AMALGAMS OF $L^p$ AND $L^q$

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ABSTRACT

An amalgam of $L^p$ and $\ell^q$ is a Banach space $(L^p, \ell^q)(G)$ $(1 \leq p, q \leq \infty)$ of (classes of) functions on a locally compact abelian group $G$ which belong locally to $L^p$ and globally to $\ell^q$. Similarly, the space of unbounded measures of type $q$ is a Banach space $M_q(G)$ $(1 \leq q \leq \infty)$ of unbounded measures which belong locally to the space of bounded, regular, Borel measures on $G$ and globally to $\ell^q$.

The Fourier transform of functions in $(L^p, \ell^q)$ and measures in $M_q$ is defined to be a linear functional on the subspace $\mathcal{A}_c(G)$ of the Fourier algebra $A(G)$, and its relation with other known definitions of Fourier transforms is established.

We introduce the space of strong resonance class of functions relative to the test space $\Phi_q$ and find its relation with respect to the linear space generated by the positive definite functions for $(L^q, \ell^1)$.

We generalize known results for amalgam spaces on the real line to locally compact abelian groups, extend some results in the theory of $L^p$ spaces to amalgams and develop a theory of multipliers for amalgam spaces and spaces of unbounded measures of type $q$.
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To my mother and sisters
To my husband

To the truly honest
A los honestos de corazón

A mi madre y hermanas
A mi esposo
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INTRODUCTION

Briefly, an amalgam of $L^p$ and $L^q$ is a Banach space $(L^p, L^q)(G)$ $(1 \leq p, q \leq \infty)$ of measurable (classes of) functions on a locally compact abelian group $G$ which belong locally to $L^p$ and have $L^q$ behavior at infinity.

Several authors have introduced special cases of amalgams during the last half century. Among others N. Wiener [54], P. Szeptycki [53], T. S. Liu, A. van Rooij and J. K. Wang [42], H. E. Krogstad [39] and H. G. Feichtinger [23]. (For a historical background of amalgams see [27]).

The first systematic study of amalgams on the real line was undertaken by F. Holland [34] and their generalization to locally compact groups was done independently by J. P. Bertrandias, C. Datry and C. Dupuis [8], J. Stewart [49], and R. C. Busby and H. A. Smith [12]. In Section 1 we give their definitions and prove that they are equivalent.

J. Stewart's definition suits best our needs and we use it throughout the work.

The study of amalgam spaces leads to the study of the Banach space $M_q$ $(1 \leq q \leq \infty)$ of unbounded measures of type $q$. The particular case $M_\infty$, has been studied by L. Argabright and J. Gil de Lamadrid [1].

In Chapter I we study the properties of duality and reflexivity of $(L^p, L^q)$, density and translation invariance of $(L^p, L^q)$ and $M_q$. 
and the product and convolution operations on \((L^p, \ell^q)\) and \(M_q\).

Chapter II is about the Fourier transform of functions in \((L^p, \ell^q)\) and measures in \(M_q\).

In Section 5 we study the Fourier transform of functions in \((L^p, \ell^q)\) for \(1 \leq p \leq \infty, 1 \leq q \leq 2\) following Holland's ideas in [34].

In Section 6 we give W. Bloom's definition of the Fourier transform of functions in \((L^p, \ell^q)\) for \(1 \leq p \leq \infty, 2 < q < \infty\) and extend his definition to measures in \(M_q (1 \leq q \leq \infty)\). This approach is different from that of Bertrandias and Dupuis [7], and Feichtinger [25].

We study the relation between the Fourier transform of unbounded measures of type \(q\) and the Fourier transform of transformable measures as it was defined by Argabright and Gil de Lamadrid in [1].

The next thing we try to do is to generalize to locally compact abelian groups some of Holland's results that appeared in [34] and [35].

In Section 7 we give Simon's generalization of Cesaro summability for locally compact abelian groups and study some of its properties related to amalgams in order to generalize Theorem 9 of [34].

In Section 8 we introduce the space \(SR(\Phi_q) (1 \leq q \leq \infty)\) of functions strongly resonant relative to the test space \(\Phi_q\) based on Holland's definition of the space \(R(\Phi_q) (1 \leq q \leq \infty)\) of functions resonant relative to \(\Phi\). We prove (Theorem 8.20) that \(< P(L^q', \ell^1) >\) is dense in \(SR(\Phi_q)\) for \(1 \leq q < 2\), \(< P(L^2, \ell^1) >\) is equal to \(SR(\Phi_2)\) and \(SR(\Phi_2)\) is included in \(< P(L^q', \ell^1) >\) for \(2 < q \leq \infty\), where \(< P(L^q', \ell^1) >\) is the linear space generated by the space of positive definite functions for \((L^q', \ell^1)\). Also we give a representa-
tion of functions in $SR(\Phi_q)$ ($1 \leq q < 2$) in terms of the Fourier transform of measures of type $q$ (Theorem 8.25). This representation is similar to the one given by F. Holland in [35].

An important aspect of the theory of amalgam spaces is that the $L^p(G)$ spaces are particular cases of amalgams. This opens immediately the possibility of extending known results in the theory of $L^p$ spaces to amalgams. Chapters IV and V are in this direction.

J. J. F. Fournier [22, Theorems 1 and 2] proved that:

a) If $\hat{G}$ is not discrete and $E$ is not locally null then for $1 < p \leq 2$
$$L^p(E) \subset \bigcup_{q > p'} L^q(E).$$

b) If $\hat{G}$ is not compact and $1 \leq p \leq 2$ then
$$L^p(G) \hat{\subset} \bigcup_{q < p'} L^q(\hat{G}).$$

c) If $\hat{G}$ is neither compact nor discrete and $1 < p \leq 2$ then
$$L^p(G) \hat{\subset} \bigcup_{q > p'} L^q(\hat{G}).$$

In Chapter IV we prove the following.

A) If $\hat{G}$ is not discrete and $E$ is not locally null then for $1 < p \leq 2$
$$(L^\infty, \ell^p)^\hat{\subset} \bigcup_{q > p'} (L^q, \ell^\infty)(E).$$

B) If $\hat{G}$ is not compact and $1 \leq p \leq 2$ then
$$(L^p, \ell^1)(G) \hat{\subset} \bigcup_{q < p'} (L^1, \ell^q)(\hat{G}).$$

C) If $\hat{G}$ is neither compact nor discrete and $1 < p \leq 2$ then
$$L^p(G) \hat{\subset} \bigcup_{q > p'} (L^q, \ell^\infty) \cap (L^1, \ell^q)$$
Since \((L^\infty, \ell^p)\) and \((\ell^p, \ell^1)\) are proper subspaces of \(L^p\) for \(1 \leq p \leq \infty\), and \(L^q\) is a proper subspace of \((L^q, \ell^\infty), (\ell^1, \ell^q)\) for \(1 \leq q \leq \infty\) and \((L^q, \ell^\infty) \cap (\ell^1, \ell^q)\) for \(1 < q < \infty\), \(A\), \(B\), \(C\) extend \(a\), \(b\), \(c\) respectively.

Chapter V is about the theory of multipliers for amalgam spaces and spaces of unbounded measures of type \(q\).

If \(A\) and \(B\) are any amalgam space or any spaces of measures \(M_q\) then we define \(M(A,B)\) to be the linear space of multipliers from \(A\) to \(B\) (continuous linear operators from \(A\) to \(B\) which commute with translations).

If \(A\) is \((L^p, \ell^1)\) \((1 \leq p \leq \infty)\), \((C_0, \ell^1)\) or \(M_1\) and \(B\) is as above then \(c-M(A,B)\) is the linear space of convolution multipliers from \(A\) to \(B\) (continuous linear operators from \(A\) to \(B\) which commute with convolution). In this case we establish an inclusion relation between \(M(A,B)\) and \(c-M(A,B)\). For some \(A\) and \(B\) we prove that \(M(A,B)\) is equal to \(c-M(A,B)\).

We also look for a characterization of the elements \(T\) of \(M(A,B)\) or \(c-M(A,B)\) of the type:

1. For all \(f \in A\), \(Tf = \mu * f\) for a unique \(\mu\) in some linear space.
2. For all \(f \in A\), \((Tf)^\wedge = \varphi \hat{f}\) for a unique \(\varphi\) in some linear space, where \((Tf)^\wedge\) and \(\hat{f}\) are the Fourier transforms of \(Tf\) and \(f\) respectively.

If \(A = L^1(G)\) and \(B\) is any amalgam space or any space \(M_q\) then the characterizations (1) and (2) of elements of \(c-M(L^1, B)\) hold.
and they are equivalent when $B$ is equal to $(L^p, \ell^q)$ ($1 < p, q \leq \infty$), $(C_0, \ell^q)$, $(L^p, c_0)$, $(L^q, \ell^1)$ ($1 < q < \infty$), $(L^1, \ell^q)$ ($1 \leq q < \infty$) or $(C_0, \ell^1)$ (Section 13).

If $A = M_1$ and $B$ is any amalgam space or any space $M_q$ then the characterizations (1) and (2) of elements of $c-M(A, B)$ hold and are equivalent (Section 14).

If $B = L^\infty$ and $A$ is equal to $(L^p, \ell^1)$ ($1 < p < \infty$) or $(C_0, \ell^1)$ then the characterizations (1) and (2) hold for elements of $M(A, L^\infty)$; if $A$ is equal to $(L^p, \ell^q)$ ($1 < p, q < \infty$), $(L^p, c_0)$ ($1 < p < \infty$), $(L^1, \ell^q)$ ($1 < q \leq \infty$) or $(C_0, \ell^q)$ ($1 \leq q < \infty$) then (1) holds for the elements of $M(A, L^\infty)$ (Section 15).

Finally in Section 16 we prove (Theorem 16.11) that if $A$ is equal to $(L^p, \ell^1)$ ($1 \leq p \leq \infty$) or $(C_0, \ell^1)$ and $B$ is any amalgam space or any space of unbounded measures of type $q$ then the characterizations (1) and (2) for elements of $c-M(A, B)$ hold and are equivalent.

This implies a characterization for elements of $M(A, B)$ ($A$, $B$ any amalgam space or any space $M_q$) similar to (1) and (2). Specifically, $A$ contains an algebra $S_A$ such that for $T$ in $M(A, B)$ there exist unique $\mu$ and $\varphi$ in some linear spaces such that $Tf = \mu f$ and $(Tf) = \varphi f$ for all $f \in S_A$ (Theorem 16.18).
CHAPTER I

AMALGAMS OF \( L^p \) AND \( L^q \)

We begin this chapter by introducing the notation we will use throughout the whole work.

For a locally compact compact group \( G \) with Haar measure \( m \), \( L^p(G) = L^p \), \( 1 \leq p < \infty \), will be, as usual, the Banach space of measurable (classes of) functions \( f \) such that
\[
\int_G |f(x)|^p \, dm(x) \text{ is finite}
\]
and \( L^\infty(G) = L^\infty \) will be the Banach space of measurable (classes of) functions which are essentially bounded. For a subset \( E \) of \( G \) the quantity \( \sup_{E} |f(x)| \) will mean \( \text{ess sup}_{E} |f(x)| \).

The integration of a measurable function \( f \) on \( G \) will be always with respect to \( m \) and we might write \( \int f \) or \( \int f(x) \, dx \) instead of \( \int_G f(x) \, dm(x) \).

\( C_c(G) = C_c, C^0(G) = C^0 \) will denote the linear space of continuous functions on \( G \), which have compact support, vanish at infinity, respectively.

The characteristic function of a subset \( E \) of \( G \) will be denoted by \( \chi_E \).

If \( J \) is a linear space of functions on \( G \) then \( J_c, J_{loc} \) will be the set of functions \( f \) in \( J \) such that \( f \) has compact support, \( f \) restricted to any compact subset \( E \) of \( G \), that is, \( f\chi_E = f|E \) belongs to \( J \). For any subset \( E \) of \( G \), \( J|E = \{ f|E \mid f \in J \} \).
By a measure $\mu$ on $G$ we will mean a complex-valued set function on $G$ which locally is a complex measure. That is, for any compact subset $E$ of $G$, $\mu_E(B) = \mu(B \cap E)$ is a complex measure as in [43, Chapter 6].

This is consistent with the functional approach of Bourbaki because the function $\mu(f) = \int f \, d\mu = \int_E f \, d\mu_E$ ($f \in C_c(G)$ with support $E$) is a continuous linear functional on $C_c(G)$ topologized as the internal inductive limit of the spaces $C_E(G) = \{f \in C_c(G) | \text{supp } f \subseteq E\}$.

$\mathcal{V}(G)$ will denote the linear space of (Radon) measures on $G$.

We will use additive notation for the operation on abelian groups and its identity will be denoted by $0$.

If $\hat{G}$ is the dual group of $G$ then for $\hat{x} \in \hat{G}$ we will write $[x, \hat{x}]$ instead of $\hat{x}(x)$ $(x \in G)$.

The difference of two sets $A, B$ will be denoted by $A \cap B$, that is, $A \cap B = \{x \in A | x \notin B\}$.

If $f$ is a function on $G$ then $f', \tilde{f}$ will be the functions on $G$ defined by $f'(x) = f(-x)$, $\tilde{f}(x) = \overline{f(-x)}$, respectively.

If $\mu$ is a measure on $G$ then $L^P(\mu)$ will always mean $L^P(|\mu|)$. 
§ 1. AMALGAMS OF $L^p$ AND $\ell^q$ AND SPACES OF UNBOUNDED MEASURES
OF TYPE $q$.

Several definitions of amalgams of $L^p$ and $\ell^q$ as well as of unbounded measures of type $q$ have appeared recently as a consequence of research done in different areas. We shall give here these definitions in chronological order of publication and immediately after we will proceed to prove their equivalence.

First in 1975 F. Holland [34] defined the amalgam spaces $(L^p, \ell^q)$ and the spaces of unbounded measures $M_q$ for the real line as follows:

**DEFINITION 1.1.** For $f \in L^p_{loc} (\mathbb{R})$, $1 \leq p \leq \infty$, we define

$$|| f ||_{p,q} = \left[ \frac{1}{2} \sum_{n=0}^{\infty} \left( \int_n^{n+1} |f|^p \right)^{q/p} \right]^{1/q}$$

if $1 \leq p, q < \infty$

$$|| f ||_{\infty,q} = \left[ \frac{1}{2} \sup_{[n,n+1]} |f|^{q} \right]^{1/q}$$

if $p = \infty$, $1 \leq q < \infty$

$$|| f ||_{p,\infty} = \sup_{2} \left[ \int_n^{n+1} |f|^p \right]^{1/p}$$

if $1 \leq p < \infty$, $q = \infty$.

Then the amalgam of $L^p$ and $\ell^q$ is the linear space

$$(L^p, \ell^q) = \{ f \in L^p_{loc} (\mathbb{R}) | || f ||_{p,q} < \infty \} \quad 1 \leq p, q \leq \infty.$$ 

Special cases of $(L^p, \ell^q)$ have appeared before; see for example [54] and [53].
DEFINITION 1.2. For a measure $\mu$ on $\mathbb{R}$, we define

$$
\| \mu \|_q = \left[ \sum_{Z} |\mu([n, n+1])|^q \right]^{1/q} \quad 1 \leq q < \infty
$$

$$
\| \mu \|_{\infty} = \sup_{Z} |\mu([n, n+1])|
$$

Then the space of unbounded measures of type $q$ is the linear space

$$
\mathcal{M}_q = \{ \mu \in V(\mathbb{R}) | \| \mu \|_q < \infty \} \quad 1 \leq q \leq \infty.
$$

Later in 1978 J. P. Bertrandias, C. Datry and C. Dupuis [8] defined the spaces $\ell^q(\ell^p)$, $\ell^q(M)$ as a consequence of an earlier paper due to J. P. Bertrandias [5] about the Riesz spaces $\bigcap^p$ and $\bigcup^q$. Their definition is as follows:

Let $G$ be a locally compact abelian group and $E$ be a nonempty, relatively compact, Borel set of $G$.

DEFINITION 1.3. A tiling of $G$ by $E$ is a pairwise disjoint family

$$
\{E_i \mid i \in I\} = \{t_i + E \mid t_i \in G, i \in I\}
$$

of translates of $E$.

DEFINITION 1.4. Let $P$ be the set of all tilings of $G$ by $E$ and

$$
1 \leq p, q \leq \infty. \text{ For } f \in L^p_{\text{loc}}(G), \text{ define}
$$

$$
\| f \|_{q,p} = \sup_{\{E_i \mid i \in I\} \in P} \left[ \sum_{i \in I} \left[ \int_{E_i} |f|^{q/p} \right]^{q/p} \right]^{1/q} \quad \text{if } 1 \leq q, p < \infty
$$

$$
\| f \|_{q,\infty} = \sup_{\{E_i \mid i \in I\} \in P} \left[ \sum_{i \in I} \sup_{E_i} |f|^{q} \right]^{1/q} \quad \text{if } p = \infty, 1 \leq q < \infty
$$

$$
\| f \|_{\infty,p} = \sup_{x \in G} \left[ \int_{-E} |f|^p \right]^{1/p} \quad \text{if } 1 \leq p < \infty, q = \infty
$$
Then the amalgam of $L^p$ and $\ell^q$ is the linear space
\[\ell^q(L^p) = \{ f \in L^p_{\text{loc}}(G) \mid \| f \|_{\ell^q} < \infty \} \quad 1 \leq p, q \leq \infty\]

**Definition 1.5.** For a measure $\mu$ on $G$, define
\[\| \mu \|_{q1} = \sup_{\{E_i\} \in \mathcal{P}} \left[ \sum_{i \in I} \| \mu \|(E_i)^q \right]^{1/q} \quad 1 \leq q < \infty\]
\[\| \mu \|_{q\infty} = \sup_{x \in G} \| \mu \|((x-E)\mathcal{E})\]

Then the space of unbounded measures of type $q$ is the linear space
\[\ell^q(M) = \{ \mu \in \mathcal{V}(G) \mid \| \mu \|_{q1} < \infty \} \quad 1 \leq q \leq \infty\]

**Remark.** Originally the spaces $\ell^q(L^p)$ and $\ell^q(M)$ were defined on locally compact abelian groups, but it is clear that this definition is equally valid for nonabelian groups.

The dependence on the set $E$ is in essence irrelevant because the spaces $\ell^q(L^p)$ and $\ell^q(M)$ defined from two different subsets are isomorphic [8, §7 a)].

In 1979 J. Stewart extended the definition of F. Holland to locally compact abelian groups using the Structure Theorem for locally compact groups [49].

Let $G$ be a locally compact abelian group. By the Structure Theorem [37, Theorem 24.30], $G$ is topologically isomorphic with $\mathbb{R}^d \times G_1$ where $d$ is a nonnegative integer and $G_1$ is a locally compact abelian group which contains an open compact subgroup $H$.

If $G_1$ is compact we can take $H = G_1$, if $G$ is discrete and infinite we can take $H = \{0\}$. Otherwise $H$ is arbitrary but fixed.
The Haar measure \( m_1 \) on \( G_1 \), is normalized so that \( m_1(H) = 1 \).

We then take the Haar measure \( m \) on \( G \) to be the product of the Lebesgue measure on \( \mathbb{R}^d \) and \( m_1 \).

**Definition 1.6.** Define \( L = (0,1)^d \times H \) and \( K = [0,1]^d \times H \).

Then we can write \( G \) as a union of compact sets \( G = \bigcup_{\alpha \in J} K_\alpha \) and as a union of relatively compact, Borel sets \( G = \bigcup_{\alpha \in J} L_\alpha \) where \( J = \mathbb{Z}^d \times T \), \( T \) being a transversal of \( H \) in \( G_1 \), that is \( G_1 = \bigcup_{g \in T} g + H \), and

\[
L_\alpha = \alpha + L, \quad K_\alpha = \alpha + K. \quad \text{(Note that } m(L_\alpha) = m(K_\alpha) = 1 \text{ for all } \alpha \in J).\]

**Definition 1.7.** For \( f \in L^p_{\text{loc}}(G) \) \( 1 \leq p \leq \infty \) define

\[
\| f \|_{pq} = \left( \sum_{\alpha \in J} \left[ \int_{K_\alpha} \left| f \right|^p \right]^{q/p} \right)^{1/q}
\]

if \( 1 \leq p, q < \infty \)

\[
\| f \|_{\infty q} = \left( \sum_{\alpha \in J} \sup_{K_\alpha} \left| f \right|^q \right)^{1/q}
\]

if \( p = \infty, 1 \leq q < \infty \)

\[
\| f \|_{p\infty} = \sup_{\alpha \in J} \left[ \int_{K_\alpha} \left| f \right|^p \right]^{1/p}
\]

if \( 1 \leq p < \infty, q = \infty \)

Then the amalgam of \( L^p \) and \( L^q \) is the linear space

\( (L^p, L^q) = \{ f \in L^p_{\text{loc}}(G) \mid \| f \|_{pq} < \infty \} \quad 1 \leq p, q \leq \infty \)

**Remark 1.8.** From the definition of the families \( \{L_\alpha\}, \{K_\alpha\} \) and the nature of the Haar measure \( m \), it is clear that

\[
\| f \|_{pq} = \left( \sum_{\alpha \in J} \left[ \int_{L_\alpha} \left| f \right|^p \right]^{q/p} \right)^{1/q}
\]

if \( 1 \leq p, q < \infty \)

\[
\| f \|_{\infty q} = \left( \sum_{\alpha \in J} \sup_{L_\alpha} \left| f \right|^q \right)^{1/q}
\]

if \( p = \infty, 1 \leq q < \infty \)
\[ \| f \|_q = \sup_{\alpha \in J} \left( \int_{L_{\alpha}} |f|^p \right)^{1/p} \quad \text{if } 1 \leq p < \infty, \quad q = \infty. \]

**Definition 1.9.** For a measure \( \mu \) on \( G \), define

\[ \| \mu \|_q = \left[ \sum_{\alpha \in J} |\mu|(K_{\alpha})^q \right]^{1/q} \quad 1 \leq q < \infty \]

\[ \| \mu \|_\infty = \sup_{\alpha \in J} |\mu|(K_{\alpha}) \]

Then the space of unbounded measures of type \( q \) is the linear space

\[ M_q = \{ \mu \in \mathcal{V}(G) | \| \mu \|_q < \infty \} \quad 1 \leq q \leq \infty. \]

**Remark.** If \( G = \mathbb{R} \) then the family \( \{K_{\alpha}\} \) becomes the collection of intervals \([n, n+1) \quad (n \in \mathbb{Z})\). Hence, in this case, Definition 1.1 and Definition 1.7 coincide.

Finally in 1980 R. C. Busby and H. A. Smith [12] used the space \( L_p^\pi \) to solve problems in compact operator theory. Their definition is for locally compact, not necessarily abelian, groups and is based on a so-called \((U-V)\) uniform partition of the group.

Let \( G \) be a locally compact group.

**Definition 1.10.** Let \( U, V \) be two relatively compact open neighborhoods of the identity with \( \overline{U} \subseteq V \). A partition \( \pi \) of \( B \) into disjoint Borel subsets is \((U-V)\) uniform if for each \( W \in \pi \) there exists \( x \in G \) such that \( xU \subseteq W \subseteq xV \).

**Definition 1.11.** For \( 1 \leq p, q \leq \infty \) and a \((U-V)\) uniform partition \( \pi \) on \( G \), we define, for a measurable function \( f \) such that \( f|_W \in L^p(W) \)
for all \( W \in \pi \),

\[
\| f \|_{pq}^\pi = \left[ \sum_{W \in \pi} \left[ \int_W | f |^p \right]^{q/p} \right]^{1/q}
\]

if \( 1 \leq p, q < \infty \)

\[
\| f \|_{\infty q} = \left[ \sum_{W \in \pi} \sup_W | f |^q \right]^{1/q}
\]

if \( p = \infty, 1 \leq q < \infty \)

\[
\| f \|_{p\infty} = \sup_{W \in \pi} \left[ \int_W | f |^p \right]^{1/p}
\]

if \( 1 \leq p < \infty, q = \infty \).

Then the amalgam of \( L^p \) and \( \ell^q \) is the linear space

\[
L_{pq}^\pi = \{ f \mid \| f \|_{pq}^\pi < \infty \}
\]

\( 1 \leq p, q \leq \infty \).

The dependence of \( L_{pq}^\pi \) on the partition \( \pi \) is irrelevant because for \( \pi, \pi' \) uniform partitions, the spaces \( L_{pq}^\pi, L_{pq}^{\pi'} \) are isomorphic [12, Proposition 3.8].

Each of the spaces \( \ell^q(L^p), (L^p, \ell^q), L_{pq}^\pi, \ell_q(M), M_q \) \( 1 \leq p, q \leq \infty \) is a Banach space under the norm, \( \| \cdot \|_{qp}, \| \cdot \|_{pq}, \| \cdot \|_{\pi q}, \| \cdot \|_{q1}, \| \cdot \|_q \) respectively ([18, 57, b]), [49, Theorem 3.2], [12, Definition 3.6]).

Hereafter and throughout the whole work, \( G \) will be a locally compact abelian group, the Haar measure \( m \), the sets \( L, K \) and the families \( \{ L_\alpha \}_\alpha, \{ K_\alpha \}_\alpha \) will be as in Definition 1.6 and the number \( a \) will always correspond to the nonnegative integer which appears in the decomposition of \( G \), unless otherwise stated.

There are several facts we ought to know before we proceed to prove the equivalence of \( \ell^q(L^p), (L^p, \ell^q), L_{pq}^\pi (\ell_q(M), M_q) \).

These are:
1) If $L_1 = t_1 + L$ is a translate of the set $L$, then the set $S(L_1) = \{K_\alpha \mid K_\alpha \cap L_1 \neq \emptyset\}$ has cardinality less than or equal to $2^d$.

2) Let $\pi$ be a $(U-V)$ uniform partition on $G$.

a) The sets $S(K_\alpha) = \{W \in \pi \mid K_\alpha \cap W \neq \emptyset\}$, $S(U) = \{K_\alpha \mid K_\alpha \cap U \neq \emptyset\}$ have cardinality less than or equal to the integers $n(V,K)$,

$2^d n(K,V)$, where $n(V,K)$, $n(K,V)$ are the greatest integers in the real numbers $m(K-V+U)/m(U)$, $m(V-K+U)/m(U)$ respectively.

b) Let $1 \leq p \leq \infty$, $f \in L^p_{\text{loc}}$ iff $f|_W \in L^p(W)$ for all $W \in \pi$.

Part 1) is clear. Part 2), a) follows from the proposition below; its proof can be found in [12, Propositions 3.4 and 3.5]. Part b) is clear.

**Proposition 1.12.** If $\pi$ is a $(U-V)$ uniform partition on $G$ then for each $W \in \pi$ there exists $x_w \in G$ such that $x_w + U \subseteq W \subseteq x_w + V$.

Let $F = \{x_w \mid W \in \pi\}$ and $E$ be a relatively compact Borel set.

i) Every translate of $E$ meets (and so is covered by) at most $n(V,E)$ members of $\pi$.

ii) Each translate of a member of $\pi$ meets at most $n(E,V)$ of the translates $x_w + E$ ($x_w \in F$).

**Lemma 1.13.** [12, Proposition 2.1]. Let $\{a_1, \ldots, a_n\}$ be nonnegative real numbers.

a) If $1 \leq p < \infty$ then

$$(a_1 + a_2 + \cdots + a_n)^p \leq \frac{1}{n} (a_1^p + a_2^p + \cdots + a_n^p).$$

b) If $0 < p \leq 1$ then

$$(a_1 + a_2 + \cdots + a_n)^p \leq a_1^p + \cdots + a_n^p.$$
Now we are ready to prove the next three theorems about the spaces \( \ell^q(L^p), (L^p, \ell^q), L^q_{pq}, \ell^q(M) \) and \( M_q \).

**Theorem 1.14.** The space \( \ell^q(L^p) \) defined from the set \( L \) is isomorphic (as a Banach space) to \( (L^p, \ell^q) \) for \( 1 \leq p, q \leq \infty \).

**Proof.** We note that the family \( \{L_\alpha\}_{\alpha} \) is a tiling of \( G \) by \( L \).

Then by the definition of \( \| \cdot \|_{pq} \) and Remark 1.8, we have that for all \( f \in \ell^q(L^p), \| f \|_{pq} \leq \| f \|_{qp} (1 \leq p, q \leq \infty) \). Hence \( \ell^q(L^p) \subseteq (L^p, \ell^q) \).

Now take \( f \in (L^p, \ell^q) \) and \( \{L_\alpha\}_{\alpha} \) a tiling of \( G \) by \( L \). Then for \( 1 \leq p, q < \infty \) and each \( i \in I \),

\[
\left\{ \sum_{\alpha \in S(L_\alpha)} \int_{K_\alpha} |f|^p \right\}^{q/p} \leq \sum_{\alpha \in S(L_\alpha)} \left( \sum_{i \in I} \int_{K_\alpha} |f|^q \right)^{q/p}
\]

where \( S(L_\alpha) \) is as in 1) above.

Applying Lemma 1.13 we have that

\[
\sum_{i \in I} \left[ \int_{L_i} |f|^p \right]^{q/p} \leq (2^a)^{q/p-1} \sum_{\alpha \in S(L_\alpha)} \left( \sum_{i \in I} \int_{K_\alpha} |f|^q \right)^{q/p}
\]

\[
\sum_{i \in I} \sup_{L_i} |f|^q \leq \sum_{\alpha \in S(L_\alpha)} \sup_{K_\alpha} |f|^q
\]

\[
\sup_{x \in G} \left[ \int_{x-L} |f|^p \right]^{1/p} \leq \sup_{x \in G} \sum_{\alpha \in S(L_\alpha)} \left( \int_{K_\alpha} |f|^p \right)^{1/p}
\]

This implies that for \( 1 \leq p, q < \infty \)

\[
\left[ \sum_{i \in I} \left[ \int_{L_i} |f|^p \right]^{q/p} \right]^{1/q} \leq 2^{a/p} \| f \|_{pq}
\]

\[
\left[ \sum_{i \in I} \sup_{L_i} |f|^q \right]^{1/q} \leq 2^{a/q} \| f \|_{\ell^q}
\]
\[ \sup_{x \in G} \left( \int_{x-L} |f|^{|p|} \right)^{1/p} \leq 2^a \| f \|_{p_{\infty}}. \]

Since \{L_j\} is arbitrary we conclude that for \( 1 \leq p, q < \infty \)
\[ \frac{a/p}{|| f ||_{q_p}} \leq \frac{a/q}{|| f ||_{q_{\infty}}} \leq 2 \| f \||_{q_{\infty}} \text{ and} \]
\[ \frac{a/p}{|| f ||_{\|q_{\infty}}} \leq 2 \| f \||_{p_{\infty}}. \]
Therefore \((L^p, \mathbb{R}^q) \subseteq \mathbb{R}^q(L^p)\) for \( 1 \leq p, q \leq \infty. \)

**THEOREM 1.15.** The space \( L^p_{pq} \) is isomorphic (as a Banach space) to \((L^p, \mathbb{R}^q)\).

**PROOF.** The argument is identical to the one used in the last part of the previous theorem.

Let \( \pi \) be a \((U-V)\) uniform partition on \( G \) and \( f \in (L^p, \mathbb{R}^q) \). By Lemma 1.13 and 2), a) above we have that for \( 1 \leq p, q < \infty \)
\[ \sum_{W \in \pi} \left( \int_{W} |f|^{|p|} \right)^{q/p} \leq (2^n(K, V))^q \sum_{W \in \pi} \sum_{K_\alpha \in S(W)} \left( \int_{K_\alpha} |f|^{|p|} \right)^{q/p} \]
\[ \sum_{W \in \pi} \left( \sup_{W} |f|^{|q|} \right) \leq \sum_{W \in \pi} \sum_{K_\alpha \in S(W)} \sup_{W} |f|^{|q|} \]
\[ \sup_{W \in \pi} \left( \int_{W} |f|^{|p|} \right)^{1/p} \leq \sup_{W \in \pi} \sum_{K_\alpha \in S(W)} \left( \int_{K_\alpha} |f|^{|p|} \right)^{1/p} \]

This implies that for \( 1 \leq p, q < \infty \)
\[ || f ||_{pq} \leq (2^n(K, V))^q || f ||_{pq} \]
\[ || f ||_{\|q_{\infty}} \leq (2^n(K, V))^q || f ||_{\|q_{\infty}} \]
\[ || f ||_{p_{\infty}} \leq 2^n(K, V) || f ||_{p_{\infty}}. \]
Therefore $L^\infty_{pq} \subseteq (L^p, L^q)$. \\

**Theorem 1.16.** The space $\mathcal{L}^q(M)$ defined from the set $L$ is isomorphic (as a Banach space) to $M_q$ for $1 \leq q \leq \infty$.

**Proof.** As in Theorem 1.14, the family $\{L_\alpha\}$ is a tiling of $G$ by $L$ and we have that for all $\mu \in \mathcal{L}^q(M)$ $\|\mu\|_q \leq \|\mu\|_1$ if $1 \leq q < \infty$.

On the other hand, since $K$ is covered by a finite number of translates $t_i - L$ $i = 1, \ldots, n$, each $K_\alpha = \alpha + K$ ($\alpha \in J$) is covered by the finite family $\{ \alpha + t_i - L \}$ $i = 1, \ldots, n$ of translates of $L$. So, for each $\alpha \in J$

$$|\mu|(K_\alpha) \leq \sum_{i=1}^n |\mu|(\alpha + t_i - L) \leq n \|\mu\|_{\infty}$$

if $\mu \in \mathcal{L}^\infty(M)$.

This implies that $\|\mu\|_\infty \leq n \|\mu\|_{\infty}$ and therefore $\mathcal{L}^q(M) \subseteq M_q$ for $1 \leq q \leq \infty$.

If $\mu \in M_q$ and $\{L_i\}_{i \in I}$ is a tiling of $G$ by $L$, then by 1) above

$$|\mu|(L_1) \leq \sum_{K_\alpha \in S(L_1)} |\mu|(K_\alpha)$$

$$|\mu|(x - L) \leq \sum_{K_\alpha \in S(x - L)} |\mu|(K_\alpha).$$

By Lemma 1.13 we have that for $1 \leq q < \infty$

$$\sum_{i \in I} |\mu|(L_i)^q \leq (2^q)^{q-1} \|\mu\|_q^q \quad \text{and} \quad \sup_{x \in G} |\mu|(x - L) \leq 2^q \|\mu\|_{\infty}.$$

Hence, $\|\mu\| \leq (2^q)^{1-1/q}$ if $1 \leq q < \infty$ and

$$\|\mu\|_\infty \leq 2^q \|\mu\|_1.$$

Therefore $M_q \subseteq \mathcal{L}^q(M)$.

**Remark 1.17.** 1) Note that the space $M_1$ is just the space of bounded regular measures $M(G)$. 
11) The space $M_\alpha$ corresponds to what Argabright and Gil de Lamadrid call the space of translation bounded measures [1, p. 5].

We will write $(L^p, L^q)(G)$, $M_\alpha(G)$ to emphasize the group on which the spaces $(L^p, L^q)$, $M_\alpha$ are defined.

We define $c_0(J)$ to be the linear space of nets $(a_\alpha)_J$ ($a_\alpha \in \mathbb{C}$, $\alpha \in J$) such that $\lim_{\alpha} a_\alpha = 0$. That is, given $\varepsilon > 0$ there exists a compact subset $E$ of $G$ such that $|a_\alpha| < \varepsilon$ for all $\alpha \notin E$.

**Definition 1.18.** For $1 \leq r < \infty$, $(C_0, L^r) = C_0 \cap (L^\infty, L^r)$ and $(L^r, C_0) = \{f \in (L^r, L^\infty) : \|f\|_r \leq \text{lim sup}_{\alpha} \|f\|_r(K_\alpha) \}_{J \in C_0}.$

**Proposition 1.19.** Let $1 \leq r < \infty$, $(C_0, L^r)$ is equal to the set of continuous functions in $(L^\infty, L^r)$.

**Proof:** Let $f$ be a continuous function in $(L^\infty, L^r)$, and $V = \{V_i \mid i \in I\}$ be the set of finite unions of $K_\alpha$'s ($\alpha \in J$), directed by $V_i \supseteq V_j$ iff $V_i \subseteq V_j$. Since $f \in (L^\infty, L^r)$, the series

$$\sum_{V_i} \|f\|_r(K_\alpha)$$

converges, and consequently the net $\{\sum_{V_i} \|f\|_r(K_\alpha)\}_{V_i}$ converges to zero. Then for any $\varepsilon > 0$ there exists $V_j$ such that

$$\sum_{V_i} \|f\|_r(K_\alpha) < \varepsilon$$

for all $V_i \supseteq V_j$. Therefore $|f(x)| < \varepsilon$ for all $x \in V_i$, $V_i \supseteq V_j$, and this implies that $|f(x)| < \varepsilon$ for all $x \notin V_j$.

Hence $f \in C_0$.

Definition 1.1 and Proposition 1.19 show clearly that the amalgam space $(C_0, L^1)(\mathbb{R})$ is the algebra defined by N. Wiener in [54].
We will use the next lemma to prove that the norm $||\cdot||_{pq}^\beta$ defined in Theorem 1.21 - first introduced in [8, Proposition VII] - is equivalent to the norm $||\cdot||_{pq}$.

The importance of $||\cdot||_{pq}^\beta$ will be seen in Chapter V.

**Lemma 1.20.** If $F^+ = \{(y_1, \ldots, y_\alpha, 0) \in G | y_1 \in \{0,1\} \text{ i = 1, \ldots, } \alpha\}$ and $F^- = \{(y_1, \ldots, y_\alpha, 0) \in G | y_1 \in \{0,-1\} \text{ i = 1, \ldots, } \alpha\}$ then

i) For all $x \in G$, $x + L \subseteq u(L_\alpha | x + y \in L_\alpha, y \in F^+)$

ii) For all $x \in L_\alpha$, $(\alpha \in J)$, $L_\alpha \subseteq x + F^- + L$.

**Proof.**

i) For $x = (x_1, \ldots, x_\alpha, x')$ in $G$, there exists a unique $L_\alpha = [n_1, n_1+1)^\times \ldots \times [n_\alpha, n_\alpha+1)^\times (t+H)$, $n_\alpha \in \mathbb{Z}$, $i = 1, \ldots, \alpha$, $t \in T$ (see Definition 1.6), such that $x \in L_\alpha$. Hence $n_i \leq x_i < n_i + 1$ $i = 1, \ldots, \alpha$ and $x' \in t+H$.

On the other hand, $x + L = [x_1, x_1 + 1), \ldots, [x_\alpha, x_\alpha + 1)^\times (x'+H)$, and we have that for $z = (z_1, \ldots, z_\alpha, z')$ in $x + L$, $x_1 \leq z_i < x_i + 1, i = 1, \ldots, \alpha$ and $z' \in x'+H$.

Define for $i = 1, \ldots, \alpha$

$$y_i = \begin{cases} 0 & \text{if } x_i \leq z_i < n_i + 1 \\ 1 & \text{if } n_i + 1 \leq z_i < x_i + 1 \end{cases}$$

Hence $y = (y_1, \ldots, y_\alpha, 0)$ belongs to $F^+$ and for $i = 1, \ldots, \alpha$

$$x_i + y_i = \begin{cases} x_i & \text{if } x_i \leq z_i < n_i + 1 \\ x_i + 1 & \text{if } n_i + 1 \leq z_i < x_i + 1 \end{cases}$$

Now, if $x + y \in L_\beta$, $L_\beta = [m_1, m_1 + 1)^\times \ldots \times [m_\alpha, m_\alpha + 1)^\times (t+H)$

$m_i \in \mathbb{Z}$, $i = 1, \ldots, \alpha$, $t \in T$, for some $\beta$, then $z' \in t+H$ since $x' \in t+H$ and $m_i \leq z_i < m_i + 1$ $i = 1, \ldots, \alpha$ because for $i = 1, \ldots, \alpha$
if \( x_i + y_i = x_i \) then \( m_i = n_i \) and \( m_i = n_i \leq x_i < n_i + 1 = m_i + 1 \)
and if \( x_i + y_i = x_i + 1 \) then \( m_i = n_i + 1 \) and
\[
\begin{align*}
\forall i \leq n & < m_i \leq z_i < x_i + 1 < n_i + 2 = m_i + 1; \\
& \text{in either case } m_i \leq z_i < m_i + 1 \\
i = 1, \ldots, \alpha. \text{ Therefore } z \in L_\beta \text{ and this implies that } x + L \subseteq \bigcup \{ L_\beta \mid x + y \in L_\beta, \ y \in F^+ \}.
\]

ii) First we will prove that if \( x = (x_1, \ldots, x_\alpha, x') \) belongs to \( L \) then \( L \subseteq x + F^- + L \). Let \( z = (z_1, \ldots, z_\alpha, z') \) be in \( L \). Then \( x', z' \)
belong to \( H \) and \( x_i, z_i \) are in \([0,1]\) \( i = 1, \ldots, \alpha \). So, for \( i = 1, \ldots, \alpha \),
\[
0 \leq z_i < x_i \text{ or } x_i \leq z_i < 1. \text{ Define }
\]
\[
y_i = \begin{cases}
-1 & \text{if } 0 \leq z_i < x_i \\
0 & \text{if } x_i \leq z_i < 1
\end{cases}
\]
Clearly \( y = (y_1, \ldots, y_\alpha, 0) \) belongs to \( F^- \) and we have that
\[
x_i + y_i + [0,1) = \begin{cases}
[x_i, x_i + 1) & \text{if } x_i \leq z_i < 1 \\
x_i - 1, x_i) & \text{if } 0 \leq z_i < x_i
\end{cases} \quad i = 1, \ldots, \alpha.
\]
Therefore \( z \in x + y + L \).

Now, we take any \( z \in L_\alpha \), \( L_\alpha = \alpha + L \). Then \( z = \alpha + x \) for
some \( x \in L \). By our previous result \( L \subseteq x + F^- + L \) hence
\[
L_\alpha \subseteq \alpha + x + F^- + L = z + F^- + L \text{ and the proof is complete.}
\]

**Theorem 1.21.** 1) A function \( f \) belongs to \((L^p, L^q), 1 \leq p, q \leq \infty \),
iff the function \( \| f \|_q \) on \( G \) defined by
\[
f^\|^q(x) = \| f \|_{L^p(x + L)}
\]
belongs to \( L^q \).

If \( \| f \|_{pq} = \| f \|^q_q \) then
\[
2^{-\alpha} \| f \|_{pq} \leq \| f \|^q_q \leq 2^\alpha \| f \|_{pq}.
\]
ii) A measure $\mu$ belongs to $M_q$, $1 \leq q \leq \infty$ iff the function $\mu^\theta$ defined by

$$\mu^\theta(t) = |\mu|(t + L)$$

belongs to $L^q$.

If $||\mu||_q = ||\mu^\theta||_q$ then

$$2^{-a} ||\mu||_q \leq ||\mu^\theta||_q \leq 2^a ||\mu||_q.$$  

**PROOF.** 1) For $x \in G$, $x \in L_\alpha$ for some $\alpha$, define the function $f^0(x) = ||f||_{L^P(L_\alpha)}$. Since $m(L_\alpha) = 1$ for all $\alpha$ we have that if $l \leq q < \infty$ then

$$||f^0||_q = \sum_{\alpha} \int_{L_\alpha} |f|^q = \sum_{\alpha} ||f||_{L^P(L_\alpha)}^q = \sum_{\alpha} ||f||_{L^P(L_\alpha)}^q = ||f||_{pq}^q$$

and

$$||f^0||_\infty = \sup_{G} ||f||_{L^P(L_\alpha)} = \sup_{\alpha} ||f||_{L^P(L_\alpha)} = ||f||_{pq}^\infty.$$

Therefore

$$(1) \quad ||f^0||_q = ||f||_{pq} \quad \text{for} \quad 1 \leq p, q \leq \infty.$$  

By Lemma 1.20, for any $x \in G$

$$f^\theta(x) \leq \sum_{y \in F^+} f^0(x + y) \quad \text{and} \quad f^0(x) \leq \sum_{y \in F^-} f^\theta(x + y).$$

Since the cardinality of $F^+$ and $F^-$ is $2^\alpha$, and for $t \in G$ and any measurable function $g$, $||\tau_t g||_q = ||g||_q$ ($1 \leq q \leq \infty$) where $\tau_t g(x) = g(x + t)$ ($x \in G$), we have from (1) that

$$||f^\theta||_{pq} \leq \sum_{y \in F^+} ||\tau_y f^0||_q = 2^a ||f^0||_q = 2^a ||f||_{pq}$$

$$||f||_{pq} \leq ||f^\theta||_q \leq \sum_{y \in F^-} ||\tau_y f^\theta||_q = 2^a ||f^\theta||_q = 2^a ||f||_{pq}^\theta.$$  

This implies i).
ii) As in i), for \( x \in G, \alpha \in L_{\alpha} \) for some \( \alpha \), define the function \( \mu^0(x) = |\mu|(K_\alpha) \). Then \( |\mu^0|_q = |\mu|_q \) \((1 \leq q \leq \infty)\) and by Lemma 1.20 for any \( x \in G \)

\[ \mu^0(x) \leq \sum_{y \in F^+} \mu^0(x + y). \]

Hence, for \( 1 \leq q \leq \infty \)

\[ |\mu|_q^0 = |\mu|_q^0 \leq \sum_{y \in F^+} |\tau_y \mu^0|_q = 2^a \quad |\mu^0|_q = 2^a \quad |\mu|_q^0. \]

\[ |\mu|_q = |\mu^0|_q \leq \sum_{y \in F^-} |\tau_y \mu^0|_q = 2^a \quad |\mu^0|_q = 2^a \quad |\mu|_q^0. \]

This implies ii).++

From Theorem 1.21 Bertandias, Datry and Dupuis [8] established a relation between \((L^P, \ell^q)\) and the mixed-norm spaces \(L^{pq} \) defined by Benedek and Panzone [11]. This relation is as follows:

\( L^{pq} \) is the normed linear space of measurable (classes of)

functions \( \phi \) on \( G \times G \) such that

\[ |\phi|^{pq} = \left[ \int_G \left[ \int_G |\phi(x, y)|^p \, dx \right]^{q/p} \right]^{1/q} < \infty \]

with the usual modifications if \( p = \infty \) or \( q = \infty \).

We associate to a function \( f \in (L^P, \ell^q) \) the function \( F \) in \( L^{pq} \) defined by \( F(x, y) = f(x)\chi(x - y) \) and to a function \( \phi \) in \( L^{pq} \) the function \( \phi^{pq} \) defined by \( \phi(x) = \int_{G \times G} \phi(x, y) \, dy. \)

By Theorem 1.21 \( |f|^{pq} \leq |F|^{pq} \) and as in [8, Proposition IX]

\[ |\phi|^{pq} \leq |\phi|^{pq}. \]

These imply that

a) The map \( f \mapsto F \) is an isometric, linear isomorphism from
(L^p, \ell^q) onto a linear subspace \( S_L^{pq} \) of \( L^p \), consisting of the functions \( F \) in \( L^{pq} \) of the form \( F(x,y) = f(x)\chi_L(y-x) \) with \( f \in (L^p, \ell^q) \).

b) The map \( \Phi \mapsto \phi \) is a continuous linear map from \( L^{pq} \) into \( (L^p, \ell^q) \).

c) The composition of \( f \mapsto F \) and \( \Phi \mapsto \phi \) is the identity map.

d) The composition of \( \Phi \mapsto \phi \) and \( f \mapsto F \) is a continuous linear map from \( L^{pq} \) onto \( S_L^{pq} \).

Busby and Smith have also pointed out another relation in [12, p. 316].

\textbf{REMARK 1.22.} It follows from Theorem 1.21 that the spaces \( V^p, V^{p,0}, N_p, M_p \) and \( W^p, (1 \leq p \leq \infty) \) defined in [42] are isomorphic to the spaces \( (L^1, \ell^p), (L^1, c_0), (L^\infty, \ell^p), (C_0, \ell^p) \) and \( M_p \) respectively.

\textbf{REMARK 1.23.} i) If \( G \) is compact then we can take the family of \( K_\alpha \)'s to be simply \( G \). Hence \( (L^p, \ell^q) = L^p, 1 \leq p, q \leq \infty \).

ii) If \( G \) is discrete then we can take \( K = \{0\} \) and therefore the family \( \{K_\alpha\} \) is \( Z^d \). Hence \( (L^p, \ell^q) = \ell^q, 1 \leq p, q \leq \infty \).

Then the amalgam spaces \( (L^p, \ell^q) \) are only of interest when \( G \) is neither compact nor discrete.

If \( \hat{G} \) is the dual group of \( G \) then \( \hat{G} = R^d \times \hat{G}_1 \) and \( \hat{G}_1 \) contains the open compact subgroup \( A \) which is the annihilator of \( H \), \( A = \{x \in G | [x, \hat{x}] = 1 \text{ for all } \hat{x} \in \hat{H} \} \). Hence we can choose \( A \) to define the families \( \{L^p_\beta \}_{\beta \in I}, \{K^p_\beta \}_{\beta \in I} \) in \( \hat{G} \) by \( L^p_\beta = \beta + \hat{L}, K^p_\beta = \beta + \hat{K} \) where \( I = Z^d \times T', T' \) being a transversal of \( A \) in \( \hat{G}_1 \), \( \hat{L} = [0,1] \times A \) and \( \hat{K} = [0,1] \times A \).
Using \( \{K_\beta\}_I \) we define as in Definitions 1.7 and 1.9, the amalgam space \((L^p, \ell^q)(\hat{\mathfrak{G}})\) and the space of unbounded measures of type \(q\) \(M_q(\hat{\mathfrak{G}})\).

Throughout the whole work, \(\{L_\beta\}_I\), \(\{K_\beta\}_I\), \(\hat{L}\) and \(\hat{K}\) will correspond to the sets so defined here, unless otherwise stated.
§ 2. INEQUALITIES AND INCLUSION RELATIONS

During our work we will make constant use of the inequalities and inclusion relations studied in this section.

PROPOSITION 2.1. For \( 1 \leq p \leq \infty \)

\[
(2.1) \quad (L^p, L^q) = L^p.
\]

PROOF. Definition 1.7.

We see from Proposition 2.1 that the theory of amalgam spaces on locally compact groups embraces the theory of \(L^p\) spaces.

PROPOSITION 2.2.

\[
(2.2) \quad \|\mu\|_p \leq \|\mu\|_q \quad 1 \leq q \leq p \leq \infty
\]

\[
(2.3) \quad \|f\|_{pq_2} \leq \|f\|_{pq_1} \quad 1 \leq q_1 \leq q_2 \leq \infty, 1 \leq p \leq \infty
\]

\[
(2.4) \quad \|f\|_{p_1q} \leq \|f\|_{p_2q} \quad 1 \leq p_1 \leq p_2 \leq \infty, 1 \leq q \leq \infty
\]

PROOF. Since \(|\mu|(K_\delta)^q \leq \sum_{\alpha \in J} |\mu|(K_\alpha)^q\) for all \(\delta \in J\), it is clear that

\[
\|\mu\|_\infty = \sup_{\delta \in J} |\mu|(K_\delta) \leq \left[ \sum_{\alpha \in J} |\mu|(K_\alpha)^q \right]^{1/q} = \|\mu\|_q.
\]

If \(p\) is finite then (2.2) follows from Jensen's inequality (as in [4, p.18]) with \(\chi_\alpha = |\mu|(K_\alpha).

If \(q_2 = \infty\) then

\[
\|f\|_{p^\infty} = \sup_{\alpha \in J} \|f\|_{L^p(K_\alpha)} \leq \left[ \sum_{\alpha \in J} \|f\|_{L^p(K_\alpha)}^q \right]^{1/q_1} = \|f\|_{pq_1}.
\]

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If \( q_2 < \infty \) then again by Jensen's inequality with
\[
\chi_\alpha = \left\| f \right\|_{L^P(K_\alpha)}
\]
we have that for \( q_1 < q_2 \)
\[
\left( \sum_{\alpha} \left| \frac{f}{\chi_\alpha} \right|^{q_2} \right)^{1/q_2} \leq \left( \sum_{\alpha} \left| f \right|^{q_1} \right)^{1/q_1}.
\]
Hence \( \left| f \right|_{L^p(K_\alpha)} \leq \left| f \right|_{L^q(K_\alpha)} \).

If \( p_2 = \infty \) then for each \( \alpha \in J \)
\[
\left| f \right|_{L^{1}(K_\alpha)} \leq \sup_{x \in K_\alpha} |f(x)| m(K_\alpha) = \left| f \right| \text{ since } m(K_\alpha) = 1.
\]
Therefore \( \left| f \right|_{L^1(K_\alpha)} \leq \left| f \right|_{L^\infty(K_\alpha)} \) \( 1 \leq q \leq \infty \).

If \( p_2 < \infty \) then by Hölder's inequality (as in [37, Corollary 12.5]) with \( f_1 = \left| f \right|_{K_\alpha}^{p_1}, \ f_2 = \chi_{K_\alpha}, \ \alpha_1 = \frac{p_1}{p_2}, \ \alpha_2 = 1 - \left( \frac{p_1}{p_2} \right) \)
we have that for any \( \alpha \in J \)
\[
\int_{K_\alpha} |f|^{p_1} = \int_{K_\alpha} \left| f \right|_1^{p_1} \chi_{K_\alpha} = \int_{K_\alpha} \left| f \right|_1^{p_2} \chi_{K_\alpha}^{\alpha_1} \chi_{K_\alpha}^{\alpha_2}
\]
\[
\leq \left( \int_{K_\alpha} \left| f \right|_1^{p_2} \chi_{K_\alpha} \right)^{\alpha_1} \left( \int_{K_\alpha} \chi_{K_\alpha} \right)^{\alpha_2} = \left( \int_{K_\alpha} \left| f \right|_1^{p_2} \right)^{p_1/p_2},
\]
again this is because \( m(K_\alpha) = 1 \). This implies that \( \left| f \right|_{p_1} \leq \left| f \right|_{p_2} \) for \( 1 \leq q \leq \infty \).

**COROLLARY 2.3.**

(2.5) \( (L^p, L^{q_1}) \subseteq (L^p, L^{q_2}) \) \( 1 \leq q_1 \leq q_2 \leq \infty, \ 1 \leq p \leq \infty \)

(2.6) \( (L^{p_2}, L^q) \subseteq (L^{p_1}, L^q) \) \( 1 \leq p_1 \leq p_2 \leq \infty, \ 1 \leq q \leq \infty \)

(2.7) \( (L^p, L^q) \subseteq L^p \cap L^q \) \( 1 \leq q \leq p \leq \infty \)

(2.8) \( L^p \cup L^q \subseteq (L^p, L^q) \) \( 1 \leq p \leq q \leq \infty \)

(2.9) \( \frac{M_p}{q} \subseteq M_q \) \( 1 \leq p \leq q \leq \infty \)

**PROOF.** It is clear that (2.5) and (2.6) follow from the inequalities (2.3) and (2.4) respectively, while (2.7) and (2.8) follow from
(2.5), (2.6) and (2.1). Finally (2.2) implies (2.9).

**THEOREM 2.4.** Of the inclusions below 1) and 8) are strict if \( G \) is noncompact, 2) is strict if \( G \) is nondiscrete and 3), 4) and 5) are strict if \( G \) is neither compact nor discrete.

1) \( (L^p, L^{q_1}) \subseteq (L^p, L^{q_2}) \quad 1 \leq q_1 < q_2 < \infty, 1 \leq p < \infty \)
2) \( (L^p, L^q) \subseteq (L^p, L^q) \quad 1 \leq p_1 < p_2 < \infty, 1 \leq q < \infty \)
3) \( (L^p, L^q) \subseteq L^p \cap L^q \quad 1 \leq q < p < \infty \)
4) \( L^q \subseteq (L^1, L^q) \cap (L^q, L^\infty) \quad 1 < q < \infty \)
5) \( L^p \subseteq (L^p, L^q) \cap (L^1, L^p) \quad 1 < p < q < \infty \)
6) \( (L^q, L^1) \cap (L^\infty, L^q) \subseteq L^q \quad 1 < q < \infty \)
7) \( (L^p, L^q) \cap (L^q, L^1) \subseteq L^q \quad 1 < q < p < \infty \)
8) \( L^q \subseteq (L^q, L^\infty) \quad 1 \leq q < \infty \).

**PROOF.** Let \( \{L_n\} \) be a countable subfamily of \( \{L_\alpha\} \) and \( \alpha \) be a real number in \( [1, \infty) \). Since \( m \) is regular and \( m(l_n) = 1 \) for all \( n \), given \( (1/n)^\alpha \) there exists a compact subset \( I_n \subseteq l_n \) such that

\[
m(L_n) - (1/n)^\alpha m(I_n).
\]

This implies that

\[
m(L_n \triangle I_n) = M(I_n) - m(I_n) < (1/n)^\alpha.
\]

Define \( f_n : L_n \to \mathbb{R} \) by

\[
f_n(x) = \begin{cases} n & x \in l_n \triangle I_n \\ 0 & \text{otherwise}. \end{cases}
\]

Hence for \( 1 \leq p, q < \infty \)

\[
\int_{L_n} |f_n|^p = \int_{L_n \setminus I_n} |f_n|^p = n^p m(l_n \triangle I_n) < n^p/n^\alpha = (1/n)^\alpha - p \quad \text{and}
\]

\[
\sup_{L_n} |f_n| = n^\alpha. \quad \text{Therefore for } \hat{f} = \sum_{n} f_n
\]
\[ \| f \|_{pq} = \left( \sum_{n} (1/n)^{(\alpha - p)q/p} \right)^{1/q} \]

\[ \| f \|_{p^\infty} = \sup_{N} \{(1/n)^{\alpha - p/p}\} \]

\[ \| f \|_{\infty, q} = \left( \sum_{n} n^q \right)^{1/q} \]

Then

\( f \in (L^p, l^q) \) if \((\alpha - p)q/p > 1\)

\( f \notin (L^p, l^q) \) if \((\alpha - p)q/p < 1\)

\( f \in (L^p, l^{\infty}) \) if \((\alpha - p)/p > 0\)

\( f \notin (L^p, l^{\infty}) \) if \((\alpha - p)/p < 0\)

\( f \notin (L^{\infty}, l^q) \) for \(1 \leq q < \infty\)

1) For \(1 \leq q_1 < q_2 < \infty\) and \(1 \leq p < \infty\), we take \(\alpha = (p/q_1) + p\).

Then \((\alpha - p)q_2/p = (q_2/q_1) > 1\) and \((\alpha - p)q_1/p = 1\). Therefore by a) and b) \( f \in (L^p, l^{q_2}) \) and \( f \notin (L^p, l^{q_1}) \). If \( q_2 = \infty \) and \( 1 \leq p < \infty \) then \((L^p, l^{q_1}) \subset (L^p, l^{\infty}) \) because for any \( q \), \( q_1 < q < \infty \)

\((L^p, l^{q_1}) \subset (L^p, l^{q_2}) \subseteq (L^p, l^{\infty}) \).

