DATA FUSION AND FILTERING FOR TARGET TRACKING AND IDENTIFICATION

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Abstract

This thesis explores two problems in target tracking and identification: (1) robust track state filtering, and (2) decision-level identity fusion.

In the first part of the thesis, a novel finite-horizon, discrete-time, time-varying state estimation method based on the robust semidefinite programming technique is proposed. The proposed method is robust to norm bounded parameter uncertainties in the system model as well as to uncertainties in the noise statistics. The robust performance of the proposed method is achieved by minimizing an upper bound on the worst case variance of the estimation error for all admissible systems.

In the second part of the thesis, two decision-level identity fusion models are proposed: Similar Sensor Fusion (SSF) model and Dissimilar Sensor Fusion (DSF) model. In the SSF model, sensors provide reports on a set of common characteristics of a target, and the fusion objective is to find a fusion result which is most consistent with all the sensor reports. In comparison, sensors in the DSF model explore different characteristics of a target. Their reports are fused in a manner that leads to decreased uncertainty on the target identity. In other words, these reports reinforce each other to generate increased certainty on the target identity, rather than being averaged to minimize inconsistency. Furthermore, we propose several fundamental principles for identity fusion, based on which all existing and future identity fusion methods can be evaluated and compared.

For the SSF model, two fusion methods are proposed: Convex quadratic fusion method and K-L fusion method. In the first method, inconsistencies between the fusion result and the sensor reports are measured by quadratic functions, and the problem is formulated as a convex quadratic programming problem. In the second method, Kullback-Leibler distance

is used to measure the inconsistencies among the probabilistic sensor reports. The resulting formulation leads to a generalized analytic center problem.

For the DSF model, we use a special objective function in the optimization formulation to accumulate the physical characteristics on a target explored by each sensor. The resulting fusion method involves solving an analytic center problem.

Compared with the two classical decision-level identity fusion methods: Bayesian inference method and Dempster-Shafer evidential inference method, the three new fusion methods require no *a priori* information on the target, and enjoy small computation complexity. In addition, we show that the three new fusion methods, as well as the two classical methods, all satisfy the fundamental principles for identity fusion. The performance of the proposed fusion methods are illustrated in several numerical examples.

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Chapter 1

Introduction

1.1 Introduction to data fusion

With the rapid development and proliferation of sensor technology, it is now possible to gather large amount of information in real time from a multitude of information sources. Such information must be processed (or fused) appropriately in order to generate an optimal decision or estimation. This process is usually called *data fusion*.

Data fusion has existed in the natural world for a long time. Human brains, for example, fuse five different kinds of information (sight, sound, smell, taste and touch) from five human sensors (eyes, ears, nose, tongue and skin) to achieve various fusion results. These fusion results range from identification and tracking of subjects, understanding of the surrounding environment, to movement control and so on. With training (experience from the past), human brains normally achieve optimal fusion results in real time with almost no error. Many data fusion techniques are developed to emulate the fusion process of human brains.

A multi-sensor system with data fusion can offer many benefits over the traditional single sensor systems. In the simplest sense, a multi-sensor system usually has larger space-time coverage over a single sensor system. This is because that in a multi-sensor system, when a target is out of the space-time range of one sensor, it may be sensed by another sensor. The other major benefits of multi-sensor systems include:

- increased detection performance: information from different sensors can be efficiently fused to improve the detection performance of the system;

- increased confidence: information from one sensor can be confirmed and reinforced by other sensors;

- increased robustness and reliability: the system will still function with the failure of some sensors, despite having a reduced performance.

Data fusion has been used in many different applications. Its earliest and (still) the biggest application is in the military domain. Here data fusion is used in air-to-air and surface-to-air defense, ocean surveillance, battlefield command, control, communication and intelligence, strategic warning and so on. Various tasks, including detection, identification and tracking of targets, battlefield situation assessment, threat analysis, fire control and intelligence collections, use data fusion to process information from sensors. In the non-military domain, data fusion has been used in remote sensing (to fuse information from satellites), medical diagnosis, robotics (to identify objects, interpret the environment, etc.), weather prediction, air traffic control, automatic manufacturing and so on.

Data fusion is a relatively new area of research which, at present, spans across a collection of several well established disciplines. The multidisciplinary nature of data fusion stems from the diversity of the data fusion applications and enables the application of various techniques. The major techniques used in data fusion include: statistics, signal processing, pattern recognition, artificial intelligence, expert system and information theory.

According to the widely accepted functional model developed by Joint Directors of Laboratories Data Fusion Sub-panel, data fusion can be divided into three levels of processing:

- Level-1: position and identity estimation,

- Level-2: military situation assessment,

- Level-3: hostile force threat assessment.

Among the three levels of processing, Level-1 is the most mature level and is relatively well-developed in theory as well as in practice. Level-1 processing involves mostly numerical methods such as linear and nonlinear estimation, pattern recognition and statistical analysis.

In comparison, Level-2 and Level-3 are less mature and typically use symbolic reasoning methods such as those from artificial intelligence.

Level-1 processing has four different fusion objectives: data alignment, data association, target tracking and target identification. Data alignment transforms multiple sensor inputs into a common space-time reference frame. Data association separates multiple sensor inputs into different groups according to target identities. Target tracking combines observations of positional and velocity data from a single sensor or multiple sensors to estimate position and velocity of a target. Finally, target identification fuses identity declarations or parametric information related to identity.

Level-2 processing involves situation abstraction and situation assessment. Situation abstraction represents the situation in a generalized way based on the fusion results from Level-1. Situation assessment interprets and expresses the situation, and in military context provides fusion results on plans of action, force distribution, enemy objectives and so on.

Once Level-2 processing has generated situation abstraction and assessment, Level-3 processing proceeds to determine the possible consequences and appropriate responses. In military context, this involves the assessment of vulnerability of "our force" when being attacked by the "enemy force", or the lethality and risk when "our force" attacks the "enemy force".

1.2 Multiple Target Tracking (MTT)

The basic principle of Multiple Target Tracking (MTT) was first recognized by Wax [1]. Since then great progress has been made in this field. The major breakthroughs during this process include the paper by Sittler [2] which provided a Bayesian formulation background for the later developments, and the papers by Bar-Shalom [3, 4] and Singer [5, 6] which combined correlation and Kalman filtering theory and initiated the development of modern MTT technology. So far, many MTT systems have been developed for various applications, all showing remarkable capabilities. The classic books by Bar-Shalom [7] and Blackman [8] provide an excellent introduction and reference to MTT.

Roughly speaking, a MTT system consists of a (group of) tracking sensor(s) and a tracking processor. A tracking sensor provides kinematics measurements on targets in its surveillance region. The measurements are usually inaccurate, and there is generally no indication to which target a measurement belongs. There also exist measurements which belong to no target, but are results of noise and enemy jamming signals. Using the provided measurements, the tracking processor estimates the kinematics parameters of each target and maintains a trajectory for it, which is called "track" of the target. The classical tracking theory is based on the so-called Tracking-While-Scanning (TWS) model in which the tracking sensor provides kinematics measurements periodically. A typical example of the TWS model is the traditional radar system whereby a radar scans its surveillance region periodically, providing updated measurements on a target each time it is hit by the radar beam. The word "scan" in MTT theory is therefore used to represent the arrival time of a group of new measurements provided by a tracking sensor.

Figure 1.1 gives a representation of a typical MTT system with the following functional blocks: measurement formation, gating, data association/correlation, track state filtering/predication and track management.

Specifically, the output data from the tracking sensor is first processed by the measurement formation block. Here, the data first undergoes a target detection procedure to separate targets from noise, the output of which consist of kinematics measurements of the targets or simply called observations. Two kinds of observations are possible: observations which belong to true targets, and false alarms which are the results of background noise sources, system thermal noise and possible enemy jamming signals. It is generally assumed that one target generates only one observation at any time, and one observation can correspond to only one target.

The output observations from the measurement formation block goes through a two-stage processing to form a single observation for each target and its track updating. The first stage of the processing is depicted as the gating block in Figure 1.1. This block determines for each observation whether it is a candidate for a track update or an initial observation for a potential new track. Usually, for each target we form a "gate" which is a neighborhood region centered at the most likely place predicted from the previous scan. Gating is then

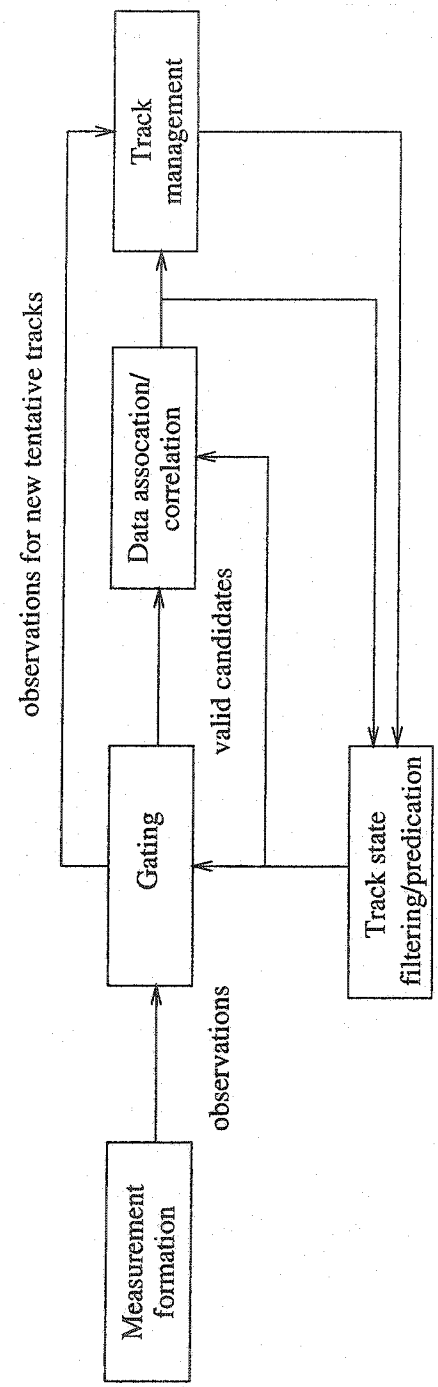

Figure 1.1: Functional blocks of a typical MTT system.

performed by comparing the observations with these gates. If an observation is within the gate of a target, it is considered as a candidate for updating the track of this target. On the other hand, if an observation is outside all the gates of existing targets, it is treated as an initial observation for a potential new track. Note that the first stage processing can generate multiple observations per gate and an observation may fall into multiple gates. Moreover, it is possible that no observation exists in a particular gate. The task of determining a single observation for updating a track is accomplished by the second stage of the processing, the so-called data association/correlation block in Figure 1.1. Specifically, this single observation can be the most likely observation among the candidates in a given gate, or a combined new observation from some or all of the candidates.

After data association and correlation, the unique observation associated with each target is processed by some adaptive filtering method in the track state filtering/prediction block to update kinematic parameter estimates for this target. A predicted observation for the next scan as well as its uncertainty are then computed for each target. This information is used to create the gate for this target in the next scan. In the case that no observation exists for a certain target, the parameter estimation and prediction are performed based on previous data.

The track management block in Figure 1.1 is responsible for track initiation, confirmation and deletion. Usually, observations not used to update the existing tracks are used to initiate new tentative tracks. The tentative tracks are either confirmed as new tracks or discarded through a confirmation procedure using observations from subsequent data scans. When a track terminates or moves out of the surveillance region of the tracking sensor, it no longer generates any observation. The lack of observation associated with an existing track can be used as evidence to terminate this track.

1.3 Identity fusion

Another important objective of Level-1 data fusion processing is target identification. Here, sensor data is processed to obtain identity estimates of targets. In military applications, these identity estimates may be in the form of either a target class (e.g., friend or foe), membership within a class (e.g., F-18 or MIG-29), or specific "serial number " within a class.

Let us first take a look at the process of target identity declaration using a single sensor. In typical military applications, the sensor can be a radar, an Infrared (IR) sensor, a video camera and so on. The resulting radar cross-section data, infrared or visible spectra, or imagery data are then processed to extract features on the target such as size information, kinematics parameters, movement pattern and shape patterns. In the end, an identity declaration is made through a pattern recognition process.

When there are multiple sensors capable of making identity estimates on a target, identity fusion is needed to combine identity information from the sensors to obtain a joint identity declaration. In general, the identity declaration obtained through identity fusion should be more specific and accurate than that from any individual sensor.

Identity fusion can be performed on three different levels: raw data level (before target feature extraction), feature level (before target identity estimation), or decision level (after target identity estimation). These choices are shown in Figure 1.2 in an illustrative two-sensor identity fusion configuration. The choice of at which level to perform fusion depends upon a number of factors such as sensor type and quality, available computing power, data transmission capability, required operator intervention, and most importantly, optimization of the fusion result.

Compared with MTT, target identification is a much broader problem. Physical models for identity declaration are generally very difficult to develop and often do not exist. So far, numerous techniques have been proposed, most of which are heuristic and lack mathematical rigor. Moreover, there is no universal way to compare and evaluate the existing identity fusion methods. This has in fact hindered the development of new fusion techniques.

The existing identity fusion techniques can be roughly classified into three categories: physical models, parametric classification and cognitive-based models. In principle, Physical model methods make identity declarations by matching the actual data with certain physical models. Unfortunately, such physical models, though conceptually possible, are very difficult to obtain. Parametric classification methods make identity declarations based on parametric data without the help of any physical models. Methods within this category

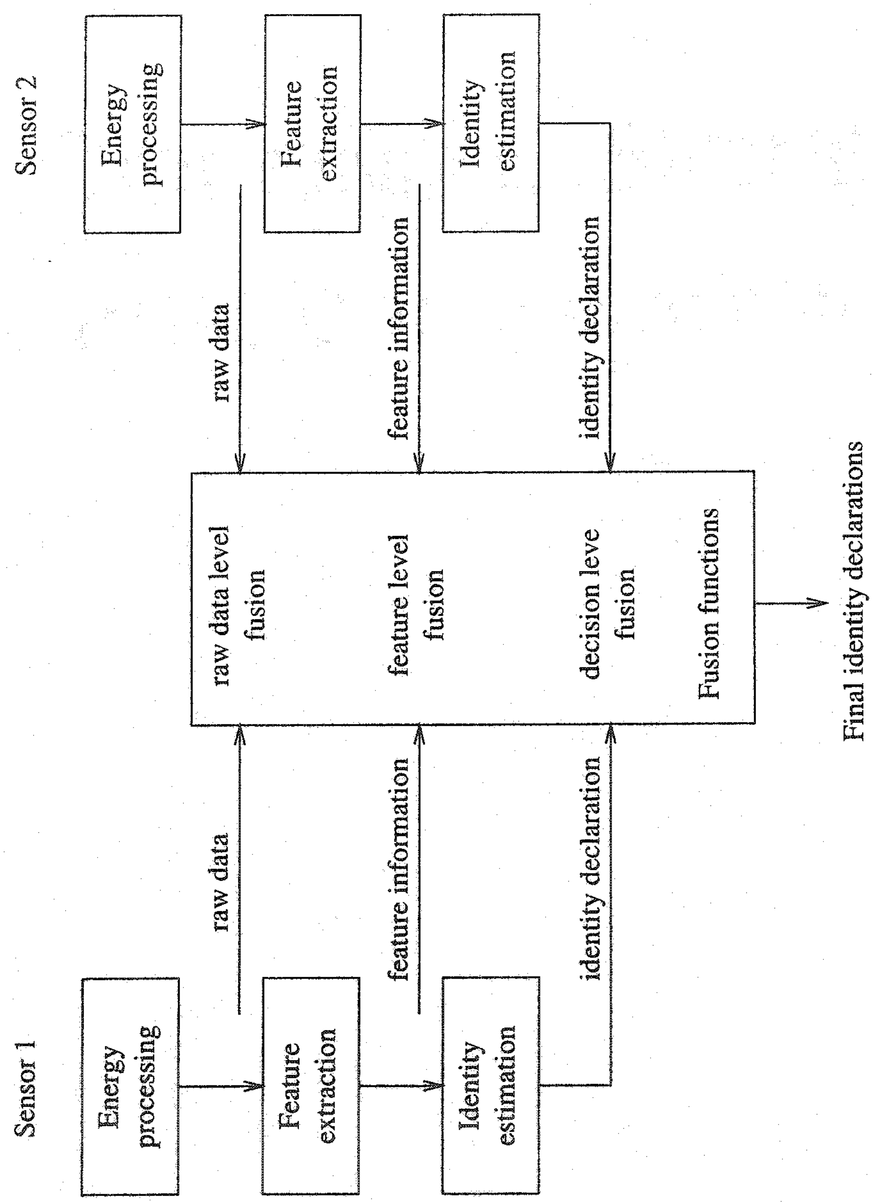

Figure 1.2: Identity fusion architecture with two sensors.

include classical inference, Bayesian inference, Dempster-Shafer evidential inference and information theoretic methods. Finally, cognitive-based model methods attempt to mimic the human reasoning process to make identity declarations. Methods within this category include logical templates, expert systems and fuzzy set theory.

1.4 Major contributions of the thesis

This thesis studies the Level-1 data fusion problem. The major contributions of the thesis are twofold: (1) robust track state filtering via semidefinite programming, (2) decision level identity fusion models, principles and new fusion methods.

As introduced in section 1.2, in the track state filtering/prediction part of a target tracking system, adaptive filters are deployed to update (for each scan) track states using new observations. The most popular adaptive filter used here is the Kalman filter. When the system model and noise statistics are known accurately, Kalman filter achieves optimal filtering in the minimal mean squared error sense. However, in applications including target tracking, there exist uncertainties in the system model and noise statistics. Under such circumstances, Kalman filter may suffer substantial performance loss. Therefore, it is necessary to develop filtering methods which are robust to small deviations in the system model and noise statistics.

In the first part of the thesis, a robust filtering method based on the robust semidefinite programming technique is proposed. The method is robust to norm bounded parameter uncertainties in the system model and noise statistics. Its robust performance is achieved by minimizing an upper bound on the worst case variance of the estimation error for all admissible systems. Our method is recursive in the sense that each subproblem has a fixed size, and it guarantees robust performance with respect to uncertainties that are known to lie within a certain *a priori* bounds. The latter is in contrast to the earlier robust H^∞ designs which accommodate all bounded energy uncertainties and therefore often lead to overly conservative solutions. From simulation results, the new method compares favorably in performance with some of the existing robust filtering approaches [11, 21, 23]. When applied to target tracking, the new method has led to a significant improvement in tracking performance.

The second part of the thesis focuses on the decision level identity fusion problem using parameter classification techniques. Here, the most widely used methods in the literature are Bayesian inference and Dempster-Shafer evidential inference theory [30,33]. The former method is well developed in statistical decision theory, but requires the knowledge of *a priori* distribution of the possible identity propositions as well as the knowledge of the conditional probabilities of all possible sensor reports. Such *a priori* knowledge is difficult if not impossible, to obtain. In contrast, Dempster-Shafer evidential inference does not require such *a priori* knowledge. However, it suffers from exponentially growing computational complexity. It is highly desirable to develop new fusion methods which can overcome the drawbacks of the existing methods. Moreover, a more important issue in identity fusion is the lack of a common framework in which fusion methods can be evaluated and compared with each other. In our opinion, this is a major problem in the development of new identity fusion methods.

In this thesis, we propose a theoretical framework for decision level identity fusion. This framework includes two fusion models and several decision fusion principles with which we can evaluate existing and future fusion methods. Specifically, two decision-level identity fusion models are proposed: Similar Sensor Fusion (SSF) model and Dissimilar Sensor Fusion (DSF) model. In the Similar Sensor Fusion model, sensors provide reports on a set of common characteristics of a target, and the fusion objective is to find a fusion result which is most consistent with all sensor reports. In comparison, sensors in the Dissimilar Sensor Fusion model explore different characteristics of a target. Their reports are fused in a manner that decreases the uncertainty in target identity. In other words, these reports reinforce each other to increase the certainty on the target identity, rather than being averaged to minimize inconsistency. In addition, we propose several identity fusion principles. These principles are based on mathematical characterizations of some common requirements of identity fusion systems such as reinforcement requirement. The aforementioned two classical fusion methods: Bayesian inference and Dempster-Shafer evidential inference, are both for the DSF model. We also establish that these two methods satisfy the identity fusion principles.

In addition, we propose three new decision level identity fusion methods, two for the SSF model and one for the DSF model. All three methods have been shown to satisfy the identity fusion principles as outlined above. In addition, they are highly efficient computationally and require no *a priori* knowledge of the targets.

Specifically, the two fusion methods for the SSF model are based on convex quadratic formulation and K-L formulation respectively. The basic idea of these two methods is to represent the inconsistency between the fusion result and each sensor report by a cost function, and to minimize the sum of all the cost functions. The difference between the two methods lies in the choice of cost functions: the first method uses convex quadratic functions, while the second method uses Kullback-Leibler distance measure to construct the cost functions. It turns out the K-L formulation leads to a generalized analytic center problem in linear programming, while the former corresponds to a convex quadratic programming problem. Both problems can be efficiently solved by interior point methods.

For the DSF model, a new method based on analytic centers and convex optimization is proposed. Although this new method is also optimization based, it differs from the previous two in that its objective function no longer represents inconsistencies between the fusion result and sensor reports. Instead, here the objective function has a special form to accumulate knowledge on a target explored by sensors. As a result, sensor reports reinforce each other through the minimization of the objective function to decreases the uncertainty in target identity. The resulting fusion method involves solving an analytic center problem.

1.5 Structure of the thesis

The thesis is structured as follows. In Chapter 2, the tracking filtering problem is studied, and the robust semidefinite programming filtering method is proposed. In Chapter 3, the two decision level identity fusion models as well as the identity fusion principles are proposed. Also given in this chapter are the proofs that the two classical decision level identity fusion methods, Bayesian inference and Dempster-Shafer evidential inference, satisfy these decision fusion principles. Chapter 4 formulates the convex quadratic fusion method for the SSF model. This is followed by Chapter 5 where another fusion method for the SSF model, the

K-L fusion method, is presented. In Chapter 6, a new fusion method called the analytic center fusion method is proposed for the DSF model. Chapter 7 provides fusion examples using data provided by Defence Research and Development Canada (DRDC) Valcatier. The conclusion and possible extensions of the thesis are given in Chapter 8.

1.6 Related publications

Part of the research results presented in this thesis has been published. The research results in Chapter 2 has been published on SIAM Journal on Optimization in 2002 [35]. Part of Chapter 4 has also been published both in the Proceedings of SPIE Symposium on Sensor Fusion in 1999 and IEEE Transactions on System, Man, and Cybernetics in 2001 [36,37].

Chapter 2

Robust Filtering via SDP for Target Tracking

2.1 Background

Target state estimation, which typically includes state filtering and predication, is a basic part of a target tracking system. As introduced in section 1.2, target state estimation uses output observations from the data association/correlation block to estimate and predict target kinematics parameters such as position, velocity and acceleration.

In classical target tracking theory, state filtering and prediction is solved as an adaptive filtering problem using the standard discrete-time linear state-space model. This model can be written as

$$\begin{cases} \underline{x}_{i+1} &= \mathbf{F}_i \underline{x}_i + \mathbf{G}_i \underline{u}_i, \quad \underline{x}_0 \text{ given}, \\ \underline{y}_i &= \mathbf{H}_i \underline{x}_i + \underline{v}_i, \quad i \geq 0, \end{cases} \tag{2.1.1}$$

where $\mathbf{F}_i \in \Re^{n \times n}$, $\mathbf{G}_i \in \Re^{n \times m}$ and $\mathbf{H}_i \in \Re^{p \times n}$ are known matrices which describe the dynamic system, and $\underline{x}_i$ describes the state of the system at time i, while $\underline{u}_i$ and $\underline{v}_i$ denote the process and measurement noise terms, respectively. In a target tracking system, $\underline{x}_i$ consists of the kinematics parameters of the target which is estimated using observation $\underline{y}_i$'s. A popular solution to this problem is given by the Kalman filter [14,16,19] which, under some standard assumptions on the statistics of the noise sources and initial state, recursively updates a minimal mean squared error (MMSE) estimator $\hat{\underline{x}}_i$ of the state $\underline{x}_i$. Such MMSE estimator minimizes the trace of the error covariance matrix $\mathcal{E}\{(\underline{x}_i - \hat{\underline{x}}_i)(\underline{x}_i - \hat{\underline{x}}_i)^T\}$, where $\mathcal{E}$ denotes the statistical expectation. Moreover, the Kalman filter is recursive and computationally efficient. In its 'innovation form', the Kalman filter is given by

$$\hat{\underline{x}}_{i+1} = \mathbf{F}_i\hat{\underline{x}}_i + \mathbf{K}_{K,i}\big(\underline{y}_i - \mathbf{H}_i\hat{\underline{x}}_i\big), \quad \hat{\underline{x}}_0 = 0, \tag{2.1.2}$$

where the so-called 'Kalman gain matrix' $\mathbf{K}_{K,i}$ can be computed via the following (analytic) recursion

$$\begin{aligned} \mathbf{K}_{K,i} &= \mathbf{F}_i\mathbf{P}_i\mathbf{H}_i\left(\mathbf{R}_i + \mathbf{H}_i\mathbf{P}_i\mathbf{H}_i^T\right)^{-1}, \\ \mathbf{P}_{i+1} &= \left(\mathbf{F}_i - \mathbf{K}_{K,i}\mathbf{H}_i\right)\mathbf{P}_i\left(\mathbf{F}_i - \mathbf{K}_{K,i}\mathbf{H}_i\right)^T + \begin{bmatrix}\mathbf{G}_i & -\mathbf{K}_{K,i}\end{bmatrix}\begin{bmatrix}\mathbf{Q}_i & 0 \\ 0 & \mathbf{R}_i\end{bmatrix}\begin{bmatrix}\mathbf{G}_i^T \\ -\mathbf{K}_{K,i}^T\end{bmatrix}, \end{aligned} \tag{2.1.3}$$

where $\mathbf{Q}_i = \mathcal{E}\{\underline{u}_i\underline{u}_i^T\}$ and $\mathbf{R}_i = \mathcal{E}\{\underline{v}_i\underline{v}_i^T\}$ are the noise covariance matrices. (The statistical assumptions made here are stated in section 2.2.) The matrix $\mathbf{P}_i$ in the recursion is the error covariance matrix $\mathcal{E}\{(\underline{x}_i - \hat{\underline{x}}_i)(\underline{x}_i - \hat{\underline{x}}_i)^T\}$. However, one drawback of the Kalman filter is that it requires the precise knowledge of the system matrices $\mathbf{F}_i$, $\mathbf{G}_i$ and $\mathbf{H}_i$, and noise covariances $\mathbf{Q}_i$ and $\mathbf{R}_i$, because even a small deviation from the 'nominal' values of these matrices can induce substantial performance loss in the Kalman filter. As a result, the Kalman filter can be ineffective in practice especially when we are faced with imprecise knowledge of the dynamic system mode, or in other words, when the matrices $\mathbf{F}_i$, $\mathbf{G}_i$, and $\mathbf{H}_i$ are only known approximately. This sensitivity of the Kalman filter has led researchers to tackle *robust filtering* problems, in which the objective is to design estimators which provide acceptable performance in the presence of uncertainties in the models of the dynamic system and the noise.

One approach for robust filtering is that of H^∞ filtering (see [13] and references therein). In that approach no statistical model of the disturbances $\underline{u}_i$ and $\underline{v}_i$ is employed; they are merely assumed to have finite energy. The idea is to obtain an estimator which minimizes (or, in the suboptimal case, bounds) the maximal energy gain from the disturbances to the estimation errors. This modelling paradigm also allows us to incorporate unstructured uncertainties in the dynamic system model (2.1.1) (see for example [12, 25]). An advantage of the H^∞ approach is that the solution closely resembles the Kalman filter, and can be efficiently implemented. Therefore, in applications where statistical knowledge of the disturbances and information regarding the structure of the modelling uncertainties are difficult to acquire, H^∞ filters are appropriate choices. Unfortunately, when the system model and the noise processes are known quite accurately, the Kalman filter may actually perform substantially better than the H^∞ filter. This is because the uncertainty model for the H^∞ filter is unstructured, and hence the H^∞ filter may be attempting to provide robustness to disturbances and modelling errors which rarely, or never, occur, at the expense of filter performance in the presence of more likely disturbances and modelling errors. In many applications including target tracking, we have some knowledge on the structure of the uncertainties in the system model and partial knowledge of disturbance statistics. It is natural to expect that careful incorporation of this knowledge into the estimator will lead to appreciable improvement in estimator performance. A major challenge is to determine whether this can be done in a computationally efficient manner. From recent work in the control field, it appears that determining filters which provide optimal robustness to highly structured uncertainties can be computationally expensive [9].

An alternative to the Kalman and H^∞ filtering methods is to find a 'robust Kalman filter' which minimizes (an upper bound on) the variance of the estimation error in the presence of a system model with norm-bounded structured parametric uncertainty and bounded uncertainty in the noise statistics. Models of this type are common in control theory (e.g., [15] and references therein), and are particularly appropriate in the context of target tracking. Previous approaches to this problem, with no uncertainty in the noise statistics, have been based on analytic recursions on some performance bounds [21, 23]. Note that robust H^∞ designs which bound the worst case error energy gain in the presence of the same system model uncertainties are also available [17, 24].

In this chapter we derive a new robust filtering algorithm using the recently developed robust semidefinite programming (SDP) technique [10]. The new method is recursive in the sense that the subproblem solved at each step depends on the solution at the previous step, and is computationally efficient since each subproblem is a semidefinite program of a fixed size which can be efficiently solved by an interior point algorithm. We demonstrate the performance of the novel algorithm in a standard benchmark example and in a target tracking example, and show that it can provide superior performance to the existing approaches to this particular problem [21, 23], and to the Kalman and H^∞ approaches. Our work shows that the robust SDP technique and the interior point algorithms [20, 22] can bring about substantial benefits to a practically important engineering problem. Note that the research results presented in this chapter has been published in [35].

Throughout this chapter, for a square matrix $\mathbf{X}$, the notation $\mathbf{X} \geq 0$ (resp. $\mathbf{X} \leq 0$) means $\mathbf{X}$ is symmetric and positive semidefinite (resp. negative semidefinite).

2.2 Problem formulation

Consider the following time-varying, discrete-time, uncertain linear state-space model

$$\begin{cases} \underline{x}_{i+1} &= [\mathbf{F}_i + \Delta\mathbf{F}_i]\,\underline{x}_i + \mathbf{G}_i\underline{u}_i, \quad \underline{x}_0, \\ \underline{y}_i &= [\mathbf{H}_i + \Delta\mathbf{H}_i]\,\underline{x}_i + \underline{v}_i, \quad i \geq 0, \end{cases} \tag{2.2.4}$$

where $\mathbf{F}_i \in \Re^{n\times n}$, $\mathbf{G}_i \in \Re^{n\times m}$, and $\mathbf{H}_i \in \Re^{p\times n}$, are known matrices which describe the nominal system. The matrices $\Delta\mathbf{F}_i$ and $\Delta\mathbf{H}_i$ represent the parameter uncertainties in the dynamic model. They are assumed to have the following structure:

$$\begin{bmatrix} \Delta\mathbf{F}_i \\ \Delta\mathbf{H}_i \end{bmatrix} = \begin{bmatrix} \mathbf{C}_{1,i} \\ \mathbf{C}_{2,i} \end{bmatrix} \mathbf{Z}_i\mathbf{E}_i \quad \text{with} \quad \mathbf{Z}_i^T\mathbf{Z}_i \leq \mathbf{I}, \tag{2.2.5}$$

where $\mathbf{C}_{1,i} \in \Re^{n\times r}$, $\mathbf{C}_{2,i} \in \Re^{p\times r}$, and $\mathbf{E}_i \in \Re^{t\times n}$ are known matrices. We remark that the above model (2.2.5) of uncertainties has been used extensively in the robust control literature, (e.g., [15] and references therein). The process noise $\{\underline{u}_i\}$, the measurement noise $\{\underline{v}_i\}$, and the initial state $\underline{x}_0$ in model (2.2.4) are all assumed to be random. These random variables have known mean values, which we can take to be zero without loss of generality, and partially known covariances, as follows:

$$\mathcal{E}\left\{\begin{bmatrix} \underline{u}_i \\ \underline{v}_i \\ \underline{x}_0 \end{bmatrix}\begin{bmatrix} \underline{u}_j \\ \underline{v}_j \\ \underline{x}_0 \end{bmatrix}^T\right\} = \begin{bmatrix} \mathbf{Q}_i\delta_{ij} & 0 & 0 \\ 0 & \mathbf{R}_i\delta_{ij} & 0 \\ 0 & 0 & \Pi_0 \end{bmatrix}, \tag{2.2.6}$$

where δ_{ij} denotes the Kronecker delta function that is equal to unity for $i = j$ and zero elsewhere, $\mathbf{Q}_i = \bar{\mathbf{Q}}_i + \Delta\mathbf{Q}_i$ and $\mathbf{R}_i = \bar{\mathbf{R}}_i + \Delta\mathbf{R}_i$. The matrices $\bar{\mathbf{Q}}_i \in \Re^{m\times m}$, $\bar{\mathbf{R}}_i \in \Re^{p\times p}$, and $\Pi_0 \in \Re^{n\times n}$ are assumed to be known and describe the nominal second order statistics of the noise and the initial state. The matrices $\Delta\mathbf{Q}_i$ and $\Delta\mathbf{R}_i$ represent the uncertainties in the noise statistics and satisfy the following bounds:

$$-\epsilon\mathbf{I} \leq \Delta\mathbf{Q}_i \leq \epsilon\mathbf{I}, \quad -\epsilon\mathbf{I} \leq \Delta\mathbf{R}_i \leq \epsilon\mathbf{I}. \tag{2.2.7}$$

Notice that when there is no uncertainty in the system model (2.2.4), namely $\epsilon = 0$ and $\mathbf{E}_i = 0$, then we recover the standard linear time-varying state-space model (2.1.1).

Let us use $\Theta_i = \{\Delta\mathbf{Q}_i, \Delta\mathbf{R}_i, \mathbf{Z}_i\}$ to denote the uncertainty variable at stage i and define the uncertainty region at stage i as

$$\Omega_i = \{\ \Theta_i : \Theta_i \text{ satisfies (2.2.5) and (2.2.7) }\}. \tag{2.2.8}$$

The problem is to estimate the state-sequence $\{\underline{x}_i,\ i \geq 0\}$, or some linear combinations of these sequences $\{\underline{s}_i = \mathbf{L}_i\underline{x}_i, i \geq 0\}$ where $\mathbf{L}_i$ is a known matrix, from the corrupted measurements. The goal of the robust filter is to provide a uniformly small estimation error for any process and measurement noise satisfying (2.2.6) and (2.2.7) and for all admissible modelling uncertainties satisfying (2.2.5). These *a priori* bounds on the uncertainties represent the designer's partial knowledge of the noise statistics and system model. They are to be incorporated into the problem formulation to guarantee robust performance.

To formulate the robust filtering problem, consider the following form of state estimator

$$\hat{\underline{x}}_{i+1} = \mathbf{A}_i\hat{\underline{x}}_i + \mathbf{K}_i(\underline{y}_i - \mathbf{H}_i\hat{\underline{x}}_i), \quad \hat{\underline{x}}_0 = 0, \tag{2.2.9}$$

where $\mathbf{A}_i$, $\mathbf{K}_i$ are filtering matrices to be determined, and $\hat{\underline{x}}_i$ denotes the estimate of the state $\underline{x}_i$. The above estimator is written in an innovation form that is similar to the structure of the Kalman filter given in (2.1.2). Notice that we use the nominal innovation $(\underline{y}_i - \mathbf{H}_i\hat{\underline{x}}_i)$, even though $\Delta\mathbf{H}_i$ may be nonzero. This structure is used for convenience, but it is general enough to generate all the full-order estimators, since $\mathbf{A}_i$ and $\mathbf{K}_i$ are free parameters. The goal of a robust filtering algorithm is to choose these free parameters to minimize (a function of) the estimation error covariance $\mathcal{E}\{(\underline{x}_i - \hat{\underline{x}}_i)(\underline{x}_i - \hat{\underline{x}}_i)^T\}$.

To express that goal precisely, we consider the following augmented system, which represents the cascade of the system in (2.2.4) and the estimator in (2.2.9):

$$\bar{\underline{x}}_{i+1} = \left[\bar{\mathbf{F}}_i + \bar{\mathbf{C}}_i\mathbf{Z}_i\bar{\mathbf{E}}_i\right]\bar{\underline{x}}_i + \bar{\mathbf{G}}_i\bar{\underline{u}}_i, \tag{2.2.10}$$

where

$$\bar{\underline{x}}_i = \begin{bmatrix} \underline{x}_i \\ \hat{\underline{x}}_i \end{bmatrix}, \quad \bar{\underline{u}}_i = \begin{bmatrix} \underline{u}_i \\ \underline{v}_i \end{bmatrix}, \quad \bar{\mathbf{F}}_i = \begin{bmatrix} \mathbf{F}_i\mathbf{0} \\ \mathbf{K}_i\mathbf{H}_i\mathbf{A}_i - \mathbf{K}_i\mathbf{H}_i \end{bmatrix}, \quad \bar{\mathbf{G}}_i = \begin{bmatrix} \mathbf{G}_i\mathbf{0} \\ \mathbf{0}\mathbf{K}_i \end{bmatrix},$$
$$\bar{\mathbf{C}}_i = \begin{bmatrix} \mathbf{C}_{1,i} \\ \mathbf{K}_i\mathbf{C}_{2,i} \end{bmatrix}, \quad \bar{\mathbf{E}}_i = [\mathbf{E}_i \;\; \mathbf{0}]. \tag{2.2.11}$$

Note that the state vector of the cascade, $\bar{\underline{x}}_i$, contains both $\underline{x}_i$ (the states of the model) and the estimates $\hat{\underline{x}}_i$, and hence the dimension of the state vector is doubled. The Lyapunov equation that governs the evolution of the covariance matrix $\Sigma_i = \mathcal{E}\{\bar{\underline{x}}_i\bar{\underline{x}}_i^T\}$ can be written as

$$\Sigma_{i+1} = \left[\bar{\mathbf{F}}_i + \bar{\mathbf{C}}_i\mathbf{Z}_i\bar{\mathbf{E}}_i\right]\Sigma_i\left[\bar{\mathbf{F}}_i + \bar{\mathbf{C}}_i\mathbf{Z}_i\bar{\mathbf{E}}_i\right]^T + \bar{\mathbf{G}}_i\mathbf{W}_i\bar{\mathbf{G}}_i^T, \tag{2.2.12}$$

where $\mathbf{W}_i = \text{blockdiag}(\mathbf{Q}_i, \mathbf{R}_i)$. The error covariance $\mathbf{P}_{i+1}$ can be obtained from equation (2.2.12) by pre-multiplying $[\mathbf{I} \;\; -\mathbf{I}]$ and post-multiplying $[\mathbf{I} \;\; -\mathbf{I}]^T$; i.e.,

$$\mathbf{P}_{i+1} = \hat{\mathbf{F}}_i\Sigma_i\hat{\mathbf{F}}_i^T + \mathbf{G}_i\mathbf{Q}_i\mathbf{G}_i^T + \mathbf{K}_i\mathbf{R}_i\mathbf{K}_i^T, \tag{2.2.13}$$

where

$$\hat{\mathbf{F}}_i = \left[\begin{array}{cc} (\mathbf{F}_i + \mathbf{C}_{1,i}\mathbf{Z}_i\mathbf{E}_i - \mathbf{K}_i\mathbf{H}_i - \mathbf{K}_i\mathbf{C}_{2,i}\mathbf{Z}_i\mathbf{E}_i) & (\mathbf{K}_i\mathbf{H}_i - \mathbf{A}_i) \end{array} \right]. \tag{2.2.14}$$

Now the finite-horizon robust state estimator problem can be stated as follows:

PROBLEM. *At each stage i, choose the filtering matrices $\{\mathbf{A}_j\}_{j=0}^{i}$ and $\{\mathbf{K}_j\}_{j=0}^{i}$ so as to minimize the worst case weighted error covariance matrix $\mathbf{DP}_{i+1}$; i.e.,*

$$\min_{\mathbf{K}_j,\mathbf{A}_j \,\forall j\leq i} \; \max_{\Theta_j\in\Omega_j \,\forall j\leq i} \mathrm{Tr}\,(\mathbf{DP}_{i+1}), \tag{2.2.15}$$

or equivalently

$$\min_{\mathbf{K}_j,\mathbf{A}_j \,\forall j\leq i} \; \max_{\Theta_j\in\Omega_j \,\forall j\leq i} \mathrm{Tr}\left(\mathbf{D}\,[\mathbf{I} \quad -\mathbf{I}]\,\Sigma_{i+1}\,[\mathbf{I} \quad -\mathbf{I}]^T\right). \tag{2.2.16}$$

where $\mathrm{Tr}\,(\cdot)$ denotes the trace of a matrix $(\cdot)$, and $\mathbf{D} \in \Re^{n\times n}$ is a positive semidefinite weighting matrix.

We have stated the robust state estimation problem in a rather general weighted form which includes many special cases. If we wish to estimate $\{\underline{x}_i\}$, then choosing $\mathbf{D} = \mathbf{I}$ will suffice, whereas to estimate $\{\underline{s}_i = \mathbf{L}_i\underline{x}_i\}$, then choosing $\mathbf{D} = \mathbf{L}_i\mathbf{L}_i^T$ will suffice. We can also weight the estimation accuracy of the states as desired, or add additional terms to $\mathbf{D}$, as long as it remains positive semidefinite. As we will observe later in section 2.4, adding additional terms to $\mathbf{D}$ may improve the numerical stability of the finite horizon filtering solutions.

The above minimax formulation is intended to incorporate robustness into the filter solution. In particular, $\mathrm{Tr}\,(\mathbf{DP}_{i+1})$, as recursively defined by (2.2.13), depends on all the uncertainties $\Theta_0, \ldots, \Theta_i$ as well as on the filtering matrices $\mathbf{K}_0, \mathbf{A}_0, \ldots, \mathbf{K}_i, \mathbf{A}_i$. The maximal weighted trace of $\mathbf{P}_{i+1}$,

$$\max_{\Theta_j\in\Omega_j \,\forall j\leq i} \mathrm{Tr}\,(\mathbf{DP}_{i+1}),$$

represents the worst case weighted error covariance when subject to the prescribed uncertainties. Therefore, the goal of robust filter design is to select the filtering matrices so that the worst case weighted error covariance is minimized.

As given by (2.2.15) or (2.2.16), the robust filter design problem is non-linear and non-smooth, hence is computationally difficult. Furthermore, the problem apparently lacks convexity which is essential in the development of computationally efficient algorithms. A further difficulty with the formulation (2.2.15) or (2.2.16) is that it is non-recursive, in the sense that the problem dimension increases linearly in i. This non-recursive feature makes it necessary to solve from scratch for the filtering matrices $\mathbf{K}_0, \mathbf{A}_0, \ldots, \mathbf{K}_i, \mathbf{A}_i$ at each stage i, which is clearly undesirable and impractical.

In practice, we typically fix $\mathbf{K}_0, \mathbf{A}_0, \ldots, \mathbf{K}_{i-1}, \mathbf{A}_{i-1}$ at stage i and solve only for $\mathbf{K}_i, \mathbf{A}_i$. But such simplification only partially fixes the problem since the uncertainties $\Theta_0, \ldots, \Theta_i$ still enter into the maximization of $\operatorname{Tr}(\mathbf{DP}_{i+1})$, indicating that the problem dimension still increases linearly with i. Our objective is to reformulate the problem (2.2.15) in a recursive way such that at each stage i, we only have to determine $\mathbf{K}_i, \mathbf{A}_i$ by solving a subproblem with a fixed dimension (i.e., independent of i).

To reformulate the problem (2.2.15), we consider a sequence of matrices

$$\{\Gamma_{i+1}(\mathbf{K}_i, \mathbf{A}_i) : \; i = 1, 2, \ldots\}$$

which are *not* dependent on the uncertainties $\{\Theta_i : \; i = 1, 2, ...\}$. These matrices will serve as upper bounds for the covariance matrices $\{\Sigma_{i+1} : \; i = 1, 2, ...\}$ which *are* dependent on the uncertainty vectors $\{\Theta_i : \; i = 1, 2, ...\}$, as well as on $\mathbf{K}_i$, and $\mathbf{A}_i$. In particular, we will have

$$\Gamma_{i+1}(\mathbf{K}_i, \mathbf{A}_i) \geq \Sigma_{i+1}, \quad \forall \; \Theta_i \in \Omega_i, \; i = 1, 2, \ldots . \tag{2.2.17}$$

There are, of course, many choices for an upper bound $\Gamma_{i+1}(\mathbf{K}_i, \mathbf{A}_i)$ which satisfies (2.2.17). Our objective should be to choose the one which, together with some $\mathbf{K}_i$ and $\mathbf{A}_i$, will yield the minimal weighted error covariance $\mathbf{DP}_{i+1}$. By the relation

$$\mathbf{P}_{i+1} = \begin{bmatrix} \mathbf{I} & -\mathbf{I} \end{bmatrix} \Sigma_{i+1} \begin{bmatrix} \mathbf{I} & -\mathbf{I} \end{bmatrix}^T, \tag{2.2.18}$$

we see that an upper bound on Σ_{i+1} naturally leads to an upper bound on $\mathbf{P}_{i+1}$. Thus we can approximately minimize $\mathbf{DP}_{i+1}$ by minimizing the trace of the matrix

$$\mathbf{D}\begin{bmatrix} \mathbf{I} & -\mathbf{I} \end{bmatrix} \Gamma_{i+1}(\mathbf{K}_i, \mathbf{A}_i) \begin{bmatrix} \mathbf{I} & -\mathbf{I} \end{bmatrix}^T$$

which is an upper bound of $\mathbf{DP}_{i+1}$. In particular, we choose Γ_{i+1}, $\mathbf{K}_i$, and $\mathbf{A}_i$ to

$$\begin{aligned} &\text{minimize} \quad \operatorname{Tr}\left(\mathbf{D}\begin{bmatrix} \mathbf{I} & -\mathbf{I} \end{bmatrix} \Gamma_{i+1} \begin{bmatrix} \mathbf{I} & -\mathbf{I} \end{bmatrix}^T\right) \\ &\text{subject to} \quad \Gamma_{i+1}, \mathbf{K}_i, \mathbf{A}_i \text{ satisfying (2.2.17).} \end{aligned} \tag{2.2.19}$$

The optimization problem (2.2.19) involves the constraint (2.2.17) which involves all of the uncertainty vectors $\{\Theta_i : i = 1, 2, ...\}$ and $\{\mathbf{K}_i, \mathbf{A}_i : i = 1, 2, ...\}$, thus making the amount of computation increase with i. To resolve this issue of dimensionality increase, we shall define the constraint recursively as follows. Specifically, let $b > 0$ be a chosen scalar bound and let $\bar{\Sigma}_0 = \Sigma_0$. For $i \geq 0$, suppose $\bar{\Sigma}_i$, an upper bound on Σ_i, has been computed and is already available. Consider the following minimization problem in the matrix variables $\{\Gamma_{i+1}, \mathbf{K}_i, \mathbf{A}_i\}$:

$$\begin{aligned} &\text{minimize} \quad \operatorname{Tr}\left(\mathbf{D}\begin{bmatrix} \mathbf{I} & -\mathbf{I} \end{bmatrix} \Gamma_{i+1} \begin{bmatrix} \mathbf{I} & -\mathbf{I} \end{bmatrix}^T\right) \\ &\text{subject to} \quad \Gamma_{i+1} \geq \left[\bar{\mathbf{F}}_i + \bar{\mathbf{C}}_i\mathbf{Z}_i\bar{\mathbf{E}}_i\right] \bar{\Sigma}_i \left[\bar{\mathbf{F}}_i + \bar{\mathbf{C}}_i\mathbf{Z}_i\bar{\mathbf{E}}_i\right]^T + \bar{\mathbf{G}}_i\mathbf{W}_i\bar{\mathbf{G}}_i^T, \quad \forall\, \Theta_i \in \Omega_i, \\ &\qquad\qquad\quad \operatorname{Tr}(\Gamma_{i+1}) \leq b. \end{aligned} \tag{2.2.20}$$

We choose $\bar{\Sigma}_{i+1}$ to be the optimal value of Γ_{i+1} in (2.2.20). Therefore our reformulation of (2.2.15) can now be stated as the following:

RFORMULATION OF THE ROBUST FILTERING PROBLEM. *Let* $\bar{\Sigma}_0 = \Sigma_0$. *For each* $i \geq 0$, *compute, recursively, the matrix* $\bar{\Sigma}_{i+1}$ *and the robust filtering matrices* $\mathbf{A}_i$ *and* $\mathbf{K}_i$ *as the minimizing solution of* (2.2.20).

We remark that the second constraint in (2.2.20), $\operatorname{Tr}(\Gamma_{i+1}) \leq b$, is used to ensure that the matrix Γ_{i+1} is bounded. This is important because otherwise the optimal solution of (2.2.20), $\bar{\Sigma}_{i+1}$, may become progressively ill-conditioned as i becomes large. An alternative way of preventing ill-conditioning is to impose the following structure on Γ_{i+1}

$$\Gamma_{i+1} = \begin{bmatrix} \bar{\Gamma} + \hat{\Gamma} & \bar{\Gamma} \\ \bar{\Gamma} & \bar{\Gamma} \end{bmatrix}, \quad \text{for some symmetric matrices } \bar{\Gamma},\, \hat{\Gamma}, \tag{2.2.21}$$

and to use the following constraint:

$$\mathrm{Tr}\,(\hat{\Gamma}) \geq \beta\, \mathrm{Tr}\,(\bar{\Gamma}), \tag{2.2.22}$$

where $\beta > 0$ is a constant. The above structure (2.2.21) for Γ_{i+1} mimics the structure of the joint covariance matrix of the state of a system and its optimal estimate in the Kalman sense, and is maintained in [21]. The bound (2.2.22) is used to ensure that the condition number of Γ_{i+1} does not become unbounded when $\bar{\Gamma}$ and $\hat{\Gamma}$ become large. Indeed, notice that

$$\Gamma_{i+1} = \begin{bmatrix} \bar{\Gamma} + \hat{\Gamma} & \bar{\Gamma} \\ \bar{\Gamma} & \bar{\Gamma} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{I} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \hat{\Gamma} & \mathbf{0} \\ \mathbf{0} & \bar{\Gamma} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{I} & \mathbf{I} \end{bmatrix}, \tag{2.2.23}$$

so we only need to bound the condition number for the matrix blockdiag$\{\hat{\Gamma}, \bar{\Gamma}\}$. By the above factorization of Γ_{i+1} and the fact that the right hand side of the first constraint in (2.2.20) is bounded from below by a positive definite matrix, we obtain that blockdiag$\{\hat{\Gamma}, \bar{\Gamma}\}$ is bounded from below by a positive definite matrix. Thus, the smallest eigenvalue of the matrix blockdiag$\{\hat{\Gamma}, \bar{\Gamma}\}$ is bounded away from zero. In the meantime, the constraint (2.2.22) and the fact we are minimizing $\hat{\Gamma}$ implies the largest eigenvalue of the matrix blockdiag$\{\hat{\Gamma}, \bar{\Gamma}\}$ is also bounded. This implies the boundedness of the condition number of Γ_{i+1} at optimal solution.

As a result of above discussion, we have the following alternative formulation (to (2.2.20)):

$$\begin{aligned} &\text{minimize} \quad \mathrm{Tr}\left(\mathbf{D}\begin{bmatrix} \mathbf{I} & -\mathbf{I} \end{bmatrix}\Gamma_{i+1}\begin{bmatrix} \mathbf{I} & -\mathbf{I} \end{bmatrix}^T\right) \\ &\text{subject to} \quad \Gamma_{i+1} \geq \left[\bar{\mathbf{F}}_i + \bar{\mathbf{C}}_i\mathbf{Z}_i\bar{\mathbf{E}}_i\right]\Sigma_i\left[\bar{\mathbf{F}}_i + \bar{\mathbf{C}}_i\mathbf{Z}_i\bar{\mathbf{E}}_i\right]^T + \bar{\mathbf{G}}_i\mathbf{W}_i\bar{\mathbf{G}}_i^T, \quad \forall\, \Theta_i \in \Omega_i, \\ &\qquad\qquad\ \Gamma_{i+1} \text{ satisfying (2.2.21) and (2.2.22).} \end{aligned} \tag{2.2.24}$$

In the remainder of this chapter, we will focus on the first formulation (2.2.19), but that the second formulation (2.2.24) can also be treated in an analogous fashion.

We point out that the dimension of problem (2.2.20) is fixed rather than growing linearly with i. Moreover, it will be shown that (2.2.20) is convex and can be reformulated as a semidefinite program. The latter can be solved very efficiently via interior point methods [18, 20, 22]. Before we explain how to solve (2.2.20), we need to show that $\bar{\Sigma}_i$ defined by (2.2.20) does provide an upper bound for Σ_i, for all $i \geq 0$. We have the following theorem.

Theorem 2.2.1 *Let $\bar{\Sigma}_0 = \Sigma_0$. For $i \geq 1$, let $\bar{\Sigma}_i$ be defined* (2.2.20). *Then there holds*

$$\bar{\Sigma}_i \geq \Sigma_i, \quad \forall\, \Theta_j \in \Omega_j, \; j = 1, 2, ..., i-1. \tag{2.2.25}$$

Proof. The theorem can be proved by mathematical induction. In particular, for $i = 0$, we have $\bar{\Sigma}_0 = \Sigma_0$. Suppose (2.2.25) holds for $i = k$. Since $\bar{\Sigma}_{k+1}$ is the optimal solution of (2.2.20), it follows from the constraint of (2.2.20) that

$$\bar{\Sigma}_{k+1} \geq \left[\bar{\mathbf{F}}_k + \bar{\mathbf{C}}_k \mathbf{Z}_k \bar{\mathbf{E}}_k\right] \bar{\Sigma}_k \left[\bar{\mathbf{F}}_k + \bar{\mathbf{C}}_k \mathbf{Z}_k \bar{\mathbf{E}}_k\right]^T + \bar{\mathbf{G}}_k \mathbf{W}_k \bar{\mathbf{G}}_k^T, \qquad \forall\, \Theta_k \in \Omega_k. \tag{2.2.26}$$

By the inductive hypothesis we have

$$\bar{\Sigma}_k \geq \Sigma_k, \quad \forall\, \Theta_j \in \Omega_j, \; j = 1, 2, ..., (k-1). \tag{2.2.27}$$

Combining this with (2.2.26) we obtain

$$\begin{aligned} \bar{\Sigma}_{k+1} &\geq \left[\bar{\mathbf{F}}_k + \bar{\mathbf{C}}_k \mathbf{Z}_k \bar{\mathbf{E}}_k\right] \Sigma_k \left[\bar{\mathbf{F}}_k + \bar{\mathbf{C}}_k \mathbf{Z}_k \bar{\mathbf{E}}_k\right]^T + \bar{\mathbf{G}}_k \mathbf{W}_k \bar{\mathbf{G}}_k^T, \\ &= \Sigma_{k+1}, \qquad \forall\, \Theta_j \in \Omega_j, \; j = 1, \ldots, k, \end{aligned} \tag{2.2.28}$$

where the last step is due to (2.2.12) for the particular value of Θ_j which represents the actual error in the model. This completes the induction proof. □

In common with the existing approaches to the finite-horizon robust filtering problem, we do not have a sufficient condition for the convergence of the estimator $\bar{\Sigma}_i$ as i tends to infinity. However, we now provide some necessary conditions. (These conditions are analogous to those in [21].)

Theorem 2.2.2 *Suppose the system* (2.2.4)–(2.2.7) *is time invariant in the sense that the data matrices* $\mathbf{H}_i$, $\mathbf{C}_{1,i}$, $\mathbf{C}_{2,i}$, $\mathbf{G}_i$, $\mathbf{E}_i$, $\bar{\mathbf{R}}_i$ *and* $\bar{\mathbf{Q}}_i$ *are fixed and independent of* i. *Then the solution* $\bar{\Sigma}_i$ *converges to some* $\bar{\Sigma}$ *only if the set of uncertain systems* (2.2.4)–(2.2.5) *is quadratically stable.*

Proof. Let $\underline{u}_i$ and $\underline{v}_i$ be zero. By the constraint (2.2.20) and the fact that $\bar{\Sigma}_i \to \bar{\Sigma}$, we have

$$\bar{\Sigma} \geq \left[\bar{\mathbf{F}} + \bar{\mathbf{C}}\mathbf{Z}_i\bar{\mathbf{E}}\right] \bar{\Sigma} \left[\bar{\mathbf{F}} + \bar{\mathbf{C}}\mathbf{Z}_i\bar{\mathbf{E}}\right]^T, \quad \forall\, \mathbf{Z}_i \text{ with } \|\mathbf{Z}_i\| \leq 1. \tag{2.2.29}$$

This shows that the augmented linear system (2.2.10) is quadratically stable. This is because the above relation easily implies that the quadratic Lyapunov function $V(\bar{\underline{x}}, i) = -\bar{\underline{x}}_i^T \bar{\Sigma} \bar{\underline{x}}_i \geq 0$ and that for all admissible systems, $V(\bar{\underline{x}}, i+1) \leq V(\bar{\underline{x}}, i)$ if the process noise $\bar{\underline{u}}_i = 0$. By construction, $\underline{x}_i$ is a component of $\bar{\underline{x}}_i$, therefore the quadratic stability of (2.2.10) (in this time-invariant case) implies the quadratic stability of (2.2.4)–(2.2.5) for all admissible systems. □

2.3 Robust semidefinite programming solution

In this section, we shall develop a semidefinite programming (SDP) [22] formulation for the robust state estimator problem (in particular, the problem (2.2.20)). This will then allow for efficient numerical solutions via recent interior point methods.

We begin by noting that the finite-horizon robust state estimator problem (2.2.20) has a constraint of the form

$$\Gamma_{i+1} \geq \left[\bar{\mathbf{F}}_i + \bar{\mathbf{C}}_i\mathbf{Z}_i\bar{\mathbf{E}}_i\right]\bar{\Sigma}_i\left[\bar{\mathbf{F}}_i + \bar{\mathbf{C}}_i\mathbf{Z}_i\bar{\mathbf{E}}_i\right]^T + \bar{\mathbf{G}}_i\mathbf{W}_i\bar{\mathbf{G}}_i^T,\ \forall\, \Theta_i = (\Delta\mathbf{Q}_i, \Delta\mathbf{R}_i, \mathbf{Z}_i) \in \Omega_i \tag{2.3.30}$$

which contains an uncertainty vector $\Theta_i = (\Delta\mathbf{Q}_i, \Delta\mathbf{R}_i, \mathbf{Z}_i)$. Recall that $\mathbf{W}_i = \text{blockdiag}(\bar{\mathbf{Q}}_i + \Delta\mathbf{Q}_i, \bar{\mathbf{R}}_i + \Delta\mathbf{R}_i)$, and that by (2.2.7), we have

$$-\epsilon\mathbf{I} \leq \Delta\mathbf{Q}_i \leq \epsilon\mathbf{I}, \quad -\epsilon\mathbf{I} \leq \Delta\mathbf{R}_i \leq \epsilon\mathbf{I}. \tag{2.3.31}$$

Therefore, by choosing the upper-bound for $\mathbf{W}_i$, the constraint (2.3.30) holds for all $\Theta_i = (\Delta\mathbf{Q}_i, \Delta\mathbf{R}_i, \mathbf{Z}_i) \in \Omega_i$ if and only if the following holds

$$\Gamma_{i+1} \geq \left[\bar{\mathbf{F}}_i + \bar{\mathbf{C}}_i\mathbf{Z}_i\bar{\mathbf{E}}_i\right]\bar{\Sigma}_i\left[\bar{\mathbf{F}}_i + \bar{\mathbf{C}}_i\mathbf{Z}_i\bar{\mathbf{E}}_i\right]^T + \bar{\mathbf{G}}_i\bar{\mathbf{W}}_i\bar{\mathbf{G}}_i^T,\quad \forall\ \mathbf{Z}_i \text{ with } \|\mathbf{Z}_i\| \leq 1, \tag{2.3.32}$$

where

$$\bar{\mathbf{W}}_i = \text{blockdiag}(\bar{\mathbf{Q}}_i + \epsilon\mathbf{I}, \bar{\mathbf{R}}_i + \epsilon\mathbf{I}). \tag{2.3.33}$$

We re-arrange the above inequality as follows:

$$\Gamma_{i+1} - \left[\bar{\mathbf{F}}_i + \bar{\mathbf{C}}_i\mathbf{Z}_i\bar{\mathbf{E}}_i \quad \bar{\mathbf{G}}_i\right]\begin{bmatrix}\bar{\Sigma}_i & \mathbf{0}\\ \mathbf{0} & \bar{\mathbf{W}}_i\end{bmatrix}\left[\bar{\mathbf{F}}_i + \bar{\mathbf{C}}_i\mathbf{Z}_i\bar{\mathbf{E}}_i \quad \bar{\mathbf{G}}_i\right]^T \geq \mathbf{0},\quad \forall\ \mathbf{Z}_i : \|\mathbf{Z}_i\| \leq 1. \tag{2.3.34}$$

By the Schur complement, the above constraint is equivalent to

$$\begin{bmatrix}\bar{\Sigma}_i^{-1} & \mathbf{0} & (\bar{\mathbf{F}}_i + \bar{\mathbf{C}}_i\mathbf{Z}_i\bar{\mathbf{E}}_i)^T\\ \mathbf{0} & \bar{\mathbf{W}}_i^{-1} & \bar{\mathbf{G}}_i^T\\ (\bar{\mathbf{F}}_i + \bar{\mathbf{C}}_i\mathbf{Z}_i\bar{\mathbf{E}}_i) & \bar{\mathbf{G}}_i & \Gamma_{i+1}\end{bmatrix} \geq \mathbf{0},\quad \forall\ \mathbf{Z}_i \text{ with } \|\mathbf{Z}_i\| \leq 1. \tag{2.3.35}$$

Note that both $\bar{\Sigma}_i$ and $\bar{\mathbf{W}}_i$ are positive definite and hence invertible.

