THE EFFECT OF CONTEXT ON CHILDREN'S ARITHMETIC PROBLEM SOLVING

BY

© BRUCE A. LINDER

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AUTHOR: Bruce A. Linder, B.E.S. (University of Minnesota)

SUPERVISOR: Professor L.S.Siegel

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ABSTRACT

Several studies are presented which show that the way in which a child solves an arithmetic problem on a test is influenced by the nature of the other items comprising the test, the so-called "context" problems. These context effects pose a challenge to existing models of children's arithmetic problem solving which assume that children learn and are guided by internalized procedural rules or algorithms that are exclusively applied according to the characteristics of the current to-be-solved problem. Context effects are also demonstrated for arithmetic problems at grade levels 2 through 5 and a variety of systematic computational errors. For example, at grade 2 children made significantly more algorithm errors of a particular type when the target problems (2-by-1-digit subtraction fact problems - e.g. 12-4= or 14-8=) were located in a test composed of 2-digit-by-2-digit subtraction problems (e.g. 54-21=) than when located in a test composed of 1-digit-by-1-digit subtraction problems (e.g. 5-4=). Studies are also presented demonstrating context effects with 2-digit-by-1-digit addition and subtraction target problems (grade 3), 2-digit-by-2-digit multiplication problems (grade 4), and 2-digit-by-1-digit division problems (grade 5). An analysis of the relationship between the problem solving strategies used to solve the context and target problems suggests an alternative view of children's arithmetic learning and problem solving. Children solve target problems by applying procedures that are similar or analogous to procedures used to solve the context problems. That is, children's arithmetic problem solving is often governed by a contextually driven analogy mechanism, similar to that proposed in some contemporary models of reading (Glushko, 1979; Kay & Marcel, 1981) and classification (Brooks, 1978). Several studies are also presented which demonstrate that children's tendencies to error in this way is dependent upon certain kinds of arithmetic training practices commonly followed in primary school arithmetic instruction: so-called "unmixed drills" in which the practice worksheets are composed entirely of problems to be solved using the same problem solving strategy.

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CHILDREN'S ARITHMETIC LEARNING

These studies will be concerned with the general problem of what causes children to make computational errors in solving basic arithmetic problems. More specifically, they will show that the way in which primary school-age children solve many basic arithmetic computation problems is significantly determined by the way in which the test is constructed. In particular, these studies will demonstrate that the way in which an individual problem on a test is solved is affected by the nature of the other problems comprising the test in orderly and systematic ways; that is, problem solving activity to individual test items is not independent of one another. Independence of problem solving activity among items on a test is a tacit assumption of current educational and psychological theories of children's arithmetic learning. These theories describe children's arithmetic problem solving as guided by internalized procedural rules or algorithms that are exclusively applied according to the characteristics of the current to-be-solved problem. The present studies, by demonstrating interproblem dependence in test problem solving, challenge these theoretical orientations. Moreover, a careful analysis of the nature of the interproblem effects suggests an alternative conception of children's arithmetic learning and problem solving. Children solve problems by applying procedures that are similar or analogous to procedures used to solve other test problems. Several studies will also demonstrate that children's tendency to error in this way is dependent upon certain kinds of arithmetic training practices commonly followed in schools.

Prior to presenting the evidence for these claims, a brief review of the

relevant educational and psychological literature will be presented to establish that (1) current views of children's arithmetic learning and problem solving have, indeed, neglected the study of interproblem test effects, and (2) abundant evidence and theory outside the children's arithmetic literature suggests that such effects should exist.

TERMINOLOGY

Several terms will be used repeatedly in this thesis.

The term "arithmetic problem solving" will be used to refer exclusively to the written solutions children produce when solving the standard primary school arithmetic forms, such as, for example, the vertically displayed 2-digit-by-2-digit addition, subtraction, and multiplication problems. During the primary school years, a large portion of the mathematics curriculum is dedicated to teaching a variety of problem-solving procedures, or algorithms, to proficiency.

The terms "context" and "target" will be used to refer to two types of problems comprising the tests used in the thesis studies. Target problems are select problems on a test the solutions to which are the primary focus of the study. Context problems are all other problems comprising the test. It will be the purpose of the thesis studies to show that target problem solving is affected by the nature of the context problems. When such interproblem-dependency effects are demonstrated it will said that "the context affected target problem solving".

The necessary background for the thesis studies will be presented in two sections corresponding to the two assertions mentioned above. The first section will be concerned with current arithmetic learning theories, the other with evidence outside of the arithmetic literature dealing with contextual effects in general.

In this first section, it will be argued that educational practices and psychological research and theory in arithmetic learning tacitly assume that children's arithmetic problem solving is uninfluenced by problem context, that is, is "context-free" or "context-independent".

CHILDREN'S ARITHMETIC LEARNING

The bulk of our current understanding of children's arithmetic learning and problem-solving stems from analyses of the kinds of computation errors children commit when solving standard arithmetic problems. As in other areas of psychology, errors are thought to be especially informative in revealing the factors underlying psychological processes controlling performance. Therefore, considerable attention has been devoted to cataloging and counting children's computation errors in an effort to better understand the learning processes in arithmetic.

One main conclusion from these studies has been important in determining the nature of theorizing in this area: a substantial portion of children's computation errors are not random or capricious, but are consistent and systematic (Ashlock, 1976; Brown & Burton, 1978; Brown & VanLehn, 1980, 1982; Cox, 1975; Engelhardt, 1977; Friend, 1972; Janke, 1980; Owston, 1981; Roberts, 1968; Young & O'Shea, 1981). For example, a child may systematically solve all vertically displayed multidigit subtraction problems by subtracting the smallest from largest number in each vertical column irrespective of whether the smallest number is located in the minuend or subtrahend (i.e. top and bottom number, respectively). This can be seen in Table 1a. Because the error is consistently displayed across many different numerical instances of multidigit subtraction problems and is systematic in the sense that a description of the child's behavior can be given in terms of a rule, this error is typically interpreted as indicating that the child is following an incorrect rule - i.e. subtract the smallest from largest.

TABLE 1
EXAMPLES OF SYSTEMATIC COMPUTATION ERRORS

a-INVERSION ERRORS		b-INCREMENT ERRORS	
		7	6
563	56	5~~6~~3	~~5~~6
-27	-18	-27	-18
544	42	556	58

Another example of systematic errors is seen in Table 1b where the child, when attempting to borrow during subtraction, always increments, rather than decrements, the number being borrowed from (cf. Ashlock, 1976; Brown & Burton, 1978; Brown & VanLehn, 1980, 1982). Systematic errors, of which these are only two examples, are common; in fact, one investigator (Cox, 1975) reports as many as 256 different types among children in the primary and secondary grade levels.

The principal result of this emphasis on the systematicness of children's problem solving behavior has been the inference that children's knowledge of arithmetic can be adequately represented in the form of procedural rules or internal algorithms that are thought to remain reasonably stable between instructional episodes. This view of children's arithmetic learning and problem solving will be refered to as the "Fixed Rule" approach. According to the Fixed Rule approach, children's arithmetic learning involves the internalization of sets of procedural rules for different types of arithmetic problems and arithmetic problem solving involves following these internalized rules. These rules are problem-type-specific, that is, they specify certain problem solving actions for certain problem types. The internalization of procedural rules is thought to occur during classroom instruction in which procedural rules are explicitly verbalized by the teacher and during practice drill exercises in which the child is given problems to solve "following the rules" explicitly taught during prior instruction. According to the Fixed Rule approach, once the rules are internalized they can be accessed by presenting to the student the relevant problems. The child may fail to recognize the problem as of a particular type, e.g. 2-by-2-digit multiplication, and thus, perform an incorrect procedure. For example, in the case of a 2-by-2-digit multiplication problem, the child may falsely identify the problem as an addition problem and, thus, may solve the problem by adding column-by-column. But, according to the Fixed Rule approach, once the correct recognition of problem type is made, the internalized procedural

rules for that type of problem will be accessed and used. According to this approach, something may go wrong during the learning process such that the child internalizes an incorrect set of procedural rules, but the children are viewed as learning and following rules nonetheless. It is important to note that according to the Fixed Rule approach, systematic computations in general result from learning procedural rules, and systematic errors, in particular, result from learning incorrect rules (see footnote 1).

The thesis studies will show that systematic errors and problem solving activities are not fixed between instructional episodes as assumed by the Fixed Rule approach, but fluctuate depending upon the contextual conditions present in tests administered between those episodes. In fact, it will be shown in all of the studies that under one set of contextual conditions children will answer a problem type correctly, and, during the same testing period but under different contextual conditions, will answer the same problems consistently incorrectly. This implies that arithmetic problem solving is not purely a result of applying knowledge in the form of fixed rules. Prior to discussing these studies we will turn to specific examples of the Fixed Rule approach in education and psychology.

THE FIXED RULE APPROACH IN EDUCATION

The Fixed Rule approach is the modal conception of arithmetic learning and problem solving tacitly used in current educational practices. According to one of the most popular current educational arithmetic teaching approaches, the so-called "Diagnostic-Prescriptive" approach (Ashlock, 1972,1976; Brueckner,1930; Reisman,1972), teachers are instructed to follow 3 basic steps in remedial instruction. First, they are to design an assessment instrument for an arithmetic problem type by writing at least 5 numerical instances of that type for the child to solve. Then, after the child has taken the test they are to count the number of

problems solved correctly to determine if the child has the assessed skill in question, and analyze the errors to determine how the child has misunderstood the procedure, that is, what incorrect rules the child has somehow learned. Assessing the performance and interpreting the errors for the underlying rules the child is following is considered the diagnostic portion of the remedial instruction. Correcting the misunderstanding is the goal of the remedial instruction that is to follow, which can take many different forms. Following remediation, a final assessment is given to determine if in fact the remediation has been successful.

Two aspects of the Diagnostic-Prescriptive Approach should be recognized. First, the approach involves direct assessment of the problem solving skill, and interpretation of systematic errors is in terms of fixed internalized procedural rules. Second, the architecture of arithmetic knowledge as underlying fixed rules presumes that problem solving is context-independent; that is, that direct access to that knowledge can be made by presenting the relevant arithmetic problems without consideration to the contextual conditions under which the original learning and diagnostic assessment takes place. Arithmetic knowledge is assumed to be acquired as context-independent or "decontextualized" information about procedural rules.

In education, the assumptions of the Fixed Rule approach are also revealed in theories used to construct standardized educational achievement tests. Most of these theories share with the diagnostic-prescriptive approach to arithmetic teaching the assumption that arithmetic knowledge is directly accessible by presenting problems irrespective of the contextual conditions in which the problems are presented. For example, in 2-factor Item Response Theory (Lord, 1980; Rasch,1966a,b; Wright,1977), problem solving to arithmetic test items is assumed to be a function of 2 parameters: the item's difficulty and the student's ability. An item's difficulty is defined in terms of the percentage of children at a certain ability level (defined by age or grade) that pass the item. An item is defined in

terms of the problem's internal characteristics such as what numerals are used and what operation is required. The tacit assumption in the use of the item difficulty parameter is that performance on the item is determined only by the internal characteristics of that item, such that the percentage of children at a given ability level that pass the item will not vary as a function of the context in which the item is presented. In other words, the difficulty of a particular item for a given ability level is assumed to be a fixed, context-independent property of the item. Item Response Theory is a highly developed formal theory considered by some (Lord,1980) to be an extension of classical error theory (see Nunnally,1967), and is currently very important in the development of many standardized arithmetic achievement tests, such as the Key Math Diagnostic Test (Connolly, Nachtman, & Pritchett, 1971).

THE FIXED RULE APPROACH IN PSYCHOLOGICAL THEORY

The assumptions of the Fixed Rule approach, which are tacit in educational theory and practice, are made much more explicit in the more detailed psychological theories recently developed to explain children's arithmetic learning (Brown & Burton, 1978; Brown & VanLehn, 1980, 1982; Young & O'Shea, 1981). In an effort to formalize the underlying rules, cognitive psychologists have adapted computer models of general problem solving to fit children's arithmetic problem solving. Three of these adaptations will be described: Bug theory, Repair theory, and Frame theory.

Bug Theory. Young & O'Shea (1981) have argued that children's arithmetic problem solving is adequately modeled by "production systems" theory. A production system is a computer-language model of a problem solver which contains, among other things, a "production memory". A production memory contains a collection of production rules each of which can be represented in the form of a simple

conditional proposition, e.g. if X then A, sometimes called a "condition-action" statement, where the left hand side of the conditional is a "condition" which may or may not exist in the environment at any given point in time, and the righthand side of the conditional is an "action" which may be taken if and only if the condition is currently satisfied. The contents of production memories are represented by computer program lines, and when certain additional information processing performance rules are added (e.g. "working" memory, and "conflict resolution" heuristics), the computer program can "run" when certain data input statements are given (the arithmetic problem). The program is presumed to model problem solving activity. According to such models of arithmetic problem solving, systematic computation errors "can be accounted for in a principled way by simple changes in the production system, such as the omission of individual rules or the inclusion of rules appropriate to other arithmetical tasks " (Young & O'Shea,1981, p. 153). Systematic errors are considered to be like "bugs" in a computer program, in the sense that they can be conceptualized as resulting from faulty "written" (learned, internalized) programming lines; and, problem solving activity can be considered as like program execution in the sense that sequences of problem solving actions "run off" like programming lines are executed when conditions defined by the program are met. For the production system model of Young & O'Shea, two factors determine problem solving: the data input statement (i.e. the particular structural characteristics of the to-be-currently solved problem), and the contents of the production memory of the problem solver (i.e. condition-action rules). Because the conditions of the condition-actions rules only pertain to conditions in the currently to-be-solved problem, this model depicts arithmetic problem solving as context-independent.

A similar theoretical approach to modeling children's arithmetic problem solving has been discussed by Brown & Burton (1978). In their approach, they assume, as Young & O'Shea (1981), that children learn and follow a fixed set of procedural

rules. They employ a "procedural network" representation of these rules instead of a production system representation. According to a procedural network model, problem solving is represented as a collection of subprocedures in which the "control structure" between subprocedures is indicated graphically by arrows. Arrows between subprocedures indicate a "calling relationship" between these components of the overall procedure. Thus, similar to the production system modeling, the operation of the procedural network is analogous to a computer program that enables actions ("methods", "implementations") to be "called" when certain conditions exist during problem solving and certain "calling relationships" exist within the network. Systematic errors are considered "bugs" in the computer program. As Brown & Burton note "...the possible misconceptions in this skill are represented in the network by incorrect implementations associated with subprocedures in its (skill) decomposition called 'bugs'. Each buggy version contains incorrect actions taken in place of the correct ones" (p. 161).

In addition to the buggy-program interpretation of errors, Brown & Burton (1978) compare arithmetic problem solving to "diagnosing a deep structure" from the "surface manifestations of a deep structure misconception" in children's arithmetic procedures. Such concepts as "deep" and "surface structure" are derived from Chomsky's (1957) modeling of language knowledge and performance in terms of a deep structure of grammatical rules which are transformed into a surface structure of language. Arithmetic learning and performance is seen by Brown & Burton as analogous to language learning and performance, both of which are assumed to be adequately represented as decontextualized information in the form of underlying rules. The key features of Bug Theory which is important for appreciating the studies described below is the tacit assumption that systematic errors and problem solving activities are determined by fixed internalized rules; and which rules are used in solving a problem is determined excluxively by characteristics of the currently to-be-solved

problems.

Repair Theory. Brown & VanLehn (1980, 1982) have recently extended their buggy-program theory of systematic errors in an effort to better explain why some errors actually occur and other potential rule misconceptions do not occur. In this extension, the problem solver is seen as behaving somewhat differently than a computer running a buggy program. As noted by Brown & VanLehn, a buggy program will usually reach an "impasse" during execution and consequently stop running; children, however, will usually finish solving the problems on which a buggy program is assumed to be run. Brown & VanLehn reason that children may be faithfully following an "impoverished" program, but when they reach an impasse, they will not stop but will "repair" the program so that it will finish the problem. According to them, "...bugs can best be explained as "patches" derived from repairing a procedure that has encountered an impasse while solving a particular problem" (1980, p.381). The resulting theory, which they call "Repair" Theory depicts arithmetic problem solving as involving two essential parts: a series of operations that generate an incomplete procedure, and a separate series of operations that repair the procedure so that it can proceed. The first part enables one to "write" an incomplete procedure (which will later require repair during actual problem solving) by applying a set of "deletion principles" that operate on a correct procedure. This deleting represents the results of mislearning the rule or forgetting. The resulting incomplete program then "runs" on some numerical problems at which time impasses are confronted. To repair the procedure, at this point, certain "repair heuristics" generate possible repairs to the core procedure which are subsequently "tested" by a "critic" for adequacy as a repair. A systematic error, or bug, is thus a result of rule-governed repairing procedures performed on incomplete core procedures.

Repair Theory is clearly more complex than Bug Theory in postulating the existence of repair rules and critics that are separate from the core procedures. In

principle, arithmetic problem solving as conceptualized by Repair Theory can be more flexible than as conceptualized by Bug Theory. For example, different patches of the same impasse can occur at different times between instructional episodes. However, these "spontaneous" fluctuations are explained as the result of random sampling of repair heuristics at each impasse, and , sometimes, short term memory for recently used patches. These mechanisms provide no opportunity for contextual influences on arithmetic problem solving. Thus, Repair Theory shares with Bug Theory the assumption that arithmetic learning and problem solving is context independent.

Frame Theory. Davis & McKnight (1979) and Davis (1980) have developed what they call a "Frame Theory" of children's arithmetic problem solving. Frame Theory is an application of "schema" theories developed recently in the field of computer comprehension of natural languages (e.g. Minsky, 1975; Rummelhart & Ortony, 1977). Among current views, Frame Theory comes closest to acknowledging general contextual influences in arithmetic problem solving, but provides no specific predictions about the types of contextual events which may be important.

According to Frame Theory, arithmetic knowledge can be represented in two different ways: as procedural rules with sequential processing properties, much like the previous models discussed above; and as procedural rules with integrated processing properties, so-called "frames". The notion of frames is what distinguishes this model from Bug and Repair theory. Frames are procedural rules that have been integrated into an organized information unit which can not only sequentially guide problem solving activity via the procedural rules, but also can more actively interpret the incoming problem data and add new information that is consistent with the overall frame being used in solving the current problem. The interpretive function of frames explains observations in the literature of mathematics that problem solvers will often confront a task which allows for differing interpretations by using an incorrect interpretation and solving the

problem using procedures appropriate for that interpretation. For example, high-school students learning the calculus may interprete the symbol for differentiation, "d/dx", as indicating division, and thus solve many differentiation problems by using division algorithms (Davis & McKnight,1979, p.109). According to a Frame Theory interpretation, the students are using a division-problem frame to process the d/dx symbol and this frame's division procedures are consequently used to solve the differentiation problems. According to Davis, frames can be retrieved by specific cues in the current problem, much as production systems or procedural networks are called by data characteristics of the current problem, but also can be retrieved by "general situational similarity" (Davis,1980, p.196). Once retrieved, frames serve as "assimilation schema for organizing input data" (p. 193), that is, guide the system to look for the input data essental for subsequent problem solving manipulations. The ability of general situational cues to retrieve frames leaves open the possibility of contextual influences on problem solving activity, and sets Frame Theory apart from the production systems and procedural network models as well as Repair Theory discussed above. Davis, however, is not at all specific about the kind of situational cues important to children's arithmetic problem solving, and no research designed to explore contextual influences has been reported in the literature. Before leaving Frame Theory more should be said about how arithmeic frames are thought to be developed.

According to Davis, arithmetic frames are said to be created in experiences where the sequential procedures are practiced and in which such procedures are "analyzed" by "looking ahead" devices and pattern recognition mechanisms, etc. resulting in a integrated procedure. The integrated procedures can then be thought about without going through the entire sequence of steps; in Davis' words "one can 'see the whole process in the mind's eye - at least in general - without actually writing it down on paper". When holistically experienced in this

way, "the specific numerals probably must be suppressed, but the general pattern can be visualized clearly" (Davis & McKnight, 1979, p. 95). When appropriate metalanguage is developed to describe and label procedural frames, strategic problem solving can be executed. Frames are formed according to two general rules - a rule of overgeneralization in which the frame's procedures are "written" in the most general form to meet the current learning demands, i.e. discriminations are not made where they are not yet needed; and a rule of frame perseveration in which an originally written frame is never erased for subsequent rewriting, but is essentially "overwritten" by current frames such that older frames can still occassionally be retrieved and influence current problem solving.

Despite claims that the concept of arithmetic frames is significantly different from the previous concepts of sequential processes by procedural rules, a look at specific frames that Davis contends are learned by primary school children does not support this claim. One of the earliest learned frames, according to Davis (1980), is the "undifferentiated binary-operation" frame. This frame is created from extensive drilling in the basis single-digit addition facts (e.g. 2+2=4, 6+8=14) which are originally the only problems presented for instruction. Because there is no need to attend to the operation sign + at this time, Davis argues, based on the overgeneralization principle for frame formation, that a frame is learned which has the following procedural steps: (1) Input two numbers, (2) Set A= first number, B= second number, (3) Output A+B. The child will, thus, show a tendency to commit "wrong operation" errors by adding problems that require other binary operations. Note that the "frame" consists of procedural steps in a manner identical with other rule-oriented descriptions of arithmetic problem solving (Brown & Burton,1978; Young & O'Shea,1980), and the program written is undifferentiated with respect to the operation sign given as potential data in the problem; that is, discriminations are not written which are not needed for the materials and tasks used during training.

According to Davis, subsequent experienced training with the other binary operations leads to adding new frames, not deleting the old undifferentiated binary operation frame. Specifically, Davis argues that the child creates a "binary-operation selection frame" the constituent procedures of which serve as a kind of screening mechanism that attends to the sign and "calls up" the appropriate "calculational frame" (Davis, 1980,p.178). Other primary-grade frames are discussed by Davis, but these two, the undifferentiated binary-operation frame and the binary-operation selection frame, illustrates the actually working of the theory.

In summary, Frame Theory expands upon Bug and Repair Theory in (1) providing principles of learning procedures (overgeneralization and frame preservation), and in (2) noting retrieval mechanisms in procedural usage. Retrieval of procedures by general situation cues implies certain kinds of contextual effects, but this implication has not been derived by researchers influenced by Frame Theory, nor has any theoretical emphasis been given to different types of situation cues as especially important in learning and retrieval. Investigators in this field have been predominantly influenced by a computer metaphor for arithmetic learning and performance in which internalized procedural rules can be directly accessed by certain external cues denoted in the rules themselfves, just as computer actions are "run off" when computer memories are accessed by input data denoted in the memories themselves.

The main assertion being made here about these approaches is that they depict the child arithmetic problem solver as following a clearly defined, fairly determinant set of procedural rules which are called forth by certain critical structural characteristics of the current problem being solved. These fixed rules may come in the form of production system condition-action propositions, procedural network nodes, deep structure generative rules, repair heuristics, internalized faulty algorithms, or procedural frames. The form of the terminology used to

describe the arithmetic knowledge is inconsequential in so far as the claim being made here: that is, none of these representations explicitly acknowledge the role of contextually driven processes in arithmetic problem solving, especially in the form of influences on current problem solving that may derive from solving other problems in the current testing situation. The studies reported in this thesis will explore this kind of contextual influence.

Although there exists virtually no acknowledgement of contextual influences on arithmetic problem solving in the literature, there exists abundant evidence in other areas of psychology which show contextual influences on basic cognitive processes. This literature raises the possibility that such influences may be involved in children's arithmetic problem solving, as well. In the next chapter we shall turn to this literature in order to gain an appreciation of the pervasiveness of contextual effects in information processing tasks.

Chapter 2

CONTEXTUAL EFFECTS IN GENERAL PSYCHOLOGY

The purpose of this chapter is to document context effects in a variety of areas of information processing, some very similar and some very different than arithmetic. The point of this review is that context effects are so widespread across different levels of information processing that one should expect them in arithmetic as well, irrespective of what theory of arithmetic one believes to be true. In many of these areas, the demonstration of context effects has stimulated investigators to examine basic, and often, implicit assumptions being made in their research.

In this review, a context effect will be said to have been demonstrated whenever the processing of a target stimulus is shown to change when some aspect of the surrounding context in which the stimulus is embedded is changed. Research that shows such effects will be reviewed in the areas of basic psychophysical scaling, basic perception, speech perception, general language processing, reading, classification, and memory. It will not be the intent of this review to exhaustively survey all empirical and theoretical aspects of these research domains, but, rather, to discuss in each area the central and basic demonstrations of contextual effects and some theoretical formulations developed to explain these effects.

SEQUENTIAL EFFECTS IN PSYCHOPHYSICAL SCALING TASKS

A starting point for this survey is with what is often considered the

most basic information processing task: the so-called psychophysical scaling tasks. There are a large variety of different type of psychophysical scaling tasks. They have in common the fact that they are intended to yield a quantitative description of how some physical dimension of a stimulus, such as, acoustic wave amplitude or stimulus object weight, is related to some psychological dimension of perception, such as tonal pitch or object heaviness. The methods by which these descriptions are obtained often involve subjects making simple comparative judgements between stimuli which differ to varying degrees along these physical continua or require subjects to attach different numbers along some subjective scale to different stimuli. Although the tasks are really quite simple procedurally, it has been known for sometime that the results are not. In particular, it has been discovered that the numerical scaling given to a particular stimulus along some physical continuum depends not only upon the stimulus' value along that continuum but also, somewhat surprisingly, upon the values of the stimuli which are arranged to precede that stimulus in the sequence of stimulus trials presented to the human subject to scale. These effects have been refered to as "sequential effects" in psychophysical scaling tasks and have been recognized in this literature for a considerable period of time (Ferberger,1920; Garner,1953; Holland & Lockhead,1968; Jesteadt,Luce,& Green,1957; Lockhead & King,1983; McGill,1957; Marshall & Degner,1966; Staddon,King, & Lockhead,1980; Ward,1973,1975; Ward & Lockhead,1970).

Sequential effects are of two forms, typically refered to as "assimilation" and "contrast". Assimilation refers to a tendency for a response on trial n to be positively correlated with the response and stimulus value on trial n-1. This means that a given pair of stimuli are judged more similar when they are presented adjacently in a trial sequence than when they are separated by other stimuli. Thus, the scaling of these stimuli depend upon the way in which the trial

sequence in the scaling task is constructed. Conversely to assimilation, "contrast" refers to a tendency for a response on trial n to be negatively correlated with the response and stimulus value on trial n-2. That is, a given pair of stimuli are judged more dissimilarly when they are separated by a single stimulus than when they are separated by several stimuli. Both of these sequential phenomena operate simultaneously in the scaling of a stimulus in a trial sequence: assimilation to the previously presented stimulus and contrast to the stimulus presented two stimuli before. In this way, the scaling of a stimulus in a psychophysical scaling task is not purely a function of the physical magnitude of the stimulus being currently scaled, illustrating the presence of intertrial dependencies in psychophysical scaling. In the thesis studies to be described below intertrial dependencies in arithmetic problem solving will be demonstrated among children.

In the literature on psychophysical scaling, some theorists, notably Lockhead & King (1983), have interpreted these sequential effects as challenging so-called "direct-scaling" models of psychophysical judgement (e.g. Stevens, 1957). Direct-scaling models assume that the perceived magnitude of a stimulus is solely related to the physical intensity of that stimulus; that is, stimulus intensity is invariantly mapped directly to perceived magnitude. Sequential trial effects pose a challenge to this assumption by demonstrating a much greater degree of variability and trial-context specificity to psychophysical judgements than would be predicted. In the studies reported in this thesis, an analogous argument will be made in relation to children's arithmetic problem solving; that is, intertrial dependencies in arithmetic problem solving will be viewed as challenging models of children's arithmetic which assume exclusive access to internalized arithmetic algorithms or procedural rules.

SEQUENTIAL EFFECTS IN BASIC VISUAL PERCEPTION - PERCEPTUAL SET

Examples of sequential effects in basic information processing have also been demonstrated in the area of visual perception. These effects have been refered to as "perceptual set" effects and have been recognized by investigators in this area for a considerable period of time (cf. Allport,1955; Dashiell,1940,1949; Gibson,1941 for early reviews). A particularly simple demonstration can be found in Bugelski & Alampay (1961). Using an ambiguous figure which could be seen as either a rat or a person's face, they demonstrated that subjects who were asked to view this ambiguous figure following the viewing of an unambiguous figure of a women saw the ambiguous figure as a man. In fact, 73% of his subjects in this condition had a "man" perception. On the other hand, when this same ambiguous figure was shown to another groups of subjects who viewed an unambiguous figure of a cat prior to the target figure, as many as 74% saw the ambiguous target as a cat. Thus, visual perception of this figure was not determined exclusively by its physical characteristics but also by the trial context in which the figure was presented.

Bugelski & Alampay discovered further that the effects of a single context experience was not significantly reversed when followed by an extended alternative context experience. Specifically, they found that subjects receiving the woman context figure followed by the ambiguous target, which was then immediately followed by a context series of 6 different drawings of animals followed by the same ambiguous target again, persisted in seeing the second target as a man even though this target's immediately preceding context should have "set" a cat perception; in fact, 63% persisted in this way. Conversely, of those subjects receiving the animal context first followed by the people context, 67% persisted in seeing the second target as a rat. Thus, Bugelski & Alampay found that recent contextual influences could be overcome by the effects of more remote prior trial contexts. However, they

further found that this effect of prior context, or first interpretation of the stimulus, was short-lived; that is, if a 1/2 hour lecture was interpolated between the context tests, no effect of prior context was observed.

The point of the Bugelsi & Alampay study is that visual perception of figures can be substantially affected by the nature of other figures which are experienced in relatively close temporal contiguity to the critical figure - that is, by the local temporal context for the target figure. This is interesting because the common sense view of visual perception is that it consists of a direct translation of current visual information into object perceptions, through, as an example, a kind of "template-matching" process (Gibson, 1963).

Additional examples of perceptual-set phenomena can be found in the studies of Rock & Aronson (1979) and Bradley & Petry (1977). Their focus was the role that trial contextual factors can play in the perception of so-called "subjective contours". Subjective contours, sometimes also refered to as "illusory contours", are contours or edges to objects which are perceived under visual conditions in which the usual physical cues associated with or defining edges are not present-i.e. abrupt changes in luminance. The purpose of the Rock & Anson study was to ascertain if "sets" to perceive contours could be induced in their subjects by having them perceive a series of pictures, with varying charactistics, prior to exposure to a critical target stimulus. In one condition, the subjects viewed 5 cards each of which involved the perception of alignments of several pictoral stimuli. This was hypothesized to induce a set to perceive lines. Following the presentation of these set-inducing cards the critical target stimulus was presented which was composed of a standard stimulus array (of filled semi-circles) which are well known to lead to the perception of subjective contours. In a control condition no cards were presented prior to the target stimulus. Under the control conditions, only 2 of the 20 subjects saw an illusory contour in the target stimulus; whereas,

under the setting conditions, the same target stimulus was seen as possessing an illusory contour by as many as 10 of the 20 subjects. Thus, trial context was important in determining the perception of subject contours which lead Rock & Anson to argue that visual perception of objects was not a passive process of invariably translating external visual features into object percepions, but was an active process of problem solving in which the problem is to reach an interpretation of the external visual data and the solutions are partly, at least, "suggested" (consciously or unconsciously) by the context in which the visual data are embedded. For this review, the important point of this argument is that it is not possible to explain object perception only in terms of invariable translation relationships between external visual features in the contemporaneous array and perceptual features of an object perception. This is an analogous argument to what will be advanced in the research studies reported in this theis with respect to children's arithmetic problem solving.

PERCEPTUAL RELATIVITY

In perceptual psychology, "perceptual relativity" phenomena can be cited as examples of contextual effects in information processing (cf. Restle & Greeno, 1970, chapter 4 for an excellent review). Perceptual relativity phenomena are examples of contextual effects in a slightly different way than the sequential or trial-context effects discussed up to this point. Relativity effects involve demonstrating changes in the perception of a stimulus feature embedded in a single array as a function of changes in the remaining features of that array. As is sometimes said, the perception of a stimulus feature depends upon the entire "sensory field" present at the time at which the stimulus feature is perceived; that is, the perception of the feature is influenced by the spatial field surrounding the feature.

If we choose to view the field as a "context" for the feature then it can be said that the perception of the feature is context-dependent and, thus, demonstrations of perceptual relativity can be considered further examples of perceptual contextual effects. It is important to notice that in the relativity-effect demonstrations the context is present simultaneously with a target feature; on the other hand, in the sequential or trial-context effects discussed above, the context consists of the trials of stimuli experienced prior to the presentation of the target stimulus.

Generally, it has been understood for some time that the perceived magnitude of a stimulus along some dimension, such as height, width, brightness, colour, or texture is determined not only the actual physical magnitude of the particular stimulus being judged but is also determined by characteristics of the remaining sensory field in which the stimulus is embedded. Several well-known examples will serve to illustrate this principle. Consider the perception of the length of a line embedded in different size squares (Kunnapas,1955). Identical lines are perceived as differing in length depending upon the size of the square frame in which the lines are embedded; as the frame size increases, the perceived line length decreases. In the sense of the term "context" we are using in this discussion, the perception of line length under these conditions is context-dependent. Likewise, consider the well known Muller-Lyer and Ponzo illusions (Gregory, 1973). Here the length of lines are judged either with different orienting "fins" at their ends, or at different positions along a picture of a receding railroad track. In both cases, the perceived length of the embedded line is dependent upon the nature of the stimuli making up the line's immediate sensory surround. Thus, again, perceived line length is context-dependent.

These types of perceptual relativity phenomena are abundant and can be extended to include so-called "simultaneous colour contrast" (Ratliff,1965); "brightness contrast" (Wallach,1963); "subject contour" perception; and Gestalten

whole-part perception, etc. The general point is that within perceptual psychology it is commonly recognized that the perception of stimuli is not "absolute", but is "relative" to the stimulus context in which the stimulus is embedded; and this has been demonstrated by an abundant variety of compelling perceptual phenomena.

PERCEPTUAL THEORIES THAT RECOGNIZE CONTEXTUAL EFFECT

The recognition of contextual effects in perception is, perhaps, no better illustrated than in the theory of Helson (1964), commonly referred to as "adaptation" theory of perception. In this theory, Helson attempt to quantify the contextual principle by proposing that the judged magnitude of a stimulus is a function of the ratio of the physical magnitude of that stimulus, S, to the "adaptation level" of the organism, A, where A represents a so-called "standard" or "reference" to which stimuli are compared. In Helson's theory, A is equal to a weighted geometric mean of the physical magnitudes of all the effective stimuli <u>BOTH PAST AND PRESENT</u>, and, as such, provides the psychological mechanism by which trial-sequencing contextual effects and perceptual relativity contextual effects can occur. A contextual stimuli, either past or present, has its characteristic effect on target perception by contributing quantitatively to the adaptation level of the organism. In the typical multi-trial perceptual scaling experiments discussed above, each trial stimulus may cause changes in the adaptation level, and, by doing so, changes subsequent perceptual judgements. Thus, the same stimulus presented within different trial sequences may be perceived differently. With such a theory, well known effects of trial-contextual variables as stimulus frequency and stimuli range in a trial sequence (Cuddy, Pinni, & Simms, 1973; John,1971; Parducci,Calfee,Marshall, & Davidson,1960; Pollack & Boynton,1963) can be explained.

The clear implication of Helson's adaptation theory of perception is that perception is not an invariable translation or direct mapping of external stimulus to internal perception. Other more recently developed theories of perception that also recognize the role of contextual effects are Parducci's "range-frequency" theory (Parducci,1965,1974) and Anderson's "information integration" theory (Anderson,1974a,1974b,1975).

