem [57]. If there exists a cyclic SQS(q + 1), where q is a prime power, then there exists a cyclic $SQS(q^n + 1)$ for all $n > 0$.

The smallest new system which results from these theorems is a cyclic SQS(28). Also, it is worth remarking that the above theorems allow for numerous non-isomorphic cyclic $SQS(q^n + 1)$'s (the exact number being determined in part by the number of distinct SQS(q + 1)'s).

In this chapter, Section 2 provides direct constructions of cyclic SQS which have been given by Köhler [42, 43] and later Diener [22]. In Section 3, we show that if there exists a cyclic SQS(v) where $v \equiv 2$ or 10 (mod 12) then there exists a cyclic SQS(2v) , which is appeared in [9]. In the same section, we include a generalized version [13] of our doubling construction above. By combining methods of Sections 2 and 3, we construct directly a S-cyclic SQS(v) for v = 52, 68, 122, 130, 146, 170, 250, 290 and 370 , and a non-S-cyclic SQS(v) for v = 26, 28, 34, 50, 58, 76, 80, 88, 92, 98 and 124 , which are listed in the Appendices. Finally, in this section we establish a table of recent results on the known spectrum for cyclic SQS(v) for $v \leq 400$. In Section 4, we discuss 1-rotational SQS which are studied by Phelps [55].

Section 2. Direct Constructions of Cyclic SQS.

First of all, let us discuss the general existence problem for cyclic SQS. Let $P_k(Z_v)$ be the collection of all k-subsets of Z_v. We define the *difference triple* (a,b,c) of a triple $\{x,y,z\}$ in $P_3(Z_v)$ with $x < y < z$ as follows:

$$a \equiv y - x, \quad b \equiv z - y, \quad c \equiv x - 2 \pmod{v} .$$

Two difference triples are *equivalent* if one is a cyclic shift of the other. Under the action of $\alpha = (0 \ldots v - 1)$, two triples of $P_3(Z_v)$ are in the same orbit if and only if their difference triples are equivalent.

In a similar manner, for each quadruple $\{x,y,z,u\}$ in $P_4(Z_v)$, where $x < y < z < u$, we can define the *difference quadruple* (a,b,c,d) where:

$$a \equiv y - x, \quad b \equiv z - y, \quad c \equiv u - z, \quad d \equiv x - u \pmod{v} .$$

A difference quadruple (a,b,c,d) (mod v) determines the four difference triples (mod v) (not necessarily distinct), namely,

$$(a,b,v-a-b), \ (b,c,v-b-c), \ (c,d,v-c-d), \ (d,a,v-d-a) .$$

It is easily seen that two quadruples are in the same orbit if and only if they have the same difference triples.

Thus, we can characterize the orbits of a cyclic SQS(v) in three ways:

(1) We can choose a quadruple from each orbit (called a base quadruple);

(2) With each orbit we can associate a difference quadruple;

(3) Finally, with each orbit we can associate a set of difference triples.

We will represent a cyclic SQS(v) by base quadruples or difference quadruples or sets of difference triples, whichever will be convenient.

Before moving on, consider a difference quadruple (a,b,c,d). If either

(1) $a = c$ (or $b = d$) or

(2) $a = b$ and $c = d$ (or $b = c$ and $d = a$),

then the difference quadruple is called symmetric. A cyclic SQS all of whose difference quadruples are symmetric is called S-cyclic. It is simple to show that each orbit of a S-cyclic SQS(v) is invariant under the mapping $\beta: x \to -x \pmod{v}$. We require the following definition: given a

SQS (V,B), if we choose any point $p \in V$ and delete that point from the set V and from all quadruples which contain it then the resulting system $(V_p, B(p))$, where

$$V_p = V \setminus \{p\} \quad \text{and} \quad B(p) = \{b' = b \setminus \{p\} \mid b \in B \text{ and } p \in b\},$$

will be a STS. Such a STS is called a <u>derived STS</u>. Now, we can easily see that S-cyclic SQS(v)'s are only in classes A and B, that is, $v \equiv 2, 4, 10$ or $20 \pmod{24}$; since a S-cyclic SQS(v) has also an automorphism $\beta: x \to -x \pmod{v}$, the derived STS(v - 1)'s of the S-cyclic SQS(v) must be reverse STS(v - 1)'s and these only exist for $v - 1 \equiv 1, 3, 9$ or $19 \pmod{24}$ (see Section 3 of Chapter 1). Recently, Diener [22], Grannell and Griggs [28] proved that if v is an admissible order and if a S-cyclic SQS(2v) exists then the S-cyclic SQS(2v) must contain a S-cyclic SQS(v) as a subsystem. This result much restricts the above necessary condition for S-cyclic SQS. Since there is no S-cyclic SQS(v) in class C, that is, $v \equiv 14$ or $22 \pmod{24}$, the condition becomes $v \equiv 2$ or $10 \pmod{24}$ or $v \equiv 4$ or $20 \pmod{48}$. There is still further restriction on the necessary condition. First it is not too hard to see the following.

2.1 <u>Remark</u>. If there exists a cyclic SQS(v), then the number of non-equivalent difference triples of each difference quadruple must be either one, two or four.

Suppose that there is a symmetric difference quadruple containing the difference triple (a,2a,4a) (mod v). Then it is possibly equivalent to either the difference quadruple (a,2a,a,3a) or (a,2a,2a,2a). But both contain three non-equivalent difference triples. Thus we can conlude that there is no S-cyclic SQS(v) containing the difference triple (a,2a,4a) (mod v), that is, $v \equiv 0 \pmod 7$. Summarizing this we have:

2.2 Theorem. A necessary condition for the existence of a S-cyclic SQS(v) is that $v \equiv 2, 4, 10, 20, 26$ or $34 \pmod{48}$, except for $v \equiv 98, 154, 196$ or $308 \pmod{336}$.

We are now going to introduce a direct construction of cyclic SQS. To begin with the direct method employs S-cyclic SQS since symmetry of difference quadruples makes it easier to construct such a system. Also, the method can be modified to construct non-S-cyclic SQS. Here, we will consider all possible orders $v \equiv 2, 4, 10$ or $20 \pmod{24}$. The idea of direct method is to construct a graph, associated with a S-cyclic SQS(v). It originates from Fitting [25] and has recently been taken up by Köhler [42, 43] and Diener [22]. The graphical terminology and notation that are used in this remaining section are those from [35], unless they are defined or explained here.

To construct a S-cyclic SQS(v), let us find a number of symmetric difference quadruples mod v such that any difference triple mod v is contained in exactly one such quadruple. Let $D_3(Z_v)$ denote the set of all non-equivalent difference triples mod v. Since there are $\binom{v}{3}$ 3-subsets of Z_v, we have:

2.3 Remark. $|D_3(Z_v)| = \binom{v}{3}\Big/v = \frac{1}{6}(v - 1)(v - 2)$.

By Remark 2.1 and a simple consideration, we have the following.

2.4 Remark. If a S-cyclic SQS(v) exists, then it must contain the following difference quadruples:

$$(a,\ a,\ \tfrac{v}{2} - a,\ \tfrac{v}{2} - a), \quad a = 1, \ldots, [\tfrac{v}{4}].$$

Thus, to construct a S-cyclic SQS(v) we may delete the difference triples, which are produced by the difference quadruples $(a, a, \frac{v}{2} - a, \frac{v}{2} - a)$, from the set $D_3(Z_v)$; the resulting set is denoted by $D_3(Z_v)^*$. Each $(a, a, \frac{v}{2} - a, \frac{v}{2} - a)$ yields 4 difference triples, except for $a = \frac{v}{4}$; if $a = \frac{v}{4}$ it produces only one. So we have:

2.5 Remark. (1) If $v \equiv 2$ or 10 (mod 24) then

$$|D_3(Z_v)^*| = \frac{(v-1)(v-2)}{6} - 4\,\frac{v-2}{4} = \frac{(v-2)(v-7)}{6} .$$

(2) If $v \equiv 4$ or 20 (mod 24) then

$$|D_3(Z_v)^*| = \frac{(v-1)(v-2)}{6} - 4\,\frac{v-4}{4} - 1 = \frac{(v-4)(v-5)}{6} .$$

Since we have orbits of the form (a,a,b,b) with $2(a+b) = v$, the problem of constructing an S-cyclic SQS(v) is equivalent to the problem of packing orbits of the form $(a, b, a, v-(2a+b))$ from the difference triples in $D_3(Z_v)^*$. The difference triples contained in such an orbit are

$$(a,\ b,\ v-(a+b)) , \quad (a,\ a+b,\ v-(2a+b)) ,$$
$$(b,\ a,\ v-(a+b)) , \quad (a+b,\ a,\ v-(2a+b)) .$$

So, we define a subset $E(v)$ of $D_3(Z_v)^*$ as follows: for each element $(a,b,c) \in D_3(Z_v)^*$,

$$(a,b,c) \in E(v) \longleftrightarrow (b,a,c) \notin E(v) .$$

We represent the elements of $E(v)$ as $\{a,b,c\}$. Thus if $\{a,b,c\} \in E(v)$ then either $(a,b,c) \in E(v)$ or $(b,a,c) \in E(v)$, but not both; according to convenience we may consider $(a,b,c) \in E(v)$ or $(b,a,c) \in E(v)$. Therefore, $E(v)$ can be defined as the set

$$\{\{a,b,c\} \mid a,b,c \in \{1,\ldots,v-3\} \setminus \{\tfrac{v}{2}\},\ a < b < c,\ a+b+c = v\}.$$

By the definition of $E(v)$, $|E(v)| = \frac{1}{2}|D_3(Z_v)^*|$. Thus:

2.6 Remark. (1) If $v \equiv 2$ or $10 \pmod{24}$, $|E(v)| = \frac{1}{12}(v-2)(v-7)$.

(2) If $v \equiv 4$ or $20 \pmod{24}$, $|E(v)| = \frac{1}{12}(v-4)(v-5)$.

Now, define a graph $H(v)$ as follows: the vertex-set of $H(v)$ is $E(v)$, and two vertices $\{a,b,c\}$ and $\{a',b',c'\}$ are joined by an edge in $H(v)$ whenever $a' = a$, $b' = a + b$, $c' = v - (2a + b)$. It is easy to see that the degrees of $H(v)$ are ≤ 3 and $H(v)$ has no isolated vertices unless $v \equiv 0 \pmod{7}$. Each edge $[\{a,b,c\}, \{a',b',c'\}]$ of $H(v)$ determines the difference quadruple $(a,b,a,c') \pmod{v}$. Obviously, the following theorem gives a sufficient condition for the existence of a S-cyclic SQS.

2.7 Theorem. If $H(v)$ contains a 1-factor, then there exists a S-cyclic SQS(v).

Now, set $E_2(v) = \{\{a,b,c\} \in E(v) | a,b,c$ are even$\}$ and $E_1(v) = E(v) \setminus E_2(v)$. For $i = 1, 2$, consider the subgraph $H_i(v)$ of $H(v)$ whose vertex set is $E_i(v)$ and whose edge set is the set of those edges of $H(v)$ that have both ends in $E_i(v)$. By the definition of edges of $H(v)$, $H(v) = H_1(v) \cup H_2(v)$ is disjoint union (to find a cyclic SQS, it is not necessarily edge-disjoint union). Moreover, the graphs $H_2(v)$ and $H(v/2)$ are isomorphic. A basic counting argument provides the number of vertices of $E_i(v)$, $i = 1, 2$. So, we have:

2.8 <u>Remark</u>. (1) If $v \equiv 2$ or $10 \pmod{24}$ then

(i) $|E_1(v)| = \dfrac{(v-2)(v-6)}{16}$, (ii) $|E_2(v)| = \dfrac{(v-2)(v-10)}{48}$.

(2) If $v \equiv 4$ or $20 \pmod{24}$ then

(i) $|E_1(v)| = \dfrac{(v-2)(v-4)}{16}$, (ii) $|E_2(v)| = \dfrac{(v-4)(v-14)}{48}$.

It is worth noting that if $v \equiv 28$ or $44 \pmod{48}$ then $|E_2(v)|$ is odd. This implies that a S-cyclic SQS(v) cannot exist for $v \equiv 28$ or $44 \pmod{48}$. Also, note that $H_1(v)$ has no isolated vertices.

Summarizing, we have:

2.9 Theorem. Let $v \equiv 2, 4, 10, 20, 26$ or 34 (mod 48) and $v \not\equiv 0$ (mod 7). Then a S-cyclic SQS(v) exists if and only if both $H_1(v)$ and $H_2(v)$ have a 1-factor, respectively.

Recently, Grannell and Griggs [28], and Diener [22] have constructed a S-cyclic SQS(52). We construct a S-cyclic SQS(v) for $v = 52, 68, 122$ and 146.

2.10 Theorem. If $v \equiv 2$ or 10 (mod 24), then $H_1(v)$ contains a 1-factor.

Proof. For each $i = 1, \ldots, \frac{v-6}{4}$, set

$$F_i = \{[\{2i-1, 2t, v-2i-2t+1\}, \{2i-1, 2i-1+2t, v-4i-2t+2\}] \mid t = 1, \ldots, \frac{v-2}{4} - i\}.$$

Then $\bigcup\{F_i \mid i = 1, \ldots, \frac{v-6}{4}\}$ is a 1-factor of $H_1(v)$.

We can conclude that:

2.11 Corollary. Let $v \equiv 2$ or 10 (mod 24) and $v \not\equiv 0$ (mod 7). Then a S-cyclic SQS(v) exists if and only if $H_2(v)$ contains a 1-factor.

Köhler [43] was able to show that $H_2(v)$ contains a 1-factor for $v = 50, 58, 74$ and 82.

Recall that $H_2(v)$ and $H(v/2)$ are isomorphic. So we will take $H(v/2)$ instead of $H_2(v)$ in the following discussion. First, we can easily see that:

2.12 Remark. If μ is a unit of $Z_{(v/2)}$, then the mapping $\alpha_\mu : E(v/2) \to E(v/2)$ given by $\{a,b,c\} \mapsto \{\mu a, \mu b, \mu c\}$ is an automorphism of $H(v/2)$; such an automorphism is called a multiplier automorphism.

Thus α_μ permutes the difference triples, whence we obtain a partition of the difference triples into equivalence classes under the action of α_μ. $E(v/2)/\sim$ denotes the set of all equivalence classes. Define a graph $H(v/2)/\sim$ as follows: the vertex set of $H(v/2)/\sim$ is $E(v/2)/\sim$ and two equivalence classes $\bar{\alpha}$ and $\bar{\beta}$ in $E(v/2)/\sim$ are joined by an edge in $H(v/2)/\sim$ if there exists a difference triple $\{a,b,c\}$ in $\bar{\alpha}$ and a difference triple $\{a',b',c'\}$ in $\bar{\beta}$ such that $a' = a$, $b' = a + b$ and $c' = v - (2a + b)$. With this notation, we have the following theorem:

2.13 Theorem. Let $v \equiv 2$ or $10 \pmod{24}$ and $v \not\equiv 0 \pmod{7}$. Then a S-cyclic SQS(v) exists if and only if $H(v/2)/\sim$ contains a 1-factor.

We were able to show that $H(v/2)$ contains a 1-factor for $v = 130, 170, 250, 290$ and 370.

