MODULES IN WHICH COMPLEMENTS ARE SUMMANDS

By

MAHMOUD AHMED KAMAL-ELDEEN, B.Sc., M.Sc.

A Thesis
Submitted to the School of Graduate Studies
in Partial Fulfilment of the Requirements
for the Degree
Doctor of Philosophy

McMaster University

MODULES IN WHICH COMPLEMENTS ARE SUMMANDS

DOCTOR OF PHILOSOPHY (1986)
(Mathematics)

McMaster University
Hamilton, Ontario

TITLE: Modules in which Complements are Summands

AUTHOR: Mahmoud A. Kamal-Eldeen, B.Sc. (Ain Shams Univ., Cairo, Egypt)
M.Sc. (Ain Shams Univ., Cairo, Egypt)

SUPERVISOR: Professor Bruno J. Mueller

NUMBER OF PAGES: v, 110

ABSTRACT

This thesis studies modules over commutative integral domains with the property that every closed submodule is a direct summand (we denote this property by (C_1)). It is shown that any non-torsion module with property (C_1) is a direct sum of an injective submodule and a finite direct sum of uniform torsion free reduced submodules. This reduces the study of the problem to finite direct sums of uniform torsion free reduced modules and to torsion modules. Then we characterize finite direct sums of uniform torsion free reduced modules over commutative (Prüfer, Noetherian of Krull dimension one, Dedekind) domains which have property (C_1). We also characterize finite direct sums of uniform torsion modules with local endomorphism rings over Noetherian domains which have property (C_1). Finally, we classify all modules with property (C_1) over Dedekind domains.

TO

My Wife, Hoda

and

My Son, Mostafa

My Daughter, Noha

ACKNOWLEDGEMENTS

The author would like to express his deep gratitude to Professor Bruno J. Mueller for his patience, criticism, advice and support during the preparation of this work. His course and his research have stimulated my interest in Rings and Modules, and his supervision has helped me make my start in Mathematical research. I am also grateful for his careful review of the manuscript.

I also thank Professors B. Banaschewski and S.H. Mohamed for their encouragement and great help.

I am thankful to the Egyptian Government for providing me with generous financial support.

I am also thankful to the Department of Mathematics and Statistics at McMaster University for making my Ph.D. program such an enjoyable learning experience.

My thanks go to Ms. Antoinette Spinosa, whose patience and competence while typing this dissertation were outstanding.

Finally, I am indebted to my wife, Hoda, for many arduous hours she spent to make this degree possible for me, and I wish to thank her for her constant support, confidence and patience during the years of my study.

TABLE OF CONTENTS

INTRODUCTION

1. A submodule N of a module M has no proper essential extension in M, if and only if there is another submodule N' such that N is maximal with respect to $N \cap N' = 0$. In the literature, such submodules N are called *closed*, or *complements*.

A module is said to have *property* (C_1), if every closed submodule is a direct summand. The property (C_1) holds in particular if the module is (quasi) injective, or more generally (quasi) continuous.

2. J. Von Neumann [19] showed that his continuous geometries can be coordinatized by continuous regular rings. Y. Utumi [18] studied regular rings, and he proved that a regular ring is continuous if and only if it has property (C_1) for right and for left ideals.

Later A.W. Chatters and C.R. Hajarnavis [21] investigated rings with chain conditions in which every complement right ideal is a direct summand.

L. Fuchs, A. Kertesz and T. Szele [22] discussed abelian groups in which every pure subgroup is a direct summand. In the case of torsion free abelian groups pure subgroups are the same as closed subgroups, but in the torsion case pure subgroups need not be closed.

S. Mohamed, B.J. Mueller and S. Singh (1983) characterized arbitrary abelian groups with property (C_1).

M. Harada and K. Oshiro [23] considered modules with extending properties, which are closely related to the property (C_1). M. Harada [24] described modules with (C_1) over Dedekind domains; an unfortunate misapplication of ([23], theorem 10) kept him from obtaining the full characterization, which we give here.

3. The present thesis studies arbitrary modules with (C_1), over commutative integral domains.

The first chapter reduces the study of the property (C_1) to the cases of torsion modules and of torsion free reduced modules, by showing that any non-torsion module with (C_1) is a direct sum of an injective submodule and a torsion free reduced submodule (th. (1.15), (1.16)).

The second chapter investigates torsion free reduced modules with (C_1). We prove that torsion free reduced modules with (C_1) are finite direct sums of uniform submodules (th. (2.16)).

We also give a necessary and sufficient condition for the direct sum of a pair of uniform torsion free reduced modules to have (C_1) (th. (2.26)). We conjecture that a finite direct sum of uniform torsion free reduced modules has (C_1) if and only if the direct sum of each pair has (C_1). We prove this conjecture in the following cases: i) general commutative domains and the uniform summands have local endomorphism rings; ii) Prüfer domains; iii) Noetherian domains and the uniform summands are finitely generated; iv) one dimensional Noetherian domains.

Finally we give a complete description of torsion free reduced modules with (C_1) over Dedekind domains, and over one dimensional Noetherian domains. The description in the first case is much simpler than in the second one.

The third chapter considers torsion modules over Noetherian domains. It was proved, by K. Oshiro in a letter to S. Rizvi, that a module with (C_1) over a Noetherian ring is a direct sum of uniform submodules. We have only obtained results in the case that all these uniform summands have local endomorphism rings.

We provide a necessary and sufficient condition for the direct sum of a pair of uniform torsion modules to have (C_1), provided that they have distinct associated primes and arbitrary endomorphism rings, or the same associated prime and local endomorphism rings.

We also prove that a finite direct sum of uniform torsion modules with local endomorphism rings has (C_1) if and only if the direct sum of each pair has (C_1).

We end this chapter by giving a full characterization of arbitrary torsion modules with (C_1) over Dedekind domains.

4. Our analysis of the property (C_1) in the torsion free case was easier and led to more complete results than in the torsion case. One reason may be that a torsion free injective module is always the direct sum of copies of the quotient field, while there is no good structure theorem for torsion injectives except over Noetherian rings. This may also explain why we had to confine ourselves, in the torsion case, to modules over Noetherian domains.

Another interesting point is that we use a characterization of (C_1) for torsion free modules (2.32) in the study of (C_1) for torsion modules (3.18). The torsion free result is applied to certain factor modules which turn out to be torsion free over an appropriate factor ring.

CHAPTER I

MODULES WITH PROPERTY (C_1) : REDUCTIONS

We recall that R is always a commutative integral domain, and that a module has property (C_1) if every closed submodule is a direct summand.

In this chapter we reduce the study of property (C_1) to the cases of torsion modules and of torsion free reduced modules.

§1. PRELIMINARIES.

In this section we list some well known facts for later use. Some of them are valid for modules over arbitrary rings, while others require our standing assumption that R is a commutative integral domain.

Definition 1.1: A submodule $M \subset E$ is said to be essential in E (denoted by $M \subset' E$), if for every submodule N of E, $M \cap N = 0$ implies $N = 0$. A module E is called uniform if it is non-zero and every non-zero submodule of E is essential in E. A submodule M is called closed in E if it has no proper essential extension in E. By Zorn's lemma any submodule A of E has a maximal essential extension in E.

Definition 1.2: A module E is said to be injective, if for every monomorphism $f: A \to B$ and every homomorphism $\varphi: A \to E$, there exists a homomorphism $\hat{\varphi}: B \to E$ such that $\hat{\varphi} f = \varphi$.

This concept was introduced by Baer [4] and Nakayama [15]. It was shown by Eckmann and Schopf [7] that every module can be embedded in an injective module. In fact they showed that, for any module M, there is an injective overmodule E of M which is essential over M. This overmodule is unique up to isomorphism over M, and is called the injective hull of M and denoted by $E(M)$.

The concept of injectivity was generalized to that of relative injectivity by Azumaya [2], and Azumaya, Mbuntum and Varadarajan [3].

Definition 1.3: A module M is said to be N-injective, if for every submodule L of N, every homomorphism $f: L \to M$ can be extended to $\hat{f}: N \to M$.

Definition 1.4: Let M be an R-module. The set $t(M) = \{x \in M : xr = 0 \text{ for some } 0 \neq r \in R\}$ is a submodule, called the torsion submodule of M. If $t(M) = M$, then M is called torsion; if $t(M) = 0$, then M is called torsion free.

Note that $t(M)$ is a closed submodule of M.

By Baer's well-known criterion for injectivity one can show that an R-module is injective if and only if it is $\mathcal{F}$-injective for some torsion free R-module $\mathcal{F}$.

Definition 1.5: A module M is said to have property (C_1), if every closed submodule of M is a direct summand. A module M is said to have property $(1-C_1)$, if every uniform closed submodule is a direct summand.

If N is a submodule of M, then $N \overset{\oplus}{\subset} M$ will signify that N is a direct summand of M.

Lemma 1.6: If an R-module M has property (C_1), then each direct summand N has again property (C_1).

Proof: Let $M = N \oplus T$, and let A be a closed submodule of N.

We show that A is closed in M. Let $A \subset' X \subset M$. Then $A = \pi A \subset \pi X \subset N$, where $\pi : M \to N$ is the projection onto N. We claim that $A \subset' \pi X$. Let $0 \neq \alpha \in \pi X$, hence $\alpha = \pi x$, $x \in X$. By essentiality of X over A, there exists $r \in R$ such that $0 \neq xr \in A$. If follows that $\alpha r = (\pi x)r = \pi(xr) = xr \in A$. Therefore $A \subset' \pi X$.

Since A is closed in N and $A \subset' \pi X \subset N$, we have $A = \pi X$.

Now let $x \in X$ be an arbitrary. Then $\pi x \in A$, i.e. $\pi x = a$, $a \in A$. Hence $\pi x = a = \pi a$. Thus $\pi(x-a) = 0$, i.e. $x-a \in \ker \pi = T$. Since $A \cap T = 0$ and $A \subset' X$, we have $X \cap T = 0$. Then $x-a \in T \cap X = 0$. It follows that $x = a \in A$. Then A has no proper essential extension in M, hence A is closed in M.

By (C_1) $M = A \oplus B$ for some submodule B of M. By the modular law $N = A \oplus B \cap N$, i.e. $A \overset{\oplus}{\subset} N$. Therefore N has (C_1). □

Note that the same proof shows that $(1-C_1)$ is inherited by direct summands.

Lemma 1.7: (i) A direct summand of a closed submodule of a module M is a closed submodule of M. If N is closed in M, then N is closed in any submodule of M containing N.
(ii) Let $M = X \oplus Y$ be a module and $\varphi: X \to Y$ be an arbitrary homomorphism. Then $X^* =: \{x + \varphi(x) : x \in X\}$ is a submodule of M isomorphic to X via $x \to x + \varphi(x)$, $x \in X$; and $M = X^* \oplus Y$. □

Definition 1.8: An R-module M is divisible, if for every element $x \in M$ and for every $0 \neq r \in R$ there exists an element $y \in M$ such that $x = yr$. This definition means that every element of M is divisible by every non-zero element r of R or, alternatively, $rM = M$. A module which has no non-trivial divisible submodule is called reduced. Every injective module is divisible.

Lemma 1.9: Every divisible, torsion free module is injective. □

It is clear that any direct sum of divisible modules is divisible.

Definition 1.10: Let M be a torsion free R-module. Let $D(M) = \{ x \in M :$ for all $0 \neq r \in R$ there exists $y \in M$ such that $x = yr \}$. According to [17], $D(M)$ is the largest divisible submodule of M and $M/D(M)$ does not contain any non-zero divisible submodule. We shall call $M/D(M)$ the reduct of M.

Remarks 1.11: (i) If A is a submodule of a torsion free R-module M, then the maximal essential extension of A in M is uniquely determined; this submodule of M consists of all those elements x of M for which $xr \in A$ holds with a suitable non-zero element $r \in R$.

(ii) Every essential extension of a uniform module is uniform.

Lemma 1.12: Every non-zero torsion free R-module F which has property (C_1) contains a uniform direct summand F_1.

Proof: Let $0 \neq x \in F$. Since the commutative integral domain R is uniform as module over itself, $xR \cong R$ is a uniform submodule of F. Let F_1 be the maximal essential extension of xR in F, then F_1 is a closed submodule of F, and by (C_1) F_1 is a direct summand. Since uniformity is preserved by essential extensions, F_1 is a uniform direct summand of F. □

Note that the lattice of submodules of every module M satisfies the modular law, i.e. if A, B and C are submodules of M such that $B \subset A$, then $A \cap (B + C) = B + A \cap C$ holds. □

§2. NON-TORSION MODULES

Lemma 1.13: In every R-module with property (C_1) the torsion submodule is a direct summand.

Proof: Obvious since $t(M)$ is closed, c.f. (1.4). □

The next theorem gives the necessary and sufficient condition for non-torsion module to have (C_1).

First we give the following lemma, which will be used in the proof of the theorem.

Lemma 1.14: Let $M = T \oplus F$, where F is non-zero torsion free and T is torsion. Let F' be a non-zero torsion free submodule of M such that $M = F' \oplus X$. Then $T \subset X$. If F is uniform, then $X = T$.

Proof: Let $t \in T$; then $t = f' + x$, where $f' \in F'$ and $x \in X$. Since T is torsion, there exists a non-zero element r in R such that $0 = tr = f'r + xr$. Since the sum is direct, it follows that $f'r = xr = 0$. But F' is torsion free, hence $f' = 0$, and $t \in X$. Therefore $T \subset X$. By the modular law $X = T \oplus (F \cap X)$, hence $M = F' \oplus T \oplus (F \cap X)$. It follows that $F \cong M/T \cong F' \oplus (F \cap X)$.

Now if F is uniform, then $F \cap X = 0$ (since F' is non-zero), and we get $X = T$.

Theorem 1.15: Let M be an R-module which is not torsion. Then M has (C_1) if and only if $t(M)$ is injective and the torsion free factor module $M/t(M)$ has (C_1).

Proof: Let M have (C_1). By lemma (1-13) we have $M = t(M) \oplus F$, where F is a non-zero torsion free submodule of M.

Since (C_1) is inherited by direct summands, F has (C_1). By (1.12), there exists a non-zero uniform direct summand F_1 of F, i.e. $F = F' \oplus F_1$ and $M = F' \oplus F_1 \oplus t(M)$.

Now $M' =: F_1 \oplus t(M)$ again has (C_1). To show that $t(M)$ is injective, by Baer's criterion for injectivity, it is enough to show that $t(M)$ is F_1-injective.

Now let $\varphi : X \to t(M)$ be a homomorphism from a submodule X of F_1 into $t(M)$. Consider $X' =: \{x - \varphi(x) : x \in X\}$. By (C_1) for M', there exists a submodule X^* of M' such that $X' \subset' X^* \overset{\oplus}{\subset} M'$. Since X is torsion free and $X \cong X' \subset' X^*$ and essential extensions of torsion free modules are torsion free, X^* is a uniform torsion free direct summand of M'. By lemma (1.14) we get $M' = X^* \oplus t(M)$.

Let $M' \overset{\pi}{\to} t(M)$ be the projection of M' onto $t(M)$. Since $x - \varphi(x) \in X' \subset X^*$, we have $0 = \pi(x - \varphi(x)) = \pi(x) - \pi\varphi(x) = \pi(x) - \varphi(x)$ for all $x \in X$, i.e. $\pi(x) = \varphi(x)$ for all $x \in X$.

Therefore $\pi|_{F_1} : F_1 \to t(M)$ extends φ, which shows that $t(M)$ is F_1-injective.

Conversly let $t(M)$ be injective and $M/t(M)$ have (C_1). Then $t(M) \overset{\oplus}{\subset} M$, and we can write $M = t(M) \oplus F$ where $M/t(M) \cong F$ is non zero torsion free and has (C_1).

Now let A be a closed submodule of M and $t(A)$ be its torsion submodule. We claim that $t(A)$ is closed submodule of $t(M)$. Let $t(A) \subset' Y \subset t(M)$. We show that $A \subset' A + Y$. To this end consider any $0 \neq y + a \in Y + A$, where $y \in Y$ and $a \in A$. If $a \in t(A)$, then $y + a \in Y$, and by essentiality of Y over $t(A)$ we can find $r \in R$ such that $0 \neq (y + a)r \in t(A) \subset A$. If $a \notin t(A)$, since Y is torsion, there exists $0 \neq s \in R$ such that $ys = 0$, and we have $(y + a)s = as \neq 0$ $(\in A)$.

Since A is closed in M, we conclude $A = Y + A$, and hence $Y = t(A)$, which shows that $t(A)$ has no proper essential extension in $t(M)$ and is thus closed. Now the injectivity of $t(M)$ implies that $t(A)$ is injective, and hence A can be written as $A = t(A) \oplus B$, where B is a torsion free submodule of A.

Let π_1, π_2 be the projections of M onto $t(M)$ and F respectively. Since B is torsion free, $B \xrightarrow{\pi_{2|B}} F$ is a monomorphism. Since $t(M)$ is injective, there exists a homomorphism $\psi : F \to t(M)$ such that $\psi\pi_{2|B} = \pi_{1|B}$, i.e. $\psi\pi_2(b) = \pi_1(b)$ for all $b \in B$.

Now let $F^* =: \{f + \psi(f) : f \in F\}$. Then $F^* \cong F$ has (C_1). Consider any element $b \in B$. b can be written as $b = \pi_2(b) + \pi_1(b) = \pi_2(b) + \psi(\pi_2(b)) \in F^*$, i.e. $B \subset F^*$, By (i) in (1.7), B is a closed submodule of F^*. By (C_1), $F^* = B \oplus B'$. Since $t(A)$ is an injective submodule of $t(M)$, then $t(M) = t(A) \oplus C$ for some submodule C of $t(M)$. By (ii) in (1.7), $M = F^* \oplus t(M) = B \oplus B' \oplus t(A) \oplus C = A \oplus B' \oplus C$, i.e. $A \overset{\oplus}{\subset} M$. Hence M has (C_1). □

§3. NON-REDUCED TORSION FREE MODULES

From (1.9), a torsion free R-module is divisible if and only if it is injective.

Recall the definition of the reduct of a torsion free module given in (1.10) : Let M be a torsion free R-module, and let E be the maximal injective submodule of M. The factor module M/E is called the reduct of M; it does not contain any injective submodule except zero.

The following theorem reduces the problem of studing property (C_1) from torsion free modules to torsion free reduced modules.

<u>Theorem 1.16:</u> A torsion free R-module has (C_1) if and only if its reduct has (C_1).

<u>Proof:</u> Let M has (C_1), and let E be its largest injective submodule. Then $M = E \oplus C$, where C is a reduced submodule of M. Since (C_1) is inherited by direct summands, the reduct $M/E \cong C$ has (C_1).

Conversely let $C \cong M/E$, the reduct of M, have (C_1). Let A be a closed submodule of M. We claim that $D(A)$, the largest injective submodule of A, is exactly $A \cap E$. It is clear that $D(A) \subset E \cap A$. Now let $x \in E \cap A$. Since E is divisible, for any $0 \neq r \in R$ there exists an element $e \in E$ such that $er = x \in A$. Since A is closed submodule of M, then $e \in A$ by (i) in (1.11). It follows that $E \cap A$ is a divisible hence injective submodule of A, and therefore $E \cap A \subset D(A)$. Then $D(A) = E \cap A$.

Now A can be written as $A = E \cap A \oplus B$, where B is reduced. From (i) in (1.7), B is a closed submodule of M.

Let $M \xrightarrow{\pi} C$ and $M \xrightarrow{\pi'} E$ be the projections of M onto C and E respectively. Since $E \cap B = 0$ ($x \in E \cap B$ implies $x \in A \cap E \cap B = 0$), $B \xrightarrow{\pi|B} C$ is a monomorphism. From the injectivity of E, there exists a homomorphism $\varphi\colon C \to E$ such that $\varphi\pi|_B = \pi'|_B$, i.e. $\varphi\pi(b) = \pi'(b)$ for all $b \in B$.

Let $C^* =: \{\varphi(c) + c : c \in C\}$; then $C^* \cong C$ has (C_1). Now let $b \in B$; then $b = \pi'(b) + \pi(b) = \varphi\pi(b) + \pi(b) \in C^*$ $(\pi(b) \in C)$. From (i) of (1.7), B is a closed submodule of C^*. By (C_1), $C^* = B \oplus Y$ for some submodule Y of C^*. By (ii) of (1.7), $M = C^* \oplus E = B \oplus Y \oplus E$. Since $A \cap E$ is an injective submodule of E, we have $M = B \oplus Y \oplus A \cap E \oplus Z = A \oplus Y \oplus Z$, where Z is a submodule of E. Therefore $A \overset{\oplus}{\subset} M$; i.e. M has (C_1). □

CHAPTER II

TORSION FREE REDUCED MODULES WITH PROPERTY (C_1).

In this chapter we characterize torsion free reduced modules with property (C_1). First we show that such modules are finite direct sums of uniform submodules. Then we give, in general, a necessary and sufficient condition for the direct sum of a pair of uniform torsion free reduced modules to have (C_1). Finally we prove, in certain cases, that a finite direct sum of uniform torsion free reduced modules has (C_1) if and only if the direct sum of each pair has (C_1).

All modules in this chapter are torsion free reduced.

We recall again that R is always a commutative integral domain; we denote its quotient field by K, and we call any ring between R and K an overring of R.

§1. PRELIMINARIES

Definition 2.1.: A valuation ring is a commutative integral domain with the property that any two ideals are comparable.

It is clear that V is a valuation if and only if V contains x or x^{-1} for every non-zero element x of the quotient field of V. Consequently, each overring of a valuation ring is a valuation ring. By a discrete valuation ring we mean a valuation ring with discrete value group.

A commutative integral domain is a discrete rank one valuation ring, if and only if it is a Noetherian valuation ring, if and only if it is a local principal ideal domain.

<u>Definition 2.2</u>: A fractional ideal I is a non-zero R-submodule of K, such that $x\,I \subset R$ holds for some $0 \neq x \in R$. By $(R:I)$ we mean the set of all $x \in K$ with $x\,I \subset R$; $(R:I)$ is again a fractional ideal. We say that I is invertible if $I(R:I) = R$.

It is easy to see that any invertible ideal in a local domain is principal.

<u>Definition 2.3</u>: A Prüfer domain is a commutative integral domain in which every non-zero finitely generated ideal is invertible.

<u>Lemma 2.4</u> ([11], Theorem 64). The following statements are equivalent for a commutative integral domain R:

(1) R is Prüfer ;

(2) For every prime ideal P, R_P is a valuation ring ;

(3) For every maximal ideal M, R_M is a valuation ring.

<u>Lemma 2.5</u> ([11], Theorem 65). Let R be Prüfer domain, and let V be a valuation over ring of R. Then $V = R_P$ for some prime ideal P in R.

Each over ring of a Prüfer domain is a Prüfer domain.

Definition 2.6 ([11], Theorem 96) A commutative domain R satisfying any (hence all) of the following equivalent conditions is called a Dedekind domain;

(1) Every non-zero ideal of R is invertible ;

(2) R is Noetherian, integrally closed, and of dimension one ;

(3) R is Noetherian, and for each maximal ideal M, R_M is a discrete valuation ring.

In a Dedekind domain any non-zero ideal is uniquely a product of prime ideals.

Any overring of a Dedekind domain is a Dedekind domain, and is a localization of R at a set of prime ideals (this follows readily from corollary (6.2) in [12] and theorem (3.4) in [16]).

Definition 2.7: Let T be an overring of R. The conductor D of R in T is the set of all elements $x \in R$ such that $xT \subset R$. The conductor D is an ideal of R and also an ideal of T; more precisely D is the largest ideal of R which is also an ideal of T. If R is Noetherian and $D \neq 0$, then T is a finitely generated R-module.

Lemma 2.8 ([11], Theorem 17) Let T be a commutative algebra over R. The following statements are equivalent: (1) T is finitely generated R-module, (2) T is finitely generated ring over R and is integral over R.

Let M be an R-submodule of K. Then $M = \bigcap_P M_P$, where P runs over all maximal ideal of R (see [5]).

Definition 2.9: A non-zero R-module M is simple if the only R-submodules of M are 0 and M. A module M is semisimple if it is a (direct) sum of simple modules.

Lemma 2.10: Let X be an R-module such that X_P is simple as R_P-module, for one maximal ideal P of R, and $X_Q = 0$ for all other maximal ideals Q. Then X is a simple R-module.

Proof: From proposition (3.2) in [12], $X_S \cong X \otimes_R R_S$, as R_S-module, for any miltiplicatively closed set S in R. Also by [12] theorem (3.3), R_S is flat. Then for any short exact sequence $0 \to Y \to X \to Z \to 0$, $0 \to Y_P \to X_P \to Z_P \to 0$ and $0 \to Y_Q \to X_Q \to Z_Q \to 0$ are exact sequences. Since X_P is simple and $X_Q = 0$ for all $Q \neq P$, we have $Y_P = 0$ or $Z_P = 0$, and $Y_Q = 0$ and $Z_Q = 0$ for all $Q \neq P$. Then $Y = 0$ or $Z = 0$. Hence X is simple. □

Definition 2.11: A chain of prime ideals has length n if it contains $n+1$ distinct members. The dimension of a ring R is the supremum of the lengths of all chains of prime ideals.

Note that our dimension for rings (which usually called Krull dimension) is different from our dimension for modules (which is usually called Goldie dimension).

Lemma 2.12: ([11], Theorem 93). Let R be a one dimensional

Noetherian domain, and T be an overring of R. Then T is again Noetherian and its dimension is at most 1.

Lemma 2.13: A ring R is Artinian if and only if R is Noetherian and 0-dimensional.

Lemma 2.14: Let $M = \bigoplus_{i \in I} M_i$, where the M_i are uniform R-submodules of K. Then A is closed in M if and only if $A = \bigoplus_{j \in J} a_j K \cap M$, for some K-linearly independent subset $\{a_j\}_{j \in J}$ of $\bigoplus_{i \in I} K$.

Proof: Let A be a closed submodule of M. Since any direct sum of divisible modules is divisible, we have $E(M) = \bigoplus_{i \in I} K$. Since each injective R-module is K-vector space, $E(A)$ is a subspace of the vector space $\bigoplus_{i \in j} K$. Then $E(A) = \bigoplus_J a_j K$, for some linearly independent subset $\{a_j\}_{j \in J}$ of $\bigoplus_{i \in I} K$. Since $A \subset' E(A)$, we have $A \subset' E(A) \cap M$. Since A is closed; then $A = E(A) \cap M = \bigoplus_J a_j K \cap M$.

Conversely let $B = \bigoplus_{j \in J} a_j K \cap M$, $\{a_j\}_{j \in J}$ a linearly independent subset of $\bigoplus_I K$. Let $B \subset' B' \subset M$. Then for any $x \in B'$, there exists $0 \neq r \in R$ such that $xr \in B$, i.e. $xr = \sum_{j \in J} a_i k_j \in M$, where $k_j \in K$. Hence $x = \sum_{j \in J} a_j (\frac{k_i}{r}) \in \bigoplus_J a_j K \cap M = B$. Therefore $B = B'$. Then B has no proper essential extension in M; i.e. B is closed in M. □

Corollary 2.15: Let $M = \bigoplus_{i \in I} M_i$, where the M_i are uniform

R-submodules of K. Then A is closed uniform in M if and only if $A = \{(q_i x)_{i \in I} : x \in K,\ q_i x \in M_i \text{ for all } i\}$ for an element $0 \neq (q_i)_{\exists i \in I} \in \oplus_I K$.

Proof: From (2-14), $A = \oplus_{j \in J} a_j K \cap M$. Since A is uniform, we have that E(A) is uniform. Then $|J| = 1$; i.e. $A = aK \cap M$, $0 \neq a \in \oplus_I K$. Let $a = (q_i)_{i \in I}$, $q_i \in K$. Hence $A = \{(q_i x)_{i \in I} : x \in K, q_i x \in M_i \text{ for all } i\}$.

