AN EXTENSION OF GREECHIE'S ATOMISTIC LOOP LEMMA

By

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SCOPE AND CONTENTS: This thesis extends Greechie's atomistic loop lemma to a "principal section" loop lemma.

ABSTRACT

One of the major deficiencies of the theory of orthomodular lattices is the lack of accessible examples with which to test and develop conjectures. R. J. Greechie has developed two methods ([1], [2]) of obtaining new orthmodular lattices by "pasting" old ones together. This thesis gives an extension of one of these, the atomistic loop lemma. Briefly, a non-empty set of Boolean lattices with common bounds which either intersect trivially or in disjoint principal sections form an orthocomplemented poset under set-theoretic union. Necessary and sufficient conditions are given for this poset to be orthomodular.

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1. Introduction.

In [2] Greechie presents a method for constructing orthocomplemented posets by pasting Boolean lattices along atomistic sections. The degree of "intertwining" of these Boolean lattices determines whether this poset is orthomodular, and in case it is, whether it is a lattice. This paper extends the construction and the accompanying loop lemma to pastings along principal sections.

An <u>orthocomplemented poset</u> is a bounded poset $P = (|P|, \leq, 0, 1)$ together with an orthocomplementation, ie. a map $': P \to P$ (we commit the common crime of writing P rather than $|P|$) such that, for all $a, b \in P$:

(1) $a \vee a' = 1$, $a \wedge a' = 0$.

(2) If $a \leq b$ then $b' \leq a'$.

(3) $(a')' = a$.

An <u>orthomodular poset</u>, abbreviated OMP, is an orthocomplemented poset P which also satisfies, for all $a, b \in P$:

(4) If $a \leq b$ then $a \vee b'$ exists.

(5) If $a \leq b$ then $b = a \vee (a \vee b')'$.

A sublattice L of an OMP P is called a <u>subortholattice</u> of P whenever L is closed under '. If P, an OMP, is itself a lattice then it is called an <u>orthomodular lattice</u>, abbreviated OML.

A maximal Boolean subortholattice of an OMP P is called a block of P. The set of all blocks of P is written $\mathcal{B}(P)$. $C(P) = \bigcap \mathcal{B}(P)$ is called the center of P.

Let P be an OMP and $x, y \in P$, x commutes with y, written $x \, C \, y$, iff $x \wedge y$, $x \wedge y'$ and $(x \wedge y) \vee (x \wedge y')$ all exist and $x = (x \wedge y) \vee (x \wedge y')$. If $x \, C \, y$ and $y \, C \, x$ then $\{x, y\}$ generates a Boolean subortholattice of P and hence x and y share a block. Conversely, if x and y share a block then $x \, C \, y$ and $y \, C \, x$. If P is an OML then $x \, C \, y$ iff $y \, C \, x$. That this remains true in an arbitrary OMP is the following claim.

Claim 0. Let P be an OMP and $x, y \in P$. If $x \, C \, y$ then $y \, C \, x$.

Remark. de Morgan's laws hold in any orthocomplemented poset. That is, if $x \wedge y$ exists then $x' \vee y'$ exists and $(x \wedge y)' = x' \vee y'$, and dually.

Proof (of claim 0). Since $x \wedge y \leq y$ we have $(x' \vee y') \wedge y = ((x \wedge y) \vee y')'$ exists. We will show that $x' \wedge y = (x' \vee y') \wedge y$.

Since $(x' \vee y') \wedge y \leq y$ and $x' = ((x \wedge y) \vee (x \wedge y'))' = (x' \vee y') \wedge (x' \vee y) \geq (x' \vee y') \wedge y$, $(x' \vee y') \wedge y$ is a lower bound of $\{x', y\}$. Let z be any lower bound of $\{x', y\}$. Since $z \leq x', y$ we have $z \leq (x' \vee y') \wedge y$. Hence $x' \wedge y = (x' \vee y') \wedge y$.

Since $x \wedge y \leq y$, we have $y = (x \wedge y) \vee (y \wedge (x' \vee y'))$ $= (x \wedge y) \vee (x' \wedge y)$.

(

2. The Construction.

<u>Convention 1</u>. Let $\mathcal{B}$ be a non-empty set of Boolean lattices such that:

(1) For all B, C $\in \mathcal{B}$, if $B \subseteq C$ then B = C.