Let us describe another of the sufficient conditions obtained by Kohler [42, 43]. Let $p = \frac{v}{2} \equiv 1$ or $5 \pmod{12}$

be a prime. Let F be the Galois field $GF(p)$, and for $\sigma \in F' = F\setminus\{0, 1, \frac{1}{2}(p-1), p-2, p-1\}$, define $\bar{\sigma} = \{\sigma_1, \sigma_1^*, \sigma_2, \sigma_2^*, \sigma_3, \sigma_3^*\}$ where $\sigma_1 = \sigma$, $\sigma_2 = \sigma^{-1}$, $\sigma_3 = -\sigma/(\sigma + 1)$, $\sigma_i^* = -\sigma_i - 1$, $i = 1, 2, 3$. Thus, for $\sigma, \delta \in F'$, either $\bar{\sigma} = \bar{\delta}$ or $\bar{\sigma} \cap \bar{\delta} = \phi$. Define a graph $B(p)$ as follows: the vertex set of $B(p)$ is the set $V = \{\bar{\sigma} | \sigma \in F'\}$, and two vertices $\bar{\sigma}, \bar{\delta}$ are joined by an edge in $B(p)$ if there exists $\sigma \in \bar{\sigma}$ and $\delta \in \bar{\delta}$ with $\sigma = \delta + 1$ or $\sigma = \delta - 1$. In this case, each vertex $\bar{\sigma}$ in $B(p)$ generates $\frac{1}{2}(p - 1)$ difference triples under the action of a given multiplier automorphism of $B(p)$. If $[\bar{\sigma}, \bar{\delta}]$ with $\sigma = \delta + 1$ (or $\sigma = \delta - 1$) is an edge of $B(p)$, then $\{\{1, \sigma, \sigma^*\}, \{1, \delta, \delta^*\}\}$ is a generator of difference quadruples. Summarizing, we have the following sufficient condition:

2.14 <u>Theorem</u>. If the graph $B(p)$ contains a 1-factor, then there exists a S-cyclic $SQS(2p)$.

Köhler [43] has shown that $B(p)$ contains a 1-factor for $p = 89, 101, 113, 137, 149$ and 233.

Let p be a prime of the form $p = 120t + 53$ or $p = 120t + 77$. Then Köhler [43] has shown that the graph $B(p)$ has exactly two vertices of degree 2 and all others have degree 3. By an application of Petersen's Theorem (in [64]), such a graph contains a 1-factor if it is bridgeless. Thus, we have:

2.15 <u>Theorem</u>. Let $p \equiv 53$ or $77 \pmod{120}$ be a prime. Then if $B(p)$ is bridgeless, then there exists a S-cyclic SQS(2p).

Köhler [42] has shown that the graph $B(p)$ is bridgeless for $p = 53, 173, 197$ and 317.

The first few orders for which the existence of S-cyclic SQS(v)'s remains open are $v = 100, 116, 148$.

Section 3. Recursive Constructions of Cyclic SQS.

In this section, we provide a recursive construction of cyclic SQS and then introduce Colbourn and Colbourn's [12] generalization of our construction.

Recall that Z_v is the set of elements of our cyclic SQS(v) and $\alpha = (0 \ldots v-1)$ is its cyclic automorphism. For the time being, we assume that $v \equiv 2$ or $10 \pmod{12}$. If (a,b,c,d) is a difference quadruple mod v then we cannot have $a = b = c = d$ because of $v \equiv 2$ or $10 \pmod{12}$. Thus we have the following two lemmas:

3.1 <u>Lemma</u>. If (a,b,c,d) is a difference quadruple of a cyclic SQS(v), then

$$(3.1.1) \qquad (a, \quad b, \quad c, \quad v+d),$$

$$(3.1.2) \qquad (a, \quad v+b, \quad c, \quad d),$$

$$(3.1.3) \qquad (a+b, \quad v-b, \quad b+c, \quad d),$$

$$(3.1.4) \qquad (a+d, \quad b, \quad d+c, \quad v-d),$$

are difference quadruples of a partial cyclic SQS(2v).

3.2 <u>Lemma</u>. For $i = 1, \ldots, v/2$, $(i, v-i, i, v-i)$ are difference quadruples of a partial cyclic SQS(2v).

In a SQS, "partial" means that each 3-subset of elements is contained in at most one quadruple. We will say that a set of difference quadruples is <u>consistent</u> if it generates a partial SQS. Two difference quadruples are <u>equivalent</u> if they contain exactly the same set of difference triples; equivalently, one is a cyclic shift of the other. The following remarks are worthy of notice.

3.3 <u>Remark</u>. In Lemma 3.1, if $a = b = c = d$ then the set of difference quadruples mod $2v$ would not be consistent. Also, observe that if $(a,b,c,d) = (x,y,x,y)$ then Lemma 3.1 should just yield two non-equivalent difference quadruples mod $2v$.

3.4 <u>Remark</u>. In Lemma 3.1, two equivalent difference quadruples mod v (a,b,c,d) and (c,d,a,b), as well as (b,c,d,a) and (d,a,b,c), must give exactly the same set of difference quadruples mod $2v$.

3.5 <u>Remark</u>. Note that if (a,b,c,d) is a difference quadruple mod v then the difference triples contained in (a,b,c,d) are represented in the following two ways:

$$(a,\ b,\ v-a-b) = (a,\ b,\ c+d),$$
$$(b,\ c,\ v-b-c) = (b,\ c,\ d+a),$$
$$(c,\ d,\ v-c-d) = (c,\ d,\ a+b),$$
$$(d,\ a,\ v-d-a) = (d,\ a,\ b+c).$$

3.6 Lemma. For any two equivalent difference quadruples mod v, the difference quadruples mod $2v$ constructed via Lemma 3.1 always contain the same set of difference triples.

Proof. Suppose that (a,b,c,d) is a difference quadruple mod v. From Remark 3.4, it suffices to show that for two equivalent difference quadruples (a,b,c,d) and (b,c,d,a), the difference quadruples mod $2v$ constructed via Lemma 3.1 contain the same set of difference triples. Then Lemma 3.1, when applied to (a,b,c,d) and (b,c,d,a), gives

(1) $(a, \quad b, \quad c, \quad v+d)$,

(2) $(a, \quad v+b, \quad c, \quad d)$,

(3) $(a+b, \quad v-b, \quad b+c, \quad d)$,

(4) $(a+d, \quad b, \quad d+c, \quad v-d)$,

and

(1)' $(b, \quad c, \quad d, \quad v+a)$,

(2)' $(b, \quad v+c, \quad d, \quad a)$,

(3)' $(b+c, \quad v-c, \quad c+d, \quad a)$,

(4)' $(b+a, \quad c, \quad a+d, \quad v-a)$, respectively. The corresponding difference triples are

(1)	(a, b, v+c+d) (b, c, v+a+d) (c, v+d, a+b) (v+d, a, b+c)	(1)'	(b, c, v+a+d) (c, d, v+a+b) (d, v+a, b+c) (v+a, b, c+d)
(2)	(a, v+b, c+d) (v+b, c, a+d) (c, d, v+a+b) (d, a, v+b+c)	(2)'	(b, v+c, a+d) (v+c, d, a+b) (d, a, v+b+c) (a, b, v+c+d)
(3)	(a+b, v-b, b+c+d) (v-b, b+c, a+b+d) (b+c, d, v+a) (d, a+b, v+c)	(3)'	(b+c, v-c, a+c+d) (v-c, c+d, a+b+c) (c+d, a, v+b) (a, b+c, v+d)
(4)	(a+d, b, v+c) (b, d+c, v+a) (d+c, v-d, a+b+d) (v-d, a+d, b+c+d)	(4)'	(b+a, c, v+d) (c, a+d, v+b) (a+d, v-a, a+b+c) (v-a, b+a, a+c+d) ,

respectively. Using Remark 3.5, a 1-1 correspondence between difference triples (1) ~ (4) and (1)' ~ (4)' is easily established, and hence the proof is complete.

We now show that the existence of a cyclic SQS(v) implies the existence of a cyclic SQS(2v) .

3.7 <u>Theorem</u> [9]. If a cyclic SQS(v) exists, where $v \equiv 2$ or $10 \pmod{12}$, then there exists a cyclic SQS(2v) .

<u>Proof</u>. Suppose that a given cyclic SQS(v) has m difference quadruples of type (a,b,a,b) and n difference quadruples of any other types. Then

(3.7.1) $$vn + (\tfrac{1}{2}v)m = \frac{1}{24}v(v-1)(v-2) .$$

Applying Lemma 3.1 to each of m + n difference quadruples, it would give 4n + 2m difference quadruples mod 2v each having length 2v . Thus, these give 2v(4n + 2m) quadruples. Add to these quadruples from Lemma 3.2 giving us v/2 - 1 difference quadruples of length v and 1 difference quadruple of length v/2 . From (3.7.1), the total number of quadruples is

$$2v(4n + 2m) + v(v/2 - 1) + v/2 = \frac{1}{24}2v(2v-1)(2v-2)$$

that is the correct number of quadruples of a SQS(2v) .

Let (Z_{2v}, q) now be the newly constructed system. Suppose that (a,b,c) is any difference triple mod $2v$. Then $a + b + c = 2v$; so at most one of a, b, c is greater than or equal to v. We distinguish two cases:

<u>Case 1</u>. One of a, b, c is greater than or equal to v, say, $c \geq v$. Let us divide it into two subcases:

<u>Subcase A</u>. $c = v$. Then $a + b = v$. If $a \leq b$ then $a \leq v/2$. Since $(a, v-a, v) = (a,b,c)$, obviously, every difference triple of this form will occur in a difference quadruple that is produced by Lemma 3.2. Similarly, if $b < a$ then $b < v/2$. So $(v-b, b, v) = (a,b,c)$ will be contained in some difference quadruple constructed by Lemma 3.2.

<u>Subcase B</u>. $c > v$. Then $a + b < v$ and hence $(a, b, v-a-b)$ is a difference triple mod v. Thus, it is in some difference quadruple mod v, that is, either

(B.1) (a, b, x, y)

(B.2) $(a, v-a-b, x, y)$ or

(B.3) $(a, x, y, v-a-b)$

is a difference quadruple mod v for the given cyclic

SQS(v), where $x + y = v - a - b$ in (B.1), $x + y = a$ in (B.2) and $x + y = b$ in (B.3). By Lemma 3.6, we know that we may assume that the difference quadruples mod v are in the above specified order when we apply Lemma 3.1. Thus,

(B.1) yields $(a, b, x, x+y)$ by (3.1.1),

(B.2) yields $(b, 2v-a-b, x, y)$ by (3.1.2),

(B.3) yields $(a, x, y, 2v-a-b)$ by (3.1.1).

In each case, it is clear that (a,b,c), where $c = 2v - a - b$, is in an appropriate difference quadruple mod $2v$.

<u>Case 2</u>. $a < v$, $b < v$ and $c < v$. We may assume that $b,c \geq v/2$. Let $b = v - j$, $c = v - i$ and $a = i + j$ for some i, j. Since $i + j < v$, we know that $(i, j, v-i-j)$ is a difference triple mod v and thus it must be contained in some difference quadruple of the given cyclic SQS(v). Again, this means we have either:

(1) (i,j,x,y) is in the cyclic SQS(v). Hence, by (3.1.3), we have $(i+j, v-j, j+x, y)$ in (Z_{2v}, q). Since $a = i + j$, $b = v - j$, this implies (a,b,c) is contained in this difference quadruple mod $2v$,

(2) $(x, j, v-i-j, y)$ mod v becomes $(x+j, v-j, v-i, y)$ mod $2v$ by (3.1.3). Since $b = v - j$,

$c = v - i$, (b,c,a) is in this difference quadruple mod $2v$ or

(S) $(v-i-j, i, x, y)$ mod v becomes $(v-j, v-i, i+x, y)$ mod $2v$ by (3.1.3) and since $b = v - j$, $c = v - i$ we again conclude (b,c,a) is in this difference quadruple mod $2v$.

From Cases 1 and 2, we conclude that every difference triple mod $2v$ is contained in some difference quadruple mod $2v$. Thus, every triple of Z_{2v} is contained in at least one quadruple of q , from this plus the fact that $|q|$ is $\frac{1}{24} 2v(2v-1)(2v-2)$, we conclude that (Z_{2v}, q) is a SQS(2v) and obviously is cyclic. This completes the proof.

3.8 Example. The difference quadruple (1,2,1.6) , (1,1,4,4) and (2,2,3,3) mod 10 give a cyclic SQS(10) .

Lemma 3.2 gives 5 difference quadruples (1,9,1,9) , (2,8,2,8) , (3,7,3,7) , (4,6,4,6) and (5,5,5,5) (mod 20) .

Lemma 3.1 applied to (1,2,1.6) , (1,1,4,4) and (2,2,3,3) gives:

A_1	B_1	C_1
(1, 2, 1, 16)	(1, 1, 4, 14)	(2, 2, 3, 13)
(1, 12, 1, 6)	(1, 11, 4, 4)	(2, 12, 3, 3)
(3, 8, 3, 6)	(2, 9, 5, 4)	(4, 8, 5, 3)
(7, 2, 7, 4)	(5, 1, 8, 6)	(5, 2, 6, 7) .

Again, Lemma 3.1 applied to (2,1.6,[illegible]) , (1,4,4,1) and (2,3,3,2) gives:

A_2	B_2	C_2
(2, 1, 6, 11)	(1, 4, 4, 11)	(2, 3, 3, 12)
(2, 11, 6, 1)	(1, 14, 4, 1)	(2, 13, 3, 2)
(3, 9, 7, 1)	(5, 6, 8, 1)	(5, 7, 6, 2)
(3, 1, 7, 9)	(2, 4, 5, 9)	(4, 3, 5, 8) .

By Lemma 3.6, A_1 and A_2 , B_1 and B_2 , and C_1 and C_2 are interchangeable, so $A_i \cup B_j \cup C_k$, where $i,j,k \in \{1,2\}$, together with the difference quadruples constructed via Lemma 3.2 give a cyclic SQS(20) .

Lemma 3.6 and Theorem 3.7 together yield:

3.8 Corollary. If a cyclic SQS(v) has n difference quadruples, where $v \equiv 2$ or 10 (mod 12) , then there exist at least 2^n pairwise distinct cyclic SQS(2v)'s .

Note that cyclic SQS(v)'s in classes B and D , that is, $v \equiv 4$ or 8 (mod 12) , necessarily contain the unique difference quadruple of the form (a,a,a,a) . So, by Remark 3.3, Theorem 3.7 does not guarantee the existence of a cyclic SQS(2v) for $v \equiv 4$ or 8 (mod 12) . Recently, however, Colbourn and Colbourn [13] realized that our doubling construction can be generalized for some special orders in classes B and D . Let us describe their methods.

Following Colbourn and Colbourn [13], a difference triple (a,b,c) is said to be an m-triple if $v \equiv 0$ (mod m) and if $a \equiv 0$ (mod v/m) , $b \equiv 0$ (mod v/m) and $c \equiv 0$ (mod v/m) . A cyclic SQS(v) has a head of order m if and only if $v \equiv 0$ (mod m) and every difference quadruple contains only m-triples or none at all. On the same lines, an m-beheaded cyclic SQS(v) is a collection of difference quadruples for which each difference triple which is not an m-triple is contained in exactly one of the difference quadruples, and no m-triples are contained in a difference quadruple. An SQS(v) with a head of order m will be denoted SQS(v, m) and an m-beheaded SQS(v) will be denoted SQS(v, -m) . It is immediate that the existence of both a cyclic SQS(2v, -2m) and a cyclic SQS(2m) necessitate the existence of a cyclic SQS(2v) .

