BLIND SIGNAL SEPARATION

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To my parents
and
my parents in law

Abstract

This thesis addresses the blind signal separation (BSS) problem. The essence of the BSS problem is to recover a set of source signals from a group of sensor observations. These observations can be modeled as instantaneous or convolutive mixtures of the sources. Accordingly, the BSS problem is known as blind separation of instantaneously mixed signals or blind separation of convolutively mixed signals.

In this thesis, we tackle both problems. For blind separation of instantaneously mixed signals, we first cast the separation problem as an optimization problem using mutual information based criterion, and solve it with an extended Newton's method on the Stiefel manifold. Then, for a special case in which the sources are *constant modulus* (CM) signals, we formulate the separation problem a constrained minimization problem utilizing the constant modulus property of the signal. Again, we solve it using the Newton's method on the Stiefel manifold.

For the problem of blind separation of convolutively mixed signals, which is also known as blind deconvolution problem, we first propose a time domain method. We cast the separation problem as an optimization problem using a mutual information based criterion and solve it using a *sequential quadratic programming* (SQP) method. Then, we propose a set of higher-order statistics (HOS) based criteria for blind deconvolution. We also discuss the relationship of our proposed criteria and other HOS based criteria. We then propose a frequency domain HOS based blind channel identification approach. In this approach, we identify the channel frequency response by jointly diagonalizing a set of so called *polyspectrum matrices*. Finally, we propose a second-order statistics (SOS) based method for blind channel identification. Assuming the channel inputs are cyclostationary signals, we identify the channel frequency response through the singular value decomposition (SVD) of a cyclic cross-spectrum based matrix.

Numerical simulations are used throughout this thesis to compare our proposed methods with other methods from the literature and to demonstrate the validity and competitiveness of our proposed methods.

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Glossary of Abbreviations

ACMA	Analytic constant modulus algorithm
BPSK	Binary phase shift keying
BSS	Blind signal separation
CM	Constant modulus
DOA	Direction of arrival
ECG	Electrocardiogram
EEG	Electroencephalographic
FECG	Foetal electrocardiogram
FHR	Foetal's heartbeat rate
FIR	Finite impulse response
FLOPs	Floating point operations
fMRI	Functional magnetic resonance imaging
HOS	Higher-order statistics
ICA	Independent component analysis
LP	Linear programming
LTI	Linear time invariant
MEG	Magnetoencephalographic
MIMO	Multi-input multi-output
MPSK	M-ary phase shift keying
ONMSE	Overall normalized mean-square error
QP	Quadratic programming
SIMO	Single-input multiple-output
SISO	Single-input single-output
SNR	Signal-to-noise-ratio
SOS	Second-order statistics
SQP	Sequential quadratic programming
SVD	Singular value decomposition
TITO	Two-input two-output

Chapter 1

Introduction to Blind Signal Separation Problem

1.1 Blind Signal Separation Problem and Its Potential Applications

The blind signal separation (BSS) problem has been the subject of extensive research for many years. The essence of the BSS problem is to recover a set of source signals from a group of sensor observations which are mixtures of the sources. The term "blind" comes from the fundamental assumption that no *a priori* knowledge of either the sources or the mixture is available. The best example to illustrate the BSS problem is the classic "cocktail party" problem which assumes that several people are speaking simultaneously in a room while several microphones at different locations are recording the mixtures of the conversations. The goal is to recover each individual speaker's conversation from the recorded mixtures.

The BSS problem arises from a diversity of fields, including wireless communication, biomedical signal processing, image processing, speech processing and financial data analysis. Some representative examples of the BSS applications will be considered in turn.

In the biomedical signal processing area, one interesting application is the foetal electrocardiogram (FECG) extraction problem. During the mother's pregnancy, it is necessary to monitor the foetal's heartbeat rate (FHR) because it contains valuable information for evaluation of the health of the foetus. One noninvasive approach for obtaining the FHR is the FECG examination. The usual procedure for obtaining the FECG is to attach several skin electrodes to the mother's body at different positions and record the ECG signals. Normally, the measurement of each electrode contains contributions of not only the FECG, but also

the maternal ECG as well as other random disturbances. As a result, the FECG may not be able to be extracted from the mixtures when the other contributions are severe. In such a situation, we are compelled to use blind signal separation method to extract the FECG [81]. Similar problems exist in the processing of other biomedical signals, such as the electrocardiogram (ECG), functional magnetic resonance imaging (fMRI), electroencephalographic (EEG) and magnetoencephalographic (MEG) [57, 75, 76].

In the area of array signal processing, beamforming is an important technique that reconstructs the source signals from the outputs of a sensor array. Traditionally, the direction of the desired signal, called the direction of arrival (DOA), is estimated first, then a beamformer is designed based on this information. This normally requires array calibration. If the sources arrive in several directions, however, the DOA estimation can be difficult in this case. Now the array processing problem can be formed as a BSS problem that contains beamforming as a special case. Different from the traditional beamformer, the BSS based beamformer relies only on the property of the sources and does not require the DOA estimation or knowledge of the array geometry. This technique has been studied and tested in the multi-input multi-output (MIMO) wireless communication system [66, 73].

The BSS problem also finds application in the area of feature extraction. Recent studies on the receptive field properties of neurons show that, in reception of input data, neurons tend to reduce the redundancy in the input data and represent it with independent components. Such a representation or code is very useful in pattern or speech recognition. The goal of feature extraction is to mimic the behaviour of neurons to extract independent components, or basis, from input data like image or speech. Blind source separation is a natural tool for this purpose and has been used by many authors in feature extraction [8, 38, 45].

In financial market data analysis, one question that interests both researchers and traders is, "what drives the moments of financial time series?" Some researchers have recently tried to answer this question by applying BSS to the analysis of the daily returns of some major stocks [6]. In their experiments, the daily stock return was interpreted as the instantaneous mixture of a set of underlying factors which they desired to separate out. They hope the application of the BSS technique might yield a new way of analyzing and forecasting financial time series.

More examples of the BSS problem can be found in speech recognition, image reconstruction and restoration, radar and remote sensing signal processing, semiconductor manufacturing, and circuit testing and diagnosis [9, 17, 28, 49, 70, 82]. These examples illustrate the nature of the BSS problem as well as difficulties in studying and solving the problem.

To study the BSS problem, it is important to establish the mathematical model for the problem. The simplest model is the so called *instantaneous model.* It assumes that the

sensor received signal is an instantaneous linear mixture of the sources. This model describes a large range of problems in which the source signals have the same time delays. However, it is more common in practical problems that the signals have different time delays. In this case, a more general and sophisticated model, which is known as the *convolutive model*, can be applied. In the following sections both models will be introduced in detail.

The remainder of the chapter is organized as follows. In Section 1.2, we give the mathematical model for the problem of blind separation of instantaneous mixtures. Some previous work on this problem is also included. Section 1.3 presents the problem of blind separation of convolutive mixtures. Finally, in Section 1.4, a preview of our contributions and an outline of the remainder of the thesis are given to conclude the chapter.

1.2 Blind Separation of Instantaneous Mixtures

As stated in the previous section, the sensor received signals can be modeled as linear instantaneous mixtures of the sources when the signal transmission delays are equal. For a general n-source and m-sensor system, the received signals can be mathematically modeled as

$$\mathbf{x}(k) = \mathbf{A}\mathbf{s}(k), \tag{1.2.1}$$

where $\mathbf{s}(k) = [s_1(k), \ldots, s_n(k)]^T$ is the vector of n sources at time k, $\mathbf{x}(k) = [x_1(k), \ldots, x_m(k)]^T$ is the vector of m received signals, and $\mathbf{A}$ is a constant $m \times n$ unknown mixing matrix. The blind signal separation problem is to recover the source signals $\mathbf{s}(k)$ form the received signals $\mathbf{x}(k)$. Typically, a separation matrix $\mathbf{B}$ is identified such that the output signals,

$$\mathbf{y}(k) = \mathbf{B}\mathbf{x}(k), \tag{1.2.2}$$

equals $\mathbf{s}(k)$ up to a permutation and a scaling. In order for the sources to be recoverable, we require that $m \geq n$ and $\mathbf{A}$ has full column rank.

This problem was first addressed by Herault and Jutten under the name of "independent component analysis" (ICA) [37]. In the ICA study, the following statistical model is assumed:

$$\mathbf{x} = \mathbf{A}\mathbf{s}$$

where $\mathbf{s}$ is a random vectors with finite covariance, zero mean and statistically independent components, $\mathbf{A}$ is a matrix with at most as many columns as rows, and $\mathbf{x}$ is a random vector whose components are mixtures of that of $\mathbf{s}$. The ICA problem can be summarized as follows:

Given T realizations of $\mathbf{x}$, it is desired to estimate both the matrix $\mathbf{A}$ and the corresponding realizations of $\mathbf{s}$ [21, 48]. If we view signals $\mathbf{x}(k)$ and $\mathbf{s}(k)$ in (1.2.1) as the realizations of some random vectors $\mathbf{x}$ and $\mathbf{s}$, we can see that the two problems are equivalent.

Ever since Herault and Jutten's pioneering work, a great deal of work has been done in this area under the name of ICA or BSS. Most noticeably, Comon in [21] laid out the theoretical foundation for the ICA/BSS problem, in which he studied the ICA problem with square $n \times n$ mixing matrix and gave the identifiability condition for problem. Comon's identifiability theorem can be summarized as follows. *Let* $\mathbf{s}$ *be a vector with independent components, of which at most one is Gaussian. Suppose that the matrix* $\mathbf{A}$ *is invertible. If the entries of* $\mathbf{y}$ *are statistically independent, then* $\mathbf{y}$ *is identical to* $\mathbf{s}$ *up to a permutation and change of sign.* This result is fundamental because it implies that the solution to the ICA/BSS problem can be devised by simply enforcing the independence of the entries of $\mathbf{y}$. In fact, many ICA/BSS algorithms are designed, explicitly or implicitly, based on this result.

Some representative algorithms are given here in turn. Bell and Sejnowski in [7] formulated the BSS problem in an information-theoretic framework and proposed a neural network based algorithm, namely *InfoMax*. Cardoso in [14] proposed a higher-order statistics (fourth-order cumulant) based algorithm called *JADE* that estimates the mixing matrix by jointly diagonalizing a set of so called *cumulant matrices*. Amari in [2] proposed a *natural gradient* based algorithm which estimates the mixing matrix by minimizing the mutual information of the output signals. Recently, Hyvärinen proposed a fixed-point algorithm, called *FastICA*, which demonstrates remarkable convergence rate in determining the separation matrix [39]. For complete introduction of the ICA problem and algorithms, the reader is referred to [15, 42].

The instantaneous model covers a wide range of applications. However, there are also many problems that can not be accurately characterized by the instantaneous model. In this case, the more complicated model, convolutive model, should be applied. This will be illustrated in the next section.

1.3 Blind Separation of Convolutive Mixtures

Consider a two-input two-output (TITO) system illustrated in Figure 1.1, in which $x_1(t)$ and $x_2(t)$ are outputs of a 2×2 linear time invariant (LTI) system and $s_1(t)$ and $s_2(t)$ are the inputs. Mathematically, the received signals, in the absence of noise, can be expressed

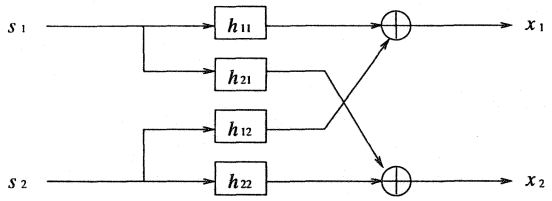

Figure 1.1: Two-input two-output convolutive channel.

as

$$\mathbf{x}(k) = \sum_{\ell=0}^{L-1} \mathbf{H}(\ell)\mathbf{s}(k-\ell), \tag{1.3.1}$$

where $\mathbf{s}(k) = [s_1(k), s_2(k)]^T$ is the vector of sources, $\mathbf{x}(k) = [x_1(k), x_2(k)]^T$ is the vector of received signals, and

$$\mathbf{H}(\ell) = \begin{bmatrix} h_{11}(\ell) & h_{12}(\ell) \\ h_{21}(\ell) & h_{22}(\ell) \end{bmatrix}, \tag{1.3.2}$$

for $\ell = 0, \ldots, L-1$, is a sequence of 2×2 matrices representing the system impulse response. It is assumed that $\mathbf{H}(\ell) = \mathbf{0}$ for $\ell < 0$ (*i.e.*, the system is causal). Here $\{h_{11}(\ell)\}_{\ell=0}^{L-1}$ and $\{h_{22}(\ell)\}_{\ell=0}^{L-1}$ are called the *direct channels*, and $\{h_{12}(\ell)\}_{\ell=0}^{L-1}$ and $\{h_{21}(\ell)\}_{\ell=0}^{L-1}$ are called the *cross channels*.

Let the channel transfer function be defined as $\boldsymbol{\mathcal{H}}(z) \triangleq \sum_{\ell=0}^{L-1} \mathbf{H}(\ell)z^{-\ell}$. The goal of blind signal separation, or blind MIMO deconvolution, as is often called in this case, is to determine a separation system $\boldsymbol{\mathcal{W}}(z) \triangleq \sum_{\ell=0}^{Lw-1} \mathbf{W}(\ell)z^{-\ell}$ such that its outputs,

$$\mathbf{y}(k) \triangleq \sum_{\ell=0}^{Lw-1} \mathbf{W}(\ell)\mathbf{x}(k-\ell), \tag{1.3.3}$$

recover the input signals $\mathbf{s}(k)$. By "recover", we mean mathematically that the combined system satisfies $\boldsymbol{\mathcal{W}}(z)\boldsymbol{\mathcal{H}}(z) = \mathbf{P}\boldsymbol{\mathcal{D}}(z)$, where $\mathbf{P}$ is a permutation matrix and $\boldsymbol{\mathcal{D}}(z)$ is diagonal matrix filter. This implies that the output signals are identical to the source signals up to a permutation and a filtering.

The blind MIMO deconvolution problem is a fairly new research subject and has attracted increasing attention in recent years. The problem is closely related to the traditional blind deconvolution, blind equalization and blind identification problems for single-input single-output (SISO) and single-input multiple-output (SIMO) systems. These problems have been

intensively studied in the past decades. For a complete review of these problems, the reader is referred to [26, 36] and the references therein.

The blind MIMO deconvolution algorithms can be roughly categorized into two groups. Algorithms in the first group focus on the identification of the convolutive MIMO channel from the channel output and then design an equalizer accordingly. Both time-domain and frequency-domain identification algorithms can be found in the literature [18,59]. Algorithms in the second group, on the other hand, directly compute the equalizer of the MIMO system. Algorithm of this class typically consists of two parts: a criterion and a method to optimize this criterion. The criterion is normally based on either the second-order or higher-order statistics of the sources. Some references are [2, 20, 21, 47, 53, 58, 61, 79].

1.4 Contributions and Outline

This thesis deals with both problems of blind separation of instantaneously mixed signals and blind separation of convolutively mixed signals. Its contributions include not only theoretical separation criteria, but also new practical algorithms for the BSS problem.

For blind separation of instantaneously mixed signals, we first propose in Chapter 2 a general method that works for arbitrary non-Gaussian independent signals. We formulate the separation problem as a constrained optimization problem based on mutual information criterion, and solve it with Edelman's extended Newton's method on the Stiefel manifold [27]. Since Newton's method is employed in our solution, our algorithm enjoys local quadratic convergence rate.

We then focus our attention on the problem of blind separation of *constant modulus* (CM) communication signals. This time we formulate the separation problem as a constrained optimization problem utilizing the CM property of the signal. Again we solve it with Edelman's method. Compared with other CM algorithms, our method demonstrates strong robustness to additive noise, achieves lower bit error rate (BER) and enjoys local quadratic convergence rate. This work is presented in Chapter 3.

For blind separation of convolutively mixed signals, we first propose a time domain mutual information based approach in Chapter 4. In this approach, we present a matrix polynomial factorization based method to decorrelate the convolutively mixed signals. Then we formulate a constrained optimization problem based on the mutual information criterion, and solve it with a *sequential quadratic programming* (SQP) method. Our approach not only separates the mixed signals but also equalizes the separated signals.

In Chapter 5, we develop a set of polyspectrum based criteria for blind deconvolution.

We extend the concept of *Mth-order uncorrelatedness*, which is defined by Zhu, Cao and Ding in [83] for scalar random variables, to the case of stationary random processes. This uncorrelatedness constraint is weaker than the usual statistical independence condition. We show that the convolutive MIMO system with Mth-order uncorrelated inputs can be deconvolved by minimizing our proposed criteria. The relations between our proposed criteria and other higher-order statistics (HOS) based criteria are also discussed in this chapter.

As an important component of our work on blind deconvolution, a frequency domain polyspectrum based approach is proposed in Chapter 6. This approach estimates the channel frequency response by jointly diagonalizing a set of so called *polyspectrum matrices*. To eliminate the phase ambiguities in the estimated channel frequency responses, a simple and elegant method is proposed. Our method can be viewed as a natural extension of Cardoso's JADE algorithm to the convolutive case.

In Chapter 7, we tackle the problem of blind separation of cyclostationary signals. We propose a second-order statistics (SOS) based channel identification method. Assuming the cycle frequencies of the sources are known and mutually distinct, we identify the channel frequency response through the singular value decomposition (SVD) of a cyclic cross-spectrum based matrix. Similar to Chapter 6, a simple and efficient method is proposed to eliminate the phase and amplitude ambiguities in the estimated channel frequency response.

Finally, we summarize the thesis and give some discussions on the future works in Chapter 8.

Chapter 2

Blind Separation of Instantaneously Mixed Signals

2.1 Overview

We propose a new approach for blind separation of instantaneously mixed signals. We formulate the separation problem as a constrained optimization problem using the mutual information criterion, and solve it using Edelman's extended Newton's method on the Stiefel manifold [27]. The proposed method has two unique features: i) Edelman's extended Newton's method is employed in the solution, thus our method enjoys local quadratic convergence rate, ii) a kernel function is derived to estimate the marginal density function of the output signal. The simulation results clearly indicate the competitiveness of our proposed method. The work has been published in [52].

2.2 Introduction

The problem of blind separation of instantaneously mixed signals has been studied extensively in the past decades because of the potential applications discussed in Chapter 1 [6, 8, 38, 45, 57, 66, 73, 75, 76]. Comon in [21] studied the separability condition for this problem, and pointed out that for statistically independent non-Gaussian sources, the separation can be achieved by simply restoring the independence. He also proposed using mutual information as a tool to measure the independence of the output signals, and to use an Edgeworth expansion to approximate the probability density function in the mutual information criterion. Later, Pham in [62] adopted the same criterion, but estimated the

density function with a kernel estimator. Recently, Boscolo *et al* [11] applied the same idea to the pre-whitened received signals. They formed a constrained optimization problem in which the same mutual information criterion is minimized while constraining the separation matrix to have normalized row vectors. Since the separation matrix is known to be orthonormal after pre-whitening, the optimization model proposed in [11] can be viewed as a relaxation of the problem that has orthonormal constraint. It was shown that the optimal solution of the relaxed problem is also a separation matrix. The optimization problem was eventually transformed into an unconstrained problem and solved with Quasi-Newton or Conjugate Gradient methods.

In this chapter, we also apply the mutual information criterion to the pre-whitened received signals. However, we work directly on the optimization model with orthonormal matrix constraints. The space defined by the orthonormal matrix constraints is called the *Stiefel manifold*. Instead of relaxing or changing the structure of the problem, we solve it directly using an extended Newton's method on the Stiefel manifold. In so doing, a previously difficult constrained optimization problem can be solved directly on the Stiefel manifold as an unconstrained problem. Moreover, our proposed method enjoys quadratic local convergence rate.

The remainder of this chapter is organized as follows. In Section 2.3, a detailed derivation of our new approach is presented. In Section 2.4, Edelman's extended Newton's method on the Stiefel manifold is introduced. Section 2.5 gives the computer simulation results with both synthetic signals and real speech signals. Some concluding remarks are given in Section 2.6 to close the chapter.

2.3 Problem Formulation

Consider an n-input m-output ($n \leq m$) memoryless channel described by

$$\mathbf{x}(k) = \mathbf{A}\mathbf{s}(k), \tag{2.3.1}$$

where $\mathbf{x}$ is a $m \times 1$ vector of the received signals, $\mathbf{s}$ is a $n \times 1$ vector of the sources, and $\mathbf{A} \in \Re^{m \times n}$ is the unknown mixing matrix. Our goal is to determine a separation matrix $\mathbf{B} \in \Re^{n \times m}$ such that $\mathbf{y}(k) \triangleq \mathbf{B}\mathbf{x}(k)$ recovers the source signal $\mathbf{s}(k)$ up to a permutation and a scaling. For simplicity of presentation, we assume that the number of sources is equal to the number of observations, *i.e.*, $n = m$. Extending the proposed method to case where $n < m$ is straightforward. We also assume that:

A1. the source signals are zero-mean, unit-variance, statistically independent, and at most one of them is Gaussian, and

A2. the mixing matrix, $\mathbf{A}$, is invertible.

In common with many existing BSS methods, our method also starts with the pre-processing step of whitening the received signals. This can be done by taking the Cholesky factorization of the cross-correlation matrix of $\mathbf{x}$ and using the inverse of the factor as pre-whitening matrix. An important advantage of this pre-whitening step is that it reduces the original unknown mixing matrix to an orthonormal matrix. Specifically, suppose the cross-correlation matrix of $\mathbf{x}$ is given by

$$\mathbf{R}_x \triangleq E[\mathbf{x}\mathbf{x}^T]. \tag{2.3.2}$$

Substituting (2.3.1) into (2.3.2) and applying the assumptions on the sources, we have

$$\mathbf{R}_x = \mathbf{A}\mathbf{A}^T. \tag{2.3.3}$$

On the other hand, by taking Cholesky factorization of $\mathbf{R}_x$, we obtain

$$\mathbf{R}_x = \mathbf{G}\mathbf{G}^T, \tag{2.3.4}$$

where $\mathbf{G}$ is some lower triangular matrix. Let $\mathbf{G}^{-1}$ be the pre-whitening matrix. Then the pre-whitened signal is given by $\hat{\mathbf{x}} = \mathbf{G}^{-1}\mathbf{x} = \mathbf{G}^{-1}\mathbf{A}\mathbf{s}$. Denote $\mathbf{U} \triangleq \mathbf{G}^{-1}\mathbf{A}$ and the pre-whitened signal as $\hat{\mathbf{x}} = \mathbf{U}\mathbf{s}$. From (2.3.3) and (2.3.4), it is easy to verify that matrix $\mathbf{U}$ is orthonormal. Consequently, our goal is to determine an orthonormal separation matrix $\mathbf{B}$ such that $\mathbf{y}(k) = \mathbf{B}\hat{\mathbf{x}}(k)$ equals the source vector $\mathbf{s}(k)$ up to a permutation and a scaling.

Pham in [62] proposed minimizing the following mutual information based criterion to determine the separation matrix $\mathbf{B}$:

$$\sum_{k=1}^{n} H(y_k) - \log|\det(\mathbf{B})|, \tag{2.3.5}$$

where $H(y_k)$ is the marginal entropy of y_k, the kth entry of $\mathbf{y}$. Let $q(y_k)$ be the marginal density function of y_k. Then the marginal entropy $H(y_k)$ can also be expressed as $H(y_k) = -E[\log(q(y_k))]$. Since matrix $\mathbf{B}$ is orthonormal for our problem, the last term in (2.3.5) is zero. Therefore, we can state our problem as

$$\begin{aligned} &\text{minimize} \quad -\sum_{k=1}^{n} E[\log(q(y_k))] \\ &\text{subject to} \quad \mathbf{B}^T\mathbf{B} = \mathbf{I}. \end{aligned} \tag{2.3.6}$$

The density function $q(y_k)$ is generally unknown and should be estimated from samples. Similar to [11, 62], we estimate $q(y_k)$ with a kernel estimator [65]:

$$\hat{q}_N(y) = \frac{1}{N}\sum_{i=1}^{N} \frac{1}{h_N\sqrt{\pi}} e^{-\left(\frac{y-y(i)}{h_N}\right)^2}, \tag{2.3.7}$$

where h_N is the window width and N is the number of samples (See also [77] for other information on kernel estimators). In reference [65], the optimal window width selection problem was solved under the assumption that $q(y)$ is twice continuously differentiable and by using the minimum mean integrated square error criterion (defined as $E[\int |\hat{q}_N(y)-q(y)|^2 dy]$). Unfortunately, the resulting optimal window width depends on the unknown density $q(y)$, thus making it inconvenient to use in practice. In contrast, our window width is chosen based on the point-wise mean squared error criterion and is independent of the density function. Moreover, we only require $q(y)$ to be continuously differentiable. We justify our choice and give an estimate of the convergence rate of our estimator in the following theorem.

Theorem 2.3.1 *Let $q(y)$ be the probability density function of the random variable $y \in \Re$. Suppose $q(y)$ is continuously differentiable at a point $\bar{y}$. Let $\{y(i), i = 1, \cdots, N\}$ be a sequence of independently and identically distributed random samples drawn from the distribution of $q(y)$. Let $h_N = N^{-\frac{1}{2}}(\log N)^{-\frac{1}{4}}$. Then the estimator*

$$\hat{q}_N(\bar{y}) = \frac{1}{N}\sum_{i=1}^{N} \frac{1}{h_N\sqrt{\pi}} e^{-\left(\frac{\bar{y}-y(i)}{h_N}\right)^2} \tag{2.3.8}$$

converges to $q(\bar{y})$ in the mean squared error sense at a rate of $O(N^{-\frac{2}{3}}(\log N)^{\frac{1}{2}})$ as $N \to \infty$, i.e., $E[|\hat{q}_N(\bar{y}) - q(\bar{y})|^2] = O(N^{-\frac{2}{3}}(\log N)^{\frac{1}{2}})$.

Details of the proof are relegated to Appendix A.1. Notice that Theorem 2.3.1 requires the unknown density function $q(y)$ be continuously differentiable. For those points y where $q(y)$ does not satisfy this property, the estimator $\hat{q}_N(y)$ may fail to converge to $q(y)$ as $N \to \infty$. However, the work of Amari *et al* [3] has shown that the estimation of $\mathbf{B}$ from (2.3.6) is asymptotically consistent even if the estimation of $q(y)$ is not consistent. Of course, the estimator $\mathbf{B}$ becomes more accurate if we use a consistent density estimator for $q(y)$. In Section 2.5 we will demonstrate the successful blind separation of Binary Phase Shift Keying (BPSK) signals using the estimator given by (2.3.6), even though the smoothness condition of Theorem 2.3.1 does not hold in this case.

Substituting (2.3.7) into (2.3.6), and omitting unnecessary constant terms, we can approximate our problem as

$$\begin{aligned} \text{minimize} \quad & -\sum_{k=1}^{n} E\left[\log\left(\sum_{i=1}^{N} e^{-\left(\frac{y_k(i)-y_k}{h_N}\right)^2}\right)\right] \\ \text{subject to} \quad & \mathbf{B}^T\mathbf{B} = \mathbf{I}. \end{aligned} \tag{2.3.9}$$

Next we use a sample average to approximate the expectation operation in (2.3.9). In particular, suppose a total of M samples, $y_k(j)$, $j = N+1, \ldots, N+M$, are available, then

(2.3.9) can further be approximated as

$$\begin{aligned} \text{minimize} \quad & -\frac{1}{M}\sum_{k=1}^{n}\sum_{j=N+1}^{N+M}\log\left(\sum_{i=1}^{N} e^{-\left(\frac{y_k(i)-y_k(j)}{h_N}\right)^2}\right) \\ \text{subject to} \quad & \mathbf{B}^T\mathbf{B} = \mathbf{I}. \end{aligned} \tag{2.3.10}$$

Since $\mathbf{y} = \mathbf{B}\hat{\mathbf{x}}$, samples $y_k(i)$, $i = 1, \ldots, N+M$, can actually be computed from the samples of $\hat{\mathbf{x}}$ by $y_k(i) = \mathbf{b}_k^T\hat{\mathbf{x}}(i)$, where $\mathbf{b}_k^T$ is the kth row of $\mathbf{B}$. Substituting the expression of $y_k(i)$ into (2.3.10), we obtain the following formulation of our problem:

$$\begin{aligned} \text{minimize} \quad & -\frac{1}{M}\sum_{k=1}^{n}\sum_{j=N+1}^{N+M}\log\left(\sum_{i=1}^{N} e^{-\frac{1}{h_N^2}\left[\mathbf{b}_k^T(\hat{\mathbf{x}}(i)-\hat{\mathbf{x}}(j))\right]^2}\right) \\ \text{subject to} \quad & \mathbf{B}^T\mathbf{B} = \mathbf{I}, \end{aligned} \tag{2.3.11}$$

in which the decision variables are the entries of the separation matrix $\mathbf{B}$, and $\hat{\mathbf{x}}(i)$, $i = 1, \ldots, N+M$, are the vectors of the pre-whitened signals. Since the orthonormal constraint $\mathbf{B}^T\mathbf{B} = \mathbf{I}$ is cumbersome to handle computationally, we shall apply the extended Newton's method over the Stiefel manifold to this problem. Details of this algorithm will be given next.

2.4 Newton's Method on the Stiefel Manifold

Edelman, Arias and Smith in [27] studied the following general optimization problem:

$$\begin{aligned} \text{minimize} \quad & \phi(\mathbf{B}) \\ \text{subject to} \quad & \mathbf{B}^T\mathbf{B} = \mathbf{I}, \end{aligned} \tag{2.4.1}$$

where $\phi(\mathbf{B})$ is an objective function and $\mathbf{B}$ is an $n \times p$ matrix. The feasible set defined by constraint $\mathbf{B}^T\mathbf{B} = \mathbf{I}$ is called the Stiefel manifold. This problem is generally hard to solve because of the quadratic equality constraints. Instead of trying to solve it in Euclidean space, Edelman and his co-workers developed a Newton's method on the Stiefel manifold and solved it as an unconstrained problem on this manifold. Moreover, the extended Newton's method inherits the local quadratic convergence property of the original Newton's method.

