TOPOLOGICAL ALGEBRAS WITH ORTHOGONAL M-BASES

By

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ABSTRACT

An M-basis in a topological vector space is a special case of the extended Markushevich basis, and a generalization of the unconditional basis. We study orthogonal bases and orthogonal M-bases in topological algebras, with emphasis on locally convex algebras. It turns out that an orthogonal basis or an orthogonal M-basis in a topological algebra is necessarily Schauder. We characterize some concrete topological algebras with orthogonal bases or orthogonal M-bases, up to a topological isomorphism. We introduce and study two classes of locally convex algebras: the class of "Φ-algebras" which includes, for example $\mathbb{C}^{\Gamma}$, $c_0(\Gamma)$, $C^*_\beta(\Gamma)$ and $H(\mathbb{D})$ (with the Hadamard multiplication); and the larger class of "locally convex s-algebras" which also includes - among other examples - ℓ_p, $1 \leq p < \infty$, and the Arens algebra $L^\omega[0,1]$. A Φ-algebra is not necessarily locally m-convex, and a locally m-convex algebra is not necessarily a locally convex s-algebra. We give two examples of Banach algebras with orthogonal bases which are not unconditional and we prove that an orthogonal basis in a B_0 algebra is unconditional iff the algebra is a locally convex s-algebra. We also study the conversion of a Fréchet space with an unconditional basis into a Fréchet algebra with the basis under consideration as an orthogonal basis and we obtain a necessary and sufficient condition for this to be possible, revising and extending a result of Husain and Watson obtained for Banach spaces.

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INTRODUCTION

Let L be a vector space over a field $\mathbb{K}$ (which is either $\mathbb{R}$ or $\mathbb{C}$). A "Hamel basis" in L is a subset $S \subset L$ with the following property: for every non-zero $x \in L$ there exist unique finite subsets $\{e_1, e_2, \ldots, e_n\} \subset S$ and $\{\lambda_1, \lambda_2, \ldots, \lambda_n\} \subset \mathbb{C}$ with $\lambda_k \neq 0$, $k = 1, 2, \ldots, n$, such that $x = \sum_{k=1}^{n} \lambda_k e_k$. The existence of a Hamel basis in every vector space is established as a direct consequence of the Zorn's lemma.

In most examples of topological vector spaces which arise in functional analysis and its applications, Hamel bases do not play an important role. However, the topology makes it possible to define the following important concept: a sequence $\{e_n\}$ in a topological vector space E is called a "topological basis" or, simply, a "basis" if for every $x \in E$ there exists a unique sequence $\{\lambda_n\}$ of scalars such that $x = \sum_{n=1}^{\infty} \lambda_n e_n = \lim_N \sum_{n=1}^{N} \lambda_n e_n$. The existence of a basis in a topological vector space facilitates the study of the space to a large extent, and reveals a considerable amount of information about its structure.

It is immediate that every topological vector space with a basis is separable. On the other hand, not every separable topological vector space has a basis. Indeed, a separable F-space with a trivial topological dual such as $L_p[0,1]$, $0 < p < 1$, cannot have a basis (see Remark

I.28 (iii)). It is, however, true that every separable Hilbert space has a basis (a complete orthonormal system). The case of a separable Banach space is more involved, in fact, since Banach (in 1932) posed the problem of whether every separable Banach space has a basis (known as the "basis problem"), this problem remained unsolved until Enflo [9] in 1973 constructed an example of a separable Banach space with no basis. The basis problem generated a great interest in the study of bases in Banach spaces which brought about a vast literature in the subject (for a detailed account, see [35]). Bases in topological vector spaces which are more general than Banach spaces, and generalizations of the concept of a basis were also studied by Arsove, Edwards, Marti and others (see [2], [27]).

During the last ten years, Husain, Liang and Watson initiated a study of the basis theory in the context of topological algebras. Motivated by several examples of topological vector spaces with bases, each of which is a topological algebra for some natural multiplication, the above authors defined two main types of bases according to the behaviour of the basis elements with respect to the multiplication operation: a basis $\{e_n\}$ in a topological algebra A is said to be "orthogonal" if $e_m e_n = \delta_{mn} e_m$ (where δ_{mn} is the Kronecker's delta), and is said to be "cyclic with the element $z \in A$ as a generator" if $e_n = z^{n-1}$, $n \in \mathbb{N}$ ($e_1 = z^0$ is understood to be an identity in A, whose existence is implied by the existence of a cyclic basis). The canonical basis in each of the Banach algebras ℓ_p, $1 \leq p < \infty$ with the coordinatewise operations is an orthogonal basis; and the Fréchet algebra $H(\mathbb{D})$ of analytic functions on the open unit disc $\mathbb{D}$ with the pointwise operations and the compact-open topology has the sequence $\{e_n\}$ – where $e_n(t) = t^{n-1}$, $t \in \mathbb{D}$ – as a cyclic basis.

Some examples of topological vector spaces and topological algebras carry the following type of generalized basis, which is a special case of the extended Markushevich basis, and a generalization of the unconditional basis (using the notations of [27]): an M-basis in a topological vector space E is a map $\alpha \to e_\alpha$ (which we denote by $\{e_\alpha\}_{\alpha\in\Gamma}$) of a set Γ into E such that for every $x \in E$ there exists a unique map $\alpha \to e_\alpha^*(x)$ of Γ into the scalar field $\mathbb{K}$ with $x = \sum_{\alpha\in\Gamma} e_\alpha^*(x)e_\alpha = \lim_{J\in\Omega} \sum_{\alpha\in J} e_\alpha^*(x)e_\alpha$, where Ω is the set of all finite subsets of Γ directed by inclusion, and $\{\sum_{\alpha\in J} e_\alpha^*(x)e_\alpha\}_{J\in\Omega}$ is regarded as a net in E. In the case of a topological algebra A, an M-basis $\{e_\alpha\}_{\alpha\in\Gamma}$ is said to be orthogonal if $e_\alpha e_\beta = \delta_{\alpha\beta} e_\alpha$, $\alpha, \beta \in \Gamma$. This thesis is – in the most part – concerned with the study of topological algebras with orthogonal M-bases. Examples of such algebras include the algebra $\mathbb{C}^\Gamma$ of all complex valued functions on a set Γ with the pointwise operations and the product topology; the algebra $C_\beta^*(\Gamma)$ of all bounded complex-valued functions on a discrete topological space Γ with the pointwise operations and the strict topology as defined by R.C. Buck in [3]; and the Banach algebra $\ell_p(\Gamma)$, $1 \le p < \infty$ of all complex-valued functions x on a set Γ with $\|x\|_p = (\sum_{\alpha\in\Gamma} |x(\alpha)|^p)^{\frac{1}{p}} < \infty$. In addition to the study of orthogonal M-bases, we include a chapter (Chapter II) on orthogonal bases. As in the case of a basis in a topological vector space, the existence of an orthogonal basis, or an orthogonal M-basis in a topological algebra is instrumental in studying the structure of the topological algebra under consideration.

Specifically, Chapter I is devoted to basic definitions, terminology and background information needed throughout the thesis.

In Chapter II, we discuss two main points concerning orthogonal bases in topological algebras: (i) When does a Fréchet space with an unconditional basis become a Fréchet algebra with the given basis being orthogonal?... (ii) Is an orthogonal basis in a B_0 algebra necessarily unconditional?... The answer to question (i) is given by Theorem II.3 and the answer to question (ii) is not - in general - in the affirmative (see Examples II.8). However, the consideration of question (ii) leads to the new concept of a locally convex s-algebra, and it turns out that an orthogonal basis in a B_0 algebra A is unconditional iff A is a B_0 s-algebra (Theorem II.12).

Chapter III starts with some general results about orthogonal bases and orthogonal M-bases in topological algebras. For instance, we show that an orthogonal basis or an orthogonal M-basis in any topological algebra is always Schauder (Theorem III.1). It is worth noticing that analogues of this result, in the context of basis theory in topological vector spaces, require the conditions of the Open Mapping theorem, which are not assumed here, in the present setting. The rest of the chapter is devoted to a study of different types of locally convex algebras with orthogonal bases or orthogonal M-bases. The types considered are: locally m-convex algebras, A-convex algebras, locally convex s-algebras (introduced in Chapter II) and "Φ-algebras", a type of locally convex algebras with orthogonal M-bases which generalizes $\mathbb{C}^\Gamma$ and $C_\beta^*(\Gamma)$. We characterize some of these algebras (e.g., Theorem III.6 and Theorem III.13) and we study their structural properties. We close the chapter with a remark (Remark III.27) about a possible generalization of bases and M-bases in topological vector spaces and topological algebras.

In Chapter IV we focus on the Banach algebras $\ell_1(\Gamma)$ and $c_0(\Gamma)$, where Γ is an arbitrary set, and we derive some characterizations of Banach algebras with orthogonal M-bases which are topologically isomorphic with either of these two algebras. The results in §1 of Chapter IV – which are modified extended versions of Husain's results in [14] and [15] – characterize $\ell_1(\Gamma)$ and $c_0(\Gamma)$ by M-basis properties, while the results in §2 characterize c_0 and $c_0(\Gamma)$, mainly, by the existence of elements with certain properties (Theorem IV.9 and Corollary IV.13).

CHAPTER I

PRELIMINARIES

In this chapter, we briefly present the main background material needed throughout the thesis. In general, proofs are omitted since they are available in standard books and research papers. Basic definitions and facts from the general theory of topological vector spaces and topological algebras are given in §1 and §2. Background material from the theory of bases in topological vector spaces is presented in §3, and some results from the theory of orthogonal bases in topological algebras, recently introduced and studied by T. Husain, J. Liang and S. Watson in [14], [15], ..., [20] is given in §4. The definition of an orthogonal M-basis in a topological algebra is also given in §4, and various examples of topological algebras with orthogonal bases or orthogonal M-bases are presented.

§1. Topological vector spaces. (For proofs and further details see [12], [23], [32], [33], [34].)

A topological vector space is a vector space E over the field of complex (or real) numbers $\mathbb{C}$ (or $\mathbb{R}$), with a topology such that each of the addition and the scalar multiplication is jointly continuous. Unless otherwise is stated, the scalar field will always be $\mathbb{C}$. The topology of a topological vector space is determined by a 0-neighbourhood base $\{U\}$, indeed, for $x \in E$, $\{x + U\}$ is an x-neighbourhood base.

A subset $B \subset E$ is said to be <u>bounded</u> if for every $U \in \{U\}$ there exists $\lambda > 0$ such that $\mu \in \mathbb{C}$, $|\mu| \geq \lambda \Rightarrow B \subset \mu U$. E is Hausdorff iff $\bigcap_{U \in \{U\}} U = \{0\}$. In any topological vector space E, there exists a 0-neighbourhood base $\{U\}$ consisting of closed sets such that (i) each U is <u>circled</u> (i.e., $\lambda \in \mathbb{C}$, $|\lambda| \leq 1 \Rightarrow \lambda U \subset U$) and <u>absorbing</u> (i.e., for each $x \in E$ there exists $\lambda > 0$ such that $\mu \in \mathbb{C}$, $|\mu| \geq \lambda \Rightarrow x \in \mu U$), (ii) for each $U \in \{U\}$ there exists $V \in \{U\}$ such that $V + V \subset U$, and (iii) $\bigcap_{U \in \{U\}} U = \{0\}$. Conversely, if $\{U\}$ is a filter base in a vector space E satisfying (i), (ii) and (iii) then $\{U\}$ is a 0-neighbourhood base for a unique topology that makes E a Hausdorff topological vector space. It follows that a subset $B \subset E$ is bounded iff for every neighbourhood W of 0, there exists $\lambda > 0$ with $B \subset \lambda U$. In the sequel, all topological vector spaces will be Hausdorff.

E is metrizable (i.e. its topology can be defined by a translation invariant metric) iff there exists a countable 0-neighbourhood base.

A seminorm on a vector space L is a function $p: L \to \mathbb{R}$ such that $p(x) \geq 0$, $p(x + y) \leq p(x) + p(y)$ and $p(\lambda x) = |\lambda| p(x)$ for all $x, y \in L$ and $\lambda \in \mathbb{C}$. If, in addition, $p(x) = 0 \Rightarrow x = 0$, p is called a norm. A linear functional on L is a map $f: L \to \mathbb{C}$ such that $f(x + y) = f(x) + f(y)$ and $f(\lambda x) = \lambda f(x)$ for all $x, y \in L$ and $\lambda \in \mathbb{C}$ (the notation $\langle x, f \rangle$ for $f(x)$ is useful in some situations). Under the pointwise operations, the set of all linear functionals on L is a vector space called the <u>algebraic dual</u> of L and is denoted by L^*. If E is a topological vector space the set $E' = \{f \in E^*: f \text{ is continuous}\}$ is a vector subspace of E^*, called the <u>topological dual</u> of E.

I.1 Theorem. (The Hahn-Banach Extension Theorem.)

Let p be a seminorm on a vector space L and let L_0 be a vector subspace of L. If $f_0 \in L_0^*$ is such that $|f_0(x)| \leq p(x)$ for all $x \in L_0$, then there exists $f \in L^*$ such that $f(x) = f_0(x)$ for all $x \in L_0$ and $|f(x)| \leq p(x)$ for all $x \in L$.

I.2 Corollary. Let p be a seminorm on a vector space L. For every $a \in L$ there exists $f_a \in L^*$ such that $f_a(a) = p(a)$ and $|f(x)| \leq p(x)$ for all $x \in L$.

A subset W of a vector space L is called convex if for every $x,y \in W$ we have $\{\lambda x + (1 - \lambda)y: \ 0 \leq \lambda \leq 1\} \subset W$. A locally convex space is a topological vector space in which a 0-neighbourhood base $\{U\}$ can be found such that each U is convex. In any locally convex space E, one can find a 0-neighbourhood base $\{U\}$ such that each U is closed, convex, circled and absorbing. A subset $V \subset E$ with these properties is called a barrel, thus, in any locally convex space E one can find a 0-neighbourhood base consisting of barrels. If every barrel in a locally convex space E is a neighbourhood of 0, E is called a barrelled space.

If $\{U\}$ is a 0-neighbourhood base in a locally convex space E, consisting of convex, circled and absorbing sets, then each U gives rise to a continuous seminorm p_U on E (called the gauge of U) given by $p_U(x) = \inf\{\lambda > 0: \ x \in \lambda U\}$. Conversely, starting with the family $\{p_U\}$ of seminorms, the sets $\{x \in E: \ p_U(x) < 1\}$ are convex, circled and absorbing, and they form a 0-neighbourhood base for a topology on E

equivalent to the original one given by $\{U\}$. Hence the topology of a locally convex space can be defined by a family of seminorms $P = \{p\}$. A subset $B \subset E$ is bounded iff for each $p \in P$, $p(B)$ is a bounded subset of $\mathbb{R}$. E is Hausdorff (as it will always be assumed) iff for each $x \in E$ with $x \neq 0$, there exists $p \in P$ with $p(x) \neq 0$. Different families of seminorms may define the same topology and they are called equivalent in this case. Two families P and Q are equivalent iff for each $p \in P$ there exists $q \in Q$ and $p' \in P$ such that $p(x) \leq q(x) \leq p'(x)$ for all $x \in E$.

Let $f \in E^*$, then $f \in E'$ iff there exists $p \in P$ such that $|f(x)| \leq p(x)$ for all $x \in E$. In the general case of a topological vector space E, it may happen that $E' = \{0\}$. This, however, cannot happen if E is locally convex. Indeed, the existence of non-zero members in E' is guaranteed by Corollary I.2 which, in addition, shows that there are enough members in E' to separate points of E in the sense that for $x,y \in E$ with $x \neq y$ there exists $f \in E'$ with $f(x) \neq f(y)$. The importance of locally convex spaces stems from this abundance of continuous linear functionals on them.

Let E be a locally convex space. For each fixed $x \in E$, we can define an element in E'^*, which we identify with x, by $x''(x') = x'(x)$, $x' \in E'$. From Corollary I.2, different elements of E define different elements of E'^* and so E is (algebraically) isomorphic with a subspace of E'^* (or, by identification, $E \subset E'^*$). The weak*-topology on E', denoted by $\sigma(E',E)$ is the coarsest topology on E' such that for every $x \in E$ the linear functional on E' defined by x is continuous. It is given by the family of seminorms

$p_S(x') = \sup_{x \in S} |\langle x',x\rangle|$, where S ranges over the set of all finite subsets of E. The topological dual of E' for the topology $\sigma(E',E)$ is precisely E. The <u>weak topology $\sigma(E,E')$</u> on E is the coarsest topology for which every $x' \in E'$ is continuous. Clearly $\sigma(E,E')$ is coarser than the original topology of E. The <u>strong topology</u> $\beta(E',E)$ on E' is given by the seminorms $p_S(x') = \sup_{x \in S} |\langle x',x\rangle|$, where S ranges over all the $\sigma(E,E')$-bounded subsets of E. E' with $\beta(E',E)$ is called the <u>strong dual</u> of E. If E is a normed space, then $\beta(E',E)$ coincides with the norm topology given by $\|x'\| = \sup_{\|x\| \le 1} |x'(x)|$. The strong topology $\beta(E,E')$ on E is given by the seminorms $p_{S'}(x) = \sup_{x' \in S'} |\langle x,x'\rangle|$, where S' runs through all the $\sigma(E',E)$-bounded subsets of E'. E is barrelled iff its topology is $\beta(E,E')$. [12], [23], [32], [34]. E is called <u>reflexive</u> if the strong dual of its strong dual is again E.

A locally convex space E is metrizable iff it has a countable defining family of seminorms. In this case, one can find an increasing (i.e. $p_n(x) \le p_{n+1}(x)$ for all $x \in A$, $n \in \mathbb{N}$) defining sequence $\{p_n\}$ of seminorms; and a metric defining the topology is given by

$d(x,y) = \sum_{n=1}^{\infty} 2^{-n} \frac{p_n(x-y)}{1 + p_n(x-y)}$. E is said to be <u>normable</u> if its topology is defined by a single norm via the metric $d(x,y) = \|x - y\|$, and the pair $(E,\|\cdot\|)$ is called a <u>normed space</u>. A locally convex space is normable iff it has a bounded neighbourhood of 0.

A net $\{x_\alpha\}$ in a topological vector space E is called a <u>Cauchy net</u> if for every neighbourhood U of 0 there exist α_0 such that

$\beta \geq \alpha_0$ and $\gamma \geq \alpha_0 \Rightarrow x_\beta - x_\gamma \in U$. $\{x_\alpha\}$ is said to be convergent to a limit $x \in E$ and we write $\lim x_\alpha = x$ if for every neighbourhood U of 0 there exists α_0 such that $\alpha \geq \alpha_0 \Rightarrow x - x_\alpha \in U$. (When E is Hausdorff, a net can converge to at most one limit). Clearly, every convergent net is Cauchy, but the converse is not always true. E is said to be complete if every Cauchy net is convergent and is said to be semi-complete (or sequentially complete) if every Cauchy sequence is convergent. A metrizable topological vector space is complete iff it is sequentially complete. A complete metrizable topological vector space is called an F-space, while a complete metrizable locally convex space is called a Fréchet space and a complete normed space is called a Banach space. A net $\{x_\alpha\}$ in a locally convex space E with a defining family P of seminorms is Cauchy iff for every $p \in P$ and $\varepsilon > 0$ there exists α_0 such that $\beta \geq \alpha_0$ and $\gamma \geq \alpha_0 \Rightarrow p(x_\beta - x_\gamma) < \varepsilon$, and is convergent to $x \in E$ iff for every $p \in P$ and $\varepsilon > 0$ there exists α_0 such that $\alpha \geq \alpha_0 \Rightarrow p(x - x_\alpha) < \varepsilon$. If E is a normed space and F is a Banach space then $B(E,F) = \{T: E \to F: T$ is linear and continuous$\}$ is a Banach space with the pointwise operations and the norm $\|T\| = \sup_{\|x\|=1} \|Tx\|$. Hence, if E is a normed space, then E' with the norm $\|f\| = \sup_{\|x\|=1} |f(x)|$ is a Banach space.

I.3 Theorem. Every Fréchet space is barrelled. ([12], page 19).

Let $\{x_n\}$ be a sequence in a topological vector space E and set $S_n = \sum_{k=1}^{n} x_k$, $n \in \mathbb{N}$. If $\{S_n\}$ is a Cauchy sequence, we say that

$\sum_{n=1}^{\infty} x_n$ is a Cauchy series. If there exists $x \in E$ with $\lim S_n = x$, we say that the series $\sum_{n=1}^{\infty} x_n$ is convergent (or summable) with sum x and we write $\sum_{n=1}^{\infty} x_n = x$. If $\sum_{n=1}^{\infty} x_{\sigma(n)} = x$ for every permutation $\sigma: \mathbb{N} \to \mathbb{N}$, $\sum_{n=1}^{\infty} x_n$ is said to be unconditionally convergent with sum x. If $\sum_{n=1}^{\infty} x_n$ is a Cauchy series, then the sequence $\{S_n\}$ is bounded and $\lim x_n = 0$.

I.4 Definitions. Let Γ be an arbitrary set with Ω as the set of all finite subsets of Γ directed by inclusion. Let $\alpha \to x_\alpha$ be a map (which we denote by $\{x_\alpha\}_{\alpha\in\Gamma}$) of Γ into a topological vector space E and for each $J \in \Omega$ set $S_J = \sum_{\alpha\in J} x_\alpha$. We say that $\{x_\alpha\}_{\alpha\in\Gamma}$ is summable with sum $\sum_{\alpha\in\Gamma} x_\alpha = x$ if the net $\{S_J\}$ converges to $x \in E$.

Clearly, if $\{x_\alpha\}_{\alpha\in\Gamma}$ is summable, then the net $\{S_J\}$ is Cauchy. The converse is true if E is complete.

I.5 Proposition. With the same notations as above, we have

(i) $\{S_J\}$ is Cauchy iff for every neighbourhood U of 0 there exists $J \in \Omega$ such that $I \in \Omega$, $I \cap J = \phi \Rightarrow S_I \in U$.

(ii) If $\{S_J\}$ is Cauchy, then for every neighbourhood U of 0 there exists $J \in \Omega$ such that $x_\alpha \in U$ for all $\alpha \in \Gamma \backslash J$.

(iii) If $\{S_J\}$ is Cauchy, then the set $\{S_J: J \in \Omega\}$ is bounded.

Proof: (i) If $\{S_J\}$ is Cauchy, then for every neighbourhood U of 0 there exists $J \in \Omega$ such that $J_1, J_2 \in \Omega$, $J \subset J_1$, $J \subset J_2 \Rightarrow S_{J_1} - S_{J_2} \in U$.

For $I \in \Omega$, $I \cap J = \phi$, take $J_1 = I \cup J$, $J_2 = J$, then clearly $J \subset J_1$, $J \subset J_2$ and hence $S_I = S_{J_1} - S_{J_2} \in U$. The converse follows since for J_1, J_2 with $J \subset J_1$, $J \subset J_2$ we have $S_{J_1} - S_{J_2} = S_{J_1 \setminus J_2} - S_{J_2 \setminus J_1}$ and $(J_1 \setminus J_2) \cap J = \phi$, $(J_2 \setminus J_1) \cap J = \phi$.

(ii) This follows from the "only if" part of (i), by letting I range over the singletons $\{\alpha\}$, $\alpha \in \Gamma \setminus J$.

(iii) Let $\{S_J\}$ be Cauchy and let U be a circled neighbourhood of 0. Find a circled neighbourhood V of 0 such that $V + V \subset U$. From (i), there exists $J_0 \in \Omega$ such that $S_I \in V$ for all $I \in \Omega$ with $I \cap J_0 = \phi$. Since the set $\{S_{I'}: I' \subset J_0\}$ is <u>finite</u>, there exists $\lambda > 0$ such that $S_{I'} \in \lambda V$ for all $I' \subset J_0$. Now, for an arbitrary $J \in \Omega$ we have $S_J = S_{J \setminus J_0} + S_{J \cap J_0} \in V + \lambda V$. Since V is circled, we have $V \subset \mu V$ and $\lambda V \subset \mu V$, where $\mu = \max\{1, \lambda\}$ and so $S_J \subset \mu V + \mu V = \mu(V + V) \subset \mu U$ for all $J \in \Omega$. This completes the proof.

Next, let L be a vector space. If M and N are two vector subspaces of L such that the linear hull of $M \cup N$ is L and $M \cap N = \{0\}$, we say that L is the <u>algebraic direct sum</u> of M and N and we write $L = M + N$. It follows that each $x \in L$ is uniquely represented as $x = x_1 + x_2$, $x_1 \in M$, $x_2 \in N$. Now let E be a topological vector space which is the algebraic direct sum of two vector subspaces E_1 and E_2. It follows from the continuity of the addition in E that the map $\psi: (x_1, x_2) \to x_1 + x_2$ of $E_1 \times E_2$ onto E is continuous in addition to being one-one (by the uniqueness of the representation $x = x_1 + x_2$). If ψ is open (and hence a topological isomorphism) we say that E is the topological direct sum of E_1 and E_2 and we write $E = E_1 \oplus E_2$. We also say that E_1 is complemented in E and that E_2 is complementary to E_1 in E. The maps $\pi_i: E \to E_i$ given by $\pi_i x = x_i$, $i = 1,2$, where $x = x_1 + x_2$, $x_1 \in E_1$, $x_2 \in E_2$ are called the projections of E onto E_i, $i = 1,2$.

I.6 Lemma. Let E be a topological vector space and let $E = E_1 + E_2$, then $E = E_1 \oplus E_2$ iff the projections π_1, π_2 are continuous. ([34], page 20).

Theorems I.7, I.9 and Corollary I.8 below will be frequently used in the sequel. Their proofs can be found, for instance, in [12], page 41, 42 or [34], page 77, 78.

I.7 Theorem. (Open Mapping) Let E, F be F-spaces. Then every continuous linear mapping of E onto F is open.

I.8 Corollary. Let E be an F-space for each of the topologies τ_1, τ_2 and assume that $\tau_1 \subset \tau_2$, then $\tau_1 = \tau_2$.

I.9 Theorem. (Closed Graph) Let E and F be F-spaces and let $g: E \to F$ be a linear mapping whose graph is closed in $E \times F$, then g is continuous.

§2. Topological algebras. (For proofs and further details see [16], [28], [36], [41] for topological algebras and [25], [29], [31], [33], [41] for Banach algebras.)

An _algebra_ is a vector space L with a function $(x,y) \to xy$ of $L \times L$ into L, called _multiplication_, satisfying the properties: $x(yz) = (xy)z$, $(x + y)z = xz + yz$, $z(x + y) = zx + zy$ and $(\lambda x)(\mu y) = \lambda\mu(xy)$ for all $x,y,z \in L$, $\lambda,\mu \in \mathbb{C}$. If the multiplication is commutative (i.e. $xy = yx$ for all $x,y \in L$), L is said to be a _commutative algebra_. An _identity_ in L (which may or may not exist) is an element $e \in L$ such that $xe = ex = x$ for all $x \in L$.

A _topological algebra_ is an algebra A with a topology for which the underlying vector space is a topological vector space and the multiplication is jointly continuous. The topological structure makes it possible to define a more general type of identity: An _approximate identity_ in a topological algebra A is a net $\{x_\alpha\}$ in A such that $\lim_\alpha xx_\alpha = x = \lim_\alpha x_\alpha x$ for every $x \in A$. The case of a _bounded approximate identity_ i.e., when the net $\{x_\alpha\}$ is bounded, will be of special interest. Clearly, every topological algebra with an identity has a bounded approximate identity but the converse is not, in general, true. However, it is always possible to adjoin an identity to any topological algebra A as follows: Put $A^+ = A \times \mathbb{C}$ with the product topology and define the algebraic operations on A^+ by $\alpha(x,\lambda) = (\alpha x,\alpha\lambda)$, $(x,\lambda) + (y,\mu) = (x + y, \lambda + \mu)$ and $(x,\lambda)\cdot(y,\mu) = (xy + \mu x + \lambda y,\lambda\mu)$. With these operations, A^+ is a topological algebra with identity $(0,1)$, moreover, the map $x \to (x,0)$ is a topological isomorphism of A onto the subalgebra $\{(x,0): x \in A\}$ of A^+.

If $\{U\}$ is a 0-neighbourhood base in a topological algebra A, then for every $U \in \{U\}$ there exists $V \in \{U\}$ such that $VV \subset U$ (this is equivalent to the joint continuity of the multiplication). A _locally_ convex algebra is a topological algebra A whose underlying topological vector space is locally convex. Equivalently, a locally convex algebra A is a topological algebra whose topology can be generated by a family P of seminorms satisfying the condition: for every $p \in P$ there exists $q \in P$ such that $p(xy) \leq q(x)q(y)$ for all $x,y \in A$. (the existence of such a family is equivalent to the joint continuity of multiplication).

I.10 Definitions. (i) A subset U of an algebra A is said to be left (right) multiplicatively absorbing if for every $a \in A$ there exists $\lambda_0 = \lambda_0(a,U) > 0$ $(\mu_0 = \mu_0(a,U) > 0)$ such that $aU \subset \lambda U$ for all $\lambda \in \mathbb{C}$ with $|\lambda| \geq \lambda_0$ $(Ua \subset \mu U$ for all $\mu \in \mathbb{C}$ with $|\mu| \geq \mu_0)$. U is said to be multiplicatively absorbing if it is left and right multiplicatively absorbing.