If we apply Lemma 3.1 to all difference quadruples of a cyclic SQS(2v, -2m) , where $v \equiv m$ (mod 2) , and add all difference quadruples of the form $(i, 2v-i, i, 2v-i)$ except those containing 4m-triples, then we obtain the following main result:

3.9 Theorem [13]. If a cyclic $SQS(2v, -2m)$ exists, where $v \equiv m \pmod 2$, then there exists a cyclic $SQS(4v, -4m)$.

Let us describe some applications of Theorem 3.9. First of all, a cyclic $SQS(16, -8)$ has difference quadruples $(1,1,9,5)$, $(1,2,3,10)$, $(1,3,9,3)$, $(1,4,7,4)$, $(1,6,7,2)$, $(1,7,1,7)$, $(2,5,6,3)$, and $(3,5,3,5)$. Thus, there exists a cyclic $SQS(2^n, -2^{n-1})$ for all $n \geq 4$. Grannell and Griggs [30] construct a cyclic $SQS(32)$. So we have:

3.10 Theorem. For all $n \geq 5$, there exists a cyclic $SQS(2^n)$.

A cyclic $SQS(4m, -4)$ guarantees the existence of a cyclic $SQS(8m, -8)$ by Theorem 3.9. Repeating this $n - 2$ times, we obtain a cyclic $SQS(2^n m, -2^n)$. Thus, by Theorem 3.10, we conclude that:

3.11 Corollary. If a cyclic $SQS(4m, -4)$ exists, then there exists a cyclic $SQS(2^n m)$ for all $n \geq 5$.

Observe that the cyclic $SQS(40)$ constructed by Colbourn and Phelps [15] is a cyclic $SQS(40, 20)$. Thus, by Theorem 3.9, there exists a cyclic $SQS(2^n \times 5, -2^{n-1} \times 5)$ for all $n \geq 3$. In Section 1, we have already seen that there exists a cyclic $SQS(v)$ for $v = 10$ and 20. These imply that:

3.12 <u>Lemma</u> [13]. There exists a cyclic $SQS(2^n \times 5)$ for all $n \geq 1$.

This convincing evidence for the utility of Theorem 3.9 is tempered somewhat by the computational difficulty of finding a cyclic $SQS(40, -20)$. We require a straightforward (or, at least, computationally feasible) method of finding initial cases to which Theorem 3.9 can be profitably applied. One of such techniques is at hand, and was given by Grannell and Griggs [28].

3.13 <u>Lemma</u> [28]. An S-cyclic $SQS(v)$ has a head of order $2m$ whenever $2m$ divides v.

In fact, the head is an S-cyclic $SQS(2m)$, but we do not need it to be S-cyclic here. From Section 2, we have a S-cyclic $SQS(v)$ for $v = 50, 130, 170, 250, 290$ and 370. Also, Grannell and Griggs [31] have constructed a S-cyclic $SQS(v)$ for $v = 130, 170, 250$ and 290. This gives:

3.14 <u>Lemma</u>. For all $n \geq 1$ and $m = 5, 13, 17, 25, 29$ and 37, there exists a cyclic $SQS(2^n \times 5m)$.

It is clear that, by Lemma 3.1, all cyclic SQS constructed in this section are non-S-cyclic.

Below we present the table that summarizes the known spectrum for cyclic and S-cyclic SQS(v) for $v \le 400$.

TABLE

CYCLIC SQSs OF ORDER ≤ 400

Order	Type	Existence	S-cyclic	Reference
8	D	NO	-	[27]
10	A	YES	YES	[1]
14	C	NO	-	[27]
16	D	NO	-	[27]
20	B	YES	YES	[39], [58]
22	C	YES	NO	[21], [57]
26	A	YES	YES	[25], [29], Appendix II
28	B	YES	NO	[57], Appendix II
32	D	YES	NO	[30]
34	A	YES	YES	[25], Appendix II
38	C	YES	NO	[15]
40	D	YES	NO	[15]
44	B	YES	NO	2 × 22
46	C	?	NO	
50	A	YES	YES	[43], [45], Appendix II
52	B	YES	YES	[9],[22],[29], Appendix I
56	D	?	NO	
58	A	YES	YES	[43], Appendix II
62	C	?	NO	
64	D	YES	NO	[30]
68	B	YES	YES	[9], Appendix I

(continued)

Order	Type	Existence	S-cyclic	Reference
70	C	?	NO	
74	A	YES	YES	[43]
76	B	YES	NO	2 × 38, Appendix II
80	D	YES	NO	[13], Appendix II
82	A	YES	YES	[43]
86	C	?	NO	[43], [57]
88	D	YES	NO	Appendix II
92	B	YES	NO	Appendix II
94	C	?	NO	
98	A	YES	NO	Appendix II
100	B	YES	?	[9], [13]
104	D	?	NO	
106	A	YES	YES	[42]
110	C	?	NO	
112	D	?	NO	
116	B	YES	?	[9]
118	C	?	NO	
122	A	YES	YES	Appendix I
124	B	YES	NO	Appendix II
128	D	YES	NO	[30]
130	A	YES	YES	[31], Appendix I
134	C	?	NO	
136	D	?	NO	

(continued)

Order	Type	Existence	S-cyclic	Reference
140	B	?	NO	
142	C	?	NO	
146	A	YES	YES	Appendix I
148	B	YES	?	[9]
152	D	?	NO	
154	A	?	NO	
158	C	?	NO	
160	D	YES	NO	[13]
164	B	YES	?	[9]
166	C	?	NO	
170	A	YES	YES	[31], Appendix I
172	B	?	NO	
176	D	?	NO	
178	A	YES	YES	[43]
182	C	?	NO	
184	D	?	NO	
188	B	?	NO	
190	C	?	NO	
194	A	?	?	
196	B	YES	NO	[9]
200	D	YES	NO	[13]
202	A	YES	YES	[43]
206	C	?	NO	

(continued)

Order	Type	Existence	S-cyclic	Reference
208	D	?	NO	
212	B	YES	?	[9]
214	C	?	NO	
218	A	?	?	
220	B	?	NO	
224	D	?	NO	
226	A	YES	YES	[43]
230	C	?	NO	
232	D	?	NO	
236	B	?	NO	
238	C	?	NO	
242	A	?	?	
244	B	YES	?	[9]
248	D	?	NO	
250	A	YES	YES	[31], Appendix I
254	C	?	NO	
256	D	YES	NO	[30]
260	B	YES	?	[13]
262	C	?	NO	
266	A	?	NO	
268	B	?	NO	
272	D	?	NO	
274	A	YES	YES	[43]

(continued)

Order	Type	Existence	S-cyclic	Reference
278	C	?	NO	
280	D	?	NO	
284	B	?	NO	
286	C	?	NO	
290	A	YES	YES	[31], Appendix I
292	B	?	?	
296	D	?	NO	
298	A	YES	YES	[43]
302	C	?	NO	
304	D	?	NO	
308	B	?	NO	
310	C	?	NO	
314	A	?	?	
316	B	?	?	
320	D	YES	NO	[13]
322	A	?	NO	
326	C	?	NO	
328	D	?	NO	
332	B	?	NO	
334	C	?	NO	
338	A	?	?	
340	B	YES	?	2 × 170
344	D	?	NO	

(continued)

Order	Type	Existence	S-cyclic	Reference
346	A	YES	YES	[42]
350	C	?	NO	
352	D	?	NO	
356	B	?	?	
358	C	?	NO	
362	A	?	?	
364	B	?	NO	
368	D	?	NO	
370	A	YES	YES	Appendix I
374	C	?	NO	
376	D	?	NO	
380	B	?	NO	
382	C	?	NO	
386	A	?	?	
388	B	?	?	
392	D	?	NO	
394	A	YES	YES	[42]
398	C	?	NO	
400	D	YES	NO	[13]

Section 4. Rotational Steiner Quadruple Systems.

It is natural that we should consider exactly the same problems for SQS that were considered in STS. In this section, we contemplate 1-rotational SQS. Since Phelps' paper [55] appeared in 1977 it seems to us that there has been no further results concerning rotational SQS yet. Here, we only summarize his results [55].

A SQS(v) is k-rotational if it admits an automorphism consisting of exactly one fixed element and k disjoint cycles of the same length. Clearly a 1-rotational SQS(v) must have a cyclic STS(v - 1) as a derived system. In particular, if (Q,q) is the SQS(v) and ∞ is the element fixed by the 1-rotational automorphism then $(Q_\infty, q(\infty))$, where $Q_\infty = Q\setminus\{\infty\}$ and $q(\infty) = \{b\setminus\{\infty\} \mid b \in q \text{ and } \infty \in b\}$, must be cyclic. In Chapter 1, we have already seen that a cyclic STS(v) exists if and only if $v \equiv 1$ or $3 \pmod 6$ and $v \neq 10$. These conditions though are not entirely sufficient. Mendelsohn and Hung [50] have established that there are exactly 4 nonisomorphic SQS(14)'s and none of these are 1-rotational. However, for many other orders there do exist 1-rotational SQS(v)'s. As a first step Phelps [55] obtained that:

4.1 <u>Theorem</u> [55]. For all n, there exists a 1-rotational $SQS(2^n)$.

<u>Proof</u>. Let Q be the Galois field $GF(2^n)$ and $q = \{\{a,b,c,d\} | a,b,c,d \in Q$ and $a+b+c+d = 0\}$. Then (Q,q) is a 1-rotational $SQS(2^n)$ with the permutation $x \to \sigma x$ as a 1-rotational automorphism fixing the element zero, where σ is a generating element of the multiplicative group.

The next construction is a modified version of Hanani's construction 3.2 [33].

4.2 <u>Construction</u> [55]. Let (Q,q) be a $SQS(v + 1)$ and $(Q_\infty, q(\infty))$ be as above for some $\infty \in Q$. Also let $q' = \{b \in q | \infty \notin b\}$ and $P = (Z_3 \times Q_\infty) \cup \{\infty\}$. Define a collection of quadruples p on P as follows:

(1) For each distinct quadruple $\{x,y,z,w\} \in q'$ we have for $i \in Z_3$

$\{(i, x), (i, y), (i+1, z), (i+2, w)\}$

$\{(i, x), (i, y), (i+2, z), (i+1, w)\}$

$\{(i+1, x), (i+2, y), (i, z), (i, w)\}$

$\{(i+2, x), (i+1, y), (i, z), (i, w)\}$

$\{(i, x), (i+1, y), (i+1, z), (i, w)\}$

$\{(i+1, x), (i, y), (i, z), (i+1, w)\}$

$\{(i, x), (i+1, y), (i, z), (i+1, w)\}$

$\{(i+1, x), (i, y), (i+1, z), (i, w)\}$

$\{(i, x), (i, y), (i, z), (i, w)\}$

where we assume that the elements of each quadruple are in some fixed arbitrary) order.

(2) $\{\infty, (i,x), (j,y), (k,z)\}$ with $\{x,y,z\} \in q(\infty)$ and $i + j + k = 0$.

(3) $\{(i,x), (i,y), (i+1, z), (i+2, z)\}$ for $i \in Z_3$ and $\{x,y,z\} \in q(\infty)$ with all possible orderings of the elements x, y, z.

(4) $\{(i,x), (i,y), (i+1, x), (i+1, y)\}$, $i \in Z_3$ and $x,y \in Q_\infty$, $x \neq y$.

(5) $\{\infty, (0,x), (1,x), (2,x)\}$ for all $x \in Q_\infty$.

Then (P,p) is a $SQS(3v + 1)$.

If we have a 1-rotational $SQS(v + 1)$ (Q,q) with fixed element ∞ then for each of the quadruples of q' we can order the elements of that quadruple so that the 1-rotational automorphism is order preserving. To see this note that each quadruple of q' must have a full orbit. Thus by arbitrarily ordering one quadruple in each orbit we can extend this ordering to each other quadruple in q' via the 1-rotational automorphism.

4.3 <u>Theorem</u> [55]. If a 1-rotational SQS(v + 1) exists where $v \equiv 1 \pmod 6$ then there exists a 1-rotational SQS(3v + 1).

<u>Proof</u>. Let (Q,q) be a 1-rotational SQS(v + 1), $v \equiv 1 \pmod 6$ with fixed element ∞, and let α be the 1-rotational automorphism. Order each of the quadruples of q' so that the 1-rotational automorphism is order preserving. With that ordering we apply the Construction 4.2 giving us a SQS(3v + 1) (P,p) where $P = (Z_3 \times Q_\infty) \cup \{\infty\}$. Clearly this SQS(3v + 1) has an automorphism the permutation $(i,x) \to (i+1, x)$ which is cyclic of order 3 having ∞ as its fixed element. Also the permutation $(i,x) \to (i, \alpha(x))$ is an automorphism due to the construction. Obviously then the permutation $\alpha': (i,x) \to (i+1, \alpha(x))$ will be a 1-rotational automorphism of order 3v that fixes ∞, since 3 and v are relatively prime.

The following construction generalizes the previous Construction 4.2.

4.4 <u>Construction</u> [55]. Let (Q,q) and (P,p) be SQS(v + 1) and SQS(u + 1), respectively, and let $(Q_\infty, q(\infty))$, $P_{\infty'}, p(\infty'))$, $q' = \{b \in q \mid \infty \notin b\}$,

$p' = \{b \in p \mid \infty' \notin b\}$ be defined as before. Set $V = \{\infty\} \cup (Q_\infty \times P_{\infty'})$ and define a collection of quadruples B on V as follows:

(1) $\{(a,x), (b,y), (c,z), (d,w)\}$ for all $\{a,b,c,d\} \in q'$ and $\{x,y,z,w\} \in p'$.

(2) For each $\{a_0,a_1,a_2\} \in q(\infty)$ and each $\{x,y,z,w\} \in p'$ where the order of the elements in each quadruple is (arbitrarily) fixed we have for $i \in Z_3$:

$$\{(a_i, x), (a_i, y), (a_{i+1}, z), (a_{i+2}, w)\}$$
$$\{(a_i, x), (a_i, y), (a_{i+2}, z), (a_{i+1}, w)\}$$
$$\{(a_{i+1}, x), (a_{i+2}, y), (a_i, z), (a_i, w)\}$$
$$\{(a_{i+2}, x), (a_{i+1}, y), (a_i, z), (a_i, w)\}$$
$$\{(a_i, x), (a_{i+1}, y), (a_i, z), (a_{i+1}, w)\}$$
$$\{(a_{i+1}, x), (a_i, y), (a_{i+1}, z), (a_i, w)\}$$
$$\{(a_i, x), (a_{i+1}, y), (a_{i+1}, z), (a_i, w)\}$$
$$\{(a_{i+1}, x), (a_i, y), (a_i, z), (a_{i+1}, w)\}.$$

(2)' For each $\{x_0,x_1,x_2\} \in p(\infty')$ and each $\{a,b,c,d\} \in q'$ where the order of the elements in each quadruple is (arbitrarily) fixed we form quadruples as in (2).

(3) For each $r \in Q_\infty$ and $s \in P_{\infty'}$ and each $\{x,y,z,w\} \in p'$ and each $\{a,b,c,d\} \in q'$ we form quadruples $\{(r,x), (r,y), (r,z), (r,w)\}$ and $\{(a,s), (b,s), (c,s), (d,s)\}$.

