AMALGAMS OF L^p AND ℓ^q

By

MARIA LUISA TORRES DE SQUIRE, B.Sc., M.Sc.

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AUTHOR: Maria Luisa Torres de Squire, B. Sc. (Universidad Nacional Autónoma de México)

M. Sc. (McMaster University)

SUPERVISOR: Professor J. Stewart

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ABSTRACT

An amalgam of L^p and ℓ^q is a Banach space $(L^p, \ell^q)(G)$ $(1 \leq p, q \leq \infty)$ of (classes of) functions on a locally compact abelian group G which belong locally to L^p and globally to ℓ^q. Similarly, the space of unbounded measures of type q is a Banach space $M_q(G)$ $(1 \leq q \leq \infty)$ of unbounded measures which belong locally to the space of bounded, regular, Borel measures on G and globally to ℓ^q.

The Fourier transform of funcions in (L^p, ℓ^q) and measures in M_q is defined to be a linear functional on the subspace $A_c(G)$ of the Fourier algebra $A(G)$, and its relation with other known definitions of Fourier transforms is established.

We introduce the space of strong resonance class of functions relative to the test space Φ_q and find its relation with respect to the linear space generated by the positive definite funcions for (L^q, ℓ^1).

We generalize known results for amalgam spaces on the real line to locally compact abelian groups, extend some results in the theory of L^p spaces to amalgams and develop a theory of multipliers for amalgam spaces and spaces of unbounded measures of type q.

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To my mother and sisters

To my husband

To the truly honest

A los honestos de corazón

A mi madre y hermanas

A mi esposo

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INTRODUCTION

Briefly, an amalgam of L^p and ℓ^q is a Banach space $(L^p, \ell^q)(G)$ $(1 \leq p, q \leq \infty)$ of measurable (classes of) functions on a locally compact abelian group G which belong locally to L^p and have ℓ^q behavior at infinity.

Several authors have introduced special cases of amalgams during the last half century. Among others N. Wiener [54], P. Szeptycki [53], T. S. Liu, A. van Rooij and J. K. Wang [42], H. E. Krøgstad [39] and H. G. Feichtinger [23]. (For a historical background of amalgams see [27]).

The first systematic study of amalgams on the real line was undertaken by F. Holland [34] and their generalization to locally compact groups was done independently by J. P. Bertrandias, C. Datry and C. Dupuis [8], J. Stewart [49], and R. C. Busby and H. A. Smith [12]. In Section 1 we give their definitions and prove that they are equivalent.

J. Stewart's definition suits best our needs and we use it throughout the work.

The study of amalgam spaces leads to the study of the Banach space M_q $(1 \leq q \leq \infty)$ of unbounded measures of type q. The particular case M_∞, has been studied by L. Argabright and J. Gil de Lamadrid [1].

In Chapter I we study the properties of duality and reflexivity of (L^p, ℓ^q), density and translation invariance of (L^p, ℓ^q) and M_q, and the product and convolution operations on (L^p, ℓ^q) and M_q.

Chapter II is about the Fourier transform of functions in (L^p, ℓ^q) and measures in M_q.

In Section 5 we study the Fourier transform of functions in (L^p, ℓ^q) for $1 \leq p \leq \infty$, $1 \leq q \leq 2$ following Holland's ideas in [34].

In Section 6 we give W. Bloom's definition of the Fourier transform of functions in (L^p, ℓ^q) for $1 \leq p \leq \infty$, $2 < q \leq \infty$ and extend his definition to measures in M_q $(1 \leq q \leq \infty)$. This approach is different from that of Bertrandias and Dupuis [7], and Feichtinger [25].

We study the relation between the Fourier transform of unbounded measures of type q and the Fourier transform of transformable measures as it was defined by Argabright and Gil de Lamadrid in [1].

The next thing we try to do is to generalize to locally compact abelian groups some of Holland's results that appeared in [34] and [35].

In Section 7 we give Simon's generalization of Cesaro summability for locally compact abelian groups and study some of its properties related to amalgams in order to generalize Theorem 9 of [34].

In Section 8 we introduce the space $SR(\Phi_q)$ $(1 \leq q \leq \infty)$ of functions strongly resonant relative to the test space Φ_q based on Holland's definition of the space $R(\Phi_q)$ $(1 \leq q \leq \infty)$ of functions resonant relative to Φ_q. We prove (Theorem 8.20) that $< P(L^{q'}, \ell^1) >$ is dense in $SR(\Phi_q)$ for $1 \leq q < 2$, $< P(L^2, \ell^1) >$ is equal to $SR(\Phi_2)$ and $SR(\Phi_q)$ is included in $< P(L^{q'}, \ell^1) >$ for $2 < q \leq \infty$, where $< P(L^{q'}, \ell^1) >$ is the linear space generated by the space of positive definite functions for $(L^{q'}, \ell^1)$. Also we give a representation of functions in $SR(\Phi_q)$ $(1 \leq q < 2)$ in terms of the Fourier transform of measures of type q (Theorem 8.25). This representation is similar to the one given by F. Holland in [35].

An important aspect of the theory of amalgam spaces is that the $L^p(G)$ spaces are particular cases of amalgams. This opens immediately the possibility of extending known results in the theory of L^p spaces to amalgams. Chapters IV and V are in this direction.

J. J. F. Fournier [22, Theorems 1 and 2] proved that:

a) If $\hat{G}$ is not discrete and E is not locally null then for $1 < p \leq 2$

$$L^{p\wedge}|E \not\subset \bigcup_{q>p'} L^q(E).$$

b) If $\hat{G}$ is not compact and $1 \leq p \leq 2$ then

$$L^p(G)^\wedge \not\subset \bigcup_{q<p'} L^q(\hat{G}).$$

c) If $\hat{G}$ is neither compact nor discrete and $1 < p \leq 2$ then

$$L^p(G)^\wedge \not\subset \bigcup_{q\neq p'} L^q(\hat{G}).$$

In Chapter IV we prove the following.

A) If $\hat{G}$ is not discrete and E is not locally null then for $1 < p \leq 2$

$$(L^\infty, \ell^p)^\wedge|E \not\subset \bigcup_{q>p'} (L^q, \ell^\infty)(E).$$

B) If $\hat{G}$ is not compact and $1 \leq p \leq 2$ then

$$(L^p, \ell^1)(G)^\wedge \not\subset \bigcup_{q<p'} (L^1, \ell^q)(\hat{G}).$$

C) If $\hat{G}$ is neither compact nor discrete and $1 < p \leq 2$ then

$$L^p(G)^\wedge \not\subset \bigcup_{q\neq p'} (L^q, \ell^\infty) \cap (L^1, \ell^q)$$

Since (L^{∞}, ℓ^{p}) and (L^{p}, ℓ^{1}) are proper subspaces of L^{p} for $1 \leq p \leq \infty$, and L^{q} is a proper subspace of (L^{q}, ℓ^{∞}), (L^{1}, ℓ^{q}) for $1 \leq q \leq \infty$ and $(L^{q}, \ell^{\infty}) \cap (L^{1}, \ell^{q})$ for $1 < q < \infty$, A), B), C) extend a), b), c) respectively.

Chapter V is about the theory of multipliers for amalgam spaces and spaces of unbounded measures of type q.

If A and B are any amalgam space or any spaces of measures M_q then we define $M(A,B)$ to be the linear space of multipliers from A to B (continuous linear operators from A to B which commute with translations).

If A is (L^{p}, ℓ^{1}) $(1 \leq p \leq \infty)$, (C_0, ℓ^{1}) or M_1 and B is as above then $c\text{-}M(A,B)$ is the linear space of convolution multipliers from A to B (continuous linear operators from A to B which commute with convolution). In this case we establish an inclusion relation between $M(A,B)$ and $c\text{-}M(A,B)$. For some A and B we prove that $M(A,B)$ is equal to $c\text{-}M(A,B)$.

We also look for a characterization of the elements T of $M(A,B)$ or $c\text{-}M(A,B)$ of the type:

(1) For all $f \in A$, $Tf = \mu * f$ for a unique μ in some linear space.

(2) For all $f \in A$, $(Tf)^{\wedge} = \varphi\hat{f}$ for a unique φ in some linear space, where $(Tf)^{\wedge}$ and $\hat{f}$ are the Fourier transforms of Tf and f respectively.

If $A = L^{1}(G)$ and B is any amalgam space or any space M_q then the characterizations (1) and (2) of elements of $c\text{-}M(L^{1},B)$ hold and they are equivalent when B is equal to (L^p, ℓ^q) $(1 < p, q \leq \infty)$, (C_0, ℓ^q), (L^p, c_0), (L^q, ℓ^1) $(1 < q < \infty)$, (L^1, ℓ^q) $(1 \leq q < \infty)$ or (C_0, ℓ^1) (Section 13).

If $A = M_1$ and B is any amalgam space or any space M_q then the characterizations (1) and (2) of elements of c-M(A,B) hold and are equivalent (Section 14).

If $B = L^\infty$ and A is equal to (L^p, ℓ^1) $(1 < p < \infty)$ or (C_0, ℓ^1) then the characterizations (1) and (2) hold for elements of $M(A,L^\infty)$; if A is equal to (L^p, ℓ^q) $(1 < p, q < \infty)$, (L^p, c_0) $(1 < p < \infty)$, (L^1, ℓ^q) $(1 < q \leq \infty)$ or (C_0, ℓ^q) $(1 \leq q \leq \infty)$ then (1) holds for the elements of $M(A,L^\infty)$ (Section 15).

Finally in Section 16 we prove (Theorem 16.11) that if A is equal to (L^p, ℓ^1) $(1 \leq p \leq \infty)$ or (C_0, ℓ^1) and B is any amalgam space or any space of unbounded measures of type q then the characterizations (1) and (2) for elements of c-M(A,B) hold and are equivalent.

This implies a characterization for elements of M(A,B) (A, B any amalgam space or any space M_q) similar to (1) and (2). Specifically, A contains an algebra S_A such that for T in M(A,B) there exist unique μ and φ in some linear spaces such that $Tf = \mu * f$ and $(Tf)^\wedge = \varphi\hat{f}$ for all $f \in S_A$ (Theorem 16.18).

CHAPTER I

AMALGAMS OF L^p AND ℓ^q

We begin this chapter by introducing the notation we will use throughout the whole work.

For a locally compact compact group G with Haar measure m, $L^p(G) = L^p$, $1 \leq p < \infty$, will be, as usual, the Banach space of measurable (classes of) functions f such that $\int_G |f(x)|^p \, dm(x)$ is finite and $L^\infty(G) = L^\infty$ will be the Banach space of measurable (classes of) functions which are essentially bounded. For a subset E of G the quantity $\sup_E |f(x)|$ will mean $\operatorname{ess\,sup}_E |f(x)|$.

The integration of a measurable function f on G will be always with respect to m and we might write $\int f$ or $\int f(x)\,dx$ instead of $\int_G f(x)\,dm(x)$.

$C_c(G) = C_c$, $C_0(G) = C_0$ will denote the linear space of continuous functions on G, which have compact support, vanish at infinity, respectively.

The characteristic function of a subset E of G will be denoted by χ_E.

If J is a linear space of functions on G then J_c, J_{loc} will be the set of functions f in J such that f has compact support, f restricted to any compact subset E of G, that is, $f\chi_E = f|E$ belongs to J. For any subset E of G, $J|E = \{f|E \mid f \in J\}$.

By a measure μ on G we will mean a complex-valued set function on G which locally is a complex measure. That is, for any compact subset E of G, $\mu_E(B) = \mu(B\cap E)$ is a complex measure as in [43, Chapter 6].

This is consistent with the functional approach of Bourbaki because the function $\mu(f) = \int f\, d\mu = \int_E f\, d\mu_E$ ($f \in C_c(G)$ with support E) is a continuous linear functional on $C_c(G)$ topologized as the internal inductive limit of the spaces $C_E(G) = \{f \in C_c(G) \mid \text{supp } f \subset E\}$.

$V(G)$ will denote the linear space of (Radon) measures on G.

We will use additive notation for the operation on abelian groups and its identity will be denoted by 0.

If $\hat{G}$ is the dual group of G then for $\hat{x} \in \hat{G}$ we will write $[x,\hat{x}]$ instead of $\hat{x}(x)$ $(x \in G)$.

The difference of two sets A, B will be denoted by $A \sim B$, that is, $A \sim B = \{x \in A \mid x \notin B\}$.

If f is a function on G then f', $\tilde{f}$ will be the functions on G defined by $f'(x) = f(-x)$, $\tilde{f}(x) = \overline{f(-x)}$ respectively.

If μ is a measure on G then $L^p(\mu)$ will always mean $L^p(|\mu|)$.

§ 1. AMALGAMS OF L^p AND ℓ^q AND SPACES OF UNBOUNDED MEASURES OF TYPE q.

Several definitions of amalgams of L^p and ℓ^q as well as of unbounded measures of type q have appeared recently as a consequence of research done in different areas. We shall give here these definitions in chronological order of publication and immediately after we will proceed to prove their equivalence.

First in 1975 F. Holland [34] defined the amalgam spaces (L^p, ℓ^q) and the spaces of unbounded measures M_q for the real line as follows:

DEFINITION 1.1. For $f \in L^p_{loc}(\mathbb{R})$, $1 \leqq p \leqq \infty$, we define

$$|| f ||_{pq} = \left[\sum_{\mathbf{Z}} \left[\int_n^{n+1} | f |^p \right]^{q/p} \right]^{1/q} \qquad \text{if } 1 \leqq p,q < \infty$$

$$|| f ||_{\infty q} = \left[\sum_{\mathbf{Z}} \sup_{[n,n+1]} | f |^q \right]^{1/q} \qquad \text{if } p = \infty,\ 1 \leqq q < \infty$$

$$|| f ||_{p\infty} = \sup_{\mathbf{Z}} \left[\int_n^{n+1} | f |^p \right]^{1/p} \qquad \text{if } 1 \leqq p < \infty,\ q = \infty$$

Then the amalgam of L^p and ℓ^q is the linear space

$$(L^p, \ell^q) = \{f \in L^p_{loc}(\mathbb{R}) \mid || f ||_{pq} < \infty\} \qquad 1 \leqq p,q \leqq \infty.$$

Special cases of (L^p, ℓ^q) have appeared before; see for example [54] and [53].

DEFINITION 1.2. For a measure μ on $\mathbb{R}$, we define

$$||\mu||_q = \left[\sum_{Z} |\mu|\Big([n, n+1]\Big)^q\right]^{1/q} \qquad 1 \leq q < \infty$$

$$||\mu||_\infty = \sup_{Z} |\mu| [n, n+1]$$

Then the space of unbounded measures of type q is the linear space

$$M_q = \{ \mu \in V(\mathbb{R}) \mid ||\mu||_q < \infty \} \qquad 1 \leq q \leq \infty.$$

Later in 1978 J. P. Bertrandias, C. Datry and C. Dupuis [8] defined the spaces $\ell^q(L^p)$, $\ell^q(M)$ as a consequence of an earlier paper due to J. P. Bertrandias [5] about the Riesz spaces $\bigcap^p$ and $\bigcup^q$. Their definition is as follows:

Let G be a locally compact abelian group and E be a nonempty, relatively compact, Borel set of G.

DEFINITION 1.3. A tiling of G by E is a pairwise disjoint family $\{E_i \mid i \in I\} = \{t_i + E \mid t_i \in G, i \in I\}$ of translates of E.

DEFINITION 1.4. Let $\mathcal{P}$ be the set of all tilings of G by E and $1 \leq p,q \leq \infty$. For $f \in L^p_{loc}(G)$, define

$$||f||_{qp} = \sup_{\{E_i\} \in \mathcal{P}} \left[\sum_{i \in I}\left[\int_{E_i} |f|^p\right]^{q/p}\right]^{1/q} \qquad \text{if } 1 \leq q,p < \infty$$

$$||f||_{q\infty} = \sup_{\{E_i\} \in \mathcal{P}} \left[\sum_{i \in I} \sup_{E_i} |f|^q\right]^{1/q} \qquad \text{if } p = \infty,\ 1 \leq q < \infty$$

$$||f||_{\infty p} = \sup_{x \in G}\left[\int_{x-E} |f|^p\right]^{1/p} \qquad \text{if } 1 \leq p < \infty,\ q = \infty$$

Then the amalgam of L^p and ℓ^q is the linear space

$$\ell^q(L^p) = \{f \in L^p_{loc}(G) \mid \|f\|_{qp} < \infty\} \qquad 1 \leq p,q \leq \infty$$

DEFINITION 1.5. For a measure μ on G, define

$$\|\mu\|_{q1} = \sup_{\{E_i\} \in P} \left[\sum_{i \in I} |\mu|(E_i)^q \right]^{1/q} \qquad 1 \leq q < \infty$$

$$\|\mu\|_{\infty 1} = \sup_{x \in G} |\mu|(x - E)$$

Then the space of unbounded measures of type q is the linear space

$$\ell^q(M) = \{ \mu \in V(G) \mid \|\mu\|_{q1} < \infty \} \qquad 1 \leq q \leq \infty.$$

REMARK. Originally the spaces $\ell^q(L^p)$ and $\ell^q(M)$ were defined on locally compact abelian groups, but it is clear that this definition is equally valid for nonabelian groups.

The dependence on the set E is in essence irrelevant because the spaces $\ell^q(L^p)$ and $\ell^q(M)$ defined from two different subsets are isomorphic [8, §7 a)].

In 1979 J. Stewart extended the definition of F. Holland to locally compact abelian groups using the Structure Theorem for locally compact groups [49].

Let G be a locally compact abelian group. By the Structure Theorem [37, Theorem 24.30], G is topologically isomorphic with $\mathbb{R}^a \times G_1$ where a is a nonnegative integer and G_1 is a locally compact abelian group which contains an open compact subgroup H.

If G_1 is compact we can take $H = G_1$, if G is discrete and infinite we can take $H = \{0\}$. Otherwise H is arbitrary but fixed.

The Haar measure m_1 on G_1, is normalized so that $m_1(H) = 1$. We then take the Haar measure $\underline{m}$ on G to be the product of the Lebesgue measure on $\mathbb{R}^a$ and m_1.

DEFINITION 1.6. Define $\underline{L} = [0,1)^a \times H$ and $\underline{K} = [0,1]^a \times H$. Then we can write G as a union of compact sets $G = \bigcup_{\alpha \in J} K_\alpha$ and as a union of relatively compact, Borel sets $G = \bigcup_{\alpha \in J} L_\alpha$ where $J = Z^a \times T$, T being a transversal of H in G_1, that is $G_1 = \bigcup_{g \in T} g + H$, and $\underline{L_\alpha} = \alpha + L$, $\underline{K_\alpha} = \alpha + K$. (Note that $m(L_\alpha) = m(K_\alpha) = 1$ for all $\alpha \in J$).

DEFINITION 1.7. For $f \in L^p_{loc}(G)$ $1 \leq p \leq \infty$ define

$$|| f ||_{pq} = \left[\sum_{\alpha \in J} \left[\int_{K_\alpha} | f |^p \right]^{q/p} \right]^{1/q} \qquad \text{if } 1 \leq p,q < \infty$$

$$|| f ||_{\infty q} = \left[\sum_{\alpha \in J} \sup_{K_\alpha} | f |^q \right]^{1/q} \qquad \text{if } p = \infty,\ 1 \leq q < \infty$$

$$|| f ||_{p\infty} = \sup_{\alpha \in J} \left[\int_{K_\alpha} | f |^p \right]^{1/p} \qquad \text{if } 1 \leq p < \infty,\ q = \infty$$

Then the amalgam of L^p and ℓ^q is the linear space

$$(L^p, \ell^q) = \{ f \in L^p_{loc}(G) \mid || f ||_{pq} < \infty \} \qquad 1 \leq p,q \leq \infty$$

REMARK 1.8. From the definition of the families $\{L_\alpha\}$, $\{K_\alpha\}$ and the nature of the Haar measure m, it is clear that

$$|| f ||_{pq} = \left[\sum_{\alpha \in J} \left[\int_{L_\alpha} | f |^p \right]^{q/p} \right]^{1/q} \qquad \text{if } 1 \leq p,q < \infty$$

$$|| f ||_{\infty q} = \left[\sum_{\alpha \in J} \sup_{L_\alpha} | f |^q \right]^{1/q} \qquad \text{if } p = \infty,\ 1 \leq q < \infty$$

$$\|f\|_{\infty q} = \sup_{\alpha \in J} \left[\int_{L_\alpha} |f|^p \right]^{1/p} \qquad \text{if } 1 \leq p < \infty,\ q = \infty$$

DEFINITION 1.9. For a measure μ on G, define

$$\|\mu\|_q = \left[\sum_{\alpha \in J} |\mu|(K_\alpha)^q \right]^{1/q} \qquad 1 \leq q < \infty$$

$$\|\mu\|_\infty = \sup_{\alpha \in J} |\mu|(K_\alpha)$$

Then the space of unbounded measures of type q is the linear space

$$M_q = \{ \mu \in V(G) \mid \|\mu\|_q < \infty \} \qquad 1 \leq q \leq \infty.$$

REMARK. If $G = \mathbb{R}$ then the family $\{K_\alpha\}$ becomes the collection of intervals $[n, n+1]$ $(n \in \mathbb{Z})$. Hence, in this case, Definition 1.1 and Definition 1.7 coincide.

Finally in 1980 R. C. Busby and H. A. Smith [12] used the space L^π_{pq} to solve problems in compact operator theory. Their definition is for locally compact, not necessarily abelian, groups and is based on a so called (U-V) uniform partition of the group.

Let G be a locally compact group.

DEFINITION 1.10. Let U, V be two relatively compact open neighborhoods of the identity with $\overline{U} \subseteq V$. A partition π of B into disjoint Borel subsets is (U-V) uniform if for each $W \in \pi$ there exists $x \in G$ such that $xU \subseteq W \subseteq xV$.

DEFINITION 1.11. For $1 \leq p,q \leq \infty$ and a (U-V) uniform partition π on G, we define, for a measurable function f such that $f|W \in L^p(W)$

for all $W \in \pi$,

$$\|f\|^{\pi}_{pq} = \left[\sum_{W \in \pi} \left[\int_W |f|^p \right]^{q/p} \right]^{1/q} \qquad \text{if } 1 \leq p,q < \infty$$

$$\|f\|_{\infty q} = \left[\sum_{W \in \pi} \sup_W |f|^q \right]^{1/q} \qquad \text{if } p = \infty,\ 1 \leq q < \infty$$

$$\|f\|_{p\infty} = \sup_{W \in \pi} \left[\int_W |f|^p \right]^{1/p} \qquad \text{if } 1 \leq p < \infty,\ q = \infty.$$

Then the amalgam of L^p and ℓ^q is the linear space

$L^{\pi}_{pq} = \{ f \mid \|f\|^{\pi}_{pq} < \infty \}$ $\qquad 1 \leq p,q \leq \infty$.

The dependence of L^{π}_{pq} on the partition π is irrelevant because for π, π' uniform partitions, the spaces L^{π}_{pq}, $L^{\pi'}_{pq}$ are isomorphic [12, Proposition 3.8].

Each of the spaces $\ell^q(L^p)$, (L^p, ℓ^q), L^{π}_{pq}, $\ell^q(M)$, M_q $1 \leq p,q \leq \infty$ is a Banach space under the norm, $\|\cdot\|_{qp}$, $\|\cdot\|_{pq}$, $\|\cdot\|^{\pi}_{pq}$, $\|\cdot\|_{q1}$, $\|\cdot\|_q$ respectively ([8, §7, b)], [49, Theorem 3.2], [12, Definition 3.6]).

Hereafter and throughout the whole work, G will be a locally compact abelian group, the Haar measure m, the sets L, K and the families $\{L_\alpha\}_J$, $\{K_\alpha\}_J$ will be as in Definition 1.6 and the number a will always correspond to the nonnegative integer which appears in the decomposition of G, unless otherwise stated.

There are several facts we ought to know before we proceed to prove the equivalence of $\ell^q(L^p)$, (L^p, ℓ^q), L^{π}_{pq} $(\ell^q(M), M_q)$.

These are:

1) If $L_i = t_i + L$ is a translate of the set L, then the set $S(L_i) = \{K_\alpha |\ K_\alpha \cap L_i \neq 0\}$ has cardinality less than or equal to 2^a.

2) Let π be a (U-V) uniform partition on G.

a) The sets $S(K_\alpha) = \{W \in \pi |\ K_\alpha \cap W \neq \phi\}$, $S(W) = \{K_\alpha |\ K_\alpha \cap W \neq \phi\}$ have cardinality less than or equal to the integers $n(V,K)$, $2^a n(K,V)$, where $n(V,K)$, $n(K,V)$ are the greatest integers in the real numbers $m(K - V + U)/m(U)$, $m(V - K + U)/m(U)$ respectively.

b) Let $1 \leq p \leq \infty$. $f \in L^p_{loc}$ iff $f|W \in L^p(W)$ for all $W \in \pi$.

Part 1) is clear. Part 2), a) follows from the proposition below; its proof can be found in [12, Propositions 3.4 and 3.5]. Part b) is clear.

<u>PROPOSITION 1.12</u>. If π is a (U - V) uniform partition on G then for each $W \in \pi$ there exists $x_W \in G$ such that $x_W + U \subseteq W \subseteq x_W + V$. Let $F = \{x_W |\ W \in \pi\}$ and E be a relatively compact Borel set.

i) Every translate of E meets (and so is covered by) at most n(V,E) members of π.

ii) Each translate of a member of π meets at most n(E,V) of the translates $x_W + E$ ($x_W \in F$).

<u>LEMMA 1.13</u>. [12, Proposition 2.1]. Let $\{a_1,\ldots a_n\}$ be nonnegative real numbers.

a) If $1 \leq p < \infty$ then

$$(a_1 + a_2 + \cdots + a_n)^p \leq n^{p-1} (a_1{}^p + a_2{}^p + \cdots + a_n{}^p).$$

b) If $0 < p \leq 1$ then

$$(a_1 + a_2 + \cdots + a_n)^p \leq a_1{}^p + \cdots + a_n{}^p.$$

Now we are ready to prove the next three theorems about the spaces $\ell^q(L^p)$, (L^p, ℓ^q), L^π_{pq}, $\ell^q(M)$ and M_q.

THEOREM 1.14. The space $\ell^q(L^p)$ defined from the set L is isomorphic (as a Banach space) to (L^p, ℓ^q) for $1 \leqq p,q \leqq \infty$.

PROOF. We note that the family $\{L_\alpha\}_J$ is a tiling of G by L. Then by the definition of $||\cdot||_{pq}$, $||\cdot||_{qp}$ and Remark 1.8, we have that for all $f \in \ell^q(L^p)$, $||f||_{pq} \leqq ||f||_{qp}$ $(1 \leqq p,q \leqq \infty)$. Hence $\ell^q(L^p) \subseteq (L^p, \ell^q)$.

Now take $f \in (L^p, \ell^q)$ and $\{L_i\}_I$ a tiling of G by L. Then for $1 \leqq p,q < \infty$ and each $i \in I$,

$$\int_{L_i} |f|^p \leqq \sum_{K_\alpha \in S(L_i)} \int_{K_\alpha} |f|^p \quad \text{and} \quad \sup_{L_i} |f|^q \leqq \sum_{K_\alpha \in S(L_i)} \sup_{K_\alpha} |f|^q$$

where $S(L_i)$ is as in 1) above.

Applying Lemma 1.13 we have that

$$\sum_{i \in I} \left[\int_{L_i} |f|^p \right]^{q/p} \leqq (2^a)^{q/p-1} \sum_{i \in I} \sum_{K_\alpha \in S(L_i)} \left[\int_{K_\alpha} |f|^p \right]^{q/p}$$

$$\sum_{i \in I} \sup_{L_i} |f|^q \leqq \sum_{i \in I} \sum_{K_\alpha \in S(L_i)} \sup_{K_\alpha} |f|^q$$

$$\sup_{x \in G} \left[\int_{x-L} |f|^p \right]^{1/p} \leqq \sup_{x \in G} \sum_{K_\alpha \in S(L_i)} \left[\int_{K_\alpha} |f|^p \right]^{1/p}$$

This implies that for $1 \leqq p,q < \infty$

$$\left[\sum_{i \in I} \left[\int_{L_i} |f|^p \right]^{q/p} \right]^{1/q} \leqq 2^{a/p} \, || f ||_{pq}$$

$$\left[\sum_{i \in I} \sup_{L_i} |f|^q \right]^{1/q} \leqq 2^{a/q} \, || f ||_{\infty q}$$

$$\sup_{x \in G} \left[\int_{x-L} |f|^p \right]^{1/p} \leqq 2^{a} \| f \|_{p\infty}.$$

Since $\{L_i\}$ is arbitrary we conclude that for $1 \leqq p,q < \infty$

$\| f \|_{qp} \leqq 2^{a/p} \| f \|_{pq}$, $\| f \|_{q\infty} \leqq 2^{a/q} \| f \|_{\infty q}$ and

$\| f \|_{\infty p} \leqq 2^{a} \| f \|_{p\infty}$. Therefore $(L^p, \ell^q) \subseteq \ell^q(L^p)$ for $1 \leqq p,q \leqq \infty$. †

THEOREM 1.15. The space L^{π}_{pq} is isomorphic (as a Banach space) to (L^p, ℓ^q).

PROOF. The argument is identical to the one used in the last part of the previous theorem.

Let π be a (U-V) uniform partition on G and $f \in (L^p, \ell^q)$. By Lemma 1.13 and 2), a) above we have that for $1 \leqq p,q < \infty$

$$\sum_{W \in \pi} \left[\int_W |f|^p \right]^{q/p} \leqq (2^{a} n(K,V))^{q/p-1} \sum_{W \in \pi} \sum_{K_\alpha \in S(W)} \left[\int_{K_\alpha} |f|^p \right]^{q/p}$$

$$\sum_{W \in \pi} \sup_W |f|^q \leqq \sum_{W \in \pi} \sum_{K_\alpha \in S(W)} \sup_{K_\alpha} |f|^q$$

$$\sup_{W \in \pi} \left[\int_W |f|^p \right]^{1/p} \leqq \sup_{W \in \pi} \sum_{K_\alpha \in S(W)} \left[\int_{K_\alpha} |f|^p \right]^{1/p}$$

This implies that for $1 \leqq p,q < \infty$

$$\| f \|^{\pi}_{pq} \leqq (2^{a} n(K,V))^{1/p} \| f \|_{pq}$$

$$\| f \|^{\pi}_{\infty q} \leqq (2^{a} n(K,V))^{1/q} \| f \|_{\infty q}$$

$$\| f \|^{\pi}_{p\infty} \leqq 2^{a} n(K,V) \| f \|_{p\infty}.$$

Therefore $L^{\pi}_{pq} \subseteq (L^p, \ell^q)$.†

THEOREM 1.16. The space $\ell^q(M)$ defined from the set L is isomorphic (as a Banach space) to M_q for $1 \leq q \leq \infty$.

PROOF. As in Theorem 1.14, the family $\{L_\alpha\}$ is a tiling of G by L and we have that for all $\mu \in \ell^q(M)$ $|| \mu ||_q \leq || \mu ||_{q1}$ if $1 \leq q < \infty$.

On the other hand, since K is covered by a finite number of translates $t_i - L$ $i = 1,..,n$, each $K_\alpha = \alpha + K$ $(\alpha \in J)$ is covered by the finite family $\{ \alpha + t_i - L \}$ $i = 1,\ldots,n$ of translates of L. So, for each $\alpha \in J$

$$|\mu|(K_\alpha) \leq \sum_{i=1}^{n} |\mu|(\alpha + t_i - L) \leq n\, ||\mu||_{\infty 1} \quad \text{if} \quad \mu \in \ell^\infty(M).$$

This implies that $||\mu||_\infty \leq n\, ||\mu||_{\infty 1}$ and therefore $\ell^q(M) \subseteq M_q$ for $1 \leq q \leq \infty$.

If $\mu \in M_q$ and $\{L_i\}_{i \in I}$ is a tiling of G by L, then by 1) above

$$|\mu|(L_i) \leq \sum_{K_\alpha \in S(L_i)} |\mu|(K_\alpha)$$

$$|\mu|(x - L) \leq \sum_{K_\alpha \in S(x-L)} |\mu|(K_\alpha).$$

By Lemma 1.13 we have that for $1 \leq q < \infty$

$$\sum_{i \in I} |\mu|(L_i)^q \leq (2^a)^{q-1} ||\mu||_q^q \quad \text{and} \quad \sup_{x \in G} |\mu|(x - L) \leq 2^a\, ||\mu||_\infty.$$

Hence, $||\mu|| \leq (2^a)^{1-1/q}$ if $1 \leq q < \infty$ and $||\mu||_{\infty 1} \leq 2^a\, ||\mu||_\infty$. Therefore $M_q \subseteq \ell^q(M)$.†

REMARK 1.17. i) Note that the space M_1 is just the space of bounded regular measures M(G).

ii) The space M_∞ corresponds to what Argabright and Gil de Lamadrid call the space of translation bounded measures [1, p. 5].

We will write $(L^p, \ell^q)(G)$, $M_q(G)$ to emphasize the group on which the spaces (L^p, ℓ^q), M_q are defined.

We define $c_0(J)$ to be the linear space of nets $(a_\alpha)_J$ $(a_\alpha \in C$, $\alpha \in J)$ such that $\lim\limits_\alpha a_\alpha = 0$. That is, given $\varepsilon > 0$ there exists a compact subset E of G such that $|a_\alpha| < \varepsilon$ for all $\alpha \notin E$.

DEFINITION 1.18. For $1 \leq r \leq \infty$, $(C_0, \ell^r) = C_0 \cap (L^\infty, \ell^r)$ and $(L^r, c_0) = \{f \in (L^r, \ell^\infty) \mid (||f||_{L^r(K_\alpha)})_J \in c_0(J)\}$.

PROPOSITION 1.19. Let $1 \leq r < \infty$, (C_0, ℓ^r) is equal to the set of continuous functions in (L^∞, ℓ^r).

PROOF. Let f be a continuous function in (L^∞, ℓ^r), and $V = \{V_i \mid i \in I\}$ be the set of finite unions of K_α's $(\alpha \in J)$, directed by $V_i \geq V_j$ iff $V_i \subset V_j$. Since $f \in (L^\infty, \ell^r)$, the series $\sum\limits_{V_i} ||f||_{L^\infty(K_\alpha)}$ converges, and consequently the net $\{\sum\limits_{V_i} ||f||_{L^\infty(K_\alpha)}\}_V$ converges to zero. Then for any $\varepsilon > 0$ there exists V_j such that $\sum\limits_{V_i} ||f||_{L^\infty(K_\alpha)} < \varepsilon$ for all $V_i \geq V_j$. Therefore $|f(x)| < \varepsilon$ for all $x \in V_i$, $V_i \geq V_j$, and this implies that $|f(x)| < \varepsilon$ for all $x \notin V_j$. Hence $f \in C_0$.+

Definition 1.1 and Proposition 1.19 show clearly that the amalgam space $(C_0, \ell^1)(\mathbb{R})$ is the algebra defined by N. Wiener in [54].

We will use the next lemma to prove that the norm $||\cdot||^{\#}_{pq}$ defined in Theorem 1.21 -first introduced in [8 , Proposition VIII]- is equivalent to the norm $||\cdot||_{pq}$.

The importance of $||\cdot||^{\#}_{pq}$ will be seen in Chapter V.

LEMMA 1.20. If $F^+ = \{(y_1,..,y_a,0) \in G|\ y_i \in \{0,1\}\ i = 1,..,a\}$ and $F^- = \{(y_1,..,y_a,0) \in G|\ y_i \in \{0,-1\}\ i = 1,..,a\}$ then

i) For all $x \in G$, $x + L \subseteq \cup\{L_\alpha|\ x + y \in L_\alpha,\ y \in F^+\}$

ii) For all $x \in L_\alpha$, $(\alpha \in J)$, $L_\alpha \subseteq x + F^- + L$.

PROOF. i) For $x = (x_1,..,x_a,x')$ in G, there exists a unique $L_\alpha = [n_1,n_1+1)\times\cdots\times[n_a,n_a+1)\times(t+H)$, $n_i \in Z$ $i = 1,..,a$, $t \in T$ (see Definition 1.6), such that $x \in L_\alpha$. Hence $n_i \leqq x_i < n_i+1$ $i = 1,..,a$ and $x' \in t+H$.

On the other hand, $x+L = [x_1, x_1+1),..,[x_a, x_a+1)\times(x'+H)$, and we have that for $z = (z_1,..,z_a,z')$ in $x+L$, $x_i \leqq z_i < x_i+1, i = 1,..,a$ and $z' \in x'+H$.

Define for $i = 1,..,a$

$$y_i = \begin{cases} 0 & \text{if } x_i \leqq z_i < n_i+1 \\ 1 & \text{if } n_i+1 \leqq z_i < x_i+1 \end{cases}$$

Hence $y = (y_1,..,y_a,0)$ belongs to F^+ and for $i = 1,..,a$

$$x_i + y_i = \begin{cases} x_i & \text{if } x_i \leqq z_i < n_i+1 \\ x_i+1 & \text{if } n_i+1 \leqq z_i < x_i+1 \end{cases}$$

Now, if $x + y \in L_\beta$, $L_\beta = [m_1, m_1+1)\times\cdots\times[m_a, m_a+1)\times(t+H)$ $m_i \in Z$ $i = 1,..,a$, $t \in T$, for some β, then $z' \in t+H$ since $x' \in t+H$ and $m_i \leqq z_i < m_i+1$ $i = 1,..,a$ because for $i = 1,..,a$

if $x_i + y_i = x_i$ then $m_i = n_i$ and $m_i = n_i \leqq x_i < n_i+1 = m_i+1$ and if $x_i + y_i = x_i + 1$ then $m_i = n_i + 1$ and $m_i = n_i+1 \leqq z_i < x_i+1 < n_i+2 = m_i+1$; in either case $m_i \leqq z_i < m_i+1$ $i = 1,..,a$. Therefore $z \in L_\beta$ and this implies that $x + L \subseteq \cup \{ L_\beta | \ x + y \in L_\beta, \ y \in F^+ \}$.

ii) First we will prove that if $x = (x_1,..,x_a,x')$ belongs to L then $L \subseteq x + F^- + L$. Let $z = (z_1,..,z_a,z')$ be in L. Then x',z' belong to H and x_i,z_i are in $[0,1)$ $i = 1,..,a$. So, for $i = 1,..,a$, $0 \leqq z_i < x_i$ or $x_i \leqq z_i < 1$. Define

$$y_i = \begin{cases} -1 & \text{if } 0 \leqq z_i < x_i \\ 0 & \text{if } x_i \leqq z_i < 1 \end{cases} .$$

Clearly $y = (y_1,..,y_a,0)$ belongs to F^- and we have that

$$x_i + y_i + [0,1) = \begin{cases} [x_i,x_i+1) & \text{if } x_i \leqq z_i < 1 \\ [x_i-1,x_i) & \text{if } 0 \leqq z_i < x_i \end{cases} \qquad i = 1,..,a.$$

Therefore $z \in x + y + L$.

Now, we take any z in L_α, $L_\alpha = \alpha + L$. Then $z = \alpha + x$ for some $x \in L$. By our previous result $L \subseteq x + F^- + L$ hence $L_\alpha \subseteq \alpha + x + F^- + L = z + F^- + L$ and the proof is complete.$\dagger$

<u>THEOREM 1.21</u>. i) A function f belongs to (L^p, ℓ^q), $1 \leqq p,q \leqq \infty$, iff the function $\underline{f^\#}$ on G defined by

$$f^\#(x) = || \ f \ ||_{L^p(x + L)}$$

belongs to L^q.

If $|| \ f \ ||^\#_{pq} = || \ f^\# \ ||_q$ then

$$2^{-a} \ || \ f \ ||_{pq} \leqq || \ f \ ||^\#_{pq} \leqq 2^a \ || \ f \ ||_{pq}.$$

ii) A measure μ belongs to M_q $1 \leq q \leq \infty$ iff the function $\underline{\mu^{\#}}$ defined by

$$\mu^{\#}(t) = |\mu|(t + L)$$

belongs to L^q.

If $|| \mu ||_q^{\#} = || \mu^{\#} ||_q$ then

$$2^{-a} || \mu ||_q \leq || \mu ||_q^{\#} \leq 2^{a} || \mu ||_q.$$

PROOF. i) For $x \in G$, $x \in L_\alpha$ for some α, define the function $f^0(x) = ||f||_{L^p(L_\alpha)}$. Since $m(L_\alpha) = 1$ for all α we have that if $1 \leq q < \infty$ then

$$||f^0||_q^q = \sum_\alpha \int_{L_\alpha} |f|^q = \sum_\alpha ||f||^q_{L^p(L_\alpha)} m(L_\alpha) = \sum_\alpha ||f||^q_{L^p(L_\alpha)} = ||f||^q_{pq} \quad \text{and}$$

$$||f^0||_\infty = \sup_G |f^0| = \sup_\alpha ||f||_{L^p(L_\alpha)} = ||f||_{p\infty}.$$

Therefore

(1) $$||f^0||_q = ||f||_{pq} \qquad \text{for } 1 \leq p,q \leq \infty.$$

By Lemma 1.20, for any $x \in G$

$$f^{\#}(x) \leq \sum_{y \in F^+} f^0(x + y) \quad \text{and} \quad f^0(x) \leq \sum_{y \in F^-} f^{\#}(x + y).$$

Since the cardinality of F^+ and F^- is 2^a, and for $t \in G$ and any measurable function g, $||\tau_t g||_q = ||g||_q$ $(1 \leq q \leq \infty)$ where $\tau_t g(x) = g(x + t)$ $(x \in G)$, we have from (1) that

$$||f||^{\#}_{pq} = ||f^{\#}||_q \leq \sum_{y \in F^+} ||\tau_y f^0||_q = 2^a ||f^0||_q = 2^a ||f||_{pq}$$

$$||f||_{pq} = ||f^0||_q \leq \sum_{y \in F^-} ||\tau_y f^{\#}||_q = 2^a ||f^{\#}||_q = 2^a ||f||^{\#}_{pq}.$$

This implies i).

ii) As in i), for $x \in G$, $x \in L_\alpha$ for some α, define the function $\mu^0(x) = |\mu|(K_\alpha)$. Then $||\mu^0||_q = ||\mu||_q$ $(1 \leq q \leq \infty)$ and by Lemma 1.20 for any $x \in G$ $\mu^{\#}(x) \leq \sum_{y \in F^+} \mu^0(x + y)$ and

$\mu^0(x) \leq \sum_{y \in F^-} \mu^{\#}(x + y)$.

Hence, for $1 \leq q \leq \infty$

$$||\mu||_q^{\#} = ||\mu^{\#}||_q \leq \sum_{y \in F^+} ||\tau_y \mu^0||_q = 2^a \, ||\mu^0||_q = 2^a \, ||\mu||_q$$

$$||\mu||_q = ||\mu^0||_q \leq \sum_{y \in F^-} ||\tau_y \mu^{\#}||_q = 2^a \, ||\mu^{\#}||_q = 2^a \, ||\mu||_q^{\#}.$$

This implies ii).+

From Theorem 1.21 Bertrandias, Datry and Dupuis [8] established a relation between (L^p, ℓ^q) and the mixed-norm spaces L^{pq} defined by Benedek and Panzone [11].This relation is as follows:

L^{pq} is the normed linear space of measurable (classes of) functions Φ on $G \times G$ such that

$$||\Phi||^{pq} = \left[\int_G \left[\int_G |\Phi(x,y)|^p \, dx \right]^{q/p} \right]^{1/q} < \infty$$

with the usual modifications if $p = \infty$ or $q = \infty$.

We associate to a function $f \in (L^p, \ell^q)$ the function F in L^{pq} defined by $F(x,y) = f(x)\chi_L(x - y)$ and to a function Φ in L^{pq} the function ϕ defined by $\phi(x) = \int_{x+L} \Phi(x,y)dy$.

By Theorem 1.21 $||f||_{pq}^{\#} = ||F||^{pq}$ and as in [8, Proposition IX] $||\phi||_{pq}^{\#} \leq ||\Phi||^{pq}$.

These imply that

a) The map $f \longmapsto F$ is an isometric, linear isomorphism from

(L^p, ℓ^q) onto a linear subspace SL^{pq} of L^{pq} consisting of the functions F in L^{pq} of the form $F(x,y) = f(x)\chi_L(y - x)$ with $f \in (L^p, \ell^q)$.

b) The map $\Phi \longmapsto \phi$ is a continuous linear map from L^{pq} into (L^p, ℓ^q).

c) The composition of $f \longmapsto F$ and $\Phi \longmapsto \phi$ is the identity map.

d) The composition of $\Phi \longmapsto \phi$ and $f \longmapsto F$ is a continuous linear map from L^{pq} onto SL^{pq}.

Busby and Smith have also pointed out another relation in [12, p. 316].

REMARK 1.22. It follows from Theorem 1.21 that the spaces $\mathcal{V}_p$, $\mathcal{V}_{\infty 0}$, $\mathcal{N}_p$, $\mathcal{M}_p$ and $\mathcal{W}_p$ $(1 \leq p \leq \infty)$ defined in [42] are isomorphic to the spaces (L^1, ℓ^p), (L^1, c_0), (L^∞, ℓ^p), (C_0, ℓ^p) and M_p respectively.

REMARK 1.23. i) If G is compact then we can take the family of K_α's to be simply G. Hence $(L^p, \ell^q) = L^p$, $1 \leq p,q \leq \infty$.

ii) If G is discrete then we can take $K = \{0\}$ and therefore the family $\{K_\alpha\}$ is Z^a. Hence $(L^p, \ell^q) = \ell^q$, $1 \leq p,q \leq \infty$.

Then the amalgam spaces (L^p, ℓ^q) are only of interest when G is neither compact nor discrete.

If $\hat{G}$ is the dual group of G then $\hat{G} = \mathbb{R}^a \times \hat{G}_1$ and $\hat{G}_1$ contains the open compact subgroup A which is the annihilator of H, $\underline{A} = \{x \in G \mid [x,\hat{x}] = 1 \text{ for all } \hat{x} \in H\}$. Hence we can choose A to define the families $\{L_\beta\}_{\beta \in I}$, $\{K_\beta\}_{\beta \in I}$ in $\hat{G}$ by $L_\beta = \beta + \hat{L}$, $K_\beta = \beta + \hat{K}$ where $I = Z^a \times T'$, T' being a transversal of A in $\hat{G}_1$, $\hat{L} = [0,1) \times A$ and $\hat{K} = [0,1] \times A$.

Using $\{K_\beta\}_I$ we define as in Definitions 1.7 and 1.9, the amalgam space $(L^p, \ell^q)(\hat{G})$ and the space of unbounded measures of type q $M_q(\hat{G})$.

Throughout the whole work, $\{L_\beta\}_I$, $\{K_\beta\}_I$, $\hat{L}$ and $\hat{K}$ will correspond to the sets so defined here, unless otherwise stated.

§ 2. INEQUALITIES AND INCLUSION RELATIONS

During our work we will make constant use of the inequalities and inclusion relations studied in this section.

PROPOSITION 2.1. For $1 \leqq p \leqq \infty$

$$(L^p, \ell^p) = L^p. \tag{2.1}$$

PROOF. Definition 1.7.

We see from Proposition 2.1 that the theory of amalgam spaces on locally compact groups embraces the theory of L^p spaces.

PROPOSITION 2.2.

$$||\mu||_p \leqq ||\mu||_q \qquad 1 \leqq q \leqq p \leqq \infty \tag{2.2}$$

$$||f||_{pq_2} \leqq ||f||_{pq_1} \qquad 1 \leqq q_1 \leqq q_2 \leqq \infty,\ 1 \leqq p \leqq \infty \tag{2.3}$$

$$||f||_{p_1 q} \leqq ||f||_{p_2 q} \qquad 1 \leqq p_1 \leqq p_2 \leqq \infty,\ 1 \leqq q \leqq \infty \tag{2.4}$$

PROOF. Since $|\mu|(K_\delta)^q \leqq \sum_{\alpha \in J} |\mu|(K_\alpha)$ for all $\delta \in J$, it is clear that

$$||\mu||_\infty = \sup_{\delta \in J} |\mu|(K_\delta) \leqq \left[\sum_{\alpha \in J} |\mu|(K_\alpha)^q\right]^{1/q} = ||\mu||_q.$$

If p is finite then (2.2) follows from Jensen's inequality (as in [4, p.18]) with $x_\alpha = |\mu|(K_\alpha)$.

If $q_2 = \infty$ then

$$||f||_{p\infty} = \sup_{\alpha \in J} ||f||_{L^p(K_\alpha)} \leqq \left[\sum_{\alpha \in J} ||f||^q_{L^p(K_\alpha)}\right]^{1/q_1} = ||f||_{pq_1}.$$

If $q_2 < \infty$ then again by Jensen's inequality with $x_\alpha = ||f||_{L^p(K_\alpha)}$ we have that for $q_1 < q_2$

$$\left[\sum_\alpha ||f||^{q_2}_{L^p(K_\alpha)}\right]^{1/q_2} \leqq \left[\sum_\alpha ||f||^{q_1}_{L^p(K_\alpha)}\right]^{1/q_1}. \text{ Hence } ||f||_{pq_2} \leqq ||f||_{pq_1}.$$

If $p_2 = \infty$ then for each $\alpha \in J$

$$||f||_{L^{p_1}(K_\alpha)} \leqq \sup_{x \in K_\alpha} |f(x)|m(K_\alpha) = ||f||_{L^\infty(K_\alpha)} \quad \text{since } m(K_\alpha) = 1.$$

Therefore $||f||_{p_1 q} \leqq ||f||_{\infty q}$ $\quad 1 \leqq q \leqq \infty$.

If $p_2 < \infty$ then by Hölder's inequality (as in [37, Corollary 12.5]) with $f_1 = |f|K_\alpha|^{p_2}$, $f_2 = \chi_{K_\alpha}$, $\alpha_1 = p_1/p_2$, $\alpha_2 = 1 - (p_1/p_2)$ we have that for any $\alpha \in J$

$$\int_{K_\alpha} |f|^{p_1} = \int_G |f|K_\alpha|^{p_1}\chi_{K_\alpha} = \int_G |f|K_\alpha|^{p_2\alpha_1}\chi_{K_\alpha}^{\alpha_2}$$

$$\leqq \left[\int_G |f|K_\alpha|^{p_2}\right]^{\alpha_1} \left[\int_G \chi_{K_\alpha}\right]^{\alpha_2} = \left[\int_{K_\alpha} |f|^{p_2}\right]^{p_1/p_2},$$

again this is because $m(K_\alpha) = 1$. This implies that $||f||_{p_1 q} \leqq ||f||_{p q}$ for $1 \leqq q \leqq \infty$.†

COROLLARY 2.3.

(2.5) $\quad (L^p, \ell^{q_1}) \subseteq (L^p, \ell^{q_2}) \quad 1 \leqq q_1 \leqq q_2 \leqq \infty,\ 1 \leqq p \leqq \infty$

(2.6) $\quad (L^{p_2}, \ell^q) \subseteq (L^{p_1}, \ell^q) \quad 1 \leqq p_1 \leqq p_2 \leqq \infty,\ 1 \leqq q \leqq \infty$

(2.7) $\quad (L^p, \ell^q) \subseteq L^p \cap L^q \quad 1 \leqq q \leqq p \leqq \infty$

(2.8) $\quad L^p \cup L^q \subseteq (L^p, \ell^q) \quad 1 \leqq p \leqq q \leqq \infty$

(2.9) $\quad M_p \subseteq M_q \quad 1 \leqq p \leqq q \leqq \infty$

PROOF. It is clear that (2.5) and (2.6) follow from the inequalities (2.3) and (2.4) respectively, while (2.7) and (2.8) follow from

(2.5), (2.6) and (2.1). Finally (2.2) implies (2.9).

THEOREM 2.4. Of the inclusions below 1) and 8) are strict if G is noncompact, 2) is strict if G is nondiscrete and 3), 4) and 5) are strict if G is neither compact nor discrete.

1) $(L^p, \ell^{q_1}) \subset (L^p, \ell^{q_2})$ $\quad 1 \le q_1 < q_2 \le \infty,\ 1 \le p \le \infty$

2) $(L^{p_2}, \ell^q) \subset (L^{p_1}, \ell^q)$ $\quad 1 \le p_1 < p_2 \le \infty,\ 1 \le q \le \infty$

3) $(L^p, \ell^q) \subset L^p \cap L^q$ $\quad 1 < q < p \le \infty$

4) $L^q \subset (L^1, \ell^q) \cap (L^q, \ell^\infty)$ $\quad 1 < q < \infty$

5) $L^p \subset (L^p, \ell^q) \cap (L^1, \ell^p)$ $\quad 1 < p < q < \infty$

6) $(L^q, \ell^1) \cap (L^\infty, \ell^q) \subset L^q$ $\quad 1 < q < \infty$

7) $(L^p, \ell^q) \cap (L^q, \ell^1) \subset L^q$ $\quad 1 < q < p < \infty$

8) $L^q \subset (L^q, \ell^\infty)$ $\quad 1 \le q < \infty$.

PROOF. Let $\{L_n\}$ be a countable subfamily of $\{L_\alpha\}$ and α be a real number in $[1,\infty)$. Since m is regular and $m(L_n) = 1$ for all n, given $(1/n)^\alpha$ there exists a compact subset $I_n \subset L_n$ such that $m(L_n) - (1/n)^\alpha \quad m(I_n)$. This implies that $m(L_n \sim I_n) = M(I_n) - m(I_n) < (1/n)^\alpha$. Define $f_n\colon L_n \longrightarrow \mathbb{R}$ by

$$f_n(x) = \begin{cases} n & x \quad L_n \sim I_n \\ 0 & \text{otherwise.} \end{cases}$$

Hence for $1 \le p, q < \infty$

$$\int_{L_n} |f_n|^p = \int_{L_n \sim I_n} |f_n|^p = n^p\, m(L_n \sim I_n) < n^p/n^\alpha = (1/n)^{\alpha - p}$$ and

$\sup_{L_n} |f_n| = n$. Therefore for $f = \sum_{\mathbb{N}} f_n$

$$||f||_{pq} = \left[\sum_{\mathbb{N}} (1/n)^{(\alpha-p)q/p}\right]^{1/q}$$

$$||f||_{p\infty} = \sup_{\mathbb{N}} \left\{(1/n)^{\alpha-p/p}\right\}$$

$$||f||_{\infty q} = \left[\sum_{\mathbb{N}} n^q\right]^{1/q} .$$

Then

a) $f \in (L^p, \ell^q)$ if $(\alpha - p)q/p > 1$

b) $f \notin (L^p, \ell^q)$ if $(\alpha - p)q/p \leq 1$

c) $f \in (L^p, \ell^\infty)$ if $(\alpha - p)/p > 0$

d) $f \notin (L^p, \ell^\infty)$ if $(\alpha - p)/p < 0$

e) $f \notin (L^\infty, \ell^q)$ for $1 \leq q < \infty$

1) For $1 \leq q_1 < q_2 < \infty$ and $1 \leq p < \infty$, we take $\alpha = (p/q_1) + p$. Then $(\alpha - p)q_2/p = (q_2/q_1) > 1$ and $(\alpha - p)q_1/p = 1$. Therefore by a) and b) $f \in (L^p, \ell^{q_2})$ and $f \notin (L^p, \ell^{q_1})$. If $q_2 = \infty$ and $1 \leq p < \infty$ then $(L^p, \ell^{q_1}) \subseteq (L^p, \ell^\infty)$ because for any q, $q_1 < q < \infty$ $(L^p, \ell^{q_1}) \subsetneq (L^p, \ell^q) \subseteq (L^p, \ell^\infty)$.

Suppose $p = \infty$, $1 \leq q_1 < q_2 < \infty$ and take α such that $1/q_2 < \alpha < 1/q_1$, and define $g: G \longrightarrow R$ by

$$g(x) = \begin{cases} (1/n)^\alpha & x \in L_n \\ 0 & \text{otherwise} \end{cases}$$

Hence $\sup_{L_n} |g(x)| = (1/n)^\alpha$ and this implies that

$$||g||_{\infty q_1} = \left[\sum_{\mathbb{N}} (1/n)^{\alpha q_1}\right]^{1/q_1} \quad \text{and} \quad ||g||_{\infty q} = \left[\sum_{\mathbb{N}} (1/n)^{\alpha q_2}\right]^{1/q_2}$$

Therefore $g \notin (L^\infty, \ell^{q_1})$ since $\alpha q_1 < 1$ and $g \in (L^\infty, \ell^{q_2})$ since $\alpha q_2 > 1$.

Suppose $p = q_2 = \infty$ and $1 \leq q_1 < \infty$. Define the function $g: G \longrightarrow R$ by

$$g(x) = \begin{cases} (1/n)^{1/q_1} & x \in L_n \\ 0 & \text{otherwise} \end{cases}$$

Hence, $||g||_\infty = \sup_N (1/n)^{1/q_1} < \infty$ and

$||g||_{\infty q_1} = \left[\sum_N 1/n \right]^{1/q_1} = \infty$. Therefore $g \notin (L^\infty, \ell^{q_1})$ and $g \in L^\infty$.

2) For $1 \leq p_1 < p_2 < \infty$ and $1 \leq q < \infty$, take $\alpha = (p_1/q) + p_2$. Then $(\alpha - p_2)q/p_2 = 1$ and $\alpha > (p_1/q) + p_1$. This means that $(\alpha - p_1)q/p_1 > 1$. Therefore by a) and b) $f \notin (L^{p_2}, \ell^q)$ and $f \in (L^{p_1}, \ell^q)$.