The converse is trivial. □

§2. REDUCTION TO FINITE GOLDIE DIMENSION

The following theorem shows that any torsion free reduced module with property (C_1) is a finite direct sum of uniform submodules.

Theorem 2.16: Let M be a torsion free reduced R-module. If M has (C_1), then $M = \overset{n}{\underset{i=o}{\oplus}} M_i$, where the M_i are uniform submodules of M.

Proof: Let $M \neq o$ have (C_1). By (1.12) M contains a uniform direct summand M_o. Then $M = M_o \oplus U_o$, where U_o has (C_1). Again by (1.12), if $U_o \neq o$, U_o contains a uniform direct summand M_1 and hence $M = M_o \oplus M_1 \oplus U_1$.

Continuing in this manner we obtain $M = \overset{n}{\underset{i=o}{\oplus}} M_i \oplus U_n$ as long as the U_i are non-zero.

If M is finite dimensional, then $U_n = o$ for some n, and hence $M = \bigoplus_{i=o}^{n} M_i$, as claimed.

If M is infinite dimensional, we shall derive a contradiction. In this case, U_n is infinite dimensional hence non-zero for all n, and hence $M \supset \bigoplus_{i=o}^{\infty} M_i$.

We show first that $\bigoplus_{i=o}^{\infty} M_i$ is closed in M. Let $\bigoplus_{i=o}^{\infty} M_i \subset' M^* \subset M$. Since $M = \bigoplus_{i=o}^{n} M_i \oplus U_n$ for all $n \geq o$, by the modular law $M^* = \bigoplus_{i=o}^{n} M_i \oplus U_n \cap M^*$. Since $M_i \subset U_n$ for all $i > n$, we have $\bigoplus_{i=n+1}^{\infty} M_i \subset U_n \cap M^*$. We claim that $U_n \cap M^*$ is essential over $\bigoplus_{i=n+1}^{\infty} M_i$. Let $o \neq x \in U_n \cap M^*$. Since $\bigoplus_{i=o}^{\infty} M_i \subset' M^*$, there exists $o \neq r \in R$ such that $xr \in \bigoplus_{i=o}^{\infty} M_i$. Hence $xr = \sum_{i=o}^{\infty} x_i$ and $\sum_{i=o}^{n} x_i = -\sum_{i=n+1}^{\infty} x_i + xr \in \bigoplus_{i=o}^{n} M_i \cap (U_n \cap M^*) = o$. It follows that $xr = \sum_{i=n+1}^{\infty} x_i \in \bigoplus_{i=n+1}^{\infty} M_i$. Therefore $\bigoplus_{i=n+1}^{\infty} M_i \subset' U_n \cap M^*$.

Now since any direct sum of divisible modules is divisible, $\bigoplus_{i=o}^{\infty} E(M_i)$ is divisible hence injective. It follows that $E(M^*) = E(\bigoplus_{i=o}^{\infty} M_i) = \bigoplus_{i=o}^{\infty} E(M_i)$.

Let $\bigoplus_{i=o}^{\infty} E(M_i) \xrightarrow{\pi_i} E(M_i)$ be the projection onto $E(M_i)$. For each $n \geq o$ we have $\pi_n(M^*) = \pi_n(\bigoplus_{i=o}^{n} M_i) + \pi_n(U_n \cap M^*) = M_n + \pi_n(U_n \cap M^*)$. We show that $\pi_n(U_n \cap M^*) = o$. Let $z \in U_n \cap M^*$. By essentiality over $\bigoplus_{i=n+1}^{\infty} M_i$, there exists $o \neq s \in R$ such that $zs \in \bigoplus_{i=n+1}^{\infty} M_i$. Hence

$o = \pi_n(zs) = \pi_n(z)s$. It follows that $\pi_n(z) = o$ for all $z \in U_n \cap M^*$. Then $\pi_n(M^*) = M_n$ for all $n \geq o$. Hence $M^* = \bigoplus_{i=o}^{\infty} M_i$, i.e. $\bigoplus_{i=o}^{\infty} M_i$ has no proper essential extension in M. Thus $\bigoplus_{i=o}^{\infty} M_i$ is closed in M.

By (C_1) $\bigoplus_{i=o}^{\infty} M_i \overset{\oplus}{\subset} M$; therefore by (1.6) $\bigoplus_{i=o}^{\infty} M_i$ also has (C_1). Since K is divisible hence injective, we have $E(M_i) \cong K$. Also $M_i \supset y_i R \cong R$, for any $o \neq y_i \in M_i$. Thus without loss of generality, we may assume that $R \subset M_i \subset K$ for all i, and therefore $\bigoplus_{i=o}^{\infty} R \subset \bigoplus_{i=o}^{\infty} M_i \subset \bigoplus_{i=o}^{\infty} K$.

Now let $o \neq r_n \in R$ $(n \geq o)$ be an arbitrary sequence of elements of R, but with $r_o = 1$. Let $a_n = e_o - e_n r_n$ $(n \geq 1)$, where $e_n = (\delta_{ni})_{i=o}^{\infty} \in \bigoplus_{i=o}^{\infty} K$. It is easy to show that $\{a_n\}_{n=1}^{\infty}$ is a linearly independent subset of $\bigoplus_{i=o}^{\infty} K$. By (2.14) $A =: \bigoplus_{n=1}^{\infty} a_n K \cap (\bigoplus_{i=o}^{\infty} M_i)$ is a closed submodule of $\bigoplus_{i=o}^{\infty} M_i$. By (C_1) $A \overset{\oplus}{\subset} \bigoplus_{i=o}^{\infty} M_i$, i.e. $\bigoplus_{i=o}^{\infty} M_i = A \oplus B$ for some submodule B of $\bigoplus_{i=o}^{\infty} M_i$. Define $f: \bigoplus_{i=o}^{\infty} M_i \to K$ by

$$f\left(\sum_{i=o}^{\infty} e_i k_i\right) = \sum_{i=o}^{\infty} \frac{k_i}{r_i}, \quad \sum_{i=o}^{\infty} e_i k_i \in \bigoplus_{i=o}^{\infty} M_i.$$

<u>Claim</u>: $\ker f = A$

If $a \in A$, then $a = \sum_{n=1}^{\infty} a_n k_n = e_o \sum_{n=1}^{\infty} k_n - \sum_{n=1}^{\infty} e_n k_n r_n$. Hence

$f(a) = \sum_{n=1}^{\infty} k_n - \sum_{n=1}^{\infty} k_n = o$. Thus $A \subset \ker f$.

If $x \in \ker f$, then $x = \sum_{i=o}^{\infty} e_i k_i$ such that $\sum_{i=o}^{\infty} \frac{k_i}{r_i} = o$. Then $k_o = -\sum_{i=1}^{\infty} \frac{k_i}{r_i}$. Let $k_i' =: \frac{k_i}{r_i} (i \geq 1)$. Then $x = -e_o \sum_{i=1}^{\infty} k_i' + \sum_{i=1}^{\infty} e_i k_i' r_i = -\sum_{i=1}^{\infty} k_i'(e_o - e_i r_i) = -\sum_{i=1}^{\infty} k_i' a_i \in \sum_{i=1}^{\infty} a_i K \cap \bigoplus_{i=o}^{\infty} M_i = A$.

Thus $\ker f \subset A$. Therefore $A = \ker f$.

It follows that $B \cong (\bigoplus_{i=o}^{\infty} M_i)/A \cong f(\bigoplus_{i=o}^{\infty} M_i) \subset K$. It is easy to show that $e_o \notin A$. Hence B is a uniform closed submodule of M. By (2.15) $B = bK \cap M$ for some $o \neq b = (b_i)_{i=o}^{\infty} \in \bigoplus_{i=o}^{\infty} K$.

Since $\bigoplus_{i=o}^{\infty} R \subset \bigoplus_{i=o}^{\infty} M_i$, we have $e_m \in \bigoplus_{i=o}^{\infty} M_i$, for all m. Then we have the following:

$e_m = \sum_{n=1}^{\infty} a_n k_{nm} + bk_m$, where $k_{nm}, k_m \in K$ and $\sum_{n=1}^{\infty} a_n k_{nm} \in A$, $bk_m \in B$, i.e. $e_m = e_o \sum_{n=1}^{\infty} k_{nm} - \sum_{n=1}^{\infty} e_n k_{nm} r_n + \sum_{i=o}^{\infty} e_i b_i k_m$

for all $m \geq o$.

We deduce

1) $i = m = o$: $1 = \sum_{n=1}^{\infty} k_{no} + b_o k_o$ where $\sum_{n=1}^{\infty} k_{no}$, $b_o k_o \in M_o$.

2) $m = o$, $i \geq 1$: $o = -k_{io} r_i + b_i k_o$ where $k_{io} r_i$, $b_i k_o \in M_i$.

3) $i = o$, $m \geq 1$: $o = \sum_{n=1}^{\infty} k_{nm} + b_o k_m$ where $\sum_{n=1}^{\infty} k_{nm}$, $b_o k_m \in M_o$.

4) $i \geq 1$, $m \geq 1$, $i \neq m$: $o = - r_i k_{im} + b_i k_m$ where $r_i k_{im}$, $b_i k_m$ ε M_i.

5) $m = i \geq 1$: $1 = - k_{ii} r_i + b_i k_i$ where $k_{ii} r_i$, $b_i k_i \varepsilon M_i$.

Now we obtain from 2) $k_{io} = \dfrac{b_i k_o}{r_i}$ $(i \geq 1)$.

From 1) we get $1 = \sum_{n=1}^{\infty} \dfrac{b_n k_o}{r_n} + b_o k_o = k_o \sum_{n=o}^{\infty} \dfrac{b_n}{r_n}$; hence

$k_o = 1/D$ where $D =: \sum_{n=o}^{\infty} \dfrac{b_n}{r_n} \neq o$.

From 4) we have $k_{im} = \dfrac{b_i k_m}{r_i}$ $(i \neq m \geq 1)$, and from 5)

$k_{ii} = \dfrac{b_i k_i - 1}{r_i}$ $(i \geq 1)$. Finally 3) yields $o = \sum_{n=1}^{\infty} \dfrac{b_n k_m}{r_n} - \dfrac{1}{r_m} + b_o k_m$ $(m \geq 1)$,

hence $\dfrac{1}{r_m} = \sum_{n=1}^{\infty} \dfrac{b_n k_m}{r_n} + b_o k_m = \sum_{n=o}^{\infty} \dfrac{b_n k_m}{r_n} = k_m D$ and therefore $k_m = \dfrac{1}{r_m D}$

for all $m \geq o$.

Since $b_i k_m \varepsilon M_i$, we have $\dfrac{b_i}{r_m D} \varepsilon M_i$ for all $i, m \geq o$. It follows that $\dfrac{b_i}{D} \varepsilon \bigcap_{m=o}^{\infty} r_m M_i$ for all $i \geq o$. Let $x_i =: \dfrac{b_i}{D}$. Then

$D = \sum_{i=o}^{\infty} \dfrac{b_i}{r_i} = \sum_{i=o}^{\infty} \dfrac{Dx_i}{r_i} = D \sum_{i=o}^{\infty} \dfrac{x_i}{r_i}$, hence $\sum_{i=o}^{\infty} \dfrac{x_i}{r_i} = 1$, where all but a finite number of $\dfrac{x_i}{r_i}$ are zero (since $b = (b_i)_{i=o}^{\infty} \varepsilon \bigoplus_{i=o}^{\infty} K$). It follows that $1 = \sum_{i=o}^{\infty} \dfrac{x_i}{r_i} \varepsilon \sum_{i=o}^{\infty} \bigcap_{m=o}^{\infty} \dfrac{r_m}{r_i} M_i$.

Since $o \neq r_i \in R$ were chosen arbitrarily, we may take $r_i = r^i$ for an arbitrary element $o \neq r \in R$. Then $1 \in \sum_{i=o}^{\infty} \bigcap_{m=o}^{\infty} r^{m-i} M_i \subset r \sum_{i=o}^{\infty} M_i$, i.e. $1/r \in \sum_{i=o}^{\infty} M_i$ for all $o \neq r \in R$. It follows that $\sum_{i=o}^{\infty} M_i$ is a divisible R-module, i.e. $\sum_{i=o}^{\infty} M_i = K$.

Now define $g: \bigoplus_{i=o}^{\infty} M_i \to \sum_{i=o}^{\infty} M_i = K$ by $g(m_i)_{i=o}^{\infty} = \sum_{i=o}^{\infty} m_i$, where $(m_i)_{i=o}^{\infty} \in \bigoplus_{i=o}^{\infty} M_i$. It is clear that g is an epimorphism.

We claim that $\ker g$ is closed in $\bigoplus_{i=o}^{\infty} M_i$. Let $\ker g \subset' Y \subset \bigoplus_{i=o}^{\infty} M_i$. Consider $o \neq y = (y_i)_{i=o}^{\infty} \in Y$. By the essentiality of Y over $\ker g$, there exists $o \neq r \in R$ such that $yr \in \ker g$, i.e. $\sum_{i=o}^{\infty} y_i r = o$. It follows that $\sum_{i=o}^{\infty} y_i = o$, i.e. $y \in \ker g$. Hence $\ker g = Y$. Therefore $\ker g$ has no proper essential extension in $\bigoplus_{i=o}^{\infty} M_i$.

By (C_1) $\bigoplus_{i=o}^{\infty} M_i = \ker g \oplus X$ for some submodule X of M. Since g is epimorphism, we have $X \cong \bigoplus_{i=o}^{\infty} M_i/\ker g \cong K$. This contradicts the fact that M is reduced. □

The following proposition shows that, for finite dimensional torsion free reduced modules, (C_1) is equivalent to $(1\text{-}C_1)$.

Recall that a module is said to have property $(1\text{-}C_1)$, if every uniform closed submodule is a direct summand.

Proposition 2.17: Let M be a finite dimensional R-module. Then M has (C_1) if and only if M has $(1\text{-}C_1)$.

Proof: Obviously (C_1) implies $(1\text{-}C_1)$. We show the converse by induction over the dimension.

For dimension ≤ 2 the claim is clear. Now assume that it holds true for dimension $< n$, and let M be a module with $(1\text{-}C_1)$ of dimension n. Using the note after (1.6), we can show, in the same way as in the beginning of the proof of (2.16), that M can be written as $M = \bigoplus_{i=1}^{n} M_i$ where the M_i are uniform submodules. Let A be a closed submodule of M with $1 < \dim(A) < n$.

Now if $A \cap \bigoplus_{i=1}^{n-1} M_i = o$, then $A \xrightarrow{\pi_n|A} M_n$ is a monomorphism, where π_n is the projection of M onto M_n. It follows that A is uniform, which contradicts the assumption. Hence $A \cap \bigoplus_{i=1}^{n-1} M_i \neq o$.

We claim that $A \cap \bigoplus_{i=1}^{n-1} M_i$ is closed in $\bigoplus_{i=1}^{n-1} M_i$. Let N be a submodule of $\bigoplus_{i=1}^{n-1} M_i$ such that $A \cap \bigoplus_{i=1}^{n-1} M_i \subset' N$. Let $x \in N$ be arbitrary. By essentiality there exists $o \neq r \in R$ such that $xr \in A \cap \bigoplus_{i=1}^{n-1} M_i$. Since A is closed, we obtain $x \in A$. Hence $x \in A \cap \bigoplus_{i=1}^{n-1} M_i$. It follows that $A \cap \bigoplus_{i=1}^{n-1} M_i = N$. Then $A \cap \bigoplus_{i=1}^{n-1} M_i$ has no proper essential extension in

$\overset{n-1}{\underset{i=1}{\oplus}} M_i$.

By induction $A \cap \overset{n-1}{\underset{i=1}{\oplus}} M_i \overset{\oplus}{\subset} \overset{n-1}{\underset{i=1}{\oplus}} M_i$, i.e. $\overset{n-1}{\underset{i=1}{\oplus}} M_i = A \cap \overset{n-1}{\underset{i=1}{\oplus}} M_i \oplus X$, where dim $(X) \leq n-2$. Then $M = \overset{n-1}{\underset{i=1}{\oplus}} M_i \oplus M_n = A \cap \overset{n-1}{\underset{i=1}{\oplus}} M_i \oplus X \oplus M_n$. By the modular law $A = (A \cap \overset{n-1}{\underset{i=1}{\oplus}} M_i) \oplus A \cap [X \oplus M_n]$. By the same argument as in the claim we can show that $A \cap [X \oplus M_n]$ is a closed submodule of $X \oplus M_n$. Since $(1\text{-}C_1)$ is inherited by direct summands, $X \oplus M_n$ has $(1\text{-}C_1)$ with dim $(X \oplus M_n) \leq n-1$. By induction $X \oplus M_n$ has (C_1). Then $A \cap [X \oplus M_n] \overset{\oplus}{\subset} X \oplus M_n$, i.e. $X \oplus M_n = A \cap (X \oplus M_n) \oplus Y$. Therefore $M = (A \cap \overset{n-1}{\underset{i=1}{\oplus}} M_i) \oplus X \oplus M_n = (A \cap \overset{n-1}{\underset{i=1}{\oplus}} M_i) \oplus A \cap (X \oplus M_n) \oplus Y = A \oplus Y$. Hence M has (C_1). □

In the proof of (2.16) we have observed that each uniform torsion free R-module can be embedded into K in such a way that its image contains R.

From now on we consider each torsion free uniform R-module as an R-submodule of K containing R.

<u>Definition 2.18</u>: Let M_i $(i = 1,2,\ldots,n)$ be R-submodules of K. By $\mathcal{O}(M_i)$ we mean the set of all $x \in K$ such that $xM_i \subset M_i$. $\mathcal{O}(M_i)$ is an overring of R isomorphic to end (M_i). By A_{ij} we mean the set of all $x \in K$ such that $xM_i \subset M_j$. A_{ij} is R-submodule of K isomorphic to

$\hom_R(M_i, M_j)$; we sometime denote it by $(M_j : M_i)$. It is clear that $A_{ii} = \mathcal{O}(M_i)$.

The next proposition gives a necessary and sufficient condition for a finite direct sum of uniform modules to have (C_1).

<u>Proposition 2.19</u>: Let $M = \bigoplus_{i=1}^{n} M_i$ be an R-module with all the M_i uniform. Then the following statements are equivalent:

1) M has (C_1).
2) For all $q_1, q_2, \ldots, q_n \in K$ (not all zero), there exist $\alpha_1, \alpha_2, \ldots, \alpha_n \in K$ such that $\sum_{i=1}^{n} \alpha_i = 1$ and $\alpha_i q_j M_i \subset q_i M_j$ for all i, j.

Proof: 1) $\Rightarrow$ 2). Let M have (C_1). Let $q_1, q_2, \ldots q_n$ be arbitrary elements of K, not all zero. By (2.15) $A =: \{(q_i x)_{i=1}^{n} : x \in K$ and $q_i x \in M_i$ for all $i\}$ is a closed uniform submodule of M. By (C_1) $A \subset^{\oplus} M$, i.e. $M = A \oplus B$ where B is an $(n-1)$ - dimensional submodule. Since (C_1) is inherited by direct summands, B has (C_1). Then B can be written as $B = \bigoplus_{j=1}^{n-1} B_j$, where the B_j are uniform. Again by (2.15)

$B_j = \{(t_{ij} x_j)_{i=1}^{n} : x_j \in K$ and $t_{ij} x_j \in M_i$, $i = 1, 2, \ldots, n\}$ for some $t_{ij} \in K$ not all zero, $i = 1, 2, \ldots, n$.

Now $A \oplus B = M$ implies that for all $m_j \in M_j$ the following system of equations has a unique solution with $t_{ij} x_j \in M_i$ and $q_i x_n \in M_i$, for each i:

$$t_{11}\,x_1 + t_{12}\,x_2 + \dots + t_{1n-1}\,x_{n-1} + q_1\,x_n = \delta_{i1}\,m_1$$

$$t_{21}\,x_1 + t_{22}\,x_2 + \dots + t_{2n-1}\,x_{n-1} + q_2\,x_n = \delta_{i2}\,m_2$$

....

....

$$t_{n1}\,x_1 + t_{n2}\,x_2 + \dots + t_{nn-1}\,x_{n-1} + q_n\,x_n = \delta_{in}\,m_n$$

Take $m_1 = m_2 = \dots = m_n = 1 \in R \subset M_i$. Since $\overset{n}{\underset{i=1}{\oplus}} R \subset M$, we have that each member e_i of the natural basis of $\overset{n}{\underset{i=1}{\oplus}} K$ is a linear combination of the $(t_{1j}, t_{2j}, \dots, t_{nj})$, $j = 1,2,\dots,n-1$; $(q_1, q_2, \dots, q_n)$. Then $\{(t_{1j},\dots,t_{nj}), (q_1, q_2, \dots, q_n)\}_{j=1}^{n-1}$ is generating and hence a linearly independent subset of $\overset{n}{\underset{i=1}{\oplus}} K$. It follows that

$$\Delta = \begin{vmatrix} t_{11} & t_{12} & \cdots & t_{1n-1} & q_1 \\ t_{21} & t_{22} & \cdots & t_{2n-1} & q_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ t_{n1} & t_{n2} & \cdots & t_{nn-1} & q_n \end{vmatrix} \neq 0.$$

Now take arbitrary $m_j \in M_j$. Call the unique solution of the ith equations $x_{i1}, x_{i2}, \dots, x_{in}$. Then, by Cramer's Rule, $x_{in} = \dfrac{(-1)^{n+i}\,\Delta_{in}\,m_i}{\Delta}$, where Δ_{in} is the minor obtain by deleting the ith row and the nth column of Δ. Since $q_j x_{in} \in M_j$, we have $q_j \frac{\Delta_{in}}{\Delta} m_i \in M_j$, hence

$\alpha_i q_j M_i \subset q_i M_j$ for all i,j, where $\alpha_i = \frac{q_i \Delta_{in}}{\Delta}$. Then

$$\sum_{i=1}^{n} \alpha_i = \sum_{i=1}^{n} \frac{q_i \Delta_{in}}{\Delta} = \frac{\Delta}{\Delta} = 1.$$

2) $\rightarrow$ 1). Let condition 2) be satisfied for arbitrary $q_1, q_2, \ldots, q_n \in K$ where not all of q_i are zero. The proof will be by induction on n.

The case $n = 1$ is trivial.

Now assume $\underset{i \in F}{\oplus} M_i$ has (C_1) for every proper subset F of $\{1,2,\ldots,n\}$. To show that M has (C_1), by (2.17), it is enough to show that each closed uniform submodule of M is a direct summand.

Let A be a closed uniform submodule of M. By (2.15) $A = \{(q_i x)_{i=1}^{n} : x \in K$ and $q_i x \in M_i$ for all $i\}$, where $q_i \in K$ and not all of them are zero. By assumption condition 2) is satisfied for $q_1, q_2, \ldots, q_n$.

Case 1:- At least one of the q_i is zero. Note that since not all q_i are zero, it follow that whenever $q_i = o$, then $\alpha_i = o$ (due to $\alpha_i q_j M_i \subset q_i M_j = o$). Therefore condition 2) holds for the proper subset $F = \{i : q_i \neq o\}$. Since $A \subset \underset{i \in F}{\oplus} M_i$, by induction $A \overset{\oplus}{\subset} \underset{i \in F}{\oplus} M_i \overset{\oplus}{\subset} M$.

Case 2:. All q_i are not zero. By condition 2), there exist $\alpha_1, \alpha_2, \ldots, \alpha_n \in K$ such that $\sum_{i=1}^{n} \alpha_i = 1$ and $\alpha_i q_i^{-1} M_i \subset q_j^{-1} M_j$ for all i,j.

Let $\Delta_{i1} := \alpha_i q_i^{-1}$; then $\sum_{i=1}^{n} q_i \Delta_{i1} = \sum_{i=1}^{n} \alpha_i = 1$. It is clear that not all Δ_{i1} are zero. Without loss of generality assume $\Delta_{11} \neq o$.

Let $B := \{(-\Delta_{21} y_2 - \sum_{i=3}^{n} \frac{\Delta_{i1}}{\Delta_{11}} y_i,\ \Delta_{11} y_2, y_3, \ldots, y_n) : y_i \in K$; and $-\Delta_{21} y_2 - \sum_{i=3} \frac{\Delta_{i1}}{\Delta_{11}} y_i \in M_1,\ \Delta_{11} y_2 \in M_2$ and $y_i \in M_i \quad (i \geq 3)\}$. We have

$$\begin{vmatrix} q_1 & -\Delta_{21} & -\frac{\Delta_{31}}{\Delta_{11}} & \cdots\cdots & -\frac{\Delta_{n1}}{\Delta_{11}} \\ q_2 & \Delta_{11} & o & \cdots\cdots & o \\ q_3 & o & 1 & o \cdots\cdots & o \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ q_n & o & \cdots\cdots & o & 1 \end{vmatrix} = \sum_{i=1}^{n} q_i \Delta_{i1} = 1.$$

Then, for each i, the following system of equations has a unique solution for all $m_j \in M_j$:

$$\begin{aligned} q_1 x - \Delta_{21} y_2 - \cdots\cdots - \frac{\Delta_{n1}}{\Delta_{11}} y_n &= \delta_{i1} m_1 \ ; \\ q_2 x + \Delta_{11} y_2 + o + \ldots + o &= \delta_{i2} m_2 \ ; \\ q_3 x + o + y_3 + o, \ldots, + o &= \delta_{i3} m_3 \ ; \\ \cdots\cdots\cdots\cdots\cdots\cdots & \\ \cdots\cdots\cdots\cdots\cdots\cdots & \\ q_n x + o + \cdots\cdots + y_n &= \delta_{in} m_n \ . \end{aligned}$$

Let $\{x_i, y_{2i}, \ldots, y_{ni}\}$ be the solution of the ith system of equations. Since, by Cramer's Rule, $x_i = \Delta_{i1} m_i = q_i^{-1} \alpha_i m_i \in q_j^{-1} M_j$, we have $q_j x_i \in M_j$ for all i,j. It follows that $-\Delta_{21} y_2 - \sum_{i=3}^{n} \frac{\Delta_{i1}}{\Delta_{11}} y_i \in M_1$, $\Delta_{11} y_2 \in M_2$, and $y_j \in M_j$ for all $j \geq 3$. Therefore $A \oplus B = M$; i.e. $A \subset^{\oplus} M$. Hence M has (C_1). □

Condition 2) in (2.19) is difficult to verify directly in concrete examples. But we shall use it in §.3-6 in the proof that, in certain cases, a finite direct sum of uniform modules has (C_1) if and only if the direct sum of each pair has (C_1).

Corollary 2.20: Let $M = M_1 \oplus M_2$ where the M_i are uniform. Then M has (C_1) if and only if for every $0 \neq q \in K$ there exist $\alpha_1, \alpha_2 \in \mathcal{O}(M_1) \cap \mathcal{O}(M_2)$ such that $\alpha_1 + \alpha_2 = 1$ and $\alpha_1 q M_1 \subset M_2$, $\alpha_2 M_2 \subset q M_1$.

Proof: Let M have (C_1). By (2.19), with $q_1 = 1$ and $q_2 = q$, there exist $\alpha_1, \alpha_2 \in K$ such that $\alpha_1 + \alpha_2 = 1$ and $\alpha_i q_j M_i \subset q_i M_j$ for all i,j. Then $\alpha_1 q M_1 \subset M_2$, $\alpha_1 M_1 \subset M_1$ and $\alpha_2 M_2 \subset qM_1$, $\alpha_2 M_2 \subset M_2$. Therefore, $\alpha_1, \alpha_2 \in \mathcal{O}(M_1) \cap \mathcal{O}(M_2)$ with $\alpha_1 q M_1 \subset M_2$ and $\alpha_2 M_2 \subseteq q M_1$.