For each fixed $\mathbf{Z}_i$ with $\|\mathbf{Z}_i\| \leq 1$, the above constraint (2.3.35) is a so-called linear matrix inequality (LMI) in the matrix variables $\{\Gamma_{i+1}, \mathbf{A}_i, \mathbf{K}_i\}$ which is convex. (Recall that the matrix variables $\{\mathbf{A}_i, \mathbf{K}_i\}$ are buried, linearly, in $\bar{\mathbf{F}}_i$, $\bar{\mathbf{G}}_i$ and $\bar{\mathbf{C}}_i$.) Thus the feasible region described by the above constraint is the intersection of convex regions described by an infinite number of linear matrix inequalities parameterized by $\mathbf{Z}_i$. This implies the feasible region of (2.2.20) is convex. It is now clear that the original robust filtering problem (2.2.20) is equivalent to

$$\begin{aligned} &\text{minimize} \quad \operatorname{Tr}\left(\mathbf{D}\begin{bmatrix}\mathbf{I} & -\mathbf{I}\end{bmatrix}\Gamma_{i+1}\begin{bmatrix}\mathbf{I} & -\mathbf{I}\end{bmatrix}^T\right) \\ &\text{subject to} \quad \{\Gamma_{i+1}, \mathbf{A}_i, \mathbf{K}_i\} \text{ satisfying (2.3.35)}, \\ &\qquad\qquad\quad \operatorname{Tr}(\Gamma_{i+1}) \le b. \end{aligned} \tag{2.3.36}$$

The formulation (2.3.36) is given as an SDP, except that the data matrices are subject to uncertainty $\mathbf{Z}_i$. Therefore it cannot be solved by standard SDP methods. The constraints in (2.3.36) imply that the solution must remain feasible for all allowable perturbations. This is precisely the intent of a robust filter solution. An SDP problem for which the data matrices are uncertain is called a robust SDP. In the next subsection, we introduce a technique for converting a robust SDP into a standard SDP which can then be solved efficiently by the recent interior point methods.

2.3.1 The robust SDP

Semidefinite programming is a convex optimization problem and can be solved in polynomial time using efficient algorithms such as the primal-dual interior point methods [18,20,22]. An SDP consists of minimizing a linear objective subject to an LMI constraint,

$$\begin{aligned} &\text{minimize} \quad \underline{c}^T\underline{\alpha} \\ &\text{subject to} \quad \mathbf{B}(\underline{\alpha}) = \mathbf{B}_0 + \sum_{k=1}^{q} \alpha_k \mathbf{B}_k \ge \mathbf{0}, \end{aligned} \tag{2.3.37}$$

where $\underline{c} \in \mathcal{R}^q$, $\underline{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_q)^T$ and the symmetric matrices $\mathbf{B}_k = \mathbf{B}_k^T \in \Re^{l\times l}$, $k = 0, \ldots, q$, are some given data matrices. In our case, these data matrices are subject to uncertainty. We can incorporate some linear uncertainty in $\mathbf{B}(\underline{\alpha})$ in the following way: Let $\mathbf{B}(\underline{\alpha}, \Delta)$ be a symmetric matrix-valued function of two variables $\underline{\alpha}$ and Δ of the form

$$\mathbf{B}(\underline{\alpha}, \Delta) = \mathbf{B}(\underline{\alpha}) + \mathbf{N}\Delta\mathbf{M}(\underline{\alpha}) + \mathbf{M}(\underline{\alpha})^T\Delta^T\mathbf{N}^T, \tag{2.3.38}$$

where $\mathbf{B}(\underline{\alpha})$ is defined in (2.3.37), $\mathbf{N}$ and $\mathbf{M}(\underline{\alpha})$ are given matrices, Δ is a perturbation which is unknown but bounded. We define the robust feasible set by

$$\mathcal{A} = \{\underline{\alpha} \in \mathcal{R}^q \mid \mathbf{B}(\underline{\alpha}, \Delta) \geq 0 \text{ for every } \Delta \text{ with } \|\Delta\| \leq 1\}. \tag{2.3.39}$$

The robust Semidefinite Problem is then defined as

$$\begin{array}{ll} \text{minimize} & \underline{c}^T\underline{\alpha} \\ \text{subject to} & \underline{\alpha} \in \mathcal{A}. \end{array} \tag{2.3.40}$$

The following lemma shows how such a robust SDP can be solved using a conventional SDP. It is a simple corollary of a classical result on quadratic inequalities referred to as the $\mathcal{S}$-procedure and its proof is detailed in [10].

Lemma 2.3.1 *Let $\mathbf{B} = \mathbf{B}^T$, $\mathbf{N}$, and $\mathbf{M}$ be real matrices of appropriate size. We have*

$$\mathbf{B} + \mathbf{N}\Delta\mathbf{M} + \mathbf{M}^T\Delta^T\mathbf{N}^T \geq \mathbf{0} \tag{2.3.41}$$

for every Δ, $\|\Delta\| \leq 1$, if and only if there exists a scalar ρ such that

$$\begin{bmatrix} \mathbf{B} - \rho\mathbf{N}\mathbf{N}^T & \mathbf{M}^T \\ \mathbf{M} & \rho\mathbf{I} \end{bmatrix} \geq \mathbf{0}. \tag{2.3.42}$$

As a consequence, the robust semidefinite problem (2.3.40) *can be formulated as the following standard SDP in variables* $\underline{\alpha}$ *and* ρ*:*

$$\begin{aligned} &\textit{minimize} \quad \underline{c}^T\underline{\alpha} \\ &\textit{subject to} \quad \begin{bmatrix} \mathrm{B}(\underline{\alpha}) - \rho \mathrm{N}\mathrm{N}^T & \mathrm{M}(\underline{\alpha})^T \\ \mathrm{M}(\underline{\alpha}) & \rho \mathbf{I} \end{bmatrix} \geq 0. \end{aligned} \tag{2.3.43}$$

We now return to the problem (2.3.36) and factorize the LMI constraint matrix (2.3.35) according to the structure in (2.3.38). In such factorization, the decision variable $\underline{\alpha}$ of (2.3.38) will correspond to a concatenation of the elements of the matrix variables Γ_{i+1}, $\mathbf{A}_i$ and $\mathbf{K}_i$ of (2.3.35), and the perturbation Δ of (2.3.38) will correspond to $\mathbf{Z}_i$ of (2.3.35). The factorization is given by:

$$\mathrm{B}(\underline{\alpha}) = \begin{bmatrix} \bar{\Sigma}_i^{-1} & \mathbf{0} & \bar{\mathbf{F}}_i^T \\ \mathbf{0} & \bar{\mathbf{W}}_i^{-1} & \bar{\mathbf{G}}_i^T \\ \bar{\mathbf{F}}_i & \bar{\mathbf{G}}_i & \Gamma_{i+1} \end{bmatrix}, \tag{2.3.44}$$

where

$$\bar{\mathbf{F}}_i = \begin{bmatrix} \mathbf{F}_i & \mathbf{0} \\ \mathbf{K}_i\mathbf{H}_i & \mathbf{A}_i - \mathbf{K}_i\mathbf{H}_i \end{bmatrix}, \quad \bar{\mathbf{G}}_i = \begin{bmatrix} \mathbf{G}_i & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_i \end{bmatrix}, \quad \bar{\mathbf{W}}_i = \begin{bmatrix} \bar{\mathbf{Q}}_i + \epsilon\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \bar{\mathbf{R}}_i + \epsilon\mathbf{I} \end{bmatrix}, \tag{2.3.45}$$

and

$$\mathrm{N}\Delta\mathrm{M}(\underline{\alpha}) + \mathrm{M}(\underline{\alpha})^T\Delta^T\mathrm{N}^T = \begin{bmatrix} \mathbf{0} & \mathbf{0} & (\bar{\mathbf{C}}_i\mathbf{Z}_i\bar{\mathbf{E}}_i)^T \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \bar{\mathbf{C}}_i\mathbf{Z}_i\bar{\mathbf{E}}_i & \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{2.3.46}$$

with

$$\bar{\mathbf{C}}_i\mathbf{Z}_i\bar{\mathbf{E}}_i = \begin{bmatrix} \mathbf{C}_{1,i}\mathbf{Z}_i\mathbf{E}_i & \mathbf{0} \\ \mathbf{K}_i\mathbf{C}_{2,i}\mathbf{Z}_i\mathbf{E}_i & \mathbf{0} \end{bmatrix}. \tag{2.3.47}$$

The matrices $\mathbf{N}$ and $\mathbf{M}(\underline{\alpha})$ are given by

$$\mathbf{M}(\underline{\alpha}) = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{C}_{1,i}^T & \mathbf{C}_{2,i}^T\mathbf{K}_i^T \end{bmatrix}, \quad \mathbf{N} = \begin{bmatrix} \mathbf{E}_i^T \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}. \tag{2.3.48}$$

Now we are in a position to apply Lemma 2.3.1 to convert the robust SDP (2.3.36) to the following standard SDP in the variables Γ_{i+1}, $\mathbf{A}_i$, $\mathbf{K}_i$ and ρ:

$$\begin{aligned} &\text{minimize} \quad \mathrm{Tr}\left(\mathbf{D}\begin{bmatrix} \mathbf{I} & -\mathbf{I} \end{bmatrix}\Gamma_{i+1}\begin{bmatrix} \mathbf{I} & -\mathbf{I} \end{bmatrix}^T\right) \\ &\text{subject to} \quad \begin{bmatrix} \mathbf{B}(\underline{\alpha}) - \rho\mathbf{N}\mathbf{N}^T & \mathbf{M}(\underline{\alpha})^T \\ \mathbf{M}(\underline{\alpha}) & \rho\mathbf{I} \end{bmatrix} \geq \mathbf{0}, \\ &\qquad\qquad\quad \mathrm{Tr}\left(\Gamma_{i+1}\right) \leq b, \end{aligned} \tag{2.3.49}$$

where the variable $\underline{\alpha}$ contains columns of the matrices Γ_{i+1}, $\mathbf{K}_i$ and $\mathbf{A}_i$, and the matrices $\mathbf{B}(\underline{\alpha})$, $\mathbf{N}$ and $\mathbf{M}(\underline{\alpha})$ are given by (2.3.44) and (2.3.48), respectively.

Note that, for each i, the problem (2.3.49) is fixed in dimension (i.e., does not grow with i). It is a standard SDP problem which has a unique solution and satisfies the usual regularity condition provided that the primal and dual of (2.3.49) are strictly feasible and for every $\underline{\alpha}$, $\mathbf{M}(\underline{\alpha}) \neq \mathbf{0}$ and $\begin{bmatrix} \mathbf{N} & \mathbf{M}^T(\underline{\alpha}) \end{bmatrix}^T$ is full column-rank. As such, the problem can be solved very efficiently by an interior point method, in particular, by the homogeneous self-dual method [18,20]. In our computational experience, the number of iterations required to solve each SDP is fixed (no more than 8), and therefore the proposed technique can be regarded as a recursive filtering method.

To make a formal comparison of the computational complexity of our robust filtering method with those of [21, 23], we need to recall the notations of our model (2.2.4): n denotes the number of states, m denotes the number of inputs, and p denotes the number of measured outputs. Then, Xie's method [23] is a "one-shot" method, and hence the robust observer matrix is only calculated once. The cost of this computation is $O((n+p)^3)$. However, Xie's method [23] works only for time-invariant systems. On the other hand, Theodor's method [21] is iterative. The cost per iteration $O((n+p)^3+n^2m)$. Our method is also iterative. Using a general purpose interior point Semi-definite programming solver requires $O\left((n+m+p)^{5/2}(n^2+np)^2\right)$ per filtering iteration. It is interesting to examine the above costs as the number of states in the model, n, grows. In that case, the total computational cost of Xie's method [23] is $O(n^3)$, while the the cost per (filtering) iteration of Theodor's method [21] and our proposed method are $O(n^3)$ and $O(n^{6.5})$ respectively. It is also interesting to examine the above costs as the number of measured outputs, p, grows. In that case, the total cost of Xie's method [23] is $O(p^3)$, while the cost per (filtering) iteration of Theodor's method [21] and our method are $O(p^3)$ and $O(p^{4.5})$ respectively. We believe it is possible to reduce the complexity per iteration for our method by exploiting the sparsity structure present in our problem. This is interesting issue for future investigation.

We now make an observation regarding the scaling of the matrices $\bar{\mathbf{C}}_i$ and $\bar{\mathbf{E}}_i$. In particular, these two matrices can be scaled and replaced by $\bar{\mathbf{C}}_i/\mu$ and $\mu\bar{\mathbf{E}}_i$ respectively. Such a scaling does not change the formulation of (2.3.36), nor does it affect the formulation (2.3.49), because the latter is completely equivalent to the former. This shows that the solutions to our reformulated robust filtering problem are independent of the scaling factor μ. This property is in contrast to the robust filter proposed in [21] where the solutions are "highly sensitive" [21] to the choice of μ. The scale invariance of our method with the choice of μ is a clear advantage.

However, our method also has a disadvantage in that it is sensitive to the choice of b in the second constraint in (2.3.49), $\mathrm{Tr}\,(\Gamma_{i+1}) \leq b$. This constraint is used to ensure that the matrix Γ_{i+1} is bounded. This is important because otherwise the optimal solution of (2.3.49), $\bar{\Sigma}_{i+1}$, may become progressively ill-conditioned as i becomes large. This phenomenon has been observed in computer simulations. In general, large values of b will allow the matrices $\{\bar{\Sigma}_i : i = 1, 2, ...\}$ to become rather ill-conditioned, while small values of b may render the subproblem (2.3.49) infeasible. The same remark applies to the alternative formulation (2.2.24) where a value of $\beta > 0$ needs to be selected. Through computer experiments we found both formulations led to filters with similar behavior and performance.

2.4 Numerical examples

In this section, the performance of the proposed robust state estimation method is illustrated via simulation results. Two numerical examples are given here, the first one is the same problem as that used in [21, 23], and the second one is a target tracking problem.

2.4.1 Example 1

In this example the following discrete-time linear uncertain state-space model is used

$$\begin{aligned} \underline{x}_{i+1} &= \begin{bmatrix} 0 & -0.5 \\ 1 & 1+\delta \end{bmatrix} \underline{x}_i + \begin{bmatrix} -6 \\ 1 \end{bmatrix} u_i, \quad |\delta| < 0.3, \\ y_i &= \begin{bmatrix} -100 & 10 \end{bmatrix} \underline{x}_i + v_i, \\ s_i &= \begin{bmatrix} 1 & 0 \end{bmatrix} \underline{x}_i, \end{aligned} \tag{2.4.50}$$

where u_i and v_i are uncorrelated zero-mean white noise signals with variances $\bar{Q} = 1$ and $\bar{R} = 1$, respectively. The value of ϵ in (2.2.7) is set to zero, so that there is no uncertainty in the knowledge of noise statistics. The uncertainty in (2.4.50) is described by the matrices

$$\mathbf{C}_1 = [0 \;\; 10]^T, \quad \mathbf{C}_2 = 0, \quad \mathbf{E} = [0 \;\; 0.03] \tag{2.4.51}$$

and the scalar parameter z, $|z| \le 1$.

To determine the robust filter at each instant i, we use the MATLAB toolbox SeDuMi [20] to solve the robust SDP (2.3.49). This code requires no initialization since it is based on the self-dual formulation of the SDP. Solving the SDP (2.3.49) at each instant i with $b = 900$ and $\mathbf{D} = \text{diag}(1, 5)$ yields a robust state estimator [of the form (2.2.9)] which converges to

$$\mathbf{A} = \begin{bmatrix} -0.1711 & -0.4624 \\ 1.4080 & 1.1786 \end{bmatrix} \quad \text{and} \quad \mathbf{K} = \begin{bmatrix} -0.0051 \\ 0.0047 \end{bmatrix}. \tag{2.4.52}$$

Note that for stability reasons the estimator weights (as seen in $\mathbf{D}$) the second component

Filter	$\delta = -0.3$	$\delta = 0$	$\delta = 0.3$
Nominal Kalman filter	551.2	36.0	8352.8
Nominal H^∞ filter	96.0	47.2	893.9
The robust filter of [21]	51.4	51.3	54.4
The robust filter of [23]	64.0	61.4	64.4
The robust filter of [11]	51.5	49.1	53.8
Proposed robust filter	46.2	45.6	51.9

Table 2.1: Steady-state estimation error variances for different filters.

of $\underline{x}_i$ more heavily than the first component even though the goal is to estimate the first component of $\underline{x}_i$. In our simulation studies, the proposed technique is compared with the Kalman and H^∞ filters and the robust filters of [21, 23]. For this purpose, steady-state Kalman and H^∞ filters are designed for the nominal system of (2.4.50), i.e., $\delta = 0$. We then apply these filters to system in (2.4.50) with $\delta = 0$, $\delta = 0.3$ and $\delta = -0.3$. The steady-state estimation error variances (i.e., $\mathcal{E}\{(s_i - \hat{s}_i)^2\}$ for sufficiently large i) for the filters are displayed in Table 2.1 where results are averaged over 100 runs. It is clear from the table that the proposed robust filter performs far better than the nominal Kalman and H^∞ filters in the presence of modelling error.

Both our method and the methods of [21,23] require the tuning of a certain parameter. In our case, we need to adjust the parameter b in order to prevent the iterates from becoming ill-conditioned, and the diagonal elements of $\mathbf{D}$ in order to get the best estimator. The methods of [21, 23] require the adjustment of the factor μ in the scaling of $\mathbf{C}_i/\mu$ and $\mu\mathbf{E}_i$. Our experiments suggest that our method works for $b \in [880, 5000]$, while the method of [21] converges for $\mu \in (0, 1.703]$, and diverges for values outside this range. The best performance is achieved with $\mu = 1.703$. (Note that the authors of [21] reported their choice of $\mu = 2.2$, but our own implementation of their method showed this value of μ leads to divergence instead.)

The filter performance for the robust filter of [23] stated in Table 2.1 is quoted from [21]. We should point out that we could not reproduce the design of the robust filter [21] using their design method. With our own (simple) Matlab implementation of their method, we could only produce a filter with ($\mu = 1.703$) whose error covariances are 51.4, 51.3 and 54.4,

Filter	$\delta = -0.09$	$\delta = 0$	$\delta = 0.09$
The robust filter of [11]	37.75	38.19	41.47
Proposed robust filter	37.38	37.78	40.31

Table 2.2: Steady-state estimation error variances.

rather than 46.6, 45.2 and 54.1 (as claimed in [21]) for model errors of $\delta = -0.3$, 0 and 0.3, respectively. From Table 2.1, we can see that the performance of the robust filters [21, 23] are inferior to the filter designed by the robust SDP method: the worst case performance (for $\delta = -0.3,\ 0,\ 0.3$) is 51.9 for our proposed robust filter, and is 54.2 and 64.4 respectively for the robust filters of [21] and [23]. From this example, it appears that our robust filter design is slightly superior.

Recently our approach has been further extended by Fu *et. al.* [11] who introduced multiple scaling factors in the SDP formulation and showed performance improvement when compared to the single scaling factor case. It should be pointed out that the single scaling factor case of [11] corresponds to the algorithm considered in this thesis, except that we have an additional boundedness constraint $\mathrm{Tr}(\Gamma_{i+1}) \leq b$ in our SDP subproblem (2.3.49). We simulated the single scaling factor case of [11] in Table 2.2 for comparison. From the simulation results, our method is slightly superior to the method of [11] in the single scaling factor case. This is due to the differences in the way the ill-conditioning of the bound on the covariance matrix is handled. The simulation results stated in [11] are for $\mathbf{C}_1 = [0\ \ 3]^T$ (instead of $\mathbf{C}_1 = [0\ \ 10]^T$), which means the simulated cases in [11] have only 30% of the uncertainty considered in Table 2.1. We also compared our method with the method of [11] for the case $\mathbf{C}_1 = [0\ \ 3]^T$, and the simulation results show that our method is still slightly superior to the method of [11] (Table 2.2).

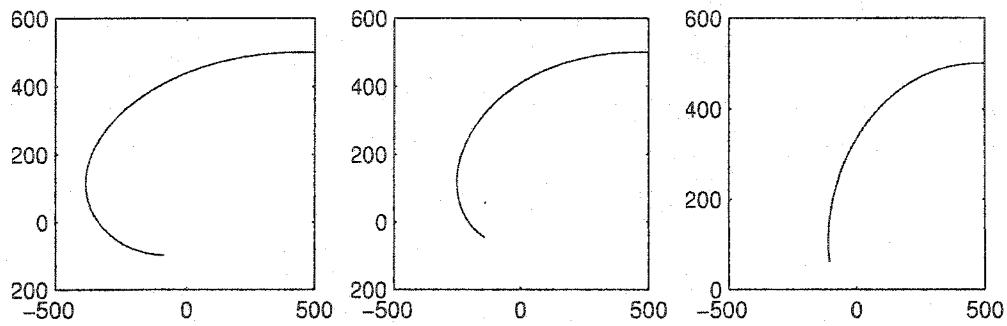

Figure 2.1: Target trajectories: $\delta = -0.05$ (left), $\delta = 0$ (middle), $\delta = 0.05$ (right).

2.4.2 A target tracking example

In this example a target tracking case is considered. The discrete-time state-space model is given by

$$\begin{aligned} \underline{x}_{i+1} &= \begin{bmatrix} 0.95 & -0.1+\delta \\ 0.05 & 0.95 \end{bmatrix} \underline{x}_i + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u_i, \quad |\delta| < 0.05, \\ y_i &= \begin{bmatrix} 1 & 0 \end{bmatrix} \underline{x}_i + v_i, \\ s_i &= \begin{bmatrix} 1 & 0 \end{bmatrix} \underline{x}_i, \end{aligned} \tag{2.4.53}$$

where u_i and v_i are uncorrelated zero-mean white noise signals with variances $\bar{Q} = 1$ and $\bar{R} = 1$, respectively. The value of ϵ in (2.2.7) is set to zero, so that there is no uncertainty in the knowledge of noise statistics. The uncertainty in (2.4.53) is described by the matrices

$$\mathbf{C}_1 = [0.05 \;\; 0]^T, \quad \mathbf{C}_2 = 0, \quad \mathbf{E} = [0 \;\; 1], \tag{2.4.54}$$

and the uncertainty parameter z, $|z| \leq 1$.

In this model, the state vector $\underline{x}_i$ represents the position of a target in a two-dimensional coordinate system, and the observation y_i is a noise-corrupted version of the first coordinate. The target is making a counter-clockwise turn starting from the position $\underline{x}_0 = [500, 500]^T$. The unknown parameter δ describes the uncertainty in the turning rate of the trajectory. Three possible trajectories from this model are shown in Figure 2.1.

Filter	$\delta = -0.05$	$\delta = 0$	$\delta = 0.05$
Nominal Kalman filter	6425.2	1.4	11404.0
The robust filter of [21]	199.7	53.6	703.5
The robust filter of [23]	1309.6	666.9	549.2
Proposed robust filter	187.9	52.8	693.4

Table 2.3: Steady-state estimation error variances for the tracking example.

Solving the SDP (2.3.49) for each value of i, with $b = 1100$ and $\mathbf{D} = \text{diag}(1, 7)$ yields a robust state estimator [of the form (2.2.9)] which converges to

$$\mathbf{A} = \begin{bmatrix} 0.9500 & -0.1016 \\ 0.0500 & 0.9644 \end{bmatrix} \quad \text{and} \quad \mathbf{K} = \begin{bmatrix} 0.7560 \\ 0.0130 \end{bmatrix}. \tag{2.4.55}$$

We have compared our method with the methods of [21,23], as well as the nominal Kalman filter. The is shown in Table 2.3 where results are averaged over 100 runs. From the simulation results, it appears that the filter designed by our method is superior to the filters obtained via the methods of [21,23]. In designing the filters by the methods of [21,23], we have adjusted their corresponding adjustable parameters (e.g., the parameter μ in the scaling of $\mathbf{C}_i/\mu$ and $\mu\mathbf{E}_i$), and picked the filters which generate the best performance guarantees. The method of [23] requires an additional parameter, denoted ϵ in [23] to be tuned. We tuned this parameter to a value of 10 in our implementation. Note that, in the presence of uncertainty, the nominal Kalman filter performs far worse than the robust filters, as expected.

We have also compared our robust filter design to the robust filters of [21,23] in higher dimensional cases. We found that the relative steady-state performance of these filters is similar to that in the above examples. From computational standpoint, our method is quite efficient as the semidefinite program solved at each instant has a fixed dimension and the interior point method used to solve it is fast. However, our method does incur a greater per-sample computational cost than methods based on *analytic* recursions, such as the Kalman filter and the robust Kalman filter in [21]. (The robust filter in [23] is a 'one-shot' filter which does not vary with i.) For example, on a 200MHz Pentium Pro workstation, using a general purpose SDP solver [20] under the MATLAB environment, the per-sample computation time of our method in the above examples was around 1s, where as that of the method in [21] was around 5ms. (Recall, however, that the performance of the method in [21] is "highly sensitive" to the parameter which must be tuned.) In future work, it will be useful to design special purpose interior point algorithms which exploit the matrix structure of the SDP in (2.3.49) to reduce the per-sample computational complexity of our new method. Such a reduction of computational complexity is essential if one is to implement the proposed robust filtering algorithm on a DSP (digital signal processing) chip for a real time filtering application.

2.5 Remarks

In this chapter, we proposed a new state estimator for linear uncertain systems. The method is robust to norm bounded parameter uncertainties on the system model as well as bounded uncertainties on the noise statistics. In the new technique, the estimation problem was formulated as a convex optimization problem, which is then solved using the recent primal-dual self-dual interior point method. This requires at most 8 iterations (or matrix inversions), and therefore, can be regarded as a recursive filtering method. The formulation guarantees the existence of robust solutions via a semidefinite program and, under some conditions, the solution to that semidefinite program is unique. The proposed technique compared favorably with the well known Kalman and H^∞ filters and the 'robust' filters of [21, 23]. When applied to the problem of target tracking, the new method has led to a significant improvement in tracking performance.

Chapter 3

Decision Fusion Models and Principles

3.1 Introduction to decision level identity fusion

As introduced in Chapter 1, identity information can be fused on three levels: data level, feature level and decision level. Among these levels, decision level identity fusion (decision fusion) fuses identity estimates from individual sensors to generate joint target identity declarations. Figure 3.1 illustrates the basic architecture of decision level identity fusion through a configuration of N sensors and a fusion center. Here, the N sensors make observations on a target. Each sensor performs feature extraction and identity estimation, and sends its resulting identity estimates on the target (called a sensor report) to the fusion center. The fusion center then performs decision level fusion on these sensor reports to develop the final joint identity declaration (called a fusion result).

Compared with fusion on the other two levels, decision level identity fusion has advantages such as requiring minimal data transmission between the fusion center and individual sensors, and less processing at the fusion center. In addition, it can fuse information from different types of sensors, which is impossible on the other two levels. Decision level identity fusion has been used in many applications including:

- multiple-sensor target detection,
- threat-warning system on tactical aircraft to identify targets,

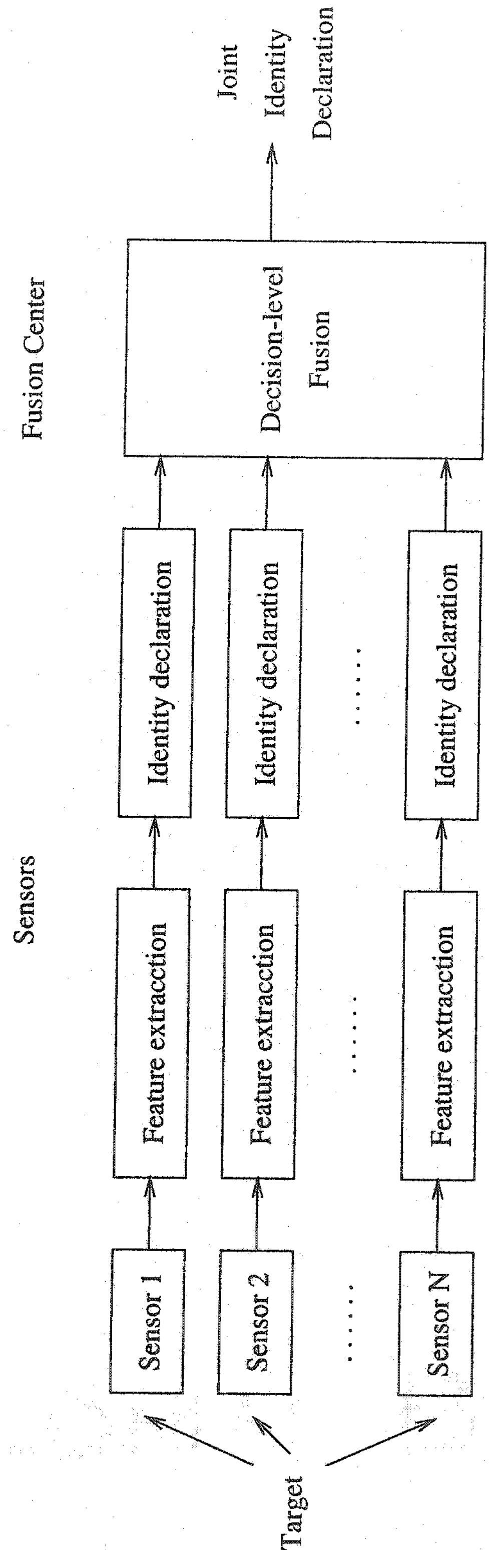

Figure 3.1: Architecture of decision level identity fusion.

- naval and aerial target identification system on battle ship,
- image processing for robotic vision.

The most widely used decision level identity fusion approaches in the literature are Bayesian inference method and Dempster-Shafer evidential inference method [30, 33]. The former is well developed in statistical decision theory, but requires the knowledge of *a priori* distribution of the possible identity propositions as well as the knowledge of conditional probabilities of all possible sensor reports. Such *a priori* knowledge is difficult, if not impossible, to obtain. In contrast, Dempster-Shafer evidential inference method does not require such *a priori* knowledge. However, it suffers from exponentially growing computational complexity. It is highly desirable to develop new fusion methods which can overcome the drawbacks of these existing methods.

A more important issue in identity fusion is the lack of common framework in which fusion methods can be evaluated and compared with each other. In our opinion, this is a major problem in the development of new identity fusion methods. A common framework for decision fusion may consist of some fusion models, each associated with a number of fusion principles characterizing the model. With such a framework, fusion methods can be categorized under fusion models, and their performance can be evaluated and compared using the fusion principles.

In the following we will formulate the decision fusion problem, after which detailed discussion on Bayesian inference method and Dempster-Shafer evidential inference method will be given in sections 3.2 and 3.3.

3.1.1 Decision fusion problem

Suppose there are a set of N exhaustive and mutually exclusive propositions for the target identity,

$$\Omega = \{a_1, \dots, a_N\}. \tag{3.1.1}$$

For example, these propositions may take the form

$$\begin{cases} a_1 = \text{the target is a friendly fighter,} \\ a_2 = \text{the target is a friendly bomber,} \\ a_3 = \text{the target is a hostile fighter,} \\ a_4 = \text{the target is a hostile bomber.} \end{cases} \tag{3.1.2}$$

A sensor report may consist of some subsets of Ω and associated likelihood values. In addition, a sensor report may be associated with a so-called Degree of Confidence (DoC) value if the sensors have varying degree of reliability. A DoC value is typically represented as a non-negative number. The objective of decision fusion is to generate a fusion result which assigns appropriate likelihood values to subsets of Ω based on available sensor reports.

For example, suppose $\Omega = \{a_1, a_2, a_3, a_4\}$, where the a_i's are given by (3.1.2). Also suppose that there are two sensors, each sending a sensor report to the fusion center. Let the report from sensor 1 (denoted by D_1 for "decision from sensor 1") be given by $D_1 = \{P_1(\Omega) = 0.3,\ P_1(a_1 \vee a_3) = 0.7\}$, where $P_1(a_1 \vee a_3)$ denotes the probability assigned to the disjunction of propositions a_1 and a_3, and $P_1(\Omega)$ denotes the probability assigned to the whole set which represents the uncertainty in report D_1. This report indicates that sensor 1 believes with 0.7 probability the target is a fighter, and with 0.3 probability the target identity is uncertain (can be any proposition in Ω). The report from sensor 2 is in the same probability form, given as $D_2 = \{P_2(\Omega) = 0.2,\ P_2(a_3 \vee a_4) = 0.8\}$. These two sensor reports can be fused by Dempster-Shafer evidential inference method (to be explained later in section 3.3), which yields the following fusion result:

$$D_f = \{P_f(\Omega) = 0.06,\ P_f(a_1 \vee a_3) = 0.14,\ P_f(a_3) = 0.56,\ P_f(a_3 \vee a_4) = 0.24\}. \tag{3.1.3}$$

Note that D_f consists of probability assignments to subsets of Ω. Here the suffix f denotes that the probability assignments are from the fusion result.

In this thesis, a sensor report is denoted by the notation D_i where i is a number. A probability assignment to subset ω in report D_i is denoted by $P_i(\omega)$. Similarly, a fusion result is denoted by D_{f_i}, and a probability assignment to subset ω in D_{f_i} is denoted by $P_{f_i}(\omega)$.

We also define two fusion modes as follows.

- **Sequential fusion mode**

  The fusion result is achieved by fusing the sensor reports in a sequential manner. Each time a new sensor report is fused with the current interim fusion result.

- **Batch fusion mode**

  For a group of sensor reports, their fusion result is obtained by fusing all the reports simultaneously.

The two fusion modes may be jointly used in decision fusion. For example, we can separate a group of sensor reports into some sub-groups, obtain intermediate fusion results for the sub-groups using the sequential fusion mode, then fuse these intermediate fusion results using the batch mode to reach the final fusion result.

Note that if a fusion method can operate in the batch fusion mode, it can also operate in the sequential fusion mode. However, there are fusion methods which can only operate in the sequential fusion mode because of their inability to fuse more than two sensor reports at the same time. Such methods cannot operate in the batch fusion mode.

Obviously, for the sequential mode, only a interim fusion result needs to be maintained for future fusion steps, while in the batch mode all sensor reports (or some functions keeping identity information in these reports) need to be kept. Consider a scenario where new sensor reports arrive in a dynamic fashion. The sequential fusion mode will fuse each new report one at a time with the current interim fusion result, while the batch mode will fuse all available sensor reports simultaneously. Intuitively, it seems that the sequential fusion mode has the advantage of requiring less memory and/or computation than the batch fusion model. However, in practice this may not be true, since keeping a interim fusion result may not be more memory efficient than keeping all the sensor reports. A good example is Dempster-Shafer evidential inference method (to be introduced in section 3.3) in which a interim fusion result may have more subsets in its representation than the total of all sensor reports and thus requires more memory to maintain. Moreover, replacing individual sensor reports with a interim fusion result aims to data compression which can result in the lose of important information on the history of the target and on the surveillance system. Secondly, from the computational point of view, it is not necessarily true that the sequential mode is more efficient than the batch mode since some fusion methods have the same computational complexity when they try to fuse two or more sensor reports. See for example, the convex quadratic fusion method to be introduced in Chapter 4.

For some fusion methods, the sequential fusion mode and the batch fusion mode give the same final fusion results. However, there are also fusion methods for which the two modes of operation will give different fusion results. We will see examples of such fusion methods in later chapters where we describe the differences of the two modes of operation for these fusion methods in some detail.

3.2 Bayesian inference method

The Bayesian inference method is based on the knowledge of *a priori* probability distribution $P_0(a_i)$ over Ω (defined in (3.1.1)), and the knowledge of conditional probabilities of the form:

$$P(D_k|a_i) = \textit{probability of receiving report } D_k \textit{ given the true proposition } a_i. \qquad (3.2.4)$$

These conditional probabilities are assumed to be known and constant (time invariant).

Bayesian inference method can operate in both the sequential fusion mode and the batch fusion mode (proposed in the previous section). In the sequential fusion mode, the new interim fusion result $D_f' = \{P_f'(a_i),\ \ i = 1, \dots, N\}$ is obtained by Bayes' rule:

$$P_f'(a_i) = \begin{cases} P_0(a_i), & \text{when no sensor report is available,} \\ \dfrac{P(D_k|a_i)P_f(a_i)}{P(D_k)}, & \text{when sensor report } D_k \text{ is fused,} \end{cases} \qquad (3.2.5)$$

where $P(D_k) = \sum_{j=1}^{N} P(D_k|a_j)P_f(a_j)$, and $D_f = \{P_f(a_i),\ \ i = 1, \dots, N\}$ is the interim fusion result before the fusion of sensor report D_k. In other words, Bayes' rule can be implemented recursively as new sensor reports arrive.

Alternatively, we can fuse the reports all at once using the batch fusion mode. Using this approach, for K sensor reports: $D_1, \ldots, D_K$, Bayesian inference method assigns the following probability to proposition a_i in the fusion result:

$$P(a_i|D_1,\ldots,D_K) = \frac{P_0(a_i)P(D_1|a_i)\cdots P(D_K|a_i)}{\sum_{j=1}^{N} P_0(a_j)P(D_1|a_j)\cdots P(D_K|a_j)}, \quad i=1,\ldots,N. \tag{3.2.6}$$

Here we assume that the sensor reports are conditionally independent with each other.

Bayesian inference method yields the same fusion result using the sequential fusion mode and the batch fusion mode, which can be verified by induction. First, Eqs. (3.2.6) and (3.2.5) are identical when $K=0$. Secondly, if this is true for K, then for $K+1$, Eq. (3.2.5) yields

$$\begin{aligned} P'_f(a_i) &= \frac{P(D_{K+1}|a_i)P_f(a_i)}{\sum_{j=1}^{N} P(D_{K+1}|a_j)P_f(a_j)} \\ &= \frac{P(D_{K+1}|a_i)\frac{P_0(a_i)P(D_1|a_i)\cdots P(D_K|a_i)}{\sum_{m=1}^{N} P_0(a_m)P(D_1|a_m)\cdots P(D_K|a_m)}}{\sum_{j=1}^{N} P(D_{K+1}|a_j)\frac{P_0(a_j)P(D_1|a_j)\cdots P(D_K|a_j)}{\sum_{m=1}^{N} P_0(a_m)P(D_1|a_m)\cdots P(D_K|a_m)}} \\ &= \frac{P_0(a_i)P(D_1|a_i)\cdots P(D_{K+1}|a_i)}{\sum_{j=1}^{N} P_0(a_j)P(D_1|a_j)\cdots P(D_{K+1}|a_j)}, \end{aligned} \tag{3.2.7}$$

which matches Eq. (3.2.6).

In practice, the *a priori* probability distribution $P_0(a_i)$ is determined based on the fusion center's past experience on the target identity. In the case that no past experience is available, $P_0(a_i)$'s should be chosen as uniform distribution.

As we can see, Bayes' rule requires the knowledge of prior probabilities and conditional probabilities (3.2.4). In practice these probabilities are usually estimated using past experience. As a result, they can sometimes be ad hoc and difficult to justify. Another limitation of Bayesian inference method is that it has no explicit mechanism to differentiate the qualities of individual sensor reports. This is unfortunate because in practical applications sensor report qualities are varied because sensors usually have different levels of reliability.

3.3 Dempster-Shafer evidential inference method

Another method of combining possibly conflicting information from various sources is to use Dempster-Shafer evidential inference theory [30,33]. This method has been shown to be effective in medical diagnosis, approximate reasoning and artificial intelligence.

Dempster-Shafer evidential inference method can assign a probability to any of the original N propositions in Ω or to a union of the propositions. For example, a union of a_1 and a_2 (denoted by $a_1 \vee a_2$) may be assigned a probability $P(a_1 \vee a_2)$. The probability of a subset of Ω (called a probability mass) is defined as the sum of the probabilities of individual propositions in the subset. Note that there are a total of $2^N - 1$ different possible subsets in Ω that may be assigned probabilities. Mathematically, the knowledge of sensor k on the target identities is summarized in its sensor report D_k, in the form of a probability mass $P_k(\omega_k^\ell)$, i.e.,

$$D_k = \{P_k(\omega_k^\ell), \quad \omega_k^\ell \subseteq \Omega, \ \ell = 1, \dots , L_k\}, \tag{3.3.8}$$

where $P_k(\omega_k^\ell) \in [0,1]$ and $\sum_{\ell=1}^{L_k} P_k(\omega_k^\ell) = 1$. Not all of the possible subsets must be assigned probabilities. The assigned probabilities must sum to unity. We emphasis here that in the Dempster-Shafer model a probability mass $P_k(\omega_k^\ell)$ generally is not the total probability of subset ω_k^ℓ, since some other probability mass $P_k(\omega_k^j)$ with $\omega_k^\ell \cap \omega_k^j \neq \emptyset$ ($\emptyset$ denotes the null set) will also contribute to the total probability of ω_k^ℓ. It should also be pointed out that this general form of representation is different from the Bayesian approach in which the probabilities are assigned only to individual propositions rather than subsets.

In the Dempster-Shafer model, the total probability 1 is assigned, part by part, to propositions or unions of propositions according to the available evidence. It is possible that due to the lack of evidence, the total probability 1 is not completely assigned. The reserved probability which is not assigned to any proposition or union of propositions, brings uncertainty to the corresponding sensor report. This probability, represented by $P(\Omega) = P(a_1 \vee \cdots \vee a_N)$, is therefore called the probability of uncertainty. $P(\Omega)$ has a inverse relationship with the Degree of Confidence (DoC): the bigger the $P(\Omega)$, the smaller the DoC. For example, when

$P(\Omega) = 0$ the corresponding sensor report has maximal DoC, since zero uncertainty in the sensor report implies complete reliability. In contrast, if $P(\Omega) = 1$, the corresponding sensor report has minimal DoC, in other words, nothing said in the report is reliable.

In Eq. (3.3.8), the DoC of sensor report D_k (represented by $P_k(\Omega)$) is not separated from the rest of the report. Specifically, if $P_k(\Omega)$ changes, the rest of D_k also changes no matter whether the identity information contained in D_k changes or not. As a remedy, we define

$$P'_k(\omega_k^\ell) = \frac{P_k(\omega_k^\ell)}{1 - P_k(\Omega)}, \quad \forall\, \omega_k^\ell \neq \Omega. \tag{3.3.9}$$

This is a normalization process, therefore $\sum_{\ell:\ \omega_k^\ell \neq \Omega} P'_k(\omega_k^\ell) = 1$ and the $P'_k(\omega_k^\ell)$'s remain constant when $P_k(\Omega)$ changes. We believe that the $P'_k(\omega_k^\ell)$'s contain all the identity information in report D_k, while $P_k(\Omega)$ represents the DoC of the report. For a better understanding, let us look at two subsets ω_k^i and ω_k^j in report D_k. The identity information concerning *only* these two subsets is represented in their relative likelihood ratio: $P_k(\omega_k^i) : P_k(\omega_k^j)$, which should not change with $P_k(\Omega)$. Because $P'_k(\omega_k^i) : P'_k(\omega_k^j) = P_k(\omega_k^i) : P_k(\omega_k^j)\ \forall\ i, j$, the identity information concerning only ω_k^i and ω_k^j is also contained in the normalized probabilities $P'_k(\omega_k^\ell)$'s. Moreover, the $P'_k(\omega_k^\ell)$'s do not contain any DoC information because they do not change with $P_k(\Omega)$.

Dempster-Shafer evidential inference method has two useful concepts: support $spt(\omega^i)$ and plausibility $pls(\omega^i)$, for any subset $\omega^i \subseteq \Omega$. They are defined as

$$\begin{cases} spt(\omega^i) = \sum_{\omega \subseteq \omega^i} P(\omega), \\ pls(\omega^i) = 1 - \sum_{\omega \cap \omega^i = \emptyset} P(\omega). \end{cases} \tag{3.3.10}$$

It can be seen that $spt(\omega^i) \leq pls(\omega^i)$ for all $\omega^i \subseteq \Omega$. $spt(\omega^i)$ and $pls(\omega^i)$ serve as lower and upper bounds for the total probability of ω^i respectively.

Similar to Bayesian inference method, Dempster-Shafer evidential inference method can operate in both the sequential fusion mode and the batch fusion mode. In the sequential fusion mode, to obtain the fusion result $D'_f = \{P'_f(\omega^\ell),\ \ \ell = 1, \ldots, L'_f\}$ of sensor report D_k

and interim fusion result D_f, we first compute

$$\mu(\omega^\ell) = \sum_{\ell_f, \ell_k:\ \omega_f^{\ell_f} \cap \omega_k^{\ell_k} = \omega^\ell} P_f(\omega_f^{\ell_f}) P_k(\omega_k^{\ell_k}), \tag{3.3.11}$$

and then perform the following normalization to obtain probabilities for all $\omega^\ell \neq \emptyset$ ($\emptyset$ denotes the null set):

$$P_f'(\omega^\ell) = \frac{\mu(\omega^\ell)}{1 - \mu(\emptyset)}. \tag{3.3.12}$$

Here, $D_f = \{P_f(\omega_f^{\ell_f}),\ \ell_f = 1, \ldots, L_f\}$ is the interim fusion result before the fusion of sensor report D_k. In other words, the fusion result is updated recursively as new sensor reports arrive.

In the batch fusion mode, K sensor reports $D_1, \ldots, D_k$ are fused all at once. Let us denote the final fusion result for the reports as $D_{f_K} = \{P_{f_K}(\omega_{f_K}^{\ell_{f_K}}),\ \ell_{f_K} = 1, \ldots, L_{f_K}\}$. Then,

$$P_{f_K}(\omega_{f_K}^{\ell_{f_K}}) = \frac{\mu_{f_K}(\omega_{f_K}^{\ell_{f_K}})}{1 - \mu_{f_K}(\emptyset)}, \tag{3.3.13}$$

where

$$\mu_{f_K}(\omega_{f_K}^{\ell_{f_K}}) = \sum_{\substack{\ell_1, \ldots, \ell_K : \\ \omega_1^{\ell_1} \cap \cdots \cap \omega_K^{\ell_K} = \omega_{f_K}^{\ell_{f_K}}}} P_1(\omega_1^{\ell_1}) \cdots P_K(\omega_K^{\ell_K}). \tag{3.3.14}$$

We now prove that the sequential fusion mode gives the same fusion result with the batch fusion mode regardless of the order in which the sensor reports are fused. First of all, when $K = 1$, both Eqs. (3.3.11) and (3.3.14) yields $\mu_{f_1}(\omega_1^{\ell_1}) = P_1(\omega_1^{\ell_1})$. In other words, D_{f_1} is identical to D_1. Next, suppose the fusion result (given by Eqs. (3.3.13) and (3.3.14)) matches that given by the batch fusion mode for some $K > 1$. Then, to fuse a new sensor report

D_{K+1}, we use Eq. (3.3.11) to compute

$$\mu_{f_{K+1}}(\omega_{f_{K+1}}^{\ell_{f_{K+1}}}) = \sum_{\substack{\ell_{f_K},\ell_{K+1}: \\ \omega_{f_K}^{\ell_{f_K}} \cap \omega_{K+1}^{\ell_{K+1}} = \omega_{f_{K+1}}^{\ell_{f_{K+1}}}}} P_{f_K}(\omega_{f_K}^{\ell_{f_K}}) P_{K+1}(\omega_{K+1}^{\ell_{K+1}})$$

$$= \sum_{\substack{\ell_{f_K},\ell_{K+1}: \\ (\omega_1^{\ell_1} \cap \cdots \cap \omega_K^{\ell_K}) \cap \omega_{K+1}^{\ell_{K+1}} = \omega_{f_{K+1}}^{\ell_{f_{K+1}}}}} \left(\sum_{\substack{\ell_1,\ldots,\ell_K: \\ \omega_1^{\ell_1} \cap \cdots \cap \omega_K^{\ell_K} = \omega_{f_K}^{\ell_{f_K}}}} P_1(\omega_1^{\ell_1}) \cdots P_K(\omega_K^{\ell_K}) \right) P_{K+1}(\omega_{K+1}^{\ell_{K+1}})$$

$$= \sum_{\substack{\ell_1,\ldots,\ell_{K+1}: \\ \omega_1^{\ell_1} \cap \cdots \cap \omega_{K+1}^{\ell_{K+1}} = \omega_{f_{K+1}}^{\ell_{f_{K+1}}}}} P_1(\omega_1^{\ell_1}) \cdots P_{K+1}(\omega_{K+1}^{\ell_{K+1}}), \tag{3.3.15}$$

which establishes Eq. (3.3.14) for $K+1$. This completes the induction proof.

Since there are exponentially $(2^N - 1)$ many subsets of Ω, the above fusion rule suffers from exponentially growing complexity as N and K increases. For large values of N and K, the computation may be difficult to perform in real time. The numerical example in subsection 3.1.1 illustrates how Dempster-Shafer evidential inference method fuses identity declarations from two sensors.

3.4 Decision level identity fusion models

In this section we propose two decision level identity fusion models: the Similar Sensor Fusion (SSF) model and the Dissimilar Sensor Fusion (DSF) model. We first illustrate the basic concepts of these models through two simple examples.

Example 3.4.1 *Suppose a target is either a fighter aircraft or a bomber aircraft, i.e., $\Omega = \{a_1, a_2\}$ with*

$$\begin{cases} a_1 = \textit{the target is a bomber aircraft,} \\ a_2 = \textit{the target is a fighter aircraft.} \end{cases} \tag{3.4.16}$$

Also suppose there are three independent sensors: a radar, an Infrared (IR) sensor and an Electronic Supporting Measurement (ESM) sensor. Each sensor sends a report to the fusion center.

The radar explores size, speed and maneuvering pattern characteristics on the target. It observes that the target has a large size, a low speed, and few maneuverings. All these characteristics favor bombers more than fighters. As a result, the radar sends the following report to the fusion center assigning more probability to a_1 *than to* a_2*:* $D_1 = \{P_1(a_1) = 0.70,\ P_1(a_2) = 0.30\}$*. In comparison, the IR sensor identifies the target according to its IR appearance. Because the IR appearance of the target looks more like a bomber than a fighter, the IR sensor sends a report similar to the one from the radar. The report is given as* $D_2 = \{P_2(a_1) = 0.65,\ P_2(a_2) = 0.35\}$*. Finally, the ESM sensor classifies radar signals emitted from the target and finds that the target has a radar signature more commonly used by bombers than by fighters. As a result, the ESM sensor assigns more probability to* a_1 *with the following report:* $D_3 = \{P_3(a_1) = 0.75,\ P_3(a_2) = 0.25\}$*.*

To obtain a reasonable fusion result, we take into consideration that these sensors all favor proposition a_1*. Moreover, each sensor explores different physical characteristics of the target, their reports provide independent confirmation of the target identity. Such cross validation of sensor reports should lead to increased assurance that the target is a bomber other than a fighter. This increased assurance is represented by a high probability assigned to* a_1 *in the fusion result. For example, using Bayesian inference method and assuming an equal* a priori *probability distribution, we obtain the following fusion result:* $D_f = \{P_f(a_1) = 0.938,\ P_f(a_2) = 0.062\}$*. Note that proposition* a_1 *has a much larger probability assigned to it in the fusion result than in any of the sensor reports.*

Example 3.4.2 *Suppose the target proposition set* Ω *is the same as in Example 3.4.1. However, instead of having sensors of different types, here we have three identical and independent radars. This scenario happens, for example, when three battle ships of the same type form a fleet and their identity declarations on a target are fused.*

For comparison purposes, we assume that the three sensor reports sent to the fusion center by these radars are identical to the reports D_1*,* D_2 *and* D_3 *in Example 3.4.1 respectively. All* *three radars explore common physical characteristics of the target: size, speed and maneuvering pattern. The radars observe that the target has big size, low speed and few maneuverings, which result in a higher probability assigned to* a_1 *than to* a_2 *in their reports.*

Although the three sensor reports are the same as in the last example, they are based on common physical characteristics of the target rather than the different characteristics explored in Example 3.4.1. As a result, these reports should agree with each other except for possible noise corruption, human factors, weather conditions and so on. An appropriate fusion result for this case should be the one that is most consistent with all the sensor reports, e.g., $D_f = \{P_f(a_1) = 0.70,\ P_f(a_2) = 0.30\}$. *Note that* $P_f(a_1)$ *is within the range spanned by* $P_1(a_1)$, $P_2(a_1)$ *and* $P_3(a_1)$.

The above examples illustrate that the certainty in target identity in the fusion result should be based on the fusion center's knowledge on available characteristics attributes of the target. The more characteristics attributes explored, the more certainty can be built up at the fusion center. In contrast, sensor reports on common characteristics do not increase the certainty in target identity at the fusion center. They only help to eliminate inconsistencies among reports.

In fact, this idea is used everyday by human brains in their fusion process. For example, we identify a friend (Bob) by characteristics such as sound, appearance, clothes and movement. When all the information is consistent with the characteristics of Bob, we can identify Bob with more assurance. However, it is not easy to identity Bob just by talking to him over the phone since in this case only the sound characteristic is available. Moreover, having more people to listen to the person on the phone generally does not help much. This is a typical fusion example with identical sensors.

Based on the above discussion, we propose two decision level identity fusion models as follows.

- **Dissimilar Sensor Fusion (DSF) model**

  In this fusion model, dissimilar and independent sensors explore different characteristics of a target. Reports from these sensors can reinforce each other to decrease the uncertainty in target identity. The fusion objective of this model is to find a fusion *result which best represents the increased certainty in target identity from the sensor reports.*

- **Similar Sensor Fusion (SSF) model**

  In this fusion model, independent sensors are similar to each other, and they explore a set of common characteristics of a target. The sensors can only confirm each other's reports instead of decrease the uncertainty in target identity. The fusion objective of this model is to find a fusion result which is most consistent with all the sensor reports.

Obviously, the above Example 3.4.1 belongs to the DSF model, while Example 3.4.2 belongs to the SSF model.

Note that we use "similar/dissimilar sensors" instead of "same/different type sensors". This is because that two sensors of the same type may not explore the same target characteristics. For example, two radars of different frequencies may explore quite different target characteristics because a target may have different reflection patterns on the two frequencies. In comparison, two sensors of different types may explore the same target characteristics. For example, an IR sensor and an imaging radar may all provide information on target size and front appearance. As a result of these considerations, we classify sensors by the target characteristics they explore instead of the types they belong to. Sensors exploring a set of common characteristics are classified as similar sensors, while sensors exploring different target characteristics are classified as dissimilar sensors.

The definition of the two fusion models are fundamental to the research of decision level identity fusion methods because for different models the fusion goals are completely different. A fusion method which provides appropriate fusion results under one model may be useless under the other model. In the following section 3.6, we define some decision fusion principles which further reveal the commonalities and differences of these two models.

It should be pointed out that these two models should be viewed as "extreme cases" of decision level identity fusion. There are many practical cases in which the sensors are neither completely similar nor completely dissimilar. These cases are not discussed in this thesis and will be left as a topic of future research.

3.5 Degree of Confidence (DoC)

In decision level identity fusion, Degree of Confidence (DoC) is an important concept. DoC is usually a single positive value associated with a sensor report. It represents the extent to which this sensor report can be trusted, i.e., how accurate the report is. For example, in Dempster-Shafer evidential inference model, DoC can be defined in terms of $P(\Omega)$ (called the probability of uncertainty). Since fusion of inaccurate sensor reports naturally generates inaccurate fusion results, a fusion result should also have an associated DoC level.

Although DoC can be expressed in terms of probability assignment, it should not be confused with the probability assignments in the corresponding sensor report. The reason is as follows. In decision level identity fusion, each sensor explores certain physical characteristics of a target. On the one hand, because physical characteristics are usually associated with more than one subset of Ω, identity declarations in a sensor report are associated with probabilities to represent the likelihood of occurrence. On the other hand, factors such as weather and human factor reduce the accuracy of sensor reports. Such inaccuracy is not based on any physical characteristics, and as a result does not help to identify the target. In fact, it generates inaccuracy in the fusion result if the corresponding sensor report is fused. It is this inaccuracy that DoC attempts to model.

Let DoC_{min} and DoC_{max} denote the minimal and maximal DoC values respectively. If a sensor report has its DoC value equal to DoC_{min}, it is completely unreliable (inaccurate). The fusion result should not change after the fusion of this report, i.e., the sensor report should be ignored by the fusion process. In comparison, if a sensor report has its DoC equals to DoC_{max}, the report is completely reliable. In other words, the report is fully accurate based on the target physical characteristics it explores. In the SSF model, sensors explore common target characteristics, their reports confirm each other on the target identity. If a sensor report D_1 has DoC_{max} as its DoC value, the fusion result D_f should agree with it. That is, for any subsets ω_1^i and ω_1^j in D_1, $P_1(\omega_1^i) : P_1(\omega_1^j) = P_f(\omega_1^i) : P_f(\omega_1^j)$. In the DSF model, sensors explore different target characteristics, their reports reinforce each other to decrease the uncertainty in target identity. As a result, if a sensor report D_1 has maximal DoC value, it has maximal reinforcement effort on the fusion result. However, the fusion result in general does not agree with D_1.

It should be pointed out that Bayesian inference method does not have any mechanism to represent DoC. Not surprisingly, this is viewed as a major drawback of the method.

3.6 Decision fusion principles

In this section we continue our discussion on the decision fusion models by summarizing the mathematical characteristics of the two models in terms of some decision fusion principles. These principles are essential to a decision fusion model and should be satisfied by all decision fusion methods.

Ideally, to ensure that a fusion method is applicable to a decision fusion model, we should test it for all possible fusion scenarios using either experimental data or data generated from appropriate physical models. However, due to the large number of decision fusion applications and the difficulty in obtaining physical models for some of these applications, this evaluation approach is impractical. As a remedy, we propose to use the aforementioned decision fusion principles to benchmark all the existing and future decision fusion methods. As will be seen later, the proposed decision fusion principles are reasonably broad so that we can use this benchmarking process to ensure the appropriateness of decision fusion methods.

It should be pointed out that if a fusion method fails to satisfy a fundamental principle, the method may still be used in practice. However, this failure indicates that this method has some shortage in the characteristics summarized by the failed principle, and the extent of the shortage depends on how the method fails the principle.

3.6.1 Fundamental principle 1: fusion result consistency with single sensor report

This principle considers the situation in which only one sensor report is available. In addition, the fusion center has no past experience on the target identity. In other words, the *a priori* probabilities on the target identity, if any, are uniformly distributed. As a result, all the information contained in the fusion result comes from this single report, implying that the fusion result should have its probability assignments agree with those in the single report. We define this property as follows.

Fundamental principle 1: fusion result consistency with single sensor report

Consider the case where there is no a priori knowledge on the target or the prior is uniformly distributed. Suppose there is only one sensor report D_1 *to fuse, and let* D_f *denote the fusion result. For any subset* ω_1^i *of* Ω *in* D_1*, let* $P_1(\omega_1^i)$ *and* $P_f(\omega_1^i)$ *denote the probabilities assigned to* ω_1^i *in* D_1 *and* D_f *respectively. Then,* $P_f(\omega_1^i) = P_1(\omega_1^i)$*. In addition, if* D_1 *is associated with DoC, then* D_f *has the same DoC value. In other words, the fusion result should be consistent with the single report.*

3.6.2 Fundamental principle 2: fusion result consistency with different fusion orders

This principle considers the case in which a number of sensor reports $D_1, \ldots, D_K$ are fused in different orders. For example, using a fusion method in the sequential mode, we can fuse D_1 with D_2 to obtain a interim fusion result which is subsequently fused with D_3, and so on until all reports are fused. Alternatively, we can use the inverse order which begins with D_K and finishes with D_1. Similarly, in the batch mode the indexing of the sensor reports can be permuted. However, no matter what order we use to fuse the sensor reports in the sequential mode, or what indexing is used in the batch mode, the final fusion result should always be the same. The principle is defined as follows.

Fundamental principle 2: fusion result consistency with different fusion orders

Suppose there are a number of sensor reports to fuse. The fusion result should remain the same when the indexing of sensor reports is arbitrarily permuted.

The principle is illustrated in the following fusion example using Dempster-Shafer evidential inference method (discussed in section 3.3).