If $p_2 = \infty$, $1 \leq p_1 < \infty$ and $1 \leq q < \infty$ then $(L^\infty, \ell^q) \subsetneq (L^{p_1}, \ell^q)$ because for any p, $p_1 < p < \infty$ $(L^\infty, \ell^q) \subseteq (L^p, \ell^q) \subsetneq (L^{p_1}, \ell^q)$.

Suppose $q = \infty$, $1 \leq p_1 < p_2 < \infty$. Take $p_1 < \alpha < p_2$, so $(\alpha - p_1)/p_1 > 0$ and $(\alpha - p_2)/p_2 < 0$. Hence by c) and d), $f \in (L^p, \ell^\infty)$ and $f \notin (L^{p_2}, \ell^\infty)$.

If $q = p_2 = \infty$ then $L^\infty \subsetneq (L^{p_1}, \ell^\infty)$ because for $p_1 < p < \infty$ $L^\infty \subseteq (L^p, \ell^\infty) \subsetneq (L^p, \ell^\infty)$.

3) For $1 \leq q < p < \infty$ take $\alpha = (p/q) + p$. Hence $(\alpha - p)q/p = 1$, $\alpha - p = p/q > 1$, $\alpha > 1 + q$, so $\alpha - q > 1$. Then according to a) and b) $f \in L^p \cap L^q$ and $f \notin (L^p, \ell^q)$.

Suppose $p = \infty$. Consider the function $g: G \longrightarrow R$ defined by

$$g(x) = \begin{cases} (1/n)^{1/q} & x \in L_n \\ 0 & \text{otherwise} \end{cases}$$

Clearly $g \in L^{\infty}$, $\sup_{L_n} |g(x)|^q = 1/n$ and

$\int_{L_n} g^q = \int_{L_n} 1/n = 1/n^2$. Therefore $||g||_{\infty q} = \sum_{N} 1/n$ and

$||g||_q = \left[\sum_{N} 1/n^2\right]^{1/q}$. This implies that $g \notin (L^{\infty}, \ell^q)$ and $g \in L^q \cap L^{\infty}$.

4) Let $\alpha = 1 + q$, so $(\alpha - 1)q = q^2 > 1$. $(\alpha - q)q = 1/q > 0$ and $\alpha - q = 1$. Hence by a), b) and c) $f \in (L^1, \ell^q) \cap (L^q, \ell^{\infty})$ and $f \notin L^q$.

5) Let $\alpha = p + 1$, so $(\alpha - p)q/p = q/p > 1$, $(\alpha - 1)p = p^2 > 1$ and $\alpha - p = 1$. Hence by a) and b) $f \in (L^p, \ell^q) \cap (L^1, \ell^p)$ and $f \notin L^p$.

6) Let $\alpha = 2q$, so $(\alpha - q)/q = 1$, $\alpha - q = q > 1$. Hence by b) $f \notin (L^q, \ell^1)$, by e) $f \notin (L^{\infty}, \ell^q)$ and by a) $f \in L^q$.

7) Let $\alpha = \min(\ p/q + p,\ 2q\ )$, so $\alpha \leq p/q + p$ and $\alpha \leq 2q$; this implies that $(\alpha - p)q/p \leq 1$ and $(\alpha - q)/q \leq 1$. Hence by b) $f \notin (L^p, \ell^q)$, $f \notin (L^q, \ell^1)$. But $\alpha > 1 + q$ and this means that $\alpha - q > 1$, therefore by a) $f \in L^q$.

8) Consider the function $g\colon G \longrightarrow R$ defined by

$$g(x) = \begin{cases} (1/n)^{1/q} & x \in L_n \\ 0 & \text{otherwise} \end{cases}$$

Then $\int_{L_n} g^q = 1/n$, and we have that $||g||_q = \left[\sum_{\mathbb{N}} 1/n\right]^{1/q}$,

$||g||_{q\infty} = \sup_{N} 1/n$. Therefore $g \notin L^q$ and $g \in (L^q, \ell^{\infty})$. †

<u>COROLLARY 2.5</u>. The following inclusions are strict if G is noncompact.

9) $(L^p, \ell^1) \subseteq L^p \qquad 1 < p \leq \infty$

10) $L^p \subseteq (L^p, \ell^q) \qquad 1 \leq p < q \leq \infty$

11) $L^q \subseteq (L^1, \ell^q) \qquad 1 < q \leq \infty$

12) $L^q \subseteq (L^p, \ell^q) \qquad 1 \leq p < q \leq \infty$

13) $(L^\infty, \ell^q) \subseteq L^q \qquad 1 \leq q < \infty$.

PROOF. 9) and 10) follow from 1). While 11) and 12) are consequences of 2). Since $(L^\infty, \ell^q) \subseteq (L^p, \ell^q) \subsetneq L^q$ for $1 \leq q < p < \infty$, we see that 13) holds. †

REMARK 2.6. From (2.5) and (2.6) we see that $(L^\infty, \ell^1) \subseteq (L^p, \ell^q) \subseteq (L^1, \ell^\infty)$ $1 \leq p,q \leq \infty$. In other words (L^∞, ℓ^1) is the smallest and (L^1, ℓ^∞) is the biggest of the amalgam spaces. At the same time, M_1 is the smallest and M_∞ is the biggest of the spaces of unbounded measures of type q.

A function f in (L^1, ℓ^q) $(1 \leq q \leq \infty)$ considered as the measure fm, where $\int g\,dfm = \int gf\,dm$, belongs to M_q and $||f||_{1q} = ||fm||_q$. Hence $f \longmapsto fdm$ is a natural embedding from (L^1, ℓ^q) into M_q. In this sense, we say that

(2.10) $\quad (L^1, \ell^q) \subseteq M_q \quad 1 \leq q \leq \infty.$

Note that for $1 \leq p,q \leq \infty$ and $f \in (L^p, \ell^q)$

(2.11) $\quad (L^p, \ell^q) \subseteq (L^1, \ell^q) \subseteq M_q$

(2.12) $\quad ||fm||_q = ||f||_{1q} \leq ||f||_{pq}.$

§ 3. PROPERTIES OF (L^p, ℓ^q) AND M_q.

The results presented here are not new. However for the sake of completness and uniformity we shall prove them using the definition of the amalgam space (L^p, ℓ^q). Alternative proofs can be found in [8].

Let $\{E_\alpha\}_J$ be a family of Banach spaces. Following G. Köthe [38, § 26 p. 359] $\ell^q(E_\alpha)$ is the linear space of nets $(x_\alpha)_J$, $x_\alpha \in E_\alpha$ such that $\sum_J \|x_\alpha\|^q_{E_\alpha} < \infty$. $\ell^q(E_\alpha)$ is a Banach space under the norm $\|(x_\alpha)\| = \left[\sum_J \|x_\alpha\|_{E_\alpha}\right]^{1/q}$ if q is finite and $\|(x_\alpha)\| = \sup_J \|x_\alpha\|_{E_\alpha}$ if q is infinite.

We see that (L^p, ℓ^q), M_q, $1 \leq p,q \leq \infty$ are particular cases of $\ell^q(E_\alpha)$. Indeed, if $E_\alpha = L^p(L_\alpha)$ $(\alpha \in J)$ then the map $f \longmapsto (f_\alpha)_J$, $f_\alpha = f|L_\alpha$ is an isometric isomorphism from (L^p, ℓ^q) onto $\ell^q(E_\alpha)$. Similarly, if $E_\alpha = M(K_\alpha)$ $(\alpha \in J)$ then M_q is isometrically isomorphic to $\ell^q(E_\alpha)$ via the map $\mu \longmapsto (\mu_\alpha)_J$, $\mu_\alpha(B) = \mu(B \cap K_\alpha)$ (B a Borel subset of G).

This fact together with § 26, 8 of [38] implies the next result.

<u>THEOREM 3.1</u>. Let $1 \leq p,q < \infty$. $(L^{p'}, \ell^{q'})$ ($(L^{p'}, \ell^1)$) is isometrically isomorphic to $(L^p, \ell^q)^*$ ($(L^p, c_0)^*$) via the map $g \longmapsto \langle f,g \rangle$, $\langle f,g \rangle = \int_G fg\,dx$, $g \in (L^{p'}, \ell^{q'})$ ($(L^{p'}, \ell^1)$), $f \in (L^p, \ell^q)$ ((L^p, c_0)).

Hence,

$$(3.1) \qquad |\langle f,g \rangle| \leq ||f||_{pq}||g||_{p'q'} \qquad 1 < p,q < \infty$$

$$(3.2) \qquad |\langle f,g \rangle| \leq ||f||_{p1}||g||_{p'\infty} \qquad 1 \leq p < \infty.$$

PROOF. The case $p = q = 1$ follows from [18, IV 8.5, p. 290] and [45, Appendix E. 10].

The proof for $(L^{p'}, \ell^1)$ is identical to that for the case $(L^{p'}, \ell^{q'})$ in [38, §26, p. 359]. $\dagger$

THEOREM 3.2. Let $1 \leq q \leq \infty$. If T is a linear functional on (C_0, ℓ^q) then there exists a unique measure $\mu \in M_{q'}$ such that

$$T(f) = \int_G f \, d\mu \qquad (f \in (C_0, \ell^q))$$

and

$$||T|| \leq ||\mu||_{q'} \leq 2^{a} ||T|| \qquad \text{if } 1 \leq q < \infty$$

$$||T|| = ||\mu||_1 \qquad \text{if } q = \infty.$$

Moreover

$$(3.3) \qquad |\langle f,g \rangle| = \left| \int_G fg \, dx \right| \leq ||f||_{\infty q}||g||_{1q'}$$

$f \in (C_0, \ell^q)$, $g \in (L^1, \ell^{q'})$.

PROOF. The case $q = \infty$ is the Riesz Representation Theorem.

For $1 \leq q < \infty$, the first part is Theorem 4.3 in [49]. What follows is a sketch of the proof.

Let E_α be the space of continuous functions on K_α ($\alpha \in J$) with the usual topology and let $\ell^q(E_\alpha)$ be the linear space of nets $(f_\alpha)_J$,

$f_\alpha \in E_\alpha$ such that $|| \, ||f_\alpha|| \, ||_q < \infty$.

If $S = \{(f_\alpha)_J \in \ell^q(E_\alpha) \mid f_\alpha = f_\beta$ on $K_\alpha \cap K_\beta\}$ then (C_0, ℓ^q) is isometrically isomorphic to S via $f \longmapsto (f_\alpha)_J$ where $f_\alpha = f|K_\alpha$.

Since $T \in (C_0, \ell^q)^* = S^*$, T can be extended without changing its norm to a linear functional T in $\ell^q(E_\alpha)^*$.

By [38, §26, 1)], $\ell^q(E_\alpha)^* = \ell^{q'}(E_\alpha{}^*)$ and this implies that there exists $T_\alpha \in E_\alpha{}^*$, $\alpha \in J$, such that

$T(f) = \sum_\alpha T_\alpha(f_\alpha) \quad (f \in (C_0, \ell^q))$ and $||T|| = || \, ||T_\alpha|| \, ||_{q'}$.

By the Riesz Representation Theorem there exists a unique measure $\mu_\alpha \in K_\alpha$ such that $||\mu_\alpha|| = ||T_\alpha||$ and

$T_\alpha(g) = \int_{K_\alpha} g \, d\mu_\alpha \quad (g \in E_\alpha)$.

Therefore $T(f) = \sum_\alpha \int_{K_\alpha} f_\alpha \, d\mu_\alpha \quad (f \in (C_0, \ell^q))$.

The measure $\mu(E) = \sum_\alpha \mu_\alpha(E \cap K_\alpha)$ (E a Borel subset of G) belongs to $M_{q'}$, $T(f) = \int f d\mu \quad (f \in (C_0, \ell^q))$ and $||T|| = || \, ||\mu_\alpha|| \, ||_{q'}$.

Now, $|\mu|(K_\gamma) \leq \sum_\alpha |\mu_\alpha|(K_\alpha \cap K_\gamma) \leq \sum_{K_\alpha \in S(K_\gamma)} |\mu_\alpha|(K_\alpha)$ where $S(K_\gamma) = \{K_\alpha \mid K_\alpha \cap K_\gamma \neq \phi\}$.

So, by Lemma 1.13

$$|\mu|(K_\gamma)^{q'} \leq (2^a)^{q'-1} \sum_{K_\alpha \in S(K_\gamma)} |\mu_\alpha|(K_\alpha)^{q'} \quad \text{for} \quad 1 \leq q' < \infty .$$

Since the cardinality of $S(K_\gamma)$ is 2^a we conclude that

$$\sum_\alpha |\mu|(K_\gamma) \leq 2^{aq'} \sum_\alpha ||\mu_\alpha||^{q'} \quad \text{if} \quad 1 < q' < \infty \text{ and also}$$

$\sup_{\gamma} |\mu|(K_\gamma) \leq 2^a \sup_{\alpha} ||\mu_\alpha||.$

This implies that $||\mu||_{q'} \leq 2^a \; || \; ||\mu_\alpha|| \; ||_{q'} = 2^a ||T||$ for $1 \leq q < \infty$.

On the other hand, by the Hölder inequality, we have that for $f \in (C_0, \ell^q)$

$$|T(f)| \leq \int |f| \, d|\mu| \leq \sum_\alpha \int_{K_\alpha} |f| \, d|\mu| \leq \sum_\alpha \sup_{K_\alpha} |f| \; |\mu|(K_\alpha)$$

$$\leq \left[\sum_\alpha \sup_{K_\alpha} |f|^q \right]^{1/q} \left[\sum_\alpha |\mu|(K_\alpha)^{q'} \right]^{1/q'} = ||f||_{\infty q} ||\mu||_{q'} .$$

Therefore $||T|| \leq ||\mu||_{q'}$.

Finally if $g \in (L^1, \ell^{q'})$ then by (2.10) $gm \in M_{q'}$ and by our previous result and (2.12)

$$|\langle f,g \rangle| = |\int f dgm| = |\int fg dx| \leq ||f||_{\infty q} ||gm||_{q'} = ||f||_{\infty q} ||g||_{1q'} . \dagger$$

The expressions (3.1), (3.2) and (3.3) will be called <u>Hölder inequality for amalgams</u>.

<u>REMARK</u>. In Theorem 3.1 $||T|| \neq ||\mu||_{q'}$ for $1 \leq q < \infty$. To see this consider the linear map T on (C_0, ℓ^q) defined by $T(f) = f(0)$. Then $T \in (C_0, \ell^q)^*$ and the measure associated to T is δ_0. Since $||T|| = 2^{-a/q}$ and $||\delta_0||_{q'} = 1 = (2^{a/q})(2^{-a/q})$ we conclude that $||T|| = 2^{-a/q} \neq ||\delta_0||_{q'}$.

<u>COROLLARY 3.3</u>. The amalgam space (L^p, ℓ^q) $(1 < p,q < \infty)$ is a reflexive Banach space.

Note that if E is a compact subset of G, then E is covered by a finite number of translates of K, because the interior of K is nonempty. Since each translate of K meets at most $2^a K_\alpha$'s we conclude that E is covered by a finite number of K_α's. Therefore the cardinality $|S(E)|$ of the set $S(E) = \{K_\alpha |\ K_\alpha \cap E \neq \emptyset\}$ is finite.

PROPOSITION 3.4. If $g \in L^p_c$, $1 \leq p < \infty$, and E is its compact support then for $1 \leq q < \infty$, $||g||_{pq} \leq |S(E)|^{1/q}\ ||g||_p$ and $||g||_{p\infty} \leq |S(E)|\ ||g||_p$.

PROOF

$$||g||_{pq} = \left[\sum_\alpha ||g||^q_{L^p(K_\alpha)}\right]^{1/q} = \left[\sum_{K_\alpha \in S(E)} ||g||^q_{L^p(K_\alpha)}\right]^{1/q}$$

$$\leq |S(E)|^{1/q}\ ||g||_p$$

$$||g||_{p\infty} = \sup_\alpha ||g||_{L^p(K_\alpha)} = \sup_{K_\alpha \in S(E)} ||g||_{L^p(K_\alpha)} \leq |S(E)|\,.\,||g||_p\ .$$

DEFINITION 3.5. For $\mu \in M_q$, $1 \leq q \leq \infty$, and E a compact subset of G; μ_E will be the bounded measure defined by $\mu_E(B) = \mu(E \cap B)$ B a Borel set of G. M^q_c will denote the linear subspace of M_1 consisting of the measures μ_E, $E \subseteq G$ compact and $\mu \in M_q$.

THEOREM 3.6. i) L^p_c is a dense subspace of (L^p, ℓ^q) for $1 \leq p \leq \infty$, $1 \leq q < \infty$.

ii) L^p_c is a dense subspace of (L^p, c_0) for $1 \leq p \leq \infty$.

iii) M^q_c is a dense subspace of M_q for $1 \leq q < \infty$.

PROOF. Let $V = \{V_i |\ i \in I\}$ be the set of finite unions of K_α's. If $f \in (L^p, \ell^q)$ then $f = \sum f_\alpha$ where $f_\alpha = f|L_\alpha$ and

$$||f||_{pq} = \left[\sum_\alpha ||f_\alpha||^q_{L^p(L_\alpha)}\right]^{1/q}$$; this implies that $\lim\limits_{V} \sum\limits_{V_i} f_\alpha = f$ in

(L^p, ℓ^q). Since $\sum_{V_i} f_\alpha$ belongs to L^p_c for each $V_i \in V$, L^p_c is dense in (L^p, ℓ^q).

To prove ii) take a function f in the closure of L^p_c in (L^p, ℓ^∞). So given $\varepsilon > 0$ there exists $g \in L^p_c$ such that $||f - g||_{p\infty} < \varepsilon$. This implies by the definition of $||\cdot||_{p\infty}$ that for all $\alpha \in J$, $||f - g||_{L^p(L_\alpha)} < \varepsilon$. Hence $||f||_{L^p(K_\alpha)} < \varepsilon + ||g||_{L^p(K_\alpha)}$ for all $\alpha \in J$.

Now, since g has compact support, $||g||_{L^p(K_\alpha)}$ is zero for all but finitely many K_α's, therefore $(||g||_{L^p(K_\alpha)})_J \in c_0$. Since ε is independent of α, this implies that $\lim_\alpha ||f||_{L^p(K_\alpha)} \leq \varepsilon$ and we conclude that $\lim_\alpha ||f||_{L^p(K_\alpha)} = 0$. In other words $f \in (L^p, c_0)$ and this proves ii).

If $\mu \in M_q$ then $\lim_V \sum_{V_i} |\mu|(K_\alpha)^q = ||\mu||_q^q$. This means that $\lim_V \sum_{V_i} \mu_\alpha = \mu$ in M_q where $\mu_\alpha = \mu_{K_\alpha}$. Since $\sum_{V_i} \mu_\alpha$ belongs to M^q_c for each $V_i \in V$, the proof is complete. +

<u>THEOREM 3.7</u>. i) C_c is dense in (C_0, ℓ^q) for $1 \leq q \leq \infty$.

ii) C_c is dense in (L^p, ℓ^q) for $1 \leq p,q < \infty$.

iii) C_c is dense in (L^p, c_0) for $1 \leq p < \infty$.

<u>PROOF</u>. First we note that C_c is included in all amalgam spaces. Let f be a function in the closure of C_c in (L^∞, ℓ^q). Hence, there exists a sequence $\{\phi_n\}$ in C_c such that $\lim ||\phi_n - f||_{\infty q} = 0$.

This means that given $\varepsilon > 0$ there exists n_0 such that for all

$n \geq n_0 \quad ||\phi_n - f||^q_{\infty q} = \sum_\alpha \sup_{K_\alpha} |\phi_n(x) - f(x)|^q < \varepsilon$. Since

$|\phi_n(x) - f(x)| \leq ||\phi - f||_{\infty q}$ for all $x \in G$, ϕ_n converges uniformly to f on G, therefore f is continuous and by Proposition 1.19, $f \in (C_0, \ell^q)$ if q is finite. If q is infinite $(C_0, \ell^\infty) = C_0$ and it is well known that C_c is dense in C_0.

Let $f \in (L^p, \ell^q)$. By Theorem 3.6 i), given $\varepsilon > 0$ there exists $g \in L^p_c$ such that $||f - g||_{pq} < \varepsilon/2$.

If E is the compact support of g then there exists h in $C_c(E)$ such that $||g - h||_p < \varepsilon/2|S(E)|^{1/q}$, because $C_c(E)$ is dense in $L^p(E)$. Hence by Proposition 3.4, $||g - h||_{pq} \leq |S(E)|^{1/q} \, ||g - h||_p$ therefore $||g - h||_{pq} < \varepsilon/2$. This implies that
$||f - h||_{pq} \leq ||f - g||_{pq} + ||g - h||_{pq} < \varepsilon$. Similarly, since L^p_c is dense in (L^p, c_0) (Theorem 3.6 ii)) and $C_c(E)$ is dense in $L^p(E)$ for all compact subset E of G, we conclude as in ii) that C_c is dense in (L^p, c_0).+

<u>COROLLARY 3.8</u>. i) (C_0, ℓ^r) is dense in (L^p, ℓ^q) for $1 \leq p < \infty$, $1 \leq r \leq q < \infty$.

ii) (L^r, ℓ^s) is dense in (L^p, ℓ^q) for $1 \leq p \leq r < \infty$, $1 \leq s \leq q < \infty$.

iii) (L^∞, ℓ^s) is dense in (L^∞, ℓ^q) for $1 \leq s < q \leq \infty$.

iv) (L^r, c_0) is dense in (L^p, c_0) for $1 \leq p < r < \infty$.

<u>PROOF</u>. These are direct consequences of (2.5) and (2.6), Theorem 3.7 and Theorem 3.6.+

REMARK 3.9. C_0 is a (dense) subspace of (L^p, c_0) for $1 < p \leq \infty$ $(1 \leq p < \infty)$. Indeed, if $f \in C_0$ and $\varepsilon > 0$ then there exists a compact set E such that $|f(x)| < \varepsilon$ for all $x \notin E$. Since $E \subseteq \cup\{ K_{\alpha_i} \mid i = 1,..,n \}$ n finite, $||f||_{L^p(K_\alpha)} < \varepsilon$ for all $\alpha \neq \alpha_i$ $i = 1,..,n$. Therefore $(||f||_{L^p(K_\alpha)})_J$ belongs to c_0. By Theorem 3.7 C_0 is dense in (L^p, c_0) for $1 \leq p < \infty$.

DEFINITION 3.10. Let A be any of the amalgam spaces (L^p, ℓ^q), (C_0, ℓ^q), (L^p, c_0), $1 \leq p,q \leq \infty$. For each $t \in G$, τ_t will denote the translation operator on A or on M_s, $1 \leq s \leq \infty$, defined by

$$\tau_t f(s) = f(s - t) \qquad (f \in A,\ s \in G)$$

$$\tau_t \mu(B) = \mu(-t + B) \qquad (\mu \in M_s,\ B \text{ a Borel set of } G).$$

The next theorem shows that for each $t \in G$, τ_t is a bounded operator.

THEOREM 3.11. Let $1 \leq p,q \leq \infty$. For each $t \in G$, $f \in (L^p, \ell^q)$ and $\mu \in M_q$

i) $||\tau_t f||_{pq} \leq 2^a \, ||f||_{pq}$.

ii) $||\tau_t \mu||_q \leq 2^a ||\mu||_q$.

PROOF. For K_γ $(\gamma \in J)$ and $t \in G$ let

$S(t + K\ ) = \{K_\alpha \mid K_\alpha \cap K_\gamma \neq \emptyset \}$.

$$||\tau_t f||_{L^\infty(K_\gamma)} = \sup_{K_\gamma} |f(x - t)| = \sup_{t+K_\gamma} |f(x)| \leq \sum_{S(t+K_\alpha)} \sup_{K_\alpha} |f(x)|$$

$$= \sum_{S(t+K_\gamma)} ||f||_{L^\infty(K_\alpha)}$$

and for $1 \leq p < \infty$

$$||\tau_t f||_{L^p(K_\gamma)} = \left[\int_{K_\gamma} |f(x-t)|^p\right]^{1/p} = \left[\int_{t+K_\gamma} |f(x)|^p\right]^{1/p}$$

$$\leq \left[\sum_{S(t+K_\gamma)} \int_{K_\alpha} |f(x)|^p\right]^{1/p} \leq \sum_{S(t+K_\gamma)} \left[\int_{K_\alpha} |f|^p\right]^{1/p}.$$

By Lemma 1.13 and the fact that cardinality of $S(t + K_\gamma)$ is less than or equal to 2^a we have that

$$(1) \quad ||\tau_t f||^q_{L^\infty(K_\gamma)} \leq (2^a)^{q-1} \sum_{S(t+K_\gamma)} ||f||^q_{L^\infty(K_\alpha)}$$

$$(2) \quad ||\tau_t f||^q_{L^p(K_\gamma)} \leq (2^a)^{q-1} \sum_{S(t+K\ )} ||f||^q_{L^p(K\ )}$$

This implies that for $1 \leq p,q < \infty$

$$||\tau_t f||_{\infty q} = \left[\sum_\gamma ||\tau_t f||^q_{L^\infty(K_\gamma)}\right]^{1/q} \leq \left[(2^a)^{q-1} \sum_\gamma \sum_{S(t+K_\gamma)} ||f||^q_{L^\infty(K_\alpha)}\right]^{1/q}$$

$$\leq 2^a \left[\sum_\alpha ||f||^q_{L^\infty(K_\alpha)}\right]^{1/q} = 2^a ||f||_{\infty q}$$

$$||\tau_t f||_{p\infty} = \sup_\gamma ||\tau_t f||_{L^p(K_\gamma)} \leq \sup_\gamma \sum_{S(t+K_\gamma)} ||f||_{L^p(K_\alpha)}$$

$$\leq 2^a \sup_\alpha ||f||_{L^p(K_\alpha)} = 2^a ||f||_{p\infty}$$

$$||\tau_t f||_{pq} = \left[\sum_\gamma ||\tau_t f||^q_{L^p(K_\gamma)}\right]^{1/q} \leq \left[(2^a)^{q-1} \sum_\gamma \sum_{S(t+K_\gamma)} ||f||^q_{L^p(K_\alpha)}\right]^{1/q}$$

$$\leq 2^a \left[\sum_\alpha ||f||^q_{L^p(K_\alpha)}\right]^{1/q} = 2^a ||f||_{pq}.$$

Since $|\tau_t\mu|(K_\gamma) = |\mu|(-t + K_\gamma) \leq \sum_{S(-t+K_\gamma)} |\mu|(K_\alpha)$,

$$||\tau_t\mu||_\infty = \sup_\gamma |\tau_t\mu|(K_\gamma) \leq 2^a \sup_\alpha |\mu|(K_\alpha) = 2^a ||\mu||_\infty.$$

Now, if $1 \leq q < \infty$ then once more by Lemma 1.13

$$||\tau_t\mu||_q = \left[\sum_\gamma |\mu|(K_\gamma)^q\right]^{1/q} \leq \left[(2^a)^{q-1} \sum_\gamma \sum_{S(-t+K_\gamma)} |\mu|(K_\alpha)^q\right]^{1/q}$$

$$\leq 2^a \left[\sum_\alpha |\mu|(K_\alpha)^q\right]^{1/q} = 2^a ||\mu||_q . \dagger$$

The next couple of lemmàs will be used to prove Theorem 3.14.

<u>LEMMA 3.12</u>. Let $1 \leq p \leq \infty$ and $g \in L_c^p$. If E is the support of g and E_t is the support of $\tau_t g - g$ $(t \in G)$ then

$$\sup_{t \in G} |S(E_t)| \leq (2^a + 1)|S(E)|$$

where $|S(E)|$ is as in page 36.

<u>PROOF</u>. Clearly $E_t \subseteq (t + E) \cup E$. Since $t + E \subseteq \cup\{t + K_\alpha | K_\alpha \in S(E)\}$ and $|S(t + K_\alpha)| \leq 2^a$ for each $t \in G$, we conclude that $|S(t + E)| \leq 2^a |S(E)|$ for all $t \in G$. Therefore $\sup_{t \in G} |S(E_t)| \leq 2^a |S(E)| + |S(E)|$. $\dagger$

<u>LEMMA 3.13</u>. i) If $f \in C_0$ then $\lim_{t \to 0} ||\tau_t f - f||_\infty = 0$.

ii) if $f \in L^p$ $(1 \leq p < \infty)$ then $\lim_{t \to 0} ||\tau_t f - f||_p = 0$.

<u>PROOF</u>. i) follows from [37, Theorem 15.4]. Indeed, given $\varepsilon > 0$ there exists a neighborhood U of 0 such that $|f(x) - f(y)| < \varepsilon$ for

all $y - x \in U$. So for $t \in U$ and $x \in G$

$|f(x - t) - f(x)| = |\tau_t f(x) - f(x)| < \varepsilon$ because $t = x - (x - t)$.

Since x is arbitrary and U does not depend on x we conclude that

$||\tau_t f - f||_\infty < \varepsilon$ for all $t \in U$.

ii) is a well known fact, see for example [37, Theorem 20.4]. +

L. Argabright and J. Gil de Lamadrid have showed [2, p. 3-20] that the next theorem does not hold for functions in (L^1, ℓ^∞).

THEOREM 3.14. Let $1 \leq p,q < \infty$. If f belongs to (L^p, ℓ^q), to (L^p, c_0) or to (C_0, ℓ^s), $1 \leq s \leq \infty$, then the map $t \longrightarrow \tau_t f$ is continuous on G.

PROOF. If $f \in (C_0, \ell^q)$ and $\varepsilon > 0$ then by Theorem 3.7 i), there exists $g \in C_c$ such that

(3) $$||f - g||_{\infty q} < \varepsilon.$$

Let E be the support of g. By Lemma 3.13 i) there exists a neighborhood U of 0 such that for all $t \in U$ $||\tau_t g - g||_\infty < \varepsilon/((2^a + 1)|S(E)|)$. If E_t is as in Lemma 3.12 then $||\tau_t g - g||_{pq} \leq |S(E_t)| \, ||\tau_t g - g||_\infty \leq (2^a + 1)|S(E)| \, ||\tau_t g - g||_\infty < \varepsilon$.

This together with (3) and Theorem 3.11 implies that for all $t \in U$,

$$||\tau_t f - f||_\infty \leq ||\tau_t f - \tau_t g||_{\infty q} + ||\tau_t g - g||_{\infty q} + ||g - f||_{\infty q}$$

$$< 2^a \, ||f - g||_{\infty q} + \varepsilon + \varepsilon < (2^a + 2)\varepsilon.$$

Therefore $\lim\limits_{t \to 0} ||\tau_t f - f||_{\infty q} = 0$. The proof for $f \in (L^p, c_0)$ is similar.

Now, since (C_0, ℓ^q) is dense in (L^p, ℓ^q) (Corollary 3.8) given $f \in (L^p, \ell^q)$ and $\varepsilon > 0$ there exists $g \in (C_0, \ell^q)$ such that $||f - g||_{pq} < \varepsilon$.

So, by Theorem 3.11 and inequality (2.4)

$$||\tau_t f - f||_{pq} \leq ||\tau_t f - \tau_t g||_{pq} + ||\tau_t g - g||_{pq} + ||g - f||_{pq}$$
$$\leq 2^{a} ||f - g||_{pq} + ||\tau_t g - g||_{\infty q} + ||g - f||_{pq}$$
$$< (2^{a} + 1)\varepsilon + ||\tau_t g - g||_{\infty q}.$$

Since ε does not depend on t, we have that $||\tau_t f - f||_{pq} \leq ||\tau_t g - g||_{\infty q}$. Hence, by our previous result

$\lim_{t \to 0} ||\tau_t f - f||_{pq} = 0$. The case $q = \infty$ is Lemma 3.13.

We have shown, up to here, that the map $t \longmapsto \tau_t f$ is continuous at 0, but this is enough because by Theorem 3.11, for $t, t_0 \in G$

$$||\tau_t f - \tau_{t_0} f||_{pq} = ||\tau_t(\tau_{t_0 - t} f - f)||_{pq} \leq 2^{a} ||\tau_{t_0 - t} f - f||_{pq}.$$

§ 4. CONVOLUTION AND POINTWISE PRODUCT

In this section we introduce two operations on the amalgam spaces: pointwise product and convolution; and two operations on the spaces of unbounded measures of type q: product and convolution.

These operations have been studied previously ([8],[12]) and with the exception of Theorem 4.8, the results presented here are not new.

Two important facts for our study of multipliers are (1) that under convolution all amalgam spaces and all spaces of unbounded measures of type q are L^1 and M_1 Banach modules, and (2) that under convolution and the norm $||\cdot||^{\#}_{p1}$ the amalgam spaces (L^p, ℓ^1) $(1 \leq p < \infty)$, (C_0, ℓ^1) are Segal algebras.

Our first result is an easy generalization of the pointwise product of L^p spaces.

PROPOSITION 4.1. If $1 \leq p,q,r,s \leq \infty$ are such that $1/p + 1/r = 1/m \leq 1$ and $1/q + 1/s = 1/n \leq 1$ then

a) $(L^p, \ell^q)(L^r, \ell^s) \subseteq (L^m, \ell^n)$

b) $(C_0, \ell^q)(C_0, \ell^s) \subseteq (C_0, \ell^n)$

c) $(L^p, c_0)(L^r, c_0) \subseteq (L^m, c_0)$.

Moreover, if $f \in (L^p, \ell^q)$ and $g \in (L^r, \ell^s)$ then $||fg||_{mn} \leq ||f||_{pq}\, ||g||_{rs}$.

PROOF. We will prove the case when p,q,r,s are finite. The remaining cases are proved (mutatis mutandis) in the same way.

Let $f \in (L^p, \ell^q)$, $g \in (L^r, \ell^s)$. We apply twice the Hölder inequality (as in [37, Corollary 12.5]). First with $\alpha_1 = m/p$, $\alpha_2 = m/r$, $f_1 = |f|^p$ and $f_2 = |g|^r$ we have that

$$\int_{K_\alpha} |fg|^m = \int_{K_\alpha} (|f|^p)^{m/p} (|g|^r)^{m/r} \leqq \left[\int_{K_\alpha} |f|^p\right]^{m/p} \left[\int_{K_\alpha} |g|^r\right]^{m/r}.$$

Second with $\alpha_1 = n/q$, $\alpha_2 = n/s$, $f_1 = ||f||^q_{L^p(K_\alpha)}$ and $f_2 = ||g||^s_{L^r(K_\alpha)}$ we conclude that

$$||fg||^n_{mn} = \left[\sum_\alpha \int_{K_\alpha} |fg|^m\right]^{n/m} \leqq \sum_\alpha \left[||f||^q_{L^p(K_\alpha)}\right]^{n/q} \left[||g||^s_{L^r(K_\alpha)}\right]^{r/s}$$

$$\leqq \left[\sum_\alpha ||f||^q_{L^p(K_\alpha)}\right]^{n/q} \left[\sum_\alpha ||g||^s_{L^r(K_\alpha)}\right]^{n/s}.$$

Therefore $||fg||_{mn} \leqq ||f||_{pq}\ ||g||_{rs}$.

b) follows from a) and Proposition 1.19 (the case when $q = s = \infty$ is well known).

Now, if $f \in (L^p, c_0)$ and $g \in (L^r, c_0)$, then by a) $fg \in (L^m, \ell^\infty)$ and $||fg||_{L^m(K_\alpha)} \leqq ||f||_{L^p(K_\alpha)} ||g||_{L^r(K_\alpha)}$. This implies that $\lim_\alpha ||fg||_{L^m(K_\alpha)} \leqq \lim_\alpha ||f||_{L^p(K_\alpha)} ||g||_{L^q(K_\alpha)} = 0$. Therefore $fg \in (L^m, c_0)$. ∓

To define the space of unbounded measures of type q on G×G we use the family of compact sets $\{K_{\alpha\gamma}\}_{J\times J}$ where $K_{\alpha\gamma} = K_\alpha \times K_\gamma$. It is clear that this is consistent with Definition 1.7 and for ν, μ measures on G, $\mu\times\nu(K_{\alpha\gamma}) = \mu(K_\alpha)\nu(K_\gamma)$.

This last fact implies our next result, which is stated but not proved in [8].

PROPOSITION 4.2. If $1 \leq q,s \leq \infty$ then $M_q(G) \times M_s(G) \subseteq M_n(G \times G)$ where $n = \max(q,s)$ and for $\mu \in M_q$, $\nu \in M_s$,

$||\mu \times \nu||_n \leq ||\mu||_q \; ||\nu||_s$.

PROOF. Let $\mu \in M_q$ and $\nu \in M_s$ and $K_{\alpha\gamma} \in \{K_{\alpha\gamma} | \; (\alpha,\gamma) \in J \times J\}$.

By inequality (2.2) we have that

$$\sum_{\alpha\gamma} |\mu \times \nu|(K_{\alpha\gamma})^n = \sum_{\alpha\gamma} |\mu|(K_\alpha)^n \, |\nu|(K_\gamma)^n = \sum_\alpha |\mu|(K_\alpha)^n \sum_\gamma |\nu|(K_\gamma)^n$$

$$= ||\mu||_n^n \; ||\nu||_n^n \leq ||\mu||_q^n \; ||\nu||_s^n.$$

Therefore $||\mu \times \nu||_n \leq ||\mu||_q \; ||\nu||_s$. †

Since the convolution of two measures does not always exist, we establish (as in [10, Chapter 8]) that two measures μ, ν are convolvable if for all $g \in C_c(G)$ the function $g^0(x,y) = g(x + y)$ on $G \times G$ is $|\mu| \times |\nu|$-integrable. In this case $\mu * \nu$ is defined by the equation

$$\mu*\nu(g) = \int g \, d\mu*\nu = \iint g(x + y) \, d\mu(x)d\nu(y) = \iint g(x + y) \, d\nu(y)d\mu(x)$$

for $g \in C_c(G)$.

THEOREM 4.3. Let $1 \leq q,s \leq \infty$ be so that $1/q + 1/s - 1 = 1/n \leq 1$. If $\mu \in M_q$ and $\nu \in M_s$ then μ, ν are convolvable and $\mu*\nu \in M_n$.

Moreover $||\mu*\nu||_n \leq 2^{2a} \; ||\mu||_q \; ||\nu||_s$.

PROOF. We will prove the case when q and s are finite. The remaining cases are proved (mutatis mutandis) using the same argument.

First we note that $1/q = 1/n + 1/s'$ and $1/s = 1/n + 1/q'$, these imply that

(1) $q(1/n + 1/s') = 1; \quad s(1/n + 1/q') = 1; \quad n'(1/q' + 1/s') = 1.$

Now, since $K + K$ is covered by 2^a K_α's (see Definition 1.6) we have that for each pair K_α, K_γ, $\alpha, \gamma \in J$,

$K_\alpha + K_\gamma = \alpha + \gamma + K + K \subseteq \bigcup_{i=1}^{2^a} \alpha + \gamma + \alpha_i + K$. That is $K_\alpha + K_\gamma$ is covered by 2^a K_α's.

So, if $g \in C_c(G)$ then

$$(2) \quad \sup\{|g(x+y)| \mid x + y \in K_\alpha + K_\gamma\} \leq \sum_{i=1}^{2^a} ||g||_{L^\infty(\alpha+\gamma+\alpha_i+K)}$$

$$= \sum_{i=1}^{2^a} ||g||_{L^\infty(K_{\alpha_i})}.$$

For $g \in C_c(G)$ (1) implies that

$$\int_{K_\alpha}\int_{K_\gamma} |g(x+y)|d\mu(x)d\nu(y) \leq \sup_{K_\alpha+K_\gamma} |g(x+y)| \; |\mu|(K_\gamma)|\nu|(K_\alpha)$$

$$= ||g^0||_{L^\infty(K_{\alpha\gamma})} |\mu|(K_\gamma)|\nu|(K_\alpha)$$

$$= ||g^0||_{L^\infty(K_{\alpha\gamma})}^{n'/q'} ||g^0||_{L^\infty(K_{\alpha\gamma})}^{n'/s'} |\mu|(K_\gamma)^{q/n}|\mu|(K_\gamma)^{q/s'}|\nu|(K_\alpha)^{s/n}|\nu|(K_\alpha)^{s/q'}$$

$$= \left[||g^0||_{L^\infty(K_{\alpha\gamma})}^{n'} |\nu|(K_\alpha)^s \right]^{1/q'} \left[||g^0||_{L^\infty(K_{\alpha\gamma})}^{n'} |\mu|(K_\gamma)^q \right]^{1/s'}$$

$$\left[|\mu|(K_\gamma)^q |\nu|(K_\alpha)^s \right]^{1/n}.$$

Applying the Hölder inequality (as in [37, Corollary 12.5])
with $\alpha_1 = 1/q'$, $\alpha_2 = 1/s'$, $\alpha_3 = 1/n$, $f_1(\alpha,\gamma) = ||g^0||^{n'}_{L^\infty(K_{\alpha\gamma})} |\nu|(K_\alpha)^s$,

$f_2(\alpha,\gamma) = ||g^0||^{n'}_{L^\infty(K_{\alpha\gamma})} |\mu|(K_\gamma)^q$, $f_3(\alpha,\gamma) = |\mu|(K_\gamma)^q |\nu|(K_\alpha)^s$

we see that

$$\sum_\alpha \sum_\gamma \int_{K_\alpha} \int_{K_\gamma} |g(x+y)| d|\mu|(x) d|\nu|(y)$$

$$\leq \left[\sum_{\alpha,\gamma} f_1\right]^{\alpha_1} \left[\sum_{\alpha,\gamma} f_2\right]^{\alpha_2} \left[\sum_{\alpha,\gamma} f_3\right]^{\alpha_3}.$$

On the other hand, by (2) and Lemma 1.13

$$\sum_{\alpha\gamma} f_1 = \sum_\alpha \sum_\gamma \sup\{\, |g(x+y)| \mid x+y \in K_\gamma + K_\alpha \,\}^{n'} |\nu|(K_\alpha)^{s'}$$

$$\leq \sum_\alpha \sum_\gamma 2^{an'-1} \sum_{i=1}^{2^a} ||g||^{n'}_{L^\infty(\alpha+\gamma+\alpha_i+K)} |\nu|(K_\alpha)^s$$

$$\leq 2^{an'-1} \sum_\alpha 2^a ||g||^{n'}_{\infty n'} |\nu|(K_\alpha)^s$$

$$= 2^{an'} ||g||^{n'}_{\infty n'} \sum_\alpha |\nu|(K_\alpha)^s = 2^{an'} ||g||^{n'}_{n'} ||\nu||^s_s .$$

Similarly, $\sum_{\alpha\gamma} f_2 \leq 2^{an'} ||g||^{n'}_{\infty n'} ||\mu||^q_q$ and

$$\sum_{\alpha\gamma} f_3 = \sum_\alpha \sum_\gamma |\mu|(K_\gamma)^q |\nu|(K_\alpha)^s = ||\mu||^q_q \, ||\nu||^s_s .$$

This implies by (1) that

$$\int\int |g(x+y)| d|\mu|(x) d|\nu|(y) \leq \sum_{\alpha\gamma} \int_{K_\alpha}\int_{K_\gamma} |g(x+y)| d|\mu|(x) d|\nu|(y)$$

$$\leq 2^a (||g||^{n'}_{\infty n'} ||\nu||^s_s)^{1/q'} (||g||^{n'}_{\infty n'} ||\mu||^q_q)^{1/s'} (||\mu||^q_q ||\nu||^s_s)^{1/n}$$

$\leq 2^{a} \; ||g||_{\infty n'} ||\nu||_{s} \; ||\mu||_{q}$.

Therefore μ, ν are convolvable and the linear functional $T(g) = \int g d\mu*\nu \quad (g \in C_c(G))$ is such that

$|T(g)| \leq 2^{a} \; ||g||_{\infty n'} ||\nu||_{s} ||\mu||_{q}$. Since C_c is dense in $(C_0, \ell^{n'})$ T has a unique continuous extension T in $(C_0, \ell^{n'})^*$ such that $||T|| \leq 2^{a} \; ||\nu||_{s} ||\mu||_{q}$.

Finally by Theorem 3.2 we conclude that $2^{-a} \; || \mu*\nu || \leq 2^{a} \; ||\mu||_{q} \; ||\nu||_{s}$. Hence $||\mu*\nu||_{n} \leq 2^{2a} \; ||\mu||_{q} ||\nu||_{s}$.†

COROLLARY 4.4. If $f \in (L^1, \ell^q)$, $1 \leq q \leq \infty$, $\mu \in M_s$, $1 \leq s \leq \infty$, and $1/q + 1/s - 1 = 1/n \leq 1$ then f as a measure is convolvable with μ, $f*\mu$ is an absolutely continuous measure with density

(3) $\quad f*\mu(t) = \int f(t - x) d\mu(x)$

$f*\mu$ belongs to (L^1, ℓ^n) and $||f*\mu||_{1n} \leq 2^{2a} \; ||f||_{1n} ||\mu||_{s}$.

PROOF. f as a measure belongs to M_q (see (2.10)), so by Theorem 4.3 f is convolvable with μ and $f*\mu \in M_n$.

On the other hand, it is well known (see for example [1, Proposition 1.1]) that $f*\mu$ is a function given by (3).

Finally, by (2.12) $||f*\mu||_{1n} = ||(f*\mu)m||_{n}$ and this ends the proof.†

COROLLARY 4.5. Let $1 \leq q,s \leq \infty$. If $f \in (L^1, \ell^q)$, $g \in (L^1, \ell^s)$ and $1/p + 1/s = 1/n \leq 1$ then $f*g \in (L^1, \ell^n)$,

$$f*g(t) = \int f(t - x) g(x) dx$$

and $||f*g||_{1n} \leq 2^{2a}\ ||f||_{1q}||g||_{1s}$.

PROOF. (2.10), (2.12) and Corollary 4.4.†

COROLLARY 4.6. If $f \in L^1(G)$ and $\mu \in M_q$ $(1 \leq q \leq \infty)$, then $f*\mu \in (L^1, \ell^q)$ and $||f*\mu||^{\#}_{1q} \leq ||f||_1||\mu||^{\#}_q$ where $||\cdot||^{\#}_q$ is the norm defined in Theorem 1.21.

PROOF. The first part follows from Corollary 4.4.

Now,

$$
\begin{aligned}
(f*\mu)^{\#}(t) &= \int_{t+L} |f*\mu|(x)dx \leq \int_{t+L} \int |f(x-s)|d|\mu|(s)dx \\
&= \int\int |f(x-s)|\ \chi_L(x-t)\ dx\ d|\mu|(s) \\
&= \int |f(u)| \int \chi_L(u+s-t)\ d|\mu|(s)\ du \\
&= \int |f(u)|\ |\mu|(t-u+L)\ du \\
&= \int |f(u)|\ \mu^{\#}(t-u)\ du = |f|*\mu^{\#}(t).
\end{aligned}
$$

Since $\mu^{\#} \in L^q$ we have by the Young inequality that

$$||f*\mu||^{\#}_{1q} = ||(f*\mu)^{\#}||_q \leq ||\ |f|*\mu^{\#}||\ \leq ||f||_1||\mu^{\#}||_q = ||f||_1||\mu||^{\#}_q. \dagger$$

The next theorem is basically §7 i) of [8] and Theorem 4.2 of [12] but with the improvement that in Young's inequality for amalgams (4.1) and (4.2) there is an explicit constant coming from the Structure Theorem.

THEOREM 4.7. If p, q, r, s are exponents such that $1/p + 1/r - 1 = 1/m \leq 1$ and $1/q + 1/s - 1 = 1/n \leq 1$ then

i) $(L^p, \ell^q) * (L^r, \ell^s) \subseteq (L^m, \ell^n)$

ii) $(L^p, \ell^q) * (L^{p'}, \ell^{q'}) \subseteq C_0$ $\quad 1 \le p \le \infty,\ 1 < q < \infty$

iii) $(L^p, \ell^q) * (L^{p'}, \ell^s) \subseteq (C_0, \ell^n)$ $\quad 1 \le p,s \le \infty,\ 1 < q < \infty$

iv) $(L^p, \ell^q) * (L^r, \ell^{q'}) \subseteq (L^m, c_0)$ $\quad 1 \le p,r \le \infty,\ 1 < q < \infty.$

Moreover if $f \in (L^p, \ell^q)$ and $g \in (L^r, \ell^s)$ then

$$(4.1)\qquad ||f*g||_{mn} \le 2^{a}\,||f||_{pq}||g||_{rs} \quad \text{if } m \neq 1$$

$$(4.2)\qquad ||f*g||_{1n} \le 2^{2a}\,||f||_{1q}||g||_{1s}.$$

PROOF. If $p = r = 1$ then i) and (4.2) follow from Corollary 4.5.

The argument to prove i) is similar to the proof of Theorem 4.3 and for the same reason given in that theorem we will prove i) for $1 < p,q,r,s < \infty$.

First we note that

$$(4)\qquad p(1/m + 1/r') = 1;\quad r(1/m + 1/p') = 1;\quad m'(1/p' + 1/r') = 1$$

$$(5)\qquad q(1/n + 1/s') = 1;\quad s(1/n + 1/q') = 1;\quad n'(1/q' + 1/s') = 1.$$

For $\phi \in C_c(G)$, $\alpha,\gamma \in J$ we have by (4) that

$$\int_{K_\alpha}\int_{K_\gamma} |\phi(x+y)|\ |f(x)|\ |g(y)|\ dxdy$$

$$= \int_{K_\alpha}\int_{K_\gamma} (|f(x)|^p|g(y)|^r)^{1/m}(|\phi(x+y)|^{m'}|f(x)|^p)^{1/r'}$$

$$(|\phi(x+y)|^{m'}|g(y)|^r)^{1/p'}\,dxdy.$$

The Hölder inequality (as in [37, Corollary 12.5]) with $\alpha_1 = 1/m$, $\alpha_2 = 1/r'$, $\alpha_3 = 1/p'$, $f_1(x,y) = |f(x)|^p|g(y)|^r$, $f_3(x,y) = |\phi(x+y)|^{m'}|g(y)|^r$, $f_2(x,y) = |\phi(x+y)|^{m'}|f(x)|^p$ implies

$$\int_{K_\alpha}\int_{K_\gamma}|\phi(x+y)|\ |f(x)|\ |g(y)|\ dxdy$$

$$\leq \left[\int_{K_\alpha}\int_{K_\gamma} f_1(x,y)dxdy\right]^{\alpha_1}\left[\int_{K_\alpha}\int_{K_\gamma} f_2(x,y)dxdy\right]^{\alpha_2}\left[\int_{K_\alpha}\int_{K_\gamma} f_3(x,y)dxdy\right]^{\alpha_3}.$$

But

$$\int_{K_\alpha}\int_{K_\gamma} f_1(x,y)dxdy = \int_{K_\alpha}|g(y)|^r dy\int_{K_\gamma}|f(x)|^p dx = ||g||^r_{L^r(K_\alpha)}\,||f||^p_{L^p(K_\gamma)}$$

$$\int_{K_\alpha}\int_{K_\gamma} f_2(x,y)dxdy = \int_{K_\alpha}\int_{K_\gamma}|\phi(x+y)|^{m'}|f(x)|^p dxdy$$

$$= \int_{K_\gamma}|f(x)|^p\int_{K_\alpha}|\phi'(-x-y)|^{m'}dydx$$

$$= \int_{K_\gamma}|f(x)|^p\int_{x+K_\alpha}|\phi'(-y)|^{m'}dydx$$

$$\leq ||\phi||^{m'}_{L^{m'}(K_\gamma+K_\alpha)}\,||f||^p_{L^p(K_\gamma)}.$$

Similarly $\displaystyle\int_{K_\alpha}\int_{K_\gamma} f_3(x,y)dxdy \leq ||\phi||^{m'}_{L^{m'}(K_\alpha+K_\gamma)}\,||g||^r_{L^r(K_\alpha)}.$

Hence by (4) and (5)

$$\int_{K_\alpha}\int_{K_\gamma}|\phi(x+y)|\ |f(x)|\ |g(y)|\ dxdy$$

$$\leq \left[||f||^p_{L^p(K_\gamma)}\,||g||^r_{L^r(K_\alpha)}\right]^{1/m}\left[||\phi||^{m'}_{L^{m'}(K_\alpha+K_\gamma)}\,||f||^p_{L^p(K_\gamma)}\right]^{1/r'}$$

$$\left[||\phi||^{m'}_{L^{m'}(K_\alpha+K_\gamma)}\,||g||^r_{L^r(K_\alpha)}\right]^{1/p'}$$

$$= ||\phi||_{L^{m'}(K_\alpha+K_\gamma)}\,||f||_{L^p(K_\gamma)}\,||g||_{L^r(K_\alpha)}$$

$$= \left[||f||^q_{L^p(K_\gamma)} ||g||^s_{L^r(K_\alpha)} \right]^{1/n} \left[||\phi||^{n'}_{L^{m'}(K_\alpha+K_\gamma)} ||f||^q_{L^p(K_\gamma)} \right]^{1/s'}$$

$$\left[||\phi||^{n'}_{L^{m'}(K_\alpha+K_\gamma)} ||g||^s_{L^r(K_\alpha)} \right]^{1/q'} .$$

Again applying the Hölder inequality, this time with $\alpha_1 = 1/n$, $\alpha_2 = 1/s'$, $\alpha_3 = 1/q'$, $f_1(\alpha,\gamma) = ||f||^q_{L^p(K_\gamma)} ||g||^s_{L^r(K_\alpha)}$,

$$f_2(\alpha,\gamma) = ||\phi||^{n'}_{L^{m'}(K_\alpha+K_\gamma)} ||f||^q_{L^p(K_\gamma)},$$

$$f_3(\alpha,\gamma) = ||\phi||^{n'}_{L^{m'}(K_\alpha+K_\gamma)} ||g||^s_{L^r(K_\alpha)}$$ we have that

$$\sum_\alpha \sum_\gamma \int_{K_\alpha} \int_{K_\gamma} |\phi(x+y)|\ |f(x)|\ |g(y)|\ dxdy$$

$$\leq \left[\sum_\alpha \sum_\gamma f_1(\alpha,\gamma) \right]^{\alpha_1} \left[\sum_\alpha \sum_\gamma f_2(\alpha,\gamma) \right]^{\alpha_2} \left[\sum_\alpha \sum_\gamma f_3(\alpha,\gamma) \right]^{\alpha_3} .$$

Now, as in Theorem 4.3, $K_\alpha + K_\gamma \subseteq \bigcup_{i=1}^{2^a} (\alpha + \gamma + \alpha_i + K)$, hence by Lemma 1.13

$$||\phi||^{n'}_{L^{m'}(K_\alpha+K_\gamma)} \leq (2^a)^{n'-1} \sum_{i=1}^{2^a} ||\phi||^{n'}_{L^{m'}(\alpha+\gamma+\alpha_i+K)} .$$

This implies that

$$\sum_\alpha \sum_\gamma f_2(\alpha,\gamma) = \sum_\alpha \sum_\gamma ||\phi||^{n'}_{L^{m'}(K_\alpha+K_\gamma)} ||f||^q_{L^p(K_\gamma)}$$

$$\leq (2^a)^{n'-1} \sum_\alpha \sum_\gamma \sum_{i=1}^{2^a} ||\phi||^{n'}_{L^{m'}(\alpha+\gamma+\alpha_i+K)} ||f||^q_{L^p(K_\gamma)}$$

$$\leq 2^{an'} \sum_\gamma ||\phi||^{n'}_{m'n'} ||f||^q_{L^p(K_\gamma)}$$

$$= 2^{an'} \, ||\phi||^{n'}_{m'n'} \, ||f||^{q}_{pq}.$$

Similarly $\sum_{\alpha} \sum_{\gamma} f_3(\alpha,\gamma) \leq 2^{an'} \, ||\phi||^{n'}_{m'n'} \, ||g||^{s}_{rs}$.

Since $\sum_{\alpha} \sum_{\gamma} f_1(\alpha,\gamma) = ||f||^{q}_{pq} \, ||g||^{s}_{rs}$ we conclude that

$$\int \int |\phi(x + y)| \, |f(x)| \, |g(y)| \, dxdy$$

$$\leq \sum_{\alpha} \sum_{\gamma} \int_{K_\alpha} \int_{K_\gamma} |\phi(x + y)| \, |f(x)| \, |g(y)| \, dxdy$$

$$\leq 2^{a} \left[||\phi||^{n'}_{m'n'} ||f||^{q}_{pq} \right]^{1/s'} \left[||\phi||^{n'}_{m'n'} ||g||^{s}_{rs} \right]^{1/q'} \left[||f||^{q}_{pq} ||g||^{s}_{rs} \right]^{1/n}$$

$$= 2^{a} \, ||\phi||_{m'n'} \, ||f||_{pq} \, ||g||_{rs}.$$

This means that the linear functional $T(\phi) = \int \phi(t) \, f*g(t) \, dt$ $(\phi \in C_c(G))$ satisfies $|T(\phi)| \leq 2^{a} \, ||\phi||_{m'n'} ||f||_{pq} ||g||_{rs}$. Since $C_c(G)$ is dense in $(L^{m'}, \ell^{n'})$ (in $(L^{m'}, c_0)$ if $n = 1$), T has a unique continuous extension T in $(L^{m'}, \ell^{n'})^* = (L^m, \ell^n)$ (in $(L^{m'}, c_0)^* = (L^m, \ell^1)$) such that $||T|| \leq 2^{a} \, ||f||_{pq} ||g||_{rs}$. By Theorem 3.1 we have that $||f*g||_{mn} \leq 2^{a} \, ||f||_{pq} ||g||_{rs}$; this proves i) and (4.1).