Conversely, let $q_1, q_2 \in K$ be given such that $q_1 \neq 0$ or $q_2 \neq 0$. If $q_i = 0$, then take $\alpha_i = 0$ and $\alpha_j = 1$ $(i \neq j)$. If $q_i \neq 0$ $(i = 1,2)$, then let $q = q_1^{-1} q_2$. Therefore condition 2) in (2.19) is satisfied for all $q_1, q_2 \in K$ with q_1 or $q_2 \neq 0$. Hence by (2.19) M has (C_1). □

Corollary 2.21: Let $M = M_1 \oplus M_2$ where the M_i are uniform. Let $\mathcal{O}(M_1)$ or $\mathcal{O}(M_2)$ be local. Then M has (C_1) if and only if for every $o \neq q \in K$, qM_1 and M_2 are comparable.

Proof: Let M have (C_1). Let $\mathcal{O}(M_2)$ be local. By (2.20), for every $o \neq q \in K$, there exist $\alpha_1, \alpha_2 \in \mathcal{O}(M_1) \cap \mathcal{O}(M_2)$ such that $\alpha_1 + \alpha_2 = 1$ and $\alpha_1 \, qM_1 \subset M_2$, $\alpha_2 \, M_2 \subset qM_1$. Since $\alpha_1, \alpha_2 \in \mathcal{O}(M_2)$ with $\alpha_1 + \alpha_2 = 1$ and $\mathcal{O}(M_2)$ is local, we get that α_1 or α_2 is unit in $\mathcal{O}(M_2)$. If α_1 is unit in $\mathcal{O}(M_2)$, then $\alpha_1^{-1} M_2 = M_2$. It follows that $qM_1 \subset M_2$. Similarly if α_2 is unit in $\mathcal{O}(M_2)$, then $\alpha_2 \, M_2 = M_2$; hence $M_2 \subset qM_1$. Therefore qM_1 and M_2 are comparable for every $o \neq q \in K$.

The converse is obvious. □

Corollary 2.22: Let $M = \bigoplus_{i=1}^{n} M_i$ where the M_i are uniform. Then M has (C_1) if and only if for all $q_1, q_2, \ldots, q_n \in K$ not all zero, we have $1 \in \sum_{i \in F} \bigcap_{j \in F} q_i q_j^{-1} A_{ij}$, where $F = \{i : q_i \neq o\}$.

Proof: Let M have (C_1). Let $q_i \in K$ $(i = 1,2,\ldots,n)$ be given such that not all of them are zero. Let $F =: \{i : q_i \neq o\}$; this is a non-empty subset of $\{1,2,\ldots,n\}$. By (2.19) there exist $\alpha_i \in K$ $(i = 1,2,\ldots,n)$ such that $\sum_{i=1}^{n} \alpha_i = 1$ and $\alpha_i \, q_j \, M_i \subset q_i \, M_j$ for all i,j.

We have observed in (2.19) that $\alpha_j = o$ if $j \notin F$. It follows that $\sum_{i \in F} \alpha_i = 1$ and $\alpha_i q_j q_i^{-1} M_i \subset M_j$ for all $i, j \in F$. Hence $\alpha_i q_j q_i^{-1} \in A_{ij}$ for all $i, j \in F$; i.e. $\alpha_i \in \bigcap_{j \in F} q_i q_j^{-1} A_{ij}$ for all $i \in F$. Therefore $1 = \sum_{i \in F} \alpha_i \in \sum_{i \in F} \bigcap_{j \in F} q_i q_j^{-1} A_{ij}$.

Conversely, let $q_i \in K$ $(i = 1,2,\ldots,n)$, not all zero. If $1 \in \sum_{i \in F} \bigcap_{j \in F} q_i q_j^{-1} A_{ij}$, where $F = \{i : q_i \neq o\}$, then $1 = \sum_{i \in F} \alpha_i$ for some $\alpha_i \in \bigcap_{j \in F} q_i q_j^{-1} A_{ij}$ for all $i \in F$. Hence $\alpha_i q_j M_i \subset q_i M_j$ for all $i, j \in F$. If we take $\alpha_j = o$ for all $j \notin F$, then condition 2) in (2.19) is satisfied for the q_i $(i = 1,2\ldots,n)$. Therefore M has (C_1).

An immediate consequence of (2.20) and (2.22) is the following:

Corollary 2.23: Let $M = M_1 \oplus M_2$ where the M_i are uniform. Then M has (C_1) if and only if for every $o \neq q \in K$, $q^{-1} A_{12} \cap S + q A_{21} \cap S = S$, where $S = \mathcal{O}(M_1) \cap \mathcal{O}(M_2)$. □

Corollary 2.24: If $M = M_1 \oplus M_2$ has (C_1) with M_i uniform, then M_i can be embedded in M_j $(i \neq j)$.

Proof: Let M have (C_1). By (2.21), for each $o \neq q \in K$ there exist $\alpha_1, \alpha_2 \in K$ such that $\alpha_1 + \alpha_2 = 1$ and $\alpha_1 q^{-1} M_1 \subset M_2$, $\alpha_2 M_2 \subset q^{-1} M_1$.

If M_1 can not be embedded in M_2, then $\alpha_1 = o$, $\alpha_2 = 1$ for each $o \neq q \in K$. It follows that $qM_2 \subset M_1$ for each $q \in K$. Hence M_1 is M_2-injective. Since M_2 is torsion free, by Baer's criterion for injectivity, we have that M_1 is injective, which contradicts the fact that M is reduced. Therefore M_1 can be embedded in M_2,

<u>Corollary 2.25</u>: If $M = M_1 \oplus M_2$ has (C_1) with M_1 uniform, then $A_{12} A_{21} \oplus A_{12}$ and $A_{12} A_{21} \oplus A_{21}$ have (C_1).

<u>Proof</u>: Let M have (C_1). By (2.23) $q^{-1}A_{12} \cap S + qA_{21} \cap S = S$ for each $o \neq q \in K$. Since $A_{12} \subset (A_{12} : A_{12}A_{21})$, $A_{21} \subset (A_{12}A_{21} : A_{12})$, and $S \subset \mathcal{O}(A_{12}A_{21}) \cap \mathcal{O}(A_{12}) = \mathcal{O}(A_{12})$, we have $q^{-1}(A_{12} : A_{12}A_{21}) \cap \mathcal{O}(A_{12}) + q\ (A_{12}A_{21} : A_{12}) \cap \mathcal{O}(A_{12}) = \mathcal{O}(A_{12})$, for every $o \neq q \in K$. Hence, by (2.23), $A_{12}A_{21} \oplus A_{12}$ has (C_1).

Similarly we can show $A_{12}A_{21} \oplus A_{21}$ has (C_1). □

§3. <u>GOLDIE DIMENSION TWO</u>

In this section we characterize all torsion free reduced modules of dimension two which have property (C_1). Such modules are direct sums of two uniform submodules.

Let M_1 and M_2 be uniform modules. According to (2.18) let $A = A_{12} = (M_2 : M_1)$ and $B = A_{21} = (M_1 : M_2)$. Denote $\mathcal{O}(M_1) \cap \mathcal{O}(M_2)$ by S. Then we have the following theorem.

Theorem 2.26: Let M be a torsion free reduced R-module of dimension two. Then the following statements are equivalent:

(1) M has (C_1) ;

(2) $M = M_1 \oplus M_2$, where the M_i are uniform submodules of M. For each $0 \neq q \in K$, $q^{-1}A \cap S + qB \cap S = S$;

(3) $M = M_1 \oplus M_2$, where the M_i are uniform and for each maximal ideal P of S. $\mathcal{O}(A_P)$ coincides with $\mathcal{O}(B_P)$, and is a valuation ring with maximal ideal $\mathfrak{M} \subset A_P B_P$. If $A_P \cong \mathfrak{M} \cong B_P$, then $\mathcal{O}(A_P) = \mathcal{O}(B_P)$ is discrete.

Proof: (We note, for later use, that the proof of the equivalence of 2) and 3) works equally well when we replace S by $\mathcal{O}(M_1)$.)

1) $\Rightarrow$ 2) Let M have (C_1). As observed earlier, $M = M_1 \oplus M_2$ where the M_i are uniform submodules. By (2.24), A and B are fractional ideals of S. By (2.23) $q^{-1}A \cap S + qB \cap S = S$, for each $0 \neq q \in K$.

2) $\Rightarrow$ 1): Is clear, by (2.23).

2) $\Rightarrow$ 3): Let $q^{-1}A \cap S + qB \cap S = S$. It is easy to see that A and B are fractional ideals of S, and $AB \subset S \subset \mathcal{O}(A)$. Since $B \subset (S:A)$ and $A \subset (S:B)$, we have, by (2.23), that $A \oplus S$ and $B \oplus S$ have (C_1). It follows that $A_P \oplus S_P$ and $B_P \oplus S_P$ have (C_1), for each maximal ideal P of S.

Case 1: $P \not\supset AB$. Then $A_P B_P = S_P$, i.e. B_P, A_P are invertible fractional ideals of S_P and hence principal. Since $B_P \oplus S_P$ has (C_1) and $B_P \cong S_P$, it follows that $S_P \oplus S_P$ has (C_1). By (2.21), $qS_P \subset S_P$

or $S_P \subset qS_P$ for every $0 \neq q \in K$. Then S_P is a valuation ring. Since $A_P \cong B_P \cong S_P$, we have that $\mathcal{O}(A_P) = \mathcal{O}(B_P) = S_P$ is a valuation ring, with maximal ideal $\mathfrak{M} = P_P \subset S_P = A_P B_P$. If $A_P \cong \mathfrak{M} \cong B_P$, then $\mathfrak{M} = P_P$ is principal, and hence $\mathcal{O}(A_P) = \mathcal{O}(B_P)$ is discrete. Therefore we have (3).

<u>Case 2</u>: $P \supset AB$. Then $A_P B_P \subset P_P$

<u>Claim 1</u>: $A_P \subsetneq (S_P : B_P)$ if and only if B_P is principal as an S_P-module.

Let $A_P \subsetneq (S_P : B_P)$ and $x \in (S_P : B_P)$, $x \notin A_P$. Since $x^{-1}A \cap S + x B \cap S = S$, we have $x^{-1}A_P \cap S_P + x B_P \cap S_P = S_P$. It follows that $x B_P \cap S_P = S_P$ (if $x^{-1}A_P \cap S_P = S_P$, then $S_P \subset x^{-1}A_P$; i.e. $x \in A_P$ which is a contradiction). Then $S_P \subset x B_P \subset S_P$ (due to $x \in (S_P : B_P)$). Therefore $B_P = x^{-1}S_P$ is principal.

Now let B_P be principal. Then B_P is invertible; i.e. $B_P(S_P : B_P) = S_P$. Hence $A_P \subsetneq (S_P : B_P)$ (otherwise $A_P B_P = (S_P : B_P) B_P = S_P$, which contradicts the fact that $A_P B_P \subset P_P$).

Similarly we can show that $B_P \subsetneq (S_P : A_P)$ iff A_P is principal.

<u>Subcase 2a</u>: B_P or A_P is principal. Let B_P be principal hence invertible. The same argument as in case (1) shows that $\mathcal{O}(B_P) = S_P$

is a valuation ring. By claim (1) $A \subsetneq (S_p : B_p)$. Now let $y \in (S_p : B_p)\backslash A_p$ be arbitrary. Since $y^{-1}A_p \cap S_p + yB_p \cap S_p = S_p$, it follows that $yB_p = S_p$. Hence $(S_p : B_p) = yS_p$. Then $(S_p : B_p)/A_p$ is a simple S_p-module. Since S_p is a valuation ring, we have that $P_p(S_p : B_p)$ is the unique maximal S_p-submodule of $(S_p:B_p)$. Therefore $A_p = P_p(S_p:B_p)$ i.e. $P_p = A_pB_p$. It is clear that if $A_p \cong P_p \cong B_p$, then $\mathcal{O}(A_p) = \mathcal{O}(B_p)$ is discrete.

To verify (2) it remains to show that $\mathcal{O}(A_p) = \mathcal{O}(B_p)$. If A_p is principal, then $\mathcal{O}(A_p) = S_p = \mathcal{O}(B_p)$. If A_p is not principal, then by claim (1), $B_p = (S_p : A_p)$. Let $x \in \mathcal{O}(A_p)$ be arbitrary. Then $xA_p \subset A_p$. It follows that $x\,b_p A_p \subset b_p A_p \subset S_p$ for all $b_p \in B_p$. Hence $x\,b_p \in (S_p : A_p) = B_p$ for all $b_p \in B_p$; i.e. $x \in \mathcal{O}(B_p)$. Therefore $\mathcal{O}(A_p) \subset \mathcal{O}(B_p) = S_p \subset \mathcal{O}(A_p)$. Hence $\mathcal{O}(A_p) = \mathcal{O}(B_p)$.

Similarly if A_p is principal, then condition (3) is satisfied.

<u>Subcase 2b</u>: A_p and B_p are not principal. From claim (1) $A_p = (S_p : B_p)$ and $B_p = (S_p : A_p)$. Since $A_p \oplus S_p$ and $B_p \oplus S_p$ have (C_1), by (2.21), we have $A_p \subset qS_p$ or $qS_p \subset A_p$, and $B_p \subset qS_p$ or $qS_p \subset B_p$, for all $0 \neq q \in K$. It follows that A_p and B_p are comparable with all S_p-submodules of K. Hence $\mathcal{O}(A_p)$ and $\mathcal{O}(B_p)$ are valuation rings.

By the same argument as in subcase 2a, we can show that $\mathcal{O}(A_p) = \mathcal{O}(B_p) =: \mathcal{O}$.

Now let $\mathfrak{M}$ be the maximal ideal of $\mathcal{O}$. Let $x \in \mathfrak{M}$ be arbitrary. Then $x^{-1} \notin \mathcal{O}$; i.e. $x^{-1}B_p \not\subset B_p$. Then $B_p \subsetneq x^{-1}B_p$. Hence there exists $b_p \in B_p$ such that $x^{-1}b_p \notin B_p = (S_p : A_p)$, i.e. $x^{-1}b_p A_p \not\subset S_p$. Then $S_p \subset x^{-1}b_pA_p$, i.e. $x \in b_pA_p \subset A_pB_p$. It follows that $\mathfrak{M} \subset A_pB_p$. On the other hand, $A_pB_p \subsetneq \mathcal{O}$, is an ideal of $\mathcal{O}$, and hence $A_pB_p \subset \mathfrak{M}$. Then $\mathfrak{M} = A_pB_p$.

If $A_p \cong \mathfrak{M} \cong B_p$, then $yA_p = \mathfrak{M}$ for some $o \neq y \in K$. Since $\mathfrak{M} = A_pB_p \subset S_p$ we have $y \in (S_p : A_p) = B_p$; i.e. $y\mathcal{O} \subset B_p$. On the other hand $A_p = y^{-1}\mathfrak{M} = y^{-1}A_pB_p$, hence $(y^{-1}B_p)A_p \subset A_p$; i.e. $y^{-1}B_p \subset \mathcal{O}$. It follows that $y\mathcal{O} = B_p$; i.e. B_p is principal as $\mathcal{O}$-module. Hence $\mathfrak{M} \cong B_p$ is principal, i.e. $\mathcal{O}$ is discrete.

3) $\Rightarrow$ 2) : Let $M = M_1 \oplus M_2$ with (3). We want to show that for every $o \neq q \in K$, $q^{-1}A \cap S + qB \cap S = S$.

Now let $o \neq q \in K$ be arbitrary. Let P be an arbitrary maximal ideal of S. If $P \neq AB$, then $A_pB_p = S_p$, i.e. B_p is invertible hence principal, and $A_p = (S_p : B_p)$. By (2) $\mathcal{O}(A_p) = \mathcal{O}(B_p) \supsetneq S_p$ is a valuation ring. Hence all S_p-submodules of K are comparable. Now if

$q^{-1}A_p \neq S_p$, then $q \notin A_p = (S_p : B_p)$, hence $qB_p \notin S_p$. Thus $S_p \subset qB_p$. It follows that $q^{-1}A_p \cap S_p + qB_p \cap S_p = S_p$.

If $P \supset AB$, then $A_pB_p \subset P_p$.

Case (a): A_p or B_p is principal. Let B_p be a principal (hence invertible) fractional ideal of S_p. By (2), $S_p = \mathcal{O}(B_p) = \mathcal{O}(A_p)$ is a valuation ring and $A_pB_p = P_p$. Then $P_p(S_p : B_p) = A_p$, the unique maximal S_p-submodule of $(S_p : B_p)$. Note that $A_p \subsetneq (S_p : B_p)$, since otherwise $A_pB_p = S_p$ which is a contradiction. Hence $(S_p : B_p)/A_p$ is a simple S_p-module.

Consider any $q \notin A$. If $q \in (S_p : B_p)$, then $(S_p : B_p)/A_p = qS_p$, hence $(S_p : B_p) = qS_p + A_p$. Since $q \notin A_p$ and qS_p, A_p are comparable, we get $(S_p : B_p) = qS_p$. Then $S_p = B_p(S_p : B_p) = qB_p$.

On the other hand if $q \notin (S_p : B_p)$, then $qB_p \notin S_p$, hence $S_p \subset qB_p$.

It follows that $S_p \subset qB_p$ holds for all $q \notin A_p$. Therefore $q^{-1}A_p \cap S_p + qB_p \cap S_p = S_p$.

Case (b): A_p and B_p are not principal. By (2) $\mathcal{O}(A_p) = \mathcal{O}(B_p) =: \mathcal{O}$ is a valuation ring with maximal ideal $\mathfrak{M} = A_pB_p$.

We claim that each of A_p and B_p is comparable with all S_p-submodules of K. Let $x \notin A_p$. Since $\mathcal{O}$ is a valuation ring, we get

$A_p \subsetneq x\mathcal{O}$; i.e. $x^{-1}A_p \subsetneq \mathcal{O}$. Then $x^{-1}A_p \subset \mathfrak{M} = A_pB \subset S_p$; i.e. $A_p \subset x\,S_p$. It follows that A_p is comparable with all S_p-submodules of K. Similarly we see that B_p is comparable with all S_p-submodules of K.

Claim (2): If $A_p \subsetneq (S_p : B_p)$, then $A_p, B_p \cong \mathfrak{M}$ and $\mathcal{O}$ is not discrete. Let $x \in (S_p : B_p) \setminus A_p$. Then $xB_p \subsetneq S_p \subset \mathcal{O}$ (due to B_p being not principal). It follows that $xB_p \subset \mathfrak{M}$. On the other hand $A_p \subsetneq x\,S_p$ (due to A_p being comparable with xS_p and $x \notin A_p$). Then $A_pB_p \subset x\,B_p$. Therefore $\mathfrak{M} = A_pB_p \subset xB_p \subset \mathfrak{M}$; i.e. $\mathfrak{M} = xB_p \cong B_p$.

Now $x^{-1} \in (S_p : A_p) \setminus B_p$, i.e. $B_p \subsetneq (S_p : A_p)$. By the same arguement we can show that $\mathfrak{M} \cong A_p$.

Now if $\mathcal{O}$ is discrete, then $\mathfrak{M}$, and hence B_p, are principal. Then $xB_p = \mathfrak{M} = A_pB_p$ implies $x \in x\,\mathcal{O} = A_p$, which contradicts the choice of x. Therefore $\mathcal{O}$ is not discrete.

By claim (2) and condition (3) we obtain now $A_p = (S_p : B_p)$. Therefore if $q \notin A_p$, then $S_p \subset qB_p$. Hence $q^{-1}A_p \cap S_p + qB_p \cap S_p = S_p$.

These two cases together show that $q^{-1}A_p \cap S_p + qB_p \cap S_p = S_p$ holds for every maximal ideal P of S. Then $q^{-1}A \cap S + qB \cap S = S$. □

Corollary 2.27: Let $M = M_1 \oplus M_2$ have (C_1), where the M_i are uniform. Then $\mathcal{O}(A)$ coincides with $\mathcal{O}(B)$, and is integrally closed.

Proof: Let M have (C_1). Then by (2.26), $\mathcal{O}(A_P) = \mathcal{O}(B_P)$ is a valuation ring for every maximal ideal P of S. Since any valuation ring is integrally closed, and the intersection of integrally closed domains is integrally closed, we get that $\bigcap_P \mathcal{O}(A_P) = \bigcap_P \mathcal{O}(B_P)$ is integrally closed.

We know that $\mathcal{O}(A) \subset \bigcap_P \mathcal{O}(A_P)$, where P ranges over all maximal ideals of S. Now let $x \in \bigcap_P \mathcal{O}(A_P)$. Then $x \in \mathcal{O}(A_P)$ for every P, i.e. $xA_P \subset A_P$. Since $A \subset A_P$, $xA \subset A_P$ for every P. Hence $xA \subset \bigcap_P A_P = A$; i.e. $x \in \mathcal{O}(A)$. Therefore $\mathcal{O}(A) = \bigcap_P \mathcal{O}(A_P)$.

Similarly we can show that $\mathcal{O}(B) = \bigcap_P \mathcal{O}(B_P)$.

Hence $\mathcal{O}(A) = \mathcal{O}(B)$ is integrally closed.

As a first application we analyze now the condition (C_1) in two very special cases.

Corollary 2.28: Let P be a maximal ideal of R. Then the following statements are equivalent:

(1) $P \oplus R$ has (C_1).

(2) $\mathcal{O}(P)$ is a Prüfer domain and P is a maximal ideal of $\mathcal{O}(P)$. R_Q is a valuation ring for all maximal ideals Q of R different from P.

Proof: 1) $\Rightarrow$ 2): Let $P \oplus R$ have (C_1). Since $P_Q = R_Q$ for all maximal ideals Q different from P, by (2.26), $R_Q = \mathcal{O}(P_Q)$ is a valuation ring.

To show that $\mathcal{O}(P)$ is a Prüfer domain and P is a maximal ideal of $\mathcal{O}(P)$, we shall consider two cases. Since $P \subset (R:P)P \subset R$, we have $(R:P)P = P$ or $(R:P)P = R$.

<u>Case 1</u>: $(R:P)P = R$. Then $\mathcal{O}(P) = R$ and P_P is invertible hence principal. By (2.25) $R_P = \mathcal{O}(P_P)$ is a valuation ring. It follows that R_M is a valuation ring for all maximal ideals M of R. Thus $\mathcal{O}(P) = R$ is a Prüfer domain.

<u>Case 2</u>: $(R:P)P = P$. Then $\mathcal{O}(P) = (R:P)$. By (2.24) $(R:P)P \oplus P = P \oplus P$ has (C_1). By (2.26) $\mathcal{O}(P)_M$ is a valuation ring, for every maximal ideal M of $\mathcal{O}(P)$. Hence, by (2.4), $\mathcal{O}(P)$ is a Prüfer domain.

Since $P \oplus R$ has (C_1), $\mathcal{O}(P)_P = (R:P)_P = \mathcal{O}((R:P)_P)$ is a valuation ring with maximal ideal $(R:P)_P P_P = P_P$, by (2.26). It follows that $\mathcal{O}(P)_P/P_P \cong (\mathcal{O}(P)/P)_P$ is a simple R_P-module, and $(\mathcal{O}(P)/P)_Q = 0$ for every maximal ideal Q different from P. Then, by (2.10), $\mathcal{O}(P)/P$ is a simple R-module, i.e. P is a maximal ideal $\mathcal{O}(P)$.

(2) $\Rightarrow$ (1) is obvious. □

We don't know a similar description of (C_1) for $I \oplus R$, for an arbitrary non-zero ideal of R, except if R is local or Noetherian (cf (2.32) and (2.62)).

Corollary 2.29: Let N be a non-zero uniform R-module. Then $N \oplus N$ has (C_1) if and only if $\mathcal{O}(N)$ is a Prüfer domain.

Proof: Obvious since $A = B = \mathcal{O}(N) = S$. □

The following example shows, in contrast to (2.29), that if $M_1 \not\cong M_2$, then neither $\mathcal{O}(M_2)$ nor $\mathcal{O}(M_1) \cap \mathcal{O}(M_2)$ need to be a Prüfer domain.

Example 2:30: Let $\mathcal{O}$ be a valuation ring with maximal ideal P. Let F be a subfield of the field $\mathcal{O}/P$; and let S be the full inverse image of F under the natural homomorphism : $\mathcal{O} \to \mathcal{O}/P$, i.e. $S \subset \mathcal{O}$ and $S/P = F$. Then S is a local domain with maximal ideal P. (Indeed, let $x \in S\backslash P$. Then x is a unit in $\mathcal{O}$, i.e. $x^{-1} \in \mathcal{O}\backslash P$. Since $o \neq \bar{x} \in S/p = F$, there exists $y \in F$ such that $\bar{x}y = \bar{1}$. It follows that $\overline{x^{-1}} = y \in F$, hence $x^{-1} \in S$. Then x is a unit in S.) It is clear that $\mathcal{O}(P) = \mathcal{O}$. By (2.28), $P \oplus S$ has (C_1).

Now we choose $\mathcal{O} = K[[t]]$, the ring of formal power series in t over a field K. Let k be a proper subfield of K. Then we obtain $S = k + t\,K[[t]]$, the full inverse image of k under the natural homomorphism : $K[[t]] \to K[[t]]/tK[[t]]$. From the previous discussion $M_1 \oplus M_2 := P \oplus S$ has (C_1), where $P = tK[[t]]$. It is clear that $S = \mathcal{O}(M_2) = \mathcal{O}(M_1) \cap \mathcal{O}(M_2)$ is local but not a valuation ring, hence not a Prüfer domain. □

Our second example shows that the statement "if $A_p \cong ℍ \cong B_p$, then

$\mathcal{O}(A)_P = \mathcal{O}(B)_P$ is discrete" in Theorem (2.26) is not avoidable, in other words, does not follow from the rest of (2).

Example 2.31: Let $\mathcal{O}$ be a valuation ring which is not discrete, with maximal ideal $\mathfrak{M}$. Let $\mathcal{O}/\mathfrak{M} = Q$ be the field of rational numbers. Let S be the full inverse image of the ring of integers $\mathbb{Z}$ under the natural homomorphism : $\mathcal{O} \to \mathcal{O}/\mathfrak{M} = Q$, i.e. $S/\mathfrak{M} = \mathbb{Z}$. Choose additive groups M_1 and M_2, $\mathfrak{M} \subset M_1, M_2 \subset \mathcal{O}$ such that $M_1/\mathfrak{M}$, $M_2/\mathfrak{M}$ can not be embedded in each other, and $\mathcal{O}(M_1) \cap \mathcal{O}(M_2) = S$.

Since $\mathfrak{M} M_1 \subset M\mathcal{O} = \mathfrak{M} \subset M_2$, we have $\mathfrak{M} \subset A = (M_2 : M_1)$. Now let $x \in A$ be arbitrary. Then $x\mathfrak{M} \subset \mathcal{O}$. Since $\mathfrak{M}^2 = \mathfrak{M}$, it follows that $x\mathfrak{M} \subset \mathfrak{M}$, i.e. $x \in \mathcal{O}(\mathfrak{M}) = \mathcal{O}$. If $x \not\in \mathfrak{M}$, then x is a unit in $\mathcal{O}$. Then $x\mathfrak{M} = \mathfrak{M}$ and hence $M_1/\mathfrak{M} \cong xM_1/x\mathfrak{M} = xM_1/\mathfrak{M} \subset M_2/\mathfrak{M}$, which contradicts the fact that $M_1/\mathfrak{M}$ can not be embedded in $M_2/\mathfrak{M}$. Thus $x \in \mathfrak{M}$, hence $A = \mathfrak{M}$.

Similarly we can show that $B = (M_1 : M_2) = \mathfrak{M}$.

Now let P be any maximal ideal of S. If $P \not\supset AB = \mathfrak{M}^2 = \mathfrak{M}$, then $S_P = A_PB_P = \mathfrak{M}_P$. It follows that $\mathfrak{M}_P = \mathcal{O}_P = S_P = \mathcal{O}(A_P) = \mathcal{O}(B_P)$ is a valuation ring with $P_P \subset A_PB_P$. On the other hand if $P \supset AB = \mathfrak{M}$, then $\mathcal{O}_P = \mathcal{O}$ hence $A_P = A = \mathfrak{M}$, $B_P = B = \mathfrak{M}$. It follows that $\mathcal{O}(A_P) = \mathcal{O}(B_P) = \mathcal{O}(\mathfrak{M}) = \mathcal{O}$ is a valuation ring with maximal ideal $\mathfrak{M} = A_PB_P$.