(4) $\{\infty, (a,x), (b,y), (c,z)\}$ for each $\{a,b,c\} \in q(\infty)$ and each $\{x,y,z\} \in p(\infty')$ for all possible orderings of the elements. Also include quadruples $\{\infty, (r,x), (r,y), (r,z)\}$, $\{\infty, (a,s), (b,s), (c,s)\}$ for each $r \in Q_\infty$ and each $s \in P_{\infty'}$.

(5) For each $\{a_0,a_1,a_2\} \in q(\infty)$ and $\{x,y,z\} \in p(\infty')$ we have $\{(a_i,x), (a_i,y), (a_{i+1},z), (a_{i+2},z)\}$ for $i \in Z_3$.

(6) $\{(a,x), (a,y), (b,x), (b,y)\}$ for all pairs $\{a,b\} \subseteq Q_\infty$ and all pairs $\{x,y\} \subseteq P_{\infty'}$.

Then (V,B) is a $SQS(vu + 1)$.

If we assume that (Q,q) and (P,p) are 1-rotational SQS with fixed elements ∞ and ∞' respectively then again we note that the quadruples of q' and p' can be ordered (when necessary) so that the respective 1-rotational automorphisms are order preserving. Similarly the triples of $q(\infty)$ and $p(\infty')$ can be ordered so that the respective 1-rotational automorphisms are order preserving or at worst will cyclically permute the ordered elements. With this in mind we have as a generalization of Theorem 4.3:

4.5 <u>Theorem</u> [55]. If a 1-rotational $SQS(v + 1)$ and a 1-rotational $SQS(u + 1)$ exist where v and u are relatively prime then there exists a 1-rotational $SQS(vu + 1)$.

Proof. Let (Q,q), (P,p) be 1-rotational SQS of orders $v+1$ and $u+1$ respectively with fixed elements $\infty \in Q$ and $\infty' \in P$. From $(Q_\infty, q(\infty))$, $(P_{\infty'}, p(\infty'))$ and q', p' construct (V,B) as in Construction 4.4. When necessary order the elements of each quadruple of q', p', as well as those of $q(\infty)$, $p(\infty')$ so that the 1-rotational automorphisms of each system are order preserving. If α and α' are 1-rotational automorphisms of (Q,q) and (P,p) respectively then $\bar{\alpha}: (x,y) \to (\alpha(x), \alpha'(y))$ and $\bar{\alpha}(\infty) = \infty$ for $(x,y) \in V = \{\infty\} \cup (Q_\infty \times P_{\infty'})$ will be a 1-rotational automorphism of (V,B).

The first order v for which the existence of a 1-rotational $SQS(v)$ is in doubt, is $v = 20$.

CHAPTER 7. CONCLUDING REMARKS

The fundamental problem in combinatorics is that of arranging objects according to given rules, and enumerating the number of ways to do this. In this thesis, an attempt has been made to construct various kinds of designs with a given automorphism type, namely, a cyclic, rotational, regular or involutory automorphism. One of the most important methods to construct such designs comes from the application of the theory of modified difference families. However, the success of these methods is due largely to the fact that the block size of these designs is 3 . For block size greater than 3 , the difficulties start to mount [see 16]. Thus, we have also restricted our attention to block size 3 .

Phelps and Rosa [59] constructed 1- and 2-rotational STSs . By combining our constructions of 3- and 4-rotational STSs with their results [59], we have shown that there exists at least one k-rotational STS(v) for all admissible orders v and some $k \le 4$. Turning to k-rotational STS with $k > 4$, the state of affairs is as follows. For $k = 5$, the existence problem is unresolved for $v \equiv 1$ or 91 (mod 120) . At present we can conclude that for $k = 6t$, $t \ge 1$, a k-rotational STS(v) exists if and only if $v \equiv 1$ (mod k) and $v \equiv 24i - 23$, $24i - 17$ or $24i - 5$ (mod 4k) ,

$i = 1, 2, \ldots, t$. However, the general existence problem for k-rotational STS with $k > 6$ remains open.

As for orders $v \equiv 3 \pmod{6}$, a different construction for an $STS(v)$ with an involutory automorphism fixing precisely 3 elements is given in Section 5 of Chapter 1. For orders $v \equiv 1 \pmod{6}$, no such construction is known.

In Chapter 2, we entirely resolve the existence problem for 1-rotational $TS_{\lambda}(v)$'s with $\lambda > 1$. As an easy consequence of this result, we settle completely the general existence problem for k-rotational $TS_{\lambda}(v)$'s with $\lambda > 1$ for all $k \geq 1$.

Integer partitioning techniques turn out to be highly applicable to constructing ETS . We construct several such partitions. However, the complete resolution of the existence problem for rotational ETS as well as for regular ETS is a long way off. It is quite possible that in the remaining open cases such systems can be constructed by appropriate integer partitioning methods.

The general existence problem for k-rotational DTS is settled completely in Chapter 4. However, in the case of rotational MTS , the existence of k-rotational $MTS(v)$'s is unsettled for the order $v = 30t + 6$, $t \neq 0$, and $k = 5$ or $6t + 1$. In the case of rotational EMTS , only the 1-rotational systems are completely determined.

In Chapter 6, we develop methods that are entirely different from those of previous chapters, namely, we deal with a recursive technique to construct cyclic SQS. Of course, our doubling construction, that is, Theorem 3.7 in Chapter 6, does yield new (non-S-) cyclic SQS(v)'s for several orders v for which the existence of cyclic SQS was previously unknown. However, a weak point of our original construction is that it is "not applicable" to the base block of the short orbit of length $\frac{v}{4}$. Recently, however, Colbourn and Colbourn [13] found a way around this difficulty, and succeeded in generalizing this construction. While the spectrum for cyclic SQS has not been completely determined, recent research has made significant progress on this question.

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APPENDICES

In these appendices, we list a S-cyclic SQS(v) for $v = 52, 68, 122, 130, 146, 170, 250, 290$ and 370, and a non-S-cyclic SQS(v) for $v = 26, 28, 34, 50, 58, 76, 80, 88, 92, 98$ and 124.

The terminology and notation used here are the same as in Chapter 6, unless they are explained below. Also, we will represent cyclic SQS as either difference quadruples or sets of difference triples.

Denote by ϕ a multi-valued function from difference quadruples mod v into difference quadruples mod $2v$ as follows: if (a,b,c,d) is a difference quadruple mod v then it is mapped into four difference quadruples mod $2v$

$$\begin{array}{llll} (a, & b, & c, & v+d)\ , \\ (a, & v+b, & c, & d)\ , \\ (a+b, & v-b, & b+c, & d)\ , \\ (a+d, & b, & d+c, & v-d)\ . \end{array}$$

For simplicity, we use the following notations:

$$D_1(v) = \{(i,\ i,\ \tfrac{v}{2}-i,\ \tfrac{v}{2}-i) \mid i = 1,\ldots,[\tfrac{v}{4}]\}\ .$$

$D_2(v) = \{(2i-1, 2j, 2i-1, v+2-4i-2j) \mid i = 1, \ldots, \frac{v-6}{4};$
$j = 1, \ldots, \frac{v-2}{4} - i\}$.

SD(v) = the set of all difference quadruples of an S-cyclic SQS(v) .

$nX = \{nt \mid t \in X\}$ where n is an integer called a multiplier and X is a set of integers.

$D_3(v) = \{(i, \frac{v}{2} - i, i, \frac{v}{2} - i) \mid i = 1, \ldots, [\frac{v}{4}]\}$.

$\phi(D)$ = the image under ϕ of D .

1. S-cyclic SQS

The difference quadruples contained in the S-cyclic SQS(v), which are omitted in the following list, are shown in the table below.

Order	Difference quadruples
52	$D_1(52)$, $2\{SD(26)\setminus D_1(26)\}$
68	$D_1(68)$, $2\{SD(34)\setminus D_1(34)\}$
122	$D_1(122)$, $D_2(122)$
130	$D_1(130)$, $D_2(130)$, $5\{SD(26)\setminus(D_1(26)\cup D_2(26))\}$
146	$D_1(146)$, $D_2(146)$
170	$D_1(170)$, $D_2(170)$, $5\{SD(34)\setminus(D_1(34)\cup D_2(34))\}$
250	$D_1(250)$, $D_2(250)$, $5\{SD(50)\setminus(D_1(50)\cup D_2(50))\}$
290	$D_1(290)$, $D_2(290)$, $5\{SD(58)\setminus(D_1(58)\cup D_2(58))\}$
370	$D_1(370)$, $D_2(370)$, $5\ SD(74)\setminus(D_1(74)\cup D_2(74))$

v = 52:

1. (1,4,1,46)
2. (1,6,1,44)
3. (1,8,1,42)
4. (1,10,1,40)
5. (1,14,1,36)
6. (1,16,1,34)
7. (1,20,1,30)
8. (1,22,1,28)
9. (2,1,2,47)
10. (2,7,2,41)
11. (2,11,2,37)
12. (2,17,2,31)
13. (3,1,3,45)
14. (3,5,3,41)
15. (3,7,3,39)
16. (3,11,3,35)
17. (3,17,3,29)
18. (3,15,3,31)
19. (4,5,4,39)
20. (4,7,4,37)
21. (4,15,4,29)
22. (4,17,4,27)
23. (5,2,5,40)
24. (5,6,5,36)
25. (5,10,5,32)
26. (5,12,5,30)
27. (5,14,5,28)
28. (5,18,5,24)
29. (6,3,6,37)
30. (6,11,6,29)
31. (6,13,6,27)
32. (6,15,6,25)
33. (7,6,7,32)
34. (7,7,7,30)
35. (7,10,7,28)
36. (7,16,7,22)
37. (8,5,8,31)
38. (8,9,8,27)
39. (8,11,8,25)
40. (9,3,9,31)
41. (9,5,9,29)
42. (9,7,9,27)
43. (9,10,9,24)
44. (9,11,9,23)
45. (9,13,9,21)
46. (10,13,10,19)
47. (10,15,10,17)
48. (11,7,11,23)
49. (11,10,11,20)
50. (11,14,11,16)
51. (11,17,11,13)
52. (12,1,12,27)
53. (12,3,12,25)
54. (12,7,12,21)
55. (12,17,12,11)
56. (13,1,13,25)
57. (13,3,13,23)
58. (13,4,13,22)
59. (13,5,13,21)
60. (14,7,14,17)
61. (15,2,15,20)
62. (15,8,15,14)
63. (15,9,15,13)
64. (16,3,16,17)
65. (16,5,16,15)
66. (18,1,18,15)
67. (18,7,18,9)
68. (19,1,19,13)
69. (19,3,19,11)
70. (20,5,20,7)
71. (21,2,21,8)
72. (21,3,21,7)
73. (22,3,22,5)
74. (23,2,23,4)
75. (24,1,24,3).

<u>v = 68</u>:

1. (1,2,1.64)
2. (1,4,1,62)
3. (1,6,1,60)
4. (1,8,1,58)
5. (1,10,1,56)
6. (1,14,1,52)
7. (1,21,1,45)
8. (1,28,1,38)
9. (2,3,2,61)
10. (2,9,2,55)
11. (2,13,2,51)
12. (2,17,2,47)
13. (2,21,2,43)
14. (2,25,2,39)
15. (2,29,2,35)
16. (3,4,3,58)
17. (3,5,3,57)
18. (3,9,3,53)
19. (3,13,3,49)
20. (3,21,3,41)
21. (3,25,3,37)
22. (3,26,3,36)
23. (3,27,3,35)
24. (4,5,4,55)
25. (4,7,4,53)
26. (4,13,4,47)
27. (4,15,4,45)
28. (4,21,4,39)
29. (4,29,4,31)
30. (5,6,5,52)
31. (5,9,5,49)
32. (5,15,5,43)
33. (5,18,5,40)
34. (5,19,5,39)
35. (5,21,5,37)
36. (5,22,5,36)
37. (5,25,5,33)
38. (6,3,6,53)
39. (6,7,6,49)
40. (6,15,6,41)
41. (6,19,6,37)
42. (6,27,6,29)
43. (7,2,7,52)
44. (7,5,7,49)
45. (7,8,7,46)
46. (7,10,7,44)
47. (7,11,7,43)
48. (7,13,7,41)
49. (7,16,7,38)
50. (7,19,7,35)
51. (7,22,7,32)
52. (8,5,8,47)
53. (8,17,8,35)
54. (8,19,8,33)
55. (8,23,8,29)
56. (9,8,9,42)
57. (9,13,9,37)
58. (9,14,9,36)
59. (9,18,9,32)
60. (9,19,9,31)
61. (9,20,9,30)
62. (9,24,9,26)
63. (10,3,10,45)
64. (10,5,10,43)
65. (10,9,10,39)
66. (10,11,10,37)
67. (10,23,10,25)
68. (11,3,11,43)
69. (11,6,11,40)
70. (11,8,11,38)
71. (11,9,11,37)
72. (11,13,11,33)
73. (11,15,11,31)
74. (11,16,11,30)
75. (11,18,11,28)
76. (12,1,12,43)
77. (12,5,12,39)
78. (12,9,12,35)
79. (12,11,12,33)
80. (12,9,12,35)
81. (12,19,12,25)
82. (13,1,13,41)
83. (13,5,13,37)
84. (13,16,13,26)
85. (13,17,13,25)
86. (13,19,13,23)
87. (14,3,14,37)
88. (14,7,14,33)
89. (14,15,14,25)
90. (15,3,15,35)
91. (15,8,15,30)
92. (15,9,15,29)
93. (15,16,15,22)
94. (15,25,15,13)
95. (16,1,16,35)
96. (16,5,16,31)

(v = 68 continued)

97. (16,9,16,27)
98. (17,1,17,33)
99. (17,3,17,31)
100. (17,5,17,29)
101. (17,6,17,28)
102. (17,10,17,24)
103. (17,15,17,19)
104. (18,1,18,31)
105. (18,3,18,29)
106. (19,1,19,29)
107. (19,3,19,27)
108. (19,14,19,16)
109. (20,1,20,27)
110. (20,3,20,25)
111. (20,13,20,15)
112. (21,7,21,19)
113. (21,9,21,17)
114. (21,11,21,15)
115. (22,3,22,21)
116. (22,13,22,11)
117. (23,1,23,21)
118. (23,3,23,19)
119. (23,4,23,18)
120. (23,6,23,16)
121. (23,8,23,14)
122. (24,1,24,19)
123. (24,7,24,13)
124. (25,1,25,17)
125. (25,7,25,11)
126. (26,1,26,15)
127. (27,1,27,13)
128. (27,4,27,10)
129. (28,5,28,7)
130. (30,1,30,7)
131. (31,1,31,5)
132. (32,1,32,3).

v = 122:

1. (2,6,2,112)
2. (2,10,2,108)
3. (2,14,2,104)
4. (2,18,2,100)
5. (22,2,22,76)
6. (12,26,12,72)
7. (2,24,2,94)
8. (32,2,32,56)
9. (2,38,2,80)
10. (42,2,42,36)
11. (44,2,44,32)
12. (46,2,46,28)
13. (2,48,2,70)
14. (2,52,2,66)
15. (2,56,2,62)
16. (4,56,4,58)
17. (4,48,4,66)
18. (6,8,6,102)
19. (10,8,10,94)
20. (6,20,6,90)
21. (10,12,10,90)
22. (4,2,4,112)
23. (22,32,22,46)
24. (10,38,10,64)
25. (10,44,10,58)
26. (10,24,10,78)
27. (10,4,10,98)
28. (14,12,14,82)
29. (14,4,14,90)
30. (26,28,26,42)
31. (12,48,12,50)
32. (4,18,4,96)
33. (34,2,34,52)
34. (36,2,36,48)
35. (30,4,30,58)
36. (12,24,12,74)
37. (34,4,34,50)
38. (18,16,18,70)
39. (24,36,24,38)
40. (18,22,18,64)
41. (22,26,22,52)
42. (14,24,14,70)
43. (26,4,26,66)
44. (14,42,14,52)
45. (22,20,22,58)
46. (40,16,40,26)
47. (22,12,22,66)
48. (24,34,24,40)
49. (16,34,16,56)
50. (16,26,16,64)
51. (8,18,8,88)
52. (28,14,28,52)
53. (32,10,32,48)
54. (10,42,10,60)
55. (6,32,6,78)
56. (20,40,20,42)
57. (12,42,12,56)
58. (44,12,44,22)
59. (12,6,12,92)
60. (22,38,22,40)
61. (24,18,24,56)
62. (14,32,14,62)
63. (12,20,12,78)
64. (36,22,36,28)
65. (24,8,24,66)
66. (16,8,16,82)
67. (12,34,12,64)
68. (30,12,30,50)
69. (6,28,6,82)
70. (40,14,40,28)
71. (30,22,30,40)
72. (26,34,26,36)
73. (8,42,8,64)
74. (34,8,34,46)
75. (18,42,18,44)
76. (10,36,10,66)
77. (32,20,32,38)
78. (16,32,16,58)
79. (20,30,20,52)
80. (28,10,28,56)
81. (18,28,18,58)
82. (20,26,20,56)
83. (32,8,32,50)
84. (8,48,8,58)
85. (10,40,10,62)
86. (10,20,10,82)
87. (26,32,26,38)
88. (40,8,40,34)
89. (14,34,14,60)
90. (20,14,20,68)
91. (12,16,12,82)
92. (12,40,12,58)
93. (6,46,6,64)
94. (6,50,6,60)
95. (6,48,6,62)
96. (20,16,20,66)

(v = 122 continued)

97. (24,28,24,46)
98. (46,4,46,26)
99. (4,38,4,76)
100. (24,26,24,48)
101. (4,50,4,64)
102. (6,36,6,74)
103. (28,32,28,34)
104. (6,18,6,92)
105. (18,20,18,66)
106. (30,6,30,56)
107. (18,32,18,54)
108. (16,22,16,68)
109. (40,6,40,36)
110. (18,30,18,56)
111. (24,30,24,44)
112. (16,36,16,54)
113. (36,18,36,32)
114. (26,18,26,52)
115. (20,28,20,54)
116. (20,38,20,44)
117. (44,4,44,30)
118. (40,4,40,38)
119. (4,32,4,82)
120. (4,24,4,90)
121. (20,4,20,78)
122. (4,12,4,102)
123. (8,4,8,102)
124. (8,20,8,86)
125. (8,36,8,70)
126. (8,52,8,54)
127. (8,38,8,68)
128. (30,8,30,54)
129. (8,14,8,92)
130. (14,22,14,72)
131. (14,44,14,50)
132. (14,16,14,78)
133. (16,30,16,60)
134. (16,28,16,62)
135. (44,6,44,28)
136. (28,2,28,64)
137. (30,2,30,60)
138. (10,16,10,86)
139. (6,10,6,100)
140. (22,6,22,72).

v = 130:

(a) Multiplier: 2×3^i, $i = 0, 1, \ldots, 11$.

1. {{1,3,61}, {1,4,60}}
2. {{2,1,62}, {2,3,60}}
3. {{5,1,59}, {5,6,54}}
4. {{1,6,58}, {1,7,57}}
5. {{8,1,56}, {8,9,48}}
6. {{1,9,55}, {1,10,54}}
7. {{12,1,52}, {12,13,40}}
8. {{17,1,47}, {17,18,30}}
9. {{1,18,46}, {1,19,45}}
10. {{1,30,34}, {1,31,33}}
11. {{2,6,57}, {2,8,55}}
12. {{2,11,52}, {2,13,50}}

(b) Multiplier: 2×3^i, $i = 0, 1, \ldots, 5$.

13. {{1,13,51}, {1,14,50}}
14. {{2,26,37}, {2,28,35}}.

v = 146:

1. (36,26,36,48)
2. (32,26,32,56)
3. (24,32,24,66)
4. (30,26,30,60)
5. (26,38,26,56)
6. (8,12,8,118)
7. (44,8,44,50)
8. (8,52,8,78)
9. (8,62,8,68)
10. (8,46,8,84)
11. (8,30,8,100)
12. (8,14,8,116)
13. (6,8,6,126)
14. (20,6,20,100)
15. (18,24,18,86)
16. (6,26,6,108)
17. (38,6,38,64)
18. (6,44,6,90)
19. (18,50,18,60)
20. (32,18,32,64)
21. (46,20,46,34)
22. (42,24,42,38)
23. (34,30,34,48)
24. (30,22,30,64)
25. (22,50,22,52)
26. (12,40,12,82)
27. (12,16,12,106)
28. (16,28,16,86)
29. (16,54,16,60)
30. (20,40,20,66)
31. (28,22,28,68)
32. (40,28,40,38)
33. (30,38,30,48)
34. (18,30,18,80)
35. (18,44,18,66)
36. (32,38,32,44)
37. (40,22,40,44)
38. (18,22,18,88)
39. (18,52,18,58)
40. (18,16,18,94)
41. (34,16,34,62)
42. (28,34,28,56)
43. (16,22,16,92)
44. (46,14,46,40)
45. (14,18,14,100)
46. (6,56,6,78)
47. (6,66,6,68)
48. (6,54,6,80)
49. (6,42,6,92)
50. (32,36,32,46)
51. (14,58,14,60)
52. (14,30,14,88)
53. (16,14,16,100)
54. (36,6,36,68)
55. (6,24,6,110)
56. (6,12,6,122)
57. (12,18,12,104)
58. (12,42,12,80)
59. (12,56,12,66)
60. (16,56,16,58)
61. (12,32,12,90)
62. (20,12,20,94)
63. (20,52,20,54)
64. (34,20,34,58)
65. (24,34,24,64)
66. (40,24,40,42)
67. (22,20,22,82)
68. (22,38,22,64)
69. (46,18,46,36)
70. (28,18,28,72)
71. (26,46,26,48)
72. (24,48,24,50)
73. (26,24,26,70)
74. (26,18,26,76)
75. (28,30,28,60)
76. (32,28,32,54)
77. (22,32,22,70)
78. (22,26,22,76)
79. (52,2,52,40)
80. (38,34,38,36)
81. (40,32,40,34)
82. (42,30,42,32)
83. (44,28,44,30)
84. (44,2,44,56)
85. (46,2,46,52)
86. (2,48,2,94)
87. (2,54,2,88)
88. (2,58,2,84)
89. (2,62,2,80)
90. (2,64,2,76)
91. (20,42,20,64)
92. (34,26,34,52)
93. (48,8,48,42)
94. (40,16,40,50)
95. (16,8,16,106)
96. (8,24,8,106)

(v = 146 continued)

97. (40,8,40,58)
98. (42,8,42,54)
99. (8,56,8,74)
100. (8,58,8,72)
101. (50,8,50,38)
102. (34,8,34,70)
103. (14,20,14,98)
104. (14,48,14,70)
105. (14,42,14,76)
106. (28,14,28,76)
107. (28,20,28,70)
108. (20,48,20,58)
109. (20,18,20,88)
110. (10,48,10,78)
111. (38,10,38,60)
112. (8,18,8,112)
113. (10,8,10,118)
114. (28,10,28,80)
115. (38,18,38,52)
116. (18,54,18,56)
117. (36,18,36,56)
118. (52,14,52,28)
119. (20,16,20,90)
120. (20,50,20,56)
121. (20,10,20,96)
122. (10,30,10,96)
123. (10,14,10,112)
124. (34,10,34,68)
125. (20,24,20,82)
126. (24,44,24,54)
127. (30,24,30,62)
128. (32,30,32,52)
129. (44,10,44,48)
130. (36,16,36,58)
131. (10,50,10,76)
132. (10,56,10,70)
133. (10,54,10,72)
134. (10,52,10,74)
135. (10,32,10,94)
136. (10,12,10,114)
137. (70,2,70,4)
138. (2,6,2,136)
139. (2,10,2,132)
140. (2,14,2,128)
141. (2,18,2,124)
142. (2,22,2,120)
143. (2,26,2,116)
144. (2,30,2,112)
145. (2,34,2,108)
146. (2,38,2,104)
147. (42,2,42,60)
148. (4,2,4,136)
149. (4,10,4,128)
150. (4,18,4,120)
151. (4,26,4,112)
152. (4,34,4,104)
153. (4,42,4,96)
154. (4,50,4,88)
155. (58,4,58,26)
156. (4,62,4,76)
157. (14,40,14,78)
158. (14,50,14,68)
159. (36,14,36,60)
160. (22,14,22,88)
161. (22,44,22,58)
162. (24,28,24,70)
163. (4,64,4,74)
164. (4,56,4,82)
165. (4,48,4,90)
166. (4,40,4,98)
167. (4,32,4,106)
168. (4,24,4,114)
169. (4,16,4,122)
170. (4,8,4,130)
171. (14,24,14,94)
172. (10,36,10,90)
173. (10,16,10,110)
174. (6,10,6,124)
175. (6,22,6,112)
176. (6,34,6,100)
177. (6,46,6,88)
178. (6,58,6,76)
179. (12,58,12,64)
180. (36,34,36,40)
181. (30,40,30,46)
182. (24,22,24,76)
183. (22,46,22,56)
184. (22,12,22,90)
185. (12,34,12,88)
186. (42,16,42,46)
187. (26,16,26,78)
188. (52,16,52,26)
189. (16,46,16,68)
190. (38,24,38,46)
191. (24,36,24,62)
192. (12,24,12,98)

193. (12,48,12,74)
194. (12,50,12,72)
195. (12,26,12,96)
196. (14,12,14,106)
197. (26,40,26,54)
198. (28,26,28,64)
199. (28,8,28,82)
200. (36,8,36,66)
201. (30,36,30,50)
202. (16,50,16,64)
203. (16,32,16,82)
204. (32,34,32,48).

<u>v = 170</u>:

(a) Multiplier: 2×3^i, $i = 0, 1, \ldots, 15$.

1. {{2,1,82}, {2,3,80}}
2. {{1,3,81}, {1,4,80}}
3. {{1,5,79}, {1,6,78}}
4. {{1,7,77}, {1,8,76}}
5. {{1,9,76}, {1,10,74}}
6. {{1,13,71}, {1,14,70}}
7. {{17,1,67}, {17,18,50}}
8. {{1,22,62}, {1,23,61}}
9. {{1,24,60}, {1,25,59}}
10. {{30,1,54}, {30,31,24}}
11. {{1,31,53}, {1,32,52}}
12. {{1,33,51}, {1,34,50}}
13. {{1,38,46}, {1,39,45}}
14. {{1,40,44}, {1,41,43}}
15. {{2,6,77}, {2,8,75}}
16. {{2,11,72}, {2,13,70}}

(b) Multiplier: 2×3^i, $i = 0, 1, \ldots, 7$.

17. {{1,15,69}, {1,16,68}}
18. {{2,30,53}, {2,32,51}}.

v = 250:

(a) Multiplier: 2×3^i , $i = 0, 1, \ldots, 49$.

1. {{1,2,122}, {1,3,121}}
2. {{1,4,120}, {1,5,119}}
3. {{6,1,118}, {6,7,112}}
4. {{7,1,117}, {7,8,110}}
5. {{1,8,116}, {1,9,115}}
6. {{10,1,114}, {10,11,104}}
7. {{11,1,113}, {11,12,102}}
8. {{15,1,109}, {15,16,94}}
9. {{1,23,101}, {1,24,100}}
10. {{29,1,95}, {29,30,66}}
11. {{1,27,97}, {1,28,96}}
12. {{1,48,76}, {1,49,75}} .

v = 290:

(a) Multiplier: 2×3^i, $i = 0, 1, \ldots, 13$.

1. {{1,2,142}, {1,3,141}}
2. {{1,4,140}, {1,5,139}}
3. {{17,1,127}, {17,18,110}}
4. {{1,18,126}, {1,19,125}}
5. {{1,20,124}, {1,21,123}}
6. {{1,6,138}, {1,7,137}}
7. {{1,8,136}, {1,9,135}}
8. {{1,10,134}, {1,11,133}}
9. {{12,11,122}, {12,23,110}}
10. {{1,22,122}, {1,23,121}}
11. {{24,1,120}, {24,25,96}}
12. {{1,25,119}, {1,26,118}}
13. {{27,1,117}, {27,28,90}}
14. {{1,28,116}, {1,29,115}}
15. {{30,1,114}, {30,31,84}}
16. {{31,1,113}, {31,32,82}}
17. {{1,32,112}, {1,33,111}}
18. {{34,1,110}, {34,35,76}}
19. {{1,37,107}, {1,38,106}}
20. {{39,1,105}, {39,40,66}}
21. {{1,40,104}, {1,41,103}}
22. {{45,1,99}, {45,46,54}}
23. {{49,1,95}, {49,50,46}}
24. {{50,1,94}, {50,51,44}}
25. {{51,1,93}, {51,52,42}}
26. {{1,52,92}, {1,53,91}}
27. {{1,54,90}, {1,55,89}}
28. {{56,1,88}, {56,57,32}}
29. {{1,57,87}, {1,58,86}}
30. {{1,59,85}, {1,60,84}}
31. {{1,61,83}, {1,62,82}}
32. {{1,66,78}, {1,67,77}}
33. {{1,68,76}, {1,69,75}}
34. {{1,70,74}, {1,71,73}}
35. {{6,2,137}, {6,8,131}}
36. {{2,13,130}, {2,15,128}}
37. {{2,17,126}, {2,19,124}}
38. {{2,20,123}, {2,22,121}}
39. {{2,52,91}, {2,54,89}}
40. {{2,56,87}, {2,58,85}}
41. {{2,21,122}, {2,23,120}}
42. {{2,26,117}, {2,28,115}}
43. {{2,29,114}, {2,31,112}}
44. {{2,38,105}, {2,40,103}}
45. {{2,42,101}, {2,44,99}}
46. {{2,53,90}, {2,55,88}}
47. {{60,2,83}, {60,62,23}}
48. {{2,61,82}, {2,63,80}}
49. {{65,2,78}, {65,67,13}}
50. {{5,21,119}, {5,26,114}}
51. {{5,28,112}, {5,33,107}}
52. {{41,5,99}, {41,46,58}}
53. {{5,57,85}, {5,62,78}}
54. {{5,53,87}, {5,58,82}}
55. {{5,56,84}, {5,61,79}}
56. {{7,14,124}, {7,21,117}}
57. {{7,19,119}, {7,26,112}}
58. {{14,28,103}, {14,42,89}}.

v = 370:

(a) Multiplier: 2×3^i, $i = 0, 1, \ldots, 17$.