Let $f \in (L^p, \ell^q)$ and $g \in (L^{p'}, \ell^{q'})$. By i) $f*g \in L^\infty$. Assume that p is finite. We have

$$f*g(t) = \int f(t - x)g(s)ds = \int \tau_t f(s) g'(s) ds = < \tau_t f, g' >$$

so by Theorem 3.1,

$$|f*g(t) - f*g(s)| = |< \tau_t f, g' > - < \tau_s f, g' >| = |< \tau_t f - \tau_s f, g' >|$$
$$\leq ||g'||_{p'q'} \, ||\tau_t f - \tau_s f||_{pq}.$$

Since $\lim_{t \to s} ||\tau_t f - \tau_s f||_{pq} = 0$ by Theorem 3.14 we conclude that $f*g$ is a continuous function.

Now, since C_c is dense in (L^p, ℓ^q) and $L_c^{p'}$ is dense in $(L^{p'}, \ell^{q'})$ given $\varepsilon > 0$ there exists $\phi \in C_c$ and $h \in L_c^{p'}$ such that $||\phi - f||_{pq} < \varepsilon/||g||_{p'q'}$ and $||h - g||_{p'q'} < \varepsilon/||\phi||_{pq}$. These imply by (4.1) that

$$||\phi*h - f*g||_\infty \leq ||\phi*h - \phi*g||_\infty + ||\phi*g - f*g||_\infty$$
$$\leq 2^a \, ||\phi||_{pq} ||h - g||_{p'q'} + 2^a \, ||g||_{p'q'} ||\phi - f||_{pq}$$
$$< 2^a \, \varepsilon.$$

Since ε is arbitrary and $\phi*h \in C_c(G)$, this means that $f*g$ is in the closure of C_c in the space of continuous functions on G, since C_c is dense in C_0 we conclude that $f*g \in C_0$.

If p is infinite then this proof with the roles of f and g exchanged yields the same result.

Let $f \in (L^p, \ell^q)$ and $g \in (L^{p'}, \ell^s)$. By i) $f*g \in (L^\infty, \ell^n)$. Since $1/s = 1/n + 1/q' \geq 1/q'$, $(L^{p'}, \ell^s) \subseteq (L^{p'}, \ell^{q'})$ (relation (2.5)); by ii) $f*g \in C_0$. Then iii) is proved.

Let $f \in (L^p, \ell^q)$ and $g \in (L^r, \ell^{q'})$. Since L_c^p, L_c^r are dense in (L^p, ℓ^q), (L^r, ℓ^s) respectively, given $\varepsilon > 0$ there exists $\phi \in L_c^p$ and $\psi \in L_c^r$ such that

$$||f - \phi||_{pq} < \varepsilon/||g||_{rq'} \quad \text{and} \quad ||g - \psi||_{rs} < \varepsilon/||\phi||_{pq}.$$

Hence by (4.1)

$$||f*g - \phi*\psi||_{m\infty} \leq ||f*g - *g||_{m\infty} + ||\phi*g - \phi*\psi||_{m\infty}$$
$$\leq 2^{a} ||g||_{rq'} ||f - \phi||_{pq} + 2^{a} ||\phi||_{pq} ||g - \psi||_{rq'}$$
$$< 2^{a}\varepsilon.$$

Since ε is arbitrary and $\phi*\psi \in L_c^m$, this means that $f*g$ is in the closure of L_c^m in (L^m, ℓ^∞) because by i) $f*g \in (L^m, \ell^\infty)$; since L_c^m is dense in (L^m, c_0) we conclude that $f*g \in (L^m, c_0)$. †

THEOREM 4.8. Let $1 \leq p,q,s \leq \infty$. If $1/q + 1/s - 1 = 1/n \leq 1$ then

i) $(L^p, \ell^q) * M_s \subseteq (L^p, \ell^n)$

ii) $(L^p, \ell^q) * M_{q'} \subseteq (L^p, c_0)$ $\quad 1 \leq p \leq \infty,\ 1 < q < \infty$

iii) $(C_0, \ell^q) * M_s \subseteq (C_0, \ell^n)$ $\quad 1 \leq q \leq \infty$.

Hence $(C_0, \ell^q) * M_{q'} \subseteq C_0$ $\quad 1 \leq q \leq \infty$.

Moreover, if $f \in (L^p, \ell^q)$ and $\mu \in M_s$ then

(4.3) $\quad ||f*\mu||_{pn} \leq 2^{a} ||f||_{pq} ||\mu||_s$ $\quad$ if $p \neq 1$.

(4.4) $\quad ||f*\mu||_{1n} \leq 2^{2a} ||f||_{1q} ||\mu||_s$.

PROOF. If $p = 1$ then i) and (4.4) follow from Corollary 4.4.

We will prove i) for $1 < p,q,s < \infty$ using the same argument of Theorem 4.7 i). The remaining cases are proved (mutatis mutandis) in the same way.

Again we have that

(6) $\quad q(1/n + 1/s') = 1;\quad n'(1/q' + 1/s') = 1;\quad s(1/n + 1/q') = 1.$

Let $\phi \in C_c(G)$ and consider the following:

$$\int_{K_\alpha}\int_{K_\gamma} |\phi(x+y)|\ |f(x)|\ dx d|\mu|(y)$$

$$= \int_{K_\alpha}\left[\int_{K_\gamma}|\phi(x+y)|^{p'}dx\right]^{1/p'}\left[\int_{K_\gamma}|f(x)|^{p}\right]^{1/p} d|\mu|(y)$$

$$= ||f||_{L^p(K_\gamma)}\int_{K_\alpha}\left[\int_{y+K_\gamma}|\phi'(-x)|^{p}dx\right]^{1/p} d|\mu|(y)$$

$$\leqq ||f||_{L^p(K_\gamma)}\int_{K_\alpha}\left[\int_{K_\alpha+K_\gamma}|\phi(x)|^{p}dx\right]^{1/p} d|\mu|(y)$$

$$\leqq ||f||_{L^p(K_\gamma)}||\phi||_{L^{p'}(K_\alpha+K_\gamma)}|\mu|(K_\alpha)$$

$$= \left[||f||^q_{L^p(K_\gamma)}|\mu|(K_\alpha)^s\right]^{1/n}\left[||f||^q_{L^p(K_\gamma)}||\phi||^{n'}_{L^{p'}(K_\alpha+K_\gamma)}\right]^{1/s'}$$

$$\left[||\phi||^{n'}_{L^{p'}(K_\alpha+K_\gamma)}|\mu|(K_\gamma)^s\right]^{1/q'}.$$

Applying the Hölder inequality with $\alpha_1 = 1/n$, $\alpha_2 = 1/s'$, $\alpha_3 = 1/q'$, $f_1(\alpha,\gamma) = ||f||^q_{L^p(K_\gamma)}|\mu|(K_\alpha)^s$,

$f_2(\alpha,\gamma) = ||f||^q_{L^p(K_\gamma)}||\phi||^{n'}_{L^{p'}(K_\alpha+K_\gamma)}$, $f_3(\alpha,\gamma) = ||\phi||^{n'}_{L^{p'}(K_\alpha+K_\gamma)}|\mu|(K_\gamma)^s$

we see that

$$\sum_\alpha\sum_\gamma\int_{K_\alpha}\int_{K_\gamma}|\phi(x+y)|\ |f(x)|\ dx\ d|\mu|(y)$$

$$\leqq \left[\sum_\alpha\sum_\gamma f_1(\alpha,\gamma)\right]^{\alpha_1}\left[\sum_\alpha\sum_\gamma f_2(\alpha,\gamma)\right]^{\alpha_2}\left[\sum_\alpha\sum_\gamma f_3(\alpha,\gamma)\right]^{\alpha_3}.$$

As in Theorem 4.3 i),

$$||\phi||^{n'}_{L^{p'}(K_\alpha+K_\gamma)} \leqq (2^a)^{n'-1}\sum_{i=1}^{2^a}||\phi||^{n'}_{L^{p'}(\alpha+\gamma+\alpha_i+K)}.$$

$$\text{So,}\quad \sum_{\alpha}\sum_{\gamma} f_1(\alpha,\gamma) = \sum_{\alpha}\sum_{\gamma} ||f||^q_{L^p(K_\gamma)} |\mu|(K_\alpha)^s = ||f||^q_{pq}||\mu||^s_s$$

$$\sum_{\alpha}\sum_{\gamma} f_2(\alpha,\gamma) = \sum_{\alpha}\sum_{\gamma} ||f||^q_{L^p(K_\gamma)} ||\phi||^{n'}_{L^{p'}(K_\alpha+K_\gamma)}$$

$$\leqq (2^a)^{n'-1} \sum_{\gamma}\sum_{\alpha}\sum_{i=1}^{2^a} ||\phi||^{n'}_{L^{p'}(\alpha+\gamma+\alpha_i+K)} ||f||^q_{L^p(K_\gamma)}$$

$$\leqq 2^{an'} \sum_{\gamma} ||\phi||^{n'}_{p'n'} ||f||^q_{L^p(K_\gamma)} \leqq 2^{an'} ||\phi||^{n'}_{p'n'}||f||^q_{pq}$$

$$\sum_{\alpha}\sum_{\gamma} f_3(\alpha,\gamma) = 2^{an'} ||\phi||^{n'}_{p'n'}||\mu||^s_s.$$

Therefore

$$\int\int |\phi(x+y)|\ |f(x)|\ dxd|\mu|(y) \leqq \sum_{\alpha}\sum_{\gamma}\int_{K_\alpha}\int_{K_\gamma} |\phi(x+y)|\ |f(x)|\ dxd|\mu|(y)$$

$$\leqq \left[||f||^q_{pq}||\mu||^s_s\right]^{1/n'}\left[2^{an'}||f||^q_{pq}||\phi||^{n'}_{p'n'}\right]^{1/s'}\left[2^{an'}||\phi||^{n'}_{p'n'}||\mu||^s_s\right]^{1/q'}$$

$$= 2^a ||\phi||_{p'n'}||f||_{pq}||\mu||_s.$$

Since

$$\int \phi(y)df*\mu(y) = \int \phi*f'(y)d\mu(y) = \int\int \phi(x+y)f(x)dxd\mu(y)$$

(see [1, Proposition 1.4]) we conclude that the linear functional $T(\phi) = \int \phi(y)\ df*\mu(y) \quad (\phi \in C_c)$ is such that $|T(\phi)| \leqq 2^a ||\phi||_{p'n'}||f||_{pq}||\mu||_s$. Since C_c is dense in $(L^{p'}, \ell^{n'})$ T has a unique continuous extension T on $(L^{p'}, \ell^{n'})$ such that $||T|| \leqq 2^a ||f||_{pq}||\mu||_s$. By Theorem 3.1 $||f*\mu||_{pn} \leqq 2^a ||f||_{pq}||\mu||_s$.

This proves i) and (4.3).

Let $f \in (L^p, \ell^q)$ and $\mu \in M_{q'}$. By Theorem 3.6, given $\varepsilon > 0$ there exists $g \in L_c^p$ and $\nu \in M_c^{q'}$ such that

$$\|f - g\|_{pq} < \varepsilon/\|\mu\|_{q'} \quad \text{and} \quad \|\nu - \mu\|_{q'} < \varepsilon/\|g\|_{pq}.$$

Similarly to Theorem 4.7 i) these imply that $\|f*\mu - g*\nu\|_{p\infty} < 2^a\varepsilon$. Now, by i) $f*\mu \in (L^p, \ell^\infty)$ and $g*\nu \in L_c^p$. Since (L^p, c_0) is a closed subspace of (L^p, ℓ^∞) and L_c^p is dense in (L^p, c_0) we conclude that $f*\mu \in (L^p, c_0)$.

Finally if $f \in (C_0, \ell^q)$ and $\mu \in M_s$ then by Theorems 3.6 and 3.7 given $\varepsilon > 0$ there exists $g \in C_c$ and $\nu \in M_c^s$ such that

$$\|f - g\|_{\infty q} < \varepsilon/\|\mu\|_s \quad \text{and} \quad \|\nu - \mu\|_s < \varepsilon/\|g\|_{\infty q}.$$

Again by (4.1) $\|f*\mu - g*\nu\|_{\infty n} < 2^a\varepsilon$. By i) $f*g \in (L^\infty, \ell^n)$ and $g*\nu \in C_c$ because $g*\nu$ has compact support and for t, s in G

$$\begin{aligned} |g*\nu(t) - g*\nu(s)| &= \left|\int (g(t - x) - g(s - x))d\nu(x)\right| \\ &= \left|\int \tau_{-t}g(-x) - \tau_{-s}g(-x)d\nu(x)\right| \\ &= |(\tau_{-t}g - \tau_{-s}g)*\nu(0)| \le \|(\tau_{-t}g - \tau_{-s}g)*\nu\|_\infty \\ &\le \|\tau_{-t}g - \tau_{-s}g\|_\infty \|\nu\|_1. \end{aligned}$$

Since $t \longmapsto \tau_{-t}g$ from G to C_0 is continuous (Theorem 3.14) this implies that $g*\nu$ is continuous.

Therefore $f*\nu$ is in the closure of C_c in (L^∞, ℓ^n), that is, $f*\nu \in (C_0, \ell^n)$ (Theorem 3.7).$\dagger$

<u>DEFINITION 4.9</u>. [19, Definition 14.1] Let A be either $L^1(G)$ or $M_1(G)$. A Banach space B is said to be a <u>Banach A-module</u> if there exists a bilinear operation $\cdot : A \times B \longrightarrow B$ such that

(B-1) $(f*g)\cdot b = f\cdot(g\cdot b)$ for all $f,g \in A$, $b \in B$

(B-2) For some constant $C \geq 1$

$$||f\cdot b||_B \leq C\,||f||_A\,||b||_B \quad \text{for all } f \in A,\ b \in B.$$

It follows from Theorems 4.3, 4.7, 4.8 that all M_s $(1 \leq s \leq \infty)$ spaces and all amalgam spaces (L^p, ℓ^q), (C_0, ℓ^q), (L^p, c_0) $(1 \leq p,q \leq \infty)$ satisfy the condition (B-2) for L^1 and M_1.

Also, if $\mu_1, \mu_2 \in M_1$ and $\nu \in M_s$ $(1 \leq s \leq \infty)$ then by [10, VIII §3 Proposition 1], $(\mu_1*\mu_2)*\nu = \mu_1*(\mu_2*\nu)$. Hence by (2.11) M_q, (L^p, ℓ^q), (C_0, ℓ^q), (L^p, c_0) $(1 \leq p,q \leq \infty)$ satisfy the condition (B-1) for L^1 and M_1. Therefore all these spaces are Banach A- modules.

<u>DEFINITION 4.10</u>. Let B be a Banach L^1-module and $B_{abs} = \{f\cdot b \mid f \in L^1,\ b \in B\}$. B is said to be an <u>essential L^1-module</u> if the linear subspace generated by B_{abs} is dense in B. If $B = B_{abs}$ then B is an <u>absolutely continuous L^1-module</u>.

The definition of absolutely continuous L^1-module is due to Gulick, Liu and Rooij [31].

If B is a Banach L^1-module then its dual B^* becomes a Banach L^1-module under the operation

$$f\cdot b^*(b) = b^*(f\cdot b) \qquad (b^* \in B^*,\ b \in B,\ f \in L^1).$$

<u>DEFINITION 4.11</u>. A net $\{e_\alpha\}$ in a commutative, normed algebra A is an <u>approximate identity</u>, abbreviated <u>a.i.</u>, if for all $a \in A$ $\lim_\alpha e_\alpha a = a$ in A.

The next theorem shows the equivalence of essential and absolutely continuous L^1-modules and gives two characterizations of essential L^1-modules.

THEOREM 4.12. If B is a Banach L^1-module then the following statments are equivalent

i) B is an essential L^1-module.

ii) B is an absolutely continuous L^1-module.

iii) $\lim_{\alpha} ||e_\alpha \cdot b - b||_B = o$ for all $b \in B$ and any a.i. $\{e_\alpha\}$ in L^1.

iv) B^* is order free (if for $b^* \in B^*$, $f \cdot b^* = 0$ for all $f \in L^1$ then $b^* = 0$).

PROOF. The equivalence of i), ii) and iii) was proved by M. A. Rieffel in [48, p. 453].

Suppose that $B = B_{abs}$ and for $b^* \in B^*$, $f \cdot b^* = 0$ for all $f \in L^1$. If $b \in B$ then $b = f \cdot b_1$ for some $f \in L^1$ and some $b_1 \in B$. So, $b^*(b) = b^*(f \cdot b_1) = f \cdot b^*(b_1) = 0$ since $f \cdot b^* = 0$. Therefore $b^* = 0$. This shows that ii) implies iv).

Now consider the inclusion map $i: B_{abs} \longrightarrow B$. Since i is linear and continuous its adjoint map $i^*: B^* \longrightarrow B_{abs}{}^*$ is also continuous. If $i^*(b^*) = 0$ then for all $f \in L^1$ and $b \in B$, $i^*(b^*)(f \cdot b) = b^*(f \cdot b) = f \cdot b^*(b) = 0$, this implies that $f \cdot b^* = 0$ for all $f \in L^1$. Hence, if iv) holds then $b^* = 0$. This means that i^* is injective and this implies that B_{abs} is dense in B [44, Corollary 4.12 p. 94]. Therefore B is an essential L^1-module. That is, iv) implies i) and the proof is complete.

PROPOSITION 4.13. The following amalgam spaces are absolutely continuous L^1-modules.

i) (L^p, ℓ^q) $1 \leq p,q < \infty$

ii) (C_0, ℓ^q) $1 \leq q \leq \infty$

iii) (L^p, c_0) $1 \leq p < \infty$.

PROOF. Let A be any of the spaces listed in the statment of the theorem. Suppose that for $a^* \varepsilon A^*$, $f*a^* = 0$ for all $f \varepsilon L^1$. In particular $g*a^* = 0$ for all $g \varepsilon C_c$. This implies that $< g,a^* > = g*a^*(0) = 0$ for all $g \varepsilon C_c$. Since C_c is dense in A (Theorem 3.7), we conclude that $< a,a^* > = 0$ for all $a \varepsilon A$. Hence $a^* = 0$ and the conclusion follows from Theorem 4.12.$_\dagger$

COROLLARY 4.14. Let A be any of the amalgam spaces listed in Proposition 4.13. If $\{e_\alpha\}$ is an approximate identity in L^1 then

$$\lim_\alpha ||e_\alpha*f - f||_A = 0$$

for all $f \varepsilon A$.

PROOF. Theorem 4.12 and Proposition 4.13.$_\dagger$

NOTATION. We will denote by $\underline{(L^p, \ell^q)^{\#}}$, $\underline{(C_0, \ell^q)^{\#}}$, $\underline{(L^p, c_0)^{\#}}$, $\underline{M_q^{\#}}$ $(1 \leq p,q \leq \infty)$ the spaces (L^p, ℓ^q), (C_0, ℓ^q), (L^p, c_0), M_q with the norm $||\cdot||^{\#}_{pq}$, $||\cdot||^{\#}_{\infty q}$, $||\cdot||^{\#}_{p\infty}$, $||\cdot||^{\#}_{q}$ defined in Theorem 1.21 respectively.

DEFINITION 4.15. A linear subspace S of L^1 is said to be a Segal algebra if it satisfies the following conditions

(S-0) S is dense in L^1.

(S-1) S is a Banach space under some norm $||\cdot||_S$, and there exists a constant C such that for all $f \in S$

$$||f||_1 \leq C\, ||f||_S$$

(S-2) S is invariant under translations ($f \in S$ implies $\tau_s f \in S$ for all $s \in G$) and for all $f \in S$ the mapping $s \longmapsto \tau_s f$ from G to S is continuous.

(S-3) The norm $||\cdot||_S$ is invariant in the sense that

$$||\tau_s f||_S = ||f||_S \quad \text{for all } s \in G,\ f \in S.$$

<u>THEOREM 4.16</u>. $(L^p, \ell^1)^\#$ $(1 < p < \infty)$ and $(C_0, \ell^1)^\#$ are Segal algebras.

<u>PROOF</u>. By Corollary 3.8, (L^p, ℓ^1) and (C_0, ℓ^1) are dense in L^1. $(L^p, \ell^1)^\#$ satisfies (S-1) ((2.4) and Theorem 1.21) and (S-2) (Theorem 3.11 and Theorem 3.14) for $1 < p \leq \infty$.

Finally if $s \in G$ and $f \in (L^p, \ell^1)$ $(1 < p \leq \infty)$ then

$$(\tau_s f)^\#(t) = ||\tau_s f||_{L^p(t + L)} = ||f||_{L^p(s + t + L)} = f^\#(s + t) = \tau_{-s} f^\#(t).$$

This implies that

$$||\tau_s f||^\#_{p1} = ||(\tau_s f)^\#||_1 = ||\tau_{-s} f^\#||_1 = ||f^\#||_1 = ||f||^\#_{p1}.$$

Hence (L^p, ℓ^1) $(1 < p \leq \infty)$ implies (S-3) and the proof is complete because $(C_0, \ell^1) \subseteq (L^\infty, \ell^1)$. †

<u>PROPOSITION 4.17</u>. $(L^p, \ell^1)^\#$, $1 < p < \infty$, and $(C_0, \ell^1)^\#$ satisfy the following

i) For all $f \in L^1$, $\mu \in M_1$, $h \in (L^p, \ell^1)$, $g \in (C_0, \ell^1)$

$$||f*h||^\#_{p1} \leq ||f||_1 ||h||^\#_{p1} \ ; \quad ||f*g||^\#_{\infty 1} \leq ||f||_1 ||g||^\#_{\infty 1}$$

$$||\mu * h||^{\#}_{p1} \leq ||\mu||_1 ||h||^{\#}_{p1} \;;\quad ||\mu * g||^{\#}_{\infty 1} \leq ||\mu||_1 ||g||^{\#}_{\infty 1}.$$

ii) $(C_0, \ell^1)^{\#}$ has an a.i. $\{e_n\}$ such that $||e_n||_1 = 1$ for all n. Hence $\{e_n\}$ is an a.i. in $(L^p, \ell^1)^{\#}$ $(1 < p < \infty)$.

PROOF. i) follows from [46, §4 Proposition 1 i) and Proposition 2]. The first part of ii) follows from [46, §8 Proposition 1 ii)] and the second part is a consequence of inclusions (2.6) and inequality (2.4).†

CHAPTER II

THE FOURIER TRANSFORM ON (L^p, ℓ^q) AND M_q

§ 5. THE FOURIER TRANSFORM ON (L^p, ℓ^q) $(1 \le p,q \le 2)$

In order to study the Fourier transform of functions in (L^p, ℓ^q) for $1 \le p,q \le 2$ we proceed as F. Holland has done for the real case [34, Theorem 8]. First we generalize the Riesz-Thorin Theorem for amalgam spaces and then we use Theorems 3.4 and 3.5 of [49] to locate the image of (L^p, ℓ^q) under the Fourier transform and to establish the corresponding Hausdorff-Young inequality.

Bertrandias and Dupuis [7 , §3] used another method. They find the Fourier transform of functions in the extreme cases (L^2, ℓ^1), (L^1, ℓ^2) and use a particular case of the Riesz-Thorin Theorem for mixed-norm spaces established by Benedek and Panzone.

For the definition of the (inverse) Fourier-Stieltjes transform of a measure $\mu \in M_1$ we follow [37, Definition 23.9 (Definition 31.2)].

DEFINITION 5.1. The Fourier-Stieltjes (inverse Fourier-Stieltjes) transform of a measure μ in $M_1(G)$ $(M_1(\hat{G}))$ is a function $\hat{\mu}$ $(\check{\mu})$ on $\hat{G}$ (G) defined by

$$\hat{\mu}(\hat{x}) = \int_G \overline{[x,\hat{x}]}\, d\mu(x) = \int_G [-x,\hat{x}]\, d\mu(x)$$

$$\left(\check{\mu}(x) = \int_{\hat{G}} [x,\hat{x}]\, d\mu(\hat{x})\right).$$

Throughout the rest of our work we will make constant use of the next theorem due to J. Stewart.

THEOREM 5.2. [49, Theorem 3.1]. Let $1 \leq p,q \leq \infty$. If E is a compact subset of G then there exists a function $g \in C_c(G)$ such that $g \equiv 1$ on E and $\hat{g} \in (L^p, \ell^q)(\hat{G})$ ($\hat{g} \in (C_0, \ell^q)(\hat{G})$ if $p = \infty$).

THEOREM 5.3. [49,Theorem 3.4]. Let $f \in L^p(G)$, $1 \leq p \leq 2$. If supp $f \subseteq E$, E being a compact subset of G, then $\hat{f} \in (C_0, \ell^{p'})(\hat{G})$ and there exists a constant C_p depending only on E and p such that

$$||\hat{f}||_{\infty p'} \leq C_p \ ||\hat{f}||_{p'}.$$

The next theorem was proved for the real line by Holland [35, Theorem 8] and for locally compact abelian groups by Stewart [49, Theorem 3.5].

THEOREM 5.4. If $f \in (L^p, \ell^1)(G)$, $1 \leq p \leq 2$, then $\hat{f} \in (C_0, \ell^{p'})(\hat{G})$ and there exists a constant C_p depending only on p such that

$$||\hat{f}||_{\infty p'} \leq C_p ||f||_{p1}.$$

NOTATION. $(L_\alpha, B_\alpha, m_\alpha)$ $(\alpha \in J)$ will be the measurable subspace of (G, B, m) where $\{L_\alpha\}$ is the family of sets defined in Definition 1.6 and B_α, m_α are the restrictions of the σ-algebra B and the Haar measure m on L_α respectively.

Similarly $(L_\beta, B_\beta, m_\beta)$ $(\beta \in I)$ (see p.23) is the corresponding measurable subspace of $(\hat{G}, \hat{B}, \mu)$.

For $1 \leq r \leq \infty$, $\ell^r_F(J)$ will denote the subspace of $\ell^r(J)$ consisting of all nets $(a_\alpha)_J$ such that $a_\alpha = 0$ for all but finetely many α.

The direct sum $\bigoplus_{\alpha \in J} S_\alpha$ of the linear spaces S_α of m_α-measurable simple functions on L_α, will be denoted by $\mathcal{L}(m)$. That is, $(x_\alpha)_J \in \mathcal{L}(m)$ iff $x_\alpha \in S_\alpha$ and $x_\alpha = 0$ for all but finetely many α. $\mathcal{L}(\mu)$ is defined similarly.

REMARK 5.5. i) The space S_c of measurable simple functions with compact support can be identified with $\mathcal{L}(m)$. Indeed, if $s \in S_c$ then the support of s meets finetely many L_α's, so the net $(s_\alpha)_J$ where $s_\alpha = s|L_\alpha$ belongs to $\mathcal{L}(m)$. Conversely if $(s_\alpha) \in \mathcal{L}(m)$ then $s_\alpha \neq 0$ for $\alpha_1,\ldots,\alpha_n$ say, so the function $s = \sum_{i=1}^{n} s_{\alpha_i}$ is a simple function whose support is contained in $\bigcup_{i=1}^{n} K_{\alpha_i}$ and therefore $s \in S_c$.

ii) S_c is dense in (L^p, ℓ^q) for $1 \leq p \leq \infty$, $1 \leq q < \infty$, because S_s is dense in L^p_c and L^p_c is dense in (L^p, ℓ^q) (Theorem 3.6).

THEOREM 5.6. (Riesz-Thorin). Let $1 \leq p_i, q_i, r_i, s_i \leq \infty$ $i = 1, 2$. Given $0 \leq \theta \leq 1$ set $1/r = \theta/r_1 + (1-\theta)/r_2$, $1/s = \theta/s_1 + (1-\theta)/s_2$, $1/p = \theta/p_1 + (1-\theta)/p_2$ and $1/q = \theta/q_1 + (1-\theta)/q_2$.

Let T be a linear operator on $\mathcal{L}(m)$ such that $T\mathcal{L}(m) \subseteq (L^{q_i}, \ell^{s_i})(\hat{G})$ $i = 1, 2$ and suppose that there exist positive constants C_i $i = 1, 2$ such that

$$||Tx||_{q_i s_i} \leq C_i \; ||x||_{p_i q_i} \qquad i = 1, 2.$$

Then $T\mathcal{L}(m) \subseteq (L^q, \ell^s)(\hat{G})$.

Furthermore for all $x \in \mathcal{L}(m)$

$$||Tx||_{qs} \leq C_1^{\theta} C_2^{1-\theta} \; ||x||_{pr} \qquad \text{if} \quad 1 < q \leq \infty, \; 1 \leq s \leq \infty$$

$$||Tx||_{1s} \leq C_1^{\theta} C_2^{1-\theta} \; 2^{a} \; ||x||_{pr} \quad 1 \leq s \leq \infty.$$

PROOF. Consider the following diagrams (for notation see §3 p. 32).

$$\begin{tikzcd}
\bigoplus_\alpha S_\alpha \arrow[r, "T"] & \ell^{s_i}(L^{q_i}(L_\beta)) \arrow[d, "\Pi_\beta"] \\
S_\alpha \arrow[u] \arrow[r, "T_{\alpha\beta}"] & L^{q_i}(L_\beta)
\end{tikzcd}
\qquad
\begin{tikzcd}
\overline{x_\alpha} \arrow[r, mapsto] & T(\overline{x_\alpha}) \arrow[d, mapsto] \\
x_\alpha \arrow[u, mapsto] \arrow[r, mapsto] & \Pi_\beta o T(\overline{x_\alpha})
\end{tikzcd}$$

where $x_\alpha \mapsto \overline{x_\alpha}$ is the canonical inclusion map and Π_β is the β-th projection.

Since each $x \in \bigoplus_\alpha S_\alpha$ can be expressed as $\sum_\alpha \overline{x_\alpha}$ $(x_\alpha \in S_\alpha)$ we have that $Tx = \left[\sum_\alpha T_{\alpha\beta}(x_\alpha)\right]_{\beta \in I}$. That is, $\Pi_\beta o Tx = \sum_\alpha T_{\alpha\beta}(x_\alpha)$. We think of T as a $J \times I$ matrix of operators $[T_{\alpha\beta}]$. For $0 \leq \theta \leq 1$ we have the numbers p, q, r, s and for these we define $A(\theta)$ to be the set of (a, y, b, x) in $\ell_F^{s'}(I) \oplus \mathcal{L}(\mu) \oplus \ell_F^{r}(J) \oplus \mathcal{L}(m)$ such that $||a||_{s'} \leq 1$. $||y_\beta||_{q'} \leq 1$ for each $\beta \in I$, $||b||_r \leq 1$ and $||x_\alpha||_p \leq 1$ for each $\alpha \in J$.

Now we define for $0 \leq \theta \leq 1$

$$M(\theta) = \sup_{A(\theta)} \left| \sum_{\alpha\beta} a_\beta\, b_\alpha \int_{L_\beta} y_\beta\, T_{\alpha\beta}(x_\alpha)\, d\mu_\beta \right|.$$

Note that if $(a, y, b, x) \in A(\theta)$ for some θ then $ay = (a_\beta y_\beta)_I$ belongs to $\ell^{s'}(L^{q'}(L_\beta))$ and $||ay||_{q's'} \leq 1$. Indeed

$$\sum_\beta \left[\int_{L_\beta} |a_\beta y_\beta|^{q'} \right]^{s'/q'} = \sum_\beta |a_\beta|^{s'} \left[\int_{L_\beta} |y_\beta|^{q'} \right]^{s'/q'} = \sum_\beta |a_\beta|^{s'} ||y_\beta||_{q'}^{s'}$$

$$\leq \sum_\beta |a_\beta|^{s'} = ||a||_{s'}^{s'} \leq 1.$$

Similarly $bx = (b_\alpha x_\alpha)_J$ belongs to $\ell^r(L^p(L_\alpha))$ and $||bx||_{pr} \leq 1$. So, by the Hölder inequality

$$\left| \sum_{\alpha\beta} a_\alpha b_\beta \int_{L_\beta} y_\beta\, T_{\alpha\beta}(x_\alpha)\, d\mu_\beta \right| \leq \sum_{\alpha\beta} \int_{L_\beta} |a_\beta y_\beta|\, |T_{\alpha\beta}(b_\alpha x_\alpha)|\, d|\mu|_\beta$$

$$\leq \sum_\beta ||a_\beta y_\beta||_{L^{q'}(L_\beta)} ||\Pi_\beta o T(bx)||_{L^q(L_\beta)}$$

$$\leq \left[\sum_\beta ||a_\beta y_\beta||^{s'}_{L^{q'}(L_\beta)} \right]^{1/s'} \left[\sum_\beta ||\Pi_\beta o T(bx)||^{s}_{L^q(L_\beta)} \right]^{1/s}$$

$$= ||ay||_{q's'} ||T(bx)||_{qs}$$

$$\leq ||T(bx)||_{qs}.$$

Since this is true for any $(a, y, b, x) \in A(\theta)$, $M(\theta) \leq ||T(bx)||_{qs}$. This implies that $M(0) \leq C_2$ and $M(1) \leq C_1$. By [18, Lemma 7 p. 522] the function $\log M(\theta)$ on $[0,1]$ is convex and we conclude that $M(\theta) \leq C_1^\theta\, C_2^{1-\theta}$.

We will see next that

$$(1) \qquad \left| \sum_{\beta} \int_{L_\beta} y_\beta \, \Pi_\beta \circ T(x) \, d\mu_\beta \right| \leq C_1^{\theta} \, C_2^{1-\theta} \, ||y||_{q's'} ||x||_{pr}$$

for all $x, y \in \mathcal{L}(m) \times \mathcal{L}(\mu)$.

The case when $x = 0$ or $y = 0$ is trivial. So we take nonzero x, y in $\mathcal{L}(m)$, $\mathcal{L}(\mu)$ and consider

$$y_\beta' = \begin{cases} y_\beta / ||y_\beta||_{q'} & y_\beta \neq 0 \\ 0 & \text{otherwise} \end{cases} \qquad a_\beta = ||y_\beta||_{q'} / ||y||_{q's'}$$

$$x_\alpha' = \begin{cases} x_\alpha / ||x_\alpha||_p & x_\alpha \neq 0 \\ 0 & \text{otherwise} \end{cases} \qquad b_\alpha = ||x_\alpha||_p / ||x||_{pr}$$

Then $a = (a_\beta)_I$, $y' = (y_\beta')_I$, $b = (b_\alpha)_J$, $x = (x_\alpha')_J$ belong to $A(\theta)$ and we have that

$$C_1^{\theta} \, C_2^{1-\theta} \geq M(\theta) \geq \left| \sum_{\alpha\beta} a_\beta b_\alpha \int_{L_\beta} y_\beta' \, \Pi_\beta \circ T(x') \, d\mu_\beta \right|$$

$$= \left| \sum_{\alpha\beta} \int_{L_\beta} a_\beta y_\beta \, T_{\alpha\beta}(b_\alpha x_\alpha') \, d\mu_\beta \right|$$

$$= \left| \sum_{\alpha\beta} \int_{L_\beta} y_\beta / ||y||_{q's'} \, T_{\alpha\beta}(x_\alpha / ||x||_{pr}) \, d\mu_\beta \right|$$

$$= 1/||y||_{q's'} \, 1/||x||_{pr} \left| \sum_{\alpha\beta} \int_{L_\beta} y_\beta \, T_{\alpha\beta}(x_\alpha) \, d\mu_\beta \right|.$$

(1) implies that the map $Tx: \mathcal{L}(\mu) \longrightarrow \mathbf{C}$ given by $y \longmapsto \sum_{\beta} \int_{L_\beta} y_\beta \, \Pi_\beta \circ T(x) \, d\mu_\beta$ is a continuous linear functional on $(\mathcal{L}(\mu), ||\cdot||_{q's'})$ and $||Tx|| \leq C_1^{\theta} \, C_2^{1-\theta} \, ||x||_{pr}$ $(x \in \mathcal{L}(m))$.

Since $\mathcal{L}(\mu)$ is a linear subspace of $(L^{q'}, \ell^{s'})$, $(C_0, \ell^{s'})$, $(L^{q'}, c_0)$ and the duals of these spaces are (L^q, ℓ^s), M_s, (L^q, ℓ^1) respectively we conclude that $T\mathcal{L}(m) \subseteq (L^q, \ell^s)$.

Finally if $1 \leq s \leq \infty$ then by Theorem 3.1 $||Tx||_{qs} = ||Tx|| \leq C_1^{\theta} C_2^{1-\theta} ||x||_{pr}$ if $1 < q \leq \infty$ and by Theorem 3.2 $||Tx||_{1s} \leq 2^{a} ||Tx|| \leq C_1^{\theta} C_2^{1-\theta} 2^{a} ||x||_{pr}$. †

The next theorem was proved for the real line by Holland [34, Theorem 8] and for locally compact abelian groups by Bertrandias and Dupuis [7, Theorem II].

<u>THEOREM 5.7</u>. If $1 \leq p,q \leq 2$ then the Fourier transform $\hat{f}$ of a function $f \in (L^p, \ell^q)(G)$ belongs to

$(L^{q'}, \ell^{p'})(\hat{G})$ if $1 < p,q \leq 2$

$(C_0, \ell^{p'})(\hat{G})$ if $q = 1$, $1 \leq p \leq 2$

$(L^{q'}, c_0)(\hat{G})$ if $p = 1$, $1 < q \leq 2$.

Moreover there exists a constant C_{pq} depending only on either p or q such that

$$||\hat{f}||_{q'p'} \leq C_{pq}^{\theta} ||f||_{pq} \qquad \text{(Hausdorff-Young inequality)}$$

where

$$\theta = \begin{cases} p'(1/q - 1/p) & \text{if } 1 < q < p \\ q'(1/p - 1/q) & \text{if } 1 < p < q \\ 1 & \text{if } p = 1 \text{ or } q = 1. \end{cases}$$

<u>PROOF</u>. Case 1) $q = 1$, $1 \leq p \leq 2$. By Theorem 5.4, if

$f \in (L^p, \ell^1)(G)$ then $\hat{f} \in (C_0, \ell^{p'})(\hat{G})$ and there exists a constant C_p depending on p such that

(2) $\quad ||\hat{f}||_{\infty p'} \leqq C_p\ ||f||_{p1}.$

Case 2) $1 < q \leq p$. If $f \in (L^p, \ell^q)(G)$ then by (2.7) $f \in L^p(G)$ and by the Hausdorff-Young inequality

(3) $\quad ||\hat{f}||_{p'} \leqq ||f||_p.$

Applying Theorem 5.6 to $f \in S_c$ with $q_1 = \infty$, $q_2 = s_1 = s_2 = p'$, $r_2 = p_1 = p_2 = p$, $r_1 = 1$, $\theta = p'(1/q - 1/p)$, $C_1 = C_p$ (as in (2)) and $C_2 = 1$ (as in (3)), we have that $\hat{f} \in (L^{q'}, \ell^{p'})(\hat{G})$ and

(4) $\quad ||\hat{f}||_{q'p'} \leqq C_p^{\theta}\ ||f||_{pq}.$

Since S_c is dense in (L^p, ℓ^q) the Fourier transform can be extended to all (L^p, ℓ^q) and case 2) is proved.

Case 3) $1 \leqq p < q$. Take $f \in L_c^p$ and $g \in (L^q, \ell^p)(\hat{G})$, hence $g \in L^p(\hat{G})$ by (2.7). So by a generalization of Parseval's identity (as in [37, 31.48 a)]).

$$\int_{\hat{G}} \hat{f}(\hat{x})\ \overline{g(\hat{x})}\ d\hat{x} = \int_G \overline{\check{g}(x)}\ f(x)\ dx.$$

On the other hand by Theorem 5.3, $\hat{f} \in (C_0, \ell^{p'})(\hat{G})$ and by case 2) $\check{g} \in (L^{p'}, \ell^{q'})(G)$, therefore by Theorem 3.1

$$\left| \int_{\hat{G}} \hat{f}(\hat{x})\ \overline{g(\hat{x})}\ d\hat{x} \right| \leqq \int_G |\check{g}(x)|\ |f(x)|\ dx \leqq ||\check{g}||_{p'q'} ||f||_{pq}.$$

This implies by (2) and (4) that

$$\left| \int_{\hat{G}} \hat{f}(\hat{x})\, g(\hat{x})\, d\hat{x} \right| \leq \begin{cases} C_q \, ||g||_{q1} \, ||f||_{1q} \\ C_q \, ||g||_{qp} \, ||f||_{pq} & \text{if } p \neq 1 \end{cases}$$

where C_q is a constant depending only on q and $\theta = q'(1/p - 1/q)$. Hence the map $g \longmapsto \int_{\hat{G}} \overline{g(\hat{x})}\hat{f}(\hat{x})d\hat{x}$ is a linear functional on $(L^q, \ell^p)(\hat{G})$. By Theorem 3.1, $\hat{f} \in (L^{q'}, \ell^{p'})(\hat{G})$ and

$$||\hat{f}||_{q'p'} \leq \begin{cases} C_q \, ||f||_{1q} \\ C_q^{\theta} \, ||f||_{pq} & \text{if } p \neq 1 \end{cases}$$

Moreover, if $p = 1$ then $\hat{f} \in C_0$ and by Remark 3.9 $\hat{f} \in (L^{q'}, c_0)(\hat{G})$.

Since L^p_c is dense in (L^p, ℓ^q), the Fourier transform can be extended to all of (L^p, ℓ^q) and we have that $(L^p, \ell^q)^\wedge \subseteq (L^{q'}, \ell^{p'})$ if $p \neq 1$ and $(L^1, \ell^q)^\wedge \subseteq (L^{q'}, c_0)$ (remember that $(L^{q'}, c_0)$ is a closed subspace of $(L^{q'}, \ell^\infty)$).

<u>REMARK 5.8</u>. If $f \in (L^p, \ell^q)$ $1 \leq q \leq 2$, $2 < p \leq \infty$ then by (2.6) $f \in (L^2, \ell^q)$ and by Theorem 5.7 $\hat{f} \in (L^{q'}, \ell^2)$.

§ 6. THE FOURIER TRANSFORM ON (L^p, ℓ^q) $(2 < q \le \infty)$ AND M_s $(1 \le s \le \infty)$

We start this section with the definitions of G. I. Gaudry [28, §1 p. 478] and W. R. Bloom [3, p. 206] for the Fourier transform of functions in $L^r(G)$, $1 \leq r \leq \infty$, and $(L^p, \ell^q)(G)$, $1 \leq p,q \leq \infty$, respectively. We will see that when $p = q$ both definitions coincide.

We will then proceed to extend Bloom's definition to the space M_s, $1 \leq s \leq \infty$, and study the relation between the transformable measures defined by L. Argabright and J. Gil de Lamadrid [1], and the measures M_s.

Finally we will give a brief account of other definitions of the Fourier transform of measures in (L^p, ℓ^q), $1 \leq p,q \leq \infty$, and M_s, $1 \leq s \leq \infty$; one due to J. P. Bertrandias and C. Dupuis, and another due to H. G. Feichtinger.

NOTATION. For a compact subset E of G, $C_c^E = C_c^E(G)$ will be the linear space of functions $f \in C_c$ such that $\operatorname{supp} f \subseteq E$, endowed with the supremum norm.

$\underline{D_E(G)}$ will denote the linear space

$$D_E(G) = \{h \mid h = \sum_{Z} f_i * g_i,\ \ f_i,\ g_i \in C_c^E(G),\ \ \sum_{Z} \|f_i\|_\infty \|g_i\|_\infty < \infty\}$$

and $\underline{A_E}(G) = \{g \in C_c(G) \mid g = \overset{\vee}{g},\ g \in L^1(\hat{G})$ and $\operatorname{supp} g \subseteq E\}$.

For $h \in D_E(G)$ and $g \in A_E(G)$ we define

$$||h||_E = \inf \{\sum_Z ||f_i||_\infty ||g_i||_\infty \mid h = \sum_Z f_i * g_i,\ f_i,\ g_i \in C_c^E,\ \sum_Z ||f_i||_\infty ||g_i||_\infty < \infty\}$$

$$||g||_{A_E} = ||\check{g}\,||_{A_E} = ||g\,||_1 .$$

REMARK 6.1. a) $D_E(G) \subseteq C_c^{E+E}(G)$ and $||\cdot||_E$ is well defined.

b) $||h||_\infty \leqq m(E)\ ||h||_E \quad (h \in D_E(G))$

c) $||g||_\infty \leqq ||g\,||_1 = ||g||_{A_E} \quad (g \in A_E(G))$

d) $(A_E(G), ||\cdot||_{A_E})$ and $(D_E(G), ||\cdot||_E)$ are Banach spaces [40, Theorem 5.1.1.].

DEFINITION 6.2. D(G) is the internal inductive limit of the Banach spaces $D_E(G)$. That is, $D(G) = \cup\{D_E(G) \mid E \subseteq G$ compact$\}$ and a basic neighborhood of the origen is of the type

$U_\varepsilon = \bigcup_E \{h \in D_E(G) \mid ||h||_E < \varepsilon\} \quad (\varepsilon > 0)$.

Similarly $\underline{A_c(G)}$ and $\underline{C_c(G)}$ are the internal inductive limits of the Banach spaces $A_E(G)$ and $C_c^E(G)$ respectively.

REMARK 6.3. It is known [16, Theorem 3.1] that the spaces D(G) and $A_c(G)$ are homeomorphic and isomorphic as spaces of functions on G, and that D(G) is dense in $C_c(G)$ [40, Theorem 5.1.2], hence so is $A_c(G)$.

LEMMA 6.4. $A_c(G) = \{\phi \in C_c(G) \mid \hat{\phi} \in (C_0, \ell^1)\}$. Hence by (2.5) and (2.6) $\hat{\phi} \in (L^p, \ell^q)(\hat{G})$, $1 \leqq p,q \leqq \infty$, for all $\phi \in A_c(G)$.

PROOF. If $\phi \in C_c$ and $\hat{\phi} \in (C_0, \ell^1)$ then by (2.6) $\hat{\phi} \in L^1(\hat{G})$ and this implies that $\phi = \check{\hat{\phi}}$. Therefore $\phi \in A_c(G)$.

Now take $g \in A_E(G)$ and let $f_E \in (C_0, \ell^1)(\hat{G})$ such that $\check{f}_E \equiv 1$ on E and $\check{f}_E \in C_c(G)$ (Theorem 5.2). Since $g = \check{g}$, $g \in L^1(\hat{G})$ we have that $g = \check{g}\check{f}_E = (g*f_E)^\vee$. By Theorem 4.7 $g*f_E \in (C_0, \ell^1)$, so $\hat{g} = \widehat{(g*f_E)^\vee} = g*f_E$ because $(g*f_E)^\vee \in L^1(\hat{G})$. Hence $\hat{g} \in (C_0, \ell^1)$ and therefore $g \in \{\phi \in C_c | \hat{\phi} \in (C_0, \ell^1)\}$.†

PROPOSITION 6.5. Let A be any of the following amalgam spaces:

(L^p, ℓ^q) $\quad 1 \leqq p,q < \infty$

(C_0, ℓ^q) $\quad 1 \leqq q \leqq \infty$

(L^p, c_0) $\quad 1 \leqq p < \infty$.

Then $D(G)$ is dense in $(C_c, ||\cdot||_A)$. Hence by Theorem 3.7 and Remark 6.3 $D(G)$ and $A_c(G)$ are dense in A and the topology on D(G) and $A_c(G)$ is stronger than the one induced by A.

PROOF. Let $\{e_\alpha\}$ be an a.i. in L^1 such that for all α, $||e_\alpha||_1 = 1$ and $e_\alpha \in C_c^E(G)$ for a fixed E. Since A is an absolutely continuous L^1-module, for all $f \in A$ $\lim ||f*e_\alpha - f||_A = 0$ (Corollary 4.14). In particular for all $\phi \in C_c(G)$ $\lim ||\phi*e_\alpha - \phi||_A = 0$. Since $\phi*e_\alpha \in D(G)$ for all α we conclude that D(G) is dense in $(C_c, ||\cdot||_A)$. Now, if $\phi \in C_c^E(G)$ then by Proposition 3.4 there exists a constant C_E depending on E and q such that $||\phi||_{\infty q} \leqq C_E ||\phi||_\infty$. Since $||\phi||_{p\infty} \leqq ||\phi||_\infty$ and $||\phi||_{pq} \leqq ||\phi||_{\infty q}$ by (2.4), the rest of the proof follows from Remark 6.1 parts b) and c).†

DEFINITION 6.6. The dual of $D(G)$ is called the space of quasimeasures and it is denoted by $Q(G)$.

The pairing between $D(G)$ and $Q(G)$ will be written as

$$< h,\sigma > \qquad (h \in D(G),\ \sigma \in Q(G)).$$

DEFINITION 6.7. [28, 1.1 p. 478]. Let $1 < p \leq \infty$. The Fourier transform of $f \in L^p(G)$ is a quasimeasure $\hat{f}$ defined by

$$< h,\hat{f} > = < \check{h}',f > = \int_G \check{h}(-t)\, f(t)\, dt \qquad (h \in D(\hat{G})).$$

DEFINITION 6.8. [3 , p. 266]. Let $1 \leq p,q \leq \infty$. The Fourier transform $\hat{f}$ of $f \in (L^p, \ell^q)(G)$ is an element of the dual of $A_c(\hat{G})$ defined by

$$< g,\hat{f} > = < \check{g}',f > = \int_G \check{g}(-t)\, f(t)\, dt \qquad (g \in A_c(\hat{G})).$$

Note that by Lemma 6.4, $\check{g} \in (L^p, \ell^q)(G)$ for all $g \in A_E(\hat{G})$ and $\check{g} = g*f_E$, $g \in L^1(\Omega)$, $f_E \in (C_0, \ell^1)(G)$. Hence $< \check{g}',f >$ is well defined and by Theorem 4.7

$$|< g,\hat{f} >| = |< \check{g}',f >| \leq ||\check{g}||_{p'q'}||f||_{pq} \leq ||g*f_E||_{\infty 1}||f||_{pq}$$

$$\leq 2^{2a}\, ||g||_{\Gamma}||f_E||_{\infty 1}||f||_{pq} = 2^{2a}\, ||f_E||_{\infty 1}||f||_{pq}||g||_{A_E}.$$

Therefore $\hat{f} \in A_c(G)^*$.

If $p = q$ then it follows from Remark 6.3 that Definition 6.7 and Definition 6.8 are equivalent.

If $1 \leq p,q \leq 2$ then the Fourier transform of $f \in (L^p, \ell^q)$ as an element of A_c^* coincides with the function $\hat{f}$ in $(L^{q'}, \ell^{p'})$ given by Theorem 5.7. Indeed, by the Hausdorff-Young inequality (Theorem 5.7) there exists a constant C such that for all $g \in A_c(\hat{G})$

$$|< g,\hat{f} >| = |< \check{g}',f >| \leq ||\check{g}||_{p'q'}||f||_{pq} \leq C\ ||g||_{qp}||f||_{pq}.$$

This implies that $\hat{f} \in (A_c, ||\cdot||_{qp})^*$. Since $A_c(\hat{G})$ is dense in $(L^q, \ell^p)(\hat{G})$ (Proposition 6.5), $\hat{f}$ has a unique continuous extension $\hat{f}$ on (L^q, ℓ^p) and by Theorem 3.1, $\hat{f} \in (L^{q'}, \ell^{p'})(\hat{G})$.

<u>DEFINITION 6.9</u>. For $\mu \in M_s(G)$, $1 \leq s \leq \infty$, we define its <u>Fourier transform</u> $\underline{\hat{\mu}}$ as an element of $A_c(\hat{G})^*$ defined by

$$< g,\hat{\mu} > = < \check{g}',\mu > = \int_G \check{g}(-t)\ d\mu(t) \qquad (g \in A_c(\hat{G})).$$

Again, by Lemma 6.4, $\check{g} \in (C_0, \ell^{s'})(G)$ for all $g \in A_c(\hat{G})$ and we have as above that

$$|< g,\hat{\mu} >| \leq ||\check{g}||_{\infty s'}||\mu||_s = ||g*f_E||_{\infty s'}||\mu||_s \leq ||\ \ ||_1||f_E||_{\infty s'}||\mu||$$
$$= ||f_E||_{\infty s'}||\mu||_s||g||_{A_E}.$$

Therefore $\hat{\mu} \in A_c(\hat{G})^*$.

In particular if $\mu \in M_s$, $1 \leq s \leq 2$, then $\hat{\mu} \in (L^{s'}, \ell^\infty)$ (as it was proved in [49, Theorem 4.2 ii)]. Indeed, if $g \in A_c(\hat{G})$ then by Theorem 5.7 there exists a constant C_s depending on s such that

$$|< g,\hat{\mu} >| = |< \check{g}',\mu >| \leq ||\check{g}||_{\infty s'}||\mu||_s \leq C_s\ ||g||_{s1}||\mu||_s.$$

This means that $\hat{\mu} \in (A_c(\hat{G}), ||\cdot||_{s1})^*$. Again since $A_c(\hat{G})$ is dense in (L^s, ℓ^1) (Proposition 6.5) $\hat{\mu}$ has a unique continuous extension $\hat{\mu}$ on (L^s, ℓ^1) and by Theorem 3.1,

$$(6.1) \qquad \hat{\mu} \in (L^{s'}, \ell^{\infty}) \quad \text{and} \quad ||\hat{\mu}||_{s'\infty} \leq C_s \, ||\mu||_s .$$

REMARK 6.10. If $f \in (L^p, \ell^q)$ $(1 \leq p,q \leq \infty)$ then its Fourier transform as an element of (L^p, ℓ^q) and as an element of M_q coincide.

If there exists a constant C such that for all $g \in A_c(\hat{G})$ $|< g,\hat{\mu} >| \leq C \, ||g||_{\infty q'}$ for some $1 \leq q \leq \infty$, then by the density of $A_c(\hat{G})$ in $(C_0, \ell^{q'})(\hat{G})$, $\hat{\mu}$ has a unique continuous extension on $(C_0, \ell^{q'})(\hat{G})$ and by Theorem 3.2, there exists a unique measure $\hat{\mu} \in M_q(\hat{G})$ such that for all $h \in (C_0, \ell^{q'})(\hat{G})$

$$< h,\hat{\mu} > = \int_{\hat{G}} h(\hat{x}) \, d\hat{\mu}(\hat{x}).$$

In this case for $\hat{\mu}$ considered as a measure we say that $\hat{\mu} \in M_q$. That is:

PROPOSITION 6.11. Let $\mu \in M_s(G)$, $1 \leq s \leq \infty$. $\hat{\mu} \in M_q$ for some $1 \leq q \leq \infty$ iff there exists a constant C such that for all $g \in A_c(\hat{G})$

$$|< g,\hat{\mu} >| \leq C \, ||g||_{\infty q'} .$$

Moreover, for all $h \in (C_0,\ell^{q'})(\hat{G})$

$$< h,\hat{\mu} > = \int_{\hat{G}} h(\hat{x}) \, d\hat{\mu}(\hat{x})$$

and for all $g \in A_c(\hat{G})$

$$\int_G \check{g}(-x) \, d\mu(x) = < \check{g}',\mu > = < g,\hat{\mu} > = \int_{\hat{G}} g(\hat{x}) \, d\hat{\mu}(\hat{x}).$$

PROPOSITION 6.12. Let $\nu \in M_1(G)$, $h \in (C_0,\ell^q)(\hat{G})$, $1 \leq q \leq \infty$, and $g \in A_c(\hat{G})$. Then

i) $h\hat{\nu} \in (C_0, \ell^q)(\hat{G})$

ii) $g\hat{\nu} \in A_c(\hat{G})$.