(2) For all B, C $\in \mathcal{B}$, $0_B = 0_C$ and $1_B = 1_C$ (hence we define $0 = 0_C$, $1 = 1_C$).

(3) For any distinct B, C $\in \mathcal{B}$, either $B \cap C = \{0, 1\}$ or there exists $a \in (B \cap C) \setminus \{0\}$ such that $B \cap C = [0, a]_B \cup [a'^B, 1]_B = [0, a]_C \cup [a'^C, 1]_C$ and $\leq_B|_{B \cap C} = \leq_C|_{B \cap C}$, $'^B|_{B \cap C} = {'^C}|_{B \cap C}$, in which case we write B -a C.

(4) For any pairwise distinct B, C, D $\in \mathcal{B}$ such that B -a C -b D, either a = b or $a <_B b'^B$.

Henceforth $\mathcal{B}$ will be a set of Boolean lattices satisfying convention 1. A subscript on an interval, operation or partial ordering indicates the Boolean lattice in which the interval, operation of partial ordering is being taken. If $y = x'^B$ and $z = x'^C$ then $x \in B \cap C$ and $y = x'^B = x'^C = z$. Thus the subscripts on the orthocomplementation are unnecessary. ie. $' : \cup\mathcal{B} \to \cup\mathcal{B}$ is well defined.

<u>Claim 1</u>. (a) For distinct B, C $\in \mathcal{B}$ with B -a C the following hold:

(1) If x, y $\in B \cap C$ then $x \vee_B y = x \vee_C y$.

(2) There exists $b \in B$ such that $a <_B b <_B 1$.

(3) a is unique.

(4) $[0, a]_B = [0, a]_C$ and $[a', 1]_B = [a', 1]_C$.

(b) If $x, y \in B \in \mathcal{B}$ and there exists $D \in \mathcal{B}$ with $x \leq_D y$ then $x \leq_B y$.

<u>Proof</u>. Re(b). If $x, y \in \{0, 1\}$ then since $0 \leq_B 1$ we have $x \leq_B y$. Otherwise $B \text{ -b } D$ for some $b \in B \cap D$ and $x, y \in B \cap D$. By (3) of convention 1, $x \leq_B y$.

Re(a.1). By (3) of convention 1 $B \cap C$ is a Boolean subortholattice of B and of C.

Re(a.2). Since $B \nsubseteq C$, $[0, a]_B \cup [a', 1]_B \subsetneq B$. Hence a is not a coatom of B and $a \neq 1$. So there exists $b \in B$ with $a <_B b <_B 1$.

Re(a.3). Assume $B \text{ -b } C$. Then $b \in B \cap C$ so $b \in [0, a]_B$ or $b \in [a', 1]_B$. By (a.2) there exists $d \in B$ such that $0 <_B d <_B a'$. If $b \in [a', 1]_B$ we have $d \in [0, b]_B \subseteq B \cap C$ and $d \notin [0, a]_B \cup [a', 1]_B = B \cap C$, a contradiction. Hence $b \in [0, a]_B$. By a symmetric argument $a \in [0, b]_B$.

Re(a.4). If $y \in [0, a]_B \cap [a', 1]_C$ then $a' \leq_C y \leq_B a$. Since $a, y \in B \cap C$ we have $a' \leq_B y \leq_B a$ which implies $a = 1$, contradicting (a.2). By a symmetric argument $[0, a]_C \cap [a', 1]_B = \emptyset$.

<u>Claim 2</u>. For pairwise distinct $B, C, D \in \mathcal{B}$, if $B \cap C \cap D \neq \{0, 1\}$ then there exists $a \in C$ such that $B \text{ -a } C \text{ -a } D$.

<u>Proof</u>. If $B \cap C \cap D \neq \{0, 1\}$ then there exists $a \in C \cap D$, $b \in B \cap C$ such that $B \text{ -b } C \text{ -a } D$. Since $B \cap C \cap D$

$\neq \{0, 1\}$, $a \not\leq_C b'$ so $a = b$.

Define an "ordering" and "complementation" on $\cup\mathcal{B}$ as:

(1) $x \leq y$ iff there exists $B \in \mathcal{B}$ such that $x \leq_B y$.

(2) $' : \cup\mathcal{B} \to \cup\mathcal{B}$ as in the remark below convention 1.

Let $L = (\cup\mathcal{B}, \leq, ', 0, 1)$.

Lemma 1. L is an orthocomplemented poset.