1. {{1,2,182}, {1,3,181}}
2. {{1,4,180}, {1,5,179}}
3. {{1,6,178}, {1,7,177}}
4. {{8,1,176}, {8,9,168}}
5. {{9,1,175}, {9,10,166}}
6. {{1,10,174}, {1,11,173}}
7. {{1,12,172}, {1,13,171}}
8. {{1,14,170}, {1,15,169}}
9. {{1,16,168}, {1,17,167}}
10. {{18,1,166}, {18,19,148}}
11. {{1,19,165}, {1,20,164}}
12. {{1,24,160}, {1,25,159}}
13. {{26,1,158}, {26,27,132}}
14. {{27,1,157}, {27,28,130}}
15. {{1,28,156}, {1,29,155}}
16. {{30,1,154}, {30,31,124}}
17. {{1,31,153}, {1,32,152}}
18. {{33,1,151}, {33,34,118}}
19. {{1,34,150}, {1,35,149}}
20. {{1,36,148}, {1,37,147}}
21. {{38,1,146}, {38,39,108}}
22. {{1,39,145}, {1,40,144}}
23. {{1,41,143}, {1,42,142}}
24. {{44,1,140}, {44,45,96}}
25. {{1,47,137}, {1,48,136}}
26. {{1,49,135}, {1,50,134}}
27. {{1,53,131}, {1,54,130}}
28. {{55,1,129}, {55,56,74}}
29. {{1,58,126}, {1,59,125}}
30. {{1,60,124}, {1,61,123}}
31. {{65,1,119}, {65,66,54}}
32. {{1,66,118}, {1,67,117}}
33. {{1,68,116}, {1,69,115}}
34. {{70,1,114}, {70,71,44}}
35. {{1,71,113}, {1,72,112}}
36. {{1,73,111}, {1,74,110}}
37. {{1,75,109}, {1,76,108}}
38. {{77,1,107}, {77,78,30}}
39. {{1,78,106}, {1,79,105}}
40. {{1,82,102}, {1,83,101}}
41. {{1,87,97}, {1,88,96}}
42. {{1,89,95}, {1,90,94}}
43. {{7,3,175}, {7,10,168}}
44. {{6,3,176}, {6,9,170}}
45. {{9,3,173}, {9,12,164}}
46. {{3,12,170}, {3,15,167}}
47. {{3,18,164}, {3,21,161}}
48. {{3,24,158}, {3,27,155}}
49. {{30,3,152}, {30,33,122}}
50. {{3,33,149}, {3,36,146}}
51. {{3,39,143}, {3,42,140}}
52. {{3,45,137}, {3,48,134}}
53. {{17,3,165}, {17,20,148}}
54. {{3,19,163}, {3,22,160}}
55. {{3,25,157}, {3,28,154}}
56. {{3,31,151}, {3,34,148}}
57. {{3,35,147}, {3,38,144}}
58. {{3,50,132}, {3,53,129}}
59. {{3,59,123}, {3,62,120}}
60. {{3,60,122}, {3,63,119}}

(v = 370 continued)

61. {{3,74,108}, {3,77,105}}
62. {{3,76,106}, {3,79,103}}
63. {{82,3,100}, {82,85,18}}
64. {{3,84,98}, {3,87,95}}
65. {{5,17,163}, {5,22,158}}
66. {{5,31,149}, {5,36,144}}
67. {{5,49,131}, {5,54,126}}
68. {{5,67,113}, {5,72,108}}
69. {{5,74,106}, {5,79,101}}
70. {{5,77,103}, {5,82,98}}
71. {{63,5,117}, {63,68,54}}
72. {{9,18,158}, {9,27,149}}
73. {{9,33,143}, {9,42,134}}
74. {{9,54,122}, {9,63,113}}.

2. Non-S-cyclic SQS

The difference quadruples contained in the non-S-cyclic SQS(v), which are omitted in the following list, are shown in the table below.

Order	Difference quadruples
26	$D_1(26)$, $D_2(26)$
28	$D_1(28)$
34	$D_1(34)$, $D_2(34)$
50	$D_1(50)$, $D_2(50)$
58	$D_1(58)$, $D_2(58)$
76	$D_3(76)$, $\phi(D_1(38))$
80	$D_1(80)\setminus\{(i,i,40-i,40-i) \mid i = 2,6,10,14,18\}$
88	$D_1(88)\setminus\{(i,i,44-i,44-i) \mid i = 4,8,12,16,20\}$
92	$D_3(92)$, $\phi(D_1(46))$
98	$D_1(98)$, $D_2(98)$
124	$D_3(124)$, $\phi(D_1(62))$

v = 26:

1. (2,4,12,8)
2. (2,6,4,14)
3. (2,8,12,4)
4. (2,14,4,6).

v = 28:

1. (4,8,5,11)
2. (4,11,5,8)
3. (1,3,1,23)
4. (1,9,1,17)
5. (1,11,1,15)
6. (2,1,2,23)
7. (2,4,2,20)
8. (2,8,2,16)
9. (2,9,2,15)
10. (3,4,3,18)
11. (3,5,3,17)
12. (3,9,3,13)
13. (4,5,4,15)
14. (5,1,5,17)
15. (5,2,5,16)
16. (6,1,6,15)
17. (6,3,6,13)
18. (6,4,6,12)
19. (7,1,7,13)
20. (7,2,7,12)
21. (7,4,7,10)
22. (8,1,8,11)
23. (10,3,10,5).

v = 34:

1. (2,4,20,8)
2. (2,6,4,22)
3. (2,8,20,4)
4. (2,22,4,6)
5. (10,6,10,8)
6. (12,4,12,6)
7. (12,8,12,2)
8. (14,2,14,4).

<u>v = 50</u>:

1. (2,4,8,36)
2. (2,36,8,4)
3. (2,6,2,40)
4. (2,14,2,32)
5. (2,18,2,28)
6. (4,6,4,36)
7. (4,12,4,30)
8. (4,14,4,28)
9. (6,12,6,26)
10. (8,14,8,20)
11. (8,16,8,18)
12. (10,2,10,28)
13. (10,6,10,24)
14. (10,8,10,22)
15. (12,8,12,18)
16. (14,6,14,16)
17. (14,10,14,12)
18. (16,6,16,12)
19. (20,4,20,6)
20. (22,2,22,4)

<u>v = 58</u>:

1. (2,18,28,10)
2. (2,10,28,18)
3. (4,2,4,48)
4. (2,6,2,48)
5. (10,16,10,22)
6. (6,10,6,36)
7. (26,2,26,4)
8. (2,22,2,32)
9. (4,18,4,32)
10. (4,10,4,40)
11. (12,10,12,24)
12. (10,8,10,30)
13. (18,8,18,14)
14. (2,12,2,42)
15. (8,16,8,26)
16. (16,2,16,24)
17. (14,10,14,20)
18. (6,22,6,24)
19. (6,12,6,34)
20. (16,12,16,14)
21. (20,2,20,16)
22. (12,14,12,20)
23. (6,14,6,32)
24. (8,6,8,36)
25. (8,20,8,22)
26. (8,4,8,38)
27. (4,12,4,38)
28. (4,20,4,30)

v = 76:

1. (14,26,16,20)
2. (14,20,16,26)
3. (4,24,4,44)
4. (32,4,32,8)
5. (24,8,24,20)
6. (12,8,12,44)
7. (4,8,4,60)
8. (4,16,4,52)
9. (16,8,16,36)
10. (8,20,8,40)
11. (12,24,12,28)
12. (16,12,16,32)
13. (8,2,8,58)
14. (10,2,10,54)
15. (10,22,10,34)
16. (2,30,2,42)
17. (2,26,2,46)
18. (18,12,18,28)
19. (6,12,6,52)
20. (24,6,24,22)
21. (6,30,6,34)
22. (28,6,28,14)
23. (22,6,22,26)
24. (4,22,4,46)
25. (18,4,18,36)
26. (14,8,14,40)
27. (10,26,10,30)
28. (10,6,10,50)
29. (4,2,4,66)
30. (4,10,4,58)
31. (18,14,18,26)
32. (14,16,14,32)
33. (2,12,2,60)
34. (16,2,16,42)
35. (8,18,8,42)
36. (6,2,6,62)
37. (6,14,6,50)
38. (20,10,20,26)
39. (24,10,24,18)
40. (12,22,12,30)
41. (20,2,20,34)
42. (2,22,2,50)
43. (2,1,2,71)
44. (1,3,1,71)
45. (6,23,6,41)
46. (1,7,1,67)
47. (1,9,1,65)
48. (1,11,1,63)
49. (1,13,1,61)
50. (1,15,1,59)
51. (17,1,17,41)
52. (20,1,20,35)
53. (21,1,21,33)
54. (22,1,22,31)
55. (23,1,23,29)
56. (24,1,24,27)
57. (1,25,1,49)
58. (1,27,1,47)
59. (29,1,29,17)
60. (1,30,1,44)
61. (1,32,1,42)
62. (1,34,1,40)
63. (2,5,2,67)
64. (2,9,2,63)
65. (13,2,13,48)
66. (15,2,15,44)
67. (4,3,4,65)
68. (11,4,11,50)
69. (7,11,7,51)
70. (7,25,7,37)
71. (7,23,7,39)
72. (16,7,16,37)
73. (9,7,9,51)
74. (11,9,11,45)
75. (15,18,15,28)
76. (13,11,13,39)
77. (13,22,13,28)
78. (26,11,26,13)
79. (11,17,11,37)
80. (5,7,5,59)
81. (11,24,11,30)
82. (9,28,9,30)
83. (4,9,4,59)
84. (13,7,13,43)
85. (5,1,5,65)
86. (5,11,5,55)
87. (5,21,5,45)
88. (5,31,5,35)
89. (5,25,5,41)
90. (6,1,6,63)
91. (13,17,13,33)
92. (6,11,6,53)
93. (7,3,7,59)
94. (10,3,10,53)
95. (9,13,9.45)
96. (13,3,13,47)

(v = 76 continued)

97. (13,21,13,29)
98. (8,13,8,47)
99. (29,8,29,10)
100. (17,14,17,28)
101. (29,3,29,15)
102. (26,3,26,21)
103. (5,4,5,62)
104. (22,15,22,17)
105. (11,18,11,36)
106. (5,17,5,49)
107. (9,27,9,31)
108. (8,23,8,37)
109. (18,9,18,31)
110. (23,10,23,20)
111. (15,9,15,37)
112. (13,18,13,32)
113. (3,20,3,50)
114. (17,3,17,39)
115. (23,4,23,26)
116. (9,12,9,46)
117. (11,14,11,40)
118. (3,11,3,59)
119. (21,6,21,28)
120. (29,2,29,16)
121. (11,23,11,31)
122. (14,23,14,25)
123. (12,25,12,27)
124. (25,4,25,22)
125. (9,14,9,44)
126. (15,6,15,40)
127. (9,24,9,34)
128. (3,22,3,48)
129. (3,28,3,42)
130. (35,2,35,4)
131. (12,21,12,31)
132. (14,15,14,33)
133. (21,4,21,30)
134. (25,8,25,18)
135. (5,24,5,42)
136. (10,17,10,39)
137. (21,16,21,18)
138. (27,5,27,17)
139. (5,32,5,34)
140. (3,34,3,36)
141. (7,8,7,54)
142. (7,22,7,40)
143. (7,20,7,42)
144. (7,26,7,36)
145. (8,9,8,51)
146. (15,20,15,26)
147. (17,9,17,33)
148. (7,28,7,34)
149. (12,17,12,35)
150. (23,13,23,17)
151. (3,30,3,40)
152. (3,24,3,46)
153. (3,15,3,55)
154. (3,9,3,61)
155. (15,12,15,34)
156. (6,3,6,61)
157. (11,10,11,44)
158. (11,22,11,32)
159. (8,27,8,33)
160. (9,26,9,32)
161. (15,16,15,30)
162. (2,31,2,41)
163. (25,9,25,17)
164. (6,27,6,37)
165. (15,10,15,36)
166. (25,10,25,16)
167. (13,14,13,36)
168. (6,25,6,39)
169. (20,5,20,31)
170. (5,10,5,56)
171. (12,11,12,41)
172. (10,21,10,35)
173. (7,14,7,48)
174. (14,21,14,27)
175. (27,2,27,20)
176. (21,2,21,32)
177. (2,23,2,49)
178. (23,5,23,25)
179. (3,5,3,65)
180. (5,8,5,58)
181. (18,5,18,35)
182. (17,7,17,35)
183. (16,17,16,27)
184. (21,3,21,31)
185. (4,29,4,39)
186. (4,27,4,41)

v = 80:

1. (6,34,6,34)
2. (10,30,10,30)
3. (14,26,14,26)
4. (2,3,2,73)
5. (1,2,1,76)
6. (1,4,1,74)
7. (6,1,6,67)
8. (2,7,2,69)
9. (2,11,2,65)
10. (2,15,2,61)
11. (2,19,2,57)
12. (23,13,23,21)
13. (13,18,13,36)
14. (5,7,5,63)
15. (5,17,5,53)
16. (5,32,5,38)
17. (5,28,5,42)
18. (5,13,5,57)
19. (5,3,5,67)
20. (7,12,7,54)
21. (11,13,11,45)
22. (6,11,6,57)
23. (11,23,11,35)
24. (1,7,1,71)
25. (6,13,6,55)
26. (6,25,6,43)
27. (12,19,12,37)
28. (6,23,6,45)
29. (6,33,6,35)
30. (9,8,9,54)
31. (1,9,1,69)
32. (11,1,11,57)
33. (10,11,10,49)
34. (12,13,12,43)
35. (26,9,26,19)
36. (8,3,8,61)
37. (5,6,5,64)
38. (9,11,9,51)
39. (26,7,26,21)
40. (5,16,5,54)
41. (7,13,7,53)
42. (27,5,27,21)
43. (8,13,8,51)
44. (8,29,8,35)
45. (8,19,8,45)
46. (5,26,5,44)
47. (16,21,16,27)
48. (13,15,13,39)
49. (23,5,23,29)
50. (11,16,11,42)
51. (11,20,11,38)
52. (23,2,23,32)
53. (25,2,25,28)
54. (13,20,13,34)
55. (15,17,15,33)
56. (7,27,7,39)
57. (16,17,16,31)
58. (27,2,27,24)
59. (29,2,29,20)
60. (21,13,21,25)
61. (9,13,9,49)
62. (11,21,11,37)
63. (2,31,2,45)
64. (2,35,2,41)
65. (4,35,4,37)
66. (4,29,4,43)
67. (4,21,4,51)
68. (17,4,17,42)
69. (11,15,11,43)
70. (18,11,18,33)
71. (31,4,31,14)
72. (23,4,23,30)
73. (19,4,19,38)
74. (13,4,13,50)
75. (4,5,4,67)
76. (7,25,7,41)
77. (19,14,19,28)
78. (5,9,5,61)
79. (9,14,9,48)
80. (8,17,8,47)
81. (6,15,6,53)
82. (13,19,13,35)
83. (15,21,15,29)
84. (9,27,9,35)
85. (1,12,1,66)
86. (14,1,14,51)
87. (15,16,15,34)
88. (27,6,27,20)
89. (17,20,17,26)
90. (10,29,10,31)
91. (11,3,11,55)
92. (11,17,11,41)
93. (9,29,9,33)
94. (3,14,3,60)
95. (8,31,8,33)
96. (8,15,8,49)

(v = 80 continued)