PROOF. i) Since $\hat{\nu}$ is a uniformly bounded function on $\hat{G}$ [37, Theorem 31.5], $h\hat{\nu} \in (C_0, \ell^q)(\hat{G})$.

ii) By Lemma 6.4 $\check{g} \in (C_0, \ell^1)$ and $(C_0, \ell^1) \subseteq L^1$. So by [37, Theorem 31.27] $g\hat{\nu} = (\check{g}*\nu)\hat{}$. Since $\check{g}*\nu \in (C_0, \ell^1)$ (Theorem 4.8) and $g\hat{\nu} \in C_c(\hat{G})$, because g has compact support, we conclude that $(g\hat{\nu})^\vee = \check{g}*\nu$ and therefore $g\hat{\nu} \in A_c(\hat{G})$. †

Proposition 6.12 allows us to define the product of an element of $A_c(\hat{G})^*$ $(M_q(\hat{G}))$ and $\hat{\nu}$ $(\nu \in M_1)$ as follows:

DEFINITION 6.13. If $F \in A_c(\hat{G})^*$ $(M_q(\hat{G}),\ 1 \leq q \leq \infty)$ and $\nu \in M_1(G)$ then $\underline{F\hat{\nu}}$ is an element of $A_c(\hat{G})^*$ $(M_q(\hat{G}))$ defined by

$$< g, F\hat{\nu} > = < g\hat{\nu}, F > \qquad g \in A_c(\hat{G})\ (g \in (C_0, \ell^{q'})(\hat{G})).$$

PROPOSITION 6.14. Let $\mu \in M_s(G)$ $(1 \leq s \leq \infty)$ and $f \in L^1(G)$.

i) $(\mu*\nu)\hat{} = \hat{\mu}\hat{\nu}$ for all $\nu \in M_1(G)$. Hence $(\mu*f)\hat{} = \hat{\mu}\hat{f}$.

ii) If $\hat{\mu} \in M_q(\hat{G})$, $1 \leq q \leq \infty$, Then $(\mu*f)\hat{} \in M_q(\hat{G})$ and for all $h \in (C_0, \ell^{q'})(\hat{G})$

$$\int_{\hat{G}} h(\hat{x})\ d(\mu*f)\hat{}(\hat{x}) = \int_{\hat{G}} h(\hat{x})\ \hat{f}(\hat{x})\ d\hat{\mu}(\hat{x})$$

and for all $g \in A_c(\hat{G})$

$$\int_G \check{g}(-x)\ \mu*f(x)\ dx = \int_{\hat{G}} g(\hat{x})\ \hat{f}(\hat{x})\ d\hat{\mu}(\hat{x}).$$

PROOF. i) Let $g \in A_c(\hat{G})$. Since $g\hat{\nu} = (g*\nu)^\wedge$ for all $\nu \in M_1(G)$ (see the proof of Proposition 6.12) we have that for $\nu \in M_1(G)$

$$\begin{aligned} < g, \hat{\mu}\hat{\nu} > &= < g\hat{\nu}, \hat{\mu} > = < (\check{g}*\nu)^\wedge, \hat{\mu} > = < (\check{g}*\nu)', \hat{\mu} > \\ &= \int \check{g}*\nu(-t)\, d\mu(t) = \int\int \check{g}(-t-s)\, d\nu(s)\, d\mu(t) \\ &= < \check{g}', \nu*\mu > = < g, (\nu*\mu)^\wedge >. \end{aligned}$$

Since this is true for all $g \in A_c(\hat{G})$ we conclude that $\hat{\mu}\hat{\nu} = (\mu*\nu)^\wedge$ for all $\nu \in M_1(G)$.

ii) By i), Proposition 6.11 and Proposition 4.1 there exists a constant C such that for all $g \in A_c(\hat{G})$

$$\begin{aligned} |< g, (\mu*f)^\wedge >| = |< g, \hat{\mu}\hat{f} >| = |< g\hat{f}, \hat{\mu} >| &\leq C\, ||g\hat{f}||_{\infty q'} \\ &\leq C\, ||\hat{f}||_\infty ||g||_{\infty q'}. \end{aligned}$$

Therefore by Proposition 6.11 $(\mu*f)^\wedge \in M_q(\hat{G})$.

Finally by the same Proposition 6.11

$$\begin{aligned} \int_{\hat{G}} h(\hat{x})\, d(\mu*f)^\wedge(\hat{x}) &= < h, (\mu*f)^\wedge > = < h, \hat{\mu}\hat{f} > = < h\hat{f}, \hat{\mu} > \\ &= \int_{\hat{G}} h(\hat{x})\, \hat{f}(\hat{x})\, d\hat{\mu}(\hat{x}) \end{aligned}$$

for all $h \in (C_0, \ell^{q'})(G)$ and

$$\int_G \check{g}(-x)\, \mu*f(x)\, dx = < \check{g}', \mu*f > = < g, (\mu*f)^\wedge > = \int_{\hat{G}} g(\hat{x})\, \hat{f}(\hat{x})\, d\hat{\mu}(\hat{x})$$

for all $g \in A_c(\hat{G})$.

(Remember that $\mu*f \in (L^1, \ell^s)$ by Corollary 4.4) +

DEFINITION 6.15. The inverse of the Fourier transform $\check{\mu}$ of a measure μ in M_s $(1 \leq s \leq \infty)$ is an element of $A_c(G)^*$ defined by

$$< \phi, \check{\mu} > = < \hat{\phi}', \mu > = \int_{\hat{G}} \hat{\phi}(-\hat{x})\, d\mu(\hat{x}) \qquad (\phi \in A_c(\hat{G}))$$

Before we continue we should remember that for $1 \le p,q \le \infty$ $(L^p, \ell^q) \subseteq (L^1, \ell^q) \subseteq M_q \subseteq M_\infty$ and therefore Proposition 6.11, Proposition 6.12, Proposition 6.14, Definition 6.15 and the results that follow hold for elements of any amalgam space and any measure space of type q.

Also, when considering an element, ϕ, of $D(G)$ we should bear in mind that $D(G) \subseteq A_c(G)$ and by Lemma 6.4
$A_c(G) = \{ \phi \in C_c(G) \mid \hat{\phi} \in (C_0, \ell^1)(\hat{G}) \} = \Phi(G)$. Thought of as an element of $A_c(G)$, $\phi = \check{\phi}^\circ$, $\phi^\circ \in L^1(\hat{G})$; and thought of as an element of $\Phi(G)$, $\hat{\phi} \in (C_0, \ell^1)(\hat{G})$.

THEOREM 6.16. (Inversion Theorem). Let $\mu \in M_\infty(G)$. If $\hat{\mu} \in M_\infty(\hat{G})$ and for all $\phi \in A_c(G)$

$$< \phi,\mu > = \int_G \phi(x)\ d\mu(x) = \int_{\hat{G}} \hat{\phi}(-\hat{x})\ d\hat{\mu}(\hat{x}) = < \hat{\phi}',\hat{\mu} >$$

then $\mu = \check{\hat{\mu}}$.

PROOF. Let $\phi \in A_c(G)$. By definition of $\check{\hat{\mu}}$ we have that $< \phi,\check{\hat{\mu}} > = < \hat{\phi}',\hat{\mu} > = < \phi,\mu >$. Since $\mu \in M_\infty$ and $A_c(G)$ is dense in $(C_0, \ell^1)(G)$ we conclude that $\check{\hat{\mu}} \in M_\infty$ (Proposition 6.11) and $\mu = \check{\hat{\mu}}$. †

Our next goal will be to see which measures in M_∞ satisfy the condition of Theorem 6.16. To this end we introduce the concept of transformable measure as defined by L. Argabright and J. Gil de Lamadrid [1].

DEFINITION 6.17. A measure ν is transformable if there exists a measure $\hat{\nu}$ on $\hat{G}$ such that for all $\phi \in C_c(G)$, the function $\hat{x} \longmapsto |\hat{\phi}|^2(-\hat{x})$ on $\hat{G}$ belongs to $L^1(\hat{\nu})$ and

$$\int_G \phi * \tilde{\phi}(x)\ d\nu(x) = \int_{\hat{G}} |\hat{\phi}|^2(-\hat{x})\ d\ulcorner\nu\urcorner(\hat{x}).$$

$M_T(G) = M_T$ will denote the set of transformable measures and $C_2(G) = \{ \phi * \tilde{\phi} \mid \phi \in C_c(G) \}$.

Note that if f, h belong to $C_c(G)$ then $f*h \in \langle C_2(G) \rangle$, the linear subspace of $C_c(G)$ spanned by $C_2(G)$. Clearly $\langle C_2(G) \rangle \subset D(G)$.

REMARK 6.18. i) If $\nu \in M_T$ then $\ulcorner\nu\urcorner$ is unique and $\ulcorner\nu\urcorner \in M_\infty(\hat{G})$ [1, Theorem 2.1 and Theorem 2.5].

ii) If $\nu \in M_T$ then $\hat{\phi} \in L^1(\ulcorner\nu\urcorner)$ for all $\phi \in \Phi(G)$, because $\hat{\phi} \in (C_0, \ell^1)(\hat{G})$. Therefore by [1, Corollary 3.1]

$$\int_G \phi(x)\ d\nu(x) = \int_{\hat{G}} \hat{\phi}(-\hat{x})\ d\ulcorner\nu\urcorner(\hat{x}) \quad \text{for all } \phi \in \Phi(G).$$

Therefore a measure $\mu \in M_\infty$ satisfies the condition of Theorem 6.16 if μ belongs to M_T and $\hat{\mu} = \ulcorner\mu\urcorner$.

The next two theorems will show that all measures in M_q for $1 \leq q \leq 2$, and all functions $\mu * f$, where $f \in L^1$ and $\mu \in M_\infty$ is such that $\hat{\mu} \in M_\infty$, satisfy these conditions.

THEOREM 6.19. (Inclusion Theorem). If $\mu \in M_q$, $1 \leq q \leq 2$ then $\mu \in M_T$ and $\ulcorner\mu\urcorner = \hat{\mu}$. Hence $\mu = \check{\hat{\mu}}$.

PROOF. Let $\nu \in M_c^q$ (see Definition 3.5). By the Extended Parseval Formula [49, Lemma 4.1] for all $h \in A_c(G)$

$$\int_G h(x)\ d\nu(x) = \int_{\hat{G}} \hat{h}(-\hat{x})\ \hat{\nu}(\hat{x})\ d\hat{x}.$$

Now, the linear functionals $T(\nu) = \int h(x)\ d\nu(x)$ and

$\overline{T}(\nu) = \int \hat{h}(-\hat{x})\ \hat{\nu}(\hat{x})\ d\hat{x}$ on M_q are continuous because

$|T(\nu)| \leq ||\nu||_q ||h||_{\infty q}$, by Theorem 3.2 and

$|\overline{T}(\nu)| \leq ||\hat{h}||_{q1} ||\hat{\nu}||_{q'\infty} \leq ||\hat{h}||_{q1}\ C_q\ ||\nu||_q$ (see (6.1) page 79), where

C_q is a constant depending only on q. Since M_c^q is dense in M_q (Theorem 3.6) and $T = \overline{T}$ on M_c^q we conclude that $T = \overline{T}$ on M_q and this implies that

$$\int_G h(x)\, d\mu(x) = \int_G \hat{h}(-\hat{x})\ \hat{\mu}(\hat{x})\ d\hat{x} \qquad \text{for all } h \in A_c(G).$$

Since $C_2(G) \subseteq A_c(G)$, $\mu \in M_T$ and $\overline{\mu} = \hat{\mu}$. †

THEOREM 6.20. Let $\mu \in M_\infty$ and $f \in L^1(G)$. If $\hat{\mu} \in M_\infty(\hat{G})$ then $f{*}\mu \in M_T$ and $(\mu{*}f)\hat{} = \overline{(\mu{*}f)}$. Hence $\mu{*}f = (\mu{*}f)^{\vee}$.

PROOF. Let $\phi \in A_c(G)$ and $\{e_\alpha\}$ be an a.i. in $L^1(G)$ such that $\{\hat{e}_\alpha\} \subseteq C_c(G)$ [29, II 7.1] and [45, 2.6.6]. In the next chapter we will actually construct such an approximate identity having further properties (p. 94).

By Proposition 4.1

$$||\hat{\phi}\hat{f}\hat{e}_\alpha - \hat{\phi}\hat{f}||_{\infty 1} \leq ||\hat{\phi}||_{\infty 1}||\hat{f}\hat{e}_\alpha - \hat{f}||_\infty \leq ||\hat{\phi}||_{\infty 1}||f{*}e_\alpha - f||_1.$$

This implies that $\lim ||\hat{\phi}\hat{f}\hat{e}_\alpha - \hat{\phi}\hat{f}||_{\infty 1} = 0$. So, by Proposition 6.14 $(\mu{*}f)\hat{} \in M_\infty(\hat{G})$ and we have that

$$(1) \qquad \int_{\hat{G}} \hat{\phi}(\hat{x})d(\mu{*}f)\hat{}(\hat{x}) = <\hat{\phi},(f{*}\mu)\hat{}> = <\hat{\phi},\hat{\mu}\hat{f}> = <\hat{\phi}\hat{f},\hat{\mu}>$$

$$= \lim <\hat{\phi}\hat{f}\hat{e}_\alpha,\hat{\mu}> = \lim <\hat{\phi}\hat{e}_\alpha,\hat{\mu}\hat{f}>$$

$$= \lim <\hat{\phi}\hat{e}_\alpha,(\mu{*}f)\hat{}>.$$

Now, $\hat{\phi}\hat{e}_\alpha \in A_c(\hat{G})$ because $\hat{\phi} \in C_0$, $\hat{e}_\alpha \in C_c$ and $\phi * e_\alpha = (\hat{\phi}\hat{e}_\alpha)^\vee$. Then by definition of $(\mu * f)\hat{}$

$$(2) \qquad \lim \langle \hat{\phi}\hat{e}_\alpha, (\mu*f)\hat{}\ \rangle = \lim \langle (\phi * e_\alpha)', \mu * f \rangle$$

$$= \lim \int_G \phi * e_\alpha(-x)\ \mu*f(x)\ dx$$

$$= \int_G \phi(-x)\ \mu*f(x)\ dx.$$

Remember that $\mu * f \in (L^1, \ell^\infty)$ by Theorem 4.8. The last equality holds because $\lim \|\phi * e_\alpha - \phi\|_{\infty 1} = 0$ by Corollary 4.14.

Hence from (1) and (2) we conclude that

$$\int_G \phi(-x)\ \mu*f(x)\ dx = \int_{\hat{G}} \hat{\phi}(\hat{x})\ d(\mu*f)\hat{}(\hat{x})$$

for all $\phi \in A_c(G)$. Since $C_2(G) \subseteq A_c(G)$, this implies that $\mu * f \in M_T$ and $(\mu*f)\hat{} = \overline{(\mu*f)}$. †

THEOREM 6.21. Let $\mu \in M_\infty$. If $\hat{\mu} \in M_\infty(\hat{G})$ then $\mu \in M_T$ and $\hat{\mu} = \overline{\mu}$. Hence $\mu = \check{\hat{\mu}}$.

PROOF. Let $f \in A_c(G)$ and $g \in D(G)$. By Corollary 4.4 $f*\mu$ as a measure belongs to $M_\infty(G)$, so by Proposition 6.14 and Theorem 6.20 $(f*\mu)\hat{} \in M_\infty(\hat{G})$ and

$$\int_G g(x)\ f*\mu(x)\ dx = \int_{\hat{G}} \hat{g}(-\hat{x})\ d(f*\mu)\hat{}(\hat{x}) = \int_{\hat{G}} \hat{g}(-\hat{x})\ \hat{f}(\hat{x})\ d\hat{\mu}(\hat{x}).$$

Note that by Lemma 6.4 $\hat{g} \in (C_0, \ell^1)(\hat{G})$.

But $\int \hat{g}(-x) f(x)\ d\ (x) = \int \int [x.\hat{x}]\ g(x)\ dx\ \hat{f}(\hat{x})\ d\hat{\mu}(\hat{x})$

$$= \int g(x) \int [x.\hat{x}]\ \hat{f}(\hat{x})\ d\hat{\mu}(\hat{x})\ dx.$$

We can apply Fubini's theorem because $g \in L^1$ and
$\int |[x,\hat{x}]\ \hat{f}(\hat{x})|\ d|\hat{\mu}|(\hat{x}) \leq ||\hat{f}||_{\infty 1}||\hat{\mu}||_{\infty}$

Then we conclude ~~that~~ for all $g \in D(G)$

$$\int_G g(x)\cdot f*\mu(x)\ dx = \int_G g(x) \int_{\hat{G}} [x,\hat{x}]\ \hat{f}(\hat{x})\ d\hat{\mu}(\hat{x})\ dx.$$

Since D(G) is dense in $C_c(G)$ [40, Theorem 5.1.2], this implies that

$$f*\mu(x) = \int_G f(x-y)\ d\mu(y) = \int_{\hat{G}} [x,\hat{x}]\ \hat{f}(\hat{x})\ d\hat{\mu}(\hat{x})$$

locally almost everywhere.

Now, $\hat{f} \in (C_0, \ell^1)$ and therefore $\int [x,\hat{x}]\ \hat{f}(\hat{x})\ d\hat{\mu}(\hat{x})$ is a continuous function on G. On the other hand $f*\mu$ is also a continuous function on G. Therefore $f*\mu(x) = \int [x,\hat{x}]\ \hat{f}(\hat{x})\ d\hat{\mu}(\hat{x})$ for all $x \in G$. So, for $x = 0$ we have that

$$\int_G f(x-y)\ d\mu(y) = \int_{\hat{G}} \hat{f}(\hat{x})\ d\hat{\mu}(\hat{x}) \qquad \text{for all } f \in A_c(G).$$

This implies that $\mu \in M_T$ and $\hat{\mu} = \overline{\mu}$. †

<u>REMARK 6.22</u>. From the proof of theorem 6.21 we see that for $\mu \in M_\infty(G)$ such that $\hat{\mu} \in M_\infty(\hat{G})$

$$\int_G f(x-y)\ d\mu(y) = \int_{\hat{G}} [x,\hat{x}]\ \hat{f}(\hat{x})\ d\hat{\mu}(\hat{x})$$

for all $f \in A_c(G)$ and all $x \in G$.

<u>THEOREM 6.23</u>. If $\mu \in M_T \cap M_\infty$ then $\hat{\mu} = \overline{\mu}$.

<u>PROOF</u>. By Remark 6.18 $\mu \in M_\infty(\hat{G})$ and for all $g \in A_c(G)$

$$< g,\check{\mu} > = \int_G g(x)\, d\mu(x) = \int_{\hat{G}} g(-\hat{x})\, d\overline{\mu}(\hat{x}) = < g,\check{\overline{\mu}} >.$$

Then by Proposition 6.11 and the density of $A_c(G)$ in $(C_0, \ell^1)(G)$ this implies that $\mu = \check{\overline{\mu}}$. Hence by Theorem 6.21, $\overline{\mu} = \hat{\check{\overline{\mu}}} = \hat{\mu}$.†

L. Argabright and J. Gil de Lamadrid defined the set $\mathcal{L}(G) = \{ \nu \in M_T(G) \mid \overline{\nu} \in M_T(\hat{G}) \}$ and proved that if $\nu \in \mathcal{L}(G)$ then $\nu = \overline{\overline{\nu}}$.

What follows is a characterization of $\mathcal{L}(G)$. This was also proved by a different method by H. G. Feichtinger [25, Theorem C1].

<u>THEOREM 6.24</u>. $\mathcal{L}(G) = \{ \mu \in M_\infty \mid \hat{\mu} \in M_\infty \}$.

<u>PROOF</u>. Take $\nu \in \mathcal{L}(G)$. Since $\nu = \overline{\overline{\nu}}$, ν belongs to $M_\infty(G)$. So by Theorem 6.23 $\hat{\nu} = \overline{\nu}$. Therefore $\nu \in \{\mu \in M_\infty \mid \hat{\mu} \in M_\infty\}$.

Now take $\mu \in M_\infty$ such that $\hat{\mu} \in M_\infty$. By Theorem 6.21 $\mu \in M_T$ and μ $\check{\hat{\mu}} = \check{\overline{\mu}}$. So, $\overline{\mu} \in M_\infty(\hat{G})$ and $\check{\overline{\mu}} \in M_\infty(G)$. Again by Theorem 6.21 $\overline{\mu} \in M_T(\hat{G})$.†

<u>REMARK 6.25</u>. We see that Theorem 6.24 implies "the Second Inclusion Theorem" proved in [1, Theorem 3.5].

Moreover, (L^p, ℓ^q), $1 \leq p \leq \infty$, $1 \leq q \leq 2$, and M_s, $1 \leq s \leq 2$, are included in $\mathcal{L}(G)$.

I. Richards provided an example of a transformable measure which is not in M_∞ [1, §7]. This implies that $M_T \nsubseteq M_\infty$ and $\mathcal{L}(G)$ is a proper subset of M_T. That is, there exists $\nu \in M_T$ such that $\overline{\nu} \notin M_T$. Since $\overline{\nu} \in M_\infty$, this means that $M_\infty \nsubseteq M_T$. Moreover, from Theorem 6.24

we have that $\mathcal{L}(G) = M_\infty \cap M_T$ (see [30, Theorem 3] and [25, Theorem C1]).

F. Holland [35, §4] defined the test spaces Φ_q, $1 \leq q \leq \infty$, on the real line. Its generalizations to locally compact groups are due to J. P. Bertrandias and C. Dupuis [7, §2 c)].

DEFINITION 6.26. Let $1 \leq q \leq \infty$. $\Phi_q = \Phi_q(G)$ is the linear space $\{ \phi \in C_c(G) \mid \hat{\phi} \in (C_0, \ell^1) \}$ endowed with the norm $\phi \longmapsto \|\hat{\phi}\|_{\infty q}$.

By (2.5) it is clear that $\Phi_1 \subseteq \Phi_q \subseteq \Phi_\infty$, $1 < q < \infty$, and by Theorem 5.7, there exists a continuous linear isomorphism from C_c onto Φ_q for $2 \leq q \leq \infty$.

F. Holland proved that for $1 \leq q < 2$ $\Phi_q(R) \subsetneq C_c(R)$ [35, §4 p. 350].

DEFINITION 6.27. [7, §4 a)]. Let g be a function in (L^p, ℓ^q) $(1 \leq p,q \leq \infty)$ or a measure in M_s $(1 \leq s \leq \infty)$. The Fourier transform $F_\Phi(g)$ of g is an element of the continuous dual of Φ_1 defined by

$$< \phi, F_\Phi(g) > = < \check{\phi}', g > = \int \check{\phi}(-t)\, g(t)\, dt \qquad (\phi \in \Phi_1(\hat{G})).$$

Since $(L^p, \ell^q) \subseteq (L^1, \ell^q) \subseteq M_q \subseteq M_\infty$, $1 \leq p,q \leq \infty$, and $\check{\phi} \in (C_0, \ell^1)$ for all $\phi \in \Phi_1$, $F_\Phi(g)$ is well defined and

$$|< \phi, F_\Phi(g) >| = |< \check{\phi}', g >| \leq C\, \|\check{\phi}\|_{\infty 1}$$

where $C = \|g\|_\infty$ if $g \in M_s$ or $C = \|gm\|_\infty$ if $g \in (L^p, \ell^q)$. Therefore $F_\Phi(g)$ belongs to $\Phi_1(\hat{G})^*$.

By Lemma 6.4, $A_c(G) = \Phi_1(G)$. Moreover for all $\phi \in A_E(\hat{G})$,

$\check{\phi} = \phi^{\circ} * f_E$ where $\phi = \hat{\phi}^{\circ}$, $\phi^{\circ} \in L^1(G)$, $f_E \in (C_0, \ell^1)(G)$ and $\hat{f}_E \equiv 1$ on E (see the proof of Lemma 6.4). So, by (2.3) and Theorem 4.7 we see that

$$(6.2) \qquad ||\phi||_{A_E} = ||\check{\phi}||_1 \leq ||\check{\phi}||_{\infty 1} = ||\phi^{\circ} * f_E||_{\infty} \leq 2^a \, ||\phi^{\circ}||_1 ||f_E||_{\infty 1}$$

$$= 2^a \, ||f_E||_{\infty 1} ||\phi||_{A_E}.$$

This implies that the embedding of $A_c(\hat{G})$ onto $\Phi_1(\hat{G})$ is continuous and the norms $f \longmapsto ||\check{f}||_{\infty 1}$, $f \longmapsto ||f||_{A_E}$ on $A_E(\hat{G})$ are equivalent. Therefore $\Phi_1(\hat{G})^* \subseteq A_c(\hat{G})^*$ and we have that $\{ F_\phi g \mid g \in M_\infty \} \subseteq \{ \hat{g} \mid g \in M_\infty \}$.

H. G. Feichtinger [25] has given another alternative definition for the Fourier transform $F_0\mu$ of a measure $\mu \in M_\infty(G)$, as an element of the dual $S_0(G)^*$ of a rather special Segal algebra $S_0(G)$ defined in [26]. This algebra $S_0(G)$ has the following properties:

1) $A_c(G)$ is dense in $S_0(G)$.

2) The inclusion of $A_c(G)$ into $S_0(G)$ is continuous.

3) $S_0(G)^\wedge = S_0(\hat{G})$.

4) $M_\infty \cup M_T \subseteq S_0(G)^* \subseteq Q(G)$.

Then Feichtinger's definition is as follows:

DEFINITION 6.28. [25, Theorem B2]. Let $\sigma \in S_0(G)^*$. Then its Fourier transform $F_0\sigma$ is an element of $S_0(G)$ defined by

$$< f, F_0\sigma > = < \check{f}, \sigma > \qquad (f \in S_0(\hat{G})).$$

By property 4) above it is clear that $F_0\sigma$ is well defined and

$F_0\sigma$ indeed belongs to $S_0(G)^*$. In particular if $\mu \in M_\infty$ we have by 2) that $F_0\mu\,|A_c(G) = \hat{\mu}$, so
$\{ F_0\mu \mid \mu \in M_\infty \} \subseteq \{ \hat{\mu} \mid \mu \in M_\infty \}$; and $\hat{\mu} = F_0\mu$ iff there exists a constant C such that for all $f \in A_c(\hat{G})$, $|< \check{f},\mu >| < C\,||f||_{S_0}$ (property 1)).

Feichtinger also proved [25, Theorem C1 ii)] that
$\{ \mu \in M_T \mid F_0\mu \in M_T \} = M_T \cap M_\infty = \{ \mu \in M_\infty \mid F_0\mu \in M_\infty \}$. Hence, we conclude from Theorem 6.24 that if $\mu \in \mathcal{L}(G)$ then $\hat{\mu} = F_0\mu$.

CHAPTER III

REPRESENTATION OF FUNCTIONS AS FOURIER TRANSFORMS

OF MEASURES IN M_q

§ 7. SIMON'S GENERALIZATION OF CESÀRO SUMMABILITY

In this section we will generalize to locally compact abelian groups the following theorem proved by F. Holland [34, §7 Theorem 9] for the real line,

THEOREM 7.1. Let $1 \leq q \leq 2$ and $\mu \in M_q$. Then as $N \to \infty$

$$\frac{1}{\sqrt{2\pi}} \int_{-N}^{N} e^{-ixt} \, d\mu(t)$$

converges in the norm of $(L^{q'}, \ell^{\infty})$ to a function $\hat{\mu}$ and

$$\int h(x) \, \hat{\mu}(x) \, dx = \int h(x) \, d\mu(x) \qquad (h \in (L^q, \ell^1))$$

Further

$$\sqrt{2\pi} \, \hat{\mu}(x) = (C,1) \int e^{-ixt} \, d\mu(t)$$

almost everywhere.

(C,1) means that the integral on the right is summable by the Cesàro method of order 1 to the value $\sqrt{2\pi} \, \hat{\mu}(x)$.

We note immediately that if $\mu \in M_q(G)$, $1 \leq q \leq 2$, then by Theorem 3.6 there exists a sequence μ_n of measures in $M_c^q(G)$, hence in

$M_1(G)$, such that $\lim \|\mu_n - \mu\|_q = 0$. Then by (6.1) $\lim \|\hat{\mu}_n - \hat{\mu}\|_{q;\infty} = 0$. Therefore the first part of Theorem 7.1 is a particular case of this fact.

To generalize the rest of the theorem we introduce A. B. Simon's generalization of Cesàro summability [15] and study some of its properties related to amalgam spaces.

The set of basic neighborhoods of an element x of G in G will be denoted by $N_x(G)$.

We consider G to be the product $\mathbb{R}^a \times G_1$, where a, G_1 and H are as in page 10.

Since H is an open subgroup of G_1, any $U \in N_0(G)$ contains a product neighborhood $(-\delta_1,\delta_1)\times\cdots\times(-\delta_a,\delta_a)\times U_H$ where $U_H \in N_0(G_1)$, $U_H \subseteq H$ and $\delta_i > 0$, $i = 1,..,a$.

For $i = 1,..,a$ set $U_i = (-\delta_i,\delta_i)$ and $N_i = 1/\delta_i$. Then define

$$\alpha_{U_i}(t) = \frac{1 - \cos(N_i t)}{\pi N_i t^2} \qquad (t \in R).$$

We see that

(7.1) Each α_{U_i} is a continuous, nonnegative function on $\mathbb{R}$, $\alpha_{U_i} \in L^1(\mathbb{R})$ and $\|\alpha_{U_i}\| = 1$.

Now we define $\alpha_U\colon \mathbb{R}^a \longrightarrow \mathbb{R}$ to be

$$\alpha_U(t) = \prod_{i=1}^{a} \alpha_{U_i}(t_i) \qquad t = (t_1,\ldots,t_a).$$

From (7.1) we see that

(7.2) α_U is a continuous, nonnegative function on $\mathbb{R}^a$.

(7.3) $\alpha_U \in L^1(\mathbb{R}^a)$ and $||\alpha_U||_1 = 1$.

Since m is a regular measure, given $m(U_H)$ there exists $V \in N_0(H)$ compact such that $V \subseteq U_H$ and $m(U_H) - m(U_H)^2 < m(V)$.

So, by the normality of H (H is Hausdorff and compact), we can define for this V, a continuous function $g: H \longrightarrow R$ as follows

$$g(s) = \begin{cases} \frac{1}{m(U_H)} & s \in V \\ m(U_H) & s \in H \sim U_H \\ m(U_H) \leq g(s) \leq \frac{1}{m(U_H)} & s \in U_H \sim V \end{cases}$$

Applying the Stone-Weierstrass Theorem to the space of continuous functions on the compact space H, we get a trigonometric polynomial P, (P is a finite linear combination of characteres) such that $||g - P||_\infty < m(U_H)$.

If r is the real part of P then $r = \sum_{j=1}^{m} \lambda_j \hat{s}_j$, $\lambda_j \in \mathbb{R}$, $\hat{s}_j: H \longrightarrow R$ $j = 1,..,m$ and $||g - r||_\infty < m(U_H)$.

Hence for all $s \in H$, $-m(U_H) < r(s) - g(s)$ and this implies that $0 \leq g(s) - m(U_H) < r(s)$ for all $s \in H$.

We then define the function $\beta_U: G \longrightarrow R$ to be

$$\beta_U(s) = \begin{cases} \frac{1}{||r||_1} r(s) & s \in H \\ 0 & s \notin H \end{cases}$$

It is clear that

(7.4) β_U is a continuous, nonnegative function on G_1, $\beta_U \in L^1(G_1)$ and $||\beta_U||_1 = 1$.

(7.5) $\sup_{s \in G} |\beta_U(s)| = B_U < \infty$.

Finally we define

$$\phi_U(t,s) = \alpha_U(t) \cdot \beta_U(s) \qquad (t,s) \in G.$$

THEOREM 7.2. Each ϕ_U $(U \in N_0(G))$ has the following properties:

i) ϕ_U is a real-valued, continuous, nonnegative function on G.

ii) $\phi_U \in L^1(G)$ and $||\phi_U||_1 = 1$.

iii) $\hat{\phi}_U \in C_c(\hat{G})$ and $||\hat{\phi}_U||_\infty \leq 1$.

iv) $\phi_U(x) = \int_{\hat{G}} \hat{\phi}_U(\hat{x})\, [x,\hat{x}]\, d\hat{x}$ ie. $\phi_U = \check{\hat{\phi}}$.

Moreover

v) For $\varepsilon > 0$ and $U \in N_0(G)$ given, we can find a V such that if $V' \subseteq V$ then $\int_{G\sim U} \phi_{V'} < \varepsilon$.

vi) $\lim_U \hat{\phi}_U(\hat{x}) = 1$.

vii) $\{\phi_U\}$ is an a.i. in $L^1(G)$.

PROOF. i) and ii) follow from (7.2), (7.3) and (7.4). For a proof of iii), iv), v), vi) and vii) see [15].

PROPOSITION 7.3. Let $1 \leq p \leq \infty$. For each $U \in N_0(G)$, $\alpha_U \in (L^p, \ell^1)(\mathbb{R}^a)$.

PROOF. By (2.5) it is enough to prove that $\alpha_U \in (L^\infty, \ell^1)(\mathbb{R}^a)$.

First we note that since α_{U_i}, $1,..,a$, is an even function, for $n > 0$

$$\sup_{t\,\varepsilon\,[0,1]} \alpha_{U_i}(t+n) = \sup_{t\,\varepsilon\,[0,1]} \alpha_{U_i}(1-t+n)$$

$$= \sup_{t\,\varepsilon\,[0,1]} \alpha_{U_i}(t-(1+n)) \quad \text{for } i = 1,..,a.$$

Then we conclude that $\sup_{t\,\varepsilon\,[0,1]} \alpha_{U_i}(t+n) \leq \dfrac{2}{N_i\pi}\,\dfrac{1}{n^2}$,

$(i = 1,..,a)$, for all $n \,\varepsilon\, Z \sim \{0,-1\}$.

Also $\sup_{t\,\varepsilon\,[0,1]} \alpha_{U_i}(t+n) = \dfrac{N_i}{\pi} \sup_{t\,\varepsilon\,[0,1]} \dfrac{1-\cos N_i(n+t)}{(N_i(n+t))^2}$

$$\leq \frac{N_i}{\pi}\, C_i \quad .$$

$(i = 1,..,a)$ if $n \,\varepsilon\, \{0,-1\}$ for a constant C_i, because

$$\lim_{t \to -n} \frac{1-\cos N_i(n+t)}{(N_i(n+t))^2} \quad \text{exists.}$$

Therefore for all $i = 1,..,a$ and all $n \,\varepsilon\, Z$

$\sup_{t\,\varepsilon\,[0,1]} |\alpha_{U_i}(t+n)| \leq C\, a_n$ where $C = \max_{1 \leq i \leq a} (2/(N_i\pi), N_iC_i/\pi)$

and

$$a_n = \begin{cases} 1/n^2 & n \,\varepsilon\, Z \sim \{0,-1\} \\ 1 & n \,\varepsilon\, \{0,1\} \end{cases}$$

So for $i = 1,..,a$

$$||\alpha_{U_i}||_{\infty 1} = \sum_{Z} \sup_{t\,\varepsilon\,[n,n+1]} |\alpha_{U_i}(t)| = \sum_{Z} \sup_{t\,\varepsilon\,[0,1]} |\alpha_{U_i}(t+n)|$$

$$\leq C.\sum_{Z} a_n < \infty \; .$$

This means that $\{\alpha_{U_i}\} \subseteq (L^\infty, \ell^1)(R)$.

Now,

$$||\alpha_U||_{\infty 1} = \sum_{n \in Z^a} \sup_{t \in [0,1]^a} |\alpha_U(t+n)|$$

$$= \sum_{n \in Z^a} \prod_{i=1}^{a} \sup_{t \in [0,1]} |\alpha_{U_i}(t+n_i)|$$

$$= \sum_{n_1 \in Z} \cdots\cdots\cdots \sum_{n_a \in Z} \prod_{i=1}^{a} \sup_{t \in [0,1]} |\alpha_{U_i}(t+n_i)|$$

$$= \prod_{i=1}^{a} ||\alpha_{U_i}||_{\infty 1} < \infty \quad \dagger$$

COROLLARY 7.4. Let $1 \leq p \leq \infty$. Each $\phi_U \in (L^p, \ell^1)(G)$ $(U \in N_0(G))$. Hence $\{\phi_U\} \subseteq (C_0, \ell^1)$.

PROOF. By definition of ϕ_U, for all $(t,s) \in G$

$\phi_U(s,t) = \alpha_U(t)\ \beta_U(s) \leq B_U\alpha_U(t)$ where $\sup_{s \in G_1} |\beta_U(s)| = B_U$ (see (7.5)).

Hence

$$||\phi_U||_{\infty 1} = \sum_{\alpha} \sup_{(t,s) \in K_\alpha} |\alpha_U(t)\beta_U(s)| \leq B_U \sum_{n \in Z^a} \sup_{t+n \in [0,1]^a} |\alpha_U(t)|$$

$$= B_U\ ||\alpha_U||_{\infty 1}.$$

Therefore by Proposition 7.3, $\phi_U \in (L^\infty, \ell^1)$ and by (2.5) $\phi_U \in (L^p, \ell^1)$. $\dagger$

We will use the following lemma to prove Theorem 7.6.

LEMMA 7.5. Let U, V be two elements of $N_0(G)$ of the form

$U = (-\delta_1,\delta_1)\times \cdots \times(-\delta_a,\delta_a)\times U_H$, $\delta_i > 0$, $U_H \subseteq H$, $U_H \in N_0(G_1)$ and

$V = [-\gamma_1,\gamma_1]\times \cdots \times[-\gamma_a\ \gamma_a]\times V_H$, $\gamma_i > 0$, $V_H \subseteq H$, $V_H \in N_0(G_1)$ compact.

If for $1 < p < \infty$, $\eta_i = \min(\delta_i^{2p}, \gamma_i)$ $i = 1,..,a$ and $W_H = \operatorname{int} V_H$ then $W = [-\eta_1,\eta_1]\times\cdots\times[-\eta_a,\eta_a]\times W_H$ belongs to $N_0(G)$ and for a fixed $y = (y_0,s_0)$ in G, $y_0 = (y_1,\ldots,y_a)$, $W_y = y + W$ has the following properties:

1) $W_y \subseteq y + V$.

2) If $\Pi a = [-\eta_1+y_1,\eta_1+y_1]\times\cdots\times[-\eta_a+y_a,\eta_a+y_a]$ then

$$\left[\int_{\Pi a} \alpha_U(y_0 - x)^p\, dx\right]^{1/p} = O\Big(\prod_{i=1}^{a} \delta_i\Big).$$

3) $\mathbb{R}^a \sim \Pi a \subseteq \bigcup_N I_n$, $\{I_n\}$ being a countable family of compact subsets of R^a and

$$\sum_{\mathbb{N}} \left[\int_{I_n} \alpha_U(y_0 - x)^p\, dx\right]^{1/p} = O\Big(\prod_{i=1}^{a} \delta_i\Big).$$

4) There exists a constant C such that $\sup_N \wp(I_n) \leq C$ where $\wp(I_n)$ is the cardinality of the set $\{m \in \mathbb{Z}^a \mid (m + [0,1]^a) \cap I_n \neq \phi\}$.

PROOF. Several constants will appear during the proof and since their specific value is irrelevant for our needs we just write C_1, C_2..

From the definition of W, 1) is clear.

Remember that for $i = 1,..,a$, $\alpha_{U_i}(t) = \dfrac{N_i}{\pi}\,\dfrac{1 - \cos N_i t}{(N_i t)^2}$ is continuous by (7.1), hence α_{U_i} is bounded on $[-\eta_i,\eta_i]$ and we have that for $J_i = [-\eta_i + y_i, \eta_i + y_i]$, $i = 1,..,a$

$$(1) \qquad \left[\int_{J_i} \alpha_{U_i}(y_i - x)^p\, dx\right]^{1/p} = \left[\int_{-\eta_i}^{\eta_i} \alpha_{U_i}(x)^p\, dx\right]^{1/p}.$$

$$\leq C_i\ N_i\ \eta_i^{1/p} \leq C_i(1/\delta_i)\ \delta_i^{\,2} = C_i\ \delta_i.$$

This implies 2) because

$$\left[\int_{\Pi a} \alpha_U(y - x)^p\ dx\right]^{1/p} = \prod_{i=1}^{a} \left[\int_{J_i} \alpha_{U_i}(y_i - x)^p\ dx\right]^{1/p} \leq C_2 \prod_{i=1}^{a} \delta_i.$$

Let $J_i = [-\eta_i + y_i, \eta_i + y_i]$ and $\Pi_i = J_1 \times \cdot \cdot \cdot \times J_i$, $i = 1,.,a$. Observe that

$$R \sim J_i = (-\infty, -\eta_i + y_i) \cup (\eta_i + y_i, \infty)$$

$$\subseteq \cup [-n - 1 - \eta_i + y_i, -n - \eta_i + y_i]$$

$$\bigcup_{N} [n + \eta_i + y_i, n + 1 + \eta_i + y_i]$$

$$= \bigcup_{N} I_n^- \cup \bigcup_{N} I_m^+ \quad \text{and}$$

$$\int_{I_n^-} \alpha_{U_i}(y_i - x)^p dx \leq \left(\frac{2}{\pi N_i}\right)^p \int_{I_n^-} \frac{dx}{(y_i - x)^{2p}} = C_3\ \delta_i^p\ a_n$$

where $a_n = \dfrac{1}{(y_i + n)^{2p-1}} - \dfrac{1}{(y_i + n + 1)^{2p-1}}$.

Since $\sum_N a_n^{1/p}$ converges we conclude that

$$\sum_N \left[\int_{I_n^-} \alpha_{U_i}(y_i - x)^p\ dx\right]^{1/p} \leq C_4 \delta_i.$$

Similarly

$$\int_{I_m^+} \alpha_{U_i}(y_i - x)^p\ dx = C_5\ \delta_i^p \int_{I_m^+} \frac{dx}{(y_i - x)^{2p}} = C_5\ \delta_i^p\ a_n$$

and therefore $\displaystyle\sum_{N} \left[\int_{I_m^+} \alpha_{U_i}(y_i - x)^p\ dx\right]^{1/p} \leq C_6\ \delta_i.$

Clearly $\sup_{\mathbb{N}} \mathcal{C}(I_n^-)$ and $\sup_{\mathbb{N}} \mathcal{C}(I_m^+)$ are less than or equal to 2. Hence, for $i = 1,..,a$

(2) $\quad R \sim J_i = \bigcup_{\mathbb{N}} I_n, \quad I_n$ compact, $\sup_{\mathbb{N}} \mathcal{C}(I_n) \leq 2$ and

$$\sum_{\mathbb{N}} \left[\int_{I_n} \alpha_{U_i}(y_i - x)^p \, dx \right]^{1/p} = O(\delta_i).$$

Since $\mathbb{R} = (\mathbb{R} \sim J_a) \cup J_a$ and J_a is compact, by (1) and (2) we see that $\mathbb{R} = \bigcup_{N} \overline{I_n}$, $\overline{I_n}$ compact for all n,

$\sup_{N} \mathcal{C}(\overline{I_n}) \leq \max (2, \mathcal{C}(J_a)) = C_7$ and

(3) $$\sum_{\mathbb{N}} \left[\int_{\overline{I_n}} \alpha_{U_a}(y_a - x)^p \, dx \right]^{1/p} = O(\delta_a).$$

We will prove 3) and 4) by induction on a. The case $a = 1$ follows from (2). Suppose that 3) and 4) hold for $a - 1$. That is, $R^{a-1} \sim \Pi a{-}1 \subseteq \bigcup_{\mathbb{N}} I_n$, $I_n \subseteq \mathbb{R}^{a-1}$ compact, $\sup_{\mathbb{N}} \mathcal{C}(I_n) \leq C_8$ and

(4) $$\sum_{\mathbb{N}} \left[\int_{I_n} \prod_{i=1}^{a-1} \alpha_{U_i}(y_i - x_i)^p \, dx \right]^{1/p} = O\left(\prod_{i=1}^{a-1} \delta_i \right) \quad x = (x_1,..,x_a).$$

By (2) with $i = a$ we have that $\mathbb{R} \sim J_a \subseteq \bigcup_{N} I_k$, $I_k \subseteq \mathbb{R}$ compact

$\sup_{N} \mathcal{C}(I_k) \leq 2$ and

(5) $$\sum_{\mathbb{N}} \left[\int_{I_k} \alpha_{U_a}(y_a - x)^p \, dx \right]^{1/p} = O(\delta_a).$$

Then $\mathbb{R}^{a} \sim \Pi a = (\mathbb{R}^{a-1} \times \mathbb{R}) \sim (\Pi a\text{-}1 \times J_a)$

$$= (\mathbb{R}^{a-1} \sim \Pi a\text{-}1) \times \mathbb{R} \cup \Pi a\text{-}1 \times (\mathbb{R} \sim J_a)$$

$$\subseteq \Big(\bigcup_{N} I_n \times \mathbb{R}\Big) \cup \Big(\Pi a\text{-}1 \times \big(\bigcup_{N} I_k\big)\Big)$$

$$= \bigcup_{n,m} (I_n \times \overline{I_m}) \cup \bigcup_{N} (\Pi a\text{-}1 \times I_k).$$

$I_n \times \overline{I_m}$ and $\Pi a\text{-}1 \times I_k$ are compact subsets of $\mathbb{R}$, for all n, m, k in $\mathbb{N}$. Since $\Pi a\text{-}1$ is compact, $\mathcal{C}(\Pi a\text{-}1) = C_9$ and we have that $\sup_{N\times N} \mathcal{C}(I_n \times \overline{I_m}) \leq C_8\, C_7$ and $\sup_{N} \mathcal{C}(\Pi a\text{-}1 \times I_k) \leq C_9 2 = C_{10}$. Therefore 4) holds with $C = \max(C_8C_7, C_{10})$.

Finally, by (3) and (4)

$$\sum_{n,m}\left[\int_{I_n\times \overline{I_m}} \alpha_U(y_0 - x)^p\, dx\right]^{1/p} =$$

$$= \sum_{n,m}\left[\int_{I_n} \prod_{i=1}^{a-1} \alpha_{U_i}(y_i - x_i)^p\, dx\right]^{1/p} \left[\int_{\overline{I_m}} \alpha_{U_a}(y_a - x_a)^p\, dx_a\right]^{1/p}$$

$$= \sum_{N}\left[\int_{I_n} \prod_{i=1}^{a-1} \alpha_{U_i}(y_i - x_i)^p\, dx\right]^{1/p} \sum_{N}\left[\int_{\overline{I_m}} \alpha_{U_a}(y_a - x_a)^p\, dx_a\right]^{1/p}$$

$$= O\Big(\prod_{i=1}^{a} \delta_i\Big).$$

By (1) and (5)

$$\sum_{N}\left[\int_{\Pi a-1\; I_k} \alpha_U(y_0 - x)^p\, dx\right]^{1/p}$$

$$= \sum_{N}\left[\prod_{i=1}^{a-1} \int_{J_i} \alpha_{U_i}(y_i - x)^p\, dx\right]^{1/p} \left[\int_{I_k} \alpha_{U_a}(y_a - x)^p\, dx\right]^{1/p}$$

$$= \prod_{i=1}^{a-1}\left[\int_{J_i} \alpha_{U_i}(y_i - x)^p \, dx\right]^{1/p} \sum_N \left[\int_{J_k} \alpha_{U_a}(y_a - x)^p \, dx\right]^{1/p}$$

$$= O\left(\prod_{i=1}^{a} \delta_i\right). \dagger$$

THEOREM 7.6. Let $f \in (L^p, \ell^\infty)$, $1 < p \leq \infty$, and $y \in G$. For each $U \in N_0(G)$ and all $V_y \in N_y(G)$

$$\int_{G \sim V_y} \phi_U(y - x)\, f(x)\, dx \longrightarrow 0 \quad \text{as } U \to 0.$$

PROOF. We write $y = (y_0, s_0)$, $y_0 = (y_1, .., y_a)$ in $\mathbb{R}^a$, $s_0 \in G_1$. Let $V_y \in N_y(G)$ and take $V \in N_0(G)$ such that $y + V \subseteq V_y$ and V has the form $[-\gamma_1,\gamma_1]\times \cdots \times[-\gamma_a,\gamma_a]\times V_H$, $\gamma_i > 0$, $i = 1,..,a$, $V_H \subseteq H$ compact and $V_H \in N_0(G_1)$. Also, U contains a product neighborhood $(-\delta_1,\delta_1)\times \cdots \times(-\delta_a,\delta_a)\times U_H$, $\delta_i > 0$, $i = 1,\ldots,a$, $U_H \subseteq H$ and $U_H \in N_0(G\ )$.

Set $\eta_i = \min(\delta_i^{2p'}, \gamma_i)$, $i = 1,..,a$, and $W_H = \operatorname{int} V_H$. Then $W = [-\eta_1,\eta_1]\times \cdots \times[-\eta_a,\eta_a]\times W_H$ satisfies the conditions listed in Lemma 7.5.

By 1) of Lemma 7.5, $W_y \subseteq V_y$ and therefore it is enough to prove that

$$\int_{G \sim W_y} \phi_U(y - x)\, f(x)\, dx \longrightarrow 0 \quad \text{as } U \to 0.$$

By 2) of Lemma 7.5, $W_y = \Pi a \times (s_0 + W_H)$. Hence

$$G \sim W_y = (\mathbb{R}^a \times G_1) \sim (\Pi a \times (s_0 + W_H)) = (\mathbb{R}^a \sim \Pi a) \times G_1$$
$$\cup\ \Pi a \times (G_1 \sim (s_0 + W_H)).$$

This implies that

$$(6)\qquad \int_{G \sim W_y} \phi_U(y - x)\, f(x)\, dx$$

$$= \int_{(R^{a}\sim \Pi a)\times G_1} \phi_U(y - x)f(x)dx + \int_{\Pi a\times (G_1 \sim (s_0+W_H))} \phi_U(y - x)f(x)dx.$$

If $x = (t,s)$ in G then by definition of ϕ_U,
$\phi_U(y - x) = \alpha_U(y_0 - t)\beta_U(s_0 - s) = 0$ if $s_0 - s \notin H$. Hence,

$$\int_{(R^{a}\sim \Pi a)\times G_1} \phi_U(y - x)f(x)dx = \int_{(R^{a}\sim \Pi a)\times (s_0+H)} \phi_U(y - x)f(x)dx$$

$$\int_{\Pi a\times (G_1 \sim (s_0+W_H))} \phi_U(y - x)f(x)dx = \int_{\Pi a\times (s_0+(H\sim W_H))} \phi_U(y - x)f(x)dx.$$

Let $I_n \in \{I_n\}_{\mathbb{N}}$, $\{I_n\}_{\mathbb{N}}$ as in 3) above. So, by Hölder's inequality

$$\int_{I_n\times (s_0+H)} \phi_U(y - x)f(x)dx \leq ||f\chi_{I_n\times (s_0+H)}||_p \left[\int_{I_n\times (s_0+H)} \phi_U(y - x)^{p'}dx\right]^{1/p'}$$

$$\leq ||f\chi_{I_n\times (s_0+H)}||_p \; B_U \left[\int_{I_n} \alpha_U(y_0 - x)^{p'}dx\right]^{1/p'}$$

where B_U is the constant in (7.5).

By 4) $|S(I_n \times (s_0 + H)| \leq C$ for all $n \in \mathbb{N}$ (see page 36 and Definition 1.6). This implies that for all $n \in N$
$||f\chi_{I_n\times (s_0+H)}||_p \leq |S(I_n \times (s_0+ H)|\; ||f||_{p\infty} \leq C\; ||f||_{p\infty}$. Then we conclude from 3) that

$$(7) \qquad \int_{(R^{a}\sim\Pi a)\times G_1} \phi_U(y - x)f(x)dx \leq \sum_N \int_{I_n\times(s_0+H)} \phi_U(y - x) \; |f(x)| \; dx$$

$$\leq C \; ||f||_\infty \; B_U \sum_N \left[\int_{I_n} \alpha_U(y_0 - x)^{p'} dx \right]^{1/p'}$$

$$= O\left(\prod_{i=1}^{a} \delta_i \; B_U\right).$$

Applying again the Hölder inequality

$$\int_{\Pi a\times(s_0+(H\sim W_H))} \phi_U(y - x) \; |f(x)| \; dx$$

$$= ||f\chi_{\Pi a\times(s_0+(H\sim W_H))}||_p \left[\int_{\Pi a\times(s_0+(H\sim W_H))} \phi_U(y - x)^{p'} dx \right]^{1/p'}$$

$$\leq B_U \; ||f||_{p\infty} \; |S(\Pi a\times(s_0+(H\sim W_H))| \left[\int_{\Pi a} \alpha_U(y_0 - x)^{p'} dx \right]^{1/p'}.$$

Note that $\Pi a \times (s_0 + (H \sim W_H))$ is compact (H is compact and $H \sim W_H$ is closed) and that by definition of β_U (pp. 93 and 94). $B_U \to 0$ as $U \to 0$.

Now, since $\Pi a \to y$ as $U \to 0$ and $s_0 + (H \sim W_H) \subseteq s_0 + H$ is independent of U, we have that $|S(\Pi a \times (s_0 + (H \sim W_H))| \to 1$ as $U \to 0$. Therefore by 2)

$$(8) \qquad \int_{\Pi a\times(s_0+(H\sim W_H))} \phi_U(y - x) \; |f(x)| \; dx \to 0 \quad \text{as} \quad U \to 0.$$

Hence we conclude from (6), (7) and (8) that

$$\int_{G\sim W_y} \phi_U(y - x)\, f(x)\, dx \to 0 \quad \text{as} \quad U \to 0. \dagger$$

<u>COROLLARY 7.7</u>. For all $f \in (L^p, \ell^\infty)$, $2 \le p \le \infty$,

$$\lim_{U \to 0} \int_G \phi_U(y - x)\, f(x)\, dx = f(y) \qquad \text{a.e.}$$

In other words, $\lim_{U \to 0} \phi_U * f(y) = f(y)$ a.e.

<u>PROOF</u>. Let $V_y \in N_y(G)$ compact. Since

$$\int_G \phi_U(y - x)\, f(x)\, dx = \int_{G\sim V_y} \phi_U(y - x)\, f(x)\, dx + \int_{V_y} \phi_U(y - x)\, f(x)\, dx$$

and by Theorem 7.6 $\int_{G\sim V_y} \phi_U(y - x)\, f(x)\, dx \to 0$ as $U \to 0$ we just have to prove that $\int_{V_y} \phi_U(y - x)\, f(x)\, dx \to f(y)$ as $U \to 0$.

By (2.5) $(L^p, \ell^\infty) \subseteq (L^2, \ell^\infty)$, so $f \in L^2_{loc}$ and the function $g = f|V_y = f\chi_{V_y}$ belongs to $L^2(G)$.

By Corollary 7.4 and (2.7), $\{\phi_U\} \subseteq L^2(G)$ and by the Parseval identity [37, Theorem 31.19]

$$\int_{V_y} \phi_U(y - x)\, f(x)\, dx = \int_G \phi_U(y - x)\, g(x)\, dx = \int_G \hat{\phi}_U(\hat{x})\, \hat{g}(-\hat{x})\overline{[y,\hat{x}]}d\hat{x}.$$

Since $\lim_{U \to 0} \hat{\phi}_U(\hat{x}) = 1$ $(\hat{x} \in \hat{G})$, $\sup_{\hat{x} \in \hat{G}} |\hat{\phi}_U(\hat{x})| \le 1$, $\hat{\phi}_U \in L^1(\hat{G})$ (Theorem 7.2) and $\hat{g} \in L^\infty(\hat{G})$, we can apply Lebesgue's Domonated Convergence Theorem and we see that

$$\lim_{U \to 0} \int_{V_y} \phi_U(y - x)\ f(x)\ dx = \lim_{U \to 0} \int_{\hat{G}} \hat{\phi}_U(\hat{x})\ \hat{g}(-\hat{x})\ \overline{[y,\hat{x}]}\ d\hat{x}$$

$$= \int_{\hat{G}} \hat{g}(-\hat{x})\ \overline{[y,\hat{x}]}\ d\hat{x} = \check{\hat{g}}(y)$$

$$= g(y) \quad \text{a.e.}$$

Note that $\check{g} = g$ a.e. by [37, 31.44 a) i)].†

The next theorem is the generalization of the Theorem 7.1.

<u>THEOREM 7.8</u>. Let $\mu \in M_q$, $1 \leq q \leq 2$.

i) $\displaystyle\int_{\hat{G}} \overline{f(\hat{x})}\ \hat{\mu}(\hat{x})\ d\hat{x} = \int_G \overline{\check{f}(x)}\ d\mu(x)$ for all $f \in (L^q, \ell^1)(\hat{G})$.

ii) (C.1) $\displaystyle\int_G \overline{[x,\hat{x}]}\ d\mu(x) := \lim_{U \to 0} \int_G \overline{\check{\phi}_U(x)}\ \overline{[x,\hat{x}]}\ d\mu(x) = \hat{\mu}(\hat{x})$ a.e.

<u>PROOF</u>. We pointed out at the beginning of this section that there exists a sequence (μ_n) in $M_1(G)$ such that $\lim ||\mu_n - \mu||_q = 0$ and $\lim ||\hat{\mu}_n - \hat{\mu}||_{q'\infty} = 0$.

By the Extended Parseval Formula (as in [49, Lemma 4.1]), for all $f \in (L^q, \ell^1)(\hat{G})$

(9) $$\int_{\hat{G}} \overline{f(\hat{x})}\ \hat{\mu}_n(\hat{x})\ d\hat{x} = \int_G \overline{\check{f}(x)}\ d\mu_n(x).$$

Now, by the Hölder inequality and Theorem 5.7,

$$\int_{\hat{G}} |f(\hat{x})|\ |\hat{\mu}_n - \hat{\mu}(\hat{x})|\ d\hat{x}$$

$$\leq \sum_\beta \left[\int_{K_\beta} |f(\hat{x})|^q\ d\hat{x} \right]^{1/q} \left[\int_{K_\beta} |\hat{\mu}_n(\hat{x}) - \hat{\mu}(\hat{x})|^{q'}\ d\hat{x} \right]^{1/q'}$$

$\leq ||f||_{q1} \cdot ||\hat{\mu}_n - \hat{\mu}||_{q', \infty}$.

Similarly $\int_G |\check{f}(x)| \, d|\mu_n - \mu|(x) \leq ||\check{f}||_{\infty q'} \cdot ||\mu_n - \mu||_q$.

Therefore the right side of (9) converges to $\int_G \check{f}(x) \, d\mu(x)$ and the left side converges to $\int_{\hat{G}} \overline{f(\hat{x})} \; \hat{\mu}(\hat{x}) \, d\hat{x}$. This proves i).

By Corollary 7.4 $\{\phi_U\} \subseteq (L^q, \ell^1)(\hat{G})$, so from i)

$$\int_{\hat{G}} \phi_U(\hat{y} - \hat{x}) \; \hat{\mu}(\hat{x}) \, d\hat{x} = \int_G \overline{[x,\hat{y}] \; \check{\phi}_U(x)} \, d\mu(x).$$

(Remember that ϕ_U is a real function).