Example 3.6.1 *Let $\Omega = \{a_1, a_2, a_3\}$. Suppose there are three sensor reports:*

$$\begin{aligned} D_1 &= \{P_1(a_1) = 0.4,\ P_1(a_2) = 0.3,\ P_1(a_3) = 0.3\}, \\ D_2 &= \{P_2(a_1 \vee a_2) = 0.6,\ P_2(a_3) = 0.4\}, \\ D_3 &= \{P_3(a_1 \vee a_2) = 0.8,\ P_3(a_3) = 0.2\}. \end{aligned} \tag{3.6.17}$$

We can first fuse D_1 with D_2, which gives $D_{f_{12}} = \{P_{f_{12}}(a_1) = 0.4,\ P_{f_{12}}(a_2) = 0.3,\ P_{f_{12}}(a_3) = 0.3\}$, then fuse the result $D_{f_{12}}$ with D_3 to generate the final fusion result $D_{f_A} = \{P_{f_A}(a_1) = 0.53,\ P_{f_A}(a_2) = 0.40,\ P_{f_A}(a_3) = 0.07\}$. Alternatively, we can first fuse D_2 with D_3 to obtain $D_{f_{23}} = \{P_{f_{23}}(a_1 \vee a2) = 0.86,\ P_{f_{23}}(a_3) = 0.14\}$, then fuse $D_{f_{23}}$ with D_1 to generate the following final fusion result $D_{f_B} = \{P_{f_B}(a_1) = 0.53,\ P_{f_B}(a_2) = 0.40,\ P_{f_B}(a_3) = 0.07\}$. Note that D_{f_A} is identical to D_{f_B}. In other words, the fusion result of these reports is not changed for the two fusion orders. It will be shown later in section 3.8 that Dempster-Shafer evidential inference method always satisfies the fundamental principle 2.

3.6.3 Fundamental principle 3: unbiasedness

First we define the notion of Cyclic sensor reports.

Definition 3.6.1 Cyclic sensor reports

Let $\Omega = \{a_1, \ldots, a_N\}$. Suppose there is no a priori knowledge on the target identity, or the prior is uniformly distributed. Suppose there are N sensor reports $D_1, \ldots, D_N$ given as

$$\begin{array}{llllll} D_1 = \{P_1(a_1) = r_1, & P_1(a_2) = r_2, & P_1(a_3) = r_3, & \ldots, & P_1(a_N) = r_N & \}, \\ D_2 = \{P_2(a_1) = r_N, & P_2(a_2) = r_1, & P_2(a_3) = r_2, & \ldots, & P_2(a_N) = r_{N-1} & \}, \\ D_3 = \{P_3(a_1) = r_{N-1}, & P_3(a_2) = r_N, & P_3(a_3) = r_1, & \ldots, & P_3(a_N) = r_{N-2} & \}, \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ D_N = \{P_N(a_1) = r_2, & P_N(a_2) = r_3, & P_N(a_3) = r_4, & \ldots, & P_N(a_N) = r_1 & \}, \end{array} \tag{3.6.18}$$

where $0 \leq r_1, \ldots, r_N \leq 1$ and $\sum_{i=1}^{N} r_i \leq 1$. In addition, these sensor reports are associated with the same Degree of Confidence value. We say that these sensor reports are cyclic, based on the fact that they form a cyclic pattern.

For example, let $\Omega = \{a_1, a_2\}$. The following two reports form a group of cyclic sensor reports:

$$D_1 = \{P_1(a_1) = 0.7,\ P_1(a_2) = 0.3\},\quad D_2 = \{P_2(a_1) = 0.3,\ P_2(a_2) = 0.7\}. \tag{3.6.19}$$

Although very rare in real fusion scenarios, cyclic sensor reports do exist. The importance of these sensor reports lies in their cyclic structure. This unique structure makes each proposition in Ω equally supported. In other words, all the propositions should have the same probability assignments in the fusion result. This is summarized in the following fundamental principle.

Fundamental principle 3: unbiasedness

Let $\Omega = \{a_1, \ldots, a_N\}$. Suppose there are a group of cyclic reports from independent sensors as defined in Definition 3.6.1. The fusion result of these reports should be

$$D_f = \{P_f(a_i) = r_{f0},\quad i = 1, \ldots, N\}, \tag{3.6.20}$$

where $r_{f0} \in [0, 1]$ is a constant. In other words, the fusion result should be unbiased when reports are cyclic.

Satisfaction of the above fundamental principle ensures that there is no inherent bias in the fusion method.

3.6.4 Fundamental principle 4: Sensor report reinforcement

The three fundamental principles introduced so far are applicable to both the DSF and SSF fusion models. They reveal some commonalities of the two fusion models. In contrast, the following two fundamental principles explore the differences between the two models. As a result, these two principles have different definitions for the two models.

As introduced in section 3.4, sensor reports in the DSF model are fused in a manner that leads to decreased uncertainty in target identity, while in the SSF model sensor reports are averaged to eliminate inconsistencies among them. Such characteristics are expressed mathematically in the so-called sensor report reinforcement principle which explores the influence of a sensor report on the fusion result. The influence is revealed by comparing fusion results before and after the fusion of the sensor report.

We first define the sensor report reinforcement principle for the DSF model as follows.

Fundamental principle 4 for DSF model: Sensor report reinforcement

Consider the Dissimilar Sensor Fusion model. Let $\Omega = \{a_1, \dots, a_N\}$ *and let* D_1 *denote a sensor report with DoC value* $w_1 \in [DoC_{min}, DoC_{max}]$. *Further denote the fusion result before and after the fusion of* D_1 *as* D_{f_1} *and* D_{f_2} *respectively. Also, for any subset* $\omega_1^i \subset \Omega$, *we denote the total probabilities assigned to* ω_1^i *in* D_{f_1} *and* D_{f_2} *as* $P_{f_1}(\omega_1^i)$ *and* $P_{f_2}(\omega_1^i)$ *respectively. Let* D_1 *be given as* $D_1 = \{P_1(\omega_1^\ell), \quad \ell = 1, \dots, L_1\}$, *where* $\omega_1^i \cap \omega_1^j = \emptyset$ *for* $i \neq j$. *In other words, all the subsets are mutually exclusive. Suppose that in* D_1,

$$P_1(\omega_1^1) > P_1(\omega_1^\ell), \quad \forall\, \ell \neq 1, \tag{3.6.21}$$

then in the fusion result,

$$\begin{aligned} P_{f_2}(\omega_1^1) &= P_{f_1}(\omega_1^1), \quad w_1 = DoC_{min}, \\ P_{f_2}(\omega_1^1) &> P_{f_1}(\omega_1^1), \quad w_1 > DoC_{min}. \end{aligned} \tag{3.6.22}$$

Moreover, if w_1 *increases,* $P_{f_2}(\omega_1^1)$ *also increases.*

This principle indicates that if in sensor report D_1, subset ω_1^1 is the most likely subset, then after the fusion of D_1 the probability of ω_1^1 increases. Moreover, when the DoC value of D_1 increases, $P_{f_2}(\omega_1^1)$ also increases, showing that the impact of D_1 on the fusion result increases.

The above principle only concerns the most likely subset ω_1^1 in report D_1. The idea can be extended to all subsets in D_1, which results in the following restricted reinforcement principle.

Restricted fundamental principle 4 for DSF model: Sensor report reinforcement

Consider the Dissimilar Sensor Fusion model. Let $\Omega = \{a_1, \dots, a_N\}$ *and let* D_1 *denote a sensor report with DoC value* $w_1 \in [DoC_{min}, DoC_{max}]$. *Further denote the fusion result before and after the fusion of* D_1 *as* D_{f_1} *and* D_{f_2} *respectively. Also, for any subset* $\omega_1^i \subset \Omega$, *we denote the total probabilities assigned to* ω_1^i *in* D_{f_1} *and* D_{f_2} *as* $P_{f_1}(\omega_1^i)$ *and* $P_{f_2}(\omega_1^i)$ *respectively.*

Suppose in D_1, there are two mutually exclusive subsets ω_1^i and ω_1^j, each of them is also mutually exclusive with other subsets in D_1, i.e.,

$$\omega_1^i \cap \omega_1^\ell = \emptyset,\ \forall\ \ell \neq i \quad and \quad \omega_1^j \cap \omega_1^\ell = \emptyset,\ \forall\ \ell \neq j. \tag{3.6.23}$$

Also suppose that ω_1^i and ω_1^j are assigned probabilities $P_1(\omega_1^i)$ and $P_1(\omega_1^j)$ respectively. These two probabilities are assigned to ω_1^i and ω_1^j directly, not to their subsets. Then,

$$\frac{P_{f_2}(\omega_1^i)}{P_{f_2}(\omega_1^j)} = \frac{P_{f_1}(\omega_1^i)}{P_{f_1}(\omega_1^j)}, \quad w_1 = DoC_{min}, \tag{3.6.24}$$

and

$$\left.\begin{array}{l} P_1(\omega_1^i) > P_1(\omega_1^j) \ \Rightarrow\ \dfrac{P_{f_2}(\omega_1^i)}{P_{f_2}(\omega_1^j)} > \dfrac{P_{f_1}(\omega_1^i)}{P_{f_1}(\omega_1^j)}, \\ P_1(\omega_1^i) < P_1(\omega_1^j) \ \Rightarrow\ \dfrac{P_{f_2}(\omega_1^i)}{P_{f_2}(\omega_1^j)} < \dfrac{P_{f_1}(\omega_1^i)}{P_{f_1}(\omega_1^j)}, \end{array}\right. \quad w_1 > DoC_{min}. \tag{3.6.25}$$

Moreover, if D_1 changes so that $|P_1(\omega_1^i) - P_1(\omega_1^j)|$ or w_1 increases, $\left|\frac{P_{f_2}(\omega_1^i)}{P_{f_2}(\omega_1^j)} - \frac{P_{f_1}(\omega_1^i)}{P_{f_1}(\omega_1^j)}\right|$ also increases.

Note that the restricted principle is a special case of the fundamental principle 4 for DSF model.

In comparison with the fundamental principle 4 for DSF model, the above restricted principle requires that if in sensor report D_1, subset ω_1^i is more (or less) likely than ω_1^j, then the fusion of D_1 with D_{f_1} increases (or decreases) the relative likelihood ratio of ω_1^i to ω_1^j. This change in relative likelihood ratio reflects the impact of D_1. Moreover, when the DoC value of D_1 increases, the impact of D_1 on the fusion result also increases.

Although it is desirable for a fusion result to satisfy the above restricted principle, it may become a great restriction on the fusion method to request the relative likelihood relationships (3.6.25) to be satisfied between any pair of subsets. In practice, satisfaction of this restriction is not absolutely necessary, since it does not matter if (3.6.25) is violated between two subsets with trivial probability assignments. As a result, satisfaction of the fundamental principle 4 for DSF model is more important than satisfaction of the above restricted principle.

Next, we define the reinforcement principle for the SSF model.

Fundamental principle 4 for SSF model: Sensor report reinforcement

Consider the Similar Sensor Fusion model. Let $\Omega = \{a_1, \ldots, a_N\}$ *and let* D_1 *denote a sensor report with DoC value* $w_1 \in [DoC_{min}, DoC_{max}]$. *Further denote the fusion results before and after the fusion of* D_1 *as* D_{f_1} *and* D_{f_2} *respectively. For any subset* $\omega_1^i \subset \Omega$, *we denote the total probabilities assigned to* ω_1^i *in* D_1, D_{f_1} *and* D_{f_2} *as* $P_1(\omega_1^i)$, $P_{f_1}(\omega_1^i)$ *and* $P_{f_2}(\omega_1^i)$ *respectively. Then,*

$$\begin{aligned}
&dist(D_{f_2}, D_{f_1}) = 0, && w_1 = DoC_{min}, \\
&dist(D_1, D_{f_1}) \geq dist(D_1, D_{f_2}), && DoC_{max} > w_1 > DoC_{min}, \\
&dist(D_1, D_{f_2}) = 0, && w_1 = DoC_{max},
\end{aligned} \tag{3.6.26}$$

for some properly chosen distance measure $dist(\cdot,\cdot)$. *Moreover, if* w_1 *increases,* $dist(D_1, D_{f_2})$ *decreases.*

The distance measure $dist(\cdot,\cdot)$ used in the above principle should satisfy the following properties:

1. $dist(D_1, D_2)$ is a continuous function of $P_1(\omega_1^{\ell_1})$'s and $P_1(\omega_2^{\ell_2})$'s.
2. $dist(D_1, D_2) \geq 0$, and $dist(D_1, D_2) = 0$ if and only if D_1 and D_2 are identical with each other.
3. $dist(D_1, D_2)$ is a convex function of $P_1(\omega_1^{\ell_1})$'s and $P_1(\omega_2^{\ell_2})$'s.
4. $dist(D_1, D_2)$ is independent of the DoC values of D_1 and D_2 (if exist).

See here we do not insist on the symmetry condition, i.e., $dist(D_1, D_2) = dist(D_2, D_1)$. Such a condition appears to be restrictive and unnecessary. For example the well-known Kullback-Leibler's measure of cross-entropy (K-L measure) does not satisfy the symmetry condition.

The above principle can be rationalized as follows. For the SSF model the fusion result should have minimal inconsistencies with sensor reports. Therefore when a new sensor report is fused, the current fusion result should "move" towards this new report to minimize inconsistencies. When w_1 increases, D_1 is more reliable and accurate, and the fusion result "moves" closer to D_1. Eventually, the fusion result completely "agrees" with D_1 when w_1 reaches its maximal value.

The above principle uses some properly chosen distance measure for the differences between sensor reports and the fusion result. If we use probability to measure these differences, we have the following restricted principle.

Restricted fundamental principle 4 for SSF model: Sensor report reinforcement

Consider the Similar Sensor Fusion model. Let $\Omega = \{a_1, \ldots, a_N\}$ and let D_1 denote a sensor report with DoC value $w_1 \in [DoC_{min}, DoC_{max}]$. Further denote the fusion results before and after the fusion of D_1 as D_{f_1} and D_{f_2} respectively. For any subset $\omega_1^i \subset \Omega$, we denote the total probabilities assigned to ω_1^i in D_1, D_{f_1} and D_{f_2} as $P_1(\omega_1^i)$, $P_{f_1}(\omega_1^i)$ and $P_{f_2}(\omega_1^i)$ respectively. Then for subset ω_1^i,

$$\begin{aligned} P_{f_2}(\omega_1^i) &= P_{f_1}(\omega_1^i), \quad w_1 = DoC_{min}, \\ P_{f_2}(\omega_1^i) &= P_1(\omega_1^i), \quad w_1 = DoC_{max}, \end{aligned} \tag{3.6.27}$$

and

$$\left.\begin{aligned} P_1(\omega_1^i) > P_{f_1}(\omega_1^i) &\Rightarrow P_1(\omega_1^i) > P_{f_2}(\omega_1^i) > P_{f_1}(\omega_1^i), \\ P_1(\omega_1^i) < P_{f_1}(\omega_1^i) &\Rightarrow P_1(\omega_1^i) < P_{f_2}(\omega_1^i) < P_{f_1}(\omega_1^i), \end{aligned}\right. \quad DoC_{max} > w_1 > DoC_{min}. \tag{3.6.28}$$

Moreover, if D_1 changes so that $|P_1(\omega_1^i) - P_{f_1}(\omega_1^i)|$ or w_1 increases, $|P_{f_2}(\omega_1^i) - P_{f_1}(\omega_1^i)|$ also increases.

Note that the above restricted principle is a special case of the fundamental principle 4 for SSF model.

Recall that in decision level identity fusion, the sensor reports and fusion results are usually expressed in the form of probability assignments. The above restricted principle is desirable because it also uses probability to measure the differences between sensor reports and fusion results. However, because probability is not the only appropriate choice of the distance measure, satisfaction of the restricted principle is less important than the satisfaction of the fundamental principle 4 for SSF model.

Note that for both the DSF and SSF models, when w_1 is at its minimal value, the fusion result remains unchanged. In other words, sensor report D_1 with minimal DoC is completely ignored. Also note that the above principles concern subsets ω_1^i and ω_1^j. For identity fusion methods such as Bayesian inference and the analytic center fusion method (to be introduced in Chapter 6), only original propositions of Ω are permitted. As a result, the above principles should be applied to propositions a_i and a_j instead of subsets ω_1^i and ω_1^j.

3.6.5 Fundamental principle 5: Asymptotic fusion result with identical sensor reports

In this subsection we continue to study the influence of a sensor report on the fusion result by defining the convergence of the fusion result with an increasing number of identical sensor reports. This is summarized in the following fundamental principle of asymptotic fusion result with identical sensor reports. This principle further characterizes the extent of reinforcement in light of infinite number of identical sensor reports.

We first define the principle for the DSF model.

Fundamental principle 5 for DSF model: Asymptotic fusion result with identical sensor reports

Consider the Dissimilar Sensor Fusion model. Let $\Omega = \{a_1, \ldots, a_N\}$. *Suppose there are* K *identical reports from dissimilar and independent sensors, each given by* $D_1 = \{P_1(\omega_1^i), i = 1, \ldots, L_1\}$, *where* $\omega_1^i \subset \Omega$ *and* $P_1(\omega_1^i) \in [0,1]$ *for* $i = 1, \ldots, L_1$, $\omega_1^i \cap \omega_1^j = \emptyset$ *for* $i \neq j$. *Let* w_1 *denote the DoC associated with* D_1 *and suppose that* $w_1 > DoC_{min}$ *(the minimal DoC value). Also suppose* $P_1(\omega_1^1) > P_1(\omega_1^i)$, $\forall\, i \neq 1$. *We further denote the fusion result of these sensor reports as* $D_{f_K}^{w_1}$, *the probability assigned to* ω_1^1 *in* $D_{f_K}^{w_1}$ *as* $P_{f_K}^{w_1}(\omega_1^1)$, *and the DoC associated with* $D_{f_K}^{w_1}$ *as* $w_{f_K}^{w_1}$. *Then,*

$$\lim_{K \to \infty} D_{f_K}^{w_1} = D_{f_\infty}, \tag{3.6.29}$$

and

$$\lim_{K\to\infty} w_{f_K}^{w_1} = DoC_{max}, \tag{3.6.30}$$

where

$$D_{f_\infty} = \left\{ \begin{array}{l} P_{f_\infty}(\omega_1^1) = 1, \\ P_{f_\infty}(\omega_1^i) = 0, \ \forall\, i \neq 1 \end{array} \right\}. \tag{3.6.31}$$

Moreover, suppose w_1 changes from w_{1a} to w_{1b}, then

$$w_{1b} > w_{1a} \ \Rightarrow\ P_{f_K}^{w_{1b}}(\omega_1^1) > P_{f_K}^{w_{1a}}(\omega_1^1), \quad \forall\, K \geq 1. \tag{3.6.32}$$

The above principle can be easily rationalized as follows. When an infinite number of identical sensor reports (from dissimilar and independent sensors) all point to ω_1^1 as the most likely subset, the fusion result should make ω_1^1 as the only likely target identity. In other words, these sensor reports corroborate each other's findings to generate a fusion outcome ω_1^1. The bigger the DoC values of these sensor reports, the stronger the reinforcement, and therefore the faster the fusion result convergence. Moreover, when K increases to infinity, more and more identical sensor reports support the final fusion result, resulting in the convergence of $w_{f_K}^{w_1}$ to DoC_{max}.

Note that in the above principle, the subsets ω_1^i's are required to be mutually exclusive. If this is not the case, their intersection may have a probability as the sum of all their probabilities, which may be larger than $P_1(\omega_1^1)$. As a result, when $K \to \infty$, D_{f_K} may not converge to D_{f_∞} given in Eq. (3.6.31). This phenomenon is illustrated in the following example using Dempster-Shafer evidential inference method.

Example 3.6.2 *Let $\Omega = \{a_1, a_2, a_3, a_4\}$. Suppose there are K identical sensor reports (each denoted by D_1) from dissimilar and independent sensors. We consider the following two cases.*

Case 1: *D_1 is given by $D_1 = \{P_1(a_1) = 0.4,\ P_1(a_2) = 0.3,\ P_1(a_3 \vee a_4) = 0.3\}$. By Dempster-Shafer evidential inference method, the fusion result is $D_{f_K} = \{P_{f_K}(a_1),\ P_{f_K}(a_2),$*

$P_{f_K}(a_3 \vee a_4)\}$, *where*

$$\begin{aligned}
P_{f_K}(a_1) &= \frac{P_1(a_1)^K}{P_1(a_1)^K + P_1(a_2)^K + P_1(a_3 \vee a_4)^K} = \frac{0.4^K}{0.4^K + 0.3^K + 0.3^K}, \\
P_{f_K}(a_2) &= \frac{P_1(a_2)^K}{P_1(a_1)^K + P_1(a_2)^K + P_1(a_3 \vee a_4)^K} = \frac{0.3^K}{0.4^K + 0.3^K + 0.3^K}, \\
P_{f_K}(a_3 \vee a_4) &= \frac{P_1(a_3 \vee a_4)^K}{P_1(a_1)^K + P_1(a_2)^K + P_1(a_3 \vee a_4)^K} = \frac{0.3^K}{0.4^K + 0.3^K + 0.3^K}.
\end{aligned} \tag{3.6.33}$$

Obviously, $\lim_{K\to\infty} D_{f_K} = D_{f_\infty}$, *where* $D_{f_\infty} = \{P_{f_\infty}(a_1) = 1,\ P_{f_\infty}(a_2) = 0,\ P_{f_\infty}(a_3 \vee a_4) = 0\}$.
Case 2: D_1 *is given by* $D_1 = \{P_1(a_1) = 0.4,\ P_1(a_2 \vee a_3) = 0.3,\ P_1(a_2 \vee a_4) = 0.3\}$. *When* $K > 1$, *fusion result* D_{f_K} *is given by* $D_{f_K} = \{P_{f_K}(a_1),\ P_{f_K}(a_2),\ P_{f_K}(a_2 \vee a_3),\ P_{f_K}(a_2 \vee a_4)\}$, *where*

$$\begin{aligned}
P_{f_K}(a_1) &= \frac{P_1(a_1)^K}{1-\mu_{f_K}(\emptyset)} = \frac{0.4^K}{1-\mu_{f_K}(\emptyset)}, \\
P_{f_K}(a_2) &= \frac{1}{1-\mu_{f_K}(\emptyset)}\left(P_1(a_2 \vee a_3)P_{f_{K-1}}(a_2 \vee a_4) + P_1(a_2 \vee a_4)P_{f_{K-1}}(a_2 \vee a_3)\right. \\
&\quad \left. + P_{f_{K-1}}(a_2)(P_1(a_2 \vee a_3) + P_1(a_2 \vee a_4))\right) \\
&= \frac{0.3^K + 0.3^K + 0.6P_{f_{K-1}}(a_2)}{1-\mu_{f_K}(\emptyset)}, \\
P_{f_K}(a_2 \vee a_3) &= \frac{P_1(a_2 \vee a_3)^K}{1-\mu_{f_K}(\emptyset)} = \frac{0.3^K}{1-\mu_{f_K}(\emptyset)}, \\
P_{f_K}(a_2 \vee a_4) &= \frac{P_1(a_2 \vee a_4)^K}{1-\mu_{f_K}(\emptyset)} = \frac{0.3^K}{1-\mu_{f_K}(\emptyset)},
\end{aligned} \tag{3.6.34}$$

and $\mu_{f_K}(\emptyset) = 1 - \left((0.3^K + 0.3^K + 0.6P_{f_{K-1}}(a_2)) + 0.3^K + 0.3^K + 0.4^K\right)$ *is the probability assigned to the null set in* D_{f_K} *before normalization. It is easy to verify that* $\lim_{K\to\infty} D_{f_K} = D_{f_\infty}$, *where* $D_{f_\infty} = \{P_{f_\infty}(a_1) = 0,\ P_{f_\infty}(a_2) = 1,\ P_{f_\infty}(a_2 \vee a_3) = 0,\ P_{f_\infty}(a_2 \vee a_4) = 0\}$.

In Case 1, the subsets are mutually exclusive. The converged fusion result assigns all the probability to the most likely proposition a_1 as expected. In contrast, the subsets in Case 2 are not mutually exclusive. The converged fusion result assigns all the probability to the intersection of subsets $a_2 \vee a_3$ and $a_2 \vee a_4$ instead of a_1 which has the highest probability in report D_1.

Next, we turn our attention to the SSF model.

Fundamental principle 5 for SSF model: Asymptotic fusion result with identical sensor reports

Consider the Similar Sensor Fusion model. Suppose there are K identical sensor reports from similar and independent sensors, each denoted by D_1. Let w_1 denote the DoC associated with D_1 and suppose that $w_1 > DoC_{min}$ (the minimal DoC value). Let D_{f_0} and $D_{f_K}^{w_1}$ (with DoC value $w_{f_K}^{w_1}$) denote the fusion results before and after the fusion of these identical reports. Then,

$$\lim_{K \to \infty} D_{f_K}^{w_1} = D_1, \tag{3.6.35}$$

and

$$\lim_{K \to \infty} w_{f_K}^{w_1} = DoC_{max}. \tag{3.6.36}$$

Moreover, suppose w_1 changes from w_{1a} to w_{1b}, then

$$w_{1b} > w_{1a} \quad \Rightarrow \quad dist(D_1, D_{f_K}^{w_{1b}}) \leq \; dist(D_1, D_{f_K}^{w_{1a}}). \tag{3.6.37}$$

for some properly chosen distance measure $dist(\cdot, \cdot)$.

Here, the distance measure $dist(\cdot, \cdot)$ should satisfy the same properties as it does for the fundamental principle 4 for SSF model (see page 58).

For the SSF model, the fusion objective is to find a fusion result that is most consistent with all the sensor reports. This results in the convergence of the fusion result to D_1 as K increases to infinity. The higher the DoC value associated with the identical sensor reports, the faster the convergence. Moreover, when K increases to infinity, more and more identical sensor reports confirm with the fusion result, resulting in the convergence of $w_{f_K}^{w_1}$ to DoC_{max}.

The above principles concerns subsets ω_1^i's. For identity fusion methods such as Bayesian inference and the analytic center fusion method (to be introduced in Chapter 6), only original propositions of Ω are permitted. As a result, the above principles should be applied to propositions a_i's instead of subsets ω_1^i's.

3.7 Additional DoC principles

In this section, we propose some additional principles associated with Degree of Confidence. First, we consider the DSF model and define a DoC principle for this fusion model.

DoC principle 1 for DSF mode

Let D_f denote the fusion result of sensor reports $D_1, \ldots, D_K$. Let w_k denote the DoC value of report D_k, and w_f denote the DoC value of D_f. Then, if any w_k increases, w_f also increases.

The above principle can be rationalized as following. As a joint identity declaration based on all sensor reports, the fusion result naturally has its accuracy based on the accuracy of each fused report. In other words, the DoC value of the fusion result is based on the DoC values of all sensor reports. In the DSF model, each sensor explores some unique target characteristics, and the fusion result is therefore based on all the characteristics explored by the sensors. If the DoC value of a sensor report increases, it indicates that this sensor has an increased accuracy on the explored target characteristics, and in turn indicates an increased accuracy on the overall characteristics upon which the fusion result is based. This naturally leads to a higher DoC value for the fusion result.

In contrast, sensors in the SSF model explore common target characteristics. Their reports confirm each other in the fusion process. As a result, the DoC of the fusion result depends not only on the DoC values of the sensor reports, but also on the extent to which each report agrees with the fusion result. We explore this idea in the following DoC principles for the SSF model.

DoC principle 1 for SSF mode

Let D_{f_1} denote the fusion result of sensor reports $D_{11}, \ldots, D_{1K}$, and D_{f_2} denote the fusion result of sensor reports $D_{21}, \ldots, D_{2K}$. Let w_{ij} denote the DoC value associated with report D_{ij}, and w_{f_i} denote the DoC value associated with fusion result D_{f_i}. Assume that $D_{f_1} = D_{f_2}$, and $w_{1k} = w_{2k}$ for $k = 1, \ldots, K$. We further assume that sensor reports D_{1k} and D_{2k} have the same subsets. Then,

$$dist(D_{1k}, D_{f_1}) > dist(D_{2k}, D_{f_2}), \quad k = 1, \ldots, K \quad \Rightarrow \quad w_{f_1} < w_{f_2}, \tag{3.7.38}$$

for some properly chosen distance measure $dist(\cdot,\cdot)$.

Here, the distance measure $dist(\cdot,\cdot)$ should satisfy the same properties as it does for the fundamental principle 4 for SSF model in the previous section (see page 58).

This principle indicates that when the DoC values associated with the reports remain unchanged but the reports drift away from the fusion result, the reports no longer confirm with the fusion result as strongly as before. Therefore, the DoC value of the fusion result decreases.

DoC principle 2 for SSF mode

Let D_{f_1} denote the fusion result of sensor reports $D_1^{w_1}, \ldots, D_K^{w_K}$, where w_k is the DoC associated with the k'th report. Let D_{f_2} denote the fusion result of $D_1^{\beta w_1}, \ldots, D_K^{\beta w_K}$. In other words, the DoC of the k'th report changes from w_k to βw_k, where $\beta > 0$ is a constant. Also let w_{f_i} denote the DoC value associated with fusion result D_{f_i}. Then,

$$\beta > 1 \quad \Rightarrow \quad w_{f_2} > w_{f_1}. \tag{3.7.39}$$

In other words, when the DoC values of all the sensor reports increase with the same extent, the fusion result should have a higher Degree of Confidence level.

3.8 Two classical methods revisited

In the previous sections, we proposed two fusion models, some fundamental principles and additional DoC principles. In this section, we analyze Bayesian inference method and Dempster-Shafer evidential inference method using these principles, which shows that the two classical methods are for the DSF model.

3.8.1 Analysis of Bayesian inference method

Theorem 3.8.1 *Bayesian inference method satisfies fundamental principles 1, 2, 3, restricted fundamental principle 4 and fundamental principle 5 for DSF model.*

Proof. Let $\Omega = \{a_1, \ldots, a_N\}$. Suppose all the *a priori* probabilities $P_0(a_1), \ldots, P_0(a_N)$ are known. Let $D_k = \{P(D_k|a_i), \quad i = 1, \ldots, N\}$ denote the report from sensor k, and assume that all sensor reports are conditionally independent. We examine the decision fusion principles one by one.

a) **Fundamental principle 1:** Suppose the *a priori* probability distribution is uniformly distributed, i.e., $P_0(a_i) = 1/N$ for $i = 1, \ldots, N$. Also suppose there is only one sensor report D_1 available. According to Bayes' rule (Eq. (3.2.5)), the fusion result assigns the following probability to proposition a_i: $P(a_i|D_1) = P(D_1|a_i)$ for $i = 1, \ldots, N$. Thus, the fusion result is a restatement of sensor report D_1.

b) **Fundamental principle 2:** This is straightforward. Bayesian inference method yields the same fusion result in both the sequential fusion mode and the batch fusion mode. In particular, for a group of sensor reports $D_1, \ldots, D_K$, the fusion result assigns $P(a_i|D_1, \ldots, D_K)$ to proposition a_i, which is given by Eq. (3.2.6):

$$P(a_i|D_1, \ldots, D_K) = \frac{P_0(a_i)P(D_1|a_i)\cdots P(D_K|a_i)}{\sum_{j=1}^{N} P_0(a_j)P(D_1|a_j)\cdots P(D_K|a_j)}, \quad i = 1, \ldots, N. \tag{3.8.40}$$

This is obviously independent of the fusion order in the sequential mode and the indexing of sensor reports in the batch mode. As a result, in both the sequential and batch fusion modes, Bayesian inference method satisfies the fundamental principle 2.

c) **Fundamental principle 3:** Suppose the *a priori* probability distribution is uniformly distributed, i.e., $P_0(a_i) = 1/N$ for $i = 1, \ldots, N$. Also suppose there are N cyclic sensor reports $D_1, D_2, \ldots, D_N$. The above assumptions imply the following "cyclic" conditional probabilities:

$$\begin{array}{lllll}
P(D_1|a_1) = r_1, & P(D_1|a_2) = r_2, & P(D_1|a_3) = r_3, \ldots, & P(D_1|a_N) = r_N; \\
P(D_2|a_1) = r_N, & P(D_2|a_2) = r_1, & P(D_2|a_3) = r_2, \ldots, & P(D_2|a_N) = r_{N-1}; \\
P(D_3|a_1) = r_{N-1}, & P(D_3|a_2) = r_N, & P(D_3|a_3) = r_1, \ldots, & P(D_3|a_N) = r_{N-2}; \\
\vdots & \vdots & \vdots \quad\quad \vdots & \vdots \\
P(D_N|a_1) = r_2, & P(D_N|a_2) = r_3, & P(D_N|a_3) = r_4, \ldots, & P(D_N|a_N) = r_1.
\end{array} \tag{3.8.41}$$

Using the fact that the sensor reports are conditionally independent with each other, we obtain the probability assigned to a_i in the fusion result as

$$\begin{aligned} P(a_i|D_1, D_2, \ldots, D_N) &= \frac{P_0(a_i)P(D_1|a_i)\cdots P(D_N|a_i)}{\sum_{j=1}^{N} P_0(a_j)P(D_1|a_j)\cdots P(D_N|a_j)} \\ &= \frac{\frac{1}{N}\prod_{j=1}^{N} r_j}{N(\frac{1}{N}\prod_{j=1}^{N} r_j)} = \frac{1}{N}, \end{aligned} \tag{3.8.42}$$

In other words, all the propositions have the same probability assignment in the fusion result.

d) Restricted fundamental principle 4 for DSF model: In Bayesian inference method, all the sensor reports are considered as equally reliable. In other words, these reports have the same DoC value which is definitely greater than DoC_{min}. As a result, we do not consider Eq. (3.6.24) which involves varying DoC values. Also note that Bayesian inference method does not permit disjunctions of propositions a_i's, therefore here we only consider basic propositions.

Suppose there are K sensor reports D_1, D_2, ..., D_K. Further denote the fusion result before the fusion of D_1 as $D_{f_1} = \{P_{f_1}(a_i), \quad i = 1, \ldots, N\}$, and the fusion result after the fusion of D_1 as $D_{f_2} = \{P_{f_2}(a_i), \quad i = 1, \ldots, N\}$. Note that D_{f_2} can be viewed as the fusion result of D_{f_1} and report D_1. By Bayes' rule (Eq. 3.2.5), the probability assignments in D_{f_2} are given as

$$P_{f_2}(a_i) = \frac{P(D_1|a_i)P_{f_1}(a_i)}{\sum_{\ell=1}^{N} P(D_1|a_\ell)P_{f_1}(a_\ell)}, \quad i = 1, \ldots, N, \tag{3.8.43}$$

and the relative likelihood ratio of proposition a_i to a_j is

$$\frac{P_{f_2}(a_i)}{P_{f_2}(a_j)} = \frac{P(D_1|a_i)}{P(D_1|a_j)} \cdot \frac{P_{f_1}(a_i)}{P_{f_1}(a_j)}. \tag{3.8.44}$$

Obviously,

$$\begin{aligned} P(D_1|a_i) > P(D_1|a_j) \quad &\Rightarrow \quad \frac{P_{f_2}(a_i)}{P_{f_2}(a_j)} > \frac{P_{f_1}(a_i)}{P_{f_1}(a_j)}, \\ P(D_1|a_i) < P(D_1|a_j) \quad &\Rightarrow \quad \frac{P_{f_2}(a_i)}{P_{f_2}(a_j)} < \frac{P_{f_1}(a_i)}{P_{f_1}(a_j)}. \end{aligned} \tag{3.8.45}$$

Moreover, it can be seen from (3.8.45) that if D_1 changes so that $|P(D_1|a_i) - P(D_1|a_j)|$ or w_1 increases, $\left|\frac{P_{f_2}(a_i)}{P_{f_2}(a_j)} - \frac{P_{f_1}(a_i)}{P_{f_1}(a_j)}\right|$ also increases.

e) Fundamental principle 5 for DSF model: As previously mentioned, in Bayesian inference method all sensor reports have equal DoC. As a result, we do not consider Eq. (3.6.32) which involves varying DoC values. In addition, here we do not consider disjunctions of propositions because they are not permitted in Bayesian inference method.

Suppose there are K identical sensor reports from dissimilar and independent sensors, each denoted by D_1. Suppose that D_1 favors proposition a_1 more than other propositions, i.e.,

$$r_1 \geq \frac{P(D_1|a_1)}{P(D_1|a_i)} \geq r_2 > 1, \quad \forall\, i \neq 1, \tag{3.8.46}$$

and let

$$r_{01} \geq \frac{P_0(a_1)}{P_0(a_i)} \geq r_{02} > 0, \quad \forall\, i \neq 1, \tag{3.8.47}$$

where r_1, r_2, r_{01} and r_{02} are some positive constants.

In the fusion result, the probability assigned to proposition a_1 is given by

$$P_{f_K}(a_1) = \frac{P_0(a_1)P(D_1|a_1)^K}{\sum_{i=1}^N P_0(a_i)P(D_1|a_i)^K}. \tag{3.8.48}$$

It is easy to verify that

$$P_{f_K}(a_1) = \frac{1}{1+\sum_{i=2}^N \frac{P_0(a_i)}{P_0(a_1)} \cdot \frac{P(D_1|a_i)^K}{P(D_1|a_1)^K}} \leq \frac{1}{1+\sum_{i=2}^N \frac{1}{r_{01}} \cdot \frac{1}{r_1^K}} = \frac{1}{1+(N-1)\frac{1}{r_{01}} \cdot \frac{1}{r_1^K}}, \tag{3.8.49}$$

and

$$1 - P_{f_K}(a_1) \geq \frac{1}{1 + \frac{1}{N-1} r_{01} r_1^K}. \tag{3.8.50}$$

Similarly,

$$1 - P_{f_K}(a_1) \leq \frac{1}{1 + \frac{1}{N-1} r_{02} r_2^K}. \tag{3.8.51}$$

Since $r_1 \geq r_2 > 1$, (3.8.50) and (3.8.51) guarantee that $\lim_{K \to \infty} P_{f_K}(a_1)$ converges to 1 exponentially. □

Note that in Bayesian inference method all sensor reports have equal DoC, the additional DoC principles do not apply to this fusion method.

3.8.2 Analysis of Dempster-Shafer evidential inference method

We first introduce a Lemma which will be used in later analysis.

Lemma 3.8.1 *If* $y_1, y_2, z_1, z_2 > 0$, $\frac{y_1}{y_2} > \frac{z_1}{z_2}$ *and* $\gamma_1 > \gamma_2 > 1$, *then*

$$\frac{\gamma_1 y_1 + z_1}{\gamma_2 y_2 + z_2} > \frac{y_1 + z_1}{y_2 + z_2}. \tag{3.8.52}$$

Proof. Because all the variables are positive, Inequality (3.8.52) is equivalent to

$$(\gamma_1 y_1 + z_1)(y_2 + z_2) > (\gamma_2 y_2 + z_2)(y_1 + z_1), \tag{3.8.53}$$

which can be expanded as

$$\gamma_1 y_1 y_2 + y_2 z_1 + \gamma_1 y_1 z_2 + z_1 z_2 > \gamma_2 y_1 y_2 + y_1 z_2 + \gamma_2 y_2 z_1 + z_1 z_2. \tag{3.8.54}$$

This can be further simplified as $y_2 z_1 + \gamma_1 y_1 z_2 > y_1 z_2 + \gamma_2 y_2 z_1$, or $(\gamma_1 - 1) y_1 z_2 > (\gamma_2 - 1) y_2 z_1$. The inequality obviously holds in light of the assumption $\frac{y_1}{y_2} > \frac{z_1}{z_2}$. This completes the proof of Lemma 3.8.1. □

We now propose the following theorem for Dempster-Shafer evidential inference method.

Theorem 3.8.2 *Dempster-Shafer evidential inference method satisfies fundamental principles 1, 2, 3, restricted fundamental principle 4 and fundamental principle 5 for DSF model. Moreover, the fusion method satisfies DoC principle 1 for DSF model.*

Proof. Let $\Omega = \{a_1, \dots, a_N\}$. Suppose all available sensors are independent with each other, and a sensor report D_k takes the following form:

$$D_k = \{P_k(\omega_k^\ell), \quad \ell = 1, \dots, L_k\}, \tag{3.8.55}$$

where $\omega_k^\ell \subseteq \Omega$, $P_k(\omega_k^\ell) \in [0, 1]$ and $\sum_{\ell=1}^{L_k} P_k(\omega_k^\ell) = 1$. We examine the decision fusion principles one by one.

a) Fundamental principle 1: By definition, if there is only one sensor report available, the fusion result will be the same with the sensor report.

b) Fundamental principle 2: As shown in section 3.3, Dempster-Shafer evidential inference method yields the same fusion result in both the sequential and batch fusion modes. In particular, suppose there are K sensor reports $P_1, \dots, P_K$. The corresponding fusion result $D_{f_K} = \{P_{f_K}(\omega_{f_K}^{\ell_{f_K}}), \quad \ell_{f_K} = 1, \dots, L_{f_K}\}$ is given by equations (3.3.13) and (3.3.14), which are restated here as follows:

$$P_{f_K}(\omega_{f_K}^{\ell_{f_K}}) = \frac{\mu_{f_K}(\omega_{f_K}^{\ell_{f_K}})}{1 - \mu_{f_K}(\emptyset)}, \tag{3.8.56}$$

where

$$\mu_{f_K}(\omega_{f_K}^{\ell_{f_K}}) = \sum_{\substack{\ell_1, \dots, \ell_K : \\ \omega_1^{\ell_1} \cap \dots \cap \omega_K^{\ell_K} = \omega_{f_K}^{\ell_{f_K}}}} P_1(\omega_1^{\ell_1}) \cdots P_K(\omega_K^{\ell_K}). \tag{3.8.57}$$

Both (3.8.56) and (3.8.57) are independent of the fusion order. As a result, in both the sequential and batch fusion modes, Dempster-Shafer evidential inference method satisfies the fundamental principle 2.

c) **Fundamental principle 3:** Suppose there are N cyclic sensor reports $D_1, \ldots, D_N$ with D_k given as

$$D_k = \left\{ \begin{array}{l} P_k(\Omega) = r_\Omega, \\ P_k(a_i) = r_{i \oplus k}, \quad i = 1, \ldots, N \end{array} \right\}, \tag{3.8.58}$$

where $0 \leq r_1, \ldots, r_N, r_\Omega \leq 1$, $r_\Omega + \sum_{i=1}^{N} r_i = 1$ and $i \oplus k = i + k - 1 \mod N$.

Because all the sensor reports assign probabilities only to Ω and basic propositions a_i's, the fusion result D_f can be denoted as

$$D_f = \left\{ \begin{array}{l} P_f(\Omega), \\ P_f(a_i), \quad i = 1, \ldots, N \end{array} \right\}, \tag{3.8.59}$$

where $P_f(a_i) = \dfrac{\mu_f(a_i)}{1 - \mu_f(\emptyset)}$ and $\mu_f(a_i)$ denotes the probability assigned to a_i in D_f before normalization (see page 46). By Eq. (3.3.14), $\mu_f(a_i)$ is given as

$$\mu_f(a_i) = \sum_{\substack{\ell_1, \ldots, \ell_N : \\ \omega_1^{\ell_1} \cap \cdots \cap \omega_N^{\ell_N} = a_i}} P_1(\omega_1^{\ell_1}) \cdots P_K(\omega_N^{\ell_N}). \tag{3.8.60}$$

Since the $\omega_j^{\ell_j}$'s in the above equation can only be Ω or a_i, we can rewrite the above equation as $\mu_f(a_i) = \sum_{m=1}^{N} \mu_m(a_i)$, where

$$\mu_m(a_i) = \sum_{j_1 \neq \cdots \neq j_N} P_{j_1}(a_i) \cdots P_{j_m}(a_i) P_{j_{m+1}}(\Omega) \cdots P_{j_N}(\Omega). \tag{3.8.61}$$

In other words, $\mu_f(a_i)$ consists of N terms. The m'th term (denoted by $\mu_m(a_i)$) represents the case that m of the N subsets $\omega_1^{\ell_1}, \ldots, \omega_N^{\ell_N}$ in Eq. (3.8.60) are equal to a_i, and the rest are equal to Ω. Using the cyclic structure of the sensor reports, we can compute $\mu_m(a_i)$ as

$$\mu_m(a_i) = \sum_{\substack{j_1 = \cdots = j_m = 1 \\ j_1 \neq \cdots \neq j_m}}^{N} r_{j_1} \cdots r_{j_m} r_\Omega^{N-m}. \tag{3.8.62}$$

Now it is clear that $\mu_m(a_1)$ is independent of i, implying that $\mu_m(a_1) = \cdots = \mu_m(a_N)$. This further implies that $\mu_f(a_1) = \cdots = \mu_f(a_N)$. As a result, the probability assigned to a_i in the fusion result D_f is given by

$$P_f(a_i) = \frac{\mu_f(a_i)}{\mu_f(\Omega) + \sum_{j=1}^{N} \mu_f(a_j)} = \frac{\mu_f(a_1)}{\mu_f(\Omega) + N\mu_f(a_1)}, \quad i = 1, \ldots, N. \tag{3.8.63}$$

In other words, all the propositions have the same probability in the fusion result.

d) Restricted fundamental principle 4 for DSF model: Suppose sensor report D_1 is given as $D_1 = \{P_1(\omega_1^\ell), \ \ \ell = 1, \ldots, L_1\}$. Without any loss of generality, we consider two mutually exclusive subsets ω_1^1 and ω_1^2, each of them is also mutually exclusive with other subsets in D_1, i.e.,

$$\omega_1^1 \cap \omega_1^\ell = \emptyset, \ \forall\, \ell \neq 1 \quad \text{and} \quad \omega_1^2 \cap \omega_1^\ell = \emptyset, \ \forall\, \ell \neq 2. \tag{3.8.64}$$

Note that $P_1(\omega_1^1)$ and $P_1(\omega_1^2)$ are the total probabilities assigned to ω_1^1 and ω_1^2 respectively. Also suppose that the fusion result before the fusion of D_1 is

$$D_{f_1} = \left\{ \begin{array}{ll} P_{f_1}(\omega_{f_{11}}^\ell), & \ell = 1, \ldots, L_{f_{11}}, \\ P_{f_1}(\omega_{f_{12}}^\ell), & \ell = 1, \ldots, L_{f_{12}}, \\ P_{f_1}(\omega_{f_1}^\ell), & \ell = 1, \ldots, L_{f_1} \end{array} \right\}, \tag{3.8.65}$$

where

$$\begin{array}{ll} \omega_{f_{11}}^\ell \subseteq \omega_1^1, & \ell = 1, \ldots, L_{f_{11}}, \\ \omega_{f_{12}}^\ell \subseteq \omega_1^2, & \ell = 1, \ldots, L_{f_{12}}, \\ \omega_1^1 \cap \omega_{f_1}^\ell = \emptyset, \ \ \omega_1^2 \cap \omega_{f_1}^\ell = \emptyset, & \ell = 1, \ldots, L_{f_1}. \end{array} \tag{3.8.66}$$

In D_{f_1}, the total probabilities assigned to ω_1^1 and ω_1^2 are

$$P_{f_1}^T(\omega_1^1) = \sum_{\ell=1}^{L_{f_{11}}} P_{f_1}(\omega_{f_{11}}^\ell) \quad \text{and} \quad P_{f_1}^T(\omega_1^2) = \sum_{\ell=1}^{L_{f_{12}}} P_{f_1}(\omega_{f_{12}}^\ell). \tag{3.8.67}$$

Note that D_{f_1} may have a probability mass $P_{f_1}(\omega_1^1)$ which has a different meaning with $P_{f_1}^T(\omega_1^1)$. In the Dempster-Shafer model, $P_{f_1}(\omega_1^1)$ means that this probability is assigned to (or add to) subset ω_1^1 instead of being the total probability of ω_1^1. Therefore, the total probability of ω_1^1 is the sum of all probabilities assigned to ω_1^1 and its subsets, as presented by $P_{f_1}^T(\omega_1^1)$ in the above equation. The fusion result after the fusion of D_1 can be denoted as

$$D_{f_2} = \left\{ \begin{array}{ll} P_{f_2}(\omega_{f_{11}}^\ell), & \ell = 1, \ldots, L_{f_{11}}, \\ P_{f_2}(\omega_{f_{12}}^\ell), & \ell = 1, \ldots, L_{f_{12}}, \\ P_{f_2}(\omega_{f_2}^\ell), & \ell = 1, \ldots, L_{f_2} \end{array} \right\}, \tag{3.8.68}$$

where

$$\omega_1^1 \cap \omega_{f_2}^\ell = \emptyset, \quad \omega_1^2 \cap \omega_{f_2}^\ell = \emptyset, \quad \ell = 1, \ldots, L_{f_2}. \tag{3.8.69}$$

In D_{f_2}, the total probabilities assigned to ω_1^1 and ω_1^2 are

$$P_{f_2}^T(\omega_1^1) = \sum_{\ell=1}^{L_{f_{11}}} P_{f_2}(\omega_{f_{11}}^\ell) \quad \text{and} \quad P_{f_2}^T(\omega_1^2) = \sum_{\ell=1}^{L_{f_{12}}} P_{f_2}(\omega_{f_{12}}^\ell). \tag{3.8.70}$$

Also note that there is no probability of uncertainty in D_1, D_{f_1} or D_{f_2}.

Using Dempster-Shafer evidential inference method, we have

$$P_{f_2}^T(\omega_{f_{11}}^\ell) = \mu P_1(\omega_1^1) P_{f_1}(\omega_{f_{11}}^\ell) \quad \text{and} \quad P_{f_2}^T(\omega_{f_{12}}^\ell) = \mu P_1(\omega_1^2) P_{f_1}(\omega_{f_{12}}^\ell), \tag{3.8.71}$$

where μ is the normalization factor, and

$$\frac{P_{f_2}^T(\omega_1^1)}{P_{f_2}^T(\omega_1^2)} = \frac{\sum_{\ell=1}^{L_{f_{11}}} P_{f_2}(\omega_{f_{11}}^\ell)}{\sum_{\ell=1}^{L_{f_{12}}} P_{f_2}(\omega_{f_{12}}^\ell)} = \frac{P_1(\omega_1^1)\sum_{\ell=1}^{L_{f_{11}}} P_{f_1}(\omega_{f_{11}}^\ell)}{P_1(\omega_1^2)\sum_{\ell=1}^{L_{f_{12}}} P_{f_1}(\omega_{f_{12}}^\ell)} = \frac{P_1(\omega_1^1)}{P_1(\omega_1^2)} \cdot \frac{P_{f_1}^T(\omega_{f_1}^1)}{P_{f_1}^T(\omega_{f_1}^2)}. \tag{3.8.72}$$

It is now obvious that

$$
\begin{aligned}
P_1(\omega_1^1) > P_1(\omega_1^2) &\Rightarrow \frac{P_{f_2}^T(\omega_1^1)}{P_{f_2}^T(\omega_1^2)} > \frac{P_{f_1}^T(\omega_1^1)}{P_{f_1}^T(\omega_1^2)}, \\
P_1(\omega_1^1) < P_1(\omega_1^2) &\Rightarrow \frac{P_{f_2}^T(\omega_1^1)}{P_{f_2}^T(\omega_1^2)} < \frac{P_{f_1}^T(\omega_1^1)}{P_{f_1}^T(\omega_1^2)}.
\end{aligned} \tag{3.8.73}
$$

It is also obvious from Eq. (3.8.72) that when D_1 changes so that $|P_1(\omega_1^1) - P_1(\omega_1^2)|$ increases, $\left|\frac{P_{f_2}(\omega_1^1)}{P_{f_2}(\omega_1^2)} - \frac{P_{f_1}(\omega_1^1)}{P_{f_1}(\omega_1^2)}\right|$ also increases.

e) Fundamental principle 5 for DSF model: Suppose that there are K identical sensor reports from dissimilar and independent sensors, each given as

$$
D_1 = \left\{ \begin{array}{l} P_1(\Omega), \\ P_1(\omega_1^\ell), \quad \ell = 1, \dots, L_1 \end{array} \right\}, \tag{3.8.74}
$$

where $P_1(\Omega) + \sum_{\ell=1}^{L_1} P_1(\omega_1^\ell) = 1$, $\omega_1^\ell \subset \Omega$, and $\omega_1^i \cap \omega_1^j = \emptyset$ for $i \neq j$. In other words, subsets ω_1^ℓ's are mutually exclusive. To separate the DoC of D_1 from the rest of the report, we use Eq. (3.3.9) to define

$$
P_1'(\omega_1^\ell) = \frac{P_1(\omega_1^\ell)}{1 - P_1(\Omega)}, \quad \ell = 1, \dots, L_1. \tag{3.8.75}
$$

Note that $\sum_{\ell=1}^{L_1} P_1'(\omega_1^\ell) = 1$. As discussed in section 3.3 (see page 44), the $P_1'(\omega_1^\ell)$'s represent the identity information contained in D_1, while $P_1(\Omega)$ reflects the DoC of D_1. Without any loss of generality, let

$$
P_1(\omega_1^1) > P_1(\omega_1^\ell), \quad \forall\, \ell \neq 1. \tag{3.8.76}
$$

In other words, $r_1 \geq \frac{P_1(\omega_1^1)}{P_1(\omega_1^\ell)} \geq r_2 > 1$ for $\ell \neq 1$, where r_1 and r_2 are some positive constants. The fusion result of these reports can be denoted as

$$D_{f_K} = \left\{ \begin{array}{ll} P_{f_K}(\Omega), & \\ P_{f_K}(\omega_1^\ell), & \ell = 1, \ldots, L_1 \end{array} \right\}. \tag{3.8.77}$$

Obviously, the probability of uncertainty in D_{f_K} before normalization (see page 46) is

$$\mu_{f_K}(\Omega) = P_1(\Omega)^K. \tag{3.8.78}$$

We claim that the probability assigned to subset ω_1^ℓ in D_{f_K} before normalization is

$$\mu_{f_K}(\omega_1^\ell) = \left(P_1(\omega_1^\ell) + P_1(\Omega)\right)^K - P_1(\Omega)^K, \tag{3.8.79}$$

which is proven by induction as follows.

When $K = 1$, Eq. (3.8.79) yields

$$\mu_{f_1}(\omega_1^\ell) = \left(P_1(\omega_1^\ell) + P_1(\Omega)\right) - P_1(\Omega) = P_1(\omega_1^\ell), \tag{3.8.80}$$

which holds trivially. Assume that Eq. (3.8.79) is correct for K identical sensor reports D_1's, and consider the case where another report D_1 is fused. By Dempster-Shafer fusion rule and using the fact that the ω_1^ℓ's are mutually exclusive, we obtain

$$\begin{aligned} \mu_{f_{K+1}}(\omega_1^\ell) &= P_1(\omega_1^\ell)\mu_{f_K}(\Omega) + \left(P_1(\omega_1^\ell) + P_1(\Omega)\right)\mu_{f_K}(\omega_1^\ell) \\ &= P_1(\omega_1^\ell)P_1(\Omega)^K + \left(P_1(\omega_1^\ell) + P_1(\Omega)\right)\left(\left(P_1(\omega_1^\ell) + P_1(\Omega)\right)^K - P_1(\Omega)^K\right) \\ &= \left(P_1(\omega_1^\ell) + P_1(\Omega)\right)^{K+1} - P_1(\Omega)^{K+1}, \end{aligned} \tag{3.8.81}$$

which establishes Eq. (3.8.79) for $K + 1$.

Using Eqs. (3.8.78) and (3.8.79), we compute the probability assigned to ω_1^1 in D_{f_K} (after normalization) as

$$P_{f_K}(\omega_1^1) = \frac{\left(P_1(\omega_1^1) + P_1(\Omega)\right)^K - P_1(\Omega)^K}{\sum_{j=1}^{L_1}\left(\left(P_1(\omega_1^j) + P_1(\Omega)\right)^K - P_1(\Omega)^K\right) + P_1(\Omega)^K}. \tag{3.8.82}$$

The largest exponential term in both the denominator and numerator is $\left(P_1(\omega_1^1) + P_1(\Omega)\right)^K$. Therefore, $\lim_{K\to\infty} P_{f_K}(\omega_1^1)$ converges to 1 exponentially. Similarly,

$$\lim_{K\to\infty} P_{f_K}(\Omega) = \lim_{K\to\infty} \frac{P_1(\Omega)^K}{\sum_{j=1}^{L_1}\left(\left(P_1(\omega_1^j) + P_1(\Omega)\right)^K - P_1(\Omega)^K\right) + P_1(\Omega)^K} = 0, \tag{3.8.83}$$

which establishes the property (3.6.30).

It remains to prove the property (3.6.32). We assume that the DoC value of D_1 changes from w_{1a} to w_{1b}, with $w_{1b} > w_{1a}$. Because the DoC of D_1 has a inverse relationship with $P_1(\Omega)$, this implies that $P_1(\Omega)$ changes from $P_1^{w_{1a}}(\Omega)$ to $P_1^{w_{1b}}(\Omega)$, with $P_1^{w_{1a}}(\Omega) > P_1^{w_{1b}}(\Omega)$. Note that the normalized probabilities $P_1'(\omega_1^i)$'s remain unchanged.