Suppose \( p = \infty \), \(1 \leq q_1 < q_2 < \infty\), and take \( \alpha \) such that

\( 1/q_2 < \alpha < 1/q_1 \), and define \( g : G \rightarrow R \) by

\[ g(x) = \begin{cases} 
(1/n)^{\alpha} & \text{if } x \in L_n \\
0 & \text{otherwise} 
\end{cases} \]

Hence \( \sup_{L_n} |g(x)| = (1/n)^{\alpha} \) and this implies that

\[ \| g \|_{\alpha, q_1} = \left( \sum_{n} (1/n)^{\alpha q_1} \right)^{1/q_1} \]

and

\[ \| g \|_{\alpha, q_2} = \left( \sum_{n} (1/n)^{\alpha q_2} \right)^{1/q_2} \]

Therefore \( g \notin (L^\infty, l^{q_1}) \) since \( \alpha q_1 < 1 \) and \( g \in (L^\infty, l^{q_2}) \) since \( \alpha q_2 > 1 \).
Suppose \( p = q_2 = \infty \) and \( 1 \leq q_1 < \infty \). Define the function \( g: G \rightarrow \mathbb{R} \) by
\[
g(x) = \begin{cases} 
\frac{1}{q_1} & x \in \mathcal{L}_n \\
0 & \text{otherwise}
\end{cases}
\]

Hence, \( \|g\|_\infty = \sup_{x_n} \frac{1}{q_1} < \infty \) and
\[
\|g\|_{q_1} = \left[ \frac{1}{n} \right]^{1/q_1} = \infty. \text{ Therefore } g \notin (L^\infty, \mathcal{L}^{q_1}) \text{ and } g \in L^\infty.
\]

2) For \( 1 \leq p_1 < p_2 < \infty \) and \( 1 \leq q < \infty \), take \( \alpha = (p_1/q) + p_2 \).

Then \( (\alpha - p_2)q/p_2 = 1 \) and \( \alpha > (p_1/q) + p_1 \). This means that

\( (\alpha - p_1)q/p_1 > 1 \). Therefore by a) and b) \( f \notin (L^{p_2}, \mathcal{L}^q) \) and

\( f \in (L^{p_1}, \mathcal{L}^q) \).

If \( p_2 = \infty \), \( 1 \leq p_1 < \infty \) and \( 1 \leq q < \infty \) then

\( (L^\infty, \mathcal{L}^q) \subseteq (L^{p_1}, \mathcal{L}^q) \) because for any \( p \), \( p_1 < p < \infty \)

\( (L^\infty, \mathcal{L}^q) \subseteq (L^p, \mathcal{L}^q) \subseteq (L^{p_1}, \mathcal{L}^q) \).

Suppose \( q = \infty \), \( 1 \leq p_1 < p_2 < \infty \). Take \( p_1 < \alpha < p_2 \), so

\( (\alpha - p_1)p_1 > 0 \) and \( (\alpha - p_2)p_2 < 0 \). Hence by c) and d), \( f \in (L^p, \mathcal{L}^\infty) \)

and \( f \notin (L^{p_2}, \mathcal{L}^\infty) \).

If \( q = p_2 = \infty \) then \( L^\infty \subseteq (L^{p_1}, \mathcal{L}^\infty) \) because for \( p_1 < p < \infty \),

\( L^\infty \subseteq (L^p, \mathcal{L}^\infty) \subseteq (L^{p_1}, \mathcal{L}^\infty) \).

3) For \( 1 \leq q < p < \infty \) take \( \alpha = (p/q) + p \). Hence \( (\alpha - p)q/p = 1 \),

\( \alpha - p = p/q > 1 \), \( \alpha > 1 + q \), so \( \alpha - q > 1 \). Then according to a) and b)

\( f \in L^p \cap L^q \) and \( f \notin (L^p, \mathcal{L}^q) \).

Suppose \( p = \infty \). Consider the function \( g: G \rightarrow \mathbb{R} \) defined by
\[
g(x) = \begin{cases} 
\frac{1}{q} & x \in \mathcal{L}_n \\
0 & \text{otherwise}
\end{cases}
\]
Clearly \( g \in L^\infty \), \( \sup_{L_n} |g(x)|^q = 1/n \) and
\[
\int_{L_n} g^n = \int_{L_n} 1/n = 1/n^2. \text{ Therefore } ||g||_{\infty} = \sum_{N} 1/n \text{ and }
\]
\[
||g||_q = \left[ \sum_{N} 1/n^2 \right]^{1/q}. \text{ This implies that } g \in (L^\infty, \ell^q) \text{ and } g \in L^q \cap L^\infty.
\]

4) Let \( \alpha = 1 + q \), so \( (\alpha - 1)q = q^2 > 1 \). \( (q - q)q = 1/q > 0 \)
and \( \alpha - q = 1 \). Hence by a), b) and c) \( f \in (L^1, \ell^q) \cap (L^q, \ell^\infty) \) and
\( f \notin L^q \).

5) Let \( \alpha = p + 1 \), so \( (\alpha - p)q/p = q/p > 1 \), \( (\alpha - 1)p = p^2 > 1 \)
and \( \alpha - p = 1 \). Hence by a) and b) \( f \in (L^p, \ell^q) \cap (L^1, \ell^p) \) and
\( f \notin L^p \).

6) Let \( \alpha = 2q \), so \( (\alpha - q)/q = 1 \), \( \alpha - q = q > 1 \). Hence by b)
\( f \notin (L^q, \ell^1) \), by c) \( f \notin (L^\infty, \ell^q) \) and by a) \( f \in L^q \).

7) Let \( \alpha = \min( p/q + p, 2q ) \), so \( \alpha \leq p/q + p \) and \( \alpha \leq 2q \);
this implies that \( (\alpha - p)q/p \leq 1 \) and \( (\alpha - q)/q \leq 1 \). Hence by b)
\( f \notin (L^p, \ell^q) \), \( f \notin (L^q, \ell^1) \). But \( \alpha > 1 + q \) and this means that
\( \alpha - q > 1 \), therefore by a) \( f \in L^q \).

8) Consider the function \( g: G \to \mathbb{R} \) defined by
\[
g(x) = \begin{cases} 
(1/n)^{1/q} & \text{if } x \in L_n \\
0 & \text{otherwise}
\end{cases}
\]
Then \( \int_{L_n} g^n = 1/n \), and we have that \( ||g||_q = \left[ \sum_{N} 1/n \right]^{1/q} \),
\[
||g||_{\infty} = \sup_{N} 1/n. \text{ Therefore } g \notin L^q \text{ and } g \in (L^q, \ell^\infty).
\]

**Corollary 2.5.** The following inclusions are strict if \( G \) is noncompact.
9) \((L^p, \ell^1) \subseteq L^p\) \(1 < p \leq \infty\)

10) \(L^p \subseteq (L^p, \ell^q)\) \(1 \leq p < \infty\)

11) \(L^q \subseteq (L^1, \ell^q)\) \(1 < q \leq \infty\)

12) \(L^q \subseteq (L^p, \ell^q)\) \(1 \leq p < q \leq \infty\)

13) \((L^\infty, \ell^q) \subseteq L^q\) \(1 \leq q < \infty\).

**Proof.** 9) and 10) follow from 1). While 11) and 12) are consequences of 2). Since \((L^\infty, \ell^q) \subseteq (L^p, \ell^q) \subseteq L^q\) for \(1 \leq q < p < \infty\), we see that 13) holds.

**Remark 2.6.** From (2.5) and (2.6) we see that 
\((L^\infty, \ell^1) \subseteq (L^p, \ell^q) \subseteq (L^1, \ell^\infty)\) \(1 \leq p, q \leq \infty\). In other words \((L^\infty, \ell^1)\) is the smallest and \((L^1, \ell^\infty)\) is the biggest of the amalgam spaces. At the same time, \(M_1\) is the smallest and \(M_\infty\) is the biggest of the spaces of unbounded measures of type \(q\).

A function \(f\) in \((L^1, \ell^q)\) \((1 \leq q \leq \infty)\) considered as the measure \(fm\), where \(\int g dm = \int g f dm\), belongs to \(M_q\) and \(\|f\|_{L^q} = \|fm\|_{M_q}\).

Hence \(f \mapsto fdm\) is a natural embedding from \((L^1, \ell^q)\) into \(M_q\). In this sense, we say that

\[(2.10)\] \((L^1, \ell^q) \subseteq M_q\) \(1 \leq q \leq \infty\).

Note that for \(1 \leq p, q \leq \infty\) and \(f \in (L^p, \ell^q)\)

\[(2.11)\] \((L^p, \ell^q) \subseteq (L^1, \ell^q) \subseteq M_q\)

\[(2.12)\] \(\|fm\|_{M_q} = \|f\|_L^q \leq \|f\|_{L^p}^{pq}\).
§ 3. PROPERTIES OF \((L^p, \ell^q)\) AND \(M_q\).

The results presented here are not new. However for the sake of completeness and uniformity we shall prove them using the definition of the amalgam space \((L^p, \ell^q)\). Alternative proofs can be found in [36].

Let \(\{E_\alpha\}_J\) be a family of Banach spaces. Following G. Köthe [38, § 26 p. 359] \(\ell^q(E_\alpha)\) is the linear space of nets \((x_\alpha)_J\), \(x_\alpha \in E_\alpha\) such that \(\sum \|x_\alpha\|^q \leq \infty\). \(\ell^q(E_\alpha)\) is a Banach space under the norm \(\|(x_\alpha)\| = \left(\sum \|x_\alpha\|^q\right)^{1/q}\) if \(q\) is finite and \(\|(x_\alpha)\| = \sup J \|x_\alpha\|\) if \(q\) is infinite.

We see that \((L^p, \ell^q), M_q\), \(1 \leq p, q \leq \infty\) are particular cases of \(\ell^q(E_\alpha)\). Indeed, if \(E_\alpha = L^p(L_\alpha)\) \((\alpha \in J)\) then the map \(f \mapsto (f_\alpha)_J\), \(f_\alpha = f|L_\alpha\) is an isometric isomorphism from \((L^p, \ell^q)\) onto \(\ell^q(E_\alpha)\).

Similarly, if \(E_\alpha = M(K_\alpha)\) \((\alpha \in J)\) then \(M_q\) is isometrically isomorphic to \(\ell^q(E_\alpha)\) via the map \(\mu \mapsto (\mu_\alpha)_J\), \(\mu_\alpha(B) = \mu(B \cap K_\alpha)\) \((B\) a Borel subset of \(G)\).

This fact together with § 26, 8 of [36] implies the next result.

**Theorem 3.1.** Let \(1 \leq p, q < \infty\). \((L^p', \ell^q')\) \((L^p, \ell^q)\) is isometrically isomorphic to \((L^p, \ell^q)^*\) \((L^p, c_0)^*\) via the map \(g \mapsto <f, g>, <f, g> = \int_G f dx\), \(g \in (L^p', \ell^q')\) \((L^p', \ell^1)\), \(f \in (L^p, \ell^q)\) \((L^p, c_0)\).
Hence,

\[(3.1) \quad |< f, g >| \leq |f|_{p,q} |g|_{p',q'} \quad 1 < p, q < \infty \]

\[(3.2) \quad |< f, g >| \leq |f|_{p,1} |g|_{p',\infty} \quad 1 \leq p < \infty. \]

**Proof.** The case \( p = q = 1 \) follows from [18, IV 8.5, p. 290] and [45, Appendix E. 10].

The proof for \((L^p, L^1)\) is identical to that for the case \((L^p, L^{q'})\) in [38, §26, p. 359].

**Theorem 3.2.** Let \( 1 < q < \infty \). If \( T \) is a linear functional on \((C_0, L^q)\) then there exists a unique measure \( \mu \in M_q^* \) such that

\[ T(f) = \int_G f \, d\mu \quad (f \in (C_0, L^q)) \]

and

\[ |T| \leq |\mu|_q \leq 2^q |T| \quad \text{if } 1 \leq q < \infty \]

\[ |T| = |\mu|_1 \quad \text{if } q = \infty. \]

Moreover,

\[(3.3) \quad |< f, g >| = \int_G fg \, dx \leq |f|_{\alpha q} |g|_{1 q'} \]

\( f \in (C_0, L^q), g \in (L^1, L^{q'}) \).

**Proof.** The case \( q = \infty \) is the Riesz Representation Theorem.

For \( 1 \leq q < \infty \), the first part is Theorem 4.3 in [49]. What follows is a sketch of the proof.

Let \( E_\alpha \) be the space of continuous functions on \( K_\alpha (\alpha \in J) \) with the usual topology and let \( L^q(E_\alpha) \) be the linear space of nets \((f_\alpha)_J, \)
$f_\alpha \in E_\alpha$ such that $\|f_\alpha\|_q < \omega$.

If $S = \{(f_\alpha) \in l^q(E_\alpha) \mid f_\alpha = f_\beta \text{ on } K_\alpha \cap K_\beta\}$ then $(C_\alpha, l^q)$ is isometrically isomorphic to $S$ via $f \mapsto (f_\alpha)_\alpha$ where $f_\alpha = f|_{K_\alpha}$.

Since $T \in (C_0, l^q)^\ast = S^\ast$, $T$ can be extended without changing its norm to a linear functional $T$ in $l^q(E_\alpha)^\ast$.

By [30, §26, 1], $l^q(E_\alpha)^\ast = l^q'(E_\alpha^\ast)$ and this implies that there exists $T_\alpha \in E_\alpha^\ast, \alpha \in J$, such that

$$T(f) = \sum_{\alpha} T_\alpha(f_\alpha) \quad (f \in (C_0, l^q)) \text{ and } \|T\| = \|T_\alpha\|_q.$$

By the Riesz Representation Theorem there exists a unique measure $\mu_\alpha \in K_\alpha$ such that $\|\mu_\alpha\| = \|T_\alpha\|$ and

$$T_\alpha(g) = \int_{K_\alpha} g \text{ d}\mu_\alpha \quad (g \in E_\alpha).$$

Therefore, $T(f) = \sum_{\alpha} \int_{K_\alpha} f_\alpha \text{ d}\mu_\alpha \quad (f \in (C_0, l^q)).$

The measure $\mu(E) = \sum_{\alpha} \mu_\alpha(E \cap K_\alpha)$ ($E$ a Borel subset of $G$) belongs to $M_q$, $T(f) = \int_E \text{ d}\mu \quad (f \in (C_0, l^q))$ and $\|T\| = \|\mu\|_q$.

Now, $|\mu|(K_\gamma)^{q'} \leq \sum_{\alpha} |\mu_\alpha|(K_\alpha \cap K_\gamma)^{q'} \leq \sum_{K_\alpha \in S(K_\gamma)} |\mu_\alpha|(K_\alpha)^{q'}$ where $S(K_\gamma) = \{K_\alpha \mid K_\alpha \cap K_\gamma \neq \emptyset\}$.

So, by Lemma 1.13

$$|\mu|(K_\gamma)^{q'} \leq (2^d)^{q' - 1} \sum_{K_\alpha \in S(K_\gamma)} |\mu_\alpha|(K_\alpha)^{q'} \quad \text{for } 1 \leq q' < \infty.$$

Since the cardinality of $S(K_\gamma)$ is $2^d$ we conclude that

$$\sum_{\alpha} |\mu|(K_\gamma) \leq 2^{da} \sum_{\alpha} |\mu_\alpha|^{q'} \quad \text{if } 1 < q' < \infty \text{ and also}$$
\[ \sup_{\gamma} |\mu|_{(K_\gamma)} \leq 2^a \sup_{\alpha} |\mu_\alpha| \cdot \]

This implies that \[ ||\mu||_{q'} \leq 2^a \sup_{\alpha} ||\mu_\alpha||_{q'} = 2^a ||T|| \]
for \( 1 \leq q < \infty \).

On the other hand, by the Hölder inequality, we have that for \( f \in (C_0, L^q) \)
\[
|T(f)| \leq \left( \int_K |f| \ d|\mu| \right)^{1/q} \leq \sum_{\alpha} \left( \int_{K_\alpha} |f| \ d|\mu| \right)^{1/q} \leq \sum_{\alpha} \sup_{K_\alpha} |f| \cdot \sup_{K_\alpha} |\mu|_{K_\alpha},
\]

\[
\leq \left[ \sum_{\alpha} |f|_{q'} \right]^{1/q} \left[ \sum_{\alpha} |\mu|_{(K_\alpha)^q} \right]^{1/q'} = ||f||_{\infty q} \cdot ||\mu||_{q'},
\]

Therefore \[ ||T|| \leq ||\mu||_{q'} . \]

Finally if \( g \in (L^1, L^{q'}) \) then by (2.10) \( g \mu \in M_q \), and by our previous result and (2.12)
\[
|<f, g>| = |\int f g \ d\mu| = |\int f g \ dx| \leq ||f||_{\infty q} ||g\mu||_{q'} = ||f||_{\infty q} ||g||_{1 q'} .
\]

The expressions (3.1), (3.2) and (3.3) will be called Hölder inequality for amalgams.

**REMARK.** In Theorem 3.1 \[ ||T|| \neq ||\mu||_{q'} \] for \( 1 \leq q < \infty \). To see this consider the linear map \( T \) on \( (C_0, L^q) \) defined by \( T(f) = f(0) \).

Then \( T \in (C_0, L^q)^* \) and the measure associated to \( T \) is \( \delta_0 \). Since \[ ||T|| = 2^{-a/q} \] and \[ ||\delta_0||_{q'} = 1 = (2^a/q)(2^{-a/q}) \] we conclude that \[ ||T|| = 2^{-a/q} ||\delta_0||_{q'} \].

**COROLLARY 3.3.** The amalgam space \( (L^1, L^q) \) \( (1 < p, q < \infty) \) is a reflexive Banach space.

Note that if \( E \) is a compact subset of \( G \), then \( E \) is covered by a finite number of translates of \( K \), because the interior of \( K \) is
nonempty. Since each translate of $K$ meets at most $2^qK_\alpha$'s we conclude that $E$ is covered by a finite number of $K_\alpha$'s. Therefore the cardinality $|S(E)|$ of the set $S(E) = \{K_\alpha | K_\alpha \cap E \neq \emptyset\}$ is finite.

**PROPOSITION 3.4.** If $g \in L_c^p$, $1 \leq p < \infty$, and $E$ is its compact support then for $1 \leq q < \infty$, $\|g\|_{pq} \leq |S(E)|^{1/q} \|g\|_p$ and $\|g\|_{p^{\infty}} \leq |S(E)| \|g\|_p$.

**PROOF**

\[
\|g\|_{pq} = \left( \sum_\alpha |g|_{L^p(K_\alpha)}^q \right)^{1/q} = \left( \sum_\alpha \|g\|_{L^p(K_\alpha)}^q \right)^{1/q} \leq |S(E)|^{1/q} \|g\|_p
\]

$\|g\|_{p^{\infty}} = \sup_\alpha \|g\|_{L^p(K_\alpha)} = \sup_\alpha \|g\|_{L^p(K_\alpha)} \leq |S(E)| \|g\|_p$.

**DEFINITION 3.5.** For $\mu \in M_q$, $1 \leq q < \infty$, and $E$, a compact subset of $G$; $\mu_E$ will be the bounded measure defined by $\mu_E(B) = \mu(EnB)$ $B$ a Borel set of $G$. $M_c^q$ will denote the linear subspace of $M_c^1$ consisting of the measures $\mu_E$, $E \subseteq G$ compact and $\mu \in M_q$.

**THEOREM 3.6.** i) $L_c^p$ is a dense subspace of $(L^p, L^q)$ for $1 \leq p \leq \infty$, $1 \leq q < \infty$.

ii) $L_c^p$ is a dense subspace of $(L^p, c_0)$ for $1 \leq p \leq \infty$.

iii) $M_c^q$ is a dense subspace of $M_q$ for $1 \leq q < \infty$.

**PROOF.** Let $V = \{V_{i} | i \in I\}$ be the set of finite unions of $K_\alpha$'s. If $f \in (L^p, L^q)$ then $f = \sum f_\alpha$ where $f_\alpha = f|L_\alpha$ and

\[
\|f\|_{pq} = \left( \sum_\alpha \|f_\alpha\|_{L^p(L_\alpha)}^q \right)^{1/q} \; ; \text{ this implies that } \lim_{V \in V_1} \sum f_\alpha = f \text{ in }
\]
\((L^p, l^q)\). Since \(\sum f_\alpha\) belongs to \(L^p_c\) for each \(v_1 \in V\), \(L^p_c\) is dense in \((L^p, l^q)\).

To prove ii) take a function \(f\) in the closure of \(L^p_c\) in \((L^p, l^q)\). So given \(\varepsilon > 0\) there exists \(g \in L^p_c\) such that
\[ ||f - g||_{p_\infty} < \varepsilon.\]
This implies by the definition of \(|| \cdot ||_{p_\infty}\) that for all \(\alpha \in J\),
\[ ||f - g||_{L^p(l_\alpha)} < \varepsilon.\]
Hence \(||f||_{L^p(K_\alpha)} < \varepsilon + ||g||_{L^p(K_\alpha)}\) for all \(\alpha \in J\).

Now, since \(g\) has compact support, \(||g||_{L^p(K_\alpha)}\) is zero for all but finitely many \(K_\alpha\)'s, therefore \((||g||_{L^p(K_\alpha)})_{\alpha} \in c_0\). Since \(\varepsilon\) is independent of \(\alpha\), this implies that \(\lim_{\alpha} ||f||_{L^p(K_\alpha)} \leq \varepsilon\) and we conclude that \(\lim_{\alpha} ||f||_{L^p(K_\alpha)} = 0\). In other words \(f \in (L^p, c_0)\) and this proves ii).

If \(\mu \in M_q\), then \(\lim \sum_{v_i} |\mu|(K_\alpha)^q = ||\mu||_q^q\). This means that
\[ \lim \sum_{v_i} \mu_\alpha = \mu\] in \(M_q\) where \(\mu_\alpha = \mu_{K_\alpha}\). Since \(\sum \mu_\alpha\) belongs to \(M^q_c\) for each \(v_i \in V\), the proof is complete.+

**Theorem 3.7.** i) \(C_c\) is dense in \((C_0, l^q)\) for \(1 \leq q \leq \infty\).

ii) \(C_c\) is dense in \((L^p, l^q)\) for \(1 \leq p, q < \infty\).

iii) \(C_c\) is dense in \((L^p, c_0)\) for \(1 \leq p < \infty\).

**Proof.** First we note that \(C_c\) is included in all amalgam spaces. Let \(f\) be a function in the closure of \(C_c\) in \((L^\infty, l^0)\). Hence, there exists a sequence \(\{\phi_n\}\) in \(C_c\) such that \(\lim ||\phi_n - f||_{l^q} = 0\).

This means that given \(\varepsilon > 0\) there exists \(n_0\) such that for all
\[ n \geq n_0 \quad \| \phi_n - f \|_{q}^{q} = \sum_{\alpha} \sup_{x \in K_{\alpha}} |\phi_n(x) - f(x)|^q < c. \] Since

\[ |\phi_n(x) - f(x)| \leq \| \phi - f \|_{q}^{q} \quad \text{for all } x \in G, \phi \text{ converges uniformly to } f \text{ on } G, \text{ therefore } f \text{ is continuous and by Proposition 1.19, } f \in (C_0, \ell^q) \]

if \( q \) is finite. If \( q \) is infinite \( (C_0, \ell^\infty) = C_0 \) and it is well known that \( C_c \) is dense in \( C_0 \).

Let \( f \in (L^p, \ell^q). \) By Theorem 3.6 i), given \( \epsilon > 0 \) there exists \( g \in L^p_c \) such that \( \| f - g \|_{pq} < \epsilon/2. \)

If \( E \) is the compact support of \( g \) then there exists \( h \in C_c(\Xi) \) such that \( \| g - h \|_p < \epsilon/2 \| S(\Xi) \|^{1/q}, \) because \( C_c(\Xi) \) is dense in \( L^p(\Xi). \) Hence by Proposition 3.4, \( \| g - h \|_{pq} \leq \| S(\Xi) \|^{1/q} \| g - h \|_p \)

therefore \( \| g - h \|_{pq} < \epsilon/2. \) This implies that

\[ \| f - h \|_{pq} \leq \| f - g \|_{pq} + \| g - h \|_{pq} < \epsilon. \] Similarly, since \( L^p_c \) is dense in \( (L^p, c_0) \) (Theorem 3.6 ii)) and \( C_c(\Xi) \) is dense in \( L^p(\Xi) \) for all compact subset \( E \) of \( G, \) we conclude as in ii) that \( C_c \) is dense in \( (L^p, c_0). \)

\[ \text{COROLLARY 3.8. } i) \quad (C_0, \ell^r) \text{ is dense in } (L^p, \ell^q) \text{ for } 1 \leq p < \infty, 1 \leq r \leq q < \infty. \]

\[ ii) \quad (L^r, \ell^s) \text{ is dense in } (L^p, \ell^q) \text{ for } 1 \leq p < r < \infty, 1 \leq s \leq q < \infty. \]

\[ iii) \quad (L^s, \ell^q) \text{ is dense in } (L^p, \ell^q) \text{ for } 1 \leq s < q < \infty. \]

\[ iv) \quad (L^r, c_0) \text{ is dense in } (L^p, c_0) \text{ for } 1 \leq p < r < \infty. \]

\[ \text{PROOF. } \text{These are direct consequences of (2.5) and (2.6), Theorem 3.7 and Theorem 3.6.} \]
REMARK 3.9. \( C_0 \) is a (dense) subspace of \((L^p, c_0)\) for \( 1 < p \leq \infty \) \((1 < p < \infty)\). Indeed, if \( f \in C_0 \) and \( \varepsilon > 0 \) then there exists a compact set \( E \) such that \( |f(x)| < \varepsilon \) for all \( x \notin E \). Since \( E \subseteq \bigcup_i \{ K_{\alpha_i} \mid i = 1, \ldots, n \} \) \( n \) finite, \( \|f\|_{L^p(K^\alpha)} < \varepsilon \) for all \( \alpha \neq \alpha_i \) \( i = 1, \ldots, n \). Therefore \( \|f\|_{L^p(K^\alpha)} \) \( \mu \) belongs to \( C_0 \). By Theorem 3.7, \( C_0 \) is dense in \((L^p, c_0)\) for \( 1 \leq p < \infty \).

DEFINITION 3.10. Let \( A \) be any of the amalgam spaces \((L^p, l^q)\), \((C_0, l^q)\), \((L^p, c_0)\), \( 1 \leq p, q \leq \infty \). For each \( t \in G \), \( \tau_t \) will denote the translation operator on \( A \) or on \( M_s \), \( 1 \leq s \leq \infty \), defined by

\[
\tau_t f(s) = f(s - t) \quad (f \in A, s \in G)
\]

\[
\tau_t \mu(B) = \mu(-t + B) \quad (\mu \in M_s, B \text{ a Borel set of } G).
\]

The next theorem shows that for each \( t \in G \), \( \tau_t \) is a bounded operator.

THEOREM 3.11. Let \( 1 \leq p, q \leq \infty \). For each \( t \in G \), \( f \in (L^p, l^q) \) \( \mu \in M_q \),

1) \( \| \tau_t f \|_{pq} \leq 2^d \| f \|_{pq} \)

2) \( \| \tau_t \mu \|_q \leq 2^d \| \mu \|_q \)

PROOF. For \( K_{\gamma} \) \((\gamma \in J)\) and \( t \in G \) let

\[ S(t + K) = \{ K_{\alpha} \mid K_{\alpha} \cap K_{\gamma} \neq \emptyset \} \]

\[
\| \tau_t f \|_\infty = \sup_{K_{\gamma}} |f(x - t)| = \sup_{t+K_{\gamma}} |f(x)| \leq \sum_{S(t+K_{\alpha})} \sup_{K_{\alpha}} |f(x)|
\]
\[
\frac{\sum_{S(t+K_{\gamma})} ||f||}{L^{\infty}(K_{\alpha})} = \left[ \int_{K_{\gamma}} |f(x-t)|^p \right]^{1/p} = \left[ \int_{S(t+K_{\gamma})} |f(x)|^p \right]^{1/p} \leq \left[ \sum_{S(t+K_{\gamma})} \int_{K_{\alpha}} |f(x)|^p \right]^{1/p} \leq \sum_{S(t+K_{\gamma})} \left[ \int_{K_{\alpha}} |f(x)|^p \right]^{1/p}.
\]

By Lemma 1.13 and the fact that cardinality of \( S(t+K_{\gamma}) \) is less than or equal to \( 2^a \), we have that:

\( \|\tau_t f\|_q \overset{L^\infty(K_{\gamma})}{\leq} (2^a)^{q-1} \sum_{S(t+K_{\gamma})} ||f||_q \)

\( \|\tau_t f\|_q \overset{L^p(K_{\gamma})}{\leq} (2^a)^{q-1} \sum_{S(t+K_{\gamma})} ||f||_q \)

This implies that for \( 1 \leq p, q < \infty \)

\( ||\tau_t f||_{\infty} = \left[ \sum_{\gamma} ||\tau_t f||_{\infty}(K_{\gamma}) \right]^{1/q} \leq \left[ (2^a)^{q-1} \sum_{\gamma} \sum_{S(t+K_{\gamma})} ||f||_{\infty}(K_{\alpha}) \right]^{1/q} \leq 2^a \left[ \sum_{\alpha} ||f||_{\infty}(K_{\alpha}) \right]^{1/q} = 2^a ||f||_{\infty} \)

\( ||\tau_t f||_p = \sup_{\gamma} ||\tau_t f||_{L^p(K_{\gamma})} \leq \sup_{\gamma} \sum_{S(t+K_{\gamma})} ||f||_{L^p(K_{\alpha})} \)

\( \leq 2^a \sup_{\alpha} \left[ ||f||_{L^p(K_{\alpha})} \right]^{1/q} = 2^a ||f||_p \)

\( ||\tau_t f||_{pq} = \left[ \sum_{\gamma} ||\tau_t f||_{L^p(K_{\gamma})} \right]^{1/q} \leq \left[ (2^a)^{q-1} \sum_{\gamma} \sum_{S(t+K_{\gamma})} ||f||_{L^p(K_{\alpha})} \right]^{1/q} \leq 2^a \left[ \sum_{\alpha} ||f||_{L^p(K_{\alpha})} \right]^{1/q} = 2^a ||f||_{pq} \)
Since \( |\tau_t \mu|(K_\gamma) = |\mu|(t + K_\gamma) \leq \sum_{S(-t+K_\gamma)} |\mu|(K_\alpha) \),

\[
|\tau_t \mu|_\infty = \sup_\gamma |\tau_t \mu|(K_\gamma) \leq 2^a \sup_\alpha |\mu|(K_\alpha) = 2^a |\mu|_\infty.
\]

Now, if \( 1 \leq q \leq \infty \), then once more by Lemma 1.13

\[
|\tau_t \mu|_q = \left[ \sum_\gamma |\mu|(K_\gamma)^q \right]^{1/q} \leq \left[ (2^a)^{q-1} \sum_\gamma \sum_\alpha |\mu|(K_\alpha)^q \right]^{1/q} \leq 2^a \left[ \sum_\alpha |\mu|(K_\alpha)^q \right]^{1/q} = 2^a |\mu|_q.
\]

The next couple of lemmas will be used to prove Theorem 3.14.

**Lemma 3.12.** Let \( 1 \leq p \leq \infty \) and \( g \in L^p \). If \( E \) is the support of \( g \) and \( E_t \) is the support of \( \tau_t g \) \( (t \in G) \) then

\[
\sup_{t \in G} |S(E_t)| \leq (2^a + 1) |S(E)|
\]

where \( |S(E)| \) is as in page 36.

**Proof.** Clearly \( E_t \subseteq (t + E) \cup E \). Since

\[
t + E \subseteq v(t + K_\alpha, K_\alpha \in S(E)) \quad \text{and} \quad |S(t + K_\alpha)| = 2^a \text{, for each } t \in G,
\]

we conclude that

\[
|S(t + E)| \leq 2^a |S(E)| \text{ for all } t \in G. \text{ Therefore}
\]

\[
\sup_{t \in G} |S(E_t)| \leq 2^a |S(E)| + |S(E)|.
\]

**Lemma 3.13.** i) If \( f \in C_0 \) then \( \lim_{t \to 0} ||\tau_t f - f||_\infty = 0. \)

ii) if \( f \in L^p \ (1 \leq p < \infty) \) then \( \lim_{t \to 0} ||\tau_t f - f||_p = 0. \)

**Proof.** i) follows from [37, Theorem 15.4]. Indeed, given \( \epsilon > 0 \) there exists a neighborhood \( U \) of \( 0 \) such that \( |f(x) - f(y)| < \epsilon \) for all \( x, y \in U \).
all \( y \), \(-x \in U \). So for \( t \in U \) and \( x \in G \),

\[
|f(x - t) - f(x)| = |\tau_t f(x) - f(x)| < \varepsilon \text{ because } t = x - (x - t).
\]

Since \( x \) is arbitrary and \( U \) does not depend on \( x \) we conclude that

\[
||\tau_t f - f||_{\omega} < \varepsilon \text{ for all } t \in U.
\]

ii) is a well known fact, see for example [37, Theorem 20.4].

L. Argabright and J. Gil de Lamadrid have showed [2, p. 3-20] that the next theorem does not hold for functions in \((L^1, \omega)\).

**Theorem 3.14.** Let \( 1 \leq p, q < \omega \). If \( f \) belongs to \((L^p, \omega^q)\), to \((L^p, c_0)\) or to \((C_0, \omega^s)\), \( 1 \leq s \leq \omega \), then the map \( t \mapsto \tau_t f \) is continuous on \( G \).

**Proof.** If \( f \in (C_0, \omega^q) \) and \( \varepsilon > 0 \) then by Theorem 3.7 i), there exists \( g \in C_0 \) such that

\[
(3) \quad ||f - g||_{\omega} < \varepsilon.
\]

Let \( E \) be the support of \( g \). By Lemma 3.13 i) there exists a neighborhood \( U \) of 0 such that for all \( t \in U \)

\[
||\tau_t g - g||_{\omega} < \varepsilon / (2^a + 1) |S(E)|.
\]

If \( E_t \) is as in Lemma 3.12 then

\[
||\tau_t g - g||_{pq} \leq |S(E_t)| ||\tau_t g - g||_{\omega} \leq (2^a + 1) |S(E)| ||\tau_t g - g||_{pq} < \varepsilon.
\]

This together with (3) and Theorem 3.11 implies that for all \( t \in U \),

\[
||\tau_t f - f||_{\omega} \leq ||\tau_t f - \tau_t g||_{\omega} + ||\tau_t g - g||_{\omega} + ||g - f||_{\omega} < 2^a ||f - g||_{\omega} + \varepsilon + \varepsilon < (2^a + 2)\varepsilon.
\]

Therefore \( \lim_{t \to 0} ||\tau_t f - f||_{\omega} = 0 \). The proof for \( f \in (L^p, c_0) \)

is similar.
Now, since \( (C^q, \mathbb{L}^q) \) is dense in \( (L^p, \mathbb{L}^q) \) (Corollary 3.8) given \( f \in (L^p, \mathbb{L}^q) \) and \( \varepsilon > 0 \) there exists \( g \in (C^q, \mathbb{L}^q) \) such that

\[
\|f - g\|_{pq} < \varepsilon.
\]

So, by Theorem 3.11 and inequality (2.4)

\[
\|\tau_t f - f\|_{pq} \leq \|\tau_t f - \tau_t g\|_{pq} + \|\tau_t g - g\|_{pq} + \|g - f\|_{pq}
\]

\[
\leq 2^\alpha \|f - g\|_{pq} + \|\tau_t g - g\|_{\omega q} + \|g - f\|_{pq}
\]

\[
< (2^\alpha + 1)\varepsilon + \|\tau_t g - g\|_{\omega q}.
\]

Since \( \varepsilon \) does not depend on \( t \), we have that

\[
\|\tau_t f - f\|_{pq} \leq \|\tau_t g - g\|_{\omega q}. \quad \text{Hence, by our previous result}
\]

\[
\lim_{t \to 0} \|\tau_t f - f\|_{pq} = 0. \quad \text{The case } q = \infty \text{ is Lemma 3.13.}
\]

We have shown, up to here, that the map \( t \mapsto \tau_t f \) is continuous at 0, but this is enough because by Theorem 3.11, for \( t, t_0 \in \mathbb{L} \)

\[
\|\tau_t f - \tau_{t_0} f\|_{pq} = \|\tau_t(\tau_{t_0} f - f)\|_{pq} \leq 2^\alpha \|\tau_{t_0} f - f\|_{pq}.
\]
§ 4. CONVOLUTION AND POINTWISE PRODUCT

In this section we introduce two operations on the amalgam spaces: pointwise product and convolution; and two operations on the spaces of unbounded measures of type q: product and convolution.

These operations have been studied previously ([8],[12]) and with the exception of Theorem 4.8, the results presented here are not new.

Two important facts for our study of multipliers are (1) that under convolution all amalgam spaces and all spaces of unbounded measures of type q are $L^1$ and $M_1$ Banach modules, and (2) that under convolution and the norm $\|\cdot\|_{p1}$ the amalgam spaces $(L^p, L^q)$, $(C_0, L^q)$ are Segal algebras.

Our first result is an easy generalization of the pointwise product of $L^p$ spaces.

**Proposition 4.1.** If $1 \leq p,q,r,s \leq \infty$ are such that $1/p + 1/r = 1/m \leq 1$ and $1/q + 1/s = 1/n \leq 1$ then

a) $(L^p, L^q)(L^r, L^s) \subseteq (L^m, L^n)$

b) $(C_0, L^q)(C_0, L^s) \subseteq (C_0, L^n)$

c) $(L^p, C_0)(L^r, C_0) \subseteq (L^m, C_0)$.

Moreover, if $f \in (L^p, L^q)$ and $g \in (L^r, L^s)$ then

$\|fg\|_{mn} \leq \|f\|_{pq} \|g\|_{rs}$.

**Proof.** We will prove the case when $p,q,r,s$ are finite. The remaining cases are proved (mutatis mutandis) in the same way.
Let $f \in (L^p, L^q)$, $g \in (L^r, L^s)$. We apply twice the Hölder inequality (as in [3], Corollary 12.5]). First with $\alpha_1 = m/p$, $\alpha_2 = m/r$, $f_1 = |f|^p$ and $f_2 = |g|^r$ we have that
\[ \left( \int_{K_\alpha} |fg|^m \right)^{m/r} \leq \left( \int_{K_\alpha} |f|^m \right)^{m/p} \left( \int_{K_\alpha} |g|^r \right)^{m/r} . \]
Second with $\alpha_1 = n/q$, $\alpha_2 = n/s$, $f_1 = \|f\|^q_{L^p(K_\alpha)}$ and $f_2 = \|g\|^s_{L^r(K_\alpha)}$ we conclude that
\[ \|fg\|^n_{mn} = \left[ \sum_{\alpha} \int_{K_\alpha} |fg|^m \right]^{n/m} \leq \left[ \sum_{\alpha} \|f\|^q_{L^p(K_\alpha)} \right]^{n/q} \left[ \sum_{\alpha} \|g\|^s_{L^r(K_\alpha)} \right]^{n/s} . \]
Therefore, $\|fg\|^n_{mn} \leq \|f\|^q_{L^p(K_\alpha)} \|g\|^s_{L^r(K_\alpha)}$.

b) follows from a) and Proposition 1.19 (the case when $q = s = \infty$ is well known).

c) Now, if $f \in (L^p, c_0)$ and $g \in (L^r, c_0)$, then by a)
\[ fg \in (L^m, c_0^\infty) \text{ and } \|fg\|^m_{L^m(K_\alpha)} \leq \|f\|^p_{L^p(K_\alpha)} \|g\|^r_{L^r(K_\alpha)} . \]
This implies that $\lim_{\alpha} \|fg\|^m_{L^m(K_\alpha)} = 0$. Therefore
\[ fg \in (L^m, c_0) . \]

To define the space of unbounded measures of type $\mathcal{A}$ on $G \times G$ we use the family of compact sets $\{K_{\alpha \gamma}\}_{\alpha \gamma}$ where $K_{\alpha \gamma} = K_\alpha \times K_\gamma$. It is clear that this is consistent with Definition 1.7 and for $\nu$, $\mu$ measures on $G$, $\mu \times \nu(K_{\alpha \gamma}) = \mu(K_\alpha) \nu(K_\gamma)$.

This last fact implies our next result, which is stated but not
proved in \cite{[8]}.  

**PROPOSITION 4.2.** If \(1 \leq q, s \leq \infty\) then \(\mathcal{M}_q(G) \times \mathcal{M}_s(G) \subseteq \mathcal{M}_n(G \times G)\)  

where \(n = \max(q, s)\) and for \(\mu \in \mathcal{M}_q, \nu \in \mathcal{M}_s\),  

\[
||\mu \times \nu||_n \leq ||\mu||_q ||\nu||_s.
\]

**PROOF.** Let \(\mu \in \mathcal{M}_q\) and \(\nu \in \mathcal{M}_s\) and \(K_{\alpha \gamma} \in \{K_{\alpha \gamma}, (\alpha, \gamma) \in J\times J\}\).  

By inequality (2.2) we have that  

\[
\sum_{\alpha \gamma} |\mu(\alpha) \nu(\gamma)| = \sum_{\alpha} |\mu(\alpha)| \sum_{\gamma} |\nu(\gamma)| = \sum_{\alpha} |\mu(\alpha)| \sum_{\gamma} |\nu(\gamma)|
\]

Therefore \(||\mu \times \nu||_n \leq ||\mu||_q ||\nu||_s\).  

Since the convolution of two measures does not always exist, we establish (as in \cite{[10], Chapter 8}) that two measures \(\mu, \nu\) are convolvable if for all \(g \in C_c(G)\) the function \(g^\circ(x, y) = g(x + y)\) on \(G \times G\) is \(|\mu| \times |\nu|\)-integrable. In this case \(\mu \ast \nu\) is defined by the equation  

\[
\mu \ast \nu(g) = \int g \, d\mu \ast \nu = \iint g(x + y) \, d\mu(x) \, d\nu(y) = \iint g(x + y) \, d\nu(y) \, d\mu(x)
\]

for \(g \in C_c(G)\).  

**THEOREM 4.3.** Let \(1 \leq q, s \leq \infty\) be such that \(1/q + 1/s - 1 = 1/n \leq 1\).  

If \(\mu \in \mathcal{M}_q\) and \(\nu \in \mathcal{M}_s\) then \(\mu, \nu\) are convolvable and \(\mu \ast \nu \in \mathcal{M}_n\).  

Moreover \(||\mu \ast \nu||_n \leq 2^{2n} ||\mu||_q ||\nu||_s\).  

**PROOF.** We will prove the case when \(q\) and \(s\) are finite. The remaining cases are proved (mutatis mutandis) using the same argument.
First we note that \( 1/q = 1/n + 1/s' \) and \( 1/s = 1/n + 1/q' \). These imply that

1. \( q(1/n + 1/s') = 1; \ s(1/n + 1/q') = 1; \ n'(1/q' + 1/s') = 1. \)

Now, since \( K + K \) is covered by \( 2^a K_\alpha \)'s (see Definition 1.6), we have that for each pair \( K_\alpha, K_\gamma, \alpha, \gamma \in J \),

\[
K_\alpha + K_\gamma = \alpha + \gamma + K + K \subseteq \bigcup_{i=1}^{2^a} \alpha + \gamma + \alpha_i + K. \]

That is, \( K_\alpha + K_\gamma \) is covered by \( 2^a \) \( K_\alpha \)'s.

So, if \( g \in C_c(G) \) then

2. \[
\sup \{ |g(x + y)| \mid x + y \in K_\alpha + K_\gamma \} \leq \sum_{i=1}^{2^a} \|g\|_{L^\infty(\alpha_\gamma + \alpha_i + K)} = \sum_{i=1}^{2^a} \|g\|_{L^\infty(K_{\alpha_i})}.
\]

For \( g \in C_c(G) \), (1) implies that

\[
\left( \int_{K_\alpha} \int_{K_\gamma} |g(x + y)| \mu(x) \nu(y) \right) \leq \sup_{K_\alpha + K_\gamma} |g(x + y)| \|\mu(K_\gamma)\|_\nu(K_\alpha)
\]

\[
= \left( \|g^0\|_{L^\infty(K_{\alpha \gamma})} \|\mu(K_\gamma)\|_\nu(K_\alpha) \right)
\]

\[
= \left( \|g^0\|_{L^\infty(K_{\alpha \gamma})} \|g^0\|_{L^\infty(K_{\alpha \gamma})}^{n'/s'} \|\mu(K_\gamma)\|_{L^\infty(K_{\alpha \gamma})}^{q/n} \|\mu(K_\gamma)\|_{L^\infty(K_{\alpha \gamma})}^{q/s'} \|\nu(K_\alpha)\|_{L^\infty(K_{\alpha \gamma})}^{s/n} \|\nu(K_\alpha)\|_{L^\infty(K_{\alpha \gamma})}^{s/q'} \right)
\]

\[
= \left[ \|g^0\|_{L^\infty(K_{\alpha \gamma})} \|\mu(K_\gamma)\|_\nu(K_\alpha) \right]^{1/q'} \left[ \|g^0\|_{L^\infty(K_{\alpha \gamma})} \|\mu(K_\gamma)\|_\nu(K_\alpha) \right]^{1/s'} \left[ \|\mu(K_\gamma)\|_\nu(K_\alpha) \right]^{1/n}
\]
Applying the Hölder inequality (as in [37, Corollary 12.5]),

with $\alpha_1 = 1/q'$, $\alpha_2 = 1/s'$, $\alpha_3 = 1/n$, $f_1(\alpha, \gamma) = \|g^\gamma\|_{n'}^{\alpha_1} \|v\|_{(K^\alpha)^s}$,

\[ f_2(\alpha, \gamma) = \|g^\gamma\|_{n'}^{\alpha_1} \|v\|_{(K^\alpha)^q} \quad f_3(\alpha, \gamma) = \|v\|_{(K^\alpha)^q} \|v\|_{(K^\alpha)^s}, \]

we see that

\[
\sum_{\alpha} \sum_{\gamma} \mathcal{f}_1 \leq \left( \sum_{\alpha, \gamma} f_1 \right)^{\alpha_1} \left( \sum_{\alpha, \gamma} f_2 \right)^{\alpha_2} \left( \sum_{\alpha, \gamma} f_3 \right)^{\alpha_3}.
\]

On the other hand, by (2) and Lemma 1.13

\[
\sum_{\alpha} f_1 = \sum_{\alpha} \sup_{\gamma} \{ g(x + y) \mid x + y \in K^\gamma + K^\alpha \}^{n'} \|v\|_{(K^\alpha)^s}.
\]

\[
\leq \sum_{\alpha} \left[ \sum_{\gamma} 2^a \|g\|_{n'}^{\alpha_1} \|v\|_{(K^\alpha)^s} \right]
\leq 2^a \sum_{\alpha} \|g\|_{n'}^{\alpha_1} \|v\|_{(K^\alpha)^s}
\]

\[
= 2^a \sum_{\alpha} \|g\|_{n'}^{\alpha_1} \sum_{\alpha} \|v\|_{(K^\alpha)^s} = 2^a \|g\|_{n'} \|v\|_{s}.
\]

Similarly, $\sum_{\alpha} \sum_{\gamma} f_2 \leq 2^a \|g\|_{n'}^{\alpha_1} \|v\|_{q}^{\alpha_2}$ and

\[
\sum_{\alpha} \sum_{\gamma} f_3 = \sum_{\alpha} \sum_{\gamma} \|v\|_{q}^{\alpha_3} \|v\|_{(K^\alpha)^s} = \|v\|_{q} \|v\|_{s}.
\]

This implies by (1) that

\[
\int_{K^\alpha} \int_{K^\gamma} g(x + y) \mu(x) \, d\nu(y) \leq \sum_{\alpha} \sum_{\gamma} \mathcal{f}_1 \leq 2^a \|g\|_{n'}^{\alpha_1} \|v\|_{s}^{\alpha_2} \|v\|_{q}^{\alpha_3} \|v\|_{s}^{\alpha_3} \leq 2^a \|g\|_{n'}^{\alpha_1} \|v\|_{q}^{\alpha_2} \|v\|_{s}^{\alpha_3}.
\]
\[ |T(g)| \leq 2^a \left\| u \right\|_{c} \left\| v \right\|_{s} \left\| \mu \right\|_{q}. \]

Therefore \( u, v \) are convolvable and the linear functional \( T(g) = \int g \cdot \mu \ast v \quad (g \in C_c(G)) \) is such that

\[ \left\| T(g) \right\| \leq 2^a \left\| g \right\|_{c} \left\| v \right\|_{s} \left\| \mu \right\|_{q}. \]

Since \( C_c \) is dense in \( (C_0, \ell^n) \), \( T \) has a unique continuous extension \( T \) in \( (C_0, \ell^n)^* \) such that

\[ \left\| T \right\| \leq 2^a \left\| v \right\|_{s} \left\| \mu \right\|_{q}. \]

Finally by Theorem 3.2 we conclude that

\[ 2^{-a} \left\| \mu \ast v \right\| \leq 2^a \left\| \mu \right\|_{q} \left\| v \right\|_{s}. \]

Hence

\[ \left\| \mu \ast v \right\|_{n} \leq 2^{2a} \left\| \mu \right\|_{q} \left\| v \right\|_{s}. \]

**COROLLARY 4.4.** If \( f \in (L^1, \ell^q) \), \( 1 \leq q \leq \infty \), \( \mu \in M_\sigma \), \( 1 \leq s \leq \infty \), and \( 1/q + 1/s - 1 = 1/n \leq 1 \) then \( f \) as a measure is convolvable with \( \mu \), \( f \ast \mu \) is an absolutely continuous measure with density

\[ (3) \quad f \ast \mu(t) = \int f(t - x) d\mu(x). \]

\( f \ast \mu \) belongs to \( (L^1, \ell^q) \) and

\[ \left\| f \ast \mu \right\|_{L^n} \leq 2^{2a} \left\| f \right\|_{L^n} \left\| \mu \right\|_{s}. \]

**PROOF.** \( f \) as a measure belongs to \( M_q \) (see (2.10)), so by Theorem 4.3 \( f \) is convolvable with \( \mu \) and \( f \ast \mu \in M_n \).

On the other hand, it is well known (see for example [1, Proposition 1.1]) that \( f \ast \mu \) is a function given by (3).

Finally, by (2.12)

\[ \left\| f \ast \mu \right\|_{L^n} = \left\| f \ast \mu_{\mu} \right\|_{n} \]

and this ends the proof.

**COROLLARY 4.5.** Let \( 1 \leq q, s \leq \infty \). If \( f \in (L^1, \ell^q) \), \( g \in (L^1, \ell^s) \) and \( 1/p + 1/s = 1/n \leq 1 \) then \( f \ast g \in (L^1, \ell^n) \),

\[ f \ast g(t) = \int f(t - x) g(x) dx. \]
and \[ ||f*\mu||_{L^q} \leq 2^{2d} ||f||_{L^q} ||\mu||_{L^q}. \]

**Proof.** (2.10), (2.12) and Corollary 4.4.

**Corollary 4.6.** If \( f \in L^1(G) \) and \( \mu \in M_q \ (1 \leq q \leq \infty) \), then \( f*\mu \in (L^1, \mathbb{L}^q) \) and \[ ||f*\mu||_{L^q} \leq ||f||_{L^1} ||\mu||_{L^q} \] where \( ||\cdot||_{L^q} \) is the norm defined in Theorem 1.21.

**Proof.** The first part follows from Corollary 4.4.

Now,

\[
(f*\mu)^\theta(t) = \int_{t+L} |f*\mu(x)| dx \leq \int_{t+L} \int |f(x-s)||\mu|(s) dx
\]

\[
= \int \int |f(x-s)| \chi_L(x-t) dx \, d\mu(s)
\]

\[
= \int |f(u)| \int \chi_L(u+s-t) \, d\mu(s) \, du
\]

\[
= \int |f(u)| \mu(t-u+L) \, du
\]

\[
= \int |f(u)| \mu^\theta(t-u) \, du = ||f*\mu||_{L^q}^\theta(t).
\]

Since \( \mu^\theta \in \mathbb{L}^q \) we have by the Young inequality that

\[
||f*\mu||_{L^q}^\theta = ||(f*\mu)^\theta||_{L^q} \leq ||f*\mu||_{L^q} \leq ||f||_{L^1} ||\mu||_{L^q} \]

The next theorem is basically 57 i) of [8] and Theorem 4.2 of [12] but with the improvement that in Young's inequality for amalgams (4.1) and (4.2) there is an explicit constant coming from the Structure Theorem.

**Theorem 4.7.** If \( p, q, r, s \) are exponents such that

\[
\frac{1}{p} + \frac{1}{r} - 1 = \frac{1}{m} \leq 1 \quad \text{and} \quad \frac{1}{q} + \frac{1}{s} - 1 = \frac{1}{n} \leq 1\]

then

\[
i) (L^p, \mathbb{L}^q) * (L^r, \mathbb{L}^s) \subseteq (L^m, \mathbb{L}^n)
\]
11) \((L^p, L^q) \ast (L^{p'}, L^{q'}) \subseteq C_0\) \(1 \leq p \leq \infty, 1 < q < \infty\)

111) \((L^p, L^q) \ast (L^{p'}, L^{q'}) \subseteq (C_0, L^p)\) \(1 \leq p, s \leq \infty, 1 < q < \infty\)

iv) \((L^p, L^q) \ast (L^r, L^q') \subseteq (L^m, C_0)\) \(1 \leq p, r \leq \infty, 1 < q < \infty\).

Moreover if \(f \in (L^p, L^q)\) and \(g \in (L^r, L^s)\) then

\[
(4.1) \quad \left\| \mathbf{f} \ast g, \mathbf{g} \right\|_{L^n} \leq 2^d \left\| f \right\|_{L^p, L^q} \left\| g \right\|_{L^r, L^s} \quad \text{if } m \neq 1
\]

\[
(4.2) \quad \left\| f \ast g \right\|_{L^n} \leq 2^{2d} \left\| f \right\|_{L^p, L^q} \left\| g \right\|_{L^r, L^s} \quad \text{if } m = 1
\]

**Proof.** If \(p = r = 1\) then 1) and (4.2) follow from Corollary 4.5.

The argument to prove 1) is similar to the proof of Theorem 4.3 and for the same reason given in that theorem we will prove 1) for \(1 < p, q, r, s < \infty\).