SUMMARY AND CONCLUSION OF CLASSICAL PERCEPTUAL STUDIES

The above review of the classical perceptual literature included sequential effects in psychophysical scaling, perceptual set effects, and simultaneous perceptual relativity phenomena. They illustrate how important contextual processes are in determining perceptual judgements. These demonstrations are especially important to the claim being made in this literature review on context effects, because perception is often regarded as involving the processing of low-level information through our most basic human information analyzing processes, and, for that reason, is often thought to function in a simple and "direct" manner, that is, as direct translation of current sensory stimulation to perceptual phenomenal experience. Classical perceptual context effects illustrate the limitations of this "direct-translation" view, and highlight the broad extent of contextually-sensitive processing in human psychological functioning. Further, less classical, but perhaps even more robust perceptual demonstrations come from studies of speech perception to which we will turn next.

SPEECH PERCEPTION

Several different types of contextual effects in the perception of speech

have been studied. For our purposes, three will be described: phoneme perceptual contrast effects, phoneme restoration effects, and what will be called general "context-specific" phoneme perception effects.

PHONEME PERCEPTUAL CONTRAST EFFECTS

Studies in so-called "contrast" effects in phoneme perception have been conducted since the early 1960's (see Sawusch, Nusbaum, & Schwab, 1980 for a recent review). These contrast effects illustrate that the categorical perception of phonemes are substantially influenced by the local phoneme environment in which perception of target phonemes occur. Although different methodologies have been used to study phoneme perceptual contrast effects, the results have been consistent in showing that a target phonemic stimulus is perceived as more extreme along various acoustic frequency continua from contextual stimuli that are present than when no such contextual stimuli are present. For example, Repp,Healy, & Crowder (1979) used a discrimination procdure in which subjects were serially presented with two brief phonetic stimuli, separated by either a 300 or 1920 msec interval, which differed systematically along 3 format frequency continua such that they approximated, to varying degrees, the phonemes /i/, /I/, or /e/ (the vowel sounds in beet, bit, and bet, respectively). The subjects task was to identify each stimulus from among the 3 possible phoneme categories. Contrast was demonstrated by finding a target stimulus were more likely to be identified as a member of a lower phoneme category (e.g. /i/ relative to /I/ or /e/) when the other phonetic stimulus making up the discrimination pair (context stimulus) was higher along the phoneme category continuum, and vice versa (e.g. /e/ vs /I/). Thus, the perceptual identification of the target phonetic stimulus depended, to some degree, upon its relative position to the context phonetic stimulus with which it was paired. Under some conditions this effect was

substantial, with a doubling of the probability that a stimulus was labeled with a phonetic category when the context stimulus' relative position to the target was shifted from lower than the target to higher than the target.

Such contrast effects have been demonstrated by a number of researchers using the same methodology (e.g. Eimas,1963;Fry,Abramson,Eimas,& Liberman,1962; Repp,Healy, & Crowder,1979) as well as by those using quite different and more ecologically valid methodologies, including changing format frequency of context sentences and target words with certain embedded vowel phonemes (Broadbent & Ladefoged,1960; Ladefoged & Broadbent,1957). Thus, these types of contrast effects are not circumscribed to laboratory discrimination procedures, but may well operate during the normal perception of continuous speech; in any event, they demonstrate that there is no invariant relationship between physical aspects of the acoustic speech signal and phoneme perception.

INTRAWORD CONTEXT EFFECTS IN PHONEME PERCEPTION

In addition to contrast effects in the perception of isolated phonetic stimuli, contextual effects in speech perception have been known to exist in the identification of phonemes occurring in continous speech as well. Demonstrations were initially provided in research conducted at Haskins laboratories during the late 1950s and early 1960s, summarized in the now classic article by Liberman, Cooper, Shankweiler, & Studdert-Kennedy (1967). They showed that the perception of consonant phonemes in continuous speech, i.e. combined with other consonants and vowels in natural productions of syllables, depended less on the acoustic cues present at the time a consonant percept was heard to occur and more on the types of phonemes preceding and following the consonant percept. For example, in the case of the perception of /d/ in the consonant-vowel (CV) syllables /di/ and /du/, the acoustic

cues present at the beginning of these syllables are vastly different, and in fact, bare virtually no resemblance to each other. If the initial acoustic segments are presented in isolation from the vowel segments which follow them, they are not heard as /d/ consonants at all nor even as similar to one another. The initial acoustic segment in the /di/ syllable (an approximately 75 msec. rise in frequency from 2200 cps to 2600cps in the 2nd formant) sounds like a "rapidly rising whistle or glissando on high pitches" (p. 435); and the initial acoustic segment in the /du/ syllable (an approximately 100 msec. fall from about 1200 to 700 cps.) sounds like a "rapidly falling whistle on low pitches" (p. 435). These data showed that the consonant percept is in part cued in the acoustic stream by acoustic information contained in the acoustic context for that percept.

In addition to the Liberman et al (1967) demonstrations of context-specificity in phoneme perception, Schatz (1954) showed that the same identical acoustic signal (a single burst of noise of 1440 cps.) is heard as /p/ when presented before /i/ and as /k/ when presented before /a/. Also, Fodor, Bever, & Garrett (1967, p. 292) demonstrated that in some dialects of American English the perception of /d/ versus /t/ in, for example, the words rider vs writer or ladder vs latter, is determined not be differences in the acoustic cues present during the consonant segment of the acoustic stream, to the extent that one can be identified, but by the duration of the vowel segment preceding the consonant, i.e. vowel context. Thus, the nature of the preceding context determined consonant perception.

The lack of a one-to-one correspondence between acoustic segment and phoneme perception and the role of context in determining phoneme perception is further highlighted by the studies showing that brief silent intervals can be perceived as a wide variety of phonemes depending upon context (e.g. Bastian,Eimas, & Liberman,1961; Liberman,Harris,Eimas, & Lisker, 1961). Liberman et al showed that a 75 msec. silent interval imposed between the s and l consonant in the word "slit" was

perceived as a p cononant resulting in the perception of the word "split". Bastian et al imposed a silent interval between s and o in "sore" to create the perception of "t" resulting in the perception of the word "store".

SEMANTIC AND SYNTACTIC CONTEXT EFFECTS ON SPEECH PERCEPTION

The studies discussed above illustrate context effects in speech perception when the target phoneme was located within a word or syllable and the context was the remaining acoustical information <u>within</u> the word or syllable. Other speech perceptual research has identified contextual effects with respect to the nature of the speech <u>outside of the word</u> on with-in word phoneme identification - i.e. the extra-word context. These type of context effects broadens our appreciation of the fact that contextually sensitive processes are involved at many levels of language processing, a point further extended in the next section on language comprehension.

That the syntax and semantics comprising a sentence in which a target word is located influences the perception of phonemes within that target word was demonstrated by R.W. Warren in his research of the so-called "phoneme restoration effect" (Obusek & Warren,1973; Warren,1970; Warren & Obusek, 1971; Warren & Warren, 1970). (Apparently, however, this effect was orginally demonstrated by William Chandler Bagley - cf. Cole & Rudnicky,1983). In these studies critical consonants are deleted from words and these incomplete words are then presented in isolation or in sentence contexts for identification. Differences in the perception of the same incomplete word presented in these different contexts is taken as evidence of contextual influences in word perception. The result typically observed is that these incomplete target words are often heard as the words which would be appropriate for the sentence context in which they are embedded. For example, Warren & Warren

(1970) presented the word "*eel", where the asterisk indicates a missing consonant, in four different sentences that were made different by changing only the last word in the sentence (e.g. It was found that the *eel was on the ____ (axle,shoe,table,orange)). The identical silent interval before "eel" was heard as either wh, h, m, or p depending upon the type of word appearing at the end of the sentence. Thus, phoneme perception depends on the nature of the context.

In a slightly different demonstration of "phoneme restoration", Warren (1970) substituted a brief 120 msec. cough sound or tone stimulus in the place of "s" in the word "legislature" which was embedded in the sentence "The state governors met with their respective legi*latures convening in the capital city". Subjeccts were told that a cough or tone would be contained somewhere in the sentence and they were to identify where it occurred. Nonetheless, nineteen of the 20 subjects reported that the sentences were complete and the one subject who reported hearing a missing phoneme identified the wrong one. Thus, our perception of natural speech is heavily determined by our expectations about what words should exist in that speech.

A variant of the phoneme restoration demonstrations which also shows the capacity of extra-word context to influence word perception are found in the so-called "mispronounciation-detection" studies (Cole,1973; Cole & Jakimik,1979; Marslen-Wilson & Welsh,1978). In this task, subjects are presented with connected spoken discourse and they are asked to indicate as quickly as possible whenever they hear a mispronounced word. As Cole (1973) originally demonstrated, the quickness with which a mispronounciation is detected is a function of the syllable position in the word and the number of acoustic features changed to produce the mispronounciation: specifically, slower for initial syllable mispronouciation and fewer number of feature changes (e.g. p to b vs p to m). However, as demonstrated by Marslen-Wilson & Welsh (1978), the degree of context-based predictability of the word also significantly influences the probability that a mispronounciation would be

detected. For example, one-feature mispronounciations in the initial syllable of target words were detected only 52% of the time when the target word was highly predicted by its preceding sentence context, but was detected 65% of the time when not well predicted by preceding context. Thus, the semantic and syntactic information provided by the sentence context greatly influenced the accuracy of word perception, demonstrating that in-context speech perception is at least partially contextually-dependent or, as is sometimes said, "contextually-driven".

Similar effects of sentence context have been shown using a slightly different methodology from the mispronounciation detection task. In so-called "phoneme monitoring tasks" subjects are required to detect as quickly as possible the occurrence of certain target phonemes in connected speech discourse (e.g. Foss, 1969, 1970; Foss & Jenkins, 1973; Foss & Lynch, 1969; Foss & Swinney, 1973; Morton & Long, 1976; Sarin & Bever, 1970). Of particular relevance here is the Morton & Long (1976) finding that the reaction time to target phonemes was shorter when the phoneme was in a word that was highly predictable from the preceding sentence context than when it was not as predictable. Thus, predictability influences ease of perceptual identification. Notice that predictability has different affects on phoneme perception in these phoneme monitoring tasks than in the mispronounciation detection tasks: in the monitoring tasks, predictability increases perceptual identification; in the mispronounciation detection, predictability decreased phoneme identification. These types of interactions between the effects of context and type of task are not unusual and highlight the complexity of modeling speech perception with simple single-parameter models, particularly those that emphasize invariant translation of external acoustic data to phoneme perceptions (speech sounds).

The facilitative role of sentence context on phoneme perception has also been demonstrated using one's ability to repeat words that are heard as the dependent measure (Miller, Heize, & Lichten,1951; Miller & Isard,1963; Pollack & Pickett,1964;

Rosenberg & Jarvella,1970). In the classic study of this type, Miller et al (1951) showed that the intelligibility of words, as measured by accuracy of repeating words that were heard, was greater when the words were presented in grammatical sentences than when scrambled and presented as isolated items on a list. In further attempts to assess the relative importance of semantic and syntactic contributions to this effect, Miller & Isard (1963) compared intelligibility of different types of sentences, and found that grammatical sentences were more intelligible than grammatical but meaningless sentences (e.g. Gadgets kill passengers from the eyes) which were, in turn, more intelligible than ungrammatical sentences which, of course, were also meaningless (e.g. Between gadgets highways passengers the steal). This finding suggests that both syntactic and semantic knowledge are made use of in the facilitation of word perception in sentence contexts. However, interestingly, sentence context may also provide additional auditory information about characteristics of the speaker's speech which can aid in word perception (e.g. voice pitch, vocal intonation, voice stress, voice quality, etc.). Pollack & Pickett (1964) have shown that identification of target words was more accurate when more subsequent sentence words were spoken after the targets even under conditions in which the subjects were told in advance what the subsequent context words would be, i.e. when the semantic information in the context was controlled. They interpreted these results as indicating that sentence context provides an "auditory context" for word perception in the form of information about the speaker's style of speech.

These type of studies indicate that continuous speech perception and word recognition are not totally contextually-independent processes engaged or driven only by the immediate sensory acoustic data present, but rather are contextually-dependent in that information provided by the contextual "surround", in the form of acoustic, semantic, and syntactic information, influences the perception of the acoustic stimuli which are embedded in that context. The studies discussed above illustrate

these contextual influences in isolated phoneme perception and the perception of phonemes embedded in words and sentences.

Because of these contextual studies in speech perception, theoretical models of speech perception have needed to incorporate processes that are somehow contextually sensitive. For example, Morton (1969,1970) has proposed the existence of hypothetical "logogens" in the mental lexicon which contain information about the various perceptual defining characteristics of a word and its meaning along semantic dimensions. During on-line speech perception the logogen "counts" perceptual information which matches the defining set and semantic/syntactic information provided by the context in order to identify a word. When the count reaches a critical threshold value, the response associated with that logogen is made available. In the logogen model the role of context is clearly recognized as an independent contributor to the perception and recognition of spoken words from data provided by sensory analyzers.

Another example of modeling of contextual effects in speech perception is the well-known HEARSAY computer model (Reddy & Newell, 1974). Designed for the understanding of speech in a highly restricted domain of discourse, i.e moves in a game of chess, HEARSAY uses acoustic data, and acoustic-phonetic, syntactic and semantic information knowledge sources pertaining to chess-game utterances. In a process partly based on contextual information, "knowledge sources" hypothesize about the likelihood that acoustic data are the results of utterances of certain words and reject hypotheses based on the specialized knowledge. Eventually, the best hypothesis emerges and becomes the "perceive and understood" speech. By this model, speech perception is essentially hypothesis testing and problem solving. In effect, the knowledge sources <u>interact</u> to determine speech perception, in contrast to systems that attempt to identify speech with only the acoustic data and acoustic-phonetic knowledge. Using the terminology of information processing theory,

it can be said that the later types of systems are referred to as "bottom-up" systems, in the sense that eventual understanding of a speech message is attained only after lower-level acoustic analyzers achieve a fully resolved identification of a word and is passed on to the semantic and syntactic analyzers and knowledge sources for eventual abstraction of meaning. In the interactive HEARSAY system and others based on it, higher-level ("top") analyzers and knowledge sources can influence lower level processing prior to word identification by restricting the types of acoustic information needed to be analyzed for speech recognition: specifically, only the acoustic data that differentiates between various syntactically and semantically based hypothesized words. The HEARSAY type system has been extremely influential in the contempory modeling of speech perception and other information processing tasks; for example, it has been adapted to explain recognizing and understanding language presented visually, i.e. reading (Rumelhart, 1977). In all of the HEARSAY models, the effect of sentence context on word perception is represented in the effect that synactic and semantic knowledge sources have on information processing of acoustic data by generating and rejecting hypotheses about the words presented.

A final example of how contemporary modeling of speech perception has incorporated processes that are contextually sensitive is the recent work by McClelland and Rumelhart (1981; Rumelhart & Mclelland, 1982). They refer to their model as an "interactive activation" model which has been developed for visual word recognition and speech recognition. They hypothesize multiple levels of information processing including the acoustic-feature level, phoneme level, and word level. The levels contain nodes or units corresponding to each possible element at that level, e.g. possible words at the word level, possible phonemes for each phoneme position within a word, or possible acoustic features at each phoneme position. The model has been developed to explain intraword context effects in letter and phoneme perception but theorectically can be extended to account for extraword context effects as well.

Each node at all levels has an activation level, and excitatory and inhibitory connections exist between nodes within and between adjacent levels. Activation of a node excites all other nodes with which it is consistent (e.g. the "t" letter node is consistent with the "the" word node and vice versa) and inhibits those with which it is not consistent (e.g. the "t" phoneme node in 1st position is inconsistent with the "our" word node). Activation at a node decays across time according to this model, but the interaction of excitation and inhibition between nodes will lead to a stabilized state of activation among subsets of nodes eventually. The eventually stabilized pattern of activated nodes determines the percept experienced. In this model, context effects in speech perception are represented in the excitatory and inhibitory connections between adjacent levels, where, for example, a word node excited by higher syntactic and semantic nodes can excite certain phoneme nodes at certain positions and inhibit others, and in turn these excited phonemes can excite certain acoustic features an inhibit others. In the case of speech perception, as pointed out by Rumelhart & McMlelland (1982, p. 92), because of how phonemes appear to be encoded in the acoustic stream, informaton from the input that excites certain phonemes would be spread over the nominal locations of several successive phonemic segments in that input.

This concludes are brief review of contextual effects in perception of phonemes and words, and theoretical models which have attempted to explain such effects. Although the models differ in their postulated specific mechanisms, they have in common a critical attribute for our discussion; that is, they incorporate some mechanism(s) to account for contextual influences in the processing of spoken language. Their existence in the literature lends credence to the possibility that contextual processes are operative in children's arithmetic information processing as well. Before leaving the information processing of speech and considering contextual studies of other information processing activities, we shall turn to some research in

language comprehension and reading.

LANGUAGE COMPREHENSION

Up to this point we have considered the processing of spoken language in terms of perceiving phonemes and words, and not yet in terms of more complete analyzes of language at the level of word meaning and sentence meaning. The extraction of word and sentence meaning, or the "comprehension" of language, is the eventual goal of language processing. At this level, studies of contextual effects have also been conducted, often times using as a demonstrative method the disambiguation of word and sentence meaning by surrounding language context. We shall look briefly at these so-called lexical and syntactic ambiguity studies.

The role context plays in word identification is clearly demonstrated in our comprehension of ambiguous words (e.g. Lackner, & Garrett,1972; Simpson, 1981; Swinney,1979; Swinney & Hakes,1976; Tanenhaus,Leisman, & Seidenberg,1979). These so-called "lexical ambiguity" studies have shown that words with more than one possible meaning are processed according to one or another meaning as a result of contextual cues present in the linguistic discourse surrounding the ambiguous word. For example, the word "bug" can be interpreted as refering to an insect or a survelliance device depending upon the sentence in which the word is embedded. The sentence can be said to provide the context in which the word is understood, and the word can be seen as ambiguous outside of any sentence context to indicate its meaning. In this sense, the meaning of the word "bug" can be said to depend upon the context in which its embedded. Swinney (1979) has shown that in the sentence "The man was not surprised when he found several BUGS in the corner of his room", "bugs" was interpreted as a survelliance device given the story in which this sentence was embedded.

In addition to the importance of context in disambiguating the meaning of words and facilitating access to those meanings, there is evidence for its role in disambiguating the syntax of syntactically ambiguous phrases, e.g. "challenging teachers can be threatening" (Rumelhart,1977). The phrase "challenging teachers" can be interpreted as either teachers who are challenging or as the act of challenging teachers; that is, the syntax of this sentence is ambiguous in that "challenging" can be an action verb or adjective. In comprehending a story in which this ambiguous sentence is embedded, the individual may use the prevailing story to determine which of these two possibilities will be processed; that is, the semantic story context influences the syntactic processing of this ambiguous sentence.

Tyler & Marslen-Wilson (1977) have shown how ambiguous phrases, such as the one described above, are interpreted in different semantic context. For example, consider the following two sentences: (1) "If you walk too near the runway, landing planes..." and (2) " If you've been trained as a pilot, landing planes...". "Landing planes" is ambiguous as "challenging teachers" was in the previous example. Tyler & Marslen-Wilson determined the extent to which these ambiguous phrases were processed consistently with a biasing syntactic context by presenting either of these sentences followed by a word appropriate or inappropriate to the interpretations. For example, in this case the next word presented was either "is" or "are"; "is" was appropriate to the second sentence given above and inappropriate to the first and vice versa for the word "are". The critical dependent measure was the latency to repeat the probe word. They found that repetition latencies were longer for the inappropriate than appropriate probes indicating that, indeed, syntactic and semantic context had syntactically disambiguated the target phrases. Thus, these studies have provided evidence of contextual effects in the processing of word and sentence meaning.

READING

There are several demonstrations of contextual effects in the research literature on reading processes that parallel effects in the literature on processing spoken language. For example, it can be shown that many handwritten cursive letters are perceived as different letters depending upon the word in which the handwritten letter is embedded (Rumelhart,1977). Such a letter-recognition context effect was shown by Bruner & Minturn (1955) in a classic perceptual-set study in which an imperfectly written block-letter target (e.g. B) was seen as "B" when in a context of letters and as a "13" when in a context of numerals.

In addition, it is known that the speed with which a letter is identified is enhanced when the letter is embedded in a word as compared to when presented in a random sequence of letters - the so-called "Word Superiority Effect" (e.g. Baron,1978). Thus, as has been discussed in speech perception, letter perception is influenced by contextual factors.

Evidence for contextual effects in reading can also be found at the word level where it has been repeated demonstrated that target words are read faster when they are preceded by an appropriate sentence context as compared to when presented alone, e.g. the word "ball" preceded by the sentence "the boy hit the----" (Perfetti, Goldman & Hogaboam, 1979; Schvaneveldt, Ackerman, & Semlear, 1977; Stanovich & West, 1979, 1980; Tulving & Gold, 1963). Target words are also recognized more slowly when the sentence context is inappropriate - so-called contextual interference effects. Thus, word recognition is at least partly driven by contextual processes. So powerful are such effects that current models of reading explicitly recognized the role of context-sensitive processes by postulating that

reading results from the simultaneous and interactive operation of "top-down" and "bottom-up" processes (e.g. McClelland & Rumelhart, 1982; Rumelhart, 1977; Rumelhart & McClelland, 1981) in an analogous way to how interactive processes of these kinds operate in the comprehension of spoken language discussed above.

Subtle forms of contextual effects in word recognition during reading are also found in so-called "semantic priming" phenomena (Meyer & Schvaneveldt,1971; Meyer,Schvaneveldt,&Ruddy,1975). Here, individuals are presented with words one at a time and they are to decide whether each stimulus is a word or not - the so-called "lexical decision" task; in other versions the subjects are simply asked to read the words aloud. The basic finding is that certain words are read or identified faster if they are immediately preceded by a semantically related word; for example, the word "butter" is read or identified faster if the preceding stimulus word is "bread" than "nurse". Thus visual word recognition processes are facilitated by preceding semantic context. Such effects are quit robust and have been demonstrated when the context item or "prime" , and target are presented in different sensory modalities, as when the prime is a picture or object an the target is a written word (Kroll & Potter,1984; Vanderwart,1984).

Priming-like effects have also recently been demonstrated bases of similarity between preceding context items and target words is orthographic (Kay & Marcel,1981). Specifically, it has shown that the pronounciation of certain target pseudowords (e.g. chave, bint, etc.) is determined by the nature of the pronounciations given to visually similar priming words. For example, the target "bint" will be pronounced as rhyming with "mint" or "pint" depending upon which of the two words has been previously experienced. Priming effects such as these are examples of how word identification processes in reading are influenced by contextual conditions, and, as pointed out by Kay & Marcel (1981), challenge conceptions of

reading which exclusively emphasize the acquisition of formal knowledge about spelling-sound correspondence rules, and support conceptions that recognize a role for so-called "lexical analogies" wherein the reading process is conceived, at least partly, as the result of drawing analogies in pronounciation between similarily appearing words and parts of words. By this conception, context can influence word pronounciation by tacitly suggesting a basis for an analogy. This kind of argument to account for priming effects in reading is very similar to that which will be advanced in connection with the contextual studies of arithmetic to be reported in this thesis.

CLASSIFICATION

The contextual studies reviewed above have concentrated on language processing and perception. Contextual processes have also been recognized in classification tasks, that is, tasks in which the subject is required to classifiy a set of objects or learn a new classification of novel stimuli. Early models of concept formation and classification emphasized algorithm and rule use and the learning of criterial features for concepts as the basis for concept learning and classification performance (e.g. Bruner, Goodnow, & Austin,1956; Hull,1920). These models were bottom-up models and did not incorporate processes that were context-sensitive. A variety of recent data and reanalysis of the structure of natural concepts has called into question such models (Brooks,1978; Medin & Schaffer,1978; Rosch,1973,1975,1978; Rosch & Mervis,1975;Roth & Shoben,1983). With respect to the issue of contextual effects, Labov's (1973) study is considered classic.

In a simple demonstration, Labov asked subjects to classifiy several

cup-like objects of varying size and shape under different contextual conditions. The different contextual conditions were created by asking the subjects to imagine different situations in which the classification task could take place. For example, under various conditions subjects were to classify the objects while imagining that coffee was being drunk from the object, or that the object contained food and was sitting on a table, or that the object was merely in the person's hand empty (neutral context). Labov found that the probability with which many of the different cup-like objects were classified as cups varied substantially depending upon the type of context in which the object was being imaged. Thus, classification was shown to be determined not exclusively by the stimulus features of the to-be-classified object, a prediction of rule-learning, criterial-feature models of classification, but was also determined by the context.

MEMORY

Contextual effects have been given, perhaps, the most research attention in the literature on human memory where two types of contextual influence have been studied: one in which the context serves to actually affect the types of encoding processes on a stimulus and it is shown that remembering that item depends on how much the retrieval context recreates those same processes; and a second type of contextual influence on memory in which stimuli normally considered incidental to processing an item are shown to be encoded in that processing such that future remembering is affected by whether those incidental stimuli are part of the retrieval context.

The classic study by Thomson & Tulving (1970) illustrates the first type

of context effect. They showed that memory for words was greately affected by the degree of correspondence between the linguistic context present during study (cue words in a paired associates task) and the linguistic context present during retrieval (cue words used for cued recall), even when the linguistic context present at retrieval was highly suggestive of the previously studied words (i.e. high associates). For example, the subjects might study the word "cold" in the context of the word "ground" and later, during retrieval, might receive the same cue word "ground" as a retrieval cue or, in an alternative condition, a highly associated cue word for "cold" such as "hot". It was observed that retrieval success was higher when the study cue word was presented than even when the highly associated but different cue word was presented during retrieval. Thus, the original cue word had created a linguistic context which lead to the processing of the target word in a way which was different from the processing encouraged by the different cue word presented during retrieval. The different retrieval cue word failed to retrieve the original word because, evidently, it encourage processing which was different from the original cue word. This study illustrates how contexts promotes different processing of stimuli embedded in those contexts, a finding discussed above with respect to language comprehension, and that remembering depends on recreating the original encoding processing via the retrieval context.

Later studies of this type (Tulving & Thomson,1973) even showed that words experiencing contextually-based processing would not be subsequently recognized when they were generated by the subjects under alternative linguistic contexts, but could be recalled when the original context was again presented. This suggests that recognition is not necessarily "easier" than recall, but that it is the correspondence between contextually-driven processing during study and retrieval which is important in determining the "difficulty" of remembering.

In the second type of contextual effect in memory, the contextual stimuli are incidental to the encoding of the study items. Several demonstrations of these incidental context effects now exist (Godden & Baddley,1975; Greenspoon & Ranyard,1975; Smith,1979). In the Greenspoon & Ranyard (1957) and Smith (1979) studies the incidental contextual stimuli manipulated were the nature of the rooms in which studying and testing took place. Smith asked his subjects to study and learn words in one room. Subsequently, they were led to a distinctively different room in which they were to draw this room from different perspectives. Following this, they were taken to a waiting room for several minutes. Half of the subjects were then taken back to the original study room for a surprise recall test (same-context); the other half were taken to the second room for the same test (different context). The same-context group recalled 25% more words than the different-context group, indicating that the incidental contextual stimuli making up the room during study had been encoded and somehow participated in the original learning experience even though stimuli of these kind are normally regarded as irrelevant to the studying task. And, further, this study shows that the match between the study and test contextual stimuli is important in memory processes.

In an interesting extension of the room study, Smith (1979) showed that subjects could generate their own "context" to aid retrieval by simply remembering the room in which original studying had occurred. By doing this, recall levels were elevated to those found when testing was actually conducted in the original room itself. In aggreement with Smith (1979), Greenspoon & Ranyard (1957) demonstrated effects of room stimuli by showing that the usual negative interference effect of learning a second word list on recall of a different list learned earlier could be greatly reduced when the second list was learned in a distinctively different room. Moreover, recall of the original list was doubled when testing was conducted in the

same room as original learning than when in a different room.

In another interesting demonstration of incidental contextual effects in memory, Godden & Baddley (1975) showed that divers who learned underwater or on land recalled approximately 50% more words if they were tested in the same conditions present during original learning.

In research related to these incidental studies, Bower (1981) has shown that hypnotically-induced moods present during study and retrieval of words can act in much the same way as the room stimuli in the Smith study; and, Eich (1980) has reviewed the vast amount of research done with drug states as incidental retrieval cues much in this same way.

Thus, the literature on memory contains many examples of contextual effects in information processing, and taken in conjunction with the other contextual studies discussed above in the areas of language comprehension, speech perception, reading, classification, basic perception and psychophysics, a strong claim can be made for the tenablity of contextual effects in children's arithmetic learning and problems solving as well.

SUMMARY OF ARITHMETIC AND OTHER LITERATURE

Two points have been made in these first two review chapters of the literature on children's arithmetic learning and on general psychology. First, contemporary models of children's arithmetic learning and problems solving are based on a "Fixed Rule" conception of what children learn. According to these models, rule-following is the way in which children problem solve. The frequently overlooked property of rule-learning and algorithm-following models is that by themselves they do not allow for the possibility of contextual influences. Both psychological and education researchers, as wells as practicing teachers, have overlooked this

possibility.

The second point made in this review is that there is abundant evidence in literatures concerned with other human information processing tasks that show robust and pervasive effects of context and top-down processing.

Following from these two assertions, the thesis is directed at the possibility that children's arithmetic learning and problem solving is influenced by contextual variables. The thesis will show this by demonstrating contextual effects at different grade levels with different types of arithmetic material. The generality of these effects across grade level and material will be taken to indicate the robustness of the phenomena and the necessity to incorporate those phenomenona in current conceptions of children's arithmetic problem solving. Accordingly, an alternative conception of children's arithmetic problem solving will be developed which acknowledges these contextual influences. Specifically, it will be argued that a more useful heuristic is that children solve standard elementary problem forms by drawing analogies (not necessarily consciously) between different solution strategies for different types of problems; and that the basis of these analogies often stem from problem solving episodes experienced in the current context. The thesis studies will be presented by grade level starting with grade-2 level arithmetic material and continuing to grade 5 material. At the same time, several studies will explore, in addition, the contributions that certain current classroom instructional practices make in determining the effect of context on certain common computational errors.

THE RATIONALE OF THE THESIS STUDIES

Prior to beginning a description of the present studies, the basis rationale for the studies will be summarized here.

As discussed in chapter 1, school learning of arithmetic involves learning a complex set of procedural algorithms. For example, children are taught to solve multi-digit subtraction problems by using a procedure that involves a series of different sub-procedural steps. When solving subtraction problems displayed in vertical form children are taught to subtract the bottom number in the units column from the top number in the units column, and then go to the tens column and subtract those two numbers. If the top number in the units column has a value which is less than the bottom number, then a "borrowing" procedure is to be performned which involves the familiar "crossing-out", "subtract-one", and "borrow-one" actions.

Children do have trouble learning these computational algorithms, and much of their problem involves using incorrect procedures, rather than not following any procedure at all. Such erroneous and faithfully followed procedures have been called "systematic" computational errors (Ashlock, 1976;Brown & Burton, 1978; Brown & VanLehn, 1980, 1982; Cox, 1975; Engelhardt, 1977; Friend, 1972; Janke, 1980; Owston, 1981; Roberts, 1968; Young & O'Shea, 1981). In an attempt to explain these systematic errors, theorists have constructed rule descriptions of the basic computational algorithms, and have attempted to describe erroneous computations as resulting from subtle errors in learning these rule descriptions (Brown & Burton,1978; Brown & VanLehn,1980,1982; Young & O'Shea,1980). In chapter 1, these rule descriptions were described as part of a Fixed Rule approach to children's arithmetic learning. One key assumption of these rule descriptions, when used as process models of chidrens' computations, is that the child's problem solving activity is solely guided by critical stimulus features in the display of the problem

to be solved. For example, a vertically displayed multi-digit subtraction problem is assumed to automatically call forth a column subtraction "production system" in which a set of "condition-action" rules guide problem solving. Subtraction actions are performed on units numbers, tens numbers, and so forth, depending on the external conditions present at any particular time during the execution of the production system. The "borrowing" operation is performed only under the condition that the value of the top number in the units column is less than the value of the bottom number in the units column. In this way the problem solving activity used is assumed to be completely guided by specific features of the problem to be currently solved. Which specific internal characteristics of the problems are important and which procedures will be performed depends entirely on the rule structure (i.e. production system) acquired by the problem solver.

However, problem solving activity may not only be determined by characterisitcs of the current problem to be solved, but also by characteristics of problems preceding and subsequent to the current problem. That is, arithmetic problem solving may be affected by contextual variables just as many other cognitive activities (see chapter 2 for review). It will be the purpose of the experiments reported herein to demonstrate that arithmetic problem solving is also context-driven, and therefore, that current models of arithmetic problem solving in children are inadequate.

The following studies will investigate the computational strategies used by children while solving certain arithmetic problems (so-called "target" problems) as a function of the context of problems in which those problems are embedded (so-called "context" problems). Basically, four types of target problems will be studied: 2-digit subtract 1-digit problems presented in vertical form involving only 1-digit correct answers ("2-digit subtraction facts"); 2-digit-by-1-digit addition

and subtraction problems presented in vertical form involving 2-digit-answers; 2-digit-by-2-digit multiplication; 2-digit-by-1-digit division involving remainders (e.g. $7\overline{)29}$). At various grade levels these target problems are solved in various ways, and it will be the purpose of the following context studies to show that which problem solving strategy is performed depends on the characteristics of the problems preceding the target problems, i.e. the context problems. These demonstrations will serve as a basis for a different conception of children's arithmetic problem solving, one based on the notion that children solve problems by analogies between the solution strategies for different but visually similar arithmetic problems and that context serves to suggest possible problem solutions for the analogies. This idea will be more fully discussed in chapters 6-8 and 13.

Chapter 3

STUDY 1: THE ROLE OF CONTEXT IN DETERMINING ARITHMETIC PROBLEM SOLVING OF 2-DIGIT SUBTRACTION FACTS IN GRADE 2 CHILDREN.

The purpose of the first study was demonstrate that grade 2 children will answer vertically displayed, 2-digit-by-1-digit subtraction fact problems differently as a function of the type of problems that make up the context of problems in which those targets are embedded.

The target problems were 2-digit-by-1-digit subtraction problems involving 1-digit answers (e.g. 13-5= ;17-9= ;14-8=). These type of problems are called 2-digit subtraction "facts" since they are taught as facts to be memorized for easy recall. On the test used in this study, these 2-digit subtraction facts were interspersed among either (1) 1-digit-by-1-digit subtraction problems, or (2) 2-digit-by-2-digit subtraction problems involving 2-digit answers. Thus, two different contexts were constructed for the target problems. The rationale for choosing these particular contexts is based on observations of grade 2 children's problem solving.

Observations of grade 2 children solving 2-digit-by-1-digit subtraction facts reveal that they solve these problems in 3 different ways. On some occasions children recall the answer immediately, although this is rare for children at grade 2 who are just beginning to learn these problems. On other occasions, children use one of many different counting strategies; for example, for the problem 13-8= they may start with the number 8 and count up to 13 using their fingers to represent each of the digits from 9 to 13 or count down from 13. And, on still other occasions, grade 2 children will solve the 2-digit subtraction facts by subtracting the units number

of the 2-digit number (e.g. 3 of 13) from the 1-digit number positioned below (e.g. 8 of the problem 13-8=). The result is placed in the units place of the answer. Following this, the child will "drop-down" the one number occupying the tens place of the 2-digit number in the problem. This kind of units-tens-column stragegy leads to an erroneous answer (e.g. 13-8=15 or 14-8=14). Because the subtraction operation performed in the units column of numbers requires that the child subtract the top from the bottom number, instead of the bottom from top, this part of the strategy can be called an "inversion" error. The other part of the erroneous procedure involving "drop-down" of the 1, can be called the "drop-down" error. The composite response can then be called the "drop-down-inversion" error or DI error. Examples are shown in Table 2. The production-system models and procedural network models of children's arithmetic problem solving, discussed above in chapter 1 (e.g. the Bug Theory of Young & O'Shea,1981), would assume that the child's 2-digit subtraction-fact problem solving knowledge consisted of a set of procedural rules; for example: (1) if the problem is a 2-digit subtraction fact then go to the units column; (2) if in the units column subtract the smallest digit from the largest digit and write result in units-place of answer; (3) if finished with the units column, proceed to tens column; and (4) if in tens column and if there is only one number on top, write that number in the tens-place of answer.