The probability assigned to subset ω_1^ℓ in $D_{f_K}^{w_{1a}}$ is

$$\begin{aligned} P_{f_K}^{w_{1a}}(\omega_1^\ell) &= \frac{\left(P_1'(\omega_1^\ell)\left(1 - P_1^{w_{1a}}(\Omega)\right) + P_1^{w_{1a}}(\Omega)\right)^K - P_1^{w_{1a}}(\Omega)^K}{\sum_{j=1}^{L_1}\left(\left(P_1'(\omega_1^j)\left(1 - P_1^{w_{1a}}(\Omega)\right) + P_1^{w_{1a}}(\Omega)\right)^K - P_1^{w_{1a}}(\Omega)^K\right) + P_1^{w_{1a}}(\Omega)^K} \\ &= \frac{(c_\ell + 1)^K - 1}{\sum_{j=1}^{L_1}\left((c_j + 1)^K - 1\right) + 1}, \end{aligned} \tag{3.8.84}$$

where $c_\ell = \dfrac{1 - P_1^{w_{1a}}(\Omega)}{P_1^{w_{1a}}(\Omega)} \cdot P_1'(\omega_1^\ell)$ for $\ell = 1, \ldots, L_1$. Note that by Eq. (3.8.76), $c_1 > c_\ell > 0$ for $\ell \neq 1$. Similarly, we can write

$$P_{f_K}^{w_{1b}}(\omega_1^\ell) = \frac{(\beta c_\ell + 1)^K - 1}{\sum_{j=1}^{L_1}\left((\beta c_j + 1)^K - 1\right) + 1}, \tag{3.8.85}$$

where $\beta = \dfrac{P_1^{w_{1a}}(\Omega)}{P_1^{w_{1b}}(\Omega)} \cdot \dfrac{1 - P_1^{w_{1b}}(\Omega)}{1 - P_1^{w_{1a}}(\Omega)}$. Note that $\beta > 1$. We observe that for all $\ell \neq 1$,

$$\frac{P_{f_K}^{w_{1b}}(\omega_1^1)}{P_{f_K}^{w_{1b}}(\omega_1^\ell)} = \frac{(\beta c_1 + 1)^K - 1}{(\beta c_\ell + 1)^K - 1} = \frac{c_1}{c_\ell} \cdot \frac{\sum_{k=0}^{K-1} (\beta c_1 + 1)^k}{\sum_{k=0}^{K-1} (\beta c_\ell + 1)^k}, \tag{3.8.86}$$

and

$$\frac{P_{f_K}^{w_{1a}}(\omega_1^1)}{P_{f_K}^{w_{1a}}(\omega_1^\ell)} = \frac{(c_1 + 1)^K - 1}{(c_\ell + 1)^K - 1} = \frac{c_1}{c_\ell} \cdot \frac{\sum_{k=0}^{K-1} (c_1 + 1)^k}{\sum_{k=0}^{K-1} (c_\ell + 1)^k}. \tag{3.8.87}$$

Define

$$f_m = \frac{(c_1 + 1)^m \sum_{k=1}^{K-m-1} (\beta c_1 + 1)^k + \sum_{k=0}^{m} (c_1 + 1)^k}{(c_\ell + 1)^m \sum_{k=1}^{K-m-1} (\beta c_\ell + 1)^k + \sum_{k=0}^{m} (c_\ell + 1)^k}, \quad m = 0, \ldots, K-1. \tag{3.8.88}$$

Then, we can write

$$\frac{P_{f_K}^{w_{1b}}(\omega_1^1)}{P_{f_K}^{w_{1b}}(\omega_1^\ell)} = \frac{c_1}{c_\ell} \cdot \frac{\sum_{k=1}^{K-1} (\beta c_1 + 1)^k + 1}{\sum_{k=1}^{K-1} (\beta c_\ell + 1)^k + 1} = \frac{c_1}{c_\ell} f_0, \tag{3.8.89}$$

and

$$\frac{P_{f_K}^{w_{1a}}(\omega_1^1)}{P_{f_K}^{w_{1a}}(\omega_1^\ell)} = \frac{c_1}{c_\ell} \cdot \frac{\sum_{k=0}^{K-1} (c_1 + 1)^k}{\sum_{k=0}^{K-1} (c_\ell + 1)^k} = \frac{c_1}{c_\ell} f_{K-1}. \tag{3.8.90}$$

We now claim that

$$f_m > f_{m+1}, \quad m = 0, \ldots, K-2. \tag{3.8.91}$$

To prove this, we derive that

$$\begin{aligned} f_m &= \frac{(c_1 + 1)^m \sum_{k=1}^{K-m-1} (\beta c_1 + 1)^k + \sum_{k=0}^{m} (c_1 + 1)^k}{(c_\ell + 1)^m \sum_{k=1}^{K-m-1} (\beta c_\ell + 1)^k + \sum_{k=0}^{m} (c_\ell + 1)^k} \\ &= \frac{(\beta c_1 + 1)(c_1 + 1)^m \sum_{k=0}^{K-m-2} (\beta c_1 + 1)^k + \sum_{k=0}^{m} (c_1 + 1)^k}{(\beta c_\ell + 1)(c_\ell + 1)^m \sum_{k=0}^{K-m-2} (\beta c_\ell + 1)^k + \sum_{k=0}^{m} (c_\ell + 1)^k}, \end{aligned} \tag{3.8.92}$$

and

$$
\begin{aligned}
f_{m+1} &= \frac{(c_1+1)^{m+1}\sum_{k=1}^{K-m-2}(\beta c_1+1)^k + \sum_{k=0}^{m+1}(c_1+1)^k}{(c_\ell+1)^{m+1}\sum_{k=1}^{K-m-2}(\beta c_\ell+1)^k + \sum_{k=0}^{m+1}(c_\ell+1)^k} \\
&= \frac{(c_1+1)^{m+1}\sum_{k=1}^{K-m-2}(\beta c_1+1)^k + (c_1+1)^{m+1} + \sum_{k=0}^{m}(c_1+1)^k}{(c_\ell+1)^{m+1}\sum_{k=1}^{K-m-2}(\beta c_\ell+1)^k + (c_\ell+1)^{m+1} + \sum_{k=0}^{m}(c_\ell+1)^k} \\
&= \frac{(c_1+1)(c_1+1)^m\sum_{k=0}^{K-m-2}(\beta c_1+1)^k + \sum_{k=0}^{m}(c_1+1)^k}{(c_\ell+1)(c_\ell+1)^m\sum_{k=0}^{K-m-2}(\beta c_\ell+1)^k + \sum_{k=0}^{m}(c_\ell+1)^k}.
\end{aligned} \tag{3.8.93}
$$

Now we are in a position to invoke Lemma 3.8.1. In particular, inequality (3.8.91) is equivalent to inequality (3.8.52), i.e.,

$$
\frac{\gamma_1 y_1 + z_1}{\gamma_2 y_2 + z_2} > \frac{y_1 + z_1}{y_2 + z_2}, \tag{3.8.94}
$$

where

$$
\begin{aligned}
\gamma_1 &= \beta c_1 + 1, \quad y_1 = (c_1+1)^m \sum_{k=0}^{K-m-2} (\beta c_1+1)^k, \quad z_1 = \sum_{k=0}^{m}(c_1+1)^k, \\
\gamma_2 &= \beta c_\ell + 1, \quad y_2 = (c_\ell+1)^m \sum_{k=0}^{K-m-2} (\beta c_\ell+1)^k, \quad z_2 = \sum_{k=0}^{m}(c_\ell+1)^k.
\end{aligned} \tag{3.8.95}
$$

Note that the assumptions in Lemma 3.8.1 are all satisfied. In particular, $\gamma_1 > \gamma_2 > 1$, $y_1, y_2, z_1, z_2 > 0$, and $\frac{y_1}{y_2} > \frac{z_1}{z_2}$ which is readily seen because

$$
\frac{y_1}{y_2} = \frac{(c_1+1)^m\sum_{k=0}^{K-m-2}(\beta c_1+1)^k}{(c_\ell+1)^m\sum_{k=0}^{K-m-2}(\beta c_\ell+1)^k} > \frac{(c_1+1)^m}{(c_\ell+1)^m}, \tag{3.8.96}
$$

and

$$
\frac{z_1}{z_2} = \frac{\sum_{k=0}^{m}(c_1+1)^k}{\sum_{k=0}^{m}(c_\ell+1)^k} < \frac{(c_1+1)^m}{(c_\ell+1)^m}. \tag{3.8.97}
$$

Combining Eqs. (3.8.89), (3.8.90) and inequality (3.8.91), we obtain

$$\frac{P_{f_K}^{w_{1b}}(\omega_1^1)}{P_{f_K}^{w_{1b}}(\omega_1^\ell)} = \frac{c_1}{c_\ell} f_0 > \frac{c_1}{c_\ell} f_1 > \cdots > \frac{c_1}{c_\ell} f_{K-1} = \frac{P_{f_K}^{w_{1a}}(\omega_1^1)}{P_{f_K}^{w_{1a}}(\omega_1^\ell)}. \tag{3.8.98}$$

Finally, we use (3.8.84), (3.8.85) and (3.8.98) to derive that

$$\begin{aligned} P_{f_K}^{w_{1b}}(\omega_1^1) &= \frac{1}{1 + \sum_{\ell=2}^{L_1} \frac{(\beta c_\ell + 1)^K - 1}{(\beta c_1 + 1)^K - 1} + \frac{1}{(\beta c_1 + 1)^K - 1}} \\ &> \frac{1}{1 + \sum_{\ell=2}^{L_1} \frac{(c_\ell + 1)^K - 1}{(c_1 + 1)^K - 1} + \frac{1}{(c_1 + 1)^K - 1}} = P_{f_K}^{w_{1a}}(\omega_1^1), \end{aligned} \tag{3.8.99}$$

which establishes the property (3.6.32).

f) DoC principle 1 for DSF model: As proven earlier, Dempster-Shafer evidential inference method satisfies the fundamental principle 2, which implies that the fusion of report D_1 with a group of other reports is equivalent to the fusion of D_1 with the fusion result of the same group of reports. Therefore, to prove the principle, we only need to consider the simple case of two sensor reports. In particular, Let D_f denote the fusion result of reports D_1 and D_2. Then, if w_1 increases, w_f also increases, i.e., if $P_1(\Omega)$ decreases, $P_f(\Omega)$ also decreases.

Let the two reports be given as

$$D_k = \left\{ \begin{array}{ll} P_k(\Omega), & \\ P_k(\omega_k^\ell), & \ell = 1, \ldots, L_k \end{array} \right\}, \quad k = 1, 2, \tag{3.8.100}$$

where $\omega_k^\ell \subset \Omega$ for $\ell = 1, \ldots, L_k$. To separate the identity information in D_k from the DoC (represented by $P_k(\Omega)$), we use Eq. (3.3.9) to define

$$P_k'(\omega_k^\ell) = \frac{P_k(\omega_k^\ell)}{1 - P_k(\Omega)}, \quad \ell = 1, \ldots, L_k, \; k = 1, 2, \tag{3.8.101}$$

and

$$\mu'_f(\emptyset) = \sum_{\ell_1,\ell_2:\omega_1^{\ell_1}\cap\omega_2^{\ell_2}=\emptyset} P'_1(\omega_1^{\ell_1})P'_2(\omega_2^{\ell_2}). \tag{3.8.102}$$

Note that the $P'_k(\omega_k^\ell)$'s and $\mu'_f(\emptyset)$ are independent of $P_1(\Omega)$ and $P_2(\Omega)$. The probability of uncertainty in the fusion result is given as $P_f(\Omega) = \dfrac{P_1(\Omega)P_2(\Omega)}{1-\mu_f(\emptyset)}$, where

$$\mu_f(\emptyset) = \sum_{\ell_1,\ell_2:\omega_1^{\ell_1}\cap\omega_2^{\ell_2}=\emptyset} P_1(\omega_1^{\ell_1})P_2(\omega_2^{\ell_2}) = (1-P_1(\Omega))\,(1-P_2(\Omega))\,\mu'_f(\emptyset). \tag{3.8.103}$$

Suppose that the probability of uncertainty for D_1 decreases from $P_1(\Omega)$ to $\beta P_1(\Omega)$, where $\beta \in [0,1)$. Then, the probability of uncertainty for D_f changes from $P_f(\Omega)$ to $P_f^*(\Omega)$ which can be written as

$$P_f^*(\Omega) = \frac{\beta P_1(\Omega)P_2(\Omega)}{1-(1-\beta P_1(\Omega))\,(1-P_2(\Omega))\,\mu'_f(\emptyset)} = \frac{P_1(\Omega)P_2(\Omega)}{\frac{1}{\beta} - \frac{\frac{1}{\beta}-P_1(\Omega)}{1-P_1(\Omega)}\mu_f(\emptyset)}. \tag{3.8.104}$$

We can derive that

$$\begin{aligned}
P_f^*(\Omega) &= \frac{P_1(\Omega)P_2(\Omega)}{1-\mu_f(\emptyset)+\left(\frac{1}{\beta}-1\right)+\left(1-\frac{\frac{1}{\beta}-P_1(\Omega)}{1-P_1(\Omega)}\right)\mu_f(\emptyset)} \\
&= \frac{P_1(\Omega)P_2(\Omega)}{1-\mu_f(\emptyset)+\left(\frac{1}{\beta}-1\right)+\left(1-\frac{1-\frac{1}{\beta}}{1-P_1(\Omega)}\right)\mu_f(\emptyset)} \\
&= \frac{P_1(\Omega)P_2(\Omega)}{1-\mu_f(\emptyset)+\left(\frac{1}{\beta}-1\right)\frac{1-P_1(\Omega)-\mu_f(\emptyset)}{1-P_1(\Omega)}} \\
&< \frac{P_1(\Omega)P_2(\Omega)}{1-\mu_f(\emptyset)} = P_f(\Omega).
\end{aligned} \tag{3.8.105}$$

□

Further discussion on DoC: In Dempster-Shafer evidential inference method, the DoC of a sensor report D_k is represented by the probability of uncertainty $P_k(\Omega)$ which can be assigned to any subset in D_k (see page 44). If $P_k(\Omega) \neq 0$, the total probabilities of the subsets ω_k^ℓ's in D_k cannot be determined. Therefore in the proof of the restricted fundamental principle 4 for DSF model which involves total probabilities of subsets, we assume that there is no probability of uncertainty in any sensor report or fusion result. However, from the introduction in section 3.3, we know that all the identity information contained in report D_k (as given in Eq. (3.3.8)) is represented by the normalized probabilities $P'_k(\omega_k^\ell)$'s (as given by Eq. (3.3.9)). As a result, when $P_k(\Omega) > 0$, we should use $P'_k(\omega_k^\ell)$'s to compute the total probabilities of ω_k^ℓ's as well as the relative likelihood of any two subsets ω_k^i and ω_k^j, and treat $P_k(\Omega)$ only as representing the DoC of D_k. In particular, for the fundamental principle 4 for DSF model, we assume that a sensor report D_1 is given as

$$D_1 = \left\{ \begin{array}{ll} P_1(\Omega), & \\ P_1(\omega_1^\ell), & \ell = 1, \ldots, L_1 \end{array} \right\}, \tag{3.8.106}$$

where $\omega_1^\ell \subset \Omega$, $\omega_1^i \cap \omega_1^j = \emptyset$ for $i \neq j$, and suppose that in D_1, $P_1(\omega_1^1) > P_1(\omega_1^\ell)$ for $\ell \neq 1$. We also assume that the fusion result before the fusion of D_1 is given as

$$D_{f_1} = \left\{ \begin{array}{ll} P_{f_1}(\Omega), & \\ P_{f_1}(\omega_{f_1}^\ell), & \ell = 1, \ldots, L_{f_1} \end{array} \right\}, \tag{3.8.107}$$

and the fusion result after the fusion of D_1 is given as

$$D_{f_2} = \left\{ \begin{array}{ll} P_{f_2}(\Omega), & \\ P_{f_2}(\omega_{f_2}^\ell), & \ell = 1, \ldots, L_{f_2} \end{array} \right\}. \tag{3.8.108}$$

Then, according to the fundamental principle 4 for DSF model,

$$\begin{array}{ll} P'_{f_2}(\omega_1^1) = P'_{f_1}(\omega_1^1), & P_1(\Omega) = 0, \\ P'_{f_2}(\omega_1^1) > P'_{f_1}(\omega_1^1), & P_1(\Omega) > 0, \end{array} \tag{3.8.109}$$

where the normalized probabilities are given as $P'_{f_i}(\omega_1^1) = \dfrac{P_{f_i}(\omega_1^1)}{1 - P_{f_i}(\Omega)}$ for $i = 1, 2$. As will be shown in the following example, Dempster-Shafer evidential inference method does not satisfy the inequality in (3.8.109). In other words, if total probabilities are computed using the normalized probabilities (e.g., $P'_k(\omega_k^\ell)$'s for D_k), the fundamental principle 4 for DSF model is not satisfied.

Example 3.8.1 *Let $\Omega = \{a_1, a_2\}$. Suppose D_1 and the fusion result before the fusion of D_1 (D_{f_1}) are given as $D_1 = \{P_1(a_1) = 0.4,\ P_1(a_2) = 0.3,\ P_1(\Omega) = 0.3\}$ and $D_{f_1} = \{P_{f_1}(a_1) = 0.6,\ P_{f_1}(a_2) = 0.4\}$. By Dempster-Shafer evidential inference method, the fusion result after the fusion of D_1 is $D_{f_2} = \{P_{f_2}(a_1),\ P_{f_2}(a_2)\}$, where*

$$\begin{aligned} P_{f_2}(a_1) &= \frac{P_{f_1}(a_1)\,(P_1(a_1) + P_1(\Omega))}{P_{f_1}(a_1)\,(P_1(a_1) + P_1(\Omega)) + P_{f_1}(a_2)\,(P_1(a_2) + P_1(\Omega))} \\ &= \frac{0.6 \times (0.4 + 0.3)}{0.6 \times (0.4 + 0.3) + 0.4 \times (0.3 + 0.3)} = 0.54, \end{aligned} \tag{3.8.110}$$

and $P_{f_2}(a_2) = 1 - P_{f_2}(a_1) = 0.46$. The fusion result violates the fundamental principle 4 for DSF model which requires that

$$P_1(a_1) > P_1(\omega_1^\ell),\ \forall\, \omega_1^\ell \neq a_1 \quad \Rightarrow \quad P_{f_2}(a_1) > P_{f_1}(a_1). \tag{3.8.111}$$

Recall that in the proof of the restricted fundamental principle 4 for DSF model, we showed that when there is no probability of uncertainty, the identity information in sensor reports are fused together very well. Therefore the above example only indicates that there is some problem with the way DoC values are introduced in Dempster-Shafer evidential inference method.

Chapter 4

Convex Quadratic Fusion Method

In the previous chapter, we have proposed two models for decision level identity fusion: the Dissimilar Sensor Fusion (DSF) model and the Similar Sensor Fusion (SSF) model. In this chapter, we introduce a new decision fusion method for the SSF model, which we call the convex quadratic fusion method. The new method is based on the minimization of inconsistencies between the fusion result and sensor reports, which yields a probability distribution with an overall best "fit" of the potentially inconsistent sensor reports. It turns out that this formulation leads to a convex quadratic minimization problem which can be solved efficiently in polynomial time. This compares favorably to Dempster-Shafer evidential inference method which has an exponential complexity. In addition, our method can handle probabilistic, inconsistent or incomplete sensor reports without the knowledge of prior probabilities or conditional probabilities. Note that part of the research results presented in this chapter has been published [36, 37].

4.1 Sensor reports

The proposed convex quadratic fusion method fuses the so-called ratio type sensor reports. A sensor report D_k is of the form

For subsets $\omega_k^1, \ldots, \omega_k^{L_k}$ of Ω, their relative likelihood are given by $r_k^1, \ldots, r_k^{L_k}$ respectively, with $r_k^\ell \geq 0$ for $\ell = 1, \ldots, L_k$. The summation $r_k^1 + \cdots + r_k^{L_k} = w_k$, *which is required to lie in $[0, \infty)$, represents the Degree of Confidence (DoC) of the report.*

In other words, the sensor declares, with certainty w_k, that the probabilities of $\omega_k^1, \ldots, \omega_k^{L_k}$ satisfy $P_k(\omega_k^1) : \cdots : P_k(\omega_k^{L_k}) = r_k^1 : \cdots : r_k^{L_k}$. Note that $DoC_{min} = 0$ and $DoC_{max} = \infty$. This form has no restriction imposed on how the subsets are selected. The selected subsets are not necessarily mutually exclusive or exhaustive.

4.2 Problem formulation

Suppose there are a total of K sensors making observations on a target within a surveillance region. All the sensors explore some common physical characteristics of the target. Each sensor, based on its own observation and local data processing, generates a ratio type report which is sent to the fusion center. Let us denote these reports by

$$D_k = \left\{ \begin{array}{l} w_k = r_k^1 + \cdots + r_k^{L_k}, \\ P_k(\omega_k^1) : \cdots : P_k(\omega_k^{L_k}) = r_k^1 : \cdots : r_k^{L_k} \end{array} \right\}, \quad k = 1, ..., K. \tag{4.2.1}$$

The fusion goal is to determine a set of probabilities $P_f(a_i) = p_i$ for $i = 1, \ldots, N$ that best fits the given sensor reports. Clearly, the chosen probabilities must satisfy

$$p_1 + \cdots + p_N = 1, \quad p_i \geq 0, \quad i = 1, \ldots, N. \tag{4.2.2}$$

The basic idea of our fusion method is to first set up a cost function for each sensor report, and then minimize a weighted sum of all the cost functions subject to the probability constraint (4.2.2). Intuitively, the cost function for D_k should measure the discrepancy between the probability given to subset ω_k^ℓ in sensor report D_k and that in the fusion result D_f. The former is given by $P_k(\omega_k^\ell)$, and the later, denoted as $P_f(\omega_k^\ell)$, is given by

$$P_f(\omega_k^\ell) = \sum_{j:\ a_j \in \omega_k^\ell} p_j. \tag{4.2.3}$$

In particular, the cost should reach its minimal value when $P_k(\omega_k^\ell) = P_f(\omega_k^\ell)$ and should increase as $P_k(\omega_k^\ell)$ drifts away from $P_f(\omega_k^\ell)$.

We propose to use the following cost function $C_k(P)$ for sensor report D_k:

$$C_k(P) = c_k(P)^2 \sum_{\ell=1}^{L_k} \left(\frac{r_k^\ell}{w_k} - \frac{P_f(\omega_k^\ell)}{c_k(P)} \right)^2, \tag{4.2.4}$$

where $P_f(\omega_k^\ell)$ is given by (4.2.3), and

$$c_k(P) = P_f(\omega_k^1) + \cdots + P_f(\omega_k^{L_k}). \tag{4.2.5}$$

In the cost function $C_k(P)$, the term $\sum_{\ell=1}^{L_k} \left(\frac{r_k^\ell}{w_k} - \frac{P_f(\omega_k^\ell)}{c_k(P)} \right)^2$ is the normalized discrepancy between sensor report D_k and fusion result D_f. This discrepancy is then weighted by $c_k(P)^2$ to reflect the likely importance of the report.

It can be seen that $C_k(P) \geq 0$ and is zero whenever $P_k(\omega_k^\ell) = P_f(\omega_k^\ell)$. The latter is because if $P_k(\omega_k^\ell) = P_f(\omega_k^\ell)$, then the condition $r_k^1 : \cdots : r_k^{L_k} = P_f(\omega_k^1) : \cdots : P_f(\omega_k^{L_k})$ implies

$$\frac{r_k^\ell}{w_k} = \frac{r_k^\ell}{r_k^1 + \cdots + r_k^{L_k}} = \frac{P_k(\omega_f^\ell)}{P_f(\omega_k^1) + \cdots + P_f(\omega_k^{L_k})} = \frac{P_f(\omega_k^\ell)}{c_k(P)}. \tag{4.2.6}$$

This further implies $C_k(P) = 0$. Also note that $C_k(P)$ satisfies all the properties for the distance measure $dist(\cdot, \cdot)$ in section 3.6.4 (see page 58).

The overall cost function for the K sensor reports is defined as

$$C_{f_K}(P) = \sum_{k=1}^{K} (w_k)^2 C_k(P) = \sum_{k=1}^{K} \sum_{\ell=1}^{L_k} \left(c_k(P) r_k^\ell - w_k P_f(\omega_k^\ell) \right)^2. \tag{4.2.7}$$

Here, each individual cost function $C_k(P)$ is weighted by $(w_k)^2$ to reflect the DoC of report D_k. Also note that $C_{f_K}(P)$ is a convex quadratic function of the variables $P = \{p_1, \ldots, p_N\}$.

Now we can formulate the identity fusion problem as the following convex quadratic programming problem:

$$\begin{aligned} &\text{minimize} \quad C_{f_K}(P) = \sum_{k=1}^{K} (w_k)^2 C_k(P) = \sum_{k=1}^{K} \sum_{\ell=1}^{L_k} \left(c_k(P) r_k^\ell - w_k P_f(\omega_k^\ell) \right)^2 \\ &\text{subject to} \quad \sum_{i=1}^{N} p_i = 1, \quad p_i \geq 0, \ i = 1, 2, \ldots, N. \end{aligned} \tag{4.2.8}$$

The above linearly constrained convex quadratic programming problem has N variables and a simplex constraint. As such, it can be solved very efficiently (i.e., in polynomial time) [32]. This is in contrast to Dempster-Shafer evidential inference approach which suffers from exponentially growing computational complexity as K and N increase.

Next, we discuss the uniqueness of the fusion result. Problem (4.2.8) has a convex objective function and a simplex constraint, which ensures its optimal solution. The uniqueness of the solution dependents on the Hessian matrix of (4.2.8). Formally, (4.2.8) can be written in the following standard form:

$$\begin{aligned} &\text{minimize} \quad \mathbf{p}^T \mathbf{H}_K \mathbf{p} + 2\mathbf{f}_K^T \mathbf{p} \\ &\text{subject to} \quad \mathbf{A}^T \mathbf{p} = \mathbf{b}, \quad \mathbf{p} \geq \mathbf{0}, \end{aligned} \tag{4.2.9}$$

where $\mathbf{H}_K \in \Re^{N\times N}$, $\mathbf{f} \in \Re^{N\times 1}$, $\mathbf{A} \in \Re^{N\times M}$, $\mathbf{p} = [p_1 \ \cdots \ p_N] \in \Re^{N\times 1}$ and $\mathbf{b} \in \Re^{M\times 1}$. It is easy to verify that for the convex quadratic formulation (4.2.8), $\mathbf{A} = [1 \ \cdots \ 1]^T$, $\mathbf{b} = 1$, $\mathbf{f}_K = \mathbf{0}$, M=1, and $\mathbf{H}_K$ is given by

$$\mathbf{H}_K = \sum_{k=1}^{K} \sum_{\ell=1}^{L_k} \mathbf{H}_k^\ell (\mathbf{H}_k^\ell)^T, \tag{4.2.10}$$

where $\mathbf{H}_k^\ell \in \Re^{N\times 1}$, and the i'th element of $\mathbf{H}_k^\ell$ is given by

$$(\mathbf{H}_k^\ell)_i = \begin{cases} 0, & a_i \notin \cup_{j=1}^{L_k} \omega_k^\ell, \\ -m_k r_k^\ell, & a_i \in \cup_{j=1}^{L_k} \omega_k^\ell, \ \ a_i \notin \omega_k^\ell, \\ w_k - m_k r_k^\ell, & a_i \in \omega_k^\ell, \end{cases} \tag{4.2.11}$$

and m_k is the number of all ω_k^ℓ's ($\ell = 1, \ldots, L_k$) with $a_i \in \omega_k^\ell$. The solution to problem (4.2.8) is unique if and only if $\mathbf{H}_K > 0$, which is equivalent to the condition that $\mathbf{H}_K$ is of full rank ($\mathbf{H}_K \neq 0$).

In the special case where all the K sensors assign probabilities to all and only the basic propositions, the k'th sensor report can be written as $D_k = \{P_k(a_1) : \cdots : P_k(a_N) = w_k p_k^1 : \cdots : w_k p_k^N\}$, where w_k is the DoC of D_k, and $P_k(a_i) = p_k^i$ represents the probability assigned to proposition a_i in report D_k, with $\sum_{i=1}^{N} p_k^i = 1$. The objective function is

$$\begin{aligned} C_f(P) &= \sum_{k=1}^{K}(w_k)^2 \sum_{i=1}^{N}(p_i - p_k^i)^2 = \sum_{k=1}^{K}(w_k)^2 \sum_{i=1}^{N}\left(p_i^2 - 2p_k^i p_i + (p_k^i)^2\right) \\ &= \sum_{i=1}^{N}\left(\left(\sum_{k=1}^{K}(w_k)^2\right)p_i^2 - 2\left(\sum_{k=1}^{K}(w_k)^2 p_k^i\right)p_i + \sum_{k=1}^{K}(w_k)^2\left(p_k^i\right)^2\right) \\ &= \sum_{i=1}^{N}\left(\sum_{k=1}^{K}(w_k)^2\right)(p_i - r_f^i)^2 + \sum_{i=1}^{N}\sum_{k=1}^{K}(w_k)^2\left(p_k^i\right)^2 - \sum_{i=1}^{N}(r_f^i)^2, \end{aligned} \tag{4.2.12}$$

where $r_f^i = \dfrac{\sum_{k=1}^{K}(w_k)^2 p_k^i}{\sum_{k=1}^{K}(w_k)^2}$. Because $\sum_{i=1}^{N} r_f^i = \dfrac{1}{\sum_{k=1}^{K}(w_k)^2}\sum_{i=1}^{N}\sum_{k=1}^{K}(w_k)^2 p_k^i = 1$, the fusion result for this case is simply

$$P_f(a_i) = r_f^i = \frac{\sum_{k=1}^{K}(w_k)^2 p_k^i}{\sum_{k=1}^{K}(w_k)^2}, \quad i = 1, \ldots, N. \tag{4.2.13}$$

This solution has an attractive interpretation: the optimal fusion result is obtained by averaging sensor reports using DoC values w_k^2's. Such an averaging minimizes the inconsistencies in the sensor reports. Therefore, this special case provides a positive justification for the new method.

The convex quadratic fusion method can operate in both the sequential and batch fusion modes. However, the fusion method does not generate the same final fusion results for the two fusion modes. This is because if we use the sequential fusion mode to fuse the interim fusion result with a new sensor report, the corresponding cost function is different from the one using the batch mode, and therefore does not correctly represent the discrepancies among the sensor reports. Moreover, when operating in the sequential fusion mode, the method does not satisfy most of the decision fusion principles. For this reason, we do not advise the use of the convex quadratic fusion method in the sequential fusion mode. Note that fusing sensor reports all at once incurs little extra computation as the number of sensor reports increases. The reason is that the fusion involves solving a convex optimization problem whose worst case complexity is independent of the number of sensor reports. In addition, the overall cost function to be minimized can be recursively updated as new sensor reports are fused. In particular, if we denote the overall cost function for sensor report $D_1, \ldots, D_K$ as $C_{f_K}(P)$, then, when a new sensor report D_{K+1} is fused, the overall cost function is updated as

$$C_{f_{K+1}}(P) = \sum_{k=1}^{K+1} C_k(P) = C_{f_K}(P) + C_{K+1}(P). \tag{4.2.14}$$

Alternatively, we can recursively update the Hessian matrix $\mathbf{H}_K$ in (4.2.9), i.e.,

$$\mathbf{H}_{K+1} = \mathbf{H}_K + \sum_{\ell=1}^{L_K} \mathbf{H}_K^\ell (\mathbf{H}_K^\ell)^T. \tag{4.2.15}$$

The above update equations suggest that it is unnecessary to keep the cost functions $C_k(P)$'s for the fusion (keeping the overall cost function $C_{f_K}(P)$ or its Hessian matrix $\mathbf{H}_k$ is sufficient). However, as will be introduced in the next section, computation of the DoC of the fusion result requires these cost functions. As a result, all the cost functions need to be maintained for future fusion steps.

4.3 Degree of Confidence for the fusion result

The DoC of the fusion result should be determined in such a way that the two DoC principles for SSF model (proposed in section 3.7) can be satisfied. In particular, we choose the quadratic cost function given in Eq. (4.2.4) as the proper distance measure in the DoC principle 1 for SSF model. The DoC of the fusion result should also satisfy property (3.6.36) in the fundamental principle 5 for SSF model. Moreover, as stated in the fundamental principle 1, when there is only one sensor report to fuse, the DoC of the fusion result should be the same with that of the single report.

Let w_f denote the DoC associated with the fusion result D_f. We propose to define w_f as follows:

$$w_f = \sum_{k=1}^{K} w_k \cdot \frac{C_{k_{max}} - C_k(D_f)}{C_{k_{max}}}, \tag{4.3.16}$$

where $C_k(D_f)$ is the value of cost function $C_k(P)$ in Eq. (4.2.4) evaluated using the probability assignments in fusion result D_f, and $C_{k_{max}}$ is given by:

$$\begin{aligned} &\max_{r^1,\dots,r^{L_k}} \quad c_k(P)^2 \sum_{\ell=1}^{L_k} \left(\frac{r^\ell}{w_k} - \frac{P_f(\omega_k^\ell)}{c_k(P)} \right)^2 \\ &\text{subject to} \quad \sum_{\ell=1}^{L_k} r^\ell = w_k, \quad r^\ell \geq 0, \ \ell = 1, \dots, L_k. \end{aligned} \tag{4.3.17}$$

Since (4.3.17) is a concave optimization problem, the optimal solution is at one of the vertices of the feasible region. Therefore,

$$C_{k_{max}} = \max_{\substack{i=1,\dots,N: \\ r^i = 1, \ r^j = 0, \ \forall j \neq i}} c_k(P)^2 \sum_{\ell=1}^{L_k} \left(\frac{r^\ell}{w_k} - \frac{P_f(\omega_k^\ell)}{c_k(P)} \right)^2. \tag{4.3.18}$$

It is easy to verify that w_f determined by Eq. (4.3.16) satisfies all the requirements discussed above.

Eq. (4.3.16) can be rationalized as follows. In the SSF model, w_f is determined by the DoC values of all the reports and how strong each report confirms with D_f. Specifically, w_f should be the sum of contributions from each w_k. The stronger D_k agrees with D_f, the more w_k should contribute to w_f. In the extreme case that D_k fully agrees with D_f ($C_k(D_f) = 0$), all the confidence associated with D_k should be contributed to D_f. In contrast, if D_f completely disagree with D_f ($C_k(D_f) = C_{k_{max}}$), fusion of D_k does not affect the confidence of the fusion result at all. In other words, no part of w_k should be added to w_f. These two cases are illustrated in the following example.

Example 4.3.1 *Let $\Omega = \{a_1, a_2\}$. Suppose there are two similar and independent sensors* *each sending a report to the fusion center. We consider the following two cases.*

Case 1: *The two sensor reports are identical to each other except their DoC values. The reports are given as*

$$D_1 = \left\{ \begin{array}{l} w_1 = 1, \\ P_1(a_1) = 0.3,\ P_1(a_2) = 0.7 \end{array} \right\}, \quad D_2 = \left\{ \begin{array}{l} w_2 = 2, \\ P_2(a_1) = 0.3,\ P_2(a_2) = 0.7 \end{array} \right\}. \quad (4.3.19)$$

Obviously, the fusion result should be the same with the two reports: $D_f = \{P_f(a_1) = 0.3,\ P_f(a_2) = 0.7\}$. *In addition,* D_f *should have all the confidence associated with* D_1 *and* D_2, *i.e.,*

$$w_f = \sum_{k=1}^{2} w_k \cdot \frac{C_{k_{max}} - C_k(D_f)}{C_{k_{max}}} = \sum_{k=1}^{2} w_k \cdot \frac{C_{k_{max}}}{C_{k_{max}}} = w_1 + w_2 = 3. \quad (4.3.20)$$

Case 2: *The two sensor reports are given by*

$$D_1 = \left\{ \begin{array}{l} w_1 = 1, \\ P_1(a_1) = 1,\ P_1(a_2) = 0 \end{array} \right\}, \quad D_2 = \left\{ \begin{array}{l} w_2 = 1, \\ P_2(a_1) = 0,\ P_2(a_2) = 1 \end{array} \right\}. \quad (4.3.21)$$

Note that D_1 *and* D_2 *are cyclic sensor reports. In the convex quadratic fusion method, the cost functions for report* D_1 *and* D_2 *are*

$$C_1(P) = (p_1 - P_1(a_1))^2 + (p_1 - P_2(a_2))^2 = (p_1 - 1)^2 + (p_2)^2, \quad (4.3.22)$$

and

$$C_2(P) = (p_1 - P_2(a_1))^2 + (p_2 - P_2(a_2))^2 = (p_1)^2 + (p_2 - 1)^2. \quad (4.3.23)$$

By minimizing $C_1(P) + C_2(P)$ *with the probabilistic constraints (4.2.2), we obtain the fusion result as* $D_f = \{P_f(a_1) = 0.5,\ P_f(a_2) = 0.5\}$. *We also compute that* $C_1(D_f) = (0.5 - 1)^2 +$

$0.5^2 = 0.5$, $C_2(D_f) = 0.5^2 + (0.5-1)^2 = 0.5$, *and*

$$C_{k_{max}} = \max_{\substack{i=1,2:\\ r^i=1,\ r^j=0,\ i\neq j}} (r^1-0.5)^2 + (r^2-0.5)^2 = (1-0.5)^2 + (0-0.5)^2 = 0.5. \quad (4.3.24)$$

Therefore, the DoC of D_f *is* $w_f = \sum_{k=1}^{2} w_k \cdot \frac{C_{k_{max}} - C_k(D_f)}{C_{k_{max}}} = 0$. *Here,* D_1 *and* D_2 *completely disagree with each other, and both of them disagree with the fusion result to the maximal extent. As a result,* D_f *has no confidence at all.*

Note that in Eq. (4.3.16), both $C_{k_{max}}$ and $D_k(D_f)$ varies with the fusion result D_f. Therefore, when a new sensor report is fused, we need to compute the $C_{k_{max}}$ and $D_k(D_f)$ again for each D_k using the new fusion result. In other words, w_f can not be recursively updated like the cost function $C_{f_K}(P)$ (Eq. (4.2.14)). Moreover, we need to maintain the cost function forms ($C_k(P)$'s) for future fusion steps.

4.4 Appropriate sensor report form

As introduced in section 4.1, a ratio type sensor report can always be expressed in the form of the following comparison:

$$P_k(\omega_k^1) : \cdots : P_k(\omega_k^{L_k}) = r_k^1 : \cdots : r_k^{L_k}. \quad (4.4.25)$$

From probabilistic point of view, this comparison can be expressed in various forms using intersections and unions of the ω_k^ℓ's without changing the probabilities of the original subsets. It can also be broken into multiple comparisons. For example, $P_1(a_1) : P_1(a_2) : P_1(a_3) = 1 : 1 : 1$ can be expressed as $P_1(a_1 \vee a_2) : P_1(a_2) : P_1(a_3) = 2 : 1 : 1$, or be broken into two comparisons: $P_1(a_1) : P_1(a_2) = 1 : 1$ and $P_1(a_1) : P_1(a_3) = 1 : 1$. Although the above forms are equivalent in the probability sense, their corresponding cost functions are different in the convex quadratic fusion method, leading to different fusion results. This is illustrated in the following example.

Example 4.4.1 *Let $\Omega = \{a_1, a_2, a_3\}$. Suppose there are two similar and independent sensors, each of them sends a report to the fusion center. Report D_1 is given by $D_1 = \{P_1(a_1) : P_1(a_2) : P_1(a_3) = 0.70 : 0.15 : 0.15\}$, and report D_2 is given in the following two forms:*

$$\begin{aligned} D_{21} &= \{P_{21}(a_1) : P_{21}(a_2) : P_{21}(a_3) = 0.90 : 0.05 : 0.05\}, \\ D_{22} &= \{P_{22}(a_1) : P_{22}(a_2) = 0.90 : 0.05, \quad P_{22}(a_1) : P_{22}(a_3) = 0.90 : 0.05\}. \end{aligned} \tag{4.4.26}$$

Note that D_{21} and D_{22} are equivalent in the probabilistic sense. By the convex quadratic fusion method, the fusion result of D_1 and D_{21} is $D_{f_1} = \{P_{f_1}(a_1) = 0.80,\ P_{f_1}(a_2) = 0.10,\ P_{f_1}(a_3) = 0.10\}$, and the fusion result of D_1 and D_{22} is $D_{f_2} = \{P_{f_2}(a_1) = 0.7850,\ P_{f_2}(a_2) = 0.1075,\ P_{f_2}(a_3) = 0.1075\}$. Note that D_{f_1} and D_{f_2} are different.

It is now clear that we need to choose an appropriate sensor report form. To do this, we break the comparison in (4.4.25) into the following group of comparisons:

$$P_k(\omega_k^\ell) : \sum_{j=1}^{L_k} P_k(\omega_k^j) = r_k^\ell : w_k, \quad \ell = 1, \dots, L_k. \tag{4.4.27}$$

The weighted cost function for (4.4.25), i.e., the cost term used in the optimization problem (4.2.8), is

$$C_k'(P) = (w_k)^2 C_k(P) = \sum_{\ell=1}^{L_k} \left(c_k(P) r_k^\ell - w_k P_f(\omega_k^\ell)\right)^2, \tag{4.4.28}$$

while the weighted cost function for (4.4.27) is

$$\begin{aligned} C_k''(P) &= (w_k + r_k^\ell)^2 C_k'(P) = \sum_{\ell=1}^{L_k} \Big(\left(\left(c_k(P) + P_f(\omega_k^\ell)\right) r_k^\ell - (w_k + r_k^\ell) P_f(\omega_k^\ell) \right)^2 \\ &\quad + \left(\left(c_k(P) + P_f(\omega_k^\ell)\right) w_k - (w_k + r_k^\ell) c_k(P) \right)^2 \Big) = 2C_k'(P). \end{aligned} \tag{4.4.29}$$

The equivalence of $C_k'(P)$ to $C_k(P)$ shown in Eq. (4.4.29) means that comparison (4.4.25) can be replaced by the group of comparisons in (4.4.27). In other words, including a subset into a comparison is equivalent to comparing it with the union of all subsets in that comparison.

Recall that the basic idea of the convex quadratic fusion method is to minimize the weighted summation of normalized errors between the fusion result and all sensor reports. Therefore if we break a comparison into multiple comparisons, the union of all subsets changes; if we use intersections or unions of the original subsets, we change the original subsets. In both cases, the normalized errors of the original subsets change, and the new cost function no longer represents the same information on discrepancy between the fusion result and sensor reports as it does before.

It is now clear that the appropriate sensor report form is to generate a single ratio type report based on each observed evidence, and the subsets in the comparison should be the ones directly involved in the observation. For example, if a sensor detects whether a target is hostile (a_1), neutral (a_2) or friendly (a_3), then its report should be of the following form:

$$P_1(a_1) : P_1(a_2) : P_1(a_3) = r_1^1 : r_1^2 : r_1^3. \tag{4.4.30}$$

The report should not be represented in the following alternative forms:

$$P_1(a_1 \vee a_2) : P_1(a_2) : P_1(a_3) = (r_1^1 + r_1^2) : r_1^2 : r_1^3, \tag{4.4.31}$$

$$P_1(a_1) : P_1(a_2) = r_1^1 : r_1^2 \text{ and } P_1(a_1) : P_1(a_3) = r_1^1 : r_1^3. \tag{4.4.32}$$

The reason is that in (4.4.31) subset $a_1 \vee a_2$ is not directly used in the observation, and in (4.4.32) the unions of subsets, i.e., $a_1 \vee a_2$ and $a_1 \vee a_3$ are not the one ($a_1 \vee a_2 \vee a_3$) used in the observation. As such, these alternative forms cannot appropriately represent the identity information from the sensor.

4.5 Fusion of Dempster-Shafer type sensor reports

In Dempster-Shafer evidential inference method, a sensor assigns probabilities to some subsets of Ω. A sensor report D_k is in the form of

$$D_k = \left\{ \begin{array}{ll} P_k(\Omega), & \\ P_k(\omega_k^\ell), & \ell = 1, \ldots, L_k \end{array} \right\}, \tag{4.5.33}$$

where $P_k(\Omega) + \sum_{\ell=1}^{L_k} P_k(\omega_k^\ell) = 1$, $\omega_k^\ell \subset \Omega$, $P_k(\Omega) \in [0,1]$ and $P_k(\omega_k^\ell) \in [0,1]$.

As discussed in section 3.3, in the above sensor report a probability mass $P_k(\omega_k^\ell)$ may only be part of the total probability for subset ω_k^ℓ. Other probability mass $P_k(\omega_k^j)$ with $\omega_k^\ell \cap \omega_k^j \neq \emptyset$ will also contribute to the total probability for ω_k^ℓ (see page 44). This is different from a ratio type sensor report which compares the total probabilities of subsets. As a result, a Dempster-Shafer type sensor report cannot be directly fused using the convex quadratic fusion method. This is illustrated in the following example.

Example 4.5.1 *Let $\Omega = \{a_1, a_2, a_3\}$. Consider the following Dempster-Shafer type sensor report: $D_1 = \{P_1(a_1) = 0.6,\ P_1(a_2) = 0.2,\ P_1(a_3) = 0.1,\ P_1(a_1 \vee a_2) = 0.1\}$. Note that the probability mass $P_1(a_1 \vee a_2)$ in the above report is not the total probability of $a_1 \vee a_2$. In other words, $P_1(a_1 \vee a_2)$ is only added to $a_1 \vee a_2$. If directly treated as a ratio type sensor report, D_1 should be replaced by the following comparison: $P_1^T(a_1) : P_1^T(a_2) : P_1^T(a_3) : P_1^T(a_1 \vee a_2) = 0.6 : 0.2 : 0.1 : 0.1$, where we use $P_1^T(\omega_1^\ell)$ to represent the total probability of subset ω_1^ℓ. This comparison is incorrect because $P_1^T(a_1) : P_1^T(a_1 \vee a_2) = 0.6 : 0.1$ is never true.*

As a remedy, we propose the following procedure to transform Dempster-Shafer type sensor reports into ratio type reports, so that they can be fused by the convex quadratic fusion method. Consider a Dempster-Shafer type sensor report D_k given in (4.5.33). Suppose a_i is included in each of the subsets $\omega_k^{\ell_1}, \ldots, \omega_k^{\ell_{M_i}}$. In the transformation procedure, we first introduce for each proposition a_i the so-called sub-propositions $a_i^{\ell_1}, \ldots, a_i^{\ell_{M_i}}$, where $a_i^{\ell_j} \in \omega_k^{\ell_j}$ for $j = 1, \ldots, M_i$, $a_i^{\ell_j} \cap a_i^{\ell_m} = \emptyset$ for $j \neq m$, and $a_i^{\ell_1} \cup \cdots \cup a_i^{\ell_{M_i}} = a_i$. In other words, a_i is separated into some mutually exclusive and exhaustive sub-propositions $a_i^{\ell_j}$'s. A subset ω_k^ℓ is then replaced by $\bigcup_{i:a_i\in\omega_k^\ell} a_i^\ell$ and its probability mass $P_k(\omega_k^\ell)$ is replaced by $\sum_{i:a_i\in\omega_k^\ell} P_k(a_i^\ell)$, where $P_k(a_i^\ell)$ is the total probability of sub-proposition a_i^ℓ in report D_k. Now we can transform report D_k into the following ratio type report:

$$\bigcup_{i:a_i\in\omega_k^1} a_i^1 : \cdots : \bigcup_{i:a_i\in\omega_k^{L_k}} a_i^{L_k} = r_k^1 : \cdots : r_k^{L_k}, \tag{4.5.34}$$

where $r_k^\ell = w_k \cdot \dfrac{P_k(\omega_k^\ell)}{1-P_k(\Omega)}$, and $w_k = -\ln P_k(\Omega)$ represents the DoC of report D_k. Note that w_k and $P_k(\Omega)$ has a inverse relationship. Specifically, when $P_k(\Omega) = 1$, $w_k = 0$ represents minimal DoC. In contrast, when $P_k(\Omega) = 0$, $w_k \to \infty$ represents maximal DoC.

If the transformed report is fused by the convex quadratic fusion method, its corresponding cost function is given by

$$C_k(P) = (c_k'(P))^2 \sum_{\ell=1}^{L_k} \left(\frac{r_k^\ell}{w_k} - \frac{P_f'(\omega_k^\ell)}{c_k'(P)} \right)^2, \tag{4.5.35}$$

where $c_k'(P) = \sum_{\ell=1}^{L_k} \sum_{i:\ a_i\in\omega_k^\ell} p(a_i^\ell)$, and $P_f'(\omega_k^\ell) = \sum_{i:\ a_i\in\omega_k^\ell} p_i^\ell$ for $\ell = 1, \ldots, L_k$. Here, $p_i^{\ell_j}$ is the probability assigned to sub-proposition $a_i^{\ell_j}$ in the fusion result. In addition, the following new probability constraints must be added into (4.2.8):

$$\sum_{j=1}^{M_i} p_i^{\ell_j} = p_i,\ i = 1, \ldots, N, \quad p_i^{\ell_i} \ge 0,\ \forall\ i, j. \tag{4.5.36}$$

The transformation procedure is illustrated in the following example.

Example 4.5.2 *Let us reconsider sensor report D_1 in Example 4.5.1. Using the above procedure, we introduce sub-propositions a_1^1 and a_1^3 such that $a_1^1 \cap a_1^3 = \emptyset$ and $a_1^1 \cup a_1^3 = a_1$. Similarly, we introduce sub-propositions a_2^1 and a_2^3 such that $a_2^1 \cap a_2^3 = \emptyset$ and $a_2^1 \cup a_2^3 = a_2$. Also let p_i^j denote the probability assigned to sub-proposition a_i^j in the fusion result. Then,*

D_1 can be transformed as the following ratio type report:

$$P_1(a_1^1) : P_1(a_2^1) : P_1(a_3) : P_1(a_1^3 \vee a_2^3) = 0.6 : 0.2 : 0.1 : 0.1. \tag{4.5.37}$$

By the convex quadratic fusion method, the cost function for D_1 is given by

$$C_1(P) = (p_1^1 - 0.6)^2 + (p_2^1 - 0.2)^2 + (p_3 - 0.1)^2 + (p_1^3 + p_2^3 - 0.1)^2. \tag{4.5.38}$$

If only D_1 is fused by the convex quadratic fusion method, the fusion result is given by

$$\begin{aligned} \textit{maximize} \quad & C_1(P) = (p_1^1 - 0.6)^2 + (p_2^1 - 0.2)^2 + (p_3 - 0.1)^2 + (p_1^3 + p_2^3 - 0.1)^2 \\ \textit{subject to} \quad & p_1 + p_2 + p_3 = 1, \;\; p_1^1 + p_1^3 = p_1, \;\; p_2^1 + p_2^3 = p_2, \\ & p_i \geq 0, \; i = 1,2,3, \quad p_i^j \geq 0, \; i,j = 1,2, \end{aligned} \tag{4.5.39}$$

which is $D_f = \{p_1^1 = 0.6, \; p_2^1 = 0.2, \; p_3 = 0.1, \; p_1^3 + p_2^3 = 0.1\}$. As expected, the fusion result is a restatement of D_1.

4.6 Robustness analysis

In this section, we analyze the robustness of the convex quadratic fusion method. In particular, we consider the situation where all the $P_k(\omega_k^\ell)$'s are perturbed versions of $\bar{P}(\omega_k^\ell)$'s which represent the true probabilities, and study the sensitivity of the resulting solution $P_f(\omega_k^\ell)$'s to the perturbation. Assume that

$$P_k(\omega_k^\ell) = \bar{P}(\omega_k^\ell) + n_k^\ell, \tag{4.6.40}$$

where n_k^ℓ is a stochastic disturbance with zero mean. We assume that the disturbances $\{n_k^\ell\}$ are independent. Then the sensor report from the k'th sensor is

$$D_k = \left\{ \begin{array}{l} w_k = r_k^1 + \cdots + r_k^{L_k}, \\ P_k(\omega_k^1) : \cdots : P_k(\omega_k^{L_k}) = r_k^1 : \cdots : r_k^{L_k} \end{array} \right\}, \tag{4.6.41}$$

with $r_k^\ell = \beta_k P_k(\omega_k^\ell) = \beta_k(\bar{P}(\omega_k^\ell) + n_k^\ell)$ for some $\beta_k \geq 0$. Note that the DoC of D_k can be written as

$$w_k = \sum_{\ell=1}^{L_k} r_k^\ell = \beta_k \sum_{\ell=1}^{L_k} \left(\bar{P}(\omega_k^\ell) + n_k^\ell\right) = \beta_k \left(c_k(\bar{P}) + \sum_{\ell=1}^{L_k} n_k^\ell \right), \tag{4.6.42}$$

where $c_k(\bar{P}) = \bar{P}(\omega_k^1) + \cdots + \bar{P}(\omega_k^{L_k})$. The overall cost function $C_{f_K}(P)$ is

$$\begin{aligned}
C_{f_K}(P) &= \sum_{k=1}^{K} (w_k)^2 C_k(P) = \sum_{k=1}^{K} \sum_{\ell=1}^{L_k} \left(c_k(P) r_k^\ell - w_k P(\omega_k^\ell) \right)^2 \\
&= \sum_{k=1}^{K} \sum_{\ell=1}^{L_k} \left(c_k(P)\beta_k \left(\bar{P}(\omega_k^\ell) + n_k^\ell\right) - \beta_k \left(c_k(\bar{P}) + \sum_{j=1}^{L_k} n_k^j \right) P_f(\omega_k^\ell) \right)^2 \\
&= \sum_{k=1}^{K} \beta_k^2 \sum_{\ell=1}^{L_k} \left(c_k(P)\bar{P}(\omega_k^\ell) + c_k(P) n_k^\ell - c_k(\bar{P}) P_f(\omega_k^\ell) - P_f(\omega_k^\ell) \sum_{j=1}^{L_k} n_k^j \right)^2 \\
&= \sum_{k=1}^{K} \beta_k^2 \sum_{\ell=1}^{L_k} \left(\left(c_k(P)\bar{P}(\omega_k^\ell) - c_k(\bar{P}) P_f(\omega_k^\ell) \right) + \left(c_k(P) n_k^\ell - P_f(\omega_k^\ell) \sum_{j=1}^{L_k} n_k^j \right) \right)^2
\end{aligned} \tag{4.6.43}$$

Here, the third step follows from (4.6.42). Taking expectation in the above expression with respect to $\{n_k^\ell\}$ and noting that $\{n_k^\ell\}$ have zero mean, we obtain

$$\begin{aligned}
E\left(C_{f_K}(P)\right) &= \sum_{k=1}^{K} \beta_k^2 \sum_{\ell=1}^{L_k} \Bigg(\left(c_k(P)\bar{P}(\omega_k^\ell) - c_k(\bar{P}) P_f(\omega_k^\ell) \right)^2 \\
&\quad + 2 \left(c_k(P)\bar{P}(\omega_k^\ell) - c_k(\bar{P}) P_f(\omega_k^\ell) \right) \left(c_k(P) E(n_k^\ell) - P_f(\omega_k^\ell) \sum_{j=1}^{L_k} E(n_k^j) \right) \\
&\quad + E \left(c_k(P) n_k^\ell - P_f(\omega_k^\ell) \sum_{j=1}^{L_k} n_k^j \right)^2 \Bigg) \\
&= \sum_{k=1}^{K} \beta_k^2 \sum_{\ell=1}^{L_k} \left(c_k(P)\bar{P}(\omega_k^\ell) - c_k(\bar{P}) P_f(\omega_k^\ell) \right)^2 \\
&\quad + \sum_{k=1}^{K} \beta_k^2 \sum_{\ell=1}^{L_k} E \left(c_k(P) n_k^\ell - P_f(\omega_k^\ell) \sum_{j=1}^{L_k} n_k^j \right)^2,
\end{aligned} \tag{4.6.44}$$

where $E\left(C_{f_K}(P)\right)$ denotes the expectation of $C_{f_K}(P)$ with respect to P. Note that

$$\sum_{\ell=1}^{L_k} E\left(c_k(P)n_k^\ell - P_f(\omega_k^\ell)\sum_{j=1}^{L_k} n_k^j\right)^2 = \sum_{\ell=1}^{L_k} E\left(\left(c_k(P) - P_f(\omega_k^\ell)\right) n_k^\ell - P_f(\omega_k^\ell) \sum_{j=1, j\neq\ell}^{L_k} n_k^j\right)^2$$
$$= \sum_{\ell=1}^{L_k}\left(\left(c_k(P) - P_f(\omega_k^\ell)\right)^2 E[(n_k^\ell)^2] + (P_f(\omega_k^\ell))^2 \sum_{j=1, j\neq\ell}^{L_k} E[n_k^j]^2\right), \quad (4.6.45)$$

where all the cross terms vanish because the n_k^ℓ's are independent with each other and have zero mean. Now suppose

$$E[n_k^1]^2 = \cdots = E[n_k^{L_k}]^2 = \sigma_k^2. \quad (4.6.46)$$

Then we can further simplify (4.6.45) as

$$\sum_{i=1}^{L_k} E\left(c_k(P)n_k^\ell - P_f(\omega_k^\ell)\sum_{j=1}^{L_k} n_k^j\right)^2$$
$$= \sum_{\ell=1}^{L_k}\left(\left(c_k(P) - P_f(\omega_k^\ell)\right)^2 E[(n_k^\ell)^2] + (P_f(\omega_k^\ell))^2 \sum_{j=1, j\neq\ell}^{L_k} E[(n_k^j)^2]\right)$$
$$= \sigma_k^2\left(\sum_{\ell=1}^{L_k}\left(c_k(P) - P_f(\omega_k^\ell)\right)^2 + (L_k - 1)\sum_{\ell=1}^{L_k}(P_f(\omega_k^\ell))^2\right)$$
$$= \sigma_k^2\left((L_k - 2)(c_k(P))^2 + L_k\sum_{\ell=1}^{L_k}(P_f(\omega_k^\ell))^2\right). \quad (4.6.47)$$

We can now substitute (4.6.47) to (4.6.44) to obtain

$$E[C_{f_K}(P)] = \sum_{k=1}^{K}\beta_k^2\left(\sum_{\ell=1}^{L_k}\left(c_k(P)\bar{P}(\omega_k^\ell) - c_k(\bar{P})P_f(\omega_k^\ell)\right)^2\right)$$
$$+\sum_{k=1}^{K}\beta_k^2\sigma_k^2\left((L_k - 2)(c_k(P))^2 + L_k\sum_{\ell=1}^{L_k}(P_f(\omega_k^\ell))^2\right). \quad (4.6.48)$$

It can be seen from the above expression that $E(C_{f_K}(P))$ consists of two terms, the second of which is proportional to the disturbance variance σ_k^2's. Minimizing the first term will make

$P_f(\omega_k^\ell)$ proportional to $\bar{P}(\omega_k^\ell)$ (the true distribution), while minimizing the second term will drive $P_f(\omega_k^\ell)$ (and thus $c_k(P)$) to zero. This implies that the deviation of our estimate P (given as the optimal solution of (4.2.8)) from the true distribution $\bar{P}$ is proportional to the magnitude σ_k's. This result is intuitively appealing: unreliable reports will lead to an unreliable fusion result. On the other hand, if the disturbances n_k^ℓ's are absent, then the second term of (4.6.48) vanishes, and minimizing the cost function $E(C_{f_K}(P))$ will lead to the identification of the true probability distribution $P_f(\omega_k^\ell) = \bar{P}(\omega_k^\ell)$ for all k and ℓ.

4.7 Satisfaction of decision fusion principles

In this section, we analyze the convex quadratic fusion method using the decision fusion principles proposed in the previous chapter.

Theorem 4.7.1 *The convex quadratic fusion method satisfies fundamental principles 1, 2, 3, and fundamental principles 4 and 5 for SSF model. The fusion method also satisfies DoC principles 1 and 2 for SSF model.*

Proof. Let $\Omega = \{a_1, \ldots, a_N\}$. Suppose a sensor report D_k takes the following form:

$$D_k = \left\{ \begin{array}{l} w_k = r_k^1 + \cdots + r_k^{L_k}, \\ P_k(\omega_k^1) : \cdots : P_k(\omega_k^{L_k}) = r_k^1 : \cdots : r_k^{L_k} \end{array} \right\}, \tag{4.7.49}$$

where $r_k^\ell \geq 0$ for $\ell = 1, \ldots, L_k$, and w_k is the DoC of D_k. Also define the distance measure $dist(D_k, D_f)$ as the cost function $C_k(D_f)$ for D_k (4.2.4), where $C_k(D_f)$ denotes the value of cost function $C_k(P)$ using the probability assignments in D_f. As introduced in the previous chapter, $dist(D_k, D_f)$ is used in the fundamental principles 4 and 5 for SSF model, and DoC principle 1 for SSF model. Note that $C_k(D_f)$ satisfies the properties proposed in section 3.6.4 (see page 58). We Now examine the decision fusion principles one by one.

a) Fundamental principle 1: Suppose there is only one sensor report D_1 to fuse. The overall cost function is $C_{f_1}(P) = (w_1)^2 C_1(P) = \sum_{\ell=1}^{L_1} \left(c_1(P) r_1^\ell - w_1 P_f(\omega_1^\ell)\right)^2$. Obviously, if we choose

$$P_f(\omega_1^\ell) = \frac{r_1^\ell}{w_1}, \qquad \ell = 1, \ldots, L_1, \tag{4.7.50}$$

the cost function reaches its minimal value:

$$C_{f_1}(D_f) = \sum_{\ell=1}^{L_1} \left(\sum_{m=1}^{L_1} \left(\frac{r_1^m}{w_1} \right) \cdot r_1^\ell - w_1 \cdot \left(\frac{r_1^\ell}{w_1} \right) \right)^2 = 0. \tag{4.7.51}$$

In other words, (4.7.50) is the optimal fusion result. The DoC of the fusion result can be calculated as $w_f = w_1 \cdot \dfrac{C_{1_{max}} - C_1(D_f)}{C_{1_{max}}} = w_1$. As such, the fusion result is identical with D_1.

b) Fundamental principle 2: This is straight forward. As discussed on page 87, we only use the convex quadratic fusion method in the batch fusion mode. Specifically, sensor reports are fused all at once by minimizing a cost function (Eq. (4.2.7)) which is constructed using original sensor reports. The cost function is obviously invariant with the permutation of the indexing of the sensor reports, and so is the fusion result.

c) Fundamental principle 3: Suppose there are N cyclic sensor reports $D_1, \ldots, D_N$ with D_k given as

$$D_k = \left\{ \begin{array}{l} w_r = \sum_{i=1}^{N} r_{i \oplus k} = \sum_{i=1}^{N} r_i, \\ P_k(a_i) = r_{i \oplus k}, \quad i = 1, \ldots, N \end{array} \right\}, \tag{4.7.52}$$

where $r_i \geq 0$ for $i = 1, \ldots, N$, and $i \oplus k = i + k - 1 \mod N$. Note that the DoC values of all sensor reports are the same. The overall cost function is

$$\begin{aligned} C_{f_N}(P) &= \sum_{k=1}^{N} (w_k)^2 C_k(P) = \sum_{k=1}^{N}\sum_{i=1}^{N} (r_{i\oplus k} - w_i p_i)^2 \\ &= \sum_{k=1}^{N}\sum_{i=1}^{N} \left(r_{i\oplus k}^2 - 2r_{i\oplus k} w_i p_i + (w_i)^2 p_i^2\right) \\ &= N\sum_{i=1}^{N}(r_i)^2 - 2N(w_1)^2 + N(w_1)^2 \sum_{i=1}^{N} p_i^2. \end{aligned} \tag{4.7.53}$$

Obviously, minimization of the above cost function yields the following fusion result: $p_i = 1/N$ for $i = 1, \ldots, N$, In other words, the convex quadratic fusion method satisfies the fundamental principle 3.

d) Fundamental principle 4 for SSF model: Suppose there are K sensor reports $D_1, \ldots, D_K$. Let D_{f_1} and D_{f_2} denote the fusion results before and after the fusion of sensor report D_1. Let us denote the cost function before the fusion of D_1 as $C_{f_1}(P)$, which is given as

$$C_{f_1}(P) = \sum_{k=2}^{K} (w_k)^2 C_k(P), \tag{4.7.54}$$

and the cost function after the fusion of D_1 as $C_{f_2}(P)$, which is given as

$$C_{f_2}(P) = \sum_{k=1}^{K} (w_k)^2 C_k(P) = (w_1)^2 C_1(P) + C_{f_1}(P). \tag{4.7.55}$$

Also let $C_k(D_{f_j})$ and $C_{f_i}(D_{f_j})$ $(i, j = 1, 2)$ denote cost functions $C_k(P)$ and $C_{f_i}(P)$ using the probability assignments in D_{f_j}. Then, it is obvious that D_{f_2} is identical with D_{f_1} when $w_1 = 0$, and is identical with D_1 (except the DoC values) when $w_1 \to \infty$. Therefore,

$$\begin{aligned} dist(D_{f_2}, D_{f_1}) = 0, &\quad w_1 = DoC_{min}, \\ dist(D_1, D_{f_2}) = 0, &\quad w_1 = DoC_{max}. \end{aligned} \tag{4.7.56}$$

Moreover, $C_{f_1}(P)$ reaches its minimal value at D_{f_1}, which gives $C_{f_1}(D_{f_1}) \leq C_{f_1}(D_{f_2})$. Similarly, $C_{f_2}(D_{f_1}) \geq C_{f_2}(D_{f_2})$, or

$$C_{f_2}(D_{f_1}) = (w_1)^2 C_1(D_{f_1}) + C_{f_1}(D_{f_1}) \quad \geq \quad C_{f_2}(D_{f_2}) = (w_1)^2 C_1(D_{f_2}) + C_{f_1}(D_{f_2}). \quad (4.7.57)$$

It is obvious from the above inequalities that $C_1(D_{f_1}) \geq C_1(D_{f_2})$, or

$$dist(D_1, D_{f_1}) \geq dist(D_1, D_{f_2}). \quad (4.7.58)$$

Note that the above inequality applies to any cost function which is additive.