The extended Newton's method on the Stiefel manifold is in principle the same as the Newton's method in Euclidean space except that the updating of vectors on the Stiefel manifold, which can be imagined as a sphere, follows a geodesic path that is the shortest curve between two points (see Figure 2.1). Such an updating scheme guarantees that the updated point is still on the Stiefel manifold. Thus the constraints are always satisfied.

The Newton's method on the Stiefel manifold is a general method that can be applied to any optimization problem under the constraint $\mathbf{B}^T\mathbf{B} = \mathbf{I}$ for some $n \times p$ matrix $\mathbf{B}$. A key advantage of the Newton-type method over other traditional BSS methods is its ability to treat general Stiefel manifold constraints. The latter arises naturally when we are interested in separating the n output signals into p independent groups. In this case, we are faced with a nonsquare matrix $\mathbf{B}$ which must satisfy $\mathbf{B}^T\mathbf{B} = \mathbf{I}$, and the generality of the Newton method plays an essential role. Further details of this method can be found in [27].

When specialized to our problem, the extended Newton's method [27] can be summarized as follows:

Step 1: Choose an initial separation matrix $\mathbf{B}$ such that $\mathbf{B}^T\mathbf{B} = \mathbf{I}$;

Step 2: Compute the gradient of $\phi(\mathbf{B})$ at point $\mathbf{B}$, which is given by $\mathbf{G} = \phi_B - \mathbf{B}\phi_B^T\mathbf{B}$, where ϕ_B is the $n \times n$ matrix of partial derivatives of $\phi(\mathbf{B})$ with respect to the elements of $\mathbf{B}$, *i.e.*, $(\phi_B)_{ij} = \frac{\partial \phi}{\partial b_{ij}}$;

Step 3: Compute the Newton's direction Δ such that $\mathbf{B}^T\Delta = -\Delta^T\mathbf{B}$ and

$$\phi_{BB}(\Delta) - \mathbf{B}\text{skew}(\phi_B^T\Delta) - \text{skew}(\Delta\phi_B^T)\mathbf{B} - \frac{1}{2}\Pi\Delta\mathbf{B}^T\phi_B = -\mathbf{G}, \tag{2.4.2}$$

where $\text{skew}(\mathbf{X}) \triangleq (\mathbf{X} - \mathbf{X}^T)/2$, $\Pi \triangleq \mathbf{I} - \mathbf{B}\mathbf{B}^T$, and $\phi_{BB}(\Delta)$ is defined by $(\phi_{BB}(\Delta))_{kl} = 2\sum_{ij}\frac{\partial^2\phi}{\partial b_{ij}\partial b_{kl}}\Delta_{ij}$;

Step 4: Move along the Newton's direction Δ from $\mathbf{B}$ to $\mathbf{B}(t)$ using the geodesic formula $\mathbf{B}(t) = \mathbf{B}\exp(t\mathbf{B}^T\Delta)$, where t is the step size which is determined via an Armijo-type line search;

Step 5: Repeat from step 2 until the norm of the Newton's direction Δ is smaller than a pre-set threshold.

In Step 3, a conjugate gradient method on the Stiefel manifold is employed in computing the Newton direction. To implement the above Newton's method, we need to calculate the gradient vector and the Hessian matrix at each iteration. We develop these expressions as follows. First we denote $\hat{\mathbf{x}}_{ij} = \hat{\mathbf{x}}(i) - \hat{\mathbf{x}}(j)$. The first order partial derivatives of $\phi(\mathbf{B})$ with respect to the coefficients of $\mathbf{B}$ are given by

$$\frac{\partial \phi}{\partial b_{pq}} = \frac{2}{Mh_N^2}\sum_{j=N+1}^{N+M}\left(\frac{\sum_{i=1}^{N} e^{-\frac{1}{h_N^2}\left(\mathbf{b}_p^T\hat{\mathbf{x}}_{ij}\right)^2}\left(\mathbf{b}_p^T\hat{\mathbf{x}}_{ij}\right)\hat{\mathbf{x}}_{ij}(q)}{\sum_{i=1}^{N} e^{-\frac{1}{h_N^2}\left(\mathbf{b}_p^T\hat{\mathbf{x}}_{ij}\right)^2}}\right) \tag{2.4.3}$$

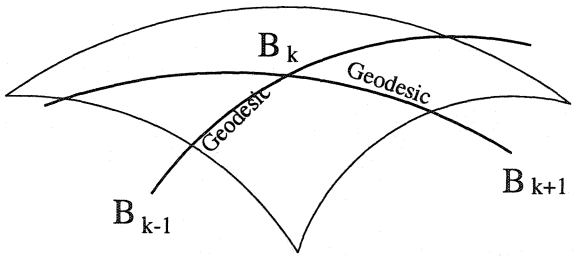

Figure 2.1: Newton's direction on the Stiefel manifold.

for $p, q = 1, \cdots, n$, where $\hat{\mathbf{x}}_{ij}(q)$ is the qth entry of vector $\hat{\mathbf{x}}_{ij}$. Denote the numerator in the bracket in (2.4.3) as $f(\mathbf{B})$ and the denominator as $g(\mathbf{B})$. Then the second order partial derivatives of $\phi(\mathbf{B})$ are given by

$$\frac{\partial^2 \phi}{\partial b_{uv} \partial b_{pq}} = \frac{2}{M h_N^2} \sum_{j=N+1}^{N+M} \left(\frac{\partial f(\mathbf{B})}{\partial b_{uv}} g(\mathbf{B}) - f(\mathbf{B}) \frac{\partial g(\mathbf{B})}{\partial b_{uv}} \right) \frac{1}{g(\mathbf{B})^2}$$

for $p, q, u, v = 1, \cdots, n$, where

$$\frac{\partial f(\mathbf{B})}{\partial b_{uv}} = \sum_{i=1}^{N} \left(e^{-\frac{1}{h_N^2}\left(\mathbf{b}_p^T \hat{\mathbf{x}}_{ij}\right)^2} \left(1 - \frac{2}{h_N^2} \left(\mathbf{b}_p^T \hat{\mathbf{x}}_{ij}\right)^2 \right) \hat{\mathbf{x}}_{ij}(q) \hat{\mathbf{x}}_{ij}(v) \delta(u-p) \right)$$

and

$$\frac{\partial g(\mathbf{B})}{\partial b_{uv}} = -\frac{2}{h_N^2} \sum_{i=1}^{N} \left(e^{-\frac{1}{h_N^2}\left(\mathbf{b}_p^T \hat{\mathbf{x}}_{ij}\right)^2} \left(\mathbf{b}_p^T \hat{\mathbf{x}}_{ij}\right) \hat{\mathbf{x}}_{ij}(v) \delta(u-p) \right).$$

Here $\delta(k)$ denotes the Kronecker delta function with $\delta(0) = 1$ and $\delta(k) = 0$ for $k \neq 0$.

2.5 Simulation Results

We performed four computer simulations to compare our proposed method with Cardoso's JADE algorithm [14] and Hyvärinen's FastICA algorithm [39]. The Matlab codes for JADE and FastICA were downloaded from web sites [16] and [43] respectively. In these simulations, both synthetic signals and real speech signals were utilized. The mixing matrix $\mathbf{A}$ was randomly generated such that the columns of $\mathbf{A}$ are normalized to 1, and the correlation between columns is not greater than 0.95. The initial separation matrix was a randomly

chosen orthonormal matrix and the performance index was adopted from [2]:

$$E_1 = \sum_{i=1}^{n}\left(\sum_{j=1}^{n}\frac{|p_{ij}|}{\max\limits_{k}|p_{ik}|}-1\right)+\sum_{j=1}^{n}\left(\sum_{i=1}^{n}\frac{|p_{ij}|}{\max\limits_{k}|p_{kj}|}-1\right), \tag{2.5.1}$$

where $\mathbf{P} = (p_{ij}) = \mathbf{BU}$. Ideally, if the separation is successful, the matrix $\mathbf{P}$ should be a permutation matrix, and index E_1 should be zero. Thus, it is clear that the smaller the value for E_1, the better the separation result.

Our algorithm was developed under the assumption of no noise, however we consider the impact of Gaussian additive noise in the simulations. The simulations were performed at signal-to-noise-ratios (SNR) of 15, 20 and 25 dB. For each of the four simulations, a total of 1000 samples were used for all three methods. In particular, we used $N = 500$ samples to estimate the marginal density function $q(y_k)$ and $M = 500$ samples to approximate $E[\log(q(y_k))]$. A total of 50 Monte Carlo runs were performed to obtain the averaged results.

Experiment 1: The source signals used for this experiment were a sine wave, a square wave and a Gaussian process. They were downloaded from Cardoso's home page [16] for JADE demo and are given by

$$\begin{aligned} s_1(t) &= \cos(0.026\pi t), \\ s_2(t) &= \text{sign}(\cos(0.04\pi t)), \\ s_3(t) &= \text{normally distributed random signal}, \end{aligned}$$

where $t = 1, 2, \cdots, 1000$. The simulation results are summarized in Table 2.1.

SNR	15 dB	20 dB	25 dB
New	0.4232	0.2901	0.2558
JADE	0.4067	0.2812	0.2510
FastICA	0.4507	0.3337	0.2714

Table 2.1: Performance index E_1 for Experiment 1.

Experiment 2: The source signals used for this experiment were taken from [2] and are given by

$$\begin{aligned} s_1(t) &= 0.1\sin(400t)\cos(30t), \\ s_2(t) &= 0.01\text{sign}(\sin(500t + 9\cos(40t))), \\ s_3(t) &= \text{noise source uniformly distributed in [-1,1]}. \end{aligned}$$

The time duration for the signals above is 0.1 second and the sampling rate is 10kHz. The simulation results are summarized in Table 2.2.

SNR	15 dB	20 dB	25 dB
New	2.4305	1.9775	1.9426
JADE	2.1536	1.7890	1.5902
FastICA	2.2121	1.9658	1.7284

Table 2.2: Performance index E_1 for Experiment 2.

Experiment 3: Two female voice signals were used as sources for this experiment. The sampling rate for both signals is 8kHz. The simulation results are summarized in Table 2.3.

SNR	15 dB	20 dB	25 dB
New	0.5890	0.5399	0.5347
JADE	0.7801	0.7242	0.7182
FastICA	0.7614	0.6952	0.6891

Table 2.3: Performance index E_1 for Experiment 3.

Experiment 4: Two simulated binary phase shift keying (BPSK) communication signals (*i.e.*, binary sequences taking values ± 1 with probability 0.5 each) were used for this experiment. The simulation results are summarized in Table 2.4.

SNR	15 dB	20 dB	25 dB
New	0.1402	0.0752	0.0510
JADE	0.1397	0.0752	0.0509
FastICA	0.1681	0.0888	0.0582

Table 2.4: Performance index E_1 for Experiment 4.

The simulation results in the above tables show that our method has performance similar to that of JADE and FastICA. Figure 2.2 shows the local quadratic convergence of the Newton's method on the Stiefel manifold. The *Error* in this figure is defined as $\|\mathbf{B}_i - \hat{\mathbf{B}}\|_F$ (the Frobenius norm), where $\mathbf{B}_i$ is the separation matrix output at the ith iteration and $\hat{\mathbf{B}}$ is the final output separation matrix. The current implementation of our method is notably slower than the JADE or FastICA code, mostly due to the Matlab `for` loops used in calculating the cost function and its derivatives. It takes up to 14 iterations on average for our method to converge. More efficient implementations of our method may be possible using improved data structures and exploiting linear algebra structure. This remains a topic of future study. During the simulation, our method may converge to a local minimum in very rare occasions. Starting with a different (random) initial point, however, the method always converged to a correct solution. Overall, the local minimum problem does not appear

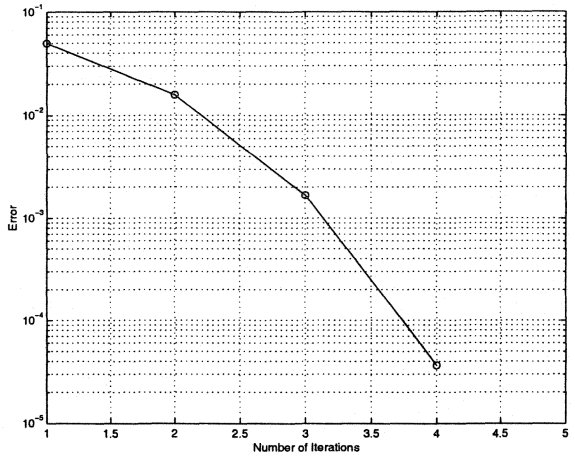

Figure 2.2: Local convergence of the extended Newton's method.

to materially affect the final separation results. JADE and FastICA, although not suffering from the local convergence problem, also produce incorrect results occasionally.

2.6 Discussion

In this chapter, we have presented a new approach for blind separation of instantaneously mixed signals. The separation problem was formulated as a constrained optimization problem based on the mutual information criterion. There are two key features that distinguish the proposed method from other BSS methods: i) the optimization problem was solved using Edelman's extended Newton's method on the Stiefel manifold, ii) the kernel estimator we used in estimating the marginal density function was developed based on the mean squared error criterion. The simulation results of our method demonstrate similar performance to that of JADE and FastICA, although our current implementation is computationally less efficient. However, our method is more general than JADE type algorithms since it works for arbitrary independent sources and does not require non-vanishing higher-order statistics of the sources as the other methods do. In addition, our method enjoys local quadratic convergence rate. We will show in the next chapter, more efficient algorithms can be designed if more information (*e.g.*, constant modulus property) about the source signals is available.

Chapter 3

Blind Separation of Constant Modulus Signals

3.1 Overview

We propose a novel approach for the separation of instantaneous mixtures of *constant modulus* (CM) signals. As in Chapter 2, we also formulate the separation problem as an optimization problem. Instead of using the statistical property of the sources, we formulate the problem using the constant modulus property of the signals. We show that the optimal solution of the formulated problem indeed yields a separation matrix, and we solve this problem using Edelman's extended Newton's method. The simulation results demonstrate the competitiveness of our proposed method. The work has been published in [51].

3.2 Introduction

In the previous chapter, we addressed the BSS problem for general independent signals. There we formulated an optimization problem using the statistical property of the sources. In this chapter, we focus our attention on a particular type of BSS problems in which the sources are constant modulus communication signals. We consider the specific scenario of digital communication whereby d independent binary signals are to be transmitted to an antenna array of M antennas ($d \leq M$). Assuming that the sources are synchronized, transmission delays are equal and the delay spread is negligible, we have the received signals at the antenna array given by

$$\mathbf{x}(n) = \mathbf{A}\mathbf{s}(n) + \mathbf{v}(n), \tag{3.2.1}$$

where $\mathbf{x}(n)$ is the vector of M received signals at the antenna array, $\mathbf{A} \in \Re^{M \times d}$ is the unknown mixing matrix, $\mathbf{s}(n)$ is the vector of d transmitted signals and $\mathbf{v}(n)$ is the vector of additive noise at the antenna array. The goal is to determine a separation matrix $\mathbf{B}$ such that $\mathbf{Bx}(n)$ resembles $\mathbf{s}(n)$. By "resemble" we mean that as the influence of noise decreases, $\mathbf{Bx}(n)$ approaches $\mathbf{s}(n)$ up to a permutation and change of signs.

For the general BSS problem, the sources are arbitrary statistically independent signals. Typically, higher-order statistics of the sources are used in determining the separation matrix, and this normally requires large amount of data and computation. However, for digital communication signals, the sources are not only statistically independent, but also constrained to a finite alphabet. This special property of the sources makes it possible for us to design more efficient and simpler separation algorithms. Recent works exploiting this property includes [4, 34, 35, 67, 71, 72, 78].

In reference [67], Talwar *et al* proposed two algorithms to solve the BSS problem. Unfortunately, the problem formulation in [67] is non-convex, hence, no global convergence is guaranteed, and the algorithms must include careful detection and management of local optimal solution. Anand *et al* [4] proposed a clustering based approach that first classifies the vectors of the received signals into 2^d groups and then maps each group to a d-dimensional vector with ± 1 entries. A major difficulty with this method is that it is sensitive to the additive noise. An analytic constant modulus algorithm (ACMA) was proposed by van der Veen [71, 72] for the separation of CM signals. But it too suffers from noise sensitivity and occasional divergence. The work of Hansen and Xu [34, 35] exploits the geometric structure of the constellation of the source signals, and casts the BSS problem as a hypercube de-rotation problem. The drawback of this de-rotation algorithm is that a sequence of non-convex optimization problems must be solved.

In this chapter, we formulate the BSS problem as a constrained optimization problem over the Stiefel manifold utilizing the CM property of the sources, and solve it using Edelman's extension of Newton's method. The simulation results show that our proposed method has several appealing features: it is robust to the additive noise, achieves low bit error rate (BER) and enjoys local quadratic convergence rate.

The remainder of the chapter is organized as follows. In Section 3.3, a detailed derivation of our new approach is presented. Some simulation results are given in Section 3.4. We then close the chapter with some discussion in Section 3.5.

3.3 Problem Formulation

For clarity in the exposition, we will restrict our discussion to the case of binary phase shift keying (BPSK) signals for which all elements of $\mathbf{s}(n)$ are ± 1. However, our method can be easily generalized to the general M-ary phase shift keying (MPSK) modulation case.

We consider the model given in (3.2.1) and assume that the mixing matrix $\mathbf{A}$ have full column rank. Suppose N vectors of data samples have been collected at the antenna array, then the model (3.2.1) can also be written in matrix form as

$$\mathbf{X} = \mathbf{AS} + \mathbf{V}, \tag{3.3.1}$$

where $\mathbf{X} = [\mathbf{x}(1), \ldots, \mathbf{x}(N)]$ and $\mathbf{V} = [\mathbf{v}(1), \ldots, \mathbf{v}(N)]$ are $M \times N$ matrices, and $\mathbf{S} = [\mathbf{s}(1), \ldots, \mathbf{s}(N)]$ is $d \times N$ matrix.

Similar to our BSS algorithm proposed in Chapter 2, this method also pre-whitens the received signals in the first step. In Chapter 2, we pointed out that pre-whitening the received signals has the advantage of removing correlations among the signals and reducing the original unknown mixing matrix to an orthonormal matrix. It can also reduce the problem dimension from $M \times d$ to $d \times d$. This will become clear in the derivation.

Let the signal cross-correlation matrix be denoted by:

$$\mathbf{R}_x = \frac{1}{N}\mathbf{X}\mathbf{X}^T.$$

Then, the pre-whitening step can be carried out by first computing the eigendecomposition of $\mathbf{R}_x$,

$$\mathbf{R}_x = \mathbf{U}\boldsymbol{\Sigma}^2\mathbf{U}^T = [\mathbf{U}_d \ \mathbf{U}_{M-d}] \begin{bmatrix} \boldsymbol{\Sigma}_d^2 & 0 \\ 0 & \boldsymbol{\Sigma}_{M-d}^2 \end{bmatrix} \begin{bmatrix} \mathbf{U}_d^T \\ \mathbf{U}_{M-d}^T \end{bmatrix}, \tag{3.3.2}$$

where $\mathbf{U}$ is a $M \times M$ orthonormal matrix, $\mathbf{U}_d$ and $\mathbf{U}_{M-d}$ are formed by the first d and last $(M-d)$ columns of $\mathbf{U}$ respectively, and $\boldsymbol{\Sigma}^2$ is a nonnegative $M \times M$ diagonal matrix with diagonal entries arranged in nonincreasing order. As well $\boldsymbol{\Sigma}_d^2$ and $\boldsymbol{\Sigma}_{M-d}^2$ are diagonal matrices formed by the first d and last $(M-d)$ diagonal elements of $\boldsymbol{\Sigma}^2$. The pre-whitening matrix is then

$$\mathbf{W} \triangleq \boldsymbol{\Sigma}_s^{-1}\mathbf{U}_d^T,$$

where $\boldsymbol{\Sigma}_s$ is $\boldsymbol{\Sigma}_d$ if $M = d$ or $(\boldsymbol{\Sigma}_d^2 - \hat{\sigma}^2\mathbf{I}_d)^{1/2}$ if $M > d$, where $\hat{\sigma}^2$ is an estimate of the noise power. The pre-whitened signal is then given by

$$\hat{\mathbf{X}} \triangleq \mathbf{WX} = \boldsymbol{\Sigma}_s^{-1}\mathbf{U}_d^T(\mathbf{AS} + \mathbf{V}).$$

Denoting $\mathbf{C} \triangleq \mathbf{\Sigma}_s^{-1}\mathbf{U}_d^T\mathbf{A}$, which is a $d \times d$ matrix, and $\tilde{\mathbf{V}} \triangleq \mathbf{\Sigma}_s^{-1}\mathbf{U}_d^T\mathbf{V}$, we express the pre-whitened signal as

$$\hat{\mathbf{X}} = \mathbf{CS} + \tilde{\mathbf{V}}.$$

Since the sources are assumed to be zero-mean, unit-variance and statistically independent, it can be checked that matrix $\mathbf{C}$ is orthonormal. Our goal then is to determine an orthonormal separating matrix $\mathbf{B}$ such that

$$\mathbf{Y} = \mathbf{B}\hat{\mathbf{X}} = \mathbf{BCS} + \mathbf{B}\tilde{\mathbf{V}} \tag{3.3.3}$$

resembles the source signal $\mathbf{S}$. Mathematically, by "resemble" we mean

$$\mathbf{BC} = \mathbf{DP}, \tag{3.3.4}$$

where $\mathbf{D}$ is a diagonal matrix with diagonal entries being ± 1 and $\mathbf{P}$ is a permutation matrix. Consequently, we have

$$\mathbf{Y} = \mathbf{DPS} + \mathbf{B}\tilde{\mathbf{V}},$$

implying that in the noise free case, $\mathbf{Y}$ and $\mathbf{S}$ are identical up to a row permutation and change of signs. Since the separated signals in $\mathbf{Y}$ should only have the values of ± 1, or constant modulus in general, we propose to solve the BSS problem using the following constant modulus formulation:

$$\begin{aligned} \text{minimize} \quad & \sum_{i=1}^{d}\sum_{j=1}^{N}(|y_{ij}|^2 - 1)^2 \\ \text{subject to} \quad & \mathbf{B}^T\mathbf{B} = \mathbf{I}_d, \end{aligned} \tag{3.3.5}$$

where y_{ij} is the (i,j) entry of matrix $\mathbf{Y}$. Let $\mathbf{B} = [b_{ij}]$ and $\hat{\mathbf{X}} = [\hat{x}_{ij}]$. From (3.3.3) we have $y_{ij} = \sum_{k=1}^{d} b_{ik}\hat{x}_{kj}$. Substituting this expression into (3.3.5), we have

$$\begin{aligned} \text{minimize} \quad & \sum_{i=1}^{d}\sum_{j=1}^{N}\left(\left|\sum_{k=1}^{d} b_{ik}\hat{x}_{kj}\right|^2 - 1\right)^2 \\ \text{subject to} \quad & \mathbf{B}^T\mathbf{B} = \mathbf{I}_d. \end{aligned} \tag{3.3.6}$$

The following proposition indicates that every optimal solution of problem (3.3.6) is a separating matrix when N is large enough. Therefore, we can solve this minimization problem (3.3.6) for the separating matrix $\mathbf{B}$.

Proposition 3.3.1 *Suppose the noise power is zero. Let* $\mathbf{S}$ *be a* $d \times 2^d$ *matrix containing all* 2^d *possible combinations of* ± 1 *in its columns. Let* $\mathbf{B}$ *be an optimal solution of (3.3.6). Then* $\mathbf{B}$ *is a separating matrix,* i.e.,

$$\mathbf{BC} = \mathbf{DP}$$

for some diagonal matrix $\mathbf{D}$ *whose diagonal elements are* ± 1*, and for some permutation matrix* $\mathbf{P}$.

Proof: Denote $\mathbf{E} = \mathbf{BC}$. Then $\mathbf{Y} = \mathbf{ES}$. Since $\mathbf{B}$ is the optimal solution of (3.3.6), $\mathbf{B}$ must be orthonormal and the entries of $\mathbf{Y}$ equal to ± 1. Recall that $\mathbf{C}$ is orthonormal. It then follows that $\mathbf{E}$ is also orthonormal. We will show that there is only one non-zero element in each row and column of $\mathbf{E}$, and this element can only be ± 1. Thus $\mathbf{E} = \mathbf{DP}$.

To show this, we only need to show that there is only one non-zero element on each row of $\mathbf{E}$. Once this is established, we can use the property $\mathbf{EE}^T = \mathbf{I}$ to conclude that all other elements on the same column as this non-zero element must be zero, and this non-zero element can only be ± 1.

Let $\mathbf{e}^T$ be a row of matrix $\mathbf{E}$. The corresponding row in $\mathbf{Y}$ will then be

$$\mathbf{y}^T = \mathbf{e}^T\mathbf{S}, \quad \text{or} \quad \mathbf{y} = \mathbf{S}^T\mathbf{e}, \tag{3.3.7}$$

where $\mathbf{y}$ is $2^d \times 1$, $\mathbf{e}$ is $d \times 1$ and $\mathbf{S}$ is $d \times 2^d$. We will use the method of induction to show that there exists only one non-zero entry in vector $\mathbf{e}$.

In the case of $d = 1$, (3.3.7) reduces to $\mathbf{y} = \mathbf{s}e$, where $\mathbf{y}$ and $\mathbf{s}$ are 2×1 vectors and e is a scalar. Since the entries of $\mathbf{y}$ and $\mathbf{s}$ are ± 1, e can only be ± 1. The proposition thus holds true in this case.

Now suppose that the proposition holds true for $d = k$, *i.e.*, if

$$\mathbf{y}_k = \mathbf{S}_k^T\mathbf{e}_k, \tag{3.3.8}$$

then $\mathbf{e}_k$ has only one non-zero element. Here we use index k to indicate that the matrix and vectors are for the case $d = k$.

Now we shall prove that the statement is true for the $d = k + 1$ case, *i.e.*, for

$$\mathbf{y}_{k+1} = \mathbf{S}_{k+1}^T\mathbf{e}_{k+1}, \tag{3.3.9}$$

there is only one non-zero entry in vector $\mathbf{e}_{k+1}$. Rewrite (3.3.9) as

$$\begin{bmatrix} \mathbf{y}^1 \\ \mathbf{y}^2 \end{bmatrix} = \begin{bmatrix} 1 & \mathbf{S}_k^T \\ -1 & \mathbf{S}_k^T \end{bmatrix} \begin{bmatrix} e_1 \\ \mathbf{e}_2 \end{bmatrix}, \tag{3.3.10}$$

where $\mathbf{y}^1$ and $\mathbf{y}^2$ are $2^d \times 1$ vectors, $\mathbf{1}$ is a $2^d \times 1$ vector with all its entries being 1's and e_1 is the first element of $\mathbf{e}_{k+1}$. The vector $\mathbf{e}_2$ is formed by the remaining entries of $\mathbf{e}_{k+1}$. Here we have used the recursive structure between $\mathbf{S}_{k+1}$ and $\mathbf{S}_k$. Equation (3.3.10) can now be written as

$$\mathbf{y}^1 = \mathbf{1}e_1 + \mathbf{S}_k^T\mathbf{e}_2 \tag{3.3.11}$$

$$\mathbf{y}^2 = -\mathbf{1}e_1 + \mathbf{S}_k^T\mathbf{e}_2. \tag{3.3.12}$$

If $e_1 = 0$, (3.3.11) becomes $\mathbf{y}^1 = \mathbf{S}_k^T\mathbf{e}_2$. From the assumption above we know that there is only one non-zero entry in $\mathbf{e}_2$. Therefore, only one element in $\mathbf{e}_{k+1}$ can be non-zero, proving the proposition in this case.

If $e_1 \neq 0$, subtracting (3.3.12) from (3.3.11) yields

$$\mathbf{y}^1 - \mathbf{y}^2 = 2e_1\mathbf{1}. \tag{3.3.13}$$

Since the entries of $\mathbf{y}^1$ and $\mathbf{y}^2$ are ± 1, e_1 can only be ± 1. Suppose $e_1 = 1$, then from (3.3.13) we must have $\mathbf{y}^1 = \mathbf{1}$ and $\mathbf{y}^2 = -\mathbf{1}$. Substituting $e_1 = 1$ and $\mathbf{y}^1 = \mathbf{1}$ into (3.3.11), we get

$$\mathbf{1} = \mathbf{1} + \mathbf{S}_k^T\mathbf{e}_2.$$

This implies $\mathbf{e}_2 = \mathbf{0}$ because $\mathbf{S}_k^T$ has full column rank. Therefore, vector $\mathbf{e}_{k+1}$ has only one non-zero entry e_1, proving that the proposition is true in this case. The case $e_1 = -1$ can be treated similarly. This completes the induction on d and the proof of proposition. **Q.E.D.**

In practice, when N is large enough, the columns of $\mathbf{S}$ cover all 2^d possible distinct d dimensional binary vectors. Then the solution of (3.3.6) is a separating matrix.

Notice that the objective function in (3.3.6) is smooth in $\mathbf{B}$ and the constraints are Stiefel manifold constraints. This makes it possible to apply Edelman's extended Newton's method on the Stiefel manifold to solve (3.3.6).