Proof. We first show that $\leq$ is a partial ordering. If $x \in L$ then there exists $B \in \mathcal{B}$ such that $x \in B$, so $x \leq_B x$ and $\leq$ is reflexive. If $x \leq y \leq x$ then there exist $B, C \in \mathcal{B}$ such that $x \leq_B y \leq_C x$. By claim 1 $y \leq_B x$ so $x = y$ and $\leq$ is antisymmetric. If $x \leq y \leq z$ then there exist $B, C \in \mathcal{B}$ such that $x \leq_B y \leq_C z$. If $\{x, y, z\} \cap \{0, 1\} \neq \emptyset$ or $B = C$ then $x \leq z$. Otherwise $B -a\ C$ for some $a \in B \cap C$ and $y \in B \cap C$. If $y \in [0, a]_B$ then $x \in C$ and $x \leq_C y \leq_C z$. If $y \in [a', 1]_C$ then $z \in B$ and $x \leq_B y \leq_B z$. Hence $\leq$ is transitive.

To see that ' is an orthocomplementation we first note that if $x, y \in L$ and $x \leq y$ then there exists $B \in \mathcal{B}$ such that $x, y \in B$, so $(x')' = x$ and $y' \leq x'$. It remains to show that ' is a complementation. If $d \geq x, x'$ then there exists $B \in \mathcal{B}$ with $d' \leq_B x \leq_B d$. This implies $d = 1$. By the dual argument $x \wedge x' = 0$.

Claim 3. For $B, C \in \mathcal{B}$ with $B -a\ C$, $[0, a]_B = \{x \in L: 0 \leq x \leq a\}$ ($= [0, a]$, by definition) and $[a', 1]_B = \{x \in L: a' \leq x \leq 1\}$ ($= [a', 1]$, by definition).

Proof. Clearly $[0, a]_B \subseteq [0, a]$. Let $b \in [0, a]$.

There exists $D \in \mathcal{B}$ with $b \leq_D a$. If $b = 0$ then $b \in [0, a]_B$. If $D = B$ then $b \in [0, a]_B$. Otherwise $b \in (B \cap C \cap D) \setminus \{0, 1\}$ and B -a D. Hence $b \in [0, a]_D = [0, a]_B$. The second part follows dually.

Definitions. (1) The elements of $\mathcal{B}$ are called the initial blocks of $\mathcal{B}$.

(2) For $M \subseteq L$, $U(M) = \{x \in L : x \geq m, \text{ for all } m \in M\}$.

(3) For $x, y \in L$:

$$J_{x,y} = \{B \in \mathcal{B} : x, y \in B\}$$

$$I_{x,y} = \begin{cases} \{0, 1\}, & \text{if } J_{x,y} = \emptyset \\ \cup J_{x,y}, & \text{otherwise} \end{cases}.$$

Claim 4. If $U(\{x, y\}) \subseteq I_{x,y}$ then $x \vee y$ exists. If, furthermore, $B \in J_{x,y}$ then $x \vee y = x \vee_B y$.

Proof. If $U(\{x, y\}) \subseteq I_{x,y} = \{0, 1\}$ then $U(\{x, y\}) = \{1\}$ and $x \vee y = 1$. Otherwise let $B \in J_{x,y}$ and $s \in U(\{x, y\}) \subseteq I_{x,y}$. There exist $C, D \in J_{x,y}$ such that $s \geq_C x$, $s \geq_D y$. So $s \in C \cap D$ and $y \in C \cap D$ which gives $s \geq_C x, y$. $x, y \in B \cap C$ and $s \geq_C x, y$ imply $s \geq x \vee_C y = x \vee_B y$. Since $x, y \leq x \vee_B y$ the claim follows.

Claim 5. For distinct $B, C \in \mathcal{B}$, if $x, y \in B \cap C$ then $x \vee y = x \vee_B y$.

Proof. Since $0 \vee y = y$ and $1 \vee y = 1$, we may assume that $\{x, y\} \cap \{0, 1\} = \emptyset$. If $1 > s \geq_D x$ for some $D \in \mathcal{B} \setminus \{B, C\}$ then $x \in (B \cap C \cap D) \setminus \{0, 1\}$. By claim 2 there exists $a \in L$ with B -a C -a D; $y \in B \cap C = C \cap D$ implies $D \in J_{x,y}$. By claim 4 $x \vee y = x \vee_B y$.