Now assume that the DoC value of D_1 changes from w_{1a} to w_{1b}, with $w_{1b} > w_{1a}$. Let $D_1^{w_{1i}}$ denote D_1 with DoC value w_{1i}, and $D_{f_2}^{w_{1i}}$ denote the fusion result after the fusion of $D_1^{w_{1i}}$ $(i = a, b)$. We also define normalized ratios $\bar{r}_1^\ell$'s as

$$\bar{r}_1^\ell = \frac{r_1^\ell}{w_1}, \quad \ell = 1, \ldots, L_1. \quad (4.7.59)$$

Note that $\sum_{\ell=1}^{L_1} \bar{r}_1^\ell = 1$, i.e., the $\bar{r}_1^\ell$'s are independent of w_1. The $\bar{r}_1^\ell$'s contain all the identity information in D_1, because the identity information is expressed in the relative likelihood ratios between the subsets ω_1^ℓ's, which is also maintained in the $\bar{r}_1^\ell$'s. This is similar to the normalized probabilities $P_k'(\omega_k^\ell)$'s (as given in Eq. (3.3.9)) in Dempster-Shafer evidential inference method. The weighted cost function corresponding to $D_1^{w_{1b}}$, i.e., the cost term corresponding to $D_1^{w_{1b}}$ in the overall cost function $C_{f_K}(P)$, can be written as

$$\begin{aligned} C_1^{w_{1b}\prime}(P) &= (w_{1b})^2 C_1^{w_{1b}}(P) = (c_1(P))^2 w_{1b}^2 \sum_{\ell=1}^{L_1} \left(\bar{r}_1^\ell - \frac{P_f(\omega_1^\ell)}{c_1(P)} \right)^2 \\ &= (c_1(P))^2 w_{1a}^2 \sum_{\ell=1}^{L_1} \left(\bar{r}_1^\ell - \frac{P_f(\omega_1^\ell)}{c_1(P)} \right)^2 + (c_1(P))^2 (w_{1b}^2 - w_{1a}^2) \sum_{\ell=1}^{L_1} \left(\bar{r}_1^\ell - \frac{P_f(\omega_1^\ell)}{c_1(P)} \right)^2 . \end{aligned} \quad (4.7.60)$$

Let $D_1^{w_{1c}}$ denote a new sensor report which is identical to D_1 except that its DoC value is $w_{1c} = \sqrt{w_{1b}^2 - w_{1a}^2}$. Then from the above equation, the fusion of $D_1^{w_{1b}}$ is equivalent to the fusion of $D_1^{w_{1a}}$ and $D_1^{w_{1c}}$. In other words, increasing the DoC of D_1 from w_{1a} to w_{1b} is equivalent to the fusion of a new sensor report $D_1^{w_{1c}}$. Similar to (4.7.58), we have

$$dist(D_1, D_{f_2}^{w_{1a}}) \geq dist(D_1, D_{f_2}^{w_{1b}}), \tag{4.7.61}$$

i.e., when w_1 increases, $dist(D_1, D_{f_2})$ decreases.

e) Fundamental principle 5 for SSF model: Suppose there are K identical sensor reports from similar and independent sensors, each denoted by $D_1^{w_1}$, where w_1 denotes the DoC value and suppose that $w_1 > DoC_{min}$. Let D_{f_0} and $D_{f_K}^{w_1}$ (with DoC value $w_{f_K}^{w_1}$) denote the fusion results before and after the fusion of these reports. We further denote the cost function before and after the fusion of these reports as $C_{f_0}(P)$ and $C_{f_K}^{w_1}(P)$ respectively. Then,

$$C_{f_K}^{w_1}(P) = K(w_1)^2 C_1^{w_1}(P) + C_{f_0}(P), \tag{4.7.62}$$

where $C_1^{w_1}(P)$ is the cost function for report $D_1^{w_1}$. Note that $C_1^{w_1}(D_1) = 0$, where $C_1^{w_1}(D_1)$ is the value of $C_1^{w_1}(P)$ using the probability assignments in $D_1^{w_1}$. $C_{f_K}^{w_1}(P)$ reaches its minimal value at $D_{f_K}^{w_1}$, i.e.,

$$K(w_1)^2 C_1^{w_1}(D_1) + C_{f_0}(D_1) \quad \geq \quad K(w_1)^2 C_1^{w_1}(D_{f_K}^{w_1}) + C_{f_0}(D_{f_K}^{w_1}), \tag{4.7.63}$$

which implies $C_{f_0}(D_1) \geq K(w_1)^2 C_1^{w_1}(D_{f_K})$, or

$$\frac{1}{K} C_{f_0}(D_1) \quad \geq \quad \left(c_1(D_{f_K}^{w_1})\right)^2 (w_1)^2 \sum_{\ell=1}^{L_1} \left(\frac{r_1^\ell}{w_1} - \frac{P_{f_K}^{w_1}(w_1^\ell)}{c_1(D_{f_K}^{w_1})} \right)^2. \tag{4.7.64}$$

Note that $C_{f_0}(D_1)$ is a positive constant and $\lim_{K\to\infty} \frac{1}{K} C_{f_0}(D_1) = 0$. Eq. (4.7.64) shows that if $\lim_{K\to\infty} c_1(D_{f_K}^{w_1}) \neq 0$, then

$$\lim_{K\to\infty} \left(P_{f_K}^{w_1}(\omega_1^1) : \cdots : P_{f_K}^{w_1}(\omega_1^{L_1})\right) = r_1^1 : \cdots : r_1^{L_1}, \tag{4.7.65}$$

or

$$\lim_{K\to\infty} D_{f_K}^{w_1} = D_1. \tag{4.7.66}$$

In the trivial case that $\lim_{K\to\infty} c_1(D_{f_K}^{w_1}) = 0$, it is obvious that $\lim_{K\to\infty} P_{f_K}^{w_1}(\omega_1^\ell) = 0$ for $\ell = 1, \dots, L_1$, and the comparison $P_{f_\infty}^{w_1}(\omega_1^1) : \cdots : P_{f_\infty}^{w_1}(\omega_1^{L_1})$ is undetermined. As a remedy, we define that when $P_{f_\infty}^{w_1}(\omega_1^\ell) = 0$ for $\ell = 1, \dots, L_1$,

$$P_{f_\infty}^{w_1}(\omega_1^1) : \cdots : P_{f_\infty}^{w_1}(\omega_1^{L_1}) = r_1^1 : \cdots : r_1^{L_1}. \tag{4.7.67}$$

Therefore the principle is also satisfied for this trivial case.

To prove property (3.6.36), we write the DoC of $D_{f_K}^{w_1}$ as

$$w_{f_K}^{w_1} = Kw_1 \cdot \frac{C_{1max} - C_1(D_{f_K}^{w_1})}{C_{1max}} + w_{f_0}(D_{f_K}), \tag{4.7.68}$$

where the first part represents the contribution from the K identical reports, and the second part represents the contribution from other reports in the fusion. Eq. (4.7.66) implies that

$$\lim_{K\to\infty} C_1(D_{f_K}^{w_1}) = \lim_{K\to\infty} C_1(D_1) = 0, \tag{4.7.69}$$

which helps to derive that

$$\lim_{K\to\infty} w_{f_K}^{w_1} \geq \lim_{K\to\infty} Kw_1 \cdot \frac{C_{1max} - C_1(D_{f_K}^{w_1})}{C_{1max}} = \lim_{K\to\infty} Kw_1 = \infty, \tag{4.7.70}$$

thus establishes property (3.6.36).

To prove property (3.6.37), suppose that the DoC value of w_1 changes from w_{1a} to w_{1b}, with $w_{1b} > w_{1a}$. From the proof of the fundamental principle 4 for SSF model (see page 102), the fusion of a sensor report $D_1^{w_{1b}}$ is equivalent to the fusion of two sensor reports: a report $D_1^{w_{1a}}$ and a report $D_1^{w_{1c}}$ with DoC value $w_{1c} = \sqrt{w_{1b}^2 - w_{1a}^2}$. As a result, increasing w_1 from w_{1a} to w_{1b} is equivalent to the fusion of an additional group of K identical reports $D_1^{w_{1c}}$'s. Similar to (4.7.61), we have $dist(D_1, D_{f_K}^{w_{1a}}) \geq dist(D_1, D_{f_K}^{w_{1b}})$.

f) DoC principle 1 for SSF model: Here we follow the assumptions made in DoC principle 1 for SSF model (see page 64). That is, let D_{f_1} denote the fusion result of sensor reports $D_{11}, \ldots, D_{1K}$, and D_{f_2} denote the fusion result of sensor reports $D_{21}, \ldots, D_{2K}$. Let w_{ij} denote the DoC value associated with report D_{ij}, and w_{f_i} denote the DoC value associated with fusion result D_{f_i}. Assume that $D_{f_1} = D_{f_2}$, and $w_{1k} = w_{2k}$ for $k = 1, \ldots, K$. We further assume that sensor reports D_{1k} and D_{2k} have the same subsets. Moreover, assume that

$$dist(D_{1k}, D_{f_1}) > dist(D_{2k}, D_{f_2}), \quad k = 1, \ldots, K, \tag{4.7.71}$$

and note that $dist(D_{ik}, D_{f_i}) = C_{ik}(D_{f_i})$ for $i = 1, 2$, we derive that

$$w_{f_2} = \sum_{k=1}^{K} w_{2k} \frac{C_{k_{max}} - C_{2k}(D_{f_2})}{C_{k_{max}}} > \sum_{k=1}^{K} w_{1k} \frac{C_{k_{max}} - C_{1k}(D_{f_1})}{C_{k_{max}}} = w_{f_1}. \tag{4.7.72}$$

g) DoC principle 2 for SSF model: This is straight forward. Follow the assumptions made in DoC principle 2 for SSF model (see page 65). That is, let D_{f_1} denote the fusion result of sensor reports $D_1^{w_1}, \ldots, D_K^{w_K}$, where w_k is the DoC associated with the k'th report. Let D_{f_2} denote the fusion result of $D_1^{\beta w_1}, \ldots, D_K^{\beta w_K}$. In other words, the DoC of the k'th report changes from w_k to βw_k. Also let w_{f_i} denote the DoC value associated with fusion result D_{f_i}. Then, it is obvious that by the convex quadratic fusion method, D_{f_1} and D_{f_2} are identical to each other except their DoC values. Since both $C_k(D_f)$ and $C_{k_{max}}$ are independent of w_1, we can compute that

$$w_{f_2} = \sum_{k=1}^{K} \beta w_k \frac{C_{k_{max}} - C_k(D_{f_2})}{C_{k_{max}}} = \beta \sum_{k=1}^{K} w_k \frac{C_{k_{max}} - C_k(D_{f_1})}{C_{k_{max}}} = \beta w_{f_1}, \tag{4.7.73}$$

which establishes property (3.7.39), i.e., $\beta > 1 \Rightarrow w_{f_2} > w_{f_1}$. □

Discussion on the restricted fundamental principle 4 for SSF model

Theorem 4.7.1 shows that the convex quadratic fusion method satisfies the decision fusion principles proposed in Chapter 3. However, the method only satisfies the restricted fundamental principle 4 for SSF model in the special case where the K sensors assign probabilities to *all and only* the basic propositions a_i's. To prove this, we observe that in this case the k'th sensor report can be written as

$$D_k = \left\{ \begin{array}{l} DoC = w_k, \\ P_k(a_1) : \cdots : P_k(a_N) = w_k p_k^1 : \cdots : w_k p_k^N \end{array} \right\}, \tag{4.7.74}$$

where $P_k(a_i) = p_k^i$ represents the probability assigned to proposition a_i in report D_k, with $\sum_{i=1}^{N} p_k^i = 1$. Denote the fusion result before and after the fusion of report D_1 as D_{f_1} and D_{f_2}. Then, from (4.2.13), D_{f_1} is given by $D_{f_1} = \{P_{f_1}(a_1), \ldots, P_{f_1}(a_N)\}$, where

$$P_{f_1}(a_i) = \frac{\sum_{k=2}^{K} (w_k)^2 p_k^i}{\sum_{k=2}^{K} (w_k)^2}, \quad i = 1, \ldots, N. \tag{4.7.75}$$

Similarly, D_{f_2} is given by $D_{f_2} = \{P_{f_2}(a_1), \ldots, P_{f_2}(a_N)\}$, where

$$P_{f_2}(a_i) = \frac{\sum_{k=1}^{K} (w_k)^2 p_k^i}{\sum_{k=1}^{K} (w_k)^2}, \quad i = 1, \ldots, N. \tag{4.7.76}$$

It is easy to verify that D_{f_2} is identical to D_{f_1} when $w_1 = 0$, and is identical to D_1 when $w_1 \to \infty$. This establishes property (3.6.27).

To prove property (3.6.28), we first derive that

$$\begin{aligned} P_{f_2}(a_i) - P_{f_1}(a_i) &= \frac{\sum_{k=2}^{K} (w_k)^2 p_k^i + (w_1)^2 p_1^i}{\sum_{k=2}^{K} (w_k)^2 + (w_1)^2} - \frac{\sum_{k=2}^{K} (w_k)^2 p_k^i}{\sum_{k=2}^{K} (w_k)^2} \\ &= \frac{1}{\sum_{k=2}^{K} (w_k)^2 + (w_1)^2} \left(\left(\sum_{k=2}^{K} (w_k)^2 p_k^i + (w_1)^2 p_1^i \right) - \frac{\sum_{k=2}^{K} (w_k)^2 p_k^i}{\sum_{k=2}^{K} (w_k)^2} \left(\sum_{k=2}^{K} (w_k)^2 + w_1^2 \right) \right) \\ &= \frac{1}{\sum_{k=2}^{K} (w_k)^2 + w_1^2} \left((w_1)^2 p_1^i - \frac{\sum_{k=2}^{K} (w_k)^2 p_k^i}{\sum_{k=2}^{K} (w_k)^2} (w_1)^2 \right) \\ &= \frac{(w_1)^2}{\sum_{k=1}^{K} (w_k)^2} \left(P_1(a_i) - P_{f_1}(a_i) \right). \end{aligned} \tag{4.7.77}$$

From (4.7.77), it is obvious that if $w_1 \in (0, \infty)$ and $P_1(a_i) > P_{f_1}(a_i)$, then $P_{f_2}(a_i) > P_{f_1}(a_i)$. Moreover, $P_{f_2}(a_i) < P_1(a_i)$ since

$$P_{f_2}(a_i) - P_{f_1}(a_i) = \frac{(w_1)^2}{\sum_{k=1}^{K}(w_k)^2}\left(P_1(a_i) - P_{f_1}(a_i)\right) < P_1(a_i) - P_{f_1}(a_i). \tag{4.7.78}$$

Similarly, we can prove that

$$P_1(a_i) < P_{f_1}(a_i) \quad \Rightarrow \quad P_1(a_i) < P_{f_2}(a_i) < P_{f_1}(a_i), \quad w_1 \in (0, \infty). \tag{4.7.79}$$

In addition, it is obvious from (4.7.77) that if D_1 changes so that $|P_1(a_i) - P_{f_1}(a_i)|$ or w_1 increases, $|P_2(a_i) - P_{f_1}(a_i)|$ also increases.

In general, the convex quadratic fusion method does not satisfy the restricted fundamental principle 4 for SSF model. This is illustrated in the following example. However, this should not be viewed as a weakness of the new fusion method since the principle is not absolutely necessary in practice (see discussion on page 59).

Example 4.7.1 *Let $\Omega = \{a_1, a_2, a_3\}$. Suppose there are three similar and independent sensors each sending a report to the fusion center. The sensor reports are given as*

$$\begin{aligned} D_1 &= \left\{ \begin{array}{l} w_1 = 1, \\ P_1(a_1) : P_1(a_2) : P_1(a_3) = 0.70 : 0.15 : 0.15 \end{array} \right\}, \\ D_2 &= \left\{ \begin{array}{l} w_2 = 1, \\ P_2(a_1) : P_2(a_2 \vee a_3) = 0.20 : 0.80 \end{array} \right\}, \\ D_3 &= \left\{ \begin{array}{l} w_3 = 1, \\ P_3(a_1) : P_3(a_2) : P_3(a_3) = 0.20 : 0.20 : 0.60 \end{array} \right\}. \end{aligned} \tag{4.7.80}$$

By the convex quadratic fusion method, the fusion result before the fusion of D_1, i.e., the fusion result of D_2 and D_3, is given by $D_{f_1} = \{P_{f_1}(a_1) = 0.20,\ P_{f_1}(a_2) = 0.20,\ P_{f_1}(a_3) = 0.60\}$, and the fusion result after the fusion of D_1, i.e., the fusion result of D_1, D_2 and D_3, is $D_{f_2} = \{P_{f_2}(a_1) = 0.350,\ P_{f_2}(a_2) = 0.225,\ P_{f_2}(a_3) = 0.4250\}$. Here, the restricted fundamental principle 4 for SSF model is not satisfied, since $P_1(a_2) = 0.15 < P_{f_1}(a_2) = 0.20$

but $P_{f_2}(a_2) = 0.225 > P_{f_1}(a_2) = 0.20$, *violating property (3.6.28).*

4.8 Numerical example

In this section, a numerical example is given to illustrate the performance of the convex quadratic fusion method.

Example 4.8.1

Let $\Omega = \{a_1, a_2, a_3, a_4\}$. Suppose the corresponding true probability distribution is $\bar{P}(\Omega) = \{0.05, 0.20, 0.60, 0.15\}$. Suppose there are three independent sensors making observations on a target. All the sensors explore common physical characteristics of the target. Assume each sensor files one report to the fusion center which are given as follows:

$$D_1 = \left\{ \begin{array}{l} w_1 = r_1^1 + r_1^2 = 1, \\ P_1(a_3 \vee a_4) : P_1(a_1 \vee a_2) = r_1^1 : r_1^2 = 0.7 : 0.3 \end{array} \right\},$$
$$D_2 = \left\{ \begin{array}{l} w_2 = r_2^1 + r_2^2 = 0.8, \\ P_2(a_3) : P_2(a_1 \vee a_4) = r_2^1 : r_2^2 = 0.6 : 0.2 \end{array} \right\}, \quad (4.8.81)$$
$$D_3 = \left\{ \begin{array}{l} w_3 = r_3^1 + r_3^2 = 0.7, \\ P_3(a_1) : P_3(a_4) = r_3^1 : r_3^2 = 0.28 : 0.42 \end{array} \right\}.$$

Using the convex quadratic fusion method, we first construct the following cost function as

$$\begin{aligned} C_{f_3}(P) &= (w_1)^2 C_1(P) + (w_2)^2 C_2(P) + (w_3)^2 C_3(P) \\ &= \left(((p_3 + p_4) - 0.7)^2 + ((p_1 + p_2) - 0.3)^2\right) \\ &\quad +0.64\left((0.75(p_1 + p_3 + p_4) - p_3)^2 + (0.25(p_1 + p_3 + p_4) - (p_1 + p_4))^2\right) \\ &\quad +0.49\left((0.4(p_1 + p_4) - p_1)^2 + (0.6(p_1 + p_4) - p_4)^2\right), \end{aligned} \quad (4.8.82)$$

where we have used the fact $p_1 + p_2 + p_3 + p_4 = 1$. The convex quadratic programming problem (4.2.8) then becomes

$$\begin{aligned} \text{minimize} \quad & C_{f_3}(P) \\ \text{subject to} \quad & p_1 + p_2 + p_3 + p_4 = 1, \\ & p_i \geq 0, \quad i = 1, 2, 3, 4. \end{aligned} \tag{4.8.83}$$

The resulting optimal fused probabilities are shown in Table 4.4. They closely match the actual probabilities $\bar{P}$ as expected.

Propositions	a_1	a_2	a_3	a_4
True probabilities	0.05	0.20	0.6	0.15
Optimally fused probabilities	0.078	0.222	0.583	0.117

Table 4.4: Fusion results for Example 4.8.1.

Finally, The DoC of the fusion result is calculated as

$$\begin{aligned} w_f &= \sum_{k=1}^{3} w_k \cdot \frac{C_{k_{max}} - C_k(D_f)}{C_{k_{max}}} \\ &= 1 \cdot \frac{1.49 - 0.09}{1.49} + 0.8 \cdot \frac{1.125 - 0}{1.125} + 0.7 \cdot \frac{0.72 - 0}{0.72} = 2.44. \end{aligned} \tag{4.8.84}$$

The fusion is performed on a 800MHz Pentium PC under the MATLAB environment. The CPU time used is less than 0.5 second, showing that the fusion method is computationally efficient.

Chapter 5

K-L Fusion Method

In the previous chapter, we have introduced the convex quadratic fusion method for the Similar Sensor Fusion (SSF) model. In this chapter, we continue our study of decision fusion and propose another method for the SSF model: a fusion method based on Kullback-Leibler's measure of cross-entropy or called K-L fusion method in short. Similar to the convex quadratic fusion method, the K-L fusion method is based on the minimization of inconsistencies between the fusion result and sensor reports. However, instead of using quadratic cost function to measure the inconsistencies, we use Kullback-Leibler's measure of cross-entropy. The new formulation leads to a generalized analytic center problem in linear programming which can be solved efficiently in polynomial time. Moreover, the method does not require any prior knowledge on the target. This is in contrast to Dempster-Shafer evidential inference method which suffers from exponential complexity, and Bayesian inference method which requires prior knowledge on the target.

5.1 Kullback-Leibler's measure of cross-entropy

Kullback-Leibler's measure of cross-entropy (also called K-L measure) is a fundamental measure of cross-entropy between two probability distributions. It provides discrimination information or directed divergence information on two probability distributions. Let $\mathbf{p} = \{p_1, \ldots, p_n\}$ and $\mathbf{q} = \{q_1, \ldots, q_n\}$ be two probability distributions. The K-L measure is defined as

$$D(\mathbf{p}:\mathbf{q}) = \sum_{i=1}^{n} p_i \ln \frac{p_i}{q_i}, \tag{5.1.1}$$

where we assume p_i is zero whenever $q_i = 0$, and define $0 \ln \frac{0}{0} = 0$. The following are some important properties of K-L measure:

1. $D(\mathbf{p}:\mathbf{q})$ is a continuous function of $\mathbf{p} = \{p_1, \ldots, p_n\}$ and $\mathbf{q} = \{q_1, \ldots, q_n\}$.

2. $D(\mathbf{p}:\mathbf{q})$ is permutationally symmetric, which means that the value of K-L measure does not change if the pairs $(p_1, q_1), \cdots, (p_n, q_n)$ are permuted among themselves.

3. $D(\mathbf{p}:\mathbf{q}) \geq 0$, and $D(\mathbf{p}:\mathbf{q}) = 0$ if and only if $\mathbf{p} = \mathbf{q}$.

4. $D(\mathbf{p}:\mathbf{q})$ is a convex function of both $\mathbf{p}$ and $\mathbf{q}$.

Note that the K-L measure $D(\mathbf{p}:\mathbf{q})$ can be used as the distance measure $dist(D_1, D_2)$ proposed in section 3.6.4 (see page 58). Here we assume that reports D_1 and D_2 are represented by probability distributions $\mathbf{p}$ and $\mathbf{q}$ respectively. Obviously, $D(\mathbf{p}:\mathbf{q})$ satisfies all the properties for $dist(D_1, D_2)$. Also note that K-L measure does not satisfy the symmetry condition, i.e., in general, $D(\mathbf{p}:\mathbf{q}) \neq D(\mathbf{q}:\mathbf{p})$.

5.2 Sensor reports

The proposed K-L fusion method fuses the so-called partition type sensor reports. A sensor report D_k is of the form

> *For a group of mutually exclusive and exhaustive subsets in Ω, their relative likelihood are given by $r_k^1, \ldots, r_k^{L_k}$ respectively, with $r_k^\ell \geq 0$, $\ell = 1, \ldots, L_k$. The summation $r_k^1 + \cdots + r_k^{L_k} = w_k$, which is required to lie in $[0, \infty)$, represents the Degree of Confidence (DoC) of the report.*

In other words, a sensor can declare, with DoC equals to w_k, that the probabilities of ω_k^1, $\ldots, \omega_k^{L_k}$ satisfy $P(\omega_k^1) : \cdots : P(\omega_k^{L_k}) = r_k^1 : \cdots : r_k^{L_k}$, where k is the sensor index, and $P(\omega_k^\ell)$

is the probability of occurrence of subset ω_k^ℓ. Partition type sensor reports are special cases of the ratio type reports introduced in section 4.1. While the latter put no restriction on the subsets, the former require the selected subsets to be mutually exclusive and exhaustive (thus forming a partition of Ω).

5.3 Problem formulation

Suppose that in the SSF model, K independent sensors observe a target in a surveillance region. The sensors explore some common target characteristics. Each sensor summarizes its identity declaration on the target in a sensor report which is sent to the fusion center. Let us denote these reports by

$$D_k = \left\{ \begin{array}{l} w_k = r_k^1 + \cdots + r_k^{L_k}, \\ P_k(\omega_k^1) : \cdots : P_k(\omega_k^{L_k}) = r_k^1 : \cdots : r_k^{L_k} \end{array} \right\}, \quad k = 1, ..., K, \tag{5.3.2}$$

where $\omega_k^i \cap \omega_k^j = \emptyset$ for $i \neq j$, and $\omega_k^1 \cup \cdots \cup \omega_k^{L_k} = \Omega$.

Similar to the convex quadratic fusion method, our fusion goal is to determine a set of probabilities

$$P_f(a_i) = p_i, \quad i = 1, \ldots, N \tag{5.3.3}$$

that best fit the sensor reports. To achieve this, we first use a cost function to measure the discrepancy between the fusion result and each sensor report, and then minimize a weighted sum of these cost functions to obtain the optimal fusion result (denoted by D_f). The cost function we choose for report D_k is

$$C_k(P) = \sum_{\ell=1}^{L_k} \frac{r_k^\ell}{w_k} \ln \left(\frac{\frac{r_k^\ell}{w_k}}{P_f(\omega_k^\ell)} \right), \tag{5.3.4}$$

where $P_f(\omega_k^\ell) = \sum_{j:\ a_j \in \omega_k^\ell} p_j$ is the probability assigned to ω_k^ℓ in the fusion result D_f. $C_k(P)$ represents the normalized discrepancy in the sensor report. It is the K-L measure between probability distributions $\{\frac{r_k^1}{w_k}, \ldots, \frac{r_k^{L_k}}{w_k}\}$ and $\{P_f(\omega_k^1), \ldots, P_f(\omega_k^{L_k})\}$. The overall cost function for the K sensor reports is a weighted sum of the $C_k(P)$'s, i.e.,

$$C_{f_K}(P) = \sum_{k=1}^{K} w_k C_k(P) = \sum_{k=1}^{K} w_k \sum_{\ell=1}^{L_k} \frac{r_k^\ell}{w_k} \ln\left(\frac{r_k^\ell}{w_k}\right) - \sum_{k=1}^{K} w_k \sum_{\ell=1}^{L_k} \frac{r_k^\ell}{w_k} \ln\left(P_f(\omega_k^\ell)\right). \quad (5.3.5)$$

Here, each $C_k(P)$ is weighted by w_k to reflect the DoC of report D_k.

Now we can formulate the identity fusion problem as the following convex optimization problem:

$$\begin{aligned} \text{minimize} \quad & C_{f_K}(P) = \sum_{k=1}^{K} w_k \sum_{\ell=1}^{L_k} \frac{r_k^\ell}{w_k} \ln\left(\frac{r_k^\ell}{w_k}\right) - \sum_{k=1}^{K} w_k \sum_{\ell=1}^{L_k} \frac{r_k^\ell}{w_k} \ln\left(P_f(\omega_k^\ell)\right) \\ \text{subject to} \quad & \sum_{i=1}^{N} p_i = 1, \;\; p_i \geq 0, \; i = 1, \ldots, N. \end{aligned} \quad (5.3.6)$$

Since the first part of the above objective function is constant, the above formulation is equivalent to

$$\begin{aligned} \text{minimize} \quad & -\sum_{k=1}^{K} \sum_{\ell=1}^{L_k} r_k^\ell \ln\left(P_f(\omega_k^\ell)\right) \\ \text{subject to} \quad & \sum_{i=1}^{N} p_i = 1, \;\; p_i \geq 0, \; i = 1, \ldots, N. \end{aligned} \quad (5.3.7)$$

We now consider the special case where K sensor reports are available, each assigns probabilities to *all and only* the a_i's. We can write the k'th sensor report as $D_k = \{P_k(a_1) : \cdots : P_k(a_N) = w_k p_k^1 : \cdots : w_k p_k^N\}$, where $P_k(a_i) = p_k^i$ represents the probability assigned to proposition a_i in report D_k, with $\sum_{i=1}^{N} p_k^i = 1$. The overall cost function for these reports can be written as

$$C_{f_K}(P) = -\sum_{k=1}^{K} \sum_{i=1}^{N} w_k p_k^i \ln p_i = -\sum_{i=1}^{N} r_f^i \ln p_i, \quad (5.3.8)$$

where $r_f^i = \sum_{k=1}^{K} w_k p_k^i$. Minimization of this cost function yields

$$P_f(a_i) = \frac{r_f^i}{r_f^1 + \cdots + r_f^N} = \frac{\sum_{k=1}^{K} w_k p_k^i}{\sum_{k=1}^{K} w_k}, \quad i = 1, \ldots, N. \tag{5.3.9}$$

In other words, the optimal fusion result is obtained by averaging sensor reports using DoC values w_k's. Similar to Eq. (4.2.13), this special case provides a positive justification for the new method.

Like the convex quadratic fusion method, the K-L fusion method also minimizes an additive cost function. Therefore the discussion on page 87 also applies to the K-L fusion method. Specifically, although the K-L fusion method can operate in both the sequential and batch fusion modes, we do not advise the use of the method in the sequential fusion mode. Moreover, the method requires little extra computation as more sensor reports are fused. Similar to the convex quadratic fusion method, the overall cost function (5.3.5) can be updated when a new sensor report D_{K+1} arrives, i.e.,

$$C_{f_{K+1}}(P) = \sum_{k=1}^{K+1} w_k C_k(P) = C_{f_K}(P) + C_{K+1}(P). \tag{5.3.10}$$

However, as will be seen in section 5.5, computing the DoC of the fusion result requires the cost functions. As a result, we need to maintain the cost functions for future fusion steps.

5.4 Interior point method solution

The optimization problem given in (5.3.7) is a special case of the following general optimization problem:

$$\begin{aligned} &\text{minimize} \quad -\sum_{m=1}^{M} r_m \ln \mathbf{c}_m^T \mathbf{x} \\ &\text{subject to} \quad \mathbf{A}\mathbf{x} = \mathbf{b}, \quad \mathbf{x} \geq \mathbf{0}, \end{aligned} \tag{5.4.11}$$

where $\mathbf{x} \in \Re^{N\times 1}$ is the decision vector, vector $\mathbf{c}_m \in \Re^{N\times 1}$, matrix $\mathbf{A} \in \Re^{L\times N}$, $\mathbf{b} \in \Re^{L\times 1}$, and $r_m > 0$ for $m = 1, \dots, M$. This is actually a generalized analytic center problem in linear programming [34]. For the K-L formulation (5.3.7), we have

$$\begin{aligned} &\mathbf{x} = [p_1 \; \cdots \; p_N]^T, \quad \mathbf{A} = [1 \; \dots \; 1], \quad \mathbf{b} = 1, \quad L = 1, \\ &r_m = r_k^\ell, \quad \text{where } m = L_1 + \cdots + L_{k-1} + \ell \text{ and } 1 \le \ell \le L_k, \\ &\mathbf{c}_m = [c_{m1} \dots c_{mN}]^T, \quad \text{where } c_{mi} = \begin{cases} 1, & \text{if } a_i \in \omega_k^\ell, \\ 0, & \text{otherwise,} \end{cases} \end{aligned} \tag{5.4.12}$$

and M is the number of different subsets in the K sensor reports. The optimal solution to (5.4.11) can be found by interior point methods [34]. In the following, we first derive the optimality conditions for (5.4.11), then define the Newton step and the starting point.

The Lagrangian for (5.4.11) is

$$L(\mathbf{x}, \mathbf{y}) = -\sum_{m=1}^{M} r_m \ln \mathbf{c}_m^T \mathbf{x} - \mathbf{y}^T(\mathbf{A}\mathbf{x} - \mathbf{b}), \tag{5.4.13}$$

where $\mathbf{y} \in \Re^{L\times 1}$ is a multiplier vector. We know that if $\mathbf{x}$ takes the optimal solution to (5.4.11), then $\nabla L(\mathbf{x}, \mathbf{y}) = 0$, i.e.,

$$-\sum_{m=1}^{M} \frac{r_m c_{mi}}{\mathbf{c}_m^T \mathbf{x}} - (\mathbf{y}^T \mathbf{A})_i = 0, \quad i = 1, \dots, N, \tag{5.4.14}$$

where $(\mathbf{y}^T\mathbf{A})_i$ is the i'th element of $\mathbf{y}^T\mathbf{A}$. Let

$$s_m = \frac{r_m}{\mathbf{c}_m^T \mathbf{x}}, \qquad m = 1, \dots, M. \tag{5.4.15}$$

Note that $s_m \ge 0$ for $m = 1, \dots, M$. Then $r_m = \mathbf{c}_m^T \mathbf{x} s_m$ for $m = 1, \dots, M$, or $\mathbf{r} = \mathbf{S}\mathbf{C}^T\mathbf{x}$ where $\mathbf{S} = diag(s_1, \dots, s_M)$, $\mathbf{C} = [\mathbf{c}_1 \; \dots \; \mathbf{c}_M]$ and $\mathbf{r} = [r_1 \; \dots \; r_M]^T$. Using (5.4.15), we rewrite (5.4.14) as

$$-\sum_{m=1}^{M} c_{mi} s_m - (\mathbf{y}^T\mathbf{A})_i = 0, \quad i = 1, \dots, N, \tag{5.4.16}$$

or simply $\mathbf{Cs} + \mathbf{A}^T\mathbf{y} = \mathbf{0}$, where $\mathbf{s} = [s_1 \ \ldots \ s_M]^T$. The optimality conditions for (5.4.11) can be summarized as follows:

$$\mathbf{SC}^T\mathbf{x} = \mathbf{r}, \tag{5.4.17}$$

$$\mathbf{Ax} = \mathbf{b}, \quad \mathbf{x} \geq \mathbf{0}, \tag{5.4.18}$$

$$\mathbf{Cs} + \mathbf{A}^T\mathbf{y} = \mathbf{0}, \quad \mathbf{s} \geq \mathbf{0}. \tag{5.4.19}$$

In addition, when $\nabla L(\mathbf{x}, \mathbf{y}) = 0$, the Lagrangian is

$$\begin{aligned} L &= -\sum_{m=1}^{M} r_m \ln \mathbf{c}_m^T\mathbf{x} - \mathbf{y}^T(\mathbf{Ax} - \mathbf{b}) = -\sum_{m=1}^{M} r_m \ln \frac{r_m}{s_m} - \mathbf{y}^T\mathbf{Ax} + \mathbf{y}^T\mathbf{b} \\ &= \mathbf{b}^T\mathbf{y} + \sum_{m=1}^{M} r_m \ln s_m - \sum_{m=1}^{M} r_m \ln r_m + \sum_{m=1}^{M} r_m, \end{aligned} \tag{5.4.20}$$

and the dual problem of (5.4.11) can be derived as

$$\begin{aligned} &\text{minimize} \quad \mathbf{b}^T\mathbf{y} + \sum_{m=1}^{M} r_m \ln s_m \\ &\text{subject to} \quad \mathbf{Cs} + \mathbf{A}^T\mathbf{y} = \mathbf{0}, \quad \mathbf{s} \geq \mathbf{0}. \end{aligned} \tag{5.4.21}$$

To derive the Newton step, we assume that $(\mathbf{x}, \mathbf{s}, \mathbf{y})$ is updated to $(\mathbf{x} + d\mathbf{x}, \mathbf{s} + d\mathbf{s}, \mathbf{y} + d\mathbf{y})$, both of which satisfy the primal-dual feasibility conditions (5.4.18) and (5.4.19). Let $(\mathbf{S} + d\mathbf{S})\mathbf{C}^T(\mathbf{x} + d\mathbf{x}) = \mathbf{r}$, which can be expanded as

$$\mathbf{SC}^T\mathbf{x} + \mathbf{SC}^T d\mathbf{x} + d\mathbf{S} \cdot \mathbf{C}^T\mathbf{x} + d\mathbf{S} \cdot \mathbf{C}^T \cdot d\mathbf{x} = \mathbf{r}. \tag{5.4.22}$$

By defining $\mathbf{H}_1 = \mathbf{SC}^T$, $\mathbf{h}_2 = \mathbf{C}^T\mathbf{x}$, $\mathbf{H}_2 = diag(h_{21}, \ldots, h_{2M})$ where h_{2m} is the m'th element of $\mathbf{h}_2$, and ignoring the trivial term $d\mathbf{S} \cdot \mathbf{C}^T \cdot d\mathbf{x}$, we can simplify (5.4.22) as

$$\mathbf{H}_1 d\mathbf{x} + \mathbf{H}_2 d\mathbf{s} = \mathbf{r} - \mathbf{SC}^T\mathbf{x}. \tag{5.4.23}$$

Linearizing (5.4.18) and (5.4.19) for $(\mathbf{x} + d\mathbf{x}, \mathbf{s} + d\mathbf{s}, \mathbf{y} + d\mathbf{y})$, and using primal-dual feasibility

$\mathbf{Ax} = \mathbf{b}$ and $\mathbf{Cs} + \mathbf{A}^T\mathbf{y} = \mathbf{0}$, we obtain $\mathbf{A}d\mathbf{x} = \mathbf{0}$ and $\mathbf{C}d\mathbf{s} + \mathbf{A}^T d\mathbf{y} = \mathbf{0}$. The Newton step can be summarized as

$$\begin{bmatrix} \mathbf{H}_1 & \mathbf{0} & \mathbf{H}_2 \\ \mathbf{A} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}^T & \mathbf{C} \end{bmatrix} \begin{bmatrix} d\mathbf{x} \\ d\mathbf{y} \\ d\mathbf{s} \end{bmatrix} = \begin{bmatrix} \mathbf{r} - \mathbf{S}\mathbf{C}^T\mathbf{x} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}. \tag{5.4.24}$$

The starting point of the iterative algorithm (denoted by $(\mathbf{x}_0, \mathbf{y}_0, \mathbf{s}_0)$) should be a primal-dual feasible point satisfying the following conditions:

$$\begin{aligned} &\mathbf{Ax}_0 = \mathbf{b}, \quad \mathbf{x}_0 > \mathbf{0}, \\ &\mathbf{Cs}_0 + \mathbf{A}^T\mathbf{y}_0 = \mathbf{0}, \quad \mathbf{s}_0 > \mathbf{0}. \end{aligned} \tag{5.4.25}$$

A qualified starting point can be easily found by existing linear programming methods [34].

Problem (5.4.11) has a convex objective function and a simplex constraint, which ensures the existence of the optimal solution. For the solution to be unique, the objective function has to be strictly convex, or equivalently, its Hessian matrix has to be positive definite. Let us denote the Hessian matrix as $\mathbf{H}(\mathbf{x})$, and let $\mathbf{H}_{ij}(\mathbf{x})$ denote its element at the i'th row and j'th column. Then,

$$\mathbf{H}_{ij}(\mathbf{x}) = \frac{d^2\left(-\sum_{m=1}^{M} r_m \ln \mathbf{c}_m^T\mathbf{x}\right)}{dx_i dx_j} = \sum_{m=1}^{M} \frac{r_m c_{mi} c_{mj}}{(\mathbf{c}_m^T\mathbf{x})^2}. \tag{5.4.26}$$

Therefore, the fusion result D_f is unique if $\mathbf{H}(D_f) > 0$, where $\mathbf{H}(D_f)$ is the value of $\mathbf{H}(\mathbf{x})$ using the probabilities in D_f.

5.5 Degree of Confidence for the fusion result

Since the K-L fusion method is for the SSF model, its generated fusion result should have a DoC satisfying the two DoC principles for SSF model proposed in section 3.7. Here we choose the K-L measure in Eq. (5.3.4) as the distance function $dist(D_k, D_f)$ in DoC principle 1 for SSF model.

Let w_f denote the DoC associated with the fusion result D_f. We propose to define w_f as follows:

$$w_f = \sum_{k=1}^{K} w_k e^{-C_k(D_f)}, \tag{5.5.27}$$

where $C_k(D_f)$ is the value of cost function $C_k(P)$ evaluated using the probability assignments in the fusion result D_f.

Eq. (5.5.27) can be motivated in the same way as Eq. (4.3.16) for the convex quadratic fusion method. Basically, w_f is accumulated from the DoC values of all sensor reports. The stronger D_k agrees with D_f, the more w_k should contribute to w_f. In the extreme case that sensor report D_k fully agrees with D_f ($C_k(D_f) = 0$), all the confidence associated with D_k is contributed to D_f. However, if D_f largely disagrees with D_f ($C_k(D_f) \to \infty$), little of w_k is added to w_f.

It is easy to verify that w_f given by Eq. (5.5.27) satisfies the two DoC principles for SSF model. In addition, as required in the fundamental principle 1, when there is only one sensor report D_1 to fuse, the DoC of the fusion result is the same as that of the single report. In section 5.7, we will prove that w_f determined by Eq. (5.5.27) satisfies Eq. (3.6.36) in the fundamental principle 5 for SSF model as well.

Note that when a new sensor report D_{K+1} is fused, the DoC of the new fusion result D'_f is given by

$$w'_f = \sum_{k=1}^{K+1} w_k e^{-C_k(D'_f)} = \sum_{k=1}^{K} w_k e^{-C_k(D'_f)} + w_{K+1} e^{-C_{K+1}(D'_f)}, \tag{5.5.28}$$

which cannot be recursively updated from w_k. Therefore, we need to maintain the function form $f(D) = \sum_{k=1}^{K} w_k e^{-C_k(D)}$ for future fusion steps.

5.6 Discussion on sensor report form

The K-L fusion method uses K-L measure to represent the discrepancies between sensor reports and the fusion result. As introduced in section 5.1, K-L measure applies to a pair of probability distributions and measures the cross-entropy between them. As a result, the subsets in a sensor report are required to be mutually exclusive and exhaustive. Otherwise, property 3 in section 5.1 is no longer true, and the new fusion method may generate incorrect fusion results (as illustrated in the following example). This is the reason why the K-L fusion method only fuses partition type reports instead of general ratio type reports.

Example 5.6.1 *Let $\Omega = \{a_1, a_2, a_3, a_4, a_5\}$. Suppose there is a single sensor report D_1 given by*

$$D_1 = \{P_1(a_1 \vee a_2) : P_1(a_2) : P_1(a_3) : P_1(a_4) : P_1(a_5) = 0.55 : 0.15 : 0.15 : 0.1 : 0.2\}. \qquad (5.6.29)$$

Note that D_1 is a ratio type report but not a partition type report, since the subsets in D_1 are not mutually exclusive. According to the fundamental principle 1, the fusion result of this single report should be:

$$D_f = \{P_f(a_1) = 0.40,\ P_f(a_2) = 0.15,\ P_f(a_3) = 0.15,\ P_f(a_4) = 0.10,\ P_f(a_5) = 0.20\}. \qquad (5.6.30)$$

However, if we use the K-L fusion method to fuse D_1, the fusion result is

$$D'_f = \{P'_f(a_1) = 0.004,\ P'_f(a_2) = 0.604,\ P'_f(a_3) = 0.131,\ P'_f(a_4) = 0.087,\ P'_f(a_5) = 0.174\}, \qquad (5.6.31)$$

which is incorrect because it is not consistent with D_1. Note that if we use the convex quadratic fusion method, the fusion result will be the same with (5.6.30).

Dempster-Shafer type sensor reports can be transformed into partition type reports by the transformation procedure proposed in section 4.5, and therefore can be properly fused by the K-L fusion method. Specifically, a Dempster-Shafer type sensor report D_k in the form of

$$D_k = \left\{ \begin{array}{ll} P_k(\Omega), & \\ P_k(\omega_k^\ell), & \ell = 1, \ldots, L_k \end{array} \right\} \qquad (5.6.32)$$

is transformed into the following partition type report:

$$\bigcup_{i:a_i\in\omega_k^1} a_i^1 : \cdots : \bigcup_{i:a_i\in\omega_k^{L_k}} a_i^{L_k} = r_k^1 : \cdots : r_k^{L_k}, \tag{5.6.33}$$

where

$$\begin{aligned} &a_i^{\ell_j} \in \omega_k^{\ell_j},\ j = 1,\ldots,M_i, \quad a_i^{\ell_1}\cup\cdots\cup a_i^{\ell_{M_i}} = a_i,\ i = 1,\ldots,N, \\ &a_i^{\ell_j}\cap a_i^{\ell_m} = \emptyset,\ \forall j\neq m, \quad r_k^{\ell} = w_k\cdot\frac{P_k(\omega_k^{\ell})}{1-P_k(\Omega)},\ w_k = -\ln P_k(\Omega). \end{aligned} \tag{5.6.34}$$

If the subsets are not exhaustive, i.e., $\omega_k^1\cup\cdots\cup\omega_k^{L_k}\neq\Omega$, the following equivalent form should be used:

$$\bigcup_{i:a_i\in\omega_k^1} a_i^1 : \cdots : \bigcup_{i:a_i\in\omega_k^{L_k}} a_i^{L_k} : \bigcup_{i:a_i\notin\cup_{\ell=1}^{L_k}\omega_k^{\ell}} a_i = r_k^1 : \cdots : r_k^{L_k} : 0. \tag{5.6.35}$$

Note that the following additional constraints must be added into (5.3.7) when the transformed report (5.6.33) is fused by the K-L fusion method:

$$\sum_{j=1}^{M_i} p_i^{\ell_j} = p_i,\ i = 1,\ldots,N; \quad p_i^{\ell_j}\geq 0,\ \forall\, i,j. \tag{5.6.36}$$

Here, $p_i^{\ell_j}$ is the probability assigned to sub-proposition $a_i^{\ell_j}$ in the fusion result.

5.7 Satisfaction of decision fusion principles

In this section, we analyze the K-L fusion method using the decision fusion principles proposed in Chapter 3.

Theorem 5.7.1 *The K-L fusion method satisfies fundamental principles 1, 2, 3, and fundamental principles 4 and 5 for SSF model. The fusion method also satisfies DoC principles 1 and 2 for SSF model.*

Proof. Let $\Omega = \{a_1, \dots, a_N\}$. Suppose a partition type sensor report D_k takes the following form:

$$D_k = \left\{ \begin{array}{l} w_k = r_k^1 + \cdots + r_k^{L_k}, \\ P_k(\omega_k^1) : \cdots : P_k(\omega_k^{L_k}) = r_k^1 : \cdots : r_k^{L_k} \end{array} \right\}, \tag{5.7.37}$$

where $\omega_k^i \cap \omega_k^j = \emptyset$ for $i \neq j$, and $\omega_k^1 \cup \cdots \cup \omega_k^{L_k} = \Omega$. In addition, $r_k^\ell \geq 0$ for $\ell = 1, \dots, L_k$, and w_k is the DoC of D_k. Also define the distance measure $dist(D_k, D_f)$ as

$$dist(D_k, D_f) = C_k(D_f) = \sum_{\ell=1}^{L_k} \frac{r_k^\ell}{w_k} \ln \left(\frac{\frac{r_k^\ell}{w_k}}{P_f(\omega_k^\ell)} \right), \tag{5.7.38}$$

where $C_k(D_f)$ denotes the value of cost function $C_k(P)$ using the probability assignments in D_f. Obviously the $dist(D_k, D_f)$ defined above satisfies the properties proposed for the distance function in section 3.6.4 (see page 58). We examine the decision fusion principles one by one.

a) Fundamental principle 1: Suppose there is only one sensor report D_1 to fuse. The overall cost function is $C_{f_1}(P) = w_1 \sum_{\ell=1}^{L_1} \frac{r_1^\ell}{w_1} \ln \left(\frac{\frac{r_1^\ell}{w_1}}{P_f(\omega_1^\ell)} \right)$. Obviously, if we choose $P_f(\omega_1^\ell) = \frac{r_1^\ell}{w_1}$ for $\ell = 1, \dots, L_1$, the cost function reaches its minimal value: $C_1(P) = 0$. In other words, this is the optimal fusion result. The DoC of the fusion result is $w_f = w_1 \cdot e^{-\ln C_1(D_f)} = w_1$. As such, the fusion result is identical to D_1.

b) Fundamental principle 2: This is straight forward. The K-L fusion method is only used in the batch fusion mode (see discussions on page 114), i.e., sensor reports are fused all at once by minimizing a cost function which is constructed using original sensor reports. The principle is satisfied because the cost function for a group of sensor reports is invariant with the permutation of sensor report indexing.

c) Fundamental principle 3: Suppose there are N cyclic sensor reports $D_1, \dots, D_N$ with D_k given as

$$D_k = \left\{ \begin{array}{l} w_r = \sum_{i=1}^{N} r_{i\oplus k} = \sum_{i=1}^{N} r_i, \\ P_k(a_i) = r_{i\oplus k}, \quad i = 1, \dots, N \end{array} \right\}, \tag{5.7.39}$$

where $r_i \geq 0$ for $i = 1, \dots, N$, and $i \oplus k = i + k - 1 \mod N$. Note that the DoC values of all the reports are the same. We can write the overall cost function as

$$\begin{aligned} C_{f_N}(P) &= \sum_{k=1}^{N} w_k C_k(P) = \sum_{k=1}^{N} w_k \sum_{i=1}^{N} \frac{r_{i\oplus k}}{w_k} \ln \left(\frac{\frac{r_{i\oplus k}}{w_k}}{p_i} \right). \\ &= w_1 \sum_{k=1}^{N} \sum_{i=1}^{N} r_{i\oplus k} \ln \left(\frac{r_{i\oplus k}}{w_k} \right) - N w_1 \sum_{i=1}^{N} \ln p_i, \end{aligned} \tag{5.7.40}$$

which naturally leads to the following desired fusion result: $p_i = \dfrac{1}{N}$ for $i = 1, \dots, N$.

d) Fundamental principle 4 for SSF model: Suppose there are K sensor reports $D_1, \dots, D_K$. Let D_{f_1} and D_{f_2} denote the fusion results before and after the fusion of sensor report D_1. Denote the cost function before the fusion of D_1 as

$$C_{f_1}(P) = \sum_{k=2}^{K} w_k C_k(P), \tag{5.7.41}$$

and the cost function after the fusion of D_1 as

$$C_{f_2}(P) = \sum_{k=1}^{K} w_k C_k(P) = w_1 C_1(P) + C_{f_1}(P). \tag{5.7.42}$$

Also let $C_k(D_{f_j})$ and $C_{f_i}(D_{f_j})$ $(i, j = 1, 2)$ denote cost functions $C_k(P)$ and $C_{f_i}(P)$ using the probability assignments in D_{f_j}. Obviously, D_{f_2} is identical with D_{f_1} when $w_1 = 0$, and is identical with D_1 (except the DoC values) when $w_1 \to \infty$, i.e.,

$$\begin{aligned} dist(D_{f_2}, D_{f_1}) = 0, \quad w_1 = DoC_{min}, \\ dist(D_1, D_{f_2}) = 0, \quad w_1 = DoC_{max}. \end{aligned} \tag{5.7.43}$$

Note that the overall cost functions (5.7.41) and (5.7.42) are additive. Therefore we can follow the same proof as that in section 4.7 (see page 102) for the convex quadratic fusion method. In particular, we have $C_1(D_{f_1}) \geq C_1(D_{f_2})$, or

$$dist(D_1, D_{f_1}) \geq dist(D_1, D_{f_2}). \tag{5.7.44}$$

This establishes the desired property (3.6.26).

Now assume that the DoC of D_1 changes from w_{1a} to w_{1b}, with $w_{1b} > w_{1a}$. Let $D_1^{w_{1i}}$ denote D_1 with DoC value w_{1i}, and $D_{f_2}^{w_{1i}}$ denote the fusion result after the fusion of $D_1^{w_{1i}}$, $i = a, b$. Also define normalized ratios $\bar{r}_1^\ell$'s as in Eq. (4.7.59), i.e.,

$$\bar{r}_1^\ell = \frac{r_1^\ell}{w_1}, \quad i = 1, \ldots, L_1. \tag{5.7.45}$$

As discussed on page 102, $\bar{r}_1^\ell$ is independent of w_1. In the overall cost function, the term corresponding to $D_1^{w_{1b}}$ can be written as

$$\begin{aligned} C_1^{w_{1b}\prime} &= w_{1b} C_1^{w_{1b}}(P) = w_{1b} \sum_{\ell=1}^{L_1} \bar{r}_1^\ell \ln\left(\frac{\bar{r}_1^\ell}{P_f(\omega_1^\ell)}\right) \\ &= w_{1a} \sum_{\ell=1}^{L_1} \bar{r}_1^\ell \ln\left(\frac{\bar{r}_1^\ell}{P_f(\omega_1^\ell)}\right) + (w_{1b} - w_{1a}) \sum_{\ell=1}^{L_1} \bar{r}_1^\ell \ln\left(\frac{\bar{r}_1^\ell}{P_f(\omega_1^\ell)}\right). \end{aligned} \tag{5.7.46}$$

The above equation suggests that increasing the DoC of D_1 from w_{1a} to w_{1b} is equivalent to the fusion of another sensor report $D_1^{w_{1c}}$ which is identical to D_1 except that its DoC value is $w_{1c} = w_{1b} - w_{1a}$. Similar to inequality (5.7.44), we have

$$dist(D_1, D_{f_2}^{w_{1a}}) \geq dist(D_1, D_{f_2}^{w_{1b}}), \tag{5.7.47}$$

i.e., when w_1 increases, $dist(D_1, D_{f_2})$ decreases.

e) Fundamental principle 5 for SSF model: Suppose there are K identical sensor reports from similar and independent sensors, each denoted by $D_1^{w_1}$, where w_1 denotes the DoC associated with D_1 and suppose that $w_1 > DoC_{min}$. Let D_{f_0} and $D_{f_K}^{w_1}$ (with DoC value $w_{f_K}^{w_1}$) denote the fusion results before and after the fusion of these reports. We further denote the cost function before and after the fusion of these reports as $C_{f_0}(P)$ and $C_{f_K}^{w_1}(P)$ respectively. Then,

$$C_{f_K}^{w_1}(P) = Kw_1C_1^{w_1}(P) + C_{f_0}(P), \tag{5.7.48}$$

where $C_1^{w_1}(P)$ is the cost function for report $D_1^{w_1}$. Note that $C_1^{w_1}(D_1) = 0$, where $C_1^{w_1}(D_1)$ is the value of $C_1^{w_1}(P)$ using the probability assignments in $D_1^{w_1}$.

$C_{f_K}^{w_1}(P)$ reaches its minimal value at $D_{f_K}^{w_1}$, i.e.,

$$Kw_1C_1^{w_1}(D_1) + C_{f_0}(D_1) \;\geq\; Kw_1C_1^{w_1}(D_{f_K}^{w_1}) + C_{f_0}(D_{f_K}^{w_1}), \tag{5.7.49}$$

which implies that $C_{f_0}(D_1) \geq Kw_1C_1^{w_1}(D_{f_K})$, or

$$\frac{1}{K}C_{f_0}(D_1) \;\geq\; w_k \sum_{\ell=1}^{L_1} \frac{r_1^\ell}{w_1} \ln\left(\frac{\frac{r_1^\ell}{w_1}}{P_f(\omega_1^\ell)}\right). \tag{5.7.50}$$

Since $\lim_{K\to\infty} \frac{1}{K}C_{f_0}(D_1) = 0$, we have

$$\lim_{K\to\infty} \left(P_{f_K}^{w_1}(\omega_1^1) : \cdots : P_{f_K}^{w_1}(\omega_1^{L_1})\right) = r_1^1 : \cdots : r_1^{L_1}, \tag{5.7.51}$$

or

$$\lim_{K\to\infty} D_{f_K}^{w_1} = D_1. \tag{5.7.52}$$

To prove property (3.6.36), we note that DoC of $D_{f_K}^{w_1}$ is

$$w_{f_K}^{w_1} = Kw_1 e^{-C_1(D_{f_K}^{w_1})} + w_{f_0}(D_{f_K}^{w_1}), \tag{5.7.53}$$

where the first part represents the contribution from the K identical reports, and the second part represents the contribution from other reports in the fusion. Since Eq. (5.7.52) implies that

$$\lim_{K\to\infty} C_1(D_{f_K}^{w_1}) = \lim_{K\to\infty} C_1(D_1) = 0, \tag{5.7.54}$$

we have

$$\lim_{K\to\infty} w_{f_K}^{w_1} \geq \lim_{K\to\infty} Kw_1 e^{-C_1(D_{f_K}^{w_1})} = \lim_{K\to\infty} Kw_1 = \infty, \tag{5.7.55}$$

i.e., $\lim_{K\to\infty} w_{f_K}^{w_1} = DoC_{max}$.

To prove property (3.6.37), suppose that the DoC value of w_1 changes from w_{1a} to w_{1b}, with $w_{1b} > w_{1a}$. Using the same rational for inequality (5.7.47), we know that increasing the DoC of the K identical sensor reports from w_{1a} to w_{1b} is equivalent to the fusion of an additional group of K identical reports $D_1^{w_{1c}}$'s. Similar to inequality (5.7.47), we have $dist(D_1, D_{f_K}^{w_{1a}}) \geq dist(D_1, D_{f_K}^{w_{1b}})$. This establishes the desired property (3.6.37).

f) DoC principle 1 for SSF model: Here we follow the assumptions made in DoC principle 1 for SSF model in section 3.7. That is, let D_{f_1} denote the fusion result of sensor reports $D_{11}, \ldots, D_{1K}$, and D_{f_2} denote the fusion result of sensor reports $D_{21}, \ldots, D_{2K}$. Let w_{ij} denote the DoC value associated with report D_{ij}, and w_{f_i} denote the DoC value associated with fusion result D_{f_i}. Assume that $D_{f_1} = D_{f_2}$, and $w_{1k} = w_{2k}$ for $k = 1, \ldots, K$. We further assume that sensor reports D_{1k} and D_{2k} have the same subsets. Then, inequalities

$$dist(D_{1k}, D_{f_1}) > dist(D_{2k}, D_{f_2}), \quad k = 1, \ldots, K \tag{5.7.56}$$

imply that

$$w_{f_2} = \sum_{k=1}^{K} w_{2k} e^{-C_{2k}(D_{f_2})} > \sum_{k=1}^{K} w_{1k} e^{-C_{1k}(D_{f_1})} = w_{f_1}, \tag{5.7.57}$$

which proves property (3.7.38).

g) DoC principle 2 for SSF model: This is straight forward. Suppose the assumptions made in DoC principle 2 for SSF model are satisfied. That is, let D_{f_1} denote the fusion result of sensor reports $D_1^{w_1}, \ldots, D_K^{w_K}$, where w_k is the DoC associated with the k'th report. Let D_{f_2} denote the fusion result of $D_1^{\beta w_1}, \ldots, D_K^{\beta w_K}$. In other words, the DoC of the k'th report changes from w_k to βw_k. Also let w_{f_i} denote the DoC value associated with fusion result D_{f_i}. Then by the K-L fusion method, D_{f_1} and D_{f_2} are identical to each other except their DoC values. Since $C_k(D_f)$ is independent of w_k, we have

$$w_{f_2} = \sum_{k=1}^{K} \beta w_k e^{-C_k(D_f)} = \beta \sum_{k=1}^{K} w_k e^{-C_k(D_f)} = \beta w_{f_1}, \tag{5.7.58}$$

which establishes property (3.7.39). □

Discussion on the restricted fundamental principle 4 for SSF model

Theorem 5.7.1 shows that the K-L fusion method satisfies the decision fusion principles proposed in Chapter 3. However, the method only satisfies the restricted fundamental principle 4 for SSF model in the special case where the K sensors assign probabilities to *all and only* the basic propositions a_i's. To prove this, we write the sensor reports in the following ratio type form:

$$D_k = \left\{ \begin{array}{l} DoC = w_1, \\ P_k(a_1) : \cdots : P_k(a_N) = w_k p_k^1 : \cdots : w_k p_k^N \end{array} \right\}, \quad k = 1, \ldots, K, \tag{5.7.59}$$

where $P_k(a_i) = p_k^i$ represents the probability assigned to proposition a_i in report D_k, with $\sum_{i=1}^{N} p_k^i = 1$. Denote the fusion result before and after the fusion of report D_1 as D_{f_1} and D_{f_2}.