To understand the choice of context items in this study it is necessary to compare these 3 strategies for solving the 2-digit subtraction facts: recall answer, counting, and DI procedure. Both the recall answer and counting strategies involve treating the 2-digit subtraction fact problem as if it were an instance of the total class of subtraction facts. The total class of subtraction facts includes 1-digit subtraction facts (e.g. 5-2= ; 9-4=), as well as the 2-digit subtraction facts. The grade 2 arithmetic curriculum involves teaching these types of problems together as

facts to be eventually memorized. As an aid to be used when the facts are not memorized, the counting strategies are explicitly taught. These counting stragtegies are also used to teach the "meaning" of the subtraction operation. Therefore, recall of answer and counting are related in that they both involve treating the 2-digit-by-1-digit subraction fact problem as a subtraction fact.

In contrast, the DI error involves treating the 2-digit subtraction fact as if it were an instance the class of vertically displayed multi-digit subtraction problems: specifically, 2-digit-by-1-digit subtraction problems involving 2-digit answers or the class of 2-digit-by-2-digit subtraction problems. In the grade 2 arithmetic curriculum, these types of problems are taught separately from the 2-digit subtraction facts (which are taught together with 1-digit facts). For the 2-digit subtraction problems with 2-digit answers, the units-tens procedure is taught in which the child must learn to subtract the numbers in the units column for the units answer, and the numbers in the tens column for the tens answer. This approach or problem solving strategy is called the "column-subtraction" procedure. Because the arithmetic curriculum for grade 2 does not include teaching the subtraction "borrowing" or "regrouping" procedure, grade 2 children learning the units-tens procedure are never confronted with 2-digit-by-2-digit subtraction problems in which the top number in the units column has a value less than the bottom number in the units column. The DI error seems to be a result of applying the grade-2 units-tens subtraction procedure to the 2-digit subtraction facts.

If context problem solving activity does influence current arithmetic problem solving, it was hypothesized that which of the two types of problem solving activity would be used for the 2-digit subtraction facts would be determined by the type of problems the child had solved immediately prior to being confronted with these facts. If the grade 2 child had been solving 1-digit subtraction facts prior

TABLE 2
EXAMPLES OF DROP-DOWN-INVERSION ERRORS (DI)

12	14	11
-4	-6	-5
12	12	14

to confronting the 2-digit subtraction fact, he would likely treat the 2-digit subtraction fact as if it were an instance of the general class of subtraction facts, and would solve it as a fact for which the recall-of-answer or counting strategies are used. If, on the other hand, the grade 2 child had been solving 2-digit subtraction problems involving 2-digit answers prior to confronting the 2-digit subtraction fact, he would likely treat the problem as if it were an instance of the general class of 2-digit subtraction problems, and would solve it as a problem for which the units-tens procedure is used. In this later case, the drop-down-inversion error would result.

The following experiment tested grade 2 children for their response to 2-digit subtraction facts embedded in either the context of 1-digit or 2-digit subtraction problems. The view of arithmetic problem solving in children as involving context-driven processes predicts that the target problems should be treated differently in the two contexts. More specifically, it would be predicted that for grade 2 children 2-digit subtraction facts would be answered correctly in the 1-digit context, and incorrectly in the 2-digit context. In the 2-digit context, the DI error should be committed.

METHOD

SUBJECTS

The subjects in this experiment were an entire class of grade 2 students at a local school in the Hamilton Board of Education. Twenty-seven pupils participated in the experiment. Of these 27, 7 were 3rd grade pupils.

MATERIALS

The materials consisted of 2 mimeographed worksheets consisting of 30 subtraction problems each, presented in vertical form, and arranged as a 5 problem X 6 problem rectangle display. Both sheets contained 5, 2-digit-by-1-digit subtraction facts (12-4= ; 14-9= ; 17-9= ; 16-8= ; 15-7=). On each sheet, these target problems occupied the same positions. If one counts the positions from the top left, moving to the right along the rows, the target items occupied positions 5,12,14,23,28. The remaining items on the worksheets were the context items.

On the single-digit worksheet (see Appendix I for test items), the context problems were single-digit subtraction facts selected randomly from the admissible combinations of 0-9 digits. On the double-digit worksheet (see Appendix II), the context problems were made up of 22 2-digit-by-2-digit and 3. 2-digit-by-1-digit subtraction problems with 2-digit answers selected randomly from the combinations of 10-99 double-digits with the constraint that the numbers selected for the top double digit were greater in value than the corresponding numbers in the bottom double digit number. This constraint guaranteed that no 2-digit subtraction problem in the context required "borrowing" or "regrouping" (e.g. 45-19=).

PROCEDURE

All subjects were required to do both worksheets with a random half receiving the single-digit test first and the other half receiving the double-digit test first.

The first worksheet was handed out to the class face down, and the instruction was given to write their name and grade on the back of the worksheet. Also, they were requested to print the number "1" on the back to indicate that this was the first test they were taking.

Following this, they were told that all the problems on the test were

subtraction problems, and that when the signal was given to begin they were to begin with the problem in the top left-hand corner of the test and to do the problems from left-to-right, row-by-row. A test form was held up to the class during this instruction, and the left-to-right, row-by-row, instruction was demonstrated by moving a finger over the problems.

Following this, the class was asked if there were any questions, and following question answering, they were told to begin. The experimenter walked up and down the rows of the classroom to check to make sure that the children were working the problems in the correct order, and to discourage any cheating that might be taking place.

As the individual children finished test #1, they were asked to indicate this by raising their hands and turning their test paper over, at which time the experimenter gave them the appropriate second test, requesting that they write their name, grade, and the number "2" on the back before beginning.

SCORING

Seven different error categories were used to score the responses to the 2-digit subtraction facts problems. These categories were chosen because they succeeded in scoring a very high percentage of the answers given by the children. It should be noted, as well, that these categories are based on a theory of how the children are generating their solutions; a theory derived from observations and interviews of grade 2 children while solving such problems.

1. CORRECT RESPONSE(CR). A correct response was defined as the correct single digit answer to the 5 subtraction fact problems.

2. FACT ERROR (FE). A fact error was defined as a single digit answer which deviated from the correct answer by +1 or -1. (e.g. 12-4=9, 12-4=7). A +1 and -1 deviation was used because observations and interviews suggested that a very large

proportion of such responses are derive through the use of an erroneous counting procedure. Counting procedures are only rarely so erroneous as to derive solutions of greater deviations.

3. DROP-DOWN RESPONSE (D). A drop-down response was defined as a single-digit answer of 1 which corresponds to the tens value of all of the 2-digit subtraction fact problems. (e.g. 12-4=1, 16-8=1).

4. INVERSION ERROR (I). An inversion error is defined as a single digit response equal in value to the difference between the units value of the 2-digit number and the 1-digit number making up the 2-digit subtraction fact. (e.g. 12-4=2, 14-8=4).

5. INVERSION FACT ERROR (IFE). An inversion fact error is defined as a single digit response equal in value to +1 or -1 of an inversion error, but not including the number 1 which was scored as a drop-down error. (e.g. 12-4=3, 16-8=3).

6. DROP-DOWN-INVERSION ERROR (DI). A 2-digit answer involving an inversion error as the units value and the number 1 as the tens value. (e.g. 12-4=12).

7. DROP-DOWN-CORRECT-RESPONSE (DCR). A 2-digit answer involving a correct response as the units value and the number 1 as the tens value. (e.g. 12-4=18).

RESULTS

Of the total possible number of responses to the 2-digit subtraction facts on the two worksheets, 47% of the responses were correct responses, 22.2% were DI errors, 7.4% were FE, 5.2% and 4.4% were D and I responses, respectively, and 2.6% and 2.2% were IFE and DCR responses. The remaining 9% consisted of responses not falling in any of the above categories, including single-digit answers deviating more than 1 digit from the correct answer and not classifiable as an inversion error (e.g. 17-9=4, 15-7=5); double-digit answers not classifiable as DI or DCR responses (e.g. 17-9=16, 12-4=10); and wrong operation errors (12-4=16, 14-8=22, 15-7=22). These unclassified miscellaneous responses were infrequent.

Because the hypothesis being tested in this study concerns whether the single-digit vs double-digit problem context can change the response to 2-digit subtraction facts from single-digit subtraction fact answers to double-digit answers, several response categories were grouped together to yield measures of these basic response types. CR and FE responses involve the single-digit subtraction fact procedure, whereas, DI and DCR involve double-digit answers and the use of a column-subtraction procedure. D, I, and IFE involve single digit answers resulting from only one or the other part of the 2-place procedure. Thus, 3 responses categories were used to present the data: CR and FE responses (single-digit answer subtraction-fact strategy); DI and DCR responses (double-digit answer column-subtraction strategy), and D,I,IFE responses (miscellaneous responses).

The results of the study in terms of the means, medians, and variances for the 3 response categories for the 2 groups under the 2 context conditions are displayed in Table 3. In addition, the percentage of subjects making at least one response for each type of response is given (n(rsp)). Because the response distributions were very skewed, an analysis of variance was considered inappropriate for the data, and statistical significance was determined by nonparametric tests.

WITHIN-GROUP CONTEXT EFFECT

Consider first the effect of change in problem context in the group receiving the 1-digit context first followed by the 2-digit context (Group 1-2). The median number of combined CR and FE responses to the 5, 2-digit subtraction fact problems embedded in the 1-digit context was 4.0 or 80%. When these same subtraction fact problems are embedded in a 2-digit problem context, the median number of CR and FE responses drops to 2.0 or 40% (Wilcoxon signed ranked test, 1-tailed, $p<.02$). Thus, changing from a 1-digit to 2-digit context significantly reduced single-digit subtraction-fact strategies.

TABLE 3
Response distributions to 2-by-1 subtraction facts as a function of context condition for groups 1-2 and 2-1 in Study 1

		CONTEXT					
		1-DIGIT			2-DIGIT		
GROUP	n	CR,FE	D,I,IFE	DI,DCR	CR,FE	D,I,IFE	DI,DCR
1-2	16						
MEAN		3.8	0.81	0.0	2.3	0.81	1.4
MEDIAN		4.0	0.0	0.0	2.0	0.0	0.0
S.D.		1.28	1.22	0.0	2.41	1.56	2.19
n(rsp)		16	7	0	8	5	6
2-1	11						
MEAN		2.6	0.0	1.8	1.8	0.6	2.2
MEDIAN		2.5	0.0	0.0	0.0	0.0	1.5
S.D		2.38	0.0	2.5	2.36	1.43	2.18
(rsp)		7	0	4	5	2	7

n(rsp)= number of subjects making at least one response of each type.
see text for definitions of response types.

CR=correct response
FE=fact error
D= drop-down error
I= inversion error
IFE=inversion fact error
DI=drop-down-inversion error
DCR=drop-down-correct-response-error

In contrast to the decline in single-digit response strategies, there was an increase in 2-place column-subtraction response strategies, i.e. DI, and DCR. Two-place responses were distributed bimodally with the modes at the extremes of the scale, i.e. 0 and 5. Therefore, neither means or medians were considered appropriate descriptive statistics. Instead, the percentage of subjects making at least one response of a particular type was used as a dependent variable. Figure 1 displays the percentage of subjects in each group under each contextual condition making 2-place, column-subtraction errors to the targets. In the single-digit context, none of the 16 subjects made DI,DCR responses, whereas, in the double-digit context, 6 of the 16 subjects or 37.5% made at least one DI,DCR response to the target problems (Wilcoxon signed rank test, 1-tailed, $p<.02$).

BETWEEN-GROUP CONTEXT EFFECT

The effect of context on 2-digit subtraction-fact problem solving can be further tested by comparing the responses between groups. The median number of CR,FE responses for group 1-2 in the single-digit context was 4.0 (80%), as compared to 0.0 CR,FE responses for group 2-1 in the double-digit context. All 16 of group 1-2 subjects made at least one CR,FE response, as compared to 5 of 11 or 45.5% of the subjects in group 2-1 (Fisher exact probability, 1-tailed, $p<.001$). (In all the statistical tests of percent differences, Siegel's, 1956 guidelines were followed especially in relation to choices of Chi-square or Fisher exact probability tests, (see p. 104).

In addition to the hypothesized decrease in CR,FE responses between groups, there was a significant increase in the use of the 2-place column-subtraction response strategies. No group 1-2 subject made a DI,DCR error, whereas 7 of the 11 group 2-1 subjects, or 63.6%, made at least one of these errors (Fischer exact probability, 1-tailed, $p<.005$).

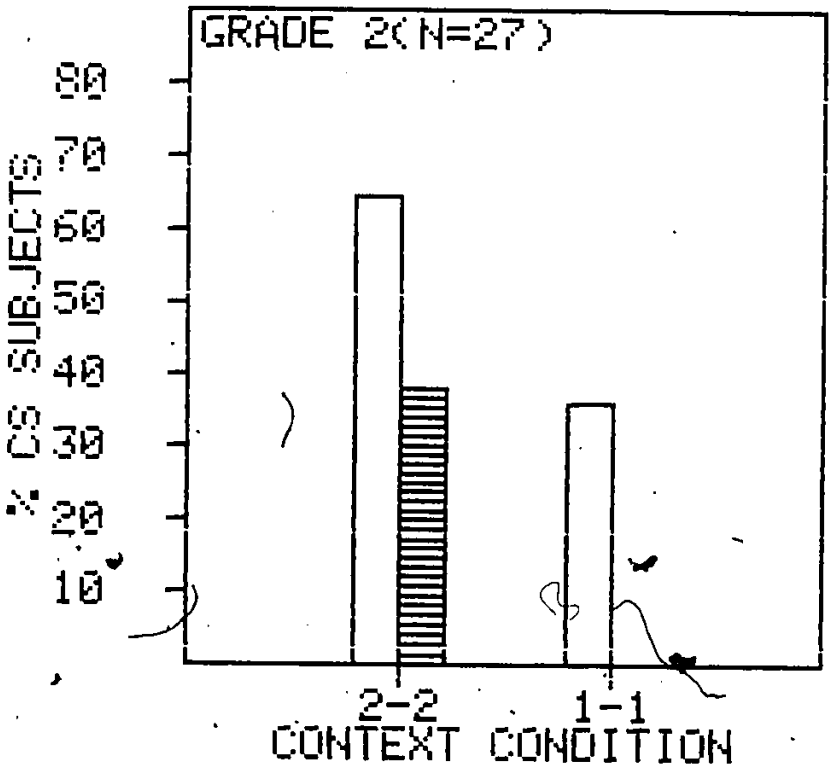

Figure 1. Percentage of subjects committing column-subtraction errors (DI,DCR) in single-digit (1-1) and double-digit (2-2) context tests for Group 2-1 (open rectangles) and Group 1-2 (hatched rectangles).

Context carry-over effect. Somewhat unexpectedly, there appeared to be evidence for an interaction between context changes and the order in which the tests were given. Group 1-2 showed a change in CR,FE and DI,DCR responding as a result of context change; however, no evidence could be found that for Group 2-1 there was such a change. Neither single-digit nor 2-place response strategies were different for this group 2-1 between the two context conditions (Wilcoxon signed rank tests). Thus, the possibility of different types of carry-over from one test to the other was considered.

Carry-over effects can be demonstrated by comparing the two groups within a specific context condition. For example, a comparison of DI,DCR responding among group 1-2 and 2-1 subjects under the single-digit context condition reveals the effects of the prior double-digit context test (group 2-1).

In fact, when such analyzes were performed, there was evidence for a carrying-over of DI,DCR responding from the the double-digit context to the single-digit context causing higher rates of DI,DCR for group 2-1 relative to group 1-2 in the single-digit context; but no group differences were observed in the double-digit context. The carry-over from double- to single-digit context worksheets was indicated by the fact that in the single digit context 4 of 11 Group 2-1 subjects or 36.4% made at least one DI,DCR response in comparison to 0 of 16 Group 1-2 subjects or 0% (Fisher exact probability, 2-tailed, $p<.05$). However, in the double-digit context, Groups 1-2 and 2-1 made 37.5% (6 of 16) and 63.6% (7 of 11) DI,DCR responses which was nonsignificant.

Thus, changing the context of items in which a 2-digit subtraction fact problem is embedded from 1-digit subtraction facts to 2-digit subtraction problems caused an increase in DI,DCR responses, both between and within subjects, but only if the single-digit context is experienced before the double-digit context. When the

double-digit context is experienced first, there is a substantial carry-over of DI,DCR responding which results in nonsignificant differences in responding between contexts, and significantly higher DI,DCR responding in the subsequent single-digit situation.

DISCUSSION

The results of this study support the hypothesis that arithmetic problem solving in grade 2 children is not determined solely by the internal characteristics of a problem to be solved, but, also, is determined by characteristics of problems embedded in the context within which a problem is to be solved. In this study, 2nd grade children solved 2-digit subtraction fact problems as single-digit subtraction facts using recall-of-answer or counting strategies which resulted in answers that were correct or within one digit of the correct answer, but only when those problems were presented to the children along with other single-digit subtraction fact problems. If those other problems were predominantly 2-digit-by-2-digit subtraction problems, the children solved the 2-digit subtraction fact problems as double-digit subtraction problems using a units-tens, column subtraction procedure or 2-place-answer procedure which resulted in answers which were incorrect in systematic ways. Thus, 2nd grade subtraction fact problem solving is influenced by context variables, and is, at least partly, context-driven.

These data not only question the adequacy of most current models of arithmetic problem solving (Brown & Burton, 1978; Brown & VanLehn,1980,1982; Young & O'Shea,1980), but also raise important questions about the adequacy of several models used to development psychometric tests of children's arithmetic knowledge (e.g. Item Response Theory, Lord, 1980, discussed in chapter 1). These test models use two parameters to desribe the arithmetic performance of children. The first is an

arithmetic aptitude parameter which is intended to be estimated from the performance of individuals on certain arithmetic items. The second parameter is a item difficulty parameter which is assigned to arithmetic items based on the performance of preselected groups of children who are presumed to be at various levels of arithmetic knowledge and aptitude. If a particular item is passed by 50% of the 2nd grade population then it is assigned an item difficulty value reflecting that success percentage, and this becomes an invariant characertistic of that item. In the study reported here, an identical sample of 2-digit-by-1-digit subtraction fact problems were either answered correctly with a mean percentage of 57.5 or 34.5 depending upon the problem context in which they were embedded. Obviously, an item does not possess an invariant item difficulty value that can be assigned to it independently from the other items making up the test. One particular diagnostic test, the Key Math Diagnostic Test (Connolly, Nachtman, & Pritchett, 1971), positions a 2-digit subtraction fact immediately after a 2-by-2 digit subtraction problem which in turn is preceded by 2 single-digit subtraction facts. The data reported in this study suggest that this 2-digit subtraction fact would be responded to very differently if its position were exchanged with the 2-digit subtraction problem preceding it.

In connection with this psychometric model is the assessment issue of what constitutes evidence of proficiency at solving a particular arithmetic problem. In Group 1-2 of this study, 37.5% of the subjects passed all 5 target items correctly when they were embedded in the single digit context, but failed all of them when presented in the double digit context. Which test result is indicative of their "true knowledge"? In spite of claims made by empirical test constructors, one can see that this issue of assessment is impossible to resolve without a model of childrens' arithmetic knowledge and performance.

The finding of a carry-over of erroneous response strategies from the

double digit to single digit context in Group 2-1, suggests that it is not only the current context of problems that determines arithmetic problem solving, but also the "recent" past's context that can be important. This carry-over was not present for both response strategies equally. That is, there was no evidence of higher rates of correct responding and fact errors or lower rates of DI,DCR errors in the double digit context for Group 1-2 as compared to Group 2-1, as there was evidence for higher rates of drop-down-inversion (DI) and drop-down-correct-response (DCR) errors and lower rates of CR and FE responses in the single digit context for Group 2-1 as compared to Group 1-2. Thus, there were differential carry-over effects for these response strategies.

The performance of one subject in this study suggested that the effect of these contexts on 2-digit subtraction fact problem solving may be different as a function of arithmetic level. One subject in Group 1-2, made 4 of 5 target responses as correct responses or fact errors in the single digit context. Subsequently, on the double digit context test, this subject, made 5 of 5 target responses as correct responses but solved the target problems using a different technique than any of the other 26 pupils. She engaged in a so-called "borrowing" or "regrouping" operation prior to reaching the solution. That is, she placed a slash mark through the "1" of the 2-digit number, printed the number "0" over the slashed "1", and printed a "1" near the units number of the 2-digit number, thus engaging in an "unnecessary" borrowing operation which did not result in a different problem from the one originally present. The double-digit context had apparently induced her to respond to the 2-digit subtraction fact problem as if it were a 2-digit subtraction problem, much in the same way it did in many of the other grade 2 children, but part of her 2-digit- subtraction procedural knowledge included the borrowing operation. Thus, she borrowed and answered correctly, instead of committing the drop-down-inversion or

drop-down-correct-response error.

This case is worthy of note in that this child was one of the 3rd graders in the sample, and therefore, suggests that a context effect for the 2-digit subtraction fact problems may also exist at higher levels of arithmetic knowledge, but may involve other response strategies not yet used at grade 2. The developmental course of this particular context effect was explored in the next study.

Chapter 4

STUDY 2: THE ROLE OF CONTEXT IN DETERMINING UNNECESSARY BORROWING TO 2-BY-1 DIGIT SUBTRACTION FACTS.

The previous study demonstrated that 2-digit-by-1-digit subtraction fact problem solving in grade 2 children is influenced by problem context, as indicated by the higher incidence of drop-down-inversion and drop-down-correct response errors in the 2-digit subtraction context than in the 1-digit subtraction context. Thus, these type of errors on these problems were context determined. Study 2 explores the possibility that other problem solving activities in subtraction are context determined. Specifically, the hypothesis is tested that the so-called "unnecessary borrowing" operation to 2-digit subtraction fact problems, which was observed in one child in the first study, is influenced by the same context as the DI and DCR errors. Such a finding would challenge directly the assumption made by current production system models of children's subtraction problem solving (Brown & Burton, 1978; Brown & VanLehn, 1980, 1982; Young & O'Shea, 1980) that the borrowing operation is determined solely by internal characteristics of subtraction problems - a vertical column of numbers in which the top number is less in numerical value than the bottom number, and would extend the observations made at grade 2 to grade 3 skill levels.

The subjects in this study were an entire class of grade 2 pupils who had been given instruction in the borrowing procedure. Normally, the borrowing operation is not taught until the 3rd grade, but the particular teacher of this class taught borrowing in addition to the grade 2 curriculum objective of 2-digit subtraction without borrowing. Such a class was thought to be ideal for this study because the children would be about the same age as the children in the previous study- i.e.

grade 2 age, but would have been taught the borrowing procedure in subtraction.

This grade 2 class was tested under identical conditions to the previous study - single-digit vs double-digit subtraction problem contexts. The procedure was a replication of the procedure used in the previous study: i.e counterbalanced order of presenting the two context worksheets among two groups. It was hypothesized that unnecessary borrowing to 2-by-1 digit subtraction facts would be more likely to occur in the double-digit subtraction context than in the single-digit context.

METHOD

SUBJECTS

The subjects were an entire class of grade 2 students in a local Hamilton pulic school. Seventeen pupils participated.

MATERIALS

Identical to study 1 (see Appendix I and II).

PROCEDURE

Identical to study 1.

SCORING

Same as study 1, except an extra category was added, making 8 response categories in all.

8. <u>UNNECESSARY BORROWING (UB)</u>: A problem with the familiar "slash" over the ten's number, and/or a 1 next to the units number. Usually, the slash and 1 are accompanied by a zero written above the slash.

RESULTS

Of the 17 subjects, 9 were in Group 1-2, 8 in Group 2-1. The worksheets

for these 17 subjects provided the data for this study.

Of the 170 total possible 2-digit subtraction fact responses made on the worksheets, 46 (27.0%) were correct responses not involving UB, 79 (46.5%) were UB responses, and 16 (8.1%) were DI. Only 5 (2.9%), 2 (1.2%), and 2 (1.2%) were DCR, I, and FE responses respectively. Twenty (11.8%) were not classifiable using the existing error categories, but 17 of these 20 were found to be unusual variants of the column-subtraction strategy (see footnote 2).

As with the previous study, the response distributions were non-normal, tending toward bimodality at the more extreme scores. Therefore, nonparametric statistics were used. Figure 2 shows the results of this study in terms of the percentage of subjects making unnecessary borrowing responses (UB) under the two context conditions for the two groups: those receiving the single-digit context first (1-2) and those receiving the double-digit context first (2-1).

In agreement with the previous study, a comparison between the groups on their first worksheets (1-digit vs 2-digit) revealed a large between-groups context effect. Of Group 1-2 subjects, 6 of 9 made CR or FE responses without unnecessary borrowing; no Group 2-1 subjects made any such responses. Because of the low number of subjects and the low expected frequencies in some of the cells in the contingency table, Fisher Exact probabilities were calculated, following Siegel's (1956) recommendations. This between-group comparison was significant (1-tailed, $p<.01$). Two of the remaining 3 Group 1-2 subjects made UB responses in the single-digit worksheet, and one of these subjects also made DI responses on 2 of the 5 targets. The third subject made an unusual error not seen in study 1, which will be called a "0I" error, to be discussed below. Among the 8 Group 2-1 subjects, 6 made UBs, 1 made DI responses and 1 made a "00" error (see footnote 2). Comparison of UB-making subjects between groups was significant (Fisher, 1-tailed, $p<.05$). Thus, there was

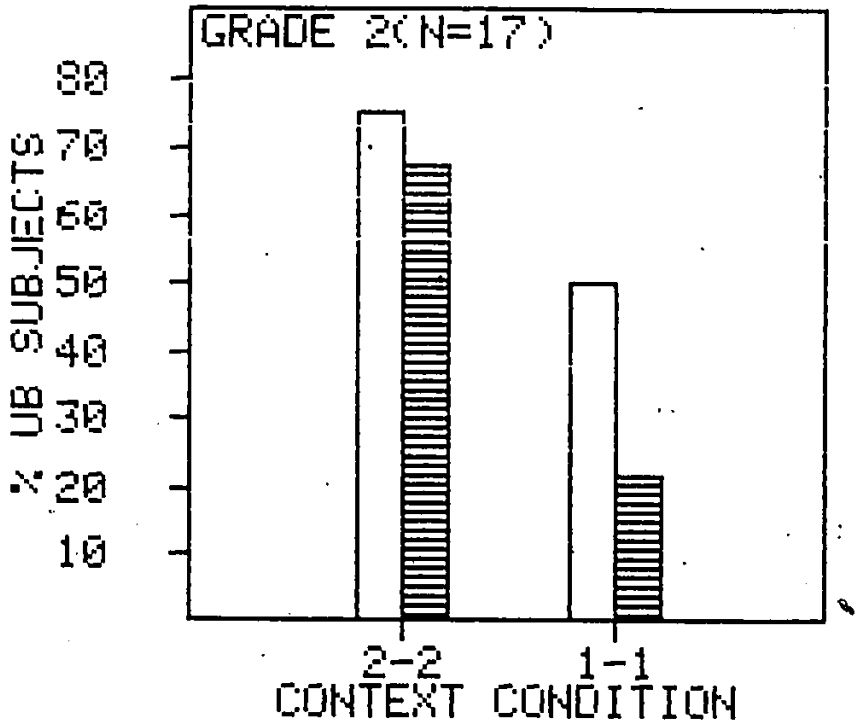

Figure 2. Percentage of subjects committing unnecessary-borrowing (UB) to targets in single-digit (1-1) and double-digit (2-2) contexts for Group 2-1 (open rectangles) and Group 1-2 (hatched rectangles).

evidence between groups that the 2-digit subtraction problem context enhanced UB responses and reduced CR,FE responses.

In addition to the between-groups effect, there was a within-groups effect, but only for Group 1-2. Of the 6 subjects who made UB responses in the double-digit context, only 2 made such responses in the single-digit context as well; 5 of these 6 increased UB responses from single-to-double-digit context, and 1 did not change (Sign Test, 1-tailed, $p=.031$). Two subjects in this group made DI responses in the double-digit context, and one made DCR responses. Interestingly, one subject made an unusual response pattern in the single-digit context - an inversion error in the units column and a placing of the number "0" in the tens place of the answer (OI response, e.g. 14-8=04; 17-9=02, see footnote 2); this child changed his problem solving activity in the subsequently administered double-digit context to 3 UB respones, 1 DI, and 1 UB with an answer of 10. Thus, although this subject's single-digit-test response pattern was an unusual result of the units-tens subtraction strategy, his double-digit response pattern generated more actual correct answers but with unnecessary borrowings. This highlights the general observation in this study that within a context computations were highly consistent, however, between the contexts computations to the same problems were very different. Such an observation is hard to explain for a Fixed Rule approach to children's arithmetic problem solving.

As was found with the within-group comparisons of study 1, there were no within-groups context effect for Group 2-1. Of the 6 subjects making UB responses in the double-digit context, 4 continued to do so in the subsequent single-digit context worksheet (Sign Test, 1-tailed, $p>.05$). A direct comparison of Group 1-2 and 2-1 on the single-digit worksheet, although suggestive of a carry-over effect, as found in study 1, with 2 of 9 (22.2%) vs 4 of 8 (50%) making UB responses, was not

significant (Fisher exact probability, 1-tailed, $p < .05$).

DISCUSSION

Thus, as with the previous study on the context specificity of DI,DCR errors, this experiment demonstrated that UB procedures are also influenced by context. In addition, there was some evidence of a differential carry-over of single digit versus double-digit context effects. The effects of double digit context problem solving continues in the single digit context, but not the reverse, as indicated in a significant change in target UB responding for Group 1-2 only.

These results, combined with those from the previous study, strengthen the challenge to existing Fixed Rule approaches to children's arithmetic problem solving (Brown & Burton, 1976; Brown & VanLehn, 1980,1982; Young & O'Shea, 1980) that attribute problem solving to internalized procedural rules that remained fixed between instructional episodes. These models describe arithmetic problem solving as determined solely by the internal characteristics of the to-be-solved arithmetic problem, and totally ignore context-based processes.

In particular, these models explicitly assume that the borrowing procedure in subtraction occurs if and only if there is, contained within the to-be-solved subtraction problem, a vertical column of numbers of which the top is less in numerical value that the bottom. Two-digit-by-1-digit subtraction fact problems possess this characteristic, but are treated as appropriate problems for the borrowing procedure only if they occur in double-digit subtraction problem contexts. In the context of single-digit subtraction facts, they are treated as subtraction facts for which alternative problem solving activities are commonly used - recall-of-answer or counting. Thus, the units-tens, column-subtraction procedure is.

not invariably "called" solely by internal characteristics of the currently to-be-solved subtraction problem.

The asymmetrical nature of the carry-over effect, i.e. that double-digit-context problem solving continues to have an effect on subsequent problem solving to targets embedded in a single-digit context but not vice versa, rules out a simple response-learning explanation of the carry-over. According to this kind of explanation, a DI (drop-down-inversion error) or DCR error (drop-down-correct-response error) may be more likely to occur in a single-digit context when preceded by a double-digit context because the subject may have learned a specific answer or a specific problem-solving strategy (e.g. units-tens column vs counting up or down) to the targets, which is then emitted in response to the targets on the subsequent worksheet. The "specific-response learning" interpretation, however, predicts a single-to-double-digit context carry-over effect, i.e. a decreased incidence of double-digit column subtraction strategies to targets in a double-digit context when preceded by a single-digit context, and this was not observed in either of the first two studies.

The differential or asymetrical carry-over effect observed in the first two studies requires explanation. One explanation of this is in terms of the relative recency of double-digit, units-tens, drill and practice as compared to 2-digit subtaction fact drill and practice in the arithmetic curriculum. Unit-tens column subtraction is taught after 2-digit subtraction facts, and therefore, has been more recently practiced. According, it should be an easier problem solving activity to induce by context, and a more difficult activity to inhibit once induced. And this is what was observed in the differential carry-over effect.

It is also possible that this differential carry-over effect results because the units-tens column subtraction procedure is a procedure that includes the

subtraction fact procedures of recall-of-answer and counting. That is, during subtraction problem solving involving multi-digit numbers, the units-tens column procedure is engaged first, followed by subtraction fact procedures. In a sense, the subtraction fact procedures are used as subprocedures by a unit-tens-column "superprocedure". Thus, once a context of double-digit subtraction problems orients the child to subtraction problems as requiring the units-tens superprocedure, subsequently presented contexts of single-digit problems <u>that require only the subprocedures</u> (counting or recalling answer) may not be effective in inhibiting the superprocedure orientation. Thus, the single-digit context problems are not effective in re-orienting the children away from the use of the unit-tens procedure after it has been induced by the previously experienced double-digit context. Because sub-procedural skills are always taught first in arithmetic curriculae, these 2 explanations are difficult to evaluate critically.

Study 2 examined the development of 2-by-1-digit subtraction-fact problem solving by examining the influence of learning borrowing procedures on context-driven, 2nd grade problem solution strategies. Study 3 explores this development further by examining the effect that learning grade 4 2-by-1-digit division problem solving has on solving 2-by-1-digit subtraction facts, which must be solved in order to execute the division algorithm.

Chapter 5

STUDY 3: THE EFFECT OF CONTEXT ON THE SOLUTION OF 2-BY-1 SUBTRACTION FACTS EMBEDDED IN DIVISION PROBLEMS AMONG GRADE 4 CHILDREN.

The present study concerns the possibility that there are limitations to the effect of single-digit contexts on 2-digit subtraction fact problem solving. In particular, the effect of embedding 2-digit subtraction fact targets within division problems is studied, a problem-solving situation introduced in the grade 4 curriculum. In this study, 2-digit-by-1-digit division problems were viewed as a different way in which children are presented with 2-digit subtraction facts, and, as such, could be used as a way of presenting 2-by-1-digit subtraction facts as targets within a single-digit subtraction fact context. The following description of these problems will make clear the rational for this study.

Consider a type of 2-digit-by-1-digit <u>division</u> problem displayed in so-called "bracketed" form and depicted in Figure 3a. The problem is $8\overline{)14}$. Children at the grade 4 level in the Hamilton curriculum are taught to approach these problems as division problems with "remainders", that is, as composed of one group of 8 with 6 remaining. Figure 3a shows the division algorithm that is typically taught to arrive at this answer. First, the child is to determine what multiplication fact involving 8 will yield an answer less than or equal to 14; 8 X 1=8 is such a fact. Following the determination of this, the child is to write the number 1 above the 14, write the answer to 8 X 1 below the 14 and subtract 14-8. The answer of 6 is called a "remainder" and is placed above the 14 next to the 1, and prefixed by "R." to indicate that it represents the number remaining after 1 group of 8 is "taken out of" 14. Notice that the subtraction problem to be solved in this particular division

FIGURE 3a

Example of 2-by-1-digit division in "bracket" form and its embedded 2-by-1-digit subtraction fact

2-by-1-digit division problem:

```
     1 r.6
8 | 14
  - 8
    6
```

Typical performance of grade 4 children:
(includes unnecessary borrowing)

```
     1 r.6
8 | °1'4
  - 8
    6
```

"Embedded" 2-by-1-digit subtraction fact=

```
  14
- 8
```

FIGURE 3b

Example of division problem with embedded 2-by-2-digit subtraction problem:

```
     7 r.7
8 | 63
  -56
    7
```

problem is a 2-digit-by-1-digit subtraction fact in vertical form, identical to the target problems used in previous context studies. In this study, 2-digit-by-1-digit subtraction facts which are solved during division problem solving in this way will be referred to as "embedded" subtraction facts, because the fact is contained or embedded within the solution of another problem. This type of embedding should not be confused with embedding a target problem within a series of other problems, as has been done in studies 1 and 2. The behavior of grade 4 students in solving these embedded 2-digit subtraction facts provided the motivation for this study. Casual observations showed that children solve these problems by making an unneccessary borrowing response. In most cases during these observations, no multi-digit subtraction problems were present in the context to provide a possible basis for expecting unnecessary borrowing. Therefore, there seemed to be no factor in the context that could account for this response. It was therefore hypothesized that embedded 2-digit subtraction facts may be relatively unaffected by contexts known to affect unembedded 2-digit subtraction facts. Specifically, if embedding a 2-digit subtraction fact in a division problem somehow protects the fact from affects arising from context problems, then we should find that more unnecessary borrowing responses would be made to embedded 2-digit subtraction-fact targets presented in a context of single-digit subtraction facts than for identical but unembedded facts presented in a single-digit-subtraction context. The comparison of these two conditions was one goal of this study.