Definition. A loop of order n ($n \geq 3$) is a sequence of initial blocks $(B_0, B_1, \ldots, B_{n-1})$ such that:

(1) For any distinct $i, j \in \{0, \ldots, n-1\}$ $B_i \cap B_j = \{0, 1\}$ if $|j-i| \notin \{1, n-1\}$, and there exists $a_i \in L$ with $B_i -a_i\, B_j$ if $j-i \in \{1, n-1\}$.

(2) For pairwise distinct $i, j, k \in \{0, \ldots, n-1\}$ $B_i \cap B_j \cap B_k = \{0, 1\}$.

If $n > 3$ then condition (2) is a consequence of condition (1). If $n = 3$ then condition (2) is equivalent to the condition that any two of the a_i are different.

Claim 6. If $J_{x,y} \neq \emptyset$ and $U(\{x, y\}) \notin I_{x,y}$ then $\mathcal{B}$ admits a loop of order 3.

Proof. If $x = 0$ then $U(\{x, y\}) \subseteq U(\{y\}) \subseteq I_{0,y}$. If $x = 1$ then $U(\{x, y\}) = \{1\} \subseteq I_{1,y}$. So we may assume that $\{x, y\} \cap \{0, 1\} = \emptyset$. Let $a \in U(\{x, y\}) \setminus I_{x,y}$ and $B_1 \in J_{x,y}$. There exist $B_0, B_2 \in \mathcal{B} \setminus J_{x,y}$ such that $a \geq_{B_0} x$ and $a \geq_{B_2} y$. Clearly B_0, B_1 and B_2 are pairwise distinct. Since $x \in (B_0 \cap B_1) \setminus \{0, 1\}$, $y \in (B_1 \cap B_2) \setminus \{0, 1\}$ and $a \in (B_0 \cap B_2) \setminus \{0, 1\}$, $B_0 -a_0\, B_1 -a_1\, B_2 -a_2\, B_0$ for some $a_0, a_1, a_2 \in L$. Since $a \notin B_1$, $a_0 \neq a_1 \neq a_2 \neq a_0$ and (B_0, B_1, B_2) is a loop of order 3.

Claim 7. If $B \in J_{x,y}$ and $x \vee y$ exists then $x \vee y = x \vee_B y$.

Proof. If $x = 0$ then $x \vee y = y = x \vee_B y$. If $x = 1$

then $x \vee y = 1 = x \vee_B y$. If $\{B\} \neq J_{x,y}$ then $x \vee y = x \vee_B y$ by claim 5. If $x \vee y \in B$ then $x \vee_B y = x \vee y$. Assume none of the above conditions hold. There exist $C, D \in \mathcal{B} \setminus \{B\}$ such that $x \leq_C x \vee y \leq_C x \vee_B y$ and $y \leq_D x \vee y \leq_D x \vee_B y$. If $C = D$ then $B \neq C \in J_{x,y}$ contrary to the assumption that $\{B\} = J_{x,y}$. Thus $B -a\ C -b\ D -e\ B$ for some $a, b, e \in L$. If $x \vee_B y \neq 1$ then $x \vee_B y \in (B \cap C \cap D) \setminus \{0, 1\}$ and $a = b = e$. This gives $C \in J_{x,y}$, again a contradiction of our assumptions. Hence $x \vee_B y = 1$. If $x \vee_B y = 1$ then $x \geq_B y'$. If $u \in U(\{x, y\})$ then $u \in U(\{y, y'\})$ which implies $u = 1$. Hence $x \vee y = 1 = x \vee_B y$.

<u>Lemma 2</u>. L is an OMP iff $\mathcal{B}$ admits no loop of order 3.

<u>Proof</u>. ($\Leftarrow$). Assume $\mathcal{B}$ admits no loop of order 3. L is an OMP iff for all $x, y \in L$:

(1) If $x \leq y'$ then $x \vee y$ exists.

(2) If $x \leq y$ then $y = x \vee (x \vee y')'$.

Re(1). If $x \leq y'$ then $J_{x,y} \neq \emptyset$. By claim 6 $U(\{x, y\}) \subseteq I_{x,y}$. Let $B \in J_{x,y}$ then by claim 4 $x \vee y = x \vee_B y$.

Re(2). If $x \leq y$ then $x \leq_B y$ for some $B \in J_{x,y}$. From Re(1). $x \vee y' = x \vee_B y'$. Also $x \leq_B x \vee_B y'$ and $B \in J_{x, x \vee_B y'}$. So again from Re(1). $x \vee (x \vee y')' = x \vee_B (x \vee_B y'')' = y$, since B is Boolean.