Then, from Eq. (5.3.9), $D_{f_1} = \{P_{f_1}(a_1), \ldots, P_{f_1}(a_N)\}$ where

$$P_f(a_i) = \frac{\sum_{k=2}^{K} w_k p_k^i}{\sum_{k=2}^{K} w_k}, \quad i = 1, \ldots, N. \tag{5.7.60}$$

Similarly, $D_{f_2} = \{P_{f_2}(a_1), \ldots, P_{f_2}(a_N)\}$ where

$$P_f(a_i) = \frac{\sum_{k=1}^{K} w_k p_k^i}{\sum_{k=1}^{K} w_k}, \quad i = 1, \ldots, N. \tag{5.7.61}$$

Note that Eqs. (5.7.60) and (5.7.61) are identical to Eqs. (4.7.75) and (4.7.76) respectively except the weighting factors. Therefore we can follow the same proof for the convex quadratic fusion method (see pages 106 to 107). Specifically, similar to Eq. (4.7.77), here we have

$$P_{f_2}(a_i) - P_{f_1}(a_i) = \frac{w_1}{\sum_{k=1}^{K} w_k}(P_1(a_i) - P_{f_1}(a_i)), \tag{5.7.62}$$

from which we can establish properties (3.6.27) and (3.6.28). It can also be seen from Eq. (5.7.62) that if D_1 changes so that $|P_1(a_i) - P_{f_1}(a_i)|$ or w_1 increases, $|P_{f_2}(a_i) - P_{f_1}(a_i)|$ also increases.

In general, the K-L fusion method does not satisfy the fundamental principle 4 for SSF model. This is illustrated in the following example. However, this should not be viewed as a weakness of the new fusion method since the principle is not absolutely necessary in practice (see discussion on page 59).

Example 5.7.1 *Let* $\Omega = \{a_1, a_2, a_3\}$. *Suppose there are three similar and independent sensors each sending a report to the fusion center. The sensor reports are the same as in*

Example 4.7.1, i.e.,

$$
\begin{aligned}
D_1 &= \left\{ \begin{array}{l} w_1 = 1, \\ P_1(a_1) : P_1(a_2) : P_1(a_3) = 0.70 : 0.15 : 0.15 \end{array} \right\}, \\
D_2 &= \left\{ \begin{array}{l} w_2 = 1, \\ P_2(a_1) : P_2(a_2 \vee a_3) = 0.20 : 0.80 \end{array} \right\}, \\
D_3 &= \left\{ \begin{array}{l} w_3 = 1, \\ P_3(a_1) : P_3(a_2) : P_3(a_3) = 0.20 : 0.20 : 0.60 \end{array} \right\}.
\end{aligned}
\tag{5.7.63}
$$

By the K-L fusion method, the fusion result before the fusion of D_1, i.e., the fusion result of D_2 and D_3, is given by $D_{f_1} = \{P_{f_1}(a_1) = 0.20,\ P_{f_1}(a_2) = 0.20,\ P_{f_1}(a_3) = 0.60\}$, and the fusion result after the fusion of D_1, i.e., the fusion result of D_1, D_2 and D_3, is $D_{f_2} = \{P_{f_2}(a_1) = 0.3667,\ P_{f_2}(a_2) = 0.2015,\ P_{f_2}(a_3) = 0.4318\}$. Here, property (3.6.28) is not satisfied, since $P_1(a_2) = 0.15 < P_{f_1}(a_2) = 0.20$, but $P_{f_2}(a_2) = 0.2015 > P_{f_1}(a_2) = 0.20$.

5.8 Numerical examples

In this section, a numerical example is given to illustrate the performance of the K-L fusion method.

Example 5.8.1

Let $\Omega = \{a_1, a_2, a_3, a_4\}$. Suppose there are three independent sensors making observations on a target. All the sensors explore common physical characteristics of the target. Assume that each sensor sends a report to the fusion center given as follows:

$$D_1 = \left\{ \begin{array}{l} w_1 = r_1^1 + r_1^2 = 0.6, \\ P_1(a_1 \vee a_3) : P_1(a_2 \vee a_4) = r_1^1 : r_1^2 = 0.6 : 0 \end{array} \right\},$$
$$D_2 = \left\{ \begin{array}{l} w_2 = r_2^1 + r_2^2 = 0.7, \\ P_2(a_1 \vee a_2) : P_2(a_3 \vee a_4) = r_2^1 : r_2^2 = 0 : 0.7 \end{array} \right\}, \tag{5.8.64}$$
$$D_3 = \left\{ \begin{array}{l} w_3 = r_3^1 + r_3^2 + r_3^3 + r_3^4 = 0.8, \\ P_3(a_1) : P_3(a_2) : P_3(a_3) : P_3(a_4) = r_3^1 : r_3^2 : r_3^3 : r_3^4 = 0.1 : 0.2 : 0.3 : 0.2 \end{array} \right\}.$$

In the K-L fusion method, the overall cost function is

$$C_{f_3}(P) = C_1(P) + C_2(P) + C_3(P) = -0.6\ln(p_1 + p_3) - 0.7\ln(p_3 + p_4)$$
$$-0.8\left(0.125\ln p_1 + 0.25\ln p_2 + 0.375\ln p_3 + 0.25\ln p_4\right). \tag{5.8.65}$$

To achieve the optimal fusion result, the method minimizes $C_{f_3}(P)$ subject to the probability constraint, i.e.,

$$\begin{array}{ll} \text{minimize} & C_{f_3}(P) \\ \text{subject to} & p_1 + p_2 + p_3 + p_4 = 1, \\ & p_i \geq 0, \quad i = 1, 2, 3, 4. \end{array} \tag{5.8.66}$$

The fusion result is shown in Table 5.5. Also shown in the table is the fusion result by the convex quadratic fusion method. Note that the results from the two fusion methods match with each other very well.

Propositions	a_1	a_2	a_3	a_4
fusion result: K-L fusion method	0.082	0.108	0.630	0.180
fusion result: convex quadratic fusion method	0.078	0.222	0.583	0.117

Table 5.5: Fusion results for Example 5.8.1.

By the K-L fusion method, the DoC of the fusion result is

$$w_f = \sum_{k=1}^{3} w_k \cdot e^{C_k(D_f)} = 0.6 \cdot e^{-0.204} + 0.7 \cdot e^{-1.163} + 0.8 \cdot e^{-0.857} = 1.048, \tag{5.8.67}$$

while the DoC of the fusion result by the convex quadratic fusion method is

$$\begin{aligned} w_f &= \sum_{k=1}^{3} w_k \cdot \frac{C_{k_{max}} - C_k(D_f)}{C_{k_{max}}} \\ &= 0.6 \cdot \frac{0.874 - 0.230}{0.874} + 0.7 \cdot \frac{0.605 - 0.527}{0.605} + 0.8 \cdot \frac{1.253 - 0.064}{1.253} = 1.292. \end{aligned} \tag{5.8.68}$$

The fusion is performed on a 800MHz Pentium PC under the MATLAB environment. The CPU time used is less than one second, showing that the fusion method is computationally efficient.

Chapter 6

Analytic Center Fusion Method

In the two previous chapters, we have proposed two fusion methods for the Similar Sensor Fusion (SSF) model. In this chapter, we turn our attention to the Dissimilar Sensor Fusion (DSF) model and propose a so-called analytic center fusion method which is based on analytic centers and optimization. Similar to the two previous methods, the new method also minimizes a cost function to achieve the optimal fusion result. However, the cost function we choose for the new method has a special structure to accumulate identity information from all sensor reports, enabling the reinforcement among the reports. As a result, the fusion result obtained by the new method represents the decreased uncertainty in target identity from the sensor reports, which is a crucial feature desired in the DSF model. Moreover, the new method has merits similar to the previously proposed methods: it can be solved efficiently in polynomial time, and does not need the knowledge of prior probabilities or conditional probabilities.

6.1 Problem formulation

Suppose there are a total of K independent sensors making observations on a target within a surveillance region. Each sensor explores some different target characteristics and sends its own target identity declarations in a sensor report to the fusion center. Let us denote these reports by

$$D_k = \left\{ \begin{array}{l} \text{Degree of Confidence} = w_k, \\ \\ P_k(a_i) = r_k^i, \quad i = 1, \ldots, N \end{array} \right\}, \quad k = 1, ..., K, \tag{6.1.1}$$

where $r_k^i \in [0,1]$ for $i = 1, \ldots, N$, $\sum_{i=1}^{N} r_k^i = 1$, and $w_k \in [0,1]$. In other words, a probability is assigned to each of the N basic propositions in Ω.

Note that $w_k \in [0,1]$, i.e., $DoC_{max} = 1$. This is different from the ratio type sensor reports introduced in section 4.1, where we define $DoC_{max} = \infty$. The different definitions for DoC_{max} result from the different natures of the DSF model and the SSF model. In the SSF model, sensors explore a set of common target characteristics. The sensor reports confirm each other on the target identity, and the fusion result should have minimal discrepancies with the sensor reports. If a sensor report has maximal DoC, the fusion result must agree with it. As a result, we define $DoC_{max} = \infty$. In the DSF model, each sensor explores some different target characteristics. The sensor reports reinforce each other to generate a fusion result which represents the decreased uncertainty in target identity. If a sensor report has maximal DoC, it means that the report is fully accurate based on the target characteristics it explores. However, in this case the fusion result does not necessarily agree with the report, i.e., the probability assignments in the fusion result generally do not match with the ones in the sensor report. This is because that the fusion result is based on target characteristics explored by all sensor reports. Therefore a report with a DoC value equals to $DoCmax$ does not carry as much weight in the DSF model as it would in the SSF model. This motivates us to define a finite $DoCmax$ value for the DSF model and an infinite $DoCmax$ for the SSF model.

It should be pointed out that a Dempster-Shafer type sensor report D_k as defined in (4.5.33) can be transformed into the report form in (6.1.1) and be fused by the new fusion method. Specifically, let a Dempster-Shafer type sensor report D_k be given as $D_k = \{P_k(\omega_k^\ell),\ \ell = 1, \ldots, L_k\}$, where $\sum_{\ell=1}^{L_k} P_k(\omega_k^\ell) = 1$, $\omega_k^\ell \subseteq \Omega$ and $P_k(\omega_k^\ell) \in [0,1]$. Let m_k^ℓ denote the number of propositions included in ω_k^ℓ. Then D_k can be transformed into the following report D'_k:

$$D'_k = \left\{ \begin{array}{l} \text{Degree of Confidence} = w'_k, \\ P'_k(a_i) = r_k^{i\,\prime}, \quad i = 1, \ldots, N \end{array} \right\}, \quad k = 1, ..., K, \tag{6.1.2}$$

where

$$r_k^{i\,\prime} = \begin{cases} \sum_{\ell: a_i \in \omega_k^\ell} \frac{1}{m_k^\ell} P_k(\omega_k^\ell), & \text{if } a_i \in \bigcup_{\ell=1}^{L_k} \omega_k^\ell, \\ 0, & \text{otherwise}, \end{cases} \tag{6.1.3}$$

and $w'_k = 1 - P_k(\Omega)$. Basically, each probability mass $P_k(\omega_k^\ell)$ is evenly separated among the propositions included in ω_k^ℓ, i.e., probability $\mu_k^\ell(a_i) = \frac{1}{m_k^\ell} P_k(\omega_k^\ell)$ is assigned to each $a_i \in \omega_k^\ell$. the total probability for each a_i is the normalized of all the $\mu_k^\ell(a_i)$'s for $\ell = 1, \ldots, L_k$. Note that $\sum_{i=1}^{N} r_k^{i\,\prime} = 1$, and DoC w'_k increases inversely with $P_k(\Omega)$.

For the new fusion method, the fusion goal is to determine a set of probabilities

$$P_{f_K}(a_i) = p_i, \qquad i = 1, \ldots, N \tag{6.1.4}$$

that best represent the decreased uncertainty in target identity from the given sensor reports. Similar to the two fusion methods introduced in the previous chapters, we formulate the fusion problem as an optimization problem in which a cost function is minimized subject to some probability constraints. We propose to use the following cost function:

$$C_{f_K}(P) = \sum_{k=1}^{K} w_k \sum_{i=1}^{N} \frac{1}{r_k^i} p_i - \sum_{i=1}^{N} \ln p_i. \tag{6.1.5}$$

In this cost function, the first part accumulates identity information from all the sensor reports. Specifically, the contribution of the k'th sensor report is summarized in the term $w_k \sum_{i=1}^{N} \frac{1}{r_k^i} p_i$. Here, the weight w_k signifies the importance of the term $\sum_{i=1}^{N} \frac{1}{r_k^i} p_i$. The coefficient of p_i, i.e., $1/r_k^i$, is used to guide the assignment of probability p_i so that a large r_k^i will tend to give rise to a large p_i in the final fusion result. The second part in the above cost function is also called the logarithmic barrier function, its value increases when any p_i approaches $p_i = 0$. Moreover, the closer p_i approaches $p_i = 0$, the faster the value increases. When $C_{f_K}(P)$ is minimized, this part keeps p_i away from $p_i = 0$. Note that the second function part is independent of the sensor reports, it is used to balance the first cost function part so that the fusion result can appropriately represent the reinforcement among sensor reports.

Now we can formulate the identity fusion problem as the following convex optimization problem:

$$\begin{aligned} \text{minimize} \quad & C_{f_K}(P) = \sum_{k=1}^{K} w_k \sum_{i=1}^{N} \frac{1}{r_k^i} p_i - \sum_{i=1}^{N} \ln p_i \\ \text{subject to} \quad & \sum_{i=1}^{N} p_i = 1, \quad p_i \geq 0, \; i = 1, \ldots, N. \end{aligned} \tag{6.1.6}$$

Let w_{f_K} denote the DoC associated with fusion result D_{f_K}. We propose to define w_{f_K} as follows:

$$w_{f_K} = 1 - \prod_{k=1}^{K} (1 - w_k). \tag{6.1.7}$$

Note that this definition of DoC is independent of the fusion result. This is in contrast to the definition of DoC of the fusion result in the SSF model (Eqs. (4.3.16) and (5.5.27)). As will be shown later in section 6.3, w_{f_K} determined by Eq. (6.1.7) satisfies the fundamental principle 1, property (3.6.30) in the fundamental principle 5 for DSF model and DoC principle 1 for DSF model.

The analytic center fusion method can operate in both the batch and the sequential fusion modes (introduced in section 3.1.1). However, similar to the two fusion methods proposed in previous chapters (see discussion on page 87 and page 114), we only recommend to use the new fusion method in the batch fusion mode. In addition, both the objective function in (6.1.6) and the DoC of the fusion result (6.1.7) can be recursively updated when new sensor reports are fused. Specifically,

$$C_{f_{K+1}}(P) = \sum_{k=1}^{K+1} w_k \sum_{i=1}^{N} \frac{1}{r_k^i} p_i - \sum_{i=1}^{N} \ln p_i = C_{f_K}(P) + w_{K+1} \sum_{i=1}^{N} \frac{1}{r_{K+1}^i} p_i, \tag{6.1.8}$$

and

$$w_{f_{K+1}} = 1 - \prod_{k=1}^{K+1} (1 - w_k) = 1 - (1 - w_{f_K})(1 - w_{K+1}). \tag{6.1.9}$$

Therefore we only need to maintain $C_{f_K}(P)$ and w_{f_K} for future fusion steps. In other words, compared with fusing sensor reports sequentially, fusing all sensor reports simultaneously does not require any extra memory. It does not incur any extra computation either, since the new method involves solving an optimization problem whose worst case complexity is invariant with the number of sensor reports to be fused.

6.2 Solving the analytic center formulation

The convex optimization problem in (6.1.6) is actually a special case of the following analytic center problem in linear programming [34]:

$$\begin{aligned} \text{minimize} \quad & \mathbf{c}^T\mathbf{x} - \mu \sum_{j=1}^{N} \ln x_j \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} = \mathbf{b}, \quad \mathbf{x} \geq \mathbf{0}, \end{aligned} \tag{6.2.10}$$

where $\mathbf{x} \in \Re^{N\times1}$, $\mathbf{c} \in \Re^{N\times1}$, $\mathbf{A} \in \Re^{M\times N}$, $\mathbf{b} \in \Re^{M\times1}$, and $\mu > 0$ is a scalar. The relationships between the analytic center formulation (6.1.6) and the above (6.2.10) are:

$$\begin{aligned} &\mathbf{c} = [\frac{1}{r_1} \quad \ldots \quad \frac{1}{r_N}]^T, \quad \text{where } \frac{1}{r_i} = \sum_{k=1}^{K} w_k \frac{1}{r_k^i}, \\ &\mathbf{x} = [p_1 \quad \ldots \quad p_N]^T, \quad \mu = 1, \quad \mathbf{A} = [1 \quad \ldots \quad 1], \quad \mathbf{b} = 1, \quad M = 1. \end{aligned} \tag{6.2.11}$$

To find the optimal solution to (6.2.10), we first derive its optimality conditions. Here, the Lagrangian for (6.2.10) is

$$L(\mathbf{x}, \mathbf{y}) = \mathbf{c}^T\mathbf{x} - \mu \sum_{j=1}^{N} \ln x_j - \mathbf{y}^T(\mathbf{A}\mathbf{x} - \mathbf{b}), \tag{6.2.12}$$

where $\mathbf{y} = [y_1, \ldots, y_M]^T \in \Re^{M\times 1}$ is an unknown vector. For $\mathbf{x}$ to be the optimal solution of (6.2.10), we must have $\nabla L(\mathbf{x}, \mathbf{y}) = 0$, i.e.,

$$c_i - \mu\frac{1}{x_i} - (\mathbf{y}^T\mathbf{A})_i = 0, \quad i = 1, \ldots, N, \tag{6.2.13}$$

where c_i and $(\mathbf{y}^T\mathbf{A})_i$ are the i'th elements of $\mathbf{c}$ and $\mathbf{y}^T\mathbf{A}$ respectively. Let

$$s_i = \frac{\mu}{x_i}, \qquad i = 1, \ldots, N. \tag{6.2.14}$$

Note that $s_i \geq 0$ for $i = 1, \ldots, N$. Then $x_i s_i = \mu$ for $i = 1, \ldots, N$, or $\mathbf{X}\mathbf{s} = \mu\mathbf{e}$, where $\mathbf{X} = diag(x_1, \ldots, x_N) \in \Re^{N\times N}$, $\mathbf{e} = [1 \ \ldots \ 1]^T \in \Re^{N\times 1}$. Using (6.2.14), we can rewrite (6.2.13) as

$$c_i - s_i - (\mathbf{y}^T\mathbf{A})_i = 0, \quad i = 1, \ldots, N, \tag{6.2.15}$$

or simply $\mathbf{A}^T\mathbf{y} + \mathbf{s} = \mathbf{c}$. The optimality conditions for (6.2.10) can be summarized as follows:

$$\begin{aligned} &\mathbf{X}\mathbf{s} = \mu\mathbf{e}, \\ &\mathbf{A}\mathbf{x} = \mathbf{b}, \quad \mathbf{x} \geq \mathbf{0}, \\ &\mathbf{A}^T\mathbf{y} + \mathbf{s} = \mathbf{c}, \quad \mathbf{s} \geq \mathbf{0}. \end{aligned} \tag{6.2.16}$$

In addition, when $\nabla L(\mathbf{x}, \mathbf{y}) = 0$, the Lagrangian is

$$\begin{aligned} L &= \mathbf{c}^T\mathbf{x} - \mu \sum_{j=1}^{N} \ln x_j - \mathbf{y}^T(\mathbf{A}\mathbf{x} - \mathbf{b}) \\ &= (\mathbf{A}^T\mathbf{y} + \mathbf{s})^T\mathbf{x} - \mu \sum_{j=1}^{N} \ln \frac{1}{s_j} - \mathbf{y}^T(\mathbf{A}\mathbf{x} - \mathbf{b}) = \mathbf{b}^T\mathbf{y} + \mu \sum_{j=1}^{N} \ln s_j + N, \end{aligned} \tag{6.2.17}$$

and the dual problem of (6.2.10) can be formed as

$$\begin{aligned} \text{maximize} \quad & \mathbf{b}^T\mathbf{y} + \mu \sum_{j=1}^{N} \ln s_j \\ \text{subject to} \quad & \mathbf{A}^T\mathbf{y} + \mathbf{s} = \mathbf{c}, \quad \mathbf{s} \geq \mathbf{0}. \end{aligned} \tag{6.2.18}$$

There are several algorithms which can solve the above analytic center problem efficiently, among which the primal-dual interior point algorithm [34] is the most popular and efficient one. In the following, we discuss the primal-dual algorithm in three steps: in subsection 6.2.1 we discuss the primal-dual Newton procedure which converges to the optimal solution $(\mathbf{x}^a, \mathbf{y}^a, \mathbf{s}^a)$, in subsection 6.2.2 we discuss the primal-dual potential algorithm which provides an initial point for the primal-dual Newton procedure, and finally in subsection 6.2.3 we discuss the generation of a starting point for the primal-dual potential algorithm.

6.2.1 Primal-dual Newton procedure

The primal-dual interior point algorithm starts from a primal-dual feasible point in the feasible region of (6.2.10) and (6.2.18), and iteratively takes the so-called Newton steps, moving an iterate towards the optimal solution. Note that in each iteration, the iterate $(\mathbf{x}, \mathbf{y}, \mathbf{s})$ is always a primal-dual feasible point, i.e., it satisfies

$$\mathbf{A}\mathbf{x} = \mathbf{b}, \quad \mathbf{x} > \mathbf{0}, \quad \mathbf{A}^T\mathbf{y} + \mathbf{s} = \mathbf{c}, \quad \mathbf{s} > \mathbf{0}. \tag{6.2.19}$$

At the next iteration, $(\mathbf{x}, \mathbf{y}, \mathbf{s})$ is updated to $(\mathbf{x}^+, \mathbf{y}^+, \mathbf{s}^+)$:

$$\mathbf{x}^+ = \mathbf{x} + d\mathbf{x}, \quad \mathbf{y}^+ = y + d\mathbf{y}, \quad \mathbf{s}^+ = s + d\mathbf{s}. \tag{6.2.20}$$

The update should satisfy the following conditions (cf. (6.2.16)):

$$\begin{aligned} \mathbf{S}d\mathbf{x} + \mathbf{X}d\mathbf{s} &= \mu\mathbf{e} - \mathbf{X}\mathbf{s}, \\ \mathbf{A}d\mathbf{x} &= \mathbf{0}, \\ -\mathbf{A}^T d\mathbf{y} - d\mathbf{s} &= \mathbf{0}, \end{aligned} \tag{6.2.21}$$

from which $d\mathbf{x}$, $d\mathbf{y}$, $d\mathbf{s}$ can be derived as

$$\begin{aligned} d\mathbf{y} &= -(\mathbf{A}\mathbf{S}^{-1}\mathbf{X}\mathbf{A}^T)^{-1}\mathbf{A}\mathbf{S}^{-1}(\mu\mathbf{e} - \mathbf{X}\mathbf{s}), \\ d\mathbf{s} &= -\mathbf{A}^T d\mathbf{y}, \\ d\mathbf{x} &= \mathbf{S}^{-1}(\mu\mathbf{e} - \mathbf{X}\mathbf{s}) - \mathbf{S}^{-1}\mathbf{X}d\mathbf{s}, \end{aligned} \tag{6.2.22}$$

where $\mathbf{S} = diag(s_1, \ldots, s_M)$. To verify that $(d\mathbf{x}, d\mathbf{y}, d\mathbf{s})$ satisfies (6.2.21), we observe that

$$\begin{aligned} \mathbf{S}d\mathbf{x} + \mathbf{X}d\mathbf{s} &= \mathbf{S}\left(\mathbf{S}^{-1}(\mu\mathbf{e} - \mathbf{X}\mathbf{s}) - \mathbf{S}^{-1}\mathbf{X}d\mathbf{s}\right) + \mathbf{X}d\mathbf{s} \\ &= ((\mu\mathbf{e} - \mathbf{X}\mathbf{s}) - \mathbf{X}d\mathbf{s}) + \mathbf{X}d\mathbf{s} = \mu\mathbf{e} - \mathbf{X}\mathbf{s}, \end{aligned} \tag{6.2.23}$$

and

$$\begin{aligned} \mathbf{A}d\mathbf{x} &= \mathbf{A}\left(\mathbf{S}^{-1}(\mu\mathbf{e} - \mathbf{X}\mathbf{s}) - \mathbf{S}^{-1}\mathbf{X}d\mathbf{s}\right) = \mathbf{A}\mathbf{S}^{-1}(\mu\mathbf{e} - \mathbf{X}\mathbf{s}) + \mathbf{A}\mathbf{S}^{-1}\mathbf{X}\mathbf{A}^T d\mathbf{y} \\ &= \mathbf{A}\mathbf{S}^{-1}(\mu\mathbf{e} - \mathbf{X}\mathbf{s}) - \mathbf{A}\mathbf{S}^{-1}\mathbf{X}\mathbf{A}^T\left((\mathbf{A}\mathbf{S}^{-1}\mathbf{X}\mathbf{A}^T)^{-1}\mathbf{A}\mathbf{S}^{-1}(\mu\mathbf{e} - \mathbf{X}\mathbf{s})\right) = \mathbf{0}. \end{aligned} \tag{6.2.24}$$

The iterative process is terminated when the iterate is close enough to the optimal solution. A generally used termination criterion is $\eta(\mathbf{x}, \mathbf{s}) \leq \varepsilon$, where ε is a very small positive value and $\eta(\mathbf{x}, \mathbf{s}) = \| \mathbf{X}\mathbf{s} - \mu\mathbf{e} \|$. Here, $\eta(\mathbf{x}, \mathbf{s})$ measures the difference between the temporary solution and the optimal solution. Note that $\eta(\mathbf{x}, \mathbf{s}) = 0$ when $(\mathbf{x}, \mathbf{y}, \mathbf{s}) = (\mathbf{x}^+, \mathbf{y}^+, \mathbf{s}^+)$.

It is proven in [34] that if we choose an initial point satisfying $\eta(\mathbf{x}, \mathbf{s}) < \frac{2}{3}$, the above Newton procedure will generate a iterate sequence $\{\mathbf{x}^k, \mathbf{y}^k, \mathbf{s}^k\}$ converging to the optimal solution $(\mathbf{x}^a, \mathbf{y}^a, \mathbf{s}^a)$.

6.2.2 Primal-dual potential algorithm

The primal-dual Newton procedure discussed in the previous subsection needs an initial point $(\mathbf{x}, \mathbf{y}, \mathbf{s})$ which is primal-dual feasible and satisfies $\eta(\mathbf{x}, \mathbf{s}) < 2/3$. In this subsection, we discuss the primal-dual potential algorithm which can provide such a point.

The primal-dual potential algorithm starts from a primal-dual feasible point $(\mathbf{x}^0, \mathbf{y}^0, \mathbf{z}^0)$ with ${\mathbf{x}^0}^T\mathbf{s}^0 = N$. It takes iterative steps to converge to a required solution. At each iteration, the current primal-dual interior point $(\mathbf{x}, \mathbf{y}, \mathbf{s})$ is updated to $(\mathbf{x}^+, \mathbf{y}^+, \mathbf{s}^+)$, with

$\mathbf{x}^+ = \mathbf{x} + \theta d\mathbf{x}$, $\mathbf{y}^+ = \mathbf{y} + \theta d\mathbf{y}$, $\mathbf{s}^+ = \mathbf{s} + \theta d\mathbf{s}$, where $d\mathbf{x}$, $d\mathbf{y}$, $d\mathbf{s}$ are calculated through (6.2.22), and θ is given by

$$\theta = \frac{\alpha\sqrt{\min(\mathbf{X}\mathbf{s})}}{\| (\mathbf{X}\mathbf{S})^{-\frac{1}{2}}(\mathbf{e} - \mathbf{X}\mathbf{s}) \|}. \qquad (6.2.25)$$

Here $\alpha \in (0,1)$ is a constant, and $\min(\mathbf{v} \in \Re^n) = \min_j\{v_j | j = 1, \ldots, N\}$.

It is proven in [34] that if $\eta(\mathbf{x},\mathbf{s}) > \eta$ for a positive constant $\eta < 1$, we can choose α such that $\psi_N(\mathbf{x} + \theta d\mathbf{x}, \mathbf{s} + \theta d\mathbf{s}) - \psi_N(\mathbf{x},\mathbf{s}) \leq -\delta$, for a positive constant δ. Here $\psi_N(\mathbf{x},\mathbf{s}) = N\ln(\mathbf{x}^T\mathbf{s}) - \sum_{j=1}^{N}\ln(x_j s_j)$ is the so-called primal-dual potential function. Therefore, in $O(\psi_N(\mathbf{x}^0,\mathbf{s}^0) - N\ln N)$ iterations, the algorithm can generate a pair $(\mathbf{x},\mathbf{s})$ such that $\eta(\mathbf{x},\mathbf{s}) < \eta < 1$.

6.2.3 Starting point for primal-dual potential algorithm

The above primal-dual potential algorithm needs a primal-dual feasible point $(\mathbf{x},\mathbf{y},\mathbf{s})$ with $\mathbf{x}^T\mathbf{s} = N$. In other words, the starting point $(\mathbf{x},\mathbf{y},\mathbf{s})$ should satisfy

$$\mathbf{A}\mathbf{x} = \mathbf{b}, \quad \mathbf{x} > \mathbf{0}, \quad \mathbf{A}^T\mathbf{y} + \mathbf{s} = \mathbf{c}, \quad \mathbf{s} > \mathbf{0}, \quad \mathbf{x}^T\mathbf{s} = N. \qquad (6.2.26)$$

For the analytic center formulation, the above conditions can be rewritten as

$$\begin{aligned}
p_1 + \cdots + p_N &= 1, && (6.2.27)\\
p_i &> 0, \quad i = 1, \ldots, N, && (6.2.28)\\
s_i &> 0, \quad i = 1, \ldots, N, && (6.2.29)\\
y + s_i &= \frac{1}{r_i}, \quad i = 1, \ldots, N, && (6.2.30)\\
\sum_{i=1}^{N} p_i s_i &= N. && (6.2.31)
\end{aligned}$$

Let us assume without losing any generality that $r_1 \geq r_j$ for $j \neq 1$. We suggest the following procedure to generate the required starting point:

Step 1: choose Δp such that

$$0 < \Delta p < \min \left\{ \frac{1}{N-1}, \frac{1}{\sum_{i=2}^{N} \left(\frac{1}{r_i} - \frac{1}{r_1} \right)} \right\}. \tag{6.2.32}$$

Step 2: compute p_i's as

$$p_i = \begin{cases} 1 - (N-1)\Delta p, & i = 1, \\ \Delta p, & \text{otherwise.} \end{cases} \tag{6.2.33}$$

Step 3: compute y as

$$y = \frac{1}{r_1} + \Delta p \sum_{i=2}^{N} \left(\frac{1}{r_i} - \frac{1}{r_1} \right) - N. \tag{6.2.34}$$

Step 4: compute s_i's as

$$s_i = \frac{1}{r_i} - y. \tag{6.2.35}$$

It is easy to verify that the starting point generated by the above procedure satisfies conditions (6.2.27) through (6.2.31). First of all, it is obvious that the p_i's computed by (6.2.33) satisfy the probability constraints in (6.2.27) and (6.2.28). The condition (6.2.29) is also satisfied since

$$s_i = \frac{1}{r_i} - y \geq \frac{1}{r_1} - y = N - \Delta p \sum_{i=2}^{N} \left(\frac{1}{r_i} - \frac{1}{r_1} \right) > 0. \tag{6.2.36}$$

The condition (6.2.30) is identical with (6.2.35). Finally, (6.2.31) is satisfied because

$$\begin{aligned} \sum_{i=1}^{N} p_i s_i &= (1 - (N-1)\Delta p) \left(\frac{1}{r_1} - y \right) + \sum_{i=2}^{N} \Delta p \left(\frac{1}{r_i} - y \right) \\ &= \frac{1}{r_1} - y + \Delta p \sum_{i=2}^{N} \left(\frac{1}{r_i} - \frac{1}{r_1} \right) = N. \end{aligned} \tag{6.2.37}$$

6.3 Satisfaction of decision fusion principles

In this section, we analyze the analytic center fusion method using the decision fusion principles proposed in Chapter 3. We start by introducing two propositions which will be used in subsequent analysis.

Proposition 6.3.1 *Let $(p_1^a, \ldots, p_N^a)$ denote the optimal solution of the following optimization problem:*

$$\begin{aligned} \textit{minimize} \quad & \sum_{i=1}^{N} \frac{1}{r_i} p_i - \sum_{i=1}^{N} \ln p_i \\ \textit{subject to} \quad & \sum_{i=1}^{N} p_i = 1, \quad p_i \geq 0, \; i = 1, \ldots, N, \end{aligned} \tag{6.3.38}$$

and $(P_1^{a'}, \ldots, p_N^{a'})$ denote the optimal solution of the following optimization problem:

$$\begin{aligned} \textit{minimize} \quad & \sum_{i=1}^{N} \frac{1}{r'_i} p'_i - \sum_{i=1}^{N} \ln p'_i \\ \textit{subject to} \quad & \sum_{i=1}^{N} p'_i = 1, \quad p'_i \geq 0, \; i = 1, \ldots, N, \end{aligned} \tag{6.3.39}$$

where

$$r'_1 = r_1 + \Delta r_1, \quad r'_i = r_i, \quad i = 2, \ldots, N. \tag{6.3.40}$$

Then

$$\Delta r_1 > 0 \quad \Rightarrow \quad p_1^{a'} > p_1^a, \; p_i^{a'} < p_i^a, \quad i = 2, \ldots, N, \tag{6.3.41}$$

$$\Delta r_1 < 0 \quad \Rightarrow \quad p_1^{a'} < p_1^a, \; p_i^{a'} > p_i^a, \quad i = 2, \ldots, N. \tag{6.3.42}$$

In other words, increasing r_1 results in increasing P_1^a and decreasing all the other P_i^a's $(i = 2, \ldots, N)$ in the optimal solution.

Proof. In the following we prove (6.3.41). (6.3.42) can be proven in a similar way.

Assume $\Delta r_1 = r_1' - r_1 > 0$. We start by observing that the above problem (6.3.38) is a special case of problem (6.2.10). Therefore, we can modify (6.2.18) to obtain the dual problem of (6.3.38):

$$\begin{array}{ll} \text{maximize} & y + \sum_{j=1}^{N} \ln s_j \\ \text{subject to} & y + s_j = \frac{1}{r_i}, \quad s_j \geq 0, \ i = 1, \dots, N, \end{array} \tag{6.3.43}$$

and modify (6.2.16) to obtain the optimality conditions for (6.3.38) and its dual problem (6.3.43):

$$p_i s_i = 1, \quad i = 1, \dots, N, \tag{6.3.44}$$

$$\sum_{i=1}^{N} p_i = 1, \quad p_i \geq 0, \quad i = 1, \dots, N, \tag{6.3.45}$$

$$y + s_i = \frac{1}{r_i}, \quad s_i \geq 0, \quad i = 1, \dots, N. \tag{6.3.46}$$

Similarly, the optimality conditions for problem (6.3.39) are given by:

$$p_i' s_i' = 1, \quad i = 1, \dots, N; \tag{6.3.47}$$

$$\sum_{i=1}^{N} p_i' = 1, \quad p_i' \geq 0, \quad i = 1, \dots, N, \tag{6.3.48}$$

$$y' + s_i' = \frac{1}{r_i'}, \quad s_i' \geq 0, \quad i = 1, \dots, N, \tag{6.3.49}$$

where the s_i''s are the dual variables. Let $\Delta y = y' - y$. It is obvious from (6.3.46) and (6.3.49) that

$$s_i' = \frac{1}{r_i'} - y' = \frac{1}{r_i'} - y' + \left(\frac{1}{r_i} - y\right) - \left(\frac{1}{r_i} - y\right) = s_i + \left(\frac{1}{r_i'} - \frac{1}{r_i}\right) - \Delta y, \quad i = 1, \dots, N, \tag{6.3.50}$$

or

$$s_1' = s_1 + \left(\frac{1}{r_1'} - \frac{1}{r_1}\right) - \Delta y, \quad s_i' = s_i - \Delta y, \ \forall\, i \neq 1. \tag{6.3.51}$$

Note that $\frac{1}{r_1'} - \frac{1}{r_1} < 0$. There are three possible cases for Δy, i.e.,

- case A: $\Delta y \geq 0$,
- case B: $\Delta y < 0$ and $\left(\frac{1}{r_1'} - \frac{1}{r_1}\right) - \Delta y \geq 0$,
- case C: $\Delta y < 0$ and $\left(\frac{1}{r_1'} - \frac{1}{r_1}\right) - \Delta y < 0$.

We now show that case A and case B cannot hold, and case C implies the desired result (6.3.41).

Case A: It is obvious from Eq. (6.3.50) and condition (6.3.40) that

$$s_1' < s_1, \qquad s_i' \leq s_i, \quad i = 2, \ldots, N, \tag{6.3.52}$$

which by Eqs. (6.3.44) and (6.3.47) imply that $p_1' > p_1$, $p_i' \geq p_i$ for $i = 2, \ldots, N$. This means that Eqs. (6.3.45) and (6.3.48) cannot be both true. As a result, $\Delta y \geq 0$ cannot hold.

Case B: In this case, we have $s_1' \geq s_1$, $s_i' > s_i$ for $i = 2, \ldots, N$, and $p_1' \leq p_1$, $p_i' < p_i$ for $i = 2, \ldots, N$. Similar to case A, this case is impossible either.

Case C: Here we have

$$\begin{aligned} s_1' &= s_1 + \left(\frac{1}{r_1'} - \frac{1}{r_1}\right) - \Delta y < s_1, \\ s_i' &= s_i - \Delta y > s_i, \quad i = 2, \ldots, N, \end{aligned} \tag{6.3.53}$$

and $p_1' > p_1$, $p_i' < p_i$ for $i = 2, \ldots, N$, which is exactly the desired result (6.3.41). □

Proposition 6.3.2 *Let p_1^K denote the solution to the following optimization problem:*

$$\begin{aligned} &\textit{minimize} \quad C_K(p_1) = K\left(\frac{p_1}{r_1} + \frac{1 - p_1}{r_2}\right) - (\ln p_1 + (N-1)\ln(1 - p_1)) \\ &\textit{subject to} \quad 1 \geq p_1 \geq 0, \end{aligned} \tag{6.3.54}$$

where $r_1 > r_2 > 0$, $N \geq 2$, $K \geq 1$. *Then,*

$$1 - p_1^K \leq c_1 \cdot \frac{1}{K}, \tag{6.3.55}$$

where $c_1 = \dfrac{N-1}{\frac{1}{r_2} - \frac{1}{r_1}}$, *and*

$$K \geq \max(K_1, K_2) \quad \Rightarrow \quad 1 - p_1^K \geq c_2 \cdot \frac{1}{K}, \tag{6.3.56}$$

where $K_1 = \dfrac{N-2}{\frac{1}{r_2} - \frac{1}{r_1}}$, $K_2 = \dfrac{N^2}{2(2N-3)\left(\frac{1}{r_2} - \frac{1}{r_1}\right)}$, $c_2 = \dfrac{1}{2\left(\frac{1}{r_2} - \frac{1}{r_1}\right)}$.

Proof. We start by observing that $1 > p_1^K > 0$ since $C_K(p_1)$ is a convex function in p_1 and $\lim_{p_1 \to 0,1} C_K(p_1) = \infty$. Therefore, the optimal solution p_1^K satisfies

$$\frac{dC_K(p_1)}{dp_1} = 0, \tag{6.3.57}$$

where

$$\begin{aligned} \frac{dC_K(p_1)}{dp_1} &= K\left(\frac{1}{r_1} - \frac{1}{r_2}\right) - \left(\frac{1}{p_1} - (N-1)\frac{1}{1-p_1}\right) \\ &= \frac{1}{p_1(1-p_1)}\left(-\beta p_1(1-p_1) - (1-p_1) + (N-1)p_1\right) \\ &= \frac{1}{p_1(1-p_1)}\left(\beta p_1^2 - (\beta - N)p_1 - 1\right). \end{aligned} \tag{6.3.58}$$

Here, $\beta = K\left(\dfrac{1}{r_2} - \dfrac{1}{r_1}\right)$. Note that $\beta > 0$ by assumption $r_1 > r_2 > 0$. Now condition (6.3.57) can be rewritten as $\beta p_1^2 - (\beta - N)p_1 - 1 = 0$. The optimal solution is equal to the positive root of the equation, i.e.,

$$p_1^K = \frac{\beta - N + \sqrt{(\beta - N)^2 + 4\beta}}{2\beta}. \tag{6.3.59}$$

Now we can derive that

$$\begin{aligned} 1-p_1^K &= 1-\frac{\beta-N+\sqrt{(\beta-N)^2+4\beta}}{2\beta}=\frac{\beta+N-\sqrt{(\beta-N)^2+4\beta}}{2\beta} \\ &< \frac{\beta+N-(\beta-N+2)}{2\beta}=\frac{N-1}{\beta}=\frac{N-1}{\frac{1}{r_2}-\frac{1}{r_1}}\cdot\frac{1}{K}, \end{aligned} \tag{6.3.60}$$

which proves inequality (6.3.55). Here the inequality follows from the fact that $N \geq 2$.

To prove (6.3.56), let

$$y=\frac{N^2}{2\beta}-N+2. \tag{6.3.61}$$

When

$$K \geq K_1=\frac{N-2}{\frac{1}{r_2}-\frac{1}{r_1}}, \tag{6.3.62}$$

we have

$$\beta+y=\beta+\left(\frac{N^2}{2\beta}-N+2\right)>\beta-N+2\geq K_1\left(\frac{1}{r_2}-\frac{1}{r_1}\right)-N+2=0, \tag{6.3.63}$$

and

$$(\beta+y)^2=\beta^2+2\beta y+y^2\geq\beta^2+2\beta y=\beta^2+2\beta\left(\frac{N^2}{2\beta}-N+2\right)=(\beta-N)^2+4\beta. \tag{6.3.64}$$

Combining (6.3.63) and (6.3.64), we have

$$\sqrt{(\beta-N)^2+4\beta}\leq\beta+y, \tag{6.3.65}$$

when y is given in Eq. (6.3.61), and K satisfies condition (6.3.62).

With (6.3.65), we can derive that

$$\begin{aligned} 1-p_1^K &= \frac{\beta+N-\sqrt{(\beta-N)^2+4\beta}}{2\beta}\geq\frac{\beta+N-(\beta+y)}{2\beta}=\frac{N-y}{2\beta} \\ &= \frac{1}{2\beta}\left(N-\left(\frac{N^2}{2\beta}-N+2\right)\right)=\frac{1}{2\beta}\left(2N-2-\frac{N^2}{2\beta}\right). \end{aligned} \tag{6.3.66}$$

Under the condition that $K \geq K_2 = \dfrac{N^2}{2(2N-3)(\frac{1}{r_2}-\frac{1}{r_1})}$, the above inequality (6.3.66) becomes

$$\begin{aligned}
1-p_1^K &\geq \frac{1}{2\beta}\left(2N-2-\frac{N^2}{2\beta}\right) = \frac{1}{2\beta}\left(2N-2-\frac{N^2}{2}\cdot\frac{1}{K\left(\frac{1}{r_2}-\frac{1}{r_1}\right)}\right) \\
&\geq \frac{1}{2\beta}\left(2N-2-\frac{N^2}{2}\cdot\frac{1}{K_2\left(\frac{1}{r_2}-\frac{1}{r_1}\right)}\right) = \frac{1}{2\beta}\left(2N-2-(2N-3)\right) \\
&= \frac{1}{2\left(\frac{1}{r_2}-\frac{1}{r_1}\right)}\cdot\frac{1}{K},
\end{aligned} \tag{6.3.67}$$

which proves inequality (6.3.56). □

With the help of Propositions 6.3.1 and 6.3.2, we now prove the following theorem.

Theorem 6.3.1 *The analytic center fusion method satisfies fundamental principles 2,3, and fundamental principles 4 and 5 for DSF model. In general, it does not satisfy fundamental principle 1. Moreover, the fusion method satisfies DoC principle 1 for DSF model.*

Proof. Let $\Omega = \{a_1, \dots, a_N\}$. Suppose a sensor report D_k takes the following form:

$$D_k = \left\{\begin{array}{l} \text{Degree of Confidence} = w_k, \\ P_k(a_i) = r_k^i, \quad i = 1, \dots, N \end{array}\right\}, \quad k = 1, ..., K, \tag{6.3.68}$$

where $r_k^i \in [0,1]$ for $i = 1, \dots, N$, $\sum_{i=1}^{N} r_k^i = 1$, and $w_k \in [0,1]$. We examine the decision fusion principles one by one.

a) Fundamental principle 1: Suppose there is only one sensor report D_1 to fuse, and assume without any loss of generality that $r_1^1 \geq r_1^2 \geq \cdots \geq r_1^N$. The analytic center fusion method solves the following convex optimization problem:

$$\begin{aligned} &\text{minimize} \quad C_{f_1}(P) = w_1 \sum_{i=1}^{N} \frac{1}{r_1^i} p_i - \sum_{i=1}^{N} \ln p_i \\ &\text{subject to} \quad \sum_{i=1}^{N} p_i = 1, \;\; p_i \geq 0, \; i = 1, \ldots, N. \end{aligned} \tag{6.3.69}$$

Let $D_f = \{P_f(a_i) = p_i, \;\; i = 1, \ldots, N\}$ denote the fusion result, then the optimality conditions can be obtained from Eqs. (6.2.11) and (6.2.16):

$$p_i s_i = 1, \;\; i = 1, \ldots, N, \tag{6.3.70}$$

$$p_1 + \cdots + p_N = 1, \;\; p_i \geq 0, \;\; i = 2, \ldots, N, \tag{6.3.71}$$

$$y + s_i = c_i, \;\; s_i \geq 0, \;\; i = 1, \ldots, N, \tag{6.3.72}$$

where $c_i = w_1 \cdot \frac{1}{r_1^i}$. Combination of Eqs. (6.3.70) and (6.3.72) gives

$$p_i = \frac{1}{c_i - y}, \;\; i = 1, \ldots, N, \tag{6.3.73}$$

which implies that a bigger r_1^i corresponds to a bigger value of p_i, i.e., $p_1 \geq p_2 \geq \cdots \geq p_N$.

We now prove that increasing w_1 results in the increasing of relative likelihood ratio $\frac{p_i}{p_j}$ for $i < j$. We start by combining Eq. (6.3.71) with Eq. (6.3.73) to obtain

$$\frac{1}{c_1 - y} + \cdots + \frac{1}{c_N - y} = 1. \tag{6.3.74}$$

Suppose that the DoC of D_1 increases from w_1 to $w_1' = \beta w_1$ with $\beta > 1$, and let $D_f' = \{P_f'(a_i) = p_i', \;\; i = 1, \ldots, N\}$ denote the new fusion result. Then, the new optimal fusion result $p_1', \ldots, p_N'$ should satisfy $p_i' = \frac{1}{\beta c_i - y'}$ for $i = 1, \ldots, N$, and

$$\frac{1}{\beta c_i - y'} + \cdots + \frac{1}{\beta c_N - y'} = 1. \tag{6.3.75}$$

It is obvious that $y' > \beta y$, for otherwise the condition $y' \leq \beta y$ would imply

$$\frac{1}{\beta c_i - y'} + \cdots + \frac{1}{\beta c_N - y'} \leq \frac{1}{\beta c_i - \beta y} + \cdots + \frac{1}{\beta c_N - \beta y} = \frac{1}{\beta} < 1, \tag{6.3.76}$$

violating Eq. (6.3.75).

Let $i < j$, then $r_1^i > r_1^j$, $c_i < c_j$, $p_i > p_j$ and $p'_i > p'_j$. Moreover,

$$\begin{aligned} \left(\frac{p'_i}{p'_j}\right) / \left(\frac{p_i}{p_j}\right) &= \left(\frac{\frac{1}{\beta c_i - y'}}{\frac{1}{\beta c_j - y'}}\right) / \left(\frac{\frac{1}{c_i - y}}{\frac{1}{c_j - y}}\right) = \frac{(\beta c_j - y')(c_i - y')}{(\beta c_i - y')(c_j - y')} \\ &= \frac{\beta c_i c_j - y' c_i - \beta c_j y + y y'}{\beta c_i c_j - y' c_j - \beta c_i y + y y'} > 1, \end{aligned} \tag{6.3.77}$$

where the last step follows the inequality $(c_j - c_i)(y' - \beta y) > 0$.

We have now characterized the behavior of fused result p_i as w_1 increases within $[0, 1]$. Let us consider two extreme cases. First, $w_1 = 1$, i.e., $w_1 = DoC_{max}$. Here problem (6.3.69) becomes

$$\begin{aligned} &\text{minimize} \quad C_{f_1}(P) = \sum_{i=1}^{N} \frac{1}{r_1^i} p_i - \sum_{i=1}^{N} \ln p_i \\ &\text{subject to} \quad \sum_{i=1}^{N} p_i = 1, \quad p_i \geq 0, \; i = 1, \ldots, N, \end{aligned} \tag{6.3.78}$$

which yields optimal fusion result $p_i = r_1^i$ for $i = 1, \ldots, N$. This obviously satisfies the fundamental principle 1. Secondly, $w_1 = 0$ (DoC_{min}). In this case, problem (6.3.69) becomes

$$\begin{aligned} &\text{minimize} \quad C_{f_0}(P) = -\sum_{i=1}^{N} \ln p_i \\ &\text{subject to} \quad \sum_{i=1}^{N} p_i = 1, \quad p_i \geq 0, \; i = 1, \ldots, N, \end{aligned} \tag{6.3.79}$$

and the fusion result is $p_i = 1/N$ for $i = 1, \ldots, N$. Note that in all cases the DoC of the fusion result is always $w_f = 1 - (1 - w_1) = w_1$.

The above analysis can be summarized as follows.

- When $w_1 = 1$, the fusion result is identical with D_1.

- Assume that $r_1^1 \geq \cdots \geq r_1^N$. Then, when w_1 decreases, $P_f(a_i)/P_f(a_j)$ decreases for $i < j$. However, it is always true that $P_f(a_i) \geq P_f(a_j)$ for $i < j$.

- When $w_1 = 0$, the fusion result is an equal probability distribution.

In other words, the analytic center fusion method satisfies the fundamental principle 1 in the special case of $w_1 = 1$. However, in general the fusion method does not satisfy the principle.

b) Fundamental principle 2: This is straight forward. The analytic center fusion method only operates in the batch fusion mode (see discussion on page 134). Similar to the two fusion methods proposed in previous chapters, the analytic center fusion method also minimizes a cost function constructed using original sensor reports. The principle is satisfied since the cost function is invariant with permutation on sensor report indexing.

c) Fundamental principle 3: Suppose there are N symmetric sensor reports $D_1, \ldots, D_N$, with D_k given as

$$D_k = \left\{ \begin{array}{l} DoC = w_r, \\ P_k(a_i) = r_{i \oplus k}, \quad i = 1, \ldots, N \end{array} \right\}, \tag{6.3.80}$$

where $r_i \geq 0$ for $i = 1, \ldots, N$, and $i \oplus k = i + k - 1 \mod N$. Note that the DoC values of all the reports are the same. The overall cost function can be written as

$$C_{f_N}(P) = \sum_{k=1}^{N} w_r \sum_{i=1}^{N} \frac{1}{r_{i \oplus k}} p_i - \sum_{i=1}^{N} \ln p_i = N w_r \sum_{i=1}^{N} \frac{1}{r_i} - \sum_{i=1}^{N} \ln p_i, \tag{6.3.81}$$

which obviously leads to the following fusion result: $p_i = 1/N$ for $i = 1, \ldots, N$. In other words, the analytic center fusion method satisfies the fundamental principle 3.

d) Fundamental principle 4 for DSF model: As introduced in section 6.1, the analytic center fusion method does not permit disjunctions of propositions. Therefore we only consider the basic propositions a_i's. Denote the fusion result before and after the fusion of report D_1 as D_{f_1} and D_{f_2}. Then, $D_{f_1} = \{P_{f_1}(a_i), \ i = 1, \ldots, N\}$ is the solution to the following optimization problem:

$$\begin{aligned} \text{minimize} \quad & C_{f_1}(P) = \sum_{i=1}^{N} \frac{1}{r^i_{f_1}} p_i - \sum_{i=1}^{N} \ln p_i \\ \text{subject to} \quad & \sum_{i=1}^{N} p_i = 1, \ \ p_i \geq 0, \ i = 1, \ldots, N. \end{aligned} \tag{6.3.82}$$

Similarly, $D^{w_1}_{f_2} = \{P^{w_1}_{f_2}(a_i), \ \ i = 1, \ldots, N\}$ is the solution to the following optimization problem:

$$\begin{aligned} \text{minimize} \quad & C^{w_1}_{f_2}(P) = \sum_{i=1}^{N} \frac{1}{r^i_{f_2}} p_i - \sum_{i=1}^{N} \ln p_i \\ \text{subject to} \quad & \sum_{i=1}^{N} p_i = 1, \ \ p_i \geq 0, \ i = 1, \ldots, N, \end{aligned} \tag{6.3.83}$$

where $\dfrac{1}{r^i_{f_2}} = \dfrac{1}{r^i_{f_1}} + w_1 \dfrac{1}{r^i_1}$ for $i = 1, \ldots, N$. Without any loss of generality, we assume that in D_1, $r^1_1 > r^2_1 \geq \cdots \geq r^N_1$. Then, it is obvious that when $w_1 = 0$, problem (6.3.82) is identical to problem (6.3.83), which establishes the first part of property (3.6.22). To prove the second part of property (3.6.22), we use the following procedure. We start by increasing the value of r^N_1 to be equal to r^2_1. As a result, problem (6.3.83) is modified to be the following optimization problem:

$$\begin{aligned} \text{minimize} \quad & C'_f(P) = \left(\frac{1}{r^1_{f_1}} + w_1 \frac{1}{r^1_1}\right) p_1 + \sum_{i=2}^{N-1} \left(\frac{1}{r^i_{f_1}} + w_1 \frac{1}{r^i_1}\right) p_i + \left(\frac{1}{r^N_{f_1}} + w_1 \frac{1}{r^2_1}\right) p_N - \sum_{i=1}^{N} \ln p_i \\ \text{subject to} \quad & \sum_{i=1}^{N} p_i = 1, \ \ p_i \geq 0, \ i = 1, \ldots, N. \end{aligned} \tag{6.3.84}$$

Let $P'_f(a_1)$ denote the probability of a_1 in the solution of the above new problem. Then according to Proposition 6.3.1,

$$P^{w_1}_{f_2}(a_1) \geq P'_f(a_1). \tag{6.3.85}$$

We continue this procedure iteratively, increasing one by one the values of $r_1^3, \ldots, r_1^{N-1}$ to be equal to r_1^2. Eventually, problem (6.3.83) is modified to be the following optimization problem:

$$\begin{aligned} \text{minimize} \quad & C_f''(P) = \left(\frac{1}{r_{f_1}^1} + w_1 \frac{1}{r_1^1}\right) p_1 + \sum_{i=2}^{N} \left(\frac{1}{r_{f_1}^i} + w_1 \frac{1}{r_1^2}\right) p_i - \sum_{i=1}^{N} \ln p_i \\ \text{subject to} \quad & \sum_{i=1}^{N} p_i = 1, \quad p_i \geq 0, \; i = 1, \ldots, N. \end{aligned} \tag{6.3.86}$$

Let $P_f''(a_1)$ denote the probability of a_1 in the solution of the above optimization problem, then similar to (6.3.85), we have

$$P_{f_2}^{w_1}(a_1) \geq P_f''(a_1). \tag{6.3.87}$$

The cost function in (6.3.86) can be simplified as

$$\begin{aligned} C_f''(P) &= \sum_{i=1}^{N} \frac{1}{r_{f_1}^i} p_i - \sum_{i=1}^{N} \ln p_i + w_1 \left(\frac{1}{r_1^1} p_1 + \sum_{i=2}^{N} \frac{1}{r_1^2} p_i\right) \\ &= C_{f_1}(P) + w_1 \left(\frac{1}{r_1^1} - \frac{1}{r_1^2}\right) p_1 + w_1 \frac{1}{r_1^2} \\ &= \left(\frac{1}{r_{f_1}^1} + w_1 \left(\frac{1}{r_1^1} - \frac{1}{r_1^2}\right)\right) p_1 + \sum_{i=2}^{N} \frac{1}{r_{f_1}^i} p_i - \sum_{i=1}^{N} \ln p_i + w_1 \frac{1}{r_1^2}, \end{aligned} \tag{6.3.88}$$

where $\frac{1}{r_1^1} - \frac{1}{r_1^2} < 0$. according to Proposition 6.3.1,

$$P_f''(a_1) > P_{f_1}(a_1). \tag{6.3.89}$$

Summarizing (6.3.87) and (6.3.89), we have $P_{f_2}^{w_1}(a_1) > P_{f_1}(a_1)$, which is the desired result.

Moreover, if w_1 increases from w_{1_a} to w_{1_b}, i.e., $w_{1_b} > w_{1_a}$, then

$$\begin{aligned} C_{f_2}^{w_{1_b}}(P) &= C_{f_1}(P) + w_{1_b}\sum_{i=1}^{N}\frac{1}{r_1^i}p_i = C_{f_1}(P) + w_{1_a}\sum_{i=1}^{N}\frac{1}{r_1^i}p_i + (w_{1_b} - w_{1_a})\sum_{i=1}^{N}\frac{1}{r_1^i}p_i \\ &= C_{f_2}^{w_{1_a}}(P) + (w_{1_b} - w_{1_a})\sum_{i=1}^{N}\frac{1}{r_1^i}p_i. \end{aligned} \quad (6.3.90)$$

Let $D_1^{w_{1c}}$ denote a new sensor report which is identical to D_1 except that its DoC value is $w_{1c} = w_{1b}^2 - w_{1a}^2$. Then from the above equation, the fusion of $D_1^{w_{1b}}$ is equivalent to the fusion of $D_1^{w_{1a}}$ and $D_1^{w_{1c}}$. That is, increasing the DoC of D_1 from w_{1a} to w_{1b} is equivalent to the fusion of a new sensor report $D_1^{w_{1c}}$. Similar to the above derivation, we can prove that $P_{f_2}^{w_{1_b}}(a_1) > P_{f_2}^{w_{1_a}}(a_1)$. In other words, if w_1 increases, $P_{f_2}^{w_1}(a_1)$ also increases.

e) Fundamental principle 5 for DSF model: As mentioned before, we only consider basic propositions a_i's since the analytic center fusion method does not permit disjunctions of a_i's. Suppose there are K identical sensor reports from dissimilar and independent sensors, each denoted by $D_1^{w_1}$, where w_1 is the DoC associated with D_1 and suppose that $w_1 > DoC_{min}$. Let $D_{fK}^{w_1} = \{P_{fK}^{w_1}(a_i),\ \ i = 1, \ldots, N\}$ (with DoC value $w_{fK}^{w_1}$) denote the fusion result of these identical reports. We further assume without any loss of generality that in report D_1, $r_1^1 > r_1^2 \geq \cdots \geq r_1^N$. Then, $D_{fK}^{w_1}$ is the solution to the following optimization problem:

$$\begin{aligned} &\text{minimize} \quad Kw_1\left(\frac{1}{r_1^1}p_1 + \cdots + \frac{1}{r_1^N}p_N\right) - \sum_{i=1}^{N}\ln p_i \\ &\text{subject to} \quad \sum_{i=1}^{N}p_i = 1, \quad p_i \geq 0,\ i = 1, \ldots, N. \end{aligned} \quad (6.3.91)$$

If we decrease the value of r_1^{N-1} to r_1^N, problem (6.3.91) is transformed to the following optimization problem:

$$\begin{aligned} &\text{minimize} \quad Kw_1\left(\frac{1}{r_1^1}p_1 + \cdots + \frac{1}{r_1^N}p_{N-1} + \frac{1}{r_1^N}p_N\right) - \sum_{i=1}^{N}\ln p_i \\ &\text{subject to} \quad \sum_{i=1}^{N}p_i = 1, \quad p_i \geq 0,\ i = 1, \ldots, N. \end{aligned} \quad (6.3.92)$$

Let $P'_f(a_1)$ denote the probability of a_1 in the new fusion result (the solution of the above problem (6.3.92)). Then according to Proposition 6.3.1,

$$P_{f_K}^{w_1}(a_1) \leq P'_f(a_1). \tag{6.3.93}$$

We continue this procedure iteratively, reducing one by one the values of $r_1^2, r_1^3, \ldots, r_1^{N-1}$ to be equal to r_1^N. As a result, we have the following optimization problem:

$$\begin{array}{ll} \text{minimize} & Kw_1\left(\frac{1}{r_1^1}p_1 + \frac{1}{r_1^N}p_2 + \cdots + \frac{1}{r_1^N}p_N\right) - \sum_{i=1}^{N} \ln p_i \\ \text{subject to} & \sum_{i=1}^{N} p_i = 1, \;\; p_i \geq 0, \; i = 1, \ldots, N. \end{array} \tag{6.3.94}$$

Let $P''_f(a_1)$ denote the probability of a_1 in the solution of the above optimization problem, then similar to (6.3.93), we have

$$P_{f_K}^{w_1}(a_1) \leq P''_f(a_1). \tag{6.3.95}$$

In (6.3.94), variables $p_2, \ldots, p_N$ are treated equally, which implies that in the solution, $p_2 = \ldots = p_N$. Therefore, the problem (6.3.94) can be simplified as

$$\begin{array}{ll} \text{minimize} & Kw_1\left(\frac{1}{r_1^1}p_1 + (N-1)\frac{1}{r_1^N}p_N\right) - (\ln p_1 + (N-1)\ln p_N) \\ \text{subject to} & p_1 + (N-1)p_N = 1, \;\; p_1 \geq 0, \; p_N \geq 0, \end{array} \tag{6.3.96}$$

or equivalently,

$$\begin{array}{ll} \text{minimize} & Kw_1\left(\frac{1}{r_1^1}p_1 + \frac{1-p_1}{r_1^N}\right) - (\ln p_1 + (N-1)\ln(1-p_1)) \\ \text{subject to} & 1 \geq p_1 \geq 0. \end{array} \tag{6.3.97}$$

According to Proposition 6.3.2,

$$K \geq \max(K_1, K_2) \;\;\Rightarrow\;\; 1 - p''_f(a_1) \geq c_2 \cdot \frac{1}{K}, \tag{6.3.98}$$

where

$$K_1 = \frac{N-2}{w_1\left(\frac{1}{r_1^N} - \frac{1}{r_1^1}\right)}, \quad K_2 = \frac{N^2}{2w_1(2N-3)\left(\frac{1}{r_1^N} - \frac{1}{r_1^1}\right)}, \quad c_2 = \frac{1}{2w_1\left(\frac{1}{r_1^N} - \frac{1}{r_1^1}\right)}. \tag{6.3.99}$$

Similarly, if in problem (6.3.91), we increase the values of $r_1^3, \ldots, r_1^N$ to r_1^2, then we have the following problem:

$$\begin{array}{ll} \text{minimize} & Kw_1\left(\frac{1}{r_1^1}p_1 + \frac{1-p_1}{r_1^2}\right) - (\ln p_1 + (N-1)\ln(1-p_1)) \\ \text{subject to} & 1 \geq p_1 \geq 0. \end{array} \tag{6.3.100}$$

Let $P_f'''(a_1)$ denote the probability of a_1 in the solution of the above problem, then

$$P_{f_K}^{w_1}(a_1) \geq P_f'''(a_1). \tag{6.3.101}$$

Moreover, from Proposition 6.3.2, there holds

$$1 - p_f'''(a_1) \leq c_1 \cdot \frac{1}{K}, \tag{6.3.102}$$

where

$$c_1 = \frac{N-1}{w_1\left(\frac{1}{r_1^2} - \frac{1}{r_1^1}\right)}. \tag{6.3.103}$$

Summarizing (6.3.95), (6.3.98), (6.3.101) and (6.3.102), we have

$$K \geq \max(K_1, K_2) \quad \Rightarrow \quad c_1 \cdot \frac{1}{K} \geq 1 - P_{f_K}^{w_1}(a_1) \geq c_2 \cdot \frac{1}{K}, \tag{6.3.104}$$

where K_1, K_2, c_1 and c_2 are given in (6.3.99) and (6.3.103). In other words, $\lim_{K\to\infty} p_{f_1}^K(a_1) = 1$, with $1 - P_{f_1}^K(a_1)$ decreases at the rate of $O(1/K)$. This establishes property (3.6.29). Moreover,

$$\lim_{K\to\infty} w_{f_K}^{w_1} = \lim_{K\to\infty} 1 - \prod_{k=1}^{K}(1 - w_k) = \lim_{K\to\infty} 1 - (1 - w_k)^K = 1, \tag{6.3.105}$$

which proves property (3.6.30).