Manton in [55] studied the complex version of the following problem,

$$\begin{array}{ll} \text{minimize} & \phi(\mathbf{B}) \\ \text{subject to} & \mathbf{B}^T\mathbf{B} = \mathbf{I}, \end{array} \tag{3.3.14}$$

and proposed two algorithms, a steepest descent algorithm and a Newton algorithm, to solve the complex-valued optimization problem. The Newton type method proposed by Manton is different from that of Edelman in [27]. Edelman's method considers the problem completely in Stiefel manifold. It defines the gradient and Hessian of the objective function in this manifold, and eventually developed Newton's method in the manifold. By doing

so, the updating of vector is along a geodesic and the new vector is still in the manifold. Manton's method, on the other hand, considers the problem in Euclidean space. It computes the gradient and Hessian of the objective function in Euclidean space. When the vector is updated, it projects the new vector back to the complex Stiefel manifold so that the constraints are still satisfied. Both methods preserve the local quadratic convergence rate of original Newton's method. For details of Manton's methods, the readers are referred to [55] and the references therein. In this work, we use Edelman's extended Newton's method for our problem.

3.4 Simulation Results

In this section, we compare our proposed method with two other methods that exploit the geometry of the BPSK signals, namely Hansen's *hypercube* method [34, 35] and Xavier's *polyhedral* method [78].

After pre-whitening, the constellation of the received signal can be viewed as a d dimensional hypercube in $\Re^d$. It can be shown that the normal vector to each face of this hypercube is actually a column in the separating matrix. Hansen's method computes the normal vectors by solving a sequence of constrained maximization problem using a gradient based method. Since the optimization problem is non-convex, the hypercube algorithm suffers from the problem of local convergence. To detect local convergence, we follow [67] and compare the residual, $\|\mathbf{X} - \hat{\mathbf{A}}\mathbf{Y}\|_F^2$, to the noise level, $NM\sigma^2$. If the residual is larger than the noise floor, we declare local convergence and restart the algorithm with a new initial guess. A maximum of 100 repetitions was allowed, and if this upper bound was reached, the solution with the lowest residual was chosen.

Xavier's method, on the other hand, defined a polyhedral using the received signal points and has shown that each vertex of this polyhedral is in fact a row of the separating matrix (called an "equalizer" in [78]). Xavier's method computes the first equalizer by solving a linear programming (LP) problem in which an arbitrary non-zero vector must be chosen in advance. Starting from this first equalizer, the remaining equalizers can be acquired sequentially by solving a series of quadratic programming (QP) problems for which the objective functions are defined based on the previous computed equalizers. A major weakness of Xavier's method is that if the arbitrarily chosen non-zero vector in the LP problem happened to be a normal vector of the polyhedral, then solving the LP problem yields an incorrect equalizer. The remaining equalizers, which depend on the correct calculation of the first equalizer, are then incorrect. This problem becomes especially severe in the noisy environment.

We compare our method with these two methods on both the bit error rate (BER) and the overall computation cost (kilo-flops). We consider the same scenario as in [34], *i.e.*, 3 BPSK signals, 5 antennas and 100 data samples. The mixing matrix $\mathbf{A}$ was randomly generated such that the columns of $\mathbf{A}$ are normalized to 1 and the correlation between columns is not greater than 0.95. A total of 5000 Monte Carlo runs were performed for SNR levels of $(10, 15, 20, 25)$ dB as suggested in [34].

Since these three blind methods are 'zero-forcing' in nature, we provide results for a 'zero forcing' separator with full channel knowledge as a benchmark. In that case, $\mathbf{Y} = \mathbf{S} + \mathbf{A}^{\dagger}\mathbf{V}$, where $\mathbf{A}^{\dagger}$ is the pseudo inverse of the matrix $\mathbf{A}$. The bit error rate then can be calculated by quantizing $\mathbf{Y}$.

Figure 3.1 shows the raw bit error rates (BERs) of the three blind methods of interest as well as the reference zero-forcing method. It is clear that our method is significantly closer to the benchmark than the other two methods. However, the raw BER can be distorted by 'failures' of the blind methods; *i.e.*, when $\mathbf{BC}$ is not close to $\mathbf{DP}$ (see 3.3.4). By computing the Frobenius norm of the difference between $\mathbf{BC}$ and the nearest matrix of the form $\mathbf{DP}$ we can identify such failures and remove the corresponding records. The resulting BER curves are shown in Figure 3.2 and the corresponding failure rates are shown in Figure 3.3. Again our method appears to have a significant advantage.

Finally, we compare the number of floating point operations (FLOPs) required to calculate $\mathbf{B}$ in our implementations of the blind methods (see Figure 3.4). Our method appears to have a significant advantage here too. It is interesting to note that Hansen's hypercube methods requires more FLOPs at higher SNR. This is because the hypercube method employs an SNR dependent statistical test to determine whether a globally optional solution to the embedded non-convex optimization problem has been achieved. If the test is negative, the optimization routine is repeated and incurs additional computational cost.

3.5 Discussion

In this chapter, we have proposed a new approach for blind separation of BPSK signals. We formulated the source separation problem as an optimization problem using the constant modulus (CM) property of the signals, and solved this constrained optimization problem with Edelman's extended Newton's method on the Stiefel manifold. The simulation results clearly demonstrate the competitiveness of our proposed method.

The BSS algorithms proposed in this and the previous chapters are based on the instantaneous model of the received signals in which synchronized sources and equal transmission

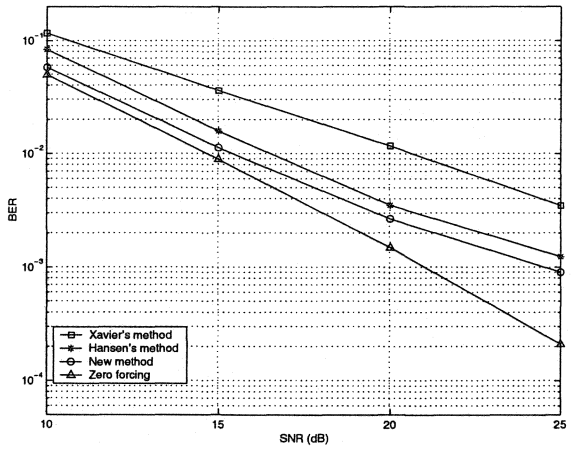

Figure 3.1: Raw bit error rates

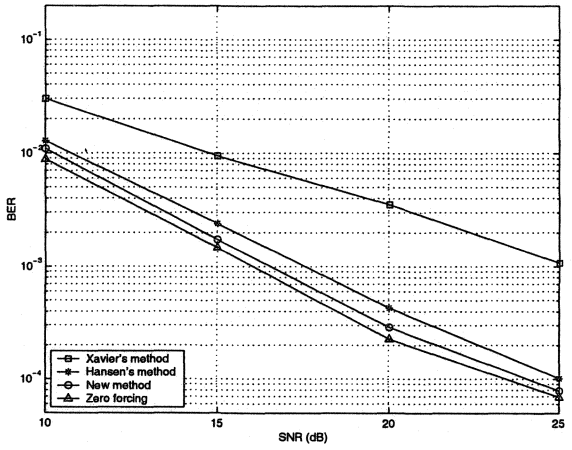

Figure 3.2: Bit error rates excluding records for which any blind method failed.

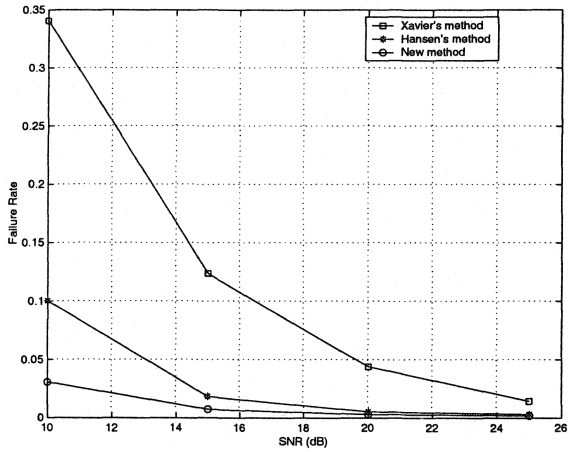

Figure 3.3: Failure rates.

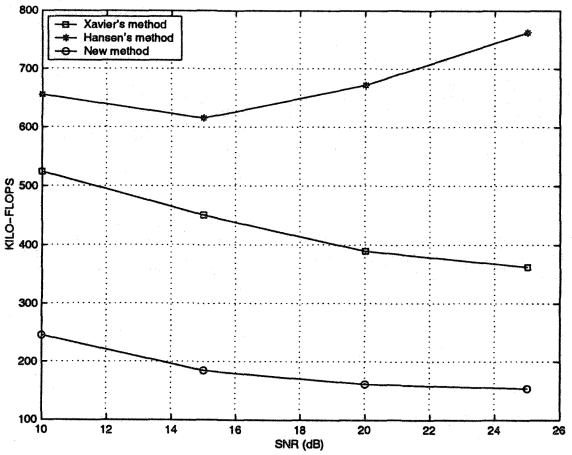

Figure 3.4: Floating point operation counts.

delays are assumed. However, for a large number of applications, such assumptions are not valid. In this case, the general convolutive model should be applied in characterizing the received signals. In the next chapter we will tackle the problem of blind separation of convolutively mixed signals.

Chapter 4

Blind Separation of Convolutively Mixed Signals

4.1 Overview

Here we address the problem of blind separation of convolutively mixed independent signals. We formulate the separation problem as a constrained optimization problem that minimizes a mutual information based criterion and solve it using a *sequential quadratic programming* (SQP) method. In this approach, we first decorrelate the convolutive mixtures by performing a spectral matrix factorization and then apply the mutual information based criterion to the decorrelated signals. Our approach not only separates independent sources, but also equalizes the extracted signals. The simulation results demonstrate the validity of our proposed method. The work has been published in [52].

4.2 Introduction

In the previous two chapters, we addressed the problem of blind separation of instantaneously mixed signals. The instantaneous model covers a wide range of practical problems in which the signal transmission delays can be assumed equal and the delay spread is negligible. However, there are more cases in which the instantaneous model can not accurately characterize the received signals. In these cases, the received signals are no longer the simple linear mixtures of the sources, instead they are convolutive mixtures of the sources. In this chapter, we will address the problem of blind separation of convolutively mixed signals, which is also known as blind deconvolution problem.

The single channel blind deconvolution problem has been extensively studied. However, the blind deconvolution for MIMO system is a fairly new research topic and is receiving more and more attentions recently. As discussed in Chapter 1, the blind deconvolution can be achieved in two ways: i) identify the channel from the channel output and design an equalizer accordingly; ii) design an equalizer blindly from the channel output through minimizing some criteria. The method we propose in this chapter belongs to the latter.

Pham in [63] proposed a mutual information based criterion for blind deconvolution of MIMO system with spatially independent and temporally i.i.d. sources, but did not provide any algorithm to optimize the criterion. There are two major difficulties that prevent the design of such an algorithm: i) the criterion itself is complicated and not easy to optimize, ii) little is known about the mixing system. In this chapter, we will propose an algorithm to minimize this criterion. Instead of applying the criterion directly over the received signals as in [63], we first decorrelate the received signals and then apply the criterion on the decorrelated signals. We propose a matrix polynomial factorization based method to decorrelate the convolutively mixed signals. In so doing, the criterion is significantly simplified and the mixing/separation system is narrowed down to a paraunitary matrix. The aforementioned two difficulties are alleviated, and this makes it possible for us to design algorithms to solve this problem. With the simplified criterion, we formulate the separation problem as a constrained optimization problem with paraunitary matrix constraint and solve it with a sequential quadratic programming (SQP) method.

The remainder of the chapter is organized as follows. Section 4.3 introduces in detail the matrix polynomial factorization based decorrelation method, and the formulation of the constrained optimization problem using Pham's criterion. To demonstrate the validity of our proposed method, some numerical simulations results are given in Section 4.4. Finally in Section 4.5, some brief remarks on the new approach are given to conclude the chapter.

4.3 Problem Formulation

Consider the general n-input n-output linear time invariant (LTI) finite impulse response (FIR) system described by the following equation,

$$\mathbf{x}(k) = \sum_{\ell=0}^{L-1} \mathbf{H}(\ell)\mathbf{s}(k-\ell), \tag{4.3.1}$$

where $\mathbf{s}(k) = [s_1(k), \ldots, s_n(k)]^T$ is the vector of n sources, $\mathbf{x}(k) = [x_1(k), \ldots, x_n(k)]^T$ is the vector of n received signals and $\{\mathbf{H}(\ell)\}_{\ell=0}^{L-1}$ is a FIR matrix filter representing the channel impulse response. Let $\mathcal{H}(z) \triangleq \sum_{\ell=0}^{L-1} \mathbf{H}(\ell)z^{-\ell}$ be the channel transfer function. Our goal

is to determine a FIR separation matrix filter, $\mathcal{W}(z) \triangleq \sum_{\ell=0}^{Lw-1} \mathbf{W}(\ell)z^{-\ell}$, such that its output $\mathbf{y}(k) = \sum_{\ell=0}^{Lw-1} \mathbf{W}(\ell)\mathbf{x}(k-\ell)$ recovers the source signals up to a permutation and a filtering, *i.e.*, the combined system is given by $\mathcal{W}(z)\mathcal{H}(z) = \mathbf{P}\mathcal{D}(z)$, where $\mathbf{P}$ is a constant permutation matrix and $\mathcal{D}(z)$ is a diagonal matrix filter.

Through out this chapter, we will make the following assumptions:

A1. the sources are statistically independent stationary random processes with each source is a sequence of zeros-mean unit-variance i.i.d. random variables, and

A2. the channel transfer function $\mathcal{H}(z)$ is invertible for all $|z| = 1$.

As mentioned earlier, the decorrelation of the received convolutively mixed signals plays an important role in our proposed method. It not only simplifies the objective function, but also narrows down our search space for the separation filter $\mathcal{W}(z)$ to the set of paraunitary matrices, *i.e.*, $\{\mathcal{W}(z) \,|\, \mathcal{W}^T(z^{-1})\mathcal{W}(z) = \mathbf{I}\}$. For the instantaneous case, decorrelation can be easily achieved by factorizing the cross-correlation matrix of the received signals. However, for the convolutive case, decorrelation of convolutive mixtures is by no means an easy task. In this work, we propose a method that factorizes the cross-power spectrum of the observed signals using a matrix polynomial factorization method. Details of this method are given below.

Suppose the cross-power spectrum of $\mathbf{x}$ is given by

$$\mathcal{R}_x(z) \triangleq \sum_{k=-\infty}^{\infty} \mathbf{R}_x(k)z^{-k}, \tag{4.3.2}$$

where $\mathbf{R}_x(k) \triangleq E[\mathbf{x}(n)\mathbf{x}(n-k)^T]$ is the cross-correlation matrix of $\mathbf{x}$. Substituting (4.3.1) into (4.3.2) and applying our assumptions on the sources, the cross-correlation matrix can be expressed as

$$\mathcal{R}_x(z) = \mathcal{H}(z)\mathcal{H}(z^{-1})^T, \tag{4.3.3}$$

where $\mathcal{H}(z)$, which is the transfer function of the channel, is a matrix polynomial of order $L-1$. Therefore, by (4.3.3), the highest (lowest) order of matrix polynomial $\mathcal{R}_x(z)$ is $L-1$ $(-L+1)$. Hence the summation in (4.3.2) is actually from $-L+1$ to $L-1$. Our objective is now to factorize $\mathcal{R}_x(z)$ as

$$\mathcal{R}_x(z) = \mathcal{G}(z)\mathcal{G}(z^{-1})^T \tag{4.3.4}$$

for some $L-1$ order matrix polynomial $\mathcal{G}(z)$, and then use $\mathcal{G}^{-1}(z)$ as the decorrelating filter.

There are several approaches that can be employed for matrix polynomial factorization. We choose Aliyev's efficient method [1] for this job. The main idea of Aliyev's method is summarized in the following theorem.

Theorem 4.3.1 *Suppose $\mathcal{R}(z)$ is an $m \times m$ matrix polynomial given by*

$$\mathcal{R}(z) = \mathbf{R}_0 z^n + \mathbf{R}_1 z^{n-1} + \cdots + \mathbf{R}_{n-1} z^1 + \mathbf{R}_n + \mathbf{R}_{n-1}^T z^{-1} + \cdots + \mathbf{R}_0^T z^{-n},$$

whose determinant has no zero on the unit circle, and the constant term $\mathbf{R}_n$ is a symmetric positive-definite matrix. Then the $m \times m$ matrix polynomial $\mathcal{V}(z)$, which is given in (4.3.5) below, is a factor of $\mathcal{R}(z)$, i.e., *$\mathcal{R}(z) = \mathcal{V}^T(z^{-1})\mathcal{V}(z)$, moreover, $\mathcal{V}^{-1}(z)$ has no poles inside the unit disk:*

$$\mathcal{V}(z) = (\mathbf{\Gamma}^T \mathbf{S} \mathbf{\Gamma})^{\frac{1}{2}} + (\mathbf{\Gamma}^T \mathbf{S} \mathbf{\Gamma})^{-\frac{1}{2}} \mathbf{\Gamma}^T \mathbf{S} \mathbf{\Psi} \mathbf{N}(z), \tag{4.3.5}$$

where

$$\mathbf{\Psi} = \begin{bmatrix} 0 & \cdots & 0 & 0 \\ \mathbf{I}_m & & & 0 \\ & \ddots & & \vdots \\ & & \mathbf{I}_m & 0 \end{bmatrix}, \quad \mathbf{\Gamma} = \begin{bmatrix} \mathbf{I}_m \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathbf{N}(z) = \begin{bmatrix} \mathbf{I}_m z \\ \vdots \\ \vdots \\ \mathbf{I}_m z^{n+1} \end{bmatrix},$$

$\mathbf{I}_m$ is the $m \times m$ identity matrix, and matrix $\mathbf{S}$ is a solution of the discrete algebraic Riccati equation

$$\mathbf{S} = \mathbf{\Psi}^T \mathbf{S} \mathbf{\Psi} - \mathbf{\Psi}^T \mathbf{S} \mathbf{\Gamma} (\mathbf{\Gamma}^T \mathbf{S} \mathbf{\Gamma})^{-1} \mathbf{\Gamma}^T \mathbf{S} \mathbf{\Psi} + \mathbf{R}, \tag{4.3.6}$$

with

$$\mathbf{R} = \begin{bmatrix} \frac{1}{n+1}\mathbf{R}_n & \mathbf{R}_{n-1} & \cdots & \mathbf{R}_0 \\ \mathbf{R}_{n-1}^T & \frac{1}{n+1}\mathbf{R}_n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{R}_0^T & 0 & \cdots & \frac{1}{n+1}\mathbf{R}_n \end{bmatrix}.$$

Here $\mathbf{S}$ is chosen to be the solution of (4.3.6) *such that $\mathbf{\Psi} - \mathbf{\Gamma}(\mathbf{\Gamma}^T \mathbf{S} \mathbf{\Gamma})^{-1} \mathbf{\Gamma}^T \mathbf{S} \mathbf{\Psi}$ has eigenvalues inside the unit disk.*

An alternative semidefinite programming based method [24] can also be employed for this matrix factorization. This method exploits the problem convexity and enjoys polynomial time complexity and is computationally efficient and stable.

Suppose $\mathcal{R}_x(z)$ has been factorized as in (4.3.4) using the method provided in Theorem 4.3.1. Let $\mathcal{G}^{-1}(z)$ be the decorrelating filter. Then the decorrelated signals are given by

$$\hat{\mathbf{x}}(k) \triangleq [\mathcal{G}^{-1}(z)]\mathbf{x}(k) = [\mathcal{G}^{-1}(z)\mathcal{H}(z)]\mathbf{s}(k), \tag{4.3.7}$$

where, for any signal $\mathbf{f}(k)$, the notation $[\mathcal{F}(z)]\mathbf{f}(k)$ stands for $\sum_\ell \mathbf{F}(\ell)\mathbf{f}(k-\ell)$. From (4.3.3) and (4.3.4), it is easy to verify that matrix $\mathcal{G}^{-1}(z)\mathcal{H}(z)$ is paraunitary. Our goal then reduces to determining a paraunitary separation filter $\mathcal{W}(z) = \sum_{\ell=0}^{Lw-1} \mathbf{W}(\ell)z^{-\ell}$ such that $\mathbf{y}(k) = \sum_{\ell=0}^{Lw-1} \mathbf{W}(\ell)\hat{\mathbf{x}}(k-\ell)$ recovers the source signals up to a permutation and a filtering.

For notational simplicity, we drop the time index k in $\mathbf{y}(k)$ and $\mathbf{s}(k)$ and consider the entries of $\mathbf{s}$ and $\mathbf{y}$ as random variables. Pham in [63] proposed the following mutual information based criterion for the separation of i.i.d. sources:

$$\sum_{k=1}^{n} H(y_k) - \frac{1}{2\pi}\int_{-\pi}^{\pi} \log\left|\det\left(\sum_{\ell=-\infty}^{\infty} \mathbf{W}(\ell)e^{i\ell\lambda}\right)\right| d\lambda, \tag{4.3.8}$$

where y_k is the kth entry of $\mathbf{y}$ and $H(y_k)$ is the marginal entropy of y_k. It was shown in [63] that the minimization of this objective function not only separates the sources but also deconvolves the extracted signals. In other words, the optimal $\mathcal{W}(z)$ determined by minimizing (4.3.8) is in fact an equalizer of the convolutive MIMO channel. It is easy to see that the criterion in (4.3.8) is not smooth because of the last term. Fortunately, the matrix filter $\mathcal{W}(z)$ is paraunitary in our case and the matrix $\sum_{\ell=-\infty}^{\infty} \mathbf{W}(\ell)e^{i\ell\lambda}$ is unitary for any frequency λ. Therefore, the last term in (4.3.8) is zero and the criterion is greatly simplified. Our problem can now be stated as the following:

$$\begin{aligned} \text{minimize} \quad & \sum_{k=1}^{n} H(y_k) = -\sum_{k=1}^{n} E[\log(q(y_k))] \\ \text{subject to} \quad & \mathcal{W}^T(z^{-1})\mathcal{W}(z) = \mathbf{I}, \end{aligned} \tag{4.3.9}$$

where $q(y_k)$ is the marginal density function of y_k. This expression is the same as (2.3.6). However, the decision variables for the two problems are different. For this problem, the decision variables are the coefficients of paraunitary matrix $\mathcal{W}(z)$. We now need to express the objective function in terms of the coefficients of $\mathcal{W}(z)$. Similar to what we did in Chapter 2, we estimate the density function $q(y_k)$ with the kernel estimator given in (2.3.7), approximate the expectation operation in (4.3.9) with a sample average and express y_k in terms of $\hat{\mathbf{x}}$ and $\mathcal{W}(z)$. After these mathematical steps, our problem can eventually be expressed as

$$\begin{aligned} \text{minimize} \quad & -\frac{1}{M}\sum_{j=N+1}^{N+M}\sum_{k=1}^{n} \log\left(\sum_{i=1}^{N} e^{-\frac{1}{h_N^2}\left[\sum_{\ell=1}^{n} \mathbf{w}_{k\ell}^T(\hat{\mathbf{x}}_\ell(i)-\hat{\mathbf{x}}_\ell(j))\right]^2}\right) \\ \text{subject to} \quad & \mathcal{W}^T(z^{-1})\mathcal{W}(z) = \mathbf{I}, \end{aligned} \tag{4.3.10}$$

where $\mathbf{w}_{k\ell} = [w_{k\ell}(0), \ldots, w_{k\ell}(L_W - 1)]^T$ is the vector of coefficients of $\mathbf{w}_{k\ell}(z)$, which is the (k, ℓ) entry of $\mathcal{W}(z)$, $\hat{\mathbf{x}}_\ell(i) = [\hat{x}_\ell(i), \ldots, \hat{x}_\ell(i - L_W + 1)]^T$, $\hat{x}_\ell(k)$ is the ℓth entry of $\hat{\mathbf{x}}(k)$, and N and M are the number of samples used in the estimation of the density function and the expectation respectively. Since the space defined by $\mathcal{W}^T(z^{-1})\mathcal{W}(z) = \mathbf{I}$ is not a Stiefel manifold, Edelman's extended Newton's method can no longer be applied to this problem. With the gradients of the objective function and constraints being calculated analytically, we solve this constrained optimization problem with a sequential quadratic programming (SQP) method written in Matlab. The SQP method is a well known nonlinear programming method with extensive research [10,33,64]. The method closely mimics Newton's method for unconstrained optimization. At each step, a local quadratic program (QP) problem is constructed and solved, yielding a step toward the solution of the original problem. The method is capable of global convergence and local quadratic convergence (under mild assumptions). An overview of SQP method can be found in [29].

In particular, the gradient vector is calculated as follows. Denote $\hat{\mathbf{x}}_\ell^{(ij)} = \hat{\mathbf{x}}_\ell(i) - \hat{\mathbf{x}}_\ell(j)$. Then the partial derivatives of the objective function, denoted by $\phi(\mathcal{W}(z))$, with respect to the coefficients of $\mathcal{W}(z)$ are given by

$$\frac{\partial \phi(\mathcal{W}(z))}{\partial w_{pq}(u)} = \frac{2}{Mh_N^2} \sum_{j=N+1}^{N+M} \left(\frac{\sum_{i=1}^{N} e^{-\frac{1}{h_N^2}\left(\sum_{\ell=1}^{n} \mathbf{w}_{p\ell}^T \hat{\mathbf{x}}_\ell^{(ij)}\right)^2} \left(\sum_{\ell=1}^{n} \mathbf{w}_{p\ell}^T \hat{\mathbf{x}}_\ell^{(ij)}\right) \hat{x}_q^{(ij)}(u)}{\sum_{i=1}^{N} e^{-\frac{1}{h_N^2}\left(\sum_{\ell=1}^{n} \mathbf{w}_{p\ell}^T \hat{\mathbf{x}}_\ell^{(ij)}\right)^2}} \right)$$

for $p, q = 1, \ldots, n$, and $u = 0, \ldots, L_B - 1$, where $\hat{x}_q^{(ij)}(u)$ is the uth entry of vector $\hat{\mathbf{x}}_q^{(ij)}$.

4.4 Simulation Results

We present one computer simulation to demonstrate the validity of our proposed method. We consider the following 2-input 2-output FIR system

$$\mathcal{H}(z) = \begin{bmatrix} -0.1750 - 0.4156z^{-1} + 0.1753z^{-2}, & -0.3031 + 1.6381z^{-1} - 0.0762z^{-2} \\ -0.1744 - 1.1654z^{-1} - 0.1895z^{-2}, & -0.7470 - 0.3932z^{-1} - 0.0565z^{-2} \end{bmatrix}.$$

The sources were two binary phase shift keying (BPSK) signals with signal lengths being $T = 3000$. Even though our method was developed in the absence of noise, in this simulation we assume the observed signals are mixed with additive white Gaussian noise. Each with the same input signal-to-noise-ratio (SNR) of 20 dB. The order of the equalizer $\mathcal{W}(z)$ is set to $L_W = 5$. We use $N = 500$ samples to estimate the marginal density function $q(y_k)$ and $M = 500$ samples to approximate $E[\log(q(y_k))]$. The initial value for $\mathcal{W}(z)$ is a randomly chosen paraunitary matrix. After 32 iterations of the SQP method, the desired separation

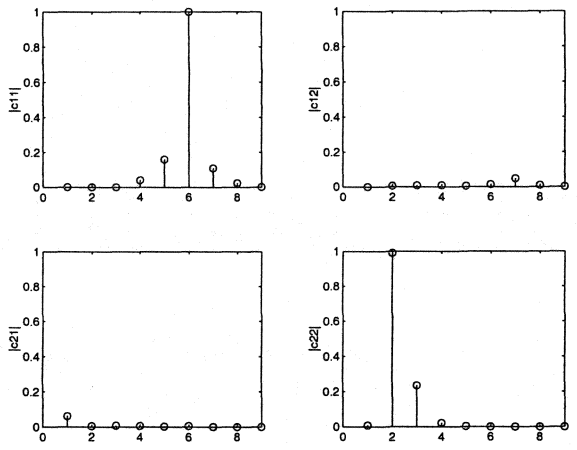

Figure 4.1: Impulse responses of the equalized system.

filter $\mathcal{W}(z)$ was found. Figure 4.1 shows the combined impulse response of the channel and the equalizer and it can be seen closing to two delta functions. Figure 4.2 shows the constellation of the two BPSK source signals. The signal constellations before and after equalization are given in Figure 4.3 and Figure 4.4 respectively. Clearly, the channel is successfully equalized and that demonstrates the effectiveness of our proposed method.

4.5 Discussion

In this chapter, we have considered the problem of blind separation of convolutively mixed i.i.d. signals. We first decorrelated the convolutively mixed signals by factorizing the cross-power spectrum matrix of the received signals and then formulated the source separation problem as a constrained minimization problem using the mutual information criterion. Finally, we solved this problem with a SQP method. This approach not only separates the sources, but also equalizes the separated signals. The simulation result indicates the validity of the proposed method.

The mutual information criterion used in this chapter is specific to statistically independent i.i.d. sources. For general sources that are not spatially independent or temporally i.i.d., different criteria should be used. In the next chapter, we will propose a set of higher-order statistics (HOS) based criteria for this purpose.

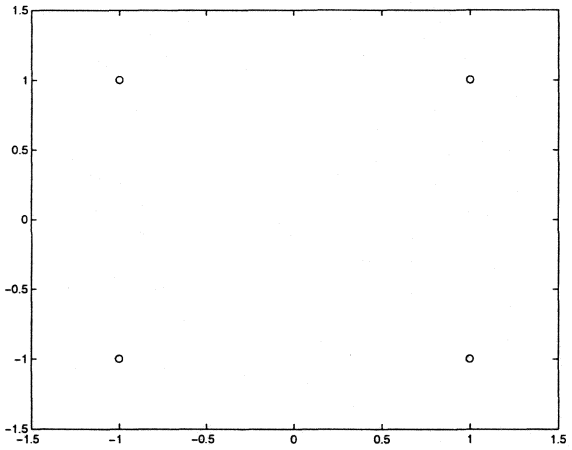

Figure 4.2: Constellation of the source signals.