If such an effect were obtained, then a second goal of the study was to see if another context could be constructed in which the rate of unnecessary borrowing to division problem targets would exceed that obtained under the single-digit context conditions. Constructing such a context would be important to the basic proposition being considered in the thesis, that is, that most or all of

children's arithmetic problem solving, including grade-4 division problem solving, is sensitive to context processes.

A particluar set of contrasting problems was chosen as context problems for the division problem targets on an assumption that visual and/or arithmetic-operator similarity between context and target problem may be important in determining the effectiveness of context problems to influence target problem solving activity. Specifically, a context was chosen that consisted of division problems similar in form to the division target problems - i.e. 2-by-1-digit division problems in bracketed form. These problems were selected to contain 2-by-2-digit subtraction problems in their solutions in the same way as the targets were selected to contain 2-by-1-digit subtraction facts. Figure 3b shows an example problem, $8\overline{)63}$, which involves 63-56 in its solution. These problems are referred to as "embedded double-digit subtraction" problems for the same reason that the division targets are referred to as "embedded double-digit subtraction-fact" problems. Such embedded double-digit subtraction problems were chosen as context problems because it was thought that these problems would be effective in promoting UB target responding on the assumption that double-digit subtraction problems promote unit-tens solution strategies and that the factor of context-target problem similarity is important in determining the effectiveness of context problems to affect target problem solving activity.

The design of study 3 involved 2 days of testing. On day 1, as in Study 1 and 2, 2 groups of grade 4 children were given single-digit and double-digit context worksheets in counterbalanced order; the target problems were the same as in the previous studies, i.e. 2-by-1-digit subtraction facts presented in grade-2 form. One day 2, the two groups were given 2 new tests containing as targets a set of division problems with embedded 2-digit subtraction facts. The context problems in one test

consisted of a set of single-digit subtraction fact problems identical to those used in the single-digit context test given on day 1, and the context problems in the other test consisted of a set of division problems containing embedded 2-by-2-digit subtraction problems, half of which required borrowing. It should be noted that it was not the purpose of the study to directly evaluate the importance of context-target-problem similarity per se, but only to show that embedded double-digit subtraction-fact problem solving is influenced by context processes.

The predictions for this study were as follows. As in previous studies, it was predicted that double-digit context problems should promote unnecessary borrowing (UB) responses to 2-digit subtraction fact targets, whereas single-digit context problems should promote subtraction-fact strategies. Further, based on casual observations, it was predicted that UB responding would be subtantially higher to 2-digit subtraction facts that were embedded in division-problem targets, relative to the same targets that were not so embedded, even though those targets would be surrounded by single-digit subtraction-fact context problems. And finally, based on the hypothesis that embedded 2-digit subtraction fact problem solving is contextually sensitive, as Study 1 and 2 had shown for unembedded problems, it was predicted that UB responding to embedded subtraction facts would be increased by a context containing other division problems which include double-digit subtraction problems requiring borrowing in their solutions. Thus, the present study extends the results of Study 1 and 2 by exploring the developmental course of 2-by-1-digit subtraction-fact problem solving to grade 4, at which level the division algorithm is taught, an algorithm which procedurally subsumes subtraction-fact problem solving.

METHOD

SUBJECTS

An entire class of grade 4 pupils at a local public school in the Hamilton Board of Education participated in this study. Twenty-nine children were tested.

MATERIALS

Four mimeographed worksheets were used. One of them was identical to ones used in previous studies: single-digit subtraction-fact context test (Test 1-1). Test 2,3 and 4 were new.

2. DOUBLE-DIGIT SUBTRACTION-WITH-BORROWING CONTEXT TEST (2-2B).

This test was identical to the double-digit subtraction test used in previous studies except that 12 of the 22 2-digit context subtraction problems were changed by reversing the top and bottom numbers in the unit column so that borrowing would be required. In one problem, the top numbers in the unit and tens column were decremented by one to insure that borrowing would be required. In all, 13 of the 22 2-by-2-digit subtraction problems required borrowing, and each problem that immediately preceded the targets was a problem requiring borrowing (see Appendix III).

3. SINGLE-DIGIT CONTEXT WITH 2-DIGIT SUBTRACTION-FACT TARGETS EMBEDDED IN DIVISION (SD).

The context problems making up this test were identical to the single-digit subtraction problems used in past tests. The target problems were five 2-digit-by-1-digit division problems chosen such that 2-digit subtraction facts were embedded in their solutions ($7\overline{)13}$,$8\overline{)14}$,$9\overline{)17}$,$6\overline{)11}$,$3\overline{)11}$) and were positioned in the same locations on the worksheets as the 2-digit subtraction-fact targets from

previous studies (see Appendix IV).

4. DIVISION-PROBLEM CONTEXT WITH 2-DIGIT SUBTRACTION-FACT TARGETS EMBEDDED IN DIVISION (DD).

The same five target division problems used in the SD test (test 3 above) were used in this test, but the remaining context items were 2-by-1 division problems. These context division problems did not involve embedded 2-digit subtraction facts, but involved other types of subtraction problems, e.g. 4|21, 8|63. The problem preceeding each target problem was a division problem involving an embedded 2-digit-by-2-digit subtraction problem requiring borrowing, e.g. 8|63 involves 63-54. Of the remaining 20 context problems, 5 were division facts (9|45,5|30,7|28,9|45,8|64), 10 were 2-by-1 division problems involving embedded 2-digit-by-2-digit subtraction problems not requiring borrowing (e.g. 8|69 involves 69-64), and 5 additional division problems required borrowing during 2-digit subtraction. These 20 context items were placed in random order (see Appendix V).

PROCEDURE

Testing was conducted on 2 adjacent days. On day 1, the single digit subtraction context test (1-1) and double-digit subtraction-with-borrowing context test (2-2B) were administered in counterbalanced order to 2 groups of randomly selected individuals from the 29 pupils. The procedure was identical to that used in previous studies. Because these groups are essentially identical to those used in Study 1 and 2, they are referred to as Group 1-2 and 2-1.

On day 2, the SD and DD tests were administered in counterbalanced order to the 2 groups of pupils: Group 1-2 received the DD test first, Group 2-1 received the SD test first (one Group 1-2 subject was mistakenly given SS first and was eliminated from day-2 data evaluation). Again, the same administration procedure was used.

RESULTS

DAY 1: SINGLE- AND DOUBLE-DIGIT SUBTRACTION TESTS

The results on day 1 clearly demonstrated the context effect found in previous studies. Figure 4a shows the percentage of subjects making at least one unnecessary borrowing (UB) response to the 2-digit subtraction-fact targets as a function of context condition for the group receiving the single-digit subtraction context test first (Group 1-2) and second (Group 2-1).

The effect of context condition is significant both between and within groups. The between groups comparison of 7.7% and 60% (1 of 13 vs 9 of 15 subjects) of the subjects making a UB reponse was significant (Fisher, 1-tailed, $p<.01$). Sign Tests for changes in UB responses were significant for subjects given the single-digit test first: 10 of 13 subjects changed by increasing UB responses (Sign Test, N=10, 1-tailed, $p=.001$), and for those given the double-digit subtraction test first, 8 of 15 changed by decreasing to the single-digit subtraction test, (Sign Test, N=8, 1-tailed, $p=.004$). The difference in UB responses under 1-digit context conditions for groups 2-1 vs 1-2 of 20.0% vs 7.7% (3 of 15 vs 1 of 13 subjects) was not significant, indicating no evidence of a carry-over of the effects of the double-digit subtraction problem context to the single-digit context situation at this grade level. This absence of carry-over effects contrasts with what was found in study 2 using younger subjects.

DAY 2: DIVISION TARGET TESTS

One subject in Group 1-2 was dropped from the analysis of day 2 results because she was accidentally assigned to the wrong group. Figure 4b shows the percentage of subjects making a UB response to the targets under the various

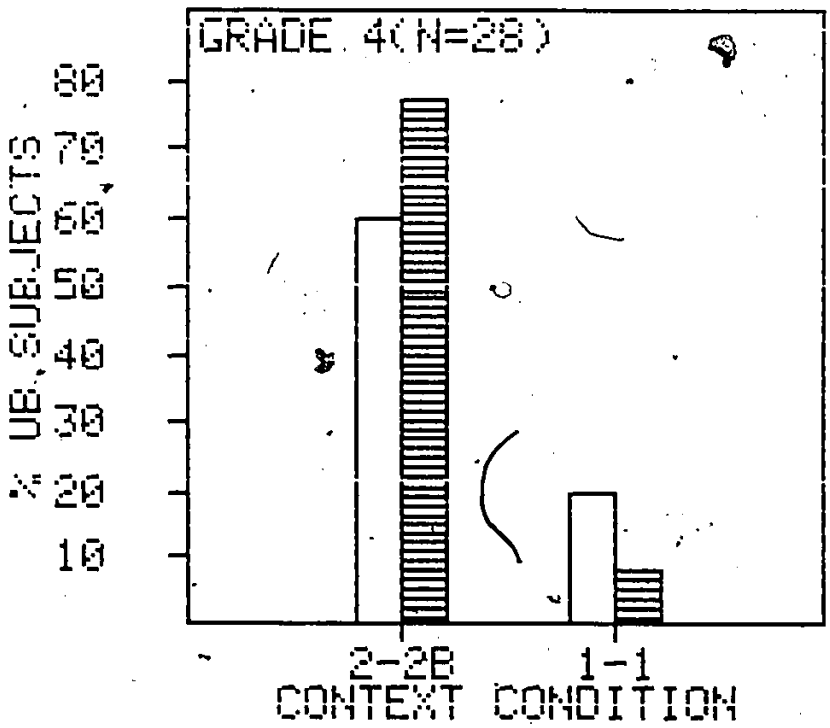

Figure 4a. Percentage of subjects committing unnecessary-borrowing (UB) to targets in single-digit (1-1) and double-digit (2-2B) subtraction contexts for Group 2-1 (open rectangles) and Group 1-2 (hatched rectangles) in Study 3

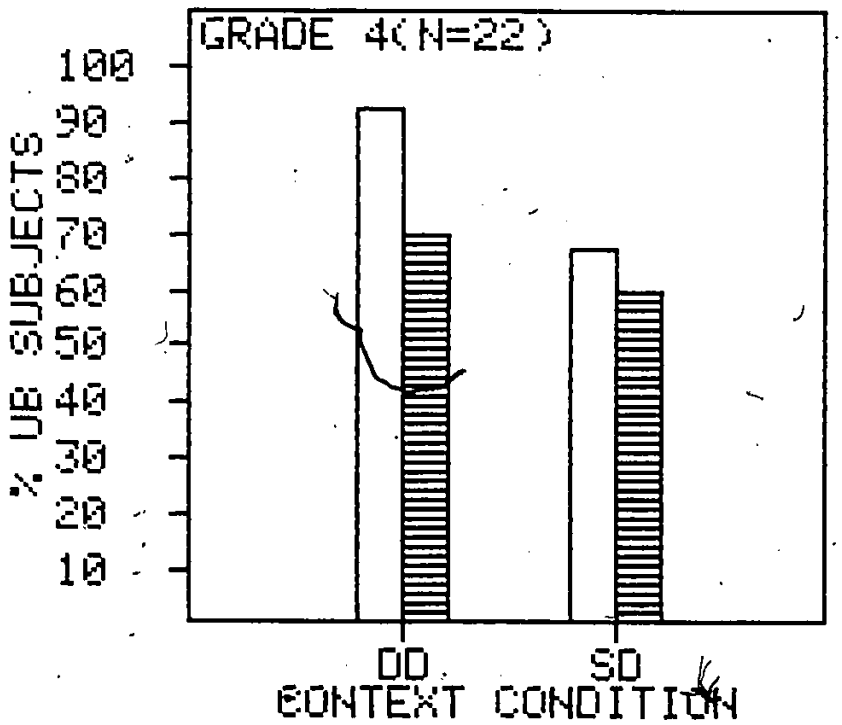

Figure 4b. Percentage of subjects committing unnecessary-borrowing (UB) to division targets in single-digit (SD) and division problem contexts (DD) for Group 2-1 (hatched rectangles) and Group 1-2 (open rectangles) on Day 2 of Study 3.

conditions for the 22 subjects who completed both tests. The percentage of subjects making a UB response to the 2-digit-subtraction facts embedded in 2-by-1 division problems under single-digit subtraction-fact context condition (SD) was 67% and 60% (8 of 12 vs 6 of 10 subjects) in the two groups. Notice that this constitutes a substantial increase in UB responding over the 7.7% or 20.0% seen under the same context conditions but with unembedded 2-digit subtraction-fact targets (test 1-1 of day 1). Of the 26 subjects who took these 2 tests, 14 (51.8%) increased UB responses in the division target condition (Sign Test, n=15, 1-tailed, $p<.004$) suggesting that single-digit subtraction fact context problems promote subtraction fact solution strategies of recall-of-answer or counting to 2-digit subtraction fact targets only when those targets are not embedded in 2-by-1 division problems.

Evidence was also found that embedded 2-digit subtraction facts were solved differently when confronted in a context of embedded 2-digit subtraction problems requiring borrowing (test DD) than in a context of unembedded single-digit subtraction facts (test SD). The between-groups context effect was not significance when only the data from the 22 subjects completing both tests were analyzed, 60% (6 of 10) vs 91.6% (11 of 12 subjects)(Fisher Exact probability, 1-tailed, $p<.10$). However, 26 subjects completed the first test given on day 2; the between-group analysis on these complete data reveals a significant effect, 50% (7 of 14) vs 91.6% (11 of 12) (Fisher exact probability, 1-tailed, $p<.05$). The within-groups comparion for Group 1-2 was also significant: the 5 subjects who changed responding across conditions all changed in the predicted direction (Sign Test, N=5, 1-tailed, $p=.031$).

DISCUSSION

The major finding of this study is that single-digit subtraction-fact contexts do not invariably lead to the solution of 2-digit subtraction facts by subtraction-fact strategies, such as recall-of-answer and counting. This was a conclusion reached from previous studies using samples of grade 2 children. It was discovered in this study that if the 2-digit subtraction fact is solved in the course of solving a simple 2-by-1 digit division problem then such facts are solved using an unnecessary borrowing procedure even when the division problem is embedded in a context of single-digit subtraction facts. Therefore, embedding these 2-digit subtraction facts within division problems eliminates, or at least reduces, the normal effect of single-digit subtraction-fact context problems on these facts.

The second finding of significance is that division-problem-embedded 2-by-1-digit subtraction-fact problem solving was affected by a surrounding problem-solving context consisting of similar division problems which, however, contained in their solution 2-by-2-digit subtraction problems requiring borrowing. Therefore, embedded 2-by-1-digit subtraction-fact problem solving is contextually sensitive, which, together with the results of Study 1 and 2, further challenge Fixed-Rule models of children's arithmetic problem solving (Brown & Burton, 1978; Brown & VanLehn, 1980, 1982; Young & O'Shea, 1980), and demonstrates the continuing role which problem solving context plays in determining arithmetic problems solving, even at higher grade levels.

With regard to the finding that 2-by-1-digit subtraction facts embedded in division problems are not as affected by single-digit subtraction fact context problems as unembedded problems, a brief consideration of the differences between

the single-digit subtraction test containing subtraction targets (test 1-1) and the single-digit subtraction test containing division targets (test SD) may be illuminating.

In the 1-1 test, the 2-digit-subtraction-fact targets are immediately preceded by single-digit subtraction facts. Therefore the subject moves immediately from subtraction-fact solving to the 2-digit target. In the SD test, the subject solves the embedded 2-digit subtraction fact after he has performed the division operation that leads to the embedded subtraction problem. At least two subject-generated events occur before solving the 2-digit subtraction fact: (1) the division operation, and (2) writing the single-digit number below the 2-digit dividend to "set up" the 2-digit subtraction fact. Either one or both of these events may diminish the effectiveness of the previously solved single-digit subtraction facts to promote solving the 2-digit subtraction fact by subtraction-fact strategies. Perhaps the effect depends on short-term memory for the single-digit subtraction-fact strategies used on the context items which is reduced due to interference from the division/writing operations.

Alternatively, the embeddedness effect may be more perceptually based than memory based. That is, embedded 2-digit subtraction facts may be sufficiently perceptually dissimilar to unembedded 2-digit subtraction facts that the typical effect of single-digit subtraction fact context does not occur. This implies that procedural embeddedness does not interact with context to determine solution strategies to 2-digit subtraction fact targets, but perceptual similarity between context and target is important. The perceptual embeddedness of the 2-digit subtraction facts within the division brackets and divisor may perceptually separate the single-digit context from the target problems. This argument is supported by the observed enhancement of UB when the context problems were 2-by-1 division problems

involving 2-digit subtraction requiring borrowing in test DD.

A third finding of this study is that the carry-over of UB from the dbuble- to single-digit context was not as pronounced here as in study 1 & 2 with 2nd and 3rd graders. This lack of carry-over may be due to the higher level of skill in subtraction which these 4th graders possess relative to the 3rd graders and/or the fact that, in this study, unnecessary borrowing was promoted by subtraction problems requiring borrowing as compared to the subtraction problems not requiring borrowing used in study 1 & 2. The differential skill interpretation of the lack of carry-over implies that carry-over of context effects should diminish with arithmetic skill. Therefore, children having difficulty learning arithmetic may evidence greater carry-over effects.

This study, together with studies 1 and 2, clearly document that 2-by-1-digit subtraction-fact problem solving is much more flexible and contextually sensitive than as predicted by Fixed Rule conceptions of children's arithmetic learning. The next two studies investigate two variables which could be important in determining the grade-2 error of using column-subtraction problem solving strategies with 2-digit subtraction facts.

Chapter 6

STUDY 4: THE TEMPORAL DURABILITY OF THE GRADE 2 CONTEXT EFFECT

The previous demonstration of the context specificity of grade 2 column-subtraction-based errors (Drop-down-inversion, drop-down-correct response) included the unexpected result that considerable column-subtraction-based errors occurred in the single-digit context when the context was immediately preceded by a double-digit context. That is, there was a carry-over of the effects of the double-digit context to the single-digit context.

One possible interpretation of this carry-over effect is that it results from the action of some type of temporary 'priming' mechanism similar to that thought to underlie such a phenomenon as semantic priming in word recognition (Meyer & Schvaneveldt, 1971; Meyer, Schvaneveldt, & Ruddy, 1975). That is, the double-digit context may activate the use of a column subtraction algorithm by exciting that portion of semantic memory given over to storing the column subtractin algorithm. This excitation-of-semantic-memory hypothesis has been used to explain priming effects in word recognition (e.g. Morton, 1969, 1971), and could be applied to this situation involving arithmetic problem solving by assuming that information about computational algorithms is stored in semantic memory for the purposes of doing arihmetic just as some models of word recognition assume that information about words is stored in semantic memory for the purposes of reading. According to an excitation hypothesis, activating the column subtraction algorithm should temporarily bias problem solving so that problems which are similar to those for which a column subtraction algorithm are appropriate will be solved using that algorithm. The key assumption of this excitation view is that the excitation decays across time so that

the problem solving bias is only temporary. If this kind of arithmetic view is correct, then it would be expected that the carry-over effect would be short-lived, so that if the single-digit context worksheet were given much later after the double-digit context worksheet, little or no column-subtraction-based errors would occur.

This study was conducted to examine the temporal durability of the carry-over effect on column-subtraction-based errors by delaying the administration of the single-digit worksheet for 24 hours after the double-digit context worksheet. A three-group design was used. Two of the groups were treated identically to those in the basic demonstration (study 1). That is, they were administered the single- and double-digit context worksheets in counterbalanced order on day 1 of testing, to replicate the previous findings of context specificity. The third group was given the double-digit context worksheet on day 1 and the single-digit worksheet on day 2, 24 hours later. No arithmetic instruction occurred between day 1 and 2. In addition, the non-delayed groups were also given the single-digit context worksheet on day 2 to provide another test of the durability of the context effects from day 1.

METHODS

SUBJECTS

The subjects consisted of 3 grade 2 classes located in 2 different schools (Hamilton Board of Education, Hamilton, Ontario). In total 56 children completed all tests: 27 from school 1, 19 from classroom 1 school 2, and 10 from classroom 2 school 2. The 10 children from classroom 2, school 2 were part of a combined grade 1 and 2 class, but received a normal grade 2 curriculum.

MATERIALS

The materials were identical to those used in the original demonstraction of context effects at grade 2 (study 1, see Appendix I and II). Briefly, 2 separate worksheets were composed of an identical set of 5 2-by-1-digit subtraction facts; one worksheet contained 25 1-by-1-digit subtraction facts as context items for the 5 targets (single-digit context); the other contained 22 2-by-2-digit and 3 2-by-1-digit subtraction problems (without borrowing) all containing 2-digit answers as context items for the 5 targets (double-digit context).

PROCEDURE

The same procedure was used for each classroom and was similar to that used in the original demonstraction study (study 1). Three groups were randomly created in each classroom. To one of those groups, the single-digit worksheet was given first, after which the double-digit context worksheet was given on day 1 (Group 1-2). For the remaining subjects in the room, the double-digit context worksheet was given first; a random half of these subjects received the single-digit context worksheet immediately upon completely their double-digit worksheet (Group 2-1-Immediate), whereas the other half were given no worksheet following the completion of their first worksheet on day 1 (Group 2-1-Delayed). The random assignment of children to Groups 2-1-Immediate or Delayed was determined by alternating assignments between groups as the children completely their first double-digit worksheet; this insured that the time to completing the double-digit worksheets was comparable between these two groups.

On Day 2 all children received the single-digit context worksheet.

The test administration procedures for this study were identical to those used in study 1.

RESULTS

The worksheets were scored as in the basic demonstraction study and DI, D0, and DCR errors were defined as column-subtraction-based errors (CS errors). So-called "D0" errors were not present in the basic demonstraction study, but occurred in this study, and are defined as any answer in which the units place numeral is a 0 and the ten's place numeral is a 1. These are scored as CS errors because interview data revealed that children arrive at these answers by using a units-tens column procedure and reasoning that since the top number (TN) in the units column is smaller than the bottom number (BN) in the units column TN-BN=0.

The results were clear and can be seen in Figure 5 where the percentage of subjects making column-subtraction-based errors under the various conditions are displayed. Large context effects were found both within and between groups, where, for example, 13 of 19 or 68.4% of Group 1-2 subjects made CS errors to the 2-by-1-digit subtraction facts on the double-digit worksheet whereas only 1 of 19 or 5.3% of these same subjects made such errors to the same problems on the immediately preceding single-digit context worksheet. Of these 13 Group 1-2 subjects making CS errors, 6 made DI errors, 2 made DCR errors, 3 made D0 errors, and 2 made a mixture of DI and DCR errors. This large change in response to the identical target problems replicates the previously observed within-subject context effect.

A significant between-groups context effect was also obtained as seen in the higher incidence of CS errors among Group 2-1-Immediate and Group 2-1-Delayed subjects given the double-digit context test first: 12 of 19 (63%) and 10 of 18 (56%) made CS errors, respectively. Combining these data and comparing it to Group 1-2 yields an incidence of CS errors of 22 of 37 subjects in Group 2-1-Immediate and -Delayed (59.5%) relative to 1 of 19 subjects in Group 1-2 (5.3%) (Corrected Chi-square=13.08, 1-tailed, $p<.0005$). Of the total 22 subjects making CS errors,

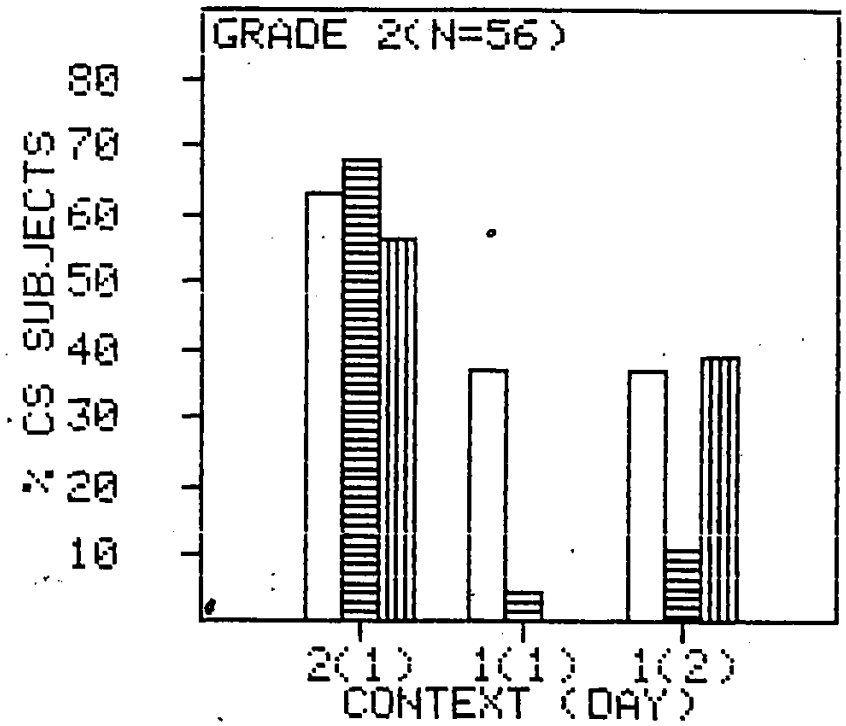

Figure 5. Percentage of subjects committing column-subtraction errors (CS) to targets in single-digit (1) and double-digit (2) subtraction contexts on Day 1 and 2 for Groups 2-1-Immediate (open rectangles), 2-1-Delayed (vertically hatched rectangles), and 1-2 (horizontally hatched rectangles) of Study 4.

17 made DI errors, 2 made D0 errors, 1 made a mixture of DI and DCR errors, and 2 made DI and D0 errors.

The double-digit-to-single-digit-context carry-over effect was also replicated as seen in the result that 7 of 19 or 37% of Group 2-1-Immediate subjects made CS errors to the targets on the single-digit worksheet, which for them was experienced after the double-digit worksheet, in comparison to only 5.3% of Group 1-2 subjects for whom the single-digit worksheet was experienced first (Correct Chi-Square= 3.96, 1-tailed, $p<.025$). Of these 7 Group 2-1-Immediate subjects making CS errors, 4 made DI errors, 1 made D0 errors, 1 made DI and DCR errors, and 1 made DCR and D0 errors. Clearly, having just immediately completed the double-digit context worksheet greatly increased CS errors to targets embedded in the single-digit context. Notice should be taken of the fact that the carry-over effects were not symmetrical; that is, CS errors were not reduced in the double-digit context when a single-digit context had just previously been completed. This replicates trends seen in previous studies.

The result of main interest in this study is the performance of Group 2-1-Delayed on day 2. As can be seen in Figure 5, 7 of 18 or 38.8% of these subjects made CS errors on the single-digit context worksheet when it was experienced 24 hours after the double-digit context worksheet. This represents a significant carry-over when compared to the incidence of errors for Group 1-2 on the single-digit context (Correct Chi-Square=4.34, 1-tailed, $p<.025$), and is, of course not significantly different in magnitude to the immediate carry-over effect. Similar to the error distributions observed in the other groups, of these 7 Group 2-1-Delay subjects making CS errors, 5 made DI errors, 1 made D0 errors, and 1 made DI and D0 errors. Thus, there was no evidence that delaying the administraction of the single-digit-context worksheet reduced the effect of the previously administered

double-digit context worksheet. This finding indicates that the immediate carry-over effect, seen here and in previous experiments, is not temporally restricted to within 24 hours.

An interesting unexpected result can be seen in the performance of Group 1-2 sujects on day 2. One day 2, these subjects experienced the single-digit context worksheet, the same one they had already experienced as their first worksheet on day 1. Since the double-digit context worksheet had been the second worksheet experienced on day 1, one might have expected a delayed carry-over of this double-digit context worksheet to the single-digit worksheet completed on day 2. However, no such delayed carry-over was observed. Only 2 of 19 or 10.5% of Group 1-2 subjects made CS errors to the single-digit context worksheet given on day 2. Thus, the delayed effect of double-digit context to single-digit context was eliminated by prior experience with the single-digit context worksheet.

DISCUSSION

These data replicate the previous study in showing the importance of current context, and prior context on 2-by-1-digit subtraction-fact problem solving among grade 2 children. In addition, it extends the previous study by showing that the effectiveness of prior double-digit context (1) lasts at least 24 hours, and (2) can be eliminated by previous exposure to the same single-digit context.

Taken together, these findings rule out a simple priming-of-semantic memory explanation of the context effects, since that explanation predicts the carry-over effect to be short lived. No evidence could be found to support that prediction; indeed, the carry-over effect seemed to be unaffected by a delay of 24 hours between double- and single-digit context worksheets. These data indicate that incorrect problem solving can be induced from the experiences of only 1 worksheet,

and once induced, are rather durable.

The lack of symmetry in the carry-over effects replicates past findings in Study 1 and 2 and rules out a simple explanation of the carry over effect; that is, that the column-subtraction-based errors are learned to the targets on the first worksheet, such that the second presentation of the same targets on the subsequent worksheet merely provide an opportunity for showing this learning. This kind of argument predicts that the carry-over should be symmetrical; that is, whatever is learned to targets on the first worksheet should be seen to the same targets on the second worksheet. Contrary to this prediction, clear asymmetry in carry-over effects was seen in this study. The only way to reconcile these data with this kind of explanation is to assume that only the column-subtraction-based errors are being learned to the targets when the double-digit context worksheet is presented first, and the correct responses made to the same targets when the single-digit context worksheet is presented first are not being learned to the same extent or at all. This reconciliation is post-hoc and has no good independent theoretical or empirical grounding, and therefore will be rejected.

A more likely theoretical possibility for the results in this study is one which acknowledges the possibility that other types of contextual stimuli, besides problem context, may influence the original learning experience. That is, the delayed carry-over effect may result from the retrieval of the prior learning experience by incidental stimuli such as the test administrator and other incidental stimuli that define the experience during day 2 as identical to the experience on day 1. Subjects receiving the double-digit worksheet first may being learning a column-subtraction-based strategy to the context which is partly, and importantly, experienced as including the particular person giving the tests, a person who, it should be remembered, is a stranger different from their teacher. These children

might be learning that "this is the guy who gave me the 'big' subtraction problems yesterday". With the 'big' subtraction problems "in mind" (i.e. double-digit problems requiring the column-subtraction-based strategy), these children might be as susceptible to making column-subtraction-based errors when the targets are presented in the single-digit context, as they are when the column-subtraction-based strategy is in mind as a result of immediately prior experience with the double-digit context. Such an explanation predicts that if a different person administered the single-digit test on the second day, the delayed carry-over would not occur, a prediction worthy for future research.

One difficulty with this contextual explanation is that it implies that Group 1-2 subjects should also make column-subtraction-based errors to the single-digit context test on day 2, a prediction not supported in the current study. That is, it could be argued that these subjects should have learned the column-subtraction-based strategy to the test administrator when they were given the double-digit test after the single-digit context test. The contextual explanation could be preserved by recognizing that the problem contexts themselves may also participate in the learning experience. That is, in the case of Group 1-2 subjects problem solving on day 2 to single-digit contexts, the single-digit context worksheet had been experienced previously on day 1 under conditions in which the subtraction fact strategy had been used to the targets. Such an experience may have lead to learning the single-digit strategy to the single-digit context situation, a learning experience that could lead to low column-subtraction-based error rates in the presence of the single-digit problems on day 2. Making the assumption that the problem context participates in the learning experience does not rule out the possibility that other contextual stimuli also participate in the learning experience for these Group 1-2 subjects during day 1 testing; conceivably, changing the test

administrator on day 2 for these subjects could have the effect of eliminating apparent carry-over of prior single-digit context learning to day 2 problem solving.

In summary, this study has produced evidence that questions a simple priming-of-semantic memory explanation of context effects: delayed carry-over effects were clearly observed. The pattern of the results suggests an alternative view of arithmetic learning that acknowledges the decisive role of both problem context and situational context as determiners of target problem solving and as participants in the learning experience they can serve to retreive information from the prior experience. In addition, this alternative recognizes the potential role of other contextual events, such as the test administrator, as a participant in the learning episode such that they too can retreive information from the prior learning experience. The explicit recognition of contextual influences of these kinds in arithmetic problem solving is not contained in current educational and psychological theories of arithetic learning in children which view the learning process as one of abstracting from instruction and drilling a set of procedural rules that are relatively fixed between explicit instructional experiences (Brown & Burton, 1978; Brown & VanLehn, 1980, 1982; Young & O'Shea, 1981).

The next study focuses on the role of classroom instructional practices in determining column-subtraction-based errors under double-digit context conditions.

Chapter 7

STUDY 5: THE EFFECTS OF MIXED-DRILL INSTRUCTION ON SUBTRACTION-FACT PROBLEM SOLVING AMONG GRADE 2 CHILDREN

Previous studies in this thesis have shown that for grade 2 and 4 children a problem context of double-digit subtraction problems greatly increases the probability of column-subtraction based errors (CS errors) to 2-by-1-digit subtraction facts over that observed to such problems in a problem context of single-digit subtraction-fact problems. One possible reason for this context effect may be based in the type of instruction given to grade 2 children. Teachers typically report that they give drill exercises on these problems with drill sheets that either (1) contain only 2-by-1-digit subtraction facts (so-called unmixed drills), or (2) contain single-digit subtraction facts intermingled with 2-by-1-digit subtraction facts. Also, for drilling 2-by-2-digit subtraction problems, unmixed drills are constructed containing only those types of problems. Unmixed drills of these kind permit the child to learn that problems on a drill sheet are solvable using a single computational strategy. For example, when all the problems are double-digit subtraction problems the child may learn that a drill sheet containing double-digit subtraction problems can be solved in its entirety by a column-subtraction, units-tens, procedure. Such unmixed drills do not provide the child with learning experiences in which he must change his problem solving strategy within a drill sheet, where he must, for example, solve a problem using a subtraction-fact-finding strategy such as counting-up, counting-down, or recall-from-memory in the context of previously solved problems for which a column-subtraction, units-tens, procedure was used. This "mixed" type of situation

is identical to the double-digit subtraction context test used in previous studies in which 2-by-1-digit subtraction facts, which require a subtraction fact-finding strategy, are interspered among 2-by-2-digit subtraction problems, which require the column subtraction strategy.

The possibility that column-subtraction-based errors seen in the double-digit context worksheet result from the absence of experiences with mixed drills in the classroom was one of the hypotheses tested in this study. A classroom of 2nd graders were given short mixed drills irregularly spaced over a 2-week period of time in their classroom by their teacher involving 2-by-2-digit subtraction problems and 2-by-1-digit subtraction facts. After this experience with mixed drills, they were administered the single-digit- and double-digit-subtraction context worksheets to assess if the teacher-administered classroom drills were sufficient to eliminate the basic context effect.