To prove property (3.6.32), consider problem (6.3.91). Following the same argument as that in the proof of the fundamental principle 1, we have

$$w_{1b} > w_{1a} \quad \Rightarrow \quad \frac{P_{f_K}^{w_{1b}}(a_1)}{P_{f_K}^{w_{1b}}(a_j)} > \frac{P_{f_K}^{w_{1a}}(a_1)}{P_{f_K}^{w_{1a}}(a_j)}, \quad j = 2, \ldots, N. \tag{6.3.106}$$

This establishes property (3.6.32), i.e., $w_{1b} > w_{1a} \Rightarrow P_{f_K}^{w_{1b}}(a_1) > P_{f_K}^{w_{1a}}(a_1)$.

f) DoC principle 1 for DSF model: This is straight forward. Let D_f denote the fusion result of sensor reports $D_1, \ldots, D_K$, w_k denote the DoC value of report D_k, and w_f denote the DoC value of D_f. Then, from Eq. (6.1.7), w_f is computed as $w_f = 1 - \prod_{k=1}^{K}(1 - w_k)$. It is obvious from the above equation that if any w_k increases, w_f also increases. □

Discussion on the restricted fundamental principle 4 for DSF model

Theorem 6.3.1 shows that the analytic center fusion method satisfies the decision fusion principles proposed in Chapter 3. However, the following example illustrates that the method in general does not satisfy the restricted fundamental principle 4 for DSF model. This should not be viewed as a weakness of the new fusion method since the principle is not absolutely necessary in practice (see discussion on page 57).

Example 6.3.1 *Let* $\Omega = \{a_1, a_2, a_3\}$. *Suppose there are two dissimilar and independent sensors each sending a report to the fusion center. The sensor reports are given as*

$$D_1 = \left\{ \begin{array}{l} w_1 = 1, \\ P_1(a_1) = 0.8,\ P_1(a_2) = 0.11,\ P_1(a_3) = 0.09 \end{array} \right\},$$
$$D_2 = \left\{ \begin{array}{l} w_2 = 1, \\ P_2(a_1) = 0.4,\ P_2(a_2) = 0.5,\ P_2(a_3) = 0.1 \end{array} \right\}. \tag{6.3.107}$$

Note that D_2 *serves as the fusion result* D_{f_1} *before the fusion of* D_1. *By the analytic center fusion method, the fusion result after the fusion of* D_1 *is* $D_{f_2} = \{P_{f_2}(a_1) = 0.829,\ P_{f_2}(a_2) = 0.117,\ P_{f_2}(a_3) = 0.054\}$. *Here, the restricted fundamental principle 4 for SSF model is not satisfied, since* $P_1(a_2) = 0.11 > P_1(a_3) = 0.09$, *but* $\frac{P_{f_2}(a_2)}{P_{f_2}(a_3)} = \frac{0.117}{0.054} < \frac{P_{f_1}(a_2)}{P_{f_1}(a_3)} = \frac{0.5}{0.1}$, *violating property (3.6.25).*

6.4 Numerical examples

In this section, two numerical examples are given. The first one is a simple two-dimensional example which illustrates the behavior of the analytic center fusion method when there is no sensor report, a single sensor report or multiple sensor reports. The second example compares the analytic center fusion method with Dempster-Shafer evidential inference method.

Example 6.4.1

Suppose there are three independent sensors, each explores some unique characteristics of a target. Assume each sensor files one report to the fusion center which are given as follows:

$$D_1 = \left\{ \begin{array}{l} w_1 = 0.8, \\ P_1(a_1) = 0.8,\ P_1(a_2) = 0.2 \end{array} \right\},\quad D_2 = \left\{ \begin{array}{l} w_2 = 0.9, \\ P_2(a_1) = 0.7,\ P_2(a_2) = 0.3 \end{array} \right\},$$
$$D_3 = \left\{ \begin{array}{l} w_3 = 0.8, \\ P_3(a_1) = 0.4,\ P_3(a_2) = 0.6 \end{array} \right\}. \tag{6.4.108}$$

We consider the following four cases:

- **case A:** there is no sensor report available at the fusion center,
- **case B:** only D_1 is fused,
- **case C:** D_1 and D_2 are fused,
- **case D:** all three sensor reports, D_1, D_2 and D_3, are fused.

We examine the four cases one by one.

Case A: The fusion result for this case is

$$D_{f_1} = \left\{ \begin{array}{l} w_{f_1} = 0, \\ P_{f_1}(a_1) = 0.5, \; P_{f_1}(a_2) = 0.5 \end{array} \right\}. \tag{6.4.109}$$

In other words, D_{f_1} assigns equal probabilities to each propositions because of lack of information on the target identity.

Case B: The fusion result is

$$D_{f_2} = \left\{ \begin{array}{l} w_{f_2} = 0.8, \\ P_{f_2}(a_1) = 0.768, \; P_{f_2}(a_2) = 0.232 \end{array} \right\}, \tag{6.4.110}$$

which is not consistent with the single sensor report D_1. However, it preserves the information in D_1 that a_1 is much more likely than a_2.

Case C: The fusion result is

$$D_{f_3} = \left\{ \begin{array}{l} w_{f_3} = 0.98, \\ P_{f_3}(a_1) = 0.831, \; P_{f_3}(a_2) = 0.169 \end{array} \right\}. \tag{6.4.111}$$

Here, both D_1 and D_2 declare that a_1 is more likely to happen than a_2. As a result, the fusion result assigns a higher probability to a_1 than to a_2. In addition, $P_{f_3}(a_1) > \max\{P_1(a_1), P_2(a_1)\}$ because of the reinforcement between D_1 and D_2.

Case D: The fusion result for this case is

$$D_{f_4} = \left\{ \begin{array}{l} w_{f_4} = 0.996, \\ P_{f_4}(a_1) = 0.811, \ P_{f_4}(a_2) = 0.189 \end{array} \right\}. \tag{6.4.112}$$

Compared with Case C, the new sensor report D_3 provides evidence that a_2 is more likely to happen than a_1. However, in D_3 the relative likelihood $\frac{P_3(a_1)}{P_3(a_2)} = \frac{0.6}{0.4}$ is not big, showing that the evidence provided in D_3 is not strong. As a result, in D_{f_4}, the probability of a_1 decreases slightly (from $P_{f_3}(a_1) = 0.85$) to $P_{f_4}(a_1) = 0.83$.

Note that in all these cases the analytic center formulation involves only one variable, and can be solved analytically.

Example 6.4.2

Suppose there are two independent sensors exploring different characteristics of a target. Each sensor sends a Dempster-Shafer type sensor report to the fusion center. The sensor reports are given as follows:

$$\begin{aligned} D_1 &= \{P_1(a_1 \vee a_2) = 0.2, \ P_1(a_3) = 0.4, \ P_1(\Omega) = 0.4\}, \\ D_2 &= \{P_2(a_1) = 0.1, \ P_2(a_2) = 0.2, \ P_2(a_3 \vee a_4) = 0.5, \ P_2(\Omega) = 0.2\}. \end{aligned} \tag{6.4.113}$$

The two sensor reports can be fused by Dempster-Shafer evidential inference, which gives the following fusion result:

$$D_{f_1} = \left\{ \begin{array}{l} P_{f_1}(a_1) = 0.077, \ P_{f_1}(a_2) = 0.154, \ P_{f_1}(a_1 \vee a_2) = 0.051, \\ P_{f_1}(a_3) = 0.359, \ P_{f_1}(a_3 \vee a_4) = 0.256, \ P_{f_1}(\Omega) = 0.103 \end{array} \right\}. \tag{6.4.114}$$

We can also use the procedure introduced in section 6.1 (see page 133) to transform the reports into the form of (6.1.1) and use the analytic center fusion method to fuse them. Let D_1' and D_2' denote the transformed reports of D_1 and D_2. Then,

$$D_1' = \left\{ \begin{array}{l} w_1' = 0.6, \\ P_1'(a_1) = 0.2, \ P_1'(a_2) = 0.2, \ P_1'(a_3) = 0.5, \ P_1'(a_4) = 0.1 \end{array} \right\}, \tag{6.4.115}$$

$$D_2' = \left\{ \begin{array}{l} w_2' = 0.8, \\ \\ P_2'(a_1) = 0.15,\ P_2'(a_2) = 0.25,\ P_2'(a_3) = 0.3,\ P_2'(a_4) = 0.3 \end{array} \right\}, \tag{6.4.116}$$

and the fusion result is

$$D_{f_2} = \left\{ \begin{array}{l} w_{f_2} = 0.92, \\ \\ P_{f_2}(a_1) = 0.153,\ P_{f_2}(a_2) = 0.226,\ P_{f_1}(a_3) = 0.476,\ P_{f_2}(a_3) = 0.145 \end{array} \right\}. \tag{6.4.117}$$

The two fusion results match with each other very well, showing that the two fusion methods are compatible with each other for this example.

Chapter 7

Fusion Using DRDC Valcatier Data Set

In the previous chapters, we proposed two identity fusion models (the DSF model and the SSF model) and some decision fusion principles. We also proposed two fusion methods for the SSF model and one fusion method for the DSF model. In this chapter, we perform fusion on a simulated data set provided by Defence Research and Development Canada (DRDC) Valcartier. Since the data is from DSF scenarios, we fuse them using Dempster-Shafer evidential inference method and the analytic center fusion method. As can be seen later in the chapter, the fusion results by these two methods match with each other very well.

7.1 Introduction to DRDC Valcatier data set

The simulated data used in this chapter is provided by Defence Research and Development Canada (DRDC) Valcartier of the Department of National Defence of Canada. There are a total of 15 groups of data, each from a target tracking and identification scenario. The sensors in each scenario are independent, and explore some different physical characteristics of a target. In other words, all the scenarios belong to the DSF model. A target belongs to a set Ω which has 142 propositions, some of which are listed in Table 7.6. As can be

PROPOSITION NO.	IDENTITY	TYPE	SUBTYPE	ACRO
017	VIRGINIA	SURMILI	CRUISER	USAM
040	NIMITZ	SURMILI	CARRIER	USAM
045	KARA-AZOV	SURMILI	CRUISER	RUSS
075	TU22MA	AIRMILI	BOMBERS	RUSS
093	F16-FALCON	AIRMILI	FIGHTIN	ISRA
095	BOEING-747-400	AIRCOMM	JETPROP	VAR
102	CONCORDE	AIRCOMM	JETPROP	FRAN

Table 7.6: Examples of propositions in Ω.

seen from the table, a target proposition is typically a specific target identity belonging to a particular party. For example, proposition no. 093 is a F16-Falcon fighter aircraft of Isreal. In some cases a target proposition is so unique that it corresponds to only one target, e.g., proposition no. 040 is actually the U.S. aircraft carrier Nimitz. In the table, the target types include surface-military (SURMILI), subsurface (SURSURF), air-military (AIRMILI) and air-commute (AIRCOMM). The target is further specified by a target subtype. The last column in the table corresponds to the party (usually a country) each proposition belongs to. Here, we see parties like U.S.A. miliary (USAM), Rusia (RUSS), Isreal (ISRA), France (FRAN) and various (VARI). There are also some other parameters associated with each proposition, e.g, list of electro-magnetic emitters, length, hight, velocity range and so on. These parameters are not important for our fusion purpose and are not listed in the table.

Each group of data consists of a number of Dempster-Shafer type sensor reports which provide identity estimates on a target. As introduced in section 3.3, a Dempster-Shafer type sensor report consists of some probability masses assigned to subsets in Ω. Note that each sensor only has one report in the data. In order to have a better understanding of the original data, we provide the ninth data group in Table 7.7. Let a_i represent the i'th proposition in Ω. Then, the two subsets in Table 7.7 are

$$\begin{aligned} \omega_1 &= a_{76} \vee a_{93} \vee a_{109} \vee a_{114} \vee a_{121}, \\ \omega_2 &= a_{72} \vee a_{75} \vee \cdots \vee a_{127} \vee a_{131} \vee a_{135} \vee a_{137} \vee \cdots \vee a_{142}. \end{aligned} \tag{7.1.1}$$

Here, ω_1 represents a military aircraft belonging to a country other than U.S.A and Russia, and ω_2 represents an airborne target. Note that $\omega_1 \subset \omega_2$. From the table we see that the data group has seven Dempster-Shafer type sensor reports, e.g., the second report is given by $D_2 = \{P_2(\Omega) = 0.08,\ P_2(\omega_1) = 0.6,\ P_2(\omega_2) = 0.32\}$. It is clear from the table that all sensors believe the target is a military aircraft belonging to some country other than U.S.A. and Russia (subset ω_1). Therefore, the fusion result should assign a very high probability to ω_1. We will see from Table 7.9 that this is true for both Dempster-Shafer evidential inference method and the analytic center fusion method.

D_k	$P_k(\omega_1)$	$P_k(\omega_2)$	$P_k(\Omega)$
D_1	0.600		0.400
D_2	0.600	0.320	0.080
D_3	0.840	0.128	0.032
D_4	0.837	0.153	0.010
D_5	0.929	0.061	0.010
D_6	0.922	0.068	0.010
D_7	0.914	0.076	0.010

Table 7.7: The ninth data group.

As introduced previously, all the 15 groups of data are for the DSF model. Therefore we use Dempster-Shafer evidential inference method and the proposed analytic center fusion method to fuse them. Note that Bayesian inference method, which is also for the DSF model, can not be directly used to fuse these Dempster-Shafer type reports (see section 3.2). Also note that before applying the analytic center fusion method, the data need to be transformed using the procedure proposed in section 6.1 (see page 133). For example, for the ninth data group shown in Table 7.7, we first construct the following new proposition set $\Omega' = \{a_1', a_2', a_3'\}$ where

$$\begin{cases} a_1' = \text{a military aircraft belonging to countries other than U.S.A and Russia,} \\ a_2' = \text{an airborne target other than } a_2, \\ a_3' = \text{a non-airborne target.} \end{cases} \tag{7.1.2}$$

Then, the sensor reports can be transformed into the form shown in Table 7.8. The reason for us to construct a new Ω' is that the three propositions in Ω' are the ones actually used in the sensor reports. Therefore further dividing them into smaller propositions has no meaning, it can even lead to incorrect fusion results since it unreasonably changes the relative likelihood ratios in the sensor reports.

report	$P_k(a'_1)$	$P_k(a'_2)$	$P_k(a'_3)$	w_k
D'_1	0.7333	0.1333	0.0133	0.600
D'_2	0.7867	0.1867	0.0267	0.920
D'_3	0.9147	0.0747	0.0107	0.968
D'_4	0.9168	0.0799	0.0033	0.990
D'_5	0.9629	0.0338	0.0033	0.990
D'_6	0.9592	0.0375	0.0033	0.990
D'_7	0.9555	0.0412	0.0033	0.990

Table 7.8: The transformed sensor reports in the ninth data group.

7.2 Fusion results and discussion

In this section, we introduce the fusion results of the 15 data groups by Dempster-Shafer evidential inference method and the analytic center fusion method. We compare the fusion results by these methods in two ways: (1) the fusion results of all sensor reports in each data group, (2) the convergence performances assuming the sensor reports arrive in a dynamic fashion. First, we present the fusion result of all sensor reports in each data group in Table 7.9. Note that for each data group only the most probable target identity and its likelihood is shown in the table. Also shown in the table are the true target identities in the data groups. Here, subsets ω_1 and ω_2 are given by Eq. (7.1.1). Note that the fusion results in the table are rounded to four decimal places. Therefore, a zero value for $P_f(\Omega)$ actually means that $P_f(\Omega)$ is a very small positive value, and probability 1 is actually a value smaller but very close to 1. Similarly, w_f in the analytic center fusion method, if having a value 1, means that the DoC of the fusion result is very close to 1.

From Table 7.9, we see that the fusion results by the two fusion methods match exactly for all but two data groups (data group 5 and 13). For data group 5, the sensor reports are given in Table 7.10. Here, six sensor reports prefer subset ω_2 ($P_k(\omega_2) > 0.5$) while another seven sensor reports support proposition a_{74} ($P_k(a_{74}) > 0.5$). The fusion result by Dempster-Shafer evidential inference method is $D_{f_1} = \{P_{f_1}(\omega_2) = 0.7255,\ P_{f_1}(a_{74}) = 0.2745\}$, while

data group	no. of reports	no. of subsets	Dempster-Shafer			Analytic			true identity
			$P_f(\Omega)$	prob.	subset	w_f	prob.	subset	
1	40	32	0	1	a_{69}	1	0.9992	a_{69}	a_{19}
2	3	1	0.0001	0.9999	ω_2	0.9999	0.9960	ω_2	ω_2
3	61	39	0	1	a_{45}	1	0.9977	a_{45}	a_{45}
4	56	29	0	1	a_{19}	1	0.9996	a_{19}	a_{64}
5	13	4	0	0.7255	ω_2	1	0.9977	ω_2	a_{74}
6	3	1	0.0001	1	ω_2	0.9999	0.9960	ω_2	ω_2
7	18	3	0	1	a_{139}	1	0.9989	a_{139}	a_{139}
8	11	3	0	1	ω_1	1	0.9904	ω_1	a_{96}
9	7	2	0	1	ω_1	1	0.9900	ω_1	ω_1
10	7	2	0	0.988	ω_1	1	0.9871	ω_1	ω_1
11	7	2	0	1	ω_1	1	0.9934	ω_1	ω_1
12	7	2	0	1	ω_1	1	0.9870	ω_1	ω_1
13	4	3	0	0.6043	$a_{95} \vee a_{96}$	1	0.6811	ω_2	a_{96}
14	11	3	0	1	ω_1	1	0.9934	ω_1	a_{96}
15	7	2	0	1	ω_1	1	0.9916	ω_1	ω_1

Table 7.9: Fusion results of all data groups.

the fusion result by the analytic center fusion method is $D_{f_2} = \{P_{f_2}(\omega_2) = 0.9977\}$. Here in D_{f_1} and D_{f_2} we omitted all other subsets with negligible probability. Obviously both fusion methods believe that ω_2 is the true target identity. However, due to the contradictions among sensor reports, the final fusion results by the two methods assign different probabilities to ω_2. Therefore, we believe that for this data group, the two fusion methods essentially agree with each other, and the difference between the final assigned probabilities to ω_2 in D_{f_1} and D_{f_2} is acceptable. The situation for data group 13 is similar: two of the four sensor reports support subset ω_1 while the other two support subset $a_{95} \vee a_{96}$. As a result, the two fusion methods support the two subsets respectively. Note that the corresponding probabilities are not very big: $P_{f_1}(a_{95} \vee a_{96}) = 0.6043$ by Dempster-Shafer evidential inference method, and $P_{f_2}(\omega_1) = 0.6811$ by the analytic center fusion method. This shows that both methods think the evidence contained in the data group is not strong.

From Table 7.9, we also notice that the most probable target identity does not always match with the true target identity for each data group. This is because in the data sets enemy counter-measures are also simulated. As a result, we may have incorrect sensor reports and in turn wrong fusion results.

D_k	$P_k(\omega_2)$	$P_k(a_{74})$	$P_k(a_{64} \vee a_{65} \vee a_{68})$	$P_k(a_{32} \vee a_{43} \vee a_{44} \vee a_{64} \vee a_{65} \vee a_{68})$	$P_k(\Omega)$
1	0.8				0.2
2	0.96				0.04
3	0.99				0.01
4	0.99				0.01
5	0.9516	0.0384			0.01
6	0.7958	0.1942			0.01
7	0.4360	0.5540			0.01
8	0.1330	0.8570			0.01
9	0.1278	0.8237	0.0384		0.01
10	0.1069	0.6889	0.1608	0.0335	0.01
11	0.0280	0.9113	0.0420	0.0087	0.01
12	0.0059	0.9733	0.0089	0.0019	0.01
13	0.0012	0.9866	0.0018	0.0004	0.01

Table 7.10: Sensor reports in the fifth data group.

We also compare the convergence performance of the two fusion methods in Figures 7.1 to 7.3. Here, we look at the probability of the most likely target identity in the fusion result, and show how it changes when more and more sensor reports are fused. Recall that in section 3.3, we have proven that Dempster-Shafer evidential inference fusion method converges exponentially when fusing a increasing number of sensor reports (see page 76). In contrast, the analytic center fusion method converges at the rate of $O(1/K)$ when fusing K sensor reports (see page 155). From Figure 7.1, we see that the convergence curves for data group 4 match with the above theoretical results. However, for the other two data groups (Figures 7.2 and 7.3), since the sensor reports are far from identical with each other, the convergence curves by the two methods do not show too much difference.

Our fusion is performed on a 800MHz Pentium PC under the MATLAB environment. We self-coded the two fusion methods as well as the data input and output interfaces. In Figure 7.4, we show the CPU time consumed by the two methods fusing data group 4. Here, two CPU times are shown: the overall CPU time to fuse K sensor reports, and the time to fuse a new sensor report along with $K-1$ other sensor reports. Let us denote the former by $T_{all}(K)$ and the latter by $T_1(K)$. It can be seen from the figure that Dempster-Shafer evidential inference method has exponentially increasing $T_{all}(K)$ and $T_1(K)$ when K increases. In contrast, the analytic center fusion method has an almost constant $T_{all}(K)$ when

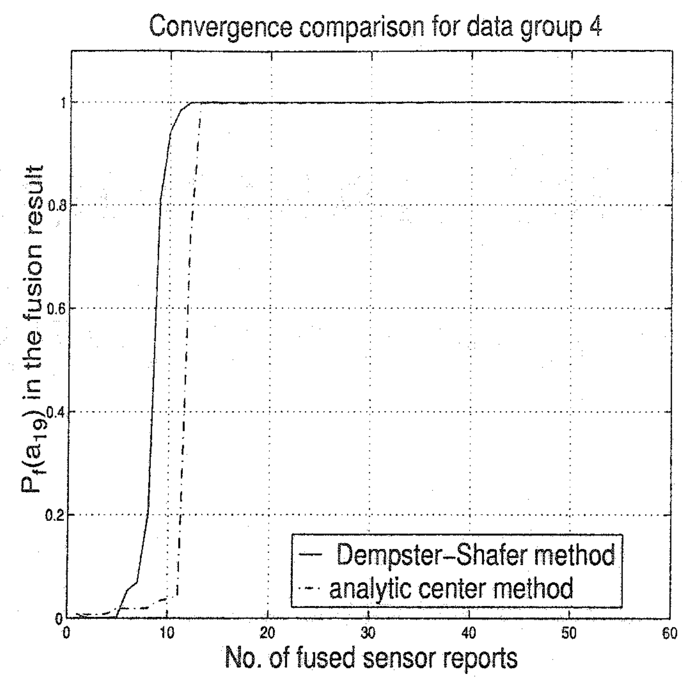

Figure 7.1: Convergence comparison: data group 4.

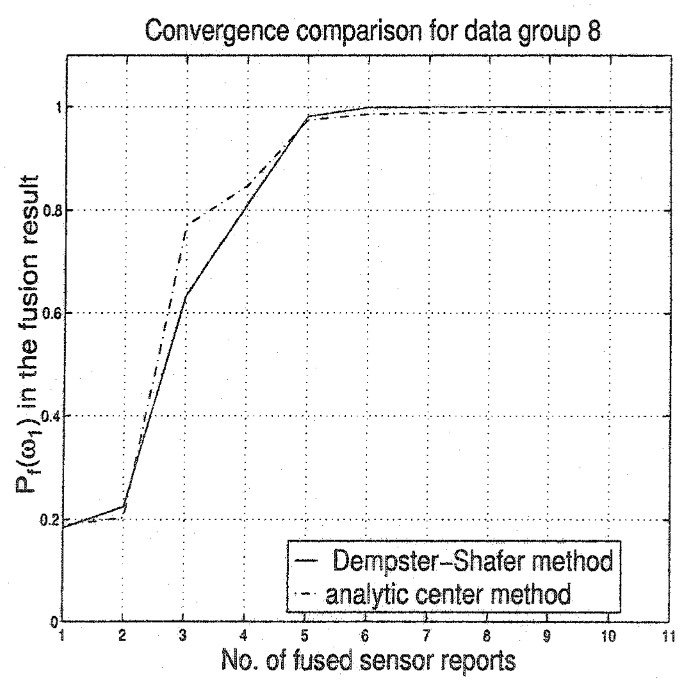

Figure 7.2: Convergence comparison: data group 8.

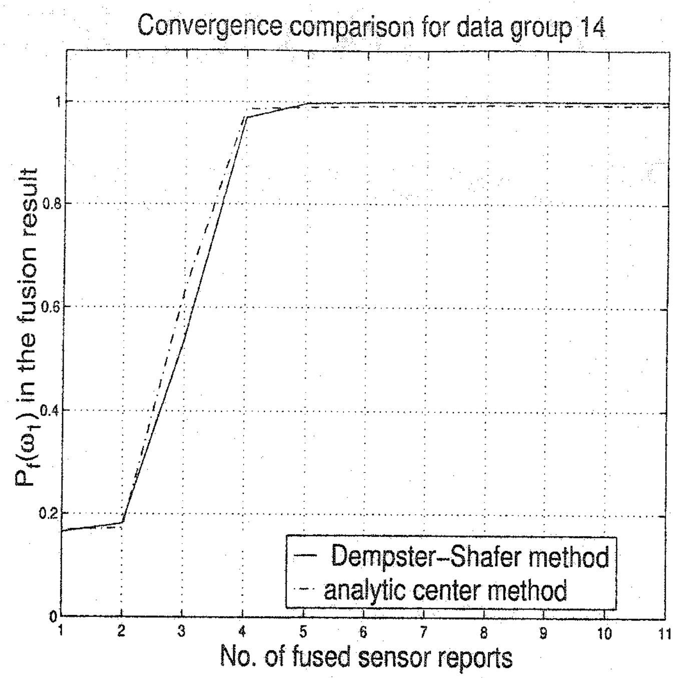

Figure 7.3: Convergence comparison: data group 14.

K is bigger than a small number. This has a simple explanation. Basically, the analytic center fusion method solves the fusion problem in three steps: transforms the Dempster-Shafer type sensor reports, constructs the objective function in (6.1.6), and solves the analytic center formulation (6.1.6). Here, the analytic center formulation (6.1.6) has a fixed worst case complexity which is invariant with the number of fused sensor reports (K). In addition, when K increases, fewer new subsets are involved in the new sensor reports, and the increased computation to transform the new reports and construct their cost function terms only consumes a very small part of $T_{all}(K)$. As a result, $T_{all}(K)$ almost stops increasing when K is big enough. Also note that the $T_1(K)$ for the analytic center fusion method is very small (less than 0.01 second). This is because that we can iteratively update the cost function in the analytic center formulation and the DoC for the fusion result when new sensor reports arrive (see page 135). Moreover, we can initiate the interior-point method using the fusion result before the fusion of the new sensor report, reducing the time to reach the optimal fusion result dramatically.

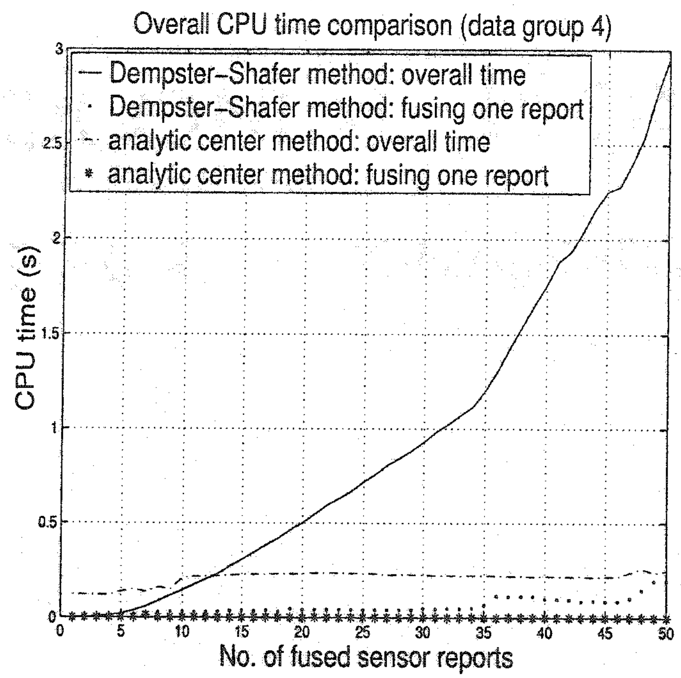

Figure 7.4: CPU time: data group 4.

Chapter 8

Conclusion and Future Work

8.1 Concluding remarks

This thesis studies two important problems in target tracking and identification: (1) robust track state filtering, and (2) decision-level identity fusion.

In Chapter 2 of the thesis, we have proposed a robust filtering method which minimizes an upper bound on the worst case variance of the estimation error for all admissible systems. The method is based on robust semidefinite programming technique and is robust to norm bounded parameter uncertainties in the system model and noise statistics. Our method is recursive, with each subproblem having a fixed size and can be solved efficiently in polynomial time. As shown in the simulation results, the new method compares favorably with some of the existing robust filtering approaches. When applied to the MTT problem to update track states using new observations, the new method has led to a significant improvement in tracking performance.

The major part of the thesis (Chapters 3 to 7) focuses on the decision fusion problem. In Chapter 3, we have proposed two decision fusion models: the Similar Sensor Fusion (SSF) model and the Dissimilar Sensor Fusion (DSF) model. In the SSF model, sensors explore a set of common target characteristics. As a result, the sensor reports can confirm with each other, and the fusion can eliminate inconsistencies among sensor reports to achieve a more accurate target declaration. In comparison, sensors in the DSF model explore some different target characteristics. Therefore when their reports are fused, decreased uncertainty in target identity can be achieved. We also proposed five fundamental principles and three additional DoC principles for decision fusion. These principles reveal the commonalities and differences of the two fusion models. Fusion methods can be classified and compared with each other using these principles. This axiomatic approach has provided a rigorous mathematical framework for the study, analysis and comparison of existing fusion methods. Moreover, we hope it will facilitate future development of new decision fusion methods.

Using the proposed decision fusion principles, we have analyzed the two classical decision fusion methods: Bayesian inference method and Dempster-Shafer evidential inference method. The analysis shows that the two methods are all for the DSF model, and satisfy all the proposed principles. However, as shown at the end of Chapter 3, Dempster-Shafer evidential inference method has some problem introducing DoC values into the fusion process.

In Chapters 4 and 5, we have proposed two new decision fusion methods for the SSF model: the convex quadratic fusion method and the K-L fusion method. The two methods are all based on the minimization of inconsistencies between the fusion result and sensor reports. The difference between the two methods lies in the functions they use to measure the inconsistencies: the former method uses the quadratic function while the latter method uses Kullback-Leibler's cross entropy. We have proven that the two methods satisfy all the decision fusion principles we proposed in Chapter 3. This suggests that the methods should have good fusion performances.

In Chapter 6, we have proposed the analytic center fusion method for the DSF model. Similar to the two new methods for the SSF model, this method also minimizes a cost function constructed using sensor reports. However, the cost function we choose for the analytic center fusion method has a unique structure to enable the reinforcement among sensor reports. In this way, the analytic center fusion method is able to decrease the uncertainty in target identity, which is the fusion goal of the DSF model. We have analyzed the new method using the proposed decision fusion principles proposed in Chapter 3. In deed, it is shown that the method satisfies all principles except fundamental principle 1. The latter is the main weakness of the method.

8.2 Future work

This thesis has laid a mathematical foundation for decision fusion. However, much more remains to be done. Below we enumerate a number of research directions for future investigation.

1. The new robust adaptive filtering method proposed in Chapter 2 minimizes an upper bound on the variance of the estimation error for all admissible systems. While this approach guarantees robustness to the norm bounded uncertainties in the system model and noise statistics, the performance of the resulting filters depends on how tight the minimized bound is. Basically, the tighter the bound, the better the performance of the resulting filters. Therefore, it is important to explore approaches which further tighten the upper bound. One possible way is to explore the structure of the estimation error covariance matrix. Intuitively, this should lead to better filters since it only minimizes an upper bound for a subset of all possible error covariance matrices.

2. The proposed robust adaptive filtering method is a recursive method. The convergence of the method may require many iterations. Therefore it is important to study ways to reduce the number of iterations before convergence. In other words, we suggest as a future study the analysis of transient performance of the method. Such transient performance is closely related with the efficiency of the method, and largely decides the tracking performance of the method when system model changes.

3. For the decision fusion problem, one important future research topic is to further develop the mathematical framework proposed in Chapter 3. For one thing, the two proposed fusion models (the DSF model and the SSF model) can be viewed as extreme fusion cases (as mentioned in section 3.4). There are many scenarios where the sensors are neither fully dissimilar nor fully similar. These scenarios should be studied in future research. New fusion methods should be developed for these scenarios. A potential approach is to intelligently combine the two classes of fusion methods developed for the DSF model and the SSF model. Furthermore, modified versions of decision fusion principles for such hybrid fusion environment should be studied. In addition, there

may be other important decision fusion principles which have not been incorporated in the current framework. We should continue to explore possible new principles or new mathematical ways to characterize practical fusion processes.

4. Last but least, a major task for decision fusion research is to find new fusion methods which can generate better fusion results while remain computationally efficient and easy to apply in practice (e.g., does not require prior target information which can be difficult to obtain). This is always an important future research topic.

Appendix A

DRDC Valcatier Data Set

This appendix lists the data set provided by Defence Research and Development Canada (DRDC) Valcartier, which is used in Chapter 7. The data set consists of 15 data groups, each has a number of Dempster-Shafer type sensor reports. All the sensor reports are based on a common proposition set Ω with 142 propositions. A sensor report has some probability masses each occupying a line. The data are shown in three columns. The *no.* column enumerate the probability masses in each sensor report. It begins from 0 and increases. Therefore if we find on the next line the *no.* value returns to 0, it indicates that the current report is over and the next report begins. The second column lists the probabilities of the probability masses. Note that for each sensor report, all the assigned probabilities sum up to 1. The last column lists the subsets in the probability masses. Each line represents one subset. Its 40-digit Hex value, when extended to binary format, has 160 digits, each is either 1 or 0. Here, the first 142 binary digits are corresponding to the 142 propositions in Ω. If a digit is 1, it indicates that the corresponding proposition is included in the subset, and vise versa. For example, a subset $\omega = a_1 \vee a_8 \vee a_{132}$ is represented by 9000 0000 0000 0000 0000 0000 0000 0000 1000 0000, and Ω is represented by a group of 40 F's.

no.	prob.	subset				
0	1.000000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
0	0.200000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.800000	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
0	0.040000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.960000	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.990000	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.990000	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.951553	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
2	0.038447	00000000	00000000	68000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.795784	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
2	0.160764	00000000	00000000	68000000	00000000	00000000
3	0.033452	00000301	80300001	F8000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.436018	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
2	0.440422	00000000	00000000	68000000	00000000	00000000
3	0.113560	00000301	80300001	F8000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.290828	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
2	0.293765	00000000	00000000	68000000	00000000	00000000
3	0.075745	00000301	80300001	F8000000	00000000	00000000
4	0.329662	00000001	00300001	90000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.114798	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
2	0.715175	00000000	00000000	68000000	00000000	00000000
3	0.029899	00000301	80300001	F8000000	00000000	00000000
4	0.130127	00000001	00300001	90000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.025084	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
2	0.781344	00000000	00000000	68000000	00000000	00000000
3	0.041405	00000301	80300001	F8000000	00000000	00000000
4	0.142167	00000001	00300001	90000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.005079	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
2	0.158197	00000000	00000000	68000000	00000000	00000000
3	0.008383	00000301	80300001	F8000000	00000000	00000000
4	0.143921	00000001	00300001	90000000	00000000	00000000
5	0.008099	00000005	90318001	B8000000	00000000	00000000
6	0.632788	00000000	00000000	28000000	00000000	00000000
7	0.033533	00000001	80300001	B8000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.001012	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000

Table A.11: Data group 1 (page 1).

no.	prob.	subset				
2	0.031514	00000000	00000000	68000000	00000000	00000000
3	0.001670	00000301	80300001	F8000000	00000000	00000000
4	0.143351	00000001	00300001	90000000	00000000	00000000
5	0.016035	00000005	90318001	B8000000	00000000	00000000
6	0.756338	00000000	00000000	28000000	00000000	00000000
7	0.040080	00000001	80300001	B8000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.913267	00000000	00000000	28000000	00000000	00000000
2	0.032182	00000001	00300001	90000000	00000000	00000000
3	0.014399	00000004	10018000	28000000	00000000	00000000
4	0.008998	00000001	80300001	B8000000	00000000	00000000
5	0.008980	00207F04	70018002	28000000	00000000	00000000
6	0.007075	00000000	00000000	68000000	00000000	00000000
7	0.003600	00000005	90318001	B8000000	00000000	00000000
8	0.001500	00000300	00000000	28000000	00000000	00000000
0	0.010695	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.913267	00000000	00000000	28000000	00000000	00000000
2	0.025745	00000001	00000000	90000000	00000000	00000000
3	0.014399	00000004	10018000	28000000	00000000	00000000
4	0.008000	00200307	900F8000	F8000000	00000000	00000000
5	0.007198	00000001	80000000	B8000000	00000000	00000000
6	0.007184	00200304	10018000	28000000	00000000	00000000
7	0.007075	00000000	00000000	68000000	00000000	00000000
8	0.006436	00000001	00300001	90000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.942946	00000000	00000000	28000000	00000000	00000000
2	0.014684	00000004	10018000	28000000	00000000	00000000
3	0.013852	00200304	10018000	28000000	00000000	00000000
4	0.008725	00207F04	70018002	28000000	00000000	00000000
5	0.005251	00000001	00000000	90000000	00000000	00000000
6	0.001632	00200307	900F8000	F8000000	00000000	00000000
7	0.001468	00000001	80000000	B8000000	00000000	00000000
8	0.001443	00000000	00000000	68000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.937110	00000000	00000000	28000000	00000000	00000000
2	0.032503	00000004	10018000	28000000	00000000	00000000
3	0.007941	00000005	90318001	B8000000	00000000	00000000
4	0.005212	00000001	00000000	90000000	00000000	00000000
5	0.002750	00200304	10018000	28000000	00000000	00000000
6	0.001732	00207F04	70018002	28000000	00000000	00000000
7	0.001457	00000001	80000000	B8000000	00000000	00000000
8	0.001296	00000005	90018000	B8000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.934659	00000000	00000000	28000000	00000000	00000000
2	0.039738	00000004	10018000	28000000	00000000	00000000

Table A.12: Data group 1 (page 2).

no.	prob.	subset				
3	0.001582	00000005	90318001	B8000000	00000000	00000000
4	0.001038	00000001	00000000	90000000	00000000	00000000
5	0.002739	00200304	10018000	28000000	00000000	00000000
6	0.009695	00207F04	70018002	28000000	00000000	00000000
7	0.000290	00000001	80000000	B8000000	00000000	00000000
8	0.000258	00000005	90018000	B8000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.960262	00000000	00000000	28000000	00000000	00000000
2	0.007949	00001F00	00000001	B8000000	00000000	00000000
3	0.007897	00000004	10018000	28000000	00000000	00000000
4	0.007706	00001F00	00000000	28000000	00000000	00000000
5	0.002177	00000300	00000000	28000000	00000000	00000000
6	0.001926	00207F04	70018002	28000000	00000000	00000000
7	0.001258	00000000	00000001	B8000000	00000000	00000000
8	0.000825	00000000	00000000	90000000	00000000	00000000
0	0.010332	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.916537	00000000	00000000	28000000	00000000	00000000
2	0.045534	00006000	40000000	00000000	00000000	00000000
3	0.007587	00001F00	00000001	B8000000	00000000	00000000
4	0.007537	00000004	10018000	28000000	00000000	00000000
5	0.007355	00001F00	00000000	28000000	00000000	00000000
6	0.002078	00000300	00000000	28000000	00000000	00000000
7	0.001839	00207F04	70018002	28000000	00000000	00000000
8	0.001200	00000000	00000001	B8000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.910204	00000000	00000000	28000000	00000000	00000000
2	0.045172	00006000	40000000	00000000	00000000	00000000
3	0.001505	00001F00	00000001	B8000000	00000000	00000000
4	0.007477	00000004	10018000	28000000	00000000	00000000
5	0.013318	00001F00	00000000	28000000	00000000	00000000
6	0.002062	00000300	00000000	28000000	00000000	00000000
7	0.010024	00207F04	70018002	28000000	00000000	00000000
8	0.000238	00000000	00000001	B8000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.758500	00000000	00000000	08000000	00000000	00000000
2	0.188080	00000000	00000000	28000000	00000000	00000000
3	0.019293	00000900	00000000	08000000	00000000	00000000
4	0.009334	00006000	40000000	00000000	00000000	00000000
5	0.008265	00000900	00100001	58000000	00000000	00000000
6	0.002752	00001F00	00000000	28000000	00000000	00000000
7	0.002071	00207F04	70018002	28000000	00000000	00000000
8	0.001704	00000100	00000000	08000000	00000000	00000000
0	0.011254	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.729327	00000000	00000000	08000000	00000000	00000000
2	0.180846	00000000	00000000	28000000	00000000	00000000

Table A.13: Data group 1 (page 3).

no.	prob.	subset				
3	0.018551	00000900	00000000	08000000	00000000	00000000
4	0.038462	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
5	0.008975	00006000	40000000	00000000	00000000	00000000
6	0.007947	00000900	00100001	58000000	00000000	00000000
7	0.002646	00001F00	00000000	28000000	00000000	00000000
8	0.001992	00207F04	70018002	28000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.607973	00000000	00000000	08000000	00000000	00000000
2	0.150755	00000000	00000000	28000000	00000000	00000000
3	0.015464	00000900	00000000	08000000	00000000	00000000
4	0.197835	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
5	0.007482	00006000	40000000	00000000	00000000	00000000
6	0.006625	00000900	00100001	58000000	00000000	00000000
7	0.002206	00001F00	00000000	28000000	00000000	00000000
8	0.001660	00207F04	70018002	28000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.330467	00000000	00000000	08000000	00000000	00000000
2	0.081944	00000000	00000000	28000000	00000000	00000000
3	0.008406	00000900	00000000	08000000	00000000	00000000
4	0.559414	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
5	0.004067	00006000	40000000	00000000	00000000	00000000
6	0.003601	00000900	00100001	58000000	00000000	00000000
7	0.001199	00001F00	00000000	28000000	00000000	00000000
8	0.000902	00207F04	70018002	28000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.716467	00000000	00000000	08000000	00000000	00000000
2	0.029650	00000000	00000000	28000000	00000000	00000000
3	0.018249	00000900	00000000	08000000	00000000	00000000
4	0.202414	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
5	0.001471	00006000	40000000	00000000	00000000	00000000
6	0.020988	00000900	00100001	58000000	00000000	00000000
7	0.000434	00001F00	00000000	28000000	00000000	00000000
8	0.000327	00207F04	70018002	28000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.385563	00000000	00000000	08000000	00000000	00000000
2	0.015956	00000000	00000000	28000000	00000000	00000000
3	0.009821	00000900	00000000	08000000	00000000	00000000
4	0.566165	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
5	0.000792	00006000	40000000	00000000	00000000	00000000
6	0.011295	00000900	00100001	58000000	00000000	00000000
7	0.000233	00001F00	00000000	28000000	00000000	00000000
8	0.000176	00207F04	70018002	28000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.369202	00000000	00000000	08000000	00000000	00000000
2	0.015279	00000000	00000000	28000000	00000000	00000000

Table A.14: Data group 1 (page 4).

no.	prob.	subset				
3	0.009404	00000900	00000000	08000000	00000000	00000000
4	0.542141	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
5	0.042767	00006000	40000000	00000000	00000000	00000000
6	0.010815	00000900	00100001	58000000	00000000	00000000
7	0.000224	00001F00	00000000	28000000	00000000	00000000
8	0.000168	00207F04	70018002	28000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.500265	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
2	0.340685	00000000	00000000	08000000	00000000	00000000
3	0.039920	00000000	00000001	10000000	00000000	00000000
4	0.014099	00000000	00000000	28000000	00000000	00000000
5	0.039464	00006000	40000000	00000000	00000000	00000000
6	0.036910	00000001	00000001	90000000	00000000	00000000
7	0.009980	00000900	00100001	58000000	00000000	00000000
8	0.008677	00000900	00000000	08000000	00000000	00000000
0	0.034075	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.599568	00000000	00000000	08000000	00000000	00000000
2	0.176082	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
3	0.070255	00000000	00000001	10000000	00000000	00000000
4	0.051967	00000000	00000001	90000000	00000000	00000000
5	0.024812	00000000	00000000	28000000	00000000	00000000
6	0.015271	00000900	00000000	08000000	00000000	00000000
7	0.013890	00006000	40000000	00000000	00000000	00000000
8	0.014079	00001F00	00000001	B8000000	00000000	00000000
0	0.028590	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.503051	00000000	00000000	08000000	00000000	00000000
2	0.147737	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
3	0.058946	00000000	00000001	10000000	00000000	00000000
4	0.043601	00000000	00000001	90000000	00000000	00000000
5	0.020818	00000000	00000000	28000000	00000000	00000000
6	0.012813	00000900	00000000	08000000	00000000	00000000
7	0.172631	00006000	40000000	00000000	00000000	00000000
8	0.011813	00001F00	00000001	B8000000	00000000	00000000
0	0.029484	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.382587	00002000	00000000	00000000	00000000	00000000
2	0.278717	00000000	00000000	08000000	00000000	00000000
3	0.095647	00006000	40000000	00000000	00000000	00000000
4	0.081854	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
5	0.063361	00202000	00000002	00000000	00000000	00000000
6	0.032659	00000000	00000001	10000000	00000000	00000000
7	0.011534	00000000	00000000	28000000	00000000	00000000
8	0.024157	00000000	00000001	90000000	00000000	00000000
0	0.012067	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.430325	00002000	00000000	00000000	00000000	00000000
2	0.313494	00000000	00000000	08000000	00000000	00000000

Table A.15: Data group 1 (page 5).

no.	prob.	subset				
3	0.107581	00006000	40000000	00000000	00000000	00000000
4	0.071267	00202000	00000002	00000000	00000000	00000000
5	0.012973	00000000	00000000	28000000	00000000	00000000
6	0.026531	00207F04	70018002	28000000	00000000	00000000
7	0.018414	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
8	0.007347	00000000	00000001	10000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.436061	00002000	00000000	00000000	00000000	00000000
2	0.317673	00000000	00000000	08000000	00000000	00000000
3	0.109015	00006000	40000000	00000000	00000000	00000000
4	0.072217	00202000	00000002	00000000	00000000	00000000
5	0.013146	00000000	00000000	28000000	00000000	00000000
6	0.036666	00207F04	70018002	28000000	00000000	00000000
7	0.003732	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
8	0.001489	00000000	00000001	10000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.592894	00002000	00000000	00000000	00000000	00000000
2	0.148224	00006000	40000000	00000000	00000000	00000000
3	0.098191	00202000	00000002	00000000	00000000	00000000
4	0.086385	00000000	00000000	08000000	00000000	00000000
5	0.039883	00206000	60000002	00000000	00000000	00000000
6	0.010877	00206002	60060002	00000000	00000000	00000000
7	0.009971	00207F04	70018002	28000000	00000000	00000000
8	0.003575	00000000	00000000	28000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.718384	00002000	00000000	00000000	00000000	00000000
2	0.187408	00006000	00000000	00000000	00000000	00000000
3	0.018485	00000000	00000000	08000000	00000000	00000000
4	0.031717	00006000	40000000	00000000	00000000	00000000
5	0.021011	00202000	00000002	00000000	00000000	00000000
6	0.008534	00206000	60000002	00000000	00000000	00000000
7	0.002328	00206002	60060002	00000000	00000000	00000000
8	0.002134	00207F04	70018002	28000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.723343	00002000	00000000	00000000	00000000	00000000
2	0.188702	00006000	00000000	00000000	00000000	00000000
3	0.003722	00000000	00000000	08000000	00000000	00000000
4	0.031936	00006000	40000000	00000000	00000000	00000000
5	0.021156	00202000	00000002	00000000	00000000	00000000
6	0.010312	00206000	60000002	00000000	00000000	00000000
7	0.010399	00206002	60060002	00000000	00000000	00000000
8	0.000430	00207F04	70018002	28000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.895312	00002000	00000000	00000000	00000000	00000000
2	0.037550	00006000	00000000	00000000	00000000	00000000

Table A.16: Data group 1 (page 6).

no.	prob.	subset				
3	0.000741	00000000	00000000	08000000	00000000	00000000
4	0.006355	00006000	40000000	00000000	00000000	00000000
5	0.045836	00202000	00000002	00000000	00000000	00000000
6	0.002052	00206000	60000002	00000000	00000000	00000000
7	0.002069	00206002	60060002	00000000	00000000	00000000
8	0.000085	00207F04	70018002	28000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.925059	00002000	00000000	00000000	00000000	00000000
2	0.037271	00006000	00000000	00000000	00000000	00000000
3	0.000147	00000000	00000000	08000000	00000000	00000000
4	0.017589	00006000	40000000	00000000	00000000	00000000
5	0.009099	00202000	00000002	00000000	00000000	00000000
6	0.000407	00206000	60000002	00000000	00000000	00000000
7	0.000411	00206002	60060002	00000000	00000000	00000000
8	0.000017	00207F04	70018002	28000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.924974	00002000	00000000	00000000	00000000	00000000
2	0.059536	00006000	00000000	00000000	00000000	00000000
3	0.000029	00000000	00000000	08000000	00000000	00000000
4	0.003490	00006000	40000000	00000000	00000000	00000000
5	0.001805	00202000	00000002	00000000	00000000	00000000
6	0.000081	00206000	60000002	00000000	00000000	00000000
7	0.000082	00206002	60060002	00000000	00000000	00000000
8	0.000003	00207F04	70018002	28000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.967598	00002000	00000000	00000000	00000000	00000000
2	0.011812	00006000	00000000	00000000	00000000	00000000
3	0.000006	00000000	00000000	08000000	00000000	00000000
4	0.000692	00006000	40000000	00000000	00000000	00000000
5	0.009859	00202000	00000002	00000000	00000000	00000000
6	0.000016	00206000	60000002	00000000	00000000	00000000
7	0.000016	00206002	60060002	00000000	00000000	00000000
8	0.000001	00207F04	70018002	28000000	00000000	00000000

Table A.17: Data group 1 (page 7).

no.	prob.	subset				
0	1.000000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
0	0.200000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.800000	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
0	0.040000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.960000	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.990000	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000

Table A.18: Data group 2.

no.	prob.	subset				
0	1.000000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
0	0.200000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.800000	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
0	0.040000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.960000	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.990000	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.990000	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.951553	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
2	0.038447	00000002	A03E0000	00000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.795784	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
2	0.032153	00000002	A03E0000	00000000	00000000	00000000
3	0.033452	00000301	80300001	F8000000	00000000	00000000
4	0.128611	00000000	80300000	00000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.436018	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
2	0.110001	00000002	A03E0000	00000000	00000000	00000000
3	0.018329	00000301	80300001	F8000000	00000000	00000000
4	0.425652	00000000	80300000	00000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.132984	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
2	0.033550	00000002	A03E0000	00000000	00000000	00000000
3	0.005590	00000301	80300001	F8000000	00000000	00000000
4	0.129823	00000000	80300000	00000000	00000000	00000000
5	0.012200	00000900	00100001	58000000	00000000	00000000
6	0.653492	00000000	00100000	00000000	00000000	00000000
7	0.022361	00000100	00100001	58000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.113086	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
2	0.028530	00000002	A03E0000	00000000	00000000	00000000
3	0.004754	00000301	80300001	F8000000	00000000	00000000
4	0.110397	00000000	80300000	00000000	00000000	00000000
5	0.010374	00000900	00100001	58000000	00000000	00000000
6	0.555709	00000000	00100000	00000000	00000000	00000000
7	0.019015	00000100	00100001	58000000	00000000	00000000
8	0.148134	00000002	200E0000	00000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.024672	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
2	0.039849	00000002	A03E0000	00000000	00000000	00000000
3	0.001037	00000301	80300001	F8000000	00000000	00000000
4	0.124578	00000000	80300000	00000000	00000000	00000000
5	0.002263	00000900	00100001	58000000	00000000	00000000

Table A.19: Data group 3 (page 1).

no.	prob.	subset				
6	0.631856	00000000	00100000	00000000	00000000	00000000
7	0.004149	00000100	00100001	58000000	00000000	00000000
8	0.161595	00000002	200E0000	00000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.645063	00000000	00100000	00000000	00000000	00000000
2	0.130916	00000000	00080000	00000000	00000000	00000000
3	0.126998	00000000	80300000	00000000	00000000	00000000
4	0.040385	00000000	80380000	00000000	00000000	00000000
5	0.032729	00000002	200E0000	00000000	00000000	00000000
6	0.008071	00000002	A03E0000	00000000	00000000	00000000
7	0.004997	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
8	0.000840	00000100	00100001	58000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.430323	00000000	00080000	00000000	00000000	00000000
2	0.340128	00000000	00100000	00000000	00000000	00000000
3	0.107143	00000002	000E0000	00000000	00000000	00000000
4	0.066963	00000000	80300000	00000000	00000000	00000000
5	0.021294	00000000	80380000	00000000	00000000	00000000
6	0.017257	00000002	200E0000	00000000	00000000	00000000
7	0.004256	00000002	A03E0000	00000000	00000000	00000000
8	0.002635	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
0	0.014101	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.592904	00000000	00080000	00000000	00000000	00000000
2	0.166644	00000002	000E0000	00000000	00000000	00000000
3	0.093726	00000000	00100000	00000000	00000000	00000000
4	0.073810	00000000	80000000	00000000	00000000	00000000
5	0.023472	00000000	80080000	00000000	00000000	00000000
6	0.018453	00000000	80300000	00000000	00000000	00000000
7	0.011022	00200307	900F8000	F8000000	00000000	00000000
8	0.005868	00000000	80380000	00000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.636495	00000000	00080000	00000000	00000000	00000000
2	0.178896	00000002	000E0000	00000000	00000000	00000000
3	0.020123	00000000	00100000	00000000	00000000	00000000
4	0.095084	00000000	80000000	00000000	00000000	00000000
5	0.030237	00000000	80080000	00000000	00000000	00000000
6	0.003962	00000000	80300000	00000000	00000000	00000000
7	0.023943	00200307	900F8000	F8000000	00000000	00000000
8	0.001260	00000000	80380000	00000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.726553	00000000	00080000	00000000	00000000	00000000
2	0.196432	00000002	000E0000	00000000	00000000	00000000
3	0.020881	00000000	80000000	00000000	00000000	00000000
4	0.021032	00000002	000E0000	28000000	00000000	00000000
5	0.008784	00000002	200E0000	28000000	00000000	00000000

Table A.20: Data group 3 (page 2).

no.	prob.	subset				
6	0.006640	00000000	80080000	00000000	00000000	00000000
7	0.005258	00200307	900F8000	F8000000	00000000	00000000
8	0.004419	00000000	00100000	00000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.741698	00000000	00080000	00000000	00000000	00000000
2	0.220386	00000002	000E0000	00000000	00000000	00000000
3	0.015229	00000002	200E0000	00000000	00000000	00000000
4	0.004263	00000002	000E0000	28000000	00000000	00000000
5	0.004232	00000000	80000000	00000000	00000000	00000000
6	0.001780	00000002	200E0000	28000000	00000000	00000000
7	0.001346	00000000	80080000	00000000	00000000	00000000
8	0.001066	00200307	900F8000	F8000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.736172	00000000	00080000	00000000	00000000	00000000
2	0.222129	00000002	000E0000	00000000	00000000	00000000
3	0.016530	00000002	200E0000	00000000	00000000	00000000
4	0.007940	00000002	A03E0000	00000000	00000000	00000000
5	0.004201	00000000	80000000	00000000	00000000	00000000
6	0.001336	00000000	80080000	00000000	00000000	00000000
7	0.000846	00000002	000E0000	28000000	00000000	00000000
8	0.000846	00000002	800E0000	00000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.731015	00000000	00080000	00000000	00000000	00000000
2	0.234376	00000002	000E0000	00000000	00000000	00000000
3	0.007944	00201F02	803F8000	00000000	00000000	00000000
4	0.006308	00000002	803E0000	00000000	00000000	00000000
5	0.004171	00000000	80000000	00000000	00000000	00000000
6	0.003283	00000002	200E0000	00000000	00000000	00000000
7	0.001577	00000002	A03E0000	00000000	00000000	00000000
8	0.001326	00000000	80080000	00000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.729070	00000000	00080000	00000000	00000000	00000000
2	0.238439	00000002	000E0000	00000000	00000000	00000000
3	0.007967	00001606	700F0000	00000000	00000000	00000000
4	0.006329	00001602	000F0000	00000000	00000000	00000000
5	0.004526	00000002	200E0000	00000000	00000000	00000000
6	0.001582	00201F02	803F8000	00000000	00000000	00000000
7	0.001256	00000002	803E0000	00000000	00000000	00000000
8	0.000831	00000000	80000000	00000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.724899	00000000	00080000	00000000	00000000	00000000
2	0.241674	00000002	000E0000	00000000	00000000	00000000
3	0.007954	00000006	000F8000	00000000	00000000	00000000
4	0.006337	00000006	000F0000	00000000	00000000	00000000
5	0.005034	00000002	000F0000	00000000	00000000	00000000

Table A.21: Data group 3 (page 3).