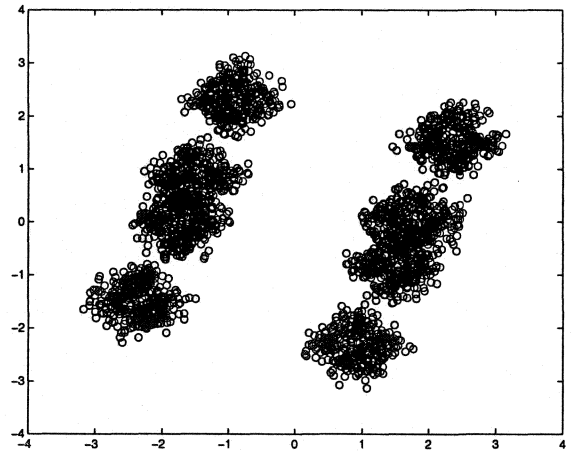

Figure 4.3: Constellation of the observed signals.

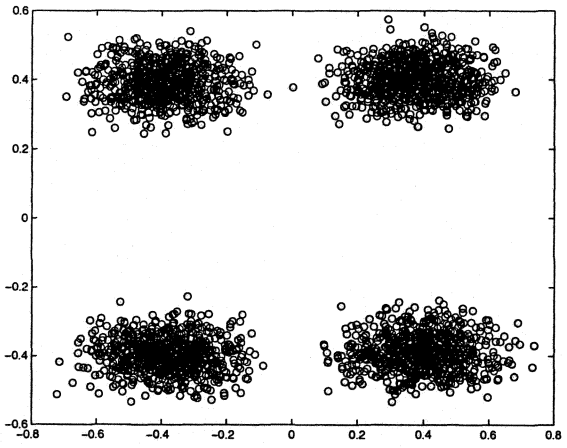

Figure 4.4: Constellation of the equalized signals.

Chapter 5

Criteria for Blind Deconvolution

5.1 Overview

Zhu, Cao and Ding in [83] introduced the concept of *Mth-order uncorrelatedness* for random variables and proposed a set of cumulant based criteria for blind separation of instantaneously mixed Mth-order uncorrelated signals. In this chapter, we extend this concept to stationary random processes and propose a set of polyspectrum based criteria for blind separation of convolutively mixed Mth-order uncorrelated signals. Our work can be viewed as an natural extension of the work in [83]. The relationship of our proposed criteria and other higher-order statistics based blind deconvolution criteria is also discussed in this chapter.

5.2 Introduction

In the previous chapter, we have successfully solved the blind deconvolution problem for spatially independent and temporally i.i.d. sources. A mutual information based criterion was minimized in determining the separation system. In this chapter, we will studied a more general class of blind deconvolution problem in which the sources may not be spatially independent nor temporally i.i.d. A set of polyspectrum based criteria will be proposed for solving the problem.

As mentioned in the previous chapter, blind MIMO deconvolution is a fairly new research subject. Solutions to the problem can be categorized into two major groups: one focuses on channel identification, the other concentrates on channel equalizer design. Solutions in the latter group typically involves a HOS based criterion and an optimization algorithm

that minimizes or maximizes the criterion. Several representative algorithms can be found in [22, 44, 58, 79].

Comon in [22] proposed two rth-order ($r > 2$) cumulant based criteria. Inouye in [44] proposed two fourth-order cumulant based criteria. Similar criteria were presented by Moreau and Thirion [58] for system with complex sources. In all three methods, the sources were assumed to be spatially independent and temporally i.i.d. To the best of our knowledge, Yellin and Weinstein [79] are the only authors who do not make such an assumption. Yellin and Weinstein [79] proposed a polyspectrum based criteria in which the sources were assumed to have non-zero third- (or fourth-) order auto-polyspectrum and have zero third- (or fourth-) order cross-polyspectrum. This assumption is weaker than the usual independence assumption. However, the criteria were developed for the restrictive case of 2×2 system with the direct channels being unity. Moreover, the polyspectrum used was limited to bispectrum or trispectrum. Detail discussion of works in [22, 79] will be given in Section 5.4.

In this chapter, we consider the general $n \times n$ LTI FIR MIMO channel. We first extend the concept of Mth-order uncorrelatedness, which was introduced by Zhu, Cao and Ding for random variables [83], to the case of stationary random processes. We then propose a set of polyspectrum based criteria for blind deconvolution of system with Mth-order uncorrelated stationary inputs. Our criteria are more general than Yellin's criteria in that they are developed for general MIMO channel and there is no limitation on the order of polyspectrum. In fact, we can show that our proposed criteria contains Yellin's criteria as a special case.

The rest of the chapter is organized as follows. Section 5.3 is an introduction to higher-order statistics, particulars cumulants and the polyspectrum, together with their properties. Section 5.4 gives some HOS based blind deconvolution criteria. In particular, Comon's cumulant based criteria [22] and Yellin's polyspectrum based criteria [79] are introduced. Our polyspectrum based criteria are presented in Section 5.5. In Section 5.6, the relationship of our proposed criteria and other HOS based criteria is analyzed. Finally, some brief remarks are made in Section 5.7 to close this chapter.

5.3 Cumulant and Polyspectrum

In this section, we will introduce two higher-order statistics, *i.e.*, cumulant and polyspectrum, and some of their useful properties. For complete review of higher-order statistics and their properties, readers are referred to [13].

5.3.1 Cumulant and Properties

There are several ways to define the joint cumulant. For example, it can be defined in terms of the *joint characteristic function* of random variables. Here we define it using moments of random variables. An advantage of this definition is that it clearly shows the relationship between cumulants and moments, and also provide a method for computing the cumulant.

Definition 1. Consider n random variables $\{y_1, \ldots, y_n\}$ with $E[|y_j|^n] < \infty$, $j = 1, \ldots, n$, where the y_j are real or complex. The nth-order joint cumulant of $\{y_1, \ldots, y_n\}$ is given by

$$\text{cum}(y_1, \ldots, y_n) \triangleq \sum (-1)^{p-1}(p-1)!\, E\left[\prod_{j \in \mu_1} y_j\right] \cdots E\left[\prod_{j \in \mu_p} y_j\right],$$

where the summation extends over all partitions $\{\mu_1, \ldots, \mu_p\}$, $p = 1, \ldots, n$, of $\{1, \ldots, n\}$.

The following are some special cases of cumulants for zero-mean real random variables, which are commonly encountered in the literature:

$$\begin{aligned} \text{cum}(y_1, y_2) &= E[y_1 y_2]; \\ \text{cum}(y_1, y_2, y_3) &= E[y_1 y_2 y_3]; \\ \text{cum}(y_1, y_2, y_3, y_4) &= E[y_1 y_2 y_3 y_4] - E[y_1 y_2]E[y_3 y_4] - E[y_1 y_3]E[y_2 y_4] - E[y_1 y_4]E[y_2 y_3]. \end{aligned}$$

Some important properties of cumulant are summarized as follows [13]:

P1. $\text{cum}(a_1 y_1, \ldots, a_n y_n) = a_1 \cdots a_n \text{cum}(y_1, \ldots, y_n)$ for $a_1, \ldots, a_n$ constant;

P2. $\text{cum}(y_1, \ldots, y_n)$ is symmetric in its arguments;

P3. If any group of random variables $\{y_1, \ldots, y_n\}$ are independent of the remaining, then $\text{cum}(y_1, \ldots, y_n) = 0$;

P4. For random variables $\{z, y_1, \ldots, y_n\}$, $\text{cum}(z + y_1, y_2, \ldots, y_n) = \text{cum}(z, y_2, \ldots, y_n) + \text{cum}(y_1, y_2, \ldots, y_n)$.

These properties will be used in Chapter 6, but we have introduced them here for convenience.

5.3.2 Polyspectrum and Properties

With the joint cumulant defined above, the polyspectrum for stationary random processes, which is also known as cumulant spectrum, is defined as below [13].

Definition 2. Let $x_1(t), \cdots, x_n(t)$ be n jointly stationary random processes. Let $\ell_1, \cdots, \ell_k \in \{1, 2, \cdots, n\}$ be a set of indices. Assume that

$$\sum_{\tau_1, \cdots, \tau_{k-1}=-\infty}^{\infty} |\text{cum}(x_{\ell_1}(t), x_{\ell_2}(t+\tau_1), \cdots, x_{\ell_k}(t+\tau_{k-1}))| < \infty.$$

Then the associated kth-order polyspectrum is defined by

$$P_{x_{\ell_1}, \cdots, x_{\ell_k}}(\omega_1, \cdots, \omega_{k-1}) = \sum_{\tau_1, \cdots, \tau_{k-1}=-\infty}^{\infty} \text{cum}(x_{\ell_1}(t), x_{\ell_2}(t+\tau_1), \cdots, x_{\ell_k}(t+\tau_{k-1})) e^{-j\sum_{i=1}^{k-1}\omega_i \tau_i},$$

where cum$(\cdot)$ is the joint cumulant of the random variables in the brackets.

Here we can see that the polyspectrum defined above is in fact the multidimensional Fourier transform of the corresponding cumulant sequence. Some special cases of the polyspectrum are given below.

Power Spectrum:

$$P_{x,x}(\omega) = \sum_{\tau=-\infty}^{\infty} \text{cum}(x(t), x(t+\tau)) e^{-j\omega\tau}.$$

Bispectrum:

$$P_{x_i, x_j, x_k}(\omega_1, \omega_2) = \sum_{\tau_1, \tau_2=-\infty}^{\infty} \text{cum}(x_i(t), x_j(t+\tau_1), x_k(t+\tau_2)) e^{-j(\omega_1\tau_1 + \omega_2\tau_2)}.$$

Trispectrum:

$$P_{x_i, x_j, x_k, x_\ell}(\omega_1, \omega_2, \omega_3) = \sum_{\tau_1, \tau_2, \tau_3=-\infty}^{\infty} \text{cum}(x_i(t), x_j(t+\tau_1), x_k(t+\tau_2), x_\ell(t+\tau_3)) e^{-j\left(\sum_{v=1}^{3}\omega_v\tau_v\right)}.$$

The following are some important properties of polyspectrum that will be frequently used in this chapter and the next:

P1. If $\{x_{\ell_1}(t), \ldots, x_{\ell_k}(t)\}$ can be divided into two or more subsets of statistically independent processes, then $P_{x_{\ell_1}, \ldots, x_{\ell_k}}(\omega_1, \ldots, \omega_{k-1}) = 0$.

P2. If $x_{\ell_1}(t), \ldots, x_{\ell_k}(t)$ are jointly Gaussian random processes and $k > 2$, then $P_{x_{\ell_1}, \ldots, x_{\ell_k}}(\omega_1, \ldots, \omega_{k-1}) = 0$.

P3. Let $y_i(t) = \sum_{j=1}^{m} h_{ij}(t) \otimes x_j(t)$, for $i = 1, 2, \ldots, n$, where $\otimes$ denotes the convolution operation and h_{ij} is the impulse response of a stable LTI system with frequency response $\mathcal{H}_{ij}(\omega)$. Then, for any $\ell_1, \ldots, \ell_k \in \{1, 2, \ldots, n\}$,

$$P_{y_{\ell_1},\ldots,y_{\ell_k}}(\omega_1, \ldots, \omega_{k-1}) = \sum_{j_1=1}^{m} \cdots \sum_{j_n=1}^{m} \mathcal{H}_{\ell_1 j_1}\left(-\sum_{i=1}^{k-1} \omega_i\right) \mathcal{H}_{\ell_2 j_2}(\omega_1) \cdots \mathcal{H}_{\ell_k j_k}(\omega_{k-1}) P_{x_{j_1},\ldots,x_{j_{k-1}}}(\omega_1, \ldots, \omega_{k-1}).$$

P4. The power spectrum $P_{x,x}(\omega)$ is real and non-negative.

5.4 HOS Based Criteria for Blind MIMO Deconvolution

In this section, several higher-order statistics based criteria for blind deconvolution are introduced. In particular, Comon's cumulant based criteria [22] and Yellin's polyspectrum based criteria [79] are described in detail.

5.4.1 Comon's Cumulant Based Criteria

Comon in [22] considered the following $n \times n$ LTI FIR MIMO system

$$\mathbf{y}(k) = \sum_{\ell=0}^{L-1} \mathbf{G}(\ell)\mathbf{s}(k-\ell),$$

where $\mathbf{s}(k)$ is the vector of sources, $\{\mathbf{G}(\ell)\}_{\ell=0}^{L-1}$ is a matrix filter and $\mathbf{y}(k)$ is the vector of output signals. Assuming the following properties on the sources and the matrix filter:

A1. The sources $\{s_i(t)\}_{i=1}^{n}$ are statistically independent stationary random processes with each source being a sequence of i.i.d. random variables with non-zero variance;

A2. Both columns and rows of the transfer matrix $\mathcal{G}(z) \triangleq \sum_{\ell=0}^{L-1} \mathbf{G}(\ell) z^{-\ell}$ are normalized over the whole frequency band, *i.e.*, $\mathrm{Diag}\left\{\oint \mathcal{G}(z)\mathcal{G}(z)^{\dagger} dz\right\} = I$ and $\mathrm{Diag}\left\{\oint \mathcal{G}(z)^{\dagger}\mathcal{G}(z) dz\right\} = I$, where function $\mathrm{Diag}(\mathbf{A})$ equals the diagonal matrix formed by the diagonal elements of matrix $\mathbf{A}$ and $\dagger$ stands for the transpose conjugate operation;

Comon proposed the following two criteria.

Theorem 5.4.1 *The functions*

$$(1)\quad \gamma_{1,r}(\mathbf{y}) \triangleq \sum_{j=1}^{n} |\operatorname{cum}(\underbrace{y_j(t),\ldots,y_j(t)}_{r})|, \tag{5.4.1}$$

$$(2)\quad \gamma_{2,r}(\mathbf{y}) \triangleq \sum_{j=1}^{n} \operatorname{cum}(\underbrace{y_j(t),\ldots,y_j(t)}_{r})^2, \tag{5.4.2}$$

for $r > 2$, reach their maximum if and only if $\mathcal{G}(z) = \mathcal{D}(z)P$, where $\mathcal{D}(z)$ is a diagonal filter and P is a permutation matrix.

The criteria above are also known as *contrast functions*. Similar criteria were developed in [44, 58] under the assumption that the combined system $\mathcal{G}(z)$ is paraunitary.

5.4.2 Yellin's Polyspectrum Based Criteria

Yellin in [79] considered the following 2×2 LTI system

$$x_1(t) = s_1(t) + [\mathcal{H}_{12}(z)] \cdot s_2(t),$$
$$x_2(t) = s_2(t) + [\mathcal{H}_{21}(z)] \cdot s_1(t),$$

where $s_1(t)$ and $s_2(t)$ are two sources, $x_1(t)$ and $x_2(t)$ are channel outputs and $\mathcal{H}_{12}$ and $\mathcal{H}_{21}$ are two LTI filters. The notation $[\mathcal{H}_{ij}(z)] \cdot s_j(t)$ stands for the discrete convolution of series $s_j(t)$ by sequence $\mathcal{H}_{ij}(k)$. The separation system is assumed to be

$$y_1(t) = x_1(t) - [\mathcal{W}_{12}(z)] \cdot x_2(t),$$
$$y_2(t) = x_2(t) - [\mathcal{W}_{21}(z)] \cdot x_1(t),$$

with $y_1(t)$ and $y_2(t)$ being the separated signals and $\mathcal{W}_{12}$ and $\mathcal{W}_{21}$ being two LTI filters. Denote the frequency responses of the channel and the separation system by

$$\mathcal{H}(\omega) = \begin{bmatrix} 1 & \mathcal{H}_{12}(\omega) \\ \mathcal{H}_{21}(\omega) & 1 \end{bmatrix} \quad \text{and} \quad \mathcal{W}(\omega) = \begin{bmatrix} 1 & -\mathcal{W}_{12}(\omega) \\ -\mathcal{W}_{21}(\omega) & 1 \end{bmatrix}, \tag{5.4.3}$$

respectively. In order that $y_1(t)$ and $y_2(t)$ recover the source signals $s_1(t)$ and $s_2(t)$, the combined system $\mathcal{G}(\omega) \triangleq \mathcal{W}(\omega)\mathcal{H}(\omega)$ should be one of the following forms:

$$\mathcal{G}(\omega) = \begin{bmatrix} \mathcal{G}_{11}(\omega) & 0 \\ 0 & \mathcal{G}_{22}(\omega) \end{bmatrix} \quad \text{or} \quad \mathcal{G}(\omega) = \begin{bmatrix} 0 & \mathcal{G}_{12}(\omega) \\ \mathcal{G}_{21}(\omega) & 0 \end{bmatrix}. \tag{5.4.4}$$

The following two theorems provide conditions under which the combined system has the above forms.

Theorem 5.4.2 *Let $s_1(t)$ and $s_2(t)$ be jointly stationary random processes such that*

$$P_{s_i^*,s_i,s_i}(\omega_1,\omega_2) \neq 0, \quad \forall\omega_1,\omega_2, \quad i=1,2, \tag{5.4.5}$$

$$P_{s_i^*,s_j,s_k}(\omega_1,\omega_2) = 0, \quad \forall\omega_1,\omega_2, \quad \forall i,j,k\in\{1,2\} \text{ except } i=j=k, \tag{5.4.6}$$

where $(*)$ denotes complex conjugate. Suppose that $\det(\boldsymbol{\mathcal{G}}(0)) \neq 0$. Then $\boldsymbol{\mathcal{G}}(\omega)$ is one of the forms in (5.4.4) if

$$P_{y_1^*,y_1,y_2}(\omega_1,\omega_2) = 0, \quad \forall\omega_1,\omega_2, \tag{5.4.7}$$

$$P_{y_2^*,y_2,y_1}(\omega_1,\omega_2) = 0, \quad \forall\omega_1,\omega_2. \tag{5.4.8}$$

Theorem 5.4.3 *Let $s_1(t)$ and $s_2(t)$ be jointly stationary random processes such that*

$$P_{s_i^*,s_i,s_i,s_i^*}(\omega_1,\omega_2,\omega_3) \neq 0, \ \forall\omega_1,\omega_2,\omega_3, \ i=1,2, \tag{5.4.9}$$

$$P_{s_i^*,s_j,s_k,s_\ell^*}(\omega_1,\omega_2,\omega_3) = 0, \ \forall\omega_1,\omega_2,\omega_3, \ \forall i,j,k,\ell\in\{1,2\} \text{ except } i=j=k=\ell. \tag{5.4.10}$$

Suppose that $\det(\boldsymbol{\mathcal{G}}(0)) \neq 0$. Then $\boldsymbol{\mathcal{G}}(\omega)$ is one of the forms in (5.4.4) if

$$P_{y_1^*,y_1,y_1,y_2^*}(\omega_1,\omega_2,\omega_3) = 0, \quad \forall\omega_1,\omega_2,\omega_3, \tag{5.4.11}$$

$$P_{y_1^*,y_1,y_2,y_2^*}(\omega_1,\omega_2,\omega_3) = 0, \quad \forall\omega_1,\omega_2,\omega_3. \tag{5.4.12}$$

These criteria can only be employed for blind deconvolution of 2×2 system with the direct channels being unity. In the next section, we will present our polyspectrum based criteria. Comparing with the criteria above, our criteria are more general that they can be used for blind deconvolution of general $n\times n$ MIMO FIR system, and the polyspectrum used in the criteria are not limited to second- or third-order.

5.5 The New Polyspectrum Based Criteria

In this section, we will first define the concept of *Mth-order uncorrelatedness* for stationary random processes. Then we will propose a set of polyspectrum based criteria for blind deconvolution of MIMO system that has Mth-order uncorrelated inputs.

Definition 3. Let $s_1(t),\cdots,s_n(t)$ be n jointly stationary random processes. These processes $\{s_i(t)\}_{i=1}^n$ are said to be mutually Mth-order uncorrelated if for any time indices $t_i, i=1,\cdots,n$, the random variables $\{s_i(t_i)\}_{i=1}^n$ are mutually Mth-order uncorrelated, *i.e.*, for any nonnegative integers p_i, $i=1,\cdots,n$, such that $p_1+\cdots+p_n\leq M$ and at least two of these integers are non-zero, we have

$$\text{cum}(\underbrace{s_1(t_1),\cdots,s_1(t_1)}_{p_1},\cdots,\underbrace{s_n(t_n),\cdots,s_n(t_n)}_{p_n}) = 0.$$

Here we have used the definition of the Mth-order uncorrelatedness for random variables [83]. It is easy to see that the Mth-order uncorrelatedness condition is weaker than the usual definition of statistical independence. By the definition of Mth-order uncorrelatedness, the following property holds:

P1'. For Mth-order uncorrelated stationary processes $s_1(t), \cdots, s_n(t)$, if indices $\ell_1, \ldots, \ell_k \in \{1, 2, \ldots, n\}$ $(k \leq M)$ are not all equal, then $P_{s_{\ell_1}, \ldots, s_{\ell_k}}(\omega_1, \ldots, \omega_{k-1}) = 0$.

Since this property is similar to property **P1** in Section 5.3.2, we denote it by **P1'**.

Consider the following general $n \times n$ MIMO LTI FIR system

$$\mathbf{y}(k) = \sum_{\ell=0}^{L-1} \mathbf{G}(\ell)\mathbf{s}(k-\ell) \tag{5.5.1}$$

where $\mathbf{s}(k) = [s_1(k), \ldots, s_n(k)]^T$ is the vector of n sources, $\{\mathbf{G}(\ell)\}$ is a sequence of $n \times n$ matrices representing the impulse responses of the combined system and $\mathbf{y}(k) = [y_1(k), \ldots, y_n(k)]^T$ is the vector of n output signals. Equation (5.5.1) can be expressed in frequency domain as

$$\mathbf{y}(\omega) = \boldsymbol{\mathcal{G}}(\omega)\mathbf{s}(\omega), \tag{5.5.2}$$

where $\mathbf{y}(\omega)$, $\boldsymbol{\mathcal{G}}(\omega)$ and $\mathbf{s}(\omega)$ are the frequency responses of $\mathbf{y}(k)$, $\mathbf{G}(k)$ and $\mathbf{s}(k)$ respectively. The following two theorems give two sets of polyspectrum based criteria under which $\boldsymbol{\mathcal{G}}(\omega)$ can be shown to have only one non-zero element on each of its row and column, *i.e.*, the signals are separated in $\mathbf{y}(\omega)$. The two theorems are given separately according to the order of polyspectrum being odd or even.

Theorem 5.5.1 *Let $s_1(t), \cdots, s_n(t)$ be jointly stationary random processes. Assume these processes are mutually Mth-order uncorrelated. Let k be an odd integer, $2 < k \leq M$. Assume also that*

$$P_{s_i,s_i}(\omega) \neq 0, \ \forall\, \omega, \tag{5.5.3}$$

$$P_{\underbrace{s_i, \cdots, s_i}_{k}}(\omega_1, \cdots, \omega_{k-1}) \neq 0, \ \forall\, \omega_1, \cdots, \omega_{k-1}, \tag{5.5.4}$$

for $i = 1, 2, \cdots, n$, and $\boldsymbol{\mathcal{G}}(0)$ is invertible. If

$$P_{y_i,y_j}(\omega) = 0, \ \forall\, \omega, \tag{5.5.5}$$

$$P_{\underbrace{y_i, \cdots, y_i}_{k-1}, y_j}(\omega_1, \cdots, \omega_{k-1}) = 0, \ \forall\, \omega_1, \cdots, \omega_{k-1}, \tag{5.5.6}$$

for $i, j = 1, 2, \cdots, n$, $i \neq j$, then the source signals are separated in $\mathbf{y}(\omega)$.

Theorem 5.5.2 *Let $s_1(t), \cdots, s_n(t)$ be jointly stationary random processes. Assume these processes are mutually Mth-order uncorrelated. Let k and p be two even integers, $2 \leq p < k \leq M$. Assume also that*

$$P_{s_i,s_i}(\omega) \neq 0, \quad \forall \omega, \tag{5.5.7}$$

$$P_{\underbrace{s_i, \cdots, s_i}_{k}}(\omega_1, \cdots, \omega_{k-1}) \neq 0, \quad \forall \omega_1, \cdots, \omega_{k-1}, \tag{5.5.8}$$

for $i = 1, \cdots, n$, and $\mathcal{G}(0)$ is invertible. If

$$P_{y_i,y_j}(\omega) = 0, \quad \forall\, \omega, \tag{5.5.9}$$

$$P_{\underbrace{y_i, \cdots, y_i}_{k-1},y_j}(\omega_1, \cdots, \omega_{k-1}) = 0, \quad \forall\, \omega_1, \cdots, \omega_{k-1}, \tag{5.5.10}$$

$$P_{\underbrace{y_i, \cdots, y_i}_{k-p}\underbrace{y_j, \cdots, y_j}_{p}}(\omega_1, \cdots, \omega_{k-1}) = 0, \quad \forall\, \omega_1, \cdots, \omega_{k-1}, \tag{5.5.11}$$

for $i, j = 1, 2, \cdots, n$, $i \neq j$, then the source signals are separated in $\mathbf{y}(\omega)$.

The criteria are developed for the Mth-order uncorrelated stationary processes. For statistically independent stationary process, the conclusions in the theorems still hold because the Mth-order uncorrelatedness condition are naturally satisfied. The proofs of Theorem 5.5.1 and Theorem 5.5.2 are given in Appendix A.2 and A.3 respectively.

5.6 Relationship to Other HOS Based Criteria

In the previous section, we have presented two sets of polyspectrum based blind deconvolution criteria. In this section, we will discuss the relationship of our proposed criteria with the criteria given in [22, 79], which we have introduced in Section 5.4. We will show that our proposed criteria, when limited to particular scenario, are equivalent to those criteria in [22, 79]. We will show this only for $n = 4$ case under the following assumptions. The case of general n can be shown similarly.

A1. The sources $\{s_i(t)\}_{i=1}^4$ are statistically independent with each source is a sequence of i.i.d. random variables with non-zero variance;

A2. The combined system $\mathcal{G}(z)$ is paraunitary.

The second assumption is not restrictive because we can always decorrelate the received signals using the method introduced in Chapter 4. We will show the equivalence of the criteria by introducing a series of statements.

Statement 1. The quantity $C \triangleq \sum_{ijkl=1}^{n} \sum_{\tau_1\tau_2\tau_3} |\operatorname{cum}(y_i(t), y_j(t+\tau_1), y_k(t+\tau_2), y_l(t+\tau_3))|^2$ is constant.

Proof: Substituting (5.5.1) into the expression of quantity C, we have

$$C = \sum_{ijk\ell=1}^{n} \sum_{\tau_1\tau_2\tau_3} \left| \operatorname{cum} \left(\sum_{p=1}^{n} \sum_{\ell_1=0}^{L-1} g_{ip}(\ell_1) s_p(t-\ell_1), \sum_{q=1}^{n} \sum_{\ell_2=0}^{L-1} g_{jq}(\ell_2) s_q(t+\tau_1-\ell_2), \right.\right.$$
$$\left.\left. \sum_{q=1}^{n} \sum_{\ell_2=0}^{L-1} g_{kr}(\ell_3) s_r(t+\tau_2-\ell_3), \sum_{q=1}^{n} \sum_{\ell_2=0}^{L-1} g_{\ell s}(\ell_4) s_s(t+\tau_3-\ell_4) \right) \right|^2,$$

where $g_{ij}(\ell)$ is the (i,j) entry of $\mathbf{G}(\ell)$. Applying the linearity property of cumulants and assumption **A1**, we can simplify the above equation as

$$C = \sum_{i,j,k,\ell=1}^{n} \sum_{\tau_1\tau_2\tau_3} \left| \sum_{p=1}^{n} \sum_{\ell=0}^{L-1} g_{ip}(\ell) g_{jp}(\ell+\tau_1) g_{kp}(\ell+\tau_2) g_{\ell p}(\ell+\tau_3) k_{4,s_p} \right|^2,$$

where $k_{4,s_p} \triangleq \operatorname{cum}(s_p(t), s_p(t), s_p(t), s_p(t))$. This equation can further be expressed as

$$C = \sum_{i,j,k,\ell=1}^{n} \sum_{\tau_1\tau_2\tau_3} \left(\sum_{p=1}^{n} \sum_{\ell=0}^{L-1} g_{ip}(\ell) g_{jp}(\ell+\tau_1) g_{kp}(\ell+\tau_2) g_{\ell p}(\ell+\tau_3) k_{4,s_p} \right)$$
$$\left(\sum_{q=1}^{n} \sum_{m=0}^{L-1} g_{iq}(m) g_{jq}(m+\tau_1) g_{kq}(m+\tau_2) g_{\ell q}(m+\tau_3) k_{4,s_q} \right).$$

Rearranging the terms and the order of the summations, we have

$$C = \sum_{p,q=1}^{n} k_{4,s_p} k_{4,s_q} \sum_{\ell,m=0}^{L-1} \left(\sum_{i=1}^{n} g_{ip}(\ell) g_{iq}(m) \right) \left(\sum_{\tau_1} \sum_{j=1}^{n} g_{jp}(\ell+\tau_1) g_{jq}(m+\tau_1) \right)$$
$$\left(\sum_{\tau_2} \sum_{k=1}^{n} g_{kp}(\ell+\tau_2) g_{kq}(m+\tau_2) \right) \left(\sum_{\tau_3} \sum_{\ell=1}^{n} g_{\ell p}(\ell+\tau_3) g_{\ell q}(m+\tau_3) \right). \quad (5.6.1)$$

Noticing that the combined system $\boldsymbol{\mathcal{G}}(z)$ is paraunitary, *i.e.*, $\boldsymbol{\mathcal{G}}(z)\boldsymbol{\mathcal{G}}^T(1/z) = \mathbf{I}$, we have

$$\sum_{\tau_1} \sum_{j=1}^{n} g_{jp}(\ell+\tau_1) g_{jq}(m+\tau_1) = \delta_{pq}\delta_{\ell m},$$
$$\sum_{\tau_2} \sum_{k=1}^{n} g_{kp}(\ell+\tau_2) g_{kq}(m+\tau_2) = \delta_{pq}\delta_{\ell m},$$
$$\sum_{\tau_3} \sum_{\ell=1}^{n} g_{\ell p}(\ell+\tau_3) g_{\ell q}(m+\tau_3) = \delta_{pq}\delta_{\ell m},$$

where δ_{ij} denotes the Kronecker delta function, which equals 1 if $i = j$ and 0 otherwise. Thus, (5.6.1) can be simplified as

$$C = \sum_{p=1}^{n} |k_{4,s_p}|^2,$$

which is constant and only depends on the source signals. **Q.E.D.**

Statement 2. The criteria in Theorem 5.4.2 are equivalent to the criteria in Theorem 5.5.1 for $M = k = 3$, and the criteria in Theorem 5.4.3 are equivalent to the criteria in Theorem 5.5.2 for $M = 4$ and $k = p = 2$.