(ii) A seminorm p on an algebra A is said to be left (right) absorbing if for every $a \in A$ there exists $M = M(a,p) > 0$ $(N = N(a,p) > 0)$ such that $p(ax) \leq Mp(x)$ $(p(xa) \leq Np(x))$ for all $x \in A$. p is said to be absorbing if it is left and right absorbing.

It is easy to check that a convex, circled, absorbing set U is left (right) multiplicatively absorbing iff its gauge is a left (right) absorbing seminorm, and that a seminorm p is left (right) absorbing iff the convex, circled, absorbing set $U_p := \{x \in A: p(x) \leq 1\}$ is left (right) multiplicatively absorbing.

(iii) An A-convex (or absorbing convex) algebra is a locally convex algebra A with a 0-neighbourhood base $\{U\}$ such that each $U \in \{U\}$ is convex, circled (absorbing), and multiplicatively absorbing. Equivalently, an A-convex algebra is a locally convex algebra A with a defining family P of absorbing seminorms.

In an A-convex algebra A, one can find a 0-neighbourhood base $\{U\}$ such that each $U \in \{U\}$ is a multiplicatively absorbing barrel.

I.11 Definitions. (i) A subset U of an algebra A is said to be idempotent if $UU \subset U$.

(ii) A seminorm p on an algebra A is said to be submultiplicative if $p(xy) \leq p(x)p(y)$ for all $x,y \in A$.

It is easy to check that a convex, circled, absorbing set U is idempotent iff its gauge is a submultiplicative seminorm, and that a seminorm p is submultiplicative iff the convex, circled, absorbing set $U_p = \{x \in A: p(x) \leq 1\}$ is idempotent.

(iii) A locally m-convex algebra is a locally convex algebra A with a 0-neighbourhood base $\{U\}$ such that each $U \in \{U\}$ is convex, circled (absorbing) and idempotent. Equivalently, a locally m-convex algebra is a locally convex algebra A with a defining family P of submultiplicative seminorms.

In a locally m-convex algebra A one can find a 0-neighbourhood base $\{U\}$ such that each $U \in \{U\}$ is an idempotent barrel.

It is easy to see that every locally m-convex algebra is A-convex. The converse is not, in general, true as will be shown by Example I.41, but it is true for barrelled algebras, indeed we have:

I.12 Theorem. Every barrelled A-convex algebra is locally m-convex.

Proof: Let A be a barrelled A-convex algebra. As an A-convex algebra, A has a 0-neighbourhood base $\{U\}$ where each U is a multiplicatively absorbing barrel. For each fixed $U \in \{U\}$ we shall construct an idempotent barrel V with $V \subset U$. This will prove the theorem since, by the barrelledness of A, $\{V\}$ will be a collection of 0-neighbourhoods and, consequently, a 0-neighbourhood base in A since $\{U\}$ is.

Set $V = \{y \in A: yx + \lambda y \in U$ for all $x \in U$, $\lambda \in \mathbb{C}$, $|\lambda| \leq 1\}$. If $y \in V$ and $\alpha \in \mathbb{C}$ with $|\alpha| \leq 1$ then for $\lambda \in \mathbb{C}$ with $|\lambda| \leq 1$ and $x \in U$ we have $(\alpha y)x + \lambda(\alpha y) = \alpha(yx + \lambda y) \in U$, since $yx + \lambda y \in U$ and U is circled. Hence $\alpha y \in V$ and so V is circled. To show that V is convex, let $y_1, y_2 \in V$ and $\alpha, \beta > 0$ with $\alpha + \beta = 1$, then for $x \in A$ and $|\lambda| \leq 1$ we have $(\alpha y_1 + \beta y_2)x + \lambda(\alpha y_1 + \beta y_2) = \alpha(y_1 x + \lambda y_1) + \beta(y_2 x + \lambda y_2) \in U$, since $y_i x + \lambda y_i \in U$, $i = 1,2$ and U is convex. This shows that $\alpha y_1 + \beta y_2 \in V$ and so V is convex. For each $x \in A$ and $\lambda \in \mathbb{C}$, the map $f_{x,\lambda}: y \to yx + \lambda y$ is continuous and hence, since U is closed, $f^{-1}_{x,\lambda}(U)$ is also closed. This shows that V is closed since, clearly, $V = \bigcap_{\substack{x \in U \\ |\lambda| \leq 1}} f^{-1}_{x,\lambda}(U)$. To prove that V is a barrel, it remains only to show that it is absorbing, and this is where the hypothesis that U is multiplicatively absorbing (and absorbing), is needed. For an arbitrary $y \in A$, there exist $\mu_i = \mu_i(y,U) > 0$, $i = 1,$ such that $|\mu| \geq \mu_1 \Rightarrow yU \subset \mu U$ and $|\mu| \geq \mu_2 \Rightarrow y \in \mu U \Rightarrow \lambda y \in \mu U$ for all $\lambda \in \mathbb{C}$ with $|\lambda| \leq 1$. Hence, for $x \in U$, $|\lambda| \leq 1$ and $\frac{|\mu|}{2} \geq \max\{\mu_1, \mu_2\}$ we have

$$yx + \lambda y \in \frac{\mu}{2}U + \frac{\mu}{2}U = \mu(\tfrac{1}{2}U + \tfrac{1}{2}U) \subset \mu U$$

(where the last inclusive follows from the convexity of U), and so $\frac{y}{\mu}x + \lambda\frac{y}{\mu} \in U$. This shows that $\frac{y}{\mu} \in V$ whenever $|\mu| \geq 2\max\{\mu_1, \mu_2\}$ i.e., V is absorbing.

To complete the proof, we must show that V is idempotent and

$V \subset U$. Indeed, for $y_1, y_2 \in V$, $x \in U$ and $|\lambda| \le 1$ we have

$$(y_1 y_2)x + \lambda(y_1 y_2) = y_1(y_2 x + \lambda y_2) = y_1 x'$$

where $x' = y_2 x + \lambda y_2 \in U$ since $y_2 \in V$. Now $(y_1 y_2)x + \lambda(y_1 y_2) = y_1 x'$ $= y_1 x' + 0y_1 \in U$, since $y_1 \in V$ and so $y_1 y_2 \in V$, proving that V is idempotent. Finally, for each $y \in V$ we have $y \cdot 0 + 1 \cdot y \in U$, but then $y \in U$ since $y = y \cdot 0 + 1 \cdot y$, hence $V \subset U$.

The above proof is essentially the same as Michael's proof ([28] Proposition 4.3) which makes use of the idea of the adjunction of an identity.

I.13 Definitions. (i) A complete metrizable locally convex algebra is called a B_0 algebra.

(ii) A complete metrizable locally m-convex algebra is called a Fréchet algebra.

(iii) An algebra A together with a submultiplicative norm $\|\cdot\|$ is called a normed algebra, and a complete normed algebra is called a Banach algebra.

In our definition of a topological algebra we required the multiplication to be jointly continuous. Some authors, however, require separate continuity only. The two definitions coincide, at least, in the complete metrizable case. Indeed we have,

I.14 Proposition. Let A be an F-space which is an algebra for a separately continuous multiplication. Then the multiplication is jointly continuous. ([36], page 112).

Next, let A be a commutative algebra. An element $f \in A^*$ is said to be a multiplicative linear functional (or a complex homomorphism) if $f(xy) = f(x)f(y)$ for all $x,y \in A$. A subalgebra I of A with $I \neq \{0\}$, $I \neq A$ is said to be an ideal if $IA \subset I$. An ideal I in A is said to be maximal if whenever J is an ideal in A and $I \subset J \subset A$ we must have $J = I$ or $J = A$. I is said to be regular if A has an identity modulo I, i.e., there exists $e_I \in A$ such that $e_I x - x \in I$ for all $x \in A$. Clearly, in an algebra with identity, every ideal is regular. The radical of A denoted by $\mathrm{rad}(A)$ is defined to be the intersection of all regular maximal ideals. A is said to be semisimple if $\mathrm{rad}(A) = \{0\}$. The set of all non-zero complex homomorphisms on A will be denoted by $\Delta(A)$ and the set of all regular maximal ideals will be denoted by $M(A)$.

I.15 Theorem. Let A be a commutative Banach algebra. Then

(i) For every $h \in \Delta(A)$, h is continuous and $\|h\| \leq 1$. If A has an identity e with $\|e\| = 1$, then $\|h\| = 1$ for all $h \in \Delta(A)$.

(ii) For every $h \in \Delta(A)$, $\mathrm{Ker}(h) = h^{-1}(\{0\}) \in M(A)$, and for every $M \in M(A)$, there exists a unique $h_M \in \Delta(A)$ such that $M = \mathrm{Ker}(h_M)$. Consequently, $h \to \mathrm{Ker}(h)$ is a bijection of $\Delta(A)$ onto $M(A)$. ([25], page 68; [29], page 19).

I.16 Definition. Let A be a commutative Banach algebra. The Gel'fand topology on $\Delta(A)$ is the relative $\sigma(A',A)$-topology (the weak*-topology) on $\Delta(A)$ as a topological subspace of A', or, equivalently, it is the relative product topology on $\Delta(A)$ as a topological subspace of $\mathbb{C}^A$. $\Delta(A)$ with the Gel'fand topology is called the maximal ideal space of A. In the sequel, $\Delta(A)$ will mean the maximal ideal space of A, rather than the set of non-zero complex homomorphisms on A.

I.17 Theorem. Let A be a commutative Banach algebra. Then,

(i) $\Delta(A)$ is a locally compact Hausdorff space.

(ii) Set $\Delta'(A) = \Delta(A) \cup \{0\}$ (where 0 is the zero complex homomorphism). Then $\Delta'(A)$ with the Gel'fand topology (i.e., as a subspace of $\mathbb{C}^A$) is the one point compactification of $\Delta(A)$.

(iii) If A has an identity, then $\Delta(A)$ is already compact, and 0 is an isolated point of $\Delta'(A)$.

(iv) For each fixed $x \in A$, set $\hat{x}(h) = h(x)$, $h \in \Delta(A)$. Then $\hat{x} \in C_0(\Delta(A))$ and $\phi: x \to \hat{x}$ is a norm-reducing (and hence continuous) algebra homomorphism of A into $C_0(\Delta(A))$, moreover $\|\hat{x}\| = \lim \|x^n\|^{1/n}$.

(v) The subalgebra $\hat{A}_0 = \phi(A)$ of $C_0(\Delta(A))$ separates the points of $\Delta(A)$, and ϕ is injective iff A is semisimple. ([29], page 22). [$C_0(\Delta(A))$ is the Banach algebra of all continuous complex-valued functions on $\Delta(A)$ vanishing at ∞ with the sup norm.]

I.18 Definitions. With the same notations as in Theorem I.17, $\hat{x}$ is called the Gel'fand transform of x, ϕ is called the Gel'fand map, and $\hat{A}_0$ is called the Gel'fand representation of A. $\hat{A}_0$ will always mean

$\phi(A)$ with the norm of $C_0(\Delta(A))$.

I.19 Theorem. Let A be a semisimple commutative Banach algebra. Then the following are equivalent:

(i) $\hat{A}_0$ is a closed subalgebra of $C_0(\Delta(A))$.

(ii) There exists $K > 0$ such that $\|x\|^2 \leq K\|x^2\|$ for all $x \in A$.

(iii) $\phi: A \to \hat{A}_0$ is a topological isomorphism. ([25], page 131).

I.20 Definition. A commutative Banach algebra A is said to be self-adjoint if $\bar{\hat{x}} \in \hat{A}$ for all $x \in A$ (where the bar denotes the complex conjugation).

I.21 Theorem. Let A be a self-adjoint semisimple commutative Banach algebra. Then the Gel'fand map is a topological isomorphism of A onto $C_0(\Delta(A))$ iff there exists $K > 0$ such that $\|x\|^2 \leq K\|x^2\|$ for all $x \in A$. ([25], page 133).

I.22 Theorem. Let A, B be commutative semisimple Banach algebras and let $\gamma: A \to B$ be an algebraic isomorphism. Then γ is a topological isomorphism. Consequently, if a commutative semisimple algebra A is a Banach algebra for each of the norms $\|\cdot\|_1$ and $\|\cdot\|_2$, then the two norms are equivalent. Hence, the topology of a commutative semisimple Banach algebra is completely determined by its algebraic structure. ([31], page 75).

§3. Bases and M-bases in topological vector spaces. (For proofs and details see [6], [27], [35].)

I.23 Definitions. A basis in a topological vector space E is a sequence $\{e_n\}$ in E such that for every $x \in E$, there exists a unique sequence $\{e_n^*(x)\}$ of scalars with $x = \sum_{n=1}^{\infty} e_n^*(x)e_n$. For each $x \in E$ and $n \in \mathbb{N}$, $e_n^*(x)$ is called the nth coordinate (or nth coefficient) of x with respect to the basis $\{e_n\}$. If the convergence of $\sum_{n=1}^{\infty} e_n^*(x)e_n$ is unconditional, i.e., if $x = \sum_{n=1}^{\infty} e_{\sigma(n)}^*(x)e_{\sigma(n)}$ for every permutation $\sigma: \mathbb{N} \to \mathbb{N}$ and every $x \in E$, the basis $\{e_n\}$ is said to be unconditional.

I.24 Definition. An M-basis in a topological vector space E is a map $\alpha \to e_\alpha$ (which we denote by $\{e_\alpha\}_{\alpha\in\Gamma}$) of a set Γ into E such that for every $x \in E$, there exists a unique map $\alpha \to e_\alpha^*(x)$ into $\mathbb{C}$ with $x = \sum_{\alpha\in\Gamma} e_\alpha^*(x)e_\alpha$, as in Definition I.4. For each $x \in E$ and $\alpha \in \Gamma$, $e_\alpha^*(x)$ is called the αth coordinate (or αth coefficient) of x with respect to the M-basis $\{e_\alpha\}_{\alpha\in\Gamma}$.

I.25 Proposition. If $\{e_\alpha\}_{\alpha\in\Gamma}$ is a countable M-basis in a topological vector space E, then for every bijection $\phi: \mathbb{N} \to \Gamma$, $\{e_{\phi(n)}\}$ is an unconditional basis in E.

Proof. If $\{e_\alpha\}_{\alpha\in\Gamma}$ is a countable M-basis in E, then for every permutation $\sigma: \mathbb{N} \to \mathbb{N}$, the sequence of finite partial sums of the form $S_N(x,\phi,\sigma) = \sum_{n=1}^{N} e_{\phi(\sigma(n))}^*(x)\, e_{\phi(\sigma(n))}$, $N = 1,2,\ldots$ converges to x, since such a sequence is a subnet of the net $\{S_J(x)\}_{J\in\Omega}$, $S_J(x) = \sum_{\alpha\in J} e^*_\alpha(x)e_\alpha$, which converges to n (see Definition I.4 for Ω).

It follows immediately from the uniqueness condition in Definitions I.23 and I.24 that each of the sets $\{e_n: n \in \mathbb{N}\}$ and $\{e_\alpha: \alpha \in \Gamma\}$ is linearly independent, in particular, $e_n \neq 0$ for all $n \in \mathbb{N}$ and $e_\alpha \neq 0$ for all $\alpha \in \Gamma$. It is also clear that each of the maps $e^*_n: E \to \mathbb{C}$, $n \in \mathbb{N}$ and $e^*_\alpha: E \to \mathbb{C}$, $\alpha \in \Gamma$ is a linear functional. We call them the nth coordinate functional and the αth coordinate functional, respectively.

I.26 Definition. A basis $\{e_n\}$ (An M-basis $\{e_\alpha\}_{\alpha\in\Gamma}$) in a topological vector space E is called a Schauder basis (Schauder M-basis) if $e^*_n \in E'$ for all $n \in \mathbb{N}$ ($e^*_\alpha \in E'$ for all $\alpha \in \Gamma$).

I.27 Theorem. Every basis in an F-space is Schauder. ([2], Theorem 2).

I.28 Remarks. (i) Theorem I.27 is not, in general, true for normed spaces. For an example of a normed space with a basis which is not Schauder see Example 16.1 page 160 and Example 6.1 page 50 in [35].

(ii) Let E be a topological vector space such that $E' = \{0\}$ (by Corollary I.2, E cannot be locally convex). If $\{e_n\}$ were a Schauder basis in E, it would follow that $e^*_n \in E' = \{0\}$ for all $n \in \mathbb{N}$ and consequently $E = \{0\}$. Hence, such a space cannot have a Schauder basis.

(iii) Let E be a topological vector space. If E has a basis $\{e_n\}$, then E is separable since the set $\{\sum_{k=1}^{n} r_k e_k: r_1, r_2, \ldots, r_n \in \mathbb{Q}, n \in \mathbb{N}\}$ is a countable subset of E which is, clearly, dense in E. The converse is <u>not</u> in general true. For example, let E be a separable F-space such that $E' = \{0\}$ ($L_p[0,1]$, $0 < p < 1$ is such a space [33], page 35). If $\{e_n\}$ is a basis in E, then $\{e_n\}$ is Schauder by Theorem I.27, which is impossible by Remark I.28 (ii). Hence E cannot have a basis. The question of whether every separable Banach space has a basis, known as the "<u>basis problem</u>", was posed by Banach, and has recently been answered, in the negative by Enflo [9]. The basis problem is equivalent to the problem of whether the topological dual E' of an arbitrary separable Banach space E, has a weak*-Schauder basis. This follows from the following theorem: "A sequence $\{f_n\}$ in the topological dual E' of a Banach space E is a basis in E' for the weak*-topology iff E has a basis $\{e_n\}$ with $e_n^* = f_n$ for all $n \in \mathbb{N}$". ([27], page 33; [35], page 155).

I.29 <u>Definition</u>. A basis $\{e_n\}$ in a Banach space E is said to be <u>shrinking</u> if for every $x' \in E'$ we have

$$\lim_n \sup_{\|x\| \le 1} \left|\langle x - \sum_{k=1}^{n} e_k^*(x) e_k , \ x' \rangle\right| = 0 .$$

I.30 <u>Theorem</u>. A basis $\{e_n\}$ in a Banach space E is shrinking iff $\{e_n^*\}$ is a basis in E' for the norm topology ([27], page 35).

I.31 <u>Definition</u>. A basis $\{e_n\}$ in a Banach space E is said to be <u>boundedly complete</u> if for every sequence $\{\lambda_n\}$ of scalars such that the set $\{\sum_{n=1}^{N} \lambda_n e_n : N \in \mathbb{N}\}$ is bounded, there exists $x \in E$ with $\lambda_n = e_n^*(x)$ for all $n \in \mathbb{N}$.

I.32 Theorem. (James). A Banach space with a basis is reflexive iff the basis is both shrinking and boundedly complete ([27], page 72).

I.33 Examples. Let c_0 be the Banach space of all sequences $x = (x(n))$ of scalars converging to 0, with the norm $\|x\| = \sup_n |x(n)|$, and let ℓ_p, $1 \le p < \infty$ be the Banach space of sequences $x = (x(n))$ of scalars such that $\sum_{n=1}^{\infty} |x(n)|^p < \infty$ with the norm $\|x\| = (\sum_{n=1}^{\infty} |x(n)|^p)^{\frac{1}{p}}$. Also ℓ_∞ will denote the Banach space of all bounded sequences $x = (x(n))$ of scalars with the norm $\|x\| = \sup_n |x(n)|$. The elements e_n, $n \in \mathbb{N}$ where $e_n(k) = \delta_{nk}$ (Kronecker's delta) constitute a basis in each of c_0 and ℓ_p, $1 \le p < \infty$, while ℓ_∞ has no basis since it is not separable. When the basis $\{e_n\}$ is considered as a basis in E, where E is either c_0 or ℓ_p, $1 < p < \infty$, the coordinate functionals $\{e_n^*\}$ can be identified with $\{e_n\}$ in the topological dual, which is ℓ_1 or ℓ_q $(\frac{1}{p} + \frac{1}{q} = 1)$ respectively. It follows from Theorem I.30 that $\{e_n\}$ is shrinking, as a basis in c_0, ℓ_p, $1 < p < \infty$. Since $\ell_1' = \ell_\infty$ and ℓ_∞ has no basis, $\{e_n\}$ is not a shrinking basis in ℓ_1. It is easy to check, either directly or by Theorem I.32, that $\{e_n\}$ is boundedly complete in ℓ_p, $1 \le p < \infty$, but not in c_0. [In c_0, notice that $\|\sum_{n=1}^{N} e_n\| = 1$ for all $N \in \mathbb{N}$ but there is no element x in c_0 with $e_n^*(x) = 1$ for all $n \in \mathbb{N}$]. It is also easy to see that $\{e_n\}_{n\in\mathbb{N}}$ is an M-basis and hence, by Proposition I.25, $\{e_n\}$ is an unconditional basis in c_0 and ℓ_p, $1 \le p < \infty$. More examples of unconditional and non-unconditional bases will be discussed in Examples II.8 and II.13.

§4. Bases and M-bases in topological algebras.

In this section, we give some background material about orthogonal bases in topological algebras, recently introduced and studied by T. Husain, J. Liang and S. Watson in [12], [13],..., [20]. Some of the given results are stated for both orthogonal bases and orthogonal M-bases, since their proofs admit straightforward extensions to the latter case. Some other results are not given here, since improved and generalized versions of them will be given in the following chapters, e.g., Proposition 3.1, Lemma 3.2, Theorem 3.3 and Theorem 3.4 in [19]. Various examples of topological algebras with either orthogonal bases or orthogonal M-bases are given.

I.34 Definitions. A basis $\{e_n\}$ (an M-basis $\{e_\alpha\}_{\alpha\in\Gamma}$) in the underlying topological vector space of a topological algebra A is said to be a quasiorthogonal basis (quasiorthogonal M-basis) if there exists a sequence $\{c_n\}$ of positive numbers such that $e_m e_n = \delta_{mn} c_m e_m$ for all $m,n \in \mathbb{N}$ (a map $\alpha \to c_\alpha$ of Γ into the positive real numbers such that $e_\alpha e_\beta = \delta_{\alpha\beta} c_\alpha e_\alpha$ for all $\alpha, \beta \in \Gamma$), where δ_{mn} $(\delta_{\alpha\beta})$ is the Kronecker's delta. A quasiorthogonal basis with $c_n = 1$ for all $n \in \mathbb{N}$ is said to be an orthogonal basis, and a quasiorthogonal M-basis is said to be an orthogonal M-basis if $c_\alpha = 1$ for all $\alpha \in \Gamma$.

In the sequel, Ω will always denote the set of all finite subsets of Γ (or $\mathbb{N}$), and it will be ordered by inclusion whenever convergence is considered.

I.35 Proposition. Let A be a topological algebra with an orthogonal basis $\{e_n\}$ (orthogonal M-basis $\{e_\alpha\}_{\alpha\in\Gamma}$). Then for every $x,y \in A$ we have:

(i) $e_n x = e_n^*(x)e_n$ for all $n \in \mathbb{N}$ ($e_\alpha x = e_\alpha^*(x)e_\alpha$ for all $\alpha \in \Gamma$).

(ii) $\sum_{n=1}^{\infty} e_n^*(x)e_n^*(y)e_n$ converges ($\sum_{\alpha\in\Gamma} e_\alpha^*(x)e_\alpha^*(y)e_\alpha$ converges) and $xy = \sum_{n=1}^{\infty} e_n^*(x)e_n^*(y)e_n$ ($xy = \sum_{\alpha\in\Gamma} e_\alpha^*(x)e_\alpha^*(y)e_\alpha$). Consequently, each e_n^* (respectively e_α^*) is multiplicative.

(iii) If A has an identity e, then $\sum_{n=1}^{\infty} e_n$ converges and $e = \sum_{n=1}^{\infty} e_n$ ($\sum_{\alpha\in\Gamma} e_\alpha$ converges and $e = \sum_{\alpha\in\Gamma} e_\alpha$). Conversely, if $\sum_{n=1}^{\infty} e_n$ ($\sum_{\alpha\in\Gamma} e_\alpha$) converges, then it serves as an identity in A.

Proof: We write a proof for the case of a basis, the case of an M-basis is similar.

(i) $e_n x = e_n \lim_N \sum_{m=1}^{N} e_m^*(x)e_m = \lim_N \sum_{m=1}^{N} e_m^*(x)e_n e_m = e_n^*(x)e_n$, where the second equality follows from the continuity of multiplication and the third equality follows from the orthogonality of the basis.

(ii) For each $N \in \mathbb{N}$ we have

$$\sum_{n=1}^{N} e_n^*(x)e_n^*(y)e_n = \sum_{n=1}^{N} e_n^*(x)e_n y = \Big(\sum_{n=1}^{N} e_n^*(x)e_n\Big)y .$$

From the continuity of multiplication we have

$$\lim_N \Big[\Big(\sum_{n=1}^{N} e_n^*(x)e_n\Big)y\Big] = xy \quad \text{and hence} \quad \lim_N \sum_{n=1}^{N} e_n^*(x)e_n^*(y)e_n = xy .$$

Consequently, $e_n^*(xy) = e_n^*(x)e_n^*(y)$ for all $n \in \mathbb{N}$, $x,y \in A$ and so each e_n^* is multiplicative.

(iii) If e is an identity in e then from (i) we have

$$e_n^*(e)e_n = e_n e = e_n .$$

Since $e_n \neq 0$, we must have $e_n^*(e) = 1$ for all $n \in \mathbb{N}$ and so $e = \sum_{n=1}^{\infty} e_n$. The converse follows from (ii).

I.36 Corollary. Every topological algebra with an orthogonal basis (orthogonal M-basis) is commutative.

Proof: This follows immediately from Proposition I.35 (ii).

I.37 Theorem. An orthogonal basis (orthogonal M-basis) in a topological algebra A is unique up to a permutation of the indexing set.

Proof: We give a proof for the case of an orthogonal M-basis. This proof is essentially the same as that in [19] for the case of an orthogonal basis.

Let $\{e_\alpha\}_{\alpha\in\Gamma}$ and $\{a_h\}_{h\in H}$ be two orthogonal M-bases in A. We shall establish a bijection $\phi: H \to \Gamma$ such that $a_h = e_{\phi(h)}$ for all $h \in H$. Pick an $h \in H$. Since $a_h \neq 0$, there exists $\alpha \in \Gamma$ with $e_\alpha^*(a_h) \neq 0$ (for otherwise, $a_h = \sum_{\alpha\in\Gamma} e_\alpha^*(a_h)e_\alpha = 0$). From Proposition I.35 (ii) and Corollary I.36 we have:

$$a_h^*(e_\alpha)a_h = a_h e_\alpha = e_\alpha a_h = e_\alpha^*(a_h)e_\alpha .$$

Since $e_\alpha^*(a_h) \neq 0$ and $e_\alpha \neq 0$, we can now write $\lambda a_h = e_\alpha$, where $0 \neq \lambda = \lambda(h,\alpha) = [e_\alpha^*(\alpha_h)]^{-1} \cdot a_h^*(e_\alpha)$. Now we have,

$$\lambda^2 a_h = \lambda^2 a_h^2 = (\lambda a_h)^2 = e_\alpha^2 = e_\alpha$$

and so $\lambda(1-\lambda)a_h = \lambda a_h - \lambda^2 a_h = e_\alpha - e_\alpha = 0$. Since $a_h \neq 0$ and $\lambda \neq 0$ we must have $\lambda = 1$ and so $e_\alpha = a_h$. There can be no other $\beta \in \Gamma$ with $e_\beta = a_h$, because of the linear independence of $\{e_\alpha : \alpha \in \Gamma\}$. Set $\alpha = \phi(h)$, then $\phi: H \to \Gamma$ is one-one. Similarly, for each $\alpha \in \Gamma$ we can find $h \in H$ such that $a_h = e_\alpha$ or; equivalently, $\phi(h) = \alpha$ and hence ϕ is onto. This completes the proof since we now have $\{a_h : h \in H\} = \{e_\alpha : \alpha \in \Gamma\}$ and $\phi: H \to \Gamma$ is a bijection.

The following examples will be of special importance:

I.38 <u>Example</u>. For an arbitrary nonempty set Γ, consider the algebra $\mathbb{C}^\Gamma$ of all $x: \Gamma \to \mathbb{C}$ with the pointwise algebraic operations. We shall denote an element $x \in \mathbb{C}^\Gamma$ by $(x(\alpha))_{\alpha\in\Gamma}$. Endowed with the product topology (pointwise convergence topology), $\mathbb{C}^\Gamma$ is a complete locally m-convex algebra with the defining family of submultiplicative seminorms $\{p_J : J \in \Omega\}$ where $p_J(x) = \max_{\alpha\in J}|x(\alpha)|$, $x \in \mathbb{C}^\Gamma$, $J \in \Omega$. The elements $e_\alpha \in \mathbb{C}^\Gamma$ where $e_\alpha(\beta) = \delta_{\alpha\beta}$, $\alpha,\beta \in \Gamma$, where $\delta_{\alpha\beta}$ is the Kronecker's delta, constitute an orthogonal M-basis in $\mathbb{C}^\Gamma$ and the coordinate functionals e_α^*, $\alpha \in \Gamma$ are given by $e_\alpha^*(x) = x(\alpha)$, $x \in \mathbb{C}^\Gamma$. Indeed, for $J \in \Omega$, $p_J(x - \sum_{\alpha\in I} x(\alpha)e_\alpha) = 0$ for all $I \in \Omega$ with $J \subset I$, and the orthogonality is clear. Whenever this M-basis is considered as an

M-basis in $\mathbb{C}^{\Gamma}$ or in any topological algebra or topological vector space which is, algebraically, a subalgebra or a subspace of $\mathbb{C}^{\Gamma}$, it will be called the coordinate unit vector M-basis, the standard M-basis or the canonical M-basis. $\mathbb{C}^{\Gamma}$ is metrizable (and hence a Fréchet algebra) iff Γ is countable.