($\Rightarrow$). Assume $\mathcal{B}$ admits a loop of order 3, $B_0 -a_0\ B_1 -a_1\ B_2 -a_2\ B_0$. Since $a_2 < a_0'$, $a_2 < a_1'$ we have

$U(\{a_0, a_1\}) \supseteq \{a_0 \vee_{B_1} a_1, a_2', 1\}$. To show that $a_0 \vee a_1$ does not exist it is sufficient to show that (1) $a_0 \vee_{B_1} a_1 \nleq a_2'$ and (2) if $a_0 \vee a_1$ does exist then $a_0 \vee a_1 = a_0 \vee_{B_1} a_1$.

Re(2). (2) is a direct application of claim 7.

Re(1). Assume $a_0 \vee_{B_1} a_1 \leq_C a_2'$ for some $C \in \mathcal{B}$. Since $a_2' \notin B_1$ and $0 < a_0 \vee_{B_1} a_1 < 1$, $B_1 -b\ C$ for some $b \in B_1$. If $a_0 \vee_{B_1} a_1 \in [0, b]$ then $a_0 <_{B_1} a_0 \vee_{B_1} a_1 \leq_{B_1} b$, a contradiction. If $a_0 \vee_{B_1} a_1 \in [b', 1]$ then $b' \leq_C a_0 \vee_{B_1} a_1 <_C a_2'$, again a contradiction. Hence $a_0 \vee_{B_1} a_1 \nleq a_2'$.

Henceforth assume $\mathcal{B}$ admits no loop of order 3.

Claim 8. If $d, t \in U(\{x\})$ and $J_{d,t} \neq \emptyset$ then there exists $B \in \mathcal{B}$ such that $x \leq_B d, t$.

Proof. Assume there exist $B_0, B_1, B_2 \in \mathcal{B}$ such that $x \leq_{B_0} d$, $x \leq_{B_1} t$ and $B_2 \in J_{d,t}$. If $x = 0$ then $x \leq_{B_2} d, t$. If $x = 1$ then $1 = x = d = t$. If $t = 1$ then $x \leq_{B_0} d, t$. If $d = 1$ then $x \leq_{B_1} d, t$. If $B_i = B_j$ for some $i \neq j$ then $x \leq_{B_i} d, t$. If none of the above conditions hold then $x \in (B_0 \cap B_1) \setminus \{0, 1\}$, $d \in (B_0 \cap B_2) \setminus \{0, 1\}$ and $t \in (B_1 \cap B_2) \setminus \{0, 1\}$. Hence there exist $a_0, a_1, a_2 \in L$ with $B_0 -a_0\ B_1 -a_1\ B_2 -a_2\ B_0$. Since (B_0, B_1, B_2) is not a loop of order 3 $a_0 = a_1 = a_2$ and $x \in B_2$. This gives $x \leq_{B_2} d, t$.

Claim 9. If $x \vee y$ does not exist then there exist $d, t \in U(\{x, y\})$ such that $J_{d,t} = \emptyset$.

Proof. If $U(\{x, y\}) = \{1\}$ then $x \vee y = 1$. Assume $x \vee y$ does not exist. By claims 4 and 6, $J_{x,y} = \emptyset$. Let

$t \in U(\{x, y\}) \setminus \{1\}$, since $J_{x,y} = \emptyset$ there exist distinct $B_0, B_1 \in \mathcal{B}$ with $x \leq_{B_0} t$, $y \leq_{B_1} t$ and $B_0 - a\ B_1$ for some $a \in L$. Since $x \notin B_1$, $t \in [a', 1]$. Let $\hat{x} = x \vee_{B_0} a$, $\hat{y} = y \vee_{B_1} a$ and $\hat{t} = \hat{x} \vee \hat{y}$ $(= \hat{x} \vee_{B_1} \hat{y})$. If $U(\{x, y\}) \subseteq B_0 \cap B_1$ then $U(\{x, y\}) = U(\{\hat{x}, \hat{y}\})$ and $x \vee y = \hat{t}$. So there exists $d \in U(\{x, y\}) \setminus (B_0 \cap B_1)$. Assume $d \notin B_1$. If there exists $B_2 \in \mathcal{B}$ with $y \leq_{B_2} d, t$ then we have $t \in (B_0 \cap B_1 \cap B_2) \setminus \{0, 1\}$ and $B_0 \neq B_1 \neq B_2 \neq B_0$. By claim 2 $B_1 \cap B_2 = B_0 \cap B_1$ and hence $y \in B_0$, a contradiction. Therefore no such B_2 exists and by claim 8 $J_{d,t} = \emptyset$. If we assume $d \notin B_0$ the same result is obtained.