This statement is obvious for real signals. For complex signals, criteria similar to those in Theorem 5.5.1 and Theorem 5.5.2 can be developed, and Statement 2 can be shown still hold.

Statement 3. The criteria in Theorem 5.5.2 for $M = 4$ and $k = p = 2$, which are given below,

$$P_{y_i,y_j}(\omega) = 0, \tag{5.6.2}$$

$$P_{y_i,y_i,y_i,y_j}(\omega_1,\omega_2,\omega_3) = 0, \tag{5.6.3}$$

$$P_{y_i,y_i,y_j,y_j}(\omega_1,\omega_2,\omega_3) = 0, \tag{5.6.4}$$

are equivalent to the criterion (5.4.2) in Theorem 5.4.1 for $r = 4$, which is

$$\gamma_{2,4}(\mathbf{y}) = \sum_j \text{cum}(y_j(t), y_j(t), y_j(t), y_j(t))^2. \tag{5.6.5}$$

Proof: By the pre-whitening assumption, (5.6.2) is always true. By the definition of polyspectrum, the criteria in (5.6.3) and (5.6.4) are equivalent to

$$\text{cum}(y_i(t), y_i(t+\tau_1), y_i(t+\tau_2), y_j(t+\tau_3)) = 0, \quad \forall \tau_1, \tau_2, \tau_3, \tag{5.6.6}$$

$$\text{cum}(y_i(t), y_i(t+\tau_1), y_j(t+\tau_2), y_j(t+\tau_3)) = 0, \quad \forall \tau_1, \tau_2, \tau_3. \tag{5.6.7}$$

We will show that (5.6.6), (5.6.7) imply (5.6.5) and *vice versa.*

From Statement 1, we know that the quantity

$$C = \sum_{ijk\ell=1}^{n} \sum_{\tau_1\tau_2\tau_3} |\text{ cum}(y_i(t), y_j(t+\tau_1), y_k(t+\tau_2), y_\ell(t+\tau_3))|^2, \tag{5.6.8}$$

is a constant. Quantity C can also be expressed as

$$\begin{aligned} C &= \sum_{i=1}^{n} |\text{cum}(y_i(t), y_i(t), y_i(t), y_i(t))|^2 + \\ &\quad \sum_{ijk\ell\tau_1\tau_2\tau_3 \in \Omega} |\text{ cum}(y_i(t), y_j(t+\tau_1), y_k(t+\tau_2), y_\ell(t+\tau_3))|^2 \end{aligned} \tag{5.6.9}$$

where $\Omega = \{i, j, k, \ell = 1 : n,\ \tau_1, \tau_2, \tau_3 = -\infty : +\infty,\ \textit{except}\ i = j = k = \ell,\ \textit{and}\ \tau_1 = \tau_2 = \tau_3 = 0\}$. Here we can see that to maximize the first term, which is the criterion in (5.6.5), is equivalent to minimizing the second term, which implies (5.6.6) and (5.6.7).

On the other hand, criteria (5.6.6) and (5.6.7) implies the sources are separated in $y_1(t), \ldots, y_n(t)$. Thus, they are fourth-order uncorrelated, which means the second term in (5.6.9) is zero, or equivalently, the first term of (5.6.9) is maximized. **Q.E.D.**

5.7 Discussion

In this chapter, we first extended the concept of the Mth-order uncorrelatedness to stationary random processes. We then proposed two sets of polyspectrum based criteria for blind deconvolution of MIMO system that has Mth-order uncorrelated inputs. Our work can be viewed as a natural extension of the work in [83]. We also analyzed the relationship of our proposed criteria and other HOS based criteria. We have shown that our proposed criteria, when limited to some special scenarios, are equivalent to those HOS based criteria given in [22,79]. But our criteria are more general in that they do not require the spatial independence and temporal i.i.d. assumptions and have no limitation on the order of polyspectrum. Our contribution in this chapter is an algebraic blind deconvolution principle instead of algorithm. Developing of practical blind deconvolution algorithm will be left for future research work.

From Chapter 1, we know that the blind deconvolution can also be achieved through blind channel identification. In the next chapter, we will propose a frequency domain polyspectrum based blind channel identification method.

Chapter 6

Blind Channel Identification Using Higher-Order Statistics

6.1 Overview

We propose an eigenstructure based algorithm for blind identification of MIMO convolutive channel. We define a fourth-order polyspectrum based matrix, called the *polyspectrum matrix*, such that the channel frequency response is uniquely determined by the eigendecomposition of this matrix. We also present an efficient method to determine the phase ambiguities in the estimated frequency responses. This method is general and can be used in conjunction with any other eigenstructure based methods in eliminating phase ambiguities. The simulation results clearly indicate the competitiveness of our proposed method.

6.2 Introduction

In the previous chapter, we have proposed a set of polyspectrum based criteria for blind deconvolution. As discussed in Chapter 1, the blind deconvolution can also be achieved via blind channel identification. In this chapter, we will address the blind MIMO channel identification problem.

A general $n \times n$ LTI FIR MIMO system, in the absence of noise, can be described by

$$\mathbf{x}(k) = \sum_{\ell=0}^{L-1} \mathbf{H}(\ell)\mathbf{s}(k-\ell), \tag{6.2.1}$$

where $\mathbf{s}(k) = [s_1(k), \ldots, s_n(k)]^T$ is the vector of n sources, $\mathbf{x}(k) = [x_1(k), \ldots, x_n(k)]^T$ is

the vector of n received signals and $\{\mathbf{H}(\ell)\}_{\ell=0}^{L-1}$ is a sequence of $n \times n$ matrices representing the system impulse response. The objective of blind MIMO identification is to identify the channel impulse response $\{\mathbf{H}(\ell)\}_{\ell=0}^{L-1}$ given only the received signal $\mathbf{x}(k)$.

Several eigenstructure based frequency domain blind identification algorithms have been proposed recently [5,18,25]. The basic idea of these algorithms can be summarized as follows:

1. First, a special SOS or HOS based matrix, denoted by $\boldsymbol{\mathcal{A}}(\omega)$, is defined such that the channel frequency response $\boldsymbol{\mathcal{H}}(\omega)$ can be estimated from the eigenvectors of this matrix;

2. Secondly, the channel frequency response $\boldsymbol{\mathcal{H}}(\omega_k)$ are estimated by the eigendecomposition of $\boldsymbol{\mathcal{A}}(\omega_k)$ for $\omega_k = \frac{2\pi k}{K}$, $k = 0, 1, \ldots, K-1$ $(K \geq L)$;

3. Finally, the channel impulse responses $\{\mathbf{H}(\ell)\}_{\ell=0}^{L-1}$ are computed by performing the usual inverse Fourier transform on $\boldsymbol{\mathcal{H}}(\omega_k)$, $k = 0, 1, \ldots, K-1$.

Several comments about such blind identification algorithms are in order. First, in order that $\boldsymbol{\mathcal{H}}(\omega)$ is uniquely determined, the eigenvalues of the specially designed matrix $\boldsymbol{\mathcal{A}}(\omega)$ must be mutually distinct. Secondly, the frequency response estimated by the eigendecomposition generally has the form $\hat{\boldsymbol{\mathcal{H}}}(\omega_k) = \boldsymbol{\mathcal{H}}(\omega_k)\mathbf{P}(\omega_k)e^{j\boldsymbol{\Phi}(\omega_k)}$, where $\mathbf{P}(\omega_k)$ is a permutation matrix and $\boldsymbol{\Phi}(\omega_k)$ is a diagonal matrix. Since both $\mathbf{P}(\omega_k)$ and $\boldsymbol{\Phi}(\omega_k)$ are frequency dependent, $\{\hat{\boldsymbol{\mathcal{H}}}(\omega_k)\}$ may not be coherent for all frequencies and thus can not be combined directly to obtain the channel impulse response. Therefore, an important task in these algorithms is to eliminate the permutation and phase ambiguities. By "eliminate" we mean the estimated frequency response can eventually be expressed as $\hat{\boldsymbol{\mathcal{H}}}(\omega_k) = \boldsymbol{\mathcal{H}}(\omega_k)\mathbf{P}e^{j\boldsymbol{\Phi}}$ for some constant permutation matrix $\mathbf{P}$ and constant phase matrix $\boldsymbol{\Phi}$.

The authors of [25] define matrix $\boldsymbol{\mathcal{A}}(\omega)$ as the second-order spectra correlation matrix. By assuming a special structure of the 2×2 FIR channel, the phase ambiguity problem is avoided and the permutation ambiguity problem is solved with the help of two invariant indices. Chen *et al* in [18] define matrix $\boldsymbol{\mathcal{A}}(\omega)$ using the bispectra (or trispectra) slice of the received signals. The eigenvalues of matrix $\boldsymbol{\mathcal{A}}(\omega)$ can be shown to be independent of frequency ω. Therefore, the permutation ambiguity problem is bypassed by simply ordering the eigenvalues assuming they are mutually distinct. For the phase ambiguity problem, the authors of [18] propose a method that computes the phase ambiguity by using the phase information in a sequence of bispectrum (or trispectrum) matrices. Matrix $\boldsymbol{\mathcal{A}}(\omega)$ in [5] is defined to be the product of two cyclic spectral matrices. The permutation and phase ambiguity problem is resolved in the same manner as in [18].

In this chapter, we identify the channel frequency response by the eigendecomposition

of a so called *polyspectrum matrix*. Our method is primarily motivated by Cardoso's JADE algorithm for blind separation of instantaneously mixed signals [14]. Since the eigenvalues of our polyspectrum matrix can be shown to be independent of frequency ω, the permutation ambiguity problem can be easily resolved by sorting the eigenvalues, again, assuming they are mutually distinct. To resolve the phase ambiguity, we propose an efficient method that calculates the phase ambiguities by solving a linear system of equations.

The work in this chapter and that of [18] are both polyspectrum based methods. There are some fundamental differences between the two methods, however.

1. Our polyspectrum matrix is defined using the fourth-order polyspectrum (trispectrum) of the received signals and an arbitrary non-zero matrix $\mathbf{M}$ (similar to Cardoso [14]), while the matrix used in [18] is defined in terms of bispectrum (or trispectrum) information only. This arbitrary matrix $\mathbf{M}$ provides us with more choices of polyspectrum matrices in the subsequent joint diagonalization operation.

2. Unlike [18], which made special assumptions on the channel to ensure the distinctiveness of the eigenvalues of the bispectrum matrix, we provide a systematic way to choose matrix $\mathbf{M}$, and it guarantees the distinctiveness of the eigenvalues of the corresponding polyspectrum matrix.

3. Our proposed method for phase ambiguity elimination uses only the phase information in the estimated channel frequency response $\{\hat{\boldsymbol{\mathcal{H}}}(\omega_k)\}_{k=0}^{K-1}$, while [18] used a different set of bispectrum matrices, the estimation of which can be computationally expensive.

The remainder of this chapter is organized as follows. In Section 6.3, a detailed derivation of our eigenstructure based blind MIMO channel identification algorithm is presented. Solutions for resolving the permutation and phase ambiguities are also proposed. Section 6.4 compares the proposed method and the method given in [18]. Some simulation results are given to demonstrate the competitiveness of the proposed method. In Section 6.5, some brief remarks are given to conclude this chapter.

6.3 Problem Formulation

Through out this chapter, we will make the following assumptions.

A1. The sources $s_i(t)$, $i = 1, \ldots, n$ are statistically independent stationary random processes.

A2. Each source is a sequence of zero-mean, unit-variance and i.i.d. random variables.

A3. The channel transfer function, defined by $\mathcal{H}(z) \triangleq \sum_{\ell=0}^{L-1} \mathbf{H}(\ell) z^{-\ell}$, is invertible for all $|z| = 1$, and the elements in each column of $\mathcal{H}(z)$ are coprime, *i.e.*, they do not share common zeros.

6.3.1 Eigenstructure Based Identification Method

In this section, we will derive our fourth-order polyspectrum based blind MIMO identification method. We first define the polyspectrum matrix as follows.

Definition 1. Let $x_1(t), \ldots, x_n(t)$ be n stationary random processes and $\mathbf{M} = [m_{k\ell}]$ be any non-zero $n \times n$ matrix. The associated polyspectrum matrix $\mathcal{P}_{\mathbf{M}}^{\mathbf{X}}(\omega_1, \omega_2, \omega_3)$ is defined component-wise by

$$\left[\mathcal{P}_{\mathbf{M}}^{\mathbf{X}}(\omega_1, \omega_2, \omega_3)\right]_{ij} \triangleq \sum_{k,\ell=1}^{n} P_{x_i,x_j,x_k,x_\ell}(\omega_1, \omega_2, \omega_3) m_{k\ell}, \tag{6.3.1}$$

where $P_{x_i,x_j,x_k,x_\ell}(\omega_1, \omega_2, \omega_3)$ is the fourth-order polyspectrum of processes $x_i(t)$, $x_j(t)$, $x_k(t)$ and $x_\ell(t)$.

From property **P3** of the polyspectrum in Section 5.3.2, we know that the polyspectrum $P_{x_i,x_j,x_k,x_\ell}(\omega_1, \omega_2, \omega_3)$ for received signals $x_i(t)$, $x_j(t)$, $x_k(t)$ and $x_\ell(t)$ can also be expressed as

$$P_{x_i,x_j,x_k,x_\ell}(\omega_1, \omega_2, \omega_3) = \sum_{p,q,r,s=1}^{n} h_{ip}\left(-\sum_{v=1}^{3}\omega_v\right) h_{jq}(\omega_1) h_{kr}(\omega_2) h_{\ell s}(\omega_3) P_{s_p,s_q,s_r,s_s}(\omega_1, \omega_2, \omega_3), \tag{6.3.2}$$

where $h_{ij}(\omega)$ is the (i,j) entry of $\mathcal{H}(\omega)$, and $P_{s_p,s_q,s_r,s_s}(\omega_1, \omega_2, \omega_3)$ is the fourth-order polyspectrum of source signals $s_p(t)$, $s_q(t)$, $s_r(t)$ and $s_s(t)$.

Applying assumptions **A1**, **A2** and the cumulant properties in Section 5.3.1, we have polyspectrum $P_{s_p,s_q,s_r,s_s}(\omega_1, \omega_2, \omega_3)$ given by

$$P_{s_p,s_q,s_r,s_s}(\omega_1, \omega_2, \omega_3) = \mathrm{cum}(s_p(0), s_p(0), s_p(0), s_p(0))\delta_{pq}\delta_{pr}\delta_{ps}, \tag{6.3.3}$$

where δ_{ij} denotes the Kronecker delta function, which is 1 if $i = j$ and 0 otherwise.

Substituting (6.3.3) into (6.3.2), we have

$$P_{x_i,x_j,x_k,x_\ell}(\omega_1, \omega_2, \omega_3) = \sum_{p=1}^{n} h_{ip}(-\omega_1 - \omega_2 - \omega_3) h_{jp}(\omega_1) h_{kp}(\omega_2) h_{\ell p}(\omega_3) k(s_p(0)), \tag{6.3.4}$$

where $k(s_p(0)) \triangleq \text{cum}(s_p(0), s_p(0), s_p(0), s_p(0))$ is called the *kurtosis* of random variable $s_p(0)$.

Substituting the expression of the polyspectrum in (6.3.4) into (6.3.1), we have

$$\left[\mathcal{P}_{\mathbf{M}}^{\mathbf{X}}(\omega_1, \omega_2, \omega_3)\right]_{ij} = \sum_{p=1}^{n} h_{ip}(-\omega_1 - \omega_2 - \omega_3) h_{jp}(\omega_1) k(s_p(0)) \left(\sum_{k,\ell=1}^{n} h_{kp}(\omega_2) h_{\ell p}(\omega_3) m_{k\ell} \right),$$

which can be expressed in matrix form as

$$\mathcal{P}_{\mathbf{M}}^{\mathbf{X}}(\omega_1, \omega_2, \omega_3) = \mathcal{H}(-\omega_1 - \omega_2 - \omega_3) \Delta_{\mathbf{M}}(\omega_2, \omega_3) \mathcal{H}^T(\omega_1), \tag{6.3.5}$$

where $\Delta_{\mathbf{M}}(\omega_2, \omega_3)$ is a diagonal matrix with its (i,i) entry being $\mathbf{h}_i^T(\omega_2)\mathbf{M}\mathbf{h}_i(\omega_3)k(s_i(0))$. Here $\mathbf{h}_i(\omega)$ is the ith column of $\mathcal{H}(\omega)$.

In particular, choosing $\omega_1 = -\omega$ and $\omega_2 = \omega_3 = 0$ in (6.3.5), we get

$$\mathcal{P}_{\mathbf{M}}^{\mathbf{X}}(-\omega, 0, 0) = \mathcal{H}(\omega) \Delta_{\mathbf{M}}(0,0) \mathcal{H}^H(\omega), \quad \forall \omega. \tag{6.3.6}$$

This is similar to the standard eigendecomposition expression. In order for the right-hand-side of (6.3.6) to be an eigendecomposition, we need to replace $\mathcal{H}(\omega)$ with an orthonormal matrix. This can be done through the pre-whitening operation described below.

Let the cross-power spectrum of the received signal $\mathbf{x}$ be given by

$$\mathcal{R}_{\mathbf{x}}(\omega) \triangleq \sum_{k=-\infty}^{\infty} \mathbf{R}_{\mathbf{x}}(k) e^{-j\omega k}, \tag{6.3.7}$$

where $\mathbf{R}_{\mathbf{x}}(k) \triangleq E[\mathbf{x}(n)\mathbf{x}(n-k)^H]$ is the cross-correlation matrix of $\mathbf{x}$. Substituting (6.2.1) into (6.3.7) and applying assumptions **A1** and **A2**, the cross-power spectrum can be expressed as

$$\mathcal{R}_{\mathbf{x}}(\omega) = \mathcal{H}(\omega) \mathcal{H}^H(\omega). \tag{6.3.8}$$

On the other hand, using a Cholesky factorization, we can factorize $\mathcal{R}_{\mathbf{x}}(\omega)$ as

$$\mathcal{R}_{\mathbf{x}}(\omega) = \mathbf{V}(\omega) \mathbf{V}^H(\omega), \tag{6.3.9}$$

where $\mathbf{V}(\omega)$ is a lower triangular matrix. Denote $\mathbf{U}(\omega) \triangleq \mathbf{V}^{-1}(\omega)\mathcal{H}(\omega)$. Then from (6.3.8) and (6.3.9), it is easy to check that $\mathbf{U}(\omega)$ is an orthonormal matrix.

Multiplying (6.3.6) from left by $\mathbf{V}^{-1}(\omega)$ and from right by $\mathbf{V}^{-H}(\omega)$, we have

$$\hat{\mathcal{P}}_{\mathbf{M}}^{\mathbf{X}}(-\omega, 0, 0) \triangleq \mathbf{V}^{-1}(\omega) \mathcal{P}_{\mathbf{M}}^{\mathbf{X}}(-\omega, 0, 0) \mathbf{V}^{-H}(\omega) = \mathbf{U}(\omega) \Delta_{\mathbf{M}}(0,0) \mathbf{U}^H(\omega), \tag{6.3.10}$$

which is the desired eigendecomposition expression. A similar pre-whitening procedure was taken in [18] and [5].

As stated in Section 6.2, the eigenvalues of $\hat{\mathcal{P}}_{\mathbf{M}}^{\mathbf{X}}(-\omega,0,0)$ must be mutually distinct in order for the matrix $\mathbf{U}(\omega)$ to be uniquely determined. Hence, the frequency response $\mathcal{H}(\omega) = \mathbf{V}(\omega)\mathbf{U}(\omega)$ is uniquely determined. The proposition below states that by properly choosing the matrix $\mathbf{M}$, the eigenvalues of $\hat{\mathcal{P}}_{\mathbf{M}}^{\mathbf{X}}(-\omega,0,0)$, *i.e.* the diagonal elements in $\Delta_{\mathbf{M}}(0,0)$, can be made mutually distinct. In [18], however, the distinctiveness of eigenvalues is ensured by a special assumption on the channel.

Proposition 6.3.1 *Assume $\mathcal{H}(\omega)$ is invertible for all ω and the kurtosis $\{k(s_i(0))\}_{i=1}^n$ are non-zero. Let $\{m_\ell\}_{\ell=1}^N$ be N distinct non-zero scalars, and define vector $\mathbf{m}_\ell = [1, m_\ell^1, \ldots, m_\ell^{n-1}]^T$ and matrix $\mathbf{M}_\ell = \mathbf{m}_\ell\mathbf{m}_\ell^T$ for $\ell = 1,\ldots,N$. If $N \geq n(n-1)^2+1$, then there must exist a matrix $\mathbf{M}_\ell$ such that the diagonal elements of $\Delta_{\mathbf{M}_\ell}(0,0)$ are mutually distinct.*

In practice, matrix $\mathbf{U}(\omega)$ is normally obtained by jointly diagonalizing a set of polyspectrum matrices $\hat{\mathcal{P}}_{\mathbf{M}}^{\mathbf{X}}(-\omega,0,0)$ for different choices of $\mathbf{M}$. In this way, an accurate and robust estimation of $\mathbf{U}(\omega)$ can be obtained [14, 32]. Here we can see that matrix $\mathbf{M}$ plays an important role in our proposed method. The proof of this Proposition 6.3.1 is given in Appendix A.4.

Recall that the estimation of matrix $\mathbf{U}(\omega)$ obtained from (6.3.10) through the eigendecomposition (or joint diagonalization) is normally of the form $\hat{\mathbf{U}}(\omega) = \mathbf{U}(\omega)\mathbf{P}(\omega)e^{j\Phi(\omega)}$ for some permutation matrix $\mathbf{P}(\omega)$ and some phase matrix $\Phi(\omega)$. Consequently, the estimated channel frequency response is given by $\hat{\mathcal{H}}(\omega) = \mathcal{H}(\omega)\mathbf{P}(\omega)e^{j\Phi(\omega)}$.

Suppose we have estimated the channel frequency responses $\hat{\mathcal{H}}(\omega_k) = \mathcal{H}(\omega_k)\mathbf{P}(\omega_k)e^{j\Phi(\omega_k)}$ for $\omega_k = \frac{2\pi k}{K}$, $k = 0,\ldots,K-1$ $(K \geq L)$. Our goal is to estimate the channel impulse response by taking the inverse Fourier transform of $\{\hat{\mathcal{H}}(\omega_k)\}_{k=0}^{K-1}$. However, because the permutation matrix $\mathbf{P}(\omega_k)$ and diagonal matrix $\Phi(\omega_k)$ are frequency dependent, we can not combine $\{\hat{\mathcal{H}}(\omega_k)\}_{k=0}^{K-1}$ directly to get the channel impulse response. Therefore, matrices $\mathbf{P}(\omega_k)$ and $\Phi(\omega_k)$, which we call permutation and phase ambiguities, must be eliminated from the estimated channel frequency responses.

Since the eigenvalues in $\Delta_{\mathbf{M}}(0,0)$ are mutually distinct and independent of frequency ω (see (6.3.10)), the permutation ambiguity can be easily resolved by sorting the eigenvalues and rearranging the corresponding eigenvectors. As a result $\mathbf{P}(\omega_k)$ equals to a constant permutation matrix $\mathbf{P}$ for all ω_k. In the next section, we will develop a method to resolve the phase ambiguity problem.

6.3.2 Resolving Phase Ambiguity

We first prove the following theorem which is essential to the algorithm we are going to develop.

Theorem 6.3.1 *Let* $\{\mathbf{h}(\ell)\}_{\ell=0}^{L-1}$ *be an* $n \times 1$ $(n \geq 2)$ *vector filter with its transfer function defined as* $\mathbf{h}(z) \triangleq \sum_{\ell=0}^{L-1} \mathbf{h}(\ell) z^{-\ell}$. *Assume that the elements in* $\mathbf{h}(z)$ *are coprime,* i.e., *they do not share common zeros. Define* $n(K-L) \times K$ *matrix*

$$\boldsymbol{\mathcal{E}} \triangleq \begin{bmatrix} \mathbf{h}(\omega_0)e^{j\omega_0 L}, & \ldots, & \mathbf{h}(\omega_{K-1})e^{j\omega_{K-1}L} \\ \vdots & \ddots & \vdots \\ \mathbf{h}(\omega_0)e^{j\omega_0(K-1)}, & \ldots, & \mathbf{h}(\omega_{K-1})e^{j\omega_{K-1}(K-1)} \end{bmatrix}$$

for some positive integer K *and* $\omega_k = \frac{2\pi k}{K}$, $k = 0, \ldots, K-1$. *Then* $\text{rank}(\boldsymbol{\mathcal{E}}) = K-1$ *whenever* $K \geq 2L-1$.

The proof of Theorem 6.3.1 is given in Appendix A.5.

We now proceed to resolving the phase ambiguity problem. By re-indexing if necessary, we can generally assume the permutation matrix $\mathbf{P}$ to be the identity matrix. The estimated frequency responses are then given by

$$\hat{\boldsymbol{\mathcal{H}}}(\omega_k) = \boldsymbol{\mathcal{H}}(\omega_k)e^{j\boldsymbol{\Phi}(\omega_k)}, \quad k = 0, 1, \ldots, K-1. \tag{6.3.11}$$

Our goal is to determine the ambiguity phases $\{\boldsymbol{\Phi}(\omega_k)\}_{k=0}^{K-1}$. Rewriting (6.3.11) as

$$\hat{\boldsymbol{\mathcal{H}}}(\omega_k)e^{-j\boldsymbol{\Phi}(\omega_k)} = \boldsymbol{\mathcal{H}}(\omega_k), \quad k = 0, 1, \ldots, K-1, \tag{6.3.12}$$

and applying the inverse Fourier transform to both sides of (6.3.12), we obtain

$$\frac{1}{K}\sum_{k=0}^{K-1} \hat{\boldsymbol{\mathcal{H}}}(\omega_k)e^{-j\boldsymbol{\Phi}(\omega_k)}e^{j\omega_k\ell} = \begin{cases} \mathbf{H}(\ell), & \ell = 0, \ldots, L-1, \\ 0, & \ell = L, \ldots, K-1. \end{cases} \tag{6.3.13}$$

We will show that the information provided by the equations above for $\ell = L, \ldots, K-1$ is sufficient to determine the phases $\{\boldsymbol{\Phi}(\omega_k)\}_{k=0}^{K-1}$ up to a constant phase.

Let $\boldsymbol{\Phi}(\omega_k) = \text{diag}\{\phi_k^{(1)}, \ldots, \phi_k^{(n)}\}$ and $\hat{\boldsymbol{\mathcal{H}}}(\omega_k) = [\hat{\mathbf{h}}_1(\omega_k), \ldots, \hat{\mathbf{h}}_n(\omega_k)]$. Then (6.3.13) can be written column-wise as

$$\sum_{k=0}^{K-1} \hat{\mathbf{h}}_i(\omega_k)e^{j\omega_k\ell}e^{-j\phi_k^{(i)}} = 0, \quad \ell = L, \ldots, K-1, \quad i = 1, \ldots, n. \tag{6.3.14}$$

Expressing (6.3.14) in matrix form, we get

$$\begin{bmatrix} \hat{\mathbf{h}}_i(\omega_0)e^{j\omega_0 L}, & \ldots, & \hat{\mathbf{h}}_i(\omega_{K-1})e^{j\omega_{K-1}L} \\ \vdots & \ddots & \vdots \\ \hat{\mathbf{h}}_i(\omega_0)e^{j\omega_0(K-1)}, & \ldots, & \hat{\mathbf{h}}_i(\omega_{K-1})e^{j\omega_{K-1}(K-1)} \end{bmatrix} \begin{bmatrix} e^{-j\phi_0^{(i)}} \\ \vdots \\ e^{-j\phi_{K-1}^{(i)}} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \tag{6.3.15}$$

for $i = 1, \ldots, n$. For simplicity, we denote the above equation by $\hat{\boldsymbol{\mathcal{E}}}_i \mathbf{e}_i = \mathbf{0}$, where $\hat{\boldsymbol{\mathcal{E}}}_i$ is the $n(K-L) \times K$ matrix of the known coefficients of the estimated channel frequency response and $\mathbf{e}_i$ is the solution vector containing the desired phases. We will show that the rank of matrix $\hat{\boldsymbol{\mathcal{E}}}_i$ is $K-1$ when K, the number of frequencies, is large enough. This is easy to see because by (6.3.11) we can always express the matrix $\hat{\boldsymbol{\mathcal{E}}}_i$ as

$$\hat{\boldsymbol{\mathcal{E}}}_i = \begin{bmatrix} \mathbf{h}_i(\omega_0)e^{j\omega_0 L}, & \ldots, & \mathbf{h}_i(\omega_{K-1})e^{j\omega_{K-1}L} \\ \vdots & \ddots & \vdots \\ \mathbf{h}_i(\omega_0)e^{j\omega_0(K-1)}, & \ldots, & \mathbf{h}_i(\omega_{K-1})e^{j\omega_{K-1}(K-1)} \end{bmatrix} \begin{bmatrix} e^{j\phi_0^{(i)}} & & \\ & \ddots & \\ & & e^{j\phi_{K-1}^{(i)}} \end{bmatrix},$$

where the first matrix, by Theorem 6.3.1 and assumption **A3**, is rank $K-1$ when $K \geq 2L-1$, and the second matrix is full rank. This implies that the null space of $\hat{\boldsymbol{\mathcal{E}}}_i$ has dimension 1. Therefore, by solving the linear equation system (6.3.15), we can obtain non-zero solution whose normalization will yield the desired phases $\{\phi_0^{(i)}, \ldots, \phi_{K-1}^{(i)}\}$ up to a constant phase shift. Here we determine the phase ambiguity using only the information in $\{\hat{\boldsymbol{\mathcal{H}}}(\omega_k)\}_{k=0}^{K-1}$, while in [18], a second set of bispectrum matrices have to be evaluated. Moreover, the method we have presented is general and can be used in conjunction with any eigenstructure based method in determining the phase ambiguity.