First we note that

\[
(4) \quad p(1/m + 1/r') = 1; \quad r(1/m + 1/p') = 1; \quad m'(1/p' + 1/r') = 1
\]

\[
(5) \quad q(1/n + 1/s') = 1; \quad s(1/n + 1/q') = 1; \quad n'(1/q' + 1/s') = 1.
\]

For \(\phi \in C_c(G), \alpha, \gamma \in J\) we have by (4) that

\[
\left. \int_{K_\alpha} \int_{K_\gamma} |\phi(x + y)| \left| f(x) \right| \left| g(y) \right| \, dx \, dy \right.
\]

\[
= \int_{K_\alpha} \int_{K_\gamma} \left( \left| f(x) \right|^p \left| g(y) \right|^r \right)^{1/m} \left( \left| \phi(x + y) \right|^{m'} \left| f(x) \right|^{p'} \right)^{1/r'}
\]

\[
\left( \left| \phi(x + y) \right|^{m'} \left| g(y) \right|^{r'} \right)^{1/p'} \, dx \, dy.
\]

The Hölder inequality (as in [37, Corollary 12.5]) with

\[
\alpha_1 = 1/m, \quad \alpha_2 = 1/r', \quad \alpha_3 = 1/p', \quad f_1(x, y) = \left| f(x) \right|^p \left| g(y) \right|^r,
\]

\[
f_2(x, y) = \left| \phi(x + y) \right|^{m'} \left| g(y) \right|^{r'}, \quad f_3(x, y) = \left| \phi(x + y) \right|^{m'} \left| f(x) \right|^p
\]

implies
\[ \int_{K_\alpha} \int_{K_\gamma} |\phi(x + y)| |f(x)| |g(y)| \, dx \, dy \]

\[ \leq \left[ \left( \frac{\|f\|^p}{L^p(K_\gamma)} \right)^{1/m} \left( \frac{\|g\|^r}{L^r(K_\alpha)} \right)^{1/m'} \right]^{1/r} \left[ \left( \frac{\|\phi\|^m'}{L^{m'}(K_\alpha + K_\gamma)} \right)^{1/p'} \left( \frac{\|f\|^p}{L^p(K_\gamma)} \right)^{1/m} \right]^{1/r'} \]

\[ = \frac{\|\phi\|}{L^{m'}(K_\alpha + K_\gamma)} \frac{\|f\|}{L^p(K_\gamma)} \frac{\|g\|}{L^r(K_\alpha)} \]
\[ \left[ \frac{\|f\|_q^{\alpha_1}}{L^p(K_{\alpha})} \frac{\|g\|_s^{\alpha_2}}{L^r(K_{\alpha})} \right]^{1/ \alpha_2} \left[ \frac{\|\phi\|_m^{\alpha_1}}{L^{m'}(K_{\alpha}+K_{\gamma})} \frac{\|f\|_q^{\alpha_2}}{L^p(K_{\gamma})} \right]^{1/ \alpha_2} \left[ \frac{\|\phi\|_m^{\alpha_1}}{L^{m'}(K_{\alpha}+K_{\gamma})} \frac{\|g\|_s^{\alpha_2}}{L^r(K_{\alpha})} \right]^{1/ \alpha_2} \]

Again, applying the Hölder inequality, this time with \( \alpha_1 = 1/n \), \( \alpha_2 = 1/s' \), \( \alpha_3 = 1/q' \),

\[ f_1(\alpha, \gamma) = \left\| \frac{\|f\|_q^{\alpha_1}}{L^p(K_{\gamma})} \frac{\|g\|_s^{\alpha_2}}{L^r(K_{\alpha})} \right\|_b \]

\[ f_2(\alpha, \gamma) = \left\| \frac{\|\phi\|_m^{\alpha_1}}{L^{m'}(K_{\alpha}+K_{\gamma})} \frac{\|\phi\|_m^{\alpha_1}}{L^{m'}(K_{\alpha}+K_{\gamma})} \right\|_b \]

\[ f_3(\alpha, \gamma) = \left\| \frac{\|\phi\|_m^{\alpha_1}}{L^{m'}(K_{\alpha}+K_{\gamma})} \frac{\|g\|_s^{\alpha_2}}{L^r(K_{\alpha})} \right\|_b \]

we have that

\[ \sum_{\alpha} \sum_{\gamma} \int_{K_{\alpha}} \int_{K_{\gamma}} |\phi(x + y)| |f(x)| |g(y)| \, dx \, dy \]

\[ \leq \left[ \sum_{\alpha} \sum_{\gamma} f_1(\alpha, \gamma) \right]^{\alpha_1} \left[ \sum_{\alpha} \sum_{\gamma} f_2(\alpha, \gamma) \right]^{\alpha_2} \left[ \sum_{\alpha} \sum_{\gamma} f_3(\alpha, \gamma) \right]^{\alpha_3} \]

Now, as in Theorem 4.3, \( K_{\alpha} + K_{\gamma} \subseteq K_{(\alpha + \gamma + \alpha_1 + K)} \), hence

by Lemma 1.13

\[ \left\| \phi \right\|_m^{\alpha_1} \left\| f \right\|_q^{\alpha_2} \leq (2^a)^{n'-1} \sum_{i=1}^{2^a} \left\| \phi \right\|_m^{\alpha_1} \left\| f \right\|_q^{\alpha_2} \]

This implies that

\[ \sum_{\alpha} \sum_{\gamma} f_2(\alpha, \gamma) = \sum_{\alpha} \sum_{\gamma} \left\| \phi \right\|_m^{\alpha_1} \left\| f \right\|_q^{\alpha_2} \]

\[ \leq (2^a)^{n'-1} \sum_{\alpha} \sum_{\gamma} \left\| \phi \right\|_m^{\alpha_1} \left\| f \right\|_q^{\alpha_2} \]

\[ \leq 2^a n' \sum_{\gamma} \left\| \phi \right\|_m^{\alpha_1} \left\| f \right\|_q^{\alpha_2} \]
\[ = 2^{an'} \left| |\phi|^{n'}_{m'n'} \right| |f|^{q}_{pq} \right| \\
\text{Similarly} \sum_{\alpha} \sum_{\gamma} f_\alpha(\alpha,\gamma) \leq 2^{an'} \left| |\phi|^{n'}_{m'n'} \right| |g|^{s}_{rs} \right| \\
\text{Since} \sum_{\alpha} \sum_{\gamma} f_\alpha(\alpha,\gamma) = ||f|^{q}_{pq} \left| |g|^{s}_{rs} \right| \text{we conclude that} \\
\int \int |\phi(x + y)| |f(x)| |g(y)| \, dx \, dy \\
\leq \sum_{\alpha} \sum_{\gamma} \int_{K_\alpha} \int_{K_\gamma} |\phi(x + y)| |f(x)| |g(y)| \, dx \, dy \\
\leq 2^{a} \left[ \left| |\phi|^{n'}_{m'n'} \right| |f|^{q}_{pq} \right]^{1/1'} \left[ \left| |\phi|^{n'}_{m'n'} \right| |g|^{s}_{rs} \right]^{1/1'} \\
= 2^{a} \left| |\phi|^{n'}_{m'n'} \right| |f|^{q}_{pq} \left| |g|^{s}_{rs} \right| \\
\text{This means that the linear functional} \ T(\phi) = \int f(t) \, f^*g(t) \, dt \\
(\phi \in \mathcal{C}_c(G)) \text{ satisfies} \ |T(\phi)| \leq 2^{a} \left| |\phi|^{n'}_{m'n'} \right| |f|^{q}_{pq} \left| |g|^{s}_{rs} \right|. \text{Since} \ \mathcal{C}_c(G) \text{is dense in} (L^{m'}, \ell^{1'}) \text{in} (L^{m'}, \mathcal{C}_0) \text{if} \ n = 1, \ T \text{has a unique continuous extension} \ T \text{in} (L^{m'}, \ell^{1'}) \text{in} (L^{m'}, \mathcal{C}_0)^* = (L^{m}, \ell^1) \text{such that} \ |T| \leq 2^{a} \left| |f|^{q}_{pq} \right| \left| |g|^{s}_{rs} \right|. \text{By Theorem 3.1 we have that} \\
|f^*g|_{mn} \leq 2^{a} \left| |f|^{q}_{pq} \right| \left| |g|^{s}_{rs} \right|; \text{this proves i) and (4.1).} \\
\text{Let} \ f \in (L^p, \ell^q) \text{and} \ g \in (L^{p'}, \ell^{q'}) \text{. By i)} \ f^*g \in L^\infty. \text{Assume that} \ p \text{is finite. We have} \\
f^*g(t) = \int f(t - x)g(s) \, ds = \int \tau_{-t} f(s)g'(s) \, ds = < \tau_{-t} f, g' > \\
\text{so by Theorem 3.1,}
\[ |f^*g(t) - f^*g(s)| = |< \tau^*_t f, g' > - < \tau^*_s f, g' >| = |< \tau^*_t f - \tau^*_s f, g' >| \]
\[ < |g'|_{p'}^{q'} \cdot |\tau^*_t f - \tau^*_s f|_{pq} >. \]

Since \( \lim_\tau \tau^*_t f - \tau^*_s f |_{pq} = 0 \) by Theorem 3.14 we conclude that \( f^*g \) is a continuous function.

Now, since \( C_\infty \) is dense in \( (L^p, L^q) \) and \( L^p_\infty \) is dense in \( (L^p', L^{q'}) \)
given \( \varepsilon > 0 \) there exists \( \phi \in C_\infty \) and \( h \in L^p_\infty \) such that
\[ ||\phi - f||_{pq} < \varepsilon/||g||_{P'q'} \quad \text{and} \quad ||h - g||_{p'q'} < \varepsilon/||\phi||_{pq}. \]
These imply by (4.1) that
\[ ||\phi^*h - f^*g||_{\infty} \leq ||\phi^*h - \phi^*g||_{\infty} + ||\phi^*g - f^*g||_{\infty} \]
\[ \leq 2^{\alpha} ||\phi||_{pq} ||h - g||_{p'q'} + 2^{\alpha} ||g||_{p'q'} ||\phi - f||_{pq} \]
\[ < 2^{\alpha} \varepsilon. \]

Since \( \varepsilon \) is arbitrary and \( \phi^*h \in C_\infty(G) \), this means that \( f^*g \) is in the closure of \( C_\infty \) in the space of continuous functions on \( G \), since \( C_\infty \) is dense in \( C_0 \) we conclude that \( f^*g \in C_\infty \).

If \( p \) is infinite then this proof with the roles of \( f \) and \( g \) exchanged yields the same result.

Let \( f \in (L^p, L^q) \) and \( g \in (L^{p'}, L^{q'}) \). By i) \( f^*g \in (L^\infty, L^n) \).

Since \( 1/s - 1/n + 1/q' \geq 1/q', \quad (L^{p'}, L^{q'}) \subseteq (L^{p'}, L^{q'}) \) (relation (2.5)); by ii) \( f^*g \in C_\infty \). Then iii) is proved.

Let \( f \in (L^p, L^q) \) and \( g \in (L^{r}, L^{q'}) \). Since \( L^r_\infty \) are dense in \( (L^p, L^q) \), \( (L^{r}, L^{q'}) \) respectively, given \( \varepsilon > 0 \) there exists \( \phi \in L^p_\infty \) and \( \psi \in L^r_\infty \) such that
\[ ||\xi - \phi||_{pq} < \varepsilon/||g||_{r'q'} \quad \text{and} \quad ||g - \psi||_{rs} < \varepsilon/||\phi||_{pq}. \]
Hence by (4.1)
\[ ||f^*g - \phi \psi||_{m_{\infty}} \leq ||f^*g - \psi||_{m_{\infty}} + ||\phi^*g - \phi \psi||_{m_{\infty}} \]
\[ \leq 2^a ||g||_{r_1} ||f - \phi||_{p_2} + 2^a ||\psi||_{p_2} ||g - \psi||_{r_1} \]
\[ \leq 2^a c. \]

Since \( \epsilon \) is arbitrary and \( \phi^* \psi \in L^m_c \), this means that \( f^*g \) is in the closure of \( L^m_c \) in \( (L^m, \langle \cdot, \cdot \rangle) \) because by i) \( f^*g \in (L^m, \langle \cdot, \cdot \rangle) \); since \( l^m_c \) is dense in \( (L^m, c_0) \) we conclude that \( f^*g \in (L^m, c_0). \)

**THEOREM 4.8.** Let \( 1 \leq p, q, s < \infty \). If \( \frac{1}{q} + \frac{1}{s} - 1 = \frac{1}{n} < 1 \) then

i) \( (L^p, l^q) \ast M_s \subseteq (L^p, l^n) \)

ii) \( (L^p, l^q) \ast M_s \subseteq (L^p, c_0) \) \( \frac{1}{q} < \infty \)

iii) \( (C_0, l^q) \ast M_s \subseteq (C_0, l^n) \) \( \frac{1}{q} < \infty \).

Hence \( (C_0, l^q) \ast M_s \subseteq C_0 \) \( \frac{1}{q} < \infty \).

Moreover, if \( f \in (L^p, l^q) \) and \( \mu \in M_s \) then

\[ (4.3) \quad ||f^*\mu||_{p_2} \leq 2^a ||f||_{p_2} ||\mu||_s \quad \text{if } p \geq 1. \]

\[ (4.4) \quad ||f^*\mu||_{l^n} \leq 2^{2a} ||f||_{l^q} ||\mu||_s. \]

**PROOF.** If \( p = 1 \) then i) and (4.4) follow from Corollary 4.4.

We will prove i) for \( 1 < p, q, s < \infty \) using the same argument of Theorem 4.7 i). The remaining cases are proved (mutatis mutandis) in the same way.

Again we have that

\[ q(1/n + 1/s') = 1; \quad n'(1/q' + 1/s') = 1; \quad s(1/n + 1/q') = 1. \]

Let \( \phi \in C_c(G) \) and consider the following:
\[
\int_{K_\alpha} \int_{K_\gamma} |\phi(x + y)| |f(x)| \, dx \, d|u|(y)
\]

\[
= \int_{K_\alpha} \left[ \int_{K_\gamma} |\phi(x + y)|^p \, dx \right]^{1/p'} \left[ \int_{K_\gamma} |f(x)|^p \, dx \right]^{1/p} \, d|u|(y)
\]

\[
= \|f\|_{L^p(K_\gamma)} \int_{K_\alpha} \left[ \int_{K_\gamma} |\phi'(-x)|^p \, dx \right]^{1/p} \, d|u|(y)
\]

\[
\leq \|f\|_{L^p(K_\gamma)} \int_{K_\alpha} \left[ \int_{K_\alpha + K_\gamma} |\phi(x)|^p \, dx \right]^{1/p} \, d|u|(y)
\]

\[
\leq \|f\|_{L^p(K_\gamma)} \|\phi\|_{L^{p'}(K_\alpha + K_\gamma)} \|u\|_{L^{s'}(K_\alpha)}
\]

\[
= \left[ \|f\|_{L^p(K_\gamma)}^q \|u\|_{L^{s'}(K_\alpha)}^q \right]^{1/n} \left[ \|f\|_{L^p(K_\gamma)}^{q'} \|\phi\|_{L^{p'}(K_\alpha + K_\gamma)}^{n'} \right]^{1/q'}
\]

Applying the Hölder inequality with \( \alpha_1 = 1/n \), \( \alpha_2 = 1/s' \),

\( \alpha_3 = 1/q' \), \( f_1(\alpha, \gamma) = \|f\|_{L^p(K_\gamma)}^q \|u\|_{L^{s'}(K_\alpha)}^q \),

\( f_2(\alpha, \gamma) = \|f\|_{L^p(K_\gamma)}^{q'} \|\phi\|_{L^{p'}(K_\alpha + K_\gamma)}^{n'} \), \( f_3(\alpha, \gamma) = \|\phi\|_{L^{p'}(K_\alpha + K_\gamma)}^{n'} \|u\|_{L^{s'}(K_\alpha)}^s \),

we see that

\[
\sum_{\alpha} \sum_{\gamma} \int_{K_\alpha} \int_{K_\gamma} |\phi(x + y)| |f(x)| \, dx \, d|u|(y)
\]

\[
\leq \left[ \sum_{\alpha} \sum_{\gamma} f_1(\alpha, \gamma) \right]^\alpha_1 \left[ \sum_{\alpha} \sum_{\gamma} f_2(\alpha, \gamma) \right]^\alpha_2 \left[ \sum_{\alpha} \sum_{\gamma} f_3(\alpha, \gamma) \right]^\alpha_3
\]

As in Theorem 4.3.1,

\[
\|\phi\|_{L^{p'}(K_\alpha + K_\gamma)}^{n'} \left( \sum_{i=1}^{2\alpha} \|\phi\|_{L^{p'}(\alpha + \gamma + \alpha_1 + K)}^{n'} \right)
\]

\[
\leq \left[ \sum_{\alpha} \sum_{\gamma} f_1(\alpha, \gamma) \right]^\alpha_1 \left[ \sum_{\alpha} \sum_{\gamma} f_2(\alpha, \gamma) \right]^\alpha_2 \left[ \sum_{\alpha} \sum_{\gamma} f_3(\alpha, \gamma) \right]^\alpha_3
\]

\[
= \left[ \sum_{\alpha} \sum_{\gamma} f_1(\alpha, \gamma) \right]^\alpha_1 \left[ \sum_{\alpha} \sum_{\gamma} f_2(\alpha, \gamma) \right]^\alpha_2 \left[ \sum_{\alpha} \sum_{\gamma} f_3(\alpha, \gamma) \right]^\alpha_3
\]
So, \[ \sum_{\alpha} \left( \int_{K_{\gamma}} \phi \: d\mu \right)^s = \left( \int_{\mathbb{R}^n} \phi \: d\mu \right)^s \leq \left( \int_{\mathbb{R}^n} |f|^q \: d\mu \right)^s \leq \left( \int_{\mathbb{R}^n} |f|^q \: d\mu \right)^s \leq 2^{an^1} \left( \int_{\mathbb{R}^n} |\phi|^{n^1} \: d\mu \right)^s. \]

Therefore

\[ \left[ \int \left( \int_{\mathbb{R}^n} |f|^q \: d\mu \right)^s \right]^{1/s^1} \leq \left[ \left( \int_{\mathbb{R}^n} |f|^q \: d\mu \right)^s \right]^{1/s^1} \leq 2^{an} \left( \int_{\mathbb{R}^n} |\phi|^{n^1} \: d\mu \right)^s. \]

Since

\[ \int \phi(y) \, df \ast \mu(y) = \int \phi \ast f(y) \, d\mu(y) = \int \int \phi(x+y) f(x) \, dx \, d\mu(y) \]

(see [1, Proposition 1.4]) we conclude that the linear functional \( T(\phi) = \int \phi(y) \, df \ast \mu(y) \) \( (\phi \in C_c) \) is such that

\[ |T(\phi)| \leq 2^a \left[ \int_{\mathbb{R}^n} |f|^q \: d\mu \right]^s. \]

Since \( C_c \) is dense in \( (L^p', \mathbb{R}^n') \) \( T \) has a unique continuous extension \( T \) on \( (L^p', \mathbb{R}^n') \) such that

\[ ||T|| \leq 2^a \left[ \int_{\mathbb{R}^n} |f|^q \: d\mu \right]^s. \]
This proves i) and (4.3).

Let \( f \in (L^p, \ell^q) \) and \( \mu \in M_q \). By Theorem 3.6, given \( \varepsilon > 0 \) there exists \( g \in L^p_c \) and \( \nu \in M_q^* \) such that
\[
||f - g||_p < \varepsilon / ||\mu||_q, \text{ and } ||\nu - \mu||_q < \varepsilon / ||g||_{pq}.
\]

Similarly to Theorem 4.7 i) these imply that
\[
||f^* \mu - g^* \nu||_{pq} < 2^d \varepsilon. \text{ Now, by i) } f^* \mu \in (L^p, \ell^\infty), \text{ and } g^* \nu \in L^p_c. \text{ Since } (L^p, c_0^*) \text{ is a closed subspace of } (L^p, \ell^\infty) \text{ and } L^p_c \text{ is dense in } (L^p, c_0), \text{ we conclude that } f^* \mu \in (L^p, c_0).
\]

Finally if \( f \in (C_0, \ell^q) \) and \( \mu \in M_S \), then by Theorems 3.6 and 3.7 given \( \varepsilon > 0 \) there exists \( g \in C_c \) and \( \nu \in M_S^c \) such that
\[
||f - g||_\infty < \varepsilon / ||\mu||_S, \text{ and } ||\nu - \mu||_S < \varepsilon / ||g||_{\infty S}.
\]

Again by (4.1) \( ||f^* \mu - g^* \nu||_{\infty S} < 2^d \varepsilon. \text{ By i) } f^* g \in (L^\infty, \ell^\infty) \) and \( g^* \nu \in C_c \) because \( g^* \nu \) has compact support and for \( t, s \) in \( G \)
\[
g^* \nu(t) - g^* \nu(s) = |\int (g(t - x) - g(s - x))d\nu(x)|
\]
\[
= |\int_{t-g} g(-x) - \tau_{-s} g(-x)d\nu(x)|
\]
\[
= |(\tau_{-g} - \tau_{-s})^* \nu(0)| \leq ||(\tau_{-g} - \tau_{-s})^* \nu||_1.
\]

Since \( t \rightarrow \tau_t g \) from \( G \) to \( C_0 \) is continuous (Theorem 3.14) this implies that \( g^* \nu \) is continuous.

Therefore \( f^* \nu \) is in the closure of \( C_c \) in \( (L^\infty, \ell^\infty) \), that is, \( f^* \nu \in (C_0, \ell^\infty) \) (Theorem 3.7).†

**Definition 4.9.** [19, Definition 14.1] Let \( A \) be either \( L^1(G) \) or \( M_1(G) \), a Banach space \( B \) is said to be a Banach \( A \)-module if there exists a bilinear operation \( : A \times B \rightarrow B \) such that
\[ (f \ast g) \ast b = f \ast (g \ast b) \quad \text{for all } f, g \in A, \ b \in B \]

(B-2) For some constant \( C \geq 1 \)
\[
||f \ast b||_B \leq C ||f||_A ||b||_B \quad \text{for all } f \in A, \ b \in B.
\]

It follows from Theorems 4.3, 4.7, 4.8 that all \( M_s \) \((1 \leq s \leq \infty)\) spaces and all amalgam spaces \((L^p, \ell^q), \ (C_0, \ell^q), \ (L^p, c_0) \) \((1 \leq p, q \leq \infty)\) satisfy the condition (B-2) for \( L^1 \) and \( M_s \).

Also, if \( \mu_1, \mu_2 \in M_1 \) and \( \nu \in M_s \) \((1 \leq s < \infty)\) then by [10, VIII §3 Proposition 1], \((\mu_1 \ast \mu_2) \ast \nu = \mu_1 \ast (\mu_2 \ast \nu)\). Hence by (2.11) \( M_q, \ (L^p, \ell^q), \ (C_0, \ell^q), \ (L^p, c_0) \) \((1 \leq p, q \leq \infty)\) satisfy the condition (B-1) for \( L^1 \) and \( M_s \). Therefore all these spaces are Banach \( A \)-modules.

**Definition 4.10.** Let \( B \) be a Banach \( L^1 \)-module and
\[
B_{\text{abs}} = \{ f \ast b \mid f \in L^1, \ b \in B \}. \ B \text{ is said to be an essential } L^1 \text{-module if}
\]
the linear subspace generated by \( B_{\text{abs}} \) is dense in \( B \). If \( B = B_{\text{abs}} \) then
\( B \) is an absolutely continuous \( L^1 \)-module.

The definition of absolutely continuous \( L^1 \)-module is due to Gulick, Liu and Rooij [31].

If \( B \) is a Banach \( L^1 \)-module then its dual \( B^* \) becomes a Banach \( L^1 \)-module under the operation
\[
f \ast b^*(b) = b^*(f \ast b) \quad (b^* \in B^*, \ b \in B, \ f \in L^1).
\]

**Definition 4.11.** A net \( (e_\alpha) \) in a commutative, normed algebra \( A \) is an approximate identity, abbreviated a.i., if for all \( a \in A \)
\[
\lim_{\alpha \to} e_\alpha a = a \quad \text{in } A.
\]

The next theorem shows the equivalence of essential and abso-
lutely continuous $L^1$-modules and gives two characterizations of essential $L^1$-modules.

**THEOREM 4.12.** If $B$ is a Banach $L^1$-module then the following statements are equivalent:

i) $B$ is an essential $L^1$-module.

ii) $B$ is an absolutely continuous $L^1$-module.

iii) $\lim_{n \to \infty} ||e_n^* - b||_B = 0$ for all $b \in B$ and any a.i. $(e_n)$ in $L^1$.

iv) $B^*$ is order free (if for $b^* \in B^*$, $f \cdot b^* = 0$ for all $f \in L^1$ then $b^* = 0$).

**PROOF.** The equivalence of i), ii) and iii) was proved by M. A. Rieffel in [48, p. 453].

Suppose that $B = B_{abs}$ and for $b^* \in B^*$, $f \cdot b^* = 0$ for all $f \in L^1$. If $b^* \in B$ then $b = \text{supp} b_1$ for some $f \in L^1$ and some $b_1 \in B$. So,

$$b^*(b) = b^*(f \cdot b_1) = f \cdot b^*(b_1) = 0$$

since $f \cdot b^* = 0$. Therefore $b^* = 0$. This shows that ii) implies iv).

Now consider the inclusion map $i : B_{abs} \hookrightarrow B$. Since $i$ is linear and continuous its adjoint map $i^* : B^* \to B^*_{abs}$ is also continuous. If $i^*(b^*) = 0$ then for all $f \in L^1$ and $b \in B$,

$$i^*(b^*)(f \cdot b) = b^*(f \cdot b) = f \cdot b^*(b) = 0$$

this implies that $f \cdot b^* = 0$ for all $f \in L^1$. Hence, if iv) holds then $b^* = 0$. This means that $i^*$ is injective and this implies that $B_{abs}$ is dense in $B$ [44, Corollary 4.12, p. 94]. Therefore $B$ is an essential $L^1$-module. That is, iv) implies i) and the proof is complete.
PROPOSITION 4.13. The following amalgam spaces are absolutely continuous \( L^1 \)-modules.

i) \((L^p, L^q)\) \(1 \leq p, q < \infty\)

ii) \((C_0, L^q)\) \(1 \leq q < \infty\)

iii) \((L^p, c_0)\) \(1 \leq p < \infty\).

PROOF. Let \( A \) be any of the spaces listed in the statement of the theorem. Suppose that for \( a^* \in A^* \), \( f a^* = 0 \) for all \( f \in L^1 \). In particular \( g a^* = 0 \) for all \( g \in C_c \). This implies that \( \langle g, a^* \rangle = g a^*(0) = 0 \) for all \( g \in C_c \). Since \( C_c \) is dense in \( A \) (Theorem 3.7), we conclude that \( \langle a, a^* \rangle = 0 \) for all \( a \in A \). Hence \( a^* = 0 \) and the conclusion follows from Theorem 4.12.

COROLLARY 4.14. Let \( A \) be any of the amalgam spaces listed in Proposition 4.13. If \( \{e_\alpha\} \) is an approximate identity in \( L^1 \), then

\[
\lim_{\alpha} \|e_\alpha f - f\|_A = 0
\]

for all \( f \in A \).


NOTATION. We will denote by \((L^p, L^q)\), \((C_0, L^q)\), \((L^p, c_0)\), \(M_q\) \((1 \leq p, q < \infty)\) the spaces \((L^p, L^q)\), \((C_0, L^q)\), \((L^p, c_0)\), \(M_q\) with the norm \(\|\cdot\|_{pq}^\#, \|\cdot\|_{\alpha q}^\#, \|\cdot\|_{p_\infty}^\#, \|\cdot\|_q^\#\) defined in Theorem 1.21 respectively.

DEFINITION 4.15. A linear subspace \( S \) of \( L^1 \) is said to be a Segal algebra if it satisfies the following conditions

(S-0) \( S \) is dense in \( L^1 \).
(S-1) S is a Banach space under some norm $|| \cdot ||_S$, and there exists a constant $C$ such that for all $f \in S$

$$||f||_1 \leq C ||f||_S$$

(S-2) S is invariant under translations ($f \in S$ implies $\tau_s f \in S$ for all $s \in G$) and for all $f \in S$ the mapping $s \mapsto \tau_s f$ from $G$ to $S$ is continuous.

(S-3) The norm $|| \cdot ||_S$ is invariant in the sense that $||\tau_s f||_S = ||f||_S$ for all $s \in G, f \in S$.

**THEOREM 4.16.** $(L^p, \ell^1)^\#$ $(1 < p < \infty)$ and $(C_0, \ell^1)^\#$ are Segal algebras.

**PROOF.** By Corollary 3.8, $(L^p, \ell^1)$ and $(C_0, \ell^1)$ are dense in $L^1$.

$(L^p, \ell^1)^\#$ satisfies (S-1) ((2.4) and Theorem 1.21) and (S-2) (Theorem 3.11 and Theorem 3.14), for $1 < p < \infty$.

Finally if $s \in G$ and $f \in (L^p, \ell^1)$ $(1 < p < \infty)$ then

$$\left(\tau_s f\right)^\#(t) = \left|\tau_s f\right|_1 = \int_{\mathbb{R}} f(t+s) \, ds = f^{\#}(s+t) = \tau_{-s} f^{\#}(t).$$

This implies that

$$||\tau_s f||_{\ell^1} \leq ||(\tau_s f)^\#||_1 = ||\tau_{-s} f^\#||_1 = ||f^\#||_1 = ||f||_{\ell^1}.$$  

Hence $(L^p, \ell^1)$ $(1 < p < \infty)$ implies (S-3) and the proof is complete because $(C_0, \ell^1) \subseteq (L^\infty, \ell^1)$.

**PROPOSITION 4.17.** $(L^p, \ell^1)^\#$ $(1 < p < \infty)$ and $(C_0, \ell^1)^\#$ satisfy the following.

1) For all $f \in L^1$, $\mu \in M_1$, $h \in (L^p, \ell^1)$, $g \in (C_0, \ell^1)$

$$||f^\#h||_{\ell^1} \leq ||f||_1 ||h||_{\ell^1} \quad ; \quad ||f^\#g||_{\ell^\infty} \leq ||f||_1 ||g||_{\ell^\infty}$$
\( \left| \left| u \ast h \right| \right|_p \leq \left| \left| u \right| \right|_1 \left| |h| \right|_p \); \quad \left| \left| u \ast g \right| \right|_\infty \leq \left| \left| u \right| \right|_1 \left| |g| \right|_\infty . \)

ii) \((C_0, \ell^1)\) has an a.i. \(\{e_n\}\) such that \(\left| |e_n| \right|_1 = 1\) for all \(n\).

Hence \(\{e_n\}\) is an a.i. in \((\ell^p, \ell^1)\) \((1 < p < \infty)\).

PROOF. i) follows from \([46, \S 4 Proposition 1 i)\) and Proposition 2. The first part of ii) follows from \([46, \S 6 Proposition 1 ii)]\) and the second part is a consequence of inclusions \((2.6)\) and inequality \((2.4)\).
CHAPTER II

THE FOURIER TRANSFORM ON \((L^p, L^q)\) AND \(M\)

§ 5. THE FOURIER TRANSFORM ON \((L^p, L^q)\) \((1 \leq p, q \leq 2)\)

In order to study the Fourier transform of functions in \((L^p, L^q)\) for \(1 \leq p, q \leq 2\) we proceed as F. Holland has done for the real case [34, Theorem 8]. First we generalize the Riesz-Thorin Theorem for amalgam spaces and then we use Theorems 3.4 and 3.5 of [49] to locate the image of \((L^p, L^q)\) under the Fourier transform and to establish the corresponding Hausdorff-Young inequality.

Bertrandias and Dupuis [7, §3] used another method. They find the Fourier transform of functions in the extreme cases \((L^2, L^1)\), \((L^1, L^2)\) and use a particular case of the Riesz-Thorin Theorem for mixed-norm spaces established by Benedek and Panzone.

For the definition of the (inverse) Fourier-Stieltjes transform of a measure \(\mu \in M_1\) we follow [37, Definition 23.9 (Definition 31.2)].

DEFINITION 5.1. The Fourier-Stieltjes (inverse Fourier-Stieltjes) transform of a measure \(\mu\) in \(M_1(G) (M_1(\hat{G}))\) is a function \(\hat{\mu}(\hat{\xi})\) on \(\hat{G}\) (G) defined by

\[
\hat{\mu}(\hat{\xi}) = \int_G \frac{[x, \hat{\xi}]}{[x, \hat{\xi}]} d\mu(x) = \int_G \frac{[-x, \hat{\xi}]}{[-x, \hat{\xi}]} d\mu(x)
\]

\[
\hat{\mu}(\hat{\xi}) = \int_G [x, \hat{\xi}] d\mu(x)
\]

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Throughout the rest of our work we will make constant use of the next theorem due to J. Stewart.

THEOREM 5.2. [49, Theorem 3.1]. Let $1 \leq p, q \leq \infty$. If $E$ is a compact subset of $G$ then there exists a function $g \in \mathcal{C}_c(G)$ such that $g = 1$ on $E$ and $\hat{g} \in (L^p, L^q)(\hat{G})$ ($\hat{g} \in (C_0, L^q)(\hat{G})$ if $p = \infty$).

THEOREM 5.3. [49, Theorem 3.4]. Let $f \in L^p(G)$, $1 \leq p \leq 2$. If $\text{supp } f \subseteq E$, $E$ being a compact subset of $G$, then $\hat{f} \in (C_0, \mathcal{L}^p)(\hat{G})$ and there exists a constant $C_p$ depending only on $E$ and $p$ such that

$$||\hat{f}||_{\mathcal{L}^p} \leq C_p ||f||_p.$$

The next theorem was proved for the real line by Holland [15, Theorem 8] and for locally compact abelian groups by Stewart [49, Theorem 3.5].

THEOREM 5.4. If $f \in (L^p, L^1)(G)$; $1 \leq p \leq 2$, then $\hat{f} \in (C_0, \mathcal{L}^p)(\hat{G})$ and there exists a constant $C_p$ depending only on $p$ such that

$$||\hat{f}||_{\mathcal{L}^p} \leq C_p ||f||_1.$$

NOTATION. $(L_\alpha, B_\alpha, m_\alpha)$ ($\alpha \in J$) will be the measurable subspace of $(G, B, m)$ where $\{L_\alpha\}$ is the family of sets defined in Definition 1.6 and $B_\alpha, m_\alpha$ are the restrictions of the $\sigma$-algebra $B$ and the Haar measure $m$ on $L_\alpha$ respectively.

Similarly $(L_\beta, B_\beta, m_\beta)$ ($\beta \in I$) (see p.23) is the corresponding measurable subspace of $(\hat{G}, \hat{B}, \mu)$. 
For $1 \leq r < \infty$, $\ell^r(J)$ will denote the subspace of $\ell^r(J)$ consisting of all nets $(a_\alpha)_J$ such that $a_\alpha = 0$ for all but finitely many $\alpha$.

The direct sum $\bigoplus_{\alpha \in J} S_\alpha$ of the linear spaces $S_\alpha$ of $m_\alpha$-measurable simple functions on $L_\alpha$ will be denoted by $\mathcal{L}(m)$. That is, $(x_\alpha)_J \in \mathcal{L}(m)$ iff $x_\alpha \in S_\alpha$ and $x_\alpha = 0$ for all but finitely many $\alpha$.

$\mathcal{L}(m)$ is defined similarly.

**REMARK 5.5.** i) The space $S_C$ of measurable simple functions with compact support can be identified with $\mathcal{L}(m)$. Indeed, if $s \in S_C$ then the support of $s$ meets finitely many $L_\alpha$'s, so the net $(s_\alpha)_J$ where $s_\alpha = s|L_\alpha$ belongs to $\mathcal{L}(m)$. Conversely if $(s_\alpha)_J \in \mathcal{L}(m)$ then $s_\alpha \neq 0$ for $\alpha_1, \ldots, \alpha_n$ say, so the function $s = \sum_{i=1}^{n} s_{\alpha_i}$ is a simple function whose support is contained in $\bigcup_{i=1}^{n} K_{\alpha_i}$ and therefore $s \in S_C$.

ii) $S_C$ is dense in $(L^p, L^q)$ for $1 \leq p \leq \infty$, $1 \leq q < \infty$, because $S_C$ is dense in $L^p_C$ and $L^p_C$ is dense in $(L^p, L^q)$ (Theorem 3.6).

**THEOREM 5.6.** (Riesz-Thorin). Let $1 \leq p_i, q_i, r_i, s_i \leq \infty$ for $i = 1, 2$. Given $0 \leq \theta \leq 1$ set $1/r\theta = \theta/r_1 + (1-\theta)/r_2$,

$1/s\theta = \theta/s_1 + (1-\theta)/s_2$, $1/p = \theta/p_1 + (1-\theta)/p_2$ and $1/q = \theta/q_1 + (1-\theta)/q_2$.

Let $T$ be a linear operator on $\mathcal{L}(m)$ such that $T \mathcal{L}(m) \subseteq (L^{q_1}, L^{s_1}_G)(G)$ for $i = 1, 2$ and suppose that there exist positive constants $C_i$ for $i = 1, 2$ such that
\[ | | Tx | |_{q_i s_i} \leq C_i | | x | |_{p_i q_i}, \quad i = 1, 2. \]

Then \( T \mathcal{L}(m) \subseteq (L^q, L^s) \hat{G} \).

Furthermore for all \( x \in \mathcal{L}(m) \)
\[
\begin{align*}
| | Tx | |_{q_0} & \leq C_1 C_2 | | x | |_{pr} \quad \text{if } 1 < q \leq \infty, 1 \leq s \leq \infty \\
\theta | | Tx | |_{1_{B^0}} & \leq C_1 C_2 2^d | | x | |_{pr} \quad 1 \leq s \leq \infty.
\end{align*}
\]

**PROOF.** Consider the following diagrams (for notation see §3 p. 32).

\[
\begin{array}{ccc}
\bigoplus_{\alpha} S_\alpha & \xrightarrow{T} & L^q_i(\mathcal{L}(L_B)) \\
\uparrow & & \downarrow \Pi_B \\
S_\alpha & \xrightarrow{T_{\alpha \beta}} & L^q_i(\mathcal{L}(L_B)).
\end{array}
\quad
\begin{array}{ccc}
x_\alpha & \xrightarrow{\Pi_B} & T(x_\alpha) \\
x_\alpha & \xrightarrow{\Pi_B \circ T} & \Pi_B \circ T(x_\alpha)
\end{array}
\]

where \( x_\alpha \mapsto \overline{x_\alpha} \) is the canonical inclusion map and \( \Pi_B \) is the \( \beta \)-th projection.

Since each \( x \in \bigoplus S_\alpha \) can be expressed as \( \sum_{\alpha} x_\alpha \) (\( x_\alpha \in S_\alpha \))

we have that \( Tx = \left( \sum_{\alpha} T_{\alpha \beta}(x_\alpha) \right) \beta \in I \). That is, \( \Pi_B \circ Tx = \sum_{\alpha} T_{\alpha \beta}(x_\alpha) \). We think of \( T \) as a \( J \times I \) matrix of operators \([T_{\alpha \beta}]\). For \( 0 \leq \theta \leq 1 \) we have the numbers \( p, q, r, s \) and for these we define \( A(\theta) \) to be the set of \((a, y, b, x)\) in \( L^s_p(1) \oplus L^q(1) \oplus L^r(1) \oplus \mathcal{L}(m) \) such that

\[ | | a | |_s \leq 1, \quad | | y_\beta | |_q \leq 1 \text{ for each } \beta \in I, \quad | | b | |_r \leq 1 \text{ and } | | x_\alpha | |_p \leq 1 \]

for each \( \alpha \in J \).

Now we define for \( 0 \leq \theta \leq 1 \)...
\[ M(\theta) = \sup_{A(\theta)} \left\| \sum_{\alpha \beta} a_\beta b_\alpha \int_{L_\beta^q} y_\beta T_\alpha(x_\alpha) \, d\nu_\beta \right\|_{q'} \]

Note that if \((a, y, b, x) \in A(\theta)\) for some \(\theta\) then \(ay = (a_\beta y_\beta)_I\) belongs to \(L^{q'}(L^q_\beta)\) and \(\|ay\|_{q'} \leq 1\). Indeed,

\[ \sum_{\beta} \left( \int_{L_\beta^q} \left| a_\beta y_\beta \right| \right)^{q'/q'} \leq \sum_{\beta} |a_\beta|^s \left( \int_{L_\beta^q} |y_\beta|^{q'/q'} \right)^{s'/q'} \leq \sum_{\beta} |a_\beta|^s \|y_\beta\|_{q'} \leq 1. \]

Similarly \(bx = (b_\alpha x_\alpha)_I\) belongs to \(L^p(L^p_\alpha)\) and \(\|bx\|_{pr} \leq 1\). So, by the Hölder inequality

\[ \left\| \sum_{\alpha \beta} a_\beta b_\alpha \int_{L_\beta^q} y_\beta T_\alpha(x_\alpha) \, d\nu_\beta \right\|_{q'} \leq \sum_{\beta} \left( \int_{L_\beta^q} |y_\beta| \right)^{q'/q'} \left( \sum_{\alpha} |a_\beta| |T_\alpha(b_\alpha x_\alpha)| \right)^{1/q} \]

\[ \leq \sum_{\beta} \|a_\beta y_\beta\|_{L^q(L_\beta^q)} \|\Pi S^b(b_\alpha x_\alpha)\|_{L^q(L_\beta^q)} \]

\[ \leq \left[ \sum_{\beta} |a_\beta y_\beta|^s \right]^{s'/s} \left[ \sum_{\beta} \|\Pi S^b(b_\alpha x_\alpha)\|^s \right]^{1/s} \]

\[ = \|ay\|_{q'} \|T(bx)\|_{qs} \]

\[ \leq \|T(bx)\|_{qs}. \]

Since this is true for any \((a, y, b, x) \in A(\theta)\),

\[ M(\theta) \leq \|T(bx)\|_{qs}. \]

This implies that \(M(0) \leq C_2\) and \(M(1) \leq C_1\). By [18, Lemma 7 p. 522] the function \(\log M(\theta)\) on \([0,1]\) is convex and we conclude that \(M(\theta) \leq C_1^\theta C_2^{1-\theta}\).

We will see next that
for all \( x, y \in \mathcal{L}(m) \times \mathcal{L}(\mu) \).

The case when \( x = 0 \) or \( y = 0 \) is trivial. So we take nonzero \( x, y \) in \( \mathcal{L}(m) \), \( \mathcal{L}(\mu) \) and consider

\[
y_{\beta}' = \begin{cases} \frac{y_{\beta}}{||y_{\beta}|| q'_s}, & y_{\beta} \neq 0 \\ 0, & \text{otherwise} \end{cases}
\]

\[
a_{\beta} = \frac{||y_{\beta}|| q'_s}{||y|| q'_s},
\]

\[
x_{\alpha}' = \begin{cases} \frac{x_{\alpha}}{||x_{\alpha}|| p}, & x_{\alpha} \neq 0 \\ 0, & \text{otherwise} \end{cases}
\]

\[
b_{\alpha} = \frac{||x_{\alpha}|| p}{||x|| pr},
\]

Then \( a = (a_{\beta})_I \), \( y' = (y_{\beta}')_I \), \( b = (b_{\alpha})_J \), \( x \neq (x_{\alpha}')_J \) belong to \( A(\theta) \) and we have that

\[
C_1^\theta C_2^{1-\theta} \geq M(\theta) \geq \sum_{\alpha\beta} a_{\beta} b_{\alpha} \int_{L_{\beta}} y_{\beta}' |y_{\beta}'| q'_s T_{\alpha\beta}(x_{\alpha}') |x||_{pr} \, d\mu_{\beta}
\]

\[
= \sum_{\alpha\beta} \int_{L_{\beta}} a_{\beta} y_{\beta}' T_{\alpha\beta}(b_{\alpha} x_{\alpha}') \, d\mu_{\beta}
\]

\[
= \sum_{\alpha\beta} \int_{L_{\beta}} y_{\beta}' |y_{\beta}'| q'_s T_{\alpha\beta}(x_{\alpha}') |x||_{pr} \, d\mu_{\beta}
\]

\[
= 1/||y|| q'_s, 1/||x||_{pr} \sum_{\alpha\beta} \int_{L_{\beta}} y_{\beta} T_{\alpha\beta}(x_{\alpha}) \, d\mu_{\beta}
\]

(1) implies that the map \( T: \mathcal{L}(\mu) \to \mathcal{C} \) given by

\[
y \mapsto \sum_{\beta} \int_{L_{\beta}} y_{\beta} |y_{\beta}| q'_s \, d\mu_{\beta}
\]

is a continuous linear functional on \( (\mathcal{L}(\mu), ||\cdot|| q'_s) \) and

\[
||T|| \leq C_1^\theta C_2^{1-\theta} ||x||_{pr} \quad (x \in \mathcal{L}(m)).
\]
Since $\mathcal{L}(\mu)$ is a linear subspace of $(L^q, \ell^s)$, $(C_0, \ell^s)$, $(L^q, c_0)$ and the duals of these spaces are $(L^q, \ell^s)$, $M_\infty$, $(L^q, \ell^1)$ respectively we conclude that $T \mathcal{L}(\mu) \subseteq (L^q, \ell^s)$.

Finally if $1 \leq s \leq \infty$ then by Theorem 3.1

$$||Tx||_{qs} \leq C_1 C_2^{1-\theta} ||x||_{pr} \quad \text{if} \quad 1 < q \leq \infty \quad \text{and by Theorem 3.2}$$

$$||Tx||_{1s} \leq 2^a ||Tx|| \leq C_1 C_2^{1-\theta} 2^a ||x||_{pr}$$

The next theorem was proved for the real line by Hollander [34, Theorem 8] and for locally compact abelian groups by Bertrandias and Dupuis [7, Theorem II].

**Theorem 5.7.** If $1 \leq p, q \leq 2$ then the Fourier transform $\hat{f}$ of a function $f \in (L^p, \ell^q)(G)$ belongs to

- $(L^{q'}, \ell^{p'})(\hat{G})$ if $1 < p, q \leq 2$
- $(C_0, \ell^{p'})(\hat{G})$ if $q = 1$, $1 \leq p \leq 2$
- $(L^{q'}, c_0)(\hat{G})$ if $p = 1$, $1 < q \leq 2$.

Moreover there exists a constant $C_{pq}$ depending only on either $p$ or $q$ such that

$$||\hat{f}||_{q', p'} \leq C_{pq} ||f||_{pq} \quad \text{(Hausdorff-Young inequality)}$$

where

$$\theta = \begin{cases} 
    p'(1/q - 1/p) & \text{if } 1 < q < p \\
    q'(1/p - 1/q) & \text{if } 1 < p < q \\
    1 & \text{if } p = 1 \text{ or } q = 1.
\end{cases}$$

**Proof.** Case 1) $q = 1$, $1 \leq p \leq 2$. By Theorem 5.4, if
\( f \in (L^p, \lambda^1)(G) \) then \( \hat{f} \in (C_0, \lambda^{p'})(\hat{G}) \) and there exists a constant \( C_p \) depending on \( p \) such that

\[
(2) \quad ||f||_{\omega p} \leq C_p ||f||_{p^1}.
\]

Case 2) \( 1 < q < p \). If \( f \in (L^p, \lambda^q)(G) \) then by (2.7) \( f \in L^p(G) \) and by the Hausdorff-Young inequality

\[
(3) \quad ||f||_{p^1} \leq ||f||_p.
\]

Applying Theorem 5.6 to \( f \in S_c \) with \( q_1 = \omega, q_2 = s_1 = s_2 = p' \), \( r_2 = p_1 = p_2 = p, r_1 = 1, \theta = p'(1/q - 1/p), C_1 = C_p \) (as in (2)) and \( C_2 = 1 \) (as in (3)), we have that \( \hat{f} \in (L^{q'}, \lambda^{p'})(\hat{G}) \) and

\[
(4) \quad ||f||_{q'p'} \leq C_p^\theta ||f||_{pq}.
\]

Since \( S_c \) is dense in \( (L^p, \lambda^q) \) the Fourier transform can be extended to all \( (L^p, \lambda^q) \) and case 2) is proved.

Case 3) \( 1 \leq p < q \). Take \( f \in L^p_c \) and \( g \in (L^q, \lambda^p)(\hat{G}) \), hence \( g \in L^p(\hat{G}) \) by (2.7). So by a generalization of Parseval's identity (as in [37, 31.48 a])

\[
\int_G \hat{f}(\hat{x}) \overline{g(\hat{x})} \, d\hat{x} = \int_G \overline{g(x)} f(x) \, dx.
\]

On the other hand by Theorem 5.3, \( \hat{f} \in (C_0, \lambda^{p'})(\hat{G}) \) and by case 2) \( \hat{g} \in (L^{p'}, \lambda^{q'})(G) \), therefore by Theorem 3.1

\[
\left| \int_G \hat{f}(\hat{x}) \overline{g(\hat{x})} \, d\hat{x} \right| \leq \int_G |\hat{g}(x)| |f(x)| \, dx \leq ||\hat{g}||_{p', q'} ||f||_{p q'}.
\]

This implies by (2) and (4) that
\[ \left| \int_{\mathbb{R}^d} \hat{f}(\xi) \, g(\xi) \, d\xi \right| \leq C_q \| g \|_{q_1} \| f \|_{l^q} \]

\[ \leq C_q \| g \|_{q_p} \| f \|_{l^p} \quad \text{if } p \neq 1 \]

where \( C_q \) is a constant depending only on \( q \) and \( \theta = q'(1/p - 1/q) \). Hence the map \( g \mapsto \int_{\mathbb{R}^d} \hat{g}(\xi) \hat{f}(\xi) \, d\xi \) is a linear functional on \((L^q, l^p)(\hat{G})\). By Theorem 3.1, \( \hat{f} \in (L^{q_1}, l^{p_1})(\hat{G}) \) and

\[ \| \hat{f} \|_{q_1'p_1'} \leq \begin{cases} C_q \| f \|_{l^q} & \text{if } p \neq 1 \\ C_q^\theta \| f \|_{l^p} & \text{if } p = 1 \end{cases} \]

Moreover, if \( p = 1 \) then \( \hat{f} \in C_0 \) and by Remark 3.9 \( \hat{f} \in (L^{q_1}, c_0)(\hat{G}) \).

Since \( L_c^p \) is dense in \((L^p, l^q)\), the Fourier transform can be extended to all of \((L^p, l^q)\) and we have that \((L^p, l^q) \subset (L^{q_1'}, l^{p_1'})\) if \( p \neq 1 \) and \((L^1, l^q) \subset (l^{q_1'}, c_0)\) (remember that \((L^{q_1'}, c_0)\) is a closed subspace of \((L^{q_1'}, l^\infty)\)).

**Remark 5.8.** If \( f \in (L^p, l^q) \) \( 1 \leq q \leq 2, 2 < p \leq \infty \) then by (2.6) \( f \in (L^2, l^q) \) and by Theorem 5.7 \( \hat{f} \in (l^{q_1}, l^2) \).
§ 6. **THE FOURIER TRANSFORM ON \((L^p, L^q) (2 < q \leq \infty)\)**

AND \(M_s (1 \leq s \leq \infty)\)

We start this section with the definitions of G. L. Gaudry [28, §1 p. 478] and W. R. Bloom [3, p. 206] for the Fourier transform of functions in \(L^r(G), 1 \leq r \leq \infty,\) and \((L^p, L^q)(G), 1 \leq p, q \leq \infty,\) respectively. We will see that when \(p = q\) both definitions coincide.

We will then proceed to extend Bloom's definition to the space \(M_s, 1 \leq s \leq \infty,\) and study the relation between the transformable measures defined by L. Argabright and J. Gil de Lamadrid [1], and the measures \(M_s.\)

Finally we will give a brief account of other definitions of the Fourier transform of measures in \((L^p, L^q), 1 \leq p, q \leq \infty,\) and \(M_s, 1 \leq s \leq \infty;\) one due to J. P. Bertrandias and C. Dupuis, and another due to H. C. Feichtinger.

**NOTATION.** For a compact subset \(E\) of \(G, C^E_c = C^E_c(G)\) will be the linear space of functions \(f \in C^E_c\) such that \(\text{supp } f \subseteq E,\) endowed with the supremum norm.

\(D^E(G)\) will denote the linear space

\[
D^E(G) = \{ h | h = \sum_{i=1}^{\infty} f_i * g_i, f_i, g_i \in C^E_c(G), \sum_{i=1}^{\infty} \| f_i \|_\infty \| g_i \|_\infty < \infty \}
\]

and \(A^E(G) = \{ g \in C^E_c(G) | g = g^*, g \in L^1(\hat{G})\) and \(\text{supp } g \subseteq E\}.\)
For \( h \in \mathcal{D}_E(G) \) and \( g \in \mathcal{A}_E(G) \) we define

\[
\|h\|_E = \inf \left\{ \sum \|f_i\|_\infty \|g_i\|_\infty \mid h = \sum f_i \ast g_i, f_i, g_i \in \mathbb{C}_c^E, \sum \|f_i\|_\infty \|g_i\|_\infty < \infty \right\}
\]

\[
\|g\|_{\mathcal{A}_E} = \|g\|_1
\]

**Remark 6.1.**

a) \( \mathcal{D}_E(G) \subseteq \mathbb{C}^{E+E}_c(G) \) and \( \|\cdot\|_E \) is well defined.

b) \( \|h\|_\infty \leq m(\mathcal{E}) \|h\|_E \) \( (h \in \mathcal{D}_E(G)) \)

c) \( \|g\|_\infty \leq \|g\|_1 = \|g\|_{\mathcal{A}_E} \) \( (g \in \mathcal{A}_E(G)) \)

d) \( (\mathcal{A}_E(G), \|\cdot\|_{\mathcal{A}_E}) \) and \( (\mathcal{D}_E(G), \|\cdot\|_E) \) are Banach spaces [40, Theorem 5.1.1.].

**Definition 6.2.** \( \mathcal{D}(G) \) is the internal inductive limit of the Banach spaces \( \mathcal{D}_E(G) \). That is, \( \mathcal{D}(G) = \cup \{\mathcal{D}_E(G) \mid E \subseteq G \text{ compact}\} \) and a basic neighborhood of the origen is of the type

\[
U_c = \cup \{h \in \mathcal{D}_E(G) \mid \|h\|_E < \epsilon\} \quad (\epsilon > 0).
\]

Similarly \( \mathbb{C}^E_c(G) \) and \( \mathbb{C}^{E+E}_c(G) \) are the internal inductive limits of the Banach spaces \( \mathcal{A}_E(G) \) and \( \mathbb{C}_c^{E}(G) \) respectively.

**Remark 6.2.** It is known [16, Theorem 3.1] that the spaces \( \mathcal{D}(G) \) and \( \mathbb{C}_c(G) \) are homeomorphic and isomorphic as spaces of functions on \( G \), and that \( \mathcal{D}(G) \) is dense in \( \mathbb{C}_c(G) \) [40, Theorem 5.1.2], hence so is \( \mathbb{C}_c(G) \).

**Lemma 6.4.** \( \mathbb{C}_c(G) = \{\phi \in \mathbb{C}_c(G) \mid \hat{\phi} \in (\mathbb{L}_p, \mathbb{L}_q^1)\} \). Hence by (2.5) and (2.6) \( \hat{\phi} \in (\mathbb{L}_p, \mathbb{L}_q^1)(\hat{G}) \), \( 1 \leq p, q \leq \infty \) for all \( \phi \in \mathbb{C}_c(G) \).
PROOF. If \( \phi \in C_c \) and \( \hat{\phi} \in (C_0, \mathcal{L}^1) \) then by (2.6) \( \hat{\phi} \in L^1(\hat{\mathcal{G}}) \)
and this implies that \( \phi = \hat{\phi} \). Therefore \( \phi \in A_c(\mathcal{G}) \).

Now take \( g \in A_c(\mathcal{G}) \) and let \( f_E \in (C_0, \mathcal{L}^1)(\hat{\mathcal{G}}) \) such that
\( \hat{f}_E \equiv 1 \) on \( \mathcal{E} \) and \( \hat{f}_E \in C_c(\mathcal{G}) \) (Theorem 5.2). Since \( g = \hat{g} \) on \( \mathcal{G} \),
\( g \in L^1(\hat{\mathcal{G}}) \) we have that \( g = \hat{g} = (\hat{g} \cdot \hat{f}_E) \). By Theorem 4.7 \( \hat{g} \cdot \hat{f}_E \in (C_0, \mathcal{L}^1) \), so \( \hat{g} = (\hat{g} \cdot \hat{f}_E) \). Because
\( \hat{g} \cdot \hat{f}_E \in L^1(\hat{\mathcal{G}}) \). Hence \( \hat{g} \in (C_0, \mathcal{L}^1) \) and therefore \( g \in \{ \phi \in C_c | \hat{\phi} \in (C_0, \mathcal{L}^1) \} \).

**Proposition 6.5.** Let \( A \) be any of the following amalgam spaces:

\[
(L^p, \mathcal{L}^q) \quad 1 \leq p, q < \infty
\]

\[
(C_0, \mathcal{L}^q) \quad 1 \leq q \leq \infty
\]

\[
(L^p, c_0) \quad 1 \leq p < \infty
\]

Then \( D(\mathcal{G}) \) is dense in \( (C_c, \| \cdot \|_A) \). Hence by Theorem 3.7 and
Remark 6.3 \( D(\mathcal{G}) \) and \( A_c(\mathcal{G}) \) are dense in \( A \) and the topology on \( D(\mathcal{G}) \)
and \( A_c(\mathcal{G}) \) is stronger than the one induced by \( A \).

**PROOF.** Let \( \{ e_\alpha \} \) be an a.i. in \( L^1 \) such that for all \( \alpha \),
\( \| e_\alpha \|_1 = 1 \) and \( e_\alpha \in C_c(\mathcal{G}) \) for a fixed \( \mathcal{G} \). Since \( A \) is an absolutely
continuous \( L^1 \)-module, for all \( f \in A \) \( \lim \| f \cdot e_\alpha - f \|_A = 0 \) (Corollary 4.14). In particular for all \( \phi \in C_c(\mathcal{G}) \) \( \lim \| \phi \cdot e_\alpha - \phi \|_A = 0 \).

Since \( \phi \cdot e_\alpha \in D(\mathcal{G}) \) for all \( \alpha \) we conclude that \( D(\mathcal{G}) \) is dense in
\( (C_c, \| \cdot \|_A) \). Now, if \( \phi \in C_c(\mathcal{G}) \) then by Proposition 3.4 there exists a
constant \( C_E \) depending on \( \mathcal{G} \) and \( q \) such that
\( \| \phi \|_{\infty q} \leq C_E \| \phi \|_{\infty} \). Since
\( \| \phi \|_{p\infty} \leq \| \phi \|_\infty \) and \( \| \phi \|_{pq} \leq \| \phi \|_{\infty q} \) by (2.4), the rest of the
proof follows from Remark 6.1 parts b) and c).
DEFINITION 6.6. The dual of \( D(G) \) is called the space of \textit{quasimeasures} and it is denoted by \( Q(G) \).

The pairing between \( D(G) \) and \( Q(G) \) will be written as

\[ \langle h, \sigma \rangle = (h \in D(G), \sigma \in Q(G)) \]

DEFINITION 6.7. [28, 1.1 p. 478]. Let \( 1 < p \leq \infty \). The Fourier transform of \( f \in L^P(G) \) is a quasimeasure \( \hat{f} \) defined by

\[ \langle h, \hat{f} \rangle = \langle h', f \rangle = \int_G h(t) \overline{f}(t) \, dt \quad (h \in D(\hat{G})). \]

DEFINITION 6.8. [3, p. 266]. Let \( 1 \leq p, q \leq \infty \). The Fourier transform \( \hat{f} \) of \( f \in (L^P, L^q)(G) \) is an element of the dual of \( A_C(\hat{G}) \) defined by

\[ \langle g, \hat{f} \rangle = \langle \overline{g'}, f \rangle = \int_G \overline{g}(t) \, f(t) \, dt \quad (g \in A_C(\hat{G})). \]

Note that by Lemma 6.4, \( \overline{g} \in (L^P, L^q)(G) \) for all \( g \in A_B(\hat{G}) \) and \( \overline{g} = g \hat{f}_E \), \( g \in L^1(G) \), \( f_E \in (C_0, L^1)(G) \). Hence \( \langle \overline{g'}, f \rangle \) is well defined and by Theorem 4.7

\[ |\langle g, \hat{f} \rangle| = |\langle \overline{g'}, f \rangle| \leq ||g||_{p,q} ||f||_{p,q} \leq ||g \hat{f}_E||_{\infty} ||f||_{p,q} \]

\[ \leq 2^{2a} ||g||_{\infty} ||f_E||_{\infty} ||f||_{p,q} = 2^{2a} ||f_E||_{\infty} ||f||_{p,q} \]

Therefore \( \hat{f} \in A_C(G)^* \).

If \( p = q \) then it follows from Remark 6.3 that Definition 6.7 and Definition 6.8 are equivalent.

If \( 1 \leq p, q \leq \infty \) then the Fourier transform of \( f \in (L^P, L^q) \) as
an element of $A_c^*$ coincides with the function $\hat{f}$ in $(L^{q'}, \mathbb{R}^p)$ given by Theorem 5.7. Indeed, by the Hausdorff-Young inequality (Theorem 5.7) there exists a constant $C$ such that for all $g \in A_c(\hat{G})$

$$\langle g, \hat{f} \rangle = \langle \hat{g'}, f \rangle \leq \|g\|_{p'q'} \|f\|_{pq} \leq C \|g\|_{pq} \|f\|_{pq}.$$  

This implies that $\hat{f} \in (A_c, \|\cdot\|_{pq})^*$. Since $A_c(\hat{G})$ is dense in $(L^q, L^p)(\hat{G})$ (Proposition 6.5), $\hat{f}$ has a unique continuous extension $\hat{f}$ on $(L^q, L^p)$ and by Theorem 3.1, $\hat{f} \in (L^{q'}, L^p)(\hat{G})$.

**DEFINITION 6.9.** For $\mu \in M_s(\hat{G})$, $1 \leq s \leq \infty$, we define its Fourier transform $\hat{\mu}$ as an element of $A_c(\hat{G})^*$ defined by

$$\langle g, \hat{\mu} \rangle = \langle \hat{g'}, \mu \rangle = \int_{G} g(-t) \ d\mu(t) \quad (g \in A_c(\hat{G})).$$

Again, by Lemma 6.4, $\hat{g} \in (C_0, \mathcal{L}^s)(\hat{G})$ for all $g \in A_c(\hat{G})$ and we have as above that

$$\langle g, \hat{\mu} \rangle \leq \|g\|_{s} \|\mu\|_{s} = \|g \ast f_E\|_{s} \|\mu\|_{s} \leq \|f_E\|_{s} \|\mu\|_{s}.$$

Therefore $\hat{\mu} \in A_c(\hat{G})^*$.