Therefore by Corollary 7.7

$$\hat{\mu}(\hat{y}) = \lim_{U \to 0} \phi_U * \hat{\mu}(\hat{y}) = \lim_{U \to 0} \int_G \check{\phi}_U(x) \, [x,\hat{y}] \, d\mu(x)$$

almost everywhere. †

§ 8. STRONG RESONANCE CLASS OF FUNCTIONS

F. Holland [35] introduced the space $R(\Phi_q)$ $(1 \leq q \leq \infty)$ of functions resonant relative to the space Φ_q for the real line and established a correspondence between the elements of $R(\Phi_q)$ and the space of unbounded measures M_q.

We will define the space $SR(\Phi_q)$ $(1 \leq q \leq \infty)$ of functions strongly resonant relative to the space Φ_q for locally compact abelian groups, characterize $SR(\Phi_q)$ and $R(\Phi_q)$ in terms of transformable measures, study the relation of $SR(\Phi_q)$ with the set of positive definite functions for $(L^{q'}, \ell^1)$ and prove two theorems for $SR(\Phi_q)$ $(1 \leq q < 2)$, similar to Theorems 7 and 9 of [35]. Furthermore we will show that both representations are equivalent.

From now on and for the rest of our work $\{\phi_U\}$ will be the summability kernel defined in §7.

First we will recall the definition of the spaces Φ_q and prove some of their already known properties [7,§2].

<u>DEFINITION 8.1</u>. Let $1 \leq q \leq \infty$. Φ_q is the linear subspace of C_c of functions ϕ such that $\hat{\phi} \in (C_0, \ell^q)$ endowed with the norm $\phi \longmapsto ||\hat{\phi}||_{\infty q}$.

We write Φ for Φ_1.

<u>REMARK 8.2</u>. If $\phi \in \Phi_q$, $1 \leq q \leq \infty$, then it is clear that ϕ', $\tilde{\phi}$, $\tau_t\phi$ $(t \in G)$, $\hat{x}\phi$ $(\hat{x} \in \hat{G})$ also belong to Φ_q.

PROPOSITION 8.3. i) $\Phi \subseteq \Phi_q \subseteq \Phi_\infty$, $1 < q < \infty$.

ii) $\Phi_q = C_c$ for $2 \le q \le \infty$.

iii) Φ_q is dense in $\check{C}_c$ for $1 \le q \le \infty$.

iv) $(\Phi_q)^{\wedge}$ is dense in (C_0, ℓ^s) $(1 \le s \le \infty)$, (L^r, c_0) $(1 \le r < \infty)$, and (L^r, ℓ^s) $(1 \le r,s < \infty)$ for $1 \le q \le \infty$.

v) $T \in \Phi_q(G)^*$ $(1 \le q \le \infty)$ iff there exists a unique measure $\mu \in M_{q'}(\hat{G})$ such that for all $\phi \in \Phi_q(G)$

$$T(\phi) = \int_{\hat{G}} \hat{\phi}(-\hat{x})\, d\mu(\hat{x}).$$

PROOF. i) follows from (2.5) and ii) is a direct consequence of the Hausdorff-Young inequality (Theorem 5.7).

By Remark 6.3 and Lemma 6.4, $D(G)$ is dense in $\check{C}_c(G)$ and $D(G) \subseteq \Phi$. So, by i) Φ_q is dense in $\check{C}_c(G)$.

Let A be any of the amalgam spaces listed in iv). Take $f \in C_c$. Since $\{\phi_U\}$ is an a.i. in L^1 (Theorem 7.2) $\lim \phi_U * f = f$ in A (Corollary 4.14). But $\{\phi_U * f\} \subseteq (\Phi_q)^{\wedge}$ because $\{\hat{\phi}_U \hat{f}\} \subseteq C_c(\hat{G})$, $\phi_U * f = (\hat{\phi}_U \hat{f})^{\vee}$ and $\phi_U * f \in (C_0, \ell^1)$ (Corollary 7.4). Since C_c is dense in A (Theorem 3.7) this proves iv).

The necessity part of v) follows from Theorem 3.2. If $T \in \Phi_q^*$ then the map $\overline{T}(\phi) = T(\phi')$ belongs to $((\Phi_q)^{\wedge}, \|\cdot\|_{\infty q})^*$. Since $(\Phi_q)^{\wedge}$ is dense in (C_0, ℓ^q), $\overline{T}$ has a unique continuous extension $\overline{T}$ on (C_0, ℓ^q). By Theorem 3.2 there exists a unique $\mu \in M_{q'}(\hat{G})$ such that for all $f \in (C_0, \ell^q)$, $\overline{T}(f) = \int_G f(\hat{x})\, d\mu(\hat{x})$. Therefore for $\phi \in \Phi_q$

$$T(\phi) = \overline{T}(\hat{\phi}') = \int_{\hat{G}} \hat{\phi}(-\hat{x})\, d\mu(\hat{x}). \dagger$$

DEFINITION 8.4. Let $1 \leqq q \leqq \infty$. A measurable function f on G is strongly resonant relative to the space Φ_q if

(R-1) $f\phi \in L^q(G)$ for all $\phi \in \Phi$

(R-2) The linear functional $\phi \longmapsto \int f\phi$ on Φ_q is continuous.

That is, there exists a constant C such that for all $\phi \in \Phi_q$

$$\left| \int f\phi \right| \leqq C \, ||\hat{\phi}||_{\infty_q}.$$

The linear space of funcions strongly resonant relative to Φ_q will be denoted by $SR(\Phi_q)$.

REMARK 8.5. If $f \in SR(\Phi_q)$ $(1 \leqq q \leqq \infty)$, then by Remark 8.2, f', $\tilde{f}$, $\tau_t f$ $(t \in G)$, $\hat{x}f$ $(\hat{x} \in \hat{G})$, also belong to $SR(\Phi_q)$.

THEOREM 8.6. i) $SR(\Phi_\infty) \subseteq SR(\Phi_q) \subseteq SR(\Phi)$ $1 < q < \infty$.

ii) A measurable function f satisfies the condition (R-1) iff $f \in L^q_{loc}$.

iii) $f \in SR(\Phi_q)$ $(1 \leqq q \leqq \infty)$ iff $f \in L^q_{loc}$, $f \in M_T$ and $\hat{f} \in M_{q'}(\hat{G})$.

iv) $SR(\Phi_q) \subseteq (L^q, \ell^\infty)$ for $2 \leqq q \leqq \infty$

v) $f \in SR(\Phi_q)$ for $2 \leqq q \leqq \infty$ iff $f \in M_T$ and $\hat{f} \in M_{q'}$

vi) If $f \in SR(\Phi)$ and $f \geqq 0$ then $f \in (L^1, \ell^\infty)$.

PROOF. i) follows immediately from Proposition 8.3, i).

The sufficiency part of ii) is clear. Suppose $f\phi \in L^q(G)$ for all $\phi \in \Phi$. Let $E \subseteq G$ compact and take $\phi \in \Phi$ such that $\phi \equiv 1$ on E (Theorem 5.2). Since $f\phi \in L^q$ and $||f||_{L^q(E)} \leqq ||f\phi||_{L^q}$ we conclude that $f \in L^q_{loc}$. Let $f \in SR(\Phi_q)$, $1 < q < \infty$. Then the map $T(\phi) = \int \phi f$ $(\phi \in \Phi_q)$ belongs to Φ_q^*. So by Proposition 8.3, v) there exists a unique measure $\mu \in M_{q'}$ such that for all $\phi \in \Phi_q$

$$\int_G f(x)\ \phi(x)\ dx = T(\phi) = \int_{\hat{G}} \hat{\phi}(-\hat{x})\ d\mu(\hat{x}).$$

Since $D(G) \subseteq \Phi \subseteq \Phi_q$ we conclude that $f \in M_T$ and $\overline{f} = \mu$. Conversely, if $f \in M_T$ and $\overline{f} \in M_{q'}$ then for $\phi \in \Phi_q$, $\hat{\phi} \in L^1(\overline{f})$ and by [1 , Corollary 3.1]

$$\int_G \phi(x)\ f(x)\ dx = \int_{\hat{G}} \hat{\phi}(-\hat{x})\ d\overline{f}(\hat{x}).$$

This implies that $|\int f\phi| \leq ||\overline{f}||_{q'}||\hat{\phi}||_{\infty q}$ for all $\phi \in \Phi_q$. Hence f satisfies the condition (R-2). This proves iii).

If $f \in M_T$ and $\overline{f} \in M_{q'}$, $1 \leq q' \leq 2$, then $\overline{f} \in \hat{M}_T$ and $\overline{\overline{f}} = \hat{\overline{f}}$ (Theorem 6.19). But $f = \overline{\overline{f}}$ (Remark 6.25) and $\hat{\overline{f}} \in (L^q, \ell^\infty)$. Therefore $f \in (L^q, \ell^\infty)$. This together with iii) implies iv) and v).

Now take $f \in SR(\Phi)$, $f \geq 0$. Let $g \in \Phi$ such that $g \equiv 1$ on K. Then $\tau_\alpha g \equiv 1$ on K_α for all $\alpha \in J$, $\tau_\alpha g \in \Phi$ and $||(\tau_\alpha g)\hat{}\,||_{\infty 1} = ||\hat{g}||_{\infty 1}$ for all $\alpha \in J$.

Since $f \in M_T$ we have that

$$\int_{K_\alpha} f(x)\ dx = \int_{K_\alpha} f(x)\ |\tau_\alpha g(x)|^2\ dx \leq \int_G f(x)\ |\tau_\alpha g(x)|^2 dx$$

$$= \int_{\hat{G}} (\tau_\alpha g) * (\overline{\tau_\alpha g})\hat{}\ \ d\overline{f}(\hat{x}).$$

Therefore by Theorem 4.7, for each $\alpha \in J$

$$\int_{K_\alpha} f(x)\,dx \leq ||\mathcal{F}||_\infty ||(\tau_\alpha g)^\wedge * (\overline{\tau_\alpha g})^\wedge||_{\infty 1}$$

$$\leq ||\mathcal{F}||_\infty\, 2^a\, ||(\tau_\alpha g)^\wedge||_{\infty 1}\, ||(\overline{\tau_\alpha g})^\wedge||_1$$

$$= ||\mathcal{F}||_\infty\, 2^a\, ||\hat{g}||_{\infty 1}\, ||\hat{g}||_1 .$$

This implies that $f \in (L^1, \ell^\infty)$. +

REMARK 8.7. We deduce from Theorem 8.6 that

i) $SR(\Phi) = L^1_{loc} \cap M_T$ (part iii) and Remark 6.18).

ii) If $f \in SR(\Phi_q)$, $1 \leq q \leq \infty$, then there exists a unique measure $\mathcal{F} \in M_q$, such that for all $\phi \in \Phi_q$

$$\int_G f(x)\,\phi(x)\,dx = \int_{\hat{G}} \hat{\phi}(-\hat{x})\,d\mathcal{F}(\hat{x}).$$

(See the proof of part iii)).

Following Holland [35, §5] we define:

DEFINITION 8.8. A measurable function f on G is resonant relative to the space Φ_q, $1 \leq q \leq \infty$, if

(R-1)' $f\phi \in L^1(G)$ for all $\phi \in \Phi_q$

and f satisfies (R-2) of Definition 8.4.

The linear space of functions resonant relative to Φ_q will be denoted by $R(\Phi_q)$.

The proof of the next theorem is very similar to Theorem 8.6 and it will be omitted.

THEOREM 8.9. i) $R(\Phi_\infty) \subseteq R(\Phi_q) \subseteq R(\Phi)$ for $1 < q < \infty$

ii) A measurable function f satisfies the condition (R-1)' iff $f \in L^1_{loc}$.

iii) $f \in R(\Phi_q)$, $1 \leq q \leq \infty$, iff $f \in L^1_{loc}$, $f \in M_T$ and $\hat{f} \in M_{q'}$.

iv) $R(\Phi_q) \subseteq (L^q, \ell^\infty)$ for $2 \leq q \leq \infty$.

v) $f \in R(\Phi_q)$, $2 \leq q \leq \infty$, iff $f \in M_T$ and $\hat{f} \in M_{q'}$.

We conclude easily from Theorem 8.6 and Theorem 8.9 that $R(\Phi) = SR(\Phi)$, $R(\Phi_q) = SR(\Phi_q)$, $2 \leq q \leq \infty$, $SR(\Phi_q) \subseteq R(\Phi_q)$, $1 < q < 2$, and $SR(\Phi_q) = R(\Phi_q) \cap L^q_{loc}$, $1 < q < 2$.

We do not know if strong resonant relative to Φ_q is equivalent to resonant relative to Φ_q for $1 < q < \infty$. In this direccion we have a partial result (see Remark 8.21).

We modify Holland's definition because by imposing the condition (R-1) we have that $SR(\Phi_q) \subseteq L^q_{loc}$ for all $1 < q < 2$, and this allow us to endow $SR(\Phi_q)$ with a locally convex topology with respect to which the linear space spanned by the set of positive definite functions for $(L^{q'}, \ell^1)$ is dense in $SR(\Phi_q)$. Also we see that Theorem 3 and Theorem 7 of [35] for $R(\Phi_q)$, $1 \leq q < 2$ are valid for $SR(\Phi_q)$, $1 \leq q < 2$.

Observe that vi) of Theorem 8.6 is a generalization of [35, Theorem 8 i)].

The next theorem was proved for the real line by Holland [35, Theorem 6] and for locally compact abelian groups by Stewart [49, Theorem 4.4].

<u>THEOREM 8.10</u>. Let $2 \leq q < \infty$. $f \in SR(\Phi_q)$ iff there exists a unique measure $\mu \in M_{q'}(\hat{G})$ such that $f = \hat{\mu}$.

<u>PROOF</u>. The necessity part follows from the proof of Theorem 8.6 iv). Suppose $f = \hat{\mu}$, $\mu \in M_{q'}$. Then $f \in M_T$ because $M_{q'} \subseteq \mathcal{A}(\hat{G})$ (Remark 6.25). Since $\hat{\mu} \in (L^q, \ell^\infty) \subseteq M_\infty$, $f \in M_\infty \cap M_T$.

So by Theorem 6.23 and Theorem 6.21, $\overline{f} = \hat{f} = \check{\mu} = \mu$. Therefore $f \in SR(\Phi_q)$ (Theorem 8.6 v)). †

<u>REMARK 8.11</u>. It follows from [1 , Theorem 3.3] that for $f \in SR(\Phi_q)$, $1 \leq q \leq \infty$.

i) If $\phi \in L^1(G)$, ϕ is convolvable with f and $\hat{\phi} \in L^1(\overline{f})$ then for locally almost all $x \in G$

$$f*\phi(x) = \int_G f(y)\ \phi(y - x)\ dy = \int_{\hat{G}} \hat{\phi}(\hat{x})\ [x,\hat{x}]\ d\overline{f}(\hat{x}).$$

ii) For any $u \in G$ such that the integral on the left is a continuous function of x in the neighborhood of u, the formula in i) is valid for $x = u$. Under this hypothesis for $u = 0$

$$\int_G f(y)\ \phi(-y)\ dy = \int_{\hat{G}} \hat{\phi}(\hat{x})\ d\overline{f}(\hat{x}).$$

<u>DEFINITION 8.12</u>. Let F be a set of complex-valued functions. A complex valued function f is <u>positive definite</u> for F if the integral

$$\int\int f(x - y)\ \phi(x)\ \overline{\phi(y)}\ dx\ dy$$

exists as a Lebesgue integral over the product set $G \times G$ and is nonnegative for all $\phi \in F$.

$\underline{P(F)}$ will denote the set of positive definite functions for F.

We list in the following remark the properties and results about positive definite functions we will need to prove Theorem 8.20.

REMARK 8.13. i) It is clear that if $F_1 \subseteq F_2$ then $P(F_2) \subseteq P(F_1)$.

ii) $P(L_c^p) \subseteq L_{loc}^{p'}$ and $P((L^p, \ell^1)) = \mathcal{P} \cap (L^{p'}, \ell^\infty)$, $1 \leq p < \infty$, where $\mathcal{P} = \{ \hat{\mu} \mid \mu \in M_\infty, \ \mu \geq 0 \}$ [17, §4 Theorem III].

iii) If $f \in P(C_c)$ then $f \in M_T$ and $\hat{F} \geq 0$ [1, Theorem 4.1].

iv) By i) of Remark 8.7 and iii), $P(C_c) \subseteq SR(\Phi)$.

DEFINITION 8.14. For $E \subseteq G$ compact we define the seminorm ρ_E on $SR(\Phi_q)$, $1 \leq q \leq \infty$, by $\rho_E(f) = ||f||_{L^q(E)}$, and endow $SR(\Phi_q)$ with the locally convex topology generated by the family $\{ \rho_E \mid E \subset G \text{ compact} \}$.

We want to prove that $< P(L^{q'}, \ell^1) >$, $1 \leq q \leq 2$, the linear space spanned by $P(L^{q'}, \ell^1)$, is dense in $SR(\Phi_q)$. For that we need to introduce the concept of summability function of type I and establish some results.

DEFINITION 8.15. Given $U \in N_0(G)$ compact, let ψ_U be a function with the following properties.

(8.1) ψ_U is continuous with support U

(8.2) $\psi_U \geq 0$ and $\int \psi_U = 1$

(8.3) $\hat{\psi}_U \geq 0$ and $\hat{\psi}_U \in L^1(\hat{G})$

Then ψ_U will be called a <u>summability function of type I</u> on $\hat{G}$.

<u>DEFINITION 8.16</u>. If $U \in N_0(G)$, let $V \in N_0(G)$ compact such that $V - V \subseteq U$ and $V = V_1 \times V_2$ where $V_1 \subseteq R^a$ and $V_2 \subseteq G_1$.

For $i = 1, 2$ let β_i be a positive L^2 function having support in V_i and integral equal to 1.

Set $\beta_U(s,t) = \beta_1(s)\beta_2(t)$ and $\psi_U = \beta_U * \tilde{\beta}_U$. Then $\hat{\psi}_U$ is a summability function of type I on $\hat{G}$. Moreover

(8.4) $\quad \{\psi_U\} \subseteq \Phi$

(8.5) $\quad ||\hat{\psi}_U||_\infty \leq 1$

Indeed, $\beta_U \in L^2_c$ and therefore $\beta_U \in (L^2, \ell^1)$. Hence by Theorem 5.4 $\hat{\beta}_U \in (C_0, \ell^2)$. This implies that $\hat{\psi}_U = \hat{\beta}_U\bar{\hat{\beta}}_U$ belongs to (C_0, ℓ^1) (Proposition 4.1).

Since $||\hat{\psi}_U||_\infty \leq ||\psi_U||_1$ we have by (8.2) that $||\hat{\psi}_U||_\infty \leq 1$.

For a proof of the next theorem see [50, II §2 Theorem 4].

<u>THEOREM 8.17</u>. Any summability kernel $\{\psi_U\}$ of type I has the following properties:

(8.6) $\quad \lim\limits_{U \to 0} ||\psi_U * f - f||_p = 0$ for any $f \in L^p(G)$, $1 \leq p < \infty$

(8.7) $\quad \hat{\psi}_U$ converges pointwise to 1 on $\hat{G}$ as $U \to 0$

Hereafter and throughout the whole work $\{\psi_U\}$ will be the summability kernel of type I as defined in Definition 8.16.

PROPOSITION 8.18. If $f \in L^q_{loc}$ and there exists a measure $\mu \in M_{q'}(\hat{G})$ $(1 < q \leq \infty)$ such that

$$f(x) = \lim_{U \to 0} \int_{\hat{G}} \hat{\psi}_U(\hat{x}) \, [x,\hat{x}] \, d\mu(\hat{x})$$

where the limit exists in L^q over any compact subset of G, then $f \in SR(\Phi_q)$ and $\hat{f} = \mu$.

PROOF. Let $\phi \in \Phi$, $E = \text{supp } \phi$. Set $F_U(x) = \int \hat{\psi}_U(\hat{x}) \, [x,\hat{x}] \, d\mu(\hat{x})$. Since F_U converges to f in $L^q(E)$ and $\phi \in L^{q'}(E)$

$$\int_G f(x) \, \phi(x) \, dx = \lim_{U \to 0} \int_G F_U(x) \, \phi(x) \, dx$$

$$= \lim_{U \to 0} \int_G \int_{\hat{G}} \hat{\psi}_U(\hat{x}) \, [x,\hat{x}] \, d\mu(\hat{x}) \, \phi(x) \, dx$$

$$= \lim_{U \to 0} \int_{\hat{G}} \hat{\psi}_U(\hat{x}) \int_G \phi(x) \, [x,\hat{x}] \, dx \, d\mu(\hat{x})$$

$$= \lim_{U \to 0} \int_{\hat{G}} \hat{\psi}_U(\hat{x}) \, \hat{\phi}(-\hat{x}) \, d\mu(\hat{x})$$

$$= \int \hat{\phi}(-x) \, d\mu(\hat{x}).$$

The last equality follows from the Lebesgue Dominated Convergence Theorem. Remember that $\hat{\phi} \in L^1(\mu)$, $||\hat{\psi}_U||_\infty \leq 1$ and $\hat{\psi}_U(\hat{s}) \to 1$ as $U \to 0$ ((8.5), (8.7)).

Since ϕ is arbitrary, this implies that $f \in M_T$ and $\hat{f} = \mu$. Therefore by Theorem 8.6 ii) $f \in SR(\Phi_q)$.†

The next proposition is the converse of Proposition 8.18.

PROPOSITION 8.19. If $f \in SR(\Phi_q)$, $1 < q < \infty$, then

$$f(x) = \lim_{U \to 0} \int_{\hat{G}} \hat{\psi}_U(\hat{x}) [x,\hat{x}] \, d\hat{f}(\hat{x})$$

where the limit exists in L^q over any compact subset of G.

PROOF. Since $\{\psi_U\} \subseteq \Phi \subseteq \Phi_q$ by (8.4) we have by Remark 8.7 ii) that

$$f*\psi_U(x) = \int_G f(y)\, \psi_U(x-y)\, dy = \int_{\hat{G}} \hat{\psi}_U(\hat{x}) [x,\hat{x}] \, d\hat{f}(\hat{x}).$$

By Theorem 8.17, (8.6), $f*\psi_U$ converges to f in L^q over any compact subset of G because $f \in L^q_{loc}$. †

If μ is a real-valued measure in M_q ($1 \le q \le \infty$) then $\mu = \mu^+ - \mu^-$ where $\mu^+ = \sup(0,\mu)$, $\mu^- = \sup(0,-\mu)$. Hence $\mu^+ \le |\mu|$, $\mu^- \le |\mu|$ and therefore μ^+, μ^- belong to M_q. Also $|\mu| = \mu^+ + \mu^-$.

Moreover, if μ is a complex-valued measure in M_q ($1 \le q \le \infty$) then $\mu = \mu_1 + i\mu_2$ where μ_i, $i = 1, 2$, is a real-valued measure in M_q and $|\mu| = |\mu_1| + |\mu_2|$ [52, I 37].

THEOREM 8.20. i) Let $1 \le q < 2$. The linear space spanned by $P(L^{q'}, \ell^1)$, $< P(L^{q'}, \ell^1) >$, is dense in $SR(\Phi_q)$.

ii) $< P(L^2, \ell^1) > = SR(\Phi_2)$

iii) $SR(\Phi_q) \subseteq < P(L^{q'}, \ell^1) >$ for $2 < q \le \infty$.

PROOF. By Remark 8.13 i) and iv) $P(L^\infty, \ell^1) \subseteq P(C_c) \subseteq SR(\Phi)$.

Let $f \in P(L^{q'}, \ell^1)$, $1 < q \le 2$. Again by Remark 8.13 ii) and iii) $f \in (L^q, \ell^\infty)$, $f \in M_T$ and $\hat{f} \ge 0$.

Let $g \in C_c(\hat{G})$ such that $g \equiv 1$ on $\hat{K}$ and $\check{g} \in (C_0, \ell^1)$. Set $h = \check{g} * \tilde{\check{g}}$. Since $\check{g} \in (C_0, \ell^1)$, $h \in (C_0, \ell^1)$ and $\hat{h} = g\bar{g} = |g|^2$. Therefore $\hat{h} \geq 0$, $\hat{h} \in C_c$ and $\hat{h} \equiv 1$ on $\hat{K}$, so for $\hat{x} \in \hat{G}$, and $\mathcal{F}^{\#}$ as defined in Theorem 1.21.

$$\mathcal{F}^{\#}(\hat{x}) = \mathcal{F}(\hat{x} + \hat{K}) = \int_{\hat{x}+\hat{K}} d\mathcal{F}(\hat{x}) = \int_{\hat{x}+\hat{K}} \tau_{\hat{x}}\hat{h}(\hat{s})\, d\mathcal{F}(\hat{s})$$

$$< \int_{\hat{G}} \tau_{\hat{x}}\hat{h}(\hat{s})\, d\mathcal{F}(\hat{s}).$$

Now, $h \in L^1$, h is convolvable with f and $\hat{h} \in L^1(\mathcal{F})$, (remember that $\mathcal{F} \in M_\infty$) and $f*h$ is continuous on G. Indeed, note that for x, s in G

$$|f*h(x) - f*h(s)| \leq \int |f(y)|\, |h(x - y) - h(s - y)|\, dy$$

$$\leq ||f||_{q\infty}\, ||\tau_x h' - \tau_s h'||_{q'1}$$

$$\leq ||f||_{q\infty}\, ||\tau_x h' - \tau_s h'||_{\infty 1}$$

and the function $x \longmapsto \tau_x h'$ is continuous on G (Theorem 3.14).

By Remark 8.11 we have that

$$\int \tau_{\hat{x}}\, \hat{h}(\hat{s})\, d\mathcal{F}(\hat{s}) = \int f(x)\, h'(x)\, [-x,\hat{s}]\, dx = (fh')^{\wedge}(\hat{s}).$$

Since $f \in (L^q, \ell^\infty)$, $h' \in (C_0, \ell^1)$, $fh' \in (L^q, \ell^1) \subseteq L^q$, so $(fh')^{\wedge} \in L^{q'}$ and this implies that $\mathcal{F}^{\#} \in L^{q'}$. By Theorem 1.21 we conclude that $\mathcal{F} \in M_{q'}$. By Theorem 8.6 iii), $f \in SR(\Phi_q)$ and therefore $< P(L^{q'}, \ell^1) > \subseteq SR(\Phi_q)$ for $1 \leq q \leq 2$.

Now, take $f \in SR(\Phi_q)$, $1 \leq q \leq \infty$. Then $\mathcal{F} \in M_{q'}$ and by our previous comment $\widehat{f} = \mu_1 - \mu_2 + i(\mu_3 - \mu_4)$ where $\mu_i \in M_{q'}$ and $\mu_i \geq 0$, $i = 1,..,4$. If $2 \leq q \leq \infty$ then $f = \hat{\widehat{f}} = \hat{\mu}_1 - \hat{\mu}_2 + i(\hat{\mu}_3 - \hat{\mu}_4)$ (see the proof of Theorem 8.6 iii)). Since $\mu_i \in \mathcal{P} \cap (L^q, \ell^\infty)$, $i = 1,..,4$ we conclude by Remark 8.13 that $f \in < P(L^{q'}, \ell^1) >$ and this proves ii) and iii).

Assume $1 \leq q < 2$, and set

$$F_U^{\,i}(x) = \int_{\hat{G}} \hat{\psi}_U(\hat{x}) \; [x,\hat{x}] \; d\mu_i(\hat{x}) \qquad i = 1,..,4.$$

We will show that $F_U^{\,i} \in P(L^{q'},\ell^1)$ and this will imply that $F_U(x) = \int_{\hat{G}} \psi_U(\hat{x}) \; [x,\hat{x}] \; d\, f\,(\hat{x})$ belongs to $< P(L^{q'}, \ell^1) >$ since

$F_U = F_U^{\,1} - F_U^{\,2} + i(F_U^{\,3} - F_U^{\,4})$. Hence by Proposition 8.19 we conclude that $f \in < P(L^{q'}, \ell^1) >$.

Let $\phi \in (L^{q'}, \ell^1)$. Since $|F_U^{\,i}(x)| \leq ||\mu_i||_{q'} ||\hat{\psi}_U||_{\infty q}$ for all $x \in G$, $F_U^{\,i} \in L^\infty$ and we can apply Fubini's theorem because $\phi * \tilde{\phi} \in (L^{q'}, \ell^1) \subset L^1$, so

$$\int_G F_U^{\,i}(x) \; \phi*\tilde{\phi}(x) \; dx = \int_G \int_{\hat{G}} \hat{\psi}_U(\hat{x}) \; [x,\hat{x}] \; d\mu_i(\hat{x}) \; \phi*\tilde{\phi}(x) \; dx$$

$$= \int_{\hat{G}} \hat{\psi}_U(\hat{x}) \int_G \phi*\tilde{\phi}(x) \; [x,\hat{x}] \; dx \; d\mu_i(\hat{x})$$

$$= \int_{\hat{G}} \hat{\psi}_U(\hat{x}) \; |\hat{\phi}|^2(-\hat{x}) \; d\mu_i(\hat{x}).$$

By (8.3) we have that

$$\int \int F_U^{\,i}(x - y) \; \phi(x) \; \tilde{\phi}(y) \; dx \; dy = \int F_U^{\,i}(x) \; \phi*\tilde{\phi}(x) \; dx$$

is nonnegative and therefore $F_U^{\,i} \in P(L^{q'}; \ell^1)$. †

REMARK 8.21. We see from the proof of Theorem 8.20 that $< P(L^{q'}, \ell^1) > \subset R(\Phi_q)$, $1 \leq q < 2$.

Since $< P(L^{q'}, \ell^1) > \subset (L^q, \ell^\infty)$, $1 < q < 2$, (Remark 8.13) we have that $R(\Phi_q)$, $1 < q < 2$, contains a linear subspace of (L^q, ℓ^∞).

THEOREM 8.22. $SR(\Phi_\infty)$ is dense in $SR(\Phi_q)$, $1 < q < \infty$. Hence $SR(\Phi_q)$ is dense in $SR(\Phi_s)$, $1 \leq s < q < \infty$.

PROOF. Let $f \in SR(\Phi_q)$. Set $F_U(x) = \int \hat{\psi}_U(\hat{x})\ [x,\hat{x}]\ d\,\overline{F}(\hat{x})$. Since $f \in M_T$ and $\{\psi_U\} \subset \Phi$ by (8.4) we have by Remark 8.7 ii) that

$$\psi_U{}'*f(x) = \int f(y)\ \psi_U(y - x)\ dy = \int \hat{\psi}_U(\hat{x})\ [x,\hat{x}]\ d\,\overline{F}(\hat{x}) = F_U(x).$$

$\psi_U{}'*f$ belongs to L^1 because $f \in L^1_{loc}$ and $\{\psi_U\} \subset C_c$, so by Theorem 6.19 $\psi_U{}'*f \in M_T$ and $(\psi_U{}'*f)^{\frown} = (\psi_U{}'*f)^{\wedge} = (\psi_U{}')^{\wedge}\ (f\chi_U)^{\wedge}$. Since $(f\chi_U)^{\wedge} \in C_0$ and $(\psi_U{}')^{\wedge} \in (C_0, \ell^1)$, $(\psi_U{}'*f)^{\frown} \in M_\infty$. Therefore $\psi_U{}'*f$ belongs to $SR(\Phi_\infty)$ (Theorem 8.6 v)). This implies by Proposition 8.19 that $SR(\Phi_\infty)$ is dense in $SR(\Phi_q)$.

Finally by Theorem 8.6 i) we conclude that $SR(\Phi_q)$ is dense in $SR(\Phi_s)$ if $1 \leq s < q < \infty$. +

The next two theorems are generalizations of Theorems 3 and 7 of [35].

THEOREM 8.23. If $f \in SR(\Phi_q)$, $1 < q \leq \infty$, then

$$\int_G f(x)\ \phi(x)\ dx = \int_{\hat{G}} \hat{\phi}(-\hat{x})\ d\,\overline{F}(\hat{x})$$

for all $\phi \in L^{q'}_c$ such that $\hat{\phi} \in (C_0, \ell^q)$.

Hence

$$\left|\int_G f(x)\,\phi(x)\,dx\right| \leq ||\hat{f}||_{q'}\,||\hat{\phi}||_{\infty q}.$$

PROOF. If $\phi \in L_c^{q'}$ and $\hat{\phi} \in (C_0, \ell^q)$ then $\phi \in L^1$, ϕ is convolvable with f and $\hat{\phi} \in L^1(\hat{f})$. This means that ϕ satisfies i) of Remark 8.11.

Let $V_1 \in N_0(G)$ compact, $s \in V_1$ and $E = \text{supp}\,\phi$.

Case 1) $1 < q < \infty$.

Since $\phi \in L_c^{q'}$, the map $x \longmapsto \tau_x\phi'$ is continuous on G (Theorem 3.14). So given $\varepsilon > 0$ there exists $V_2 \in N_0(G)$ such that for all $x \in V_2$

$$||\tau_x\phi' - \tau_s\phi'||_{q'} < \varepsilon/||f\chi_{V_1 \sim E}||_q.$$

Then for $x \in V_1 \cap V_2$ we have by the Hölder inequality that

$$|f*\phi(x) - f*\phi(s)| \leq \int |f(y)|\,|\phi(x-y) - \phi(s-y)|\,dy$$

$$= \int_{V_1 \sim E} |f(y)||\phi'(y-x) - \phi'(y-s)|\,dy$$

$$\leq ||f\chi_{V_1 \sim E}||_q\,||\tau_x\phi' - \tau_s\phi'||_{q'} < \varepsilon.$$

Therefore $f*\phi$ is continuous at s.

Case 2) $q = 1$.

Similarly to case 1), the map $x \longmapsto \tau_x(f\chi_{V_1 \sim E})'$ is continuous on G. So given $\varepsilon > 0$ there exists $V_2 \in N_0(G)$ such that for all $x \in V_2$

$$||\tau_x(f\chi_{V_1 \sim E})' - \tau_s(f\chi_{V_1 \sim E})'||_1 < \varepsilon/||\phi||_\infty.$$

Then for $x \in V_1 \cap V_2$

$$|f*\phi(x) - f*\phi(s)| \leqq \int_E |\phi(y)|\ |f(x - y) - f(s - y)|\ dy$$

$$\leqq \int_G |\phi(y)|\ |f\chi_{V_1\sim E} - f\chi_{V_1\sim E}(s - y)|\ dy$$

$$\leqq ||\phi||_\infty\ ||\tau_x(f\chi_{V_1\sim E})' - \tau_s(f\chi_{V_1\sim E})'||_1 < \varepsilon.$$

Again $f*\phi$ is continuous at s.

Applying Remark 8.11 we conclude that

$$\int_G f(x)\ \phi(x)\ dx = \int_{\hat{G}} \hat{\phi}(-\hat{x})\ d\,\overline{F}(\hat{x}).\dagger$$

THEOREM 8.24. Let $1 \leqq q < 2$. If $f \in SR(\Phi_q)$ then

$$\text{(C.1)} \quad \int_G \check{h}(x)\ f(x)\ dx = \int_{\hat{G}} h(-\hat{x})\ d\,\overline{F}(\hat{x})$$

for all $h \in (C_0, \ell^q)(\hat{G})$.

Furthermore if $p = \dfrac{2q}{2q - 1}$ then

$$\int\int f(x - y)\ \phi(x)\ \overline{\psi(y)}\ dx\ dy = \int \hat{\phi}(\hat{x})\ \overline{\hat{\psi}}(\hat{x})\ d\,\overline{F}(\hat{x})$$

for all $\phi, \psi \in (L^p, \ell^1)$. The double integral exists not necessarily as a Lebesgue integral but as the limit of the integral

$$\int_{V_\beta} \int_{V_\alpha} f(x - y)\ \phi(x)\ \overline{\psi(y)}\ dx\ dy \qquad \text{over } \alpha, \beta$$

where V_α, V_β are finite unions of the sets K_α $(\alpha \in J)$.

That is, as the sum of the absolutely convergent series

$$\sum_\beta \sum_\alpha \int_{V_\alpha} \int_{V_\beta} f(x-y)\, \phi(x)\, \overline{\psi(y)}\ dx\, dy.$$

PROOF. If $h \in (C_0, \ell^q)(\hat{G})$ then $\check{h} \in (L^{q'}, \ell^2)$ (Remark 5.8). Since $\{\phi_U\} \subset C_c$ (Theorem 7.2) $\check{h}\check{\phi}_U \in L^{q'}_c$ and $h*\phi_U = (\check{h}\check{\phi}_U)^\wedge$. So by Theorem 8.23

$$\int_G \check{\phi}_U(x)\, \check{h}(x)\, f(x)\, dx = \int_{\hat{G}} h*\phi_U(-\hat{x})\ d\ulcorner f^\urcorner(\hat{x}).$$

Since $\ulcorner f^\urcorner \in M_q$, and $\lim_{U \to 0} ||h*\phi_U - h||_{\infty q} = 0$ (Theorem 7.2 and Corollary 4.14), the integral on the right converges to $\int h(-\hat{x})\ d\ulcorner f^\urcorner(\hat{x})$. Therefore

$$\text{(C.1)} \quad \int_G \check{h}(x)\, f(x)\, dx = \int_{\hat{G}} h(-\hat{x})\ d\ulcorner f^\urcorner(\hat{x})$$

Let $\phi, \psi \in L^p_c$. Since $p = \dfrac{2q}{2q-1}$ and $1 \leq q < 2$ we have that $\dfrac{1}{p} = \dfrac{2q-1}{2q} = 1 - \dfrac{1}{2q}$. Since $\dfrac{1}{4} < \dfrac{1}{2q} \leq \dfrac{1}{2}$, this implies that $\dfrac{1}{2} \leq 1 - \dfrac{1}{2q} = \dfrac{1}{p} < \dfrac{3}{4}$, therefore $1 < p \leq 2$. If $\phi \in L^p_c$ then $\hat{\phi} \in (C_0, \ell^{p'})$, because $\phi \in (L^p, \ell^1)$. Hence by Proposition 4.1 $(\phi*\psi)^\wedge = \hat{\phi}\hat{\psi}$ belongs to (C_0, ℓ^q) because $\dfrac{1}{p'} = 1 - (1 - \dfrac{1}{2q}) = \dfrac{1}{2q}$, that is, $p' = 2q$. So by Theorem 8.23

$$\text{(1)} \qquad \int\int f(x-y)\, \phi(x)\, \overline{\psi(y)}\ dxdy = \int f(x)\, \phi*\tilde{\psi}(x)\ dx$$

$$= \int \hat{\phi}(\hat{x})\, \overline{\hat{\psi}}(\hat{x})\ d\ulcorner f^\urcorner(\hat{x}).$$

Set $B(\phi,\psi) = \int\int f(x-y)\,\phi(x)\,\overline{\psi(y)}\,dxdy \qquad (\phi,\ \psi \in L^p_c)$.

By the Hausdorff-Young inequality (Theorem 5.7) we have that for all $\phi,\ \psi \in L^p_c$

$$\begin{aligned}|B(\phi,\psi)| &\le \|\overline{f}\|_{q'}\|\hat{\phi}\hat{\psi}\|_{\infty q} \le \|\overline{f}\|_{q'}\|\hat{\phi}\|_{\infty p'}\|\hat{\psi}\|_{\infty p'}\\ &\le \|\overline{f}\|_{q'}\,C_p\,\|\phi\|_{p1}\|\psi\|_{p1}\end{aligned}$$

where C_p is a constant depending on p and a.

If $h \in (L^p, \ell^1)$ then we can write $h = \sum_\alpha h_\alpha$ where $h_\alpha = h|L_\alpha$ $(\alpha \in J)$. So $\|h\|_{p1} = \sum_\alpha \|h_\alpha\|_p$. Then for $\phi,\ \psi \in (L^p, \ell^1)$

$$|B(\phi_\beta,\psi_\alpha)| \le C_p\,\|\overline{f}\|_{q'}\|\phi_\beta\|_{p1}\|\psi_\alpha\|_{p1}.$$

So the double series $\sum_{\alpha,\beta} B(\phi_\beta,\psi_\alpha)$ is absolutely convergent. That is, the left side of (1) exists as the sum of the absolutely convergent series

$$\sum_\beta\sum_\alpha \int_{K_\beta}\int_{K_\alpha} f(x-y)\,\phi(x)\,\overline{\psi(y)}\,dxdy.$$

Moreover since $\hat{h} = \sum_\alpha h_\alpha$ is uniformly convergent and $\sum_\alpha h_\alpha$ converges in the norm of $(C_0, \ell^{p'})$ to $\hat{h}$ we have that

$$\sum_\alpha\sum_\beta \int \hat{\phi}_\beta(x)\,\overline{\hat{\psi}}_\alpha(x)\,d\overline{f}(x) = \int \hat{\phi}(\hat{x})\,\overline{\hat{\psi}}(\hat{x})\,d\overline{f}(\hat{x}).$$

This ends the proof. †

THEOREM 8.25. Let $1 < q < 2$. The following statments are equivalent:

i) $f \in SR(\Phi_q)$

ii) $f \in L^q_{loc}$ and there exists a unique measure $\overline{f}$ in M_q, such that for all $\phi, \psi \in (L^p, \ell^1)$ where $p = 2q/(2q-1)$ the integral

$$\int\int f(x-y)\,\phi(x)\,\overline{\psi(y)}\,dxdy = \int \hat{\phi}(\hat{x})\,\overline{\hat{\psi}}(\hat{x})\,d\overline{f}(\hat{x})$$

exists, in the sense of the Theorem 8.24.

iii) $f \in L^q_{loc}$ and there exists a unique measure $\overline{f} \in M_q$, such that

$$f(x) = \lim_{U \to 0} \int \hat{\psi}_U(\hat{x})\,[x,\hat{x}]\,d\overline{f}(\hat{x})$$

where the limit exists in L^q over any compact subset of G.

PROOF. i) implies ii) follows from Theorem 8.22.

iii) implies i) is Proposition 8.18. It remains to prove that ii) implies iii).

$\psi_U = \beta_U * \tilde{\beta}_U$, $\beta_U \in L^2_c$ (Definition 8.16). So by ii) there exists a unique $\overline{f} \in M_q$, such that for all $t \in G$

$$\int\int f(x-y)\,\tau_t\beta_U(x)\,\overline{\beta}_U(y)\,dxdy = \int (\tau_t\beta_U)^{\wedge}(\hat{x})\,\overline{\hat{\beta}}_U(\hat{x})\,d\overline{f}(\hat{x})$$

Now, the left side of this equality is equal to

$$\int f(x)\,\beta_U * \tilde{\beta}_U(t-x)\,dx = \int f(x)\,\psi_U(t-x)\,dx = f*\psi_U(t)$$

and the right side is equal to

$$\int \hat{\beta}_U(\hat{x})\,\overline{\hat{\beta}}_U(\hat{x})\,[t,\hat{x}]\,d\,\overline{f}(\hat{x}) = \int \hat{\psi}_U(\hat{x})\,[t,\hat{x}]\,d\overline{f}(\hat{x}).$$

Since $\psi_U * f$ converges to f in the sense of the part ii) we conclude that ii) implies iii).+

Compare Theorem 8.25 with [51, Theorem 4.1 and Theorem 4.2].

CHAPTER IV

GENERALIZATIONS OF FOURNIER'S THEOREMS ON LOCAL COMPLEMENTS TO THE HAUSDORFF-YOUNG THEOREM

§ 9. THE CASE WHERE $\hat{G}$ IS NOT DISCRETE

For a subset E of $\hat{G}$, the Fourier transform of a function f restricted to E will be denoted by $\hat{f}|E$.

$(L^p, \ell^q)^\wedge$ will be the set of Fourier transforms of functions in $(L^p, \ell^q)^\wedge$ and $(L^p, \ell^q)^\wedge|E$ will be the set of functions in $(L^p, \ell^q)^\wedge$ restricted to E.

We will keep this notation for the rest of our work.

J. Fournier [22, Theorem 1] proved the following theorem.

THEOREM 9.1. If $\hat{G}$ is nondiscrete and $E \subset \hat{G}$ is not locally null then for $1 < p \leq 2$

$$L^{p\wedge}|E \not\subset \bigcup_{q>p'} L^q(E).$$

Here we shall see that under the same conditions

$$(L^\infty, \ell^p)^\wedge|E \not\subset \bigcup_{q>p'} (L^q, \ell^\infty)(E). \tag{1}$$

If G is neither compact nor discrete then (1) extends Theorem 9.1 because (L^∞, ℓ^p) and L^q are proper subspaces of L^p and (L^q, ℓ^∞) $(1 \leq p, q \leq \infty)$ respectively.(Theorem 2.4).

THEOREM 9.2. If $\hat{G}$ is nondiscrete and $E \subset \hat{G}$ is not locally null, then for $1 < p \leq 2$

$$(L^\infty, \ell^p)^\wedge |E \not\subset \bigcup_{q>p'} (L^q, \ell^\infty)(E).$$

PROOF. Since E is not locally null, it contains a subset of positive measure. By the inner regularity of the Haar measure this subset contains a compact set of positive measure. Therefore it is enough to prove the theorem for compact sets E of positive measure.

In this case $(L^q, \ell^\infty)(E)$ is equal to $L^q(E)$, but it will be convenient for our proof to consider $(L^q, \ell^\infty)(E)$.

Suppose that

$$(L^\infty, \ell^p)^\wedge |E \subset (L^q, \ell^\infty)(E) \quad \text{for some} \quad q \in (p',\infty).$$

Take $f \in L^p(G)$ and let $\phi \in C_c(\hat{G})$ such that $\phi \equiv 1$ on E and $\check{\phi} \in (L^{p'}, \ell^1)(G)$ (Theorem 5.2). By (2.6) $(L^{p'}, \ell^1) \subset L^1$, so the Fourier transform $\hat{\check{\phi}}$ of $\check{\phi}$ is equal to ϕ [37, 31.44 b)].

Applying Theorem 4.7, we have that $f*\check{\phi} \in (L^\infty, \ell^p)(G)$. Therefore by our assumption $(f*\phi)^\wedge |E \in (L^q, \ell^\infty)(E)$, hence $\hat{f}|E \in L^q(E)$ (see above). Since f is arbitrary, we conclude that $L^{p\wedge}|E \subset L^q(E)$. This contradicts Theorem 9.1. Therefore

(2) $$(L^\infty, \ell^p)^\wedge |E \not\subset (L^q, \ell^\infty)(E) \quad \text{for all} \quad p' < q < \infty.$$

For $p' < q < \infty$, define the function F on (L^∞, ℓ^p) by

(3) $$F(f) = || \hat{f}|E ||_{q\infty}.$$

By (2) F takes the infinite value.

Clearly $F(\alpha f) = \alpha F(f)$ for all nonnegative real α and for all $f, g \in (L^\infty, \ell^p)$, $F(f - g) < F(f) + F(g)$. These properties of F imply that for all real α, the set $V_\alpha = \{f \in (L^\infty, \ell^p) \mid F(f) > \alpha\}$ is dense in (L^∞, ℓ^p). Indeed, suppose that V_α is not dense for some real α. Then $\alpha \neq 0$ since $V_0 = (L^\infty, \ell^p) \sim \{0\}$.

Take $g \in (L^\infty, \ell^p) \sim \overline{V_\alpha}$. Then there exists $\varepsilon > 0$ such that for all $||f||_{\infty p} \leq \varepsilon$, $f + g \notin V_\alpha$. That is

(4) $$F(f + g) \leq \alpha \qquad \text{for all } ||f||_{\infty p} \leq \varepsilon.$$

Let $\overline{f}$ be a function in (L^∞, ℓ^p) such that $F(\overline{f}) = \infty$. Then $||\overline{f}||_{\infty p} > 0$ and $f = (\varepsilon/||\overline{f}||_{\infty p})\overline{f}$ belongs to (L^∞, ℓ^p). Since $||f||_{\infty p} = \varepsilon$ and $f = f + g - g$ we have by (4) that $(\varepsilon/||\overline{f}||_{\infty p})F(\overline{f}) = F(f) \leq F(f + g) + F(g) \leq \alpha + \alpha = 2\alpha$.

This contradiction shows that V_α is dense in (L^∞, ℓ^p) for all α.

Moreover F is lower semicontinuous. Indeed, let $U = \{g \in (L^p, \ell^1) \mid ||g||_{q'1} \leq 1\}$ and define for each $g \in (L^p, \ell^1)$ the function F_g on (L^∞, ℓ^p) by

$$F_g(f) = \left|\int_E \hat{f} g\right|.$$

Since $(L^\infty, \ell^p) \subset L^p$ by (2.6) and $p \leq 2$, the Fourier transform $\hat{f}$ of f belongs to $L^{p'}$. Then for $g \in U$, $g|E \in L^p$. So, by the Hausdorff-Young inequality we have that

$$\begin{aligned} F_g(f) &\leq ||\hat{f}|E||_{p'}||g|E||_p \leq ||\hat{f}||_{p'}||g||_p \leq ||f||_p||g||_p \\ &\leq ||f||_{\infty p}\ ||g||_{p1} \end{aligned}$$

where the last inequality is due to (2.4) and (2.3). Therefore F_g is continuous for all $g \in U$.

We want to prove next that

$$(5) \qquad F = \sup \{F_g \mid g \in U\}.$$

First we note that $(L^p, \ell^1) \subset (L^{q'}, \ell^1)$ as $p > q'$. Since (L^p, ℓ^1) is dense in $(L^{q'}, \ell^1)$ (Corollary 3.8) U is dense in the unit ball B of $(L^{q'}, \ell^1)$ and therefore

$$\sup \{F_g \mid g \in U\} = \sup \{F_g \mid g \in B\}.$$

By the converse of the Hölder inequality (as in [36, 191 p.142] with $G = 1$, $k' = q'$, $k = q$, $F^{1/q} = \sup \{F_g \mid g \in B\}$) we have that for all β

$$\left[\int_{L_\beta} |\hat{f}|E|^q\right]^{1/q} \leq \sup \{F_g(f) \mid g \in B\}.$$

This implies that $||\hat{f}|E||_q \leq \sup \{F_g(f) \mid g \in U\}$.

Now, if $||\hat{f}|E||_{q\infty} < \infty$, that is $\hat{f}|E \in (L^q, \ell^\infty)(E)$, then for all $g \in U$

$$F_g(f) = \left| \int_E \hat{f}\, g \right| \leq ||\hat{f}|E||_{q\infty} ||g||_{q'1} \leq ||\hat{f}|E||_{q\infty}.$$

Therefore $F_g(f) \leq ||\hat{f}|E||_{q\infty}$. Hence (5) holds and this implies that F is lower semicontinuous.

By Baire's theorem $\{f \in (L^\infty, \ell^p) \mid F(f) = \infty\} = \bigcup_N V_n$ is a set of type G_δ.

Now, choose a strictly decreasing sequence $\{q_n\}$ converging to p'. Again by Baire's theorem (as in [43, Corollary of Theorem 5.6]) the set $\{f \in (L^\infty, \ell^p) \mid ||\hat{f}|E||_{q_n\infty} = \infty$ for all $n \in N\}$ is a dense G_δ set because is equal to $\bigcap_N \{f \in (L^\infty, \ell^p) \mid \|\hat{f}|E\|_{q_n\omega} = \infty\}$. Therefore nonempty.

Take f in this set. Since $(L^\infty, \ell^p) \subset (L^1, \ell^q)$ (see (2.6)) we have by Theorem 5.7 that $\hat{f}|E \in (L^{q'}, \ell^\infty)(E)$. If also $\hat{f}|E \in (L^q, \ell^\infty)$ then by (2.6), $\hat{f}|E \in (L^{q_n}, \ell^\infty)(E)$ for all sufficiently large n and this contradicts the choice of f. Therefore $\hat{f}|E \notin (L^q, \ell^\infty)$ for all $q > p'$ and this ends the proof.+

COROLLARY 9.3. If $\hat{G}$ is nondiscrete and $1 < p \leq 2$ then

$$(L^\infty, \ell^p)^\wedge \not\subset \bigcup_{q>p'} (L^q, \ell^\infty)(\hat{G}).$$

W. Bloom proved [3 , Theorem 1] the following theorem.

THEOREM 9.4. If Ω is a nonempty open subset of G and $\hat{G}$ is noncompact, then for $1 \leq p < q \leq \infty$ there exists $f \in (L^\infty, \ell^q)(\hat{G})$ such that $\check{f} - \check{g}$ does not vanish on Ω for all $g \in (L^1, \ell^p)(\hat{G})$.

In other words, the local inclusion $(L^1, \ell^p)^\wedge \subset (L^\infty, \ell^q)^\wedge$ is strict.

For the particular case $1 < q \leq 2$, Theorem 9.2 proves Theorem 9.4. Indeed, if $1 \leq p < q \leq 2$, $\hat{G}$ is noncompact and Ω is a nonempty open set of G, then G is nondiscrete and Ω is not locally null. So, Theorem 9.2 with p = q and q = r says that

$(L^\infty, \ell^q)^\vee|\Omega \not\subset \bigcup_{r>q'} (L^r, \ell^\infty)(\Omega)$. That is, there exists $f \in (L^\infty, \ell^q)$, such that $\check{f}|\Omega \notin (L^r, \ell^\infty)(\Omega)$ for all $r > q'$. So, for $g \in (L^1, \ell^p)(\hat{G})$, $\check{g} \in (L^{p'}, \ell^\infty)(G)$ (Theorem 5.7) and therefore $\check{f} - \check{g}$ does not vanish on Ω since $p' > q'$.

§ 10. THE CASE WHERE $\hat{G}$ IS NONCOMPACT

Theorem 2, b) of [22] is as follows.

THEOREM 10.1. If $\hat{G}$ is noncompact and $1 \leq p \leq 2$ then

$$L^{p\wedge} \not\subset \bigcup_{q<p'} L^q(\hat{G}).$$

We will generalize this result by proving the next theorem.

THEOREM 10.2. If $\hat{G}$ is noncompact and $1 \leq p \leq 2$ then

$$(L^p, \ell^1)^{\wedge} \not\subset \bigcup_{q<p'} (L^1, \ell^q)(\hat{G}).$$

Theorem 10.2 indeed extends Theorem 10.1 if G is neither compact nor discrete, because (L^p, ℓ^1) and L^q are proper subspaces of L^p and (L^1, ℓ^q) respectively (Theorem 2.4) for $1 \leq p, q \leq \infty$.

In order to prove Theorem 10.2 we need the following two results which appeared in [7, p. 194 and Theorem IV].

PROPOSITION 10.3. Let $\Phi = \{\phi \in C_c \mid \hat{\phi} \in (C_0, \ell^1)\}$ endowed with the norm $\phi \longmapsto \|\hat{\phi}\|_{pq}$, $1 \leq p, q < \infty$. If $T \in \Phi^*$ then there exists a unique function h in $(L^{p'}, \ell^{q'})(\hat{G})$ such that for all $\phi \in \Phi$

$$T(\phi) = \int_{\hat{G}} \hat{\phi}(-\hat{x})\, h(\hat{x})\, d\hat{x}.$$

PROOF. We proceed as in Proposition 8.3. If $T \in \Phi^*$ then the map $\overline{T}(\hat{\phi}) = T(\phi')$ belongs to $(\Phi^{\wedge}, \|\cdot\|_{pq})^*$. Since $\Phi^{\wedge}$ is dense in $(L^p, \ell^q)(\hat{G})$ (Proposition 8.3, iv)) there exists a unique continuous extension $\overline{T}$ on (L^p, ℓ^q). By Theorem 3.1 there exists a unique h in $(L^{p'}, \ell^{q'})(\hat{G})$ such that for all f in $(L^p, \ell^q)(\hat{G})$; $\overline{T}(f) = \int f(\hat{x})h(\hat{x})d\hat{x}$.

So for all $\phi \in \Phi$ we have that

$$T(\phi) = \overline{T}(\hat{\phi}') = \int \hat{\phi}(-\hat{x})\, h(\hat{x})\, d\hat{x} \;.\dagger$$

PROPOSITION 10.4. Let $\mu \in M_\infty$. If there exists a constant C such that for all $\phi \in \Phi$

$$\left|\int_G \phi(x)\, d\mu(x)\right| \leq C\|\hat{\phi}\|_{pq} \qquad (1 \leq p,q < \infty)$$

then $\hat{\mu} \in (L^{p'}, \ell^{q'})$.

PROOF. Set $T(\phi) = \int\phi(x)d\mu(x)$. Then $T \in \Phi^*$, Φ as in Proposition 10.3. So by our previous result there exists a unique h in $(L^{p'}, \ell^{q'})(\hat{G})$ such that $T(\phi) = \int\phi(x)d\mu(x) = \int\phi(-\hat{x})h(\hat{x})d\hat{x}$. This implies that $\mu \in M_T$ and $\overline{\mu} = h$ (Definition 6.1). By Theorem 6.23 we conclude that $\hat{\mu} = \overline{\mu}$ and therefore $\hat{\mu} \in (L^{p'}, \ell^{q'})$.$\dagger$

PROOF OF THEOREM 10.2. We consider two cases.

Case 1) $p = 2$. Let E be a compact subset of G of positive measure with interior Ω, and $1 \leq q < 2$. Since $\hat{G}$ is noncompact and Ω is nonempty, there exists f in $(L^\infty, \ell^2)(\hat{G})$ such that $\check{f} - \check{g}$ does not vanish on Ω for all $g \in (L^1, \ell^q)(\hat{G})$ (Theorem 9.4).