Therefore condition (2) of (2.26), except for "if $A_P \cong \mathfrak{M} \cong B_P$, then $\mathcal{O}(A_P)$ ($= \mathcal{O}(B_P)$) is discrete", is satisfied. In particular $M_1 \oplus M_2$ does not have (C_1).

Note that since $\mathfrak{M} \subsetneq M_1 \subset \mathcal{O}$ are additive group, we have that

$M_i/\mathfrak{M} \subset \mathcal{O}/\mathfrak{M} = Q$ are rank one torsion free abelian groups, of certain types τ_i (see [6]). It is easy to see that $\mathcal{O}(M_i)/\mathfrak{M} \cong \mathcal{O}(M_i/\mathfrak{M}) = \mathbb{Z}_{(i)} \subset Q$ where $\mathbb{Z}_{(i)} = \mathbb{Z}_{\{p : \tau_i(p) \neq \infty\}}$, and that $\mathcal{O}(M_i)$ is the inverse image of $\mathcal{O}(M_i)/\mathfrak{M}$. It follows that $\mathcal{O}(M_1) \cap \mathcal{O}(M_2)$ is the full inverse image of $\mathbb{Z}_{(1)} \cap \mathbb{Z}_{(2)}$. Therefore $\mathcal{O}(M_1) \cap \mathcal{O}(M_2) = S$ if and only if $\mathbb{Z}_{(1)} \cap \mathbb{Z}_{(2)} = \mathbb{Z}$.

To construct M_1, M_2 as required in the example, choose M_1, M_2 such that $M_1/\mathfrak{M}$, $M_2/\mathfrak{M}$ are of incomparable types τ_1, τ_2, and such that $\tau_1(p)$, $\tau_2(p)$ are not both ∞, for any prime number p. □

§4. CONDITIONS ON THE ENDOMORPHISM RINGS

In this section we characterize all finite direct sums of torsion free uniform reduced modules with local, or comparable, endomorphism rings which have property (C_1).

Let $A = A_{12} = (M_2 : M_1)$ and $B = A_{21} = (M_1 : M_2)$. We have the following:

Proposition 2.32: Let $M = M_1 \oplus M_2$ be an R-module, where the M_i are uniform. Let $\mathcal{O}(M_1)$ or $\mathcal{O}(M_2)$ be local. Then the following statements are equivalent:

1) M has (C_1);

2) $\mathcal{O}(A)$ coincides with $\mathcal{O}(B)$, and is a valuation ring with maximal ideal $\mathfrak{M} \subset AB$. If $A \cong \mathfrak{M} \ncong B$, then $\mathcal{O}(A)(= \mathcal{O}(B))$ is discrete.

Proof: Let $\mathcal{O}(M_1)$ be a local domain with maximal ideal P. We show that M has (C_1) if and only if $q^{-1}A \cap \mathcal{O}(M_1) + qB \cap \mathcal{O}(M_1) = \mathcal{O}(M_1)$, for each $o \neq q \in K$. From (2.23), it is clear that if M has (C_1), then $q^{-1}A \cap \mathcal{O}(M_1) + qB \cap \mathcal{O}(M_1) = \mathcal{O}(M_1)$ for each $o \neq q \in K$. Now let $q^{-1}A \cap \mathcal{O}(M_1) + qB \cap \mathcal{O}(M_1) = \mathcal{O}(M_1)$ for every $o \neq q \in K$. Since $\mathcal{O}(M_1)$ is local, we have $q^{-1}A \cap \mathcal{O}(M_1) = \mathcal{O}(M_1)$ or $qB \cap \mathcal{O}(M_1) = \mathcal{O}(M_1)$. It follows that $q \in A$ or $q^{-1} \in B$ and hence qM_1 and M_2 are comparable, for each $o \neq q \in K$. Then M has (C_1), by (2.21).

By using the note in the beginning of the proof of Theorem (2.26), and by noting that $A_p = A$, $B_p = B$, we have that 1) is equivalent to 2). □

Proposition 2.33: Let $M = \overset{n}{\underset{i=1}{\oplus}} M_i$ be an R-module, where the M_i are uniform. Let all $\mathcal{O}(M_i)$ but possibly two be local. Then M has (C_1) if and only if $M_i \oplus M_j$ has (C_1) for all $i \neq j$.

Proof: Let $M_i \oplus M_j$ have (C_1) for all $i \neq j$. Without loss of generality assume that $\mathcal{O}(M_i)$ is local for all $i \geq 3$. Let $q_1, q_2, \ldots, q_n \in K$ (not all zero) be given arbitrarily. Let $F = \{i : q_i \neq o\}$. By (2.20) $q_i^{-1}M_i$ and $q_j^{-1}M_j$ are comparable for all $i,j \in F \setminus \{1\}$. Hence $\{q_i^{-1}M_i\}_{i \in F \setminus \{1\}}$ forms a chain of R-submodules of K. Let $q_{i_o}^{-1} M_{i_o}$ be the smallest member. Now if $q_1 = o$, then $q_j M_{i_o} \subset q_{i_o} M_j$ for all $j = 1,2,\ldots,n$.

Therefore condition 2) in (2.19) is satisfied for the $q_1,\ldots,q_n \in K$ with $\alpha_j = o$ for all $j \neq i_o$ and $\alpha_{i_o} = 1$. If $q_1 \neq o$ then, by (2.20), there exist $\alpha_1, \alpha_{i_o} \in \mathcal{O}(M_1) \cap \mathcal{O}(M_{i_o})$ such that $\alpha_1 + \alpha_{i_o} = 1$ and $\alpha_1 q_1^{-1} M_1 \subset q_{i_o}^{-1} M_{i_o}$ and $\alpha_{i_o} q_{i_o}^{-1} M_{i_o} \subset q_1^{-1} M_1$. It follows that $\alpha_1 q_1^{-1} M_1 \subset q_{i_o}^{-1} M_{i_o} \subset q_j^{-1} M_j$ and $\alpha_{i_o} q_{i_o}^{-1} M_{i_o} \subset q_j^{-1} M_j$ for all $j \in F$. Hence $\alpha_1 q_j M_1 \subset q_1 M_j$ and $\alpha_{i_o} q_j M_{i_o} \subset q_{i_o} M_j$, $j = 1,2,\ldots,n$. Therefore condition 2) of (2.19) is satisfied for the $q_1,\ldots,q_n \in K$ with $\alpha_j = o$ for all $j \notin F$. Then M has (C_1).

An immediate consequence of (2.32) and (2.33) is the following:

<u>Theorem 2.34</u>: Let $M = \overset{n}{\underset{i=1}{\oplus}} M_i$ be an R-module, where the M_i are uniform torsion free reduced with local endomorphism rings. Then the following statements are equivalent.

1) M has (C_1).

2) For all $i \neq j$, $\mathcal{O}(A_{ij})$ coincides with $\mathcal{O}(A_{ji})$, and is a valuation ring with maximal ideal $\mathfrak{M}_{ij} \subset A_{ij}A_{ji}$. If $A_{ij} \cong \mathfrak{M} \cong A_{ji}$, then $\mathcal{O}(A_{ij}) (= \mathcal{O}(A_{ji}))$ is discrete. □

In the following proposition we show that $M = \overset{n}{\underset{i=1}{\oplus}} M_i$ has (C_1) if and only if $M_i \oplus M_j$ has (C_1) for all $i \neq j$, where the M_i are uniform and $\mathcal{O}(M_i)$, for some i, is contained in all $\mathcal{O}(M_j)$ with one possible exception.

<u>Proposition 2.35</u>: Let $M = \bigoplus_{i=1}^{n} M_i$ be an R-module with all M_i uniform. Let, for some i, $\mathcal{O}(M_i) \subset \mathcal{O}(M_j)$ for all j but possibly one. Then M has (C_1) if and only if $M_i \oplus M_j$ has (C_1) for all $i \neq j$.

<u>Proof</u>: Let $M_i \oplus M_j$ have (C_1) for all $i \neq j$. The proof will be by induction on n.

For $n \leq 2$ the claim is clear. Now assume that $\bigoplus_{i \in F} M_i$ has (C_1) for any proper subset F of $\{1,2,\ldots,n\}$, and let $M = \bigoplus_{i=1}^{n} M_i$. Without loss of generality assume that $\mathcal{O}(M_1) \subset \mathcal{O}(M_j)$, $j = 1,2,\ldots,n-1$. Then $S = \bigcap_{i=1}^{n} \mathcal{O}(M_i) = \mathcal{O}(M_1) \cap \mathcal{O}(M_n)$.

Let $q_1, q_2, \ldots, q_n \in K$ (not all zero) be given arbitrarily, and let $F = \{i : q_i \neq o\}$. We show that $1 \in \sum_{i \in F} \bigcap_{j \in F} q_i q_j^{-1} A_{ij}$, i.e. $S \subset \sum_{i \in F} \bigcap_{j \in F} q_i q_j^{-1} A_{ij}$ (since all A_{ij} are S-modules). If $F \subsetneq \{1,2,\ldots n\}$, then by induction $\bigoplus_{i \in F} M_i$ has (C_1). By (2.25) $S \subset \sum_{i \in F} \bigcap_{j \in F} q_i q_j^{-1} A_{ij}$.

Now let all q_i not be zero. Since $M_1 \oplus M_n$ has (C_1) and $S = \mathcal{O}(M_1) \cap \mathcal{O}(M_n)$, by (2.23), we have $q_1 q_n^{-1} A_{1n} \cap S + q_1^{-1} q_n A_{n1} \cap S = S$. Hence for each maximal ideal P of S, $q_1 q_n^{-1} (A_{1n})_P \cap S_P + q_1^{-1} q_n (A_{n1})_P \cap S_P = S_P$.

Since S_P is local, we have $q_1q_n^{-1}(A_{1n})_P \cap S_P = S_P$ or $q_1^{-1}q_n(A_{n1})_P \cap S_P = S_P$; i.e. $S_P \subset q_1q_n^{-1}(A_{1n})_P$ or $S_P \subset q_1^{-1}q_n(A_{n1})_P$.

<u>Case 1</u>: $S_P \subset q_1q_n^{-1}(A_{1n})_P$. Then $q_nq_1^{-1} \in (A_{1n})_P$. Since $(A_{i1})_P(A_{1n})_P \subset (A_{in})_P$ for all i, we have $q_nq_1^{-1}(A_{i1})_P \subset (A_{in})_P$. It follows that $q_iq_1^{-1}(A_{i1})_P \subset q_iq_n^{-1}(A_{in})_P$ for all i. Hence $\bigcap_{j=1}^{n} q_iq_j^{-1}(A_{ij})_P = \bigcap_{j=1}^{n-1} q_iq_j^{-1}(A_{ij})_P$. It follows that

$$\sum_{i=1}^{n} \bigcap_{j=1}^{n} q_iq_j^{-1}(A_{ij})_P = \sum_{i=1}^{n-1} \bigcap_{j=1}^{n-1} q_iq_j^{-1}(A_{ij})_P + \bigcap_{j=1}^{n-1} q_nq_j^{-1}(A_{nj})_P.$$

By induction $\bigoplus_{i=1}^{n-1} M_i$ has (C_1). Hence, by (2.22), $1 \in \sum_{i=1}^{n-1} \bigcap_{j=1}^{n-1} q_iq_j^{-1}A_{ij}$. Thus $S_P \subset \sum_{i=1}^{n} \bigcap_{j=1}^{n} q_iq_j^{-1}(A_{ij})_P$.

<u>Case 2</u>: $S_P \subset q_nq_1^{-1}(A_{n1})_P$. Then $q_1q_n^{-1} \in (A_{n1})_P$. By the same argument as in case (1), we can show that $q_iq_n^{-1}(A_{in})_P \subset q_iq_1^{-1}(A_{i1})_P$ for all i. Hence $\bigcap_{j=1}^{n} q_iq_j^{-1}(A_{ij})_P = \bigcap_{j=2}^{n} q_iq_j^{-1}(A_{ij})_P$. It follows that

$$\sum_{i=1}^{n} \bigcap_{j=1}^{n} q_iq_j^{-1}(A_{ij})_P = \bigcap_{j=2}^{n} q_iq_j^{-1}(A_{ij})_P + \sum_{i=2}^{n} \bigcap_{j=2}^{n} q_iq_j^{-1}(A_{ij})_P.$$

Again by induction $\bigoplus_{i=2}^{n} M_i$ has (C_1). By (2.22), $1 \in \sum_{i=2}^{n} \bigcap_{j=2}^{n} q_iq_j^{-1}A_{ij}$. Then $S_P \subset \sum_{i=1}^{n} \bigcap_{j=1}^{n} q_iq_j^{-1}(A_{ij})_P$.

These two cases together show that $S_P \subset \sum_{i=1}^{n} \bigcap_{j=1}^{n} q_i q_j^{-1}(A_{ij})_P$ holds for every maximal ideal P of S. Then $S \subset \sum_{i=1}^{n} \bigcap_{j=1}^{n} q_i q_j^{-1} A_{ij}$.

We have shown that $S \subset \sum_{i\in F} \bigcap_{j\in F} q_i q_j^{-1} A_{ij}$ for all $q_1, q_2, \ldots, q_n \in K$ (not all zero) where $F = \{i : q_i \neq o\}$. Therefore, by (2.22) M has (C_1).

The converse is obvious. □

An immediate consequence of (2.29) and (2.35) is the following:

<u>Corollary 2.36</u>: Let N be a uniform R-module. Then $\bigoplus_{i=1}^{n} N$ has (C_1), for any integer $n \geq 2$, if and only if $\mathcal{O}(N)$ is Prüfer.

The following theorem is an immediate consequence of (2.26) and (2.35).

<u>Theorem 2.37</u>: Let $M = \bigoplus_{i=1}^{n} M_i$ be an R-module, where the M_i are uniform torsion free reduced and for some i, $\mathcal{O}(M_i) \subset \mathcal{O}(M_j)$ for all j but possibly one. Then the following statements are equivalent:

1) M has (C_1);
2) For each maximal ideal P of $S = \bigcap_{i=1}^{n} \mathcal{O}(M_i)$, $\mathcal{O}((A_{ij})_P)$ coincides with $\mathcal{O}((A_{ji})_P)$, and is a valuation ring with maximal ideal $\mathfrak{M}_{ij} \subset (A_{ij})_P (A_{ji})_P$ for all $i \neq j$. If $(A_{ij})_P \cong \mathfrak{M}_{ij} \cong (A_{ji})_P$, then $\mathcal{O}((A_{ij})_P)$ $(= \mathcal{O}((A_{ji})_P))$ is discrete. □

§5. PRÜFER DOMAINS

In this section we characterize all torsion free reduced modules over Prüfer domains, which have property (C_1).

The following proposition is an immediate consequence of (2.26).

Proposition (2.38): Let $M = M_1 \oplus M_2$ be an R-module, where the M_i are uniform. Let $S =: \mathcal{O}(M_1) \cap \mathcal{O}(M_2)$ be a Prüfer domain. Then the following statements are equivalent:

1) M has (C_1).

2) For each maximal ideal P of S, $\mathcal{O}(A_P)$ coincides with $\mathcal{O}(B_P)$, and its maximal ideal $\mathfrak{M}$ is contained in $A_P B_P$. If $A_P \cong \mathfrak{M} \cong B_P$, then $\mathcal{O}(A_P) = \mathcal{O}(B_P)$ is a discrete valuation ring. □

Proposition 2.39: Let R be a Prüfer domain and let P be a maximal ideal of R. Then $P \oplus R$ has (C_1).

Proof: We show that $\mathcal{O}(P) = R$. Since R_P is a valuation ring with maximal ideal P_P, we have that $\mathcal{O}(P_P) = R_P$. It is clear that $\mathcal{O}(P_Q) = R_Q$ for each maximal ideal Q different from P. It follows that $\mathcal{O}(P_M) = R_M$ for every maximal ideal M of R, hence

$$\mathcal{O}(P) = \bigcap_M \mathcal{O}(P_M) = \bigcap_M R_M = R.$$

By (2.28), we get that $P \oplus R$ has (C_1). □

The following proposition is an immediate consequence of (2.36):

Proposition 2.40: Let N be a uniform module over a Prüfer domain R. Then $\overset{n}{\underset{i=1}{\oplus}} N$ has (C_1), for any positive integer n. □

Proposition 2.41: Let $M = \overset{n}{\underset{i=1}{\oplus}} M_i$ be an R-module, where the M_i are uniform. Let $S =: \overset{n}{\underset{i=1}{\cap}} \mathcal{O}(M_i)$ be a Prüfer domain. Then M has (C_1) if and only if $M_i \oplus M_j$ has (C_1) for all i, j.

Proof: Let $M_i \oplus M_j$ have (C_1) for all i, j. Let $q_1, q_2, \ldots, q_n \in K$ (not all zero) be arbitrary. Let $F = \{ i : q_i \neq o \}$. Denote $\mathcal{O}(M_i) \cap \mathcal{O}(M_j)$ by S_{ij}.

We show that $S \subset \sum_{i \in F} \bigcap_{j \in F} q_i q_j^{-1} A_{ij}$. Since $M_i \oplus M_j$ has (C_1), by (2.23), we have $q_i q_j^{-1} A_{ij} \cap S_{ij} + q_j q_i^{-1} A_{ji} \cap S_{ij} = S_{ij}$ for all $i, j \in F$. It follows that $q_i q_j^{-1} (A_{ij})_P \cap (S_{ij})_P + q_j q_i^{-1} (A_{ji})_P \cap (S_{ij})_P = (S_{ij})_P$ for each maximal ideal P of S. Since S is Prüfer, we have that S_P is a valuation ring. Hence all S_P-submodules of K are comparable. It follows that $q_i q_j^{-1} (A_{ij})_P \cap (S_{ij})_P = (S_{ij})_P$ or $q_j q_i^{-1} (A_{ij})_P \cap (S_{ij})_P = (S_{ij})_P$. Hence $S_P \subset q_i q_j^{-1} (A_{ij})_P$ or $S_P \subset q_j q_i^{-1} (A_{ji})_P$ for all $i, j \in F$. Now for each $i \in F$, $\{q_i q_j^{-1} (A_{ij})_P\}_{j \in F}$ forms a chain of S_P-submodules of K; let $q_i q_{\alpha(i)}^{-1} (A_{i\alpha(i)})_P$ be the smallest member. Then

$\bigcap_{j\in F} q_i q_j^{-1}(A_{ij})_P = q_i q^{-1}_{\alpha(i)}(A_{i\alpha(i)})_P$, and hence $\sum_{i\in F} \bigcap_{j\in F} q_i q_j^{-1}(A_{ij})_P$

$= \sum_{i\in F} q_i q^{-1}_{\alpha(i)}(A_{i\alpha(i)})_P$.

Note that the α introduced above defines a function from F into itself. Then we can form powers of α.

Assume that $q_i q^{-1}_{\alpha(i)}(A_{i\alpha(i)})_P \subset P_P$ for all $i \in F$. Then $S_P \subset q_{\alpha(i)} q_i^{-1}(A_{\alpha(i)i})_P$ for all $i \in F$. Since F is a finite set, there exists an integer $k \geq 2$ and an element $i \in F$ such that $\alpha^k(i) = i$. Since $S_P \subset q_{\alpha^\ell(i)} q^{-1}_{\alpha^{\ell-1}(i)} (A_{\alpha^\ell(i)\alpha^{\ell-1}(i)})_P$ for all $\ell \leq k$, we obtain that

$q_{\alpha^{\ell-1}(i)} q^{-1}_{\alpha^\ell(i)} \in (A_{\alpha^\ell(i)\,\alpha^{\ell-1}(i)})_P$. Since

$(A_{\alpha^k(i)\alpha^{k-1}(i)})_P\, (A_{\alpha^{k-1}(i)\alpha^{k-2}(i)})_P \quad \cdots \quad (A_{\alpha^3(i)\,\alpha^2(i)})_P\, (A_{\alpha^2(i)\,\alpha(i)})_P$

$\subset (A_{i\,\alpha(i)})_P$. It follows that $q_{\alpha^2(i)} q_i^{-1} (A_{\alpha^2(i)\,\alpha(i)})_P \subset (A_{i\,\alpha(i)})_P$,

hence $q_{\alpha^2(i)} q^{-1}_{\alpha(i)} (A_{\alpha^2(i)\,\alpha(i)})_P \subset q_i q^{-1}_{\alpha(i)} (A_{i\,\alpha(i)})_P$. It follows that

$S_P \subset q_{\alpha^2(i)} q^{-1}_{\alpha(i)} (A_{\alpha^2(i)\alpha(i)})_P \subset q_i q^{-1}_{\alpha(i)} (A_{i\,\alpha(i)})_P \subset P_P$, which is a contradiction. Therefore $S_P \subset q_i q^{-1}_{\alpha(i)}(A_{i\,\alpha(i)})_P$ for some i. Hence

$S_P \subset \sum_{i\in F} \bigcap_{j\in F} q_i q_j^{-1}(A_{ij})_P$ for every maximal ideal P of S.

Then $S \subset \sum_{i \in F} \bigcap_{j \in F} q_i q_j^{-1} A_{ij}$.

By (2.22) M has (C_1).

The converse is obvious. □

Let $S_{ij} = \mathcal{O}(M_i) \cap \mathcal{O}(M_j)$ for all i, j. Then we have the following:

Theorem 2.42: Let M be a torsion free reduced module over a Prüfer domain R. Then the following statements are equivalent:

1) M has (C_1);

2) $M = \bigoplus_{i=1}^{n} M_i$ with all M_i uniform. For every maximal ideal P_{ij} of S_{ij}, $\mathcal{O}((A_{ij})_{P_{ij}})$ coincides with $\mathcal{O}((A_{ji})_{P_{ij}})$, and its maximal ideal $\mathfrak{m}_{ij}$ is contained in $(A_{ij})_{P_{ij}}$ $(A_{ji})_{P_{ij}}$, for all i,j.

If $(A_{ij})_{P_{ij}} \cong \mathfrak{m}_{ij} \cong (A_{ji})_{P_{ji}}$, then $\mathcal{O}((A_{ij})_{P_{ij}}) = \mathcal{O}((A_{ji})_{P_{ij}})$ is a discrete valuation ring.

Proof: 1) → 2): Let M have (C_1). By (2.16) we have $M = \bigoplus_{i=1}^{n} M_i$, where the M_i are uniform submodules of M. Since any overring of R is a Prüfer domain, we have that the S_{ij} are Prüfer domains for all i, j. Since (C_1) is inherited by direct summands, and $M_i \oplus M_i$ has (C_1) for all i by (2.40), we have $M_i \oplus M_j$ has (C_1) for all i, j. By (2.38) we get 2).

2) → 1): Let 2) be given. Since the S_{ij} are Prüfer, by (2.38),

$M_i \oplus M_j$ has (C_1) for all i, j. Since $S =: \bigcap_{i=1}^{n} \mathcal{O}(M_i)$ is an overring of R, we have that S is a Prüfer domain.

By (2.41), M has (C_1).

§6. DEDEKIND DOMAINS:

In this section we characterize all torsion free reduced modules over Dedekind domains which have property (C_1).

We first mention some properties of Dedekind domains. Let R be a Dedekind domain with quotient field K. The factor module K/R is a torsion divisible R-module, analogously to the case of torsion abelian groups, it can be written as a direct sum of P-primary submodules; i.e. $K/R = \bigoplus_P T_P$ where $T_P = \{x \in K/R : P^n x = o$ for some $n\}$, and P ranges over the maximal ideals of R (T_P is the natural generalization of the divisible torsion abelian group $C(P^\infty)$. Define T_P^n by $T_P^n = \{x \in T_P : P^n x = o\}$ if $n < \infty$, $T_P^\infty = T_P$. $\{T_P^n\}_{n<\infty}$ forms a chain of submodules of T_P such that $T_P = \bigcup_{n<\infty} T_P^n$. If $\infty > n \geq m$, then $P^{n-m} T_P^n = T_P^m$. If X and Y are submodules of K/R with $X \subset Y$, then X and Y can be written as $X = \bigoplus_P T_P^{n_P}$, $Y = \bigoplus_P T_P^{m_P}$ where $n_P \leq m_P \leq \infty$ for all P.

Let $A = (M_2 : M_1)$, $B = (M_1 : M_2)$ and $S = \mathcal{O}(M_1) \cap \mathcal{O}(M_2)$. Then we have the following:

Lemma 2.43: Let M_1 and M_2 be R-submodules of K such that $1 \in M_1 \subset M_2$, and M_2 can be embedded in M_1. Let S be Dedekind and

$M_1/S = \bigoplus_P T_P^{n_P}$, $M_2/S = \bigoplus_P T_P^{m_P}$. Then $B = \prod_P P^{m_P - n_P}$, where P ranges over the maximal ideals of S.

Proof: Since S is Dedekind and B is a non-zero ideal of S, we have $B = \prod_P P^{b_P}$, where $b_P \geq 0$. Since $\prod_P P^{b_P} M_2 = B\, M_2 \subset M_1$, it follows that $\prod_P P^{b_P} \overline{M}_2 \subset \overline{M}_1$ where $\overline{M}_1 = M_1/S$. Then $\bigoplus_P T_P^{m_P - b_P} = \prod_P P^{b_P} \bigoplus_P T_P^{m_P} \subset \bigoplus_P T_P^{n_P}$, and hence $m_P - b_P \leq n_P$, i.e. $b_P \geq m_P - n_P$.

Now since $\prod_P P^{m_P - n_P} \bigoplus_P T_P^{m_P} = \bigoplus_P T_P^{n_P}$, we have $\prod_P P^{m_P - n_P} M_2 \subset \prod_P P^{m_P - n_P} (M_2 + S) \subset M_1 + S = M_1$ (since $S \subset M_1$), and hence $\prod_P P^{m_P - n_P} \subset B = \prod_P P^{b_P}$. Therefore $m_P - n_P \geq b_P \geq m_P - n_P$, and it follows that $b_P = m_P - n_P$ and $B = \prod_P P^{m_P - n_P}$.

Lemma 2.44: Let M_1 and M_2 be modules over a Dedekind domain R, where the M_i are uniform and can be embedded in each other. Then $\mathcal{O}(M_1) = \mathcal{O}(M_2)$.

Proof: Since M_i can be embedded in M_j, we have that $(M_i)_P$ can be embedded in $(M_j)_P$, for any maximal ideal P of R. Since R is Dedekind, we have that R_P is a rank one discrete valuation ring. Since $\mathcal{O}((M_i)_P)$ is an overring of R_P, it follows that $\mathcal{O}((M_i)_P) = R_P$ or $\mathcal{O}((M_i)_P) = K$. Since $(M_i)_P$ can be embedded in $(M_j)_P$, we have $\mathcal{O}((M_1)_P) = K$ if and only if $\mathcal{O}((M_2)_P) = K$.

Now by the same argument as in (2.27), we can show that $\mathcal{O}(M_i) = \bigcap_P \mathcal{O}((M_i)_P)$, $i = 1,2$. Then $\mathcal{O}(M_1) = \bigcap_P \mathcal{O}((M_1)_P) = \bigcap_{\mathcal{O}((M_1)_P) \neq K} R_P =$

$$= \bigcap_{\mathcal{O}((M_2)_P) \neq K} R_P = \bigcap_P \mathcal{O}((M_2)_P) = \mathcal{O}(M_2). \ \square$$

<u>Proposition 2.45</u>: Let $M = M_1 \oplus M_2$ be an R-module, where the M_i are uniform. Let S be Dedekind. Then the following statements are equivalent:

1) M has (C_1).

2) M_i can be embedded in M_j $(i,j = 1,2)$.

3) There exists a fractional ideal I of S such that $M_2 = IM_1$.

<u>Proof</u>: 1) $\Rightarrow$ 2) This is clear by (2.24).