In particular if $\mu \in M_s$, $1 \leq s \leq 2$, then $\hat{\mu} \in (L^s, \ell^\infty)$ (as it was proved in [49, Theorem 4.2 ii]). Indeed, if $g \in A_c(\hat{G})$ then by Theorem 5.7 there exists a constant $C_s$ depending on $s$ such that

$$\langle g, \hat{\mu} \rangle = \langle \hat{g'}, \mu \rangle \leq \|g\|_{s} \|\mu\|_{s} \leq C_s \|g\|_{1} \|\mu\|_{s}.$$  

This means that $\hat{\mu} \in (A_c(\hat{G}), \|\cdot\|_{s})^*$. Again since $A_c(\hat{G})$ is dense in $(L^s, \ell^1)$ (Proposition 6.5) $\hat{\mu}$ has a unique continuous extension $\hat{\mu}$ on $(L^s, \ell^1)$ and by Theorem 3.1,
(6.1) \( \hat{\mu} \in (L^{s', \infty}) \) and \( ||\hat{\mu}||_{s', \infty} \leq c_s ||\mu||_s \).

**Remark 6.10.** If \( f \in (L^p, L^q) \) \( (1 \leq p, q \leq \infty) \) then its Fourier transform as an element of \( (L^p, L^q) \) and as an element of \( M_q \) coincide.

If there exists a constant \( C \) such that for all \( g \in A_c(\hat{G}) \)
\[ |<g, \hat{\mu}>| \leq C ||g||_{\infty} \] for some \( 1 \leq q \leq \infty \), then by the density of \( A_c(\hat{G}) \) in \( (C_0, L^q')(\hat{G}) \), \( \hat{\mu} \) has a unique continuous extension on \( (C_0, L^q')(\hat{G}) \) and by Theorem 3.2, there exists a unique measure \( \hat{\mu} \in M_q(\hat{G}) \) such that for all \( h \in (C_0, L^q')(\hat{G}) \)
\[ <h, \hat{\mu}> = \int_G h(\hat{x}) \, d\hat{\mu}(\hat{x}). \]

In this case for \( \hat{\mu} \) considered as a measure we say that \( \hat{\mu} \in M_q \). That is:

**Proposition 6.11.** Let \( \mu \in M_s(G) \), \( 1 \leq s \leq \infty \). \( \hat{\mu} \in M_q \) for some \( 1 \leq q \leq \infty \) iff there exists a constant \( C \) such that for all \( g \in A_c(\hat{G}) \)
\[ |<g, \hat{\mu}>| \leq C ||g||_{\infty} \]

Moreover, for all \( h \in (C_0, L^q')(\hat{G}) \)
\[ <h, \hat{\mu}> = \int_G h(\hat{x}) \, d\hat{\mu}(\hat{x}) \]
and for all \( g \in A_c(\hat{G}) \)
\[ \int_G \hat{g}(\hat{x}) \, d\hat{\mu}(\hat{x}) = <\hat{g}', \hat{\mu}> = <g, \hat{\mu}> = \int_G g(\hat{x}) \, d\hat{\mu}(\hat{x}). \]
PROPOSITION 6.12. Let \( \nu \in M_1(G) \), \( h \in (C_0, \ell^q \hat{G}) \), \( 1 \leq q \leq \infty \), and \( g \in A_c(\hat{G}) \). Then
\[
\begin{align*}
\text{i)} & \quad h\hat{\nu} \in (C_0, \ell^q \hat{G}) \\
\text{ii)} & \quad g\hat{\nu} \in A_c(\hat{G}).
\end{align*}
\]

PROOF. i) Since \( \hat{\nu} \) is a uniformly bounded function on \( \hat{G} \) [37, Theorem 31.5], \( h\hat{\nu} \in (C_0, \ell^q \hat{G}) \).

ii) By Lemma 6.4 \( \hat{g} \in (C_0, \ell^1) \) and \( (C_0, \ell^1) \subseteq L^1 \). So by [37, Theorem 31.27] \( \hat{g} \hat{\nu} = (\hat{g} * \hat{\nu})^{-} \). Since \( \hat{g} * \hat{\nu} \in (C_0, \ell^1) \) (Theorem 4.8) and \( g\hat{\nu} \in A_c(\hat{G}) \), because \( g \) has compact support, we conclude that \( (g\hat{\nu})^{-} = \hat{g} * \hat{\nu} \) and therefore \( g\hat{\nu} \in A_c(\hat{G}) \).

Proposition 6.12 allows us to define the product of an element of \( A_c(\hat{G})^* \) \( (M_q(\hat{G})) \) and \( \hat{\nu} \) \( (\nu \in M_1) \) as follows:

DEFINITION 6.13. If \( F \in A_c(\hat{G})^* \) \( (M_q(\hat{G}), 1 \leq q \leq \infty) \) and \( \nu \in M_1(G) \) then \( \hat{F}\hat{\nu} \) is an element of \( A_c(\hat{G})^* \) \( (M_q(\hat{G})) \) defined by
\[
\langle \hat{g}, \hat{F}\hat{\nu} \rangle = \langle g\hat{\nu}, F \rangle \quad g \in A_c(\hat{G}) \quad (g \in (C_0, \ell^q \hat{G})).
\]

PROPOSITION 6.14. Let \( \mu \in M_s(G) \) \( (1 \leq s \leq \infty) \) and \( f \in L^1(G) \).

i) \( (\mu * \nu) = \hat{\mu} \hat{\nu} \) for all \( \nu \in M_1(G) \). Hence \( (\mu * f) = \hat{\mu} \hat{f} \).

ii) If \( \hat{\mu} \in M_q(\hat{G}) \), \( 1 \leq q \leq \infty \), then \( (\mu * f) \in M_q(\hat{G}) \) and for all \( h \in (C_0, \ell^q \hat{G}) \)
\[
\int_{\hat{G}} h(\hat{\xi}) d(\mu * f)(\hat{\xi}) = \int_{\hat{G}} h(\hat{\xi}) \hat{f}(\hat{\xi}) d\hat{\mu}(\hat{\xi})
\]
and for all \( g \in A_c(\hat{G}) \)
\[
\int_{\hat{G}} g(\hat{\xi}) \mu * f(\hat{\xi}) d\hat{\mu}(\hat{\xi}) = \int_{\hat{G}} g(\hat{\xi}) \hat{f}(\hat{\xi}) d\hat{\mu}(\hat{\xi}).
\]
PROOF. i) Let \( g \in A_c(\hat{G}) \). Since \( g \hat{\nu} = (g \ast \nu)^\wedge \) for all \( \nu \in M_1(G) \), (see the proof of Proposition 6.12) we have that for \( \nu \in M_1(G) \)

\[
\langle g, \hat{\mu} \hat{\nu} \rangle = \langle g \hat{\nu}, \hat{\mu} \rangle = \langle (g \ast \nu)^\wedge, \hat{\mu} \rangle = \langle (g \ast \nu)' \hat{\nu}, \hat{\mu} \rangle = \int \int g^*(t-s) \nu(s) \, ds \, du(t) \]

\[
= \langle \hat{g}' \nu, \nu \hat{\mu} \rangle = \langle g, (\nu \hat{\mu})^\wedge \rangle.
\]

Since this is true for all \( g \in A_c(\hat{G}) \) we conclude that

\( \hat{\mu} \hat{\nu} = (\nu \hat{\mu})^\wedge \) for all \( \nu \in M_1(G) \).

ii) By i), Proposition 6.11 and Proposition 4.1 there exists a constant \( C \) such that for all \( g \in A_c(\hat{G}) \)

\[
|\langle g, (\nu \hat{\mu})^\wedge \rangle| = |\langle g, \hat{\mu} \hat{\nu} \rangle| = |\langle \hat{g} \hat{\nu}, \hat{\mu} \rangle| \leq C \|g \hat{\nu}\|_\infty \|\hat{\mu}\|_\infty.
\]

Therefore by Proposition 6.11 \( (\nu \hat{\mu})^\wedge \in M_q(\hat{G}) \).

Finally by the same Proposition 6.11

\[
\int_{\hat{G}} h(\hat{x}) d(\nu \hat{\mu})^\wedge(\hat{x}) = \langle h, (\nu \hat{\mu})^\wedge \rangle = \langle h, \hat{\mu} \hat{\nu} \rangle = \langle h \hat{\nu}, \hat{\mu} \rangle = \int_{\hat{G}} h(\hat{x}) \hat{f}(\hat{x}) \, d\hat{u}(\hat{x})
\]

for all \( h \in (C_0, \ell^q)'(G) \) and

\[
\int_{\hat{G}} \hat{g}(\hat{x}) \hat{\mu}(\hat{x}) \, d\hat{x} = \langle \hat{g}' \hat{\nu}, \hat{\mu} \rangle = \langle g, (\nu \hat{\mu})^\wedge \rangle = \int_{\hat{G}} g(\hat{x}) \hat{f}(\hat{x}) \, d\hat{u}(\hat{x})
\]

for all \( g \in A_c(\hat{G}) \).

(Remember that \( \mu \hat{\nu} \in (L^1, \ell^q) \) by Corollary 4.4).

**DEFINITION 6.15.** The inverse of the Fourier transform \( \hat{\nu} \) of a measure \( \mu \) in \( M_1(1 \leq q \leq \infty) \) is an element of \( A_c(\hat{G})^* \) defined by

\[
\langle \phi, \hat{\nu} \rangle = \langle \phi', \mu \rangle = \int_{\hat{G}} \phi(\hat{x}) \, d\hat{u}(\hat{x}) \quad (\phi \in A_c(\hat{G}))
\]
Before we continue we should remember that for \(1 \leq p, q \leq \infty\)
\((L^p, \ell^q) \subseteq (L^1, \ell^1) \subseteq M_q \subseteq M_\infty\) and therefore Proposition 6.11, Proposition 6.12, Proposition 6.14, Definition 6.15 and the results that follow hold for elements of any amalgam space and any measure space of type \(q\).

Also, when considering an element, \(\phi\), of \(D(G)\) we should bear in mind that \(D(G) \subseteq A_C(G)\) and by Lemma 6.4
\[A_C(G) = \{ \phi \in C_c(G) \mid \hat{\phi} \in (C_0, \ell^1)(\hat{G}) \} = \Phi(G).\]
Thought of as an element of \(A_c(G)\), \(\phi = \phi^0\), \(\phi^0 \in L^1(\hat{G})\); and thought of as an element of \(\Phi(G)\), \(\hat{\phi} \in (C_0, \ell^1)(\hat{G})\).

**Theorem 6.16. (Inversion Theorem).** Let \(\mu \in M_\infty(G)\). If \(\hat{\mu} \in M_\infty(\hat{G})\) and for all \(\phi \in A_C(G)\)
\[< \phi, \mu > = \int_G \phi(x) \, d\mu(x) = \int_{\hat{G}} \hat{\phi}(-\hat{x}) \, d\hat{\mu}(\hat{x}) = < \hat{\phi}, \hat{\mu} >\]
then \(\mu = \hat{\mu}\).

**Proof.** Let \(\phi \in A_C(G)\). By definition of \(\mu\) we have that
\[< \phi, \mu > = < \hat{\phi^t}, \hat{\mu} > = < \phi, \mu >.\]
Since \(\mu \in M_\infty\) and \(A_C(G)\) is dense in \((C_0, \ell^1)(G)\) we conclude that \(\mu \in M_\infty\) (Proposition 6.11) and \(\mu = \hat{\mu}\).

Our next goal will be to see which measures in \(M_\infty\) satisfy the condition of Theorem 6.16. To this end we introduce the concept of **transformable measure** as defined by L. Argabright and J. Gil de Lamadrid [1].

**Definition 6.17.** A measure \(\nu\) is **transformable** if there exists a measure \(\tilde{\nu}\) on \(\hat{G}\) such that for all \(\phi \in C_0(G)\), the function
\[\hat{x} \mapsto |\hat{\phi}|^2(-\hat{x})\] on \(\hat{G}\) belongs to \(L^1(\tilde{\nu})\) and...
\[ \int_G \phi^* \hat{\phi}(x) \, d\nu(x) = \int_G \hat{\phi}^2(-\hat{x}) \, d\nu(\hat{x}). \]

\( M_T(G) = M_T \) will denote the set of transformable measures and
\[ C_2(G) = \{ \phi^* \hat{\phi} \mid \phi \in C_c(G) \}. \]

Note that if \( f, h \) belong to \( C_c(G) \) then \( f^* h \in < C_2(G) > \), the linear subspace of \( C_c(G) \) spanned by \( C_2(G) \). Clearly \( < C_2(G) > \subset D(G) \).

**Remark 6.18.** 1) If \( \nu \in M_T \) then \( \nabla \nu \) is unique and \( \nabla \nu \in M_\infty(G) \)
[1, Theorem 2.1 and Theorem 2.5].

2) If \( \nu \in M_T \) then \( \hat{\phi} \in L^1(\nabla \nu) \) for all \( \phi \in \Phi(G) \), because \( \hat{\phi} \in (C_0, L^1(\hat{G})) \). Therefore by \[ \int_G \phi(x) \, d\nu(x) = \int_G \hat{\phi}(-\hat{x}) \, d\nabla \nu(\hat{x}) \] for all \( \phi \in \Phi(G) \).

Therefore a measure \( \mu \in M_\infty \) satisfies the condition of Theorem 6.16 if \( \mu \) belongs to \( M_T \) and \( \hat{\mu} = \mu \).

The next two theorems will show that all measures in \( M_q \) for \( 1 \leq q \leq 2 \), and all functions \( \mu^* f \), where \( f \in L^1 \) and \( \mu \in M_\infty \) is such that \( \hat{\mu} \in M_\infty \), satisfy these conditions.

**Theorem 6.19.** (Inclusion Theorem). If \( \mu \in M_q \), \( 1 \leq q \leq 2 \), then \( \mu \in M_T \) and \( \hat{\mu} = \mu \). Hence \( \mu = \hat{\mu} \).

**Proof.** Let \( \nu \in M_q \) (see Definition 3.5). By the Extended Parseval Formula [49, Lemma 4.1] for all \( h \in A_c(G) \)
\[ \int_G h(x) \, d\nu(x) = \int_G \hat{h}(-\hat{x}) \nu(\hat{x}) \, d\hat{x}. \]

Now, the linear functionals \( T(\nu) = \int h(x) \, d\nu(x) \) and
\[ T(\nu) = \int h(-\hat{x}) \hat{\nu}(\hat{x}) \; d\hat{x} \quad \text{on} \; M_q \quad \text{are continuous because} \]

\[ |T(\nu)| \leq ||\nu||_q ||\hat{h}||_{\omega q}^\prime \] by Theorem 3.2 and

\[ |\tilde{T}(\nu)| \leq ||\hat{h}||_q ||\nu||_q \quad \varphi \leq ||\hat{h}||_q \; C_q \; ||\nu||_q \] (see (6.1) page 79), where

C_q is a constant depending only on q. Since \( M^{q+}_c \) is dense in \( M_q \) (Theo-

rem 3.6) and \( \tilde{T} = \tilde{T} \) on \( M^{q+}_c \) we conclude that \( T = \tilde{T} \) on \( M_q \) and this

implies that

\[ \int_G h(\omega) \; d\mu(\omega) = \int_G h(-\hat{x}) \hat{\mu}(\hat{x}) \; d\hat{x} \quad \text{for all} \; h \in A_c(G). \]

Since \( C_2(G) \subseteq A_c(G) \), \( \mu \in M_T \) and \( \tilde{\mu} = \hat{\mu}, \) \( 

\textbf{THEOREM 6.20.} \) Let \( \mu \in M_{\omega} \) and \( f \in L^1(G). \) If \( \hat{\mu} \in M_{\omega}(\hat{G}) \) then

\( f*\mu \in M_T \) and \( (u*f)^\wedge = \frac{u*\hat{f}}{\hat{\mu}}. \) Hence \( \mu*\hat{\nu} = (u*\hat{f})^\wedge. \n

\textbf{PROOF.} \) Let \( \phi \in A_c(G) \) and \( \{e_\alpha\} \) be an a.i. in \( L^1(G) \) such that

\( \{\hat{e}_\alpha\} \subseteq C_0(G) \) [29, II 7.1] and [45, 2.6.6]. In the next chapter we will

actually construct such an approximate identity having further properties (p. 94).

By Proposition 4.1

\[ ||\hat{\phi}e_\alpha - \hat{\phi}f||_{\omega 1} \leq ||\hat{\phi}||_{\omega 1} ||\hat{e}_\alpha - \hat{f}||_{\omega \infty} \leq ||\hat{\phi}||_{\omega 1} ||f*\alpha - \hat{f}||_1. \]

This implies that \( \lim ||\hat{\phi}e_\alpha - \hat{\phi}f||_{\omega 1} = 0. \) So, by Propo-

sition 6.14 \( (u*\hat{f})^\wedge \in M_{\omega}(\hat{G}) \) and we have that

(1) \[ \int_G \hat{\phi}(\omega) \; d(\mu*\hat{f})(\omega) = \langle \hat{\phi}, (f*\mu)^\wedge \rangle = \langle \hat{\phi}, \hat{\mu} \rangle = \langle \hat{\phi}f, \hat{\mu} \rangle = \lim \langle \hat{\phi}e_\alpha, \hat{\mu} \rangle = \lim \langle \hat{\phi}e_\alpha, \hat{\mu} \rangle = \lim \langle \hat{\phi}e_\alpha, \hat{\mu} \rangle = \lim \langle \hat{\phi}e_\alpha, (f*\mu)^\wedge \rangle. \]
Now, \( \hat{\phi e_\alpha} \in A_c(\hat{G}) \) because \( \hat{\phi} \in C_c \), \( \hat{e_\alpha} \in C_c \) and 
\( \phi^* e_\alpha = (\hat{\phi e_\alpha})^\vee \). Then by definition of \((\mu^* f)^\vee\)

\[
\lim \left< \hat{\phi e_\alpha}, (\mu^* f)^\vee \right> = \lim \left< (\phi^* e_\alpha)', \mu^* f \right> \\
= \lim \int_G \phi^* e_\alpha(-x) \mu^* f(x) \, dx \\
= \int_G \phi^* (\mu^* f(x)) \, dx.
\]

Remember that \( \mu^* f \in L^1(\mathbb{R}^n) \) by Theorem 4.8. The last equality holds because \( \lim ||\phi^* e_\alpha - \phi||_{L^1} = 0 \) by Corollary 4.14.

Hence from (1) and (2) we conclude that

\[
\int_G \phi(-x) \mu^* f(x) \, dx = \int_G \hat{\phi}(\hat{x}) \, d(\mu^* f)^\vee(\hat{x})
\]

for all \( \phi \in A_c(G) \). Since \( C_c(G) \subseteq A_c(G) \), this implies that \( \mu^* f \in M_T \) and 
\( (\mu^* f)^\vee = (\mu^* f)^\vee \).

**Theorem 6.21.** Let \( \mu \in M_\infty \). If \( \hat{\mu} \in M_\infty(\hat{G}) \) then \( \mu \in M_T \) and \( \hat{\mu} = \overline{\mu} \). Hence \( \mu = \hat{\mu} \).

**Proof.** Let \( f \in A_c(G) \) and \( g \in D(G) \). By Corollary 4.4 \( f^* \mu \) as a measure belongs to \( M_\infty(G) \), so by Proposition 6.14 and Theorem 6.20 
\( f^* \mu \in M_\infty(\hat{G}) \) and

\[
\int_G g(x) f^* \mu(x) \, dx = \int_G \hat{g}(-\hat{x}) \, d(f^* \mu)^\vee(\hat{x}) = \int_G \hat{g}(-\hat{x}) \, \hat{f}(\hat{x}) \, d\hat{\mu}(\hat{x}).
\]

Note that by Lemma 6.4 \( g \in (C_0, L^1)(\hat{G}) \).

But \( \int g(-x) f(x) \, d(x) = \int \int [x, \hat{x}] g(x) \, dx \, \hat{f}(\hat{x}) \, d\hat{\mu}(\hat{x}) \)

\[
= \int g(x) \int [x, \hat{x}] \, \hat{f}(\hat{x}) \, d\hat{\mu}(\hat{x}) \, dx.
\]
We can apply Fubini's theorem because \( g \in L^1 \) and

\[
\int [x, \hat{x}] \hat{f}(\hat{x}) \, d\hat{\mu}(\hat{x}) \leq \|\hat{f}\|_{\infty} \|\hat{\mu}\|_{\infty}
\]

Then we conclude that for all \( g \in D(G) \)

\[
\int_G g(x) \cdot f^\mu(x) \, dx = \int_G \int_G [x, \hat{x}] \hat{f}(\hat{x}) \, d\hat{\mu}(\hat{x}) \, dx.
\]

Since \( D(G) \) is dense in \( C_c(G) \) [40, Theorem 5.1.2], this implies that

\[
f^\mu(x) = \int_G f(x - y) \, d\mu(y) = \int_G [x, \hat{x}] \hat{f}(\hat{x}) \, d\hat{\mu}(\hat{x})
\]

locally almost everywhere.

Now, \( \hat{f} \in (C_0, \ell^1) \) and therefore \( \int [x, \hat{x}] \hat{f}(\hat{x}) \, d\hat{\mu}(\hat{x}) \) is a continuous function on \( G \). On the other hand \( f^\mu \) is also a continuous function on \( G \). Therefore \( f^\mu(x) = \int [x, \hat{x}] \hat{f}(\hat{x}) \, d\hat{\mu}(\hat{x}) \) for all \( x \in G \).

So, for \( x = 0 \) we have that

\[
\int_G f(x - y) \, d\mu(y) = \int_G \hat{f}(\hat{x}) \, d\hat{\mu}(\hat{x}) \quad \text{for all } f \in A_c(G).
\]

This implies that \( \mu \in M_T \) and \( \hat{\mu} = \underline{\mu}^* \).

**Remark 6.22.** From the proof of theorem 6.21 we see that for \( \mu \in M_\infty(G) \) such that \( \hat{\mu} \in M_\infty(\hat{G}) \)

\[
\int_G f(x - y) \, d\mu(y) = \int_G [x, \hat{x}] \hat{f}(\hat{x}) \, d\hat{\mu}(\hat{x})
\]

for all \( f \in A_c(G) \) and all \( x \in G \).

**Theorem 6.23.** If \( \mu \in M_T \cap M_\infty \) then \( \hat{\mu} = \underline{\mu}^* \).

**Proof.** By Remark 6.18 \( \mu \in M_\infty(\hat{G}) \) and for all \( g \in A_c(G) \)
\[ < g, \mu > = \int g(x) \, d\mu(x) = \int g(-\mathbf{x}) \, d\nu(\mathbf{x}) = < g, \nu >. \]

Then by Proposition 6.11 and the density of \( A_\mathbb{C}(G) \) in \( (\mathcal{C}_0, \mathcal{A}_1)(G) \), this implies that \( \nu = \hat{\mu} \). Hence by Theorem 6.21, \( \hat{\nu} = \hat{\mu} = \hat{\mu} \).

L. Argabright and J. Gil de Lamadrid defined the set
\[ \mathcal{I}(G) = \{ \nu \in M_T(G) | \nu \in M_T(\hat{G}) \} \]
and proved that if \( \nu \in \mathcal{I}(G) \) then \( \nu = \hat{\nu} \).

What follows is a characterization of \( \mathcal{I}(G) \). This was also proved by a different method by H. G. Feichtinger [25, Theorem Cl].

**Theorem 6.24.** \( \mathcal{I}(G) = \{ \mu \in M_\infty | \hat{\mu} \in M_\infty \} \).

**Proof.** Take \( \nu \in \mathcal{I}(G) \). Since \( \nu = \hat{\nu} \), \( \nu \) belongs to \( M_\infty(G) \).

So by Theorem 6.23 \( \hat{\nu} = \hat{\nu} \). Therefore \( \nu \in \{ \mu \in M_\infty | \hat{\mu} \in M_\infty \} \).

Now take \( \mu \in M_\infty \) such that \( \hat{\mu} \in M_\infty \). By Theorem 6.21 \( \hat{\mu} \in M_T \) and \( \hat{\mu} = \hat{\mu} \).

So, \( \hat{\mu} \in M_\infty(\hat{G}) \) and \( \hat{\mu} \in M_\infty(G) \). Again by Theorem 6.21 \( \hat{\mu} \in M_T(G) \).

**Remark 6.25.** We see that Theorem 6.24 implies "the Second Inclusion Theorem" proved in [1, Theorem 3.5].

Moreover, \( (L^p, \ell^q), 1 \leq p < \infty, 1 \leq q < \infty, \) and \( L^s, 1 \leq s \leq 2 \), are included in \( \mathcal{I}(G) \).

I. Richards provided an example of a transformable measure which is not in \( M_\infty \) [1, §7]. This implies that \( M_T \notin M_\infty \) and \( \mathcal{I}(G) \) is a proper subset of \( M_T \). That is, there exists \( \nu \in M_T \) such that \( \nu \notin M_T \).

Since \( \nu \in M_\infty \), this means that \( M_\infty \notin M_T \). Moreover, from Theorem 6.24,
we have that \( \hat{L}(G) = M_\omega \cap M_T \) (See [30, Theorem 3] and [25, Theorem C1]).

F. Holland [35, §4] defined the test spaces \( \Phi_q, 1 \leq q \leq \omega \), on the real line. Its generalizations to locally compact groups are due to J. P. Bertrandias and C. Dupuis [7, §2 c]).

**Definition 6.26.** Let \( 1 \leq q \leq \omega \). \( \Phi_q = \Phi_q(G) \) is the linear space \( \{ \phi \in C_0(G) \mid \hat{\phi} \in (C_0, \ell^1) \} \) endowed with the norm \( \phi \mapsto ||\hat{\phi}||_{\omega q} \).

By (2.5) it is clear that \( \Phi_1 \subset \Phi_q \subset \Phi_\omega \), \( 1 < q < \omega \), and by Theorem 5.7, there exists a continuous linear isomorphism from \( C_0 \) onto \( \Phi_q \) for \( 2 \leq q \leq \omega \).

F. Holland proved that for \( 1 \leq q < 2 \) \( \Phi_q(R) \subseteq C_0(R) \) [35, §4 p. 350].

**Definition 6.27.** [7, §4 a)]. Let \( g \) be a function in \( (L^p, \ell^q) \) (\( 1 \leq p, q \leq \omega \)) or a measure in \( M_\omega \) (\( 1 \leq s \leq \omega \)). The Fourier transform \( F_\phi(g) \) of \( g \) is an element of the continuous dual of \( \Phi_1 \) defined by

\[
< \phi, F_\phi(g) > = < \hat{\phi}', g > = \int \hat{\phi}(-t) g(t) \, dt \quad (\phi \in \Phi_1(G)).
\]

Since \( (L^p, \ell^q) \subseteq (L^1, \ell^q) \subseteq M_q \subseteq M_\omega \), \( 1 \leq p, q \leq \omega \), and \( \hat{\phi} \in (C_0, \ell^1) \) for all \( \phi \in \Phi_1 \), \( F_\phi(g) \) is well defined and

\[
|< \phi, F_\phi(g) >| = |< \hat{\phi}', g >| \leq C ||\phi||_{\omega 1}
\]

where \( C = ||g||_\omega \) if \( g \in M_\omega \) or \( C = ||gm||_\omega \) if \( g \in (L^p, \ell^q) \).

Therefore \( F_\phi(g) \) belongs to \( \Phi_1(\hat{G})^* \).

By Lemma 6.4, \( A_\omega(G) = \Phi_1(G) \). Moreover for all \( \phi \in A_\omega(G) \),
\[ \psi = \phi^* f_E \] where \( \phi = \tilde{\phi}^* \), \( \phi^* \in L^1(G) \), \( f_E \in (C_0, \chi^1(G)) \) and \( f_E = 1 \) on \( E \) (see the proof of Lemma 6.4). So, by (2.3) and Theorem 4.7 we see that

\[ \|\psi\|_{A_E} = \|\phi\|_1 < \|\phi\|_{\omega_1} = \|\phi^* f_E\|_{\omega_1} < 2^d \|\phi^*\|_1 \|f_E\|_{\omega_1} \]

\[ = 2^d \|f_E\|_{\omega_1} \|\phi\|_{A_E}. \]

This implies that the embedding of \( A_c(\hat{G}) \) onto \( \Phi_1(\hat{G}) \) is continuous and the norms \( f \mapsto \|f\|_{\omega_1} \), \( f \mapsto \|f\|_{A_E} \) on \( A_E(\hat{G}) \) are equivalent. Therefore \( \Phi_1(\hat{G})^* \subseteq A_c(\hat{G})^* \) and we have that

\[ \{ F_{g \hat{g}} \mid g \in M_\omega \} \subseteq \{ \hat{g} \mid g \in M_\omega \}. \]

H. G. Feichtinger [25] has given another alternative definition for the Fourier transform \( \hat{F}_\mu \) of a measure \( \mu \in M_\omega(G) \), as an element of the dual \( S_\infty(G)^* \) of a rather special Segal algebra \( S_\infty(G) \) defined in [26]. This algebra \( S_\infty(G) \) has the following properties:

1) \( A_c(G) \) is dense in \( S_\infty(G) \).
2) The inclusion of \( A_c(G) \) into \( S_\infty(G) \) is continuous.
3) \( S_\infty(G) \hat{=} S_\infty(\hat{G}) \).
4) \( M_\omega \cup M_T \subseteq S_\infty(G)^* \subseteq Q(G) \).

Then Feichtinger's definition is as follows:

**DEFINITION 6.28.** [25, Theorem B2]. Let \( \sigma \in S_\infty(G)^* \). Then its Fourier transform \( \hat{F}_\sigma \) is an element of \( S_\infty(G) \) defined by

\[ < f, \hat{F}_\sigma > = < \hat{f}, \sigma > \quad (f \in S_\infty(G)). \]

By property 4) above it is clear that \( \hat{F}_\sigma \) is well defined and
Indeed belongs to $S_0(G)^\ast$. In particular if $\mu \in \mathcal{M}_\omega$, we have by 2) that $F_0\mu \mid \mathcal{A}_c(G) = \hat{\mu}$, so

\[
\{ F_0\mu \mid \mu \in \mathcal{M}_\omega \} \subseteq \{ \hat{\mu} \mid \mu \in \mathcal{M}_\omega \}; \quad \text{and} \quad \hat{\mu} = F_0\mu \quad \text{iff there exists a constant C such that for all } f \in \mathcal{A}_c(\hat{G}), \langle \check{f}, \mu \rangle < C \| f \|_{S_0} \quad (property \ 1)).
\]

Feichtinger also proved \([25, \text{Theorem C1 ii}]\) that

\[
\{ \mu \in \mathcal{M}_T \mid F_0\mu \in \mathcal{M}_T \} = \mathcal{M}_T \cap \mathcal{M}_\omega = \{ \mu \in \mathcal{M}_\omega \mid F_0\mu \in \mathcal{M}_\omega \}. \quad \text{Hence, we conclude from Theorem 6.24 that if } \mu \in \mathcal{A}(G) \text{ then } \hat{\mu} = F_0\mu.
\]
CHAPTER III

REPRESENTATION OF FUNCTIONS AS FOURIER TRANSFORMS

OF MEASURES IN $M_q$

§ 7. SIMON'S GENERALIZATION OF CESÁRO SUMMABILITY

In this section we will generalize to locally compact abelian groups the following theorem proved by F. Holland [34, §7 Theorem 9] for the real line,

**THEOREM 7.1.** Let $1 \leq q \leq 2$ and $\mu \in M_q$. Then as $N \to \infty$

$$
\frac{1}{\sqrt{2\pi}} \int_{-N}^{N} e^{-i\pi t} \nu(t) dt
$$

converges in the norm of $(L^q, L^\infty)$ to a function $\hat{\mu}$ and

$$
\int h(x) \hat{\mu}(x) dx = \int h(y) d\nu(y) \quad (h \in (L^q, L^1))
$$

Further

$$
\sqrt{2\pi} \hat{\mu}(x) = (C,1) \int e^{-i\pi t} d\nu(t)
$$

almost everywhere.

$(C,1)$ means that the integral on the right is summable by the Cesàro method of order 1 to the value $\sqrt{2\pi} \hat{\mu}(x)$.

We note immediately that if $\mu \in M_q(G)$, $1 \leq q \leq 2$, then by Theorem 3.6 there exists a sequence $\mu_n$ of measures in $M_q^c(G)$, hence in
\[ M_1(G), \text{ such that } \lim ||u_n - u||_q = 0. \text{ Then by (6.1)} \]
\[ \lim ||u_n - \hat{u}||_q = 0. \text{ Therefore the first part of Theorem 7.1 is a} \]
particular case of this fact.

To generalize the rest of the theorem we introduce A. B. Simon's
generalization of Cesàro summability [15] and study some of its proper-
ties related to amalgam spaces.

The set of basic neighborhoods of an element \( x \) of \( G \) in \( G \) will
be denoted by \( N_x(G) \).

We consider \( G \) to be the product \( R^\alpha \times G_1 \), where \( \alpha, G_1 \) and \( H \) are
as in page 10.

Since \( H \) is an open subgroup of \( G_1 \), any \( U \in N_0(G) \) contains
a product neighborhood \((-\delta_1, \delta_1) \times \cdots \times (-\delta_\alpha, \delta_\alpha) \times U_H\) where \( U_H \in N_0(G_1) \),
\( U_H \subseteq H \) and \( \delta_i > 0, i = 1, \ldots, \alpha. \)

For \( i = 1, \ldots, \alpha \) set \( U_i = (-\delta_i, \delta_i) \) and \( N_i = 1/\delta_i \). Then define
\[ \alpha_{U_i}(t) = \frac{1 - \cos(N_it)}{\pi N_i t^2} \quad (t \in R). \]

We see that

(7.1) Each \( \alpha_{U_i} \) is a continuous, nonnegative function on \( R \),
\( \alpha_{U_i} \in L^1(R) \) and \( ||\alpha_{U_i}|| = 1. \)

Now we define \( \alpha_U : R^\alpha \to R \) to be
\[ \alpha_U(t) = \prod_{i=1}^{\alpha} \alpha_{U_i}(t_i) \quad t = (t_1, \ldots, t_\alpha). \]

From (7.1) we see that

(7.2) \( \alpha_U \) is a continuous, nonnegative function on \( R^\alpha \).
\[ (7.3) \quad \alpha_U \in L^1(\mathbb{R}^d) \quad \text{and} \quad \|\alpha_U\|_1 = 1. \]

Since \( m \) is a regular measure, given \( m(U_H) \) there exists \( V \in N_0(H) \) compact such that \( V \subseteq U_H \) and \( m(U_H) - m(U_H)^2 < m(V) \).

So, by the normality of \( H \) (\( H \) is Hausdorff and compact), we can define for this \( V \), a continuous function \( g: H \rightarrow \mathbb{R} \) as follows

\[
g(s) = \begin{cases} 
\frac{1}{m(U_H)} & s \in V \\
m(U_H) & s \in H \setminus U_H \\
m(U_H) < g(s) \leq \frac{1}{m(U_H)} & s \in U_H \setminus V
\end{cases}
\]

Applying the Stone-Weierstrass Theorem to the space of continuous functions on the compact space \( H \), we get a trigonometric polynomial \( P \) (\( P \) is a finite linear combination of characters) such that

\[ \|g - P\|_\infty < m(U_H). \]

If \( r \) is the real part of \( P \) then \( r = \sum_{j=1}^{\infty} \lambda_j \hat{s}_j \), \( \lambda_j \in \mathbb{R}, \hat{s}_j: H \rightarrow \mathbb{R}, \ j = 1, \ldots, m \) and \( \|g - r\|_\infty < m(U_H) \).

Hence for all \( s \in H \), \(-m(U_H) < r'(s) - g(s) \) and this implies that \( 0 \leq g(s) - m(U_H) < r(s) \) for all \( s \in H \).

We then define the function \( \beta_U: G \rightarrow \mathbb{R} \) to be

\[
\beta_U(s) = \begin{cases} 
\frac{1}{\|r\|_1} & s \in H \\
0 & s \notin H
\end{cases}
\]

It is clear that
\[(7.4) \quad \beta_U \text{ is a continuous, nonnegative function on } G, \quad \beta_U \in L^1(G) \text{ and } \|\beta_U\|_1 = 1.\]

\[(7.5) \quad \sup_{s \in G} |\beta_U(s)| = B_U < \infty.\]

Finally we define

\[\phi_U(t, s) = \alpha_U(t) \cdot \beta_U(s) \quad (t, s) \in G.\]

**Theorem 7.2.** Each \(\phi_U^U (U \in N_0(G))\) has the following properties:

i) \(\phi_U\) is a real-valued, continuous, nonnegative function on \(G\).

ii) \(\phi_U \in L^1(G)\) and \(\|\phi_U\|_{L^\infty} = 1\).

iii) \(\hat{\phi}_U \in C_c(\hat{G})\) and \(\|\hat{\phi}_U\|_{\infty} \leq 1\).

iv) \(\phi_U(x) = \int_{\hat{G}} \hat{\phi}_U(\hat{x}) \, [x, \hat{x}] \, d\hat{x}\) \(\text{ i.e. } \phi_U = \hat{\phi}\).

Moreover

v) For \(\varepsilon > 0\) and \(U \in N_0(G)\) given, we can find a \(V\) such that if \(V' \subseteq V\) then \(\int_{G \cap U} \phi_{V'} < \varepsilon\).

vi) \(\lim_{U} \hat{\phi}_U(\hat{x}) = 1\).

vii) \(\{\phi_U\}\) is an a.i. in \(L^1(G)\).

**Proof.** i) and ii) follow from (7.2), (7.3) and (7.4). For a proof of iii), iv), v), vi) and vii) see [15].

**Proposition 7.3.** Let \(1 \leq p \leq \infty\). For each \(U \in N_0(G)\), \(\alpha_U \in (L^p, L^1)(\mathbb{R}^d)\).

**Proof.** By (2.5) it is enough to prove that \(\alpha_U \in (L^\infty, L^1)(\mathbb{R}^d)\).
First we note that since \( \alpha_{U_i}, \ldots, \alpha_i \), is an even function, for \( n > 0 \)

\[
\sup_{t \in [0,1]} \alpha_{U_i}(t + n) = \sup_{t \in [0,1]} \alpha_{U_i}(1 - t + n) = \sup_{t \in [0,1]} \alpha_{U_i}(t - (1 + n)) \quad \text{for } i = 1, \ldots, \alpha.
\]

Then we conclude that

\[
\sup_{t \in [0,1]} \alpha_{U_i}(t + n) \leq \frac{2}{N_i \pi} \frac{1}{n^2},
\]

(\( i = 1, \ldots, \alpha \)) for all \( n \in \mathbb{Z} \setminus \{0, -1\} \).

Also

\[
\sup_{t \in [0,1]} \alpha_{U_i}(t + n) = \sup_{n \in \mathbb{Z} \setminus \{0, -1\}} \frac{1 - \cos N_i(n + t)}{(N_i(n + t))^2} \leq \frac{N_i}{\pi} C_i
\]

(\( i = 1, \ldots, \alpha \)) for \( n \in \{0, -1\} \) for a constant \( C_i \), because

\[
\lim_{t \to -n} \frac{1 - \cos N_i(n + t)}{(N_i(n + t))^2} \quad \text{exists.}
\]

Therefore for all \( i = 1, \ldots, \alpha \) and all \( n \in \mathbb{Z} \)

\[
\sup_{t \in [0,1]} |\alpha_{U_i}(t + n)| \leq C a_n \quad \text{where } C = \max_{1 \leq i \leq \alpha} \left(\frac{2}{N_i \pi}, N_i C_i / \pi\right)
\]

and

\[
a_n = \begin{cases} 
1/n^2 & \text{if } n \in \mathbb{Z} \setminus \{0, -1\}, \\
1 & \text{if } n \in \{0, 1\}.
\end{cases}
\]

So for \( i = 1, \ldots, \alpha \)

\[
\|\alpha_{U_i}\|_\infty = \sum_{\mathbb{Z}} \sup_{t \in [n, n+1]} |\alpha_{U_i}(t)| = \sum_{\mathbb{Z}} \sup_{t \in [0,1]} |\alpha_{U_i}(t + n)| \leq C \sum_{\mathbb{Z}} a_n < \infty.
\]
This means that \( \{u_{ij}\} \subseteq (L^\infty, l^1)(R) \).

Now,
\[
\left\| a_{ij} \right\|_{l^1} = \sum_{n \in Z^a} \sup_{t \in [0,1]^a} |a_{ij}(t + n)|
\]
\[
= \sum_{n_1 \in Z} \cdots \sum_{n_a \in Z} \prod_{i=1}^a \sup_{t \in [0,1]} |a_{ij}(t + n_i)|
\]
\[
= \prod_{i=1}^a \left\| a_{ij} \right\|_{l^1} < \infty \quad \dagger
\]

**Corollary 7.4.** Let \( 1 \leq p \leq \infty \). Each \( \phi_U \in (L^p, l^1)(G) \) and \( U \in N_0(G) \). Hence \( \{\phi_U\} \subseteq (C_0, l^1) \).

**Proof.** By definition of \( \phi_U \), for all \( (t,s) \in G \)
\[
\phi_U(t,s) = a_U(t) \beta_U(s) \leq B_U a_U(t) \quad \text{where} \quad \sup_{s \in G} |\beta_U(s)| = B_U \text{ (see (7.5))}.
\]

Hence
\[
\left\| \phi_U \right\|_{l^1} = \sum_{\alpha} \sup_{(t,s) \in K_\alpha} |a_U(t)\beta_U(s)| \leq B_U \sum_{n \in Z^a} \sup_{t+n \in [0,1]^a} |a_U(t)|
\]
\[
= B_U \left\| a_U \right\|_{l^1}.
\]

Therefore by Proposition 7.3, \( \phi_U \in (L^\infty, l^1) \) and by (7.5)
\[
\phi_U \in (L^p, l^1). \quad \dagger
\]

We will use the following lemma to prove Theorem 7.6.

**Lemma 7.5.** Let \( U, V \) be two elements of \( N_0(G) \) of the form
\[
U = (-\delta, \delta_1) \times \cdots \times (-\delta, \delta_a) \times V_H, \quad \delta_1 > 0, \quad U_H \subseteq H, \quad U_H \in N_0(G_1) \quad \text{and}
\]
\[
V = [-\gamma, \gamma_1] \times \cdots \times [-\gamma, \gamma_a] \times V_H, \quad \gamma_1 > 0, \quad V_H \subseteq H, \quad V_H \in N_0(G_1) \quad \text{compact.}
\]
If for \( 1 < p < \infty \), \( \eta_i = \min(\frac{\delta_i^{2p}}{\lambda_i}, \gamma_i) \) \( i = 1, \ldots, \alpha \) and
\[ W_H = \text{int} V_H \text{ then } W = [-\eta_1, \eta_1] \times \cdots \times [-\eta_\alpha, \eta_\alpha] \times W_H \text{ belongs to } N_0(G) \text{ and for a fixed } y = (y_0, s_0) \in G, y = (y_1, \ldots, y_\alpha), W_y = y + W \text{ has the following properties: } \]

1) \( W_y \subseteq y + V \).

2) If \( \Pi_a = [-\eta_1 + y_1, \eta_1 + y_1] \times \cdots \times [-\eta_\alpha + y_\alpha, \eta_\alpha + y_\alpha] \) then
\[
\left( \int_{\Pi_a} \frac{\alpha(y_0 - x)^p}{\lambda_i} \, dx \right)^{1/p} = O \left( \prod_{i=1}^{\alpha} \delta_i \right).
\]

3) \( \mathbb{R}^a \cap \Pi_a \subseteq \bigcup_{n=1}^{N} I_n \), \( \{ I_n \} \) being a countable family of compact subsets of \( \mathbb{R}^a \) and
\[
\left( \sum_{n=1}^{N} \left( \int_{I_n} \frac{\alpha(y_0 - x)^p}{\lambda_i} \, dx \right)^{1/p} \right)^{1/p} = O \left( \prod_{i=1}^{\alpha} \delta_i \right).
\]

4) There exists a constant \( C \) such that \( \sup_{n} \gamma(I_n) \leq C \) where \( \gamma(I_n) \) is the cardinality of the set \( \{ m \in \mathbb{Z}^2 \mid (m + [0,1] \mathbb{Z}) \cap I_n \neq \emptyset \} \).

**PROOF.** Several constants will appear during the proof and since their specific value is irrelevant for our needs we just write \( C_1, C_2 \).

From the definition of \( W \), 1) is clear.

Remember that for \( i = 1, \ldots, \alpha \), \( \alpha U_i(t) = \frac{N_i}{\pi} \frac{1 - \cos N_i t}{(N_i t)^2} \)
is continuous by (7.1), hence \( \alpha U_i \) is bounded on \([-\eta_i, \eta_i]\) and we have that for \( J_i = [-\eta_i + y_i, \eta_i + y_i] \), \( i = 1, \ldots, \alpha \)

\[
\left( \int_{J_i} \alpha U_i(y_i - x)^p \, dx \right)^{1/p} = \left( \int_{-\eta_i}^{\eta_i} \alpha U_i(x)^p \, dx \right)^{1/p}. \]
\[ C_1^{1/p} \leq C_1 \left( \frac{1}{\delta_1} \right) \delta_1^2 = C_1 \delta_1. \]

This implies 2) because

\[
\left[ \int_{\mathbb{R}^d} \alpha_y(y-x)^p \, dx \right]^{1/p} = \prod_{i=1}^{a} \left[ \int_{J_1} \alpha_{U_i}(y_1-x)^p \, dx \right]^{1/p} \leq C_2 \prod_{i=1}^{a} \delta_i.
\]

Let \( J_1 = [-\eta_1 + y_1, \eta_1 + y_1] \) and \( \prod_{i=1}^{a} J_1 \times \cdots \times J_1 \), \( i = 1, \ldots, a \).

Observe that \( R \cap J_1 = (-\infty, -\eta_1 + y_1] \cup [\eta_1 + y_1, \infty) \)

\[ \leq u \left[ -n - 1 - \eta_1 + y_1, -n - \eta_1 + y_1 \right] \cup \left[ n + \eta_1 + y_1, n + 1 + \eta_1 + y_1 \right] \]

\[ = u \bigcup_{N} I_{m}^{-} \cup u \bigcup_{N} I_{m}^{+} \text{ and} \]

\[
\int_{I_{m}^{-}}^{I_{m}^{+}} \alpha_{U_i}(y_1-x)^p \, dx \frac{2^p}{\left( \frac{2}{\pi \eta_1} \right)^p} \int_{I_{m}^{-}}^{I_{m}^{+}} \frac{dx}{(y_1-x)^{2p}} = C_3 \delta_1^p a_n.
\]

where

\[ a_n = \frac{1}{(y_1+n)^{2p-1}} \frac{1}{(y_1+n+1)^{2p-1}}. \]

Since \( \sum a_n^{1/p} \) converges we conclude that

\[
\sum_{N} \left[ \int_{I_{m}^{-}}^{I_{m}^{+}} \alpha_{U_i}(y_1-x)^p \, dx \right]^{1/p} \leq C_4 \delta_1.
\]

Similarly

\[
\int_{I_{m}^{+}} \alpha_{U_i}(y_1-x)^p \, dx = C_5 \delta_1^p \int_{I_{m}^{+}} \frac{dx}{(y_1-x)^{2p}} = C_5 \delta_1^p a_n.
\]

and therefore

\[
\sum_{N} \left[ \int_{I_{m}^{+}} \alpha_{U_i}(y_1-x)^p \, dx \right]^{1/p} \leq C_6 \delta_1.
\]
Clearly $\sup_N \mathcal{E}(I_n^-)$ and $\sup_N \mathcal{E}(I_n^+)$ are less than or equal to

2. Hence, for $i = 1, \ldots, a$

\[(2) \quad R \sim J_i = \cup_{i} I_n, \quad I_n \text{ compact}, \quad \sup_N \mathcal{E}(I_n) \leq 2 \text{ and}\]

\[
\frac{1}{N} \left[ \int_{I_n} \alpha u_i (y_i - x)^p \, dx \right]^{1/p} = O(\delta_i).
\]

Since $R = (R \sim J_i \cup J_\alpha)$ is compact, by (1) and (2) we see that $R \cup \overline{I_{\alpha}}$ is compact for all $n$,

\[
\sup_N \mathcal{E}(\overline{I_n}) \leq \max (2, \mathcal{E}(J_\alpha)) = C_7 \text{ and}
\]

\[(3) \quad \frac{1}{N} \left[ \int_{\overline{I_n}} \alpha u_i (y_i - x)^p \, dx \right]^{1/p} = O(\delta_\alpha).
\]

We will prove 3) and 4) by induction on $\alpha$. The case $\alpha = 1$

follows from (2). Suppose that 3) and 4) hold for $\alpha - 1$. That is,

\[R^{\alpha-1} \sim \cap_{\alpha-1} \cup_{N} I_n, \quad I_n \subseteq \mathbb{R}^{\alpha-1} \text{ compact}, \quad \sup_N \mathcal{E}(I_n) \leq C_9 \text{ and}\]

\[(4) \quad \frac{1}{N} \left[ \prod_{i=1}^{\alpha-1} \int_{I_n} \alpha u_i (y_i - x_i)^p \, dx \right]^{1/p} = O(\prod_{i=1}^{\alpha-1} \delta_i).
\]

By (2) with $i = \alpha$ we have that $R \sim J_\alpha \subseteq \cup_{N} I_k, \quad I_k \subseteq \mathbb{R}$ compact

\[
\sup_N \mathcal{E}(I_k) \leq 2 \quad \text{and}
\]

\[(5) \quad \frac{1}{N} \left[ \int_{I_k} \alpha u_\alpha (y_\alpha - x)^p \, dx \right]^{1/p} = O(\delta_\alpha).
\]
Then \( R^a \times \Pi a = (R^{a-1} \times R) \times (\Pi a-1 \times J_a) \)

\[
= (R^{a-1} \times \Pi a-1) \times R \times \Pi a-1 \times (R \times J_a)
\]

\[
\leq (\bigcup_{n} I_n \times R) \cup (\Pi a-1 \times (\bigcup_{n} I_k))
\]

\[
= \bigcup_{n,m} (I_n \times \overline{I_m}) \cup \bigcup_{n,k} (\Pi a-1 \times I_k).
\]

\( I_n \times \overline{I_m} \) and \( \Pi a-1 \times I_k \) are compact subsets of \( R \), for all \( n, m, k \) in \( N \). Since \( \Pi a-1 \) is compact, \( C(\Pi a-1) = C_9 \) and we have that

\[
\sup_{N \times N} C(I_n \times \overline{I_m}) \leq C_9 C_7 \quad \text{and} \quad \sup_{N \times N} C(\Pi a-1 \times I_k) \leq C_9 2 = C_{10}.
\]

Therefore 4) holds with \( C = \max(C_9 C_7, C_{10}) \).

Finally, by (3) and (4)

\[
\sum_{n,m} \left[ \int_{I_n \times \overline{I_m}} \alpha_a(y_0 - x)^p \, dx \right]^{1/p} =
\]

\[
\sum_{n,m} \left[ \int_{I_n} \Pi_{i=1}^{a-1} \alpha a(y_i - x_i)^p \, dx \right]^{1/p} \left[ \int_{\overline{I_m}} \alpha a(y_a - x_a)^p \, dx_a \right]^{1/p}
\]

\[
= \sum_{n} \left[ \int_{I_n} \Pi_{i=1}^{a-1} \alpha a(y_i - x_i)^p \, dx \right]^{1/p} \sum_{k} \left[ \int_{\overline{I_k}} \alpha a(y_a - x_a)^p \, dx_a \right]^{1/p}
\]

\[
= \left( \prod_{i=1}^{a} \delta_i \right).
\]

By (1) and (5)

\[
\sum_{N} \left[ \int_{\Pi a-1 \times I_k} \alpha a(y_0 - x)^p \, dx \right]^{1/p}
\]

\[
= \sum_{N} \left[ \int_{J_{i=1}} \Pi_{i=1}^{a-1} \alpha a(y_i - x_i)^p \, dx \right]^{1/p} \left[ \int_{I_k} \alpha a(y_a - x)^p \, dx \right]^{1/p}
\]
\[ \begin{align*}
\alpha & - 1 \left[ \int J_1 \alpha_{U_1}(y_1 - x)^p \, dx \right]^{1/p} \sum_{\alpha} \left[ \int J_2 \alpha_{U_2}(y_2 - x)^p \, dx \right]^{1/p} \\
& = O \left( \prod_{i=1}^{\alpha} \delta_i \right) \dagger
\end{align*} \]

**Theorem 7.6.** Let \( f \in (L^p, L^\infty), \ 1 < p < \infty, \) and \( y \in G. \) For each \( U \in N_0(G) \) and all \( V_y \in N_y(G) \)
\[ \int_{G \sim V_y} \phi_U(y - x) f(x) \, dx \longrightarrow 0 \quad \text{as } U \to 0. \]

**Proof.** We write \( y = (y_0, s_0), \ y_0 = (y_1, \ldots, y_\alpha) \) in \( R^\alpha, \ s_0 \in G. \)

Let \( V_y \in N_y(G) \) and take \( V \in N_0(G) \) such that \( y + V \in V_y \) and \( V \) has the form \( [-Y_1, Y_1] \times \cdots \times [-Y_\alpha, Y_\alpha] \times V_H, \ Y_i > 0, \ i = 1, \ldots, \alpha, \ V_H \subset H \) compact and \( V_H \in \mathcal{N}_0(G_1). \) Also, \( U \) contains a product neighborhood \( (-\delta_1, \delta_1) \times \cdots \times (-\delta_\alpha, \delta_\alpha) \times U_H, \ \delta_i > 0, \ i = 1, \ldots, \alpha, \ U_H \in H \) and \( U_H \in \mathcal{N}_0(G_1). \)

Set \( \eta_i = \min (\delta_i^{2p'}, \ Y_i), \ i = 1, \ldots, \alpha, \) and \( W_H = \text{int} V_H. \)

Then \( W = [-\eta_1, \eta_1] \times \cdots \times [-\eta_\alpha, \eta_\alpha] \times W_H \) satisfies the conditions listed in Lemma 7.5.

By 1) of Lemma 7.5, \( W \subset V_y \) and therefore it is enough to prove that
\[ \int_{G \sim W} \phi_U(y - x) f(x) \, dx \longrightarrow 0 \quad \text{as } U \to 0. \]

By 2) of Lemma 7.5, \( W = \Pi \alpha \times (s_0 + W_H). \) Hence
\[ G \sim W_y = (R^\alpha \times G_1) \sim (\Pi \alpha \times (s_0 + W_H)) = (R^\alpha \sim \Pi \alpha) \times G_1 \]
\[ U \sim \Pi \alpha \times (G_1 \sim (s_0 + W_H)). \]
This implies that

\[(6) \quad \int_{G \cap W_y} \phi_U(y - x) f(x) \, dx = \int_{(\mathbb{R}^\alpha \cap \alpha) \times G_1} \phi_U(y - x) f(x) \, dx + \int_{\mathbb{M} \times (G \cap (s_0 + \mathbb{H}))} \phi_U(y - x) f(x) \, dx.\]

If \( x = (t, s) \) in \( G \) then by definition of \( \phi_U \),

\[\phi_U(y - x) = \alpha_U(y_0 - t) \beta_U(s_0 - s) = 0 \text{ if } s_0 - s \notin \mathbb{H}. \text{ Hence,}\]

\[\int_{(\mathbb{R}^\alpha \cap \alpha) \times G_1} \phi_U(y - x) f(x) \, dx = \int_{(\mathbb{R}^\alpha \cap \alpha) \times (s_0 + \mathbb{H})} \phi_U(y - x) f(x) \, dx\]

\[\int_{\mathbb{M} \times (G \cap (s_0 + \mathbb{H}))} \phi_U(y - x) f(x) \, dx = \int_{\mathbb{M} \times (s_0 + (H \cap \mathbb{H}))} \phi_U(y - x) f(x) \, dx.\]

Let \( I_n \in \{I_1\} \), \( \{I_n\} \) as in 3) above. So, by Hölder's inequality

\[\int_{I_n \times (s_0 + \mathbb{H})} \phi_U(y - x) f(x) \, dx \leq \left( \int_{I_n \times (s_0 + \mathbb{H})} \chi_{I_n \times (s_0 + \mathbb{H})} \, dx \right)^{1/p'} \left( \int_{I_n \times (s_0 + \mathbb{H})} \phi_U(y - x)^{P'} \, dx \right)^{1/p'} \]

\[\leq \left( \int_{I_n \times (s_0 + \mathbb{H})} \chi_{I_n \times (s_0 + \mathbb{H})} \, dx \right)^{1/p'} B_U \left( \int_{I_n} \alpha_U(y_0 - x)^{P'} \, dx \right)^{1/p'}\]

where \( B_U \) is the constant in (7.5).

By 4) \( |S(I_n \times (s_0 + \mathbb{H})| \leq C \) for all \( n \in \mathbb{N} \) (see page 36 and Definition 1.6). This implies that for all \( n \in \mathbb{N} \)

\[\left| \int_{I_n \times (s_0 + \mathbb{H})} \phi_U(y - x) f(x) \, dx \right| \leq |S(I_n \times (s_0 + \mathbb{H})| \cdot \left| \int_{I_n} f \, dx \right|_{P'} \leq C \left| f \right|_{P'} \cdot \left| f \right|_{P'}. \]

Then we conclude from 3) that
\begin{equation}
\int_{\Pi \times (s_0 + (H \cdot W_H))} \phi_U(y - x)|f(x)| \, dx \leq \sum_{N} \int_{\Pi \times (s_0 + H)} \phi_U(y - x)|f(x)| \, dx
\end{equation}

\begin{align*}
&\leq C \left\| f \right\|_{\infty} B_U \sum_{N} \left[ \int_{\Pi} \alpha(y_0 - x)^{p'} \, dx \right]^{1/p'} \\
&= O \left( \prod_{i=1}^{a} \delta_i B_U \right).
\end{align*}

Applying again the H"older inequality

\begin{align*}
\int_{\Pi \times (s_0 + (H \cdot W_H))} \phi_U(y - x)|f(x)| \, dx &= \left\| f \right\|_{\Pi \times (s_0 + (H \cdot W_H))}^{1/p} \left[ \int_{\Pi \times (s_0 + (H \cdot W_H))} \phi_U(y - x)^{p'} \, dx \right]^{1/p'} \\
&\leq B_U \left\| f \right\|_{\infty} \left[ S(\Pi \times (s_0 + (H \cdot W_H))) \right]^{1/p} \left[ \int_{\Pi \times (s_0 + (H \cdot W_H))} \alpha(y_0 - x)^{p'} \, dx \right]^{1/p'}.
\end{align*}

Note that \( \Pi \times (s_0 + (H \cdot W_H)) \) is compact (\( H \) is compact and \( H \cdot W_H \) is closed) and that by definition of \( B_U \) (pp. 93 and 94), \( B_U \to 0 \) as \( U \to 0 \).