no.	prob.	subset				
6	0.001584	00001606	700F0000	00000000	00000000	00000000
7	0.001259	00001602	000F0000	00000000	00000000	00000000
8	0.001259	00000002	000F8000	00000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.720225	00000000	00080000	00000000	00000000	00000000
2	0.240116	00000002	000E0000	00000000	00000000	00000000
3	0.007948	00001806	000F8000	00000000	00000000	00000000
4	0.007903	00000006	000F8000	00000000	00000000	00000000
5	0.006296	00000006	000F0000	00000000	00000000	00000000
6	0.005002	00000002	000F0000	00000000	00000000	00000000
7	0.001259	00001006	000F0000	00000000	00000000	00000000
8	0.001250	00000002	000F8000	00000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.718297	00000000	00080000	00000000	00000000	00000000
2	0.191578	00000002	000A0000	00000000	00000000	00000000
3	0.047895	00000002	000E0000	00000000	00000000	00000000
4	0.012647	00000006	000B8000	00000000	00000000	00000000
5	0.007979	00000006	000B8001	80000000	00000000	00000000
6	0.006028	00000006	000B0000	00000000	00000000	00000000
7	0.003991	00000002	000B0000	00000000	00000000	00000000
8	0.001585	00001806	000F8000	00000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.712765	00000000	00080000	00000000	00000000	00000000
2	0.190103	00000002	000A0000	00000000	00000000	00000000
3	0.047526	00000002	000E0000	00000000	00000000	00000000
4	0.018883	00000006	000B8000	00000000	00000000	00000000
5	0.009197	00000006	000F8000	00000000	00000000	00000000
6	0.005982	00000006	000B0000	00000000	00000000	00000000
7	0.003960	00000002	000B0000	00000000	00000000	00000000
8	0.001583	00000006	000B8001	80000000	00000000	00000000
0	0.011138	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.685351	00000000	00080000	00000000	00000000	00000000
2	0.182791	00000002	000A0000	00000000	00000000	00000000
3	0.045698	00000002	000E0000	00000000	00000000	00000000
4	0.038462	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
5	0.018157	00000006	000B8000	00000000	00000000	00000000
6	0.008843	00000006	000F8000	00000000	00000000	00000000
7	0.005752	00000006	000B0000	00000000	00000000	00000000
8	0.003808	00000002	000B0000	00000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.571482	00000000	00080000	00000000	00000000	00000000
2	0.152421	00000002	000A0000	00000000	00000000	00000000
3	0.038105	00000002	000E0000	00000000	00000000	00000000
4	0.197506	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
5	0.015140	00000006	000B8000	00000000	00000000	00000000

Table A.22: Data group 3 (page 4).

no.	prob.	subset				
6	0.007374	00000006	000F8000	00000000	00000000	00000000
7	0.004796	00000006	000B0000	00000000	00000000	00000000
8	0.003175	00000002	000B0000	00000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.310857	00000000	00080000	00000000	00000000	00000000
2	0.082909	00000002	000A0000	00000000	00000000	00000000
3	0.020727	00000002	000E0000	00000000	00000000	00000000
4	0.558924	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
5	0.008236	00000006	000B8000	00000000	00000000	00000000
6	0.004011	00000006	000F8000	00000000	00000000	00000000
7	0.002609	00000006	000B0000	00000000	00000000	00000000
8	0.001727	00000002	000B0000	00000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.094237	00000000	00080000	00000000	00000000	00000000
2	0.025134	00000002	000A0000	00000000	00000000	00000000
3	0.006284	00000002	000E0000	00000000	00000000	00000000
4	0.859319	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
5	0.002497	00000006	000B8000	00000000	00000000	00000000
6	0.001216	00000006	000F8000	00000000	00000000	00000000
7	0.000791	00000006	000B0000	00000000	00000000	00000000
8	0.000524	00000002	000B0000	00000000	00000000	00000000
0	0.010567	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.724687	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
2	0.097641	00000002	00020000	00000000	00000000	00000000
3	0.079473	00000000	00080000	00000000	00000000	00000000
4	0.025298	00000002	00060000	00000000	00000000	00000000
5	0.021196	00000002	000A0000	00000000	00000000	00000000
6	0.033733	00206002	60060002	00000000	00000000	00000000
7	0.005299	00000002	000E0000	00000000	00000000	00000000
8	0.002105	00000006	000B8000	00000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.344033	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
2	0.343863	00000002	00020000	00000000	00000000	00000000
3	0.188641	00000000	00080000	00000000	00000000	00000000
4	0.060375	00000002	000A0000	00000000	00000000	00000000
5	0.012010	00000002	00060000	00000000	00000000	00000000
6	0.020066	00000006	000B8001	80000000	00000000	00000000
7	0.016014	00206002	60060002	00000000	00000000	00000000
8	0.004998	00000006	000B8000	00000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.599679	00000000	00080000	00000000	00000000	00000000
2	0.160556	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
3	0.160477	00000002	00020000	00000000	00000000	00000000
4	0.028177	00000002	000A0000	00000000	00000000	00000000
5	0.018668	00000000	80380000	00000000	00000000	00000000

Table A.23: Data group 3 (page 5).

no.	prob.	subset				
6	0.009365	00000006	000B8001	80000000	00000000	00000000
7	0.007474	00206002	60060002	00000000	00000000	00000000
8	0.005605	00000002	00060000	00000000	00000000	00000000
0	0.012590	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.688056	00000000	00080000	00000000	00000000	00000000
2	0.184128	00000002	00020000	00000000	00000000	00000000
3	0.032329	00000002	000A0000	00000000	00000000	00000000
4	0.036844	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
5	0.021419	00000000	80380000	00000000	00000000	00000000
6	0.009179	00201F02	803F8000	00000000	00000000	00000000
7	0.008596	00000002	000B8000	00000000	00000000	00000000
8	0.006860	00200002	00060000	00000000	00000000	00000000
0	0.012085	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.726608	00000000	00080000	00000000	00000000	00000000
2	0.189720	00000002	00020000	00000000	00000000	00000000
3	0.033311	00000002	000A0000	00000000	00000000	00000000
4	0.010378	00001606	700F0000	00000000	00000000	00000000
5	0.007593	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
6	0.007566	00001602	000F0000	00000000	00000000	00000000
7	0.007086	00000002	000B0000	00000000	00000000	00000000
8	0.005655	00000002	00060000	00000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.727442	00000000	00080000	00000000	00000000	00000000
2	0.194466	00000002	00020000	00000000	00000000	00000000
3	0.033349	00000002	000A0000	00000000	00000000	00000000
4	0.013154	00000002	000B0000	00000000	00000000	00000000
5	0.009679	00000006	000B8001	80000000	00000000	00000000
6	0.008312	00000006	000B0000	00000000	00000000	00000000
7	0.002078	00001606	700F0000	00000000	00000000	00000000
8	0.001520	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.722491	00000000	00080000	00000000	00000000	00000000
2	0.193143	00000002	00020000	00000000	00000000	00000000
3	0.033122	00000002	000A0000	00000000	00000000	00000000
4	0.013064	00000002	000B0000	00000000	00000000	00000000
5	0.017559	00000006	000B8001	80000000	00000000	00000000
6	0.009906	00000006	000B0000	00000000	00000000	00000000
7	0.000413	00001606	700F0000	00000000	00000000	00000000
8	0.000302	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.716917	00000000	00080000	00000000	00000000	00000000
2	0.191653	00000002	00020000	00000000	00000000	00000000
3	0.065040	00000002	000A0000	00000000	00000000	00000000
4	0.008266	00000002	200E0000	00000000	00000000	00000000
5	0.003485	00000006	000B8001	80000000	00000000	00000000

Table A.24: Data group 3 (page 6).

no.	prob.	subset				
6	0.002593	00000002	000B0000	00000000	00000000	00000000
7	0.001966	00000006	000B0000	00000000	00000000	00000000
8	0.000082	00001606	700F0000	00000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.711170	00000000	00080000	00000000	00000000	00000000
2	0.190116	00000002	00020000	00000000	00000000	00000000
3	0.071078	00000002	000A0000	00000000	00000000	00000000
4	0.001640	00000002	200E0000	00000000	00000000	00000000
5	0.011393	00000006	000B8001	80000000	00000000	00000000
6	0.002572	00000002	000B0000	00000000	00000000	00000000
7	0.002015	00000006	000B0000	00000000	00000000	00000000
8	0.000016	00001606	700F0000	00000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.705469	00000000	00080000	00000000	00000000	00000000
2	0.188592	00000002	00020000	00000000	00000000	00000000
3	0.070508	00000002	000A0000	00000000	00000000	00000000
4	0.001627	00000002	200E0000	00000000	00000000	00000000
5	0.002260	00000006	000B8001	80000000	00000000	00000000
6	0.002551	00000002	000B0000	00000000	00000000	00000000
7	0.011040	00000006	000B0000	00000000	00000000	00000000
8	0.007952	00001606	700F0000	00000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.700042	00000000	00080000	00000000	00000000	00000000
2	0.187142	00000002	00020000	00000000	00000000	00000000
3	0.082549	00000002	000A0000	00000000	00000000	00000000
4	0.015543	00000002	000E0000	00000000	00000000	00000000
5	0.002191	00000006	000B0000	00000000	00000000	00000000
6	0.001578	00001606	700F0000	00000000	00000000	00000000
7	0.000506	00000002	000B0000	00000000	00000000	00000000
8	0.000449	00000006	000B8001	80000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.694431	00000000	00080000	00000000	00000000	00000000
2	0.185641	00000002	00020000	00000000	00000000	00000000
3	0.094222	00000002	000A0000	00000000	00000000	00000000
4	0.003084	00000002	000E0000	00000000	00000000	00000000
5	0.003426	00000006	000B0000	00000000	00000000	00000000
6	0.000313	00001606	700F0000	00000000	00000000	00000000
7	0.000502	00000002	000B0000	00000000	00000000	00000000
8	0.008381	00000006	000B8001	80000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.688864	00000000	00080000	00000000	00000000	00000000
2	0.184153	00000002	00020000	00000000	00000000	00000000
3	0.093467	00000002	000A0000	00000000	00000000	00000000
4	0.003059	00000002	000E0000	00000000	00000000	00000000
5	0.010049	00000006	000B0000	00000000	00000000	00000000

Table A.25: Data group 3 (page 7).

no.	prob.	subset				
6	0.008246	00001606	700F0000	00000000	00000000	00000000
7	0.000498	00000002	000B0000	00000000	00000000	00000000
8	0.001663	00000006	000B8001	80000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.683410	00000000	00080000	00000000	00000000	00000000
2	0.182695	00000002	00020000	00000000	00000000	00000000
3	0.102418	00000002	000A0000	00000000	00000000	00000000
4	0.014482	00000002	200E0000	00000000	00000000	00000000
5	0.003035	00000002	000E0000	00000000	00000000	00000000
6	0.001994	00000006	000B0000	00000000	00000000	00000000
7	0.001636	00001606	700F0000	00000000	00000000	00000000
8	0.000330	00000006	000B8001	80000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.677932	00000000	00080000	00000000	00000000	00000000
2	0.181231	00000002	00020000	00000000	00000000	00000000
3	0.103441	00000002	000A0000	00000000	00000000	00000000
4	0.002873	00000002	200E0000	00000000	00000000	00000000
5	0.023737	00000002	000E0000	00000000	00000000	00000000
6	0.000396	00000006	000B0000	00000000	00000000	00000000
7	0.000325	00001606	700F0000	00000000	00000000	00000000
8	0.000065	00000006	000B8001	80000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.672639	00000000	00080000	00000000	00000000	00000000
2	0.179816	00000002	00020000	00000000	00000000	00000000
3	0.102633	00000002	000A0000	00000000	00000000	00000000
4	0.025832	00000002	000E0000	00000000	00000000	00000000
5	0.007938	00201F02	803F8000	00000000	00000000	00000000
6	0.000570	00000002	200E0000	00000000	00000000	00000000
7	0.000314	00000002	000B0000	00000000	00000000	00000000
8	0.000258	00001602	000F0000	00000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.667457	00000000	00080000	00000000	00000000	00000000
2	0.178431	00000002	00020000	00000000	00000000	00000000
3	0.101843	00000002	000A0000	00000000	00000000	00000000
4	0.025633	00000002	000E0000	00000000	00000000	00000000
5	0.007938	00001606	700F0000	00000000	00000000	00000000
6	0.006557	00001602	000F0000	00000000	00000000	00000000
7	0.001575	00201F02	803F8000	00000000	00000000	00000000
8	0.000566	00000002	200E0000	00000000	00000000	00000000
0	0.010159	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.641786	00000000	00080000	00000000	00000000	00000000
2	0.171568	00000002	00020000	00000000	00000000	00000000
3	0.097926	00000002	000A0000	00000000	00000000	00000000
4	0.038462	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
5	0.024647	00000002	000E0000	00000000	00000000	00000000

Table A.26: Data group 3 (page 8).

no.	prob.	subset				
6	0.007633	00001606	700F0000	00000000	00000000	00000000
7	0.006305	00001602	000F0000	00000000	00000000	00000000
8	0.001515	00201F02	803F8000	00000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.536481	00000000	00080000	00000000	00000000	00000000
2	0.143417	00000002	00020000	00000000	00000000	00000000
3	0.081858	00000002	000A0000	00000000	00000000	00000000
4	0.194723	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
5	0.020603	00000002	000E0000	00000000	00000000	00000000
6	0.006381	00001606	700F0000	00000000	00000000	00000000
7	0.005270	00001602	000F0000	00000000	00000000	00000000
8	0.001266	00201F02	803F8000	00000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.293614	00000000	00080000	00000000	00000000	00000000
2	0.078492	00000002	00020000	00000000	00000000	00000000
3	0.044801	00000002	000A0000	00000000	00000000	00000000
4	0.554748	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
5	0.011276	00000002	000E0000	00000000	00000000	00000000
6	0.003492	00001606	700F0000	00000000	00000000	00000000
7	0.002884	00001602	000F0000	00000000	00000000	00000000
8	0.000693	00201F02	803F8000	00000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.526368	00000000	00080000	00000000	00000000	00000000
2	0.198902	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
3	0.140713	00000002	00020000	00000000	00000000	00000000
4	0.080315	00000002	000A0000	00000000	00000000	00000000
5	0.020215	00000002	000E0000	00000000	00000000	00000000
6	0.014342	00000006	000F8000	00000000	00000000	00000000
7	0.005008	00000006	000F0000	00000000	00000000	00000000
8	0.004137	00000002	000F0000	00000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.285442	00000000	00080000	00000000	00000000	00000000
2	0.560999	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
3	0.076307	00000002	00020000	00000000	00000000	00000000
4	0.043554	00000002	000A0000	00000000	00000000	00000000
5	0.010962	00000002	000E0000	00000000	00000000	00000000
6	0.007777	00000006	000F8000	00000000	00000000	00000000
7	0.002716	00000006	000F0000	00000000	00000000	00000000
8	0.002243	00000002	000F0000	00000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.514543	00000000	00080000	00000000	00000000	00000000
2	0.202254	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
3	0.137552	00000002	00020000	00000000	00000000	00000000
4	0.078510	00000002	000A0000	00000000	00000000	00000000
5	0.019761	00000002	000E0000	00000000	00000000	00000000

Table A.27: Data group 3 (page 9).

no.	prob.	subset				
6	0.028441	00000006	000F8000	00000000	00000000	00000000
7	0.004896	00000006	000F0000	00000000	00000000	00000000
8	0.004044	00000002	000F0000	00000000	00000000	00000000
0	0.010340	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.613869	00000000	00080000	00000000	00000000	00000000
2	0.164105	00000002	00020000	00000000	00000000	00000000
3	0.093666	00000002	000A0000	00000000	00000000	00000000
4	0.048259	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
5	0.027144	00000002	000F8000	00000000	00000000	00000000
6	0.023575	00000002	000E0000	00000000	00000000	00000000
7	0.009544	00201F02	803F8000	00000000	00000000	00000000
8	0.009497	00000002	000F0000	00000000	00000000	00000000
0	0.027217	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.568640	00000000	00080000	00000000	00000000	00000000
2	0.152014	00000002	00020000	00000000	00000000	00000000
3	0.086765	00000002	000A0000	00000000	00000000	00000000
4	0.044704	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
5	0.038314	00001F00	00000001	B8000000	00000000	00000000
6	0.035364	00001F00	00000000	00000000	00000000	00000000
7	0.025145	00000002	000F8000	00000000	00000000	00000000
8	0.021838	00000002	000E0000	00000000	00000000	00000000
0	0.013592	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.589730	00000000	00080000	00000000	00000000	00000000
2	0.157652	00000002	00020000	00000000	00000000	00000000
3	0.089983	00000002	000A0000	00000000	00000000	00000000
4	0.068464	00001F00	00000000	00000000	00000000	00000000
5	0.026077	00000002	000F8000	00000000	00000000	00000000
6	0.022648	00000002	000E0000	00000000	00000000	00000000
7	0.022581	00201F02	803F8000	00000000	00000000	00000000
8	0.009272	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.628641	00000000	00080000	00000000	00000000	00000000
2	0.168054	00000002	00020000	00000000	00000000	00000000
3	0.115234	00000002	000A0000	00000000	00000000	00000000
4	0.041495	00000002	000B8000	00000000	00000000	00000000
5	0.014596	00001F00	00000000	00000000	00000000	00000000
6	0.011591	00000006	000B8001	80000000	00000000	00000000
7	0.005560	00000002	000F8000	00000000	00000000	00000000
8	0.004829	00000002	000E0000	00000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.897731	00000000	00080000	00000000	00000000	00000000
2	0.039105	00000002	00020000	00000000	00000000	00000000
3	0.026814	00000002	000A0000	00000000	00000000	00000000
4	0.009655	00000002	000B8000	00000000	00000000	00000000
5	0.009308	00000000	80380000	00000000	00000000	00000000

Table A.28: Data group 3 (page 10).

no.	prob.	subset				
6	0.003396	00001F00	00000000	00000000	00000000	00000000
7	0.002697	00000006	000B8001	80000000	00000000	00000000
8	0.001294	00000002	000F8000	00000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.901095	00000000	00080000	00000000	00000000	00000000
2	0.038928	00000002	00020000	00000000	00000000	00000000
3	0.036531	00000002	000A0000	00000000	00000000	00000000
4	0.007964	00000002	200E0000	00000000	00000000	00000000
5	0.001922	00000002	000B8000	00000000	00000000	00000000
6	0.001853	00000000	80380000	00000000	00000000	00000000
7	0.001030	00000002	000E0000	00000000	00000000	00000000
8	0.000676	00001F00	00000000	00000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.894478	00000000	00080000	00000000	00000000	00000000
2	0.038642	00000002	00020000	00000000	00000000	00000000
3	0.036262	00000002	000A0000	00000000	00000000	00000000
4	0.007941	00201F02	803F8000	00000000	00000000	00000000
5	0.007347	00000002	000E0000	00000000	00000000	00000000
6	0.001908	00000002	000B8000	00000000	00000000	00000000
7	0.001840	00000000	80380000	00000000	00000000	00000000
8	0.001581	00000002	200E0000	00000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.889056	00000000	00080000	00000000	00000000	00000000
2	0.038408	00000002	00020000	00000000	00000000	00000000
3	0.037560	00000002	000A0000	00000000	00000000	00000000
4	0.007952	00000002	A03E0000	00000000	00000000	00000000
5	0.007303	00000002	000E0000	00000000	00000000	00000000
6	0.006314	00000002	803E0000	00000000	00000000	00000000
7	0.001828	00000000	80380000	00000000	00000000	00000000
8	0.001579	00201F02	803F8000	00000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.883984	00000000	00080000	00000000	00000000	00000000
2	0.054412	00000002	000A0000	00000000	00000000	00000000
3	0.038126	00000002	00020000	00000000	00000000	00000000
4	0.007941	00000006	000B8001	80000000	00000000	00000000
5	0.001579	00000002	A03E0000	00000000	00000000	00000000
6	0.001450	00000002	000E0000	00000000	00000000	00000000
7	0.001254	00000002	803E0000	00000000	00000000	00000000
8	0.001254	00000002	000B8000	00000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.876898	00000000	00080000	00000000	00000000	00000000
2	0.057374	00000002	000A0000	00000000	00000000	00000000
3	0.037821	00000002	00020000	00000000	00000000	00000000
4	0.015814	00000006	000B8001	80000000	00000000	00000000
5	0.000313	00000002	A03E0000	00000000	00000000	00000000

Table A.29: Data group 3 (page 11).

no.	prob.	subset				
6	0.000288	00000002	000E0000	00000000	00000000	00000000
7	0.000249	00000002	803E0000	00000000	00000000	00000000
8	0.001244	00000002	000B8000	00000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.869869	00000000	00080000	00000000	00000000	00000000
2	0.057589	00000002	000A0000	00000000	00000000	00000000
3	0.037518	00000002	00020000	00000000	00000000	00000000
4	0.023623	00000006	000B8001	80000000	00000000	00000000
5	0.000062	00000002	A03E0000	00000000	00000000	00000000
6	0.000057	00000002	000E0000	00000000	00000000	00000000
7	0.000049	00000002	803E0000	00000000	00000000	00000000
8	0.001234	00000002	000B8000	00000000	00000000	00000000

Table A.30: Data group 3 (page 12).

no.	prob.	subset				
0	1.000000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
0	0.200000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.800000	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
0	0.040000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.960000	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.990000	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.990000	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.951553	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
2	0.038447	00006000	40000000	00000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.795784	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
2	0.194216	00006000	40000000	00000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.436018	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000

Table A.31: Data group 4 (page 1).

no.	prob.	subset				
2	0.532066	00006000	40000000	00000000	00000000	00000000
3	0.021916	00207F04	70018002	28000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.386214	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
2	0.471290	00006000	40000000	00000000	00000000	00000000
3	0.019413	00207F04	70018002	28000000	00000000	00000000
4	0.035431	00000005	90318001	B8000000	00000000	00000000
5	0.077652	00000004	10018000	28000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.127757	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
2	0.779496	00006000	40000000	00000000	00000000	00000000
3	0.006422	00207F04	70018002	28000000	00000000	00000000
4	0.011720	00000005	90318001	B8000000	00000000	00000000
5	0.025687	00000004	10018000	28000000	00000000	00000000
6	0.013232	00206002	60060002	00000000	00000000	00000000
7	0.025687	00206000	60000002	00000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.028238	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
2	0.861471	00006000	40000000	00000000	00000000	00000000
3	0.015938	00207F04	70018002	28000000	00000000	00000000
4	0.002591	00000005	90318001	B8000000	00000000	00000000
5	0.038750	00000004	10018000	28000000	00000000	00000000
6	0.002925	00206002	60060002	00000000	00000000	00000000
7	0.040087	00206000	60000002	00000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.724434	00002000	00000000	00000000	00000000	00000000
2	0.181108	00006000	40000000	00000000	00000000	00000000
3	0.057981	00202000	00000002	00000000	00000000	00000000
4	0.008427	00206000	60000002	00000000	00000000	00000000
5	0.008147	00000004	10018000	28000000	00000000	00000000
6	0.005937	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
7	0.003351	00207F04	70018002	28000000	00000000	00000000
8	0.000615	00206002	60060002	00000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.872198	00002000	00000000	00000000	00000000	00000000
2	0.036342	00006000	40000000	00000000	00000000	00000000
3	0.076147	00202000	00000002	00000000	00000000	00000000
4	0.001691	00206000	60000002	00000000	00000000	00000000
5	0.001635	00000004	10018000	28000000	00000000	00000000
6	0.001191	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
7	0.000672	00207F04	70018002	28000000	00000000	00000000
8	0.000123	00206002	60060002	00000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.867171	00002000	00000000	00000000	00000000	00000000
2	0.036132	00006000	40000000	00000000	00000000	00000000

Table A.32: Data group 4 (page 2).

no.	prob.	subset				
3	0.075708	00202000	00000002	00000000	00000000	00000000
4	0.002216	00206000	60000002	00000000	00000000	00000000
5	0.000325	00000004	10018000	28000000	00000000	00000000
6	0.000237	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
7	0.000134	00207F04	70018002	28000000	00000000	00000000
8	0.008077	00206002	60060002	00000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.860608	00002000	00000000	00000000	00000000	00000000
2	0.035859	00006000	40000000	00000000	00000000	00000000
3	0.075135	00202000	00000002	00000000	00000000	00000000
4	0.002305	00206000	60000002	00000000	00000000	00000000
5	0.000065	00000004	10018000	28000000	00000000	00000000
6	0.000047	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
7	0.000027	00207F04	70018002	28000000	00000000	00000000
8	0.015955	00206002	60060002	00000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.853741	00002000	00000000	00000000	00000000	00000000
2	0.035573	00006000	40000000	00000000	00000000	00000000
3	0.074536	00202000	00000002	00000000	00000000	00000000
4	0.014949	00206000	60000002	00000000	00000000	00000000
5	0.000064	00000004	10018000	28000000	00000000	00000000
6	0.000009	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
7	0.007962	00207F04	70018002	28000000	00000000	00000000
8	0.003166	00206002	60060002	00000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.906101	00002000	00000000	00000000	00000000	00000000
2	0.063922	00006000	40000000	00000000	00000000	00000000
3	0.014789	00202000	00000002	00000000	00000000	00000000
4	0.002966	00206000	60000002	00000000	00000000	00000000
5	0.000013	00000004	10018000	28000000	00000000	00000000
6	0.000002	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
7	0.001580	00207F04	70018002	28000000	00000000	00000000
8	0.000628	00206002	60060002	00000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.910585	00002000	00000000	00000000	00000000	00000000
2	0.062770	00006000	00000000	00000000	00000000	00000000
3	0.012682	00006000	40000000	00000000	00000000	00000000
4	0.002934	00202000	00000002	00000000	00000000	00000000
5	0.000588	00206000	60000002	00000000	00000000	00000000
6	0.000313	00207F04	70018002	28000000	00000000	00000000
7	0.000125	00206002	60060002	00000000	00000000	00000000
8	0.000003	00000004	10018000	28000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.963165	00002000	00000000	00000000	00000000	00000000
2	0.012453	00006000	00000000	00000000	00000000	00000000

Table A.33: Data group 4 (page 3).

no.	prob.	subset				
3	0.002516	00006000	40000000	00000000	00000000	00000000
4	0.011661	00202000	00000002	00000000	00000000	00000000
5	0.000117	00206000	60000002	00000000	00000000	00000000
6	0.000062	00207F04	70018002	28000000	00000000	00000000
7	0.000025	00206002	60060002	00000000	00000000	00000000
8	0.000000	00000004	10018000	28000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.967325	00002000	00000000	00000000	00000000	00000000
2	0.002471	00006000	00000000	00000000	00000000	00000000
3	0.000499	00006000	40000000	00000000	00000000	00000000
4	0.019665	00202000	00000002	00000000	00000000	00000000
5	0.000023	00206000	60000002	00000000	00000000	00000000
6	0.000012	00207F04	70018002	28000000	00000000	00000000
7	0.000005	00206002	60060002	00000000	00000000	00000000
8	0.000000	00000004	10018000	28000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.929759	00002000	00000000	00000000	00000000	00000000
2	0.018901	00202000	00000002	00000000	00000000	00000000
3	0.038447	00000001	00000001	90000000	00000000	00000000
4	0.002375	00006000	00000000	00000000	00000000	00000000
5	0.000480	00006000	40000000	00000000	00000000	00000000
6	0.000022	00206000	60000002	00000000	00000000	00000000
7	0.000012	00207F04	70018002	28000000	00000000	00000000
8	0.000005	00206002	60060002	00000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.951634	00002000	00000000	00000000	00000000	00000000
2	0.019346	00202000	00000002	00000000	00000000	00000000
3	0.007870	00000001	00000001	90000000	00000000	00000000
4	0.002431	00006000	00000000	00000000	00000000	00000000
5	0.000491	00006000	40000000	00000000	00000000	00000000
6	0.000032	00206000	60000002	00000000	00000000	00000000
7	0.000002	00207F04	70018002	28000000	00000000	00000000
8	0.008193	00206002	60060002	00000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.861383	00002000	00000000	00000000	00000000	00000000
2	0.029664	00000002	00020000	00000000	00000000	00000000
3	0.028495	00000000	00000001	80000000	00000000	00000000
4	0.017511	00202000	00000002	00000000	00000000	00000000
5	0.036206	00000006	000B8001	80000000	00000000	00000000
6	0.007416	00206002	60060002	00000000	00000000	00000000
7	0.007124	00000001	00000001	90000000	00000000	00000000
8	0.002200	00006000	00000000	00000000	00000000	00000000
0	0.011731	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.828253	00002000	00000000	00000000	00000000	00000000
2	0.028523	00000002	00020000	00000000	00000000	00000000

Table A.34: Data group 4 (page 4).

no.	prob.	subset				
3	0.027399	00000000	00000001	80000000	00000000	00000000
4	0.016838	00202000	00000002	00000000	00000000	00000000
5	0.038462	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
6	0.034814	00000006	000B8001	80000000	00000000	00000000
7	0.007131	00206002	60060002	00000000	00000000	00000000
8	0.006850	00000001	00000001	90000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.689608	00002000	00000000	00000000	00000000	00000000
2	0.023749	00000002	00020000	00000000	00000000	00000000
3	0.022813	00000000	00000001	80000000	00000000	00000000
4	0.014019	00202000	00000002	00000000	00000000	00000000
5	0.199185	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
6	0.028986	00000006	000B8001	80000000	00000000	00000000
7	0.005937	00206002	60060002	00000000	00000000	00000000
8	0.005703	00000001	00000001	90000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.495166	00002000	00000000	00000000	00000000	00000000
2	0.102315	00000002	00020000	00000000	00000000	00000000
3	0.098283	00000000	00000001	80000000	00000000	00000000
4	0.010066	00202000	00000002	00000000	00000000	00000000
5	0.143023	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
6	0.132788	00000006	000B8001	80000000	00000000	00000000
7	0.004263	00206002	60060002	00000000	00000000	00000000
8	0.004095	00000001	00000001	90000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.611485	00002000	00000000	00000000	00000000	00000000
2	0.257534	00000002	00020000	00000000	00000000	00000000
3	0.024274	00000000	00000001	80000000	00000000	00000000
4	0.012431	00202000	00000002	00000000	00000000	00000000
5	0.035324	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
6	0.032796	00000006	000B8001	80000000	00000000	00000000
7	0.015144	00206002	60060002	00000000	00000000	00000000
8	0.001011	00000001	00000001	90000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.516838	00002000	00000000	00000000	00000000	00000000
2	0.217672	00000002	00020000	00000000	00000000	00000000
3	0.020517	00000000	00000001	80000000	00000000	00000000
4	0.010507	00202000	00000002	00000000	00000000	00000000
5	0.183091	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
6	0.027720	00000006	000B8001	80000000	00000000	00000000
7	0.012800	00206002	60060002	00000000	00000000	00000000
8	0.000855	00000001	00000001	90000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.290331	00002000	00000000	00000000	00000000	00000000
2	0.122276	00000002	00020000	00000000	00000000	00000000

Table A.35: Data group 4 (page 5).

no.	prob.	subset				
3	0.011525	00000000	00000001	80000000	00000000	00000000
4	0.005902	00202000	00000002	00000000	00000000	00000000
5	0.536723	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
6	0.015572	00000006	000B8001	80000000	00000000	00000000
7	0.007190	00206002	60060002	00000000	00000000	00000000
8	0.000480	00000001	00000001	90000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.252061	00002000	00000000	00000000	00000000	00000000
2	0.106159	00000002	00020000	00000000	00000000	00000000
3	0.104107	00000000	00000001	80000000	00000000	00000000
4	0.005124	00202000	00000002	00000000	00000000	00000000
5	0.465976	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
6	0.013519	00000006	000B8001	80000000	00000000	00000000
7	0.006242	00206002	60060002	00000000	00000000	00000000
8	0.036812	00000001	00000001	90000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.598789	00002000	00000000	00000000	00000000	00000000
2	0.050437	00000002	00020000	00000000	00000000	00000000
3	0.049462	00000000	00000001	80000000	00000000	00000000
4	0.043041	00202000	00000002	00000000	00000000	00000000
5	0.221391	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
6	0.006423	00000006	000B8001	80000000	00000000	00000000
7	0.002966	00206002	60060002	00000000	00000000	00000000
8	0.017490	00000001	00000001	90000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.821240	00002000	00000000	00000000	00000000	00000000
2	0.013835	00000002	00020000	00000000	00000000	00000000
3	0.013568	00000000	00000001	80000000	00000000	00000000
4	0.073257	00202000	00000002	00000000	00000000	00000000
5	0.060728	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
6	0.001762	00000006	000B8001	80000000	00000000	00000000
7	0.000814	00206002	60060002	00000000	00000000	00000000
8	0.004797	00000001	00000001	90000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.732659	00002000	00000000	00000000	00000000	00000000
2	0.066808	00000000	00000001	80000000	00000000	00000000
3	0.065355	00202000	00000002	00000000	00000000	00000000
4	0.054177	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
5	0.012343	00000002	00020000	00000000	00000000	00000000
6	0.035686	00000001	00300001	90000000	00000000	00000000
7	0.021400	00000001	00000001	90000000	00000000	00000000
8	0.001572	00000006	000B8001	80000000	00000000	00000000
0	0.011127	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.704480	00002000	00000000	00000000	00000000	00000000
2	0.064238	00000000	00000001	80000000	00000000	00000000

Table A.36: Data group 4 (page 6).

no.	prob.	subset				
3	0.062842	00202000	00000002	00000000	00000000	00000000
4	0.052094	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
5	0.011868	00000002	00020000	00000000	00000000	00000000
6	0.038462	00000000	00000000	68000000	00000000	00000000
7	0.034313	00000001	00300001	90000000	00000000	00000000
8	0.020577	00000001	00000001	90000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.913276	00002000	00000000	00000000	00000000	00000000
2	0.015546	00000000	00000001	80000000	00000000	00000000
3	0.010771	00006000	00000000	00000000	00000000	00000000
4	0.015208	00202000	00000002	00000000	00000000	00000000
5	0.012607	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
6	0.009308	00000000	00000000	68000000	00000000	00000000
7	0.008304	00000001	00300001	90000000	00000000	00000000
8	0.004980	00000001	00000001	90000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.957947	00002000	00000000	00000000	00000000	00000000
2	0.011149	00006000	00000000	00000000	00000000	00000000
3	0.008281	00006000	40000000	00000000	00000000	00000000
4	0.003218	00000000	00000001	80000000	00000000	00000000
5	0.003148	00202000	00000002	00000000	00000000	00000000
6	0.002610	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
7	0.001927	00000000	00000000	68000000	00000000	00000000
8	0.001719	00000001	00300001	90000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.903422	00002000	00000000	00000000	00000000	00000000
2	0.010515	00006000	00000000	00000000	00000000	00000000
3	0.007810	00006000	40000000	00000000	00000000	00000000
4	0.015176	00000000	00000001	80000000	00000000	00000000
5	0.002969	00202000	00000002	00000000	00000000	00000000
6	0.002461	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
7	0.001817	00000000	00000000	68000000	00000000	00000000
8	0.045830	00000001	00300001	90000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.703021	00002000	00000000	00000000	00000000	00000000
2	0.173781	00000001	00000001	90000000	00000000	00000000
3	0.059049	00000000	00000001	80000000	00000000	00000000
4	0.035664	00000001	00300001	90000000	00000000	00000000
5	0.008182	00006000	00000000	00000000	00000000	00000000
6	0.006077	00006000	40000000	00000000	00000000	00000000
7	0.002311	00202000	00000002	00000000	00000000	00000000
8	0.001915	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.892727	00002000	00000000	00000000	00000000	00000000
2	0.044019	00000001	00000001	90000000	00000000	00000000

Table A.37: Data group 4 (page 7).

no.	prob.	subset				
3	0.014957	00000000	00000001	80000000	00000000	00000000
4	0.009034	00000001	00300001	90000000	00000000	00000000
5	0.026653	00006000	00000000	00000000	00000000	00000000
6	0.001539	00006000	40000000	00000000	00000000	00000000
7	0.000585	00202000	00000002	00000000	00000000	00000000
8	0.000485	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
0	0.016496	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.680411	00002000	00000000	00000000	00000000	00000000
2	0.134201	00000000	00000001	10000000	00000000	00000000
3	0.045600	00000000	00000001	00000000	00000000	00000000
4	0.020314	00006000	00000000	00000000	00000000	00000000
5	0.011400	00000000	00000001	80000000	00000000	00000000
6	0.027541	00000000	00100001	10000000	00000000	00000000
7	0.033550	00000001	00000001	90000000	00000000	00000000
8	0.030487	00000900	00100001	58000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.876867	00002000	00000000	00000000	00000000	00000000
2	0.011753	00000000	00000001	00000000	00000000	00000000
3	0.034590	00000000	00000001	10000000	00000000	00000000
4	0.026179	00006000	00000000	00000000	00000000	00000000
5	0.017007	00206002	60060002	00000000	00000000	00000000
6	0.008647	00000001	00000001	90000000	00000000	00000000
7	0.007858	00000900	00100001	58000000	00000000	00000000
8	0.007099	00000000	00100001	10000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.921506	00002000	00000000	00000000	00000000	00000000
2	0.002470	00000000	00000001	00000000	00000000	00000000
3	0.007270	00000000	00000001	10000000	00000000	00000000
4	0.050218	00006000	00000000	00000000	00000000	00000000
5	0.003575	00206002	60060002	00000000	00000000	00000000
6	0.001818	00000001	00000001	90000000	00000000	00000000
7	0.001652	00000900	00100001	58000000	00000000	00000000
8	0.001492	00000000	00100001	10000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.837882	00002000	00000000	00000000	00000000	00000000
2	0.011231	00000000	00000001	00000000	00000000	00000000
3	0.044485	00000000	00000001	10000000	00000000	00000000
4	0.045660	00006000	00000000	00000000	00000000	00000000
5	0.003250	00206002	60060002	00000000	00000000	00000000
6	0.044633	00000001	00000001	90000000	00000000	00000000
7	0.001502	00000900	00100001	58000000	00000000	00000000
8	0.001357	00000000	00100001	10000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.906131	00002000	00000000	00000000	00000000	00000000
2	0.002429	00000000	00000001	00000000	00000000	00000000

Table A.38: Data group 4 (page 8).

no.	prob.	subset				
3	0.009622	00000000	00000001	10000000	00000000	00000000
4	0.060843	00006000	00000000	00000000	00000000	00000000
5	0.000703	00206002	60060002	00000000	00000000	00000000
6	0.009654	00000001	00000001	90000000	00000000	00000000
7	0.000325	00000900	00100001	58000000	00000000	00000000
8	0.000293	00000000	00100001	10000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.871190	00002000	00000000	00000000	00000000	00000000
2	0.058497	00006000	00000000	00000000	00000000	00000000
3	0.038458	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
4	0.009281	00000001	00000001	90000000	00000000	00000000
5	0.009251	00000000	00000001	10000000	00000000	00000000
6	0.002335	00000000	00000001	00000000	00000000	00000000
7	0.000676	00206002	60060002	00000000	00000000	00000000
8	0.000312	00000900	00100001	58000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.728549	00002000	00000000	00000000	00000000	00000000
2	0.048919	00006000	00000000	00000000	00000000	00000000
3	0.194255	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
4	0.007762	00000001	00000001	90000000	00000000	00000000
5	0.007736	00000000	00000001	10000000	00000000	00000000
6	0.001953	00000000	00000001	00000000	00000000	00000000
7	0.000565	00206002	60060002	00000000	00000000	00000000
8	0.000261	00000900	00100001	58000000	00000000	00000000
0	0.010400	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.699825	00002000	00000000	00000000	00000000	00000000
2	0.186596	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
3	0.046991	00006000	00000000	00000000	00000000	00000000
4	0.038423	00000000	00000000	68000000	00000000	00000000
5	0.007456	00000001	00000001	90000000	00000000	00000000
6	0.007431	00000000	00000001	10000000	00000000	00000000
7	0.001876	00000000	00000001	00000000	00000000	00000000
8	0.001003	00000000	00000000	48000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.908589	00002000	00000000	00000000	00000000	00000000
2	0.011580	00006000	00000000	00000000	00000000	00000000
3	0.010251	00202000	00000002	00000000	00000000	00000000
4	0.045982	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
5	0.009468	00000000	00000000	68000000	00000000	00000000
6	0.001837	00000001	00000001	90000000	00000000	00000000
7	0.001831	00000000	00000001	10000000	00000000	00000000
8	0.000462	00000000	00000001	00000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.740985	00002000	00000000	00000000	00000000	00000000
2	0.009444	00006000	00000000	00000000	00000000	00000000

Table A.39: Data group 4 (page 9).

no.	prob.	subset				
3	0.008360	00202000	00000002	00000000	00000000	00000000
4	0.220121	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
5	0.007722	00000000	00000000	68000000	00000000	00000000
6	0.001498	00000001	00000001	90000000	00000000	00000000
7	0.001493	00000000	00000001	10000000	00000000	00000000
8	0.000377	00000000	00000001	00000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.383974	00002000	00000000	00000000	00000000	00000000
2	0.004894	00006000	00000000	00000000	00000000	00000000
3	0.004332	00202000	00000002	00000000	00000000	00000000
4	0.591053	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
5	0.004001	00000000	00000000	68000000	00000000	00000000
6	0.000776	00000001	00000001	90000000	00000000	00000000
7	0.000774	00000000	00000001	10000000	00000000	00000000
8	0.000195	00000000	00000001	00000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.568208	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
2	0.369132	00002000	00000000	00000000	00000000	00000000
3	0.038454	00000000	80380000	00000000	00000000	00000000
4	0.004704	00006000	00000000	00000000	00000000	00000000
5	0.004165	00202000	00000002	00000000	00000000	00000000
6	0.003847	00000000	00000000	68000000	00000000	00000000
7	0.000746	00000001	00000001	90000000	00000000	00000000
8	0.000744	00000000	00000001	10000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.542998	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
2	0.352755	00002000	00000000	00000000	00000000	00000000
3	0.036748	00000000	80380000	00000000	00000000	00000000
4	0.004496	00006000	00000000	00000000	00000000	00000000
5	0.003980	00202000	00000002	00000000	00000000	00000000
6	0.003676	00000000	00000000	68000000	00000000	00000000
7	0.041792	00000001	00000001	90000000	00000000	00000000
8	0.003555	00000000	00000001	10000000	00000000	00000000
0	0.013358	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.392600	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
2	0.255050	00002000	00000000	00000000	00000000	00000000
3	0.151083	00000001	00000001	90000000	00000000	00000000
4	0.106278	00000000	80300000	00000000	00000000	00000000
5	0.012851	00000000	00000001	10000000	00000000	00000000
6	0.013289	00000000	00000000	68000000	00000000	00000000
7	0.028921	00000301	80300001	F8000000	00000000	00000000
8	0.026570	00000000	80380000	00000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.214560	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
2	0.139388	00002000	00000000	00000000	00000000	00000000

Table A.40: Data group 4 (page 10).

no.	prob.	subset				
3	0.505265	00000001	00000001	90000000	00000000	00000000
4	0.058082	00000000	80300000	00000000	00000000	00000000
5	0.035116	00000000	00000001	10000000	00000000	00000000
6	0.007263	00000000	00000000	68000000	00000000	00000000
7	0.015806	00000301	80300001	F8000000	00000000	00000000
8	0.014521	00000000	80380000	00000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.059430	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
2	0.038608	00002000	00000000	00000000	00000000	00000000
3	0.699753	00000001	00000001	90000000	00000000	00000000
4	0.096527	00000000	80300000	00000000	00000000	00000000
5	0.048633	00000000	00000001	10000000	00000000	00000000
6	0.010058	00000000	00000000	68000000	00000000	00000000
7	0.032969	00000301	80300001	F8000000	00000000	00000000
8	0.004022	00000000	80380000	00000000	00000000	00000000
0	0.012381	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.759306	00000001	00000001	90000000	00000000	00000000
2	0.108233	00000000	80300000	00000000	00000000	00000000
3	0.052771	00000000	00000001	10000000	00000000	00000000
4	0.028620	00000001	80300001	B8000000	00000000	00000000
5	0.012898	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
6	0.008732	00000000	00000000	28000000	00000000	00000000
7	0.008681	00000005	90318001	B8000000	00000000	00000000
8	0.008379	00002000	00000000	00000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.677276	00000000	00000001	90000000	00000000	00000000
2	0.169319	00000001	00000001	90000000	00000000	00000000
3	0.058838	00000000	00000001	10000000	00000000	00000000
4	0.024135	00000000	80300000	00000000	00000000	00000000
5	0.033271	00000000	00000001	B8000000	00000000	00000000
6	0.011043	00001F00	00000001	B8000000	00000000	00000000
7	0.009735	00000000	00000000	28000000	00000000	00000000
8	0.006382	00000001	80300001	B8000000	00000000	00000000

Table A.41: Data group 4 (page 11).

no.	prob.	subset				
0	1.000000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
0	0.200000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.800000	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
0	0.040000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.960000	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.990000	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.990000	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.951553	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
2	0.038447	00000000	00000000	00400000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.795784	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
2	0.194216	00000000	00000000	00400000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.436018	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
2	0.553982	00000000	00000000	00400000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.132984	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
2	0.857016	00000000	00000000	00400000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.127820	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
2	0.823733	00000000	00000000	00400000	00000000	00000000
3	0.038447	00000001	00000001	90000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.106896	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
2	0.688888	00000000	00000000	00400000	00000000	00000000
3	0.160764	00000001	00000001	90000000	00000000	00000000
4	0.033452	00000001	00300001	90000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.027955	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
2	0.911253	00000000	00000000	00400000	00000000	00000000
3	0.042043	00000001	00000001	90000000	00000000	00000000
4	0.008748	00000001	00300001	90000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.005920	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
2	0.973324	00000000	00000000	00400000	00000000	00000000
3	0.008903	00000001	00000001	90000000	00000000	00000000
4	0.001853	00000001	00300001	90000000	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.001190	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
2	0.986647	00000000	00000000	00400000	00000000	00000000
3	0.001790	00000001	00000001	90000000	00000000	00000000
4	0.000373	00000001	00300001	90000000	00000000	00000000

Table A.42: Data group 5.

no.	prob.	subset				
0	1.000000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
0	0.200000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.800000	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
0	0.040000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.960000	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.990000	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000

Table A.43: Data group 6.

no.	prob.	subset				
0	1.000000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
0	0.400000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.600000	00000000	00000000	00100008	00084080	00000000
0	0.080000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.600000	00000000	00000000	00100008	00084080	00000000
2	0.320000	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
0	0.032000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.840000	00000000	00000000	00100008	00084080	00000000
2	0.128000	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.836957	00000000	00000000	00100008	00084080	00000000
2	0.153043	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.830247	00000000	00000000	00100008	00084080	00000000
2	0.159753	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.926484	00000000	00000000	00100008	00084080	00000000
2	0.063516	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.919057	00000000	00000000	00100008	00084080	00000000
2	0.070943	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF

Table A.44: Data group 7 (page 1).

no.	prob.	subset				
1	0.692561	00000000	00000000	00100008	00084080	00000000
2	0.053459	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
3	0.243980	00000000	00000000	00000000	00000000	00200000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.308879	00000000	00000000	00100008	00084080	00000000
2	0.023843	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
3	0.657279	00000000	00000000	00000000	00000000	00200000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.541680	00000000	00000000	00100008	00084080	00000000
2	0.015693	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
3	0.432627	00000000	00000000	00000000	00000000	00200000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.189943	00000000	00000000	00100008	00084080	00000000
2	0.005503	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
3	0.794554	00000000	00000000	00000000	00000000	00200000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.044452	00000000	00000000	00100008	00084080	00000000
2	0.001288	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
3	0.944260	00000000	00000000	00000000	00000000	00200000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.009145	00000000	00000000	00100008	00084080	00000000
2	0.000265	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
3	0.980590	00000000	00000000	00000000	00000000	00200000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.001828	00000000	00000000	00100008	00084080	00000000
2	0.000053	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
3	0.988119	00000000	00000000	00000000	00000000	00200000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.000363	00000000	00000000	00100008	00084080	00000000
2	0.000011	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
3	0.989626	00000000	00000000	00000000	00000000	00200000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.000072	00000000	00000000	00100008	00084080	00000000
2	0.000002	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
3	0.989926	00000000	00000000	00000000	00000000	00200000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.000014	00000000	00000000	00100008	00084080	00000000
2	0.000000	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
3	0.989985	00000000	00000000	00000000	00000000	00200000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.000003	00000000	00000000	00100008	00084080	00000000
2	0.000000	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
3	0.989997	00000000	00000000	00000000	00000000	00200000

Table A.45: Data group 7 (page 2).

no.	prob.	subset				
0	1.000000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
0	0.200000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.800000	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
0	0.080000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.320000	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
2	0.600000	00000000	00000000	00100008	00084080	00000000
0	0.016000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.384000	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
2	0.600000	00000000	00000000	00100008	00084080	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.153043	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
2	0.836957	00000000	00000000	00100008	00084080	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.159753	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
2	0.830247	00000000	00000000	00100008	00084080	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.063516	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
2	0.926484	00000000	00000000	00100008	00084080	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.070943	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
2	0.919057	00000000	00000000	00100008	00084080	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.053459	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
2	0.692561	00000000	00000000	00100008	00084080	00000000
3	0.243980	00000000	00000000	00000003	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.024917	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
2	0.851365	00000000	00000000	00100008	00084080	00000000
3	0.113718	00000000	00000000	00000003	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.015568	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
2	0.531922	00000000	00000000	00100008	00084080	00000000
3	0.442510	00000000	00000000	00000003	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.005385	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
2	0.183978	00000000	00000000	00100008	00084080	00000000
3	0.800637	00000000	00000000	00000003	00000000	00000000

Table A.46: Data group 8.

no.	prob.	subset				
0	1.000000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
0	0.400000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.600000	00000000	00000000	00100008	00084080	00000000
0	0.080000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.600000	00000000	00000000	00100008	00084080	00000000
2	0.320000	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
0	0.032000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.840000	00000000	00000000	00100008	00084080	00000000
2	0.128000	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.836957	00000000	00000000	00100008	00084080	00000000
2	0.153043	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.929151	00000000	00000000	00100008	00084080	00000000
2	0.060849	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.921703	00000000	00000000	00100008	00084080	00000000
2	0.068297	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.914315	00000000	00000000	00100008	00084080	00000000
2	0.075685	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000

Table A.47: Data group 9.

no.	prob.	subset				
0	1.000000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
0	0.200000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.800000	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
0	0.080000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.320000	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
2	0.600000	00000000	00000000	00100008	00084080	00000000
0	0.016000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.384000	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
2	0.600000	00000000	00000000	00100008	00084080	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.153043	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
2	0.836957	00000000	00000000	00100008	00084080	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.159753	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
2	0.830247	00000000	00000000	00100008	00084080	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.063516	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
2	0.926484	00000000	00000000	00100008	00084080	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.070943	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
2	0.919057	00000000	00000000	00100008	00084080	00000000

Table A.48: Data group 10.

no.	prob.	subset				
0	1.000000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
0	0.400000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.600000	00000000	00000000	00100008	00084080	00000000
0	0.080000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.600000	00000000	00000000	00100008	00084080	00000000
2	0.320000	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
0	0.032000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.840000	00000000	00000000	00100008	00084080	00000000
2	0.128000	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.836957	00000000	00000000	00100008	00084080	00000000
2	0.153043	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.929151	00000000	00000000	00100008	00084080	00000000
2	0.060849	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.965807	00000000	00000000	00100008	00084080	00000000
2	0.024193	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.958065	00000000	00000000	00100008	00084080	00000000
2	0.031935	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000

Table A.49: Data group 11.

no.	prob.	subset				
0	1.000000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
0	0.200000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.800000	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
0	0.080000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.320000	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
2	0.600000	00000000	00000000	00100008	00084080	00000000
0	0.016000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.384000	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
2	0.600000	00000000	00000000	00100008	00084080	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.153043	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
2	0.836957	00000000	00000000	00100008	00084080	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.159753	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
2	0.830247	00000000	00000000	00100008	00084080	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.063516	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
2	0.926484	00000000	00000000	00100008	00084080	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.070943	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
2	0.919057	00000000	00000000	00100008	00084080	00000000

Table A.50: Data group 12.

no.	prob.	subset				
0	1.000000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
0	0.200000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.800000	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
0	0.080000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.320000	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
2	0.600000	00000000	00000000	00100008	00084080	00000000
0	0.030769	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.123077	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
2	0.230769	00000000	00000000	00100008	00084080	00000000
3	0.615385	00000000	00000000	00000003	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.147121	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
2	0.229876	00000000	00000000	00100008	00084080	00000000
3	0.613003	00000000	00000000	00000003	00000000	00000000

Table A.51: Data group 13.

no.	prob.	subset				
0	1.000000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
0	0.200000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.800000	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
0	0.080000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.320000	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
2	0.600000	00000000	00000000	00100008	00084080	00000000
0	0.016000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.384000	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
2	0.600000	00000000	00000000	00100008	00084080	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.153043	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
2	0.836957	00000000	00000000	00100008	00084080	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.159753	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
2	0.830247	00000000	00000000	00100008	00084080	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.063516	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
2	0.926484	00000000	00000000	00100008	00084080	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.070943	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
2	0.919057	00000000	00000000	00100008	00084080	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.028206	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
2	0.961794	00000000	00000000	00100008	00084080	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.024434	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
2	0.833178	00000000	00000000	00100008	00084080	00000000
3	0.132388	00000000	00000000	00000003	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.014596	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
2	0.497708	00000000	00000000	00100008	00084080	00000000
3	0.477696	00000000	00000000	00000003	00000000	00000000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.004818	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
2	0.164289	00000000	00000000	00100008	00084080	00000000
3	0.820893	00000000	00000000	00000003	00000000	00000000

Table A.52: Data group 14.

no.	prob.	subset				
0	1.000000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
0	0.400000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.600000	00000000	00000000	00100008	00084080	00000000
0	0.080000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.600000	00000000	00000000	00100008	00084080	00000000
2	0.320000	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
0	0.032000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.840000	00000000	00000000	00100008	00084080	00000000
2	0.128000	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.836957	00000000	00000000	00100008	00084080	00000000
2	0.153043	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.830247	00000000	00000000	00100008	00084080	00000000
2	0.159753	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.926484	00000000	00000000	00100008	00084080	00000000
2	0.063516	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000
0	0.010000	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF	FFFFFFFF
1	0.964747	00000000	00000000	00100008	00084080	00000000
2	0.025253	00000000	00000000	013FFFFF	FFFFFFFE	22FC0000

Table A.53: Data group 15.

Bibliography

[1] N. WAX, "Signal-to-Noise Improvement and the Statistics of Tracking Populations", *Journal of Applied Physics*, Vol. 26, pp. 586–595, May 1955.

[2] R. W. SITTLER, "An Optimal Data Association Problem in Surveillance Theory", *IEEE Transactions on Millitary Electronics*, MIL-8, pp. 125–139, Apr 1964.

[3] A. J. JAFFER AND Y. BAR-SHALOM, "On Optimal Tracking in Multiple Target Environments", *Proceedings of the Third Symposium on Non-Linear Estimation Theory and Its Applications, San Diego, CA*, MIL-8, pp. 112–117, Sept. 1972.

[4] Y. BAR-SHALOM AND E. TSE, "Tracking in a Cluttered Environment with Probabilistic Data Association", *Automatica*, Vol. 11, pp. 451–460, Sept. 1975.

[5] R. A. SINGER AND J. J. STEIN, "An Optimal Tracking Filter for Surveillance Systems", *Proceedings of the 1971 IEEE Conference on Decision and Control, Miami Beach, FL*, pp. 171–175, Dec. 1971.

[6] R. A. SINGER, R. G. SEA AND K. B. HOUSEWRIGHT, "Derivation and Evaluation of Improved Tracking Filters for Use in Dense Multi-Target Environments ", *IEEE Transactions on Information Theory*, IT-20, pp. 423–432, Jul. 1974.

[7] Y. Bar-Shalom, *Multi-Target Multi-Sensor Tracking: Applications and Advances*, Artech House, Boston, MA, 1992.

[8] S. S. Blackman, *Multi-Target Tracking with Radar Application*, Artech House, Boston, MA, 1986.

[9] V. D. BLONDEL AND J. N. TSTITSIKLIS, "A Survey of Computational Complexity Results in Systems and Control", *Automatica*, Vol. 36, No. 4, pp. 1249–1274, September 2000.

[10] L. EL GHAOUI AND H. LEBRET, "Robust Solution to Least-Squares Problems with Uncertain data matrices", *SIAM J. Matrix Anal. Appl.*, Vol. 18, No. 4, pp. 1035–1064, 1997.

[11] M. FU, C. E. DE SOUZA AND Z.-Q. LUO, "Finite Horizon Robust Kalman Filter Design", Accepted for publication in *IEEE Trans. Signal Processing.*

[12] M. Green and D. J. N. Limebeer, *Linear Robust Control*, Prentice Hall, Englewood Cliffs, NJ, 1995.

[13] B. HASSIBI, A.H. SAYED AND T. KAILATH, "Linear Estimation in Krein Spaces-Part I: Theory & Part II: Applications", *IEEE Trans. Automatic Control*, Vol. 41, No. 1, pp. 18–49, January 1996.

[14] S. HAYKIN, *Adaptive Filter Theory*, third edition, Prentice-Hall, Englewood Cliffs, NJ, 1996.

[15] D. HINRICHSEN AND A. J. PRITCHARD, "Robustness Measures for Linear Systems with Application to Stability Radii of Hurwitz and Schur Polynomials", *Int. J. Control*, Vol. 55, No. 4, pp. 809–844, April 1992.

[16] T. KAILATH, A. H. SAYED, AND B. HASSIBI, *Linear Estimation*, Prentice-Hall, Upper Saddle River, NJ, 2000.

[17] H. LI AND M. FU, "A Linear Matrix Inequality Approach to Robust H_∞ Filtering", *IEEE Trans. Signal Processing*, Vol. 45, No. 9, pp. 2238–2250, September 1997.

[18] Z.-Q. LUO, J. F. STURM AND S. ZHANG, "Conic Convex Programming and Self-dual Embedding", Accepted for publication in *Optimization Methods and Software.*

[19] A. H. SAYED AND T. KAILATH, "A State-Space Approach to Adaptive RLS Filtering", *IEEE Signal Processing Magazine*, pp. 18–60, July 1994.

[20] J. F. Sturm, " Using SeDuMi 1.02, a MATLAB Toolbox for Optimization over Symmetric Cones", *Optimization Methods and Software*, Vol. 11–12, pp. 625–653, 1999. Available on the World Wide Web: `http://www.unimaas.nl/~sturm/software/sedumi.html`.

[21] Y. Theodor and U. Shaked, "Robust Discrete-Time Minimum-Variance Filtering", *IEEE Trans. Signal Processing*, Vol. 44, No. 2, pp. 181-189, February 1996.

[22] L. Vandenberghe and S. Boyd, "Semidefinite Programming", *SIAM Review*, Vol. 38, pp. 49-95, 1996.

[23] L. Xie, Y. C. Soh, and C. E. de Souza, "Robust Kalman Filtering for Uncertain Discrete-Time Systems", *IEEE Trans. Automatic Control*, Vol. 39., No. 6, pp. 1310-1314, June 1994.

[24] L. Xie, C. E. de Souza, and M. Fu, "H_∞ estimation for linear discrete-time uncertain systems", *Int. J. Robust Nonlinear Control*, Vol. 1, pp. 111-123, 1991.

[25] K. Zhou, with J. C. Doyle and K. Glover, *Robust and Optimal Control*, Prentice-Hall, Englewood Cliffs, NJ, 1996.

[26] Bar-Shalom, Y., *Tracking And Data Association*, Boston: Academic Press, 1988.

[27] Bar-Shalom, Y., *Multi-target and Multi-sensor Tracking: Advanced Applications*, Boston: Academic Press, 1990.

[28] Bar-Shalom, Y., *Multi-target Multi-sensor Tracking: Applications and Advances*, Boston: Artech house, 1992.

[29] Blackman, S.S., *Multiple-Target Tracking with Radar Applications*, Artech House, Boston, Massachusetts, 1986.

[30] Dempster, A.P., "A Generalization of Bayesian Inference," *Journal of Royal Statistical Society,* Series B, Vol. 30, pp. 205–247, 1968.

[31] REID, D.B., "The Application of Multiple Target Theory to Ocean Surveillance," in *Proceedings of the 1979 IEEE Conference on Decision and Control,* Fort Lauderdale, FL, pp. 1046–1051, 1979.

[32] ROOS, C., TERLAKY, T. AND VIAL, J.-PH., *Theory and Algorithms for Linear Optimization, An Interior Point Approach*, John Wiley& Sons, 1997.

[33] SHAFER, G., *A Mathematical Theory of Evidence,* Princeton, NJ: Princeton University Press, 1976.

[34] YE, Y., *Interior Point Algorithms: Theory and Analysis,* New York: Wiley, 1997.

[35] LI, L., LUO, Z-Q, DAVIDSON, T. N., WONG, K. M. AND BOSSE, E., "Robust Filtering via Semidefinite Programming With Applications to Target Tracking," in *SIAM Journal on Optimization,* Vol. 12, No. 3, pp. 740-755, 2002.

[36] LI, L., LUO, Z-Q, WONG, K. M. AND BOSSE, E., "Convex Optimization Approach to the Fusion of Identity Information," in *Proceedings of the 1999 SPIE Symposium on Sensor Fusion: Architecture, Algorithms, and Applications,* Orlando, FL, Vol. 3719, pp. 389-397, April 1999.

[37] LI, L., LUO, Z-Q, WONG, K. M. AND BOSSE, E., "Convex Optimization Approach to Identity Fusion for Multisensor Target Tracking," in *IEEE Trans. System, Man, and Cybernetics, Part A,* Vol. 31, No. 3, pp. 172-178, May 2001.