Let $\phi \in C_c(G)$ such that $\phi \equiv 1$ on E. Now $\check{f} \in L^2$ as $f \in (L^\infty, \ell^2)$ and $(L^\infty, \ell^2) \subset L^2$. Since $\phi \in (L^\infty, \ell^2)$ we have by Proposition 4.1 that $\check{f}\phi \in (L^2, \ell^1)(G)$. So $\check{f}\phi \in L^2 \cap L^1$ by (2.7). This implies that the inverse of the Fourier transform of $(\check{f}\phi)\hat{}$ is equal to $\check{f}\phi$. Therefore $(\check{f}\phi)\hat{} \notin (L^1, \ell^q)(\hat{G})$ because $\check{f} - \check{f}\phi$ does vanish on Ω. So, there exists a function $\check{f}\phi \in (L^2, \ell^1)(G)$ such that $(\check{f}\phi)\hat{} \notin (L^1, \ell^q)(G)$. This means that

(1) $$(L^2, \ell^1)(G)^\wedge \not\subset (L^1, \ell^q)(\hat{G}) \quad \text{for all } 1 \le q < 2.$$

Furthermore, we note that $(\check{f}\phi)^\wedge|E \notin (L^1, \ell^q)(E)$. Therefore

(2) $$(L^2, \ell^1)(E)^\wedge \not\subset (L^1, \ell^q)(E) \quad \text{for all } q < 2$$

and compact E of positive measure.

Case 2) $p < 2$. We want to prove that

(3) $$(L^p, \ell^1)^\wedge \not\subset (L^1, \ell^q)(\hat{G}) \quad \text{for all } 1 \le q < p'.$$

If $1 < q < 2$, we know by (1) that (3) holds, because $p < 2$ and this implies that $(L^2, \ell^1) \subset (L^p, \ell^1)$. So we will consider the case when $2 \le q < p'$. Suppose not, that is, $(L^p, \ell^1)^\wedge \subset (L^1, \ell^q)(\hat{G})$ for some $2 \le q < p'$. By the Closed Graph Theorem (as in [32, Corollary p. 70]) the map $T: (L^p, \ell^1) \longrightarrow (L^1, \ell^q)(\hat{G})$ given by $T(f) = \hat{f}$ is bounded. Indeed, let $\{f_n\}$ be a sequence in (L^p, ℓ^1) such that $f_n \to 0$. Then $\lim ||f_n||_{p1} = 0$.

Assume $\hat{f}_n \to g$ in $(L^1, \ell^q)(\hat{G})$ and take $\phi \in C_c$, since $\phi \in (L^\infty, \ell^{q'}) \cap (L^1, \ell^p)$ we have that $|< g,\phi >| = \lim |< \hat{f}_n,\phi >|$ (remember that $(L^\infty, \ell^{q'}) = (L^1, \ell^q)^*$).

But by Theorem 3.1

$$|< \hat{f}_n,\phi >| = |\int \hat{f}_n(\hat{x})\phi(\hat{x})d\hat{x}| \le ||\hat{f}_n||_{\infty p'}||\phi||_{1p}.$$

So, by the Hausdorff-Young inequality for amalgams

$$|< g,\phi >| = \lim |< \hat{f}_n,\phi >| \le \lim ||\hat{f}_n||_{\infty p'}||\phi||_{1p}$$
$$\le ||\phi||_{1p}\, C_p \lim ||f_n||_{p1} = 0.$$

We conclude that $< g,\phi > = 0$ for all $\phi \in C_c$. Since any g in (L^1, ℓ^q) can be considered as an element of M_q, $(C_0, \ell^{q'})^* = M_q$, and C_c is dense in $(C_0, \ell^{q'})$, we conclude that $< g,f > = 0$ for all f in $(C_0, \ell^{q'})$ and this implies that $g = 0$. Hence T is continuous.

Let $g \in (L^{\infty}, \ell^{q'})(\hat{G})$ and $\phi \in \Phi(\hat{G})$. Then

$$|< g,\phi >| = |\int g(\hat{x})\phi(\hat{x})d\hat{x}| = |\int g(\hat{x})\hat{\check{\phi}}(\hat{x})d\hat{x}| = |\int g(\hat{x})T(\check{\phi})(\hat{x})d\hat{x}|$$
$$\leq ||g||_{\infty q'}||T\check{\phi}||_{1q} \leq ||g||_{\infty q'}||T|| \; ||\check{\phi}||_{p1}.$$

By Proposition 10.4, $\check{g} \in (L^{p'}, \ell^{\infty})(G)$. This implies that $(L^{\infty}, \ell^{q'})(\hat{G})^{\vee} \subset (L^{p'}, \ell^{\infty})(G)$.

(We have given an alternative proof of this last inclusion based on ideas of Fournier [22, p. 269]in Remark 10.5).

Now, $\hat{G}$ noncompact implies G nondiscrete and by Corollary 9.3, $(L^{\infty}, \ell^{q'})(\hat{G})^{\vee} \not\subset (L^{p'}, \ell^{\infty})(G)$ because $q' \leq 2$ and $p' > q = (q')'$. This contradiction shows (3).

From cases 1) and 2) we conclude that for $1 \leq p \leq 2$

(4) $\qquad (L^p, \ell^1)(G)^{\wedge} \not\subset (L^1, \ell^q)(\hat{G}) \qquad$ for all $1 \leq q < p'$.

For $q \in [1,p')$ define the function F on $(L^p, \ell^1)(G)$ by

$$F(f) = || \hat{f} ||_{1q}.$$

By (4) F takes the infinite value, $F(\alpha f) = \alpha F(f)$ for all nonnegative real α and all $f,g \in (L^p, \ell^1)$, $F(f - g) \leq F(f) + F(g)$.

Similarly to Theorem 9.2 these properties of F imply that the set $V_{\alpha} = \{f \in (L^p, \ell^1) | \; F(f) > \alpha\}$ is dense in (L^p, ℓ^1) for all real α.

Also, F is lower semicontinuous. Indeed, for a finite subset E of J, define the function F_E on (L^p, ℓ^1) by

$$F_E(f) = \sum_{\alpha \in E} || \hat{f}_{\alpha} ||_1^q$$

where $\hat{f}_{\alpha} = \hat{f}\chi_{L_{\alpha}}$

Since $||\hat{f}||^q_{1q} = \sum_\alpha ||\hat{f}_\alpha||^q_1$ we have that $\lim F_E = F^q$.

Note that $\{F_E\}$ is an increasing net of functions and hence $F = \sup\{F_E \mid E \subset J \text{ finite}\}$.

$1 \leq q < p'$ implies that $L^{p'}(L_\alpha) \subset L^q(L_\alpha)$ for all α, so by the Hausdorff-Young inequality, we have that for all α

$||\hat{f}_\alpha||^q_1 \leq ||\hat{f}_\alpha||^q_{p'} \leq ||\hat{f}||^q_{p'} \leq ||f||^q_p < ||f||^q_{p1}$

Therefore $F_E(f) \leq |E|\ ||f||^q_{p1}$ where $|E|$ is the cardinality of E. This shows that each F_E is continuous and we conclude that F^q is lower semicontinuous, so is F. Hence, the set

$\{f \in (L^p, \ell^1) \mid F(f) = \infty\} = \bigcap_{\mathbb{N}} \{f \in (L^p, \ell^1) \mid F(f) > n\}$ is a dense set of type G_δ. If $\{q_n\}$ is a strictly increasing sequence converging to p', then by Baire's theorem the set $\{f \in (L^p, \ell^1) \mid ||\hat{f}||_{1q_n} = \infty \text{ for all } n\}$ is a dense set of type G_δ. Hence it is nonempty. Take f in this set; let $q \in [1, p')$. Since $f \in (L^p, \ell^1)$, $\hat{f} \in (C_0, \ell^{p'})$ (Theorem 5.4), if also $\hat{f} \in (L^1, \ell^q)$ then by (2.5) $\hat{f} \in (L^1, \ell^{q_n})(\hat{G})$ for all sufficiently large n and this contradicts the choice of f. Therefore $\hat{f} \notin (L^1, \ell^q)(\hat{G})$ for all $1 \leq q < p'$ and this proves the theorem.†

REMARK 10.5. Let $1 \leq p < 2$, $2 \leq q < p'$. If the map $T: (L^p, \ell^1) \longrightarrow (L^1, \ell^q)(\hat{G})$ defined by $Tf = \hat{f}$ is continuous then

$$(L^\infty, \ell^{q'})^\vee \subset (L^{p'}, \ell^\infty).$$

Indeed, if T is continuous then its dual map $T^*: (L^\infty, \ell^{q'}) \longrightarrow (L^{p'}, \ell^\infty)$ is also continuous [44, Theorem 4.10]. Let $g \in L^\infty_c(\hat{G})$ with support E. Hence $g \in L^p(E)$ and by Theorem 3.1 we have that for $f \in (L^p, \ell^1)(G)$

$$T^\star(\bar{g})(f) = \int_{\hat{G}} \hat{f}(\hat{x})\ \bar{g}(\hat{x})\ d\hat{x}.$$

Since $f \in L^1$ by (2.7) we apply the Parseval's identity (as in [37 , 31.48 a)]) and we get

$$T^\star(\bar{g})(f) = \int_G f(x)\ \overline{\check{g}}(x)\ dx.$$

Hence $T^\star(\bar{g}) = \overline{\check{g}}$. Since T is linear $T^\star(g) = \check{g}$. So for all g in $L_c^\infty(\hat{G})$

(5) $$|| \check{g} ||_{p'\infty} \leq || T^\star ||\ || g ||_{\infty q}.$$

Let $g \in (L^\infty, \ell^{q'})(\hat{G})$. Since L_c^∞ is dense in $(L^\infty, \ell^{q'})$ (Theorem 3.6) there exists a sequence $\{g_n\}$ in L_c^∞ such that $\lim g_n = g$ in $(L^\infty, \ell^{q'})$. Since $q' \leq 2$ and $(L^\infty, \ell^{q'}) \subset (L^1, \ell^q)$, we have by the Hausdorff-Young inequality for amalgams that $\lim ||\check{g}_n - \check{g}||_{q\infty} = 0$.

Now, by (5) $\{g_n\}$ is a Cauchy sequence in $(L^{q'}, \ell^\infty)$. Therefore there exists $h \in (L^{p'}, \ell^\infty)$ such that $\lim ||\check{g}_n - h||_{p'\infty} = 0$. This implies that $\lim \check{g}_n = h$ in (L^q, ℓ^∞) by (2.4), as $p' > q$. Hence $h = \check{g}$ and we conclude that $(L^\infty, \ell^{q'})^\vee \subset (L^{p'}, \ell^\infty)$, since g is arbitrary.+

COROLLARY 10.6. If $E \subset G$ is not locally null, $\hat{G}$ is not compact and $1 \leq p \leq 2$ then

(6) $$(L^p, \ell^1)(E)^\wedge \not\subset \bigcup_{q<p'} (L^1, \ell^q)(\hat{G})$$

PROOF. As in Theorem 9.2 it is enough to prove the corollary for compact sets of positive measure.

By (2) of the case 1) of Theorem 10.2.

$$(L^p, \ell^1)(E)^\wedge \not\subset (L^1, \ell^q)(\hat{G}) \quad \text{for all } q < 2.$$

Using the same argument as in the case 2) of the proof of

Theorem 10.2 and Theorem 9.2 we have that

$$(L^p, \ell^1)(E)^\wedge \not\subset (L^1, \ell^q)(\hat{G}) \quad \text{for all } 2 \leq q < p'.$$

Again the function F on $(L^p, \ell^1)(E)$ defined by $F(f) = ||\hat{f}||_{1q}$ is lower semicontinuous and by a Baire's categorical argument we conclude (6).+

§ 11. $\hat{G}$ IS NEITHER COMPACT NOR DISCRETE

Finally we will generalize the following theorem [22, Theorem 2, c)].

THEOREM 11.1. If $\hat{G}$ is neither compact nor discrete and $1 < p \leq 2$ then

$$L^{p\wedge} \not\subset \bigcup_{q \neq p'} L^q(\hat{G}).$$

We will prove that under the same hypothesis

$$L^{p\wedge} \not\subset \bigcup_{q \neq p'} (L^q, \ell^\infty) \cap (L^1, \ell^q).$$

This improves the right side of Theorem 11.1, because L^q is a proper subspace of $(L^q, \ell^\infty) \cap (L^1, \ell^q)$ for $1 < q < \infty$ (Theorem 2.4).

THEOREM 11.2. If $\hat{G}$ is neither compact nor discrete then for $1 < p \leq 2$

$$L^{p\wedge} \not\subset \bigcup_{q \neq p'} (L^q, \ell^\infty) \cap (L^1, \ell^q).$$

PROOF. By corollary 9.3 there exists $f \in (L^\infty, \ell^p)$ such that

(1) $$\hat{f} \notin \bigcup_{q > p'} (L^q, \ell^\infty).$$

By Theorem 10.2 there exists $h \in (L^p, \ell^1)$ such that

(2) $$\hat{h} \notin \bigcup_{q < p'} (L^1, \ell^q).$$

We shall see that one of the three functions $\hat{f}$, $\hat{h}$, $\hat{f} + \hat{h}$ in

L^p (note that (L^∞, ℓ^p) and (L^p, ℓ^1) are included in L^p by (2.5) and (2.6)) does not belong to $\bigcup_{q \neq p'} (L^q, \ell^\infty) \cap (L^1, \ell^q)$.

Suppose not, that is, $\hat{f} + \hat{h} \in (L^{q_0}, \ell^\infty) \cap (L^1, \ell^{q_0})$, $\hat{f} \in (L^{q_1}, \ell^\infty) \cap (L^1, \ell^{q_1})$ and $\hat{h} \in (L^{q_2}, \ell^\infty) \cap (L^1, \ell^{q_2})$ for some q_0, q_1, q_2 distinct from p'.

Since $\hat{f} \notin (L^{q_1}, \ell^\infty)$ if $q_1 > p'$ and $\hat{h} \notin (L^1, \ell^{q_2})$ if $q_2 < p'$ we have that $q_1 < p' < q_2$. So,

a) If $p' < q_0 \leqq q_2$ then $\hat{h} \in (L^{q_2}, \ell^\infty) \subset (L^{q_0}, \ell^\infty)$. Hence $\hat{f} = (\hat{f} + \hat{h}) - \hat{h} \in (L^{q_0}, \ell^\infty)$. This contradicts (1) as $q_0 > p'$.

b) If $q_1 < q_2 < q_0$ then $\hat{f} + \hat{h} \in (L^{q_0}, \ell^\infty) \subset (L^{q_2}, \ell^\infty)$. Hence $\hat{f} = (\hat{f} + \hat{h}) - \hat{h} \in (L^{q_2}, \ell^\infty)$. This contradicts (1) as $q_2 > p'$.

c) If $q_1 \leqq q_0 < p'$ then $\hat{f} \in (L^1, \ell^{q_1}) \subset (L^1, \ell^{q_0})$. Hence $\hat{h} = (\hat{f} + \hat{h}) - \hat{f} \in (L^1, \ell^{q_0})$. This contradicts (2) as $q_0 < p'$.

d) If $q_0 < q_1 < p'$ then $\hat{f} + \hat{h} \in (L^1, \ell^{q_0}) \subset (L^1, \ell^{q_1})$. Hence $\hat{h} = (\hat{f} + \hat{h}) - \hat{f} \in (L^1, \ell^{q_1})$. This contradicts (2) as $q_1 < p'$.

It follows from a) – d) that $q_0 = p'$. This contradiction proves the theorem. †

CHAPTER V

MULTIPLIERS

In this chapter we will characterize the multipliers from A to B, where A and B are any amalgam space or any space of unbounded measures of type q.

Specifically, for a continuous linear operator $T: A \longrightarrow B$ such that for all $s \in G$, $T\tau_s = \tau_s T$ where τ_s is the translation operator on A and B respectively, defined in Definition 3.10, we want to find a " μ " such that

(1) $\qquad Tf = \mu * f \quad$ for all $f \in A$.

In the particular case when $A = B = L^1(G)$ it is known that such a μ belongs to $M_1(G)$. Moreover, we have the following theorem [40, Thoerem 0.1.1 and Corollary 0.1.1].

<u>THEOREM I</u>. If $T: L^1(G) \longrightarrow L^1(G)$ is a continuos linear operator, then the following are equivalent:

i) $T\tau_s = \tau_s T$ for all $s \in G$

ii) $T(f*g) = Tf*g$ for all $f, g \in L^1(G)$

iii) There exists a unique continuous bounded function φ on G such that $(Tf)^\wedge = \varphi\hat{f}$ for all $f \in L^1(G)$

iv) There exists a unique $\mu \in M_1(G)$ such that $Tf = \mu * f$ for all $f \in L^1(G)$.

Furthermore, the linear space of multipliers from L^1 to L^1 is isometric and linearly isometric with M_1.

i) and ii) say that T commutes with translations and convolution respectively while iii) justifies the name multiplier. So, the reason we pursue a characterization like (1) is because once we have found such a μ, in some cases, we shall be able to vindicate our choice of the word multiplier by taking the Fourier transform φ of μ and conclude that for all $f \in A$, $(Tf)^{\wedge} = \varphi\hat{f}$.

Currently there are different definitions of a multiplier according to the spaces involved. Indeed,

1) If A and B are translation invariant topological linear spaces of functions on G (for all $s \in G$ and all $f \in A$, the function $\tau_s f(t) = f(t - s)$ belongs to A) then a multiplier from A to B is a continuous linear operator $T: A \longrightarrow B$ such that T commutes with translations. That is, $T\tau_s = \tau_s T$ for all $s \in G$.

2) If A is a topological algebra and B is a topological A-module then a multiplier from A to B is a continuous linear operator $T: A \longrightarrow B$ such that T commutes with convolution. That is, $T(f*g) = Tf*g$ for all $f, g \in A$. (Note that $*$ on the left denotes the operation on A, and on the right the module operation on B).

3) If A, B are semisimple, commutative, Banach algebras then a multiplier from A to B is a function φ on the regular maximal ideal space of A such that $\varphi\hat{x} \in \hat{B}$ whenever $\hat{x} \in \hat{A}$, where $\hat{x}$ is the Gelfand transform of x.

It is clear from Definition 3.10 that the first definition is meaningful for all amalgam spaces and all spaces M_q. For this reason we have chosen this as our definition of a multiplier on amalgams and measure spaces M_q.

However, the second definition is also meaningful when A is any of the spaces (C_0, ℓ^1), (L^p, ℓ^1) ($1 \leq p \leq \infty$) or M_1 and B is any amalgam space or any measure space M_q, because in this case A is a subalgebra of L^1 ((2.6) and Theorem 4.7) or $A = M_1$ and B is both a L^1 and M_1-module (§4 p. 60).

So, in order to distinguish between these two definitions we will say that if A and B are as above and T is as in the second definition then T is a convolution multiplier, abbreviated, c-multiplier.

In general multipliers and c-multipliers are not the same [29, pp. 89 and 94]. Hence, whenever the definition of a c-multiplier makes sense, we will be interested in knowing the relation between these two concepts.

In this direction, we should observe that if a multiplier (c-multiplier) T has the form (1) for some μ then by the properties of convolution T is a c-multiplier (multiplier).

As in Theorem I, once we characterize the multipliers for certain amalgam or measure spaces A, B, we will try to establish a linear isomorphism between the linear space of multipliers from A to B and some linear space C.

Since the $L^p(G)$ spaces are particular cases of amalgams (see (2.1)) it is natural for us to follow very closely the theorey of multipliers from L^p to L^q and through generalizations try to develop and characterize a theory of multipliers for amalgams and M_q spaces.

Our main source of information about the theory of multipliers for $L^p(G)$ spaces will be [40].

§ 12. SPACES OF MULTIPLIERS

Let B be a linear space of functions on G. For $s \in G$ and $f \in B$, $\tau_s f$ is the function on G defined by $\tau_s f(t) = f(t - s)$. If $\tau_s f \in B$ for all $f \in B$ then B is said to be translation invariant.

If B is translation invariant then the linear operator $\tau_s\ B \longmapsto B$, $f \longmapsto \tau_s f$ is called a translation operator.

DEFINITION 12.1. Let A, B be two translation invariant linear spaces. A multiplier T from A to B is a continuous linear operator $T: A \longrightarrow B$ which commutes with translations. That is, for all $s \in G$, $T\tau_s = \tau_s T$ where τ_s is the translation operator on A and B.

The linear space of multipliers from A to B will be denoted by $M(A,B)$.

Since $(L^\infty, \ell^q) = (L^1, \ell^{q'})^*$, $1 < q \leq \infty$ and $(L^\infty, \ell^1) = (L^1, c_0)^*$ (Theorem 3.1), (L^∞, ℓ^q) can be endowed with the weak*-topology induced by $(L^1, \ell^{q'})$ if $1 < q \leq \infty$, (L^1, c_0) if $q = 1$. We will write $(L^\infty, \ell^q)^w$ for this space. Similarly $(L^p, \ell^\infty)^w$, $1 < p \leq \infty$, $M_q{}^w$, $1 \leq q \leq \infty$ are the spaces (L^p, ℓ^∞), M_q endowed with the weak*-topology induced by $(L^{p'}, \ell^1)$ and $(C_0, \ell^{q'})$ respectively.

By [40, Theorem D.4.1] the continuous linear functionals on $(L^\infty, \ell^q)^w$, $(L^p, \ell^\infty)^w$ and $M_q{}^w$ can be identified with $(L^1, \ell^{q'})$ if $1 < q \leq \infty$, (L^1, c_0) if $q = 1$, $(L^{p'}, \ell^1)$ and $(C_0, \ell^{q'})$ respectively by the formula

$$(12.1) \qquad < f,g > = \int_G f(-x)\, g(x)\, dx.$$

$f \in (L^{\infty}, \ell^{q})$, $g \in (L^{1}, \ell^{q'})$ $1 < q < \infty$; $f \in (L^{\infty}, \ell^{1})$, $g \in (L^{1}, c_0)$;

$f \in (L^{p}, \ell^{\infty})$, $g \in (L^{p'}, \ell^{1})$ $1 < p < \infty$.

$$(12.2) \qquad < \mu, f > = \int_G f(-x)\, d\mu(x)$$

$$\mu \in M_q, \ f \in (C_0, \ell^{q'}).$$

We should mention that this is one way to represent a continuous linear functional and we choose this because in this case $< f,g > = f*g(0)$, $< \mu,f > = \mu*f(0)$, (see Corollary 4.4 and Corollary 4.5).

PROPOSITION 12.2. Let A,B be any of the following spaces

(L^p, ℓ^q) $\quad 1 \leq p,q < \infty$

(C_0, ℓ^q) $\quad 1 \leq q \leq \infty$

(L^p, c_0) $\quad 1 \leq p < \infty$

$(L^{\infty}, \ell^q)^w$ $\quad 1 \leq q \leq \infty$

$(L^p, \ell^{\infty})^w$ $\quad 1 < p \leq \infty$

M_q^w $\quad 1 \leq q \leq \infty$

If T is a multiplier from A to B then its adjoint T^* is a multiplier from B^* to A^*.

PROOF. $T: A \longrightarrow B$ is a continuous linear operator and its adjoint T^* is a continuous linear operator $T^*: B^* \longrightarrow A^*$ defined by $< f,T^*g > = < Tf,g >$ $\quad f \in A$, $g \in B^*$ [44, Theorem 4.10]

So, for $s \in G$, $f \in A$, $g \in B^*$ we have that

$$\begin{aligned} < f,T^*\tau_s g > &= < Tf,\tau_s g > = \int Tf(-x)\tau_s g(x)dx = \int Tf(-x)g(x - s)dx \\ &= \int Tf(-x - s)g(x)dx = \int \tau_s Tf(-x)g(x)dx \\ &= < \tau_s Tf,g > = < T\tau_s f,g > = < \tau_s f,T^*g > \\ &= < f,\tau_s T^*g >. \end{aligned}$$

Since this holds for any f in A, we conclude that $T^*\tau_s g = \tau_s T^* g$ for all $g \in B^*$. Hence $T^* \in M(B^*, A^*)$.+

The next two results are generalizations of Lemma 3.5.1 and Theorem 5.2.5 of [40] and their proofs are based on Hormander's original theorem for R^n [33] Theorem 1.1 .

LEMMA 12.3. Assume G is noncompact. Then

i) If $f \in (L^p, \ell^q)$, $1 \leq p,q < \infty$, then $\lim_{s \to \infty} ||f + \tau_s f||_{pq} = 2^{1/q}||f||_{pq}$.

ii) If $f \in (L^\infty, \ell^q)$ or $f \in (C_0, \ell^q)$, $1 \leq q < \infty$, then
$$\lim_{s \to \infty} ||f + \tau_s f||_{\infty q} = 2^{1/q}||f||_{\infty q}.$$

iii) If $f \in (L^p, c_0)$, $1 \leq p < \infty$, or $f \in C_0$ then $\lim_{s \to \infty}||f + \tau_s f||_{p\infty} = ||f||_{p\infty}$

iv) If $\mu \in M_q$, $1 \leq q < \infty$, then $\lim_{s \to \infty} ||\mu + \tau_s \mu||_q = 2^{1/q}||\mu||_q$.

PROOF. If $g \in L^p_c(G)$, $E = \text{supp } g$ and $\mathcal{E} = \cup\{K_\alpha | K_\alpha \cap E \neq \phi\}$ then for $s \notin \mathcal{E} - \mathcal{E}$, g and $\tau_s g$ have disjoint support. So
$$\int_{K_\alpha} |g + \tau_s g|^p = \int_{K_\alpha \cap (E \cup s+E)} |g + \tau_s g|^p = \int_{K_\alpha \cap E} |g + \tau_s g|^p + \int_{K_\alpha \cap (s+E)} |g + \tau_s g|^p$$
$$= \int_{K_\alpha} |g|^p + \int_{-s+K_\alpha} |g|^p$$
and
$$\sup_{K_\alpha} |g + \tau_s g| = \sup \{ |g + \tau_s g| \mid (K_\alpha \cap E) \cup (K_\alpha \cap (s + E)) \}$$
$$= \max (\sup_{K_\alpha \cap E} |g + \tau_s g| , \sup_{K_\alpha \cap (s+E)} |g + \tau_s g|)$$
$$= \max (\sup_{K_\alpha \cap E} |g| , \sup_{K_\alpha \cap (s+E)} |\tau_s g|)$$
$$= \max (\sup_{K_\alpha} |g| , \sup_{-s+K_\alpha} |g|)$$

By the definition of the norm $||\cdot||_{pq}$ it follows

(1) $$||g + \tau_s g||_{pq} = 2^{1/q}\, ||g||_{pq} \quad 1 \leq p \leq \infty,\ 1 \leq q < \infty$$

(2) $$||g + \tau_s g||_{p\infty} = ||g||_{p\infty} \quad 1 \leq p \leq \infty$$

Now, if $\mu \in M_q$ and $\nu \in M_c^q$, $\nu(A) = \mu(A \cap E)$, $E \subset G$ compact (see Definition 3.5) then for $s \notin E - E + L - L$

(3) $$(\nu + \tau_s \nu)^{\#}(t) = \begin{cases} \nu^{\#}(t) & t \in E - L \\ \tau_s \nu^{\#}(t) & t \in s + E - L \\ 0 & \text{otherwise} \end{cases}$$

(see Theorem 1.21).

To see this we consider two cases.

Case 1) ν is a real valued measure. By the decomposition of $\nu + \tau_s \nu$ (p.117) we have that

$(\nu + \tau_s \nu)^{\#}(t) = |\nu + \tau_s \nu|(t + L) = (\nu + \tau_s \nu)^{+}(t + L) + (\nu + \tau_s \nu)^{-}(t + L)$ where $(\nu + \tau_s \nu)^{+} = \max\,(\nu + \tau_s \nu, 0)$ and $(\nu + \tau_s \nu)^{-} = \max\,(-\nu - \tau_s \nu, 0)$.

Now, $(\nu + \tau_s \nu)(t + L) = \mu((t + L) \cap E) + \mu((-s + t + L) \cap E)$, since $s \notin E - E + L - L$, $(t + L) \cap E \neq \phi$ and $(-s + t + L) \cap E = \phi$ if $t \in E - L$; $(t + L) \cap E = \phi$ and $(-s + t + L) \cap E \neq \phi$ if $t \in s + E - L$; $(t + L) \cap E = (-s + t + L) \cap E = \phi$ otherwise. Hence

$$(\nu + \tau_s \nu)^{\#}(t) = |\nu + \tau_s \nu|(t + L) = \begin{cases} |\nu|(t + L) & t \in E - L \\ |\nu|(-s + t + L) & t \in s + E - L \\ 0 & \text{otherwise} \end{cases}$$

Therefore (3) holds.

Case 2) ν is a complex valued measure. Then $\nu = \nu_1 + i\nu_2$, ν_i real valued measure $i = 1,2$. So, $\nu + \tau_s \nu = \nu_1 + \tau_s \nu_1 + i(\nu_2 + \tau_s \nu_2)$ and

$|\nu + \tau_s\nu| = |\nu_1 + \tau_s\nu_1| + |\nu_2 + \tau_s\nu_2|$.

Hence $(\nu + \tau_s\nu)^{\#} = (\nu_1 + \tau_s\nu_1)^{\#} + (\nu_2 + \tau_s\nu_2)^{\#}$. From case 1) we have that

$$(\nu + \tau_s\nu)^{\#}(t) = \begin{cases} \nu_1^{\#}(t) + \nu_2^{\#}(t) & t \in E - L \\ \tau_s(\nu_1^{\#} + \nu_2^{\#})(t) & t \in s + E - L \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \nu^{\#}(t) & t \in E - L \\ \tau_s\nu^{\#}(t) & t \in s + E - L \\ 0 & \text{otherwise} \end{cases}$$

By (3) we have that for $1 \leq q < \infty$,

$$\int_G (\nu + \tau_s\nu)^{\#}(t)^q\, dt = \int_{E-L} (\nu + \tau_s\nu)^{\#}(t)^q\, dt + \int_{s+E-L} (\nu + \tau_s\nu)^{\#}(t)^q\, dt$$

$$= \int_{E-L} \nu^{\#}(t)^q\, dt + \int_{s+E-L} \tau_s\nu^{\#}(t)^q\, dt$$

$$= \int_{E-L} \nu^{\#}(t)^q\, dt + \int_{E-L} \nu^{\#}(t)^q\, dt$$

$$= 2\int_{E-L} \nu^{\#}(t)^q\, dt$$

This implies that

(4) $$||\, \nu + \tau_s\nu\, ||_q^{\#} = 2^{1/q}\, ||\, \nu\, ||_q^{\#}.$$

Let $\varepsilon > 0$. If $f \in (L^p, \ell^q)$ $1 \leq p \leq \infty$, $1 \leq q < \infty$, then there exists g in L_c^p such that $||f - g||_{pq} < \varepsilon/3(2^a)$ (Theorem 3.6).

For $s \notin E - E$, E as in the beginning of the proof, we have by (1) and Theorem 3.11 that

$$\left|\, \|f + \tau_s f\|_{pq} - 2^{1/q}\|f\|_{pq} \,\right|$$

$$\leq \left|\, \|f + \tau_s f\|_{pq} - \|g + \tau_s g\|_{pq} \,\right| + \left|\, 2^{1/q}\|g\|_{pq} - 2^{1/q}\|f\|_{pq} \,\right|$$

$$\leq \|f - g\|_{pq} + \|\tau_s f - \tau_s g\|_{pq} + 2^{1/q}\|g - f\|_{pq}$$

$$< \varepsilon/3(2^{a}) + 2^{a}(\varepsilon/3(2^{a})) + 2^{1/q}(\varepsilon/3(2^{a})) < \varepsilon.$$

This proves i).

Similarly if $p = \infty$ we have by (1) and Theorem 3.11 that

$$\left|\, \|f + \tau_s f\|_{\infty q} - 2^{1/q}\|f\|_{\infty q} \,\right|$$

$$\leq \left|\, \|f + \tau_s f\|_{\infty q} - \|g + \tau_s g\|_{\infty q} \,\right| + \left|\, 2^{1/q}\|g\|_{\infty q} - 2^{1/q}\|f\|_{\infty q} \,\right|$$

$$\leq \|f - g\|_{\infty q} + \|\tau_s f - \tau_s g\|_{\infty q} + 2^{1/q}\|g - f\|_{\infty q} < \varepsilon.$$

Therefore ii) holds for $f \in (L^{\infty}, \ell^{q})$ and hence for $f \in (C_0, \ell^{q})$. The proof of iii) is the same, but taking $g \in C_c$ and using (2). The case $f \in C_0$ is [40, Lemma 3.5.1].

Finally to prove iv) we keep in mind that the norms $\|\cdot\|_q^{\#}$ and $\|\cdot\|_q$ are equivalent.

If $\mu \in M_q$ then by Theorem 3.6 there exists $\nu \in M_c^q$ such that $\nu(A) = \mu(A \cap E)$, $E \subset G$ compact and $\|\mu - \nu\|_q^{\#} < \varepsilon/3(2^{a})$ where $\varepsilon > 0$ is given.

As before, we have by (4) that for all $s \notin E - E + L - L$

$$\left|\, \|\mu - \tau_s \mu\|_q^{\#} - 2^{1/q}\|\mu\|_q^{\#} \,\right| < \varepsilon.$$

<u>THEOREM 12.4</u>. If G is noncompact then the following linear spaces of multipliers are trivial. That is, the zero multiplier is the only element in these spaces.

i) $M((L^p, \ell^q),(L^r, \ell^s))$ $\quad 1 \leq r,\ p \leq \infty,\ 1 \leq s < q < \infty$

ii) $M((L^p, \ell^q),(C_0, \ell^s))$ $\quad 1 \leq p \leq \infty,\ 1 \leq s < q < \infty$

iii) $M((C_0, \ell^q),(L^r, \ell^s))$ $\quad 1 \leq r \leq \infty,\ 1 \leq s < q < \infty$

iv) $M((C_0, \ell^q),(C_0, \ell^s))$ $\quad 1 \leq s < q < \infty$

v) $M((L^p, c_0),(L^r, \ell^s))$ $\quad 1 \leq p,\ r \leq \infty,\ 1 \leq s < \infty$

vi) $M((L^p, c_0),(C_0, \ell^s))$ $\quad 1 \leq p \leq \infty,\ 1 \leq s < \infty$

vii) $M(M_q,M_s)$ $\quad 1 \leq s < q < \infty$

viii) $M(M_q,(L^r, \ell^s))$ $\quad 1 \leq r \leq \infty,\ 1 \leq s < q < \infty$

ix) $M(M_q,(C_0, \ell^s))$ $\quad 1 \leq s < q < \infty$

x) $M(L^p, \ell^q),M_s)$ $\quad 1 \leq p \leq \infty,\ 1 \leq s < q < \infty$

xi) $M((C_0, \ell^q),M_s)$ $\quad 1 \leq s < q < \infty$

xii) $M((L^p, c_0),M_s)$ $\quad 1 \leq p \leq \infty,\ 1 \leq s < \infty$

xiii) $M((L^\infty, \ell^q)^w,(L^r, \ell^s))$ $\quad 1 \leq p < \infty,\ 1 < s < q < \infty$

xiv) $M((L^\infty, \ell^q)^w,(C_0, \ell^s))$ $\quad 1 < s < q \leq \infty$

xv) $M((L^\infty, \ell^q)^w,(L^\infty, \ell^s)^w)$ $\quad 1 < s < q \leq \infty$

xvi) $M((L^p, \ell^q),(L^\infty, \ell^s)^w)$ $\quad 1 \leq p < \infty,\ 1 < s < q < \infty$

xvii) $M((C_0, \ell^q),(L^\infty, \ell^s)^w)$ $\quad 1 < s < q \leq \infty$ or $1 \leq s < q < \infty$

xviii) $M((L^p, c_0),(L^\infty, \ell^s)^w)$ $\quad 1 \leq p < \infty,\ 1 < s < \infty$

xix) $M((L^p, \ell^\infty)^w,(L^r, \ell^s))$ $\quad 1 < p \leq \infty,\ 1 \leq r < \infty,\ 1 < s < \infty$

xx) $M((L^p, \ell^\infty)^w,(C_0, \ell^s))$ $\quad 1 < p \leq \infty,\ 1 < s < \infty.$

<u>PROOF</u>. The proof of each of the first twelve cases are the same; so we will prove only i).

Suppose $T \in M((L^p, \ell^q),(L^r, \ell^s))$ and $T \neq 0$. Then for $f \in (L^p, \ell^q)$ and $s \in G$ we have that

$$||Tf + \tau_s Tf||_{rs} = ||Tf + T\tau_s f||_{rs} \leq ||T||\ ||f + \tau_s f||_{pq}.$$

Taking the limit on both sides we have by Lemma 12.3 that

$$2^{1/s}\ ||Tf||_{rs} \leq 2^{1/q}\ ||T||\ ||f||_{pq}.$$

This implies that $||T|| < 2^{1/q - 1/s}\ ||T||$ and we have a contradiction because $2^{1/q - 1/s}$ is strictly less than one. Therefore $T \equiv 0$.

The proofs of the remaining cases are similar to each other. So we will prove only xiii).

Let $T \in M((L^\infty, \ell^q)^w,(L^r, \ell^s))$. By Proposition 12.2 its adjoint T^* belongs to $M((L^{r'}, \ell^{s'}),(L^1, \ell^{q'}))$. Since $1 < q' < s' < \infty$, $T^* \equiv 0$ by case i). This implies that $T \equiv 0$.+

The next theorem corresponds to Theorem 5.2.1 of [40] and the proof is the same.

THEOREM 12.5. Let $1 < p, q, r, s < \infty$. $M((L^p, \ell^q),(L^r, \ell^s))$ is isometrically isomorphic to $M((L^{r'}, \ell^{s'}),(L^{p'}, \ell^{q'}))$.

PROOF. By Proposition 12.2, $T \longrightarrow T^*$ defines a linear map from $M((L^p, \ell^q),(L^r, \ell^s))$ into $M((L^{r'}, \ell^{s'}),(L^{p'}, \ell^{q'}))$. Moreover since $||T|| = ||T^*||$ [44, Theorem 4.10] and (L^p, ℓ^q), (L^r, ℓ^s) are reflexive (Corollary 3.3) the map is continuous and onto.

Similarly to Theorem 5.3.1 of [40] we will apply the Riesz-Thorin Theorem for amalgams (Theorem 5.6) to prove the next result.

THEOREM 12.6. Let $1 \leq p_i, q_i, r_i, s_i \leq \infty$, $i = 1, 2$. Suppose that, for some $0 < \theta < 1$,

$$\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2} \ ; \quad \frac{1}{q} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2} \ ; \quad \frac{1}{r} = \frac{1-\theta}{r_1} + \frac{\theta}{r_2} \ ; \quad \frac{1}{s} = \frac{1-\theta}{s_1} + \frac{\theta}{s_2} \ .$$

If $T \in M((L^{p_i}, \ell^{q_i}),(L^{r_i}, \ell^{s_i}))$, $i = 1, 2$, then T defines a unique element of

$M((L^p, \ell^q),(L^r, \ell^s))$ if $1 \leq p < \infty$, $1 \leq q < \infty$

$M((L^p\ \ell^\infty)^w, (L^r, \ell^s))$ if $1 \leq p \leq \infty$, $q = \infty$

$M((L^p, c_0),(L^r, \ell^s))$ if $1 \leq p < \infty$, $q = \infty$

$M(C_0,(L^r, \ell^s))$ if $p = q = \infty$.

PROOF. Let $\mathcal{L}(m)$ be as in §5 p.67. Since $\mathcal{L}(m) \subset (L^{p_i}, \ell^{q_i})$, $i = 1, 2$, T restricted to $\mathcal{L}(m)$ is a continuous operator from $\mathcal{L}(m)$ to (L^{r_i}, ℓ^{s_i}), $i = 1, 2$, which commutes with translations. So, if $||T||_i$, $i = 1, 2$, is the norm of T in $M((L^{p_i}, \ell^{q_i}),(L^{r_i}, \ell^{s_i}))$ then for all $x \in \mathcal{L}(m)$

$$||Tx||_{r_i s_i} \leq ||T||_i \ ||x||_{p_i q_i} \ , \ i = 1, 2.$$

By the Riesz-Thorin Theorem $T\,\mathcal{L}(m) \subset (L^r, \ell^s)$ and for all $x \in \mathcal{L}(m)$

$$||Tx||_{rs} \leq ||T||_1^\theta \ ||T||_2^{1-\theta} \ ||x||_{pq} \quad \text{if } 1 < r \leq \infty, \ 1 \leq s \leq \infty$$

$$||Tx||_{rs} \leq ||T||_1^\theta \ ||T||_2^{1-\theta} \ 2^a \ ||x||_{pq} \quad \text{if } r = 1, \ 1 \leq s \leq \infty.$$

Therefore T restricted to $\mathcal{L}(m)$ defines a continuous linear operator from $(\mathcal{L}(m), \|\cdot\|_{pq})$ to (L^r, ℓ^s) which commutes with translations. Since $\mathcal{L}(m)$ is dense in (L^p, ℓ^q) if $1 \leq p \leq \infty$, $1 \leq q < \infty$ (Remark 5.5), $(L^p, \ell^\infty)^w$ if $1 \leq p \leq \infty$, (L^p, c_0) if $1 \leq p < \infty$, and C_0, T has a unique continuous extension, also called T, of the same norm on these spaces.

By the continuity of τ_s on (L^p, ℓ^q), $1 \leq p, q \leq \infty$ (Theorem 3.11) and on $(L^p, \ell^\infty)^w$, we conclude that T is a multiplier.

To see that τ_s is in fact continuous on $(L^p, \ell^\infty)^w$, take $\{f_n\}$, f in (L^p, ℓ^∞) such that $\lim f_n = f$ in $(L^p, \ell^\infty)^w$.

Let $g \in (L^{p'}, \ell^1)$ and $\varepsilon > 0$. Since $\tau_s g \in (L^{p'}, \ell^1)$, $|< \tau_s g, f_n - f >| < \varepsilon$ for all $n \geq N$. This implies that, for $n \geq N$, $|< g, \tau_s f_n - \tau_s f >| = |< \tau_s g, f_n - f >| < \varepsilon$. Therefore the translation operator τ_s on $(L^p, \ell^\infty)^w$ is continuous.+

<u>COROLLARY 12.7</u>. $M((L^p, \ell^q),(L^p, \ell^q)) \subset M(L^2, L^2)$, $1 < p, q \leq 2$.

<u>PROOF</u>. Let $T \in M((L^p, \ell^q),(L^p, \ell^q))$. By Theorem 12.5 T belongs to $M((L^{p'}, \ell^{q'}),(L^{p'}, \ell^{q'}))$. Applying Theorem 12.6 with $p_1 = r_1 = p$, $p_2 = r_2 = p'$, $q_1 = s_1 = q$, $q_2 = s_2 = q'$ we have that for $\theta = 1/2$; $1/2 = (1 - \theta)/p + \theta/p' = (1 - \theta)/q + \theta/q'$.

Therefore T defines a unique T in $M(L^2, L^2)$.+

<u>DEFINITION 12.8</u>. Let A be a Banach algebra and B be a Banach A-module. A continuous linear operator $T: A \longrightarrow B$ is a <u>c-multiplier</u> from A to B if T commutes with convolution. That is, for all $f, g \in A$, $T(f*g) = Tf*g$.

The linear space of c-multipliers from A to B will be denoted by c-M(A,B).

THEOREM 12.9. Let A be any of the spaces (C_0, ℓ^1), $(L^\infty, \ell^1)^w$, (L^p, ℓ^1) $(1 \leq p < \infty)$, $M_1{}^w$; and let B be any of the spaces (L^p, ℓ^q) $(1 \leq p, q < \infty)$, (C_0, ℓ^q) $(1 \leq q \leq \infty)$, (L^p, c_0) $(1 \leq p < \infty)$, $(L^\infty, \ell^q)^w$ $(1 \leq p \leq \infty)$, $(L^p, \ell^\infty)^w$ $(1 \leq p \leq \infty)$, $M_q{}^w$ $(1 \leq q \leq \infty)$, $(L^p, \ell^q)^w$ $(1 \leq p, q < \infty)$. If T is a linear operator from A to B such that T commutes with translations then T commutes with convolution.

PROOF. Note that A^* is M_∞, (L^1, c_0), $(L^{p'}, \ell^\infty)$ or (C_0, ℓ^∞) and B^* is either an amalgam space or a measure space $M_q (1 \leq q \leq \infty)$.

Suppose A, B^* are amalgam spaces. Let $T^*: B^* \longrightarrow A^*$ be the adjoint of T, f, g in A and h in B^*. By Theorem 3.1 or (12.1)

$$\int Tf(t)\ h(-t)\ dt = < Tf,h > = < f,T^*h > = \int f(t)\ T^*(-t)\ dt.$$

So, we have that

$$< Tf*g,h > = \int Tf*g(t)\ h(-t)\ dt = \int \int Tf(t-s)\ g(s)\ ds\ h(-t)\ dt$$

$$= \int g(s) \int \tau_s Tf(t)\ h(-t)\ dt\ ds$$

$$= \int g(s) \int T\tau_s f(t)\ h(-t)\ dt\ ds$$

$$= \int g(s) \int \tau_s f(t)\ T^*h'(t)\ dt\ ds$$

$$= \int \int f(t-s)\ g(s)\ ds\ T^*h'(t)\ dt$$

$$= \int f*g(t)\ T^*h'(t)\ dt = \int T(f*g)(t)\ h(-t)\ dt$$

$$= < T(f*g),h >.$$

We can apply Fubini's theorem because f, g are in A, hence g is in L^1, and T^*h is in A^*.

Since this holds for all $h \in B^*$ we conclude by the Hahn-Banach theorem that $Tf*g = T(f*g)$ for all $f, g \in A$.

The proof of the remaining cases is similar. †

§ 13. MULTIPLIERS FROM L^1 TO AMALGAM SPACES AND SPACES OF MEASURES M_q

The multipliers which have the most satisfactory characterization are those from L^1 to amalgam spaces and spaces of measures M_q. This is so because of the nature of the algebra L^1.

We will consider the cases: c-multipliers from L^1 to (L^p, ℓ^1), $1 \leqq p \leqq \infty$, c-multipliers from L^1 to (L^p, ℓ^q), $1 < p, q \leqq \infty$, and c-multipliers from L^1 to (L^1, ℓ^q), $1 < q \leqq \infty$.

Our first theorem is an extension of Theorem 3.11 of [40] first introduced by R. E. Edwards [21, Theorem 1].

THEOREM 13.1. Let B be any of the spaces (L^p, ℓ^q), $1 < p \leqq \infty$, $1 \leqq q \leqq \infty$, M_q, $1 \leqq q \leqq \infty$. If $T \in c\text{-}M(L^1,B)$ then there exists a unique $\mu \in B$ such that $Tf = \mu * f$ for all $f \in L^1(G)$. Hence $c\text{-}M(L^1,B) \subset M(L^1,B)$.

PROOF. Let A be $(L^{p'}, \ell^{q'})$ if $B = (L^p, \ell^q)$, $1 < p, q \leqq \infty$, $(L^{p'}, c_0)$ if $B = (L^p, \ell^1)$, $1 < p \leqq \infty$, $(C_0, \ell^{q'})$ if $B = M_q$, $1 \leqq q \leqq \infty$. Then $A^* = B$ and C_c is dense in A (Theorem 3.7). Let $\{\phi_U\}$ be the a. i. in $L^1(G)$ defined in §7. So for $f \in L^1$ we have that

$$||Tf - T\phi_U * f||_B = ||Tf - T(\phi_U * f)||_B \leqq ||T||\ ||f - \phi_U * f||_1.$$

This shows that

(1) $$\lim T\phi_U * f = Tf \quad \text{in } B.$$

On the other hand, $||T\phi_U||_B \leqq ||T||\ ||\phi_U||_1 = ||T||$ for all U. Therefore $\{T\phi_U\}$ lies in a norm-bounded subset of $B = A^*$. By Alaoglu's theorem there exists a subnet $\{T\phi_V\}$ of $\{T\phi_U\}$ and $\mu \in B$ such that $\{T\phi_V\}$ converges to μ in the weak*-topology $\sigma(B,A)$. That is, for each $h \in A$

(2) $$\lim_{V \to 0} < h, T\phi_V > = < h, \mu >.$$

Now, by (1) and the fact that $C_c \subset A \subset B^*$ we have that for all f, g in C_c

$$\lim_{V \to 0} < g, T\phi_V * f > = < g, Tf >.$$

This together with (2) implies that for all f, g in C_c.

$$< g,Tf > = \lim < g,T\phi_V*f > = \lim g*(T\phi_V*f)(0) = \lim T\phi_V*(f*g)(0)$$
$$= < f*g, T\phi_V > = < f*g,\mu > = < g,\mu*f >.$$

Therefore $Tf = \mu*f$ for all $f \in C_c$, because C_c is dense in A.

Then $Tf = \mu*f$ for all $f \in L^1$ because C_c is dense in L^1 and convolution with μ and T are continuous linear transformations from L^1 to B (Theorem 4.7).

Suppose that $\mu*f = 0$ for all $f \in L^1$. In particular for all $f \in C_c$, $< f,\mu > = \mu*f(0)$ for all $f \in L^1$. In particular for all $f \in C_c$, $< f,\mu > = \mu*f(0)$. This implies that $\mu = 0$ since $\mu \in A^*$ and C_c is dense in A. Hence μ is unique.

Finally, if $f \in L^1$ and s, $t \in G$ then for $T \in c\text{-}M(L^1,B)$

$$\tau_s(Tf)(t) = Tf(t - s) = \mu*f(t - s) = \int f(t - s - x)\, d\mu(x)$$
$$= \int \tau_s f(t - x)\, d\mu(x) = \mu*\tau_s f(t) = T(\tau_s f)(t).$$

Therefore T commutes with translations.†

<u>COROLLARY 13.2</u>. Let $1 < p < \infty$, $1 \leq q < \infty$. If $T: L^1 \longrightarrow (L^p, \ell^q)$ is a bounded linear operator then the following are equivalent:

i) $T\tau_s = \tau_s T$ for all $s \in G$.

ii) $T(f*g) = Tf*g$ for all $f, g \in L^1$.

iii) There exists a unique $\mu \in (L^p, \ell^q)$ such that $Tf = \mu*f$ for all $f \in L^1$.

Hence $c\text{-}M(L^1,(L^p, \ell^q)) = M(L^1,(L^p, \ell^q))$.

<u>PROOF</u>. i) implies ii) follows from Theorem 12.9. By Theorem 13.1 ii) implies iii) implies i).+

Theorem 3.1.1 of [40] says that T belongs to $M(L^1,L^\infty)$ iff there exists a unique $\mu \in L^\infty$ such that $Tf = \mu*f$ for all $f \in L^1$.

This together with Theorem 13.1 and the properties of convolution implies the following result.

<u>THEOREM 13.3</u>. Let $T: L^1 \longrightarrow L^\infty$ be a bounded linear operator. Then the following are equivalent:

i) $T\tau_s = \tau_s T$ for all $s \in G$.

ii) $Tf*g = T(f*g)$ for all $f, g \in L^1$.

iii) There exists a unique $\mu \in L^\infty$ such that $Tf = \mu*f$ for all $f \in L^1$.

Hence $c\text{-}M(L^1,L^\infty) = M(L^1,L^\infty)$.

Now we are able to characterize the multipliers from L^1 to (L^p, ℓ^1), $1 < p < \infty$, in terms of bounded functions on G and establish an isometric algebra isomorphism between $c\text{-}M(L^1,(L^p, \ell^1)^\#)$ and the Segal algebra $(L^p, \ell^1)^\#$ (see Theorems 1.21 and 4.16).

In [46] is proved that a Segal algebra S is a semisimple, regular, commutative, Banach algebra with maximal ideal space homeomorphic to $\hat{G}$, such that the Gelfand transform is the Fourier transform restricted to S. Hence, for f, g in S, $||\hat{f}||_\infty \leqq ||f||_1$ and if $\hat{f}(\hat{x}) = \hat{g}(\hat{x})$ for all $\hat{x} \in \hat{G}$ then $f = g$.

Theorem 13.4. Let $T: L^1 \longrightarrow (L^p, \ell^1)$, $1 < p < \infty$, be a linear operator. Then the following are equivalent:

i) $T \in M(L^1,(L^p, \ell^1))$.

ii) There exists a unique $\mu \in (L^p, \ell^1)$ such that $Tf = \mu * f$ for all $f \in L^1$.

iii) There exists a unique $\varphi \in (C_0, \ell^{p'})(\hat{G})$ if $1 < p \leq 2$, or in (C_0, ℓ^2) if $2 < p < \infty$, such that $(Tf)\hat{} = \varphi\hat{f}$ for all $f \in L^1$.

The correapondence between T and μ establishes an isometric algebra isomorphism from $M(L^1,(L^p, \ell^1)^\#)$ onto $(L^p, \ell^1)^\#$.

PROOF. i) implies ii) implies iii) follow from Corollary 13.2 and the properties of the Fourier transform on (L^p, ℓ^1) (Theorem 5.4, Remark 5.8, Proposition 6.14).

Suppose iii).

Since $(\tau_s f)\hat{} = [s,\cdot]\,\hat{f}$ for all $s \in G$ and $f \in L^1$ we have that $(T\tau_s f) = \varphi(\tau_s f)\hat{} = [s,\cdot]\varphi\hat{f} = (\tau_s Tf)$. Therefore $T\tau_s = \tau_s T$. This means that T commutes with translations.

Let $\{f_n\}$, f be in L^1 such that $\lim ||f_n - f||_1 = 0$ and $\lim ||Tf_n - g||_{p1} = 0$.

So by the previous observation and (2.4) we have that

$$||(Tf)^{\wedge} - \hat{g}||_\infty \leq ||(Tf)^{\wedge} - (Tf_n)^{\wedge}||_\infty + ||(Tf_n)^{\wedge} - \hat{g}||_\infty$$
$$\leq ||\varphi(f - f_n)^{\wedge}||_\infty + ||Tf_n - g||_1$$
$$\leq ||\varphi||_\infty \, ||f - f_n||_1 + ||Tf_n - g||_{p1}.$$

This implies that $(Tf)^{\wedge} = \hat{g}$. Hence $Tf = g$ and by the Closed Graph Theorem we conclude that T is continuous. Therefore T belongs to $M(L^1,(L^p, \ell^1))$.

Since $M(L^1,(L^p, \ell^1)) = M(L^1, (L^p, \ell^1)^{\#})$ given $\mu \in (L^p, \ell^1)^{\#}$ the linear operator T defined by $Tf = \mu * f$ $(f \in L^1)$ belongs to $M(L^1,(L^p, \ell^1)^{\#})$ and by Proposition 4.17

$$||Tf||^{\#}_{p1} = ||\mu * f||^{\#}_{p1} \leq ||f||_1 \; ||\mu||^{\#}_{p1} \quad \text{for all } f \in L^1.$$

This implies that $||T|| \leq ||\mu||^{\#}_{p1}$.

On the other hand, if $\{e_n\}$ is the a. i. in $(L^p, \ell^1)^{\#}$ as in Proposition 4.17 ii) then $\lim ||e_n * \mu - \mu||^{\#}_{p1} = 0$. So given $\varepsilon > 0$ there exits an e_n such that $||e_n * \mu||^{\#}_{p1} > ||\mu||^{\#}_{p1} - \varepsilon$. Since $e_n \in L^1$ and $||e_n||_1 = 1$ we conclude that $||T|| = ||\mu||^{\#}_{p1}$.†

<u>REMARK 13.5</u>. Theorem 13.4 is a particular case of a more general result proved by Rieffel [48]. This was restated in terms of absolutely continuous Banach L^1-modules by Gulick, Liu and van Rooij [31, Theorem 5.2]. This says that if B is an absolutely continuous L^1-module then the relation $\mu \longmapsto T_\mu$, $T_\mu f = \mu * f$ $(f \in L^1)$ establishes an isometric linear homeomorphism from B^* onto $c\text{-}M(L^1, B^*)$. Indeed, (L^p, ℓ^1) is the dual of the absolutely continuous L^1-module $(L^{p'}, c_0)$ (Proposition 4.13).

We will make use of the work done by Burnham and Goldberg [9] about c-multipliers from L^1 to Segal algebras to characterize c-$M(L^1,(C_0, \ell^1))$.

In what follows S will be a Segal algebra (Definition 4.15) and $||f||_S$ will be the norm of f in S.

<u>DEFINITION 13.6</u>. [14, Definition 3]. The <u>relative completion</u> $\tilde{S}$ of S is the set of all $f \in L^1$ such that

$$f \in \bigcup_{x > 0} \overline{B_S(x)}^1$$

where $B_S(x) = \{f \in S \mid ||f||_S \leq x\}$ and $\overline{E}^1$ is the closure of E in L^1.

Thus, $f \in \tilde{S}$ iff there exists a sequence $\{f_n\}$ in S such that $\sup_n ||f_n||_S \leq x < \infty$ and $||f_n - f||_1 \to 0$.

For $f \in \tilde{S}$ we define $|||f||| = \inf \{x \mid f \in \overline{B_S(x)}^1\}$. Then $(\tilde{S}, |||\cdot|||)$ is a Banach algebra, S is a closed ideal of $\tilde{S}$ and the embedding of $(S, |||\cdot|||)$ into $(\tilde{S}, |||\cdot|||)$ is an isometry [14].

<u>LEMMA 13.7</u>. [14, Theorem 5]. $f \in \tilde{S}$ iff $f \in L^1$ and $\sup ||f*e_n||_S < \infty$ where $\{e_n\}$ is an a. i. in S.