If Γ is countable with $n \leftrightarrow \alpha_n$ as a 1-1 correspondence, we may relabel the orthogonal M-basis described above as $\{e_n\}_{n\in\mathbb{N}}$. The algebra $\mathbb{C}^{\mathbb{N}}$ is usually denoted by s, the algebra of all sequences, and by Proposition I.25, $\{e_n\}$ is an orthogonal unconditional basis in s. Whenever this basis is considered as a basis in s or in any topological algebra or topological vector space which is, algebraically, a subalgebra or a subspace of s, it will be called the coordinate unit vector basis, the standard basis or the canonical basis.

I.39 Example. Under the coordinatewise multiplication $xy = (x(n)y(n))$, each of the Banach spaces c_0 and ℓ_p, $1 \le p < \infty$ becomes a Banach algebra with the canonical basis $\{e_n\}$ as an orthogonal unconditional basis.

Replacing $\mathbb{N}$ by an arbitrary set Γ, we obtain the Banach algebras $c_0(\Gamma)$ and $\ell_p(\Gamma)$, $1 \le p < \infty$, with the canonical orthogonal M-basis $\{e_\alpha\}_{\alpha\in\Gamma}$. $c_0(\Gamma)$ is the Banach algebra of all $x: \Gamma \to \mathbb{C}$ such that given $\varepsilon > 0$ there exists $J \in \Omega$ with $|x(\alpha)| < \varepsilon$ for all $\alpha \in \Gamma\backslash J$, with the sup norm (we can also think of $c_0(\Gamma)$ as the Banach space of all $x: \Gamma \to \mathbb{C}$ vanishing at ∞, when Γ is viewed as a discrete topological space), and $\ell_p(\Gamma)$ is the Banach algebra of all $x: \Gamma \to \mathbb{C}$ such that $\sum_{\alpha\in\Gamma} |x(\alpha)|^p < \infty$ with the norm $\|x\|_p = (\sum_{\alpha\in\Gamma} |x(\alpha)|^p)^{\frac{1}{p}}$.

(Summations are in the sense of Definition I.4 with $E = \mathbb{R}$).

I.40 Example. Let $1 \le p < \infty$, and let $\mathbf{T}$ denote the torus (circle) group. The set $L_p(\mathbf{T})$ of all germs of functions $x: \mathbf{T} \to \mathbb{C}$ with $\int_0^{2\pi} |x(t)|^p dt < \infty$, endowed with the pointwise addition and scalar multiplication, the convolution multiplication

$$(xy)(t) = \frac{1}{2\pi}\int_0^{2\pi} x(t-u)y(u)du \quad ; \quad x,y \in L_p(\mathbf{T})$$

and the norm

$$\|x\|_p = \left(\frac{1}{2\pi}\int_0^{2\pi} |x(t)|^p\right)^{\frac{1}{p}}$$

is a Banach algebra. The maximal ideal space of $L_p(\mathbf{T})$ is homeomorphic with the dual group of $\mathbf{T}$, which is the additive group $\mathbf{Z}$ with the discrete topology [13], [25]. Let the sets $\{e_n: n = 0,1,2,\ldots\} \subset L_p(\mathbf{T})$ and $\{e_n^*: n = 0,1,2,\ldots\} \subset L_p'(\mathbf{T})$ be given by $e_0(t) = 1$, $e_0^*(x) = \hat{x}(0)$; $e_{2k-1}(t) = e^{-ikt}$, $e_{2k-1}^*(x) = \hat{x}(-k)$; $e_{2k}(t) = e^{ikt}$, $e_{2k}^*(x) = \hat{x}(k)$, $t \in \mathbf{T}$, $x \in A$, $k = 1,2,\ldots$ where $\hat{x}: \mathbf{Z} \to \mathbb{C}$ is the Gel'fand (Fourier) transform of x as in I.18. For $k \in \mathbf{Z}$ we have

$\|e^{ik(\cdot)}\|_p = \left(\frac{1}{2\pi}\int_0^{2\pi} |e^{ikt}|^p\right)^{\frac{1}{p}} = 1$ and so $\|e_n\|_p = 1$ for $n = 0,1,2,\ldots$.

For $1 < p < \infty$ we also have

$$\text{(I.40.1)} \qquad \lim_{k\to\infty} \|x - \sum_{j=-k}^{k} \hat{x}(j)\, e^{ij(\cdot)}\|_p = 0$$

(See [21].) It follows from (I.40.1) and Theorem I.17 (iv) that for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$|\hat{x}(k)| < \frac{\varepsilon}{2} , \quad \|x - \sum_{j=-k}^{k} \hat{x}(j)e^{ij(\cdot)}\|_p < \frac{\varepsilon}{2}$$

for all $k \in \mathbb{N}$ with $|k| > \frac{N}{2}$. Hence if n is even with $n > N$ we have

$$\text{(I.40.2)} \qquad \|x - \sum_{j=0}^{n} e_j^*(x)e_j\|_p = \|x - \sum_{j=-\frac{n}{2}}^{\frac{n}{2}} \hat{x}(j)e^{ij(\cdot)}\|_p < \frac{\varepsilon}{2} < \varepsilon$$

and if n is odd with $n > N$ we have

$$\text{(I.40.3)} \qquad \|x - \sum_{j=0}^{n} e_j^*(x)e_j\|_p = \|x - \sum_{j=0}^{n+1} e_j^*(x)e_j + e_{n+1}^*(x)e_{n+1}\|_p \le$$

$$\le \|x - \sum_{j=0}^{n+1} e_j^*(x)e_j\|_p + |e_{n+1}^*(x)| \; \|e_{n+1}\|_p =$$

$$= \|x - \sum_{j=-\frac{n+1}{2}}^{\frac{n+1}{2}} \hat{x}(j)e^{ij(\cdot)}\|_p + |\hat{x}(\tfrac{n+1}{2})| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

From (I.40.2) and (I.40.3) we have $x = \lim_n \sum_{j=1}^{n} e_j^*(x)e_j$ in the L_p-norm and hence $\{e_n\}_{n=0}^{\infty}$ is a basis in $L_p(\mathbb{T})$, $1 < p < \infty$. This basis is orthogonal, indeed for $m,n \in \mathbb{N} \cup \{0\}$ we have

$$(e_m e_n)(t) = \frac{1}{2\pi} e^{ijt} \int_0^{2\pi} e^{-i(j-k)u} du = \delta_{jk} e^{ijt} = \delta_{mn} e_m(t) ,$$

where j,k are the two integers corresponding with m,n as in the definition of the e_n's.

It will be shown in Chapter II that for $1 < p < \infty$, $p \neq 2$ the orthogonal basis described above is not unconditional. It is, however, unconditional in $L_2(\mathbf{T})$ since the Fourier transform is an isometric algebra isomorphism of $L_2(\mathbf{T})$ onto $\ell_2(\mathbf{Z})$ [21].

I.41 Example. For an arbitrary locally compact Hausdorff space T, let $C^*_\beta(T)$ be the algebra of all bounded continuous functions $x: T \to \mathbb{C}$ with the "strict topology" and the pointwise operations. The strict topology β, which was introduced by R.C. Buck in [3], is defined by the family $\{p_\phi\}$ of seminorms:

$$p_\phi(x) = \sup_{t \in T} |\phi(t)x(t)|$$

where ϕ ranges over $C_0(T)$ (continuous functions vanishing at ∞). $C^*_\beta(T)$ is a complete A-convex algebra with identity, which is locally m-convex iff T is compact [3], [22].

Now let Γ be an infinite discrete space and let $\{e_\alpha\}_{\alpha \in \Gamma}$ be the canonical orthogonal M-basis in $\mathbb{C}^\Gamma$. $\{e_\alpha\}_{\alpha \in \Gamma}$ is also an orthogonal M-basis in $C^*_\beta(\Gamma)$. Indeed, for $x \in C^*_\beta(\Gamma)$, $\phi \in C_0(\Gamma)$ and $\varepsilon > 0$, there exists $J_0 \in \Omega$ (the set of all finite subsets of Γ) such that $|\phi(\alpha)| < \varepsilon$ for all $\alpha \in \Gamma \backslash J_0$ and hence, for $J \in \Omega$, $J_0 \subset J$ we have

$$p_\phi(x - \sum_{\alpha \in J} x(\alpha)e_\alpha) = \sup_{\alpha \in \Gamma \backslash J} |\phi(\alpha)x(\alpha)| \leq (\sup_{\alpha \in \Gamma} |x(\alpha)|) \cdot \varepsilon$$

thus proving that $\{e_\alpha\}_{\alpha \in \Gamma}$ is an M-basis in $C^*_\beta(\Gamma)$, and the orthogonality is clear.

Since $C_{\beta}^{*}(\Gamma)$ is complete and A-convex, it follows from Theorems I.3 and I.12 that $C_{\beta}^{*}(\Gamma)$ is neither metrizable nor barrelled.

More examples of topological algebras with orthogonal bases or orthogonal M-bases will be given in the sequel.

CHAPTER II

LOCALLY CONVEX S-ALGEBRAS AND THE UNCONDITIONALITY OF ORTHOGONAL BASES IN B_0 ALGEBRAS

In many of the most known examples of B_0 algebras with orthogonal bases (e.g. $\ell_p(\mathbb{N})$, $1 \leq p < \infty$, $C_0(\mathbb{N})$ and s), the existing orthogonal basis is unconditional. This, however, is not always the case. Examples II.8 (i), (ii) show that even in the special case of a Banach algebra, an orthogonal basis need not be unconditional. Furthermore, we introduce the notions of a squarely submultiplicative seminorm and a locally convex s-algebra and we prove that being a locally convex s-algebra is a characterizing property of those B_0 algebras in which an orthogonal basis (if it exists) is necessarily unconditional (Theorem II.12). This is followed by a brief study of locally convex s-algebras, in §4. In §2 we examine the possibility of introducing a multiplication on a Fréchet space with an unconditional basis, so as to make the Fréchet space into a Fréchet algebra in which the basis is orthogonal, using the idea of Proposition 4.1 in [19]. In §1 we give some facts necessary for §2 and §3.

§1. Unconditional bases in Fréchet spaces.

Theorem II.1 and Proposition II.2 of this section are direct generalizations of results known in the Banach space case ([35], Theorem 16.1) to the case of a Fréchet space. However, in the Banach space case, statement (iv) in Theorem II.1 does not arise in the generality stated here, since it coincides with statement (v) in the same theorem.

II.1 Theorem. Let E be a Fréchet space whose topology is generated by the family $\{\|\cdot\|_\nu : \nu \in \mathbb{N}\}$ of seminorms, and let $\{e_n\}$ be a basis in E. Then the following statements are equivalent:

(i) The basis $\{e_n\}$ is unconditional.

(ii) $\{e_n\}$ is an M-basis, i.e. $x = \lim_{J \in \Omega} \sum_{n \in J} e_n^*(x) e_n$ (as in Definition I.4) for every $x \in E$.

(iii) If $\Lambda_\nu = \{f \in E' : \|f\|_\nu \equiv \sup_{\|x\|_\nu \leq 1} |f(x)| \leq 1\}$, $\nu \in \mathbb{N}$, then for each fixed $x \in E$, $\sum_{n=1}^{\infty} |e_n^*(x)| |f(e_n)|$ converges uniformly on each Λ_ν, i.e., for $x \in E$, $\nu \in \mathbb{N}$ and $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $\sum_{k=m+1}^{n} |e_k(x)| |f(e_k)| < \varepsilon$ for every $f \in \Lambda_\nu$ and $n > m \geq N$.

(iv) Let $T_\nu = \{n \in \mathbb{N} : \|e_n\|_\nu \neq 0\}$, $\nu \in \mathbb{N}$, then for every $x \in E$ and every sequence $\{\beta_n\}$ of scalars such that each of the sets $\{\beta_n : n \in T_\nu\}$, $\nu \in \mathbb{N}$ is bounded, the series $\sum_{n=1}^{\infty} \beta_n e_n^*(x) e_n$ converges.

(v) For every $x \in E$ and every bounded sequence $\{\beta_n\}$ of scalars, $\sum_{n=1}^{\infty} \beta_n e_n^*(x) e_n$ converges.

(vi) For every $x \in E$ and every sequence $\{\varepsilon_n\}$ of scalars with $\varepsilon_n = \pm 1$ for all $n \in \mathbb{N}$, $\sum_{n=1}^{\infty} \varepsilon_n e_n^*(x) e_n$ converges.

(vii) Every subseries $\sum_{k=1}^{\infty} \lambda_{n_k} e_{n_k}$ of a convergent series $\sum_{n=1}^{\infty} \lambda_n e_n$ converges.

Proof: (i) ⇒ (ii) Assume (ii) is false, then there exists $\nu \in \mathbb{N}$, $x \in E$ and $\varepsilon_0 > 0$ such that for every $I \in \Omega$ there exists $J \in \Omega$, $J \cap I = \phi$ with $\|\sum_{n\in J} e_n^*(x)e_n\|_\nu \geq \varepsilon_0$. Let $I_1 = \{1\}$ and find $J_1 \in \Omega$, $J_1 \cap \{1\} = \phi$ such that $\|\sum_{n\in J_1} e_n^*(x)e_n\|_\nu \geq \varepsilon_0$. Let $I_2 = \{1,2,\ldots, \max J_1\}$. Find $J_2 \in \Omega$, $J_2 \cap I_2 = \phi$ such that $\|\sum_{n\in J_2} e_n^*(x)e_n\|_\nu \geq \varepsilon_0$ and set $I_3 = \{1,2,\ldots, \max J_2\}$. Having obtained I_k, find $J_k \in \Omega$, $I_k \cap J_k = \phi$ and set $I_{k+1} = \{1,2,\ldots, \max J_k\}$. Clearly $I_1 \subsetneqq I_2 \subsetneqq \ldots \subsetneqq I_k \subsetneqq \ldots$, $\bigcup_{k=0}^{\infty}(I_{k+1}\setminus I_k) = \mathbb{N}$ (with $I_0 = \phi$) and $\phi \neq J_k \subset I_{k+1}\setminus I_k$ for all $k \in \mathbb{N}$.

Let $\sigma: \mathbb{N} \to \mathbb{N}$ be a permutation such that $\sigma(I_{k+1}\setminus I_k) = I_{k+1}\setminus I_k$ and $\sigma(\{|I_k| + 1, |I_k| + 2,\ldots, |I_k| + |J_k|\}) = J_k$ (where $|S|$ denotes the cardinality of a set S). It is easy to see that

$$\Big\|\sum_{n=|I_k|+1}^{|I_k|+|J_k|} e_{\sigma(n)}^*(x)e_{\sigma(n)}\Big\|_\nu = \Big\|\sum_{n\in J_k} e_n^*(x)e_n\Big\|_\nu \geq \varepsilon_0$$

for all $k \in \mathbb{N}$ and hence $\sum_{n=1}^{\infty} e_{\sigma(n)}^*(x)e_{\sigma(n)}$ does not converge.

(ii) ⇒ (iii) Assume (ii) and let $x \in E$, $\nu \in \mathbb{N}$ and $f \in \Lambda_\nu$. For $\varepsilon > 0$ there exists $J \in \Omega$ such that $\|\sum_{n\in I} e_n^*(x)e_n\|_\nu < \frac{\varepsilon}{4}$ whenever $I \in \Omega$ with $I \cap J = \phi$. For such I we have

$$\text{(II.1.1)} \quad \sum_{n\in I} |e_n^*(x)f(e_n)| \le \sum_{n\in I} |Re[e_n^*(x)f(e_n)]| + \sum_{n\in I} |Im[e_n^*(x)f(e_n)]| =$$

$$= \sum_{n\in I_1} |Re[e_n^*(x)f(e_n)]| + \sum_{n\in I_2} |Re[e_n^*(x)f(e_n)]| +$$

$$+ \sum_{n\in I_3} |Im[e_n^*(x)f(e_n)]| + \sum_{n\in I_4} |Im[e_n^*(x)f(e_n)] ,$$

where $I_1 = \{n \in I: Re[e_n^*(x)f(e_n)] \ge 0\}$, $I_2 = \{n \in I: Re[e_n^*(x)f(e_n)] < 0\}$, $I_3 = \{n \in I: Im[e_n^*(x)f(e_n)] \ge 0\}$ and $I_4 = \{n \in I: Im[e_n^*(x)f(e_n)] < 0\}$. Since $I_1 \cap J = \phi$, we also have,

$$\sum_{n\in I_1} |Re[e_n^*(x)f(e_n)]| = |Ref(\sum_{n\in I_1} e_n^*(x)e_n| \le |f(\sum_{n\in I_1} e_n^*(x)e_n)| \le \|\sum_{n\in I_1} e_n^*(x)e_n\|_\nu < \frac{\varepsilon}{4} .$$

Similarly we can show that each of the other three terms in the right hand side of (II.1.1) is smaller than $\frac{\varepsilon}{4}$ and hence $\sum_{n\in I} |e_n^*(x)|\cdot|f(e_n)| < \varepsilon$ for $I \in \Omega$ with $I \cap J = \phi$. Take $N = \max J$.

(iii) ➡ (iv) Assume (iii) and let $\{\beta_n\}$ be as in (iv). For each $\nu \in \mathbb{N}$ put $M_\nu = \sup_{n\in T_\nu} |\beta_n|$. For $x \in E$, $\nu \in \mathbb{N}$ and $\varepsilon > 0$ let $N \in \mathbb{N}$ be such that $\sum_{k=m+1}^{n} |e_k^*(x)|\cdot|f(e_k)| < \frac{\varepsilon}{M_\nu}$ whenever $n > m \ge N$ and $f \in \Lambda_\nu$.
For every such m,n, it follows from a well-known corollary of the Hahn-Banach theorem (Corollary I.2) that there exists $g_{m,n} \in \Lambda_\nu$ such that

$$g_{m,n}(\sum_{k=m+1}^{n} \beta_k e_k^*(x)e_k) = \|\sum_{k=m+1}^{n} \beta_k e_k^*(x)e_k\|_\nu$$

and hence

$$\Big\| \sum_{k=m+1}^{n} \beta_k e_k^*(x) e_k \Big\|_\nu \le \sum_{k=m+1}^{n} |\beta_k| |e_k^*(x)| |g_{m,n}(e_k)| = \sum_{\substack{k=m+1 \\ k\in T_\nu}}^{n} |\beta_k| |e_k^*(x)| |g_{m,n}(e_k)| ,$$

where the last equality holds since for $k \in \mathbb{N}\setminus T_\nu'$ we have $|g_{m,n}(e_k)| \le \|e_k\|_\nu = 0$. It follows that

$$\Big\| \sum_{k=m+1}^{n} \beta_k e_k^*(x) e_k \Big\|_\nu \le \sum_{\substack{k=m+1 \\ k\in T_\nu}}^{n} |\beta_k| \cdot |e_k^*(x)| \cdot |g_{m,n}(e_k)| \le M_\nu \sum_{\substack{k=m+1 \\ k\in T_\nu}}^{n} |e_k^*(x)| \cdot |g_{m,n}(e_k)| \le$$

$$\le M_\nu \sum_{k=m+1}^{n} |e_k^*(x)| \cdot |g_{m,n}(e_k)| < M_\nu \frac{\varepsilon}{M_\nu} = \varepsilon$$

whenever $n > m \ge N$. Since E is complete, $\sum_{n=1}^{\infty} \beta_n e_n^*(x) e_n$ converges.

(iv) ⇒ (v) ⇒ (vi) Obvious.

(vi) ⇒ (vii) Let $\sum_{k=1}^{\infty} \lambda_{n_k} e_{n_k}$ be a subseries of a convergent series $\sum_{n=1}^{\infty} \lambda_n e_n$ and let $\varepsilon_n = 1$ for all $n \in \{n_k : k \in \mathbb{N}\}$, $\varepsilon_n = -1$ for all $n \in \mathbb{N}\setminus\{n_1, n_2, \ldots\}$. Under the hypothesis of (vi), $\sum_{n=1}^{\infty} \varepsilon_n \lambda_n e_n$ is convergent, and hence so is $\sum_{k=1}^{\infty} \lambda_{n_k} e_{n_k} = \frac{1}{2}\big(\sum_{n=1}^{\infty} \varepsilon_n \lambda_n e_n + \sum_{n=1}^{\infty} \lambda_n e_n \big)$.

(vii) ⇒ (i) If (i) is false, then for some $x \in E$ and a permutation $\sigma : \mathbb{N} \to \mathbb{N}$, $\sum_{n=1}^{\infty} e_{\sigma(n)}^*(x) e_{\sigma(n)}$ does not converge. Hence there exists

$\nu \in \mathbb{N}$, $\varepsilon > 0$ and $m_1, n_1 \in \mathbb{N}$, $n_1 > m_1 > 1$, $\max_{m_1 \le k \le n_1} \sigma(k) > \sigma(1)$ such that $\|\sum_{n=m_1+1}^{n_1} e^*_{\sigma(n)}(x) e_{\sigma(n)}\|_\nu \ge \varepsilon$. Having obtained m_j, n_j, find m_{j+1}, n_{j+1}; $n_{j+1} > m_{j+1} > n_j$; $\min_{m_{j+1} \le k \le n_{j+1}} \sigma(k) \ge \max_{m_j \le k \le n_j}$ such that $\|\sum_{n=m_{j+1}}^{n_{j+1}} e^*_{\sigma(n)}(x) e_{\sigma(n)}\|_\nu \ge \varepsilon$. Let $r_1 < r_2 < \ldots < r_i < \ldots$ be the elements of $\bigcup_{j=1}^{\infty} \{\sigma(k) : m_j \le k \le n_j\}$. The subseries $\sum_{i=1}^{\infty} e^*_{r_i}(x) e_{r_i}$ of $\sum_{n=1}^{\infty} e^*_n(x) e_n$ does not converge.

II.2 Proposition. Let $(E, \{\|\cdot\|_\nu : \nu \in \mathbb{N}\})$ be a Fréchet space with an unconditional basis $\{e_n\}$. For each $\nu \in \mathbb{N}$ put

$$\|x\|'_\nu = \sup_{f \in \Lambda_\nu} \sum_{n=1}^{\infty} |e^*_n(x)| \cdot |f(e_n)|$$

$$\|x\|''_\nu = \sup_{J \in \Omega} \|\sum_{n \in J} e^*_n(x) e_n\|_\nu .$$

Then each of $\{\|\cdot\|'_\nu : \nu \in \mathbb{N}\}$ and $\{\|\cdot\|''_\nu : \nu \in \mathbb{N}\}$ is a family of seminorms on E generating the original topology and $\|x\|_\nu \le \|x\|''_\nu \le \|x\|'_\nu$ for all $x \in E$ and $\nu \in \mathbb{N}$.

Proof: Let $\nu \in \mathbb{N}$, $x \in E$. From the equivalence of (i) and (iii) in Theorem II.1 it is easy to see that $\|x\|'_\nu = \sup_{f \in \Lambda_\nu} \sum_{n=1}^{\infty} |e^*_n(x)| \cdot |f(e_n)| < \infty$.

Let $J \in \Omega$. It follows from Corollary I.2 that there exists $g \in \Lambda_\nu$ such that

$$\|\sum_{n\in J} e_n^*(x)e_n\|_\nu = g(\sum_{n\in J} e_n^*(x)e_n) \le \sum_{n\in J} |e_n^*(x)|\cdot|g(e_n)| \le$$

$$\le \sup_{f\in\Lambda_\nu} \sum_{n=1}^{\infty} |e_n^*(x)|\cdot|f(e_n)| = \|x\|_\nu'$$

and hence

$$\|x\|_\nu = \lim_{J\in\Omega} \|\sum_{n\in J} e_n^*(x)e_n\|_\nu \le \sup_{J\in\Omega} \|\sum_{n\in J} e_n^*(x)e_n\|_\nu \; (= \|x\|_\nu'') \le \|x\|_\nu' .$$

It is easy to see that for each $\nu \in \mathbb{N}$, $\|\cdot\|_\nu'$ and $\|\cdot\|_\nu''$ are seminorms on E. If τ, τ', τ'' are the topologies on E generated by $\{\|\cdot\|_\nu : \nu \in \mathbb{N}\}$, $\{\|\cdot\|_\nu' : \nu \in \mathbb{N}\}$ and $\{\|\cdot\|_\nu'' : \nu \in \mathbb{N}\}$ respectively, then from the above inequality we have $\tau \subset \tau'' \subset \tau'$. The equivalence of τ, τ'' and τ' will follow from the Open Mapping theorem (Theorem I.7) once we show that E is complete for the topology τ'. Let $\{x_k\}$ be a Cauchy sequence in (E,τ'), then for $\varepsilon > 0$ and $\nu \in \mathbb{N}$ there exists $N \in \mathbb{N}$ such that $\|x_k - x_j\|_\nu' < \varepsilon$ whenever $k > j \ge N$. Let $J \in \Omega$ and $f \in \Lambda_\nu$ then

(II.2.1) $$\sum_{n\in J} |e_n^*(x_k - x_j)|\cdot|f(e_n)| \le \|x_k - x_j\|_\nu' < \varepsilon$$

for $k > j \ge N$. Since the sequence $\{x_k\}$ is Cauchy in (E,τ') it is Cauchy in (E,τ) and by the completeness of (E,τ) there exists $x \in E$ such that $x = \lim_k x_k$ (in (E,τ)). Since every basis in a Fréchet space is Schauder (Theorem I.27), we have $\lim_{k\to\infty} e_n^*(x_k) = e_n^*(x)$ for all $n \in \mathbb{N}$ and hence by taking the limit in (II.2.1) as $k \to \infty$ we get

$$\sum_{n\in J} |e_n^*(x - x_j)|.|f(e_n)| \le \varepsilon$$

for all $J \in \Omega$, $f \in \Lambda_\nu$ and $j \ge N$ and consequently

$$\|x - x_j\|_\nu' = \sup_{f\in\Lambda_\nu} \sum_{n=1}^{\infty} |e_n^*(x - x_j)||f(e_n)| \le \varepsilon .$$

This shows that $x = \lim_{k\to\infty} x_k$ in (E,τ') and completes the proof.

As we shall see (Lemma III.17), the equivalence of τ' and τ'' remains true under the weaker hypothesis that E is a locally convex space.

§2. Fréchet algebra structure on Fréchet spaces with unconditional bases.

Let E be a topological vector space with a basis $\{e_n\}$. It is natural to think of using the series expression of the elements of E in terms of the basis to introduce a multiplication on E by the formula

$$xy = \sum_{n=1}^{\infty} e_n^*(x)e_n^*(y)e_n$$

provided that the series in the right hand side converges for every pair of elements $x,y \in E$. It follows immediately from this definition that

(i) The introduced multiplication is commutative.

(ii) $e_m e_n = \delta_{mn} e_n$ for all $m,n \in \mathbb{N}$.

In the present section we discuss conditions, necessary and sufficient for converting a Fréchet space with an unconditional basis $\{e_n\}$ into a Fréchet algebra with the basis $\{e_n\}$ as an orthogonal or quasi-orthogonal (unconditional) basis. In Theorem II.3 we generalize a result due to Husain and Watson ([19], Proposition 4.1) from Banach spaces to Fréchet spaces, we also simplify the proof and avoid the use of the closed graph theorem. It should be pointed out that the unconditional basis in the hypothesis of Proposition 4.1 in [19] should be assumed to be bounded away from zero as we will show by Corollary II.6 and Example II.7 (i).

II.3 Theorem. Let $\{e_n\}$ be an unconditional basis in a Fréchet space E. The following statements are equivalent:

(i) The topology of E can be defined by a family $\{\|\cdot\|_\nu : \nu \in \mathbb{N}\}$ of seminorms such that for every $\nu \in \mathbb{N}$, zero is not a limit point of the set $\{\|e_n\|_\nu : n \in \mathbb{N}\}$.

(ii) E can be endowed with a multiplication that makes E into a Fréchet algebra with $\{e_n\}$ as an orthogonal (unconditional) basis.

Proof: (ii) ⇒ (i) If $\{\|\cdot\|_\nu : \nu \in \mathbb{N}\}$ is a defining family of submultiplicative seminorms, then for every $\nu \in \mathbb{N}$, $n \in \mathbb{N}$ we have $\|e_n\|_\nu = \|e_n^2\|_\nu \le \|e_n\|_\nu^2$ and so $\|e_n\|_\nu = 0$ or $\|e_n\|_\nu \ge 1$.