Now, since \( \Pi \times y \) as \( U \to 0 \) and \( s_0 + (H \cdot W_H) \leq s_0 + H \) is independent of \( U \), we have that \( S(\Pi \times (s_0 + (H \cdot W_H))) \to 1 \) as \( U \to 0 \).

Therefore by 2)

\begin{equation}
\int_{\Pi \times (s_0 + (H \cdot W_H))} \phi_U(y - x)|f(x)| \, dx \to 0 \quad \text{as} \quad U \to 0.
\end{equation}

Hence we conclude from (6), (7) and (8) that
\[ \int_{G \setminus V_y} \phi_U(y - x) f(x) \, dx \to 0 \quad \text{as} \quad U \to 0. \]

**Corollary 7.7.** For all \( f \in (L^p, \ell^\infty), \) \( 2 < p < \infty, \)

\[ \lim_{U \to 0} \int_G \phi_U(y - x) f(x) \, dx = f(y) \quad \text{a.e.} \]

In other words, \( \lim_{U \to 0} \phi_U f(y) = f(y) \) a.e.

**Proof.** Let \( V_y \in N_y(G) \) compact. Since

\[ \int_G \phi_U(y - x) f(x) \, dx = \int_{G \setminus V_y} \phi_U(y - x) f(x) \, dx + \int_{V_y} \phi_U(y - x) f(x) \, dx \]

and by Theorem 7.6, \( \int_{G \setminus V_y} \phi_U(y - x) f(x) \, dx \to 0 \) as \( U \to 0. \) we just

have to prove that \( \int_{V_y} \phi_U(y - x) f(x) \, dx \to f(y) \) as \( U \to 0. \)

By (2.5) \( (L^p, \ell^\infty) \subseteq (L^2, \ell^\infty), \) so \( f \in L^2_{\text{loc}} \) and the function

\( g = f|_{V_y} = f_{x|V_y} \) belongs to \( L^2(G). \)

By Corollary 7.4 and (2.7), \( \{\phi_U\} \subseteq L^2(G) \) and by the Parseval identity [37, Theorem 31.19]

\[ \int_{V_y} \phi_U(y - x) f(x) \, dx = \int_G \phi_U(y - x) g(x) \, dx = \int_G \hat{\phi_U}(\hat{x}) \hat{g}(-\hat{x}) \, d\hat{x}. \]

Since \( \lim_{U \to 0} \hat{\phi_U}(\hat{x}) = 1 \quad (\hat{x} \in \hat{G}), \quad \sup_{\hat{x} \in \hat{G}} |\hat{\phi_U}(\hat{x})| \leq 1, \)

\( \hat{\phi_U} \in L^1(\hat{G}) \) (Theorem 7.2) and \( \hat{g} \in L^\infty(\hat{G}), \) we can apply Lebesgue's

Dominated Convergence Theorem and we see that
\[
\lim_{U \to 0} \int_{\frac{U}{y}} \phi_U(y - x) f(x) \, dx = \lim_{U \to 0} \int_G \phi_U(x) \hat{g}(-\hat{x}) \frac{y}{x} \, d\hat{x} = \int_G \hat{g}(-\hat{x}) \frac{y}{x} \, d\hat{x} = \frac{y}{x} \, g(y) = g(y) \text{ a.e.}
\]

Note that \( g = g \text{ a.e.} \) by \([37, 31.44 \text{ a)] \).†

The next theorem is the generalization of the Theorem 7.1.

**Theorem 7.8.** Let \( u \in M^1_q, 1 \leq q < 2 \).

1) \( \int_G \overline{\hat{f}(\hat{x})} \hat{\mu}(\hat{x}) \, d\hat{x} = \int_G \frac{y}{x} \, f(x) \, d\mu(x) \text{ for all } f \in (L^q, \ell^1)(\hat{G}). \)

ii) \( \int_G \overline{\phi_U(x)} [x, \hat{x}] \, d\mu(x) := \lim_{U \to 0} \int_G \phi_U(x) \frac{y}{x} \, d\mu(x) = \hat{\mu}(\hat{x}) \text{ a.e.} \)

**Proof.** We pointed out at the beginning of this section that there exists a sequence \((u_n)\) in \( M^1(G) \) such that

\[
\lim ||u_n - u||_q = 0 \text{ and } \lim ||\hat{u}_n - \hat{u}||_{q'} = 0.
\]

By the Extended Parseval Formula (as in \([49, \text{ Lemma 4.1}]\)), for all \( f \in (L^q, \ell^1)(\hat{G}) \)

\[
(9) \quad \int_G \overline{\hat{f}(\hat{x})} \hat{u}_n(\hat{x}) \, d\hat{x} = \int_G \frac{y}{x} \, f(x) \, d\mu_n(x).
\]

Now, by the Hölder inequality and Theorem 5.7,

\[
\int_G |\hat{f}(\hat{x})| |\hat{u}_n - \hat{\mu}(\hat{x})| \, d\hat{x} \leq \int_G \frac{y}{x} \, |f(x)|^q \, d\mu(x)^{1/q} \left[ \int_{K_B} |\hat{u}_n(\hat{x}) - \hat{\mu}(\hat{x})|^{q'} \, d\hat{x} \right]^{1/q'}
\]
\[ < ||f||_q, \hat{\mu}_n - \hat{\mu}||_q' > \]

Similarly \[ \int_G |\hat{\gamma}(x)| \, d\mu_n - \mu(x) \leq ||\hat{\gamma}||_{\alpha q'} ||\mu_n - \mu||_q. \]

Therefore the right side of (9) converges to \[ \int_G \hat{\gamma}(x) \, d\mu(x) \]
and the left side converges to \[ \int_G \hat{\mu}(\hat{\gamma}) \, d\hat{\gamma}. \] This proves 1).

By Corollary 7.4 \( \{\phi_U\} \in (L^q, \ell^1)(\hat{G}) \), so from 1)

\[ \int_\hat{G} \phi_U(\hat{\gamma} - \hat{\xi}) \hat{\mu}(\hat{\xi}) \, d\hat{\xi} = \int_G [x, \hat{\gamma}] \phi_U(x) \, d\mu(x). \]

(Remember that \( \phi_U \) is a real function).

Therefore by Corollary 7.7

\[ \hat{\mu}(\hat{\gamma}) = \lim_{U \to 0} \phi_U \ast \hat{\mu}(\hat{\gamma}) = \lim_{U \to 0} \int_G [x, \hat{\gamma}] \phi_U(x) \, d\mu(x) \]

almost everywhere. \( \dagger \)
§ 8. STRONG RESONANCE CLASS OF FUNCTIONS

F. Holland [35] introduced the space $R(\Phi_q)$ ($1 \leq q \leq \infty$) of functions resonant relative to the space $\Phi_q$ for the real line and established a correspondence between the elements of $R(\Phi_q)$ and the space of unbounded measures $M_q$.

We will define the space $SR(\Phi_q)$ ($1 \leq q \leq \infty$) of functions strongly resonant relative to the space $\Phi_q$ for locally compact abelian groups, characterize $SR(\Phi_q)$ and $R(\Phi_q)$ in terms of transformable measures, study the relation of $SR(\Phi_q)$ with the set of positive definite functions for $(L^q, l^1)$ and prove two theorems for $SR(\Phi_q)$ ($1 \leq q < 2$), similar to Theorems 7 and 9 of [35]. Furthermore we will show that both representations are equivalent.

From now on and for the rest of our work $\{\Phi_q\}$ will be the summability kernel defined in §7.

First we will recall the definition of the spaces $\Phi_q$ and prove some of their already known properties [7, §2].

**DEFINITION 8.1.** Let $1 \leq q \leq \infty$. $\Phi_q$ is the linear subspace of $C_c$ of functions $\phi$ such that $\hat{\phi} \in (C_0, l^q)$ endowed with the norm $\phi \mapsto ||\hat{\phi}||_{\alpha_q}$.

We write $\Phi$ for $\Phi_1$.

**REMARK 8.2.** If $\phi \in \Phi_q$, $1 \leq q \leq \infty$, then it is clear that $\phi'$, $\hat{\phi}$, $\tau_t \phi$ ($t \in G$), $x \phi$ ($x \in \hat{G}$) also belong to $\Phi_q$. 107
Proposition 8.3. i) \( \Phi \subset \Phi_q \subset \Phi_\infty \), \( 1 < q < \infty \).

ii) \( \Phi_q = \mathcal{C}_q \) for \( 2 \leq q \leq \infty \).

iii) \( \Phi_q \) is dense in \( \mathcal{C}_q \) for \( 1 \leq q \leq \infty \).

iv) \( (\Phi_q') \) is dense in \( (C_0, \ell^q) \) (\( 1 < s < \infty \)), \( (L^r, c_q) \) (\( 1 \leq r < \infty \)), \( (L^r, \ell^s) \) (\( 1 \leq r, s < \infty \)) for \( 1 < q < \infty \).

v) \( T \in \Phi_q(G) \) (\( 1 \leq q \leq \infty \)) iff there exists a unique measure \( \mu \in M_q(\hat{G}) \) such that for all \( \phi \in \Phi_q(G) \)

\[
T(\phi) = \int_G \phi(\hat{x}) \, d\mu(\hat{x}).
\]

Proof. i) follows from (2.5) and ii) is a direct consequence of the Hausdorff-Young inequality (Theorem 5.7).

By Remark 6.3 and Lemma 6.4, \( D(G) \) is dense in \( \mathcal{C}_q(G) \) and \( D(G) \subset \Phi \). So, by i) \( \Phi_q \) is dense in \( \mathcal{C}_q(G) \).

Let \( A \) be any of the amalgam spaces listed in iv). Take \( f \in C_c \). Since \( \{\phi_u\} \) is an a.i. in \( L^1 \) (Theorem 7.2), \( \lim \phi_u^*f = f \) in \( A \) (Corollary 4.14). But \( \{\phi_u^*f\} \subset (\Phi_q)^\vee \) because \( \{\hat{\phi_u}^\vee\} \subset C_\infty(\hat{G}) \), \( \phi_u^*f = (\hat{\phi_u})^\vee \) and \( \phi_u^*f \in (C_0, \ell^1) \) (Corollary 7.4). Since \( \mathcal{C}_q \) is dense in \( A \) (Theorem 3.7) this proves iv).

The necessity part of v) follows from Theorem 3.2. If \( T \in \Phi_q^* \) then the map \( \tilde{T}(\phi) = T(\phi') \) belongs to \( (\Phi_q)^\vee, ||\cdot||_\infty \). Since \( (\Phi_q)^\vee \) is dense in \( (C_0, \ell^q) \), \( \tilde{T} \) has a unique continuous extension \( \tilde{T} \) on \( (C_0, \ell^q) \). By Theorem 3.2 there exists a unique \( \mu \in M_q(\hat{G}) \) such that for all \( f \in (C_0, \ell^q) \), \( \tilde{T}(f) = \int_G f(\hat{x}) \, d\mu(\hat{x}) \). Therefore for \( \phi \in \Phi_q \)

\[
T(\phi) = \tilde{T}(\hat{\phi}) = \int_G \hat{\phi}(\hat{x}) \, d\mu(\hat{x}).
\]
DEFINITION 8.4. Let $1 \leq q \leq \infty$. A measurable function $f$ on $G$ is strongly resonant relative to the space $\Phi_q$ if

(R-1) $f\phi \in L^q(G)$ for all $\phi \in \Phi$

(R-2) The linear functional $\phi \mapsto \int f\phi$ on $\Phi_q$ is continuous.

That is, there exists a constant $C$ such that for all $\phi \in \Phi_q$

$$\left| \int f\phi \right| \leq C \|\phi\|_{\omega_q}.$$  

The linear space of functions strongly resonant relative to $\Phi_q$ will be denoted by $\text{SR}(\Phi_q)$.

REMARK 8.5. If $f \in \text{SR}(\Phi_q)$ ($1 \leq q \leq \infty$), then by Remark 8.2, $f'$, $\tau_t e f$ ($t \in G$), $\mathcal{A} f$ ($\mathcal{A} \in \mathcal{G}$), also belong to $\text{SR}(\Phi_q)$.

THEOREM 8.6. i) $\text{SR}(\Phi_q) \subseteq \text{SR}(\Phi_{q'}) \subseteq \text{SR}(\Phi)$ for $1 < q < \infty$.

ii) A measurable function $f$ satisfies the condition (R-1) iff $f \in L^q_{\text{loc}}$.

iii) $f \in \text{SR}(\Phi_q)$ ($1 \leq q \leq \infty$) iff $f \in L^q_{\text{loc}}$, $f \in M_T$, and $\mathcal{F} f \in M_{q'}(\mathcal{G})$.

iv) $\text{SR}(\Phi_q) \subseteq (L^q, \omega^\infty)$ for $2 \leq q \leq \infty$.

v) $f \in \text{SR}(\Phi_q)$ for $2 \leq q \leq \infty$ iff $f \in M_T$ and $\mathcal{F} f \in M_{q'}$.

vi) If $f \in \text{SR}(\Phi)$ and $f \geq 0$ then $f \in (L^1, \omega^\infty)$.

PROOF. i) follows immediately from Proposition 8.3, i).

The sufficiency part of ii) is clear. Suppose $f\phi \in L^q(G)$ for all $\phi \in \Phi$. Let $E \subseteq G$ compact and take $\phi \in \Phi$ such that $\phi \equiv 1$ on $E$ (Theorem 5.2). Since $f\phi \in L^q$ and $\|f\|_{L^q(E)} \leq \|f\phi\|_{L^q}$, we conclude

$$\text{SR}(\Phi_q) \subseteq (L^q, \omega^\infty).$$
that $f \in L^q_{\text{loc}}$. Let $f \in \text{SR}(\Phi_q)$, $1 < q < \omega$. Then the map $T(\phi) = \int f\phi$ ($\phi \in \Phi_q$) belongs to $\Phi_q^*$. So by Proposition 8.3, v) there exists a unique measure $\mu \in M_q'$ such that for all $\phi \in \Phi_q$

$$
\int_G f(x) \phi(x) \, dx = T(\phi) = \int_G \hat{\phi}(-\hat{x}) \, d\mu(\hat{x}).
$$

Since $D(G) \subseteq \Phi \subseteq \Phi_q$ we conclude that $f \in \mathcal{M}_T$ and $\mathcal{F} = \mu$.

Conversely, if $f \in \mathcal{M}_T$ and $\mathcal{F} \in M_q'$ then for $\phi \in \Phi_q$, $\hat{\phi} \in L^1(\mathcal{F})$ and by [1, Corollary 3.1]

$$
\int_G \phi(x) f(x) \, dx = \int_G \hat{\phi}(-\hat{x}) \, d\mathcal{F}(\hat{x}).
$$

This implies that $|ff\phi| \leq ||\mathcal{F}||_q, ||\hat{\phi}||_{\omega_q}$ for all $\phi \in \Phi_q$.

Hence $f$ satisfies the condition (R-2). This proves iii).

If $f \in \mathcal{M}_T$ and $\mathcal{F} \in M_q'$, $1 \leq q' \leq 2$, then $\mathcal{F} \in \mathcal{M}_T$ and $\mathcal{F} = \mathcal{F}$ (Theorem 6.19). But $f = \mathcal{F}$ (Remark 6.25) and $\mathcal{F} \in (L^q, L^\infty)$. Therefore $f \in (L^q, L^\infty)$. This together with iii) implies iv) and v).

Now take $f \in \text{SR}(\Phi)$, $f \geq 0$. Let $g \in \Phi$ such that $g \equiv 1$ on $K$. Then $\tau_\alpha g \equiv 1$ on $K_\alpha$ for all $\alpha \in J$, $\tau_\alpha g \in \Phi$ and $||(\tau_\alpha g)\hat{\phi}||_{\omega_1} = ||\hat{g}\phi||_{\omega_1}$ for all $\alpha \in J$.

Since $f \in \mathcal{M}_T$ we have that

$$
\int_{K_\alpha} f(x) \, dx = \int_{K_\alpha} f(x) |\tau_\alpha g(x)|^2 \, dx \leq \int_G f(x) |\tau_\alpha g(x)|^2 \, dx
$$

$$
= \int_G (\tau_\alpha g)^\wedge(\tau_\alpha \hat{g}) \, d\mathcal{F}(\hat{x}).
$$

Therefore by Theorem 4.7, for each $\alpha \in J$
\[
\int_{K_{\alpha}} f(x) \, dx \leq || f ||_{\infty} ||(\tau_{q}g) \hat{\phi} ||_{1}
\]
\[
= || f ||_{\infty} 2^a ||(\tau_{q}g) \hat{\phi} ||_{1}||\hat{\phi}||_{1}
\]

This implies that \( f \in (L^1, l^\infty). \)

**Remark 8.7.** We deduce from Theorem 8.6 that

i) \( \mathcal{SR}(\phi) = L_{1, \text{loc}}^{1} \cap \mathcal{M}_{T} \) (part iii) and Remark 6.18).

ii) If \( f \in \mathcal{SR}(\phi_q), \) \( 1 \leq q \leq \infty, \) then there exists a unique measure \( f \in \mathcal{M}_{q}, \) such that for all \( \phi \in \phi_q \)

\[
\int_{G} f(x) \phi(x) \, dx = \int_{G} \hat{\phi}(\xi) \, d \tau(\xi).
\]

(See the proof of part iii).

Following Holland [35, 55] we define:

**Definition 8.8.** A measurable function \( f \) on \( G \) is resonant relative to the space \( \Phi_q, \) \( 1 \leq q \leq \infty, \) if

(R-1)' \( f \phi \in L^1(G) \) for all \( \phi \in \phi_q \)

and \( f \) satisfies (R-2) of Definition 8.4.

The linear space of functions resonant relative to \( \phi_q \) will be denoted by \( \mathcal{R}(\phi_q). \)

The proof of the next theorem is very similar to Theorem 8.6 and it will be omitted.
THEOREM 8.9. i) $R(\phi_\omega) \subseteq R(\phi_q) \subseteq R(\phi)$ for $1 < q < \infty$

ii) A measurable function $f$ satisfies the condition (R-1)''

iff $f \in L^1_{loc}$

iii) $f \in R(\phi_q), 1 \leq q \leq \infty$, iff $f \in L^1_{loc}, f \in M_T$ and $\overline{f} \in M_q$.

iv) $R(\phi_q) \subseteq (L^q, \ell^\infty)$ for $2 \leq q \leq \infty$.

v) $f \in R(\phi_q), 2 \leq q < \infty$, iff $f \in M_T$ and $\overline{f} \in M_q$.

We conclude easily from Theorem 8.6 and Theorem 8.9 that $R(\phi) = SR(\phi), R(\phi_q) = SR(\phi_q), 2 \leq q \leq \infty, SR(\phi_q) \subseteq R(\phi_q), 1 < q < 2$, and $SR(\phi_q) = R(\phi_q) \cap L^q_{loc}, 1 < q < 2$.

We do not know if strong resonant relative to $\phi_q$ is equivalent to resonant relative to $\phi_q$ for $1 < q < \infty$. In this direction we have a partial result (see Remark 8.21).

We modify Holland's definition because by imposing the condition (R-1) we have that $SR(\phi_q) \subseteq L^q_{loc}$ for all $1 < q < 2$, and this allow us to endow $SR(\phi_q)$ with a locally convex topology with respect to which the linear space spanned by the set of positive definite functions for $(L^q, \ell^1)$ is dense in $SR(\phi_q)$. Also we see that Theorem 3 and Theorem 7 of [35] for $R(\phi_q), 1 \leq q < 2$ are valid for $SR(\phi_q), 1 \leq q < 2$.

Observe that vi) of Theorem 8.6 is a generalization of [35, Theorem 8 i]).

The next theorem was proved for the real line by Holland [35, Theorem 6] and for locally compact abelian groups by Stewart [49, Theorem 4.4].
THEOREM 8.10. Let \( 2 < q < \infty \). \( f \in \text{SR}(\Phi_q) \) iff there exists a unique measure \( \mu \in M_q(\hat{G}) \) such that \( f = \hat{\mu} \).

PROOF. The necessity part follows from the proof of Theorem 8.6 iv). Suppose \( \hat{f} = \hat{\mu}, \mu \in M_q(G) \). Then \( f \in M_T \) because \( M_q \subseteq L_1(\hat{G}) \) (Remark 6.25). Since \( \hat{\mu} \in (L_1^q, \mathbb{L}_1^q) \subseteq M_w, \ f \in M_w \cap M_T \).

So by Theorem 6.23 and Theorem 6.21, \( \int f \hat{\mu} = \int f \hat{\mu} = \int \hat{f} \hat{\mu} \). Therefore \( f \in \text{SR}(\Phi_q) \) (Theorem 8.6 v)).

REMARK 8.11. It follows from [1, Theorem 3.3] that for \( f \in \text{SR}(\Phi_q), 1 \leq q \leq \infty \).

1) If \( \phi \in L_1^1(G), \phi \) is convolvable with \( f \) and \( \hat{\phi} \in L_1^1(\hat{G}) \), then for locally almost all \( x \in G \)

\[
\mathcal{f} \ast \phi(x) = \int_G \mathcal{f}(y) \phi(y - x) \, dy = \int_G \hat{\phi}(\hat{x}) \mathcal{f}(\hat{x}) \, d\hat{f}(\hat{x}).
\]

ii) For any \( u \in G \) such that the integral on the left is a continuous function of \( x \) in the neighborhood of \( u \), the formula in i) is valid for \( x = u \). Under this hypothesis for \( u = 0 \)

\[
\int_G \mathcal{f}(y) \phi(-y) \, dy = \int_G \hat{\phi}(\hat{x}) \, d\hat{f}(\hat{x}).
\]

DEFINITION 8.12. Let \( F \) be a set of complex-valued functions. A complex valued function \( f \) is **positive definite** for \( F \) if the integral

\[
\int \int f(x - y) \phi(x) \overline{\phi(y)} \, dx \, dy
\]

exists as a Lebesgue integral over the product set \( G \times G \) and is nonnegative.
tive for all $\phi \in F$.

$P(F)$ will denote the set of positive definite functions for $F$.

We list in the following remark the properties and results about positive definite functions we will need to prove Theorem 8.20.

**Remark 8.13.**

1. It is clear that if $F_1 \subseteq F_2$ then $P(F_2) \subseteq P(F_1)$.

2. $P(L^p_C) \subseteq L^p_{\text{loc}}$ and $P((L^p, \ell^1)) = P(\mathcal{P} \cup (L^p, \ell^\infty), 1 \leq p < \infty$, where $\mathcal{P} = \{ \mu | \mu \in M_\omega, \mu > 0 \}$ (17, §4 Theorem III).

3. If $f \in P(C_c)$ then $f \in M_T$ and $|T| \geq 0$ (1, Theorem 4.1).

4. By i) of Remark 8.7 and iii), $P(C_c) \subseteq SR(\phi)$.

**Definition 8.14.** For $E \subseteq G$ compact we define the seminorm $\rho_E$ on $SR(\phi_q)$, $1 \leq q \leq \infty$, by $\rho_E(f) = \|f\|_{L^q(E)}$, and endow $SR(\phi_q)$ with the locally convex topology generated by the family

$\{ \rho_E | E \subseteq G \text{ compact} \}$.

We want to prove that $P(L^q, \ell^1)$, $1 \leq q \leq 2$, the linear space spanned by $P(L^q, \ell^1)$, is dense in $SR(\phi_q)$. For that we need to introduce the concept of summability function of type I and establish some results.

**Definition 8.15.** Given $U \in N_0(G)$ compact, let $\psi_U$ be a function with the following properties.

1. $\psi_U$ is continuous with support $U$.
2. $\psi_U \geq 0$ and $\int \psi_U = 1$.
3. $\hat{\psi}_U \geq 0$ and $\hat{\psi}_U \in L^1(G)$.
Then \( \psi_U \) will be called a summability function of type 1 on \( G \).

**Definition 8.16.** If \( U \subset N_0(G) \), let \( V \subset N_0(G) \) compact such that \( V - V \subset U \) and \( V = V_1 \times V_2 \) where \( V_1 \subset R^d \) and \( V_2 \subset G_1 \).

For \( i = 1, 2 \) let \( \beta_i \) be a positive \( L^2 \) function having support in \( V_i \) and integral equal to 1.

Set \( \beta_U(s,t) = \beta_1(s)\beta_2(t) \) and \( \psi_U = \hat{\psi}_U^* \beta_U \). Then \( \hat{\psi}_U \) is a summability function of type 1 on \( \hat{G} \). Moreover

\[
\{ \psi_U \} \subset \Phi
\]

\[
\| \hat{\psi}_U \|_\infty \leq 1
\]

Indeed, \( \beta_U \in L^2_c \) and therefore \( \beta_U \in (L^2, L^1) \). Hence by Theorem 5.4, \( \hat{\beta}_U \in (C_0, L^2) \). This implies that \( \hat{\psi}_U = \beta_U \hat{\psi}_U \) belongs to \( (C_0, L^1) \) (Proposition 4.1).

Since \( \| \hat{\psi}_U \|_\infty \leq \| \hat{\psi}_U \|_1 \) we have by (8.2) that \( \| \hat{\psi}_U \|_\infty \leq 1 \).

For a proof of the next theorem see [50, II §2 Theorem 4].

**Theorem 8.17.** Any summability kernel \( \{ \psi_U \} \) of type I has the following properties:

\[
\lim_{U \to 0} \| \psi_U f - f \|_p = 0 \text{ for any } f \in \mathcal{L}_p(G), 1 < p < \infty
\]

\[
\hat{\psi}_U \text{ converges pointwise to } \hat{f} \text{ on } \hat{G} \text{ as } U \to 0
\]

Hereafter and throughout the whole work \( \{ \psi_U \} \) will be the summability kernel of type I as defined in Definition 8.16.
PROPOSITION 8.18. If \( f \in L^q_{\text{loc}} \) and there exists a measure \( \mu \in M^q (\hat{G}) \) \((1 < q < \infty)\) such that

\[
f(x) = \lim_{U \to 0} \int_{\hat{G}} \hat{\psi}_U(\hat{x}) \left[ x, \hat{x} \right] d\mu(\hat{x})
\]

where the limit exists in \( L^q \) over any compact subset of \( G \), then \( f \in SR(\phi_q) \) and \( \overline{T} = \mu \).

**PROOF.** Let \( \phi \in \phi, E = \text{supp } \phi \). Set \( F_U(x) = \int_{\hat{G}} \hat{\psi}_U(\hat{x}) \left[ x, \hat{x} \right] d\mu(\hat{x}) \).

Since \( F_U \) converges to \( f \) in \( L^q(E) \) and \( \phi \in L^{q'}(E) \)

\[
\int_G f(x) \phi(x) \, dx = \lim_{U \to 0} \int_G F_U(x) \phi(x) \, dx
\]

\[
= \lim_{U \to 0} \int_{\hat{G}} \int_G \hat{\psi}_U(\hat{x}) \left[ x, \hat{x} \right] d\mu(\hat{x}) \phi(x) \, dx
\]

\[
= \lim_{U \to 0} \int_{\hat{G}} \hat{\psi}_U(\hat{x}) \int_G \phi(x) \left[ x, \hat{x} \right] dx \, d\mu(\hat{x})
\]

\[
= \lim_{U \to 0} \int_{\hat{G}} \hat{\psi}_U(\hat{x}) \hat{\phi}(\hat{x}) \, d\mu(\hat{x})
\]

\[
= \int \hat{\phi}(-\hat{x}) \, d\mu(\hat{x}).
\]

The last equality follows from the Lebesgue Dominated Convergence Theorem. Remember that \( \hat{\phi} \in L^1(\mu), ||\hat{\psi}_U||_{\infty} \leq 1 \) and \( \hat{\psi}_U(\hat{x}) \to 1 \) as \( U \to 0 \) \((8.5), (8.7)\)).

Since \( \phi \) is arbitrary, this implies that \( f \in M_T \) and \( \overline{T} = \mu \).

Therefore by Theorem 8.6 ii) \( f \in SR(\phi_q) \).

The next proposition is the converse of Proposition 8.18.
PROPOSITION 8.19. If $f \in \text{SR}(\phi_q)$, $1 < q < \infty$, then
\[
f(x) = \lim_{U \to 0} \int \hat{\psi}_U(\hat{R}) \, |x, \hat{R}| \, d \mathcal{F}(\hat{R})
\]
where the limit exists in $L^q$ over any compact subset of $G$.

PROOF. Since $\{\psi_U\} \subseteq \phi \subseteq \phi_q$ by (8.4) we have by Remark 8.7 ii) that
\[
f * \psi_U(x) = \int_G f(y) \psi_U(x - y) \, dy = \int_G \hat{\psi}_U(\hat{R}) \, |x, \hat{R}| \, d \mathcal{F}(\hat{R}).
\]

By Theorem 8.17, (8.6), $f * \psi_U$ converges to $f$ in $L^q$ over any compact subset of $G$ because $f \in L^q_{\text{loc}}$.

If $\mu$ is a real-valued measure in $M_q$, $1 < q < \infty$, then
\[
\mu = \mu^+ - \mu^- \quad \text{where} \quad \mu^+ = \text{sup} (0, \mu), \quad \mu^- = \text{sup} (0, -\mu).
\]
Hence $\mu^+ < |\mu|$, $\mu^- < |\mu|^1$, and therefore $\mu^+$, $\mu^-$ belong to $M_q$. Also $|\mu| = \mu^+ + \mu^-$. Moreover, if $\mu$ is a complex-valued measure in $M_q$, $1 < q < \infty$ then $\mu = \mu_1 + i\mu_2$ where $\mu_i$, $i = 1, 2$, is a real-valued measure in $M_q$ and $|\mu| = |\mu_1| + |\mu_2|$ [52, 147].

THEOREM 8.20. i) Let $1 < q < 2$. The linear space spanned by $P(L^{q'}, \ell^1)$, $< P(L^{q'}, \ell^1) >$, is dense in $\text{SR}(\phi_q)$.

ii) $P(L^2, \ell^1) = \text{SR}(\phi_2)$

iii) $\text{SR}(\phi_q) \subseteq < P(L^{q'}, \ell^1) >$ for $2 < q < \infty$.

PROOF. By Remark 8.13 i) and iv) $P(L^\infty, \ell^1) \subseteq P(C_c) \subseteq \text{SR}(\phi)$.

Let $f \in P(L^{q'}, \ell^1)$, $1 < q < 2$. Again by Remark 8.13 ii) and iii) $f \in (L^q, \ell^\infty)$, $f \in M_q$, and $\mathcal{F} > 0$. 

Let \( g \in C_c(\hat{G}) \) such that \( g \) is the spherical function associated to \( g^* \in (C_c, \mathbb{L}^1) \). Set \( h = \hat{g}^* \hat{g} \). Since \( \hat{g} \in (C_c, \mathbb{L}^1) \), \( h \in (C_c, \mathbb{L}^1) \) and \( \hat{h} = \hat{g}^* \hat{g} = |g|^2 \). Therefore \( \hat{h} > 0 \), \( \hat{h} \in C_c \) and \( \hat{h} \leq 1 \) on \( \hat{K} \), so for \( \hat{x} \in \hat{G} \) and \( \hat{\mathcal{T}} \theta \) as defined in Theorem 1.21.

\[
\hat{\mathcal{T}}^\theta(\hat{x}) = \hat{\mathcal{T}}(\hat{x} + \hat{K}) = \int_{\hat{\mathcal{T}}(\hat{x} + \hat{K})} \hat{\mathcal{T}}(\hat{\mathcal{T}}(\hat{s})) \, d\hat{\mathcal{T}}(\hat{s}) = \int_{\hat{\mathcal{T}}(\hat{x} + \hat{K})} \tau_{\hat{x}}^\theta(\hat{s}) \, d\hat{\mathcal{T}}(\hat{s}) \\
\leq \int_{\hat{G}} \tau_{\hat{x}}^\theta(\hat{s}) \, d\hat{\mathcal{T}}(\hat{s}).
\]

Now, \( h \in L^1 \), \( h \) is convolvolable with \( f \) and \( \hat{h} \in L^1(\hat{\mathcal{T}}) \), (remember that \( \hat{\mathcal{T}} \in M_\infty \)) and \( f \ast h \) is continuous on \( G \). Indeed, note that for \( x, s \) in \( G \)

\[
|f \ast h(x) - f \ast h(s)| \leq \int |f(y)| \, |h(x - y) - h(s - y)| \, dy \\
\leq ||f||_{q\infty} \, ||\tau_x h' - \tau_s h'||_{q'1}
\]

and the function \( x \mapsto \tau_x h' \) is continuous on \( G \) (Theorem 3.14).

By Remark 8.11 we have that

\[
\int \tau_x^\theta h(\hat{s}) \, d\hat{\mathcal{T}}(\hat{s}) = \int f(x) \, h'(x) \left[-x, \hat{s}\right] \, dx = (fh') \hat{\ast} (\hat{s}).
\]

Since \( f \in (L^q, \mathbb{L}^\infty) \), \( h' \in (C_c, \mathbb{L}^1) \), \( fh' \in (L^q, \mathbb{L}^1) \) \( \subseteq L^q \), so \( (fh') \hat{\ast} \in L^q \) and this implies that \( \hat{\mathcal{T}}^\theta \in L^q \). By Theorem 1.21 we conclude that \( \hat{\mathcal{T}} \in M_{q'} \). By Theorem 8.6 iii), \( f \in SR(\varphi_q) \) and therefore

\[
< P(L^q, \mathbb{L}^1) \subseteq SR(\varphi_q) \text{ for } 1 \leq q \leq 2.
\]

Now, take \( f \in SR(\varphi_q) \), \( 1 \leq q \leq \infty \). Then \( \hat{\mathcal{T}} \in M_{q'} \) and by our
previous comment \( T^i = \mu_1 - \mu_2 + i(\mu_3 - \mu_4) \) where \( \mu_1, \mu_2, \mu_3, \) and \( \mu_4 \geq 0, i = 1, \ldots, 4 \). If \( 2 \leq q < \infty \) then \( f = \hat{T} = \hat{\mu}_1 - \hat{\mu}_2 + i(\mu_3 - \mu_4) \) (see the proof of Theorem 8.6 iii)). Since \( \mu_1 \in \mathcal{P} \cap (L^q, L^{\infty}), \)

\( i = 1, \ldots, 4 \) we conclude by Remark 8.13 that \( f \in \mathcal{P}(L^{q'}, L^1) \) and this proves ii) and iii).

Assume \( 1 < q < 2 \), and set

\[
F_U^i(x) = \int_G \hat{\psi}_U(\hat{x}) \ [x, \hat{x}] \ d\mu_i(\hat{x}) \quad i = 1, \ldots, 4.
\]

We will show that \( F_U^i \in \mathcal{P}(L^{q'}, L^1) \) and this will imply that \( F_U(x) = \int_G \hat{\psi}_U(\hat{x}) \ [x, \hat{x}] \ d\mu(\hat{x}) \) belongs to \( \mathcal{P}(L^{q'}, L^1) \) since

\[
F_U = F_U^1 - F_U^2 + i(F_U^3 - F_U^4). \]

Hence by Proposition 8.19 we conclude that \( f \in \mathcal{P}(L^{q'}, L^1) \).

Let \( \phi \in (L^{q'}, L^1) \). Since \( |F_U^i(x)| \leq ||\mu_i||_{q'} ||\psi_U||_{\infty} \) for all \( x \in G, F_U^i \in L^{\infty} \) and we can apply Fubini's theorem because \( \phi \ast \tilde{\phi} \in (L^{q'}, L^1) \subset L^1 \), so

\[
\int_G F_U^i(x) \phi \ast \tilde{\phi}(x) \ dx = \int_G \int_G \hat{\psi}_U(\hat{x}) \ [x, \hat{x}] \ d\mu_1(\hat{x}) \ \phi \ast \tilde{\phi}(x) \ dx
\]

\[
= \int_G \hat{\psi}_U(\hat{x}) \int_G \phi \ast \tilde{\phi}(x) \ [x, \hat{x}] \ dx \ d\mu_1(\hat{x})
\]

\[
= \int_G \hat{\psi}_U(\hat{x}) \ |\hat{\phi}|^2(-\hat{x}) \ d\mu_1(\hat{x}).
\]

By (8.3) we have that

\[
\int \int F_U^i(x - y) \ \phi(x) \ \tilde{\phi}(y) \ dx \ dy = \int F_U^i(x) \ \phi \ast \tilde{\phi}(x) \ dx
\]

is nonnegative and therefore \( F_U^i \in \mathcal{P}(L^{q'}, L^1) \).
REMARK 8.21. We see from the proof of Theorem 8.20 that
\[ \langle P(L^q'), L^1 \rangle \subset R(\Phi_q), \; 1 < q < 2. \]
Since \( \langle P(L^q'), L^1 \rangle \subset (L^q, L^\infty) \), \( 1 < q < 2 \), (Remark 8.13) we have that \( R(\Phi_q), \; 1 < q < 2 \), contains a linear subspace of \( (L^q, L^\infty) \).

THEOREM 8.22. \( SR(\Phi_\infty) \) is dense in \( SR(\Phi_q) \), \( 1 < q < \infty \). Hence \( SR(\Phi_q) \) is dense in \( SR(\Phi_s) \), \( 1 < s < q < \infty \).

PROOF. Let \( f \in SR(\Phi_q) \). Set \( F_U(x) = \int \hat{\psi}_U(\xi) \{x, \xi\} d \hat{\mathcal{P}}(\xi) \).
Since \( f \in M_T \) and \( \{\psi_U\} \subset \phi \) by (8.4) we have by Remark 8.7 ii) that \( \psi_U* f(x) = \int f(y) \psi_U(y-x) dy = \int \hat{\psi}_U(\xi) \{x, \xi\} d \hat{\mathcal{P}}(\xi) = F_U(x) \).
\( \psi_U* f \) belongs to \( L^1 \) because \( f \in L^1_{loc} \) and \( \{\psi_U\} \subset C_c \), so by Theorem 6.19 \( \psi_U* f \in M_T \) and
\( (\psi_U* f)^\vee = (\psi_U^* f)^\vee = (f_{\chi_U})^\vee \). Since \( (f_{\chi_U})^\vee \in C_0 \) and 
\( (\psi_U^* f)^\vee \in (C_0, L^1) \), \( (\psi_U^* f)^\vee \in M_\infty \). Therefore \( \psi_U* f \) belongs to \( SR(\Phi_\infty) \)
(Theorem 8.6 v)). This implies by Proposition 8.19 that \( SR(\Phi_\infty) \) is dense in \( SR(\Phi_q) \).

Finally by Theorem 8.6 vi) we conclude that \( SR(\Phi_q) \) is dense in \( SR(\Phi_s) \) if \( 1 < s < q < \infty \).

The next two theorems are generalizations of Theorems 3 and 7 of [35].

THEOREM 8.23. If \( f \in SR(\Phi_q) \), \( 1 < q < \infty \), then
\[ \int_G f(x) \phi(x) dx = \int_G \hat{\phi}(-\xi) d \hat{\mathcal{P}}(\xi) \]
for all \( \phi \in L^q_c \) such that \( \hat{\phi} \in (C_0, L^q) \).
Hence

\[ \left| \int_G f(x) \phi(x) \, dx \right| \leq \| \tau_x \|_q \cdot \| f \|_q. \]

**Proof.** If \( \phi \in L^q_c \) and \( \hat{\phi} \in C^0 \), then \( \phi \in L^1_c \). \( \hat{\phi} \) is convolvable with \( f \) and \( \hat{\phi} \in L^1(\Gamma) \). This means that \( \phi \) satisfies i) of Remark 8.11.

Let \( V_1 \subset N_0(G) \) compact, \( s \in V_1 \) and \( E = \text{supp } \phi \).

**Case 1)** \( 1 < q < \infty \).

Since \( \phi \in L^q_c \), the map \( x \mapsto \tau_x \phi \) is continuous on \( G \) (Theorem 3.14). So given \( \varepsilon > 0 \) there exists \( V_2 \subset N_0(G) \) such that for all \( x \in V_2 \)

\[ \| \tau_x \phi - \tau_s \phi \|_q < \varepsilon / \| f_{xV_1 \cup E} \|_q. \]

Then for \( x \in V_1 \cap V_2 \) we have by the Hölder inequality that

\[ \| f \ast \phi \ast (s) - f \ast \phi (s) \| \leq \int |f(y)| \| \phi(x - y) - \phi(s - y) \| \, dy \]

\[ = \int_{V_1 \cup E} |f(y)| |\phi'(y - x) - \phi'(y - s)| \, dy \]

\[ < \varepsilon / \| f_{xV_1 \cup E} \|_q \cdot \| \tau_x \phi' - \tau_s \phi' \|_q < \varepsilon. \]

Therefore \( f \ast \phi \) is continuous at \( s \).

**Case 2)** \( q = 1 \).

Similarly to case 1), the map \( x \mapsto \tau_x (f_{xV_1 \cup E})' \) is continuous on \( G \). So given \( \varepsilon > 0 \) there exists \( V_2 \subset N_0(G) \) such that for all \( x \in V_2 \)

\[ \| \tau_x (f_{xV_1 \cup E}) - \tau_s (f_{xV_1 \cup E}) \|_1 < \varepsilon / \| \phi \|_\infty. \]

Then for \( x \in V_1 \cap V_2 \)
\[ |f*\phi(x) - f*\phi(s)| \leq \int_E |\phi(y)| |f(x - y) - f(s - y)| \, dy \]
\[ \leq \int_G |\phi(y)| |f_{\mathcal{V}_1 \cap E} - f_{\mathcal{V}_1 \cap E}(s - y)| \, dy \]
\[ \leq \|\phi\|_\infty \|\tau_x(f_{\mathcal{V}_1 \cap E})' - \tau_s(f_{\mathcal{V}_1 \cap E})'\|_1 < \varepsilon. \]

Again \( f*\phi \) is continuous at \( s \).

Applying Remark 8.11 we conclude that
\[ \int_G f(x) \phi(x) \, dx = \int \hat{\phi}(\hat{x}) \, d\hat{F}(\hat{x}) \].

**THEOREM 8.24.** Let \( 1 \leq q < 2 \). If \( f \in SR(\phi_q) \) then
\[ (C.1) \int_G \hat{h}(x) f(x) \, dx = \int \hat{h}(\hat{x}) \, d\hat{F}(\hat{x}) \]
for all \( h \in (\mathcal{C}_0, \mathbb{L}^q)(G) \).

Furthermore if \( p = \frac{2q}{2q - 1} \) then
\[ \int \int f(x - y) \phi(x) \overline{\psi(y)} \, dx \, dy = \int \hat{\phi}(\hat{x}) \hat{\psi}(\hat{x}) \, d\hat{F}(\hat{x}) \]
for all \( \phi, \psi \in (L^p, \mathbb{L}^1) \). The double integral exists not necessarily as a Lebesgue integral but as the limit of the integral
\[ \int_{V_\beta} \int_{V_\alpha} f(x - y) \phi(x) \overline{\psi(y)} \, dx \, dy \]
over \( \alpha, \beta \)

where \( V_\alpha, V_\beta \) are finite unions of the sets \( K_\alpha (\alpha \in J) \).

That is, as the sum of the absolutely convergent series
\[ \sum_{\beta} \int_{B_{\alpha}} \int_{V_{\beta}} \int_{B_{\beta}} f(x - y) \varphi(x) \bar{\psi(y)} \, dx \, dy. \]

**Proof.** If \( h \in (C_0, \mathcal{L}^q)(\hat{G}) \) then \( h \in (L^q, \mathcal{L}^2) \) (Remark 5.8). Since \( \{\phi_U\} \subseteq C_c \) (Theorem 7.2) \( \mathcal{V} \mathcal{V} \mathcal{h} \phi_U \in L^q \) and \( h^* \phi_U = (h \phi_U)^* \). So by Theorem 8.23

\[ \int_G \varphi_U(x) \mathcal{V} \mathcal{h}(x) \, dx = \int_G h^* \phi_U(-\hat{x}) \, d\mathcal{F}(\hat{x}). \]

Since \( \mathcal{F} \in M^q \) and \( \lim_{U \to 0} \|h^* \phi_U - h\|_{\mathcal{M}^q} = 0 \) (Theorem 7.2 and Corollary 4.14), the integral on the right converges to \( \int \mathcal{F}(-\hat{x}) \, d\mathcal{F}(\hat{x}) \). Therefore

\[ (C.1) \int_G \mathcal{V} \mathcal{h}(x) \, dx = \int_G h(-\hat{x}) \, d\mathcal{F}(\hat{x}). \]

Let \( \phi, \psi \in L^P_c \). Since \( p = \frac{2q}{2q - 1} \) and \( 1 < q < 2 \) we have that \( \frac{1}{p} = \frac{2q - 1}{2q} = 1 - \frac{1}{2q} \), this implies that \( \frac{1}{2} < 1 - \frac{1}{2q} < \frac{1}{2} \).

\[ \frac{1}{2} < 1 - \frac{1}{2q} = \frac{1}{p} < \frac{3}{4}, \] therefore \( 1 < p < 2 \). If \( \phi \in L^P_c \) then \( \hat{\phi} \in (C_0, \mathcal{L}^q) \), because \( \phi \in (L^P, \mathcal{L}^1) \). Hence by Proposition 4.1

\[ (\hat{\phi} \hat{\psi}) = \hat{\varphi \psi} \] belongs to \((C_0, \mathcal{L}^q)\) because \( \frac{1}{p^*} = 1 - (1 - \frac{1}{2q}) = \frac{1}{2q} \), that is, \( p^* = 2q \). So by Theorem 8.23

\[ (1) \int_{\mathcal{F}} f(x - y) \, dx \, dy = \int_{\mathcal{F}} f(x) \, \hat{\phi \psi}(x) \, dx \]

\[ = \int \hat{\phi}(\hat{x}) \overline{\psi(\hat{x})} \, d\mathcal{F}(\hat{x}). \]
Set $B(\phi, \psi) = \int \int f(x - y) \phi(x) \overline{\psi(y)} \ dx \ dy \quad (\phi, \psi \in L^p_c)$.

By the Hausdorff-Young inequality (Theorem 5.7) we have that for all $\phi, \psi \in L^p_c$

$$|B(\phi, \psi)| \leq \|T\|_{q', q} \|\hat{\phi}\|_{\omega_q} \|\hat{\psi}\|_{\omega_q'} \leq \|T\|_{q', q} C_p \|\phi\|_{p_1} \|\psi\|_{p_1},$$

where $C_p$ is a constant depending on $p$ and $\alpha$.

If $h \in (L^p, \ell^1)$ then we can write $h = \sum h_{\alpha}$ where $h_{\alpha} = h|_{\ell^\alpha}$ (a.e.). So $\|h\|_{p_1} = \sum \|h_{\alpha}\|_p$. Then for $\phi, \psi \in (L^p, \ell^1)$

$$|B(\phi, \psi)| \leq C_p \|T\|_{q', q} \|\phi\|_{p_1} \|\psi\|_{p_1}.$$

So the double series $\sum_{\alpha, \beta} B(\phi_{\beta}, \psi_{\alpha})$ is absolutely convergent. That is, the left side of (1) exists as the sum of the absolutely convergent series

$$\sum_{\alpha, \beta} \int_{K_{\beta}} \int_{K_{\alpha}} f(x - y) \phi(x) \overline{\psi(y)} \ dx \ dy.$$

Moreover since $\hat{h} = \sum h_{\alpha}$ is uniformly convergent and $\sum h_{\alpha}$ converges in the norm of $(C_\alpha, \ell^p)$ to $\hat{h}$, we have that

$$\sum_{\alpha, \beta} \int_{K_{\beta}} \int_{K_{\alpha}} f(x) \overline{\psi_{\alpha}(x)} \ d\overline{\psi_{\alpha}(x)} = \int \hat{\phi}(\xi) \overline{\hat{\psi}(\xi)} \ d\overline{\psi}(\xi).$$

This ends the proof.
THEOREM 8.25. Let $1 < q < 2$. The following statements are equivalent:

i) $f \in \mathcal{L}(q)$

ii) $f \in L^q_{\text{loc}}$ and there exists a unique measure $F$ in $M_q$, such that for all $\phi, \psi \in (L^p, L^1)$ where $p = 2q/(2q - 1)$ the integral

$$
\int \int f(x - y) \phi(x) \psi(y) \, dx \, dy = \int \hat{\phi}(\hat{\xi}) \hat{\psi}(\hat{\xi}) \, dF(\hat{\xi})
$$

exists in the sense of the Theorem 8.24.

iii) $f \in L^q_{\text{loc}}$ and there exists a unique measure $F \in M_q$, such that

$$
f(x) = \lim_{U \to 0} \int \hat{\psi}_U(\hat{\xi}) [x, \hat{\xi}] \, dF(\hat{\xi})
$$

where the limit exists in $L^q$ over any compact subset of $G$.

PROOF. i) implies ii) follows from Theorem 8.22.

iii) implies i) is Proposition 8.18. It remains to prove that

ii) implies iii).

$$
\psi_U = \beta_U * \beta_U, \quad \beta_U \in L^2_c \text{ (Definition 8.16). So by ii) there exists a unique } F \in M_q, \text{ such that for all } t \in G
$$

$$
\int \int f(x - y) \tau_t \beta_U(x) \beta_U(y) \, dx \, dy = \int (\tau_t \beta_U)(\hat{\xi}) \beta_U(\hat{\xi}) \, dF(\hat{\xi})
$$

Now, the left side of this equality is equal to

$$
\int f(x) \beta_U * \beta_U(t - x) \, dx = \int f(x) \psi_U(t - x) \, dx = f^* \psi_U(t)
$$

and the right side is equal to

$$
\int \hat{\psi}_U(\hat{\xi}) \beta_U(\hat{\xi}) \, [t, \hat{\xi}] \, dF(\hat{\xi}) = \int \hat{\psi}_U(\hat{\xi}) \, [t, \hat{\xi}] \, dF(\hat{\xi}).
$$
Since $\psi_* f$ converges to $f$ in the sense of the part ii) we conclude that ii) implies iii).†

Compare Theorem 8.25 with [51, Theorem 4.1 and Theorem 4.2].
CHAPTER IV

GENERALIZATIONS OF FOURNIER'S THEOREMS ON LOCAL
COMPLEMENTS TO THE HAUSDORFF-YOUNG THEOREM

§ 9. THE CASE WHERE \( \hat{G} \) IS NOT DISCRETE

For a subset \( E \) of \( \hat{G} \), the Fourier transform of a function \( f \) restricted to \( E \) will be denoted by \( \hat{f}|_E \).

\( (L^p, L^q)^\wedge \) will be the set of Fourier transforms of functions in \( (L^p, L^q)^\wedge \) and \( (L^p, L^q)^\wedge |E \) will be the set of functions in \( (L^p, L^q)^\wedge \) restricted to \( E \).

We will keep this notation for the rest of our work.

J. Fournier [22, Theorem 1] proved the following theorem.

THEOREM 9.1. If \( \hat{G} \) is nondiscrete and \( E \subset \hat{G} \) is not locally null then for \( 1 < p \leq 2 \)

\[ L^p|_E \subset \bigcup_{q>p} L^q(E). \]

Here we shall see that under the same conditions

\[ (L^\infty, L^p)^\wedge |E \subset \bigcup_{q>p} (L^q, L^\infty)(E). \]  

(1)

If \( G \) is neither compact nor discrete then (1) extends Theorem 9.1 because \( (L^\infty, L^p) \) and \( L^q \) are proper subspaces of \( L^p \) and \( (L^q, L^\infty) \) \( (1 \leq p, q \leq \infty) \) respectively (Theorem 2.4).
THEOREM 9.2. If \( \hat{G} \) is nondiscrete and \( E \subset \hat{G} \) is not locally null, then for \( 1 < p < 2 \)

\[
(L^\infty, L^p) \ | \ E \cap \bigcup_{q > p'} (L^q, L^\infty)(E).
\]

PROOF. Since \( E \) is not locally null, it contains a subset of positive measure. By the inner regularity of the Haar measure this subset contains a compact set of positive measure. Therefore it is enough to prove the theorem for compact sets \( E \) of positive measure.

In this case \((L^q, L^\infty)(E)\) is equal to \(L^q(E)\), but it will be convenient for our proof to consider \((L^q, L^\infty)(E)\).

Suppose that

\[
(L^\infty, L^p) \ | \ E \subset (L^q, L^\infty)(E) \quad \text{for some} \quad q \in (p', \infty).
\]

Take \( f \in L^p(G) \) and let \( \phi \in C_c(\hat{G}) \) such that \( \phi \equiv 1 \) on \( E \) and \( \check{\phi} \in (L^{p'}, L^1)(G) \) (Theorem 5.2). By (2.6) \((L^{p'}, L^1) \subset L^1\), so the Fourier transform \( \check{\phi} \) of \( \check{\phi} \) is equal to \( \phi \) \([37, 31.44 \ b]\).

Applying Theorem 4.7, we have that \( f \check{\phi} \in (L^\infty, L^p)(G) \). Therefore by our assumption \((f \check{\phi}) \in (L^q, L^\infty)(E)\), hence \( \check{f} \in L^q(E) \) (see above). Since \( f \) is arbitrary, we conclude that \( L^p \mid E \subseteq L^q(E) \).

This contradicts Theorem 9.1. Therefore

\[
(2) \quad (L^\infty, L^p) \mid E \cap (L^q, L^\infty)(E) \quad \text{for all} \quad p' < q < \infty.
\]

For \( p' < q < \infty \), define the function \( F \) on \((L^\infty, L^p)\) by

\[
(3) \quad F(f) = \| \hat{f} \mid E \|_{q \infty}.
\]

By (2) \( F \) takes the infinite value.
Clearly \( F(\alpha f) = \alpha F(f) \) for all nonnegative real \( \alpha \) and for all \( f, g \in (L^\infty, L^p) \), \( F(f - g) < F(f) + F(g) \). These properties of \( F \) imply that for all real \( \alpha \), the set \( V_\alpha = \{ f \in (L^\infty, L^p) \mid F(f) > \alpha \} \) is dense in \( (L^\infty, L^p) \). Indeed, suppose that \( V_\alpha \) is not dense for some real \( \alpha \). Then \( \alpha > 0 \) since \( V_0 = (L^\infty, L^p) \cap \{0\} \).

Take \( g \in (L^\infty, L^p) \cap V_\alpha \). Then there exists \( \varepsilon > 0 \) such that for all \( \|f\|_{\omega p} \leq \varepsilon \), \( f + g \not\in V_\alpha \). That is

\[
(4) \quad F(f + g) > \alpha \quad \text{for all} \quad \|f\|_{\omega p} \leq \varepsilon.
\]

Let \( \overline{f} \) be a function in \( (L^\infty, L^p) \) such that \( F(\overline{f}) = \infty \). Then

\[
\|\overline{f}\|_{\omega p} > 0 \quad \text{and} \quad \overline{f} = (\varepsilon / \|\overline{f}\|_{\omega p}) \overline{f} \quad \text{belongs to} \quad (L^\infty, L^p).
\]

Since \( \|\overline{f}\|_{\omega p} = \varepsilon \) and \( \overline{f} = f + g - g \), we have by (4) that

\[
(\varepsilon / \|\overline{f}\|_{\omega p}) F(\overline{f}) = F(f) \leq F(f + g) + F(g) \leq \alpha + \alpha = 2\alpha.
\]

This contradiction shows that \( V_\alpha \) is dense in \( (L^\infty, L^p) \) for all \( \alpha \).

Moreover, \( F \) is lower semicontinuous. Indeed, let

\( U = \{ g \in (L^p, L^1) \mid \|g\|_{p,1} \leq 1 \} \) and define for each \( g \in (L^p, L^1) \)

the function \( F_g \) on \( (L^\infty, L^p) \) by

\[ F_g(f) = \left| \int_E \hat{f} \hat{g} \right|. \]

Since \( (L^\infty, L^p) \subset L^p \) by (2.6) and \( p \leq 2 \), the Fourier transform \( \hat{f} \) of \( f \) belongs to \( L^p \). Then for \( g \in U, \|g\|_E \in L^p \). So, by the Hausdorff-Young inequality we have that

\[
F_g(f) \leq \|\hat{f}\|_p \|g\|_p \leq \|\hat{f}\|_p \|g\|_p \leq \|f\|_p \|g\|_p \leq \|f\|_{\omega p} \|g\|_p.
\]
where the last inequality is due to (2.4) and (2.3). Therefore $F_g$ is continuous for all $g \in U$.

We want to prove next that

$$F = \sup \{ F_g \mid g \in U \}.$$  

First we note that $(L^p, \ell^1) \subset (L^{q'}, \ell^1)$ as $p > q'$. Since $(L^p, \ell^1)$ is dense in $(L^{q'}, \ell^1)$ (Corollary 3.8) $U$ is dense in the unit ball $B$ of $(L^{q'}, \ell^1)$ and therefore

$$\sup \{ F_g \mid g \in U \} = \sup \{ F_g \mid g \in B \}.$$  

By the converse of the Hölder inequality (as in [36, 191 p.142]) with $G = 1$, $k' = q'$, $k = q$, $F^{1/q} = \sup \{ F_g \mid g \in B \}$ we have that for all $g$

$$\left[ \int_{E} \left| \hat{f} \right| E \right]^{1/q} \leq \sup \{ F_g(\hat{f}) \mid g \in B \}.$$  

This implies that $\left| \left| \hat{f} \right| E \right|_q \leq \sup \{ F_g(\hat{f}) \mid g \in U \}$.  

Now, if $\left| \left| \hat{f} \right| E \right|_q < \infty$, that is $\hat{f} \in (L^q, \ell^\infty)(E)$, then for all $g \in U$

$$F_g(\hat{f}) = \left| \int_{E} \hat{f} \, g \right| \leq \left| \left| \hat{f} \right| E \right|_q \left| g \right|_{q'} \leq \left| \left| \hat{f} \right| E \right|_q \infty.$$  

Therefore $F_g(\hat{f}) \leq \left| \left| \hat{f} \right| E \right|_q \infty$. Hence (5) holds and this implies that $F$ is lower semicontinuous.

By Baire's theorem $\{ f \in (L^\infty, \ell^P) \mid F(f) = \infty \} = U \cap \cup N$ is a set of type $G_\delta$.