A second hypothesis was investigated in this study. If the mixed drills have been successful in eliminating column-subtraction-based errors to targets in the double-digit subtraction problem context, one could argue that the context effect for these children had been eliminated as well; that is, for these children, there would be no difference between problem solving to 2-by-1-digit subtraction-facts when they are embedded in double-digit subtraction-problem contexts or single-digit subtraction-fact contexts. This could then be interpreted to indicated that for these children such problem solving had become context-independent or "decontextualized", and that, moreover, these children's knowledge about 2-by-1-digit subtraction fact problem solving could be represented in the form of internalized, context-independent, procedural rules. This decontextualization hypothesis about the effects of mixed drills implies that no problem context could be found in which column-subtraction-based errors would occur to 2-by-1-digit subtraction facts after

the mixed drills have been experienced. In contrast, the alternative view of arithmetic learning discussed in connection with the previous 24-hour delay study (study 4), emphasized the continual role of learning context in constraining the generality of the effects of learning experiences and interprets the mixed drill experience as providing opportunities for learning to make subtraction-fact problem solving strategies in the context of 2-by-2-digit subtraction problems, but does not guarantee that such correct problem solving will be seen to targets embedded in problem contexts which are different from the 2-by-2-digit subtraction problems and which are not present in the mixed-drill experiences. These alternative views of the effects of mixed drills were tested in this study by giving further follow-up tests to this classroom that were designed to vary the problem contexts for 2-by-1-digit subtaction-fact targets from the 2-by-2-digit subtraction problems used in the mixed drills. This enabled an assessment of the generality of the effects of mixed drills, and, thus, of the appropriateness of one or another of these two views of arithmetic learning in children.

METHOD

SUBJECTS

The subjects in this experiment were an entire class of grade 2 children in another school in the city of Hamilton. Twenty-seven pupils were in the study. Of these 27, 24 completed all 3 days of testing.

MATERIALS

The materials consisted of 4 mimeographed worksheets consisting of 30 subtraction problems each presented in vertical form. Three of the worksheets were identical to materials used in previous studies. The single-digit and double-digit subtraction-problem context worksheets which are describe in detail above (p.52 and

displayed in Appendix I and II) were two of these worksheets.

In addition to the single- and double-digit context worksheets, a third worksheet was identical to one use in study 3 (see Appendix III) and a fourth new context worksheet was used (see Appendix VI). Both of these contained context problems requiring regrouping (borrowing) and will be described here. These tests contained the same 5 target problems located in the same spatial positions as in the single- and double-digit context worksheets. One of the sheets contained as context for these targets 13 2-by-2 subtraction problems requiring regrouping (e.g. 92-44=48; 93-67=26), 9 problems that do not require regrouping, and 3 2-by-1-digit subtraction problems involving double-digit numbers between 10 and 19 and double-digit answers (16-5=11;18-4=14;18-5=13). This sheet is dominated by 2-by-2-digit subtraction problems requiring regrouping and henceforth will be referred to as the "2-2B" context test in this study (2-by-2-digit with borrowing) (see Appendix III).

The second new worksheet involved 25 2-by-1-digit subtraction problems in the context: 13 2-by-1-digit subtraction problems requiring regrouping (e.g. 82-4=78;87-9=78), and 12 not requiring regrouping. This worksheet will be referred to as the "2-1B" context test (2-by-1-digit with borrowing) (see Appendix VI).

DESIGN

The phases of the study are shown in Table 4. In phase I, all children were given mixed drills by their teacher (discussed more below). In phase 2, the results of the mixed drills were assessed by administering the standard double-digit context test (2-2 Context Test), the test of primary interest in this study, and the single-digit context test. In fact, during phase 2, the design and procedure was identical to that used in study 4 (see p. 82). Essentially, two groups of randomly chosen subjects received the single-digit and double-digit worksheets (2-2) in counterbalanced order (Group 1-2 and 2-1-Immediate); a third group received the

double-digit context worksheet only (Group 2-1-Delayed).

On day 2 of this testing, phase 3 of the study, Groups 1-2 and 2-1-Immediate received the double-digit-regrouping worksheet (2-2B), and Group 2-1-Delayed received the delayed administration of the single-digit worksheet as designed in study 4.

On day 3 of testing, phase 4 of the study, all three groups received the 2-by-1-digit-regrouping worksheet (2-1B).

In order to assess the results of the mixed drills, the results of phase 2 testing can be directly compared to the data collected from previous classrooms (study 4) tested in an identical manner.

PROCEDURE

The test administration procedures were identical to previous studies. The subjects were always told in advanced that they were working on sheets containing "all subtraction problems", and that they were to write their names and test number on the back of the sheet, and to solve the problems working from left-to-right, top-to-bottom, skipping only problems they absolutely could not do.

The teacher designed her own mixed drills which included problems in which the children were asked to copy 3-5 problems from the black board and do them at their desks on their own paper. The problems included a mixture of 2-by-2-digit subtraction problems and 2-by-1-digit subtraction facts. These drills were typically given before each arithmetic lesson.

RESULTS

The effect of mixed drills on the basic context effect can be assessed by comparing Groups 1-2, 2-1-Immediate, and 2-1-Delayed in this experiment with the same groups from the previous study (study 4). Table 5 gives the percentage of subjects

TABLE 4

DESIGN OF MIXED DRILL STUDY

PHASE	DESCRIPTION	PROBLEMS		
1	Mixed Drills (2 weeks)	56 -11	12 - 4	
2	2-2 context test	68 -25	12 - 4	
3	2-2B context test	34 -21	65 -18	12 - 4
4	2-1B context test	34 - 2	63 - 8	12 - 4

making column-subtraction-based errors (i.e. DI, DCR, or DO errors) for the three groups collapsed into context conditions. As can be seen, little or no CS errors were observed under double-digit context conditions. The best estimate of the incidence of CS errors can be obtained by collapsing across the groups under the double-digit context conditions; the result being that only 4 of 24 subjects or 16.6% in this classroom of children made CS errors. This contrasts significantly from the comparable estimate of CS errors obtained from the previous study of 35 of 56 subjects or 62.5% (Corrected Chi-Square= 9.58, 1-tailed, $p<.005$). Thus teacher-administered, in-classroom mixed drills involving double-digit subtraction problems (without regrouping) interspersed with 2-by-1-digit subtraction facts significantly decreased the occurrence of CS errors under the double-digit context conditions. This result suggests that the absence of mixed drills of this kind is an important determiner of the basic context effect observed among grade 2 children in previous studies.

Table 5 gives the results from day 2 and 3 of testing in which the generality of the effects of the mixed drills was assessed. On day 2, a context predominately composed of double-digit subtraction problems that required regrouping was used in which 2-by-1-digit subtracfion-fact target problems were presented (test 2-2B). Of the 16 subjects in Group 1-2 and 2-1-Immediate who took this test, only 4 or 25% made CS errors. This is not significantly different from the 16.6% incidence observed on day 1 under the double-digit subtraction context not requiring regrouping (Sign Test, N=4, 1-tailed, n.s.), and indicates that the effects of mixed drills do generalize to this problem context.

On day 3, a context composed entirely of 2-by-1-digit subtraction problems was used, the majority of the problems requiring regrouping. The results were dramatic. Of the 24 subjects taking this test, 12 or 50% made CS errors. This

TABLE 5

RESULTS OF MIXED-DRILL STUDY 6

	CONTEXT		
	2-2	2-2B	2-1B
Number of CS Subjects	4	4	12
Total Subjects	24	16	24
Percentage	14	25	50

CS = Column-subtraction-based errors
2-2 Test = 2-by-2-digit subtraction context
2-2B Test = 2-by-2-digit subtraction (borrowing context)
2-1B Test = 2-by-1-digit subtraction (borrowing context)

increase is significantly different from the 2-2B test results (Sign Test, N=5,1-tailed, $p=.031$) as well as the 2-2 test results (Sign Test, N=12, 1-tailed, $p<.003$) Thus, the generality of the effects of mixed drills was limited: a context condition requiring 2-by-1-digit subtraction with and without regrouping was sufficient to promote CS errors to the 2-by-1-digit subtraction-fact targets in this group of children for whom mixed drills had eliminated CS errors under the double-digit subtraction contextual conditions.

DISCUSSION

There were three results obtained from this study. First, teacher-administered, in-classroom mixed drills involving double-digit subtraction problems not requiring regrouping and 2-by-1-digit subtraction facts were sufficient to reduce the typically observed ability of double-digit subtraction contexts to induce column-subtraction-based errors to 2-by-1-digit subtraction-fact targets among 2nd graders. Second, the generality of the effects of this type of mixed drills extended to a problem context containing double-digit subtraction problems requiring regrouping, but, third, not to a context containing 2-by-1-digit subtraction problems predominantly requiring regrouping.

Concerning the interpretation of the follow-up test results of day 2 and 3 it could be argued that time since mixed drills or number of prior exposures to targets was confounded with the type of problem context for the targets, and thus, that the results are ambiguous. However, consider the probable results of a single-digit context test given after day 3 testing. Although this test was not administered due to scheduling problems, the data from the previously discussed 24-hour delay study suggests that the results of such testing would have shown low column-subtraction-based error rates. Recall the results of testing Group 1-2 on day

2 with a single-digit context worksheet in Study 4; there it was observed that low column-subtraction-based error rates occurred in spite of the fact that an intervening double-digit context worksheet had been given on day 1 in which column-subtraction-based error rates were high. This suggests that repeated exposure to CS-error-promoting contexts is not sufficient to eliminate context specificity of target responding, and that the correct responding to targets embedded in a single-digit context extends over time (24 hours). Although the number and type of intervening CS-error-promoting contexts is different between this study and the 24-hour delay study and the time parameters are different, the conjecture is supported that a single-digit context presented on day 4 of this study would have produced the low rates of column-subtraction-based responding necessary to unambiguously demonstrate context specificity of target problem solving after the mixed drills.

The first result, that mixed drills can eliminate column-subtraction-based errors in double-digit context, indicates that context-specific column-subtraction-based errors result from the absence of learning experiences in which subtraction-fact problem solving strategies (such as, counting-down, counting-up, and recal-from-memory) are required in the context of item sequences requiring column-subtraction problem solving. Such experiences are necessary in order to insure that grade 2 children learn to shift problem solving strategies in item sequences for which this is required. Models of children's arithmetic knowledge that only assume the internalization of context-independent procedural rules are silent on the issue of what learning experiences are necessary for the development of fully differentiated procedural knowledge and discriminative arithmetic problem solving.

The second and third findings of the current study concerning the

generality of the effects of mixed drills indicate that the effects of such discriminative learning experiences are specific to certain aspects of the problems that are part of the drills. The fact that mixed-drill effects generalized to 2-by-2-digit subtraction-with-regrouping contexts but not to 2-by-1-digit subtraction-with-regrouping contexts implies that the visual similarity of the problem form in which training is given, i.e. the 2-by-2 rectangular array, and the form used as the context to which the training effects are being tested, is an important determiner of the extent of generalization. When similarity is high, generalization is virtually complete. In contrast, a change in the relative magnitudes of the unit's top and bottom numbers is not as an important factor, i.e. change from top>bottom to bottom>top. Although increasing physical dissimilarity between training and testing context may be an important determiner of generalization observed here, notice should be taken of the fact that the context changes used in the generalization testing not only involve increases in physical differences from context training conditions but also increases in physical similarities to the target problems. These two types of changes are confounded in this study, and it is certainly possible that making the context problems more similar to the targets while maintaining the context's ability to provoke use of a column-subtraction-based problem solving strategy is responsible for the observed increase in column-subtraction-based errors in the 2-by-1-digit subtraction-with-regrouping context. Further studies must be conducted to exclude this possibility. For example, post-drill testing using 3-by-2 or 3-by-3 subtraction context problems would involve decreases in training-and-testing-context similarity but no increases in context-target similarity. If column-subtraction-based errors were observed under these conditions, the conclusion would be favored that training-test context similarity is the more important determiner of the generalization of mixed drill

effects.

The lack of decontextualization of 2-by-1-digit subtraction-fact knowledge lends further support to the general proposition being illustrated in this dissertation; that is, that contextual influences are substantial in determining children's arithmetic learning and problem solving, and that current models are inadequate to explaining this influence.

STUDIES 6-8: CONTEXTUAL AND INSTRUCTIONAL INFLUENCES ON THE GRADE-3 DIAGONAL ERROR

The analysis of the errors made in grade 2, discussed in the above studies, that showed the importance of contextual and instructional influences was extended to grade 3 errors in three studies. Evidence was collected for contextual and instructional influences in determining a very dominant error at grade 3 - the so-called "diagonal error ". Extending the analysis to other systematic errors at other grade levels would lend considerable support to the view being advanced here, that is, that arithmetic learning and problem solving is, at least partly, contextually driven. The following discussion provides a description of the diagonal error, some views which attempt to explain it, and an alternative interpretation that includes the possibility of contextual influences. The three studies will be briefly described in which evidence for the contextually-based interpretation is examined.

The diagonal error is a systematic error seen among primary school children made to 2-by-1-digit addition and subtraction problems. It involves the use of a diagonal procedure in calculating the numerals to be placed in the tens and, if necessary, hundreds columns of an answer. Consider problems displayed in Table 6 (23+4=67 and 56-4=12). The addition problem was solved by (1) adding the numbers in the units column - 3+4=7, and (2) adding the tens number in the top double-digit number, 2, to the single-digit addend of the problem, 4, yielding the number 6 which is placed in the tens place of the answer. The second step in this procedure is the diagonal subprocedure because it involves operating on numbers situated along an imaginary diagonal line relative to 90 degrees vertical or horizontal. As can be

seen in Table 6, a diagonal error can also be committed during the solution of a 2-by-1-digit subtraction problem (56-4=12) were the number 1, appearing in the tens place of the answer, was derived by subtracting the numbers lying on th diagonal, 5-4=1.

The diagonal error has been identified elsewhere in descriptions of children's systematic computational errors (Cox, 1975; Friend, 1979), were some investigators have interpreted it to be the result of the child's use of a so-called "binary operation rule" (Davis & Knight,1979; Davis,1980). The rule states that numbers placed in an answer must result from arithmetic operations performed on at least 2 numbers in the problem. The binary operation rule is assumed to be learned by children when they learn to solve 2-by-2-digit problems using a unit-tens column procedure. When the column procedure is used on these so-called "rectangular" problems such as 2-by-2-digit or 3-by-3-digit problems, answers for each of the columns are invariably derived by performing an operation on the 2 numbers in each vertical column. According to the binary-operation-rule hypothesis, children mislearn from column addition and subtraction instruction and drilling with these rectangular forms that all addition and subtraction problems must be answered by using a column procedure that requires that column answers result from operating on <u>2 numbers</u>. It is assumed that for 2-by-1-digit, nonrectangular problems, the diagonal error results because children are attempting to apply the binary operation rule in the tens column in which only one number is located. Faced with this impasse in the execution of the procedure, the children improvise by using the single-digit addend as the second number with which to perform the operation that yields the tens column answer.

In a related theoretical analysis, Brown & VanLehn (1980, p.394) interpret this error as resulting from an impasse reached because a column operation

TABLE 6

EXAMPLES OF GRADE 3 DIAGONAL ERRORS

23	56	1 78	7 1 ~~8~~3
+ 4	− 4	+ 5	− 5
67	12	133	28

rule that handles blanks in the bottom row has not be internalized. The impasse calls forth a special problem-solving repair heuristic - Refocus Right. This heuristic is said to search horizontally from the blank space to the nearest nonblank space and read the number. This number can then be operated on by the usual binary-based column operation rule. Both of these interpretations are a Fixed Rule approaches to the analysis of the diagonal error, and do not predict that such an error should be contextually sensitive.

An alternative interpretation of the diagonal error, compatable with the contextual effects reported in this thesis, views it as resulting from a problem solver who solves the 2-by-1-digit addition and subtraction problems by analogy to the the 2-by-1-digit multiplication algorithm. That is, the typically taught 2-by-1-digit multiplication procedure contains a diagonal column procedure, where the tens-place number in the top double-digit number is multiplied to the single-digit bottom number (multiplier) to yield the tens (and hundreds) place number for the answer. Perhaps children make the diagonal error to the 2-by-1-digit addition and subtraction problems because they are solving these problems analogously to the manner in which they solve the 2-by-1-digit multiplication problems. The analogical derivation is encouraged perhaps because of the obvious visual similarity between the 2-by-1-digit addition, subtraction, and multiplication problems.

The binary operation rule hypothesis and the analogy-to-mulitiplication hypothesis can be differentiated by considering the different classroom learning experiences assumed to provide a basis of the error. The binary operation rule hypothesis assumes that learning experiences associated with 2-by-2-digit addition and subtraction are important in determining the error, while the analogy-to-multiplication hypothesis assumes that learning experiences associated with 2-by-1-digit multiplication are important. In the arithmetic curriculum in the

Hamilton schools, the column addition and subtraction algorithm is taught during grade 2; the multiplication algorithm is taught during grade 3; therefore, the two hypotheses make differing predictions about the developmental onset of diagonal errors for Hamilton taught grade 3 children. The first diagonal study was a cross-sectional developmental study that traced the developmental course of diagonal errors with special emphasis on the grade at which diagonal errors are first to be observed.

The second study attempted to assess the role of 2-by-1-digit multipication classroom instruction as a determiner of diagonal errors by measuring diagonal errors immediately before classroom instruction in the multiplication algorithm was given and immediately after it terminated, during the spring portion of the grade-3 arithmetic curriculum in Hamilton. Higher rates of diagonal errors during post-lesson testing would implicate multiplication instruction as an important factor.

Two other variables were studied in this second experiment as extensions of the contextual and instructional arguments made for grade 2 errors. The grade 2 studies (studies 1-5) showed the importance of local problem context in determining computation errors. If problem context is a generally important determiner of arithmetic learning and problem solving at the primary school level, then the context of problems surrounding 2-by-1-digit addition and subtraction problems (problem targets for the diagonal error) should be important in determining the occurrence of diagonal errors. In particular, a context containing 2-by-1-digit multiplication problems might increase diagonal errors if the analogy-to-muliplication hypothesis of diagonal errors is correct. This prediction about contextual effects is made more plausible by considering how this hypothesis views the problem solver - i.e. solving by analogy. Because there can be many different bases for an analogy, it might well

be expected from this view that a problem solver operating by a principle of analogy would change his problem solving approach under circumstances which change the basis of his analogy. Problem context may operate to determine the probable basis for the analogy used to solve target problems; therefore, target problem solving would be expected to change with the problem context. Two-by-1-digit multiplication problem contexts may set the probable basis for the analogy used to solve 2-by-1-digit addition and subtraction problems embedded in that context such that a high rate of diagonal errors would be expected. Moreover, if correct responding to 2-by-1-digit addition and subtraction problems is desired, then a problem context must be constructed that provides a basis for an analogy that promotes correct responding. In the case of 2-by-1-digit addition and subtraction targets, a correct column procedure would be a vertical procedure, analogous to that performed on rectangular problems. Therefore, a problem context that would provide an analogy basis promoting correct responding would be a rectangular problem context. This type of context was used in the second study to promote correct responding and to serve as an alternative to the 2-by-1-digit multiplication context, which was expected, by the analogy view, to promote high rates of diagonal errors.

It is important to note that in order for the analogy point of view to predict diagonal errors it is a necessary assumption that certain aspects or components of 2-by-1-digit multiplication problem solving be separable. That is, it must be the case that the operation of multiplying numbers is separable from the choosing of numbers along a diagonal; only then would it be possible that the child could add or subtract numbers chosen along a diagonal portion of 2-by-1-digit addition or subtraction problems. Such an assumption is of particular interest in that it is contrary to a hidden assumption of Fixed Rule approaches to modeling children's arithmetic knowledge and problem solving; the assumption being that

procedural rules that comprise an algorithm or "core" procedure are tightly and inseparably organized - i.e. once the algorithm is learned, the subprocedures are assumed to be linked to one another inseparably unless changed by explicit instructional experiences. Thus, a demonstration of context driven diagonal errors would be especially difficult to explain for Fixed Rule approaches.

Not only did the second study focus on the potential role of problem context in determining diagonal errors, but it also tested the possibility, suggested by the grade 2 studies, that the absence of mixed drills in the classroom may be an important instructional variable determining diagonal errors. Specifically, the possibility was entertained that diagonal errors occurred also because unmixed drills were being exclusively used in the classroom during 2-by-1-digit multiplication instruction. Unmixed drills of 2-by-1-digit multiplication problem solving would not provide learning experiences in which children could learn to make vertical-column responses to 2-by-1-digit addition and sobtraction problems in the problem context of making diagonal responses to 2-by-1-digit multiplication problems. Therefore, similar to the grade 2 mixed drill study, a group of children were exposed to teacher-administered, in-classroom mixed drills over the course of their normally occurring 2-by-1-digit multiplication instruction period in which 2-by-1-digit multplication problem solving was interspersed with 2-by-1-digit addition and subtraction problem solving. Post-lesson comparisons of this specially treated group of children to the untreated classrooms served as a measure of the importance of unmixed drills in determining diagonal errors.

The third and final study of the diagonal error concerned the question of the generality and robustness of contextual influences on diagonal errors. Diagonal errors were assessed using a short well-known standardized arithmetic test (the Wide Range Achievement Test, Jastak & Jastak, 1978) that contains a problem sequence in

which a 2-by-1-digit multiplication problem precedes a 2-by-1-digit addition problem. An alternative test form was constructed by eliminating the multiplication problem to see if its presence was necessary for the occurrence of diagonal errors to the target addition problem. A contextual view of the diagonal error based on an analogy-to-multiplication interpretation predicts that the presence of the multiplication problem should be necessary.

Together, these studies are designed to achieve 3 main objectives. First, to evaluate the contributions made by contextual factors in determining diagonal errors. Second, to determine the type of arithmetic instruction that is responsible for diagonal errors. And third, to evaluate the effectiveness of a type of classroom training in eliminating diagonal errors. The hypothesis for these studies is that arithmetic learning and problem solving is importantly influenced by problem context, a proposition hetherto ignored in contemporary educational and psychological theories of children's arithmetic learning and problem solving, but suggested by the previously presented studies of problem solving of grade 2 children.

Chapter 8

STUDY 6: A CROSS-SECTIONAL DEVELOPMENTAL STUDY OF THE DIAGONAL ERROR

This study addressed the issue of whether diagonal errors result from (1) the use of a binary operation rule learned from instructional and drills in column addition and subtraction (see above for details), or (2) the use of a 2-by-1-digit multiplication strategy based on learning the mulitplication algorithm. Because instruction and drills in column addition and subtraction begins at grade 2, and in 2-by-1-digit multiplication in grade 3, these alternative hypotheses for the error can be differentiated by a cross-sectional developmental research design that compares diagonal errors among groups of grade 2 and 3 children. This study provided a complete analysis of primary school-level children by comparing grade 2,3,4,5, and 6 children with respect to the incidence of diagonal errors to an item on a standardized arithmetic test - the arithmetic subtest of the Wide Range Achievement test (Jastak & Jastak,1978)

METHOD

SUBJECTS

The subjects included 282 children from 16 separate classes of 4 different schools representing grades 2 to 6 from the Hamilton Board of Education. The 4 different schools were located within a 3 mile radius, and all were in the same middle class socieconomic strata. The number of classes per grade were: 2 from grade 2, 2 from grade 3, 4 from grade 4, 3 from grade 5, and 5 from grade 6. All classes were tested during the last month (June) of the school year.

MATERIALS

The test used to assess diagonal errors was the Arithmetic subtest of the Wide Range Achievement Test level 1 (Jastak & Jastak, 1978) which is a 10-minute test composed of a single worksheet of 8 rows of different types of problems mixed together. The target problem of interest in this study was one problem located on the first line of problems - a 2-by-1-digit addition problem (75+8=83).

PROCEDURE

Standardized administration procedures were used for this test which included verbal instruction that emphasized that (1) the children would be given 10 minutes to do as many problems on the page as possible, (2) there were many different types of problems on the test, (3) they should look at each problem carefully to figure out what it asked them to do - add, subtract, multiply, or divide, (4) they could skip over problems that they could not do, and (5) they should start at the top, working left-to-right.

The children were tested in groups of one class at a time in their normal classrooms with the teacher sometimes absent from the room.

RESULTS AND DISCUSSION

Table 7 gives the number of children who made a diagonal error to the target problem, 75+8=83, for each classroom tested at each grade level. A diagonal error was defined as any response that contained the number 16 +/-1 in the hundreds-tens or thousands-hundreds places of the answer. The thousands-hundreds place was included because some children failed to "carry" in the units column by placing the 2-digit result of adding the units numbers (i.e.5+8=13) in the tens-units place of the answer, and then used a diagonal procedure on the ten-place number of the 2-digit addend (7+8=15) and placing the result in the thousands-hundreds place of the answer. The responses scored as diagonal errors were 75+8=153 or 163 or 1513.

TABLE 7

NUMBER OF DIAGONAL RESPONDERS
ON WRAT TEST FOR GRADES 2 TO 6

	Grade 2			Grade 3			Grade 4				
Class	1	2	Total	1	2	Total	1	2	3	4	Total
Dia.	0	0	0	4	8	12	3	6	5	2	16
N	24	25	49	13	17	30	11	9	26	20	66
%	0	0	0	31	47	40	27	67	19	10	24

	Grade 5				Grade 6					
Class	1	2	3	Total	1	2	3	4	5	Total
Dia.	0	7	5	12	1	7	3	1	0	12
N	5	30	26	61	26	26	27	22	24	125
%	0	23	19	20	4	27	11	5	0	10

Diagonal responder = Subject committing a diagonal error on 2-by-1-digit addition problem.
DIA = number of diagonal responders in group.

All of the children given the WRAT attemtped to answer the target problem. And, as indicated in Table 7, the percentage of children at each grade level from 2 to 6 committing a diagonal error was 0, 40, 24, 20, 10, thus identifying grade 3 as the grade at which diagonal errors are first seen. Corrected Chi-Square tests (one-tailed) on grade 3 vs 4, 3 vs 5, and 3 vs 6 yielded significant differences for 3 vs 5 and 3 vs 6 only (Chi-Square= 1.77, $p<.1$; 3.29, $p<.05$; 14.84, $p<.0005$, respectively).

Post-hoc examin ation of the variability in diagonal errors among classrooms within a grade level failed to reveal significant classroom effects. Two statistical comparisons were attempted on classrooms at the grade 3 and 4 level following Siegel's (1956) guidelines that chi-squares are meaningful only if no expected frequences are less than 1. Neither test was significant; at grade 3, the corrected Chi-Square=0.062, and at grade 4, the Chi-Square=5.77 (df=2, 2-tailed, $p<.1$).

Thus, the results are clear in indicating that diagonal errors do not arise until grade 3, after which reliable decreases in diagonal errors occurs. Interestingly, even at as late as grade 6, evidence could be found for diagonal errors (10%) on this test. No evidence, however, was found for classroom effects.

These results support the analogy-to-multiplication interpretation of diagonal errors in predicting that such errors should not be seen until grade 3. The next study further localized the time within grade 3 at which this error becomes evident, and tests for the possible role of contextual and instructional variables in determining its occurrence.

Chapter 9

STUDY 7: EFFECTS OF MULTIPLICATION LESSONS, MIXED DRILLS, AND PROBLEM CONTEXT ON DIAGONAL ERRORS

The purposes of the present study at grade 3 are to confirm (1) the necessity and sufficiency of 2-by-1-digit multiplication lessons in determining diagonal errors, (2) the role of unmixed drill practices in enabling such errors to arise, and (3) the effect of problem context in influencing diagonal errors in an analogous manner to what had been previously shown for column-subtraction-based errors to 2-by-1-digit subtraction facts at grade 2.

To serve these purposes, the teachers of 3 classrooms of grade 3 children were asked if their classes would participate in a study evaluating the effects of instruction in the 2-by-1-digit multiplication algorithm. Pretests given one day before instruction in this algorithm were scheduled to begin and post-tests given one day after a 2-4-week instruction period terminated were designed to evaluate the importance of the intervening instruction in determining diagonal errors. These tests were also designed to enable an evaluation of the importance of problem context in determining diagonal errors. Two problem contexts for 2-by-1-digit addition and subtraction target problems were constructed, one containing 2-by-1-digit multiplication problems and another containing 2-by-2-digit addition and subtraction problems. If diagonal errors result from drawing analogies to multiplication problem solving (and if the operation of multiplying two numbers is separable from the operation of selecting numbers along a diagonal), and if problem contexts can determine the predominant basis for these analogies, then diagonal errors should be more prevalent in the multiplication context than in the context containing

rectangular addition and subtraction problems, on the basis of which a vertical column procedure should result from the analogy.

The role of classroom drill practices was also investigated in this study by including a classroom in which mixed drills were programmed to be administered during multiplication instruction. The mixed drills included teacher-constructed and -administered worksheets of 2-by-1-digit addition, subtraction, and multiplication problems. Prelesson and postlesson testing was conducted with this classroom to provide a comparison to the other classrooms in which mixed drills were not explicitly programmed. If the lack of mixed drills is responsible for diagonal errors, then the mixed-drill classroom should evidence significantly less diagonal errors during postlesson testing.

METHOD

SUBJECTS

The subjects in this study included 3 classrooms of students scheduled to be given normal classroom instruction in the 2-by-1-digit multiplication algorithm, a grade 3 curriculum objective of the Hamilton Board of Education typically scheduled for the early spring portion of the year. Two classrooms were located in the same school - a grade 2 and 3 classroom (N=17 and 21), and a third classroom, grade 3, (N=27) was located in a different school approximately 2 miles away but in a similar socieconomic area.

MATERIALS

Two worksheets comprised the prelesson and postlesson test materials administered to all children in the study. Each test consisted of 36 problems presented in vertical form, 24 target problems and 12 context items.

The 24 target items consisted of 12 addition and subtraction problems,

half of each requiring regrouping (carrying, cancelling). Different numbers were used to construct the target items for each worksheet.

The 12 context items were of different types for the two worksheet. For context-test M (for multiplication) the 12 items were 2-by-1-digit multiplication problems, half requiring regrouping. For context-test AS (for addition and subtraction), the problems were 2-by-2-digit addition and subtraction problems.

The tests were constructed so that similar targets and context problems were located in the same positions on each worksheet. For example, all 2-by-1-digit addition and subtraction problems requiring regrouping were located in the same position between the two worksheets. In addition, wherever a multiplication context item was located on test M, a rectangular addition or subtraction item was substituted in the same place on test AS. Thus, the same sequence of context and target types of problems occurred in both worksheets; only the type of context problems changed across worksheets.

PROCEDURE

Teachers of the 3 classrooms were approached to volunteer their classroom for a study of the "effects of 2-by-1-digit multiplication learning on arithmetic problem solving". For each classroom, the day before the lessons were normally scheduled to begin was scheduled as the prelesson test day. During the following weeks of multiplication instruction, the teacher was periodically contacted to determine exactly when instruction would be completed and, thus, when the first day after instruction would occur. Postlesson testing, using the identical tests as prelesson testing, was scheduled for this first day after multiplication instruction.

The grade 2 classroom was chosen to experience mixed drills during the course of multiplication instructions as determined by the teacher. The other two grade 3 classrooms followed normal instructional practices programmed by their

teachers; these two teachers did not know the purpose of the study until postlesson testing.

The mixed-drill grade 2 classroom teacher was informed of the purpose of the study and the potential benefits of mixed drills on diagonal errors. (Because the experimenter did not have the authority within the schools to program the drills himself, telling the teacher of the benefits of mixed drills and purpose of the study was the technique used to apply the specialized training). Accordingly, she designed a multiplication instruction plan that included periodic drills with 2-by-1-digit addition, subtraction and multiplication problems combined. The way in which the teacher implemented her mixed-drill plan was not monitored; the teacher was simply told of the beneficial effects of mixed drills and encouraged to incorporate them into their instruction. Following the study she was asked to submit a written description of her plan (see footnote 3).

The prelesson and postlesson testing procedure was identical and involved administering both context worksheets to two randomly selected groups counterbalanced for order of testing. One half received test M first, the other half received test AS. All children were told before testing that they were going to be given two worksheets containing addition, subtraction, and multiplication problems, that they should put their name and test number (1 or 2) on the back of their test before starting. They were further told to work the problems starting at the top left, working across from left-to-right, line-by-line down the page. The test administrator circulated around the room during testing to insure all children were following directions; if a child was not, he was quietly reminded of the rule, and told where on his page to continue solving problems. For prelesson testing, they were also told that they may come across problems they do not know how to do yet, and if this happens to skip the problem. For postlesson testing, they were told they

should not skip any problem unless they absolutely could not "figure out how to do it".

RESULTS AND DISCUSSION

Prelesson and postlesson tests were scored for the occurrence of diagonal errors on the target problems. A diagonal error was defined as any response in the tens and hundreds place of the answer which equaled (+/-1) the sum of the diagonal numbers in the addition problems (i.e. tens-place numeral in the top 2-digit numeral and the single-digit addend, as well as any 'carried' numeral) or the difference of the diagonal numerals in the subtraction problems (the tens-place numeral could be any numeral written above by the child as a result of a 'borrowing' operation).

The duration of the classroom lessons was approximately 1 month for the two grade 3 classrooms, and 2 weeks for the mixed drill grade-2 classroom.

PRELESSON TEST RESULTS

The prelesson results concerning diagonal errors to 2-by-1-digit addition and subtraction target problems were clear in showing that virtually no such responding occurred before multiplication lessons; thus, providing further support for the contention that diagonal errors result from multiplication learning experiences. Of the 62 children who completed both tests from the 3 classrooms, only 3 (4.8%) showed any evidence of diagonal errors, and the response occurred very infrequently to the targets in these cases - only once in each case. Furthermore, all of these children showed some evidence of actually knowing the multiplication algorithm before the lessons, as indicated in correct responding to the multiplication context problems - 58% correct, 67%, and 100% correct. Therefore, these subject's diagonal responding prior to classroom instruction in the multiplication algorithm does not constitute evidence of diagonal errors that

contradicts the hypothesis that such errors result from multiplication learning experiences.

In order to answer the 2 main questions addressed in this study, i.e. do diagonal errors result from normal multiplication instruction and are they eliminated by mixed drills?, the prelesson data be must be inspected for 2 attributes. First, the children in the classrooms must show a lack of knowing the multiplication algorithm prior to being giving the lesson. And second, the classrooms must be comparable in overall levels of performance on the target and context items prior to the lessons, so that any subsequently observed postlesson differences can be attributed to differences in the nature of the lessons themselves. Table 8a,b,and c give the results of the prelesson testing in terms of mean and standard deviation errors on target and context items for the two worksheets for the 2 test-order groups among the 3 classrooms.

With regard to prelesson responding on the multiplication context items on worksheet M, there is clear evidence of a lack of knowing the multiplication algorithm. The mean number of errors on the 12 context items was 8.12, 8.16, and 8.30 for the grade 2 and grade 3 classrooms respectively. This represents an average percent error of 68%, 68%, and 69% among the classrooms, and indicates that the classrooms were comparable in their prelesson test multiplication performance. These percentages are not higher because some of the 2-by-1-digit multiplication problems were solved by addition (e.g. 23X3= 23+23+23=69), rather than by using any multiplication algorithm, an observation which could be made because of their scratch work written clearly next to the problems.