97.	(19,15,19,27)	131.	(14,23,14,29)	165.	(19,17,19,25)
98.	(19,10,19,32)	132.	(6,3,6,65)	166.	(11,25,11,33)
99.	(5,34,5,36)	133.	(14,13,14,39)	167.	(22,11,22,25)
100.	(26,13,26,15)	134.	(1,15,1,63)	168.	(3,16,3,58)
101.	(15,23,15,27)	135.	(17,1,17,45)	169.	(3,22,3,52)
102.	(9,10,9,52)	136.	(18,9,18,35)	170.	(28,3,28,21)
103.	(5,19,5,51)	137.	(3,9,3,65)	171.	(3,31,3,43)
104.	(13,24,13,30)	138.	(25,14,25,16)	172.	(9,25,9,37)
105.	(15,9,15,41)	139.	(19,20,19,22)	173.	(9,7,9,55)
106.	(18,7,18,37)	140.	(12,9,12,47)	174.	(7,16,7,50)
107.	(16,19,16,29)	141.	(12,15,12,41)	175.	(7,30,7,36)
108.	(25,13,25,17)	142.	(7,8,7,58)	176.	(1,35,1,43)
109.	(13,16,13,38)	143.	(21,9,21,29)	177.	(1,37,1,41)
110.	(4,11,4,61)	144.	(24,9,24,23)	178.	(1,33,1,45)
111.	(7,4,7,62)	145.	(12,23,12,33)	179.	(1,31,1,47)
112.	(3,4,3,70)	146.	(23,16,23,18)	180.	(30,1,30,19)
113.	(3,10,3,64)	147.	(21,18,21,20)	181.	(1,28,1,50)
114.	(10,13,10,47)	148.	(22,13,22,23)	182.	(1,26,1,52)
115.	(10,27,10,33)	149.	(1,18,1,60)	183.	(27,4,27,22)
116.	(17,10,17,36)	150.	(1,20,1,58)	184.	(5,10,5,60)
117.	(7,10,7,56)	151.	(1,22,1,56)	185.	(5,20,5,50)
118.	(7,31,7,35)	152.	(1,24,1,54)	186.	(30,5,30,15)
119.	(24,7,24,25)	153.	(22,15,22,21)	187.	(20,15,20,25)
120.	(30,3,30,17)	154.	(3,24,3,50)	188.	(10,15,10,45)
121.	(17,14,17,32)	155.	(3,18,3,56)	189.	(2,2,70,6)
122.	(28,7,28,17)	156.	(26,3,26,25)	190.	(2,8,50,20)
123.	(17,22,17,24)	157.	(21,14,21,24)	191.	(2,14,14,50)
124.	(19,18,19,24)	158.	(7,14,7,52)	192.	(2,20,18,40)
125.	(23,3,23,31)	159.	(11,19,11,39)	193.	(2,32,2,44)
126.	(9,30,9,32)	160.	(3,29,3,45)	194.	(2,48,26,4)
127.	(15,3,15,47)	161.	(3,33,3,41)	195.	(4,10,32,34)
128.	(29,5,29,17)	162.	(3,35,3,39)	196.	(4,22,12,42)
129.	(12,17,12,39)	163.	(7,22,7,44)	197.	(6,6,18,50)
130.	(20,3,20,37)	164.	(28,9,28,15)	198.	(6,16,10,48)

(v = 80 continued)

199. (6,28,22,24)
200. (8,14,48,10)
201. (10,12,34,24)
202. (12,26,24,18)
203. (2,4,62,12)
204. (2,10,14,54)
205. (2,16,34,28)
206. (2,26,34,18)
207. (2,36,18,24)
208. (2,62,6,10)
209. (4,14,44,18)
210. (4,26,10,30)
211. (6,8,46,20)
212. (6,20,46,8)
213. (6,30,14,30)
214. (6,46,10,18)
215. (8,18,16,38)
216. (10,16,26,28)
217. (10,38,14,18)
218. (12,30,24,14)
219. (2,6,58,14)
220. (2,12,10,56)
221. (2,18,22,38)
222. (2,30,22,26)
223. (2,42,14,22)
224. (4,6,60,10)
225. (4,18,36,22)
226. (4,30,8,38)
227. (6,14,38,22)
228. (6,22,42,10)
229. (6,32,6,36)
230. (8,10,20,42)
231. (8,22,32,18)
232. (10,26,10,34)
233. (12,22,16,30)
234. (14,16,18,32)
235. (8,16,8,48)
236. (4,12,4,60)
237. (4,20,4,52)
238. (8,4,8,60)
239. (8,20,8,44)
240. (12,16,36,16)
241. (12,20,28,20)
242. (12,24,12,32)
243. (16,20,24,20)
244. (4,28,4,44) .

v = 88:

1. (4,40,4,40)
2. (4,28,4,52)
3. (8,16,32,32)
4. (8,36,16,28)
5. (12,24,24,28)
6. (4,4,8,72)
7. (4,36,12,36)
8. (8,20,40,20)
9. (8,44,24,12)
10. (16,20,32,20)
11. (4,16,4,64)
12. (4,60,16,8)
13. (8,24,16,40)
14. (12,12,44,20)
15. (4,24,48,12)
16. (8,12,20,48)
17. (8,28,28,24)
18. (12,16,16,44)
19. (2,4,2,80)
20. (2,8,2,76)
21. (2,12,2,72)
22. (2,16,2,68)
23. (2,20,2,64)
24. (2,24,2,60)
25. (2,28,2,56)
26. (2,32,2,52)
27. (2,36,2,48)
28. (40,2,40,6)
29. (4,6,4,74)
30. (4,14,4,66)
31. (22,4,22,40)
32. (26,4,26,32)
33. (10,14,10,54)
34. (30,4,30,24)
35. (14,12,14,48)
36. (24,18,24,22)
37. (26,12,26,24)
38. (28,14,28,18)
39. (14,18,14,42)
40. (14,20,14,40)
41. (18,16,18,36)
42. (16,22,16,34)
43. (14,24,14,36)
44. (22,14,22,30)
45. (20,22,20,26)
46. (22,6,22,38)
47. (12,6,12,58)
48. (16,14,16,42)
49. (6,18,6,58)
50. (6,30,6,46)
51. (6,28,6,48)
52. (6,10,6,66)
53. (16,10,16,46)
54. (10,26,10,44)
55. (10,22,10,46)
56. (22,12,22,32)
57. (8,34,8,38)
58. (12,30,12,34)
59. (18,22,18,40)
60. (18,20,18,32)
61. (6,26,6,50)
62. (6,14,6,62)
63. (8,6,8,66)
64. (8,22,8,50)
65. (20,10,20,38)
66. (10,30,10,38)
67. (10,18,10,50)
68. (8,10,8,62)
69. (26,8,26,28)
70. (4,34,4,46)
71. (11,55,10,12)
72. (12,10,55,11)
73. (1,2,1,84)
74. (1,4,1,82)
75. (1,6,1,80)
76. (1,8,1,78)
77. (1,10,1,76)
78. (1,12,1,74)
79. (14,1,14,59)
80. (1,15,1,71)
81. (1,17,1,69)
82. (1,19,1,67)
83. (1,21,1,65)
84. (1,23,1,63)
85. (1,25,1,61)
86. (1,27,1,59)
87. (1,29,1,57)
88. (1,31,1,55)
89. (33,1,33,21)
90. (1,34,1,52)
91. (1,36,1,50)
92. (1,38,1,48)
93. (1,40,1,46)
94. (42,1,42,3)
95. (2,3,2,81)
96. (2,7,2,77)

(v = 88 continued)

97. (2,11,2,73)
98. (2,15,2,69)
99. (2,19,2,65)
100. (2,23,2,61)
101. (2,27,2,57)
102. (2,31,2,53)
103. (2,35,2,49)
104. (2,39,2,45)
105. (7,9,7,65)
106. (7,23,7,51)
107. (21,16,21,30)
108. (9,21,9,49)
109. (9,31,9,39)
110. (9,13,9,57)
111. (22,13,22,31)
112. (13,27,13,35)
113. (14,13,14,47)
114. (14,19,14,41)
115. (19,17,19,33)
116. (17,18,17,36)
117. (14,29,14,31)
118. (17,14,17,40)
119. (23,17,23,25)
120. (15,25,15,33)
121. (18,15,18,37)
122. (19,18,19,32)
123. (13,19,13,43)
124. (30,13,30,15)
125. (15,13,15,45)
126. (13,28,13,34)
127. (13,8,13,54)
128. (17,13,17,41)
129. (24,17,24,23)
130. (18,23,18,29)
131. (12,17,12,47)
132. (17,25,17,29)
133. (21,4,21,42)
134. (4,13,4,67)
135. (4,5,4,75)
136. (9,5,9,65)
137. (23,9,23,33)
138. (9,32,9,38)
139. (3,35,3,47)
140. (32,3,32,21)
141. (29,3,29,27)
142. (26,3,26,33)
143. (3,20,3,62)
144. (3,14,3,68)
145. (3,8,3,74)
146. (5,3,5,75)
147. (5,23,5,55)
148. (23,14,23,28)
149. (5,14,5,64)
150. (24,5,24,35)
151. (5,33,5,45)
152. (5,35,5,43)
153. (30,5,30,23)
154. (12,23,12,41)
155. (6,29,6,47)
156. (6,17,6,59)
157. (6,5,6,71)
158. (5,11,5,67)
159. (5,21,5,57)
160. (31,5,31,21)
161. (5,36,5,42)
162. (5,32,5,46)
163. (5,22,5,56)
164. (18,13,18,39)
165. (29,5,29,25)
166. (34,5,34,15)
167. (24,15,24,25)
168. (4,25,4,55)
169. (4,33,4,47)
170. (10,31,10,37)
171. (10,11,10,57)
172. (9,15,9,55)
173. (9,33,9,37)
174. (9,19,9,51)
175. (10,9,10,59)
176. (29,10,29,20)
177. (9,11,9,59)
178. (19,20,19,30)
179. (21,18,21,28)
180. (22,15,22,29)
181. (15,21,15,37)
182. (29,9,29,21)
183. (8,21,8,51)
184. (8,35,8,37)
185. (8,11,8,61)
186. (27,8,27,26)
187. (11,21,11,45)
188. (11,23,11,43)
189. (23,8,23,34)
190. (8,31,8,41)
191. (8,25,8,47)
192. (8,9,8,63)
193. (9,17,9,53)
194. (11,19,11,47)
195. (25,5,25,33)
196. (5,15,5,63)
197. (10,5,10,63)
198. (10,25,10,43)

(v = 88 continued)

199. (33,10,33,12)
200. (12,19,12,45)
201. (31,7,31,19)
202. (7,36,7,38)
203. (29,7,29,23)
204. (27,6,27,28)
205. (19,22,19,28)
206. (7,15,7,59)
207. (8,7,8,65)
208. (23,15,23,27)
209. (15,20,15,38)
210. (20,13,20,35)
211. (13,29,13,33)
212. (16,13,16,43)
213. (16,11,16,45)
214. (17,21,17,33)
215. (25,3,25,35)
216. (21,20,21,26)
217. (5,12,5,66)
218. (7,5,7,69)
219. (7,19,7,55)
220. (7,33,7,41)
221. (7,27,7,47)
222. (20,7,20,41)
223. (7,6,7,68)
224. (6,13,6,63)
225. (11,25,11,41)
226. (14,11,14,49)
227. (14,21,14,39)
228. (7,14,7,60)
229. (7,28,7,46)
230. (7,32,7,42)
231. (25,6,25,32)
232. (7,18,7,56)
233. (11,7,11,59)
234. (11,29,11,37)
235. (6,31,6,45)
236. (6,33,6,43)
237. (19,21,19,29)
238. (21,6,21,40)
239. (16,17,16,39)
240. (23,16,23,26)
241. (13,26,13,36)
242. (13,10,13,52)
243. (26,11,26,25)
244. (11,4,11,62)
245. (15,26,15,32)
246. (17,15,17,39)
247. (22,17,22,27)
248. (12,27,12,37)
249. (12,13,12,51)
250. (13,25,13,37)
251. (24,13,24,27)
252. (11,13,11,53)
253. (11,31,11,35)
254. (20,11,20,37)
255. (17,20,17,34)
256. (3,31,3,51)
257. (3,37,3,45)
258. (3,36,3,46)
259. (3,30,3,52)
260. (27,3,27,31)
261. (3,21,3,61)
262. (3,15,3,67)
263. (12,3,12,61)
264. (3,6,3,76)
265. (6,9,6,67)
266. (11,27,11,39)
267. (11,17,11,49)
268. (17,26,17,28)
269. (19,24,19,26)
270. (21,22,21,24)
271. (23,20,23,22)
272. (25,18,25,20)
273. (18,9,18,43)
274. (27,9,27,25)
275. (9,34,9,36)
276. (9,16,9,54)
277. (25,16,25,22)
278. (16,15,16,41)
279. (15,27,15,31)
280. (19,23,19,27)
281. (10,17,10,51)
282. (7,10,7,64)
283. (24,7,24,33)
284. (28,3,28,29)
285. (16,19,16,37)
286. (19,15,19,35)
287. (12,9,12,55)
288. (4,39,4,41)
289. (4,31,4,49)
290. (4,23,4,57)
291. (4,15,4,65)
292. (4,3,4,77)
293. (3,7,3,75)
294. (3,13,3,69)
295. (3,19,3,63)
296. (5,13,5,65)

y = 92:

1. (4,30,10,48)
2. (4,48,10,30)
3. (1,31,1,59)
4. (1,33,1,57)
5. (29,2,29,32)
6. (3,29,3,57)
7. (26,3,26,37)
8. (1,29,1,61)
9. (25,2,25,40)
10. (27,2,27,36)
11. (15,25,15,37)
12. (2,31,2,57)
13. (3,30,3,56)
14. (3,36,3,50)
15. (3,42,3,44)
16. (3,38,3,48)
17. (35,3,35,19)
18. (29,4,29,30)
19. (11,15,11,55)
20. (11,33,11,37)
21. (22,11,22,37)
22. (27,3,27,35)
23. (35,2,35,20)
24. (10,5,10,67)
25. (15,24,15,38)
26. (37,2,37,16)
27. (15,5,15,57)
28. (20,5,20,47)
29. (27,1,27,37)
30. (28,1,28,35)
31. (4,33,4,51)
32. (41,4,41,6)
33. (4,39,4,45)
34. (2,41,2,47)
35. (39,2,39,12)
36. (35,4,35,18)
37. (12,33,12,35)
38. (4,27,4,57)
39. (25,10,25,32)
40. (10,35,10,37)
41. (30,7,30,25)
42. (5,25,5,57)
43. (35,1,35,21)
44. (36,1,36,19)
45. (1,38,1,52)
46. (1,40,1,50)
47. (1,42,1,48)
48. (5,38,5,44)
49. (5,34,5,48)
50. (35,5,35,17)
51. (5,40,5,42)
52. (5,32,5,50)
53. (5,22,5,60)
54. (5,12,5,70)
55. (7,5,7,73)
56. (19,12,19,42)
57. (19,7,19,47)
58. (7,26,7,52)
59. (7,38,7,40)
60. (7,24,7,54)
61. (7,10,7,68)
62. (10,17,10,55)
63. (27,6,27,32)
64. (3,34,3,52)
65. (7,37,7,41)
66. (17,24,17,34)
67. (7,27,7,51)
68. (29,12,29,22)
69. (15,7,15,55)
70. (7,22,7,56)
71. (11,29,11,41)
72. (31,12,31,18)
73. (19,15,19,39)
74. (15,28,15,34)
75. (13,15,13,51)
76. (13,25,13,41)
77. (12,13,12,55)
78. (37,6,37,12)
79. (31,6,31,24)
80. (17,22,17,36)
81. (33,5,33,21)
82. (18,1,18,55)
83. (4,11,4,73)
84. (7,4,7,74)
85. (7,18,7,60)
86. (5,24,5,58)
87. (5,14,5,68)
88. (29,10,29,24)
89. (21,7,21,43)
90. (21,1,21,49)
91. (7,28,7,50)
92. (7,36,7,42)
93. (6,19,6,61)
94. (31,5,31,25)
95. (27,16,27,22)
96. (19,9,19,45)