Now, choose a strictly decreasing sequence $\{ q_n \}$ converging to $p'$. Again by Baire's theorem (as in [43, Corollary of Theorem 5.6]) the set $\{ f \in (L^\infty, \ell^P) \mid \left| \left| \hat{f} \right| E \right|_{q_n} \infty = \infty \text{ for all } n \in \mathbb{N} \}$ is a dense $G_\delta$ set.
because is equal to \( \bigcap_n \{ f \in (L^\infty, L^p) \mid \| \hat{f} \|_{L^{\infty}} = \infty \} \). Therefore nonempty.

Take \( f \) in this set. Since \((L^\infty, L^p) \subset (L^1, L^p)\) (see (2.6)) we have by Theorem 5.7 that \( \hat{f} \in (L^q, L^\infty)(E) \). If also \( \hat{f} \in (L^q, L^\infty) \) then by (2.6), \( \hat{f} \in (L^q, L^\infty)(E) \) for all sufficiently large \( n \) and this contradicts the choice of \( f \). Therefore \( \hat{f} \in (L^q, L^\infty) \) for all \( q > p' \) and this ends the proof.†

**Corollary 9.3.** If \( \hat{G} \) is nondiscrete and \( 1 < p < 2 \) then

\[
(D^\infty, L^p) \uparrow \bigcup_{q>p'} (L^q, L^\infty)(\hat{G}).
\]

W. Bloom proved [3, Theorem 1] the following theorem.

**Theorem 9.4.** If \( \Omega \) is a nonempty open subset of \( G \) and \( \hat{G} \) is noncompact, then for \( 1 \leq p < q \leq \infty \) there exists \( f \in (L^\infty, L^q)(\hat{G}) \) such that \( \forall g \), \( \forall \bar{g} \) does not vanish on \( \Omega \) for all \( g \in (L^1, L^p)(\hat{G}) \).

In other words, the local inclusion \((L^1, L^p) \subset (L^\infty, L^q)\) is strict.

For the particular case \( 1 < q < 2 \), Theorem 9.2 proves Theorem 9.4. Indeed, if \( 1 \leq p < q < 2 \), \( \hat{G} \) is noncompact and \( \Omega \) is a nonempty open set of \( G \), then \( G \) is nondiscrete and \( \Omega \) is not locally null. So, Theorem 9.2 with \( p = q \) and \( q = r \) says that

\[
(L^\infty, L^q)^{\vee} \mid \Omega \uparrow \bigcup_{r>q'} (L^\infty, L^q)(\Omega). \quad \text{That is, there exists } f \in (L^\infty, L^q), \quad r>q'
\]

such that \( \forall \bar{g} \), \( \forall g \in (L^1, L^p)(\hat{G}), \forall g \in (L^p, L^\infty)(G) \) (Theorem 5.7) and therefore \( \forall f - g \) does not vanish on \( \Omega \) since \( p' > q' \).
§ 10. **THE CASE WHERE \( \hat{G} \) IS NONCOMPACT**

Theorem 2, b) of [22] is as follows.

**THEOREM 10.1.** If \( \hat{G} \) is noncompact and \( 1 \leq p \leq 2 \) then

\[
L^p \hat{\phi} \cup \bigcup_{q < p} L^q(\hat{G}).
\]

We will generalize this result by proving the next theorem.

**THEOREM 10.2.** If \( \hat{G} \) is noncompact and \( 1 \leq p \leq 2 \) then

\[
(L^p, \ell^1)^\perp \cup \bigcup_{q < p} (L^1, \ell^q)(\hat{G}).
\]

Theorem 10.2 indeed extends Theorem 10.1 if \( G \) is neither compact nor discrete, because \((L^p, \ell^1)\) and \(L^q\) are proper subspaces of \(L^p\) and \((L^1, \ell^q)\) respectively (Theorem 2.4) for \(1 \leq p, q \leq \infty\).

In order to prove Theorem 10.2 we need the following two results which appeared in [7, p. 194 and Theorem IV].

**PROPOSITION 10.3.** Let \( \phi = \{\phi \in C_c \mid \hat{\phi} \in \ell^1(G)\} \) endowed with the norm \( \phi \mapsto \|\hat{\phi}\|_{pq}^1 \leq p, q < \infty\). If \( T \in \mathcal{B}(X) \) then there exists a unique function \( h \in (L^p, \ell^q)(\hat{G}) \) such that for all \( \phi \in X \)

\[
T(\phi) = \int_{\hat{G}} \hat{\phi}(\hat{\xi}) h(\hat{\xi}) d\hat{\xi}.
\]

**PROOF.** We proceed as in Proposition 8.3. If \( T \in \mathcal{B}(X) \) then the map \( \hat{T}(\hat{\phi}) = T(\hat{\phi}') \) belongs to \((\hat{\phi}', \|\cdot\|_{pq})^*\). Since \( \hat{\phi}' \) is dense in \((L^p, \ell^q)(\hat{G})\) (Proposition 8.3, iv)) there exists a unique continuous extension \( \hat{T} \) on \((L^p, \ell^q)\). By Theorem 3.1 there exists a unique \( h \in (L^p, \ell^q)(\hat{G}) \) such that for all \( f \) in \((L^p, \ell^q)(\hat{G})\); \( \hat{T}(f) = \int f(\hat{\xi})h(\hat{\xi})d\hat{\xi} \).
So for all $\phi \in \Phi$ we have that

$$T(\phi) = \hat{T}(\phi') = \int \hat{\phi}(-\hat{\sigma})h(\hat{\sigma})d\hat{\sigma}.$$  

**Proposition 10.4.** Let $\mu \in M_T$. If there exists a constant $C$ such that for all $\phi \in \Phi$

$$\left\| \int_{\Gamma} \phi(x) d\mu(x) \right\| \leq C \left\| \hat{\phi} \right\|_{pq} \quad (1 < p, q < \infty)$$

then $\hat{\mu} \in (L^p, L^q)$.

**Proof.** Set $T(\phi) = \int \phi(x)d\mu(x)$. Then $T \in \Phi^*$, $\phi$ as in Proposition 10.3. So by our previous result there exists a unique $h$ in $(L^p, L^q) \hat{\text{(G)}}$ such that $T(\phi) = \int \phi(x)d\mu(x) = \int \phi(-\hat{\sigma})h(\hat{\sigma})d\hat{\sigma}$. This implies that $\mu \in M_T$ and $\hat{\mu} = h$ (Definition 6.1). By Theorem 6.23 we conclude that $\hat{\mu} \in \Phi$ and therefore $\Phi \in (L^p, L^q)$.

**Proof of Theorem 10.2.** We consider two cases.

Case 1) $p = 2$. Let $E$ be a compact subset of $G$ of positive measure with interior $\Omega$, and $\frac{p}{2} < q < 2$. Since $\hat{G}$ is noncompact and $\hat{\Omega}$ is nonempty, there exists $f$ in $(L^\infty, L^2) \hat{\text{(G)}}$ such that $\hat{f} = \hat{\mu}$ does not vanish on $\hat{\Omega}$ for all $g \in (L^1, L^q) \hat{\text{(G)}}$ (Theorem 9.4).

Let $\phi \in C_c(G)$ such that $\hat{\phi} = 1$ on $E$. Now $\hat{\phi} \in L^2$ as $f \in (L^\infty, L^2)$ and $(L^\infty, L^2) \subset L^2$. Since $\hat{\phi} \in (L^\infty, L^2)$ we have by Proposition 4.1 that $\hat{f}\phi \in (L^2, L^1) \hat{\text{(G)}}$. So $\hat{f}\phi \in L^2 M_1$ by (2.7). This implies that the inverse of the Fourier transform of $(f\phi)^*$ is equal to $\hat{f}\phi$. Therefore $(f\phi)^* \in (L^1, L^q) \hat{\text{(G)}}$ because $\hat{f} = \hat{f}\phi$ does vanish on $\hat{\Omega}$. So, there exists a function $\hat{f}\phi \in (L^2, L^1) \hat{\text{(G)}}$ such that $(f\phi)^* \neq (L^1, L^q) \hat{\text{(G)}}$. This means that...
(1) \((L^2, \ell^1)(\hat{G}) \not\subset (L^1, \ell^q)(\hat{G})\) for all \(1 < q < 2\).

Furthermore, we note that \((\ell^p)^* \not\subset (L^1, \ell^q)(E)\). Therefore

(2) \((L^2, \ell^1)(E) \not\subset (L^1, \ell^q)(E)\) for all \(q < 2\)

and compact \(E\) of positive measure.

Case 2) \(p < 2\). We want to prove that

(3) \((L^p, \ell^1)^* \subset (L^1, \ell^q)(\hat{G})\) for all \(1 \leq q < p\).

If \(1 < q < 2\), we know by (1) that (3) holds, because \(p < 2\) and this implies that \((L^2, \ell^1) \subset (L^p, \ell^1)\). So we will consider the case when \(2 \leq q < p\). Suppose not, that is, \((L^p, \ell^1)^* \subset (L^1, \ell^q)(\hat{G})\) for some \(2 \leq q < p\). By the Closed Graph Theorem (as in [32, Corollary p. 70]), the map \(T: (L^p, \ell^1) \to (L^1, \ell^q)(\hat{G})\) given by \(T(f) = \hat{f}\) is bounded. Indeed, let \(\{f_n\}\) be a sequence in \((L^p, \ell^1)\) such that \(f_n \to 0\). Then

\[
\lim ||f_n||_{L^p} = 0.
\]

Assume \(\hat{f}_n \to g\) in \((L^1, \ell^q)(\hat{G})\) and take \(\phi \in C_c\), since \(\phi \in (L^\infty, \ell^{q'}) \cap (L^1, \ell^p)\) we have that

\[
|< g, \phi >| = \lim |< \hat{f}_n, \phi >| (remember that \((L^\infty, \ell^{q'}) = (L^1, \ell^q)^*\)).
\]

But by Theorem 3.1

\[
|< \hat{f}_n, \phi >| = \int \hat{f}_n(\xi)\phi(\xi) d\xi \leq ||\hat{f}_n||_{\ell^{q'}} ||\phi||_{L^p}.
\]

So, by the Hausdorff–Young inequality for amalgams

\[
|< g, \phi >| = \lim |< \hat{f}_n, \phi >| \leq \lim ||\hat{f}_n||_{\ell^{q'}} ||\phi||_{L^p} \leq ||\phi||_{L^p} C_p \lim ||f_n||_{L^p} = 0.
\]

We conclude that \(< g, \phi > = 0\) for all \(\phi \in C_c\). Since any \(g\) in \((L^1, \ell^q)\) can be considered as an element of \(M^\prime_q\), \((C_0, \ell^{q'})^* = M_q\) and \(C_c\) is dense in \((C_0, \ell^{q'})\), we conclude that \(< g, f > = 0\) for all \(f\) in \((C_0, \ell^{q'})\) and this implies that \(g = 0\). Hence \(T\) is continuous.
Let $g \in (L^\infty, L^{q'})(\hat{G})$ and $\phi \in \Phi(\hat{G})$. Then

$$\langle g, \phi \rangle = |\int g(\hat{x})\phi(\hat{x})d\hat{x}| = |\int \hat{g}(\hat{x})\hat{\phi}(\hat{x})d\hat{x}| = |\int \hat{g}(\hat{x})T(\hat{\phi})(\hat{x})d\hat{x}| \leq ||g||_{L^{q'}}||\hat{\phi}||_{L^q} \leq ||g||_{L^{q'}}||T|| ||\hat{\phi}||_{L^p}.$$ 

By Proposition 10.4, $g \in (L^p, L^\infty)(\hat{G})$. This implies that $$(L^\infty, L^{q'})(\hat{G})^\vee \subset (L^p, L^\infty)(G).$$

We have given an alternative proof of this last inclusion based on ideas of Fournier [22, p. 269] in Remark 10.5.

Now, $\hat{G}$ noncompact implies $G$ nondiscrete and by Corollary 9.3, $(L^\infty, L^{q'})(\hat{G})^\vee \not\subset (L^p, L^\infty)(G)$ because $q' < 2$ and $p' > q = (q')'$. This contradiction shows (3).

From cases 1) and 2) we conclude that for $1 \leq p \leq 2$

$$\langle L^p, L^1 \rangle (G)^\vee \not\subset (L^1, L^q)(\hat{G}) \quad \text{for all } 1 \leq q < p'.$$

For $q \in [1, p')$ define the function $F$ on $(L^p, L^1)(G)$ by

$$F(f) = ||\hat{f}||_{L^q}.$$ 

By (4) $F$ takes the infinite value, $F(\alpha f) = \alpha F(f)$ for all nonnegative real $\alpha$ and all $f, g \in (L^p, L^1)$, $F(f - g) \leq F(f) + F(g)$.

Similarly to Theorem 9.2 these properties of $F$ imply that the set $V_\alpha = \{f \in (L^p, L^1) | F(f) > \alpha\}$ is dense in $(L^p, L^1)$ for all real $\alpha$.

Also, $F$ is lower semicontinuous. Indeed, for a finite subset $E$ of $J$, define the function $F_E$ on $(L^p, L^1)$ by

$$F_E(f) = \sum_{\alpha \in E} ||\hat{f}_\alpha||_{L^q}^q,$$

where $\hat{f}_\alpha = \hat{f} \chi_{L_\alpha}$.
Since \( \left\| f \right\|_{L^q} = \sum \left\| f_\alpha \right\|_{L^q} \), we have that \( \lim F_{E^n} = F^q \).

Note that \( \{ F_{E^n} \} \) is an increasing net of functions and hence

\[ F = \sup \{ F_{E^n} \mid E \subset J \text{ finite} \}. \]

\( 1 \leq q < p' \) implies that \( L^p(E) \subset L^q(E) \) for all \( \alpha \), so by the Hausdorff-Young inequality, we have that for all \( \alpha \)

\[ \left\| \hat{f}_\alpha \right\|_{L^q} \leq \left\| f_\alpha \right\|_{L^q} \leq \left\| f \right\|_{L^p} \leq \left\| f \right\|_{L^q} \]

Therefore \( F_{E^n}(f) \leq \left\| f \right\|_{L^q} \) where \( |E| \) is the cardinality of \( E \). This shows that each \( F_{E^n} \) is continuous and we conclude that \( F^q \) is lower semicontinuous, so is \( F \). Hence, the set

\[ \{ f \in (L^p, L^1) \mid F(f) = \infty \} = \bigcap \{ f \in (L^p, L^1) \mid F(f) > n \} \]

is a dense set of type \( C_\delta \). If \( \{ q_n \} \) is a strictly increasing sequence converging to \( p' \), then by Baire's theorem the set \( \{ f \in (L^p, L^1) \mid \left\| f \right\|_{L^{q_n}} = \infty \} \) is a dense set of type \( C_\delta \). Hence it is nonempty. Take \( f \) in this set; let \( q \in [1, p') \). Since \( f \in (L^p, L^1) \), \( \hat{f} \in (C_\delta, L^{p'}) \) (Theorem 5.4), if also \( \hat{f} \in (L^1, L^{q}) \) then by (2.5) \( \hat{f} \in (L^1, L^{q_n}) \) for all sufficiently large \( n \) and this contradicts the choice of \( f \). Therefore \( \hat{f} \notin (L^1, L^{q}) \) for all \( 1 \leq q < p' \) and this proves the theorem.

**Remark 10.5.** Let \( 1 \leq p < 2, 2 \leq q < p' \). If the map

\[ T : (L^p, L^1) \longrightarrow (L^1, L^q)(\hat{G}) \]

defined by \( Tf = \hat{f} \) is continuous then

\[ (L^\infty, L^q)' \subset (L^p', L^{q_n}). \]

Indeed, if \( T \) is continuous then its dual map

\[ T^* : (L^\infty, L^{p'}) \longrightarrow (L^p, L^\infty) \]

is also continuous [44, Theorem 4.10]. Let \( g \in L^\infty_c(\hat{G}) \) with support \( E \). Hence \( g \in L^p(E) \) and by Theorem 3.1 we have that for \( f \in (L^p, L^1)(\hat{G}) \).
\[ T^*(\overline{g})(f) = \int_G \overline{f(\overline{x})} \overline{g(\overline{x})} \, dx. \]

Since \( f \in L^1 \) by (2.7) we apply the Parseval's identity (as in [37, 31.48 a]) and we get
\[ T^*(\overline{g})(f) = \int_G f(x) \overline{g(x)} \, dx. \]

Hence \( T^*(\overline{g}) = \overline{g} \). Since \( T \) is linear \( T^*(g) \overset{\lambda}{=} \overline{g} \). So for all \( g \) in \( L^\infty_c(\hat{G}) \)
\[ \| g \|_{L^p} \leq \| T^* \| \| g \|_{L^q}. \]

Let \( g \in (L^\infty, L^{q'})(\hat{G}) \). Since \( L^\infty_c \) is dense in \( (L^\infty, L^{q'}) \) (Theorem 3.6) there exists a sequence \( \{g_n\} \) in \( L^\infty_c \) such that \( \lim g_n = g \) in \( (L^\infty, L^{q'}) \). Since \( q' < 2 \) and \( (L^\infty, L^{q'}) \subset (L^1, L^q) \), we have by the Hausdorff-Young inequality for amalgams that
\[ \| T^* g_n - \overline{g} \|_{L^{q'}} = 0. \]

Now, by (5) \( \{g_n\} \) is a Cauchy sequence in \( (L^{q'}, L^\infty) \). Therefore there exists \( h \in (L^{p'}, L^\infty) \) such that \( \lim \| g_n - h \|_{L^{p'}} = 0 \). This implies that \( \lim g_n = h \) in \( (L^q, L^\infty) \) by (2.4), as \( p' > q \). Hence \( h = \overline{g} \) and we conclude that \( (L^\infty, L^{q'}) \subset (L^{p'}, L^\infty) \), since \( g \) is arbitrary.

**COROLLARY 10.6.** If \( E \subset G \) is not locally null, \( \hat{G} \) is not compact and \( 1 \leq p \leq 2 \) then
\[ (L^p, L^1)(E) \subset \bigvee_{q \leq p'} (L^1, L^q)(\hat{G}) \]

**PROOF.** As in Theorem 9.2 it is enough to prove the corollary for compact sets of positive measure.

By (2) of the case 1) of Theorem 10.2.
\[ (L^p, L^1)(E) \cap \bigvee_{q \leq p} (L^1, L^q)(\hat{G}) \]

for all \( q < 2 \).

Using the same argument as in the case 2) of the proof of
Theorem 10.2 and Theorem 9.2 we have that

\[(L^p, L^1)(E)^* \ni (L^1, L^q)(G) \quad \text{for all } 2 \leq q < p'.\]

Again the function \( F \) on \((L^p, L^1)(E)\) defined by \( F(\xi) = \|\hat{\xi}\|_q \)
is lower semicontinuous and by a Baire's categorical argument we conclude \( (6) \).
§ 11. \( \hat{G} \) IS NEITHER COMPACT NOR DISCRETE

Finally we will generalize the following theorem [22, Theorem 2, c)].

**THEOREM 11.1.** If \( \hat{G} \) is neither compact nor discrete and \( 1 < p < 2 \) then

\[
L^p \subset \bigcup_{q \neq p'} L^q(\hat{G}).
\]

We will prove that under the same hypothesis

\[
L^p \subset \bigcup_{q \neq p'} (L^q, \infty) \cap (L^1, \ell^q).
\]

This improves the right side of Theorem 11.1, because \( L^q \) is a proper subspace of \( (L^q, \infty) \cap (L^1, \ell^q) \) for \( 1 < q < \infty \) (Theorem 2.4).

**THEOREM 11.2.** If \( \hat{G} \) is neither compact nor discrete then for \( 1 < p \leq 2 \)

\[
L^p \subset \bigcup_{q \neq p'} (L^q, \infty) \cap (L^1, \ell^q).
\]

**PROOF.** By corollary 9.3 there exists \( f \in (L^\infty, \ell^p) \) such that

(1) \( \hat{f} \subset \bigcup_{q > p'} (L^q, \infty) \).

By Theorem 10.2 there exists \( h \in (L^p, \ell^1) \) such that

(2) \( \hat{h} \subset \bigcup_{q < p'} (L^1, \ell^q) \).

We shall see that one of the three functions \( \hat{f}, \hat{h}, \hat{f} + \hat{h} \) in
(note that \(L^\infty, \ell^p\)) and \((\ell^p, \ell^1)\) are included in \(L^p\) by (2.5) and (2.6) does not belong to \(\bigcup_{q \neq p'} (L^q, \ell^\infty) \cap (L^1, \ell^q)\).

Suppose not, that is, \(\hat{f} + \hat{h} \in (L^{q_0}, \ell^\infty) \cap (L^1, \ell^{q_1})\), \(\hat{f} \in (L^{q_1}, \ell^\infty) \cap (L^1, \ell^{q_1})\) and \(\hat{h} \in (L^{q_2}, \ell^\infty) \cap (L^1, \ell^{q_2})\) for some \(q_0, q_1, q_2\) distinct from \(p'\).

Since \(\hat{f} \not\in (L^{q_1}, \ell^\infty)\) if \(q_1 > p'\) and \(\hat{h} \not\in (L^1, \ell^{q_2})\) if \(q_2 < p'\) we have that \(q_1 < p' < q_2\). So,

a) If \(p' < q_0 \leq q_2\) then \(\hat{h} \in (L^{q_2}, \ell^\infty) \subset (L^{q_0}, \ell^\infty)\). Hence
\[
\hat{f} = (\hat{f} + \hat{h}) - \hat{h} \in (L^{q_0}, \ell^\infty).
\]
This contradicts (1) as \(q_0 > p'\).

b) If \(q_1 < q_2 < q_0\) then \(\hat{f} + \hat{h} \in (L^{q_0}, \ell^\infty) \subset (L^{q_2}, \ell^\infty)\). Hence
\[
\hat{f} = (\hat{f} + \hat{h}) - \hat{h} \in (L^{q_2}, \ell^\infty).
\]
This contradicts (1) as \(q_2 > p'\).

c) If \(q_1 \leq q_0 < p'\) then \(\hat{f} \in (L^1, \ell^{q_1}) \subset (L^1, \ell^{q_0})\). Hence
\[
\hat{h} = (\hat{f} + \hat{h}) - \hat{f} \in (L^1, \ell^{q_0}).
\]
This contradicts (2) as \(q_0 < p'\).

d) If \(q_0 < q_1 < p'\) then \(\hat{f} + \hat{h} \in (L^1, \ell^{q_0}) \subset (L^1, \ell^{q_1})\). Hence
\[
\hat{h} = (\hat{f} + \hat{h}) - \hat{f} \in (L^1, \ell^{q_1}).
\]
This contradicts (2) as \(q_1 < p'\).

It follows from a) - d) that \(q_0 = p'\). This contradiction proves the theorem.\(\dagger\)
CHAPTER V

MULTIPLIERS

In this chapter we will characterize the multipliers from A to B, where A and B are any amalgam space or any space of unbounded measures of type q.

Specifically, for a continuous linear operator \( T: A \rightarrow B \) such that for all \( s \in G \), \( T \circ \tau_s = \tau_s \circ T \) where \( \tau_s \) is the translation operator on A and B respectively, defined in Definition 3.10, we want to find a "\( \mu \)" such that

\[ Tf = \mu f \quad \text{for all } f \in A. \]

In the particular case when \( A = B = L^1(G) \) it is known that such a \( \mu \) belongs to \( M_1(G) \). Moreover, we have the following theorem [40, Theorem 0.1.1 and Corollary 0.1.1].

**THEOREM 1.** If \( T: L^1(G) \rightarrow L^1(G) \) is a continuous linear operator, then the following are equivalent:

i) \( T \circ \tau_s = \tau_s \circ T \) for all \( s \in G \)

ii) \( T(f \ast g) = Tf \ast g \) for all \( f, g \in L^1(G) \)

iii) There exists a unique continuous bounded function \( \varphi \) on \( G \) such that \( \hat{Tf} = \hat{\varphi} \hat{f} \) for all \( f \in L^1(G) \)

iv) There exists a unique \( \mu \in M_1(G) \) such that \( Tf = \mu f \) for all \( f \in L^1(G) \).

Furthermore, the linear space of multipliers from \( L^1 \) to \( L^1 \) is isometric and linearly isometric with \( M_1 \).
1) and 2) say that \( T \) commutes with translations and convolution respectively while 3) justifies the name multiplier. So, the reason we pursue a characterization like (1) is because once we have found such a \( \mu \), in some cases, we shall be able to vindicate our choice of the word multiplier by taking the Fourier transform \( \hat{\varphi} \) of \( \mu \) and conclude that for all \( f \in A \), \( \hat{Tf} = \hat{\varphi} f \).

Currently there are different definitions of a multiplier according to the spaces involved. Indeed,

1) If \( A \) and \( B \) are translation invariant topological-linear spaces of functions on \( G \) (for all \( s \in G \) and all \( f \in A \), the function \( \tau_s f(t) = f(t - s) \) belongs to \( A \)) then a multiplier from \( A \) to \( B \) is a continuous linear operator \( T : A \to B \) such that \( T \) commutes with translations. That is, \( T \tau_s = \tau_s T \) for all \( s \in G \).

2) If \( A \) is a topological \( \mathbb{C} \)-algebra and \( B \) is a topological \( A \)-module then a multiplier from \( A \) to \( B \) is a continuous linear operator \( T : A \to B \) such that \( T \) commutes with convolution. That is, \( T(f \ast g) = T f \ast g \) for all \( f, g \in A \). (Note that * on the left denotes the operation on \( A \), and on the right the module operation on \( B \)).

3) If \( A, B \) are semisimple, commutative, Banach algebras then a multiplier from \( A \) to \( B \) is a function \( \varphi \) on the regular maximal ideal space of \( A \) such that \( \varphi \hat{x} \in B \) whenever \( \hat{x} \in \hat{A} \), where \( \hat{x} \) is the Gelfand transform of \( x \).

It is clear from Definition 3.10 that the first definition is meaningful for all amalgam spaces and all spaces \( M_q \). For this reason we have chosen this as our definition of a multiplier on amalgams and measure spaces \( M_q \).
However, the second definition is also meaningful when $A$ is any of the spaces $(C_0, l^1), (L^p, l^1) (1 \leq p \leq \infty)$ or $M_1$ and $B$ is any amalgam space or any measure space $M_q$, because in this case $A$ is a subalgebra of $L^1$ (§2.6 and Theorem 4.7) or $A = M_1$ and $B$ is both a $L^1$ and $M_1$-module (§4 p. 60).

So, in order to distinguish between these two definitions we will say that if $A$ and $B$ are as above and $T$ is as in the second definition then $T$ is a convolution multiplier, abbreviated, c-multiplier.

In general multipliers and c-multipliers are not the same [29, pp. 89 and 94]. Hence, whenever the definition of a c-multiplier makes sense, we will be interested in knowing the relation between these two concepts.

In this direction, we should observe that if a multiplier (c-multiplier) $T$ has the form (1) for some $u$ then by the properties of convolution $T$ is a c-multiplier (multiplier).

As in Theorem 1, once we characterize the multipliers for certain amalgam or measure spaces $A, B$, we will try to establish a linear isomorphism between the linear space of multipliers from $A$ to $B$ and some linear space $C$.

Since the $L^p(G)$ spaces are particular cases of amalgams (see (2.1)) it is natural for us to follow very closely the theory of multipliers from $L^p$ to $L^q$ and through generalizations try to develop and characterize a theory of multipliers for amalgams and $M_q$ spaces.

Our main source of information about the theory of multipliers for $L^p(G)$ spaces will be [40].
§ 12. **SPACES OF MULTIPLIERS**

Let $B$ be a linear space of functions on $G$. For $s \in G$ and $f \in B$, $	au_s f$ is the function on $G$ defined by $\tau_s f(t) = f(t - s)$. If $\tau_s f \in B$ for all $f \in B$ then $B$ is said to be translation invariant.

If $B$ is translation invariant then the linear operator

$$
\tau_s : B \rightarrow B, \quad f \mapsto \tau_s f
$$

is called a translation operator.

**DEFINITION 12.1.** Let $A$, $B$ be two translation invariant linear spaces. A **multiplier** $T$ from $A$ to $B$ is a continuous linear operator $T : A \rightarrow B$ which commutes with translations. That is, for all $s \in G$,

$$
T \tau_s = \tau_s T
$$

where $\tau_s$ is the translation operator on $A$ and $B$.

The linear space of multipliers from $A$ to $B$ will be denoted by $M(A, B)$.

Since $(L^\infty, L^q) = (L^1, L^q)^*$, $1 < q \leq \infty$ and $(L^\infty, x^1) = (L^1, c_0)^*$ (Theorem 3.1), $(L^\infty, L^q)$ can be endowed with the weak*-topology induced by $(L^1, L^{q'})$ if $1 < q \leq \infty$, $(L^1, c_0)$ if $q = 1$. We will write $(L^\infty, L^q)^w$ for this space. Similarly $(L^p, L^\infty)^w$, $1 < p \leq \infty$, $M_q^w$, $1 \leq q \leq \infty$ are the spaces $(L^p, L^{q'})$, $M_q$ endowed with the weak*-topology induced by $(L^p', L^1)$ and $(C_0, L^{q'})$ respectively.

By [40, Theorem D.4.1] the continuous linear functionals on $(L^\infty, L^q)^w$, $(L^p, L^\infty)^w$ and $M_q^w$ can be identified with $(L^1, L^{q'})$ if $1 < q \leq \infty$, $(L^1, c_0)$ if $q = 1$, $(L^p', L^1)$ and $(C_0, L^{q'})$ respectively by the formula

$$
(12.1) \quad < f, g > = \int_G f(-x) g(x) \, dx.
$$

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\[ f \in (L^\infty, L^q), \ g \in (L^1, L^{q'}) \quad 1 < q < \infty; \ f \in (L^\infty, L^1), \ g \in (L^1, c_0); \]
\[ f \in (L^p, L^\infty), \ g \in (L^{p'}, L^1) \quad 1 < p < \infty. \]
(12.2) \quad \langle \mu, f \rangle = \int_G f(-x) \, d\nu(x)
\mu \in M_q, \ f \in (C_0, L^{q'})

We should mention that this is one way to represent a continuous linear functional and we choose this because in this case
\[ \langle f, g \rangle = f*g(0), \ \langle \mu, f \rangle = \mu*f(0), \] (see Corollary 4.4 and Corollary 4.5).

PROPOSITION 12.2. Let \( A, B \) be any of the following spaces
\[
\begin{align*}
(L^p, L^q) &\quad 1 < p, q < \infty \\
(C_0, L^q) &\quad 1 < q < \infty \\
(L^p, c_0) &\quad 1 < p < \infty \\
(L^\infty, L^q)^w &\quad 1 < q < \infty \\
(L^p, L^\infty)^w &\quad 1 < p < \infty \\
M_q &\quad 1 < q < \infty
\end{align*}
\]

If, \( T \) is a multiplier from \( A \) to \( B \) then its adjoint \( T^* \) is a multiplier from \( B^* \) to \( A^* \).

PROOF. \( T: A \rightarrow B \) is a continuous linear operator and its adjoint \( T^* \) is a continuous linear operator \( T^*: B^* \rightarrow A^* \) defined by
\[ \langle f, T^* g \rangle = \langle T f, g \rangle \quad f \in A, \ g \in B^* \] [44, Theorem 4.10]

So, for \( s \in G, f \in A, g \in B^* \) we have that
\[ \langle f, T^*_s g \rangle = \langle T f, \tau_s g \rangle = \int_T f(-x) \tau_s g(x) \, dx = \int f(-x) g(x - s) \, dx \]
\[ = \int_T f(-x - s) g(x) \, dx = \int \tau_s T f(-x) g(x) \, dx \]
\[ = \langle \tau_s T f, g \rangle = \langle T \tau_s f, g \rangle = \langle \tau_s f, T^* g \rangle \]
\[ = \langle f, \tau_s T^* g \rangle. \]
Since this holds for any $t$ in $\Lambda$, we conclude that $T^g_s g = \tau_s g$ for all $g \in \mathcal{B}$. Hence $T^g_s \in \mathcal{M}(\mathcal{B}, \Lambda)$.

The next two results are generalizations of Lemma 3.5.1 and Theorem 5.2.5 of [40] and their proofs are based on Hörmander's original theorem for $\mathbb{R}^n$ [33] Theorem 1.1.

**Lemma 12.3.** Assume $G$ is noncompact. Then

i) If $f \in (L^p, \lambda^q)$, $1 \leq p, q < \infty$, then $\lim_{s \to \infty} \|f + \tau_s g\|_{pq} = 2^{1/q} \|f\|_{pq}$.

ii) If $f \in (L^\infty, \lambda^q)$ or $f \in (C_0, \lambda^q)$, $1 \leq q < \infty$, then $\lim_{s \to \infty} \|f + \tau_s f\|_{\omega q} = 2^{1/q} \|f\|_{\omega q}$.

iii) If $f \in (L^p, C_0)$, $1 < p \leq \infty$, or $f \in C_0$ then $\lim_{s \to \infty} \|f + \tau_s f\|_{p^\infty} = \|f\|_{p^\infty}$.

iv) If $u \in \Lambda_q$, $1 \leq q < \infty$, then $\lim_{s \to \infty} \|u + \tau_s u\|_q = 2^{1/q} \|u\|_q$.

**Proof.** If $g \in L_c^p(G)$, $E = \text{supp } g$ and $E = \cup(\Lambda \cap E + \phi)$ then for $s \notin E - E$, $g$ and $\tau_s g$ have disjoint support. So

$$\int_{K_\alpha \setminus K_\alpha \cap (E + s)}|g + \tau_s g|_p^p = \int_{K_\alpha \cap (E + s)}|g + \tau_s g|_p^p = \int_{K_\alpha \cap E} |g + \tau_s g|_p^p + \int_{K_\alpha \setminus (s + E)} |g + \tau_s g|_p^p$$

and

$$\sup_{K_\alpha} |g + \tau_s g| = \max \left( \sup_{K_\alpha \cap E} |g + \tau_s g|, \sup_{K_\alpha \setminus (s + E)} |g + \tau_s g| \right)$$

for $s \notin E - E$, $g$ and $\tau_s g$ have disjoint support. So
By the definition of the norm \( \| \cdot \|_{pq} \) it follows
\begin{align*}
(1) \quad \| g + \tau_s g \|_{pq} = 2^{1/q} \| g \|_{pq} & \quad 1 \leq p \leq \infty, \ 1 \leq q < \infty \\
(2) \quad \| g + \tau_s g \|_{p\infty} = \| g \|_{p\infty} & \quad 1 \leq p \leq \infty
\end{align*}

Now, if \( \mu \in M_q \) and \( \nu \in M_c \), \( \nu(A) = \mu(A \cap E) \), \( E \in \mathcal{G} \) compact (see Definition 3.5) then for \( s \in E - E + L - L \)
\[ (\nu + \tau_s \nu)^\theta(t) = \begin{cases} 
\nu^\theta(t) & t \in E - L \\
\tau_s \nu^\theta(t) & t \in s + E - L \\
0 & \text{otherwise}
\end{cases} \]
(see Theorem 1.21).

To see this we consider two cases.

Case 1) \( \nu \) is a real valued measure. By the decomposition of \( \nu + \tau_s \nu \)
(p.117) we have that
\[ (\nu + \tau_s \nu)^\theta(t) = |\nu + \tau_s \nu|(t + L) = (\nu + \tau_s \nu)^+(t + L) + (\nu + \tau_s \nu)^-(t + L) \]
where \( (\nu + \tau_s \nu)^+ = \max(\nu + \tau_s \nu, 0) \) and \( (\nu + \tau_s \nu)^- = \max(-\nu - \tau_s \nu, 0) \).

Now, \( (\nu + \tau_s \nu)(t + L) = \mu((t + L) \cap E) + \mu((-s + t + L) \cap E) \),
since \( t \in E - E + L - L, \ (t + L) \cap E = \emptyset \) and \( (-s + t + L) \cap E = \emptyset \) if \( t \in E - L; \ (t + L) \cap E = \emptyset \) and \( (-s + t + L) \cap E = \emptyset \) if \( t \in s + E - L; \)
\( (t + L) \cap E = (-s + t + L) \cap E = \emptyset \) otherwise. Hence
\[ (\nu + \tau_s \nu)^\theta(t) = \begin{cases} 
|\nu|(t + L) & t \in E - L \\
|\nu|(-s + t + L) & t \in s + E - L \\
0 & \text{otherwise}
\end{cases} \]

Therefore (3) holds.

Case 2) \( \nu \) is a complex valued measure. Then \( \nu = \nu_1 + i\nu_2 \), \( \nu_1 \) real valued measure \( i = 1, 2 \). So, \( \nu + \tau_s \nu = \nu_1 + \tau_s \nu_1 + i(\nu_2 + \tau_s \nu_2) \) and
\[ |v + \tau_s v| = |v_1 + \tau_s v_1| + |v_2 + \tau_s v_2|.\]

Hence \((v + \tau_s v)^\# = (v_1 + \tau_s v_1)^\# + (v_2 + \tau_s v_2)^\#\). From case \(1\) we have that

\[
(v + \tau_s v)^\#(t) = \begin{cases} 
v_1^\#(t) + v_2^\#(t) & t \in E - L \\
\tau_s(v_1^\# + v_2^\#)(t) & t \in s + E - L \\
0 & \text{otherwise} \\
\end{cases}
\]

\[
= \begin{cases} 
\tau_s v^\#(t) & t \in E - L \\
0 & \text{otherwise} \\
\end{cases}
\]

By \((3)\) we have that for \(1 < q < \infty\),

\[
\int_G (v + \tau_s v)^\#(t) \, dt = \int_{E-L} (v + \tau_s v)^\#(t) \, dt + \int_{s+E-L} (v + \tau_s v)^\#(t) \, dt
\]

\[
= \int_{E-L} v^\#(t) \, dt + \int_{s+E-L} \tau_s v^\#(t) \, dt
\]

\[
= \int_{E-L} v^\#(t) \, dt + \int_{E-L} v^\#(t) \, dt
\]

\[
= 2 \int_{E-L} v^\#(t) \, dt
\]

This implies that

\[
(4) \quad \| v + \tau_s v \|^\#_q = \frac{1}{q} \| v \|^\#_q.
\]

Let \(\varepsilon > 0\). If \(f \in L^p, \ell^q\) \(1 \leq p \leq \infty, 1 \leq q < \infty\), then there exists \(g\) in \(L^p_c\) such that \(\|f - g\|_{pq} < \varepsilon/3(2^d)\) (Theorem 3.6).
For \( s \notin E - E \), \( E \) as in the beginning of the proof, we have by (1) and Theorem 3.11 that
\[
\| \| f + \tau_s f \|_{pq} - 2^{1/q} \| f \|_{pq} \| \leq \| f + \tau_s f \|_{pq} - \| g + \tau_s g \|_{pq} + 2^{1/q} \| g \|_{pq} - 2^{1/q} \| f \|_{pq} \|
\]
\[
\leq \| f - g \|_{pq} + \| \tau_s f - \tau_s g \|_{pq} + 2^{1/q} \| g - f \|_{pq} \|
\]
\[
< \varepsilon/3(2^a) + 2^a(\varepsilon/3(2^a)) + 2^{1/q}(\varepsilon/3(2^a)) < \varepsilon. \]

This proves i).

Similarly if \( p = \infty \) we have by (1) and Theorem 3.11 that
\[
\| \| f + \tau_s f \|_{\infty} - 2^{1/q} \| f \|_{\infty} \| \leq \| f + \tau_s f \|_{\infty} - \| g + \tau_s g \|_{\infty} + 2^{1/q} \| g \|_{\infty} - 2^{1/q} \| f \|_{\infty} \|
\]
\[
\leq \| f - g \|_{\infty} + \| \tau_s f - \tau_s g \|_{\infty} + 2^{1/q} \| g - f \|_{\infty} < \varepsilon.
\]

Therefore ii) holds for \( f \in (L^\infty, \ell^q) \) and hence for \( f \in (C_c, \ell^q) \).

The proof of iii) is the same, but taking \( g \in C_c \) and using (2). The case \( f \in C_c \) is [40, Lemma 3.5.1].

Finally to prove iv) we keep in mind that the norms \( \| \cdot \|_q \) and \( \| \cdot \|_{\ell^q} \) are equivalent.

If \( \mu \in M_q \), then by Theorem 3.6 there exists \( \nu \in M^q_c \) such that
\[
\nu(A) = \mu(A \cap E), \ E \subset G \text{ compact and } \| \mu - \nu \|_q < \varepsilon/3(2^a) \text{ where } \varepsilon > 0
\]
is given.

As before, we have by (4) that for all \( s \notin E - E + L - L \)
\[
\| \| u - \tau_s u \|_q - 2^{1/q} \| u \|_q \| < \varepsilon. \]
THEOREM 12.4. If \( G \) is noncompact then the following linear spaces of multipliers are trivial. That is, the zero multiplier is the only element in these spaces.

i) \( M((L^p, L^q), (L^r, L^s)) \quad 1 \leq r, p \leq \infty, 1 \leq s < q < \infty \)

ii) \( M((L^p, L^q), (C_0, L^s)) \quad 1 \leq p \leq \infty, 1 \leq s < q < \infty \)

iii) \( M((C_0, L^q), (L^r, L^s)) \quad 1 \leq r \leq \infty, 1 \leq s < q < \infty \)

iv) \( M((C_0, L^q), (C_0, L^s)) \quad 1 \leq s < q < \infty \)

v) \( M((L^p, C_0), (L^r, L^s)) \quad 1 \leq p, r \leq \infty, 1 \leq s < \infty \)

vi) \( M((L^p, C_0), (C_0, L^s)) \quad 1 \leq p \leq \infty, 1 \leq s < \infty \)

vii) \( M((M_q, M_s)) \quad 1 \leq s < q < \infty \)

viii) \( M(M_q, (L^r, L^s)) \quad 1 \leq r \leq \infty, 1 \leq s < q < \infty \)

ix) \( M(M_q, (C_0, L^s)) \quad 1 \leq s < q < \infty \)

x) \( M(L^p, L^q), M_s \quad 1 \leq p \leq \infty, 1 \leq s < q < \infty \)

xi) \( M((C_0, L^q), M_s) \quad 1 \leq s < q < \infty \)

xii) \( M((L^p, C_0), M_s) \quad 1 \leq p \leq \infty, 1 \leq s < \infty \)

xiii) \( M((L^\infty, L^q)_w, (L^r, L^s)) \quad 1 \leq p < \infty, 1 \leq s < q < \infty \)

xiv) \( M((L^\infty, L^q)_w, (C_0, L^s)) \quad 1 \leq s < q < \infty \)

xv) \( M((L^\infty, L^q)_w, (L^\infty, L^s)_w) \quad 1 \leq s < q < \infty \)

xvi) \( M((L^p, L^q), (L^\infty, L^s)_w) \quad 1 \leq p \leq \infty, 1 \leq s < \infty \)

xvii) \( M((C_0, L^q), (L^\infty, L^s)_w) \quad 1 \leq s < q \leq \infty \) or \( 1 \leq s < q < \infty \)

xviii) \( M((L^p, c_0), (L^\infty, L^s)_w) \quad 1 \leq p < \infty, 1 \leq s < \infty \)

xix) \( M((L^p, L^\infty)_w, (L^r, L^s)) \quad 1 \leq p \leq \infty, 1 \leq r \leq \infty, 1 \leq s < \infty \)

xx) \( M((L^p, L^\infty)_w, (C_0, L^s)) \quad 1 \leq p \leq \infty, 1 \leq s < \infty \).

**Proof.** The proof of each of the first twelve cases are the same; so we will prove only i).
Suppose $T \in M((L^P, \ell^q), (L^r, \ell^s))$ and $T \neq 0$. Then for $f \in (L^P, \ell^q)$ and $s \in G$ we have that

$$||Tf + \tau_sTf||_{rs} = ||Tf + T\tau_sf||_{rs} \leq ||f|| \cdot ||\tau_s f||_{pq}.$$

Taking the limit on both sides we have by Lemma 12.3 that

$$2^{1/s} ||Tf||_{rs} \leq 2^{1/q} ||T|| \cdot ||f||_{pq}.$$

This implies that $||T|| < 2^{1/q - 1/s} ||T||$ and we have a contradiction because $2^{1/q - 1/s}$ is strictly less than one. Therefore $T \equiv 0$.

The proofs of the remaining cases are similar to each other. So we will prove only xiii).

Let $T \in M((L^\infty, \ell^q), (L^r, \ell^s))$. By Proposition 12.2 its adjoint $T^*$ belongs to $M((L^{r'}, \ell^{s'}), (L^1, \ell^{q'}))$. Since $1 < q' < s' < \infty$, $T^* \equiv 0$ by case i). This implies that $T \equiv 0$.

The next theorem corresponds to Theorem 5.2.1 of [40] and the proof is the same.

THEOREM 12.5. Let $1 < p, q, r, s < \infty$. $M((L^P, \ell^q), (L^r, \ell^s))$ is isometrically isomorphic to $M((L^{r'}, \ell^{s'}), (L^P, \ell^q'))$.

PROOF. By Proposition 12.2, $T \rightarrow T^*$ defines a linear map from $M((L^P, \ell^q), (L^r, \ell^s))$ into $M((L^{r'}, \ell^{s'}), (L^P, \ell^q'))$. Moreover since $||T|| = ||T^*||$ [44, Theorem 4.10] and $(L^P, \ell^q)$, $(L^r, \ell^s)$ are reflexive (Corollary 3.3) the map is continuous and onto.
Similarly to Theorem 5.3.1 of [40], we will apply the Riesz-Thorin Theorem for amalgams (Theorem 5.6) to prove the next result.

**Theorem 12.6.** Let \( 1 \leq p_i, q_i, r_i, s_i \leq \infty \), \( i = 1, 2 \). Suppose that, for some \( 0 < \theta < 1 \),

\[
\frac{1}{p} = \frac{1 - \theta}{p_1} + \frac{\theta}{p_2} ; \quad \frac{1}{q} = \frac{1 - \theta}{q_1} + \frac{\theta}{q_2} ; \quad \frac{1}{r} = \frac{1 - \theta}{r_1} + \frac{\theta}{r_2} ; \quad \frac{1}{s} = \frac{1 - \theta}{s_1} + \frac{\theta}{s_2}.
\]

If \( T \in \mathcal{M}((L^{p_i}, L^{q_i}),(L^{r_i}, L^{s_i})) \), then \( T \) defines a unique element of

\[
\mathcal{M}(L^p, L^q) \quad \text{if} \quad 1 \leq p < \infty, 1 \leq q < \infty
\]

\[
\mathcal{M}((L^p, L^q), \mathbb{C}^n) \quad \text{if} \quad 1 \leq p < \infty, q = \infty
\]

\[
\mathcal{M}((L^p, L^q), (L^r, L^s)) \quad \text{if} \quad 1 \leq p < \infty, q = \infty
\]

\[
\mathcal{M}(C_0, (L^r, L^s)) \quad \text{if} \quad p = q = \infty.
\]

**Proof.** Let \( \mathcal{L}(m) \) be as in §5 p.67. Since \( \mathcal{L}(m) \subset (L^{p_i}, L^{q_i}) \), \( i = 1, 2 \), \( T \) restricted to \( \mathcal{L}(m) \) is a continuous operator from \( \mathcal{L}(m) \) to \( (L^{r_i}, L^{s_i}) \), \( i = 1, 2 \), which commutes with translations. So, if \( ||T||_i \), \( i = 1, 2 \), is the norm of \( T \) in \( \mathcal{M}((L^{p_i}, L^{q_i}),(L^{r_i}, L^{s_i})) \) then for all \( x \in \mathcal{L}(m) \)

\[
||Tx||_{r_is_i} \leq ||T||_i ||x||_{p_i q_i}, \quad i = 1, 2.
\]

By the Riesz-Thorin Theorem, \( T \mathcal{L}(m) \subset (L^r, L^s) \) and for all \( x \in \mathcal{L}(m) \)

\[
||Tx||_{rs} \leq ||T||_1 \theta ||T||_2 \frac{1 - \theta}{pq} ||x||_{pq} \quad \text{if} \quad 1 \leq r \leq \infty, 1 \leq s \leq \infty
\]

\[
||Tx||_{rs} \leq ||T||_1 \theta ||T||_2 \frac{1 - \theta}{2^r} ||x||_{pq} \quad \text{if} \quad r = 1, 1 \leq s \leq \infty.
\]

Therefore \( T \) restricted to \( \mathcal{L}(m) \) defines a continuous linear
operator from \( (\mathcal{L}(m), \| \cdot \|_{pq}) \) to \((L^r, \ell^s)\) which commutes with translations. Since \( \mathcal{L}(m) \) is dense in \((L^p, \ell^q)\) if \(1 \leq p \leq \infty, 1 \leq q \leq \infty\) (Remark 5.5), \((L^p, \ell^q)^W\) if \(1 \leq p < \infty, (L^p, c_0)\) if \(1 \leq p < \infty, \) and \(c_0, T\) has a unique continuous extension, also called \(T,\) of the same norm on these spaces.

By the continuity of \(\tau_s\) on \((L^p, \ell^q), 1 \leq p, q \leq \infty\) (Theorem 3.11) and on \((L^p, \ell^q)^W\), we conclude that \(T\) is a multiplier.

To see that \(\tau_s\) is in fact continuous on \((L^p, \ell^q)^W,\) take \(\{f_n\}, f \in (L^p, \ell^q)\) such that \(\lim f_n = f \) in \((L^p, \ell^q)^W.\)

Let \(g \in (L^p', \ell^q)\) and \(\varepsilon > 0.\) Since \(\tau_s g \in (L^p', \ell^q),\)
\[ |\langle \tau_s g, f_n - f \rangle| < \varepsilon \text{ for all } n \geq N.\] This implies that, for \(n > N,\)
\[ |\langle g, \tau_s f_n - \tau_s f \rangle| < \varepsilon.\] Therefore the translation operator \(\tau_s\) on \((L^p, \ell^q)^W\) is continuous.*

**COROLLARY 12.7.** \(M((L^p, \ell^q), (L^p, \ell^q)) \subset M(L^2, L^2), 1 \leq p, q \leq 2.\)

**PROOF.** Let \(T \in M((L^p, \ell^q), (L^p, \ell^q)).\) By Theorem 12.5 \(T\) belongs to \(M((L^p, \ell^q), (L^p, \ell^q)).\) Applying Theorem 12.6 with \(p_1 = r_1 = p, q_1 = s_1 = q, p_2 = r_2 = p', q_2 = s_2 = q',\) we have that for \(\theta = 1/2; 1/2 = (1 - \theta)/p + \theta/p' = (1 - \theta)/q + \theta/q'.\)

Therefore \(T\) defines a unique \(T\) in \(M(L^2, L^2).\)

**DEFINITION 12.8.** Let \(A\) be a Banach algebra and \(B\) be a Banach \(A\)-module. A continuous linear operator \(T: A \rightarrow B\) is a \(c\)-multiplier from \(A\) to \(B\) if \(T\) commutes with convolution. That is, for all \(f, g \in A,\)
\[T(f \ast g) = Tf \ast g.\]
The linear space of multipliers from A to B will be denoted by $c-N(A,B)$.

**Theorem 12.9.** Let $A$ be any of the spaces $(C_0, \ell^1)$, $(L^\infty, \ell^1)^w$, $(L^p, \ell^1)$ ($1 \leq p < \infty$), $M_1^w$; and let $B$ be any of the spaces $(L^p, \ell^q)$ ($1 \leq p, q < \infty$), $(C_0, \ell^q)$ ($1 \leq q < \infty$), $(L^p, \ell^q)$ ($1 \leq p < \infty$), $(L^\infty, \ell^q)^w$, $(L^p, \ell^\infty)^w$ ($1 < p \leq \infty$), $M_q^w$ ($1 \leq q < \infty$), $(L^p, \ell^q)^w$ ($1 \leq p, q < \infty$). If $T$ is a linear operator from $A$ to $B$ such that $T$ commutes with translations then $T$ commutes with convolution.

**Proof.** Note that $A^*$ is $M_\infty$, $(L^1, c_0)$, $(L^p, \ell^\infty)$ or $(C_0, \ell^\infty)$ and $B^*$ is either an amalgam space or a measure space $M_q(1 \leq q \leq \infty)$.

Suppose $A$, $B^*$ are amalgam spaces. Let $T^*: B^* \rightarrow A^*$ be the adjoint of $T$, $f$, $g$ in $A$ and $h$ in $B^*$. By Theorem 3.1 or (12.1)

$$\int Tf(t) h(-t) \, dt = \langle T^*f, h \rangle = \langle f, T^*h \rangle = \int f(t) T^*(-t) \, dt.$$ 

So, we have that

$$\langle T^*f g, h \rangle = \int T^*f g(t) h(-t) \, dt = \int \int Tf(t - s) g(s) \, ds \, h(-t) \, dt$$

$$= \int g(s) \int T^s f(t) h(-t) \, dt \, ds$$

$$= \int g(s) \int T^s f(t) h(-t) \, dt \, ds$$

$$= \int g(s) \int T^s f(t) T^*h'(t) \, dt \, ds$$

$$= \int f(t - s) \int g(s) \, ds \, T^*h'(t) \, dt$$

$$= \int f^* g(t) \, T^*h'(t) \, dt = \int T(f^* g)(t) \, h(-t) \, dt$$

$$= \langle T(f^* g), h \rangle.$$
We can apply Fubini's theorem because \( f, g \) are in \( A \), hence \( g \) is in \( L^1 \) and \( T^* h \) is in \( A^* \).

Since this holds for all \( h \in B^* \) we conclude by the Hahn-Banach theorem that \( Tf \ast g = T(f \ast g) \) for all \( f, g \in A \).

The proof of the remaining cases is similar.
§ 13. MULTIPLIERS FROM $L^1$ TO AMALGAM SPACES AND SPACES OF MEASURES $M_q$

The multipliers which have the most satisfactory characterization are those from $L^1$ to amalgam spaces and spaces of measures $M_q$. This is so because of the nature of the algebra $L^1$.

We will consider the cases: c-multipliers from $L^1$ to $(L^p, L^q)$, $1 \leq p \leq \infty$, c-multipliers from $L^1$ to $(L^p, L^q)$, $1 < p < \infty$, and c-multipliers from $L^1$ to $(L^q, L^q)$, $1 < q \leq \infty$.

Our first theorem is an extension of Theorem 3.11 of [40] first introduced by R. E. Edwards [21, Theorem 1].

THEOREM 13.1. Let $B$ be any of the spaces $(L^p, L^q)$, $1 < p \leq \infty$, $1 \leq q \leq \infty$, $M_q$. If $T \in c-M(L^1, B)$ then there exists a unique $u \in B$ such that $Tf = u * f$ for all $f \in L^1(G)$. Hence $c-M(L^1, B) \subset M(L^1, B)$.

PROOF. Let $A$ be $(L^p', L^q')$ if $B = (L^p, L^q)$, $1 < q \leq \infty$, $(L^p, c_0)$ if $B = (L^p, L^1)$, $1 < p \leq \infty$, $(C_0, L^q)$ if $B = M_q$, $1 \leq q \leq \infty$.

Then $A^* = B$ and $C_C$ is dense in $A$ (Theorem 3.7). Let $(\phi_U)$ be the a.e. in $L^1(G)$ defined in §7. So for $f \in L^1$ we have that

$$||Tf - T\phi_U * f||_B = ||Tf - T(\phi_U * f)||_B = ||T||\left(\sum ||\phi_U||_B^2\right)^{1/2} \leq ||T|| \cdot ||f - \phi_U * f||_1.$$ (1)

This shows that

$$\lim_{U} T\phi_U * f = Tf \quad \text{in } B.$$ (2)

On the other hand, $||T\phi_U||_B \leq ||T||\sum ||\phi_U||_B = ||T||$ for all $U$. Therefore $(T\phi_U)$ lies in a norm-bounded subset of $B = A^*$. By Alaoglu's
Theorem there exists a subnet \( \{ T \Phi_v \} \) of \( \{ T \Phi U \} \) and \( \mu \in B \) such that \( \{ T \Phi_v \} \) converges to \( \mu \) in the weak*-topology \( \sigma(B, A) \). That is, for each \( h \in A \)

\[
\lim_{V \to 0} \langle h, T \Phi_v \rangle = \langle h, \mu \rangle.
\]

Now, by (1) and the fact that \( C \subset A \subset B^* \) we have that for all \( f, g \) in \( C \)

\[
\lim_{V \to 0} \langle g, T \Phi_v \ast f \rangle = \langle g, T f \rangle.
\]

This together with (2) implies that for all \( f, g \) in \( C \)

\[
\langle g, T f \rangle = \lim_{V \to 0} \langle g, T \Phi_v \ast f \rangle = \lim_{V \to 0} g^* (T \Phi_v \ast f)(0) = \lim_{V \to 0} T \Phi_v \ast (f \ast g)(0)
\]

\[
= \langle f \ast g, T \Phi_v \rangle = \langle f \ast g, \mu \rangle = \langle g, \mu \ast f \rangle.
\]

Therefore \( T f = \mu \ast f \) for all \( f \in C \), because \( C \) is dense in \( A^* \).

Then \( T f = \mu \ast f \) for all \( f \in L^1 \) because \( C \) is dense in \( L^1 \) and convolution with \( \mu \) and \( T \) are continuous linear transformations from \( L^1 \) to \( B \) (Theorem 4.7).

Suppose that \( \mu \ast f = 0 \) for all \( f \in L^1 \). In particular for all \( f \in C \), \( \langle f, \mu \rangle = \mu \ast f(0) \) for all \( f \in L^1 \). In particular for all \( f \in C \), \( \langle f, \mu \rangle = \mu \ast f(0) \). This implies that \( \mu = 0 \) since \( \mu \in A^* \) and \( C \) is dense in \( A \). Hence \( \mu \) is unique.

Finally, if \( f \in L^1 \) and \( s, t \in G \) then for \( T \in c_{\mathcal{M}}(L^1, B) \)

\[
\tau_s (T f)(t) = T f(t - s) = \mu \ast f(t - s) = \int f(t - s - x) \, d\mu(x)
\]

\[
= \int \tau_s f(t - x) \, d\mu(x) = \mu \ast \tau_s f(t) = T(\tau_s f)(t).
\]

Therefore \( T \) commutes with translations.
COROLLARY 13.2. Let $1 < p < \omega$, $1 < q < \omega$. If $T : L^1 \to (L^p, \ell^q)$ is a bounded linear operator then the following are equivalent:

i) $T \tau_s = \tau_s T$ for all $s \in G$.

ii) $T(f \ast g) = Tf \ast g$ for all $f, g \in L^1$.

iii) There exists a unique $\mu \in (L^p, \ell^q)$ such that $Tf = \mu \ast f$ for all $f \in L^1$.

Hence $c-M(L^1, (L^p, \ell^q)) = M(L^1, (L^p, \ell^q))$.