<u>Claim 10</u>. If $x \wedge y$, $x \wedge y'$, $(x \wedge y) \vee (x \wedge y')$ exist and $J_{x,y} = \emptyset$ then $x \neq (x \wedge y) \vee (x \wedge y')$.

<u>Proof</u>. $J_{x,y} = \emptyset$ implies $\{x, y\} \cap \{0, 1\} = \emptyset$. There exist $B_0, B_1 \in \mathcal{B}$ such that $x \wedge y' \leq_{B_0} y'$ and $x \wedge y \leq_{B_1} y$. Since $J_{x,y} = \emptyset$, $B_0 \neq B_1$. $y \in (B_0 \cap B_1) \setminus \{0, 1\}$ so there exists $a \in L$ with $B_0 - a\ B_1$. If $y' \in [0, a]$ then $x \wedge y' \leq_{B_1} y'$, if $y' \in [a', 1]$ then $y \in [0, a]$ and $x \wedge y \leq_{B_0} y$. So there exists $A \in \mathcal{B}$ such that $\{x \wedge y', x \wedge y, y, y'\} \subseteq A$. Since $(x \wedge y') \vee (x \wedge y)$ exists we have by claim 7 that $(x \wedge y') \vee (x \wedge y) = (x \wedge y') \vee_A (x \wedge y)$. But $J_{x,y} = \emptyset$ so $x \notin A$. Hence $x \neq (x \wedge y') \vee (x \wedge y)$.

<u>Lemma 3</u>. The blocks of L are precisely the initial blocks.

<u>Proof</u>. From claims 4, 6 and 7 we have that the

initial blocks are Boolean subortholattices of L. Therefore, it is sufficient to show that if A is a Boolean subortholattice of L then $A \subseteq B$ for some $B \in \mathcal{B}$.

Let A be a Boolean subortholattice of L and $x, y \in A \setminus \{0, 1\}$. By claim 10 there exists $B_0 \in J_{x,y}$. If $A \subseteq B_0$ then we are finished. Otherwise, there exists $z \in A \setminus B_0$. By claim 10 there exist $B_1 \in J_{x,z}$, $B_2 \in J_{y,z}$. If $\{B_1, B_2\} \cap J_{x,z} \cap J_{y,z} = \emptyset$ then $x \in (B_0 \cap B_1) \setminus B_2$, $y \in (B_0 \cap B_2) \setminus B_1$ and $z \in (B_1 \cap B_2) \setminus B_0$. This implies that (B_0, B_1, B_2) is a loop of order 3, a contradiction. Hence $\{B_1, B_2\} \cap J_{x,z} \cap J_{y,z} \neq \emptyset$. If $B_1 \in J_{x,z} \cap J_{y,z}$ and $w \in A \setminus B_1$ then, again by claim 10, there exist $B_3 \in J_{w,x}$, $B_4 \in J_{w,z}$. Since $w \in B_3 \setminus B_1$ and $x \in (B_0 \cap B_1 \cap B_3) \setminus \{0,1\}$, by claim 2 $B_1 \cap B_3 = B_0 \cap B_1$. Since $z \in B_1$ and $z \notin B_0$, we have $z \notin B_3$. Since $w \in B_4 \setminus B_1$ and $z \in (B_1 \cap B_2 \cap B_4) \setminus \{0, 1\}$, $B_1 \cap B_4 = B_1 \cap B_2$. Since $x \notin B_2$ and $x \in B_1$, $x \notin B_4$. We now have $w \in (B_3 \cap B_4) \setminus B_1$, $z \in (B_1 \cap B_4) \setminus B_3$ and $x \in (B_1 \cap B_3) \setminus B_4$. But this implies that (B_1, B_3, B_4) is a loop of order 3, a contradiction. Hence, if $B_1 \in J_{x,z} \cap J_{y,z}$ then $A \subseteq B_1$. By a symmetric argument, if $B_2 \in J_{x,z} \cap J_{y,z}$ then $A \subseteq B_2$. $\{B_1, B_2\} \cap J_{x,z} \cap J_{y,z} \neq \emptyset$, therefore $A \subseteq B_1$ or $A \subseteq B_2$.