(i) ⇒ (ii) From Proposition II.2 we have

$$|e_n^*(x)|\ \|e_n\|_\nu = \|e_n^*(x)e_n\|_\nu \le \sup_{J\in\Omega} \|\sum_{n\in J} e_n^*(x)e_n\|_\nu = \|x\|_\nu'' \le \|x\|_\nu'$$

for every $x \in E$ and $\nu, n \in \mathbb{N}$. By hypothesis, $\delta_\nu = \inf_{\substack{n \in \mathbb{N} \\ \|e_n\|_\nu \neq 0}} \|e_n\|_\nu > 0$ and hence for each fixed $\nu \in \mathbb{N}$,

$$|e_n^*(x)| \leq \frac{\|x\|'_\nu}{\|e_n\|_\nu} \leq \frac{\|x\|'_\nu}{\delta_\nu} < \infty$$

for every $n \in \mathbb{N}$ with $\|e_n\|_\nu \neq 0$. Thus the sequence $\{e_n^*(x)\}$ of scalars satisfies the condition described in statement (iv) of Theorem II.1. It follows that for $x = \sum_{n=1}^{\infty} e_n^*(x)e_n$, $y = \sum_{n=1}^{\infty} e_n^*(y)e_n \in E$ the series $\sum_{n=1}^{\infty} e_n^*(x)e_n^*(y)e_n$ converges in E. We define a multiplication on E by $xy = \sum_{n=1}^{\infty} e_n^*(x)e_n^*(y)e_n$.

Under this multiplication, E is a Fréchet algebra. Indeed, for $\nu \in \mathbb{N}$ and $x, y \in E$ we have

$$\|xy\|'_\nu = \sup_{f \in \Lambda_\nu} \sum_{n=1}^{\infty} |e_n^*(x)|\cdot|e_n^*(y)|\cdot|f(e_n)| = \sup_{f \in \Lambda_\nu} \sum_{\substack{n=1 \\ n \in T_\nu}}^{\infty} |e_n^*(x)|\cdot|e_n^*(y)|\cdot|f(e_n)| \leq$$

$$\leq \frac{\|x\|'_\nu}{\delta_\nu} \sup_{f \in \Lambda_\nu} \sum_{\substack{n=1 \\ n \in T_\nu}}^{\infty} |e_n^*(y)|\cdot|f(e_n)| = \frac{\|x\|'_\nu}{\delta_\nu} \sup_{f \in \Lambda_\nu} \sum_{n=1}^{\infty} |e_n^*(y)|\cdot|f(e_n)| =$$

$$= \frac{\|x\|'_\nu}{\delta_\nu} \|y\|'_\nu$$

where the second equality from the left and the second equality from the right hold since $|f(e_n)| \leq \|e_n\|_\nu = 0$ for $f \in \Lambda_\nu$ and $n \notin T = \{n \in \mathbb{N}: \|e_n\|_\nu \neq 0\}$. Set $p_\nu(z) = \frac{\|z\|'_\nu}{\delta_\nu}$ for $\nu \in \mathbb{N}$ and $z \in E$ then $\{p_\nu: \nu \in \mathbb{N}\}$ is a family of seminorms generating the original topology on E, since $\{\|\cdot\|'_\nu: \nu \in \mathbb{N}\}$ is such a family by Proposition II.2. Moreover, each p_ν is submultiplicative since from the last inequality we have

$$p_\nu(xy) = \frac{1}{\delta_\nu}\|xy\|'_\nu \leq \frac{1}{\delta_\nu}\frac{\|x\|'_\nu\|y\|'_\nu}{\delta_\nu} = p_\nu(x)p_\nu(y) .$$

The proof is completed by the simple observation that the basis $\{e_n\}$ is indeed orthogonal under the multiplication introduced.

Without the hypothesis of statement (i) in Theorem II.3, one can still endow a Fréchet space E which has an unconditional basis $\{e_n\}$, with a multiplication that makes E into a Fréchet algebra, but the basis $\{e_n\}$ will no longer be, necessarily, orthogonal. Indeed we have:

II.4 Theorem. Let E be a Fréchet space with an unconditional basis $\{e_n\}$. Then E can be endowed with a multiplication that makes E into a Fréchet algebra with $\{e_n\}$ as a quasiorthogonal (unconditional) basis.

Proof: Let $\|\cdot\|_1, \|\cdot\|_2, \ldots, \|\cdot\|_\nu, \ldots$ be an increasing sequence of seminorms which generates the topology of E. For each fixed $n \in \mathbb{N}$ put $\gamma_n = \|e_n\|_{\nu_n}$ where ν_n is the smallest positive integer such that

$\|e_n\|_{\nu_n} \neq 0$ and set $e_n' = \gamma_n^{-1} e_n$. Then for each fixed $\nu \in \mathbb{N}$ we have $\|e_n'\|_\nu = \gamma_n^{-1}\|e_n\|_\nu \geq 1$ for all $n \in \mathbb{N}$ with $\|e_n'\|_\nu \neq 0$. Thus statement (i) in Theorem II.3 is true for the unconditional basis $\{e_n'\}$ and the defining family $\{\|\cdot\|_\nu : \nu \in \mathbb{N}\}$ of seminorms and hence E can be endowed with a multiplication under which E is a Fréchet algebra with $\{e_n'\}$ as an orthogonal basis. Now $e_n^2 = (\gamma_n e_n')^2 = \gamma_n^2 e_n'^2 = \gamma_n^2 e_n' = \gamma_n(\gamma_n e_n') = \gamma_n e_n$ and $e_m e_n = \gamma_m e_m' \cdot \gamma_n e_n' = 0$ for $m \neq n$.

II.5 Remark. Multiplication in Theorem II.4 can be expressed in terms of the basis $\{e_n\}$ by $xy = \sum_{n=1}^{\infty} \gamma_n e_n^*(x) e_n^*(y) e_n$. For indeed,

$$xy = \left(\sum_{n=1}^{\infty} e_n^*(x) e_n\right) \cdot \left(\sum_{n=1}^{\infty} e_n^*(y) e_n\right) = \left(\sum_{n=1}^{\infty} e_n^*(x) \gamma_n e_n'\right) \cdot \left(\sum_{n=1}^{\infty} e_n^*(y) \gamma_n e_n'\right) =$$

$$= \sum_{n=1}^{\infty} e_n^*(x) e_n^*(y) \gamma_n^2 e_n' \quad \text{(by orthogonality of the basis } \{e_n'\}) =$$

$$= \sum_{n=1}^{\infty} \gamma_n e_n^*(x) e_n^*(y) e_n .$$

For the special case when E is a Banach space, Theorem II.3 reduces to the following

II.6 Corollary. A Banach space $(E, \|\cdot\|)$ with an unconditional basis $\{e_n\}$ can be made into a Banach algebra with $\{e_n\}$ an orthogonal (unconditional) basis iff $\inf_{n \in \mathbb{N}} \|e_n\| > 0$.

II.7 Examples: (i) For an example of a Banach space with an unconditional basis $\{e_n\}$ which cannot be made into a Banach algebra with $\{e_n\}$ as an orthogonal basis we consider E to be the Banach space of all sequences $x = (x(n))_n$ of complex numbers such that $\|x\| = \sum_{n=1}^{\infty} \frac{|x(n)|}{n} < \infty$. Clearly the coordinate unit vector basis $\{e_n\}$ is an unconditional basis in E. Under a multiplication that makes E into a Banach algebra in which the basis $\{e_n\}$ is orthogonal we must have $xy = \sum_{n=1}^{\infty} x(n)y(n)e_n = (x(n)y(n))_n$ for all $x,y \in E$. But this multiplication is not possible for if $x(n) = n^{\frac{1}{3}}$ if $n = k^3$ for some $k \in \mathbb{N}$ and $x(n) = 0$ otherwise, then $\sum_{n=1}^{\infty} \frac{|x(n)|}{n} = \sum_{k=1}^{\infty} \frac{k}{k^3} < \infty$ and so $x = (x(n))_n \in E$. However, x^2 is undefined since $\sum_{k=1}^{\infty} \frac{k^2}{k^3} = \infty$.

Following Remark II.5, the multiplication $xy = \sum_{n=1}^{\infty} \frac{x(n)y(n)}{n} e_n = (\frac{x(n)y(n)}{n})_n$ converts E into a Banach algebra with $\{e_n\}$ as a quasi-orthogonal basis.

(ii) The Fréchet space $H(\mathbb{D})$ of all functions holomorphic on the open unit disc $\mathbb{D}$ with the compact-open topology, has the unconditional basis $\{e_n\}$ where $e_n(z) = z^n$, $n \in \mathbb{N}$. The multiplication $xy = \sum_{n=1}^{\infty} e_n^*(x)e_n^*(y)e_n$ is well defined on $H(\mathbb{D})$ and it makes $H(\mathbb{D})$ into a B_0 algebra (which is not a Fréchet algebra) in which the basis $\{e_n\}$ is orthogonal. (For details, see Example III.12 (iii) and Remark III.25). The increasing sequence $\{q_\nu\}$ of seminorms where $q_\nu(x) = \sup_{|z| \le \frac{\nu}{1+\nu}} |x(z)|$, $x \in H(D)$, define the topology of $H(D)$ and clearly $q_1(e_n) = \sup_{|z| \le \frac{1}{2}} |z^n| = (\frac{1}{2})^n$.

Hence, using the same notation as in Theorem II.4 and Remark II.5, $\gamma_n = q_1(e_n) = 2^{-n}$ for all $n \in \mathbb{N}$ and the multiplication $xy = \sum_{n=1}^{\infty} 2^{-n} e_n^*(x) e_n^*(y) e_n$ converts $H(D)$ into a Fréchet algebra with $\{e_n\}$ as a quasiorthogonal basis.

§3. Unconditionality of orthogonal bases in B_0 algebras.

For every $x = (x(n))_n$ and $y = (y(n))_n$ in $\ell_1(\mathbb{N})$ we have

$$\sum_{n=1}^{\infty} |x(n)y(n)| \le \left(\sum_{n=1}^{\infty} |x(n)|^2\right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} |y(n)|^2\right)^{\frac{1}{2}}$$

which is the Cauchy-Schwarz inequality. This can be written in the form

$$\|xy\| \le \|x^2\|^{\frac{1}{2}} \|y^2\|^{\frac{1}{2}} .$$

A norm (or a seminorm) on an algebra, with this property will be called "squarely submultiplicative". It turns out that squarely submultiplicative seminorms play an important rôle in the study of the unconditionality of orthogonal bases in B_0 algebras. In fact, as we shall prove, an orthogonal basis in a B_0 algebra is unconditional iff the topology can be defined by a family of squarely submultiplicative seminorms (Theorem II.12).

First, we have two examples of Banach algebras which admit non-unconditional orthogonal bases.

II.8 Examples.

(i) For $1 < p < \infty$, $p \neq 2$ the orthogonal basis in $L_p(\mathbf{T})$ (Example I.40) is not unconditional. Indeed, there exists $x \in L_p(\mathbf{T})$ and $\varepsilon: \mathbf{Z} \to \{-1,+1\}$ such that $\varepsilon\hat{x}: \dot{\mathbf{Z}} \to \mathbf{C}$ is not the Fourier transform of any $y \in L_p(\mathbf{T})$ (see [7], page 301), contrary to statement (vi) of Theorem II.1.

(ii) Let ω_0 (also denoted by bv_0) be the set of all sequences $x = (x(n))$ of complex numbers such that $\lim_n x(n) = 0$ and $\sum_{n=1}^{\infty} |x(n) - x(n+1)| < \infty$. The function $\|\cdot\|: \omega_0 \to \mathbf{R}$ given by $\|x\| = \sup_n |x(n)| + \sum_{n=1}^{\infty} |x(n) - x(n+1)|$ is a norm on ω_0 such that $(\omega_0, \|\cdot\|)$ is a Banach algebra under the coordinatewise operations, which is (algebraically) a subalgebra of c_0 [39]. It is easy to see that the coordinate unit vectors $\{e_n\}$ form an orthogonal basis in ω_0. By Theorem II.1, this basis is not unconditional since $\sum_{n=1}^{\infty} \frac{1}{n} e_n$ converges in ω_0 while its subseries $\sum_{n=1}^{\infty} \frac{1}{2n-1} e_{2n-1}$ does not. (Notice that $\sum_{n=1}^{\infty} |\frac{1}{n} - \frac{1}{n+1}| = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} < \infty$ while $\sum_{n=1}^{\infty} |\frac{\varepsilon_n}{n} - \frac{\varepsilon_{n+1}}{n+1}| = \infty$, where $\varepsilon_n = 1$, n odd, $\varepsilon_n = 0$, n even).

II.9 Definitions. Let A be an algebra (no topological structure assumed).

(i) A subset $S \subset A$ is said to be a squarely idempotent set (in short, s.i. set) if $xy \in S$ whenever $x^2, y^2 \in S$.

(ii) A seminorm $\|\cdot\|$ on A is said to be squarely submultiplicative if $\|xy\|^2 \le \|x^2\| \, \|y^2\|$ for all $x, y \in A$.

II.10 Proposition. Let A be as in Definition II.9 and let S be a circled, convex and absorbing subset of A with gauge $\|\cdot\|$. Then S is a squarely idempotent set iff the seminorm $\|\cdot\|$ is squarely submultiplicative.

Proof: Assume that $\|\cdot\|$ is squarely submultiplicative and let $x, y \in A$ be such that $x^2, y^2 \in S$, then $\|xy\| \le \|x^2\|^{\frac{1}{2}} \, \|y^2\|^{\frac{1}{2}} \le 1 \cdot 1 = 1$ and so $xy \in S$.

Conversely, if S is an s.i. set and $x, y \in A$ such that $\|x^2\| \neq 0$ and $\|y^2\| \neq 0$, then $\frac{x^2}{\|x^2\|}, \ \frac{y^2}{\|y^2\|} \in S$ and so $\frac{x}{\|x^2\|^{\frac{1}{2}}} \, \frac{y}{\|y^2\|^{\frac{1}{2}}} \in S$.

Hence $\left\| \frac{xy}{\|x^2\|^{\frac{1}{2}} \, \|y^2\|^{\frac{1}{2}}} \right\| \le 1$ and consequently $\|xy\| \le \|x^2\|^{\frac{1}{2}} \, \|y^2\|^{\frac{1}{2}}$. If one of $\|x^2\|$ and $\|y^2\|$ (say $\|x^2\|$) is zero, then for any $\lambda > 0$ we have $\frac{x^2}{\lambda^2/\mu_0} \in S$, where $\mu_0 > 0$ is such that $\frac{y^2}{\mu_0} \in S$. Hence $\frac{xy}{\lambda} = \frac{x}{\lambda/\sqrt{\mu_0}} \cdot \frac{y}{\sqrt{\mu_0}} \in S$ and so $xy \in \lambda S$. Since $\lambda > 0$ is arbitrary, we have $\|xy\| = 0 \le \|x^2\|^{\frac{1}{2}} \|y^2\|^{\frac{1}{2}}$.

II.11 Definitions. (i) A locally convex algebra A is said to be a locally convex s-algebra if A has a 0-neighbourhood base {U} such that each $U \in \{U\}$ is an s.i. set, in addition to being circled, convex and closed.

From Proposition II.10 we easily see that a locally convex s-algebra can equivalently be defined as a locally convex algebra whose topology can be generated by a family of squarely submultiplicative seminorms.

(ii) A locally m-convex algebra A is said to be a locally m-convex s-algebra if A has a 0-neighbourhood base {U} such that each $U \in \{U\}$ is an s.i. set, in addition to being circled, convex, idempotent and closed.

Clearly, the topology of a locally m-convex s-algebra can be generated by a family of submultiplicative, squarely submultiplicative seminorms.

Moreover, a B_0 s-algebra is a complete metrizable locally convex s-algebra, a Fréchet s-algebra is a complete metrizable locally m-convex s-algebra, a normed s-algebra is an algebra which is topologized by a submultiplicative, squarely submultiplicative norm and a Banach s-algebra is a complete normed s-algebra.

Now we prove our main result in this chapter.

II.12 Theorem. An orthogonal basis $\{e_n\}$ in a B_0 algebra A is unconditional iff A is a B_0 s-algebra.

Proof: Let $\{\|\cdot\|_\nu : \nu \in \mathbb{N}\}$ be a generating family of squarely submultiplicative seminorms on A. If $I \subset J \subset \mathbb{N}$, J is finite and λ_k, $k \in J$ are scalars, then because of the orthogonality of the basis $\{e_n\}$ we have $\sum_{k\in I} \lambda_k e_k = \sum_{k\in I} \lambda_k^{\frac{1}{2}} e_k \cdot \sum_{k\in J} \lambda_k^{\frac{1}{2}} e_k$ and so for every $\nu \in \mathbb{N}$,

(II.12.1) $$\|\sum_{k\in I} \lambda_k e_k\|_\nu^2 \le \|\sum_{k\in I} \lambda_k e_k\|_\nu \cdot \|\sum_{k\in J} \lambda_k e_k\|_\nu .$$

If $\|\sum_{k\in I} \lambda_k e_k\|_\nu = 0$, then we trivially have

(II.12.2) $$\|\sum_{k\in I} \lambda_k e_k\|_\nu \le \|\sum_{k\in J} \lambda_k e_k\|_\nu .$$

If $\|\sum_{k\in I} \lambda_k e_k\|_\nu \neq 0$, then (II.12.2) follows by dividing both sides of (II.12.1) by $\|\sum_{k\in I} \lambda_k e_k\|_\nu$. Let $\sum_{k=1}^{\infty} \lambda_k e_k$ be a convergent series in A and let $\sum_{j=1}^{\infty} \lambda_{k_j} e_{k_j}$ be a subseries. Given $\nu \in \mathbb{N}$ and $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $\|\sum_{k=m+1}^{n} \lambda_k e_k\|_\nu < \varepsilon$ whenever $n > m \ge N$. Let $N' \in \mathbb{N}$ be so large that $k_j \ge N$ for $j \ge N'$, then for $j > i \ge N'$ we have $k_j > k_i \ge N$ and consequently

$$\left\| \sum_{r=i+1}^{j} \lambda_{k_r} e_{k_r} \right\|_\nu \le \left\| \sum_{k=k_i+1}^{k_j} \lambda_k e_k \right\|_\nu < \varepsilon$$

where the first inequality follows from (II.12.2). Hence the sequence of partial sums of the subseries $\sum_{j=1}^{\infty} \lambda_{k_j} e_{k_j}$ is Cauchy and, from the completeness of A, is convergent. It follows from Theorem II.1 that the basis $\{e_n\}$ is unconditional.

Conversely, if the basis $\{e_n\}$ is unconditional, then by Proposition II.2 we have an equivalent family $\{\|\cdot\|'_\nu : \nu \in \mathbb{N}\}$ of seminorms on A given by

$$\|x\|'_\nu = \sup_{f \in \Lambda_\nu} \sum_{n=1}^{\infty} |e_n^*(x)| . |f(e_n)| , \quad x \in A , \quad \nu \in \mathbb{N}$$

(see Theorem II.1 for Λ_ν). We show that each of the seminorms $\|\cdot\|'_\nu$ is squarely submultiplicative. For each $\nu \in \mathbb{N}$ and $f \in \Lambda_\nu$ we can define a measure μ_f on the power set of $\mathbb{N}$ by $\mu_f(T) = \sum_{n \in T} |f(e_n)|$, and each $x \in A$ can be thought of as a function $x: \mathbb{N} \to \mathbb{C}$ given by $x(n) = e_n^*(x)$, $n \in \mathbb{N}$. Since

$$\sum_{n=1}^{\infty} |e_n^*(x)|^2 |f(e_n)| = \sum_{n=1}^{\infty} |e_n^*(x^2)| |f(e_n)| \le \|x^2\|'_\nu < \infty$$

for all $x \in A$, it follows from the Cauchy-Schwarz inequality that for $x, y \in A$,

$$\sum_{n=1}^{\infty} |e_n^*(xy)| \cdot |f(e_n)| = \sum_{n=1}^{\infty} |e_n^*(x)| |e_n^*(y)| |f(e_n)| \leq$$

$$\leq \left(\sum_{n=1}^{\infty} |e_n^*(x)|^2 |f(e_n)|\right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} |e_n^*(y^2)|^2 |f(e_n)|\right)^{\frac{1}{2}} =$$

$$= \left(\sum_{n=1}^{\infty} |e_n^*(x^2)| |f(e_n)|\right)^{\frac{1}{2}} \cdot \left(\sum_{n=1}^{\infty} |e_n^*(y^2)| |f(e_n)|\right)^{\frac{1}{2}} .$$

Whence we have:

$$\|xy\|_\nu' = \sup_{f \in \Lambda_\nu} \sum_{n=1}^{\infty} |e_n^*(xy)| |f(e_n)| \leq$$

$$\leq \left(\sup_{f \in \Lambda_\nu} \sum_{n=1}^{\infty} |e_n^*(x^2)| |f(e_n)|\right)^{\frac{1}{2}} \left(\sup_{f \in \Lambda_\nu} \sum_{n=1}^{\infty} |e_n^*(y^2)| |f(e_n)|\right)^{\frac{1}{2}} =$$

$$= \|x^2\|_\nu'^{\frac{1}{2}} \|y^2\|_\nu'^{\frac{1}{2}} ,$$

thus proving that A is a B_0 s-algebra. (Notice how the Cauchy-Schwarz inequality, in its classical form, was explicitly involved in establishing the square submultiplicativity of the seminorms $\|.\|_\nu'$).

II.13 Examples.

(i) Since the canonical orthogonal basis in each of the Banach algebras ℓ_p, $1 \leq p < \infty$ is unconditional, these are B_0 s-algebras by Theorem

II.12. To verify this fact directly, we see that for $x = (x(n))$, $y = (y(n)) \in \ell_p$ we have:

$$\|xy\|_p = \left(\sum_{n=1}^{\infty} |x(n)y(n)|^p\right)^{\frac{1}{p}} \le \left[\left(\sum_{n=1}^{\infty} |x(n)|^{2p}\right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} |y(n)|^{2p}\right)^{\frac{1}{2}}\right]^{\frac{1}{p}}$$

$$= \left[\left(\sum_{n=1}^{\infty} |x^2(n)|^p\right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} |y^2(n)|^p\right)^{\frac{1}{p}}\right]^{\frac{1}{2}} = \|x^2\|_p^{\frac{1}{2}} \|y^2\|_p^{\frac{1}{2}} .$$

and hence ℓ_p is a Banach s-algebra. The same is true for the Banach algebra c_0 because $\|xy\|_\infty = \sup_n |x(n)y(n)| \le \sup_n |x(n)| \cdot \sup_n |y(n)| =$

$$= (\sup_n |x(n)|^2)^{\frac{1}{2}} (\sup_n |y(n)|^2)^{\frac{1}{2}} = \|x^2\|_\infty^{\frac{1}{2}} \|y^2\|_\infty^{\frac{1}{2}} .$$

(ii) The Banach algebras $L_p(T)$, $1 \le p < \infty$, $p \ne 2$ (Example II.8 (i)) are not B_0 s-algebras. For the case $p \ne 1$, this follows from Theorem II.12 since $L_p(T)$, $1 < p < \infty$, $p \ne 2$ has an orthogonal basis which is not unconditional. For the case $p = 1$, we notice that the Banach algebra $L_1(T)$ has a bounded approximate identity [25]. If $\{x_\alpha\}$ is such an identity with $\|x_\alpha\| \le K$ for all α and some $K > 0$ and if $|||\cdot|||$ were an equivalent squarely submultiplicative norm on $L_1(T)$ with $M_1\|x\| \le |||x||| \le M_2\|x\|$ for some $M_1, M_2 > 0$ and for all $x \in L_1(T)$, then

$$\|x\| \le \frac{1}{M_1} |||x||| = \frac{1}{M_1} |||\lim_\alpha xx_\alpha||| = \frac{1}{M_1} \lim_\alpha |||xx_\alpha||| \le \frac{1}{M_1} \sup_\alpha |||xx_\alpha||| \le$$

$$\le \frac{1}{M_1} \sup_\alpha |||x^2|||^{\frac{1}{2}} |||x_\alpha^2|||^{\frac{1}{2}} \le \frac{(M_2\|x^2\|)^{\frac{1}{2}}}{M_1} \sup_\alpha (M_2\|x_\alpha^2\|)^{\frac{1}{2}} =$$

$$= \frac{M_2}{M_1} \|x^2\|^{\frac{1}{2}} \sup_\alpha \|x_\alpha^2\|^{\frac{1}{2}} \le \frac{M_2}{M_1} \|x^2\|^{\frac{1}{2}} \sup_\alpha \|x_\alpha\| \le \frac{KM_2}{M_1} \|x^2\|^{\frac{1}{2}} ,$$

where the second inequality from the right follows from the submultiplicativity of $\|\cdot\|$. It follows that $\|x\|^2 \le (\frac{KM_2}{M_1})^2 \|x^2\|$, thus leading to the false conclusion that the Fourier transform $L_1(\mathbf{T}) \to \widehat{L_1(\mathbf{T})} \subset C_0(\mathbf{Z})$ is a topological isomorphism. $L_2(\mathbf{T})$ is, however, a Banach s-algebra since the Fourier transform is an isometric algebra isomorphism of $L_2(\mathbf{T})$ onto $\ell_2(\mathbf{Z})$, which is a Banach s-algebra as in Example II.13 (i).

(iii) For an example of a complete A-convex s-algebra with an orthogonal basis, which is not locally m-convex, consider the algebra $C^*_\beta(\mathbf{N})$ of all bounded sequences $x = \{x(n)\}$, with the strict topology (Example I.41 with $\Gamma = \mathbf{N}$). The fact that $C^*_\beta(\mathbf{N})$ is an s-algebra can be shown as follows (with the same notations as in Example I.41). For $x,y \in C^*_\beta(\mathbf{N})$ and $\phi \in C_0(\mathbf{N})$ we have

$$p_\phi(xy) = \sup_n |\phi(n)x(n)y(n)| = (\sup_n |\phi^2(n)x^2(n)y^2(n)|)^{\frac{1}{2}} =$$

$$= (\sup_n |\phi(n)x^2(n)\cdot\phi(n)y^2(n)|)^{\frac{1}{2}} \le (\sup_n |\phi(n)x^2(n)|)^{\frac{1}{2}} (\sup_n |\phi(n)y^2(n)|)^{\frac{1}{2}} =$$

$$= p_\phi^{\frac{1}{2}}(x^2)\ p_\phi^{\frac{1}{2}}(y^2) .$$

Theorem 2.12 does not apply in this case since B is not metrizable. The orthogonal basis is, however, unconditional.

(iv) For any set Γ, it is easy to see that the algebra $\mathbb{C}^\Gamma$. (Example I.38) is a complete locally m-convex s-algebra. Indeed each of the seminorms $P_J = \sup_{\alpha\in J} |x(\alpha)|$, $J \in \Omega$ is a squarely submultiplicative seminorm.

(v) The Arens algebra L^ω is defined as $L^\omega = \bigcap_{1\le p<\infty} L_p[0,1]$ with pointwise algebraic operations, topologized by the family $\{\|.\|_p : 1 \le p < \infty\}$ of seminorms where $\|x\|_p = (\int_0^1 |x(t)|^p)^{\frac{1}{p}}$, $x \in L^\omega$, $1 \le p < \infty$ [1]. L^ω is a B_0 algebra. If $x,y \in L^\omega$ and $1 \le p < \infty$ then $x,y \in L_{2p}[0,1]$ and so $x^{2p}, y^{2p} \in L_1[0,1]$ but then $x^p, y^p \in L_2[0,1]$. It follows from the Cauchy-Schwarz inequality that

$$\|xy\|_p^p = \int_0^1 |x(t)|^p \, |y(t)|^p dt \le \left(\int_0^1 |x(t)|^{2p}\right)^{\frac{1}{2}} \cdot \left(\int_0^1 |y(t)|^{2p}\right)^{\frac{1}{2}} =$$

$$= \left[\int_0^1 |x^2(t)|^p dt \cdot \int_0^1 |y^2(t)|^p dt\right]^{\frac{1}{2}} = \left[\|x^2\|_p^p \, \|y^2\|_p^p\right]^{\frac{1}{2}}$$

and so $\|xy\|_p \le \|x^2\|_p^{\frac{1}{2}} \|y^2\|_p^{\frac{1}{2}}$. Hence L^ω is a B_0 s-algebra. However, L^ω is not locally m-convex and hence is not a Fréchet algebra [1].

(vi) The Banach algebra ω_0 (Example II.8 (ii)) is not a B_0 s-algebra by Theorem II.12 since it has an orthogonal basis which is not unconditional.

(vii) It is easy to verify that the algebra $\ell^{(1)} = \bigcap_{p>1} \ell_p$ topologized by all the ℓ_p-norms, $1 < p \le \infty$ is a Fréchet s-algebra, and that the coordinate unit vectors form an orthogonal basis in $\ell^{(1)}$.

§4. Locally convex s-algebras.

In this section we briefly discuss some consequences of the definition of a locally convex s-algebra. Further details of such algebras will be studied elsewhere. We assume in this section that all algebras are commutative.

II.14 Proposition. Let $\|\cdot\|$ be a squarely submultiplicative seminorm on an algebra A. For every $n \in \mathbb{N}$ and $x \in A$, set $\|x\|_n = \|(x)^{2^n}\|^{\frac{1}{2^n}}$.