PROOF. i) implies ii) follows from Theorem 12.9. By Theorem 13.1 ii) implies iii) implies i).†

Theorem 3.1.1 of [40] says that $T$ belongs to $M(L^1, L^\infty)$ iff there exists a unique $\mu \in L^\infty$ such that $Tf = \mu \ast f$ for all $f \in L^1$.

This together with Theorem 13.1 and the properties of convolution implies the following result.

THEOREM 13.3. Let $T : L^1 \to L^\infty$ be a bounded linear operator. Then the following are equivalent:

i) $T \tau_s = \tau_s T$ for all $s \in G$.

ii) $T(f \ast g) = T(f \ast g)$ for all $f, g \in L^1$.

iii) There exists a unique $\mu \in L^\infty$ such that $Tf = \mu \ast f$ for all $f \in L^1$.

Hence $c-M(L^1, L^\infty) = M(L^1, L^\infty)$.

Now we are able to characterize the multipliers from $L^1$ to $(L^p, \ell^1)$, $1 < p < \omega$, in terms of bounded functions on $G$ and establish an isometric algebra isomorphism between $c-M(L^1, (L^p, \ell^1))$ and the Segal algebra $(L^p, \ell^1)$# (see Theorems 1.21 and 4.16).
In [46] is proved that a Segal algebra $S$ is a semisimple, regular, commutative, Banach algebra with maximal ideal space homeomorphic to $\hat{G}$, such that the Gelfand transform is the Fourier transform restricted to $S$. Hence, for $f, g$ in $S$, $||\hat{f}||_\infty \leq ||f||_1$ and if $\hat{f}(\xi) = \hat{g}(\xi)$ for all $\xi \in \hat{G}$ then $f = g$.

Theorem 13.4. Let $T: \mathcal{L}^1(\mathbb{R}) \to (L^p, \mathcal{L}^1)$, $1 < p < \infty$, be a linear operator. Then the following are equivalent:

i) $T \in \mathcal{M}(L^1, (L^p, \mathcal{L}^1))$.

ii) There exists a unique $\mu \in (L^p, \mathcal{L}^1)$ such that $Tf = \mu \ast f$ for all $f \in L^1$.

iii) There exists a unique $\psi \in (C_0, \mathcal{L}^p)(\hat{G})$ if $1 < p \leq 2$, or in $(C_0, \mathcal{L}^2)$ if $2 < p < \infty$, such that $(Tf)^\sim = \psi \hat{f}$ for all $f \in L^1$.

The correspondence between $T$ and $\mu$ establishes an isometric algebra isomorphism from $\mathcal{M}(L^1, (L^p, \mathcal{L}^1))$ onto $(L^p, \mathcal{L}^1)^\#$.

**Proof.** i) implies ii) implies iii) follow from Corollary 13.2 and the properties of the Fourier transform on $(L^p, \mathcal{L}^1)$ (Theorem 5.4, Remark 5.8, Proposition 6.14).

Suppose iii).

Since $(\tau_s f)^\sim = [s, \cdot] \hat{f}$ for all $s \in G$ and $f \in L^1$ we have that $(T \tau_s f)^\sim = \psi(\tau_s f)^\sim = [s, \cdot] \psi \hat{f} = (\tau_s T f)$. Therefore $T \tau_s = \tau_s T$. This means that $T$ commutes with translations.

Let $\{f_n\}, f$ be in $L^1$ such that $\lim ||f_n - f||_1 = 0$ and $\lim ||Tf_n - g||_p = 0$.

So by the previous observation and (2.4) we have that
\[ \| (Tf) - \hat{g} \|_\infty \leq \| (Tf_n) - (Tf_n)^* \|_\infty + \| (Tf_n)^* - \hat{g} \|_\infty \]
\[ \leq \| \varphi (f - f_n) \|_\infty + \| Tf_n - g \|_1 \]
\[ \leq \| \varphi \|_\infty \| f - f_n \|_1 + \| Tf_n - g \|_{p_1}. \]

This implies that \( (Tf) = \hat{g} \). Hence \( Tf = g \) and by the Closed Graph Theorem we conclude that \( T \) is continuous. Therefore \( T \) belongs to \( M(L^1, (L^p, l^1)) \).

Since \( M(L^1, (L^p, l^1))^* = M(L^1, (L^p, l^1)^\#) \) given \( \mu \in (L^p, l^1)^\# \) the linear operator \( T \) defined by \( Tf = \mu * f \) (\( f \in L^1 \)) belongs to \( M(L^1, (L^p, l^1)^\#) \) and by Proposition 4.17
\[ \| Tf \|_{p_1}^\# = \| \mu * f \|_{p_1}^\# \leq \| f \|_1 \| \mu \|_{p_1}^\# \]
for all \( f \in L^1 \).

This implies that
\[ \| T \| \leq \| \mu \|_{p_1}^\#. \]

On the other hand, if \( \{ e_n \} \) is the a. i. in \( (L^p, l^1)^\# \) as in Proposition 4.17 ii) then \( \lim \| e_n * \mu - \mu \|_{p_1}^\# = 0 \). So given \( \varepsilon > 0 \) there exits an \( e_n \) such that \( \| e_n * \mu \|_{p_1}^\# > \| \mu \|_{p_1}^\# - \varepsilon \). Since \( e_n \in L^1 \) and \( \| e_n \|_1 \leq \varepsilon \) we conclude that \( \| T \| = \| \mu \|_{p_1}^\#. \)

REMARK 13.5. Theorem 13.4 is a particular case of a more general result proved by Rieffel [48]. This was restated in terms of absolutely continuous Banach \( L^1 \)-modules by Gulick, Llinas, and van Rooij [31, Theorem 5.2]. This says that if \( B \) is an absolutely continuous \( L^1 \)-module then the relation \( \mu \mapsto T_\mu, \ T_\mu f = \mu * f \) (\( f \in L^1 \)) establishes an isometric linear homeomorphism from \( \overline{B}^* \) onto \( c\overline{M}(L^1, B^*) \). Indeed, \( (L^p, l^1) \) is the dual of the absolutely continuous \( L^1 \)-module \( (L^p, c_0) \) (Proposition 4.13).
We will make use of the work done by Burnham and Goldberg [9] about $c$-multipliers from $L^1$ to Segal algebras to characterize $c$-$M(L^1, (G, L^1))$.

In what follows $S$ will be a Segal algebra (Definition 4.15) and $\|f\|_S$ will be the norm of $f$ in $S$.

**DEFINITION 13.6.** [14, Definition 3]. The relative completion $\tilde{S}$ of $S$ is the set of all $f \in L^1$ such that

$$f \in \bigcup_{x > 0} B_S(x)^1$$

where $B_S(x) = \{f \in S \mid \|f\|_S \leq x\}$ and $E^1$ is the closure of $E$ in $L^1$.

Thus, $f \in \tilde{S}$ iff there exists a sequence $(f_n)$ in $S$ such that

$$\sup_n \|f_n\|_S \leq x < \infty \quad \text{and} \quad \|f_n - f\|_1 \to 0.$$  

For $f \in \tilde{S}$, we define $\|f\| = \inf \{x \mid f \in B_S(x)^1\}$. Then $(S, |||\cdot|||)$ is a Banach algebra, $S$ is a closed ideal of $\tilde{S}$ and the embedding of $(S, |||\cdot|||)$ into $(\tilde{S}, |||\cdot|||)$ is an isometry [14].

**LEMMA 13.7.** [14, Theorem 5]. $f \in \tilde{S}$ iff $f \in L^1$ and

$$\sup \|f^* e_n\|_S < \infty$$

where $(e_n)$ is an a. i. in $S$.

Moreover $|||f||| = \sup \|f^* e_n\|_S$.

Since $S$ is a subalgebra of $L^1$ (Definition 4.15) it is clear that a $c$-multiplier $T$ from $L^1$ to $S$ must be a $c$-multiplier from $L^1$ to $L^1$. Hence there exists a unique measure $\mu \in M_1(G)$ such that

$$Tf = \mu^* f \quad \text{for all} \quad f \in L^1 \quad \text{(Theorem I).}$$

If $A \subset M_1$, we will write $M(L^1, S) \subset A$ if every $T \in c$-$M(L^1, S)$. 

\[\text{\exists} \]
has the form (3) for some \( \mu \in A \). Thus \( c-M(L^1, S) \subseteq M_1(G) \).

**Theorem 13.8.** ([9, Theorem 2.6] If \( c-M(L^1, S) \subseteq L^1 \) then \( c-M(L^1, S) = \tilde{S} \). In this case, if for \( \mu \in \tilde{S} \) we define \( T_{\mu} \), by \( T_{\mu} f = \mu^* f \) \( (f \in L^1) \) then the correspondence \( \mu \mapsto T_{\mu} \) is an isometric algebra isomorphism from \( \tilde{S} \) onto \( c-M(L^1, S) \).

**Theorem 13.9.** ([9, Theorem 2.3] Let \( \mu \in M_1 \) and \( \{ e_n \} \) as in Lemma 13.7. Then the following are equivalent:

i) \( \sup ||\mu^* e_n||_S < \infty \).

ii) \( \mu \in c-M(L^1, S) \).

iii) \( \mu \in c-M(L^1, \tilde{S}) \).

Hereafter and for the rest of our work \( \{ e_n \} \) will be the a.i. in \( (C^0, \ell^1) \), hence the a.i. in \( (L^p, \ell^1) \), \( 1 < p < \infty \), given by Proposition 4.17.

Applying Theorem 13.8 and Theorem 13.9 to the Segal algebra \( (L^p, \ell^1) \), we have the following theorem.

**Theorem 13.10.** Let \( 1 < p < \infty \).

i) \( (L^p, \ell^1) \), \( (L^p, \ell^1) \). Hence \( ||f||_p^p = ||f||_1 \) for all \( f \) in \( (L^p, \ell^1) \).

ii) \( f \in (L^p, \ell^1) \) iff \( f \in \ell^1 \) and \( \sup ||f^* e_n||_p < \infty \).

Moreover \( ||f||_p^p = \sup ||f^* e_n||_p^p \) for all \( f \in \ell^1 \).

iii) Let \( \mu \in M_1(G) \), \( \mu \in c-M(L^1, (L^p, \ell^1)) \) iff \( \sup ||\mu^* e_n||_p < \infty \).
PROOF. By Theorem 13.4, \( c-M(L^1, (L^p, L^1)) \subseteq (L^p, L^1) \subseteq L^1 \). Then

1) follows from Theorem 13.8. ii) is a direct consequence of 1) and

Lemma 13.7. iii) follows from i) and Theorem 13.9.

THEOREM 13.11. [9, Theorem 4.5]. \((C_0, L^1)^\theta = (L^\infty, L^1)\) and the norms

\[ ||| \cdot ||| \text{ and } ||| \cdot |||_{\infty} \text{ are equivalent in } (L^\infty, L^1). \]

COROLLARY 13.12. \( f \in (L^\infty, L^1) \) iff \( f \in L^1 \) and

\[ \sup ||| f \cdot e_n |||_{\infty} \theta < \infty. \]

PROOF. Lemma 13.7 and Theorem 13.11.

Similarly to [9, Theorem 4.6] we will characterize

\( c-M(L^1, (C_0, L^1)). \)

THEOREM 13.13. Let \( S^\infty = ((L^\infty, L^1), ||| \cdot |||) \). If \( T \) is in

\( c-M(L^1, (C_0, L^1)) \), then there exists a unique \( \mu \in (L^\infty, L^1) \) such that

\( Tf = \mu \cdot f \) for all \( f \in L^1 \) and the correspondence of \( T \) and \( \mu \) defines an

isometric algebra isomorphism from \( c-M(L^1, (C_0, L^1)) \) onto \( S^\infty \).

PROOF. Let \( T \in c-M(L^1, (C_0, L^1)). \) Then there exists a unique

\( \mu \in L^1 \) such that \( Tf = \mu \cdot f \) for all \( f \in L^1 \) (see (3) above). On the

other hand, \( (C_0, L^1) \subseteq L^\infty \) and therefore \( T \in c-M(L^1, L^\infty) \), so by Theo-

rem 13.3 there exists a unique \( \varphi \in L^\infty \) such that \( Tf = \varphi \cdot f \) for all

\( f \in L^1 \). This implies that for all \( f \in L^1 \),

\[ \int f(-t) \, d\mu(t) = f \cdot \mu(0) = f \cdot \varphi(0) = \int f(-t) \, \varphi(t) \, dt. \]

In particular for a measurable \( E \) in \( G \).
\[ \mu(E) = \int \chi_E \, d\mu = \int \varphi(x) \, dx. \]

This means that \( \mu \) is absolutely continuous. Therefore \( \varphi \in L^1 \)
and \( \varphi dx = \mu \). So \( c-M(L^1, (C_0, \ell^1)) \subset L^\infty \cap L^1 \subset L^1 \). Then the conclusion
follows from Theorem 13.11 and Theorem 13.8.

\textbf{Corollary 13.14.} Let \( \mu \in M_1 \). Then the following are equivalent:

i) \( \sup ||\mu * e_n||_{L^1} < \infty \).

ii) \( \mu \in c-M(L^1, (C_0, \ell^1)) \).

iii) \( \mu \in c-M(L^1, (L^\infty, \ell^1)) \).

iv) \( \mu \in M(L^1, (C_0, \ell^1)) \).

\textbf{Proof.} i), ii) and iii) are equivalent by Theorem 13.9 and Theorem 13.13. By Theorem 13.13 and the properties of convolution, ii) implies iv). Finally by Theorem 12.9, iv) implies ii).

\textbf{Theorem 13.15.} Let \( T: L^1 \longrightarrow (C_0, \ell^1) \) be a linear operator.
Then the following are equivalent:

i) \( T \in M(L^1, (C_0, \ell^1)) \).

ii) There exists a unique \( \mu \in (L^\infty, \ell^1) \) such that \( Tf = \mu * f \) for all \( f \in L^1 \).

iii) There exists a unique \( \varphi \in (C_0, \ell^2) \) such that \( (Tf)^{\hat{\cdot}} = \varphi \hat{f} \)
for all \( f \in L^1 \).

\textbf{Proof.} i) implies ii) follows from Corollary 13.14 and Theorem 13.13. Since \( (L^\infty, \ell^1) \subset (L^2, \ell^1) \), the Fourier transform \( \varphi \) of \( \mu \)
in \( (L^\infty, \ell^1) \) belongs to \( (C_0, \ell^2) \) (Theorem 5.4). Therefore if ii) holds,
then \( (Tf)^{\hat{\cdot}} = \varphi \hat{f} \) for all \( f \in L^1 \). Clearly \( \varphi \) is unique.

The proof of iii) implies i) is the same as in Theorem 13.4.
REMARK 13.16. Note that by Corollary 13.14
\[ c^{-}\text{M}(L^1, (C_0, \mathcal{L}^1)) = \text{M}(L^1, (C_0, \mathcal{L}^1)), \quad \text{and} \quad c^{-}\text{M}(L^1, (L^\infty, \mathcal{L}^1)) \subset \text{M}(L^1, (L^\infty, \mathcal{L}^1)). \]

Similarly to [40, Theorem 0.1.2] we have the following. Let \( \mu \in (L^p, \mathcal{L}^1), 1 < p < \infty \). \( T_\mu \) will be the c-multiplier from \( L^1 \) to \( (L^p, \mathcal{L}^1) \) defined by \( T_\mu g = \mu * g \).

THEOREM 13.17. For each \( T \in c^{-}\text{M}(L^1, (L^p, \mathcal{L}^1)), 1 < p < \infty \), there exists a net \( \{i_n\} \) in \( (C_0, \mathcal{L}^1) \) such that
\[ \lim ||T_{i_n}f - Tf||_{L^1} = 0 \quad \text{for all} \quad f \in L^1. \]

That is, \( \{T_{i_n}\} \) is strong operator convergent to \( T \) and therefore \( \{T_\mu | \mu \in (C_0, \mathcal{L}^1)\} \) is strong operator dense in \( c^{-}\text{M}(L^1, (L^p, \mathcal{L}^1)) \).

PROOF. Take \( f \in L^1 \) and \( \epsilon > 0 \). Then there exists \( \phi \in C_0 \) such that \( ||f - \phi||_1 < \epsilon/4 \). Since \( \{e_n\} \) is an a. i. in \( (C_0, \mathcal{L}^1) \), there exists \( N \) such that, for all \( n > N \), \( ||e_n*\phi - \phi||_\infty < \epsilon/2 \). By (2.4),
\[ ||e_n*\phi - \phi||_1 < \epsilon/2 \quad \text{for all} \quad n > N. \]

Then for all \( n > N \)
\[ ||e_n*f - f||_1 \leq ||e_n*\phi - e_n*f||_1 + ||e_n*\phi - \phi||_1 + ||\phi - f||_1 \]
\[ \leq ||e_n||_1 ||\phi - f||_1 + ||e_n*\phi - \phi||_1 + ||\phi - f||_1 \]
\[ = 2 ||\phi - f||_1 + ||e_n*\phi - \phi||_1 < \epsilon. \]

Therefore
\[ \lim ||e_n*f - f||_1 = 0. \]

This means that \( \{e_n\} \) is an a.i. in \( L^1 \).

Let \( \mu \) be the element in \( (L^p, \mathcal{L}^1) \) associated to \( T \) and set \( i_n = e_n*\mu \). By Theorem 4.7 \( \{i_n\} \subset (C_0, \mathcal{L}^1) \) and
\[ \| T_{in}f - Tf \|_p = \| i_n * f - \mu * f \|_p = \| u_n * \mu * f - \mu * f \|_p \]
\[ \leq 2^{q} \| \mu \|_p \| e_n * f - f \|_1. \]

By (4) we conclude that \( \lim \| T_{in}f - Tf \|_p = 0. \)

Now we will characterize the \( c \)-multipliers from \( L^1 \to (L^p, \ell^q) \)
\( 1 < p, q \leq \infty. \)

We know that the amalgam \( (L^p, \ell^q) \), \( 1 \leq p, q \leq \infty \), is a Banach \( L^1 \)-module (§4 p. 60). Then there exists an equivalent norm \( \| \cdot \|_{pq} \) in \( (L^p, \ell^q) \) such that for all \( f \in L^1, \mu \in (L^p, \ell^q) \)
\[ \| f * \mu \|_{pq} \leq \| f \|_1 \| \mu \|_{pq} \]

We will write \( (L^p, \ell^q)' \) for \( (L^p, \ell^q) \) endowed with the norm \( \| \cdot \|_{pq} \) [19, 14.4].

**Theorem 13.18.** Let \( 1 < p, q \leq \infty \). If \( T: L^1 \to (L^p, \ell^q) \) is a linear operator then the following are equivalent:

i) \( T \in c-M(L^1, (L^p, \ell^q)). \)

ii) There exists a unique \( \mu \in (L^p, \ell^q) \) such that \( Tf = \mu * f \) for all \( f \in L^1. \)

The correspondence between \( T \) and \( \mu \) defines a continuous linear isomorphism from \( c-M(L^1, (L^p, \ell^q)) \) onto \( (L^p, \ell^q) \).

If \( 1 < p, q \leq \infty \) then the isomorphism is an isometry from \( c-M(L^1, (L^p, \ell^q)) \) onto \( (L^p, \ell^q)' \).

**Proof.** The case \( p = q = \infty \) is Theorem 13.3.

By Theorem 13.1, i) implies ii).
If $\mu \in (L^P, L^q)$ and $Tf = \mu * f$ ($f \in L^1$) then clearly $T$ commutes with convolution and by Theorem 4.7

\[ ||Tf||_{pq} = ||\mu * f||_{pq} \leq 2^d \ ||f||_1 \ ||\mu||_{pq} \text{ for all } f \in L^1. \]

Hence $T$ is continuous and $||T|| \leq 2^d \ ||\mu||_{pq}$. Therefore $T \in c-M(L^1, (L^P, L^q))$ and the equation $Tf = \mu * f$ defines a continuous linear isomorphism from $c-M(L^1, (L^P, L^q))$ onto $(L^P, L^q)$.

If $1 < p, q < \infty$ and $T \in c-M(L^1, (L^P, L^q))$ then for all $f$ in $L^1$, $||Tf||_{pq} = ||\mu * f||_{pq} \leq ||f||_1 \ ||\mu||_{pq}$. Hence $||T|| \leq ||\mu||_{pq}$.

Now, by Corollary 4.14 and Theorem 7.2,

\[ \lim_{U \to 0} ||\phi_U * \mu - \mu||_{pq} = 0. \]

So given $\varepsilon > 0$ there exists a $\phi_U$ such that $||\phi_U * \mu - \mu||_{pq} < \varepsilon$.

This implies that $||\phi_U * \mu||_{pq} > ||\mu||_{pq} - \varepsilon$. Since $\phi_U \in L^1$ and $||\phi_U||_1 = 1$ we conclude that $||T|| = ||\mu||_{pq}^\dagger$.

**Remark 13.19.** By Theorem 4.7, $f * \mu \in (C_0, L^q)$ ($(L^q, c_0)$) for all $f \in L^1$ and $\mu \in (L^\infty, L^q)$ ($(L^q, L^\infty)$) ($1 < q < \infty$). Hence by Theorem 13.18 for $1 < q < \infty$, $c-M(L^1, (L^\infty, L^q)) = c-M(L^1, (C_0, L^q))$ ($c-M(L^1, (L^q, L^\infty)) = c-M(L^1, (L^q, C_0))$).

This implies the next theorem whose first part is already known [42, Theorem 4.2].

**Theorem 13.20.** Let $1 < q < \infty$. If $T: L^1 \to (C_0, L^q)$ ($(L^q, c_0)$) is a linear operator then the following are equivalent:

1) $T \in c-M(L^1, (C_0, L^q))$ ($c-M(L^1, (L^q, c_0)$)
ii) There exists a unique \( \mu \in (L^\infty, \mathbb{L}^q) \) such that \( Tf = \mu \ast f \) for all \( f \in L^1 \).

The correspondence between \( T \) and \( \mu \) defines a continuous isomorphism from \( c-M(L^1, (C_0, \mathbb{L}^q)) \) onto \( (C_0, \mathbb{L}^q) \).

**Remark 13.21.** From Theorem 12.9 and Theorem 13.20 we conclude that for \( 1 < q < \infty \), \( c-M(L^1, (C_0, \mathbb{L}^q)) = M(L^1, (C_0, \mathbb{L}^q)) \) and \( c-M(L^1, (L^q, c_0)) = M(L^1, (L^q, c_0)) \). Then by Remark 13.19, for \( 1 < q < \infty \), \( c-M(L^1, (L^\infty, \mathbb{L}^q)) = c-M(L^1, (C_0, \mathbb{L}^q)) = M(L^1, (C_0, \mathbb{L}^q)) \) and \( c-M(L^1, (L^q, \mathbb{L}^\infty)) = c-M(L^1, (L^q, c_0)) = M(L^1, (L^q, c_0)) \).

The next theorem is the counterpart to the uniqueness theorem in the theory of \( L^p \) spaces and we will use it to characterize \( c-M(L^1, (L^p, \mathbb{L}^q)) \) in terms of functions on \( \hat{G} \).

**Theorem 13.22.** (Uniqueness Theorem for Amalgams)

i) Let \( \mu, \nu \) be in \( M_\infty(G) \). If \( \hat{\mu} = \hat{\nu} \) then \( \mu = \nu \).

ii) Let \( f, g \) be in \( (L^p, \mathbb{L}^q) \), \( 1 \leq p, q \leq \infty \). If \( \hat{f} = \hat{g} \) (as linear functionals on \( A_0(G) \) if \( q > 2 \)) then \( f = g \) a.e.

**Proof.** i) If \( \hat{\mu} = \hat{\nu} \) then by definition of the Fourier transform (Definition 6.9) for all \( \phi \in \Phi(\hat{G}) \) (see Lemma 6.4)

\[
\langle \phi', \mu \rangle = \langle \phi, \hat{\mu} \rangle = \langle \phi, \hat{\nu} \rangle = \langle \phi', \nu \rangle.
\]

Since \( \Phi(\hat{G})^\vee \) is dense in \( (C_0, L^1) \) (Proposition 8.3 iv)) and \( \mu, \nu \) belong to \( M_\infty = (C_0, \mathbb{L}^1) \ast \) (Theorem 3.2) we conclude that \( \mu = \nu \).

ii) \( f, g \) as measures belong to \( M_\infty \). That is, \( \mu = \int fdx, \nu = gdx \).
belong to $M_q$, hence to $M_w$ by (2.9). Since the Fourier transform of $f$ as a function of $(L^P, \ell^q)$ and as a measure of $M_q$ is the same $(\S 6, p. 79)$ $\hat{\mu} = (fdx)^\wedge = \hat{f} = \hat{g} = (gdx)^\wedge = \hat{\nu}$. By the uniqueness of the Fourier transform $\hat{\mu} = \hat{\nu}$ as $M_q$ and $M_w$ transform. Therefore by i) $\mu = \nu$. This implies that $fdx = gdx$ and we conclude that $f = g$ a.e. 

**Theorem 13.13.** Let $1 < p, q \leq \infty$. If $T: L^1 \rightarrow (L^P, \ell^q)$ is a linear operator then the following are equivalent:

i) $T \in C(M(L^1, L^P, \ell^q))$.

ii) There exists a unique $\varphi$ in $A_c(\hat{G})^*$, in $(L^{q'}, \ell^2')(\hat{G})$ if $1 < p, q \leq 2$, in $(L^{q'}, L^2)(\hat{G})$ if $1 < q \leq 2, 2 < p \leq \infty$, such that $\varphi f = \hat{f}$ for all $f \in L^1$. (See 95 pp. 71 and 73.)

**Proof.** If $T \in C(M(L^1, L^P, \ell^q))$, then by Theorem 13.18, there exists a unique $\mu \in (L^P, \ell^q)$ such that $Tf = \mu * f$ for all $f \in L^1$. Then by Proposition 6.14, $(Tf)^\wedge = \hat{\mu} f = \varphi f, \hat{\mu} = \varphi$ belongs to $(L^{q'}, \ell^2')$ if $1 < q \leq 2$ (Theorem 5.7), to $(L^{q'}, L^2)$ if $1 < q \leq 2, 2 < p \leq \infty$ (Remark 5.8). Clearly $\varphi$ is unique. Therefore i) implies ii).

If ii) holds then for $f, g$ in $L^1$

$$(Tf * g)^\wedge = (T)^\wedge g = (\varphi f)^\wedge g = \varphi(f * g)^\wedge = (T(f * g))^\wedge.$$ 

By Theorem 13.22, $Tf * g = T(f * g)$ for all $f, g$ in $L^1$. That is, $T$ commutes with convolution.

To prove that $T$ is continuous, take $\{f_n\}, f$ in $L^1$ such that $$\lim ||f_n - f||_1 = 0$$ and suppose that $$\lim ||Tf_n - g||_{pq} = 0.$$ 

In any case we will think of $\varphi$ as a linear functional on $A_c(\hat{G})$. So for $\psi \in A_c(\hat{G}), \hat{\psi} \in (C_0, \ell^1)$ (Lemma 6.4) and by Definition 6.6 and
the Hölder inequality for amalgams (Theorem 3.1) we have that

\[ |\langle \psi, (Tf)^\wedge - \hat{g} \rangle| \leq |\langle \psi, (Tf)^\wedge - (Tf_n)^\wedge \rangle| + |\langle \psi, (Tf_n)^\wedge - \hat{g} \rangle|\]

\[ = |\langle \psi, \varphi \wedge - \varphi \wedge_n \rangle| + |\langle \psi, Tf_n - \hat{g} \rangle|\]

\[ \leq |\langle \psi, (f - f_n)^\wedge, \varphi \rangle| + \| \psi \|_{L^p, \infty} \| Tf_n - \hat{g} \|_{pq}\]

\[ \leq \| \psi \| \| \psi (f - f_n)^\wedge \|_{L^A_c} + \| \psi \|_{L^\infty} \| Tf_n - \hat{g} \|_{pq}\]

\[ = \| \psi \| \| \psi (f - f_n)^\wedge \|_{L^A_c} + \| \psi \|_{L^\infty} \| Tf_n - \hat{g} \|_{pq}\]

\[ = \| \psi \| \| \psi \|_{L^\infty} \| f - f_n \|_{L^1} + \| \psi \|_{L^\infty} \| Tf_n - \hat{g} \|_{pq}\]

This implies that \( |\langle \psi, (Tf)^\wedge - \hat{g} \rangle| = 0 \) for all \( \psi \in A_c(\hat{G}) \)

Hence \( (Tf)^\wedge = \hat{g} \) and by Theorem 13.22 \( Tf = g \) a.e. Therefore by the Closed Graph Theorem \( T \) is continuous and ii) implies i).†

**Corollary 13.24.** Let \( 1 < q < \infty \). If \( T: L^1 \rightarrow (C_0, L^q) \)

\( (T: L^1 \rightarrow (L^q, c_0)) \) is a linear operator, then the following are equivalent:

i) \( T \in \text{M}(L^1, (C_0, L^q)) \) \( (\text{M}(L^1, (L^q, c_0))^\wedge \)

ii) There exists a unique \( \varphi \) in \( A_c(\hat{G})^\wedge (A_c(\hat{G}))^\wedge \), in \( (L^q, \ell^2) \) if \( 1 < q \leq 2 \), such that \( (Tf)^\wedge = \varphi \wedge \) for all \( f \in L^1 \).

**Proof.** Corollary 13.2 and Theorem 13.23 (remember that \( (C_0, L^q) \subset (L^\infty, L^q) \) and \( (L^q, c_0) \subset (L^q, L^\infty) \)).†

To characterize the \( c \)-multipliers from \( L^1 \) to \( M_q \) we need the following lemma.
**Lemma 13.25** Let \( \mu \in M_q, 1 \leq q \leq \infty \).

i) If \( f \in L^1 \) and \( \xi \geq 0 \) then \( f \ast \mu^\xi = (f \ast |\mu|)^\xi \).

ii) If \( f = \alpha \chi_E \) where \( \alpha \) is a nonnegative real number and \( E \) is a measurable subset of \( G \) then \( (f \ast \mu)^\# = f \ast \mu^\# \).

**Proof.**

i) \[
\begin{align*}
\left( f \ast \mu^\xi \right)(x) &= \int f(x - t)^\xi \mu^\xi(t) \, dt = \int f(x - t) \, |\mu|(t + L) \, dt \\
&= \int \int f(x - t) \chi_{E-L}(s) \, d|\mu|(s) \, dt \\
&= \int \int f(u - s) \chi_L(u - x) \, du \, d|\mu|(s) \\
&= \int \int f(u - s) \chi_{x+L}(u) \, du \, d|\mu|(s) \\
&= \int \chi_{x+L}(u) \int f(u - s) \, d|\mu|(s) \, du \\
&= \int_{x+L} f \ast \mu(u) \, du = (f \ast |\mu|)^\#(x).
\end{align*}
\]

ii) First we note that \( f \ast \mu(t) = \alpha \int \chi_t(t - x) \, d\mu(x) = \alpha \mu(t - E) \).

If \( \mu \) is real-valued then by the definition of \( (f \ast \mu)^+ \) (see p.117) we have that for \( t \in G \)

\[
(f \ast \mu)^+(t) = \sup (f \ast \mu, 0) = \alpha \sup (\mu(t - E), 0) = \alpha \mu^+(t - E) = \alpha (\chi_E \ast \mu^+(t)) = f \ast \mu^+(t).
\]

Similarly \( (f \ast \mu)^- = f \ast \mu^- \). Hence

\[
(6) \quad |f \ast \mu| = f \ast \mu^+ + f \ast \mu^- = f \ast |\mu|.
\]

Thus, by part i) \( (f \ast \mu)^\# = (f \ast |\mu|)^\# = f \ast \mu^\# \).

If \( \mu \) is complex-valued then \( \mu = \mu_1 + i\mu_2 \) where \( i = -1, \ 2 \)

is a real valued measure in \( M_q \). So, by (6)
\[ f^*|\mu| = f^*|\mu_1| + f^*|\mu_2| = |f^*\mu_1| + |f^*\mu_2| = |f^*\mu|. \]

Again by part i) \((f^*\mu)^\theta = f^* \mu^\theta)\).

**Theorem 13.26.** Let \(1 \leq q < \infty\). If \(T : L^1 \to M_q\) is a linear operator then the following are equivalent:

i) \(T \in c\v M(L^1, M_q)\).

ii) There exists a unique \(\mu \in M_q\) such that \(Tf = \mu^*f\) for all \(f \in L^1\).

The correspondence between \(T\) and \(\mu\) defines an isometric linear isomorphism from \(c\v M(L^1, M_q)\) onto \(M_q^\theta\).

**Proof.** The case \(q = \infty\) was proved by Feichtinger [24, Theorem 1.3]. It should be mentioned that the definition of a multiplier used throughout [24] corresponds to what we call \(c\v\)-multiplier and not the one given in [24, p. 342].

Assume \(1 \leq q < \infty\). By Theorem 13.1, i) implies ii).

Now, if \(\mu \in M_q\) and \(Tf = \mu^*f\) \((f \in L^1)\), then clearly \(T\) commutes with convolution and by Corollary 4.6

\[ ||Tf||_q^\theta = ||f^*\mu||_q^\theta = ||f^*\mu||_q^\theta \leq ||f||_1 ||\mu||_q^\theta \text{ for all } f \in L^1. \]

This shows that \(T\) is continuous, hence \(T \in c\v M(L^1, M_q)\), and

\[ ||T|| \leq ||\mu||_q^\theta. \]

Since \(\mu^\theta \in L^q\) (see Theorem 1.21) and \(q\) is finite, given \(\varepsilon > 0\) there exists a neighborhood \(U\) of 0 such that

\[ ||h^*\mu^\theta - \mu^\theta||_q < \varepsilon \text{ for all } h \in L^1, ||h||_1 = 1, h \geq 0 \text{ and } \int_{C^1} h = 0. \]

[37, Theorem 20.15]. Let \(f = 1/\mu(X_U)\). Clearly \(f\) satisfies all the above conditions and therefore

\[ ||f^*\mu^\theta - \mu^\theta||_q < \varepsilon. \]

By Lemma 13.25 we have that...
\[ \| Tf \|_{L^q} = \| f^\#u \|_{L^q} = \| (f^\#u)^\# \|_{L^q} = \| f^\#u \|_{L^q} > \| \mu \|_{L^q} = \varepsilon. \]

Since \( \| \mu \|_{L^q} = \| \mu \|_{L^q} \) we conclude that \( \| T \| = \| \mu \|_{L^q} \).

**Remark 13.27.** If \( T \in c-M(L^1, M_q), 1 \leq q \leq \infty \), then \( Tf = u^f \) for some \( u \in M_q \). So by Corollary 4.4 \( Tf \in (L^1, L^q) \) for all \( f \in L^1 \).

Hence \( c-M(L^1, M_q) \subseteq c-M(L^1, (L^1, L^q)) \). Now, since \( f \mapsto \int f \, dx \) from \( (L^1, L^q) \) to \( L^1 \) is a natural embedding which is an isometry we conclude that \( c-M(L^1, M_q) = c-M(L^1, (L^1, L^q)) \). This implies the next theorem which is already known [42, Corollary 6.3].

**Theorem 13.28.** Let \( 1 \leq q \leq \infty \). If \( T : L^1 \to (L^1, L^q) \) is a linear operator then the following are equivalent:

1) \( T \in c-M(L^1, (L^1, L^q)) \).

2) There exists a unique \( u \in M_q \) such that \( Tf = u^f \) for all \( f \in L^1 \).

The correspondence between \( T \) and \( u \) defines an isometric linear isomorphism from \( c-M(L^1, (L^1, L^q)^\#) \) onto \( M_q^\# \).

**Remark.** An alternative approach to Theorem 13.28 might be to use Feichtinger's Theorem 1.5 in [24] and show that \( (L^1, L^q)^\# = M_q \).

**Remark 13.29.** From Theorems 13.26 and 13.28, for \( 1 \leq q \leq \infty \),

\( c-M(L^1, M_q) \subseteq M(L^1, M_q) \) and \( c-M(L^1, (L^1, L^q)) \subseteq M(L^1, (L^1, L^q)) \). Then by Theorem 12.9, \( c-M(L^1, (L^1, L^q)) = M(L^1, (L^1, L^q)) \) for \( 1 \leq q < \infty \). Therefore for \( 1 \leq q < \infty \)

\( M_q \cong c-M(L^1, M_q) \subseteq c-M((L^1, (L^1, L^q)) = M(L^1, (L^1, L^q)). \)

We do not know if for \( 1 \leq q \leq \infty \), \( c-M(L^1, M_q) = M(L^1, M_q) \).
THEOREM 13.30. Let $1 \leq q < \infty$. If $T : L^1 \to (L^1, \ell^q)$ is a linear operator then the following are equivalent:

1) $T \in c-M(L^1, (L^1, \ell^q))$.

2) There exists a unique $\varphi$ in $A_c(\hat{G})^*$, in $(L^{q'}, c_0(\hat{G}))$ if $1 \leq q \leq 2$, such that $\hat{Tf} = \varphi \hat{f}$ for all $f \in L^1$.

The proof is similar to the proof of Theorem 13.23.

THEOREM 13.31. Let $1 \leq q \leq \infty$. If $T : L^1 \to M_q$ is a linear operator then the following are equivalent:

1) $T \in c-M(L^1, M_q)$

2) There exists a unique $\varphi$ in $A_c(\hat{G})^*$, in $(L^{q'}, \ell^\infty)(\hat{G})$ if $1 \leq q \leq 2$, such that $(Tf) = \varphi f$ for all $f \in L^1$.

The proof of this theorem is also similar to the proof of Theorem 13.23 but using the Hölder inequality as in Theorem 3.2.

THEOREM 13.32. Let $\mu \in M_q$, $1 \leq q \leq \infty$, and $f \in (L^1, \ell^s)$, $1 \leq q' \leq s \leq \infty$. If $\hat{\mu} = \hat{f}$ on $A_c(\hat{G})^*$ then $f \in (L^1, \ell^q)$, $\mu$ is the image of $f$ under the embedding of Remark 13.27 and $d\mu = fdx$.

PROOF. Let $T_f$, $T_\mu$ be the c-multipliers in $c-M(L^1, (L^1, \ell^q))$, $c-M(L^1, M_q)$ associated to $f$ and $\mu$ respectively. Then by hypothesis

$$(Tfg) = (f \ast g) = \hat{f} \hat{g} = \hat{\mu g} = \hat{(ug)} = (T_\mu g)$$

for all $g \in L^1$ (see Definition 6.13). Since $\mu g \in (L^1, \ell^q)$ and $(L^1, \ell^q) \subset (L^1, \ell^s)$, $(Tfg)$ as $(L^1, \ell^s)$ transform for all $g \in L^1$. By Theorem 13.22 $Tfg = f \ast g = \mu g = T_\mu g$ for all $g \in L^1$. This implies that $d\mu = fdx$ and therefore $||\mu||_q = ||f||_{1q}$. Hence $f \in (L^1, \ell^q)$. 

REMARK 13.33. In the particular case \( u \in M_q' \), \( 1 \leq q \leq 2 \), \( f \in (L^1, L^2) \), Theorem 13.32 says that if \( \hat{\mu} = \hat{f} \) a.e. on \( \mathcal{G} \) then \( f \) belongs to \((L^1, L^q)\) and \( \mu \) is the image of \( f \) under the embedding of remark 13.27 and \( d\mu = fd\mu \). This generalizes the result in [37, Theorem 31.33] because when \( q = 1 \) and \( f \in L^p \), \( 1 \leq p \leq 2 \), then \( f \) belongs to \((L^1, L^2)\) as \((L^p, L^p) \subset (L^1, L^2) \). So by Theorem 13.32 \( f \in L^1, \mu \) is absolutely continuous and \( d\mu = fd\mu \).

COROLLARY 13.34. Let \( 1 \leq q \leq \infty \), \( 1 \leq q \leq \infty \). If \( g \in (L^1, L^q) \), \( f \in (L^1, L^s) \) and \( \hat{g} = \hat{f} \) on \( A_c(\mathcal{G}) \), then \( f = g \) a.e.

PROPOSITION 13.35. Let \( 1 \leq p \leq r \leq \infty \), \( 1 \leq s \leq q \leq \infty \). If \( f \in (L^p, L^q) \), \( g \in (L^r, L^s) \) and \( \hat{f} = \hat{g} \) on \( A_c(\mathcal{G}) \), then \( f = g \) a.e.

PROOF. Similarly to Theorem 13.32, let \( T_f, T_g \) be in \( c-M(L^1, (L^p, L^q)) \), \( c-M(L^1, (L^r, L^s)) \) associated to \( f, g \) respectively.

By hypothesis \( (T_f h) \hat{=} (f \hat{\star} h) \hat{=} \hat{f} \hat{\star} \hat{h} = \hat{g} \hat{\star} \hat{h} = (g \hat{\star} h) \hat{=} (T_g h) \hat{=} \), for all \( h \in L^1 \). Since \((L^r, L^s) \subset (L^p, L^q) \), \((T_f h) \hat{=} (T_g h) \hat{=} \) as \((L^p, L^q) \) transform, so by Theorem 13.22, for all \( h \in L^1 \),

\[ T_f h = f \hat{\star} h = g \hat{\star} h = T_g h. \]

This implies that \( f = g \) a.e.

We will end this section with the following conjecture.

By Theorem 13.18 \( c-M(L^1, (L^\infty, L^q)) \subset M(L^1, (L^\infty, L^q)) \) and \( c-M(L^1, (L^q, L^\infty)) \subset M(L^1, (L^q, L^\infty)) \) for \( 1 < q < \infty \). We believe that \( c-M(L^1, (L^\infty, L^q)) \neq M(L^1, (L^\infty, L^q)) \) and \( c-M(L^1, (L^q, L^\infty)) \neq M(L^1, (L^q, L^\infty)) \) for all \( 1 < q < \infty \).
§ 14. MULTIPLIERS FROM $M_1$ TO AMALGAM SPACES AND SPACES

OF MEASURES $M_q$

The $c$-multipliers from $M_1(G)$ to any Banach $M_1$-module $A$ (Definition 4.9) are easily characterized. Indeed, if $T : M_1 \rightarrow A$ is a linear operator and $T$ has the form $T\nu = \nu \ast \psi$ ($\psi \in M_1$) for some $\psi \in A$ then by the properties of convolution and $M_1$-module (see (B-1) p. 60 ) $T$ is a $c$-multiplier from $M_1$ to $A$.

Conversely if $T$ commutes with convolution and $\delta$ is the identity in $M_1$ then, for $\psi \in M_1$, $T\nu = T(\delta \ast \nu) = T\delta \ast \nu = \psi \ast \nu$ with $\mu = T\delta$.

Hence, $T \in c \cdot M(M_1, A)$ iff there exists a unique $\mu \in A$ such that $T\nu = \mu \ast \nu$ for all $\nu \in M_1$.

By §4 p. 60 we immediately have the following theorem.

THEOREM 14.1. Let $A$ be any amalgam space $(L^p, \ell^q)$, $(C_0, \ell^q)$, $(L^p, c_0)$, $1 \leq p, q \leq \infty$, or a measure space $M_1$, $1 \leq s \leq \infty$. If $T : M_1 \rightarrow A$ is a linear operator then the following are equivalent:

i) $T \in c \cdot M(M_1, A)$.

ii) There exists a unique $\mu \in A$ such that $T\nu = \mu \ast \nu$ for all $\nu \in M_1$.

iii) There exists a unique $\psi \in A(\hat{G})^*$ such that $(T\nu)^* = \psi \hat{\nu}$ for all $\nu \in M_1$.

$\varphi \in (L^1, \ell^1)(\hat{G})$ if $A = (L^p, \ell^q)$, $1 \leq p, q \leq 2$.

$\varphi \in (L^p, \ell^q)'(\hat{G})$ if $A = (L^p, \ell^q)$, $2 < p \leq \infty$, $1 \leq q \leq 2$, or $A = (c_0, \ell^q)$, $1 \leq q \leq 2$; $\varphi \in (L^p, \ell^\infty)'$ if $A = M_1$, $1 \leq s \leq 2$.
The correspondence between $T$ and $\mu$ establishes an isometric linear isomorphism from $c-M(M_1,A')$ onto $A'$ (see §13 p. 166).

**Proof.** As we have already said, i) is equivalent to ii). If ii) holds then by Proposition 6.14, i) $(Tv)^\wedge = (\mu^*v)^\wedge = \hat{\tilde{\nu}} = \Phi^\wedge$ for all $v \in M_1$, where $\Phi = \hat{\mu}$. Clearly $\Phi$ is unique. The remaining clauses in iii) follow from Theorem 5.7 and Remark 5.8.

Now suppose that $(Tv)^\wedge = \Phi^\wedge$ for all $v \in M_1$, $\Phi \in A_c(\hat{\mathcal{G}})^*$. Then for $v, \eta$ in $M_1$

$$(Tv^*\eta)^\wedge = (Tv)^\wedge = (\Phi^\wedge)\eta = \Phi(\hat{\tilde{\nu}}\eta) = \Phi(v^*\eta)^\wedge = (T(v^*\eta))^\wedge.$$ 

Hence by Theorem 13.22 $Tv^*\eta = T(v^*\eta)$. This means that $T$ commutes with convolution. Therefore $Tv = \mu^*v$ ($v \in M_1$) with $\mu = T\delta$ as we saw at the beginning of the section. This shows that ii) is equivalent to iii).

Finally, by the $M_1$-modularity of $A'$, for $v \in M_1$,

$$||Tv||_{A'} = ||\mu^*v||_{A'} \leq ||\mu||_{A'} ||v||_1.$$ 

Hence $||T|| \leq ||\mu||_{A'}$. Since $\mu = T\delta$ and $||\delta||_1 = 1$,

$$||\mu||_{A'} \leq ||T||.$$ 

Therefore $||T|| = ||\mu||_{A'}$ and the proof is complete.

**Remark 14.2.** It follows from Theorem 14.1 that $c-M(M_1,A) \subseteq M(M_1,A)$. But we know that there exists a $T$ in $M(M_1,M_1)$ such that $T$ is not defined by the convolution with an element of $M_1$ [29, p. 94]. So $c-M(M_1,M_q) \neq M(M_1,M_q)$ for $1 < q \leq \infty$. Indeed, if $c-M(M_1,M_q) = M(M_1,M_q)$ then for $T \in M(M_1,M_1)$, $T$ belongs to $M(M_1,M_q)$ since $M_1 \subseteq M_q$ by (2.9). Therefore $T \in c-M(M_1,M_q)$. So $Tv = \mu^*v$ ($v \in M_1$) where $\mu = T\delta$, hence $\mu \in M_1$. This contradiction proves our
claim. (We do not know if \( c\sim M(M_1, A) = M(M_1, A) \) for some amalgam space \( A \)).

However the situation is different when we consider \( M_1 \overset{w}{\rightarrow} A \) (see §12 p. 144).

**Theorem 14.3.** Suppose \( A \) is an amalgam or measure space of type \( q \) such that \( \hat{A} \) satisfies the following two conditions:

(14-1) There exists a Banach space \( B \) such that \( B^* = A \).

(14-2) For all \( f \in B \) and \( g \in A \), \( f \star g \in C_0 \).

If \( T: M_1 \overset{w}{\rightarrow} A \) is a linear operator then the following are equivalent:

i) \( T \in M(M_1, A) \).

ii) \( T \in c\sim M(M_1, A) \).

iii) There exists a unique \( \mu \in A \) such that \( Tv = \mu \star v \) for all \( v \in M_1 \).

iv) There exists a unique \( \varphi \in A(G)^* \) such that \( (Tv)^* = \varphi \varphi \) for all \( v \in M_1 \).

The correspondence between \( T \) and \( \mu \) defines a linear isomorphism from \( M(M_1, A) \) onto \( A \).

**Proof.** By Theorem 12.9, i) implies ii). As in Theorem 14.1 ii) implies iii) with \( \mu = T\delta \) and iii) is equivalent to iv).

Now suppose iii) By the properties of convolution \( T \) commutes with translations.

To prove that \( T \) is continuous take \( \{v_n\} \) in \( M_1 \) such that \( \lim v_n = v \) in \( M_1 \overset{w}{\rightarrow} A \). That is, for all \( h \in C_0 \), \( \lim <v_n, h> = <v, h> \).

Let \( f \in B \). By condition (14-2), \( f \star \mu \in C_0 \) and we have that
\[ <TV_n, f> = <μ_n f, f> = <f, f> \text{ lim } <μ_n f, f> = \text{ lim } <μ_n f, f> = \text{ lim } <TV_n, f>. \]

This implies that \( T \) is continuous and therefore \( T \in M(M^w_1, A^w). \)

The rest of the theorem is clear. 

**Proposition 14.4.** The following spaces satisfy conditions (14-1) and (14-2).

1. \((L^p, L^q)\) \(1 < p, q < \infty\)
2. \(M_{eq}\) \(1 < q < \infty\)
3. \((L^p, L^1)\) \(1 < p < \infty\)
4. \((L^p, L^\infty)\) \(1 < p < \infty\)
5. \((L^\infty, L^q)\) \(1 < q < \infty\)

**Proof.** a) and b) follow from Theorems 4.7 and 4.8 with \( B = (L^p, L^q)\) and \( B = (C_0, L^q)\) respectively.

b) \((L^p, L^1)\) satisfies (14-1) with \( B = (L^p, C_0)\) (Theorem 3.1). Let \( h \in C_0\) with support \( \mathcal{E}\) and \( g \in (L^p, L^1)\). Then there exists a sequence \( \{g_n\}\) in \( L^p\) such that \( \lim g_n = g\) in \( (L^p, L^1)\) (Theorem 3.6). By a), \( g_n h \in C_0\) and this implies that \( \{g_n h\} \in C_0\), because \( C_0\) is a closed subspace of \( L^\infty\) and \( \lim g_n h = g h\) in \( L^\infty\). Since \( C_0\) is dense in \( (L^p, C_0)\) and \( f \to f g\) from \( (L^p, C_0)\) to \( L^\infty\) is continuous, we conclude that \( f g \in C_0\).

c) \((L^\infty, L^1)\) satisfies (14-1) with \( B = (L^1, C_0)\). Let \( f \in (L^1, C_0)\) and \( g \in (L^\infty, L^1)\). By Theorem 13.13, \( g\) defines a \( c\)-multiplier \( T_g\) from \( L^1\) to \((C_0, L^1)\). That is, \( T_g h = g h\) belongs to \((C_0, L^1)\) for
all $h \in L^1$. In particular, for all $\phi \in C_c$, $g * \phi \in (C_0, L^1)$ and hence is in $C_0$. Since $C_c$ is dense in $(L^1, c_0)$ and convolution with $g$ is a continuous linear map from $(L^1, c_0)$ into $L^\infty$ we conclude that $f * g \in C_0$.

$(L^\infty, L^q)$ satisfies (14-1) with $B = (L^1, L^{q'})$ for $1 < q < \infty$, and by Theorem 4.7, $(L^\infty, L^q)$ satisfies (14-2).+ 

**Remark 14.5.** We should mention that in Theorem 14.3 the function $\phi$ of iv) belongs to $(L^{q'}, L^{p'})$ if $A = (L^p, L^q)$, $1 < p \leq 2$, $1 \leq q \leq 2$; to $(L^{q'}, L^2)$ if $A = (L^p, L^q)$, $2 < p \leq \infty$, $1 \leq q \leq 2$; or to $(L^{q'}, L^\infty)$ if $A = M_q$, $1 \leq q \leq 2$. (See the proof of Theorem 14.1).
§ 15. MULTIPLIERS FROM AMALGAM SPACES AND SPACES OF MEASURES

Mₚ TO L∞

PROPOSITION 15.1. Let A be (Lₚ, L₀), 1 < p, q < ∞, or (Lₚ, C₀)
1 < p < ∞. Then A* has the following properties:

(15-1) A* = (Lₚ', L₀'), 1 < p', q' < ∞, or (Lₚ', L'), 1 < p' < ∞.

(15-2) For all f ∈ A and g ∈ A* f* g ∈ C₀ and

||f* g||₁ ≤ ||f||₁ ||g||₁

(15-3) If T ∈ M(L₁, A*) then there exists a unique μ ∈ A* such that

Tf = μ*f for all f ∈ L₁.

PROOF. (15-1) follows from Theorem 3.1. (15-2) is a direct
consequence of the Hölder inequality for amalgams (Theorem 3.1) and
Proposition 14.4. (15-3) follows from Corollary 13.2.

The next result extends Edwards' theorem for Lₚ spaces [21, The-
orem 3], and its proof is similar to his.

THEOREM 15.2. Let A be as in Proposition 15.1. If T: A → L∞
is a linear operator then the following are equivalent:

i) T ∈ M(A, L∞).

ii) There exists a unique μ ∈ A* such that T f = μ*f for all
    f ∈ A.

Moreover, the correspondence between T and μ defines and iso-
metric isomorphism from M(A, L∞) onto A* M(A, L∞) = M(A, C₀) and μ = T*
where $T^*$ is the adjoint operator of $T$ and $\delta$ is the identity of $M_1$.

**Proof.** If ii) holds then $T$ commutes with translations and by (15-2) $T$ is continuous. Hence ii) implies i).

Suppose i). Let $T^*: L^{\infty} \rightarrow A^*$ be the adjoint operator of $T$. We shall see that $T^*$ restricted to $L^1$ belongs to $M(L^1, A^*)$. Since $L^1 \subset L^*$, $<f, T^*h> = <Tf, h>$ for all $h \in L^1$ and $f \in A$. Then for $s \in G$, $f \in A$ and $h \in L^1$

$$<f, T^*T^*h> = <Tf, T^*h> = <T^*f, h> = <Tf, T^*h> = <T^*T^*f, h>.$$ 

Therefore $T^*T^*h = T^*T^*h$. That is, $T^*$ commutes with translations. This implies that $T^*|L^1 \in M(L^1, A^*)$. So by (15-3)

(1) There exists a unique $\mu \in A^*$ such that $T^*f = \mu f$ for all $f \in L^1$.

Now take $f \in A$, $h \in L^1$ and consider the following

(2) $<Tf, h> = <f, T^*h> = <f, \mu f> = <\mu f, h>.$

Therefore $Tf = \mu f$ for all $f \in A$. Clearly $\mu$ is unique. Hence i) implies ii).

Now, by (15-2) $M(A, L^\infty) = M(A, C_0)$ and if $T \in M(A, C_0)$ then $||T|| \leq ||\mu||_{A^*}$ and $T^*: M_1 \rightarrow A^*$.

To prove that $T^* \in c-M(M_1, A^*)$ we take $\nu, \eta$ in $M_1$ and $h \in A$ and we see that

(3) $<T^*(\nu \eta), h> = <\nu \eta, Th> = <\nu \eta, \mu h> = <\nu, \mu h \eta> 
= <\nu, T(h \eta)> = <T^*\nu, h \eta> = <T^*\nu \eta, h>.$

Therefore $T^*(\nu \eta) = T^*\nu \eta$. This implies that $T^* \in c-M(M_1, A^*)$.

So by Theorem 14.1 $T^*\nu = T^*\delta \nu$ for all $\nu \in M_1$. In particular for
f \in L^1, \quad T^*f = T^* \delta \ast f. \text{ Then it follows from the uniqueness of } \mu \text{ in (1)}

that \( \mu = T^* \delta \) and we have that

\[ ||\mu||_{A^*} = ||T^* \delta||_{L^\infty} \leq ||T^*|| ||\delta||_1 = ||T^*|| = ||T||. \]

By ii) and (15-2) we conclude that \( ||T|| \leq ||\mu||_{A^*} \). Therefore

\[ ||T|| = ||\mu||_{A^*} \] and the proof is complete. \( \dagger \)

**Theorem 15.3.** Let \( 1 < q \leq \infty \). If \( T: (L^1, \ell^q) \rightarrow L^\infty \) is a linear operator then the following are equivalent:

i) \( T \in M((L^1, \ell^q), L^\infty) \).

ii) There exists a unique \( \mu \in (L^\infty, \ell^{q'}) \) such that \( Tf = \mu \ast f \) for all \( f \in (L^1, \ell^q) \).

The correspondence between \( T \) and \( \mu \) defines a continuous linear isomorphism from \( M((L^1, \ell^q), L^\infty) \) onto \( (L^\infty, \ell^{q'}) \). If \( 1 < q < \infty \) then

\( M((L^1, \ell^q), L^\infty) = M((L^1, \ell^q), C_0) \), \( \mu = T^* \delta \) where \( T^* \) is the adjoint operator of \( T \) and the isomorphism is an isometry.

**Proof.** If ii) holds then by the properties of convolution \( T \) commutes with translations and \( T \) is bounded because

\[ ||T^*f||_\infty \leq ||f||_{1q} ||\mu||_{\infty q'} \] for all \( f \in (L^1, \ell^q), \mu \in (L^\infty, \ell^{q'}) \)

(Theorem 3.1).

Suppose i). Since for all \( 1 < q \leq \infty \) \( (L^1, \ell^q)^* \subset (L^\infty, \ell^{q'}) \)

(note that \( (L^1, c_0)^* = (L^\infty, \ell^1) \) and \( (L^1, \ell^\infty)^* \subset (L^1, c_0)^* \))

and \( (L^\infty, \ell^{q'}) \subset L^\infty \) we can consider the adjoint operator \( T^*: L^{\infty^*} \rightarrow (L^1, \ell^q)^* \) of \( T \), to be a linear continuous operator from \( L^{\infty^*} \) into \( L^\infty \). So, \( T^*: L^{\infty^*} \rightarrow L^\infty \) and \( T^*|L^1 \) commutes with translations because if \( s \in G \) and \( f, g \in L^1 \) then
\[ \langle T^* \tau_s f, g \rangle = \langle \tau_s f, T g \rangle = \langle f, \tau_s T g \rangle = \langle f, \tau_s \tau g \rangle = \langle T^* f, \tau_s g \rangle = \langle \tau_s T^* f, g \rangle. \]

This implies that \( T^* \tau_s f = \tau_s T^* f \) for all \( f \in L^1 \). Using the same argument as in Theorem 12.9 we shall prove that \( T^* \mid L^1 \) commutes with convolution. Let \( f, g \in L^1 \). Then

\[ \langle T^* f * g, h \rangle = \int T^* f * g(t) \, h(-t) \, dt = \int \int T^* f(t - s) \, g(s) \, ds \, h(-t) \, dt \]
\[ = \int g(s) \int T^* f(t - s) \, h(-t) \, dt \, ds \]
\[ = \int g(s) \int \tau_s T^* f(t) \, h'(t) \, dt \, ds \]
\[ = \int g(s) \int T^* \tau_s f(t) \, h'(t) \, dt \, ds \]
\[ = \int g(s) \int \tau_s f(t) \, Th'(t) \, dt \, ds \]
\[ = \int \int f(t - s) \, g(s) \, ds \, Th'(t) \, dt \]
\[ = \int f * g(t) \, Th'(t) \, dt = \int T^* (f * g) \, h(-t) \, dt \]
\[ = \langle T^* (f * g), h \rangle. \]

We can apply Fubini's theorem because \( Th \in L^\infty \) and \( f, g \in L^1 \).