Lemma 4. L is an OML iff admits no loop of order 4.

Proof. ($\Leftarrow$). Assume $x \vee y$ does not exist. By claims 9, 4 and 6 there exist $c, d \in U(\{x, y\}) \setminus \{0, 1\}$ such that $J_{c,d} = \emptyset$ and $J_{x,y} = \emptyset$. There exist $B_0, B_1, B_2, B_3 \in \mathfrak{B}$ such that $x \leq_{B_0} c$, $x \leq_{B_1} d$, $y \leq_{B_2} d$ and $y \leq_{B_3} c$. $J_{c,d} = \emptyset$ implies $B_0 \neq B_1 \neq B_3 \neq B_2 \neq B_0$, $J_{x,y} = \emptyset$ implies $B_1 \neq B_2$ and $B_3 \neq B_0$. $x \in (B_0 \cap B_1) \setminus \{0, 1\}$, $d \in (B_1 \cap B_2) \setminus \{0, 1\}$ $y \in (B_2 \cap B_3) \setminus \{0, 1\}$ and $c \in (B_0 \cap B_3) \setminus \{0, 1\}$. So there exist $a_0, a_1, a_2, a_3 \in L$ such that $B_0 - a_0\ B_1 - a_1\ B_2 - a_2\ B_3 - a_3\ B_0$. To prove that (B_0, B_1, B_2, B_3) is a loop of order 4 it remains to show that $\{0, 1\} = B_0 \cap B_2 = B_1 \cap B_3$. Since (B_0, B_1, B_3) is not a loop of order 3 either $B_1 \cap B_3 = \{0, 1\}$ or $a_0 = a_3$. If $a_0 = a_3$ then $B_0 \cap B_1 = B_0 \cap B_3$ which gives $c \in B_1$. But $c \in B_1$ implies $B_1 \in J_{c,d}$, a contradiction. Hence $B_1 \cap B_3 = \{0, 1\}$. By a symmetric argument $B_0 \cap B_2 = \{0, 1\}$.

($\Rightarrow$). Assume $\mathfrak{B}$ admits a loop of order 4, $B_0 - a_0\ B_1 - a_1\ B_2 - a_2\ B_3 - a_3\ B_0$. $a_0 < a_1', a_3'$ and $a_2 < a_1', a_3'$ so $U(\{a_0, a_2\}) \supseteq \{a_1', a_3', 1\}$. Hence if $a_0 \vee a_2$ exists then there exists $a \in L$ such that $a_0, a_2 \leq a \leq a_1', a_3'$. To prove $a_0 \vee a_2$ does not exist we will assume that such an a exists and derive a contradiction. Since $0 < a_0 \leq a \leq a_1' < 1$, $a \notin \{0, 1\}$. If $a \notin B_1$ there exists $C \in \mathfrak{B}$ such that $a_0 \leq_C a \leq_C a_1'$ and $B_1 - b\ C$ for some $b \in L$. Since $a_0 \in (B_0 \cap B_1 \cap C) \setminus \{0, 1\}$, $a_0 = b$. Since $a_1 \in (B_1 \cap B_2 \cap C) \setminus \{0, 1\}$, $a_1 = b$. But since (B_0, B_1, B_2, B_3) is a loop of order 4, $a_0 \neq a_1$. Hence $a \in B_1$. By a symmetric argument

$a \in B_3$. But since (B_0, B_1, B_2, B_3) is a loop of order 4, $a \in ((B_1 \cap B_3) \setminus \{0, 1\}) = \emptyset$, a contradiction. Hence $a_0 \vee a_2$ does not exist.

3. Concluding Remarks.

The results of the previous section are summarized in the following lemma.

Loop lemma. Let $\mathcal{B}$ be a set of Boolean lattices satisfying convention 1. L is an OMP (OML) iff $\mathcal{B}$ admits no loop of order less than 4 (5). If L is an OMP then the blocks of L are precisely the initial blocks.

A simple example of an OML which is obtainable via this construction but not by pasting is given. Let B_0, B_1, B_2, B_3, B_4 be arbitrary Boolean lattices such that $\{B_0, \ldots, B_4\}$ satisfies convention 1 and such that $B_0 -a_0\ B_1 -a_1\ B_2 -a_2\ B_3 -a_3 B_4 -a_4\ B_0$ with $a_i \neq a_j$, for $i \neq j$. Then $(B_0, B_1, B_2, B_3, B_4)$ is a loop of order 5 and $\bigcup_{i=0}^{4} B_i$ is an OML.