2) $\Rightarrow$ 3): Without loss of generality assume that $S \subset M_1 \subset M_2$. Since S is Dedekind, we have, by (2.44), $\mathcal{O}(M_1) = \mathcal{O}(M_2) = S$. Since $S \subset M_1 \subset M_2 \subset K$, M_1/S and M_2/S as S-submodule of K/S can be written as $M_1/S = \bigoplus_P T_P^{n_P}$ and $M_2/S = \bigoplus_P T_P^{m_P}$, where P ranges over the maximal ideals of S and $\infty \le n_P \le m_P$. By (2.43), $B = \prod_P P^{m_P - n_P}$. Now let $o \neq b \in B$. We have $bM_2 \subset M_1$, and it follows that $S \subset M_2 \subset b^{-1}M_1 \subset K$. Hence we obtain that $b^{-1}M_1/S = \bigoplus_P T_P^{m_P + \ell_P}$, where $\ell_P \ge o$. It is easy to see that $(M_2 : b^{-1}M_1) = bA$, and hence, by (2.43), $bA = \prod_P P^{m_P + \ell_P - m_P} = \prod_P P^{\ell_P}$.

On the other hand since $b \in B \subset S$; i.e. $bM_1 \subset M_1$, we have $S \subset M_1 \subset b^{-1}M_1 \subset K$. It follows that $\prod_P P^{m_P+\ell_P-n_P} = (M_1 : b^{-1}M_1) = b\,\mathcal{O}(M_1) = bS$. Hence $b^{-1}S = \prod_P P^{n_P-m_P-\ell_P}$. Therefore $A = b^{-1}\prod_P P^{\ell_P} = b^{-1}S \prod_P P^{\ell_P}$

$= \prod_P P^{n_P-m_P-\ell_P} \prod_P P^{\ell_P} = \prod_P P^{n_P-m_P}$. It follows that $AB = S$.

Now $M_2 = ABM_2 \subset AM_1 \subset M_2$, and we obtain that $M_2 = AM_1$.

3) $\Rightarrow$ 1) : Let $M_2 = IM_1$. Since S is Dedekind and I is a fractional ideal of S, we have that I is invertible and its inverse is $(S:I)$. It follows that $\mathcal{O}(M_1) = \mathcal{O}(M_2) = S$ and $A = I$, $B = (S:I)$. Then $\mathcal{O}(A) = \mathcal{O}(I) = \mathcal{O}(S:I) = \mathcal{O}(B)$ is Dedekind. Hence for every maximal ideal P of S, $\mathcal{O}(A_P) = \mathcal{O}(B_P) = S_P$ is a rank one discrete valuation ring with maximal ideal $P_P \subset S_P = A_PB_P$. Therefore, by (2.26), M has (C_1). □

Theorem 2.46: Let $M = M_1 \oplus M_2$ be a module over a Dedekind domain R, where the M_i are uniform torsion free reduced. Then the following statements are equivalent:

1) M has (C_1);

2) M_i can be embedded in M_j $(i,j = 1,2)$;

3) There exists a fractional ideal I of R such that $M_2 = IM_1$.

Proof: 1) $\Rightarrow$ 2): Is trivial.

2) $\Rightarrow$ 3): By (2.44), we have $\mathcal{O}(M_1) = \mathcal{O}(M_2) = S$. Since any overring of

R is a Dedekind domain and is a localization of R at a set of prime ideals of R, we have S is Dedekind and $S = R_*$, a localization of R at a set of prime ideals of R. By (2.45), we have $M_2 = JM_1$ where J is a fractional ideal of S. Hence $J = I_*$ for some fractional ideal I of R. Now $JM_1 = I_*M_1 = IR_*M_1 = ISM_1 = IM_1$. It follows that $M_2 = IM_1$.

3) $\Rightarrow$ 1) : Let $M_2 = IM_1$ where I is a fractional ideal of R. Then $M_2 = ISM_1$ where IS is a fractional ideal of S and hence by (2.45) we have condition 1). □

<u>Corollary 2.47</u>: If R is a principal ideal domain and M_1, M_2 are torsion free reduced R-modules, then $M_1 \oplus M_2$ has (C_1) if and only if M_1 is isomorphic to M_2.

Proof: Since any fractional ideal of R is principal. □

The following proposition is an immediate consequence of (2.41).

<u>Proposition 2.48</u>: Let $M = \bigoplus_{i=1}^{n} M_i$ be an R-module, where the M_i are uniform. Let $S =: \bigcap_{i=1}^{n} \mathcal{O}(M_i)$ be Dedekind. Then M has (C_1) if and only if $M_i \oplus M_j$ has (C_1) for all i,j. □

<u>Theorem 2.49</u>: Let M be a torsion free reduced module over a Dedekind domain R. Then the following statements are equivalent:

1) M has (C_1)

2) $M \cong \bigoplus_{i=1}^{n} I_iN$, where N is a proper R-submodule of K and the I_i

are fractional ideals of R.

Proof: 1) $\Rightarrow$ 2): Let M have (C_1). By (2.16) we have $M = \oplus_{i=1}^{n} M_i$, where the M_i are uniform submodules of M. Since (C_1) is inherited by direct summands, we have that $M_1 \oplus M_i$ has (C_1) for all i. By (2.46) we have $M_i = I_i M_1$ where the I_i are fractional ideal of R with $I_1 = R$. Then $M = \oplus_{i=1}^{n} I_i M_1$.

2) $\Rightarrow$ 1): Let $M \cong \oplus_{i=1}^{n} I_i N$ with $N \subsetneq K$ and the I_i fractional ideals of R. Let $M_i =: I_i N$. It follows that $I_i^{-1} M_i = N = I_j^{-1} M_j$, and hence $M_j = I_j I_i^{-1} M_i$. Then by (2.46), $M_i \oplus M_j$ has (C_1) for all i, j. By (2.48), M has (C_1). □

§7. NOETHERIAN DOMAINS OF KRULL DIMENSION ONE.

In this section we characterize all torsion free reduced modules over one dimensional Noetherian domains which have property (C_1).

Let $A = (M_2 : M_1)$, $B = (M_1 : M_2)$ and $S = \mathcal{O}(M_1) \cap \mathcal{O}(M_2)$. Then we have the following:

Proposition 2.50: Let $M = M_1 \oplus M_2$ be an R-module, where the M_i are uniform. Let S be Noetherian. Then the following statements are equivalent:

1) M has (C_1);

2) $\mathcal{O}(A)$ coincides with $\mathcal{O}(B)$, and is a Dedekind domain $\mathcal{O}$. AB is a product of distinct maximal ideals of $\mathcal{O}$. There is a one-to-one correspondence between the maximal ideals of $\mathcal{O}$ and the maximal

ideals of S, via contraction.

Proof: 1) $\Rightarrow$ 2): Let M have (C_1). By (2.27) $\mathfrak{O}(A) = \mathfrak{O}(B) =: \mathfrak{O}$ is integrally closed. By (2.24), AB is a non-zero ideal of S which is also an ideal of $\mathfrak{O}$. Then AB is contained in the conductor D of S in $\mathfrak{O}$. Since S is Noetherian and $D \neq o$, by (2.7), we have that $\mathfrak{O}$ is finitely generated S-module. It follows that $\mathfrak{O}$ is Noetherian and integral over S, and hence $\mathfrak{O}$ is the integral closure of S.

Since A and B are finitely generated fractional ideals of S, it follows that $\mathfrak{O}(A_P) = \mathfrak{O}(A)_P = \mathfrak{O}(B)_P = \mathfrak{O}(B_P)$, for every maximal ideal P of S. By (2.26) and since $\mathfrak{O}$ is Noetherian, we have that $\mathfrak{O}_P = \mathfrak{O}(B_P) = \mathfrak{O}(A_P)$ is a rank one discrete valuation ring. Since $\mathfrak{O}_P$ is integral over S_P, it follows that S_P is a one.dimensional Noetherian domain for every maximal ideal P of S. Then S is a one dimensional Noetherian domain. Since $\mathfrak{O}$ is the integral closure of S, it follows that $\mathfrak{O}$ is a Dedekind domain.

We show that for each maximal ideal P of S there exists a unique maximal ideal $\mathfrak{P}$ of $\mathfrak{O}$ such that $P = \mathfrak{P} \cap S$. Let P be a maximal ideal of S. Since $\mathfrak{O}_P$ is a rank one discrete valuation ring we have, by (2.5), $\mathfrak{O}_P = \mathfrak{O}_{\mathfrak{P}}$ for some maximal ideal $\mathfrak{P}$ of $\mathfrak{O}$ with $\mathfrak{P} \cap S = P$. Now let $\mathfrak{P}_1$ be any maximal ideal of $\mathfrak{O}$ such that $P = \mathfrak{P}_1 \cap S$. Then $\mathfrak{O}_{\mathfrak{P}} = \mathfrak{O}_P \subset \mathfrak{O}_{\mathfrak{P}_1}$. It follows that $\mathfrak{P}_{1_{\mathfrak{P}_1}} \subset \mathfrak{P}_{\mathfrak{P}}$, hence $\mathfrak{P}_1 \subset \mathfrak{P}$ and hence $\mathfrak{P}_1 = \mathfrak{P}$. Therefore there is a one-to-one correspondence between the maximal ideals of $\mathfrak{O}$ and the maximal ideals of S, via contraction.

We show that AB is a product of distinct maximal ideals of $\mathfrak{O}$. We have $\mathfrak{O}_P = \mathfrak{O}_{\mathfrak{P}}$ for every maximal ideal P [illegible] S, where $\mathfrak{P}$ is the maximal ideal of $\mathfrak{O}$ with $P = S \cap \mathfrak{p}$. Then $(AB)_P = AB\,\mathfrak{O}_P = AB\mathfrak{O}_{\mathfrak{p}} = (AB)_{\mathfrak{p}}$. By (2.27) we have that for every maximal ideal $\mathfrak{p}$ of $\mathfrak{O}$ containing AB, $(AB)_{\mathfrak{p}} = A_P B_P$ is the maximal ideal of $\mathfrak{O}_P = \mathfrak{O}_{\mathfrak{p}}$, i.e. $(AB)_{\mathfrak{p}} = \mathfrak{p}_{\mathfrak{p}}$. On the other hand, since $\mathfrak{O}$ is a Dedekind domain we have $AB = \Pi\,\mathfrak{p}^{n(\mathfrak{p})}$. It follows that $(AB)_{\mathfrak{p}} = \mathfrak{p}_{\mathfrak{p}}^{\,n(\mathfrak{p})}$. By comparison we conclude that $n(\mathfrak{p}) = 1$, hence $AB = \Pi\,\mathfrak{p}$.

2) $\Rightarrow$ 1): Let condition 2) be satisfied. The one-to-one correspondence means that, for every maximal ideal P of S, there exists a unique maximal ideal $\mathfrak{p}$ of $\mathfrak{O}$ such that $\mathfrak{p} \cap S = P$. Since $\mathfrak{O}$ is Dedekind and S is Noetherian, we have that $\mathfrak{O}(A_P) = \mathfrak{O}(B_P) = \mathfrak{O}_P = \mathfrak{O}_{\mathfrak{p}}$ is a rank one discrete valuation ring with maximal ideal $\mathfrak{p}_{\mathfrak{p}} = \mathfrak{p}_P$. Now let $AB = \prod_{i=1}^{n} \mathfrak{p}_i$, where the $\mathfrak{p}_i$ are maximal ideals of $\mathfrak{O}$ with $\mathfrak{p}_i \cap S = P_i$. Then for all i, $(AB)_{P_i} = (\prod_{i=1}^{n} \mathfrak{p}_i)_{P_i} = \prod_{i=1}^{n} (\mathfrak{p}_i)_{P_i} = (\mathfrak{p}_i)_{P_i} = \mathfrak{p}_{i\mathfrak{p}_i}$; and for every $P \neq P_i$ $(i = 1,2,\ldots,n)$, we have $S_P = (AB)_P = (\prod_{i=1}^{n} \mathfrak{p}_i)_P = \prod_{i=1}^{n} (\mathfrak{p}_i)_P = \Pi\,\mathfrak{O}_P \supsetneq \mathfrak{O}_P$. Therefore condition 2) of (2.26) is satisfied. Hence M has (C_1). □

The following theorem is useful, in concrete examples, for constructing a direct sum of two uniform modules with property (C_1).

<u>Theorem 2.51</u>: Let $M = M_1 \oplus M_2$ be an R-module, where the M_i are uniform torsion free reduced. Let $S =: \mathfrak{O}(M_1) \cap \mathfrak{O}(M_2)$ be Noetherian. Then the following statements are equivalent:

1) M has (C_1);

2) The integral closure S' of S is Dedekind and is a (maximal) equivalent order. There is a one-to-one correspondence, via contraction (and extension), between the maximal ideals of S and of S'. The conductor D of S in S' is a product of distinct maximal ideals of S' (or S). A and B are S'-modules and $AB = D$.

Proof: 1) $\Rightarrow$ 2): Let M have (C_1). In the beginning of the proof of (2.43) we have shown that $\mathfrak{O}(A) = \mathfrak{O}(B) =: \mathfrak{O}$ is a Dedekind domain and is the integral closure of S. It follows that $\mathfrak{O}$ is a maximal equivalent order.

Now let D be the conductor of S in $\mathfrak{O}$. Since AB is a non-zero ideal of S' which is also an ideal of $\mathfrak{O}$, we have that $AB \subset D$. We show that $AB = D$. Since A and B are non-zero, it follows that $M_1\mathfrak{O}$ and $M_2\mathfrak{O}$ can be embedded in each other. Since $\mathfrak{O}$ is Dedekind, we have, by (2.46), that $M_1\mathfrak{O} \oplus M_2\mathfrak{O}$ has (C_1) as $\mathfrak{O}$-module, and $M_1\mathfrak{O} = M_2 I$, where I is a fractional ideal of $\mathfrak{O}$. Since $AB \subset D \subset S \subset \mathfrak{O}$, we have $M_i D \subset M_i$ $(i = 1,2)$. It follows that $M_2 DI = M_1 D \subset M_1$, and hence $DI \subset B$. Similarly $M_1 DI^{-1} = M_2 D \subset M_2$, hence $DI^{-1} \subset A$. Then $D^2 \subset AB \subset D$. By (2.43) $AB = \Pi\, \mathfrak{p}_i$, where the $\mathfrak{p}_i$ are maximal ideals of $\mathfrak{O}$. Since $AB \subset D$, we have $D = \Pi \mathfrak{p}_i^{n_i}$; $n_i \leq 1$. But $D^2 = \Pi \mathfrak{p}_i^{2n_i} \subset AB = \Pi \mathfrak{p}_i$, hence $n_i \geq 1$. It follows that $n_i = 1$ for all i, i.e. $AB = D$.

By (2.50), there is a one-to-one correspondence, via contraction, between the maximal ideals of S and of $\mathcal{O}$. Now we show that for any maximal ideal P of S, $P\mathcal{O} = \mathfrak{p}$ where $\mathfrak{p}$ is the unique maximal ideal of $\mathcal{O}$ lying over P, i.e. the inverse of the one-to-one correspondence via contraction is given by extension. If $P \not\supset D$, then $\mathfrak{p} \not\supset D$. Since $\mathcal{O}$ is Dedekind, we have $P\mathcal{O} = \mathfrak{p}^n$, $n \geq 1$. It follows that $\mathfrak{p} D \subset \mathfrak{p} \cap S$ $= P \subset P\mathcal{O} = \mathfrak{p}^n$. Then $n = 1$ (otherwise $D \subset \mathfrak{p}^{n-1} \subset \mathfrak{p}$, which is a contradiction), i.e. $P\mathcal{O} = \mathfrak{p}$. On the other hand if $P \supset D = \Pi \mathfrak{p}_i$, then $P =: P_i = \mathfrak{p}_i \cap S$ for a unique $\mathfrak{p}_i$. Since $\mathcal{O}$ is Dedekind, we have $P_i\mathcal{O} = \mathfrak{p}_i^{n_i}$, $n_i \geq 1$. Since $\Pi \mathfrak{p}_i \subset P_i\mathcal{O} = \mathfrak{p}_i^{n_i}$, it follows that $n_i = 1$, and hence $P\mathcal{O} = P_i\mathcal{O} = \mathfrak{p}_i$.

Now $D = D \cap S = (\Pi \mathfrak{p}_i) \cap S = (\bigcap_i \mathfrak{p}_i) \cap S = \bigcap_i (\mathfrak{p}_i \cap S) = \bigcap_i P_i = \prod_i P_i$ (since the P_i are distinct maximal ideals of S).

2) $\Rightarrow$ 1): Since A and B are S'-modules, we have $S' \subset \mathcal{O}(A), \mathcal{O}(B)$. Since $D = AB$ is a non-zero ideal of $\mathcal{O}(A)$ and of $\mathcal{O}(B)$ which is also an ideal of S', we have that $\mathcal{O}(A)$ and $\mathcal{O}(B)$ are finitely generated S'-modules, and hence integral over S'. Since S' is integrally closed, we have that $\mathcal{O}(A) = \mathcal{O}(B) = S'$ is a Dedekind domain. Hence, by (2.50), M has (C_1). □

The following is an example of two uniform torsion free reduced modules M_1 and M_2 for which $M_1 \oplus M_2$ has (C_1). It shows that if $M_1 \oplus M_2$ has (C_1), then $\mathcal{O}(M_1)$ and $\mathcal{O}(M_2)$ need not be comparable.

Example 2.52: Let $\mathcal{O} = F[t]$ be the polynomial ring in t over a field F. Let k be a proper subfield of F such that F is a finite dimensional vector space over k. Let M_1 and M_2 be the full inverse images of k under the natural homomorphisms $\mathcal{O} \twoheadrightarrow \mathcal{O}/t\mathcal{O} \cong F$, and $\mathcal{O} \twoheadrightarrow \mathcal{O}/(t-1)\mathcal{O} \cong F$ respectively, i.e. $M_1 = k + t\mathcal{O}$ and $M_2 = k + (t-1)\mathcal{O}$. It is clear that $\mathcal{O}_i := \mathcal{O}(M_i) = M_i$, and it is easy to see that $S =: \mathcal{O}_1 \cap \mathcal{O}_2 = k + kt + t(t-1)\mathcal{O}$. Now $\mathcal{O}$ is the integral closure of S, and hence a maximal equivalent order. The conductor D of S in $\mathcal{O}$ is equal to $t(t-1)\mathcal{O}$. Since $S/D \cong k + kt \cong k[t]/t(t-1)k[t]$, it follows that the only maximal ideals of S which contain D are $P_1 := kt + t(t-1)\mathcal{O}$ and $P_2 := k(t-1) + t(t-1)\mathcal{O}$. Hence $t\mathcal{O} \cap S = P_1$ and $(t-1)\mathcal{O} \cap S = P_2$, $P_1\mathcal{O} = t\mathcal{O} + t(t-1)\mathcal{O} = t\mathcal{O}$ and $P_2\mathcal{O} = (t-1)\mathcal{O} + t(t-1)\mathcal{O} = (t-1)\mathcal{O}$. We have shown, in the proof of (2.51), that there is a one-to-one correspondence, via contraction (and extension), between the maximal ideals of $\mathcal{O}$ and of S, which are not containing the conductor D. It is clear that $A = (t-1)\mathcal{O}$ and $B = t\mathcal{O}$ are $\mathcal{O}$-modules and $AB = D$. Therefore condition 2) of (2.51) is satisfied. Hence $M_1 \oplus M_2$ has (C_1). Note that $\mathcal{O}_1$ and $\mathcal{O}_2$ are not comparable. □

The second example shows that if the integral closure $\mathcal{O}$ of S is a Dedekind domain, and is a maximal equivalent order; and there is a one-to-one correspondence, via contraction and extension, between the maximal ideals of $\mathcal{O}$ and of S, then the conductor of S in $\mathcal{O}$ need not be a product of distinct maximal ideals of $\mathcal{O}$ (nor of S).

<u>Example 2.53</u>: Let $\mathcal{O} = F[t]$ be the polynomial ring in t over

a field F. Let k be a proper subfield of F such that F is a finite dimensional vector space over k. Let $S = k + kt + t^2\mathcal{O}$. Then the conductor D of S in $\mathcal{O}$ is $t^2\mathcal{O}$, and hence $\mathcal{O}$ is a (maximal) equivalent order. Since $S/t^2\mathcal{O} \cong k[t]/tk[t]$, we have that $P := kt + t^2\mathcal{O}$ is the only maximal ideal of S containing D. It is easy to show that $P\mathcal{O} = t\mathcal{O}$ and $t\mathcal{O} \cap S = P$. As in the previous example, this suffices to establish a one-to-one correspondence, via contraction and extension, between all the maximal ideals of $\mathcal{O}$ and of S. But $D = t^2\mathcal{O}$ is not a product of distinct maximal ideals of $\mathcal{O}$ (nor of S). □

The next example shows that the statement "A and B are S′-modules" does not follow from the rest of condition 2) of (2.51).

<u>Example 2.54</u>: Let $\mathcal{O} = F[t]$ be the polynomial ring in t over a field F. Let k be a proper subfield of F such that $\dim_k F < \infty$. Let S be the full inverse image of k under the natural homomorphism $: \mathcal{O} \to \mathcal{O}/t\mathcal{O} = F$, i.e. $S = k + t\mathcal{O}$. It is clear that there is a one-to-one correspondence, via contraction (and extension) between the maximal ideals of $\mathcal{O}$ and of S, that the conductor D of S in $\mathcal{O}$ is $t\mathcal{O}$, and that $S' = \mathcal{O}$. Now let V be a proper k-subspace of F such that $\dim_k V \geq 2$, and let $M_1 =: Vt + t^2\mathcal{O}$ and $M_2 = S$.

Then $B = M_1$ and $A = (S : B)_{\mathcal{O}}$ Since $B\mathcal{O} = (Vt + t^2\mathcal{O})\mathcal{O} = t\mathcal{O} \subset S$, we have $\mathcal{O} \subset A$. Now let $a \in A$, i.e. $aB \subset S$; it follows that $at^2\mathcal{O} \subset S$ and hence $at^2 \in D = t\mathcal{O}$. Then $at \in \mathcal{O}$, and hence $at = x + yt$ where $x \in F$ and $y \in \mathcal{O}$. On the other hand $atV = (x + yt)V \subset S$; it follows that $xV \subset k$. Since $\dim_k V \geq 2$, it follows that $x = o$ and $at = yt \in t\mathcal{O}$, i.e. $a \in \mathcal{O}$. Hence $A = \mathcal{O}$ and $AB = t\mathcal{O} = D$. Therefore condition 2) of (2.51) is satisfied except that B is not an $\mathcal{O}$-module, and hence $M_1 \oplus M_2$ does not have (C_1). □

The last example shows that $AB = D$ does not follow from the rest of condition 2) of (2.51).

Example 2.55: Let $\mathcal{O}$ and S be as in example (2.54). Let L_i be subfields of F such that $k \subset L_i \subset F$ and the L_i are not comparable, $i = 1,2$. Let M_i be the full inverse image of L_i under the natural homomorphism: $\mathcal{O} \twoheadrightarrow \mathcal{O}/t\mathcal{O} \cong F$, i.e. $M_i = L_i + t\mathcal{O}$. Since the L_i are not comparable, we have $A = B = t\mathcal{O}$. It is clear that A and B are $\mathcal{O}$-modules, and that $AB = t^2\mathcal{O} \subsetneq t\mathcal{O} = D$. Therefore condition 2) of (2.51) holds except that $AB = D$. □

Remark 2.56: We have shown in (2.50) that if $M_1 \oplus M_2$ has (C_1) where S is Noetherian, then S is a one dimensional domain. If S is not Noetherian, then S need not be one dimensional. For example, let R be a Prüfer domain, which is not one dimensional, and let P be a maximal ideal of R. By (2.39), we have that $P \oplus R$ has (C_1), while $S = R$ is not one dimensional. □

We observe now that, in the special case where the domain R is Noetherian and the M_i are finitely generated R-modules, (C_1) for $M_1 \oplus M_2$

forces R(and S) to have Krull dimension one.

Proposition 2.57: Let R ba a Noetherian domain. Let M_1, M_2 be finitely generated uniform R-modules. If $M_1 \oplus M_2$ has (C_1), then R has Krull dimension one.

Proof: Define $g : R^n \longrightarrow M_1$ by $g(r_i)_{i=1}^n = \sum_{i=1}^n r_i x_i$, where $x_1, x_2, \ldots, x_n$ are the generators of M_1. This induces an embedding $R \hookrightarrow S \hookrightarrow \mathcal{O}(M_1) \cong \hom_R (M_1, M_1) \rightarrowtail \hom_R (R^n, M_1) \cong M_1^n$. Since M_1 is finitely generated and R is Noetherian, we have that M_1 is Noetherian, and hence M_1^n is Noetherian. It follows that S is a Noetherian R-module. Thus S is a Noetherian ring and integral over R, and hence Krull dim(S) = Krull dim(R).

Now if $M_1 \oplus M_2$ has (C_1), then, by (2.50), S is one dimensional, and therefore so is R. □

Remark 2.58: Note that if $M_1 \oplus M_2$ has (C_1) where the M_i are infinitely generated uniform modules over a Noetherian domain R, then R need not be one dimensional. For example, let R be Noetherian but not one dimensional. There exists a valuation overring V of R (cf, Theorem 56, [11]). Hence by (2.21), $V \oplus V$ has (C_1) as an R-module.

Proposition 2.59: Let $M = \bigoplus_{i=1}^n M_i$ be an R-module, where the M_i are uniform. Let $S =: \bigcap_{i=1}^n \mathcal{O}(M_i)$ be Noetherian. Then M has (C_1) if and only if $M_i \oplus M_j$ has (C_1) for all $i \neq j$.

Proof: Let $M_i \oplus M_j$ have (C_1) for all $i \neq j$. Denote $\mathcal{O}(M_i)$ by $\mathcal{O}_i$ and $\mathcal{O}(M_i) \cap \mathcal{O}(M_j)$ by S_{ij}. It is easy to see that $I = A_{12}A_{23}\ldots A_{n1}$ is a non-zero ideal of S, which is also an ideal of S_{ij}. Hence the conductor D_{ij} of S in S_{ij} contains I, for all $i \neq j$. Since S is Noetherian and $D_{ij} \neq o$, it follows that the S_{ij} are finitely generated S-modules. Then the S_{ij} are Noetherian and integral over S, and hence Krull $\dim(S)$ = Krull $\dim(S_{ij})$.

Since $M_i \oplus M_j$ has (C_1) for all $i \neq j$, we have, by (2.50), that $\mathcal{O}(A_{ij}) = \mathcal{O}(A_{ji})$ is Dedekind, and is the integral closure of S_{ij}; there is a one-to-one correspondence, via contraction, between the maximal ideals of $\mathcal{O}(A_{ij})$ and S_{ij}. Since the S_{ij} are integral over S for all i,j, we have that $\mathcal{O}(A_{ij})$ is the integral closure of S, and hence the $\mathcal{O}(A_{ij})$ all coincide; we denote this ring by $\mathcal{O}$.

Now $\mathcal{O}_i/I \subset \mathcal{O}/I$ are o-dimensional and have the same number m of maximal ideals (since there is a one-to-one correspondence between the maximal ideals, containing I, of $\mathcal{O}$ and of $\mathcal{O}_i$). By (2.13) $\mathcal{O}_i/I$ and $\mathcal{O}/I$ are artinian. Thus there exists a complete set of orthogonal idempotents $e_1, e_2, \ldots, e_m \in \mathcal{O}/I$. From the one-to-one correspondence, it follows that $e_j \in \mathcal{O}_i/I$ for all i, and hence $e_j \in \bigcap_{i=1} \mathcal{O}_i/I = S/I$; $j = 1,2,\ldots,m$. Then the number of maximal ideals of S, which contain I, is m. Therefore there is a one-to-one correspondence, via contraction, between the maximal ideals, containing I, of $\mathfrak{O}$ and of S.