Then:

(i) Each $\|.\|_n$, $n \in \mathbb{N}$ is a squarely submultiplicative seminorm on A.

(ii) If, in addition, $\|.\|$ is submultiplicative, so is every $\|.\|_n$, $n \in \mathbb{N}$, and $\|x\| \geq \|x\|_1 \geq \|x\|_2 \geq \dots$ for all $x \in A$.

Proof: (i) Clearly $\|0\|_n = 0$ and $\|\lambda x\|_n = |\lambda|\, \|x\|_n$ for all $n \in \mathbb{N}$, $\lambda \in \mathbb{C}$ and $x \in A$. It remains to show that each $\|.\|_n$ is squarely submultiplicative and satisfies the triangular inequality. For this, we use an induction argument. Assume that for some $n \in \mathbb{N}$, $\|.\|_n$ satisfies the triangular inequality and is squarely submultiplicative, then

$$\|x + y\|_{n+1} = \|(x + y)^{2^{n+1}}\|^{\frac{1}{2^{n+1}}} = \left(\|[(x + y)^2]^{2^n}\|^{\frac{1}{2^n}}\right)^{\frac{1}{2}} = \|(x + y)^2\|_n^{\frac{1}{2}} =$$

$$= \|x^2 + 2xy + y^2\|_n^{\frac{1}{2}} \leq \left(\|x^2\|_n + 2\|xy\|_n + \|y^2\|_n\right)^{\frac{1}{2}} \leq$$

$$\leq \left(\|x^2\|_n + 2\|x^2\|_n^{\frac{1}{2}} \|y^2\|_n^{\frac{1}{2}} + \|y^2\|_n\right)^{\frac{1}{2}} = \|x^2\|_n^{\frac{1}{2}} + \|y^2\|_n^{\frac{1}{2}} .$$

Since

$$\text{(II.14.1)} \qquad \|x\|_{n+1} = \|(x)^{2^{n+1}}\|^{\frac{1}{2^{n+1}}} = \left(\|(x^2)^{2^n}\|^{\frac{1}{2^n}}\right)^{\frac{1}{2}} = \|x^2\|_n^{\frac{1}{2}} ,$$

it follows from the last inequality that, $\|x + y\|_{n+1} \leq \|x\|_{n+1} + \|y\|_{n+1}$,

which is the triangular inequality for $\|.\|_{n+1}$. From (II.14.1) and the square submultiplicativity of $\|.\|_n$ we have $\|xy\|_{n+1} = \|x^2y^2\|_n^{\frac{1}{2}} \le$
$\le \left(\|(x^2)^2\|_n^{\frac{1}{2}} \|(y^2)^2\|_n^{\frac{1}{2}}\right)^{\frac{1}{2}} = \left(\|x^2\|_{n+1}\|y^2\|_{n+1}\right)^{\frac{1}{2}}$ and so $\|.\|_{n+1}$ is squarely submultiplicative. This completes the proof of part (i) since, by hypothesis, the original norm $\|.\|$ $(= \|.\|_0)$ is squarely submultiplicative.

(ii) If, in addition, $\|.\|$ is submultiplicative, then for $n \in \mathbb{N}$, $x, y \in A$ we have $\|xy\|_n = \|(xy)^{2^n}\|^{\frac{1}{2^n}} = \|(x)^{2^n}(y)^{2^n}\|^{\frac{1}{2^n}} \le \left(\|(x)^{2^n}\|.\|(y)^{2^n}\|\right)^{\frac{1}{2^n}} =$
$= \|x\|_n \|y\|_n$ and so each $\|.\|_n$ is submultiplicative. It then follows from (II.14.1) that $\|x\|_{n+1} = \|x^2\|_n^{\frac{1}{2}} \le (\|x\|_n \|x\|_n)^{\frac{1}{2}} = \|x\|_n$ for all $x \in A$, $n \in \mathbb{N}$.

II.15 Corollary. Let U be a squarely idempotent, convex, circled and absorbing subset of an algebra A. For each $n \in \mathbb{N}$ define the function f_n on A by $f_n: x \to (x)^{2^n}$, then $f_n^{-1}(U)$ is also squarely idempotent, convex, circled and absorbing.

Proof: Let $\|.\|$ be the gauge of U. From Proposition II.10 we see that $\|.\|$ is a squarely submultiplicative seminorm on A and hence so is every $\|.\|_n$, by Proposition II.14. Thus, for every $n \in \mathbb{N}$ we have $f_n^{-1}(U) =$
$= \{x \in A: (x)^{2^n} \in U\} = \{x \in A: \|(x)^{2^n}\| \le 1\} = \{x \in A: \|(x)^{2^n}\|^{\frac{1}{2^n}} \le 1\} =$
$= \{x \in A: \|x\|_n \le 1\}$ is a squarely idempotent, convex, circled and absorbing subset of A.

II.16 Theorem. Let P be a family of squarely submultiplicative seminorms on an algebra A. Let P_1 be the family of squarely submultiplicative seminorms given by $P_1 = \{p_1 : p \in P\}$, $p_1(x) = p^{\frac{1}{2}}(x^2)$. Then A is a topological algebra under P iff P generates a stronger topology on A than P_1. Moreover, if A is a topological algebra under P, it is also a topological algebra under P_1.

Proof: If multiplication is jointly continuous under P then for every $p \in P$ there exists $q \in P$ such that $p(xy) \le q(x)q(y)$ for all $x, y \in A$. In particular, $p(x^2) \le (q(x))^2$ and so $p_1(x) = p^{\frac{1}{2}}(x^2) \le q(x)$ for all $x \in A$.

Conversely, if P generates a stronger topology on A than P_1, the joint continuity of multiplication under P follows from $p(xy) \le p^{\frac{1}{2}}(x^2)\, p^{\frac{1}{2}}(y^2) = p_1(x)\, p_1(y)$, $x, y \in A$, $p \in P$. The last assertion follows from $p_1(xy) = [p(x^2y^2)]^{\frac{1}{2}} \le [q(x^2)q(y^2)]^{\frac{1}{2}} = q_1(x)q_1(y)$, for some $q \in P$.

II.17 Corollary. Let A be a locally convex s-algebra with P as a generating family of squarely submultiplicative seminorms. Then each of the families $P_n = \{p_n : p \in P\}$, $p_n(x) = (p[(x)^{2^n}])^{\frac{1}{2^n}}$, is equivalent to P, provided that A has a bounded approximate identity.

Proof: From (II.14.1) we have $p_{n+1}(x) = p_n^{\frac{1}{2}}(x^2)$ for all $p \in P$ and $x \in A$ and so P_{n+1} is derived from P_n the same way as P_1 is derived from P. Hence, the proof for all $n \in \mathbb{N}$ follows inductively once we have a proof for $n = 1$. From Proposition II.14 (i) and Theorem II.16, each p_1, $p \in P$ is a continuous squarely multiplicative seminorm and hence if $\{x_\alpha\}$ is a bounded approximate identity in A then for each $p_1 \in P_1$ there exists $M_{p_1} > 0$ such that $p_1(x_\alpha) \le M_{p_1}$ for all α. Thus for each $p \in P$ and $x \in A$ we have $p(x) = p(\lim_\alpha xx_\alpha) =$

$$= \lim_\alpha p(xx_\alpha) \le \sup_\alpha p(xx_\alpha) \le \sup_\alpha p^{\frac{1}{2}}(x^2)\, p^{\frac{1}{2}}(x_\alpha^2) = p_1(x) \sup_\alpha p_1(x_\alpha) \le M_{p_1} p_1(x) .$$

Hence P_1 generates a stronger topology on A than P and so the two topologies are equivalent by Theorem II.16.

II.18 <u>Proposition</u>. Let p be a squarely submultiplicative seminorm on an algebra A. Then p is submultiplicative iff $p_1(x) = p^{\frac{1}{2}}(x^2) \le p(x)$ for all $x \in A$.

<u>Proof</u>: If $p_1(x) = p^{\frac{1}{2}}(x^2) \le p(x)$ for all $x \in A$, then $p(xy) \le$ $\le p^{\frac{1}{2}}(x^2)\, p^{\frac{1}{2}}(y^2) = p_1(x)p_1(y) \le p(x)p(y)$ for all $x,y \in A$. Conversely, if $p(xy) \le p(x)p(y)$ for all $x,y \in A$ then, in particular, $p(x^2) \le p^2(x)$ and so $p_1(x) = p^{\frac{1}{2}}(x^2) \le p(x)$ for all $x \in A$.

II.19 <u>Corollary</u>. Let A be a semisimple Banach s-algebra. Consider the following:

(i) A has a bounded approximate identity.

(ii) $(A, \|\cdot\|_n)$ is complete for some $n \in \mathbb{N}$.

(iii) $\|\cdot\|$ and $\|\cdot\|_1$ are equivalent.

(iv) All the norms $\|\cdot\|$ and $\|\cdot\|_n$, $n \in \mathbb{N}$ are equivalent.

(v) The Gel'fand map $\phi: A \to \hat{A}$ is a topological isomorphism.

Then (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v).

Proof: By Proposition II.14, $(A, \|\cdot\|_n)$ is a normed s-algebra for each $n \in \mathbb{N}$. (ii) $\Rightarrow$ (iii) follows from the inequality $\|x\| \geq \|x_1\| \geq \|x\|_n$ for all $x \in A$ (Proposition II.14 (ii)) and the Banach Open Mapping Theorem (Theorem I.7). (iii) $\Rightarrow$ (iv) follows inductively from (II.14.1) as in the proof of Corollary II.17, and (iv) $\Rightarrow$ (ii) is trivial. The equivalence of (iii) and (v) follows from the fact that the Gel'fand map $\phi: A \to \hat{A}_0$ is a topological isomorphism iff there exists $M > 0$ such that $\|x\|^2 \leq M\|x^2\|$ for all $x \in A$ (Theorem I.19). Finally, the implication (i) $\Rightarrow$ (iv) follows from Corollary II.17.

II.20 Corollary. If in addition to the hypothesis of Corollary II.19 A has an orthogonal basis $\{e_n\}$, then (v) can be replaced by:

(vi) A is topologically isomorphic with c_0.

Furthermore, the statements (i), (ii), (iii), (iv) and (vi) are equivalent in this case and the hypothesis of semisimplicity can be deleted (since it follows from the existence of an orthogonal basis [19]).

Proof: Indeed, the maximal ideal space of A is $\mathbb{N}$ with the discrete topology and consequently $\hat{A}_0$ is closed in $C_0(\mathbb{N}) = c_0$. Moreover, $\hat{A}_0$ contains the set of all elements of c_0 with finitely many non-zero coordinates which is dense in c_0. Hence $\hat{A}_0 = c_0$. To complete the proof, we show that (vi) $\Rightarrow$ (i). Since the sequence $\{u_n\}$ in $C_0(\mathbb{N})$, where $u_n(k) = 1$ for $0 \leq k \leq n$ and $u_n(k) = 0$ for $k > n$, is a bounded approximate identity in $C_0(\mathbb{N})$ (vi) $\Rightarrow$ (i) follows.

II.21 Remark. Corollary II.20 characterizes c_0 (up to a topological isomorphism) as the Banach s-algebra with an orthogonal basis and a bounded approximate identity. Notice that the Banach algebra ω_0 (Example II.8 (ii) and Example II.13 (vi)) has all these properties except being a Banach s-algebra. (It is easy to see that $\{u_n\}$ in Corollary II.20 remains as a bounded approximate identity in ω_0). Equivalently (by Theorem II.12) c_0 is the Banach algebra with an unconditional orthogonal basis and a bounded approximate identity.

We conclude this chapter with the following remark.

II.22 Remark. In some locally convex s-algebras (with or without orthogonal bases) the topology is defined by a sequence of (squarely submultiplicative) seminorms $\|.\|_0$, $\|.\|_1$, $\|.\|_2, \ldots$ derived from a single squarely submultiplicative seminorm by

$$\|x\|_n = \|(x)^{2^n}\|^{\frac{1}{2^n}} \quad , \quad n = 0, 1, 2, \ldots$$

Evidently, such algebras are necessarily metrizable. Consider an algebra A of this type:

(i) If $\|.\| = \|.\|_0$ is submultiplicative, then by Proposition II.14 (ii) we have $\|x\| = \|x\|_0 \geq \|x\|_1 \geq \|x\|_2 \geq \ldots$ for all $x \in A$ and so A is normable since the seminorm $\|.\|$ (which must then be a norm because A is Hausdorff as we always assume) defines the topology. Examples of this case include ℓ_p with $\|.\|$ as the ℓ_p-norm, $1 \leq p \leq \infty$, and normed s-algebras in general.

(ii) In some other cases, the sequence $\|.\|_0, \|.\|_1, \|.\|_2, \ldots$ is strictly increasing, in the sense that $\|.\|_{n+1}$ is a strictly stronger seminorm than $\|.\|_n$. Examples of such algebras include: the Arens algebra L^{ω} (Example II.13 (v)), with $\|.\|$ as the L_1-norm since, clearly, the family of seminorms given by

$$\|x\|_n = \|(x)^{2^n}\|^{\frac{1}{2^n}} = \left(\int_0^1 |x(t)|^{2^n} dt\right)^{\frac{1}{2^n}} \quad , \quad n = 0, 1, 2, \ldots$$

defines the topology of L^{ω}. Another example is the algebra $H(\mathbb{D})$ briefly discussed in Example II.7 (ii) (for more details, see Example III.12 (iii)) with $\|.\|$ as the seminorm

$$\|f\| = \sup_n \left|\frac{f^{(n)}(0)}{n!} 2^{-n}\right|$$

and $\|f\|_1 = \|f\|$, $\|f\|_k = \|(f)^{2^k}\|^{\frac{1}{2^k}} = \left(\sup_n \left[\frac{f^{(n)}(0)}{n!}\right]^{2^k} 2^{-n}\right)^{\frac{1}{2^k}} =$

$$= \sup_n \left|\frac{f^{(n)}(0)}{n!} 2^{-n/2^k}\right| , \quad k = 2, 3, \ldots .$$

The algebra $\ell^{(1)}$ (Example II.13 (vii)) is a metrizable locally convex s-algebra which is not of the type discussed in this remark. Indeed, if $|||.|||_p$ denotes the ℓ_p-norm, $1 < p \leq \infty$ and if $\|.\|$ were a squarely submultiplicative seminorm such that the family $\|x\|_n = \|(x)^{2^n}\|^{\frac{1}{2^n}}$, $n = 0, 1, 2, \ldots$ defines the topology of $\ell^{(1)}$, then there exists p with $1 < p \leq \infty$ such that $\|x\| \leq |||x|||_p$ for all $x \in \ell^{(1)}$, but then for each $n = 0, 1, 2, \ldots$ we would have

$$\text{(II.22.1)} \qquad \|x\|_n = \|(x)^{2^n}\|^{\frac{1}{2^n}} \leq |||(x)^{2^n}|||_p^{\frac{1}{2^n}} \leq |||x|||_p$$

for all $x \in \ell^{(1)}$, where the last inequality in (II.22.1) follows from Proposition II.14 (ii) since $|||\cdot|||_p$ is a submultiplicative, squarely submultiplicative seminorm. This completes the argument since (II.22.1) entails the false conclusion that a single norm, namely $|||\cdot|||_p$, defines the topology of $\ell^{(1)}$.

It would be interesting to pursue this discussion and obtain necessary or sufficient conditions for a locally convex s-algebra to be as in (ii).

CHAPTER III

LOCALLY CONVEX ALGEBRAS WITH ORTHOGONAL M-BASES

In §1 of this chapter, we obtain some results concerning topological algebras with orthogonal bases or orthogonal M-bases. Interestingly, it turns out that every orthogonal basis or orthogonal M-basis in a topological algebra is Schauder (Theorem III.1) and, as a consequence, every topological algebra with an orthogonal basis or orthogonal M-basis has a faithful (i.e., one-one) continuous representation as a dense subalgebra of $\mathbb{C}^\Gamma$, for a suitable Γ (Theorem III.3 (i)). The relationship between this representation and the Gel'fand representation, in the Banach algebra case, is also discussed in Corollary III.4 and Remark III.5. In §2 we focus on locally m-convex algebras with orthogonal bases or orthogonal M-bases and we show that the existence of an identity in an algebra of this kind makes the faithful continuous representation mentioned above, a homeomorphic one, thus uniquely determining the topology (Theorem III.6).

In §3 we discuss and characterize a certain type of locally convex algebras which we call Φ-algebras (Theorem III.13). Examples of Φ-algebras include $\mathbb{C}^\Gamma$, $C^*_\beta(\Gamma)$ and $H(\mathbb{D})$. Finally, in §4 we study locally convex s-algebras and $\mathcal{B}$-convex algebras with orthogonal M-bases and we obtain a necessary and sufficient condition for a locally convex s-algebra to be A-convex (Proposition III.22), and a characterization of the algebra s of all sequences (Theorem III.24), different from the characterization which follows from Theorem III.6 with $\Gamma = \mathbb{N}$.

§1. Topological algebras with orthogonal bases or orthogonal M-bases.

One of the interesting questions in the classical basis theory is whether every basis $\{e_n\}$ in a topological vector space of a certain type is a Schauder basis. Positive answers to this question have been established in situations where the Open Mapping theorem holds, for instance, when E is an F-space [2].

In the context of topological algebras, Husain and Watson showed that an orthogonal basis in a locally m-convex algebra (a setting where the Open Mapping theorem does not necessarily hold), is always a Schauder basis (see [19], Proposition 3.1). Here we extend this result to its "most general" setting, indeed, Theorem III.1, which is our main result in this section, states that an orthogonal basis (an orthogonal M-basis) in any topological algebra is necessarily a Schauder basis (a Schauder M-basis). As we shall see, the proof follows from Proposition I.35 (i) which links the coordinate functionals with the multiplication operation, which is continuous. It is interesting to compare this result of Theorem III.1 with the fact that even in a Banach algebra, an orthogonal basis need not be unconditional (see Examples II.8).

The rest of this section deals with some general facts about topological algebras with orthogonal bases or orthogonal M-bases, which are versions of Theorem 2.1, Lemma 3.2, Theorem 3.3 and Theorem 3.4 in [19] extended to the case of an orthogonal M-basis and improved in the light of Theorem III.1.

III.1 <u>Theorem</u>. An orthogonal basis $\{e_n\}$ (orthogonal M-basis $\{e_\alpha\}_{\alpha\in\Gamma}$) in a topological algebra A is a Schauder basis (Schauder M-basis).

<u>Proof</u>: We give a proof for the case of an orthogonal M-basis. The case of an orthogonal basis is similar. For an arbitrary $\alpha \in \Gamma$ we have $e^*_\alpha(x)e_\alpha = e_\alpha x$ for all $x \in A$ (Proposition I.35 (i)). Since $e_\alpha \neq 0$ and A is Hausdorff, there exists a circled absorbing neighbourhood U of 0 such that $e_\alpha \notin U$. From the continuity of multiplication, there exists a circled absorbing neighbourhood V of 0 with $VV \subset U$. We have $e_\alpha \notin V$ for otherwise $e_\alpha = e_\alpha^2 \in VV \subset U$, a contradiction. Since U is absorbing, the number $\mu_0 = \inf\{\mu > 0: e_\alpha \in \mu U\}$ is finite, and since U is circled and $e_\alpha \notin U$ we have $\mu_0 \geq 1 > 0$. Since V is also absorbing, we can find $\lambda > 0$ with $e_\alpha \in \lambda V$. Now let $\varepsilon > 0$ be given and consider the neighbourhood $W = \frac{\mu_0\varepsilon}{\lambda} V$. We prove the theorem by showing that $|e^*_\alpha(x)| \leq \varepsilon$ for all $x \in W$. Indeed for $x \in W$ we have

$$\frac{e^*_\alpha(x)}{\mu_0\varepsilon} e_\alpha = \frac{1}{\mu_0\varepsilon} xe_\alpha = \frac{\lambda}{\mu_0\varepsilon} x \cdot \frac{1}{\lambda} e_\alpha \in VV \subset U$$

and hence, since U is circled we also have $\frac{|e^*_\alpha(x)|}{\mu_0\varepsilon} e_\alpha \in U$. If $e^*_\alpha(x) \neq 0$ then $e_\alpha \in \frac{\mu_0\varepsilon}{|e^*_\alpha(x)|} U$ and hence $|e^*_\alpha(x)| \leq \varepsilon$ (notice that $\frac{\mu_0\varepsilon}{|e^*_\alpha(x)|} \geq \mu_0$ by the definition of μ_0). This completes the proof since $|e^*_\alpha(x)| \leq \varepsilon$ is trivial if $e^*_\alpha(x) = 0$.

III.2 Notations. Let A be a topological algebra with an orthogonal basis $\{e_n\}$ (orthogonal M-basis $\{\hat{e}_\alpha\}_{\alpha\in\Gamma}$). For each $x \in A$, let $\hat{x}$ be the element in s (in $\mathbb{C}^\Gamma$) given by $\hat{x}(n) = e_n^*(x)$, $n \in \mathbb{N}$. ($\hat{x}(\alpha) = e_\alpha^*(x)$, $\alpha \in \Gamma$). By $\hat{A}$ we mean the set $\{\hat{x}: x \in A\}$ endowed with the relative topology as a topological subspace of s (of $\mathbb{C}^\Gamma$), and by σ we mean the map $x \to \hat{x}$ of A onto $\hat{A}$.

The symbol $\hat{x}$ was used in Theorem I.17 and Definitions I.18 to denote the Gel'fand transform of an element x in a Banach algebra A, which is the map of the maximal ideal space $\Delta(A)$ into $\mathbb{C}$ given by $\hat{x}(h) = h(x)$, $h \in \Delta(A)$. In the case of a Banach algebra with an orthogonal basis (orthogonal M-basis), the two meanings of $\hat{x}$ coincide, as will be shown by Theorem III.3 and Corollary III.4. Throughout the proofs, the symbol $\hat{x}$ will be used as in Notations III.2 and $\phi(x)$ will denote the Gel'fand transform of x, as in Definitions I.18. We write the proof for the case of an orthogonal M-basis. The case of an orthogonal basis is similar.

III.3 Theorem. Let A be a topological algebra with an orthogonal basis $\{e_n\}$ (orthogonal M-basis $\{e_\alpha\}_{\alpha\in\Gamma}$). Then,

(i) $\hat{A}$ is a dense subalgebra of s (of $\mathbb{C}^\Gamma$) and $\sigma: A \to \hat{A}$ is a continuous algebra isomorphism.

(ii) If f is a non-zero multiplicative linear functional on A, then f is continuous iff $f = e_n^*$ for some $n \in \mathbb{N}$ ($f = e_\alpha^*$ for some $\alpha \in \Gamma$). Furthermore,

(iii) If A is functionally continuous (i.e., every multiplicative linear functional on A is continuous [16]) then $f = e_n^\star$ for some $n \in \mathbb{N}$ ($f = e_\alpha^\star$ for some $\alpha \in \Gamma$), and the space $\Delta(A)$ of all multiplicative linear functionals with the Gel'fand topology - which is the topology of pointwise convergence - is homeomorphic with the discrete space $\mathbb{N}$ (the discrete space Γ).

<u>Proof</u>: (i) From Proposition I.35 (ii) we have $(\widehat{xy})(\alpha) = e_\alpha^\star(xy) = e_\alpha^\star(x)e_\alpha^\star(y) = \hat{x}(\alpha)\cdot\hat{y}(\alpha)$ for every $x,y \in A$ and $\alpha \in \Gamma$ and hence $\hat{A}$ is a subalgebra of $\mathbb{C}^\Gamma$ and σ is an algebra homomorphism of A onto $\hat{A}$, and it follows from the uniqueness of the expansion of an element in A in terms of the M-basis that σ is an algebra isomorphism, moreover, $\hat{A}$ is dense in $\mathbb{C}^\Gamma$ since it contains all $t \in \mathbb{C}^\Gamma$ with finite support. For each $\alpha \in \Gamma$ let $\pi_\alpha: \mathbb{C}^\Gamma \to \mathbb{C}$ be the αth projection: $\pi_\alpha(t) = t(\alpha)$, $t \in \mathbb{C}^\Gamma$. Clearly $e_\alpha^\star = \pi_\alpha \circ \sigma$ for all $\alpha \in \Gamma$ and so σ is continuous, since each $e_\alpha^\star$ is continuous (Theorem III.1) and $\mathbb{C}^\Gamma$ has the product topology.

(ii) From Proposition I.35 (i) we have

$$\text{(III.3.1)} \qquad f(x)f(e_\alpha) = f(xe_\alpha) = f(e_\alpha^\star(x)e_\alpha) = e_\alpha^\star(x)\cdot f(e_\alpha)$$

for all $x \in A$ and $\alpha \in \Gamma$. If f is continuous, then $f(e_{\alpha_0}) \neq 0$ for some $\alpha_0 \in \Gamma$, for otherwise $f(x) = f(\sum_{\alpha\in\Gamma} e_\alpha^\star(x)e_\alpha) = \sum_{\alpha\in\Gamma} e_\alpha^\star(x)f(e_\alpha) = 0$ for all $x \in A$, contrary to the hypothesis that f is non-zero. From (III.3.1) with $\alpha = \alpha_0$ we have $f(x) = e_{\alpha_0}^\star(x)$ for all $x \in A$. The converse follows from Theorem III.1.

(iii) That $f = e_\alpha^*$ for some $\alpha \in \Gamma$ follows immediately from part (ii) above. Now we have $\Delta(A) = \{e_\alpha^* : \alpha \in \Gamma\}$, which can be identified with Γ – as sets – via the bijection $e_\alpha^* \to \alpha$. For $\alpha_0, \beta \in \Gamma$ the set

$$V(\alpha_0,\beta,1) = \{\alpha \in \Gamma : |e_{\alpha_0}^*(e_\beta) - e_\alpha^*(e_\beta)| < 1\}$$

is a neighbourhood of α_0 in the topology on Γ transferred from $\Delta(A)$ via the bijection $e_\alpha^* \to \alpha$. Since $e_{\alpha_0}^*(e_\alpha) = \delta_{\alpha_0\alpha}$, we have $V(\alpha_0,\alpha_0,1) = \{\alpha_0\}$ and hence $\Delta(A)$ is homeomorphic with the discrete space Γ.

III.4 Corollary. Let A be a Banach algebra with an orthogonal basis $\{e_n\}$ (an orthogonal M-basis $\{e_\alpha\}_{\alpha\in\Gamma}$). Then $\Delta(A)$ is homeomorphic with the discrete space $\mathbb{N}$ (with the discrete space Γ) and $(\phi(x))(n) = \hat{x}(n)$ for all $n \in \mathbb{N}$ $((\phi(x))(\alpha) = \hat{x}(\alpha)$ for all $\alpha \in \Gamma)$.

Proof. From Theorem I.15 (i) we see that A is functionally continuous and hence by Theorem III.3 (iii), $\Delta(A)$ is homeomorphic with the discrete space Γ. With the identification $e_\alpha^* \to \alpha$ we have

$$\phi(x)(\alpha) = \phi(x)(e_\alpha^*) = e_\alpha^*(x) = \hat{x}(\alpha)$$

for all $\alpha \in \Gamma$ and $x \in A$, which completes the proof of the corollary.

III.5 Remark. Notice that although $\phi(x)$ and $\hat{x}$ are now identified in the Banach algebra case, $\hat{A}_0$ (as in Definition I.18) and $\hat{A}$ are not, since $\hat{A}_0$ carries the norm topology of $C_0(\Delta(A)) \approx C_0(\Gamma)$ (which is the sup norm), while $\hat{A}$ carries the relative topology as a subspace of $\mathbb{C}^\Gamma$ with the product topology which is a strictly weaker topology than that of $\hat{A}_0$ when Γ is infinite. Thus, in the case of a Banach algebra A, the continuity of $\sigma: A \to \hat{A}$ proved in Theorem III.3 (i) can be replaced with the continuity of $\phi: A \to \hat{A}_0$, which is a stronger conclusion.

§2. Locally m-convex algebras with orthogonal bases or orthogonal M-bases.

In [19] Husain and Watson showed that every locally m-convex algebra A with an orthogonal basis and an identity is metrizable. They were able to give a full characterization of such algebras only under the additional condition of completeness, since their method of proof relied on the Open Mapping theorem. Indeed, a locally m-convex algebra A with an identity and an orthogonal basis is metrizable ([19], Theorem 3.3) and under the additional hypothesis of completeness, $\sigma: A \to s$ is onto ([19], Lemma 3.2) and the Open Mapping theorem then ensures that σ (which is already continuous) is a topological isomorphism.

Using a method of proof which avoids the Open Mapping theorem, we show that for a locally m-convex algebra A with an identity e and an orthogonal basis or an orthogonal M-basis, the map $\sigma: A \to \hat{A}$ is a topological isomorphism (Theorem III.6), thus giving a full characterization of such algebras without the assumptions of completeness and metrizability, needed for the application of the Open Mapping theorem. Theorem III.6 and Remarks III.8 are stated and proved for the case of an orthogonal M-basis, however, they are also true (using the same method of proof) with the orthogonal M-basis replaced by an orthogonal basis.