6.3.3 Algorithm

We summarize our algorithm as follows:

1. Select $K \geq 2L-1$ and a set of $n \times n$ non-zero matrices $\mathcal{M} = \{\mathbf{M}_1, \ldots, \mathbf{M}_p\}$.
2. For $k = 0, 1, \ldots, K-1$,
   (a) let $\omega_k = \frac{2\pi k}{K}$;
   (b) compute p fourth-order polyspectrum matrices $\{P^{\mathbf{x}}_{\mathbf{M}_i}(-\omega_k, 0, 0)\}_{i=1}^{p}$;
   (c) estimate the frequency response $\hat{\boldsymbol{\mathcal{H}}}(\omega_k)$ by jointly diagonalizing the polyspectrum matrices $\{P^{\mathbf{x}}_{\mathbf{M}_i}(-\omega_k, 0, 0)\}_{i=1}^{p}$;

3. Solve (6.3.15) to determine the phase ambiguities and remove them from the estimated frequency responses $\{\hat{\mathcal{H}}(\omega_k)\}_{k=0}^{K-1}$;

4. Apply the inverse Fourier transform to the phase corrected frequency responses, denoted by $\{\tilde{\mathcal{H}}(\omega_k)\}_{k=0}^{K-1}$, to obtain the estimated channel impulse response $\{\tilde{H}(\ell)\}_{\ell=0}^{L-1}$.

6.4 Simulation Results

In this section, two computer simulations are carried out to demonstrate the validity and competitiveness of our proposed method. In the first example, we compare our method with the method in [18]. In the second example, we apply our method to binary phase shift keying (BPSK) signals. In both examples, the channel transfer function is the same as in [18]:

$$\mathcal{H}(z) = \begin{bmatrix} 1 - 1.5537z^{-1} - 0.0363z^{-2} + 0.5847z^{-3} + 0.5093z^{-4}, & 1 + 2.2149z^{-1} + 1.0828z^{-2} - 1.1731z^{-3} - 0.8069z^{-4} \\ 1 + 0.9295z^{-1} + 0.2453z^{-2} - 0.7510z^{-3} + 0.3717z^{-4}, & 1 - 0.7137z^{-1} - 1.5079z^{-2} + 1.6471z^{-3} - 1.2443z^{-4} \end{bmatrix}.$$

Similar to [18], we evaluate the performance in terms of the overall normalized mean-square error (ONMSE) defined below:

$$\text{ONMSE} = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \text{NMSE}_{ij},$$

where

$$NMSE_{ij} = \frac{M_c^{-1} \sum_{\ell=1}^{M_c} \sum_{k=0}^{L_e-1} (\hat{h}_{ij}^{\ell}(k) - h_{ij}(k))^2}{\sum_{k=0}^{L_e-1} h_{ij}(k)^2}.$$

Here, M_c is the number of Monte Carlo runs and L_e is the estimated channel length ($L_e > L$). In both simulations we choose $M_c = 50$ and $L_e = 11$.

Example 1. We consider the same scenario as presented in [18], *i.e.*, 2×2 linear time invariant (LIT) channel, with two independent exponentially distributed signals and additive white Gaussian noise. To simulate the method in [18], we use the Matlab code downloaded from the authors' web site [19]. In the authors' simulation, the identification method is followed by a single-input single-output (SISO) equalizer [69] in order to achieve better identification result. We use the same equalizer in our code for comparison. Also, we choose $K = 128$ for the method in [18] as suggested by the paper and $K = 32$ for our method. The method in [18] is implemented using the bispectrum.

Table 6.1 shows the averaged ONMSE's of both methods for different signal-to-noise-ratios (SNR) and different signal lengths T. We can see that our method generally performs better than the method in [18], especially in low SNR and short data scenario.

signal length	T=2048			T=4096			T=8192		
SNR	10 dB	20 dB	30 dB	10 dB	20 dB	30 dB	10 dB	20 dB	30 dB
New method	0.1350	0.1161	0.1146	0.0802	0.0649	0.0627	0.0495	0.0499	0.0507
method in [18]	0.2060	0.1695	0.1384	0.1094	0.0854	0.0780	0.0775	0.0479	0.0482

Table 6.1: ONMSE of both methods for exponentially distributed sources.

Example 2. In the previous example, a SISO equalizer [69] is used following the identification method in [18]. It is a general equalizer and can be applied to any system driven by temporally i.i.d. non-Gaussian stationary signal. In this example, we consider two binary phase shift keying (BPSK) source signals, and we use a SISO equalizer that is specially designed for system with BPSK inputs [56]. Table 6.2 shows the identification results of our method. We can see that better identification results can be achieved when the property of signal is considered.

SNR	10 dB	20 dB	30 dB
T=2048	0.1621	0.0259	0.0196
T=4096	0.0438	0.0087	0.0067
T=8192	0.0172	0.0033	0.0034

Table 6.2: ONMSE of our method for BPSK sources.

6.5 Discussion

In this chapter, we have proposed a frequency domain eigenstructure based method for blind identification of convolutive MIMO channel. We estimated the channel frequency response by jointly diagonalizing a set of polyspectrum matrices. We also proposed an elegant method to eliminate the phase ambiguities in the estimated channel frequency responses by solving a linear system of equations. This phase ambiguity elimination method is a general one that can be used in conjunction with any other eigenstructure based method. Our algorithm can be viewed as a direct extension of Cardoso's JADE algorithm to the convolutive case. The simulation results clearly demonstrate the effectiveness and competitiveness of our proposed method.

The method we proposed in this chapter is a general one that works for any spatially independent and temporally i.i.d. sources. In the next chapter, we will focus our attention on statistically uncorrelated cyclostationary signals. We will propose a second-order statistics based algorithm for blind identification of convolutive MIMO channel with cyclostationary inputs.

Chapter 7

Blind Identification of MIMO Channel with Cyclostationary Inputs Using Second-Order Statistics

7.1 Overview

We propose a second-order statistics (SOS) based algorithm for blind identification of a convolutive MIMO channel with cyclostationary inputs. We estimate the channel frequency response through the singular value decomposition (SVD) of a cyclic cross-spectrum based matrix assuming that the cycle frequencies of the sources are known and distinct. Similar to Chapter 6, we also propose an efficient method in determining the phase and amplitude ambiguities in the estimated channel frequency response. The simulation results show the validity and effectiveness of our proposed method.

7.2 Introduction

In the previous chapter, we proposed a higher-order statistics (fourth-order polyspectrum) based approach for blind channel identification. However, the estimation of higher-order statistics normally requires a large amount of data and involves complex computations. Therefore, second-order statistics based algorithms tend to be preferable in practice. In this chapter, we propose a second-order statistics based method for blind identification of convolutive MIMO channel with cyclostationary inputs.

Blind channel identification using second-order statistics of cyclostationary sources is a fairly new research topic. Gardner in [31] proposes a method for single-input single-output (SISO) channel identification. In this method, the channel frequency response is identified by exploiting the spectral correlation property of the received signal. Recent works on MIMO channel identification can be found in [5,12,20]. Antoni *et al* in [5] define a cyclic spectrum based matrix and identify the channel frequency response by the eigendecomposition of this matrix. In their method, the sources are assumed to be both cyclostationary and white and have equal magnitude. However, these assumptions imply that the sources are stationary. Moreover, under these assumptions, the distinctiveness of the eigenvalues of the cyclic spectrum based matrix can not be ensured. Unique identification of the channel frequency response may, therefore, not be achievable. The authors of [12] apply a blind identification method for the linear memoryless channel [68] to the identification of a convolutive channel in the frequency domain. With special assumptions on the channel, the authors propose an approach to eliminate the permutation ambiguity using some invariant indices. But this approach is complicated when the number of sources is large. Chevreuil and Loubaton in [20] propose a SOS based subspace method for blind identification of convolutive MIMO channel. In their method, the sources are assumed to be modulated stationary signals with distinct modulation frequencies, and the channel is identified column-wisely by solving a sequence of constrained minimization problems. However, this method can be computationally expansive.

In this chapter, we propose a SOS based algorithm that identifies the channel frequency response by the SVD's of a set of cyclic cross-spectrum based matrices. The channel impulse response can then be obtained by applying the inverse Fourier transform to the estimated channel frequency responses. Similar to what we have seen in Chapter 6, the estimated channel frequency response is generally corrupted by some phase and amplitude ambiguities, and can not be used directly in obtaining the channel impulse response. In this chapter, we propose a simple and efficient method that eliminates these ambiguities by solving a linear system of equations. Our method can be viewed as an extension of Gardner's method [31] to the MIMO case.

The remainder of this chapter is organized as follows. In Section 7.3, we present the detailed derivation of our cyclic cross-spectrum based algorithm together with the method for eliminating the phase and amplitude ambiguities. In Section 7.4, some computer simulations are performed to demonstrate the validity of the proposed method. Finally in Section 7.5, some brief remarks are given to conclude this chapter.

7.3 Problem Formulation

We consider the general $n \times n$ LTI FIR system described by

$$\mathbf{x}(k) = \sum_{\ell=0}^{L-1} \mathbf{H}(\ell)\mathbf{s}(k-\ell) + \mathbf{v}(k), \tag{7.3.1}$$

where $\mathbf{s}(k) = [s_1(k), \ldots, s_n(k)]^T$ is the vector of n sources, $\mathbf{v}(k) = [v_1(k), \ldots, v_n(k)]^T$ is the vector of n additive Gaussian noises, $\mathbf{x}(k) = [x_1(k), \ldots, x_n(k)]^T$ is the vector of n received signals and $\{\mathbf{H}(\ell)\}_{\ell=0}^{L-1}$ is a sequence of $n \times n$ matrices representing the system impulse response. The objective is to identify the channel impulse response $\{\mathbf{H}(\ell)\}_{\ell=0}^{L-1}$ given only the received signal $\mathbf{x}(k)$.

Throughout this chapter, we make the following assumptions.

A1. The sources $s_i(t)$, $i = 1, \ldots, n$ are mutually uncorrelated cyclostationary random processes, and their cycle frequencies, denoted by α_i, $i = 1, \ldots, n$, are assumed to be known, non-zero and mutually distinct.

A2. The noises $v_i(t)$, $i = 1, \ldots, n$, are zero-mean Gaussian stationary random processes which are mutually uncorrelated and are uncorrelated with the sources.

A3. The channel transfer function, defined by $\boldsymbol{\mathcal{H}}(z) \triangleq \sum_{\ell=0}^{L-1} \mathbf{H}(\ell)z^{-\ell}$, is invertible for all $|z| = 1$, and the elements in each column of $\boldsymbol{\mathcal{H}}(z)$ are coprime, *i.e.*, they do not share common zeros.

7.3.1 Second-Order Statistics Based Blind Identification Method

Our proposed blind identification method is based on the cyclic cross-spectrum of the received signals. Similar to [30], we define the *cyclic cross-correlation matrix* and the *cyclic cross-spectrum matrix* for vector random process $\mathbf{x}(n)$ as follows.

Definition 1. Let $\mathbf{R}_{\mathbf{x}}(m+k, m) \triangleq E[\mathbf{x}(m+k)\mathbf{x}^H(m)]$ be the *cross-correlation matrix* for vector random process $\mathbf{x}(m)$. The *cyclic cross-correlation matrix* with cycle frequency α is defined as

$$\mathbf{R}_{\mathbf{x}}^{\alpha}(k) \triangleq \lim_{N\to\infty} \frac{1}{2N+1} \sum_{m=-N}^{N} \mathbf{R}_{\mathbf{x}}(m+k, m)e^{-j(m+\frac{k}{2})\alpha}. \tag{7.3.2}$$

With the cyclic cross-correlation matrix defined above, we define the cyclic cross-spectrum matrix as the Fourier transform of the cyclic cross-correlation matrix.

Definition 2. The *cyclic cross-spectrum matrix* at frequency ω is defined by

$$\mathbf{S}_{\mathbf{x}}^{\alpha}(\omega) \triangleq \sum_{k=-\infty}^{\infty} \mathbf{R}_{\mathbf{x}}^{\alpha}(k) e^{-jk\omega}. \tag{7.3.3}$$

Substituting (7.3.1) into the expression of the cross-correlation matrix, we have

$$\mathbf{R}_{\mathbf{x}}(m+k,m) = E\left[\left(\sum_{\ell_1=0}^{L-1} \mathbf{H}(\ell_1)\mathbf{s}(m+k-\ell_1) + \mathbf{v}(m+k)\right)\left(\sum_{\ell_2=0}^{L-1} \mathbf{H}(\ell_2)\mathbf{s}(m-\ell_2) + \mathbf{v}(m)\right)^{H}\right].$$

Applying assumption **A2** to above equation, we have the cross-correlation matrix expressed as

$$\mathbf{R}_{\mathbf{x}}(m+k,m) = E\left[\left(\sum_{\ell_1=0}^{L-1} \mathbf{H}(\ell_1)\mathbf{s}(m+k-\ell_1)\right)\left(\sum_{\ell_2=0}^{L-1} \mathbf{H}(\ell_2)\mathbf{s}(m-\ell_2)\right)^{H}\right] + \delta(k)\sigma^2\mathbf{I}, \tag{7.3.4}$$

where $\delta(k)$ is the Kronecker delta function, σ^2 is the variance of the additive noise and $\mathbf{I}$ is the $n \times n$ identity matrix.

Substituting (7.3.4) into (7.3.2), and after some simplifications, we have the cyclic cross-correlation matrix given by

$$\mathbf{R}_{\mathbf{x}}^{\alpha}(k) = \sum_{\ell_1=0}^{L-1}\sum_{\ell_2=0}^{L-1} \mathbf{H}(\ell_1)\mathbf{R}_{\mathbf{s}}^{\alpha}(k+\ell_2-\ell_1)\mathbf{H}(\ell_2)e^{-j\pi\alpha(\ell_1+\ell_2)} + \delta(\alpha)\delta(k)\sigma^2\mathbf{I}, \tag{7.3.5}$$

where $\mathbf{R}_{\mathbf{s}}^{\alpha}(k)$ is the cyclic cross-correlation matrix of random process $\mathbf{s}(k)$. Since the cycle frequency α is always non-zero for our problem, the last term in the above equation is always zero, meaning that the cyclic cross-correlation matrix is not affected by the additive Gaussian noise. Therefore, our method, which is developed based on the cyclic cross-correlation matrix, is immune to the additive Gaussian noise.

Substituting (7.3.5) into (7.3.3), and after some simple algebra, we have the cyclic cross-spectrum matrix expressed as

$$\mathbf{S}_{\mathbf{x}}^{\alpha}(\omega) = \boldsymbol{\mathcal{H}}\left(\omega+\pi\alpha\right)\mathbf{S}_{\mathbf{s}}^{\alpha}(\omega)\boldsymbol{\mathcal{H}}\left(\omega-\pi\alpha\right)^{H}, \tag{7.3.6}$$

where $\boldsymbol{\mathcal{H}}(\omega) \triangleq \sum_{\ell=0}^{L-1}\mathbf{H}(\ell)e^{-j\ell\omega}$ is the channel frequency response at ω, and $\mathbf{S}_{\mathbf{s}}^{\alpha}(\omega)$ is the cyclic cross-spectrum matrix of random process $\mathbf{s}(k)$. By assumption **A1**, the sources are mutually uncorrelated, thus the cross-correlation matrix $\mathbf{R}_{\mathbf{s}}(m+k,m)$ is diagonal. Consequently, the cyclic cross-correlation matrix $\mathbf{R}_{\mathbf{s}}^{\alpha}(k)$, and the corresponding cyclic cross-spectrum matrix $\mathbf{S}_{\mathbf{s}}^{\alpha}(\omega)$, are diagonal. It is easy to see that the diagonal elements of $\mathbf{S}_{\mathbf{s}}^{\alpha}(\omega)$

are actually the cyclic spectra of the sources at cycle frequency α, which we denote by $\{S_{s_1}^{\alpha}(\omega), \ldots, S_{s_n}^{\alpha}(\omega)\}$.

Specially, if we choose α to be the cycle frequency of a particular signal $s_i(k)$, then the cyclic spectra of other sources at α are all zero, *i.e.*, $S_{s_j}^{\alpha_i}(\omega) = 0$, $\forall j \neq i$. This is because α_i does not equal the cycle frequency of other sources. Therefore, the cyclic cross-spectrum matrix $\mathbf{S}_{\mathbf{s}}^{\alpha_i}(\omega)$ has only one non-zero diagonal element, $S_{s_i}^{\alpha_i}(\omega)$, and (7.3.6) can be expressed as

$$\mathbf{S}_{\mathbf{x}}^{\alpha_i}(\omega) = S_{s_i}^{\alpha_i}(\omega)\mathbf{h}_i\left(\omega + \pi\alpha_i\right)\mathbf{h}_i\left(\omega - \pi\alpha_i\right)^H, \tag{7.3.7}$$

where $\mathbf{h}_i(\omega)$ is the ith column of $\boldsymbol{\mathcal{H}}(\omega)$. Multiplying (7.3.7) by its Hermitian, we have

$$\mathbf{S}_{\mathbf{x}}^{\alpha_i}(\omega)\mathbf{S}_{\mathbf{x}}^{\alpha_i}(\omega)^H = \left\{\left|S_{s_i}^{\alpha_i}(\omega)\right|^2 \left\|\mathbf{h}_i\left(\omega - \pi\alpha_i\right)\right\|^2\right\}\mathbf{h}_i\left(\omega + \pi\alpha_i\right)\mathbf{h}_i\left(\omega + \pi\alpha_i\right)^H. \tag{7.3.8}$$

Specially, setting $\omega = \omega - \pi\alpha_i$ in (7.3.8), we have

$$\mathbf{S}_{\mathbf{x}}^{\alpha_i}\left(\omega - \pi\alpha_i\right)\mathbf{S}_{\mathbf{x}}^{\alpha_i}\left(\omega - \pi\alpha_i\right)^H = \left\{\left|S_{s_i}^{\alpha_i}\left(\omega - \pi\alpha_i\right)\right|^2 \left\|\mathbf{h}_i\left(\omega - 2\pi\alpha_i\right)\right\|^2\right\}\mathbf{h}_i(\omega)\mathbf{h}_i(\omega)^H. \tag{7.3.9}$$

This implies that vector $\mathbf{h}_i(\omega)$ can be estimated from the singular value decomposition (SVD) of the matrix $\mathbf{S}_{\mathbf{x}}^{\alpha_i}(\omega - \pi\alpha_i)\mathbf{S}_{\mathbf{x}}^{\alpha_i}(\omega - \pi\alpha_i)^H$ up to a complex scalar, *i.e.*, the estimation is given by $\hat{\mathbf{h}}_i(\omega) = c_i(\omega)\mathbf{h}_i(\omega)$ for some scalar $c_i(\omega) \in \mathbb{C}$. The complex scalar $c_i(\omega)$ can be viewed as ambiguity in both phase and amplitude to the true frequency response $\mathbf{h}_i(\omega)$.

Suppose we have obtained estimations

$$\hat{\mathbf{h}}_i(\omega_k) = c_i(\omega_k)\mathbf{h}_i(\omega_k), \tag{7.3.10}$$

for $\omega_k = \frac{2k\pi}{K}$, $k = 0, 1, \ldots, K-1$, $(K \geq L)$. Ideally, if the ambiguities $\{c_i(\omega_k)\}_{k=0}^{K-1}$ are all equal to a constant c_i, *i.e.*, the estimations are given by $\hat{\mathbf{h}}_i(\omega_k) = c_i\mathbf{h}_i(\omega_k)$ for all k, we can always apply the inverse Fourier transform to $\{\hat{\mathbf{h}}_i(\omega_k)\}_{k=0}^{K-1}$ to obtain the time domain impulse response. Unfortunately, the ambiguities are generally frequency dependent, thus, $\{\hat{\mathbf{h}}_i(\omega_k)\}_{k=0}^{K-1}$ can not be combined directly in obtaining the time domain impulse response. In the next section, we will propose a method that can eliminate the ambiguities $\{c_i(\omega_k)\}_{k=0}^{K-1}$ up to a constant factor.

7.3.2 Resolving Phase and Amplitude Ambiguities

In this section, we develop an algorithm that can determine the ambiguities $\{c_i(\omega_k)\}_{k=0}^{K-1}$ up to a constant. Our method here is similar to the one proposed in Chapter 6 for phase ambiguity determination.

Rewriting (7.3.10) as

$$\hat{\mathbf{h}}_i(\omega_k)c_i^{-1}(\omega_k) = \mathbf{h}_i(\omega_k), \quad k = 0, 1, \ldots, K-1, \tag{7.3.11}$$

and applying the inverse Fourier transform to both sides of (7.3.11), we obtain

$$\frac{1}{K}\sum_{k=0}^{K-1} \hat{\mathbf{h}}_i(\omega_k)c_i^{-1}(\omega_k)e^{j\omega_k\ell} = \begin{cases} \mathbf{h}_i(\ell), & \ell = 0, \ldots, L-1, \\ 0, & \ell = L, \ldots, K-1. \end{cases} \tag{7.3.12}$$

We will identify the ambiguities $\{c_i(\omega_k)\}_{k=0}^{K-1}$ by solving the above equation for $\ell = L, \ldots, K-1$.

Expressing (7.3.12) in matrix form, we get

$$\begin{bmatrix} \hat{\mathbf{h}}_i(\omega_0)e^{j\omega_0 L}, & \ldots, & \hat{\mathbf{h}}_i(\omega_{K-1})e^{j\omega_{K-1}L} \\ \vdots & \ddots & \vdots \\ \hat{\mathbf{h}}_i(\omega_0)e^{j\omega_0(K-1)}, & \ldots, & \hat{\mathbf{h}}_i(\omega_{K-1})e^{j\omega_{K-1}(K-1)} \end{bmatrix} \begin{bmatrix} c_i^{-1}(\omega_0) \\ \vdots \\ c_i^{-1}(\omega_{K-1}) \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix}. \tag{7.3.13}$$

For simplicity, we denote the above equation by $\hat{\boldsymbol{\mathcal{E}}}_i\mathbf{c}_i = \mathbf{0}$, where $\hat{\boldsymbol{\mathcal{E}}}_i$ is the $n(K-L)\times K$ matrix of the known coefficients of the estimated channel frequency response and $\mathbf{c}_i$ is the solution vector containing the desired ambiguities. From Theorem 6.3.1 in Chapter 6, we know that when $K \geq 2L-1$, the rank of matrix $\hat{\boldsymbol{\mathcal{E}}}_i$ is $K-1$. This implies that the null space of $\hat{\boldsymbol{\mathcal{E}}}_i$ is one dimensional. Thus, by solving the linear equation system (7.3.13) we can obtain the desired ambiguities $\{c_i^{-1}(\omega_0), \ldots, c_i^{-1}(\omega_{K-1})\}$ up to a constant factor. Multiplying (7.3.10) by the solutions of (7.3.13), we obtain the ambiguity eliminated estimations as $\hat{\mathbf{h}}_i(\omega_k) = c_i\mathbf{h}_i(\omega_k)$ for $k = 0, 1, \ldots, K-1$. Applying the inverse Fourier transform to $\{\hat{\mathbf{h}}_i(\omega_k)\}_{k=0}^{K-1}$, we have the time-domain impulse response estimated by $\hat{\mathbf{h}}_i(\ell) = c_i\mathbf{h}_i(\ell)$ for $\ell = 0, \ldots, L-1$. At this point, we have estimated one column, the ith column, of the channel $\boldsymbol{\mathcal{H}}(z)$.

By repeating the above procedure for α equaling the cyclic frequencies of the other sources, we can estimate other columns of $\boldsymbol{\mathcal{H}}(z)$. Suppose we have all the estimations given by $\hat{\mathbf{h}}_i(\ell) = c_i\mathbf{h}_i(\ell)$ for $i = 1, \ldots, n$ and $\ell = 0, \ldots, L-1$, then we have the time-domain channel impulse response estimated by

$$\hat{\mathbf{H}}(\ell) = [c_1\mathbf{h}_1(\ell), \ldots, c_n\mathbf{h}_n(\ell)] = \mathbf{H}(\ell)\mathbf{C}, \tag{7.3.14}$$

for $\ell = 0, \ldots, L-1$, where $\mathbf{C}$ is a diagonal matrix with diagonal elements being $\{c_1, \ldots, c_n\}$. The channel transfer function is estimated by $\boldsymbol{\mathcal{H}}(z)\mathbf{C}$. We can always apply the inverse of the estimated channel $\boldsymbol{\mathcal{H}}(z)\mathbf{C}$ to the received signal $\mathbf{x}(z) = \boldsymbol{\mathcal{H}}(z)\mathbf{s}(z)$ to get the separated signal as $\mathbf{y}(z) = \mathbf{C}^{-1}\mathbf{s}(z)$.

7.3.3 Algorithm

We can summarize our algorithm as the following:

1. Select $K \geq 2L-1$.
2. For $i = 1, \ldots, n$,
   (a) compute the cyclic cross-spectrum matrix $\mathbf{S}_{\mathbf{x}}^{\alpha_i}(\omega_k - \pi\alpha_i)$ for $\omega_k = \frac{2k\pi}{K}$, $k = 0, 1, \ldots, K-1$;
   (b) perform an SVD on the matrix $\mathbf{S}_{\mathbf{x}}^{\alpha_i}(\omega_k - \pi\alpha_i)\mathbf{S}_{\mathbf{x}}^{\alpha_i}(\omega_k - \pi\alpha_i)^H$ and obtain the estimate $\hat{\mathbf{h}}_i(\omega_k)$, which is the singular vector corresponding to the only non-zero singular value;
   (c) with $\{\hat{\mathbf{h}}_i(\omega_k)\}_{k=0}^{K-1}$ available, solve (7.3.13) to determine the ambiguities and remove them from $\{\hat{\mathbf{h}}_i(\omega_k)\}_{k=0}^{K-1}$;
   (d) apply the inverse Fourier transform to the ambiguity eliminated frequency responses to obtain an estimate of the ith column of the channel impulse response $\{\hat{\mathbf{h}}_i(\ell)\}_{\ell=0}^{L-1}$.
3. Combine $\{\hat{\mathbf{h}}_i(\ell)\}_{\ell=0}^{L-1}$ for $i = 1, \ldots, n$, to get the estimated channel impulse response $\{\hat{\mathbf{H}}(\ell)\}_{\ell=0}^{L-1}$.

7.4 Simulation Results

In this section, two computer simulations were performed to demonstrate the validity of our proposed method. In these simulations, we consider a two-input two-output convolutive FIR channel that was adopted from [18]:

$$\mathcal{H}(z) = \begin{bmatrix} 1 - 1.5537z^{-1} - 0.0363z^{-2} + 0.5847z^{-3} + 0.5093z^{-4}, & 1 + 2.2149z^{-1} + 1.0828z^{-2} - 1.1731z^{-3} - 0.8069z^{-4} \\ 1 + 0.9295z^{-1} + 0.2453z^{-2} - 0.7510z^{-3} + 0.3717z^{-4}, & 1 - 0.7137z^{-1} - 1.5079z^{-2} + 1.6471z^{-3} - 1.2443z^{-4} \end{bmatrix},$$

where the filter length is $L = 5$. Two cyclostationary inputs with cycle frequencies being $\alpha_1 = 0.5$ and $\alpha_2 = 0.2$ and two additive white Gaussian noise sources were considered. Similar to the simulations in Chapter 6, we evaluate the performance in terms of the overall normalized mean-square error (ONMSE) for SNR's of 15, 20 and 25 dB and for different signal lengths. 50 Monte Carlo runs were performed to obtain the averaged result. The number of frequencies was chosen to be $K = 32$, which obviously satisfies the condition $K \geq 2L - 1$.

Example 1. Two amplitude-modulated (AM) sine waves were used for this experiment. Table 7.1 shows the averaged ONMSE's of our proposed method for different SNR's and different signal lengths T. We can see that the additive noise has little effect on the identification results. This confirms our derivation in Section 7.3.1. We also can see that the data length is important in the proposed method. This is because the longer the data length, the better the estimation of the cyclic cross-spectrum matrix, hence the better the channel identification result.

SNR	15 dB	20 dB	25 dB
T=5120	0.1524	0.1522	0.1423
T=10240	0.0589	0.0569	0.0530

Table 7.1: Averaged ONMSE of our proposed method for AM sources.