To show that M has (C_1), let $q_1, q_2, \ldots, q_n \in K$ (not all zero) be arbitrary. Let $F = \{i : q_i \neq 0\}$. Since $M_i \oplus M_j$ has (C_1) for all $i \neq j$, it follows that, by (2.23), $q_i q_j^{-1} A_{ij} \cap \mathfrak{O} + q_j q_i^{-1} A_{ji} \cap \mathfrak{O} = \mathfrak{O}$ for all $i \neq j \in F$. Then for every maximal ideal P of S,

$q_i q_j^{-1}(A_{ij})_P \cap \mathfrak{O}_P + q_j q_i^{-1}(A_{ji})_P \cap \mathfrak{O}_P = \mathfrak{O}_P$.

Now if $P \supset I$, then there exists a unique maximal ideal $\mathfrak{P}$ of $\mathfrak{O}$ such that $P = \mathfrak{P} \cap S$. It follows that $\mathfrak{O}_P$ is local hence a rank one discrete valuation ring (since $\mathfrak{O}$ is Dedekind). On the other hand if $P \not\supset I$, then $I_P = S_P$ and hence $S_P = \mathfrak{O}_P$ is a rank one discrete valuation ring. Then $S_P \subset q_i q_j^{-1}(A_{ij})_P$ or $S_P \subset q_j q_i^{-1}(A_{ji})_P$ holds for each $i \neq j$, and $\{q_i q_j^{-1}(A_{ij})_P\}_{j \in F}$ forms a chain of submodules of K for each $i \in F$. By the same argument as in (2.41), we can show that $S \subset \sum_{i \in F} \bigcap_{j \in F} q_i q_j^{-1} A_{ij}$. Therefore M has (C_1).

The converse is obvious. □

<u>Theorem 2.60</u>: Let $M = \bigoplus_{i=1}^{n} M_i$ be an R-module, where the M_i are uniform torsion free reduced. Let S be Noetherian. Then the following statements are equivalent:

1) M has (C_1);

2) The integral closure S' of S is a Dedekind domain, and is a (maximal) equivalent order. There is a one-to-one correspondence,

via contraction (and extension), between the maximal ideals of S' and of S. The conductor D of S in S' is a product of distinct maximal ideals of S' (or S). For all $i \neq j$, the A_{ij} are S'-modules, and $A_{ij}A_{ji} \supset D$.

Proof: 1) $\Rightarrow$ 2): Let M have (C_1). Since (C_1) is inherited by direct summands, we have $M_i \oplus M_j$ has (C_1) for all $i \neq j$. From the proof of (2.59), we have that all $\mathcal{O}(A_{ij})$ coincide, and that this ring is a Dedekind domain $\mathcal{O}$; there is a one-to-one correspondence, via contraction, between the maximal ideals of $\mathcal{O}$ and of S. Let D be the conductor of S in $\mathcal{O}$, and D_{ij} be the conductor of S_{ij} in $\mathcal{O}$. Since $M_i \oplus M_j$ has (C_1), we have, by (2.50), $D_{ij} = A_{ij}A_{ji}$; and hence $D \subset D_{ij} = A_{ij}A_{ji}$.

We show that $(S_{ij}:D)D = D_{ij}$. Since $(S_{ij}:D)D$ is an ideal of S_{ij}, which is also an ideal of $\mathcal{O}$, we have $(S_{ij}:D)D \subset D_{ij}$. Now let $x \in D_{ij}$ be arbitrary. For any $y \in (\mathcal{O}:D)$, we have $yD \subset \mathcal{O}$, and hence $xyD \subset x\mathcal{O} \subset S_{ij}$. It follows that $xy \in (S_{ij}:D)$, hence $x(\mathcal{O}:D) \subset (S_{ij}:D)$. Then $x \in x\mathcal{O} = x(\mathcal{O}:D)D \subset (S_{ij}:D)D$, i.e. $D_{ij} \subset (S_{ij}:D)D$. Therefore $D_{ij} = (S_{ij}:D)D$.

From (C_1) for $M_i \oplus M_j$, we have that $(S_{ij}:D)D = D_{ij}$ is a product of distinct maximal ideal of $\mathcal{O}$; and there is a one-to-one correspondence between the maximal ideal of $\mathcal{O}$ and of S_{ij}. It is easy to show that $\mathcal{O}(D) = \mathcal{O}(S_{ij}:D) = \mathcal{O}$. Therefore, by (2.50), $D \oplus S_{ij}$ has (C_1).

We show that $D \oplus S$ has (C_1). Since S is Noetherian, by (2.26), it is enough to show that $D_P \oplus S_P$ has (C_1), for every maximal ideal P of S.

Case 1: $P \supset D$. Since $D \oplus S_{ij}$ has (C_1) for all $i \neq j$, we have that $D_P \oplus (S_{ij})_P$ has (C_1). Since $\mathcal{O}(D_P) = \mathcal{O}(D)_P = \mathcal{O}_P$ is local, hence a rank one discrete valuation ring, we have, by (2.21), that D_P and $q(S_{ij})_P$ are comparable for every $o \neq q \in K$. If $q(S_{ij})_P \subset D_P$ for some i,j, then $q S_q \subset q(S_{ij}) \subset D_P$. On the other hand, if $D_P \subset q(S_{ij})_P$ for all $i \neq j$, then $D_P \subset \bigcap_{\substack{i=1 \\ (i \neq j)}}^{n} qS_{ij} = qS$. It follows that D_P, qS_P are comparable for every $o \neq q \in K$, and hence $D_P \oplus S_P$ has (C_1).

Case 2: $P \not\supset D$. Then $D_P = S_P = \mathcal{O}_P$ is a rank one valuation ring, and hence, by (2.21), $D_P \oplus S_P$ has (C_1).

These two cases together show that $D_P \oplus S_P$ has (C_1) for every maximal ideal P of S, and hence $D \oplus S$ has (C_1). By (2.51), the inverse of the one-to-one correspondence, (via contraction) is given by extension, between the maximal ideals of $\mathcal{O}$ and of S; and D is a product of distinct maximal ideals of $\mathcal{O}$ (or of S).

2) $\Rightarrow$ 1): Since A_{ij} are S'-modules, we have $S' \subset \mathcal{O}(A_{ij})$. Since $A_{ij}A_{ji}$ is a non-zero ideal of $\mathcal{O}(A_{ij})$ which is also an ideal of S', we have that the conductor of S' in $\mathcal{O}(A_{ij})$ is non-zero. Then $\mathcal{O}(A_{ij})$ is a finitely generated S'-module, and hence integral over S'. Since S' is integrally closed, we get $\mathcal{O}(A_{ij}) = S'$ for all $i \neq j$.

To show that M has (C_1), it is enough, by (2.60), to show that $M_i \oplus M_j$ has (C_1) for all $i \neq j$. Since $A_{ij}A_{ji} \supset D$, and D is a product of distinct maximal ideals of S', it follows that $A_{ij}A_{ji}$ is a product of distinct maximal ideals of S'. It is clear that there is a one-to-one correspondence, via contraction, between the maximal ideals of S' and of S_{ij}. Therefore condition 2) of (2.50) is satisfied for the direct sum $M_i \oplus M_j$, for all $i \neq j$. Hence $M_i \oplus M_j$ has (C_1) for all $i \neq j$. □

<u>Corollary 2.61</u>: Let $M = \bigoplus_{i=1}^{n} M_i$ be an R-module, where the M_i are uniform torsion free reduced. Let S be Noetherian. The following statements are equivalent:

1) M has (C_1);

2) S is the full inverse image of $\bigoplus_{i=1}^{n} k_i \subset \bigoplus_{i=1}^{n} F_i$ under the natural homomorphism $: \mathfrak{O} \to \mathfrak{O}/\prod_{i=1}^{n} \mathfrak{P}_i \cong \bigoplus_{i=1}^{n} F_i$, where the k_i are subfields of $F_i = \mathfrak{O}/\mathfrak{P}_i$, and $\mathfrak{O}$ is Dedekind; and $\mathfrak{P}_i$ are maximal ideals of $\mathfrak{O}$. For all $i \neq j$, the A_{ij} are $\mathfrak{O}$-modules and $A_{ij}A_{ji} \supset \prod_{i=1}^{n} \mathfrak{P}_i$.

<u>Proof</u>: 1) $\Rightarrow$ 2): Let M have (C_1). By (2.60), we have that the integral closure S' of S is a Dedekind domain, and that there is a one-to-one correspondence, via contraction, between the maximal ideals of S' and of S. We also have that the conductor D of S in S' is a product of distinct maximal ideals of S'.

Now let $D = \prod_{i=1}^{n} \mathfrak{P}_i = \bigcap_{i=1}^{n} \mathfrak{P}_i$, and let $\mathfrak{P}_i \cap S = P_i$; it follows that $D = \bigcap_{i=1}^{n} P_i$, and hence

$$\bigoplus_{i=1}^{n} k_i =: \bigoplus_{i=1}^{n} S/P_i \cong S/\bigcap_{i=1}^{n} P_i = S/\bigcap_{i=1}^{n} \mathfrak{P}_i \cap S \subset S'/\bigcap_{i=1}^{n} \mathfrak{P}_i \cong \bigoplus_{i=1}^{n} S'/\mathfrak{P}_i =: \bigoplus_{i=1}^{n} F_i,$$

where $k_i = S/P_i = S/\mathfrak{P}_i \cap S \cong (S + \mathfrak{P}_i)/\mathfrak{P}_i \subset S'/\mathfrak{P}_i = F_i$.

We have also, by (2.60), that the A_{ij} are S'-modules and $A_{ij}A_{ji} \supset \prod_{i=1}^{n} \mathfrak{P}_i$ for all $i \neq j$.

2) $\Rightarrow$ 1). Let condition 2) be given. Let $I =: \prod_{i=1}^{n} \mathfrak{P}_i$. Since I is a non-zero ideal of S which is also an ideal $\mathfrak{O}$, it follows that the conductor D of S' in $\mathfrak{O}$ is non-zero. Since S is Noetherian and $D \neq o$, we have that $\mathfrak{O}$ is integral over S, and hence Krull dim(S) = 1. Since $A_{ij}A_{ji} \supset I$, we have that $A_{ij}A_{ji}$ is a product of distinct maximal ideals of $\mathfrak{O}$.

Since $S/I \cong \bigoplus_{i=1}^{n} k_i \subset \bigoplus_{i=1}^{n} F_i = \mathfrak{O}/_I$, it follows that $S/I \subset \mathfrak{O}/I$ are o-dimensional and have the same number of maximal ideals. Hence there is a one-to-one correspondence, via contraction between the maximal ideals of $\mathfrak{O}$ and of S containing I. It is clear that $S_p = \mathfrak{O}_p$ is a rank one discrete valuation ring for any maximal ideal P not containing I. Therefore there is a one-to-one correspondence, via contraction, between all maximal ideals of $\mathfrak{O}$ and of S. Since the A_{ij} are $\mathfrak{O}$-modules, it follows that $\mathfrak{O}(A_{ij}) = \mathfrak{O}$ for all $i \neq j$. Since $S_{ij} =: \mathfrak{O}(M_i) \cap \mathfrak{O}(M_j) \supset S$, we have that condition 2) of (2.50) is satisfied for all $i \neq j$, and hence $M_i \oplus M_j$ has (C_1) for all $i \neq j$. Therefore M has (C_1), by (2.59). □

In the special case where $M_1 = R$, and R is Noetherian; (C_1)

implies that M_2 is isomorphic to an ideal of R, in this situation we have the following:

Corollary (2.62): Let R be a Noetherian domain and I be an ideal of R. Then the following statements are equivalent:

1) $R \oplus I$ has (C_1);
2) R is the full inverse image of $\bigoplus_{i=1}^{n} k_i \subset \bigoplus_{i=1}^{n} F_i$ under the natural homomorphism : $\mathcal{O} \to \mathcal{O}/\prod_{i=1}^{n} \mathfrak{p}_i \cong \bigoplus_{i=1}^{n} F_i$, where the k_i are subfields of $F_i =: \mathcal{O}/\mathfrak{p}_i$, and $\mathcal{O}$ is a Dedekind domain; and $\mathfrak{p}_i$ are maximal ideal of $\mathcal{O}$. I is an $\mathcal{O}$- module.

Proof: 1) $\Rightarrow$ 2): Is clear.

2) $\Rightarrow$ 1): Since I is an $\mathcal{O}$-module, it follows that $(R:I)$ is an $\mathcal{O}$-module. By the same argument as in (2.60) we can show that $I(R:I)$ is the conductor of R in $\mathcal{O}$ hence contains $\prod_{i=1}^{n} \mathfrak{p}_i$. Therefore, by (2.61), $R \oplus I$ has (C_1). □

Since overrings of a one dimensional Noetherian domain are Noetherian, and by (2.16), (2.60) we have the following:

Corollary 2.63: Let M be a torsion free reduced module over a Noetherian domain of Krull dimension one. Then the following statements are equivalent:

1) M has (C_1);
2) $M = \bigoplus_{i=1}^{n} M_i$, where the M_i are uniform. The integral closure S' of

$S =: \bigcap_{i=1}^{n} \mathcal{O}(M_i)$ is a (maximal) equivalent order. There is a one-to-one correspondence, via contraction (and extension), between the maximal ideals of S' and of S. The conductor D of S in S' is a product of distinct maximal ideals of S' (or S). For all $i \neq j$, the A_{ij} are S'-modules, and $A_{ij}A_{ji} \supset D$. □

§8. OPEN QUESTIONS

We list some open questions, related to this chapter, for which we could find neither proofs nor counter examples.

1) Let $M = M_1 \oplus M_2$ be an R-module, where the M_i are uniform torsion free reduced. If M has (C_1), is $\mathcal{O}(A)$ $(= \mathcal{O}(B))$ a Prüfer domain; and is it the integral closure of S?

2) Let $M = \bigoplus_{i=1}^{n} M_i$ be a module over an arbitrary commutative integral domain, where the M_i are uniform torsion free reduced. Does M have (C_1) if $M_i \oplus M_j$ has (C_1) for all $i \neq j$?

3) Are there uniform torsion free reduced modules M_1 and M_2 such that $M_1 \oplus M_2$ has (C_1) for which $S =: \mathcal{O}(M_1) \cap \mathcal{O}(M_2)$ is (one-dimensional) Noetherian but not Dedekind, and such that M_i are infinitely generated as S-module?

CHAPTER III

TORSION MODULES WITH PROPERTY (C_1)

In this chapter we study finite direct sums of uniform torsion modules with local endomorphism rings over Noetherian domains. First we give a necessary and sufficient condition for the direct sum of a pair of uniform torsion modules with local endomorphism rings to have (C_1). Then we prove that a finite direct sum of uniform torsion modules with local endomorphism rings has (C_1) if and only if the direct sum of each pair has (C_1). Finally, we characterize arbitrary torsion modules over Dedekind domains which have (C_1).

In this chapter R is always a commutative Noetherian domain, and all modules over R are torsion.

§1. PRELIMINARIES.

We recall the definition of relative injectivity given in (1.3): A module M is said to be N-injective, if for every sub-module L of N, every homomorphism $f : L \to M$ can be extended to $\hat{f} : N \to M$.

The next useful characterization of relative injectivity was given by Azumaya [2], and generalizes the result by Johnson and Wang [10] for quasi-injective modules.

Lemma 3.1.: A module M is N-injective if and only if, for every homomorphism $f : N \to E(M)$, $f(N) \subset M$ holds. □

Lemma 3.2: Let $\varphi : E(M_1) \to E(M_2)$ be an arbitrary homomorphism, and let $X = \{x \in M_1 : \varphi(x) \in M_2\}$. Let $\psi : M_1 \to M_2$ be a homomorphism such that $\psi(x) = \varphi(x)$ for all $x \in X$. Then $\psi(M_1) = \varphi(M_1)$, i.e. $X = M_1$.

Proof: If $(\varphi - \psi)(M_1) \neq o$, then by essentiality of M_2 in $E(M_2)$, we have $(\varphi - \psi)(M_1) \cap M_2 \neq o$. Hence there exists $o \neq m_2 \in M_2$ such that $m_2 = (\varphi - \psi)(m_1)$ for some $m_1 \in M_1$. It follows that $\varphi(m_1) = m_2 + \psi(m_1) \in M_2$, and hence $m_1 \in X$. Therefore $(\varphi - \psi)(m_1) = \varphi(m_1) - \psi(m_1) = o$, which is a contradiction. Thus $\varphi(M_1) = \psi(M_1)$, and $X = M_1$. □

L. Jeremy [9] defined quasi-continuous modules, as follows:

Definition 3.3: A module M is called quasi-continuous if it satisfies the following:

(C_1) : Every closed submodule of M is a direct summand.

(C_3) : If M_1 and M_2 are direct summands of M such that $M_1 \cap M_2 = o$, then $M_1 \oplus M_2$ is a direct summand of M.

It is clear that any uniform module is quasi-continuous.

Jeremy [9], also observed that a module M is quasi-continuous if and only if M is invariant under all idempotents of the endomorphism ring of its injective hull.

Müller and Rizvi [14] proved the following;

Lemma 3.4: If $M = \bigoplus_{i \in I} A_i$, with $\bigoplus_{i \in I} E(A_i)$ injective, then M is quasi-continuous if and only if the A_i are quasi-continuous and A_j-injective for all $j \neq i$. □

The extra assumption in (3.4), that $\bigoplus_{i \in I} E(A_i)$ is injective, is automatically satisfied if the index set I is finite, or if the ring R is Noetherian.

Lemma 3.5: ([8],P.22). For a module $M = \bigoplus_{i \in I} M_i$, with all $\text{end}(M_i)$ local, the following statements are equivalent:

1) the decompositon complements direct summands;
2) any local direct summand of M is a direct summand;
3) the decomposition is locally semi-T-nilpotent. □

These conditions are automatically satisfied if the index set I is finite.

It is a well known fact (due to Matlis [13]) that, if R is a commutative Noetherian ring, then the indecomposable injective modules E correspond to the prime ideals P of R in a one-to-one fashion, such that $E \cong E(R/P)$.

Lemma 3.6: ([13], Th.3.7). Let R be a commutative Noetherian ring, and $E \cong E(R/P)$ be an indecomposable injective R-module. Then E is an R_P- as well as an $\hat{R}_P$-module, where $\hat{R}_P$ is the completion of R_P.

Furthermore, if $H = \operatorname{Hom}_R(E,E)$, then H is isomorphic to $\hat{R}_P$; more precisely, every R-homorphism of E into itself can be realized by multiplication by a unique element of $\hat{R}_P$.

Definition 3.7: Let M be a non-zero R-module, and P be a prime ideal of R. We say that M has an associated prime P if and only if $P = \operatorname{ann}(N)$ for some non-zero submodule N of M such that $P = \operatorname{ann}(T)$ for all non-zero submodules T of N (the submodule N of M may then equally well be chosen to be cyclic). If R is Noetherian and M is a uniform R-module, then M has a unique associated prime P; in this case we shall denote it by $\operatorname{ass}(M) = P$.

Lemma 3.8: Let $M = \bigoplus_{i=1}^{n} M_i$ be an R-module with all M_i uniform. Then a submodule A of M is uniform and closed in M if and only if $A = \{\sum_{i=1}^{n} \varphi_i(x) : x \in B \subset' M_k$ for some M_k and some submodule B of $M_k\}$, where $\varphi_i : B \to M_i$ are homomorphisms such that $\varphi_k(b) = b$ for all $b \in B$; and φ_i are not simultaneously extendable (i.e. if $\psi_i : B_i \to M_i$ extends φ_i, $B \subset B_i \subset M_k$, for each i, then $B = \bigcap_{i=1}^{n} B_i$.)

Proof: Let A be a closed and uniform submodule of M. Let π_i be the projection of M onto M_i, $i = 1,2,\ldots,n$. Since A is uniform, we have $A \cap \ker \pi_k = o$ for some k. It follows that $\pi_{k|A} : A \to M_k$ is a monomorphism. For each i, $\pi_i(\pi_{k|A})^{-1} : \pi_k(A) \to M_i$ is a homomorphism, we denote it by φ_i. Observe that $\varphi_k(x) = x$ for all $x \in \pi_k(A)$. Now let $a \in A$ be arbitrary, as an element of M, $a = \sum_{i=1}^{n} \pi_i(a) = \sum_{i=1}^{n} \pi_i \pi_k^{-1} \pi_k(a)$

$= \sum_{i=1}^{n} \varphi_i(\pi_k a)$, where $\pi_k(a) \in \pi_k(A) \subset' M_k$. Hence $A \subset \{ \sum_{i=1}^{n} \varphi_i(x) : x \in \pi_k(A) \subset' M_k \}$.

On the other hand if $b = \sum_{i=1}^{n} \varphi_i(x)$, where $x \in \pi_k(A)$, then there exists a unique $a' \in A$ such that $x = \pi_k(a')$, and hence $b = \sum_{i=1}^{n} \pi_i \pi_k^{-1}(\pi_k(a'))$ $= \sum_{i=1}^{n} \pi_i(a') = a' \in A$. Therefore $A = \{ \sum_{i=1}^{n} \varphi_i(x) : x \in B \subset' M_k \}$, where $B := \pi_k(A)$.

Now let $\psi_i : B_i \to M_i$ extend φ_i, for each i, where $B \subset' B_i \subset M_k$. It follows that $A \subset' \{ \sum_{i=1}^{n} \psi_i(y) : y \in \bigcap_{i=1}^{n} B_i \}$. Since A is closed, we obtain that $A = \{ \sum_{i=1}^{n} \psi_i(y) : y \in \bigcap_{i=1}^{n} B_i \}$, and hence $B = \bigcap_{i=1}^{n} B_i$.

Conversely, let $X = \{ \sum_{i=1}^{n} f_i(b) : b \in B \subset' M_k \}$, where the $f_i : B \to M_i$ are not simultaneously extendable homomorphisms, and $f_k(b) = b$ for all $b \in B$. It is clear that $X \cong B$, via $b \to \sum_{i=1}^{n} f_i(b)$, and hence X is a uniform submodule of M.

We show that X is closed. Let $X \subset' X^* \subset M$. It is easy to see that $X \cap \ker \pi_k = o$, where $\pi_k : M \to M_k$ is the projection of M onto M_k. Since X^* is essential over X, we have $X^* \cap \ker \pi_k = o$, and hence $X^* \xrightarrow{\pi_k | X^*} M_k$ is a monomorphism. Since $B = \pi_k X \subsetneq \pi_k X^*$ and

$\pi_i\pi_k^{-1}(b) = \pi_i(\sum_{i=1}^{n} f_i(b))$, it follows that

$\pi_i\pi_k^{-1}: \pi_k X^* \to M_k$ extends f_i. Thus $B = \pi_k X^*$, and hence $X = X^*$.

Therefore X has no proper essential extension in M. □

Lemma 3.9: Let $M = \sum_{i=1}^{n} M_i$ be an R-module, where the M_i are uniform. Then a submodule A of M is closed of dimension $n-1$ if and only if $A = \{b + \varphi(b) : b \in B \subset' \bigoplus_{i \neq k}^{n} M_i\}$, where $\varphi: B \to M_k$ is a non-extendable homomorphism, for some k.

Proof: Let A be a closed submodule of M of dimension $n-1$. Then $A \cap M_k = o$ for some k, and hence $f =: \sum_{i \neq k} \pi_i : A \to \bigoplus_{i \neq k} M_i$ is a monomorphism, where π_i is the projection of M onto M_i. Since $f(A) \cong A$, we have $f(A) \subset' \bigoplus_{i \neq k} M_i$. By the same argument as in (3.8) we can show that $A = \{b + \varphi(b) : b \in f(A) \subset' \bigoplus_{i \neq k} M_i\}$, where $\varphi := \pi_k f^{-1}$. Now φ is not extendable (otherwise $A \subsetneq' \{x + \psi(x) : x \in X \subset' \bigoplus_{i \neq k} M_i\}$, where $\psi: X \to M_k$ is the proper extension of φ, which contradicts the assumption that A is closed).

By the same arugment as in (3.9) we can show that any submodule of M, of the form $\{(b + \psi(b) : b \in B \subset' \bigoplus_{i \neq k} M_i\}$ where $\psi : B \to M_k$ is a non-extendable homomorphism, is closed and of dimension $n-1$. □

Lemma 3.10: Let $M = X \oplus Y$ be an R-module with $\hom_R(X,Y) = o$. Then Y has a unique complementary direct summand, namely X.

Proof: Let $M = Y \oplus Z$, and let $M \xrightarrow{\pi} Y$ be the projection of M onto Y. Since $\hom_R(X,Y) = o$, we obtain that $\pi|X = o$, i.e. $\pi(x) = o$ for all $x \in X$, and hence $X \subset \ker \pi = Z$. Therefore $X = Z$. □

§2. GOLDIE DIMENSION TWO

In this section we characterize torsion modules of Goldie dimension two over Noetherian domains, which have property (C_1). First we show that (C_1), for direct sums of two uniform torsion modules with distinct associated primes is equivalent to quasi-continuity. Then we give a necessary and sufficient condition for a direct sum of two uniform torsion modules with the same associated prime, and with local endomorphism rings, to have (C_1).

Recall that R will be a commutative Noetherian domain, and any R-module will be torsion.

Proposition 3.11: Let $M = M_1 \oplus M_2$ be an R-module, where the M_i are uniform. Let the associated prime of M_1 be different from the associated prime of M_2. Then M has (C_1) if and only if M_i is M_j-injective, $i \neq j$.

Proof: Let M have (C_1). Since M_i are uniform, we have that $E(M_i)$ are indecomposable injective. Since $\operatorname{ass}(M_1) \neq \operatorname{ass}(M_2)$, it follows that $E(M_1) \cong E(R/P_1)$ and $E(M_2) \cong E(R/P_2)$, where P_1 and P_2 are non-zero distinct prime ideals of R. Without loss of generality assume that $M_1 \subset E(R/P_1)$ and $M_2 \subset E(R/P_2)$; and that $P_2 \not\subset P_1$.

Let $\varphi \in \hom_R(E(M_2), E(M_1))$, and e be an arbitrary element of $E(M_2)$. Then for some $n > o$, $eP_2^n = o$, and hence $\varphi(e)P_2^n = o$. It follows that $\varphi(e) = o$ (otherwise $P_2^n \subset \operatorname{ann}(\varphi(e)) \subset \operatorname{ass}(E(M_1)) = P_1$, i.e. $P_2 \subset P_1$ which contradicts our assumption that $P_2 \not\subset P_1$). Therefore $\hom_R(E(M_2), E(M_1)) = o$, hence, by (3.1), M_1 is M_2-injective. Similarly if $P_1 \not\subset P_2$, then M_2 is M_1-injective.

Now let $P_1 \subset P_2$, and let $\psi : X \to M_2$ be an arbitrary homomorphism from a submodule X of M_1 into M_2. Let $X' =: \{x - \psi(x) : x \in X\}$, and let X^* be a maximal essential extension of X' in M. Since X^* is closed in M, by (C_1), we have $M = X^* \oplus Y$. Since $X \cong X' \subset' X^*$, we get $E(X^*) \cong E(R/P_1)$. Since $\operatorname{end}(E(M_i))$ are local, and $E(Y) \overset{\oplus}{\subset} E(R/P_1) \oplus E(R/P_2)$, it follows that $E(Y) \oplus E(R/P_1) = E(M)$. Since $\hom_R(E(R/P_2), E(R/P_1)) = o$, by (3.10), $E(Y) = E(R/P_2)$. Since Y is closed in M, we have $Y = E(R/P_2) \cap M = M_2$, and hence $M = X^* \oplus M_2$.

Now let $\pi : X^* \oplus M_2 \to M_2$ be the projection onto M_2. Then $o = \pi(x - \psi(x)) = \pi(x) - \pi\psi(x) = \pi(x) - \psi(x)$, and hence $\pi(x) = \psi(x)$ for all $x \in X$. Therefore $\pi|M_1$ extends ψ. Hence M_2 is M_1-injective.