Moreover $|||f||| = \sup ||f*e_n||_S$.

Since S is a subalgebra of L^1 (Definition 4.15) it is clear that a c-multiplier T from L^1 to S must be a c-multiplier from L^1 to L^1. Hence there exists a unique measure $\mu \in M_1(G)$ such that

(3) $\qquad Tf = \mu * f \quad$ for all $f \in L^1$ (Theorem I).

If $A \subset M_1$, we will write $M(L^1,S) \subset A$ if every $T \in$ c-$M(L^1,S)$.

has the form (3) for some $\mu \in A$. Thus $c\text{-}M(L^1,S) \subset M_1(G)$.

THEOREM 13.8. [9, Theorem 2.6]. If $c\text{-}M(L^1,S) \subset L^1$ then $c\text{-}M(L^1,S) = \tilde{S}$. In this case, if for $\mu \in \tilde{S}$ we define T_μ, by $T_\mu f = \mu * f$ $(f \in L^1)$ then the correspondence $\mu \longmapsto T_\mu$ is an isometric algebra isomorphism from $\tilde{S}$ onto $c\text{-}M(L^1,S)$.

THEOREM 13.9. [9, Theorem 2.3]. Let $\mu \in M_1$ and $\{e_n\}$ as in Lemma 13.7. Then the following are equivalent:

i) $\sup ||\mu * e_n||_S < \infty$.

ii) $\mu \in c\text{-}M(L^1,S)$.

iii) $\mu \in c\text{-}M(L^1,\tilde{S})$.

Hereafter and for the rest of our work $\{e_n\}$ will be the a.i. in $(C_0, \ell^1)^\#$, hence the a. i. in $(L^p, \ell^1)^\#$, $1 < p < \infty$, given by Proposition 4.17.

Applying Theorem 13.8 and Theorem 13.9 to the Segal algebra $(L^p, \ell^1)^\#$ we have the following theorem.

THEOREM 13.10. Let $1 < p < \infty$.

i) $(L^p, \ell^1)^\# = (L^p, \ell^1)^\#$. Hence $||f||^\#_{p1} = |||f|||$ for all f in (L^p, ℓ^1).

ii) $f \in (L^p, \ell^1)$ iff $f \in L^1$ and $\sup ||f * e_n||^\#_{p1} < \infty$.

Moreover $||f||^\#_{p1} = \sup ||f * e_n||^\#_{p1}$ for all $f \in (L^p, \ell^1)$.

iii) Let $\mu \in M_1(G)$. $\mu \in c\text{-}M(L^1,(L^p, \ell^1))$ iff $\sup_k ||\mu * e_n||^\#_{p1} < \infty$.

PROOF. By Theorem 13.4 $c\text{-}M(L^1,(L^p, \ell^1)) \subset (L^p, \ell^1) \subset L^1$. Then i) follows from Theorem 13.8. ii) is a direct consequence of i) and Lemma 13.7. iii) follows from i) and Theorem 13.9.+

THEOREM 13.11. [9, Theorem 4.5]. $(C_0, \ell^1)^{\#} = (L^\infty, \ell^1)$ and the norms $|||\cdot|||$ and $||\cdot||_{\infty 1}$ are equivalent in (L^∞, ℓ^1).

COROLLARY 13.12. $f \in (L^\infty, \ell^1)$ iff $f \in L^1$ and $\sup ||f*e_n||^{\#}_{\infty 1} < \infty$.

PROOF. Lemma 13.7 and Theorem 13.11.+

Similarly to [9, Theorem 4.6] we will characterize $c\text{-}M(L^1,(C_0, \ell^1))$.

THEOREM 13.13. Let $S^\infty = ((L^\infty, \ell^1), |||\cdot|||)$. If T is in $c\text{-}M(L^1,(C_0, \ell^1))$, then there ~~exists~~ a unique $\mu \in (L^\infty, \ell^1)$ such that $Tf = \mu*f$ for all $f \in L^1$ and the correspondence of T and μ defines an isometric algebra isomorphism from $c\text{-}M(L^1,(C_0, \ell^1))$ onto S^∞.

PROOF. Let $T \in c\text{-}M(L^1,(C_0, \ell^1))$. Then there exists a unique $\mu \in M_1$ such that $Tf = \mu*f$ for all $f \in L^1$ (see (3) above). On the other hand $(C_0, \ell^1) \subset L^\infty$ and therefore $T \in c\text{-}M(L^1,L^\infty)$, so by Theorem 13.3 there exists a unique $\varphi \in L^\infty$ such that $Tf = \varphi*f$ for all $f \in L^1$. This implies that for all $f \in L^1$,

$$\int f(-t)\, d\mu(t) = f*\mu(0) = f*\varphi(0) = \int f(-t)\, \varphi(t)\, dt.$$

In particular for a measurable E in G.

$$\mu(E) = \int \chi_E \, d\mu = \int_E \varphi(x) \, dx.$$

This means that μ is absolutely continuous. Therefore $\varphi \in L^1$ and $\varphi dx = \mu$. So $c\text{-}M(L^1,(C_0, \ell^1)) \subset L^\infty \cap L^1 \subset L^1$. Then the conclusion follows from Theorem 13.11 and Theorem 13.8.†

COROLLARY 13.14. Let $\mu \in M_1$. Then the following are equivalent:

i) $\sup ||\mu * e_n||_{\infty 1} < \infty$.

ii) $\mu \in c\text{-}M(L^1,(C_0, \ell^1))$.

iii) $\mu \in c\text{-}M(L^1,(L^\infty, \ell^1))$.

iv) $\mu \in M(L^1,(C_0, \ell^1))$.

PROOF. i), ii) and iii) are equivalent by Theorem 13.9 and Theorem 13.13. By Theorem 13.13 and the properties of convolution, ii) implies iv). Finally by Theorem 12.9, iv) implies ii).†

THEOREM 13.15. Let $T: L^1 \longrightarrow (C_0, \ell^1)$ be a linear operator. Then the following are equivalent:

i) $T \in M(L^1, (C_0, \ell^1))$.

ii) There exists a unique $\mu \in (L^\infty, \ell^1)$ such that $Tf = \mu * f$ for all $f \in L^1$.

iii) There exists a unique $\varphi \in (C_0, \ell^2)(\hat{G})$ such that $(Tf)^\wedge = \varphi\hat{f}$ for all $f \in L^1$.

PROOF. i) implies ii) follows from Corollary 13.14 and Theorem 13.13. Since $(L^\infty, \ell^1) \subset (L^2, \ell^1)$, the Fourier transform φ of μ in (L^∞, ℓ^1) belongs to (C_0, ℓ^2) (Theorem 5.4). Therefore if ii) holds then $(Tf)^\wedge = \varphi\hat{f}$ for all $f \in L^1$. Clearly φ is unique.

The proof of iii) implies i) is the same as in Theorem 13.4.†

REMARK 13.16. Note that by Corollary 13.14
$c\text{-}M(L^1,(C_0,\ \ell^1)) = M(L^1,(C_0,\ \ell^1))$ and $c\text{-}M(L^1,(L^\infty,\ \ell^1)) \subset M(L^1,(L^\infty,\ell^1))$.

Similarly to [40, Theorem 0.1.2] we have the following. Let μ in $(L^p,\ \ell^1)$, $1 < p \leq \infty$. T_μ will be the c-multiplier from L^1 to $(L^p,\ \ell^1)$ defined by $T_\mu g = \mu * g$.

THEOREM 13.17. For each $T \in c\text{-}M(L^1,(L^p,\ \ell^1))$, $1 < p \leq \infty$, there exists a net $\{i_n\}$ in $(C_0,\ \ell^1)$ such that

$$\lim ||T_{i_n} f - Tf||_{p1} = 0 \quad \text{for all } f \in L^1.$$

That is, $\{T_{i_n}\}$ is strong operator convergent to T and therefore $\{T_\mu |\ \mu \in (C_0,\ \ell^1)\}$ is strong operator dense in $c\text{-}M(L^1,(L^p,\ \ell^1))$.

PROOF. Take $f \in L^1$, and $\varepsilon > 0$. Then there exists $\phi \in C_0$ such that $||f - \phi||_1 < \varepsilon/4$. Since $\{e_n\}$ is an a. i. in $(C_0,\ \ell^1)$, there exists N such that, for all $n > N$, $||e_n*\phi - \phi||_{\infty 1} < \varepsilon/2$. By (2.4), $||e_n*\phi - \phi||_1 < \varepsilon/2$ for all $n > N$. Then for all $n > N$

$$\begin{aligned} ||e_n*f - f||_1 &\leq ||e_n*\phi - e_n*f||_1 + ||e_n*\phi - \phi||_1 + ||\phi - f||_1 \\ &\leq ||e_n||_1\ ||\phi - f||_1 + ||e_n*\phi - \phi||_1 + ||\phi - f||_1 \\ &= 2\ ||\phi - f||_1 + ||e_n*\phi - \phi||_1 < \varepsilon. \end{aligned}$$

Therefore

$$\text{(4)} \qquad \lim ||e_n*f - f||_1 = 0.$$

This means that $\{e_n\}$ is an a.i. in L^1.

Let μ be the element in $(L^p,\ \ell^1)$ associated to T and set $i_n = e_n*\mu$. By Theorem 4.7 $\{i_n\} \subset (C_0,\ell^1)$ and

$$||T_{i_n}f - Tf||_{p1} = ||i_n*f - \mu*f||_{p1} = ||e_n*\mu*f - \mu*f||_{p1}$$

$$\leq 2^a\ ||\mu||_{p1}\ ||e_n*f - f||_1.$$

By (4) we conclude that $\lim ||T_{i_n}f - Tf||_{p1} = 0.$†

Now we will characterize the c-multipliers from L^1 to (L^p, ℓ^q) $1 < p, q \leq \infty$.

We know that the amalgam (L^p, ℓ^q), $1 \leq p, q \leq \infty$, is a Banach L^1-module (§4 p. 60). Then there exists an equivalent norm $||\cdot||'_{pq}$ in (L^p, ℓ^q) such that for all $f \in L^1$, $\mu \in (L^p, \ell^q)$

$$||f*\mu||'_{pq} \leq ||f||_1\ ||\mu||'_{pq}$$

We will write $\underline{(L^p, \ell^q)'}$ for (L^p, ℓ^q) endowed with the norm $||\cdot||'_{pq}$ [19, 14.4].

<u>THEOREM 13.18</u>. Let $1 < p, q \leq \infty$. If $T: L^1 \longrightarrow (L^p, \ell^q)$ is a linear operator then the following are equivalent:

i) $T \in c\text{-}M(L^1,(L^p, \ell^q))$.

ii) There exists a unique $\mu \in (L^p, \ell^q)$ such that $Tf = \mu*f$ for all $f \in L^1$.

The correspondence between T and μ defines a continuous linear isomorphism from $c\text{-}M(L^1,(L^p, \ell^q))$ onto (L^p, ℓ^q).

If $1 < p, q < \infty$ then the isomorphism is an isometry from $c\text{-}M(L^1,(L^p, \ell^q)')$ onto $(L^p, \ell^q)'$.

<u>PROOF</u>. The case $p = q = \infty$ is Theorem 13.3.

By Theorem 13.1, i) implies ii).

If $\mu \in (L^p, \ell^q)$ and $Tf = \mu * f$ $(f \in L^1)$ then clearly T commutes with convolution and by Theorem 4.7

$||Tf||_{pq} = ||\mu * f||'_{pq} \leq 2^a \, ||f||_1 \, ||\mu||_{pq}$ for all $f \in L^1$.

Hence T is continuous and $||T|| \leq 2^a \, ||\mu||_{pq}$. Therefore $T \in c\text{-}M(L^1,(L^p, \ell^q))$ and the equation $Tf = \mu * f$ defines a continuous linear isomorphism from $c\text{-}M(L^1,(L^p, \ell^q))$ onto (L^p, ℓ^q).

If $1 < p, q < \infty$ and $T \in c\text{-}M(L^1,(L^p, \ell^q)')$ then for all f in L^1, $||Tf||'_{pq} = ||\mu * f||'_{pq} \leq ||f||_1 \, ||\mu||'_{pq}$. Hence $||T|| \leq ||\mu||'_{pq}$.

Now, by Corollary 4.14 and Theorem 7.2,

$$\lim_{U \to 0} ||\phi_U * \mu - \mu||'_{pq} = 0.$$

So given $\varepsilon > 0$ there exists a ϕ_U such that $||\phi_U * \mu - \mu||'_{pq} < \varepsilon$. This implies that $||\phi_U * \mu||'_{pq} > ||\mu||'_{pq} - \varepsilon$. Since $\phi_U \in L^1$ and $||\phi_U||_1 = 1$ we conclude that $||T|| = ||\mu||'_{pq}$.†

<u>REMARK 13.19</u>. By Theorem 4.7, $f * \mu \in (C_0, \ell^q)$ $((L^q, c_0))$ for all $f \in L^1$ and $\mu \in (L^\infty, \ell^q)$ $((L^q, \ell^\infty))$ $(1 < q < \infty)$. Hence by Theorem 13.18 for $1 < q < \infty$, $c\text{-}M(L^1,(L^\infty, \ell^q)) = c\text{-}M(L^1,(C_0, \ell^q))$ $(c\text{-}M(L^1,(L^q, \ell^\infty)) = c\text{-}M(L^1,(L^q, c_0)))$.

This implies the next theorem whose first part is already known [42, Theorem 4.2].

<u>THEOREM 13.20</u>. Let $1 < q < \infty$. If $T: L^1 \longrightarrow (C_0, \ell^q)$ $((L^q, c_0))$ is a linear operator then the following are equivalent:

i) $T \in c\text{-}M(L^1,(C_0, \ell^q))$ $(c\text{-}M(L^1,(L^q, c_0))$

ii) There exists a unique $\mu \in (L^\infty, \ell^q)$ $((L^q, \ell^\infty))$ such that $Tf = \mu * f$ for all $f \in L^1$.

The correspondence between T and μ defines a continuous isomorphism from $c\text{-}M(L^1, (C_0, \ell^q))$ $(c\text{-}M(L^1,(L^q, c_0)))$ onto (C_0, ℓ^q) $((L^q, c_0))$.

REMARK 13.21. From Theorem 12.9 and Theorem 13.20 we conclude that for $1 < q < \infty$, $c\text{-}M(L^1,(C_0, \ell^q)) = M(L^1,(C_0, \ell^q))$ and $c\text{-}M(L^1,(L^q, c_0)) = M(L^1,(L^q, c_0))$. Then by Remark 13.19, for $1 < q < \infty$ $c\text{-}M(L^1,(L^\infty, \ell^q)) = c\text{-}M(L^1,(C_0, \ell^q)) = M(L^1,(C_0, \ell^q))$ and $c\text{-}M(L^1,(L^q, \ell^\infty)) = c\text{-}M(L^1,(L^q, c_0)) = M(L^1,(L^q, c_0))$.

The next theorem is the counterpart to the uniqueness theorem in the theory of L^p spaces and we will use it to characterize $c\text{-}M(L^1,(L^p, \ell^q))$ in terms of functions on $\hat{G}$.

THEOREM 13.22. (Uniqueness Theorem for Amalgams).

i) Let μ,ν be in $M_\infty(G)$. If $\hat{\mu} = \hat{\nu}$ then $\mu = \nu$.

ii) Let f, g be in (L^p, ℓ^q), $1 \leq p, q \leq \infty$. If $\hat{f} = \hat{g}$ (as linear functionals on $A_c(\hat{G})$ if $q > 2$) then $f = g$ a.e.

PROOF. i) If $\hat{\mu} = \hat{\nu}$ then by definition of the Fourier transform (Definition 6.9) for all $\phi \in \Phi(\hat{G})$ (see Lemma 6.4)
$< \check{\phi}',\mu > = < \phi,\hat{\mu} > = < \phi,\hat{\nu} > = < \check{\phi}',\nu >$.

Since $(\Phi(\hat{G}))^\vee$ is dense in (C_0, ℓ^1) (Proposition 8.3 iv)) and μ, ν belong to $M_\infty = (C_0, \ell^1)^*$ (Theorem 3.2) we conclude that $\mu = \nu$.

ii) f, g as measures belong to M_q. That is, $\mu = fdx, \nu = gdx$

belong to M_q, hence to M_∞ by (2.9). Since the Fourier transform of f as a function of (L^p, ℓ^q) and as a measure of M_q is the same (§6, p.79) $\hat{\mu} = (fdx)^\wedge = \hat{f} = \hat{g} = (gdx)^\wedge = \hat{\nu}$. By the uniqueness of the Fourier transform $\hat{\mu} = \hat{\nu}$ as M_q and M_∞ transform. Therefore by i) $\mu = \nu$. This implies that $fdx = gdx$ and we conclude that $f = g$ a.e.†

THEOREM 13.23. Let $1 < p, q \leq \infty$. If $T: L^1 \longrightarrow (L^p, \ell^q)$ is a linear operator then the following are equivalent:

i) $T \in c\text{-}M(L^1,(L^p, \ell^q))$.

ii) There exists a unique φ in $A_c(\hat{G})^*$, in $(L^{q'}, \ell^{p'})(\hat{G})$ if $1 < p, q \leq 2$, in $(L^{q'}, \ell^2)(\hat{G})$ if $1 < q \leq 2$, $2 < p \leq \infty$, such that $(Tf)^\wedge = \varphi\hat{f}$ for all $f \in L^1$. (See §5 pp. 71 and 73).

PROOF. If $T \in c\text{-}M(L^1,(L^p, \ell^q))$ then by Theorem 13.18, there exists a unique $\mu \in (L^p, \ell^q)$ such that $Tf = \mu * f$ for all $f \in L^1$. Then by Proposition 6.14, $(Tf)^\wedge = \hat{\mu}\hat{f} = \varphi\hat{f}, \hat{\mu} = \varphi$ belongs to $(L^{q'}, \ell^{p'})$ if $1 < p, q \leq 2$ (Theorem 5.7), to $(L^{q'}, \ell^2)$ if $1 < q \leq 2$, $2 < p \leq \infty$ (Remark 5.8). Clearly φ is unique. Therefore i) implies ii).

If ii) holds then for f, g in L^1

$$(Tf*g)^\wedge = (Tf)^\wedge\hat{g} = (\varphi\hat{f})\hat{g} = \varphi(\hat{f}\hat{g}) = \varphi(f*g)^\wedge = (T(f*g))^\wedge.$$

By Theorem 13.22, $Tf*g = T(f*g)$ for all f, g in L^1. That is, T commutes with convolution.

To prove that T is continuous, take $\{f_n\}$, f in L^1 such that $\lim \|f_n - f\|_1 = 0$ and suppose that $\lim \|Tf_n - g\|_{pq} = 0$.

In any case we will think of φ as a linear functional on $A_c(\hat{G})$. So for $\psi \in A_c(\hat{G})$, $\hat{\psi} \in (C_0, \ell^1)$ (Lemma 6.4) and by Definition 6.6 and the Hölder inequality for amalgams (Theorem 3.1) we have that

$$\begin{aligned}|<\psi,(Tf)^{\wedge}-\hat{g}>| &\leq |<\psi,(Tf)^{\wedge}-(Tf_n)^{\wedge}>| + |<\psi,(Tf_n)^{\wedge}-\hat{g}>| \\ &= |<\psi,\varphi\hat{f}-\varphi\hat{f}_n>| + |<\check{\psi}',Tf_n-g>| \\ &\leq |<\psi(f-f_n)^{\wedge},\varphi>| + ||\check{\psi}||_{p'q'}\,||Tf_n-g||_{pq} \\ &\leq ||\varphi||\;||\psi(f-f_n)^{\wedge}||_{A_c} + ||\check{\psi}||_{\infty 1}\,||Tf_n-g||_{pq} \\ &= ||\varphi||\;||\check{\psi}*(f-f_n)||_1 + ||\check{\psi}||_{\infty 1}\,||Tf_n-g||_{pq} \\ &= ||\varphi||\;||\check{\psi}||_1\;||f-f_n||_1 + ||\check{\psi}||_{\infty 1}\,||Tf_n-g||_{pq}\end{aligned}$$

This implies that $<\psi,(Tf)^{\wedge}-\hat{g}> = 0$ for all $\psi \in A_c(\hat{G})$. Hence $(Tf)^{\wedge} = \hat{g}$ and by Theorem 13.22 $Tf = g$ a.e. Therefore by the Closed Graph Theorem T is continuous and ii) implies i).†

<u>COROLLARY 13.24</u>. Let $1 < q < \infty$. If $T: L^1 \longrightarrow (C_0, \ell^q)$ $(T: L^1 \longrightarrow (L^q, c_0))$ is a linear operator, then the following are equivalent:

i) $T \in M(L^1,(C_0, \ell^q))$ $(M(L^1,(L^q, c_0)))$

ii) There exists a unique φ in $A_c(\hat{G})^*$ $(A_c(\hat{G})^*)$, in $(L^{q'}, \ell^2)$ if $1 < q \leq 2$, such that $(Tf)^{\wedge} = \varphi\hat{f}$ for all $f \in L^1$.

<u>PROOF</u>. Corollary 13.2 and Theorem 13.23 (remember that $(C_0, \ell^q) \subset (L^{\infty}, \ell^q)$ and $(L^q, c_0) \subset (L^q, \ell^{\infty})$).†

To characterize the c-multipliers from L^1 to M_q we need the following lemma.

LEMMA 13.25. Let $\mu \in M_q$, $1 \leq q \leq \infty$.

i) If $f \in L^1$ and $f \geq 0$ then $f*\mu^{\#} = (f*|\mu|)^{\#}$.

ii) If $f = \alpha\chi_E$ where α is a nonnegative real number and E is a measurable subset of G then $(f*\mu)^{\#} = f*\mu^{\#}$.

PROOF.

i)
$$f*\mu^{\#}(x) = \int f(x-t)\,\mu^{\#}(t)\,dt = \int f(x-t)\,|\mu|(t+L)\,dt$$
$$= \int\int f(x-t)\,\chi_{t-L}(s)\,d|\mu|(s)\,dt$$
$$= \int\int f(u-s)\,\chi_L(u-x)\,du\,d|\mu|(s)$$
$$= \int\int f(u-s)\,\chi_{x+L}(u)\,du\,d|\mu|(s)$$
$$= \int \chi_{x+L}(u)\int f(u-s)\,d|\mu|(s)\,du$$
$$= \int_{x+L} f*|\mu|(u)\,du = (f*|\mu|)^{\#}(x).$$

ii) First we note that $f*\mu(t) = \alpha\int\chi_E(t-x)\,d\mu(x) = \alpha\,\mu(t-E)$.

If μ is real-valued then by the definition of $(f*\mu)^+$ (see p.117) we have that for $t \in G$

$$(f*\mu)^+(t) = \sup(f*\mu, 0) = \alpha\sup(\mu(t-E), 0) = \alpha\,\mu^+(t-E)$$
$$= \alpha\,(\chi_E*\mu^+(t)) = f*\mu^+(t).$$

Similarly $(f*\mu)^- = f*\mu^-$. Hence

(6) $$|f*\mu| = f*\mu^+ + f*\mu^- = f*|\mu|.$$

Thus, by part i) $(f*\mu)^{\#} = (f*|\mu|)^{\#} = f*\mu^{\#}$.

If μ is complex-valued then $\mu = \mu_1 + i\mu_2$ where μ_i, $i = 1, 2$ is a real valued measure in M_q. So, by (6)

$f*|\mu| = f*|\mu_1| + f*|\mu_2| = |f*\mu_1| + |f*\mu_2| = |f*\mu|$.

Again by part i) $(f*\mu)^{\#} = f* \mu^{\#}$. †

THEOREM 13.26. Let $1 \leq q \leq \infty$. If $T: L^1 \longrightarrow M_q$ ia a linear operator then the following are equivalent:

i) $T \in c\text{-}M(L^1, M_q)$.

ii) There exists a unique $\mu \in M_q$ such that $Tf = \mu * f$ for all $f \in L^1$.

The correspondence between T and μ defines an isometric linear isomorphism from $c\text{-}M(L^1, M_q)$ onto $M_q^{\#}$.

PROOF. The case $q = \infty$ was proved by Feichtinger [24, Theorem 1.3]. It should be mention that the definition of a multiplier used throughout [24] corresponds to what we call c-multiplier and not the one given in [24, p. 342].

Assume $1 \leq q < \infty$. By Theorem 13.1, i) implies ii).

Now, if $\mu \in M_q$ and $Tf = \mu * f$ $(f \in L^1)$ then clearly T commutes with convolution and by Corollary 4.6

$||Tf||_q^{\#} = ||f*\mu||_q^{\#} = ||f*\mu||_{1q}^{\#} \leq ||f||_1 \; ||\mu||_q^{\#}$ for all $f \in L^1$.

This shows that T is continuous, hence $T \in c\text{-}M(L^1, M_q)$, and $||T|| \leq ||\mu||_q^{\#}$. Since $\mu^{\#} \in L^q$ (see Theorem 1.21) and q is finite, given $\varepsilon > 0$, there exists a neighborhood U of 0 such that $||h*\mu^{\#} - \mu^{\#}||_q < \varepsilon$ for all $h \in L^1$, $||h||_1 = 1$, $h \geq 0$ and $\int_{G \sim U} h = 0$

[37, Theorem 20.15]. Let $f = 1/m(U)\, \chi_U$. Clearly f satisfies all the above conditions and therefore $||f*\mu^{\#} - \mu^{\#}||_q < \varepsilon$.

By Lemma 13.25 we have that

$$||Tf||_q^{\#} = ||f*\mu||_{1_q}^{\#} = ||(f*\mu)^{\#}||_q = ||f*\mu^{\#}||_q > ||\mu^{\#}||_q - \varepsilon.$$

Since $||\mu^{\#}||_q = ||\mu||_q^{\#}$ we conclude that $||T|| = ||\mu||_q^{\#}$. †

REMARK 13.27. If $T \in c\text{-}M(L^1, M_q)$, $1 \leq q \leq \infty$, then $Tf = \mu*f$ for some $\mu \in M_q$. So by Corollary 4.4 $Tf \in (L^1, \ell^q)$ for all $f \in L^1$. Hence $c\text{-}M(L^1,M_q) \subset c\text{-}M(L^1,(L^1, \ell^q))$. Now, since $f \longmapsto fdx$ from $(L^1, \ell^q)^{\#}$ to $M_q^{\#}$ is a natural embedding which is an isometry we conclude that $c\text{-}M(L^1,M_q) = c\text{-}M(L^1,(L^1, \ell^q))$. This implies the next theorem which is already known [42, Corollary 6.3].

THEOREM 13.28. Let $1 \leq q \leq \infty$. If $T: L^1 \longrightarrow (L^1, \ell^q)$ is a linear operator then the following are equivalent:

i) $T \in c\text{-}M(L^1,(L^1, \ell^q))$.

ii) There exists a unique $\mu \in M_q$ such that $Tf = \mu*f$ for all $f \in L^1$.

The correspondence between T and μ defines an isometric linear isomorphism from $c\text{-}M(L^1,(L^1, \ell^q)^{\#})$ onto $M_q^{\#}$.

REMARK. An alternative approch to Theorem 13.28 might be to use Feichtinger's Theorem 1.5 in [24] and show that $(L^1, \ell^q)^{\sim} = M_q$.

REMARK 13.29. From Theorems 13.26 and 13.28, for $1 \leq q \leq \infty$, $c\text{-}M(L^1,M_q) \subset M(L^1,M_q)$ and $c\text{-}M(L^1,(L^1, \ell^q)) \subset M(L^1,(L^1 \ell^q))$. Then by Theorem 12.9, $c\text{-}M(L^1,(L^1, \ell^q)) = M(L^1,(L^1 \ell^q))$ for $1 \leq q < \infty$. Therefore for $1 \leq q < \infty$

$$M_q \approx c\text{-}M(L^1,M_q) \subseteq c\text{-}M(L^1,(L^1, \ell^q)) = M(L^1,(L^1, \ell^q)).$$

We do not know if for $1 \leq q \leq \infty$, $c\text{-}M(L^1,M_q) = M(L^1,M_q)$.

THEOREM 13.30. Let $1 \leq q < \infty$. If $T: L^1 \longrightarrow (L^1, \ell^q)$ is a linear operator then the following are equivalent:

i) $T \in c\text{-}M(L^1,(L^1, \ell^q))$.

ii) There exists a unique φ in $A_c(\hat{G})^*$, in $(L^{q'}, c_0)(\hat{G})$ if $1 \leq q \leq 2$, such that $(Tf)^\wedge = \varphi\hat{f}$ for all $f \in L^1$.

The proof is similar to the proof of Theorem 13.23.

THEOREM 13.31. Let $1 \leq q \leq \infty$. If $T: L^1 \longrightarrow M_q$ is a linear operator then the following are equivalent:

i) $T \in c\text{-}M(L^1,M_q)$

ii) There exists a unique φ in $A_c(\hat{G})^*$, in $(L^{q'}, \ell^\infty)(\hat{G})$ if $1 \leq q \leq 2$, such that $(Tf)^\wedge = \varphi\hat{f}$ for all $f \in L^1$.

The proof of this theorem is also similar to the proof of Theorem 13.23 but using the Hölder inequality as in Theorem 3.2.

THEOREM 13.32. Let $\mu \in M_q$, $1 \leq q \leq \infty$, and $f \in (L^1, \ell^s)$, $1 \leq q' \leq s \leq \infty$. If $\hat{\mu} = \hat{f}$ on $A_c(\hat{G})^*$ then $f \in (L^1, \ell^q)$, μ is the image of f under the embedding of Remark 13.27 and $d\mu = fdx$.

PROOF. Let T_f, T_μ be the c-multipliers in $c\text{-}M(L^1,(L^1, \ell^q))$, $c\text{-}M(L^1,M_q)$ associated to f and μ respectively. Then by hypothesis $(T_f g)^\wedge = (f*g)^\wedge = \hat{f}\hat{g} = \hat{\mu}\hat{g} = (\mu*g)^\wedge = (T_\mu g)^\wedge$ for all $g \in L^1$ (see Definition 6.13). Since $\mu*g \in (L^1, \ell^q)$ and $(L^1, \ell^q) \subset (L^1, \ell^s)$, $(T_f g)^\wedge = (T_\mu g)^\wedge$ as (L^1, ℓ^s) transform for all $g \in L^1$. By Theorem 13.22 $T_f g = f*g = \mu*g = T_\mu g$ for all $g \in L^1$. This implies that $d\mu = fdx$ and therefore $||\mu||^\#_q = ||f||^\#_{1q}$. Hence $f \in (L^1, \ell^q)$.†

REMARK 13.33. In the particular case $\mu \in M_q$, $1 \leq q \leq 2$, $f \in (L^1, \ell^2)$, Theorem 13.32 says that if $\hat{\mu} = \hat{f}$ a.e. on $\hat{G}$ then f belongs to (L^1, ℓ^q) and μ is the image of f under the embedding of remark 13.27 and $d\mu = fdx$. This generalizes the result in [37, Theorem 31.33] because when $q = 1$ and $f \in L^p$, $1 \leq p \leq 2$, then f belongs to (L^1, ℓ^2) as $(L^p, \ell^p) \subset (L^1, \ell^p) \subset (L^1, \ell^2)$. So by Theorem 13.32 $f \in L^1$, μ is absolutely continuous and $d\mu = fdx$.

COROLLARY 13.34. Let $1 \leq q \leq \infty$, $1 \leq q \leq s \leq \infty$. If $g \in (L^1, \ell^q)$, $f \in (L^1, \ell^s)$ and $\hat{g} = \hat{f}$ on $A_c(\hat{G})$ then $f = g$ a.e.

PROPOSITION 13.35. Let $1 \leq p \leq r \leq \infty$, $1 \leq s \leq q \leq \infty$. If $f \in (L^p, \ell^q)$, $g \in (L^r, \ell^s)$ and $\hat{f} = \hat{g}$ on $A_c(\hat{G})$, then $f = g$ a.e.

PROOF. Similarly to Theorem 13.32, let T_f, T_g be in $c\text{-}M(L^1,(L^p, \ell^q))$, $c\text{-}M(L^1,(L^r, \ell^s))$ associated to f, g respectively.

By hypothesis $(T_f h)^\wedge = (f*h)^\wedge = \hat{f}\hat{h} = \hat{g}\hat{h} = (g*h)^\wedge = (T_g h)^\wedge$ for all $h \in L^1$. Since $(L^r, \ell^s) \subset (L^p, \ell^q)$, $(T_f h)^\wedge = (T_g h)^\wedge$ as (L^p, ℓ^q) transform, so by Theorem 13.22, for all $h \in L^1$, $T_f h = f*h = g*h = T_g h$. This implies that $f = g$ a.e. †

We will end this section with the following conjecture.

By Theorem 13.18 $c\text{-}M(L^1,(L^\infty, \ell^q)) \subset M(L^1,(L^\infty, \ell^q))$ and $c\text{-}M(L^1,(L^q, \ell^\infty)) \subset M(L^1,(L^q, \ell^\infty))$ for $1 < q < \infty$. We believe that $c\text{-}M(L^1,(L^\infty, \ell^q)) \neq M(L^1,(L^\infty, \ell^q))$ and $c\text{-}M(L^1,(L^q, \ell^\infty)) \neq M(L^1,(L^q,\ell^\infty))$ for all $1 < q < \infty$.

§ 14. MULTIPLIERS FROM M_1 TO AMALGAM SPACES AND SPACES OF MEASURES $\dot{M}_q$

The c-multipliers from $M_1(G)$ to any Banach M_1-module A (Definition 4.9) are easily characterized. Indeed, if $T: M_1 \longrightarrow A$ is a linear operator and T has the form $T\nu = \mu * \nu$ $(\nu \in M_1)$ for some $\mu \in A$ then by the properties of convolution and M_1-module (see (B-1) p. 60) T is a c-multiplier from M_1 to A.

Conversely if T commutes with convolution and δ is the identity in M_1 then, for $\nu \in M_1$, $T\nu = T(\delta * \nu) = T\delta * \nu = \mu * \nu$ with $\mu = T\delta$.

Hence, $T \in c\text{-}M(M_1,A)$ iff there exists a unique $\mu \in A$ such that $T\nu = \mu * \nu$ for all $\nu \in M_1$.

By §4 p. 60 we immediately have the following theorem.

<u>THEOREM 14.1</u>. Let A be any amalgam space (L^p, ℓ^q), (C_0, ℓ^q), (L^p, c_0), $1 \leq p, q \leq \infty$, or a measure space M_s, $1 \leq s \leq \infty$. If $T: M_1 \longrightarrow A$ is a linear operator then the following are equivalent:

i) $T \in c\text{-}M(M_1,A)$.

ii) There exists a unique $\mu \in A$ such that $T\nu = \mu * \nu$ for all $\nu \in M_1$.

iii) There exists a unique $\varphi \in A_c(\hat{G})^*$ such that $(T\nu)^\wedge = \varphi\hat{\nu}$ for all $\nu \in M_1$.

$\varphi \in (L^{q'}, \ell^{p'})(\hat{G})$ if $A = (L^p, \ell^q)$, $1 \leq p, q \leq 2$.

$\varphi \in (L^{q'}, \ell^2)(\hat{G})$ if $A = (L^p, \ell^q)$, $2 < p \leq \infty$, $1 \leq q \leq 2$, or $A = (C_0, \ell^q)$, $1 \leq q \leq 2$; $\varphi \in (L^{q'}, \ell^\infty)$ if $A = M_s$, $1 \leq s \leq 2$.

The correspondence between T and μ establishes an isometric linear isomorphism from $c\text{-}M(M_1,A')$ onto A' (see §13 p. 166).

PROOF. As we have already said, i) is equivalent to ii). If ii) holds then by Proposition 6.14 i) $(T\nu)^\wedge = (\mu*\nu)^\wedge = \hat{\mu}\hat{\nu} = \varphi\hat{\nu}$ for all $\nu \in M_1$, where $\varphi = \hat{\mu}$. Clearly φ is unique. The remaining clauses in iii) follow from Theorem 5.7 and Remark 5.8.

Now suppose that $(T\nu)^\wedge = \varphi\hat{\nu}$ for all $\nu \in M_1$, $\varphi \in A_c(\hat{G})^*$. Then for ν, η in M_1

$$(T\nu*\eta)^\wedge = (T\nu)^\wedge\hat{\eta} = (\varphi\hat{\nu})\hat{\eta} = \varphi(\hat{\nu}\hat{\eta}) = \varphi(\nu*\eta)^\wedge = (T(\nu*\eta))^\wedge.$$

Hence by Theorem 13.22 $T\nu*\eta = T(\nu*\eta)$. This means that T commutes with convolution. Therefore $T\nu = \mu*\nu$ $(\nu \in M_1)$ with $\mu = T\delta$ as we saw at the beginning of the section. This shows that ii) is equivalent to iii).

Finally, by the M_1-modularity of A', for $\nu \in M_1$,

$$||T\nu||_{A'} = ||\mu*\nu||_{A'} \leq ||\mu||_{A'}\, ||\nu||_1.$$

Hence $||T|| \leq ||\mu||_{A'}$. Since $\mu = T\delta$ and $||\delta||_1 = 1$, $||\mu||_{A'} \leq ||T||$. Therefore $||T|| = ||\mu||_{A'}$ and the proof is complete.$\dagger$

REMARK 14.2. It follows from Theorem 14.1 that $c\text{-}M(M_1,A) \subset M(M_1,A)$. But we know that there exists a T in $M(M_1,M_1)$ such that T is not defined by the convolution with an element of M_1 [29, p. 94]. So $c\text{-}M(M_1,M_q) \neq M(M_1,M_q)$ for $1 < q \leq \infty$. Indeed, if $c\text{-}M(M_1,M_q) = M(M_1,M_q)$ then for $T \in M(M_1,M_1)$, T belongs to $M(M_1,M_q)$ since $M_1 \subset M_q$ by (2.9). Therefore $T \in c\text{-}M(M_1,M_q)$. So $T\nu = \mu*\nu$ $(\nu \in M_1)$ where $\mu = T\delta$, hence $\mu \in M_1$. This contradiction proves our claim. (We do not know if $c\text{-}M(M_1,A) = M(M_1,A)$ for some amalgam space A).

However the situation is different when we consider $M_1{}^w$ (see §12 p. 144).

<u>THEOREM 14.3</u>. Suppose A is an amalgam or measure space of type q such that A satisfies the following two conditions:

(14-1) There exists a Banach space B such that $B^* = A$.

(14-2) For all $f \in B$ and $g \in A$, $f*g \in C_0$.

If $T: M_1{}^w \longrightarrow A^w$ is a linear operator then the following are equivalent:

i) $T \in M(M_1{}^w, A^w)$.

ii) $T \in c\text{-}M(M_1{}^w, A^w)$.

iii) There exists a unique $\mu \in A$ such that $T\nu = \mu*\nu$ for all $\nu \in M_1$.

iv) There exists a unique $\varphi \in A_c(\hat{G})^*$ such that $(T\nu)^\wedge = \varphi\hat{\nu}$ for all $\nu \in M_1$.

The correspondence between T and μ defines a linear isomorphism from $M(M_1{}^w, A^w)$ onto A.

<u>PROOF</u>. By Theorem 12.9, i) implies ii). As in Theorem 14.1 ii) implies iii) with $\mu = T\delta$ and iii) is equivalent to iv).

Now suppose iii). By the properties of convolution T commutes with translations.

To prove that T is continuous take $\{\nu_n\}$, ν in M_1 such that $\lim \nu_n = \nu$ in $M_1{}^w$. That is, for all $h \in C_0$, $\lim < \nu_n, h > = < \nu, h >$. Let $f \in B$. By condition (14-2), $f*\mu \in C_0$ and we have that

$$< T\nu, f > = < \mu*\nu, f > = < \nu, f*\mu > = \lim < \nu_n, f*\mu > = \lim < \nu_n*\mu, f >$$
$$= \lim < T\nu_n, f >.$$

This implies that T is continuous and therefore $T \in M(M_1{}^w, A^w)$. The rest of the theorem is clear.†

PROPOSITION 14.4. The following spaces satisfy conditions (14-1) and (14-2).

a) (L^p, ℓ^q) $1 < p, q < \infty$

b) M_q $1 < q \leq \infty$

c) (L^p, ℓ^1) $1 < p < \infty$

d) (L^p, ℓ^∞) $1 < p < \infty$

e) (L^∞, ℓ^q) $1 \leq q < \infty$

PROOF. a) and b) follow from Theorems 4.7 and 4.8 with $B = (L^{p'}, \ell^{q'})$ and $B = (C_0, \ell^{q'})$ respectively.

c) (L^p, ℓ^1) satisfies (14-1) with $B = (L^{p'}, c_0)$ (Theorem 3.1). Let $h \in C_c$ with support E and $g \in (L^p, \ell^1)$. Then there exists a sequence $\{g_n\}$ in L_c^p such that $\lim g_n = g$ in (L^p, ℓ^1) (Theorem 3.6). By a), $g_n*h \in C_0$ and this implies that $g*h \in C_0$, because C_0 is a closed subspace of L^∞ and $\lim g_n*h = g*h$ in L^∞. Since C_c is dense in $(L^{p'}, c_0)$ and $f \longmapsto f*g$ from $(L^{p'}, c_0)$ to L^∞ is continuous, we conclude that $f*g \in C_0$.

d) is similar to c).

e) (L^∞, ℓ^1) satisfies (14-1) with $B = (L^1, c_0)$. Let f in (L^1, c_0) and $g \in (L^\infty, \ell^1)$. By Theorem 13.13, g defines a c-multiplier T_g from L^1 to (C_0, ℓ^1). That is, $T_g h = g*h$ belongs to (C_0, ℓ^1) for all $h \in L^1$. In particular for all $\phi \in C_c$, $g*\phi \in (C_0, \ell^1)$ and hence is in C_0. Since C_c is dense in (L^1, c_0) and convolution with g is a continuous linear map from (L^1, c_0) into L^∞ we conclude that $f*g \in C_0$.

(L^∞, ℓ^q) satisfies (14-1) with $B = (L^1, \ell^{q'})$ for $1 < q < \infty$, and by Theorem 4.7, (L^∞, ℓ^q) satisfies (14-2). †

REMARK 14.5. We should mention that in Theorem 14.3 the function φ of iv) belongs to $(L^{q'}, \ell^{p'})$ if $A = (L^p, \ell^q)$, $1 < p \leq 2$, $1 \leq q \leq 2$; to $(L^{q'}, \ell^2)$ if $A = (L^p, \ell^q)$, $2 < p \leq \infty$, $1 \leq q \leq 2$; or to $(L^{q'}, \ell^\infty)$ if $A = M_q$, $1 \leq q \leq 2$. (See the proof of Theorem 14.1).

§ 15. MULTIPLIERS FROM AMALGAM SPACES AND SPACES OF MEASURES

M_q TO L^∞

PROPOSITION 15.1. Let A be (L^p, ℓ^q), $1 < p, q < \infty$, or (L^p, c_0) $1 < p < \infty$. Then A^* has the following properties:

(15-1) $A^* = (L^{p'}, \ell^{q'})$, $1 < p', q' < \infty$, or $(L^{p'}, \ell^1)$, $1 < p' < \infty$ ...

(15-2) For all $f \in A$ and $g \in A^*$ $f*g \in C_0$ and

$$||f*g||_\infty \leq ||f||_A \, ||g||_{A^*}.$$

(15-3) If $T \in M(L^1, A^*)$ then there exists a unique $\mu \in A^*$ such that $Tf = \mu * f$ for all $f \in L^1$.

PROOF. (15-1) follows from Theorem 3.1. (15-2) is a direct consequence of the Hölder inequality for amalgams (Theorem 3.1) and Proposition 14.4. (15-3) follows from Corollary 13.2.†

The next result extends Edwards' theorem for L^p spaces [21, Theorem 3], and its proof is similar to his.

THEOREM 15.2. Let A be as in Proposition 15.1. If $T: A \longrightarrow L^\infty$ is a linear operator then the following are equivalent:

i) $T \in M(A, L^\infty)$.

ii) There exists a unique $\mu \in A^*$ such that $Tf = \mu * f$ for all $f \in A$.

Moreover, the correspondence between T and μ defines and isometric isomorphism from $M(A, L^\infty)$ onto A^*, $M(A, L^\infty) = M(A, C_0)$ and $\mu = T^*\delta$

where T^* is the adjoint operator of T and δ is the identity of M_1.

PROOF. If ii) holds then T commutes with translations and by (15-2) T is continuous. Hence ii) implies i).

Suppose i). Let $T^*: L^{\infty *} \longrightarrow A^*$ be the adjoint operator of T. We shall see that T^* restricted to L^1 belongs to $M(L^1,A^*)$. Since $L^1 \subset L^{\infty *}$, $< f,T^*h > = < Tf,h >$ for all $h \in L^1$ and $f \in A$. Then for $s \in G$, $f \in A$ and $h \in L^1$

$$< f,T^*\tau_s h > = < Tf,\tau_s h > = < \tau_s Tf,h > = < T\tau_s f,h > = < \tau_s f,T^*h >$$
$$= < f,\tau_s T^*h >.$$

Therefore $T^*\tau_s h = \tau_s T^*h$. That is, T^* commutes with translations. This implies that $T^*|L^1 \in M(L^1,A^*)$. So by (15-3)

(1) There exists a unique $\mu \in A^*$ such that $T^*f = \mu * f$ for all $f \in L^1$.

Now take $f \in A$, $h \in L^1$ and consider the following

(2) $< Tf,h > = < f,T^*h > = < f,\mu * h > = < \mu * f,h >.$

Therefore $Tf = \mu * f$ for all $f \in A$. Clearly μ is unique. Hence i) implies ii).

Now, by (15-2) $M(A,L^\infty) = M(A,C_0)$ and if $T \in M(A,C_0)$ then $||T|| \leq ||\mu||_{A*}$ and $T^*: M_1 \longrightarrow A^*$.

To prove that $T^* \in c\text{-}M(M_1,A^*)$ we take ν, η in M_1 and $h \in A$ and we see that

(3) $< T^*(\nu * \eta),h > = < \nu * \eta,Th > = < \nu * \eta,\mu * h > = < \nu,\mu * h * \eta >$
$$= < \nu,T(h * \eta) > = < T^*\nu,h * \eta > = < T^*\nu * \eta,h >.$$

Therefore $T^*(\nu * \eta) = T^*\nu * \eta$. This implies that $T^* \in c\text{-}M(M_1,A^*)$. So by Theorem 14.1 $T^*\nu = T^*\delta * \nu$ for all $\nu \in M_1$. In particular for

$f \in L^1$, $T^*f = T^*\delta * f$. Then it follows from the uniqueness of μ in (1) that $\mu = T^*\delta$ and we have that

$||\mu||_{A*} = ||T^*\delta||_{A*} \leq ||T^*|| \; ||\delta||_1 = ||T^*|| = ||T||$.

By ii) and (15-2) we conclude that $||T|| \leq ||\mu||_{A*}$. Therefore $||T|| = ||\mu||_{A*}$ and the proof is complete.†

THEOREM 15.3. Let $1 < q \leq \infty$. If $T: (L^1, \ell^q) \longrightarrow L^\infty$ is a linear operator then the following are equivalent:

i) $T \in M((L^1, \ell^q), L^\infty)$.

ii) There exists a unique $\mu \in (L^\infty, \ell^{q'})$ such that $Tf = \mu * f$ for all $f \in (L^1, \ell^q)$.

The correspondence between T and μ defines a continuous linear isomorphism from $M((L^1, \ell^q), L^\infty)$ onto $(L^\infty, \ell^{q'})$. If $1 < q < \infty$ then $M((L^1, \ell^q), L^\infty) = M((L^1, \ell^q), C_0)$, $\mu = T^*\delta$ where T^* is the adjoint operator of T and the isomorphism is an isometry.

PROOF. If ii) holds then by the properties of convolution T commutes with translations and T is bounded because

$||f*\mu||_\infty \leq ||f||_{1q} \; ||\mu||_{\infty q'}$, for all $f \in (L^1, \ell^q)$, $\mu \in (L^\infty, \ell^{q'})$ (Theorem 3.1).

Suppose i). Since for all $1 < q \leq \infty$ $(L^1, \ell^q)^* \subset (L^\infty, \ell^{q'})$ (note that $(L^1, c_0)^* = (L^\infty, \ell^1)$ and $(L^1, \ell^\infty)^* \subset (L^1, c_0)^*$) and $(L^\infty, \ell^{q'}) \subset L^\infty$ we can consider the adjoint operator $T^*: L^{\infty *} \longrightarrow (L^1, \ell^q)^*$ of T, to be a linear continuous operator from $L^{\infty *}$ into L^∞. So, $T^*: L^{\infty *} \longrightarrow L^\infty$ and $T^*|L^1$ commutes with translations because if $s \in G$ and $f, g \in L^1$ then

$$< T^*\tau_s f,g > = < \tau_s f,Tg > = < f,\tau_s Tg > = < f,T\tau_s g > = < T^*f,\tau_s g >$$
$$= < \tau_s T^*f,g >.$$

This implies that $T^*\tau_s f = \tau_s T^* f$ for all $f \varepsilon L^1$. Using the same argument as in Theorem 12.9 we shall prove that $T^*|L^1$ commutes with convolution. Let f, g in L^1. Then

$$< T^*f*g,h > = \int T^*f*g(t)\ h(-t)\ dt = \int\int T^*f(t - s)\ g(s)\ ds\ h(-t)\ dt$$
$$= \int g(s) \int T^*f(t - s)\ h(-t)\ dt\ ds$$
$$= \int g(s) \int \tau_s T^*f(t)\ h'(t)\ dt\ ds$$
$$= \int g(s) \int T^*\tau_s f(t)\ h'(t)\ dt\ ds$$
$$= \int g(s) \int \tau_s f(t)\ Th'(t)\ dt\ ds$$
$$= \int\int f(t - s)\ g(s)\ ds\ Th'(t)\ dt$$
$$= \int f*g(t)\ Th'(t)\ dt = \int T^*(f*g)\ h(-t)\ dt$$
$$= < T^*(f*g),h >.$$

We can apply Fubini's theorem because $Th \varepsilon L^\infty$ and f, $g \varepsilon L^1$.

Hence $T^*f*g = T^*(f*g)$ and we conclude that $T^*|L^1$ belongs to $c\text{-}M(L^1,(L^\infty, \ell^{q'}))$. By Corollary 13.14, Theorem 13.15 and Theorem 13.18 there exists a unique $\mu \varepsilon (L^\infty, \ell^{q'})$ such that $T^*f = \mu*f$ for all $f \varepsilon L^1$. Clearly μ is unique and as in the proof of Theorem 15.2 it follows that $Tf = \mu*f$. Therefore i) implies ii).

It is clear that the relation $Tf = \mu*f$ defines a linear isomorphism from $M((L^1, \ell^q),L^\infty)$ onto $(L^\infty, \ell^{q'})$ and $||T|| \leq ||\mu||_{\infty q'}$.

If $1 < q < \infty$ then $(L^\infty, \ell^{q'})$ satisfies condition (14-2) of Theorem 14.3 (Proposition 14.4). Hence $M((L^\infty, \ell^q),L^\infty) = M((L^\infty, \ell^q),C_0)$. That is, $T: (L^1, \ell^q) \longrightarrow C_0$ and this implies that $T^*: M_1 \to (L^\infty,\ell^{q'})$. Using the argument (3) of the proof of Theorem 15.2 we have that $\mu = T^*\delta$. So, $||\mu||_{\infty q'} = ||T^*\delta||_{\infty q'} \leq ||T^*|| \; ||\delta||_1 = ||T^*|| = ||T||$. Therefore $||T|| = ||\mu||_{\infty q'}$.†

<u>THEOREM 15.4</u>. Let $1 < p < \infty$. If $T: (L^p, \ell^1) \longrightarrow L^\infty$ is a linear operator then the following are equivalent:

i) $T \in M((L^p, \ell^1),L^\infty)$.

ii) $T(f*g) = Tf*g$ for all $f \in (L^p, \ell^1)$ and $g \in L^1$.

iii) There exists a unique $\mu \in (L^{p'}, \ell^\infty)$ such that $Tf = \mu*f$ for all $f \in (L^p, \ell^1)$.

The correspondence between T and μ defines an isometric isomorphism from $M((L^p, \ell^1),L^\infty)$ onto $(L^{p'}, \ell^\infty)$, $M((L^p, \ell^1),L^\infty)$ is equal to $M((L^p, \ell^1),C_0)$ and $\mu = T^*\delta$ where T^* is the adjoint operator of T.

<u>PROOF</u>. Let $T^*: L^{\infty*} \longrightarrow (L^{p'}, \ell^\infty)$ be the adjoint operator of T.

Suppose i). Using the same argument as in Theorem 12.9 we have that for $f \in (L^p, \ell^1)$ and $g, h \in L^1$ $< Tf*g,h > = < T(f*g),h >$. Therefore $Tf*g = T(f*g)$ for all $f \in (L^p, \ell^1)$ and $g \in L^1$.

Suppose ii). Take $f \in (L^p, \ell^1)$, $g, h \in L^1$. So

$$(4) \qquad < T^*g*h,f > = < T^*g,h*f > = < g,T(h*f) > = < g,Tf*h >$$
$$= < g*h,Tf > = < T^*(g*h),f >.$$

This implies that $T^*g*h = T^*(g*h)$ for all $h, g \in L^1$. Therefore $T^*|L^1 \in c\text{-}M(L^1,(L^{p'}, \ell^\infty))$. By Theorem 13.18 there exists a unique $\mu \in (L^{p'}, \ell^{\infty})$ such that $T^{*}f = \mu * f$ for all $f \in L^{1}$.

Similarly to (2) of the proof of Theorem 15.2, $Tf = \mu * f$ for all $f \in (L^{p}, \ell^{1})$. Hence ii) implies iii).

If iii) holds then by the properties of convolution T commutes with convolution and by Theorem 3.1

$||Tf||_{\infty} = ||\mu * f||_{\infty} \leq ||\mu||_{p'\infty} \, ||f||_{p1}$.

Hence T is continuous and $||T|| \leq ||\mu||_{p'\infty}$. Therefore iii) implies i).

Since $(L^{p'}, \ell^{\infty})$ satisfies condition (15-2) of Theorem 14.3 $M((L^{p}, \ell^{1}), L^{\infty}) = M((L^{p}, \ell^{1}), C_{0})$. Similarly to the argument (3) of Theorem 15.2 we have that $\mu = T^{*}\delta$ and $||T|| = ||\mu||_{p'\infty}$.†

<u>COROLLARY 15.5</u>. $M((L^{p}, \ell^{1}), L^{\infty}) = c\text{-}M((L^{p}, \ell^{1}), L^{\infty})$ for $1 < p < \infty$.

<u>PROOF</u>. By Theorem 15.4, $M((L^{p}, \ell^{1}), L^{\infty}) \subset c\text{-}M((L^{p}, \ell^{1}), L^{\infty})$ for $1 < p < \infty$. To prove the other inclusion we shall see that if T belongs to $c\text{-}M((L^{p}, \ell^{1}), L^{\infty})$ then $Tf*g = T(f*g)$ for all $f \in (L^{p}, \ell^{1})$ and $g \in L^{1}$. So by Theorem 15.4 $T \in M((L^{p}, \ell^{1}), L^{\infty})$.

Let $g \in L^{1}$. Then there exists a sequence $\{g_n\}$ in (L^{p}, ℓ^{1}) such that $\lim g_n = g$ in L^{1} (Corollary 3.8). So for $f \in (L^{p}, \ell^{1})$ $||Tf*g - Tf*g_n||_{\infty} \leq ||Tf||_{\infty} \, ||g - g_n||_{1}$. Hence $Tf*g = \lim Tf*g_n = \lim T(f*g_n)$ in L^{∞}. Since $\lim ||f*g - f*g_n||_{p1} = 0$ (Theorem 4.7) we conclude that $Tf*g = T(f*g)$ for all $f \in (L^{p}, \ell^{1})$ and $g \in L^{1}$ and the proof is complete.†

The proof of the next theorem is based on [40, Theorem 3.4.3].

THEOREM 15.6. Let $1 \leq q \leq \infty$. If $T: (C_0, \ell^q) \longrightarrow L^\infty$ is a linear operator then the following are equivalent:

i) $T \in M((C_0, \ell^q), L^\infty)$.

ii) There exists a unique $\mu \in M_{q'}$ such that $Tf = \mu * f$ for all $f \in (C_0, \ell^q)$.

Moreover, $M((C_0, \ell^q), L^\infty) = M((C_0, \ell^q), C_0)$, $\mu = T^*\delta$ where T^* is the adjoint of T and the correspondence between T and μ defines an isometric isomorphism from $M((C_0, \ell^q), L^\infty)$ onto $M_{q'}$.

PROOF. If ii) holds then T commutes with translations and T is continuous by Theorem 4.8.