Prelesson performance on the 2 remaining types of problems - 2-by-1-digit and 2-by-2-digit addition and subtraction problems were also comparable in the different classrooms, except in 2 instances. The two grade 3 classrooms differed

TABLE 8a
MEAN NUMBER OF ERRORS TO TARGET AND CONTEXT
PROBLEMS ON PRELESSON TESTS FOR CLASSROOM FP-Gd 2 IN STUDY 7

		Context Condition					
		Test M			Test AS		
Group	n	2X1	2+-1	TOTAL	2+-2	2+-1	TOTAL
M-AS	8						
MEAN		7.75	3.00	10.75	1.38	3.50	4.88
S.D.		3.28	3.74	3.54	2.33	4.44	6.75
AS-M	9						
MEAN		8.44	6.11	14.56	3.11	5.33	8.44
S.D		3.88	4.48	7.47	2.76	4.36	6.93
TOTAL	17						
MEAN		8.12	4.65	12.77	2.29	4.47	6.76
S.D		3.52	4.33	6.10	2.64	4.36	6.88

M = multiplication context test
AS = addition/subtraction context test
2X1 = 2-by-1-digit multiplication context problems
2+-1 = 2-by-1-digit addition and subtraction target problems
2+-2 = 2-by-2-digit addition and subtraction context problems
M-AS = Group given M test first followed by AS test
AS-M = Group given AS test first followed by M test

TABLE 8b
MEAN NUMBER OF ERRORS TO TARGET AND CONTEXT
PROBLEMS ON PRELESSON TESTS FOR CLASSROOM I-Gd 3 IN STUDY 7

		Context Condition					
		Test M			Test AS		
Group	n	2X1	2+-1	TOTAL	2+-2	2+-1	TOTAL
M-AS	14						
MEAN		7.07	4.79	11.86	3.86	5.36	9.22
S.D.		3.58	3.47	6.22	2.91	3.90	6.22
AS-M	11						
MEAN		9.55	7.55	18.10	4.00	6.09	10.09
S.D		2.30	5.75	7.23x	2.76	4.55	6.93
TOTAL	25						
MEAN		8.16	6.00	14.16	3.92	5.68	9.60
S.D		3.27	4.72	7.06	2.78	4.12	6.42

M = multiplication context test
AS = addition/subtraction context test
2X1 = 2-by-1-digit multiplication context problems
2+-1 = 2-by-1-digit addition and subtraction target problems
2+-2 = 2-by-2-digit addition and subtraction context problems
M-AS = Group given M test first followed by AS test
AS-M = Group given AS test first followed by M test

x due to one extreme score.

TABLE 8c
MEAN NUMBER OF ERRORS TO TARGET AND CONTEXT
PROBLEMS ON PRELESSON TESTS FOR CLASSROOM FP-G3 3 IN STUDY 7

		Context Condition					
		Test M			Test AS		
Group	n	2X1	2+-1	TOTAL	2+-2	2+-1	TOTAL
M-AS	10						
MEAN		7.80	3.20	11.00	1.30	3.20	4.50
S.D.		5.10	1.81	4.57	1.49	3.26	4.60
AS-M	10						
MEAN		9.10	2.60	11.70	2.30	4.20	6.50
S.D		4.15	2.32	5.23	2.58	2.90	5.34
TOTAL	20						
MEAN		8.30	3.00	11.35	1.80	3.70	5.50
S.D		4.33	2.05	4.79	2.12	3.05	4.96

M = multiplication context test
AS = addition/subtraction context test
2X1 = 2-by-1-digit multiplication context problems
2+-1 = 2-by-1-digit addition and subtraction target problems
2+-2 = 2-by-2-digit addition and subtraction context problems
M-AS = Group given M test first followed by AS test
AS-M = Group given AS test first followed by M test

significantly on 2-by-2-digit addition and subtraction problems, with mean errors of 3.92 vs 1.80 ($t=2.66$, 2-tailed, $p<.02$), and on 2-by-1-digit addition and subtraction on test M, with mean errors of 6.00 vs 3.00 ($t=2.58$, 2-tailed, $p<.02$). However, there were no overall differences in prelesson performance between either grade 3 class and the grade 2 class subsequently given mixed drills. And, as shown in Table 9, there are no differences between the combined grade 3 classrooms and the grade 2 classroom on any of the items, indicating that the differently treated groups of children were not different on prelesson assessment of target problem solving. As expected, prelesson performance on the 2-by-1-digit addition and subtraction targets was quite good, with average error percentages of 19.25% and 20.0% for grade 3 classrooms under the 2 test conditions, and 20.6% and 18.6% for the mixed-drill, grade-2 classroom. Two-by-2-digit problem solving was comparable to the 2-by-1-digit problem solving for both groups of children, with error percentages of 24.5% and 19.0% for grade 3 and 2 children.

Therefore, the prelesson data demonstrates that performance on the target and contexts items was comparable across the classrooms subsequently given different types of multiplication instruction, so that postlesson classroom differences can be unambiguously attributed to treatment differences and not to measured prelesson differences.

POSTLESSON TEST RESULTS

Table 10a and 10b give the results of the postlesson testing for the grade 3 classrooms combined and the grade 2 classroom in terms of diagonal errors and total errors committed on the target items under the various testing conditions for those subjects who completed both worksheets. Notice should be taken of the fact that the distributions of the error scores were asymmetrical, as indicated in the consistently obtained lower values for median than mean statistics However, they

TABLE 9
MEAN NUMBER OF ERRORS TO TARGET AND CONTEXT
PROBLEMS ON PRELESSON TESTS FOR FP-Gd 3 and I-Gd 3 IN STUDY 7

		Context Condition					
		Test M			Test AS		
Group	n	2X1	2+-1	TOTAL	2+-2	2+-1	TOTAL
M-AS	24						
MEAN		7.38	4.13	11.50	2.79	4.46	7.25
S.D.		4.19	2.95	5.50	2.70	3.73	5.98
AS-M	21						
MEAN		9.33	5.19	14.52	3.19	5.19	8.38
S.D		3.23	5.04	6.79	2.75	3.88	6.34
TOTAL	45						
MEAN		8.29	4.62	12.91	2.98	4.80	7.78
S.D		3.86	4.05	6.25	2.70	3.78	6.11

M = multiplication context test
AS = addition/subtraction context test
2X1 = 2-by-1-digit multiplication context problems
2+-1 = 2-by-1-digit addition and subtraction target problems
2+-2 = 2-by-2-digit addition and subtraction context problems
M-AS = Group given M test first followed by AS test
AS-M = Group given AS test first followed by M test

were not as bimodally distributed as the errors in previous studies, and therefore, medians were chosen as proper descriptive statistics, and non-parametric tests were used to test all statistical hypotheses.

First, with respect to the hypothesis that instruction in the 2-by-1-digit multiplication algorithm is sufficient to promote diagonal errors, there is a significant increase in diagonal errors from prelesson to postlesson testing. Only 1 subject among the 48 grade 3 subjects receiving regular multiplication instruction (2%) made diagonal errors on the prelesson test; in comparison to 40 of 48, or 83% of those same subjects made diagonal errors on the postlesson tests. Thus, diagonal errors are virtually nonexistent before regular instruction in the 2-by-1-digit multiplication algorithm, and are prevalent among the same subjects after such instruction.

This finding together with the results of the previous cross-sectional developmental study strongly argues for an interpretation of diagonal errors which stresses the use of an adapted 2-by-1-digit multiplication, diagonal subprocedure as opposed to an interpretation stressing a binary-operation-rule interpretation (Friend, 1979; Davis & McKnight, 1979; Davis, 1980. According to binary-operation-rule interpretation the diagonal error was said to occur because of earning experiences occurring during grade 2 rectangular addition and subtraction instruction.

Second, with respect to the hypothesis that mixed drills should reduce diagonal errors, it can be seen from Table 10a and 10b that indeed the children in the mixed-drill, grade-2 classroom made significantly fewer diagonal and total target errors than the regularly taught grade 3 classrooms. Under the multiplication context conditions, 21 of 25 (84%) subjects made diagonal errors in the regularly taught classrooms; among mixed-drill, grade-2 subjects only 1 of 8 (13%) made

TABLE 10a
MEAN AND MEDIAN DIAGONAL ERRORS ON POST-TESTS FOR GRADE 3 CLASSROOMS IN STUDY 7

		Context Condition			
		Test M		Test AS	
Group	n	Diagonal	Total	Diagonal	Total
M-AS	25				
Mean		8.40	13.56	6.12	11.32
Median		7.00	13.00	2.17	8.50
S.D.		7.16	7.30	7.54	7.75
AS-M	23				
Mean		3.30	7.91	5.78	9.78
Median		0.90	7.00	2.50	8.00
S.D.		5.67	7.11	7.03	7.38

TABLE 10b
MEAN AND MEDIAN DIAGONAL ERRORS ON POST-TESTS FOR MIXED DRILL GROUP IN STUDY 7

		Context Condition			
		Test M		Test AS	
Group	n	Diagonal	Total	Diagonal	Total
M-AS	8				
Mean		1.75	3.78	0.25	4.46
Median		0.14	3.50	0.25	1.33
S.D.		4.95	4.21	0.46	3.73
AS-M	8				
Mean		0.25	6.11	0.38	3.50
Median		0.14	2.00	1.06	3.25
S.D.		0.71	4.48	1.06	4.44

diagonal errors (Fisher exact probability, 1-tailed, $p=.00053$). Thus, mixed-drills interspersed with multiplication instruction greatly reduced postlesson diagonal errors, which supports the hypothesis that diagonal errors result from the exclusive use of unmixed drills during 2-by-1-digit multiplication instruction. This finding and the mixed-drill findings of grade 2 suggest that the absence of mixed drills and the use of unmixed drills by teachers is partially responsible for several dominant computational errors seen during the primary-school years.

Third, with respect to the hypothesis that diagonal errors are contextually influenced by 2-by-1-digit multiplication problems, there were differences between the context tests in the predicted direction, but these differences did not reach conventional levels of statistical significant. Figure 6 displays the median probability of diagonal and nondiagonal errors committed to the targets under the two context conditions for the two counterbalanced groups. The median number of diagonal errors committed with a multiplication context (Test M) for the group receiving this test first (Group M-AS) was 7.00, or a median probability of error of .318 (number of items=22); in contrast, the median number of diagonal errors committed with a context composed of rectangular addition and subtraction problems for the group receiving this test first (Group AS-M) was 2.50, or a median probability of error of .091. This between-group context difference in diagonal errors did not reach convention levels of significance (Mann-Whitney U test corrected for tie scores, $z=1.485$, 1-tailed, $p<.068$).

Clear evidence for symmetrical carry-over effects from one test to the subsequent test were seen in the data. Specifically, changes in context from nonrectangular multiplication problems to rectangular addition and subtraction problems resulted in a significant decrease in diagonal errors from median error scores of 7.00 to 2.17 (Wilcoxon signed-ranked test, 1-tailed, $T=31$, $p<.025$).

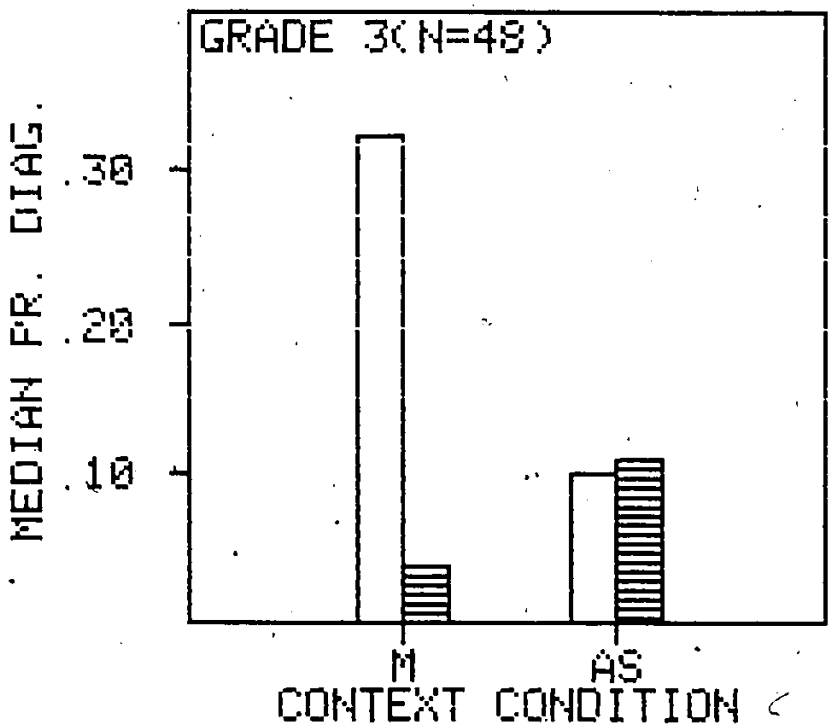

Figure 6. Median probability of a diagonal error to targets in multiplication problem contexts (M) and addition/subtraction contexts (AS) for Group M-AS (open rectangles) and Group AS-M (hatched rectangles) of Study 7.

Changes from a context of rectangular addition and subtraction problems to nonrectangular multiplication problems also resulted in a decrease in median diagonal errors from 2.50 to 0.90 (Wilcoxon, 2-tailed, $T=22$, $p<.01$). Thus, a multiplication context did not result in higher diagonal error rates if a previous context was experienced consisting of rectangular addition and subtraction problems, suggesting either or both of the following possibilities: the prior effects of the rectangular addition and subtraction problem context carried over to reduce the normal effects of the multiplication context, or continual practice with the targets lead to a progressive decline in incorrect error responding (through self-correction) since the carry-over effects were symmetrical.

Because the context effects observed for diagonal errors in this study failed to reach conventional levels of statistical significance the next study was also conducted to test the hypothesis that diagonal errors are significantly contextually influenced.

Chapter 10

STUDY 8: A FURTHER STUDY OF THE EFFECTS OF CONTEXT ON DIAGONAL ERRORS

A second study on diagonal errors was designed to test the robustness of the context effects observed in the previous study. In that study differences were observed that failed to reach conventional levels of statistical significance under conditions in which the tests were given on the first school day after 2-by-1-digit multiplication instruction was completed. The tests used were worksheets containing 24 targets and only 12 context items. Perhaps only nonsignificant differences between contexts were observed in that study because the effects of the multiplication lessons were so powerful that effects of the context variable were masked. On this assumption, one might expect significant context effects if testing were to be conducted longer after multiplication instruction.

Another grade 3 classroom was tested several weeks later in the Spring (end of May) using the original context worksheets. In addition, a different set of context conditions was also added, adapted from a standardized test, to test the generality of the hypothesized context effect. The arithmetic subtest of the Wide Range Archievement Test Level 1 (Jastak & Jastak,1978), the same test used for the cross-sectional developmental study, was given to separate groups, either in regular form which arranges a problem sequence in which a 2-by-1-digit multiplication item is followed by a 2-by-1-digit addition target (separated by a 2-by-2-digit subtraction problem), or in a slightly altered form in which the 2-by-1-digit multiplication problem is simply omitted. Based on the same logic used in the previous diagonal-error study, it was expected that diagonal errors would be reduced if the 2-by-1-digit multiplication item was omitted since the presence of this problem type

is assumed to play a role in contextually determining diagonal errors committed on the 2-by-1-digit addition problem. Of additional interest, was the fact that the WRAT test is a standardized test which is often used as a measure of arithmetic achievement in school systems (Reid & Hresko,1979), and therefore, the results would be of potential interest to many who use the test routinely.

METHOD

SUBJECTS

An entire grade 3 classroom (N=24) located in a different school from the previous diagonal study was used. This school was also in Hamilton, Ontario and was located within 2 or 3 miles of the other schools and in a similar middle-class neighborhood.

MATERIALS

Two sets of 2 different context worksheets each were used in this study. One set was identical to context worksheets test M and AS used in the previous study and is fully described there (see also Appendix VII and VIII). Briefly, 24 2-by-1-digit addition and subtraction targets were randomly interspersed among either 12 2-by-1-digit multiplication problems (Test M) or 12 2-by-2-digit addition and subtraction context problems (Test AS).

The other set of context worksheets was based on the Arithmetic subtest of the Wide Range Achievement Test, level 1 (Jastak & Jastak,1978). One test was identical to the standard WRAT form; the other was identical to the standard WRAT except the 2-by-1-digit multiplication item (23X3=69) was omitted. This left an empty space in place of the 2-by-1-digit multiplication item.

PROCEDURE

Two consecutive days of testing were conducted. On day 1, a random half of the classrooom was given the standard WRAT and simultaneously the other half was given the altered WRAT using standard instructions, discussed completely in diagonal-error study 1 above, that emphasized, among other things, that care be taken in identifying the type of arithmetic operation (add, subtract,multiply,divide) required by each problem.

On the following day, test day 2, the 2 context worksheets used in the previous study (Test M and AS) were adminstered according to the following procedure (see Table 11). All subjects given the standard WRAT on day 1 were assigned to group AS-M and were given test AS first followed by test M on day 2; conversely, all subjects given the altered WRAT on day 1 were assigned to group M-AS and were given test M first followed by test AS on day 2. This group assignment procedure was used to enhance the power of the context demonstration by arranging that the expected effect on diagonal errors of the first test given on day 2 (AS or M) was opposite to the expected effect of the type of WRAT test (standard or altered) given on day 1. For example, subjects given the standard WRAT on day 1 would be expected to exhibit elevated levels of diagonal errors compared to those receiving the altered WRAT; therefore, on day 2, they were given the AS test first because the expected effect under the AS context condition would be a reduced level of diagonal errors compared to test M. Consistency of expected context effects across days would be strong evidence for context influences.

RESULTS AND DISCUSSION

Diagonal errors for each test were defined as in previous studies using these worksheets. Three subjects were excluded because they failed to complete all tests over the two days, which left 11 subjects in group M-AS, and 10 in group AS-M

TABLE 11

DESIGN OF GRADE 3 DIAGONAL STUDY 8

	Day 1	Day 2	
		Test	
GROUP		1	2
AS-M	Standard WRAT	AS	M
M-AS	Altered WRAT	M	AS

AS = addition-subtraction context test.
M = multiplication context test.
Altered WRAT = WRAT without 2-by-1-digit multiplication problem.

for a total of 21 subjects.

On day 1, 6 of 10 subjects receiving the normal WRAT made diagonal errors to the 2-by-1-digit addition target (75+8), whereas only 1 of 11 subjects (9%) receiving the altered WRAT made such errors (Fisher exact probability, 1-tailed, $p=.021$). Thus, removing only 1 item from the test, two problems before the critical target problem, resulted in dramatically altered problem solving to the target, and changed error rates in this sample of grade 3 children from 9% to 60%. This finding strongly questions the validity of standard psychometric models of arithmetic achievement, discussed in chapter 1 and study 1, which assume that an item-difficulty parameter's value is exclusively determined by the internal structural characteristics of the item (e.g. number of digits involved, type of operation, whether regrouping is required or not, or the particular numbers used in the problem), and can be estimated by simply calculating the percentage of children at a certain grade level that pass the item under non-specific test conditions. In this study, it was clearly shown that the difficulty of an item depends extensively upon the item-sequence in which it occurs. Therefore, the theoretical value of an item-difficulty parameter is questioned, with its assumption of context-independence.

Day 2 results for study 8 are shown in Figure 7. Median diagonal errors were calculated since the distributions were positively skewed, as in the previous study. The median number of diagonal errors made by group M-AS under test M conditions was 3.00; whereas the median diagonal errors by group AS-M under AS conditions was 0.5 (Mann-Whitney U test, U=29, 1-tailed, $p<.05$). Therefore, a significant between-group context effect was obtained.

Although both within-subject context comparisons were in the predicted direction, neither were significant (Wilcoxon signed rank tests).

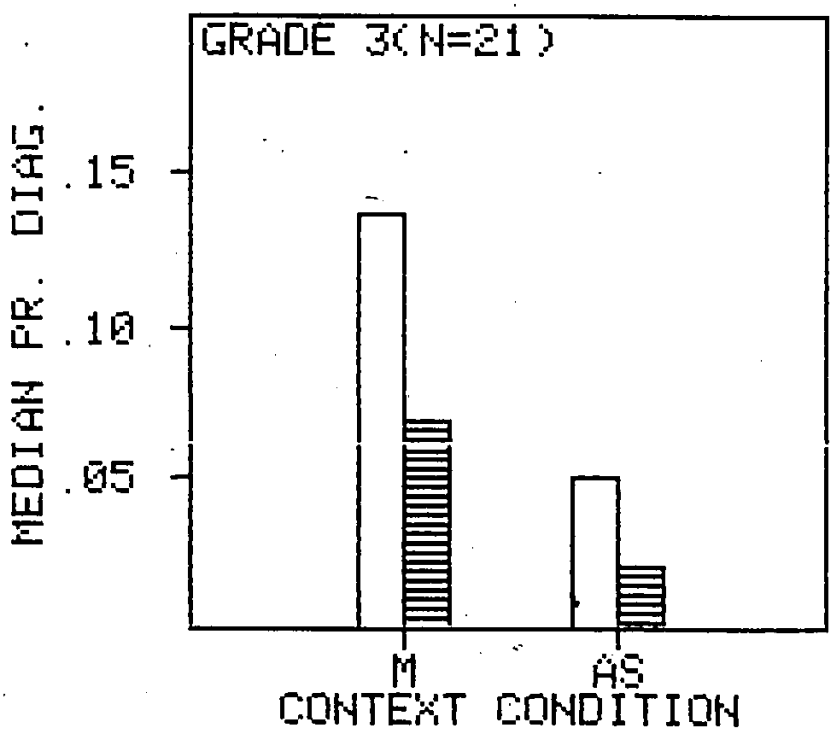

Figure 7. Median probability of a diagonal error to targets in multiplication problem contexts (M) and addition/subtraction contexts (AS) for Group M-AS (open rectangles) and Group AS-M (hatched rectangles) of Study 8.

The hypothesis was tested of whether the diagonal error rates were lower for this grade 3 classroom, tested several weeks after their 2-by-1-digit multiplication instruction, than for the grade 3 classrooms used in the previous study in which testing was conducted immediately following such instruction. Considering only the optimuum context condition with respect to promoting diagonal errors, Test M, the median rate was 2.00 here as compared to 7.00 for the postlesson testing from the previous study. This represents a significantly lower diagonal error rate (Mann-Whitnety U-test, z=3.039, 2-tailed, $p<.003$).

This study produced evidence of a significant context effect for diagonal errors. This finding is consistent with the analogy-to-multiplication interpretation and difficult for a Fixed Rule model of children's arithmetic learning. In particular, the context effect of diagonal errors questions the tacit assumption that the diagonal error represents a systematic error resulting from the learning of a faulty algorithm which is applied to problems exclusively on the basis of the current problem's internal structural characteristics.

Extension of the WRAT findings. As an extension of one of the findings in this study, the robustness of the WRAT context finding was further tested by giving the 2 forms used in this study to another group of grade 3 students, and entire classes of grade 2,4 and 5 children to also obtain information about the development of the effect. No diagonal errors occurred among grade 2 subjects (N=19) and the target item was answered predominantly correctly; for grade 3 subjects in total, including the 24 from this study, 9 of 16 or 56% made diagonal errors to the standard WRAT and only 1 of 14 or 7% made such an error to the altered WRAT (Chi-Square=6.044, 1-tailed, $p<.01$). At grade 4, nonsignificant context effects were seen; 5 of 11 or 45% made diagonal errors under standard WRAT conditions, and 1 of 10 or 10% under altered conditions (Chi-Square=1.60, 1-tailed, $p<.1$). At grade

5, diagonal errors are further reduced, with 2 of 14 or 14% making such errors to a standard WRAT and 0 of 16 doing so to the altered WRAT. Thus, the context effect found with the WRAT is consistent at the grade 3 level.

SUMMARY OF GRADE 3 CONTEXT STUDIES

The grade 3 context studies were aimed at extending the observations of context effects made with grade 2 arithmetic problem type - 2-digit-by-1-digit subtraction facts - to grade 3 material and a very common error observed at that grade level- 2-digit-by-1-digit addition and subtraction and the diagonal error. As in the grade 2 studies a type of problem was found which, as a context item, could induce the diagonal error to occur - 2-digit-by-1-digit multiplication. As with the studies of grade 2 problem solving, it was found that the use of unmixed drills during instruction of the basic algorithms was an important contributor to the occurrence of these errors. As will be discussed more fully below, one hypothesis for why teachers fail to give their students mixed drills is that they observe correct performance on unmixed tests, and interpret this to indicate successful learning and generality of correct performance to different test situations. This interpretation may be encouraged by the tacit acceptance of a Fixed Rule approach to children's arithmetic learning and problem solving.

The next two thesis studies extend the grade 2 and 3 studies by exploring the possible presence of context effects with grade 4 and 5 arithmetic problems using similar research designs.

Chapter 11

STUDY 9: THE EFFECT OF CONTEXT ON 2-BY-2-DIGIT MULTIPLICATION PROBLEM SOLVING AT GRADE 4

The purpose of this study is to extend the evidence of context-driven arithmetic problem solving to a grade 4 problem type. The main arithmetic curriculum objective for grade 4 is teaching the multi-digit multiplication algorithm, principly applied to 2-by-2-digit multiplication problems. To understand this current context study we shall first consider the multidigit multiplication algorithm, then the most often observed computation error to these types of problems, and finally the rationale behind the choice of context problems used in this study.

During grade 4, children are typically taught an algorithm that involves the calculation of "partial products" that are summed to yield the correct answer. This is a long and complex procedure that many grade 4 children find difficult to learn (Cox, 1975). The algorithm is as follows. If the vertical problem is 32X13, children are instructed to solve this problem by finding two partial products: 32X3=69 and 32X10=320, which are taught to be written underneath the vertically displayed problem in two rows. Following this, they are instructed to add these partial products by applying the already learned multi-digit addition algorithm to yield the final answer. This familiar procedure is called the partial-product algorithm. The procedure is a mechanical one in which children are to begin in the units column by using the already learned (in grade 3) 2-by-1-digit multiplication algorithm, with the diagonal subprocedure, to generate the first partial product (e.g. 32X3=69, the 1st partial product) which is to be written in the first row of

the answer space. Following this they are to immediately write down a "0" beneath the units answer of the first partial product, as a "place holder"; and multiply the tens number of the multiplier (the bottom number in the problem) to the units number of the top number (the multiplicans). This yields the tens number for the second partial product and, spatially, this subprocedure is a diagonal procedure in a mirror-image reverse orientation to the first diagonal procedure that is used to generate the first partial product. Finally, they are to multiply the tens numbers from top and bottom numbers to yield the hundreds and possibly thousands number of the second partial product. At this point, adding the two partial products, an old skill learned in grade 3, is to be executed on the two partial products. Of course, further substeps are involved when regrouping during multiplication occurs. The important point to note is that the 2-by-2-digit multipliction algorithm has many procedural steps, and therefore, can be mislearned and incorrectly performed in many different ways.

Although one observes many different types of miscalculations to 2-by-2-digit multiplication problems, the most often observed error is to multiply each vertical column separately and write the products below - i.e. 32X13=36, an error that can be referred to as a "vertical column" error (VC). The VC error can be interpreted to involve using a procedure analogous to the one used for rectangular multi-digit addition and subtraction where correct solutions are generated by adding or subtracting column-by-column. Conceivably, the VC error could be a result of drawing an analogy between the solution strategy for rectangular addition and subtraction problems and the rectangular multiplication problems, using a similar analogy process as apparently occurs with nonrectangular addition, subtraction and multiplication problems at grade 3 when the diagonal error is made. As in the grade 3 studies, this hypothesis has implications for what kind of problem contexts should

effectively influence VC errors to 2-by-2-digit multiplication targets among grade 4 children.

The hypothesis that VC errors at grade 4 result from context-driven analogies from rectangular addition and subtraction to mulitplication suggests that VC errors should be highly probable in a context containing rectangular addition and subtraction problems. Moreover, VC errors should be less probable in a context containing problems that encourage the use of the correct multiplication procedure, such as, the 2-by-1-digit multiplication problem which contains the diagonal procedure as one step in the partial-product algorithm. Following this line of reasoning, the current context study sought further evidence for context-specificity effects in arithmetic problem solving by comparing the incidence of VC errors to 2-by-2-digit multiplication target problems embedded in contexts of either 2-by-2-digit addition and subtraction problems or 2-by-1-digit multiplication problems. In addition to the counterbalanced presentation of tests containing these two types of problems, two additional modes of target item presentation were used: presenting the same targets without any other context problems, a so-called "blocked" presentation of targets; and the other involved presenting the same targets in blocked fashion but with a partially-completed example presented above the targets, a so-called "cued" test condition. The blocked and cued test conditions were added to the context tests in order to determine what conditions were necessary for optimum and correct responding.

METHOD

SUBJECTS

The subjects in this study consisted of 2 classrooms of grade 4 pupils from a school in Hamilton. One classroom contained 20 pupils, the other 25, for a

total of 45 subjects. This school was located in a middle-class neighborhood within 3 miles of all the other schools used in the present studies.

MATERIALS

The materials consisted of 4 separate mimeographed worksheets of arithmetic problems. As in all the context studies reported, all problems were written using a 3/16-inch size template to insure comparability of written numerials across problems. The 5 target problems were vertically displayed 2-by-2-digit multiplication problems: 32X11, 34X21, 43X23, 34X32, 64X12. Only one of the problems required regrouping (34X32); the regrouping was located in the calculation of the 2nd partial product. For the two worksheets containing contextual problems, the targets were located in position 5,12,14,23,and 28 of a 5-by-6-problem ~~array~~ totally 30 problems. These are the same target positions used in all of the grade 2 context studies.

One context worksheet, test AS (addition and subtraction) (see Appendix IX for items), contained 25 context problems of the 2-by-2-digit vertically displayed form (so-called "rectagular" form), including 12 addition and 13 subtraction problems in a random sequence. None of the problems required regrouping, and all numerals used in the problems were randomly selected from the numerals 0 to 6.

The other context worksheet, test M (multiplication) (see Appendix X), also contained 25 context problems, but all were 2-by-1-digit multiplication (so-called "nonrectangular" forms) including 3 requiring regrouping. All numerals used in the problems were randomly selected from numerals 0 to 9.

The Blocked and Cued test worksheets contained the same 5 targets used in the context worksheets but they were arranged in one row half-way down the page of the worksheet. The Cued test included an example problem situated above the row of targets and the example was partially solved with question-marks (?) located in

spaces left empty for the remaining numbers. The example looked like this:

```
   43
  X32
   86
+ 2 90
  ? ??
```

PROCEDURE

The two classrooms were tested separately during the month of June of 1983 after all review drills in arithmetic for the yearwere completed. By this time the students had received 2-by-2-digit multiplication algorithm instruction and drills during the school year (approximately February to April), and had received the end-of-year review and drills (end of May, early June), and thus were well acquainted with this algorithm.

Two groups of subjects were randomly chosen from the two classrooms and were adminstered the two context worksheets (Test AS and M) in counterbalanced order following procedures identical to other context studies. The children were told they would receive four worksheets, one at a time; they were to write their name on the back of the worksheet and the number of the test on which they were working (1-4) before starting. They were also told that the tests consisted of addition, subtraction, and multiplication problems, and that they were to work the problems from left-to-right, top-to-bottom. As soon as a child completed his worksheet s/he was to raise her/his hand and the next worksheet was given to her/him. All 4 worksheets took between 40-50 minutes to complete.

The two context worksheets, Test AS and M, were given first, followed by the Blocked test and then the Cued test. When the child was given the Cued Test s/he

was told that "there is an example at the top of the page, look at it, try to fill in the spaces with the question marks, and then do the problems below". If the child did not understand the example, s/he was told to "just look at it carefully, and try to figure it out; if you can't, just do the problems underneath".

SCORING

The worksheets were scored for the the following 10 types of errors, which can be grouped into 3 main classes for the purposes of this study. Reliability of scoring these categories was checked by having an independent observer score a total of 12 randomly selected subjects worksheets (Test M and AS). Of the 120 total observations, 112 or 93% were in agreement with the original scoring.

1. PARTIAL-PRODUCT SOLUTIONS. A solution that contains two partial products aligned below the problem, irrespective of how the partial products were calculated. Seven different partial-product solutions, including the correct procedure, were counted.

a. DIAGONAL-ORDER ERROR (DO). A solution in which the 2nd partial product is calculated by reversing the order in multiplying the units and tens numbers of the top number by the tens number of the bottom. In the example given in Table 12, the child in calculating the second partial product has multiplied the tens numbers in the top and bottom number (3X2=6), followed by multiplying the top units and bottom tens numbers for the thousands place of the 2nd partial product (2X4=8).

b. CRISS-CROSS ERROR (CC). A solution in which the child multiplies only diagonally for both the 1st and 2nd partial product. In the example given in Table 12, the 1st partial product is calculated by first multipling "diagonally"

TABLE 12
EXAMPLES OF MULTIPLICATION ERRORS SCORED IN STUDY 9

1. Partial-Product Solutions

a. Diagonal-order Error (Do)

```
  34
 X21
  34
 860
 894
```

b. Criss-Cross Error (CC)

```
  34      34      43
 X21     X32     X23
  83      26      72
 830     127     720
 913    1296     792
```

c. Conjoined-partial product

```
  34      43
 X21     X23
6834    8729
```

d. Place-holder error (PH)

```
  32
 X11
  32
  32
  64
```

e. Mixed-operation error (Mo)

```
  34
 X32
  68
 72
 788
```

f. Wrong-operation on partial products (WO-PP)

```
  32
 X11
  32
 320
 312
```

2. Vertical-Column Error (VC)

```
  34      43      32
 X21     X23     X11
  64      89      32
```

3. Wrong-Operation Errors (Wo)

```
  32      32
 X12     X12
  20      44
```

4. Horizontal-Column Error (HC)

```
  34      34
 X21     X32
 122     126
```

the top ten's and bottom units numbers to yield the units place in the 1st partial product (1X3=3) followed by multiplying the top units and bottom tens number to yield the tens place (2X4=8). In the 2nd partial product, the child repeats the "criss-cross" procedure or simply copies the 1st partial product, adding a "place-holding" zero. However, the second example suggests that the child is not simply copying the first partial product to arrive at the second partial product.

c. CONJOINED-PARITAL-PRODUCTS ERROR (CPP). A solution in which the two partial products are calculated correctly but they are layed out in one row of numbers instead of 2 rows. In the Table 12, the 1st example (34X21) shows the conjoined partial products on one line (4X1=4 : 1X3=3 : 2X4=8 : 2X3=6). In the second example, (43X23), we see how regrouping requirements are treated: 3X3=9, 3X4=12 and the '1' of the 12 is carried to the units column were it is added to 2X3=6 and finally 2X4=8.

d. PLACE-HOLDER ERROR (PH). A solution in which the place-holding zero is not used (see table for an example).

e. MIXED-OPERATION ERROR (MO). A solution in which partial products are calculated but a non-multiplication fact (e.g. addition or subtraction fact) is used instead of a multiplication fact. In table 12, during the solution of 34X32, the child arrives at the ten's number of the 2nd partial product by adding the top and bottom tens number (3+3=6) instead of multipying, and then adding the carried '1' to arrive at 7.

f. WRONG OPERATION ON PARTIAL PRODUCTS (WO-PP). A solution in which the partial products are subtracted instead of added (see table).

2. VERTICAL-COLUMN ERROR (VC). A solution in which the units top and bottom numbers are multiplied to yield the units answer, and the tens top and bottom numbers are multiplied to yield the tens answer. In Table 12 the problem 34X21 is solved by multiplying 4X1=4 and 3X2=6. Mixed-operations embedded in vertical column procedures were also counted as VC errors; that is, solutions in which the child uses a VC strategy but multiplies only one of the column's numbers and the other column is either added or subtracted. Only one child did this, and he added the units and multiplied the tens.

3. WRONG-OPERATION (WO). A solution in which the problem is solved using a different operation altogether - addition or subtraction (e.g. 32X12=20 or 32X12=44).

4. HORIZONTAL-COLUMN ERROR (HC). A solution in which the problem is solved by multiplying along rows rather than columns to arrive at the answer. One child did this consistently.

RESULTS AND DISCUSSION

A between-groups context effect was observed; however, the within-groups context effect was not observed.

BETWEEN-GROUP CONTEXT EFFECT:

The main prediction is that error should be more probable with problems in the addition and subtraction context, Test AS, than in the multiplication context, Test M. Figure 5 displays the proportion of subjects who

made a VC error to at least one of the 5 targets under the various test conditions for the two counterbalanced groups.