Conversely, let M_i be M_j-injective, $i \neq j$. Then, by (3.4), M is quasi-continuous. Hence M has (C_1). □

The following is an example of two uniform torsion modules M_1 and M_2 with $\mathrm{ass}(M_1) \subsetneqq \mathrm{ass}(M_2)$, for which $M_1 \oplus M_2$ has (C_1).

Example 3.12: Let R be a Noetherian domain, let $P_1 \subsetneqq P_2$ be prime ideals of R. Let $M_1 =: R/P_1$ and $M_2 = HM_1$, where $H =: \hom_R(E(R/P_1), E(R/P_2))$. It is easy to see that $HM_1 = \mathrm{ann}_{E(R/P_2)}(P_1)$. It is clear that M_i is M_j-injective, $i \neq j$, and hence $M_1 \oplus M_2$ has (C_1).

Note that M_2 is R/P_1-injective but need not be R-injective. For example let $R = k[x,y]$ be the polynomial ring in x,y over a field k, and $P_1 = \langle x \rangle \subsetneqq \langle x,y \rangle = P_2$. Then $E((R/P_2)_{R/P_1}) = E(k_{k[y]}) = k[y^{-1}]$ $\subsetneqq k[x^{-1}, y^{-1}] = E(k_{k[x,y]}) = E((R/P_2)_R)$.

Corollary 3.13: Let $M = \bigoplus_{i \in I} M_i$ be an R-module, where the M_i are uniform. Let the associated primes of all M_i be distinct. Then M has (C_1) if and only if M is quasi-continuous.

Proof: Let M have (C_1). Since (C_1) is inherited by direct summands, we have that $M_i \oplus M_j$ has (C_1) for all $i \neq j \in I$. By (3.11), M_i is M_j-injective for all $i \neq j \in I$. Since R is Noetherian and the M_i are uniform, we have, by (3.4), that M is quasi-continuous.

The converse is obvious. □

<u>Lemma 3.14</u>: Let $M = M_1 \oplus M_2$ be an R-module, where the M_i are uniform with local endomorphism rings. If M has (C_1) and M_i can not be embedded in M_j for some $i \neq j$, then M_i is M_j-injective.

<u>Proof</u>: Let M have (C_1), and let M_1 not be embeddable in M_2. Let $\varphi : X \to M_1$ be an arbitrary homomorphism from a submodule X of M_2 into M_1. Let $X' := \{ x - \varphi(x) : x \in X \}$, and X^* be a maximal essential extension of X' in M. It is easy to see that $X' \cap M_1 = o$, and hence, by essentiality of X^* over X', $X^* \cap M_1 = o$. It follows that $X^* \xrightarrow{\pi_2|X^*} M_2$ is a monomorphism, where π_2 is the projection of M onto M_2.

By (C_1), we have $X^* \subset^{\oplus} M$. Since the end(M_i) are local, we have $M = X^* \oplus M_1$ or $M = X^* \oplus M_2$. If $M = X^* \oplus M_2$, then $M_1 \cong X^* \xrightarrow{\pi_2|X^*} M_2$ is an embedding, which contradicts the assumption that M_1 can not be embedded in M_2. Hence $M = X^* \oplus M_1$.

Now let $\pi : X^* \oplus M_1 \to M_1$ be the projection onto M_1. Since $x - \varphi(x) \in X' \subset X^*$, we have $o = \pi(x - \varphi(x)) = \pi x - \pi\varphi(x) = \pi x - \varphi(x)$ for all $x \in X$; i.e. $x = \varphi(x)$ for all $x \in X$. Therefore $\pi|_{M_2}$ extends φ, and hence M_1 is M_2-injective. □

<u>Lemma 3.15</u>: Let $M = M_1 \oplus M_2$ be an R-module, where the M_i are uniform with local endomorphism rings. Let the M_i have the same associated prime. If M has (C_1), then M_i can be embedded in M_j for some $i \neq j$.

Proof: Let M have (C_1), and let M_1 not be embeddable in M_2. By (3.13), we have that M_1 is M_2-injective; and hence, by (3.1), $\varphi(M_2) \subset M_1$ for every $\varphi \in \hom_R(E(M_2), E(M_1))$. Since the M_i are uniform with the same associated prime P we have $E(M_1) \cong E(R/P) \cong E(M_2)$. Now let $\psi : E(M_2) \to E(M_1)$ be an isomorphism, it follows that $\psi|_{M_2} : M_2 \to M_1$ is an embedding; i.e. M_2 can be embedded in M_1. □

On the basis of lemma (3.14) we can assume, without loss of generality, that $M_1 \subset M_2$, in this situation we have two cases, either M_2 can be embedded in M_1 or M_2 can not be embedded in M_1. The second case will be studied in (3.16), and the first case will be studied in (3.17).

Proposition 3.16: Let $M = M_1 \oplus M_2$ be an R-module, where the M_i are uniform with local endomorphism rings. Let $M_1 \subset M_2 \subset E(R/P)$, where P is the associated prime of M_1. Let M_2 not be embeddable in M_1. Then M has (C_1) if and only if $P R_P M_2 \subset M_1 \subset R_P M_1 \subset M_2$ holds.

Proof: Let M have (C_1). Since M_2 can not be embedded in M_1, it follows, by (3.13), that M_2 is M_1-injective. By (3.6), we have that $\mathrm{end}(E(R/P)) \cong \hat{R}_P$, where $\hat{R}_P$ is the completion of R_P. We identify $\mathrm{end}(E(R/P))$ by $\hat{R}_P$. Since M_2 is M_1-injective, we have that $R_P M_1 = \hat{R}_P M_1 \subset M_2$ holds.

Now let $\varphi \in P\hat{R}_P$; i.e. φ is not unit in $\hat{R}_P$. Let $A = : \{m_2 \in M_2 : \varphi(m_2) \in M_1\}$. Since $A \subset' M_2 \subset' E(R/P)$, it follows that $\varphi|_A : A \to M_1$ is not a monomorphism. Now if $\psi : B \to M_1$ is a homomorphism such that $A \subset B \subset M_2$ and $\psi(a) = a$ for all $a \in A$, then, by (3.2), $A = B$; i.e. $\varphi|_A$ is a non-extendable homomorphism. By (3.8), $A^* =: \{a + \varphi(a) : a \in A\}$ is a closed and uniform submodule of M. Since the end(M_i) are local and $A^* \cap M_2 \neq o$, by (C_1), we have $M = A^* \oplus M_1$. Let $\pi : A^* \oplus M_1 \to M_1$ be the projection onto M_1. It follows that $o = \pi(a + \varphi(a)) = \pi a + \pi\varphi(a) = \pi a + \varphi(a)$, i.e. $-\pi a = \varphi(a)$ for all $a \in A$. Thus $-\pi|_{M_2}$ extends $\varphi|A$, and hence $A = M_2$. Therefore $\varphi M_2 \subset M_1$ for all $\varphi \in P\hat{R}_p$. Then $PR_pM_2 = P\hat{R}_pM_2 \subset M_1$, and it follows that $PR_pM_2 \subset M_1 \subset R_pM_1 \subset M_2$ holds.

Conversely, let A be a uniform closed submodule of M. By (3.8), we have $A = \{x + \varphi(x) : x \in X \subset' M_i\}$, where $\varphi : X \to M_j$ is a non-extendable homomorphism for some $i \neq j$.

Case 1: $A = \{x + \varphi(x) : x \in X \subset' M_1\}$, where $\varphi : X \to M_2$ is non-extendable. Since $\hat{R}_pM_1 = R_pM_1 \subset M_2$, we have $X = M_1$, and hence $A \oplus M_2 = M$.

Case 2: $A = \{x + \varphi(x) : x \in X \subset' M_2\}$, where $\varphi : X \to M_1$ is a non-extendable homomorphism. If φ is not a monomorphism, then there exists $\hat{\varphi} \in P\hat{R}_p$ such that $\hat{\varphi}|X = \varphi$. Since $P\hat{R}_pM_2 = PR_pM_2 \subset M_1$, it follows that $X = M_2$, and hence $M = A \oplus M_1$. On the other hand if φ is a monomorphism, then $\hat{\varphi} \in \hat{R}_p \setminus P\hat{R}_p$, i.e. $\hat{\varphi}$ is a unit in $\hat{R}_p$. Since $\hat{R}_pM_1 \subset M_2$ and $\hat{\varphi}^{-1} \in \hat{R}_p$, we have $\hat{\varphi}^{-1}(M_1) \subset M_2$. Since $X = \hat{\varphi}^{-1}(M_1) \cap M_2 = \hat{\varphi}^{-1}(M_1)$, we have $\varphi(X) = \hat{\varphi}(X) = M_1$, and hence $M = A \oplus M_2$ (for all $m_1 \in M_1$, $m_1 = \varphi(x) + x - x$, where $x \in X$, i.e. $m_1 \in A \oplus M_2$).

These two cases together show that any uniform closed submodule of M is a direct summand. Therefore M has (C_1). □

<u>Proposition 3.17</u>: Let $M = M_1 \oplus M_2$ be an R-module, where the M_i are uniform with local endomorphism rings. Let $M_1 \subset M_2 \subset E(R/P)$, where P is the associated prime of M_i, and let M_2 be embeddable in M_1. Then the following statements are equivalent:

1) M has (C_1);

2) there exists a valuation overring $\mathcal{O}$ of R/P, and an $\mathcal{O}$-submodule X of the quotient field K of R/P such that $XV_1 \subset V_2 \subset \bigcap_{q \in K \setminus X} qV_1 \subset V$,

where $V_i = M_i/PN$, $V = N/PN$, and $N = R_PM_1 = R_PM_2$.

<u>Proof</u>: 1) ⇒ 2): Let M have (C_1). Since $M_1 \subset M_2$ can be embedded in M_1, it follows that $R_PM_1 = R_PM_2$, we denote this R_P-module by N.

By the same argument as in the proof of (3.15), we can show that $PR_PM_i \subset M_j (i \neq j = 1,2)$. Hence $PN \subset M_i$. Let $V_i =: M_i/PN$ and $V = N/PN$.

Now let $x \in \hat{R}_P/P\hat{R}_P$; i.e. x is a unit in $\hat{R}_P$. Let $B =: x^{-1}M_2 \cap M_1$. If $B = M_1$, then $xM_1 \subset M_2$. Let $B \subsetneqq M_1$ and let $B^* =: \{ b + xb : b \in B \}$. Since $x : B \to M_2$ is a non-extendable homomorphism, and B^* is isomorphic to B, we have, by (3.8), that B^* is a uniform closed submodule of M. By (C_1) and since the $\text{end}(M_i)$ are local, we have $M = B^* \oplus M_1$ (if $M = B^* \oplus M_2$, then for all $m_1 \in M_1$, $m_1 = x + xb + m_2$, and hence $m_1 \in B$.

Therefore $B = M_1$ which contradicts the assumption that $B \subsetneq M_1$). It follows that for every $m_2 \in M_2$ there exists $b \in B$ such that $m_2 = xb$, and hence $M_2 \subset xB \subset xM_1$. Therefore $xM_1 \subset M_2$ or $M_2 \subset xM_1$ for all $x \in \hat{R}_P \setminus P\hat{R}_P$.

It is easy to see that $K \cong R_P/PR_P \cong \hat{R}_P/P\hat{R}_P$, and V_i are torsion free R/P-modules.

Let $X = :\{x \in K : xV_1 \subset V_2\}$, $\mathcal{O}(X) = \{y \in K : yX \subset X\}$, and $\mathcal{O}(V_i) = \{z \in K : zV_i \subset V_i\}$, and $S = \mathcal{O}(V_1) \cap \mathcal{O}(V_2)$. Since xM_1 and M_2 are comparable for all $x \in K$, it follows that xV_1 and V_2 are comparable for all $x \in K$.

We show that X is comparable with all S-submodules of K. Let $y \in K$ such that $y \notin X$. Since yV_1 and V_2 are comparable, and $y \notin X$, we have $V_2 \subset yV_1$; i.e. $y^{-1}V_2 \subset V_1$. It follows that $y^{-1}XV_1 \subset y^{-1}V_2 \subset V_1$ and $y^{-1}XV_2 \subset XV_1 \subset V_2$, and hence $y^{-1}X \subset \mathcal{O}(V_1) \cap \mathcal{O}(V_2) = S$. Therefore $X \subset yS$.

We show that $\mathcal{O}(X)$ is a valuation ring. Let $q \in K$ and $q \notin \mathcal{O}(X)$. Then $qX \not\subset X$, and hence $qx \notin X$ for some $x \in X$. Since qxS and X are comparable, we have $X \subset qxS$, i.e. $q^{-1}X \subset xS \subset X$, and hence $q^{-1} \in \mathcal{O}(X)$. Therefore $\mathcal{O}(X)$ is a valuation ring.

Since $V = N/PN$ is R_P/PR_P-module, and qV_1 and V_2 are comparable for all $q \in K$, we have that $XV_1 \subset V_2 \subset \bigcap_{q \in K \setminus X} qV_1 \subset V$ holds.

2) $\Rightarrow$ 1) Let $\mathcal{O}$ be a valuation overring of the domain R/P, and X be an $\mathcal{O}$-submodule of the quotient field K of R/P such that

$XV_1 \subset V_2 \subset \bigcap_{q \in K \setminus X} qV_1$ holds, where $V_i = M_i/PN$ and $N = R_PM_i$. Let A be a uniform closed submodule of M. Without loss of generality assume that

$A = \{y + \varphi(y) \in Y \subset' M_1\}$, where $\varphi : Y \to M_2$ is a non-extendable homomorphism. If $Y = M_1$, then $A \oplus M_2 = M$. Now let $Y \subsetneq M_1$ and let, $\hat{\varphi} \in \hat{R}_P$ such that $\hat{\varphi}|_X = \varphi$ ($\hat{\varphi}$ is not unique). Since $P\hat{R}_P M_1 = PN \subset M_2$, it follows that $\hat{\varphi} \in \hat{R}_P \setminus P\hat{R}_P$ (otherwise $Y = M_1$ which is a contradiction). Then $o \neq \bar{\hat{\varphi}} \in \hat{R}_P/P\hat{R}_P \cong K$, and hence $\bar{\hat{\varphi}}V_1$ and V_2 are comparable. But if $\bar{\hat{\varphi}}V_1 \subset V_2$, then $\hat{\varphi}M_1 \subset M_2$, and hence $Y = M_1$, which contradicts the assumption that $Y \subsetneq M_1$. Therefore $V_2 \subset \bar{\hat{\varphi}}V_1$; it follows that $M_2 \subset \hat{\varphi}M_1$, i.e. $\hat{\varphi}^{-1}M_2 \subset M_1$. Thus $Y = \hat{\varphi}^{-1}(M_2) \cap M_1 = \hat{\varphi}^{-1}(M_2)$, i.e. $\hat{\varphi}(Y) = \varphi(Y) = M_2$. Therefore $A \oplus M_1 = M$ (for all $m_2 \in M_2$ there exists $y \in Y$ such that $m_2 = \varphi(y) + y - y \in A \oplus M_1$ and since φ is an isomorphism, we have $A \cap M_1 = o$). Then M has (C_1). □

We observe now that, in the special case where N is a local R_P-module (for example if R_P is a rank one discrete valuation ring), $M_1 \oplus M_2$ has (C_1) if and only if $V_1 \oplus V_2$ has (C_1).

<u>Corollary 3.18</u>: Let $M = M_1 \oplus M_2$ be an R-module, where the M_i are uniform with local endomorphism rings. Let M_i be embeddable in M_j, and let $N := R_P M_1 = R_P M_2$ be a local R_P-module, where $P = \mathrm{ass}(M_1) = \mathrm{ass}(M_2)$. Then the following statements are equivalent:

1) M has (C_1);

2) $V_1 \oplus V_2$ has (C_1) as R/P-module where $V_i = M_i/PN$.

3) $\mathcal{O}(A)$ coincides with $\mathcal{O}(B)$, and is a valuation ring with maximal ideal $\mathfrak{M} \subset AB$. If $A \cong \mathfrak{M} \cong B$, then $\mathcal{O}(A) (= \mathcal{O}(B))$ is discrete; where $A = \hom_{R/P}(V_1, V_2)$, $B = \hom_{R/P}(V_2, V_1)$, $\mathcal{O}(A) = \hom_{R/P}(A, A)$, and $\mathcal{O}(B) = \hom_{R/P}(B, B)$.

Proof: 1) $\Rightarrow$ 2). Let M have (C_1). Since N is a local R_P-module, it follows that its unique maximal R_P-submodule is PN, and hence $N/PN =: V$ is a simple R_P-module, i.e. V is a one-dimensional R_P/PR_P-space. Since $V_i \subset V$ as R/P-modules with $R_P V_i = V$, and since $\hat{R}_P/P\hat{R}_P \cong R_P/PR_P = K$ is the quotient field of R/P, it follows that the V_i are uniform torsion free R/P-modules. By (3.16), we have that qV_1 and V_2 are comparable for all $0 \neq q \in K$. By (2.20), we have that $V_1 \oplus V_2$ has (C_1).

2) $\Rightarrow$ 1) Let $V_1 \oplus V_2$ have (C_1). It is easy to see that $\Phi(V_1) = \mathrm{end}(M_1)/P\hat{R}_P$ is local. Since V_i are uniform torsion free R/P-modules, we have, by (2.21), that qV_1 and V_2 are comparable for all $0 \neq q \in K$. By the same argument as in (3.16) we can show that $M = M_1 \oplus M_2$ has (C_1).

2) $\leftrightarrow$ 3): Is clear, by (2.32).

Corollary 3.19: Let $M = M_1 \oplus M_2$ be an R-module, where the M_i are uniform with local endomorphism rings. Let M_1 be isomorphic to M_2. Then M has (C_1) if and only if $\varphi(V_1) = \varphi(V_2)$ is a valuation ring, where $V_i = M_i/PN$ and $N = R_P M_1$; and $P = \mathrm{ass}(M_1)$.

Proof: Let M have (C_1). Without loss of generality assume that $M_1 = M_2$. Since $X = \Phi(V_i)$, we have $\Phi(X) = \Phi(V_i)$, and hence, by (3.17), $\varphi(V_i)$ is a valuation ring.

Conversely, let $\varphi(V_i)$ be a valuation ring, and let $X = \varphi(V_i)$. It follows that $XV_1 = V_1 = V_2 \subset \bigcap_{q \in K \setminus X} qV_1$ (K is the quotient of R/P), and hence, by (3.17), M has (C_1). □

The following example shows that if $M = M_1 \oplus M_2$ is an R-module, given as in (3.17), then (C_1) for M does not imply that $V_1 \oplus V_2$ has (C_1) as R/P-module.

Example (3.20): Let R be a Noetherian domain, and let P be a prime ideal of R such that R_P is not a rank one discrete valuation ring. Let N be an R_P-submodule of $E(R/P)$ such that N/PN is a two dimensional vector space over $R_P/PR_P =: K$, i.e. $N/PN \cong K \oplus K$. Let $\mathcal{O}$ be a valuation overring of R/P with maximal ideal $\mathfrak{M}$, and let $F =: \mathcal{O}/\mathfrak{M}$

Now let A be an indecomposable two dimensional L-submodule of $F \oplus F$ such that $\mathrm{end}(A) \cap F = L$, where $L \subset F$ is a local domain. Let V_1 be the full inverse image of A under the homomorphism $:\mathcal{O} \oplus \mathcal{O} \twoheadrightarrow \mathcal{O}/\mathfrak{M} \oplus \mathcal{O}/\mathfrak{M} = F \oplus F$.

Let M_1 and M_2 be the full inverse images of V_1 and $V_2 := \mathcal{O} \oplus \mathcal{O}$, (respectively) under the natural homomorphism : $N \twoheadrightarrow N/PN$.

Let $X := \mathcal{O}$. It is easy to see that $\mathcal{O} V_1 = \mathcal{O} \oplus \mathcal{O}$, and hence $XV_1 = V_2$. If $q \in K \setminus X$, then $q^{-1} \in \mathfrak{M}$. It follows that $q^{-1}(\mathcal{O} \oplus \mathcal{O}) \subset \mathfrak{M} \oplus \mathfrak{M} \subset V_1$, and hence $V_2 \subset qV_1$. Therefore $V_1 \subset XV_1 = V_2 \subset \bigcap_{q \in K\setminus X} q V_1$.

We show that $\mathrm{end}(M_1)$ is local. Let x_1, x_2 be non-unit elements in $\mathrm{end}(M_1)$, and let $\hat{x}_i \in \hat{R}_P = \mathrm{end}(E(R/P))$ be an extension of $x_i (i = 1,2)$. If $\hat{x}_i$ is a unit in $\hat{R}_P$, then $\hat{x}_i M_1 \subsetneq M_1$. On the other hand if $\hat{x}_i \in P\hat{R}_P$, then $\hat{x}_i M_1 \subset P\hat{R}_P M_1 = PN \subsetneq M_1$. Then $\overline{\hat{x}}_i V_1 \subsetneq V_1$, where $\overline{\hat{x}}_i \in R_P/P\hat{R}_P = K$, (otherwise $\overline{\hat{x}}_i V_1 = V_1$, i.e. $\hat{x}_i M_1 + PN = M_1$. Since $\hat{x}_i M_1$ and PN are comparable, it follows that $\hat{x}_1M_1 = M_1$ or $PN = M_1$, which is a contradiction). It follows that $\bar{\hat{x}}_1 \mathcal{O} V_1 \subset \mathcal{O} V_1$, i.e. $\bar{\hat{x}}_1(\mathcal{O} \oplus \mathcal{O}) \subset \mathcal{O} \oplus \mathcal{O}$, and hence $\bar{\hat{x}}_1 \in \mathcal{O}$. Since $\bar{\hat{x}}_1 V_1$ and $\mathfrak{M} \oplus \mathfrak{M}$ are comparable, and since $\bar{\hat{x}}_1 V_1 \subsetneq V_1$, it follows that $\bar{\bar{\hat{x}}}_1 A \subsetneq A$, where $\bar{\bar{\hat{x}}}_1 \in \mathcal{O}/\mathfrak{M} = F$. Therefore $\bar{\bar{\hat{x}}}_1 \in \mathrm{end}(A) \cap F = L$, and hence $\bar{\bar{\hat{x}}}_1$ are not unit in L. Since L is local, we have that $\bar{\bar{\hat{x}}}_1 + \bar{\bar{\hat{x}}}_2$ is not unit in L, and hence $(\bar{\bar{\hat{x}}}_1 + \bar{\bar{\hat{x}}}_2)\, A \subsetneq A$. We choose $\hat{x}_1 + \hat{x}_2$ to be an extension of $x_1 + x_2$ in $\mathrm{end}(E(M_1)) = \hat{R}_p$; i.e. $x_1 \hat{+} x_2 = \hat{x}_1 + \hat{x}_2$. Since $\overline{\overline{x_1 \hat{+} x_2}}\, A \subsetneq A$, it follows that $\overline{x_1 \hat{+} x_2}\, V_1 \subsetneq V_1$, and hence $(x_1 + x_2)M_1 = (x_1 \hat{+} x_2)M_1 \subsetneq M_1$. Therefore $x_1 + x_2$ is not unit in $\mathrm{end}(M_1)$. Then $\mathrm{end}(M_1)$ is local.

We show that $\mathrm{end}(M_2)$ is local. Let y_1, y_2 be non-unit elements in $\mathrm{end}(M_2)$. By the same argument as above we can show that $\bar{\hat{y}}_i V_2 \subsetneq V_2$, where $\hat{y}_i \in \hat{R}_p$ is an extension of $y_i (i=1,2)$, and $\bar{\hat{y}}_1 \in R_p/PR_p = K$. Hence $\bar{\hat{y}}_1(\mathcal{O} \oplus \mathcal{O}) \subsetneq \mathcal{O} \oplus \mathcal{O}$. It follows that $\bar{\hat{y}}_1 \in \mathfrak{M}$, and hence $\bar{\hat{y}}_1 + \bar{\hat{y}}_2 \in \mathfrak{M}$. Therefore $(\overline{y_1 \hat{+} y_2})\, V_2 \subsetneq V_2$, where $y_1 \hat{+} y_2$ is an extension of $y_1 + y_2$. It follows that $(y_1+y_2)M_2 = (y_1 \hat{+} y_2)M_2 \subsetneq M_2$, i.e. $y_1 + y_2$ is a non-unit in $\mathrm{end}(M_2)$. Hence $\mathrm{end}(M_2)$ is local.

By (3.17) $M_1 \oplus M_2$ has (C_1). Since V_1 is indecomposable, we have that $V_1 \oplus V_2$ does not have (C_1) as R/P-module, by (2.16).

Now we give two concrete examples where all the above made assumptions are satisfied.

Example 1) : Let $R = k[[x,y,z,u,v]]$ be the power series ring in x,y,z,u,v over a field k, and let $P = \langle u,v \rangle$. It follows that $R/P \cong k[[x,y,z]]$, and its quotient field $K = R/PR_P = k((x,y,z))$. Hence $E(R/P) = E(R_P/P_P) = E(K_R) = k((x,y,z))\ [u^{-1},v^{-1}]$. Let $N := \operatorname{ann}_E(P)$, hence $PN = \operatorname{ann}_E(P)$. It follows that N/PN is a two dimensional K-space, i.e. $N/PN \cong K \oplus K$. Let $\mathcal{O} = k((x,y))[[z]]$ be the valuation overring of R/P with the maximal ideal $\mathfrak{m} := \langle z \rangle$, and let $F := k((x,y)) \cong \mathcal{O}/\mathfrak{m}$. Let $L = k[[x,y]]$; hence L is a local unique factorization domain (see [20]).

Now let $A = \langle (\frac{1}{p_1^n}, o),\ (o, \frac{1}{p_2^n}),\ (\frac{1}{q}, \frac{1}{q}) : n \in \mathbb{N} \rangle$ be an L-submodule of $F \oplus F$, where P_1, P_2 and q are distinct prime elements of L. Let $E_1 =: \bigcup_{n=o}^{\infty} (\frac{1}{p_1^n}, o)L$, and $E_2 =: \bigcup_{n=o}^{\infty} (o, \frac{1}{p_2^n})L$. Hence $A = E_1 \oplus E_2 + (\frac{1}{q}, \frac{1}{q})L$.

Claim 1: A is an indecomposable L-module.

First we show that E_i is the largest P_i-divisible submodule of A, $i = 1,2$.

Let $x = (\frac{1}{p_1^n} + \frac{c}{q}, \frac{1}{p_2^n} + \frac{c}{q})$ be an element of A such that

$\frac{1}{p_1^N} x \in A$ for all $N > o$. It follows that $\left(\frac{a}{p_1^n} + \frac{c}{q} , \frac{b}{p_2^n} + \frac{c}{q} \right)$

$= p_1^N \left(\frac{\alpha}{p_1^v} + \frac{\gamma}{q} , \frac{\beta}{p_2^v} + \frac{\gamma}{q} \right)$, where α, β and $\gamma \in L$. Hence

$$a \, p_1^{v-n} q + c \, p_1^v = \alpha \, p_1^N q + \gamma \, p_1^{N+v} \qquad (1)$$

$$b \, p_2^{v-n} q + c \, p_2^v = \beta \, p_1^N q + \alpha \, p_1^N p_2^v \qquad (2)$$

Since L is a unique factorization domain, from (1), it follows that $p_1^v \mid (\alpha p_1^N - a \, p_1^{v-n})$; i.e. $\alpha \, p_1^N - a \, p_1^{v-n} = \alpha_1 \, p_1^v$, where $\alpha_1 \in L$. The $c = \alpha_1 q + \gamma p_1^N \in \bigcap_N \langle q \rangle + \langle p_1^N \rangle$. Since $\langle q \rangle$ is closed ideal in the p_1-adic topology on L, it follows that $\bigcap_N \langle q \rangle + \langle p_1^N \rangle = \langle q \rangle$, and hence $c \in \langle q \rangle$; i.e. $c = q c'$, where $c' \in L$. It follows that $x = \left(\frac{a}{p_1^n} + c' , \frac{b}{p_2^n} + c' \right) = (e_1, e_2)$, where $e_1 = \frac{a}{p_1^n} + c'$ and $e_2 = \frac{b}{p_2^n} + c'$. From (2), it follows that

$e_2 = \frac{b}{p_2^n} + c' = p_1^N \left(\frac{\beta + \gamma p_2^v}{p_2^v} \right) \in p_1^N \, T$ for all N, where

$T = \bigcup_{t=1}^{\infty} \frac{1}{p_2^t} L$. It is clear that T is Noetherian, hence $e_2 \in \bigcap_N p_1^N T = o$.