III.6 Theorem. Let A be a locally m-convex algebra with an orthogonal M-basis $\{e_\alpha\}_{\alpha\in\Gamma}$ and a family P of defining submultiplicative seminorms. Consider the following statements:

(i) $\sigma: A \to \mathbb{C}^\Gamma$ is onto.

(ii) A has an identity.

(iii) For every $p \in P$ there exists a finite $J_p \subset \Gamma$ such that $p(e_\alpha) = 0$ for all $\alpha \in \Gamma \backslash J_p$.

(iv) A is topologically isomorphic with a dense subalgebra of $\mathbb{C}^\Gamma$.

(v) A is topologically isomorphic with $\mathbb{C}^\Gamma$.

Then (i) ⇒ (ii) ⇒ (iii) ⇒ (iv). If A is complete, then all the above five statements are equivalent.

Proof: (i) ⇒ (ii) is obvious. If A has an identity e, then by Proposition I.35 (iii) $e = \sum_{\alpha\in\Gamma} e_\alpha$, whose convergence implies that for every $p \in P$ there exists $J_p \in \Omega$ (the set of all finite subsets of Γ) such that $p(e_\alpha) = p(e_\alpha^n) \leq [p(e_\alpha)]^n < (\frac{1}{2})^n$ for every $\alpha \in \Gamma\backslash J_p$ and every $n \in \mathbb{N}$ whence (ii) ⇒ (iii).

(iii) ⇒ (iv): From Theorem III.3 (i) we see that the map $\sigma: A \to \hat{A} \subset \mathbb{C}^\Gamma$ is a 1-1 continuous homomorphism onto a dense subalgebra $\hat{A}$ of $\mathbb{C}^\Gamma$. It remains to show that under (iii) σ is an open map. For $p \in P$ let $I_p = \{\alpha \in \Gamma: p(e_\alpha) \neq 0\}$, then by the hypothesis of (iii) I_p is finite. Let $x \in A$ be arbitrary. Apart from the trivial case $I_p = \phi$ we have

$$p(x) = p(\sum_{\alpha\in\Gamma\setminus I_p} e_\alpha^*(x)e_\alpha + \sum_{\alpha\in I_p} e_\alpha^*(x)e_\alpha) \le 0 + \sum_{\alpha\in I_p} |e_\alpha^*(x)|p(e_\alpha) \le$$

$$\le (|I_p| \max_{\alpha\in I_p} p(e_\alpha)) \max_{\alpha\in I_p}|e_\alpha^*(x)| = \left[|I_p| \max_{\alpha\in I_p}(p(e_\alpha))\right] \max_{\alpha\in I_p}|\hat{x}(\alpha)|$$

where $|I_p|$ is the number of elements in I_p. This proves that σ is an open map, since $\max_{\alpha\in I_p}|\hat{y}(\alpha)|$, $\hat{y}\in\hat{A}$ is a continuous seminorm on $\hat{A}$.

Clearly (v) ⇒ (i), and under the additional hypothesis that A is complete, (iv) ⇒ (v).

III.7 <u>Corollary</u>. Let A be a locally m-convex algebra with an orthogonal M-basis $\{e_\alpha\}_{\alpha\in\Gamma}$ and an identity e. Then A is metrizable iff Γ is countable.

<u>Proof</u>: If Γ is countable, then A is metrizable since it is topologically isomorphic with a subalgebra of $\mathbb{C}^{\mathbb{N}}$ by Theorem III.6. Conversely, if A is metrizable, so is $\hat{A}$. Since $\mathbb{C}^\Gamma$ is the completion of $\hat{A}$, $\mathbb{C}^\Gamma$ is also metrizable and consequently Γ is countable.

III.8 <u>Remarks</u>. (i) Let A be a locally m-convex algebra with an orthogonal M-basis $\{e_\alpha\}_{\alpha\in\Gamma}$. From Theorem III.3 (i) we see that A has a 1-1 continuous image in $\mathbb{C}^\Gamma$ which – under the additional hypothesis of the existence of an identity in A – becomes a homeomorphic image by Theorem III.6. This shows that the product topology "is" the coarsest locally m-convex topology on a topological algebra with an orthogonal M-basis and, this coarsest topology is attained when A has an identity.

(ii) Let A be a topological algebra with an orthogonal M-basis $\{e_\alpha\}_{\alpha\in\Gamma}$ and identity e. If τ is a locally m-convex topology on A coarser than the original topology, then for every $x \in A$, the convergence of $\sum_{\alpha\in\Gamma} e_\alpha^*(x)e_\alpha$ to x for the original topology implies its convergence to x for the coarser topology τ and so $\{e_\alpha\}_{\alpha\in\Gamma}$ remains an orthogonal M-basis when A is endowed with τ. It follows from Theorem III.6 that (A,τ) is topologically isomorphic with a subalgebra of $\mathbb{C}^\Gamma$. This, in the light of Theorem III.3 (i), shows that there is exactly one locally m-convex topology on A coarser than its original topology and this locally m-convex topology "is" the product topology on $\mathbb{C}^\Gamma$. Compare this with Lemma 3.1 in [4] which establishes the existence of a finest locally m-convex topology coarser than a given A-convex topology.

(iii) Theorem III.6 can be used to show that for an infinite Γ the algebra $C_\beta^*(\Gamma)$ of Example I.41 is not locally m-convex. Indeed, if $C_\beta^*(\Gamma)$ were locally m-convex then, since it is also complete and has an identity, it would follow that $\sigma: C_\beta^*(\Gamma) \to \mathbb{C}^\Gamma$ is onto, which is not the case.

§3. Φ-topologies and Φ-algebras with orthogonal M-bases.

For an arbitrary set Γ, the topology on $\mathbb{C}^\Gamma$ is generated by the family of seminorms $p_\phi(x) = \sup_{\alpha\in\Gamma}|\phi(\alpha)x(\alpha)|$, $x \in \mathbb{C}^\Gamma$, where ϕ ranges over the set $C_{00}^+(\Gamma)$ of all (non-negative) functions on Γ with finite support. Now let A be an (algebraic) vector subspace of $\mathbb{C}^\Gamma$ and let Φ be a family of non-negative functions on Γ such that for every $x \in E$ and $\phi \in \Phi$

$$p_\phi(x) = \sup_{\alpha\in\Gamma}|\phi(\alpha)x(\alpha)| < \infty .$$

Clearly each p_ϕ is a seminorm on A. We shall assume that A contains all the coordinate unit vectors e_α, $\alpha \in \Gamma$ and to ensure that the locally convex topology defined by p_ϕ, $\phi \in \Phi$ is Hausdorff, we assume that $C_{00}(\Gamma) \subset \Phi$. Any subfamily $\Psi \subset \Phi$ which defines the same topology will be called a defining family. It is easy to see that $\Psi \subset \Phi$ is defining iff for each $\phi \in \Phi$ there exists $M > 0$ and $\psi_1, \ldots, \psi_n \in \Psi$ such that $\phi(\alpha) \le M \max_{1\le k\le n} \psi_k(\alpha)$ for all $\alpha \in \Gamma$. Clearly, the topology defined by Φ is metrizable iff a countable defining subfamily Ψ exists. Throughout this section, Φ will be assumed to contain every non-negative $\lambda: \Gamma \to \mathbb{C}$ such that $p_\lambda(x) = \sup_{\alpha\in\Gamma}|\lambda(x)x(\alpha)|$ defines a seminorm on $\mathbb{C}^\Gamma$ continuous for the topology generated by Φ.

III.9 Proposition. Let A, Φ be as above, then $\{e_\alpha\}_{\alpha\in\Gamma}$ is an M-basis in A iff for every $x \in A$, $\phi \in \Phi$ and $\varepsilon > 0$ there exists a finite $J = J(\varepsilon,\phi,x) \subset \Gamma$ such that $|\phi(\alpha)x(\alpha)| < \varepsilon$ for all $\alpha \in \Gamma\setminus J$. In this case, $e_\alpha^*(x) = x(\alpha)$, $\alpha \in \Gamma$.

Proof: It is clear, since

$$p_\alpha(x - \sum_{\alpha\in J} x(\alpha)e_\alpha) = \sup_{\alpha\in\Gamma\setminus J}|\phi(\alpha)x(\alpha)|$$

for $x \in A$, $\phi \in \Phi$ and $J \in \Omega$ (the set of all finite subsets of Γ).

III.10 Theorem. Let A be an (algebraic) subalgebra of $\mathbb{C}^\Gamma$ and assume that the family Φ generates a topology on A. Then A is a topological algebra (i.e., multiplication is continuous) for the topology generated by Φ iff $\sqrt{\phi} \in \Phi$ whenever $\phi \in \Phi$.

Proof: If the multiplication is continuous, then for $\phi \in \Phi$ there exists $\psi \in \Phi$ such that $\sup_{\alpha\in\Gamma}|\phi(\alpha)x(\alpha)y(\alpha)| \le \sup_{\alpha\in\Gamma}|\psi(\alpha)x(\alpha)|\cdot\sup_{\alpha\in\Gamma}|\psi(\alpha)y(\alpha)|$ for all $x,y \in A$. For $x = y = e_\beta$, $\beta \in \Gamma$ we get $\phi(\beta) \le (\psi(\beta))^2$ and so $\sqrt{\phi} \le \psi$, whence $\sqrt{\phi} \in \Phi$. Conversely, if $\sqrt{\phi} \in \Phi$ whenever $\phi \in \Phi$ then for $x,y \in A$ we have

$$p_\phi(xy) = \sup_{\alpha\in\Gamma}|\sqrt{\phi(\alpha)}x(\alpha)\sqrt{\phi(\alpha)}y(\alpha)| \le \sup_{\alpha\in\Gamma}|\sqrt{\phi(\alpha)}x(\alpha)|\cdot\sup|\sqrt{\phi(\alpha)}y(\alpha)| = p_{\sqrt{\phi}}(x)\cdot p_{\sqrt{\phi}}(y)$$

and so the multiplication is continuous.

III.11 Definition. Let A be a topological algebra with an orthogonal M-basis $\{e_\alpha\}_{\alpha\in\Gamma}$ and consider the isomorphism $\sigma: A \to \hat{A}$. If $\hat{A}$ can be given a topology generated by a family Φ of non-negative functions on Γ via the seminorms

$$p_\phi(\hat{x}) = \sup_{\alpha\in\Gamma}|\phi(\alpha)\hat{x}(\alpha)| \quad ; \quad \hat{x} \in \hat{A}, \quad \phi \in \Phi$$

so as to make $\hat{A}$ into a topological algebra $\tilde{A}$ with $\sigma: A \to \tilde{A}$ as a topological isomorphism, we call A a Φ-algebra and the topology of A a Φ-topology.

III.12 Examples.

(i) The algebra $\mathbb{C}^{\Gamma}$ is a Φ-algebra with $\Phi = C_{00}(\Gamma)$, the set of all non-negative functions on Γ with finite support.

(ii) The algebra $C_{\beta}^{*}(\Gamma)$ of Example I.41 is a Φ-algebra with Φ as the set of all non-negative functions vanishing at ∞ on the discrete space Γ.

(iii) Let $\mathbb{D}$ be the open unit disc in the complex plane and let $H(\mathbb{D})$ be the vector space of all holomorphic functions on $\mathbb{D}$ with pointwise addition and scalar multiplication. With the compact-open topology (the topology of uniform convergence on compact subsets of $\mathbb{D}$), $H(\mathbb{D})$ is a Fréchet space. For $f \in H(\mathbb{D})$, the convergence of the Taylor series $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$ to $f(z)$ in the compact-open topology shows that the elements $e_n \in H(\mathbb{D})$ given by $e_n(z) = z^n$, $n = 0, 1, 2, \ldots$ form a basis in $H(\mathbb{D})$ and $e_n^{*}(f) = \frac{f^{(n)}(0)}{n!}$. The Hadamard multiplication in $H(\mathbb{D})$ is defined by

$$(fg)(z) = \frac{1}{2\pi i} \int_{|u|=r} f(u)g(zu^{-1})u^{-1}du , \quad z \in \mathbb{D} , \quad |z| = r < 1$$

or equivalently by

$$(fg)(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \frac{g^{(n)}(0)}{n!} z^n$$

and with this multiplication, $H(\mathbb{D})$ is a B_0 algebra (see [30]). The second expression for multiplication shows that the basis $\{e_n\}$ is orthogonal. The element $e \in H(\mathbb{D})$ given by $e(z) = \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$, $z \in \mathbb{D}$ is an identity in $H(\mathbb{D})$. An equivalent family of seminorms on $H(\mathbb{D})$ is given by

$$\text{(III.12.1)} \qquad \|f\|_r = \sup_n \left|\frac{f^{(n)}(0)}{n!} r^n\right| = \sup_n |e_n^*(f) r^n| \quad , \quad 0 < r < 1$$

([26], page 45) which can easily be seen to be a family of squarely submultiplicative seminorms and hence $H(\mathbb{D})$ is a B_0 s-algebra. From Theorem II.12 we see that the orthogonal basis $\{e_n\}$ is unconditional and hence an orthogonal M-basis. It is also clear from (III.12.1) that $H(\mathbb{D})$ is a Φ-algebra with $\Phi = \{\phi_r : 0 < r < 1\}$, where $\phi_r(n) = r^n$, $n \in \mathbb{N} \cup \{0\}$.

Notice also that the countable family of seminorms

$$\text{(III.12.2)} \qquad \|f\|_k = \sup_n \left|\frac{f^{(n)}(0)}{n!} 2^{-\frac{n}{k}}\right| \quad , \quad k = 1, 2, \ldots$$

defines the topology of $H(\mathbb{D})$. Since $H(\mathbb{D})$ is complete, metrizable and locally convex it is barrelled, by Theorem I.3. Hence, if $H(\mathbb{D})$ were A-convex, it would be locally m-convex, by Theorem I.12, but then Theorem III.6 would imply that $\sigma: H(\mathbb{D}) \to \mathbb{C}^{\mathbb{N} \cup \{0\}}$ is onto, which is not the case. Thus $H(\mathbb{D})$ is not A-convex.

III.13 Theorem. Let A be a locally convex algebra with an orthogonal M-basis $\{e_\alpha\}_{\alpha\in\Gamma}$ and an identity e. The following two statements are equivalent:

(i) A is a Φ-algebra.

(ii) A has a defining family P of seminorms satisfying the following condition: For every $x,y \in A$ and every $p,q \in P$ we have $p(x) \le q(y)$ whenever $p(e_\alpha^*(x)e_\alpha) \le q(e_\alpha^*(y)e_\alpha)$ for all $\alpha \in \Gamma$

Proof: (i) $\Rightarrow$ (ii): The family of seminorms $p(x) = \sup_{\alpha\in\Gamma}|\phi(\alpha)\hat{x}(\alpha)| = \sup_{\alpha\in\Gamma}|\phi(\alpha)e_\alpha^*(x)|$, $\phi \in \Phi$ define an equivalent family of seminorms on A via the topological isomorphism $\sigma: x \to \hat{x}$. For $x \in A$, $\phi \in \Phi$ and $\beta \in \Gamma$ we have

$$p_\phi(e_\beta^*(x)e_\beta) = \sup_{\alpha\in\Gamma}|\phi(\alpha)e_\alpha^*(e_\beta^*(x)e_\beta)| = |\phi(\beta)e_\beta^*(x)|$$

(since $e_\alpha^*(e_\beta) = \delta_{\alpha\beta}$). Hence for $x,y \in A$ and $\phi,\psi \in \Phi$: $p_\phi(e_\alpha^*(x)e_\alpha) \le p_\psi(e_\alpha^*(y)e_\alpha)$ for all $\alpha \in \Gamma \Rightarrow |\phi(\alpha)e_\alpha^*(x)| \le |\psi(\alpha)e_\alpha^*(y)|$ for all $\alpha \in \Gamma \Rightarrow p_\phi(x) = \sup_{\alpha\in\Gamma}|\phi(\alpha)e_\alpha^*(x)| \le \sup_{\alpha\in\Gamma}|\psi(\alpha)e_\alpha^*(y)| = p_\psi(y)$.

(ii) $\Rightarrow$ (i): From the convergence of $\sum_{\alpha\in\Gamma} e_\alpha^*(x)e_\alpha$ for every $x \in A$ we have $\sup_{\alpha\in\Gamma}|e_\alpha^*(x)p(e_\alpha)| = \sup_{\alpha\in\Gamma} p(e_\alpha^*(x)e_\alpha) < \infty$ for all $x \in A$ and $p \in P$. Setting $\phi_p(\alpha) = p(e_\alpha)$, $\alpha \in \Gamma$ and $\Phi = \{\phi_p : p \in P\}$, the family Φ of non-negative functions on Γ define a topology on $\hat{A}$ by the seminorms

$$\|\hat{x}\|_p = \sup_{\alpha\in\Gamma}|\phi_p(\alpha)\hat{x}(\alpha)| = \sup_{\alpha\in\Gamma}|p(e_\alpha)e_\alpha^*(x)| \quad , \quad p \in P$$

Let $\tilde{A}$ be $\hat{A}$ with this topology. We show that under the hypothesis of (ii), $\sigma: A \to \tilde{A}$ is a homeomorphism. For $x \in A$, $\alpha \in \Gamma$ and $p \in P$ put $x_\alpha = e^*_\alpha(x)e_\alpha$ then clearly $p(e^*_\beta(x_\alpha)e_\beta) = 0$ for $\beta \neq \alpha$ and $p(e^*_\alpha(x_\alpha)e_\alpha) = p(e^*_\alpha(x)e_\alpha)$ and so $p(e^*_\beta(x_\alpha)e_\beta) \le p(e^*_\beta(x)e_\beta)$ for all $\beta \in \Gamma$. It follows from the hypothesis of (ii) that

$$|\hat{x}(\alpha)\phi_p(\alpha)| = |e^*_\alpha(x)p(e_\alpha)| = p(e^*_\alpha(x)e_\alpha) = p(x_\alpha) \le p(x)$$

for all $\alpha \in \Gamma$ and hence $\|\hat{x}\|_p = \sup_{\alpha\in\Gamma}|\phi_p(\alpha)\hat{x}(\alpha)| \le p(x)$. This shows that $\sigma: A \to \tilde{A}$ is continuous. It remains to show that $\sigma^{-1}: \tilde{A} \to A$ is continuous. Let $p \in P$ and find $q \in P$ such that $p(xy) \le q(x)q(y)$ for all $x,y \in A$. We show that there exists $M > 0$ such that $p(x) \le M$ whenever $\|\hat{x}\|_q \le 1$ and this will complete the proof. Let $\hat{x} \in \tilde{A}$ with $\|\hat{x}\|_q \le 1$, then

$$q(e^*_\alpha(x)e_\alpha) = |e^*_\alpha(x)q(e_\alpha)| = |\phi_q(\alpha)\hat{x}(\alpha)| \le \sup_{\beta\in\Gamma}|\phi_q(\beta)\hat{x}(\beta)| = \|\hat{x}\|_q \le 1$$

and hence

$$p(e^*_\alpha(x)e_\alpha) = |e^*_\alpha(x)p(e_\alpha)| = |e^*_\alpha(x)p(e^2_\alpha)| \le |e^*_\alpha(x)q(e_\alpha)|q(e_\alpha) \le q(e_\alpha)$$

for all $\alpha \in \Gamma$. Since $e^*_\alpha(e) = 1$ for all $\alpha \in \Gamma$, it follows that

$$p(e^*_\alpha(x)e_\alpha) \le q(e^*_\alpha(e)e_\alpha)$$

for all $\alpha \in \Gamma$. It follows from the hypothesis of (ii) that $p(x) \leq q(e)$. Take $M = q(e)$. [Notice that $q(e) \neq 0$, for otherwise $p(x) = p(xe) \leq$ $\leq q(e)q(x) = 0$ for all $x \in A$] .

III.14 Remark. The norm of the Banach algebra $\ell_1(\Gamma)$ satisfies the condition in (ii) Theorem III.13. Indeed, if $\|e_\alpha^*(x)e_\alpha\| \leq \|e_\alpha^*(y)e_\alpha\|$ for all $\alpha \in \Gamma$, then $|e_\alpha^*(x)| = \|e_\alpha^*(x)e_\alpha\| \leq \|e_\alpha^*(y)e_\alpha\| = |e_\alpha^*(y)|$ for all $\alpha \in \Gamma$ and so $\|x\| = \sum_{\alpha\in\Gamma} |e_\alpha^*(x)| \leq \sum_{\alpha\in\Gamma} |e_\alpha^*(y)| = \|y\|$. But, for an infinite Γ, $\ell_1(\Gamma)$ is not a ϕ-algebra, for if ϕ is a non-negative function on Γ such that $\|\hat{x}\|_\phi = \sup_{\alpha\in\Gamma} |\hat{x}(\alpha)\phi(\alpha)|$ is a norm on $\widehat{\ell_1(\Gamma)}$ making $\sigma: \ell_1(\Gamma) \to \widetilde{\ell_1(\Gamma)}$ a topological isomorphism, then there exists $K > 0$ such that $\phi(\alpha) = \|\hat{e}_\alpha\|_\phi \leq K$ for all $\alpha \in \Gamma$, since the M-basis $\{e_\alpha\}_{\alpha\in\Gamma}$ is bounded in $\ell_1(\Gamma)$ ($\|e_\alpha\| = 1$ for all $\alpha \in \Gamma$) . It follows that for any element of the form $e_J = \sum_{\alpha\in J} e_\alpha$ where J is a finite subset of Γ we have $\|\hat{e}_J\|_\phi = \max_{\alpha\in J} \phi(\alpha) \leq K$, a contradiction since $\{e_J : J \subset \Gamma ,\ J$ is finite$\}$ is not bounded in $\ell_1(\Gamma)$. Notice that $\ell_1(\Gamma)$ does not have an identity. The author, however, does not know of an example of a locally convex algebra with an orthogonal M-basis and an identity where condition (ii) (or equivalently, condition (i)) of Theorem III.13 is not satisfied.

§4. A-convex algebras and locally convex s-algebras with orthogonal M-bases.

This section includes some results about A-convex algebras and locally convex s-algebras with orthogonal M-bases, and a characterization of the algebra s of all sequences as a B_0 algebra with an orthogonal M-basis. First we have:

III.15 Proposition. Let A be a locally convex algebra with an orthogonal M-basis $\{e_\alpha\}_{\alpha\in\Gamma}$ and an identity e. If A is metrizable, then Γ is countable.

Proof: Let P be a defining family of seminorms. The convergence of $\sum_{\alpha\in\Gamma} e_\alpha = e$ implies that for $p \in P$ and $k \in \mathbb{N}$ there exists a finite $J_p(k) \subset \Gamma$ such that $p(e_\alpha) < \frac{1}{k}$ for all $\alpha \in \Gamma\setminus J_p(k)$ and so $\{\alpha \in \Gamma: p(e_\alpha) \geq \frac{1}{k}\} \subset J_p(k)$. Let $J_p = \{\alpha \in \Gamma: p(e_\alpha) \neq 0\}$ then clearly $J_p = \bigcup_{k\in\mathbb{N}} \{\alpha \in \Gamma: p(e_\alpha) \geq \frac{1}{k}\} \subset \bigcup_{k\in\mathbb{N}} J_p(k)$ and hence J_p is countable since each $J_p(k)$ is finite. Since A is Hausdorff, each $\alpha \in \Gamma$ belongs to some J_p and so $\Gamma \subset \bigcup_{p\in P} J_p$. If A is metrizable, a countable defining family P can be found and hence Γ is countable.

III.16 Remarks. (i) The existence of an identity in Proposition III.15 is essential, indeed, for an arbitrary Γ, $\ell_1(\Gamma)$ is a Banach algebra and hence metrizable.

(ii) The converse of Proposition III.15 is not, in general, true. Indeed, the algebra $C_\beta^*(\Gamma)$ (Example I.41) is not metrizable, if Γ is infinite, even when Γ is countable.

The proof of the following theorem is an adaptation of the proof of Theorem 16.1 (b) in [35] (proved there for Banach spaces) to our purposes.

III.17 Theorem. Let E be a locally convex space with an M-basis $\{e_\alpha\}_{\alpha\in\Gamma}$ and a family $\{\|\cdot\|_\mu: \mu \in M\}$ of defining seminorms. For every $x \in E$ and $\mu \in M$ put $\|x\|'_\mu = \sup_{f\in\Lambda_\mu} \sum_{\alpha\in\Gamma} |e^*_\alpha(x)|\cdot|f(e_\alpha)|$ and $\|x\|''_\mu = \sup_{J\in\Omega} \|\sum_{\alpha\in J} e^*_\alpha(x)e_\alpha\|_\mu$ where $\Lambda_\mu = \{f \in E': |f(x)| \le \|x\|_\mu$ for all $x \in E\}$ and, as usual, $\Omega = \{J \subset \Gamma: J$ is finite$\}$. Then each of $\{\|\cdot\|': \mu \in M\}$ and $\{\|\cdot\|'': \mu \in M\}$ is a family of seminorms on E. If τ' and τ'' are the topologies defined by these families, respectively, and if τ is the original topology generated by $\{\|\cdot\|_\mu: \mu \in M\}$, then $\tau \subset \tau' = \tau''$.