Figure 7.1 shows the estimated and true channel impulse response for the case of $T = 10240$ and $SNR = 15$ dB.

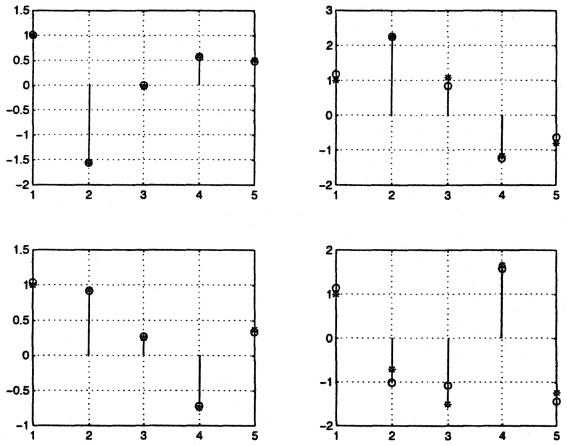

Figure 7.1: Estimated channel impulse responses (o) and true channel impulse responses (*).

Figure 7.2 shows the impulse response of the combined system (mixing and separating systems) for the case of $T = 10240$ and $SNR = 15$ dB. We can see that the sources are indeed separated.

Example 2. Two BPSK signals were used in this experiment. Similar to the simulation in Chapter 6, the single-input single-output (SISO) equalizer given in [56] was used in order to

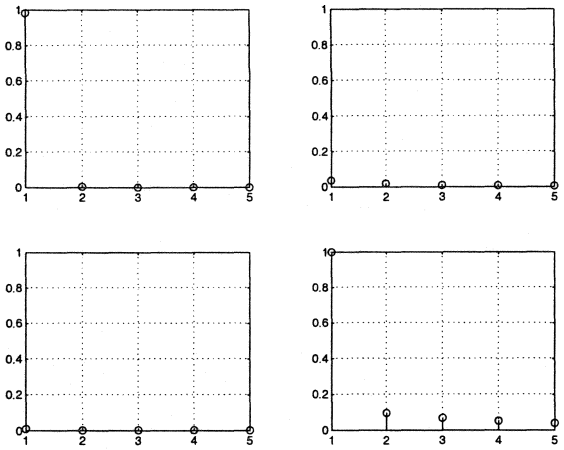

Figure 7.2: Impulse response of the combined system.

achieve better identification result. Table 7.2 shows the identification results of our method for different SNR's and different signal lengths T.

SNR	15 dB	20 dB	25 dB
T=2560	0.0725	0.0563	0.0527
T=5120	0.0423	0.0347	0.0332

Table 7.2: Averaged ONMSE of our proposed method for BPSK sources.

7.5 Discussion

In this chapter, we have proposed a SOS based frequency domain method for blind identification of convolutive MIMO channel with cyclostationary inputs. The method estimates the channel frequency response by exploiting the properties of the SVD of a cyclic cross-spectrum based matrix. We also proposed an efficient method that eliminates ambiguities in the estimated channel frequency response. Our method can determine the ambiguities up to a constant factor by solving a linear system of equations. The simulation results clearly demonstrate the effectiveness of our proposed method.

Chapter 8

Summary and Discussions

8.1 Summary

The primary objective of this thesis was to develop novel and practical algorithms for solving the blind signal separation problem.

For blind separation of instantaneous mixtures, we proposed two approaches. Our first approach is a general one that works for arbitrary statistically independent signals. In this approach, we formulated the separation problem as a constrained optimization problem by minimizing the mutual information of the output signals, and we solved this problem using Edelman's extended Newton's method on the Stiefel manifold. The application of Newton's method on the Stiefel manifold is a key feature of this approach, and because of this, our method enjoys local quadratic convergence rate. The simulation results shows that our proposed method achieved similar separation results as JADE and FastICA.

In our second approach, we considered the problem of blind separation of instantaneously mixed constant modulus (CM) signals. In this approach, we again formulated the separation problem into a constrained optimization problem by enforcing the constant modulus property of the signals. We showed that the optimal solution of this problem indeed yields the desired separation matrix, and solved this problem with the Newton's method on the Stiefel manifold. One major advantage of this approach over other CM based separation methods is that it is robust to additive noise.

In the second part of this thesis, we tackled the problem of blind separation of convolutively mixed signals. We first proposed a mutual information based approach for the separation of spatially independent and temporally i.i.d. sources. Again a constrained optimization problem was formed based on the mutual information criterion, which when

minimized not only separates the signals, but also deconvolves the separated signals. The problem was solved with a sequential quadratic programming (SQP) method. One key feature of our approach is that we proposed a matrix polynomial factorization based method to pre-whiten (decorrelate) the convolutively mixed signals. By applying the pre-whitening step, the original complicated objective function was significantly simplified.

Then we proposed two sets of higher-order statistics (polyspectrum) based criteria for blind deconvolution of Mth-order uncorrelated signals. We also analyzed the relationship between our proposed criteria and other HOS based criteria from the literature, and showed that they are equivalent under some conditions. However, we pointed out that our criteria are more general in that:

i) they are developed for signals that may not be spatially independent or temporally i.i.d while most other criteria requires the spatial independence and temporal i.i.d. assumption; and

ii) there is no limitation on the order of polyspectrum for our criteria while most other criteria uses only third- or fourth-order statistics.

As an important component of our thesis, we proposed a frequency domain polyspectrum based approach for blind identification of convolutive MIMO channel. We defined a polyspectrum matrix using the fourth-order polyspectrum of the received signals and an arbitrary non-zero matrix $\mathbf{M}$, and identified the channel frequency response by the eigendecomposition of the polyspectrum matrix (or by the joint diagonalization of a set of such defined polyspectrum matrices). We also proposed a method to eliminate the phase ambiguities in the estimated channel frequency response by solving a system of linear equations. Compared to existing polyspectrum based methods, our approach has several advantages:

i) the eigenvalues of our defined polyspectrum matrix can be made mutually distinct by properly choosing the arbitrary matrix $\mathbf{M}$, and therefore, the channel frequency response is uniquely determined;

ii) our method for phase ambiguity elimination is simple, elegant and requires little computation.

Our method can be viewed as an extension of Cardoso's JADE algorithm to the convolutive case. The simulation results indicated the competitiveness of our proposed method.

Finally, we proposed a second-order statistics based approach for blind identification of convolutive MIMO channel with cyclostationary inputs. Assuming that the sources have

distinct cycle frequencies, we identified the channel frequency response column-wisely by the singular value decomposition (SVD) of a specially defined cyclic spectrum based matrix. We eliminated the amplitude and phase ambiguities in the estimated channel frequency response with an approach similar to the one in Chapter 6. Since our approach is based on second-order statistics, it is efficient and requires less computation comparing to higher-order statistics based methods.

8.2 Further Work

The work reported in this thesis can be extended in several directions. From the point of view of practical implementation, the following needs further study:

i) currently the approach given in Chapter 2 for blind separation of instantaneous mixtures is not as computationally efficient as JADE and FastICA, and improving its efficiency is still an open problem;

ii) algorithms should be developed utilizing our blind deconvolution criteria proposed in Chapter 5;

iii) the method proposed in Chapter 6 was derived assuming the absence of additive noise. For real world application, additive noise must be incorporated to the derivation of the method.

From the view point of theoretical study, the following problems should be considered:

i) for the approach presented in Chapter 3, we have shown for 2×2 channel that "any local minimum of optimization problem (3.3.6) is in fact a global minimum". This implies that our optimization problem does not suffer from the local convergence problem (the statement and proof are not included in the thesis). Unfortunately, we have not proved the statement for the general $n \times n$ channel case. This could be a topic for future study;

ii) in Chapter 4, we solved an optimization problem that has paraunitary constraint with a SQP method. Can we develop Newton's method on the space defined by paraunitary constraint?

This thesis is our first step toward solving the blind separation problem. We hope we can answer the questions above in our future study.

Postscript

Part of the work reported in this thesis have previously appeared in the literature or submitted for publication. The work in Chapters 2 and 4 has been published in Mathematical Programming [52]. The work in Chapter 3 was published in the 28th International Conference on Acoustics, Speech, and Signal Processing (ICASSP), 2003 [51]. The work in Chapter 6 has been accepted by the 22nd Biennial Symposium on Communications, 2004, Queen's university, Kingston Ontarion, Canada. The work in Chapter 7 has been summarized in manuscript that is ready for submission.

Appendix

A.1 Proof of Theorem 2.3.1

Proof: Without loss of generality, we assume $\bar{y} = 0$. Our goal is to show that the mean square error $E[(\hat{q}_N(0) - q(0))^2] \to 0$, as $N \to \infty$. Since the mean square error can be written as

$$E\left[(\hat{q}_N(0) - q(0))^2\right] = E\left[(\hat{q}_N(0) - E\left[\hat{q}_N(0)\right])^2\right] + \left(E\left[\hat{q}_N(0)\right] - q(0)\right)^2,$$

we will show that both $E[(\hat{q}_N(0) - E[\hat{q}_N(0)])^2]$ and $(E[\hat{q}_N(0)] - q(0))^2$ converge to 0 as $N \to \infty$. We will also estimate their convergence rates separately. For notational convenience, let us denote

$$\Delta_1 = E\left[(\hat{q}_N(0) - E\left[\hat{q}_N(0)\right])^2\right], \tag{A.1.1}$$

$$\Delta_2 = \left(E\left[\hat{q}_N(0)\right] - q(0)\right)^2. \tag{A.1.2}$$

Denote also $Z_i = \frac{1}{\sqrt{\pi} h_N} e^{\left(-\frac{y(i)}{h_N}\right)^2}$, for $i = 1, \cdots, N$. Then $\hat{q}_N(0) = \frac{1}{N}\sum_{i=1}^{N} Z_i$, and Δ_1 can be expressed as

$$\Delta_1 = E\left[\left(\frac{1}{N}\sum_{i=1}^{N} Z_i - E\left[\frac{1}{N}\sum_{i=1}^{N} Z_i\right]\right)^2\right]. \tag{A.1.3}$$

Noticing that the random variables $Z_i, i = 1, \cdots, N$ are i.i.d., we can simplify (A.1.3) as

$$\Delta_1 = \frac{1}{N}\left(E[Z_1^2] - E[Z_1]^2\right). \tag{A.1.4}$$

Similarly, we can express Δ_2 as

$$\Delta_2 = (E[Z_1] - q(0))^2. \tag{A.1.5}$$

Next, we will show that both Δ_1 and Δ_2 converge to 0. We will first estimate the convergence rate of Δ_2. From (A.1.5), we have

$$\Delta_2 = \left(\int_{-\infty}^{+\infty} q(y) \frac{1}{\sqrt{\pi} h_N} E^{-\left(\frac{y}{h_N}\right)^2} dy - q(0) \right)^2. \tag{A.1.6}$$

It is easy to verify that

$$\int_{-\infty}^{+\infty} \frac{1}{\sqrt{\pi} h_N} e^{-\left(\frac{y}{h_N}\right)^2} dy = 1.$$

Therefore, (A.1.6) can be rewritten as

$$\Delta_2 = \left(\int_{-\infty}^{+\infty} (q(y) - q(0)) \frac{1}{\sqrt{\pi} h_N} e^{-\left(\frac{y}{h_N}\right)^2} dy \right)^2.$$

Hence,

$$\sqrt{\Delta_2} \leq \int_{-\infty}^{+\infty} |q(y) - q(0)| \frac{1}{\sqrt{\pi} h_N} e^{-\left(\frac{y}{h_N}\right)^2} dy. \tag{A.1.7}$$

By assumption, we know that $q(y)$ is continuously differentiable at $y = 0$, which implies that $q'(0)$ exists. Assuming $q'(0) \neq 0$, then for any ϵ such that $|q'(0)| > \epsilon > 0$, we have a $\delta_0 > 0$ such that

$$\left| \frac{q(y) - q(0)}{y} - q'(0) \right| < \epsilon$$

for all $|y| < \delta_0$. It follows that $|q(y) - q(0)| < 2|q'(0)||y|$ for all $|y| < \delta_0$. Thus, for any positive $\delta < \delta_0$ and $|y| < \delta$, we have

$$|q(y) - q(0)| < 2|q'(0)||y| < 2\delta|q'(0)|. \tag{A.1.8}$$

Similar expression can be obtained if $q'(0) = 0$.

Now rewrite (A.1.7) as the following,

$$\sqrt{\Delta_2} \leq \int_{|y|<\delta} |q(y) - q(0)| \frac{1}{\sqrt{\pi} h_N} e^{-\left(\frac{y}{h_N}\right)^2} dy + \int_{|y|\geq\delta} |q(y) - q(0)| \frac{1}{\sqrt{\pi} h_N} e^{-\left(\frac{y}{h_N}\right)^2} dy,$$

and upper bound the right-hand-side to obtain

$$\begin{aligned} \sqrt{\Delta_2} \leq & \int_{|y|<\delta} |q(y) - q(0)| \frac{1}{\sqrt{\pi} h_N} e^{-\left(\frac{y}{h_N}\right)^2} dy \\ & + \int_{|y|\geq\delta} q(y) \frac{1}{\sqrt{\pi} h_N} e^{-\left(\frac{y}{h_N}\right)^2} dy \\ & + \int_{|y|\geq\delta} q(0) \frac{1}{\sqrt{\pi} h_N} e^{-\left(\frac{y}{h_N}\right)^2} dy. \end{aligned}$$

Denote the three terms on the right-hand-side by T_1, T_2 and T_3 respectively,

$$\begin{aligned} T_1 &= \int_{|y|<\delta} |q(y)-q(0)| \frac{1}{\sqrt{\pi}h_N} e^{-\left(\frac{y}{h_N}\right)^2} dy, \\ T_2 &= \int_{|y|\geq\delta} q(y) \frac{1}{\sqrt{\pi}h_N} e^{-\left(\frac{y}{h_N}\right)^2} dy, \\ T_3 &= \int_{|y|\geq\delta} q(0) \frac{1}{\sqrt{\pi}h_N} e^{-\left(\frac{y}{h_N}\right)^2} dy. \end{aligned}$$

We will bound T_1, T_2 and T_3 separately.

Applying (A.1.8) to T_1, we have

$$T_1 \leq 2\delta|q'(0)| \int_{|y|<\delta} \frac{1}{\sqrt{\pi}h_N} e^{-\left(\frac{y}{h_N}\right)^2} dy \leq 2\delta|q'(0)|. \tag{A.1.9}$$

Using the fact that $e^{-\left(\frac{y}{h_N}\right)^2} \leq e^{-\left(\frac{\delta}{h_N}\right)^2}$ for all $|y| \geq \delta$, we can bound T_2 as

$$T_2 \leq \frac{1}{\sqrt{\pi}h_N} e^{-\left(\frac{\delta}{h_N}\right)^2} \int_{|y|\geq\delta} q(y)dy \leq \frac{1}{\sqrt{\pi}h_N} e^{-\left(\frac{\delta}{h_N}\right)^2}. \tag{A.1.10}$$

Now substituting $y = th_N$ into the expression of T_3, we obtain

$$T_3 = \frac{2q(0)}{\sqrt{\pi}} \int_{\frac{\delta}{h_N}}^{\infty} e^{-t^2} dt.$$

Since $\int_y^{\infty} e^{-t^2}dt \to 0$ at the rate $O(e^{-y^2}/y)$, as $y \to +\infty$, there exist constants $C > 0$ and $Y > 0$ such that for all $y > Y$, we have $\int_y^{\infty} e^{-t^2}dt < Ce^{-y^2}/y$. Therefore, there holds

$$T_3 = \frac{2q(0)}{\sqrt{\pi}} \int_{\frac{\delta}{h_N}}^{\infty} e^{-t^2} dt \leq \left(\frac{2q(0)C}{\sqrt{\pi}}\right) \frac{h_N}{\delta} e^{-\left(\frac{\delta}{h_N}\right)^2}, \quad \text{when } \frac{\delta}{h_N} > Y. \tag{A.1.11}$$

Combining (A.1.9), (A.1.10) and (A.1.11) above, we have

$$\begin{aligned} \sqrt{\Delta_2} &\leq T_1 + T_2 + T_3 \\ &\leq 2\delta|q'(0)| + \frac{1}{\sqrt{\pi}h_N} e^{-\left(\frac{\delta}{h_N}\right)^2} + \left(\frac{2q(0)C}{\sqrt{\pi}}\right) \frac{h_N}{\delta} e^{-\left(\frac{\delta}{h_N}\right)^2}, \end{aligned} \tag{A.1.12}$$

when $\frac{\delta}{h_N} > Y$.

Setting $\delta = \sqrt{2}N^{-\frac{1}{3}}(\log N)^{\frac{1}{4}}$ and selecting $h_N = N^{-\frac{1}{3}}(\log N)^{-\frac{1}{4}}$, we see that the condition $\frac{\delta}{h_N} > Y$ is easily satisfied with large N. Moreover, we have $e^{-(\frac{\delta}{h_N})^2} = \frac{1}{N^2}$. Substituting these choices of δ and h_N into (A.1.12), we obtain

$$\sqrt{\Delta_2} \leq C' \frac{(\log N)^{\frac{1}{4}}}{N^{\frac{1}{3}}} + C'' \frac{(\log N)^{\frac{1}{4}}}{N^{\frac{5}{3}}} + C''' \frac{1}{N^2\sqrt{\log N}},$$

where $C' = 2\sqrt{2}|q'(0)|$, $C'' = \frac{1}{\sqrt{\pi}}$ and $C''' = \frac{2q(0)C}{\sqrt{2\pi}}$. Hence,

$$\Delta_2 \leq \left(C' \frac{(\log N)^{\frac{1}{4}}}{N^{\frac{1}{3}}} + C'' \frac{(\log N)^{\frac{1}{4}}}{N^{\frac{5}{3}}} + C''' \frac{1}{N^2 \sqrt{\log N}} \right)^2, \tag{A.1.13}$$

implying that the convergence rate for Δ_2 is of order $O(N^{-\frac{2}{3}}(\log N)^{\frac{1}{2}})$.

Now we estimate the convergence rate of Δ_1. From (A.1.4) we have

$$\begin{aligned} \Delta_1 &\leq \frac{1}{N} E[Z_1^2] \\ &= \frac{1}{N} \int_{-\infty}^{+\infty} q(y) \left(\frac{1}{\sqrt{\pi} h_N} e^{-\left(\frac{y}{h_N}\right)^2} \right)^2 dy \\ &= \frac{1}{\sqrt{2\pi} N h_N} \int_{-\infty}^{+\infty} q(y) \left(\frac{\sqrt{2}}{\sqrt{\pi} h_N} e^{-2\left(\frac{y}{h_N}\right)^2} \right) dy. \end{aligned} \tag{A.1.14}$$

From (A.1.13), we know $\Delta_2 \to 0$ as $N \to \infty$, *i.e.*,

$$\int_{-\infty}^{+\infty} q(y) \frac{1}{\sqrt{\pi} h_N} e^{-\left(\frac{y}{h_N}\right)^2} dy \to q(0), \quad \text{as} \quad N \to \infty.$$

This further implies

$$\int_{-\infty}^{+\infty} q(y) \frac{\sqrt{2}}{\sqrt{\pi} h_N} e^{-2\left(\frac{y}{h_N}\right)^2} dy \to q(0), \quad \text{as} \quad N \to \infty.$$

Therefore, from (A.1.14) we know that the convergence rate of Δ_1 is $O(\frac{1}{N h_N})$, *i.e.*, $O(N^{-\frac{2}{3}}(\log N)^{\frac{1}{4}})$, which is the same as that of Δ_2. Consequently, the convergence rate for $E[(\hat{q}_N(0) - q(0))^2]$ is $O(N^{-\frac{2}{3}}(\log N)^{\frac{1}{2}})$. **Q.E.D.**

A.2 Proof of Theorem 5.5.1

Proof: We will show that in each row of $\mathcal{G}(\omega)$ there exists only one nonzero element, and this element is also the only nonzero element in its column. Without loss of generality, we will prove this statement for the first row of $\mathcal{G}(\omega)$. We denote the (i,j)'th element of $\mathcal{G}(\omega)$ by $\mathcal{G}_{i,j}(\omega)$.

Since $\mathcal{G}(0)$ is invertible, there must exists an nonzero element in the first row of $\mathcal{G}(0)$ such that its cofactor is also nonzero. Again, without loss of generality, we assume this element is $\mathcal{G}_{1,1}(0)$. We now show that $\mathcal{G}_{1,1}(0)$ is the only nonzero element in its row and column.

From (5.5.5) and property **P3** of polyspectrum in Section 5.3.2, we have

$$P_{y_i,y_j}(\omega) = \sum_{p=1}^{n}\sum_{q=1}^{n} \mathcal{G}_{i,p}(-\omega)\mathcal{G}_{j,q}(\omega)P_{s_p,s_q}(\omega) = 0, \tag{A.2.1}$$

for $\forall\omega$, and $i,j = 1,2,\cdots,n$, $i \neq j$. Noticing that $s_1(t),\cdots,s_n(t)$ are mutually M'th-order uncorrelated, we have $P_{s_p,s_q}(\omega) = 0$, for all ω, and $p \neq q$. Thus, (A.2.1) can be rewritten as

$$\sum_{p=1}^{n} \mathcal{G}_{i,p}(-\omega)\mathcal{G}_{j,p}(\omega)P_{s_p,s_p}(\omega) = 0, \tag{A.2.2}$$

for $\forall\omega$, and $i,j = 1,2,\cdots,n$, $i \neq j$. In particular, for $i = 1$, $j = 2,\cdots,n$ and $\omega = 0$ in (A.2.2), we have

$$\sum_{p=1}^{n} \mathcal{G}_{1,p}(0)\mathcal{G}_{j,p}(0)P_{s_p,s_p}(0) = 0,$$

or

$$\sum_{p=2}^{n} \mathcal{G}_{j,p}(0)\mathcal{G}_{1,p}(0)P_{s_p,s_p}(0) = -\mathcal{G}_{j,1}(0)\mathcal{G}_{1,1}(0)P_{s_1,s_1}(0),$$

for $j = 2,\cdots,n$. This can be expressed in matrix form as

$$\begin{bmatrix} \mathcal{G}_{2,2}(0), & \cdots, & \mathcal{G}_{2,n}(0) \\ \vdots & & \vdots \\ \mathcal{G}_{n,2}(0), & \cdots, & \mathcal{G}_{n,n}(0) \end{bmatrix} \begin{bmatrix} \mathcal{G}_{1,2}(0)P_{s_2,s_2}(0) \\ \vdots \\ \mathcal{G}_{1,n}(0)P_{s_n,s_n}(0) \end{bmatrix} = -\mathcal{G}_{1,1}(0)P_{s_1,s_1}(0) \begin{bmatrix} \mathcal{G}_{2,1}(0) \\ \vdots \\ \mathcal{G}_{n,1}(0) \end{bmatrix} \tag{A.2.3}$$

Denoting

$$\alpha_p \triangleq \frac{\mathcal{G}_{1,p}(0)P_{s_p,s_p}(0)}{\mathcal{G}_{1,1}(0)P_{s_1,s_1}(0)}, \quad p = 1,\ldots,n,$$

where $\mathcal{G}_{1,1}(0)$ and $P_{s_1,s_1}(0)$ are nonzero by assumption, we can express (A.2.3) as

$$\begin{bmatrix} \mathcal{G}_{2,2}(0), & \cdots, & \mathcal{G}_{2,n}(0) \\ \vdots & & \vdots \\ \mathcal{G}_{n,2}(0), & \cdots, & \mathcal{G}_{n,n}(0) \end{bmatrix} \begin{bmatrix} \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} = - \begin{bmatrix} \mathcal{G}_{2,1}(0) \\ \vdots \\ \mathcal{G}_{n,1}(0) \end{bmatrix}. \tag{A.2.4}$$

Similarly, applying properties **P3** (Section 5.3.2) and **P1'** (Section 5.5) to (5.5.6), we have

$$\sum_{p=1}^{n} \mathcal{G}_{i,p} \left(-\sum_{v=1}^{k-1} \omega_v \right) \cdots \mathcal{G}_{i,p}(\omega_{k-2}) \mathcal{G}_{j,p}(\omega_{k-1}) P_{s_p,\cdots,s_p}(\omega_1, \cdots, \omega_{k-1}) = 0, \tag{A.2.5}$$

for all $\omega_1, \cdots, \omega_{k-1}$, and $i, j = 1, 2, \cdots, n$, $i \neq j$. Specifically, we choose $\omega_v = (-1)^{v+1}\omega$ for $v < k-1$, and $\omega_{k-1} = 0$ in (A.2.5), yielding

$$\sum_{p=1}^{n} |\mathcal{G}_{i,p}(\omega)|^{k-1} \mathcal{G}_{j,p}(0) P_{s_p,\cdots,s_p}(\omega, -\omega, \cdots, \omega, 0) = 0, \tag{A.2.6}$$

for all ω, and $i, j = 1, 2, \cdots, n$, $i \neq j$. Here we used the fact that $\mathcal{G}_{u,v}(-\omega) = \mathcal{G}^*_{u,v}(\omega)$.

Now let $i = 1$, and $j = 2, \cdots, n$, in (A.2.6). This yields

$$\sum_{p=1}^{n} |\mathcal{G}_{1,p}(\omega)|^{k-1} \mathcal{G}_{j,p}(0) P_{s_p,\cdots,s_p}(\omega, -\omega, \cdots, \omega, 0) = 0, \quad j = 2, \cdots, n. \tag{A.2.7}$$

Define

$$\beta_p(\omega) \triangleq \frac{|\mathcal{G}_{1,p}(\omega)|^{k-1} P_{s_p,\cdots,s_p}(\omega, -\omega, \cdots, \omega, 0)}{|\mathcal{G}_{1,1}(0)|^{k-1} P_{s_1,\cdots,s_1}(\omega, -\omega, \cdots, \omega, 0)}, \quad p = 1, \cdots, n,$$

where $\mathcal{G}_{1,1}(0)$ and $P_{s_1,\cdots,s_1}(\omega, -\omega, \cdots, \omega, 0)$ are nonzero by assumption, then (A.2.7) can be rewritten as

$$\sum_{p=2}^{n} \mathcal{G}_{j,p}(0) \beta_p(\omega) = -\mathcal{G}_{j,1}(0), \quad j = 2, \cdots, n.$$

This can be further expressed in matrix form as

$$\begin{bmatrix} \mathcal{G}_{2,2}(0), & \cdots, & \mathcal{G}_{2,n}(0) \\ \vdots & & \vdots \\ \mathcal{G}_{n,2}(0), & \cdots, & \mathcal{G}_{n,n}(0) \end{bmatrix} \begin{bmatrix} \beta_2(\omega) \\ \vdots \\ \beta_n(\omega) \end{bmatrix} = - \begin{bmatrix} \mathcal{G}_{2,1}(0) \\ \vdots \\ \mathcal{G}_{n,1}(0) \end{bmatrix}. \tag{A.2.8}$$

Let $\omega = 0$ in (A.2.8), and compare it with (A.2.4). Noticing that the matrix on the left side of both equations is invertible because its determinant, which is the cofactor of $\mathcal{G}_{1,1}(0)$, is nonzero, we have $\alpha_p = \beta_p(0)$, $p = 2, \cdots, n$, *i.e.*,

$$\frac{\mathcal{G}_{1,p}(0)P_{s_p,s_p}(0)}{\mathcal{G}_{1,1}(0)P_{s_1,s_1}(0)} = \frac{|\mathcal{G}_{1,p}(0)|^{k-1}P_{s_p,\cdots,s_p}(0,\cdots,0)}{|\mathcal{G}_{1,1}(0)|^{k-1}P_{s_1,\cdots,s_1}(0,\cdots,0)} \tag{A.2.9}$$

for $p = 2, \cdots, n$.

From (A.2.9) we have either

$$\mathcal{G}_{1,p}(0) = 0, \quad \text{or} \quad \frac{P_{s_p,s_p}(0)}{P_{s_1,s_1}(0)} = \frac{|\mathcal{G}_{1,p}(0)|^{k-2}P_{s_p,\cdots,s_p}(0,\cdots,0)}{|\mathcal{G}_{1,1}(0)|^{k-2}P_{s_1,\cdots,s_1}(0,\cdots,0)}, \tag{A.2.10}$$

for $p = 2, \cdots, n$. Later we will show that the latter does not hold, thus the former is true. For now, we assume that the first case in (A.2.10) is true, *i.e.*, $\mathcal{G}_{1,p}(0) = 0$.

Let $j = 1$, $i = 2, \cdots, n$ in (A.2.6), and substitute $\mathcal{G}_{1,p}(0) = 0$ for $p = 2, \cdots, n$, into (A.2.6). This gives us

$$|\mathcal{G}_{i,1}(\omega)|^{k-1}G_{1,1}(0)P_{s_p,\cdots,s_p}(\omega, -\omega, \cdots, \omega, 0) = 0,$$

for all ω, which implies

$$\mathcal{G}_{i,1}(\omega) = 0, \ \ \forall\omega, \ \ i = 2, \cdots, n. \tag{A.2.11}$$

Therefore, $\mathcal{G}_{1,1}(\omega)$ is the only nonzero element in the first column of $\mathcal{G}(\omega)$. Next, we will show that $\mathcal{G}_{1,1}(\omega)$ is also the only nonzero element in its row.