Suppose i). For $f \in (C_0, \ell^q)$ and $s \in G$

$$||\tau_s Tf - Tf||_\infty = ||T(\tau_s f) - Tf||_\infty \leq ||T||\ ||\tau_s f - f||_{\infty q}.$$

Since $s \longmapsto \tau_s f$ is a continuous function from G to (C_0, ℓ^q) (Theorem 3.14 and Lemma 3.13) Tf is uniformly continuous on (C_0, ℓ^q). Hence, it makes sense to define $F(f) = Tf(0)$ $(f \in (C_0, \ell^q))$. Clearly F is linear and for $f \in (C_0, \ell^q)$

$$|F(f)| = |Tf(0)| \leq ||Tf||_\infty \leq ||T||\ ||f||_{\infty q}.$$

Therefore F is continuous. That is, $F \in (C_0, \ell^q)^*$. By Theorem 3.2 there exists a unique $\mu \in M_{q'}$ such that $F(f) = \int f(-t)\, d\mu(t)$. So, for $f \in (C_0, \ell^q)$ and $s \in G$

$$(Tf)(s) = \tau_{-s}Tf(0) = T\tau_{-s}f(0) = F(\tau_{-s}f) = \int f(s - t)\, d\mu(t) = f*\mu(s).$$

Therefore $Tf = \mu * f$ for all $f \in (C_0, \ell^q)$. By Theorem 3.2 $||T|| \leq ||\mu||_{q'}$ and μ is unique.

The rest of the proof is similar to Theorem 15.2, using the fact that (C_0, ℓ^q) has the property (15-2) of Proposition 15.2 (Theorem 4.8). †

For the special case of the Wiener algebra we have the following result.

COROLLARY 15.7.

$M((C_0, \ell^1),L^\infty) = M((C_0, \ell^1),C_0) = c\text{-}M((C_0, \ell^1),C_0) = c\text{-}M((C_0, \ell^1),L^\infty)$.

Hence $M((C_0, \ell^1),L^\infty) \approx M_\infty$.

PROOF. By Theorem 15.6 $M((C_0, \ell^1),L^\infty) = M((C_0, \ell^1),C_0)$ and $M((C_0, \ell^1),L^\infty) \subset c\text{-}M((C_0, \ell^1),L^\infty)$, $M((C_0, \ell^1),C_0) \subset c\text{-}M((C_0, \ell^1),C_0)$.

Let $T \in c\text{-}M((C_0, \ell^1),L^\infty)$ and consider its adjoint $T^*: L^{\infty *} \longrightarrow M_\infty$. For $h, f \in (C_0, \ell^1)$ and $g \in L^1$

$$< h,T^*g*f > = < h*f,T^*g > = < T(h*f),g > = < Th*f,g > = < Th,g*f >$$
$$= < h,T^*(g*f) >.$$

Hence $T^*g*f = T^*(g*f)$ for all $f \in (C_0, \ell^1)$, $g \in L^1$.

For $f \in (C_0, \ell^1)$, $g, h \in L^1$

$$< h,T(f*g) > = < T^*h,f*g > = < T^*h*g,f > = < T^*(h*g),f >$$
$$= < h*g,Tf > = < h,Tf*g >.$$

Hence $T(f*g) = Tf*g$ for all $f \in (C_0, \ell^1)$, $g \in L^1$.

For $f \in (C_0, \ell^1)$, $g, h \in L^1$

$$< f,T^*(g*h) > = < Tf,g*h > = < Tf*g,h > = < T(f*g),h >$$
$$= < f*g,T^*h > = < f,T^*h*g >.$$

Hence $T^*(f*g) = T^*f*g$ for all $f, g \in L^1$.

Therefore $T^*|L^1$ belongs to $c\text{-}M(L^1,M_\infty)$. By Remark 13.29,

$T^*|L^1 \varepsilon M(L^1,M_\infty)$. So, for $s \varepsilon G$, $f \varepsilon (C_0, \ell^1)$ and $g \varepsilon L^1$ we have that

$$< g, T\tau_s f > = < T^*g, \tau_s f > = < \tau_s T^* g, f > = < T^*\tau_s g, f > = < \tau_s g, Tf >$$
$$= < g, \tau_s Tf >.$$

Hence $T\tau_s f = \tau_s Tf$ for all $s \varepsilon G$, $f \varepsilon (C_0, \ell^1)$. Therefore $T \varepsilon M((C_0, \ell^1), L^\infty)$ and we conclude that $M((C_0, \ell^1), L^\infty) = c\text{-}M((C_0, \ell^1), L^\infty)$.

Finally, since $C_0 \subset L^\infty$ it is clear that $c\text{-}M((C_0, \ell^1), C_0) \subset c\text{-}M((C_0, \ell^1), L^\infty)$ and this ends the proof.†

<u>THEOREM 15.8</u>. Let A be any of the algebras (L^p, ℓ^1), $1 < p < \infty$, or (C_0, ℓ^1). If $T\colon A \longrightarrow L^\infty$ is a linear operator then the following are equivalent:

i) $T \varepsilon M(A, L^\infty)$.

ii) There exists a unique $\varphi \varepsilon A_c(G)^*$ such that $(Tf)^\wedge = \varphi\hat{f}$ for all $f \varepsilon A$.

<u>PROOF</u>. If $T \varepsilon M(A, L^\infty)$ then by Theorem 15.4 or Theorem 15.6, there exists a unique μ in $(L^{p'}, \ell^\infty)$ or M_∞ such that $Tf = \mu * f$ for all $f \varepsilon A$. Since $A \subset L^1$ we have by Proposition 6.14 that $(Tf)^\wedge = \hat{\mu}\hat{f} = \varphi\hat{f}$, $\varphi = \hat{\mu}$. It is clear that φ is unique. Therefore i) implies ii). The proof of ii) implies i) is the same as Theorem 13.23. Remember that by (2.4), if $\lim ||f_n - f||_A = 0$ then $\lim ||f_n - f||_1 = 0$.†

It is not known, not even for L^p spaces, if $M((L^p, \ell^q), (L^r, \ell^s))$ is isomorphic to an amalgam space (L^x, ℓ^y) for some p, q, r, s. In this direction we have a partial result, similar to [40, Theorem 5.3.3].

THEOREM 15.9. Let $1 < p \leq r < \infty$, $1 < p, s, x, y < \infty$. If $s/r = y/x$, $1/p - 1/r = 1 - 1/x$ and $1/q - 1/s = 1 - 1/y$ then there exists a continuous linear isomorphism from (L^x, ℓ^y) into $M((L^p, \ell^q),(L^r, \ell^s))$.

PROOF. By Theorem 13.18 and Theorem 15.2, the mapping $\mu \longmapsto T_\mu$, $T_\mu g = \mu * g$ on (L^x, ℓ^y) is an isometric linear isomorphism from (L^x, ℓ^y) onto $M(L^1,(L^x, \ell^y))$ and onto $M((L^{x'}, \ell^{y'}),L^\infty)$ (see Corollary 13.2). If $||T_\mu||_1$, $||T_\mu||_\infty$ are the norms of T_μ in $M(L^1,(L^x, \ell^y))$, $M((L^{x'}, \ell^{y'}),L^\infty)$ respectively then $||\mu||_{xy} = ||T_\mu||_1 = ||T_\mu||_\infty$.

By hypothesis $1/p - 1/r = 1 - 1/x$, hence $x/r = x/p - x + 1 < 1$ because $x/p - x < 0$. Therefore $0 < 1 - x/r < 1$.

Let $\theta = 1 - x/r$. Since $s/r = y/x$, $1 - x/r = \theta = 1 - y/s$. So, $1/r = (1 - \theta)/x$ and $1/s = (1 - \theta)/y$, also

$$\frac{1}{p} = \frac{1}{r} + \left[1 - \frac{1}{x}\right] = \frac{1}{r} + \frac{1}{x'} = \frac{x}{r}\left[\frac{1}{x}\right] + \frac{1}{x'} = \frac{x}{r}\left[1 - \frac{1}{x'}\right] + \frac{1}{x'} = \frac{x}{r} - \frac{x}{rx'} + \frac{1}{x'}$$

$$= \frac{x}{r} + \frac{1}{x'}\left[1 - \frac{x}{r}\right] = 1 - \theta + \frac{\theta}{x'}.$$

Similarly $1/q = 1 - \theta + \theta/y'$.

Applying Theorem 12.6 with $p_2 = q_2 = 1$, $p_1 = x'$, $q_1 = y'$, $r_2 = x$, $s_2 = y$, $r_1 = s_1 = \infty$ we conclude that $T_\mu \in M((L^p, \ell^q),(L^r, \ell^s))$ and the norm $||T_\mu||$ of T_μ in $M((L^p, \ell^q),(L^r, \ell^s))$ is such that

$$||T_\mu|| \leq ||T||_1^\theta \, ||T||_\infty^{1-\theta} = ||\mu||_{xy}^\theta \, ||\mu||_{xy}^{1-\theta} = ||\mu||_{xy}.$$

Therefore $\mu \longmapsto T_\mu$ defines a continuous linear isomorphism from (L^x, ℓ^y) into $M((L^p, \ell^q)^\prime, (L^r, \ell^s))$.†

§ 16. CHARACTERIZATION OF MULTIPLIERS IN TERMS OF ELEMENTS OF $S_0(G)^*$

In section 6 we gave a brief account of the Segal algebra $S_0(G)$ originally defined by H. G. Feichtinger [25] and studied independently by J. P. Bertrandias [6].

In this last section we will study this algebra $S_0(G)$ in more detail and characterize the c-multipliers from the algebras (L^p, ℓ^1), $1 \leqq p \leqq \infty$, (C_0, ℓ^1) and M_1 to any amalgam space or any space of measures M_q, in terms of elements of $S_0(G)^*$.

First we will prove some results about the multipliers and c-multipliers from the Segal algebras (L^p, ℓ^1) $(1 < p < \infty)$, (C_0, ℓ^1) to amalgam spaces and to the space M_q $(1 \leqq q \leqq \infty)$.

THEOREM 16.1. Let S be any of the Segal algebras (L^p, ℓ^1) $(1 < p < \infty)$, (C_0, ℓ^1) and B be any of the spaces (L^p, ℓ^q), (L^p, c_0), $1 \leqq p, q < \infty$, (C_0, ℓ^s), $1 \leqq s \leqq \infty$, Then $M(S,B) \subset c\text{-}M(S,B)$.

PROOF. Let $f, g \in S$ and $\psi \in S^*$. A simple calculation shows that

(1) $$< f*g,\psi > = \int g(s) < \tau_s f,\psi > ds.$$

Now observe that for $h \in L^1$, $k \in B$ and $F \in B^*$

(2) $$< h*k,F > = \int h(s) < \tau_s k,F > ds.$$

Take $T \in M(S,B)$. For a fixed $F \in B^*$ the map $\Lambda_F(f) = < Tf,F >$ is a bounded linear functional on S. That is,

$$< f,\Lambda_F > = < Tf,F > \qquad \text{for all } f \in S.$$

This together with (1) and (2) implies that for f, $g \in S$ and $F \in B^*$

$$< Tf*g,F > = \int g(s) < \tau_s(Tf),F > ds = \int g(s) < T(\tau_s f),F > ds$$
$$= \int g(s) < \tau_s f,\Lambda_F > ds = < f*g,\Lambda_F > = < T(f*g),F >.$$

Since F is arbitrary, $T(f*g) = Tf*g$ for all f, $g \in S$.†

THEOREM 16.2. Let S be as in Theorem 16.1 and B be any of the spaces (L^p, ℓ^q), (L^p, c_0), $1 < p, q < \infty$. Then $M(S,B) = c\text{-}M(S,B)$.

PROOF. Take $T \in c\text{-}M(S,B)$, $s \in G$, $f \in S$ and $h \in C_c$. So,

$$< T\tau_s f,h > = T\tau_s f*h(0) = T(\tau_s f*h)(0) = T(f*\tau_s h)(0) = Tf*\tau_s h(0)$$
$$= < Tf,\tau_s h > = < \tau_s Tf,h >.$$

Hence $< T\tau_s f,h > = < \tau_s Tf,h >$ for all $h \in C_c$. Since C_c is dense in B^*, (B^* is either $(L^{p'}, \ell^{q'})$ or $(L^{p'}, \ell^1)$, $1 < p',q' < \infty$) we conclude that $< T\tau_s f,h > = < \tau_s Tf,h >$ for all $h \in B^*$. Therefore $T\tau_s f = \tau_s Tf$ for all $s \in G$, $f \in S$. This means that $T \in M(S,B)$. The conclusion follows from Theorem 16.1.†

THEOREM 16.3. Let S be as in Theorem 16.1 and B be any of the spaces (L^∞, ℓ^q), M_q, $1 \leq q \leq \infty$, (L^p, ℓ^∞), (L^p, ℓ^1), $1 < p \leq \infty$. Then $M(S,B) = c\text{-}M(S,B)$.

PROOF. First we note that B is the dual of an amalgam space C. So for $h \in L^1$, $k \in \overset{\circ}{C}$ and $F \in B$

(3) $\qquad < h*k,F > = \int h(s) < \tau_s k,F > ds.$

As in the proof of Theorem 16.1, using (1) and (3) we see that if $T \in M(S,B)$ then T commutes with convolution. Hence $M(S,B)$ is included in $c\text{-}M(S,B)$.

Conversely, as in the proof of Theorem 16.2, we have that for $T \in c\text{-}M(S,B)$, $s \in G$, $f \in S$, and $h \in C_c$ $< T\tau_s f,h > = < \tau_s Tf,h>$. Since C_c is dense in C we conclude that $< T\tau_s f,h > = < \tau_s Tf,h >$ for all $h \in C$. This implies that $T \in M(S,B)$.†

DEFINITION 16.4. The Fourier algebra A(G) is the linear space of functions f in $C_0(G)$ such that $f = \check{\mathfrak{f}}$, $\mathfrak{f} \in L^1(\hat{G})$, endowed with the norm $||f||_A = ||\mathfrak{f}||_1$.

A(G) is an algebra under pointwise multiplication.

DEFINITION 16.5. Consider the following function $f: \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & \text{if } |x| \geq 1 \\ 1 - |x| & \text{if } |x| \leq 1. \end{cases}$$

The Fourier transform of f is equal to $2/\sqrt{2\pi}\ (1 - \cos t/t^2)$ and therefore $f \in A_c(\mathbb{R})$ and $\operatorname{supp} f = [-1,1]$.

For $n \in \mathbb{Z}$ we define the function f_n to be $\tau_n f$. Then it is clear that for each n $f_n \in A_c(\mathbb{R})$ and $\operatorname{supp} f_n = n + \operatorname{supp} f$.

Moreover for each $x \in \mathbb{R}$ $\sum_n f_n(x) = 1$ because if

$x \in [m - 1, m]$ for some $m \in Z$ then

$$\sum_n f_n(x) = f_{m-1}(x) + f_m(x) = (m - x) + x - (m - 1) = 1.$$

Now, by the Decomposition Theorem $G = \mathbb{R}^a \times G_1$ (see p. 10) and for $s = (x,t)$ in G, we define the function $\psi: G \longrightarrow \mathbb{R}$ by

$$\psi(s) = f(x_1) \cdot \cdot \cdot f(x_a) \times \chi_H(t) \qquad x = (x_1, .., x_a).$$

It is clear that $\psi \in A_c(G)$ and $\operatorname{supp} \psi = [-1, 1]^a \times H$. Then for $\alpha = (m_1, .., m_a, t)$ in J (J as in Definition 1.6) the function $\psi_\alpha = \tau_\alpha \psi = f_{m_1} \cdot \cdot \cdot f_{m_a} \cdot \chi_{t+H}$ has the following properties

(1) $\psi_\alpha \in A_c(G)$

(2) $\operatorname{supp} \psi_\alpha = \alpha + \operatorname{supp} \psi$

(3) $\sum_\alpha \psi_\alpha(s) = 1$ for each $s \in G$.

DEFINITION 16.6. Let $\{\psi_\alpha\}$ be the family defined in Definition 16.5. Then $S_0 = S_0(G)$ is the linear space of continuous functions $f \in A(G)$ such that $\sum_\alpha ||f\psi_\alpha||_A < \infty$, endowed with the norm $||f||_{S_0} = \sum_\alpha ||f\psi_\alpha||_A$.

It follows from [41, Proposition 1] that Definition 16.6 is equivalent to Feichtinger original definition of $S_0(G)$ in [26].

The following are some of the properties of $S_0(G)$. For a proof see [26] and [41].

1) $S_0(G)$ is a Segal algebra. Hence it is a Banach L^1 and M_1 module.

2) $A_c(G)$ is dense in $S_0(G)$.

3) $M_T \cup M_\infty \subset S_0(G)^* \subset Q(G)$.

4) $S_0(G) \subset \{f \in (C_0, \ell^1) \mid \hat{f} \in (C_0, \ell^1)\}$.

5) $S_0(G)^\wedge = S_0(\hat{G})$.

DEFINITION 16.7. The Fourier transform $F_0\sigma$ of $\sigma \in S_0(G)^*$ is an element of $S_0(\hat{G})^*$ defined by

$$< h, F_0\sigma > = < \check{h}', \sigma > = < \hat{h}, \sigma >, \qquad (h \in S_0(\hat{G})).$$

It is clear from 5) that $F_0\sigma$ is well defined.

REMARK 16.8. i) By 4) any h in $S_0(G)$ is equal to the inverse of its Fourier transform. That is, $h = \check{\hat{h}}$. Hence for any ψ_α, (as in Definition 16.5) $||h\psi_\alpha||_A = ||\hat{h}*\hat{\psi}_\alpha||_1$ and $\{h\psi_\alpha\}_J \subset (C_0, \ell^1)$.

ii) If $\sigma \in S_0(\hat{G})^*$ then its Fourier transform $F_\vee\sigma$ is defined as

$$< h, F_\vee\sigma > = < \hat{h}', \sigma > = < \check{h}, \sigma > \qquad (h \in S_0(G)).$$

Therefore by i) $F_0(F_\vee\sigma) = \sigma$ $(\sigma \in S_0(\hat{G})^*)$.

PROPOSITION 16.9. Let $\mu \in M_\infty$. Then $\hat{\mu}$ as in Definition 6.9 coincides with $F_0\mu$ as in Definition 16.7 iff there exists a constant C such that

$$|< h, \hat{\mu} >| \leq C \, ||h||_{S_0} \qquad \text{for all } h \in A_c(\hat{G}).$$

PROOF. This is a direct consequence of Definition 6.9 and property 2). $\dagger$

By property 1) we can define $\sigma * f$ $(\sigma * \nu)$ for $\sigma \in S_0(G)^*$ and

$f \in L^1(G)$ $(\nu \in M_1(G))$ to be an element of $S_0(G)^*$ given by

(16.1) $\langle h, \sigma * f \rangle = \langle h * f, \sigma \rangle$

(16.2) $\langle h, \sigma * \nu \rangle = \langle h * \nu, \sigma \rangle$ $(h \in S_0(G))$.

Moreover, if $f \in L^1(\hat{G})$ $(\nu \in M_1(\hat{G}))$ and $h \in S_0(G)$ then $h\check{f}$ $(h\check{\nu})$ belongs to $S_0(G)$ because for any $\alpha \in J$

$||h\check{f}\psi_\alpha||_A = ||\hat{h}*\hat{\psi}_\alpha * f||_1 \leqq ||f||_1 \; ||\hat{h}*\hat{\psi}_\alpha||_1$.

Similarly $||h\check{\nu}\psi_\alpha|| \leqq ||\nu||_1 \; ||\hat{h}*\hat{\psi}_\alpha||_1$.

Hence by Remark 16.8 i)

(16.3) $||h\check{f}||_{S_0} \leqq ||f||_1 \; ||h||_{S_0}$

(16.4) $||h\check{\nu}||_{S_0} \leqq ||\nu||_1 \; ||h||_{S_0}$.

Then we define $\underline{\sigma\check{f}}$ $(\underline{\sigma\check{\nu}})$ for $\sigma \in S_0(G)^*$, $f \in L^1(\hat{G})$ $(\nu \in M_1(\hat{G}))$ to be the element in $S_0(G)^*$ given by

(16.5) $\langle h, \sigma\check{f} \rangle = \langle h\check{f}, \sigma \rangle$

(16.6) $\langle h, \sigma\check{\nu} \rangle = \langle h\check{\nu}, \sigma \rangle$ $(h \in S_0(G))$.

PROPOSITION 16.10. Let $\sigma \in S_0(G)^*$, $f \in L^1(G)$ $(\nu \in M_1(G))$, and $g \in L^1(\hat{G})$ $(\eta \in M_1(\hat{G}))$. Then

i) $F_0(\sigma * f) = F_0\sigma(\hat{f})$ $(F_0(\sigma * \nu) = F_0\sigma(\hat{\nu}))$

ii) $F_0(\sigma\check{g}) = F_0\sigma * g$ $(F_0(\sigma\check{\eta}) = F_0\sigma * \eta)$.

PROOF. Let $h \in S_0(\hat{G})$. By (16.5) and the definition of $F_0\sigma$, we have that

$$\langle h, F_0\sigma(\hat{f}) \rangle = \langle h\hat{f}, F_0\sigma \rangle = \langle (h\hat{f})^{\vee\prime}, \sigma \rangle = \langle (\check{h}*f)', \sigma \rangle$$

$$= \langle \check{h}'*f, \sigma \rangle = \langle \check{h}', \sigma * f \rangle = \langle h, F_0(\sigma * f) \rangle.$$

Therefore i) holds.

Now, by Remark 16.8 i) and part i)

$F_\vee(F_0\sigma * g) = F_\vee(F_0\sigma)\check{g} = \sigma\check{g}$. This implies that $F_0\sigma * g = F_0(\sigma\check{g})$ and this proves ii).

The proof for ν and η is the same. †

THEOREM 16.11. Let $1 \leq p, q \leq \infty$. If B is any of the spaces (L^p, ℓ^q), (L^p, c_0), (C_0, ℓ^q) or M_q, S is any of the algebras (L^p, ℓ^1), (C_0, ℓ^1) and $T: S \longrightarrow B$ is a linear operator then the following are equivalent:

i) $T \in c\text{-}M(S,B)$

ii) There exits a unique $\sigma \in S_0(\hat{G})^*$ such that $(Tf)^\wedge = \sigma\hat{f}$ for all $f \in S$.

iii) There exists a unique $\mu \in S_0(G)^*$ such that $Tf = \mu * f$ for all $f \in S$.

PROOF. We will prove that i) is equivalent to ii) and ii) is equivalent to iii).

First we observe that if for all $h \in A_c(\hat{G})$ and $f \in S$, $< h,(Tf)^\wedge > = < h,\sigma\hat{f} >$ for some $\sigma \in S_0(\hat{G})^*$ then by (16.5) and (16.3)

$$|< h,(Tf)^\wedge >| = |< h,\sigma\hat{f} >| = |< h\hat{f},\sigma >| \leq ||\sigma||\ ||h\hat{f}||_{S_0}$$

$$\leq ||\sigma||\ ||f||_1\ ||h||_{S_0}.$$

Therefore by Proposition 16.9 $F_0(Tf) = (Tf)^\wedge$ for all $f \in S$. Hence if ii) holds then by Remark 16.8 ii) and Proposition 16.10 for all $f \in S$ $Tf = F_\vee(F_0(Tf)) = F_\vee(Tf)^\wedge = F_\vee(\sigma\hat{f}) = F_\vee\sigma * f$.

Then we conclude that ii) implies iii) with $\mu = F_V\sigma$.

Conversely if iii) holds then by Proposition 16.10 and (16.3) we have that for all $f \in S$ and $h \in A_c(\hat{G})$

$$|< h,(Tf)^\wedge >| = |< \check{h}',Tf >| = |< \check{h}',\mu*f >| = |< h,F_0(\mu*f) >|$$
$$= |< h,F_0\mu\hat{f} >| = |< h\hat{f},F_0\mu >| \leq ||F_0\mu|| \; ||h\hat{f}||_{S_0}$$
$$\leq ||F_0\mu|| \; ||f||_1 \; ||h||_{S_0}.$$

Again by Proposition 16.9 $(Tf)^\wedge = F_0(Tf)$.

Applying Proposition 16.10 once more we have that $(Tf)^\wedge = F_0(Tf) = F_0(\mu*f) = F_0\mu\hat{f}$ and therefore ii) holds with $\sigma = F_0\mu$.

Suppose i). Since either B^* is an amalgam space or a measure space of type q, or B is the dual of an amalgam space C, we have by the Hölder inequality for amalgams (Theorems 3.1 and 3.2) that

$$|< f,g >| \leq ||f||_{B^*} \; ||g||_B \quad (g \in B,\ f \in B^*)$$
$$|< f,g >| \leq ||f||_C \; ||g||_B \quad (g \in B,\ f \in C).$$

This implies by (2.3) and (2.4) that

(4) $\quad |< f,g >| \leq ||g||_B \; ||f||_{\infty 1}$ for all $g \in B$, $f \in (C_0, \ell^1)$.

Now, for $f, g \in S$, $Tf*g = T(f*g) = Tg*f$. So by Proposition 6.14 part i), $(Tf)^\wedge\hat{g} = (Tg)^\wedge\hat{f}$. This implies by Definition 6.13 that for all $f, g \in S$ and $h \in A_c(\hat{G})$

(5) $\quad < h\hat{g},(Tf)^\wedge > = < h,(Tf)^\wedge\hat{g} > = < h,(Tg)^\wedge\hat{f} > = < h\hat{f},(Tg)^\wedge >.$

Let $\{\psi_\alpha\} \subset A_c(\hat{G})$ be as in Definition 16.5 and $W = \text{supp}\ \psi$. To each α we associate a function λ_α in $(C_0, \ell^1)(G)$ as follows.

Take λ_W in $(C_0, \ell^1)(G)$ such that $\hat{\lambda}_W \equiv 1$ on W and $\hat{\lambda}_W \in C_c(\hat{G})$. Then $\lambda_\alpha = [\cdot,\alpha]\lambda_W$. It is clear that each λ_α has the following properties:

a) $\lambda_\alpha \varepsilon (C_0, \ell^1)(G)$.

b) $\hat{\lambda}_\alpha = \tau_\alpha \hat{\lambda}_W$. Hence by (2) $\hat{\lambda}_\alpha \equiv 1$ on $\alpha + W = \text{supp } \psi_\alpha$.

c) $\hat{\lambda}_\alpha \varepsilon C_c(\hat{G})$.

d) $||\lambda_\alpha||_{\infty 1} = ||\lambda_W||_{\infty 1}$.

We define σ on $A_c(\hat{G})$ by

$$< h,\sigma > = \sum_\alpha < h\psi_\alpha, (T\lambda_\alpha)^\wedge > \qquad (h \varepsilon A_c(\hat{G})).$$

First we note that if $h \varepsilon A_E(\hat{G})$ then $h\psi_\alpha$ belongs to $A_E(\hat{G})$ because $h\psi_\alpha \varepsilon C(E)$, $(h\psi_\alpha)^\vee = \check{h}*\check{\psi}_\alpha$ and $\check{h}*\check{\psi}_\alpha \varepsilon (C_0, \ell^1)$ (Theorem 4.7), so by Lemma 6.4 $h\psi_\alpha \varepsilon A_E(\hat{G})$. Also by b) $h\psi_\alpha = h\psi_\alpha\hat{\lambda}_\alpha$ and this implies that

$$||(h\psi_\alpha)^\wedge||_{\infty 1} = ||\check{h}*\check{\psi}_\alpha*\lambda_\alpha||_{\infty 1} \leqq ||\lambda_\alpha||_{\infty 1}\ ||\check{h}*\check{\psi}_\alpha||_1$$
$$\leqq ||\lambda_W||_{\infty_1}\ ||h\psi_\alpha||_A.$$

Therefore by (4)

$$|< h\psi_\alpha, (T\lambda_\alpha)^\wedge >| = |< (h\psi_\alpha)^{\vee\prime}, T\lambda_\alpha >| \leqq ||T\lambda_\alpha||_B\ ||(h\psi_\alpha)^\vee||_{\infty 1}$$
$$\leqq ||T||\ ||\lambda_\alpha||_S\ ||\lambda\ ||_{\infty_1}\ ||h\psi_\alpha||_A$$
$$\leqq ||T||\ ||\lambda_W||_{\infty 1}^2\ ||h\psi_\alpha||_A.$$

Hence σ is well defined and for all $h \varepsilon A_c(\hat{G})$

$|< h,\sigma >| \leqq ||T||\ ||\lambda_W||_{\infty 1}^2\ ||h||_{S_0}$.

If λ_W' is another function in $(C_0, \ell^1)(G)$ with the same properties as λ_W and $\lambda_\alpha' = [\cdot,\alpha]\lambda_W'$ then by (5) we have that for all $h \varepsilon A_c(\hat{G})$

$$< h\psi_\alpha, (T\lambda_\alpha)^\wedge > = < h\psi_\alpha \widehat{\lambda_\alpha'}, (T\lambda_\alpha)^\wedge > = < h\psi_\alpha \hat{\lambda}_\alpha, (T\lambda_\alpha')^\wedge >$$
$$= < h\psi_\alpha, (T\lambda_\alpha') >.$$

This shows that the definition of σ does not depend on the choice of the function λ_W.

By the density of $A_c(\hat{G})$ in $S_0(\hat{G})$, σ has a unique continuous extension σ on $S_0(\hat{G})$.

Now, if $h \in A_E(\hat{G})$ then $\{h\psi_\alpha\} \subset A_E(\hat{G})$, $h\psi_\alpha = 0$ for all but finitely many α's and by (3) $h = \sum_\alpha h\psi_\alpha$ pointwise. Then for $f \in S$
$$< h, (Tf)^\wedge > = < \sum_\alpha h\psi_\alpha, (Tf)^\wedge > = \sum_\alpha < h\psi_\alpha, (Tf)^\wedge >.$$

This together with (5) and (16.5) implies that for $f \in S$ and $h \in A_E(\hat{G})$
$$< h, \sigma\hat{f} > = < h\hat{f}, \sigma > = \sum_\alpha < h\hat{f}\psi_\alpha, (T\lambda_\alpha)^\wedge > = \sum_\alpha < h\psi_\alpha \hat{\lambda}_\alpha, (Tf)^\wedge >$$
$$= \sum_\alpha < h\psi_\alpha, (Tf)^\wedge > = < h, (Tf)^\wedge >.$$

By the observation we made at the beginning of the proof we conclude that $(Tf)^\wedge \in S_0(\hat{G})^*$. Since $A_c(\hat{G})$ is dense in $S_0(\hat{G})$, $(Tf)^\wedge = \sigma\hat{f}$ for all $f \in S$.

If $h \in A_E(\hat{G})$ and λ_E is a function in $(C_0, \ell^1)(\hat{G})$ such that $\hat{\lambda}_E \equiv 1$ on E then we have that

$$(6) \qquad < h, \sigma > = < h\hat{\lambda}_E, \sigma > = < h, \sigma\hat{\lambda}_E > = < h, (T\lambda_E)^\wedge >.$$

Finally if σ' is another functional in $S_0(\hat{G})^*$ such that $(Tf)^\wedge = \sigma'\hat{f}$ for all $f \in S$, then by (6) for all $h \in A_E(\hat{G})$
$$< h, \sigma > = < h, (T\lambda_E)^\wedge > = < h, \sigma\hat{\lambda}_E > = < h\hat{\lambda}_E, \sigma > = < h, \sigma >.$$

Hence $\sigma = \sigma'$ and therefore i) implies ii).

Conversely if ii) holds then for $f, g \in S$

$$(T(f*g))^\wedge = \sigma(f*g)^\wedge = \sigma(\hat{f}\hat{g}) = (\sigma\hat{f})\hat{g} = (Tf)^{\wedge}\hat{g} = ((Tf)*g)^\wedge.$$

By Theorem 13.22 $T(f*g) = Tf*g$.

To prove that T is continuous we proceed as in Theorem 13.23.

Let $\{f_n\}$, $f \in S$ such that $\lim ||f_n - f||_S = 0$ and assume that $\lim ||Tf_n - g||_B = 0$.

Take $h \in A_c(\hat{G})$. So, by (16.3) and (4) we have that

$$\begin{aligned}
|< h,(Tf)^\wedge - \hat{g} >| &\leq |< h,(Tf_n)^\wedge - (Tf)^\wedge >| + |< h,(Tf_n)^\wedge - \hat{g} >| \\
&\leq |< h,\sigma(f_n - f)^\wedge >| + |< \check{h}',Tf_n - g >| \\
&\leq |< h(f_n - f)^\wedge,\sigma >| + ||Tf_n - g||_B \, ||\check{h}||_{\infty 1} \\
&\leq ||\sigma|| \, ||h(f_n - f)^\wedge||_{S_0} + ||Tf_n - g||_B \, ||\check{h}||_{\infty 1} \\
&\leq ||\sigma|| \, ||h||_{S_0} \, ||f_n - f||_1 + ||Tf_n - g||_B \, ||\check{h}||_{\infty 1}.
\end{aligned}$$

Therefore $(Tf)^\wedge = \hat{g}$ on $A_c(\hat{G})$ and by Theorem 13.22, $Tf = g$. By the Closed Graph Theorem T is continuous and this implies that $T \in M(S,B)$ and the proof is complete.+

The proof of the next theorem is very similar to Theorem 16.11 and it will be omitted.

THEOREM 16.12. If B is as in Theorem 16.11 and $T: M_1 \longrightarrow B$ is a linear operator then the following are equivalent:

i) $T \in c\text{-}M(M_1,B)$.

ii) There exists a unique $\sigma \in S_0(\hat{G})^*$ such that $(T\nu)^\wedge = \sigma\hat{\nu}$ for all $\nu \in M_1$.

iii) There exists a unique $\mu \in S_0(G)^*$ such that $T\nu = \mu * \nu$ for all $\nu \in M_1$.

REMARK 16.13. It follows from Theorem 16.11 that $c\text{-}M(S,B) \subset M(S,B)$ (S, B as in Theorem 16.11). Indeed if $T \in c\text{-}M(S,B)$ then for $s \in G$ and $f \in S$

$$(T\tau_s f)^\wedge = \sigma(\tau_s f)^\wedge = \sigma([s,\cdot]\hat{f}) = [s,\cdot](\sigma\hat{f}) = [s,\cdot](Tf)^\wedge = (\tau_s(Tf))^\wedge .$$

Hence by Theorem 13.22 T commutes with translations and this implies that $c\text{-}M(S,B) \subset M(S,B)$.

We conclude that if S is as in Theorem 16.1 and B is as in Theorem 16.11 then $c\text{-}M(S,B) = M(S,B)$.

Observe that $c\text{-}M(S,(L^1, \ell^\infty)) \subset c\text{-}M(S,M_\infty)$, and by Theorem 16.3 $c\text{-}M(S,M_\infty) = M(S,M_\infty)$. Therefore $c\text{-}M(S,(L^1, \ell^\infty)) \subset M(S,(L^1, \ell^\infty))$. Similarly by Theorem 16.3 $c\text{-}M(S,(L^\infty, c_0)) \subset c\text{-}M(S,L^\infty) = M(S,L^\infty)$, so $c\text{-}M(S,(L^\infty, c_0)) \subset M(S,(L^\infty, c_0))$.

COROLLARY 16.14. Let S be as in Theorem 16.11 and B be any space (L^p, ℓ^q), (C_0, ℓ^q), M_q ($1 \leq p \leq \infty$, $1 \leq q \leq 2$). If $T: S \longrightarrow B$ is a linear operator then the following are equivalent:

i) $T \in c\text{-}M(S,B)$

ii) There exists a unique $\varphi \in (L^{q'}, \ell^\infty)(\hat{G})$ such that $(Tf)^\wedge = \varphi\hat{f}$ for all $f \in S$.

PROOF. Let $\hat{x} \in \hat{G}$ and $\psi_{\hat{x}}$ be a function in S such that $\psi_{\hat{x}}(\hat{x}) = 1$. Define $\varphi(\hat{x}) = (T\psi_{\hat{x}})^\wedge(\hat{x})$ $(\hat{x} \in \hat{G})$.

If $T \in c\text{-}M(S,B)$ then φ is independent of the choice of $\psi_{\hat{x}}$

because if $\psi_{\hat{x}}' \in S$ and $\psi_{\hat{x}}'(\hat{x}) = 1$ then

$$(T\psi_{\hat{x}})^{\wedge}(\hat{x}) = (T\psi_{\hat{x}})^{\wedge}(\hat{x})\psi_{\hat{x}}' = (T\psi_{\hat{x}}*\psi_{\hat{x}}')^{\wedge}(\hat{x}) = (T\psi_{\hat{x}}'*\psi_{\hat{x}})^{\wedge}(\hat{x})$$
$$= (T\psi_{\hat{x}}')^{\wedge}(\hat{x})\psi_{\hat{x}}(\hat{x}) = (T\psi_{\hat{x}}')^{\wedge}(\hat{x}).$$

Also for $f \in S$

$$\varphi\hat{f}(\hat{x}) = (T\psi_{\hat{x}})^{\wedge}(\hat{x})\hat{f}(\hat{x}) = (T\psi_{\hat{x}}*f)^{\wedge}(\hat{x}) = (Tf*\psi_{\hat{x}})^{\wedge}(\hat{x}) = (Tf)^{\wedge}(\hat{x})\psi_{\hat{x}}(\hat{x})$$
$$= (Tf)^{\wedge}(\hat{x}).$$

Let φ_T be the element in $A_c(G)^*$ associated to T by Theorem 16.11. Then by (6) of Theorem 16.11 for $h \in A_E(\hat{G})$

$$< h,\varphi_T > = < h,(T\lambda_E)^{\wedge} > = \int h(\hat{x})\ (T\lambda_E)^{\wedge}(\hat{x})\ d\hat{x}$$
$$= \int h(\hat{x})\ \varphi(\hat{x})\ \hat{\lambda}_E(\hat{x})\ d\hat{x}$$
$$= \int h(\hat{x})\ \varphi(\hat{x})\ d\hat{x}.$$

Therefore $< h,\varphi_T > = \int h(\hat{x})\ \varphi(\hat{x})\ d\hat{x}$ for all $h \in A_c(\hat{G})$.

Now take $k \in C_c(\hat{G})$ such that $k \equiv 1$ on $\hat{K}$ and $\check{k} \in S$. Then $k_\beta = \tau_\beta k$ belongs to $C_c(\hat{G})$, $k_\beta \equiv 1$ on $\hat{K}_\beta$, $\check{k}_\beta = [\cdot,\beta]\check{k}$ belongs to S and $||\check{k}_\beta||_S = ||\check{k}||_S$ for all β.

By the definition of the norm $||\cdot||_{q'r}$, $1 \leq r \leq \infty$, it is clear that $||\varphi\chi_{L_\beta}|| \leq ||\varphi\chi_{L_\beta}||_{q'r}$.

If B is the space (L^p, ℓ^q), $1 \leq p, q \leq 2$ then by Theorem 5.7

$$||\varphi\chi_{L_\beta}||_{q'} \leq ||\varphi\chi_{L_\beta}||_{q'p'} \leq ||\varphi k_\beta||_{q'p'} = ||\check{k}_\beta||_{q'p'}$$
$$= ||(T\check{k}_\beta)^{\wedge}||_{q'p'} \leq C\ ||T\check{k}_\beta||_{pq} \leq C\ ||T||\ ||\check{k}_\beta||_S$$
$$= C\ ||T||\ ||\check{k}||_S.$$

If B is the space (L^p, ℓ^q), (C_0, ℓ^q), $1 \leq q \leq 2 \leq p \leq \infty$, then $B \subset (L^2, \ell^q)$ and as in the previous case

$||\varphi\chi_{L_\beta}||_{q'} \leq ||\varphi k_\beta||_{q'2} \leq C\ ||T|| \cdot ||\check{k}||_S$.

Similarly if $B = M_q$, $||\varphi\chi_{L_\beta}||_{q'} \leq ||\varphi k_\beta||_{q'\infty} \leq C'\ ||T||\ ||\check{k}||_S$. Since this is for all α, in any case $\varphi \in (L^{q'}, \ell^\infty)$. Hence i) implies iii).

Conversely if $\varphi \in (L^{q'}, \ell^\infty)$ and $(Tf)^\wedge = \varphi\hat{f}$ $(f \in S)$ then we define φ_T on $A_c(\hat{G})$ by $< h,\varphi_T > = \int h(\hat{x})\ \varphi_T(\hat{x})\ d\hat{x}$ $(h \in A_c(\hat{G}))$. Then as above $< h,\varphi_T > = < h,(T\lambda_E)^\wedge >$. Therefore by Theorem 16.11, ii) implies i). (See (6) of the proof of Theorem 16.11).†

<u>THEOREM 16.15</u>. Let S be any of the algebras (L^p, ℓ^1), $2 \leq p \leq \infty$, or (C_0, ℓ^1). The correspondence between c-M(S,L^2) and (L^2, ℓ^∞) established by Corollary 16.14 defines a continuous isomorphism from c-M(S,L^2) onto (L^2, ℓ^∞).

<u>PROOF</u>. If $f \in S$ then $f \in (L^2, \ell^1)$. Hence by Theorem 5.2, $\hat{f} \in (C_0, \ell^2)$. By Proposition 4.1, $\hat{f} \in L^2$ for $\varphi \in (L^2, \ell^\infty)$. Then there exists a unique $Tf \in L^2$ such that $(Tf)^\wedge = \varphi\hat{f}$. It is clear that T so defined is a linear operatior from S to L^2 and the conclusion follows from Theorem 16.11 and the fact that for $f \in S$

$$||Tf||_2 = ||(Tf)^\wedge||_2 = ||\varphi\hat{f}||_2 \leq ||\varphi||_{2\infty}\ ||\hat{f}||_{\infty 2} \leq ||\varphi||_{2\infty} C\ ||f||_{21}$$

$$\leq ||\varphi||_{2\infty}\ C\ ||f||_S. \dagger$$

<u>COROLLARY 16.16</u>. Let S be as in Theorem 16.15 and B be (L^r, ℓ^2), $1 \leq r \leq 2$, or M_2. Then c-M(S,L^2) = c-M(S,B).

PROOF. Since $L^2 \subset B$ it is clear that $c\text{-}M(S,L^2) \subset c\text{-}M(S,B)$. If $T \in c\text{-}M(S,B)$ then by Corollary 16.14 there exists a unique $\varphi \in (L^2, \ell^\infty)$ such that $(Tf)^\wedge = \varphi\hat{f}$ for all $f \in S$.

By Theorem 16.15, there exists T' in $c\text{-}M(S,L^2)$ such that $(T'f)^\wedge = \varphi\hat{f}$ for all $f \in S$.

Therefore $(Tf)^\wedge = (T'f)^\wedge$ for all $f \in S$. This implies by Theorem 13.22 that $T = T'$. Hence $c\text{-}M(S,B) \subset c\text{-}M(S,L^2)$.†

COROLLARY 16.17. Let S be as in Theorem 16.15 and B be (L^p, ℓ^2), $2 < p \leq \infty$, or (C_0, ℓ^2). Then $c\text{-}M(S,B) \subset c\text{-}M(S,L^2)$.

PROOF. Let $T \in c\text{-}M(S,B)$. By Corollary 16.14 there exists a unique $\varphi \in (L^2, \ell^\infty)$ such that $(Tf)^\wedge = \varphi\hat{f}$ for all $f \in S$.

Similarly to Corollary 16.16, this implies that $T = T'$ for $T' \in c\text{-}M(S,L^2)$.†

Let A be any of the amalgam spaces (L^p, ℓ^q), (C_0, ℓ^q), (L^p, c_0) $(1 \leq p, q \leq \infty)$ and B be as in Theorem 6.11.

We associate to A the biggest algebra $\underline{S_A}$ such that $c\text{-}M(S_A,B) = M(S_A,B)$.

By Remark 16.13 we see that if A is (L^p, ℓ^q), (L^p, c_0) $(1 \leq p < \infty, 1 \leq q \leq \infty)$ then $S_A = (L^p, \ell^1)$ and $S_A = (C_0, \ell^1)$ for the remaining cases.

THEOREM 16.18. Let A be any of the spaces (L^p, ℓ^q), (C_0, ℓ^q), (L^p, c_0) $(1 \leq p, q \leq \infty)$ and B be as in Theorem 16.11. If $T \in M(A,B)$ and S_A is as above then

i) There exists a unique $\varphi \in S_0(\hat{G})^*$ such that $(Tf)^\wedge = \varphi\hat{f}$ for all $f \in S_A$.

ii) There exists a unique $\mu \in S_0(G)^*$ such that $Tf = \mu * f$ for all $f \in S_A$.

PROOF. If $T \in M(A,B)$ then $T|S_A \in M(S_A,B)$. By Remark 16.13 $T|S_A \in c\text{-}M(S_A,B)$ and the conclusions follow from Theorem 16.11.†

Compare Theorem 16.18 with [25, Theorem C2].

THEOREM 16.19. Let A be any of the spaces (L^p, ℓ^q), (C_0, ℓ^q), $1 \leq p < \infty$, $1 \leq q \leq 2$, and let B be any of the spaces (L^r, ℓ^s), (C_0, ℓ^s), $1 \leq r < \infty$, $1 \leq s \leq 2$. If $T: A \longrightarrow B$ is a linear operator then the following are equivalent:

i) $T \in M(A,B)$

ii) There exists a unique $\varphi \in (L^{s'}, \ell^\infty)(\hat{G})$ such that $(Tf)^\wedge = \varphi\hat{f}$ for all $f \in A$.

PROOF. We will prove the theorem for $A = (L^p, \ell^q)$ and $B = (L^r, \ell^s)$, $1 \leq r, s \leq 2$. The remainig cases are similar.

Suppose $T \in M(A,B)$. Then $T|(L^p, \ell^1)$ belongs to $M((L^p, \ell^1),(L^r, \ell^s))$. So by Corollary 16.14 there exists a unique $\varphi \in (L^{s'}, \ell^\infty)$ such that $(Tf)^\wedge = \varphi\hat{f}$ for all $f \in (L^p, \ell^1)$.

Now, we note that for $f \in (L^p, \ell^1)$

$$||\varphi\hat{f}||_{s'r'} = ||(Tf)^\wedge||_{s'r'} \leq C\ ||Tf||_{rs} \leq C\ ||T||\ ||f||_{pq}.$$

Therefore the map $f \longmapsto \varphi\hat{f}$ from $((L^p, \ell^1), ||\cdot||_{pq})$ to $(L^{s'}, \ell^{r'})$ is continuous. Since (L^p, ℓ^1) is dense in (L^p, ℓ^q)

(Corollary 3.8) this map has a unique continuous extension on (L^p, ℓ^q) and this implies that $(Tf)^\wedge = \varphi\hat{f}$ for all $f \in (L^p, \ell^q)$.

Conversely if ii) holds then by Corollary 16.14 and Remark 16.13 $T|(L^p, \ell^1)$ belongs to $M((L^p, \ell^1),(L^r, \ell^s))$. Again since (L^p, ℓ^1) is dense in (L^p, ℓ^q) and the map $f \longrightarrow \tau_s f$ on A and B is continuous for all $s \in G$, (Theorem 3.14) we conclude that T has a unique continuous extension $\overline{T}$ on (L^p, ℓ^q) which commutes with translations. Hence $\overline{T}$ belongs to $M((L^p, \ell^q),(L^r, \ell^s))$. So, by the previous case there exists a unique $\varphi' \in (L^{s'}, \ell^\infty)$ such that $(\overline{T}f)^\wedge = \varphi'\hat{f}$ for all $f \in (L^p, \ell^q)$. Then for $f \in (L^p, \ell^1)$, $\varphi'\hat{f} = (\overline{T}f)^\wedge = (Tf)^\wedge = \varphi\hat{f}$. For $\hat{x} \in \hat{G}$, let $f_{\hat{x}} \in (L^p, \ell^1)$ such that $f_{\hat{x}}(\hat{x}) = 1$, so $\varphi'(\hat{x}) = \varphi'\hat{f}_{\hat{x}}(\hat{x}) = \varphi\hat{f}_{\hat{x}}(\hat{x}) = \varphi(\hat{x})$. This implies that $\varphi = \varphi'$ and therefore $(Tf)^\wedge = (\overline{T}f)^\wedge$ for all $f \in (L^p, \ell^q)$. By Theorem 13.22 we conclude that $T = \overline{T}$.†

Now we will prove again (Theorem 12.4 i)) that for $1 \leqq p < \infty$, $1 \leqq s < q \leqq 2$, $M((L^p, \ell^q),(L^1, \ell^s))$ is a trivial space, to show an application of Theorem 9.2 and Theorem 16.19.

<u>PROPOSITION 16.20</u>. Let $\hat{G}$ be nondiscrete and $1 \leqq p < \infty$, $1 \leqq s < q \leqq 2$. Then the only multiplier from (L^p, ℓ^q) to (L^1, ℓ^s) is the zero multiplier.

<u>PROOF</u>. Suppose the contrary. That is, there exists a nonzero function φ such that $\varphi\hat{f} \in (L^1, \ell^s)^\wedge$ for all $f \in (L^p, \ell^q)$ Theorem 16. 19). Then for some $\varepsilon > 0$, the set

$$E = \{ \hat{x} \in \hat{G} \mid |\varphi(\hat{x})| \geqq \varepsilon \}$$

is not locally null. Since $(L^1, \ell^s)^\wedge|E \subset (L^{s'}, \ell^\infty)(E)$ by Theorem 5.7; $\varphi(L^p, \ell^q)^\wedge|E \subset (L^1, \ell^s)^\wedge|E \subset (L^{s'}, \ell^\infty)(E)$. Hence by definition of E we have the relation $(L^p, \ell^q)^\wedge|E \subset (L^{s'}, \ell^\infty)(E)$. (Note that if $f \in (L^p, \ell^q)$ then $||\hat{f}|E||_{s',\infty} \leq \varepsilon\, ||\varphi\hat{f}|E||_{s'\infty}$).

But this is a contradiction because by Theorem 9.2 there exists $f \in (L^\infty, \ell^q)$, hence in (L^p, ℓ^q), such that $\hat{f}|E \notin (L^{s'}, \ell^\infty)(E)$ as $s' > q'$. This ends the proof.†

BIBLIOGRAPHY

[1] Argabright L. and Gil de Lamadrid J. Fourier Analysis of Unbounded Measures on Locally Compact Abelian Groups. Memoirs Am. Math. Soc. v. 145 (1974).

[2] Argabright L. and Gil de Lamadrid J. Almost Periodic Measures. University of Minnesota.Mathematics Report 81 - 112.

[3] Bloom, W. R. Strict Local Inclusion Results Between Spaces of Fourier Transforms. Pac. J. Math. v. 99, No. 2 (1982) 265 - 270.

[4] Beckenback, E. F. and Bellman, K. Inequalities. Berlin-Heidelberg-New York. Springer-Verlag 1965.

[5] Bertrandias, J. P. Unions et Intersections D'espaces L^p sur un Espace Localement Compact. Bull. Sc. Math. 2^o série, 101, (1977) 209 - 247.

[6] -----------, Espace ℓ^p $(\widehat{\ell^\alpha})$. Groupe de Travail de Analyse Harmonique. Grenoble II (1984), IV.1 - IV.11.

[7] Bertrandias, J. P. and Dupuis, C. Transformation de Fourier sur les Espaces $\ell^p(L^p)$. Ann. Inst. Fourier. Grenoble. 29, 1 (1979) 189 - 206.

[8] Bertrandias, J. P., Datry, C. and Dupuis, C. Unions et Intersections D'espaces L^p Invariantes par Translation ou Convolution. Ann. Inst. Fourier. Grenoble. 28, 2 (1978) 53 - 84.

[9] Burnham, J. T. and Goldberg R. R. Multipliers from $L^1(G)$ to a Segal Algebra. Bull. Inst. Math. Acad. Sinica 2 (1974) 153 - 164.

[10] Bourbaki, N. Elements de Mathematique. Integration. Chaps. 7 - 8 (1963). Act. Sci. et Ind. 1306. Hermann et Cie, Paris.

[11] Benedek, A. and Panzone, R. The Spaces L^p, with Mixed Norm. Duke Math. J. 28 (1961) 301 - 324.

[12] Busby, R. C. and Smith, H. A. Product-Convolution Operators and Mixed-Norm Spaces. Trans. Am. Math. Soc. 263, No. 2, (1981) 309 - 341.

[14] Burnham, J. T. The Relative Completion of an A-Segal Algebra is Closed. Proc. Am. Math. Soc. v. 49. No. 1, May 1975, 116 - 122.

[15] Simon, A. B. Cesàro Summability on Groups: Characterization and Inversion of Fourier Transforms. Function Algebras. Proc. Internat. Sympoc. on Function Algebras, Tulane Univ. 1965. pp. 208 - 215. (Scott-Foresman, Chicago Illinois, 1966).

[16] Cowling, M. Some Applications of Grothendick's Theory of Topological Tensor Products in Harmonic Analysis. Math. Ann. 232 (1978) 273 - 285.

[17] Dupuis, C. Quasimesures de Type Positif. Bull. Sc. Math. 2 série 105, (1981) 169 - 180.

[18] Dunford, N. and Schwartz, J. Linear Operators. Part I: General Theory. New York. Interscience Pub. Inc. 1958.

[19] Doran, R. S. and Wichmann, J. Approximate Identities and Factorization in Banach Modules. Berlin-Heidelberg-New York. Springer-Verlag (1970) v. 768.

[21] Edwards, R. E. On Factor Functions. Pac. J. Math. 5 (1955) 367 - 378.

[22] Fournier, J. J. F. Local Complements to the Hausdorff-Young Theorem. Michigan Math. J. 20 (1973) 263 - 276.

[23] Feichtinger, H. G. A Characterization of Wiener's Algebra on Locally Compact Groups. Arch. Math. 29 (1977) 136 - 140.

[24] ----------. Multipliers from $L^1(G)$ to a Homogeneous Banach Space. J. Math. Anal. Appl. 61 (1977) 341 - 356.

[25] ----------. Un Espace de Banach de Distributions Tempéreés sur les Groups Localement Compacts Abeliens. CR. Acad. Sci. Paris. 290 (1980) série A, 791 - 794.

[26] ----------. On a New Segal Algebra. Mh. Math. 92 (1981) 269 - 289.

[27] Fournier, J. J. F. and Stewart, J. Amalgams of L^p and ℓ^q. Submitted to Bull. Am. Math. Soc.

[28] Gaudry, G. I. Multipliers of Type (p,q). Pac. J. Math. v. 18, No. 3 (1966) 477 - 488.

[29] ------. Topics in Harmonic Analysis. Lecture Notes. Department of Mathematics. Yale University. New Haven, Ct. (1969).

[30] Gil de Lamadrid, J. Application D'une Forme Generale de la Formule de Poisson a L'etude des Mesures Transformables. Séminaire de Initiation à l'Alalyse de M. G. Choquet. December 1982.

[31] Gulick, S. L., Liu, T. S. and van Rooij, A. C. M. Group Algebra Modules III. Trans. Am. Math. Soc. v. 152 (1970) 561 - 579.

[32] Hórváth, J. Topological Vector Spaces and Distributions. v. I. Massachusetts. Addison-Wesley Pub. Co. (1966).

[33] Hörmander, L. Estimates for Translation Invariant Operators in L^p Spaces. Acta. Math. 104 (1960). 93 - 140.

[34] Holland, F. Harmonic Analysis on Amalgams of L^p and L^q. J. London Math. Soc. (2) 10 (1975) 295 - 305.

[35] -------. On the Representation of Functions as Fourier Transforms of Unbounded Measures. Proc. London Math. Soc. (3) 30 (1975) 347 - 365.

[36] Hardy, G. H., Littlewood, J. E. and Pólya, G. Inequalities. Cambridge. Cambridge University Press. (1934).

[37] Hewitt, E. and Ross, K. A. Abstract Harmonic Analysis. v. I, II. Berlin-Heidelberg- New York. Springer-Verlag (1970), (1979).

[38] Köthe, G. Topological Vector Spaces. v. I. Berlin. Springer-Verlag. (1969).

[39] Krogstad, H. E. Multipliers of Segal Algebras. Math. Scand. 38, (1976) 285 - 303.

[40] Larsen, R. An Introduction to the Theory of Multipliers. Berlin. Springer-Verlag (1971).

[41] Losert, V. A Characterization of the Minimal Strongly Character Invariant Segal Algebra. Ann. Inst. Fourier, Grenoble. 30, No. 3 (1980) 129 - 139.

[42] Liu, T. S., van Rooij, A. C. M. and Wang, J. K. On Some Group Algebra of Modules Related to Wiener's Algebra M1. Pac. J. Math. v. 55, No. 2 (1974) 507 - 520.

[43] Rudin, W. Real and Complex Analysis. New York. Mc Graw Hill Inc. (1974).

[44] -----. Functional Analysis. New Delhi. Tata Mc Graw Hill. (1974).

[45] -----. Fourier Analysis on Groups. New York. Interscience (1962).

[46] Reiter, H. L^1-Algebras and Segal Algebras. Berlin-Heidelberg-New York. Springer-Verlag v. 231 (1971).

[47] ------. Classical Harmonic Analysis and Locally Compact Groups. Oxford. Oxford University Press (1968).

[48] Rieffel, M. A. Induced Banach Representations of Banach Algebras and Locally Compact Groups. J. Functional Anal. 1 (1967) 443 - 491.

[49] Stewart, J. Fourier Transforms of Unbounded Measures. Can. J. Math. v. 31, No. 6 (1979). 1281 - 1292.

[50] -------. Positive Definite Functions and Generalizations. Ph. D. Thesis. University of Toronto. (1967).

[51] -------. Unbounded Positive Definite Functions. Can. J. Math. v. 21, No. 6 (1969) 1309 - 1318.

[52] Schwartz, L. Radon Measures on Arbitrary Topological Spaces and Cylindrical Measures. Oxford. Oxfrod University Press. (1973).

[53] Szeptycki, P. On Functions and Measures Whose Fourier Transforms are Functions. Math. Ann. 179 (1968) 31 - 41.

[54] Wiener, N. The Fourier Integral and Certain of its Applications. New York. Dover, Pub. Inc. (1958).