A comparison of the percentage of VC responders on Test AS for Group AS-M and on Test M for Group M-AS yielded the context effect. As can be seen in the figure, 48% (11 of 23) of Group AS-M subjects made the VC error, whereas only 18% (4 of 22) did so among Group M-AS subjects (Corrected Chi-Square=3.213, 1-tailed, $p<.05$). Thus, VC errors were more probable to targets located in a problem context containing rectangular addition and subtraction then in one containing nonrectangular multiplication problems. This effect did not reach conventional levels of statistical significance when the difference in percentage of subjects making partial-product solution attempts (errors CD,DO,CC,CPP,PH,MO,WO-PP) under the two conditions: 73% (16 of 22 subjects) vs 52% (12 of 23) for Test M and Test AS respectively (Corrected Chi-square= 1.949, 1-tailed, $p<.1$).

This finding extends previous observations of context-sensitive problem solving to include grade 4 vertical column errors to 2-by-2-digit multiplication problems.

WITHIN-SUBJECTS CONTEXT EFFECT

Within-subjects context effects can be assessed by comparing the magnitude of change in VC responding between the two worksheets for each group separately. Such effects were not observed as indicated in the small differences in percentage VC-responders among Group M-AS subjects, 48% to 35% (test M vs As), and Group AS-M subjects, 18% to 23%.

Test carry-over effects. Test carry-over effects were tested by comparing the incidence of VC responding between the two groups for each test separately. Neither carry-over effect reached conventional levels of significance. For Test AS, 48% of Group AS-M subjects versus 23% of Group M-AS subjects made VC

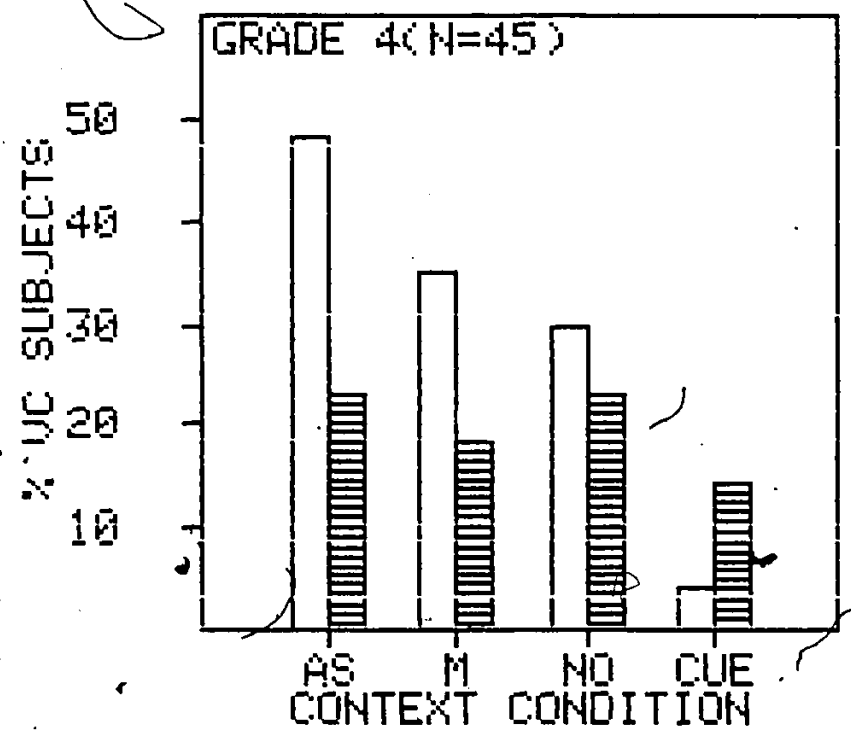

Figure 8. Percentage of subjects committing a vertical-column error (VC) to multiplication targets in addition-subtraction problem context (AS), multiplication context (M), no-context (NO), and cue-context (CUE) for Group AS-M (open rectangles) and M-AS (hatched-rectangles) of Study 9.

errors (Correct Chi-Square= 2.09, 2-tailed, p>.1); and for Test M, 18% of Group M-AS subjects versus 35% of Group AS-M made VC errors (Correct Chi-Sqaure=0.849).

Wrong-operation errors. One possible reason for the failure to observe within-subjects context effects for Group M-AS can be seen in the effect Test M had on the incidence of wrong-operation errors to targets in the subsequently administered Test AS. Ten of 22 subjects or 45% made wrong-operation errors to Test AS targets after Test M had been completed, whereas, only 2 of 23 or 9% made these errors to Test AS when it was experienced first (Correct Chi-Square= 6.003, 2-tailed, p<.02). Four of these 10 made WO errors to all 5 of the targets. Such an unexpectedly high incidence of wrong-operation errors would serve to reduce the opportunity for multiplication-based problem solving to the targets, and therefore, reduce the opportunity to observe VC target errors under Test AS context conditions when followed by test M.

This carry-over effect on wrong-operation errors adds further evidence at variance with current Fixed-Rule models which assumes that arithmetic problem solving is determined exclusively by the internal characteristics of the to-be-solved problem. Since wrong-operation errors themselves seem to result from context factors these models have no explanation for their occurrence. Most observers of children's arithmetic problems are familiar with such errors, and would acknowledge a role for problem context in their determination. If a child solves a current problem by addition when s/he should have multiplied, no one is particularly surprised if s/he had just solved an addition problem; evidently, the child had not "paid attention" to the arithmetic sign on this problem. (However, we are still left with the need to explain why attention was not adequately directed to this problem.) It should be recognized that this is not the type of context carry-over observed in this study. In this study, it was observed that children are much more likely to solve a

multiplication target by adding or subtracting when that multiplication problem is embedded in those addition and subtraction problems after having solved a test in which all the problems are multiplication, than when they had not previously experienced the all-multiplication test. That is, the homogeniety of the type of problems solved on the previous test influenced the probability that, on the current test, children would carry-over arithmetic operations from recently solved problems to currently-to-be-solved problems. A closer look at the data is helpful in understanding this effect.

Forty-five percent of Group M-AS subjects (10 of 22) made wrong-operation error to multiplication targets in Test AS as compared to only 9% of Group AS-M subjects making such errors on the same test. Of the 10 wrong-operation responders, 4 made wrong-operation errors on all 5 targets; 3 of these 4 subtracted every one of the multiplication targets, as well as virtually every addition problem contained in the context. For these 3 subjects, the widespread use of a subtraction procedure on this addition, subtraction, and multiplication test when they had previously been given a multiplication-only test can be only understood if it is assumed that the multiplication-only test "set" the children to expect homogeneous worksheets, and the items in the beginning of the mixed worksheet lead them to expect that the worksheet contained subtraction problems. In fact, the first row of the mixed test contains an addition problem and 3 consecutive subtraction problems before the first multiplication target, which could have the effect of suggesting to the children, for those having been set to expect homogeneity, that subtraction problems were the type of problems on this test worksheet. However, the conclusion that the entire sheet was made up of subtraction problems was not a common one since only 3 of the 10 wrong-operation responders made all subtraction wrong-operation errors to the context and target problems. Six made 1 or 2 wrong-operation errors on the 5 targets, and in

each case solved the target with the same operation used on the previous item (4 subtraction, 3 addition), suggesting that the effect of the previous homogeneous multiplication worksheet was to bias responding towards local homogeneity of procedure across several items even when accurate discriminative performance had occurred previously within the same test. In these later subjects, it is as if they "knew" the worksheet was not entirely homogeneous, but they were biased towards "assuming" local homogeneity in performance.

Two points are of interest in connection with this unusual test carry-over effect. First, in order to understand this effect it is necessary to assume that children are highly sensitive to the homogeneity of the worksheets they solve; not only with respect to within-test homogeneity, but with respect to between-test changes in homogeneity. Being forced to use this construct reinforces claims made in the grade 2 and 3 studies that teachers predominantly rely on homogeneously constructed drills and tests in their administration of the arithmetic curriculum; and that unmixed drills have effects on the learning and performance of arithmetic skills. Second, as emphasized in interpreting the basic context effects of grade 2 and 3, a recognition of the role that recent problem context has on current problem solving has not been made in current theorizing and experimentation in the field of children's arithmetic learning and problem solving.

BLOCKED AND CUE TEST CONDITIONS

Testing of the two groups of subjects in this study under blocked and cue test conditions produced evidence that the cue test conditions substantially reduced VC errors. As can be seen in figure 8, only 4 of the 45 grade-4 children or 8.8% made VC errors under the cue test condition where a partially-worked example was displayed above the 5 target problems. For Group AS-M subjects, only 1 of 23 or 4.3% made VC errors under these conditions, which represents a significant reduction

errors from either Test M or AS context conditions. Thus, the children could solve the targets using a partial product algorithm under conditions in which a retrieval cue was presented.

The next and final study further extended observations of context effects to division problem solving at grade 4 and 5.

Chapter 12

STUDY 10: THE EFFECT OF CONTEXT ON DIVISION PROBLEM SOLVING AMONG GRADE 4 AND 5 CHILDREN

The purpose of this study was to extend the previous demonstrations of context-sensitive processes in children' arithmetic problem solving to division problems solving during grade 4 and 5. One aspect of division problem solving will be the focus of this study - finding a "remainder". A description of whole-number division problem solving as taught in the schools will reveal the rational behind the context manipulation used in this study.

The so-called "whole-number division" algorithm is taught during grade 4 and 5 in most arithmetic curriculae, and involves the use of a "long division" technique. Figure 9 displays a grade 4 2-by-1-digit division problem in "bracket" form and the kind of long-division approach that is commonly taught. For example, 7 divided into 23 is approached by (1) finding the division fact that yields a product nearest to and less than the dividend (i.e.23), (2) writing the single-digit number which, when multiplied to the divisor, yields this nearest product above the dividend (i.e.3), and (3) writing the product (21) below the dividend, vertically matching units and tens column numbers, (4) subtracting the product from the dividend and writing the result below (23-21=2), (5) writing an "r." in the quotient space to the right of the single-digit quotient, and (6) writing the difference number to the right of the "r." (i.e.2). The correct answer is "3 r.2", to be read as "three remainder two". The long division algorithm yielding answers with remainders is taught during grade 4 and 5; later in the curriculum children are taught to convert the whole-number remainder into a fraction by using the divisor as the denominator

Figure 9

Example of "long-division" procedure:

```
     3 r. 2
7 | 23
   -21
    2
```

and the remainder as the numerator. Prior to grade 4, children are acquainted with the "bracket" form of division problems by being given experience with solving the basic division facts in bracket form ($2\overline{)14}$, $3\overline{)6}$, etc) without using the long-division procedure in which subtraction must be used to find remainders.

Observations of children's division problem solving often reveals what might be called "failure-to-find -remainders" errors. For example, on the problem 7 divided into 23, children may answer 3, either without using a long-division follow-up procedure of subtraction, or by using a subtraction follow-up procedure but writing 23 below the dividend which yields a zero remainder. In both cases the child has not found the remainder, and appears to fail because s/he thinks that 7 into 23 is 3 and 7 times 3 equals 23. That is, the failure to find a remainder ("no-remainder" error) is commonly interpreted from within a Fixed-Rule model as resulting from a <u>lack of knowledge of the division and multiplication facts</u> relevant to the problem.

This interpretation is implausible for the following reason. Not infrequently, errors of these kinds are made on relatively simple problems; that is, problems containing low-valued divisors and dividends such as 7 and 23. Inadequate fact knowledge is an unlikely basis for these error since children learn and are drilled extensively on the multiplication and division facts during grades 3, 4, and 5 before the whole-number division algorithm is taught because "fact finding fluency" is recognized as a crucial subsidiary skill for successfully using the division and multiplication algorithms.

An alternative interpretation is suggested by considering this error as a result of treating a division-with-remainder problem as if it were an instance of the class of division facts. Suppose that the child, in answering 3 to the division problem 7 into 23, is attempting to apply, for some reason, a division-fact-finding

problem solving strategy. Such as strategy is one in which the result is a single-digit number and no remainders are involved. The child may then produce 3, a single-digit number and no remainder, even though he "knows" that 3 X 7=21 or 21 divided by 7 is 3 when such knowledge is tested in alternative ways. Moreover, if the child attempts, for some other reason, to apply a find-remainder problem solving strategy to 23 divided by 7 he will produce the correct response - 3 r.2 with 7X3=21 and 23-21=2. Thus, it may be that 2-by-1-digit division problems are approached by children as either division facts for which single-digit answers and no remainders are appropriate, or as division problems with remainders for which single-digit answers, long-division, and remainders are appropriate. Accordingly, if the division fact-finding strategy is used a division fact error will occur to division problems requiring remainders; if a find-remainder strategy is used a correct response will occur. This view is analogous to what had been argued to occur with grade-2 children's various solution attempts of 2-by-1-digit subtraction facts; grade-3 children's various solutions of 2-by-1-digit addition and subtraction problems; and with grade-4 children's various solutions of 2-by-2-digit multiplication problems. As in those instances, it may be the case that problem context is a major determiner of the problem solving strategy used.

The context view suggests that the error can be made more probable when the target problem is presented in a context of problems that encourages the use of a problem solving strategy presumed to yield the error, and on the other hand, can be made less probable when the target is presented in a context of problems that encourages the use of the correct problem solving strategy. Thus, with respect to division problems requiring remainders and the no-remainder error which occurs with them, the expectation is that the nonremainder errors to 2-by-1-digit division problems requiring remainders will be more probable in a context of problems

encouraging a division-fact problem solving strategy (i.e. 2-by-1-digit division facts, e.g. 7 $\overline{)14}$, 5 $\overline{)20}$), and less probable in a context of problems encouraging a long-division-with-remainder strategy (i.e. 2-by-1-digit division problems requiring remainders, e.g. 4$\overline{)27}$, 8$\overline{)43}$). This study investigated this prediction on separate populations of grade 4 and 5 children. Because grade 5 children have had 1 year of long-division problem solving and have advanced beyond single-digit divisors, the hypothesis was tested that the context effect would only be found among the grade 4 children, since grade 5 children would be expect to be fully knowledgable about these types of problems which may lead to context-independent problem solving on them.

METHOD

SUBJECTS

The subjects included separate classes of grade 4 and 5 children in Hamilton. The classes were located in different schools in similar middle class neighborhoods. The grade 4 and 5 classes included 31 and 28 pupils, respectively.

MATERIALS

As in previous context studies, 5 targets were located in positions 5,12,14,23,and 28 of two seperate 5-by-6 problem worksheets. The 5 targets were 2-by-1-digit division problems requiring remainders presented in bracket form (8$\overline{)63}$, 6$\overline{)56}$, 7$\overline{)54}$, 7$\overline{)43}$, 4$\overline{)27}$) with the remaining 25 context problems differing between the worksheets. Two context worksheets were used (see Appendix XI and XII).

The 25 context problems on the division fact test (Test DF, Appendix XI) were 2-by-1-digit division facts presented in bracket form, none of which involved remainders (e.g. 4$\overline{)16}$, 8$\overline{)40}$, 5$\overline{)45}$), whereas, the other context test (Test DR, Appendix XII) contained division problems with remainders (e.g. 7$\overline{)12}$, 8$\overline{)25}$, 4 $\overline{)10}$).

PROCEDURE

The procedure was analogous to procedures used previous studies. All subjects received both tests in one of two orders (DF-DR, DR-DF) determined by random assignment. The children were given the first worksheet and told to write their name and the number of the test on the back of the worksheet. They were told that they were going to receive 2 worksheets of division problems and they were to work them left-to-right, top-to-bottom. When they finished the first worksheet, they were to raise their hand, at which time the second worksheet would be given to them. The two worksheets were completed in 45-60 minutes.

RESULTS

The children's worksheets were scored for whether or not the answer to a target problem contained a remainder. An answer containing a zero remainder was also scored as a non-remainder error. Because as in previous studies the errors were bimodally distributed with modes at the extreme scores (0 and 5), the preferred statistic was percentage of subjects per group making nonremainder errors. As in previous context studies, 3 main analyses were conducted: between-group, within-group context effects, as well as carry-over effects. Each analysis will be considered separately.

BETWEEN-GROUP CONTEXT EFFECT

A comparison of nonremainder errors to targets in Test DF (division-fact context test) for Group DF-DR and in Test DR (division problems with remainders in context) for Group DR-DF provides an analysis of the between-group context effect.

At grade 4, a significant between-group context effect was observed as indicated in Figure 10a. Ten of 15, or 67% of Group DF-DR subjects made nonremainder errors, whereas only 4 of 16 or 25% made such errors among Group DR-DF (Corrected

Chi-Square= 3.87, 1-tailed, $p<.025$).

At grade 5, a between-group context effect was not obtained at the .05 level of significance. As can been seen in Figure 10b, 7 of 14 or 50% of Group DF-DR subjects made nonremainder error in Test DF; only 2 of 14 or 14% of Group DR-DF subjects making such errors in Test DR (Corrected Chi-Square= 2.62, 1-tailed, $p<.1$).

Thus, for only grade 4 was there evidence of between-group context effects.

WITHIN-SUBJECT CONTEXT EFFECTS

Within-subject context effects were observed only among Group DR-DF subjects for both grade 4 and 5 classes.

For grade 4 subjects, 6 of 16 Group DR-DF children made more nonremainder errors when the context changed from one containing remainder problems to one containing no remainders, and no child showed a decrease (Sign Test, 1-tailed, $p=.016$). On the other hand, for Group DF-DR children, 5 of 15 subjects made more nonremainder errors to targets in Test DF, but 2 made fewer errors (Sign test, 1-tailed, $p=.227$) which was nonsignificant.

Within-subject context effects were observed among grade 5 children. Five of 13 Group DR-DF children showed higher rates of nonremainder errors to targets in Test DF with no children showing lower rates (Sign Test, 1-tailed, $p=.031$). And among Group DF-DR children only 3 of 14 decreased rates when given Test DR, and one increased error responding (Sign Test, 1-tailed, $p<.10$). These results suggest test carry-over effects similar to those observed at grade 2.

Test carry-over effects. The presence of test carry-over effects can also be tested by comparing target responding for the two groups under the same context conditions. Inspection of Figure 10a and b suggests that for both grade 4

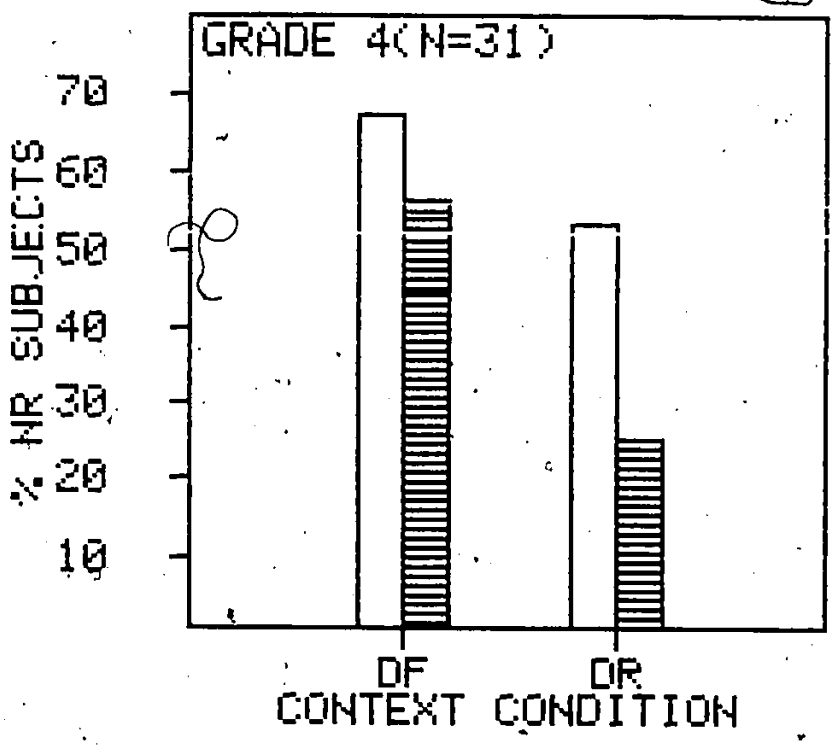

Figure 10a. Percentage of grade-4 subjects committing non-remainder errors (NR) to division targets in division-fact context (DR) and division-with-remainder context (DR) for Group DF-DR (open rectangles) and DR-DF (hatched rectangles) of Study 10.

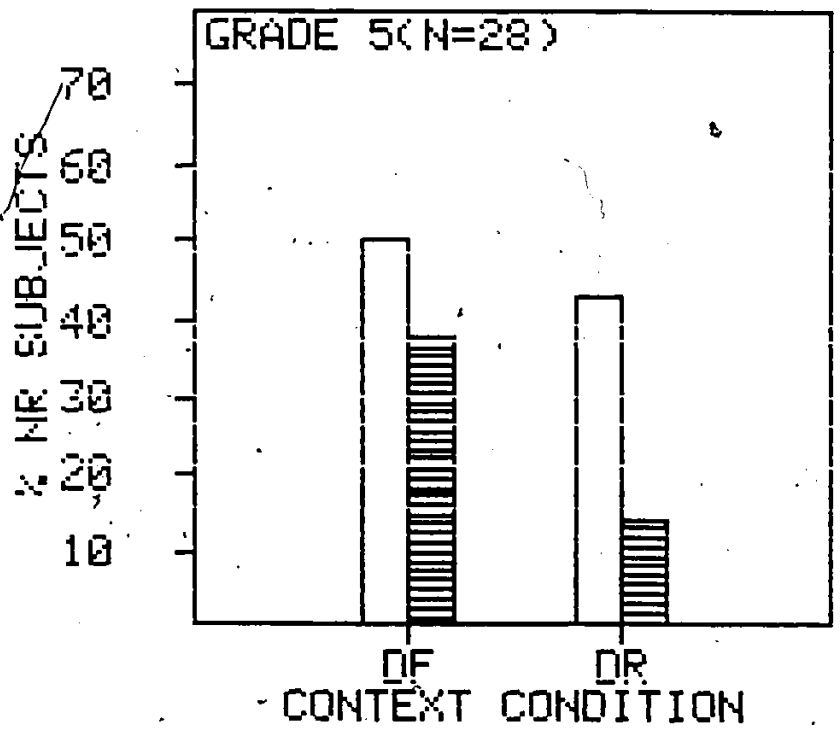

Figure 10b. Percentage of grade-5 subjects committing non-remainder errors (NR) to division targets in division-fact context (DR) and division-with-remainder context (DR) for Group DF-DR (open rectangles) and DR-DF (hatched rectangles) of Study 10.

and 5 children nonremainder errors are more probable in a remainder context if a nonremainder context was previously experienced, but nonremainder errors are not less probable in a nonremainder context if a remainder context was previously experience, i.e. the carry-over effects are asymmetrical.

Indeed, although the separate chi-square tests of target nonremainder errors in Test DR were nonsignificant for grade 4 (53% vs 25%, Chi-Square= 1.56, 1-tailed, $p>.1$) and grade 5 (43% vs 14%, Chi-square= 1.58, 1-tailed, $p>.1$), a test of nonremainder carry-over for the combined grade 4 and 5 data was significant (14 of 29, or 48%, vs 6 of 30, or 20%, correctd Chi-Square 4.07, 1-tailed, $p<.025$). The combining of grade 4 and 5 data is justified on the grounds that no significant differences between the groups in nonremainder errors were observed under any conditions.

In contrast to the significant carry-over of nonremainder errors from a previous nonremainder context, there was no evidence of a carry-over from a previous remainder context in either the separate grade chi-squares (grade 4:67% vs 56%, Chi-Square= 0.05, $p<.45$; grade 5: 50% vs 38%, Chi-Square=0.046, $p<.45$) or the combined test (17 of 29, 59% vs 14 of 29,48%, Chi-Square= 0.277, 1-tailed, $p<.45$).

CONTEXT PROBLEM SOLVING

The effects of the prior DF test experience on Test DR test performance, can not only be assessed with respect to target problem solving, but also with respect to problem solving on the context problems making up Test DR. Recall that Test DR consists of 2-by-1-digit division problems requiring remainders as targets and as <u>context problems</u>. Therefore, carry-over effects of the nonremainder context of Test DF on Test DR problem solving can be assessed by comparing nonremainder errors on Test DR context problems for children who had previously been given Test DF, Group DF-DR, and for those who had not, Group DR-DF. Figure 11 displays the

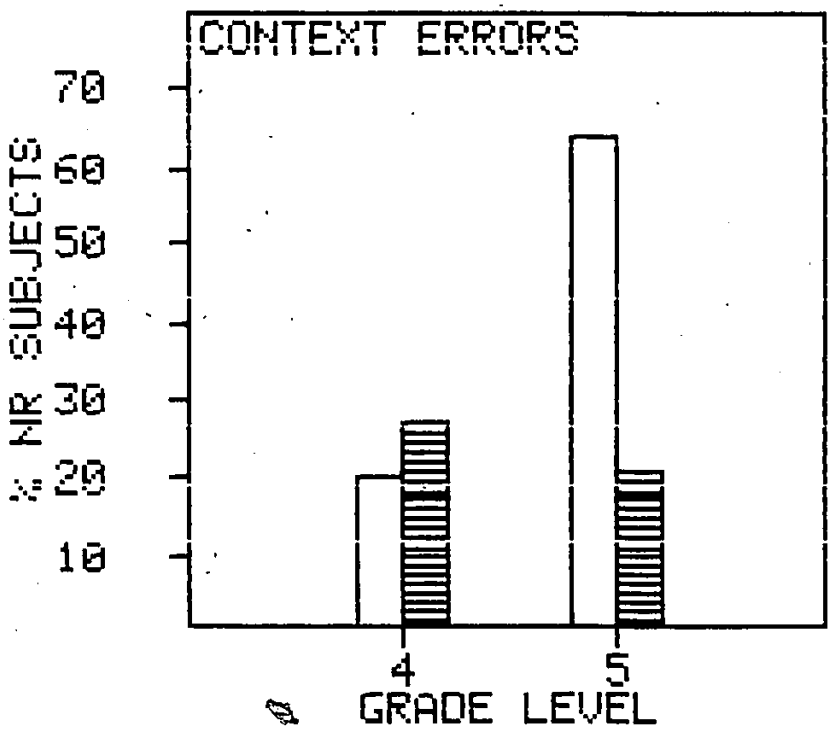

Figure 11. Percentage of grade 4 and 5 subjects committing non-remainder errors (NR) to division context problems in the division-with-remainder (DR) test for Groups DF-DR (open rectangles) and DR-DF (hatched rectangles) in Study 10.

percentage of grade 4 and 5 subjects making nonremainder errors to the context problems for these 2 groups. As can be seen, although no carry-over effect was seen at grade 4, significant effects occurred among grade 5 students. Specifically, 9 of 14 grade 5 students or 64% made nonremainder errors to Test DR context problems when they had received Test DF previously, in contrast to only 3 of 14 or 21% if they had not previously received that test (64% vs 21%, Corrected Chi-Square, 1-tailed, $p<.05$).

In addition, of the 9 nonremainder responders in Group DF-DR, 4 made nonremainder errors to all 25 context problems, in contrast to only one Group DR-DF child who made 21 nonremainder errors. Thus, the effect of the prior nonremainder test, in some cases, was to completely eliminate remainder-finding problem solving, even when an entire worksheet was composed of problems requiring remainders. For these 4 subjects, the entire 30 problem test was solved by giving single-digit answers, using long division procedures but no remainders!

DISCUSSION

The study produced evidence that for grade 4 and 5 children a context containing division facts substantially reduced the use of a find-remainder procedure to 2-by-1-digit division problems requiring remainders embedded in that context, in contrast to a context containing 2-by-1-digit division problems requiring remainders. Also, prior exposure to a division facts context increased nonremainder errors not only to targets embedded in that context, but also to the same targets embedded in a problem context containing division problems requiring remainders subsequently experienced. That is, division fact context effects carried over to target problem solving in a division-with-remainder problem context. Problem solving to the division-with-remainder contextual problems was similarly affected. However, no

evidence was found for a symmetrical carry-over effect, that is, where prior exposure to a division-with-remainder problem context affected target problem solving in a subsequent context of division facts.

These results extend previous studies in showing that context problem solving is important in determining common computation errors in division problem solving involving more "advanced" procedural algorithms being used by more advanced and experienced problem solvers, i.e. grade 4 children. Thus, context-driven computation problem solving is not confined to "simple" arithmetic problem solving by young and inexperienced children. This finding further challenges the appropriateness of Fixed Rule descriptions of children's computational problem solving; description that emphasize the exclusive role of internal characteristics of the to-be-solved problem in determining the problem solving strategies children use in solving arithmetic problems.

Comments should be made about two other findings in this study. First, the expectation that grade 5 children's division problem solving would be less contextually influenced than grade 4 children's was not supported. Although there was some evidence that overall nonremainder errors were less prevalent among grade 5 children, the context effects appeared to be comparable. Therefore, increases in expertise in division problem solving between grade 4, when single-digit-divisor problem solving is first taught, and grade 5, when division lessons move on to the double-digit-division algorithm, is not sufficient to decontextualize single-digit-divisor problem solving. In fact, there were data in this study suggesting larger context effects for grade 5 children in that when contextual influences were measured as nonremainder errors committed to context problems contained in the remainder context after previous exposure to the nonremainder context, only grade 5 children evidenced a carry-over context effect.

The preferred view of division problem solving among grade 4 and 5 children that emerges from this context study is that these children classify division problems into those that have remainders and those that do not. Those that have remainders required a "find-nearest-division-fact" process that is followed by a long-division procedure that derives the remainder via solving a vertical column subtraction problem. Problems that do not contain remainders are those for which a "find-fact" process is initiated. If a find-fact strategy is applied to a problem that contains a remainder, this study shows that the classifiation of the problem as a division fact is psychologically powerful enough to sufficiently alter the accuracy of the child' find-fact process so that a non-fact is much more likely to be falsely solved as if it were a fact. This study produces evidence that the nature of the problem context making up a sequence of division problems is sufficient to influence the classification of problems embedded in this sequence as either division facts or division-with-remainder types of problems, and accordingly, influence the type of problem solving activity applied. Such results are inconsistent with the more rigid,inflexible views of children's division problem solving, based on a Fixed Rule model, which conceptualizes it as purely data-driven algorithm activity, and absolute knowledge of division facts.

Chapter 13

GENERAL DISCUSSION

The aim of the studies was to explore the hypothesis that many systematic computation errors made by primary-school age children result from the way in which tests are constructed to measure arithmetic problem solving. In particular, the studies demonstrated that problem solving activities on specific problems within an arithmetic test, so-called "target problems", were systematically related to problem solving activities on the remaining problems comprising the test, so-called "context problems". Systematic manipulation of the type of context problems, while holding the type of target problems constant, produced reliable changes in target problem solving activity at a variety of primary-school grade levels using a variety of types of problems. Thus, the generality of arithmetic context effects among primary school children was established. In addition, the contributions of a certain style of classroom instruction to the observed context effects on systematic errors was assessed. Before discussing the psychological and educational implications of these findings, a brief summary of the results of studies 1-10 will follow.

SUMMARY OF RESULTS

Studies 1-5 examined problem solving on a type of problem taught at grade level 2 - the so-called 2-by-1-digit subtraction facts (e.g. 12-4=8, 17-9=8). Grade 2 children were commonly observed to error on these problems by treating the problem as consisting of two columns of numbers, a unit's and ten's column, resulting in answers which appear, at first glance, to be rather unusual, e.g. 11-5=14. These

answers result from starting in the units column and subtracting 1 from 5, yielding 4, followed by subtracting the numbers in the tens column, 1 and 0, yielding 1 in the tens column and a final answer of 14 (referred to as a "drop-down-inversion" or DI error). Study 1 demonstrated that at grade 2 this error was made to 2-by-1-digit subtraction facts located on a test as long as the context problems are double-digit subtraction problems requiring a column-subtraction procedure as a solution strategy; if the context is composed of single-digit subtraction facts (e.g. 5-2=3, 6-4=2), requiring any of several "subtraction fact" solution strategies, such as counting-up or -down or recall-from memory, then the same 2-by-1-digit subtraction facts are solved correctly. Thus, the solution strategy used on these grade 2 problems was determined by the solution strategies used on context problems present on the same test. In effect, it appeared that the children were using the same strategy on both context and target problems.

Study 2 expanded upon this finding by showing that a similar effect was seen at grade 3 on the same type of problems but with a different type of solution strategy - "unnecessary borrowing". Study 3 researched grade 4 children working on these same problems either presented as they had been at grade 2 or as contained in the solution algorithm of certain 2-by-1-digit division problems (e.g. $9\overline{)17}$). In both studies, the original finding was reproduced; that is, the children's style of solving the critical target problems was influenced by the style of solving the context problems comprising the tests.

In studies 6-8 further support for the role of contextual processes in the determination of children's arithmetic problem solving was sought at grade 3 in the solutions of 2-by-1-digit addition and subtraction problems. In particular, one type of error was studied - the so-called "diagonal error" (e.g. 45+3=78, 67-5=12) in which children appear to incorrectly solve the problems using a procedure that is

similar to one they are taught to use in solving 2-by-1-digit multiplication problems (e.g. 23X3=69). By an extension of the same logic used to construct the context tests for grade 2 problems, tests were constructed consisting of context problems that would be expected to promote an incorrect diagonal error to 2-by-1-digit addition and subtraction problems (2-by-1-digit multiplication), and problems which would be expected to promote correct solutions (i.e. 2-by-2-digit addition and subtraction). The results supported the hypothesis that the diagonal error was context sensitive, and, indirectly, that children solve target problems using solutions similar to those used to solve context problems.

In so far as the generality of the contexts effects were concerned, studies 9 and 10 explored two other errors at grade 4 and 5. First, it was shown in study 9 that "vertical-column" errors, often seen during the solution of 2-by-2-digit multiplication problems (e.g. 23X43=89), are more frequent in contexts of 2-by-2-digit addition and subtraction problems, for which vertical-column solution strategies are appropriate, than in contexts of 2-by-1-digit multiplication problems, for which a portion of the correct 2-by-2-digit multiplication algorithm is appropriate (i.e. the diagonal procedure).

In study 10, a common division problem error was similarly researched - the so-called "failure-to-find-remainder" error in which the child appears to be commiting a "division fact error" in answering 2-by-1-digit division problems (that require the derivation of a "remainder") by solving the problem as if it were a perfect division fact (e.g. $9\overline{)56}$ =6; instead of 6, plus remainder 2). Here it was shown with grade 5 children that such errors are much more probable in contexts containing other division facts, than in contexts containing division problems containing remainders. As in the other studies, the children seem to be solving the target problems using techniques similar to those used in solving the context

problems.

Another finding was obtained in the studies reported here at both grade 2 and 3. Specifically, the possibility was explored that the tendency of children to solve target and context problems similarly is a result of the way in which teachers construct arithmetic drills used during their instruction of various algorithms. Teachers tend to construct so-called "homogeneous" or "unmixed" drills in which all the problems on the drill worksheets are of the same type. This could be presumed to encourage the learning of a solution strategy of solving all problems on a worksheet in a similar manner. Studies 3 and 7 found evidence for this hypothesis by showing that classrooms given mixed drills in which different types of problems were presented in random order were significantly less likely to commit context-based errors to targets than regularly instructed classrooms. Study 3 showed this for the grade-2 column-subtraction errors, and study 7 for the grade-3 diagonal error.

The theoretical significance of these findings are in their implications for existing models of children's arithmetic learning and problem solving from within psychology and education.