Therefore $x = (e_1, o) \in E_1$.

Similarly we can show that E_2 is the largest p_2-divisible submodule of A.

To show that A is indecomposable, suppose $A = B \oplus C$. Since E_i is the largest p_i-divisible submodule of A, it follows that $E_i = (E_i \cap B) \oplus (E_i \cap C)$. Assume that $E_1 \subset B$ and $E_2 \subset C$. Let $(\frac{1}{q}, \frac{1}{q}) = b + c$, where $b \in B$, $c \in C$. Hence $(1,o) + (o,1) = q\,b + q\,c$. Since $(1,o) \in E_1 \subset B$ and $(o,1) \in E_2 \subset C$, it follows that $(1,o) = q\,b$; i.e. $(1,o)$ is divisible by q. Then $(1,o) = q\left(\frac{\ell_1}{p_1^n} + \frac{\ell_2}{q}, \frac{\ell_3}{p_2^n} + \frac{\ell_2}{q}\right)$, where $\ell_i \in L$.

Hence
$$p_1^n = q\ell_1 + \ell_2 p_1^n \qquad (1)$$
$$o = q\ell_3 + \ell_2 p_2^n \qquad (2)$$

From (1), we get $\ell_1 = p_1^n \ell_1'$, where $\ell_1' \in L$, and hence $1 = q\ell_1' + \ell_2$. From (2), we get $\ell_2 = q\ell_2'$, where $\ell_2' \in L$. Therefore $1 = q\ell_1' + q\ell_2' \in qL$, which is a contradiction. Then $E_1, E_2 \subset B$ or $E_1, E_2 \subset C$, and hence $B = o$ or $C = o$.

<u>Claim 2</u>: $\operatorname{end}(A) \cap F = L$

Let $\alpha \in F$ such that $\alpha A \subset A$. It follows that $(\frac{\alpha}{q}, \frac{\alpha}{q}) \in A$, and hence $(\frac{\alpha}{q}, \frac{\alpha}{q}) = \left(\frac{a}{p_1^n} + \frac{b}{q}, \frac{c}{p_2^n} + \frac{b}{q}\right)$ for some $a, b, c \in L$. Then $\frac{\alpha}{q} = \frac{a}{p_1^n} + \frac{b}{q} = \frac{c}{p_2^n} + \frac{b}{q}$, and thus $\frac{a}{p_1^n} = \frac{c}{p_2^n}$. Since L is a unique factorization domain, it follows that $a = a'p_1^n$, and $c = c'p_2^n$, where $a',c' \in L$. Therefore $\frac{\alpha}{q} = a' + \frac{b}{q}$; i.e. $\alpha = a'q + b \in L$. Then $end(A) \cap F \subset L$. It is obvious that $L \subset end(A) \cap F$, and therefore $end(A) \cap F = L$.

Let M_1 and M_2 be R-modules constructed as before. Then, by the same argument as before, we can show that $M_1 \oplus M_2$ has (C_1) with $end(M_i)$ local $(i=1,2)$, where $V_1 \oplus V_2$ does not have (C_1). □

<u>Example 2)</u>: $R = \mathbb{Z}[[x,y,z]]$ be the power series ring in x,y,z over the ring of integers $\mathbb{Z}$, and let $P = \langle x,y \rangle$. It follows that $R/P \cong \mathbb{Z}[[z]]$, and its quotient field $K := Q((z))$. Hence $E := E(R/P) = E(K_R) = Q((z))[x^{-1}, y^{-1}]$.

Let $N =: \underset{E}{\mathrm{ann}}(P^2)$, it follows that $PN = \underset{E}{\mathrm{ann}}(P)$, and hence N/PN is a two dimensional K-space; i.e. $N/PN \cong K \oplus K$. Let $\mathcal{O} := Q[[z]]$ be a valuation overring of R/P with maximal ideal $\mathfrak{M} = \langle z \rangle$, it follows that $\mathcal{O}/\mathfrak{M} \cong Q$ (the field of rational numbers).

Now let q be a prime element of $\mathbb{Z}$. Since $\mathbb{Z}_q$ is incomplete discrete valuation ring, Kaplansky, in "Infinite abelian group" theorem 19, constructed a rank two indecomposable $\mathbb{Z}_q$-submodule A of $Q \oplus Q$. Since $\mathbb{Z}_q \subset end(A) \cap Q \subsetneq Q$, it follows that $end(A) \cap Q = \mathbb{Z}_q$ is local ring.

Let M_1 and M_2 be R-modules constructed as before, it follows that $M_1 \oplus M_2$ has (C_1) with $\mathrm{end}(M_i)$ local $(i = 1,2)$, where $V_1 \oplus V_2$ does not have (C_1). □

§3. FINITE DIRECT SUMS OF UNIFORM TORSION MODULES

In this section we study finite direct sums of uniform torsion modules with local endomorphism rings over Noetherian domains. We prove that a finite direct sum of uniform torsion modules with local endomorphism rings has (C_1) if and only if the direct sum of each pair has (C_1).

Recall from Chapter II that a module is said to have $(1 - C_1)$, if every uniform closed submodule is a direct summand.

Proposition 3.20: Let $M = \bigoplus_{i=1}^{n} M_i$ be an R-module, where the M_i are uniform with local endomorphism rings. Then M has (C_1) if and only if M has $(1 - C_1)$.

Proof: Obviously (C_1) implies $(1 - C_1)$. We show the converse by induction on n.

The case $n = 2$ is trivial. Now assume that $\bigoplus_{i \in F} M_i$ has (C_1) for any proper subset F of $\{1,2,\ldots,n\}$. Let A be a closed submodule of $M = \bigoplus_{i=1}^{n} M_i$ of dimension $n-1$. By (3.9), we have

$A = \{b + \varphi(b) : b \in B \subset' \bigoplus_{i \neq k} M_i\}$, where $\varphi : B \to M_k$ is a non-extendable homormorphism, for some k.

Without loss of generality assume that $A = \{b + \varphi(b) : B \subset' \bigoplus_{i=1}^{n-1} M_i\}$, where $\varphi : B \to M_n$, is a non-extendable homomorphism. Let $\hat{\varphi} : \oplus_{i=1}^{n-1} E(M_i) \to E(M_n)$ be an extension of φ to the injective hulls, i.e. $\hat{\varphi}| B = \varphi$. Since φ is a non-extendable homomorphism, we have $B = \{x \in \oplus_{i=1}^{n-1} M_i : \hat{\varphi}(x) \in M_n\}$.

Either $B = \oplus_{i=1}^{n-1} M_i$; then $A \oplus M_n = M$ or $B \subsetneqq \oplus_{i=1}^{n-1} M_i$; then for each $i \neq n$. Let $B_i =: \{m_i \in M_i : \hat{\varphi}_i(m_i) \in M_n\}$, where $\hat{\varphi}_i = \hat{\varphi}\pi_i$ and $\pi_i : \oplus_{i=1}^{n-1} E(M_i) \to E(M_i)$ is the projection onto $E(M_i)$. It is clear that $\oplus_{i=1}^{n-1} B_i \subset B$. Since $B \subsetneqq \oplus_{i=1}^{n-1} M_i$, we have $B_j \subsetneqq M_j$ for some j. Since (C_1) is inherited by direct summands, we have that $M_j \oplus M_n$ has (C_1). It follows, by (3.8), that $B_j^* =: \{b_j + \hat{\varphi}_j(b_j) : b_j \in B_j\}$ is a uniform closed submodule of $M_j \oplus M_n$. Since the end(M_j) are local, we have, by (C_1) for $M_j \oplus M_n$, that $M_j \oplus M_n = B_j^* \oplus M_j$ or $M_j \oplus M_n = B_j^* \oplus M_n$. Now if $M_j \oplus M_n = B_j^* \oplus M_n$, then $-\pi|_{M_j} : M_j \to M_n$ extends $\hat{\varphi}_j$, where $\pi : B_j^* \oplus M_n \to M_n$ is the projection onto M_n. By (3.2), it follows that $\hat{\varphi}_j(M_j) = -\pi(M_j)$, and hence $B_j^* = M_j$ which contradicts the assumption that $B_j \subsetneqq M_j$. Therefore $M_j \oplus M_n = B_j^* \oplus M_j$, and hence $M = B_j^* \oplus \oplus_{i=1}^{n-1} M_i$. Since $B_j^* \subset A$, by the modular law, we have $A = B_j^* \oplus A \cap (\oplus_{i=1}^{n-1} M_i)$. Since $A \cap \oplus_{i=1}^{n-1} M_i$ is a direct summand of the closed submodule A of M, it follows that $A \cap \overset{n-1}{\underset{i=1}{\oplus}} M_i$ is a closed submodule of $\overset{n-1}{\underset{i=1}{\oplus}} M_i$. By induction $\overset{n-1}{\underset{i=1}{\oplus}} M_i$ has (C_1), and hence $A \cap \overset{n-1}{\underset{i=1}{\oplus}} M_i \overset{\oplus}{\subset} \overset{n-1}{\underset{i=1}{\oplus}} M_i$, i.e. $\overset{n-1}{\underset{i=1}{\oplus}} M_i = (A \cap \overset{n-1}{\underset{i=1}{\oplus}} M_i) \oplus Y$. Then $M = B_j^* \oplus \overset{n-1}{\underset{i=1}{\oplus}} M_i = B_j^* \oplus (A \cap \overset{n-1}{\underset{i=1}{\oplus}} M_i) \oplus Y = A \oplus Y$, i.e. $A \overset{\oplus}{\subset} M$. Therefore any closed of dimension $n-1$ in M is a direct summand of M. Now let X be a closed submodule of M of dimension less than $n-1$. It is easy to see that there exists a closed submodule X' of M of dimension $n-1$ containing X. It follows that X' is a direct summand of M. Since the $\text{end}(M_i)$ are local hence the decomposition $\overset{n}{\underset{i=1}{\oplus}} M_i$ complements direct summands, it follows that X' is isomorphic to $\underset{i \in F}{\oplus} M_i$ for some subset F of $\{1,2,\ldots,n\}$ with $|F| = n-1$. By induction $\underset{i \in F}{\oplus} M_i$ has (C_1), and hence X' has (C_1). Since X is closed in M hence closed in X', it follows that, by (C_1) for X', $X \overset{\oplus}{\subset} X'$, and $X \overset{\oplus}{\subset} M$. Therefore M has (C_1). □

<u>Proposition 3.21</u>: Let $M = \overset{n}{\underset{i=1}{\oplus}} M_i$ be an R-module, where the M_i are uniform with local endomorphism rings. Then M has (C_1) if and only if $M_i \oplus M_j$ has (C_1) for all $i \neq j$.

<u>Proof</u>: Let $M_i \oplus M_j$ have (C_1) for all $i \neq j$. Let A be a uniform closed submodule of M. By (3.8), we have

$A = \{\sum_{i=1}^{n} \varphi_i(x) : x \in X \subset' M_k$ for some M_k and some submodule X of $M_k\}$, where $\varphi_i : X \to M_i$ are homomorphisms such that $\varphi_k(x) = x$ for all $x \in X$; and φ_i are not simultaneously extendable.

Without loss of generality assume that $A = \{x + \varphi_2(x) + \ldots + \varphi_n(x) : x \in X \subset' M_1\}$, where the $\varphi_i : X \to M_i$ are not simultaneously extendable, $i = 2, \ldots, n$. Either $X = M_1$; then $M = A \oplus \oplus_{i=2}^{n} M_i$ or $X \subsetneq M_1$. Let $\hat{\varphi}_i : E(M_1) \to E(M_i)$ be an extension of φ_i to the injective hulls, i.e. $\hat{\varphi}_i|_X = \varphi_i$. Let $X_i =: \{m_1 \in M_1 : \hat{\varphi}_i(m_1) \in M_i\}$; $i = 2, 3, \ldots n$. Since φ_i are not simultaneously extendable, we have $X = \bigcap_{i=2}^{n} X_i$. Let $F = \{i : X_i \subsetneq M_1\}$ (F is non-empty due to $X \subsetneq M_1$). It follows that $X_i^* =: \{x_i + \hat{\varphi}_i(x_i) : x_i \in X_i\}$ is a uniform closed submodule of $M_1 \oplus M_i$. If $\hat{\varphi}_i$ is not an isomorphism, then by (C_1) and since the end(M_i) are local, $X_i^* \oplus M_i = M_1 \oplus M_i$. By the same argument as in (3.20), we have that $X_i = M_1$. It follows that $X = \bigcap_{i \in F} X_i$, and that $\hat{\varphi}_i$ is an isomorphism for every $i \in F$. Then $E(M_1) \cong E(M_i)$ for all $i \in F$. Since $X_i^* \oplus M_1 = M_1 \oplus M_i$ for $i \in F$, it follows that $\hat{\varphi}_i(X_i) = M_i$ and hence $\hat{\varphi}_i^{-1}(M_i) = X_i \subset M_1$. Since $M_i \oplus M_j$ has (C_1) for all $i \neq j \in F$, by (3.16), (3.17), it follows that $\hat{\varphi}_i^{-1}(M_i)$ and $\hat{\varphi}_j(M_j)$ are comparable. Hence $\{X_i\}_{i \in F}$ forms a chain of submodules of M_1. Let X_k be the smallest element of that chain; it follows that $X = X_k$, and hence $\hat{\varphi}_{k|X} = \varphi_k : X \to M_k$ is an isomorphism. Then $A \oplus \underset{i \neq k}{\oplus} M_i = M$.

We have shown that any uniform closed submodule of M is a direct summand, i.e. M has $(1-C_1)$. Therefore, by (3.20), M has (C_1).

The converse is obvious. □

§4. DEDEKIND DOMAINS.

In this section we characterize all torsion modules over Dedekind domains which have property (C_1).

Lemma 3.22: Let M be a module over a Dedekind domain R such $P^n M = o$, and $P^{n-1} M \neq o$ for some maximal ideal P of R and positive integer n. Then $M = \underset{i \in I}{\oplus} M_i$, where the M_i are cyclic submodules, more precisely, $M_i \cong R/_{P}m$, $m \leq n$.

Proof: Since $P^n M = o$, it follows that M is R/P^n-module. Since R/P^n is Noetherian, it follows that M possesses a maximal R/P^n-injective submodule N (see [13], prop. (1.2)); i.e. $M = N \oplus N_1$. Since R/P^n is injective and since N is a maximal R/P^n-injective submodule of M, it follows that $P^{n-1} N_1 = o$. Since every injective R/P^n-module has a decomposition as a direct sum of indecomposable injective submodules, we have that $N = \underset{i \in I_o}{\oplus} M_i$, where the $M_i \cong R/P^n$, and hence $M = \underset{i \in I_o}{\oplus} M_i \oplus N_1$.

Let $n_1 \leq n-1$ such that $P^{n_1} N_1 = o$ and $P^{n_1 - 1} N_1 \neq o$. Consider N_1 as R/P^{n_1}-module, by the same argument as above, we get that

$N_1 = N_1' \oplus N_2$, where $N_1' = \underset{i \in I_1}{\oplus} M_{i_1}$, $M_i \cong R/P^{n_1}$ for all $i \in I_1$, and $P^{n_1 - 1} N_2 = o$. Hence $M = \underset{i \in I_o}{\oplus} M_i \oplus \underset{i \in I_1}{\oplus} M_i \oplus N_2$.

Continuing in this manner we obtain $M = \underset{i \in I}{\oplus} M_i$, where $M_i \cong R/P^m$ for all $i \in I$ and for some $m \leq n$. □

Lemma 3.23 ([5], (2.6) and (2.7)) : Let R be a Dedekind domain, and M be a torsion R-module. Let $M(P) = \{x \in M : P^n x = o$ for some $n \in \mathbb{N}\}$, where P is a maximal ideal of R. Then:

1) $M = \underset{P}{\oplus} M(P)$, where P ranges over all maximal ideals of R.

2) $M(P) \cong M_P$, hence $M(P)$ is an R_P-module. The set of R_P-submodules of M equals to the set of R-submodules of M. □

Note that any uniform torsion R-module, where R is a Dedekind domain, has local endomorphism ring.

Proposition 3.24: Let $M = M_1 \oplus M_2$ be a module over a Dedekind domain R, where the M_i are uniform and $\text{ass}(M_1) \neq \text{ass}(M_2)$. Then M is quasi-continuous.

Proof: Is clear since $\underset{R}{\text{hom}}(E(M_i), E(M_j)) = o$ for $i \neq j$ $(=1,2)$. □

Proposition 3.25: Let $M = M_1 \oplus M_2$ be a module over a Dedekind domain R, where the M_i are uniform and $\text{ass}(M_1) = \text{ass}(M_2) = P$. Then M has (C_1) if and only if either $M \cong E(R/P) \oplus E(R/P)$ or $M \cong R/P^n \oplus R/P^m$ with $|n - m| \leq 1$, for some $n, m \in \mathbb{N}$.

Proof: Let M have (C_1). By (3.15), we have that M_i can be embedded in M_j for some $i \neq j$. Without loss of generality assume that $M_1 \subset M_2$.

Case 1: M_2 can be embedded in M_1. It follows that $M_1 = M_2$ and hence either $M_1 = M_2 \cong E(R/P)$ or $M_1 = M_2 \cong R/P^n$ for some $n \in \mathbb{N}$.

Case 2: M_2 can not be embedded in M_1, i.e. $M_1 \subsetneq M_2$. By (C_1) and since M_i are R_P-modules, it follows, by (3.16), that $PM_2 \subset M_1 \subset M_2$. It is clear, in this case, that $M_i \not\cong E(R/P)$ $(i = 1,2)$. Since R is Dedekind, we have that R_P is a rank one discrete valuation ring; and hence M_2/PM_2 is a simple R_P-module. Since $M_1 \subsetneq M_2$, it follows that $PM_2 = M_1$.

Now if $M_1 \cong R/P^n$ for some $n \in \mathbb{N}$, then $M_2 \cong R/P^{n+1}$. Therefore $M \cong R/P^n \oplus R/P^{n+1}$.

Conversely, let $M \cong R/P^n \oplus R/P^m$ with $|n-m| \leq 1$, $n, m \in \mathbb{N}$.

Case a: $n = m$, i.e. $M \cong R/P^n \oplus R/P^n$. It is clear that M is R/P^n-injective. Then M has (C_1) as R/P^n-module, and hence M has (C_1) as R-module.

Case b: $m = n+1$, i.e. $M \cong R/P^n \oplus R/P^{n+1}$. By (3.16), we have that M has (C_1). □

Theorem 3.26: Let M be a torsion module over a Dedekind domain R. Then M has (C_1) if and only if either $M(P)$ is injective or

$M(P) = \bigoplus_{i \in I} M_i(P)$, where the $M_i(P) \cong R/P^n$ or R/P^{n+1} for all $i \in I$.

<u>Proof</u>: Let M have (C_1), and let $M(P)$ not be injective for some maximal ideal P of R. Let $o \neq x \in M(P)$. Since R is Dedekind hence Noetherian, it follows that xR is finite dimensional. Let $(xR)^* \supset' xR$ be a maximal essential extension of xR in $M(P)$. By (C_1), $(xR)^*$ is a direct summand of $M(P)$; i.e. $M(P) = (xR)^* \oplus N(P)$. By essentiality of $(xR)^*$ over xR, we have that $(xR)^*$ is finite dimensional. Since (C_1) is inherited by direct summands, it follows that $(xR)^*$ has (C_1); and hence $(xR)^* = \bigoplus_{i=1}^{m} U_i$, where the U_i are uniform. By (3.25), we have that $U_i \cong R/P^n$ or R/P^{n+1}, $i = 1,2,\ldots,m$. Then $P^{n+1} U_i = o$ for all i.

Now let $o \neq y \in M(P)$ be arbitrary. Hence $y = y_1 + y_2$, where $y_1 \in (xR)^*$, $y_2 \in N(P)$. By the same argument we can show that

$y_2 R \subset' (y_2 R)^* = \bigoplus_{j=1}^{s} V_j \overset{\oplus}{\subset} N(P)$. It follows that $\bigoplus_{i=1}^{m} U_i \oplus \bigoplus_{j=1}^{s} V_j$ has

(C_1), and hence, by (3.25), $V_j \cong R/P^n$ or R/P^{n+1}. Then $P^{n+1} y = o$, and therefore $P^{n+1} M(P) = o$.

By (3.22) and (3.25), we have that $M(P) = \bigoplus_{i \in I} M_i(P)$, where the $M_i(P) \cong R/P^n$ or R/P^{n+1} for all $i \in I$.

Conversely, let A be a closed submodule of M. Then $A = \bigoplus_P A(P)$, where P runs over all maximal ideals of R. To show that M has (C_1), it is enough to show that $A(P)$ is a direct summand of $M(P)$ for every P.

Since $A(P)$ is a direct summand of the closed submodule A, we have $A(P)$ is a closed submodule of $M(P)$. Either $M(P)$ is injective; hence $A(P) \overset{\oplus}{\subset} M(P)$ or $M(P) = \underset{i \in I}{\oplus} M_i(P)$, where the $M_i(P) \cong R/P^n$ or R/P^{n+1}; hence $P^{n+1} M(P) = o$. It follows that $P^{n+1} A(P) = o$, and hence, by (3.22), $A(P) = \underset{j \in J}{\oplus} A_j(P)$ where $A_j(P)$ are cyclic R-modules.

Let F be an arbitrary finite subset of J, we have that $\underset{j \in F}{\oplus} A_j(P) \subset \underset{i \in F_1}{\oplus} M_i(P)$ where F_1 is a finite subset of I. By (2.25), we have that $M_i(P) \oplus M_j(P)$ has (C_1) for all $i,j \in F_1$; and hence, by (3.21), $\underset{i \in F_1}{\oplus} M_i(P)$ has (C_1). Since $\underset{j \in F}{\oplus} A_j(P)$ is closed in $\underset{i \in F_1}{\oplus} M_i(P)$, we have, by (C_1), that $\underset{j \in F}{\oplus} A_j(P) \overset{\oplus}{\subset} \underset{i \in F_1}{\oplus} M_i(P) \overset{\oplus}{\subset} M(P)$.

We have just shown that every finite subsum of $\underset{j \in J}{\oplus} A_j(P)$ is a direct summand of $M(P) = \underset{i \in I}{\oplus} M_i(P)$; i.e. $\underset{j \in J}{\oplus} A_j(P)$ is a local direct summand of $M(P)$.

To show that $A(P) = \underset{j \in J}{\oplus} A_j(P)$ is a direct summand of $M(P) = \underset{i \in I}{\oplus} M_i(P)$, it remains to show, for applying lemma (3.5), that the decomposition $\underset{i \in I}{\oplus} M_i(P)$ is locally semi-T-nilpotent; i.e. for every sequence $f_m: M_{i_m}(P) \to M_{i_{m+1}}(P)$ $(m \in \mathbb{N})$ of non-isomorphisms, with all i_m distinct, and every $x \in M_{i_o}(P)$, there exists $m_o \in \mathbb{N}$ with $f_{m_o} \ldots f_1 f_o(x) = o$.

Since the $M_i(P) \cong R/P^n$ or R/P^{n+1} for every $i \in I$, and since non-monomorphisms reduce the lengths of all submodules of $M_i(P)$,s, it follows that $f_{2n} \dots f_1 f_o = o$ for every sequence $f_m : M_{i_m}(P) \to M_{i_{m+1}}$ of non-isomorphisms. Therefore $\underset{i \in I}{\oplus} M_i(P)$ is locally semi-T-nilpotent. Which completes the proof. □

The following theorem is an immediate consequence of (1.15), (1.16), (2.49) and (3.26).

<u>Theorem 3.27</u>: Let M be a module over a Dedekind domain R. Then M has (C_1) if and only if either

i) M is torsion and for every maximal ideal P of R, either $M(P)$ is injective or $M(P) = \underset{i \in I}{\oplus} M_i(P)$ where $M_i(P) \cong R/P^n$ or R/P^{n+1} $(n \in \mathbb{N})$ for all $i \in I$; or

ii) M is non-torsion and $M = F \oplus E$, where E is an injective submodule and F is torsion free reduced, $F \cong \overset{n}{\underset{i=1}{\oplus}} I_i N$ where N is a proper R-submodule of the quotient field K and the I_i are fractional ideals of R. □

REFERENCES

[1] F.W. Anderson and K.R. Fuller, Rings and category of modules, Springer Verlag : New York (1973).

[2] G. Azumaya, M-projective and M-injective modules, (1974), (unpublished).

[3] G. Azumaya, F. Mbuntum and K. Varadarajan, On M-projective and M-injective Modules, Pacific J. Math. V. 59, Vol. (1975), 9-16.

[4] R. Baer, Abelian groups that are direct summands of every containing abelian group, Bull. Amer. Math. Soc. 46 (1940), 800-806.

[5] W. Brandal, Commutative rings whose finitely generated modules decompose, Springer Lecture Notes in Math. 723 (1979).

[6] L. Fuchs, Abelian groups, International Series of Monographs in Pure and App. Math., Pergamon Press, New York, Vol. 12 (1960).

[7] B. Eckmann and A. Schopf, Über injective moduln, Arch der Math., 4 (1953), 75-78.

[8] M. Harada, Application of factor categories to completely decomposable modules, Publ. Dep. Math. Lyon 11 (2) (1974), 19-104.

[9] L. Jeremy, Modules et anneaux quasi-continuous, Canad. Math. Bull. 17 (2), (1974), 217-228.

[10] R.E. Johnson and E.T. Wong, Quasi-injective modules and irreducible rings, J. London Math. Soc. 36 (1961), 260-268.

[11] I. Kaplansky, Commutative rings, The Univ. of Chicago Press, Chicago, (1974).

[12] M.D. Larsen and P.J. McCarthy, Multiplicative theory of ideals, Acad. Press : New York, (1971)

[13] E. Matlis, Injective modules over Noetherian rings, Pacific J. Math. 8 (1958), 511-528.

[14] B.J. Müller and S. Tariq Rizvi, On injective and quasi-continuous modules, J. Pure Applied Algebra 28 (1983), 197-210.

[15] H. Nagao and H. Nakayama, On the structure of M_o- and M_u-modules, Math. Zeit. 59 (1953), 164-170.

[16] B. Strenström, Rings of Quotients, Springer Verlag : New York (1975).

[17] C. Tasi, Report on injective modules, Queen's Paper in Pure and Applied Math., No. 6 (1965).

[18] Y. Utumi, On continuous rings and self-injective rings, Trans. Amer. Math. Soc. 118 (1965), 158-173.

[19] J. Von Neuman, Continuous Geometry, Princeton Univ. Press (1960).

[20] O. Zariski and P. Samuel, Commutative Algebra, Springer Verlag Vol. 2 : New York (1960).

[21] A.W. Chatter and C.R. Hajarnavis, Rings in which every complement right ideal is a direct summand, Quart J. Math. Oxford (2), 28 (1977), 61-80.

[22] L. Fuchs, A. Kertesz and T. Szele, Abelian groups in which every serving subgroup is a direct summand, Publ. Math. Decrecen, 3 (1953), 95-105.

[23] M. Harada and K. Oshiro, On extending property on direct sums of uniform modules, Osaka J. Math., 18 (1981), 767-785.

[24] M. Harada, On modules with extending property, Osaka J. Math. 19 (1982), 203-215.