(v = 92 continued)

97. (28,5,28,31)
98. (18,11,18,45)
99. (9,27,9,47)
100. (18,9,18,47)
101. (15,27,15,35)
102. (20,7,20,45)
103. (8,27,8,49)
104. (8,33,8,43)
105. (5,21,5,61)
106. (26,9,26,31)
107. (19,25,19,29)
108. (10,33,10,39)
109. (25,9,25,33)
110. (10,9,10,63)
111. (13,6,13,60)
112. (9,35,9,39)
113. (17,2,17,56)
114. (1,19,1,71)
115. (20,21,20,31)
116. (10,21,10,51)
117. (5,4,5,78)
118. (9,4,9,70)
119. (13,4,13,62)
120. (17,4,17,54)
121. (13,22,13,44)
122. (14,31,14,33)
123. (19,14,19,40)
124. (22,9,22,39)
125. (19,2,19,52)
126. (31,9,31,21)
127. (17,1,17,57)
128. (13,30,13,36)
129. (14,7,14,57)
130. (14,29,14,35)
131. (1,15,1,75)
132. (14,1,14,63)
133. (16,17,16,43)
134. (17,25,17,33)
135. (9,34,9,40)
136. (1,12,1,78)
137. (17,3,17,55)
138. (13,18,13,48)
139. (8,31,8,45)
140. (8,29,8,47)
141. (2,13,2,75)
142. (11,32,11,38)
143. (11,2,11,68)
144. (15,17,15,45)
145. (11,10,11,60)
146. (20,9,20,43)
147. (17,26,17,32)
148. (1,10,1,80)
149. (1,8,1,82)
150. (11,16,11,54)
151. (9,7,9,67)
152. (5,11,5,71)
153. (9,2,9,72)
154. (21,8,21,42)
155. (13,8,13,58)
156. (2,5,2,83)
157. (5,8,5,74)
158. (17,14,17,44)
159. (13,14,13,52)
160. (9,8,9,66)
161. (30,15,30,17)
162. (13,32,13,34)
163. (6,5,6,75)
164. (17,11,17,47)
165. (7,6,7,72)
166. (8,17,8,59)
167. (15,29,15,33)
168. (21,16,21,37)
169. (13,24,13,42)
170. (11,34,11,36)
171. (3,11,3,75)
172. (11,14,11,56)
173. (1,5,1,85)
174. (7,1,7,77)
175. (3,5,3,81)
176. (2,1,2,87)
177. (16,29,16,31)
178. (1,3,1,87)
179. (16,25,16,35)
180. (11,20,11,50)
181. (3,4,3,82)
182. (13,16,13,50)
183. (3,10,3,76)
184. (16,3,16,57)
185. (11,28,11,42)
186. (32,9,32,19)
187. (20,19,20,33)
188. (21,18,21,32)
189. (11,8,11,62)
190. (19,24,19,30)
191. (9,33,9,41)
192. (27,17,27,21)
193. (25,14,25,28)
194. (30,11,30,21)
195. (9,29,9,45)
196. (9,21,9,53)
197. (14,27,14,37)
198. (25,18,25,24)

(v = 92 continued)

199. (6,15,6,65)
200. (13,26,13,40)
201. (21,24,21,26)
202. (9,6,9,68)
203. (26,15,26,25)
204. (1,24,1,66)
205. (26,1,26,39)
206. (6,33,6,47)
207. (24,9,24,35)
208. (15,21,15,41)
209. (3,6,3,80)
210. (18,15,18,41)
211. (3,12,3,74)
212. (12,15,12,53)
213. (6,29,6,51)
214. (3,18,3,68)
215. (24,3,24,41)
216. (12,9,12,59)
217. (25,4,25,38)
218. (19,3,19,51)
219. (22,3,22,45)
220. (31,3,31,27)
221. (3,25,3,61)
222. (37,1,37,17)
223. (2,4,2,84)
224. (4,6,4,78)
225. (4,14,4,70)
226. (6,10,6,70)
227. (4,22,4,62)
228. (34,4,34,20)
229. (6,22,6,58)
230. (30,6,30,26)
231. (6,36,6,44)
232. (28,2,28,34)
233. (6,32,6,48)
234. (6,18,6,62)
235. (24,14,24,30)
236. (26,2,26,38)
237. (10,28,10,44)
238. (10,14,10,58)
239. (14,20,14,44)
240. (30,14,30,18)
241. (6,8,6,72)
242. (26,18,26,22)
243. (8,2,8,74)
244. (8,18,8,58)
245. (28,14,28,22)
246. (12,6,12,62)
247. (26,6,26,34)
248. (2,30,2,58)
249. (34,2,34,22)
250. (14,22,14,42)
251. (14,26,14,38)
252. (18,24,18,32)
253. (12,14,12,54)
254. (22,18,22,30)
255. (24,2,24,42)
256. (16,10,16,50)
257. (8,14,8,62)
258. (18,16,18,40)
259. (12,30,12,38)
260. (10,22,10,50)
261. (34,8,34,16)
262. (10,2,10,70)
263. (22,16,22,32)
264. (2,12,2,76)
265. (2,16,2,72)
266. (20,2,20,50)
267. (20,10,20,42)
268. (18,20,18,36)
269. (40,2,40,10)
270. (2,36,2,52)
271. (4,32,4,52)
272. (4,24,4,60)
273. (4,16,4,68)
274. (4,8,4,76)
275. (8,12,8,64)
276. (8,28,8,48)
277. (8,32,8,44)
278. (16,8,16,52)
279. (24,8,24,36)
280. (12,24,12,44)
281. (20,12,20,40)
282. (32,12,32,16)
283. (16,12,16,48)
284. (20,24,20,28)
285. (28,12,28,24)
286. (16,20,16,40)

v = 98:

1. (14,28,24,32)
2. (14,32,24,28)
3. (4,2,4,88)
4. (10,4,10,74)
5. (4,18,4,72)
6. (4,26,4,64)
7. (34,4,34,26)
8. (4,38,4,52)
9. (46,2,46,4)
10. (44,4,44,6)
11. (4,36,4,54)
12. (4,28,4,62)
13. (14,4,14,66)
14. (16,26,16,40)
15. (10,6,10,72)
16. (10,26,10,52)
17. (16,6,16,60)
18. (10,24,10,54)
19. (20,24,20,34)
20. (4,20,4,70)
21. (4,12,4,78)
22. (22,18,22,36)
23. (12,16,12,58)
24. (8,4,8,78)
25. (40,6,40,12)
26. (34,6,34,24)
27. (6,22,6,64)
28. (22,26,22,28)
29. (16,8,16,58)
30. (16,28,16,38)
31. (8,24,8,58)
32. (8,10,8,72)
33. (2,8,2,86)
34. (8,26,8,56)
35. (8,40,8,42)
36. (10,38,10,40)
37. (10,32,10,46)
38. (18,10,18,52)
39. (24,18,24,32)
40. (2,12,2,82)
41. (2,16,2,78)
42. (2,20,2,74)
43. (12,14,12,60)
44. (14,26,14,44)
45. (14,16,14,54)
46. (18,20,18,42)
47. (22,10,22,44)
48. (22,24,22,30)
49. (12,36,12,38)
50. (16,18,16,48)
51. (22,20,22,34)
52. (12,10,12,64)
53. (26,18,26,28)
54. (34,12,34,18)
55. (28,10,28,32)
56. (20,8,20,50)
57. (20,10,20,48)
58. (36,8,36,18)
59. (30,10,30,28)
60. (16,30,16,36)
61. (24,2,24,48)
62. (38,8,38,14)
63. (28,8,28,34)
64. (20,12,20,46)
65. (14,34,14,36)
66. (8,22,8,60)
67. (14,8,14,62)
68. (14,6,14,64)
69. (20,6,20,52)
70. (44,2,44,8)
71. (20,16,20,42)
72. (30,12,30,26)
73. (6,2,6,84)
74. (26,6,26,40)
75. (6,32,6,54)
76. (6,36,6,50)
77. (30,6,30,32)
78. (24,6,24,44)
79. (6,12,6,74)
80. (18,12,18,50)
81. (12,32,12,42)
82. (32,16,32,18)
83. (2,40,2,54)
84. (38,2,38,20)
85. (2,34,2,60)
86. (2,30,2,64)
87. (2,26,2,68)
88. (24,12,24,38)

v = 124:

1. (2,46,36,40)
2. (40,36,46,2)
3. (2,32,2,88)
4. (36,2,36,50)
5. (38,2,38,46)
6. (8,30,8,78)
7. (22,8,22,72)
8. (22,28,22,52)
9. (6,16,6,96)
10. (10,6,10,98)
11. (16,26,16,66)
12. (50,8,50,16)
13. (42,8,42,32)
14. (10,22,10,82)
15. (12,10,12,90)
16. (12,34,12,66)
17. (12,42,12,58)
18. (56,2,56,10)
19. (2,52,2,68)
20. (2,48,2,72)
21. (26,24,26,48)
22. (22,26,22,54)
23. (32,22,32,38)
24. (16,22,16,70)
25. (30,12,30,52)
26. (22,34,22,46)
27. (24,22,24,54)
28. (28,14,28,54)
29. (30,24,30,40)
30. (10,20,10,84)
31. (44,6,44,30)
32. (6,32,6,80)
33. (30,26,30,38)
34. (42,14,42,26)
35. (6,20,6,92)
36. (14,6,14,90)
37. (8,6,8,102)
38. (20,30,20,54)
39. (34,20,34,36)
40. (14,40,14,56)
41. (14,12,14,84)
42. (4,6,4,110)
43. (4,14,4,102)
44. (4,22,4,94)
45. (4,30,4,86)
46. (4,38,4,78)
47. (4,46,4,70)
48. (42,2,42,38)
49. (44,2,44,34)
50. (14,22,14,74)
51. (14,46,14,50)
52. (14,18,14,78)
53. (18,32,18,56)
54. (18,20,18,78)
55. (20,38,20,46)
56. (26,20,26,52)
57. (18,36,18,52)
58. (34,18,34,38)
59. (14,38,14,58)
60. (34,26,34,30)
61. (26,12,26,60)
62. (22,38,22,42)
63. (28,18,28,50)
64. (18,42,18,46)
65. (10,8,10,96)
66. (8,18,8,90)
67. (34,8,34,48)
68. (36,22,36,30)
69. (44,14,44,22)
70. (14,16,14,80)
71. (16,18,16,74)
72. (26,28,26,44)
73. (6,18,6,94)
74. (6,30,6,82)
75. (6,42,6,70)
76. (6,54,6,58)
77. (6,46,6,66)
78. (6,34,6,78)
79. (40,10,40,34)
80. (10,50,10,54)
81. (10,34,10,70)
82. (10,14,10,90)
83. (2,4,2,116)
84. (2,8,2,112)
85. (2,12,2,108)
86. (2,16,2,104)
87. (2,20,2,100)
88. (2,24,2,96)
89. (28,2,28,66)
90. (28,10,28,58)
91. (10,38,10,66)
92. (10,36,10,68)
93. (30,18,30,46)
94. (12,6,12,94)
95. (18,40,18,48)
96. (26,32,26,40)

(v = 124 continued)

97. (24,18,24,58)
98. (34,24,34,32)
99. (4,52,4,64)
100. (4,44,4,72)
101. (4,36,4,80)
102. (4,28,4,88)
103. (4,20,4,96)
104. (4,12,4,104)
105. (8,4,8,104)
106. (8,20,8,88)
107. (8,36,8,72)
108. (8,52,8,56)
109. (8,40,8,68)
110. (8,24,8,84)
111. (16,8,16,84)
112. (40,16,40,28)
113. (16,36,16,56)
114. (20,16,20,68)
115. (20,28,20,56)
116. (24,28,24,48)
117. (28,16,28,52)
118. (28,32,28,36)
119. (40,12,40,32)
120. (20,12,20,72)
121. (12,16,12,84)
122. (12,48,12,52)
123. (12,24,12,76)
124. (24,36,24,40)
125. (20,40,20,44)
126. (24,20,24,56)
127. (32,24,32,36)
128. (12,32,12,68)
129. (32,16,32,44)
130. (16,44,16,48)

Further, apply ϕ to the following difference quadruples:

131. (1,2,1,58)
132. (1,4,1,56)
133. (1,6,1,54)
134. (1,8,1,52)
135. (1,10,1,50)
136. (12,1,12,37)
137. (13,1,13,35)
138. (14,1,14,33)
139. (1,15,1,45)
140. (1,17,1,43)
141. (19,1,19,23)
142. (2,3,2,55)
143. (2,7,2,51)
144. (2,11,2,47)
145. (2,15,2,43)
146. (19,2,19,22)
147. (21,2,21,18)
148. (2,23,2,35)
149. (5,7,5,45)
150. (5,17,5,35)
151. (5,25,5,27)
152. (20,5,20,17)
153. (3,17,3,39)
154. (3,23,3,33)
155. (29,1,29,3)
156. (3,24,3,32)
157. (3,18,3,38)
158. (15,3,15,29)
159. (3,9,3,47)
160. (8,9,8,37)
161. (7,8,7,40)
162. (13,9,13,27)
163. (15,8,15,24)
164. (6,3,6,47)
165. (9,15,9,29)
166. (9,11,9,33)
167. (13,16,13,20)
168. (17,12,17,16)
169. (12,9,12,29)
170. (14,5,14,29)
171. (5,19,5,33)
172. (5,23,5,29)
173. (5,13,5,39)
174. (1,27,1,33)
175. (1,25,1,35)
176. (1,23,1,37)
177. (1,20,1,40)
178. (13,10,13,26)
179. (10,19,10,23)
180. (9,10,9,34)
181. (25,3,25,9)

(v = 124 continued)

182. (11,18,11,22)
183. (7,22,7,26)
184. (7,12,7,36)
185. (22,1,22,17)
186. (17,4,17,24)
187. (4,9,4,45)
188. (5,4,5,48)
189. (14,9,14,25)
190. (9,21,9,23)
191. (11,10,11,30)
192. (19,11,19,13)
193. (17,13,17,15)
194. (9,7,9,37)
195. (7,16,7,32)
196. (7,18,7,30)
197. (19,6,19,18)
198. (6,7,6,43)
199. (7,13,7,35)
200. (7,21,7,27)
201. (14,7,14,27)
202. (13,15,13,21)
203. (13,11,13,25)
204. (11,16,11,24)
205. (19,8,19,16)
206. (21,5,21,15)
207. (17,9,17,19)
208. (9,18,9,26)
209. (17,10,17,18)
210. (7,4,7,44)
211. (3,4,3,52)
212. (7,10,7,38)
213. (10,15,10,27)
214. (5,10,5,42)
215. (15,12,15,20)
216. (12,11,12,27)
217. (11,17,11,23)
218. (3,10,3,46)
219. (3,16,3,40)
220. (22,3,22,15)
221. (6,15,6,35)
222. (27,2,27,6)
223. (6,17,6,33)
224. (6,5,6,45)
225. (5,11,5,41)
226. (21,4,21,16)
227. (25,4,25,8)
228. (8,13,8,33)
229. (5,3,5,49)
230. (8,3,8,43)
231. (3,11,3,45)
232. (11,14,11,26)
233. (11,4,11,36)
234. (15,4,15,28)
235. (4,19,4,35)