Hence \( T^* f * g = T^* (f * g) \) and we conclude that \( T^* \mid L^1 \) belongs to \( c-M(L^1, (L^\infty, \ell^q)) \). By Corollary 13.14, Theorem 13.15 and Theorem 13.18 there exists a unique \( \mu \in (L^\infty, \ell^q) \) such that \( T^* f = \mu * f \) for all \( f \in L^1 \). Clearly \( \mu \) is unique and as in the proof of Theorem 15.2 it follows that \( T f = \mu * f \). Therefore i) implies ii).

It is clear that the relation \( T f = \mu * f \) defines a linear isomorphism from \( M((L^1, \ell^q), L^\infty) \) onto \( (L^\infty, \ell^q) \) and \( ||T|| \leq ||\mu||_{\ell^q}. \)
If \( 1 < q < \infty \) then \((L^q, \ell^q')\) satisfies condition (14-2) of Theorem 14.3 (Proposition 14.4). Hence \( M((L^q, \ell^q), L^\omega) = M((L^\omega, \ell^q), C_0) \).

That is, \( T : (L^1, \ell^q) \to C_0 \) and this implies that \( T^* : M_1 \to (L^\omega, \ell^q') \).

Using the argument (3) of the proof of Theorem 15.2 we have that

\[ |\mu|_{\omega,q} = |T^*\delta|_{\omega,q'} \leq |T^*| \cdot |\delta|_1 = |T^*| = |T|. \]

Therefore \( |T| = |T^*|_{\omega,q} \).

**Theorem 15.4.** Let \( 1 < p < \infty \). If \( T : (L^P, \ell^1) \to L^\infty \) is a linear operator then the following are equivalent:

i) \( T \in M((L^P, \ell^1), L^\infty) \).

ii) \( T(f \ast g) = Tf \ast g \) for all \( f \in (L^P, \ell^1) \) and \( g \in L^1 \).

iii) There exists a unique \( \mu \in (L^P, \ell^\infty) \) such that \( Tf = \mu \ast f \) for all \( f \in (L^P, \ell^1) \).

The correspondence between \( T \) and \( \mu \) defines an isometric isomorphism from \( M((L^P, \ell^1), L^\infty) \) onto \( (L^P, \ell^\infty) \), \( M((L^P, \ell^1), L^\infty) \) is equal to \( M((L^P, \ell^1), C_0) \) and \( \mu = T^*\delta \) where \( T^* \) is the adjoint operator of \( T \).

**Proof.** Let \( T^* : L^\infty \to (L^P, \ell^\infty) \) be the adjoint operator of \( T \).

Suppose i). Using the same argument as in Theorem 12.9 we have that for \( f \in (L^P, \ell^1) \) and \( g, h \in L^1 \)

\[ < T^*g, h \ast f > = < T^*g, h > = < T(f \ast g), h >. \]

Therefore \( T^*g = T(f \ast g) \) for all \( f \in (L^P, \ell^1) \) and \( g \in L^1 \).

Suppose ii). Take \( \tilde{f} \in (L^P, \ell^1) \), \( g, h \in L^1 \). So

\[ < T^*g, h \ast f > = < T^*g, h > = < g, T(h \ast f) > = < g, Tf \ast h > = < g \ast h, Tf > = < T^*(g \ast h), f >. \]

This implies that \( T^*g \ast h = T^*(g \ast h) \) for all \( h, g \in L^1 \). Therefore \( T^*|L^1 \in c-M(L^1, (L^P, \ell^\infty)) \). By Theorem 13.18 there exists a
unique $\mu \in (L^p, \ell^\infty)$ such that $T^*f = \mu * f$ for all $f \in L^1$.

Similarly to (2) of the proof of Theorem 15.2, $Tf = \mu * f$ for all $f \in (L^p, \ell^1)$. Hence ii) implies iii).

If iii) holds then by the properties of convolution $T$ commutes with convolution and by Theorem 3.1

$$||Tf||_\infty = ||\mu * f||_\infty \leq ||\mu||_{p,\infty} ||f||_{p,1}.$$  

Hence $T$ is continuous and $||T|| \leq ||\mu||_{p,\infty}$. Therefore iii) implies i).

Since $(L^p, \ell^\infty)$ satisfies condition (15-2) of Theorem 14.3 $M((L^p, \ell^1), L^\infty) = M((L^p, \ell^1), C_b)$. Similarly to the argument (3) of Theorem 15.2 we have that $\mu = T^*\delta$ and $||T|| = ||\mu||_{p,\infty}.$

COROLLARY 15.5. $M((L^p, \ell^1), L^\infty) = c-M((L^p, \ell^1), L^\infty)$ for $1 < p < \infty$.

PROOF. By Theorem 15.4, $M((L^p, \ell^1), L^\infty) \subset c-M((L^p, \ell^1), L^\infty)$ for $1 < p < \infty$. To prove the other inclusion we shall see that if $T$ belongs to $c-M((L^p, \ell^1), L^\infty)$ then $Tf \ast g = T(f \ast g)$ for all $f \in (L^p, \ell^1)$ and $g \in L^1$. So by Theorem 15.4 $T \in M((L^p, \ell^1), L^\infty)$.

Let $g \in L^1$. Then there exists a sequence $(g_n)$ in $(L^p, \ell^1)$ such that $\lim g_n = g$ in $L^1$ (Corollary 3.8). So for $f \in (L^p, \ell^1)$

$$||Tf \ast g - T^*g_n||_\infty \leq ||Tf||_\infty ||g - g_n||_1.$$ Hence

$$Tf \ast g = \lim T^*g_n = \lim T(f \ast g_n) \in L^\infty.$$ Since $\lim ||f \ast g - f \ast g_n||_{p,1} = 0$ (Theorem 4.7) we conclude that $Tf \ast g = T(f \ast g)$ for all $f \in (L^p, \ell^1)$ and $g \in L^1$ and the proof is complete.

The proof of the next theorem is based on [40, Theorem 3.4.3].
THEOREM 15.6. Let \( 1 \leq q < \infty \). If \( T: (C_0, \ell^q) \rightarrow L^\infty \) is a linear operator then the following are equivalent:

i) \( T \in M((C_0, \ell^q), L^\infty) \).

ii) There exists a unique \( \mu \in M_q \), such that \( Tf = \mu \ast f \) for all \( f \in (C_0, \ell^q) \).

Moreover, \( M((C_0, \ell^q), L^\infty) = M((C_0, \ell^q), C_0) \), \( \mu = T^* \delta \) where \( T^* \) is the adjoint of \( T \) and the correspondence between \( T \) and \( \mu \) defines an isometric isomorphism from \( M((C_0, \ell^q), L^\infty) \) onto \( M_q \).

PROOF. If ii) holds then \( T \) commutes with translations and \( T \) is continuous by Theorem 4.8.

Suppose i). For \( f \in (C_0, \ell^q) \) and \( s \in G \)

\[ \| \tau_s T f - T f \|_\infty = \| T (\tau_s f) - T f \|_\infty \leq \| T \| \cdot \| \tau_s f - f \|_\infty. \]

Since \( s \mapsto \tau_s f \) is a continuous function from \( G \) to \( (C_0, \ell^q) \) (Theorem 3.14 and Lemma 3.13) \( T f \) is uniformly continuous on \( (C_0, \ell^q) \). Hence, it makes sense to define \( F(f) = T f(0) \) \( (f \in (C_0, \ell^q)) \). Clearly, \( F \) is linear and for \( f \in (C_0, \ell^q) \)

\[ |F(f)| = |T f(0)| \leq \| T f \|_\infty \leq \| T \| \cdot \| f \|_\infty. \]

Therefore \( F \) is continuous. That is, \( F \in \langle C_0, \ell^q \rangle \). By Theorem 3.2 there exists a unique \( \mu \in M_q \), such that \( F(f) = \int f(-t) \, d\mu(t) \).

So, for \( f \in (C_0, \ell^q) \) and \( s \in G \)

\[ (T f)(s) = \tau_s T f(0) = T (\tau_s f)(0) = F(\tau_s f) = \int f(s - t) \, d\mu(t) = f \ast \mu(s). \]

Therefore \( T f = \mu \ast f \) for all \( f \in (C_0, \ell^q) \). By Theorem 3.2

\[ \| T \| \leq \| \mu \|_q \] and \( \mu \) is unique.

The rest of the proof is similar to Theorem 15.2, using the
fact that \((C_0, \mathcal{L}^1)\) has the property \((15-2)\) of Proposition 15.2 (Theorem 4.8).

For the special case of the Wiener algebra we have the following result.

**Corollary 15.7.**

\[ M((C_0, \mathcal{L}^1), L^\infty) = M((C_0, \mathcal{L}^1), C_0) = c-M((C_0, \mathcal{L}^1), C_0) = c-M((C_0, \mathcal{L}^1), L^\infty). \]

Hence \[ M((C_0, \mathcal{L}^1), L^\infty) \cong M_\infty. \]

**Proof.** By Theorem 15.6 \[ M((C_0, \mathcal{L}^1), L^\infty) = M((C_0, \mathcal{L}^1), C_0) \] and \[ M((C_0, \mathcal{L}^1), L^\infty) \subset c-M((C_0, \mathcal{L}^1), L^\infty), \quad M((C_0, \mathcal{L}^1), C_0) \subset c-M((C_0, \mathcal{L}^1), C_0). \]

Let \( T \in c-M((C_0, \mathcal{L}^1), L^\infty) \) and consider its adjoint \( T^*: L^\infty \longrightarrow M_\infty. \)

For \( h, f \in (C_0, \mathcal{L}^1) \) and \( g \in L^1 \)
\[ < h, T^* g * f > = < h^* f, T^* g > = < T(h^* f), g > = < Th^* f, g > = < Th, g^* f > = < h, T^* (g^* f) >. \]

Hence \( T^* g * f = T^* (g^* f) \) for all \( f \in (C_0, \mathcal{L}^1), g \in L^1 \).

For \( f \in (C_0, \mathcal{L}^1), g, h \in L^1 \)
\[ < h, T(f * g) > = < T^* h, f * g > = < T^* h, g * f > = < T^* (h * g), f > = < h^* g, Tf > = < h, Tf * g >. \]

Hence \( T(f * g) = Tf * g \) for all \( f \in (C_0, \mathcal{L}^1), g \in L^1 \).

For \( f \in (C_0, \mathcal{L}^1), g, h \in L^1 \)
\[ < f, T^*(g * h) > = < Tf, g * h > = < Tf * g, h > = < T(f * g), h > = < f * g, T^* h > = < f, T^* h * g >. \]

Hence \( T^*(f * g) = T^* f * g \) for all \( f, g \in L^1 \).

Therefore \( T^* | L^1 \) belongs to \( c-M(L^1, M_\infty) \). By Remark 13.29,
\[ T^*|L^1 \subseteq M(L^1, \mu_0). \] So, for \( s \in G, f \in (C_0, \ell^1) \) and \( g \in L^1 \) we have that
\[
< g, \tau_s f > = < T^* g, \tau_s^* f > = < \tau_s^* g, f > = < \tau_s g, Tf > = < g, \tau_s Tf >.
\]
Hence \( \tau_s f = \tau_s Tf \) for all \( s \in G, f \in (C_0, \ell^1) \). Therefore
\[ T \in M((C_0, \ell^1), L^\infty) \] and we conclude that
\[ M((C_0, \ell^1), L^\infty) = c-M((C_0, \ell^1), L^\infty). \]
Finally, since \( C_0 \subseteq L^\infty \) it is clear that
\[ c-N((C_0, \ell^1), C_0) \subseteq c-M((C_0, \ell^1), L^\infty) \] and this ends the proof.

**Theorem 15.8.** Let \( A \) be any of the algebras \((l^p, \ell^1), 1 \lt p \lt \infty, \) or \((C_0, \ell^1). \) If \( T: A \to L^\infty \) is a linear operator then the following are equivalent:

1) \( T \in M(A, L^\infty). \)

2) There exists a unique \( \varphi \in A_c(G) \) such that \( (Tf)^\wedge = \varphi \hat{f} \) for all \( f \in A. \)

**Proof.** If \( T \in M(A, L^\infty) \) then by Theorem 15.4 or Theorem 15.6, there exists a unique \( \mu \in (l^p, l^\infty) \) or \( M^\infty \) such that \( Tf = \mu \hat{f} \) for all \( f \in A. \) Since \( A \subseteq L^1 \) we have by Proposition 6.14 that \( (Tf)^\wedge = \hat{\mu} f = \varphi f, \) \( \varphi = \hat{\mu}. \) It is clear that \( \varphi \) is unique. Therefore 1) implies 2). The proof of 2) implies 1) is the same as Theorem 13.23. Remember that by (2.4), if \( \lim |f_n - f|_A = 0 \) then \( \lim |f_n - f|_1 = 0. \)

It is not known, not even for \( L^p \) spaces, if \( M((L^p, \ell^q), (L^r, \ell^s)) \) is isomorphic to an amalgam space \((L^x, \ell^y) \) for some \( p, q, r, s. \) In this
direction we have a partial result, similar to [40, Theorem 5.3.3].

THEOREM 15.9. Let $1 < p \leq r < \infty$, $1 < s$, $x$, $y < \infty$. If $s/r = y/x$, $1/p - 1/r = 1 - 1/x$ and $1/q - 1/s = 1 - 1/y$ then there exists a continuous linear isomorphism from $(L^x, \ell^y)$ into $M((L^p, \ell^q), (L^r, \ell^S))$.

PROOF. By Theorem 13.18 and Theorem 15.2, the mapping $u \mapsto T_u$, $T_u g = u * g$ on $(L^x, \ell^y)$ is an isometric linear isomorphism from $(L^x, \ell^y)$ onto $M(L^{1/2}, (L^x, \ell^y))$ and onto $M((L_x', \ell^y'), L^\infty)$ (see Corollary 13.2). If $||T_u||_1$, $||T_u||_\infty$ are the norms of $T_u$ in $M(L^{1/2}, (L^x, \ell^y))$, $M((L_x', \ell^y'), L^\infty)$ respectively then

$$||u||_{xy} = ||T_u||_1 = ||T_u||_\infty.$$  

By hypothesis $1/p - 1/r = 1 - 1/x$, hence $x/r = x/p - x + 1 < 1$ because $x/p - x < 0$. Therefore $0 < 1 - x/r < 1$.

Let $\theta = 1 - x/r$. Since $s/r = y/x$, $1 - x/r = \theta = 1 - y/s$. So,

$$\frac{1}{r} = \frac{1}{r} + \left[\frac{1}{x'} - \frac{1}{x}\right] = \frac{1}{r} + \frac{1}{x'} = \frac{1}{r} + \frac{1}{1 - \frac{1}{x'}} = \frac{1}{r} - \frac{x}{rx'} + \frac{1}{x'}. $$

Similarly $1/q = 1 - \theta + \theta/y$.

Applying Theorem 12.6 with $p_2 = q_2 = 1$, $p_1 = x'$, $q_1 = y'$, $r_2 = x$, $s_2 = y$, $r_1 = s_1 = \infty$ we conclude that $T_u \in M((L^p, \ell^q), (L^r, \ell^S))$ and the norm $||T_u||$ of $T_u$ in $M((L^p, \ell^q), (L^r, \ell^S))$ is such that...
\[ ||T_\mu|| \leq ||T||^\theta \cdot ||T||^{1-\theta}_\infty = ||u||_{xy}^\theta \cdot ||u||_{xy}^{1-\theta} \leq ||u||_{xy}. \]

Therefore \( \mu \rightarrow T_\mu \) defines a continuous linear isomorphism from \((L^X, \ell^Y)\) into \(M((L^P, \ell^Q), (L^R, \ell^S))\).\(\dagger\)
§ 16. CHARACTERIZATION OF MULTIPLIERS IN TERMS OF ELEMENTS OF $S_0(G)^*$

In section 6 we gave a brief account of the Segal algebra $S_0(G)$, originally defined by H. G. Feichtinger [25] and studied independently by J. P. Bertrandias [6].

In this last section we will study this algebra $S_0(G)$ in more detail and characterize the c-multipliers from the algebras $(L^p, \ell^1)$, $1 < p < \infty$, $(C_0, \ell^1)$ and $M_1$ to any amalgam space or any space of measures $M_q$, in terms of elements of $S_0(G)^*$.

First we will prove some results about the multipliers and c-multipliers from the Segal algebras $(L^p, \ell^1)$ $(1 < p < \infty)$, $(C_0, \ell^1)$ to amalgam spaces and to the space $M_q$ $(1 < q < \infty)$.

THEOREM 16.1. Let $S$ be any of the Segal algebras $(L^p, \ell^1)$ $(1 < p < \infty)$, $(C_0, \ell^1)$ and $B$ be any of the spaces $(L^p, \ell^q)$, $(L^p, c_0)$, $1 < p, q < \infty$, $(C_0, \ell^S)$, $1 < s < \infty$, Then $M(S,B) \subseteq c-M(S,B)$.

PROOF. Let $f, g \in S$ and $\psi \in S^*$. A simple calculation shows that

\[ < f* g, \psi > = \int g(s) < \tau_s f, \psi > ds. \]  

Now observe that for $h \in L^1_1, k \in B$ and $F \in B^*$

\[ < h* k, F > = \int h(s) < \tau_s k, F > ds. \]
Take \( T \in M(S, B) \). For a fixed \( F \in B^* \) the map \( \Lambda_F(f) = \langle Tf, F \rangle \) is a bounded linear functional on \( S \). That is,

\[
\langle f, \Lambda_F \rangle = \langle Tf, F \rangle \quad \text{for all } f \in S.
\]

This together with (1) and (2) implies that for \( f, g \in S \) and \( F \in B^* \)

\[
\langle Tf^*g, F \rangle = \int g(s) \langle \tau_s(Tf), F \rangle \, ds = \int g(s) \langle T(\tau_s f), F \rangle \, ds
\]

\[
= \int g(s) \langle \tau_s f, \Lambda_F \rangle \, ds = \langle f^*g, \Lambda_F \rangle = \langle T(f^*g), F \rangle.
\]

Since \( F \) is arbitrary, \( T(f^*g) = Tf^*g \) for all \( f, g \in S \).

**THEOREM 16.2.** Let \( S \) be as in Theorem 16.1 and \( B \) be any of the spaces \( (L^p, \ell^q) \), \( (L^p, c_0) \), \( 1 < p, q < \infty \). Then \( M(S, B) = c-M(S, B) \).

**PROOF.** Take \( T \in c-M(S, B) \), \( s \in G, f \in S \) and \( h \in C_c \). So,

\[
\langle T_{\tau_s} f, h \rangle = \langle T_{\tau_s} f^*h(0) = T(\tau_s f^*h)(0) = T(f^*\tau_s h)(0) = Tf^*\tau_s h(0)
\]

\[
= \langle Tf, \tau_s h \rangle = \langle \tau_s Tf, h \rangle.
\]

Hence \( \langle T_{\tau_s} f, h \rangle = \langle \tau_s Tf, h \rangle \) for all \( h \in C_c \). Since \( C_c \) is dense in \( B^* \), \( B^* \) is either \( (L^{p'}, \ell^{q'}) \) or \( (L^{p'}, \ell^1) \), \( 1 < p', q' < \infty \) we conclude that \( \langle T_{\tau_s} f, h \rangle = \langle \tau_s Tf, h \rangle \) for all \( h \in B^* \). Therefore

\( T_{\tau_s} f = \tau_s Tf \) for all \( s \in G, f \in S \). This means that \( T \in M(S, B) \). The conclusion follows from Theorem 16.1.

**THEOREM 16.3.** Let \( S \) be as in Theorem 16.1 and \( B \) be any of the spaces \( (L^\infty, \ell^q) \), \( M_q \), \( 1 \leq q \leq \infty \), \( (L^p, \ell^\infty) \), \( (L^p, \ell^1) \), \( 1 < p \leq \infty \). Then \( M(S, B) = c-M(S, B) \).
**Proof.** First we note that $B$ is the dual of an amalgam space $C$.

So for $h \in L^1, k \in C$ and $F \in B$

$$< h^*k, F > = \int h(s) \langle \tau_s k, F \rangle \, ds.$$  \hspace{1cm} (3)

As in the proof of Theorem 16.1, using (1) and (3) we see that if $T \in M(S,B)$ then $T$ commutes with convolution. Hence $M(S,B)$ is included in $c\cdot M(S,B)$.

Conversely, as in the proof of Theorem 16.2, we have that for $T \in c\cdot M(S,B), s \in G, f \in S$, and $h \in C_c < \tau_s f, h > = < \tau_s Tf, h >$. Since $C_c$ is dense in $C$ we conclude that $< \tau_s f, h > = < \tau_s Tf, h >$ for all $h \in C$. This implies that $T \in M(S,B).$ \hspace{1cm} \dagger

**Definition 16.4.** The Fourier algebra $A(G)$ is the linear space of functions $f$ in $C_0(G)$ such that $f = \frac{1}{\delta}, \delta \in L^1(\hat{G})$, endowed with the norm $||f||_A = ||\delta||_1$.

$A(G)$ is an algebra under pointwise multiplication.

**Definition 16.5.** Consider the following function $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & \text{if } |x| \geq 1 \\ 1 - |x| & \text{if } |x| \leq 1. \end{cases}$$

The Fourier transform of $f$ is equal to $2/\sqrt{2\pi} \left(1 - \cos(\tau^2)\right)$ and therefore $f \in A_c(\mathbb{R})$ and $\text{supp } f = [-1,1]$.

For $n \in \mathbb{Z}$ we define the function $f_n$ to be $\tau_{nf}$. Then it is clear that for each $n$ $f_n \in A_c(\mathbb{R})$ and $\text{supp } f_n = n + \text{supp } f$.

Moreover for each $x \in \mathbb{R}$ \[ \sum_{n} f_n(x) = 1 \] because if
\[ x \in [m-1, m] \text{ for some } m \in \mathbb{Z} \text{ then} \]
\[ \sum_{n} f_{n}(x) = f_{m-1}(x) + f_{m}(x) = (m - x) + x - (m - 1) = 1. \]

Now, by the Decomposition Theorem \( G = \mathbb{R}^{\alpha} \times G_{1} \) (see p. 10) and for \( s = (x, t) \) in \( G \), we define the function \( \psi: G \rightarrow \mathbb{R} \) by
\[ \psi(s) = f(x_{1}) \cdot \ldots \cdot f(x_{\alpha}) \chi_{H}(t), \quad x = (x_{1}, \ldots, x_{\alpha}). \]

It is clear that \( \psi \in A_{c}(G) \) and \( \text{supp } \psi = [-1, 1]^{\alpha} \times \mathbb{H} \). Then for \( \alpha = (m_{1}, \ldots, m_{\alpha}, t) \) in \( J \) (\( J \) as in Definition 1.6) the function
\[ \psi_{\alpha} = \tau_{\alpha} \psi = f_{m_{1}} \cdot \ldots \cdot f_{m_{\alpha}} \chi_{t+\mathbb{H}} \]
has the following properties

1. \( \psi_{\alpha} \in A_{c}(G) \)
2. \( \text{supp } \psi_{\alpha} = \alpha + \text{supp } \psi \)
3. \( \sum_{\alpha} \psi_{\alpha}(s) = 1 \) for each \( s \in G \).

**DEFINITION 16.6.** Let \( \{\psi_{\alpha}\} \) be the family defined in Definition 16.5. Then \( S_{\alpha} = S_{\alpha}(G) \) is the linear space of continuous functions \( f \in A(G) \) such that \( \sum_{\alpha} \| f\psi_{\alpha} \|_{A} < \infty \), endowed with the norm \( \| f \|_{S_{\alpha}} = \sum_{\alpha} \| f\psi_{\alpha} \|_{A} \).

It follows from [41, Proposition 1] that Definition 16.6 is equivalent to Feichtinger original definition of \( S_{\alpha}(G) \) in [26].

The following are some of the properties of \( S_{\alpha}(G) \). For a proof see [26] and [41].

1. \( S_{\alpha}(G) \) is a Segal algebra. Hence it is a Banach \( L^{1} \) and \( M_{1} \) module.
2. \( A_{c}(G) \) is dense in \( S_{\alpha}(G) \).
3) \( M_1 \cup M_\omega \subseteq S_0(G)^\times \subseteq Q(G) \).
4) \( S_0(G) \subseteq \{ f \in (C_0, \ell^1) \mid \hat{f} \in (C_0, \ell^1) \} \).
5) \( S_0(G) = S_0(\hat{G}) \).

**Definition 16.7.** The Fourier transform \( F_0 \sigma \) of \( \sigma \in S_0(G)^\times \) is an element of \( S_0(\hat{G})^\times \) defined by

\[
< h, F_0 \sigma > = < h', \sigma > = < \hat{h}, \sigma > \quad (h \in S_0(\hat{G})).
\]

It is clear from 5) that \( F_0 \sigma \) is well defined.

**Remark 16.8.** i) By 4) any \( h \) in \( S_0(G) \) is equal to the inverse of its Fourier transform. That is, \( h = \hat{h} \). Hence for any \( \psi_\alpha \), (as in Definition 16.5) \( \| h\psi_\alpha \|_A = \| \hat{h}\hat{\psi}_\alpha \|_1 \) and \( \{ h\psi_\alpha \}_j \subseteq (C_0, \ell^1) \).

ii) If \( \sigma \in S_0(\hat{G})^\times \) then its Fourier transform \( F_\sigma \) is defined as

\[
< h, F_\sigma > = < \hat{h}', \sigma > = < \hat{h}, \sigma > \quad (h \in S_0(G)).
\]

Therefore by 1) \( F_0(F_\sigma \sigma) = \sigma \) (\( \sigma \in S_0(\hat{G})^\times \)).

**Proposition 16.9.** Let \( \mu \in M_\omega \). Then \( \hat{\mu} \) as in Definition 6.9 coincides with \( F_0 \mu \) as in Definition 16.7 iff there exists a constant \( C \) such that

\[
| < h, \hat{\mu} > | \leq C \| h \|_{S_0} \quad \text{for all } h \in A_0(\hat{G}).
\]

**Proof.** This is a direct consequence of Definition 6.9 and property 2).†

By property 1) we can define \( \sigma * \tilde{\mu} \) (\( \sigma * \tilde{\mu} \)) for \( \sigma \in S_0(G)^\times \) and
\[ f \in L^1(G) \ (\nu \in M_1(G)) \] to be an element of \( S_0(G)^* \) given by

(16.1) \[ < h, \sigma f > = < h^* f, \sigma > \]

(16.2) \[ < h, \sigma^\nu > = < h^\nu, \sigma > \quad (h \in S_0(G)). \]

Moreover, if \( f \in L^1(\hat{G}) \ (\nu \in M_1(\hat{G})) \) and \( h \in S_0(G) \) then
\[ \hat{h}^\nu \ (h^\nu) \] belongs to \( S_0(G) \) because for any \( \alpha \in J \)

\[ \| h^\nu \psi_\alpha \|_A = \| \hat{h}^\nu \psi_\alpha^* f \|_1 \leq \| f \|_1 \| \hat{h}^\nu \psi_\alpha \|_1. \]

Similarly \[ \| h^\nu \psi_\alpha \| \leq \| \nu \|_1 \| \hat{h}^\nu \psi_\alpha \|_1. \]

Hence by Remark 16.8 i)

(16.3) \[ \| h^\nu f \|_{S_0} \leq \| f \|_1 \| h \|_{S_0} \]

(16.4) \[ \| h^\nu \psi_\alpha \|_{S_0} \leq \| \nu \|_1 \| h \|_{S_0}. \]

Then we define \( \hat{\sigma}^\nu (\sigma^\nu) \) for \( \sigma \in S_0(G)^* \), \( f \in L^1(\hat{G}) \)
\( (\nu \in M_1(\hat{G})) \) to be the element in \( S_0(G)^* \) given by

(16.5) \[ < h, \sigma f > = < h^\nu, \sigma > \]

(16.6) \[ < h, \sigma^\nu > = < h^\nu, \sigma > \quad (h \in S_0(G)). \]

**PROPOSITION 16.10.** Let \( \sigma \in S_0(G)^* \), \( f \in L^1(G) \ (\nu \in M_1(G)), \) and \( g \in L^1(\hat{G}) \ (\eta \in M_1(\hat{G})). \) Then

i) \[ F_0(\sigma^\nu f) = F_0(\sigma(\hat{f}) \quad (F_0(\sigma^\nu \nu) = F_0(\sigma(\hat{\nu}) \]

ii) \[ F_0(\sigma g) = F_0(\sigma^\nu \nu) \quad (F_0(\sigma \eta) = F_0(\sigma^\nu \eta). \]

**PROOF.** Let \( h \in S_0(\hat{G}). \) By (16.5) and the definition of \( F_0(\sigma), \)
we have that

\[ < h, F_0(\sigma(\hat{f}) > = < h^\nu, F_0(\sigma > = < h^\nu, \sigma (> = < (h^\nu)^*, \sigma >

\[ = < h^\nu^*, \sigma > = < h^\nu, \sigma^\nu f > = < h, F_0(\sigma^\nu f) >. \]
Therefore i) holds.

Now, by Remark 16.8 i) and part i)
\[ F_v(F_0 \sigma g) = F_v(F_0 \sigma) g = \sigma g. \]
This implies that \( F_0 \sigma g = F_0 (\sigma g) \) and this proves ii).

The proof for \( v \) and \( \eta \) is the same.†

**THEOREM 16.11.** Let \( 1 \leq p, q \leq \infty \). If \( B \) is any of the spaces \((L^p, L^q), (L^p, c_0), (C_0, L^q) \) or \( M_q \), \( S \) is any of the algebras \((L^p, L^1), (C_0, L^1) \) and \( T: S \to B \) is a linear operator then the following are equivalent:

i) \( T \in \mathcal{M}(S, B) \)

ii) There exits a unique \( \sigma \in S_0(\hat{G})^* \) such that \( (Tf)^\wedge = \sigma f \) for all \( f \in S \). 

iii) There exists a unique \( \mu \in S_0(G)^* \) such that \( Tf = \mu f \) for all \( f \in S \).

**PROOF.** We will prove that i) is equivalent to ii) and ii) is equivalent to iii).

First we observe that if for all \( h \in A_c(\hat{G}) \) and \( f \in S \),
\[ < h, (Tf)^\wedge > = < h, \sigma \hat{f} > \text{ for some } \sigma \in S_0(\hat{G})^* \text{ then by (16.5) and (16.3)} \]
\[ |< h, (Tf)^\wedge >| = |< h, \sigma \hat{f} >| = |< \hat{h} f, \sigma >| \leq ||h||_1 ||\sigma||_1 ||f||_1. \]

Therefore by Proposition 16.9 \( F_0(Tf) = (Tf)^\wedge \) for all \( f \in S \).

Hence if ii) holds then by Remark 16.8 ii) and Proposition 16.10 for all \( f \in S \)
\[ Tf = F_v(F_0(Tf)) = F_v(Tf)^\wedge = F_v(\sigma f) = F_v(\sigma \hat{f}). \]
Then we conclude that ii) implies iii) with \( \nu = F_0\mu \).

Conversely if iii) holds then by Proposition 16.10 and (16.3) we have that for all \( f \in S \) and \( h \in \mathcal{A}_C(\hat{G}) \)

\[
|< h, (Tf)^\wedge >| = |< \hat{y}', Tf >| = |< \hat{y}', \mu^*f >| = |< h, F_0(\mu^*f) >| \\
= |< h, F_0\hat{f} >| = |< \hat{h}\hat{f}, F_0\mu >| \leq \|F_0\mu\| \|\hat{f}\|_{S_0}
\]

Again by Proposition 16.9 \((Tf)^\wedge = F_0(Tf)\).

Applying Proposition 16.10 once more we have that \((Tf)^\wedge = F_0(Tf) = F_0(\mu^*f) = F_0\hat{f}\) and therefore ii) holds with \( \sigma = F_0\mu \).

Suppose i). Since either \( B^* \) is an amalgam space or a measure space of type \( q \), or \( B \) is the dual of an amalgam space \( C \), we have by the Hölder inequality for amalgams (Theorems 3.1 and 3.2) that

\[
|< f, g >| \leq \|f\|_{B^*} \|g\|_B \quad (g \in B, f \in B^*) \\
|< f, g >| \leq \|f\|_C \|g\|_B \quad (g \in B, f \in C).
\]

This implies by (2.3) and (2.4) that

\[
|< f, g >| \leq \|g\|_B \|f\|_{s_1} \quad \text{for all } g \in B, f \in (C_0, l^1).
\]

Now, for \( f, g \in S, Tf*g = T(f*g) = Tg*f \). So by Proposition 6.14 part 1), \((Tf)^\wedge = (Tg)^\wedge \hat{f} \). This implies by Definition 6.13 that for all \( f, g \in S \) and \( h \in \mathcal{A}_C(\hat{G}) \)

\[
< h, (Tf)^\wedge g > = < h, (Tf)^\wedge > < h, (Tg)^\wedge f > = < h, (Tg)^\wedge >.
\]

Let \( \{\psi_\alpha\} \subset \mathcal{A}_C(\hat{G}) \) be as in Definition 16.5 and \( W = \text{supp } \psi \). To each \( \alpha \) we associate a function \( \lambda_\alpha \) in \((C_0, l^1)(G)\) as follows.

Take \( \lambda_W \in (C_0, l^1)(G) \) such that \( \lambda_W \equiv 1 \) on \( W \) and \( \lambda_W \in C_c(\hat{G}) \). Then \( \lambda_\alpha = [\cdot, \alpha]\lambda_W \). It is clear that each \( \lambda_\alpha \) has the fol-
lowing properties:

a) \( \lambda_\alpha \in (C_0, \ell^1)(G) \).

b) \( \hat{\lambda}_\alpha = T_\alpha^{\hat{\gamma}} \). Hence by (2) \( \hat{\lambda}_\alpha \equiv 1 \) on \( \alpha + \mathbb{W} = \text{supp } \psi_\alpha \).

c) \( \hat{\lambda}_\alpha \in C_c(\hat{G}) \).

d) \( \|\lambda_\alpha\|_1 = \|\lambda_\mathbb{W}\|_1 \).

We define \( \sigma \) on \( A_c(\hat{G}) \) by

\[
<h, \sigma> = \sum_{\alpha} <h \psi_\alpha, (T \lambda_\alpha)^\wedge> \quad (h \in A_c(\hat{G})).
\]

First we note that if \( h \in A_E(\hat{G}) \) then \( h \psi_\alpha \) belongs to \( A_E(\hat{G}) \) because \( h \psi_\alpha \in C(E) \), \( (h \psi_\alpha)^\wedge = h^* \phi_\alpha \) and \( h^* \phi_\alpha \in (C_0, \ell^1) \) (Theorem 4.7), so by Lemma 6.4 \( h \psi_\alpha \in A_E(\hat{G}) \). Also by b) \( h \psi_\alpha = h \psi_\alpha \hat{\gamma} \\alpha \) and this implies that

\[
\|h \psi_\alpha\|^\wedge_{1} = \|h^* \phi_\alpha \gamma\lambda_\alpha\|^\wedge_{1} \leq \|\lambda_\alpha\|_1 \|h^* \phi_\alpha\|_1
\]

\[
\leq \|\lambda_\mathbb{W}\|_1 \|h \psi_\alpha\|_A.
\]

Therefore by (4)

\[
< h \psi_\alpha, (T \lambda_\alpha)^\wedge > = \|h \psi_\alpha\|^\wedge_{1} \|\lambda_\alpha\|_1 \|h \psi_\alpha\|_A
\]

\[
\leq \|T\| \|\lambda_\alpha\|_\mathcal{S}^{\gamma} \|h \psi_\alpha\|_\mathcal{A}
\]

\[
\leq \|T\| \|\lambda_\mathbb{W}\|_1 \leq \|h \psi_\alpha\|_\mathcal{A}.
\]

Hence \( \sigma \) is well defined and for all \( h \in A_c(\hat{G}) \)

\[
|< h, \sigma >| \leq \|T\| \|\lambda_\mathbb{W}\|_1^2 \|h\|_\mathcal{S}.\]

If \( \lambda_\mathbb{W}^\gamma \) is another function in \( (C_0, \ell^1)(G) \) with the same properties as \( \lambda_\mathbb{W} \) and \( \lambda_\alpha^\gamma = [\cdot, \alpha] \lambda_\mathbb{W}^\gamma \) then by (5) we have that for all

\( h \in A_c(\hat{G}) \)
\[
\langle \psi_\alpha, (T\lambda_\alpha) \rangle = \langle \psi_\alpha \lambda_\alpha, (T\lambda_\alpha) \rangle = \langle \psi_\alpha \lambda_\alpha, (T\lambda_\alpha') \rangle = \langle \psi_\alpha, (T\lambda_\alpha') \rangle.
\]

This shows that the definition of $\sigma$ does not depend on the choice of the function $\lambda_\alpha$.

By the density of $A_c(\hat{G})$ in $S_0(\hat{G})$, $\sigma$ has a unique continuous extension $\sigma$ on $S_0(\hat{G})$.

Now, if $h \in A_E(\hat{G})$ then $\{h\psi_\alpha \in A_E(\hat{G}), h\psi_\alpha = 0 \text{ for all but finitely many } \alpha\}$ and by (3) $h = \sum_{\alpha} h\psi_\alpha$, pointwise. Then for $f \in S$
\[
\langle h, (Tf) \rangle = \sum_{\alpha} \langle h\psi_\alpha, (Tf) \rangle = \sum_{\alpha} \langle h\psi_\alpha, (Tf) \rangle = \langle h, (Tf) \rangle.
\]

This together with (5) and (16.5) implies that for $f \in S$ and $h \in A_E(\hat{G})$
\[
\langle h, \sigma f \rangle = \langle h\hat{f}, \sigma \rangle = \sum_{\alpha} \langle h\hat{\psi}_\alpha, (T\lambda_\alpha) \rangle = \sum_{\alpha} \langle h\psi_\alpha \lambda_\alpha, (Tf) \rangle = \sum_{\alpha} \langle h\psi_\alpha, (Tf) \rangle = \langle h, (Tf) \rangle.
\]

By the observation we made at the beginning of the proof we conclude that $(Tf) \in S_0(\hat{G})^*$. Since $A_c(\hat{G})$ is dense in $S_0(\hat{G})$,
\[
(Tf) = \sigma f \text{ for all } f \in S.
\]

If $h \in A_E(\hat{G})$ and $\lambda_\hat{E}$ is a function in $(C_0, l^1)(\hat{G})$ such that $\lambda_\hat{E} \equiv 1$ on $E$ then we have that
\[
\langle h, \sigma \rangle = \langle h\lambda_\hat{E}, \sigma \rangle = \langle h, \sigma \lambda_\hat{E} \rangle = \langle h, (T\lambda_\hat{E}) \rangle.
\]

Finally if $\sigma'$ is another functional in $S_0(\hat{G})^*$ such that $(Tf) = \sigma' f$ for all $f \in S$, then by (6) for all $h \in A_E(\hat{G})$
\[
\langle h, \sigma \rangle = \langle h, (T\lambda_\hat{E}) \rangle = \langle h, \sigma \lambda_\hat{E} \rangle = \langle h\lambda_\hat{E}, \sigma \rangle = \langle h, \sigma \rangle.
\]
Hence $\sigma = \sigma'$ and therefore i) implies ii).

Conversely if ii) holds then for $f, g \in S$
\[
(T(f*g)) = \sigma(f*g) = \sigma(fg) = (\sigma f)g = (Tf)g = ((Tf)*g).
\]

By Theorem 13.22 $T(f*g) = Tf*g$.

To prove that $T$ is continuous we proceed as in Theorem 13.23.

Let $\{f_n\}, f \in S$ such that $\lim ||f_n - f||_S = 0$ and assume that $\lim ||Tf_n - g||_B = 0$.

Take $h \in A_C(\hat{G})$. So, by (16.3) and (4) we have that
\[
|h, (Tf)^\sim - \hat{g} > | \leq |< h, (Tf_n)^\sim -(Tf)^\sim > | + \|h, (Tf_n)^\sim - \hat{g} > | \\
\leq |< h, \sigma(f_n - f) > | + |< h', T\sigma f_n - g > | \\
\leq |< h, (f_n - f)^\sim, \sigma > | + ||Tf_n - g||_B ||h||_{\sigma_1} \\
\leq ||\sigma|| \||h(f_n - f)^\sim||_{\sigma_0} + ||Tf_n - g||_B ||h||_{\sigma_1} \\
\leq ||\sigma|| \||h||_{\sigma_0} ||f_n - f||_1 + ||Tf_n - g||_B ||h||_{\sigma_1}.
\]

Therefore $(Tf)^\sim = \hat{g}$ on $A_C(\hat{G})$ and by Theorem 13.22, $Tf = g$.

By the Closed Graph Theorem $T$ is continuous and this implies that $T \in M(S, B)$ and the proof is complete.

The proof of the next theorem is very similar to Theorem 16.11 and it will be omitted.

**THEOREM 16.12.** If $B$ is as in Theorem 16.11 and $T: M_1 \longrightarrow B$ is a linear operator then the following are equivalent:

i) $T \in c-M(M_1, B)$.

ii) There exists a unique $\sigma \in S_0(\hat{G})^*$ such that $(T\upsilon)^\sim = \sigma\upsilon$ for all $\upsilon \in M_1$. 

iii) There exists a unique \( \mu \in \mathcal{S}_0(G) \) such that \( T \nu = \mu \ast \nu \) for all \( \nu \in \mathcal{M}_1 \).

**Remark 16.13.** It follows from Theorem 16.11 that \( c\mathcal{M}(S,B) \subset \mathcal{M}(S,B) \) (\( S, B \) as in Theorem 16.11). Indeed if \( T \in c\mathcal{M}(S,B) \) then for \( s \in G \) and \( f \in S \)

\[
(T_{s} f)^{\wedge} = \sigma(s)^{\wedge} \circ ([s, \cdot]^{\wedge} f) = [s, \cdot]^{\wedge} (T f)^{\wedge} = \sigma(s)^{\wedge} (T f)^{\wedge}.
\]

Hence by Theorem 13.22 \( T \) commutes with translations and this implies that \( c\mathcal{M}(S,B) \subset \mathcal{M}(S,B) \).

We conclude that if \( S \) is as in Theorem 16.1 and \( B \) is as in Theorem 16.11 then \( c\mathcal{M}(S,B) = \mathcal{M}(S,B) \).

Observe that \( c\mathcal{M}(S, (L^1, l^\infty)) \subset c\mathcal{M}(S, M_{\infty}), \) and by Theorem 16.3 \( c\mathcal{M}(S, M_{\infty}) = \mathcal{M}(S, M_{\infty}) \). Therefore \( c\mathcal{M}(S, (L^1, l^\infty)) \subset \mathcal{M}(S, (L^1, l^\infty)) \). Similarly by Theorem 16.3 \( c\mathcal{M}(S, \left( L^\infty, c_0 \right)) \subset c\mathcal{M}(S, L^\infty) = \mathcal{M}(S, L^\infty) \), so \( c\mathcal{M}(S, (L^\infty, c_0)) \subset \mathcal{M}(S, (L^\infty, c_0)) \).

**Corollary 16.14.** Let \( S \) be as in Theorem 16.11 and \( B \) be any space \((L^p, l^q), (C_0, l^q), M_q\) \((1 \leq p \leq \infty, 1 \leq q \leq 2)\). If \( T : S \rightarrow B \) is a linear operator then the following are equivalent:

i) \( T \in c\mathcal{M}(S,B) \)

ii) There exists a unique \( \phi \in (L_1, l^\infty)(\hat{G}) \) such that \( (T f)^{\wedge} = \phi \hat{f} \) for all \( f \in S \).

**Proof.** Let \( \hat{x} \in \hat{G} \) and \( \psi_x \) be a function in \( S \) such that \( \psi_x(\hat{x}) = 1 \). Define \( \phi(\hat{x}) = (T \psi_x)^{\wedge}(\hat{x}) \) \((\hat{x} \in \hat{G})\).

If \( T \in c\mathcal{M}(S,B) \) then \( \phi \) is independent of the choice of \( \psi_x \).
because if \( \psi' \in S \) and \( \psi_X'(\hat{x}) = 1 \) then

\[
(T\psi_X')^\wedge(\hat{x}) = (T\psi_X^\wedge(\hat{x}))^\wedge(\hat{x}) = (T\psi_X^\wedge(\hat{x}))^\wedge(\hat{x}) = (T\psi_X^\wedge(\hat{x}))^\wedge(\hat{x}) = (T\psi_X')^\wedge(\hat{x}) = (T\psi_X')^\wedge(\hat{x}).
\]

Also for \( f \in S \)

\[
\varphi^\wedge(\hat{x}) = (T\varphi_X')^\wedge(\hat{x})(\hat{f}) = (T\varphi_X^\wedge(\hat{x}))(\hat{f}) = (T\varphi_X^\wedge(\hat{x}))(\hat{f}) = (T\varphi_X^\wedge(\hat{x}))(\hat{f}) = (T\varphi_X^\wedge(\hat{x}))(\hat{f}) = (T\varphi_X^\wedge(\hat{x}))(\hat{f}) = (T\varphi_X')^\wedge(\hat{x}).
\]

Let \( \varphi_T \) be the element in \( A_c(\hat{G})^\wedge \) associated to \( T \) by Theorem 16.11. Then by (6) of Theorem 16.11 for \( h \in A_c(\hat{G}) \)

\[
\langle h, \varphi_T \rangle = \langle h, (T\lambda_e)^\wedge \rangle = \int h(\hat{x}) (T\lambda_e)^\wedge(\hat{x}) d\hat{x}
\]

\[
= \int h(\hat{x}) \varphi(\hat{x}) \lambda_e(\hat{x}) d\hat{x}
\]

\[
= \int h(\hat{x}) \varphi(\hat{x}) d\hat{x}.
\]

Therefore \( \langle h, \varphi_T \rangle = \int h(\hat{x}) \varphi(\hat{x}) d\hat{x} \) for all \( h \in A_c(\hat{G}) \).

Now take \( k \in C_c(\hat{G}) \) such that \( k \equiv 1 \) on \( \hat{G} \) and \( \varphi \in S \). Then \( k_B = \tau_B k \) belongs to \( C_c(\hat{G}) \), \( k_B \equiv 1 \) on \( \hat{B} \), \( \varphi \in S \) and \( \|k_B\|_S = \|k\|_S \) for all \( \beta \).

By the definition of the norm \( \|\cdot\|_{q'r} \), \( 1 \leq r \leq \infty \), it is clear that \( \|\varphi_{x_LB}\|_{\lambda} \leq \|\varphi_{x_LB}\|_{q'r} \).

If \( B \) is the space \( (L^p, L^q) \), \( 1 \leq p, q \leq 2 \) then by Theorem 5.7

\[
\|\varphi_{x_LB}\|_{q'} \leq \|\varphi_{x_LB}\|_{q'p'} \leq \|\varphi_{k_B}\|_{q'p'} = \|\varphi_{k_B}\|_{q'p'} = \|T_{k_B}\|_{q'p'} \leq c \|T_{k_B}\|_{pq} \leq c \|T\| \|k_B\|_S
\]

\[
= c \|T\| \|k\|_S.
\]
If \( B \) is the space \((L^p, l^q), (C_0, l^q)\), \(1 \leq q \leq 2 \leq p \leq \infty\), then \( B \subset (L^2, l^q) \) and as in the previous case

\[
\|\varphi_{X_{\beta}}\|_{q'} \leq \|\varphi_{X_{\beta}}\|_{q} \leq c \cdot \|T\| \cdot \|k\|_S .
\]

Similarly if \( B = M_q \), \( \|\varphi_{X_{\beta}}\|_{q'} \leq \|\varphi_{X_{\beta}}\|_{q} \leq c \cdot \|T\| \cdot \|k\|_S . \)

Since this is for all \( \alpha \), in any case \( \varphi \in (L^q', l^\infty) \). Hence (i) implies (iii).

Conversely if \( \varphi \in (L^q', l^\infty) \) and \( (Tf)^{\hat{}} = \varphi^{\hat{}} (f \in S) \) then we define \( \varphi_T \) on \( A_c (\hat{G}) \) by \( < h, \varphi_T > = \int h(\xi) \varphi_T (\xi) d\xi \) \( (h \in A_c (\hat{G})) \). Then as above \( < h, \varphi_T > = < h, (T \lambda_E)^{\hat{}} > \). Therefore by Theorem 16.11, (ii) implies (i). (See (6) of the proof of Theorem 16.11.)

**THEOREM 16.15.** Let \( S \) be any of the algebras \((L^p, l^1), 2 \leq p \leq \infty\), or \((C_0, l^1). The correspondence between \( c-M(S, L^2) \) and \((L^2, l^\infty) \) established by Corollary 16.14 defines a continuous isomorphism from \( c-M(S, L^2) \) onto \((L^2, l^\infty) \).

**PROOF.** If \( f \in S \) then \( \hat{f} \in (L^2, l^1) \). Hence by Theorem 5.2, \( \hat{f} \in (C_0, l^2) \). By Proposition 4.1, \( \hat{f} \in L^2 \) for \( \varphi \in (L^2, l^\infty) \). Then there exists a unique \( Tf \in L^2 \) such that \( (Tf)^{\hat{}} = \varphi^{\hat{}} \). It is clear that \( T \) so defined is a linear operator from \( S \) to \( L^2 \) and the conclusion follows from Theorem 16.11 and the fact that for \( f \in S \)

\[
\|Tf\|_2 = \| (Tf)^{\hat{}} \|_2 \leq \|\varphi\|_2 \|\hat{f}\|_\infty \leq \|\varphi\|_\infty C \|f\|_2 \leq \|\varphi\|_\infty C \|f\|_S.
\]

**COROLLARY 16.16.** Let \( S \) be as in Theorem 16.15 and \( B \) be \((L^r, l^2), 1 \leq r \leq 2, \) or \( M_2 \). Then \( c-M(S, L^2) = c-M(S, B) \).
PROOF. Since \( L^2 \subset B \) it is clear that \( c-M(S,L^2) \subset c-M(S,B) \). If \( T \in c-M(S,B) \) then by Corollary 16.14 there exists a unique \( \varphi \in (L^2, L^\infty) \) such that \( (Tf) = \hat{\varphi} f \) for all \( f \in S \).

By Theorem 16.15, there exists \( T' \in c-M(S,L^2) \) such that \( (T'f) = \hat{\varphi} f \) for all \( f \in S \). Therefore \( (Tf) = (T'f) \) for all \( f \in S \). This implies by Theorem 13.22 that \( T = T' \). Hence \( c-M(S,B) \subset c-M(S,L^2) \).

COROLLARY 16.17. Let \( S \) be as in Theorem 16.15 and \( B \) be \((L^p, L^2), 2 < p \leq \infty, \) or \((C_0, L^2)\). Then \( c-M(S,B) \subset c-M(S,L^2) \).

PROOF. Let \( T \in c-M(S,B) \). By Corollary 16.14 there exists a unique \( \varphi \in (L^2, L^\infty) \) such that \( (Tf) = \hat{\varphi} f \) for all \( f \in S \).

Similarly to Corollary 16.16, this implies that \( T = T' \) for \( T' \in c-M(S,L^2) \).

Let \( A \) be any of the amalgam spaces \((L^p, L^q), (C_0, L^q)\), \((L^p, c_0)\) \((1 \leq p, q < \infty)\) and \( B \) be as in Theorem 6.11.

We associate to \( A \) the biggest algebra \( \hat{A} \) such that \( c-M(S_A, B) = M(S_A, B) \).

By Remark 16.13 we see that if \( A \) is \((L^p, L^q), (L^p, c_0)\) \((1 \leq p < \infty, 1 \leq q < \infty)\) then \( S_A = (L^p, L^1) \) and \( S_A = (C_0, L^1) \) for the remaining cases.

THEOREM 16.18. Let \( A \) be any of the spaces \((L^p, L^q), (C_0, L^q), (L^p, c_0)\) \((1 \leq p, q < \infty)\) and \( B \) be as in Theorem 16.11. If \( T \in M(A,B) \) and \( S_A \) is as above then
i) There exists a unique \( \varphi \in S_0^*(\hat{G}) \) such that \((Tf)^\sim = \varphi f\) for all \( f \in S_A^* \).

ii) There exists a unique \( \mu \in S_0(G)^* \) such that \( Tf = \mu f \) for all \( f \in S_A^* \).

**Proof.** If \( T \in M(A,B) \) then \( T|_{S_A} \in M(S_A,B) \). By Remark 16.13 \( T|_{S_A} \in cM(S_A,B) \) and the conclusions follow from Theorem 16.11.†

Compare Theorem 16.18 with [25, Theorem C2].

**Theorem 16.19.** Let \( A \) be any of the spaces \((L^p, \ell^q), (C_0, \ell^q), \)

\( 1 \leq p < \infty, 1 \leq q \leq 2 \), and let \( B \) be any of the spaces \((L^r, \ell^s), (C_0, \ell^s), 1 \leq r < \infty, 1 \leq s \leq 2 \). If \( T: A \rightarrow B \) is a linear operator then the following are equivalent:

i) \( T \in M(A,B) \)

ii) There exists a unique \( \varphi \in (L^{s'}, \ell^{\infty})(\hat{G}) \) such that \((Tf)^\sim = \varphi f\) for all \( f \in A \).

**Proof.** We will prove the theorem for \( A = (L^p, \ell^q) \) and \( B = (L^r, \ell^s), 1 \leq r, s \leq 2 \). The remaining cases are similar.

Suppose \( T \in M(A,B) \). Then \( T|_{(L^p, \ell^1)} \) belongs to \( M((L^p, \ell^1),(L^r, \ell^s)) \). So by Corollary 16.14 there exists a unique \( \varphi \in (L^{s'}, \ell^{\infty}) \) such that \((Tf)^\sim = \varphi f\) for all \( f \in (L^p, \ell^1) \).

Now, we note that for \( f \in (L^p, \ell^1) \)

\[ ||\varphi f||_{s', r'} = ||(Tf)^\sim||_{s', r'} \leq C ||Tf||_{r,s} \leq C ||T|| ||f||_{p,q} . \]

Therefore the map \( f \mapsto \varphi f \) from \((L^p, \ell^1), ||\cdot||_{pq}) \) to \((L^{s'}, \ell^{r'}) \) is continuous. Since \((L^p, \ell^1) \) is dense in \((L^p, \ell^q) \)
(Corollary 3.8) this map has a unique continuous extension on \((L^p, \ell^q)\) and this implies that \((Tf) = \varphi f\) for all \(f \in (L^p, \ell^q)\).

Conversely if \(\mathcal{H}\) holds then by Corollary 16.14 and Remark 16.13 \(T| (L^p, \ell^1) \) belongs to \(M((L^P, \ell^1), (L^r, \ell^S))\). Again since \((L^p, \ell^1)\) is dense in \((L^p, \ell^q)\) and the map \(f \mapsto (s f)\) on \(A\) and \(B\) is continuous for all \(s \in G\), (Theorem 3.14) we conclude that \(T\) has a unique continuous extension \(\overline{T}\) on \((L^p, \ell^q)\) which commutes with translations. Hence \(\overline{T}\) belongs to \(M((L^P, \ell^q), (L^r, \ell^S))\). So, by the previous case there exists a unique \(\varphi' \in (L^S, \ell^\infty)\) such that \((\varphi') = \varphi f\) for all \(f \in (L^p, \ell^q)\).

Then for \(f \in (L^p, \ell^1)\), \(\varphi f = (\overline{T} f) = (\overline{f})\) for all \(f \in (L^p, \ell^q)\). For \(\hat{x} \in \hat{G}\), let \(f_{\hat{x}} \in (L^p, \ell^1)\) such that \(f_{\hat{x}}(\hat{x}) = 1\), so \(\varphi f_{\hat{x}}(\hat{x}) = \varphi f_{\hat{x}}(\hat{x}) = \varphi f_{\hat{x}}(\hat{x}) = \varphi(\hat{x})\).

This implies that \(\varphi = \varphi'\) and therefore \((Tf) = (\overline{T}f)\) for all \(f \in (L^p, \ell^q)\). By Theorem 13.22 we conclude that \(T = \overline{T}\).

Now we will prove again (Theorem 12.4 i)) that for \(1 \leq p \leq \infty, \quad 1 \leq s < q \leq 2\), \(M((L^p, \ell^q), (L^1, \ell^S))\) is a trivial space, to show an application of Theorem 9.2 and Theorem 16.19.

**PROPOSITION 16.20.** Let \(\hat{G}\) be nondiscrete and \(1 \leq p \leq \infty, \quad 1 \leq s < q \leq 2\). Then the only multiplier from \((L^p, \ell^q)\) to \((L^1, \ell^S)\) is the zero multiplier.

**PROOF.** Suppose the contrary. That is, there exists a nonzero function \(\varphi\) such that \(\varphi f \in (L^1, \ell^S)\) for all \(f \in (L^p, \ell^q)\) (Theorem 16.19). Then for some \(\varepsilon > 0\), the set

\[E = \{ \hat{x} \in \hat{G} \mid |\varphi(\hat{x})| \geq \varepsilon \} \]
is not locally null. Since \((L^1, \mathcal{L}^\infty) \quad |E \subset (L^{s'}, \mathcal{L}^\infty)\quad (E)\quad by\quad Theorem\quad 5.7;\)

\(\varphi(L^p, \mathcal{L}^q) \quad |E \subset (L^1, \mathcal{L}^\infty) \quad |E \subset (L^{s'}, \mathcal{L}^\infty)\quad (E).\quad Hence\quad by\quad definition\quad of\quad E\quad we

have\quad the\quad relation\quad \((L^p, \mathcal{L}^q) \quad |E \subset (L^{s'}, \mathcal{L}^\infty)\quad (E).\quad (Note\quad that\quad if

\(f \in (L^p, \mathcal{L}^q)\quad then\quad ||\hat{f}|\mathcal{E}|_s^{\infty} \leq \varepsilon ||\varphi|\mathcal{E}|_s^{\infty}).\)

But\quad this\quad is\quad a\quad contradiction\quad because\quad by\quad Theorem\quad 9.2\quad there\quad exists

\(f \in (L^\omega, \mathcal{L}^q),\quad hence\quad in\quad \((L^p, \mathcal{L}^q),\quad such\quad that\quad \hat{f}|\mathcal{E} \notin (L^{s'}, \mathcal{L}^\infty)\quad (E)\quad as

s' > q'.\quad This\quad ends\quad the\quad proof.\)
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