Proof: Let $x \in E$, $\mu \in M$ and $J \in \Omega$. By a corollary to the Hahn-Banach theorem (Corollary I.2), there exists $g \in \Lambda_\mu$ such that $g(\sum_{\alpha\in J} e^*_\alpha(x)e_\alpha) = \|\sum_{\alpha\in J} e^*_\alpha(x)e_\alpha\|_\mu$, but then we have

$$\|\sum_{\alpha\in J} e^*_\alpha(x)e_\alpha\|_\mu = |g(\sum_{\alpha\in J} e^*_\alpha(x)e_\alpha)| \le \sum_{\alpha\in J} |e^*_\alpha(x)|\cdot|g(e_\alpha)| \le$$

$$\le \sum_{\alpha\in\Gamma} |e^*_\alpha(x)|\cdot|g(e_\alpha)| \le \sup_{f\in\Lambda_\mu} \sum_{\alpha\in\Gamma} |e^*_\alpha(x)|\cdot|f(e_\alpha)|$$

and hence

(III.17.1) $$\sup_{J\in\Omega} \|\sum_{\alpha\in J} e^*_\alpha(x)e_\alpha\|_\mu \le \sup_{f\in\Lambda_\mu} \sum_{\alpha\in\Gamma} |e^*_\alpha(x)|\cdot|f(e_\alpha)|$$

For $f \in \Lambda_\mu$ set $\theta_\alpha = \mathrm{sign}(e^*_\alpha(x)f(e_\alpha))$, $\alpha \in \Gamma$ then for $J \in \Omega$ we have

$$\sum_{\alpha\in J}|e_\alpha^*(x)|\cdot|f(e_\alpha)| = |\sum_{\alpha\in J}\theta_\alpha e_\alpha^*(x)f(e_\alpha)| =$$

$$= |f(\sum_{\alpha\in J}\theta_\alpha e_\alpha^*(x)e_\alpha)| \le \|\sum_{\alpha\in J}\theta_\alpha e_\alpha^*(x)e_\alpha\|_\mu .$$

Let $\overline{\mathbb{D}}$ be the closed unit disc in $\mathbb{C}$ and put $B = \overline{\mathbb{D}}^\Gamma = \{b = (b_\alpha)_{\alpha\in\Gamma}: |b_\alpha| \le 1\}$ then $(\theta_\alpha)_{\alpha\in\Gamma} \in B$ and hence by taking the supremum over all $J \in \Omega$ on the leftmost handside and the supremum over all $J \in \Omega$ and $b \in B$ on the rightmost handside we get

$$\sum_{\alpha\in\Gamma}|e_\alpha^*(x)|\cdot|f(e_\alpha)| = \sup_{J\in\Omega}\sum_{\alpha\in J}|e_\alpha^*(x)|\cdot|f(e_\alpha)| \le \sup_{b\in B, J\in\Omega}\|\sum_{\alpha\in J}b_\alpha e_\alpha^*(x)e_\alpha\|_\mu$$

and consequently

(III.17.2) $$\sup_{f\in\Lambda_\mu}\sum_{\alpha\in\Gamma}|e_\alpha^*(x)|\cdot|f(e_\alpha)| \le \sup_{f\in B, J\in\Omega}\|\sum_{\alpha\in J}b_\alpha e_\alpha^*(x)e_\alpha\|_\mu$$

Next, let $b \in B$ and $I \in \Omega$. Then for every $\alpha \in \Gamma$ we have $|r_\alpha| \le 1$, $|t_\alpha| \le 1$, where $r_\alpha = \mathrm{Re}\, b_\alpha$, $t_\alpha = \mathrm{Im}\, b_\alpha$ and,

(III.17.3) $$\|\sum_{\alpha\in I}b_\alpha e_\alpha^*(x)e_\alpha\|_\mu \le \|\sum_{\alpha\in I}r_\alpha e_\alpha^*(x)e_\alpha\|_\mu + \|\sum_{\alpha\in I}t_\alpha e_\alpha^*(x)e_\alpha\|_\mu .$$

Put $n = |I|$ (the number of elements in I). Since $(r_\alpha)_{\alpha\in I}$ is in the unit cell of $\mathbb{R}^I$, which is the convex hull of its 2^n extreme points $(u_\alpha^{(k)})_{\alpha\in I}$, $k = 1, 2, \ldots, 2^n$, $u_\alpha^{(k)} = \pm 1$, there exists $\lambda^{(k)} \geq 0$, $k = 1, 2, \ldots, 2^n$, $\sum_{k=1}^{2^n} \lambda^{(k)} = 1$ such that $(r_\alpha)_{\alpha\in I} = \sum_{k=1}^{2^n} \lambda^{(k)} (u_\alpha^{(k)})_{\alpha\in I}$ and so $r_\alpha = \sum_{k=1}^{2^n} \lambda^{(k)} u_\alpha^{(k)}$, $\alpha \in I$. Hence,

$$\left\| \sum_{\alpha\in I} r_\alpha e_\alpha^*(x) e_\alpha \right\|_\mu = \left\| \sum_{\alpha\in I} \left(\sum_{k=1}^{2^n} \lambda^{(k)} u_\alpha^{(k)} \right) e_\alpha^*(x) e_\alpha \right\|_\mu =$$

$$= \left\| \sum_{k=1}^{2^n} \lambda^{(k)} \sum_{\alpha\in I} u_\alpha^{(k)} e_\alpha^*(x) e_\alpha \right\|_\mu \leq \sum_{k=1}^{2^n} \lambda^{(k)} \left\| \sum_{\alpha\in I} u_\alpha^{(k)} e_\alpha^*(x) e_\alpha \right\|_\mu \leq$$

$$\leq \max_{1\leq k\leq 2^n} \left\| \sum_{\alpha\in I} u_\alpha^{(k)} e_\alpha^*(x) e_\alpha \right\|_\mu \leq \sup_{u\in U} \left\| \sum_{\alpha\in I} u_\alpha e_\alpha^*(x) e_\alpha \right\|_\mu$$

where $U = \{-1,+1\}^\Gamma = \{u = (u_\alpha)_{\alpha\in\Gamma} : u_\alpha = \pm 1\}$ (Notice that the last inequality is an equality since for any $u \in U$ only u_α, $\alpha \in I$ affect the sum $\sum_{\alpha\in I} u_\alpha e_\alpha^*(x) e_\alpha$ on the right side of the inequality, and the maximum on the left side is taken over all the 2^n elements in $\{-1,+1\}^I$). It follows from the last inequality that

$$\left\| \sum_{\alpha\in I} r_\alpha e_\alpha^*(x) e_\alpha \right\|_\mu \leq \sup_{u\in U} \left\| \sum_{\alpha\in I} u_\alpha e_\alpha^*(x) e_\alpha \right\|_\mu \leq \sup_{u\in U, J\in\Omega} \left\| \sum_{\alpha\in J} u_\alpha e_\alpha^*(x) e_\alpha \right\|_\mu .$$

Using a similar argument for the second term in (III.17.3) we deduce from (III.17.3) that

(III.17.4) $$\|\sum_{\alpha\in I} b_\alpha e^*_\alpha(x)e_\alpha\|_\mu \le 2 \sup_{u\in U, J\in\Omega} \|\sum_{\alpha\in J} u_\alpha e^*_\alpha(x)e_\alpha\|_\mu$$

Now for $u \in U$ and $J \in \Omega$ put $J^-_u = \{\alpha \in J: u_\alpha = -1\}$ and $J^+_u = \{\alpha \in J: u_\alpha = +1\}$ then

$$\|\sum_{\alpha\in J} u_\alpha e^*_\alpha(x)e_\alpha\|_\mu = \|\sum_{\alpha\in J^+_u} e^*_\alpha(x)e_\alpha - \sum_{\alpha\in J^-_u} e^*_\alpha(x)e_\alpha\|_\mu \le$$

$$\le \|\sum_{\alpha\in J^+_u} e^*_\alpha(x)e_\alpha\|_\mu + \|\sum_{\alpha\in J^-_u} e^*_\alpha(x)e_\alpha\|_\mu \le 2 \sup_{J'\in\Omega} \|\sum_{\alpha\in J'} e^*_\alpha(x)e_\alpha\|_\mu .$$

This, together with (III.17.4) implies

$$\|\sum_{\alpha\in I} b_\alpha e^*_\alpha(x)e_\alpha\|_\mu \le 4 \sup_{J\in\Omega} \|\sum_{\alpha\in J} e^*_\alpha(x)e_\alpha\|_\mu$$

and consequently,

$$\sup_{b\in B, J\in\Omega} \|\sum_{\alpha\in J} b_\alpha e^*_\alpha(x)e_\alpha\|_\mu \le 4 \sup_{J\in\Omega} \|\sum_{\alpha\in J} e^*_\alpha(x)e_\alpha\|_\mu .$$

From the last inequality and (III.17.1), (III.17.2) we get

(III.17.5) $$\|x\|''_\mu \le \|x\|'_\mu \le 4\|x\|''_\mu .$$

From the convergence of $\sum_{\alpha\in\Gamma} e_\alpha^*(x)e_\alpha$ to x for the topology τ we have $\|x\|_\mu'' = \sup_{J\in\Omega} \|\sum_{\alpha\in J} e_\alpha^*(x)e_\alpha\|_\mu < \infty$ and $\|x\|_\mu \le \|x\|_\mu''$. This completes the proof since, clearly, $\|\cdot\|_\mu'$ and $\|\cdot\|_\mu''$, $\mu \in M$ are seminorms.

III.18 <u>Corollary</u>. If the space E in Theorem III.17 is a Fréchet space, then $\tau = \tau' = \tau''$ and $\{e_\alpha\}_{\alpha\in\Gamma}$ is a Schauder M-basis.

<u>Proof</u>: First, we notice that each e_α^* is τ''-continuous. This follows since for $\mu \in M$ and $x \in E$ we have

$$|e_\alpha^*(x)|\ \|e_\alpha\|_\mu = \|e_\alpha^*(x)e_\alpha\|_\mu \le \sup_{J\in\Omega} \|\sum_{\alpha\in J} e_\alpha^*(x)e_\alpha\|_\mu = \|x\|_\mu''$$

and hence $|e_\alpha^*(x)| \le \|e_\alpha\|_{\mu_0}^{-1} \|x\|_\mu''$, where $\mu_0 \in M$ is such that $\|e_\alpha\|_\mu \neq 0$. We complete the proof by showing that $\tau = \tau''$. This will follow from Corollary I.8 of the Open Mapping theorem once we show that (E,τ'') is complete and metrizable. The metrizability of (E,τ'') follows from that of (E,τ) by choosing the family $\{\|\cdot\|_\mu : \mu \in M\}$ to be countable. To show that (E,τ'') is complete, let $\{y_n\}$ be a Cauchy sequence in (E,τ''). Since each e_α^* is τ''-continuous, $\{e_\alpha^*(y_n)\}_n$ is a Cauchy sequence in $\mathbb{C}$, for each $\alpha \in \Gamma$. Set $t_\alpha = \lim_n e_\alpha^*(y_n)$, $\alpha \in \Gamma$. We prove that $\sum_{\alpha\in\Gamma} t_\alpha e_\alpha$ converges in (E,τ'') to some y and $y = \lim y_n$ in (E,τ''). Let $\mu \in M$ and let $\varepsilon > 0$ be given. Since $\{y_n\}$ is Cauchy in (E,τ''), there exists $N \in \mathbb{N}$ such that

$$\left\| \sum_{\alpha\in J} [e_\alpha^*(y_n) - e_\alpha^*(y_m)] e_\alpha \right\|_\mu < \varepsilon$$

for all $J \in \Omega$ and $m > n \geq N$. Letting $m \to \infty$ we get

$$\text{(III.18.1)} \qquad \left\| \sum_{\alpha\in J} e_\alpha^*(y_n) e_\alpha - \sum_{\alpha\in J} t_\alpha e_\alpha \right\|_\mu \leq \varepsilon$$

for all $J \in \Omega$ and $n \geq N$. For a fixed $n \geq N$ there exists $J_0 \in \Omega$ such that $\| \sum_{\alpha\in J} e_\alpha^*(y_n) e_\alpha \|_\mu \leq \varepsilon$ for every $J \in \Omega$ with $J \cap J_0 = \phi$ and hence, in view of (III.18.1) we have

$$\left\| \sum_{\alpha\in J} t_\alpha e_\alpha \right\|_\mu \leq \left\| \sum_{\alpha\in J} e_\alpha^*(y_n) e_\alpha - \sum_{\alpha\in J} t_\alpha e_\alpha \right\|_\mu + \left\| \sum_{\alpha\in J} e_\alpha^*(y_n) e_\alpha \right\|_\mu \leq 2\varepsilon$$

for all $J \in \Omega$ with $J \cap J_0 = \phi$. Since $\mu \in M$ is arbitrary, the net $\{ \sum_{\alpha\in J} t_\alpha e_\alpha \}_{J\in\Omega}$ is a Cauchy net in (E,τ) and hence, is convergent. Set $y = \sum_{\alpha\in\Gamma} t_\alpha e_\alpha = \lim_{J\in\Omega} \sum_{\alpha\in J} t_\alpha e_\alpha$. From (III.18.1) we have

$$\| y_n - y \|''_\mu = \sup_{J\in\Omega} \left\| \sum_{\alpha\in J} e_\alpha^*(y_n) e_\alpha - \sum_{\alpha\in J} t_\alpha e_\alpha \right\|_\mu \leq \varepsilon$$

for all $n \geq N$. Hence $y = \lim y_n$ in (E,τ''). The proof is now complete.

The same conclusion as in Corollary III.18 can be obtained in the topological algebra setting under a different hypothesis without the Open Mapping theorem. Indeed we have

III.19 Corollary. If E in Theorem III.17 is replaced by a locally convex s-algebra with $\{\|\cdot\|_\mu : \mu \in M\}$ as a family of squarely submultiplicative seminorms, then $\tau = \tau' = \tau''$. (Notice that $\{e_\alpha\}_{\alpha\in\Gamma}$ is already a Schauder M-basis by Theorem III.1).

Proof: Again, we show that $\tau = \tau''$. Let $\mu \in M$, $x \in A$ and let $\varepsilon > 0$ be given. Since $\|x\|_\mu = \lim_{J\in\Omega} \|\sum_{\alpha\in J} e_\alpha^*(x)e_\alpha\|_\mu$, there exists $J_0 \in \Omega$ such that

(III.19.1) $$\|\sum_{\alpha\in J} e_\alpha^*(x)e_\alpha\|_\mu < \varepsilon + \|x\|_\mu$$

for all $J \in \Omega$ with $J_0 \subset J$. In the same way as in (II.12.2) (Theorem II.12) we can show that

(III.19.2) $$\|\sum_{\alpha\in I} e_\alpha^*(x)e_\alpha\|_\mu \le \|\sum_{\alpha\in I\cup J_0} e_\alpha^*(x)e_\alpha\|_\mu$$

for all $I \in \Omega$. From (III.19.1) and (III.19.2) we obtain

$$\|\sum_{\alpha\in I} e_\alpha^*(x)e_\alpha\|_\mu < \varepsilon + \|x\|_\mu$$

for all $I \in \Omega$ and consequently

$$\|x\|''_\mu = \sup_{I\in\Omega} \|\sum_{\alpha\in I} e^*_\alpha(x)e_\alpha\|_\mu < \varepsilon + \|x\|_\mu .$$

Since $\varepsilon > 0$ is arbitrary, we have $\|x\|''_\mu \le \|x\|_\mu$. This completes the proof since we already have $\|x\|_\mu \le \|x\|''_\mu$.

III.20 Proposition. Let A be an A-convex algebra with an orthogonal M-basis $\{e_\alpha\}_{\alpha\in\Gamma}$ and a family P of absorbing seminorms. Then for each $x \in A$ and $p \in P$ the set $\{e^*_\alpha(x): \alpha \in \Gamma ,\ p(e_\alpha) \neq 0\}$ is bounded.

Proof: For each $p \in P$ and $x \in A$, there exists $K(p,x) > 0$ such that $p(xy) \le K(p,x)p(y)$ for all $y \in A$. In particular,

$$|e^*_\alpha(x)|p(e_\alpha) = p(e^*_\alpha(x)e_\alpha) = p(xe_\alpha) \le K(p,x)p(e_\alpha)$$

for all $\alpha \in \Gamma$. Hence, for $\alpha \in \Gamma$ with $p(e_\alpha) \neq 0$ we have $|e^*_\alpha(x)| \le K(p,x)$.

III.21 Corollary. If, in addition to the hypothesis of Proposition III.20, there exists $p \in P$ with $p(e_\alpha) \neq 0$ for all $\alpha \in \Gamma$, then for each $x \in A$, $\{e^*_\alpha(x): \alpha \in \Gamma\}$ is a bounded set of scalars.

III.22 Proposition. Let A be a locally convex s-algebra with an orthogonal M-basis $\{e_\alpha\}_{\alpha\in\Gamma}$. The following are equivalent:

(i) A is A-convex.

(ii) A has a defining family of seminorms such that for each $x \in A$ and $p \in P$ the set $\{e_\alpha^*(x): p(e_\alpha) \neq 0\}$ is bounded.

<u>Proof</u>: (i) ⇒ (ii): This follows from Proposition III.20.

(ii) ⇒ (i): First we observe that if Q is any other family of defining seminorms then Q also satisfies the condition in (ii). Indeed, for $q \in Q$ there exists $p \in P$ and $K > 0$ such that $q(x) \leq Kp(x)$ for all $x \in A$ and hence $q(e_\alpha) \neq 0 \Rightarrow p(e_\alpha) \neq 0$. Thus for each $x \in A$, $\{e_\alpha^*(x): \alpha \in \Gamma, \ q(e_\alpha) \neq 0\} \subset \{e_\alpha^*(x): \alpha \in \Gamma, \ p(e_\alpha) \neq 0\}$ which is bounded. Let $\{\|.\|_\mu: \mu \in M\}$ be a defining family of squarely submultiplicative seminorms. By the observation we have just made, for each $x \in A$ and $\mu \in M$ there exists $K(x,\mu) > 0$ such that $|e_\alpha^*(x)| \leq K(x,\mu)$ for all $\alpha \in T_\mu$, where $T_\mu = \{\alpha \in \Gamma: \|e_\alpha\|_\mu \neq 0\}$. From Corollary III.19, with the same notations, we have $\tau = \tau'$ and hence the proof will be complete once we show that each $\|.\|'_\mu$, $\mu \in M$ is absorbing. For $f \in \Lambda_\mu$ (as defined in Theorem III.17) and $\alpha \in \Gamma \setminus T_\mu$ we have $|f(e_\alpha)| \leq \|e_\alpha\|_\mu = 0$ and hence for $x \in A$

$$\|xy\|'_\mu = \sup_{f\in\Lambda_\mu} \sum_{\alpha\in\Gamma} |e_\alpha^*(x)|\cdot|e_\alpha^*(y)|\cdot|f(e_\alpha)| =$$

$$= \sup_{f\in\Lambda_\mu} \sum_{\alpha\in T_\mu} |e_\alpha^*(x)|\cdot|e_\alpha^*(y)|\cdot|f(e_\alpha)| \leq$$

$$\leq K(x,\mu) \sup_{f\in\Lambda_\mu} \sum_{\alpha\in T_\mu} |e_\alpha^*(y)|\cdot|f(e_\alpha)| =$$

$$= K(x,\mu) \sup_{f\in\Lambda_\mu} \sum_{\alpha\in\Gamma} |e_\alpha^*(y)|\cdot|f(e_\alpha)| = K(x,\mu)\ \|y\|'_\mu$$

for every $y \in A$ and so $\|\cdot\|'_\mu$ is absorbing.

III.23 Proposition. Let A and B be topological algebras with orthogonal M-bases $\{e_\alpha\}_{\alpha\in\Gamma}$ and $\{a_h\}_{h\in H}$, respectively. If $F: A \to B$ is a continuous algebra isomorphism, then $\{F(e_\alpha)\}_{\alpha\in\Gamma}$ is an orthogonal M- basis in B, which is a permutation of $\{a_h\}_{h\in H}$.

Proof: Since F is a continuous algebra isomorphism, it is easy to see that $F(x) = \sum_{\alpha\in\Gamma} e^*_\alpha(x)F(e_\alpha)$ for every $x \in A$ and that $\{F(e_\alpha)\}_{\alpha\in\Gamma}$ is an orthogonal M-basis in B, which is a permutation of $\{a_h\}_{h\in H}$ by Theorem I.37.

The following characterization of the algebra s of all sequences, is a consequence of the last two propositions.

III.24 Theorem. Let A be a B_0 algebra with an orthogonal M-basis $\{e_\alpha\}_{\alpha\in\Gamma}$ and an identity e. Then A is topologically isomorphic with the algebra s iff A has a defining family P of seminorms such that for each $x \in A$ and $p \in P$ the set $\{e^*_\alpha(x): p(e_\alpha) \neq 0\}$ is bounded.

Proof: If A is topologically isomorphic with s, then by Proposition III.23 there is a bijection $t: \mathbb{N} \to \Gamma$ and A has the family of defining seminorms $\{p_J: J \subset \mathbb{N},\ J \text{ is finite}\}$ where $p_J(x) = \max_{n\in J}|e^*_{t(n)}(x)|$. Clearly, for each J, $\{n \in \mathbb{N}: p_J(e_{t(n)}) \neq 0\} = J$ which is finite and hence $\{e^*_{t(n)}(x): p_J(e_{t(n)}) \neq 0\}$ is bounded, for each $x \in A$. To prove the converse, we notice that by Proposition III.15 Γ is countable. From Theorem II.1 we see that A has an orthogonal unconditional basis and consequently, by Theorem II.12, A is a B_0 s-algebra. If A has a defining family P of seminorms such that for each $p \in P$ the set $\{e_n^*(x): p(e_n) \neq 0\}$ is bounded, then by Proposition III.22, A is an A-convex algebra. Since A is complete and metrizable, it is also barrelled, by Theorem I.3. It follows from Theorem I.12 that A is locally m-convex. The proof is completed by an appeal to Theorem III.6.

III.25 Remark. $H(\mathbb{D})$ is a locally convex s-algebra which is not A-convex (see Example III.12 (iii)). Another way of showing that $H(\mathbb{D})$ is not A-convex - assuming it is a locally convex s-algebra - follows from Proposition III.22 by showing that $H(\mathbb{D})$ does not satisfy condition (ii) of that proposition. Consider the element $x \in H(\mathbb{D})$ given by $x(z) = \sum_{n=0}^{\infty} (n+1)z^n = \sum_{n=0}^{\infty} (n+1)e_n$, which is the derivative of the identity $e(z) = \frac{1}{1-z} = \sum_{n=0}^{\infty} e_n$. Clearly, for $0 < r < 1$ the set $\{e_n^*(x): \|e_n\|_r \neq 0\} = \{e_n^*(x): n = 0, 1, 2, \ldots\}$ is unbounded. By the observation made in the beginning of the proof of (ii) $\Rightarrow$ (i) in Theorem III.22, we see that for any other defining family P of seminorms, there exists $p \in P$ such that the set $\{e_n^*(x): p(e_n) \neq 0\}$ is unbounded and so condition (ii) of Proposition III.22 is not satisfied.

III.26 Remark. Every Φ-algebra is a locally convex s-algebra since, using the same notations as in Definition III.11, we have

$$p_\phi(\hat{x}\hat{y}) = \sup_{\alpha\in\Gamma}|\phi(\alpha)\hat{x}(\alpha)\hat{y}(\alpha)| = \left(\sup_{\alpha\in\Gamma}|\phi(\alpha)\hat{x}(\alpha)\hat{y}(\alpha)|^2\right)^{\frac{1}{2}} \leq$$

$$\leq \left(\sup_{\alpha\in\Gamma}|\phi(\alpha)(\hat{x}(\alpha))^2|\right)^{\frac{1}{2}} \cdot \left(\sup_{\alpha\in\Gamma}|\phi(\alpha)(\hat{y}(\alpha))^2|\right)^{\frac{1}{2}} =$$

$$= \left[p_\phi(\hat{x}^2)\right]^{\frac{1}{2}} \left[p_\phi(\hat{y}^2)\right]^{\frac{1}{2}}$$

and A is topologically isomorphic with $\tilde{A}$.

$H(\mathbb{D})$ is an example of a Φ-algebra which is not A-convex (Remark III.25) and ℓ_1 is a locally m-convex algebra which is not a Φ-algebra (Remark III.14).

We conclude this chapter with a remark concerning a possible generalization of the concept of an M-basis.

III.27 <u>Remark</u>. The concept of an M-basis considered here, generalizes the concept of an unconditional basis rather than that of a basis, since in the case of an M-basis or an unconditional basis, the type of convergence involved is the convergence of the net of partial sums over the members of the set Ω of all finite subsets of the indexing set ($\mathbb{N}$ or Γ), while in the concept of a basis, the convergence involved is that of the sequence of partial sums over the members of the directed subset $\{J_n : n = 1, 2, \ldots\}$ of Ω, where $J_n = \{1, 2, \ldots, n\}$. This suggests the following generalization of the concept of a basis to the case of an arbitrary indexing set Γ: $\{e_\alpha\}_{\alpha\in\Gamma}$ is an M'-basis in a topological vector space E if there exists a directed subset Ω' of Ω (satisfying some suitable conditions) such that for each $x \in E$, there exists a unique map $\alpha \to e_\alpha^*(x)$ of Γ into $\mathbb{C}$ with $x = \lim_{J\in\Omega'} \sum_{\alpha\in J} e_\alpha^*(x)e_\alpha$. It is reasonable to require Ω' to be cofinal with Ω (i.e., for each $J \in \Omega$ there exists $J' \in \Omega'$ such that $J \subset J'$), in order to eventually include every term $e_\alpha^*(x)e_\alpha$ in the representation $x = \lim_{J\in\Omega'} \sum_{\alpha\in J} e_\alpha^*(x)e_\alpha$.

In view of the remarks made by Edwards in [8] (Sections 2.9.0, 2.9.1,..., 2.9.7), it may be possible to apply this concept of an M'-basis in the case of a convolution algebra $L_p(G)$, $1 < p < \infty$; where G is a compact abelian group. In the case $G = \mathbf{T}$, the dual group of G is the discrete group $\mathbf{Z}$ and, as it was shown in Example I.40, Ω' can be chosen as the directed subset

(III.27.1) $\{\{0\}, \{0,-1\}, \{0,-1,1\}, \{0,-1,1,-2\}, \{0,-1,1,-2,2\},\ldots\}$.

of the set $\Omega(\mathbf{Z})$ of all finite subsets of $\mathbf{Z}$ and, for each $x \in L_p(\mathbf{T})$,

$$x = \lim_{J\in\Omega'} \sum_{n\in J} \hat{x}(n)e^{in(\cdot)}$$

in the $L_p(\mathbf{T})$-norm, where $\hat{x}: \mathbf{Z} \to \mathbb{C}$ is the Gel'fand (Fourier) transform of x. We may then expect Ω' – in the case of a compact abelian group G – to be a directed subset of the set $\Omega(G')$ of all finite subsets of the dual group G' (which is a discrete group), and each $x \in L_p(G)$ may be expected to have the M'-basis representation

$$x = \lim_{J \in \Omega'} \sum_{X \in J} \hat{x}(X) X(\cdot)$$

in the $L_p(G)$-norm. The structure of the directed set Ω' and its members should be related to the structure of G, or G'. For information concerning topological groups and convolution algebras see [10], [13] or [25].

Results analogous to Theorem II.1 and Theorem II.12 may also follow.

CHAPTER IV

CHARACTERIZATIONS OF $\ell_1(\Gamma)$ and $c_0(\Gamma)$

Let A be a Banach algebra with an orthogonal M-basis $\{e_\alpha\}_{\alpha\in\Gamma}$. It was shown (Corollary III.4) that the maximal ideal space of A is homeomorphic with Γ and the Gel'fand map ϕ sends $x \in A$ to the function $\hat{x}: \alpha \to \hat{x}(\alpha) = e_\alpha^*(x)$. By the uniqueness of the representation of $x \in A$ in terms of the M-basis, the Gel'fand map must be one-one (equivalently, A is semisimple) in addition to being continuous. Thus, by the Open Mapping theorem, any condition on A (or on the M-basis $\{e_\alpha\}_{\alpha\in\Gamma}$) which forces ϕ to be onto, is a sufficient condition for A to be topologically isomorphic with $c_0(\Gamma)$. In §2 of this chapter, we make use of this observation to develop characterizations of Banach algebras with orthogonal M-bases which are topologically isomorphic with $c_0(\Gamma)$.

In §1 we give characterizations - based on Husain's ideas in [14] and [15] - of Banach algebras with orthogonal M-bases which are topologically isomorphic with $\ell_1(\Gamma)$ or $c_0(\Gamma)$ (Theorems IV.2 and IV.6, respectively).

§1. The Banach algebras $\ell_1(\Gamma)$ and $c_0(\Gamma)$.

In [14] Husain proved that a Banach algebra A with an unconditional orthogonal basis $\{e_n\}$ is topologically isomorphic with the Banach algebra ℓ_1 iff: (i) the basis is boundedly complete (see Definition I.31), and

(ii) there exists $e' \in A'$ with $e'(e_n) = 1$ for all $n \in \mathbb{N}$; under the assumption that the basis is normal (i.e., $\|e_n\| = 1$ for all $n \in \mathbb{N}$).

Here we give an improved version of this result in which the "if" part, which is the significant part of the proof, follows under much weaker conditions (Lemma IV.1 and Theorem IV.2). Proofs are written for the case of an M-basis which is - by Theorem II.1 - more general than the case of an unconditional basis.

IV.1 Lemma. Let E be a Banach space with an M-basis $\{e_\alpha\}_{\alpha\in\Gamma}$ and assume that the following two conditions are satisfied:

(i) $\{e_\alpha\}_{\alpha\in\Gamma}$ is bounded, i.e., there exists $M > 0$ such that $\|e_\alpha\| \le M$ for all $\alpha \in \Gamma$.

(ii) There exists $e' \in E'$ (the topological dual of E) such that the set $\{e'(e_\alpha): \alpha \in \Gamma\}$ is bounded away from zero.

Then E is topologically isomorphic with the Banach space $\ell_1(\Gamma)$.

Proof: The map $\sigma: x \to (e^*_\alpha(x))_{\alpha\in\Gamma}$ is a vector space homomorphism of E into $\mathbb{C}^\Gamma$, which is one-one because of the uniqueness of the representation of an element $x \in E$ in terms of the M-basis. From Corollary III.18 we see that

$$\|x\|' = \sup_{f\in\Lambda} \sum_{\alpha\in\Gamma} |e^*_\alpha(x)|\,|f(e_\alpha)| \quad , \quad x \in E$$

where $\Lambda = \{f \in E': \|f\| \le 1\}$; defines an equivalent norm on E and therefore, there exists $K > 0$ such that $\|x\| \ge K\|x\|'$ for all $x \in E$. By condition (ii), there exists $e' \in E'$ and $\delta > 0$ such that

$|e'(e_\alpha)| \geq \delta$ for all $\alpha \in \Gamma$ and hence, setting $f_0 = \|e'\|^{-1}e'$, we have $f_0 \in \Lambda$ and $|f_0(e_\alpha)| = \|e'\|^{-1}|e'(e_\alpha)| \geq \|e'\|^{-1}\delta$. It follows that

$$\infty > \|x\| \geq K\|x\|' = K \sup_{f\in\Lambda} \sum_{\alpha\in\Gamma} |e_\alpha^*(x)|\cdot|f(e_\alpha)| \geq$$

$$\geq K \sum_{\alpha\in\Gamma} |e_\alpha^*(x)|\cdot|f_0(e_\alpha)| \geq K\,\|e'\|^{-1}\delta \sum_{\alpha\in\Gamma} |e_\alpha^*(x)|$$

for all $x \in E$. This shows that $\sigma(x) \in \ell_1(\Gamma)$ for all $x \in E$ and $\sigma: E \to \ell_1(\Gamma)$ is continuous. Next, we show that $\sigma: E \to \ell_1(\Gamma)$ is onto, and this is where condition (i) is needed. Let $a = (a_\alpha)_{\alpha\in\Gamma} \in \ell_1(\Gamma)$ and let $\varepsilon > 0$ be given. There exists $J_0 \in \Omega$ (the set of all finite subsets of Γ) such that $\sum_{\alpha\in\Gamma\setminus J_0} |a_\alpha| < \frac{\varepsilon}{2M}$ and therefore, for $J_1, J_2 \in \Omega$ with $J_0 \subset J_1$, $J_0 \subset J_2$ we have

$$\Big\|\sum_{\alpha\in J_2} a_\alpha e_\alpha - \sum_{\alpha\in J_1} a_\alpha e_\alpha\Big\| = \Big\|\sum_{\alpha\in J_2\setminus J_0} a_\alpha e_\alpha - \sum_{\alpha\in J_1\setminus J_0} a_\alpha e_\alpha\Big\| \leq$$

$$\leq \Big\|\sum_{\alpha\in J_2\setminus J_0} a_\alpha e_\alpha\Big\| + \Big\|\sum_{\alpha\in J_1\setminus J_0} a_\alpha e_\alpha\Big\| \leq \sum_{\alpha\in J_2\setminus J_0} |a_\alpha|\,\|e_\alpha\| + \sum_{\alpha\in J_1\setminus J_0} |a_\alpha|\,\|e$$

$$\leq M \sum_{\alpha\in J_2\setminus J_0} |a_\alpha| + M \sum_{\alpha\in J_1\setminus J_0} |a_\alpha| < M\,\frac{\varepsilon}{2M} + M\,\frac{\varepsilon}{2M} = \varepsilon$$

which shows that $\{\sum_{\alpha\in J} a_\alpha e_\alpha\}_{J\in\Omega}$ is a Cauchy net in E. Since E is complete, there exists $x \in E$ with $x = \lim_{J\in\Omega} \sum_{\alpha\in J} a_\alpha e_\alpha = \sum_{\alpha\in\Gamma} a_\alpha e_\alpha$ and, clearly, $\sigma(x) = a$. We have thus shown that $\sigma: E \to \ell_1(\Gamma)$ is a continuous isomorphism. To complete the proof, we need only to show that σ is an open map. This follows either from the Open Mapping theorem, or from

$$\|\sigma^{-1}(a)\| = \|\sum_{\alpha\in\Gamma} a_\alpha e_\alpha\| \le \sum_{\alpha\in\Gamma} |a_\alpha| \, \|e_\alpha\| \le M \sum_{\alpha\in\Gamma} |a_\alpha| = M\|a\|$$

for every $a = (a_\alpha)_{\alpha\in\Gamma} \in \ell_1(\Gamma)$.