If L is an OML whose set of blocks satisfies convention 1 then either L is a weak horizontal sum of Boolean lattices or there exist $A, B \in \mathcal{B}(L)$ such that $A \cap B = \{0, 1\}$. In either case there exist $A, B \in \mathcal{B}(L)$ such that $A \cap B = C(L)$. This is proved and extended to products of such lattices in the following claim.

Claim 11. Let $(L_i)_{i \in I}$ be a non-empty family of OML's such that for each $i \in I$, $\mathcal{B}(L_i)$ satisfies convention 1. Let $L = \prod_{i \in I} L_i$. Then there exist $A, B \in \mathcal{B}(L)$ such that $A \cap B = C(L)$.

Proof. Let $i \in I$. Since $\mathcal{B}(L_i)$ satisfies convention 1 either

(1) L_i is Boolean and $\mathcal{B}(L_i) = \{L_i\}$.

or (2) $A \text{-}a\, B$ for some $a \in L$, for all distinct $A, B \in \mathcal{B}(L_i)$.

or (3) There exist $B, C, A, D \in \mathcal{B}(L_i)$ such that $B \text{-}a\, C$, $A \text{-}b\, D$ and $a \neq b$.

Re(1). $C(L_i) = L_i = L_i \cap L_i$.

Re(2). For distinct $A, B \in \mathcal{B}(L_i)$, $A \cap B = [0, a] \cup [a', 1] = C(L_i)$.

Re(3). For B, C, A, D as in (3) we may assume that $A \neq B$. If $A \cap B \neq \{0, 1\}$ then either $A \cap C = \{0, 1\}$ or $A \text{-}a\, B$, since (A, B, C) is not a loop of order 3. If $A \cap C \neq \{0, 1\}$ then $B \cap D = \{0, 1\}$, since (B, A, D) is not a loop of order 3. $C(L_i) = \{0, 1\}$.

We have shown that for any $i \in I$ there exist $A_i, B_i \in \mathcal{B}(L_i)$ such that $A_i \cap B_i = C(L_i)$. But $C(L) = \prod_{i \in I} C(L_i) = \prod_{i \in I} (B_i \cap A_i) = (\prod_{i \in I} B_i) \cap (\prod_{i \in I} A_i)$ and $\prod_{i \in I} B_i, \prod_{i \in I} A_i \in \mathcal{B}(L)$.

It is therefore natural to ask whether all OML's have this property. A finite counterexample is given.

Let $\langle x_0, \ldots, x_n \rangle$ denote a Boolean lattice with atoms $x_0, \ldots, x_n$. Let $B_0 = \langle b_0, b_1, b_2, b_3, b_4 \rangle$, $B_1 = \langle b_0, b_1, b_5, b_6 \rangle$, $B_2 = \langle b_1, b_2, b_7, b_8 \rangle$, $B_3 = \langle b_0, b_2, b_9, b_{10} \rangle$ be Boolean lattices such that for all i, j $\in$ {0, ...10} $b_i = b_j$ iff i = j and such that for all i, j $\in$ {0, 1, 2, 3} $B_i \cap B_j$ is a Boolean sublattice of B_i. By applying Greechie's paste job (theorem 3.4 of [1]) three times we have that $L_1 = B_0 \cup B_1$ is an OML with blocks B_0, B_1, that $L_2 = L_1 \cup B_2$ is an OML with blocks B_0, B_1, B_2, and that $L = L_2 \cup B_3$ is an OML with blocks B_0, B_1, B_2, B_3. $C(L) = \{0, 1\}$ and $B_i \cap B_j \neq \{0, 1\}$ for any i, j $\in$ {0, 1, 2, 3}. Figure 1 gives the "Greechie" diagram [2] of L.

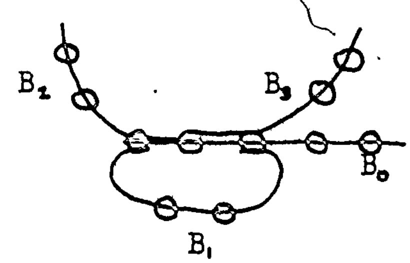

Figure 1.

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