Let $\omega = 0$ in (A.2.11) and substitute it into the right-hand-side of (A.2.8). Noticing that the matrix on the left-hand-side is invertible because its determinant, which is the cofactor of $\mathcal{G}_{1,1}(0)$, is nonzero, we obtain $\beta_p(\omega) = 0$, for $p = 2, \cdots, n$. Consequently, we have

$$\mathcal{G}_{1,p}(\omega) = 0, \ \ \forall\omega, \ \ p = 2, \cdots, n, \tag{A.2.12}$$

meaning that all entries in the first row of $\mathcal{G}(\omega)$ are zeros except $\mathcal{G}_{1,1}(\omega)$. Therefore, $\mathcal{G}_{1,1}(\omega)$ is the only nonzero entry of its row and column in $\mathcal{G}(\omega)$.

Now we will show that the second case in (A.2.10) can not happen. We will use the method of contradiction to prove it. Suppose the second case in (A.2.10) is true. Then we have

$$\mathcal{G}_{1,p}(0)P_{s_p,\cdots,s_p}(0,\cdots,0) = \frac{|\mathcal{G}_{1,1}(0)|^{k-2}P_{s_1,\cdots,s_1}(0,\cdots,0)P_{s_p,s_p}(0)}{|\mathcal{G}_{1,p}(0)|^{k-3}P_{s_1,s_1}(0)}, \tag{A.2.13}$$

for $p = 1, \cdots, n$. Here we include 1 in index p because above equation also holds for $p = 1$.

Letting $j = 1$, $i = 2, \cdots, n$, and $\omega = 0$ in (A.2.6), we have

$$\sum_{p=1}^{n} |\mathcal{G}_{i,p}(0)|^{k-1} \mathcal{G}_{1,p}(0) P_{s_p,\cdots,s_p}(0, \cdots, 0) = 0. \tag{A.2.14}$$

Substituting (A.2.13) into (A.2.14), we have

$$\sum_{p=1}^{n} |\mathcal{G}_{i,p}(0)|^{k-1} \frac{|\mathcal{G}_{1,1}(0)|^{k-2} P_{s_1,\cdots,s_1}(0, \cdots, 0) P_{s_p,s_p}(0)}{|\mathcal{G}_{1,p}(0)|^{k-3} P_{s_1,s_1}(0)} = 0, \tag{A.2.15}$$

for $i = 2, \cdots, n$, which can further be simplified as

$$\sum_{p=1}^{n} \frac{|\mathcal{G}_{i,p}(0)|^{k-1}}{|\mathcal{G}_{1,p}(0)|^{k-3}} P_{s_p,s_p}(0) = 0, \tag{A.2.16}$$

for $i = 2, \cdots, n$.

From property **P4** in Section 5.3.2 and the nonzero assumption of $P_{s_p,s_p}(\omega)$, we know that $P_{s_p,s_p}(\omega) > 0$ for $\forall \omega$. By assumption, k is odd and $k > 2$, thus every term in the summation in (A.2.16) is non-negative. Therefore, we conclude that

$$\mathcal{G}_{i,p}(0) = 0, \tag{A.2.17}$$

for $i = 2, \cdots, n$, and $p = 1, \cdots, n$, which is contradictory to the assumption that $\mathcal{G}(0)$ is invertible. Thus the second case in (A.2.10) can not hold. **Q.E.D.**

A.3 Proof of Theorem 5.5.2

Proof: The proof is similar to that of Theorem 5.5.1. Our goal is also to show that there exists only one nonzero element in each row and column of $\mathcal{G}(\omega)$. Again, we will only show this statement for the first row of $\mathcal{G}(\omega)$. For other rows, the statement can be shown in exactly the same way.

Since $\mathcal{G}(0)$ is invertible, there must exist nonzero element in the first row of $\mathcal{G}(0)$ such that its cofactor is also nonzero. Without loss of generality, we assume this element is $\mathcal{G}_{1,1}(0)$. Applying property **P3** (Section 5.3.2) and **P1'** (Section 5.5) to (5.5.9), we have

$$P_{y_i,y_j}(\omega) = \sum_{q=1}^{n} \mathcal{G}_{i,q}(-\omega)\mathcal{G}_{j,q}(\omega)P_{s_q,s_q}(\omega) = 0 \tag{A.3.1}$$

for all ω, and $i, j = 1, 2, \cdots, n$, $i \neq j$.

In particular, let $i = 1$, $j = 2, \cdots, n$, and $\omega = 0$ in (A.3.1). We then have

$$\sum_{q=1}^{n} \mathcal{G}_{1,q}(0)\mathcal{G}_{j,q}(0)P_{s_q,s_q}(0) = 0,$$

or

$$\sum_{q=2}^{n} \mathcal{G}_{j,q}(0)\mathcal{G}_{1,q}(0)P_{s_q,s_q}(0) = -\mathcal{G}_{j,1}(0)\mathcal{G}_{1,1}(0)P_{s_1,s_1}(0),$$

for $j = 2, \cdots, n$. This can be expressed in matrix form as

$$\begin{bmatrix} \mathcal{G}_{2,2}(0), & \cdots, & \mathcal{G}_{2,n}(0) \\ \vdots & & \vdots \\ \mathcal{G}_{n,2}(0), & \cdots, & \mathcal{G}_{n,n}(0) \end{bmatrix} \begin{bmatrix} \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} = - \begin{bmatrix} \mathcal{G}_{2,1}(0) \\ \vdots \\ \mathcal{G}_{n,1}(0) \end{bmatrix}, \tag{A.3.2}$$

where α_q is defined by

$$\alpha_q \triangleq \frac{\mathcal{G}_{1,q}(0)P_{s_q,s_q}(0)}{\mathcal{G}_{1,1}(0)P_{s_1,s_1}(0)}, \quad q = 2, \cdots, n.$$

Here $\mathcal{G}_{1,1}(0)$ and $P_{s_1,s_1}(0)$ are nonzero by assumption.

Similarly, applying properties **P3** (Section 5.3.2) and **P1'** (Section 5.5) to (5.5.10), we have

$$\sum_{q=1}^{n} \mathcal{G}_{i,q}\left(-\sum_{v=1}^{k-1}\omega_v\right) \cdots \mathcal{G}_{i,q}(\omega_{k-2})\mathcal{G}_{j,q}(\omega_{k-1})P_{s_q,\cdots,s_q}(\omega_1, \cdots, \omega_{k-1}) = 0 \tag{A.3.3}$$

for all $\omega_1, \cdots, \omega_{k-1}$, and $i, j = 1, 2, \cdots, n$, $i \neq j$.

In particular, let $i = 1$, $j = 2, \cdots, n$, and $\omega_v = 0$ for $v \leq k-1$ in (A.3.3). This gives us

$$\sum_{q=1}^{n} |\mathcal{G}_{1,q}(0)|^{k-1} \mathcal{G}_{j,q}(0) P_{s_q,\cdots,s_q}(0, \cdots, 0) = 0 \tag{A.3.4}$$

for $j = 2, \cdots, n$, which can also be expressed in matrix form as

$$\begin{bmatrix} \mathcal{G}_{2,2}(0), & \cdots, & \mathcal{G}_{2,n}(0) \\ \vdots & & \vdots \\ \mathcal{G}_{n,2}(0), & \cdots, & \mathcal{G}_{n,n}(0) \end{bmatrix} \begin{bmatrix} \beta_2 \\ \vdots \\ \beta_n \end{bmatrix} = - \begin{bmatrix} \mathcal{G}_{2,1}(0) \\ \vdots \\ \mathcal{G}_{n,1}(0) \end{bmatrix}, \tag{A.3.5}$$

where β_q is defined by

$$\beta_q \triangleq \frac{|\mathcal{G}_{1,q}(0)|^{k-1} P_{s_q,\cdots,s_q}(0, \cdots, 0)}{|\mathcal{G}_{1,1}(0)|^{k-1} P_{s_1,\cdots,s_1}(0, \cdots, 0)}, \quad q = 2, \cdots, n.$$

Comparing (A.3.2) and (A.3.5) and noticing that the matrix on the left-hand-side of both equations is invertible because its determinant, which is the cofactor of $\mathcal{G}_{1,1}(0)$, is nonzero, we have $\alpha_q = \beta_q$, *i.e.*,

$$\frac{\mathcal{G}_{1,q}(0) P_{s_q,s_q}(0)}{\mathcal{G}_{1,1}(0) P_{s_1,s_1}(0)} = \frac{|\mathcal{G}_{1,q}(0)|^{k-1} P_{s_q,\cdots,s_q}(0, \cdots, 0)}{|\mathcal{G}_{1,1}(0)|^{k-1} P_{s_1,\cdots,s_1}(0, \cdots, 0)} \tag{A.3.6}$$

for $q = 2, \cdots, n$.

From (A.3.6) we have either

$$\mathcal{G}_{1,q}(0) = 0, \quad \text{or} \quad \frac{P_{s_q s_q}(0)}{P_{s_1 s_1}(0)} = \frac{|\mathcal{G}_{1,q}(0)|^{k-2} P_{s_q,\cdots,s_q}(0, \cdots, 0)}{|\mathcal{G}_{1,1}(0)|^{k-2} P_{s_1,\cdots,s_1}(0, \cdots, 0)}, \tag{A.3.7}$$

for $q = 2, \cdots, n$. Later we will show that the latter case can not happen, thus only the first case is true. For now we assume the former is true, *i.e.*, $\mathcal{G}_{1,q}(0) = 0$, for $q = 2, \cdots, n$.

Applying property **P3** (Section 5.3.2) and **P1'** (Section 5.5) to (5.5.11), we have

$$\sum_{q=1}^{n} \mathcal{G}_{i,q}\left(-\sum_{v=1}^{k-1} \omega_v\right) \cdots \mathcal{G}_{i,q}(\omega_{k-p-1}) \mathcal{G}_{j,q}(\omega_{k-p}) \cdots \mathcal{G}_{j,q}(\omega_{k-1}) P_{s_q,\cdots,s_q}(\omega_1, \cdots, \omega_{k-1}) = 0$$

for all $\omega_1, \cdots, \omega_{k-1}$, and $i, j = 1, \cdots, n$, $i \neq j$. In particular, let $j = 1$, $i = 2, \cdots, n$, and

$$\omega_v = \begin{cases} (-1)^{v+1}\omega, & \text{for } v = 1, \cdots, k-p-1, \\ 0, & \text{for } v = k-p, \cdots, k-1, \end{cases}$$

in the above equation. We then have

$$\sum_{q=1}^{n} |\mathcal{G}_{i,q}(\omega)|^{k-p} |\mathcal{G}_{1,q}(0)|^{p} P_{s_q,\cdots,s_q}(\omega, -\omega, \cdots, 0) = 0 \tag{A.3.8}$$

for all ω.

Substituting $\mathcal{G}_{1,q}(0) = 0$ for $q = 2, \cdots, n$ into (A.3.8) and noticing that $\mathcal{G}_{1,1}(0)$ and $P_{s_q,\cdots,s_q}(\omega, -\omega, \cdots, 0)$ are nonzero, we have

$$\mathcal{G}_{i,1}(\omega) = 0, \tag{A.3.9}$$

for all ω, and $i = 2, \cdots, n$. Thus, all entries in the first column of $\mathcal{G}(\omega)$ are zero except $\mathcal{G}_{1,1}(\omega)$. Next we will show $\mathcal{G}_{1,1}(\omega)$ is also the only nonzero entry in its row.

Let $i = 1$, $j = 2, \cdots, n$, and $\omega_{k-1} = 0$ in (A.3.3). This gives us

$$\sum_{q=1}^{n} \mathcal{G}_{1,q}\left(-\sum_{v=1}^{k-2} \omega_v\right) \cdots \mathcal{G}_{1,q}(\omega_{k-2}) \mathcal{G}_{j,q}(0) P_{s_q,\cdots,s_q}(\omega_1, \cdots, \omega_{k-2}, 0) = 0 \tag{A.3.10}$$

for $j = 2, \cdots, n$. Noticing that by (A.3.9) $\mathcal{G}_{j,q}(0) = 0$ for $q = 1$, (A.3.10) can be expressed in matrix form as

$$\begin{bmatrix} \mathcal{G}_{2,2}(0), & \cdots, & \mathcal{G}_{2,n}(0) \\ \vdots & & \vdots \\ \mathcal{G}_{n,2}(0), & \cdots, & \mathcal{G}_{n,n}(0) \end{bmatrix} \begin{bmatrix} \gamma_2 \\ \vdots \\ \gamma_n \end{bmatrix} = - \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{A.3.11}$$

where γ_q is defined by

$$\gamma_q \triangleq \mathcal{G}_{1,q}\left(-\sum_{v=1}^{k-2} \omega_v\right) \cdots \mathcal{G}_{1,q}(\omega_{k-2}) P_{s_q,\cdots,s_q}(\omega_1, \cdots, \omega_{k-2}, 0), \quad q = 2, \cdots, n.$$

Again, since the matrix on the left-hand-side of (A.3.11) is invertible, we have $\gamma_q = 0$ for $q = 2, \cdots, n$, *i.e.*,

$$\mathcal{G}_{1,q}\left(-\sum_{v=1}^{k-2} \omega_v\right) \cdots \mathcal{G}_{1,q}(\omega_{k-2}) P_{s_q,\cdots,s_q}(\omega_1, \cdots, \omega_{k-2}, 0) = 0$$

for all $\omega_1, \cdots, \omega_{k-2}$ and $q = 2, \cdots, n$, which implies

$$\mathcal{G}_{1,q}\left(-\sum_{v=1}^{k-2} \omega_v\right) \cdots \mathcal{G}_{1,q}(\omega_{k-2}) = 0 \tag{A.3.12}$$

for all $\omega_1, \cdots, \omega_{k-2}$ and $q = 2, \cdots, n$. From (A.3.12), we will show that $\mathcal{G}_{1,q}(\omega) = 0$ for all ω and $q = 2, \cdots, n$. To show this, let $\omega_1 = \cdots = \omega_{k-3} = v_1$ and $\omega_{k-2} = v_2$ in (A.3.12). We then have

$$\mathcal{G}_{1,q}(-(k-3)v_1 - v_2)\mathcal{G}_{1,q}^{k-3}(v_1)\mathcal{G}_{1,q}(v_2) = 0 \tag{A.3.13}$$

for all v_1, v_2, and $q = 2, \cdots, n$.

Suppose the statement is not true, *i.e.*, there exists ω such that $\mathcal{G}_{1,q}(\omega) \neq 0$. Then the corresponding transfer function $\mathcal{G}_{1,q}(z)$ is a nonzero polynomial which has finite zeros. We claim that for any given v_2, we can always find v_1 such that

$$\mathcal{G}_{1,q}(-(k-3)v_1 - v_2)\mathcal{G}_{1,q}(v_1) \neq 0,$$

since otherwise it would contradict to the fact that the polynomial $\mathcal{G}_{1,q}(z_1)\mathcal{G}_{1,q}(z_2)$ has finite zeros. If this is true, then by (A.3.13), we have $\mathcal{G}_{1,q}(v_2) = 0$ for all v_2, which is in contradiction to the assumption above. This implies that the assumption is not true, *i.e.*, $\mathcal{G}_{1,q}(\omega) = 0$ for all ω and $q = 2, \cdots, n$. Thus, $\mathcal{G}_{1,1}(\omega)$ is the only nonzero entry in its row.

Now we will show that the second case in (A.3.7) can not happen. Again, we use the method of contradiction to prove it. Suppose the second case in (A.3.7) is true, then we have

$$|\mathcal{G}_{1,q}(0)|^p P_{s_q,\cdots,s_q}(0,\cdots,0) = \frac{|\mathcal{G}_{1,1}(0)|^{k-2} P_{s_1,\cdots,s_1}(0,\cdots,0) P_{s_q,s_q}(0)}{|\mathcal{G}_{1,q}(0)|^{k-p-2} P_{s_1,s_1}(0)}, \tag{A.3.14}$$

for $q = 1, \cdots, n$. Here we include 1 in index q because above equation also holds for $q = 1$.

Let $\omega = 0$ in (A.3.8). This gives us

$$\sum_{q=1}^{n} |\mathcal{G}_{i,q}(0)|^{k-p} |\mathcal{G}_{1,q}(0)|^p P_{s_q,\cdots,s_q}(0,\cdots,0) = 0 \tag{A.3.15}$$

for $i = 2, \cdots, n$. Substituting (A.3.14) into (A.3.15), we have

$$\sum_{q=1}^{n} |\mathcal{G}_{i,q}(0)|^{k-p} \frac{|\mathcal{G}_{1,1}(0)|^{k-2} P_{s_1,\cdots,s_1}(0,\cdots,0) P_{s_q,s_q}(0)}{|\mathcal{G}_{1,q}(0)|^{k-p-2} P_{s_1,s_1}(0)} = 0 \tag{A.3.16}$$

for $i = 2, \cdots, n$, which can further be simplified as

$$\sum_{q=1}^{n} \frac{|\mathcal{G}_{i,q}(0)|^{k-p}}{|\mathcal{G}_{1,q}(0)|^{k-p-2}} P_{s_p,s_p}(0) = 0, \tag{A.3.17}$$

for $i = 2, \cdots, n$. Since k and p are even and $k > p \geq 2$, and $P_{s_p,s_p}(0) > 0$ (Property 4), from (A.3.17) we have

$$\mathcal{G}_{i,p}(0) = 0,$$

for $i = 2, \cdots, n$, and $q = 1, \cdots, n$, which is contradicts the assumption that $\mathcal{G}(0)$ is invertible. This implies that the second case of (A.3.7) can not hold. **Q.E.D.**

A.4 Proof of Proposition 6.3.1

Proof: Let $\mathbf{m}_\ell = [1, m_\ell^1, \ldots, m_\ell^{n-1}]^T$ and $\mathbf{M}_\ell = \mathbf{m}_\ell \mathbf{m}_\ell^T$. Then the diagonal elements of $\Delta_{\mathbf{M}_\ell}(0,0)$ are given by

$$\mathbf{h}_i^T(0)\mathbf{M}_\ell \mathbf{h}_i(0) k(s_i(0)) = k(s_i(0))(\mathbf{m}_\ell^T \mathbf{h}_i(0))^2$$

for $i = 1, \ldots, n$, where $\mathbf{h}_i(0)$ is the ith column of $\mathcal{H}(0)$. For convenient notation, we denote $k(s_i(0))$ by k_i and $\mathbf{h}_i(0)$ by $\mathbf{h}_i$. Our goal now is to show there exists a vector $\mathbf{m}_\ell$ such that the entries of vector $[k_1(\mathbf{m}_\ell^T \mathbf{h}_1)^2, \ldots, k_n(\mathbf{m}_\ell^T \mathbf{h}_n)^2]^T$ are mutually distinct.

We will use the method of contradiction to prove it. Suppose this statement is not true, *i.e.*, for any vector $\mathbf{m}_\ell$, there are at least two entries in vector $[k_1(\mathbf{m}_\ell^T \mathbf{h}_1)^2, \ldots, k_n(\mathbf{m}_\ell^T \mathbf{h}_n)^2]^T$ which are the same. For example,

$$k_{i_\ell}(\mathbf{m}_\ell^T \mathbf{h}_{i_\ell})^2 = k_{j_\ell}(\mathbf{m}_\ell^T \mathbf{h}_{j_\ell})^2,$$

or, it can be expressed as

$$\mathbf{m}_\ell^T(\sqrt{|k_{i_\ell}|}\mathbf{h}_{i_\ell} \pm \sqrt{|k_{j_\ell}|}\mathbf{h}_{j_\ell}) = 0, \tag{A.4.1}$$

where $\pm$ depends on the signs of $\mathbf{m}_\ell^T \mathbf{h}_{i_\ell}$ and $\mathbf{m}_\ell^T \mathbf{h}_{j_\ell}$.

If we collect all the possible vectors defined by

$$\sqrt{|k_i|}\mathbf{h}_i + \sqrt{|k_j|}\mathbf{h}_j \quad \text{and} \quad \sqrt{|k_i|}\mathbf{h}_i - \sqrt{|k_j|}\mathbf{h}_j, \quad \text{for} \quad i < j,$$

in a set called $\mathcal{P}$, then the size of set $\mathcal{P}$ is $n(n-1)$. For convenience, we label the elements in $\mathcal{P}$ by $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_{n(n-1)}\}$. Then equation (A.4.1) implies that there exists a vector $\mathbf{y}_{k_\ell} \in \mathcal{P}$ such that $\mathbf{m}_\ell^T \mathbf{y}_{k_\ell} = 0$.

By our assumption, we know that for vectors $\mathbf{m}_1, \ldots, \mathbf{m}_N$, there exist vectors $\mathbf{y}_{j_1}, \ldots, \mathbf{y}_{j_N} \in \mathcal{P}$ such that $\mathbf{m}_\ell^T \mathbf{y}_{j_\ell} = 0$ for $\ell = 1, \ldots, N$. Since the size of $\mathcal{P}$ is $n(n-1)$ and $N \geq n(n-1)^2 + 1$, there must be one element in $\mathcal{P}$ such that it appears in $\{\mathbf{y}_{j_1}, \ldots, \mathbf{y}_{j_N}\}$ no less than n times. Denote this element by $\mathbf{y}_{j_k}$ and the corresponding orthogonal vectors by $\mathbf{m}_{j_{k1}}, \ldots, \mathbf{m}_{j_{kn}}$. Then we have

$$\begin{bmatrix} \mathbf{m}_{j_{k1}}^T \\ \vdots \\ \mathbf{m}_{j_{kn}}^T \end{bmatrix} \mathbf{y}_{j_k} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Noticing that the matrix on the left is of Vandermonde structure, we have $\mathbf{y}_{j_k} = 0$. This means that

$$\sqrt{|k_i|}\mathbf{h}_i + \sqrt{|k_j|}\mathbf{h}_j = 0, \quad \text{or} \quad \sqrt{|k_i|}\mathbf{h}_i - \sqrt{|k_j|}\mathbf{h}_j = 0$$

for some indices $i < j$. Either of these cases above imply that $\mathbf{h}_i$ and $\mathbf{h}_j$ are linear dependent. This contradicts to assumption **A3**. **Q.E.D.**

A.5 Proof of Theorem 6.3.1

Proof: To show rank($\mathcal{E}$) = K − 1, we will show the null space of $\mathcal{E}$ is one dimensional. For notational convenience, we first rearrange the rows in $\mathcal{E}$ by grouping together the rows that correspond to the same filter. For example, the rows that correspond to the first filter h_1 can be grouped as

$$\begin{bmatrix} h_1(\omega_0)e^{j\omega_0 L}, & \ldots, & h_1(\omega_{K-1})e^{j\omega_{K-1}L} \\ \vdots & \ddots & \vdots \\ h_1(\omega_0)e^{j\omega_0(K-1)}, & \ldots, & h_1(\omega_{K-1})e^{j\omega_{K-1}(K-1)} \end{bmatrix},$$

or equivalently

$$\begin{bmatrix} e^{j\omega_0 L}, & \ldots, & e^{j\omega_{K-1}L} \\ \vdots & \ddots & \vdots \\ e^{j\omega_0(K-1)}, & \ldots, & e^{j\omega_{K-1}(K-1)} \end{bmatrix} \begin{bmatrix} h_1(\omega_0) & & \\ & \ddots & \\ & & h_1(\omega_{K-1}) \end{bmatrix}.$$

Denote

$$\mathbf{V}^H \triangleq \begin{bmatrix} e^{j\omega_0 L}, & \ldots, & e^{j\omega_{K-1}L} \\ \vdots & \ddots & \vdots \\ e^{j\omega_0(K-1)}, & \ldots, & e^{j\omega_{K-1}(K-1)} \end{bmatrix} \quad \text{and} \quad \mathbf{H}_p \triangleq \begin{bmatrix} h_p(\omega_0) & & \\ & \ddots & \\ & & h_p(\omega_{K-1}) \end{bmatrix}$$

for $p = 1, \ldots, n$. Then the row rearranged matrix, which we still denote by $\mathcal{E}$ because its null space does not change, is given by

$$\mathcal{E} = \begin{bmatrix} \mathbf{V}^H\mathbf{H}_1 \\ \vdots \\ \mathbf{V}^H\mathbf{H}_n \end{bmatrix}. \tag{A.5.1}$$

Our goal is to show that the null space of $\mathcal{E}$ is one dimensional. To show this, we define a $K \times K$ matrix $\mathbf{E}$ as the following

$$\mathbf{E} = \begin{bmatrix} e^{-j\omega_0 0}, & \ldots, & e^{-j\omega_0(K-1)} \\ \vdots & \ddots & \vdots \\ e^{-j\omega_{K-1}0}, & \ldots, & e^{-j\omega_{K-1}(K-1)} \end{bmatrix}. \tag{A.5.2}$$

It is easy to verify that $\mathbf{E}$ is unitary up to a constant factor $K^{-\frac{1}{2}}$, and the matrix formed by the last $K - L$ columns is $\mathbf{V}$. Denote by $\mathbf{U}$ the matrix of the first L columns of $\mathbf{E}$. Then $\mathbf{E} = [\mathbf{U}\mathbf{V}]$. The space spanned by the columns of $\mathbf{U}$ and the space spanned by columns of $\mathbf{V}$ are orthogonal to each other.

Now let us consider the solution of equation $\mathcal{E}\mathbf{y} = \mathbf{0}$, *i.e.*, the solution of linear system

$$\mathbf{V}^H\mathbf{H}_p\mathbf{y} = \mathbf{0}, \quad p = 1, \ldots, n. \tag{A.5.3}$$

We will show that the only solution to (A.5.3) is $y_1 = \cdots = y_n$ where $\{y_i\}_{i=1}^n$ are the entries of $\mathbf{y}$.

Equation (A.5.3) implies that the vector $\mathbf{H}_p\mathbf{y}$ belongs to the null space of matrix $\mathbf{V}^H$, *i.e.*, the space spanned by the columns of $\mathbf{U}$. Consequently, there exists some vector $\mathbf{c}_p \in \mathbb{C}^L$ such that

$$\mathbf{H}_p\mathbf{y} = \mathbf{U}\mathbf{c}_p, \quad p = 1, \ldots, n. \tag{A.5.4}$$

Noticing that the entries of $\mathbf{U}\mathbf{c}_p$ are actually the Fourier transform of $\mathbf{c}_p$ at $\omega_0, \ldots, \omega_{K-1}$, *i.e.*,

$$\mathbf{U}\mathbf{c}_p = \begin{bmatrix} c_p(\omega_0) \\ \vdots \\ c_p(\omega_{K-1}) \end{bmatrix},$$

(A.5.4) can be expressed component-wise as

$$y_k h_p(\omega_k) = c_p(\omega_k), \quad k = 0, 1, \ldots, K-1, \quad p = 1, \ldots, n. \tag{A.5.5}$$

If $h_1(\omega_k) \neq 0$ and $h_2(\omega_k) \neq 0$, we have

$$y_k = \frac{c_1(\omega_k)}{h_1(\omega_k)} = \frac{c_2(\omega_k)}{h_2(\omega_k)},$$

or

$$c_1(\omega_k)h_2(\omega_k) = c_2(\omega_k)h_1(\omega_k). \tag{A.5.6}$$

If $h_1(\omega_k) = 0$ (or $h_2(\omega_k) = 0$), then from (A.5.5) we have $c_1(\omega_k) = 0$ (or $c_2(\omega_k) = 0$). Equation (A.5.6) still holds. Thus, we always have

$$c_1(\omega_k)h_2(\omega_k) - c_2(\omega_k)h_1(\omega_k) = 0, \quad k = 0, 1, \ldots, K-1. \tag{A.5.7}$$

Denote $G(\omega) = c_1(\omega)h_2(\omega) - c_2(\omega)h_1(\omega)$. Then $G(\omega)$ can be expressed as

$$\begin{aligned} G(\omega) &= \sum_{m=0}^{L-1} c_1(m)e^{-j\omega m} \sum_{\ell=0}^{L-1} h_2(\ell)e^{-j\omega\ell} - \sum_{m=0}^{L-1} c_2(m)e^{-j\omega m} \sum_{\ell=0}^{L-1} h_1(\ell)e^{-j\omega\ell} \\ &= \sum_{m=0}^{L-1}\sum_{\ell=0}^{L-1}[c_1(m)h_2(\ell) - c_2(m)h_1(\ell)]e^{-j\omega(m+\ell)} \\ &\triangleq \sum_{m=0}^{2(L-1)} g(m)e^{-j\omega m}. \end{aligned} \tag{A.5.8}$$

In light of (A.5.7), we know

$$G(\omega_k) = \sum_{m=0}^{2(L-1)} g(m)e^{-j\omega_k m} = 0, \quad k = 0, 1, \ldots, K-1.$$

In matrix form, we have

$$\begin{bmatrix} e^{-j\omega_0 0}, & \ldots, & e^{-j\omega_0 2(L-1)} \\ \vdots & \ddots & \vdots \\ e^{-j\omega_{K-1} 0}, & \ldots, & e^{-j\omega_{K-1} 2(L-1)} \end{bmatrix} \begin{bmatrix} g(0) \\ \vdots \\ g(2L-2) \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{A.5.9}$$

Here we can see that when $K \geq 2L-1$, the matrix on the left-hand-side has full column rank because of its Vandermonde structure. Thus, $g(0) = \cdots = g(2L-2) = 0$. Consequently, $G(\omega) = c_1(\omega)h_2(\omega) - c_2(\omega)h_1(\omega) = 0$ for all ω, *i.e.*,

$$c_1(\omega)h_2(\omega) = c_2(\omega)h_1(\omega), \quad \forall\omega. \tag{A.5.10}$$

Since $h_1(\omega)$ and $h_2(\omega)$ are coprime by assumption **A3**, (A.5.10) holds only when

$$c_1(\omega) = c_2(\omega) = 0, \quad \forall\omega,$$

or

$$c_1(\omega) = h_1(\omega)\alpha, \quad \text{and} \quad c_2(\omega) = h_2(\omega)\alpha, \quad \forall\omega,$$

for some constant $\alpha \in \mathbb{C}$. In either case, we conclude $y_0 = \cdots = y_{K-1}$. **Q.E.D.**

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