IMPLICATIONS OF FINDINGS

Chapter 1 reviewed in some detail the existing models of children's arithmetic problem solving: Bug Theory, Repair Theory, and Frame Theory. It was pointed out that they share an approach to children's arithmetic learning and problem solving referred to as the Fixed Rule approach. In this approach it is assumed, sometimes tacitly, that arithmetic learning involves internalizing a set of procedural rules that are relatively stable between instruction episodes and learning experiences. According to this approach, the procedural rules determine the set of problem solving activities directed to particular types of arithmetic problems. The selection of the procedural rules to apply to a particular type of problem is

conceptualized as resulting exclusively or predominantly from information contained within the currently to-be-solved problem itself (e.g. form, operation sign). Fixed Rule models view systematic errors as resulting from internalizing incorrect rules. By these formulations, the problem solver operates in much the same way as a computer; that is, by responding to moment-to-moment stimulus input with actions specified in the rules. The research literature which has been of primary influence in fostering this approach has stressed the consistency and systematicness of children's solutions to written problems. A fundamental implication of the studies reported in this thesis is that children's problem solving is considerably more variable in nature than is suggested by this traditional research and the Fixed Rule approach. In the studies reported here, it has been shown repeatedly that under one set of conditions, performance can be correct, and in another set of conditions, performance to the same problems can be completely erroneous. The view that one set of rules provides knowledge for solving a problem type cannot account for this extent of variability, and the only way in which the Fixed Rule approaches could explain this type of variability would be to assume that the rules or algorithms are not fixed across learning experiences but somehow vary with testing conditions.

A second implication of the thesis studies is that the procedure-selection process is not exclusively based upon information contained in the currently to-be-solved problem, as tacitly assumed by Fixed Rule approaches. Indeed, the studies have repeatedly demonstrated that problem solving activities to test problems are not independent from one another, but that the type of problems and problem solving activity occurring in the context, can significantly influence the type of problem solving activity used to target problems. Fixed Rule theorists could easily modify their view of the procedure selection process to include rules and information from current context problems to explain context-specific problem

solving. For example, with respect to the grade 2 finding that column-subtraction errors (DI,DCR,D0 errors) occur in column-subtraction problem contexts and correct solutions occur in single-digit subtraction-fact problem contexts, a Fixed Rule framework could account for this effect by adding a rule stating that the procedure to use on a particular problem should be identical to the one used on the previous problem. This rule could account for the results of study 10 also with respect to the effect of problem context on using a finding-a-remainder procedure versus a division-fact procedure in solving 2-by-1-division-with-remainder problems. However, such an additional rule or heuristic would have difficulty with the diagonal and vertical-column errors of study 8 and 9 because these errors do not involve solving a problem with a procedure fully identical to that used on other problems, but involves using a procedure that is only, in some respects, similar to procedures for other problems. That is, the diagonal error involves adding diagonally when solving non-rectangular addition problems just as one multiplies diagonally when solving nonrectangular multiplication problems; solving addition problems identically to multiplication problems would involve multiplying the number in the addition problems. The similarity between the diagonal error and the multiplication algorithm is in the spatial location of the numbers chosen for an arithmetic operation; the operation actually performed on the numbers is different. A similarity rule would have to be written to enable this kind of partial similarity in solutions.

Although it is not obvious how such a partial similarity rule might be conceptualized, one approach might be to assume that the procedural rules in the algorithms (subprocedures) are written in such a way that the rules pertaining to the choice of numbers from the problems (number-selection rules) are independent from the rules pertaining to the choice of the arithmetic operation (addition, subtraction, multiplication, or division) to be performed on those numbers (operation rules).

Then it might be further hypothesized that only the number-selection rules are the rules used on target problems when a context-target-problem similarity heuristic is used. Further post-hoc assumptions could then be made about possible reasons for why it is only the number-selection rules that are used (e.g. perceptual salience of number-selection versus operation, etc.). However, this post-hoc explanation implies a degree of separability of subprocedures in the execution of an algorithm that is at variance with Fixed-Rule views of problem solving. These views assume that the subprocedures are tightly organized and integrated within the overall algorithm. Changing this tightness-of-organization assumption would so dramatically alter the notion of "algorithm", as a set of procedures invariably applied to specified types of problems, that it would not be clear what heuristic or explanatory value would remain for such a concept. Furthermore, at this point the Fixed Rule or algorithm approach would begin to lose some of it's attractive simplicity as a description of arithmetic problem solving. In any case, the existing Fixed Rule approaches have failed to include any mechanism(s) for the learning of such "special" rules, or even to recognize that such rules may be necessary.

Fixed Rule approaches as an alternative to proposing the possibility of a partial similarity rule, could propose that context effects merely reflect the fact that the children had yet to fully learn the correct procedures for the target problem. This "partial learning" idea would, indeed, imply that performance may not be consistent across different numerical instances of a type of problem. However, the construct of "partially learned" procedures carries only the implication that performance may at some time, under some undefined conditions, be incorrect, and implies nothing about the specific errors which might be expected to be observed, or the specific conditions, if any, which might be important in determining the errors. As such, the "partial learning" construct provides little in the way of any

explanation of the basic context findings reported in these studies, or even a useful post hoc heuristic for future research.

A final implication of the context findings worthy of note is the possibility that was researched in Study 4 that the basic context effect may represent a type of temporary "priming" or biasing of procedure usage. Such a priming idea has been discussed extensively in other areas of cognitive psychology where Fixed Rule approaches to the modeling of cognitive tasks have been attempted, e.g. reading (Meyer & Schvanevelkt,1971; Meyer,Schvaneveldt, & Ruddy,1975). The finding of study 4 that the effects of a single double-digit worksheet on problem solving can be observed 24 hours after it has been experienced suggests that a temporary priming mechanism is probably not the basis for the contextual influences seen in some of these arithmetic tasks. It was suggested in study 4 that Fixed Rule approaches to children's arithmetic learning must recognize that problem solving can be dramatically altered by very limited and brief experiences. This fact is at variance with the typical conceptions of rule learning which assume that rules derive from abstracting regularities in information, regarding features of stimuli and/or procedures, from numerous experiences with those stimuli and procedures. The 24-hour carry-over context effect of study 4, based as it was on the experiences of a very small number of problems, suggests the possibility of a rather different mechanism of arithmetic learning and problem solving than that assumed by Fixed Rule approaches. More will be said about this below.

Prior to that discussion, the theoretical implications of the present studies for one specific Fixed Rule model, Repair Theory, should be discussed.

IMPLICATIONS FOR REPAIR THEORY

From among the fixed-rule approaches, Repair Theory has introduced

several ideas to explain variability in arithmetic problem solving not originally recognized in their data base (Brown & VanLehn,1982; VanLehn,1983). Recall from chapter 1 that Repair Theory depicted errors as resulting from the application of procedural repair rules or, as they called them - "repair heuristics", to problems when an impasse had been reached during the execution of an erroneous "core procedure". Core procedures were the internalized procedural rules for different types of problems. An impasse might be reached if an incorrectly internalized core procedure is being applied to a problem, e.g. trying to subtract a column of two numbers with a rule of subtracting the bottom from the topnumber on a problem in which the top number has a smaller value than the bottom number. Repair rules provide the problem solver with possible ways in which to overcome the impasse, e.g. in the above example, "switch" numbers, that is, subtract the top from the bottom. Errors are viewed, by this model, as the result of the application of a repair rule to a problem being solved with an incorrect core procedure. In the original model, errors were viewed as systematic computations. Later it was recognized that errors to particular problems were not as consistent as originally assumed. In fact, VanLehn (1981) reported that only 12% of the subtraction errors he observed remained stable between instructional episodes. This motivated the concept of "bug migration", and the full development of Repair Theory.

The concept of bug migration is an explicit recognition of the variability in erroneous computations, and therefore is highly relevant to one central point of the thesis studies. VanLehn (1983) recognized two types of bug migration: intertest and intratest. A change in an error within a test is referred to as intratest bug migration, and between tests, as intertest bug migration. These are explained as a result of applying different repairs to the same impasse resulting from applying a difficient core procedure. The rules for which repair rule are

applied and under what set of conditions are not extensively discussed, but VanLehn notes that some consistency in errors are observed and interprets these as resulting from temporary associations between an impasse and a specific repair, an association he refers to as a "patch". Patches are held in short-term memory and thus can persist during testing over short periods of time. The possibility that patches are "abstracted" and stored in long-term memory is an idea he indicates need not be assumed given the data available, and he denies the possibility that patches are incorporated into core procedures (p.210). So, theoretically, Repair theory can account for variability in problem solving while remaining within a Fixed Rule framework.

Although Repair Theory recognizes and attempts to account for changes in problem solving between instructional episodes, it cannot explain the context effects observed in the studies reported here. There are two reasons for this. First, note that bug migration refers to the use of different repairs for the same impasse in a single core procedure. The changes in problem solving observed in these studies, however, involves solving a given problem correctly on one test worksheet, and then solving that same problem incorrectly on another test worksheet. Using Repair Theory, this would be described as a migration from correct problem solving to a bug or vice versa, but not migration between two different bugs. Migration in problem solving can only occur, according to Repair Theory, when there is a fixed impasse being repaired by two different repairs. No provision is provided for changing from a core procedure which is correct to one that has an impasse, and thus, the variability seen in these studies cannot be explained by the notion of bug migration.

A second and more curcial reason the bug migration concept cannot account for the context effects observed in these studies is that their is no provision for a mechanism whereby repairs on target problems can be influenced by problem solving on

context problems. That is, repairs are made by context-independent mechanisms.

In summary, Repair Theory has recently recognized the variability in arithmetic errors, and, as such, the theory represents a clearly superior description of some aspects of children's arithmetic problem solving than had been previously offered by other Fixed Rule models, e.g. Bug Theory. However, to account for the basic context effects observed in the studies reported here, where an even greater degree of flexibility in problem solving must be recognized, the theory would have to introduce context-sensitive rules and procedures, an approach that would have no apriori justification within the existing principles of the theory and would no doubt greatly reduce the theories attractive simplicity. The following discussion concerns an extension of a recent theory used in the psychology of reading and concept formation which may be more useful as a description of children's arithmetic - what shall be referred to as 'procedural analogy' theory.

AN ALTERNATIVE CONCEPTION - PROCEDURAL ANALOGIES

An alternative view of arithmetic problems solving can be developed by extending an increasingly popular theory of reading and concept formation to arithmetic problem solving. Although several examples of such theories could be cited (Brooks,1978; Kay & Marcel,1981; Glushko,1979), the theory of Glushko's and Kay & Marcel's will serve as a departure point. Kay & Marcel (1981) propose a mechanism of reading, lexical analogies, which can be used as a basis for developing an analogy theory of arithmetic problems solving. As briefly discussed in Chapter 1, they propose and present evidence for the notion that reading pronouncable pseudowords (e.g. bint) is accomplished by drawing an analogy between this nonword and a known real word (e.g. mint or pint), in contrast to, as had been often argued in the past, calling on a set of spelling-sound correspondence rules to apply to the word. That is, the pronounciation of this nonword is accomplished by identifying a similar

looking known word and pronouncing the nonword similarly to how the known word is pronounced. In the case of bint, the word would be pronounced similarily to mint or pint depending upon which of these known words were used as the basis for the analogy. This process was referred to as using "lexical analogies". A similar process was identified much earlier by Brooks (1978) as a possible basis for learning to classify instances of concepts, that is, by identifying a similar already known instance and giving the new unknown stimulus the category label associated with the known instance. This "analogy to instance" model of categorization contrasted sharply with the prevailing rule models which conceptualized the process as resulting from learning rules of category membership. The point of the analogy accounts of cognitive performance is that behavior which is apparently regular and systematic on the surface can be a result of a mechanism which does not rely on the use of explicit rules. This possibility has never been considered with respect to arithmetic where procedural rules, algorithms and systematic performance have been thought to be so very obvious. The context studies reported here raise the possibility that an analogy mechanism may operate in children's arithmetic problem solving.

Systematic arithmetic performance can be interpreted as resulting from a mechanism whereby the child solves problems by analogy to the way in which he solves other problems. At grade 2, children may be solving 2-by-1-digit subtraction facts by analogy to either single-digit subtraction facts or double-digit subtraction problems. Which problem serves as the basis of the analogy determines whether the child is correct or makes a column-subtraction-based error. At grade 3, children may solve 2-by-1-digit addition and subtraction problems by analogy to either double-digit addition and subtraction problems or 2-by-1-digit multiplication problems. Which problem serves as the basis for the analogy determines whether the child solves the problem correctly or with a diagonal error. At grade 4, children

may solve 2-by-2-digit multiplication problems by analogy to double-digit addition and subtraction problems or to 2-by-1-digit multiplication problems; and which one serves as the basis for the analogy will determine whether a vertical-column error is committed or the problem is solved correctly. And finally, at grade 5, children may solve 2-by-1-digit bracketed division problems involving remainders by analogy to either 2-by-1-digit division facts or to other 2-by-1-digit division problems requiring remainders, and which one serves as the basis of the analogy will determine whether a division fact error occurs or the problem is solved by a correct "long division" procedure.

As in lexical analogy theory for reading and "instance" theory of categorization, one potent determiner of the basis of analogies is the nature of the context provided for the target stimulus. Recently experienced stimuli can bias the analogy mechanism towards one analogy or another. For example, as discussed in Chapter 1, if the real word mint is presented on the same reading sheet as bint, bint is more likely to be pronounced similarly to mint; if, on the other hand, pint is present on the same reading sheet, the word is more likely to be pronounced similarly to it. The point is that context stimuli can influence target stimulus processing by influencing the probable basis for the analogy used to process the target stimulus. The view of children's arithmetic problem solving that is suggested by the studies reported here is that subprocedures for problem solving are incompletely integrated and can be reorganized in various ways for the solution of arithmetic problems based upon analogies to episodes of problem solving activity contained in previously experienced contexts. In children's arithmetic, this predicts the type of context effects observed in the studies reported here.

An interesting characteristic of lexical analogies of the kind discussed above is that there is recent evidence (see Jacoby & Witherspoon, 1982 for a review)

indicating that the ability of context stimuli to influence the reading of target stimuli is not a short-lived phenomenon. Jacoby has produced data showing an effect for as long as 24 hours and more. Such a long-lasting context effect in reading corresponds nicely to the 24-hour carry-over context effects observed in Study 4.

The possibility suggested here that highly procedurally appearing arithmetic problem solving could be guided by a mechanism of analogy has been recognized in a much more limited fashion by Repair Theorists (Brown & VanLehn, 1982, p.130) and deserves comment. In this article Brown & VanLehn suggest that in order to explain certain bugs in children's subtraction problem solving an analogy heuristic must be considered. They specifically describe this heuristic in the form of a rule stating "use an operation that worked in an analogous situation". They have postulated this repair rule for a specific error seen rather rarely in subtraction: incrementing by 1 during a regrouping operation instead of decrementing by 1 (seen in Table 1b). Incrementing is the proper procedure used in the regrouping operation in addition where "carrying" a 1 is performed when the sum of a column of numbers exceeds 9. Therefore, they interpret the increment error during regrouping in subtraction as resulting from drawing an analogy to the regrouping operation in addition. They point out that this analogy heuristic is really a general purpose heuristic in the sense that it is used in situations other than arithmetic. Several aspects of their analogy concept should be noted. First, as a repair heuristic it is used only when an incorrect core procedure reaches an impasse, and, even then, is only one among many other heuristics which can be called upon for use in a repair. As such, it can not serve as a basis for correct problem solving, an aspect of the procedural analogy concept proposed here.

Second, as a repair heuristic, it is not sensitive to contextual events, and, therefore, cannot explain the context effects observed in the studies reported

here. Its use follows the rules of repair heuristic selection which have not been well defined in Repair Theory, but seem to involve random samplying, short-term memory for recent repairs (i.e. patches), and passing the "critics" acceptability criteria, none of which are processes involving context-sensitive mechanisms. Thus, the use of an analogy heuristic as defined in Repair Theory is not an acceptable explanaton of the context-sensitive problem solving presented in these studies.

In summary, their appears to be considerable merit to the idea that children solve simple arithmetic problems by an analogy mechanism which is context-sensitive. No existing modifications of Fixed Rule approaches have produced a theory that depicts the child problem solver with sufficient flexibility to account for the basic context effects reported in this thesis. A final set of comments should be made, prior to considering directions for future research, concerning the educational implications of the study findings, especially the mixed-drill effects.

IMPLICATIONS FOR TEACHING ARITHMETIC

Studies 3 and 7 produced some evidence to indicate that certain types of context-based computational errors could be dramatically reduced by providing children with experiences with arithmetic problem solving drills in which target and context types of problems are randomly intermixed - so-called "mixed-drill" effect. A question of considerable practical interest is why teachers do not routinely program such mixed drills during the course of their instruction?

First, regarding the grade 2 column-sutraction-based errors in solving 2-by-1-digit subtraction fact problems (CS). Two different reasons are possible for why grade 2 teachers do not give mixed drills, both reflecting the possible effect of tacit acceptance of a Fixed-Rule view of arithmetic learning. First, teachers appear to be often unaware of column-subtraction-based errors, as revealed in numerous

conversations with them during the collection of data for these studies. This may result from the tacit assumption that arithmetic learning is context-independent, which may lead them to erroneously conclude that learning is complete on the basis of observations of successful 2-by-1-digit subtraction-fact problem solving in unmixed test situations. Accordingly, no attempt is made to test 2-by-1-digit subtraction-fact problem solving proficiency in alternative test situations, and, thus, column-subtraction-based errors are rarely actually observed by teachers.

Second, teachers may be prone to misinterpret column-subtraction-based errors they do happen to observe. Discussions with grade 2 teachers revealed that some are aware of such errors but interpret them as indicating that the children have forgotten that 2-by-1-digit subtraction facts are solvable with a subtraction-fact-finding procedure. The remediation that is followed given this interpretation is designed to restore the forgotten association between 2-by-1-digit subtraction facts and subtraction-fact-finding activities and entails demonstrating counting-up and counting-down procedures using number lines and other concrete material. Thus, rather than view the column-subtraction-based error as indicating a failure to procedurally differentiate between 2-by-1-digit-subtraction facts and 2-by-2-digit subtraction problems, which would increase the chance of designing mixed drill remediations, these teachers view such errors as indicating a forgetting of fact finding procedures and give remedial experiences that are, in fact, contraindicated by the basic context effect which reveals that children "remember" subtraction-fact strategies in the solution of 2-by-1-digit subtraction facts but only in the single-digit subtraction context.

Similar interpretations can be given for the lack of use of mixed drills at grade 3 in relation to the diagonal error. Many grade 3 teachers casually interviewed were not aware of the diagonal error, even though it was found in these

studies that under non-rectangular multiplication context conditions these errors were made by over 80% of their students. During postlesson testing in study 7, when teachers became aware of the use of a mixed-problem-type test (Test M), they sometimes protested that this was "unfair" because it was "tricking" the children. This interpretation of such errors under the mixed conditions may reveal the nature of the learning theories implicitly held by these teachers. Rather than viewing these errors under these conditions as indicating a lack of inter-problem procedural differentiation in responding, and as a "valid" test of the limits of arithmetic knowledge, they chose to view such errors as what might be called performance perturbations and as resulting from "invalid" tests of the "true" knowledge the children possessed. Such interpretations can be viewed as reflecting their implicit view of arithmetic learning which is, in essence, the Fixed Rule perspective, in which knowledge is presumed to be either present in the child or absent, and if testing conditions are found in which the child performs correctly (unmixed drills) then it is concluded that such knowlede is present in fixed, absolute form, and no thought is given to testing the knowledge under a variety of conditions. Continuing with this argument, if subsequent "poor" performance is found it is rationalized by teachers within this implicit fixed-rule theory as an insignificant performance quirk and not a challenge to the assumption that the children have fixed knowledge.

A final few comments will be made concerning what the findings presented in these studies may suggest in terms of future research.

FURTHER RESEARCH

Although the procedural analogy concept has been fruitful in suggesting contextual manipulations for demonstrations of computational flexibility, limitations to this concept are worthy of note.

First, the notion of analogy in arithmetic problem solving implies that

analogies between pairs of problem types should be bidirectional; that is, if problem type A can serve as a basis for a procedural analogy for problem type B, so can B serve as a basis for A. In some cases this is quite reasonable, for example, with regard to the 2-by-1-digit addition, subtraction, and multiplication problems. The addition and subtraction problems are solved with diagonal errors, and the multiplication problems are also solved with vertical column procedures. However, some pairs of problem types do not exhibit bidirectional analogy errors. For example, 2-by-2-digit multiplication problems are often solved using a vertical column solution strategy, as if by analogy to 2-by-2-digit addition and subtraction; however, I have never observed 2-by-2-digit addition or subtraction problems solved as if by analogy to a multiplication problem which would involve using a modified partial-product procedure, although there is one published observation of such erroneous computations (Owston, 1981). The lack of bidirectional analogy errors suggests that there are other factors playing a role in determining the types of analogies which will actually generate solution errors.

Second, some systematic errors made by primary school children are not a result of any obvious analogies. As seen in Table 12, many of the errors observed among grade 4 children, made while solving 2-digit-by-2-digit multiplication problems, are difficult to explain as resulting from any obvious analogy. For example, consider the place-holder error (1f.) where the child solves the problem by using an otherwise completely correct partial-product procedure except he systematically does not use a place-holding 0 or blank space in the second partial product, or the diagonal order error (1a.) where the child systematicaally performs the diagonal substep and vertical substep in deriving the second partial product in the reverse order. It would be difficult to imagine any part of any other procedure that could provide a basis, by analogy, for these errors. In fact, most of the

different partial-product errors observed in this study are in this category. However, these types of errors represented a clear minority of the total errors observed on these problems: no partial product error represented more than 11% of the total errors, whereas, the vertical-column error, accountable in terms of an analogy mechanism, represented 56% of the errors. Thus, although these different errors are difficult to account for in terms of analogies to other procedures or subprocedures, they are of less importance in terms of the difficulties children typically have in multiplication problem solving. They, nonetheless, are of considerable theoretical interest. For example, are they as contextually sensitive as the apparent analogy-based errors? It is of some interest to note that the non-analogy systematic errors often involve directional confusions such as the Horizonatal-column error (4.) where the child uses a straight column procedure on each row rather than on each column. A possible basis for these non-analogy errors may be in processing errors involving information related to temporal order and spatial direction. Only future research can resolve this issue.

Another observation raises questions about the generality of an extreme analogy view of arithmetic problem solving. In the analysis of diagonal errors it was proposed that correct solutions should be more probable to 2-by-1-digit subtraction and addition targets in rectangular addition and subtraction problem contexts and, indeed, this proposal was bore out by the data. However, in one individual case observed while solving 2-by-1-digit subtraction fact targets in single-digit subtraction fact contexts and in double-digit subtraction problem contexts (rectangular), she was observed to solve the target correctly in the single digit context, and with a diagonal error in the rectangular context. This implies that in her case the diagonal procedure was part of a column-subtraction procedure, since it was the later procedure that served as the basis for analogy in the

double-digit subtraction problem context. This contradicts the predicted effect of double-digit subtraction problem contexts derived from the analogy analysis of the diagonal error and the correct vertical column procedure. Therefore, in individual cases, the organization of arithmetic problem solving can be at variance with that expected from a simple analogy perspective. Further research in these cases are of considerable interest.

A fourth problem with the procedural analogy concept as it is currently used in this thesis, is the inability to predict which problem types will evidence within-subject context effects. The 2-by-1-digit subtraction facts and the division with remainder problems of grade 2 and 5, respectively, evidenced appropriate within-subject changes in problem solving with changes in context. However, the diagonal error to 2-by-1-digit addition and subtraction problems and the vertical-column error to 2-by-2-digit multiplication problems did not show the expected within-subject changes, but, instead, showed carry-over effects from the prior context. More work needs to be done to explore possible bases for these problem type differences before a complete theory of children's arithmetic problem solving can be said to exist.

In summary, the studies reported here suggested that existing models of children's arithmetic learning and problem solving, based on a Fixed Rule perspective, are inadequate in that they imply that arithmetic problem solving is context-independent. Context effects were clearly demonstrated at several grade levels which were interpreted to indicate that, rather than following fixed rules, children solve arithmetic problems by using procedures developed by analogy to solutions of other problems, and the effect of context problem solving is to biase the analogy process toward certain problems as bases of procedural analogies. Future theory and research in this area must recognize the considerable flexibility in

computational problem solving exhibited by primary-school age children, as well as the powerful influence contextual problem solving has on the computational strategies observed to particular test problems.

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FOOTNOTES

1. The position that is presented here as the "Fixed Rule" approach is really an abstraction from a variety of specific theories of children's arithmetic learning, discussed later in this chapter, as well as from other rule-oriented views proposed outside of the arithmetic literature (e.g. in reading, and in classification), discussed in chapter 2. The emphasis of especially the arithmetic rule-learning theories is on explaining the consistency in performance. Little explicit recognition of variability in performance is made except in various specific ways to be discussed in this chapter. For example, forgetting an algorithm may occur and an assumption that retreival cues can be effective in influencing this forgetting is developed in Frame Theory. No recognition is explicitly given to variability which may result from such possible mechanisms as impulsivity or the partial learning of rules. These mechanisms probably do influence arithmetic problem solving, but they cannot easily be used to explain the type of context effects demonstrated in these studies, wherein it is shown that certain arithmetic problems in a test cause changes in problem solving on certain other specific problems in the same test, unless purely post hoc assumptions about variables that may influence these mechanisms is made. Therefore, these mechanisms have not been stressed in this representation of a "fixed rule" approach to theorizing in arithmetic learning.

2. Among these 17 responses were three types of errors. Ten of these responses were made by one child in group 2-1 and consisted of writing "00" in the answer space. The "00" response was made consistently to all 5 targets on both context worksheets. Because of the unusual quality of these responses, an interview was conducted with this child on a subsequent day. A block of 2-by-1-digit subtraction facts were presented and the child was asked to explain how she was doing these problems as she solved them ("talk outloud"). She approached every problem by going first to the units column and reading the two numbers and saying that "you can't take" the bottom number from the top number so the answer is "0" in the units answer. She then would go to the tens column and say that "one take away nothing is nothing" and write 0 in the tens place of the answer. In fact, on other 2-digit-by-1-digit subtraction problems such as 16-5 and 18-4 she answered by writing 0 in the tens place and the correct subtraction fact for the units number in the units place of the answer (16-5=01; 18-4=14). Clearly, she is using a units-tens, column-subtraction strategy but has unusual and erroneous solution strategies for the single-digit subtraction facts embedded in the 2-by-1-digit subtraction fact problems. One other child in Group 1-2 made one of these responses in the double-digit context only which is consistent with the interpretation of this solution as resulting from the use of a column-subtraction strategy that is contextually-driven.

Of the remaining unclassified responses, 5 were evidenced by one child only in the double-digit context. This child gave consistent answers in which a "0" was also written in the tens place, and and inversion error was written in the units place, yielding answers such as 12-4=02, 14-8=04; 17-9=02. This child appeared to also being using a unit-tens, column-subtraction response strategy which was context-specific to the

double-digit worksheets.

The final type of response seen was an answer of "10" to one target in the double-digit context. Consistent responses of 10 to 2-digit subtraction-fact targets were not abundant in this studys, but were observed more frequently in a classroon of grade 2 children used in study 3. Given the rational used by the child making the "00" error it is entirely plausible that this error resulted from the use of a column subtraction procedure in which units numbers with the smallest number on top are responded to by writing a "0" in the units place. The fact that this error occurred in the double-digit context supports this interpretation. The context-specificity of the "10" error is further documented in study 3, lending further credence to this interpretation.

Three responses of the 20 unclassificed were not interpretable.

The point being made in this footnote is that although the specific answers often given by grade 2 children to 2-digit subtraction facts do not appear, at first glance, to be interpretable, they often are a product of the use of a column-subtraction problem solving strategy with various interesting combinations of types of erroneous single-digit subtraction-fact problems solving strategies. Although, a rigid scoring classification scheme can be constructed, it will almost inevitably miss some of these interesting and unusual variants.

3. The precise number of days on which mixed drills were scheduled by this teacher over the course of her 2-week instruction in the 2-by-1-digit multiplication algorithm is not known. Nor was data collected about how she specifically constructed the drills. Her overall plan, as submitted to the researcher, was as follows, covering a 2-week period of time:

(1) Introduction of concept of multiplication (sets of, groups of).
(2) Drill exercises of facts up to 7 X 10. This continues throughout remainder of program.
(3) Begin 2-by-1-digit multiplication algorithm instruction - no regrouping (i.e. no "carrying").
(4) Word problems using multiplication (no regrouping). Used to promote the use of the newly learned multiplication algorithm with word problems.
(5) Mixed drills including 2-by-1-digit multiplication, addition, and subtraction.
(6) Word problems with above three types of problems combined.
(7) Introduction to 2-by-1-digit multiplication with regrouping.
(8) Mixed drills including the three types with regrouping.

APPENDIX I

PROBLEMS ON TEST 1-1 OF STUDY 1-5

9	9	6	8	12
-3	-1	-4	-6	- 4
6	2	7	4	8
-0	-2	-2	-3	-1
8	14	8	17	7
-4	- 8	-8	- 9	-5
7	5	5	9	6
-3	-4	-3	-2	-5
7	7	16	5	9
-7	-4	- 8	-0	-5
3	9	15	3	9
-3	-8	- 7	-2	-4

APPENDIX II

PROBLEMS ON TEST 2-2 OF STUDY 1,2,4,5

96	89	47	94	12
-31	-42	-16	-42	- 4
35	75	57	84	16
-15	-22	-24	-61	- 5
52	14	58	17	92
-30	- 8	-15	- 9	-41
29	20	78	67	97
-14	-10	-42	-25	-63
89	18	16	86	43
-59	- 4	- 8	-35	-12
18	69	15	35	66
- 5	-34	- 7	-14	-54

APPENDIX III

PROBLEMS ON TEST DS (OR 2-2B) OF STUDY 3 & 5

96 −31	82 −49	47 −16	92 −44	12 − 4
35 −15	72 −25	54 −27	84 −61	16 − 5
50 −33	14 − 8	55 −18	17 −9	91 −42
24 −19	20 −10	72 −48	67 −25	93 −67
78 −59	18 − 4	16 − 8	86 −35	42 −13
18 − 5	64 −39	15 − 7	35 −14	66 −54

APPENDIX IV

PROBLEMS ON TEST SD OF STUDY 3

9 − 3	9 − 1	6 − 4	8 − 6	7) 13
6 − 0	2 − 2	7 − 2	4 − 3	8 − 1
8 − 4	8) 14	8 − 8	9) 17	7 − 5
7 − 3	5 − 4	5 − 3	9 − 2	6 − 5
7 − 7	7 − 4	6) 11	5 − 0	9 − 5
3 − 0	9 − 8	3) 11	3 − 2	9 − 4

APPENDIX V

PROBLEMS ON TEST DD OF STUDY 3

$4\overline{)21}$	$8\overline{)69}$	$8\overline{)30}$	$8\overline{)63}$	$7\overline{)13}$
$4\overline{)26}$	$7\overline{)62}$	$7\overline{)28}$	$9\overline{)45}$	$3\overline{)17}$
$7\overline{)54}$	$8\overline{)14}$	$4\overline{)31}$	$9\overline{)17}$	$9\overline{)39}$
$9\overline{)80}$	$7\overline{)47}$	$5\overline{)27}$	$6\overline{)41}$	$5\overline{)30}$
$3\overline{)20}$	$6\overline{)23}$	$6\overline{)11}$	$8\overline{)64}$	$2\overline{)13}$
$6\overline{)49}$	$9\overline{)53}$	$3\overline{)11}$	$9\overline{)76}$	$3\overline{)15}$

APPENDIX VI

PROBLEMS ON TEST 2-1B OF STUDY 5

96 − 4	82 − 4	47 − 6	23 − 5	12 − 4
32 − 5	75 − 2	54 − 7	84 − 8	16 − 5
72 − 7	14 − 8	31 − 4	17 − 9	95 − 3
29 − 4	26 − 8	78 − 2	65 − 7	97 − 3
87 − 9	64 − 8	16 − 8	86 − 5	42 − 3
18 − 5	61 − 7	15 − 7	35 − 4	66 − 4

APPENDIX VII

PROBLEMS ON TEST M OF STUDY 7 & 8

39 + 4	69 − 6	32 X 3	33 − 2	42 X 2	54 − 5
48 + 0	25 X 3	23 − 7	46 + 6	63 + 8	21 X 5
83 − 6	43 + 2	54 X 4	33 − 0	32 X 6	16 − 4
51 X 3	76 − 4	21 + 5	12 X 2	40 − 6	64 − 7
76 + 3	43 X 7	39 + 5	35 + 3	16 X 5	40 + 3
50 X 0	67 − 3	54 + 5	72 − 4	54 X 0	16 + 7

APPENDIX VIII

PROBLEMS ON TEST AS OF STUDY 7 & 8

37 + 8	69 − 5	64 +11	34 − 2	65 +56	43 − 5
37 + 0	85 −43	23 − 5	48 + 6	74 + 9	58 −30
84 − 6	44 + 2	27 +82	42 − 0	71 −56	15 − 3
14 +77	87 − 4	21 + 4	75 −63	30 − 7	68 − 9
75 + 4	78 +97	59 + 3	36 + 3	44 −18	20 + 5
62 −33	58 − 3	53 + 5	72 − 6	58 +31	15 + 6

APPENDIX IX

PROBLEMS ON TEST AS OF STUDY 9

35	65	42	34	32
+21	−11	−12	−22	X11
33	13	43	34	61
+21	−12	+22	−20	+11
34	34	42	43	34
+31	X21	−22	X23	+22
45	24	31	64	33
−21	−20	+21	+12	+21
42	33	34	52	44
−12	+32	X32	−11	+21
24	32	64	42	45
−12	−11	X12	+12	−31

APPENDIX X

PROBLEMS ON TEST M OF STUDY 9

22 X 7	68 X 6	19 X 2	79 X 4	32 X11
18 X 6	87 X 8	27 X 3	74 X 7	83 X 6
38 X 8	34 X21	77 X 6	43 X23	76 X 2
96 X 5	61 X 4	18 X 3	89 X 1	32 X 7
49 X 6	87 X 9	34 X32	25 X 2	26 X 8
21 X 3	54 X 4	64 X12	37 X 9	23 X 5

APPENDIX XI

PROBLEMS ON TEST OF OF STUDY 10

$7\overline{)14}$	$4\overline{)16}$	$5\overline{)20}$	$6\overline{)24}$	$8\overline{)63}$
$2\overline{)8}$	$9\overline{)36}$	$5\overline{)45}$	$8\overline{)40}$	$2\overline{)12}$
$5\overline{)30}$	$6\overline{)56}$	$8\overline{)24}$	$7\overline{)54}$	$1\overline{)4}$
$8\overline{)16}$	$7\overline{)49}$	$3\overline{)15}$	$2\overline{)10}$	$7\overline{)35}$
$6\overline{)18}$	$3\overline{)21}$	$7\overline{)43}$	$4\overline{)8}$	$3\overline{)12}$
$5\overline{)25}$	$9\overline{)81}$	$4\overline{)27}$	$1\overline{)6}$	$2\overline{)18}$

APPENDIX XII

PROBLEMS ON TEST DR OF STUDY 10

$7\overline{)23}$	$6\overline{)44}$	$9\overline{)46}$	$8\overline{)25}$	$8\overline{)63}$
$6\overline{)55}$	$8\overline{)14}$	$4\overline{)10}$	$5\overline{)38}$	$9\overline{)31}$
$7\overline{)39}$	$6\overline{)56}$	$7\overline{)12}$	$7\overline{)54}$	$9\overline{)49}$
$4\overline{)15}$	$7\overline{)65}$	$8\overline{)79}$	$8\overline{)11}$	$9\overline{)46}$
$6\overline{)32}$	$6\overline{)22}$	$7\overline{)43}$	$5\overline{)38}$	$4\overline{)26}$
$9\overline{)31}$	$6\overline{)59}$	$4\overline{)27}$	$7\overline{)17}$	$8\overline{)29}$