IV.2 <u>Theorem</u>. Let A be a Banach algebra with an orthogonal M-basis $\{e_\alpha\}_{\alpha\in\Gamma}$. Then A is topologically isomorphic with the Banach algebra $\ell_1(\Gamma)$ of Example I.39 iff conditions (i) and (ii) of Lemma IV.1 are satisfied.

<u>Proof</u>: The "if" part follows immediately from Lemma IV.1 since $\sigma(xy) = \sigma(\sum_{\alpha\in\Gamma} e_\alpha^*(x)e_\alpha^*(y)e_\alpha) = (e_\alpha^*(x)e_\alpha^*(y))_{\alpha\in\Gamma} = \sigma(x)\sigma(y)$ for all $x,y \in A$. To prove the "only if" part, we notice - by Proposition III.23 and the continuity of σ - that $\{\sigma(e_\alpha)\}_{\alpha\in\Gamma}$ is an orthogonal M-basis in $\ell_1(\Gamma)$ which is a permutation of the canonical M-basis and so $\|\sigma(e_\alpha)\| = 1$ and $f(\sigma(e_\alpha)) = 1$ for all $\alpha \in \Gamma$, where f is the element in $\ell_1'(\Gamma) = \ell_\infty(\Gamma)$ with $f(\alpha) \equiv 1$. Hence, condition (ii) of Lemma IV.1 is satisfied, with $e' = f\circ\sigma$; and condition (i) of Lemma IV.1 is satisfied since - by the continuity of σ^{-1} - there exists $M > 0$ such that $\|x\| \le M\|\sigma(x)\|$ for all $x \in A$; in particular,

$$\|e_\alpha\| \le M\|\sigma(e_\alpha)\| = M$$

for all $\alpha \in \Gamma$. The proof is now complete.

IV.3 Example. Let E be the Banach space of all $x \in \mathbb{C}^N$ such that $\sum_{n=1}^{\infty} n|x(n)| < \infty$, with the coordinatewise operations and the norm $\|x\| = \sum_{n=1}^{\infty} n|x(n)|$. Under the coordinatewise multiplication, E becomes a Banach algebra with an unconditional orthogonal basis, which is the canonical basis $\{e_n\}$. This can easily be shown either by Remark II.5 and Corollary II.6, or directly as follows:

$$\|xy\| = \sum_{n=1}^{\infty} n|x(n)||y(n)| \le \sum_{n=1}^{\infty} n|x(n)| \cdot n|y(n)|$$

$$\le \sum_{n=1}^{\infty} n|x(n)| \cdot \sum_{n=1}^{\infty} n|y(n)| = \|x\|\cdot\|y\|$$

for all $x,y \in E$. It is easy to check the following: the basis $\{e_n\}$ is boundedly complete; condition (ii) of Lemma IV.1 is satisfied (notice that $E' = \ell_\infty$); condition (i) of Lemma IV.1 is not satisfied and E is not topologically isomorphic with ℓ_1.

IV.4 Remark. Notice that a basis $\{e_n\}$ in a normed space E can always be normalized, i.e., replaced by the normal basis $\left\{\frac{e_n}{\|e_n\|}\right\}$. However, an orthogonal basis $\{e_n\}$ in a normed algebra A cannot be normalized, with the orthogonality preserved, unless it is already normal; this follows from the uniqueness of an orthogonal basis up to a permutation (Theorem I.37).

In Theorem IV.6 below, we give a characterization of Banach algebras with orthogonal M-bases, which are topologically isomorphic with the Banach algebra $c_0(\Gamma)$. The proof is a simplified and improved version of Husain's proof in [15], written for the case of an M-basis. First we have:

IV.5 Definition. An M-basis $\{e_\alpha\}_{\alpha\in\Gamma}$ in a Banach space E is said to be shrinking if $\{e_\alpha^*\}_{\alpha\in\Gamma}$ is an M-basis in E'.

IV.6 Theorem. A Banach algebra A with an orthogonal M-basis $\{e_\alpha\}_{\alpha\in\Gamma}$ is topologically isomorphic with $c_0(\Gamma)$ iff the following two conditions are satisfied:

(i) $\{e_\alpha\}_{\alpha\in\Gamma}$ is shrinking.

(ii) There exists $e'' \in A''$ (the second topological dual of A) such that the set $\{e''(e_\alpha^*): \alpha \in \Gamma\}$ is bounded away from zero.

Proof: First, assume that conditions (i) and (ii) above are satisfied; we use Lemma IV.1 to establish a topological isomorphism $\psi: \ell_1(\Gamma) \to A'$. By condition (i) of the present theorem, $\{e_\alpha^*\}_{\alpha\in\Gamma}$ is an M-basis in the Banach space A'. We show that A' and the M-basis $\{e_\alpha^*\}_{\alpha\in\Gamma}$ satisfy conditions (i) and (ii) of Lemma IV.1. Indeed, condition (ii) of Lemma IV.1 follows immediately from condition (ii) of the present theorem; and condition (i) of Lemma IV.1 follows from the fact that each e_α^* is a multiplicative linear functional on A (Proposition I.35 (ii)) and consequently, by Theorem I.15 (i), $\|e_\alpha^*\| \le 1$ for all $\alpha \in \Gamma$. It follows from Lemma IV.1 that $\psi: a = (a_\alpha)_{\alpha\in\Gamma} \to \sum_{\alpha\in\Gamma} a_\alpha e_\alpha^*$ is a topological isomorphism of $\ell_1(\Gamma)$ onto A'. From Corollary III.4 and Remark III.5 we see that the Gel'fand map $\phi: A \to C_0(\Delta(A)) = c_0(\Gamma)$, which is a continuous linear map, is given by $\phi(x) = (e_\alpha^*(x))_{\alpha\in\Gamma}$. Using the duality notation ($<x,f>$ for $<f(x)>$) we have

$$<\phi(x),a> = <(e_\alpha^*(x))_{\alpha\in\Gamma}, (a_\alpha)_{\alpha\in\Gamma}> = \sum_{\alpha\in\Gamma} a_\alpha e_\alpha^*(x) =$$

$$= (\sum_{\alpha\in\Gamma} a_\alpha e_\alpha^*)(x) = <x, \sum_{\alpha\in\Gamma} a_\alpha e_\alpha^*> = <x,\psi(a)>$$

for every $x \in A$ and $a = (a_\alpha)_{\alpha\in\Gamma} \in \ell_1(\Gamma) = c_0'(\Gamma)$, and therefore $\psi: c_0'(\Gamma) = \ell_1(\Gamma) \to A'$ is the adjoint of $\phi: A \to c_0(\Gamma)$. Since ϕ is a continuous linear map and ψ is a topological isomorphism, it follows that $\phi: A \to c_0(\Gamma)$ is onto (see [34], page 160) and, by the Open Mapping theorem, ϕ is a topological isomorphism of A onto $c_0(\Gamma)$, which proves the "if" part.

The "only if" part follows by an argument parallel to the one used to prove the "only if" part of Theorem IV.2, in the light of the simple observation that conditions (i) and (ii) of the present theorem are satisfied by $c_0(\Gamma)$ with the canonical orthogonal M-basis. [Notice that the fact that $\{e_\alpha\}_{\alpha\in\Gamma}$ is shrinking in $c_0(\Gamma)$ can be shown in the same way as in the case of c_0 (see Examples I.33)].

§2. Other characterizations of c_0 and $c_0(\Gamma)$.

In this section, we characterize Banach algebras with orthogonal

M-bases which are topologically isomorphic with c_0 or $c_0(\Gamma)$, by the existence of elements satisfying certain properties (Lemma IV.7, Theorem IV.9 and Corollary IV.13).

IV.7 Lemma. Let A be a Banach algebra with an orthogonal M-basis $\{e_\alpha\}_{\alpha\in\Gamma}$. The following are equivalent:

(i) There exists $K > 0$ such that $\|\sum_{\alpha\in J} e_\alpha\| \leq K$ for all $J \in \Omega$ (the set of all finite subsets of Γ).

(ii) A is topologically isomorphic with $c_0(\Gamma)$.

Proof: (i) $\Rightarrow$ (ii): We show that the Gel'fand map $\phi: x \to (e_\alpha^*(x))_{\alpha\in\Gamma}$ of A into $c_0(\Gamma)$, which is one-one and continuous, is onto. From Corollary III.18 we see that

$$\|x\|' = \sup_{f\in\Lambda} \sum_{\alpha\in\Gamma} |e_\alpha^*(x)||f(e_\alpha)| \quad , \quad x \in A$$

where $\Lambda = \{f \in A': \|f\| \leq 1\}$; defines an equivalent norm on A. For every $J \in \Omega$ and $\{t_\alpha: \alpha \in J\} \subset \mathbb{C}$ we have

$$\text{(IV.7.1)} \quad \|\sum_{\alpha\in J} t_\alpha e_\alpha\|' = \sup_{f\in\Lambda} \sum_{\alpha\in J} |t_\alpha||f(e_\alpha)| \leq \sup_{f\in\Lambda} \left[\max_{\alpha\in J}|t_\alpha| \cdot \sum_{\alpha\in J} |f(e_\alpha)|\right] =$$

$$= \max_{\alpha\in J}|t_\alpha| \cdot \sup_{f\in\Lambda} \sum_{\alpha\in J} |f(e_\alpha)| = \max_{\alpha\in J}|t_\alpha| \cdot \|\sum_{\alpha\in J} e_\alpha\|' \leq K' \max_{\alpha\in J}|t_\alpha|$$

where $K' > 0$ is such that $\|\sum_{\alpha\in J} e_\alpha\|' \leq K'$ for all $J \in \Omega$. It is easy to see from (IV.7.1) that for an arbitrary $(t_\alpha)_{\alpha\in\Gamma} \in c_0(\Gamma)$, $\{\sum_{\alpha\in J} t_\alpha e_\alpha\}_{J\in\Omega}$ is a Cauchy net in A and hence, by the completeness of A, there exists $x \in A$ with $x = \sum_{\alpha\in\Gamma} t_\alpha e_\alpha$ and, clearly, $\phi(x) = (t_\alpha)_{\alpha\in\Gamma}$. Thus ϕ is onto, and by the Open Mapping theorem, it is a topological isomorphism of A onto $c_0(\Gamma)$.

(ii) ⇒ (i): This is easy to see, as in the "only if" part of Theorems IV.2 and IV.6.

IV.8 Remark. Lemma IV.7 does not hold in the more general case of a Banach space with an M-basis, as shown by the following example. Let E be the Banach space of all $x \in \mathbb{C}^{\mathbb{N}}$ such that $\sum_{n=1}^{\infty} \frac{|x(n)|}{n^2} < \infty$, with the coordinatewise operations and the norm $\|x\| = \sum_{n=1}^{\infty} \frac{|x(n)|}{n^2}$. It is easy to see that condition (i) of Lemma IV.7 is satisfied, while condition (ii) is not.

IV.9 Theorem. A Banach algebra A with a countable orthogonal M-basis $\{e_n\}_{n\in\mathbb{N}}$ is topologically isomorphic with c_0 iff there exists an element $x \in A$ satisfying the following three conditions:

(i) $e_n^*(x) \neq 0$ for all $n \in \mathbb{N}$.

(ii) For every $k \in \mathbb{N}$ there exists an element in A denoted by $x^{\frac{1}{k}}$ – which we call "a kth root of x" – such that $(x^{\frac{1}{k}})^k = x$.

(iii) There exists $M > 0$ such that $\|x^{\frac{1}{k}}\| \leq M$ for all $k \in \mathbb{N}$.

Proof: First, we observe that the element $a = (\frac{1}{n})_{n\in\mathbb{N}} \in c_0$ satisfies the above three conditions, with $M = 1$ in condition (iii) (in fact, conditions (ii) and (iii) are satisfied by every element y in c_0; we can take $M = \max\{1, \|y\|\}$, as can easily be verified). Thus, if $\psi: c_0 \to A$ is a topological isomorphism then, clearly, $\psi(a)$ satisfies conditions (ii) and (iii) and, condition (i) is also satisfied by $\psi(a)$ since - by Proposition III.23 - $\{e_n^*(\psi(a)): n \in \mathbb{N}\} = \{\frac{1}{n}: n \in \mathbb{N}\}$. This proves the "only if" part.

To prove the "if" part, assume that there exists an element $x \in A$ which satisfies the above three conditions. Using the equivalent norm $\|\cdot\|'$ (see Corollary III.18) we get

$$\sup_{f\in\Lambda} \sum_{n=1}^{\infty} |e_n^*(x^{\frac{1}{k}})|\,|f(e_n)| = \|x^{\frac{1}{k}}\|' \le M' \quad , \quad \text{for all } k \in \mathbb{N}$$

where M' is a positive number which exists because of condition (iii) and the equivalence of the norm $\|\cdot\|'$ to the original norm $\|\cdot\|$ on A. Since the coordinate functionals e_n^*, $n \in \mathbb{N}$ are multiplicative, it is easy to verify that $|e_n^*(x^{\frac{1}{k}})| = |e_n^*(x)|^{\frac{1}{k}}$ and hence for every $J \in \Omega$ (the set of all finite subsets of $\mathbb{N}$) we have

$$\text{(IV.9.1)} \quad \|\sum_{n\in J} |e_n^*(x)|^{\frac{1}{k}} e_n\|' = \sup_{f\in\Lambda} \sum_{n\in J} |e_n^*(x)|^{\frac{1}{k}} \cdot |f(e_n)| \le \sup_{f\in\Lambda} \sum_{n\in\mathbb{N}} |e_n^*(x)|^{\frac{1}{k}} \cdot |f(e_n)|$$

$$= \sup_{f\in\Lambda} \sum_{n\in\mathbb{N}} |e_n^*(x^{\frac{1}{k}})| \cdot |f(e_n)| = \|x^{\frac{1}{k}}\|' \le M'$$

for all $k \in \mathbb{N}$. From condition (i) we have $|e_n^*(x)| \neq 0$ and hence $\lim_k |e_n^*(x)|^{\frac{1}{k}} = 1$ for all $n \in \mathbb{N}$. Letting $k \to \infty$ in (IV.9.1) we get

$$\|\sum_{n\in J} e_n\|' = \lim_k \|\sum_{n\in J} |e_n^*(x)|^{\frac{1}{k}} e_n\|' \leq M'$$

for every $J \in \Omega$. The proof is completed by an appeal to Lemma IV.7', in view of the equivalence of $\|.\|'$ and the original norm $\|.\|$.

IV.10 Remark. In the "if" part of Theorem IV.9, condition (iii) is essential, as we show by an example. Consider the element $x = (e^{-n})_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$ For every $k \in \mathbb{N}$ we have

$$\sum_{n=1}^{\infty} e^{-\frac{n}{k}} = \frac{e^{-\frac{1}{k}}}{1-e^{-\frac{1}{k}}} = \frac{1}{e^{\frac{1}{k}}-1} < \infty$$

and so x is in ℓ_1, together with all its roots, thus satisfying conditions (i) and (ii). However, condition (iii) is not satisfied by x since $\|x^{\frac{1}{k}}\| = \sum_{n=1}^{\infty} e^{-\frac{n}{k}} = (e^{\frac{1}{k}} - 1)^{-1} \to \infty$ as $k \to \infty$.

IV.11 Proposition. Let E be a Banach space with an M-basis $\{e_\alpha\}_{\alpha\in\Gamma}$. Then for every $x \in E$ and every bounded function $b: \Gamma \to \mathbb{C}$ there exists $y \in E$ with $y = \sum_{\alpha\in\Gamma} b_\alpha e_\alpha^*(x) e_\alpha$, where $b_\alpha = b(\alpha)$, $\alpha \in \Gamma$.

Proof: Let $M > 0$ be such that $|b_\alpha| \le M$ for all $\alpha \in \Gamma$. Given $\varepsilon >$ (there exists $I \in \Omega$ (the set of all finite subsets of Γ) such that

$$\sup_{f\in\Lambda} \sum_{\alpha\in J} |e_\alpha^*(x)||f(e_\alpha)| = \|\sum_{\alpha\in J} e_\alpha^*(x)e_\alpha\|' < \frac{\varepsilon}{M}$$

for every $J \in \Omega$ with $J \cap I = \phi$ and hence

$$\|\sum_{\alpha\in J} b_\alpha e_\alpha^*(x)e_\alpha\|' = \sup_{f\in\Lambda} \sum_{\alpha\in J} |b_\alpha|\cdot|e_\alpha^*(x)||f(e_\alpha)| \le M \sup_{f\in\Lambda} \sum_{\alpha\in J} |e_\alpha^*(x)||f(e_\alpha)| < M\frac{\varepsilon}{M}$$

for every $J \in \Omega$ with $J \cap I = \phi$. This shows that $\{\sum_{\alpha\in J} b_\alpha e_\alpha^*(x)e_\alpha\}_{J\in\Omega}$ i a Cauchy net in E and, by the completeness of E, there exists $y \in E$ such that $y = \sum_{\alpha\in\Gamma} b_\alpha e_\alpha^*(x)e_\alpha$.

IV.12 Corollary. Let A be a Banach algebra with an orthogonal M-basis $\{e_\alpha\}_{\alpha\in\Gamma}$ and let I be a countable subset of Γ. The following statements are equivalent:

(i) There exists $L > 0$ such that $\|\sum_{\alpha\in J} e_\alpha\| \le L$ for every finite $J \subset I$.

(ii) There exists $x_0 \in A$ and $M > 0$ such that: $e_\alpha^*(x_0) \neq 0$ for all $\alpha \in I$; x_0 has a k^{th} root $x_0^{\frac{1}{k}}$ for each $k \in \mathbb{N}$; and $\|x_0^{\frac{1}{k}}\| \le M$ for all $k \in \mathbb{N}$.

(iii) A is topologically isomorphic with the direct sum $c_0(I) \oplus B$ of $c_0(I)$ and some Banach subalgebra B of A.

Proof: Let $x \in A$ be arbitrary. Taking b as in Proposition IV.11 with $b_\alpha = 1$ for $\alpha \in I$ and $b_\alpha = 0$ for $\alpha \in \Gamma\setminus I$ we see that $\sum_{\alpha\in I} e_\alpha^*(x)e_\alpha$ converges in A. Setting $Px = \sum_{\alpha\in I} e_\alpha^*(x)e_\alpha$, $x \in A$; it is easy to verify that P is a linear operator on A, which is the projection of A onto the subspace $B(I)$ of all $x \in A$ with $e_\alpha^*(x) = 0$ for all $\alpha \in \Gamma\setminus I$. Since the M-basis $\{e_\alpha\}_{\alpha\in\Gamma}$ is orthogonal, each of $B(I)$ and $B(\Gamma\setminus I)$ (where $B(\Gamma\setminus I)$ is the subspace of all $x \in A$ with $e_\alpha^*(x) = 0$ for all $\alpha \in I$) is a subalgebra of A and, clearly, A is the algebraic direct sum of $B(I)$ and $B(\Gamma\setminus I)$. To show that this algebraic direct sum is a topological one, it is enough to show that P is continuous (see Lemma I.6). This can be done using the equivalent norm $\|.\|'$ as follows:

$$\|Px\|' = \sup_{f\in\Lambda} \sum_{\alpha\in I} |e_\alpha^*(x)|\cdot|f(e_\alpha)| \le \sup_{f\in\Lambda} \sum_{\alpha\in\Gamma} |e_\alpha^*(x)|\cdot|f(e_\alpha)| = \|x\|'$$

for all $x \in A$. From Lemma IV.7 and Theorem IV.9 we see that each of the statements (i) and (ii) is equivalent to the statement that $B(I)$ is topologically isomorphic with $c_0(I)$. The proof is now complete.

IV.13 Corollary. Let A be a Banach algebra with an orthogonal M-basis $\{e_\alpha\}_{\alpha\in\Gamma}$. The following are equivalent:

(i) There exists $M > 0$ such that for every finite $J \subset \Gamma$, $\|\sum_{\alpha\in J} e_\alpha\| \le M$.

(ii) For every countable $I \subset \Gamma$ there exists $M_I > 0$ such that for every finite $J \subset I$, $\|\sum_{\alpha\in J} e_\alpha\| \le M_I$.

(iii) For every countable $I \subset \Gamma$, the subalgebra $B(I)$ of all $x \in A$ with $e_\alpha^*(x) = 0$ for all $\alpha \in \Gamma\backslash I$, is topologically isomorphic with $c_0(I)$ and $A = B(I) \oplus B(\Gamma\backslash I) = c_0(I) \oplus B(\Gamma\backslash I)$.

(iv) For every countable $I \subset \Gamma$, there exists $L_I > 0$ and $x \in A$ such that: $e_\alpha^*(x) \neq 0$ for all $\alpha \in I$; x has a k^{th} root $x^{\frac{1}{k}}$ for each $k \in \mathbb{N}$; and $\|x^{\frac{1}{k}}\| \le M$ for all $k \in \mathbb{N}$.

(v) A is topologically isomorphic with $c_0(\Gamma)$.

Proof: The equivalence of (i) and (v) is a restatement of Lemma IV.7, and the implication (i) $\Rightarrow$ (ii) is obvious. The implication (v) $\Rightarrow$ (iv) follows from Corollary IV.12 since for every (countable) $I \subset \Gamma$, $c_0(\Gamma)$ can be written as the direct sum $c_0(I) \oplus c_0(\Gamma\backslash I)$. The equivalence of (ii), (iii) and (iv) is also a consequence of Corollary IV.12. We complete the proof by showing that (iii) $\Rightarrow$ (v). Since the Gel'fand map $\phi: A \to c_0(\Gamma)$ is one-one and continuous, the implication (iii) $\Rightarrow$ (v) will follow from the Open Mapping theorem once we show that (iii) forces ϕ to be onto. For an arbitrary $y \in c_0(\Gamma)$ the set $I_y = \{\alpha \in \Gamma : y(\alpha) \neq 0\}$ is countable and therefore, by (iii), $A = c_0(I_y) \oplus B(\Gamma\backslash I_y)$. Let x be the restriction of y to I_y then clearly $x \in c_0(I_y)$ and so $x = x + 0 \in c_0(\Gamma) \oplus B(\Gamma\backslash I_y) = A$. Evidently $\phi(x) = y$ and so ϕ is onto. The proof is now complete.

BIBLIOGRAPHY

[1] Arens, R.F.; The space L^ω and convex topological rings; Bull. Amer. Math. Soc. 52 (1946), 931-935.

[2] Arsove, M.G.; The Paley-Wiener theorem in metric linear spaces. Pacific J. Math. 10, 365-379 (1960).

[3] Buck, R.C.; Bounded continuous functions on a locally compact space; Michigan Math. Jr. 5 (1958), 95-104.

[4] Cochran, A.C.; Topological algebras and Mackey topologies; Proc. Amer. Math. Soc. 30 (1971), 115-119.

[5] Cochran, A.C.; Representation of A-convex algebras; Proc. Amer. Math. Soc. 41 (1973), 473-479.

[6] Day, M.M.; Normed linear spaces; 3rd edition; Springer-Verlag, New York-Heidelberg-Berlin, 1973.

[7] Edwards, R.E.; Fourier Series, a modern introduction, Vol. II, 2nd edition; Springer-Verlag, New York-Heidelberg-Berlin, 1982.

[8] Edwards, R.E.; Integration and harmonic analysis on compact groups; London Math. Soc. Lecture Note Series 8; Cambridge Univ. Press, 1972.

[9] Enflo, P.; A counterexample to the approximation property in Banach spaces; Acta Math. 130, 309-317 (1973).

[10] Hewitt, E. and Ross, K.; Abstract harmonic analysis I; Springer-Verlag; Berlin-Göttingen-Heidelberg, 1963.

[11] Hewitt, E. and Ross, K.; Abstract harmonic analysis II; Springer-Verlag; New York-Heidelberg-Berlin, 1970.

[12] Husain, T.; The open mapping and closed graph theorems in topological vector spaces; Oxford Mathematical Monographs; Clarendon Press; Oxford, 1965.

[13] Husain, T.; Introduction to topological groups; W.B. Saunders; Philadelphia and London, 1966.

[14] Husain, T; Topological algebras isomorphic with ℓ_1 or $H(D)$; The Punjab Univ. Journal of Math. Vol. XIV-XV (1981-82), pp. 1-14.

[15] Husain, T.; Banach algebras homeomorphic with c_0; The Punjab Univ. Journal of Math., Vol. XVI (1983), pp. 1-4.

[16] Husain, T.; Multiplicative functionals on topological algebras; Research Notes in Math. #85; Pitman Advanced Publishing Program; Boston, 1983.

[17] Husain, T.; Orthogonal primitive idempotents and Banach algebras isomorphic with ℓ_2; Pacific J. Math. 117 #2, 313-327 (1985).

[18] Husain, T. and Liang, J.; Multiplicative functionals on Fréchet algebras with bases; Can. J. Math., Vol. XXIX, No. 2 (1977), 270-276.

[19] Husain, T. and Watson, S.; Topological algebras with orthogonal Schauder bases; Pacific Jr. of Math., Vol. 91, No. 2 (1980), 339-347.

[20] Husain, T. and Watson, S.; Algebras with unconditional orthogonal bases; Proc. Amer. Math. Soc. 79 (1980), 539-545.

[21] Katznelson, Y.; An introduction to harmonic analysis; John Wiley & Sons Inc.; New York, 1968.

[22] Khaleelulla, S.M.; Counterexamples in topological vector spaces; Lecture Notes in Math., No. 936, Springer-Verlag; Berlin-Heidelberg-New York, 1982.

[23] Köthe, G.; Topological vector spaces I; Springer-Verlag; New York, 1969.

[24] Köthe, G.; Topological vector spaces II; Springer-Verlag; New York-Heidelberg-Berlin, 1979.

[25] Larsen, R.; Banach algebras, an introduction; Marcel Dekker Inc.; New York, 1973.

[26] Luecking, D.H. and Rubel, L.A.; Complex analysis, a functional analysis approach; Springer-Verlag; New York-Berlin-Heidelberg-Tokyo, 1984.

[27] Marti, J.; Introduction to the theory of bases; Springer Tracts in Natural Philosophy, Vol. 18; Springer-Verlag; New York Inc., 1969.

[28] Michael, E.; Locally multiplicatively convex topological algebras; AMS Memoirs #11, 1952.

[29] Mosak, R.D.; Banach algebras; Chicago Lectures in Math., The University of Chicago Press; Chicago and London, 1975.

[30] Procelli, R.; Linear spaces of analytic functions; Rand McNally & Company; Chicago, 1966.

[31] Rickart, C.E.; General theory of Banach algebras; D. Van Nostrand; Princeton, N.J., 1960.

[32] Robertson, A.P. and Robertson, W.J.; Topological vector spaces; Cambridge University Press; Cambridge, 1963.

[33] Rudin, W.; Functional analysis; McGraw-Hill Book Company; New York, 1973.

[34] Schaefer, H.H.; Topological vector spaces; Springer-Verlag; New York-Heidelberg-Berlin, 1971.

[35] Singer, I.; Bases in Banach spaces I; Springer-Verlag; New York-Heidelberg-Berlin, 1970.

[36] Waelbroeck, L.; Topological vector spaces and algebras; Lecture Notes in Math., No. 230, Springer-Verlag; Berlin-Heidelberg-New York, 1971.

[37] Watson, S.; F-algebras with cyclic bases; Comm. Math., 23 (1982), 141-146.

[38] Willard, S.; General topology; Addison-Wesley Publishing Company; London-Amsterdam-Don Mills, Ont., 1970.

[39] Yamazaki, S.; Normed rings and bases in Banach spaces; Sci. Pap. Coll. Gen. Educ. Univ.; Tokyo 15, 1-13.

[40] Zelazko, W.; On the algebras L_p of locally compact groups; Colloq. Math. 8 (1961), 115-120.

[41] Zelazko, W.; Banach algebras; Elsevier; Amsterdam, 1973.